NAME:

MAT 105-03

Instructor: Balasuriya

Introduction To Mathematical Thought

Final Examination

Fall, 2011

- *This exam consists of Part A: Multiple Choice, and Part B: Short Answers.*
- *The exam is worth 100 points in total: 1 point per multiple-choice question, and the points awarded per short-answer question are indicated.*
- *No notes or books are permitted.*
- *A formula sheet and table of random digits appear on the last two pages of this exam.*
- ***Calculators are permitted.***

Please write and sign the Connecticut College Honor Pledge in the space below.
(I promise neither to give nor receive any aid in this examination.)

PART A (Multiple Choice): Circle the most appropriate answer. [1 point each]

1. Kenneth Arrow has determined that

(a) One cannot ensure fairness in a voting method if there are more than 2 candidates

(b) It is impossible to have a non-manipulable voting method

(c) It is impossible to traverse a graph while visiting each edge exactly once if there are nodes whose valence is odd

(d) One cannot be assured of finding the minimum-cost spanning tree in a graph by using the sorted edges algorithm

2. If a graph has a path which visits every vertex exactly once and returns to the original vertex, this path is

(a) a tree, but may not be a minimum-cost spanning tree

(b) a Hamiltonian circuit

(c) an Euler circuit

(d) a Condorcet path

3. Which of these statistical quantities measures the spread of a data set?

(a) Mean

(b) Correlation

(c) Standard deviation

(d) Median

4. If using the sorted edges algorithm on a graph, you

(a) start at a vertex and go to the vertex which is connected to the original vertex with an edge of least weight

(b) start at a vertex and choose edges which are connected to this vertex such that no circuit is formed

(c) list the edges in order of decreasing weight, and choose the edges one by one so that no circuits are formed

(d) list the edges in order of increasing weight, and choose the edges one by one so that no vertex has three edges emanating from it

5. Which of these is not a person for whom a voting *method* is named?

(a) Borda

(b) Hare

(c) Pareto

(d) Condorcet

6. Suppose the edges of a graph represent streets that must be plowed after a snowstorm. To eulerize the graph, four edges must be added. The real-world implication of this is that

(a) four streets will not be plowed

(b) four streets will be traversed twice

(c) the corresponding four streets will need to be plowed first

(d) four new streets will need to be built

7. A study published on the internet claims that a person's Body Mass Index (BMI) has a positive linear relationship to the person's daily caloric intake (C), as given by $\text{BMI} = 42 + 3.2\,\text{C}$, with a correlation value of $r = -0.98$. You find this conclusion unsatisfactory because

(a) the correlation can never be negative

(b) the relationship given between BMI and C does not demonstrate a *positive* relationship

(c) the value of the correlation indicates that the relationship is weak

(d) the slope of the regression line and the correlation have opposite signs

8. In a two candidate voting situation, the concept of manipulability is equivalent to

(a) the Pareto condition

(b) independence of irrelevant alternatives

(c) monotonicity

(d) the equal treatment of voters and candidates

9. The scale of scores on an IQ test is approximately normal, with mean 100 and standard deviation 15. The organization MENSA, which calls itself "the high IQ society," requires an IQ score of 130 or higher for membership. What percent of adults would qualify for membership?

(a) 95%
(b) 15%
(c) 5%
(d) 2.5%

10. A class has 27 students, of whom 15 are female. How many different ways can you form a committee of 3 students from this class, if at least one of the committee has to be female?

(a) $27 \times 26 \times 25$
(b) $27 \times 26 \times 15$
(c) $12 \times 11 \times 15$
(d) $15 \times 26 \times 25$

11. For which of the following cases would it be most desirable to use the technique of finding an Euler circuit?

(a) Checking all the traffic lights in a small town
(b) Checking the water mains in a small town
(c) Determining where to place fire hydrants in a small town
(d) Determining where to run electric cables in a small town

12. Which of these people has *not* made a contribution to the theory of voting methods?

(a) Leonhard Euler
(b) Mark Satterthwaite
(c) Kenneth Arrow
(d) John Banzhaf

13. In graph theory, we learnt of many methods (algorithms) for solving certain types of problems. Which of the following is guaranteed to give the correct solution, *and* is an efficient method?

(a) nearest-neighbor algorithm

(b) sorted-edges algorithm

(c) Kruskal's algorithm

(d) brute force method

14. Which of the following voting methods is monotone?

(a) Plurality with runoff

(b) Hare method

(c) Borda-count

(d) None of these

15. The traveling salesperson problem is

(a) a minimum-cost Hamiltonian circuit problem

(b) a minimum-cost spanning tree problem

(c) an order-requirement digraph problem

(d) an Euler circuit problem

16. Normal distributions

(a) have a median which is larger than the mean

(b) have 95% of observations within one standard deviation from the mean

(c) are symmetric

(d) have strong correlations

17. A critical voter in a coalition is a voter

(a) who has the most votes

(b) who has veto power

(c) who has fewer votes than the number of extra votes for the coalition

(d) whose defection changes the coalition from a winning coalition to a losing coalition

18. A data set comprises a person's sugar consumption as the exploratory variable and the person's dental health (measured as the number of healthy teeth at age 50) as the response variable. The correlation is calculated to be $r = -0.93$. *Based only on this information*, you can conclude that

(a) there is *no* linear relationship between a person's sugar consumption and dental health

(b) increased sugar consumption *causes* worse teeth

(c) people with more healthy teeth tend to have a smaller sugar consumption

(d) if you eat less sugar you *will* have more healthy teeth

19. If a connected graph has 8 vertices of odd valence and 6 vertices of even valence, what is the absolute minimum number of duplicate edges that are needed to eulerize the graph?

(a) 3

(b) 4

(c) 6

(d) 8

20. Which of these is not a fairness criterion for voting methods?

(a) the Pareto condition

(b) monotonicity

(c) correlation

(d) independence of irrelevant alternatives

21. In a weighted voting system, the power of each voter is well represented by

(a) the voter's "weight" (i.e., number of votes the voter can cast)

(b) the "quota"

(c) the voter's Banzhaf power index

(d) the voter's Borda count

22. Based on a data set of 10 observations containing x and y data, the following numbers are calculated: $\bar{x} = 2$, $\bar{y} = -2.5$, $s_x = 0.11$, $s_y = 0.25$ and $r = 0.212$. This means that:

(a) there is no strong relationship between x and y

(b) if there is a strong relationship between x and y, it is not linear

(c) the sum of all the y data is 20

(d) the x data is spread out more than the y data

23. With respect to the data given in the previous problem, the least squares regression line predicted between x and y is

(a) $y = -3.46 + 0.482x$

(b) $y = 0.482 - 3.46x$

(c) $y = -0.0933 + 0.212x$

(d) $y = 0.212 - 0.0933x$

24. The "Chair's paradox" is paradoxical because

(a) all three voters have tie-breaking power

(b) the Chair has the most power, but fares the worst

(c) no voter gets his or her first choice

(d) its voter preferences are equivalent to the Condorcet voting paradox

25. Condorcet's method

(a) is non-manipulable

(b) always produces a winner

(c) does not satisfy monotonicity

(d) does not satisfy the Pareto condition

26. Which of the following is *not* true about a data set?

(a) The standard deviation would increase if an outlier is added to a data set

(b) If a histogram is roughly symmetric, the median and the mean will be close

(c) The mean will always lie between the first and the third quartiles

(d) The box in the box-plot encompasses the region between the first and the third quartiles

27. Which of the following voting methods is manipulable according to the definition we used in class?

(a) majority rule (with two candidates)

(b) plurality method

(c) Condorcet's method

(d) Hare method

28. The number of ways of choosing 3 objects from a selection of 7 objects is

(a) 210

(b) 140

(c) 35

(d) 120

29. The shortest time of completion of a project which is described in terms of an order-requirement digraph

(a) is found from the minimum-cost spanning tree of the graph

(b) is the length of the longest directed path in the graph

(c) is the length of the shortest path in the graph which goes from a beginning vertex to an ending vertex

(d) is the length of the shortest Euler circuit in the graph

30. Consider the following preference list for the 3 voter situation, in which there are four candidates A, B, C and D.

Voter 1	Voter 2	Voter 3
C	D	A
D	B	C
B	A	D
A	C	B

If you were Voter 2, and wanted candidate D to win in a sequential pairwise voting setting, what agenda would you propose?

(a) B C D A

(b) D B A C

(c) A B C D

(d) A C D B

31. The five number summary for the test scores from a math exam is 32 38 54 68 95. Which of the following is correct about the quartiles?

(a) $Q_1 = 32$ and $Q_4 = 68$

(b) $Q_3 = 54$ and $Q_5 = 95$

(c) $Q_0 = 32$ and $Q_4 = 68$

(d) $Q_2 = 54$ and $Q_4 = 95$

32. The Borda count is non-manipulable in the special instance in which

(a) there are only 3 candidates

(b) there are only 2 voters

(c) all but one of the voters is honest

(d) the preference lists are identical to the Condorcet voting paradox

33. If a three character password system must begin with a lowercase letter of the English alphabet followed by two decimal digits that may be repeated, the number of different possible passwords is

(a) 2600

(b) 2106

(c) 46

(d) 2340

34. Independence of irrelevant alternatives refers to the fact that an election outcome with respect to two candidates A and B will not change if

(a) a voter swaps A and B in his or her preference list

(b) a voter's preferences are moved around while maintaining the relative ordering of A and B

(c) a voter can by moving B to the top of his or her preference list make B the winner in a situation in which A won the previous election

(d) a voter manipulates his or her ballot

35. In the weighted voting system [7: 3, 2, 2, 2] the number of voting combinations with exactly the quota to win is

(a) 2

(b) 3

(c) 4

(d) 5

36. A Fast Food Company was interested in knowing whether their customers were satisfied with the overall service and cleanliness of the Company's franchises. In an effort to obtain this information, The Fast Food Company randomly selected 75 of the 325 customers from one of their 25 franchise stores to fill out a survey. The sample in this situation is the

(a) 25 stores

(b) 75 selected customers

(c) 325 customers

(d) the one store chosen

37. Suppose 65% of all college students find studying for final exams a waste of time. The population proportion is $p = 0.65$. Suppose many different simple random samples of $3,000$ college students were taken. The standard deviation of the sampling distribution

(a) is 0.65

(b) is 0.0087

(c) is 0.00016

(d) needs to be calculated from the data gathered from the random samples

38. In a weighted voting system, if a voter A has veto power,

(a) A is a dictator

(b) all other voters are dummies

(c) the other voters can collectively pass a resolution in spite of A's objection to it

(d) no resolution can be passed without A voting for it

39. You conduct a randomized survey to determine the percentage of under-age college students who attend alcohol-related partying on Thursday nights, and conclude from your results that the number is 45% with a margin of error of 7%. If you double your sample size, your best guess for the margin of error in your new study is

(a) about 1.8%

(b) about 3.5%

(c) about 5%

(d) about 14%

40. Which of the following theorems which we encountered in this course enables one to do something efficiently?

(a) Euler's theorem

(b) May's theorem

(c) Arrow's theorem

(d) Central Limit theorem

PART B (Short Answer): Complete the following questions in the space provided. The points associated with each part are indicated in brackets.

1. Explain the meaning of each of the following terms, as discussed in this course, in a sentence or two. Use diagrams if appropriate. [2 points each]

(a) Confounding

(b) Spanning tree

(c) Critical voter in a coalition

(d) Least-squares regression line

(e) Critical path of an order-requirement digraph

(f) Hamiltonian circuit

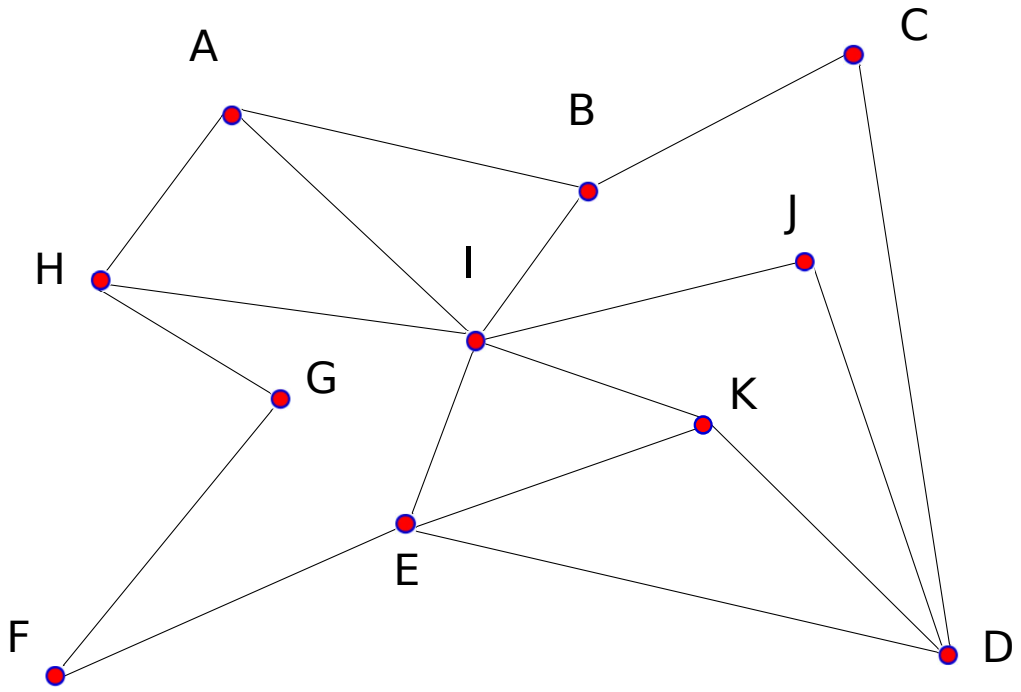
(g) monotonicity in a voting method

(h) the Pareto condition

(i) randomized comparative experiment

(j) group manipulability of an election

2. Suppose you are to traverse the following collection of roads and return to your original starting point, such that the number of times you need to backtrack in minimum. Identify which road segments you choose to backtrack on. [4 points]



3. Here is a data set:

1, 2, 2, 3, 4, 5, 7, 9, 9, 11, 14, 15, 15, 16, 17, 18, 99

(a) Find the five-number summary for this data set. [2 points]

(b) Calculate the mean and standard deviation of this data. [2 points]

(c) *Carefully explaining your procedure*, extract 5 *random* data points from the above data. [2 points]

4. Construct a weighted voting system, expressed in the standard notation we have discussed in class, which has at least 3 voters with non-zero weights, in which one of the voters is a dictator. [4 points]

5. Consider the following preference list, in which four voters are voting for 4 candidates A, B, C and D.

Voter 1	Voter 2	Voter 3	Voter 4
B	D	C	B
C	C	A	A
D	A	B	C
A	B	D	D

(a) Determine (with explanations) the winner if using (i) plurality voting, and (ii) the Hare method. [2 points]

(b) If the Borda count is being used, demonstrate how Voter 1 can improve her election outcome by voting strategically. [5 points]

6. A wizard who is based on Oz wishes to visit Narnia, Lilliput, Middle-Earth and Atlantis (in no particular order), and return to Oz. The distances between these places, in pentogramometers, are given in the following table.

	Oz	Narnia	Lilliput	Middle-Earth	Atlantis
Oz	-	3	11	9	5
Narnia	3	-	6	7	1
Lilliput	11	6	-	4	8
Middle-Earth	9	7	4	-	7
Atlantis	5	1	8	7	-

(a) Draw a weighted graph that incorporates this information. [2 points]

(b) If the wizard is greedy and knows only of the nearest neighbor algorithm, what route would he follow? [2 points]

(c) The wizard wants to establish a linked network between all these countries, a magical pipeline along which wizardry could be transmitted from any one of the countries to any other (magic may pass across other countries during this process). Assuming that shorter pipelines are cheaper, find the best links along which to construct this pipeline. [3 points]

7. Calculate the Banzhaf Power Index for each of the voters in the following weighted voting systems. To give yourself the best chance of partial-credit, make sure you show your work! [4 points each]

(a) $[5 : 2, 4, 1]$

(b) $[5 : 4, 1, 1, 1, 1, 1]$

(c) $[7 : 5, 2, 1, 1, 1, 1, 1, 1]$

(Blank page)

FORMULA SHEET

1. **Mean and standard deviation**:
   Given a data set x_1, x_2, x_3, ... x_n with n observations,
   - Mean (average): $\bar{x} = \dfrac{1}{n}\sum_i x_i = \dfrac{1}{n}\left[x_1 + x_2 + \cdots + x_n\right]$
   - Standard deviation:
     $s = \sigma_{n-1} = \sqrt{\dfrac{1}{n-1}\sum_i (x_i - \bar{x})^2} = \sqrt{\dfrac{1}{n-1}\left[(x_1-\bar{x})^2 + (x_2-\bar{x})^2 + \cdots + (x_n-\bar{x})^2\right]}$

2. **Normal (Bell) curves**:
   - A symmetric curve, with mean μ and standard deviation σ
   - 68% of observations lie within one standard deviation of the mean
   - 95% of observations lie within two standard deviations of the mean
   - 99.7% of observations lie within three standard deviations of the mean
   - The quartiles are located at $Q_1 = \mu - 0.67\sigma$ and $Q_3 = \mu + 0.67\sigma$

3. **Regression lines and correlation**:
   Given a data set with two variables x and y for n individuals,
   - Means: $\bar{x} = \dfrac{1}{n}\sum_i x_i$ and $\bar{y} = \dfrac{1}{n}\sum_i y_i$
   - Standard deviations: $s_x = \sqrt{\dfrac{1}{n-1}\sum_i (x_i - \bar{x})^2}$ and $s_y = \sqrt{\dfrac{1}{n-1}\sum_i (y_i - \bar{y})^2}$
   - Correlation:
     $$\begin{aligned} r &= \frac{1}{n-1}\sum_i \left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right) \\ &= \frac{1}{n-1}\left[\left(\frac{x_1 - \bar{x}}{s_x}\right)\left(\frac{y_1 - \bar{y}}{s_y}\right) + \left(\frac{x_2 - \bar{x}}{s_x}\right)\left(\frac{y_2 - \bar{y}}{s_y}\right) + \cdots + \left(\frac{x_n - \bar{x}}{s_x}\right)\left(\frac{y_n - \bar{y}}{s_y}\right)\right] \end{aligned}$$
   - Least squares regression line: $y = mx + b$, where
     $$m = r\,\frac{s_y}{s_x} \quad \text{and} \quad b = \bar{y} - m\,\bar{x}$$

4. **Inference**:
   Let p be a parameter in a population, which you are trying to estimate by the statistic $\hat{p}$ which you get by choosing a simple random sample of size n from the population.
   - The estimate for the parameter is $\hat{p} = \dfrac{\text{count of successes in the sample}}{n}$
   - For large sample sizes, the sampling distribution for $\hat{p}$ is approximately normal, with mean p and standard deviation $\sqrt{\dfrac{p(1-p)}{n}}$

5. **Combination formula**: $C_k^n = {}_nC_k = \dfrac{n!}{k!\,(n-k)!} = \dfrac{n\,(n-1)\,(n-2)\cdots(n-k+1)}{k\,(k-1)\,(k-2)\cdots 1}$

TABLE OF RANDOM DIGITS

Line \ Col.	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
1	10460	15011	01536	02011	81647	91646	69179	14194	62590	36207	20969	99570	91291	90700
2	22368	46573	25595	85393	30995	89198	27982	53402	93965	34095	52666	19174	39615	99505
3	24130	48360	22527	97265	76393	64809	15179	24830	49340	32081	30680	19655	63348	58629
4	42167	93093	06243	61680	07856	16376	39440	53537	71341	57004	00849	74917	97758	16379
5	37570	39975	81837	16656	06121	91782	60468	81305	49684	60672	14110	06927	01263	54613
6	77921	06907	11008	42751	27756	53498	18602	70659	90655	15053	21916	81825	44394	42880
7	99562	72905	56420	69994	98872	31016	71194	18738	44013	48840	63213	21069	10634	12952
8	96301	91977	05463	07972	18876	20922	94595	56869	69014	60045	18425	84903	42508	32307
9	89579	14342	63661	10281	17453	18103	57740	84378	25331	12566	58678	44947	05585	56941
10	85475	36857	53342	53988	53060	59533	38867	62300	08158	17983	16439	11458	18593	64952
11	28918	69578	88231	33276	70997	79936	56865	05859	90106	31595	01547	85590	91610	78188
12	63553	40961	48235	03427	49626	69445	18663	72695	52180	20847	12234	90511	33703	90322
13	09429	93969	52636	92737	88974	33488	36320	17617	30015	08272	8411	27156	30613	74952
14	10365	61129	87529	85689	48237	52267	67689	93394	01511	26358	85104	20285	29975	89868
15	07119	97336	71048	08178	77233	13916	47564	81056	97735	85677	29372	74461	28551	90707
16	51085	12765	51821	51259	77452	16308	60756	92144	49442	53900	70960	63990	75601	40719
17	02368	21382	62404	60268	89368	19885	55322	44819	01188	65255	64835	44919	05944	55157
18	01011	54092	33362	94904	31273	04146	18594	29852	71585	85030	51132	01915	92747	64951
19	52162	53916	46369	58586	23216	14513	83149	98736	23495	64350	94738	17752	35156	35749
20	07056	97628	33787	09998	42698	06691	76988	13602	51851	46104	88916	19509	25625	58104
21	48663	91245	85826	14346	09172	30168	90229	04734	59193	22178	30421	61666	99904	32812
22	54164	58492	00421	74103	47070	25306	76468	26384	58151	06646	21524	15227	96909	44592
23	32639	32363	05597	24200	13363	38005	94342	28728	35806	06912	17012	64161	18296	22851
24	29334	27001	87637	87308	58731	00256	45834	15398	46557	41135	10367	07684	36188	18510
25	02488	33062	28834	07351	19731	92420	60952	61280	50001	67658	32586	86679	50720	94953
26	81525	72295	04839	96423	24878	82651	66566	14778	76797	14780	13300	87074	79666	95725
27	29676	20591	68086	26432	46901	20849	89768	81536	86645	12659	92259	57102	80428	25280
28	00742	57392	39064	66432	84673	40027	32832	61362	98947	96067	64760	64584	96096	98253
29	05366	04213	25669	26422	44407	44048	37937	63904	45766	66134	75470	66520	34693	90449
30	91921	26418	64117	94305	26776	25940	39972	22209	71500	64568	91402	42416	07844	69618
31	00582	04711	87917	77341	42206	35126	74087	99547	81817	42607	43808	76655	62028	76630
32	00725	69884	62797	56170	86324	88072	76222	36086	84637	93161	76038	65855	77919	88006
33	69011	65795	95876	55293	18988	27354	26575	08625	40801	59920	29841	80150	12777	48501
34	25976	57948	29888	88604	67917	48708	18912	82271	65424	69774	33611	54262	85963	03547
35	09763	83473	73577	12908	30883	18317	28290	35797	05998	41688	34952	37888	38917	88050
36	91567	42595	27958	30134	04024	86385	29880	99730	00036	84855	29080	09250	79656	73211
37	17955	56349	90999	49127	20044	59931	06115	20542	18059	02008	73708	83517	36103	42791
38	46503	18584	18845	49618	02304	51038	20655	58727	28168	15475	56942	53389	20562	87338
39	92157	89634	94824	78171	84610	82834	09922	25417	44137	48413	25555	21246	35509	20468
40	14577	62765	35605	81263	39667	47358	56873	56307	61607	45918	89686	20103	77490	18062
41	98427	07523	00062	64270	01638	92477	66969	98420	04880	45585	46565	04102	46880	45709
42	34914	63976	88720	82765	34476	17032	87589	40836	32427	70002	70663	88863	77775	69348
43	70060	28277	39475	46473	23219	53416	94970	25832	69975	94884	19661	72828	00102	66794
44	53976	54914	06990	67245	68350	82948	11398	42878	80287	88267	47363	46634	06541	97809
45	76072	29515	40980	07391	58745	25774	00987	80059	39911	96189	41151	14222	60697	59583
46	90725	52210	83974	29992	65831	38857	50490	83765	55657	14361	31720	57375	56228	41546
47	64364	67412	33339	31926	14883	24413	59744	92351	97473	89286	35931	04110	23726	51900
48	08962	00358	31662	25388	61642	34072	81249	35648	56891	69352	48373	45578	78547	81788
49	95012	68379	93526	70765	10592	04542	76463	54328	02349	17247	28865	14777	62730	92277
50	15664	10493	20492	38391	91132	21999	59516	81652	27195	48223	46751	22923	32261	85653