Indian Institute of Technology Kharagpur
Department of Mathematics
MA11003 - Advanced Calculus
Problem Sheet - 11
Autumn 2022

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1. Using Beta and Gamma functions prove the following:

   (a) $\int_0^\infty \sqrt{x}\ e^{-x^3} dx = \frac{\sqrt{\pi}}{3}$

   (b) $\int_0^\infty e^{-a^2x^2} dx = \frac{\sqrt{\pi}}{2a}$

   (c) $\int_0^1 x^3\ (1-x^2)^{\frac{5}{2}}\ dx = \frac{2}{63}$

   (d) $\int_0^{\frac{\pi}{2}} \sin^m x\ dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m+2}{2})}$

   (e) $\int_0^1 \sqrt{1-x^4}\ dx = \frac{\{\Gamma(\frac{1}{4})\}^2}{6\sqrt{2\pi}}$

   (f) $\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\ dx = \frac{\pi}{\sqrt{2}}$

   (g) $\beta(m+1, n) = \frac{m}{m+n}\beta(m,n)$

   (h) $\int_a^b (x-a)^{m-1}(b-x)^{n-1}dx =$ $(b-a)^{m+n-1}\beta(m,n)$

   (i) $\int_0^1 \frac{x^n}{\sqrt{1-x^2}}\ dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})}$

   (j) $\int_0^1 x^{p-1}\ (1-x^r)^{q-1}\ dx = \frac{1}{r}\beta(\frac{p}{r}, q)$

   (k) $\int_0^1 x^{p-1}(\ln \frac{1}{x})^{\alpha-1}dx = \frac{\Gamma(\alpha)}{p^\alpha}$

   (l) $\int_0^1 \frac{dx}{(1-x^n)^{\frac{1}{n}}}\ dx = \frac{1}{n}\Gamma(\frac{1}{n})\Gamma(1-\frac{1}{n})$

2. Given $\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt,\ x > 0,\ y > 0$, show that

   (a) $\beta(x,y) = \int_0^{\frac{\pi}{2}} 2\sin^{2x-1}\theta \cos^{2y-1}\theta\ d\theta$

   (b) $\beta(x,y) = \int_0^\infty \frac{u^{x-1}}{(u+1)^{x+y}}du,\ x,\ y > 0.$

   (c) $\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

   (d) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

3. Show that

   (a) $\int_0^\infty x^m e^{-ax^n} dx = \frac{1}{n} a^{-\frac{m+1}{n}} \Gamma(\frac{m+1}{n})$, where m, n and a are positive integer .

   (b) $\int_0^\infty x^m n^{-x} dx = \frac{m!}{(\log n)^{m+1}}$, where $m \geq 0$ is an integer and $n > 0$ is a constant.

4. Show that $\sqrt{\pi}\ \Gamma(2m+1) = 2^{2m}\Gamma(m+\frac{1}{2})\Gamma(m+1)$ for any positive integer m. Hence deduce that Legendre's duplication formula $\sqrt{\pi}\Gamma(2m) = 2^{2m-1}\Gamma(m)\Gamma(m+\frac{1}{2})$.

5. Given $\beta(n,\ 1-n) = \frac{\pi}{\sin n\pi}$ if $-1 < n < 1$, prove that

$$\int_0^1 \frac{x^n + x^{-n}}{1+x^2}dx = \frac{\pi}{2}\sec\frac{n\pi}{2},\quad -1 < n < 1.$$

6. Show that $\displaystyle\int_0^\infty \frac{x^m}{x^n+a}dx = \frac{1}{n}a^{\left(\frac{m+1}{n}-1\right)}\Gamma(\tfrac{m+1}{n})\Gamma(1-\tfrac{m+1}{n})$, where $a>0$ and $0<m+1<n$.

7. Show that if m is a positive integer then

   (a) $2.\ 4.\ 6.\ 8.\ 10, \ldots., 2m = 2^{2m}\Gamma(m+1)$.

   (b) $1.\ 3.\ 5.\ 7.\ 9, \ldots., (2m-1) = \frac{2^{1-m}\Gamma(2m)}{\Gamma(m)}$

8. Evaluate the integral $\displaystyle\int_0^1 \frac{x^\alpha - 1}{\log x}\,dx$, $(\alpha > -1)$ by applying differentiating under the integral sign.

9. Using differentiation under integral sign prove the following:

   (i) $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{log(1+b\sin x)}{\sin x}\,dx = \pi\sin^{-1} b$, where $|b|<1$.

   (ii) Prove that $\displaystyle\int_0^\infty \frac{\tan^{-1}\alpha x\tan^{-1}\beta x}{x^2}\,dx = \frac{1}{2}\log\left[\frac{(\alpha+\beta)^{\alpha+\beta}}{\alpha^\alpha\beta^\beta}\right]$, $\alpha>0,\ \beta>0$.

   (iii) If $\alpha>0,\ \beta>0$, prove that $\displaystyle\int_0^{\pi/2} \log(\alpha\cos^2\theta+\beta\sin^2\theta)\,d\theta = \pi\log\frac{\sqrt{\alpha}+\sqrt{\beta}}{2}$.

10. Let $f(x,t) = (2x+t^3)^2$.

    (i) Find $\displaystyle\int_0^1 f(x,t)\,dx$.

    (ii) Prove that $\frac{d}{dt}\displaystyle\int_0^1 f(x,t)\,dx = \int_0^1 \frac{\partial}{\partial t} f(x,t)\,dx$.

11. (i) Find $F'(x)$, where $F(x) = \displaystyle\int_0^{\frac{\pi}{2}} f(x,t)\,dt$ and $f:\mathbb{R}^2\to\mathbb{R}$ is given by

$$f(x,t) = \begin{cases} \frac{\sin xt}{t} & \text{if } t\neq 0 \\ x & \text{if } t=0. \end{cases}$$

    (ii) Given $f : x \to \displaystyle\int_0^{x^2} \tan^{-1}\frac{t}{x^2}\,dt$, find $f'(x)$.

12. For any real numbers x and t, let

$$f(x,t) = \begin{cases} \frac{xt^3}{(x^2+t^2)^2} & \text{if } (x,t)\neq(0,0) \\ 0 & \text{if } (x,t)=(0,0) \end{cases}$$

and $F(t) = \displaystyle\int_0^1 f(x,t)\,dx$. Is $\frac{d}{dt}\displaystyle\int_0^1 f(x,t)\,dx = \int_0^1 \frac{\partial}{\partial t}f(x,t)\,dx$? Give the justification.

13. Find the value of the integral $\displaystyle\int_0^\infty \frac{e^{-bx}\sin ax}{x}\,dx$, where $a>0,\ b>0$ are fixed, and hence deduce the value of the integral $\displaystyle\int_0^\infty \frac{\sin ax}{x}\,dx$.

14. Find the values of the following integrals:

(i) $\displaystyle\int_0^{\frac{\pi}{2}} \log(1 - x^2 \sin^2 \theta)\, d\theta, \ |x| < 1.$

(ii) $\displaystyle\int_0^{\infty} \frac{e^{-px} \cos qx - e^{-ax} \cos bx}{x}\, dx.$

(iii) $\displaystyle\int_0^{\infty} e^{-x^2} \cos 2ax\, dx$