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SEPARATRIX SPLITTING THROUGH HIGH-FREQUENCY NON-SMOOTH PERTURBATIONS

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Abstract. Separatrices in integrable dynamical systems, if perturbed with analytic high frequency perturbations, split apart such that the splitting distance is exponentially small in the frequency parameter ω. This article utilizes a recent straightforward connection between the Melnikov function (which gives a measure of such a splitting) and a Fourier transform, to quantify such splitting under less smooth perturbations. If the perturbation is only piecewise C^k spatially, the splitting distance goes as ω^{-k-1} for large ω.
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1 Introduction

A paradigm for the generation of chaotic motion is when an integrable two-dimensional system is subjected to a time-periodic perturbation. If the original system contained a heteroclinic orbit (a connecting trajectory between two fixed points), the generic picture is that this separatrix splits apart after perturbation, creating a heteroclinic tangle and consequent chaotic motion within that phase space region. The width of this tangled region therefore provides a good measure of the amount of chaoticity in the perturbed system. Many articles have established the exponential smallness of this width with respect to the perturbing frequency [12, 21, 17, 13, 9, 16, 20, 8, 15]. These usually relate to specific versions of the forced pendulum equation [9, 16, 20, 8], analogous Hamiltonian systems [12, 21, 17, 13], or else to specific assumptions on spatial analyticity of the perturbation [15].

There have been few studies which assess the manifold splitting behavior under less smooth perturbations. The exceptions are [14] which deals with near identity mappings and flows generating such mappings, and [26] which examines a class of standard-like mappings. In each case – though not necessarily expressed in these words in these articles – the splitting distance has a reciprocal power dependence on the frequency, with the power depending on the smoothness.

This article provides a quick and easy derivation of this result in the *general* setting of continuous two-dimensional dynamical systems. The precise form of this power-law decay (including its coefficient) is obtained in terms of the smoothness of the perturbation in Theorem 4.1. The proof utilizes Fourier transform techniques, which are applicable here because the chaotic layer width is expressible in terms of a Fourier transform [2, 3]. The essential conclusion is that if the perturbation's kth derivative is piecewise Lipschitz (but its $(k+1)$st derivative is not), then the splitting distance goes as $|\omega|^{-k-1}$ for large frequencies ω. This result is useful in quantifying the competing roles of the smallness of the perturbation and of its high frequency, in obtaining a measure of the chaoticity of such systems.

2 Setting

The unperturbed (nonchaotic) flow is assumed to be given by

$$x' = \mathrm{J}\, D\, H(x)\,, \tag{1}$$

where $x \in \Omega$ which is an open subset of $\mathbb{R}^2$, $\mathrm{J} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, D is the gradient operator on Ω, and H is the Hamiltonian function. It shall be assumed that $H \in \mathrm{C}^\infty(\Omega)$, and that the solutions $x(t)$ are themselves in $\mathrm{C}^\infty(\mathbb{R})$. Suppose a and b are hyperbolic fixed points of (1), not necessarily distinct, each possessing one-dimensional stable and unstable manifolds. A branch of the unstable manifold of a, W_a^u, will be assumed to coincide with a branch of the stable manifold of b, W_b^s, to form a one-dimensional heteroclinic manifold Γ. This manifold consists of a heteroclinic trajectory $\bar{x}(t)$ which backwards asymptotes to a and forward asymptotes to b. Given the hyperbolicity of the fixed points, there is exponential decay of $\bar{x}(t)$ to a as $t \to -\infty$, and also to b as $t \to \infty$. A parametrization of Γ is possible by adopting heteroclinic coordinates [27]; $t \in \mathbb{R}$ is associated with the point $\bar{x}(-t)$.

The heteroclinic manifold serves as a flow separatrix in phase space. Under a perturbation, however, it splits. Consider a perturbation to (1) in the form

$$x' = \mathrm{J}\, D\, H(x) + \varepsilon\, g(x)\, \cos(\omega t - \beta)\,, \tag{2}$$

where $0 < \varepsilon \ll 1$, $g : \Omega \to \mathbb{R}^2$, β is some constant phase factor, and the frequency $\omega \gg 1$. The spatial part of the perturbation is assumed to satisfy the conditions (i) $g \in \mathrm{C}^2(\Omega)$, and (ii) g is uniformly bounded on Ω. With respect to the Poincaré map which samples the flow every $(2\pi/\omega)$ time units, the fixed points a and b perturb to new fixed points, while retaining their stable and unstable manifold structures [11, 7, 19]. These manifolds need not coincide, however, leading to a splitting. A sketch of the generically expected behavior appears in Figure 1, where the solid curve indicate the perturbed manifolds, with the dashed curve being Γ. The resulting *heteroclinic tangle* is a signature of strong stretching and folding in this region of phase space, leading to chaotic mixing through the Smale-Birkhoff Theorem [18].

Consider the (signed) distance $d(t,\varepsilon)$ from the stable manifold to the unstable manifold, measured in the direction of $\nabla H(\bar{x}(-t))$ at a point on Γ

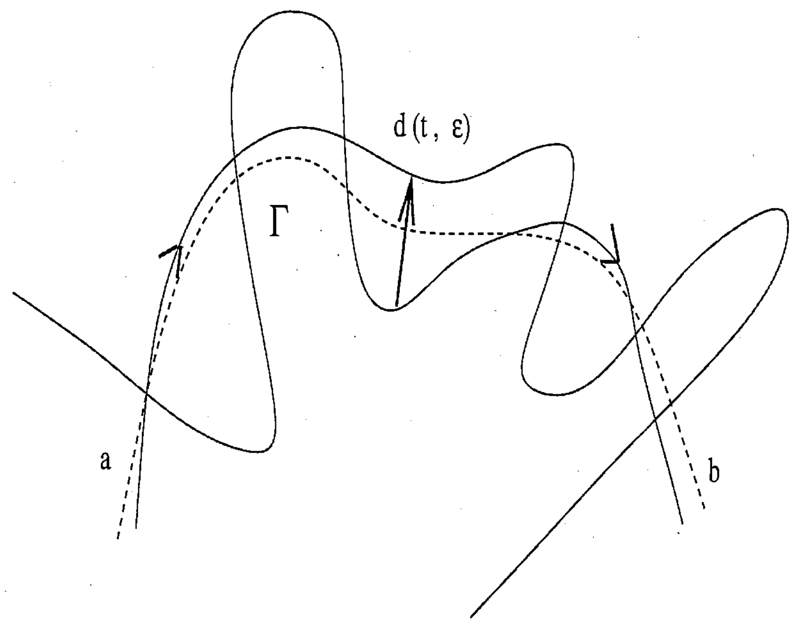

Figure 1: Heteroclinic tangle. Here Γ is the unperturbed heteroclinic (dashed curve), and the solid curves are the perturbed manifolds.

parametrized by t; see Figure 1. This can be expanded in ε as [18, 1, 27]

$$d(t,\varepsilon) = \varepsilon\, \frac{M(t)}{|D\, H(\bar{x}(-t))|} + \mathcal{O}(\varepsilon^2)\,, \tag{3}$$

where the quantity $M(t)$ is called the Melnikov function. This quantity encodes the leading-order splitting information, and is known to be given by [18, 1, 27]

$$M(t) = \int_{-\infty}^{\infty} D\, H(\bar{x}(\tau)) \cdot g(\bar{x}(\tau))\, \cos[\omega(t+\tau) - \beta]\, \mathrm{d}\tau\,. \tag{4}$$

While (3) shows that the width of the stochastic layer is given by the Melnikov function, it is also of importance in other measures of chaoticity such as lobe areas [22] and direct chaotic flux [2, 3], and is generally useful in assessing fluid transport [22, 23, 4, 5, 24, 6]. Define the quantity

$$\lambda(t) := D\, H(\bar{x}(-t)) \cdot g(\bar{x}(-t))\,, \tag{5}$$

and adopt the Fourier transform definition

$$\widehat{\lambda}(s) := \int_{-\infty}^{\infty} \lambda(t) \exp[-i\, s\, t]\, \mathrm{d}t\,.$$

This exists classically since $\lambda \in L^p(\mathbb{R})$ for all $p \in [1,\infty]$ by the exponential decay properties of DH and the boundedness of g.

Lemma 2.1 *The Melnikov function (4) is expressible as*

$$M(t) = \left|\widehat{\lambda}(\omega)\right| \cos\left[\omega t - \beta + Arg\left(\widehat{\lambda}(\omega)\right)\right] . \tag{6}$$

Proof: This is a trivial trigonometric manipulation; see [3, 2]. □

The ability of representing the leading-order splitting in terms of Fourier transforms enables more detailed investigations on its behavior than previously possible, since Fourier transform machinery can be utilized. In particular, (6) indicates that the amplitude of splitting (modulo the normalising factor in (3) which relates to the unperturbed flow speed) is governed directly by the modulus of the Fourier transform of $\lambda(t)$. Balasuriya [2, 3] in fact goes further, extending to more general time-periodicity, and demonstrating that the chaotic flux generated is directly proportional to this modulus. To investigate the size of the manifold splitting for high frequencies, one needs to analyze the behavior of $\widehat{\lambda}(\omega)$ at large ω. Such is addressed in Section 3 within a general Fourier transform setting, the results of which will be applied to $\lambda(t)$ in Section 4.

3 Fourier transform analysis

A function $f:\mathbb{R}\to\mathbb{R}$ shall be called *piecewise Lipschitz continuous (p.L.c.)* if it satisfies the following conditions:

(i) The function f is defined on all $\mathbb{R}$ except possibly at a finite number of (ordered) isolated points $\{t_j\}$, $j = 1, 2, \ldots, (N-1)$. (Extend this sequence to $j=0$ and $j=N$ by setting $t_0 = -\infty$ and $t_N = \infty$.)

(ii) In each open interval (t_{j-1}, t_j), f is Lipschitz continuous.

(iii) There exists at least one genuine jump discontinuity at one of the endpoints, i.e., at least one of the J_js defined below is non-zero:

$$J_j := \lim_{t\to t_j^+} f(t) - \lim_{t\to t_j^-} f(t) \quad , \quad j \in \{1, 2, \ldots, (N-1)\} .$$

Define the *piecewise Lipschitz continuous space of order* k by

$$\mathcal{P}^k(\mathbb{R}) := \left\{ f:\mathbb{R}\to\mathbb{R} \ \text{ such that } \ f^{(k)} \text{ is p.L.c.} \right\} ,$$

where k is a non-negative integer, and $f^{(k)}$ represents the kth derivative of f.

The Sobolev spaces for functions defined from an interval $\mathcal{I} \subset \mathbb{R}$ to $\mathbb{R}$ shall be denoted by

$$W_p^m(\mathcal{I}) := \left\{ f : \ \sum_{j=0}^{m} \left\| f^{(j)} \right\|_p < \infty \right\} ,$$

where $\|\cdot\|_p$ is the $L^p(\mathcal{I})$ norm. Define additionally the function space $\mathcal{E}^k$ for non-negative integers k by

$$\mathcal{E}^k(\mathbb{R}) := \left\{ f : \ f \in W_1^k(\mathbb{R}) \cap W_1^{k+1}((-\infty,-T)) \cap W_1^{k+1}((T,\infty)) \ \text{ for some } T > 0 \right\} .$$

The main result that will be argued in this section is as follows: if $f \in \mathcal{P}^k(\mathbb{R}) \cap \mathcal{E}^k(\mathbb{R})$, then $\left|\widehat{f}(\omega)\right| \sim |\omega|^{-k-1}$ for large $|\omega|$. A precise formula for this leading-order term will also be established.

First note that for $f \in \mathcal{P}^k(\mathbb{R}) \cap W_1^k(\mathbb{R})$, the Fourier transform exists classically for $f^{(j)}$ for $j = 0, 1, 2, \ldots, k$. Since $f^{(k)} \in L^1(\mathbb{R})$, the Riemann-Lebesgue Lemma (see for example [25]) asserts that $|\widehat{f^{(j)}}(\omega)| \to 0$ as $\omega \to \pm\infty$. By repeated integration by parts,

$$\widehat{f^{(k)}}(\omega) := \int_{-\infty}^{\infty} f^{(k)}(t) \exp(-i\,\omega\,t)\ \mathrm{d}t = (i\,\omega)^k\ \hat{f}(\omega) . \tag{7}$$

The classical result of (7) extends to the distribution $f^{(k+1)}$ as well.

Lemma 3.1 *For* $f \in \mathcal{P}^k(\mathbb{R}) \cap W_1^k(\mathbb{R})$,

$$\widehat{f^{(k+1)}}(\omega) = (i\,\omega)\ \widehat{f^{(k)}}(\omega) = (i\,\omega)^{k+1}\ \widehat{f}(\omega) . \tag{8}$$

Proof: Write

$$f^{(k)}(t) = \sum_{j=1}^{N} \phi_j(t) \left[u(t - t_{j-1}) - u(t - t_j)\right] \tag{9}$$

in the sense of distribution. The finite points of jump discontinuity are identified by the $\{t_j\}$ for $j = 1, 2, \ldots, (N-1)$, and additionally define $t_0 := -\infty$ and $t_N := \infty$. The functions ϕ_j are $f^{(k)}$'s representation in the jth interval (t_{j-1}, t_j), and are uniformly Lipschitz in each interval since $f \in \mathcal{E}^k$. By Rademacher's Theorem [10], the derivative ϕ_j' exist almost everywhere (with respect to Lebesgue measure) in each such interval. Thus the derivative of $f^{(k)}$, in a distributional sense, is

$$f^{(k+1)}(t) = \sum_{j=1}^{N} \left\{ \phi_j'(t) \left[u(t - t_{j-1}) - u(t - t_j)\right] + \phi_j(t) \left[\delta(t - t_{j-1}) - \delta(t - t_j)\right] \right\} ,$$

where $\delta(\cdot)$ is the Dirac delta distribution. The Fourier transform of $f^{(k+1)}$ is therefore given by

$$\begin{aligned}
\widehat{f^{(k+1)}}(\omega) &= C(\omega) + D(\omega)\,, \quad \text{where} \qquad (10)\\
C(\omega) &:= \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \phi_j'(t) \exp(-i\,\omega\, t)\, \mathrm{d}t\,,\\
D(\omega) &:= \sum_{j=1}^{N} \int_{-\infty}^{\infty} \phi_j(t) \left[\delta(t-t_{j-1}) - \delta(t-t_j)\right] \exp(-i\,\omega\, t)\, \mathrm{d}t\,.
\end{aligned}$$

By integrating $C(\omega)$ by parts,

$$\begin{aligned}
C(\omega) &= \sum_{j=1}^{N}\left\{ -(-i\omega)\int_{t_{j-1}}^{t_j} \phi_j(t) \exp(-i\,\omega\, t)\, \mathrm{d}t + \Big[\phi_j(t)\exp(-i\omega t)\Big]_{t_{j-1}^+}^{t_j^-}\right\}\\
&= (i\,\omega)\sum_{j=1}^{N}\int_{-\infty}^{\infty}\phi_j(t)\left[u(t-t_{j-1}) - u(t-t_j)\right] e^{-i\omega t}\,\mathrm{d}t\\
&\qquad + \sum_{j=1}^{N}\left[\phi_j(t_j^-)\exp(-i\omega t_j) - \phi_j(t_{j-1}^+)\exp(-i\omega t_{j-1})\right]\\
&= (i\,\omega)\,\widehat{f^k}(\omega) + \sum_{j=1}^{N}\left[\phi_j(t_j^-)\exp(-i\omega t_j) - \phi_j(t_{j-1}^+)\exp(-i\omega t_{j-1})\right]\\
&=: (i\,\omega)\,\widehat{f^k}(\omega) + \Xi\,.
\end{aligned}$$

where the above serves to define the quantity Ξ. Note that Ξ must be finite, since by assumption all the jumps in $f^{(k)}$ are finite. By integrating $D(\omega)$,

$$D(\omega) = \sum_{j=1}^{N}\left[\phi_j(t_{j-1}^+)\exp(-i\,\omega\, t_{j-1}) - \phi_j(t_j^-)\exp(-i\,\omega\, t_j)\right] = -\,\Xi\,. \qquad (11)$$

Substituting these results into (10),

$$\widehat{f^{(k+1)}}(\omega) = (i\,\omega)\,\widehat{f^{(k)}}(\omega) + \Xi - \Xi = (i\,\omega)\,\widehat{f^{(k)}}(\omega)\,.$$

Applying the classical result for $\widehat{f^{(k)}}$ from (7) now gives the result. □

Lemma 3.2 *If $f \in \mathcal{P}^k(\mathbb{R}) \cap \mathcal{E}^k(\mathbb{R})$, then in the limit of large $|\omega|$,*

$$\widehat{f^{(k+1)}}(\omega) \sim \sum_{j=1}^{N-1} J_j \exp(-i\,\omega\, t_j) \qquad (12)$$

where $J_j = f^{(k)}(t_j^+) - f^{(k)}(t_j^-)$ is the jump of the function $f^{(k)}$ at the jump point t_j.

Proof: Once again, utilize the splitting of $\widehat{f^{(k+1)}}$ into $C(\omega) + D(\omega)$ as given in (10). Notice that

$$C(\omega) = \int_{-\infty}^{\infty}\left(\sum_{j=1}^{N}\phi_j'(t)\left[u(t_{j-1}) - u(t_j)\right]\right)\exp(-i\,\omega\, t)\,\mathrm{d}t$$

is the Fourier transform of an $\mathrm{L}^1(\mathbb{R})$ function. Hence, by the Riemann-Lebesgue Lemma, $C(\omega) \to 0$ as $\omega \to \pm\infty$. On the other hand, from (11), since the boundary terms $\phi_1(-\infty)$ and $\phi_N(\infty)$ go to zero,

$$\begin{aligned}
D(\omega) &= \sum_{j=2}^{N}\phi_j(t_{j-1}^+)\exp(-i\,\omega\, t_{j-1}) - \sum_{j=1}^{N-1}\phi_j(t_j^-)\exp(-i\,\omega\, t_j)\\
&= \sum_{j=1}^{N-1}\left[f^{(k)}(t_j^+) - f^{(k)}(t_j^-)\right]\exp(-i\,\omega\, t_j)\\
&= \sum_{j=1}^{N-1} J_j \exp(-i\,\omega\, t_j)\,.
\end{aligned}$$

□

Corollary 3.1 *For $f \in \mathcal{P}^k(\mathbb{R}) \cap \mathcal{E}^k(\mathbb{R})$, $\left|\widehat{f^{(k+1)}}(\omega)\right|$ does not decay to zero in either of the limits $\omega \to \pm\infty$.*

Proof: For $f \in \mathcal{P}^k$, at least one of the J_js in (12) must be non-zero. Thus, $\widehat{f^{(k+1)}}(\omega)$ approaches a genuine quasi-periodic function as $\omega \to \pm\infty$. □

Proposition 3.1 *For $f \in \mathcal{P}^k(\mathbb{R}) \cap \mathcal{E}^k(\mathbb{R})$, as $\omega \to \pm\infty$,*

$$\left|\widehat{f}(\omega)\right| \sim \frac{|\psi(\omega)|}{|\omega|^{k+1}}\,, \qquad (13)$$

where $\psi(\omega)$ is the non-zero quasi-periodic function as given in (12).

Proof: From Lemmas 3.1 and 3.2 and Corollary 3.1,

$$\left|\widehat{f}(\omega)\right| = |\omega|^{-k-1}\left|\widehat{f^{(k+1)}}(\omega)\right| \sim \frac{|\psi(\omega)|}{|\omega|^{k+1}}\,,$$

where $\psi(\omega)$ is the non-zero quasi-periodic function as given in (12). □

Remark 3.1 The function $\psi(\omega)$ may be *periodic* in some cases; this depends on the commensurability of the jump points $\{t_j\}$.

Remark 3.2 If $k = \infty$, then $f \in \mathcal{S}$ (the Schwartz space), and hence $\widehat{f}(\omega)$ decays faster than any polynomial power of ω.

4 Manifold splitting

The Fourier transform results are now applied to the manifold splitting as described in Section 2. A leading-order measure of the splitting of manifolds (3) is given by the Melnikov function (6). This in turn relates to the Fourier transform of the function $\lambda(t)$ given in (5), which can be recast as

$$\lambda(t) = |D\,H\,(\bar{x}(-t))|\; g_H\,(\bar{x}(-t))\ , \tag{14}$$

where $g_H : \Gamma \to \mathbb{R}$ is the component of g in the direction of $D\,H$ (normal to Γ). Since $H \in C^\infty(\Omega)$, the smoothness of $\lambda(t)$ is affected only through g_H. Choose the arclength parametrization y on the one-dimensional curve $\bar{\Gamma}$; this is the closure of the heteroclinic manifold Γ. For functions h defined from this set to $\mathbb{R}$, form the following definitions analogous to those of Section 3.

A function $h : \bar{\Gamma} \to \mathbb{R}$ is *piecewise Lipschitz continuous* if

(i) It is defined on all $\bar{\Gamma}$ except possibly at a finite number of isolated points $\{y_j\}$, $j = 1, 2, \ldots, (N-1)$ which are ordered in the reverse direction of the heteroclinic flow. (Extend this sequence to $j = 0$ and $j = N$ by setting $y_0 = 0$ to be equivalent to b and $y_N = \text{length}(\Gamma)$ to a.)

(ii) The function h is Lipschitz continuous in each parametric interval (y_{j-1}, y_j), where $j = 1, 2, \ldots, N$.

(iii) At least one of the J_js defined below is non-zero:

$$J_j := \lim_{y \to y_j^+} h(y) - \lim_{y \to y_j^-} h(y)\ ,\quad j \in \{1, 2, \ldots, (N-1)\}\ .$$

The *Lipschitz continuous space of order* k, $\mathcal{P}^k(\bar{\Gamma})$, is the set of functions on this space whose kth derivative with respect to the arclength parameter on Γ is piecewise Lipschitz continuous.

Theorem 4.1 *Suppose the perturbation g in (2) is such that $g_H \in \mathcal{P}^k(\bar{\Gamma})$, and that $g_H^{(k)}$'s points of jump discontinuity along Γ are represented in reverse order by $\{y_j\}$, where $j = 1, 2, \ldots, N-1$. Define the values $\{t_j\}$ through $y_j = \bar{x}(-t_j)$. Then, the Melnikov function is*

$$M(t) \;=\; A(\omega)\cos\left[\omega t - \beta + Arg\left(\widehat{\lambda}(\omega)\right)\right]\ ,$$

where the amplitude $A(\omega)$ in the asymptotic limit as $|\omega| \to \infty$ is given by

$$A(\omega) \sim \frac{1}{|\omega|^{k+1}}\left|\sum_{j=1}^{N-1} |D\,H\,(y_j)|\left[g_H^{(k)}(y_j^+) - g_H^{(k)}(y_j^-)\right]\exp(i\,\omega\,t_j)\right|\ .$$

Proof: From (14), the kth time-derivative of $\lambda(t)$ is almost everywhere given by

$$\lambda^{(k)}(t) = \sum_{j=0}^{k}\binom{k}{j}\frac{d^j}{dt^j}\left[g_H\,(\bar{x}(-t))\right]\frac{d^{k-j}}{dt^{k-j}}|D\,H\,(\bar{x}(-t))|\ , \tag{15}$$

where it should be noted that $|D\,H\,(\bar{x}(-t))|$ is infinitely differentiable since $D\,H$ never becomes zero on Γ (which would correspond to a fixed point), and $H \in C^\infty$. Thus, $\lambda \in \mathcal{P}^k(\mathbb{R})$ if and only if $g_H \in \mathcal{P}^k(\bar{\Gamma})$. Now, the quantity $|D\,H\,(\bar{x}(-t))| \in W_p^m(\mathbb{R})$ for any $m \in \mathbb{N}$ and $p \in [1, \infty]$, since it approaches an exponentially decaying function in each of the limits $t \to \pm\infty$ in a C^∞ fashion. Thus in particular $\lambda(t) \in W_1^k(\mathbb{R})$.

Choose $\bar{y} \in (y_0, y_1)$. The function $g_H^{(k)}$ is Lipschitz in $(y_0, \bar{y})$, and hence its derivative $g_H^{(k+1)}$ is defined almost everywhere in this set. Thus, there exists T such that $g_H^{(k+1)}(\bar{x}(-t))$ is defined and bounded in the set $t \in (T, \infty)$ when a measure zero set is possibly excluded. The strong exponential decay properties of $|D\,H\,(\bar{x}(-t))|$ therefore ensure that $\lambda(t) \in W_1^{k+1}((T, \infty))$ for some large enough T. A similar argument works for $W_1^{k+1}((-\infty, -T))$, and thus $\lambda \in \mathcal{E}^k(\mathbb{R})$. A direct application of Proposition 3.1 is therefore possible. Using also the expression in (12),

$$\begin{aligned}\left|\widehat{\lambda}\right| &\sim \frac{1}{|\omega|^{k+1}}\left|\sum_{j=1}^{N-1}\left[\lambda^{(k)}(t_j^+) - \lambda^{(k)}(t_j^-)\right]\exp(-i\,\omega\,t_j)\right| \\ &= \frac{1}{|\omega|^{k+1}}\left|\sum_{j=1}^{N-1}|D\,H\,(y_j)|\left[g_H^{(k)}(y_j^+) - g_H^{(k)}(y_j^-)\right]\exp(i\,\omega\,t_j)\right|\ ,\end{aligned}$$

since g_H is C^{k-1}-smooth along Γ, and using (15). The result follows by using Lemma 2.1. □

Remark 4.1 In this reduced smoothness scenario, the manifold splitting as a function of high frequencies behaves like a quasi-periodic oscillation within a reciprocal power-law envelope.

Remark 4.2 If a shifted time-parametrization $\bar{x}(t-\tau)$ is used instead for Γ, there is no change in this amplitude, since this contributes the multiplicative term $|\exp(i\omega\tau)|$ to $A(\omega)$.

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