(a)

1. Let $f : [a, b] \to \mathbb{R}$ be a function, where $a > 0$. Assume that f is continuous on $[a, b]$ and differentiable in (a, b). Using Cauchy's Mean Value Theorem (CMVT) proved that there exists a point $c \in (a, b)$ such that $\dfrac{af(b) - bf(a)}{a - b} = f(c) - cf'(c)$.

Sol: It is clear that $0 \notin [a, b]$.

Consider the functions $F(x) = \frac{f(x)}{x}$ and $G(x) = \frac{1}{x}$.

It is clear that F and G both are continuous in $[a, b]$ and both are differentiable in (a, b). [2M]

[if either F or G is incorrect, then no marks.]

Using CMVT there exists $c \in (a, b)$ such that

$\frac{F(b)-F(a)}{G(b)-G(a)} = \frac{F'(c)}{G'(c)}$ [1M]

[if either F or G is incorrect, then no marks.]

$\frac{\frac{f(b)}{b} - \frac{f(a)}{(a)}}{\frac{1}{b} - \frac{1}{a}} = \frac{\frac{cf'(c)-f(c)}{c^2}}{-\frac{1}{c^2}}$

$\frac{af(b)-bf(a)}{a-b} = -cf'(c) + f(c)$

$af(b) - bf(a) = (a - b)(f(c) - cf'(c))$ [1M]

Maths I - EndSem - Advanced Calculus

Questions and Answers

Question 1: Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{(x^2+y^2)} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(b) Test the differentiability of f at $(x,y) = (0,0)$. [3]

(c) Find the partial derivative $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$ at $(x,y) = (0,0)$. [3]

Solution: (a) Using the definitions of partial derivatives, we have

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0 \quad \text{and}$$
$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0.$$

For finding these two partial derivatives. [1]

The function f is differentiable at $(x,y) = (0,0)$ because

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - 0\cdot h - 0\cdot k}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{h^3 k}{(h^2+k^2)^{3/2}}.$$

For writing the correct formula for total derivative. [1]

The limit can be obtained by setting $h = r\cos\theta$ and $k = r\sin\theta$, and then taking $r \to 0$. Showing that the limit is zero. [1]

There are other methods also. For example, showing the existence of partial derivatives of f in a domain around $(0,0)$, and their continuity in that domain. This is correct method.

However, many students showed the existence of these partial derives at $(0,0)$ only. In this situation, they will get only 1 mark.

(b) For $(x,y) \neq (0,0)$, compute

$$f_y(x,y) = \frac{(x^2+y^2)x^3 - x^3 y(2y)}{(x^2+y^2)^2} = \frac{x^5 - x^3 y^2}{(x^2+y^2)^2}.$$

Thus $f_y(h,0) = h$ when $h \neq 0$. Next show that $f_y(0,0) = 0$ separately. Up to this point. [1]

Hence

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}. \qquad [1]$$
$$= \lim_{h\to 0} \frac{h-0}{h} = 1. \qquad [1]$$

Many students found $\dfrac{\partial^2 f}{\partial x \partial y}$ as a function of (x,y) ($\neq (0,0)$), and taking limit $(x,y) \to (0,0)$ to find $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$ at $(x,y) = (0,0)$. This is not a correct method.

Many students found $\dfrac{\partial^2 f}{\partial y \partial x}(0,0) = f_{xy}(0,0)$. We did not ask this.

Q.2(a) Solve $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 2y = e^{-x}(x^2 + \sin x)$

Ans Here A.E. is $m^3 - m^2 + 2 = 0$

$\Rightarrow m = -1, 1 \pm i$ —— (1m)

or C.F. $= C_1 e^{-x} + e^{x}(C_2 \sin x + C_3 \cos x)$

Now P.I. $= \frac{1}{f(D)} e^{-x}(x^2 + \sin x)$ where $f(D) = D^3 - D^2 + 2$

$= e^{-x} \frac{1}{f(D-1)} (x^2 + \sin x)$ —— (1m)

Now $f(D-1) = D^3 - 4D^2 + 5D$

$= e^{-x} \frac{1}{D^3 - 4D^2 + 5D} (x^2 + \sin x)$

Now $\frac{1}{D^3 - 4D^2 + 5D} x^2 = \frac{1}{5D} \frac{1}{\left(1 - \frac{4D - D^2}{5}\right)} x^2$

$= \frac{1}{5D}\left[1 + \frac{4D - D^2}{5} + \left(\frac{4D - D^2}{5}\right)^2 + \cdots\right] x^2$

$= \frac{1}{5}\left[\frac{x^3}{3} + \frac{4x^2}{5} + \frac{22x}{25}\right]$ —— (1m)

Now $\frac{1}{D^3 - 4D^2 + 5D} \sin x = \frac{1}{-D + 4 + 5D} \sin x$

$= \frac{1}{4(D+1)} \sin x$

$$= \frac{D-1}{4(D^2-1)} \sin x$$

$$= \frac{\cos x - \sin x}{-8}$$

$$= \frac{1}{8}(\sin x - \cos x) \longleftarrow (1\text{ M})$$

or $$P.I. = \frac{1}{5}\left(\frac{x^3}{3} + \frac{4x^2}{5} + \frac{22}{25}x\right) + \frac{1}{8}(\sin x - \cos x)$$

or general sol^n

$$y = C.F. + P.I.$$

$$= C_1 e^{-x} + e^{x}(C_2 \sin x + C_3 \cos x)$$

$$+ \frac{1}{5}\left(\frac{x^3}{3} + \frac{4x^2}{5} + \frac{22}{25}x\right) + \frac{1}{8}(\sin x - \cos x)$$

Q(2b) If x^3 and $\ln x$ are two L.I. solutions of $f(x)y'' + g(x)y' - h(x)y = 0$, then $f(1)+g(1)+h(1) = ?$

Ans: let us assume x^3, $\ln x$ and y are solutions of the second order ODE. Then we must have

$$\left.\begin{vmatrix} y & y' & y'' \\ x^3 & 3x^2 & 6x \\ \ln x & 1/x & -1/x^2 \end{vmatrix} = 0 \right\} \text{— (1 M)}$$

as the second order ODE has atmost two L.I. sol^n. So here x^3, $\ln x$ and y must be L.D. and thus their wronskian must be zero.

Now we have

$$\text{or } x^2(1-3\ln x)y'' + x(1+6\ln x)y' - 9y = 0$$

$$\left.\begin{aligned} \text{so that } f(x) &= x^2(1-3\ln x) \\ g(x) &= x(1+6\ln x) \\ h(x) &= 9 \end{aligned}\right\} \text{—(1 m)}$$

$$\text{or } f(1) + g(1) + h(1) = 11. \quad \underline{\text{Ans}} \text{ —(1)}.$$

Q.(2) Find the general solution of the ordinary differential equation $x^2y'' - xy' - 3y = 25x^4 \ln x$ by evaluating the particular solution using the method of variation of parameters.

Ans# Given ODE is

$$x^2y'' - xy' - 3y = 25x^4 \ln x$$

by choosing $x = e^z$ or $z = \ln x$, one can get

$$(D^2 - 2D - 3)y = 25e^{4z}z \quad \text{where } D \equiv \frac{d}{dz}$$

then A.E. is $m^2 - 2m - 3 = 0$

$$\left.\begin{aligned} (m-3)(m+1) &= 0 \\ \text{or C.F. is } y &= c_1e^{3z} + c_2e^{-z} \\ &= c_1x^3 + c_2x^{-1} \end{aligned}\right\} \text{— (1 m)}$$

Now e^{3z} and e^{-z} are L.I. sol^n of Given ODE

then $W(y_1, y_2) = \begin{vmatrix} e^{3z} & e^{-z} \\ 3e^{3z} & -e^{-z} \end{vmatrix}$

$= -e^{2z} - 3e^{2z} = -4e^{2z} \neq 0 \; \forall z \in \mathbb{R}$

then particular sol^n $y_p = y_1 u_1 + y_2 u_2$

where $y_1 = e^{3z}$, $y_2 = e^{-z}$

$$u_1 = -\int \frac{y_2 (25e^{4z} z)}{W(y_1, y_2)} dz$$

$$u_2 = +\int \frac{y_1 (25e^{4z} z)}{W(y_1, y_2)} dz$$

Now $u_1 = -\displaystyle\int \frac{25 z e^{3z}}{-4e^{2z}} dz$

$$= 25/4 \int z e^{z} dz$$

$$u_1 = \frac{25}{4}\left[ze^{z} - e^{z}\right]$$

and $u_2 = \displaystyle\int \frac{e^{3z} \times 25 e^{4z} z}{-4 e^{2z}} dz$

$$= -\frac{25}{4}\left[\frac{ze^{5z}}{5} - \frac{e^{5z}}{25}\right]$$

$$u_2 = \frac{1}{4}\left[e^{5z} - 5ze^{5z}\right]$$

— (m)

so $y_p = \frac{25}{4} e^{4z}[z-1] + \frac{1}{4} e^{4z}[1-5z]$

and the general sol^n is

$$y = y_c + y_p$$
$$= c_1 e^{3z} + c_2 e^{-z} + \frac{25}{4} e^{4z}(z-1) + \frac{1}{4} e^{4z}\{1-5z)$$

$$y = c_1 x^3 + c_2 x^{-1} + \frac{x^4}{4}\left[25(\ln x - 1) + 1 - 5\ln x\right]$$

$$= c_1 x^3 + c_2 x^{-1} + \frac{x^4}{\cancel{4}}\left[\overset{5}{\cancel{20}} \ln x - \overset{6}{\cancel{24}}\right]$$

$$\boxed{y = c_1 x^3 + c_2 x^{-1} + x^4 (5\ln x - 6)}$$ — (1m)

Alternate way :-

Set $y = x^m$ then the corresponding A.E. is

$$m(m-1) - m - 3 = 0$$
$$m^2 - 2m - 3 = 0$$
$$\Rightarrow m = 3, -1$$

or C.F. $y_c = c_1 x^3 + c_2 x^{-1}$

Now $W(y_1, y_2) = \begin{vmatrix} x^3 & x^{-1} \\ 3x^2 & -1/x^2 \end{vmatrix}$

$$= -4x$$

— (1m)

Now P.I. $y_p = y_1 u_1 + y_2 u_2$

$$u_1 = \int \frac{-y_2 \ \text{R.H.S. (with leading term 1)}}{W}$$

$$= \int \frac{-\cancel{x^4}\,(25\cancel{x^2}\ln x)}{-4\cancel{x}}\,dx$$

$$u_1 = \frac{25}{4}x\left[\ln x - 1\right]$$

$$u_2 = \int \frac{y_1 \text{R.H.S. (with leading term 1)}}{W}$$

$$= \int \frac{x^3 \ 25x^2 \ln x}{-4x}$$

$\ln x \quad + \quad x^4$

$1/x \quad \quad \frac{x^5}{5}$

$$= \frac{-25}{4}\int x^4 \ln x$$

$\frac{x^5}{5}\ln x - \frac{1}{5}\frac{x^5}{5}$

$$\left.\begin{aligned}
&= -\frac{25}{4}\left[\frac{x^5}{5}\ln x - \frac{x^5}{25}\right] \\
&= -\frac{\cancel{25}\,x^5}{4\times\cancel{25}}\left[5\ln x - 1\right] \\
u_2 &= \frac{x^5}{4}\left[1 - 5\ln x\right]
\end{aligned}\right\} \text{— (1m)}$$

so $$y_p = \frac{x^4\,25}{4}(\ln x - 1) + \frac{x^4}{4}(1 - 5\ln x)$$

$$= \frac{x^4}{\cancel{4}}\left[\overset{5}{\cancel{20}}\ln x - \overset{6}{\cancel{24}}\right]$$

$$y_p = x^4 (5\ln x - 6)$$

so General Sol^n is

$$y = y_c + y_p$$

$$y = C_1 x^3 + C_2 x^{-1} + x^4 (5\ln x - 6)$$ Ans

(1m)

Mousumi Mandal

Q3 Evaluate the integrals using Beta or gamma fn.

(a) $\int_0^1 (x \log x)^4 \, dx$

(b) $\int_0^{1/2} x^3 (1-4x^2)^{1/2} \, dx$.

Ans (a) $\int_0^1 (x \log x)^4 \, dx$.

Put $\log x = -t$, $x = e^{-t}$, $dx = -e^{-t} dt$

$$= \int_\infty^0 e^{-4t} t^4 (-e^{-t}) \, dt \quad \text{—— [1]}$$

$$= \int_0^\infty t^4 e^{-5t} \, dt$$

Put $5t = z$, $5\,dt = dz$

$$= \frac{1}{5^4} \int_0^\infty z^4 e^{-z} \cdot \frac{1}{5} \, dz$$

$$= \frac{1}{5^5} \Gamma(5) = \frac{1}{5^5} \cdot 4! = \frac{24}{5^5} \quad \text{Ans.}$$

—— [1]

(b) $\int_0^{1/2} x^3 (1-4x^2)^{1/2} dx$

Put $4x^2 = t$

$x = \frac{\sqrt{t}}{2}$

$dx = \frac{1}{4}\frac{1}{\sqrt{t}} dt$

$$= \int_0^1 \left(\frac{\sqrt{t}}{2}\right)^3 (1-t)^{1/2} \cdot \frac{1}{4}\frac{1}{\sqrt{t}}\, dt$$

$$= \frac{1}{32} \int_0^1 t\,(1-t)^{1/2}\, dt \quad \text{—— [1]}$$

$$= \frac{1}{32} B\left(2, 3/2\right) \quad \text{—— [1]}$$

$$= \frac{1}{32} \cdot \frac{\Gamma(2)\,\Gamma(3/2)}{\Gamma(7/2)}$$

$$= \frac{1}{32} \cdot \frac{1!\ \Gamma(3/2)}{5/2 \cdot 3/2 \cdot \Gamma(3/2)}$$

$$= \frac{4}{32 \times 15}$$

$$= \frac{1}{120} \quad \text{Ans.} \quad \text{—— [1].}$$

END SEM

ADVANCE CALCULUS: IMPROPER INTEGRALS

Q (c)

Q. (4 marks) Test for convergence of the integral

$$\int_0^1 \frac{\sin^3(1/x)\cos(1/x)}{x^{3/2}+x^2\ln(1+1/x)}\mathrm{d}x.$$

Ans. Put $z = 1/x$ in our given integral $\int_0^1 \frac{\sin^3(1/x)\cos(1/x)}{x^{3/2}+x^2\ln(1+1/x)}\mathrm{d}x$, then $dz = \frac{-1}{x^2}$, and given integral becomes

$$\begin{aligned} & -\int_\infty^1 \frac{\sin^3(z)\cos(z)}{\sqrt{z}+\ln(1+z)}\mathrm{d}z \\ &= \int_1^\infty \frac{\sin^3(z)\cos(z)}{\sqrt{z}+\ln(1+z)}\mathrm{d}z \end{aligned}$$

The function $\frac{1}{\sqrt{z}+\ln(1+z)}$ is a monotonic decreasing function and

$$\lim_{z\to\infty} \frac{1}{\sqrt{z}+\ln(1+z)} = 0.$$

Also

$$\begin{aligned} \left|\int_1^b \sin^3(z)\cos(z)dz\right| &= \left|\int_1^b \sin^3(z)d\sin(z)\right| \\ &\le \left|\frac{sin^4(b)}{4} - \frac{sin^4(1)}{4}\right| \le \frac{1}{2} \end{aligned}$$

for all $b > 1$. Hence we can apply Dirichlet's test for improper integral to conclude the given integral is convergent.

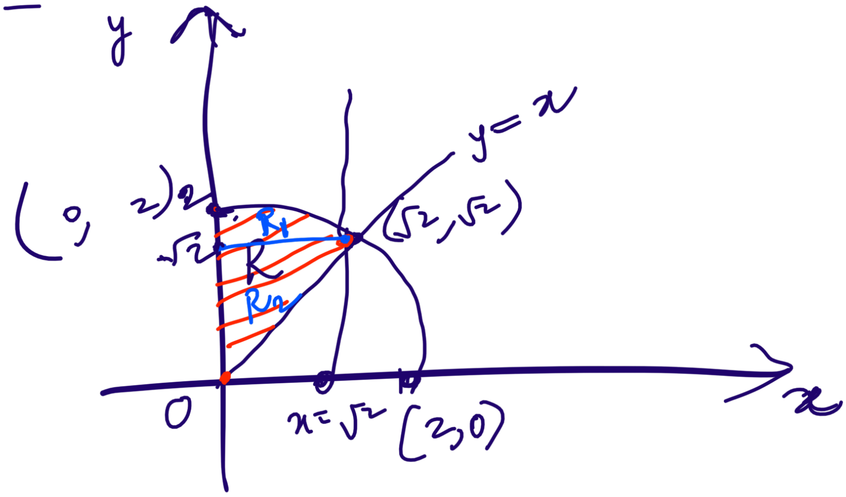
2 marks for substitution and 2 marks for Dirichlet test.

Date: February 14, 2023.

R.N

4 (R) Evaluate $\int_{x=0}^{\sqrt{2}} \int_{y=x}^{\sqrt{4-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$ by changing the order of integration. (4 m).

Sol:-



$$\int_{x=0}^{\sqrt{2}} \int_{y=x}^{\sqrt{4-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx = \iint_{R_1} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy + \iint_{R_2} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$$

$$= \int_{y=\sqrt{2}}^{2} \int_{x=0}^{\sqrt{4-y^2}} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy + \int_{y=0}^{\sqrt{2}} \int_{x=0}^{y} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy.$$

(1+1) marks

$$= \int_{y=\sqrt{2}}^{2} \left(\sqrt{x^2+y^2} \Big|_{x=0}^{\sqrt{4-y^2}} \right) dy$$

$$+ \int_{y=0}^{\sqrt{2}} \left(\sqrt{x^2+y^2} \right) \Big|_{x=0}^{y} dy$$

$$= \int_{y=\sqrt{2}}^{2} (2-y)\, dy + \int_{y=0}^{\sqrt{2}} (\sqrt{2}y - y)\, dy$$

$$= \left(2y - \frac{y^2}{2}\right) \Big|_{\sqrt{2}}^{2} + (\sqrt{2}-1) \frac{y^2}{2} \Big|_{0}^{\sqrt{2}}$$

$$= (3-2\sqrt{2}) + (\sqrt{2}-1)$$ (1 + 1 marks)

$$= 2-\sqrt{2}$$

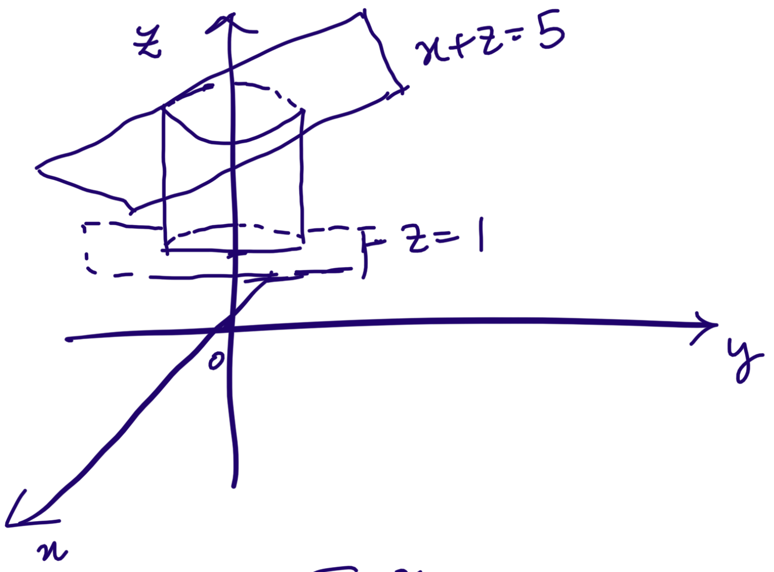
Note:-

① No marks are awarded if both the limits for the regions R_1 & R_2 are wrong.

(4) (b) Find the volume of the solid bounded by the cylinder $x^2+y^2=9$ & the planes $z=1$ & $x+z=5$.

(3 m)

Sol:-



$$\text{volume} = \int_{x=-3}^{3} \int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{z=1}^{5-x} dz\, dy\, dx. \quad (1\ m)$$

$$= \int_{x=-3}^{3} \int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4-x)\, dy\, dx$$

$$= \int_{x=-3}^{3} 2(4-x)\sqrt{9-x^2}\, dx$$

$$= \underline{36\pi} \longrightarrow (2\ m)$$

Note:- (1) No marks awarded, if the limits of the integral are wrong.

Q4 (c)

Solve the following ordinary differential equation:

$$(x^2+1)\frac{dy}{dx} + 3xy = 6x \;;\; y(0) = -1.$$

3-Marks

$$\Rightarrow \frac{dy}{dx} + \left(\frac{3x}{x^2+1}\right)y = \left(\frac{6x}{x^2+1}\right).$$

Sol

$$I.F = e^{\int P\,dx} = e^{\frac{3}{2}\int \frac{2x}{x^2+1}dx}$$

$$= e^{3/2 \log(x^2+1)} = (x^2+1)^{3/2}.$$

$$\therefore\; y \times (x^2+1)^{3/2} = \int \frac{6x}{(x^2+1)}(x^2+1)^{3/2}\,dx + c$$

$$= \int 6x\,(x^2+1)^{1/2}\,dx + c$$

Put $(x^2+1) = t \quad \therefore \; 2x\,dx = dt$

Then, $y \times t^{3/2} = 3\int t^{1/2}\,dt + c$

$$= 2t^{3/2} + c$$

$$\therefore\; y = 2 + c\,(x^2+1)^{-3/2} \longrightarrow \text{2 Marks}$$

~~when y = 0;~~ when $x = 0$, $y = -1$.

$$\therefore\; -1 = 2 + c \Rightarrow c = -3$$

$$\therefore\; y(x) = 2 - 3(x^2+1)^{-3/2} \rightarrow \text{1 Mark}$$

5 a) Calculate the curl of the vector field

$$\vec{F} = xe^{y}\,\hat{i} + ye^{z}\,\hat{j} + ze^{x}\,\hat{k}$$ 2 Mark

Soln:

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xe^{y} & ye^{z} & ze^{x} \end{vmatrix}$$ [1 Mark]

$$= \hat{i}\left(\frac{\partial}{\partial y}(ze^{x}) - \frac{\partial}{\partial z}(ye^{z})\right) + \hat{j}\left(\frac{\partial}{\partial z}(xe^{y}) - \frac{\partial}{\partial x}(ze^{x})\right) + \hat{k}\left(\frac{\partial}{\partial x}(ye^{z}) - \frac{\partial}{\partial y}(xe^{y})\right)$$

$$= -ye^{z}\,\hat{i} - ze^{x}\,\hat{j} - \cancel{xy}\,xe^{y}\,\hat{k}.$$ [1 Mark]

5 (B) Find the directional derivative of the function

$f(x, y, z) = 6xy + z^2$ at the point

$(1, -2, 2)$ in the direction from that point

toward the origin.

3 Mark.

Sol^n: Vector from that point toward the origin

$\vec{v} = (-1, 2, -2)$.

$\therefore$ unit vector $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$

Directional derivative in the direction $\vec{u}$ is

$D_u \left(f(1, -2, 2)\right)$

$= \vec{\nabla} f(1, -2, 2) \cdot \vec{u}$ ← 1 Mark

Now

$\nabla f(x,y,z) = (6y, 6x, 2z)$

$\therefore \nabla f(1,-2,2) = (-12, 6, 4)$ ← 1 Mark

$\therefore D_{\vec{u}} f(1, -2, 2)$

$= (-12, 6, 4) \cdot \left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$

$= 4 + 4 - 8/3 = \frac{24 - 8}{3} = \frac{16}{3}$. ← 1 Mark

2

5c) Evaluate $\int_0^{\frac{\pi}{2}-\alpha} \cos\theta \cdot \sin^{-1}(\sin\alpha \sec\theta)\, d\theta$.

Solution :- Let; $g(\alpha) = \int_0^{\frac{\pi}{2}-\alpha} \cos\theta \cdot \sin^{-1}(\sin\alpha \sec\theta)\, d\theta$.

Differentiating, we get,

$$g'(\alpha) = \int_0^{\pi/2-\alpha} \cos\theta \cdot \frac{\sec\theta \cdot \cos\alpha}{\sqrt{1-\sin^2\alpha \sec^2\theta}}\, d\theta$$

$$+ \frac{d}{d\alpha}\left(\frac{\pi}{2}-\alpha\right) \cos\left(\frac{\pi}{2}-\alpha\right) \sin^{-1}\left(\sin\alpha \cdot \sec\left(\frac{\pi}{2}-\alpha\right)\right)$$

$$= \int_0^{\frac{\pi}{2}-\alpha} \frac{\cos\alpha\, d\theta}{\sqrt{1-\sin^2\alpha \sec^2\theta}} - \sin\alpha \sin^{-1}(\sin\alpha \operatorname{cosec}\alpha)$$

$$= \int_0^{\frac{\pi}{2}-\alpha} \frac{\cos\alpha\, d\theta}{\sqrt{1-\sin^2\alpha \sec^2\theta}} - \sin\alpha \sin^{-1}(1)$$

$$= \int_0^{\frac{\pi}{2}-\alpha} \frac{\cos\alpha\, d\theta}{\sqrt{1-\sin^2\alpha \sec^2\theta}} - \frac{\pi}{2}\sin\alpha \quad (1)$$

$$= \int_0^{\frac{\pi}{2}-\alpha} \frac{\cos\alpha \cos\theta}{\sqrt{\cos^2\theta - \sin^2\alpha}}\, d\theta - \frac{\pi}{2}\sin\alpha$$

$$= \int_0^{\frac{\pi}{2}-\alpha} \cos\alpha \frac{\cos\theta\, d\theta}{\sqrt{\cos^2\alpha - \sin^2\theta}} - \frac{\pi}{2}\sin\alpha$$

$$= \int^{\cos\alpha} \cos\alpha \frac{dt}{} - \frac{\pi}{2}\sin\alpha.$$

Let, $t = \sin\theta$

$\therefore dt = \cos\theta\, d\theta$

θ	0	$\pi/2-\alpha$

2

$$= \cos\alpha \cdot \left[\sin^{-1}\left(\frac{t}{\cos\alpha}\right)\right]_0^{\cos\alpha} - \frac{\pi}{2}\sin\alpha$$

$$= \cos\alpha \left[\sin^{-1}(1) - \sin^{-1}(0) \right] - \frac{\pi}{2}\sin\alpha$$

$$= \cos\alpha \ \frac{\pi}{2} - \frac{\pi}{2}\sin\alpha$$

$$= \frac{\pi}{2}\cos\alpha - \frac{\pi}{2}\sin\alpha$$

Now, Integrating, we have

$$g(\alpha) = \frac{\pi}{2}\sin\alpha + \frac{\pi}{2}\cos\alpha + C \qquad [C = \text{Int. Const}]$$

2

Again,

$$g(0) = \frac{\pi}{2} + C = \int_0^{\frac{\pi}{2}} \cos\theta \sin^{-1}(0)\, d\theta = 0$$

$$\therefore C = -\pi/2$$

$$\therefore g(\alpha) = \frac{\pi}{2}\sin\alpha + \frac{\pi}{2}\cos\alpha - \pi/2$$

$$= \frac{\pi}{2}\left(\sin\alpha + \cos\alpha - 1\right)$$

1