COMP20003
Algorithms and Data Structures

Priority Queues

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Queues

A queue Q has the following operations:

- `makeQ();`
- `enQ(Q,item);`
- `deQ(Q,first);`
- `emptyQ(Q);`

Priority Queues

A priority queue PQ has the operations:

- `makePQ();`
- `enQ(PQ,item);`
- `deletemax(PQ);   /* or deletemin() */`
- `emptyPQ(PQ);`
- `changeWeight(PQ,item);`

Also `delete(PQ,item)`, `replace(PQ,item)`.

Simple implementations of priority queue

Unsorted array:

- Construct: O(n)
- Get highest priority: O(n)

Sorted array:

- Construct:O(n^2)
- Get highest priority: O(1)

Simple implementations of priority queue

Unsorted list:

- Construct: O(n)
- Get highest priority: O(n)

Sorted list:

- Construct: O(n^2)
- Get highest priority: O(1)

A better implementation of priority queue: The Heap

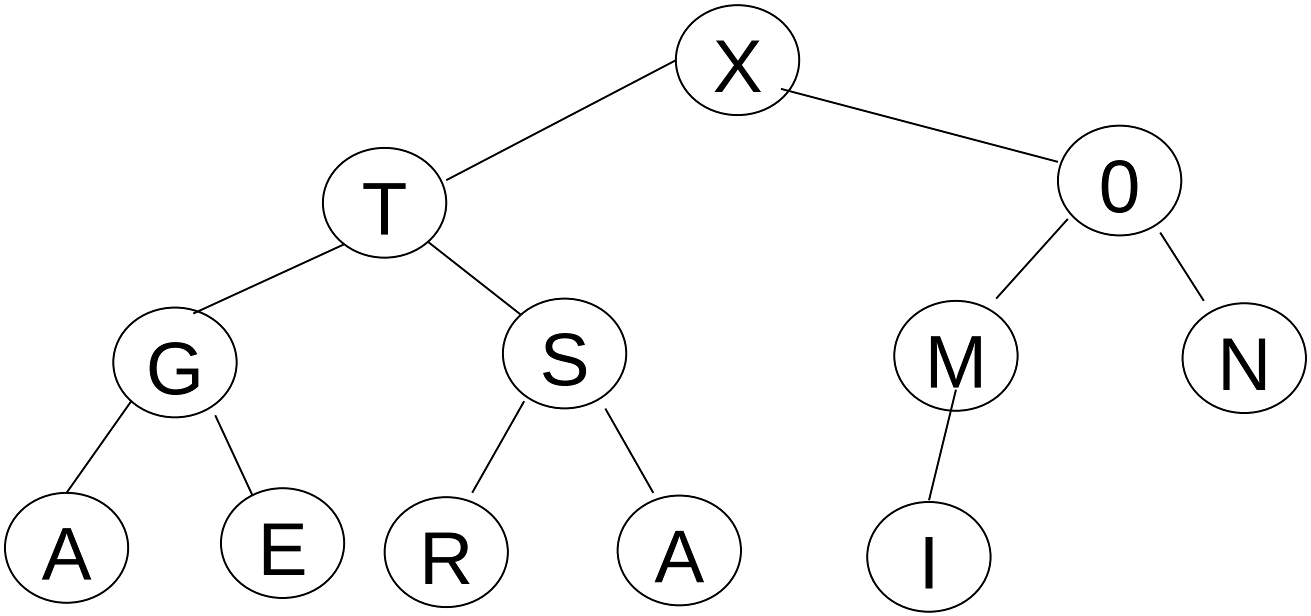
Heap data structure:

- A complete tree
  - *n.b.* a complete tree is… Look at slides about binary trees
- Every node satisfies the "heap condition":
  - **`parent->key >= child->key`**, for all children
  - Root is therefore …?

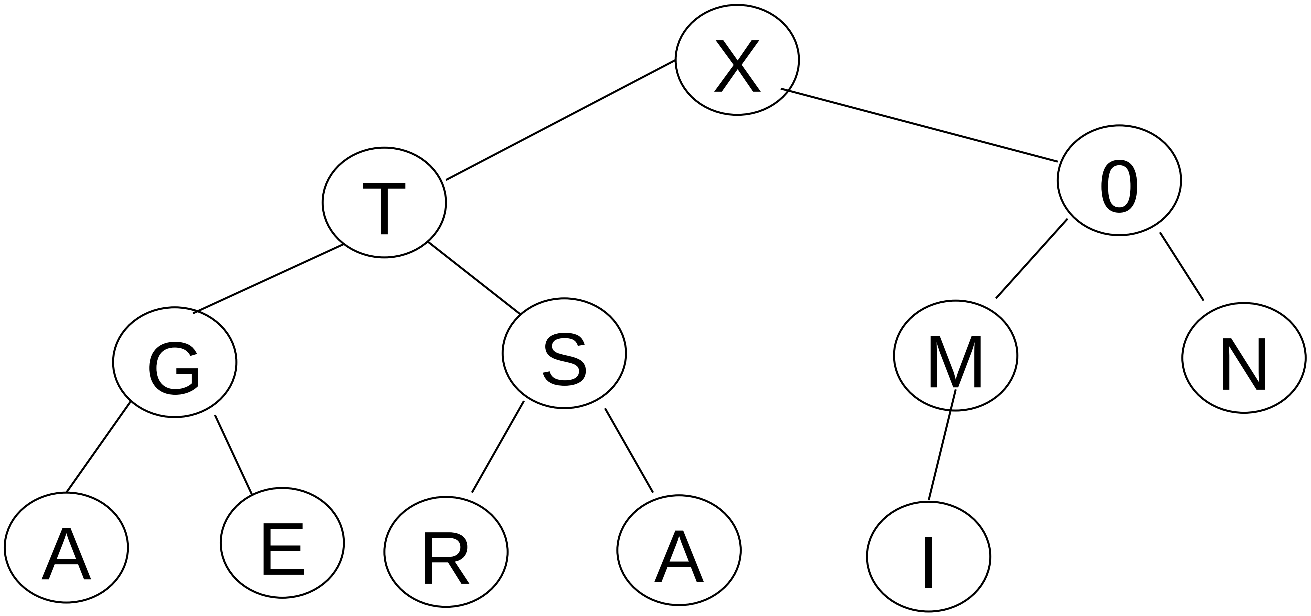
Complete tree represented as an array:

- *n.b.* we first look at binary heaps, but
  - A heap need not be binary

Example heap



Example heap



X	T	O	G	S	M	N	A	E	R	A	I

Example heap

X	T	O	G	S	M	N	A	E	R	A	I

Example heap

The arithmetic is easier if we use A[1] as the root.

Parent:
A[i], E.g. i =1

Children of A[i]:
i = parent*2, parent*2 + 1
E.g. i = 2,3

	X	T	O	G	S	M	N	A	E	R	A	I

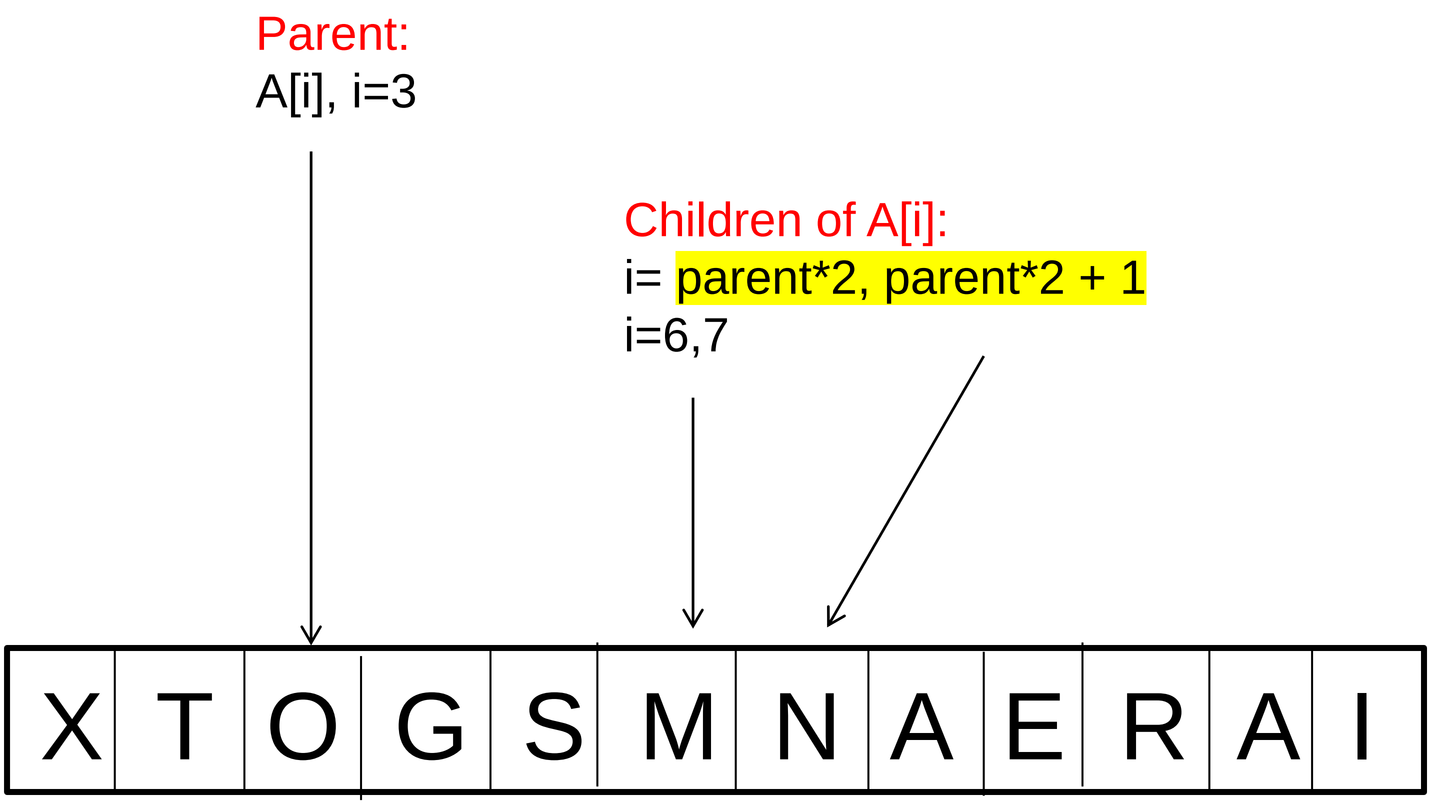
Example heap

Parent:
A[i], i=2

Children of A[i]:
i= parent*2, parent*2 + 1
i=4,5

X	T	O	G	S	M	N	A	E	R	A	I

Example heap



Example heap

Recall the heap condition:

- **parent->key >= child->key**, for all children

For array representation of Binary Heap, this means:

- A[i] >= A[2*i] && A[i] >= A[2*i+1]

X	T	O	G	S	M	N	A	E	R	A	I

`deletemax()`

1) Return highest priority item:

- Return root

2) Fix heap:

- Put last item into root position
- Reduce size of PQ by one
- Fix heap condition for root: **`downheap()`**

deletemax(): Excercise

Return highest priority item:

- Return root.

Fix heap:

- Put last item into root position.
- Reduce size of PQ by one.
- Fix heap condition for root: **`downheap()`**.

X	T	O	G	S	M	N	A	E	R	A	I	← Keys
1	2	3	4	5	6	7	8	9	10	11	12	← Array Index

downheap()

```
downheap(int[] PQ, int k)
{
    int j,v;
    v = PQ[k];              /* value, or priority */
    while( k <= n/2 )       /* A[k] has children */
    {
      /* point to children*/
      j= k*2;
      /* j set to highest child*/
            if(j<n && PQ[j]< PQ[j+1]) j++;
            if (v>= PQ[j]) break;  /* check heap OK */
       PQ[k] = PQ[j]; k = j;       /* swap and continue */
    }
    /* final position of original A[k] value*/
    PQ[k] = v;
```

I	T	O	G	S	M	N	A	E	R	A	← Keys
1	2	3	4	5	6	7	8	9	10	11	← Array Index

Complete code: https://jdoodle.com/a/70I

deletemax()

For a maxheap of integers:

```
int deletemax(int[] PQ)
{
        int v = PQ[1];
        PQ[1] = PQ[n--];
        downheap(1);
        return(v);
}
```

Exercise: construct a maxheap of pointers to struct; return a pointer to the maximum priority item

- LINK: https://jdoodle.com/a/70I

Fixing heap with upheap()

Inserting a new item into an *already-formed heap:*

```
void upheap(int* PQ, int k)
  {
    int v;
    v = PQ[k];
    PQ[0] = INT_MAX; /* sentinel, limits.h */
    while(PQ[k/2] <= v){ /* note integer arith */
        PQ[k] = PQ[k/2];
        k = k/2;
    }
    PQ[k] = v;
  }
```

`uphead()` *vs.* `downheap()`

Add new item in last place in heap:

- **`upheap()`**
- $O(\log n)$

Replace root in heap:

- **`downheap()`**
- $O(\log n)$

Heapsort

Heap suggests a method for sorting:

- Construct heap
- Swap root (max) with last element
- Remove last element from further consideration, *i.e.* decrease size of heap by 1
- Fix heap using….

  … **`downheap()`**
- Repeat until finished

Heapsort: Excercise

Heap suggests a method for sorting:

- Construct heap.
- Swap root (max) with last element.
- Remove last element from further consideration, *i.e.* decrease size of heap by 1.
- Fix heap using….

  … **`downheap()`**

- Repeat until finished.

X	T	O	G	S	M	N	A	E	R	A	I	← Keys
1	2	3	4	5	6	7	8	9	10	11	12	← Array Index

Cost of heapsort

Construct heap O(n) ?

Sucessively move max to end and fix:

- *n* * **`deletemax()`** :
- *n * O(logn) –> O(n log n)*

Making a heap: two strategies

Strategy 1:

- Insert items one-by-one into the array
- `upheap()` as each new item is inserted

Insert n items into heap of size n:

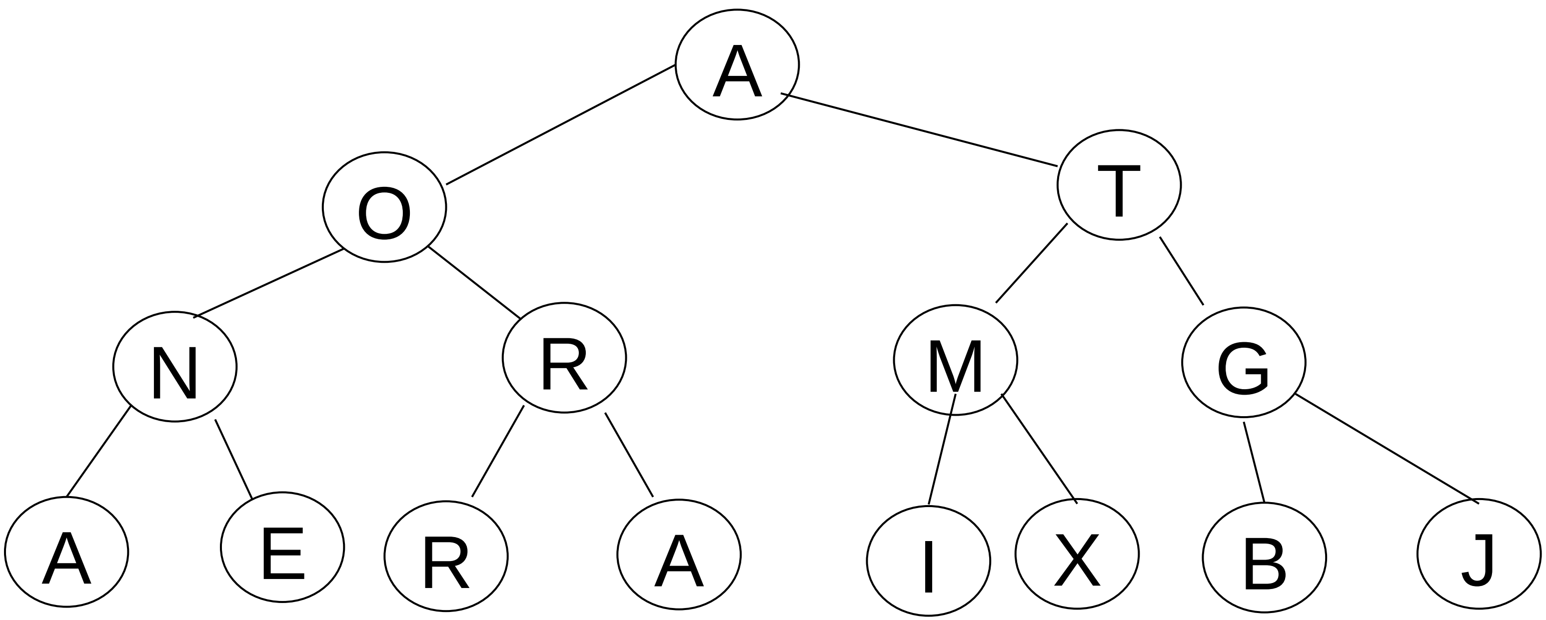
- Each insertion: $O(\log n)$
- How many insertions? $O(n)$
- Overall: $O(n \log n)$

Making a heap: two strategies

Strategy 2:

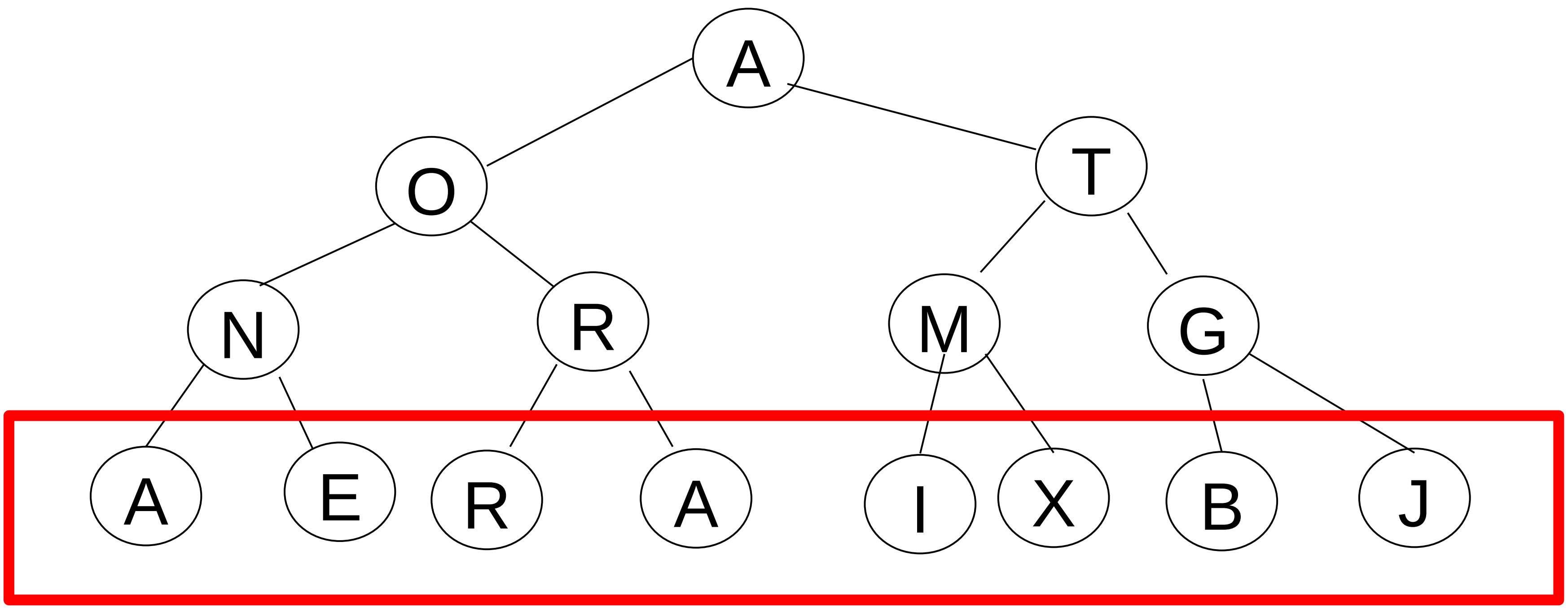
- Insert items into unordered array
- Once all items are in, **`downheap()`** for each subheap with roots from **`A[n/2]`** to **`A[1]`**

Strategy 2: How does it work?



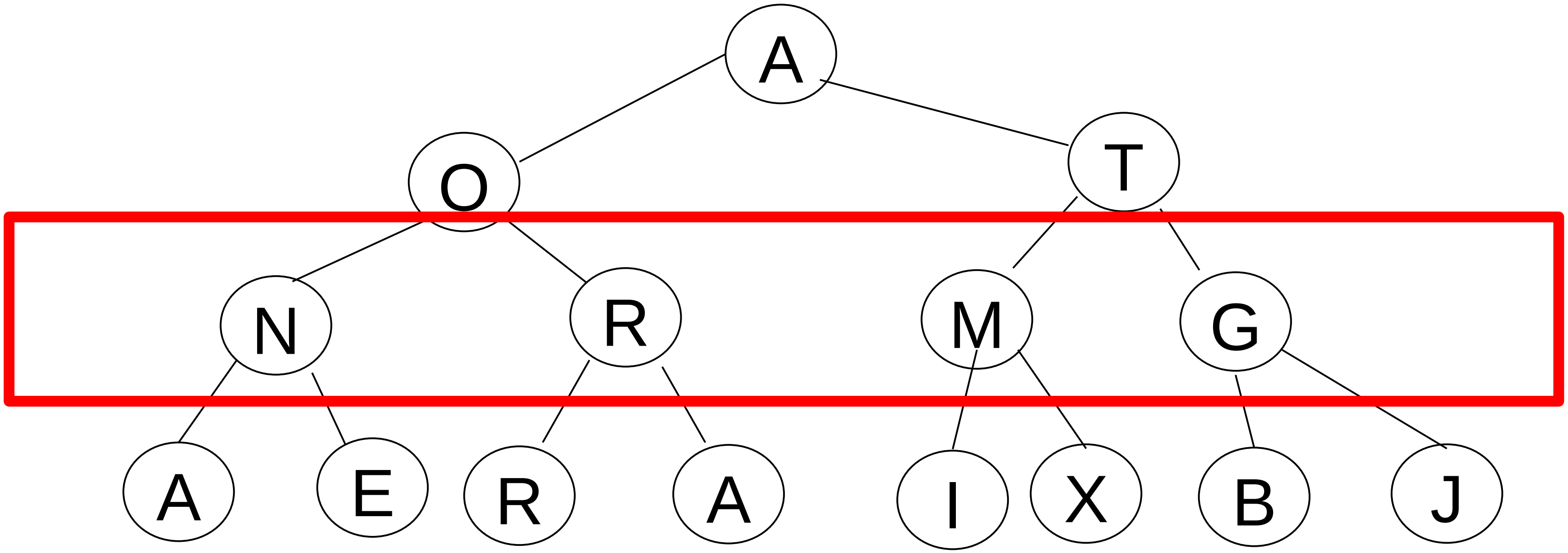
A	O	T	N	R	M	G	A	E	R	A	I	X	B	J

Strategy 2: How does it work?
downheap ( ) bottom row



A	O	T	N	R	M	G	A	E	R	A	I	X	B	J

Strategy 2: How does it work?
downheap ( ) next row up



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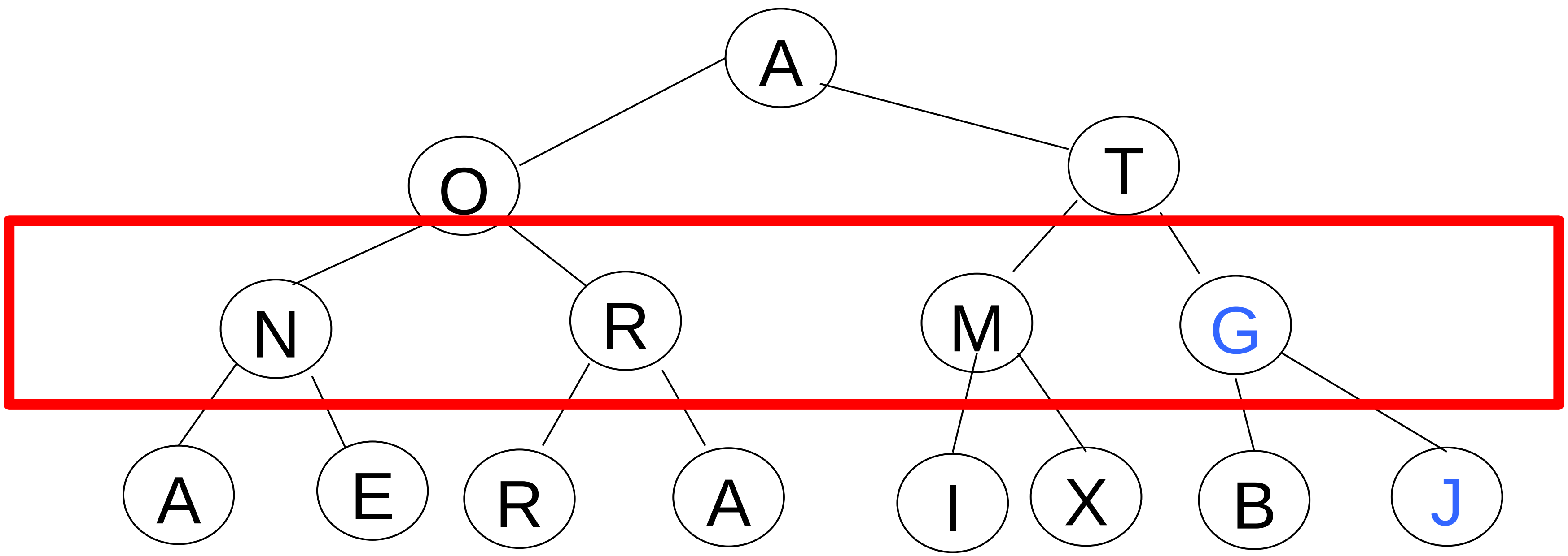
Strategy 2: How does it work?
downheap() next row up



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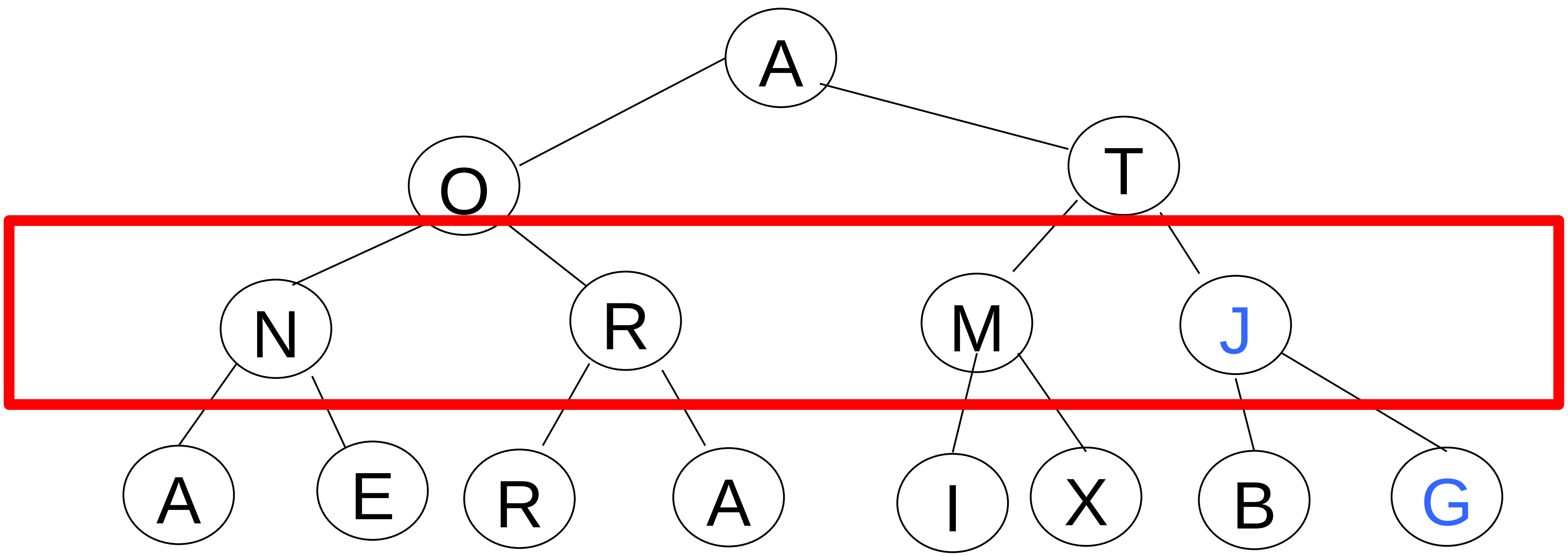

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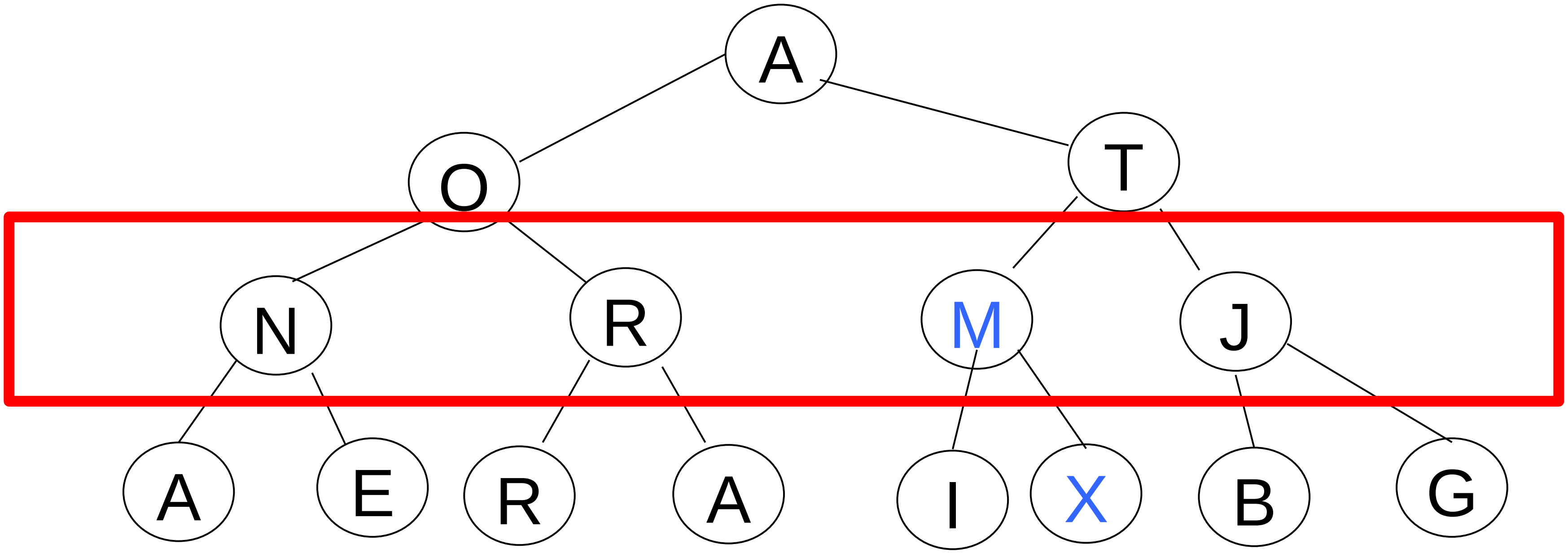

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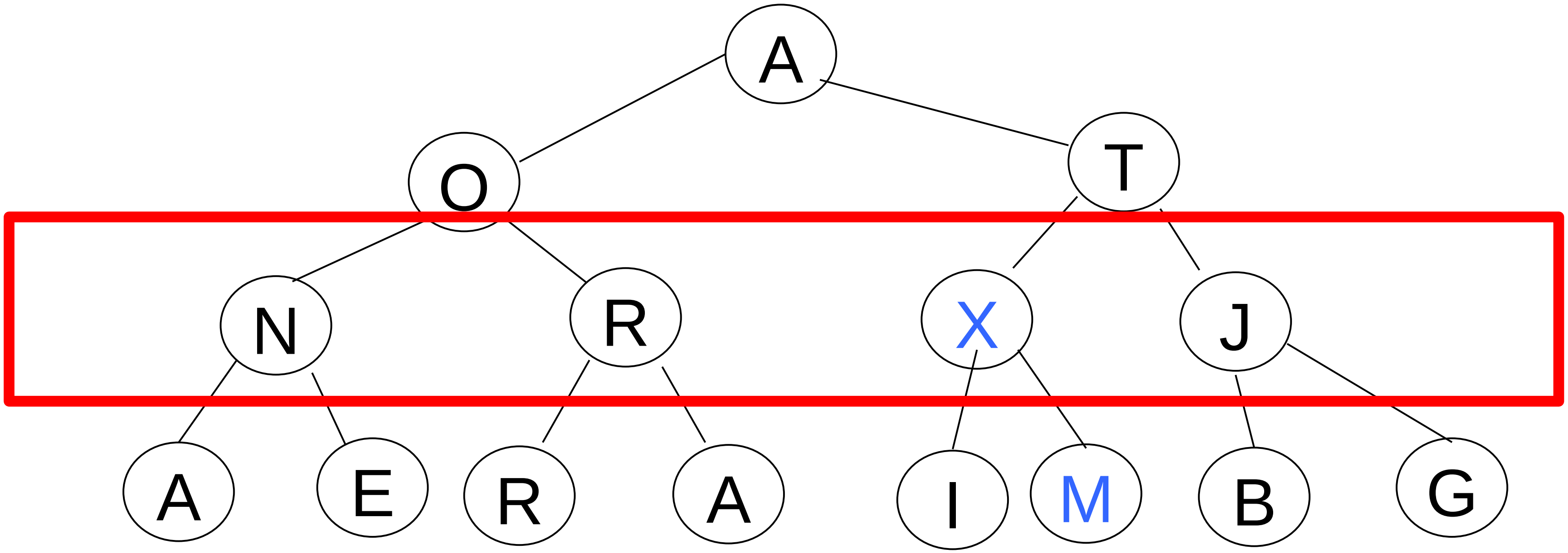

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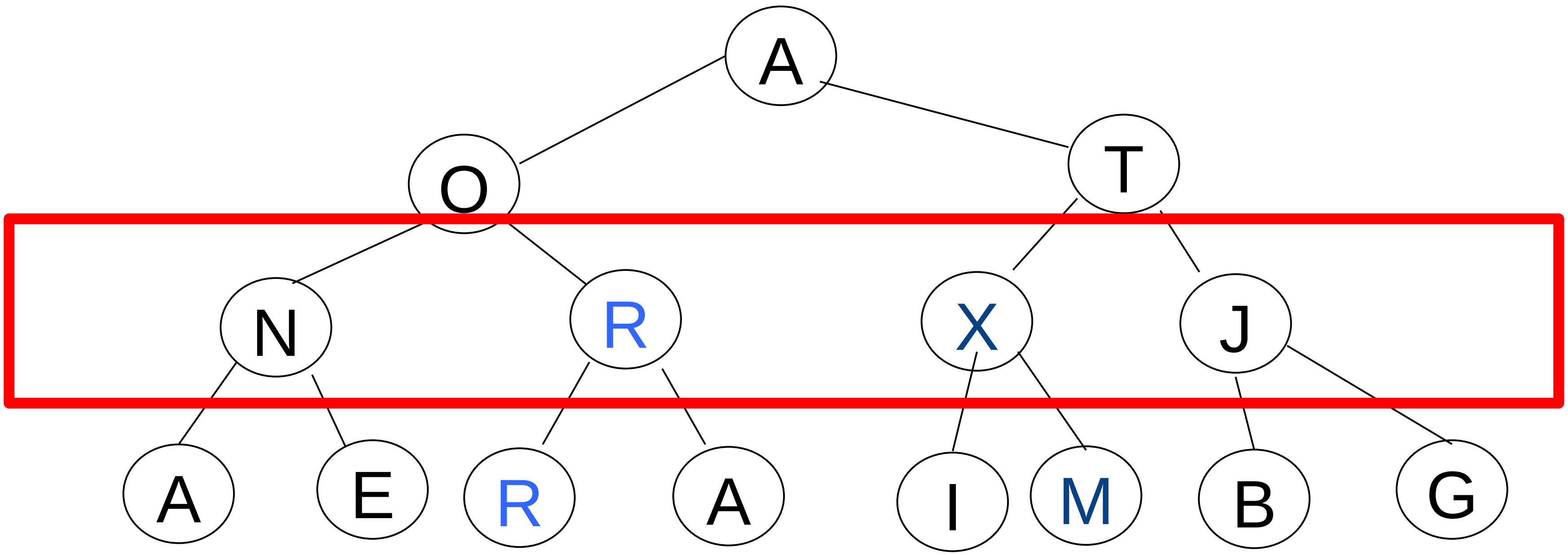
Strategy 2: How does it work?
downheap() next row up



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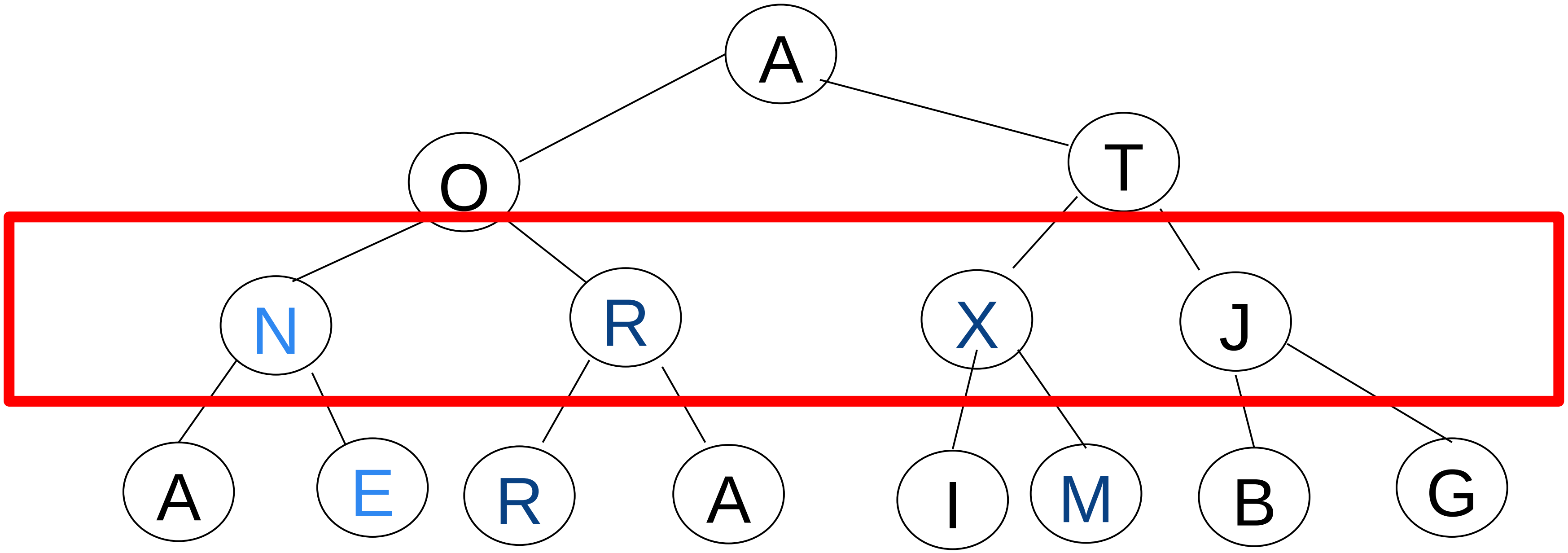

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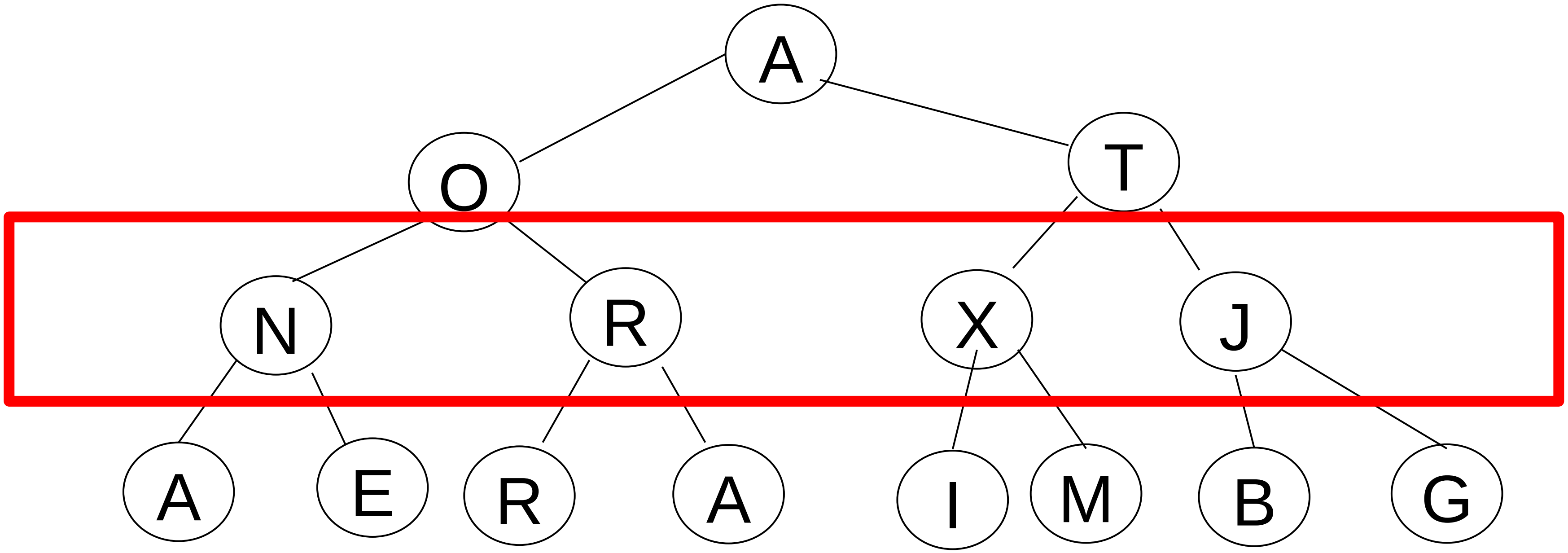
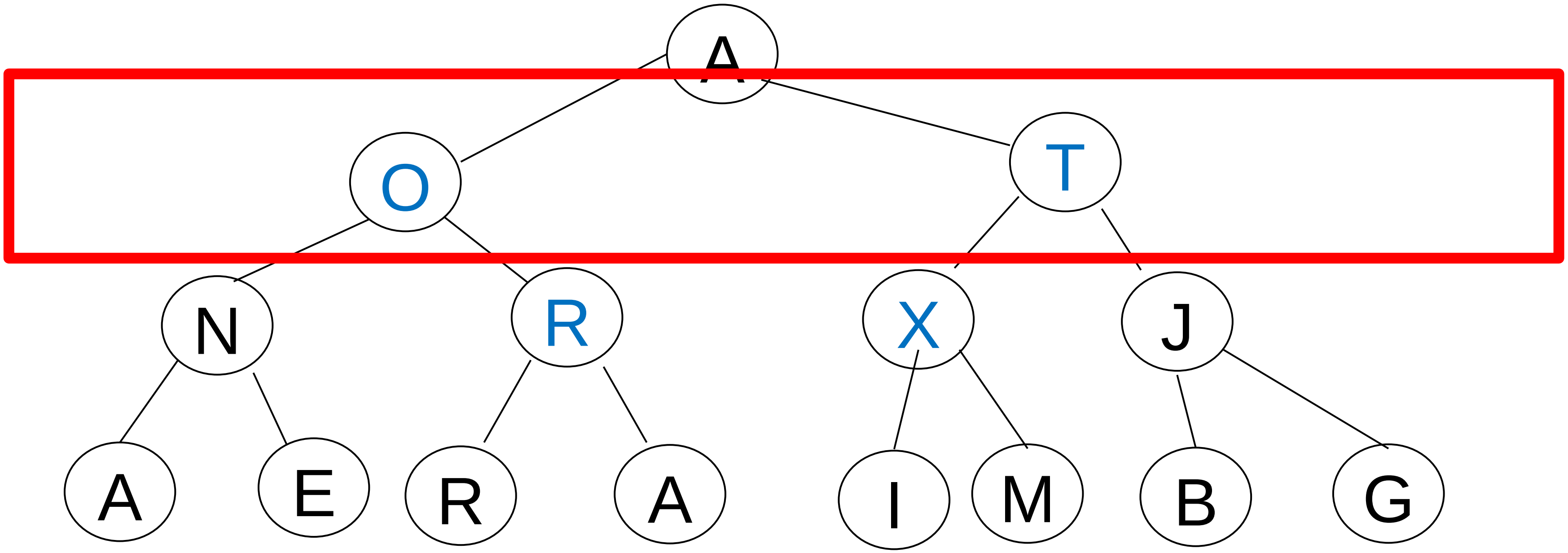


A	O	T	N	R	X	J	A	E	R	A	I	M	B	G

Strategy 2: How does it work?
downheap() next row up



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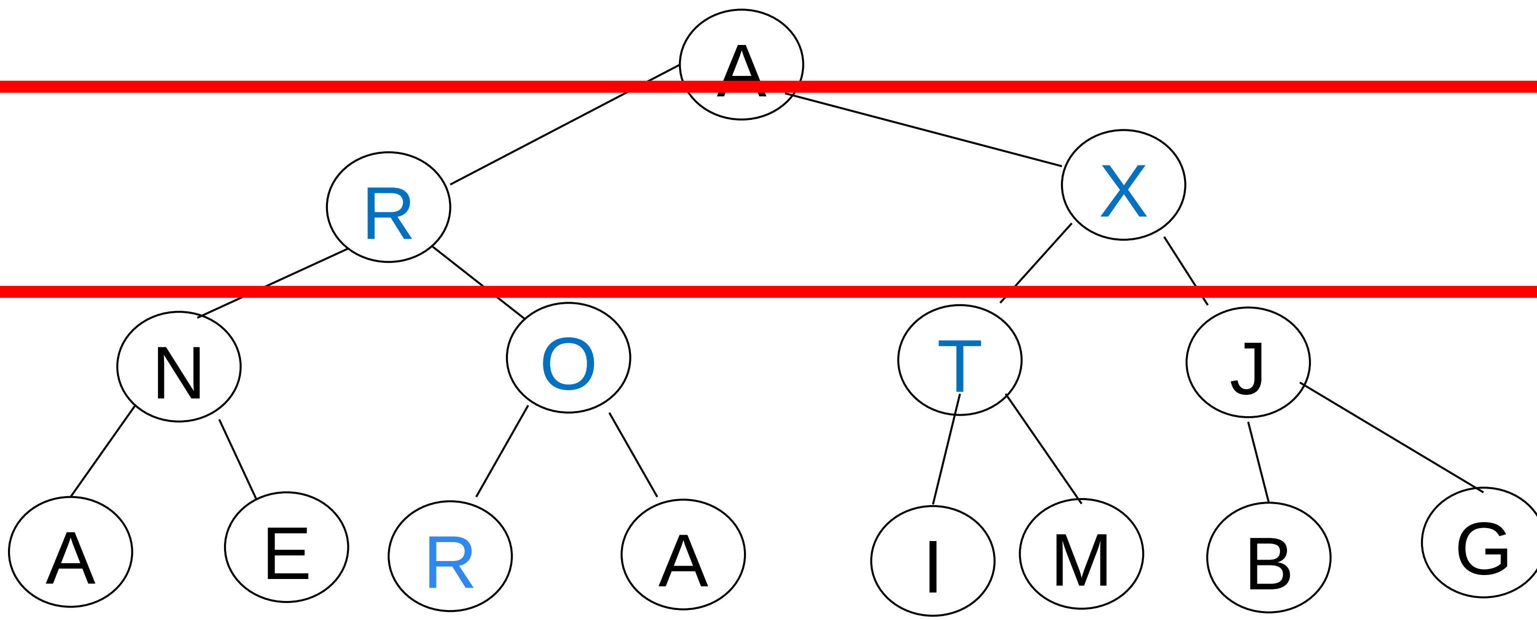

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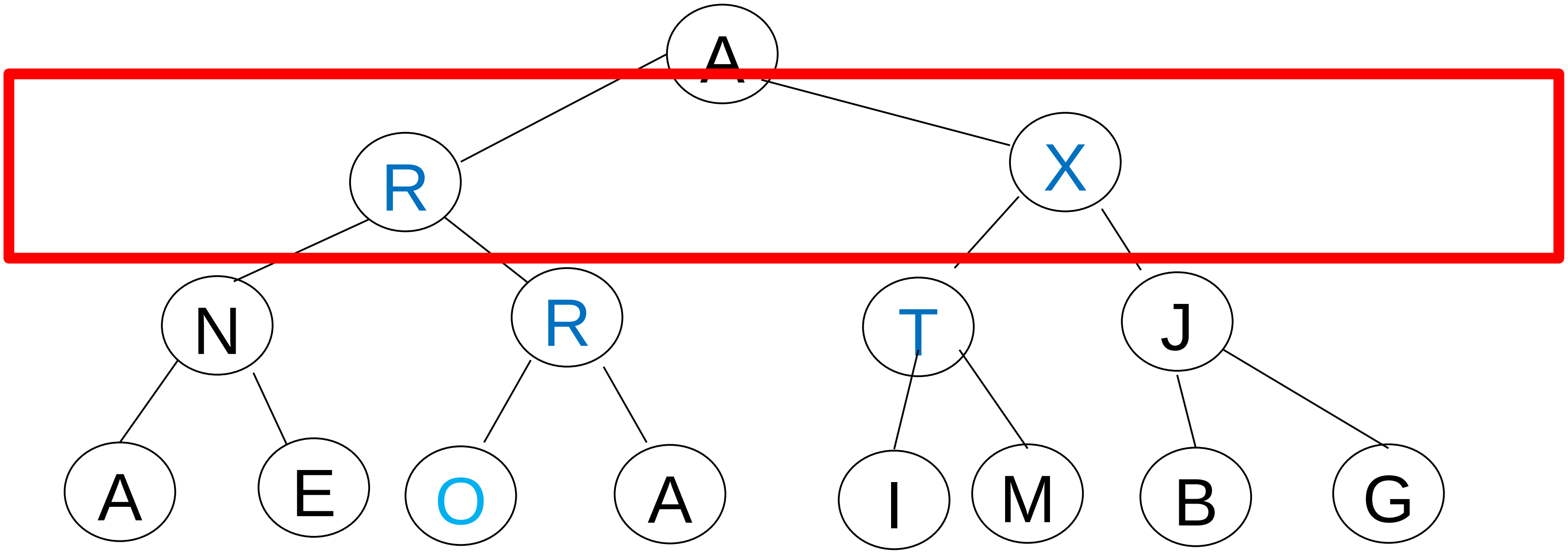
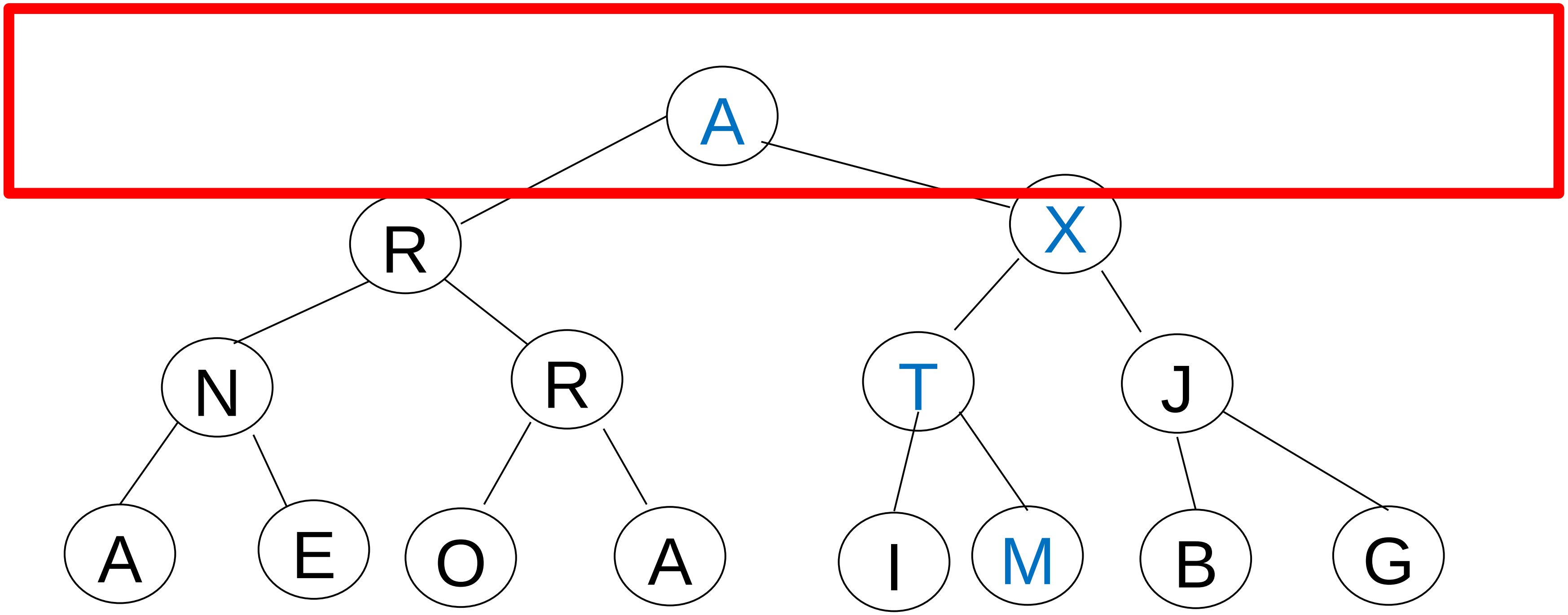
Strategy 2: How does it work?
downheap() next row up



A	R	X	N	R	T	J	A	E	O	A	I	M	B	G

Strategy 2: How does it work? downheap() next row up



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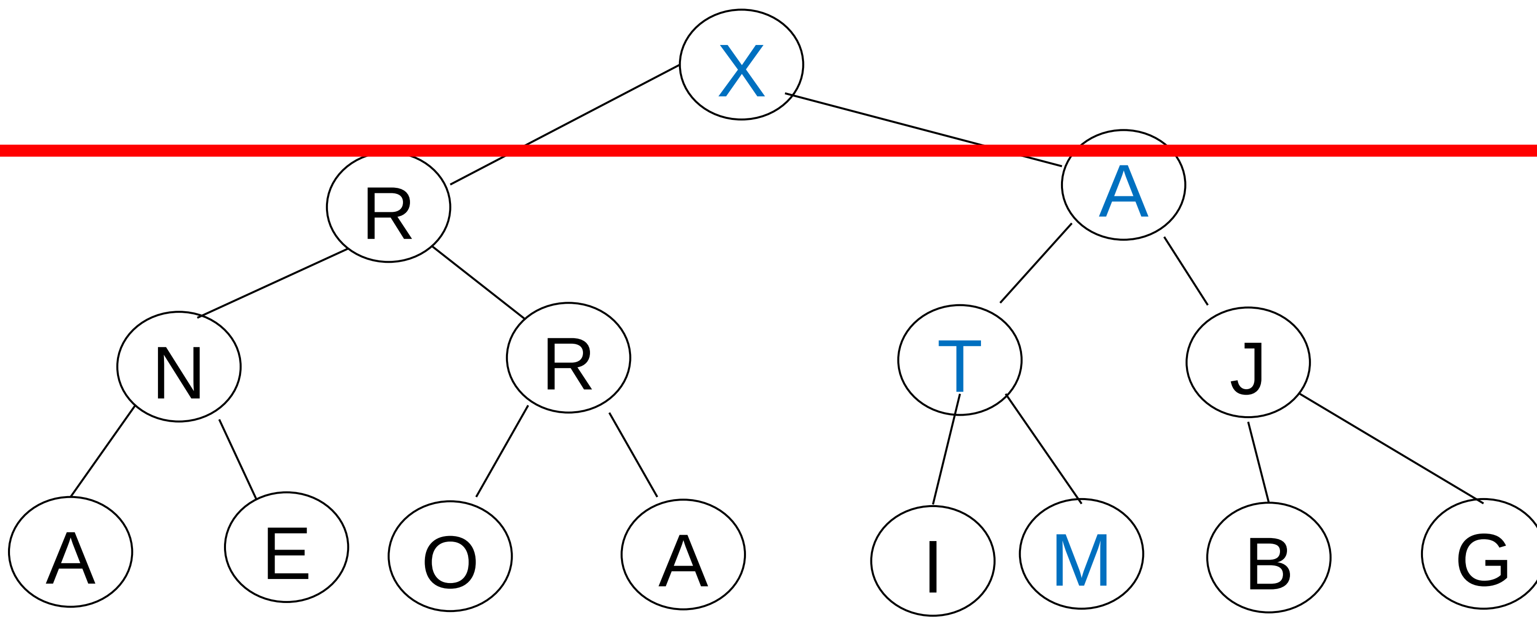
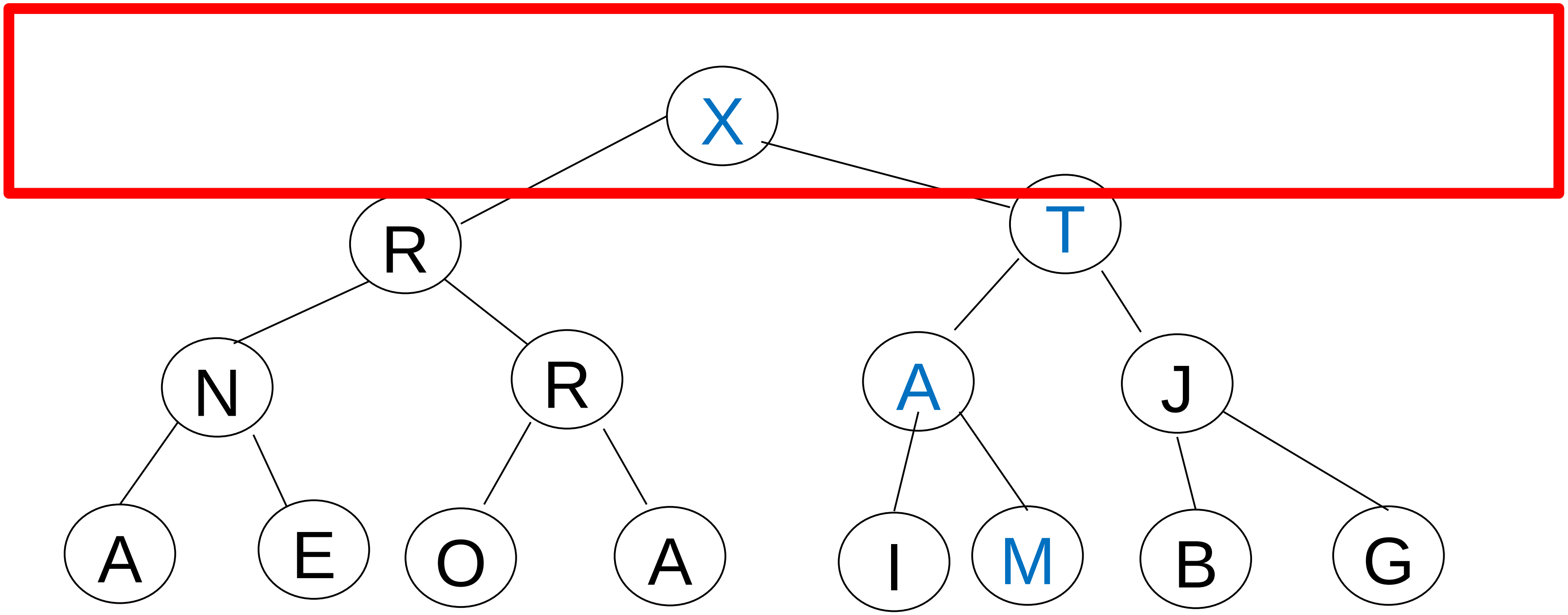
Strategy 2: How does it work?
downheap() next row up



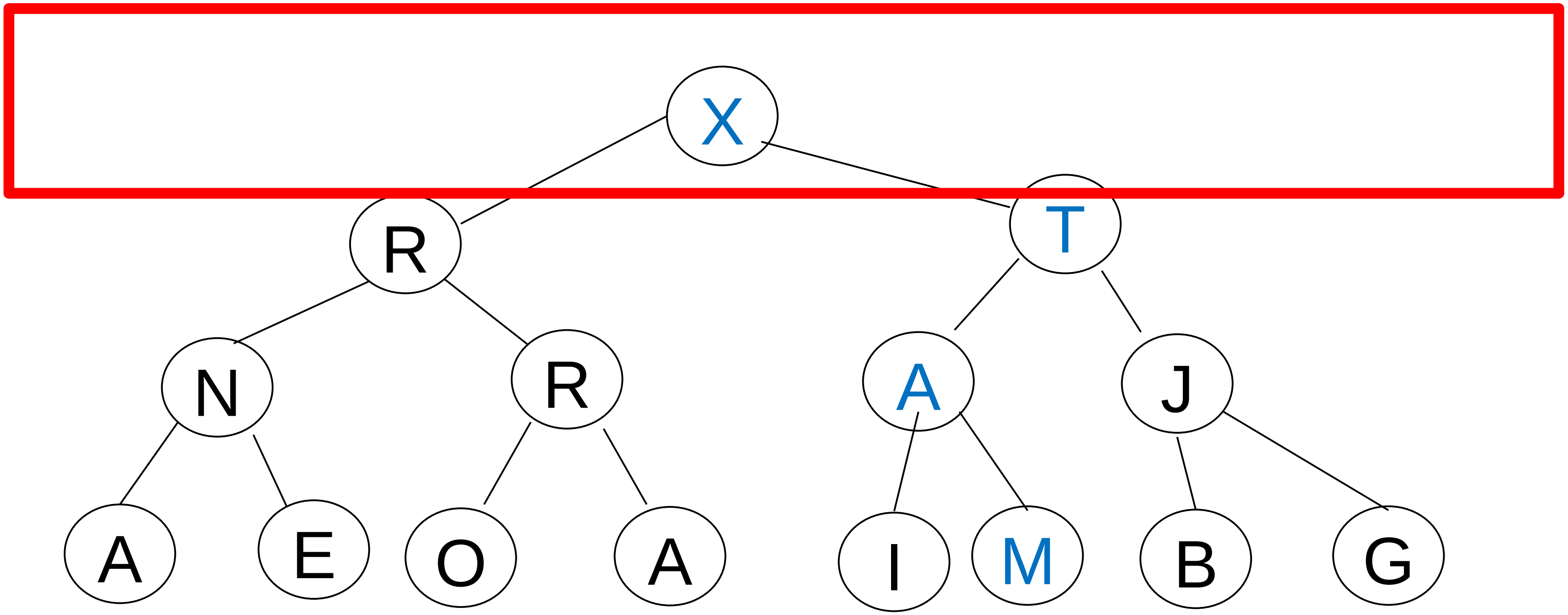
X	R	A	N	O	T	J	A	E	R	A	I	M	B	G

Strategy 2: How does it work?
downheap() next row up



X	R	T	N	O	A	J	A	E	R	A	I	M	B	G

Strategy 2: How does it work?
downheap ( ) next row up



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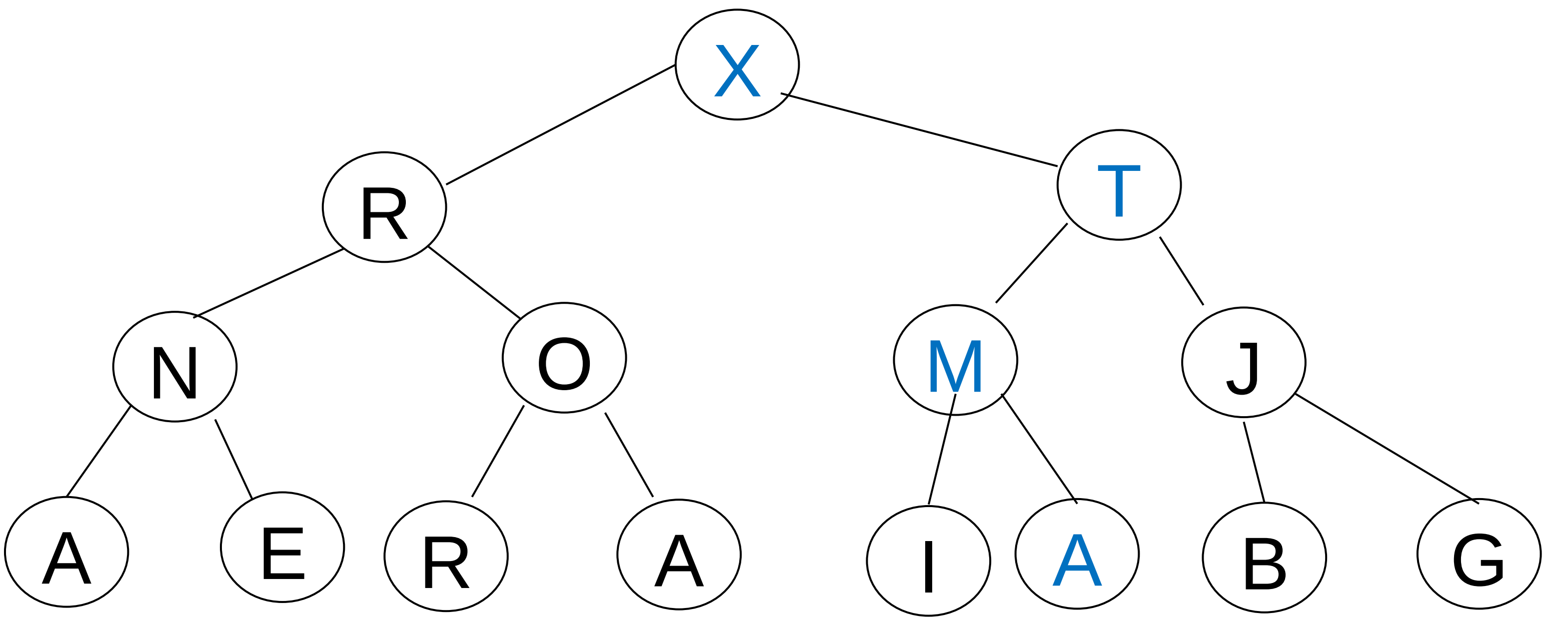
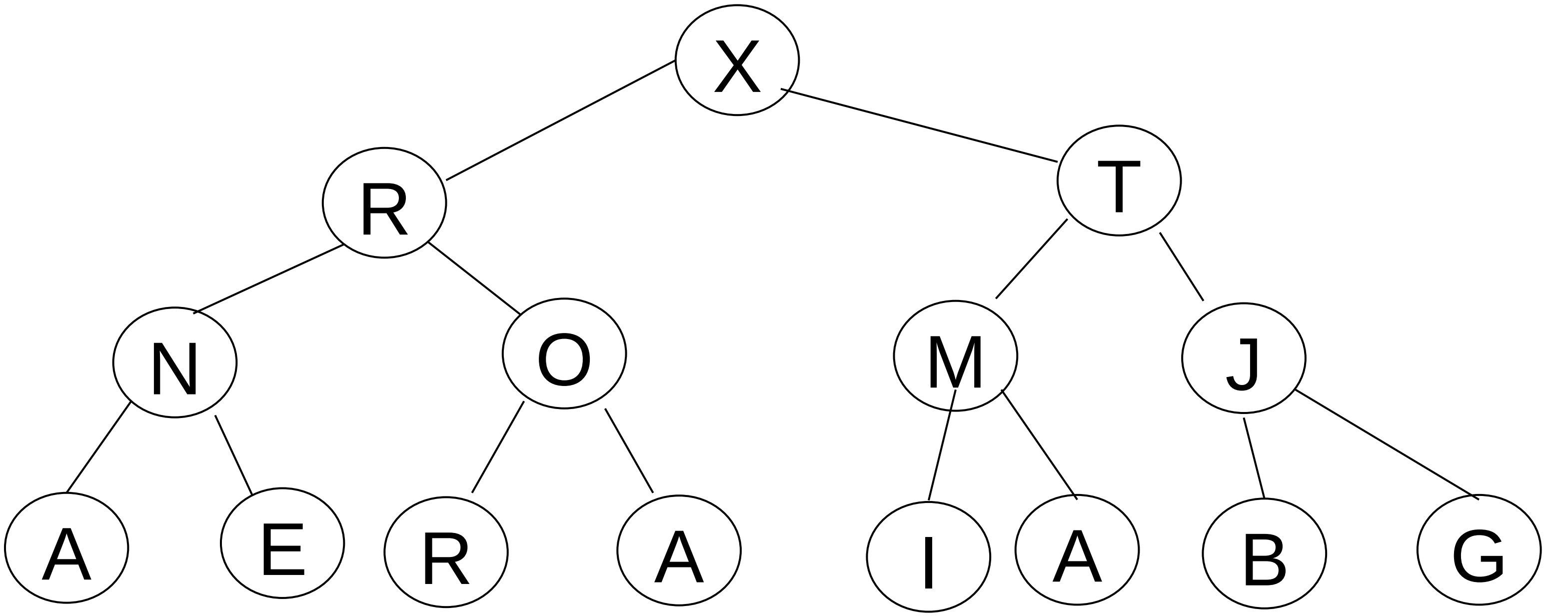
Strategy 2: How does it work?
downheap ( ) next row up



X	R	T	N	O	M	J	A	E	R	A	I	A	B	G

Strategy 2: Finished heap after bottom-up heap construction



X	R	T	N	O	M	J	A	E	R	A	I	A	B	G

Strategy 2: Analysis

Strategy 2:

- Insert items into unordered array
- Once all items are in, **`downheap()`** for each subheap with roots from **`A[n/2]`** to **`A[1]`**

Insert n items into heap of size n:

- Start with Insert into unordered array: $O(n)$
- Then **`downheap( )`** subheaps from **`A[n/2]`** to **`A[1]`**

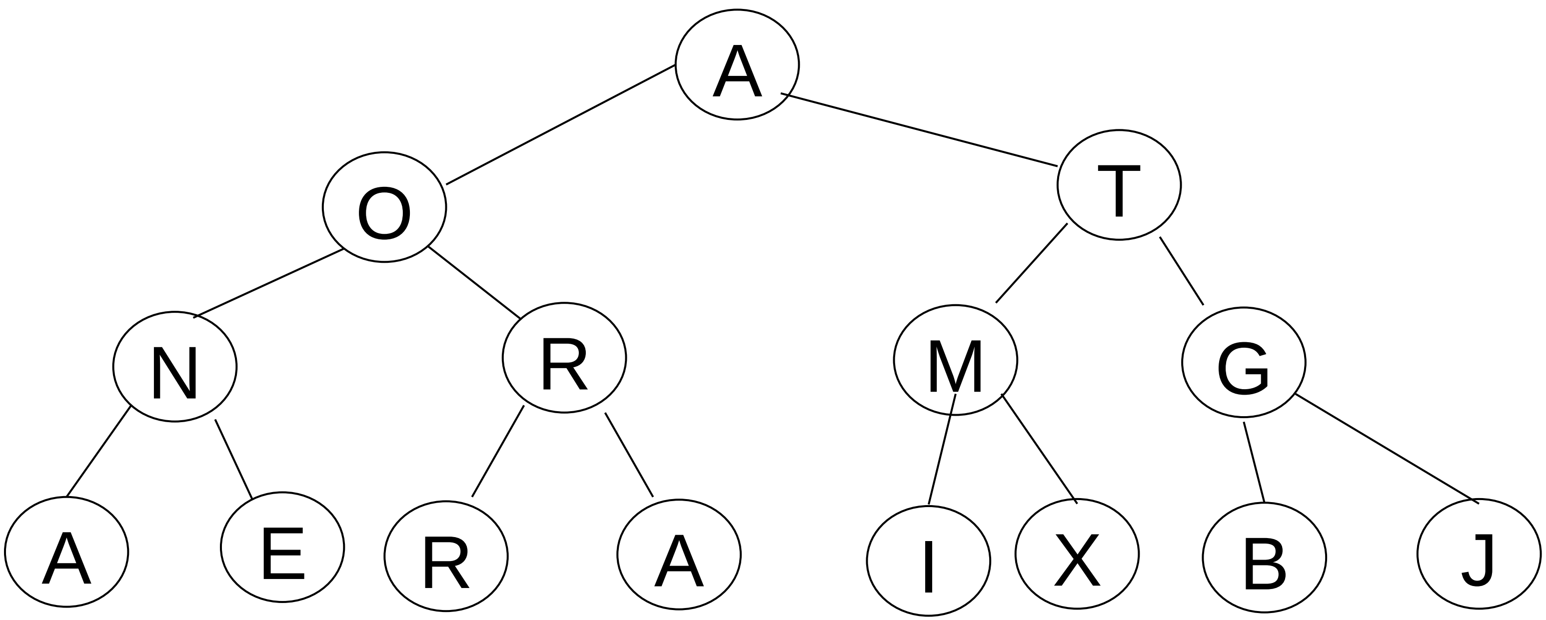
– `downheap( )` subheaps from `A[n/2]` to `A[1]`
– Bottom n/2 nodes are already heaps

Cost to fix: 0

– Next level up nodes:

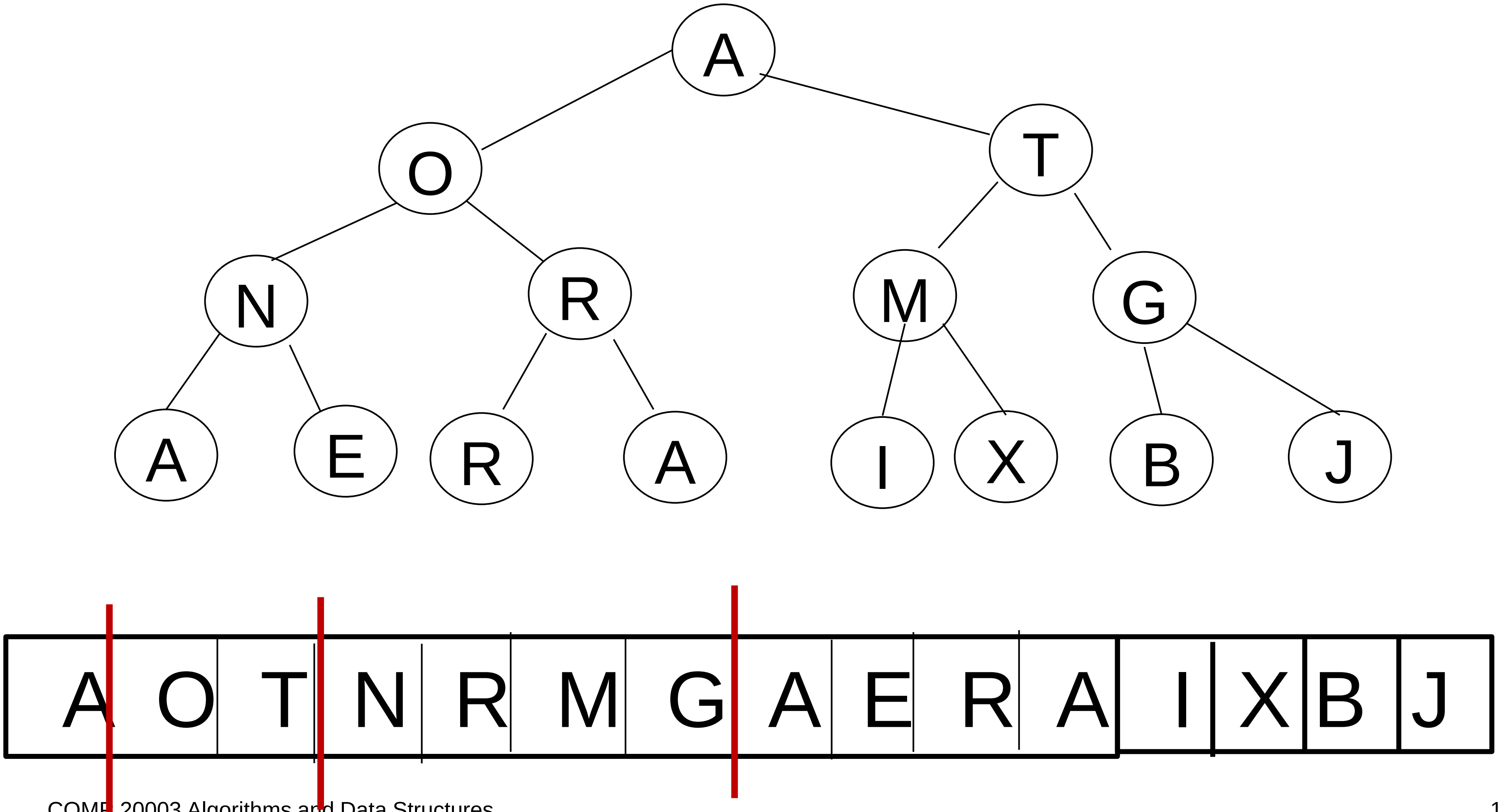
- n/4 nodes, max cost each = 2 (cmp both children)
- n/8 nodes, max cost each = 2 levels * 2 cmps
- n/16, 3 levels*2 cmps
- At the root, may need up to log n cmps to fix up
  – but there is only one node at root level

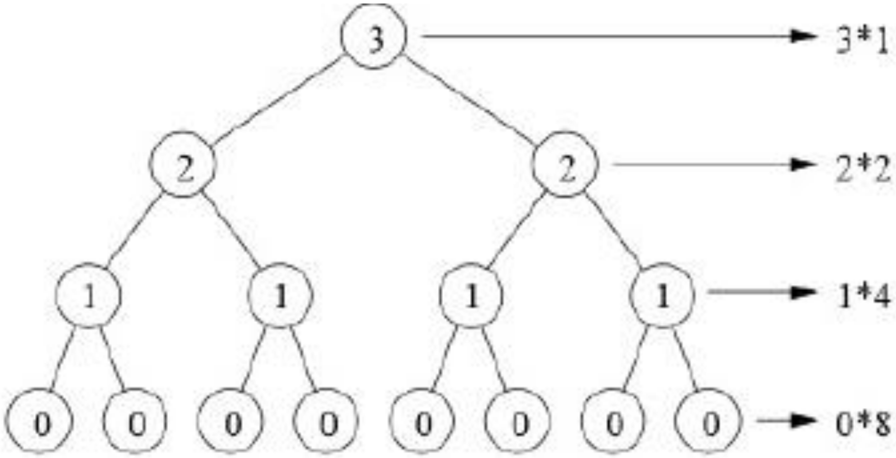
Strategy 2: Analysis



A	O	T	N	R	M	G	A	E	R	A	I	X	B	J

Strategy 2: Analysis




3
2
2
1
1
1
1
0
0
0
0
0
0
0
0
3*1
2*2
1*4
0*8

Analysis of buildheap()

Loose bound:

- **`downheap()`** *O(logn)*
- *n* operations
- On first glance: *O(n log n)*

BUT: observe

- only the root ever goes has a *log n* **`downheap()`**
- The n/2 leaves have 0 work for **`downheap()`**
- n/4 leaves at level h-1 have max 1 **`downheap()`**

Analysis of buildheap()

Overall:

- at most *ceil*(n/2$^{(h+1)}$) nodes exist at height h
- When *h = 0,* n/2 nodes
- When *h = 1,* n/4 nodes
- When *h = floor(log n),* 1 node

Total cost =

- $\sum_{(h=0 \rightarrow floor(\log n))} ceil(n/2^{(h+1)}) * O(h)$

number of nodes at this level

work for each node at this level to restore heap

Analysis of `buildheap()`

$$\sum_{(h=0 \rightarrow \text{floor}(\log n))} \text{ceil}(n/2^{(h+1)})*O(h)$$

$$= O(n \sum h/2^h)$$

(converging geometric series)

$$= O(n)$$

See Cormen, Leiserson, and Rivest for more detail

Heapsort

We will be using Priority Queues in the context of graph algorithms, a lot!

But note that the Priority Queue suggests an efficient sorting algorithm:

- Heapsort

Applications

Bandwidth Management:

- VoIP, IPTV

Shortest Path Algorithms:

- Pathfinding, navigation, games

Job Scheduling:

- OS, Clusters

Minimum Spanning Tree algorithm:

- network design

Huffman Code:

- Entropy encoding, compression jpeg, mp3