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THE UNIVERSITY OF SYDNEY
FACULTIES OF ARTS, ECONOMICS, EDUCATION,
ENGINEERING AND SCIENCE

MATH2005: FOURIER SERIES AND DIFFERENTIAL EQUATIONS

November 2004 | Time allowed: Two hours

Lecturers: S. Balasuriya
W. Gibson.

This paper consists of two parts.

PART M 40 marks	There are 20 MULTIPLE CHOICE QUESTIONS each worth 2 marks. Answers should be marked on the SEPARATE ANSWER SHEET provided. Rough working may be done on the even numbered pages (pages 2, ...14) left blank in this booklet. This working will not be marked.
PART E 60 marks	There are 3 EXTENDED ANSWER QUESTIONS, each worth 20 marks. Answers should be written in the supplied answer booklets. Answers with insufficient reasoning may receive no marks.

Non programmable University supplied calculators may be used.
One A4 sheet (written on both sides) of handwritten notes may be used.
The Table of Laplace Transforms is reproduced in this booklet on page 16.

Please check that your booklet is complete, and then sign the declaration below.
Students finding an incomplete booklet should obtain a replacement from
the Examination Supervisor immediately.

SIGNATURE: ....

THIS EXAMINATION PAPER IS NOT TO BE TAKEN FROM THE EXAMINATION ROOM.

MC	
Q1	
Q2	
Q3	

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PART M MULTIPLE CHOICE Q1 - Q20

1. If $f(x)$ is an odd, 2π-periodic function (not identically zero), then:

(a) $\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 \quad$ for $n = 0, 1, 2, \cdots$

(b) $\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 \quad$ for $n = 1, 2, \cdots$

(c) $\int_{-\pi}^{\pi} f(x)\cos nx \sin nx\,dx = 0 \quad$ for $n = 1, 2, \cdots$

(d) $\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 \quad$ for $n = 0$, but not for $n = 1, 2, \cdots$

(e) $\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 \quad$ for n even

2. The function $f(x)$ defined by

$$f(x) = \frac{x}{2}, \quad -\pi < x < \pi, \qquad\qquad f(x + 2\pi) = f(x), \quad \text{for all } x$$

is expanded in a Fourier series. At the point $x = \pi$ this series converges to:

(a) π

(b) $\frac{\pi}{2}$

(c) 0

(d) $-\frac{\pi}{2}$

(e) π .

3. A periodic function $f(x)$ has a Fourier series $g(x)$. Which of the following is always a correct statement?

(a) $f(x) = g(x)$ for all x

(b) $f(x)$ must be a continuous function

(c) $g(x)$ must be continuous function

(d) $f(x) = g(x)$ at all points at which f is discontinuous

(e) $f(x) = g(x)$ at all points at which f is continuous

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4. The general solution of the second-order differential equation

$$y''(x) - 2\,y'(x) + 5\,y(x) = 0$$

is (where A and B are arbitrary constants):

(a) $A\,e^{x} + B\,e^{2x}$
(b) $A\,\cos x + B\,\sin x$
(c) $A\,e^{2x} + B\,x\,e^{2x}$
(d) $A\,e^{x}\,\cos 2x + B\,e^{x}\,\sin 2x$
(e) $A\,e^{-x} + B\cos x$

5. A particular solution of the second order, non-homogeneous differential equation

$$y'' - y = 2\,e^{-x}$$

is:

(a) $2e^{-x}$
(b) 2
(c) xe^{-x}
(d) e^{x}
(e) $-xe^{-x}$.

6. The solution $y(x)$ of the differential equation

$$(x^2+1)\,y' + 2\,x\,y = 1$$

which satisfies the initial condition $y(0) = 1$ is:

(a) $\dfrac{x^2+x+1}{x^2+1}$
(b) $\dfrac{x+1}{x^2+1}$
(c) $\dfrac{e^{2x}}{x^2+1}$
(d) $\dfrac{e^{-2x}}{x^2+1}$
(e) $\dfrac{x}{x^2+1}$.

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7. The function $u(x,y) = \phi(x+4y)$, where ϕ is an arbitrary differentiable function, satisfies the partial differential equation:

(a) $u_x + 4u_y = 0$
(b) $4u_x - u_y = 0$
(c) $4u_x + u_y = 0$
(d) $u_x^4 - u_y = 0$
(e) $u_x\,(4+x) = u_y\,(1+4y)$.

8. The electrostatic potential $u(x,y)$ in the square $0 < x < L$, $0 < y < L$ is given by Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with three sides of the square $x = 0$, $x = L$ and $y = 0$ held at zero potential (i.e, $u(0,y) = u(L,y) = 0$ for all $y \in [0,L]$ and $u(x,0) = 0$ for all $x \in [0.L]$). The general solution satisfying these conditions takes the form (where the A_ns and B_ns are constants)

(a) $\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \sinh\frac{n\pi y}{L}$
(b) $\sum_{n=1}^{\infty} B_n \sin\frac{2n\pi x}{L}\, e^{\frac{n\pi y}{L}}$
(c) $\sum_{n=1}^{\infty} \left(A_n \sin\frac{n\pi x}{L} + B_n \cos\frac{n\pi x}{L}\right) \sin\frac{n\pi y}{L}$
(d) $\sum_{n=1}^{\infty} B_n \cosh\frac{n\pi x}{L} \cos\frac{n\pi y}{L}$
(e) $\sum_{n=1}^{\infty} \cos\frac{n\pi x}{L} \left(A_n e^{\frac{n\pi y}{L}} + B_n e^{-\frac{n\pi y}{L}}\right)$

9. The partial differential equation

$$\frac{\partial^2 u}{\partial x \partial y} = x$$

has solution $u(x,y)$ (where $f(x)$ and $g(y)$ are arbitrary function of x and y, respectively)

(a) $\dfrac{x^2 y}{2} + y f(x) + x g(y)$
(b) $xy + x f(x) + y g(y)$
(c) $xy + f(x) + g(y)$
(d) $\dfrac{x^2 y}{2} + f(x) + g(y)$
(e) $xy + y f(x) + x g(y)$.

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10. A real function $f(x)$ with period T has complex Fourier series $\displaystyle\sum_{n=-\infty}^{\infty} c_n\, e^{in2\pi x/T}$.
If $f(x)$ is an odd function, the coefficients c_n for $n = 1, 2, \cdots$ satisfy:

(a) $c_n = -c_{-n}$
(b) $c_n = c_{-n}$
(c) $c_n = 0$
(d) $c_{-n} = 0$
(e) $c_n = 0$ for n even.

11. A piecewise continuous function $g(x)$ with period 2π has a Fourier series in which $a_n = 0$ for all n, and $\displaystyle\sum_{n=1}^{\infty} b_n^2 = 2$. Then, the integral $\displaystyle\int_{-\pi}^{\pi} [g(x)]^2\, dx$ takes the value

(a) π
(b) 2π
(c) 3π
(d) 4π
(e) cannot be determined from the given information

12. The 2π-periodic function $f(x)$ is defined in its base period $[-\pi, \pi]$ by $f(x) = |x|$. Its Fourier series is given by $\displaystyle\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos[(2n-1)\,x]$.
The square error $E^* = \int_{-\pi}^{\pi} |f(x) - F_n(x)|^2\, \mathrm{d}x$ in using the Nth partial sum

$$F_N(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{N} \frac{1}{(2n-1)^2} \cos[(2n-1)\,x]$$

to approximate this function is given by

(a) $\frac{\pi}{2} - \left[\frac{1}{1^2} + \frac{1}{3^2} + \cdots + \frac{1}{(2N-1)^2}\right]$
(b) $\frac{16}{\pi^2} \sum_{n=1}^{N} \frac{1}{(2n-1)^2} \cos[(2n-1)x]$
(c) $\frac{2\pi^3}{3} - \frac{16}{\pi^2} \sum_{n=1}^{N} \frac{1}{(2n-1)^2}$
(d) $\frac{\pi^3}{6} - 16\pi \left[\frac{1}{1^4} + \frac{1}{3^4} + \cdots + \frac{1}{(2N-1)^4}\right]$
(e) $\sum_{n=1}^{N} \frac{1}{(2n-1)^2}$

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13. The function

$$y = A\cos qx + B\sin qx,$$

where A, B and q are arbitrary constants, is required to satisfy the boundary conditions

$$y(0) = 0, \quad y(\ell) = 0.$$

The appropriate choice of constants is $(n = \pm 1, \pm 2, \dots)$

(a) $A = 0$ and $q = n\pi/\ell$

(b) $B = 0$ and $q = n\pi/\ell$

(c) $A = B$ and $q = n\pi/\ell$

(d) $A = 0$ and $q = n\pi\ell$

(e) $B = 0$ and $q = n\pi\ell$.

14. The solution to

$$y''(t) - 3\,y'(t) + 2\,y(t) = 10\,\sin t$$

which satisfies the conditions $y(0) = y'(0) = 0$ is

(a) $e^{-2t} + e^{-t} + 8\sin t$

(b) $8\sin t - 8e^t + 8$

(c) $3\cos t + \sin t$

(d) $2e^{2t} - 5e^t + \sin t + 3\cos t$

(e) $e^{2t} - e^t - \sin t$

15. For the system described by the differential equations

$$\begin{aligned} \frac{dy_1}{dt} &= 2\,y_1 + y_2 \\ \frac{dy_2}{dt} &= -y_1 + 2\,y_2 \end{aligned}$$

the critical point $(y_1, y_2) = (0, 0)$ is

(a) an unstable node

(b) a stable node

(c) a centre

(d) a stable spiral

(e) an unstable spiral

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16. The solution $y(x)$ of the differential equation

$$y'' - 2\,y' + 2\,y = 0$$

satisfying $y(0) = 1$ and $y'(0) = 0$ is

(a) $e^{-x}(\cos x - \sin x)$

(b) $e^{x}(\cos x - \sin x)$

(c) $e^{-x}(\cos x + \sin x)$

(d) $e^{x}(\cos x + \sin x)$

(e) $e^{x}\cos x$

17. In the system

$$\begin{aligned} x_1' &= b\,x_1 + x_2 \\ x_2' &= -x_1 - x_2 \end{aligned}$$

where b is a constant, the critical point $(x_1, x_2) = (0, 0)$ is an stable spiral if

(a) $b < 0$

(b) $b < -1$

(c) $b = 2$

(d) $-3 < b < 1$

(e) $-2 < b < 2$

18. The inverse Laplace transform of the function $\dfrac{1}{(s-2)^3}$ is (u is the unit step function)

(a) $e^{3t}t$

(b) t^2e^{-2t}

(c) $u(t-2)e^{2t}$

(d) $u(t-2)te^{-2t}$

(e) $\frac{1}{2}t^2\,e^{2t}$

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19. The Laplace transform of the square wave defined by

$$f(t) = \begin{cases} 0 & 0 < t < 1 \\ 1 & 1 < t < 2 \\ 0 & t > 2 \end{cases}$$

is:

(a) $\frac{1}{s}\left(e^{-s} - e^{-2s}\right)$
(b) $e^{-s} - e^{-2s}$
(c) $e^{-2s} - e^{-s}$
(d) $\frac{1}{s}\left(e^{-s} + e^{-2s}\right)$
(e) $\frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2}$.

20. The general solution of the system of differential equations

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

is (here, A and B are arbitrary constants):

(a) $y_1 = Ae^x + Be^{3x}, y_2 = Ae^x - Be^{3x}$
(b) $y_1 = Ae^x + 3Be^{3x}, y_2 = 3Ae^{2x} + Be^x$
(c) $y_1 = Ae^x, y_2 = Be^{3x}$
(d) $y_1 = Ae^{-x} + Be^x, y_2 = Ae^{-3x} + Be^{3x}$
(e) $y_1 = -Ae^{-x}, y_2 = 3Ae^{3x}$

THIS IS THE END OF PART M

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Table of Laplace Transforms

Function $f(t)$	Laplace Transform $F(s) = \int_0^\infty e^{-st} f(t)dt$
e^{at}	$\frac{1}{s-a}$ $(s > a)$
1	$\frac{1}{s}$ $(s > 0)$
$\cosh at$	$\frac{s}{s^2-a^2}$ $(s > \|a\|)$
$\sinh at$	$\frac{a}{s^2-a^2}$ $(s > \|a\|)$
$\cos at$	$\frac{s}{s^2+a^2}$ $(s > 0)$
$\sin at$	$\frac{a}{s^2+a^2}$ $(s > 0)$
t	$\frac{1}{s^2}$ $(s > 0)$
$t^n, \quad n \geq 0$	$\frac{n!}{s^{n+1}}$ $(s > 0)$
$u(t-a)$ (unit step fn)	$\frac{e^{-as}}{s}$ $(s > 0)$
$af(t) + bg(t)$	$aF(s) + bG(s)$ (linearity)
$e^{at}f(t)$	$F(s-a)$ (s-shifting)
$u(t-a)f(t-a)$	$e^{-as}F(s)$ (t-shifting)
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$ (multiplication by t^n)
$\frac{\partial f(a,t)}{\partial a}$	$\frac{\partial F(a,s)}{\partial a}$ (differentiation wrt parameter a)
$f'(t)$	$sF(s) - f(0)$ (first derivative wrt t)
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$ (second derivative wrt t)

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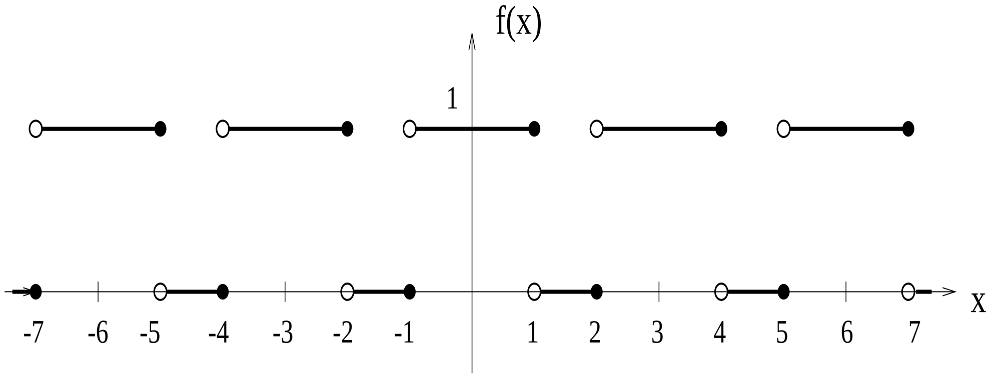
PART E EXTENDED ANSWER Q1, Q2, Q3
Q1 (page 17); Q2 (page 18); Q3 (page 19)

1. (a) Determine the Fourier series of the periodic function $f(x)$ pictured below. Note that the function has value 1 in segments of length 2, and value 0 in segments of length 1.



(b) Let $g(x) = \cosh x$ for $-\pi < x < \pi$, and $g(x + 2\pi) = g(x)$ for all x. You are given that the Fourier series of $g(x)$ is

$$\frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} \cos nx \,.$$

(You may recall that $\cosh \theta = \left(e^{\theta} + e^{-\theta}\right)/2$ and $\sinh \theta = \left(e^{\theta} - e^{-\theta}\right)/2$.)

(i) By applying Parseval's identity to $g(x)$, determine the value of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(1+n^2)^2}$.

(ii) Compute an expression for the square error resulting from $g(x)$ being approximated by the first three non-zero terms in its Fourier series. (You do not need to use your calculator to determine its exact numerical value.)

(iii) What is the complex Fourier series of $g(x)$?

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2. (a) [7 marks] *Using Laplace transforms*, solve the initial value problem

$$y'(t) - 4\,y(t) = t - 7\,u(t-1) \quad ; \quad y(0) = 1\,,$$

where u is the unit step function.

(b) Consider the ordinary differential equation

$$x\,z''(x) - (1+x)\,z'(x) + z(x) = x^2\,e^{2\,x} \tag{1}$$

for the unknown function $z(x)$.

(i) [5 marks] Write down the homogeneous equation corresponding to (1). Given that $z_1(x) = e^x$ is a solution to this homogeneous equation, use the method of reduction of order to show that $z_2(x) = (1+x)$ is another solution.

(ii) [6 marks] Use variation of parameters to obtain the fact that a particular solution to (1) is $z_p(x) = \frac{1}{2}\,(x-1)\,e^{2x}$.

(iii) [2 marks] Determine the solution to (1) which satisfies $z(1) = 0$ and $z'(1) = 0$.

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3. This question relates to solving the modified wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{L^2}\, u = \frac{1}{c^2}\,\frac{\partial^2 u}{\partial t^2} \tag{2}$$

for the unknown $u(x,t)$ within the domain $0 < x < L$ for positive times $t > 0$. Here, L and c are known positive constants. This equation is subject to the boundary and initial conditions

$$\begin{aligned} u(0,t) &= 0 \quad \text{for all} \;\; t \geq 0\,, \\ u(L,t) &= 0 \quad \text{for all} \;\; t \geq 0\,, \\ u(x,0) &= 0 \quad \text{for all} \;\; x \in [0,L]\,, \\ \frac{\partial u}{\partial t}(x,0) &= f(x) \quad \text{for all} \;\; x \in [0,L]\,, \end{aligned}$$

where $f(x)$ is a given function.

(a) [5 marks] Postulating a seperable solution of the form $u(x,t) = X(x)T(t)$, and considering all homogeneous (i.e., zero) conditions, obtain the equations

$$X''(x) - \left[\lambda - \frac{1}{L^2}\right] X(x) = 0 \;\; ; \;\; X(0) = X(L) = 0\,, \tag{3}$$

$$T''(t) - \lambda\, c^2\, T(t) = 0 \;\; ; \;\; T(0) = 0, \tag{4}$$

where λ is a separation constant.

(b) [6 marks] Considering the equation (3) for $X(x)$, obtain the fact that it has non-zero solutions only if

$$\lambda = \lambda_n = \frac{1 - n^2\,\pi^2}{L^2} \quad ; \quad n = 1, 2, 3, \ldots\,,$$

and that the corresponding (fundamental) solutions are

$$X_n(x) = \sin\left(\frac{n\,\pi}{L} x\right) \quad ; \quad n = 1, 2, 3, \ldots\,.$$

(c) [3 marks] For choices of λ as given in (b), solve equation (4) to obtain the (fundamental) solutions

$$T_n(t) = \sin\left(\frac{c\sqrt{n^2\pi^2 - 1}}{L}\, t\right) \quad ; \quad n = 1, 2, 3, \ldots$$

(d) [2 marks] Using the above parts, write down the form of the solution $u(x,t)$ which satisfies the partial differential equation (2) along with the given zero boundary and initial conditions. Your answer will depend on some number of yet to be determined constants.

(e) [4 marks] Finally, utilise the non-zero initial condition to find the constants in (d) in terms of the function $f(x)$, thereby determining the complete solution to the given initial-boundary value problem.