EE312
END TERM EXAM (SET 1)

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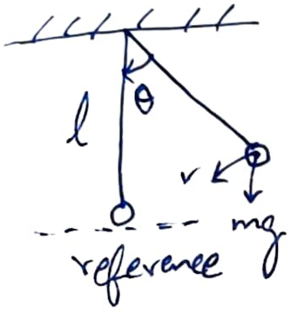
EEE (IDD)

1. Lagrangian equation is -

$$q = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) + \left(\frac{\partial P}{\partial \dot{q}}\right)$$

q → generalised force



Here we take $q = \theta$ & external torque $= \tau$

The Lagrangian equation becomes:

$$\tau = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) + \left(\frac{\partial P}{\partial \dot{\theta}}\right)$$ (P = dissipated energy)

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m(l\dot{\theta})^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

$$PE = mgl(1-\cos\theta)$$

$$L = KE - PE = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$$

$$P = \frac{1}{2}bv^2 = \frac{1}{2}b(l\dot{\theta})^2$$ (P is dissipated energy - air friction ; b = coeff. of air friction)

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \dot{\theta}}\left[\frac{1}{2}ml^2\dot{\theta}^2 - \cancelto{0}{mgl} + \cancelto{0}{mgl\cos\theta}\right]$$ $\left(\frac{\partial(mgl)}{\partial \dot{\theta}} = 0,\right.$ same for $\left.mgl\cos\theta\right)$

$$\Rightarrow \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\cancel{\frac{1}{2}ml^2\dot{\theta}^2}\right)^{0} + \frac{\partial}{\partial \theta}(mgl\cos\theta) + \frac{-\partial}{\partial \theta}\cancel{mgl}^{0}$$

$$= -mgl\sin\theta$$

$$\frac{\partial P}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}bl^2\dot{\theta}^2\right) = bl^2\dot{\theta}$$

Substituting obtained expressions in the lagrangian equation,

$$T = ml^2\ddot{\theta} + mgl\sin\theta + bl^2\dot{\theta}$$

$$\ddot{\theta} = \frac{-g\sin\theta}{l} - \frac{b}{m}\dot{\theta} + \frac{T}{ml^2}$$

Let state variables $\theta_1 = \theta$, $\theta_2 = \dot{\theta}$, $u(t) = T$

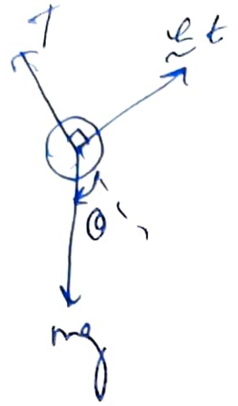
$$\dot{\theta}_1 = \theta_2$$

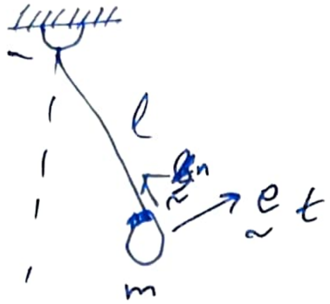
$$\dot{\theta}_2 = \frac{-g\sin\theta_1}{l} - \frac{b}{m}\theta_2 + \frac{u(t)}{ml^2}$$

→ state space model. The system is a ~~...~~ non-linear ~~...~~ system.

ii) Since the linearisation of motion of single pendulum, equilibrium points, feedback ~~model~~ not taught in class, hence the answer here will not match the physical interpretation that is taught.

Suppose an ideal pendulum, with end mass m. The end mass is assumed to be connected to support with a light ~~...~~ slender rigid rod of length l.





The differential equation of motion of the pendulum may be found by summing forces in the ~~tan~~ tangential direction ($\underset{\sim}{e}\,t$). From the free ~~period~~ body diagram, $\sum F_t = m a_t \Rightarrow -mg\sin\theta = ml\ddot{\theta}$

$$\Rightarrow \ddot{\theta} + \frac{g\sin\theta}{l} = 0$$

Equilibrium positions: These can be found by setting $\ddot{\theta} = 0$ in the above equation.

$\therefore$ The pendulum has two equilibrium posns.

$$\theta_{eq} = 0, \pi$$

Linearized equation of motion: To study small motions of the pendulum about the ~~eq~~ equilibrium position ($\theta_{eq} = 0$), we have

$\theta = \theta_{eq} + \Delta\theta$ & linearize.

Linearising the fn $f(\theta) = \sin\theta$ about $\theta = 0$

$$\Delta f = \left(\frac{df}{d\theta}\right)_{\theta=0} \Delta\theta = 1\,\Delta\theta$$

$\Rightarrow$ Approximate ~~linear~~ ~~equation~~ linear equation

$$\Delta\ddot{\theta} + \left(\frac{g}{l}\right)\Delta\theta = 0$$

If small disturbance is observed,

$$\Delta\ddot{\theta} = -\frac{g}{l}\Delta\theta$$

The acceleration is ~~towards~~ the equilibrium position. Hence for any disturbance, the pendulum will ~~s~~ swing back to equilibrium position.

Hence $\theta = 0$ is a stable equilibrium.

for $\theta = \pi$

$$\Delta f = \left(\frac{df}{d\theta}\right)_{\theta=\pi} \Delta\theta \doteq -\Delta\theta$$

$$\Delta\ddot{\theta} - \frac{g}{l}\Delta\theta = 0$$

$$\therefore \Delta\ddot{\theta} = \frac{g\Delta\theta}{l}$$

For small disturbance $\Delta\theta$, the acceleration will be positive and will move away ~~from~~ the equilibrium point. As $\Delta\theta$ will increase, it will in turn increase $\Delta\ddot{\theta}$.

Hence $\theta = \pi$ is an unstable equilibrium point.

iii) Consider pendulum equation,

$$\ddot{\theta} = -a\sin\theta - b\dot{\theta} + cT.$$

We want to stabilize pendulum at $\theta = \delta$.

To maintain ~~eq~~ at $\theta = \delta$, we must apply steady state torque ~~ss~~ such that,

$$\ddot{\theta} = -a\sin\theta - b\dot{\theta} + cT_{ss} = 0$$

$$-a\sin\theta + cT_{ss} = 0$$

Let $x_1 = \theta - \delta$, $x_2 = \dot{\theta}$ & ~~u~~ $= T - T_{ss}$

<u>State equations</u> -

$$\dot{x}_1 = x_2; \quad \dot{x}_2 = -a[\sin(x_1+\delta) - \sin\delta] - bx_2 + cu$$

Linearising around origin,

$$A = \begin{bmatrix} 0 & 1 \\ -a\cos(x_1+\delta) & -b \end{bmatrix}\Bigg|_{x_1 = 0}$$

$$= \begin{bmatrix} 0 & 1 \\ -a\cos\delta & -b \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ c \end{bmatrix}$$

Taking $K = [K_1 \; K_2]$, where $K_1 > -\frac{a\cos\delta}{c}$

$K_2 > -b/c$, $A - bK$ can be made Hurwitz.

The torque is given by $T = \frac{a\sin\delta}{c} - Kx$

$$T = \frac{a\sin\delta}{c} - K_1(\theta - \delta) - K_2\dot{\theta}$$

iv) A mathematically proof to show limitation of linearization

Let examine a simple 1st order system.

$$\dot{x} = ax^2 + u \; ; \; x(0) = x_0$$

Then a certainly linearizing control law for regulation is $u = -\hat{a}x^2 - bx$, where b is any positive constant and $\hat{a}$ is an open loop estimate of the parameter a.

The true state dynamics are -

$$\dot{x} = \epsilon x^2 - bx \; ; \; x(0) = x_0 \quad (\epsilon = a - \hat{a})$$

For $x_0 \epsilon > b$, the above would be unstable.

Similar problems ~~occur~~ occur in higher order systems also.

2. Advantages of orthogonal and ~~of the~~ orthonormal bases -

- After obtaining basis of a vector space, expressing other elements as linear combination of basis elements is very difficult task. It becomes very hard to find their coordinates in the prescribed basis.

The main advantage of orthogonal/normal bases is that the change of basis computation becomes relatively easy.

Gram Schmidt's formula :

$$V_K = W_K - \sum_{j=1}^{K-1} \frac{\langle W_K, V_j \rangle}{\|V_j\|^2} V_j \qquad K = 1, 2, \ldots$$

for $n=3$

$$V_1 = W_1$$
$$V_2 = W_2 - \frac{\langle W_2, V_1 \rangle V_1}{\|V_1\|^2}$$

$$V_3 = W_3 - \frac{\langle W_3, V_1 \rangle V_1}{\|V_1\|^2} - \frac{\langle V_3, V_2 \rangle V_2}{\|V_2\|^2}$$

c) $\underset{W_1}{\begin{pmatrix}1\\0\\1\end{pmatrix}}$ $\underset{W_2}{\begin{pmatrix}1\\1\\1\end{pmatrix}}$ $\underset{W_3}{\begin{pmatrix}-1\\2\\1\end{pmatrix}}$ $\Rightarrow V_1 = W_1$

$\|V_1\| = \sqrt{2}$

$$V_2 = W_2 - \frac{\langle W_2 . V_1 \rangle}{2} V_1$$

$$\langle W_2 . V_1 \rangle = W_2 . V_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\0\\1\end{bmatrix} = 2$$

$$\Rightarrow V_2 = \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$V_3 = W_3 - \frac{\langle W_3 V_1 \rangle V_1}{\|V_1\|^2} - \frac{\langle W_3 \; V_2 \rangle}{\|V_2\|^2} * V_2$$

$\|V_2\| = 1$; $\langle W_3 V_2 \rangle = \begin{bmatrix}-1\\2\\1\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = 2$

$$\Rightarrow V_3 = \begin{bmatrix}-1\\2\\1\end{bmatrix} - 0 - 2\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$\|V_3\| = \sqrt{2}$

Orthogonal basis -

$$\frac{V_1}{\|V_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}$$

$$\frac{V_2}{\|V_2\|} = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \frac{V_3}{\|V_3\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}$$

$$V_1 = \begin{bmatrix}0.7071\\0\\0.7071\end{bmatrix}, \quad V_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad V_3 = \begin{bmatrix}-0.7071\\0\\0.7071\end{bmatrix}$$

b) $\begin{pmatrix}1\\1\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\\-1\end{pmatrix}$ $\begin{pmatrix}1\\0\\-1\end{pmatrix}$

w_1 $\quad w_2$ $\quad w_3$

$$V_1 = w_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\|V_1\| = \sqrt{2}$$

$$V_2 = w_2 - \frac{\langle w_2 V_1\rangle}{\|V_1\|^2} V_1$$

$$\langle w_2 V_1\rangle = w_2 \cdot V_1 = \begin{bmatrix}0\\1\\-1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix} = 1$$

$$V_2 = \begin{bmatrix}0\\1\\-1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}-1/2\\+1/2\\1\end{bmatrix}; \quad \|V_2\| = \sqrt{\frac{1}{4}+\frac{1}{4}+1}$$

$$\|V_2\| = \sqrt{3/2}$$

$$V_3 = w_3 - \frac{\langle w_3 \cdot V_1\rangle}{\|V_1\|^2} V_1 - \frac{\langle w_3 \cdot V_2\rangle}{\|V_2\|^2} V_2$$

$$\langle w_3 \cdot V_2\rangle = w_3 V_2 = \begin{bmatrix}1\\0\\-1\end{bmatrix}\begin{bmatrix}-1/2\\1/2\\-1\end{bmatrix} = 1/2$$

$$\langle w_3 . V_1 \rangle = w_3 . V_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1$$

$$V_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} -1/2 \\ 1/2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ -2/3 \end{bmatrix}$$

$$\|V_3\| = \frac{2}{\sqrt{3}}$$

Orthogonal basis

$$\frac{V_1}{\|V_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.7071 \\ 0.7071 \\ 0 \end{bmatrix}$$

$$\frac{V_2}{\|V_2\|} = \frac{\sqrt{2}}{\sqrt{3}} \begin{bmatrix} -1/2 \\ +1/2 \\ -1 \end{bmatrix} = \begin{bmatrix} -0.4083 \\ 0.4083 \\ -0.8165 \end{bmatrix}$$

$$\frac{V_3}{\|V_3\|} = \frac{\sqrt{3}}{2} \begin{bmatrix} 2/3 \\ -2/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 0.5774 \\ -0.5774 \\ -0.5774 \end{bmatrix}$$

c) $\underset{w_1}{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} \underset{w_2}{\begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}} \underset{w_3}{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}}$

$V_1 = w_1 \quad \|V_1\| = \sqrt{14}$

$$V_2 = w_2 - \frac{\langle w_2 . V_1 \rangle}{\|V_1\|^2} V_1$$

$$\langle w_2 . V_1 \rangle = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 10 + 4 + 0 = 14$$

$$\rightarrow V_2 = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} - \frac{14}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}; \|V_2\| = 3\sqrt{3}$$

$$V_3 = \omega_3 - \frac{\langle w_3 V_1 \rangle V_1}{\|V_1\|^2} - \frac{\langle \omega_3 V_2 \rangle}{\|V_2\|^2} V_2$$

$$\langle \omega_3 V_1 \rangle = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2 + 6 - 3 = 5$$

$$\langle w_3 V_2 \rangle = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} = 18$$

$$V_3 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} - \frac{5}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{18}{27} \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}$$

$$V_3 = \begin{bmatrix} -5/14 \\ 2/7 \\ -1/14 \end{bmatrix} ; \quad \|V_3\| = \sqrt{\left(\frac{-5}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(\frac{-1}{14}\right)^2} = \frac{\sqrt{42}}{14}$$

For orthogonal basis

$$\frac{V_1}{\|V_1\|} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.2673 \\ 0.5345 \\ 0.8018 \end{bmatrix}$$

$$\frac{V_2}{\|V_2\|} = \frac{1}{3\sqrt{3}} \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0.5774 \\ 0.5774 \\ -0.5774 \end{bmatrix}$$

$$\frac{V_3}{\|V_3\|} = \frac{14}{\sqrt{42}} \begin{bmatrix} -5/14 \\ 2/7 \\ -1/14 \end{bmatrix} = \begin{bmatrix} -0.7715 \\ 0.6172 \\ -0.1543 \end{bmatrix}$$

3. Ways to check positive semi-definiteness of a matrix:

→ A real $n \times n$ symmetric matrix S is called positive semi-definite matrix if for every non-zero column matrix 'x' of 'n' elements satisfy

$$x^T S x \geq 0$$

→ All eigenvalues are non-negative

→ Determinants of all upper-left sub-matrices are non-negative (≥ 0) where, pivots should be found in the row echelon matrix which is obtained by Gaussian elimination.

Given $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

Using upper-left det. test,

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$D_1 = 2 \geq 0$

$D_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3 \geq 0$

$D_3 = 2(4-1) - (-1)(-2 \cdot 1) + (-1)(3)$
$= 4 \geq 0$

We can see that all upper left sub-matrices $D_1 \geq 0$, $D_2 \geq 0$ and matrix $D_3 \geq 0$, so we can say that A is positive semi-definite matrix.

4. Continuity of functions:

The notion of distance in a normed linear space enables us to define continuity of functions.

Let $(X, \|\cdot\|_X)$ & $(Y, \|\cdot\|_Y)$ be two normed linear spaces and suppose $f: X \to Y$.

Then function f is said to be continuous at $x_0 \in X$ if $\forall\ \epsilon > 0$, there exists $\delta = \delta(\epsilon, x_0)$ such that

$$\|f(x) - f(x_0)\|_Y < \epsilon \text{ whenever } \|x - x_0\|_X < \delta$$

f is said to be continuous if it is continuous at all $x \in X$. f is said to be uniformly continuous if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon)$ such that $\|f(x) - f(x_0)\|_Y < \epsilon$ whenever $\|x - x_0\|_X < \delta$

$\epsilon - \delta$ interpretation of stability

The equilibrium point $x = 0$ of $\dot{x} = f(x)$ (where $f(x)$ is continuous and is a locally Lipschitz map from a domain $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$)

i) is stable if for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that $\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon \ \forall\ t \geq 0$

ii) Unstable if not stable

iii) Asymptotically stable if it is stable and δ can be chosen such that $\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$

Continuity plays an important role in stability, because if a function is continuous at some point for every ϵ, there exist a δ such that we have a boundness behaviour across that point.

Moreover the eqn used for stability $\dot{x} = f(x)$, we can see that $\dot{x}$ comes into picture, which means x is definitely continuous, for $\dot{x}$ to exist. Hence, we can arrive at stability by using notion of continuity.

$$\dot{x}(t) = -\left(1+\sin^2(x(t))\right)x(t) \qquad x(t) \in R$$

$$\cancel{\frac{dx(t)}{dt}} \Rightarrow \int_0^t \frac{dx(t)}{x(t)} = \int_0^t -\left(1+\sin^2(x(t))\right)dt$$

$$\ln\left(\frac{x(t)}{x(0)}\right) = \int_0^t -\left(1+\sin^2(x(\tau))\right)d\tau$$

$$x(t) = x(0) \cdot e^{-\int_0^t d\tau} \cdot e^{-\int_0^t \sin^2 x(\tau)\, d\tau}$$

$$= x(0) \cdot e^{-t} \cdot e^{-\int_0^t \sin^2 x(\tau)\, d\tau}$$

$$0 \le \sin^2 x(\tau) \le 1$$

$$\Rightarrow \int_0^t \sin^2(x(\tau))\, d\tau \ge 0$$

$$\Rightarrow x(t) \le x(0) e^{-t}$$

$$\boxed{\|x(t)\| \le \|x(0)\| e^{-t}} \quad - 1$$

i) Stable: If we choose some $\epsilon > 0$, there exist $\delta > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon \quad \forall \ t \geq 0$$

$$(\because \|x(t)\| \leq \|x(0)\| e^{-t} < \epsilon)$$

$\Rightarrow$ The system is stable

ii) Asymptotically stable: It is stable and $\|x(0)\| < \delta$

$$\Rightarrow \lim_{t \to \infty} x(t) = \lim_{t \to \infty} \|x(0)\| e^{-t} = \lim_{t \to \infty} x(t) = 0$$

$\Rightarrow$ The system is asymptotically stable.

iii) Exponentially stable: An equilibrium point x is exponentially stable if there exist two strictly positive numbers 'α' & 'λ' such that

$$\|x(t)\| \leq \alpha \|x(0)\| e^{-\lambda t} \quad \forall \ t > 0$$

Comparing 1 and the above equation,

~~Here~~, $\alpha = 1$, $\lambda = 1$

Since ~~all~~ α and λ are > 0 and it satisfies the equation above, for all $t > 0$ - hence it is said to be exponentially stable.