1 Introduction

The aim of this report is to describe the design and development of different system identification techniques using data from Quanser AERO

1.1 Identification Procedure

The identification of system parameters consists of two basic steps

1.1.1 Data acquisition and Parameter estimation

The system to be modelled is first given a predetermined input and the outputs of the system are observed. The input signal must be rich. The inputs and observed outputs are then fed into the algorithms described in Section 2.
For estimation of the parameters we use the double step reference input as shown below of the tachometer sensor.
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1.1.2 Validation

Validation of the obtained model is performed by comparing the measured data to the data predicted by the model. The similarity of the parameters to the actual model is obtained by performing a Whiteness Test on the errors obtained.

$$R(0) = \frac{1}{N}\sum_{i=1}^{N}\epsilon(t)^2$$

$$RN(i) = \frac{\frac{1}{N}\sum_{i=1}^{N}\epsilon(t)\epsilon(t-i)}{R(0)}$$

The whiteness test measures the auto-correlation of the error. A perfect model will show zero auto correlation at non zero lag, however, in a practical situation, there will be some correlation at all values. If the value of the lag shifted correlation is below a threshold, the system is considered to be well designed.
A single step response of the tachometer is used as reference for validation of the estimated model.
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$$|RN(i)| < \frac{2.17}{\sqrt{N}}$$

According to the independence test criteria, a good model has residuals uncorrelated with past inputs. Evidence of correlation indicates that the model does not describe how part of the output relates to the corresponding input.
In addition to the whiteness test and independence test, the parameters estimated by the algorithms are compared with the parameters estimated by the Matlab System Identification toolbox.

2 Estimation algorithms

2.1 Recursive Least Square

Consider a discrete system with the following process model:

$$y(t+1)+a_1*y(t)+a_2*y(t-1)+...+a_{na}*y(t+1-n_a) = b_1*u(t)+b_2*u(t-1)+...+b_{nb}*u(t+1-n_b)$$

where n_a, n_b are known and $a_1, a_2, .., a_{na}, b_1, b_2, ..., b_{nb}$ are unknown parameters.
Popularly known as the *arx* model , is given by

$$y(t) = \frac{B(q^{-1})}{A(q^{-1})}u(t)$$

The values of the unknown parameters are initially taken as zero.

Using the current and previous states of the system and the inputs provided, the *prior* of the state, $\hat{y}(t+1)$ is found from the expression

$$\hat{y}(t+1) = \hat{\theta}^T\phi$$

where

$$\hat{\theta^T}(t) = [a_1, a_2, .., a_{na}, b_1, b_2, ..., b_{nb}]$$

$$\phi^T(t) = [y(t), y(t-1), ...y(t+1-a_{na}), u(t), u(t-1), u(t+1-b_{nb})]$$

The error between the *prior* and the observation ϵ^0 is then found and ϵ is the *posterior* error obtained by

$$\epsilon^0(t+1) = y(t+1) - \hat{y}(t+1)$$

$$\epsilon(t+1) = \frac{\epsilon^0(t+1)}{1+\phi^T(t)D(t)\phi(t)}$$

the values of θ and D are updated by

$$\hat{\theta}(t+1) = \hat{\theta}(t) + D(t)\phi(t)\epsilon(t+1)$$

$$D(t+1) = D(t) - \frac{D(t)\phi(t)\phi^T(t)D(t)}{1+\phi^T(t)D(t)\phi(t)}$$

2.1.1 Validation

To obtain the predicted values for validation, a new data set is used as input to the model. Given a few initial states $y(t), y(t-1)..$ and the input $u(t), u(t-1), ..,$ the expected state is obtained as

$$\hat{y}(t+1) = \theta^T \phi$$

Where θ is the vector of estimated parameters from the algorithm described above, and ϕ is defined below

$$\phi^T(t) = [y(t), y(t-1), ...y(t+1-a_{na}), u(t), u(t-1), u(t+1-b_{nb})]$$

$$\theta^T(t) = [a_1, a_2, .., a_{na}, b_1, b_2, ..., b_{nb}]$$

In order to calculate the prediction error efficiently, the Matlab function *pe* from the SysID toolbox is used, and the function *autocorr* is used to obtain the autocorrelation function of the error.
The *residual* function of Matlab is used to analyze the residuals of the estimated model

The parameters obtained by this algorithm are compared with the Matlab's inbuit *rls* estimator given by *arx* function

2.2 Extended Least Square

The Extended Least Squares method(armax) uses a different equation to describe the propagation of the parameters.

The system equation is described by

$$y(t+1) + A(q^{-1})y(t) = B(q^{-1})u(t) + C(q^{-1})v(t+1)$$

where q^{-1} is the delay operator

$$y(t) = \frac{B(q^{-1})}{A(q^{-1})}u(t) + C(q^{-1})v(t)$$

The predictor becomes

$$\hat{y}(t+1) = -a_1 y(t) + b_1 u(t) + c_1 e(t)$$

where

$$e(t) = y(t) - \hat{y}(t)$$

At time t the current estimate and measurements are known:

$$\hat{\theta}^T = [a_1, a_2, ..., b_1, b_2, ..., c_1, c_2, ...]$$

$$\phi(t) = [-y(t), -y(t-1), ..., u(t), u(t-1), ..., e(t), e(t-1), ...]$$

The prior is computed:

$$\hat{y}(t+1) = \theta^T \phi$$

at $t+1$ a new value $y(t+1)$ is measured.

$$\epsilon^0(t+1) = y(t+1) - \hat{y}^0$$

$$\epsilon(t+1) = \frac{e^0(t+1)}{1+\phi(t)^T D(t)\phi(t)}$$

The update is

$$\hat{\theta}(t+1) = \hat{\theta}(t) + D(t)\phi(t)\epsilon(t+1)$$

$$e(t+1) = \epsilon(t+1)$$

$$D(t+1) = D(t) - \frac{D(t)\phi(t)\phi^T(t)D(t)}{1+\phi^T(t)D(t)\phi(t)}$$

The parameters are found recursively in a manner similar to the one used in Section 2.1.

2.2.1 Validation

The validation of the estimated system is performed in the same method as described in Section 2.1.1. Since the model described by ELS uses the past errors to predict the current output, the error between the previous output and the predicted output is used.

$$y(t \hat{+} 1) = \theta^T \phi$$

Where θ is the vector of estimated parameters from the algorithm described above, and ϕ is defined below

$$\phi(t) = [-y(t), -y(t-1), ..., u(t), u(t-1), ..., e(t), e(t-1), ...$$

$$\theta^T(t) = [a_1, a_2, .., a_{na}, b_1, b_2, ..., b_{nb}]$$

2.3 Generalized Least Square

The Generalized Least Squares method uses a different equation to describe the effect of the error.

The system equation is given by

$$y(t+1) + A(q^{-1})y(t) = B(q^{-1})u(t) + \frac{1}{1+C(q^{-1})}v(t+1)$$

Defining

$$\alpha(t+1) = \frac{v(t+1)}{1+c_1 q^{-1}}$$

the equation becomes
This model is *ararx* and is of the form

$$y(t) = \frac{B(q^{-1})}{A(q^{-1})}u(t) + \frac{1}{C(q^{-1})}v(t)$$

$$\hat{y}(t+1) = -a_1 y(t) + b_1 u(t) - c_1 \alpha(t)$$

The parameter vectors become

$$\hat{\theta}^T = [a_1, a_2, ..., b_1, b_2, ..., c_1, c_2, ...]$$

and

$$\phi(t) = [-y(t), -y(t-1), ..., u(t), u(t-1), ..., -\alpha(t), -\alpha(t-1), ...$$

where

$$\alpha = (1+A)y(t) - Bu(t)$$

The parameters are found recursively using the equations given in 2.1.

2.3.1 Validation

Since the Generalized Least Squares method uses a similar method to the Extended Least Squares method, the validation equations are similar to the ones used in 2.2.1.

2.4 Output error method

In the RLS algorithm from Section 2.1, the expected measurement is corrected at every iteration using the latest measurement. In the output error method, the expected measurement is instead updated using the previous expected measurement

This results in a change in ϕ^T vector, which becomes

$$\phi^T(t) = [\hat{y}(t), \hat{y}(t-1), ...\hat{y}(t+1-a_{na}), u(t), u(t-1), u(t+1-b_{nb})]$$

The remaining equations are unchanged and taken directly from Section 2.1

$$\hat{\theta^T}(t) = [a_1, a_2, .., a_{na}, b_1, b_2, ..., b_{nb}]$$

$$\epsilon(t+1) = \frac{\epsilon^0(t)}{1 + \phi^T(t)D(t)\phi(t)}$$

$$\hat{\theta}(t+1) = \hat{\theta}(t) + D(t)\phi(t)\epsilon(t+1)$$

$$D(t+1) = D(t) - \frac{D(t)\phi(t)\phi^T(t)D(t)}{1 + \phi^T(t)D(t)\phi(t)}$$

2.4.1 Validation

For the validation of the OE model, we find the cross-correlation between the residual error and the estimated output.For good results we expect the cross correlation to be between certain bounds.

3 Results

The system was subjected two types on inputs: a single step and a double step. The system parameters were obtained using the double step response and the validation was performed using the single step response.
The data set used for validation is as show in figure below

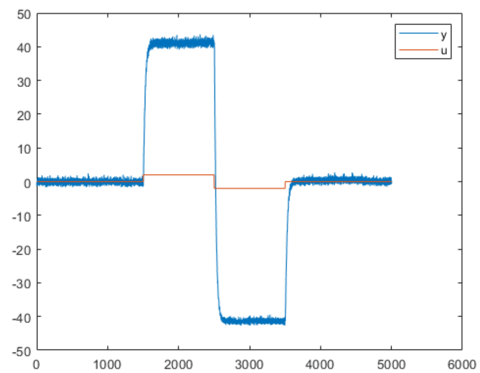


Figure 1: Data set used for estimation

The values of n_a n_b n_c were varied and the models obtained were validated. The parameters are shown in Table 1 below.
The same data was also used to estimate the parameters using inbuilt Matlab functions *arx* ,*armax* ,*oe* and *polyest*, the results of which are shown in Table 2

$Numbersofparameters$	$Method$	a_i	b_i	c_i
$n_a = 3, n_b = 2, n_c = 5$	RLS	$a_0 = 1.0000$ $a_1 = -0.8895$ $a_2 = -0.2235$ $a_3 = 0.1510$	$b_0 = 0$ $b_1 = -0.0821$ $b_2 = 0.8591$	
	ELS	$a_0 = 1.0000$ $a_1 = -1.6982$ $a_2 = 0.8625$ $a_3 = -0.1471$	$b_0 = 0$ $b_1 = -0.0692$ $b_2 = 0.4253$	$c_0 = 1.0000$ $c_1 = -1.0291$ $c_2 = 0.5976$ $c_3 = -0.3469$ $c_4 = -0.0561$ $c_5 = -0.0745$
	GLS	$a_0 = 1.0000$ $a_1 = -1.3442$ $a_2 = 0.3251$ $a_3 = -0.0441$	$b_0 = 0$ $b_1 = -0.1485$ $b_2 = 0.6647$	$c_0 = 1.0000$ $c_1 = 0.5166$ $c_2 = 0.0924$ $c_3 = 0.0743$ $c_4 = 0.1129$ $c_5 = 0.1587$

$Numbersofparameters$	$Method$	a_i	b_i	c_i
	Output error	$a_0 = 1.0000$ $a_1 = -0.3590$ $a_2 = -0.1564$ $a_3 = -0.4093$	$b_0 = 0$ $b_1 = 0.0864$ $b_2 = 1.4946$	
$n_a = 5, n_b = 3, n_c = 10$	RLS	$a_0 = 1.0000$ $a_1 = -0.8792$ $a_2 = -0.2594$ $a_3 = 0.2288$ $a_4 = 0.1695$ $a_5 = -0.2155$	$b_0 = 0$ $b_1 = 0.1227$ $b_2 = -0.0486$ $b_3 = 0.8329$	
	ELS	$a_0 = 1.0000$ $a_1 = -1.2201$ $a_2 = -0.1376$ $a_3 = 0.5645$ $a_4 = 0.1840$ $a_5 = -0.3550$	$b_0 = 0$ $b_1 = -0.0390$ $b_2 = 0.1746$ $b_3 = 0.6069$	$c_0 = 1.0000$ $c_1 = -0.6815$ $c_2 = -0.1280$ $c_3 = 0.2538$ $c_4 = 0.0018$ $c_5 = -0.3283$ $c_6 = 0.0478$ $c_7 = 0.0697$ $c_8 = -0.1377$ $c_9 = 0.0657$ $c_{10} = -0.0176$
	GLS	$a_0 = 1.0000$ $a_1 = -1.1273$ $a_2 = -0.0897$ $a_3 = 0.2696$ $a_4 = 0.4171$ $a_5 = -0.4274$	$b_0 = 0$ $b_1 = -0.0715$ $b_2 = 0.1587$ $b_3 = 0.7923$	$c_0 = 1.0000$ $c_1 = 0.4259$ $c_2 = 0.2021$ $c_3 = 0.0626$ $c_4 = -0.2421$ $c_5 = 0.0759$ $c_6 = 0.0718$ $c_7 = 0.0358$ $c_8 = 0.0869$ $c_9 = -0.1174$ $c_{10} = -0.0571$
	Output error	$a_0 = 1.0000$ $a_1 = -0.0531$ $a_2 = -0.0383$ $a_3 = -0.1597$ $a_4 = -0.2027$ $a_5 = -0.4176$	$b_0 = 0$ $b_1 = 0.0616$ $b_2 = -0.0334$ $b_3 = 2.6170$	

$Numbersofparameters$	$Method$	a_i	b_i	c_i
	RLS	$a_0 = 1.0000$ $a_1 = -0.9205$ $a_2 = -0.1308$	$b_0 = -0.4814$ $b_1 = 2.039$	

$n_a = 3, n_b = 2, n_c = 5$

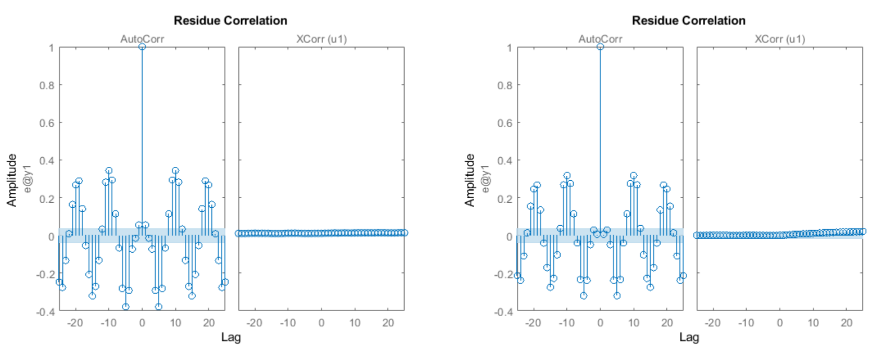
$Numbers of parameters$	$Method$	a_i	b_i	c_i
		$a_3 = 0.1268$		
	ELS	$a_0 = 1.0000$ $a_1 = -2.543$ $a_2 = 2.481$ $a_3 = -0.9244$	$b_0 = -0.4832$ $b_1 = 0.7564$	$c_0 = 1.0000$ $c_1 = -2.242$ $c_2 = 1.917$ $c_3 = -0.5988$ $c_4 = -0.1301$ $c_5 = -0.1081$
	GLS	$a_0 = 1.0000$ $a_1 = -2.538$ $a_2 = 2.421$ $a_3 = -0.8715$	$b_0 = -0.3004$ $b_1 = 0.5385$	$c_0 = 1.0000$ $c_1 = 1.953$ $c_2 = 2.21$ $c_3 = 1.765$ $c_4 = 0.9688$ $c_5 = 0.2804$
	Output error	$a_0 = 1.0000$ $a_1 = -1.155$ $a_2 = 0.1872$	$b_0 = 0.3784$ $b_1 = -1.587$ $b_2 = 1.881$	
$n_a = 5, n_b = 3, n_c = 10$	RLS	$a_0 = 1.0000$ $a_1 = -0.9012$ $a_2 = -0.1807$ $a_3 = 0.243$ $a_4 = 0.1883$ $a_5 = -0.2755$	$b_0 = -0.5056$ $b_1 = 0.315$ $b_2 = 1.719$	
	ELS	$a_0 = 1.0000$ $a_1 = -2.416$ $a_2 = 1.596$ $a_3 = 0.8528$ $a_4 = -1.571$ $a_5 = -0.5468$	$b_0 = 0.3074$ $b_1 = -0.7003$ $b_2 = 0.5693$	$c_0 = 1.0000$ $c_1 = -2.152$ $c_2 = 1.132$ $c_3 = 0.8892$ $c_4 = -1.25$ $c_5 = 0.3698$ $c_6 = 0.06445$ $c_7 = -0.03558$ $c_8 = -0.06308$ $c_9 = 0.1638$ $c_{10} = -0.08791$
	GLS	$a_0 = 1.0000$ $a_1 = -1.226$ $a_2 = -0.07046$ $a_3 = 0.6016$ $a_4 = -0.2888$ $a_5 = +0.0101$	$b_0 = 0.00421$ $b_1 = -0.5427$ $b_2 = 1.091$	$c_0 = 1.0000$ $c_1 = 0.0.8296$ $c_2 = 0.8657$ $c_3 = 0.5215$ $c_4 = 0.5788$ $c_5 = 0.5945$ $c_6 = 0.8011$ $c_7 = 0.7604$ $c_8 = 0.7235$ $c_9 = 0.4125$ $c_{10} = 0.2689$
	Output error	$a_0 = 1.0000$	$b_0 = 0.3727$	

$Numbersofparameters$	$Method$	a_i	b_i	c_i
		$a_1 = -0.8213$ $a_2 = -0.9457$ $a_3 = +0.7805$ $a_5 = -0.4176$	$b_1 = -1.052$ $b_2 = 0.6631$ $b_3 = 1.572$ $b_4 = -1.276$	

4 Whiteness Test and Independence test results

Here we have three plots for each estimation method.
The first two is the plot of *resid* function which gives the autocorrelation plot and the crosscorrelation plot with the input, which explains the whiteness test and the independence test respectively.
The third plot is by *autocorr* function of matlab which gives the autocorrelation of the error with the confidence margins.
A good model has the residual autocorrelation function inside the confidence interval of the corresponding estimates, indicating that the residuals are uncorrelated.But here we see that the autocorrelation function is outside the confidence margins.From which we can say that the model is not entirely describing the output data , hence the residual are not purely white
For Independence test,a peak outside the confidence interval for lag k means that the output y(t) that originates from the input u(t-k) is not properly described by the model.

4.1 Model 1 $n_a = 3, n_b = 2, n_c = 5$



(a) Calculated model

(b) Matlab model

Figure 2: Residuals for the RLS model

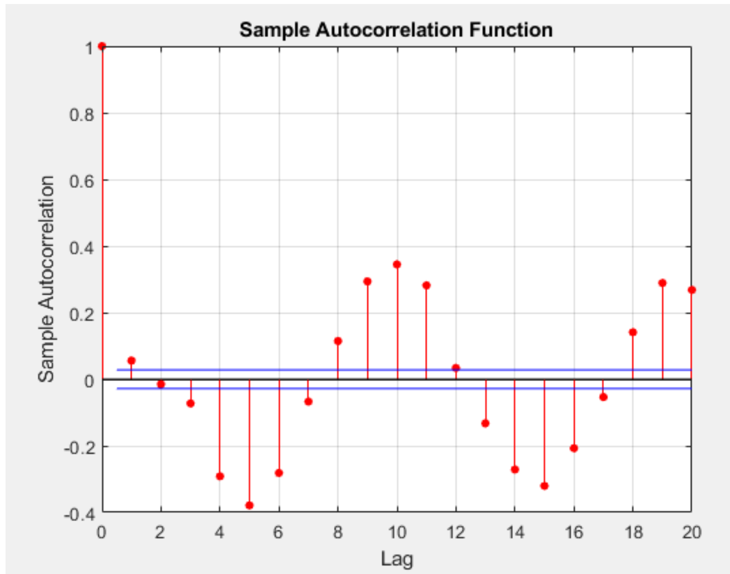
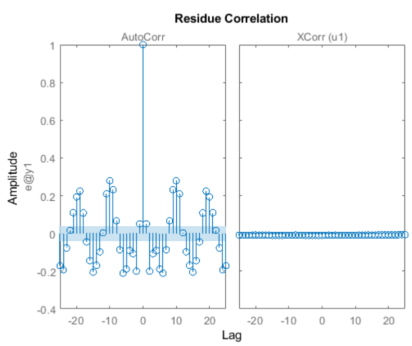
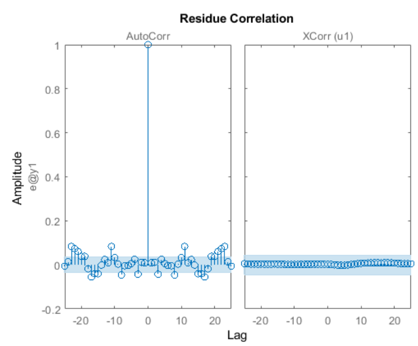


Figure 3: Whiteness test for the RLS model



(a) Calculated model



(b) Matlab model

Figure 4: Residuals for the ELS model

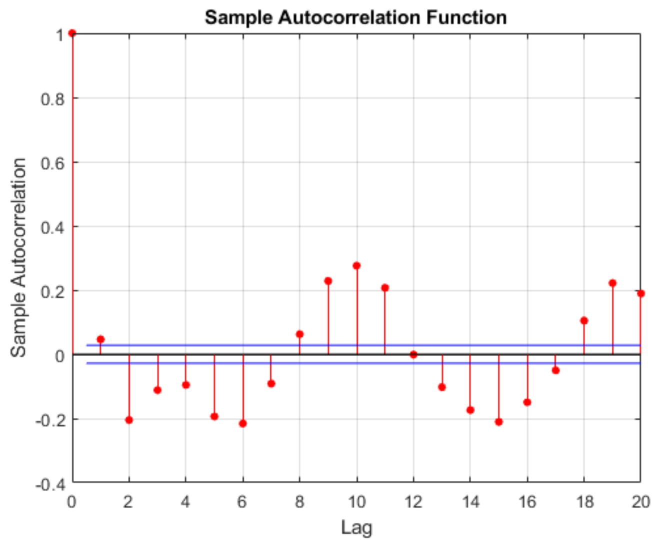
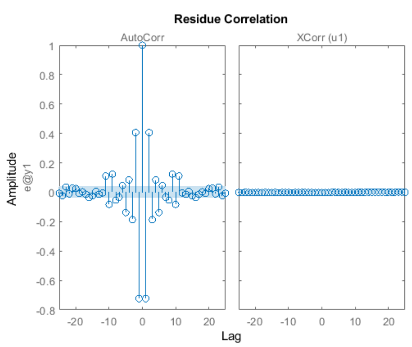
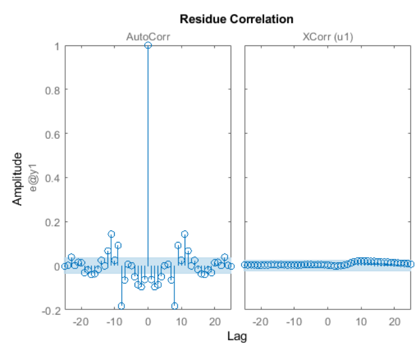


Figure 5: Whiteness test for the ELS model



(a) Calculated model



(b) Matlab model

Figure 6: Residuals for the GLS model

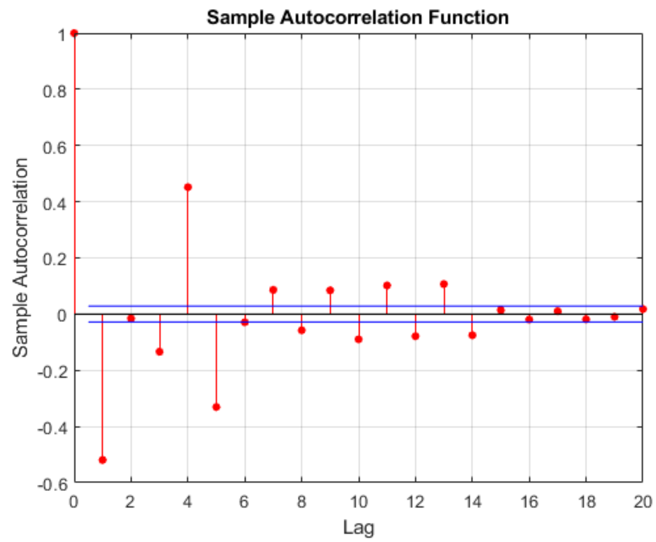
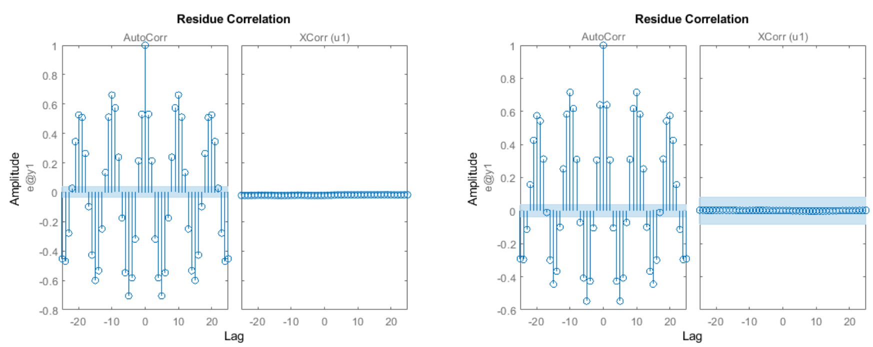


Figure 7: Whiteness test for the GLS model



(a) Calculated model

(b) Matlab model

Figure 8: Residuals for the Output Error method

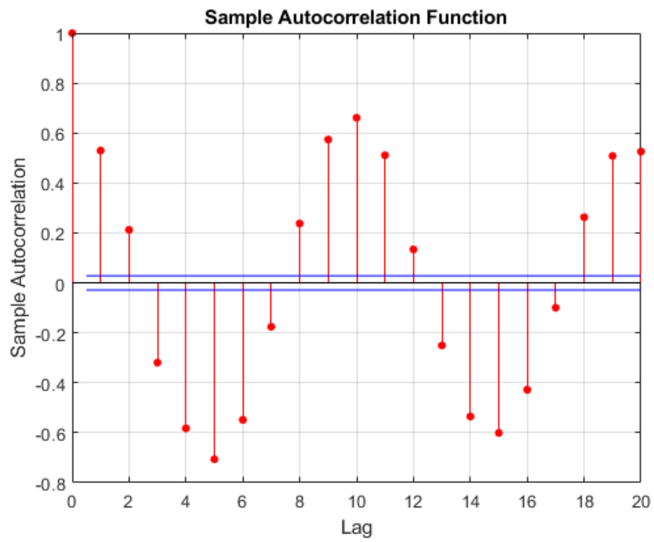
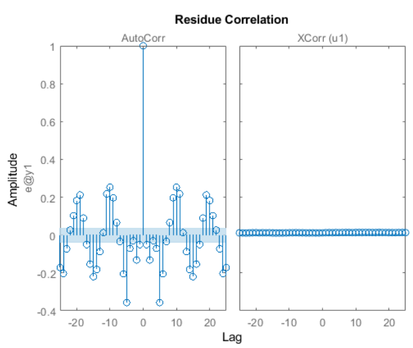
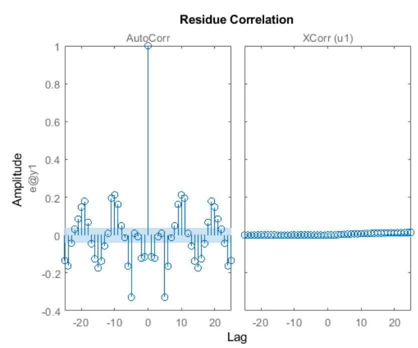


Figure 9: Whiteness test for the Output Error method

4.2 Model 2 $n_a = 5, n_b = 3, n_c = 10$



(a) Calculated model



(b) Matlab model

Figure 10: Residuals for the RLS model

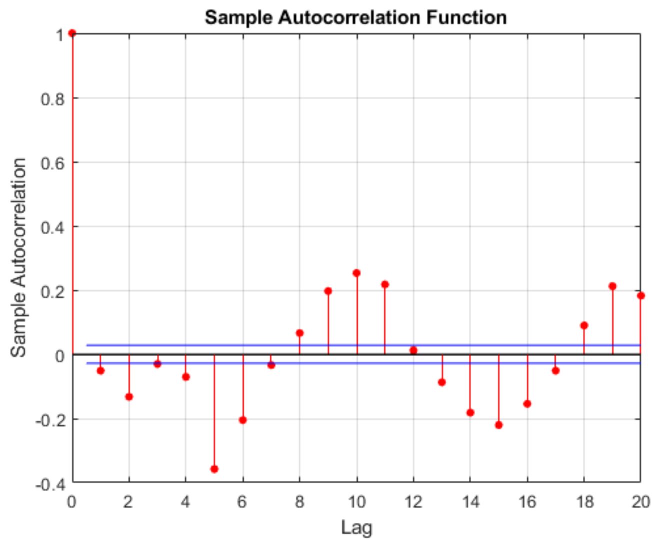
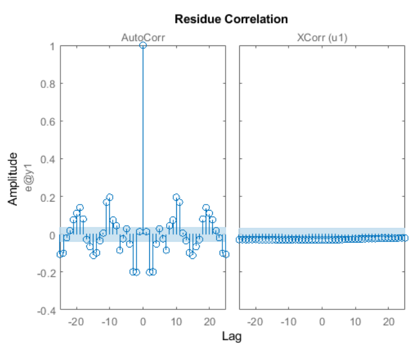
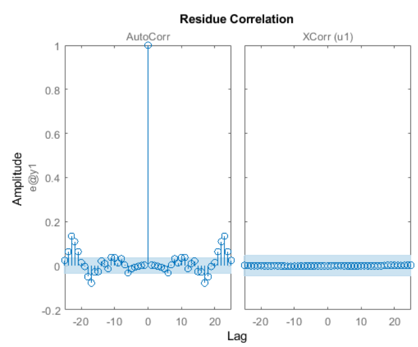


Figure 11: Whiteness test for the RLS model



(a) Calculated model



(b) Matlab model

Figure 12: Residuals for the ELS model

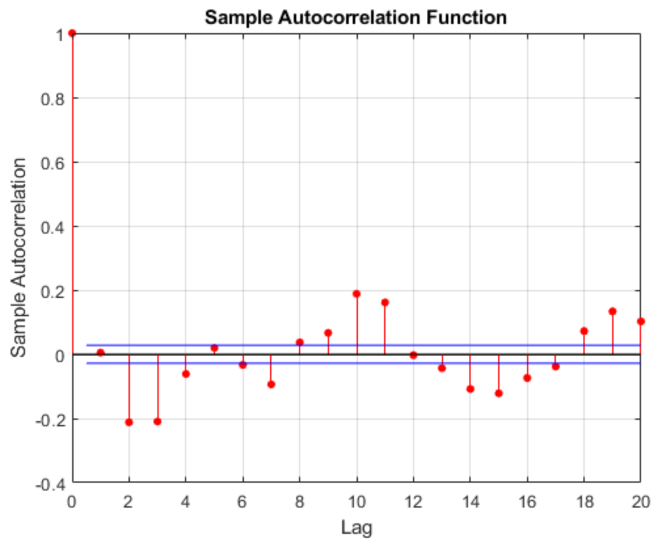
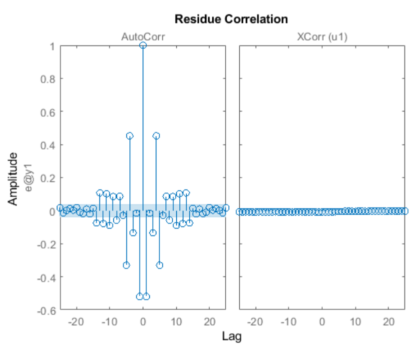
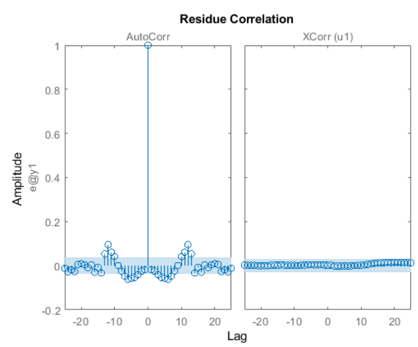


Figure 13: Whiteness test for the ELS model



(a) Calculated model



(b) Matlab model

Figure 14: Residuals for the GLS model

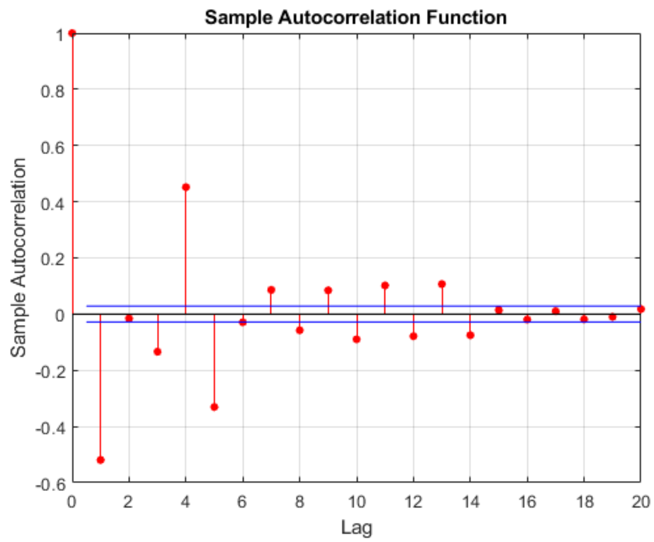
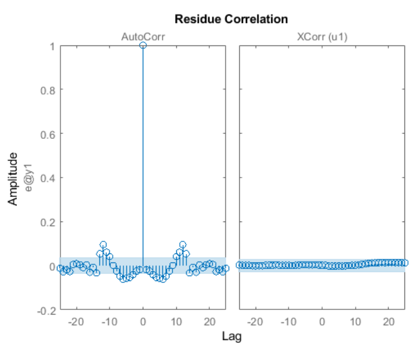
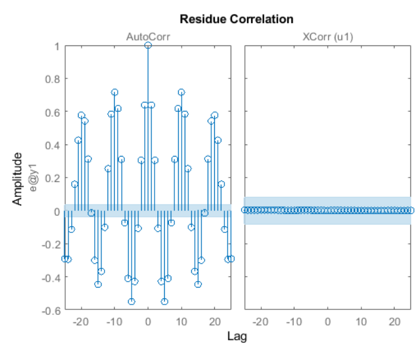


Figure 15: Whiteness test for the GLS model



(a) Calculated model



(b) Matlab model

Figure 16: Residuals for the Output Error method

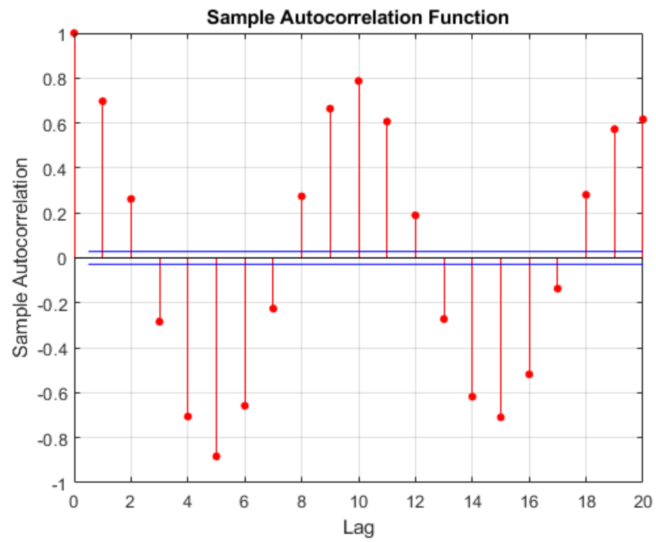
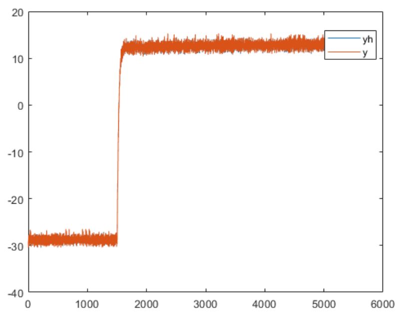
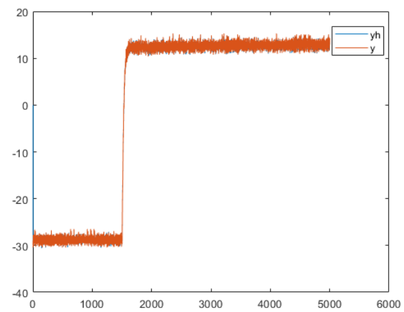


Figure 17: Whiteness test for the Output Error method

The output of the estimated models with the two different parameter sets are compared against the measured data below. we can see that with all the estimation methods, we obtained acceptable results

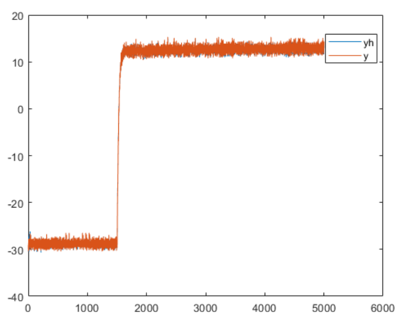


(a) Model 1

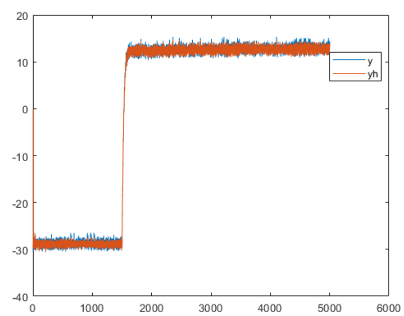


(b) Model 2

Figure 18: Outputs of the RLS models

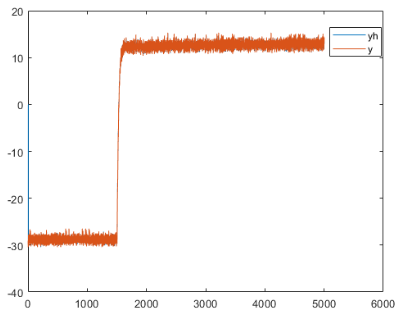


(a) Model 1

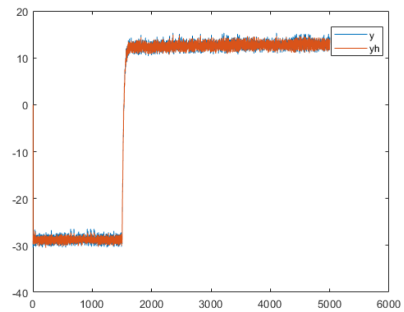


(b) Model 2

Figure 19: Outputs of the ELS models

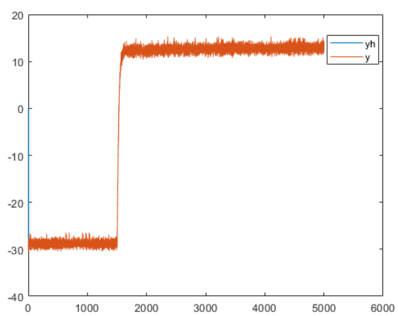


(a) Model 1

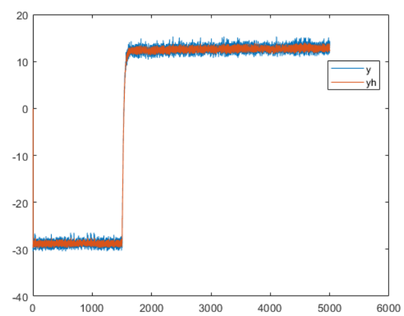


(b) Model 2

Figure 20: Outputs of the GLS models



(a) Model 1



(b) Model 2

Figure 21: Outputs of the Output error models