(By Naveen Garg)

① Let $f(x) = a_1 \sin x + a_2 \sin 2x + \text{—} + a_n \sin nx$

& $|f'(x)| \leq |\cos x| \quad \forall \ x \in \mathbb{R}$.

Then using Lagrange Mean value Theorem, prove that $|a_1 + 2a_2 + \text{—} + na_n| \leq 1$.

Proof :→ $f(x)$ is continuous as well as differentiable on $\mathbb{R}$.

Apply LMVT over $[0, x]$, where $x > 0$,

we get $\dfrac{f(x) - f(0)}{x - 0} = f'(\eta)$

$0 < \eta < x$.

① MARK TIL HERE

But $f(0) = 0$

$\therefore \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} f'(\eta)$

$0 < \eta < x$

since, $0 < \eta < x$ & $x \to 0$

$\Rightarrow \eta \to 0$

① MARK HERE

$\therefore f'(0) = \lim\limits_{\eta \to 0} \left(a_1 \cos \eta + 2a_2 \cos 2\eta + \text{—} + n a_n \cos n\eta \right)$

$= (a_1 + 2a_2 + \text{—} + na_n)$

$\Rightarrow |f'(0)| = |a_1 + 2a_2 + \text{—} + na_n|$

(Continue)

But; $|f'(x)| \leq \cos x \quad \forall \; x \in \mathbb{R}$

$\therefore \quad |f'(0)| \leq 1$

1 MARK

$\therefore \quad |a_1 + 2a_2 + \dots + na_n| \leq 1$

Q-3 : Write Maclaurin series for $\sqrt{1+x+x^2}$ ~~to~~ for first 3-terms along with Lagrange's form of remainder term. Use this approximation to estimate $\sqrt{1.75}$

Ans :

$$f(x) = \sqrt{1+x+x^2}$$

$$f'(x) = \frac{1+2x}{2(1+x+x^2)^{1/2}}$$

$$f''(x) = -\frac{1}{4}\frac{1+2x}{(1+x+x^2)^{3/2}} + \frac{1}{(1+x+x^2)^{1/2}}$$

$$= \frac{-1/4 - 1/2\,x + 1+x+x^2}{(1+x+x^2)^{3/2}} = \frac{3/4 + x/2 + x^2}{(1+x+x^2)^{3/2}}.$$

$$f'''(x) = \frac{-\frac{3}{2}(3/4 + x/2 + x^2)(1+2x)}{(1+x+x^2)^{5/2}} + \frac{\frac{1}{2}+2x}{(1+x+x^2)^{3/2}}$$

$$= \frac{-9/8 - 3x - 3x^2 - 3x^3 + \frac{1}{2} + 5x/2 + 5x^2/2 + 2x^3}{(1+x+x^2)^{5/2}}$$

$$= \frac{-5/8 - x/2 - x^2/2 - x^3}{(1+x+x^2)^{5/2}}$$

$f(0) = 1$

$f'(0) = 1/2$

$f''(0) = 3/4$

$$\sqrt{1+x+x^2} = 1 + \frac{x}{2} + \left(3/4\right)\frac{x^2}{2!}$$

$$+ \frac{-5/8 - \theta x/2 - \frac{\theta^2 x^2}{2} - \theta^3 x^3}{\left(1+\theta x+\theta^2 x^2\right)^{5/2}}$$

$$0 < \theta < 1.$$

$\sqrt{1.75} = \sqrt{1+0.5+(0.5)^2}$ So $x = 0.5$

$$\sqrt{1.75} \simeq 1 + \frac{0.5}{2} + \frac{3}{8}(0.5)^2 = 1 + \frac{1}{4} + \frac{3}{32} = 43/32$$

$$= 1.34375$$

$|\text{Error}|_{\theta=0} = 5/8$

Writing Taylor polynomial and Remainder term - 1 mark

Doing the derivatives correctly — 1 mark

$\sqrt{1.75}$ value — 1 mark

2 (a) Along the path $y^2 = x + mx^2$ ——(1)

$$\lim_{(x,y)\to(0,0)} \frac{y^2-x}{y^4+x^2} = \lim_{x\to 0} \frac{mx^2}{(x+mx^2)^2+x^2}$$

$$= \lim_{x\to 0} \frac{m\cancel{x^2}}{\cancel{x^2}[(1+mx)^2+1]}$$

$$= \frac{m}{2},$$

1 M

which is not unique. limit does not exis

Hence $\lim_{(x,y)\to(0,0)} \frac{y^2-x}{y^4+x^2} \neq f(0,0)=0$

f is discontinuous at $(0,0)$

1

Now along x axis $(y=0)$

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x} = 1$$

while along y axis, $(x=0)$

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x} = 0$$

which implies that two different paths have two different limits so the limit doesn't exist at $(0,0)$ and hence f_x is not continuous at the origin $(0,0)$.

— (1 Mark)

Alternative: Consider the path $y=mx$

$$\lim_{(x,y)\to(0,0)} \frac{\partial f(x,y)}{\partial x} = \lim_{x\to 0} \frac{\partial}{\partial x} f(x,mx)$$

$$= \lim_{x\to 0} \frac{1+3m^2}{(1+m^2)^2} = \frac{1+3m^2}{(1+m^2)^2}$$

which depends on choice of m which implies that along different lines, different limits are there. Hence limit doesn't exist and f_x is not conts. at the origin.

Ans 2 (2 b) $f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

Now $\left.\dfrac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h}$

$$= \lim_{h\to 0} \frac{\frac{h^3}{h^2} - 0}{h}$$

$$= \lim_{h\to 0} \frac{h^3}{h^3} = 1.$$

So $\boxed{f_x(0,0) = 1.}$ —— (1 mark)

Now $\left.\dfrac{\partial f}{\partial x}\right|_{(x,y)\neq(0,0)} = \dfrac{\partial}{\partial x}\left(\dfrac{x^3}{x^2+y^2}\right)$

$$= \frac{(x^2+y^2)\,3x^2 - x^3\,(2x)}{(x^2+y^2)^2}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)\neq(0,0)} = \frac{x^4+3x^2y^2}{(x^2+y^2)^2}$$

Now

$$\boxed{\frac{\partial f}{\partial x} = \begin{cases} \dfrac{x^4+3x^2y^2}{(x^2+y^2)^2} & , (x,y) \neq (0,0) \\ 1 & , (x,y) = (0,0) \end{cases}}$$ (1 mark)

Test differentiability of the following function at the origin

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^4+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

(3 Marks)

Solution :-

Here $f_x(0,0) = \lim\limits_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h} = 0$

[1 Mark]

$f_y(0,0) = \lim\limits_{K \to 0} \dfrac{f(0,K) - f(0,0)}{K} = 0$

Therefore, $df = h\, f_x(0,0) + K\, f_y(0,0) = 0$

Now, $\Delta f = f(0+h, 0+K) - f(0,0)$
$= f(h,K).$

Let $\Delta\rho = \sqrt{h^2+K^2}$

$\therefore \lim\limits_{\Delta\rho \to (0,0)} \dfrac{\Delta f - df}{\Delta \rho} = \lim\limits_{(h,K)\to(0,0)} \dfrac{f(h,K) - 0}{\sqrt{h^2+K^2}}$

$= \lim\limits_{(h,K)\to(0,0)} \dfrac{h^2K^2}{(h^4+K^2)\sqrt{h^2+K^2}}$

$\leq \lim\limits_{(h,K)\to(0,0)} \dfrac{h^2}{\sqrt{h^2+K^2}}$ $\quad [\because h^4+K^2 \geq K^2]$

$= 0.$ [2 Mark]

Hence, the given function is differentiable at $(0,0)$. [because, by Sandwich theorem on limit, $\lim\limits_{\Delta\rho \to 0} \dfrac{\Delta f - df}{\Delta \rho} = 0$]

3(b) If $z = f(x, y)$, $x = e^{2u} + e^{-2v}$, $y = e^{-2u} + e^{2v}$, then show that $\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left[x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right]$.

Sol: Using chain rule, we obtain

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 2e^{2u}\frac{\partial f}{\partial x} - 2e^{-2u}\frac{\partial f}{\partial y} \qquad \text{[1M]}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = -2e^{2v}\frac{\partial f}{\partial x} + 2e^{2v}\frac{\partial f}{\partial y} \qquad \text{[1M]}$$

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2(e^{2u} + e^{-2v})\frac{\partial f}{\partial x} - 2(e^{-2u} + e^{2v})\frac{\partial f}{\partial y}$$

Therefore, $\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2x\frac{\partial f}{\partial x} - 2y\frac{\partial f}{\partial y}$ [1M]

1. Let, $u^3 = \tan^{-1} \dfrac{x^n + y^n + z^n}{\sqrt{x^4 + y^4 + z^4}} + 3$. Then find the value of n such that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{\sin(2u^3 - 6)}{u^2}$$

3

Solution. Let. $v = \tan(u^3 - 3) = \dfrac{x^n + y^n + z^n}{\sqrt{x^4 + y^4 + z^4}}$

v is a homogeneous function of degree $n-2$

Then by Euler's th.

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = (n-2)\, v = (n-2)\tan(u^3 - 3)$$

1

or, $x \sec^2(u^3-3)\, 3u^2 \dfrac{\partial u}{\partial x} + y \sec^2(u^3-3) \cdot 3u^2 \dfrac{\partial u}{\partial y}$

$$+ z \cdot \sec^2(u^3-3) \cdot 3u^2 \frac{\partial u}{\partial z} = (n-2)\tan(u^3-3)$$

or, $3u^2 \sec^2(u^3-3)\left(x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}\right)$

$$= (n-2)\tan(u^3-3)$$

or, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = \dfrac{n-2}{3u^2}\,\dfrac{\tan(u^3-3)}{\sec^2(u^3-3)}$

$$= \frac{n-2}{3u^2}\sin(u^3-3)\cos(u^3-3)$$

or, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = \dfrac{n-2}{\cancel{3}6u^2}\sin(2u^3-6)$

2

$$\therefore \frac{n-2}{6} = 1$$

$$\Rightarrow n - 2 = 6 \Rightarrow n = \underline{8} \quad \text{(Ans)}$$

Mousumi Mandal

Compute $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0)$ for the following fn.

$$f(x,y) = \frac{xy^2(x-2y)}{x^2+y^2}.$$

$$\frac{\partial}{\partial y} f_x(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{-2k - 0}{k} = -2. \quad \text{———} \quad [1]$$

$$f_x(0,y) = \lim_{h\to 0} \frac{f(h,y) - f(0,y)}{h} = \lim_{h\to 0} \frac{hy^2(h-2y)}{(h^2+y^2)h}$$

$$= -2y. \quad \text{———} \quad [1]$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= 0 \quad \text{———} \quad [1]$$

RN

(5)(a) Let $P_2(x,y) = ax^2 + bxy + y^2/2 + x - y + 1$ be the Taylor polynomial of degree 2 about the point $(0,0)$ for the funtion $f(x,y) = e^{x-y}(4x^2-3xy+1)$, where $a,b \in \mathbb{R}$. Then find the values of a,b?

(3M)

Sol:- we have

$$P_2(x,y) = f(0,0) + \left(x f_x(0,0) + y f_y(0,0)\right)$$
$$+ \frac{1}{2!}\left(x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right)$$

(1m)

Given,

$$f(x,y) = e^{x-y}(4x^2-3xy+1), \quad f(0,0) = 1$$

$$f_x = e^{x-y}(4x^2-3xy+1+8x-3y), \quad f_x(0,0) = 1$$

$$f_y = e^{x-y}(-4x^2+3xy-1+(-3x)), \quad f_y(0,0) = -1$$

$$f_{xy} = e^{x-y}(-4x^2+3xy-1-8x+3y+(-3x-3)), \quad f_{xy}(0,0) = -4$$

$$f_{xx} = e^{x-y}(4x^2-3xy+1+8x-3y+8x-3y+8), \quad f_{xx}(0,0) = 9$$

$$f_{yy} = e^{x-y}(4x^2-3xy+1+3x+(-3x)), \quad f_{yy}(0,0) = 1$$

(1m)

$$\therefore \quad P_2(x,y) = 1 + \left(x(1) + y(-1)\right) + \frac{1}{2}\left(x^2(9) + 2xy(-4) + y^2(1)\right)$$

$$= 1 + x - y + \frac{9}{2}x^2 - 4xy + \frac{y^2}{2}$$

$$= ax^2 + bxy + y^2/2 + x - y + 1 \quad \text{(given)}$$

$$\Rightarrow \quad a = 9/2, \quad b = -4$$

$$\boxed{a = 4.5}, \quad \boxed{b = -4}$$

$\left(\frac{1}{2} m\right)$ $\left(\frac{1}{2} m\right)$

Note: Any two partial derivatives of order two are correct, $\frac{1}{2}$ mark awarded.

Maths I - MidSem - Advanced Calculus

Questions and Answers

Question 5.(b): Find and classify the points where a saddle, local maximum or local minimum occurs for the function $f(x, y) = xye^{-2(x^2+y^2)}$.

Solution: The critical points of f are given by

$$f_x(x, y) = ye^{-2(x^2+y^2)}(1 - 4x^2) = 0 \quad \text{and} \quad f_y(x, y) = xe^{-2(x^2+y^2)}(1 - 4y^2) = 0.$$

These yield the points $(0, 0)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{1}{2}, -\frac{1}{2}\right)$, $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{1}{2}\right)$. In order to classify the critical points, compute the second derivatives:

$$\begin{aligned} f_{xx}(x, y) &= 4xye^{-2(x^2+y^2)}(4x^2 - 3) \\ f_{yy}(x, y) &= 4xye^{-2(x^2+y^2)}(4y^2 - 3) \\ f_{xy}(x, y) &= (1 - 4x^2)(1 - 4y^2)e^{-2(x^2+y^2)}. \end{aligned}$$

The discriminant is given by

$$f_{xx}f_{yy} - f_{xy}^2 = e^{-4(x^2+y^2)} \left(16x^2y^2(4x^2 - 3)(4y^2 - 3) - (1 - 4x^2)^2(1 - 4y^2)^2\right).$$

At the point $(0, 0)$**,** $f_{xx}f_{yy} - f_{xy}^2 = -1 < 0$, hence f has a **saddle point** at $(0, 0)$. [0.5 + 0.5]

At all other critical points, $f_{xx}f_{yy} - f_{xy}^2 = 4e^{-2} > 0$ and $f_{xx} = 4e^{-1}(-2)xy$

At $\left(\frac{1}{2}, \frac{1}{2}\right)$ **and** $\left(-\frac{1}{2}, -\frac{1}{2}\right)$**,** $f_{xx} = -2e^{-1} < 0$, hence f has **local maximum** values. [0.5 + 0.5]

At $\left(\frac{1}{2}, -\frac{1}{2}\right)$ **and** $\left(-\frac{1}{2}, \frac{1}{2}\right)$**,** $f_{xx} = 2e^{-1} > 0$, hence f has **local minimu**m values. [0.5 + 0.5]

If they find **only the critical points** without specifying the behavior of f at those points, they will **get only 1 mark.**

For both the questions in (5), without giving justification, if someone writes only the final answer(s), we have given zero marks.