## Week 5: Phase and group velocities, polarisation, standing waves

**Phase and group velocities:** Let us go back to the plane electromagnetic wave we defined earlier. The electric field is

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_{\mathbf{0}} \cos\left(\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \omega t\right) \tag{1}$$

We had taken $\vec{\mathbf{E}}_{\mathbf{0}}$ as a constant vector. Hence, on each cophasal surface (wavefront), where $\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \omega t = constant$, the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields form two mutually perpendicular vector fields. The magnitude of the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields depend on the value of $\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \omega t$. After a spatial period of $\frac{2\pi}{|\vec{\mathbf{k}}|}$ or a temporal period $\frac{2\pi}{\omega}$ we get back the same $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ , but on a different co-phasal surface. We define the wavelength and frequency of the wave as

$$\lambda = \frac{2\pi}{|\vec{\mathbf{k}}|} \quad ; \quad \nu = \frac{\omega}{2\pi} \tag{2}$$

The vector $\vec{\mathbf{k}}$ is known as the **propagation vector** or **wave vector**, $|\vec{\mathbf{k}}|$ is called the **wave number** and $\omega$ is the **angular frequency** (we also use $2\pi\nu = \omega$ where $\nu$ is known as the **frequency**). The propagating plane wave front is shown in Fig. 1. Note that on the infinitely extended co-phasal surfaces the $\vec{\mathbf{E}}$, $\vec{\mathbf{B}}$ are there at every point. It is the **time averaged energy** of the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields which we define as the **intensity**. We **see** this intensity and our **detectors are capable of measuring this intensity in terms of current or voltage**.

What is the speed at which the cophasal surface moves? We can quickly note that this is nothing but

$$v_P = \frac{\omega}{k} \tag{3}$$

which is named as the **phase velocity** of the wave.

Another velocity, known as the **group velocity** is of great importance and significance. Though we discuss this in the context of electromagnetic waves, it is a general concept associated with wave motion. The phase velocity defined above is also a general concept and is not specific to electromagnetic waves only.

We may contemplate a situation where the $\vec{\mathbf{E}}_{\mathbf{0}}$ is no longer a constant, though we will retain that it is still directed along a specific constant vector. In other words we will try and see whether we can **modulate** the amplitude. How to do this? What functions are allowed?

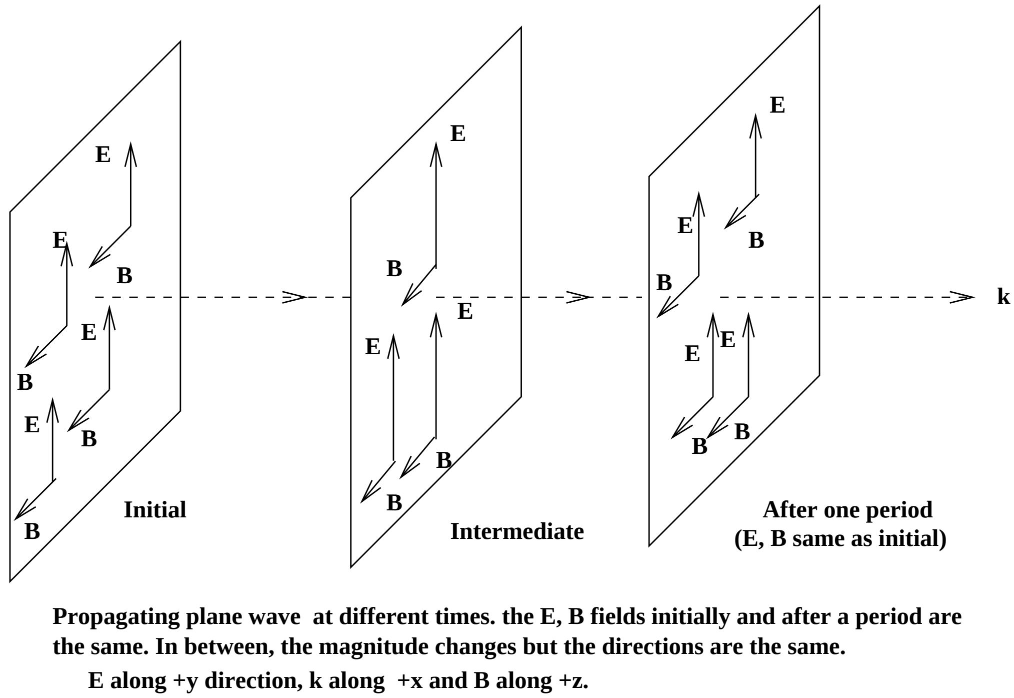


FIG. 1. Propagating plane wave

To keep things simple let us assume that the propagation vector is $\vec{\mathbf{k}} = k\hat{\mathbf{i}}$, i.e. along the $x$ direction. Further the direction of the electric field is along $\hat{\mathbf{j}}$. We can write down an expression for the electric field as

$$\vec{\mathbf{E}} = \hat{\mathbf{j}}\left[E_0 \cos\left(ax - bt\right)\right] \cos\left(kx - \omega t\right) \tag{4}$$

Here $a$ and $b$ are two constants. This will solve the wave equation for $\vec{\mathbf{E}}$ given earlier when $\frac{\omega}{k} = \frac{b}{a} = v$.

Exercise: Show the above expression for $\vec{\mathbf{E}}$ solves the wave equation for $\vec{\mathbf{E}}$ under the conditions mentioned.

It is easy to see that the above expression can also be rewritten as,

$$\vec{\mathbf{E}} = \hat{\mathbf{j}}\frac{E_0}{2}\left[\cos\left((k+a)x - (\omega+b)t\right) + \cos\left((k-a)x - (\omega-b)t\right)\right] \tag{5}$$

If we rename $k = k_0$ and also assume $a = \Delta k$, $b = \Delta\omega$, then we have,

$$\vec{\mathbf{E}} = \hat{\mathbf{j}}\left[E_0 \cos\left(\Delta k\ x - \Delta\omega\ t\right)\right] \cos\left(k_0 x - \omega_0 t\right) \tag{6}$$

Thus, the $(k_0, \omega_0)$ wave is **amplitude modulated** by the cosine function involving $(\Delta k, \Delta\omega)$ (which are much **smaller** in value than the $k$, $\omega$). The speed at which the modulation moves is given as

$$v_g = \frac{\Delta\omega}{\Delta k} \tag{7}$$

and is called the **group velocity**. Unless $\Delta k$ and $\Delta\omega$ are much smaller than $k$ and $\omega$ respectively, we will not get an amplitude modulated wave or a **wave packet** To understand this look at Fig. 2 and Fig. 3, both of which are at a fixed $t$ (say $t = 0$). Fig. 2 involves a superposition of $\cos 2.5x$ and $\cos 0.5x$–the result does not give a wave packet with a **carrier** and a **modulation**. On the other hand, Fig. 3 involves a superposition of $\cos 1.6x$ and $\cos 1.4x$ and it shows a distinct carrier and a modulation. Thus, if you wish to superpose two waves such that the superposed wave has a carrier and a modulation then the two waves which are being superposed must have their respective $k$ and $\omega$ close to each other. Why is it necessary to study such modulated waves? This is simply because one would like to transmit **information** and/or **energy**. A sinusoidal wave is somewhat **boring** – it just travels to infinity with a single $\omega$ and $k$. Thus, it is an idealisation.

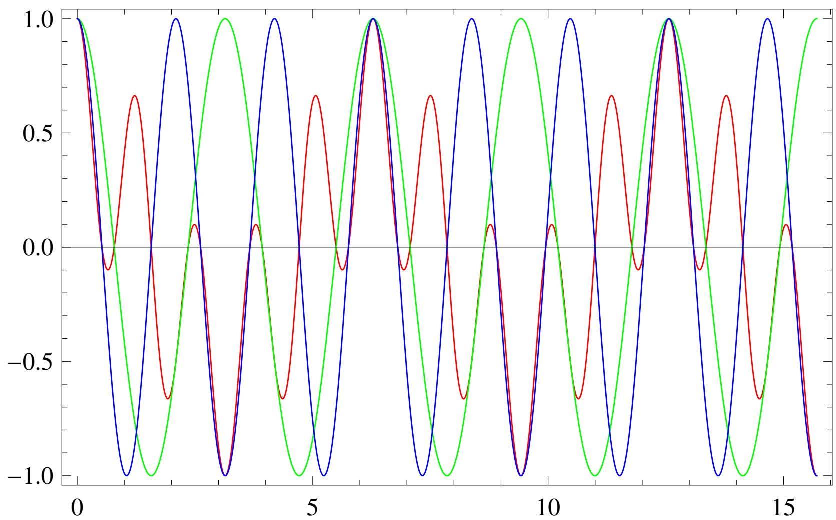


FIG. 2. The functions $\cos 2x \cos 3x$ (red), $\cos 2x$ (green), $\cos 3x$(blue)

If we look at the expression for $\vec{\mathbf{E}}$ above (the component part), then we recall that for a fixed $x$ or a fixed $t$, we had obtained a similar expression while dealing with coupled oscillations (look at the figures for $x_0$ and $x_1$ given at the end of the discussion on coupled oscillations given in Week 3). In the case of coupled oscillations too we had assumed $\Delta\omega$ to be small. $\Delta\omega$ is related to **the beat frequency**.

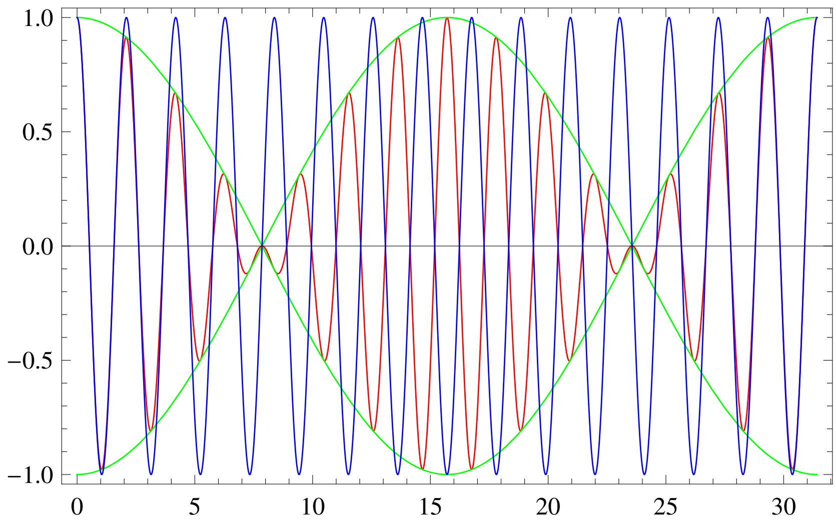


FIG. 3. The functions $\cos 0.2x \cos 3x$ (red), $\cos 0.2x$ (green), $\cos 3x$ (blue)

One can easily note that the modulation discussed above travels like a pulse or a packet, known as the **wave packet** or **wave group** and the speed of this **wave packet** is the group velocity. In the special case when $\frac{\Delta\omega}{\Delta k} = \frac{\omega}{k}$ the wave equation for $\vec{\mathbf{E}}$ is solved by the modulated wave. In general, it is not– simply because in such cases, the wave travels through a **medium**, not through vacuum and the wave equation for $\vec{\mathbf{E}}$ which we derived for vacuum, is no longer valid.

We can, in principle, add more such terms in the expression,

$$\vec{\mathbf{E}} = \hat{\mathbf{j}}\frac{E_0}{2}\left[\cos\left((k_0+\Delta k)x - (\omega_0+\Delta\omega)t\right) + \cos\left((k_0-\Delta k)x - (\omega_0-\Delta\omega)t\right)\right] \tag{8}$$

to get

$$\vec{\mathbf{E}} = \hat{\mathbf{j}}\frac{E_0}{2}\sum_{n=1}^{\infty}\left[\cos\left((k_0+n\Delta k)x - (\omega_0+n\Delta\omega)t\right) + \cos\left((k_0-n\Delta k)x - (\omega_0-n\Delta\omega)t\right)\right] \tag{9}$$

One may choose any discrete set of values of $n$ and generate a wave packet. When several $n$ values are taken, the modulation will surely be much different from the simple case discussed earlier.

It is useful to consider a superposition in the form of an integral over $k$. Thus in a general context (taking $\phi(x,t)$ as the component of the $\vec{\mathbf{E}}$ vector considered above), we can write

$$\phi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{i(kx-\omega t)}dk \tag{10}$$

where we have superposed an infinite number of plane waves of wavenumber $k$, each of amplitude $A(k)$. Technically, the above expression is known as a **Fourier transform**. Let us now assume $\omega$ as a function of $k$, $\omega(k)$. We can then expand $\omega(k)$ in a Taylor series (upto linear order in $k$), about $k_0$, to get,

$$\omega(k) = \omega(k_0) + \frac{d\omega}{dk}|_{k_0}\,(k - k_0) \tag{11}$$

where $k_0$ (and its corresponding $\omega(k_0) = \omega_0$) is a fixed (central) wave number about which variations are considered. Substituting this in the expression for $\phi(x,t)$ we get

$$\phi(x,t) = e^{i(k_0 x - \omega_0 t)}\left[\int_{-\infty}^{\infty} A(k) e^{i(\Delta k\ x - v_g \Delta k\ t)}\right] \tag{12}$$

where

$$v_g = \frac{d\omega}{dk} \tag{13}$$

is the group velocity. The term in square brackets is the modulation of the amplitude of the wave with wave number $k_0$. Thus, if we know the dispersion relation $\omega = \omega(k)$ we can find the group velocity by differentiation w.r.t. $k$. Note that dispersion gives rise to wavelength dependence of the refractive index $n = \frac{ck}{\omega(k)}$, which is the main cause behind various observed phenomena (dispersion of white light by a prism, colours in a rainbow, etc).

It is important to note that a wave-packet can be **dispersive** or **non-dispersive**. If the packet moves without change of shape, in time, then it is said to be non-dispersive. The motion of the packet is a parallel shift in time. If its shape does change with time, it is said to be dispersive. In the non-dispersive case, the group velocity will equal the phase velocity and the relation between $\omega$ and $k$ will be linear ($\omega = vk$). Note that is the relation is $\omega = vk + b$ ($b \neq 0$), then we have nondispersive propagation with $v_p$ not equal to $v_g$. In the example discussed above we have non-dispersive propagation with $v_p = v_g$.

You can see various animations illustrating nondispersive and dispersive propagation in

(i) the Wikipedia page on Group Velocity

(ii) In www.acs.psu.edu/drussell/demos/dispersion/dispersion.html

Many other sources with animations are also available in the web. Through all these animations you can get an idea of what group velocity means and also understand the difference between non-dispersive and dispersive propagation. A more detailed discussion on this topic is however beyond the scope of this course.

Exercise: Find the phase and group velocity if the dispersion relation is given as: $\omega^2 = k^2c^2 + \omega_p^2$. Note that the phase velocity is greater than $c$ but the group velocity is less than $c$.

**Polarisation:** Since $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are orthogonal to each other and to $\vec{\mathbf{k}}$ it is possible that $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ can have a infinite number of orientations on the plane perpendicular to the direction of propagation. This **freedom** is associated with an **exclusive** property of **transverse electromagnetic waves** known as **polarisation**. Note that transverse electromagnetic waves are somewhat different from the transverse waves on a string because they concern **vectorial** quantities $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$.

Let us now go back to the expression for the electric field discussed earlier and given as:

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_{\mathbf{0}} \cos\left(\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \omega t\right) \tag{14}$$

Assume $\vec{\mathbf{E}}_{\mathbf{0}} = \mathbf{E}_{\mathbf{0x}}\vec{\mathbf{i}} + \mathbf{E}_{\mathbf{0y}}\vec{\mathbf{k}}$, i.e. the $\vec{\mathbf{E}}$ vector is along a certain direction in the $xz$ plane and $\vec{\mathbf{k}}$ is along the $y$ direction. Look at the figure below which shows the electric field at different instances of time at the origin on the $xz$ plane. All these snapshots are finally superposed in the last figure. The end-result is that the electric field vector oscillates along a line on the $xz$ plane. This is known as **linear polarisation**.

This, of course, is not the only thing that the electric field vector can do as it evolves in time. It can rotate as shown in Fig. 5. In mathematical terms we can have

$$\vec{\mathbf{E}} = E_0 \left[\cos\left(ky - \omega t\right)\hat{\mathbf{i}} + \sin\left(ky - \omega t\right)\hat{\mathbf{k}}\right] \tag{15}$$

Here the magnitude of the electric field vector is unchanged during propagation. But its direction changes. If we view the electric field vector at different times, as shown in Fig. 5, we can see that its tip travels along a spiral. Superposing the snapshots on a single plane we see that the tip travels along a circle. Hence the name **circular polarisation**. If the circle is traversed anti-clockwise in forward time we have **left circular polarisation** and for clockwise we have **right circular polarisation**.

Exercise: The expression for $\vec{\mathbf{E}}$ in (15) is for left circular polarisation. Write down its counterpart for right circular polarisation. Also show that linear polarisation can be written as a superposition of left and right circular polarisations.

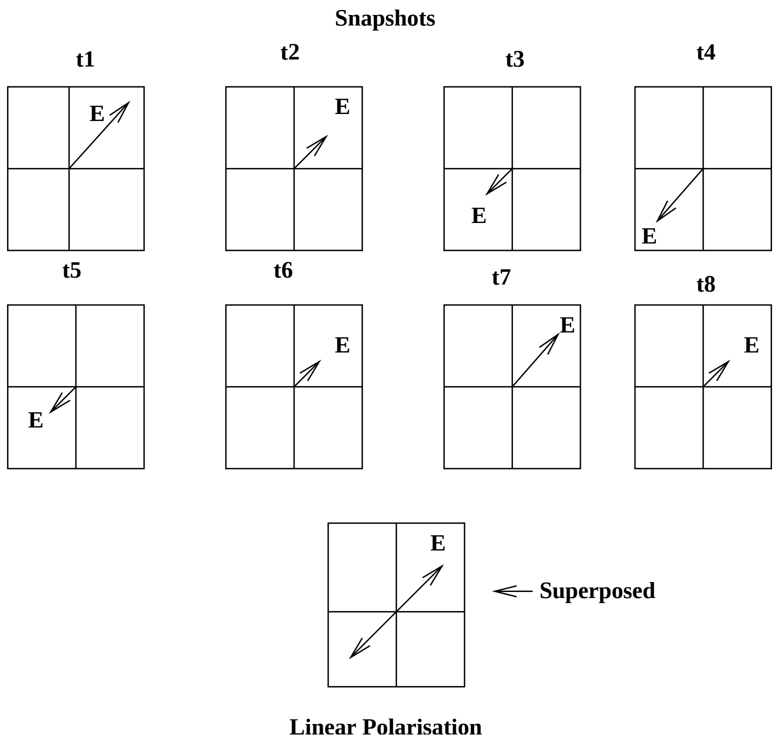


FIG. 4. Linear polarisation

A more general case, which includes all of the above is the case of **elliptic polarisation**. The expression for the $\vec{\mathbf{E}}$ would then become

$$\vec{\mathbf{E}} = \left[E_{0x} \cos\left(ky - \omega t\right) \hat{\mathbf{i}} + E_{0y} \sin\left(ky - \omega t\right) \hat{\mathbf{k}}\right] \tag{16}$$

Here, instead of a circle the tip of the $\vec{\mathbf{E}}$ vector moves along an ellipse. If $E_{0x} = E_{0y}$ then we have left circular polarisation (similarly for right circular). If either of $E_{0x}$ or $E_{0y}$ is zero we have linear polarisation.

We shall have more to say on polarisation again later on.

**Standing waves:** We now turn to a new kind of wave which is quite important in various applications. This is called the **standing wave**. To understand this we go back to the **transverse waves** on a stretched string once again. We imagine that the string is now fixed at two ends, say at $x = 0$ and $x = L$. Hence, the transverse displacement at these ends

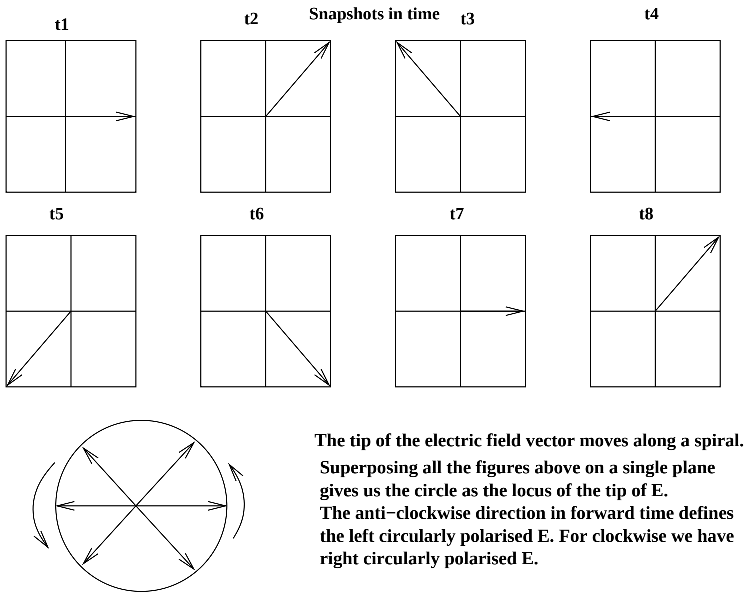


FIG. 5. Circular polarisation

is zero at all times. We must therefore solve the wave equation with the added requirement,

$$\xi(0,t) = \xi(L,t) = 0 \tag{17}$$

To solve the wave equation for $\xi(x,t)$ given by,

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c_s^2}\frac{\partial^2 \xi}{\partial t^2} \tag{18}$$

we adopt a method known as the method of **separation of variables**. First, let us assume the solution for $\xi(x,t)$ as

$$\xi(x,t) = X(x)T(t) \tag{19}$$

This way of writing the solution as a product of a function of $x$ alone ($X(x)$) and a function of $t$ alone ($T(t)$) is the basic starting point of the method. Note that the solutions of the wave equation discussed earlier, i.e. $f(x - vt)$ or $g(x + vt)$ are not **separable** in the above sense, in general.

Substituting the solution above in the wave equation, we get

$$T(t)\frac{d^2X}{dx^2} = \frac{1}{c_s^2}X(x)\frac{d^2T}{dt^2} \tag{20}$$

Note that we have written the derivatives in the above as **ordinary derivatives** because $T(t)$ and $X(x)$ are functions of $t$ and $x$, respectively. Dividing both sides by $T(t)X(x)$ we get

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c_s^2}\frac{1}{T}\frac{d^2T}{dt^2} \tag{21}$$

Thus, the L. H. S. is a function of $x$ alone, while the R. H. S. is a function of $t$ alone. Now, if a function of $x$ alone is equal to a function of $t$ alone then they can only be equal to a constant. Therefore, we write

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k^2 = \frac{1}{c_s^2}\frac{1}{T}\frac{d^2T}{dt^2} \tag{22}$$

where $-k^2$ is the constant. Why do we choose the constant to be negative? This will become clear if we write down the two ordinary differential equations for $X$ and $T$ separately, using this choice. We have

$$\frac{d^2X}{dx^2} + k^2X = 0 \tag{23}$$

$$\frac{d^2T}{dt^2} + k^2c_s^2T = 0 \tag{24}$$

both of which are simple harmonic oscillator equations. If we had not chosen the constant as negative (i.e. we had chosen $+k^2$) we would have obtained exponential solutions instead of sines and cosines. As we will see, it will not be possible to satisfy the given boundary conditions if we assume the constant as positive.

The solutions for $X(x)$ and $T(t)$ are simple and we now write them down. These are

$$X(x) = A\cos kx + B\sin kx \tag{25}$$

$$T(t) = C\cos kc_st + D\sin kc_st \tag{26}$$

Let us now implement the boundary condition. If we want $\xi(0,t) = \xi(L,t) = 0$ then we must have $X(0) = X(L) = 0$. The condition $X(0) = 0$ gives $A = 0$ and the condition $X(L) = 0$ gives $\sin kL = 0$ (using $A = 0$). Therefore, we have $kL = n\pi$. We can write the solution for $T(t)$ in the familiar form

$$T(t) = F\cos(\omega t + \phi) \tag{27}$$

where $\phi$ and $F$ are related to $C$ and $D$. Also, $\omega = kc_s$. Putting everything together, we get

$$\xi_n(x,t) = F'\sin\frac{n\pi x}{L}\cos\left(\frac{n\pi c_st}{L} + \phi_n\right) \tag{28}$$

where $F'$ is a new constant related to $B$ and $F$. Thus, the various **modes** of vibration are to be found for $n = 1, 2, 3.....$ The $n = 1$ mode, in particular is called the **fundamental** and is given as

$$\xi_1(x,t) = F' \sin\frac{\pi x}{L} \cos\left(\frac{\pi c_s t}{L} + \phi_1\right) \tag{29}$$

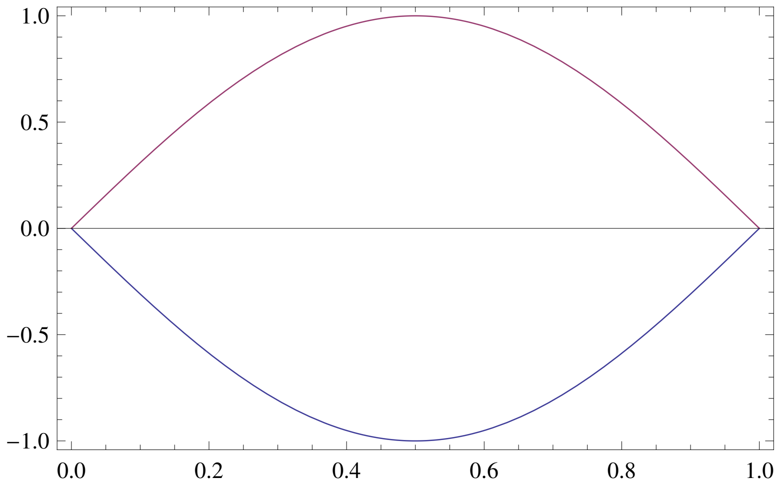


FIG. 6. Standing wave, fundamental mode with $L = 1$, $c_s = 1$,$\phi_1 = 0$, $t = 1$ (blue) and $t = 2$ (red)

Higher modes (from $n = 2$ onwards) are known as **harmonics**. The waves we have obtained **do not propagate** hence they are known as **standing waves**. The figure (Fig. 6) shows how the fundamental mode changes with time. In your experiment in the lab class on transverse waves on a string you will see such waves. However, do note that the exact analysis of the lab experiment on transverse waves will involve solving the wave equation for $\xi(x,t)$ with boundary conditions $\xi(0,t) = B\cos\omega t$ (vibrator end) and $\xi(L,t) = 0$ (pulley end). This is discussed in the book by A. P. French in Pg 168.

It is possible to write a standing wave as a superposition of two traveling waves of same frequency propagating along opposite directions.

Exercise: Show that the statement made just above is indeed correct.

Standing electromagnetic waves are created in many applications involving **waveguides and resonant cavities**. In a **laser** cavity too such waves are created.