Week 2: Forced oscillations, linear 2nd order ODEs (inhomogeneous)

While studying forced oscillations we are introduced to a class of second order, linear differential equations which are **inhomogeneous**. The equation we will deal with is given as

$$m\ddot{x} + r\dot{x} + kx = F(t) \tag{1}$$

The term $F(t)$ on the R. H. S. is known as the **inhomogeneous term** or, in the context of physics of oscillations, the forcing term. If we assume that the forcing is periodic (not necessarily sine/cosine) with some period, say T, then, by a result due to **J. Fourier**, we know that any periodic function can always be written in terms of an infinite series of sines and cosines. Such a series is called a **Fourier series**. We will not discuss the details of such a series now but use the above result to motivate ourselves in assuming a harmonic (sine/cosine) forcing function. Thus, for the simplest case, we take,

$$F(t) = F_0 \cos \Omega t \tag{2}$$

where the forcing frequency is Ω. With the usual redefinitions, we rewrite the differential equation as

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos \Omega t \tag{3}$$

There are several ways of solving such equations. One simple method is to write down its complex version

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\Omega t} \tag{4}$$

and assume a solution of the form

$$z = Ae^{i\Omega t} \tag{5}$$

with A as complex. Substituting in the equation we can get an expression for A from which the particular solution can be written down. In this way, we actually get the **particular integral** to which we must add the **complementary function** to get the full solution (the particular integral and the complimentary function are defined below). However, this method **assumes** a form of the solution which is like a **free oscillation**. How do we know that it will indeed be a free oscillation? To really **derive** it, we may employ a different method of solving such inhomogeneous differential equations known as the **method of variation of**

parameters. The advantage of this different method is that it is applicable to **any** forcing function– periodic or otherwise.

Let us first note the solutions of the homogeneous equation (i.e. when $F_0 = 0$). These are written as

$$x_1(t) = e^{-\beta t} \cos \omega t \quad ; \quad x_2(t) = e^{-\beta t} \sin \omega t \tag{6}$$

where $\omega^2 = \omega_0^2 - \beta^2$. Note that we are only considering the underdamped (damped oscillatory) case. The full solution of the inhomogeneous differential equation is written as

$$x(t) = Ax_1(t) + Bx_2(t) + x_p(t) \tag{7}$$

where $x_p(t)$ is called the **particular integral** which satisfies the full inhomogeneous equation. The first two terms in the above general expression for the solution represents the **complementary function**. A and B are the integration constants which are fixed by the initial conditions on x and $\dot{x}$. Our job therefore is to find the $x_p(t)$–the particular integral.

Let us assume the particular integral in the form

$$x_p(t) = u_1(t)x_1(t) + u_2(t)x_2(t) \tag{8}$$

where $u_1(t)$ and $u_2(t)$ are the two functions we need to find. Note that we have not assumed anything till now about the form of the solution and there is no loss of generality.

We shall now substitute this form of the solution back into the original differential equation. We have

$$\dot{x}_p = \dot{u}_1 x_1 + u_1 \dot{x}_1 + \dot{u}_2 x_2 + u_2 \dot{x}_2 \tag{9}$$

Let us now impose a condition as follows

$$\dot{u}_1 x_1 + \dot{u}_2 x_2 = 0 \tag{10}$$

Therefore, we have

$$\dot{x}_p = u_1 \dot{x}_1 + u_2 \dot{x}_2 \tag{11}$$

and

$$\ddot{x}_p = \dot{u}_1 \dot{x}_1 + u_1 \ddot{x}_1 + \dot{u}_2 \dot{x}_2 + u_2 \ddot{x}_2 \tag{12}$$

Let us further impose

$$\dot{u}_1 \dot{x}_1 + \dot{u}_2 \dot{x}_2 = f \tag{13}$$

where $f(t)$ is the inhomgeneity in the original differential equation. Putting the expressions for x_p, $\dot{x}_p$ and $\ddot{x}_p$ back into the original equation, it is easy to see that the equation is satisfied. However, we have the two conditions

$$\begin{aligned}\dot{u}_1 x_1 + \dot{u}_2 x_2 &= 0 \\ \dot{u}_1 \dot{x}_1 + \dot{u}_2 \dot{x}_2 &= f\end{aligned} \tag{14}$$

These are two algebraic equations in $\dot{u}_1$ and $\dot{u}_2$ which can be solved using **Cramer's rule**. The solution is simply written as

$$\dot{u}_1 = \frac{1}{W}\begin{vmatrix} 0 & x_2 \\ f & \dot{x}_2 \end{vmatrix}$$

$$\dot{u}_2 = \frac{1}{W}\begin{vmatrix} x_1 & 0 \\ \dot{x}_1 & f \end{vmatrix}$$

and W (known as the Wronskian) is given as,

$$W = \begin{vmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{vmatrix}$$

Therefore, if we know $f(t)$ and the solutions of the homogeneous equation, we can easily find the particular integral. It is easy to check that for the given x_1 and x_2, we can find the Wronskian as

$$W = \omega e^{-2\beta t} \tag{15}$$

The expressions for $\dot{u}_1$ and $\dot{u}_2$ are, therefore,

$$\dot{u}_1 = -\frac{f_0}{\omega_0} e^{\beta t} \sin \omega t \cos \Omega t \tag{16}$$

$$\dot{u}_2 = \frac{f_0}{\omega_0} e^{\beta t} \cos \omega t \cos \Omega t \tag{17}$$

We will now have to integrate the above two expressions to obtain u_1 and u_2. Then we can easily find $x_p(t)$.

In order to do the integrals, replace the trigonometric functions (sines and cosines) using known relations like $\cos \omega t = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$ etc. Then, on doing the integrals and after some straighforward calculations one gets,

$$\begin{aligned} u_1(t) = -\frac{f_0 e^{\beta t}}{2\omega} \Bigg[& \frac{\beta \sin(\Omega+\omega)t - (\Omega+\omega)\cos(\Omega+\omega)t}{\beta^2 + (\Omega+\omega)^2} \\ & - \frac{\beta \sin(\Omega-\omega)t - (\Omega-\omega)\cos(\Omega-\omega)t}{\beta^2 + (\Omega-\omega)^2} \Bigg] \end{aligned} \tag{18}$$

Similarly, the expression for $u_2(t)$ is given as

$$u_2(t) = \frac{f_0 e^{\beta t}}{2\omega}\left[\frac{\beta\cos(\Omega+\omega)t + (\Omega+\omega)\sin(\Omega+\omega)t}{\beta^2+(\Omega+\omega)^2}\right.$$
$$\left. + \frac{\beta\cos(\Omega-\omega)t + (\Omega-\omega)\sin(\Omega-\omega)t}{\beta^2+(\Omega-\omega)^2}\right] \qquad (19)$$

Therefore, since $x_p(t) = u_1(t)x_1(t) + u_2(t)x_2(t)$ we obtain, after some calculations,

$$x_p(t) = \frac{f_0\left[(\omega_0^2-\Omega^2)\cos\Omega t + 2\Omega\beta\sin\Omega t\right]}{(\omega_0^2-\Omega^2)^2 + 4\beta^2\Omega^2} \qquad (20)$$

In the above expressions for $u_1(t)$ and $u_2(t)$ we have assumed the integration constants to be zero. If we consider these constants as nonzero then the net effect in $x_p(t)$ would be the presence of a term like $(constant)x_1(t) + (constant)x_2(t)$ which is nothing but a solution of the homogeneous equation and this term just adds up to the complementary function with redefined constants.

Defining $\omega_0^2 - \Omega^2 = D\cos\phi$ and $2\Omega\beta = -D\sin\phi$ we finally obtain

$$x_p(t) = \frac{f_0}{\sqrt{(\omega_0^2-\Omega^2)^2 + 4\beta^2\Omega^2}}\cos(\Omega t + \phi) \qquad (21)$$

where

$$\tan\phi = \frac{-2\Omega\beta}{\omega_0^2-\Omega^2} \qquad (22)$$

As mentioned before, the above solution can also be obtained by using **the method of complex variables**. In fact, it is perhaps easier to do it that way. Recall the complex version of the forced oscillator differential equation. This is given as:

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\Omega t} \qquad (23)$$

The real part of the above equation is the actual forced oscillator equation. Using a trial solution of the form $z = Ae^{i\Omega t}$ (A is complex) in the equation we obtain

$$A = \frac{f_0}{\omega_0^2 - \Omega^2 + 2i\beta\Omega} = f_0\frac{(\omega_0^2-\Omega) - 2i\beta\Omega}{(\omega_0^2-\Omega^2)^2 + 4\beta^2\Omega^2} \qquad (24)$$

Rewriting $(\omega_0^2 - \Omega^2) - 2i\beta\Omega = A_R e^{i\phi}$ and extracting out the real part from the expression for z gives us the same $x_p(t)$ quoted above.

The full solution with initial conditions $x = x_0$ and $\dot{x} = v_0$ at $t = 0$ is therefore given as

$$x(t) = e^{-\beta t}(A\cos\omega t + B\sin\omega t) + \frac{f_0}{\sqrt{(\omega_0^2-\Omega^2)^2 + 4\beta^2\Omega^2}}\cos(\Omega t + \phi) \qquad (25)$$

with

$$A = x_0 - \frac{f_0 \cos\phi}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\beta^2\Omega^2}} \tag{26}$$

$$B = \frac{v_0 + \beta x_0}{\omega} + \frac{f_0}{\omega}\frac{\Omega \sin\phi - \beta\cos\phi}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\beta^2\Omega^2}} \tag{27}$$

Note that the first term (complimentary function) has an overall damping through $e^{-\beta t}$. The second term is a **free oscillation**. Thus after a certain time, the damping term contributes far less and the second term takes over. The forcing therefore has the effect of overcoming the effect of damping and thereby it makes the system oscillate freely at a frequency which is the same as the forcing frequency. The amplitude and phase of this free oscillation are:

$$A(\Omega) = \frac{f_0}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\beta^2\Omega^2}} \tag{28}$$

$$\phi(\Omega) = tan^{-1}\frac{-2\Omega\beta}{\omega_0^2 - \Omega^2} \tag{29}$$

It is easy to show that the amplitude has a maximum at $\Omega^2 = \Omega_{res}^2 = \omega_0^2 - 2\beta^2$.

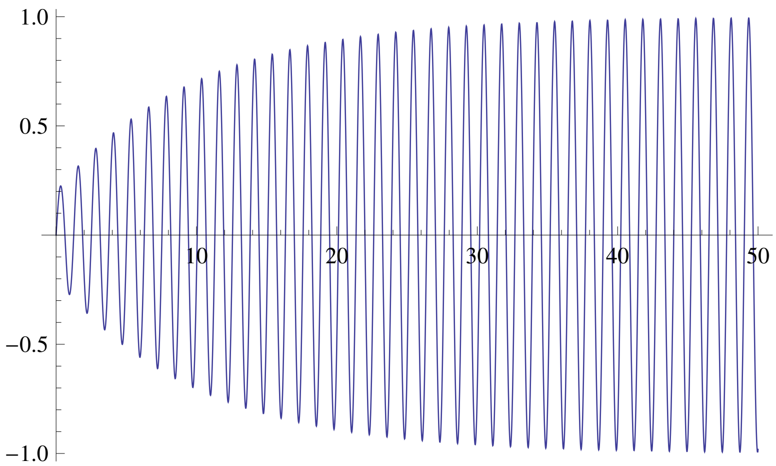

FIG. 1. $x(t)$ versus t when $\Omega = \Omega_{res}$

We will now illustrate the full solution using values of the various parameters. This will help us in understanding what is going on. Let us assume $\omega_0 = 5$, $\beta = 0.1$, $x_0 = 0$, $v_0 = 1$, $f_0 = 1$. In Fig. 1, we plot the solution $x(t)$ versus t for $\Omega = \Omega_{res}$ i.e. the value of Ω for which the amplitude of the particular integral solution is a maximum. The gradual increase of the amplitude with time and the eventual steady state behaviour is apparent in the solution.

Exercise: Plot the solution for values of Ω away from the resonance value $\Omega = \Omega_{res}$ using the same set of values of all the other parameters. Note the differences that arise. Also plot the amplitude and phase of the particular integral as a function of Ω for the same set of values of the other parameters.