21/5/25

Lec 9 & 10

Problem 6

A particle P of mass m moves in a central force field described by the force:

$$F = -\frac{\gamma m}{r^2}\hat{r},$$

where γ is a positive constant, & $\hat{r}$ is the radial unit vector. The force is attractive & follows an inverse square law, similar to gravity. We need to determine the conditions for bound & unbound orbits based on the total mechanical energy E.

An asteroid approaches the Sun from a great distance with const. speed u in a straight line, where the perpendicular distance from the Sun is p. We need to find the equation satisfied by the subsequent orbit. For the special case where

$$u^2 = \frac{4M_\odot G}{3p}$$ (where $M_\odot$ is the mass of the Sun, & G is the Gravitational const.),

find :

(i) the distance of closest approach of the asteroid to the Sun, &

(ii) the speed of the asteroid at the time of closest approach.

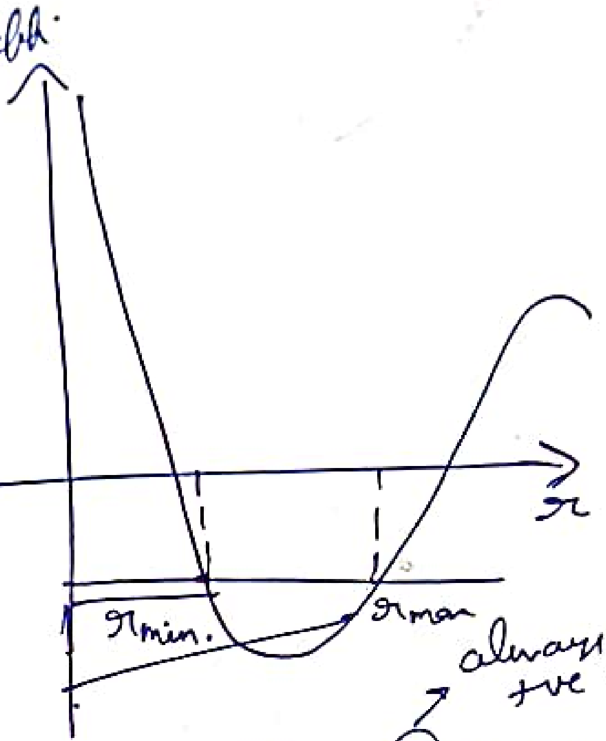
bounded.

$$L = T - V$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

$$E = \frac{1}{2} m\dot{r}^2 + \underbrace{V(r) + \frac{L^2}{2mr^2}}_{V_{eff}}$$

$\frac{-k}{r}$



$$E = T + V_{eff}.$$

$$\underbrace{(V_{eff.} - E)}_{(-ve\ or\ 0)} = -T$$

$$(V_{eff.} - E) \leq 0$$

$$V_{eff.} \leq E$$

$$V_{eff.} = E$$

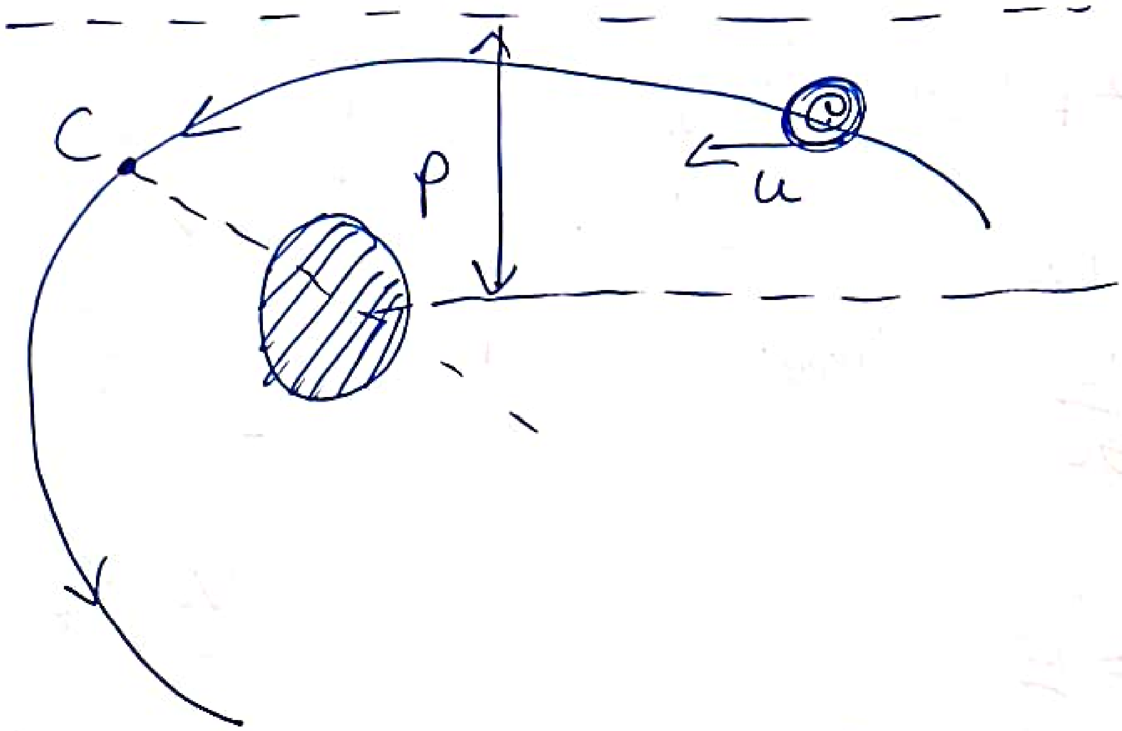
$$\frac{-V}{r} + \frac{L^2}{2mr^2} = E$$

(acc. to que.)

$\frac{-k}{r}$

Bounded means $\rightarrow E < 0$

$E > 0$



Sun in a asteroid system.

$\cancel{E = \frac{1}{2}}$ $V_{eff} = E$

$$\frac{-\gamma}{r} + \frac{L^2}{2mr^2} = E$$

$$\frac{-GM_{\odot}m_a}{r} + \frac{L^2}{2m_a r^2} = E$$

$$\tilde{L} = \frac{L}{m} = r^2\dot{\theta}^2$$

$L \Rightarrow$ conserv... quanti...

$$\frac{-GM_\odot}{r} + \frac{L^2}{2r^2} = E$$

moment of th

$L = pu$

$$\frac{-GM_\odot}{r} + \frac{p^2u^2}{2r^2} = \frac{1}{2}u^2$$

$E = \frac{1}{2}u^2$

$(\Rightarrow$ Rn Gravity$)$

$$\frac{-GM_\odot}{r} + \frac{p^2u^2}{2r^2} = \frac{1}{2}u^2$$

$$\cancel{\frac{-GM_\odot}{r} + p^2}$$

$$\frac{-2GM_\odot r + p^2u^2}{2r^2} = \frac{1}{2}u^2$$

$$-2GM_\odot r + p^2u^2 = \frac{\cancel{2}r^2u^2}{\cancel{2}}$$

$$-\frac{GM_\odot}{r} + \frac{p^2u^2}{2r^2} = \frac{1}{2}u^2$$

$$\cancel{-\frac{GM_\odot}{r} + p^2} \quad \cancel{-GM_\odot}$$

$$\frac{-2GM_\odot r + p^2u^2}{2r^2} = \frac{1}{2}u^2$$

$$-2GM_\odot r + p^2u^2 = \frac{\cancel{2}r^2u^2}{\cancel{2}}$$

$$-2GM_\odot r + p^2u^2 = r^2u^2$$

$$r^2(u^2) + (2GM_\odot)r + (-p^2u^2) = 0$$

$$r = \frac{-(2GM_\odot) \pm \sqrt{(2GM_\odot)^2 - 4(u^2)(-p^2u^2)}}{2(u^2)} \qquad \cancel{x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}$$

$$= -\frac{\cancel{2}GM_\odot}{\cancel{2}u^2} \pm \frac{\left((2GM_\odot)^2 + 4u^2 \cdot u^2 \cdot p^2\right)^{1/2}}{2u^2}$$

$$r = GM_\odot \cdot \frac{3p}{4M_\odot G} \pm \frac{\left[4G^2M_\odot^2 + 4p^2\left(\frac{4M_\odot G}{3p}\right)^{2}\right]}{2 \cdot \frac{4M_\odot G}{3p}}$$

$$= \cancel{GM_\odot} \cdot \frac{3p}{4\cancel{M_\odot G}} \pm \frac{\frac{10}{3}\cancel{GM_\odot}}{\frac{8}{3p}\cancel{M_\odot G}}$$

$$= \frac{3p}{4} \pm \frac{\cancel{10}^{5}}{\cancel{3}} \times \frac{\cancel{3}p}{\cancel{8}4} \qquad \cancel{(2GM_\odot)^2 +}$$

$$= \frac{3p}{4} \pm \frac{5p}{4}$$

$$= \frac{\cancel{8}^{2}p}{\cancel{4}_1} \text{ or } -\frac{\cancel{2}^{1}p}{\cancel{4}_2}$$

$$= 2p \text{ or } -\frac{p}{2}$$

$\boxed{r = 2p}$

so

$\boxed{r = \frac{p}{2}}$

closest pt.

$$4G^2M_\odot^2 + 4\cancel{p^2} \cdot \frac{4^2M_\odot^2G^2}{3^2\cancel{p^2}}$$

$$\sqrt{4G^2M_\odot^2\left(1 + \frac{4^2}{3^2}\right)}$$

$$2GM_\odot\sqrt{1 + \left(\frac{4}{3}\right)^2}$$

$$1 + \frac{16}{9} = \frac{9+16}{9} = \frac{25}{9}$$

$$= \frac{5}{3}$$

$$\frac{2 \times 5}{3}GM_\odot$$

$$\frac{10}{3}GM_\odot$$

Hence the closest point is $\frac{p}{2}$.

So the velocity at that point would be, ?

Angular momentum is conserved.

$$pu = \frac{p}{2}v$$

$$v = 2u$$