Solution for Problem set - 4

Maths - 1 , Autumn 2018

1. $$f(x,y) = \begin{cases} \dfrac{x^2+y^2}{|x|+|y|} , & (x,y) \neq (0,0) \\ 0 , & (x,y) = (0,0) \end{cases}$$

Let $x = r\cos\theta$, $y = r\sin\theta$

Then $(x,y) \to (0,0) \Rightarrow r \to 0$

$$\therefore \; |f(x,y) - 0| = \left| \frac{x^2+y^2}{|x|+|y|} \right| = \left| \frac{r^2}{|r\cos\theta| + |r\sin\theta|} \right|$$

$$= \left| \frac{r^2}{r(|\cos\theta| + |\sin\theta|)} \right| = \frac{r}{|\cos\theta| + |\sin\theta|}$$

$\to 0$ as $r \to 0$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

Hence $f(x,y)$ is continuous at $(0,0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h^2}{|h|}}{h} = \lim_{h\to 0} \frac{h}{|h|}$$

as $h \to 0^+$, $\displaystyle\lim_{h\to 0^+} \frac{h}{|h|} = \lim_{h\to 0^+} \frac{h}{h} = 1$

as $h \to 0^-$, $\displaystyle\lim_{h\to 0^-} \frac{h}{|h|} = \lim_{h\to 0^-} \frac{h}{-h} = -1$

Hence $f_x(0,0)$ does not exist.

Similarly, $f_y(0,0) = \lim_{k \to 0} \dfrac{f(0,k) - f(0,0)}{k}$

$$= \lim_{k \to 0} \frac{\frac{k^2}{|k|}}{k} = \lim_{k \to 0} \frac{k}{|k|}$$

which does not exist.

Hence $f_y(0,0)$ does not exist.

2. (a) $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

~~$\lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^4 + y^2}$~~

consider the limit along the path $x^2 = my$

Then $\lim_{(x,y) \to (0,0)} \dfrac{x^2 y}{x^4 + y^2}$

$$= \lim_{y \to 0} \frac{my \cdot y}{(my)^2 + y^2} = \lim_{y \to 0} \frac{my^2}{y^2(m^2+1)}$$

$$= \frac{m}{m^2+1}$$

which depends on m.

ie we get different limiting values for different values of m.

So $\lim\limits_{(x,y)\to(0,0)} f(x,y) \neq 0 = f(0,0)$

$\therefore$ $f(x,y)$ is not continuous at $(0,0)$

Now $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h}$

$= \lim\limits_{h\to 0} \dfrac{0}{h} = 0$

$f_y(0,0) = \lim\limits_{k\to 0} \dfrac{f(0,k) - f(0,0)}{k} = \lim\limits_{k\to 0} \dfrac{0}{k} = 0$

$\therefore$ Both $f_x(0,0)$ and $f_y(0,0)$ exist.

(b) $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y} & , \text{ if } x \neq y \\ 0 & , \text{ if } x = y \end{cases}$

consider the path $y = x - mx^3$

$\therefore \lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x-y} = \lim\limits_{x\to 0} \dfrac{x^3 + (x \cancel{m} - mx^3)^3}{x - x + mx^3}$

$= \lim\limits_{x\to 0} \dfrac{x^3 \left[1 + (1 - mx^2)^3 \right]}{mx^3}$

$= \lim\limits_{x\to 0} \dfrac{1 + (1-mx^2)^3}{m} = \dfrac{2}{m}$

which depends on m.

So the limit is different for different values of m

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y) \neq 0 = f(0,0)$

So $f(x,y)$ is not continuous at $(0,0)$

Now $$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{h^3/h}{h} = \lim_{h \to 0} h = 0$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{-k^3/k}{k} = -\lim_{k \to 0} k = 0$$

Therefore $f_x(0,0)$ and $f_y(0,0)$ exist.

3. (a) $f(x,y) = x^2 + y^2$

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + y^2 - (x^2+y^2)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 2xh}{h} = \lim_{h \to 0} (h + 2x) = 2x$$

$$f_y(x,y) = \lim_{k \to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

$$= \lim_{k \to 0} \frac{x^2 + (y+k)^2 - (x^2+y^2)}{k}$$

$$= \lim_{k \to 0} \frac{k^2 + 2ky}{k}$$

$$= \lim_{k \to 0} (k + 2y) = 2y$$

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(b) $f(x,y) = \sin(3x+4y)$ ⑤

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(3(x+h)+4y) - \sin(3x+4y)}{h}$$

$$= \lim_{h\to 0} \frac{2\cdot\cos\left(3x+4y+\frac{3h}{2}\right)\sin\frac{3h}{2}}{h} \quad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{h\to 0} \frac{-2\cdot\frac{3}{2}\sin\left(3x+4y+\frac{3h}{2}\right)\sin 3h/2 + 2\cdot 3/2\cos\left(3x+4y+\frac{3h}{2}\right)\cos\frac{3h}{2}}{1}$$

$$= 3\cos(3x+4y)$$

$$f_y(x,y) = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

$$= \lim_{k\to 0} \frac{\sin(3x+4(y+k)) - \sin(3x+4y)}{k}$$

$$= \lim_{k\to 0} \frac{2\cos(3x+4y+2k)\sin 2k}{k} \quad \left[\text{form } \frac{0}{0}\right.$$

$$= \lim_{k\to 0} \frac{-2\cdot 2\sin(3x+4y+2k)\sin 2k + 2\cdot 2\cos(3x+4y+2k)\cos 2k}{1}$$

$$= 4\cos(3x+4y)$$

(c) $f(x,y) = ye^{-x} + xy$

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{ye^{-(x+h)} + (x+h)y - ye^{-x} - xy}{h}$$

$$= \lim_{h\to 0} \frac{ye^{-x}(e^{-h}-1)}{h} + \lim_{h\to 0} \frac{hy}{h}$$

$$= ye^{-x} \lim_{h\to 0} (-e^{-h}) + y$$

$$= -ye^{-x} + y$$

$$f_y(x,y) = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

$$= \lim_{k\to 0} \frac{(y+k)e^{-x} + x(y+k) - ye^{-x} - xy}{k}$$

$$= \lim_{k\to 0} \frac{\cancel{k}\ ke^{-x} + kx}{k} = e^{-x} + x$$

(d) $f(x,y) = x^2 + xy + y^3$

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 + (x+h)y + y^3 - x^2 - xy - y^3}{h}$$

$$= \lim_{h\to 0} \frac{(2xh + hy + h^2)}{h} = \lim_{h\to 0} (2x+y+h)$$

$$= 2x + y$$

$$f_y(x,y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x,y)}{k}$$

$$= \lim_{k \to 0} \frac{x^2 + \cancel{6x}\, x(y+k) + (y+k)^3 - x^2 - xy - y^3}{k}$$

$$= \lim_{k \to 0} \frac{kx + 3y^2 k + 3y k^2 + k^3}{k}$$

$$= \lim_{k \to 0} \left(x + 3y^2 + 3yk + k^2 \right)$$

$$= x + 3y^2$$

(e) $f(x,y) = \cancel{8}\; x \sin y + x^2$

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)\sin y + (x+h)^2 - x\sin y - x^2}{h}$$

$$= \lim_{h \to 0} \frac{h \sin y + 2xh + h^2}{h}$$

$$= \lim_{h \to 0} \sin y + 2x + h\cancel{^2} = \sin y + 2x$$

$$f_y(x,y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x,y)}{k}$$

$$= \lim_{k \to 0} \frac{x \sin(y+k) + x^2 - x\sin y - x^2}{k}$$

$$= \lim_{k \to 0} \frac{x\left[\sin(y+k) - \sin y \right]}{k} \quad \left[\text{form } \tfrac{0}{0} \right]$$

$$= \lim_{k \to 0} x \cancel{\frac{}{}} \cos$$

$$= \lim_{k \to 0} x \cdot \cos(y+k) = x\cos y$$

(f) $f(x,y) = e^{xy} + x/y$

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

$$= \lim_{h \to 0} \frac{e^{(x+h)y} + \frac{x+h}{y} - e^{xy} - x/y}{h}$$

$$= \lim_{h \to 0} \frac{e^{xy}(e^{hy} - 1)}{h} + \lim_{h \to 0} \frac{h/y}{h}$$

$$= \lim_{h \to 0} (e^{xy} \cdot y e^{hy}) + 1/y$$

$$= y e^{xy} + 1/y$$

$$f_y(x,y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x,y)}{k}$$

$$= \lim_{k \to 0} \frac{e^{x(y+k)} + \frac{x}{y+k} - e^{xy} - \frac{x}{y}}{k.}$$

$$= \lim_{k \to 0} \frac{e^{xy}(e^{kx} - 1)}{k} + \lim_{k \to 0} \frac{\left(\frac{x}{y+k} - \frac{x}{y}\right)}{k}$$

$$= \lim_{k \to 0} e^{xy} \cdot x e^{kx} + \lim_{k \to 0} \frac{(-kx)}{yk(y+k)}$$

$$= x e^{xy} - \frac{x}{y^2}$$

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4. (a)

$$f(x,y) = \begin{cases} \dfrac{xy}{x+y} &, (x,y) \neq (0,0) \\ 0 &, (x,y) = (0,0) \end{cases}$$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \to 0} \frac{0-0}{k} = 0$$

$$f_x(0,y) = \lim_{h \to 0} \frac{f(h,y) - f(0,y)}{h} = \lim_{h \to 0} \frac{\frac{hy}{h+y} - 0}{h}$$

$$= \lim_{h \to 0} \frac{y}{h+y} = 1$$

$$f_y(x,0) = \lim_{k \to 0} \frac{f(x,k) - f(x,0)}{k} = \lim_{k \to 0} \frac{\frac{xk}{x+k} - 0}{k}$$

$$= \lim_{k \to 0} \frac{x}{x+k} = 1$$

(b) $f(x,y) = \log(1+xy)$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\log(1+h\cdot 0) - \log 1}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{\log(1+0\cdot k) - \log 1}{k}$$

$$= 0$$

$$f_x(0,y) = \lim_{h\to 0} \frac{f(h,y) - f(0,y)}{h}$$

$$= \lim_{h\to 0} \frac{\log(1+hy) - \log 1}{h} \quad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{h\to 0} \frac{y/_{1+hy}}{1} = y$$

$$f_y(x,0) = \lim_{k\to 0} \frac{f(x,k) - f(x,0)}{k}$$

$$= \lim_{k\to 0} \frac{\log(1+xk) - \log 1}{k} \quad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{k\to 0} \frac{x}{1+xk} = x$$

(c) $f(x,y) = \begin{cases} 1 & \text{if } x=0, \text{ or } y=0 \text{ or } (x,y)=(0,0) \\ 0 & \text{otherwise} \end{cases}$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{1-1}{k} = 0$$

$$f_x(0,y) = \lim_{h\to 0} \frac{f(h,y) - f(0,y)}{h} = \lim_{h\to 0} \frac{0-1}{h}$$

— does not exist

$$f_y(x,0) = \lim_{k\to 0} \frac{f(x,k) - f(x,0)}{k} = \lim_{k\to 0} \frac{0-1}{k}$$

does not exist

(d) $f(x,y) = e^{x-y} - e^{y-x}$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{e^{h} - e^{-h}}{h} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h\to 0} (e^{h} + e^{-h}) = 2$$

$$f_y(0,0) = \lim_{K\to 0} \frac{f(0,K) - f(0,0)}{K} = \lim_{K\to 0} \frac{e^{-K} - e^{K}}{K} \quad \left[\text{form } \frac{0}{0}\right.$$

$$= \lim_{K\to 0} (-e^{-K} - e^{K}) = -2$$

$$f_x(0,y) = \lim_{h\to 0} \frac{f(h,y) - f(0,y)}{h}$$

$$= \lim_{h\to 0} \frac{e^{h-y} - e^{y-h} - e^{-y} + e^{y}}{h}$$

$$= \lim_{h\to 0} \frac{e^{-y}(e^{h}-1)}{h} - \lim_{h\to 0} \frac{e^{y}(e^{-h}-1)}{h}$$

$$= e^{-y} \lim_{h\to 0} e^{h} - e^{y} \lim_{h\to 0} (-e^{-h})$$

$$= e^{-y} + e^{y}$$

$$f_y(x,0) = \lim_{K\to 0} \frac{f(x,K) - f(x,0)}{K}$$

$$= \lim_{K\to 0} \frac{e^{x-K} - e^{K-x} - e^{x} + e^{-x}}{K}$$

$$= \lim_{K\to 0} \frac{e^{x}(e^{-K}-1)}{K} - \lim_{K\to 0} \frac{e^{-x}(e^{K}-1)}{K}$$

$$= \lim_{k \to 0} e^{x}(-e^{-k}) - \lim_{k \to 0} e^{-x} e^{k}$$

$$= -e^{x} - e^{-x}$$

(e) $$f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x+y}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{h^3/h - 0}{h} = \lim_{h \to 0} h = 0$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{-k^3/k - 0}{k} = -\lim_{k \to 0} k = 0$$

$$f_x(0,y) = \lim_{h \to 0} \frac{f(h,y) - f(0,y)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h^3 - y^3}{h+y} - \frac{-y^3}{y}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h^3 - y^3}{h+y} + y^2}{h}$$

$$= \lim_{h \to 0} \frac{h^3 - \cancel{y^3} + y^2 h + \cancel{y^3}}{h(h+y)}$$

$$= \lim_{h \to 0} \frac{h^2 + y^2}{h+y} = \frac{y^2}{y} = y$$

$$f_y(x,0) = \lim_{k \to 0} \frac{f(x,k) - f(x,0)}{k}$$

$$= \lim_{K\to 0} \frac{\frac{x^3-K^3}{x+K} - \frac{x^3}{x}}{K} = \lim_{K\to 0} \frac{x^3-K^3-x^3-x^2K}{K(x+K)}$$

$$= \lim_{K\to 0} \frac{-(K^2+x^2)}{(x+K)} = -\frac{x^2}{x} = -x$$

5.(a) $$f(x,y) = \begin{cases} xy\,\frac{x^2-y^2}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{K\to 0} \frac{f(0,K)-f(0,0)}{K} = \lim_{K\to 0} \frac{0-0}{K} = 0$$

Now $$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{f(\Delta x,\Delta y) - f(0,0) - f_x(0,0)\Delta x - f_y(0,0)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \Delta x\,\Delta y \frac{(\Delta x)^2-(\Delta y)^2}{\{(\Delta x)^2+(\Delta y)^2\}^{3/2}}$$

$$= \lim_{r\to 0} r^2\sin\theta\cos\theta\,\frac{r^2(\cos^2\theta-\sin^2\theta)}{(r^2)^{3/2}}$$ [Putting $x = r\cos\theta$, $y = r\sin\theta$

$$= \lim_{r\to 0} r\sin\theta\cos\theta(\cos^2\theta-\sin^2\theta)$$

$$= 0$$

$\therefore f(x,y)$ is differentiable at $(0,0)$

(b) $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & , \quad x^2+y^2 \neq 0 \\ 0 & , \quad x^2+y^2 = 0 \end{cases}$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

Now $\displaystyle\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{f(\Delta x,\Delta y) - f(0,0) - \Delta x\, f_x(0,0) - \Delta y\, f_y(0,0)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{(\Delta x)^2 \Delta y}{\{(\Delta x)^2 + (\Delta y)^2\}^{3/2}}$$

$$= \lim_{r\to 0} \frac{r^2\cos^2\theta \cdot r\sin\theta}{(r^2)^{3/2}}$$ [Putting $x = r\cos\theta$
$y = r\sin\theta$

$$= \lim_{r\to 0} \cos^2\theta \sin\theta \quad , \text{(depends on } \theta)$$

$\neq 0$

$\therefore$ $f(x,y)$ is not differentiable at $(0,0)$

6. $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$

Let $x = r\cos\theta$, $y = r\sin\theta$

Then $(x,y) \to (0,0) \Rightarrow r \to 0$

$\therefore \left| f(x,y) - 0 \right| = \left| \frac{xy}{\sqrt{x^2+y^2}} \right| = \left| \frac{r^2 \cos\theta \sin\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}} \right|$

$= \left| \frac{r^2 \cos\theta \sin\theta}{r} \right| = r \left| \cos\theta \sin\theta \right|$

$\to 0$ as $r \to 0$

$\therefore$ $f(x,y)$ is continuous at ~~(0,0)~~ origin.

Now $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$

$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$

So, partial derivatives exist at origin.

Now $\lim_{(\Delta x, \Delta y)\to(0,0)} \frac{f(\Delta x, \Delta y) - f(0,0) - \Delta x f_x(0,0) - \Delta y f_y(0,0)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$

$= \lim_{(\Delta x, \Delta y)\to(0,0)} \frac{\Delta x \cdot \Delta y}{(\Delta x)^2 + (\Delta y)^2}$

$= \lim_{r\to 0} \frac{r^2 \cos\theta \sin\theta}{r^2}$ [putting $\Delta x = r\cos\theta$, $\Delta y = r\sin\theta$

$= \cos\theta \sin\theta \neq 0$

>0

Hence $f(x,y)$ is not differentiable at origin. <0

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7. $f(x,y) = \sqrt{|xy|}$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

Now $\displaystyle\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{f(\Delta x,\Delta y) - f(0,0) - \Delta x f_x(0,0) - \Delta y f_y(0,0)}{\sqrt{(\Delta x)^2-(\Delta y)^2}}$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\sqrt{|\Delta x\,\Delta y|}}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$= \lim_{r\to 0} \frac{\sqrt{r^2|\cos\theta\sin\theta|}}{\sqrt{r^2(\cos^2\theta+\sin^2\theta)}}$$ [putting $\Delta x = r\cos\theta$, $\Delta y = r\sin\theta$

$= \displaystyle\lim_{r\to 0} \cos\theta \sin\theta$ depends on θ

$\neq 0$

Hence $f(x,y)$ is not differentiable at $(0,0)$

for $(x,y) \neq (0,0)$

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{|x+h||y|} - \sqrt{|xy|}}{h}$$

$$= \lim_{h\to 0} \sqrt{|y|}\, \frac{|x+h|-|x|}{h\left[\sqrt{|x+h|}+\sqrt{|x|}\right]}$$

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Now as $h \to 0$, we can take

$$|x+h| = \begin{cases} x+h, & \text{when } x>0 \\ -(x+h), & \text{when } x<0 \end{cases}$$

$$\therefore f_x(x,y) = \begin{cases} \lim\limits_{h\to 0} \sqrt{|y|}\, \dfrac{(x+h)-x}{h\left[\sqrt{|x+h|}+\sqrt{|x|}\right]} & \text{when } x>0 \\ \lim\limits_{h\to 0} \sqrt{|y|}\, \dfrac{-(x+h)+x}{h\left[\sqrt{|x+h|}+\sqrt{|x|}\right]} & \text{when } x<0 \end{cases}$$

$$\Rightarrow f_x(x,y) = \begin{cases} \frac{1}{2}\sqrt{\dfrac{|y|}{|x|}}, & \text{when } x>0 \\ -\frac{1}{2}\sqrt{\dfrac{|y|}{|x|}}, & \text{when } x<0 \end{cases} \quad [\text{for } (x,y)\neq(0,0)]$$

Again

$$f_y(x,y) = \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k}$$

$$= \lim_{k\to 0} \frac{\sqrt{|x|\,|y+k|}-\sqrt{|xy|}}{k}$$

$$\Rightarrow f_y(x,y) = \begin{cases} \frac{1}{2}\sqrt{\dfrac{|x|}{|y|}}, & \text{when } y>0 \\ -\frac{1}{2}\sqrt{\dfrac{|y|}{|x|}}, & \text{when } y<0 \end{cases}$$

$[\text{for } (x,y)\neq(0,0)]$

Now for $(x,y)\neq(0,0)$

$$\lim_{(h,k)\to(0,0)} f_x(x+h,y+k) = \lim_{(h,k)\to(0,0)} \begin{cases} \frac{1}{2}\sqrt{\dfrac{|y+k|}{|x+h|}}, & x+h>0 \\ -\frac{1}{2}\sqrt{\dfrac{|y+k|}{|x+h|}}, & x+h<0 \end{cases}$$

$$= \begin{cases} \frac{1}{2}\sqrt{\dfrac{|y|}{|x|}}, & x>0 \\ -\frac{1}{2}\sqrt{\dfrac{|y|}{|x|}}, & x<0 \end{cases}$$

$$= f_x(x,y)$$

$\Rightarrow f_x$ is continuous at $(x,y) \neq (0,0)$

Similarly we can show f_y is continuous at $(x,y) \neq (0,0)$

But $\displaystyle\lim_{(x,y)\to(0,0)} f_x(x,y) = \lim_{(x,y)\to(0,0)} \begin{cases} \frac{1}{2}\sqrt{\frac{|y|}{|x|}} , & x>0 \\ -\frac{1}{2}\sqrt{\frac{|y|}{|x|}} , & x<0 \end{cases}$

$$= \lim_{x\to 0} \begin{cases} \frac{1}{2}\sqrt{\frac{|mx|}{|x|}} , & x>0 \\ -\frac{1}{2}\sqrt{\frac{|mx|}{|x|}} , & x<0 \end{cases} \quad [\text{along the path } y=mx]$$

$$= \lim_{x\to 0} \begin{cases} \frac{1}{2}\sqrt{|m|} , & x>0 \\ -\frac{1}{2}\sqrt{|m|} , & x<0 \end{cases}$$

$\neq f_x(0,0) = 0$

$\Rightarrow f_x$ is not continuous at origin.

Similarly we can show f_y is not continuous at origin.

8. (a) $f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{x^2+y^2} , & (x,y)\neq(0,0) \\ 0 , & (x,y)=(0,0) \end{cases}$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h^2\sin(\frac{1}{h^2})-0}{h}$$

$$= \lim_{h\to 0} h\sin(\frac{1}{h^2}) = 0$$

$$f_y(0,0) = \lim_{K\to 0} \frac{f(0,K) - f(0,0)}{K}$$

$$= \lim_{K\to 0} \frac{K^2 \sin 1/K^2 - 0}{K}$$

$$= \lim_{K\to 0} K \sin(1/K^2) = 0$$

NOW
$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{f(\Delta x,\Delta y) - f(0,0) - \Delta x f_x(0,0) - \Delta y f_y(0,0)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{[(\Delta x)^2 + (\Delta y)^2] \sin\left(\frac{1}{(\Delta x)^2+(\Delta y)^2}\right)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \sqrt{(\Delta x)^2 + (\Delta y)^2} \sin\left(\frac{1}{(\Delta x)^2+(\Delta y)^2}\right)$$

$$= 0$$

$\therefore$ $f(x,y)$ is differentiable at $(0,0)$

(b) $f(x,y) = \begin{cases} \dfrac{x^3 - 2y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h^3/h^2 - 0}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{-2k^3/k^2 - 0}{k} = \lim_{k \to 0} \frac{-2k}{k} = -2$$

Now
$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{f(\Delta x, \Delta y) - f(0,0) - \Delta x\, f_x(0,0) - \Delta y\, f_y(0,0)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\frac{(\Delta x)^3 - 2(\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2} - 1 \cdot \Delta x + 2 \cdot \Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^3}}$$

$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{(\Delta x)^3 - 2(\Delta y)^3 - (\Delta x)^3 + 2(\Delta x)^2 \Delta y - \Delta x(\Delta y)^2 + 2(\Delta y)^3}{\{(\Delta x)^2 + (\Delta y)^2\}^{3/2}}$$

$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{2(\Delta x)^2 \Delta y - \Delta x (\Delta y)^2}{\{(\Delta x)^2 + (\Delta y)^2\}^{3/2}}$$

$$= \lim_{r \to 0} \frac{2 r^2 \cos^2\theta\, r \sin\theta - r\cos\theta\, r^2 \sin^2\theta}{(r^2)^{3/2}}$$

$$= \lim_{r \to 0} \frac{r^3 [2\cos^2\theta \sin\theta - \cos\theta \sin^2\theta]}{r^3}$$

$$= 2\cos^2\theta \sin\theta - \cos\theta \sin^2\theta \neq 0$$

So, $f(x,y)$ is not differentiable at $(0,0)$

$$f(x,y) = \begin{cases} y\,\dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$f_{xx}(0,0) = \lim_{h \to 0} \frac{f_x(h,0) - f_x(0,0)}{h}$$

Now $f_x(0,0) = \lim_{h_1 \to 0} \dfrac{f(h_1,0) - f(0,0)}{h_1}$

$$= \lim_{h_1 \to 0} \frac{0-0}{h_1} = 0$$

$$f_x(h,0) = \lim_{h_1 \to 0} \frac{f(h+h_1,0) - f(h,0)}{h_1} = \lim_{h_1 \to 0} \frac{0-0}{h_1}$$

$$= 0$$

$$\therefore\ f_{xx}(0,0) = \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$f_{xy}(0,0) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{\frac{k(-k^2)}{k^2} - 0}{k} = -1$$

$$f_y(h,0) = \lim_{k \to 0} \frac{f(h,k) - f(h,0)}{k}$$

$$= \lim_{k \to 0} \frac{k \cdot \frac{h^2-k^2}{h^2+k^2} - 0}{k} = \lim_{k \to 0} \frac{h^2-k^2}{h^2+k^2} = 1$$

$\therefore\ f_{xy}(0,0) = \lim_{h \to 0} \dfrac{1-(-1)}{h}$ does not exist.

$$f_{yx}(0,0) = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$f_x(0,k) = \lim_{h \to 0} \frac{f(h,k) - f(0,k)}{h}$$

$$= \lim_{h \to 0} \frac{k \, \frac{h^2 - k^2}{h^2 + k^2} - k \cdot \frac{0 - k^2}{k^2}}{h}$$

$$= \lim_{h \to 0} \frac{k\,(h^2 - \cancel{k^2} + h^2 + \cancel{k^2})}{h(h^2 + k^2)}$$

$$= \lim_{h \to 0} \frac{2hk}{h^2 + k^2} = 0$$

$$\therefore \; f_{yx}(0,0) = \lim_{k \to 0} \frac{0 - 0}{k} = 0$$

$$f_{yy}(0,0) = \lim_{k \to 0} \frac{f_y(0,k) - f_y(0,0)}{k}$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = -1$$

$$f_y(0,k) = \lim_{k_1 \to 0} \frac{f(0, k+k_1) - f(0,k)}{k_1}$$

$$= \lim_{k_1 \to 0} \frac{-(k+k_1) - (-k)}{k_1} = \lim_{k_1 \to 0} \frac{-k_1}{k_1} = -1$$

$$f_{yy}(0,0) = \lim_{k \to 0} \frac{-1 - (-1)}{k} = 0$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{f(\Delta x,\Delta y) - f(0,0) - \Delta x f_x(0,0) - \Delta y f_y(0,0)}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta y \cdot \frac{(\Delta x)^2-(\Delta y)^2}{(\Delta x)^2+(\Delta y)^2} - 0 - 0 + 1\cdot\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta y\left[(\Delta x)^2-(\Delta y)^2+(\Delta x)^2+(\Delta y)^2\right]}{\left\{(\Delta x)^2+(\Delta y)^2\right\}^{3/2}}$$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{2\cdot\Delta y\,(\Delta x)^2}{\left\{(\Delta x)^2+(\Delta y)^2\right\}^{3/2}}$$

$$= \lim_{r\to 0} \frac{2r^3\cos^2\theta\sin\theta}{r^3} \quad \left[\text{putting } \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right.$$

$$= 2\sin\theta\cos^2\theta \neq 0$$

$\therefore$ $f(x,y)$ is not differentiable at the origin.

10. $$f(x,y) = \begin{cases} \dfrac{x^2y(x-y)}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$f_y(h,0) = \lim_{k \to 0} \frac{f(h,k) - f(h,0)}{k}$$

$$= \lim_{k \to 0} \frac{\frac{h^2k(h-k)}{h^2+k^2} - 0}{k} = \lim_{k \to 0} \frac{h^2(h-k)}{h^2+k^2}$$

$$= h$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \to 0} \frac{0-0}{k} = 0$$

$$\therefore \quad f_{xy}(0,0) = \lim_{h \to 0} \frac{h-0}{h} = 1$$

$$f_{yx}(0,0) = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

$$f_x(0,k) = \lim_{h \to 0} \frac{f(h,k) - f(0,k)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h^2k(h-k)}{h^2+k^2} - 0}{h} = \lim_{h \to 0} \frac{hk(h-k)}{h^2+k^2}$$

$$= 0$$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$\therefore \quad f_{yx}(0,0) = \lim_{k \to 0} \frac{0-0}{k} = 0$$

$$\therefore \quad f_{xy}(0,0) \neq f_{yx}(0,0)$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{f(\Delta x,\Delta y)-f(0,0)-\Delta x f_x(0,0)-\Delta y f_y(0,0)}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{(\Delta x)^2\Delta y\,(\Delta x-\Delta y)}{\{(\Delta x)^2+(\Delta y)^2\}^{3/2}}$$

$$= \lim_{r\to 0} \frac{r^2\cos^2\theta\; r\sin\theta\; r(\cos\theta-\sin\theta)}{(r^2)^{3/2}}$$ [putting $x=r\cos\theta$, $y=r\sin\theta$

$$= \lim_{r\to 0} r\cos^2\theta\sin\theta(\cos\theta-\sin\theta)$$

$$= 0$$

$\therefore$ $f(x,y)$ is differentiable at the origin.

11. (a) $f(x,y) = x^4\sin 3y + 5x - 6y$

$f_x(x,y) = 4x^3\sin 3y + 5$

$f_{xx}(x,y) = 12x^2\sin 3y$

$f_{yxx}(x,y) = 36x^2\cos 3y$

~~for~~ $f_x(x,y) = 4x^3\sin 3y + 5$

$f_{yx}(x,y) = 12x^3\cos 3y$

$f_{xyx}(x,y) = 36x^2\cos 3y$

(b) $f(x,y) = x^5y^3 + \log(xy) + 10x$

$f_x(x,y) = 5x^4y^3 + \frac{y}{xy} + 10$

$= 5x^4y^3 + \frac{1}{x} + 10$

$f_{xx}(x,y) = 20x^3y^3 - 1/x^2$

$f_{yxx}(x,y) = 60x^3y^2$

$f_x(x,y) = 5x^4y^3 + 1/x + 10$

$f_{yx}(x,y) = 15x^4y^2$

$f_{xyx}(x,y) = 60x^3y^2$

(c) $f(x,y) = e^{xy}\tan x + x^3y^2$

$f_x(x,y) = e^{xy}\sec^2 x + ye^{xy}\tan x + 3x^2y^2$

$f_{xx}(x,y) = e^{xy} \cdot 2\sec x \cdot \sec x \tan x + ye^{xy}\sec^2 x$

$+ ye^{xy}\sec^2 x + y^2e^{xy}\tan x + 6xy^2$

$= 2e^{xy}\sec^2 x \tan x + 2ye^{xy}\sec^2 x \cancel{+16xy^2}$

$+ y^2e^{xy}\tan x + 6xy^2$

$f_{yxx}(x,y) = 2xe^{xy}\sec^2 x \tan x + 2e^{xy}\sec^2 x$

$+ 2yxe^{xy}\sec^2 x + 2ye^{xy}\tan x$

$+ xy^2e^{xy}\tan x + 12xy$

$= 2xe^{xy}\sec^2 x \tan x + 2e^{xy}\sec^2 x + 2xye^{xy}\sec^2 x$

$+ 2ye^{xy}\tan x + xy^2e^{xy}\tan x + 12xy$

$$f_x(x,y) = e^{xy}\sec^2 x + ye^{xy}\tan x + 3x^2y^2$$

$$f_{yx}(x,y) = xe^{xy}\sec^2 x + e^{xy}\tan x + xy\,e^{xy}\tan x$$
$$+ 6x^2 y$$

$$f_{xyx}(x,y) = e^{xy}\sec^2 x + xy\,e^{xy}\sec^2 x$$
$$+ xe^{xy}\cdot 2\sec x\cdot\sec x\tan x$$
$$+ ye^{xy}\tan x + e^{xy}\sec^2 x + ye^{xy}\tan x$$
$$+ xy^2e^{xy}\tan x + xy\,e^{xy}\sec^2 x + 12xy$$
$$= 2e^{xy}\sec^2 x + 2xy\,e^{xy}\sec^2 x + 2xe^{xy}\sec^2 x\tan x$$
$$+ 2ye^{xy}\tan x + xy^2e^{xy}\tan x + 12xy$$

(d) $f(x,y) = x^3\sin y + y^3\cos x$

$$f_x(x,y) = 3x^2\sin y - y^3\sin x$$
$$f_{xx}(x,y) = 6x\sin y - y^3\cos x$$
$$f_{yxx}(x,y) = 6x\cos y - 3y^2\cos x$$

$$f_x(x,y) = 3x^2\sin y - y^3\sin x$$
$$f_{yx}(x,y) = 3x^2\cos y - 3y^2\sin x$$
$$f_{xyx}(x,y) = 6x\cos y - 3y^2\cos x$$

(e) $f(x,y) = e^x \ln y + \cos y \ln x$

$f_x(x,y) = e^x \ln y + \cos y \cdot 1/x$

$f_{xx}(x,y) = e^x \ln y - \cos y \cdot 1/x^2$

$f_{yxx}(x,y) = e^x \cdot 1/y + \sin y \cdot 1/x^2$

$f_x(x,y) = e^x \ln y + 1/x \cos y$

$f_{yx}(x,y) = e^x \cdot 1/y - 1/x \sin y$

$f_{xyx}(x,y) = e^x \cdot 1/y + 1/x^2 \sin y$

(f) $f(x,y) = x^3y^2 + 2xy^3 + \cos(xy^2)$

$f_x(x,y) = 3x^2y^2 + 2y^3 - y^2 \sin(xy^2)$

$f_{xx}(x,y) = 6xy^2 - y^4 \cos(xy^2)$

$f_{yxx}(x,y) = 12xy - 4y^3 \cos(xy^2) + y^4 \sin(xy^2) \cdot 2xy$

$= 12xy - 4y^3 \cos(xy^2) + 2xy^5 \sin(xy^2)$

$f_x(x,y) = 3x^2y^2 + 2y^3 - y^2 \sin(xy^2)$

$f_{yx}(x,y) = 6x^2y + 6y^2 - 2y \sin(xy^2)$

$- 2xy^3 \cos(xy^2)$

$f_{xyx}(x,y) = 12xy - 2y^3 \cos(xy^2) - 2y^3 \cos(xy^2)$

$+ 2xy^3 \sin(xy^2) \cdot y^2$

$= 12xy - 4y^3 \cos(xy^2) + 2xy^5 \sin(xy^2)$

12. (a) $w = x^2 + xy^2 + xy^2z^3$

Total differential of w is

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz$$

$$\frac{\partial w}{\partial x} = 2x + y^2 + y^2z^3$$

$$\frac{\partial w}{\partial y} = 2xy + 2xyz^3$$

$$\frac{\partial w}{\partial z} = 3xy^2z^2 .$$

$$dw = (2x + y^2 + y^2z^3)\,dx + (2xy + 2xyz^3)\,dy + 3xy^2z^2\,dz$$

(b) $z = \tan^{-1}(x/y)$

Total differential of z is

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

$$\frac{\partial z}{\partial x} = \frac{1}{1+(x/y)^2} \cdot 1/y = \frac{y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+(x/y)^2}\,(-x/y^2) = \frac{-x}{x^2+y^2}$$

$$\therefore\ dz = \left(\frac{y}{x^2+y^2}\right)dx - \left(\frac{x}{x^2+y^2}\right)dy$$

(c) $u = e^{x^2+y^2+z^2}$

Total differential of u is

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

$$\frac{\partial u}{\partial x} = e^{x^2+y^2+z^2} \cdot 2x$$

$$\frac{\partial u}{\partial y} = e^{x^2+y^2+z^2} \cdot 2y$$

$$\frac{\partial u}{\partial z} = e^{x^2+y^2+z^2} \cdot 2z$$

$$du = \left(2x\, e^{x^2+y^2+z^2}\right) dx + \left(2y\, e^{x^2+y^2+z^2}\right) dy + \left(2z\, e^{x^2+y^2+z^2}\right) dz$$

$$= 2e^{x^2+y^2+z^2} \left(x\,dx + y\,dy + z\,dz \right)$$

(d) $w = \sin(3x+4y) + 5e^z$

Total differential of w is

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$

$$\frac{\partial w}{\partial x} = 3\cos(3x+4y)$$

$$\frac{\partial w}{\partial y} = 4\cos(3x+4y)$$

$$\frac{\partial w}{\partial z} = 5e^z$$

$$dw = 3\cos(3x+4y)\,dx + 4\cos(3x+4y)\,dy + 5e^z\,dz$$

(e) $w = z\ln y + y\ln z + xyz$

Total differential of w is

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

$$\frac{\partial w}{\partial x} = yz$$

$$\frac{\partial w}{\partial y} = \frac{z}{y} + \ln z + xz$$

$$\frac{\partial w}{\partial z} = \ln y + \frac{y}{z} + xy$$

$$dw = yz\,dx + (z/y + \ln z + xz)\,dy + (\ln y + y/z + xy)\,dz$$

(f) $u = \sqrt{x^2+y^2+z^2}$

Total differential of u is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x$$

$$\frac{\partial u}{\partial y} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2y$$

$$\frac{\partial u}{\partial z} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2z$$

$$\therefore du = \frac{x}{\sqrt{x^2+y^2+z^2}}dx + \frac{y}{\sqrt{x^2+y^2+z^2}}dy$$

$$+ \frac{z}{\sqrt{x^2+y^2+z^2}}dz$$

(g) $w = e^x \sin(y+2z) - x^2y^2$

Total differential of w is

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$

$$\frac{\partial w}{\partial x} = e^x \sin(y+2z) - 2xy^2$$

$$\frac{\partial w}{\partial y} = e^x \cos(y+2z) - 2x^2y$$

$$\frac{\partial w}{\partial z} = 2e^x \cos(y+2z)$$

$$dw = [e^x \sin(y+2z) - 2xy^2]\, dx$$

$$+ [e^x \cos(y+2z) - 2x^2y]\, dy + 2e^x \cos(y+2z)\, dz$$

(h) $w = e^{x/y} + e^{z/y}$.

Total differential of w is.

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$

$$\frac{\partial w}{\partial x} = e^{x/y} \cdot 1/y$$

$$\frac{\partial w}{\partial y} = e^{x/y} (-x/y^2) + e^{z/y} (-z/y^2)$$

$$\frac{\partial w}{\partial z} = e^{z/y} \cdot 1/y$$

$$dw = 1/y\, e^{x/y}\, dx - 1/y^2 (x e^{x/y} + z e^{z/y})\, dy$$

$$+ 1/y\, e^{z/y}\, dz.$$

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