COMP20003
Algorithms and Data Structures

Shortest Paths

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Example weighted graph

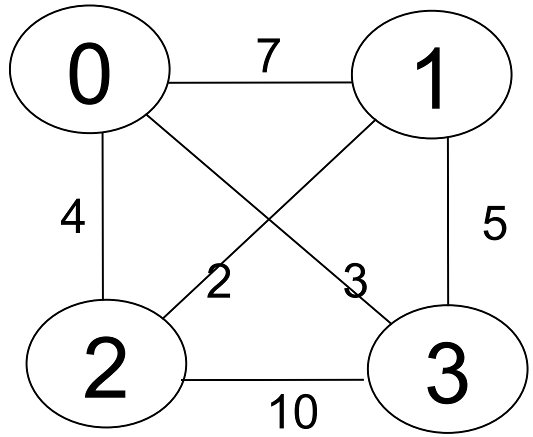
Adjacency List

0→1→2

1→0→2→3

2→0→1→3

3→1→2

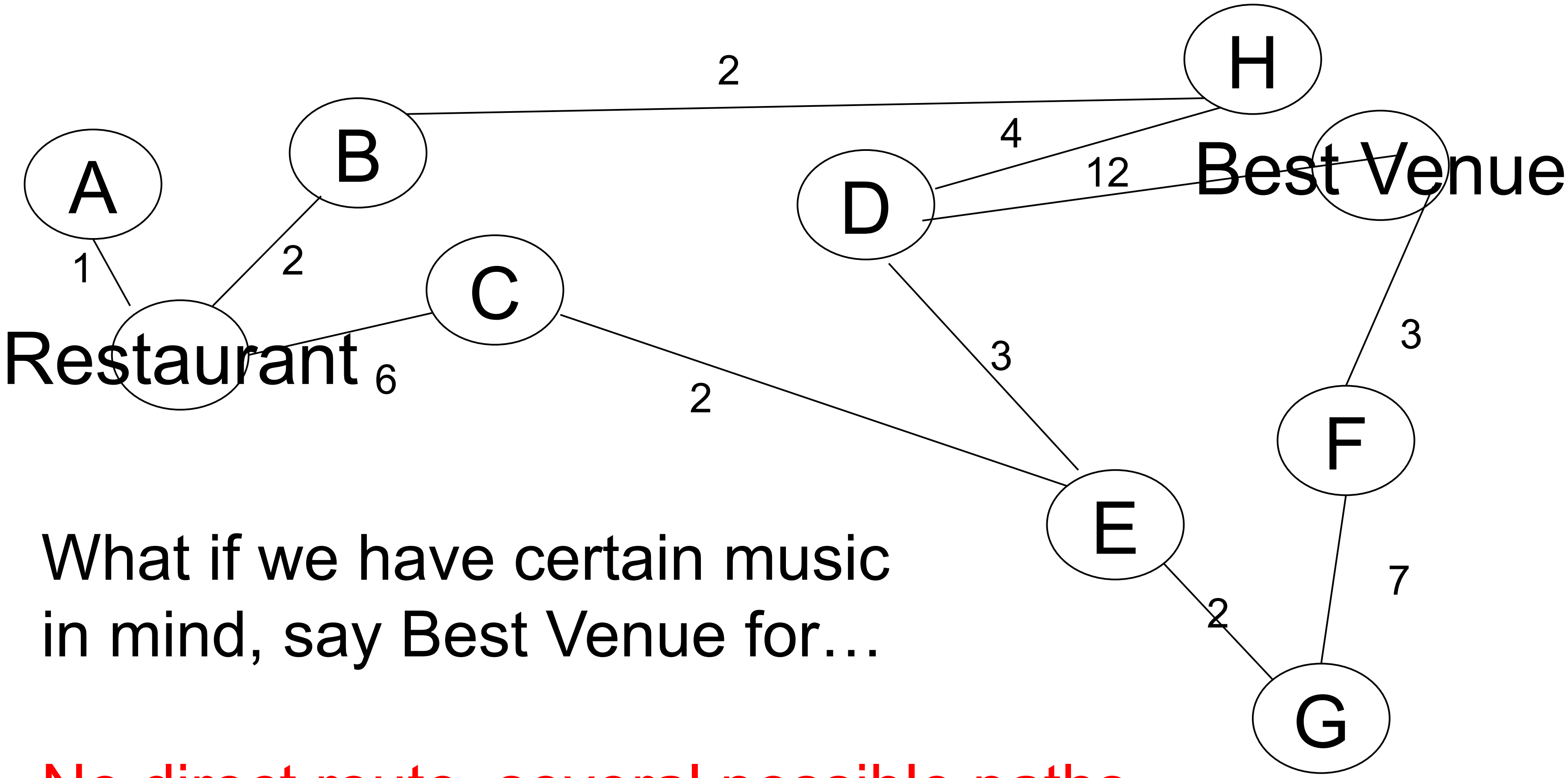


Previous visit order from node 0:

- But if these are restaurants and bars, and we want to go to a nearby bar From restaurant 0…

… in this case the answer is easy. But if you scale it…

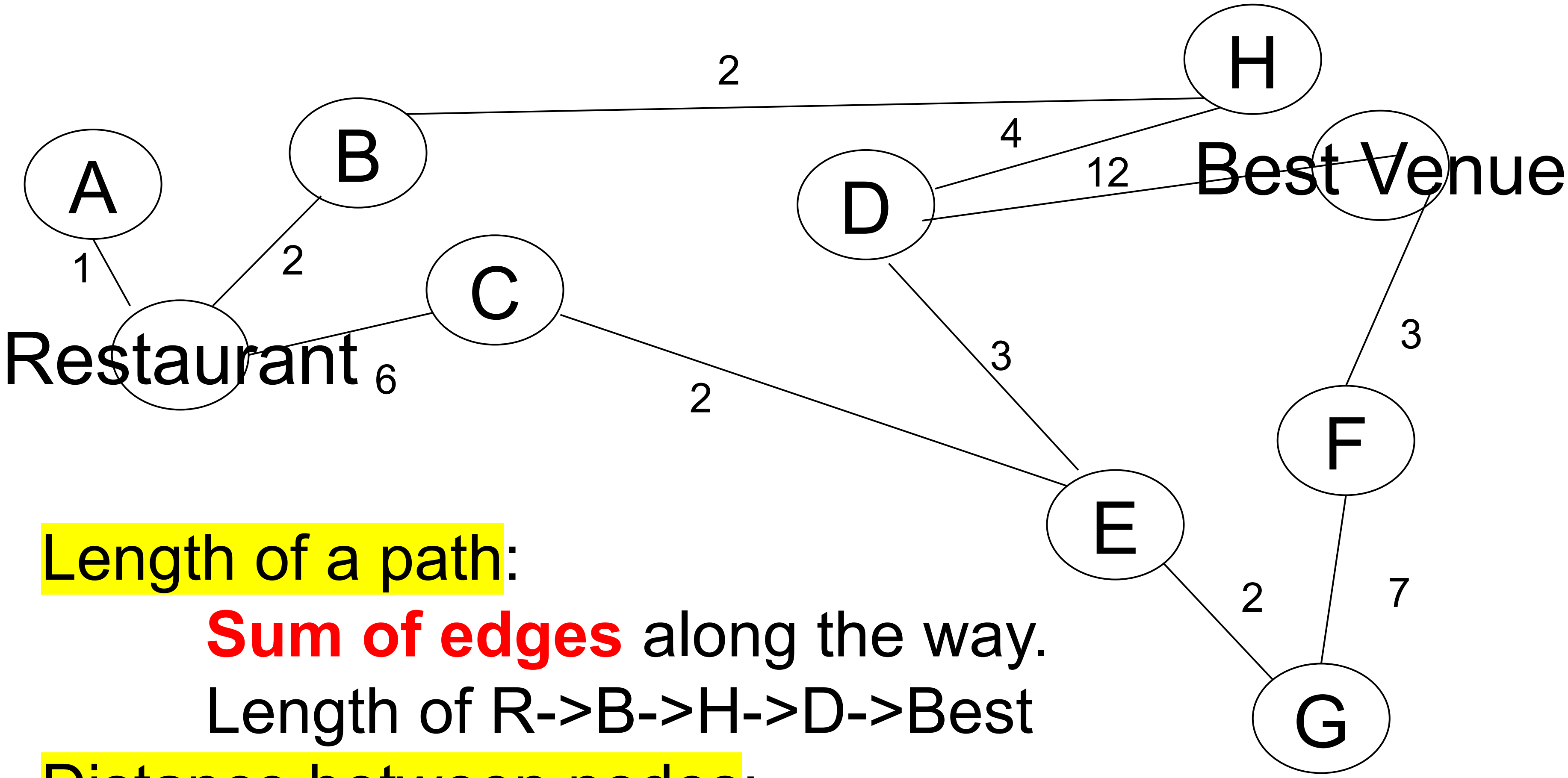
Example weighted graph



What if we have certain music in mind, say Best Venue for…

No direct route, several possible paths

Example weighted graph bfs



Length of a path:

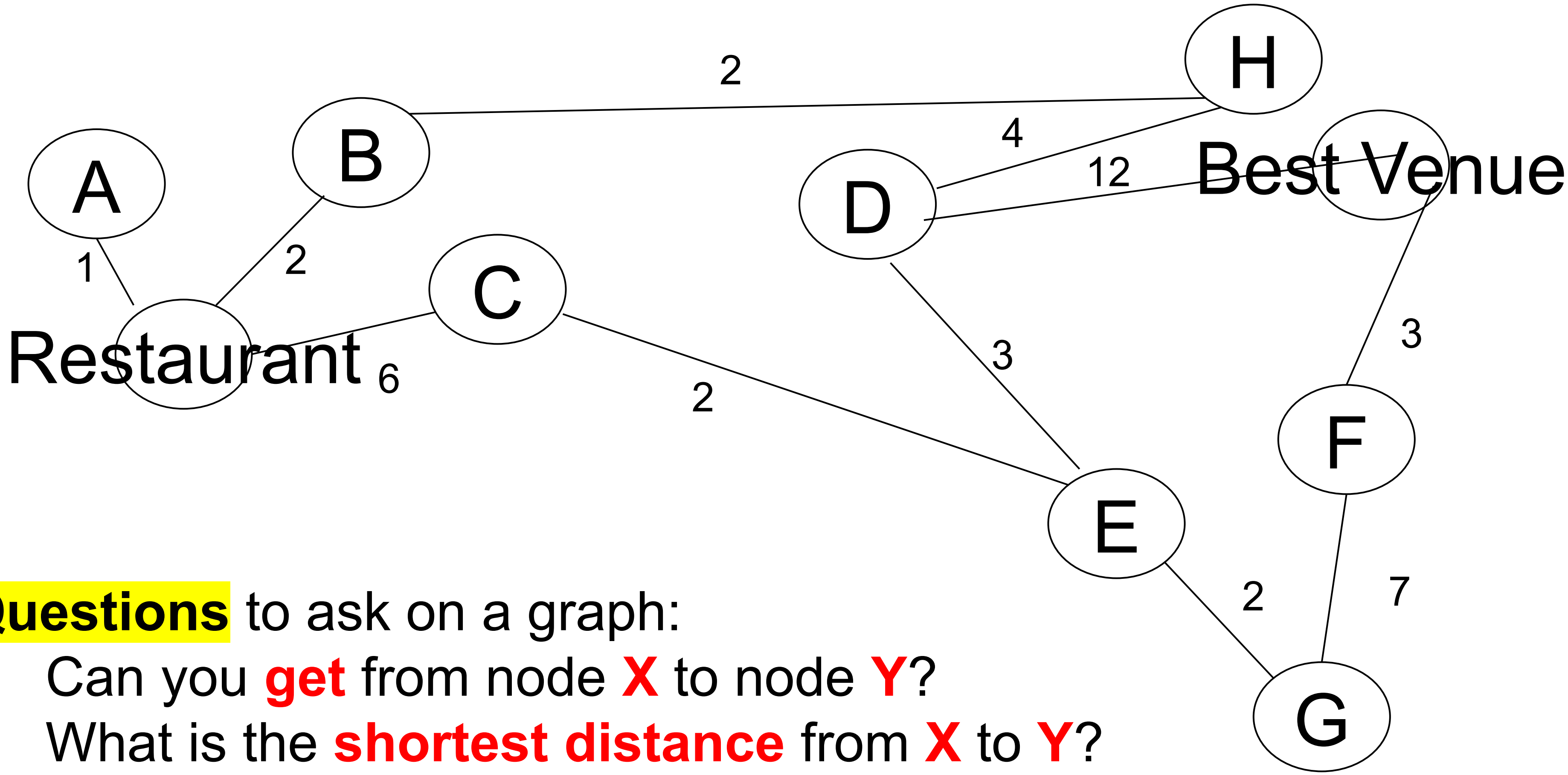
Sum of edges along the way.

Length of R->B->H->D->Best

Distance between nodes:

Length of **shortest path**.

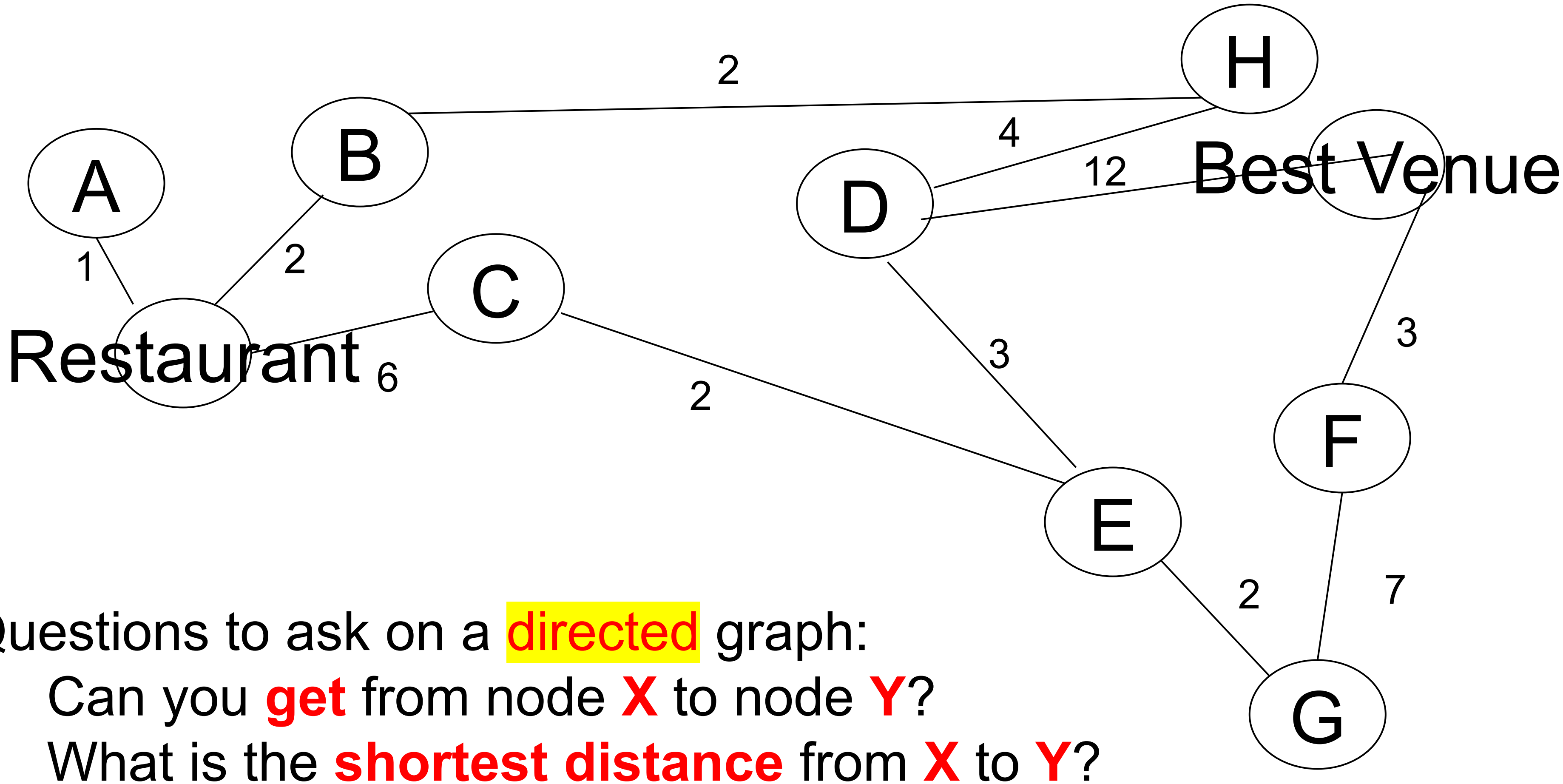
Example weighted graph



Questions to ask on a graph:

- Can you **get** from node **X** to node **Y**?
- What is the **shortest distance** from **X** to **Y**?
  What are the **shortest distances** from **X** to **any node**?
- What are the **shortest distances** from **any** node to **any** other node?

Example weighted graph



Questions to ask on a directed graph:

- Can you **get** from node **X** to node **Y**?
- What is the **shortest distance** from **X** to **Y**?

  What are the **shortest distances** from **X** to **any node**?
- What are the **shortest distances** from **any** node to **any** other node?

Single Source Shortest Path Problem

Given:

- Directed graph G(V,E)
- Source vertex s in V

Determine:

- ***Shortest*** ***distance*** path

  from s to every other vertex in V

Brute force approach

For each vertex v_i:

- Enumerate all paths from s to v_i
- Calculate cost of each path $s \rightarrow \rightarrow v_i$
- Pick minimum cost.

How many possible paths from s to v_i?

- For a dense graph O(V!)
- V=20: 2432902008176640000 paths
- Not feasible!

Dijkstra's algorithm for single source shortest path

Greedy algorithm:

- Based on idea that **any subpath** along a shortest path **is also a shortest path**

- NodeA→→→→→NodeX→NodeY
  - –If shortest path from A to Y is through X,
  - –then this path from A to X is also a shortest path

Dijkstra, E. W., *Numerische Mathematik* **1**: 269–271,1959

Dijkstra's algorithm for single source shortest path

Greedy algorithm:

- Based on idea that **any subpath** along a shortest path **is also a shortest path**

- NodeA→→→→→NodeX→NodeY
  - –If shortest path from A to Y is through X,
  - –then this path from A to X is also a shortest path

Assumes **no negative edges**, so:

- –Distance(A→X) $\leq$ Distance(A→Y)

Dijkstra's algorithm for single source shortest path

Algorithm will give us a shortest path *tree*

Root = source node

- **Every node** is **connected** to the **root** through its *shortest* path

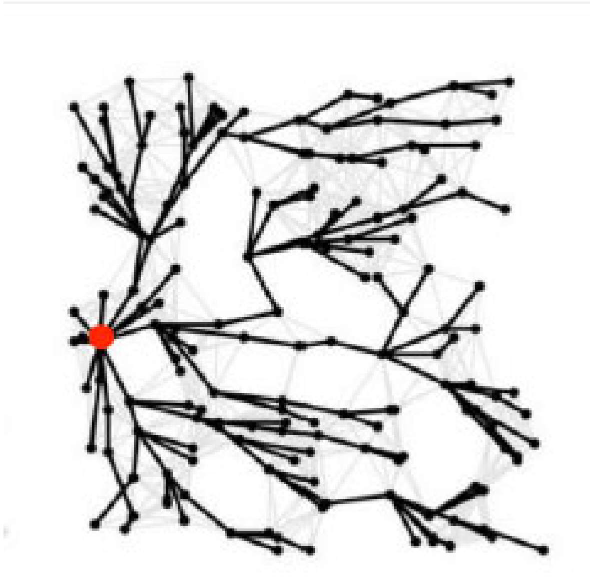


Image from R. Sedgewick, Lecture Notes http://www.cs.princeton.edu/courses/archive/fall05/cos226/lectures/shortest-path.pdf

Dijkstra's Algorithm: Overview

For every vertex v and source s, maintain *estimate* **`dist[v]`** of minimum distance $\delta(s,v)$

`dist[v]`: length of *a known path s→v*, but not necessarily the shortest path

- **`dist[v]`** $\geq \delta(s,v)$ Always
- When **`dist[v]`**$=\infty$, there is no estimate (yet)

Initially **`dist[s]`**=0, all other **`dist[v]`**$=\infty$

Dijkstra's Algorithm: Overview

Process vertices one-by-one, updating **`dist[v]`** until **`dist[v]`** $= \delta(s,v)$, for every vertex v

- Along the way, keep track of best path information in array **`pred[v]`**

When algorithm finishes:

- Have shortest distances in **`dist[]`**
- Can reconstruct shortest path from **`pred[]`**

Relaxation: Updating estimated distances

Relaxation:

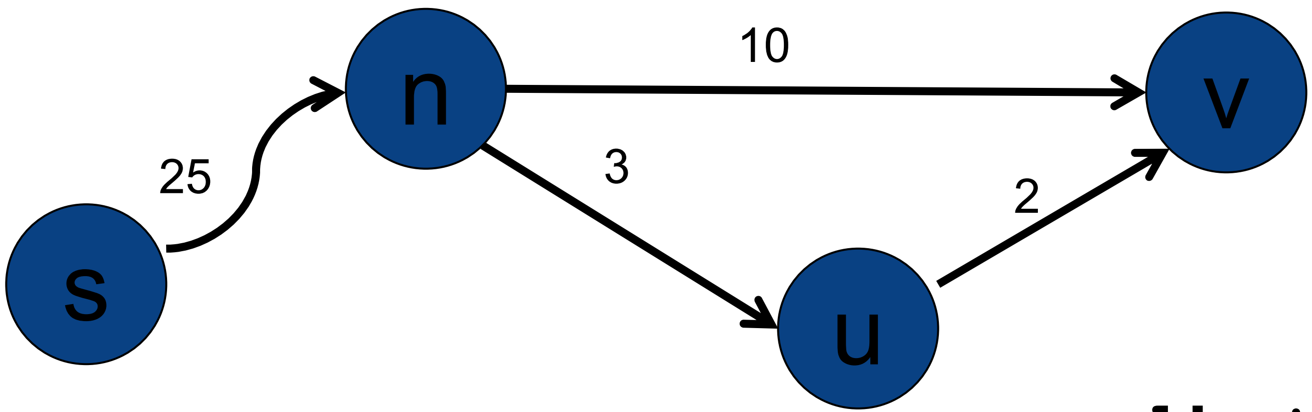
- **Estimate** the solution by answering an **easier problem** (relax the conditions)
- **`dist[]`** **Keeps updating** the relaxed estimate **until it is the solution** to the original problem

For shortest paths:

- **Estimate**: known distance of ***best*** path **so far**
- **Solution**: shortest possible distance

Relaxation: Updating estimated distances

Example:

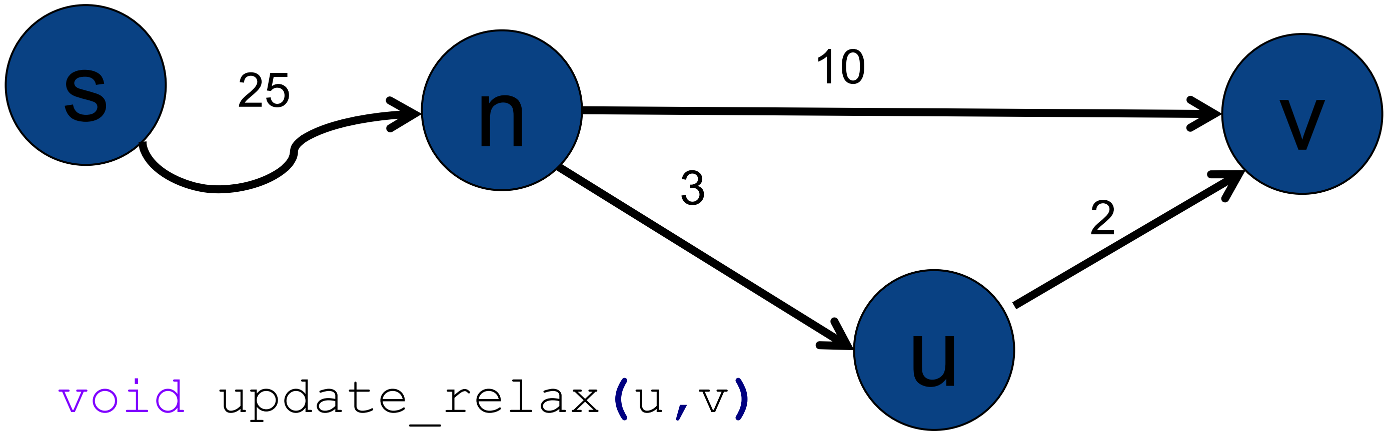


dist[u]: 28
dist[v]: 30
pred[v]: u

dist[v]: 35
pred[v]: n
pred[n]: s

Relaxation: Updating estimated distances

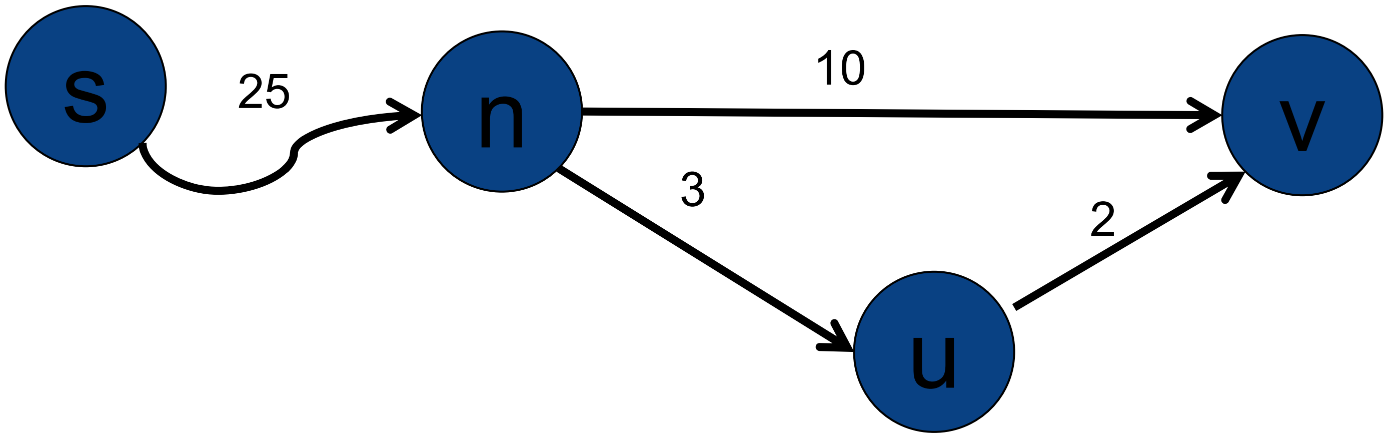
Example:




```
void update_relax(u,v)
{
    if( dist[u] + edgeweight(u,v) < dist[v] )
    {
        dist[v] = dist[u] + edgeweight(u,v);
        pred[v] = u;
    }
```

Relaxation:
Updating estimated distances

Example:



Note:

```
pred[v] = u;
pred[u] = n;
  …
pred[j] = s;
```

Reconstruct path $s \leadsto v$ going backwards through **`pred[]`**

Dijkstra's algorithm: successive relaxations

How do we **pick the next node** to look at?

Process vertices in order of **estimated closeness** to source, value of **`dist[v]`**

Priority queue to store vertex *v* and **`dist[v]`** value

Dijkstra's algorithm (C-ish pseudocode)

```
/* Find shortest paths in graph G from source s*/
/* vertices identified by number for convenience */

void dijkstra(int** G, int s)
{
    int  dist[Vsize], pred[Vsize];
    initialize(G, s, pred, dist);
    run(G, s, pred, dist);

    reconstruct(s, pred, dist);
}
```

Dijkstra’s algorithm (C-ish pseudocode)

```
void initialize(int** G, int Vsize, int s, int* pred, int* dist)
{
    int i;
    for( i = 0; i < Vsize; i++)
        dist[i] = MAX_INT;
    dist[s] = 0;
    for( i = 0; i < V; i++)
        pred[i] = NULL;
}
```

Dijkstra’s algorithm(C-ish pseudocode)

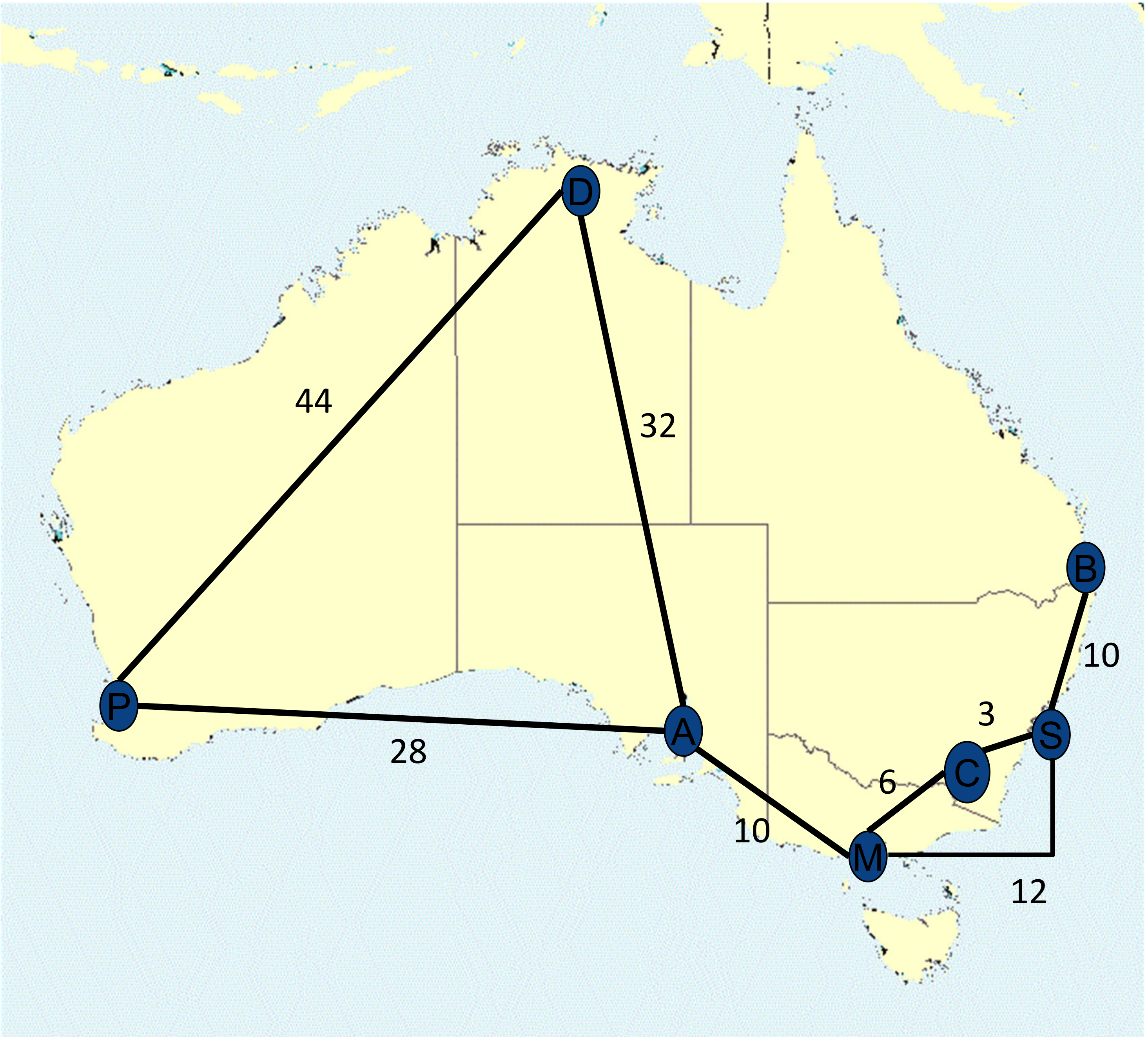
```
void run(int** G, int Vsize, int s, int* pred, int* dist)
{
  pq_node_t* pq;
  int u, v;
  pq = makePQ(G); /* vertices into min PQ, dist as priority */

  while( !emptyPQ(pq) )
  {
    u = deletemin(pq);
    /* At this point vertex u has been processed,
    * i.e. dist[u] = delta(s,u) = shortest path to u found */

    for(/*each v conneted to u */)
     if(dist[u] + edgeweight(u,v) < dist[v])
          update(v, pred, dist, pq);
   }
}
```

Dijkstra's algorithm (C-ish pseudocode)

```
void update(int v, int* pred, int* dist, pq_node_t* pq)
{
    dist[v] = dist[u] + edgeweight(u,v);
    pred[v] = u;
    decreaseweight(pq, v, dist[v]);
}
```


D
44
32
B
10
3
P
A
S
28
6
C
10
M
12

Dijkstra's algorithm (Example)

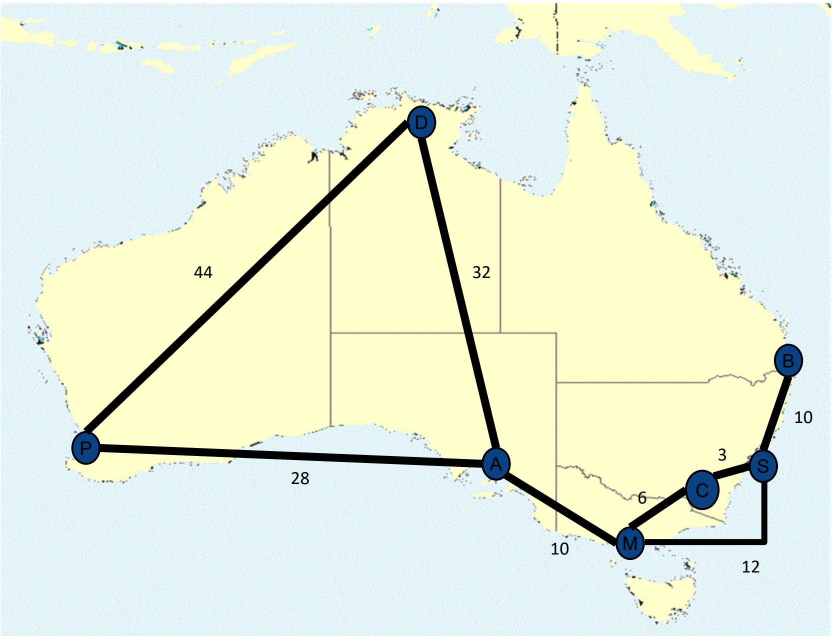
M C S B A P D

D = 　　　　　　　　**u=**

Pred = 　　　　　　**v∈{ }**

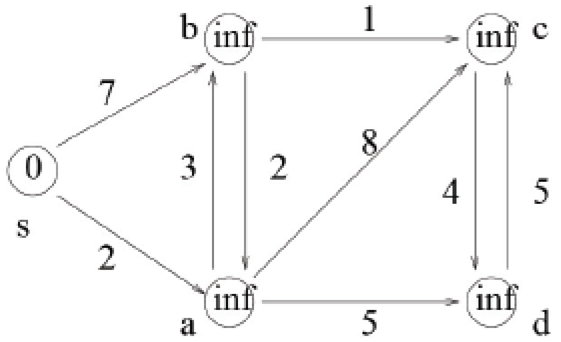
PQ =

```
if(dist[u] + edgeweight(u,v) < dist[v])
        update(v, pred, dist, pq);
```



Dijkstra's algorithm: example

Example:

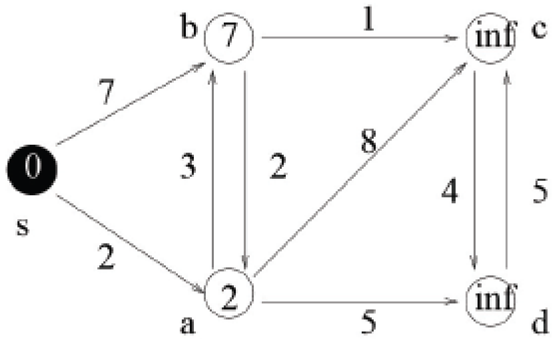


Step 0: Initialization.

v	s	a	b	c	d
$d[v]$	0	∞	∞	∞	∞
$pred[v]$	nil	nil	nil	nil	nil
$color[v]$	W	W	W	W	W

Priority Queue:

v	s	a	b	c	d
$d[v]$	0	∞	∞	∞	∞



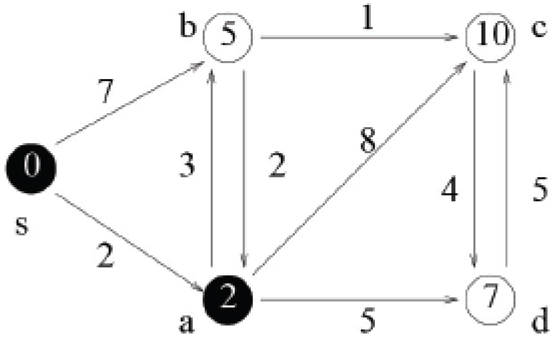
Step 1: As $Adj[s] = \{a, b\}$, work on a and b and update information.

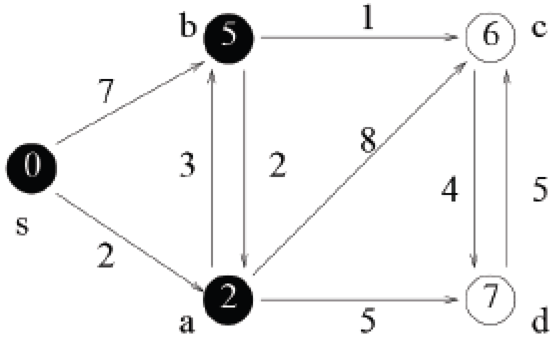
v	s	a	b	c	d
$d[v]$	0	2	7	∞	∞
$pred[v]$	nil	s	s	nil	nil
$color[v]$	B	W	W	W	W

Priority Queue:

v	a	b	c	d
$d[v]$	2	7	∞	∞

From lecture notes by Mordechai Golin, Univ Science and Technology, Hong Kong: http://www.cse.ust.hk/faculty/golin/COMP271Sp03/Notes/MyL09.pdf





Step 2: After Step 1, a has the minimum key in th priority queue. As $Adj[a] = \{b, c, d\}$, work on b, c, and update information.

v	s	a	b	c	d
$d[v]$	0	2	5	10	7
$pred[v]$	nil	s	a	a	a
$color[v]$	B	B	W	W	W

Priority Queue:

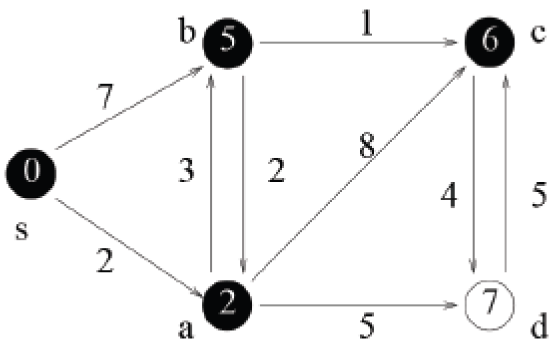
v	b	c	d
$d[v]$	5	10	7

Step 3: After Step 2, b has the minimum key in the priority queue. As $Adj[b] = \{a, c\}$, work on a, c and update information.

v	s	a	b	c	d
$d[v]$	0	2	5	6	7
$pred[v]$	nil	s	a	b	a
$color[v]$	B	B	B	W	W

Priority Queue:

v	c	d
$d[v]$	6	7

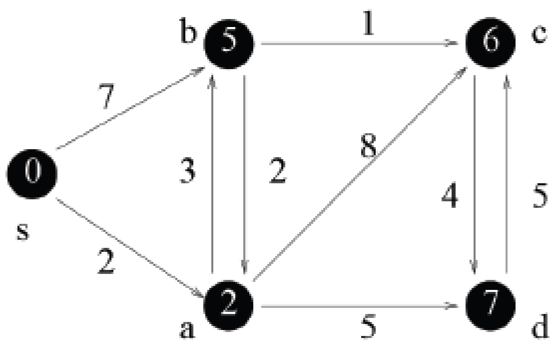


Step 4: After Step 3, c has the minimum key in the priority queue. As $Adj[c] = \{d\}$, work on d and up information.

v	s	a	b	c	d
$d[v]$	0	2	5	6	7
$pred[v]$	nil	s	a	b	a
$color[v]$	B	B	B	B	W

Priority Queue:

v	d
$d[v]$	7



Step 5: After Step 4, d has the minimum key in the priority queue. As $Adj[d] = \{c\}$, work on c and update information.

v	s	a	b	c	d
$d[v]$	0	2	5	6	7
$pred[v]$	nil	s	a	b	a
$color[v]$	B	B	B	B	B

Priority Queue: $Q = \emptyset$.

Dijkstra's algorithm(ANALYSIS)

```
void run(int** G, int Vsize, int s, int* pred, int* dist)
{
  pq_node_t* pq; /********** Assuming PQ is a minheap */
  int u, v;
  pq = makePQ(G);  /************************** big-O()???? */

  while( !emptyPQ(pq) )
  {
    u = deletemin(pq); /******************* big-O()???? */
    for(/*each v conneted to u */)
     if(dist[u] + edgeweight(u,v) < dist[v])
          update(v, pred, dist, pq); /***** big-O()???? */
   }

}
```

Dijkstra's Algorithm: Analysis

Cost depends on implementation of PQ

Using a heap:

- `makePQ()` *O(V)*

- *V* * `deletemin()` `operations` @*O(log V)*

- *O(E)* `decreaseweight()` ops @*O(log V)*

- Total: *O((V+E) log V)*

Dijkstra’s Algorithm: Limitations

Assumes no negative edges:

- Good for physical distances
- Distances are static

Negative edges:

- Use Bellman-Ford algorithm
- Cannot deal with negative cycles
- O(V*E)

Dijkstra's Algorithm: Limitations

Negative *cycles*:

- What is the shortest path?
- Problem is not well-formed, intractable
- Bellman-Ford detects negative cycles (algorithm does terminate, stops keeps shortening paths)

Tutorial:

https://www.dyclassroom.com/graph/detecting-negative-cycle-using-bellman-ford-algorithm

Applications

More applications

- Robot navigation
- Texture mapping
- Typesetting in TeX
- Urban traffic planning
- Optimal pipelining of VLSI chip
- Telemarketer operator scheduling
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP)
- Exploiting arbitrage opportunities in currency exchange
- Optimal truck routing through given traffic congestion pattern

Negative Cycle Detection: Arbitrage

Common example in CS materials is arbitrage:

– currency 1 → currency 2 → currency 3 → currency 1'

– If currency 1' > currency 1, you have made money

- Model problem as a graph:

– Vertices = currency

– Edges = - $\log_2$(exchange rate)

– Detect negative cycle and change money → get rich!

Not realistic!

– D.J.Fenn *et al.*, "The Mirage of Triangular Arbitrage in the Foreign Currency Exchange Market", *Int. J. Theoretical and Applied Finance* **12**(8), 1105-1123, 2009.

Edsger W. Dijkstra

- The question of whether computers can think is like the question of whether submarines can swim
- *Computer science is no more about computers than astronomy is about telescopes*
- How do we convince people that in programming simplicity and clarity —in short: what mathematicians call "elegance"— are not a dispensable luxury, but a crucial matter that decides between success and failure?

Elegance is not a dispensable luxury but a quality that decides between success and failure

Turing award 1972

A very nice explanation of Dijkstra's algorithm by Mordechai Golin can be found at

http://www.cse.ust.hk/faculty/golin/COMP271Sp03/Notes/MyL09.pdf