On Palindromic Squares of Non-Palindromic Numbers

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Introduction

Numbers which are unaffected by reversal of the order of their digits, such as 141 and 15351, are called *palindromes* by analogy with words or sentences having this property. These numbers have been the subject of a considerable recreational problem literature [1, 2, 3, 4, 5, 6, 7, 10, 11, 13, 14, 15] much of which is devoted to an open conjecture that the operation of successively reversing and adding any integer to itself a finite number of times will ultimately generate a palindrome [1, 6, 10, 12].

This paper originated from the observation that although there are infinitely many decimal numbers whose squares, cubes, and fourth powers are palindromes* [1] a computer search showed that there was no instance of a decimal number n with $1 < n^k < 2.8 \times 10^{14}$ for $4 < k < 11$ such that n^k was palindromic. Furthermore, in the case of decimal numbers it is not known whether there are finitely or infinitely many non-palindromic n for which n^k, $k = 2, 3,$ or 4, is palindromic, for example, $26^2 = 676$, $307^2 = 94{,}249$, and $2201^3 = 10{,}662{,}526{,}601$. In fact, the magnitude of the numbers which have already been examined suggests that for decimal numbers there may be no palindromic kth powers for $k \geqq 5$ and no palindromic powers of non-palindromic numbers for $k \geqq 4$. This also raises the question of whether these curious representational properties are restricted to decimal numbers or whether similar behavior occurs in other bases.† In this paper it is shown that this is indeed the case—by exhibiting several infinite sets of non-palindromic numbers in bases 3 and 4 whose powers are palindromic.

Property P

Since all decimal numbers of the form $1\overbrace{0\cdots0}^{i}1$ have palindromic squares, cubes and fourth powers (but not a palindromic fifth or higher power), there are infinitely many palindromic integers x^n for $x > 1$ and $n = 2, 3$ or 4. As the examples

* $1\overbrace{0\cdots0}^{i}1$ is obviously such a number for every i.

† The author is indebted to H. Hanani for suggesting this extension of earlier work [8].

mentioned in the introduction illustrate, however, there are also examples of palindromic powers of non-palindromic integers for at least squares and cubes. Table 2 of reference [8] gives an exhaustive compilation computed to the limits imposed by a CDC 6600 of the decimal palindromic squares and reveals that out of the 55 such palindromes, 16 are the squares of non-palindromic integers. We shall call this property of a palindromic integer being a power of a non-palindromic integer-property P. In decimal notation instances of palindromes possessing property P are apparently exceedingly rare for powers higher than the square. We have proven [8] and C. W. Trigg has shown independently [11] and communicated to the author the result that the only palindromic cube less than 1,953,125,000,000 whose cube root is not a palindrome is $10{,}662{,}526{,}601 = (2201)^3$. As was noted above in this connection, an exhaustive computer examination of all of the palindromes less than 2.8×10^{14} has failed to discover even a single instance of a palindrome greater than 1 whose fifth, sixth, seventh, eighth, ninth or tenth root was an integer [8] and no instance of a fourth power palindrome possessing property P. Hence, for decimal integers $(2201)^3$ is the only known instance for a power higher than a square of an integer possessing property P.

Decimal Palindromic Squares

The following four non-palindromic integers

$$\begin{array}{l} 3 \\ 307 \\ 30693 \\ 3069307 \end{array}$$

were all shown to have palindromic squares in [8]. Let N_n represent the nth such number and extend this list systematically by the recursion relation $N_n = N_{n-1} \cdot 10^2 + (-1)^n \cdot 7$. This procedure generates three new palindromic squares:

$$\begin{aligned} (306930693)^2 &= 94206450305460249 \\ (30693069307)^2 &= 942064503484305460249 \\ (3069306930693)^2 &= 9420645034800084305460249 \end{aligned}$$

The square of the last number above, with twenty-five digits, is the largest palindrome possessing property P known to the author. The next such number, $(306930693069307)^2 = 942064503480\bar{5}1\bar{4}0084305460249$ while almost a palindrome, fails in the two indicated digits, as do the other larger numbers generated by this recursion. The magnitude of the task of direct computation and search with numbers of this size makes it unlikely that this example of a twenty-five digit palindrome with a non-palindromic square root will be improved upon by direct calculation.

Binary Palindromic Powers

Table 1 gives the binary palindromic squares, N^2, for $1 < N \leqq 1{,}234{,}162$. The decimal value of the indicated number is given below its binary representation.

For binary representations we have been unable to even show whether there are infinitely many palindromic squares or not. It is worth noting that the only value of N greater than 1 and less than 65,000 whose cube is a palindrome is

$$(11)^3 = 11011.$$

TABLE 1. Binary Palindromic Squares

11 3	1001 9
1000110101011 4523	1001110000010100000111001 20457529
10111011010111 11991	1000100100011111100010010001 143784081
100011100010101 18197	10011101111001010011110111001 331130809
100010100101110011 141683	10010101100100000100000100110101001 20074072489
100001010101110001001 1092489	10001010111100100000100000100111101010001 1193532215121

Ternary Palindromic Squares

In ternary representation there are several infinite classes of palindromic squares possessing property P as we shall prove by forming the squares of the corresponding base 3 power series and then using arithmetic in base 3 to reduce the coefficients to normal form.

First consider the ternary integers of the form

$$3^{2k+i-2} + 3^{k+i-1} - 3^{k-1} + 1 \tag{1}$$

where $k \geqq 2$ and $1 \leqq i < k$.
Simple manipulation shows the square of (1) to be of the form

$$3^{4k+2i-4} + 2\cdot 3^{3k+2i-3} - 2\cdot 3^{3k+i-3} + 3^{2k+2i-2} + 3^{2k-2} + 2\cdot 3^{k+i-1} - 2\cdot 3^{k-1} + 1 \tag{2}$$

There are two cases to be considered.

Case 1. If $1 \leqq i \leqq k-2$ it is easy to verify that each of the exponents in the seven variable terms are distinct and ordered (in magnitude as shown. The two pairs of terms

$$2\cdot 3^{3k+2i-3} - 2\cdot 3^{3k+i-3} = 1\cdot 3^{3k+2i-3} + 2\cdot 3^{3k+2i-4} + \cdots + 2\cdot 3^{3k+i-2} + 1\cdot 3^{3k+i-3}$$

and

$$2\cdot 3^{k+i-1} - 2\cdot 3^{k-1} = 1\cdot 3^{k+i-1} + 2\cdot 3^{k+i-2} + \cdots + 2\cdot 3^{k} + 1\cdot 3^{k-1}$$

each contributes $i+1$ terms symmetric about the central term $0\cdot 3^{2k+i-2}$. $3^{2k+2i-2}$ and 3^{2k-2} each contribute a 1 symmetrically located i spaces from the center also. Therefore, expression (2) is a palindrome for all $1 \leqq i \leqq k-2$.

Case 2. If $i = k - 1$, then

$$3k + i - 3 = 2k + 2i - 2$$

and

$$k + i - 1 = 2k - 2$$

so that expression (2) becomes:

$$3^{6k-6} + 2\cdot 3^{5k-5} - 3^{4k-4} + 3^{2k-1} - 2\cdot 3^{k-1} + 1. \tag{3}$$

For $k \geqq 2$ the exponents in expression (3) are distinct and ordered as shown. The two pairs of terms

$$2\cdot 3^{5k-5} - 3^{4k-4} = 1\cdot 3^{5k-5} + 2\cdot 3^{5k-6} + \cdots + 2\cdot 3^{4k-3} + 2\cdot 3^{4k-4}$$

and

$$3^{2k-1} - 2\cdot 3^{k-1} = 2\cdot 3^{2k-2} + 2\cdot 3^{2k-3} + \cdots + 2.3^{k} + 1\cdot 3^{k-1}$$

each contributes k terms symmetric about 3^{3k-3}. Hence, expression (2) is also a palindrome for $i = k - 1$. Expression (2) is not a palindrome for $i \geqq k$. The following examples illustrate the foregoing result:

$$k = 4, \qquad i = 1$$

$$(10002001)^2 = 100110101011001$$

$$k = 4, \qquad i = 2$$

$$(100022001)^2 = 10012110001121001$$

$$k = 4, \qquad i = 3$$

$$(1000222001)^2 = 1001222000002221001$$

Using precisely the same method of proof, it is possible to show that expressions (4), (5) and (6) below also have squares which possess property P.

$$3^{k} + 2\cdot 3^{k-2} + 3^{2} + 2 \qquad k \geqq 7 \tag{4}$$

and

$$3^{2k} + 3^{k} - 1 \qquad k \geqq 2 \tag{5}$$

and

$$2\cdot 3^{2k+1} + 3^{k+1} + 3^{k} + 2 \qquad k \geqq 2 \tag{6}$$

The square of expression (4) is

$$(3^{k} + 2\cdot 3^{k-2} + 3^{2} + 2)^2 = \begin{cases} 3^{2k} + 3^{2k-1} + 3^{2k-2} + 3^{2k-3} + 3^{2k-4} + 2\cdot 3^{k+2} + 2\cdot 3^{k+1} \\ \quad + 2\cdot 3^{k} \\ 1 + 3 + 3^{2} + 3^{3} + 3^{4} + 2\cdot 3^{k-2} + 2\cdot 3^{k-1} \end{cases}$$

which is a palindrome, i.e.,

$$(10200102)^2 = 11111\ 22222\ 11111$$

$$(102000102)^2 = 11111\ 0222220\ 11111 \text{ etc.}$$

$+ 1.$ (3)

ordered as shown. The

${}^{k-3} + 2\cdot 3^{4k-4}$

$+ 1\cdot 3^{k-1}$

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(4)

(5)

(6)

$3^{2k-4} + 2\cdot 3^{k+2} + 2\cdot 3^{k+1}$

$3^{k-2} + 2\cdot 3^{k-1}$

etc.

TABLE 2. Palindromic Squares—Base 3 For $N < 575{,}571$ (Base 10)

N	N^2	N	N^2
1	1	1000000001	1000000002000000001
2	11	1000020001	1000110010100110001
11	121	1000222001	1001222000002221001
101	10201	1020000102	1111100222220011111
102	11111	1100112011	1211102000002011121
202	112211	2000000002	11000000022000000011
211	122221	2000110002	11001210111101210011
1001	1002001	2002212102	11102202211220220111
1021	1120211	2012102202	11211211111111211211
2002	11022011	10000000001	100000000020000000001
10001	100020001	10000022222	100001222212222100001
10022	101212101	10000220001	100012101000101210001
11012	122111221	10012011022	101011221101122110101
12201	1012112101	10122002102	111012100111001210111
20002	1100220011	10200000102	111110002222200011111
100001	10000200001	10200022121	111112100010001211111
100201	10111011101	11000120011	121011111111111110121
200002	110002200011	12100102001	1001121100220011211001
201102	111221122111	12200112201	1012100011221100012101
1000001	1000002000001	20000000002	1100000000220000000011
1000222	1001221221001	20020122102	1110000101001010000111
1002201	1012200022101	20200110202	1122200022002200022211
1011221	1101202021011	100000000001	10000000000200000000001
1101211	1221221221221	100000200001	10000110001010001100001
1211201	10101111110101	100002220001	10001221100000112210001
1212022	10110200201101	100010210001	10002112112221121120001
2000002	11000022000011	100120011022	10101102101010120110101
10000001	100000020000001	101000200101	10201111121212111110201
10002001	100110101011001	101122011221	11012010120202101021011
10200102	111112222211111	102000000102	11111000022222000011111
10201121	111212202212111	102020121021	11200102212021220100211
11011211	122110101011221	110111021211	12210001111011110001221
12212101	1021101221011201	200000000002	110000000002200000000011
20000002	1100000220000011	200001100002	110001210011110012100011
20011002	1101211111121011	200110011002	110121101220022101121011
100000001	10000000200000001	201221020202	112201102002200201102211
100002222	10001222122210001	1000000000001	1000000000000200000000001
100022001	10012110001121001	1000000222222	1000001222221222221000001
102000102	11111022222011111	1000002200001	1000012100100010012100001
102021021	11200222122200211	1000022220001	1000122220000000222210001
102110021	11212012021021211	1001112021201	1010010220112110220100101
110012011	12111121112111121	1001200011022	1010110010010100100110101
122010022	101212011110212101	1001202111201	1010122002222222002210101
200000002	110000002200000011		
202022112	120021201102120021		

The square of expression (5) is

$$(3^{2k} + 3^k - 1)^2 = 3^{4k} + 2 \cdot 3^{3k} - 3^{2k} - 2 \cdot 3^k + 1.$$

The difference $2 \cdot 3^{3k} - 2 \cdot 3^k = 1 \cdot 3^{3k} + 2 \cdot 3^{3k-1} + \cdots + 2 \cdot 3^{k+1} + 3^k$ which upon subtracting 3^{2k} gives a 1 in the central position to form a palindrome, i.e.,

$$(10022)^2 = 101212101$$

$$(1000222)^2 = 1001221221001 \text{ etc.}$$

The square of expression (6) requires a slightly different argument to show that it is a palindrome.

$$(2 \cdot 3^{2k+1} + 3^{k+1} + 3^k + 2)^2 = \begin{cases} 3^{4k+3} + 3^{4k+2} + 3^{3k+3} + 2 \cdot 3^{3k+2} + 3^{3k+1} + 3^{2k+3} \\ \quad + 3^{2k+2} \\ 1 + 3 + 3^k + 2 \cdot 3^{k+1} + 3^{k+2} + 3^{2k} + 3^{2k+1} \end{cases} \qquad (7)$$

Case 1. For $k \geqq 3$: the exponents in expression (7) are all distinct and in the order shown, hence (7) represents a palindromic square in these cases.

Case 2. For $k = 2$:

$$(201102)^2 = 111221122111$$

which is a palindromic square.

Note: When $k = 1$, the square is not a palindrome;

$$(2112)^2 = 20100021$$

hence expression (6) is the best possible.

$$(20011002)^2 = 1101211111121011$$

$$(2000110002)^2 = 11001210111101210011 \text{ etc.}$$

Ternary Palindromic Cubes

The only value of $1 < N \leqq 28{,}800$ for which N^3 is palindromic in base 3 is $2^3 = 22$, so that no table is shown for this case.

Base Four Palindromic Squares

In base four an even richer set of identities can be found which generate palindromic squares from non-palindromic factors—and hence palindromic integers which possess property P. Since the method of proof is identical to that used in the previous paragraph, we shall merely state the results without proofs or examples. The following infinite classes of non-palindromic polynomials all possess palindromic squares.

$$4^{2k} + 4^{k+1} + 3 \cdot 4^{k-1} + 1 \qquad k \geqq 4 \qquad (8)$$

and

$$4^{2k+1} + 4^{k+1} + 3 \cdot 4^k + 2 \cdot 4^{k-1} + 1 \qquad k \geqq 3 \qquad (9)$$

and

$$4^{2k+2} + 4^{2k+1} + 2 \cdot 4^k + 4 + 1 \qquad k \geqq 2 \qquad (10)$$

and

$$4^{2k+1} + 2\cdot 4^{k} - 1 \qquad k \geqq 1 \qquad (11)$$

and

$$4^{2k} + 4^{k} + 2\cdot 4^{k-1} + 1 \qquad k \geqq 2 \qquad (12)$$

and

$$4^{2k} + 4^{2k-1} + 4^{k} + 2\cdot 4^{k-1} + 4 + 1 \qquad k \geqq 3 \qquad (13)$$

TABLE 3. Palindromic Squares—Base 4 For $N < 566{,}449$ (Base 10)

N	N^2	N	N^2
1	1	100010001	10002000300020001
11	121	100012001	10003001010030001
101	10201	100103001	10021212021212001
111	12321	100301113	10121101210112101
1001	1002001	100332101	10200011011000201
1013	1032301	101002101	10202031013020201
1103	1223221	103231203	12010203030201021
10001	100020001	110002011	12101103030110121
10101	102030201	110003323	12102131013120121
10121	103101301	110012011	12103303230330121
10331	120202021	110033011	12121132223112121
100001	10000200001	110111211	12131212121213121
100133	10033233001	1000000001	1000000002000000001
1000001	1000002000001	1000013333	1000033332333300001
1001001	1002003002001	1000132001	1000330100010330001
1001201	1003010103001	1012211213	1031312213122131301
1010301	1021320231201	1033001203	1201210310130121021
1100211	1211130311121	1033020131	1201311320231131021
1100323	1212110112121	1100333011	1212132222222312121
1101211	1213332333121	1102013231	1221113300033111221
10000001	100000020000001		
10001333	100033323330001		
10013201	100331000133001		
10031113	101231222132101		
100000001	10000000200000001		

TABLE 4. Palindromic Cubes—Base 4 For $N < 28{,}800$ (Base 10)

N	N^3
1	1
11	1331
101	1030301
1001	1003003001
10001	1000300030001
100001	1000030000300001
1000001	1000003000003000001
10000001	1000000300000030000001

TABLE 5. Palindromic Squares—Base 5 For $N < 692{,}021$ (Base 10)

N	N^2
2	4
11	121
101	10201
111	12321
231	114411
1001	1002001
1111	1234321
10001	100020001
10101	102030201
11011	121242121
11204	131141131
100001	10000200001
101101	10221412201
110011	12102420121
242204	131441144131
1000001	1000002000001
1001001	1002003002001
1010101	1020304030201
1042214	1143442443411
1100011	1210024200121
2020303	4133144413314
2043122	4342230322434
2443304	13431400413431
10000001	100000020000001
10011001	100220141022001
10100101	102012040210201
11000011	121000242000121
100000001	10000000200000001
100010001	10002000300020001
100101001	10020210401202001
101000101	10201020402010201
110000011	12100002420000121
111103411	12400140104100421

TABLE 6. Palindromic Cubes—Base 5 For $N \leqq 65{,}000$ (Base 10)

N	N^3
11	1331
101	1030301
1001	1003003001
10001	1000300030001
100001	1000030000300001
1000001	1000003000003000001

Conclusion

Based on the several results given here and in spite of our failure to exhibit even a single infinite class in either binary or decimal notation, we conjecture that, independent of the base, there are infinitely many palindromic squares possessing property P. Unfortunately we have been unable to construct an infinite class of cubics with property P irrespective of the base, so that we cannot even conjecture about the behavior to be expected for higher powers with respect to property P.

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