Solution of Tutorial-7 for PH11001/Spring2018

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Optics: Week 8

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1. Assume a non-absorbing glass plate of refractive index n (which is close to 1) and thickness x is placed between a source S and an observer P. In absence of the plate the observer sees the wave as, $E = E_0 e^{-i(k_0 x - \omega t)}$. Show that in presence of the plate the wave will be of the form, $\tilde{E} \simeq E\left[1 + k_0(n-1)\Delta x e^{-i\pi/2}\right]$.

Ans. 1:

The phase difference ($\Delta\phi$) due to the presence of the glass plate of thickness Δx and refractive index n is:

$$\Delta\phi = k_0(n\Delta x - \Delta x) = k_0(n-1)\Delta x \quad \text{---------------------------- (i)}$$

Thus, in the presence of the plate, the observer sees an extra phase added to the original wave, or,

$$\tilde{E} = E_0 e^{-i(k_0 x - \omega t) - i\Delta\phi} = E e^{-i k_0 (n-1)\Delta x} \text{---------------------------- (ii)}$$

Expanding (ii) in Taylor Series and assuming Δx is small or $n \approx 1$, we get,

$$\tilde{E} \simeq E[1 - i k_0 (n-1)\Delta x] \text{ ------------------------(iii)}$$

$$\text{Again, } -i = e^{-i\pi/2}$$

$\therefore$ Eqn (iii) can be written as: $\boxed{\tilde{E} \simeq E\left[1 + k_0(n-1)\Delta x e^{-i\pi/2}\right]} \rightarrow$ Answer

2. The reflection and transmission coefficient is defined as, $r_{ij} = \frac{k_i - k_j}{k_i + k_j}$ and $t_{ij} = \frac{2k_i}{k_i + k_j}$.

(a) Prove the Stokes relation $r_{12} = -r_{21}$ and $1 - {r_{21}}^2 = t_{12} t_{21}$.

(b) Prove the same using time-reversal argument of transmission and reflection of light. Assume absorption is absent.

Ans. 2(a):

Stokes relations

$$\boxed{r_{12} = \frac{k_1 - k_2}{k_1 + k_2} = -\frac{k_2 - k_1}{k_1 + k_2} = -r_{21}}$$

$$1 - r_{21}^2 = 1 - \frac{(k_2 - k_1)^2}{(k_1 + k_2)^2} = \frac{k_1^2 + k_2^2 + 2k_1 k_2 - k_1^2 - k_2^2 + 2k_1 k_2}{(k_1 + k_2)^2}$$

$$\boxed{\text{or, } 1 - r_{21}^2 = \frac{4k_1 k_2}{(k_1 + k_2)^2} = t_{12} t_{21}}$$

Ans: 2(b):

The principle of optical reversibility:

According to this principle, in the absence of any absorption, a light ray that is reflected or refracted will retrace its original path if its direction is reversed.

Consider a light ray incident on an interface of two media of refractive indices n_1 and n_2 as shown in Fig. 2(a) (given below). Let the amplitude reflection and transmission coefficients be r_1

and t_1, respectively. Thus, if the amplitude of the incident ray is a, then the amplitudes of the reflected and refracted rays are ar_1 and at_1, respectively.

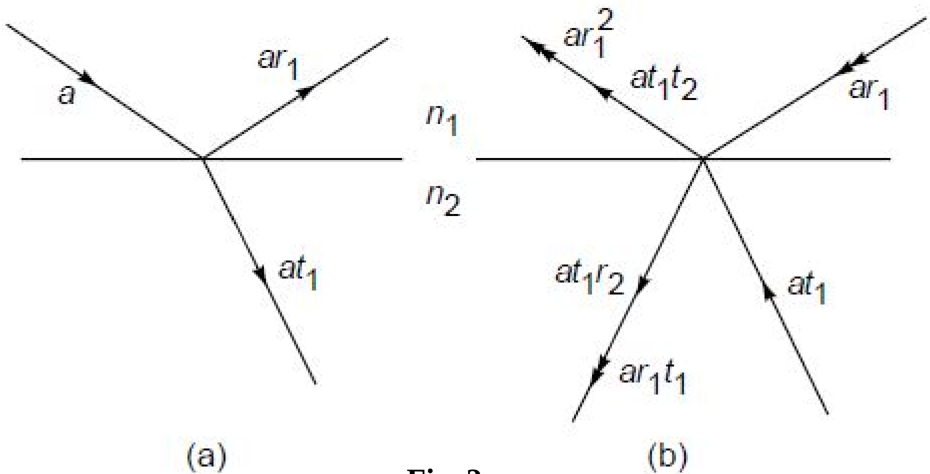


Fig. 2

We now reverse the rays, and we consider a ray of amplitude at_1 incident on medium 1 and a ray of amplitude ar_1 incident on medium 2 as shown in Fig. 2(b). The ray of amplitude at_1 will give rise to a reflected ray of amplitude at_1r_2 and a transmitted ray of amplitude at_1t_2, where r_2 and t_2 are the amplitude reflection and transmission coefficients, respectively, when a ray is incident from medium 2 on medium 1. Similarly, the ray of amplitude ar_1will give rise to a ray of amplitude ar_1^2 and a refracted ray of amplitude ar_1t_1.

According to the principle of optical reversibility, the two rays of amplitudes ar_1^2 and at_1t_2 must combine to give the incident ray of Fig. 2(a); thus

$$ar_1^2 + at_1t_2 = a$$

$$t_1t_2 = 1 - r_1^2$$

Further, the two rays of amplitudes at_1r_2 and ar_1t_1 must cancel each other, i.e.,

$$at_1r_2 + ar_1t_1 = 0$$

$$r_2 = - r_1$$

Hence, Stokes relations are proved using time-reversal argument of transmission and reflection of light.

3. Consider a medium where refractive index changes layer-wise (say, n_1, n_2, n_3.....) along z direction. The optical ray is in *xz* plane.

(a) Show that, $n_1\cos\theta_1 = n_2\cos\theta_2 = n_3\cos\theta_3 = \ldots = \beta$, where θ_i is the angle made by optical ray with x axis in i^{th} layer.

(b) When the thickness of each layer becomes infinitesimally small, the variation of refraction index is continuous. Under such condition establish the ray-path equation which is

$(dz/dx)^2 = n^2(z)/\beta^2 - 1$.

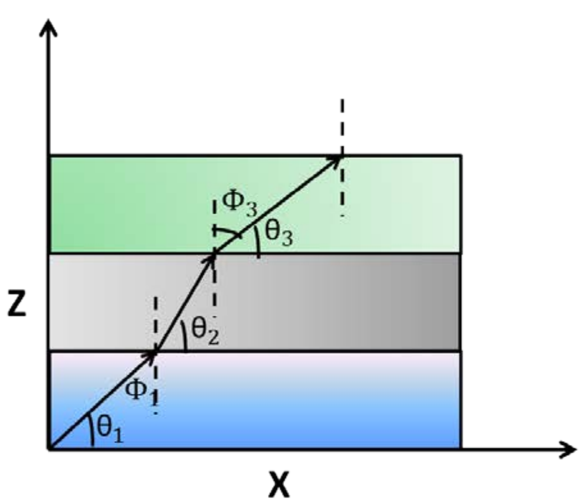
(c) Show that the ray-equation can be written in more convenient form as, $\frac{d^2z}{dx^2} = \frac{1}{2\beta^2}\frac{dn^2(z)}{dx}$.

(d) From the ray-equation find out the ray-path when $n(z) = n_0 + kz$.

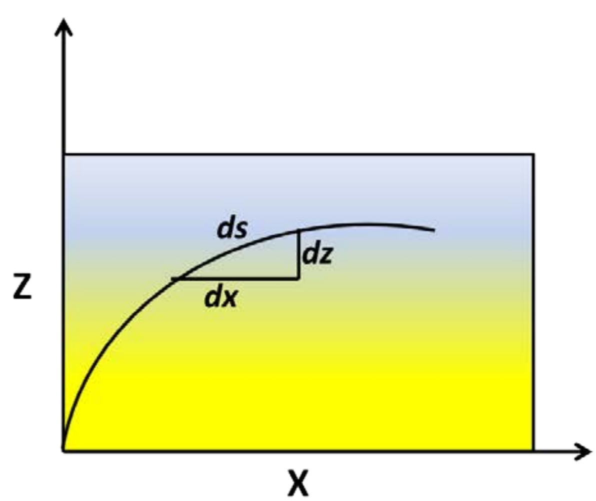
Ans 3(a):

In an inhomogeneous medium, the refractive index varies in a continuous manner and, in general, the ray paths are curved. As a limiting case, it can be thought to be consisting of a continuous set of thin slices of media of different refractive indices; see Fig. 3(a). At each interface, the light ray satisfies Snell's law, and one obtains:

$$n_1 sin\Phi_1 = n_2 sin\Phi_2 = n_3 sin\Phi_3 = \ldots\ldots\ldots$$



3(a)



3(b)

Again, in each layer, $\theta_1 + \phi_1 = 90°$. So $\sin\phi_1 = \sin(90-\theta_1) = \cos\theta_1$

Hence in the i^{th} layer, $\sin\phi_i = \cos\theta_i$.

So substituting in Snell's law we obtain,

$$n_1 \cos\theta_1 = n_2 \cos\theta_2 = n_3 \cos\theta_3 = \ldots.$$

Thus, the product $n(z)cos\theta(z)$ is an invariant of the ray path; we will denote this invariant by β. This implies that as the refractive index changes, the ray path bends in such a way that the product $n(z)cos\theta(z)$ remains constant

So we obtain,

$$\boxed{n_1 \cos\theta_1 = n_2 \cos\theta_2 = n_3 \cos\theta_3 = \ldots. \quad \beta} \rightarrow \text{Answer}$$

Ans 3(b):

In the limiting case of a continuous variation of refractive index, the piecewise straight lines shown in Fig. 3(a) form a continuous curve which is determined from the equation

$$n(z)cos\theta(z) = n_1 cos\theta_1 = \beta \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

From Fig. 3 (b), if ds represents the infinitesimal arc length along the curve, then

$$(ds)^2 = (dx)^2 + (dz)^2$$

$$\left(\frac{ds}{dx}\right)^2 - 1 = \left(\frac{dz}{dx}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Now, if we refer to Fig. 3 (b), we find that,

$$\frac{dx}{ds} = \cos\theta = \frac{\beta}{n(z)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

So substituting in (ii), we obtain,

$$\boxed{\left(\frac{dz}{dx}\right)^2 = \frac{n^2(z)}{\beta^2} - 1} \ldots\rightarrow \text{Answer} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

Ans.3(c):

For a given $n(z)$ variation, Eq. (iv) can be integrated to give the ray path $z(x)$; however, it is often more convenient to put Eq. (iv) in a slightly different form by differentiating it with respect to x

$$2\frac{dz}{dx}\frac{d^2z}{dx^2} = \frac{1}{\beta^2}\frac{dn^2(z)}{dz}\frac{dz}{dx}$$

or, $$\boxed{\frac{d^2z}{dx^2} = \frac{1}{2\beta^2}\frac{dn^2(z)}{dz}}$$ → Answer…… (v)

Ans.3 (d):

The ray paths in a medium characterized by the refractive index variation

$$n(z) = n_0 + kz$$

For the above profile, the ray equation [Eq. (v)] takes the for

$$\frac{d^2z}{dx^2} = \frac{2k(n_0 + kz)}{2\beta^2}$$

Let, $Z = z + \frac{n_0}{k}$, $K = \frac{k}{\beta}$

∴The above Eq. can be written as,

$$\frac{d^2Z}{dx^2} = K^2 Z(x)$$

This Eq. has a solution of the form,

$$Z(x) = C_1 e^{Kx} + C_2 e^{-Kx}$$, where C_1 and C_2 are constants

Thus the ray path is given by,

$$\boxed{z(x) = -\frac{n_0}{k} + C_1 e^{Kx} + C_2 e^{-Kx}}$$ →Answer

4. Suppose that the intensity of the sunlight falling on the ground on a particular day is $140 W/m^2$.

(a) What are the peak values of electric and magnetic fields associated with the incident radiation?

(b) A linearly polarised harmonic plane wave with scalar amplitude of 10V/m is propagating along a line in the xy plane at 45 degrees to the *x* axis, with the *xy* plane as the plane of vibration. Write down a vectorexpression for the wave, assuming k_x and k_y are both positive. Calculate the flux density taking the wave tobe in vacuum.

Ans.4 (a):

The peak values of electric and magnetic fields are related to the intensity of the incident radiation and are given as:

$$I = \frac{E_0^2}{2c\mu_0}$$

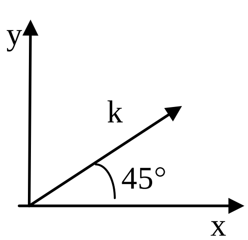
$\therefore$

$$\boxed{\begin{aligned} E_0 &= \sqrt{2c\mu_0 I} \approx 324.9\frac{V}{m} \\ B_0 &= \frac{E_0}{c} \approx 1 \times 10^{-6}\, T \end{aligned}}$$

Answer

Ans. 4(b):

As the wave propagates along a line in the xy plane 45° to the axis,



$K = k_x\hat{\imath} + k_y\hat{\jmath}$, $k_x = K\cos 45° = K/\sqrt{2}$

$k_y = K\sin 45° = K/\sqrt{2}$

The plane of vibration is also the xy plane, so the direction of vibration ($\hat{n}$) is perpendicular to $\vec{K}$.

Let $\hat{n} = \frac{a\hat{\imath}+b\hat{\jmath}}{\sqrt{a^2+b^2}}$. $\hat{n}.\vec{K} = 0 \Rightarrow \left(\frac{aK}{\sqrt{2}} + \frac{bK}{\sqrt{2}}\right) = 0 \Rightarrow b = -a$

$$\therefore \quad \hat{n} = \frac{\hat{\imath}-\hat{\jmath}}{\sqrt{2}}$$

So the vector expression for the wave is:

$$\boxed{\vec{E} = E_0\left(\frac{\hat{\imath}-\hat{\jmath}}{\sqrt{2}}\right)\cos\left(\frac{K}{\sqrt{2}}x + \frac{K}{\sqrt{2}}y - \omega t\right),\ E_0 = 10\text{V/m}}$$

The flux density is given as the magnitude of the time average of the Poynting's vector.

$\therefore$Flux density $= |\langle S\rangle| = \frac{|E||B|}{\mu_0}\langle\cos^2\left(\frac{K}{\sqrt{2}}x + \frac{K}{\sqrt{2}}y - \omega t\right)\rangle$

$$|\langle S\rangle| = \frac{|E|^2}{2c\mu_0},\ |E| = \frac{10}{\sqrt{2}}$$

$$\boxed{\therefore |\langle S\rangle| = \frac{100}{2\times2\times3\times10^8\times4\pi\times10^{-7}} \approx 66.3\ mJm^{-2}s^{-1}} \longrightarrow \text{Answer}$$