Successive Over Relaxation Method

Any second order Elliptic Equation e.g. Laplace Equation, while solving by FD method, always reduces to a equation containing $u_{i,j}$, $u_{i-1,j}$, $u_{i+1,j}$ and $u_{i,j+1}$ which for different values of (i, j) may be converted to a system of equations. Rewriting equation (1.3) again:

$$u_{i+1,j} + u_{i-1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = (\Delta x)^2 f(i\Delta x, j\Delta y).$$

Putting j =1, i =1,2,-------,N, then j =2, i =1,2,--------,N and in the same manner j = M, i =1,2,----- ,N, thus finally we obtain a system of $M \times N$ equations which are solved by any convenient method to get $u_{i,j}$ at all nodes. This is one of the Implicit scheme but however sometimes the size of the system becomes so large that it is unmanageable. Hence this method is not an efficient way for solving Elliptic PDE.

In order to increase the accuracy , the scheme (1.3) known as Five-Point Scheme, can be modified to get a Nine-Point Scheme. This scheme in the form of computational molecule can be written as:

$$\begin{array}{ccc} (1)u_{i-1,j+1} & (4)u_{i,j+1} & (1)u_{i+1,j=1} \\ (4)u_{i-1,j} & (-20)u_{i,j} & (4)u_{i+1,j} \\ (1)u_{i-1,j-1} & (4)u_{i,j-1} & (1)u_{i+1,j-1} \end{array} = 6(\Delta x)^2 f(i\Delta x, j\Delta y) + o(\Delta x)^4 \qquad (2.1)$$

Scheme (2.1) has error of o(Δx)^4 which is sufficiently less than Five-Point Scheme, but it involves more computational complexity.

Another way of increasing accuracy, is to modify the scheme for solving the equation (1.4). One of the technique used for this purpose is the Successive over relaxation (SOR) method.

To explain its implementation, we can write

$$u^{n+1}_{i,j} = u^{n}_{i,j} + \frac{\omega}{4}\left[(\Delta x)^2 f_{i,j} - u^{n+1}_{i-1,j} - u^{n+1}_{i,j-1} - u^{n}_{i,j+1} - u^{n}_{i+1,j} - 4u^{n}_{i,j}\right]$$

$$\Rightarrow u^{n+1}_{i,j} = \omega(\text{RHS of Gauss -Seidel method}) - (\omega - 1)u^{n}_{i,j}$$

ω is called as a relaxation parameter. It is verified that for over relaxation method ,1< ω <2.

For a rectangular domain, a reasonable estimate of ω can be taken as the small root of the equation

$$\left(\cos\frac{\pi}{M}+\cos\frac{\pi}{N}\right)^2\omega^2-16\omega+16=0$$

where M & N are number of sub-domains in each side. This shows $\omega=\dfrac{4}{2+\sqrt{4-k^2}}$

where, $k=\cos\dfrac{\pi}{M}+\cos\dfrac{\pi}{N}$

The following table shows the value of ω for different M & N.

M = N	ω
2	1.000
3	1.072
5	1.260
10	1.528
20	1.729
∞	2.000

It is observed that usually ω_{opt} = 1.5

For large M and N, one can approximate ω by

$$\omega_{\text{opt}}=2-\sqrt{2}\pi\left(\frac{1}{M^2}+\frac{1}{N^2}\right)^{\frac{1}{2}}\cong 2\left(1-\frac{\pi}{M}\right)\ ,M=N$$

Example: Solve ∇^2u=0, subject to Dirichlet conditions in the domain $0\le x\le 0.75$, $0\le y\le 0.75$ with Δx= Δy=0.25 , subject to the boundary conditions:
u(0,y) =0, u(0.75,y) =10 and u(x,0) =0, u(x,0.75) =10.
Obtain the solution at the internal nodes: (1) using symmetry (2) without using symmetry.
With; (a) Gauss- Seidel Method (b) SOR method.
And compare the results.
Work out this problem with Nine-Point formula also.

Solution:

We solve the problem with symmetry.

By Gauss-Seidal method ,we have 3 equations:

$$\text{(start with assumed values as 0 i.e. } u_{i,j}^{0} = 0)$$

$$u_{1,1}^{n+1} = \frac{1}{2}\left[u_{2,1}^{n}\right]$$
$$u_{2,1}^{n+1} = \frac{1}{4}\left[u_{1,1}^{n+1} + u_{2,2}^{n} + 10\right]$$
$$u_{2,2}^{n+1} = \frac{1}{2}\left[u_{2,1}^{n+1} + 10\right]$$

The values ,so computed are given in the following table:

	$u_{1,1}$	$u_{2,1}$	$u_{2,2}$
n = 0	0	2.5	6.25
n = 1	1.25	4.3750	7.1875
n = 2	2.8175	4.8438	7.4219
n = 3	2.4219	4.9610	7.4805
n = 4	2.4805	4.9902	7.4951
n = 5	2.4951	4.9976	7.4988
n=6	2.4988	4.9994	7.4997

Hence solution correct to 2d is:

$u_{1,1}$=2.50 ; $u_{2,1}$=5.00 ; $u_{3,1}$=7.50

(b) By SOR Method:

$$k = \cos\frac{\pi}{M} + \cos\frac{\pi}{N}, \quad \text{M, N} = \text{no. of sub} - \text{divisions. Here M} = 3,\ \text{N} = 3$$

The equations in the iterative form can be written as follows :

$$\therefore k = 0.5 \quad \omega = \frac{4}{2+\sqrt{4-k^2}} \quad \therefore \omega = 1.072$$

(start with assumed values as 0 i.e. $u^0_{i,j} = 0$)

$$u^{n+1}_{1,1} = 1.072\left[\frac{1}{2}\left(u^n_{2,1} + 5\right)\right] - 0.072u^n_{1,1}$$

$$u^{n+1}_{2,1} = 1.072\left[\frac{1}{4}\left(u^{n+1}_{1,1} + u^n_{2,2} + 10\right)\right] - 0.072u^n_{2,1}$$

$$u^{n+1}_{2,2} = 1.072\left[\frac{1}{2}\left(u^{n+1}_{2,1} + 10\right)\right] - 0.072u^n_{2,2}$$

The values so computed are given in the following table:

	$u_{1,1}$	$u_{2,1}$	$u_{2,2}$
n = 0	0	2.68	6.70
n = 1	1.4365	4.6676	7.3794
n = 2	2.3984	4.9644	7.4896
n = 3	2.4882	4.9844	7.4817
n=4	2.4988	4.9914	7.4917

Hence solution correct to 2d is:

$u_{1,1}$=2.50 ; $u_{2,1}$=5.00 ; $u_{3,1}$=7.50

Thus SOR Method gives faster convergence.

Let us solve the problem **without symmetry**.

(a) By Gauss-Seidal method, we have 4 equations:

(start with assumed values as 0 i.e. $u^0_{i,j} = 0$)

$$u^{n+1}_{1,1} = \frac{1}{4}\left(u^n_{1,2} + u^n_{2,1}\right)$$

$$u^{n+1}_{1,2} = \frac{1}{4}\left(u^{n+1}_{1,1} + u^n_{2,2} + 10\right)$$

$$u^{n+1}_{2,1} = \frac{1}{4}\left(u^{n+1}_{1,1} + u^n_{2,2} + 10\right)$$

$$u^{n+1}_{2,2} = \frac{1}{4}\left(u^{n+1}_{2,1} + u^n_{1,2} + 20\right)$$

The values so computed are given as follows:

	$u_{1,1}$	$u_{1,2}$	$u_{2,1}$	$u_{2,2}$
n = 0	0	2.5	2.5	6.25
n = 1	1.25	4.3750	4.3750	7.1875
n = 2	2.8175	4.8438	4.8438	7.4219
n = 3	2.4219	4.9610	4.9610	7.4805
n = 4	2.4805	4.9902	4.9902	7.4951
n = 5	2.4951	4.9976	4.9976	7.4988
n=6	2.4988	4.9994	4.9994	7.4997

Hence solution correct to 2d is:

$u_{1,1}$=2.50 ; $u_{2,1}$=5.00 ; $u_{3,1}$=7.50

(b) By SOR Method:

$$k = \cos\frac{\pi}{M} + \cos\frac{\pi}{N} \quad \mathrm{M, N = no.\ of\ sub-divisions}$$

$$\text{Here M} = 3,\ \text{N} = 3$$

$$\therefore k = 0.5$$

$$\omega = \frac{4}{2+\sqrt{4-k^2}}$$

$$\therefore \omega = 1.072$$

Hence, we have 3 equations:

$$(\text{start with assumed values as 0 i.e. } \mathrm{u}_{\mathrm{i,j}}^0 = 0)$$

$$u_{1,1}^{n+1} = 1.072\left[\frac{1}{2}\left(u_{1,2}^n + u_{2,1}^n\right)\right] - 0.072u_{1,1}^n$$

$$u_{2,1}^{n+1} = 1.072\left[\frac{1}{4}\left(u_{1,1}^{n+1} + u_{2,2}^n + 10\right)\right] - 0.072u_{2,1}^n$$

$$u_{2,1}^{n+1} = 1.072\left[\frac{1}{4}\left(u_{1,1}^{n+1} + u_{2,2}^n + 10\right)\right] - 0.072u_{2,1}^n$$

$$u_{2,2}^{n+1} = 1.072\left[\frac{1}{4}\left(u_{2,1}^{n+1} + u_{1,2}^n + 20\right)\right] - 0.072u_{2,2}^n$$

Now we record all values in table form:

	$u_{1,1}$	$u_{1,2}$	$u_{2,1}$	$u_{2,2}$
n = 0	0	2.68	2.68	6.70
n = 1	1.4365	4.6676	4.6676	7.3794
n = 2	2.3984	4.9644	4.9644	7.4896
n = 3	2.4882	4.9844	4.9844	7.4817
n=4	2.4988	4.9914	4.9914	7.4917

Hence solution correct to 2d is:

$u_{1,1}=2.50$; $u_{2,1}=5.00$; $u_{3,1}=7.50$

Hence SOR Method gives faster convergence.

Now by **Nine-Point Formula**, using symmetry, we have 3 equations:

$$20u_{1,1} - 8u_{2,1} - u_{2,2} = 0$$
$$4u_{1,1} - 19u_{2,1} + 4u_{2,2} + 55 = 0$$
$$u_{1,1} + 8u_{2,1} - 20u_{2,2} + 110 = 0$$

On solving by Gauss- Elimination method ,we have:

$u_{1,1}=2.3810$; $u_{2,1}=5.00$; $u_{3,1}=7.6190$