**Week 1: SHO, damped SHO, linear 2nd order ODEs.**

*These are incomplete notes. Not everything mentioned/discussed in the class is to be found here. You may read from other sources like books, other lecture notes to reach a more complete understanding.*

Newton's second law (for a constant mass $m$ and in three dimensional space), given as,

$$m\frac{d^2\vec{\mathbf{r}}}{dt^2} = \vec{\mathbf{F}}_{\mathbf{ext.}} \tag{1}$$

is a set of three second order ordinary differential equations (one for each component of the position vector $\vec{\mathbf{r}}(t)$), provided we are given $\vec{\mathbf{F}}_{\mathbf{ext}}$. We note that the L. H. S. of the above equation is mass $\times$ acceleration.

Therefore, given $\vec{\mathbf{F}}_{\mathbf{ext.}}$ which can be, in general, a function of $\vec{\mathbf{r}}$, $\vec{\mathbf{v}}$, we can find the *trajectory* $\vec{\mathbf{r}}(t)$ of the particle by *solving/integrating* the differential equations. Of course, we must be given the initial ($t = 0$) values, i.e. $\vec{\mathbf{r}}(t = 0)$ and $\frac{d\vec{\mathbf{r}}}{dt}(t = 0) = \vec{\mathbf{v}}(t = 0)$, in order to get a complete answer (i.e. with values of the unknown integration constants). Generically, such problems are known as *initial value problems*.

Let us now go back to Eqn. (1) and consider a very specific form of $\vec{\mathbf{F}}_{\mathbf{ext.}}$. Remember, $\vec{\mathbf{F}}_{\mathbf{ext.}}$ is the external force acting on mass $m$. Imagine the simple example of a mass-spring system. This is a one dimensional system. We choose the coordinate describing the one dimensional motion as $x(t)$. There is no gravity here. We stretch or compress the spring from its rest length and then let it go. We all know that the mass oscillates back and forth. We now want to understand this phenomenon better.

What do we write for $F_{ext.}$? Robert Hooke in 1678 taught us what we should – the *restoring force* on the *stretched/compressed spring* is linearly proportional to the displacement of the spring. Using $F_{ext.} = F_{restoring} = -kx$, one may write Newton's second law for this problem as

$$m\frac{d^2x}{dt^2} = -kx \tag{2}$$

where $k$ is a proportionality constant with dimensions $MT^{-2}$. The minus sign indicates the direction of the restoring force on the body of mass $m$. Note that $\sqrt{\frac{k}{m}}$ will have dimensions $T^{-1}$.

What really is $k$? It is called the *spring constant* and is a measurable quantity. It is a *property* of the spring you are using. Different springs made with different materials and in different ways will end up having different $k$.

To understand what is the trajectory of the particle (i.e. $x(t)$) under the influence of this restoring force, we must however, solve the above equation. We rewrite the equation as

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \tag{3}$$

where $\omega_0^2 = \frac{k}{m}$. This is now a **linear**, second order, **homogeneous** (no explicit functions of time on the R. H. S.) ordinary differential equation. The word **linear** used in the context of differential equations means that **derivatives of all orders** (including the zeroth derivative, which is the function itself) in the equation appear **linearly**. For example an equation of the form

$$\left(\frac{d^2x}{dt^2}\right)^2 + \alpha^2 x = 0 \tag{4}$$

is **not linear**–rather it is **non-linear**. If the **highest order derivative** appears linearly and at least one other lower order derivative or the zeroth derivative (i.e. the function itself) appears with a power other than 0 or 1, the equation is called **quasi-linear**. For instance, the equation

$$\frac{d^2x}{dt^2} + \alpha^2 x^2 = 0 \tag{5}$$

is **quasi-linear**. We shall however restrict ourselves to **equations in which derivatives of all orders appear linearly. Such differential equations are linear.**

Linear differential equations have the important property that if $x_1(t)$ is a solution and $x_2(t)$ is another **linearly independent** solution then $x(t) = C_1x_1 + C_2x_2$ is also a solution. There are specific ways to check linear independence of two solutions–we will not get into them. This property, though simple, is very useful in writing down solutions with specific initial conditions. The property is known as the **principle of linear superposition**.

How do we solve such equations? We will outline two methods below.

**Method 1:** You may be familiar with series solutions. If not assume that we can write down a solution of the above ordinary differential equation as

$$x(t) = a_0 + a_1 t + a_2 t^2 + ..... = \sum_{n=0}^{\infty} a_n t^n \tag{6}$$

We will substitute this in the differential equation and get

$$\sum_{n=2}^{\infty} a_n n(n-1) t^{n-2} + \omega_0^2 \sum_{n=0}^{\infty} a_n t^n = 0 \tag{7}$$

We now write down the coefficient of the nth term, $t^n$. For each $n$ we therefore get,

$$a_{n+2} = -\frac{1}{(n+2)(n+1)}\omega_0^2 a_n \tag{8}$$

Thus

$$a_2 = -\frac{1}{2}\omega_0^2 a_0 \quad ; \quad a_4 = \omega_0^2 \frac{1}{4.3} a_2 = \omega_0^4 \frac{1}{4.3.2.1} a_0 ..... \tag{9}$$

Similarly,

$$a_3 = -\frac{1}{3.2}\omega_0^2 a_1 \quad ; \quad a_5 = \omega_0^2 \frac{1}{5.4} a_3 = \omega_0^4 \frac{1}{5.4.3.2} a_1 ..... \tag{10}$$

Hence the full solution would be

$$x(t) = a_0\left(1 - \frac{(\omega_0 t)^2}{2!} + \frac{(\omega_0 t)^4}{4!} - ....\right) + \frac{a_1}{\omega_0}\left(\omega_0 t - \frac{(\omega_0 t)^3}{3!} + \frac{(\omega_0 t)^5}{5!} - ...\right) \tag{11}$$

The quantities in brackets are just cos $\omega_0 t$ and sin $\omega_0 t$ and so we have

$$x(t) = a_0 \cos\ \omega_0 t + \frac{a_1}{\omega_0} \sin\ \omega_0 t \tag{12}$$

We may write $a_0 = A\cos\ \phi$ and $\frac{a_1}{\omega_0} = -A\sin\ \phi$ which will eventually lead to

$$x(t) = A\cos\left(\omega_0 t + \phi\right) \tag{13}$$

$A$ here is known as the amplitude and $\phi$ or, rather, the whole of $\omega_0 t + \phi$ is called the phase, $\phi$ is called the initial (t=0) phase. Notice that

$$x(t=0) = a_0 = A\cos\phi \quad ; \quad \frac{dx}{dt}(t=0) = a_1 = -A\omega_0 \sin\phi \tag{14}$$

The end-result of this little calculation is that the motion of the mass $m$ under this restoring force is a repetetive motion or an oscillation. We call it a free oscillation since there are no other forces associated with it apart from the restoring force.
We can rename quantities as follows:

$$x(t=0) = x_0 = a_0 \quad ; \quad \frac{dx}{dt}(t=0) = v_0 = a_1 \tag{15}$$

Exercise: Take the mass $m$ as suspended from a spring hanging vertically from a support. Assume $z(t)$ as the coordinate. If we pull the spring and leave it what is the differential equation for motion and what is its solution? Remember to include gravity.

**Method 2.** The other method of solving the equation proceeds as follows. Start again with

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \tag{16}$$

We shall henceforth denote a derivative w.r.t $t$ with an overdot. Let us multiply both sides by $2\dot{x}$. This gives

$$2\dot{x}\ddot{x} + 2\omega_0^2 x\dot{x} = 0 \tag{17}$$

We can rewrite this as

$$\frac{d}{dt}\left[\dot{x}^2 + \omega_0^2 x^2\right] = 0 \tag{18}$$

which would imply that the quantity in square brackets is a constant. Hence, we have

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E \tag{19}$$

where we defined $E$ as the integration constant. Note that $E$ is the energy–the first term in the L. H. S is the K. E. while the second is $V(x) = \frac{1}{2}kx^2$ is the P. E.

We can rearrange the above equation to write,

$$dt = \pm\sqrt{\frac{m}{k}}\int\frac{dx}{\sqrt{\alpha^2 - x^2}} \tag{20}$$

where $\alpha^2 = \frac{2E}{k}$. The integral can easily be done with the substitution $x = \alpha\cos\theta$. This will give

$$\omega_0 t + \phi = \mp\cos^{-1}\frac{x}{\alpha} \tag{21}$$

which yields

$$x = \alpha\cos\left(\omega_0 t + \phi\right) \tag{22}$$

Here $\alpha = \sqrt{\frac{2E}{k}}$ is the same as the $A$ mentioned before. Note that $A$ is given in terms of the energy $E$ and the spring constant. If you evaluate $A$ from the earlier expressions

$$A^2 = x_0^2 + \frac{v_0^2}{\omega_0^2} = \frac{2}{k}\left(\frac{1}{2}kx_0^2 + \frac{1}{2}mv_0^2\right) = \frac{2E}{k} \tag{23}$$

Notice that this method can be used for any force $F$ which is given as

$$F = -\frac{\partial V}{\partial x} \tag{24}$$

The expression for the energy is

$$\frac{1}{2}m\dot{x}^2 + V(x) = E \tag{25}$$

This can be reduced to the expression

$$t = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx}{\sqrt{E - V(x)}} \tag{26}$$

Obtaining $x(t)$ depends on the solvability of the integral for a given $V(x)$ and the inversion of $t(x)$ to get $x(t)$.

Let us now introduce linear in speed damping ($F_{damp} = -r\dot{x}$). Here and henceforth we will use the 'overdot' notation to denote a derivative w.r.t. $t$. The differential equation of motion for the mass-spring system with damping takes the form:

$$m\ddot{x} + r\dot{x} + kx = 0 \tag{27}$$

This, with some simple redefinitions, like $\frac{r}{m} = 2\beta$ and $\frac{k}{m} = \omega_0^2$ leads to the equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{28}$$

We now need to solve this equation. Let us define

$$x(t) = a(t)b(t) \tag{29}$$

Therefore $\dot{x}$ and $\ddot{x}$ are

$$\dot{x} = \dot{a}b + a\dot{b} \quad ; \quad \ddot{x} = \ddot{a}b + 2\dot{a}\dot{b} + a\ddot{b} \tag{30}$$

Substituting these expressions in the original equation gives

$$b\ddot{a} + \left(2\dot{b} + 2\beta b\right)\dot{a} + \left(\ddot{b} + 2\beta\dot{b} + \omega_0^2 b\right)a = 0 \tag{31}$$

or, equivalently

$$\ddot{a} + 2\left(\frac{\dot{b}}{b} + \beta\right)\dot{a} + \left(\frac{\ddot{b}}{b} + 2\beta\frac{\dot{b}}{b} + \omega_0^2\right)a = 0 \tag{32}$$

Let us now demand that the coefficient of the second term (i.e. the $\dot{a}$ term) is zero. This gives $\dot{b} = -\beta b$, which is solved as

$$b(t) = e^{-\beta t} \tag{33}$$

where we have chosen an integration constant appropriately (without any loss of generality). With a little bit of calculation, we can quickly see that the equation for $a$ takes the form

$$\ddot{a} + \left(\omega_0^2 - \beta^2\right)a = 0 \tag{34}$$

where there is **no first order derivative w.r.t. t**. This method of **removing the first derivative term** applies in the general case where the coefficients of $\ddot{x}$, $\dot{x}$ or $x$ are time-dependent.

Therefore, there are three types of solutions depending on (a) $\omega_0^2 > \beta^2$, (b) $\omega_0^2 < \beta^2$ (c) $\omega_0^2 = \beta^2$ For each of these three cases we now write down the solutions from our knowledge of finding solutions of such equations. We shall, as before use the initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$.

**Case 1:** $\omega_0^2 - \beta^2 = \omega^2 > 0$**:** In this case we have,

$$a(t) = A\cos\omega t + B\sin\omega t \tag{35}$$

and hence $x(t) = e^{-\beta t}a(t)$. With the given initial conditions one can fix $A$, $B$. This finally gives

$$x(t) = e^{-\beta t}\left(x_0\cos\omega t + \frac{v_0 + \beta x_0}{\omega}\sin\omega t\right) \tag{36}$$

**Case 2:** $\beta^2 - \omega_0^2 = \omega^2 > 0$**:** In this case we have,

$$a(t) = A\cosh\omega t + B\sinh\omega t \tag{37}$$

Exercise: Solve the equation $\ddot{a} - \omega^2 a = 0$ using the power series method.

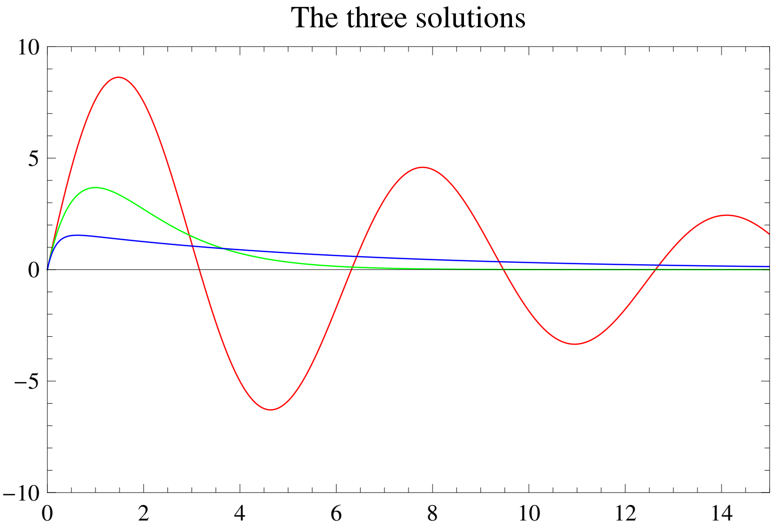


FIG. 1. Red (underdamped,$\beta = 0.1$), Green (Critically damped, $\beta = 1$), Blue(overdamped, $\beta = 3$), $\omega_0 = 1$, $x_0 = 0, v_0 = 1$

Thus we have

$$x(t) = e^{-\beta t}\left(x_0\cosh\omega t + \frac{v_0 + \beta x_0}{\omega}\sinh\omega t\right) \tag{38}$$

**Case 3:** $\beta^2 - \omega_0^2 = 0$**:** In this case we have, $a(t) = A + Bt$ and the full solution is,

$$x(t) = (x_0 + (v_0 + \beta x_0)t)\, e^{-\beta t} \tag{39}$$

We now turn to a discussion on some quantities related to damped oscillatory motion (underdamped).

**Logarithmic decrement:** Let us recall the solution for $x(t)$ in the damped oscillatory case. This is given as

$$x(t) = e^{-\beta t}\left[x_0 \cos\omega t + \frac{v_0 + \beta x_0}{\omega}\sin\omega t\right] \tag{40}$$

We can check that $\dot{x}$ and $\ddot{x}$ are given by the following expressions.

$$\dot{x} = e^{-\beta t}\left[v_0 \cos\omega t - \frac{\omega_0^2 x_0 + \beta v_0}{\omega}\sin\omega t\right] \tag{41}$$

$$\ddot{x} = -\beta\dot{x} + e^{-\beta t}\left[-v_0\omega\sin\omega t - (\beta v_0 + \omega_0^2 x_0)\cos\omega t\right] \tag{42}$$

We now wish to find the extrema of $x(t)$. These occur at $\dot{x} = 0$ which leads to the requirement for extrema at $t = t_*$ as

$$\tan\omega t_* = \frac{\omega v_0}{\omega_0^2 x_0 + \beta v_0} \tag{43}$$

The value of $\ddot{x}$ at the extrema $t = t_*$ (using $\dot{x}(t_*) = 0$ and the expression for $\tan\omega t_*$ above ) is given by

$$\ddot{x}(t_*) = -e^{-\beta t_*}\frac{\beta v_0 + \omega_0^2 x_0}{\cos\omega t_*} \tag{44}$$

Therefore, when

$$t_* = \frac{1}{\omega}\left(\tan^{-1}\frac{\omega v_0}{\omega_0^2 x_0 + \beta v_0} + 2n\pi\right) \tag{45}$$

we have maxima for $n = 0, 1, 2...$ (since $\ddot{x}(t_*) < 0$). When

$$t_* = \frac{1}{\omega}\left(\tan^{-1}\frac{\omega v_0}{\omega_0^2 x_0 + \beta v_0} + (2n+1)\pi\right) \tag{46}$$

we have minima for $n = 0, 1, 2...$ (since $\ddot{x}(t_*) > 0$).

Let us now define the ratio of the amplitudes, $A_n$ and $A_{n+1}$ corresponding to the $n$th and $(n+1)$th maxima or minima. This gives

$$\frac{A_n}{A_{n+1}} = e^{-\left(2\pi n\frac{\beta}{\omega} - 2\pi(n+1)\frac{\beta}{\omega}\right)} = e^{2\pi\frac{\beta}{\omega}} = e^{\beta T} \tag{47}$$

where $T = \frac{2\pi}{\omega}$. Therefore, we have, the logarithmic decrement $\lambda$ defined as,

$$\lambda = \beta T = \ln\frac{A_n}{A_{n+1}} \tag{48}$$

A measurement of the amplitudes at successive maxima or minima can therefore determine $\lambda$ and hence $\beta T$. Knowing the time period $T$ one can determine the $\beta$ and hence the damping coefficient $r$. This is useful while doing experiments with damped systems (for example, you will do it while performing the Pohl's pendulum experiment in your lab class).

**Quality factor:** Another useful quantity, for weakly damped systems is the **Quality factor** of an oscillator. To understand this let us go back to the equation of motion

$$m\ddot{x} + r\dot{x} + kx = 0 \tag{49}$$

Multiply both sides by $\dot{x}$ and rewrite the equation as

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = \frac{dE}{dt} = -r\dot{x}^2 = -2\beta m\dot{x}^2 \tag{50}$$

$E$ is not a constant and therefore the damped oscillator is not a conservative system. The work done by the force $F = -r\dot{x} - kx$ is simply

$$P = F\dot{x} = -kx\dot{x} - r\dot{x}^2 \tag{51}$$

When $\dot{x}$ and $x$ have the same sign the spring (restoring) force does negative work–whereas when $\dot{x}$ and $x$ have the opposite sign it may do positive work. Over a full cycle the work done by the restoring force is zero. On the other hand the work done by the damping force is always negative. The damping force thus reduces the kinetic energy of the system.
To characterise such a damped oscillator when the damping is small (i.e. $\beta << \omega_0$) we define the **Quality Factor** as

$$Q = 2\pi \frac{Energy\ \ stored\ \ in\ \ the\ \ oscillator}{Energy\ \ lost\ \ in\ \ one\ \ oscillation\ \ period} \tag{52}$$

To find an expression for $Q$ we first calculate the denominator, given as

$$\Delta E = -\int_0^T \frac{dE}{dt}dt = 2\beta m \int_0^T \dot{x}^2 dt \tag{53}$$

which turns out to be

$$\Delta E = 2\beta m \int_0^T e^{-2\beta t}\left(v_0 \cos\omega t - \frac{\omega_0^2 x_0 + \beta v_0}{\omega}\sin\omega t\right)^2 dt \tag{54}$$

Over one cycle and with $\beta$ small, we can obtain an approximate expression for $\Delta E$ by ignoring the $e^{-2\beta t}$ and setting it to one. Hence, after some simple calculations, we get,

$$\Delta E = \frac{2\pi\beta m}{\omega}\frac{\omega_0^2}{\omega^2}\left[v_0^2 + \omega_0^2 x_0^2 + 2\beta v_0 x_0\right] \tag{55}$$

where we have used the values of $\int_0^T \sin^2 \omega t\, dt = \int_0^T \cos^2 \omega t\, dt = \frac{\pi}{\omega}$ and $\int_0^T \sin \omega t \cos \omega t\, dt = 0$. Recall further that the amplitude squared of damped oscillations (ignoring the $e^{-\beta t}$ as small), is

$$A^2 = \frac{\omega_0^2 x_0^2 + v_0^2 + 2\beta v_0 x_0}{\omega^2} \tag{56}$$

Hence, we have

$$\Delta E = \frac{2\pi m \omega_0^2 \beta A^2}{\omega} \tag{57}$$

The energy stored in the oscillator is $E = \frac{1}{2} m \omega_0^2 A^2$. Thus the Quality Factor is

$$Q = \frac{\omega}{2\beta} \sim \frac{\omega_0}{2\beta} \tag{58}$$

Q very large corresponds to weak damping and Q small corresponds to relatively larger damping. However **remember that the formula above is approximate and only valid in the weak damping limit.**

Exercise: Take a LCR series circuit. Write down the differential equation for the current $I(t)$ using Kirchoff's law. Draw parallels with the damped oscilator equation and obtain an expression for the Quality Factor.

Exercise on overdamped systems: Write down the general solution for an overdamped oscillator using $x_0 = 0$ and $v_0 \neq 0$. Assume $\beta = 2\omega_0$ and $\beta = 10\omega_0$ and draw two graphs on the same graph paper (choose $\omega_0 = 1$ and $v_0 = 1$). Using your graphs explain the statement: **for heavy damping, the system can get stuck, away from equilibrium, for a long time**. Try to observe this behaviour while you do your Pohl's pendulum experiment in the lab.