Indian Institute of Technology Kharagpur
Department of Mathematics
MA11003 - Advanced Calculus
Tutorial Problems Sheet - 1
Autumn 2022

---

1. Using the Intermediate Value Theorem and the Rolle's Theorem, show that the polynomial $2x^3 + 5x - 9$ has exactly one real root.

2. Verify which of the following functions satisfy the conditions of the LMVT.

   (a) $f(x) = |x - 1|$ in $[0, 2]$.

   (b) $f(x) = 1 + x^{\frac{2}{3}}$ in $[-8, 8]$.

   (c) $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ in $\left[-\frac{2}{\pi}, \frac{2}{\pi}\right]$.

3. Calculate $\xi \in (a, b)$ in Cauchy's MVT for each of the following pairs:

   (a) $f(x) = \sin x,\ g(x) = \cos x$ on $[\frac{\pi}{4}, \frac{3\pi}{4}]$.

   (b) $f(x) = (1 + x)^{\frac{3}{2}}$, $g(x) = \sqrt{1 + x}$ on $[0, \frac{1}{2}]$.

4. Show that the formula in the Lagrange's MVT can be written as follows:

$$\frac{f(b) - f(a)}{b - a} = f'(a + \theta(b - a))$$

   where $0 < \theta < 1$.

   Substitute $a = x$ and $b = x + h$. Then $b - a = h$. Determine θ as a function of x and h for the following functions.
   (a)$f(x) = x^2$ (b)$f(x) = e^x$ (c)$f(x) = \log x$, $x > 0$.
   Keep $x \neq 0$ fixed, and find $\lim_{h \to 0} \theta$ in each case.

5. (a) Suppose, $f(x)$ is continuous on $[1, 2]$ and differentiable in $(1, 2)$ such that $f(2) = -5$ and $|f'(x)| \leq 2$. Then, what is the largest possible value of $f(1)$.

   (b) Use Lagrange's MVT to estimate $\sqrt[3]{28}$.

   (c) If $f''(x) \geq 0$ on $[a, b]$ prove that $f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{2}\left[f(x_1) + f(x_2)\right]$ for any two points x_1 and x_2 in $[a, b]$.

6. Prove that

   (a) $\frac{2x}{\pi} < \sin x < x$ for $0 < x < \frac{\pi}{2}$.

   (b) $na^{n-1}(b - a) < b^n - a^n < nb^{n-1}(b - a)$ where $0 < a < b$ and $n > 1$.

   (c) $\frac{x}{1+x} < \log(1 + x) < x$ for all $x > 0$.

7. (a) Assume f is continuous on $[a, b]$ and has a finite second derivative f'' in the open interval (a, b). Assume that the line segment joining the points $A = (a, f(a))$ and $B = (b, f(b))$ intersects the graph of f in a third point P different from A and B. Prove that $f''(\xi) = 0$ for some ξ in (a, b).

   (b) If f is differentiable on $[0, 1]$ show by Cauchy's MVT that the equation $f(1) - f(0) = \frac{f'(x)}{2x}$ has atleast one solution in $(0, 1)$.

   (c) Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = a$ and $f(b) = b$, show that there exist distinct c_1 and c_2 in (a, b) such that $f'(c_1) + f'(c_2) = 2$.

8. (a) If $f(x)$ and $\phi(x)$ are continuous on $[a, b]$ and differentiable on (a, b), then show that

$$\begin{vmatrix} f(a) & f(b) \\ \phi(a) & \phi(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(b) & f'(c) \\ \phi(b) & \phi'(c) \end{vmatrix}, a < c < b.$$

   (b) Let f be continuous on $[a, b]$ and differentiable on(a, b). Using Cauchy's MVT, show that if $a \geq 0$, then there exist $x_1, x_2, x_3 \in (a, b)$ such that

$$f'(x_1) = (b+a)\frac{f'(x_2)}{2x_2} = (b^2 + ba + a^2)\frac{f'(x_3)}{3x_3^2}.$$

9. Use CMVT to prove the following:

   (a) Show that $1 - \frac{x^2}{2!} < \cos x$ for $x \neq 0$.

   (b) Let f be continuous on $[a, b]$, $a > 0$ and differentiable on (a, b). Prove that there exist $c \in (a, b)$ such that $\frac{b^2 f(a) - a^2 f(b)}{b^2 - a^2} = \frac{1}{2}[2cf(c) - c^2 f'(c)]$.

   (c) Show that $\frac{2\ln x}{2\arcsin x - \pi} < \frac{\sqrt{1-x^2}}{x}$ for $0 < x < 1$.

10. A twice differentiable function $f(x)$ on a closed interval $[a, b]$ is such that $f(a) = f(b) = 0$ and $f(x_0) > 0$ where $a < x_0 < b$. Prove that there exists at least one value of $x = c$ between a and b for which $f''(c) < 0$

****