Tutorial 3 –Solutions

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Q1) A door shutter has a spring which, in the absence of damping, shuts the door in 0.5 seconds. The problem is that the door bangs shut with a speed 1 m/s. A damper with a damping coefficient is introduced to ensure that the door shuts gradually. What are the times required for the door to shut and the velocities of the door at the instant it shuts, if damping coefficient β= 0.9π. Note that the spring is unstretched when the door is shut.

The equation of displacement of the door shutter, in the absence of damping, is

$$x(t) = a \cos \omega_0 t + b \sin \omega_0 t \quad \text{------------} \quad \text{(i)}$$

When the door is shut, then x = 0. The shutter shuts the door in 0.5 seconds; so $T_0/4 = 0.5$ sec, where T_0 is the time period of oscillation of the shutter attached to the spring.

$$\therefore \frac{dx}{dt} = \omega_0(-a \sin \omega_0 t + b \cos \omega_0 t) \quad \text{---------------} \text{(ii)}$$

Now imposing the conditions

$$\frac{dx}{dt}(t = 0) = 0$$

$$\frac{dx}{dt}(t = T_0/4) = 1 \text{ m/s}$$

$$T_0/4 = (2\pi/4\omega_0) = 0.5 \text{ sec} \Rightarrow T_0 = 2 \text{ sec}, \boldsymbol{\omega_0 = \pi} \textbf{ rad/sec}$$

$$\therefore b\omega_0 = 0 \Rightarrow b = 0,$$

$$\text{And, } -a\pi\sin(\pi/2) = 1 \Rightarrow \mathbf{a = -\frac{1}{\pi}}$$

Now, if a damper with a damping coefficient β is introduced, then the equation for x(t) can be expressed as,

$$x(t) = e^{-\beta t}[c \cos\omega t + d \sin \omega t] \quad \text{-----------------}\text{(iii)}$$

where $\omega = (\omega_0^2 - \beta^2)^{1/2}$.

$$\text{Again, } \frac{dx}{dt} = -\beta e^{-\beta t}[c \cos\omega t + d \sin \omega t] + e^{-\beta t}[-\omega c \sin\omega t + \omega d \cos\omega t]$$

Now imposing the conditions:

$$x(t = 0) = a, \ \frac{dx}{dt}(t = 0) = 0, \text{ we get,}$$

$$\mathbf{c = a = -\frac{1}{\pi}}, \quad \boldsymbol{d = \frac{\beta}{\omega} c} \textbf{ .}$$

The equation (iii) can be written as

$$x(t) = ae^{-\beta t}\,(\cos\omega t + \frac{\beta}{\omega}\sin\omega t) \quad \text{------------------ (iv)}$$

Now, in the presence of damping, if the shutter shuts the door in time t_0, then $x(t_0) = 0$.

Using the equation (iv) we have

$$\cos\omega t_0 = -\frac{\beta}{\omega}\sin\omega t_0$$

$$\text{or, } -\tan\omega t_0 = \frac{\omega}{\beta}$$

$$\text{But, } \tan(\pi-\theta) = -\tan(\theta)$$

$$\therefore \tan(\pi-\omega t_0) = -\tan(\omega t_0)$$

$$\text{or, } \pi-\omega t_0 = \tan^{-1}\frac{\omega}{\beta} \Rightarrow t_0 = \frac{\pi}{\omega} - \frac{1}{\omega}tan^{-1}(\frac{\omega}{\beta}) \quad \text{----------- -(v)}$$

The velocity of the door at the instant it shuts is

$$\frac{dx}{dt}(t = t_0) = -a\omega e^{-\beta t_0}(1 + \frac{\beta^2}{\omega^2})\sin\omega t_0 \quad \text{---------------------(vi)}$$

For $\beta = 0.9\pi$, $\omega_0 = \pi \Rightarrow$ $\boldsymbol{\omega = \sqrt{(0.19)}\pi}$

$\therefore$ Substituting in (v), we get $\boxed{t_0 \sim 1.96\text{sec}}$

Substituting in (vi), we get $\boxed{\frac{dx}{dt}(t_0) \sim 0.004 \text{ m/s} \sim 0 \text{ m/s.}}$ → Answer

Q2) Solve the initial value problem set up by the differential equation below. Find both the complementary solution and the particular solution.

$\ddot{x} + 64x = 5\cos 3t + 10\sin 3t$

Where $x(t = 0) = (12/11)$ and $\dot{x}(t = 0) = (5/11)$.

Take complex-exponential form of the particular solution and extract the real part.

Solution:

Differential Equation:

$$\ddot{x} + 64x = 5\cos 3t + 10\sin 3t \quad \text{-----------(i)}$$

We modify the RHS of Eqn (i) and write:

$$\ddot{x} + 64x = 5\sqrt{(2^2+1)}\cos(3t+\phi)$$

$$\text{or, } \ddot{x} + 64x = 5\sqrt{5}\,\mathrm{Re}[e^{i(3t+\phi)}], \text{ --------------- (ii)}$$

$$\text{where, } \cos\phi = 1/\sqrt{5},\ \sin\phi = -2/\sqrt{5}$$

Solution of the above Eqn. can be written as:

$$x = x_{CF} + x_{PI},$$

where, x_{CF} is the complementary function and x_{PI} is the particular integral.

$$x_{CF} = A_1\cos\omega_0 t + A_2 \sin\omega_0 t$$

$$\text{with } \omega_0 = \sqrt{64} = 8$$

$$x_{PI} = 5\sqrt{5}Re\left[\frac{1}{D^2+64}e^{i(3t+\phi)}\right]$$

$$\text{or, } x_{PI} = 5\sqrt{5}Re\left[\frac{1}{-9+64}e^{i(3t+\phi)}\right]$$

$$\text{or, } x_{PI} = \frac{\sqrt{5}}{11}Re\left[e^{i(3t+\phi)}\right],$$

$$\text{or} \quad x_{PI} = \frac{\sqrt{5}}{11}\cos(3t+\phi) \qquad \text{-------------------(iii)}$$

$$\therefore x(t) = A_1\cos 8t + A_2\sin 8t + \frac{\sqrt{5}}{11}\cos(3t+\phi), \qquad \tan\phi = -2 \text{ -------------(iv)}$$

Initial values:

$$x(t=0) = (12/11) \text{ and } \dot{x}(t=0) = (5/11)$$

$$\therefore A_1 + \frac{\sqrt{5}}{11}\cos\phi = \frac{12}{11}, \quad \cos\phi = \frac{1}{\sqrt{5}}, \ \therefore \mathbf{A_1 = 1}$$

$$\text{and } 8A_2 - \frac{\sqrt{5}}{11}\times 3\sin\phi = \frac{5}{11}, \quad \sin\phi = -\frac{2}{\sqrt{5}}, \ \therefore \mathbf{A_2 = -\frac{1}{88}}$$

$$\boxed{\therefore x(t) = \cos 8t - \frac{1}{88}\sin 8t + \frac{\sqrt{5}}{11}\cos(3t+\phi), \qquad \tan\phi = -2} \rightarrow \text{Answer}$$

Q3) A 20 gm mass is suspended by a spring of spring constant k = 8 N/m. The mass performs steady state vertical oscillations of amplitude 1.0 cm due to an external harmonic force of frequency $\omega = 10s^{-1}$. The displacement lags behind the force by an angle $(\pi/4)$. Find the quality factor Q of the oscillator.

Solution

Natural frequency of oscillation $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{20\times10^{-3}}} = 20\ \text{sec}^{-1}$

Under the effect of external force, the equation of motion is that of a damped harmonic oscillator under an external force:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F}{m} \sin \omega t \quad \text{----------} \quad \text{(i)}$$

The complementary solution is a transient solution which dies after some time. The particular integral (PI) is the steady state solution.

PI can be written as:

$$x_{PI}(t) = \frac{1}{D^2+2\beta D+\omega_0^2}\frac{F}{m}\sin \omega t$$

$$x_{PI}(t) = \frac{F}{m} Im \frac{1}{D^2+2\beta D+\omega_0^2} e^{i\omega t}$$

$$\text{or,}\quad x_{PI}(t) = \frac{F}{m} Im \frac{1}{-\omega^2+2i\beta\omega+\omega_0^2} e^{i\omega t}$$

Rationalizing the denominator:

$$x_{PI}(t) = \frac{F}{m} Im \frac{e^{i\omega t}\,(\omega_0^2-\omega^2 - 2i\beta\omega)}{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}$$

Taking the imaginary part,

$$x_{PI}(t) = \frac{F}{m}\frac{1}{\sqrt{[(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2]}}\left[\frac{(\omega_0^2-\omega^2)\sin\omega t - 2\beta\omega\cos\omega t}{\sqrt{[(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2]}}\right]$$

$$\text{or}\quad x_{PI}(t) = \frac{F}{m}\frac{1}{\sqrt{[(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2]}}\sin(\omega t - \varphi)$$

$$\text{Phase angle, } \varphi = tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\right) = \pi/4$$

$$\omega = 10\ \text{sec}^{-1}, \text{then } \beta = \frac{\omega_0^2-\omega^2}{2\omega}.$$

So, the quality factor Q of the oscillator: $Q = \omega_0/2\beta$

$$\text{or } Q = \frac{\omega\omega_0}{\omega_0^2-\omega^2}$$

$\boxed{Q=0.667}$ → Answer

Q4) A transverse sine wave with amplitude of 2.50 mm and a wavelength of 1.80 m travels from left to right along a very-long, horizontal stretched string with a speed of 36.0 m/s. Take the origin at the left end of the undisturbed string. At time t=0, the left end of the string has its maximum upward displacement.

(a) What are the frequency, angular frequency and wave number of the wave?

(b) What is the function $y(x,t)$ that describes the wave?

(c) What is $y(t)$ for a particle 1.35 m to the right of the origin?

Solution:

a) Frequency $f = \frac{v}{\lambda} = \frac{36}{1.8} = 20Hz$

Angular frequency $\omega = 2\pi f = 40\pi \text{ rad/sec}$ → Answer

Wavenumber $\bar{v} = {}^{1}/_{\lambda} = 0.555\ m^{-1}$

Wavevector $k = {}^{2\pi}/_{\lambda} \sim 3.49\ m^{-1}$

b) $y(x,t) = A_0 \sin(\omega t - kx + \phi)$

$A_0 = 2.5\text{mm} = 2.5\times10^{-3}$ m, $\quad k = \frac{2\pi}{\lambda} = \frac{2\pi}{1.8}$

At $t = 0$, $y(0,0) = y_{max} = 2.5$ mm

$\therefore \sin\phi = 1 \rightarrow \phi = \pi/2$

$$\therefore \boxed{y(x,t) = 2.5x10^{-3} \cos\left(40\pi t - \frac{2\pi}{1.8}x\right)}$$ → Answer

c) $y(t)$ at x = 1.35m

$\therefore\ y(t) = 2.5x10^{-3} \cos\left(40\pi t - \frac{2\pi}{1.8}1.35\right)$

or $y(t) = 2.5x10^{-3} \cos\left(40\pi t - \frac{3\pi}{2}\right)$

$$\therefore \boxed{y(t) = -2.5x10^{-3} \sin(40\pi t)}$$ → Answer