NAME:

MAT 225 (Ordinary Differential Equations)

Examination 2

May 9, 2012

Answer all questions in the space provided. The points awarded for each problem are indicated. No notes or books are permitted. Tables of Laplace transforms are provided.

Please write and sign the Connecticut College Honor Pledge in the space below.
(I promise neither to give nor receive any aid in this examination.)

1. [5 points] Find the general solution to

$$\frac{d\mathbf{Y}}{dy} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \mathbf{Y}\,.$$

2. [6 points] Use Laplace transforms to solve the initial-value problem

$$\frac{dy}{dt} - 3y = h(t)\ ,\ y(0) = 0\ \ ,\quad \text{where}\quad h(t) = \begin{cases} 6 & \text{if}\ \ t < 3 \\ 0 & \text{if}\ \ t \geq 3 \end{cases}\ .$$

3. [5 points] Find the solution to the initial value problem

$$\frac{d^2y}{dt^2} + 4\,\frac{dy}{dt} + 3\,y = e^{-2t} \qquad ; \qquad y(0) = 1\ ,\ y'(0) = 0\,.$$

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4. [6 points] Use the idea of complexification to find a particular solution to the equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} = e^{-t}\cos t\,.$$

5. Short answer questions which are unrelated to each other. [3 points each]

(a) Sketch the phase plane of

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{Y}.$$

(b) Find $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2s+5}\right\}$.

(c) If $Y(s) = \mathcal{L}\{y(t)\}$, find a formula for $\mathcal{L}\left\{\frac{d^3y}{dt^3}\right\}$ in terms of $Y(s)$ and values of $y(t)$ and its derivatives at $t=0$.

(d) Convert the third-order differential equation

$$\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} - \frac{dy}{dt} = 7y$$

into a first-order system, and write it in matrix notation.

(e) If the solution $v(t)$ to

$$\frac{d^2v}{dt^2} + 9\,v = 2\sin(\omega t) \quad ; \quad v(0) = 1\ ,\ v'(0) = -1$$

satisfies $|v(t)| < 7$ for all $t \in \mathbb{R}$, what can you say about the value of the parameter ω?

(f) Find $\mathcal{L}\{\delta_1(t)u_2(t)\}$ (with explanations).

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Table 6.1
Frequently Encountered Laplace Transforms.

$y(t) = \mathcal{L}^{-1}[Y]$	$Y(s) = \mathcal{L}[y]$	$y(t) = \mathcal{L}^{-1}[Y]$	$Y(s) = \mathcal{L}[y]$
$y(t) = e^{at}$	$Y(s) = \dfrac{1}{s-a} \quad (s > a)$	$y(t) = t^n$	$Y(s) = \dfrac{n!}{s^{n+1}} \quad (s > 0)$
$y(t) = \sin \omega t$	$Y(s) = \dfrac{\omega}{s^2+\omega^2}$	$y(t) = \cos \omega t$	$Y(s) = \dfrac{s}{s^2+\omega^2}$
$y(t) = e^{at} \sin \omega t$	$Y(s) = \dfrac{\omega}{(s-a)^2+\omega^2}$	$y(t) = e^{at} \cos \omega t$	$Y(s) = \dfrac{s-a}{(s-a)^2+\omega^2}$
$y(t) = t \sin \omega t$	$Y(s) = \dfrac{2\omega s}{(s^2+\omega^2)^2}$	$y(t) = t \cos \omega t$	$Y(s) = \dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$y(t) = u_a(t)$	$Y(s) = \dfrac{e^{-as}}{s} \quad (s > 0)$	$y(t) = \delta_a(t)$	$Y(s) = e^{-as}$

Table 6.2
Rules for Laplace Transforms:
Given functions $y(t)$ and $w(t)$ with $\mathcal{L}[y] = Y(s)$ and $\mathcal{L}[w] = W(s)$ and constants α and a.

Rule for Laplace Transform	Rule for Inverse Laplace Transform
$\mathcal{L}\left[\dfrac{dy}{dt}\right] = s\mathcal{L}[y] - y(0) = sY(s) - y(0)$	
$\mathcal{L}[y+w] = \mathcal{L}[y] + \mathcal{L}[w] = Y(s) + W(s)$	$\mathcal{L}^{-1}[Y+W] = \mathcal{L}^{-1}[Y] + \mathcal{L}^{-1}[W] = y(t) + w(t)$
$\mathcal{L}[\alpha y] = \alpha\mathcal{L}[y] = \alpha Y(s)$	$\mathcal{L}^{-1}[\alpha Y] = \alpha\mathcal{L}^{-1}[Y] = \alpha y(t)$
$\mathcal{L}[u_a(t)y(t-a)] = e^{-as}\mathcal{L}[y] = e^{-as}Y(s)$	$\mathcal{L}^{-1}[e^{-as}Y] = u_a(t)y(t-a)$
$\mathcal{L}[e^{at}y(t)] = Y(s-a)$	$\mathcal{L}^{-1}[Y(s-a)] = e^{at}\mathcal{L}^{-1}[Y] = e^{at}y(t)$