Physics of Waves

PH11003

Tutorial 5

Waves

08 December 2022

[5.1] A transverse wave on a string is given by $y(x,t) = 2.4cos[\frac{\pi}{20}(0.5x - 40t)]$ where both x and y are in centimeters. Find: (a) the maximum particle velocity; (b) the particle velocity at x = 1.5 cm at t = 0.25 s; (c) the maximum particle acceleration; (d) the acceleration at x = 1.5 cm and t = 0.25 s.

solution 5.1

(a) $\frac{\partial y}{\partial t}_{max} = \omega A = 2\pi \times 2.4 = 15.1cm/s$

(b) $\frac{\partial y}{\partial t}_{x,t} = 4.8\pi sin(\frac{\pi}{20}(0.5x - 40t))$ = -15.0 cm/s

(c) $a_{max} = \omega^2 A = (4.8\pi)^2 A = 94.7cm/s^2$

(d) $a = -\omega^2 Acos(0.75\pi/20 - \pi/2) = -11.1cm/s^2$

[5.2] The wave function of a standing wave on a string is given by $y(x,t) = 0.02sin(0.3x)cos(25t)$ where x and y are in centimeters and t is in seconds. (a) What is the length of the string if this function represents the third harmonic? (b) At what points is the particle velocity permanently zero?

solution 5.2

Given A = 0.02, k = 0.3 cm^{-1} , ω = 25 s^{-1} so,

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.3} = 20.9cm$$

$$v = \frac{\omega}{k} = \frac{25}{0.3} = 83.3m/s$$

(a) 3rd harmonic : $L = \frac{3\lambda}{2} = 31.4cm$

(b) L/3 = 10.5 cm, 2L/3 = 20.9 cm

[5.3] A longitudinal standing wave $\xi = a\ cos(kx)\ cos(\omega t)$ is maintained in a homogeneous medium of density ρ. Find the expressions for : (a) Potential energy density and (b) Kinetic energy density.

solution
(a) Potential energy (per unit volume) is the energy of longitudinal strain $\frac{\partial \xi}{\partial x}$
so

$$E_p = \frac{1}{2}E(\frac{\partial \xi}{\partial x})^2$$
$$E_p = (1/2)Ea^2k^2sin^2(kx)cos^2(\omega t)$$

and (b) Kinetic energy density is

$$E_k = \frac{1}{2}\rho(\frac{\partial \xi}{\partial t})^2$$
$$E_k = (1/2)\rho a^2\omega^2cos^2(kx)sin^2(\omega t)$$

Total energy

$$E = E_p + E_k = (1/2)\rho a^2\omega^2sin^2(kx)$$

[5.4] The phase velocity of a surface wave on a liquid of density ρ and surface tension T is given by

$$v_p = \left(\frac{g\lambda}{2\pi} + \frac{2\pi T}{\lambda\rho}\right)^{1/2}$$

where λ is the wavelength of the wave and g is the acceleration due to gravity, (a) Find the group velocity of the surface wave, (b) Find the λ for which v_p is minimum, (c) Evaluate the minimum value of v_p and the corresponding v_g.

solution 4

(a) since

$$v_p = \frac{\omega}{k} \quad \lambda = \frac{2\pi}{k}$$
$$\frac{\omega}{k} = \left(\frac{g}{k} + \frac{kT}{\rho}\right)^{1/2}$$
$$\omega = \left(gk + \frac{k^3T}{\rho}\right)^{1/2}$$

The

$$v_g = \frac{d\omega}{dk} = \frac{g + 3k^2T/\rho}{2(gk + k^3T/\rho)^{1/2}}$$

When v_p is minimum, so is v_p^2 and the condition for this is $\frac{d}{d\lambda}(v_p^2) = 0$, then $\frac{g}{2\pi} - \frac{2\pi T}{\lambda^2\rho} = 0$ so $\lambda = 2\pi(\frac{T}{\rho g})^{1/2}$
The minimum value of v_p is $(v_p)_{min} = \sqrt{(2)}(\frac{Tg}{\rho})^{1/4}$
and the corresponding value of v_g is $\sqrt{2}(\frac{Tg}{\rho})^{1/4}$

[5.5] A linear array of particles with equal masses m are connected by identical springs whose stiffness constant is k. The equilibrium position of the nth particle is $x_n = na$, while s_n is its displacement from equilibrium. (a) Show that

$$m\frac{d^2s_n}{dt^2} = k(s_{n+1} + s_{n-1} - 2s_n)$$

(b) Show that $s_n = Asin(kx_n - \omega t)$ is a solution provided that

$$\omega^2 = \frac{4k}{m}sin^2(\frac{ka}{2})$$

solution 5.5

(a) $md^2s_n/dt^2 = -k(s_n - s_{n-1}) + k(s_{n+1} - s_n)$
$= k(s_{n+1} + s_{n-1} - 2s_n)$
Substitute function for s_n:
$-m\omega^2 A\sin(kna - \omega t) = kA\sin[k(n+1)a - \omega t] +$
$+ kA\sin[k(n-1)a - \omega t] - 2kA\sin[kna - \omega t]$
Use $\sin A + \sin B = 2\sin[(A + B)/2]\cos[(A - B)/2]$ for first two terms, then
$-m\omega^2 A\sin(kna - \omega t) = 2kA\sin(kna - \omega t)[\cos(ka) - 1] =$
$= -4kA\sin^2(ka/2)\sin(kna - \omega t)$
Thus, $\omega^2 = 4k/m \sin^2(ka/2)$

Answers:

[5.1] (a) 15.1 cm/s (b) -15.0 cm/s (c) 94.7 cm/s^2 (d) -11.1 cm/s^2

[5.2] (a) 31.4 cm (b) 10.5 cm, 20.9 cm