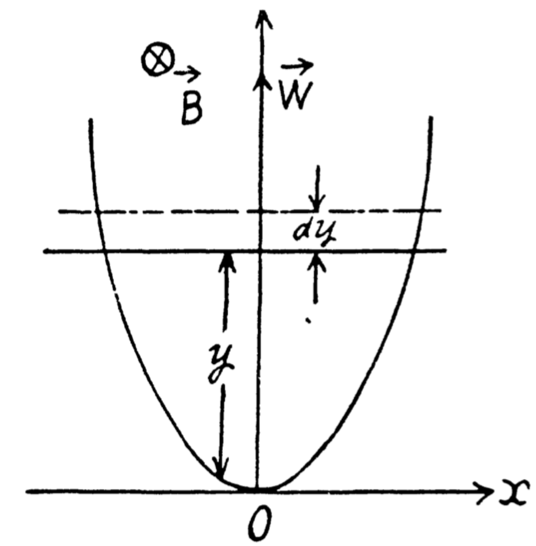
Solutions for Tutorial #5

1. A wire bent as a parabola $y = ax^2$ is located in a uniform magnetic field B oriented perpendicular to the plane $x-y$. At the moment $t=0$ a connector parallel to x axis starts sliding from the apex of the parabola with a constant acceleration w, see Figure 1. Find the emf of the electromagnetic induction in the loop as a function of y.

According to Lenz's law induced current & induced emf will be anticlockwise in the loop



We can obtain emf from Faraday's law $\epsilon_0 = |\frac{d\phi}{dt}| - (1)$

$$where,\ d\phi = \vec{B}.d\vec{S} - (2)$$

As the connector slides parallel to x-axis, area of the parabola For small displacement dy is 2xdy

$$Hence,\ d\phi = 2Bxdy - (3)$$

Wire is bent like a parabola $y = ax^2$ $Therefore,\ x = \sqrt{\frac{y}{a}} - (4)$

Also as the connector moves with constant acceleration, $y = \frac{1}{2}wt^2 - (5)$ $So, t = \sqrt{\frac{2y}{w}} - (6)$

$$From\ (5),\ \frac{dy}{dt} = wt = \sqrt{2yw} - (7)$$

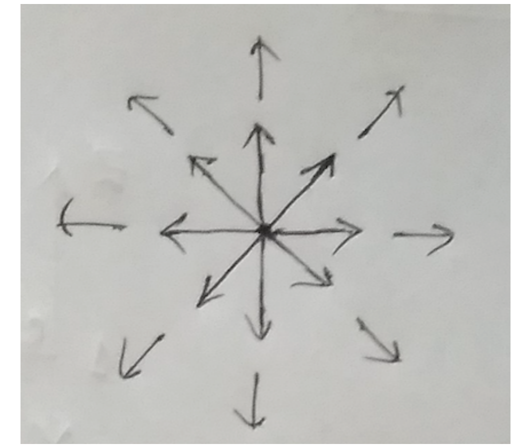
$$Hence\ From\ (1),\ \epsilon_0 = |\frac{d\phi}{dt}| = 2B\sqrt{\frac{y}{a}}\frac{dy}{dt}$$ [Using (3) & (4)]

$$\epsilon_0 = By\sqrt{\frac{8w}{a}}$$ [Using (7)]

2. Plot the vector field $\frac{\hat{r}}{r^2}$ and compute the divergence

$$\vec{\nabla}.\frac{\hat{r}}{r^2}$$

where $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$. Explain the result. Is this calculation valid everywhere?



Spherically symmetric with decreasing magnitude

$$Let,\ V = \frac{\hat{r}}{r^2}$$

$$\vec{\nabla}\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{rsin\theta}\frac{\partial}{\partial\theta}(sin\theta V_\theta) + \frac{1}{rsin\theta}\frac{\partial V_\phi}{\partial\phi}$$

$$\vec{\nabla}\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{1}{r^2}) = 0$$

So $\vec{\nabla}\cdot\vec{V}$ will be zero everywhere except at the origin r=0

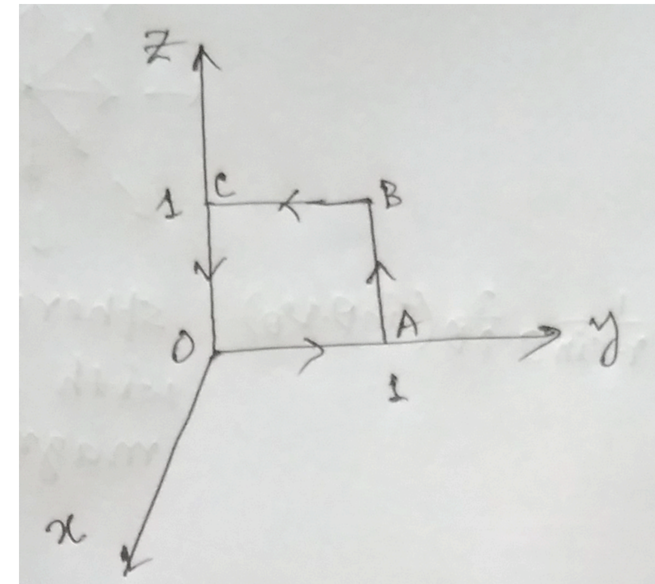
At origin consider a sphere of radius R

$$\int_V (\vec{\nabla}\cdot\vec{V})dV = \oint_s \vec{V}\cdot d\vec{S} = \int (\frac{1}{R^2}\hat{r})(R^2 sin\theta d\theta d\phi)\hat{r}$$

$$\Rightarrow \int_V (\vec{\nabla}\cdot\vec{V})dV = \int_0^\pi sin\theta d\theta \int_0^{2\pi} d\phi = 4\pi$$

The integral value of $\vec{\nabla}\cdot\vec{V}$ over a volume containing origin gives a non-zero value. So it behaves like a Dirac Delta function.

3. Check Stokes theorem for the vector field $\vec{W}(x, y, z) = (2xz + 3y^2)\hat{j} + (4yz^2)\hat{k}$ over a square surface of unit length in the $y - z$ plane positive quadrant with one edge at the origin.



$$\vec{W}(x, y, z) = (2xz + 3y^2)\hat{j} + 4yz^2\hat{k}$$

$$d\vec{S} = dydz \cdot \hat{i}$$

$$\vec{\nabla} \times \vec{W} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 2xz + 3y^2 & 4yz^2 \end{vmatrix}$$

$$\Rightarrow \vec{\nabla} \times \vec{W} = (4z^2 - 2x)\hat{i} + 2z\hat{z}$$

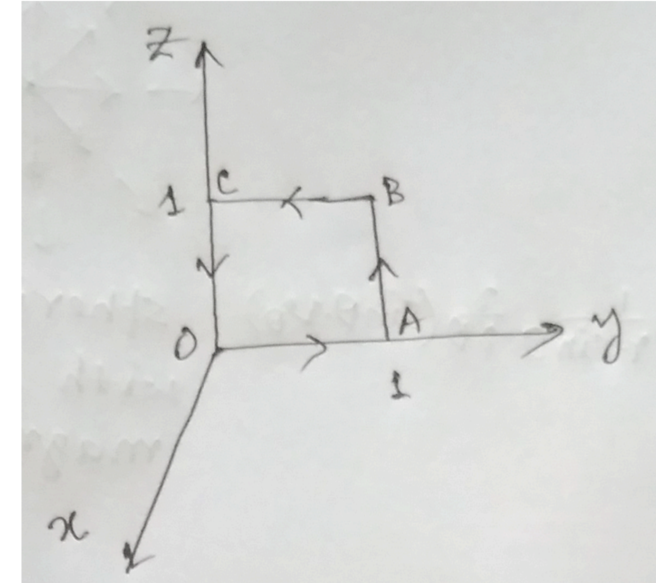
$$Therefore, \int_S (\vec{\nabla} \times \vec{W}) \cdot d\vec{S} = \int (4z^2 - 2x)dydz$$

$$\Rightarrow \int_S (\vec{\nabla} \times \vec{W}) \cdot d\vec{S} = \int_0^1 dy \int_0^1 4z^2 dz = \frac{4}{3}$$ Since [x =0]

Question 3 continued ..

Line integral,

$$d\vec{l} = dx \cdot \hat{i} + dy \cdot \hat{j} + dz \cdot \hat{k}$$



$$\oint_{OABC} \vec{W} \cdot \vec{dl} = \int_{OA} \vec{W} \cdot \vec{dl} + \int_{AB} \vec{W} \cdot \vec{dl} + \int_{BC} \vec{W} \cdot \vec{dl} + \int_{CO} \vec{W} \cdot \vec{dl}$$

$$\Rightarrow \oint_{OABC} \vec{W} \cdot \vec{dl} = \int_0^1 3y^2 dy + \int_0^1 4z^2 dz + \int_1^0 3y^2 dy + \int_1^0 0dz$$

$$\Rightarrow \oint_{OABC} \vec{W} \cdot \vec{dl} = 1 + \frac{4}{3} - 1 + 0 = \frac{4}{3}$$

$$Therefore, \ \oint_{OABC} \vec{W} \cdot \vec{dl} = \int_S (\vec{\nabla} \times \vec{W}) \cdot d\vec{S}$$

So Stokes theorem is satisfied

4. Given the electric flux law:

$$\Phi_E = \oint \vec{E}.d\vec{A} = \frac{q}{\epsilon_0}$$

Use the Gauss's divergence theorem to show

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}$$

where ρ is the volume charge density.

$$Given,\ \phi_E = \oint \vec{E}.d\vec{A} = \frac{q}{\epsilon_0}$$

According to Gauss's divergence theorem : Volume integral of the divergence Is equal to the surface integral over the boundary that encloses the volume

$$\oint \vec{E} \cdot d\vec{A} = \int_V (\vec{\nabla} \cdot \vec{E}) dV$$

$$Again, \oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0} = \frac{\int_V \rho dV}{\epsilon_0}$$ Where ρ is volume charge density

$$Thus,\ \oint \vec{E} \cdot d\vec{A} = \int_V (\vec{\nabla} \cdot \vec{E}) dV = \frac{\int_V \rho dV}{\epsilon_0}$$

$$Hence,\ \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

5. Given $\vec{E} = \rho_0 \epsilon_0 x \ \hat{i}$ find the charge density responsible for this electric field.

$$\vec{E} = \rho_0 \epsilon_0 x \cdot \hat{i}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial}{\partial x}(\rho_0 \epsilon_0 x) = \rho_0 \epsilon_0$$

According to differential form of Gauss's law,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$ Where ρ is charge density

$$So, \ \rho = \epsilon_0(\vec{\nabla} \cdot \vec{E}) = \rho_0 \epsilon_0^2$$