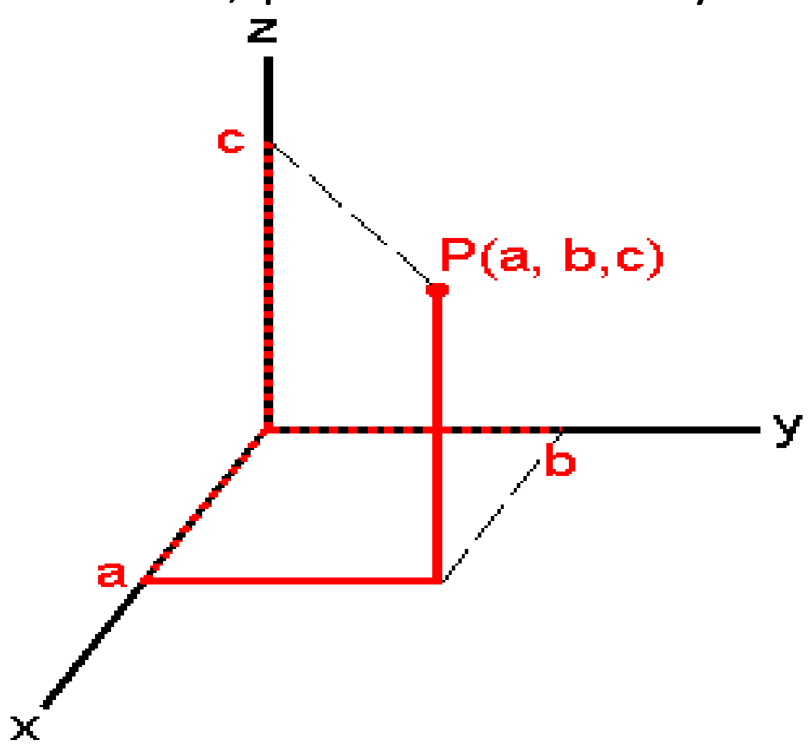
3D Geometric Transformation

Unit 4

Nipun Thapa-Computer Graphics

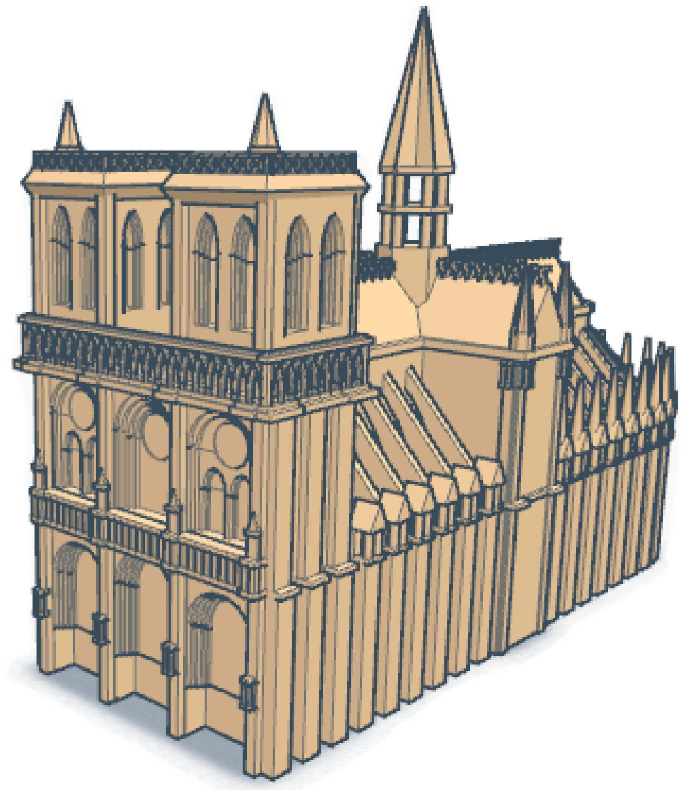
What is 3-Dimension?

- Three-dimensional space is a geometric 3-parameters model of the physical universe (without considering time) in which all known matter exists. These three dimensions can be labeled by a combination of length, breadth, and depth. Any three directions can be chosen, provided that they do not all lie in the same plane.



What is 3 Dimensional Object?

- An object that has height, width and depth, like any object in the real world is a 3 dimensional object.
- Types of objects: Geometrical shapes, trees, terrains, clouds, rocks, glass, hair, furniture, human body, etc.



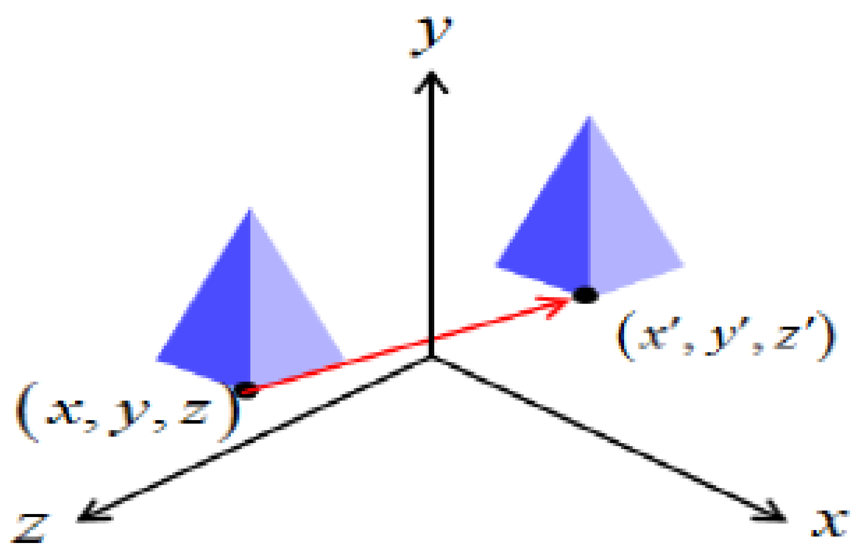
3D Transformations

- Just as 2D-transfromtion can be represented by 3x3 matrices using homogeneous co-ordinate can be represented by 4x4 matrices, provided we use homogenous co-ordinate representation of points in 3D space as well.
  1. Translation
  2. Rotation
  3. Scaling
  4. Reflection
  5. Shear

1.Translation

- Translation in 3D is similar to translation in the 2D except that there is one more direction parallel to the z-axis. If, tx, ty, and tz are used to represent the translation vectors. Then the translation of the position P(x, y, z) into the point P' (x', y', z') is done by
  - $x' = x + tx$
  - $y' = y + ty$
  - $z' = z + tz$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



$$P' = T.P$$

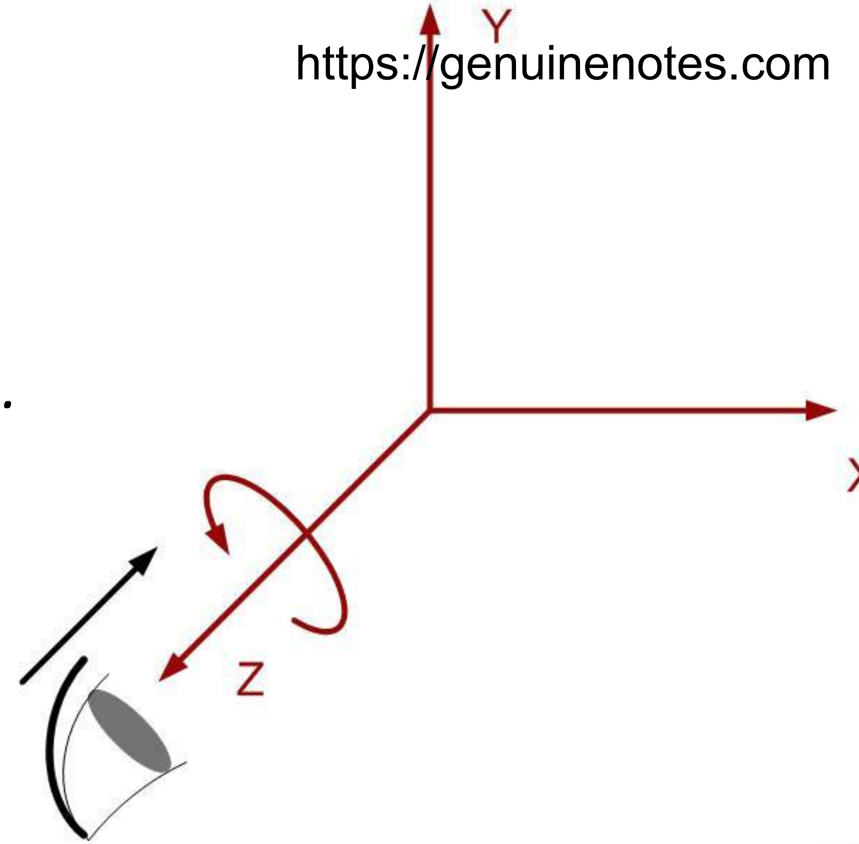
- In matrix notation using homogeneous coordinate this is performed by the matrix multiplication,

2.Rotation

i) <u>Rotation About z-axis:</u>

Z-component does not change.

$X' = X\cos\theta - Y\sin\theta$
$Y' = X\sin\theta + Y\cos\theta$
$Z' = Z$



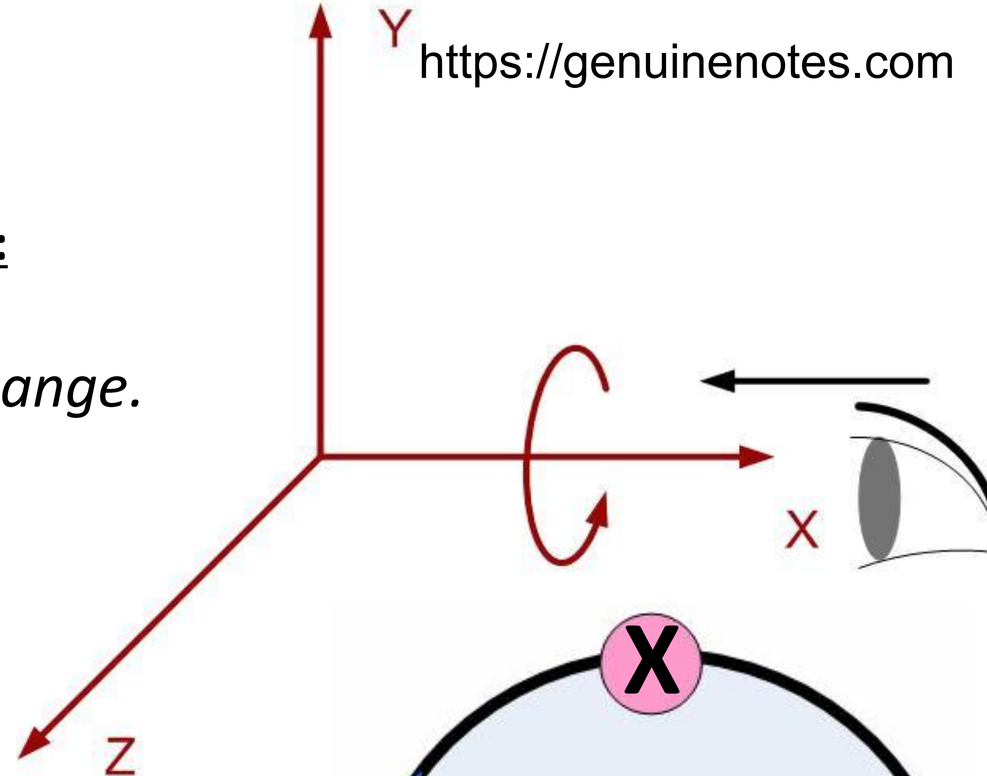
Matrix representation for rotation around z-axis,

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

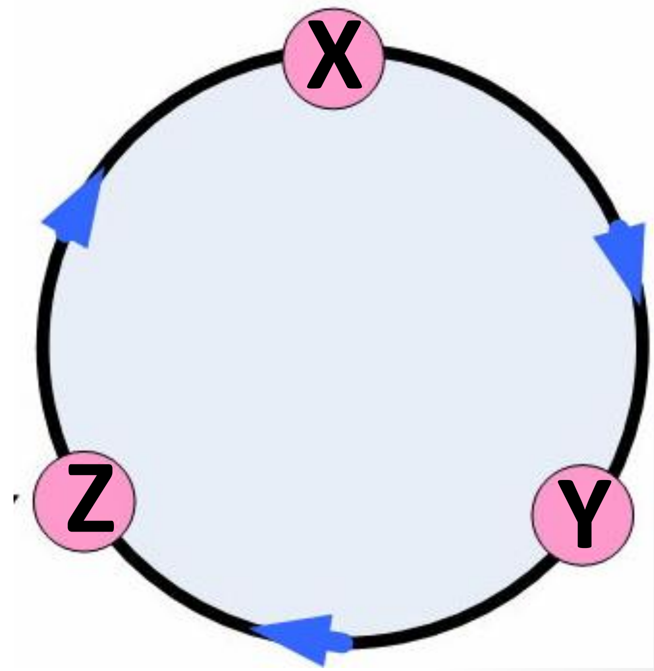
2.Rotation

ii) Rotation About x-axis:

X-component does not change.



Y' = Y cosθ - Z sinθ
Z' = Y sinθ + Z cosθ
X' = X

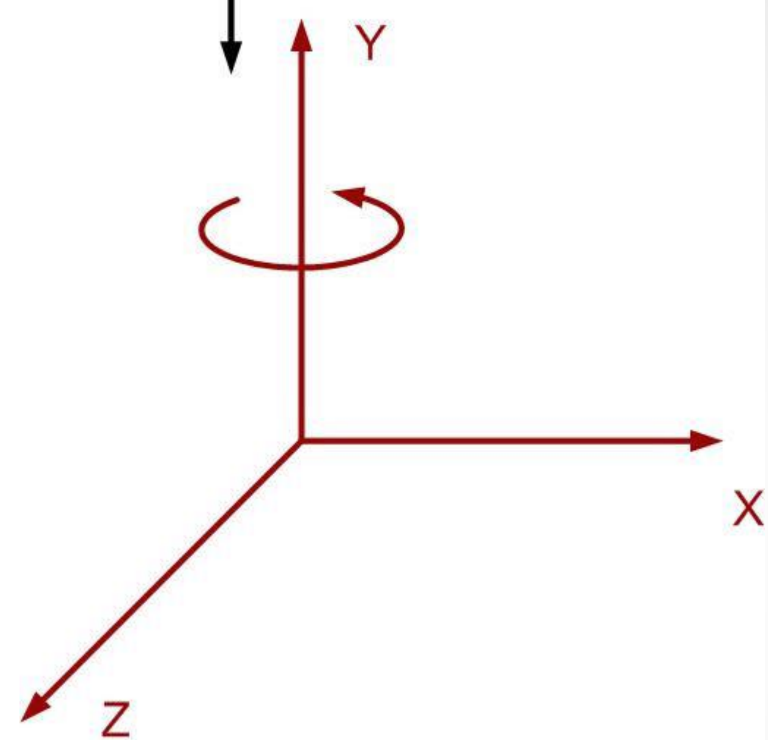


Matrix representation for rotation around X-axis,

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

2.Rotation

iii) Rotation About Y-axis:



Y-component does not change.

$Z' = Z\cos\theta - X\sin\theta$
$X' = Z\sin\theta + X\cos\theta$
$Y' = Y$

Matrix representation for rotation around Y-axis,

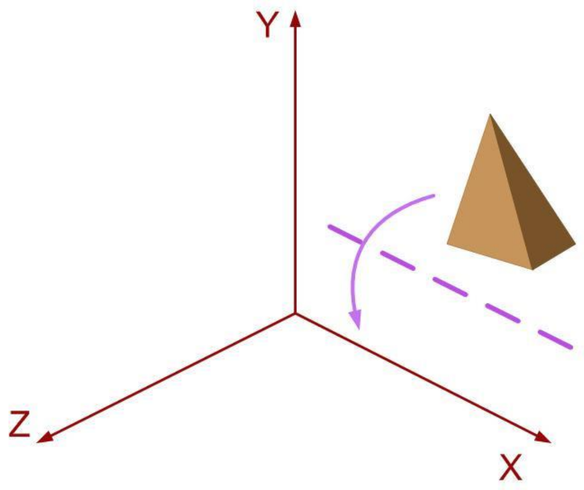
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

2.Rotation

- **General 3D Rotations:**

  (a) Rotation about an axis parallel to any of the co-axis: *When an object is to be rotated about an axis that is parallel to one of the co-ordinate axis, we need to perform series of transformation.*

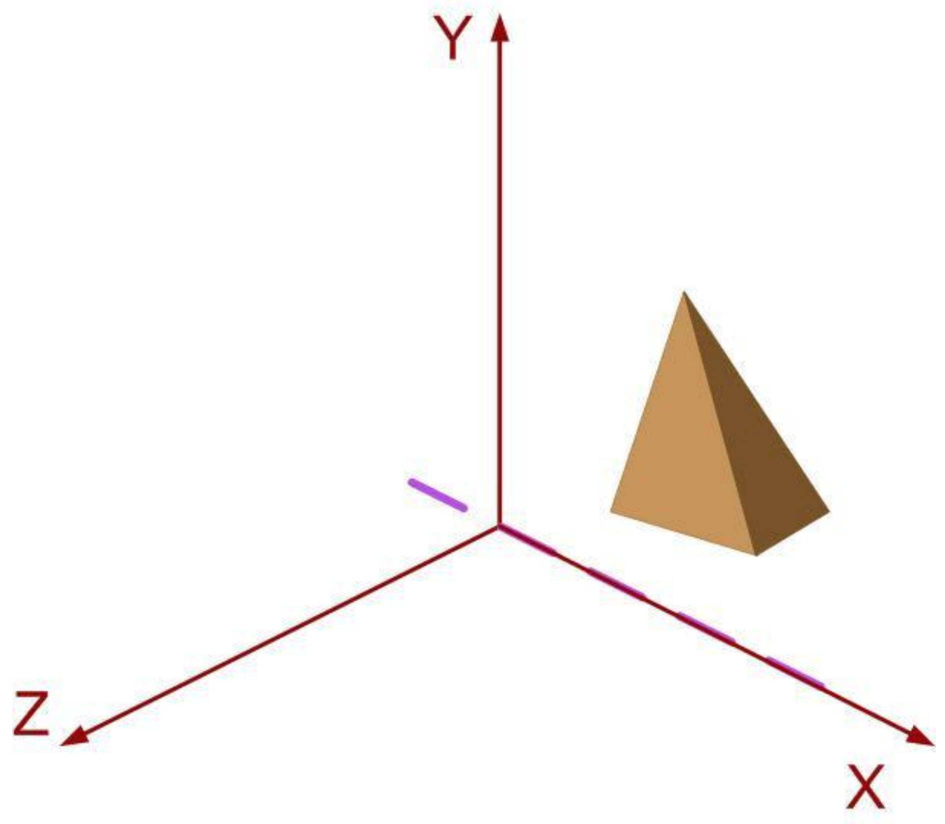
  i. Translate the object so that the rotation axis coincides with the parallel co-ordinate axis.
  ii. Perform the specified rotation about the axis.
  iii. Translate the object so that the rotation axis is moved to its original position.



a) Rotation about an axis parallel to any of the co-axis:

Step 1

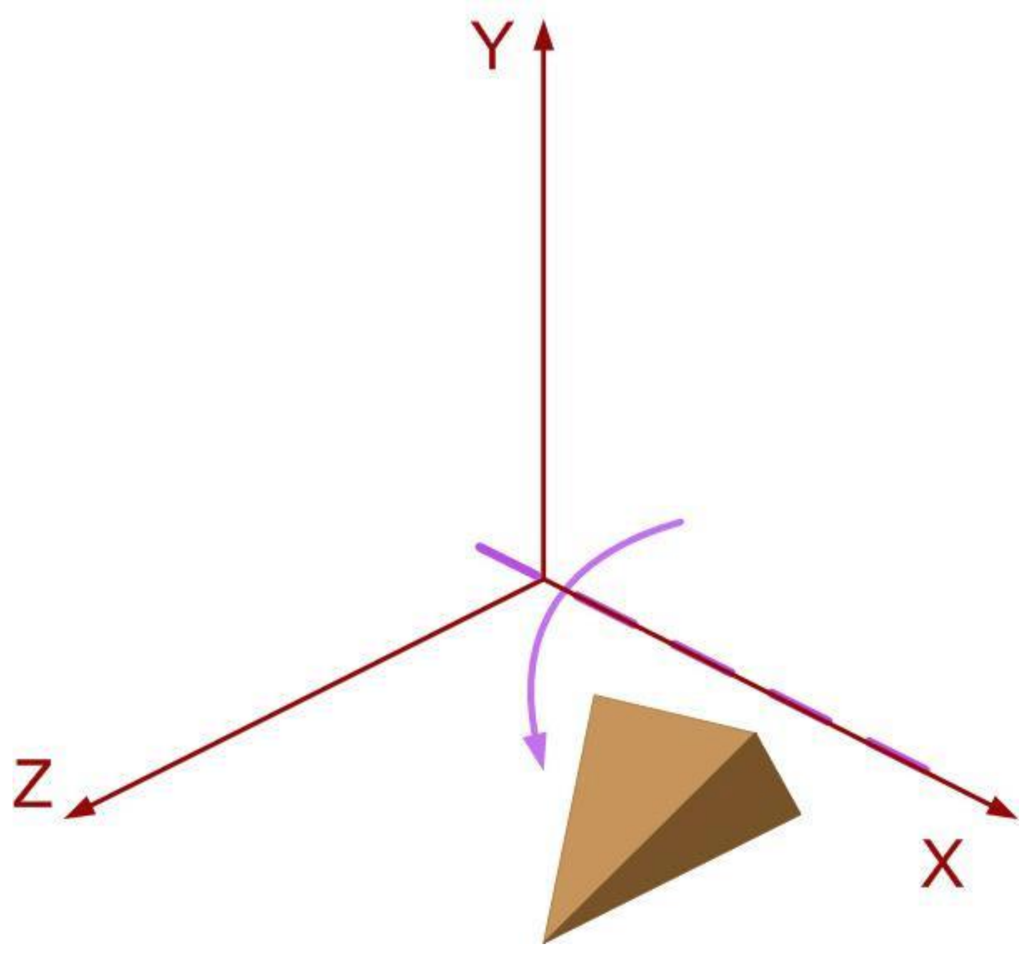
- Translate the object so that the rotation axis coincides with the parallel co-ordinate axis.



a) Rotation about an axis parallel to any of the co-axis:

Step 2

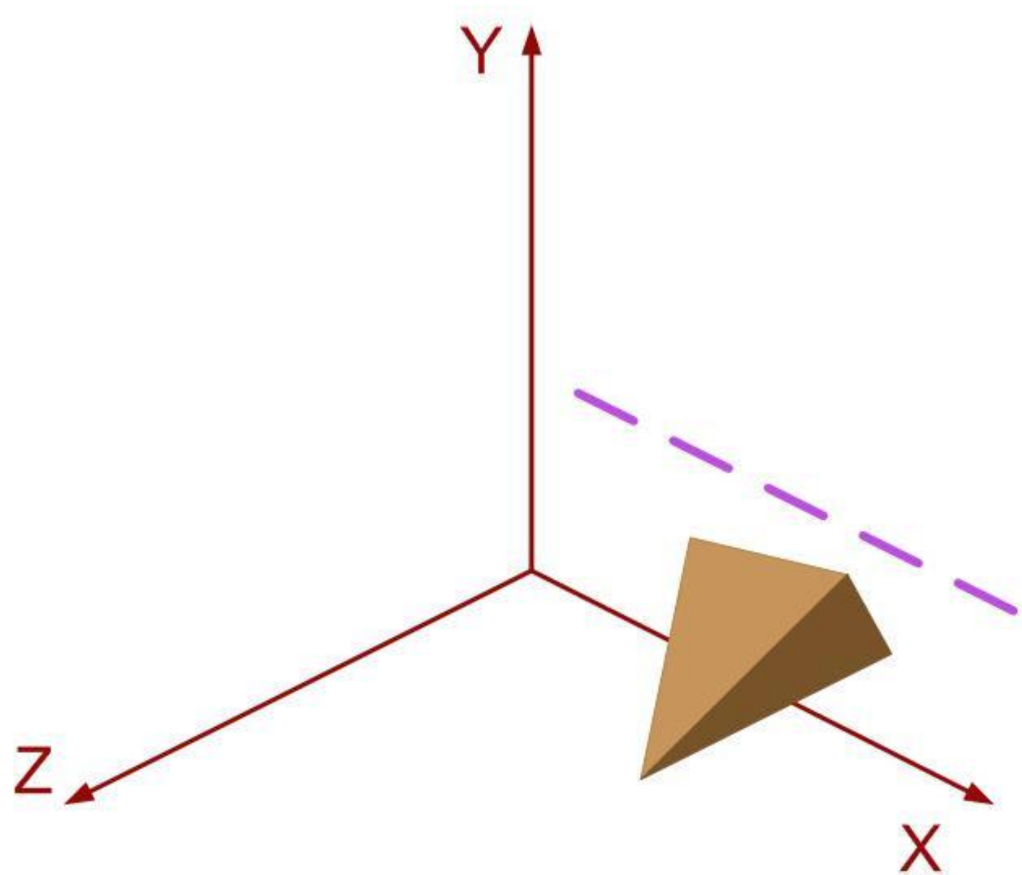
- Perform the specified rotation about the axis.



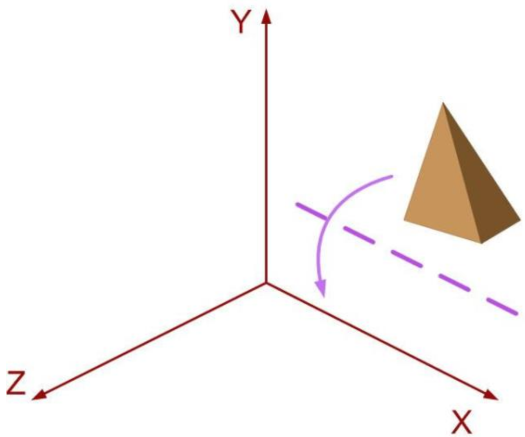
a) Rotation about an axis parallel to any of the co-axis:

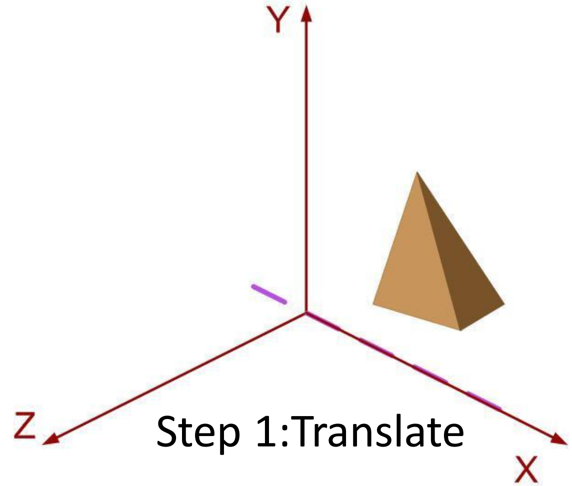
Step 3

- Translate the object so that the rotation axis is moved to its original position.

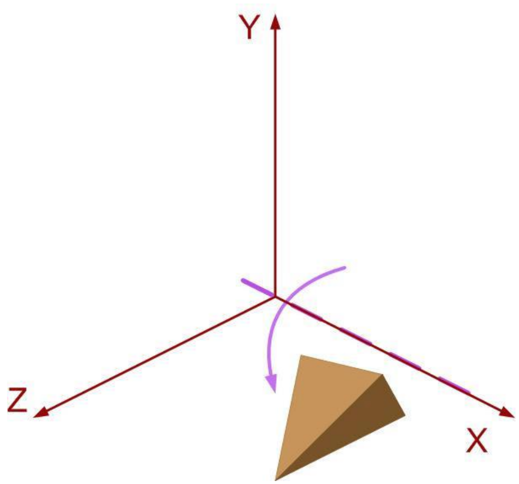


a) Rotation about an axis parallel to any of the co-axis:

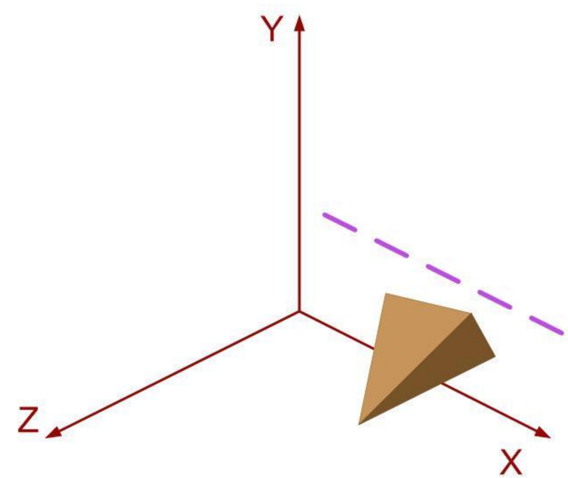




Step 1:Translate



Step 2:Rotation



Step 3:Translate to original place

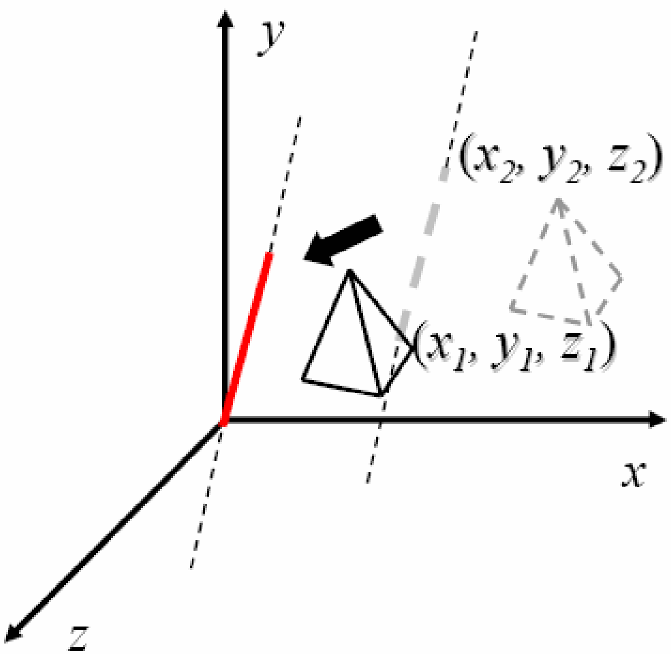
2.Rotation

- **General 3D Rotations:**
  - **(b) Rotation about an axis not parallel to any of the co-axis:**

i. Translate the object such that rotation axis passes through co-ordinate origin.
ii. Rotate the axis such that axis of rotation coincides with one of the co-ordinate axis.
iii. Perform the specific rotation about the ordinate axis.
iv. Apply inverse rotation to bring the rotation axis back to its original orientation.
v. Apply inverse translation to bring the rotation axis back to its original position.

(b) Rotation about an axis not parallel to any of the co-axis:

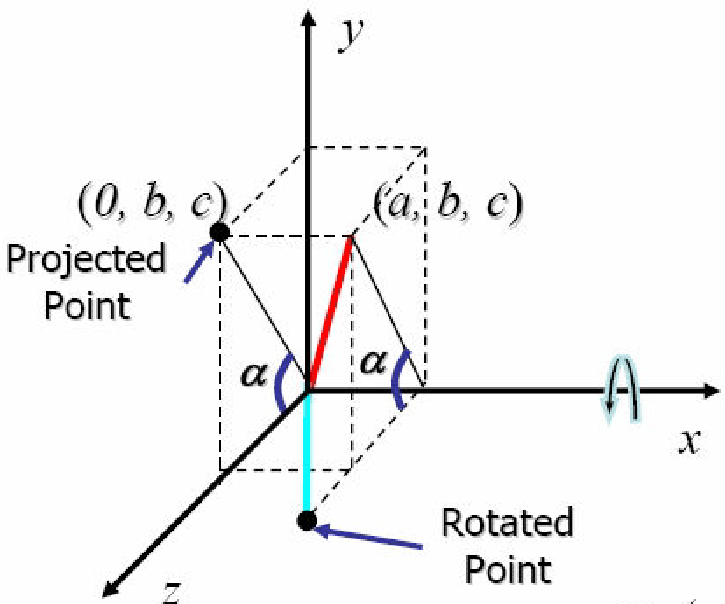
- Step 1. Translate



$$T = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Rotation about an axis not parallel to any of the co-axis:

- Step 2. Rotate about x axis by α



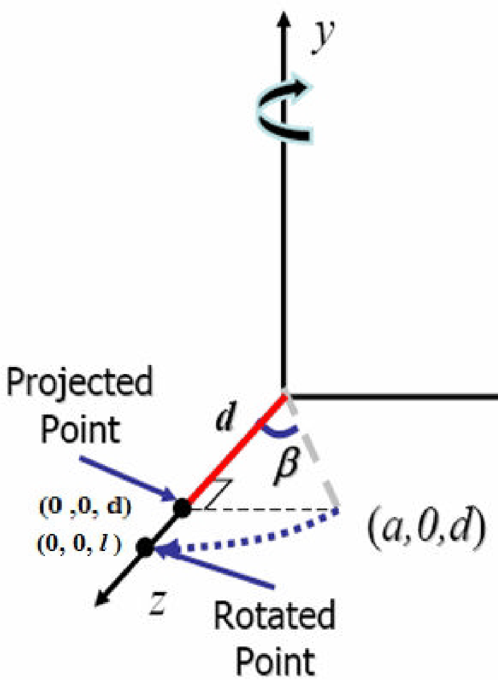
$$\sin\alpha = \frac{b}{\sqrt{b^2+c^2}} = \frac{b}{d}$$

$$\cos\alpha = \frac{c}{\sqrt{b^2+c^2}} = \frac{c}{d}$$

$$\mathbf{R}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Rotation about an axis not parallel to any of the co-axis:

- Step 3. Rotate about y axis by β (clockwise)



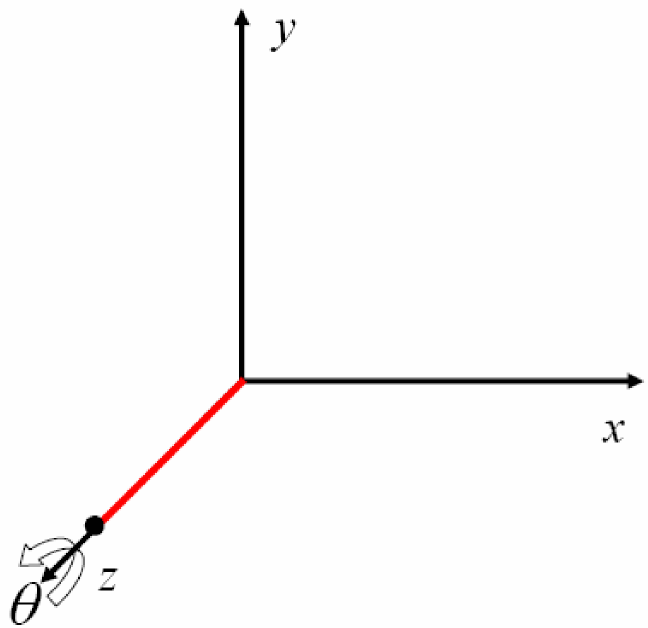
$$\sin\beta = \frac{a}{l}, \qquad \cos\beta = \frac{d}{l}$$

$$l^2 = a^2 + b^2 + c^2 = a^2 + d^2$$

$$\mathbf{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} d/l & 0 & -a/l & 0 \\ 0 & 1 & 0 & 0 \\ a/l & 0 & d/l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Rotation about an axis not parallel to any of the co-axis:

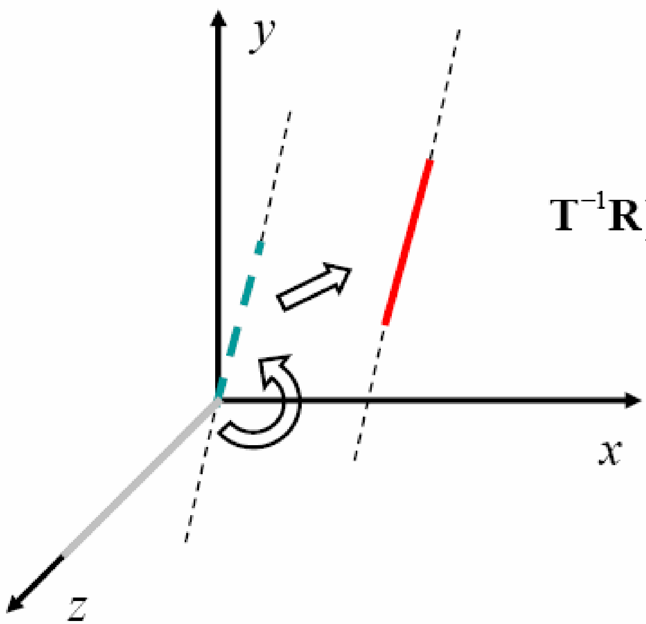
- Step 4. Rotate about z axis by the angle $\boldsymbol{\theta}$



$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Rotation about an axis not parallel to any of the co-axis:

- Step 5. Reverse transformation

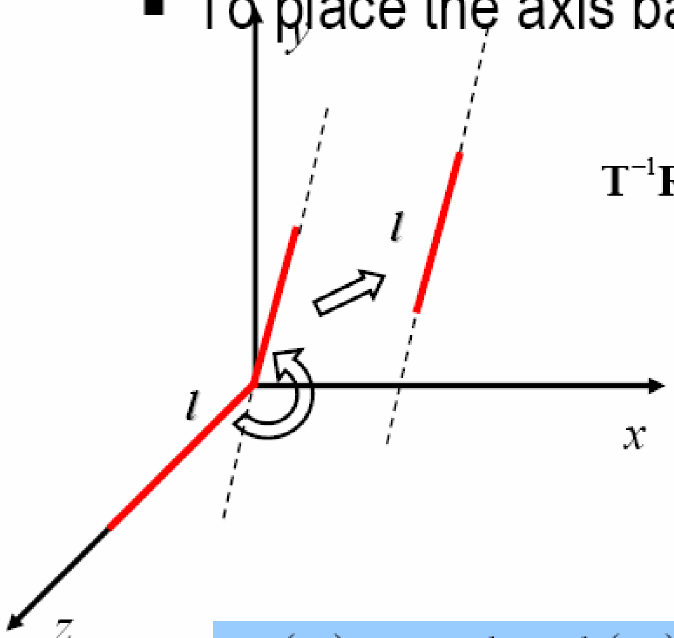


$$\mathbf{T}^{-1}\mathbf{R}_x^{-1}(\alpha)\mathbf{R}_y^{-1}(\beta)=\begin{bmatrix}1 & 0 & 0 & x_1\\ 0 & 1 & 0 & y_1\\ 0 & 0 & 1 & z_1\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\alpha & \sin\alpha & 0\\ 0 & -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\beta & 0 & \sin\beta & 0\\ 0 & 1 & 0 & 0\\ -\sin\beta & 0 & \cos\beta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\mathbf{R}(\theta)=\mathbf{T}^{-1}\mathbf{R}_x^{-1}(\alpha)\mathbf{R}_y^{-1}(\beta)\mathbf{R}_z(\theta)\mathbf{R}_y(\beta)\mathbf{R}_x(\alpha)\mathbf{T}$$

(b) Rotation about an axis not parallel to any of the co-axis:

- Step 5. Reverse transformation
  - To place the axis back in its initial position



$$\mathbf{T}^{-1}\mathbf{R}_x^{-1}(\alpha)\mathbf{R}_y^{-1}(\beta)=\begin{bmatrix}1&0&0&x_1\\0&1&0&y_1\\0&0&1&z_1\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&\cos\alpha&\sin\alpha&0\\0&-\sin\alpha&\cos\alpha&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}\cos\beta&0&\sin\beta&0\\0&1&0&0\\-\sin\beta&0&\cos\beta&0\\0&0&0&1\end{bmatrix}$$

$$\mathbf{R}(\theta)=\mathbf{T}^{-1}\mathbf{R}_x^{-1}(\alpha)\mathbf{R}_y^{-1}(\beta)\mathbf{R}_z(\theta)\mathbf{R}_y(\beta)\mathbf{R}_x(\alpha)\mathbf{T}$$

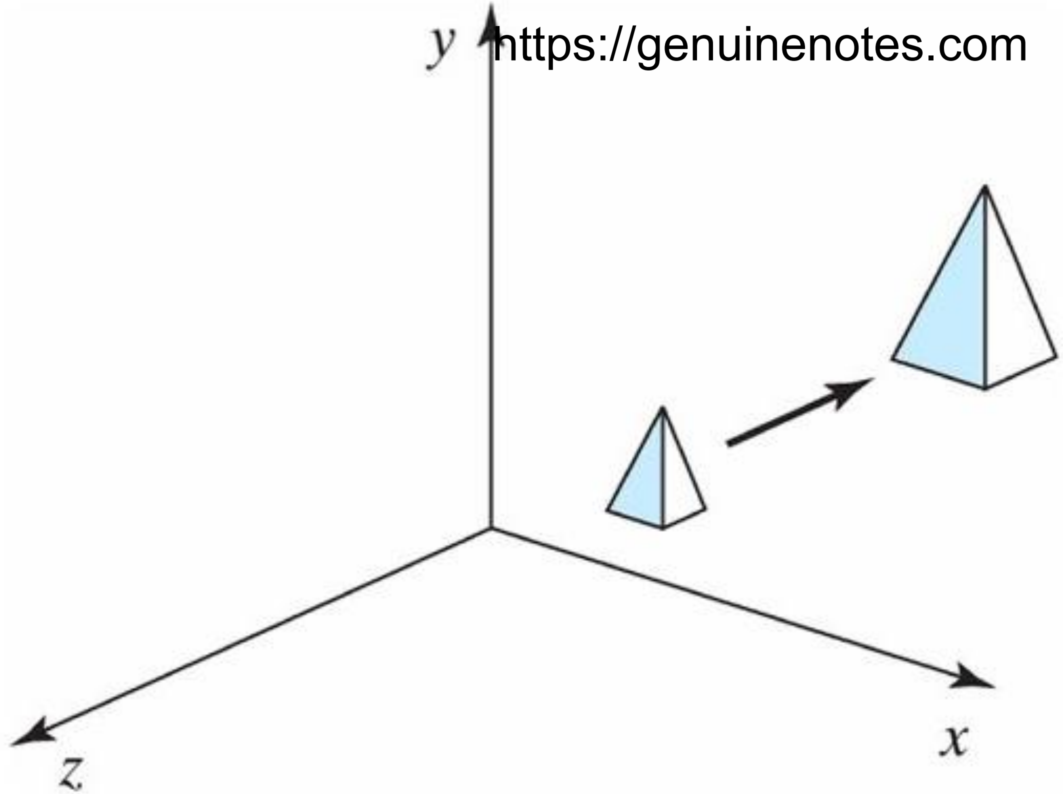
3. Scaling

Scaling about origin

X' = X . Sx

Y' = Y . Sy

Z' = Z . Sz

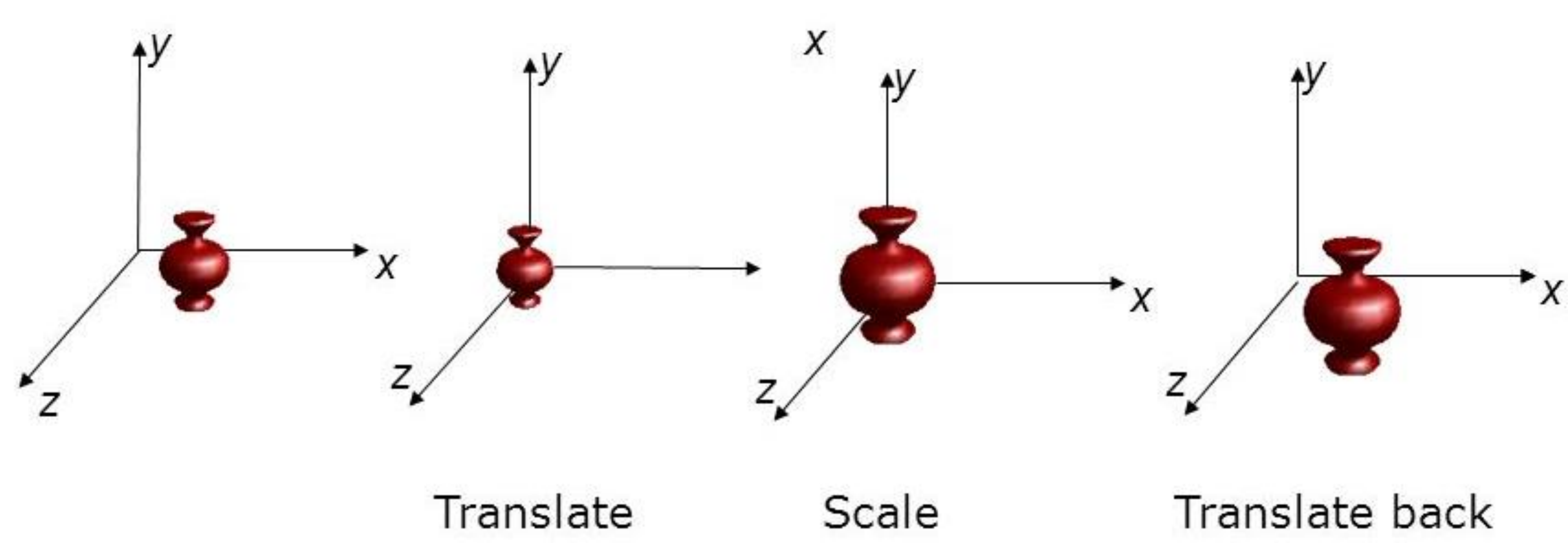


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{S} \cdot \mathbf{P}$$

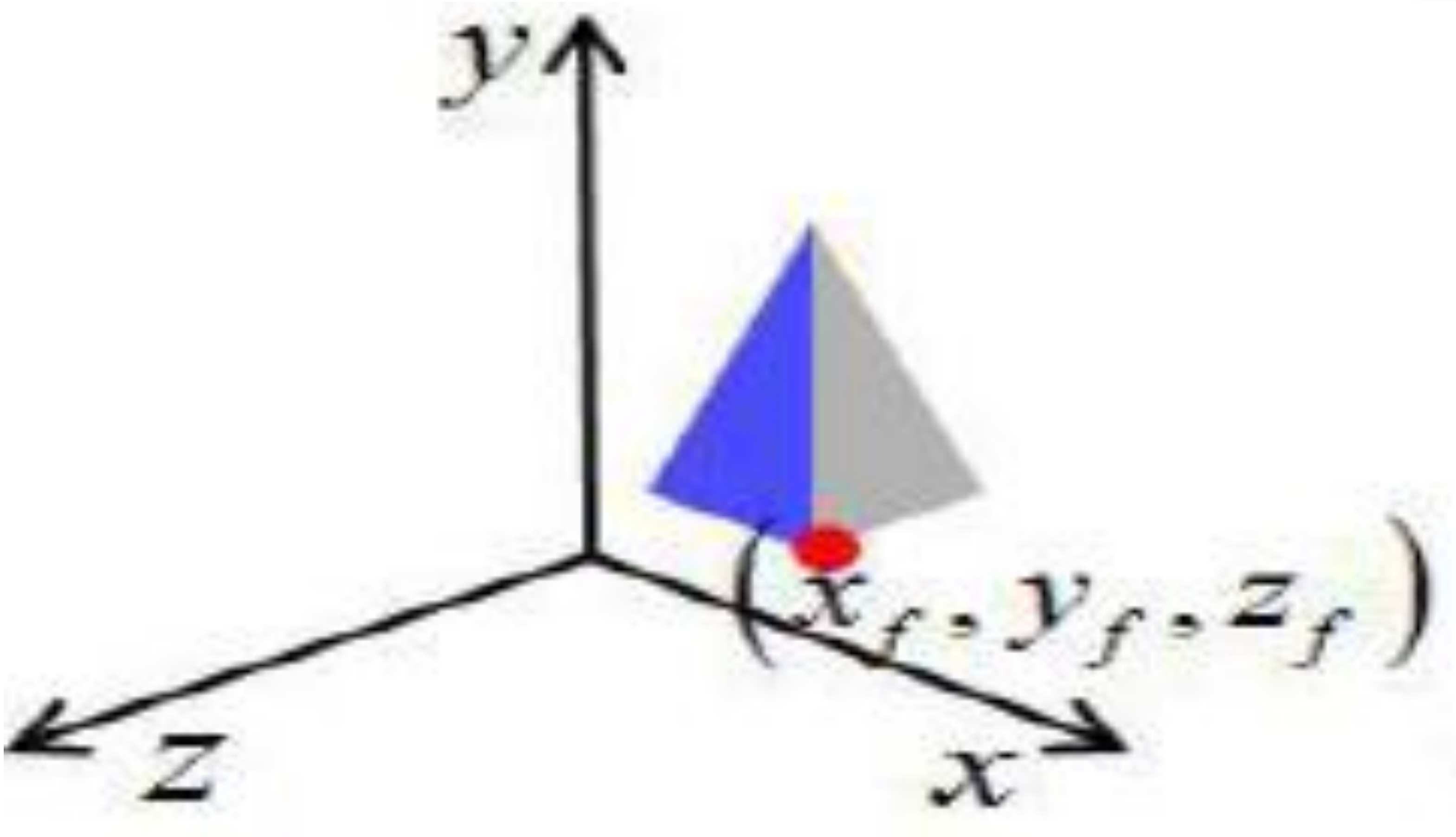
3. Scaling

- ***Scaling about an arbitrary point or Fixed point (xf, yf, zf)***



3. Scaling

- ***Scaling about an arbitrary point or Fixed point (xf, yf, zf)***



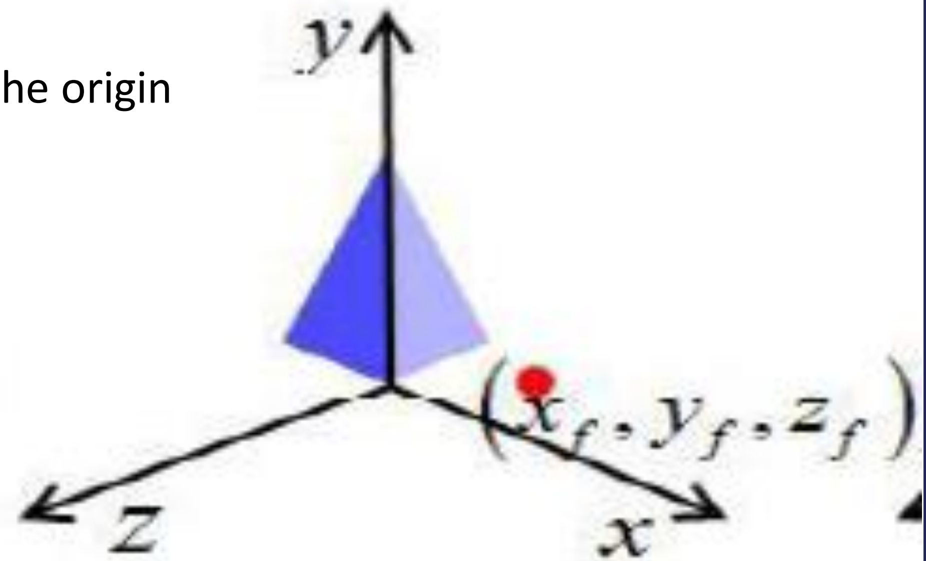
3. Scaling

- ***Scaling about an arbitrary point or Fixed point (xf, yf, zf)***

Step 1

- Translate the fixed point to the origin

$$= \begin{bmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



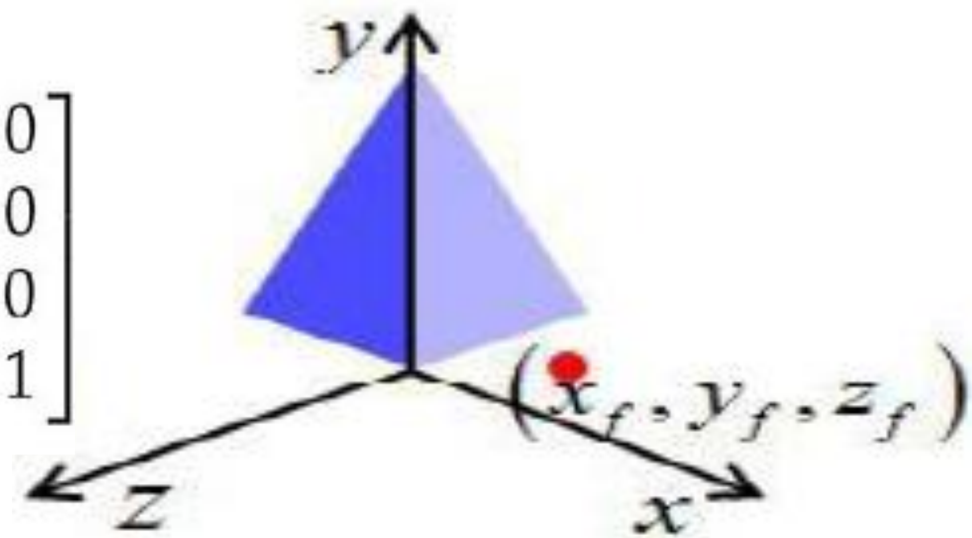
3. Scaling

- ***Scaling about an arbitrary point or Fixed point (xf, yf, zf)***

Step 2

- Scale the object relative to the coordinate origin.

$$= \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



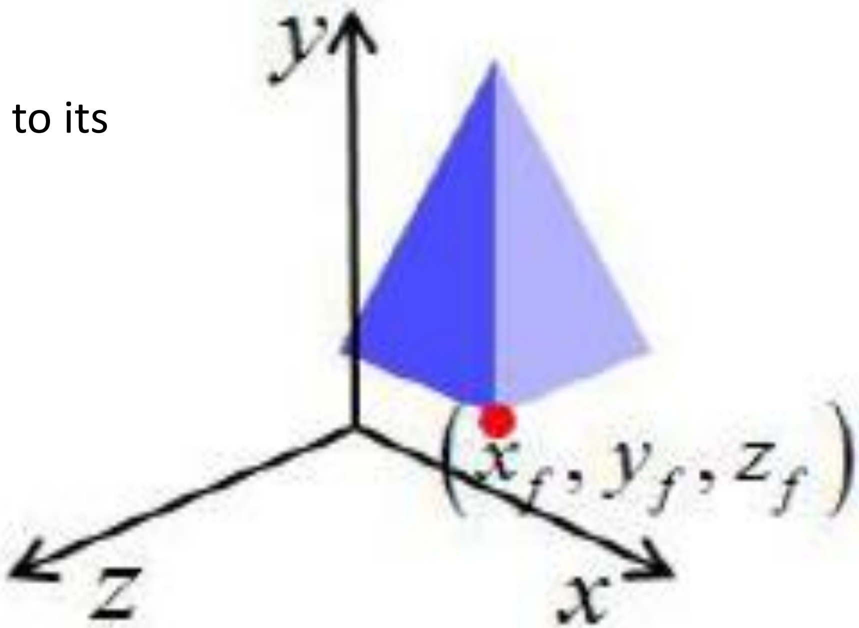
3. Scaling

- ***Scaling about an arbitrary point or Fixed point (xf, yf, zf)***

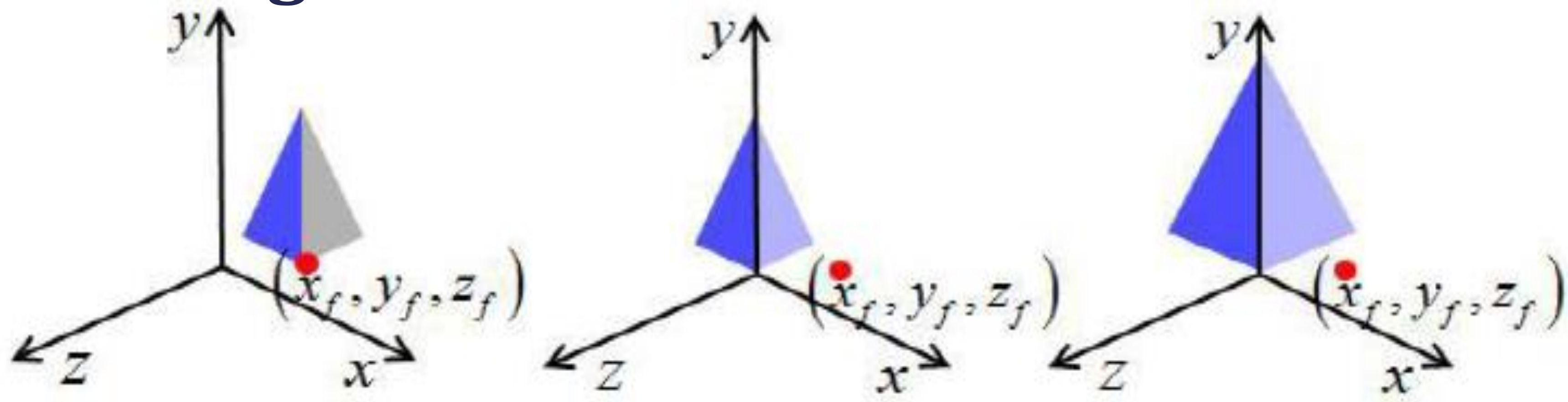
Step 3

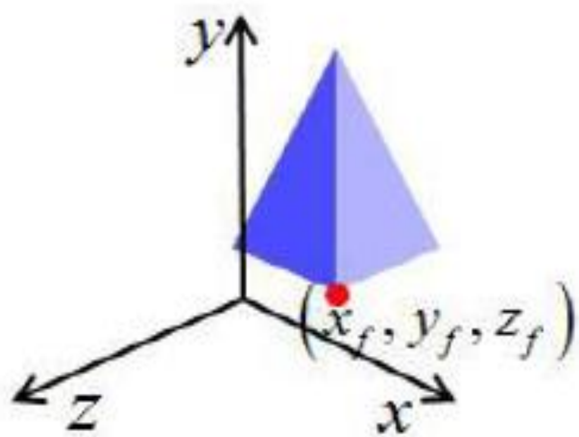
- Translate the fixed point back to its original position.

$$= \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



3. Scaling





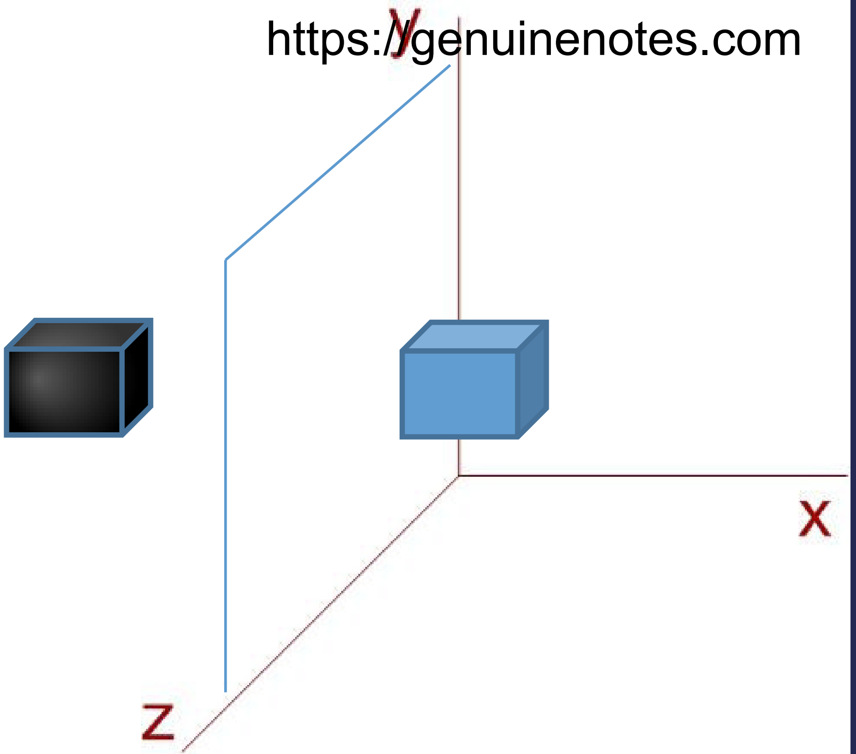
$$\mathbf{T} \cdot \mathbf{S} \cdot \mathbf{T}^{-1} = \begin{bmatrix} S_x & 0 & 0 & (1-S_x)x_f \\ 0 & S_y & 0 & (1-S_y)y_f \\ 0 & 0 & S_z & (1-S_z)z_f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

C.M. $= T(x_f, y_f, z_f).S(s_x, s_y, s_z).T(-x_f, -y_f, -z_f)$

4.Reflection

i) Reflection about yz plane

X' = -X
Y' = Y
Z' = Z



$$T_{yz} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

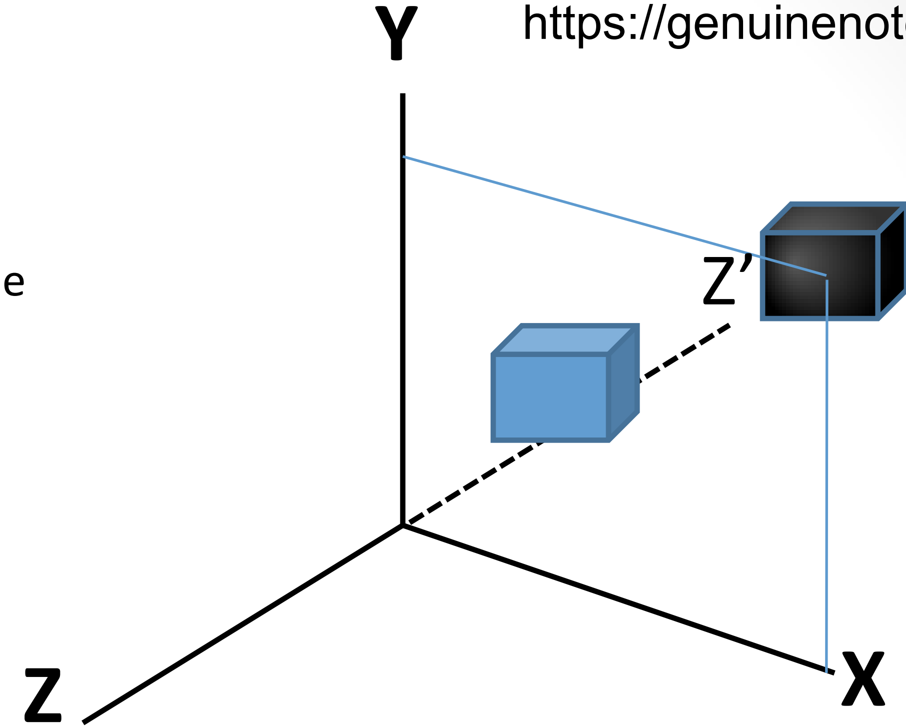
4.Reflection

ii) Reflection about XY plane

X' = X

Y' = Y

Z' = -Z



$$T_{xy} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

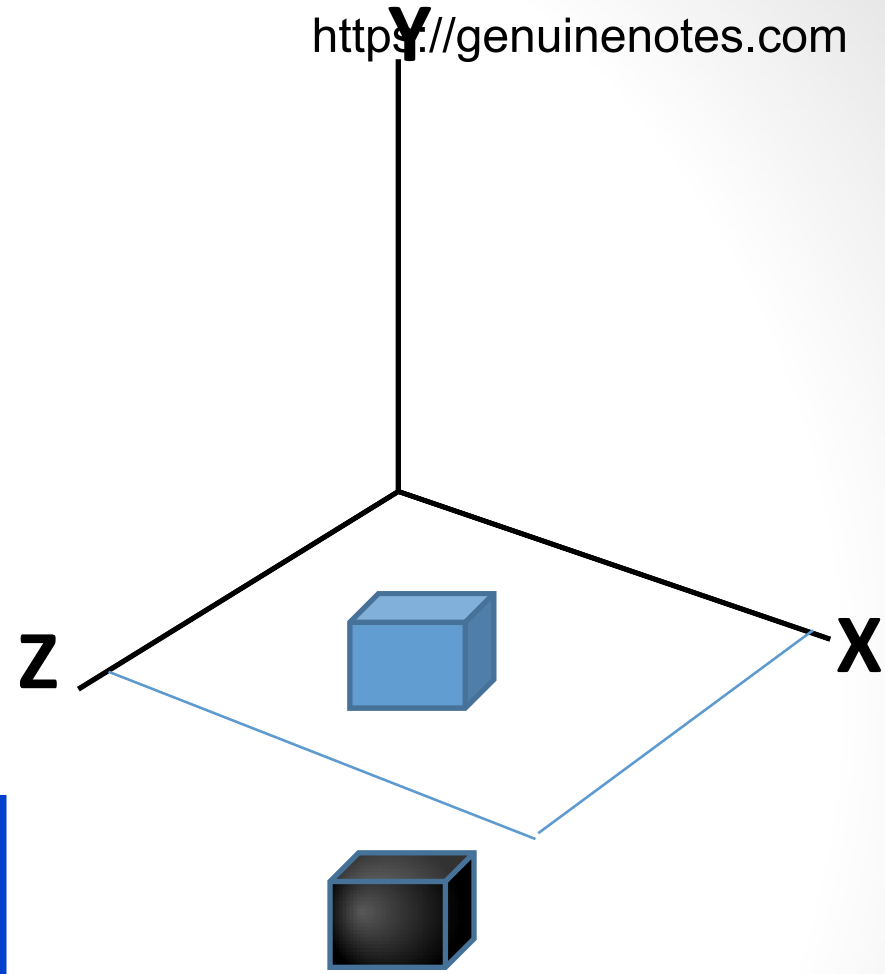
4.Reflection

iii) Reflection about XZ plane

X’ = X

Y’ = -Y

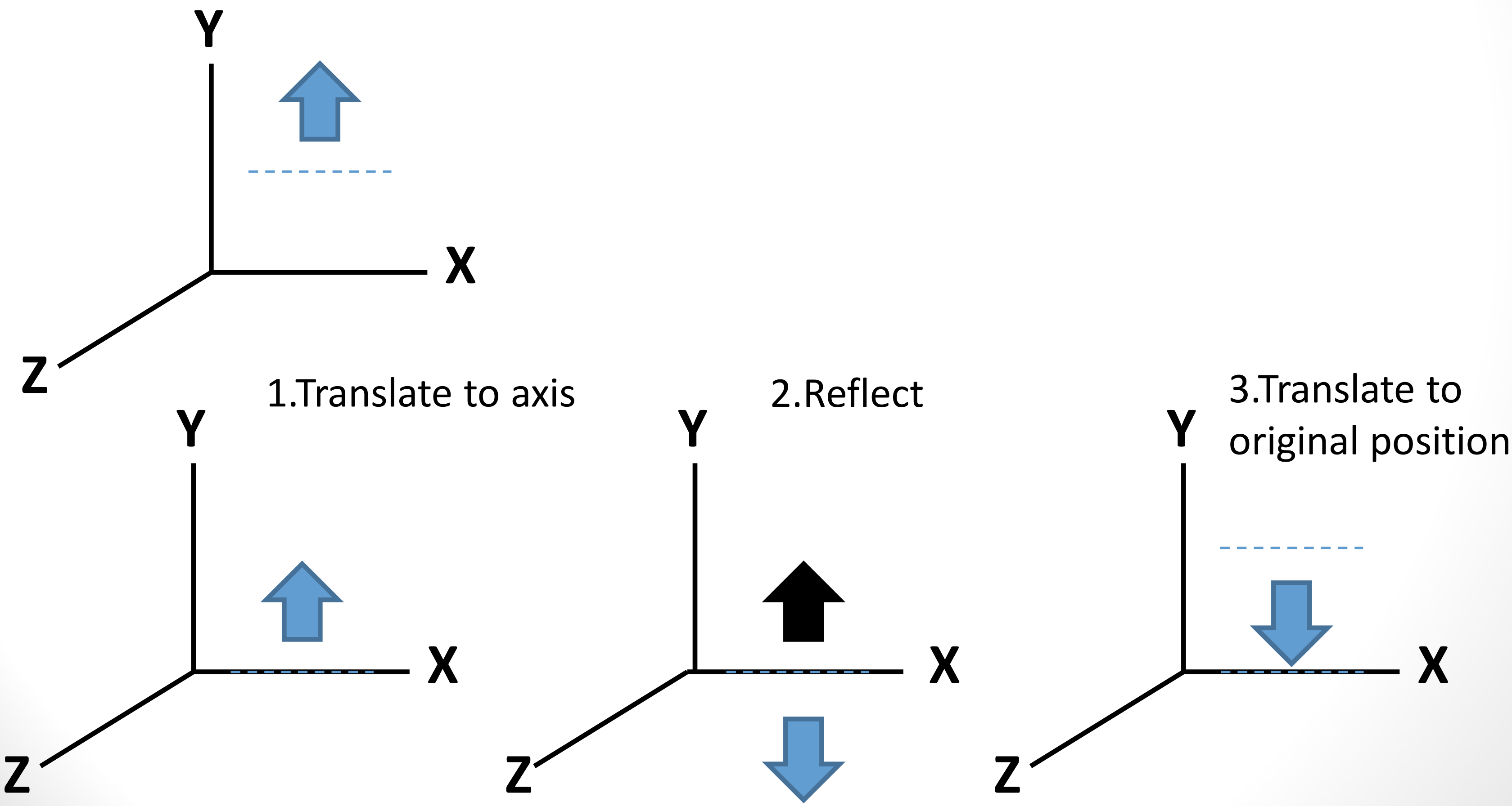
Z’ = Z



$$T_{xz} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4.Reflection

- Reflection of an object about a line that is parallel to one of the major coordinate axes



5.Shear

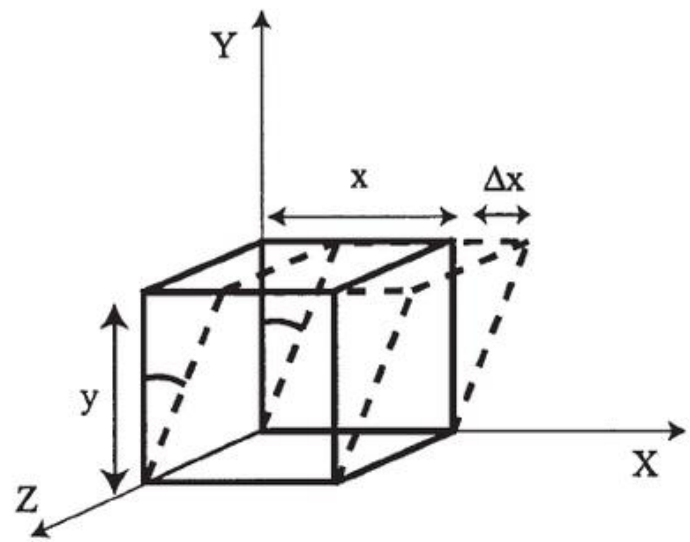
Shearing transformations can be used to modify object shapes.

Z-axis Shear

- This transformation alters x- and y-coordinate values by an amount that is proportional to the z value, while leaving the z coordinate unchanged, i.e,

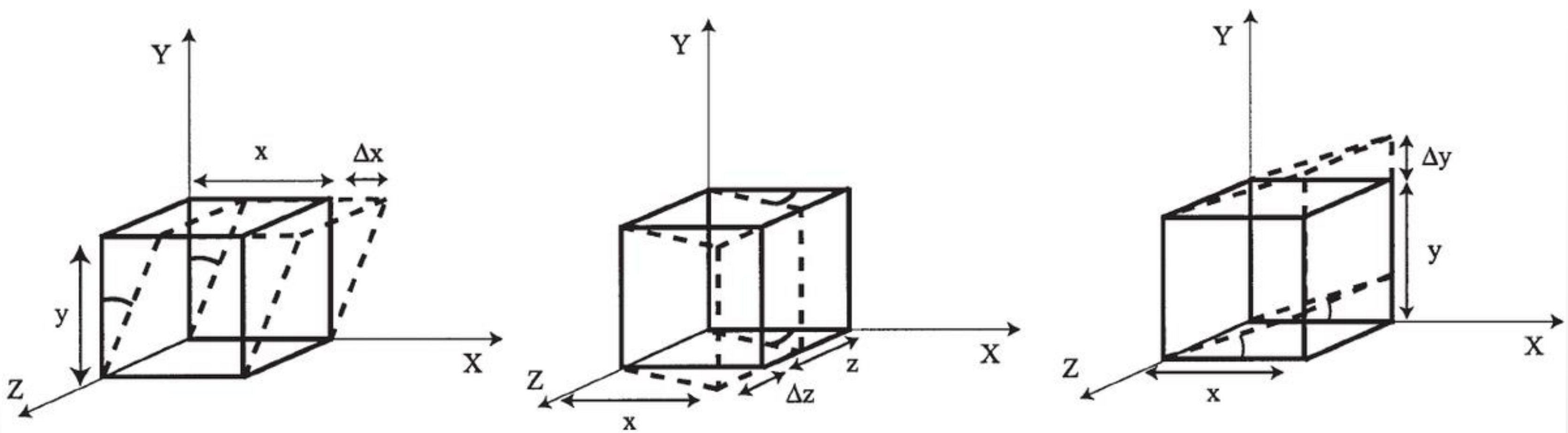
$$x' = x + S_{hx} .z$$
$$y' = y + S_{hy} .z$$
$$z' = z$$

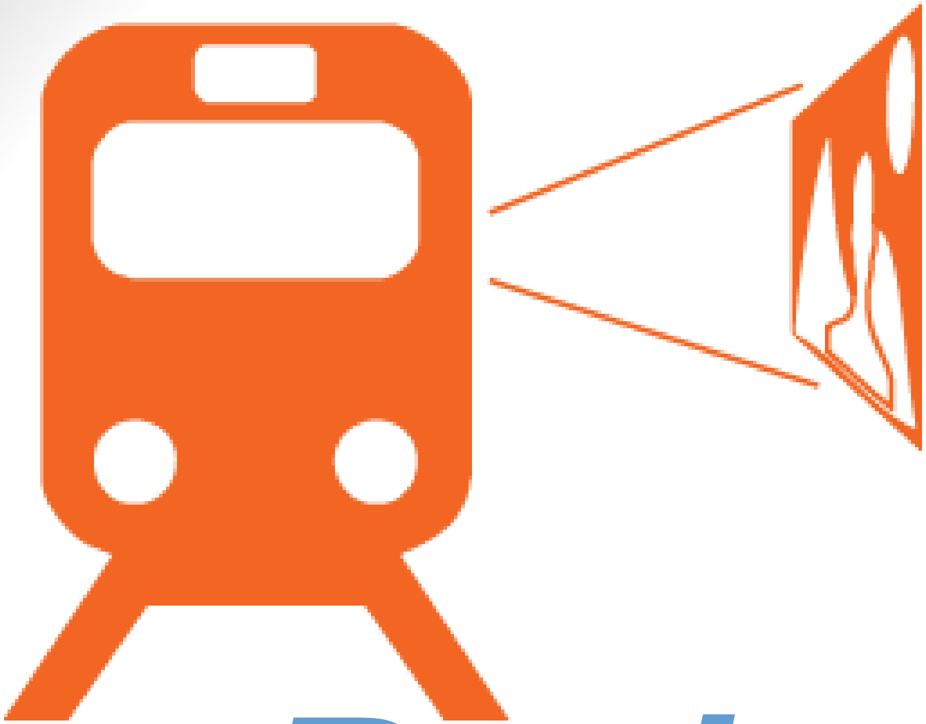
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & S_{hx} & 0 \\ 0 & 1 & S_{hy} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Similarly, we can find X-axis shear and Y-axis shear.

$$SH_z = \begin{bmatrix} 1 & 0 & S_{hx} & 0 \\ 0 & 1 & S_{hy} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad SH_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ S_{hx} & 1 & 0 & 0 \\ S_{hx} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad SH_y = \begin{bmatrix} 1 & S_{hx} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & S_{hx} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$





Projection

What is *Projection ?*

- Transformation that changes a point in n-dimensional coordinate system into a point in a coordinate system that has dimension less than n.
- Converts 3-D viewing co-ordinates to 2-D projection co-ordinates
- View Plane or Projection Plane: Two dimensional plane in which 3D objects are projected is called the view plane or projection plane. Simply it is a display plane on an output device

Types of Projection

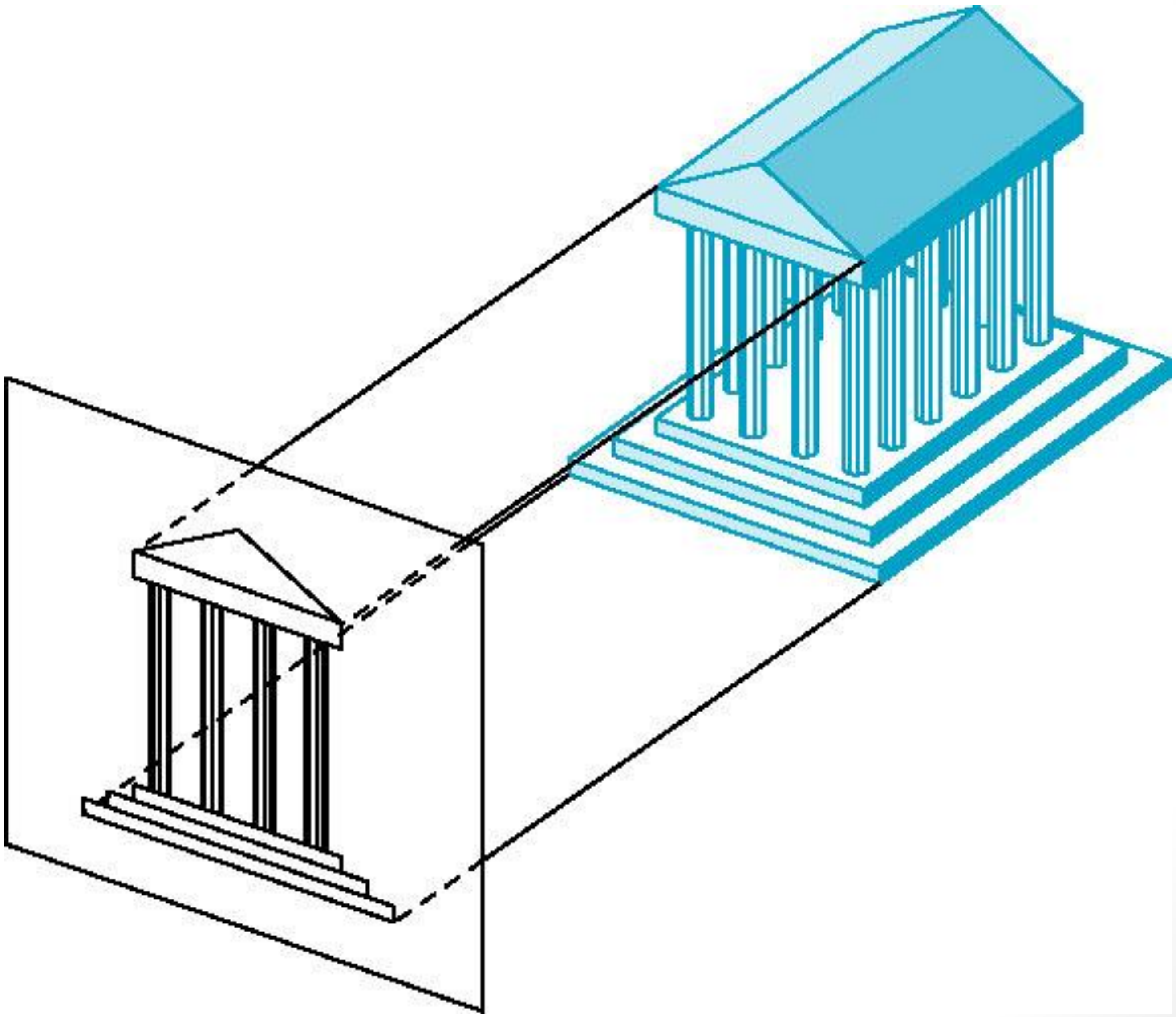
1. **Parallel Projection**
   a) **Orthographic parallel projection**
   b) **Oblique parallel projection**

2. **Perspective Projection**

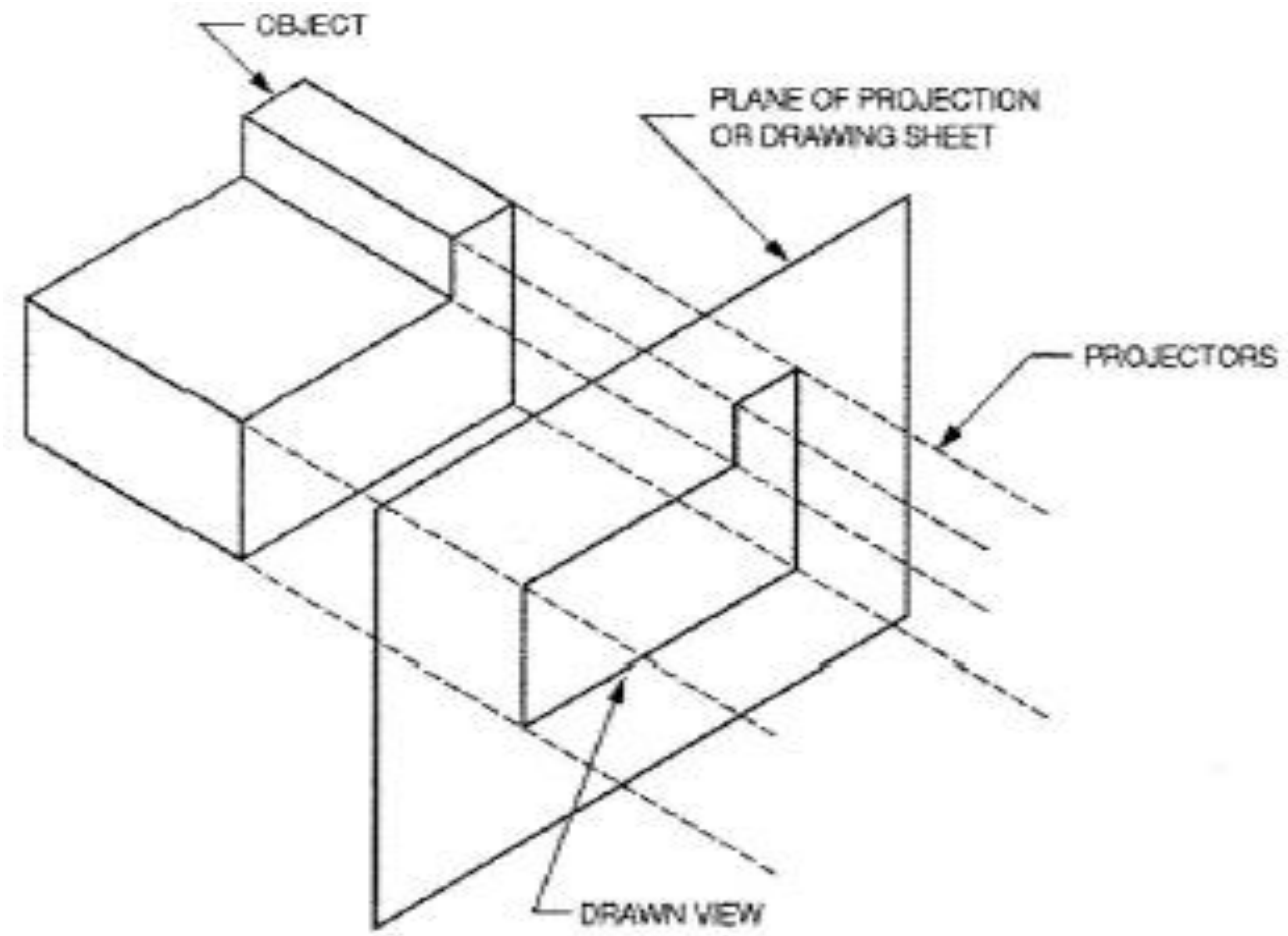
1. Parallel Projection

- Coordinate positions are transformed to view plane along parallel lines (projection lines)
- Preserves relative proportions of objects
- Accurate views of various sides of an object are obtained.
- Doesn't give realistic representation of the appearance of the 3-D object
- **Types**
  - **Orthographic-** when the projection is perpendicular to the view plane. Used to produce Front, Side and Top view of an object
  - **Oblique** – when the projection is not perpendicular to the view plane

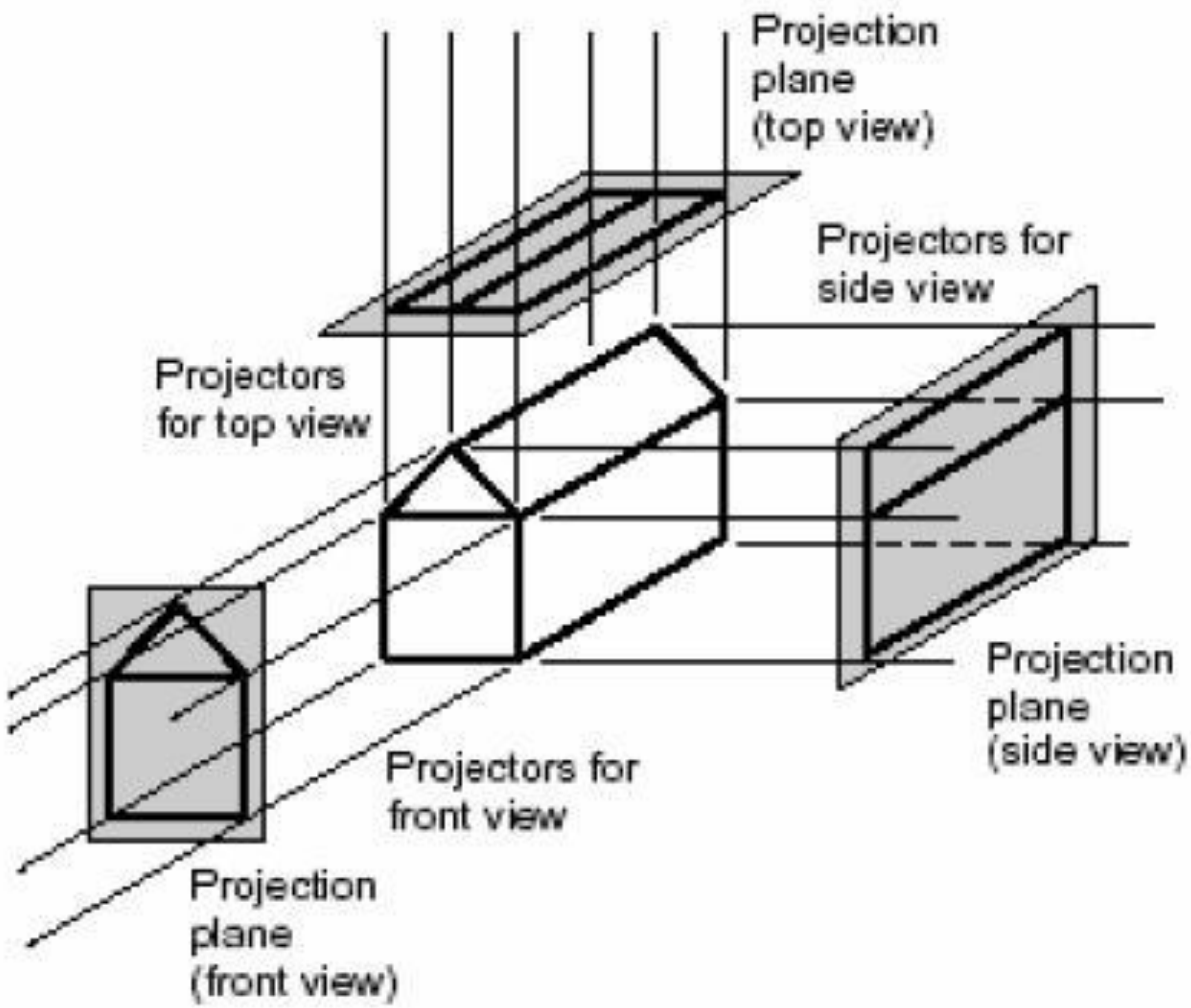
1 . Parallel Projection..



1 . Parallel Projection..

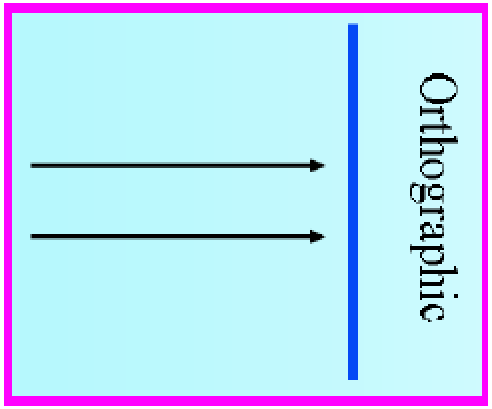


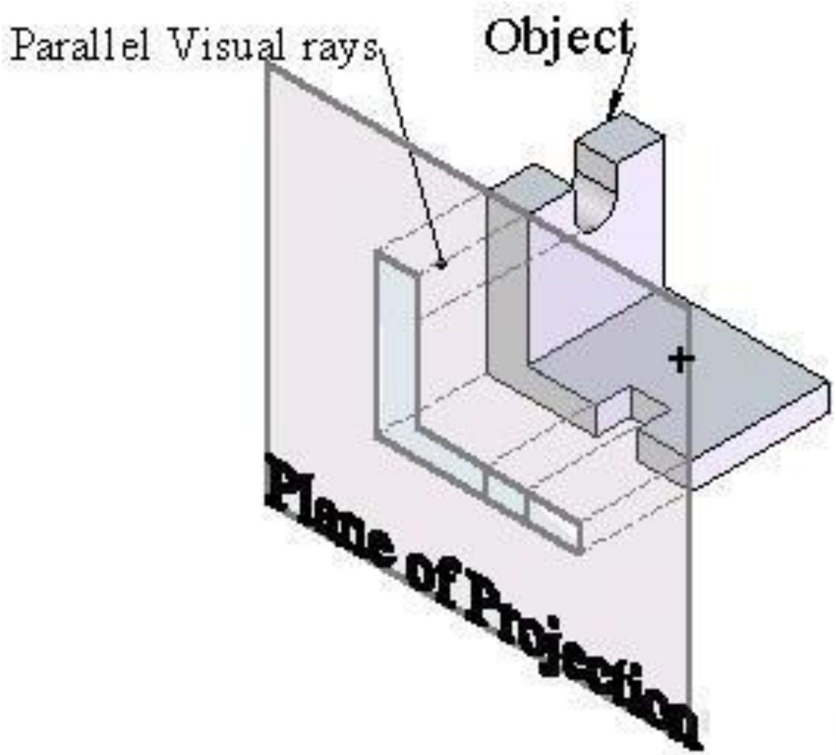
1 . Parallel Projection..



1.1. Orthographic Parallel Projection

- When projection is perpendicular to view plane then it is called orthographic parallel projection

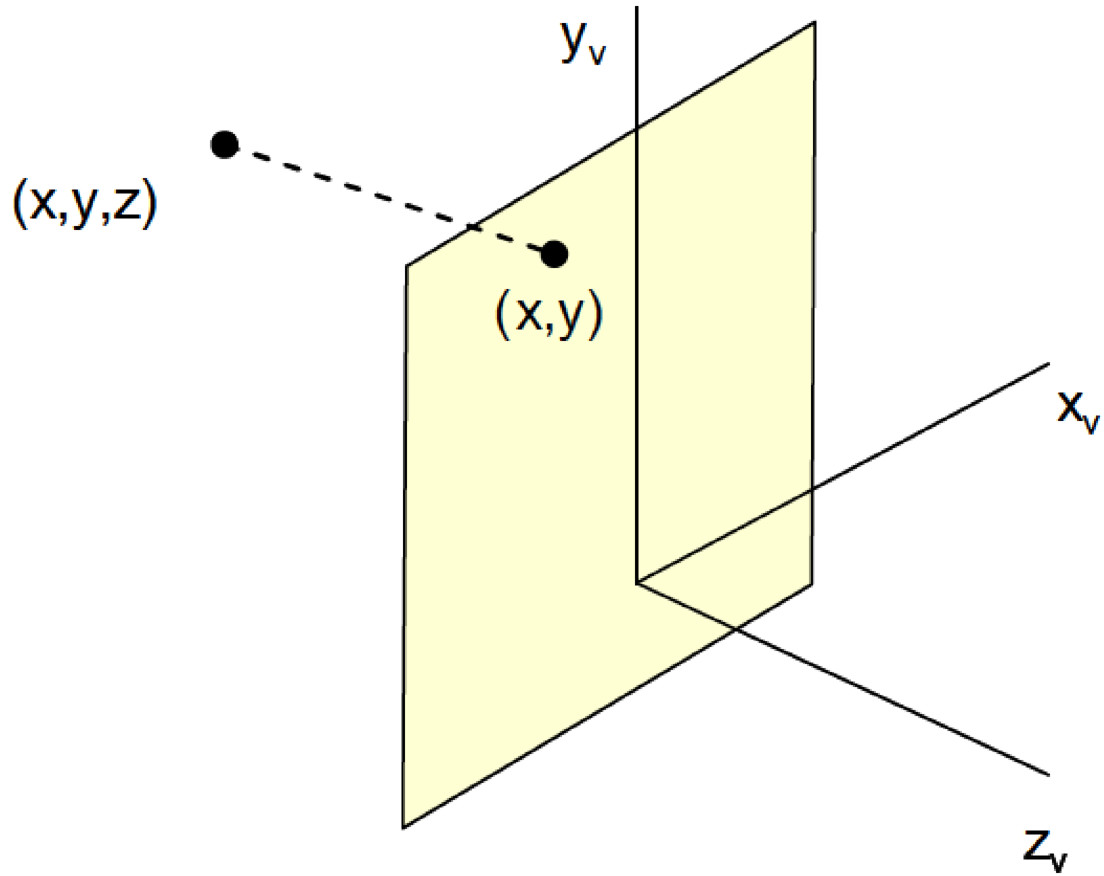




1.1. Orthographic Parallel Projection...

$X_p = X$

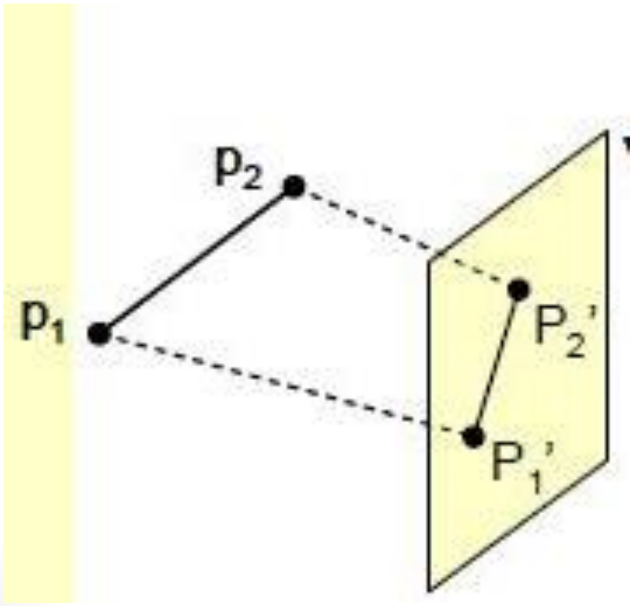
$Y_p = Y$

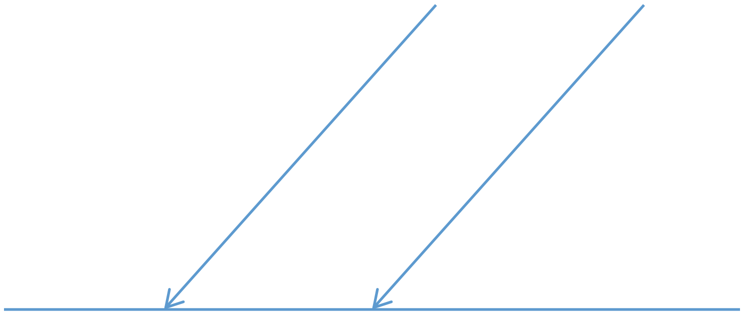


Note: Z value is preserved for the depth information needed in depth culling and visible surface determination procedur

1.2. Oblique Parallel Projection

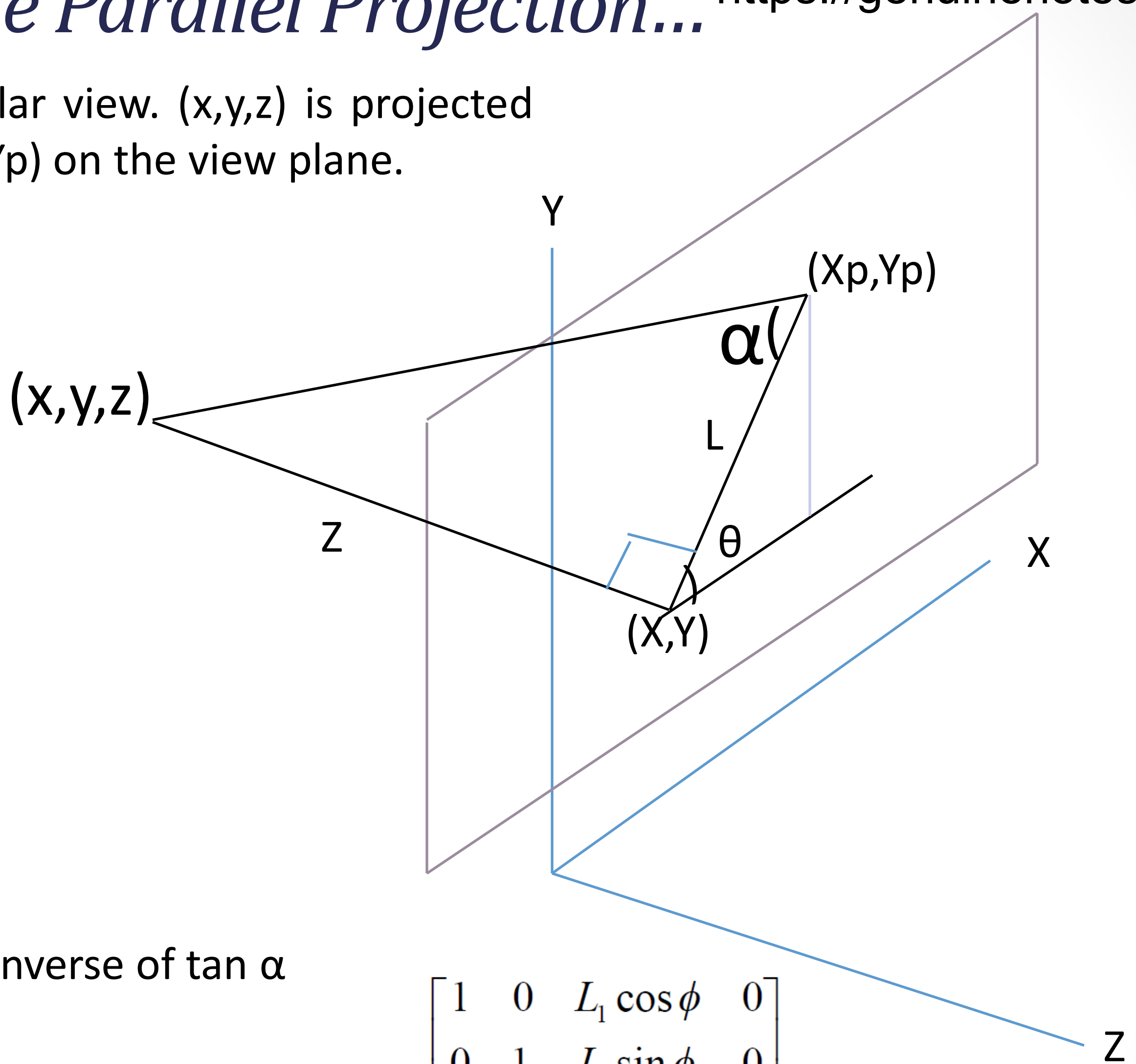
- Projectors (projection vectors) are not perpendicular to the projection plane. It preserves 3D nature of an object.





1.2. Oblique Parallel Projection…

- Not perpendicular view. (x,y,z) is projected To position (Xp,Yp) on the view plane.



$\cos\theta = Xp / L$

$Xp = L \cos\theta$

But exact position is

$$\mathbf{Xp = X + L \cos\theta}$$

similarly

$\sin\theta = Yp / L$

$$\mathbf{Yp = Y + L \sin\theta}$$

L depend on angle α

$\tan\alpha = Z / L$

$L = Z L_1$ Where L_1 is inverse of $\tan\alpha$

$$\mathbf{Xp = X + Z L_1 \cos\theta}$$

$$\mathbf{Yp = Y + Z L_1 \sin\theta}$$

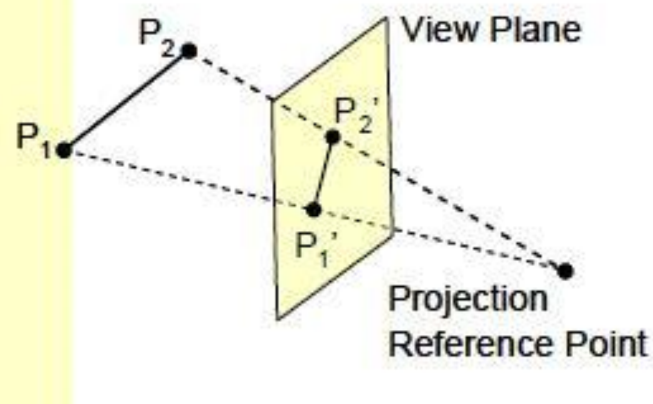
$$M_{parallel} = \begin{bmatrix} 1 & 0 & L_1\cos\phi & 0 \\ 0 & 1 & L_1\sin\phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

When $\alpha = 0$, i.e. $L_1 = 0$, it is orthographic projection

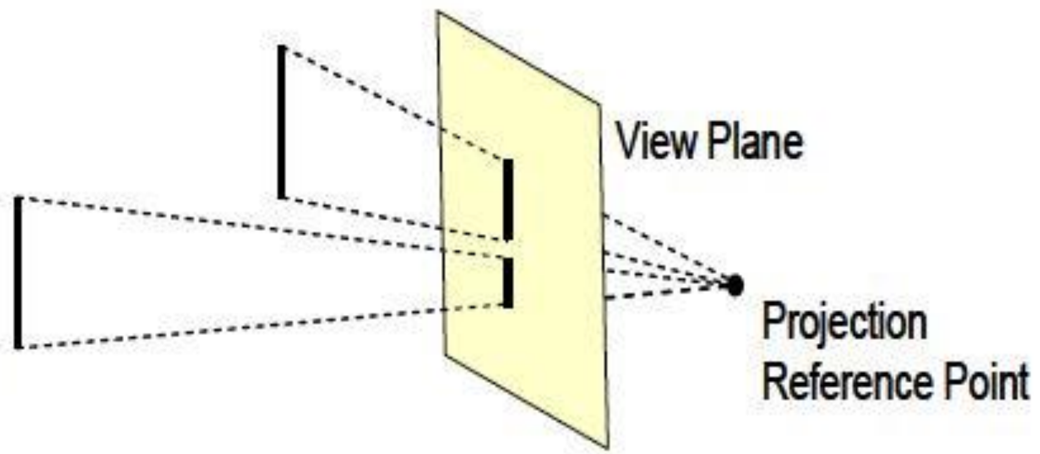
2. Perspective Projection

- Coordinate positions are transformed to view plane along lines (projection lines) that converges to a point called **projection reference point** (center of projection)
- Produce realistic view
- Does not preserve relative proportions
- Equal sized object appears in different size according as distance from view plane

2. Perspective Projection

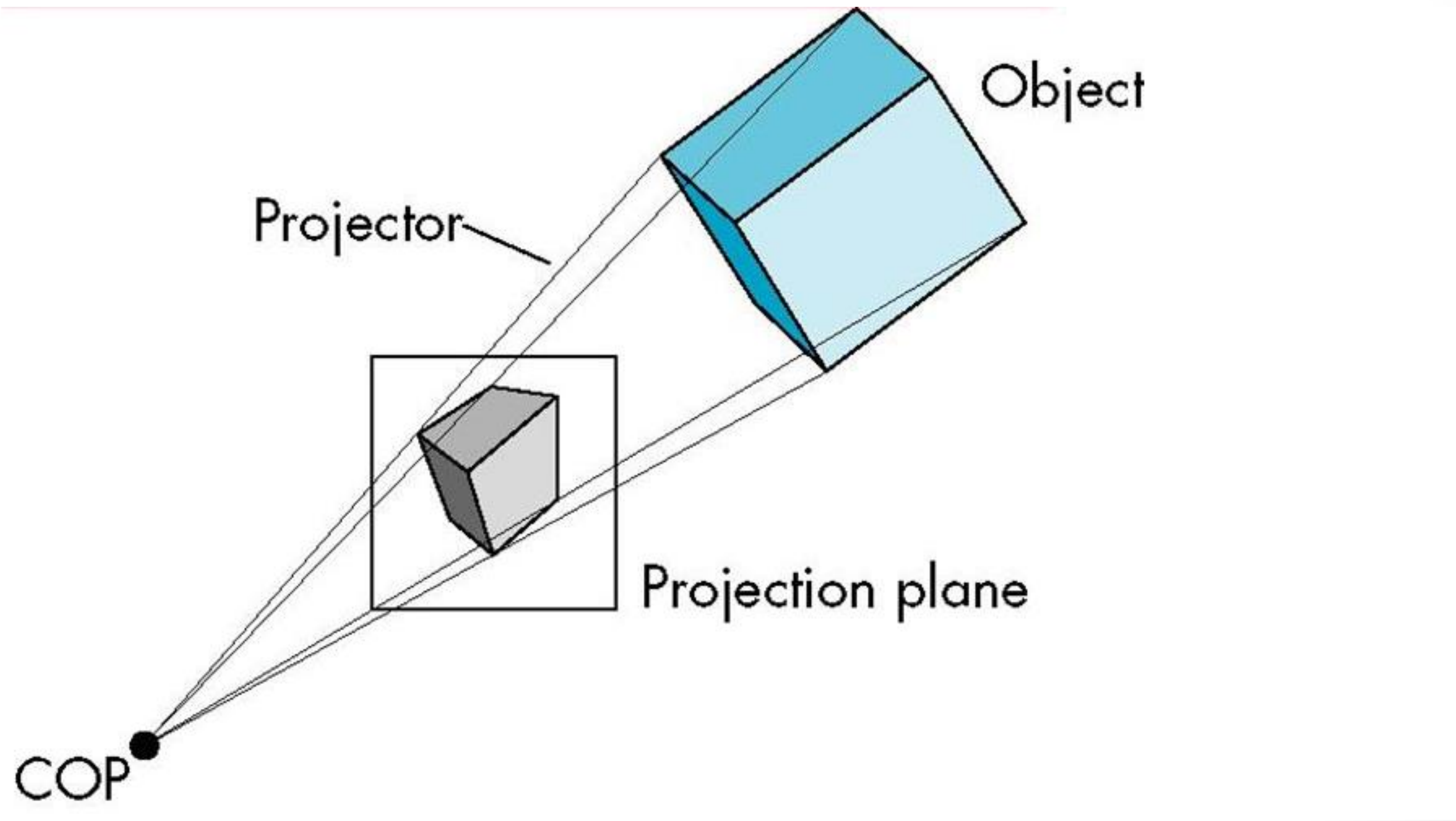


Perspective Projection



Perspective Projection of equal sized objects at different distances from the view plane

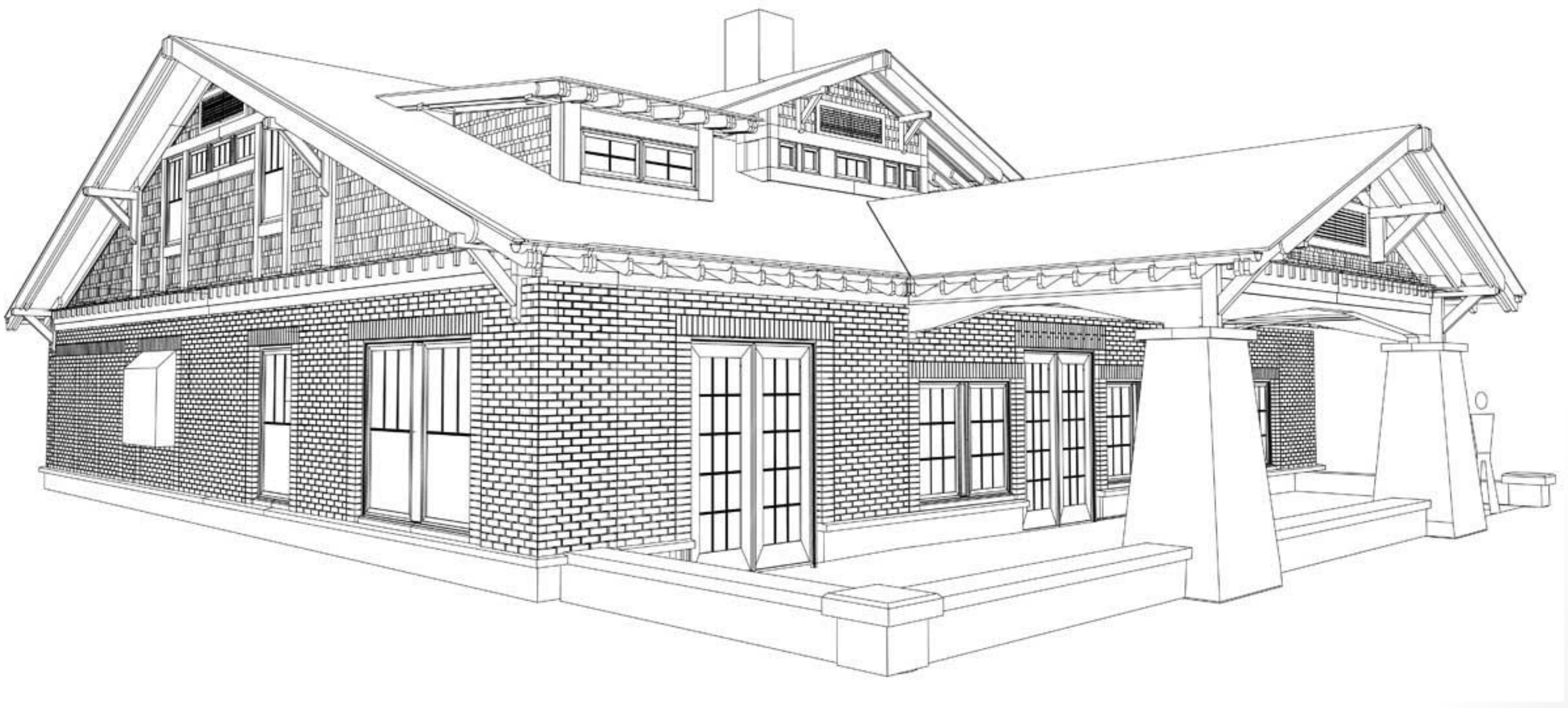
2. Perspective Projection



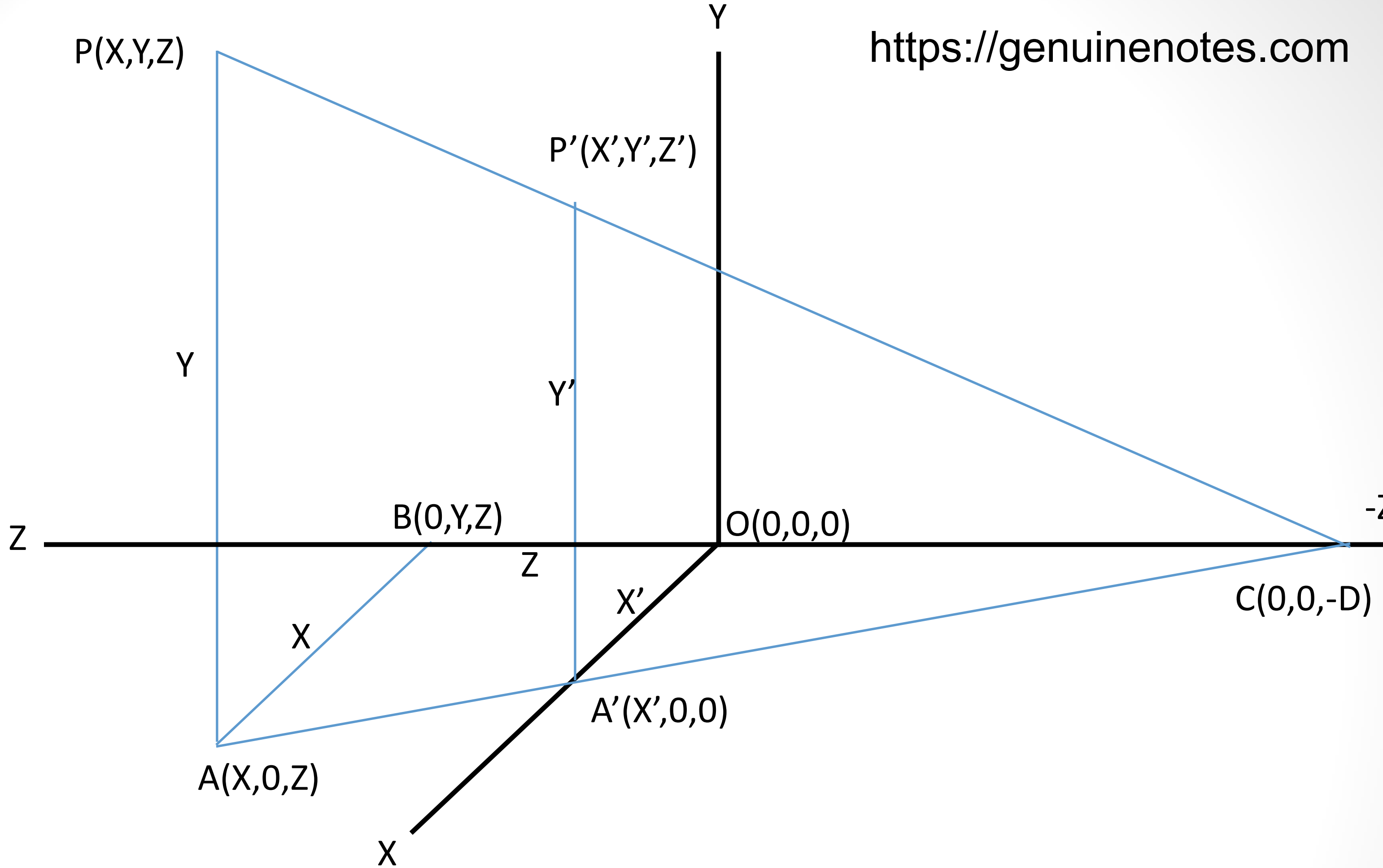
Perspective View

Perspective View

Perspective View

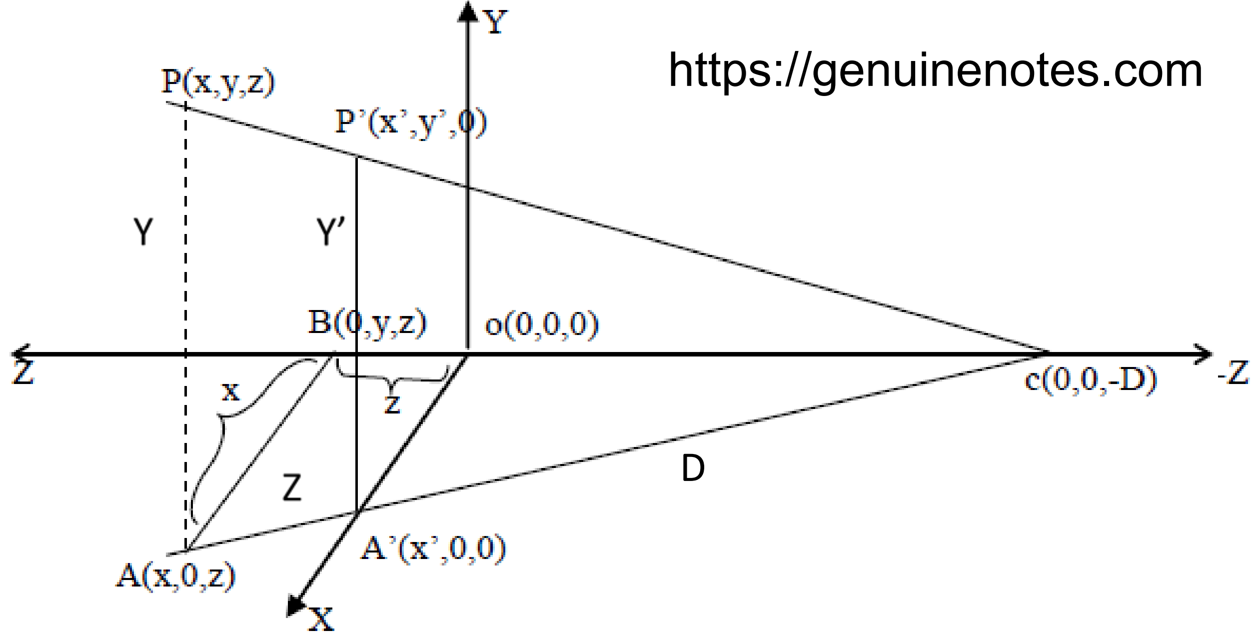


https://genuinenotes.com



Here center of Projection is c (0,0,-D) along the direction of Z axis so the reference point is taken of world coordinate space Wc and the normal vector N is aligned with the y axis.

So now the view plane vp is the xy plane and center of projection is c (0,0,-D) now from similar triangles ABC and A'OC



Triangles ABC and A'OC
(x/x') = AC/A'C = (Z+D)/D
X' = (XD)/(Z+D)
And Z'=0

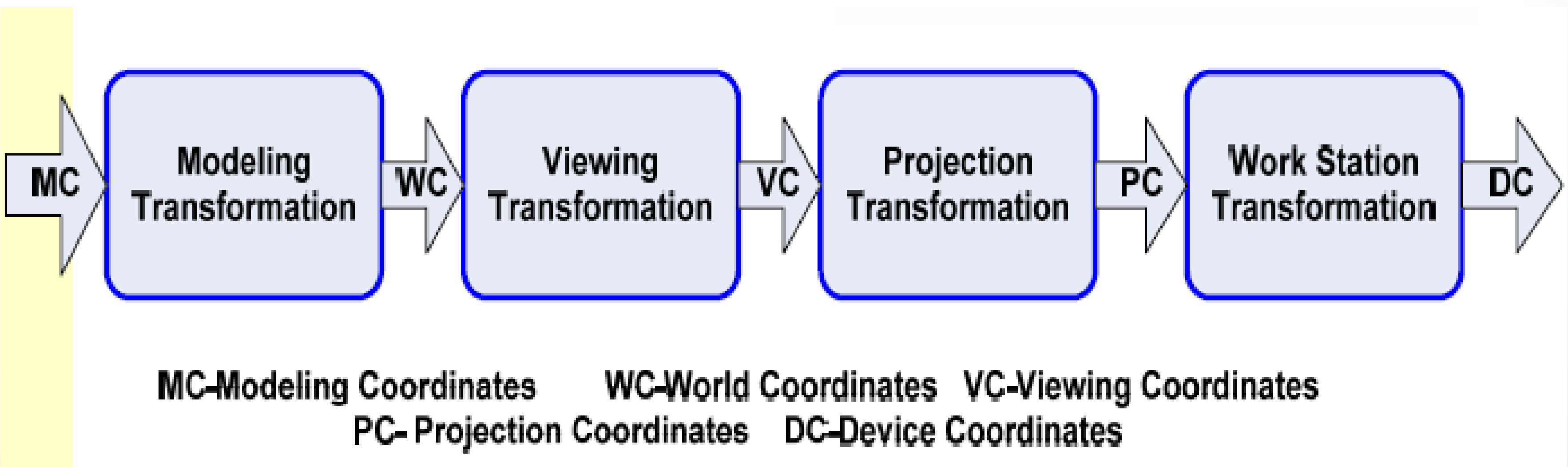
Triangles APC and A'P'C
(y/y') = (AC/AC') = (Z+D)/D
y' = (DY)/(Z+D)
And Z' =0

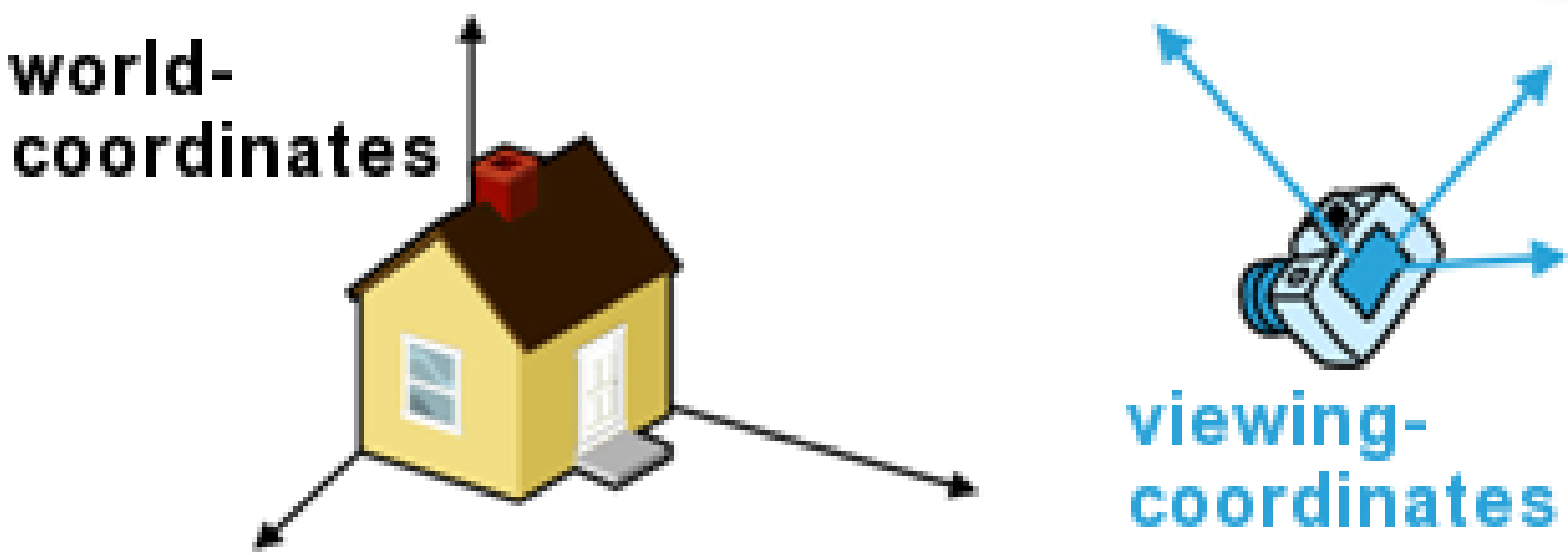
now in homogenous coordinates

$$\begin{pmatrix} X' \\ Y' \\ Z' \\ 1 \end{pmatrix} = \frac{1}{z+D} \begin{pmatrix} Dx \\ Dy \\ 0 \\ z+D \end{pmatrix} = \frac{1}{z+D} \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & D \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

3D-Viewing Pipeline

- The viewing-pipeline in 3 dimensions is almost the same as the 2D-viewing-pipeline. Only after the definition of the viewing direction and orientation (i.e., of the camera) an additional projection step is done, which is the reduction of 3D-data onto a projection plane:




world-
coordinates
viewing-
coordinates

Chapter 4 Finished