Mathematics in Education and Industry

Innovators in Mathematics Education

MEI Conference 2009

Proof

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Workshop H1

Some proofs for GCSE students

- Prove that the sum of two odd numbers must be even.

- Prove that the square of an odd number must be odd.

- Prove that the sum of two consecutive numbers must be odd.

- Prove that the sum of four consecutive numbers must be even.

- Prove that the sum of three consecutive numbers must be divisible by 3. Can you generalise this result?

- Prove that the product of 3 consecutive numbers is divisible by 3. Can you generalise this result?
- Prove that $0.\dot{9} = 1$.

- Prove that $0.\dot{a}\dot{b} = \frac{ab}{99}$. Can you generalise this result?

- Prove the quadratic formula.

- Prove that the n^{th} triangular number is $\frac{1}{2}n(n+1)$.

- Prove the circle theorems.

- Prove in at least two different ways that the interior angle of a regular polygon with n sides is $\frac{180(n-2)}{n}$.

- Understand at least two different proofs of Pythagoras's theorem.

- Prove that the area of a trapezium is a half times the sum of the lengths of the two parallel sides times the distance between them.

- Prove that any quadrilateral will tessellate with itself.

- A platonic solid is a 3 dimensional shape with faces that are congruent regular polygons and with identical vertices – the cube and the tetrahedron are examples of platonic solids. By considering the vertices of platonic solids, prove that there can only be at most 5 different types of platonic solid. Why does this argument not prove that there are exactly 5 different types of platonic solid?

Proof in C1

3 The smallest of three consecutive integers is n.

Write down the other two integers.

Prove that the sum of any three consecutive integers is divisible by 3. [3]

1 n is a positive integer. Show that $n^2 + n$ is always even. [2]

9 (i) Prove that 12 is a factor of $3n^2 + 6n$ for all even positive integers n. [3]

(ii) Determine whether 12 is a factor of $3n^2 + 6n$ for all positive integers n. [2]

Proof in C3

6 (i) Disprove the following statement.

'If $p > q$, then $\frac{1}{p} < \frac{1}{q}$.' **[2]**

(ii) State a condition on p and q so that the statement is true. **[1]**

4 Use the method of exhaustion to prove the following result.

No 1- or 2-digit perfect square ends in 2, 3, 7 or 8

State a generalisation of this result. [3]

3 The converse of the statement 'P $\Rightarrow$ Q' is 'Q $\Rightarrow$ P'.

Write down the converse of the following statement.

'n is an odd integer $\Rightarrow 2n$ is an even integer.'

Show that this converse is false. [2]

5 (i) Verify the following statement:

'$2^p - 1$ is a prime number for all prime numbers p less than 11'. [2]

(ii) Calculate 23×89, and hence disprove this statement:

'$2^p - 1$ is a prime number for all prime numbers p'. [2]

Proof by induction in FP1

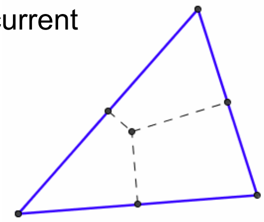
6 Prove by induction that $3+10+17+.....+(7n-4)=\frac{1}{2}n(7n-1)$ for all positive integers n. **[7]**

6	When n = 1, $\frac{1}{2}n(7n-1)=3$, so true for n = 1 Assume true for $n = k$ $3+10+17+.....+(7k-4)=\frac{1}{2}k(7k-1)$ $\Rightarrow 3+10+17+.....+(7(k+1)-4)$ $=\frac{1}{2}k(7k-1)+(7(k+1)-4)$ $=\frac{1}{2}[k(7k-1)+(14(k+1)-8)]$ $=\frac{1}{2}[7k^2+13k+6]$ $=\frac{1}{2}(k+1)(7k+6)$ $=\frac{1}{2}(k+1)(7(k+1)-1)$ But this is the given result with k + 1 replacing k. Therefore if it is true for k it is true for k + 1. Since it is true for k = 1, it is true for k = 1, 2, 3 and so true for all positive integers.	B1 E1 M1 M1 A1 E1 E1 **[7]**	 Assume true for k Add $(k+1)$th term to both sides Attempt to factorise c.a.o. with correct simplification Dependent on previous E1 and immediately previous A1 Dependent on B1 and both previous E marks

Proof by deduction, exhaustion, contradiction or disproof by counterexample?

How do you give students the insight to know where to begin?

1. A number is divisible by 9 if and only if the sum of its digits is divisible by 9.
2. If xy is of the form $3k+2$ then exactly one of the integers x, y is of this form.
3. The perpendicular bisectors of the sides of a triangle are concurrent



4. There are no integer solutions to $x^2 - 4y = 3$
5. If $2^m + 1$ is prime then m must be a power of 2.
6. For all positive integers $\frac{n}{3} + \frac{n^2}{2} + \frac{n^3}{6}$ is an integer.
7. No number in the infinite sequence 11, 111, 1111, . . . is a square number.
8. $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n-1} + \frac{1}{n}$ is not an integer for any integer $n > 1$
9. If m and n are odd positive integers and m^m is a factor of n^n then m must be a factor of n
10. For any quadrilateral, the midpoints of the sides are the vertices of a parallelogram.
11. $\sqrt{2} + \sqrt{3}$ is irrational.
12. Every positive integer can be written in the form $a^2 + b^2 - c^2$ where a, b and c are integers
13. For any polynomial equation $x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1 + a_0 = 0$ where all the coefficients are integers, if any roots are rational numbers then they must be integers.
14. An equilateral triangle in the $x - y$ plane cannot have all three vertices on grid points (i.e. points where both coordinates are integers.)
15. If you add any two square numbers and double the result the answer is also the sum of two squares.
16. Add 1 to each prime number: 3, 4, 6, 8, 12, 14, 18, … The only square number that will appear in the list is 4.

Using spreadsheets to motivate a search for a proof or a counterexample

In the following, n represents a positive integer. Set up a spreadsheet to investigate the propositions. Can you find a counterexample? If not, do the results from the spreadsheet give you an idea of how to construct a proof?

1. The product of four consecutive integers is always one less than a square.

first number	product	product +1	square root
1	24	25	5
2	120	121	11
3	360	361	19
4	840	841	29
5	1680	1681	41
6	3024	3025	55
7	5040	5041	71
8	7920	7921	89

Can you find a link between the first number and the square root and then base a proof on this?

Or do you think a counterexample will appear soon?

2. For all odd positive integers k, $n^k - n$ is divisible by k

So far, so good!

E5 f_x =(E$1^$A5-E$1)/$A5

	A	B	C	D	E	F
1		2	3	4	5	6
2	1	0	0	0	0	0
3	3	2	8	20	40	70
4	5	6	48	204	624	1554
5	7	18	312	2340	11160	39990

3. For any two positive numbers a and b the arithmetic mean, $\frac{a+b}{2}$, is never less than the geometric mean, $\sqrt{ab}$.

No negatives values of $\frac{a+b}{2} - \sqrt{ab}$ appearing yet

D6 f_x =(D$1+$A6)/2-SQRT(D$1*$A6)

	A	B	C	D	E	F
1		1	2	3	4	5
2	1	0	0.085786	0.267949	0.5	0.763932
3	2	0.085786	0	0.05051	0.171573	0.337722
4	3	0.267949	0.05051	0	0.035898	0.127017
5	4	0.5	0.171573	0.035898	0	0.027864
6	5	0.763932	0.337722	0.127017	0.027864	0

4. $\frac{n}{3}+\frac{n^2}{2}+\frac{n^3}{6}$ is an integer.

5. n^5-n is divisible by 30

6. $n(n+1)(2n+1)$ is divisible by 6

7. $\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n-1}+\frac{1}{n}$ is never an integer

8. The space diagonal of a cuboid with dimensions $n\times(n+1)\times n(n+1)$ has integer length.

9. There is an infinite number of integers which are both square and triangular (e.g. $1+2+3+4+5+6+7+8=36=6^2$)

10. n^2+n+41 is always prime

How would you test to check if a number is prime?

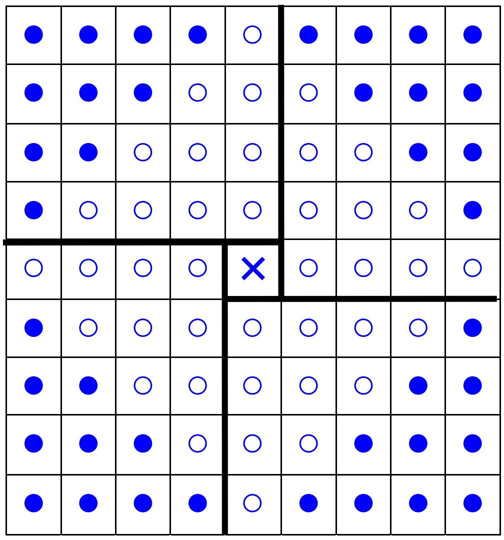
11. $n^2-79n+1601$ is always prime

Excel will express numbers as fractions where possible. To set this up, highlight the cells you want to appear as fractions and from the Format menu choose 'cells'. On the number tab choose 'fraction' and 'Upto three digits'. This is useful in investigating the following propositions.

12. $\frac{1}{3}+\frac{1}{5}=\frac{8}{15}$ and $(8,15,17)$ is a Pythagorean triple. Add the reciprocals of any two consecutive odd numbers. Prove that the resulting fraction, $\frac{x}{y}$, always generates an integer Pythagorean triple, (x,y,z).

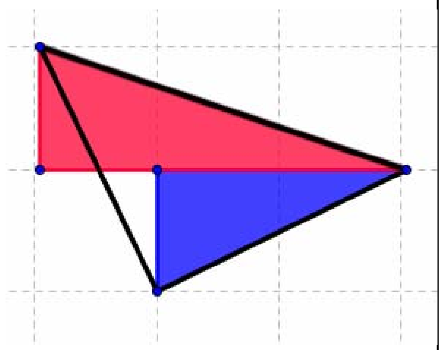
13. Find the value of $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{n}{(n+1)!}$ for some small values of n.
Find the general value of the expression and give a proof of your answer.

Every triangular number multiplied by 8 is 1 less than a square number.

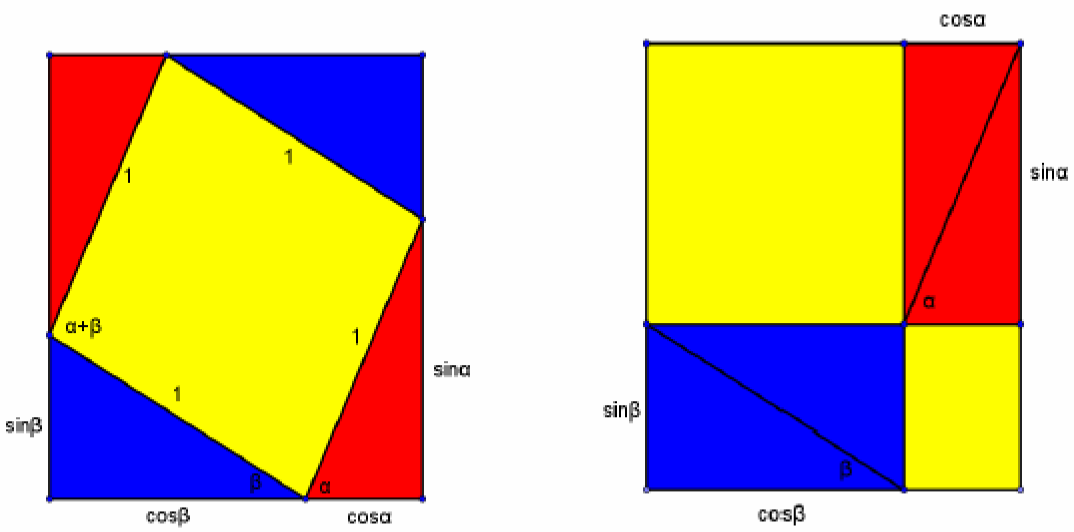


See the two books ‘Proofs without words’ by Roger B. Nelsen

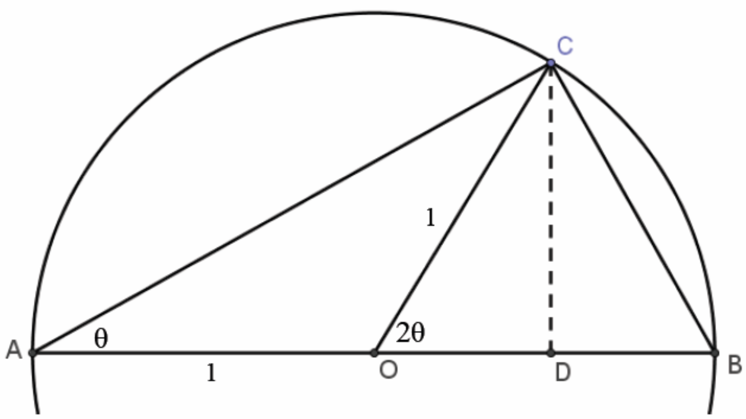
$$\arctan\frac{1}{2}+\arctan\frac{1}{3}=\frac{\pi}{4}$$



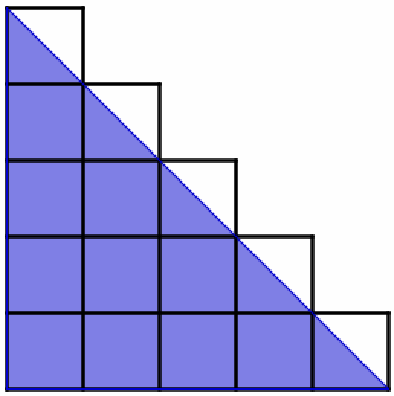
$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

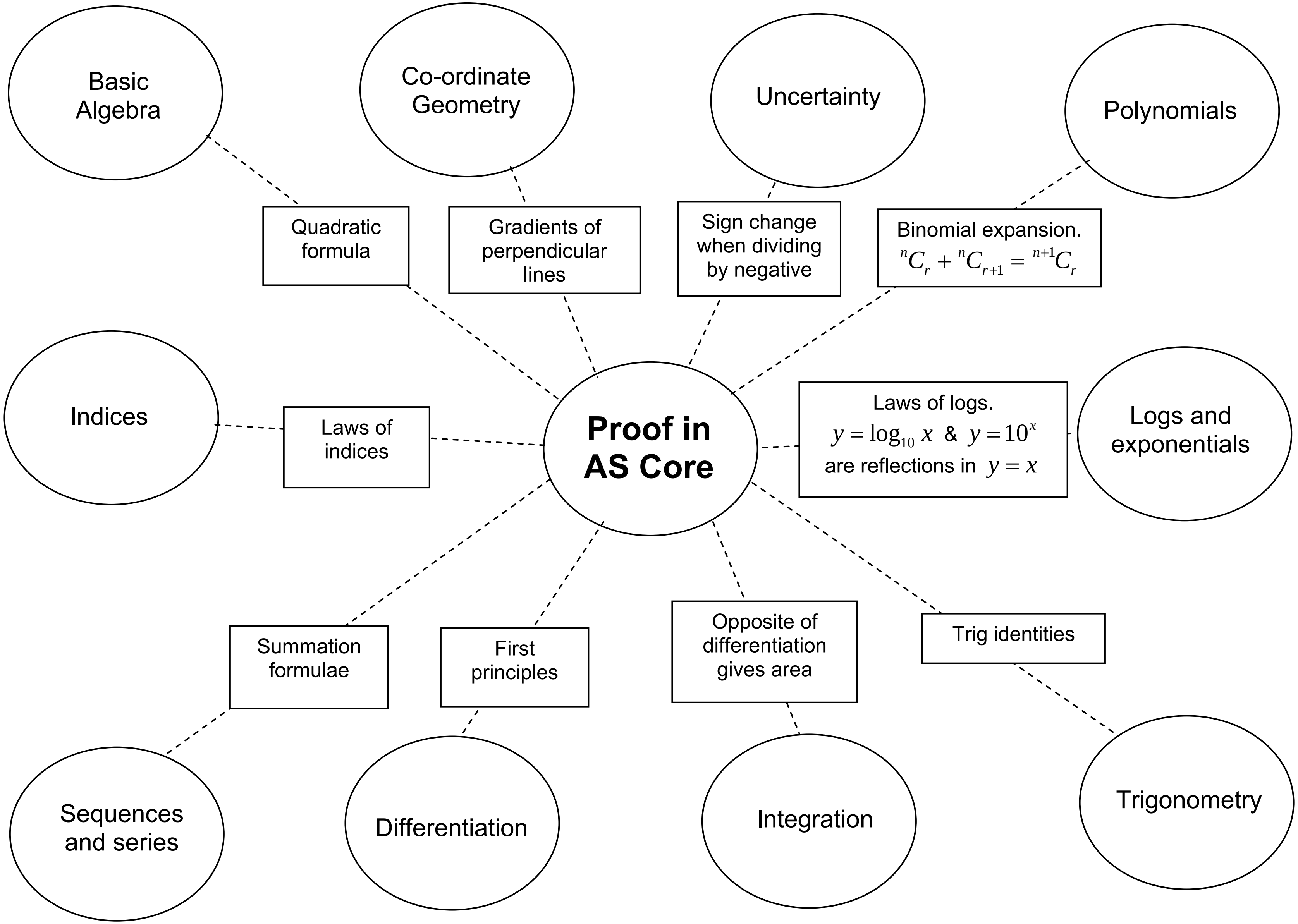


Proving the double angle formulae

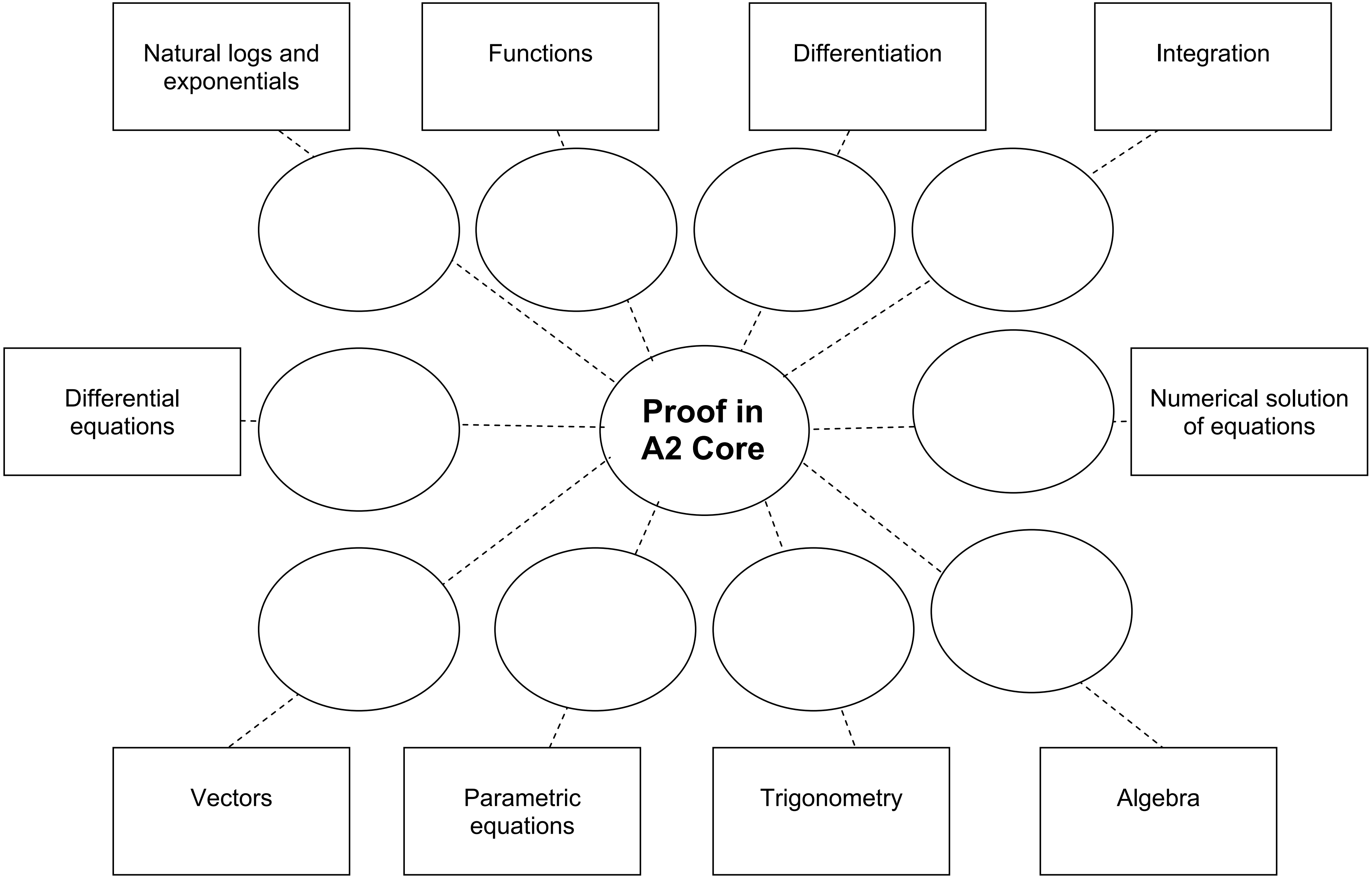


$$1+2+\ldots+n=\frac{n^2}{2}+\frac{n}{2}$$





Proof in AS Core
Basic Algebra
Quadratic formula
Co-ordinate Geometry
Gradients of perpendicular lines
Uncertainty
Sign change when dividing by negative
Polynomials
Binomial expansion. ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_r$
Indices
Laws of indices
Logs and exponentials
Laws of logs. $y=\log_{10}x$ & $y=10^x$ are reflections in $y=x$
Sequences and series
Summation formulae
Differentiation
First principles
Integration
Opposite of differentiation gives area
Trigonometry
Trig identities


Natural logs and exponentials
Functions
Differentiation
Integration
Differential equations
Proof in A2 Core
Numerical solution of equations
Vectors
Parametric equations
Trigonometry
Algebra