Solution for Tutorial sheet - 3

Maths - I . Autumn - 2018

1. Determine the limits of the following functions, if they exist:

(a) $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Sol^n: Consider the path $y = mx$. Then as $(x,y)\to(0,0)$ we get $x \to 0$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x \cdot mx}{x^2+(mx)^2}$$

$$= \lim_{x\to 0} \frac{mx^2}{x^2(1+m^2)}$$

$$= \frac{m}{1+m^2}$$

which depends on m, ie. for different values of m, we get different limiting values

Hence the limit does not exist.

(b) $$\lim_{(x,y)\to(0,0)} \log\left(\frac{y}{x}\right)$$

Sol^n: - Consider the path $y = mx$

Then $$\lim_{(x,y)\to(0,0)} \log\left(\frac{y}{x}\right) = \lim_{x\to 0} \log\left(\frac{mx}{x}\right) = \log m$$

which is different for different values of m.

Hence the limit does not exist

(c) $\lim\limits_{(x,y)\to(0,0)} \frac{|x|}{y^2} \exp\left(-\frac{|x|}{y^2}\right)$

Soln:- consider the path $x = my^2$

$$\lim_{(x,y)\to(0,0)} \frac{|x|}{y^2} \exp\left(-\frac{|x|}{y^2}\right) = \lim_{y\to 0} \frac{|my^2|}{y^2} \exp\left(-\frac{|my^2|}{y^2}\right)$$

$$= \lim_{y\to 0} |m| \exp(-|m|)$$

$$= |m| \exp(-|m|)$$

Which is different for different values of m.

$\therefore$ The limit does not exist.

(d) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+y^2}{\tan(x,y)}$

Soln:- Let $x = r\cos\theta$, $y = r\sin\theta$

Then $$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\tan(xy)} = \lim_{r\to 0} \frac{r^2}{\tan(r^2\sin\theta\cos\theta)}$$

$$= \lim_{r\to 0} \frac{r^2\cos(r^2\sin\theta\cos\theta)}{\sin(r^2\cos\theta\sin\theta)}$$

$$= \lim_{r\to 0} \frac{\cos(r^2\sin\theta\cos\theta)}{\frac{\sin(r^2\cos\theta\sin\theta)}{r^2}}$$

$$= \frac{\lim\limits_{r\to 0} \cos(r^2\sin\theta\cos\theta)}{\lim\limits_{r\to 0} \frac{\sin(r^2\cos\theta\sin\theta)}{r^2\sin\theta\cos\theta} \cdot \sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta \cos\theta}$$, which depends on θ.

$\therefore$ The limit does not exist.

(e) $$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$$

Soln:- Consider the path $y = mx^2$

Then $$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2} = \lim_{x\to 0} \frac{x^2 \cdot mx^2}{x^4+m^2x^4} = \frac{m}{1+m^2}$$

which is different ~~from~~ for different values of m.

$\therefore$ The limit does not exist.

(f) $$\lim_{(x,y)\to(0,0)} \log\left(\frac{\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2}-x}\right)$$

Soln:- Consider the path $y = mx$

$$\lim_{(x,y)\to(0,0)} \log\left(\frac{\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2}-x}\right)$$

$$= \lim_{x\to 0} \log\left(\frac{\sqrt{x^2+m^2x^2}+x}{\sqrt{x^2+m^2x^2}-x}\right)$$

$$= \log\left(\frac{\sqrt{1+m^2}+1}{\sqrt{1+m^2}-1}\right)$$

which is different for different values of m

$\therefore$ The limit does not exist.

(g) $\lim_{(x,y)\to(0,0)} \left(\sin\frac{x}{y} + \sin\frac{y}{x} \right)$

Solⁿ:- consider the path $y = mx$

Then $\lim_{(x,y)\to(0,0)} \left[\sin\frac{x}{y} + \sin\frac{y}{x} \right]$

$= \lim_{x\to 0} \left[\sin\frac{x}{mx} + \sin\frac{mx}{x} \right]$

$= \sin\left(\frac{1}{m}\right) + \sin(m)$ which depends on m,

$\therefore$ The limit does not exist.

(h) $\lim_{(x,y)\to(0,0)} \cos^3\left(\sqrt{x^2+y^2}\right)$

Solⁿ:- Let $x = r\cos\theta$, $y = r\sin\theta$

Then $(x,y)\to(0,0) \Rightarrow r\to 0$

Then $\lim_{(x,y)\to(0,0)} \cos^3\left(\sqrt{x^2+y^2}\right)$

$= \lim_{r\to 0} \cos^3(r) = \cos^3(0) = 1$

(i) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2y+xy^2)}{xy}$

Solⁿ:- $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2y+xy^2)}{xy}$

$= \lim_{(x,y)\to(0,0)} \left[(x+y)\cdot\frac{1}{x+y} \frac{\sin(xy(x+y))}{xy} \right]$

$= \lim_{(x,y)\to(0,0)} \left\{ (x+y) \frac{\sin[xy(x+y)]}{xy(x+y)} \right\}$

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$$= \lim_{(x,y)\to(0,0)} (x+y) \lim_{(x,y)\to(0,0)} \frac{\sin[xy(x+y)]}{xy(x+y)}$$

$$= 0 \cdot 1 = 0$$

(j) $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$ $y=mx$ $\lim \frac{x^3 - m^3x^3}{x^2+m^2x^2} = \lim_{x\to 0} x\left[\frac{(1-m^3)}{(1+m^2)}\right]$ $= 0$

Soln:- Let $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2}$$

$$= \lim_{r\to 0} r(\cos^3\theta - \sin^3\theta)$$

$$= 0$$

(k) $\lim_{(x,y,z)\to(0,0,0)} \frac{xy^2z^2}{x^4+y^4+z^8}$

Soln:- Consider the path $x = mz^2$, $y = nz^2$

Then $\lim_{(x,y,z)\to(0,0,0)} \frac{xy^2z^2}{x^4+y^4+z^8}$

$$= \lim_{z\to 0} \frac{mz^2 \cdot (nz^2)^2 z^2}{m^4z^8 + n^4z^8 + z^8}$$

$$= \frac{mn^2}{m^4+n^4+1}$$

which has different values for different values of m and n

$\therefore$ The limit does not exist.

(i) $\lim_{(x,y)\to(0,0)} \tan^{-1}\left(\frac{|x|+|y|}{x^2+y^2}\right)$

Soln:- Consider the path $y = mx$

$x = r\cos\theta$

$y = r\sin\theta$

$\lim_{r\to 0} \tan^{-1} \frac{|r|\left[|\cos\theta|+|\sin\theta|\right]}{r^2}$

$\lim_{r\to 0} \tan^{-1}\left[\frac{[1\ 1+1|]}{|r|}\right]$

$\pi/2$

$\therefore \lim_{(x,y)\to(0,0)} \tan^{-1}\left(\frac{|x|+|y|}{x^2+y^2}\right)$

$= \lim_{x\to 0} \tan^{-1}\left(\frac{|x| + |mx|}{x^2+m^2x^2}\right)$

$= \lim_{x\to 0} \tan^{-1}\left(\frac{|x|(1+|m|)}{x^2(1+m^2)}\right)$

Now $\lim_{x\to 0^-} \tan^{-1}\left(\frac{|x|(1+|m|)}{x^2(1+m^2)}\right)$

$= \lim_{x\to 0^{-1}} \tan^{-1}\left(\frac{-x(1+|m|)}{x^2(1+m^2)}\right)$

$= \lim_{x\to 0^{-1}} \tan^{-1}\left(\frac{-(1+|m|)}{x(1+m^2)}\right)$

$= \tan^{-1}(-\infty) = -\pi/2$

$\lim_{x\to 0^+} \tan^{-1}\left(\frac{|x|(1+|m|)}{x^2(1+m^2)}\right)$

$= \cancel{\tan^{-1}} \lim_{x\to 0^+} \tan^{-1}\left(\frac{x(1+|m|)}{x^2(1+m^2)}\right)$

$= \lim_{x\to 0^+} \tan^{-1}\left(\frac{1+|m|}{x(1+m^2)}\right) = \tan^{-1}(\infty) = \pi/2$

$\therefore$ The limit does not exist.

(m) $$\lim_{(x,y)\to(2,1)} \frac{\sin^{-1}(xy-2)}{\tan^{-1}(3xy-6)}$$

Soln:- Let $xy-2=t$, then $t\to 0$ as $(x,y)\to(2,1)$

$$\therefore \lim_{(x,y)\to(2,1)} \frac{\sin^{-1}(xy-2)}{\tan^{-1}(3xy-6)}$$

$$= \lim_{t\to 0} \frac{\sin^{-1}(t)}{\tan^{-1}(3t)} \quad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{t\to 0} \frac{\frac{1}{\sqrt{1-t^2}}}{\frac{3}{1+9t^2}} = 1/3$$

(n) $$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$$

Soln:- consider the path $y=x$

$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y} = \lim_{x\to 0} \frac{\sin(x^2)}{x+x}$$

$$= \lim_{x\to 0} \frac{\sin(x^2)}{2x} \quad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{2x\cdot\cos x}{2} = 0$$

Consider the path $y=-\sin x$

$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y} = \lim_{x\to 0} \frac{\sin(-x\sin x)}{x-\sin x} \quad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-\cos(x\sin x)[\sin x + x\cos x]}{1-\cos x} \quad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{+\sin(x\sin x)\,(\sin x + x\cos x)^2 - \cos(x\sin x)\,(\cos x + \cos x - x\sin x)}{+\sin x}$$

$$= \frac{-2}{0} = -\infty$$

$\therefore$ The limit does not exist.

(o) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2}$

Sol^n:- Consider the path along $x=1$ ~~$y=1$~~

$$\therefore \lim_{(x,y)\to(1,0)} = \lim_{y\to 0} \frac{y-y}{y^2} = 0$$

$$\lim_{x\to 1} \frac{x-1}{(x-1)^2+1} = 0$$

consider the path along $y = x-1$

$$\lim_{(x,y)\to(1,0)} = \lim_{x\to 1} \frac{x(x-1)-(x-1)}{(x-1)^2+(x-1)^2}$$

$$= \lim_{x\to 1} \frac{(x-1)^2}{2(x-1)^2} = \frac{1}{2}$$

$\therefore$ The limit does not exist.

(p) $\displaystyle\lim_{(x,y)\to(0,0)} \cos\frac{x^3-y^3}{x^2+y^2}$

Sol^n:- Let $x = r\cos\theta$, $y = r\sin\theta$, so the limit becomes

$$\therefore \lim_{r\to 0} \cos\left[\frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2}\right] = \cos(0)$$

$$= 1$$

for every $\epsilon>0$; $\exists\ \delta>0$ s.t $0<|x-a|<\delta \Rightarrow |f(x)-L|<\epsilon$

$\lim_{x\to a} f(x) = L$

2. Using ϵ-δ method, prove the followings.

(a) $$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{2x^2+y^2}} = 0$$

Soln:- $$\left|\frac{xy}{\sqrt{2x^2+y^2}} - 0\right| = \frac{|x|\,|y|}{\sqrt{2x^2+y^2}}$$

$$\le \frac{\sqrt{x^2+y^2}\cdot\sqrt{x^2+y^2}}{\sqrt{x^2+x^2+y^2}}$$

$$\le \frac{\sqrt{x^2+y^2}\,\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$$

$$= \sqrt{x^2+y^2} < \epsilon$$

$(x-1)^2$

$\lim_{x\to 2} x^2 - 4x+5 = 1$

$x+2, 4$

$\lim_{x\to 2} x^2-2x+2=1$

find $\delta>0$ s.t $|x^2-2x+1-1|<\epsilon$

$|x-2|<\delta$

$x<6$ $(x-2)^2$ x^2-4x+4

$4-4+2$

$x^2-4x+5=1$

$4-8+5$

[where ϵ is any preassigned +ve real no.]

we choose $\delta = \epsilon$ s.t $\sqrt{x^2+y^2} < \delta$

$\therefore$ $\left|\frac{xy}{\sqrt{2x^2+y^2}} - 0\right| < \epsilon$ whenever $\sqrt{x^2+y^2} < \delta$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{2x^2+y^2}} = 0$$

(b) $$\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\left(\frac{1}{xy}\right) = 0$$

Soln:- $$\left|(x^2+y^2)\sin\left(\frac{1}{xy}\right) - 0\right|$$

$$= |x^2+y^2|\,\left|\sin\left(\frac{1}{xy}\right)\right|$$

$$\le \left(\sqrt{x^2+y^2}\right)^2 < \epsilon$$ [where ϵ is any preassigned +ve real no.]

we choose $\delta = \sqrt{\epsilon}$ s.t $\sqrt{x^2+y^2} < \delta$

$\therefore \; |(x^2+y^2)\sin\left(\frac{1}{xy}\right) - 0| < \varepsilon$ wherever $\sqrt{x^2+y^2} < \delta$

$$\therefore \lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\left(\frac{1}{xy}\right) = 0$$

(c) $$\lim_{(x,y)\to(2,1)} (x^2-2y+y^2) = 3$$

Soln:- $|(x^2-2y+y^2)-3|$

$= |(x-2+2)^2 - 2(y-1+1) + (y-1+1)^2 - 3|$

$= |(x-2)^2 + 4(x-2) + 4 - 2(y-1) - 2$
$\quad + (y-1)^2 + 2(y-1) + 1 - 3|$

$= |(x-2)^2 + 4(x-2) + (y-1)^2|$

$\leq |x-2|^2 + 4|x-2| + |y-1|^2$

$< \varepsilon$ (where ε is any preassigned ~~fixed~~ +ve real no.)

We choose $2\delta^2 + 4\delta < \varepsilon$

or, $\delta^2 + 2\delta < \varepsilon/2$

or, $(\delta+1)^2 < \varepsilon/2 + 1$

or, $\delta < \sqrt{\varepsilon/2 + 1} - 1$

$\therefore \; |(x^2-2y+y^2)-3| < \varepsilon$ wherever

$|x-2| < \delta, \; |y-1| < \delta$, where $\delta < \sqrt{\frac{\varepsilon+2}{2}} - 1$

$\therefore \lim_{(x,y)\to(2,1)} (x^2 - 2y + y^2) = 3$

(d) $\lim_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+y^2} = 0$

<u>Soln:-</u> $\left| \dfrac{4xy^2}{x^2+y^2} - 0 \right| = \dfrac{4|x|\,|y|^2}{|x^2+y^2|}$

$\leq \dfrac{4\sqrt{x^2+y^2}\,(x^2+y^2)}{x^2+y^2}$

$= 4\sqrt{x^2+y^2}$

$< \varepsilon$ (where ε is any preassigned +ve real no.)

we choose $\delta = \varepsilon/4$ s.t $\sqrt{x^2+y^2} < \delta$

$\therefore \left| \dfrac{4xy^2}{x^2+y^2} - 0 \right| < \varepsilon$ whenever $\sqrt{x^2+y^2} < \delta$

$\therefore \lim_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+y^2} = 0$

(e) $\lim_{(x,y)\to(-1,1)} (xy - 2x^2) = -1$

<u>Soln:-</u> $|(xy - 2x^2) - (-1)|$

$= |xy - 2x^2 + 1|$

$= |(x+1-1)(y+1-1) - 2(x+1-1)^2 + 1|$

$= |(x+1)(y+1) - (x+1) - (y+1) + 1 - 2[(x+1)^2 - 2(x+1) + 1] + 1|$

$$= \left| (x+1)(y+1) + 3(x+1) - (y+1) - 2(x+1)^2 \right|$$

$$\leq |x+1|\,|y+1| + 3|x+1| + |y+1| + 2|x+1|^2$$

$$< \varepsilon$$ [where ε is any preassigned +ve real no.]

We choose δ s.t, $3\delta^2 + 4\delta < \varepsilon$

or, $\delta^2 + \frac{4}{3}\delta < \frac{\varepsilon}{3}$

or, $\left(\delta + \frac{2}{3}\right)^2 < \frac{\varepsilon}{3} + \frac{4}{9}$

$\therefore$ $\delta < \sqrt{\frac{3\varepsilon+4}{9}} - \frac{2}{3}$

$\therefore$ $|xy - 2x^2 + 1| < \varepsilon$ whenever $|x+1| < \delta$, $|y+1| < \delta$

where $\delta < \sqrt{\frac{3\varepsilon+4}{9}} - \frac{2}{3}$

$$\therefore \lim_{(x,y) \to (-1,-1)} (xy - 2x^2) = -1$$

(f) $$\lim_{(x,y) \to (1,0)} \frac{(x-1)^2 \ln x}{y^2 + (x-1)^2} = 0$$

Soln:- Let $x-1 = r\cos\theta$, $y = r\sin\theta$, then $(x,y) \to (1,0)$

$\Rightarrow r \to 0$

$$\therefore \left| \frac{(x-1)^2 \ln(x)}{y^2 + (x-1)^2} - 0 \right| = \left| \frac{r^2 \cos^2\theta \ln(1 + r\cos\theta)}{r^2} \right|$$

$$= |\cos^2\theta|\,|\ln(1 + r\cos\theta)|$$

$$\leq |\ln(1 + r\cos\theta)|$$

Now for $r \to 0$, we can take as $r << 1$

Now $-1 \leq \cos\theta \leq 1$, $\forall \theta$

$-r \leq r\cos\theta \leq r$

$\Rightarrow r\cos\theta \geq -r \geq -1$

Then $\left| \dfrac{(x-1)^2 \ln x}{y^2+(x-1)^2} - 0 \right|$

$\leq |\ln(1+r\cos\theta)|$

$\leq |r\cos\theta|$ $[\because r\cos\theta \geq -1]$

$\leq r < \varepsilon$

Let $\phi(x) = \log(1+x) - x$

$\therefore \phi(0) = 0$ & $\phi'(x) = \dfrac{-x}{1+x}$

$\Rightarrow \phi'(x) > 0$ if $-1 < x < 0$

& $\phi'(x) < 0$ if $x > 0$

$\Rightarrow \phi(x)$ is increasing in $(-1, 0]$ and decreasing in $[0, \infty)$

$\Rightarrow \phi(x) \leq \phi(0)$, $x \in (-1, 0]$

& $\phi(x) \leq \phi(0)$, $x \in [0, \infty)$

$\Rightarrow \phi(x) \leq \phi(0) = 0$, $x \geq -1$

$\Rightarrow \log(1+x) \leq x$, $x \geq -1$

We choose $\delta = \varepsilon$

$\therefore \left| \dfrac{(x-1)^2 \ln x}{y^2+(x-1)^2} - 0 \right| < \varepsilon$ whenever $|r| < \delta$

i.e. $\sqrt{x^2+y^2} < \delta$

$$\therefore \lim_{(x,y)\to(1,0)} \frac{(x-1)^2 \ln x}{y^2+(x-1)^2} = 0$$

(g) $$\lim_{(x,y)\to(-2,2)} \frac{x^2-y^2}{y+x} = -4$$

<u>Soln:-</u> $\left| \dfrac{x^2-y^2}{x+y} + 4 \right| = |(x-y)+4|$

$= |(x+2) - (y-2)|$

$\leq |x+2| + |y-2|$

$< \varepsilon$

Now chose $\delta = \varepsilon/2$

$$\therefore \left| \frac{x^2-y^2}{x+y} + 4 \right| < \varepsilon \text{ wherever } |x+2| < \delta$$

$$\text{and } |y-2| < \delta$$

$$\therefore \lim_{(x,y)\to(-2,1)} \frac{x^2-y^2}{x+y} = -4$$

(h) $$\lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2} = 0$$

Solⁿ:- $$\left| xy\frac{x^2-y^2}{x^2+y^2} - 0 \right| = |x||y| \left| \frac{x^2-y^2}{x^2+y^2} \right|$$

$$\leq |x|\,|y|$$

$$\leq \sqrt{x^2+y^2}\,\sqrt{x^2+y^2}$$

$$= \left(\sqrt{x^2+y^2}\right)^2 < \varepsilon$$

We chose $\delta = \sqrt{\varepsilon}$ s.t $\sqrt{x^2+y^2} < \delta$

$$\therefore \left| xy\frac{x^2-y^2}{x^2+y^2} - 0 \right| < \varepsilon \text{ wherever } \sqrt{x^2+y^2} < \delta$$

$$\therefore \lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2} = 0$$

(i) $$\lim_{(x,y)\to(0,0)} x\sin x \cos y = 0$$

Solⁿ:- $$|x\sin x\cos y - 0| = |x|\,|\sin x|\,|\cos y|$$

$$\leq |x| \leq \sqrt{x^2+y^2} < \varepsilon$$

we choose $\delta = \varepsilon$ s.t $\sqrt{x^2+y^2} < \delta$

$\therefore$ $|x \sin x \cos y - 0| < \varepsilon$ whenever $\sqrt{x^2+y^2} < \delta$

$$\therefore \lim_{(x,y)\to(0,0)} x \sin x \cos y = 0$$

(j) $$\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0$$

Solⁿ:- $$\left| \frac{x^2}{\sqrt{x^2+y^2}} - 0 \right| = \frac{|x^2|}{\sqrt{x^2+y^2}}$$

$$\leq \frac{x^2+y^2}{\sqrt{x^2+y^2}}$$

$$= \sqrt{x^2+y^2} < \varepsilon$$

we choose $\delta = \varepsilon$ s.t $\sqrt{x^2+y^2} < \delta$

$\therefore$ $\left| \frac{x^2}{\sqrt{x^2+y^2}} - 0 \right| < \varepsilon$ whenever $\sqrt{x^2+y^2} < \delta$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0$$

(k) $$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0$$

Solⁿ: $$\left| \frac{x^2y^2}{x^2+y^2} - 0 \right| = \frac{x^2 \cdot y^2}{x^2+y^2} \leq \frac{(x^2+y^2)(x^2+y^2)}{x^2+y^2}$$

$$= x^2+y^2 < \varepsilon$$

we choose $\delta = \sqrt{\varepsilon}$ s.t $\sqrt{x^2+y^2} < \delta$

$$\therefore \left| \frac{x^2y^2}{x^2+y^2} - 0 \right| < \varepsilon \quad \text{whenever } \sqrt{x^2+y^2} < \delta$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0$$

(ii) $$\lim_{(x,y)\to(1,1)} (x^2+y^2-1) = 1$$

Soln:- $$|(x^2+y^2-1)-1| = |(x-1+1)^2 + (y-1+1)^2 - 2|$$

$$= |(x-1)^2 + 2(x-1) + 1 + (y-1)^2 + 2(y-1) + 1 - 2|$$

$$\leq |x-1|^2 + 2|x-1| + |y-1|^2 + 2|y-1|$$

$$< \varepsilon$$

We choose δ s.t $2\delta^2 + 4\delta < \varepsilon$

$$\Rightarrow \delta^2 + 2\delta < \varepsilon/2$$

$$\Rightarrow (\delta+1)^2 < \varepsilon/2 + 1$$

$$\Rightarrow \delta < \sqrt{\varepsilon/2 + 1} - 1$$

$\therefore |(x^2+y^2-1)-1| < \varepsilon$ whenever $|x-1| < \delta$

and $|y-1| < \delta$ where $\delta < \sqrt{1+\varepsilon/2} - 1$

$$\therefore \lim_{(x,y)\to(0,0)} (x^2+y^2-1) = 1$$

(m) $\lim_{(x,y)\to(0,0)} \frac{x^4y - 3x^2y^3 + y^5}{(x^2+y^2)^2} = 0$

Soln:- $\left| \frac{x^4y - 3x^2y^3 + y^5}{(x^2+y^2)^2} - 0 \right|$ Let $x = r\cos\theta$, $y = r\sin\theta$

$= \left| \frac{r^5(\cos^4\theta \sin\theta - 3\cos^2\theta \sin^3\theta + \sin^5\theta)}{r^4} \right|$

$= r\,|\cos^4\theta \sin\theta - 3\cos^2\theta \sin^3\theta + \sin^5\theta|$

$\leq r\,[\,|\cos\theta|^4 |\sin\theta| + 3|\cos\theta|^2|\sin\theta|^3 + |\sin\theta|^5$

$\leq r(1+3+1) = 5\sqrt{x^2+y^2} < \varepsilon$

we choose $\delta = \varepsilon/5$ s.t $\sqrt{x^2+y^2} < \delta$

so $\left| \frac{x^4y - 3x^2y^3 + y^5}{(x^2+y^2)^2} - 0 \right| < \varepsilon$, whenever $\sqrt{x^2+y^2} < \delta$

$\therefore \lim_{(x,y)\to(0,0)} \frac{x^4y - 3x^2y^3 + y^5}{(x^2+y^2)^2} = 0$

(n) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = 0$

Soln:- $\left| \frac{xy^2}{x^2+y^2} - 0 \right| \leq \left| \frac{\sqrt{x^2+y^2}\,(x^2+y^2)}{x^2+y^2} \right|$

$= \sqrt{x^2+y^2} < \varepsilon$

we choose $\delta = \varepsilon$

so $\left| \frac{xy^2}{x^2+y^2} - 0 \right| < \varepsilon$ whenever $\sqrt{x^2+y^2} < \delta$

$\therefore \lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^2} = 0$

(o) $\lim_{(x,y)\to(0,0)} \left[y\sin\left(\frac{x}{y}\right) + x\sin\left(\frac{y}{x}\right) \right] = 0$

Soln:- $\left| y\sin\left(\frac{x}{y}\right) + x\sin\left(\frac{y}{x}\right) - 0 \right|$

$\leq |y|\left|\sin\left(\frac{x}{y}\right)\right| + |x|\left|\sin\left(\frac{y}{x}\right)\right|$

$\leq |y| + |x| \leq 2\sqrt{x^2+y^2} < \varepsilon$

we choose $\delta = \varepsilon/2$

so $\left| y\sin\left(\frac{x}{y}\right) + x\sin\left(\frac{y}{x}\right) - 0 \right| < \varepsilon$

whenever $\sqrt{x^2+y^2} < \delta$

$\therefore \lim_{(x,y)\to(0,0)} \left[y\sin\left(\frac{x}{y}\right) + x\sin\left(\frac{y}{x}\right) \right] = 0$

(p) $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{\sqrt{x^2+y^2+z^2}} \sin\left(\frac{1}{xyz}\right) = 0$

Soln! $\left| \dfrac{xyz}{\sqrt{x^2+y^2+z^2}} \sin\left(\frac{1}{xyz}\right) \right|$

$= \dfrac{|x||y||z|}{\left|\sqrt{x^2+y^2+z^2}\right|} \left|\sin\left(\frac{1}{xyz}\right)\right|$

$\leq \dfrac{\sqrt{x^2+y^2+z^2}\cdot\sqrt{x^2+y^2+z^2}\cdot\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}}$

$= \left(\sqrt{x^2+y^2+z^2}\right)^2 < \varepsilon$

we choose $\delta = \sqrt{\varepsilon}$ s.t $\sqrt{x^2+y^2+z^2} < \delta$

so $\left| \dfrac{xyz}{\sqrt{x^2+y^2+z^2}} \sin\left(\dfrac{1}{xyz}\right) - 0 \right| < \varepsilon$ whenever $\sqrt{x^2+y^2+z^2} < \delta$

$$\therefore \lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{\sqrt{x^2+y^2+z^2}} \sin\left(\frac{1}{xyz}\right) = 0$$

3. Using ε-δ method, show that the following functions are continuous:

(a) $f(x,y) = \begin{cases} xy, & (x,y) \neq (2,3) \\ 6, & (x,y) = (2,3) \end{cases}$

Soln:- $|f(x,y) - f(2,3)| = |xy - 6|$

$$= |(x-2+2)(y-3+3) - 6|$$

$$= |(x-2)(y-3) + 3(x-2) + 2(y-3)|$$

$$\leq |x-2||y-3| + 3|x-2| + 2|y-3|$$

Let $|x-2| < \delta$ and $|y-3| < \delta$

Then $|f(x,y) - f(2,3)| \leq |x-2||y-3| + 3|x-2| + 2|y-3|$

$$< \delta^2 + 5\delta$$

Now $\delta^2 + 5\delta < \varepsilon \Rightarrow (\delta + 5/2)^2 < \varepsilon + 25/4$

$$\Rightarrow \delta < \sqrt{\varepsilon + 25/4} - 5/2 > 0$$

we choose $\delta = \sqrt{\varepsilon + 25/4} - 5/2 > 0$

Then $|f(x,y) - f(2,3)| < \varepsilon$ whenever $|x-2| < \delta$ and $|y-3| < \delta$

$\therefore f(x,y)$ is continuous at $(2,3)$

(b) $f(x,y) = \begin{cases} \dfrac{5x^2y^2}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

Soln:- $|f(x,y) - f(0,0)| = \left| \dfrac{5x^2y^2}{x^2+y^2} - 0 \right|$

$$= \left| \frac{5r^2\cos^2\theta \cdot r^2\sin^2\theta}{r^2} \right|$$

Putting $x = r\cos\theta$, $y = r\sin\theta$

$\therefore |f(x,y) - f(0,0)| = |5r^2 \cos^2\theta \sin^2\theta|$

$$= 5r^2 |\cos^2\theta| |\sin^2\theta|$$

$$\leq 5r^2 = 5(x^2+y^2) < \varepsilon$$

we choose $\delta = \sqrt{\varepsilon/5}$

Then $|f(x,y) - f(0,0)| < \varepsilon$ whenever $\sqrt{x^2+y^2} < \delta$

$\therefore f(x,y)$ is continuous at $(0,0)$

(c) $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

Soln! $|f(x,y) - f(0,0)| = \left| \dfrac{xy}{\sqrt{x^2+y^2}} - 0 \right|$

$$= \left| \frac{xy}{\sqrt{x^2+y^2}} \right| = \left| \frac{r^2\cos\theta\sin\theta}{r} \right|$$

putting
$x = r\cos\theta$
$y = r\sin\theta$

$$= |r\cos\theta \sin\theta|$$

$$\leq r = \sqrt{x^2+y^2} < \varepsilon$$

we choose $\delta = \varepsilon$

$\therefore \ |f(x,y) - f(0,0)| < \varepsilon$ whenever $\sqrt{x^2+y^2} < \delta$

$\therefore \ f(x,y)$ is continuous at $(0,0)$

(d) $f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

Soln:- $|f(x,y) - f(0,0)| = \left| xy\dfrac{x^2-y^2}{x^2+y^2} - 0 \right|$

$$= \left| xy\frac{x^2-y^2}{x^2+y^2} \right|$$

$$= \left| r\cos\theta \, r\sin\theta \cdot \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} \right|$$

putting $x = r\cos\theta, \ y = r\sin\theta$

$$= |r^2 \sin\theta \cos\theta \cos 2\theta|$$

$$= r^2 |\sin\theta| \, |\cos\theta| \, |\cos 2\theta|$$

$$= r^2 = x^2+y^2 < \varepsilon$$

we chose $\delta = \sqrt{\varepsilon}$

$\therefore \ |f(x,y) - f(0,0)| < \varepsilon$ whenever $\sqrt{x^2+y^2} < \varepsilon$

$\therefore \ f(x,y)$ is continuous at $(0,0)$

(e) $f(x,y) = \begin{cases} \dfrac{y(x^2-y^2)}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

Soln:- $|f(x,y) - f(0,0)|$

$$= \left| \frac{y(x^2-y^2)}{x^2+y^2} - 0 \right| = |y| \left| \frac{x^2-y^2}{x^2+y^2} \right|$$

$$\leq |y| \leq \sqrt{x^2+y^2} < \varepsilon$$

we choose $\delta = \sqrt{\varepsilon}$

$\therefore$ $|f(x,y) - f(0,0)| < \varepsilon$ whenever $\sqrt{x^2+y^2} < \varepsilon$

$\therefore$ $f(x,y)$ is continuous at $(0,0)$

(f) $f(x,y) = \begin{cases} \dfrac{x^2+y^2}{|x|+|y|} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

Soln:- $$|f(x,y) - f(0,0)| = \left| \frac{x^2+y^2}{|x|+|y|} - 0 \right|$$

$$= \frac{|x^2+y^2|}{|x|+|y|}$$

$$= \frac{|x|^2+|y|^2}{|x|+|y|}$$

$$\leq \frac{(|x|+|y|)^2}{|x|+|y|} = |x|+|y|$$

$$\leq 2\sqrt{x^2+y^2} < \varepsilon$$

we choose $\delta = \varepsilon/2$

$\therefore$ $|f(x,y) - f(0,0)| < \varepsilon$ whenever $\sqrt{x^2+y^2} < \delta$

$\therefore$ $f(x,y)$ is continuous at $(0,0)$

Discuss the continuity of the following functions at $(0,0)$

(a) $f(x,y) = \begin{cases} \dfrac{1}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

Soln:- Let $x = r\cos\theta$, $y = r\sin\theta$

Then $(x,y) \to (0,0) \Rightarrow r \to 0$ $\forall$ values of θ.

Now $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{1}{x^2+y^2}$

$= \lim\limits_{r\to 0} \dfrac{1}{r^2\cos^2\theta + r^2\sin^2\theta}$

$= \lim\limits_{r\to 0} \dfrac{1}{r^2} \neq 0 = f(0,0)$

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y) \neq f(0,0)$

$\therefore$ $f(x,y)$ is not continuous at $(0,0)$.

(b) $f(x,y) = \begin{cases} \dfrac{x^3y^3}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

Soln:- $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y^3}{x^2+y^2}$

$= \lim\limits_{r\to 0} \dfrac{(r\cos\theta)^3(r\sin\theta)^3}{r^2}$ [Putting $x = r\cos\theta$, $y = r\sin\theta$

$= \lim\limits_{r\to 0} r^4\cos^3\theta\sin^3\theta = 0 = f(0,0)$

$\therefore$

$\therefore$ $f(x,y)$ is continuous at $(0,0)$.

(c) $f(x,y) = \begin{cases} \dfrac{|xy|}{\sqrt{x^2+y^2}} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

Soln:- $|f(x,y) - f(0,0)| = \left| \dfrac{|xy|}{\sqrt{x^2+y^2}} - 0 \right|$

$$\leq \frac{\sqrt{x^2+y^2} \cdot \sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$$

$$= \sqrt{x^2+y^2} < \varepsilon$$

Choose $\delta = \varepsilon$

Then $|f(x,y) - f(0,0)| < \varepsilon$ whenever $\sqrt{x^2+y^2} < \delta$

$\therefore$ $f(x,y)$ is continuous at $(0,0)$

(d) $f(x,y) = \begin{cases} \dfrac{|xy|}{xy} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

Soln:- Let $\varepsilon = 1/2$

Then $|f(x,y) - f(0,0)| = \left| \dfrac{|xy|}{xy} - 0 \right|$

$$= \frac{|xy|}{|xy|} = 1 \nless 1/2 \ (= \varepsilon)$$

So for $\varepsilon = 1/2$, we cannot get any $\delta(\varepsilon)$ for which $|f(x,y) - f(0,0)| < \varepsilon$ whenever $\sqrt{x^2+y^2} < \delta(\varepsilon)$

$\Rightarrow$ $f(x,y)$ is not continuous at $(0,0)$

(e) $f(x,y) = \begin{cases} \dfrac{e^{xy}}{x^2+y^2} , & (x,y) \neq (0,0) \\ 0 , & (x,y) = (0,0) \end{cases}$

Soln:- $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{e^{xy}}{x^2+y^2}$

$= \lim\limits_{r\to 0} \dfrac{e^{r^2 \sin\theta \cos\theta}}{r^2}$ [$\because$ putting $x = r\cos\theta$, $y = r\sin\theta$

$\neq 0 = f(0,0)$ as $\dfrac{e^{r^2 \sin\theta \cos\theta}}{r^2} \to \infty$ as $r \to 0$

$\therefore$ $f(x,y)$ is not continuous.

(f) $f(x,y) = \begin{cases} \dfrac{3x^2y - y^3}{x^2+y^2} , & (x,y) \neq (0,0) \\ 0 , & (x,y) = (0,0) \end{cases}$

Soln:- $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y - y^3}{x^2+y^2}$

$= \lim\limits_{r\to 0} \dfrac{3r^2\cos^2\theta \cdot r\sin\theta - r^3\sin^3\theta}{r^2}$ [putting $x = r\cos\theta$, $y = r\sin\theta$

$= \lim\limits_{r\to 0} r\left[3\sin\theta\cos^2\theta - \sin^3\theta\right]$

$= 0 = f(0,0)$

$\Rightarrow$ $f(x,y)$ is continuous at $(0,0)$

(g) $f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Solⁿ:- $|f(x,y) - f(0,0)| = \left| \dfrac{x^3}{x^2+y^2} - 0 \right|$

$$\leq \frac{(\sqrt{x^2+y^2})^3}{(\sqrt{x^2+y^2})^2}$$

$$= \sqrt{x^2+y^2} < \varepsilon$$

Choose $\delta = \varepsilon$

Then $|f(x,y) - f(0,0)| < \varepsilon$ whenever $\sqrt{x^2+y^2} < \delta$

$\Rightarrow$ $f(x,y)$ is continuous at $(0,0)$.

(h) $f(x,y) = \begin{cases} \dfrac{\sin xy}{xy}, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0) \end{cases}$

Solⁿ:- $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{xy}$

$$= \lim_{r\to 0} \frac{\sin(r^2 \sin\theta \cos\theta)}{r^2 \sin\theta \cos\theta}$$ [Putting $x = r\cos\theta,\ y = r\sin\theta$

$$= \lim_{p\to 0} \frac{\sin p}{p}$$ [Let $p = r^2 \sin\theta \cos\theta$
If $r \to 0$, then $p \to 0$

$$= 1 = f(0,0)$$

$\therefore$ $f(x,y)$ is continuous at $(0,0)$

(i) $f(x,y) = \begin{cases} x \sin 1/y + y \sin 1/x, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Soln:-

$$|f(x,y) - f(0,0)|$$

$$= |(x \sin 1/y + y \sin 1/x) - 0|$$

$$\leq |x| \, |\sin 1/y| + |y| \, |\sin 1/x|$$

$$\leq |x| + |y| < \varepsilon$$

Choose $\delta = \varepsilon/2$ and $|x| < \delta$, ~~$|x|$~~ $|y| < \delta$

Then $|f(x,y) - f(0,0)| < \varepsilon$ whenever $|x| < \delta$, $|y| < \delta$

$\Rightarrow$ $f(x,y)$ is continuous at $(0,0)$

(j) $f(x,y) = \begin{cases} \dfrac{(x-y)^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Soln:- $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{(x-y)^2}{x^2+y^2}$

$$= \lim_{r\to 0} \frac{(r\cos\theta - r\sin\theta)^2}{r^2\cos^2\theta + r^2\sin^2\theta}$$ [putting $x = r\cos\theta$, $y = r\sin\theta$

$$= \lim_{r\to 0} \frac{r^2(\cos\theta - \sin\theta)^2}{r^2}$$

$$= (\cos\theta - \sin\theta)^2 \neq 0$$

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y) \neq f(0,0)$

$\therefore$ $f(x,y)$ is not continuous at $(0,0)$.

(k) $f(x,y) = \begin{cases} \dfrac{2x^2+y^2}{3+\sin x}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Soln:- $|f(x,y) - f(0,0)| = \left| \dfrac{2x^2+y^2}{3+\sin x} - 0 \right|$

$= \dfrac{2x^2+y^2}{|3+\sin x|}$

Since $-1 \leq \sin x \leq 1$

$3-1 \leq 3+\sin x \leq 3+1$

$2 \leq 3+\sin x \leq 4$

$\frac{1}{4} \leq \dfrac{1}{3+\sin x} \leq \frac{1}{2}$

$\therefore$ $|f(x,y) - f(0,0)| \leq \dfrac{2x^2+y^2}{2}$

$\leq \dfrac{2(x^2+y^2)}{2} = x^2+y^2 < \epsilon$

Choose $\delta = \sqrt{\epsilon}$

$|f(x,y) - f(0,0)| < \epsilon$ whenever $\sqrt{x^2+y^2} < \delta$

$\therefore$ $f(x,y)$ is continuous at $(0,0)$

(l) $f(x,y) = \begin{cases} \dfrac{x^2+\sin^2 y}{2x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Soln: $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+\sin^2 y}{2x^2+y^2}$

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Taking limit along the path $x = 0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} \frac{\sin^2 y}{y^2}$$

$$= \lim_{y\to 0} \left(\frac{\sin y}{y}\right)^2 = 1$$

Again taking limit along the path $y = 0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0)$$

$\therefore$ $f(x,y)$ is not continuous at $(0,0)$

5. For what values of n, the following function f is continuous at $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{2xy}{(x^2+y^2)^n}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Soln:- For continuity of $f(x,y)$ at $(0,0)$

we have

$$\cancel{\lim_{(x,y)\to(0,0)} \frac{2xy}{(x^2+y^2)^n} = 0}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

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$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{2xy}{(x^2+y^2)^n} = 0$$

Putting $x = r\cos\theta$, $y = r\sin\theta$,

Then as $(x,y) \to (0,0) \Rightarrow r \to 0$

$$\therefore \lim_{r\to 0} \frac{2r^2 \sin\theta \cos\theta}{(r^2\cos^2\theta + r^2\sin^2\theta)^n} = 0$$

$$\Rightarrow \lim_{r\to 0} \frac{2r^2 \sin\theta \cos\theta}{r^{2n} (\cos^2\theta + \sin^2\theta)^n} = 0$$

$$\Rightarrow \lim_{r\to 0} 2\cos\theta \sin\theta \; r^{2-2n} = 0$$

$$\Rightarrow 2-2n > 0$$

$$\Rightarrow 2n < 2$$

$$\Rightarrow n < 1$$

Hence $f(x,y)$ is continuous at $(0,0)$ if $n<1$

Q6

(a)

For, $f(x,y)$ to be continuous at $(0,0)$, we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = c$$

$$\Rightarrow f(0,0) = c = \lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$$

putting $x = r\cos\theta$, $y = r\sin\theta$, $(x,y)\to(0,0) \Rightarrow r\to 0$

$$c = \lim_{r\to 0} \frac{r^4\cos^4\theta - r^4\sin^4\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \lim_{r\to 0} \frac{\cancel{r^4}^{\,2}(\cos^4\theta - \sin^4\theta)}{\cancel{r^2}(\cos^2\theta + \sin^2\theta)}$$

$$= \lim_{r\to 0} r^2(\cos^4\theta - \sin^4\theta)$$

$$= 0.$$

$$\therefore \quad c = 0$$

6 (b) for continuity of $f(x,y)$ at $(0,0)$, we must have

$$f(0,0) = \lim_{(x,y)\to(0,0)} f(x,y)$$

$$\Rightarrow c = \lim_{(x,y)\to(0,0)} x^2\log(x^2+y^2)$$

$\Rightarrow$

putting $x = r\cos\theta,\ y = r\sin\theta,\quad (x,y) \to (0,0) \Rightarrow r \to 0$

$$C = \lim_{r \to 0} r^2 \cos^2\theta \log(r^2\cos^2\theta + r^2\sin^2\theta)$$

$$= \lim_{r \to 0} r^2 \cos^2\theta \log r^2$$

$$= \lim_{r \to 0} 2r^2 \cos^2\theta \log r$$

$$= \lim_{r \to 0} \frac{2\cos^2\theta \log r}{1/r^2} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{r \to 0} \frac{\cancel{2}\cos^2\theta \cdot \frac{1}{r}}{-\frac{\cancel{2}}{r^3}} \qquad [\text{L'Hospitals rule}]$$

$$= \lim_{r \to 0} -\cos^2\theta \cdot r^2$$

$$= 0$$

$\therefore\ C = 0.$

6 (c) For continuity of $f(x,y)$ at $(0,0)$, we have

$$f(0,0) = \lim_{(x,y) \to (0,0)} f(x,y)$$

$$\Rightarrow \quad C = \lim_{(x,y) \to (0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$$

putting $x = r\cos\theta,\ y = r\sin\theta,\quad (x,y) \to (0,0), \Rightarrow r \to 0$

$$C = \lim_{r \to 0} \frac{\sin(r^2\cos^2\theta + r^2\sin^2\theta)}{r^2(\cos^2\theta + \sin^2\theta)}$$

$$= \lim_{r \to 0} \frac{\sin r^2}{r^2}$$

$$= 1$$

$\therefore\ C = 1$

(d) for continuity of $f(x,y)$ at $(0,0)$, we have

$$f(0,0) = \lim_{(x,y)\to(0,0)} f(x,y)$$

$$\Rightarrow \quad c = \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$$

putting $x = r\cos\theta$, $y = r\sin\theta$, $(x,y)\to(0,0) \Rightarrow r\to 0$

$$\Rightarrow \quad c = \lim_{r\to 0} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \lim_{r\to 0} \frac{r^{\cancel{3}}(\cos^3\theta + \sin^3\theta)}{\cancel{r^2}(\cos^2\theta + \sin^2\theta)}$$

$$= 0$$

$\therefore \; c = 0.$

6 (e) for continuity of $f(x,y)$ at $(1,0)$, we have

$$f(1,0) = \lim_{(x,y)\to(1,0)} f(x,y)$$

$$\Rightarrow \quad c = \lim_{(x,y)\to(1,0)} \frac{(x-1)^2\log x}{(x-1)^2+y^2}$$

putting $x-1 = r\cos\theta$, $y = r\sin\theta$, $(x,y)\to(0,0) \Rightarrow r\to 0$

$$\Rightarrow \quad c = \lim_{r\to 0} \frac{r^2\cos^2\theta\log(1+r\cos\theta)}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \lim_{r\to 0} \cos^2\theta\log(1+r\cos\theta)$$

$$= 0$$

$\therefore \; c = 0.$

6 (f) for the continuity of $f(x,y)$ at $(0,0)$, we have

$$f(0,0) = \lim_{(x,y)\to(0,0)} f(x,y)$$

$$\Rightarrow \quad c = \lim_{(x,y)\to(0,0)} \frac{e^{-(x^2+y^2)}-1}{x^2+y^2} = c$$

putting $x = r\cos\theta$, $y = r\sin\theta$, $(x,y)\to(0,0) \Rightarrow r\to 0$

$$\Rightarrow \quad c = \lim_{r\to 0} \frac{e^{-(r^2\cos^2\theta + r^2\sin^2\theta)}-1}{(r^2\cos^2\theta + r^2\sin^2\theta)}$$

$$= \lim_{r\to 0} \frac{e^{-r^2}-1}{r^2} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{r\to 0} \frac{-2re^{-r^2}}{2r} \qquad \{\text{by L'Hopitals rule}\}$$

$$= -1$$

$$\therefore \quad c = -1$$

6 (g) for the continuity of $f(x,y)$ at $(0,0)$, we have

$$f(0,0) = \lim_{(x,y)\to(0,0)} f(x,y)$$

$$\Rightarrow \quad c = \lim_{(x,y)\to(0,0)} \exp\left(\frac{-1}{|x-y|}\right)$$

$$= \lim_{r\to 0} \exp\left(\frac{-1}{|r\cos\theta - r\sin\theta|}\right)$$

$$= \lim_{r\to 0} \exp\left(\frac{-1}{r|\cos\theta-\sin\theta|}\right)$$

$x = r\cos\theta$
$y = r\sin\theta$
$(x,y)\to(0,0)$
$\Rightarrow r\to 0$

$$= 0$$

$$\therefore \quad c = \underline{0}.$$

$\frac{3}{3}$

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for the continuity of $f(x,y)$ at $(0,0)$, we have

$$f(0,0) = \lim_{(x,y)\to(0,0)} f(x,y)$$

$$\Rightarrow c = \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{1+x^2+y^2}$$

$$= 0$$

$\therefore$ $c = 0$

7. (a) $f(x,y) = \dfrac{x-y}{1+x+y}$

The function $f(x,y)$ is discontinuous only at the points where $1+x+y=0$

ie $f(x,y)$ is discontinuous on the line $x+y+1=0$

(b) $f(x,y) = \dfrac{x-y}{1+x^2+y^2}$

Now for all x,y, $x^2+y^2 \geq 0$

$$\Rightarrow x^2+y^2 \neq -1$$

$$\Rightarrow 1+x^2+y^2 \neq 0$$

So $\dfrac{x-y}{1+x^2+y^2}$ is defined for all points (x,y)

$\therefore$ $f(x,y)$ has no points of discontinuity.

(c) $f(x,y) = \dfrac{xy}{1+e^{x-y}}$

Notice that the given function is discontinuous only when $1+e^{x-y}=0$, but $1+e^{x-y}$ will always be non zero.

So all possible points will be in the domain and have a limit.

Therefore $f(x,y)$ has no points of discontinuity.

8) (a) $f(x,y) = \dfrac{1}{\sin^2\pi x + \sin^2\pi y}$

The function $f(x,y)$ will be discontinuous only at the points where $\sin^2\pi x + \sin^2\pi y = 0$

$\Rightarrow \sin^2\pi x = 0$ and $\sin^2\pi y = 0$

$\Rightarrow \sin\pi x = 0$ and $\sin\pi y = 0$

$\Rightarrow \pi x = m\pi$ and $\pi y = n\pi$, $n,m \in \mathbb{Z}$

$\Rightarrow x = m$ and $y = n$

(b) $f(x,y) = \dfrac{e^x+e^y}{e^{xy}-1}$

The function $f(x,y)$ will be discontinuous only at the points where $e^{xy}-1=0$

ie $e^{xy} = 1$

ie $xy = 0$

ie $x = 0$ or $y = 0$

9. $$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

$f(x,y)$ will be continuous at $(0,0)$ if we can find a limit of $f(x,y)$ at $(0,0)$

Now $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

$$= \lim_{r\to 0} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$ [putting $x = r\cos\theta$
$y = r\sin\theta$

$$= \lim_{r\to 0} \frac{r^2 \cos 2\theta}{r^2} = \cos 2\theta$$, depends on θ

we get different limits for different values of θ.

$\therefore$ $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$\therefore$ $f(x,y)$ is not continuous at $(0,0)$

10. $f(x,y) = \begin{cases} 0 , & xy \neq 0 \\ 1 , & xy = 0 \end{cases}$

(i) Since $f(x,y)$ is constantly zero along the path $y = x$ (except at the origin), we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} 0 = 0$$

(ii) Since $f(0,0) = 1$ and the limit of the function $f(x,y)$ at $(0,0)$ along the path $y = x$ is $0 \neq 1$

So f is not continuous at origin.