ECE 148 Homework 2

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Problem 1: CTFT Sampled Spectrum

We have a simple coherent signal $f(t)$

$$f(t) = \exp(j\omega_x t) \tag{1}$$

with a single peak at $\omega = \omega_x$. We sample with spacing Δt and then reconstruct with a low-pass filter with cutoff frequencies $\pm\frac{\omega_0}{2}$, where $\omega_0 = \frac{2\pi}{\Delta t}$, yielding reconstructed signal

$$g(t) = \exp(j\omega_y t) \tag{2}$$

with a single peak at $\omega = \omega_y$.

(a) Mapping ω_x to ω_y

- For properly considered ω_x and ω_0, where $\omega_x < \frac{\omega_0}{2}$, $\omega_y = \omega_x$.
- For $\frac{\omega_0}{2} < \omega_x < \frac{3\omega_0}{2}$, we have $\omega_y = -\omega_0 + \omega_x$.
- It is not until $\omega_x > \frac{3\omega_0}{2}$ that the aliasing captures frequencies from the $-2\omega_0$ "group."
- We know from the Nyquist sampling theorem that aliasing begins at $\omega_x > \frac{\omega_0}{2}$ and that the reconstructed frequency "wraps around" to $-\frac{\omega_0}{2}$.

Conclusion: The mapping from ω_x to ω_y is given by

$$\omega_y = [(\omega_x + \frac{\omega_0}{2}) \mod \omega_0] \ - \frac{\omega_0}{2} \tag{3}$$

(b) ω_x to ω_y Correspondence Curve

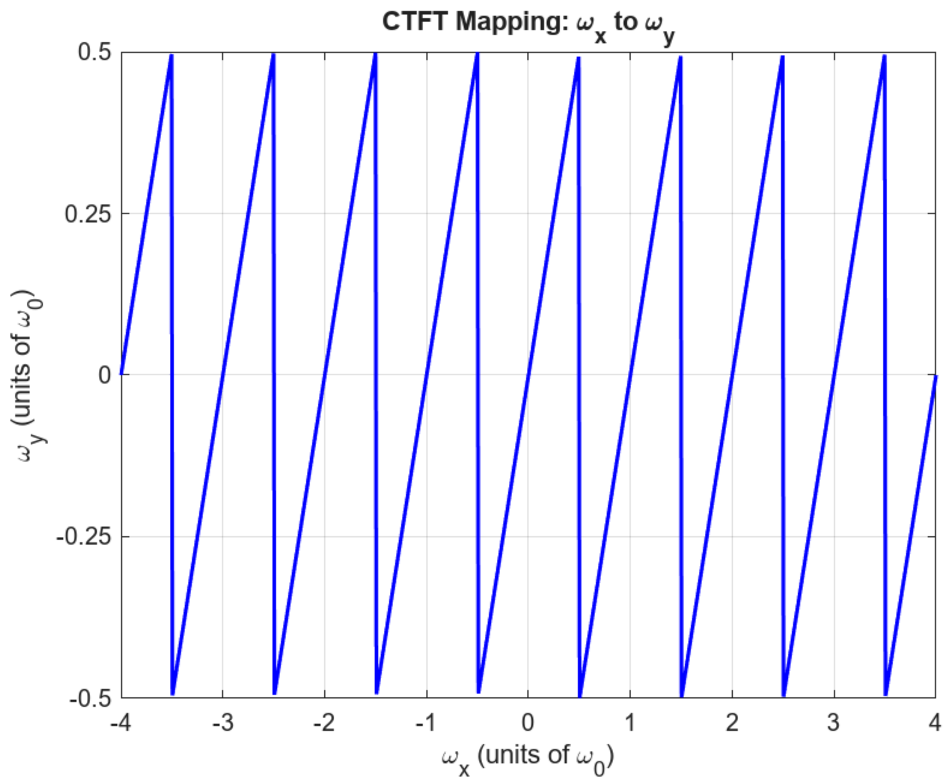

Figure 1: Sketch of ω_x to ω_y Correspondence Curve

Problem 2: DTFT Sampled Spectrum

We have a simple coherent signal $f(t)$:

$$f(t) = \exp(j\omega_x t) \tag{4}$$

with a single peak at $\omega = \omega_x$. We sample the waveform with sample spacing Δt to form a discrete sequence:

$$f(n\Delta t) = \exp(j\omega_x n\Delta t) \tag{5}$$

We compute the DTFT of the sequence

$$F(e^{j\theta}) = DTFT\{f(n\Delta t)\} = \sum_{n=-\infty}^{\infty} f(n\Delta t)\, \exp(-jn\theta) \tag{6}$$

which has a single peak at $\theta = \theta_y$.

(a) Mapping ω_x to θ_y

- We know the CTFT of a signal is given by

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, \exp(-j\omega t)$$

- Sampling (discretization), turns this into

$$\sum_{-\infty}^{\infty} f(n\Delta t)\ \exp(-j\omega\ n\Delta t)$$

- Setting this equal to our DTFT result, we have:

$$f(n\Delta t)\ \exp(-j\omega\ n\Delta t) = f(n\Delta t)\ \exp(-jn\theta)$$

$$\Rightarrow -j\omega\ n\Delta t = -jn\theta$$

$$\Rightarrow\ \ \theta = \omega\Delta t$$

Conclusion: From (3) and knowing $\omega_0 = \frac{2\pi}{\Delta t}$:

$$\theta_y = [(\omega_x \Delta t + \pi) \mod 2\pi] \ \ - \pi \tag{7}$$

(b) ω_x to θ_y Correspondence Curve

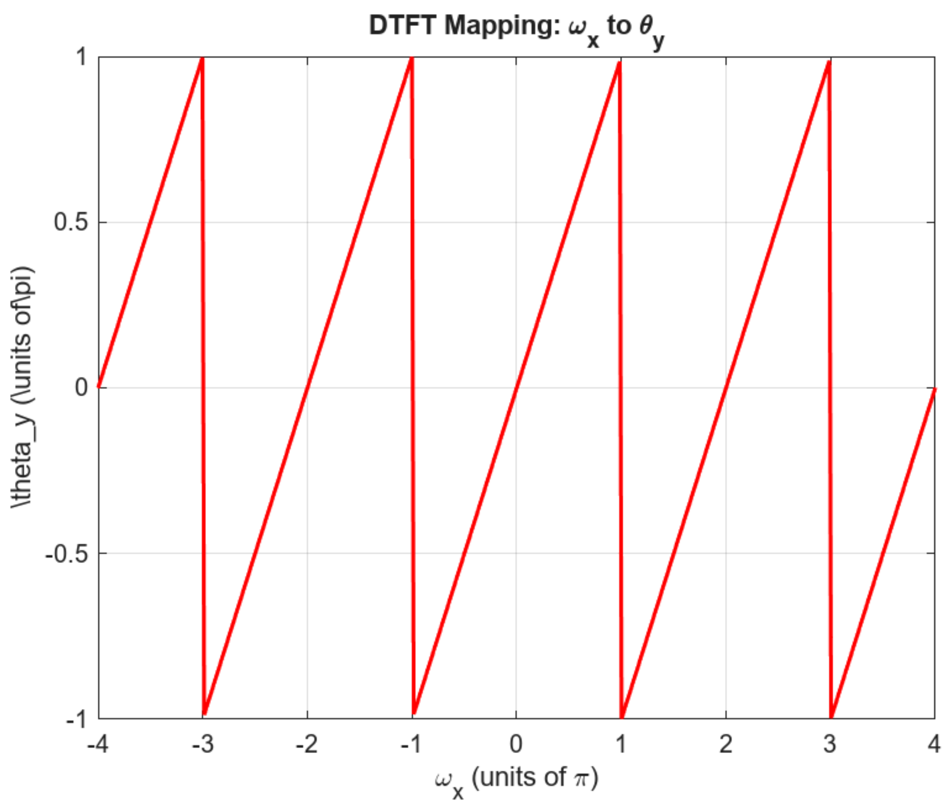

Figure 2: Sketch of ω_x to θ_y Correspondence Curve

Problem 3: DFT Sampled Spectrum

(a) Compute $F(m, k)$

```matlab
N = 128;
M = 512;

[m, n] = meshgrid(0:N-1, 0:M-1);  % Grid of indices
f = exp(1j * 2 * pi * m .* n / N);  % Compute f(m,n)

f_fft = fft(f, N, 2);  % 2 indicates operation along columns

magnitude = abs(f_fft);
```

Listing 1: MATLAB Script to Compute $F(m\,k)$

(b) Spectrogram

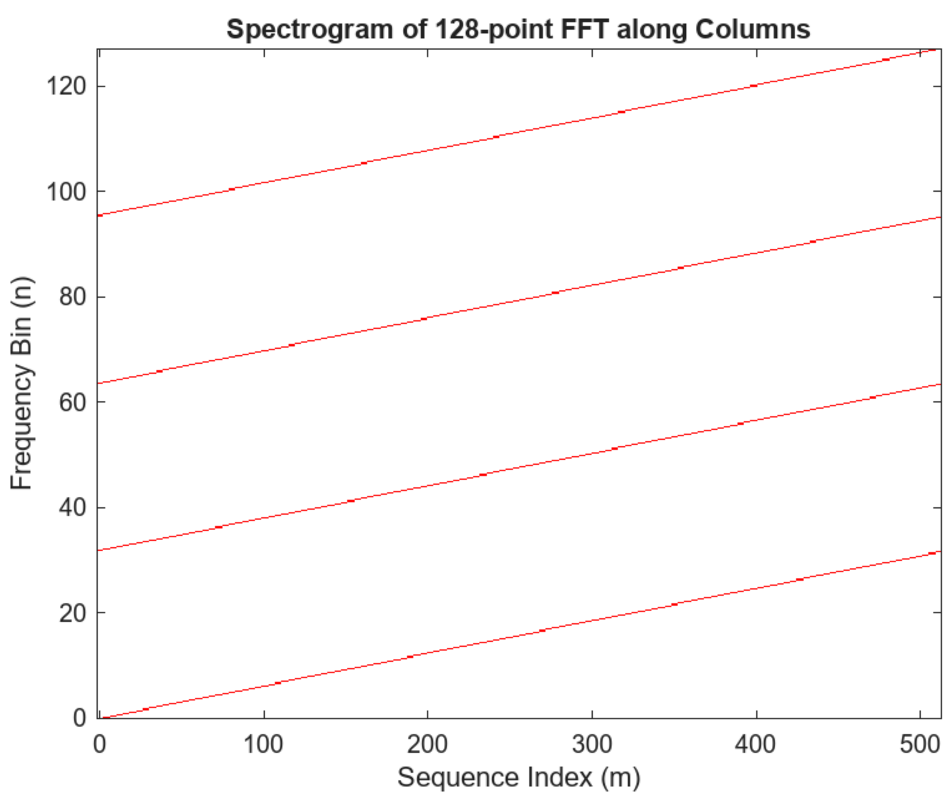

Figure 3: Spectral Image of $F(m, k)$

Note: I initially used a classic spectrogram color palette (Turbo), but changed it to just red and white afterwards for visibility after I saw what was actually of interest— the diagonal stripes throughout.

Summary

- This assignment developed understanding of the congruences of the CTFT, DTFT, and DFT, especially as it pertains to their aliasing behavior. The interplay of the $\omega \rightarrow 2\pi \rightarrow 2N+1$ transformation became especially apparent.
- The triplet plots of the aliasing stripes are truly remarkable; one can see the mapping of inputs to the FT and outputs simultaneously.
- We can also see the congruences of the DC values: all 0. This is harder to see in the FFT graph as we do not show negative frequencies.