COMP20003
Algorithms and Data Structures

Recurrences

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Semester 2

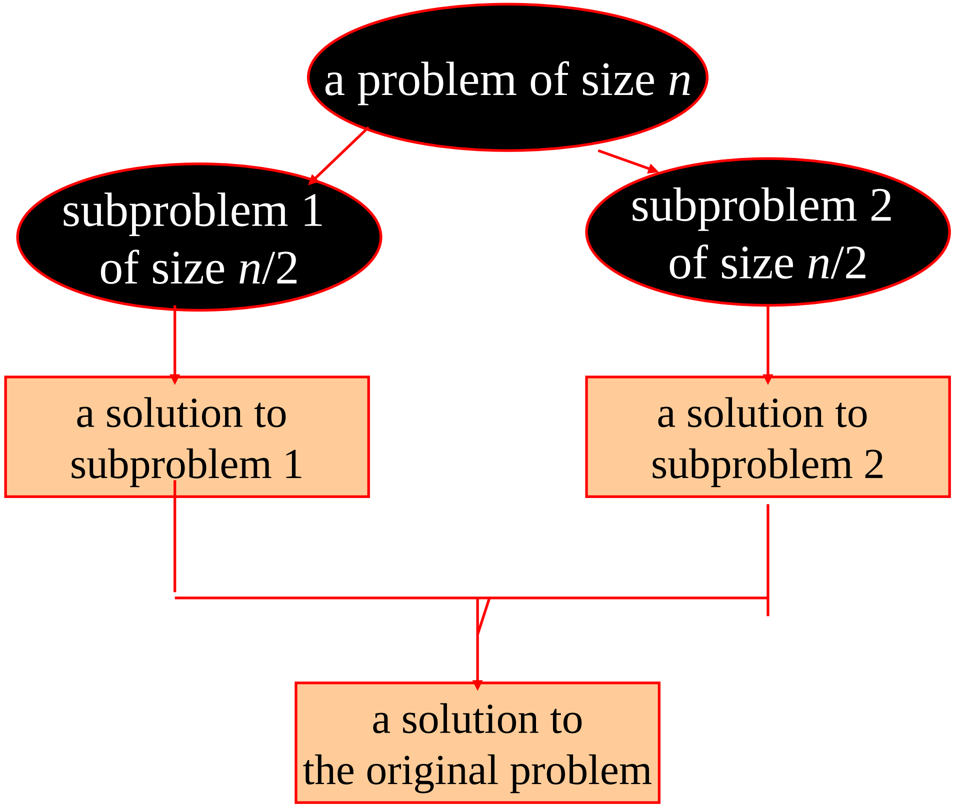
Divide and Conquer Algorithms

Mergesort and quicksort are instances of divide-and-conquer algorithms:

- **Solve the problem by continually dividing into smaller problems**

Other examples?

Split-solve-join approach:



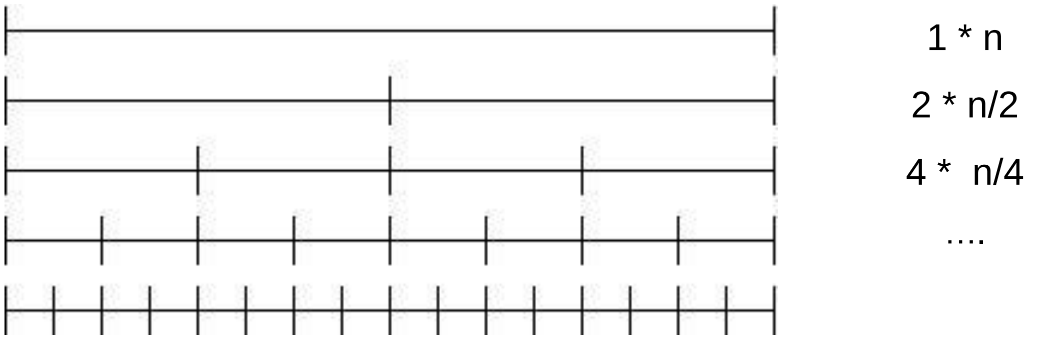
For problems where the output is a transformation of the input, need to:

- process both sub-problems, and
- join the sub-solutions after processing

Recurrence for divide and conquer sorting algorithms

One pass through the data reduces problem size by half.
Process both halves:

- Operation (process) takes constant time *c*
- Base case takes time *d*



Recurrence for divide and conquer sorting algorithms

One pass through the data reduces problem size by half. Process both halves

- Operation takes constant time *c*
- Base case takes time *d*

$$T(1) = d$$

$$T(n) = 2T(n/2) + nc$$

$$= nc + 2cn/2 + 4cn/4 ... + n/2*2c + nd$$

$$= c(n-1)\log n + nd$$

Divide and Conquer: Recurrences to Master Theorem

- Most common case:

$$T(n) = 2T(n/2) + n$$

- General case:

$$T(n) = aT(n/b) + f(n)$$

$$f(n) \in \Theta(n^d)$$

- Most common case:

$$T(n) = 2T(n/2) + n$$

$$a=2,\ b=2,\ d=1$$

Master Theorem for Divide and Conquer

- $T(n) = aT(n/b) + f(n)$

  $f(n) \in \Theta(n^d)$
- *T(n) closed form varies, depending on whether:*
  - $d > \log_b a \qquad T(n) \in \Theta(n^d)$
  - $d = \log_b a \qquad T(n) \in \Theta(n^d \log n)$
  - $d < \log_b a \qquad T(n) \in \Theta(n^{\log_b a})$

Master Theorem for Divide and Conquer

- $T(n) = aT(n/b) + f(n)$, where
  $a \geq 1$, $b > 1$, n^d *asymptotically positive*
- *$T(n)$ closed form varies, depending on whether:*
  - $d > \log_b a$ $\quad T(n) \in \Theta(n^d)$
  - $d = \log_b a$ $\quad T(n) \in \Theta(n^d \log n)$
  - $d < \log_b a$ $\quad T(n) \in \Theta(n^{\log_b a})$

Where do Θ() solutions to the Master Theorem come from?

$T(n) = aT(n/b) + f(n),\ f(n) \in \Theta(n^d)$

Size of subproblems decreases by b

- So base case reached after $\log_b n$ levels
- Recursion tree $\log_b n$ levels

Branch factor is a

- At kth level, have a^k subproblems

At level k, total work is then

- $a^k * O(n/b^k)^d$
- *(#subproblems * cost of solving one)*

Where do Θ() solutions to the Master Theorem come from?

$T(n) = aT(n/b) + f(n),\ f(n) \in \Theta(n^d)$

At level *k*, total work is then

- $a^k * O(n/b^k)^d = O(n^d) * (a/b^d)^k$

As *k* (levels) goes from *0 to* $log_b n$, this is a geometric series, with ratio a/b^d

$\Sigma\ O(n^d) * (a/b^d)^k$

Where do Θ() solutions to the Master Theorem come from?

$T(n) = aT(n/b) + f(n),\ f(n) \in \Theta(n^d)$

Geometric series: $O(n^d) * (a/b^d)^k$

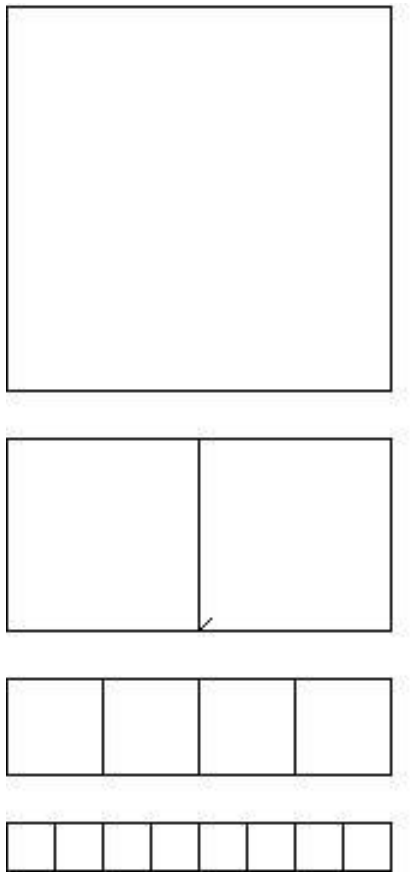
- *as* k *goes from* $0 \rightarrow \log_b n$

Case 1: *ratio* $a/b^d < 1$ *or* $d > \log_b a$

- $(a/b^d)^k$ *gets smaller as k goes from* $1 \rightarrow \log n$
- a/b^d First term is the largest, and is <1
- $O(n^d)$

Example for $a/b^d < 1$

$$T(n) = 2T(n/2) + n^2$$



Where do the solutions to the Master Theorem come from?

$T(n) = aT(n/b) + f(n),\ f(n) \in \Theta(n^d)$

Geometric series: $O(n^d) * (a/b^d)^k$

- *as* k *goes from* $0 \rightarrow \log_b n$

Case 2: *ratio* $a/b^d = 1$ *or* $d = \log_b a$

- Series is $O(n^d) + O(n^d) + \ldots$
  - For $\log_b n$ levels
- Sum = $O(n^d \log n)$

Example for most common case $a/b^d = 1$

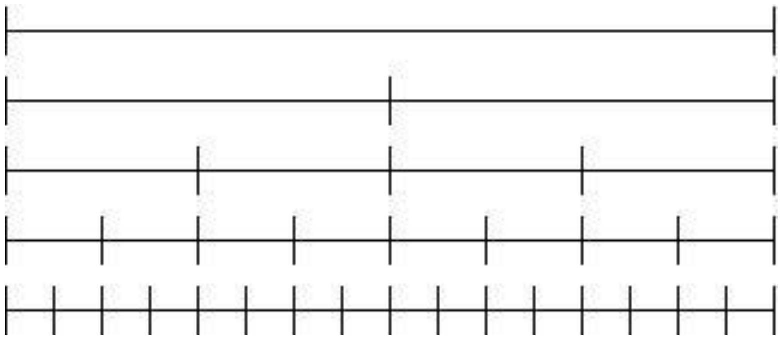
$$T(n) = 2T(n/2) + n$$

$$T(n) = 2(2T(n/4) + n/2) + n$$

$$= 4T(n/4) + n + n$$

$$= 8T(n/8) + n + n + n$$

....



1 * n

2 * n/2

4 * n/4

....

Where do Θ() solutions to the Master Theorem come from?

$T(n) = aT(n/b) + f(n),\ f(n) \in \Theta(n^d)$

Geometric series: $O(n^d) * (a/b^d)^k$

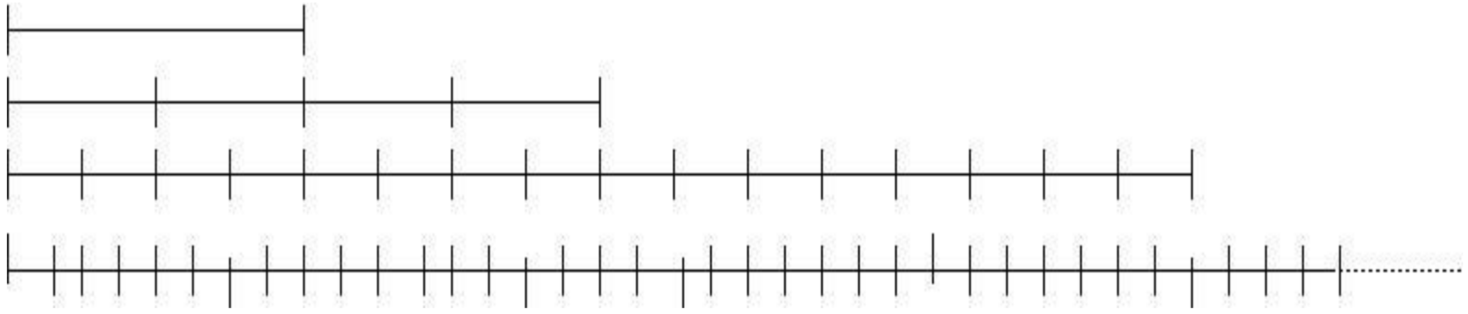
- *as* k *goes from* $0 \rightarrow \log_b n$

Case 3: ratio $a/b^d > 1$ *or* $d < \log_b a$

- $a/b^d > 1 \rightarrow$ series is *increasing*
- Sum dominated by last term:
  - $O(n^d)(a/b^d)^{\log(b)n} = n^{\log(b)a}$

Example for $a/b^d > 1$

T(n) = 4T(n/2) + n



Some pointers...

For more on geometric series, and calculation of closed form, see:

http://www.youtube.com/watch?v=JJZ-shHiayU

4 minute tutorial from Rose-Hulman Institute of Technology