1.
(a)

Given that

$$f(x,y) = x\cos y + e^x \sin y$$

$$x(t) = t^2+1 \quad , \quad y(t) = t^3+t$$

We have

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$= \left(\cos y + e^x \sin y\right)2t + \left(x(-\sin y) + e^x \cos y\right)(3t^2+1)$$

when $t=0$ we get $x=1, y=0$

So at $t=0$, $\frac{df}{dt} = (1+e\cdot 0)\cdot 0 + (0+e\cdot 1)\cdot 1 = e$

(b). Given that

$$f(x,y,z) = x^3 + xz^2 + y^3 + xyz$$

$$x(t) = e^t \quad y(t) = \cos t \quad z(t) = t^3$$

We have

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$

$$= \left(3x^2+z^2+yz\right)e^t + \left(3y^2+xz\right)(-\sin t) + \left(2xz+xy\right)\cdot 3t^2$$

At $t=0$, $x=1$, $y=1$, $z=0$

So at $t=0$, $\frac{df}{dt} = (3+0+0)\cdot 1 + (3+0)\cdot 0$
$= 3$

(e). Given that

$f(x_1, x_2, x_3) = 2x_1^2 - x_2 x_3 + x_1 x_3^2$

$x_1(t) = 2\sin t \quad x_2(t) = t^2 - t + 1 \quad x_3(t) = 3^{-t}$

We have

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial z}{\partial x_2}\cdot\frac{dx_2}{dt} + \frac{\partial z}{\partial x_3}\cdot\frac{dx_3}{dt}$$

$$= (4x_1 + x_3^2)\cdot 2\cos t + (-x_3)(2t-1) + (-x_2 + 2x_1x_3)(-3^{-t}\log 3)$$

At $t=0$, $\quad x_1 = 0$, $x_2 = 1$ $\quad x_3 = 1$

$$\therefore \frac{df}{dt} = (0+1)\,2 + (-1)(-1) + (-1+2\cdot 0)(-\log 3)$$

$$= 2+1+\log 3$$

$$= 3 + \log 3$$

2.

(a)(i) We have $f(x,y) = x^y + y^x - c = 0$

From the implicit differentiation formula, we have

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

Here $f_x = y x^{y-1} + y^x \log y$

$f_y = x^y \log x + x y^{x-1}$

So, $$\frac{dy}{dx} = -\frac{y x^{y-1} + y^x \log y}{x^y \log x + x y^{x-1}}$$

(ii). $f(x,y) = xy^2 + \exp(x)\sin(y^2) + \tan^{-1}(x+y) - c$

From the implicit differentiation, we have

$$\frac{dy}{dx} = -\frac{f_x}{f_y}.$$

Here, $f_x = y^2 + \exp(x)\sin(y^2) + \frac{1}{1+(x+y)^2}$

$f_y = 2xy + 2y\exp(x)\cos(y^2) + \frac{1}{1+(x+y)^2}$

$$\therefore \quad \frac{dy}{dx} = -\frac{[1+(x+y)^2][y^2 + \exp(x)\sin(y^2)] + 1}{[1+(x+y)^2][2xy + 2y\exp(x)\cos(y^2)] + 1}$$

$= \cancel{-} \dfrac{\cancel{y^2 + \exp(x)\sin}}{}$

(iii) $f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + 1$

From the implicit differentiation, we have $\frac{dy}{dx} = -\frac{f_x}{f_y}$

$$\frac{\partial f}{\partial x} = \frac{1}{a^2} \cdot 2x \cdot \qquad \frac{\partial f}{\partial y} = \frac{1}{b^2} \cdot 2y$$

$$\therefore \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{b^2 x}{a^2 y}$$

(iv) $f(x,y) = \ln(x^2+y^2) + \tan^{-1}(y/x)$

From the implicit differentiation

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$f_x = \frac{2x}{x^2+y^2} + \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)$$

$$= \frac{2x}{x^2+y^2} - \frac{y}{x^2+y^2} = \frac{2x-y}{x^2+y^2}$$

$$f_y = \frac{2y}{x^2+y^2} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{2y+x}{x^2+y^2}$$

$$\therefore \frac{dy}{dx} = -\frac{2x-y}{2y+x}$$

2) (b) (i) Let, $F(x,y,z) = xy^2z^2 + \sin(yz) - e^{xz^2} = 0$

Then, $\frac{\partial F}{\partial x} = y^2z^2 - z^2e^{xz^2}$

$\frac{\partial F}{\partial y} = 2xyz^2 + z\cos(yz)$

$\frac{\partial F}{\partial z} = 2xy^2z + y\cos(yz) - 2xz\,e^{xz^2}$

By implicit differentiation formula, we have

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-\partial F/\partial x}{\partial F/\partial z}$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{z^2e^{xz^2} - y^2z^2}{2xy^2z + y\cos(yz) - 2xz\,e^{xz^2}}$$

and, $$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(2xyz^2 + z\cos(yz))}{2xy^2z + y\cos(yz) - 2xz\,e^{xz^2}}$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{2xyz^2 + z\cos(yz)}{2xz\cdot e^{xz^2} - 2xy^2z - y\cos(yz)}$$

2. (b) (ii) Let, $F(x,y,z) = x\tan^{-1}\left(\frac{y}{z}\right) + y\tan^{-1}\left(\frac{z}{x}\right) + z\tan^{-1}\left(\frac{x}{y}\right) = 0$

Then, $$\frac{\partial F}{\partial x} = \tan^{-1}\left(\frac{y}{z}\right) + \frac{y}{1+\left(\frac{z}{x}\right)^2}\cdot\left(\frac{-z}{x^2}\right) + \frac{z}{1+\left(\frac{x}{y}\right)^2}\cdot\left(\frac{1}{y}\right)$$

$$= \tan^{-1}\left(\frac{y}{z}\right) - \frac{yz}{x^2+z^2} + \frac{yz}{x^2+y^2}$$

$$\frac{\partial F}{\partial y} = \frac{x}{1+(y/z)^2}\cdot\frac{1}{z} + \tan^{-1}\left(\frac{z}{x}\right) + \frac{z}{1+(x/y)^2}\cdot\left(\frac{-x}{y^2}\right)$$

$$= \tan^{-1}\left(\frac{z}{x}\right) - \frac{xz}{x^2+y^2} + \frac{xz}{y^2+z^2}$$

Similarly, $\frac{\partial F}{\partial z} = \tan^{-1}\left(\frac{x}{y}\right) - \frac{xy}{z^2+y^2} + \frac{xy}{z^2+x^2}$

Then by implicit differentiation formula,

$$\frac{\partial z}{\partial x} = \frac{-F_x}{\partial F_z} = \frac{-\left[\tan^{-1}\left(\frac{y}{z}\right) - \frac{yz}{x^2+z^2} + \frac{yz}{x^2+y^2}\right]}{\tan^{-1}\left(\frac{x}{y}\right) - \frac{xy}{y^2+z^2} + \frac{xy}{x^2+z^2}}$$

and, $$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-\left[\tan^{-1}\left(\frac{z}{x}\right) - \frac{xz}{x^2+y^2} + \frac{xz}{y^2+z^2}\right]}{\tan^{-1}\left(\frac{x}{y}\right) - \frac{xy}{y^2+z^2} + \frac{xy}{x^2+z^2}}$$

(iii). Let $F(x,y,z) = xy^2 + z^3 + \sin(xyz) = 0$

Then $\frac{\partial F}{\partial x} = y^2 + yz\cos(xyz)$

$$\frac{\partial F}{\partial y} = 2xy + xz\cos(xyz)$$

$$\frac{\partial F}{\partial z} = 3z^2 + xy\cos(xyz)$$

By implicit differentiation formula, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y^2 + yz\cos(xyz)}{3z^2 + xy\cos(xyz)}$$

and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2xy + xz\cos(xyz)}{3z^2 + xy\cos(xyz)}$$

2.b.

(iv). Let. $F(x,y,z) = x - yz + \cos(xyz) - x^2z^2 - 1 = 0$

Then $F_x = 1 - yz\sin(xyz) - 2xz^2$

$F_y = -z - xz\sin(xyz)$

$F_z = -y - xy\sin(xyz) - 2zx^2$

By implicit differentiation formula, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{1 - yz\sin(xyz) - 2xz^2}{-y - xy\sin(xyz) - 2zx^2}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

$$= -\frac{-z - xz\sin(xyz)}{-y - xy\sin(xyz) - 2zx^2}$$

(3). Given that

$$u = f(r, s, t)$$

where $r = \frac{x}{y}$ $s = \frac{y}{z}$ $t = \frac{z}{x}$

We have to show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}\left(-\frac{z}{x^2}\right)$$

$$\therefore x\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\left(\frac{x}{y}\right) + \frac{\partial u}{\partial t}\left(-\frac{z}{x}\right) \quad \text{------} \quad (1)$$

Again

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}\left(-\frac{x}{y^2}\right) + \frac{\partial u}{\partial s}\cdot\left(\frac{1}{z}\right) + \frac{\partial u}{\partial t}(0)$$

$$\therefore y\frac{\partial u}{\partial y} = -\frac{x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s} \quad \text{------} \quad (2)$$

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Also

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(-\frac{y}{z^2}\right) + \frac{\partial u}{\partial t}\left(\frac{1}{x}\right)$$

$$\therefore z\frac{\partial u}{\partial z} = -\frac{y}{z}\cdot\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t} \quad \text{———} (3)$$

Adding (1), (2) & (3) we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$$

$$= \frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\cdot\frac{\partial u}{\partial t} - \frac{x}{y}\cdot\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s} - \frac{y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t}$$

$$= 0$$

So, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

(4). $v = f(u)$, u is a homogeneous function of x and y of degree n.

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}$$

$$= x\left[\frac{dv}{du}\cdot\frac{\partial u}{\partial x}\right] + y\left[\frac{dv}{du}\cdot\frac{\partial u}{\partial y}\right]$$ [Since v is a function of u, $\frac{\partial v}{\partial u} = \frac{dv}{du}$]

$$= \left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]\frac{dv}{du}$$

$$= nu\frac{dv}{du}$$ ($\because$ u is a homogeneous function in x, y of degree n)

(5) (a).

$$f(x,y) = \tan^{-1}\frac{y}{x} + \sin^{-1}\frac{x}{y}$$

$$f(tx,ty) = \tan^{-1}\frac{ty}{tx} + \sin^{-1}\frac{tx}{ty}$$

$$= \tan^{-1}\frac{y}{x} + \sin^{-1}\frac{x}{y}$$

$$= t^0 f(x,y)$$

$\therefore$ $f(x,y)$ is homogeneous function in x and y of degree 0.

(b) $f(x,y) = \cancel{\tan^{-1}\frac{y}{x}}\ \cos^{-1}\dfrac{y}{\sqrt{x^2+y^2}}$

$$f(tx,ty) = \cos^{-1}\frac{ty}{\sqrt{t^2x^2+t^2y^2}} = \cos^{-1}\frac{y}{\sqrt{x^2+y^2}}$$

$$= t^0 f(x,y)$$

$\therefore$ f is a homogeneous function in x and y of degree 0.

(c). $f(x,y) = \dfrac{x^2}{y} + \dfrac{y^2}{x}$

$$f(tx,ty) = \frac{t^2x^2}{ty} + \frac{t^2y^2}{tx} = t\,\frac{x^2}{y} + t\,\frac{y^2}{x}$$

$$= t\left(\frac{x^2}{y} + \frac{y^2}{x}\right)$$

$$= t^1 f(x,y)$$

$\therefore$ f is a homogeneous function in x and y of degree 1.

(d) $f(x,y) = \dfrac{x}{y}\sin\left(\dfrac{y}{x}\right)$

$$f(tx,ty) = \frac{tx}{ty}\sin\left(\frac{ty}{tx}\right) = \frac{x}{y}\sin\left(\frac{y}{x}\right)$$

$$= t^0 f(x,y)$$

$\therefore$ f is a homogeneous function of degree 0.

e). $f(x,y) = x^{2/3} y^{4/3} \tan \frac{y}{x}$

$f(tx, ty) = (tx)^{2/3} (ty)^{4/3} \tan \frac{ty}{tx}$

$= t^2 x^{2/3} y^{4/3} \tan \frac{y}{x}$

$\therefore$ f is homogeneous function in x and y of degree 2.

(f). $f(x,y) = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

$= \dfrac{x^{1/4}\left(1 + \frac{y^{1/4}}{x^{1/4}}\right)}{x^{1/5}\left(1 + \frac{y^{1/5}}{x^{1/5}}\right)}$

$= x^{1/20} \phi(y/x)$

Where, $\phi(y/x) = \dfrac{1 + \left(\frac{y}{x}\right)^{1/4}}{1 + \left(\frac{y}{x}\right)^{1/5}}$

$\therefore$ f is a homogeneous function of degree $1/20$

(g). $f(x,y) = x^2y^2 + xy^3 + x^2y + x^3y$

$f(tx,ty) = t^2x^2 \cdot t^2y^2 + tx \cdot t^3y^3 + t^2x^2 \cdot ty + t^3x^3 \cdot ty$

$= t^3(tx^2y^2 + txy^3 + x^2y + tx^3y)$

$\therefore f(x,y)$ is not a homogeneous function in x, y as it can not be written as $f(tx,ty) = t^n f(x,y)$ for any n

(h). $f(x,y) = \dfrac{x^2+y^2}{x^3+y^3}$

$= \dfrac{x^2\left(1+\frac{y^2}{x^2}\right)}{x^3\left(1+\frac{y^3}{x^3}\right)}$

$= x^{-1}\phi(y/x)$

where $\phi(y/x) = \dfrac{1+\left(\frac{y}{x}\right)^2}{1+\left(\frac{y}{x}\right)^3}$

$\therefore f$ is a homogeneous function in x, y of degree -1

6.
$$f(x,y) = \frac{y}{x} + \frac{x}{y}$$

$$f(tx,ty) = \frac{ty}{tx} + \frac{tx}{ty}$$

$$= \frac{y}{x} + \frac{x}{y}$$

$$= t^0 f(x,y)$$

$\therefore$ f is a homogeneous function of degree 0.

So, by Euler's theorem

$$x f_x + y f_y = 0.$$

7.
$$u(x,y) = \frac{x^2+y^2}{\sqrt{x+y}}$$

$$u(tx,ty) = \frac{t^2x^2+t^2y^2}{\sqrt{tx+ty}} = t^{3/2}\, u(x,y)$$

$\therefore$ u is a homogeneous function of degree $3/2$

$$\therefore \; x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u$$

$\therefore$ $K = 3/2$

8. $$y = f(x+ct) + \phi(x-ct)$$

$$\therefore\ y_t = cf'(x+ct) + (-c)\phi'(x-ct)$$

$$y_{tt} = c^2 f''(x+ct) + c^2 \phi''(x-ct)$$

$$y_x = f'(x+ct) + \phi'(x-ct)$$

$$y_{xx} = f''(x+ct) + \phi''(x-ct)$$

$$\therefore\ y_{tt} = c^2 y_{xx}$$

9. $$u = e^{-mx} \sin(nt - mx)$$

$$u_t = n e^{-mx} \cos(nt - mx)$$

$$u_x = -m e^{-mx} \sin(nt-mx) - m e^{-mx} \cos(nt-mx)$$

$$u_{xx} = m^2 e^{-mx} \sin(nt-mx) + m^2 e^{-mx} \cos(nt-mx)$$
$$+ m^2 e^{-mx} \cos(nt-mx) - m^2 e^{-mx} \sin(nt-mx)$$

$$= 2m^2 e^{-mx} \cos(nt-mx)$$

$$\therefore\ \frac{n}{2m^2} u_{xx} = u_t$$

From the given condition we have

$$x\log x + y\log y + z\log z = \log k \quad (\text{to the base } e)$$

Differentiating w.r.t x we get

$$(\log x + 1) + (1+\log z)\frac{\partial z}{\partial x} = 0$$

or, $$\frac{\partial z}{\partial x} = -\frac{1+\log x}{1+\log z}$$

Similarly we get

$$\frac{\partial z}{\partial y} = -\frac{1+\log y}{1+\log z}$$

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(-\frac{1+\log y}{1+\log z}\right)$$

$$= (1+\log y)\frac{1}{(1+\log z)^2}\cdot\frac{1}{z}\cdot\frac{\partial z}{\partial x}$$

$$= (-)\frac{1+\log y}{(1+\log z)^2}\cdot\frac{1}{z}\;\frac{1+\log x}{1+\log z}$$

At $x = y = z$

$$\frac{\partial^2 z}{\partial x\partial y} = (-)\frac{(1+\log x)^2}{(1+\log x)^3}\cdot\frac{1}{x}$$

$$= -\frac{1}{x(1+\log x)}$$

$$= -\frac{1}{x\log_e(ex)}$$

$$\therefore \frac{\partial^2 z}{\partial x\partial y} = -\frac{1}{x\log_e(ex)} \quad \text{at } x = y = z.$$

11. $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$

$$u_x = 2x \tan^{-1}\frac{y}{x} + x^2 \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) - y^2 \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{1}{y}$$

$$= 2x \tan^{-1}\frac{y}{x} + \frac{x^2}{x^2+y^2}(-y) - \frac{y^3}{y^2+x^2}$$

$$= 2x \tan^{-1}\frac{y}{x} - \frac{y(x^2+y^2)}{x^2+y^2}$$

$$= 2x \tan^{-1}\frac{y}{x} - y$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

$$= 2x \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} - 1$$

$$= \frac{2x^2}{x^2+y^2} - 1 = \frac{x^2-y^2}{x^2+y^2}$$

$$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}$$

12. $$u = \tan^{-1} \frac{x^3+y^3}{x-y}$$

$$\tan u = \frac{x^3+y^3}{x-y} = x^2 \frac{1+\left(\frac{y}{x}\right)^3}{1-\frac{y}{x}}$$

$$= x^2 \phi\left(\frac{y}{x}\right)$$

So, $\tan u$ is a homogeneous function of degree 2.

By Euler's theorem

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

or, $$x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2\tan u$$

or, $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\sin u}{\cos u}.\cos^2 u$$

$$= \sin 2u$$

i.e $$x u_x + y u_y = \sin 2u$$

13. $$u = \sin^{-1}\sqrt{\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}$$

$$\sin u = \left(\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right)^{1/2} = \frac{x^{1/6}}{x^{1/4}}\left(\frac{1+\left(\frac{y}{x}\right)^{1/3}}{1+\left(\frac{y}{x}\right)^{1/2}}\right)^{1/2}$$

$$= x^{-\frac{1}{12}} f(y/x)$$

i.e $\sin u$ is a homogeneous function of x and y of degree $-\frac{1}{12}$

By Euler's theorem

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = -\frac{1}{12}\sin u$$

or, $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{12}\tan u$$

Differentiability partially w.r.t x and then multiplying by x we get

$$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x\partial y} = -\frac{1}{12}\sec^2 u\left(x\frac{\partial u}{\partial x}\right) - x\frac{\partial u}{\partial x}$$

$$= -x\frac{\partial u}{\partial x}\left(\frac{1}{12}\sec^2 u + 1\right) \quad \text{——— (1)}$$

$x^2 \cancel{\frac{\partial^2 u}{\partial x^2}} + y^2$

$n(n-1)$

$x^2\frac{\partial}{\partial x}(\sin u)$ $2xy\frac{\partial^2}{\partial x\partial y}$ $\sin$ ..

x ( )

Again differentiating partially w.r.t y and then multiplying by y we get

$$xy\frac{\partial^2 u}{\partial y \partial x} + y^2\frac{\partial^2 u}{\partial y^2} = -y\frac{\partial u}{\partial y}\left(\frac{1}{12}\sec^2 u + 1\right) \quad \text{——— (2)}$$

Adding (1) and (2) we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$$

$$= -\left(\frac{1}{12}\sec^2 u + 1\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= -\left(\frac{1}{12}\sec^2 u + 1\right)\left(-\frac{1}{12}\tan u\right)$$

$$= \frac{\tan u}{12}\left(\frac{13}{12} + \frac{1}{12}\tan^2 u\right)$$

$$\therefore \quad x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy}$$

$$= \frac{\tan u}{12}\left(\frac{13}{12} + \frac{1}{12}\tan^2 u\right)$$

14. $u(x,y) = x\log\left(\frac{y}{x}\right)$ for $xy \neq 0$

$u(tx,ty) = tx\log\left(\frac{ty}{tx}\right) = t\,u(x,y)$

$\therefore$ $u(x,y)$ is homogeneous function of degree 1.

By Euler's theorem we have

$$x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy} = n(n-1)u$$

where u is a homogeneous function of x and y of degree n.

Here $n = 1$

$$\therefore\ x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy} = 0$$

15. Let, $u = U + V$

where $U = \dfrac{(ax^3+by^3)^n}{3n(3n-1)}$ and $V = xf\left(\frac{y}{x}\right)$

$$U = \frac{x^{3n}\left\{a + b\left(\frac{y}{x}\right)^3\right\}^n}{3n(3n-1)} = x^{3n} f\left(y/x\right)$$

So, U is a homogeneous function of x and y of degree $3n$

By Euler's theorem

$$x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = 3nU \quad \text{———} \ (*)$$

Applying $\frac{\partial}{\partial x}$ we get

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (3n-1)\frac{\partial u}{\partial x}$$

Multiplying by x we have

$$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = (3n-1)x\frac{\partial u}{\partial x} \quad \text{——} (1)$$

Similarly applying $\frac{\partial}{\partial y}$ and multiplying by y on ④ we have

$$y^2\frac{\partial^2 u}{\partial y^2} + xy\frac{\partial^2 u}{\partial y \partial x} = (3n-1)y\frac{\partial u}{\partial y} \quad \text{——} (2)$$

Adding (1) and (2) we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (3n-1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= (3n-1)\,3n\,u$$

$$= (ax^3 + by^3)^n$$

Again $V = x\,f(y/x)$

V is homogeneous function of x and y

of degree 1.

By Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = V$$

Operating by $\frac{\partial}{\partial x}$ and then multiplying by x, we get

$$x^2\frac{\partial^2 v}{\partial x^2} + xy\frac{\partial^2 v}{\partial x \partial y} = 0 \quad \text{———} (3)$$

Similarly operating by $\frac{\partial}{\partial y}$ and then multiplying by y we obtain

$$y^2\frac{\partial^2 v}{\partial y^2} + xy\frac{\partial^2 v}{\partial y \partial x} = 0 \quad \text{———} (4)$$

Adding (3) & (4) we get

$$x^2\frac{\partial^2 v}{\partial x^2} + y^2\frac{\partial^2 v}{\partial y^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} = 0$$

So we get

$$\left(x^2\frac{\partial^2}{\partial x^2} + 2xy\frac{\partial^2}{\partial x\partial y} + y^2\frac{\partial^2}{\partial y^2}\right)u$$

$$= \left(x^2\frac{\partial^2}{\partial x^2} + 2xy\frac{\partial^2}{\partial x\partial y} + y^2\frac{\partial^2}{\partial y^2}\right)(v+v)$$

$$= \left(ax^3 + by^3\right)^n$$

16.

Given $z = x^m f\left(\frac{y}{x}\right) + y^n g\left(\frac{x}{y}\right)$

Let; $\alpha(x,y) = x^m f\left(\frac{y}{x}\right)$ and $\beta(x,y) = y^n g\left(\frac{x}{y}\right)$

$\alpha(x,y)$ and $\beta(x,y)$ are two homogeneous functions of degree m and n respectively.

So using Euler's theorem applying on $\alpha(x,y)$ and $\beta(x,y)$ we get

$$x\alpha_x + y\alpha_y = m\alpha \quad \text{———} (1)$$

$$x\beta_x + y\beta_y = n\beta \quad \text{———} (2)$$

$$x^2\alpha_{xx} + 2xy\,\alpha_{xy} + y^2\alpha_{yy} = m(m-1)\alpha \quad \text{———} (3)$$

$$x^2\beta_{xx} + 2xy\,\alpha_{xy} + y^2\beta_{yy} = n(n-1)\beta \quad \text{———} (4)$$

Adding (3) and (4) we get

$$x^2 z_{xx} + 2xy\,z_{xy} + y^2 z_{yy} = m(m-1)\alpha + n(n-1)\beta$$

$$[\because z = \alpha + \beta]$$

Adding (1) and (2) we get

$$xz_x + yz_y = m\alpha + n\beta \quad \text{———} (5)$$

L.H.S

$= x^2 z_{xx} + 2xy\, z_{xy} + y^2 z_{yy} + mnz$

$= m(m-1)\alpha + n(n-1)\beta + mn(\alpha+\beta)$

$= (m^2 - m + mn)\alpha + (n^2 - n + mn)\beta$

$= (m+n-1)(m\alpha + n\beta)$

$= (m+n-1)(xz_x + yz_y)$ [Using (5)]

$=$ R.H.S [Proved]

17. Let, $f(x,y)$ and $g(x,y)$ be two homogeneous functions of degree m and n respectively, where $m \neq 0$

So using Euler's theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = mf \quad \text{and}$$

$$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = ng$$

Adding these two

$$x\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right) + y\left(\frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}\right) = mf + ng$$

$$\Rightarrow \quad x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} = mf + ng$$

$$\therefore \quad mf + ng = 0.$$

$$\Rightarrow \quad f = -\frac{n}{m}g \quad [\because m \neq 0]$$

$$\therefore \quad f = \alpha g, \quad \alpha = -\frac{n}{m} \text{ is a scalar}$$

18

$$U(\xi,\eta) = u(x,y)$$

$$\xi = a + \alpha x + \beta y \cdot \quad \eta = b - \beta x + \alpha y$$

By chain rule

$$\frac{\partial u}{\partial x} = \frac{\partial U}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial U}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}$$

$$= \frac{\partial U}{\partial \xi}\cdot\alpha - \beta\frac{\partial U}{\partial \eta}$$

We may write $\frac{\partial}{\partial x} \equiv \alpha\frac{\partial}{\partial \xi} - \beta\frac{\partial}{\partial \eta}$

and similarly $\frac{\partial}{\partial y} \equiv \beta\frac{\partial}{\partial \xi} + \alpha\frac{\partial}{\partial \eta}$

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\alpha\frac{\partial U}{\partial \xi} - \beta\frac{\partial U}{\partial \eta}\right)$$

$$= \alpha\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial \xi}\right) - \beta\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial \eta}\right)$$

$$= \alpha\left[\alpha\frac{\partial}{\partial \xi}\left(\frac{\partial U}{\partial \xi}\right) - \beta\frac{\partial}{\partial \eta}\left(\frac{\partial U}{\partial \xi}\right)\right]$$

$$-\beta\left[\alpha\frac{\partial}{\partial \xi}\left(\frac{\partial U}{\partial \eta}\right) - \beta\frac{\partial}{\partial \eta}\left(\frac{\partial U}{\partial \eta}\right)\right]$$

i.e. $u_{xx} = \alpha^2 U_{\xi\xi} - 2\alpha\beta U_{\xi\eta} + \beta^2 U_{\eta\eta}$ ✓

Similarly, $u_{yy} = \beta^2 U_{\xi\xi} + 2\alpha\beta U_{\xi\eta} + \alpha^2 U_{\eta\eta}$ ✓

Now

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left[\alpha \frac{\partial U}{\partial \xi} - \beta \frac{\partial U}{\partial \eta}\right]$$

$$= \alpha \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial \xi}\right) - \beta \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial \eta}\right)$$

$$= \alpha\left[\beta \frac{\partial}{\partial \xi} + \alpha \frac{\partial}{\partial \eta}\right]\left(\frac{\partial U}{\partial \xi}\right)$$

$$- \beta\left[\beta \frac{\partial}{\partial \xi} + \alpha \frac{\partial}{\partial \eta}\right]\left(\frac{\partial U}{\partial \eta}\right)$$

$$= \alpha\beta \frac{\partial^2 U}{\partial \xi^2} + \alpha^2 \frac{\partial^2 U}{\partial \eta \partial \xi} - \beta^2 \frac{\partial^2 U}{\partial \xi \partial \eta} - \alpha\beta \frac{\partial^2 U}{\partial \eta^2}$$

i.e $u_{xy} = u_{yx} = \alpha\beta\left[U_{\xi\xi} - U_{\eta\eta}\right] + (\alpha^2 - \beta^2) U_{\xi\eta}$

Hence

$u_{xx} u_{yy} - u_{xy}^2$

$$= \left(\alpha^2 U_{\xi\xi} - 2\alpha\beta U_{\xi\eta} + \beta^2 U_{\eta\eta}\right)\left(\beta^2 U_{\xi\xi} + 2\alpha\beta U_{\xi\eta} + \alpha^2 U_{\eta\eta}\right)$$

$$-\left(\alpha\beta\left(U_{\xi\xi} - U_{\eta\eta}\right) + (\alpha^2 - \beta^2) U_{\xi\eta}\right)^2$$

$=$

$$= U_{\xi\xi} U_{\eta\eta} \left(\alpha^4 + \beta^4 + 2\alpha^2\beta^2\right)$$

$$- U_{\xi\eta}^2 \left(\alpha^4 + \beta^4 + 2\alpha^2\beta^2\right)$$

$$= \left(U_{\xi\xi} U_{\eta\eta} - U_{\xi\eta}^2\right) \left(\alpha^2 + \beta^2\right)^2$$

$$= U_{\xi\xi} U_{\eta\eta} - U_{\xi\eta}^2 \quad \left(\because \alpha^2 + \beta^2 = 1\right)$$

$\therefore$ ~~U_{xx} ...~~

$$\therefore u_{xx} u_{yy} = U_{\xi\xi} U_{\eta\eta} - U_{\xi\eta}^2$$

19.
(a).

$x = r\cos\theta \qquad y = r\sin\theta$

Using chain rules we have

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}.$$

So, $\frac{\partial}{\partial r} \equiv \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}.$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= -r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y}.$$

$\frac{\partial}{\partial \theta} \equiv -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}.$

$$\therefore \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

$$= \left(\cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}\right)^2 + \frac{1}{r^2}\left(-r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y}\right)^2$$

$$= \cos^2\theta\left(\frac{\partial z}{\partial x}\right)^2 + \sin^2\theta\left(\frac{\partial z}{\partial y}\right)^2 + 2\sin\theta\cos\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

$$+ \frac{1}{r^2}\left(r^2\sin^2\theta\left(\frac{\partial z}{\partial x}\right)^2 + r^2\cos^2\theta\left(\frac{\partial z}{\partial y}\right)^2 - 2r^2\sin\theta\cos\theta\frac{\partial z}{\partial x}\right.$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 (\cos^2\theta + \sin^2\theta) + \left(\frac{\partial z}{\partial y}\right)^2 (\cos^2\theta + \sin^2\theta)$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

19.
(b).

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right)$$

$$= \frac{\partial}{\partial r}\left(\cos\theta \frac{\partial z}{\partial x} + \sin\theta \frac{\partial z}{\partial y}\right)$$

$$= \cos\theta \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + \sin\theta \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)$$

$$= \cos\theta \left[\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}\right]\left(\frac{\partial z}{\partial x}\right) +$$

$$\sin\theta \left[\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}\right]\left(\frac{\partial z}{\partial y}\right)$$

$$= \cos^2\theta \frac{\partial^2 z}{\partial x^2} + \cos\theta \sin\theta \frac{\partial^2 z}{\partial y \partial x} + \sin\theta\cos\theta \frac{\partial^2 z}{\partial x \partial y}$$

$$+ \sin^2\theta \frac{\partial^2 z}{\partial y^2}$$

$$= \cos^2\theta \frac{\partial^2 z}{\partial x^2} + 2\sin\theta\cos\theta \frac{\partial^2 z}{\partial x \partial y} + \sin^2\theta \frac{\partial^2 z}{\partial y^2}$$

$$\left[\text{Using } \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}\right]$$

$$\frac{\partial^2 z}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial \theta}\right)$$

$$= \frac{\partial}{\partial \theta}\left(-r\sin\theta \frac{\partial z}{\partial x} + r\cos\theta \frac{\partial z}{\partial y}\right)$$

$$= \left(-r\cos\theta \frac{\partial z}{\partial x} - r\sin\theta \frac{\partial z}{\partial y}\right) - r\sin\theta \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\right) + r\cos\theta \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial y}\right)$$

$$= -r\left(\cos\theta \frac{\partial z}{\partial x} + \sin\theta \frac{\partial z}{\partial y}\right) - r\sin\theta\left[-r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y}\right] \times \left(\frac{\partial z}{\partial x}\right)$$

$$+ r\cos\theta \cdot \left[-r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y}\right]\left(\frac{\partial z}{\partial y}\right)$$

$$= -r\frac{\partial z}{\partial r} + r^2\sin^2\theta \frac{\partial^2 z}{\partial x^2} - 2r^2\sin\theta\cos\theta \frac{\partial^2 z}{\partial x \partial y} + r^2\cos^2\theta \frac{\partial^2 z}{\partial y^2}$$

$$\left[\frac{\partial z}{\partial r} = \cos\theta \frac{\partial z}{\partial x} + \sin\theta \frac{\partial z}{\partial y}\right]$$

$$\therefore \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}$$

$$= \sin^2\theta \frac{\partial^2 z}{\partial x^2} - 2\sin\theta\cos\theta \frac{\partial^2 z}{\partial x \partial y} + \cos^2\theta \frac{\partial^2 z}{\partial y^2}$$

So we have

$$\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{\partial^2 z}{\partial r^2}$$

$$= \sin^2\theta \frac{\partial^2 z}{\partial x^2} - 2\sin\theta\cos\theta \frac{\partial^2 z}{\partial x \partial y} + \cos^2\theta \frac{\partial^2 z}{\partial y^2}$$

$$+ \cos^2\theta \frac{\partial^2 z}{\partial x^2} + 2\sin\theta\cos\theta \frac{\partial^2 z}{\partial x \partial y} + \sin^2\theta \frac{\partial^2 z}{\partial y^2}$$

$$= \left(\sin^2\theta + \cos^2\theta\right)\frac{\partial^2 z}{\partial x^2} + \left(\cos^2\theta + \sin^2\theta\right)\frac{\partial^2 z}{\partial y^2}$$

$$= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

[Proved].

✓

20 Given that

$$W = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^K \quad , \text{ for } n \geq 2$$

We have,

$$\frac{\partial W}{\partial x_1} = K\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{K-1} \times 2x_1$$

$$\frac{\partial W}{\partial x_2} = K\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{K-1} \times 2x_2$$

$$\vdots$$

$$\frac{\partial W}{\partial x_n} = K\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{K-1} \times 2x_n$$

Again, We have

$$\frac{\partial^2 W}{\partial x_1^2} = 2K\Big[(K-1)\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{K-2} 2x_1^2 + \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{K-1}\Big]$$

$$= 2K\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{K-2} \times \left[2(K-1)\,x_1^2 + x_1^2 + x_2^2 + \cdots + x_n^2\right]$$

Similarly,

$$\frac{\partial^2 W}{\partial x_2^2} = 2K\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{K-2} \times \left[2(K-1)\,x_2^2 + x_1^2 + x_2^2 + \cdots + x_n^2\right]$$

$$\frac{\partial^2 W}{\partial x_1^2} = 2K\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{K-2} \times \left[2(K-1)\,x_n^2 + x_1^2 + x_2^2 + \cdots + x_n^2\right]$$

Now.

$$\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} + \cdots + \frac{\partial w}{\partial x_n^2} = 0$$

$$\Rightarrow 2k\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{k-2}\Big[2(k-1)\left(x_1^2 + x_2^2 + \cdots + x_n^2\right) + n\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)\Big] = 0$$

$$\Rightarrow 2k\left[2(k-1) + n\right]\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{k-1} = 0$$

$$\Rightarrow k\left[2(k-1) + n\right] = 0 \quad \left\{\because x_1^2 + x_2^2 + \cdots + x_n^2 \neq 0\right\}$$

$$\Rightarrow k = 0 \text{ or } k = 1 - \frac{n}{2}.$$

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21(b) We have $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$

$$= \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}$$

$$= \frac{x}{r}\frac{\partial u}{\partial x} + \frac{y}{r}\frac{\partial u}{\partial y}$$

and so, $r\frac{\partial u}{\partial r} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ —— (1)

Therefore,

$$r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$r\left[r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\right] = x\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) + y\frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial y\partial x} + xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2}$$

$$+ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$r^2\frac{\partial^2 u}{\partial r^2} + \cancel{r\frac{\partial u}{\partial r}} = x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \cancel{r\frac{\partial u}{\partial r}}$$

Therefore, $r^2\frac{\partial^2 u}{\partial r^2} = x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$ ——

Again, $\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$

$$= x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x}$$

Therefore,

$$\frac{\partial^2 u}{\partial \theta^2} = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\left(x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x}\right)$$

$$= x\frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x}\right) - y\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x}\right)$$

$$= x^2\frac{\partial^2 u}{\partial y^2} - 2xy\frac{\partial^2 u}{\partial y \partial x} + y^2\frac{\partial^2 u}{\partial x^2} - x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial^2 u}{\partial \theta^2} + r\frac{\partial u}{\partial x} = x^2\frac{\partial^2 u}{\partial y^2} - 2xy\frac{\partial^2 u}{\partial y \partial x} + y^2\frac{\partial^2 u}{\partial x^2} \quad \text{---(3)}$$

Subtracting (3) from (2) we get

$$r^2\frac{\partial^2 u}{\partial r^2} - \frac{\partial^2 u}{\partial \theta^2} - r\frac{\partial u}{\partial r}$$

$$= (x^2 - y^2)\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right) + 4xy\frac{\partial^2 u}{\partial x \partial y}.$$

Hence the required result.

Subtracting (3) from (2), we get the required result

i.e. $(x^2-y^2)\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\right)+4xy\frac{\partial^2 u}{\partial x\partial y}=r^2\frac{\partial^2 u}{\partial r^2}-r\frac{\partial u}{\partial r}-\frac{\partial^2 u}{\partial \theta^2}$ Proved.

(26)

22. We have $x=\xi\cos\alpha-\eta\sin\alpha$

$y=\xi\sin\alpha+\eta\cos\alpha.$

Now $\frac{\partial u}{\partial \xi}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \xi}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \xi}=\cos\alpha\cdot\frac{\partial u}{\partial x}+\sin\alpha\frac{\partial u}{\partial y}.$

which yields

$$\frac{\partial}{\partial \xi}(u)=\cos\alpha\frac{\partial u}{\partial x}+\sin\alpha\frac{\partial u}{\partial y}.$$
$$=\left(\cos\alpha\frac{\partial}{\partial x}+\sin\alpha\frac{\partial}{\partial y}\right)(u)$$

and so, $\frac{\partial}{\partial \xi}=\cos\alpha\frac{\partial}{\partial x}+\sin\alpha\frac{\partial}{\partial y}.$

Thus,

$$\frac{\partial^2 u}{\partial \xi^2}=\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}\right)=\left(\cos\alpha\frac{\partial}{\partial x}+\sin\alpha\frac{\partial}{\partial y}\right)\times\left(\cos\alpha\frac{\partial u}{\partial x}+\sin\alpha\frac{\partial u}{\partial y}\right)$$
$$=\cos\alpha\frac{\partial}{\partial x}\left(\cos\alpha\frac{\partial u}{\partial x}+\sin\alpha\frac{\partial u}{\partial y}\right)$$
$$+\sin\alpha\frac{\partial}{\partial y}\left(\cos\alpha\frac{\partial u}{\partial x}+\sin\alpha\frac{\partial u}{\partial y}\right).$$
$$=\cos^2\alpha\frac{\partial^2 u}{\partial x^2}+2\sin\alpha\cos\alpha\cdot\frac{\partial^2 u}{\partial x\partial y}+\sin^2\alpha\frac{\partial^2 u}{\partial y^2}\cdot\ (1)$$

One the other hand,

$$\frac{\partial u}{\partial \eta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \eta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \eta}=-\sin\alpha\frac{\partial u}{\partial x}+\cos\alpha\frac{\partial u}{\partial y}.$$

which gives $\frac{\partial}{\partial \eta}=\left(-\sin\alpha\frac{\partial}{\partial x}+\cos\alpha\frac{\partial}{\partial y}\right).$

$^2+0$

$=0$

Therefore,
$$\frac{\partial^2 u}{\partial \eta^2} = \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\right)$$
$$= \left(-\sin\alpha \frac{\partial}{\partial x} + \cos\alpha \frac{\partial}{\partial y}\right) \times \left(-\sin\alpha \frac{\partial u}{\partial x} + \cos\alpha \frac{\partial u}{\partial y}\right)$$
$$= -\sin\alpha \frac{\partial}{\partial x}\left(-\sin\alpha \frac{\partial u}{\partial x} + \cos\alpha \frac{\partial u}{\partial y}\right) + \cos\alpha \frac{\partial}{\partial y}\left(-\sin\alpha \frac{\partial u}{\partial x} + \cos\alpha \frac{\partial u}{\partial y}\right)$$
$$= \sin^2\alpha \frac{\partial^2 u}{\partial x^2} - 2\sin\alpha\cos\alpha \cdot \frac{\partial^2 u}{\partial x \partial y} + \cos^2\alpha \frac{\partial^2 u}{\partial y^2} \quad \cdots (2)$$

Adding (1) & (2), we get.
$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = \cos^2\alpha \frac{\partial^2 u}{\partial x^2} + \cancel{2\sin\alpha\cos\alpha \frac{\partial^2 u}{\partial x \partial y}} + \sin^2\alpha \frac{\partial^2 u}{\partial y^2} + \sin^2\alpha \frac{\partial^2 u}{\partial x^2} - \cancel{2\sin\alpha\cos\alpha \frac{\partial^2 u}{\partial x \partial y}} + \cos^2\alpha \frac{\partial^2 u}{\partial y^2}$$
$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \text{(proved)}$$