NAME:

MAT 301 (Real Analysis I)

Final Examination

Cover Sheet

Due: Wednesday, December 19, 2012 (noon)

Please sign in the space below, thereby acknowledging the Connecticut College Honor Pledge. This also attests that this exam represents *your own work* (using other material only as permitted in the examination rules), and that you have complied with all the examination rules.

Real Analysis 1

Final Examination

Examination Rules

- This is a take-home examination with no time limit imposed, apart from the fact that you need to hand the exam in by the due date and time.
- This is an open-book examination in the following sense. You are permitted to freely refer any *inanimate* sources. This includes your class notes, homework problems, textbook, other textbooks, the internet, the tea-leaves at the bottom of your cup, etc.
- Communication (in any sense) with any other person other than your instructor regarding this exam is prohibited, until after the exam deadline. You are therefore prohibited from using any *animate* sources, such as your classmates, a tutor, any electronic communications with a person, your aunt Mabel, your pet ferret, etc.
- You may consult the instructor regarding material on this exam. The instructor will help clarify any questions, but is not likely to provide guidance in how to solve any problems. On the other hand, the instructor can certainly be asked about course material which is not explicitly covered in this exam.
- Should you need to obtain a copy of a graded homework set from one of your homework partners, it is legitimate to contact them to make this request during the course of the exam. However, communicating with anyone in the class regarding anything else to do with this course will be forbidden from this moment until the exam deadline.
- If you choose, you may (for extra-credit) *extend* any of the problems. This could happen, for example, if you can extend the proof to a broader class of functions, if you can prove a stronger result than that requested, or if you can provide a rigorous proof when only a conceptual expression is required. While the problems in this exam are *not* designed with this in mind, the instructor will certainly reward you if you can provide additional insights in any of these ways.
- Please note that your solutions will be graded for their presentation in addition to their content. This is particularly important in a course such as this, in which careful proofs are sought. Neatness, clarity, good explanations, logical writing, elegance of proofs, etc, will be favorably viewed.
- You may provide your solutions on any paper of your choice. However, please attach the duly completed cover sheet (with the Honor Code declaration completed and signed) to your solutions when you hand it in.
- Violation of any of the rules given here, or of other aspects of the Honor Code, *will* result in referral to the Honor Council, with potentially serious consequences. If you are unsure of any of the rules, please check with your instructor for clarification.
- In keeping with course policy, *late exams will not be accepted under any circumstances.*

Real Analysis 1

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Please answer all questions, using paper of your choice. The points awarded for each problem are indicated in brackets. When handing in your exam, you would need to attach the exam cover sheet, with the Honor Code declaration duly signed.

1. [10 points] Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $\lim_{x\to a} f'(x)$ exists, then show that $f'(a)$ exists, and is equal to this quantity.
Hint: Do the hypotheses on f in the first sentence remind you of the hypotheses in an important theorem?

2. [10 points] Consider the function $h(x) = \sqrt{x}$ on the set $A = [0,1]$.

(a) Show that h is continuous on A.

(b) Explain why you can therefore also conclude that h is uniformly continuous on A.

(c) *Using the ϵ-δ characterization of uniform continuity* (Definition 4.4.5 on page 116 of your textbook), provide a direct proof that h is uniformly continuous on A.

3. [20 points] For $x \in [0,1]$, define the function f by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 1-x & \text{if } x \in \mathbb{Q}^c \end{cases}.$$

Prove that

(a) $f(f(x)) = x$ for all $x \in [0,1]$.

(b) $f(x) + f(1-x) = 1$ for all $x \in [0,1]$.

(c) f is continuous only at the point $x = 1/2$.

(d) f assumes every value between 0 and 1.

(e) $f(x+y) - f(x) - f(y)$ is rational for all x and y in $[0,1]$.

4. [3 points each] Decide whether the following statements are true or false. If true, provide an example. If false, prove that they are false.

(a) There exists a countably infinite subset of $\mathbb{R}$ that is compact.

(b) There exists a countably infinite subset of $\mathbb{R}$ that is open.

(c) There exists a countable open cover for the set $\{1, 1/2, 1/3, 1/4, 1/5, \cdots\}$ which has no finite subcover.

5. [10 points] Let ψ be a differentiable function defined on $[5,8]$ such that $\psi(5) = 1$, $\psi(6) = 2$ and $\psi(8) = 2$.

(a) Prove that the function $\tilde{\psi}(x) = \frac{9}{2}\psi(x)$ has a fixed point in the interval $(5,8)$ (see Exercise 5.3.5, which we completed for homework, for the definition of a fixed point).

(b) Show the existence of a point $c \in (5,8)$ such that $\psi'(c) = 1/4$.

(c) Show the existence of a point $d \in (5,8)$ such that $\psi'(d) = 3/4$.

6. [1 point] Write down something funny. Have a great break!