SOLUTION SHEET (Tutorial - 1)

es: (1) (i) Determine all the number(s) c which satisfy the conclusion of Rolle's Theorem for

(i) $f(x) = x^2 - 2x - 8$ on $[-1, 3]$

(ii) $g(x) = 2x - x^2 - x^3$ on $[-2, 1]$.

olution :

(i) As, $f(x)$ is a polynomial.

So, f is continuous on $[-1,3]$ and differentiable on $(-1,3)$.

Now, $f(-1) = f(3) = -5$.

Thus, it satisfies all the conditions of Rolle's theorem.

$\therefore \quad f'(x) = 2x - 2$

Applying Rolle's theorem on $f(x)$,

$\exists \; c \in (-1, 3)$ such that $f'(c) = 0$

i.e, $\quad 2c - 2 = 0$

$\Rightarrow \quad c = 1$

(ii) As, $g(x)$ is a polynomial.

So, g is continuous on $[-2, 1]$ and differentiable on $(-2, 1)$.

Now, $g(-2) = g(1) = 0.$

Thus, all the conditions of Rolle's theorem are satisfied.

$\Rightarrow \quad \exists \; c \in (-2,1)$ such that $g'(c) = 0.$

Now, $g'(t) = 2 - 2t - 3t^2$

$g'(c) = -3c^2 - 2c + 2 = 0$

$\Rightarrow \quad c = \frac{1 \pm \sqrt{7}}{-3} = -1.2153, \; 0.5486$

(iii) Cont.

Ques ② Verify Rolle's Theorem for $f(x) = x(x+3)e^{-x/2}$ in $[-3,0]$.

Solution Let $f(x) = x(x+3)e^{-x/2}$; $x \in [-3,0]$

Now, $f(0) = 0 = f(-3)$

and $f(x)$ is derivable in the interval $[-3,0]$.

We have, $f'(x) = (2x+3)e^{-x/2} + x(x+3)e^{-x/2}\left(-\frac{1}{2}\right)$

$$f'(x) = \frac{(-x^2+x+6)}{2} \cdot e^{-x/2}$$

Now, $f'(x) = 0$ iff $-x^2+x+6 = 0$.

i.e, $f'(x) = 0$ iff $x = -2, 3$.

as $-3 \notin (-3,0)$ and $-2 \in (-3,0)$.

Hence, -2 will be under consideration

Verified

Ques. (3) If $f(x) = (x-a)^m (x-b)^n$; $n, m \in \mathbb{N}$. Use Rolle's theorem to show that the point where $f'(x)$ vanishes divides the line segment $a \le x \le b$ in the ratio $m:n$

Solution: Let $f(x) = (x-a)^m (x-b)^n$.

$f(x)$ satisfies all the conditions of Rolle's theorem on $[a,b]$.
Hence, $\exists\ c \in (a,b)$ such that $f'(c) = 0$.

Now, $f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-a)^m (x-b)^{n-1}$

As, $f'(c) = 0$

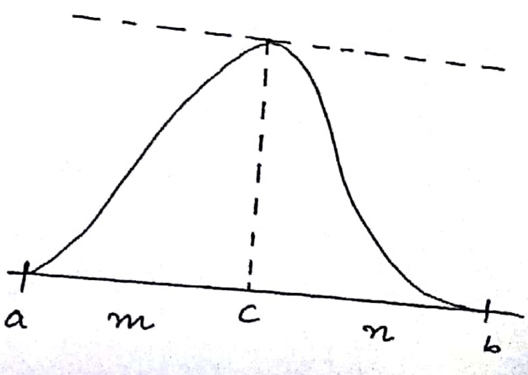
$$\Rightarrow (c-a)^{m-1}(c-b)^{n-1}\{m(c-b) + n(c-a)\} = 0.$$

$$\Rightarrow m(c-b) + n(c-a) = 0 \qquad (\because c \in (a,b))$$

$$\Rightarrow \frac{b-c}{c-a} = \frac{n}{m}$$

Thus, c divides the line segment $a \le x \le b$ in the ratio $m:n$

(6)



theorem
line

(4) Let $f(x) = (x-a)(x-b)(x-c)$, $a<b<c$

Show that $f'(x)=0$ has two roots one belonging to $]a,b[$ and other belonging to $]b,c[$.

lution: $f(x)$ is a polynomial, so f is continuous and differentiable for all real values of x.

We also have, $f(a) = f(b) = f(c) = 0$.

By Rolle's theorem, $f'(x)=0$ for atleast one value in $]a,b[$ and atleast one value / root in $]b,c[$.

But, $f'(x)=0$ is a polynomial of degree 2.

Hence, $f'(x)=0$ can not have more than 2 roots. Therefore, exactly one root in $]a,b[$ and exactly one root in $]b,c[$ for $f'(x)=0$.

Ques: ⑤ Use Rolle's theorem to prove the following:

(i) Let $f : [0,1] \longrightarrow \mathbb{R}$ be a continuous function on $[0,1]$ satisfy the condition $\int_0^1 f(x)\,dx = 0$. Then, $\exists$ $c \in (0,1)$ such that

$$f(c) = \int_0^c f(x)\,dx$$

(ii) Let $f : [a,b] \rightarrow \mathbb{R}$ be a continuous function on $[a,b]$ and $f''(x)$ exists $\forall$ $x \in [a,b)$. Let $a < c < b$, then there exists $\xi \in (a,b)$ such that

$$f(c) = \frac{b-c}{b-a} f(a) + \frac{c-a}{b-a} f(b) + \frac{1}{2}(c-a)(c-b) f''(\xi).$$

Solution:

(i) (IDEA: Replacing c by x and x by t in the problem.)

We have, $f(x) = \int_0^x f(t)\,dt$

$\Rightarrow$ $f'(x) = f(x)$ $\Rightarrow$ $\frac{f'(x)}{f(x)} = 1$

$\Rightarrow$ $\int \frac{df}{f} = \int dx + c_1$

$\Rightarrow$ $\ln f(x) = x + c_1$

$\Rightarrow$ $f(x) = e^x e^{c_1}$

$\therefore$ $e^{-x} f(x) = \text{constant}$

$\Rightarrow$ $e^{-x} \int_0^x f(t)\,dt = \text{constant}$.

Hence, we consider the function ... x.

nx ... consider the ... $f(x)$ is contin... $g(0) = ...$

⑧

Consider h...

nsider the function $g(x) = e^{-x}\int_0^x f(t)\,dt$

$g(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$.

$g(0) = 0$, $g(1) = e^{-1}\int_0^1 f(t)\,dt = 0$ (Given, $\int_0^1 f(x)\,dx = 0$)

$\therefore$ By Rolle's theorem, $\exists\ c \in (0,1)$ such that

$$g'(c) = 0 \Rightarrow -e^{-c}\int_0^c f(t)\,dt + e^{-c}f(c) = 0$$

$$\Rightarrow f(c) = \int_0^c f(t)\,dt = \int_0^c f(x)\,dx \qquad (\because e^{-c} \neq 0)$$

(ii) Let a function $\phi : [a,b] \to \mathbb{R}$ be defined by,

$$\phi(x) = f(x) - \frac{(x-b)(x-c)}{(a-b)(a-c)}f(a) - \frac{(x-c)(x-a)}{(b-c)(b-a)}f(b) - \frac{(x-a)(x-b)}{(c-a)(c-b)}f(c)$$

ϕ is continuous on $[a,b]$, since f is continuous on $[a,b]$.

Since, $f''(x)$ exists $\forall\ x \in (a,b)$, as f' is continuous on (a,b).

Thus, $\phi''(x)$ exists $\forall\ x \in (a,b)$ and ϕ' is continuous on (a,b).

$$\phi(a) = \phi(b) = \phi(c) = 0.$$

Applying Rolle's theorem to the function ϕ on $[a,c]$ and $[c,b]$.

we have, $\phi'(\xi_1) = 0$ for some $\xi_1 \in (a,c)$ and

$\phi'(\xi_2) = 0$ for some $\xi_2 \in (c,b)$.

Applying Rolle's theorem to the function ϕ' on $[\xi_1, \xi_2]$, we have, $\phi''(\xi) = 0$ for some $\xi \in (\xi_1, \xi_2)$

$\therefore$ i.e, $\phi''(\xi) = 0$ for some $\xi \in (a,b)$.

But, $$\phi''(\xi) = f''(\xi) - \frac{2f(a)}{(a-b)(a-c)} - \frac{2f(b)}{(b-a)(b-c)} - \frac{2f(c)}{(c-a)(c-b)}.$$

Hence, $$f(c) = \frac{b-c}{b-a} f(a) + \frac{c-a}{b-a} f(b) + \frac{1}{2}(c-a)(c-b) f''(\xi)$$

$; \; a < \xi < b.$

1 (E) Determine all the number(s) c which satisfy the conclusion of MVT for $f(x) = 8t + e^{-3t}$ on $[-2,3]$.

Solution:

As, f(x) is sum of polynomial and exponential function.

$\Rightarrow$ f(x) is continuous on $[-2,3]$ and differentiable on $(-2,3)$.

Therefore, the conditions of MVT are met.

Now, $f(-2) = -16 + e^{6}$

$f(3) = 24 + e^{-9}$

$f'(x) = 8 - 3e^{-3t}$

Using MVT, we have,

$$8 - 3e^{-3c} = \frac{24 + e^{-9} - (-16 + e^{6})}{3 - (-2)} = -72.6857$$

$$\Rightarrow \quad 3e^{-3c} = 80.6857$$

$$\Rightarrow \quad e^{-3c} = 26.8952$$

$$\Rightarrow \quad -3c = \ln(26.8952) = 3.29195$$

$$\Rightarrow \quad c = -1.0973.$$

Ques (7) Suppose that $f(x)$ is continuous and differentiable everywhere. And also $f(x)$ has two roots.

Then show that, $f'(x)$ must have atleast one root.

Solution:

Let a and b be the two root of $f(x)$.

Now, we know that $f(x)$ is continuous and differentiable everywhere.

$\Rightarrow$ f is continuous on $[a,b]$ and differentiable in (a,b).

Therefore, by MVT,

$\exists$ $c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

But, $f(b) = f(a) = 0$

$\Rightarrow$ $f'(c) = 0$.

$\Rightarrow$ f' has root at $x = c$.

iable

s: 8 (i) Suppose that $f(0) = -3$ and $f'(x) \le 5$ $\forall x$. Use LMVT to find the largest possible value of $f(2)$.

Solution: Using LMVT on $[0,2]$.

$$\frac{f(2) - f(0)}{2 - 0} = f'(c) \quad ; \text{ for some } c \in (0,2).$$

$$\Rightarrow \quad f(2) = f(0) + 2f'(c) \le -3 + 2 \times 5 = 7$$

Then, largest possible value of $f(2)$ is 7.

(ii) Use LMVT to estimate $\sqrt[3]{28}$.

Solution: Let $f(x) = \sqrt[3]{x}$; $x \in [27, 28]$.

Then, $f'(x) = \frac{1}{3} x^{-2/3}$

So, f satisfies all the conditions of LMVT.

Now, applying LMVT on f,

$$\frac{\sqrt[3]{28} - \sqrt[3]{27}}{28 - 27} = \frac{1}{3} c^{-2/3} \quad ; \text{ for some } c \in (27, 28).$$

$$\therefore \sqrt[3]{28} = 3 + \frac{1}{3} c^{-2/3} \le 3 + \frac{1}{3}(27)^{-2/3}$$

$$\le 3 + \frac{1}{27} \simeq 3.037.$$

$c \geq 27$

$c^{-1} \le \frac{1}{27}$

$c^{-2/3} \le \frac{1}{3^2}$

$\therefore$ The estimated value of $\sqrt[3]{28}$ is 3.037.

Ques: (9) If $f(x)$ and $\phi(x)$ are continuous on $[a,b]$ and differentiable on (a,b), then show that

$$\begin{vmatrix} f(a) & f(b) \\ \phi(a) & \phi(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(b) & f'(c) \\ \phi(b) & \phi'(c) \end{vmatrix}, \quad a<c<b.$$

Solution: Let $g(x) = \begin{vmatrix} f(x) & f(b) \\ \phi(x) & \phi(b) \end{vmatrix}$ satisfies LMVT conditions on $[a,b]$.

Applying LMVT on $[a,b]$, we have —

$$\frac{g(b)-g(a)}{b-a} = g'(c), \text{ for some } c \in (a,b) \quad \text{—— (1)}$$

and $g'(x) = \begin{vmatrix} f'(x) & f(b) \\ \phi'(x) & \phi(b) \end{vmatrix}$

Then, from (1),

$$\begin{vmatrix} f(b) & f(b) \\ \phi(b) & \phi(b) \end{vmatrix} - \begin{vmatrix} f(a) & f(b) \\ \phi(a) & \phi(b) \end{vmatrix} = (b-a)\begin{vmatrix} f'(c) & f(b) \\ \phi'(c) & \phi(b) \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} f(a) & f(b) \\ \phi(a) & \phi(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(b) & f'(c) \\ \phi(b) & \phi'(c) \end{vmatrix}$$

(10) Use Lagrange's MVT to prove Bernoulli's inequality:

$\forall\ x > 0$ and for all $n \in \mathbb{N}$, $(1+x)^n > (1+nx)$.

Solution: Let $f(t) = (1+t)^n$; $t \in [0,x]$ and $x > 0$.

Apply LMVT and we have,

$$\frac{f(x)-f(0)}{x-0} = f'(c) \quad ; \text{ for some } c \in (0,x).$$

$$\Rightarrow \frac{(1+x)^n - 1}{x} = n(1+c)^{n-1}$$

$$\Rightarrow (1+x)^n = nx(1+c)^{n-1} + 1 > (1+nx)$$

$(\because c > 0 \Rightarrow (1+c) > 1 \Rightarrow (1+c)^{n-1} > 1)$.

$$\Rightarrow (1+x)^n > (1+nx).$$

Proved.

Ques. (1) Suppose $f(x)$ is continuous on $[-7,0]$ and differentiable in $(-7,0)$, such that $f(-7) = -3$ and $f'(x) \le 2$. Then, what is the largest possible value for $f(0)$?

Solution:

As, $f(x)$ is continuous on $[-7,0]$ and differentiable in $(-7,0)$.
We can apply MVT, so we have,

$$f(0) - f(-7) = f'(c) \cdot (7+0)$$

$$\Rightarrow f(0) + 3 = 7f'(c)$$

$$\Rightarrow f(0) = 7f'(c) - 3$$

$$\Rightarrow f(0) \le 7 \times 2 - 3 = 11$$

i.e largest possible value of $f(0)$ is 11.

differentiate

(12) Using Cauchy's MVT show that,

$$\left(1 - \frac{x^2}{2}\right) < \cos x \quad ; \quad x \neq 0$$

<u>Solution:</u>

Consider $f(x) = 1 - \cos x$

and $g(x) = \frac{x^2}{2}$ on $[0, x]$

Applying Cauchy's MVT,

$$\frac{(1 - \cos x) - 0}{\left(\frac{x^2}{2} - 0\right)} = \frac{\sin c}{c} \quad , \text{for some } c \in (0, x).$$

$$\Rightarrow \frac{1 - \cos x}{(x^2/2)} < 1 \qquad (\because \sin c < c \ , \text{ for } c < 0)$$

$$\Rightarrow (1 - \cos x) < \frac{x^2}{2}$$

$$\Rightarrow \left(1 - \frac{x^2}{2}\right) < \cos x \quad , \text{for } x \neq 0$$

Ques: (13) (i) Let f be a continuous on $[a,b]$, $a>0$ and differentiable on (a,b). Prove that $\exists\ c \in (a,b)$ such that,

$$\frac{bf(a) - af(b)}{(b-a)} = f(c) - cf'(c).$$

(ii) If f is differentiable on $[0,1]$, show by Cauchy's MVT that the equation $f(1) - f(0) = \frac{f'(x)}{2x}$ has atleast one solution in $(0,1)$

(iii) Let f be continuous on $[a,b]$ and differentiable on (a,b). Using Cauchy's MVT show that if $a \geq 0$ then $\exists\ x_1, x_2, x_3 \in (a,b)$ such that

$$f'(x_1) = (b+a)\frac{f'(x_2)}{2x_2} = (b^2+ba+a^2)\frac{f'(x_3)}{3x_3^2}.$$

Solution:

(i) Consider $h(x) = \frac{f(x)}{x}$ and $g(x) = \frac{1}{x}$ on $[a,b]$; $(a>0)$

Applying Cauchy's MVT on $[a,b]$,

$$\frac{\frac{f(b)}{b} - \frac{f(a)}{a}}{\frac{1}{b} - \frac{1}{a}} = \frac{\frac{f'(c)}{c} - \frac{f(c)}{c^2}}{-\frac{1}{c^2}} \quad ; \text{ for some } c \in (a,b).$$

$$\Rightarrow \frac{af(b) - bf(a)}{a-b} = f(c) - cf'(c).$$

$$\Rightarrow \frac{bf(a) - af(b)}{b-a} = f(c) - cf'(c).$$

(ii) Consider $h(x) = f(x)$ and $g(x) = x^2$ on $[0,1]$.

Applying Cauchy's MVT on $[0,1]$,

$$\frac{f(1)-f(0)}{1-0} = \frac{f'(c)}{2c} \quad ; \text{ for some } c \in (0,1).$$

$$\Rightarrow \quad f(1) - f(0) = \frac{f'(c)}{2c}$$

$\therefore$ $f(1) - f(0) = \dfrac{f'(x)}{2x}$ has atleast one solution in $(0,1)$.

(iii) Consider $h(x) = f(x)$, $k(x) = x$ on $[a,b]$.

Applying Cauchy's MVT on $[a,b]$,

$$\frac{f(b)-f(a)}{(b-a)} = f'(x_1) \quad , \text{ for some } x_1 \in (a,b) \quad \text{——— (1)}$$

Consider $h(x) = f(x)$ and $k(x) = x^2$ on $[a,b]$.

Applying CMVT on $[a,b]$,

$$\frac{f(b)-f(a)}{b^2-a^2} = \frac{f'(x_2)}{2x_2} \quad ; \text{ for some } x_2 \in (a,b)$$

$$\Rightarrow \quad \frac{f(b)-f(a)}{(b-a)} = (b+a)\frac{f'(x_2)}{2x_2} \quad \text{——— (2)}$$

Consider $h(x) = f(x)$, $k(x) = x^3$ on $[a,b]$.

Applying CMVT on $[a,b]$,

$$\frac{f(b)-f(a)}{b^3-a^3} = \frac{f'(x_3)}{3x_3^2} \quad \text{for some } x_3 \in (a,b).$$

$$\Rightarrow \frac{f(b)-f(a)}{b-a} = (b^2+ba+a^2)\frac{f'(x_3)}{3x_3^2} \quad \text{--- (3)}$$

from (1), (2) and (3),

$$(b+a)\frac{f'(x_2)}{2x_2} = f'(x_1) = (b^2+ba+a^2)\frac{f'(x_3)}{3x_3^2}$$

$\cancel{5-1}$ 2+2 +1
6-1 8
7-2 4-0