Indian Institute of Technology Kharagpur
Department of Mathematics
Advanced Calculus
Tutorial Problem Sheet - 2
Autumn 2022

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1. Use Taylor's theorem to prove that

   a) $x - \frac{x^2}{2} < \log(1+x) < x$ for $x > 0$.

   b) $\cos x \geq 1 - \frac{x^2}{2}$ for $-\pi < x < \pi$.

   c) $1 + \frac{x}{2} - \frac{x^2}{8} < \sqrt{1+x} < 1 + \frac{x}{2}$ for $x > 0$.

2. Let $c \in \mathbb{R}$ and a real function f be such that f'' is continuous on some neighbourhood of c. Prove that
$$\lim_{h\to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$$

3. Let $a \in \mathbb{R}$ and a real function f defined on some neighbourhood $N(a)$ of a such that f'' is continuous at a and $f''(a) \neq 0$. Prove that $\lim\limits_{h\to 0} \theta = \frac{1}{2}$, where θ is given by $f(a+h) = f(a) + hf'(a+\theta h)$ $(0 < \theta < 1)$.

4. Each of the series in the following is the value of the Taylor series at $x = 0$ of a function $f(x)$ at a particular point. What function and what point? What is the sum of the series?

   a) $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \ldots$

   b) $\frac{2}{3} - \frac{4}{18} + \frac{8}{81} - \ldots$

   c) $\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} + \frac{1}{45\sqrt{3}} - \ldots$

5. Using Taylor series expansion, evaluate

   a) $\lim\limits_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

   b) $\lim\limits_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$

   c) $\lim\limits_{x\to 0} \frac{\tan x - x}{x^2 \tan x}$

   d) $\frac{\cosh x - \cos x}{x \sin x}$

6. If f is continuous at x_0, and there are constants a_0 and a_1 such that
$$\lim_{x\to x_0} \frac{f(x) - a_0 - a_1(x - x_0)}{x - x_0},$$
then prove that $a_0 = f(x_0)$, f is differentiable at x_0, and $f'(x_0) = a_1$.

7. Obtain the Maclaurin's series expansion of $f(x) = \sin(m \sin^{-1} x)$, where m is a constant.

8. For the Maclaurin's polynomial approximation of degree less than or equal to n for the function e^x, determine the value of n such that the error satisfies $|R_n(x)| \leq 0.005$, when $-1 \leq x \leq 1$.

9. (Binomial Expansion)
(a) Find the 'n-th'term for the Maclaurin series expansion of $(1+x)^h$, where h is a fixed non-zero real number.
(b) Using (a), for a positive integer m, re-prove the binomial expansion of $(1+x)^m$:

$$(1+x)^m = \binom{m}{0} + \binom{m}{1}x + \cdots + \binom{m}{k}x^k + \cdots + \binom{m}{m}.$$

10. (a) Estimate $\sqrt{1.5}$ using first three terms of the binomial expansion of $f(x) = \sqrt{1+x}$.
(b) Use Lagrange's form of remainder to bound the error.

11. Let L be the length of a pendulum that makes a maximum angle θ_0 with the vertical. The period of the pendulum T is given by the following formula:

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}},$$

where $k = \sin(\theta_0/2)$ and g denotes the accelaration due to gravity.
(a) Use only the first term of the binomial series to approximate T.
(b) Use first two terms of the binomial series to approximate T.

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