Week 4: Waves

The coupled system consisting of two masses connected by three springs can be further generalised by including more masses and springs. If we denote the longitudinal displacement of the i-th mass as $\xi_i(t)$ then its equation of motion is given as:

$$m\ddot{\xi}_i = -k\left(\xi_i - \xi_{i-1}\right) - k\left(\xi_i - \xi_{i+1}\right) \tag{1}$$

where we have assumed that all springs and masses are identical.

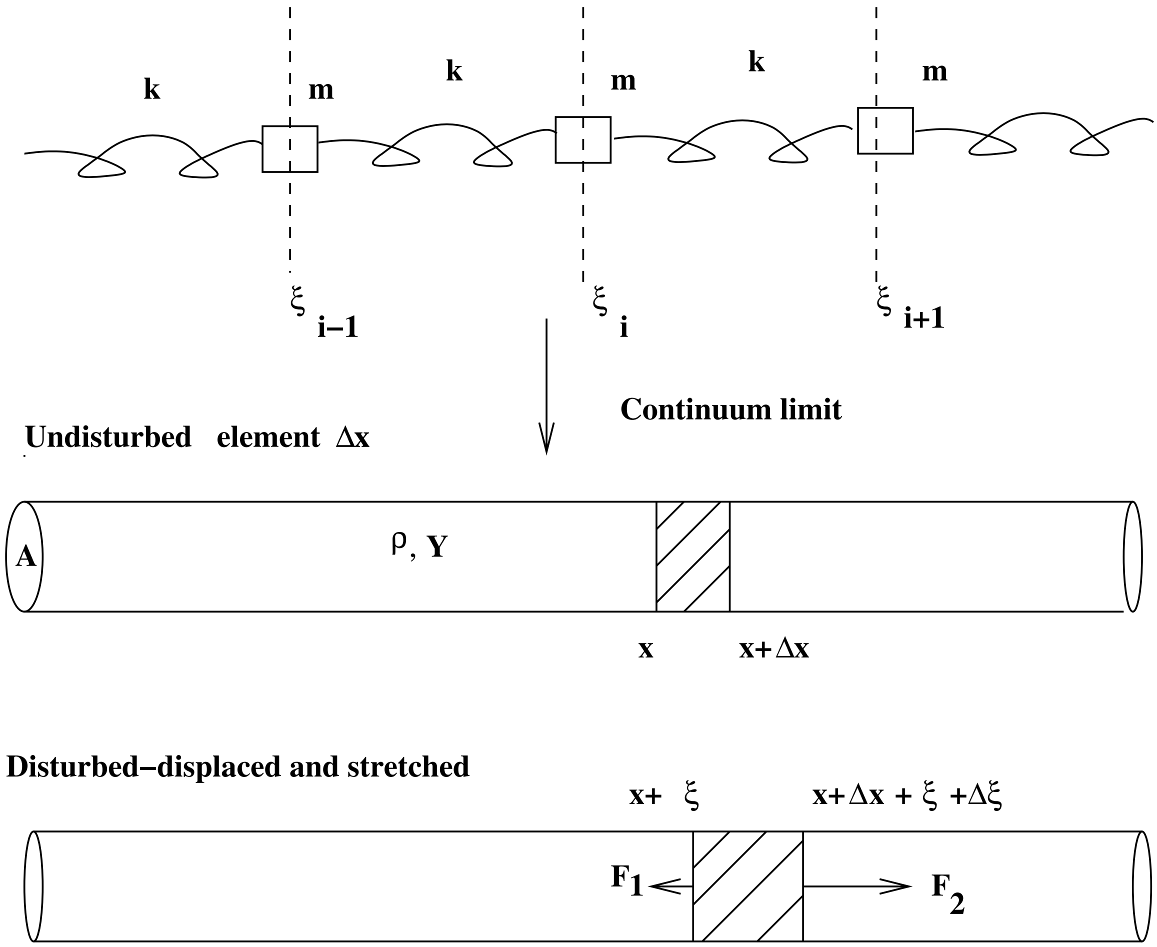

FIG. 1: Longitudinal waves

In the **continuum** limit when the masses are distributed along the line joining them and the springs are made smaller, we end up with an elastic rod. The index i of ξ_i is replaced by a continuous variable x and ξ_i goes over to $\xi(x,t)$–a function of the two variables x and t–thereby implying the necessity of using partial derivatives of ξ. Let us now consider an element of the rod between x and $x + \Delta x$. If the mass density is denoted as ρ and the cross-sectional area of the elastic rod as A, then $m \to \rho A \Delta x$ for the mass in the element of length Δx. Let us now look at the $R.H.S.$ of the above equation. What happens to k? In the continuum limit $k \to \frac{YA}{\Delta x}$ where Y is the Young's modulus of the elastic rod. This follows from the relation

$$Stress = \frac{F}{A} = Y \times Strain = Y\frac{\Delta l}{l} \tag{2}$$

from which we have

$$F = \left(\frac{YA}{l}\right)\Delta l \tag{3}$$

In our case the un-stretched length is Δx and hence the effective k in the continuum limit is $\frac{YA}{\Delta x}$.

We also note the following:

$\xi_i - \xi_{i-1} \to \xi(x,t) - \xi(x-\Delta x, t) \to \frac{\partial \xi}{\partial x}|_x \Delta x$

$\xi_{i+1} - \xi_i \to \xi(x+\Delta x,t) - \xi(x, t) \to \frac{\partial \xi}{\partial x}|_{x+\Delta x} \Delta x$

Thus, we have

$$\begin{aligned} k\left(\xi_{i+1} - \xi_i\right) - k\left(\xi_i - \xi_{i-1}\right) &\to \frac{YA}{\Delta x}\left(\frac{\partial \xi}{\partial x}|_{x+\Delta x} - \frac{\partial \xi}{\partial x}|_x\right)\Delta x \\ &= \frac{YA}{\Delta x}\frac{\partial^2 \xi}{\partial x^2}\left(\Delta x\right)^2 \end{aligned} \tag{4}$$

What about the L. H. S.? Using Newton's second law and the correspondences mentioned above, we find it to be equal to $\rho A \Delta x \frac{\partial^2 \xi}{\partial t^2}$. Thus, equating both sides we obtain,

$$\frac{Y}{\rho}\frac{\partial^2 \xi}{\partial x^2} = \frac{\partial^2 \xi}{\partial t^2} \tag{5}$$

If we define $\frac{Y}{\rho} = v^2$ then,

$$v^2\frac{\partial^2 \xi}{\partial x^2} = \frac{\partial^2 \xi}{\partial t^2} \tag{6}$$

This is known as the **wave equation** for $\xi(x,t)$. Here, since $\xi(x,t)$ is along the rod, we call the disturbance $\xi(x,t)$ as **longitudinal**. v is the speed of the disturbance as it moves along x. The above equation is an example of a **linear, second order partial differential equation**.

Surely, there is a **propagating** character of $\xi(x,t)$. How do we see that? Let try to find the general solution of the wave equation. To do this we define two new variables

$$u = x - vt \quad ; \quad w = x + vt \tag{7}$$

We then rewrite the wave equation using these new coordinates using standard rules of partial differentiation like

$$\frac{\partial \xi}{\partial t} = \frac{\partial \xi}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial \xi}{\partial w}\frac{\partial w}{\partial t} \tag{8}$$

and similar ones. The end result is that the wave equation, rewritten using the u and w become

$$\frac{\partial^2 \xi}{\partial u \partial w} = 0 \tag{9}$$

The solution of this equation is simple

$$\xi = f(u) + g(w) = f(x - vt) + g(x + vt) \tag{10}$$

where f and g are arbitrary functions.

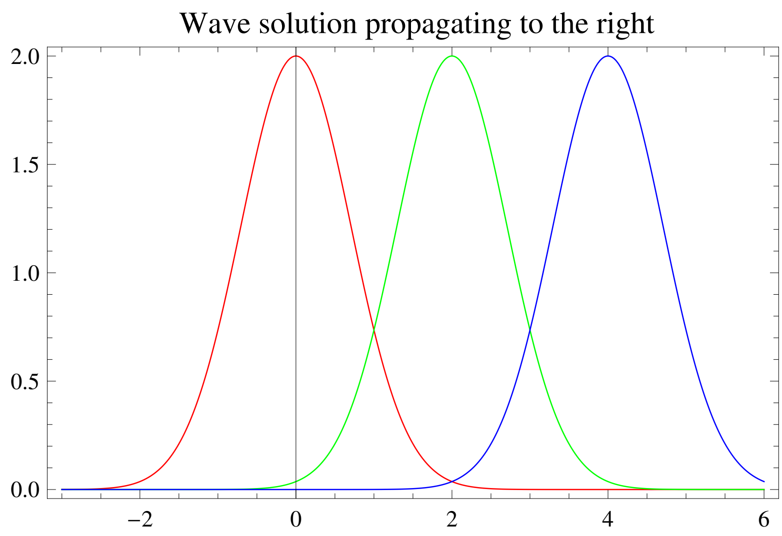

FIG. 2: Gaussian initial profile propagating to the right.

To understand the propagating aspect of this solution further, let us consider a **profile** for ξ at $t = 0$. We choose it to be

$$\xi(x, 0) = Ce^{-bx^2} \tag{11}$$

where C, b are positive, nonzero constants. At a later time the profile would look like (assuming the $f(x - vt)$ solution only)

$$\xi(x, t) = Ce^{-b(x-vt)^2} \tag{12}$$

which, obviously solves the wave equation. What happens to the initial profile in time? We can easily see that with increasing time the maximum which is at $x = 0$ at $t = 0$ **shifts** to the **right**. Similarly, if we consider the $g(x + vt)$ solution the shift, with increasing time, will be towards the **left**. Thus, we can see that the initial profile propagates in time. This is irrespective of the functional forms of f or g.

The wave equation obtained above is for longitudinal waves. For example it could be used for disturbances in a fluid, or, for the propagation of sound waves (for a discussion on sound waves in particular, see Feynman Lectures, Chapter 47).

We are, in this course, more interested in another kind of wave, known as the transverse wave. In this kind, the disturbances are orthogonal to the direction of propagation. To illustrate this let us consider the beaded string first. We can see how the normal modes look like as we keep increasing the number of beads in the string. In the continuum limit, the disturbances looks like a harmonic (sine/cosine) function (for a discussion and pictures of the modes of beaded strings see Berkeley Physics Course, Volume 3, Pg 50).

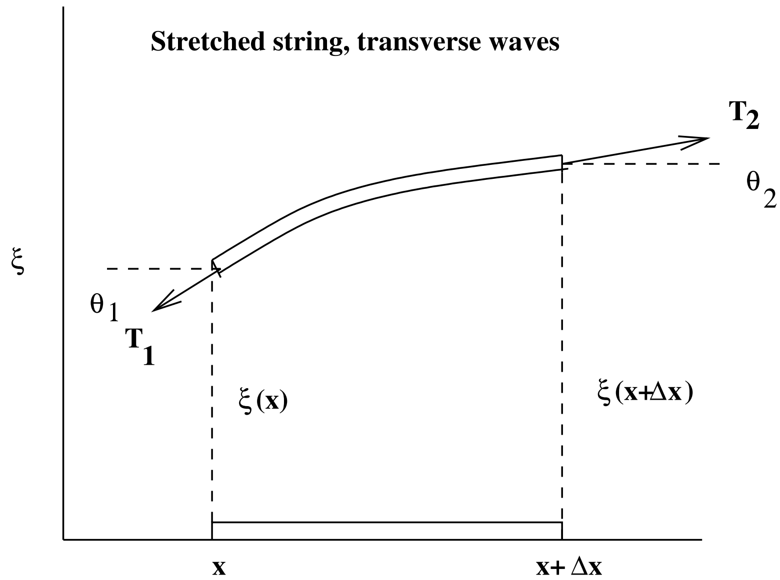

FIG. 3: A stretched string.

Let us now look at the example of a stretched string with mass per unit length μ. We will look at its transverse vibrations. Consider an element of width Δx as shown in the figure. $\xi(x, t)$ and $\xi(x + \Delta x, t)$ denote the transverse displacements at the end point. The tensions at the two ends are along the vectors $\vec{T_1}$ and $\vec{T_2}$. Resolving the tensions along the horizontal and vertical directions, we get

$$T_1 \cos\theta_1 = T_2 \cos\theta_2 = T \tag{13}$$

and

$$F = T_2 \sin\theta_2 - T_1 \sin\theta_1 = T\left(\tan\theta_2 - \tan\theta_1\right) \tag{14}$$

Using the fact that $\tan\theta = \frac{\partial\xi}{\partial x}$ we obtain

$$F = T\left(\frac{\partial\xi}{\partial x}|_{x+\Delta x} - \frac{\partial\xi}{\partial x}|_{x}\right) = T\Delta x \frac{\partial^2\xi}{\partial x^2} \tag{15}$$

for the net force along the transverse direction. Equating this force, via Newton's second law, we get, using $F = \mu\Delta x \frac{\partial^2\xi}{\partial t^2}$,

$$\frac{T}{\mu}\frac{\partial^2\xi}{\partial x^2} = \frac{\partial^2\xi}{\partial t^2} \tag{16}$$

Identifying $\frac{T}{\mu} = v^2$ (v is the speed of the transverse disturbance), we have,

$$v^2\frac{\partial^2\xi}{\partial x^2} = \frac{\partial^2\xi}{\partial t^2} \tag{17}$$

which is the **wave equation for transverse disturbances**. The equation is the same as for longitudinal waves. However the dependent variable $\xi(x,t)$ has a **different interpretation**, in the sense that ξ is now **transverse** to the direction of propagation of the wave.

One can write down a generalisation of the wave equation to three space dimensions. The equation would look like

$$\nabla^2\xi = \frac{1}{v^2}\frac{\partial^2\xi}{\partial t^2} \tag{18}$$

where ∇^2 is the Laplacian operator and

$$\nabla^2\xi = \frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2} + \frac{\partial^2\xi}{\partial z^2} \tag{19}$$

We will consider special solutions of the three dimensional wave equation known as **transverse harmonic waves**. Such solutions resemble the solutions in mechanical oscillators known as **harmonic oscillations**, which we have studied earlier (remember $x(t) = A\cos\left(\omega t + \delta_0\right)$). If we define the 'total phase' in harmonic oscillations as $\delta = \omega t + \delta_0$, then $\frac{\partial\delta}{\partial t} = \omega$, which 'defines' the angular frequency as the 'time derivative of the phase'. Similarly we can define the argument in a harmonic solution for $\xi(\mathbf{r}, t)$ as $\mathbf{k}\cdot\mathbf{r} - \omega t$ such that

$$\xi(\mathbf{r}, t) = \xi_0 \cos\left(\mathbf{k}\cdot\vec{\mathbf{r}} - \omega t + \delta_0\right) \tag{20}$$

Such a wave is named a transverse harmonic wave and defining the phase as

$$\delta = \left(\mathbf{k}\cdot\vec{\mathbf{r}} - \omega t + \delta_0\right) \tag{21}$$

we find

$$\nabla\delta = \mathbf{k} \quad ; \quad \frac{\partial\delta}{\partial t} = -\omega \tag{22}$$

$\mathbf{k}$ and ω are related to the scale of spatial and temporal oscillations. δ_0 is a fixed quantity defined as the phase at $(x, y, z) = (0, 0, 0)$ and $t = 0$. ξ_0 is a constant quantity related to the amplitude of the wave.

The quantity $\mathbf{k}$ is named the **wave vector**. It is a fixed vector with components k_x, k_y and k_z having specific numerical values for a given waveform. ω is the angular frequency. Further, $\omega = 2\pi\nu$, where ν is the frequency. Using the magntitude of the $\mathbf{k}$ vector one defines the wavelength: $\lambda = \frac{2\pi}{|\vec{\mathbf{k}}|}$.

Exercise: Show by explicit calculation that $\xi(\vec{\mathbf{r}}, t) = \xi_0 \cos(\mathbf{k} \cdot \vec{\mathbf{r}} - \omega t + \delta_0)$ is a solution of the three dimensional wave equation. You will have to calculate $\nabla\xi$, $\nabla \cdot \nabla\xi$ and then $\frac{\partial^2\xi}{\partial t^2}$. This will result in a relation between ω, k and v, if the given ξ has to be a solution. What is this relation?

A surface of constant phase (δ) defines the **wave front** or the **co-phasal surface**. One can put $\delta = constant$ and obtain the geometry, shape of the wavefront. For the above wave function, we have the wavefront as given by the surface $\mathbf{k} \cdot \vec{\mathbf{r}} - \omega t_0 + \delta_0 = const$ at a certain $t = t_0$. As 't' changes, this surface 'moves' in the $\mathbf{k}$ direction giving the notion of propagation. In this case, our wavefront is of the form of a surface $ax + by + cz = const$ in three dimensional space, at a given time – it is thus, geometrically, a **plane** and we have a planar wavefront or a plane wave propagating in space and time.

At what speed does this wave propagate? This is determined by the v_p, which is given as (obtained in the exercise just above)

$$v_p = \frac{\omega}{k} \tag{23}$$

v_p is termed as the **phase velocity** of the wave.

In your laboratory session you will perform an experiment on transverse standing waves on a vibrating string. We will discuss the theory behind such standing waves later on.

Exercise: Show that the normal to the wavefront surface is along the vector $\mathbf{k}$. Hint: find $\nabla\delta$, which is the definition of the normal.

One can also have other types of wavefronts like sections of spheres or cylinders. The wave function for such situations will have to be written using the right coordinates and the appropriate solution of the wave equation.

We will now move on towards the primary objective of this course–understanding what light is and various phenomena associated with light. Recall that in our first lecture we had mentioned that light has something to do with electric and magnetic fields which are produced at a source and which propagate to the observer at a speed in vacuum given by $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 m/s$. The fact that the observed value of c equals that of $\frac{1}{\sqrt{\mu_0 \epsilon_0}}$ gave us the hint that light is linked with $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields. We will now try to understand this better. Obviously this requires that we go back to electromagnetism. We would like to write down the basic equations of electromagnetism known as **Maxwell's equations** in differential form. To do this, we need some knowledge of vector calculus. Therefore, we will first review some aspects of vector calculus in the next lecture and then move on to electromagnetism and electromagnetic waves.