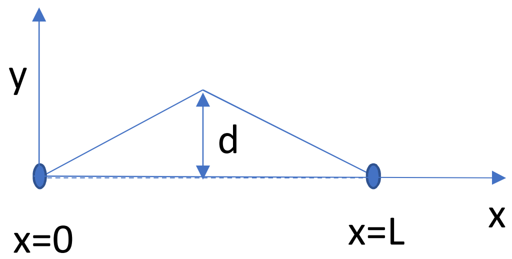
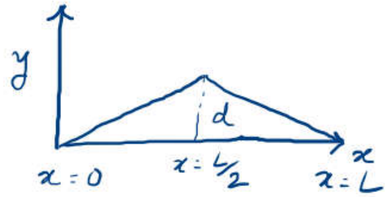
1 (a). A string of length L=1 m between two fixed points is displaced at its mid point by a distance d=10 cm and released at t=0 (as shown in the Fig.1). Find the amplitude of the fundamental mode that is excited.



Q.1: Sol

shape of the string @ t=0



$$f(x) = \frac{2d}{L}x \quad 0 \leq x \leq \frac{L}{2}$$

$$f(x) = 2d - \frac{2d}{L}x \quad \frac{L}{2} \leq x \leq L$$

The amplitude can be calculated using Fourier series

$$B_n = \frac{2}{L}\int_0^{L/2} f(x)\sin\left(\frac{n\pi}{L}x\right)dx + \int_{L/2}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

(only sin term will be there as per the boundary conditions)

$$B_n = \frac{8d}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

First normal mode: $n=1$

$$B_1 = \frac{8d}{(\pi)^2} \Rightarrow \frac{8\times 0.1}{(\pi)^2} \approx 0.081\ m$$

1 (b). A string of length L=1.5 m between two fixed points is displaced at its mid point by a distance d=15 cm and released at t=0 (as shown in the Fig.1). Find the amplitude of the fundamental mode that is excited.

$$B_1 = \frac{8d}{(\pi)^2} = \frac{8 \times 15 \times 10^{-2}}{(\pi)^2} = 0.12\, m.$$

2 (a) . Consider a sound wave travelling in a solid with Young's modulus $Y = 2 \times 10^{11}$ kg $m^{-1}s^{-2}$, and whose mass density is $\rho = 2 \times 10^3$ kg m^{-3}. The wave-solution has the form $\xi(x, t) = A \cos^2(kx-(2\pi \times 10^2)t)$, where x and t are measured in SI units. The wavelength of the given wave form is

Sol:

$$\xi(x,t) = A\left[1 + \cos 2\left[kx - (2\pi \times 10^2)t\right]\right]$$

[Constant can be neglected]

$$C = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{2\times10^{11}}{2\times10^{3}}} = 10^4 \text{ m/s}$$

$$\omega = 4\pi \times 10^2 \text{ rad } s^{-1}$$

$$\nu = \frac{\omega}{2\pi} = \frac{\cancel{4}^{2}\cancel{\pi}\times10^2}{\cancel{2}\cancel{\pi}} = 200 \text{ Hz.}$$

$$\lambda = \frac{c}{\nu} = \frac{10^4}{200} = \underline{50 \text{ m}}.$$

2 (b). Consider a sound wave travelling in a solid with Young's modulus $Y = 4 \times 10^{11}$ kg $m^{-1}s^{-2}$, and whose mass density is $\rho = 1 \times 10^3$ kg m^{-3}. The wave-solution has the form $\xi(x, t) = A\cos^2(kx - (2.5\pi \times 10^2)t)$, where x and t are measured in SI units. The wavelength of the given wave form is

Sol:

$$C = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{4\times10^{11}}{1\times10^{3}}} = \sqrt{4\times10^{8}} = 2\times10^{4} \text{ m/s}$$

$$\omega = 5\pi\times10^{2} \text{ rad/s} \Rightarrow \nu = \frac{5\pi\times10^{2}}{2\pi} = 2.5\times10^{2}$$

$$= 250 \text{ Hz}$$

$$\lambda = \frac{2\times10^{4}}{250} = \underline{\underline{80 \text{ m}}}$$

3 (a). *The phase velocity of a surface wave on a liquid of density ρ and surface tension is given by*

$$v_p = \left(\frac{g\lambda}{2\pi} + \frac{2\pi T}{\lambda\rho}\right)^{\frac{1}{2}}$$

where λ is the wavelength of the wave and g is the acceleration due to gravity.
Using the values: T= 0.07 N/m and ρ = 1000 kg/m³ and *g= 10 m/s²*

(a) Find the wavelength for which v_p is minimum

(b)The group velocity at the above wavelength

Sol. (a) $\lambda_{min} = 2\pi \left(\frac{T}{\rho g}\right)^{1/2} = 1.66 \text{ cm}$

(b) $V_g = \dfrac{g + \left(\frac{12\pi^2 T}{\rho\lambda^2}\right)}{2\left[\frac{2\pi g}{\lambda} + \frac{8\pi^3 T}{(\rho\lambda^3)}\right]^{1/2}} \approx 0.23 \text{ m/s}$

3 (b). The phase velocity of a surface wave on a liquid of density ρ and surface tension is given by

$$v_p = \left(\frac{g\lambda}{2\pi} + \frac{2\pi T}{\lambda\rho}\right)^{\frac{1}{2}}$$

where λ is the wavelength of the wave and g is the acceleration due to gravity.
Using the values: T= 0.04 N/m and ρ = 1024 kg/m³ and *g= 10 m/s²*

(a) Find the wavelength for which v_p is minimum

(b)The group velocity at the above wavelength

Sol:

$\lambda_{min} = 1.24$ cm

$V_g = 0.19$ m/s

4 (a). String 1 of linear mass density 1 g/cm is joined with a string 2 of linear mass density 4 g/cm and the combination is held under constant tension. A transverse sinusoidal wave of amplitude 3 cm and wavelength 26 cm is launched along the string 1. The amplitude of the wave when it is traveling on string 2 is

Sol:

$$\frac{A_2}{A_1} = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}} = \frac{2 \times 1}{1 + 2} = \frac{2}{3}$$

$$A_2 = \frac{2}{3} A_1 = \underline{2\,\text{cm}}$$

4 (b). String 1 of linear mass density 4 g/cm is joined with a string 2 of mass 36 g/cm and the combination is held under constant tension. A transverse sinusoidal wave of amplitude 6 cm and wavelength 15 cm is launched along the string 1. The amplitude of the wave when it is traveling on string 2 is

Sol:

$$\frac{A_2}{A_1} = \frac{2 \times 2}{2 + 6} = \frac{4}{8}$$

$$A_2 = \frac{1}{2} A_1 = \frac{1}{2} 6 \Rightarrow 3\,\text{cm}$$

5 (a). String 1 of linear mass density 4 g/cm is joined with a string 2 of mass 36 g/cm and the combination is held under constant tension. A transverse sinusoidal wave of amplitude 6 cm and wavelength 15 cm is launched along the string 1. The wavelength of the wave when it is travelling on string 2 is

<u>Sol:</u>

$$\frac{\lambda_2}{\lambda_1} = \sqrt{\frac{\cancel{4}}{\cancel{36}\,9}} = \frac{1}{3} \Rightarrow \lambda_2 = \frac{1}{3}\lambda_1 = \frac{1}{3}(15\,cm) = \underline{\underline{5\,cm}}$$

5 (b) String 1 of linear mass density 1 g/cm is joined with a string 2 of linear mass density 4 g/cm and the combination is held under constant tension. A transverse sinusoidal wave of amplitude 3 cm and wavelength 26 cm is launched along the string 1. The wavelength of the wave when it is travelling on string 2

Sol:

$$\frac{\lambda_2}{\lambda_1} = \sqrt{\frac{\mu_1}{\mu_2}} \Rightarrow \lambda_2 = \sqrt{\frac{\mu_1}{\mu_2}} \cdot \lambda_1$$

$$= \sqrt{\frac{1}{4}}\,\lambda_1 \Rightarrow \frac{\lambda_1}{2} = \underline{13\text{ cm}}$$

6 (a). The induced electric field at a location is given by $\vec{E}(x, y, z) = E_0\left(\left(\frac{z}{2}\right)^2 \hat{\imath} + \left(\frac{x}{3}\right)^2 \hat{\jmath} + \left(\frac{y}{2}\right)^2 \hat{k}\right)$ where E_0= 6 SI Units. The magnitude of the $\hat{\jmath}$ component of the rate of change with time of the magnetic field (in SI units) at a location (3,4,6) is

Sol:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (\text{we know})$$

$$\frac{\partial \vec{B}}{\partial t} = -E_0\left[\frac{2y}{y_0^2}\hat{i} + \frac{2z}{z_0^2}\hat{j} + \frac{2x}{x_0^2}\hat{k}\right]$$

$$\frac{\partial \vec{B}}{\partial t} = -6\left[\frac{2y}{16}\hat{i} + \frac{2z}{4}\hat{j} + \frac{2x}{9}\right]$$

$$\left|\frac{\partial \vec{B}}{\partial t}\right|_{(3,4,6)} = 6\times\frac{2z}{4}\Bigg|_{(3,4,6)} = \frac{6\times\cancel{2}\times\cancel{6}^{\,3}}{\cancel{4}\,2} = \underline{\underline{18}}$$

6 (b). The induced electric field at a location is given by $\vec{E}(x,y,z) = E_0\left(\left(\frac{z}{1}\right)^2\hat{\imath} + \left(\frac{x}{2}\right)^2\hat{\jmath} + \left(\frac{y}{2}\right)^2\hat{k}\right)$ where E_0= 3 SI Units.

The magnitude of the $\hat{\imath}$ component of the rate of change with time of the magnetic field (in SI units) at a location (2,4,1) is

$$\frac{\partial B}{\partial t} = -E_0\left[\frac{2y}{4}\hat{i} + \frac{2z}{1}\hat{j} + \frac{2x}{4}\hat{k}\right] \qquad x_0, y_0, z_0 = (2,2,1)$$

$$\left|\frac{\partial B}{\partial t}\right|_{(2,4,1)} = \frac{E_0 \times 2 \times y}{4} = \frac{3\times 2\times 4}{\cancel{4}} = 6$$

5 (c) . The induced electric field at a location is given by $\vec{E}(x,y,z) = E_0\left(\left(\frac{z}{1}\right)^2\hat{\imath} + \left(\frac{x}{2}\right)^2\hat{\jmath} + \left(\frac{y}{2}\right)^2\hat{k}\right)$ where E_0= 4 SI Units. The magnitude of the $\widehat{k}$ component of the rate of change with time of the magnetic field (in SI units) at a location (1,3,2) is

Sol:

$$\left|\frac{\partial B}{\partial t}\right|_{(1,3,2)} = E_0\,\frac{2x}{x_0^2} = \frac{4\times 2\times 1}{4} = 2$$

7. Consider a medium in which the phase velocity is given as $v_p = A\omega^3$, where A is a constant. The ratio of phase velocity to the group velocity is

Solution:

$$v_p = \frac{\omega}{k} = A\omega^3 \Rightarrow \omega^2 = \frac{1}{Ak}$$

$$\text{Differentiate wrt} k,\ 2\omega\frac{d\omega}{dk} = -\frac{1}{Ak^2} = -\frac{k\omega^2}{k^2}$$

$$v_g = \frac{d\omega}{dk} = -\frac{\omega}{2k} \Rightarrow \frac{v_p}{v_g} = -2$$

8. The given equation represents a 2-D wave. The ratio of phase velocity to the group velocity is

$$\xi(x,y,t) = 2\sin(1.00x + 1.00y - 2.0t)\cos(0.04x + 0.04y - 0.2t)$$

Sol:

$$\xi(x,y,t) = 2\sin(1.00x + 1.00y - 2.0t)\cos(0.04x + 0.04y - 0.2t)$$

$$k_x = 1.0,\ k_y = 1.0, \Rightarrow k = \sqrt{k_x^2 + k_y^2} = \sqrt{2}$$

$$\Delta k_x = 0.04,\ \Delta k_y = 0.04, \Rightarrow \Delta k = \sqrt{(\Delta k_x)^2 + (\Delta k_y)^2} = 0.04\sqrt{2}$$

$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{0.2}{0.04\sqrt{2}} \text{unit},\ v_p = \frac{\omega}{k} = \frac{2}{\sqrt{2}} = \sqrt{2}\ \text{unit}$$

$$\Rightarrow \frac{v_g}{v_p} = \frac{0.2}{0.08} = 2.5$$

9. Calculate the surface integral $\oiint \frac{1}{\pi}(2\,x\,\hat{\imath} + 2\,y\,\hat{\jmath} - z\,\hat{k}).\,d\vec{s}$ over a sphere of radius 2.

$$\oiint_S \frac{1}{\pi}\left[2x\,\hat{i} + 2y\,\hat{j} - z\,\hat{k}\right]\cdot d\vec{s}$$

$$= \iiint_V \frac{1}{\pi}\,\vec{\nabla}\cdot\left[2x\,\hat{i} + 2y\,\hat{j} - z\,\hat{k}\right] dv$$

$$= \iiint_V \frac{1}{\pi}\left[2+2-1\right] dv$$

$$= \frac{3}{\pi}\iiint dv = \frac{3}{\pi} \text{ volume of the sphere}$$

$$= \frac{3}{\pi}\,\frac{4}{3}\pi r^3 = 4\,2^3 = \underline{\underline{32}}$$

10. Calculate the line integral $\oint \frac{1}{\pi}(\,x\;\hat{j})\cdot d\vec{l}$ in the counter clockwise direction along the circle $x^2 + y^2 = 2$.

10) $\oint \frac{1}{\pi}(x\hat{j})\cdot d\vec{l}$

$$= \oiint_S \frac{1}{\pi}(\vec{\nabla}\times x\hat{j})\cdot d\vec{s}$$

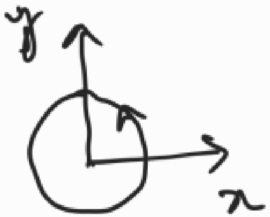
$$= \oiint_S \frac{1}{\pi}\hat{k}\cdot d\vec{s}$$

here $d\vec{s} = \hat{k}\, ds$

$$= \frac{1}{\pi}\oiint_S ds$$

$$= \frac{1}{\pi}\pi r^2$$

$$= \underline{\underline{2}}$$



$\vec{\nabla}\times x\hat{j}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x & 0 \end{vmatrix}$$

$$= \hat{k}$$

In an experiment to study coupled pendulum, where two simple pendula are coupled by a spring, it is observed that the time taken for 10 oscillations for the system to oscillate in the **in phase** mode is 13 seconds while the time taken for 10 oscillations for the system to oscillate in the **out of phase** mode is 12 seconds. In this system, if one of the pendulum is kept at rest and the other is displaced by a certain amount and released, the phenomenon of beating is observed. Find the time period of the beats.

$$T_0 = \frac{2\pi}{\omega_0} = \frac{13}{10} = 1.3 \text{ sec.}$$

$$T_1 = \frac{2\pi}{\omega_1} = \frac{12}{10} = 1.2 \text{ sec.}$$

$$T_B = \frac{4\pi}{\omega_1 - \omega_0} = \frac{2 T_0 T_1}{T_0 - T_1}$$

$$= 31.2 \text{ sec.}$$

Two simple pendula, each of length (l) having masses (m_1) and (m_2) are coupled by a spring of spring constant (k) as shown in the figure . Take l = 10 m, m_1 = 1 kg, m_2 = 2 kg, k = 1 Nm^{-1} and g = 10 ms^{-2}. Find the normal mode frequencies [slow (ω_1) and fast (ω_2)] of the system. [2 Marks]

$$m_1 \ddot{x}_1 = -m_1 \frac{g}{l} x_1 - k(x_1 - x_2)$$

$$m_2 \ddot{x}_2 = -m_2 \frac{g}{l} x_2 - k(x_2 - x_1)$$

Assumed solutions:- $x_1 = A e^{i\omega t}$, $x_2 = B e^{i\omega t}$

$$\therefore \; -m_1 \omega^2 A = -m_1 \frac{g}{l} A - k(A - B)$$

$$-m_2 \omega^2 B = -m_2 \frac{g}{l} B - k(B - A)$$

For non-trivial solutions:-

$$\begin{vmatrix} -m_1\omega^2 + m_1 \frac{g}{l} + k & -k \\ -k & -m_2\omega^2 + m_2\frac{g}{l} + k \end{vmatrix} = 0$$

$\therefore$ Putting the values of l, k, m_1 & m_2

$$\begin{vmatrix} -\omega^2 + 2 & -1 \\ -1 & -2\omega^2 + 3 \end{vmatrix} = 0$$

$\therefore \; \omega^2 = 1$ and $\omega^2 = 5/2$.

$\omega_1 = 1$ rad s^{-1}

$\omega_2 = 1.58$ rad s^{-1}

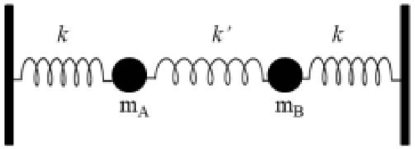
Consider the situation shown in figure where both the masses m_A and m_B are equal (i.e $m_A = m_B = m$). The spring constant of the spring coupling m_A and m_B is (k') while the spring constant of the other two springs are k. If q_1 and q_2 represent the amplitudes of normal modes of the system and E is the total energy of the system. Suppose $k' = k/2$ then which of the following is correct: [2 Marks]

(a) $E = (\frac{m}{4})(\dot{q_1}^2 + \dot{q_2}^2) + (\frac{k}{4})(q_1^2) + \frac{k}{2}(q_2)^2$

(b) $E = (\frac{m}{2})(\dot{q_1}^2 + \dot{q_2}^2) + (\frac{k}{2})(q_1^2) + \frac{k}{2}(q_2)^2$

(c) $E = (\frac{m}{4})(\dot{q_1}^2 + \dot{q_2}^2) + (\frac{k}{4})(q_1^2) + \frac{k}{4}(q_2)^2$

(d) $E = (\frac{m}{2})(\dot{q_1}^2 + \dot{q_2}^2) + (\frac{k}{4})(q_1^2) + \frac{k}{4}(q_2)^2$



Answer: Option (a)

Total K.E $\Rightarrow \frac{1}{2} m \dot{x}_0^2 + \frac{1}{2} m \dot{x}_1^2$

Total P.E $\Rightarrow \frac{1}{2} k x_0^2 + \frac{1}{2} k x_1^2 + \frac{1}{2} k (x_0 - x_1)^2$

x_0, x_1 are displacements of m_A & m_B respectively

$q_1 = x_1 + x_0$

$q_2 = x_1 - x_0$

$x_1 = \frac{q_1 + q_2}{2}$: $x_2 = \frac{q_2 - q_1}{2}$

$$E = \frac{1}{2} m \left(\frac{\dot{q}_2 - \dot{q}_1}{2}\right)^2 + \frac{1}{2} m \left(\frac{\dot{q}_1 + \dot{q}_2}{2}\right)^2 + \frac{1}{2} k \left(\frac{q_2 - q_1}{2}\right)^2 + \frac{1}{2} k \left(\frac{q_1 + q_2}{2}\right)^2 + \frac{1}{2} k' q_2^2$$

$$\therefore \boxed{E = \frac{1}{4} m (\dot{q}_1^2 + \dot{q}_2^2) + \frac{k}{4}(q_1^2 + q_2^2) + \frac{1}{2} k' q_2^2}$$

Correct soln:- (A)