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Weak finite-time Melnikov theory and 3D viscous perturbations of Euler flows

Sanjeeva Balasuriya [a,*], Igor Mezić [b,c], Christopher K.R.T. Jones [d]

[a] *School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia*
[b] *Department of Mechanical and Environmental Engineering, University of California, Santa Barbara, CA 93106, USA*
[c] *Department of Mathematics, University of California, Santa Barbara, CA 93106, USA*
[d] *Department of Mathematics, University of North Carolina, Chapel Hill, NC 27516, USA*

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Abstract

The ordinary differential equations related to fluid particle trajectories are examined through a 3D Melnikov approach. This theory assesses the destruction of 2D heteroclinic manifolds (such as that present in Hill's spherical vortex) under a perturbation which is neither differentiable in the perturbation parameter ε, nor defined for all times. The rationale for this theory is to analyse viscous flows that are close to steady Euler flows; such closeness in ε can only reasonably be expected in a weak sense for finite times. An expression characterising the splitting of the two-dimensional separating manifold is derived.

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1. Introduction

Melnikov theory has proved to be a powerful tool in fluid particle kinematics analysis (among other references [6,31,32,34,35] are particularly related to the current paper) due to the fact that it is capable of predicting the size of the splitting of separating manifolds in weakly perturbed integrable dynamical systems. The size of the splitting is directly related to the flux of fluid particles from one region of the physical space to another [34,35], and thus to mixing [42]. The analysis of the kinematics of particles is typically performed as follows: an integrable regime is identified. A perturbation from that regime is considered, which is typically time-periodic (or quasi-periodic [8]). Melnikov theory is then used to show that an transverse intersection of stable and unstable manifolds to periodic orbits (or fixed points) of the unperturbed dynamics exists, indicating chaotic dynamics. A measure of flux between

* Corresponding author. Tel.: +61-2-9351-4873; fax: +61-2-9351-4534.
E-mail address: sanjeeva@maths.usyd.edu.au (S. Balasuriya).

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zones that were separated by a separatrix of the unperturbed Hamiltonian can be obtained by integrating the Melnikov function.

Balasuriya et al. [6] have identified another set-up for use of the Melnikov method. The integrable system is a steady 2D Euler flow and the conserved quantity is vorticity (as opposed to the streamfunction mentioned previously). The perturbation is in that case provided by assuming that the integrable steady incompressible Euler flow is the initial condition for an incompressible Navier–Stokes equation solution. It is assumed, in a time-valued manner of the theoretical fluid dynamics and boundary layer theory [40] that the perturbation is small in viscosity (meaning that it vanishes as viscosity does). However, the perturbation was neither considered time-periodic, nor differentiable with respect to the perturbation parameter that is in this case viscosity. The fact that a similar set-up can be pursued in the context of three-dimensional, steady flows was indicated in [30]. The consequence of such an approach is interesting: it turns out that the magnitude of flux and thus the efficiency of mixing decreases with viscosity. In [31] a formal approach indicated such a conclusion is possible and the relevant experimental and numerical evidence was reviewed. In [23] similar ideas were explored based on earlier work in [45]. In this paper we pursue finite-time perturbations of three-dimensional, volume-preserving Euler flows by viscous perturbations to contribute a number of rigorous results in the above indicated research direction. Specifically, in this paper we investigate the implications of one possible hypothesis concerning its solutions: the [boundaryless] Navier–Stokes and the Euler velocities are $\mathcal{O}(\varepsilon)$-close for finite times (in a specific sense), where ε is the viscous parameter. This ansatz leads to a nice analytical technique for assessing the role of small viscosity towards fluid transport far away from boundaries. In developing our theory, it is first necessary to improve Melnikov theory to account for the finite-time weak perturbation that our hypothesis engenders. We will first give a rationale for following this approach.

Steady three-dimensional (nonBeltrami) Euler flows are integrable [2,3,18], with an integral of motion provided by the Bernoulli function B. Motion of fluid essentially follows laminar patterns, or more precisely, remains on torii or cylinders [4]. Two-dimensional sheets (comprising level sets of B which contain its critical points) separate regions of disparate motion. These are called separating manifolds or separatrices, and no complicated motion is to be expected under these circumstances. A classical example of this geometry is given by a bubble-vortex [20,26,27,31]. Any steady flows admitting a volume-preserving symmetry are also integrable in this fashion [18,32], and may have this geometry. In the presence of low viscosity, however, one would expect these separatrices to perturb and split, enabling complicated mixing between fluid of different antecedents. Should the inclusion of viscosity be representable as a perturbation, Melnikov theory would quantify the splitting of these separatrices, and therefore be a tool which assesses fluid transport.

There are many available results which indicate that the Euler and the Navier–Stokes velocities are ‘close’ in the case of small viscosity and no boundaries [5,14,15,22,24,28]. None of these are strong enough to validate the hypothesis of closeness that we will use in this paper—this being an outstanding problem of mathematical physics left unsolved from the previous century. Our assumption is that, in a three-dimensional flow with no boundary, these velocity fields are $\mathcal{O}(\varepsilon)$-close in a weak functional analytical sense, for finite times. This hypothesis is made precise in Section 3. Our Melnikov theory (Section 2) is set-up specifically to handle this type of weak, finite-time perturbation. We build on the approaches of [6,7,12,16,29,33,37] in developing this new theory, in Section 2. An expression determining the destruction of the separating bubble of Hill’s spherical vortex under a generic perturbation is obtained from this result.

Section 3 presents the usage of our method in considering a *viscous* perturbation of a three-dimensional steady Euler flow. This inviscid flow is assumed to possess a two-dimensional heteroclinic manifold, whose splitting under a viscous (and forcing) perturbation we intend to address. Through suitable manipulations of the Navier–Stokes equations, we are able to obtain an expression of the Melnikov function resulting from this viscous perturbation. We split the Melnikov function into three components, which we identify as resulting from viscosity, forcing, and a perturbative pressure effect, respectively. Interestingly, we are able to express *most* relevant terms of the Melnikov function entirely in terms of the Euler flow. The implications of the Melnikov expression are extracted by examining additional simplifying assumptions.

2. Melnikov theory for weak perturbations

As motivation for this section, we shall briefly describe currently available Melnikov methods, and how they fail in simultaneously addressing the properties of the problem we wish to treat.

In general, Melnikov analysis pertains to the equation

$$\dot{x} = f(x) + g(x, t, \varepsilon),$$

where $x \in \mathbb{R}^n$, $g(x, t, 0) = 0$, and the perturbation is turned on through setting the perturbation parameter $\varepsilon \neq 0$. The $\varepsilon = 0$ case is assumed to have an appropriate heteroclinic or homoclinic structure, whose destruction is analysed through the zeros of a Melnikov function. The original Melnikov theory [29] was developed for $n = 2$, in which additionally $g(x, t, \varepsilon) = \varepsilon h(x, t)$, and h is periodic in the temporal variable t and sufficiently smooth in its arguments. A nice geometrical theory for this problem has evolved [17,43]. The relevant manifolds for this case are one-dimensional connections between fixed points, and the Melnikov function measures the leading order separation between them after perturbation. Zeroes of the Melnikov function correspond exactly to the presence of heteroclinic points (i.e., of intersections between the perturbed manifolds). For $\varepsilon \neq 0$, a complicated tangling of manifolds occurs, leading to chaotic transport vis-à-vis the Smale-Birkhoff theorem (see for example [41]).

A method in general $\mathbb{R}^n$ has also been developed by Gruendler [16] for time-periodic perturbations. In common with geometric 2D versions, the taking of an ε derivative is necessary in this work, which is expressly forbidden in the Navier–Stokes case, since no justification of strong differentiability in the viscous parameter ε exists. Some additional generalisations in $\mathbb{R}^n$ are provided in [7,33,43,44]. Wiggins [43] and Yagasaki [44] need smoothness in ε. Battelli and Lazzari [7] build upon the results in Palmer [33], and use functional analytical arguments concentrating on exponential dichotomies, and in fact do not need time-periodicity in their perturbation. Nevertheless, both approaches hypothesise differentiability in ε. The Melnikov method developed in [6] relaxes this assumption by closely following the functional analytic arguments in [11,12], but is limited to two dimensions. One goal in this section is to extend this result to three dimensions, to enable analysis of the Navier–Stokes problem. The unperturbed problem is assumed three-dimensional and autonomous, while possessing a conserved quantity and an appropriate two-dimensional heteroclinic manifold. The perturbed equation is permitted to be nonautonomous, and close to the unperturbed flow only in a weak sense (in particular, differentiability in the perturbation parameter ε is not expected). Section 2.3 develops the theory under these conditions.

With respect to the fluid mechanics application, we note that when considering a viscous perturbation to an Euler flow, the velocity fields could only reasonably be expected to be close for finite times. A standard Melnikov method fails in this case, since this g is no longer a genuine perturbation (it is only $\mathcal{O}(\varepsilon)$ for finite times). Nevertheless, based on the (two-dimensional) finite-time Melnikov method developed in [37], it is possible to extend our theory to such finite-time perturbations; we do so in Section 2.4.

2.1. Unperturbed geometry

Let Ω be a smooth open subset of $\mathbb{R}^3$, and let $f : \Omega \to \mathbb{R}^3$ such that $f \in C^r(\Omega)$, $r \geq 2$. For $x \in \Omega$, consider the autonomous differential equation

$$\dot{x} = f(x). \tag{1}$$

Hypothesis 2.1. The function f is divergence-free, i.e., $\nabla \cdot f = 0$.

The above is motivated by incompressible fluid flow. It is possible to develop the theory which follows by relaxing this volume-preserving assumption (see [16] to obtain an intuition on the additional exponential term that would be necessary to include in the Melnikov integral that we derive). However, in the interests of simplicity, we will restrict attention to divergence-free velocity fields. We also assume that the flow (1) possesses an integral of motion (as is given in fluids by a Bernoulli function; see [18,32] for additional conditions under which this exists).

Hypothesis 2.2. There exists a nonconstant function $B_0 : \Omega \to \mathbb{R}$ such that $B_0 \in C^r(\Omega)$, $r \geq 2$, which is conserved along trajectories of the flow of (1).

Note that for x solving (1),

$$0 = \frac{\mathrm{d}B_0}{\mathrm{d}t}(x) = \nabla B_0(x) \cdot \dot{x} = \nabla B_0(x) \cdot f(x),$$

which establishes that the vectors ∇B_0 and f are perpendicular in Ω. The next hypothesis concerns the phase space geometry of (1).

Hypothesis 2.3. There exist hyperbolic fixed points a and b of (1), such that $\dim W_a^u = \dim W_b^s = 2$. Moreover, a branch of W_a^u coincides with a branch of W_b^s, forming a smooth two-dimensional parametrisable heteroclinic manifold Γ on which $\nabla B_0 \neq 0$. "Parametrisability" is in the sense that

$$\Gamma = \bigcup_{\alpha \in S^1} \bigcup_{\tau \in \mathbb{R}} x_\alpha(\tau),$$

where $(\alpha, \tau) \in S^1 \times \mathbb{R}$ and x_α are the collection of heteroclinic trajectories on the manifold.

The possible geometrical structures in $\mathbb{R}^3$ which satisfy Hypothesis 2.3 are illustrated in Fig. 1. In each case $\Gamma \cup \{a\} \cup \{b\}$ is topologically a sphere, and we will think of a as its north pole, while b is the south. The manifold Γ is foliated with heteroclinic trajectories x_α. Fixing a value of $\alpha \in S^1$ determines *which* heteroclinic trajectory is being considered. Moreover, fixing a value of $\tau \in \mathbb{R}$ determines the point $x_\alpha(\tau)$ on that trajectory. It is in this sense that (α, τ) parametrises Γ. Fig. 1(a) corresponds to the case where the eigenvalues at a and b are purely real, and hence trajectories on Γ in the vicinity of the fixed points travel from/towards the fixed points with no spiralling. In this case, the (α, τ) parametrisation is effectively the longitude and latitude, respectively. In (b), the eigenvalues at a and b have nonzero imaginary part, and hence spiralling is observed. In this case, the (α, τ) are "spiralling coordinates" which smoothly parametrise Γ by assumption. A simple consequence of Hypothesis 2.3 is

Lemma 2.1. $\nabla B_0(a) = \nabla B_0(b) = 0$. *Moreover,* $B_0 =$ *constant on* $\Gamma \cup \{a\} \cup \{b\}$.

Proof. The fact that the flow is confined to level surfaces of B_0 is immediate from Hypothesis 2.2; this must hold even in the infinite time limits. Since a has an attached two-dimensional manifold which asymptotes to a in backwards time, $B_0 = B_0(a)$ on that entire manifold. Thus, $B_0 = B_0(a)$ on Γ. Moreover, a possesses a stable manifold by Hypothesis 2.3, along which B_0 must equal $B_0(a)$ as well. Therefore, there are three independent directions in which B_0 does not change (two of these can be taken to be parallel to the surface of Γ, while the other can be any direction parallel to W_a^s). Hence, $\nabla B_0(a) \cdot v_i = 0$ for three independent vectors v_i, which implies that the three-dimensional vector $\nabla B_0(a) = 0$. A similar argument holds for $\nabla B_0(b)$. □

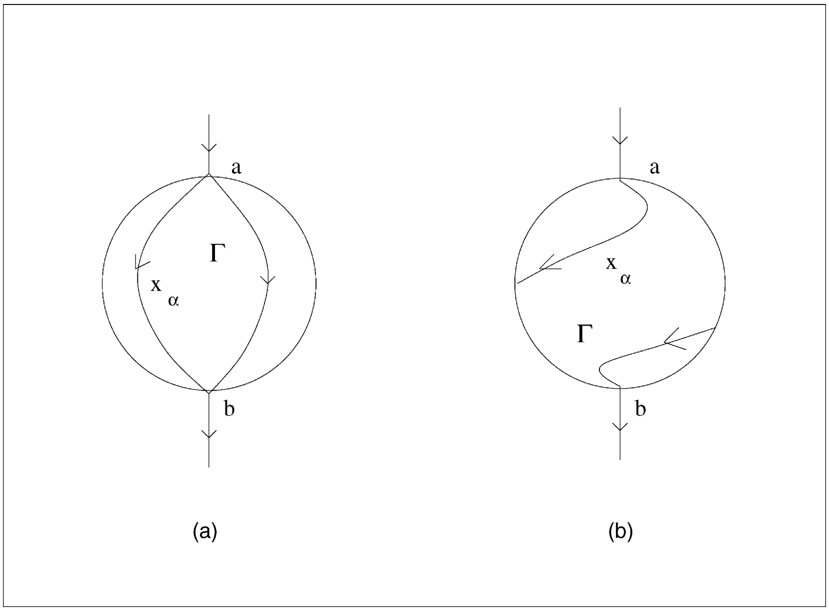

Fig. 1. Possible structures for the heteroclinic manifold Γ: (a) no swirl, and (b) swirl.

The variational equation of (1) is given by

$$\dot{z} = \nabla f(x(t))z, \tag{2}$$

where $x(t)$ is any trajectory of (1), and ∇f is the 3×3 Jacobian matrix of f. Of particular interest is the choice of a heteroclinic trajectory along which to compute the equation of variations:

$$\dot{z} = \nabla f(x_\alpha(t))z. \tag{3}$$

Since $x_\alpha \to a$ as $t \to -\infty$ (and $x_\alpha \to b$ as $t \to \infty$), and since a and b are critical points, there are exponential decay rates associated with (3). These are best expressed through *exponential dichotomies* [7,13,33] associated with fundamental matrix solutions $Z(t)$ to (3). For the decay in forward time, there exists positive constants K and θ, and a projection operator $\mathcal{P}_b$ such that

$$|Z(t)\mathcal{P}_b Z^{-1}(s)| \leq K\,\mathrm{e}^{-\theta(t-s)} \quad \text{for } 0 \leq s \leq t, \quad |Z(t)(\mathrm{id} - \mathcal{P}_b)Z^{-1}(s)| \leq K\,\mathrm{e}^{-\theta(s-t)} \quad \text{for } s \geq t \geq 0. \tag{4}$$

Similarly, for the decay in backwards time, there exists constants K and θ (which can be chosen to be the same as those above), and a projection $\mathcal{P}_a$ such that

$$|Z(t)\mathcal{P}_a Z^{-1}(s)| \leq K\,\mathrm{e}^{-\theta(s-t)} \quad \text{for } 0 \geq s \geq t, \quad |Z(t)(\mathrm{id} - \mathcal{P}_a)Z^{-1}(s)| \leq K\,\mathrm{e}^{-\theta(t-s)} \quad \text{for } s \leq t \leq 0. \tag{5}$$

Recall from [13] that $\mathcal{P}_a$ and $\mathcal{P}_b$ can be thought of as projections on to the unstable manifold of a and the stable manifold of b, respectively. Since $\mathcal{P}_a$ is defined for negative t only, this is intuitively a projection onto some version of a northern hemisphere of Γ, while $\mathcal{P}_b$ accounts for the southern part (together, they form a projection onto Γ for all times). Also associated with the flow is the cotangent equation (or adjoint variational equation)

$$\dot{z}_a = -(\nabla f)^{\mathrm{T}}(x(t))z_a, \tag{6}$$

where the superscript T denotes the matrix transpose.

Lemma 2.2. *$z_a = \nabla B_0(x(t))$ is a solution of (6).*

Proof. We apply the gradient operator to the relation $f^{\mathrm{T}}(x(t))\nabla B_0(x(t)) = 0$, which yields

$$(\nabla f)^{\mathrm{T}}(x(t))\nabla B_0(x(t)) + f^{\mathrm{T}}(x(t))\nabla[\nabla B_0(x(t))] = 0,$$

where ∇ is assumed a column vector, as is ∇B_0. Noting that the operator $f^{\mathrm{T}}\nabla$ is equivalent to $\mathrm{d}/\mathrm{d}t$ and rearranging, the desired result is obtained. □

2.2. *Weak perturbation*

We now add an additional term to the vector field in (1), and consider instead the nonautonomous differential equation

$$\dot{x} = f(x) + g(x, t, \varepsilon). \tag{7}$$

The parameter ε is considered small: $\varepsilon \in I = [0, \varepsilon_0)$ for sufficiently small ε_0. The function g shall be a *weak* perturbation, made precise in

Hypothesis 2.4. The function $g : \Omega \times \mathbb{R} \times I \to \mathbb{R}$ satisfies the following conditions:

(a) $g \in C^r(\Omega \times \mathbb{R})$, $r \geq 2$, for each $\varepsilon \in I$ with uniform bounds.
(b) $g(x, t, 0) = 0$ for all $(x, t) \in \Omega \times \mathbb{R}$.
(c) There exists C such that $|g(x, t, \varepsilon)| + |\nabla g(x, t, \varepsilon)| \leq C\varepsilon$ uniformly for $(x, t) \in \Omega \times \mathbb{R}$.

Differentiability of g in ε is *not* imposed in condition (a); instead, a Lipschitz-type condition (c) is assumed. The standard geometrical development of Melnikov theory (for example [16,17,43]) requires the taking of an ε derivative (the standard procedure is to append the equations $\dot{\varepsilon} = 0$ and $\dot{t} = 1$ to the system (7), and then consider the equation of variations in (x, t, ε) space). Condition (c) in Hypothesis 2.4 weakens this condition, so that the vector field of (7) is only *weakly* close to the vector field of (1), with less smoothness in ε than is traditional. Note also that time-periodicity, often an ingredient of such a theory ([16,43,44]), is not required. Our motivation for including a condition of the form (c) is through fluid mechanics: if ε is a measure of kinematic viscosity, in regions removed from boundaries, there is evidence to suggest such an $\mathcal{O}(\varepsilon)$-closeness between the unperturbed (inviscid) velocity field and its perturbed (viscous) counterpart [5,24]. We shall in Section 2.4 further weaken Hypothesis 2.4 to permit the Lipschitz condition to only be valid for *finite* times.

To see what happens after perturbation, consider (1) in (x, t) phase space. The fixed points correspond to specialised trajectories (a, t) and (b, t), which possess hyperbolic decay rates nearby. However, when $\varepsilon \neq 0$, these persist in the form $(a_\varepsilon(t), t)$ and $(b_\varepsilon(t), t)$, and retain their manifolds, albeit slightly perturbed. This holds through Theorem 6.1 in Hirsch et al. [19], which asserts such persistence under $\mathcal{O}(\varepsilon)$ perturbations of normally hyperbolic noncompact manifolds. However, the perturbed manifolds need no longer coincide, and our goal is to determine intersections between them, i.e., the persistence of a heteroclinic connection. To investigate this, we adopt a functional analytic approach, akin to the two-dimensional version given in [11,12]. Define the Banach space

$$\mathcal{L}(\mathbb{R}) = \{G : \mathbb{R} \to \mathbb{R}^3, \quad \text{bounded and continuous}\}$$

endowed with the norm

$$\|G\| = \sup_{t\in\mathbb{R}} |G(t)|.$$

Now, imagine a perturbation on the variational equation (3)

$$\dot{z} = \nabla f(x_\alpha(t))z + G(t). \tag{8}$$

The idea is to find conditions of $G(t)$ which will ensure that a heteroclinic trajectory will persist. This turns out to be related to whether the solutions to (8) are in $\mathcal{L}(\mathbb{R})$. Now, the variation of parameters formula for solutions of (8) can be expressed as

$$z(t) = Z(t)\left(c + \int_0^t Z^{-1}(s)G(s)\,\mathrm{d}s\right), \tag{9}$$

where $Z(t)$ is a fundamental matrix solution to (3) satisfying $Z(0) = \mathrm{id}$, and c an appropriate initial condition.

Lemma 2.3. *If $G \in \mathcal{L}(\mathbb{R})$ and $z(t)$ is the solution to (8), then the components of $z(t)$ tangential to Γ are uniformly bounded for $t \in \mathbb{R}$.*

Proof. Since $Z(t)$ is a fundamental matrix solution to the unperturbed equation of variations (3), it obeys the exponential dichotomies (4) and (5). Define the projection matrix operator $\Phi_b(t) = Z(t)\mathcal{P}_b Z^{-1}(t)$, which represents a projection onto the stable manifold of b. Then, (4) can be rewritten as

$$|\Phi_b(t)Z(t)Z^{-1}(s)| \le K\,\mathrm{e}^{-\theta(t-s)} \quad \text{for } 0 \le s \le t. \tag{10}$$

Similarly, defining the projection $\Phi_a(t) = Z(t)\mathcal{P}_a Z^{-1}(t)$ onto the unstable manifold of a, (5) can be expressed as

$$|\Phi_a(t)Z(t)Z^{-1}(s)| \le K\,\mathrm{e}^{-\theta(s-t)} \quad \text{for } 0 \ge s \ge t. \tag{11}$$

Suppose $\|G\| = M$. Consider first the case where $t \ge 0$. Premultiply (9) by $\Phi_b(t)$ to get

$$\Phi_b(t)z(t) = \Phi_b(t)Z(t)Z^{-1}(0)c + \int_0^t \Phi_b(t)Z(t)Z^{-1}(s)G(s)\,\mathrm{d}s,$$

where $Z^{-1}(0)$ has been included in the first term, since $Z^{-1}(0) = \mathrm{id}$. Hence

$$\begin{aligned}|\Phi_b(t)z(t)| &\le |\Phi_b(t)Z(t)Z^{-1}(0)||c| + \int_0^t |\Phi_b(t)Z(t)Z^{-1}(s)||G(s)|\,\mathrm{d}s \\ &\le K\,\mathrm{e}^{-\theta t}|c| + M\int_0^t K\,\mathrm{e}^{-\theta t}\,\mathrm{e}^{\theta s}\,\mathrm{d}s \le K|c|\,\mathrm{e}^{-\theta t} + \frac{KM\,\mathrm{e}^{-\theta t}}{\theta}(e^{\theta t} - 1) \\ &= K|c|\,\mathrm{e}^{-\theta t} + \frac{KM}{\theta}(1 - \mathrm{e}^{-\theta t}).\end{aligned}$$

The exponential decay (10) was used in the second inequality above. Thus,

$$|\Phi_b(t)z(t)| \le K|c| + \frac{KM}{\theta} \quad \text{for } t \ge 0.$$

Now, the case $t \le 0$ can be handled similarly, by premultiplying (9) by $\Phi_a(t)$ and using (11) to bound the terms, resulting in

$$|\Phi_a(t)z(t)| \le K|c| + \frac{KM}{\theta} \quad \text{for } t \le 0,$$

thereby giving uniform boundedness for $t \in \mathbb{R}$. □

It remains to consider the normal component of the solution $z(t)$. To this end, define the continuous projection operator $P_\alpha : \mathcal{L}(\mathbb{R}) \to \mathcal{L}(\mathbb{R})$ by

$$P_\alpha G = \frac{\nabla B_0(x_\alpha(t))}{\int_{-\infty}^{\infty} |\nabla B_0(x_\alpha(s))|^2 \,\mathrm{d}s} \int_{-\infty}^{\infty} \nabla B_0(x_\alpha(s)) \cdot G(s)\,\mathrm{d}s.$$

The denominator above is well-defined, since $\nabla B_0(x_\alpha(t))$ exponentially decays as $t \to \pm\infty$. The following Lyapunov–Schmidt type lemma now holds.

Lemma 2.4. *If $G \in \mathcal{L}(\mathbb{R})$, Eq. (8) has a solution in $\mathcal{L}(\mathbb{R})$ if and only if $P_\alpha G = 0$. If so, the solution which has an initial condition in the direction $\nabla B_0(x_\alpha(0))$ is unique. Moreover, the solution operator $Q_\alpha : (\mathrm{id} - P_\alpha)\mathcal{L}(\mathbb{R}) \to \mathcal{L}(\mathbb{R})$ is linear and continuous.*

Proof. We have shown in Lemma 2.3 that the components of $z(t)$ tangential to Γ remain bounded. Thus, it now suffices to consider the normal component. Premultiply Eq. (8) by $(\nabla B_0)^{\mathrm{T}}(x_\alpha(t))$, and integrate from 0 to T, to obtain

$$\int_0^T (\nabla B_0)^{\mathrm{T}}(x_\alpha(t))\dot{z}\,\mathrm{d}t = \int_0^T (\nabla B_0)^{\mathrm{T}}(x_\alpha(t))\nabla f(x_\alpha(t))z\,\mathrm{d}t + \int_0^{\mathrm{T}} (\nabla B_0)^{\mathrm{T}}(x_\alpha(t))G(t)\,\mathrm{d}t. \tag{12}$$

Integrating the left-hand side by parts

$$\begin{aligned}\int_0^T (\nabla B_0)^{\mathrm{T}}(x_\alpha(t))\dot{z}\,\mathrm{d}t &= (\nabla B_0)^{\mathrm{T}}(x_\alpha(t))z(t)|_0^T - \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}[(\nabla B_0)^{\mathrm{T}}(x_\alpha(t))]z(t)\,\mathrm{d}t\\ &= (\nabla B_0)^{\mathrm{T}}(x_\alpha(t))z(t)|_0^T + \int_0^T (\nabla B_0)^{\mathrm{T}}(x_\alpha(t))\nabla f(x_\alpha(t))z(t)\,\mathrm{d}t,\end{aligned}$$

where the last step is by virtue of Lemma 2.2. Substituting in (12), we obtain

$$(\nabla B_0)^{\mathrm{T}}(x_\alpha(T))z(T) = (\nabla B_0)^{\mathrm{T}}(x_\alpha(0))z(0) + \int_0^T (\nabla B_0)^{\mathrm{T}}(x_\alpha(t))G(t)\,\mathrm{d}t.$$

Dividing through by $|(\nabla B_0)(x_\alpha(T))|$, and reverting to the dot product notation

$$z(T) \cdot \frac{\nabla B_0(x_\alpha(T))}{|\nabla B_0(x_\alpha(T))|} = \frac{(\nabla B_0(x_\alpha(0)) \cdot z(0) + \int_0^T \nabla B_0(x_\alpha(t)) \cdot G(t)\,\mathrm{d}t)}{|\nabla B_0(x_\alpha(T))|}.$$

Noting that $\nabla B_0(x_\alpha(T))$ is never 0 for $T \in \mathbb{R}$ (by Hypothesis 2.3), but approaches 0 as $T \to \pm\infty$, it is clear that boundedness is achieved on $[0, \infty)$ if and only if

$$\nabla B_0(x_\alpha(0)) \cdot z(0) + \int_0^{\infty} \nabla B_0(x_\alpha(t)) \cdot G(t)\,\mathrm{d}t = 0$$

and on $(-\infty, 0]$ if and only if

$$\nabla B_0(x_\alpha(0)) \cdot z(0) + \int_0^{-\infty} \nabla B_0(x_\alpha(t)) \cdot G(t)\,\mathrm{d}t = 0.$$

Combining these statements, we obtain the condition

$$\int_{-\infty}^{\infty} \nabla B_0(x_\alpha(t)) \cdot G(t)\,\mathrm{d}t = 0,$$

which is equivalent to $P_\alpha G = 0$. The linearity and continuity of the solution operator Q_α, as well as uniqueness, are obvious from (9). □

2.3. Melnikov function

The remaining development closely follows the two-dimensional approach in [6], which itself modifies the ideas in [11] to weak perturbations. Consider a solution $x(t)$ of (7), and set

$$x(t) = x_\alpha(t-\tau) + \xi_\alpha(t-\tau).$$

In order to find 'nearby' heteroclinic orbits after perturbation, a 'small' solution is sought for $\xi_\alpha(t)$. This must satisfy

$$\begin{aligned}\dot{\xi}_\alpha(t) &= \dot{x}(t+\tau) - \dot{x}_\alpha(t) = f(x(t+\tau)) + g(x(t+\tau), t+\tau, \varepsilon) - f(x_\alpha(t)) \\ &= f(x_\alpha(t)+\xi_\alpha) + g(x_\alpha(t)+\xi_\alpha, t+\tau, \varepsilon) - f(x_\alpha(t)) = \nabla f(x_\alpha(t))\xi_\alpha + G_\alpha(\xi_\alpha, t+\tau, \varepsilon),\end{aligned} \tag{13}$$

where the function G_α is defined by

$$G_\alpha(\xi_\alpha, t+\tau, \varepsilon) := f(x_\alpha+\xi_\alpha) - f(x_\alpha) - \nabla f(x_\alpha)\xi_\alpha + g(x_\alpha+\xi_\alpha, t+\tau, \varepsilon)$$

with each of x_α and ξ_α being evaluated at t. We shall take advantage of the fact that, if ξ_α remains small, the first three terms in G_α contribute an order ξ_α^2 term, and the final term is itself small by Hypothesis 2.4. The existence of a heteroclinic point of (1) close to Γ depends on the existence of a bounded solution to (13), which can be shown in exactly the same way as in the two-dimensional case (Theorem 11.3.3 of [11]). Hence, by Lemma 2.4, the problem is equivalent to solving the pair of equations

$$P_\alpha G_\alpha(\xi_\alpha, \cdot+\tau, \varepsilon) = 0, \tag{14}$$

$$\xi_\alpha = Q_\alpha(\mathrm{id} - P_\alpha)G_\alpha(\xi_\alpha, \cdot+\tau, \varepsilon). \tag{15}$$

Lemma 2.5. *There exists a unique solution $\bar{\xi}(\alpha, \tau, \varepsilon)(t)$ to (15) for small enough ε, which satisfies*

$$|\bar{\xi}(\alpha, \tau, \varepsilon)| \le C\varepsilon$$

for a constant C uniform in (α, τ).

Proof. This is based on an application of the Banach–Cacciopoli contraction mapping principle (see for example [11]) on (15). The details of the proof are identical to that in [6], so will be omitted. □

In preparation for the statement of the main result in this section, we define the Melnikov-type *distance function* $d(\alpha, \tau, \varepsilon)$ by

$$d(\alpha, \tau, \varepsilon) := \int_{-\infty}^{\infty} \nabla B_0(x_\alpha(t)) \cdot G_\alpha(\bar{\xi}(\alpha, \tau, \varepsilon)(t), t+\tau, \varepsilon)\,\mathrm{d}t, \tag{16}$$

where $\bar{\xi}(\alpha, \tau, \varepsilon)(t)$ is the unique solution to (15) obtained from Lemma 2.5. If a geometrical development (akin to the two-dimensional version in [17,43]) is employed, the quantity in (16) would be seen to correspond to a *signed* distance function, which represents the (suitably normalised) distance between the perturbed manifolds at a point on Γ identified by (α, τ), measured in the normal direction $\nabla B_0(x_\alpha(\tau))$. In this case we have been forced to use a functional analytic method, since our perturbation need not possess the differentiability in ε necessary for the geometrical approach. Nevertheless, we shall think of d intuitively as a signed distance between perturbed manifolds; as we shall show, $d = 0$ *does* correspond to the case where the actual distance is 0.

Theorem 2.1. *Suppose the unperturbed flow satisfies Hypotheses 2.1–2.3, and that the perturbation g satisfies Hypothesis 2.4. Then, the perturbed equation (7) has a heteroclinic point (an intersection of stable and unstable manifolds) in the neighbourhood of* Γ *for* $\bar{\varepsilon} \in \mathcal{I}$ *if and only if* $(\bar{\alpha}, \bar{\tau}, \bar{\varepsilon})$ *satisfy* $d(\bar{\alpha}, \bar{\tau}, \bar{\varepsilon}) = 0$. *If*

$$\left| \frac{\partial d}{\partial \alpha}(\bar{\alpha}, \bar{\tau}, \bar{\varepsilon}) \right| + \left| \frac{\partial d}{\partial \tau}(\bar{\alpha}, \bar{\tau}, \bar{\varepsilon}) \right| \neq 0,$$

then the intersection is transverse.

Proof. The first statement has already been established from Lemmas 2.4 and 2.5, and Eq. (14); for additional details see [11]. For transversality at fixed $\bar{\varepsilon}$, it is necessary to be able to represent $d(\alpha, \tau, \bar{\varepsilon}) = 0$ locally near $(\bar{\alpha}, \bar{\tau})$ in the form $\alpha = \alpha(\tau)$ or $\tau = \tau(\alpha)$, which is possible by the implicit function theorem if either of the appropriate derivatives is nonzero at $(\bar{\alpha}, \bar{\tau})$. □

As it stands, Theorem 2.1 is useful only as a theoretical result, since the determination of G_α is impractical. To establish a more standard form, note from Hypothesis 2.4 that we may write

$$g(x, t, \varepsilon) = \varepsilon \bar{g}(x, t, \varepsilon),$$

where $\bar{g} \in C^r(\Omega \times \mathbb{R})$ for each $\varepsilon \in \mathcal{I}$, and is uniformly bounded in $\Omega \times \mathbb{R} \times \mathcal{I}$. Define the *Melnikov function* $M(\alpha, \tau, \varepsilon)$ by

$$M(\alpha, \tau, \varepsilon) := \int_{-\infty}^{\infty} \nabla B_0(x_\alpha(t)) \cdot \bar{g}(x_\alpha(t), t + \tau, \varepsilon)\, \mathrm{d}t. \tag{17}$$

Corollary 2.1. *Suppose the assumptions of Theorem 2.1 are satisfied. Then, it is possible to represent*

$$d(\alpha, \tau, \varepsilon) = \varepsilon M(\alpha, \tau, \varepsilon) + \mathcal{O}(\varepsilon^2).$$

Proof. Since ∇g satisfies a Lipschitz estimate in ε, and both f and g are sufficiently smooth in the spatial variable, we have

$$g(x_\alpha(t) + \xi_\alpha(t), t + \tau, \varepsilon) = g(x_\alpha(t), t + \tau, \varepsilon) + \mathcal{O}(\varepsilon |\xi_\alpha|) \quad \text{and}$$
$$f(x_\alpha(t) + \xi_\alpha(t)) - f(x_\alpha(t)) - \nabla f(x_\alpha(t))\xi_\alpha(t) = \mathcal{O}(|\xi_\alpha|^2)$$

and moreover, Lemma 2.5 asserts that $\xi_\alpha = \mathcal{O}(\varepsilon)$. This permits G_α to be expanded appropriately in ε, enabling the expansion of d. □

For the frequently considered special case where $\bar{g}$ is independent of ε, the Melnikov function is itself independent of ε, and the integrand of Eq. (17) takes the familiar form of ∇B_0 dotted with the perturbation. In the standard fashion, the existence of transverse zeros of the Melnikov function will imply the existence of transverse heteroclinic points. A complicated collection of zeros in M leads to complicated motion across Γ, which served as a separatrix for the $\varepsilon = 0$ flow. A special form of the formula (17) was formally argued by Holmes [20] for the time-periodic strong perturbation case, and used by Holmes [20] and Mezić [31] for analysing perturbations on a class of fluid flows. We remark that the presence of an ε in (17) at this point appears to diminish the usefulness of the Melnikov function as a predictor of intersections for small ε; we will show in Section 3 that for our fluid mechanical application much of this ε dependence disappears.

2.4. Finite-time perturbations

The previous sections developed a Melnikov method for the case where the perturbing function g in (7) was not necessarily differentiable in the perturbation parameter ε, but instead satisfied a Lipschitz-type condition in ε for all $t \in \mathbb{R}$. It turns out that these conditions can be weakened further, enabling better validity in using the theory for Navier–Stokes perturbations of Euler flows. This section briefly outlines two improvements: (i) the function g is a perturbation not for all times, but only for *finite times* and (ii) the function is $\mathcal{O}(\varepsilon^{\gamma})$, where $\gamma \in (1/2, 1]$, rather than $\mathcal{O}(\varepsilon)$. These weakenings are made specific in the following hypothesis, which is the replacement for Hypothesis 2.4.

Hypothesis 2.5. The perturbation $g(x, t, \varepsilon)$ in (7) satisfies the following conditions. There exists functions $\tau_{\pm}(\varepsilon)$, and a constant $\gamma \in (1/2, 1]$ such that

(a) $g : \Omega \times \mathcal{T} \times I \to \mathbb{R}$ is C^r, $r \geq 2$, in (x, t) for each $\varepsilon \in I$ with uniform bounds, where

$$\mathcal{T} = \left(\tau_-(\varepsilon) - \frac{2\gamma}{\theta} |\ln \varepsilon|, \tau_+(\varepsilon) + \frac{2\gamma}{\theta} |\ln \varepsilon| \right)$$

and θ is the decay rate given in the exponential dichotomies (4) and (5).

(b) $g(x, t, 0) = 0$ for all $(x, t) \in \Omega \times \mathcal{T}$.

(c) There exists $C > 0$ such that

$$|g(x, t + \tau, \varepsilon)| + |\nabla g(x, t + \tau, \varepsilon)| \leq C\varepsilon^{\gamma}$$

uniformly for $\tau \in [\tau_-(\varepsilon), \tau_+(\varepsilon)]$, $|t| \leq (2\gamma/\theta)|\ln \varepsilon|$ and $(x, \varepsilon) \in \Omega \times I$.

Based on this hypothesis, but with special emphasis on two dimensions, Sandstede et al. [37] showed the following:

- One can extend g to $\mathbb{R}$ smoothly such that the extended function $g_{\mathrm{e}} : \Omega \times \mathbb{R} \times I$ satisfies Hypothesis 2.4;
- Under this extended perturbation g_{e}, the hyperbolic trajectories persist as described in Section 2.2, as do their manifolds;
- The "finite-time manifolds" associated with the flow for $t \in [\tau_-(\varepsilon), \tau_+(\varepsilon)]$ can be defined by restricting the manifolds obtained for g_{e};
- Although there is no unique smooth extension g_{e} to $\mathbb{R}$, the manifolds associated with different extensions can be no worse that $\mathcal{O}(\varepsilon^{2\gamma})$ apart.

These proofs have validity in an n-dimensional setting, and hence hold in our case as well. Therefore, analogously following the development of Section 2.3, we define the distance function for a choice of g_{e} by

$$d(\alpha, \tau, \varepsilon) := \int_{-\infty}^{\infty} \nabla B_0(x_\alpha(t)) \cdot G_\alpha(\bar{\xi}(\alpha, \tau, \varepsilon)(t), t+\tau, \varepsilon)\, \mathrm{d}t = \int_{-\infty}^{\infty} \nabla B_0(x_\alpha(t)) \cdot g_{\mathrm{e}}(x_\alpha(t), t+\tau, \varepsilon)\, \mathrm{d}t + \mathcal{O}(\varepsilon^{2\gamma}).$$

Now it was proven in [37] that the error in using g instead of g_{e}, and thereby replacing the infinite-time integral with an finite-time one, has an error *also* of $\mathcal{O}(\varepsilon^{2\gamma})$. This enables Theorem 2.1 to be modified as

Theorem 2.2. *Suppose the unperturbed flow* (1) *satisfies* Hypotheses 2.1–2.3, *and the perturbed flow* (7) *satisfies* Hypothesis 2.5. *There exists a persistent heteroclinic connection near* $(\bar{\alpha}, \bar{\tau}, \bar{\varepsilon})$ *if and only if* $d(\bar{\alpha}, \bar{\tau}, \bar{\varepsilon}) = 0$. *If* $|\partial d/\partial \alpha| + |\partial d/\partial \tau| \neq 0$ *at this value, this corresponds to a transverse intersection of manifolds. Moreover, for* $\tau \in (\tau_-(\varepsilon), \tau_+(\varepsilon))$

$$d(\alpha, \tau, \varepsilon) = \int_{-(2\gamma/\theta)|\ln \varepsilon|}^{(2\gamma/\theta)|\ln \varepsilon|} \nabla B_0(x_\alpha(t)) \cdot g(x_\alpha(t), t + \tau, \varepsilon)\, \mathrm{d}t + \mathcal{O}(\varepsilon^{2\gamma}). \tag{18}$$

By virtue of Hypothesis 2.5(c), we may write

$$g(x, t+\tau, \varepsilon) = \varepsilon^{\gamma} \bar{g}(x, t+\tau, \varepsilon)$$

for $(x, t+\tau, \varepsilon) \in \Omega \times \mathcal{T} \times I$. Define the Melnikov function by

$$M(\alpha, \tau, \varepsilon) := \int_{-(2\gamma/\theta)|\ln \varepsilon|}^{(2\gamma/\theta)|\ln \varepsilon|} \nabla B_0(x_\alpha(t)) \cdot \bar{g}(x_\alpha(t), t+\tau, \varepsilon)\, \mathrm{d}t. \tag{19}$$

Corollary 2.2. *Suppose the assumptions of Theorem 2.2 are satisfied. Then, d can be written in the form*

$$d(\alpha, \tau, \varepsilon) = \varepsilon^{\gamma} M(\alpha, \tau, \varepsilon) + \mathcal{O}(\varepsilon^{2\gamma}).$$

Notice also that if $\bar{g}$ is smoothly extended to a function $\bar{g}_{\mathrm{e}}$ defined for $t \in \mathbb{R}$, we can define

$$M_{\mathrm{e}}(\alpha, \tau, \varepsilon) := \int_{-\infty}^{\infty} \nabla B_0(x_\alpha(t)) \cdot \bar{g}_{\mathrm{e}}(x_\alpha(t), t+\tau, \varepsilon)\, \mathrm{d}t$$

and that d can also be written in the form

$$d(\alpha, \tau, \varepsilon) = \varepsilon^{\gamma} M_{\mathrm{e}}(\alpha, \tau, \varepsilon) + \mathcal{O}(\varepsilon^{2\gamma}).$$

(This is since the error in moving to infinite time is of exactly $\mathcal{O}(\varepsilon^{2\gamma})$; see [37] for additional details.) Therefore, the Melnikov theory can be extended to the case where the perturbation is "weak" in the sense of Hypothesis 2.5. Both differentiability in ε, and existence for all times, have been weakened in this formulation.

2.5. Orbits heteroclinic to periodic orbits

In this section we quickly outline the extension to weak perturbations of volume-preserving systems possessing orbits homoclinic or heteroclinic to periodic orbits. Our starting point are symmetry considerations of the type introduced in [18,32]. We were not able to use these in the previous sections due to the fact that vector fields with fixed points cannot admit regular action of a one-dimensional symmetry group.

Consider a three-dimensional dynamical system which can be written as

$$\dot{r} = p(r), \qquad \dot{\theta} = \omega(r), \tag{20}$$

where $r \in \mathbb{R}^2$ and $\theta \in S^1$, and $\nabla \cdot p = 0$. This is a symmetry coordinate representation, which would be appropriate if the system admitted a regular action of a rotational symmetry in a certain direction. The volume-preserving symmetry implies that the existence of a nondegenerate conserved quantity B_0 [18,32]. Assume moreover that Hamiltonian part of the system possesses two hyperbolic fixed points r_i^* for $i = 1, 2$; then $p(r_i^*) = 0$ and $|\omega(r_i^*)| > 0$. Thus, $\gamma_i(t) = (r_i^*, \theta_0 + \omega(0)t)$ are periodic trajectories for (20). Suppose each of these hyperbolic periodic trajectories has two-dimensional stable and unstable manifolds, and that the stable manifold of one coincides with the unstable manifold of the other, forming a heteroclinic manifold. In addition to the structures in Fig. 1, the geometries illustrated in Fig. 2 are now possible configurations for the unperturbed flow in the original x (nonsymmetry) coordinates in $\mathbb{R}^3$. The general case is shown in (a), in which the end trajectories are periodic but distinct, and Γ is the heteroclinic manifold connecting the two. Here, $a(t)$ and $b(t)$ correspond to r_1^* and r_2^* in the symmetry coordinates, respectively. If $a(t)$ and $b(t)$ are the same, the homoclinic case of (b) results. In each figure, the other branches of the manifolds have been omitted for clarity, and a typical heteroclinic trajectory x_α lying on Γ is indicated.

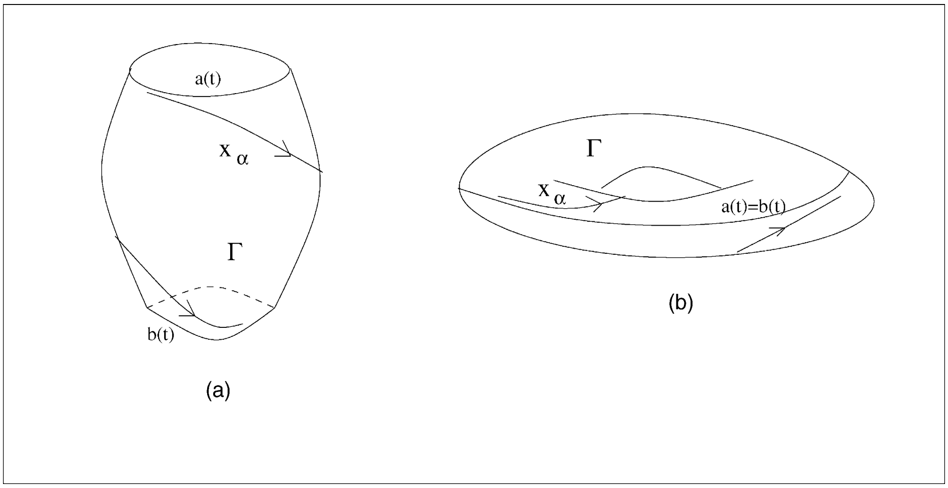

Fig. 2. Possible structures for the heteroclinic manifold Γ in the case of periodic end trajectories.

Let $r_\infty(t)$ be a trajectory of $\dot{r} = p(r)$ which is heteroclinic between r_1^* and r_2^*; this is associated with $(r_\infty(t), \theta_0 + \int_0^t \omega(r_\infty(s))\,\mathrm{d}s)$ in (r, θ) space. The variational equation along this trajectory is

$$\dot{y} = \nabla p(r_\infty(t)) \cdot y, \qquad \dot{\phi} = \nabla \omega(r_\infty(t)) \cdot y.$$

Now consider perturbing equation (20) through the addition of functions q and h in the form

$$\dot{r} = p(r) + q(r, \theta, t, \varepsilon), \qquad \dot{\theta} = \omega(r) + h(r, \theta, t, \varepsilon).$$

Here, q and h are considered $\mathcal{O}(\varepsilon)$ in the usual way. The normal hyperbolicity of $\gamma_i(t)$ ensures that each γ_i persists (though not necessarily as a periodic trajectory) for small ε, and retains its stable and unstable manifolds. This is proven in Theorem 6.1 in [19]. Now, the perturbed variational equations in the symmetry setting would take the general form

$$\dot{y} = \nabla p(r_\infty(t)) \cdot y + F(t), \qquad \dot{\phi} = \nabla \omega(r_\infty(t)) \cdot y + G(t).$$

Define the projection operator

$$PF := \frac{\dot{r}_\infty^\perp(t)}{\int_{-\infty}^{\infty} [\dot{r}_\infty^\perp(s)]^2\,\mathrm{d}s} \int_{-\infty}^{\infty} \dot{r}_\infty^\perp(s) \cdot F(s)\,\mathrm{d}s$$

and set $r(t) = z(t-\tau) + r_\infty(t-\tau)$. Our goal is to determine conditions which ensure that z remains small; this will guarantee that that trajectory lies on both the stable and unstable [perturbed] manifolds. Arguing as in [11,12] and Section 2.2, this is related to solving

$$PF(z, \cdot + \tau, \theta_\varepsilon(\cdot), \varepsilon) = 0, \tag{21}$$

$$z = Q(\mathrm{id} - P)F(z, \cdot + \tau, \theta_\varepsilon(\cdot), \varepsilon). \tag{22}$$

Notice that the difference now is that we have another function of time, $\theta_\varepsilon(t)$, inside the perturbing function F. Nevertheless, the procedure of Lemma 2.5 yields an $\mathcal{O}(\varepsilon)$ solution $\bar{z}(\tau, \varepsilon)(t)$ to (22). The nonlocal dependence of $\bar{z}$

on the angular variable θ_ε does not interfere with the smallness of $\bar{z}$. Eq. (21) then gives the condition

$$PF = \int_{-\infty}^{\infty} \dot{r}_\infty^\perp(t) \cdot F(\bar{z}, t+\tau, \theta_\varepsilon(t), \varepsilon)\,\mathrm{d}t = 0.$$

We now argue that replacing θ_ε above the unperturbed solution $\theta_0(t)$ results in a higher-order term. Note that since $h = \mathcal{O}(\varepsilon)$, θ_ε and θ_0 are $\mathcal{O}(\varepsilon)$ apart for finite-times, and moreover, cannot differ by more than 1 since $\theta \in S^1$. For a finite-time version of the integral above, therefore replacing θ_ε with the unperturbed θ gives a $\mathcal{O}(\varepsilon)$ error term; extending to infinite times using arguments analogous to those in [37] are straightforward. Therefore, in the original three-dimensional (x) coordinates, persistence of a heteroclinic connection is through solving

$$d(\alpha, \tau, \varepsilon) := \varepsilon \int_{-\infty}^{\infty} \nabla B_0(x_\alpha(t)) \cdot \bar{g}(x_\alpha(t), t+\tau, \varepsilon)\,\mathrm{d}t + \mathcal{O}(\varepsilon^2) = 0,$$

where $x_\alpha(t)$ is the heteroclinic trajectory in the original (nonsymmetry) coordinates and $\varepsilon\bar{g}$ is the transformation of the perturbation vector to the original coordinates. That is, the formula derived in Section 2.3 for the case of a heteroclinic connection between fixed points, extends to this case of heteroclinic connections between periodic trajectories.

2.6. Hill's spherical vortex

As an example, an explicit expression for the Melnikov function is derived in this section for the case where the unperturbed flow is Hill's spherical vortex. This is an exact analytical solution to Euler's equations of motion for an inviscid fluid, and possesses the geometrical structure as given in Fig. 1(a). It is convenient to express the solution in spherical polar coordinates (r, θ, ϕ). The (differentiable) integral of motion is

$$B_0(r,\theta) = \begin{cases} -\dfrac{U}{2}\left(r^2 - \dfrac{c^3}{r}\right)\sin^2\theta & \text{if } r \geq c, \\[2ex] \dfrac{3U}{4} r^2 \left(1 - \dfrac{r^2}{c^2}\right)\sin^2\theta & \text{if } r \leq c, \end{cases}$$

where U is a constant with dimensions of velocity, while c is a constant which defines the radius of the vortex [1,25]. This function serves as a Stokes streamfunction [1], enabling the (continuous) velocity field to be written in (r, θ, ϕ) coordinates as

$$u(r,\theta,\phi) = \frac{1}{r^2 \sin\theta}\frac{\partial B_0}{\partial \theta}\hat{r} - \frac{1}{r\sin\theta}\frac{\partial B_0}{\partial r}\hat{\theta} + 0\hat{\phi},$$

which results in

$$u(r,\theta,\phi) = \begin{cases} -U\left(1 - \dfrac{c^3}{r^3}\right)\cos\theta\,\hat{r} + U\left(1 + \dfrac{c^3}{2r^3}\right)\sin\theta\,\hat{\theta} & \text{if } r \geq c, \\[2ex] \dfrac{3U}{2}\left(1 - \dfrac{r^2}{c^2}\right)\cos\theta\,\hat{r} - \dfrac{3U}{2}\left(1 - \dfrac{2r^2}{c^2}\right)\sin\theta\,\hat{\theta} & \text{if } r \leq c. \end{cases}$$

This velocity field is an exact incompressible solution to Euler's [inviscid] fluid equations. The points $a \equiv (c, 0)$ and $b \equiv (c, \pi)$ in (r, θ) coordinates are the north and south poles, respectively, of the globe $r = c$, which forms the heteroclinic manifold Γ, with the behaviour as illustrated in Fig. 1(a) (or Fig. 3(a) in [27]). The manifold separates motion on torii (inside the manifold) from motion on cylinders (outside the manifold); this is a prototypical example of the phase portraits possible for steady solutions to Euler's equations in 3D as proven by Arnold [3]. The destruction of this manifold under a perturbation would result in fluid mixing across hitherto disparate flow regimes. Now, Γ is foliated with heteroclinic trajectories which traverse the longitudes. In other words, the ϕ-coordinate does not change for trajectories (this is clear from the absence of a $\hat{\phi}$ term in the velocity field above). In particular, one can index the heteroclinic trajectories with $\alpha \in S^1$ by simply setting $2\pi\alpha = \phi$, where ϕ corresponds to the longitude of that particular heteroclinic trajectory. Thus, $x_\alpha(t)$ has coordinates $(c, \bar{\theta}(t), 2\pi\alpha)$, and $\bar{\theta}$ is a solution to

$$c\frac{\mathrm{d}}{\mathrm{d}t}\bar{\theta} = \frac{3U}{2}\sin\bar{\theta}. \tag{23}$$

Eq. (23) is obtained by evaluating the θ component of the velocity field on $x_\alpha(t)$—the continuity of u across Γ enables this to be computed using either the 'inner' or 'outer' flow. Separation of variables on the differential equation (23) shows that the symmetric solution (i.e., the solution with initial condition chosen on the equator) must satisfy

$$\sin\bar{\theta}(t) = \operatorname{sech}\left(\frac{3Ut}{2c}\right). \tag{24}$$

Now, in order to use the Melnikov function we must compute

$$\nabla B_0(x_\alpha(t)) = \nabla B_0(c, \bar{\theta}(t), 2\pi\alpha) = -\frac{3Uc}{2}\sin^2\bar{\theta}(t)\hat{r} = -\frac{3Uc}{2}\operatorname{sech}^2\left(\frac{3Ut}{2c}\right)\hat{r}.$$

Now, we note that the decay rate at the poles is $3U/2c$; this is easily seen by linearising (23). If we suppose that we perturb the velocity field through the addition of a function $g(r, \theta, \phi, t; \varepsilon)$ as described in Hypothesis 2.5, the distance function is then given by

$$d(\alpha, \tau, \varepsilon) = -\frac{3Uc}{2}\int_{-(4c\gamma/3U)|\ln\varepsilon|}^{(4c\gamma/3U)|\ln\varepsilon|}\operatorname{sech}^2\left(\frac{3Ut}{2c}\right)g_r(c, \bar{\theta}(t), 2\pi\alpha, t+\tau; \varepsilon)\,\mathrm{d}t + \mathcal{O}(\varepsilon^{2\gamma}), \tag{25}$$

where $g = g_r\hat{r} + g_\theta\hat{\theta} + g_\phi\hat{\phi}$, $\gamma \in (1/2, 1]$, and the limits can be replaced with $\pm\infty$ if the perturbation is defined for all times. The function $\bar{\theta}(t)$ can be obtained by taking the inverse sine of (24), and choosing $\bar{\theta} \in (0, \pi/2)$ if $t < 0$, but in $(\pi/2, \pi)$ if $t > 0$. There has not been much confidence in the ability of Melnikov methods to *analytically* determine criteria for *genuine* solutions to three-dimensional fluids equations (see [20]); we offer (25)).

We will examine several special cases. Note that it is only necessary to specify the radial component of the perturbation. First, take the simple periodic case where

$$g_r = \varepsilon^\gamma U\sin\left(\frac{Ut}{c}\right)$$

and is valid for all times (the constants have been chosen in this manner to ensure consistent dimensions). By substituting in (25), it is possible to derive

$$d(\alpha, \tau, \varepsilon) = -\varepsilon^\gamma\frac{2\pi Uc^2}{3\sinh(\pi/3)}\sin\left(\frac{U\tau}{c}\right) + \mathcal{O}(\varepsilon^{2\gamma}),$$

hence the Melnikov function for this case is *explicitly* given by

$$M(\tau) = -\frac{2\pi Uc^2}{3\sinh(\pi/3)}\sin\left(\frac{U\tau}{c}\right).$$

It is easy to see that $M(\tau)$ has infinitely many transverse zeroes at $\tau = \pi nc/U$ for integer n. Since M is independent of α, an axisymmetric picture would be observed at each instance in time; W_a^u and W_b^s would intersect along constant latitudes (constant τ), resulting in an alternating latitudinal band structure (see Fig. 5 in [9] for such a picture). If one considers a Poincaré map of this portrait (with period $2\pi c/U$, the period of the perturbation), each band will map to a more southern one. The bands will become more tightly spaced as the poles are approached, and will compensate by stretching in the direction perpendicular to Γ to maintain incompressibility. The standard chaotic picture of a heteroclinic tangle then occurs along each and every longitude of Hill's spherical vortex under this perturbation.

An extension is possible to the case where the perturbation takes the form

$$g_r = \varepsilon^\gamma U h(r, \phi) \sin\left(\frac{Ut}{c}\right).$$

There is no real difference in the computation of the Melnikov function, which in this case becomes

$$M(\alpha, \tau) = -\frac{2\pi U c^2}{3\sinh(\pi/3)} h(c, 2\pi\alpha) \sin\left(\frac{U\tau}{c}\right).$$

Again, W_a^u and W_b^s will intersect along constant latitudes in each Poincaré time-snapshot. The added complication is that there would be a longitudinal (nonaxisymmetric) effect due to the presence of α in M. If $h(c, 2\pi\alpha)$ has no zeroes, the portrait will be qualitatively the same as the previous one. On the other hand, if it has zeroes (it will generically have an even number for $\alpha \in S^1$), in each inter-latitudinal band in which $\sin(U\tau/c)$ is sign-definite, W_a^u and W_b^s will intersect as the band is traversed. For example, if h has four zeroes, each band will consist of four regions, two of which would have W_a^u outside W_b^s. Thus, the sphere will be 'blotchy', with the manifolds intersecting in nontrivial ways. Under the Poincaré map, these regions will map to one another in complicated ways, resulting in nonaxisymmetric chaos. However, given the explicit form of the Melnikov function which has been derived, we can determine the exact locations at which the manifolds intersect.

Now consider the case of an autonomous perturbation for all time

$$g_r = \varepsilon^\gamma U h(r, \theta, \phi).$$

The Melnikov function now has the explicit form

$$M(\alpha) = -\frac{3U^2 c}{2} \int_{-\infty}^{\infty} \operatorname{sech}^2\left(\frac{3Ut}{2c}\right) h(c, \bar{\theta}(t), 2\pi\alpha)\, \mathrm{d}t.$$

The absence of the τ-variable here means that the intersection picture in each latitudinal cross-section will simply translate to other latitudes. Generically, there would be an even number of intersections of the perturbed manifolds in each $\tau =$ constant cross-section (latitude); the global picture will be of nonintersecting north–south bands foliating Γ. Alternating bands will escape from the perturbed globular surface near each pole—but no chaos would occur. If there were swirl in the original unperturbed flow, however, these bands would wind around the globe, resulting in chaotic motion (see Fig. 4 in [20]). For the (no-swirl) Hill's spherical vortex base flow we have examined in this section, however, this cannot happen under an autonomous perturbation.

3. Navier–Stokes perturbation of Euler flow

This section adapts the Melnikov theory developed in Section 2 to a class of three-dimensional fluid flows. The unperturbed flow is assumed to be a solution to the steady Euler equations, which has a natural integral of motion: the Bernoulli function. What we analyse is the effect (upon the manifolds of this flow) of the inclusion of small viscous and forcing contributions in the dynamical equations; in other words, the usage of the Navier–Stokes equations with high Reynolds number to generate the velocity field. The splitting of the manifold through using this viscous perturbation would generate fluid transport across that manifold, and therefore our method is a preliminary step in assessing fluid transport for which small viscosity can be held responsible.

3.1. The Euler flow

A three-dimensional steady incompressible inviscid flow subject to conservative forcing is governed by the *Euler equations*

$$\frac{\mathrm{d}v_0}{\mathrm{d}t} = -\frac{1}{\rho}\nabla p_0 - \nabla \Phi_0, \quad \nabla \cdot v_0 = 0,$$

where $\mathrm{d}/\mathrm{d}t$ is the derivative following the flow. If $x \in \Omega \subset \mathbb{R}^3$, the velocity field v_0, the density ρ, the pressure p_0 and the force potential Φ_0 are in general functions of x, which shall all be assumed to be in $C^r(\Omega)$, $r \geq 2$. Moreover, it is hypothesised that the flow of interest is far removed from boundaries. We now make the assumption that the flow is *isentropic*, in which case there exists an enthalpy function whose gradient equals $(1/\rho(x))\nabla p_0(x)$, see [10]. Standard cases of isentropic flows are when ρ is constant, or when isobaric ("equi-pressure") and isopycnal ("equi-density") surfaces coincide. We note that the potential forcing term can *also* be absorbed into an effective enthalpy function $H_0(x)$, which we shall define through

$$\nabla H_0(x) := \frac{1}{\rho(x)}\nabla p_0(x) + \nabla \Phi_0(x).$$

With the additional observation that, for this steady case

$$\frac{\mathrm{d}v_0}{\mathrm{d}t} = (v_0 \cdot \nabla)v_0 = \frac{1}{2}\nabla(|v_0|^2) - v_0 \times \omega_0,$$

where $\omega_0(x) := \nabla \times v_0(x)$ is the vorticity of the flow, the Euler equations can be written in the form

$$\nabla B_0 = v_0 \times \omega_0, \qquad \nabla \cdot v_0 = 0, \tag{26}$$

where the *Bernoulli function* $B_0(x)$ is given by

$$B_0(x) := H_0(x) + \tfrac{1}{2}|v_0(x)|^2.$$

Now, the fluid trajectories are governed by the ordinary differential equation

$$\dot{x} = v_0(x). \tag{27}$$

We see that

$$\frac{\mathrm{d}}{\mathrm{d}t}B_0(x) = \nabla B_0(x) \cdot \dot{x} = \nabla B_0 \cdot v_0(x) = 0$$

by Eq. (26); $B_0(x)$ is therefore conserved by the flow (27). If v_0 and ω_0 are never collinear (i.e., the flow is not a *Beltrami* flow), (26) shows that ∇B_0 can only be zero at fixed points or zero vorticity points, thereby providing a genuine integral of motion.

Thus, (27) is integrable, and the flow is confined to the *Lamb surfaces* $B_0 =$ constant (see [38]). The integrability of (27) was shown by Arnold [2] through the observation that the Lie bracket between the velocity and vorticity vanishes. In fact, such integrability, with an explicit construction of the integral of motion, can be extended to the case where it is a symmetry rather than the vorticity which provides the appropriate vector in (26) [18,32]. A (small) symmetry breaking lends itself to a Melnikov analysis akin to what we present here, for the case where the integral of motion is the Bernoulli function.

We note that Hypotheses 2.1 and 2.2 are met by Eq. (27), and additionally, we assume that Hypothesis 2.3 is satisfied. Thus, a two-dimensional heteroclinic manifold Γ is assumed to exist for the flow (27). This manifold is automatically a Lamb surface by Lemma 2.1, and its assumed topological structure (as illustrated in Fig. 1) is consistent with the proof in [2] that Lamb surfaces are toroidal or cylindrical (with the exception of separating surfaces between torii and cylinders). This topology is reminiscent of a bubble-vortex, the experimental breakdown of which is described in the review by Leibovich [26]. This has also been theoretically studied in [20,27,31]. The heteroclinic trajectories may be spiralling into the poles (the case illustrated in Fig. 1(b); an example is given in [32] called the *rotating Hill's spherical vortex*), or nonspiralling (the heteroclinic trajectories lie along the longitudes of the globe; the classical *Hill's spherical vortex* as in Fig. 1(a) and Section 2.6). The manifold Γ in either case is foliated with heteroclinic orbits, which we shall represent (as in Section 2) as $x_\alpha(\tau)$ such that (α, τ) forms a parametrisation of Γ. Moreover, let θ be the intrinsic decay rate of x_α to a or b, as expressed mathematically by the exponential dichotomies (4) and (5).

The manifold Γ serves to separate regions of disparate types of flow. There are nested torii within Γ upon which the motion occurs, whereas flow is confined to cylinders outside Γ. When viscous effects are included, Γ would generically break down in some fashion, permitting fluid transport across the hitherto separatrix. If the viscous effects are small (i.e., if the Reynolds number is large), one would expect that this transport is small. The nature of this transport depends on the way in which Γ splits upon including the viscous perturbation. This will be assessed using the Melnikov approach developed in Section 2.

3.2. The Navier–Stokes flow

We consider as a 'perturbation' a high Reynolds number Navier–Stokes flow, which has the form

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{\rho}\nabla p - \nabla \Phi_0 + \varepsilon[\nabla^2 v + \phi(x,t)], \qquad \nabla \cdot v = 0.$$

The positive parameter ε represents the kinematic viscosity, and is assumed small, while $\phi(x,t)$ is an imposed forcing, which may be conservative or not, and is assumed bounded on $\Omega \times \mathbb{R}$. The understanding is that the added viscous and forcing terms have the same magnitude of effectiveness, ε, on the scale of the momentum equation. The flow is no longer steady, and hence the velocity v and pressure p are functions of (x,t), and the acceleration term involves a partial derivative with respect to t. However, we assume that the flow remains isentropic. We shall define the (perturbed versions of the) vorticity $\omega(x,t) := \nabla \times v(x,t)$, and the effective enthalpy $H(x)$ through

$$\nabla H(x,t) := \frac{1}{\rho(x)}\nabla p(x,t) + \nabla \Phi_0(x).$$

The perturbed Bernoulli function is defined as

$$B(x,t) := H(x,t) + \tfrac{1}{2}|v(x,t)|^2.$$

Some manipulation on the Navier–Stokes equations yields

$$\nabla B = v \times \omega - \frac{\partial v}{\partial t} + \varepsilon \nabla^2 v + \varepsilon \phi, \qquad \nabla \cdot v = 0. \tag{28}$$

Of interest is the legitimacy of considering the Navier–Stokes velocity field $v(x,t)$ as a perturbation on the Euler velocity field $v_0(x)$. This subtle issue has been much researched, in a variety of geometries and functional spaces. In two dimensions *with no boundary*, convergence of these Navier–Stokes velocity fields to the Euler one is well-understood, at least for finite times.

Convergence in $L^2(\Omega)$ has been known for some time [24], while convergence in L^1 [28], L^p_s [22], H^m [14], C^1 [15] and C^3 [5] have also been shown, under smoothness hypotheses on the Euler flow. Of these, [24] and [5] show specifically that convergence occurs as $\mathcal{O}(\varepsilon)$, enabling the statement that the Navier–Stokes and Euler fields are

$\mathcal{O}(\varepsilon)$-close. The absence of boundaries is crucial in all these proofs; if not, the closeness may worsen to $\mathcal{O}(\sqrt{\varepsilon})$, as suggested by recent results [36]. Certain results in this direction also exist in three dimensions (see [21,39]), but are very restrictive, being valid only for small times. This is not surprising, given the technical difficulties associated with the three-dimensional Navier–Stokes equations. Nevertheless, motivated by these results, we shall assume

Hypothesis 3.1. There exist positive constants C_1, C_2 and κ such that

$$|v(x,t)-v_0(x)|+|\nabla v(x,t)-\nabla v_0(x)|+|\nabla^2 v(x,t)-\nabla^2 v_0(x)|\left|\frac{\partial v}{\partial t}(x,t)\right|\leq C_1\varepsilon,$$
$$|H(x,t)-H_0(x)|+\left|\frac{\partial H(x,t)}{\partial t}\right|\leq C_2\varepsilon$$

uniformly for $x\in\Omega$ and $t\in[-\varepsilon^{-\kappa}-(2/\theta)|\ln\,\varepsilon|,\varepsilon^{-\kappa}+(2/\theta)|\ln\,\varepsilon|]$.

This simply requires the $\mathcal{O}(\varepsilon)$-closeness of the Euler and Navier–Stokes velocities (respectively enthalpies), in certain types of norms. Recall that θ is the decay rate associated with the heteroclinic trajectories (see (4) and (5)). Worthy of note is that we are assuming $\mathcal{O}(\varepsilon)$ closeness between the Euler and Navier–Stokes flows not for all times, but for finite (though sufficiently large) times. An alternative interpretation is that the constants C_1 and C_2 are permitted to increase with time at a rate which is controlled by the heteroclinic decay rate θ of the Euler flow. Although Hypothesis 3.1 cannot be fully justified with the extant results in fluid mechanics, the $\mathcal{O}(\varepsilon)$-closeness for finite-times seems a reasonable hypothesis based on the available results outlined in the previous paragraph.

As a result of Hypothesis 3.1, we write

$$v(x,t;\varepsilon)=v_0(x)+\varepsilon v_1(x,t;\varepsilon),\qquad \omega(x,t;\varepsilon)=\omega_0(x)+\varepsilon\omega_1(x,t;\varepsilon),\qquad p(x,t;\varepsilon)=p_0(x)+\varepsilon p_1(x,t;\varepsilon),$$
$$H(x,t;\varepsilon)=H_0(x)+\varepsilon H_1(x,t;\varepsilon),\qquad B(x,t;\varepsilon)=B_0(x)+\varepsilon B_1(x,t;\varepsilon),$$

where each of the variables with a subscript "1" is bounded uniformly for $(x,\varepsilon)\in\Omega\times I$ and $t\in[-\varepsilon^{-\kappa}-(2/\theta)|\ln\,\varepsilon|,\varepsilon^{-\kappa}+(2/\theta)|\ln\,\varepsilon|]$, where $I=[0,\varepsilon_0)$ is a suitably small interval. Thus, the variables can be represented in a "perturbative" setting, as is appropriate for using the finite time Melnikov method of Section 2.4, where $\gamma=1$ for this particular case.

3.3. Splitting behaviour

The viscous flow trajectories are obtained from

$$\dot{x}=v(x,t;\varepsilon)=v_0(x)+\varepsilon v_1(x,t;\varepsilon), \tag{29}$$

which has a velocity field close to (27) in the sense of satisfying Hypothesis 2.5. Continuity of v_1 in ε is not imposed as there is no physical (or mathematical) argument to support this; it is this issue which required the weak Melnikov development described in Section 2. Moreover, the $\mathcal{O}(\varepsilon)$-closeness is also only assumed for finite times in a specific sense. Thus, from Corollary 2.2, $d(\alpha,\tau,\varepsilon)=\varepsilon M(\alpha,\tau,\varepsilon)+\mathcal{O}(\varepsilon^2)$, and the Melnikov function is given by

$$M(\alpha,\tau,\varepsilon)=\int_{-(2/\theta)|\ln\,\varepsilon|}^{(2/\theta)|\ln\,\varepsilon|}\nabla B_0(x_\alpha(t))\cdot v_1(x_\alpha(t),t+\tau,\varepsilon)\,\mathrm{d}t, \tag{30}$$

where $x_\alpha(t)$ is a trajectory of the Euler flow which lies on the manifold Γ. Notice that since the integrand involves $\nabla B_0\cdot v_1$, knowledge of the perturbed (Navier–Stokes) velocity is required (though, as is common with perturbation techniques, the perturbed trajectories are not needed). We shall obtain an additional simplification: that the Melnikov function can effectively be expressed mainly in terms of the Euler flow, with very little details of the Navier–Stokes velocity necessary. This is accomplished through utilising the Navier–Stokes equations (28), and the ε-expansions of the variables possible through Hypothesis 3.1. The following definitions prove useful in the statement of our main theorem. The *viscous Melnikov function* M_{v}, the *forcing Melnikov function* M_{f} and the *perturbative Melnikov function* M_{p} are defined by

$$M_{\mathrm{v}}(\alpha) := \int_{-\infty}^{\infty} v_0 \cdot \nabla^2 v_0(x_\alpha(t))\,\mathrm{d}t, \tag{31}$$

$$M_{\mathrm{f}}(\alpha,\tau) := \int_{-\infty}^{\infty} v_0 \cdot \phi(x_\alpha(t), t+\tau)\,\mathrm{d}t, \tag{32}$$

$$\begin{aligned} M_{\mathrm{p}}(\alpha,\tau,\varepsilon) :=& \int_{-(2/\theta)|\ln\varepsilon|}^{0} \left[\frac{\partial H_1}{\partial t}(x_\alpha(t), t+\tau;\varepsilon) - \frac{\partial H_1}{\partial t}(a, t+\tau;\varepsilon)\right]\mathrm{d}t \\ &+ \int_{0}^{(2/\theta)|\ln\varepsilon|} \left[\frac{\partial H_1}{\partial t}(x_\alpha(t), t+\tau;\varepsilon) - \frac{\partial H_1}{\partial t}(b, t+\tau;\varepsilon)\right]\mathrm{d}t + H_1(a,\tau;\varepsilon) - H_1(b,\tau;\varepsilon). \end{aligned} \tag{33}$$

The exponential decay of $x_\alpha(t)$ to a (respectively b) as $t \to -\infty$ (respectively $t \to \infty$), along with the boundedness assumption on ϕ, ensures that the indefinite integrals appearing above are finite. We are now in a position to present our main theorem concerning the Melnikov function related to the Navier–Stokes perturbation of the Euler equations.

Theorem 3.1. *The distance function d whose zeroes indicate intersections between perturbed manifolds can be expressed as*

$$d(\alpha,\tau,\varepsilon) = \varepsilon M(\alpha,\tau,\varepsilon) + \mathcal{O}(\varepsilon^2|\ln\varepsilon|)$$

for $(\alpha,\tau,\varepsilon) \in S^1 \times [-\varepsilon^{-\kappa}, \varepsilon^{-\kappa}] \times I$, where the Melnikov function exhibits the splitting

$$M(\alpha,\tau,\varepsilon) = M_{\mathrm{v}}(\alpha) + M_{\mathrm{f}}(\alpha,\tau) + M_{\mathrm{p}}(\alpha,\tau,\varepsilon). \tag{34}$$

The proof of Theorem 3.1 shall be postponed to Section 3.4, since it is tedious and not especially illuminating. As the result is quite general and involved, its immediate applicability depends on additional conditions that one may impose—some possible simplifications are listed below. The specific application of Theorem 3.1 to transport in the bubble-vortex problem is discussed in detail in another paper currently under preparation.

(a) The only information from the Navier–Stokes (perturbed) flow appears in M_{p}; all other quantities derive from the (unperturbed) Euler flow, and the added forcing function, which is assumed known. All ε dependence is also confined to M_{p}.

(b) If the perturbing velocity (and consequently, pressure) is steady, then $\partial H_1/\partial t = 0$. This would cause considerable simplifications in the perturbative Melnikov function, which can then be written as

$$M_{\mathrm{p}}(\varepsilon) = H_1(a;\varepsilon) - H_1(b;\varepsilon).$$

(c) If viscosity is disregarded in deference to forcing, then we can simply set $M_{\mathrm{v}} \equiv 0$ to analyse the contribution of pure forcing. Notice also that if ϕ is perpendicular to the Eulerian velocity field v_0, this forcing term does not contribute to M_{f}.

(d) The τ dependence in M_{f} will disappear from this term should ϕ have no explicit time dependence (i.e., if the forcing is steady). Thus, if the Navier–Stokes velocity *and* the forcing are steady, $M = M(\alpha,\varepsilon)$ alone.

(e) If the density ρ is constant, then the perturbative Melnikov function M_{p} can in fact be expressed purely in terms of the pressure field (an additional motivation for using the notation M_{p} for this quantity). This is achieved by writing $H_1(x,t) = (1/\rho)p_1(x,t)$, and hence

$$\rho M_{\mathrm{p}}(\alpha,\tau,\varepsilon) = \int_{-(2/\theta)|\ln\varepsilon|}^{0}\left[\frac{\partial p_1}{\partial t}(x_\alpha(t),t+\tau;\varepsilon) - \frac{\partial p_1}{\partial t}(a,t+\tau;\varepsilon)\right]\mathrm{d}t + \int_{0}^{(2/\theta)|\ln\varepsilon|}\left[\frac{\partial p_1}{\partial t}(x_\alpha(t),t+\tau;\varepsilon) - \frac{\partial p_1}{\partial t}(b,t+\tau;\varepsilon)\right]\mathrm{d}t + p_1(a,\tau;\varepsilon) - p_1(b,\tau;\varepsilon).$$

(f) Notice that $v_0(x_\alpha(t))\,\mathrm{d}t$ is dotted with either $\nabla^2 v_0$ or ϕ in M_{v} and M_{f}. This quantity is exactly the line element $\mathrm{d}l$ along the heteroclinic orbit x_α. Thus (31) and (32) can be thought of in terms of line integrals along $x_\alpha(t)$, of the vector fields $\nabla^2 v_0$ and ϕ, respectively.

(g) If the forcing ϕ is both steady and conservative, then its contribution to M_{f} is constant throughout Γ. (If $\phi = -\nabla\psi$, the integrand has the form $-\nabla\psi\cdot\mathrm{d}l = -\mathrm{d}\phi$, and hence $M_{\mathrm{f}}(\alpha,\tau) = \psi(a) - \psi(b)$.)

(h) By taking the divergence of the Euler momentum equation (26) and using the incompressibility condition on v_0, it is possible to show that

$$v_0\cdot\nabla^2 v_0 = \nabla^2 B_0 - |\omega_0|^2.$$

This may be substituted into (31) to get an alternative expression for M_{v}, in terms of the Bernoulli function and vorticity of the Euler flow.

3.4. Proof of Theorem 3.1

The details of the proof of the expression for the Melnikov function in (34) are given in this section. We begin by taking the dot product of (28) with v, and then substituting the expansions in ε guaranteed by Hypothesis 3.1 to obtain the expression

$$(v_0+\varepsilon v_1)\cdot\nabla(B_0+\varepsilon B_1) = -(v_0+\varepsilon v_1)\cdot\frac{\partial(v_0+\varepsilon v_1)}{\partial t} + \varepsilon(v_0+\varepsilon v_1)\cdot\nabla^2(v_0+\varepsilon v_1) + \varepsilon(v_0+\varepsilon v_1)\cdot\phi.$$

Grouping terms in orders of ε

$$[v_0\cdot\nabla B_0] + \varepsilon\left[v_0\cdot\nabla B_1 + v_1\cdot\nabla B_0 + v_0\cdot\frac{\partial v_1}{\partial t} - v_0\cdot\nabla^2 v_0 - v_0\cdot\phi\right] = \varepsilon^2\left[-v_1\cdot\nabla B_1 - v_1\cdot\frac{\partial v_1}{\partial t} + v_0\cdot\nabla^2 v_1 + v_1\cdot\nabla^2 v_0 + v_1\cdot\phi\right] + \varepsilon^3[v_1\cdot\nabla^2 v_1].$$

The first term in square brackets disappears by virtue of (26), and dividing the remainder through by ε and rearranging

$$v_1\cdot\nabla B_0 = \left[-v_0\cdot\nabla B_1 - v_0\cdot\frac{\partial v_1}{\partial t} + v_0\cdot\nabla^2 v_0 + v_0\cdot\phi\right] + \varepsilon^2[v_1\cdot\nabla^2 v_1] + \varepsilon\left[v_0\cdot\nabla^2 v_1 + v_1\cdot\nabla^2 v_0 - v_1\cdot\nabla B_1 - v_1\cdot\frac{\partial v_1}{\partial t} + v_1\cdot\phi\right]. \tag{35}$$

Now, evaluate (35) on a heteroclinic trajectory $(x_\alpha(t), t+\tau)$, and integrate from $t = -(2/\theta)|\ln\varepsilon|$ to $(2/\theta)|\ln\varepsilon|$. The left-hand side of the above is then precisely the Melnikov function of (30). We will keep the first term in square brackets in (35) above, and will now show that the next two terms go to 0 as $\varepsilon\to 0$. We have

$$\varepsilon^2\int_{-(2/\theta)|\ln\varepsilon|}^{(2/\theta)|\ln\varepsilon|} v_1\cdot\nabla^2 v_1(x_\alpha(t),t+\tau;\varepsilon)\,\mathrm{d}t \le \varepsilon^2\int_{-(2/\theta)|\ln\varepsilon|}^{(2/\theta)|\ln\varepsilon|} C_1C_1\,\mathrm{d}t \le C_1^2\varepsilon^2\frac{4}{\theta}|\ln\varepsilon| = \frac{4C_1^2}{\theta}\varepsilon^2\ln\frac{1}{\varepsilon}$$

which goes to 0 as $\varepsilon \to 0$, and is thus of higher order. Similarly, the term in square brackets multiplied by ε in (35) has the same behaviour, since the ε outside is sufficient to combat the $\ln(1/\varepsilon)$ contribution from integrating v_1. Thus, the second and third terms in square brackets in (35) contribute at most $\mathcal{O}(\varepsilon \ln(1/\varepsilon)) = \mathcal{O}(\varepsilon|\ln \varepsilon|)$, leaving us with

$$M(\alpha,\tau,\varepsilon) = \int_{-(2/\theta)|\ln \varepsilon|}^{(2/\theta)|\ln \varepsilon|} v_0 \cdot \nabla^2 v_0(x_\alpha(t))\,\mathrm{d}t + \int_{-(2/\theta)|\ln \varepsilon|}^{(2/\theta)|\ln \varepsilon|} v_0 \cdot \phi(x_\alpha(t), t+\tau)\,\mathrm{d}t + \int_{-(2/\theta)|\ln \varepsilon|}^{(2/\theta)|\ln \varepsilon|} \left(-v_0 \cdot \nabla B_1 - v_0 \cdot \frac{\partial v_1}{\partial t}\right)(x_\alpha(t), t+\tau, \varepsilon)\,\mathrm{d}t + \mathcal{O}(\varepsilon|\ln \varepsilon|). \tag{36}$$

Let us now analyse the last integral in (36). Notice that if the spatial argument of B_1 is a trajectory of the Euler flow, then for any τ

$$\frac{\mathrm{d}B_1}{\mathrm{d}t}(x(t), t+\tau, \varepsilon) = \frac{\partial B_1}{\partial t} + \nabla B_1 \cdot \dot{x} = \frac{\partial B_1}{\partial t} + v_0 \cdot \nabla B_1, \tag{37}$$

since v_0 is the velocity of the trajectory. However,

$$B_1 = \frac{B - B_0}{\varepsilon} = \frac{1}{\varepsilon}\left[H - H_0 + \frac{1}{2}(|v|^2 - |v_0|^2)\right] = H_1 + v_0 \cdot v_1 + \frac{\varepsilon}{2} v_1 \cdot v_1, \tag{38}$$

where $\nabla H_1 = (1/\rho)\nabla p_1$. Thus

$$\frac{\partial B_1}{\partial t} = \frac{\partial H_1}{\partial t} + v_0 \cdot \frac{\partial v_1}{\partial t} + \frac{\varepsilon}{2}\frac{\partial}{\partial t}(v_1 \cdot v_1).$$

Substituting in (37),

$$\frac{\mathrm{d}B_1}{\mathrm{d}t} = \frac{\partial H_1}{\partial t} + v_0 \cdot \frac{\partial v_1}{\partial t} + v_0 \cdot \nabla B_1 + \frac{\varepsilon}{2}\frac{\partial}{\partial t}(v_1 \cdot v_1)$$

hence

$$-v_0 \cdot \nabla B_1 - v_0 \cdot \frac{\partial v_1}{\partial t} = -\frac{\mathrm{d}B_1}{\mathrm{d}t} + \frac{\partial H_1}{\partial t} + \frac{\varepsilon}{2}\frac{\partial}{\partial t}(v_1 \cdot v_1)$$

along orbits of the Euler flow. Thus, the last integral in (36) can be rewritten as

$$\int_{-(2/\theta)|\ln \varepsilon|}^{(2/\theta)|\ln \varepsilon|} \left[-\frac{\mathrm{d}B_1}{\mathrm{d}t}(x_\alpha(t), t+\tau; \varepsilon) + \frac{\partial H_1}{\mathrm{d}t}(x_\alpha(t), t+\tau; \varepsilon)\right]\mathrm{d}t + \frac{\varepsilon}{2}\int_{-(2/\theta)|\ln \varepsilon|}^{(2/\theta)|\ln \varepsilon|} \frac{\partial}{\partial t}[v_1 \cdot v_1](x_\alpha(t), t+\tau; \varepsilon)\,\mathrm{d}t.$$

Since the last term above is $\mathcal{O}(\varepsilon|\ln \varepsilon|)$, it can be absorbed into the error term in (36). The remaining integral shall be evaluated in two steps: from $t = -(2/\theta)|\ln \varepsilon|$ to 0, and from $t = 0$ to $(2/\theta)|\ln \varepsilon|$. The first of these can be written in the form

$$\int_{-(2/\theta)|\ln \varepsilon|}^{0} \left[-\frac{\mathrm{d}B_1}{\mathrm{d}t}(x_\alpha(t), t+\tau; \varepsilon) + \frac{\mathrm{d}B_1}{\mathrm{d}t}(a, t+\tau; \varepsilon)\right]\mathrm{d}t + \int_{-(2/\theta)|\ln \varepsilon|}^{0} \left[\frac{\partial H_1}{\mathrm{d}t}(x_\alpha(t), t+\tau; \varepsilon) - \frac{\partial H_1}{\partial t}(a, t+\tau; \varepsilon)\right]\mathrm{d}t,$$

since by Eq. (38), $\mathrm{d}B_1/\mathrm{d}t$ and $\partial H_1/\partial t$ are equal (with error of order ε) on a. The error resulting from this process can be absorbed into the $\mathcal{O}(\varepsilon|\ln \varepsilon|)$ error in the main equation (36) as usual. Now, the first integral above is of a total derivative, and hence

$$\begin{aligned}
&\int_{-(2/\theta)|\ln\varepsilon|}^{0}\left[-\frac{\mathrm{d}B_1}{\mathrm{d}t}(x_\alpha(t),t+\tau;\varepsilon)+\frac{\mathrm{d}B_1}{\mathrm{d}t}(a,t+\tau;\varepsilon)\right]\mathrm{d}t\\
&\quad=-B_1(x_\alpha(0),\tau)+B_1(a,\tau)+B_1\left(x_\alpha\left(-\frac{2}{\theta}|\ln\varepsilon|\right),-\frac{2}{\theta}|\ln\varepsilon|+\tau\right)-B_1\left(a,-\frac{2}{\theta}|\ln\varepsilon|+\tau\right)\\
&\quad=-B_1(x_\alpha(0),\tau)+B_1(a,\tau)+\mathcal{O}\left(\exp\left[\theta\left(-\frac{2}{\theta}|\ln\varepsilon|\right)\right]\right)\\
&\quad=-B_1(x_\alpha(0),\tau)+B_1(a,\tau)+\mathcal{O}(\varepsilon^2).
\end{aligned}$$

The second equality above is since $x_\alpha(t)$ converges to a in the form $\mathrm{e}^{\theta t}$ for $t\to-\infty$ (see (5)), and B_1 is assumed smooth in the spatial argument. Thus,

$$\begin{aligned}
&\int_{-(2/\theta)|\ln\varepsilon|}^{0}\left[-\frac{\mathrm{d}B_1}{\mathrm{d}t}(x_\alpha(t),t+\tau;\varepsilon)+\frac{\partial H_1}{\mathrm{d}t}(x_\alpha(t),t+\tau;\varepsilon)\right]\mathrm{d}t\\
&\quad=\int_{-(2/\theta)|\ln\varepsilon|}^{0}\left[\frac{\partial H_1}{\mathrm{d}t}(x_\alpha(t),t+\tau;\varepsilon)-\frac{\partial H_1}{\partial t}(a,t+\tau;\varepsilon)\right]\mathrm{d}t-B_1(x_\alpha(0),\tau)+B_1(a,\tau)+\mathcal{O}(\varepsilon^2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{0}^{(2/\theta)|\ln\varepsilon|}\left[-\frac{\mathrm{d}B_1}{\mathrm{d}t}(x_\alpha(t),t+\tau;\varepsilon)+\frac{\partial H_1}{\mathrm{d}t}(x_\alpha(t),t+\tau;\varepsilon)\right]\mathrm{d}t\\
&\quad=\int_{0}^{(2/\theta)|\ln\varepsilon|}\left[\frac{\partial H_1}{\mathrm{d}t}(x_\alpha(t),t+\tau;\varepsilon)-\frac{\partial H_1}{\partial t}(b,t+\tau;\varepsilon)\right]\mathrm{d}t+B_1(x_\alpha(0),\tau)-B_1(b,\tau)+\mathcal{O}(\varepsilon^2).
\end{aligned}$$

Adding these together,

$$\begin{aligned}
&\int_{-(2/\theta)|\ln\varepsilon|}^{(2/\theta)|\ln\varepsilon|}\left[-\frac{\mathrm{d}B_1}{\mathrm{d}t}(x_\alpha(t),t+\tau;\varepsilon)+\frac{\partial H_1}{\mathrm{d}t}(x_\alpha(t),t+\tau;\varepsilon)\right]\mathrm{d}t\\
&\quad=\int_{-(2/\theta)|\ln\varepsilon|}^{0}\left[\frac{\partial H_1}{\mathrm{d}t}(x_\alpha(t),t+\tau;\varepsilon)-\frac{\partial H_1}{\partial t}(a,t+\tau;\varepsilon)\right]\mathrm{d}t\\
&\qquad+\int_{0}^{(2/\theta)|\ln\varepsilon|}\left[\frac{\partial H_1}{\mathrm{d}t}(x_\alpha(t),t+\tau;\varepsilon)-\frac{\partial H_1}{\partial t}(b,t+\tau;\varepsilon)\right]\mathrm{d}t+B_1(a,\tau;\varepsilon)-B_1(b,\tau;\varepsilon)+\mathcal{O}(\varepsilon^2).
\end{aligned}$$

Now, notice from (38) that $B_1(a,\tau;\varepsilon)=H_1(a,\tau;\varepsilon)+\mathcal{O}(\varepsilon)$, and hence one can replace $B_1(a,\tau;\varepsilon)$ and $B_1(b,\tau;\varepsilon)$ above with $H_1(a,\tau;\varepsilon)$ and $H_1(b,\tau;\varepsilon)$, respectively. Hence, what we have above is what is defined as the perturbative Melnikov function M_p in (33). Recalling that this messy expression was obtained by analysing the final integral in (36), we can then write (36) as

$$M(\alpha,\tau,\varepsilon)=\int_{-(2/\theta)|\ln\varepsilon|}^{(2/\theta)|\ln\varepsilon|}v_0\cdot\nabla^2 v_0(x_\alpha(t))\,\mathrm{d}t+\int_{-(2/\theta)|\ln\varepsilon|}^{(2/\theta)|\ln\varepsilon|}v_0\cdot\phi(x_\alpha(t),t+\tau)\,\mathrm{d}t+M_\mathrm{p}(\alpha,\tau,\varepsilon)+\mathcal{O}(\varepsilon|\ln\varepsilon|).$$

Our next step is to replace the finite integrals remaining in the above expression with integrals from $-\infty$ to ∞. The error corresponding to this is $\mathcal{O}(\varepsilon^2)$, as is illustrated by the following (typical) calculation, which uses the fact that v_0 decays to zero exponentially as x_α approaches b.

$$\int_{(2/\theta)|\ln\varepsilon|}^{\infty}v_0\cdot\nabla^2 v_0(x_\alpha(t))\,\mathrm{d}t\le K\int_{(2/\theta)|\ln\varepsilon|}^{\infty}\mathrm{e}^{-\theta t}\,\mathrm{d}t=\left[\frac{K}{-\theta}\,\mathrm{e}^{-\theta t}\right]_{(2/\theta)|\ln\varepsilon|}^{\infty}=\frac{K}{\theta}\exp\left[-\theta\left(\frac{2}{\theta}\ln\frac{1}{\varepsilon}\right)\right]=\frac{K\varepsilon^2}{\theta}.$$

All other error terms similarly are of order ε^2. Hence, recalling the definitions of the viscous and the forcing Melnikov functions as given in (31) and (33), we have

$$M(\alpha,\tau,\varepsilon) = M_{\rm v}(\alpha) + M_{\rm f}(\alpha,\tau) + M_{\rm p}(\alpha,\tau,\varepsilon) + \mathcal{O}(\varepsilon|\ln\,\varepsilon|).$$

Absorbing the error term into higher order terms in d, we have proven Theorem 3.1.

4. Conclusions

In this paper, we have developed a Melnikov theory for three-dimensional volume-preserving vector fields, subject to a weak, finite-time perturbation. As an example, explicit expressions for the Melnikov function under generic perturbations to Hill's spherical vortex were obtained. Moreover, the theory was applied to viscous perturbations of steady integrable Euler flows, and explicit expressions for the size of the separatrix splitting due to viscosity and external body forces were obtained. These results are a contribution towards understanding the role of viscosity in mixing and transport in laminar flows [6,23,30,31,45]. The mathematical ideas presented here should be applicable to a variety of other physical contexts such as in magnetohydrodynamics. In a paper currently under preparation, we will use the results derived here to discuss possible viscous transport mechanisms in the well-known bubble-vortex problem of fluid mechanics.

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