$\Rightarrow$

$\Rightarrow \lambda = (x^2 + y^2 + z^2)$

$\therefore x = \dfrac{\mu l}{2(1-\lambda a)}$; $y = \dfrac{\mu m}{2(1-\lambda b)}$

$z = \dfrac{\mu n}{2(1-\lambda c)}$

from (5)

$$\frac{\mu}{2}\left[\frac{l^2}{1-\lambda a} + \frac{m^2}{1-\lambda b} + \frac{n^2}{1-\lambda c}\right] = 0$$

$$\Rightarrow \frac{l^2}{1-\lambda a} + \frac{m^2}{1-\lambda b} + \frac{n^2}{1-\lambda c} = 0 \quad \ldots (6)$$

where $\lambda = x^2 + y^2 + z^2$ (Proved)

Let $f(x,y,z) = x^2+y^2+z^2$

$g(x,y,z) = ax^2+by^2+cz^2-1 = 0$

$h(x,y,z) = lx+my+nz = 0$

Construct the function $F(x,y,z)$ as

$F(x,y,z) = f - \lambda g - \mu h$

where λ, μ are the Lagrange's multipliers.

for stationary points we must have

$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$ & $g=0$, $h=0$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x - 2\lambda ax - \mu l = 0 \quad \ldots (1)$

$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y - 2\lambda by - \mu m = 0 \quad \ldots (2)$

$\frac{\partial F}{\partial z} = 0 \Rightarrow 2z - 2\lambda cz - \mu n = 0 \quad \ldots (3)$

and $ax^2+by^2+cz^2 = 1 \quad \ldots (4)$

$lx+my+nz = 0 \quad \ldots (5)$

$x \times (1) + y \times (2) + z \times (3)$

$\Rightarrow 2(x^2+y^2+z^2) - 2\lambda(ax^2+by^2+cz^2) - \mu(lx+my+nz) = 0$

using (4) & (5)

$\Rightarrow 2(x^2+y^2+z^2) - 2\lambda = 0$

and minimum value of $ax+by$

$$= \frac{-a^2}{\sqrt{a^2+b^2}} - \frac{b^2}{\sqrt{a^2+b^2}}$$

$$= -\sqrt{a^2+b^2}$$

Note: We can establish theoretically the existance of maximum or minimum values:

$L_{xx} = 2\lambda$, $L_{yy} = 2\lambda$, $L_{xy} = 0$

~~$\therefore L_{xx} \cdot L_{yy} - (L_{xy})^2 = 4\lambda^2$~~

$$\therefore d^2L = 2\lambda(dx)^2 + 2\lambda(dy)^2$$

$$= 2\lambda\left((dx)^2 + (dy)^2\right)$$

> 0 when $\lambda > 0$ i.e at $\left(\frac{-a}{\sqrt{a^2+b^2}}, \frac{-b}{\sqrt{a^2+b^2}}\right)$

< 0 when $\lambda < 0$ i.e at $\left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right)$

$\therefore$ At $\left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right)$ maximum value exists and at $\left(-a/\sqrt{a^2+b^2}, -b/\sqrt{a^2+b^2}\right)$ minimum value exists.

We construct a Lagrangian function

$$L(x,y) = ax + by + \lambda (x^2+y^2-1)$$

Where λ is a Lagrangian Undetermined Multiplier.

For stationary points, we have

$$\frac{\partial L}{\partial x} = a + 2\lambda x = 0 \Rightarrow x = -a/2\lambda$$

$$\frac{\partial L}{\partial y} = b + 2\lambda y = 0 \Rightarrow y = -b/2\lambda$$

Stationary points satisfies,

$$x^2+y^2 = 1$$

$$\Rightarrow \frac{a^2}{4\lambda^2} + \frac{b^2}{4\lambda^2} = 1$$

$$\Rightarrow a^2+b^2 = 4\lambda^2$$

$$\Rightarrow \lambda = \pm \frac{\sqrt{a^2+b^2}}{2}$$

$$\therefore x = \pm \frac{a}{\sqrt{a^2+b^2}} ; \ y = \pm \frac{b}{\sqrt{a^2+b^2}}$$

When $a, b > 0$.

$\therefore$ Stationary points are $\left(\pm \frac{a}{\sqrt{a^2+b^2}}, \pm \frac{b}{\sqrt{a^2+b^2}} \right)$.

So maximum value of $ax + by$

$$= \frac{a^2}{\sqrt{a^2+b^2}} + \frac{b^2}{\sqrt{a^2+b^2}} = \sqrt{a^2+b^2}$$

(19.) Let (x,y,z) be any point on the sphere. So we have to find the maximum and minimum of d or equivalently $d^2 = (x-1)^2 + (y-2)^2 + (z+1)^2$

subject to $x^2+y^2+z^2 = 24$

So $F(x,y,z,\lambda) = (x-1)^2 + (y-2)^2 + (z+1)^2 + \lambda(x^2+y^2+z^2-24)$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 2(x-1) + 2x\lambda = 0 \Rightarrow x = \frac{1}{1+\lambda}$

$\frac{\partial F}{\partial y} = 0 \Rightarrow 2(y-2) + 2y\lambda = 0 \Rightarrow y = \frac{2}{1+\lambda}$

$\frac{\partial F}{\partial z} = 0 \Rightarrow 2(z+1) + 2z\lambda = 0 \Rightarrow z = \frac{-1}{1+\lambda}$

Putting these values in constraints $x^2+y^2+z^2 - \cancel{24} = 0$

$\Rightarrow \left(\frac{1}{1+\lambda^2}\right)^2 = 4 \Rightarrow (1+\lambda)^2 = \frac{1}{4}$

$\Rightarrow 1+\lambda = \pm\frac{1}{2} \Rightarrow \lambda = -\frac{1}{2}$ or $\frac{-3}{2}$

When $\lambda = -\frac{1}{2}$, the point on the sphere is $(2,4,-2)$

when $\lambda = \frac{-3}{2}$, the point on the sphere is $(-2,-4,2)$

when the point is $(2,4,-2)$

$d = \sqrt{(2-1)^2 + (4-2)^2 + (-2+1)^2} = \sqrt{1+4+1} = \sqrt{6}$

& when the point is $(-2,-4,2)$

$d = \sqrt{(-2-1)^2 + (-4-2)^2 + (2+1)^2}$

$= \sqrt{9+36+9} = 3\sqrt{6}$

$\therefore$ Shortest and longest distance are $\sqrt{6}$ & $3\sqrt{6}$.

18) Let x, y, z be the length, breadth and height of the box respectively. The material for the Construction of the box is least if the area of the surface of the box is least.

Hence we have to minimize

$$S = xy + 2yz + 2zx$$

subject to the Cond^n that the volume is 32 Cm

i.e $\quad xyz = 32$

So by method of Lagrange multiplier

$$F(x, y, z, \lambda_1) = xy + 2yz + 2zx + \lambda_1 (xyz - 32)$$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow y + 2z + \lambda_1 yz = 0 \quad \text{—} \; (1)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow x + 2z + xz\lambda_1 = 0 \quad \text{—} \; (2)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow 2y + 2x + xy\lambda_1 = 0 \quad \text{—} \; (3)$$

$$xyz = 32 \quad \text{—} \; (4)$$

from (1), (2) & (3) we have

$$x = -\frac{4}{\lambda}, \quad y = -4/\lambda, \quad z = \frac{-2}{\lambda}$$

Putting these values in $xyz = 32$, we have

$$-\frac{32}{\lambda^3} = 32 \Rightarrow \lambda^3 = -1 \Rightarrow \lambda = -1$$

$$\therefore x = 4, \; y = 4, \; z = 2$$

Thus the dimension of the box are 4cm, 4Cm & 2Cm.

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Put in $\Rightarrow u = v = 0$

Constraint ②, we get

$$u^2+u^2+u^2-a^2=0 \Rightarrow 3u^2=a^2$$

$$\boxed{u = \pm a/\sqrt{3} = v = w}$$

so corresponding to $(u,v,w) = \left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$, we get

$$\frac{a}{\sqrt{3}} - x = \frac{a}{\sqrt{3}} - y = \frac{a}{\sqrt{3}} - z$$

$$\Rightarrow x = y = z$$

so constant ① $\Rightarrow 3x = 2a \Rightarrow x = \frac{2a}{3} = y = z$

Hence an extremum is obtained where $(x,y,z) = \left(\frac{2a}{3}, \frac{2a}{3}, \frac{2a}{3}\right)$

and $(u,v,w) = \left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$. The distance b/w the two points

$$d = \sqrt{\left(\frac{2a}{3} - \frac{a}{\sqrt{3}}\right)^2 + \left(\frac{2a}{3} - \frac{a}{\sqrt{3}}\right)^2 + \left(\frac{2a}{3} - \frac{a}{\sqrt{3}}\right)^2}$$

$$= \sqrt{3\left(\frac{2a}{3} - \frac{\sqrt{3}a}{3}\right)^2} = \sqrt{\frac{\cancel{3}a^2}{\cancel{9}3}(2-\sqrt{3})^2}$$

$$= \sqrt{\frac{a^2}{3}(4+3-4\sqrt{3})} = \sqrt{\frac{a^2}{3}(7-4\sqrt{3})}$$

Now, corresponding to $\left(\frac{-a}{\sqrt{3}}, \frac{-a}{\sqrt{3}}, \frac{-a}{\sqrt{3}}\right)$, we get

$(x,y,z) = \left(\frac{2a}{3}, \frac{2a}{3}, \frac{2a}{3}\right)$. So distance b/w the two points is $d^2 = 3\left(\frac{2a}{3} + \frac{a}{\sqrt{3}}\right) = \frac{a^2}{3}(7+4\sqrt{3})$

$\therefore$ Smallest distance is $\frac{a}{\sqrt{3}}\sqrt{(7-4\sqrt{3})}$

Largest distance is $\frac{a}{\sqrt{3}}\sqrt{7+4\sqrt{3}}$.

(17.) Let $P(x,y,z)$ & $Q(u,v,w)$. The smallest / largest distance b/w them is the square root of minimum/ maximum value of

$$F(x,y,u,v) = (x-u)^2 + (y-v)^2 + (z-w)^2$$

subject to

$$\phi_1(x,y,z) = x+y+z-2a=0 \; ; \; \phi_2(u,v) = u^2+v^2+w^2-a^2=0$$

$$F(x,y,z,u,v,w,\lambda_1,\lambda_2) = (x-u)^2 + (y-v)^2 + (z-w)^2$$
$$+\lambda_1(x+y+z-2a) + \lambda_2(u^2+v^2+w^2-a^2)$$

$$\frac{\partial F}{\partial x} = 2(x-u) + \lambda_1 = 0 \Rightarrow \lambda_1 = -2(x-u) = 2(u-x)$$

$$\frac{\partial F}{\partial y} = 2(y-v) + \lambda_1 = 0 \Rightarrow \lambda_1 = 2(v-u)$$

$$\frac{\partial F}{\partial z} = 2(z-w) + \lambda_1 = 0 \Rightarrow \lambda_1 = 2(w-z)$$

$$\frac{\partial F}{\partial u} = -2(x-u) + 2u\lambda_2 = 0 \Rightarrow u\lambda_2 = (x-u)$$

$$\frac{\partial F}{\partial v} = -2(y-v) + 2v\lambda_2 = 0 \Rightarrow v\lambda_2 = (y-v)$$

$$\frac{\partial F}{\partial w} = -2(z-w) + 2w\lambda_2 = 0 \Rightarrow w\lambda_2 = (z-w)$$

$$u-x = v-y = w-z \; , \quad \frac{x-u}{u} = \frac{y-v}{v} = \frac{z-w}{w}$$

$$\Rightarrow (u-x) = v-y \; , \; (v-y) = (w-z)$$
$$\& \; (w-z) = (u-x)$$

$$v(x-y) = u(y-v)$$
$$w(y-v) = v(z-w)$$
$$w(x-y) = u(z-w)$$

$$\frac{u-x}{v(u-x)} = \frac{v-y}{-u(v-y)} \Rightarrow \boxed{v=u}$$

$$\frac{v-y}{w(y-v)} = \frac{w-z}{v(z-w)} \Rightarrow \boxed{w=v}$$

$$\frac{w-z}{-u(w-z)} = \frac{u-x}{-w(u-x)} \Rightarrow \boxed{w=y}$$

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$\Rightarrow \quad b = a$

$2s = 2a + b \Rightarrow s(-b/2)(2a + b - a - 2b) = 0$

$\Rightarrow \quad a = b$

If we express Z as a function of b and c, we similarly get $b = c$

$$\therefore \quad a = b = c = \frac{2s}{3}$$

Now $$\frac{\partial^2 f}{\partial a^2} = -2s(s-b)$$

$$\frac{\partial^2 f}{\partial b^2} = -2s(s-a)$$

$$\frac{\partial^2 f}{\partial a \partial b} = s\left[-(2s-b-2a) - (s-b)\right]$$
$$= s(2a + 2b - 3s)$$

$$\therefore \left.\frac{\partial^2 f}{\partial a^2}\right|_{\left(\frac{2s}{3}, \frac{2s}{3}\right)} = -2s\left(-\frac{2s}{3} + s\right) = -\frac{2s^2}{3} < 0$$

$$\left.\frac{\partial^2 f}{\partial b^2}\right|_{\left(\frac{2s}{3}, \frac{2s}{3}\right)} = -\frac{2s^2}{3}$$

$$\left.\frac{\partial^2 f}{\partial a \partial b}\right|_{\left(\frac{2s}{3}, \frac{2s}{3}\right)} = -s^2/3$$

$$\therefore \frac{\partial^2 f}{\partial a^2} \cdot \frac{\partial^2 f}{\partial b^2} - \left(\frac{\partial^2 f}{\partial a \partial b}\right)^2 = \frac{s^4}{3} > 0$$

when $a = b = c = \frac{2s}{3}$

$\therefore$ $\Delta^2 f$ is maximum when $a = b = c$. i.e

So Δ is maximum when $a = b = c$. i.e triangle is equilateral.

(6) Let a, b, c be the sides of a triangle whose perimeter $2s$ is constant.

Then $2s = a+b+c \Rightarrow c = 2s-a-b$

$$\text{Area } \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{s(s-a)(s-b)(a+b-s)}$$

Let $z = \Delta^2 = s(s-a)(s-b)(a+b-s)$
$$= f(a,b)$$

Then
$$\frac{\partial f}{\partial a} = s(s-b)\frac{\partial}{\partial a}\left[(s-a)(a+b-s)\right]$$
$$= s(s-b)\left[(s-a)-(a+b-s)\right]$$
$$= s(s-b)(s-a-a-b+s)$$
$$= s(s-b)(2s-2a-b)$$

$$\frac{\partial f}{\partial b} = s(s-a)(2s-a-2b)$$

For maximum or minimum value of $f(a,b)$, $\frac{\partial f}{\partial a} = 0 = \frac{\partial f}{\partial b}$

$$\frac{\partial f}{\partial a} = 0 \Rightarrow s(s-b)(2s-b-2a) = 0$$
$$\Rightarrow s = b \text{ or } 2s = 2a+b$$

$$\frac{\partial f}{\partial b} = 0 \Rightarrow s(s-a)(2s-2b-a) = 0$$

Now $s = b \Rightarrow b(b-a)(2b-2b-a) = 0$

$\Rightarrow x^3 = \frac{\lambda}{2a^3}$

$\frac{\partial F}{\partial y} = 2b^3 y - \lambda/y^2 = 0 \Rightarrow \frac{1}{y} = \left(\frac{2b^3}{\lambda}\right)^{1/3}$ or $y^3 = \frac{\lambda}{2b^3}$

$\frac{\partial F}{\partial z} = 2c^3 z - \frac{\lambda}{z^2} = 0 \Rightarrow \frac{1}{z} = \left(\frac{2c^3}{\lambda}\right)^{1/3}$ or $z^3 = \frac{\lambda}{2c^3}$

Substituting this in the constraints $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$, we get,

$$\left(\frac{2a^3}{\lambda}\right)^{1/3} + \left(\frac{2b^3}{\lambda}\right)^{1/3} + \left(\frac{2c^3}{\lambda}\right)^{1/3} = 1$$

$$\Rightarrow 2^{1/3}(a+b+c) = \lambda^{1/3}$$

$$\Rightarrow \lambda = 2(a+b+c)^3$$

$\therefore x^3 = \frac{(a+b+c)^3}{a^3}, \ y^3 = \frac{(a+b+c)^3}{b^3}, \ z^3 = \frac{(a+b+c)^3}{c^3}$

$\Rightarrow x = \frac{a+b+c}{a}, \ y = \frac{a+b+c}{b}, \ z = \frac{a+b+c}{c}$

$\therefore$ Extremum value is :

$$F\left(\frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}\right)$$

$$= a(a+b+c)^2 + b(a+b+c)^2 + c(a+b+c)^2$$

$$= (a+b+c)^3$$

we get,

$$\frac{1}{4\lambda_1^2}+\frac{9}{4\lambda_1^2}=5 \quad \text{or } \lambda_1^2=\frac{1}{2} \Rightarrow \lambda_1=\pm\frac{1}{\sqrt{2}}$$

for $\lambda_1=\frac{1}{\sqrt{2}}$, we get

$$x=-\frac{\sqrt{2}}{2}, \; y=-\frac{3\sqrt{2}}{2}, \; z=1-x=\frac{2+\sqrt{2}}{2}$$

and
$$f(x,y,z)=-\sqrt{2}-\frac{9\sqrt{2}}{2}+\frac{2+\sqrt{2}}{2}=\frac{2-10\sqrt{2}}{2}$$
$$=1-5\sqrt{2}$$

for $\lambda_1=-\frac{1}{\sqrt{2}}$, we get $x=\frac{\sqrt{2}}{2}, y=\frac{3\sqrt{2}}{2}, z=\frac{2-\sqrt{2}}{2}$

and
$$f(x,y,z)=\sqrt{2}+\frac{9\sqrt{2}}{2}+\frac{2-\sqrt{2}}{2}=\frac{2+10\sqrt{2}}{2}$$
$$=1+5\sqrt{2}$$

$\therefore$ Given function has absolute maximum value $1+5\sqrt{2}$ at $\left(\frac{\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}, \frac{2-\sqrt{2}}{2}\right)$ & absolute minimum value $1-5\sqrt{2}$ at $\left(-\frac{\sqrt{2}}{2}, \frac{-3\sqrt{2}}{2}, \frac{2+\sqrt{2}}{2}\right)$.

(15.) Consider the auxilary function

$$F(x,y,z,\lambda)=a^3x^2+b^3y^2+c^3z^2+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$$

To find the stationary points

$$\frac{\partial F}{\partial x}=0, \; \frac{\partial F}{\partial y}=0, \; \frac{\partial F}{\partial z}=0$$

$$\frac{\partial F}{\partial x}=2a^3x-\lambda/x^2=0 \Rightarrow \frac{1}{x}=\left(\frac{2a^3}{\lambda}\right)^{1/3}$$

we

Similarly from ② & ③

$$\frac{2\lambda y^2}{b^2} = -8xyz = \frac{2\lambda z^2}{c^2}$$

Thus $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = K$ (say)

using ④, we have $3K = 1 \Rightarrow K = 1/3$

$\therefore x = \frac{a}{\sqrt{3}}, \ y = \frac{b}{\sqrt{3}}, \ z = \frac{c}{\sqrt{3}}$

Therefor maximum volume $= \frac{8abc}{3\sqrt{3}}$

(14.) Consider the auxilary function

$$F(x,y,z,\lambda_1,\lambda_2) = (2x+3y+z) + \lambda_1(x^2+y^2-5) + \lambda_2(x+z-1) \quad \text{—— ①}$$

for stationary points :

$\frac{\partial F}{\partial x} = 0, \ \frac{\partial F}{\partial y} = 0, \ \frac{\partial F}{\partial z} = 0$

$\frac{\partial F}{\partial x} = 2 + 2\lambda_1 x + \lambda_2 = 0 \Rightarrow 2 + 2\lambda_1 x + \lambda_2 = 0$ —— ②

$\frac{\partial F}{\partial y} = 3 + 2\lambda_1 y = 0 \Rightarrow 3 + 2\lambda_1 y = 0$ —— ③

$\frac{\partial F}{\partial z} = 1 + \lambda_2 = 0 \Rightarrow \lambda_2 = -1$ —— ④

from equation ② ③ & ④ we get

$x = -\frac{1}{2\lambda_1}$ & $y = -\frac{3}{2\lambda_1}$

Substituting these values in constraints $x^2+y^2=5$, (13.) Let $2x, 2y, 2z$ be the dimension of the required rectangle parallelapiped. By symmetry the centre of the parallelopiped coincides with that of the ellipsoid, namely the origin and its faces are parallel to the co-ordinate planes.

Also, one of the vertices of the parallelopiped has co-ordinates (x, y, z), which satisfy the eq^n of the elliposid.

Thus we have maximise

$V = 8xyz$, subject to the condition

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

So, the auxilary function is

$$F(x, y, z, \lambda) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

The stationary points of F are given by

$$F_x = 0 \Rightarrow 8yz + \frac{2\lambda x}{a^2} = 0 \quad \text{—— (1)}$$

$$F_y = 0 \Rightarrow 8xz + \frac{2\lambda y}{b^2} = 0 \quad \text{—— (2)}$$

$$F_z = 0 \Rightarrow 8xy + \frac{2\lambda z}{c^2} = 0 \quad \text{— (3)}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{—— (4)}$$

multiplying (1) by x, we get

$$8xzy + \frac{2\lambda x^2}{a^2} = 0 \Rightarrow \frac{2\lambda x^2}{a^2} = -8xyz$$

on dividing we get,

$$8y = 9x$$

substituting this in equation ⑤, we get

$$\frac{x^2}{4} + \frac{9x^2}{64} = 1 \Rightarrow x^2 = \frac{64}{25}$$

$$\therefore x = \pm 8/5 \quad \& \quad y = \pm \frac{9}{5}$$

corresponding to $x = \frac{8}{5}$, $y = \frac{9}{5}$ we get

$$\frac{8}{5} - u = 2\left(\frac{9}{5} - v\right)$$

$$\Rightarrow 2v - u = 2 \quad \text{or} \quad u = 2v - 2$$

substituting in eqⁿ ⑥, we get

$$u = 18/5 \quad \& \quad v = 14/5$$

Hence the extremum is obtained when $(x,y) = \left(\frac{8}{5}, \frac{9}{5}\right)$ and $(u,v) = \left(\frac{18}{5}, \frac{14}{5}\right)$. The distance between the two points is $\sqrt{5}$

Now Corresponding to $x = -8/5$, $y = -9/5$, we get $u - 2v = 2$. Substituting this in eqⁿ of line $2u + v = 10$, we get $u = \frac{22}{5}$, $v = 6/5$. Hence another extremum is obtained when $(x,y) = (-8/5, -9/5)$ & $(u,v) = \left(\frac{22}{5}, 6/5\right)$

The distance b/w these two points is $3\sqrt{5}$.

Hence the shortest distance between the line & ellipse is $\sqrt{5}$.

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12. Let (x, y) be a point on the ellipse and (u, v) be a point on the line. Then the shortest distance between the line and the ellipse is the square root of the minimum value of

$$F(x, y, u, v) = (x-u)^2 + (y-v)^2 \quad \text{--- (1)}$$

Subject to constraints,

$$\phi_1(x, y) = \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \;\&\; \phi_2(x, y) = 2u + v - 10 = 0$$

Define the auxilary function as

$$F(x, y, u, v, \lambda_1, \lambda_2) = (x-u)^2 + (y-v)^2 + \lambda_1\left(\frac{x^2}{4} + \frac{y^2}{9} - 1\right) + \lambda_2(2u + v - 10)$$

For stationary point

$$\frac{\partial F}{\partial x} = 2(x-u) + \frac{\lambda_1 x}{2} = 0 \Rightarrow \lambda_1 x = 4(u-x) \quad \text{--- (1)}$$

$$\frac{\partial F}{\partial y} = 2(y-v) + \frac{2\lambda_1 y}{9} = 0 \Rightarrow \lambda_1 y = 9(v-y) \quad \text{--- (2)}$$

$$\frac{\partial F}{\partial u} = -2(x-u) + 2\lambda_2 = 0 \Rightarrow \lambda_2 = (x-u) \quad \text{--- (3)}$$

$$\frac{\partial F}{\partial v} = -2(y-v) + \lambda_2 = 0 \Rightarrow \lambda_2 = 2(y-v) \quad \text{--- (4)}$$

$$\phi_1 = \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1 \quad \text{--- (5)}$$

$$\phi_2 = 2u + v - 10 = 0 \Rightarrow 2u + v = 10 \quad \text{--- (6)}$$

Eliminating λ_1 & λ_2 from equation (1) (2) (3) & (4)

we get $4(u-x)y = 9(v-y)x$ & $(x-u) = 2(y-v)$

Now $A = f_{xx} = 6$, $B = f_{xy} = 0$, $C = f_{yy} = 2$

$\therefore AC - B^2 = 12 > 0$ & $A = 6 > 0$

$\Rightarrow (1/6, 0)$ is a point of local Minimum and minimum value is $F(1/6, 0) = -\frac{1}{12}$.

On the boundary we have

$$y^2 = 1 - 2x^2, \quad -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$$

Substituting this in $F(x,y)$, we obtain

$$F(x,y) = 3x^2 + (1-2x^2) - x = 1 - x + x^2 = g(x)$$

$g'(x) = 0 \Rightarrow -1 + 2x = 0 \Rightarrow x = 1/2$. Also $g''(x) = 2 > 0$

for $x = 1/2$, we get $y^2 = \frac{1}{2}$ or $y = \pm\frac{1}{\sqrt{2}}$. Hences the points $\left(\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right)$ are points of Minimum.

The Minimum value is $F\left(\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right) = 3/4$

At the vertices we have

$$f\left(\frac{1}{\sqrt{2}}, 0\right) = \frac{3-\sqrt{2}}{2}, \quad F\left(\frac{-1}{\sqrt{2}}, 0\right) = \frac{3+\sqrt{2}}{2}$$

$$F(0, \pm 1) = 1$$

Therefore the given function has absolute minimum value $-\frac{1}{12}$ at $\left(\frac{1}{6}, 0\right)$ and absolute maximum value $\frac{3+\sqrt{2}}{2}$ at $\left(-\frac{1}{\sqrt{2}}, 0\right)$.

$\therefore$ Stationary points exist at $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$

(i) At $\pi/4$: $g(\pi/4) = 1 + \frac{1}{2}\sin\frac{\pi}{2} = 3/2$. This occurs at $x = \cos\pi/4 = \frac{1}{\sqrt{2}}$, $y = \sin\pi/4 = \frac{1}{\sqrt{2}}$

(ii) At $3\pi/4$: $g(3\pi/4) = 1 - \frac{1}{2} = \frac{1}{2}$. This occurs at

$$x = \cos 3\pi/4 = -\frac{1}{\sqrt{2}}, \quad y = \sin 3\pi/4 = \frac{1}{\sqrt{2}}$$

(iii) At $5\pi/4$: $g(5\pi/4) = 1 + \frac{1}{2} = 3/2$. This occurs at

$$x = \cos\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}; \quad y = \sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$$

(iv) At $7\pi/4$: $g(7\pi/4) = 1 - \frac{1}{2} = \frac{1}{2}$. This occurs at

$$x = \cos\frac{7\pi}{4} = \frac{1}{\sqrt{2}}, \quad y = \sin\frac{7\pi}{4} = -\frac{1}{\sqrt{2}}$$

Hence $f(x,y)$ attains absolute Maximum value $3/2$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and absolute Minimum value 0 at $(0,0)$.

(11.) $F(x,y) = 3x^2 + y^2 - x$; $R\{2x^2 + y^2 \le 1\}$

$F_x = 0 \Rightarrow 6x - 1 = 0 \Rightarrow x = 1/6$

$F_y = 0 \Rightarrow 2y = 0 \Rightarrow y = 0$

$\therefore$ only stationary point is $(1/6, 0)$

(10) $F(x,y) = x^2+xy+y^2$; $R = \{(x,y) \mid x^2+y^2 \le 1\}$

Step (1) : For all critical point of $f(x,y)$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 0, \Rightarrow 2x+y=0 \\ \frac{\partial f}{\partial y} &= 0 \Rightarrow x+2y=0 \end{aligned}\right\} \Rightarrow x=0,\ y=0$$

$\therefore$ $(0,0)$ is the only critical/stationary point which lies in R. $F(0,0)=0$

Since at $(0,0)$ $F_{xx}F_{yy} - F_{xy}^2 = 2\cdot 2 - 1 = 3 > 0$

and $F_{xx} = 2 > 0$. So $(0,0)$ is a point of local minima.

Step (2) : Examine $f(x,y)$ over the boundary of R i.e $x^2+y^2 = 1$

Let $x(t) = \cos t$, $y(t) = \sin t$; $0 \le t \le 2\pi$

$$g(t) = f(x(t), y(t))$$

$$= \cos^2 t + \cos t \sin t + \sin^2 t$$

$$\Rightarrow g(t) = 1 + \frac{1}{2}\sin 2t \ ; 0 \le t \le 2\pi$$

Now, to determine Minimum/Maximum value of $g(t)$, $g'(t) = 0$

$\Rightarrow \cos 2t = 0 \Rightarrow 2t = \pi/2, 3\pi/2, 5\pi/2, 7\pi/2$

$\Rightarrow t = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$

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Now, $\frac{d^2g}{dx^2} = 8 > 0 \Rightarrow$ At $x=1$ function has a minimum and the minimum value is $g(1)=0$.

Also at the Corners $(0,0)$, $(2,0)$, we have

$F(0,0) = g(0) = 4$; $F(2,0) = g(2) = 4$.

Similarly along other boundary lines, we have

- $x=2$: $h(y) = 9y^2 - 12y + 4$; $\frac{dh}{dy} = 18y - 12 = 0$ gives $y = 2/3$. $\frac{d^2h}{\partial y^2} = 18 > 0$. Therefore $y = 2/3$ is a point of minima and minimum values is $f(2, 2/3) = 0$. At the corner $(2,3)$, we have $f(2,3) = 49$.

- $y=3$; $g(x) = 4x^2 - 8x + 49$, $\frac{dg}{dx} = 8x - 8 = 0 \Rightarrow x = 1$

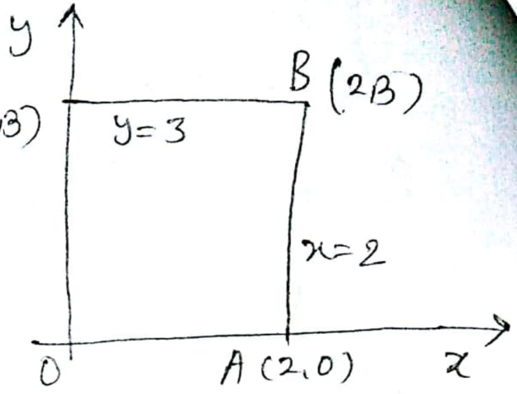
$\frac{d^2g}{dx^2} = 8 > 0$. Therefore $x=1$ is point of minima and minimum value is $f(1,3) = 45$. At the corner point $(0,3)$, we have $f(0,3) = 49$.

- $x=0$; $h(y) = 9y^2 - 12y + 4$, which is the same case as for $x=2$.

$81 - 36 + 4$

$\therefore$ Absolute minimum value is -4 which occurs at $(1, 2/3)$ and the absolute maximum value is 49 which occurs at $(2,3)$ & $(0,3)$.

(9.) $f(x,y) = 4x^2 + 9y^2 - 8x \mp 12y + 4$



$F_x = 8x - 8$

$F_y = 18y - 12$

$A = f_{xx} = 8$

$B = f_{xy} = 0$

$C = f_{yy} = 18$

To find stationary point :

$f_x = 0 \Rightarrow 8x - 8 = 0 \Rightarrow x = 1$

$f_y = 0 \Rightarrow 18y - 12 = 0 \Rightarrow y = 2/3$

$\therefore$ Stationary Point is $(1, 2/3)$

At $(1, 2/3)$; $AC - B^2 = 144 > 0$, Also $A > 0$

$\Rightarrow (1, 2/3)$ is a point of relative Minimum

and value is $f(1, 2/3) = -4$

Now, we need to check on the boundary.

(i) On the boundary line OA, we have $y = 0$

& $f(x,y) = 4x^2 - 8x + 4 = F(x,0) = g(x)$

$= 4x^2 - 8x + 4$ which is a function of one variable.

Setting $g'(x) = 0 \Rightarrow 8x - 8 = 0 \Rightarrow x = 1$

(8). $f(x,y) = x^3y^2(1-x-y)$

$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$

$f_y = 2x^3y - 2x^4y - 3x^3y^2$

$A = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$

$B = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$

$C = f_{yy} = 2x^3 - 2x^4 - 6x^3y$

To find the stationary point $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$

$\Rightarrow x^2y^2(3-4x-3y) = 0 \Rightarrow x=0$, or $y=0$
or $3-4x-3y=0$

$x^3y(2-2x-3y) = 0 \Rightarrow x=0$ or $y=0$ or
$2-2x-3y=0$

$3-4x-3y=0$
$2-2x-3y=0$ $\Rightarrow x = 1/2$, $y = 1/3$

$\Rightarrow (1/2, 1/3)$ is one of the stationary point

At $\left(\frac{1}{2}, \frac{1}{3}\right)$: $A = -\frac{1}{9}$, $B = \frac{-1}{12}$; $C = -1/8$

$\therefore AC - B^2 = \frac{1}{72} - \frac{1}{144} = \frac{1}{144} > 0$

Also $A < 0$

$\Rightarrow f(x,y)$ has a maximum at $(1/2, 1/3)$.

7(e). $f(x,y) = x^2y - xy^2 + 4xy - 4x^2 - 4y^2$ (8)

$f_x = 2xy - y^2 + 4y - 8x$

$f_y = x^2 - 2xy + 4x - 8y$

$A = f_{xx} = 2y - 8$; $B = f_{xy} = 2x - 2y + 4$

$C = f_{yy} = -2x - 8$;

To find stationary points $f_x = 0$, $f_y = 0$

$y^2 - (2x+4)y + 8x = 0$

$x^2 - (2y-4)x - 8y = 0$

$\Rightarrow y = \dfrac{(2x+4) \pm (2x-4)}{2} \Rightarrow y = 2x$ or $y = 4$

$x = \dfrac{(2y-4) \pm 2(y+2)}{2} \Rightarrow x = 2y$ or $x = -4$

$\therefore$ Stationary points are $(0,0)$, $(-4,-8)$, $(-4,4)$ & $(8,4)$

(i) At $(0,0)$: $A = -8$, $B = 4$, $C = -8$; $AC - B^2 = 48 > 0$

Also $A < 0 \Rightarrow (0,0)$ is local maxima

(ii) At $(-4,-8)$: $A = -24$, $B = 12$, $C = 0$; $AC - B^2 = -144 < 0$

$\Rightarrow (-4,-8)$ is a saddle point.

(iii) At $(-4,4)$: $A = 0$, $B = -12$, $C = 0$; $AC - B^2 = -144 < 0$

$\Rightarrow (-4,4)$ is a saddle point

(iv) At $(8,4)$: $A = 0$, $B = 12$, $C = -24$; $AC - B^2 = -144 < 0$

$\Rightarrow (8,4)$ is a saddle point.

7(d). $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$f_x = 3x^2 + 3y^2 - 30x + 72$

$f_y = 6xy - 30y$

$A = f_{xx} = 6x - 30$; $B = f_{xy} = 6y$; $C = f_{yy} = 6x - 30$

$f_x = 0$ $\Rightarrow$ $3x^2 + 3y^2 - 30x + 72 = 0$
$f_y = 0$ $\quad 6y(x-5) = 0 \Rightarrow y = 0$ or $x = 5$

$y = 0 \Rightarrow 3x^2 + 0 - 30x + 72 = 0 \Rightarrow x = 4, 6$

$x = 5 \Rightarrow 75 + 3y^2 - 150 + 72 = 0 \Rightarrow 3y^2 - 3 = 0$
$\Rightarrow y = \pm 1$

Therefore stationary points are
$(4,0)$, $(6,0)$, $(5,1)$ & $(5,-1)$

(i) At $(4,0)$; $A = -6$, $B = 0$, $C = -6$
$\therefore AC - B^2 = 36 > 0$, Also $A < 0$
$\Rightarrow (4,0)$ is local Maxima.

(ii) At $(6,0)$, $A = 6$, $B = 0$, $C = 6$; $AC - B^2 = 36 > 0$ & $A > 0$
$\Rightarrow (6,0)$ is a local Minima.

(iii) At $(5,1)$; $A = 0$, $B = 6$, $C = 0$; $AC - B^2 = -36 < 0$
$\Rightarrow (5,1)$ is a saddle point.

(iv) At $(5,-1)$; $A = 0$, $B = -6$, $C = 0$; $AC - B^2 = -36 < 0$
$\Rightarrow (5,-1)$ is a saddle point.

(7(c)) $f(x,y) = x^3 - 12x + y^3 + 3y^2 - 9y$

$\frac{\partial f}{\partial x} = 3x^2 - 12$; $\frac{\partial f}{\partial y} = 3y^2 + 6y - 9$

$A = \frac{\partial^2 f}{\partial x^2} = 6x$; $B = \frac{\partial^2 f}{\partial x \partial y} = 0$; $C = \frac{\partial^2 f}{\partial y^2} = 6y + 6$

To find the critical points ; $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

$\Rightarrow 3x^2 - 12 = 0 \Rightarrow x = \pm 2$

$3y^2 + 6y - 9 = 0 \Rightarrow y = 1, -3$

So stationary points are $(2,1)$, $(2,-3)$, $(-2,1)$, $(-2,-3)$

(i) At $(2,1)$; $A = 12$, $B = 0$, $C = 12$

$\therefore AC - B^2 = 144 > 0$; Also $A > 0$

$\Rightarrow (2,1)$ is a local maxima.

(ii) At $(2,-3)$; $A = 12$, $B = 0$, $C = -12$

$\therefore AC - B^2 = -144 < 0$

$\Rightarrow (+2,-3)$ is a saddle point

(iii) At $(-2,1)$; $A = -12$, $B = 0$, $C = 12$

$\therefore AC - B^2 = -144 < 0$

$\Rightarrow (-2,1)$ is saddle point

(iv) $(-2,-3)$; $A = -12$, $B = 0$, $C = -12$

$\therefore AC - B^2 = 144 > 0$; $A < 0$

$\Rightarrow (-2,-3)$ is a local Maxima.

(0,0)	4	−4	4	0	We have to Investigate further
$(\sqrt{2}, -\sqrt{2})$	−20	−4	−20	384 > 0	Maximum as A<0
$(-\sqrt{2}, \sqrt{2})$	−20	−4	−20	384 > 0	Local Maximum as A<0

Now, Consider the Case when $(x,y)=(0,0)$.

For point $(x,0)$ along x-axis the value of the function $= 2x^2-x^4 = x^2(2-x^2) > 0$ for points in the immediate neighbourhood of the origin.

But for the points along $y=x$, the value of ~~(0,0)~~ the function $= -2x^4 < 0$. Thus every nbd of (0,0), there are points where the function assumes positive ~~where the f~~ values i.e. $> F(0,0)$ and there are points where the function assume negative values i.e. $< F(0,0)$. Hence (0,0) is neither Maximum nor Minimum.

(iv) At $(-1, 1/2)$; $A = 1$, $B = -1$, $C = 4$

$\therefore AC - B^2 = 4 - 1 = 3 > 0$, Also $A > 1$

$\Rightarrow (-1, 1/2)$ is a local minimum point.

7(b). $f(x,y) = 2(x-y)^2 - x^4 - y^4$

$f_x = 4(x-y) - 4x^3$

$f_y = -4(x-y) - 4y^3$

$A = F_{xx} = 4 - 12x^2$; $B = f_{xy} = -4$; $C = f_{yy} = 4 - 12y^2$

for stationary points, $f_x = 0$; $F_y = 0$

$\therefore f_x = 0 \Rightarrow 4(x-y) - 4x^3 = 0$
& $f_y = 0 \Rightarrow -4(x-y) - 4y^3 = 0$ $\Big\} \Rightarrow x^3 + y^3 = 0$ or $(x+y)(x^2+y^2-xy) = 0$

$x + y = 0$ or $x^2 + y^2 - xy = 0$

$\left.\begin{array}{l} x - y - x^3 = 0 \\ x + y = 0 \end{array}\right\} \Rightarrow (0,0)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$

$\left.\begin{array}{l} x - y - x^3 = 0 \\ x^2 - xy + y^2 = 0 \end{array}\right\} \Rightarrow (0,0)$ is the only real solution.

Therefore stationary points are $(0,0)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$

When $y=0$, from ②, $x=0$ & $x=-3$

When $x=0$, from ①, $y=0$ & $y=3/2$

& $2x-2y+3=0$

$x-4y+3=0$ $\Rightarrow x=-1, y=1/2$

$\therefore$ Stationary points are $(0,0), (-3,0), (0,3/2)$ & $(-1, 1/2)$

Now, $A = f_{xx} = 2y$

$B = f_{xy} = 2x-4y+3$

$C = f_{yy} = -4x$

(i) At $(0,0)$; $A=0, B=3, C=0$

$\therefore AC-B^2 = -9 < 0$

$\Rightarrow (0,0)$ is a sadle point.

(ii) At $(-3,0)$; $A=0, B=-3, C=12$

$\therefore AC-B^2 = -9 < 0$

$\Rightarrow (-3,0)$ is a saddle point.

(iii) At $(0,3/2)$; $A=3, B=-3, C=0$

$\therefore AC-B^2 = -9 < 0$

$\Rightarrow (0,3/2)$ is a saddle point.

$f_{xyy}(x,y) = -\cos x \sin y$;

$f_{yyy}(x,y) = -\sin x \cos y$;

Then from (1), we get

$$f(x,y) = 0+0+0+0+\frac{1}{2}[0+2xy\cdot 1+0]+\frac{1}{6}[-x^3\cos\xi \sin\eta$$

$$-3x^2y \sin\xi \cos\eta - 3xy^2 \cos\xi \sin\eta - y^3 \sin\xi \cos\eta]$$

Since $\xi = \theta x$ & $\eta = \theta y$; we have

$$f(x,y) = xy - \frac{1}{6}[(x^3+3xy^2)\cos(\theta x)\sin(\theta y)$$

$$+ (y^3+3x^2y)\sin(\theta x)\cos(\theta y)]$$

$$0<\theta<1$$

Hence proved.

(7) (a) $f(x,y) = x^2y - 2xy^2 + 3xy + 4$

$$\frac{\partial f}{\partial x} = 2xy - 2y^2 + 3y \; ; \; \frac{\partial f}{\partial y} = x^2 - 4xy + 3x$$

for stationary points

$$\frac{\partial f}{\partial x} = 0 \quad \& \quad \frac{\partial f}{\partial y} = 0$$

$\therefore \frac{\partial f}{\partial x} = 0 \Rightarrow y(2x-2y+3) = 0 \Rightarrow y = 0$ or $2x-2y+3=0$ —(1)

& $\frac{\partial f}{\partial y} = 0 \Rightarrow x(x-4y+3) = 0 \Rightarrow x=0$ or $x-4y+3=0$ —(2)

$$F(x,y) = (\pi+e) + (x-1)(2\pi+e) + \frac{(x-1)^2}{2}(2\pi+e)$$
$$+ 2(x-1)(y-\pi) + \frac{(x-1)^3}{6}e^{\xi} + (x-1)^2(y-\pi)$$
$$- (y-\pi)^3 \cos\eta$$

(6.) $f(x,y) = \sin x \sin y$

Expanding using Taylor's series about $(0,0)$, we get, (taking Remainder after ~~there~~ three term)

$$f(x,y) = f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2!}[x^2 f_{xx}(0,0)$$
$$+ 2xy f_{yx}(0,0) + y^2 f_{yy}(0,0)] + \frac{1}{3!}[x^3 f_{xxx}(\xi,\eta)$$
$$+ 3x^2y f_{xxy}(\xi,\eta) + 3xy^2 f_{xyy}(\xi,\eta) + y^3 f_{yyy}(\xi,\eta)] \quad \text{—— (1)}$$

where $\xi = \theta x$; $\eta = \theta y$; $0 < \theta < 1$

Now, $f(x,y) = \sin x \cdot \sin y$; $f(0,0) = 0$

$f_x(x,y) = \cos x \cdot \sin y$; $f_x(0,0) = 0$

$f_y(x,y) = \sin x \cdot \cos y$; $f_y(0,0) = 0$

$F_{xx}(x,y) = (-\sin x) \cdot \sin y$; $f_{xx}(0,0) = 0$

$F_{xy}(x,y) = \cos x \cdot \cos y$; $f_{xy}(0,0) = 1$

$F_{yy}(x,y) = -\sin x \sin y$; $f_{yy}(0,0) = 0$

$f_{xxx}(x,y) = -\cos x \sin y$;

$f_{xxy}(x,y) = -\sin x \cos y$;

$F_{yy}(x,y) = -\sin y$; $f_{yy}(1,\pi) = 0$

$f_{xy}(x,y) = 2x$; $f_{xy}(1,\pi) = 2$

$f_{xxx}(x,y) = e^{x}$; $f_{xxx}(1,\pi) = e$

$f_{xxy}(x,y) = 2$; $f_{xxy}(1,\pi) = 2$

$f_{xyy}(x,y) = 0$; $f_{xyy}(1,\pi) = 0$

$f_{yyy}(x,y) = -\cos y$; $f_{yyy}(1,\pi) = 1$

By Taylor's - Theorem, we have

$$f(x,y) = f(1,\pi) + [(x-1) f_x(1,\pi) + (y-\pi) f_y(1,\pi)]$$
$$+ \frac{1}{2!}[(x-1)^2 f_{xx}(1,\pi) + 2(x-1)(y-\pi) f_{xy}(1,\pi)$$
$$+ (y-\pi)^2 f_{yy}(1,\pi)] + \frac{1}{3!}[(x-1)^3 F_{xxx}(\xi,\eta)$$
$$+ 3(x-1)^2 (y-z) f_{xxy}(\xi,\eta) + \frac{1}{6} 3(x-1) \cdot$$
$$(y-z)^2 f_{xyy}(\xi,\eta) + (y-\pi)^3 f_{yyy}(\xi,\eta)]$$

where $\xi = 1 + (x-1)\theta$; $\eta = \pi + (y-\pi)\theta$;

$$0 < \theta < 1$$

$$\Rightarrow f(x,y) = (\pi+e) + (x-1)(2\pi+e) + \frac{1}{2}[(x-1)^2 (2\pi+e)$$
$$+ 4(x-1)(y-\pi)] + \frac{1}{6}[(x-1)^3 e^{\xi} + 6(x-\pi)^2 \ldots$$
$$+ (y-\pi)^3 (-\cos\eta)]$$

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$F_{xx}(x,y,z) = -e^z \sin(x+y)$; $f_{xx}(0,0,0) = 0$

$F_{xy}(x,y,z) = -e^z \sin(x+y)$; $f_{xy}(0,0,0) = 0$

$f_{xz}(x,y,z) = e^z \cos(x+y)$; $f_{xz}(0,0,0) = 1$

$f_{yy}(x,y,z) = -e^z \sin(x+y)$; $f_{yy}(0,0,0) = 0$

$f_{yz}(x,y,z) = e^z \cos(x+y)$; $f_{yz}(0,0,0) = 1$

$f_{zz}(x,y,z) = e^z \sin(x+y)$; $f_{zz}(0,0,0) = 0$

Now, by Taylor's Theorem, we have

$$f(x,y,z) = f(0,0,0) + \left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right] f(0,0,0)$$

$$+ \frac{1}{2!}\left[x^2\frac{\partial^2}{\partial x^2} + y^2\frac{\partial^2}{\partial y^2} + z^2\frac{\partial^2}{\partial z^2} + 2xy\frac{\partial^2}{\partial x\partial y}\right.$$

$$\left. + 2yz\frac{\partial^2}{\partial y\partial z}\right] + 2zx\frac{\partial^2}{\partial z\partial x}\Big] F(0,0,0)$$

$$= x + y + xz + yz$$

(5.) $f(x,y) = x^2y + \sin y + e^x$; $f(1,\pi) = \pi + e$

$f_x(x,y) = 2xy + e^x$; $f_x(1,\pi) = 2\pi + e$

$f_y(x,y) = x^2 + \cos y$; $f_y(1,\pi) = 0$

$f_{xx}(x,y) = 2y + e^x$; $f_{xx}(1,\pi) = 2\pi + e$

Now Expanding $f(x,y)$ in Taylor's series upto second order term about $(\pi/2, 1)$, we get

$$f(x,y) = f(\pi/2, 1) + \left[(x-\pi/2)\frac{\partial}{\partial x} + (y-1)\frac{\partial}{\partial y}\right] f(\pi/2, 1)$$

$$+ \frac{1}{2!}\left[(x-\frac{\pi}{2})^2 \frac{\partial^2}{\partial x^2} + 2(x-\pi/2)(y-1)\frac{\partial^2}{\partial y \partial x} + (y-1)^2 + \frac{\partial^2}{\partial y^2}\right] F(\pi/2, 1)$$

$$= e + [(x-\pi/2)\cdot 0 + (y-1)\cdot e]$$

$$+ \frac{1}{2!}\left[(x-\pi/2)^2(-e) + 2(x-\pi/2)(y-1)\cdot 0 + (y-1)^2 \cdot e\right]$$

$$F(x,y) = e\left[1 + (y-1) - \frac{1}{2}(x-\pi/2)^2 + \frac{1}{2}(y-1)^2\right]$$

$$\therefore f\left(\frac{51\pi}{100}, 0.99\right) = e\left[1 + (0.99-1) - \frac{1}{2}\left(\frac{51}{100}\pi - \frac{\pi}{2}\right)^2 + \frac{1}{2}(0.99-1)\right]$$

$$= 2.68989$$

(4.) $F(x,y,z) = e^z \sin(x+y)$; $f(0,0,0) = 0$

$f_x(x,y,z) = e^z \cos(x+y)$; $f_x(0,0,0) = 1$

$f_y(x,y,z) = e^z \cos(x+y)$; $f_y(0,0,0) = 1$

$f_z(x,y,z) = e^z \sin(x+y)$; $f_z(0,0,0) = 0$

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$$F(x,y) = F(1,\pi/2) + \left[(x-1)\frac{\partial}{\partial x} + (y-\pi/2)\frac{\partial}{\partial y}\right]F(1,\pi/2)$$
$$+ \frac{1}{2!}\left[(x-1)^2\frac{\partial^2}{\partial x^2} + 2(x-1)(y-\pi/2)\frac{\partial^2}{\partial x \partial y} + (y-\pi/2)^2\frac{\partial^2}{\partial y^2}\right]F(1,\pi/2) + R$$

$$F(x,y) = 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)(y-\pi/2) - \frac{1}{2}\left(y-\frac{\pi}{2}\right)^2 + R$$

Where R is the Remainder term, given by

$$R = \frac{1}{3!}\left[(x-1)^3\frac{\partial^3}{\partial x^3} + 3(x-1)^2\left(y-\frac{\pi}{2}\right)\frac{\partial^3}{\partial x^2 \partial y} + 3(x-1)\left(y-\frac{\pi}{2}\right)^2\frac{\partial^3}{\partial x \partial y^2} + \left(y-\frac{\pi}{2}\right)^3\frac{\partial^3}{\partial y^3}\right]F(\xi,\eta)$$

Where $\xi = (1-\theta) + \theta x$; $\eta = (1-\theta)\frac{\pi}{2} + \theta y$;

$0 < \theta < 1$

(3.) $F(x,y) = e^y \sin x$; $F(\frac{\pi}{2},1) = e$

$F_x(x,y) = e^y \cos x$; $F_x(\pi/2,1) = 0$

$F_y(x,y) = e^y \sin x$; $F_y(\pi/2,1) = e$

$F_{xx}(x,y) = -e^y \sin x$; $F_{xx}(\pi/2,1) = -e$

$F_{xy}(x,y) = e^y \cos x$; $F_{xy}(\pi/2,1) = 0$

$F_{yy}(x,y) = e^y \sin x$; $F_{yy}(\pi/2,1) = e$

$$\therefore F(x,y) = 1 + x\cdot 2 + y\cdot 0 + \frac{1}{2}[x^2\cdot 4 + 2xy\cdot 1 + y^2\cdot 2]$$

$$F(x,y) = 1 + 2x + 2x^2 + xy + y^2$$

(2.) $F(x,y) = \sin(xy)$; $F(1, \pi/2) = 1$

$F_x(x,y) = y\cos(xy)$; $F_x(1, \pi/2) = 0$

$F_y(x,y) = x\cos(xy)$; $F_y(1, \pi/2) = 0$

$F_{xx}(x,y) = -y^2\sin(xy)$; $F_{xx}(1, \pi/2) = -\frac{\pi^2}{4}$

$F_{xy}(x,y) = \cos(xy) - xy\sin(xy)$;

$F_{xy}(1, \pi/2) = -\pi/2$

$F_{yy}(x,y) = -x^2\sin(xy)$; $F_{yy}(1, \pi/2) = -1$

$F_{xxx}(x,y) = -y^3\cos(xy)$; $F_{xxx}(1, \pi/2) = 0$

$F_{xxy}(x,y) = -2y\sin(xy) - xy^2\cos(xy)$;

$F_{xxy}(1, \pi/2) = -\pi$

$F_{xyy}(x,y) = -2x\sin(xy) - x^2y\cos(xy)$;

$F_{xyy}(1, \pi/2) = -2$

$F_{yyy}(x,y) = -x^3\cos(xy)$; $F_{yyy}(1, \pi/2) = 0$

Now using the Taylor's series expansion formula for $a=1$, $b=\pi/2$, we get

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(1.) $F(x,y) = e^{(2x+xy+y^2)}$

By Taylor's series expansion of $F(x,y)$ about (a,b) is given by

$$F(x,y) = F(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]F(a,b) + \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 F(a,b) + \dots$$

$$= F(a,b) + (x-a)F_x(a,b) + (y-b)F_y(a,b) + (x-a)^2 F_{xx}(a,b) + (x-a)(y-b)F_{xy}(a,b) + \frac{(y-b)^2}{2}F_{yy}(a,b) + \frac{(x-a)^3}{3!}F_{xxx}(a,b) + \dots$$

Here $F(x,y) = e^{(2x+xy+y^2)}$

and $(a,b) = (0,0)$

Now, $F_x(x,y) = (2+y)\,e^{(2x+xy+y^2)}$, $F_x(0,0) = 2$

$F_y(x,y) = (x+2y)\,e^{(2x+xy+y^2)}$, $F_y(0,0) = 0$

$F_{xx}(x,y) = (2+y)^2\,e^{(2x+xy+y^2)}$, $F_{xx}(0,0) = 4$

$F_{xy}(x,y) = e^{(2x+xy+y^2)} + (2+y)(x+2y)\,e^{(2x+xy+y^2)}$

$F_{xy}(0,0) = 1$

$F_{yy}(x,y) = (x+2y)^2\,e^{(2x+xy+y^2)} + 2e^{(2x+xy+y^2)}$

$F_{yy}(0,0) = 2$