ECE 148 Home 1

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Problem 1: Orthogonal Decomposition

(a) Orthogonality of $\{\psi_n(t)\}$

We have a set of orthogonal basis functions defined as:

$$\psi_n(t) = \text{sinc}\left(\frac{\omega_0(t - n\Delta t)}{2}\right), \quad \Delta t = \frac{2\pi}{\omega_0} \tag{1}$$

To show that the set $\{\psi_n(t)\}$ forms an orthogonal basis, we show that the inner product is zero:

$$\langle \psi_n(t), \psi_m(t) \rangle = \int_{-\infty}^{\infty} \psi_n(t)\psi_m(t)\, dt = 0 \quad \text{for } n \neq m \tag{2}$$

- $\psi_n(t)$ has Fourier transform of a rect function with a phase shift:

$$\Psi_n(\omega) = \frac{2\pi}{\omega_0} \cdot \text{rect}\left(\frac{\omega}{\omega_0}\right) e^{-jn\Delta t\omega} \tag{3}$$

- By Parseval's Relation:

$$\langle \psi_n(t), \psi_m(t) \rangle = \frac{1}{2\pi} \int_{-\omega_0/2}^{\omega_0/2} \Psi_n(\omega)\Psi_m^*(\omega)\, d\omega \tag{4}$$

- Substituting and simplifying:

$$\langle \psi_n, \psi_m \rangle = \frac{2\pi}{\omega_0^2} \int_{-\omega_0/2}^{\omega_0/2} e^{-j(n-m)\Delta t\omega}\, d\omega = \frac{2\pi}{\omega_0^2} \cdot \frac{2\sin((n-m)\pi)}{(n-m)\pi} = 0 \quad \text{for } n \neq m \tag{5}$$

Conclusion: $\sin((n-m)\pi) = 0$ for all $n - m \neq 0$. Important to note that this is only true because $n - m$ is always an integer. Thus, we have an orthogonal basis.

(b) Validity of Transformations

Forward Conversion: We define sampling of $f(t)$ at $t = n\Delta t$ as:

$$f(n\Delta t) = \langle f(t), \psi_n(t) \rangle = \int_{-\infty}^{\infty} f(t)\psi_n(t)\, dt \tag{6}$$

The sinc basis function is the IFT of an ideal lowpass filter (a rect function).

$$\Psi_n(\omega) = \frac{2\pi}{\omega_0} \cdot \text{rect}\left(\frac{\omega}{\omega_0}\right) e^{-jn\Delta t\omega} \tag{7}$$

We use Parseval's relation to compute the inner product. Because the rect is real and even, $\Psi_n^*(\omega = \Psi_n(\omega$. This bounds the integral to $\pm\omega_0/2$, throws in the scalar and the phase component. Thus:

$$f(n\Delta t) = \frac{1}{\omega_0} \int_{-\omega_0/2}^{\omega_0/2} F(\omega) e^{jn\Delta t\omega}\, d\omega \tag{8}$$

This matches the IFT of $F(\omega)$ for $t = n\Delta t$

Problem 2: Time Domain Partitioning

The time-domain function $f(t)$ is a lowpass signal. It can be partitioned into N sub-components:

$$f(t) = \sum_{n=1}^{N} f_n(t) \tag{9}$$

Each of the $f_n(t)$ has a spectrum that spans from $-\omega_0/2$ to $+\omega_0/2$, i.e. the total bandwidth is:

$$B = \omega_0 \tag{10}$$

Now suppose each subcomponent is modulated by a single frequency term:

$$\hat{f}(t) = \sum_{n=1}^{N} f_n(t) e^{jk_n\omega_0 t} \tag{11}$$

- Each of the N sub-components has an arbitrary spectrum within the bounds of $\pm\omega_0/2$. It does not matter what each spectrum looks like, just that it does not exceed those bounds.
- Each sub-component is modulated independently by $e^{jk_n\omega_0 t}$ and a copy is placed at the modulation frequency $k_n\omega_0$.
- Sampling at rate $\Delta t = 2\pi/\omega_0$ places a copy of each of those individual spectra at each integer multiple of ω_0. This renders a full copy of the $f(t)$ spectrum centered on each $k\omega_0$.

- Because we are sampling at exactly twice the maximum frequency in $f(t)$, we may overlap on the very edge of the spectrum, but this is likely neglible.

Conclusion: Applying an ideal low-pass filter with cutoff frequencies of $\pm\omega_0/2$, perfectly recovers the original signal $g(t)$.

Problem 3: Application of Aliasing — Frequency-Domain Partition

(a) Sketch the Fourier Spectrum of $g(t)$

The function $g(t)$ is a periodic lowpass signal, expressed as:

$$g(t) = \sum_{n=-N}^{N} G_n e^{jn\omega_x t} \tag{12}$$

where ω_x is the fundamental frequency and G_n are Fourier coefficients. Its bandwidth is:

$$B = 2N\omega_x < \omega_0 \tag{13}$$

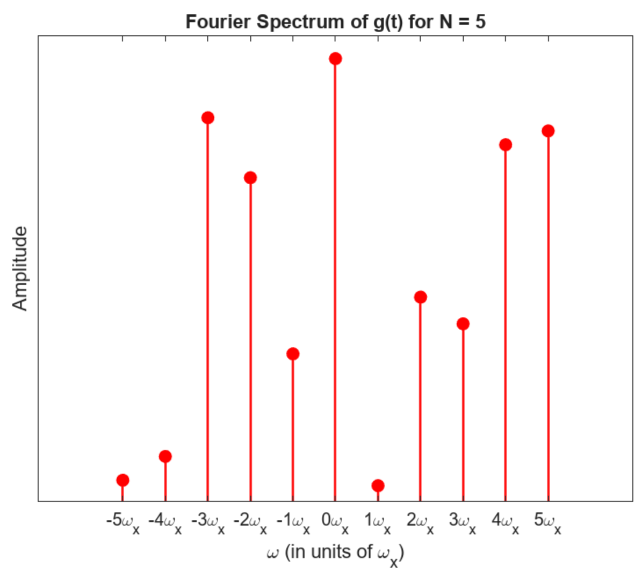

Figure 1: Sketch of Fourier spectrum of $g(t)$: Impulses at $n\omega_x$ from $-N\omega_x$ to $N\omega_x$

The Fourier spectrum of $g(t)$ looks like $2N+1$ impulses distributed evenly, with spacing ω_x, between $-N\omega_x$ and $N\omega_x$.

(b) Spectrum of $\hat{g}(t)$

Now suppose we modulate each component with a frequency term $e^{jk_n\omega_0 t}$, giving:

$$\hat{g}(t) = \sum_{n=-N}^{N} G_n e^{j(n\omega_x + k_n\omega_0)t} \tag{14}$$

$\hat{g}(x)$ places the "origin" of each of those individual components at each respective $k_n\omega_0$. That is, the location of each impulse is at $n\omega_x + k_n\omega_0$. There are still only $2N+1$ components.

(c) Sampling and Filtering

- Sampling at rate $\Delta t = 2\pi/\omega_0$ places a copy of each one of those individual impulses at each integer multiple of ω_0. This, in effect, renders a full copy of the $2N+1$ impulses centered on each $k\omega_0$.
- Because $\omega_0 > 2N\omega_x$ (the fundamental frequency of $g(t)$), there is no aliasing between adjacent sets of impulses; the sets do not overlap.

Conclusion: Passing the whole shebang through an ideal low-pass filter with cutoff frequencies of $\pm\omega_0/2$, recovers the original signal $g(t)$ perfectly.

Summary

- Problems 2 and 3 show how signals can be torn asunder, in time or frequency as we desire, and through the process of sampling and filtering, we can achieve perfect reconstruction of the original.
- In the case of Problem 2, presented with a summation of time-domain signals of varying spectra, we can modulate each component to various places along the frequency axis. As long as the modulation frequency is atleast double the maximum frequency in the base signal, they will not be aliased.
- Now. the modulated signal is sampled. Here occurs some intentional aliasing; sampling takes each of those spectra and overlays them on each integer multiple of ω_0.
- Passing the sampled signal through a low-pass filter of width $\pm\omega_0/2$ yields a perfect reconstruction of the original signal.
- Problem 3: We have a Fourier Series with 2N+1 components, which present as impulses in the frequency domain, spaced evenly ω_x apart.
- Modulating puts those impulses in the vicinity of various multiples of ω_0. Sampling puts a copy of each of these impulses together in a full set centered on each $k_n\omega_0$.
- Passing the sampled signal through a low-pass filter of width $\pm\omega_0/2$ yields a perfect reconstruction of the original signal.