Lecture 6: Fourier Series Expansion

Fourier series expansion has been commonly introduced in the engineering classes in the form of an orthogonal decomposition, along with properties and applications. In this section, a different perspective is presented in the form of the modulation property of Fourier analysis. This is the most effective way to lead into the concept of *frequency-shift keying (FSK).* In addition, the trace of a periodic function over the two-dimensional complex plane provides an interesting and illuminating viewpoint of the expansion.

1. Frequency-shift model

As described previously, when a function is periodic with period *T*, it can be represented by an infinite series, known as the *Fourier series*, in the form

$$f(t) = \sum_{n=-\infty}^{\infty} F_n \, exp(\, jn\omega_o t \,)$$

where $\omega_o = 2\pi/T$ is known as the *fundamental frequency*, and the coefficients of the expansion $\{F_n\}$ can be obtained with the formula,

$$F_n = \frac{1}{T} \int_0^T f(t) \, exp(-jn\omega_o t) \, dt$$

The coefficients $\{F_n\}$ are the governing parameters of the periodic waveform $f(t)$. The combination, given by the coefficients $\{F_n\}$, uniquely defines the characteristics of the function. Thus, the coefficients $\{F_n\}$ is fact represent the information content of the overall waveform.

Now we consider a simple example,

$$x(t) = C$$

Then we modulate this function with a carrier $exp(jk\omega_o t)$, corresponding to the carrier frequency

$$\omega = k\,\omega_o$$

such that modulated waveform is in the simple form $C \cdot exp(jk\omega_o t)$. Then, if we demodulate it according to the corresponding frequency, the parameter C can be retrieved.

$$\frac{1}{T}\int_0^T C \cdot exp(jk\omega_o t)\, exp(-jk\omega_o t)\, dt = \frac{1}{T}\int_0^T C\, dt = C$$

One way to describe the procedure is to treat the parameter C as the objective information content. The modulation process moves this information content up, places it into a *frequency bin*, and marks it with the *spectral address* $k\omega_o$, corresponding to the bin number k. The demodulation procedure first examines the *spectral address*, then moves over the corresponding *frequency bin*, and unlocks it to retrieve the information content. That means we can regard the carrier term $exp(jk\omega_o t)$ as the lock, and the matching key is $exp(-jk\omega_o t)$. The average over a window of period T can be regarded as part of the unlocking process. The unique matching correspondence between the lock and the key is established by the orthogonality among the locks and keys.

spectral-bin address	$(n-2)\omega_o$	$(n-1)\omega_o$	$n\omega_o$	$(n+1)\omega_o$	$(n+2)\omega_o$
information content	F_{n-2}	F_{n-1}	F_n	F_{n+1}	F_{n+2}

Therefore, now we can describe the Fourier series expansion of the function $f(t)$ as a collection of modulated waveforms.

$$f(t) = \sum_{n=-\infty}^{\infty} F_n\, exp(\, jn\omega_o t\,)$$

For a periodic waveform, the information content, represented by the Fourier coefficients $\{F_n\}$, are placed in a collection of spectral bins, marked by the bin number n and spectral addresses $\omega = n\omega_o$. These spectral bins are distributed uniformly with the spacing ω_o. This also means the size of the spectral bins is ω_o. If we wish to retrieve the coefficient of a particular component, say $n = k$, we examine the address $\omega = k\omega_o$ to identify the location of the particular bin, and then unlock it with the key $exp(-jk\omega_o t)$ to capture the corresponding coefficient F_k.

$$F_k = \frac{1}{T} \int_0^T f(t)\, exp(-jk\omega_o t)\, dt$$

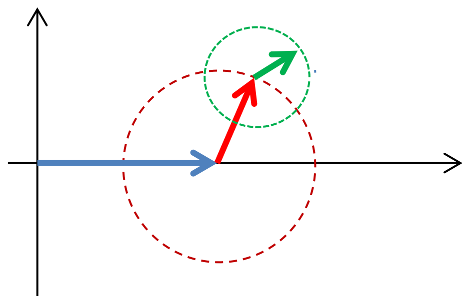
Orthogonality means the key $exp(-jk\omega_o t)$ can unlock only one bin located at $\omega = k\omega_o$. The orthogonality suggests that the locks and keys have unique one-to-one correspondences, without any ambiguity. In addition, if we shift the operating frequency, we will be able to unlock the frequency bins sequentially to reproduce the entire sequence $\{F_n\}$.

2. Observation of the Trajectory

Another interesting way to describe the Fourier series expansion is to visualize the trajectory of the time function. For that, we rewrite the expansion in the form

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} F_n\, exp(\, jn\omega_o t\,) \\ &= \cdots + F_{-1}\, exp(\, -j\omega_o t\,) \\ &\quad + F_0 \\ &\quad + F_1\, exp(\, j\omega_o t\,) \\ &\quad + F_2\, exp(\, j2\omega_o t\,) + \cdots \end{aligned}$$

Now we examine the physical behavior of the expansion, term by term. The first we elect to analyze is F_0. It is stationary point over the *s-plane*, at $s = F_0$, marked in ***blue***. The next term is $F_1\, exp(\, j\omega_o t\,)$, which is a vector F_1, marked in ***red***, rotating at the rate of $\omega = \omega_o$. The radius of the circular trace is $|F_1|$. Similarly, the term $F_2\, exp(\, j2\omega_o t\,)$ means a rotating vector a vector F_2, marked in ***green***, rotating at the rate of $\omega = 2\omega_o$. The radius of the circular trace is $|F_2|$.



s-plane

This means the function $f(t)$ can be represented by a collection of rotating vectors, with various radii $|F_n|$ and rates of the rotation $\omega = n\omega_o$. At $t = mT$, all vectors are rotated back to the starting positions. Thus the trajectory repeats itself with the period T, which illustrates the periodic nature of the function.

For the components corresponding to the negative indexes, the vectors rotate in the opposite direction. For example, the term $F_{-5}\, exp(\, -j5\omega_o t\,)$, means a rotating vector a vector F_{-5}, rotating at the rate of $\omega = 5\omega_o$, clockwise. The radius of the circular trace is $|F_{-5}|$.

3. Hermitian Symmetry

When the function $f(t)$is real,

$$f(t) = f^*(t)$$

the Fourier coefficients are Hermitian symmetrical, as illustrated in the figure, where one is marked in blue and the other in red.

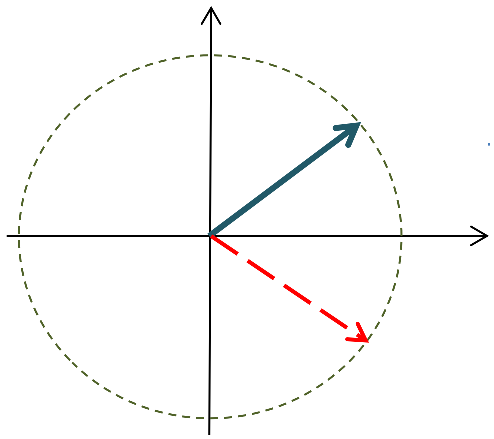
$$F_{-n} = F_n^*$$

Suppose

$$F_n = |F_n| \, exp(j\theta)$$

Then we have

$$F_{-n} = F_n^* = |F_n| \, exp(-j\theta)$$



s-plane

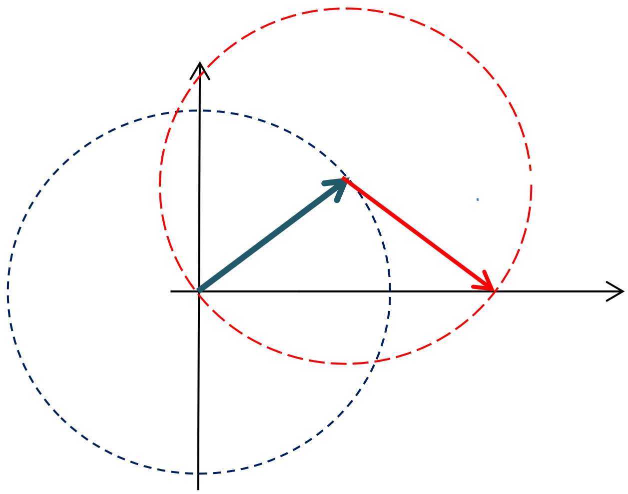
The two vectors, $F_n \, exp(jn\omega_o t)$ and $F_{-n} \, exp(-jn\omega_o t)$, are rotating along the same circle of radius $|F_n|$ at the same rate of $n\omega_o$, but in opposite directions. Now we can examine the superposition of these two components,

$$F_n \, exp(jn\omega_o t) + F_{-n} \, exp(-jn\omega_o t) = |F_n| \, exp(j\theta)exp(\, jn\omega_o t\,) + |F_n| \, exp(-j\theta)exp(-jn\omega_o t\,)$$

$$= |F_n| \, [\, exp(\, j(n\omega_o t + \theta)) + exp(-\, j(n\omega_o t + \theta))]$$

$$= 2|F_n| \, cos(n\omega_o t + \theta)$$

This shows the result of the superposition of these two rotating vectors is always real, which means the resultant superposition of these two vectors is a point along the real axis, as shown in the following figure. This also illustrates why Hermitian symmetrical Fourier series always yield real and periodic waveforms.



s-plane

From the previous equation, we realize the superposition of these two Fourier components is in fact the real part of $2F_n \, exp(jn\omega_o t)$,

$$F_n \; exp(jn\omega_o t) + F_{-n} \; exp(-jn\omega_o t) \; = \; 2\,|F_n|\,cos(n\omega_o t + \theta)$$

$$= \; Real\,\{2F_n \, exp(jn\omega_o t)\}$$

This means, when the periodic waveform is real, the trajectory is the shadow of the single-sideband version of the Fourier series expansion onto the real axis.

$$f(t) = Real\{\, \tilde{f}(t)\}$$

$$= \; F_0 + Real\,\{\sum_{n=1}^{\infty} \; 2F_n \; exp(\,jn\omega_o t\,)\}$$

where the single-sideband version of the waveform is

$$\tilde{f}(t) \; = \; F_0 + \sum_{n=1}^{\infty} \; 2F_n \; exp(\,jn\omega_o t\,)$$

A similar relationship will surface again in the topic of *Hilbert transform.*