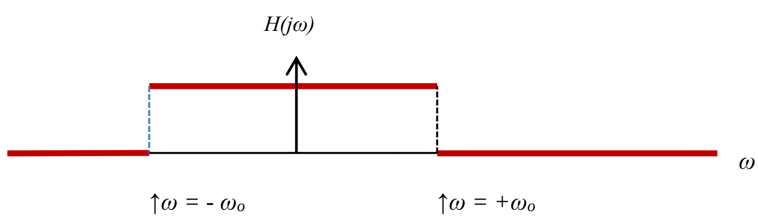
Lecture 13: Butterworth Filters

For the simplicity, Butterworth filters have been the one of the most widely applied techniques in filter design. Therefore, in the analysis of filter design procures, analog or digital, low-pass Butterworth filters have been the best starting point.

1. Low-pass Butterworth filter

The frequency response of an ideal low-pass filter is in the form

$$H(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_o \\ 0 & elsewhere \end{cases}$$



where ω_o denotes the *cutoff frequency* of the filter. For the ideal filter, the frequency response is unity within the passband and zero outside. Butterworth filter has been a popular implementation of the low-pass filter designs in practice as an approximation of the ideal low-pass filter. The magnitude of the n^{th}*-order* low-pass Butterworth filters is formulated as

$$|H_n(j\omega)|^2 = \frac{1}{1 + (\frac{\omega}{\omega_o})^{2n}}$$

where n is the order of the filter, ω_o is the cutoff frequency, and the polynomial in the denominator is known as the Butterworth polynomial. It should be noted that the magnitude of the frequency response, $|H_n(j\omega)|$, is a monotonically decreasing function and is unity at $\omega = 0$, independent of the order. In addition, the half-power point of the frequency response is always at $\omega = \omega_o$.

2. Pole Locations and Transfer Function

Then we reorganize the formula slightly to represent it as a function of $j\omega$,

$$\left|H_n(j\omega)\right|^2 = \frac{1}{1+(\frac{\omega}{\omega_o})^{2n}} = \frac{1}{1+((\frac{j\omega}{j\omega_o})^2)^n}$$

$$= \frac{1}{1+[-(\frac{j\omega}{\omega_o})^2]^n}$$

$$= \frac{1}{1+(-1)^n(\frac{j\omega}{\omega_o})^{2n}}$$

Utilizing the Hermitian symmetry, the formula can be rewritten in the form

$$\left|H_n(j\omega)\right|^2 = H_n(j\omega)\, H_n^*(j\omega) = H_n(j\omega)\, H_n(-j\omega)$$

$$= \frac{1}{1+(-1)^n(\frac{j\omega}{\omega_o})^{2n}}$$

$$= H_n(s)\, H_n(-s)\Big|_{s=j\omega}$$

Then we conclude with a simple formula

$$H_n(s)\, H_n(-s) = \frac{1}{1+(-1)^n(\frac{s}{\omega_o})^{2n}}$$

It should be noted here that this formula gives only the mathematical structure of the product $H_n(s)H_n(-s)$. Thus, the objective of the subsequent analysis is to identify the transfer function $H_n(s)$.

Since the numerator is a constant, the first logical step is to identify the poles of the transfer function by finding the roots of the Butterworth polynomial in the denominator.

$$1 + (-1)^n \left(\frac{s}{\omega_o}\right)^{2n} = 0$$

After simple rearrangement, it becomes

$$\left(\frac{s}{\omega_o}\right)^{2n} = (-1)^{n+1}$$

If *n* is odd, it is in the simple form

$$\left(\frac{s}{\omega_o}\right)^{2n} = 1 = exp(j2k\pi)$$

Thus, the $2n$ solutions to the equation are

$$\frac{s}{\omega_o} = exp(j2k\pi/2n) \qquad k = 0, 1, \ldots 2n\text{-}1$$

This indicates the $2n$ poles of the term $H_n(s)H_n(-s)$ are

$$s = \omega_o\, exp(j2k\pi/2n) \qquad k = 0, 1, \ldots 2n\text{-}1$$

$$= \omega_o\, exp(jk\Delta\theta)$$

The $2n$ poles are uniformly distributed along the circle of radius ω_o with the angular increment

$$\Delta\theta = 2\pi/2n$$

There are only two real roots at $s = \pm\,\omega_o$, corresponding to $k = 0$, and $k = n$. Then if n is even, the equation becomes slightly different,

$$\left(\frac{s}{\omega_o}\right)^{2n} = -1 = exp(j(2k+1)\pi)$$

Thus, the *2n* poles are

$$s = \omega_o \exp(j(2k+1)\pi/2n) \qquad k = 0, 1, \dots 2n-1$$

To illustrate the distribution of the poles, we partition the term and place it in the form

$$s = \omega_o \exp(j\pi/2n) \exp(j2k\pi/2n) \qquad k = 0, 1, \dots 2n-1$$

$$= \omega_o \exp(j\Delta\theta/2) \exp(jk\Delta\theta)$$

It can be seen that, for the case of even order, the *2n* roots are located along a circle of radius ω_o, equally spaced with the same angular increment $\Delta\theta=2\pi/2n$. The only difference is that the first root, corresponding to $k=0$, is located at $s = \omega_o \exp(j\pi/2n)$, instead of $s = \omega_o$, of the case of odd orders. This suggests there are two real roots at $s = \pm\, \omega_o$, when the order is odd. And there is no real root when the order is even.

The $2n^{th}$*-order* polynomial in the denominator consists of *2n* roots. As a result, the combined transfer function term has *2n* poles. As analyzed previously, these *2n* poles are equally spaced along a circle of radius ω_o.

$$H_n(s)\, H_n(-s) = \frac{1}{1 + (-1)^n\, (s)^{2n}} = \frac{\omega_o^{2n}}{(s-s_1)(s-s_2)\cdots(s-s_{2n})}$$

Because of the symmetry, the *2n* poles are in the form of *n* symmetric pairs

$$H_n(s)\, H_n(-s) = \frac{\omega_o^{2n}}{[(s-s_1)(s+s_1)]\,[(s-s_2)(s+s_2)]\cdots[(s-s_n)(s+s_n)]}$$

$$= \frac{\omega_o^{n}}{(s-s_1)(s-s_2)\cdots(s-s_n)} \cdot \frac{\omega_o^{n}}{(s+s_1)(s+s_2)\cdots(s+s_n)}$$

We thus partition the term into two components and formulate the transfer function $H_n(s)$ with the n poles on the left-half plane, for stability purpose.

$$H_n(s) = \frac{\omega_o^n}{(s-s_1)(s-s_2)\cdots(s-s_n)}$$

The numerator is given a scalar term ω_o^n in order for the transfer function to have unity gain at $\omega = 0$. When n is an even number, there are $n/2$ pairs of complex-conjugate roots. If n is odd, there are $(n-1)/2$ pairs of complex-conjugate roots, and one real root at $s = -\omega_o$.

Because of the symmetry, the poles on the left-half plane are on the circle of radius ω_o. The angles are off by π. Thus, when the order n is odd, the locations of the poles are

$$\begin{aligned} s &= \omega_o \exp(\pm j\varphi_k) = \omega_o \exp(j(\pi \pm \theta_k)) \qquad k = 0, 1, \ldots (n-1)/2 \\ &= \omega_o \exp(j(\pi \pm 2k\pi)/2n)) \\ &= -\omega_o \, [\cos\theta_k \pm j \sin\theta_k] \end{aligned}$$

where θ_k are commonly referred to as the *Butterworth angles*. If n is even, the pole locations are also in the similar form

$$s = -\omega_o \, [\cos\theta_k \pm j \sin\theta_k] \qquad k = 0, 1, \ldots n/2 - 1$$

Example:

This example tabulates the low-pass Butterworth filter for the low orders, $n = 1, 2, 3, 4$ and 5. This chart gives the angular spacing, Butterworth angles, pole locations, and transfer function of each case.

n	$\Delta\theta$	Butterworth angles θ_k	pole locations
1	180^o	$\theta = 0^o$	$s = -\omega_o$
2	90^o	$\theta = \pm 45^o$	$s = -\omega_o \exp(j45^o)$
3	60^o	$\theta = 0^o, \pm 60^o$	$s = -\omega_o, -\omega_o \exp(j60^o)$
4	45^o	$\theta = \pm 22.5^o, \pm 67.5^o$	$s = -\omega_o \exp(j22.5^o), -\omega_o \exp(j67.5^o)$
5	36^o	$\theta = 0^o, \pm 36^o, \pm 72^o$	$s = -\omega_o, -\omega_o \exp(j36^o), -\omega_o \exp(j72^o)$

n	transfer functions $H(s)$
1	$H(s) = \frac{\omega o}{s + \omega o}$
2	$H(s) = \frac{\omega_o^2}{(s^2 + 1.414\,\omega_o s + \omega_o^2)}$
3	$H(s) = \frac{\omega_o^3}{(s + \omega_o)(s^2 + \omega_o s + \omega_o^2)}$
4	$H(s) = \frac{\omega_o^4}{(s^2 + 1.848\,\omega_o s + \omega_o^2)(s^2 + 0.765\,\omega_o s + \omega_o^2)}$
5	$H(s) = \frac{\omega_o^5}{(s + \omega_o)(s^2 + 1.618\,\omega_o s + \omega_o^2)(s^2 + 0.618\,\omega_o s + \omega_o^2)}$

3. Passband Requirement

In the passband, $|\omega| \leq \omega_p$, the frequency response provides relatively high gains. In practice, the gain with the passband is not unity. Thus, to be practical, a small attenuation is allowed. Typically, the frequency response is bounded between zero and $-\alpha_{max}$ *dB*, where α_{max} denotes the maximum allowable attenuation within the passband.

$$0 \geq 20 \log|H_n(j\omega)| \geq -\alpha_{max}$$

This leads to the numerical relationship,

$$0 \geq 10 \log \frac{1}{1 + (\frac{\omega}{\omega_o})^{2n}} \geq -\alpha_{max}$$

After rearranging the terms, it becomes

$$0 \leq (\frac{\omega}{\omega_o})^{2n} \leq 10^{\alpha_{max}/10} - 1$$

Thus, for the frequency components within the passband, the basic requirement is

$$0 \leq \omega \leq \omega_o (10^{\alpha_{max}/10} - 1)^{\frac{1}{2n}}$$

Because the frequency response is a monotonically decreasing function, the maximum attenuation occurs at the passband frequency. Thus, the relationship can be simplified further,

$$\omega_p \leq \omega_o (10^{\alpha_{max}/10} - 1)^{\frac{1}{2n}}$$

This relationship gives a simple relationship between the cutoff frequency ω_o and the design specifications including passband frequency ω_p, passband attenuation limit α_{max}, and the order of the filter n.

4. Stopband Requirement

In the passband, $|\omega| \leq \omega_p$, a similar numerical relationship can be established. By definition, frequency response gives high attenuation for the removal or reduction of the frequency components. In practice, although the frequency response is not zero, large attenuation is required. Typically, a minimum attenuation, α_{min} *dB,* is assigned as part of the design specifications. This concept translates into the relationship

$$10\ log \frac{1}{1 + (\frac{\omega}{\omega_o})^{2n}} \leq -\alpha_{min}$$

A similar relationship then arrives,

$$\omega \geq \omega_o\ (\ 10^{\alpha_{min}/10} - 1\ )^{\frac{1}{2n}}$$

Because the Butterworth function is monotonically decreasing, the minimum allowable attenuation occurs at the stopband frequency,

$$\omega_s \geq \omega_o\ (\ 10^{\alpha_{min}/10} - 1\ )^{\frac{1}{2n}}$$

5. Determination of the Order of the Filter

Combining the two mathematical constraints derived from the passband and stopband frequency specification, a simple relationship can be established. This formula illustrates the fundamental interrelationships of the four design specifications as well as the order of the filter, in a simple and concise manner.

$$\frac{\omega_s}{\omega_p} \geq (\frac{10^{\alpha_{min}/10} - 1}{10^{\alpha_{max}/10} - 1})^{\frac{1}{2n}}$$

If apply the *log* operation to both side of this equation, it yields,

$$\log\frac{\omega_s}{\omega_p} \geq \frac{1}{2n} log \frac{10^{\alpha_{min}/10} - 1}{10^{\alpha_{max}/10} - 1}$$

Reorganizing the relationship, we produce one of the most useful formulas,

$$n \geq \frac{log \frac{10^{\alpha_{min}/10} - 1}{10^{\alpha_{max}/10} - 1}}{2 \log\frac{\omega_s}{\omega_p}}$$

This formula facilitates the computation of the order of the filter from the four design specifications in an effective manner. The resultant value is then round it up to an integer.

6. Identification of the Cutoff Frequency of the Filter

Again, from the two constraints from the passband and stopband requirements,

$$\omega_p \leq \omega_o \left(10^{\alpha_{max}/10} - 1 \right)^{\frac{1}{2n}}$$

and

$$\omega_s \geq \omega_o \left(10^{\alpha_{min}/10} - 1 \right)^{\frac{1}{2n}}$$

The upper and lower bounds of the cutoff frequency can be determined,

$$\frac{\omega_p}{\left(10^{\alpha_{max}/10} - 1 \right)^{\frac{1}{2n}}} \leq \omega_o \leq \frac{\omega_s}{\left(10^{\alpha_{min}/10} - 1 \right)^{\frac{1}{2n}}}$$

Any frequency within these bounds will be feasible for the filter design. The range given by the upper and lower bounds often provides useful flexibility in the design process.

Summary: Design Procedure of Butterworth Low-pass Filters

The typical design specifications for low-pass filters are in the form of four basic elements:

a. Passband frequency: ω_p *(rad/sec)*
b. Stopband frequency: ω_s *(rad/sec)*
c. Maximum passband attenuation: α_{max} *(dB)*
d. Minimum stopband attenuation: α_{min} *(dB)*

Step 1: Determine the order *n* of the filter

$$n \;\geq\; \frac{log\,\dfrac{10^{\alpha_{min}/10}-1}{10^{\alpha_{max}/10}-1}}{2\;log\,\dfrac{\omega_s}{\omega_p}}$$

Step 2: Select the cutoff frequency ω_o of the filter

$$\frac{\omega_p}{(\,10^{\alpha_{max}/10}-1\,)^{\frac{1}{2n}}} \;\leq\; \omega_o \;\leq\; \frac{\omega_s}{(\,10^{\alpha_{min}/10}-1\,)^{\frac{1}{2n}}}$$

Step 3: Identify the poles of the transfer function

If n is odd: $s = \omega_o \, exp(\,j2k\pi/2n\,)$
If n is even: $s = \omega_o \, exp(\,j(2k+1)\pi/2n\,)$

for $k = 0, 1, 2, \dots 2n\text{-}1$. Select the *n* poles on the left-half plane from the set of *2n*.

Step 4: Formulate the transfer function

$$H_n(s) \;=\; \frac{\omega_o^n}{(s-s_1)(s-s_2)\,\cdots\,(s-s_n)}$$

Step 5: Hardware implementation (Sallen-Key circuits)

Example: Butterworth Low-Pass Filter

The typical design specifications:

a. Passband frequency: $\omega_p = 10\ k$
b. Stopband frequency: $\omega_s = 24.58\ k$
c. Maximum passband attenuation: $\alpha_{max} = 0.3\ dB$
d. Minimum stopband attenuation: $\alpha_{min} = 22.0\ dB$

Step 1: Determine the order of the filter

$$n \geq \frac{log\frac{10^{\alpha_{min}/10}-1}{10^{\alpha_{max}/10}-1}}{2\ log\frac{\omega_s}{\omega_p}} = \frac{3.34}{0.781} = 4.27 \quad \text{(rounded up to } n = 5\text{)}$$

Step 2: Select the Butterworth angles

$$\varphi = 0^o, \pm 36^o, \pm 72^o$$

Step 3: Calculate the bounds of cutoff frequency ω_o of the filter

$$\frac{\omega_p}{(10^{\alpha_{max}/10} - 1)^{\frac{1}{2n}}} \leq \omega_o \leq \frac{\omega_s}{(10^{\alpha_{min}/10} - 1)^{\frac{1}{2n}}}$$

$$\frac{10k}{(10^{0.03} - 1)^{0.1}} \leq \omega_o \leq \frac{24.58k}{(10^{2.2} - 1)^{0.1}}$$

$$13.02\ k \leq \omega_o \leq 14.82\ k$$

(We choose $\omega_0 = 14k$, within the range.)

Step 4: Determine the *5* poles of the transfer function

$$s_1 = -\omega_0 = -14k$$
$$s_{2,3} = -\omega_0 \ exp(\pm j36^o) = -14k \ exp(\pm j36^o)$$
$$s_{4,5} = -\omega_0 \ exp(\pm j72^o) = -14k \ exp(\pm j72^o)$$

Step 5: Formulate the transfer function

$$H_n(s) = \frac{\omega_o}{(s-s_1)} \cdot \frac{(\omega_0)^2}{(s-s_2)\,(s-s_3)} \cdot \frac{(\omega_0)^2}{(s-s_4)(s-s_5)}$$
$$= \frac{\omega_o}{(s-s_1)} \cdot \frac{(\omega_0)^2}{(s-s_2)(s-s_2^*)} \cdot \frac{(\omega_0)^2}{(s-s_4)(s-s_4^*)}$$
$$= \frac{14k}{s+14k} \cdot \frac{(14k)^2}{s^2+(22.7k)s+(14k)^2} \cdot \frac{(14k)^2}{s^2+(8.7k)s+(14k)^2}$$