Indian Institute of Technology Kharagpur
Department of Mathematics
MA11003 - Advanced Calculus
Tutorial Problem Sheet - 3
Autumn 2022

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1. Determine the limits as $(x, y) \to (0, 0)$ of the following functions, if they exist.

(a) $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

(b) $f(x,y) = \begin{cases} \log\left(\dfrac{y}{x}\right), & xy \neq 0; \\ 0, & xy = 0. \end{cases}$

(c) $f(x,y) = \begin{cases} \dfrac{|x|}{y^2} \exp\left(-\dfrac{|x|}{y^2}\right), & y \neq 0; \\ 0, & y = 0. \end{cases}$

(d) $f(x,y) = \begin{cases} \dfrac{x^2+y^2}{\tan(xy)}, & xy \neq 0; \\ 0, & xy = 0. \end{cases}$

(e) $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

(f) $f(x,y) = \begin{cases} \log\left(\dfrac{\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2}-x}\right), & y \neq 0; \\ 0, & y = 0. \end{cases}$

(g) $f(x,y) = \begin{cases} \sin\left(\dfrac{x}{y}\right) + \sin\left(\dfrac{y}{x}\right), & xy \neq 0; \\ 0, & xy = 0. \end{cases}$

(h) $f(x,y) = \cos^3(\sqrt{x^2+y^2})$.

(i) $f(x,y) = \begin{cases} \dfrac{\sin(x^2y+xy^2)}{xy}, & xy \neq 0; \\ 0, & xy = 0. \end{cases}$

(j) $f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

(k) $f(x,y,z) = \begin{cases} \dfrac{xy^2z^2}{x^4+y^4+z^8}, & (x,y,z) \neq (0,0,0); \\ 0, & (x,y,z) = (0,0,0). \end{cases}$

2. Using ϵ - δ method, prove the following:

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy^2}{y^2+x^2} = 0,$

(b) $\displaystyle\lim_{(x,y)\to(-1,-1)} (xy - 2x^2) = -1,$

(c) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{(x-1)^2 \ln x}{y^2+(x-1)^2} = 0,$

(d) $\displaystyle\lim_{(x,y)\to(-2,2)} \frac{x^2-y^2}{y+x} = -4,$

(e) $\displaystyle\lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{y^2+x^2} = 0,$

(f) $\displaystyle\lim_{(x,y)\to(0,0)} x\sin x\cos y = 0,$

(g) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{y^2+x^2}} = 0,$

(h) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{y^2+x^2} = 0,$

(i) $\displaystyle\lim_{(x,y)\to(1,1)} (x^2+y^2-1) = 1,$

(j) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y-3x^2y^3+y^5}{(x^2+y^2)^2} = 0,$

(k) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = 0.$

(l) $\displaystyle\lim_{(x,y)\to(0,0)} \left[y\sin\left(\frac{x}{y}\right) + x\sin\left(\frac{y}{x}\right)\right] = 0.$

3. Using ϵ - δ method, show that the following functions are continuous.

(a) $f(x,y)=\begin{cases} xy, & (x,y)\neq(2,3); \\ 6, & (x,y)=(2,3). \end{cases}$

(b) $f(x,y)=\begin{cases} \dfrac{5x^2y^2}{x^2+y^2}, & (x,y)\neq(0,0); \\ 0, & (x,y)=(0,0). \end{cases}$

(c) $f(x,y)=\begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0); \\ 0, & (x,y)=(0,0). \end{cases}$

(d) $f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0); \\ 0, & (x,y)=(0,0). \end{cases}$

4. Discuss the continuity of the following functions at $(0,0)$.

(a) $f(x,y)=\begin{cases} \dfrac{1}{x^2+y^2}, & (x,y)\neq(0,0); \\ 0, & (x,y)=(0,0). \end{cases}$

(b) $f(x,y)=\begin{cases} \dfrac{x^3y^3}{x^2+y^2}, & (x,y)\neq(0,0); \\ 0, & (x,y)=(0,0). \end{cases}$

(c) $f(x,y)=\begin{cases} \dfrac{|xy|}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0); \\ 0, & (x,y)=(0,0). \end{cases}$

(d) $f(x,y)=\begin{cases} \dfrac{|xy|}{xy}, & xy\neq 0; \\ 0, & xy=0. \end{cases}$

(e) $f(x,y)=\begin{cases} \dfrac{e^{xy}}{x^2+y^2}, & (x,y)\neq(0,0); \\ 0, & (x,y)=(0,0). \end{cases}$

(f) $f(x,y)=\begin{cases} \dfrac{3x^2y-y^3}{x^2+y^2}, & (x,y)\neq(0,0); \\ 0, & (x,y)=(0,0). \end{cases}$

(g) $f(x,y)=\begin{cases} \dfrac{x^3}{x^2+y^2}, & (x,y)\neq(0,0); \\ 0, & (x,y)=(0,0). \end{cases}$

(h) $f(x,y)=\begin{cases} \dfrac{\sin xy}{xy}, & xy\neq 0; \\ 1, & xy=0. \end{cases}$

(i) $f(x,y)=\begin{cases} x\sin\dfrac{1}{y}+y\sin\dfrac{1}{x}, & xy\neq 0; \\ 0, & xy=0. \end{cases}$

5. For what values of n, the following function f is continuous at $(0,0)$:

$$f(x,y)=\begin{cases} \dfrac{2xy}{(x^2+y^2)^n}, & (x,y)\neq(0,0); \\ 0, & (x,y)=(0,0). \end{cases}$$

6. Find the values of c for which the following functions are continuous at $(0,0)$.

(a) $f(x,y)=\begin{cases} \dfrac{x^4-y^4}{x^2+y^2}, & (x,y)\neq(0,0); \\ c, & (x,y)=(0,0). \end{cases}$

(b) $f(x,y)=\begin{cases} x^2\log(x^2+y^2), & (x,y)\neq(0,0); \\ c, & (x,y)=(0,0). \end{cases}$

(c) $f(x,y)=\begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2}, & (x,y)\neq(0,0); \\ c, & (x,y)=(0,0). \end{cases}$

(d) $f(x,y)=\begin{cases} \dfrac{x^3+y^3}{x^2+y^2}, & (x,y)\neq(0,0); \\ c, & (x,y)=(0,0). \end{cases}$

(e) $f(x,y)=\begin{cases} \dfrac{(x-1)^2\ln x}{(x-1)^2+y^2}, & (x,y)\neq(1,0); \\ c, & (x,y)=(1,0). \end{cases}$

(f) $f(x,y)=\begin{cases} \dfrac{e^{-(x^2+y^2)}-1}{x^2+y^2}, & (x,y)\neq(0,0); \\ c, & (x,y)=(0,0). \end{cases}$

(g) $f(x,y)=\begin{cases} \exp\left(-\dfrac{1}{|x-y|}\right), & x\neq y; \\ c, & x=y. \end{cases}$

7. Do the following functions have any point of discontinuity? Explain!

(a) $f(x,y) = \dfrac{x-y}{1+x+y}$, (b) $f(x,y) = \dfrac{x-y}{1+x^2+y^2}$.

8. Find the points of discontinuities of the following functions.

(a) $f(x,y) = \dfrac{1}{\sin^2 \pi x + \sin^2 \pi y}$, (b) $f(x,y) = \dfrac{1}{\sin \pi x} + \dfrac{1}{\sin \pi y}$.

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