Butterworth Filter Design

Sanjot Bains

June 8, 2025

Introduction

The objective of this assignment is to construct a simple program for the design of analog lowpass Butterworth filters. The program takes as input the standard design specifications:

- Passband frequency (ω_p)
- Maximum passband attenuation (a_{max})
- Stopband frequency (ω_s)
- Minimum stopband attenuation (a_{min})

We test the program with the following specifications:

- Passband frequency: $\omega_p = 5$k rad/s
- Max. passband attenuation: $a_{max} = 0.5$ dB
- Stopband frequency: $\omega_s = 10$k rad/s
- Min. stopband attenuation: $a_{min} = 20$ dB

The results include:

1. The order of the filter
2. Cutoff frequency ω_c
3. Locations of the poles
4. The transfer function
5. Frequency-response plot (amplitude only)
6. Verification of the design at passband and stopband frequencies

Design Specifications

```
% Passband Frequency:
omega_p = 5000;      % rad/s
% Maximum Passband Attenuation:
a_max = 0.5;         % dB
% Stopband Frequency:
omega_s = 10_000;    % rad/s
% Minimum Stopband Attenuation:
a_min = 20;          % dB
```

Listing 1: Design Specifications

Step 1: Filter Order

The order n of the filter is given by:

$$n \geq \frac{\log_{10}\left(\frac{10^{a_{min}/10}-1}{10^{a_{max}/10}-1}\right)}{2\log_{10}(\omega_s/\omega_p)}$$

```
numerator = log10(
        (10^(a_min / 10) - 1) ...
      / (10^(a_max / 10) - 1)
    );
denominator = 2 * log10(omega_s / omega_p);

n = ceil(numerator / denominator); % Filter Order
    %    = 5
```

Listing 2: Order Calculation

Thus, the filter order is $n = 5$.

Step 2: Cutoff Frequency

The cutoff frequency ω_c must satisfy:

$$\frac{\omega_p}{(10^{a_{max}/10}-1)^{1/(2n)}} \leq \omega_c \leq \frac{\omega_s}{(10^{a_{min}/10}-1)^{1/(2n)}}$$

```
min_omega_c =    omega_p / (
                 (10^(a_max / 10) - 1)^(1 / (2*n))
             ); %     = 6170.6 rad/s
max_omega_c =    omega_s / (
                 (10^(a_min / 10) - 1)^(1 / (2*n))
             ); %     = 6315.9 rad/s

omega_c = 6200;     % Chosen Cutoff Frequency
```

Listing 3: Cutoff Frequency Calculation

We choose $\omega_c = 6200$ rad/s.

Step 3: Poles of the Transfer Function

For $n = 5$ (odd), the poles are:

$$s_k = \omega_c \exp\left(j \frac{2k\pi}{2n}\right), \quad k = 0, 1, ..., 2n - 1$$

We select the n poles in the left-half plane:

```
k = 0:(2*n - 1);
s = omega_c * exp( j * k * pi / n );
% The left-half plane is distinguished by the real part of s being negative
s_left = s(real(s) < 0);
```

Listing 4: Poles Calculation

The poles are:

$$\begin{aligned}
&-1915.9 + 5896.6j \\
&-5015.9 + 3644.3j \\
&-6200.0 + 0.0j \\
&-5015.9 - 3644.3j \\
&-1915.9 - 5896.6j
\end{aligned}$$

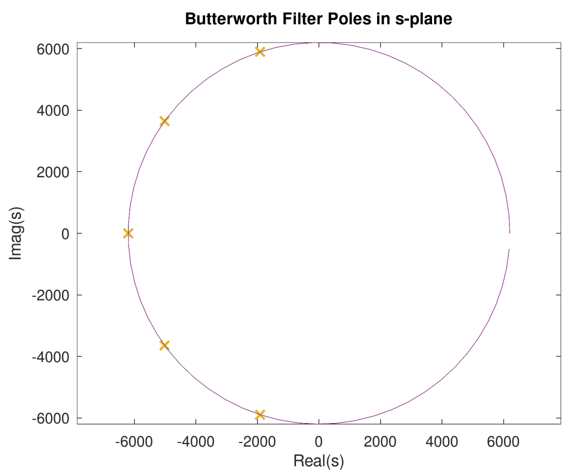

Figure 1: Butterworth Filter Poles in s-plane

Step 4: Transfer Function

The transfer function is:

$$H_n(s) = \frac{\omega_c^n}{(s - s_1)(s - s_2) \cdots (s - s_n)}$$

```
H_n = @(omega) omega_c^n ./ prod(bsxfun(@minus, j*omega(:), s_left), 2);
```

Listing 5: Transfer Function

Frequency Response

The frequency response is computed and plotted:

```
freq_range = linspace(1, 25000); %  rad/s
freq_response = H_n(freq_range);
freq_response_dB = 20*log10(abs(freq_response));
```

Listing 6: Frequency Response Plot

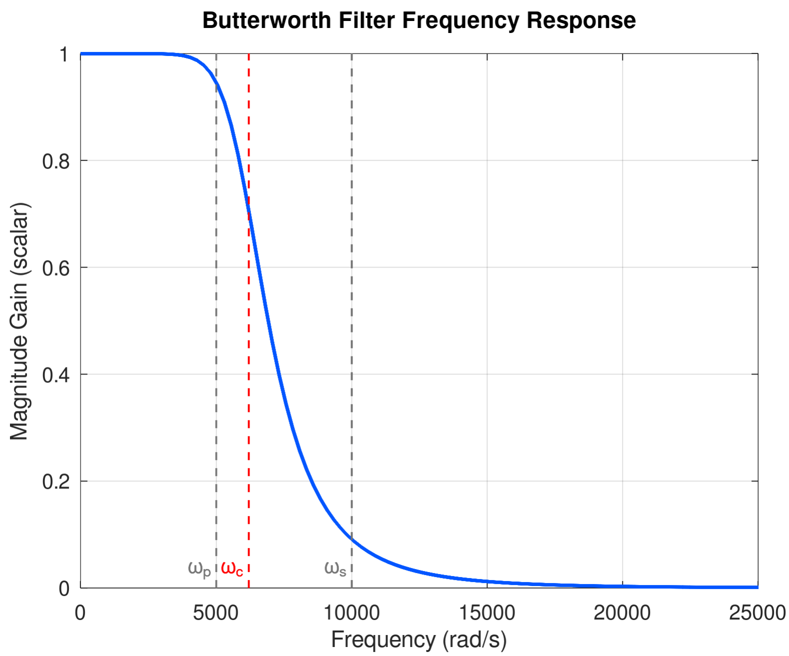

Figure 2: Butterworth Filter Frequency Response (Magnitude)

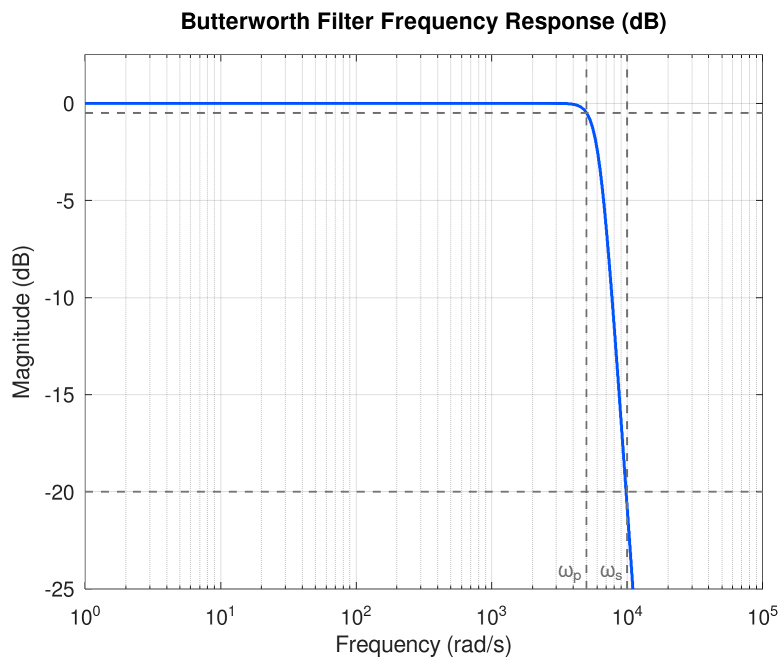

Figure 3: Butterworth Filter Frequency Response (dB)

Verification

The attenuation at the passband and stopband frequencies is:

```
omega_p_dB = 20*log10(abs(H_n(omega_p))); %      = -0.4780 dB
omega_s_dB = 20*log10(abs(H_n(omega_s))); %      = -20.797 dB
```

Listing 7: Attenuation at Key Frequencies

At $\omega_p = 5$k rad/s, attenuation = 0.4780 dB (within spec, ≤ 0.5 dB).
At $\omega_s = 10$k rad/s, attenuation = 20.797 dB (within spec, ≥ 20 dB).

Summary

- Designed a 5th-order analog lowpass Butterworth filter to meet given attenuation and frequency specifications.
- Calculated the required filter order and cutoff frequency based on design constraints.
- Determined pole locations and constructed the transfer function.
- Plotted the frequency response to verify filter characteristics.
- Verified that the filter meets the specified passband and stopband attenuation requirements.
-