Tutorial 8: Solutions

1. * (a) Light from the Sun arrives at the Earth, an average of 1.5×10^{11} m away, at the rate $1.4 \times 10^{3}\ Watts/m^{2}$ of area perpendicular to the direction of the light. Assume that sunlight is monochromatic with a frequency 5×10^{14} Hz.

(i) How many photons fall per second on each square metre of the Earth's surface directly facing the Sun?

(ii) What is power output of the Sun, and how many photons per second does it emit?

(b) A 1000 Watt radio transmitter operates at a frequency of 880 kHz. How many photons per second does it emit?

Solutions:

(a) (i) We have $E = h\nu$–so the energy in each photon is $E = 33.15 \times 10^{-20}$ J. The number of photons in the total energy of 1.4×10^{3} J is therefore given as

$$n = \frac{1.4 \times 10^{3}}{33.15 \times 10^{-20}} = 4.22 \times 10^{21} \tag{1}$$

(a) (ii) The power output of the Sun is:

$$P = 4\pi \left(1.5 \times 10^{11}\right)^{2} \times 1.4 \times 10^{3}\ W = 3.96 \times 10^{26}\ W \tag{2}$$

(b) The number of photons is:

$$n = \frac{1000}{880 \times 10^{3} \times 6.63 \times 10^{-34}} = 1.71 \times 10^{30} \tag{3}$$

2. * (a) Illuminating the surface of a metal alternately with light of wavelengths $\lambda_1 = 0.35\mu$m and $\lambda_2 = 0.54\mu$m, it was found that the corresponding maximum velocities of the photoelectron differ by a factor of 2. Find the work function of the metal.

(b) The maximum energy of photoelectrons from Alumunium is 2.3 eV for radiation of $2000\mathring{A}$ and 0.90 eV for radiation of $2580\mathring{A}$. Use these data to calculate Planck's constant and the work function of Alumunium.

Solutions:

(a) We have

$$\frac{1}{2}mv_1^2 = h\nu_1 - W \tag{4}$$

$$\frac{1}{2}m\frac{v_1^2}{4} = h\nu_2 - W \tag{5}$$

Therefore

$$W = \frac{4}{3}h\left(\nu_2 - \frac{\nu_1}{4}\right) = \frac{4}{3}hc\left(\frac{1}{\lambda_2} - \frac{1}{4\lambda_1}\right) \tag{6}$$

Subsituting the values, we get

$$W = 3.02 \times 10^{-19}\ J \tag{7}$$

(b) We have $T_{max} = h\nu - W$. Therefore

$$2.3 \times 1.6 \times 10^{-19} = \frac{hc}{2000 \times 10^{-10}} - W \tag{8}$$

$$0.90 \times 1.6 \times 10^{-19} = \frac{hc}{2580 \times 10^{-10}} - W \tag{9}$$

On solving these two equations for h and W one gets

$$h = 6.64 \times 10^{-34} \quad ; \quad W = 4.83\ eV \tag{10}$$

3. * (a) A Compton effect experiment is performed in such a way that the scattered photon and the recoil electron are detected only when their paths are at right angles ($\theta + \phi = \frac{\pi}{2}$).

(i) Show that under these conditions $\lambda' = \frac{\lambda}{\cos\phi}$

(ii) Show that the energy of the scattered photon is m_0c^2.

(iii) Find the energy of the recoil electron.

(iv) What is the minimum energy of the incident photons for which the experiment can be done ?

(b) Show that when a free electron is scattered in a direction making an angle θ with the incident photon in a Compton scattering, the kinetic energy of the electron is:

$$E_k = \frac{h\nu\,(2\alpha\cos^2\theta)}{[(1+\alpha)^2 - \alpha^2\cos^2\theta]} \tag{11}$$

where $\alpha = \frac{h\nu}{m_0c^2}$.

Solutions:

(a) We assume the angle of scattering of the photon as ϕ and that of the recoil electron as θ. Therefore, conservation of energy gives

$$h\nu + m_0c^2 = h\nu' + \sqrt{p^2c^2 + m_0^2c^4} \tag{12}$$

where p is the momentum of the recoil electron and m_0 is the rest mass of the free electron. Conservation of momentum along the horizontal and vertical directions give

$$\frac{h\nu}{c} = \frac{h\nu'}{c}\cos\phi + p\cos\theta \tag{13}$$

$$\frac{h\nu'}{c}\sin\phi = p\sin\theta \tag{14}$$

(i) Since $\theta + \phi = \frac{\pi}{2}$ we have, from the conservation of momentum equations

$$\frac{h\nu}{c} = \frac{h\nu'}{c}\cos\phi + p\sin\phi \tag{15}$$

$$\frac{h\nu'}{c}\sin\phi = p\cos\phi \tag{16}$$

Multiplying the first of the above two equations by $\cos\phi$ and the second by $\sin\phi$ and subtracting them, one gets

$$\nu' = \nu\cos\phi \tag{17}$$

which, in terms of wavelength is

$$\lambda' = \frac{\lambda}{\cos\phi} \tag{18}$$

(ii) From the momentum conservation equations obtained after using $\theta + \phi = \frac{\pi}{2}$ we can easily obtain

$$pc = h\nu\sin\phi \tag{19}$$

Using this in the energy conservation equation gives

$$h\nu + m_0c^2 = h\nu' + \sqrt{(h\nu)^2\sin^2\phi + m_0^2c^4} \tag{20}$$

Writing $\sin^2\phi = 1 - \left(\frac{\nu'}{\nu}\right)^2$ we get

$$h\nu + m_0c^2 = h\nu' + \sqrt{(h\nu)^2 - (h\nu')^2 + m_0^2c^4} \tag{21}$$

which gives, on a rearrangement and squaring

$$\left(h\nu - h\nu' + m_0c^2\right)^2 = (h\nu)^2 - (h\nu')^2 + m_0^2c^4 \tag{22}$$

This equation can be rewritten as

$$2h\left(\nu - \nu'\right)\left(h\nu' - m_0c^2\right) = 0 \tag{23}$$

Since $\nu \neq \nu'$, we have the energy of the scattered photon as

$$h\nu' = m_0c^2 \tag{24}$$

(iii) The energy of the recoil electron is obtainable from

$$h\nu + m_0 c^2 = h\nu' + E_{recoil} \tag{25}$$

Since $h\nu' = m_0 c^2$, $E_{recoil} = h\nu$.

(iv) The minimum energy of the incident photons required for the experiment to take place is given as $h\nu = m_0 c^2$, i.e. when $p = 0$. In this case however, there is no change of the frequency of the incident photon and the electron also remains at rest.

(b) The kinetic energy of the scattered electron is

$$E_k = \sqrt{p^2c^2 + m_0 c^4} - m_0 c^2 \tag{26}$$

Using the fact that $\sqrt{p^2c^2 + m_0^2 c^4} = h\nu - h\nu' + m_0 c^2$ (from energy conservation) we can rewrite E_k as

$$E_k = h\left(\nu - \nu'\right) = \frac{h^2}{m_0 \lambda} \frac{2\sin^2\frac{\phi}{2}}{\lambda + \frac{2h}{m_0 c}\sin^2\frac{\phi}{2}} \tag{27}$$

where we have converted to wavelengths and use $\lambda' - \lambda = \frac{h}{m_0 c}\left(1 - \cos\phi\right)$.

Now we will need to convert this in terms of θ. For this we need to know the relation between θ and ϕ. Using the conservation equations for momenta given as

$$\frac{h\nu}{c} = \frac{h\nu'}{c}\cos\phi + p\cos\theta \tag{28}$$

$$\frac{h\nu'}{c}\sin\phi = p\sin\theta \tag{29}$$

we get

$$\cot\theta = \frac{\nu}{\nu'}\frac{1}{\sin\phi} - \cot\phi \tag{30}$$

From the Compton scattering formula we have

$$\frac{\nu}{\nu'} = 1 + \frac{h\nu}{m_0 c^2}\left(1 - \cos\phi\right) = 1 + \alpha\left(1 - \cos\phi\right) \tag{31}$$

Therefore, the equation for $\cot\theta$ becomes

$$\cot\theta = \left(1 + \alpha\left(1 - \cos\phi\right)\right)\frac{1}{\sin\phi} - \cot\phi \tag{32}$$

Mutiplying by $\sin\phi$ on both sides and after a bit of algebra one obtains

$$\cot\theta = \left(1 + \alpha\right)\tan\frac{\phi}{2} \tag{33}$$

The above expression for $\cot\theta$ is equivalent to

$$\sin^2\frac{\phi}{2} = \frac{\cot^2\theta}{(1+\alpha)^2+\cot^2\theta} \tag{34}$$

Substituting this back in the expression for E_k and after some straightforward algebra one obtains

$$E_k = \frac{2\alpha h\nu\cos^2\theta}{(1+\alpha)^2-\alpha^2\cos^2\theta} \tag{35}$$

4. (a) Find the de Broglie wavelength of a 1 mg grain of sand blown by the wind at a speed 20 m/s.

* **(b)** Calculate the de Broglie wavelength of an electron when its energy is 1 eV, 100 eV, 1000 eV. At what value of the kinetic energy is the de Broglie wavelength of an electron equal to its Compton wavelength?

(c) Show that the ratio of the de Broglie wavelength to the Compton wavelength of the same particle is equal to $\sqrt{\left(\frac{c}{v}\right)^2-1}$.

Solutions:

(a) The de Broglie wavelength here is:

$$\lambda = \frac{h}{mv} = \frac{6.63\times10^{-34}}{10^{-3}\times 20} = 3.315\times10^{-36}\ m \tag{36}$$

(b) The de Broglie wavelength can be found from the formula $\lambda = \frac{h}{mv} = \frac{h}{\sqrt{2mE}}$. For 1 eV, 100 eV and 1000 eV, the de Broglie wavelengths are 12.28 Å, 1.228 Åand 0.39 Å.

Using $\frac{h}{m_0c} = \frac{h}{mv}$ one gets $v = \frac{m_0}{m}c$. This results in $\frac{v}{c} = \frac{1}{\sqrt{2}}$ (using $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$). Therefore, the energy of the electron is $E = \frac{m_0c^2}{\sqrt{1-\frac{1}{2}}} = \sqrt{2}m_0c^2 = \sqrt{2}\times 0.5$ MeV. Therefore the KE $= E - m_0c^2 = 0.21MeV$.

(c) This follows from the previous problem. We have

$$\frac{\lambda_{dB}}{\lambda_c} = \frac{m_0}{m}\frac{c}{v} = \sqrt{\frac{c^2}{v^2}-1} \tag{37}$$

5. * **(a)** A parallel beam of electrons accelerated by a potential difference of 25 Volts falls normally on a diaphragm with two narrow slits separated by a distance , 50 μm. Calculate the distance between neighboring maxima of the diffraction pattern on a screen located at a distance, 100 cm from the slits.

(b) A parallel beam of monoenergetic electrons falls normally on a diaphragm with narrow square slit of width 1 micrometre. Find the velocity of the electrons if the width of the central diffraction maximum formed on a screen located 50 cm from the slit is equal to 0.36 mm.

Solutions:

(a) The de Broglie wavelength of electron is

$$\begin{aligned}\lambda &= \frac{h}{p}\\ &= \frac{h}{\sqrt{2m_e E}}\\ &= \frac{h}{\sqrt{2m_e eV}}\\ &= \frac{hc}{\sqrt{2m_e c^2 eV}}\end{aligned}$$

$m_e c^2 = 0.501\,MeV$ is the rest mass energy of electron. $h = 6.625 \times 10^{-34}\,Js$, $V = 25\,V$. So $\lambda = 2.47$Angstrom. Separation between two neighboring maximum of diffraction pattern on the screen is

$$\begin{aligned}L\Delta\theta &= L\frac{\lambda}{d}\\ &= 4.9\,\mu m\end{aligned}$$

where, $L = 100\,cm$ and $d = 10\,\mu m$.

(b). Width of the central diffraction maximum is

$$\frac{2\lambda L}{a} = 0.36\,mm$$

where, λ is de Broglie wavelength of the electron, $L = 50\,cm$ and $a = 1\,\mu m$. So $\lambda = 3.6$ Angstrom. Hence speed of the electron is

$$v = \frac{h}{m_e\lambda} = 2 \times 10^6\,m/s$$

where, $m_e = 9.11 \times 10^{-31}\,kg$.

6. * (a) The position and momentum of a 1 keV electron are simultaneously determined. If the position is located to within one Angstrom what is the percentage of uncertainty in its momentum?

(b) Verify that the uncertainty principle can be written in the form $\Delta L \Delta\theta \geq \frac{\hbar}{2}$ where ΔL is the uncertainty in the angular momentum of a particle and $\Delta\theta$ is the uncertainty in its angular position.

*** (c)** An atom in an excited state has a lifetime of 12 nanoseconds and in a second excited state the lifetime is 23 nanoseconds. What is the uncertainty in energy of the photon when an electron makes a transition between these two states?

(a). Minimum uncertainty in momentum is

$$\begin{aligned}\Delta p &= \frac{h}{4\pi\Delta x} \\ &= 5.27 \times 10^{-25}\, kg - m/s\end{aligned}$$

Momentum of 1 KeV electron is

$$\begin{aligned}p &= \sqrt{2m_e E} \\ &= 1.71 \times 10^{-23} kg - m/s\end{aligned}$$

So $\frac{\Delta p}{p} = 3.08\%$

(b). The uncertainty relation between angular momentum and angular position can be verified from the uncertainty relation between linear momentum and position.

$$\Delta x \Delta p \geq \frac{h}{4\pi}$$

Consider a particle with linear momentum p moving on a circle of radius r. The particles angular momentum is given by, $L = pr$.In moving a distance x on the circle the particle sweeps out an angle $\theta = x/r$. Then for a fixed radius r, $\Delta\theta = \Delta x/r$ and $\Delta L = r\Delta p$. So,

$$\Delta L \Delta\theta = \Delta x \Delta p \geq \frac{h}{4\pi}$$

(c). We know the the energy-time uncertainty relation, $\Delta E \Delta t \geq h/4\pi$. For $\Delta t = \tau_1 = 12\, ns$, $\Delta E_1 = 4.4 \times 10^{-27}\, J$ and for $\Delta t = \tau_2 = 23\, ns$, $\Delta E_2 = 2.3 \times 10^{-27}\, J$. So the minimum uncertainty in energy of the photon is $(\Delta E_1 + \Delta E_2)$= $6.7 \times 10^{-27}\, J$.

7. (a) A particle is located in a one dimensional square potential with infinitely high walls. The width of the well is a. Find the normalised wave functions of the stationary states of the particle taking the midpoint of the well as the origin of the x coordinate.

(b) A particle is confined in a one dimensional box located between $x = 0$ and $x = L$. The potential is infinity at the walls and zero inside. Determine the probability of finding the particle in a region of width $\Delta x = 0.1L$ at each of the following locations when it is in its ground state: $x = 0, 0.25L, 0.5L.0.75L, L$. Repeat this exercise for the first excited state.

Solutions:

(a). For a particle located in a one dimensional potential, we solve the time independent Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{d^2u(x)}{dx^2} + V(x)u(x) = Eu(x)$$

where, $\psi(x,t) = e^{-iEt/\hbar}u(x)$ is the wave function of the stationary states of the particle. Now, $V(x) = \infty$ for $x \geq a/2$ and $x \leq -a/2$; $V(x) = 0$ for $-a/2 < x < a/2$. So, for $x \geq a/2$ and $x \leq -a/2$ $u(x) = 0$. For $-a/2 < x < a/2$, the above equation becomes

$$\frac{d^2u(x)}{dx^2} + k^2u(x) = 0$$

where, $k^2 = 2mE/\hbar^2$. The general solution of this equation is

$$u = A\cos\, kx + B\sin\, kx$$

Using the boundary conditions , $u(-a/2) = u(a/2) = 0$, we have two equations

$$A\cos(ka/2) + B\sin(ka/2) = 0$$
$$A\cos(ka/2) - B\sin(ka/2) = 0$$

For non trivial solution of A and B, $2\cos(ka/2)\sin(ka/2) = 0$ (determinent of the coefficient matrix is zero). So $\sin(ka) = 0$, or, $ka = n\pi$, where n is an integer.For even n, $A = 0$ and for odd n, $B = 0$. So the solution for u is

$$\begin{aligned} u &= B\sin\left(\frac{n\pi x}{a}\right) \qquad n = 2, 4, 6, ... \\ &= A\cos\left(\frac{n\pi x}{a}\right) \qquad n = 1, 3, 5, ... \end{aligned}$$

Now using the normalisation condition, we have

$$\int_{-a/2}^{a/2} B^2\sin^2\left(\frac{n\pi x}{a}\right) dx = 1$$

performing the integration we get $B = \sqrt{2/a}$. Similarly, $A = \sqrt{2/a}$. So finally the normalised wave function is

$$\psi(x,t) = \sqrt{\frac{2}{a}} e^{-iE_n t/\hbar} \sin\left(\frac{n\pi x}{a}\right) \qquad n = 2, 4, 6, ...$$
$$= \sqrt{\frac{2}{a}} e^{-iE_n t/\hbar} \cos\left(\frac{n\pi x}{a}\right) \qquad n = 1, 3, 5, ...$$

where, $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$.

(b). The probability of finding the particle in a region of width Δx , at the location x is

$$P = \Delta x |\psi(x,t)|^2 \tag{38}$$

where $\psi(x,t)$ normalised wave function. In this problem, $\psi_n(x,t) = \sqrt{\frac{2}{L}} e^{-iE_n t/\hbar} \sin\left(\frac{n\pi x}{L}\right)$. Then the expression for P becomes

$$P = \frac{2}{L} \Delta x \sin^2 \frac{\pi x}{L} \tag{39}$$

For ground state $n = 1$. Thus the probabilities at $x = 0, 0.25L, 0.5L, 0.75L, L$ are $0, 0.1, 0.2, 0.1, 0$ respectively. One can repeat this easily for the $n = 2$ eigenfunction.

8. * (a) An electron is trapped in a region of width $L = 10$ Angstroms. What are the energies of the ground state and the first two excited state? An electron in the first excited state de-excites to the ground state. What is the wavelength of the emitted radiation?

(b) The wavefunction of a particle confined in a one dimensional box, between $x = 0$ and $x = L$, is given to be

$$\psi(x,t) = \frac{3}{5}\psi_1(x,t) + \frac{4}{5}\psi_2(x,t) \tag{40}$$

where $\psi_1(x,t)$ and $\psi_2(x,t)$ are the ground and first excited state wavefunctions of the particle in a box. What are the possible outcomes and their probabilities if the energy of the particle is measured?

Solutions:

(a). The energies of an electron trapped in a region of width $L = 10$ Angstrom are

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

for ground state $n = 1$ and for first exicted state $n = 2$. So $E_1 = 0.37\, eV$ and $E_2 = 1.5\, eV$. The wave length of the emitted radiation for transition first excited state to ground state is

$$\lambda = \frac{hc}{E_2 - E_1} = 1.1\, \mu m$$

(b). If the energy of the particle is measured then after every measurement the outcome will be either ground state energy or the first excited state energy, since $\psi(x,t)$ is a combination of ground state and first excited state. $E_1 = \frac{\pi^2\hbar^2}{2mL^2}$ and $E_2 = \frac{2\pi^2\hbar^2}{mL^2}$. If the measurement is performed several times then the probability of geting the ground state energy is $P_1 = \frac{|c_1|^2}{|c_1|^2+|c_2|^2} = 9/25$ and probability of finding the first excited energy is $P_2 = \frac{|c_2|^2}{|c_1|^2+|c_2|^2} = 16/25$, where $c_1 = 3/5$ and $c_2 = 4/5$.