Tutorial Sheet-2

AUTUMN 2018

MATHEMATICS-I (MA10001)

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1. Determine the following limits using L'Hospital rule, if exist:

   a) $\lim_{x\to 0} x\log x$

   b) $\lim_{x\to 0} x^x$

   c) $\lim_{x\to 1} x^{\frac{1}{x-1}}$

   d) $\lim_{x\to\infty} x^{1/x}$

   e) $\lim_{x\to 0} \frac{\tan^{-1}x}{\sin^{-1}x}$

   f) $\lim_{x\to 0}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)$

   g) $\lim_{x\to\pi} |\sin x|^{\tan x}$

   h) $\lim_{x\to 0} |\sin x|^x$

   i) $\lim_{x\to e} \frac{\log(\log x)}{\sin(x-e)}$

   j) $\lim_{x\to -\infty} e^{x^2}\sin(e^x)$

   k) $\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$

2. By Taylor series expansion, using suitable function

   a) find the value of $\sqrt{1.5}$ approximately.

   b) show that $\sin 46^\circ \approx \frac{1}{\sqrt{2}}\left(1+\frac{\pi}{180}\right)$.

3. If $f(x) = e^x$ then using Taylor's theorem, find the smallest interval in which value of $e^{0.1}$ belong. (Take $n = 2$.)

4. Use Taylor's theorem to prove that

   a) $\cos x \geq 1 - \frac{x^2}{2}$ for $-\pi < x < \pi$.

   b) $x - \frac{x^3}{6} < \sin x < x$ for $0 < x < \pi$.

   c) $1 + x + \frac{x^2}{2} + \frac{x^3}{3!} < e^x < 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}e^x$ for all $x > 0$.

5. Prove: If f is continuous at x_0 and there are constatnts a_0 and a_1 such that

$$\lim_{x\to x_0} \frac{f(x) - a_0 - a_1(x-x_0)}{x-x_0} = 0$$

   then $a_0 = f(x_0)$, f' is differentialble at x_0, and $f'(x_0) = a_1$.

6. Using Taylor's series formula, evaluate

a) $\lim_{x \to 0} \frac{\sin x}{\sqrt{1-\cos x}}$ b) $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

c) $\lim_{x \to 0} \frac{x - \log(1+x)}{1-\cos x}$ d) $\lim_{x \to 0} \frac{xe^x - \log(1+x)}{x^2}$.

7. Find the Maclaurin's infinite series expansion for

   a) $f(x) = \cos x$ for all $x \in \mathbb{R}$.

   b) $f(x) = \log(1+x)$ for $(-1, 1]$. $\frac{(-1)^{n} \lfloor n-1}{(1+x)^n}$ $x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots +$

   c) $f(x) = e^x \cos x$ for all $x \in \mathbb{R}$

8. Can the function f(x) defined by $f(x) = e^{1/x}$ for $x \neq 0$ and f(0)=0 be expanded in ascending powers of x by Maclaurin's Theorem?

9. Write the Maclaurin's formula for the function $f(x) = \sqrt[3]{1+x}$ of degree 2. Further estimate the error of the approximate equation $\sqrt[3]{1+x} \approx 1 + \frac{1}{3}x - \frac{1}{9}x^2$ when $x = 0.3$.

10. Using Maclaurin's Theorem expand $\frac{e^x}{1+e^x}$.

Problem Set - 2 (SOLUTION)

① Determine the following limits using L'Hospital Rul if exist:

a) $\lim_{x \to 0} x \log x$

$= \lim_{x \to 0} \frac{\log x}{1/x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$

Using L'-Hospital's rule.

$= \lim_{x \to 0} \frac{1/x}{-1/x^2}$

$= \lim_{x \to 0} (-x)$

$= 0$

) $\lim_{x \to 0} x^x$

let $y = \lim_{x \to 0} x^x$

aking 'log' both sides

$\log y = \lim_{x \to 0} x \log x$

$\log y = 0 \quad$ [From 1) a)]

$\because y = 1$

$\lim_{x \to 0} x^x = 1$

) $\lim\limits_{x \to 1} x^{\frac{1}{x-1}}$

let $y = \lim\limits_{x \to 1} x^{\frac{1}{x-1}}$

$\log y = \lim\limits_{x \to 1} \frac{1}{x-1} \cdot \log x$ $\left(\frac{0}{0} \text{ form}\right)$

Using L- Hospital's Rule.

$\log y = \lim\limits_{x \to 1} \frac{1/x}{1}$

$\therefore \log y = 1$

$\therefore y = e$

$\lim\limits_{x \to 1} x^{\frac{1}{x-1}} = e$

) $\lim\limits_{x \to \infty} x^{1/x}$

let $y = \lim\limits_{x \to \infty} x^{1/x}$

$\log y = \lim\limits_{x \to \infty} \frac{1}{x} \log x$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

Using L- Hospital's Rule

$\log y = \lim\limits_{x \to \infty} \frac{1/x}{1}$

$\log y = 0$

$y = 1$

$\lim x^{1/x} = 1$

$\lim$

x

$=$

$$\lim_{x\to 0} \frac{\tan^{-1}x}{\sin^{-1}x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{1/(1+x^2)}{1/\sqrt{1-x^2}}$$

$$= 1$$

f) $$\lim_{x\to 0} \left(\frac{1}{e^x-1} - \frac{1}{x} \right)$$

$$= \lim_{x\to 0} \frac{x - e^x + 1}{x(e^x-1)} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{1-e^x}{e^x - 1 + x e^x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{-e^x}{2e^x + x e^x} \qquad (**)$$

$$= -\frac{1}{2}$$

g) $$\lim_{x\to \pi} |\sin x|^{\tan x}$$

$$= \lim_{x\to\pi} \exp\left[\tan x \log(|\sin x|)\right]$$

$$= \exp\left[\lim_{x\to\pi} \tan x \log(|\sin x|)\right]$$

$$= \exp\left[\lim_{x\to\pi} \frac{\log|\sin x|}{\cot x}\right]$$

$$= \exp\left[\lim_{x\to\pi} \frac{\cot x}{-\operatorname{cosec}^2 x}\right] = \exp\left[\lim_{x\to\pi}\left(-\overset{\sin x\cdot\cos x}{\cancel{\tan x}}\right)\right] \qquad (**)$$

$$= \exp(0) = 1 \qquad 5$$

) $\lim_{x\to 0} |\sin x|^x$

$\because \quad |\sin x|^x = \exp[x \log(|\sin x|)]$

ind.

$$\lim_{x\to 0} x\log(|\sin x|) = \lim_{x\to 0} \frac{\log(|\sin x|)}{1/x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\cos x / \sin x}{-1/x^2}$$

$$= -\left(\lim_{x\to 0} \cos x\right)\left(\lim_{x\to 0} \frac{x^2}{\sin x}\right)$$

$$= -1 \cdot \lim_{x\to 0} \frac{2x}{\cos x}$$

$$= -1 \cdot 0$$

$$= 0$$

So $\lim_{x\to 0} |\sin x|^x = \exp(0) = 1$

i) $\lim_{x\to e} \frac{\log(\log x)}{\sin(x-e)}$

$$= \lim_{x\to e} \frac{1/x \log x}{\cos(x-e)}$$

$$= \frac{1}{e}$$

$$\lim_{x\to-\infty} e^{x^2}\sin(e^x)$$

$$= \lim_{x\to-\infty} \frac{\sin e^x}{e^{-x^2}} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to-\infty} \frac{e^x \cos(e^x)}{-2x\, e^{-x^2}}$$

$$= -\frac{1}{2}\left(\lim_{x\to-\infty} \cos(e^x)\right)\left(\lim_{x\to-\infty} \frac{e^{x^2+x}}{x}\right)$$

$$= -\frac{1}{2} \lim_{x\to-\infty} \frac{(2x+1)\, e^{x^2+x}}{1}$$

$$= \infty$$

$$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\cos x - 1 + x^2/2}{5x^4} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{-\sin x + x}{20x^3} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{-\cos x + 1}{60x^2} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\sin x}{120x}$$

$$= \frac{1}{120} \lim_{x\to 0} \frac{\sin x}{x}$$

$$= \frac{1}{120}$$

2) By taylor series expansion, using suitable functio

a) Find the value of $\sqrt{1.5}$ approximately.

Consider the function, $f(x) = \sqrt{1+x}$

$$f(0) = \sqrt{1}$$
$$= 1$$

Then $f'(x) = \frac{1}{2\sqrt{1+x}} \Rightarrow f'(0) = \frac{1}{2}$

$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-3/2} \Rightarrow f''(0) = -1/4$$

$\therefore$ $$f'''(x) = \frac{3}{8}(1+x)^{-5/2} \Rightarrow f'''(0) = \frac{3}{8}$$

By taylor's series expansion with Lagranges form of remainder after 4 terms, we get,

$$f(x) = \sqrt{1+x} = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + R_3(x)$$

where, $R_3(x) = \frac{x^4}{\lfloor 4} f^{iv}(\theta x)$

$$f(x) = 1 + \frac{1}{2}x - \frac{x^2}{8} + \frac{x^3}{16} + R_3(x)$$

$\therefore$ We can approximate $\sqrt{1+x}$, by

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$$

at $x = 0.5$

$$\sqrt{1.5} = 1 + \frac{0.5}{2} - \frac{(0.5)^2}{8} + \frac{(0.5)^3}{16} = 1.22656$$

$\therefore \sqrt{1.5} \approx 1.22656$

Now, from the remainder term $R_3(x)$ at $x = 0.5$.

$$|R_3(x)| = \left| \frac{x^4}{\lfloor 4} \left(-\frac{15}{16}\right) \frac{1}{(1+\theta x)^{7/2}} \right|$$

$$= \frac{15}{16 \times 24} \left| x^4 \frac{1}{(1+\theta x)^{7/2}} \right|$$

at $x = 0.5$

$$|R_3(0.5)| = \frac{15}{16 \times 24} \cdot \frac{1}{16} \left| \frac{1}{(1+0.5\times\theta)^{7/2}} \right|$$

as $\left| \frac{1}{1+\theta \times 0.05} \right| < 1$

$0 < \theta < 1$

$$\Rightarrow |R_3(0.5)| \leq \frac{15}{16 \times 16 \times 24}$$

$$= 0.00366$$

$\therefore |R_3(0.5)| < 0.5 \times 10^{-2}$.

The value of $\sqrt{1.5}$ is correct upto two decimal places.

$\sqrt{1.5} \approx 1.22556$.

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1b) By taylor series expansion, using suitable funct show that $\sin 46^\circ \approx \frac{1}{\sqrt{2}}\left(1+\frac{\pi}{180}\right)$

let $f(x) = \sin x$, $f(45^\circ) = 1/\sqrt{2}$

$f'(x) = \cos x$, $f'(45^\circ) = 1/\sqrt{2}$

$f''(x) = -\sin x$

By taylor's Theorem, we get

$\sin x = f(45^\circ) + (x-45^\circ) f'(45^\circ) + R_2(45^\circ)$

$\sin x \approx \frac{1}{\sqrt{2}} + (x-45^\circ)\frac{1}{\sqrt{2}}$

$\therefore \sin 46^\circ \approx \frac{1}{\sqrt{2}} + (46^\circ - 45^\circ)\frac{1}{\sqrt{2}}$

$\sin 46^\circ \approx \frac{1}{\sqrt{2}}(1+1^\circ)$

$\sin 46^\circ \approx \frac{1}{\sqrt{2}}\left(1+\frac{\pi}{180}\right)$

$\because f(x) = e^x$

$\therefore f'''(x) = e^x$

By taylors theorem.

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{e^c x^3}{\lfloor 3}$$

where c is between 0 and x.

Hence

$$e^{0.1} = 1.105 + e^c \frac{(0.1)^3}{6}$$

where $0 < c < 0.1$. Since $0 < e^c < e^{0.1}$,

$$1.105 < e^{0.1} < 1.105 + \frac{e^{0.1}(0.1)^3}{6}$$

The second inequality implies that

$$e^{0.1}\left[1 - \frac{(0.1)^3}{6}\right] < 1.105$$

$$e^{0.1} < 1.1052$$

Therefore $1.105 < e^{0.1} < 1.1052$

4) a) $\cos x \geq 1 - \frac{x^2}{2}$ for $-\pi < x < \pi$

let $x \in (0, \pi)$

By taylor's theorem with Lagranges form of Remainde we have,

$\cos x = 1 - \frac{x^2}{2} + \frac{x^3}{6} \sin \theta x$, $0 < \theta x < x < 1$

for $x \in (0, \pi)$, $\sin \theta x > 0$

$\therefore \frac{x^3}{6} \sin \theta x > 0$

$\therefore \cos x > 1 - \frac{x^2}{2}$, for $0 < x < \pi$

Now, for $x \in (-\pi, 0]$

$\cos x = 1 - \frac{x^2}{2} + \frac{x^3}{6} \sin \theta x$, $-\pi < x < \theta x \leq 0$

for $x \in (-\pi, 0)$; $\sin \theta x \leq 0$

$\therefore \frac{x^3}{6} \sin \theta x \geq 0$

$\therefore \cos x \geq 1 - x^2/2$, $-\pi < x \leq 0$

$\cos x \geq 1 - \frac{x^2}{2}$ $\forall$ $-\pi < x < \pi$

b)

By taylor's theorem with Lagranges form of remainder,

$$\sin x = x - \frac{x^3}{6} + \frac{x^4}{\lfloor 4} \sin\theta x \quad ; \quad 0 < \theta x < x < \pi$$
$$\forall \; x \in (0, \pi)$$

Now, since $\forall \; x \in (0, \pi)$; $\frac{x^4}{\lfloor 4} \sin\theta x > 0$

$$\therefore \quad \sin x > x - \frac{x^3}{6} \quad \text{—— (1)}$$

Again taylors theorem with Lagranges form of remainder gives

$$\sin x = x + \frac{x^2}{\lfloor 2}(-\sin\theta x) \quad ; \quad 0 < \theta x < x < \pi$$
$$x \in (0, \pi)$$

$$\therefore \quad \sin x = x - \frac{x^2}{\lfloor 2} \sin\theta x$$

Now $\forall \; x \in (0, \pi)$, $\frac{x^2}{2} \sin\theta x > 0$

$$\therefore \quad \sin x < x \quad \text{—— (2)}$$

Combining eq^n (1) & (2), we get :

$$x - \frac{x^3}{6} < \sin x < x \quad \text{for} \quad 0 < x < \pi$$

c)

let $f(x) = e^x$

Then, $f(x) = e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} f'''(\theta x)$

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} e^{\theta x} \; ; \quad 0 < \theta < 1$$

Now, $e^0 < e^{\theta x} < e^x \quad \forall \; 0 < \theta < 1$

$\& \; 0 < x < 1$

i.e. $1 < e^{\theta x} < e^x$

$$\therefore \; 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} < 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} e^{\theta x} < 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} e^x$$

i.e. $1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} < e^x < 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} e^x$

The hypotheses implies that

(A) $f(x) = a_0 + a_1(x - x_0) + E(x)(x - x_0)$

where (B) $\lim_{x \to x_0} E(x) = 0$

Therefore $\lim_{x \to x_0} f(x) = a_0$

so $a_0 = f(x_0)$ because f is continuous at x_0.

Now (A) and (B) implies that

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = a_1$$

6) Using taylor's series formula, Evaluate,

a) $\lim\limits_{x\to 0} \dfrac{\sin x}{\sqrt{1-\cos x}}$

$= \lim\limits_{x\to 0} \dfrac{\left[x - x^3/\lfloor 3 + x^5/\lfloor 5 - \cdots\right]}{\sqrt{1 - 1 + \dfrac{x^2}{\lfloor 2} - \dfrac{x^4}{\lfloor 4} + \cdots}}$

$= \lim\limits_{x\to 0} \dfrac{\cancel{x}\left(1 - x^2/\lfloor 3 + x^4/\lfloor 5 - \cdots\right)}{\cancel{x}\sqrt{\dfrac{1}{\lfloor 2} - \dfrac{x^2}{\lfloor 4} + \cdots}}$

$= \dfrac{1-0}{\sqrt{\frac{1}{2} - 0}}$

$= \sqrt{2}$

b) $\lim\limits_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$

$= \lim\limits_{x\to 0} \left(\dfrac{\sin x - x}{x \sin x}\right)$

$= \lim\limits_{x\to 0} \dfrac{\left(x - \dfrac{x^3}{\lfloor 3} + \dfrac{x^5}{\lfloor 5} - \cdots\right) - x}{x\left(x - \dfrac{x^3}{\lfloor 3} + \dfrac{x^5}{\lfloor 5} - \cdots\right)}$

$= \lim\limits_{x\to 0} \dfrac{x^3\left(-\dfrac{1}{\lfloor 3} + \dfrac{x^2}{\lfloor 5} - \cdots\right)}{x^2\left(1 - \dfrac{x^2}{\lfloor 3} + \dfrac{x^4}{\lfloor 5} - \cdots\right)}$

$= \lim\limits_{x\to 0} \dfrac{x \cdot \left(-\dfrac{1}{\lfloor 3} + \dfrac{x^2}{\lfloor 5} - \cdots\right)}{\left(1 - \dfrac{x^2}{\lfloor 3} + \dfrac{x^4}{\lfloor 5} - \cdots\right)} = 0.$

) $\lim\limits_{x\to 0} \dfrac{x-\log(1+x)}{1-\cos x}$

$= \lim\limits_{x\to 0} \dfrac{x-\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right)}{1-\left(1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}-\cdots\right)}$

$= \lim\limits_{x\to 0} \dfrac{\frac{x^2}{\lfloor 2}-\frac{x^3}{\lfloor 3}+\cdots}{\frac{x^2}{\lfloor 2}-\frac{x^4}{\lfloor 4}+\cdots}$ (* *)

$= \lim\limits_{x\to 0} \dfrac{\frac{1}{2}-\frac{x}{3}+\cdots}{\frac{1}{\lfloor 2}-\frac{x^2}{\lfloor 4}+\cdots}$

$= \dfrac{\frac{1}{2}-0}{\frac{1}{2}-0}$

$=1$

d) $\lim\limits_{x\to 0} \dfrac{xe^x-\log(1+x)}{x^2}$

$= \lim\limits_{x\to 0} \dfrac{x\left(1+x+\frac{x^2}{\lfloor 2}+\cdots\right)-\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right)}{x^2}$

$= \lim\limits_{x\to 0} \dfrac{\left(x^2+\frac{x^3}{\lfloor 2}+\cdots\right)-\left(-\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right)}{x^2}$

$= \lim\limits_{x\to 0} \left(1+\frac{x}{\lfloor 2}+\cdots\right)-\left(-\frac{1}{2}+\frac{x}{3}+\cdots\right)$

$= 1-\left(-\frac{1}{2}\right)$

$= \dfrac{3}{2}$

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7) Find the Maclaurine's infinite series expansion fo

a) $\because f(x) = \cos x \quad \forall x \in \mathbb{R}$

$\therefore f^n(x) = \cos\left(\frac{n\pi}{2} + x\right) \; ; \; \forall n \geq 1$

By Maclaurin's theorem with Lagranges form of remainder after n terms, for any non-zero $x \in \mathbb{R}$.

$$f(x) = f(0) + x f'(0) + \cdots + \frac{x^{n-1}}{\lfloor n-1} f^{n-1}(0) + R_n$$

where, $R_n = \frac{x^n}{\lfloor n} f^n(\theta x) \; ; \; 0 < \theta < 1$

$$\therefore f(x) = \cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \cdots + \frac{x^n}{\lfloor n} \cos\left(\frac{n\pi}{2} + \theta x\right) \quad \text{--- (1)}$$

Since $\forall, n \in N$, $f^n(x)$ exist for all real x, the right hand side polynomial in (1) takes the form of infinite series as $n \to \infty$. The infinite series will converges to $f(x)$ for those non zero real x for which $\lim_{n\to\infty} R_n = 0$.

Now, $|R_n| = \left|\frac{x^n}{\lfloor n} \cos\left(\frac{n\pi}{2} + \theta x\right)\right| \leq \left|\frac{x^n}{\lfloor n}\right|$

let $u_n = \frac{|x|^n}{\lfloor n} \; ; \; x \neq 0$

Then $\lim_{n\to\infty} u_n = 0 \quad \forall$ real x.

$\therefore \lim_{n\to\infty} |R_n| = 0 \quad \forall x \neq 0$

Consequently, the infinite series $1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \cdots$ Converges to '$\cos x$' $\forall x \in \mathbb{R} \; ; \; x \neq 0$

$x = 0$, the convergence hold trivially.

$\therefore \cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \dots \quad \forall \ x \in \mathbb{R}$

b) $f(x) = \log(x+1) \quad \forall \ x \in (-1, 1]$

$\therefore f^n(x) = \frac{(-1)^{n-1} \lfloor n-1}{(1+x)^n} \quad \forall \ x \in (-1, 1] \ \& \ n \geq 1$

By Maclaurin's theorem,

$\therefore f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \dots + \frac{x^{n-1}}{\lfloor n-1} f^{n-1}(0) + R_n(x)$

where $R_n(x) = \frac{x^n}{\lfloor n} f^n(\theta x)$, $0 < \theta < 1$

$\therefore \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + (-1)^{n-2} \frac{x^{n-1}}{n-1} + R_n$

where $R_n = \frac{(-1)^{n-1}}{n} \frac{x^n}{(1+\theta x)^n}$

Since, $\forall \ n \in N$, $f^n(x)$ exists, the right hand polynomial takes the form of an infinite series and it converges to $f(x)$ if $\lim_{n \to \infty} R_n = 0$ as $n \to \infty$

$|R_n| = \frac{1}{n} \left| \frac{x}{1+\theta x} \right| \qquad x \in [0, 1]$

Now, for $x \in (-1, 1]$ and $0 < \theta < 1$, i.e. $0 < 1 - \theta < 1$

$1 > x(1-\theta)$

i.e. $1 + \theta x > x$

$\frac{x}{1+\theta x} < 1$

i.e. $0 < \left| \frac{x}{1+\theta x} \right| < 1$

Case-2 $x \in (-1, 0]$. $0 < \theta < 1$. $x < \theta x < 0$.

$\cancel{x} < 1 + \cancel{x} < 1 + \theta x < 1$

$\therefore \left| \frac{x}{1+\theta x} \right| < 1$

Then

$\lim_{n \to \infty} \left(\left| \frac{x}{1+\theta x} \right| \right)^n = 0$

and $\lim_{n \to \infty} \frac{1}{n} = 0$

$$\therefore \lim_{n\to\infty} R_n = 0$$

Thus the infinite series $x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ Converges to $\log(1+x)$ $\forall\, x \in [-1, 1]$

c) $f(x) = e^x \cos x \quad \forall\, x \in \mathbb{R}$

$$f(x) = \left(1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots\right)\left(1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \cdots\right)$$

$$= 1 + x + \left(\tfrac{1}{2} - \tfrac{1}{2}\right)x^2 + \left(\tfrac{1}{6} - \tfrac{1}{2}\right)x^3 + \left(\tfrac{2}{\lfloor 4} - \tfrac{1}{\cancel{2}4}\right)x^4 + \cdots$$

$$f(x) = 1 + x - \frac{2}{\lfloor 3}x^3 - \frac{2^2}{\lfloor 4}x^4 - \cdots$$

since, $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots$

and $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \cdots$ both are convergent series and $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots$ is absolutely Convergent series $\forall\, x \in \mathbb{R}$

$\therefore$ The product series is also Convergent and converges to $e^x \cos x$.

$$e^x \cos x = 1 - x - \frac{2}{\lfloor 3}x^3 - \frac{2^2}{\lfloor 4}x^4 - \cdots$$

$\lim_{n\to\infty} \frac{u_n v_n}{u_n} = \lim_{n\to\infty} v_n = 0$. if $\lim_{n\to\infty} u_n$ - conv. then $u_n v_n$ is conv.

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} e^{1/x}$$

$$= \lim_{x \to 0} \left[1 + \frac{1}{x} + \frac{1}{2x^2} + \cdots \right] = \infty, \text{ i.e. Not finite}$$

Hence the given function is discontinuous at $x = 0$.

Hence the function $f(x)$ cannot expand in ascending power of x by Maclaurin's Theorem.

9) $f(x) = \sqrt[3]{1+x}$, $f(0) = 1$

$f'(x) = \frac{1}{3}\left(\sqrt[3]{1+x}\right)^{-2}$, $f'(0) = \frac{-1}{3}$

$f''(x) = -\frac{2}{9}(1+x)^{-5/3}$, $f''(0) = -\frac{2}{9}$

$f'''(x) = \frac{10}{27}(1+x)^{-8/3}$

The required maclaurins polynomial of degree 2 is

$f(x) = 1 + \frac{x}{3} - \frac{2}{9}\frac{x^2}{2!} + R(x)$

$= 1 + \frac{x}{3} - \frac{x^2}{9} + R(x)$

Where, $R(x) = \frac{x^3}{3!} f'''(\theta x)$

$= \frac{x^3}{3!} \frac{10}{27} (1+\theta x)^{-8/3}$, $0 < \theta < 1$

The error of the approximate equation will be

$\left| \sqrt[3]{1+x} - \left(1 + \frac{2}{3}x - \frac{1}{9}x^2\right) \right| = |R(x)|$

$= \left| \frac{x^3}{3!} \times \frac{10}{27} \times \frac{1}{(1+\theta x)^{8/3}} \right|$

When $x = 0.3$, the estimated error is

$\left| \frac{(0.3)^3}{3!} \times \frac{10}{27} \times \frac{1}{(1+\theta \times 0.3)^{8/3}} \right|$

$< (0.3)^3 \times \frac{5}{81} = \frac{5}{3} \times 10^{-3}$

$< 5 \times 10^{-3}$

The approximation at 0.3 is accurate upto 3 decimal places.

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) let $y = \frac{e^x}{1+e^x} = 1 - \frac{1}{1+e^x}$, so that $y(0) = \frac{1}{2}$

$$y' = \frac{e^x}{(1+e^x)^2} = \frac{e^x}{1+e^x} \frac{1}{1+e^x} = y(1-y), \quad y'(0) = \frac{1}{4}$$

$$y'' = y' - 2yy', \quad y''(0) = 0$$

$$y''' = y'' - 2y'^2 - 2yy'', \quad y'''(0) = -\frac{1}{8}$$

Hence Maclaurin's Theorem yields,

$$\frac{e^x}{1+e^x} = y = y(0) + x y'(0) + \frac{x^2}{\lfloor 2} y''(0) + \frac{x^3}{\lfloor 3} y'''(0) + \cdots$$

$$= \frac{1}{2} + \frac{1}{4}x - \frac{x^3}{48} + \cdots$$