A Pedagogical Relook at Bertrand's Theorem *

Jeevitha T. U. and Sanjit Das

Bertrand's theorem is one of the landmark results in the context of the central force problem in mechanics. It leads to the conditions for closed bound orbits. This pedagogical article intends to motivate the teacher as well as the student to engage in a formal proof of the theorem. We overview the basics of central forces and the Abel integral. Thereafter, we obtain the condition for closed orbits. We provide the reader with a broad panorama of areas/topics where Bertrand's theorem arises, including interesting scenarios in the regime of Einstein's general theory of relativity.

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Introduction

One of the scarcely discussed topics in undergraduate classical mechanics is Bertrand's theorem [1] which arises in the context of central forces. The theorem is discussed in one of the most famous books – *Goldstein's Classical Mechanics*. Unfortunately, the 3rd edition of this book omits the proof. Thus, those who are interested, have to find it in the 2nd edition [2]. It is also true that very few of the other well-known books on mechanics deals with (or even mention) the theorem or its proof (Arnold [3], José [4], Fasano [5] and Chaichian [6]). The original proof given by Bertrand is quite different [1] from what appears in *Goldstein* (2nd edition). Except in the book by V. I. Arnold (where the theorem appears to be given as couple of exercises in chapter 2, section 8) José, Fasano and Chaichian mention it. However, we will discuss Tikochinsky's proof which appeared in the *American Journal of Physics* [7].

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Among central forces, it is the Kepler problem which is of utmost importance. However, Bertrand's theorem discusses closed orbits and bound trajectories under central forces, in general. Our goal in this article is two-fold. First, we will recapitulate the bare necessities on central forces, and secondly, we will revisit and expand on the proof of Bertrand's theorem, largely motivated by Tikochinsky [7].

It would not be inappropriate to say a few words about Bertrand. Joseph Louis Françios Bertrand (1822–1900) was a French polymath who worked in different fields of mathematics including analysis, number theory, probability, differential geometry, thermodynamics and celestial mechanics. If we closely see Bertrand's academic career, we realise that he has spent several years on each of these topics. Around 1865 he started taking interest in celestial mechanics and published a book titled *Les fondateurs de l'astronomie Moderne: Copernic, Tycho Brahé, Képler, Galilée, Newton*. A decade after publishing this book, he came up with his celebrated theorem which we know today as Bertrand's theorem. Bertrand's paper was published in *Comptes Rendus* of the Paris Science Academy on 20th October 1873 under the presidency of Mr Quatrefages. Bertrand himself called this theorem as a '*la loi de la nature*' (which means 'the law of nature').

Rudimentary Central Force

One of the oldest and richest topics in classical mechanics is the central force problem. The subject has been enriched by stalwarts like Copernicus and Newton. Almost all great minds have explored the central forces at some point in their career. We have Johannes Kepler framing the Kepler's laws [8], Sir Isaac Newton discovering the law of gravitation and conceptualizing the laws of motion [9], Bernoulli, Laplace, Hamilton and Lenz looking into the constants of motion for Keplerian orbits [10], Bertrand finding the condition for circular bound orbits, D'Alembert analysing the precession of Earth's axes of rotation [11], Delaunay & Hill's for lunar motion studies [12, 13], Poisson giving the differential

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equation for the gravitational potential [14], Jacobi finding the inverse square law for parabolic coordinates [15], Lagrange solving the three body problem (in limiting case) [16], Gauss's determination for the orbit of Ceres leading to the least square approximation [17] and so on. Here, we will discuss only the essentials that we need to understand the basic principles and properties of celestial mechanics.

By a central force, we mean a mass subject to an attractive or a repulsive force whose line of action always goes through a fixed point (the *center of force*). It is not that the force emerges from the origin or that there is but one force, but it is simply the fact that the resultant of all forces acting on the object always passes through this point. In precise mathematical terms, we can say that vector fields (a bunch of vectors) which point along the radial rays through the origin (centre) and which has a constant magnitude on each sphere r (= constant) is called *central*. In our case, this vector field is the *central force field* defined by

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$$\vec{F}(r) = \lambda(r)\vec{r}, \tag{1}$$

where $\lambda(r)$ is a scalar function of $\vec{r}$. The most pronounced central force examples we encounter in our undergraduate classes are related to Hooke's law of elasticity and Newton's inverse square law of gravitation. By the end of the article, you will have a better reason for appreciating the importance of these two central forces.

Central forces have two special properties. First, they are conservative and second, they are rotationally symmetric. A diagrammatic acquaintance may be taken from the *Figures* 1 and 2 where the first one is central, the second one is conservative without rotational symmetry, and the third one is pointing towards the centre but is not central.

The differential equation we look at is known as the orbit equation. Most undergraduate texts deal with the relation of unit vectors in polar and Cartesian coordinates. This helps us to derive the form of velocity and acceleration in polar coordinates, which will help in deriving the orbit differential equation. Another method

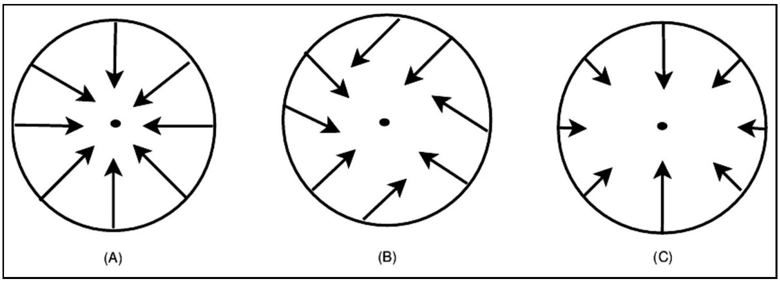


Figure 1. Different force fields (**A**) Central, (**B**) Non-conservative without rotational symmetry, (**C**) Pointing at the center but not central.

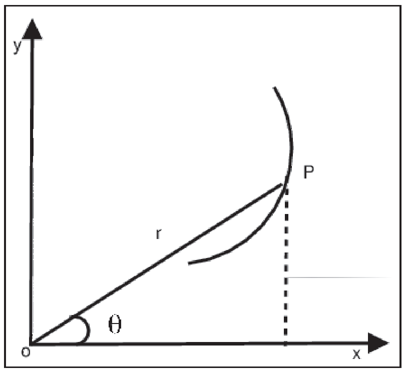


Figure 2. A schematic representation of polar coordinates.

followed by advanced texts directly uses the Lagrangian and the Euler–Lagrange equation leads us to the orbit differential equation. Without invoking either of the above two ways, let us write the acceleration in the *xy* components which look as follows

$$\frac{d^2x}{dt^2} = -\frac{fx}{r} \tag{2}$$

$$\frac{d^2y}{dt^2} = -\frac{fy}{r}, \tag{3}$$

where f is the force per unit mass, i.e. acceleration. If we change from Cartesian to polar coordinates by $x = r\cos\theta$, $y = r\sin\theta$ then the acceleration in x and y component will become

$$\frac{d^2r}{dt^2}\cos\theta - 2\frac{dr}{dt}\frac{d\theta}{dt}\sin\theta - r\left(\frac{d\theta}{dt}\right)^2\cos\theta - r\frac{d^2\theta}{dt^2}\sin\theta = -f\cos\theta \tag{4}$$

$$\frac{d^2r}{dt^2}\sin\theta - 2\frac{dr}{dt}\frac{d\theta}{dt}\cos\theta - r\left(\frac{d\theta}{dt}\right)^2\sin\theta + r\frac{d^2\theta}{dt^2}\cos\theta = -f\sin\theta\,. \tag{5}$$

For each of the equations, if we write coefficient of $\sin\theta$ and $\cos\theta$, we will get exactly,

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -f(r) \tag{6}$$

$$2r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = 0\,. \tag{7}$$

The second equation can be reduced to $r^2\frac{d\theta}{dt} = \frac{L}{m}$, which will be used to eliminate $\frac{d\theta}{dt}$ from the first equation

$$\frac{d^2r}{dt^2} = \frac{L^2}{m^2r^3} - f(r)\,. \tag{8}$$

Now let $r = \frac{1}{u}$, then the previous equation turns out to be the much desired orbital differential equation as

$$\frac{d^2u}{d\theta^2} + u = \frac{m^2}{L^2u^2}f\left(\frac{1}{u}\right). \tag{9}$$

Both Bertrand and Goldstein (though they have different types of proofs) have started from the orbital ODE. Goldstein uses the perturbative technique on the orbital ODE, which is not our approach in this article.

Consider a radial potential problem defined by the equation $\ddot{\vec{r}} = -\Phi(I)\vec{r}$, where $I = ||\vec{r}||^2 = r^2$ and $\Phi : (0, +\infty) \to \mathbb{R}$ is an analytic function. Assume that all the bounded nonrectilinear orbits are closed orbits. Then either the central force is everywhere repulsive, that is $\Phi(I) \le 0$ for all values of I and there exist no bounded orbits, or there is a $\mathcal{G} > 0$ such that $\Phi(I) = \mathcal{G}I^{-\frac{3}{2}}$ or $\Phi(I) = \mathcal{G}$.

Proof of Bertrand's Theorem

Bertrand's theorem states that only central potentials that gives birth to closed orbits are essentially of two types namely the Kepler potential and harmonic oscillator potential.

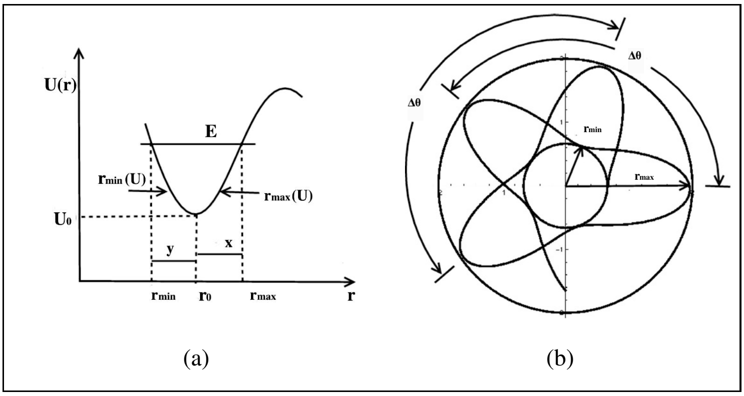


Figure 3. (**a**) Schematic diagram of effective potential vs radius. (**b**) Representation of an apsidal angle ($\Delta\theta$) of a particle moving under central force.

More precise statement can be found in [18]. We will discuss an alternative non-perturbative proof of Bertrand's theorem following Tikonchisky's approach.

If the force is derivable from scalar potential energy, then we will have an ordinary potential energy function such that,

$$\vec{F}(r) = -\vec{\nabla}V(r). \tag{10}$$

In general, the total energy, $E = \frac{1}{2}mv^2 + V(r)$, which will become in polar coordinates,

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r). \tag{11}$$

To solve the above equation, using angular momentum $L = mr^2\dot{\theta}$ to find $\dot{\theta}$, we get,

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r)\,, \tag{12}$$

where

$$U(r) = \frac{L^2}{2mr^2} + V(r)\,. \tag{13}$$

which is called the effective potential energy of the particle in one-dimension. There are two extreme radii for each closed orbit, which is r_{min} and r_{max}, that are known as apsidal points. Then the apsidal angle $\Delta\theta(E)$ is determined by the angular displacement of the particle between two following apsidal points in a boundary,

Figure 4. Niels Henrik Abel (1802–1829) was a Norwegian mathematician who has immensely contributed to various fields of mathematics.

i.e. the complete path traversed from $r_{\max}$ to $r_{\min}$ and back to $r_{\max}$. As the orbit is symmetric around the stationary point r_0 (see *Figure* 3), we have

$$\Delta\theta(E) = \frac{2L}{m} \int_{r_{\min}}^{r_{\max}} \frac{dr}{r^2 \sqrt{\frac{2}{m}(E - U(r))}}. \tag{14}$$

Little Digression on Abel Integral

This digression is to tell the story of how Abel (*Figure* 4) recovered the shape of the potential, given the periodic oscillation of a particle as a function of its energy [19].

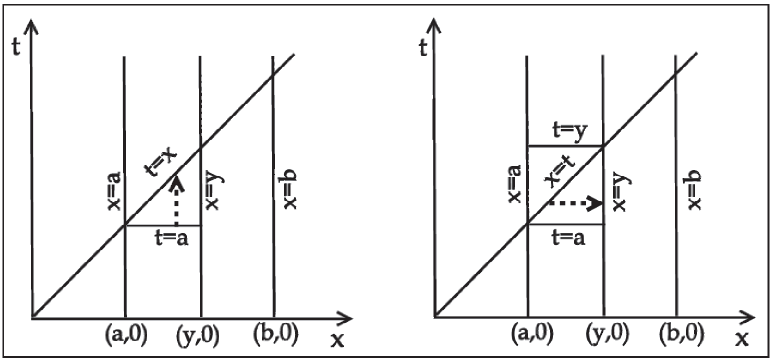
This is sometimes called Abel's mechanical problem. The problem is as follows: consider a particle of mass m which can slide down a wire under gravity but without any friction. If we know the shape of the wire, it is easy to determine the time as the function of the height from where the particle is sliding to the ground. But Abel's problem (*Résolution d'un probléme de mécanique*) is the inverse. Given the time as a function of height, we have to determine the shape of the wire. As we will see soon, this will lead to an integral equation which has to be solved. For this, let us make some mathematical prerequisites which will help us in due course. Consider the integral equation

Consider a particle of mass m which can slide down a wire under gravity but without any friction. If we know the shape of the wire, it is easy to determine the time as the function of the height from where the particle is sliding to the ground.

$$\int_a^x (x-t)^{\alpha-1} u(t)dt = f(x); \qquad x \in (a,b), \alpha \in (0,1) \tag{15}$$

In order to solve this equation we multiply both sides by the factor

Figure 5. Change of integration order. In the first figure, the integration order is $dt\,dx$ whereas in the second figure, the order changes to $dx\,dt$.



$\frac{dx}{(y-x)^\alpha}$ where $a < x < y < b$. Now integrating over (a, y) yields

$$\int_{x=a}^{x=y} \frac{dx}{(y-x)^\alpha} \left[\int_{t=a}^{t=x} (x-t)^{\alpha-1} u(t) dt \right] = \int_{x=a}^{x=y} \frac{f(x)}{(y-x)^\alpha} dx \quad (16)$$

To solve this double integral, we need to change the order of the integrals (see *Figure* 5) While changing the order of integrals the limit will also change which has to be taken care of, judiciously.

$$\int_{t=a}^{t=y} \left[\int_{x=t}^{x=y} \frac{dx}{(y-x)^\alpha (x-t)^{1-\alpha}} \right] u(t) dt = \int_{x=a}^{x=y} \frac{f(x)}{(y-x)^\alpha} dx \,. \quad (17)$$

Hint: $dx = d\lambda\,(t-y)$

	lower	upper
x	t	y
λ	1	0

$(x-t) = (1-\lambda)\,(y-t)$
$(y-x) = \lambda\,(y-t)$.

If we introduce the parametrisation $\lambda = \frac{(x-y)}{(t-y)}$, then the left hand side integral inside the bracket turns out to be $\int_0^1 \lambda^{-\alpha} (1-\lambda)^{\alpha-1} d\lambda$ which is nothing but $\Gamma(\alpha)\,\Gamma(1-\alpha)$. So we will get

$$\int_{t=a}^{t=y} u(t) dt = \frac{1}{\Gamma(\alpha)\,\Gamma(1-\alpha)} \int_{x=a}^{x=y} \frac{f(x) dx}{(y-x)^\alpha} \,. \quad (18)$$

By fundamental theorem of calculus (FTC) and the evaluation theorem, if U is the anti-derivative of u, i.e. $U'(y) = u(y)$, then

$$u(t) = \frac{1}{\Gamma(\alpha)\,\Gamma(1-\alpha)} \frac{d}{dt} \int_{x=a}^{x=t} \frac{f(x)\,dx}{(t-x)^\alpha}. \quad (19)$$

Proof of Bertrand's Theorem (Cont.)

Splitting (14) into two branches of apse (see *Figure* 3) we get

$$\Delta\theta(E) = 2\int_{r_{\min}}^{r_0} \frac{L}{mr^2}\frac{dr}{\sqrt{\frac{2}{m}(E-U(r))}} + \int_{r_0}^{r_{\max}} \frac{L}{mr^2}\frac{dr}{\sqrt{\frac{2}{m}(E-U(r))}}. \tag{20}$$

Now, applying inverse function theorem in the neighbourhood of r_0 for the above equation, we can write the two integrals together

$$\begin{aligned}\Delta\theta(E) = &\sqrt{\frac{2}{m}}L\int_E^{U_0}\frac{1}{r(U)^2}\frac{dr}{dU}\frac{dU}{\sqrt{E-U}}\\ &+\sqrt{\frac{2}{m}}L\int_{U_0}^{E}\frac{1}{r(U)^2}\frac{dr}{dU}\frac{dU}{\sqrt{E-U}},\end{aligned} \tag{21}$$

or in compact form as

$$\Delta\theta(E) = \sqrt{\frac{2}{m}}L\int_{U_0}^{E}\frac{dU}{\sqrt{(E-U)}}\frac{d}{dU}\left(\frac{1}{r_{\min}(U)}-\frac{1}{r_{\max}(U)}\right). \tag{22}$$

We can write (22) as

$$\Delta\theta(E) = \int_{U_0}^{E}F(U)\frac{dU}{\sqrt{(E-U)}}, \tag{23}$$

where,

$$F(U) = \sqrt{\frac{2}{m}}L\frac{d}{dU}\left(\frac{1}{r_{\min}(U)}-\frac{1}{r_{\max}(U)}\right). \tag{24}$$

Now, we need to solve (23) by applying Abel's integral technique. (The interested reader may consult Landau [20] where this type of integral has been dealt with, though we will strictly follow the previous mentioned technique).

Before we actually start computing the integral, let us invoke the condition for closed orbit, $\Delta\theta = 2\pi\frac{q}{p}$, where both p and q are integers. Let us clarify the meaning of p and q. After q periods (apogee), the radius vector of a particle will have made p complete revolutions and accordingly will occupy its original position

Fundamental theorem of calculus (FTC) tells us if f is a continuous function on $[a,b]$, then the function g defined by $g(x) = \int_a^x f(t)dt$, $x \in [a,b]$ is continuous on $[a,b]$ and differentiable on (a,b) , and $\frac{d}{dx}\int_a^x f(t)dt = f(x)$. i.e, $g'(x) = f(x)$.

Inverse function theorem: It says that if we have a nice function(continuously differentiable) with a non zero derivative around some point p then there is some open interval X around p and an open interval Y around $f(p)$ such that f maps X onto Y in a one-to-one fashion. In easy words it gives us sufficient condition for a function to be invertible in its domain.

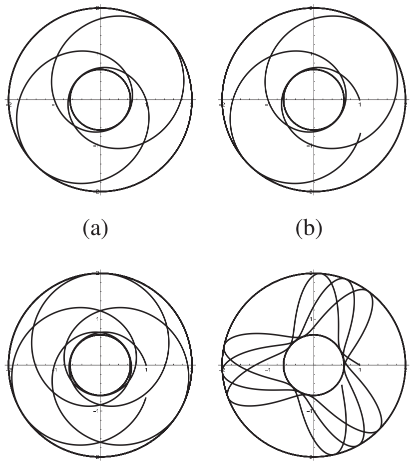


Figure 6. Examples for closed and unclosed orbits: (**a**) In the first example where $\frac{p}{q} = \frac{2}{3}$, we can see that after three revolutions and two apogees the particle returns to the starting point. (**b**) The second figure is the same as the first one where we want to emphasise the closing track. Here one clearly gets the meaning of the numbers 2 and 3. (**c**) In the same spirit, the third figure (intentionally we keep the closing track instead of full revolution) depicts $\frac{p}{q} = \frac{3}{5}$. (**d**) The last figure for unclosed orbits where $\frac{p}{q}$ is not a rational number. No matter how many revolutions it completes, it will never come back to the starting point.

i.e. the path of trajectory is closed. If $\Delta\theta \neq 2\pi\frac{q}{p}$, the trajectory is open. The corresponding *Figure* 6 will illuminate this in a better way. Those who wish may also see the animation of the closed and unclosed orbits. They can find them in the given links below, where each of animation has been done using Mathematica.

http://y2u.be/YQowmu9Qftw
http://y2u.be/6L0BKN85ZpQ
http://y2u.be/nwm_CQPnRnY

Comparing Eqn. (23) and Eqn. (15) we see $\alpha = \frac{1}{2}$, so that $\Gamma(\alpha)\,\Gamma(1-\alpha) = \pi$. From the result of Eqn. (19) we can write

$$F(U) = \frac{1}{\pi}\frac{d}{dU}\int_{U_0}^{U}\frac{\Delta\theta(E)dE}{(U-E)^{\frac{1}{2}}}. \tag{25}$$

An interesting attribute for $\Delta\theta(E)$ is it's constant value, specially when we are talking about closed orbits. To put in a different manner, we see that if $\frac{\Delta\theta(E)}{2\pi}\left(=\frac{1}{\beta}\right)$ is a continuous function of energy taking only rational values, then it must be independent of energy, i.e. a constant function. Because of this fact, we can bring β outside the integral. We assumed here when the energy varies over a range of values, β can either change in a discontinuous manner or not at all. This is the cornerstone for the proof. Now the condition for closed orbits reminds us $\Delta\theta(E) = 2\pi\frac{q}{p} = \frac{2\pi}{\beta}$ (say), which turns (25) to

$$F(U) = \frac{2}{\beta}\frac{d}{dU}\int_{E=U_0}^{E=U}\frac{dE}{(U-E)^{\frac{1}{2}}}. \tag{26}$$

Comparing the above equation with (24) we get

$$\frac{1}{r_{\min}(U)} - \frac{1}{r_{\max}(U)} = \frac{2\sqrt{2m}}{\beta L}\sqrt{U-U_0}. \tag{27}$$

The integral on the RHS is quite easy. Using a linear transformation as: $U - E = W$

E	U_0	U
W	$U-U_0$	0

$\int_{U_0}^{U}\frac{dE}{(U-E)^{\frac{1}{2}}} = 2\sqrt{U-U_0}$

Later, we will see that β will play the pivotal role in determining the shape of the potentials or rather say the types of central forces which will rise the closed orbits.

From Effective Potential to Condition of Closed Orbits

In the next step, we use Taylor's expansion on the effective potential $U(r)$ around its minimum r_0 (once again see *Figure* 3). In this section, we will use the following notation $U_0' = \frac{dU}{dr}|_{r=r_0}$, $U_0'' = \frac{d^2U}{dr^2}|_{r=r_0}$, $U_0''' = \frac{d^3U}{dr^3}|_{r=r_0}$ and so on. Now, expanding the term $\sqrt{U-U_0}$ from the (27) using Taylor series up to first non-vanishing order,

$$U - U_0 = \frac{1}{2}x^2U_0'' + ... = \frac{1}{2}y^2U_0'' + ... \tag{28}$$

From here, we can conclude that $x = y$ (upto this order). Now squaring (27) we arrive at:

$$\left[\frac{1}{r_{\min}(U)} - \frac{1}{r_{\max}(U)}\right]^2 = \left[\frac{2\sqrt{2}m}{\beta L}\sqrt{U-U_0}\right]^2 \tag{29}$$

$$\left[\frac{1}{r_{\min}(U)} - \frac{1}{r_{\max}(U)}\right]^2 = \frac{4m}{\beta^2L^2}x^2(U_0'' + ...) \tag{30}$$

From the effective potential (see *Figure* 3), we shall understand that $r_{\min}(U) = r_0 - y$ and $r_{\max}(U) = r_0 + x$, so using this condition we can solve the LHS of (30) which is

$$\left[\frac{1}{r_{\min}} - \frac{1}{r_{\max}}\right] = \left[\frac{1}{r_0 - y} - \frac{1}{r_0 + x}\right]$$
$$= \frac{r_0 + x - r_0 + y}{(r_0 - y)(r_0 + x)} \tag{31}$$
$$= \frac{x + y}{r_0^2\left(1 - \frac{y}{r_0}\right)\left(1 + \frac{x}{r_0}\right)}.$$

Hint:

$$\frac{x + y}{r_0^2\left(1 - \frac{y}{r_0}\right)\left(1 + \frac{x}{r_0}\right)} = \frac{x + x}{r_0^2\left(1 - \frac{x}{r_0}\right)\left(1 + \frac{x}{r_0}\right)}.$$

Using the fact $x = y$ (up to first non-vanishing order) the above equation turns out to be

$$\frac{1}{r_{\min}} - \frac{1}{r_{\max}} \approxeq \left(\frac{2x}{r_0^2}\right). \tag{32}$$

Putting (32) into (30), we will get

$$\beta^2 - 3 = \frac{V_0''}{V_0'} r_0. \tag{33}$$

Hint: From first derivative of effective potential
$U'(r_0) = V'(r_0) - \frac{L^2}{mr_0^3}$
$0 = V_0' - \frac{L^2}{mr_0^3}$

Here is how we have proved it. We used the fact $L^2 = V_0' m r_0^3$ and $U_0'' = V_0'' + \frac{3L^2}{m} r_0^{-4}$.

Using (32) will give,

$$\frac{\beta^2}{r_0} = \frac{U_0''}{V_0'}$$
$$= \frac{1}{V_0'}\left(V_0'' + \frac{3L^2}{m} r_0^{-4}\right)$$
$$= \frac{V_0''}{V_0'} + \frac{3}{r_0}.$$

Now integrating the above equation,

$$\frac{1}{V_0'}\frac{dV_0'}{dr_0} = \frac{\beta^2 - 3}{r_0}$$

$$\int \frac{dV_0'}{V_0'} = \int \frac{\beta^2 - 3}{r_0} dr_0$$

. We can take r_0 at any position, so solving the integral for any arbitrary r will determine the central force as

$$-f(r) = V^{'}(r) = Kr^{\beta^2-3}\,, \tag{34}$$

where k is a positive constant. Essentially we get the power law of the force from (34), with rational β (remember we defined $\beta = \frac{p}{q}$).

Our next job is to estimate the value of β which requires higher order Taylor's expansion of effective potential.

$$\begin{aligned} U - U_0 &= \frac{1}{2}x^2U_0^{''} + \frac{1}{6}x^3U_0^{'''} + \frac{1}{24}x^4U_0^{''''} + ... \\ &= \frac{1}{2}y^2U_0^{''} - \frac{1}{6}y^3U_0^{'''} + \frac{1}{24}y^4U_0^{''''} - .... \end{aligned} \tag{35}$$

We substitute $y = x(1 + ax + bx^2 + ...)$ in (35) which gives

$$\begin{aligned} U - U_0 &= \frac{1}{2}x^2U_0^{''} + \frac{1}{6}x^3U_0^{'''} + \frac{1}{24}x^4U_0^{''''} + ... \\ &= \frac{1}{2}x^2U_0^{''} + (aU_0^{''} - \frac{U_0^{'''}}{6})x^3 + (\frac{a^2U_0^{''}}{2} + U_0^{''}b - \frac{U_0^{'''}a}{2} + \frac{U_0^{''''}}{24})x^4 + ... \end{aligned} \tag{36}$$

Comparing both the lines in (36), we can say $a = \frac{U_0^{'''}}{3U_0^{''}}$ and $b = a^2$. Using this result, if we compute (31), we will reach the following steps:

$$\begin{aligned} \left[\frac{1}{r_{\min}} - \frac{1}{r_{\max}}\right]^2 &= \left[\frac{x + (x(1 + ax + a^2x^2 + ...))}{r_0^2\left(1 - \frac{x(1+ax+a^2x^2+...)}{r}\right)\left(1 + \frac{x}{r_0}\right)}\right]^2 \\ &= \frac{4x^2}{r_0^4} + \frac{4ax^3}{r_0^4} + \frac{\left(8 + 8ar_0 + 5a^2r_0^2\right)x^4}{r_0^6} + 0(x)^5 \\ &= \frac{x^2}{r_0^4}\left[4 + 4ax + \frac{\left(8 + 8ar_0 + 5a^2r_0^2\right)}{r_0^2}x^2\right] + 0(x)^5\,. \end{aligned} \tag{37}$$

Equating (27) and (37) gives us the platform to compare the same powers of x,

$$\left[\frac{1}{r_{\min}} - \frac{1}{r_{\max}}\right]^2 = \frac{x^2}{r_0^4}\left[4 + 4ax + \frac{\left(8 + 8ar_0 + 5a^2 r_0^2\right)}{r_0^2}x^2\right] \tag{38}$$
$$= \frac{4m}{\beta^2 L^2}x^2(U_0'' + \frac{x}{3}U_0''' + \frac{x^2}{12}U_0'''' ....),$$

which gives back

$$\frac{1}{r_0^4} = \left(\frac{m}{L^2\beta^2}\right)U_0'' \tag{39}$$

$$\frac{a}{r_0^4} = \left(\frac{m}{3L^2\beta^2}\right)U_0''' \tag{40}$$

$$\left(\frac{1}{r_0^4}\right)\left(5a^2 + \frac{8a}{r_0} + \frac{8}{r_0^2}\right) = \left(\frac{m}{3L^2\beta^2}\right)U_0'''' . \tag{41}$$

Final Leap to Estimate β

Hint: Make use of the fact $L^2 = V_0' m r_0^3$ and $U_0'' = V_0'' + \frac{3L^2}{mr_0^4}$.

Now as all the ingredients are ready, we can finally cook the meal. If we closely examine (39), we can see that it is a manifestation of (33). The same fate is awaiting for (40) also, we will sketch a very short proof. Replacing U_0''' by $3aU_0''$ we land up to (39). The actual information is hidden inside (41). To decipher, it we have to go through the following steps. The second, third and the fourth derivative of effective potential are respectively

$$U''(r) = V''(r) + \frac{3L^2}{mr^4}, \tag{42a}$$
$$U'''(r) = V'''(r) - \frac{12L^2}{mr^5}, \tag{42b}$$
$$U''''(r) = V''''(r) + \frac{60}{r^3}V'(r). \tag{42c}$$

Using the fact,

$$\frac{V''(r)}{V'(r)} = \frac{\beta^2 - 3}{r},$$

we can compute (42b) and (42c) which turns out to be respectively

Hint: $V''''(r) = (\beta^2-3)(\beta^2-4)(\beta^2-5)\frac{V'}{r^3}$.

$$U'''(r) = \frac{V'}{r^2}\beta^2(\beta^2 - 7), \qquad (43a)$$

$$U''''(r) = \frac{V'(r)}{r^2}\beta^2(\beta^4 - 12\beta^2 + 47). \qquad (43b)$$

Using the fact $U_0'' = \frac{V_0'}{r_0}\beta^2$ and (43a), we can compute the parameter $a(=\frac{U_0'''}{3U_0''})$, which turns out to be $\frac{(\beta^2-7)}{3r_0}$. Putting this result along with (43b) in (41), will finally give birth to a polynomial equation in β^2

Hint: $U_0'' = V_0'' + \frac{3L^2}{mr_0^4} = \frac{V_0'}{r_0}\beta^2$.

$$\beta^4 - 5\beta^2 + 4 = 0. \qquad (44)$$

The solution of the above equations are $\beta^2 = 1, 4$. This will lead us to our desired equation (34) of power law for the central force $(-f(r) = \frac{k}{r^{3-\beta^2}})$ which are respectively so,

$$\beta^2 = 1, 4$$

So, these are two potentials for which the apsidal angle is constant and the orbits are closed and bounded.

Conclusion

The story of Bertrand's theorem does not end here at all. There are still different ways to prove the theorem. At least we can give one classic reference given in Arnold's book and the same line of thought has been reproduced in José & Saletan (1998) [4], Fasano & Marimi [5] (Italian 2002, English Translation 2006) and Chaichian *et.al* (2012) [6]. It is worth to summarising the key ideas around which the proof has been presented. Without invoking the orbit differential equation, the starting emphasis has been given to the relation between apsidal angle and radial coordinate. While considering the closed motion between two turning points in the neighbourhood of the local minimum, we take advantage of inverse function theorem and express the effective potential as the function of radial coordinates in each branch of the turning points. The next job is the role of Abel integral along with the closed orbit condition which leads to the functional dependence of central force as a function of the radial coordinate, where the exponent of the radial coordinate is a function of apsidal angle. The fact β is independent of energy plays the crucial role in the proof. Later, using the Taylor series expansion, we can exactly calculate the appropriate exponents of the power law.

The whole idea of Bertrand's theorem has been given a wider perspective in the backdrop of Lorentzian curved spacetimes by Perlick [21] where all spherically symmetric metrics have been determined in which the closed trajectories are periodic. Let us also mention some recent articles ([22], [23], [24] & [25]) which capture some of the different aspects of Bertrand's theorem.

Bertrand's theorem never ceases to excite scientific minds for centuries. We hope this article will motivate younger minds to ask newer and more interesting questions related to Bertrand's theorem, in future.

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