PHYSICS OF WAVES

(PH11003)

AUTUMN 2022-23

DEPARTMENT OF PHYSICS, IIT KHARAGPUR

CREDITS: 3-1-0 = 4

Syllabus

Review of Simple Harmonic Motion, Damped and Forced oscillations, Resonance, Coupled oscillations, Normal modes (6-7)

Wave Motion, longitudinal and transverse waves, wave equation, plane waves, phase velocity, superposition, wave packets and group velocity, dispersion relations, two and three dimensional waves. Electromagnetic Waves, Energy-momentum, Poynting's theorem, reflection and refraction, Stokes relations (6-7)

Superposition of waves, Interference, Coherence, Two-beam and Multi-beam interference, Fresnel Biprism and Mirrors, Newton's rings, Michelson and Fabry-Perot Interferometers, Thin films, Diffraction, Fraunhofer single slit diffraction and Grating, Polarisation, Birefringence, Retarders (13-14)

....Syllabus

Failure of classical physics, Planck spectrum, Compton Effect, Davisson-Germer and Thomson Experiments, de Broglie waves, Uncertainty principle. Observables and Hermitian Operators, Wave function and Schrodinger equation, Probability interpretation, One dimensional problems (13-14)

References

Lecture Notes & Problems bank for Physics	**R S Saraswat and G P Sastry**
Oscillations and Waves	**S Bharadwaj and S P Khastgir**
The Physics of vibrations and Waves	**H J Pain**
The Physics of Waves	**H Georgi**
Waves (Berkeley Physics Course, Volume 3)	**F S Crawford**
Optics	**E Hecht**
Fundamentals of Optics	**F A Jenkins and H E White**
Concepts of Modern Physics	**A Beiser**
Feynman Lectures on Physics-I	**R P Feynman**

PPT slides: will be made available at regular intervals

Marks Break-up

Mid-sem:	30
End-sem:	50
Class test 1 (by tutor):	10
Class test 2 (by tutor):	10
Total:	**100**

Less than 75 % attendance (in tutorial class) would lead to subject deregistration.

Review of (Simple) Harmonic Motion

At this juncture, let's recall some

Important Notations

$$\frac{d}{dt} = \dot{}$$

$$\frac{d^2}{dt^2} = \ddot{}$$

$$\frac{dx}{dt} = \dot{x}$$

$$\frac{d^2\theta}{dt^2} = \ddot{\theta}$$

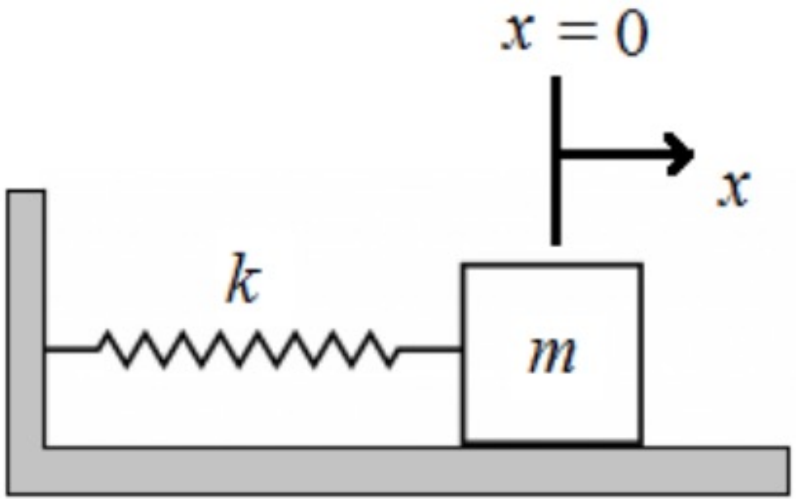
$$\frac{d}{dx} = {}'$$

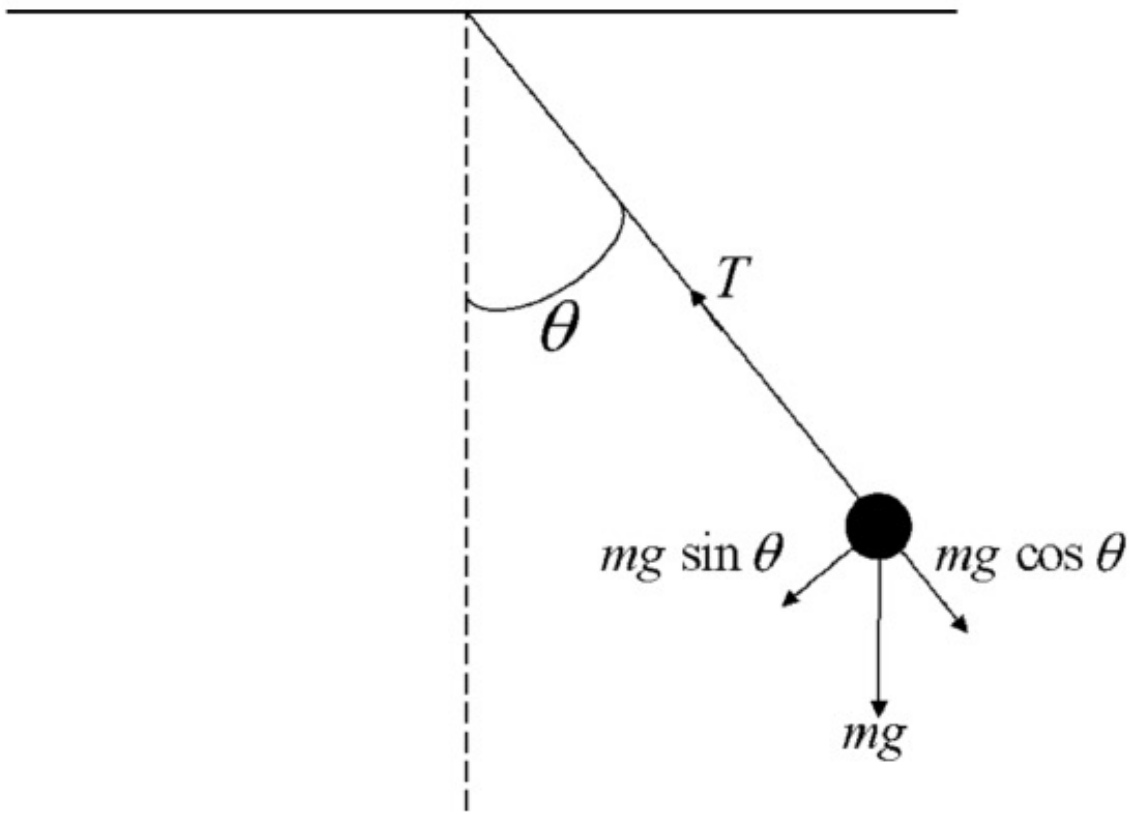
$$\frac{d^2}{dx^2} = {}''$$

$$\frac{dy}{dx} = y'$$

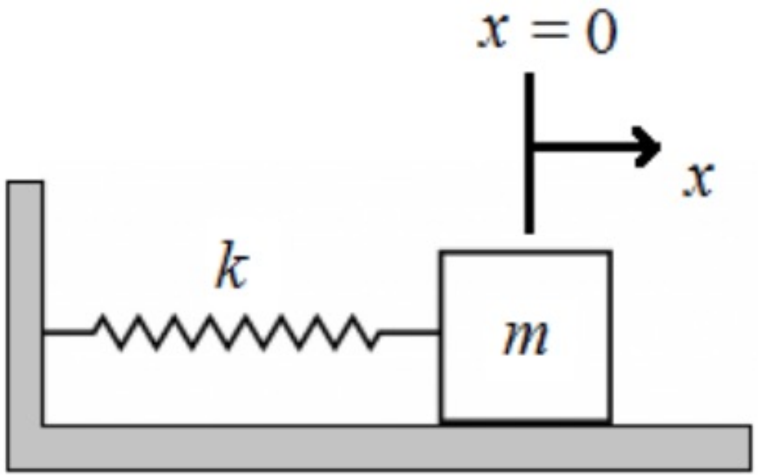
$$\frac{d^2V}{dx^2} = V''$$

SHM: Classic Examples





SHM: RECALL



Force ∝ (- displacement)

$$F(x) = m\ddot{x} = -kx$$

$$\Downarrow$$

$$\boxed{\ddot{x} = -\omega^2 x}$$

$$\omega = \sqrt{\frac{k}{m}}$$

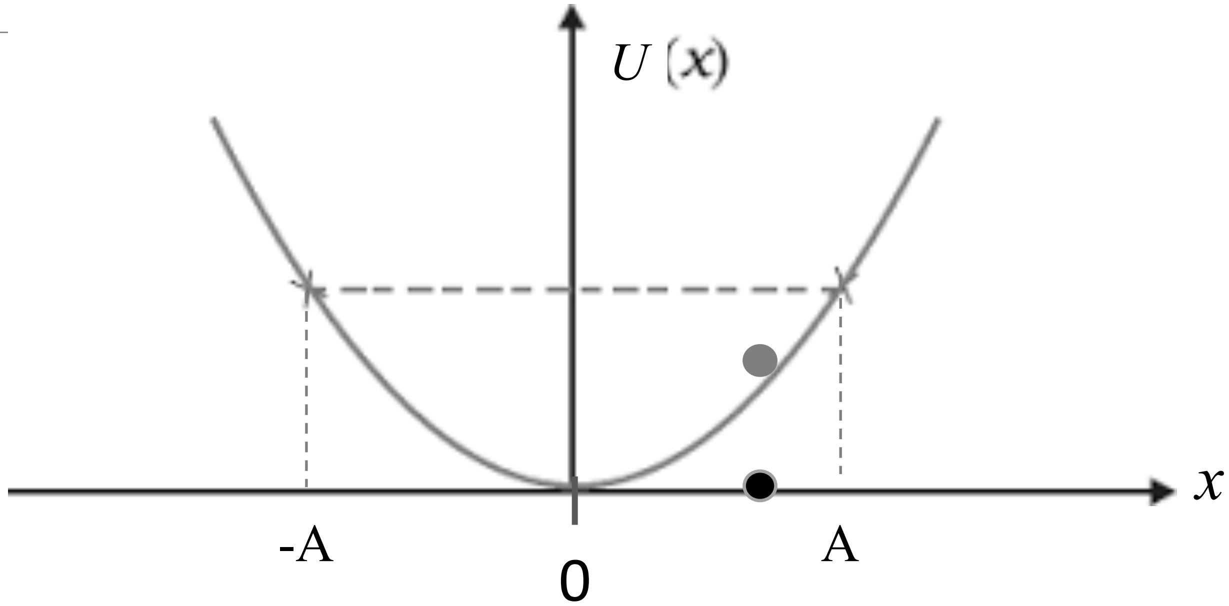
SHM: Associated Potential

$$\boxed{F(x) = m\ddot{x} = -kx}$$

$$U(x) = -\int F(x)dx$$
$$= \frac{1}{2}kx^2 + \text{constant}$$

At equilibrium (x = 0), U = 0 (energy reference)

$$\Rightarrow U(x) = \frac{1}{2}kx^2$$



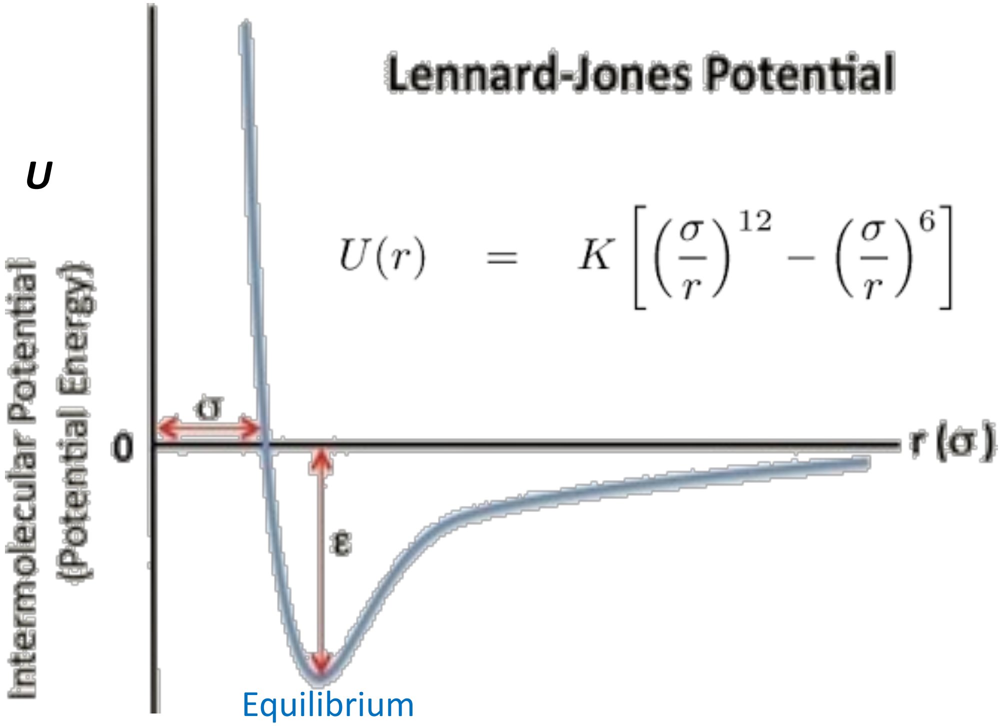
Equilibrium

$$U'(x) = \frac{dU}{dx} = 0$$

Harmonic Motion (Oscillation):
Occurs also for a general potential

$$U(x) \neq \frac{1}{2}kx^2$$

Example:



Any general potential: *Taylor expansion* around equilibrium

$$U(y) = U(y_0) + (y - y_0)U'(y_0) + \frac{1}{2}(y - y_0)^2 U''(y_0) + \cdots$$

$$U'(y_0) = 0$$

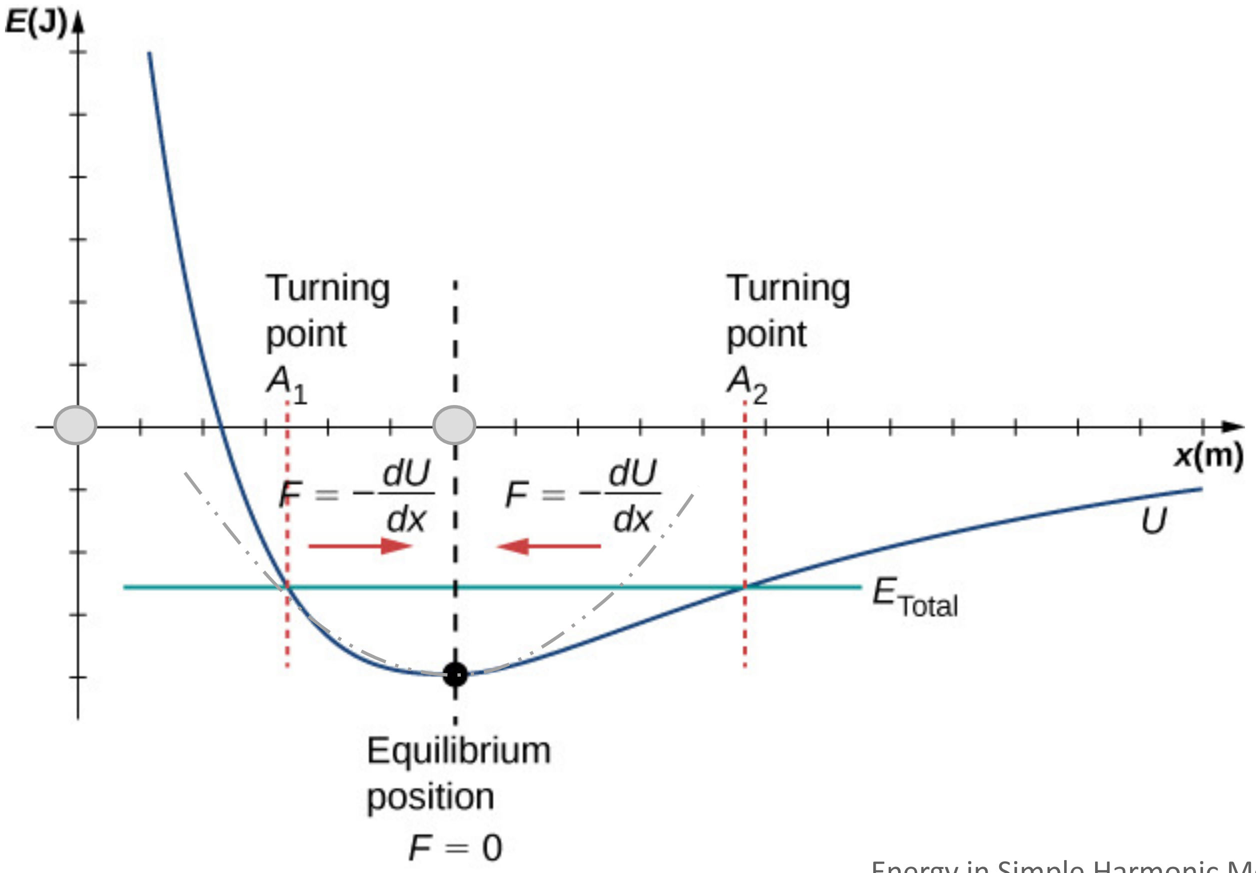
Replace $(y - y_0)$ with x; take $U''(y_0) = k$; consider the vicinity of y_0:

$$\Rightarrow U(x) \approx U(y_0) + \frac{1}{2}kx^2$$

Take $U(y_0)$ as the reference for energy

$$\Rightarrow \boldsymbol{U(x) \approx \frac{1}{2}kx^2} \text{ (near equilibrium)}$$

OSCILLATIONS APPEAR ALMOST EVERYWHERE IN PHYSICS



Energy in Simple Harmonic Motion – University Physics Volume 1 (bccampus.ca)

Examples

Obvious Oscillations

- Water waves
- Pendula
- Musical instruments
- Suspension bridges

Subtle Oscillations

- Heat in a solid
- Superconductivity
- Pulsars
- Neutron Stars

Less-Obvious Oscillations

- Shock absorbers
- Lasers
- Electronic watches
- Radio antenna
- Fiber optics

More technically,

Oscillations form the basis for understanding

waves and optics,
astrophysics,
thermal physics,
quantum mechanics,
condensed-matter physics,
mechanics,
atmospheric and planetary physics,
etc.

So, it is important to understand the nature and causes of oscillations.

Q1

A particle of mass m in the potential $U(x) = 2e^{x^2/L^2}$ J (L, a constant) is found to behave like a SHO for small displacements from equilibrium. Determine the angular frequency of this SHO.

SOLUTION

For the particle to behave like a SHO around the equilibrium position, say x_0, $\boldsymbol{U'(x_0)}$ **= 0**.

$\Rightarrow 2.\frac{1}{L^2}.2x_0.e^{x_0^2/L^2} = 0 \qquad \Rightarrow x_0 = 0.$

Next, we verify whether $x_0 = 0$ is really an equilibrium position (i.e., a minima):

$U"(x_0) = \frac{4}{L^2}\left[e^{x_0^2/L^2} + x_0.\frac{1}{L^2}.2x_0.e^{x_0^2/L^2}\right] = \frac{4}{L^2} > 0$. So, $x_0 = 0$

is really an equilibrium position.

…Q1

$U(x) = 2e^{x^2/L^2}$

Now, we derive the equation of motion:

$$\boldsymbol{F} = m\ddot{\boldsymbol{x}} = -U'(x) = -\frac{4x}{L^2}e^{\frac{x^2}{L^2}} = -\frac{4x}{L^2}\left[1 + \frac{x^2}{L^2} + \frac{x^4}{2L^4} + \cdots\right]$$

$\approx -\frac{\boldsymbol{4x}}{\boldsymbol{L^2}}$ (for small x near x = 0)

$$\boxed{\ddot{x} = -\omega^2 x}$$

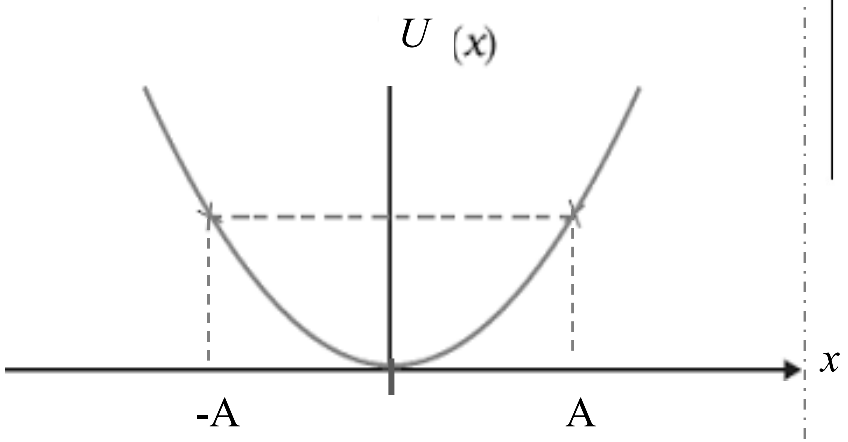
$$\Rightarrow \ \boldsymbol{\omega} = \frac{\boldsymbol{2}}{\boldsymbol{L\sqrt{m}}}$$

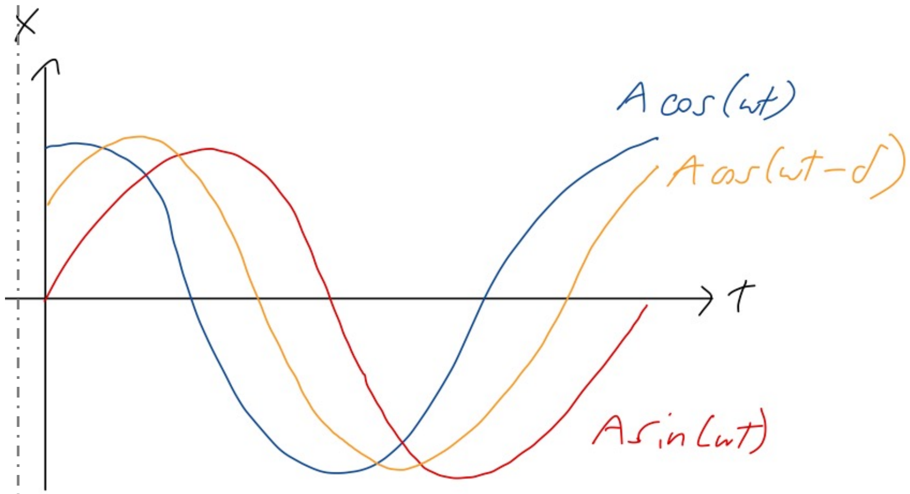
SHM ($\ddot{x} = -\omega^2 x$): The Solution

$$\boxed{\boldsymbol{x(t) = A\cos(\omega t - \delta)}}$$

OR

$$x(t) = A\sin(\omega t - \delta')$$





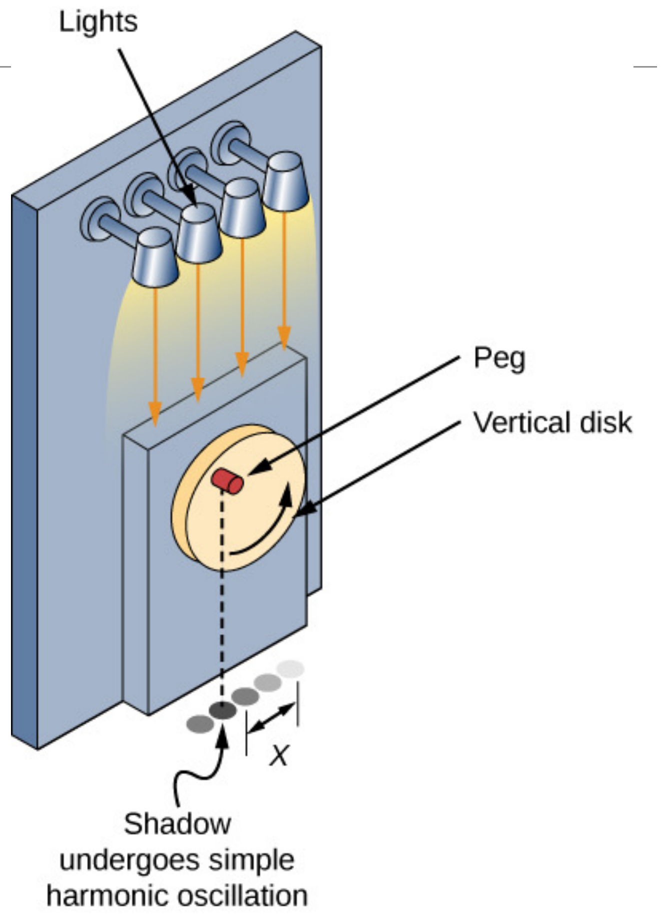
$\delta = 0$, when the motion starts from the maximum displacement.

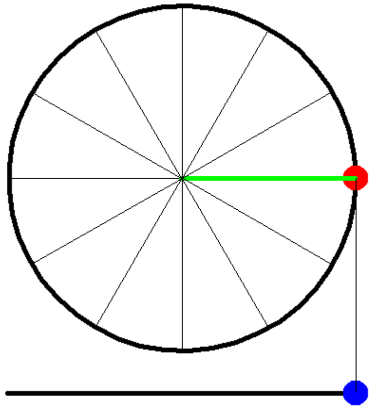
$$v = \frac{dx}{dt} = -\omega A\sin(\omega t - \delta)$$

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}k\,(A^2 - x^2)$$

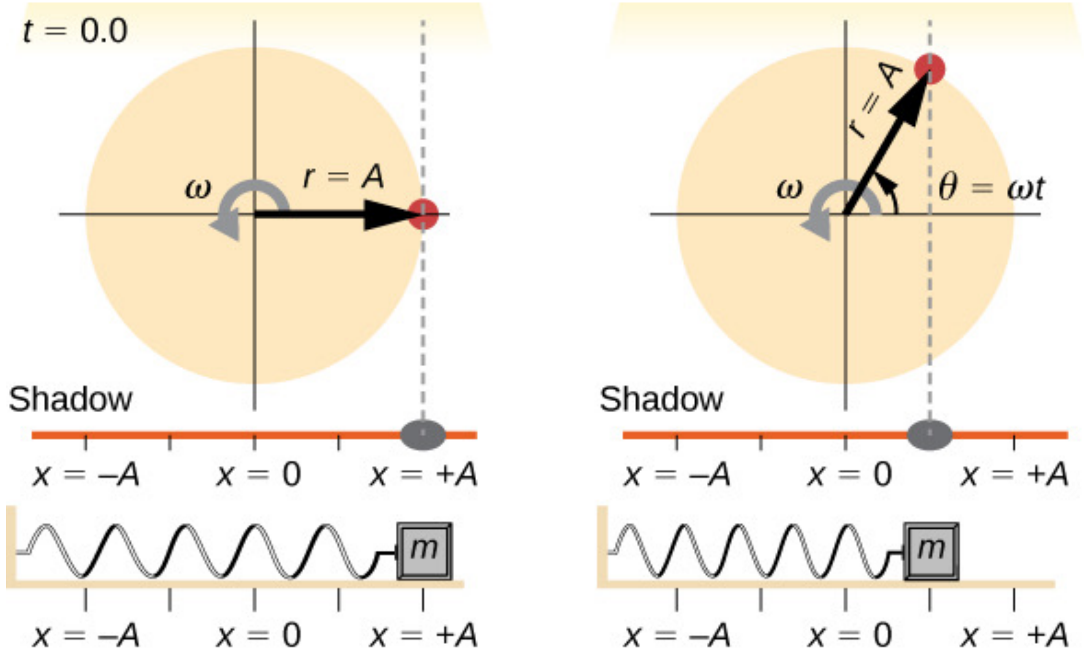
$$E = U + KE = \frac{1}{2}kA^2$$

Comparing SHM and Circular Motion





Simple Harmonic Motion and Circular Motion | Physics (uoguelph.ca)



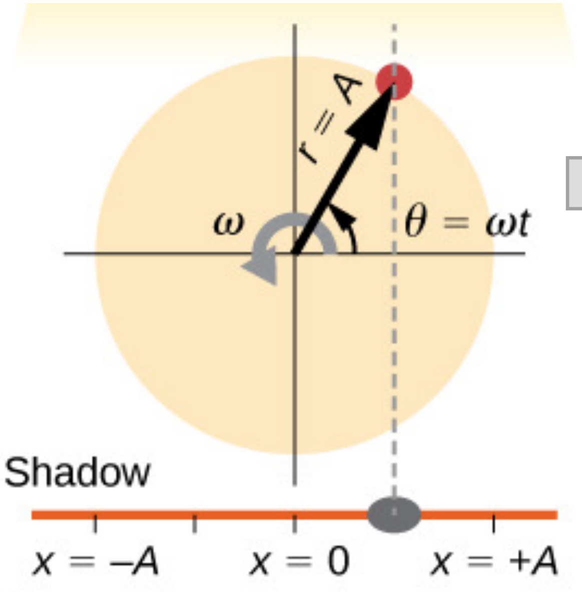
$$x(t) = A\cos(\omega t)$$

Comparing Simple Harmonic Motion and Circular Motion – University Physics Volume 1 (bccampus.ca)

2.4: Comparing Simple Harmonic Motion and Circular Motion – Physics LibreTexts

SHM and Complex Number

2.4: Comparing Simple Harmonic Motion and Circular Motion – Physics LibreTexts

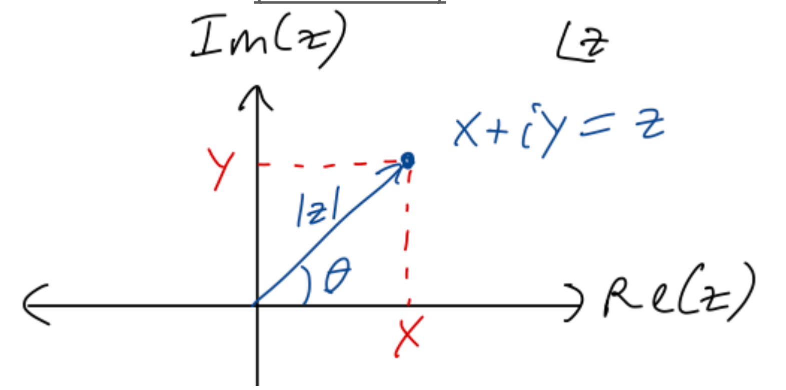


$$x(t) = A\cos(\omega t)$$
$$= Re\,[(A\cos(\omega t) + iA\sin(\omega t)]$$
$$= Re\left[A\,e^{i\omega t}\right]$$

Methodology:

Assume, $\boxed{x(t) = A\,e^{i\omega t}}$ and use real part of the final result as the answer.

Harmonic oscillators and complex numbers (colorado.edu)



$$|z| = \sqrt{x^2 + y^2} \qquad x = |z|\cos\theta$$
$$\theta \equiv \arg(z) \qquad y = |z|\sin\theta$$

$$z = x + iy$$
$$= |z|\cos\theta + i|z|\sin\theta$$
$$= |z|\,(\cos\theta + i\sin\theta)$$

Consider **Euler's formula**

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Then, $e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$

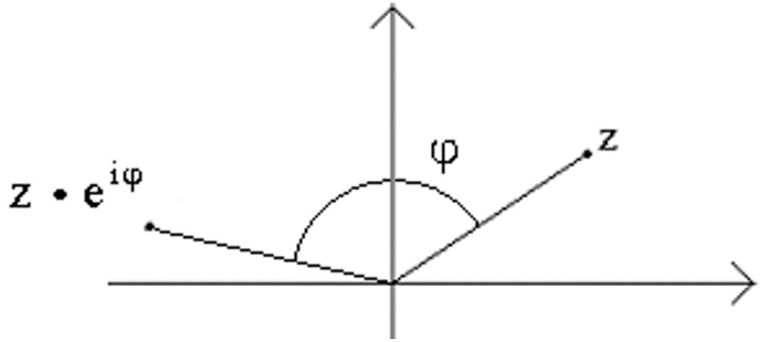
$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

$$\boxed{\Rightarrow z = |z|e^{i\theta} = |z|e^{i.\arg(z)}}$$ Polar form

Complex number properties that can be used in SHM

- Complex number of magnitude unity: Pure Phase ($e^{i\phi}$)
- **Multiplying a complex number** by a pure phase rotates the corresponding point in the complex plane counterclockwise by an angle equal to the phase.



- **Multiplication**:

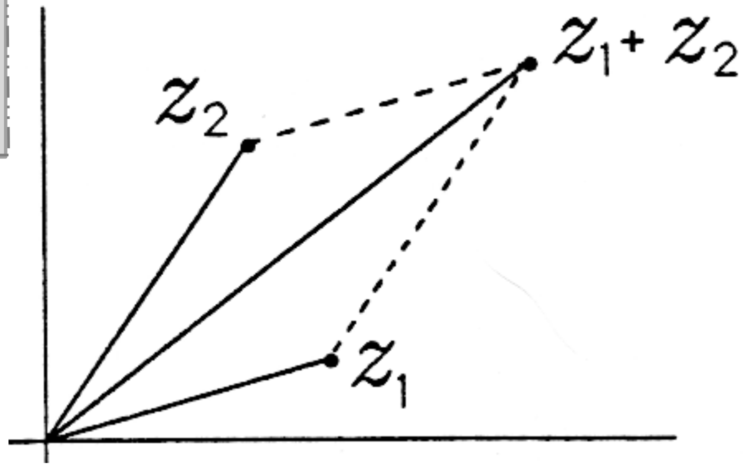
$$z_1 z_2 = \left(|z_1| e^{i\theta_1}\right)\left(|z_2| e^{i\theta_2}\right) = |z_1||z_2|\ e^{i(\theta_1+\theta_2)}$$

so that

$$|z_1 z_2| = |z_1||z_2| \ , \ \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

Addition: $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

$$|z_1 + z_2| = [(x_1 + x_2)^2 + (y_1 + y_2)^2]^{1/2}$$
$$< [(x_1)^2 + (y_1)^2]^{1/2} + [(x_2)^2 + (y_2)^2]^{1/2}$$
$$= |z_1| + |z_2|$$



Complex Numbers: Algebra of Complex Numbers – A First Course in Electrical and Computer Engineering – OpenStax CNX

SHM: Complex Representation

$$\boxed{x(t) = A\cos(\omega t) \equiv A\, e^{i\omega t}}$$

More accurately, $\boxed{x(t) = A\cos(\omega t - \delta) = A\, cos(\omega t + \phi) \;\equiv A\, e^{i(\omega t+\phi)}}$

$$\tilde{x}(t) = Ae^{i(\omega t+\phi)}$$

$$= Ae^{i\phi}\, e^{i\omega t}$$

Complex amplitude $\tilde{A}$

$$\tilde{v}(t) = i\omega\, Ae^{i(\omega t+\phi)}$$

$$= i\omega\tilde{A}e^{i\omega t}$$

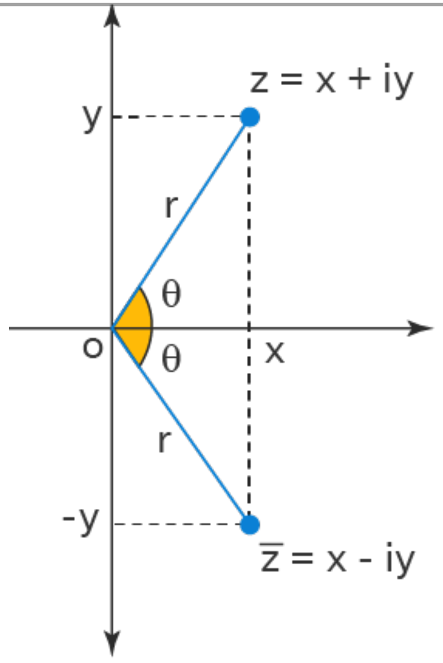
Complex Conjugates

$$z = x + iy = |z|e^{i\theta}$$
$$z^* = x - iy = |z|e^{-i\theta}$$

$$\tilde{x}^*(t) = Ae^{-i(\omega t+\phi)} = \tilde{A}^* e^{-i\omega t}$$

$$\tilde{x}.\tilde{x}^* = \tilde{A}.\tilde{A}^* = A^2$$

$$\tilde{v}.\tilde{v}^* = \omega^2\tilde{A}.\tilde{A}^* = \omega^2 A^2$$



Complex Conjugate - Theorem, Examples | Conjugate of Complex Number (cuemath.com)

SHM: (Energy and) Phase Space

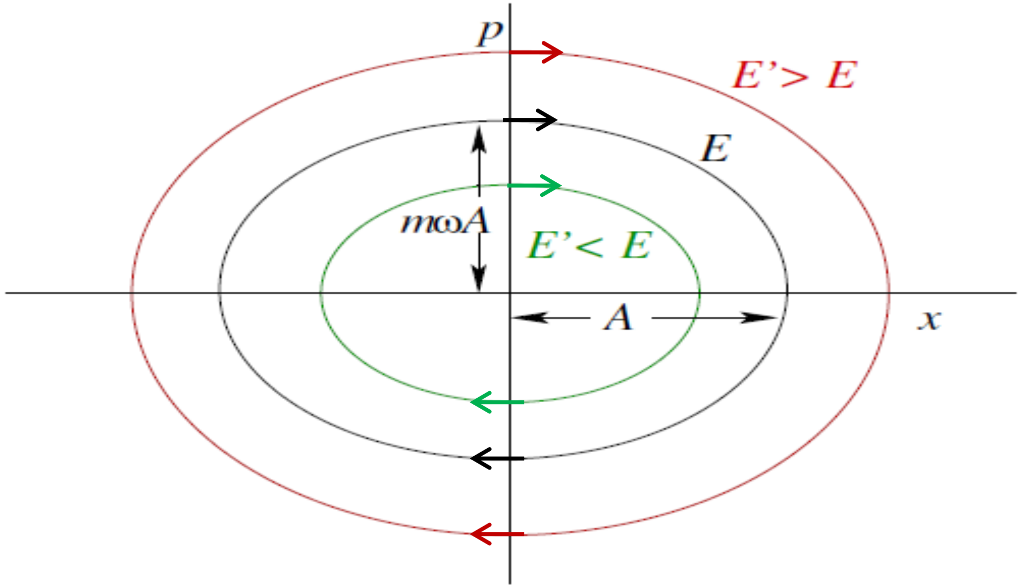
$$PE = U = \frac{1}{2}kx^2$$

$$E = U + KE = \frac{1}{2}kx^2 + \frac{p^2}{2m} = \frac{1}{2}kA^2 = \frac{1}{2}k\tilde{x}\tilde{x}^* = \frac{1}{2}m\omega^2A^2 = \frac{1}{2}m\tilde{v}\tilde{v}^*$$

$$\Rightarrow \frac{x^2}{A^2} + \frac{p^2}{mkA^2} = 1$$

$$\Rightarrow \frac{x^2}{A^2} + \frac{p^2}{m^2\omega^2A^2} = 1$$

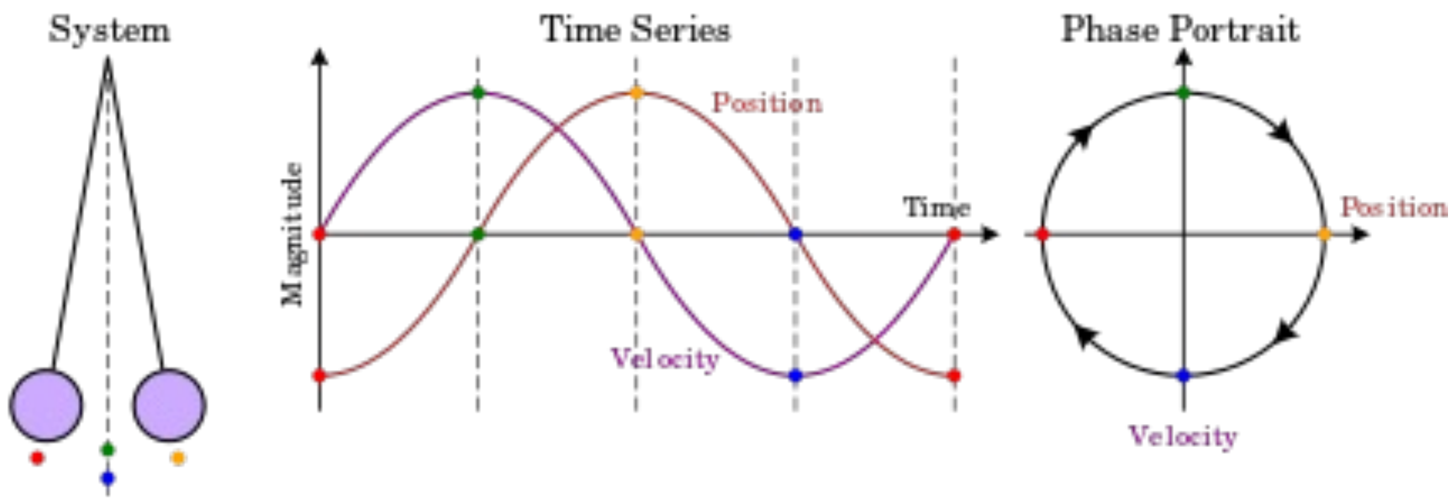
An ellipse!

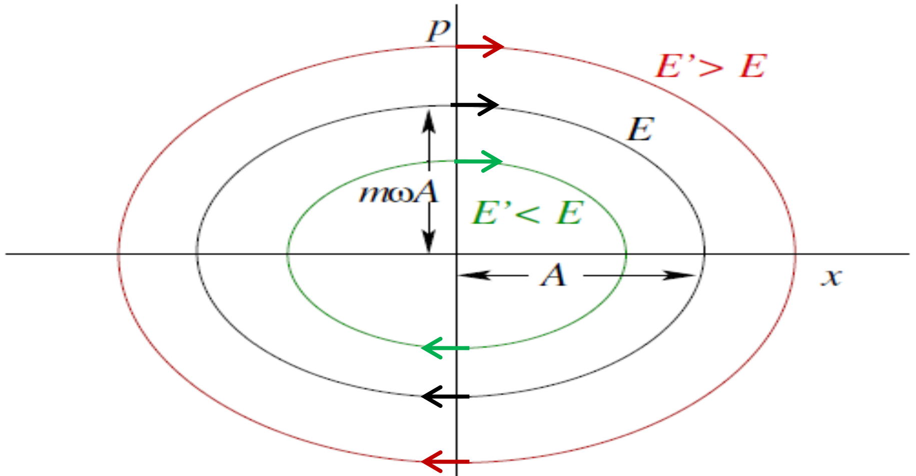


PHASE SPACE

SHM: (Energy and) Phase Space

$$\frac{x^2}{A^2} + \frac{p^2}{m^2\omega^2A^2} = 1$$





PHASE SPACE

Complex Solution(s) of SH Oscillator Equation: The Mathematics Way

$$\boxed{\ddot{x} = -\omega^2 x \Rightarrow \ddot{x} + \omega^2 x = 0}$$

NATURE: **2nd order ordinary homogenous linear** differential equation with constant coefficients, like

$$a_2\ddot{x} + a_1\dot{x} + a_0 x = 0$$

2nd order: because the highest derivative is second order

ordinary: because the derivatives are w. r. t. only one variable t

homogenous: because x or its derivative appear in every term

linear: because x or its derivatives appear separately and linearly in each term

$\boxed{\ddot{x} + \omega^2 x = 0}$

Finding the General Solution

Assume that the solution is: $x = e^{qt}$

Then, $\dot{x} = q\, e^{qt}$

$$\ddot{x} = q^2 e^{qt}$$

$\Rightarrow (q^2 + \omega^2)\, e^{qt} = 0$

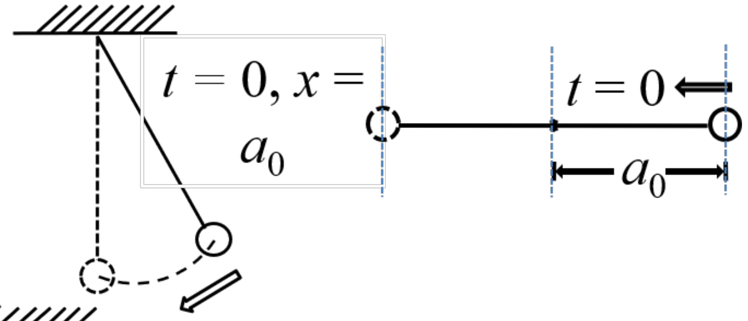
$\Rightarrow q = \pm\, i\omega$

$\Rightarrow x = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$

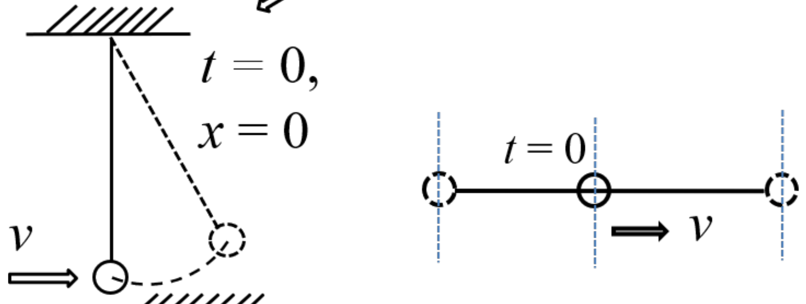
- (the general solution)
- $c_1, c_2 \equiv$ arbitrary constants (to be determined by initial
- conditions)
- $\dot{x} = i\omega c_1 e^{i\omega t} - i\omega c_2 e^{-i\omega t}$

Typical initial conditions:

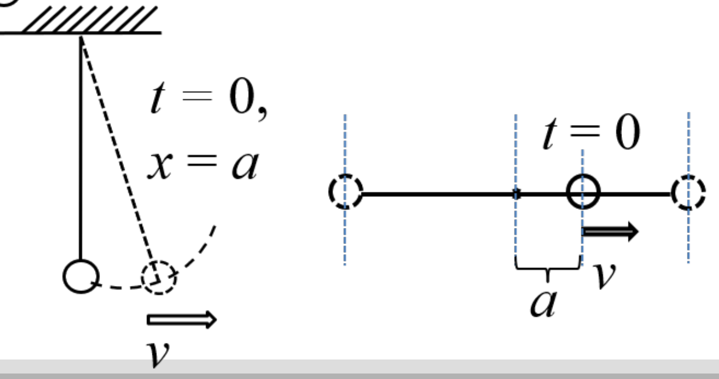
Released from extremity



Impulsed at equilibrium



Hit at intermediate position



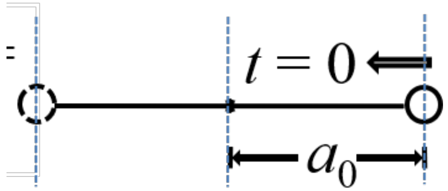
$$\boxed{\ddot{x} + \omega^2 x = 0}$$

The specific solutions

$$x = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$$

$$\dot{x} = i\omega c_1 e^{i\omega t} - i\omega c_2 e^{-i\omega t}$$

Released from extremity

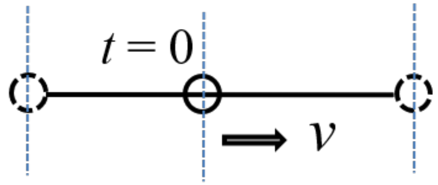


$$x(0) = a_0; \qquad \dot{x}(0) = 0$$

$$\Rightarrow x(t) = \frac{1}{2} a_0 \left(e^{i\omega t} + e^{-i\omega t}\right) = a_0 \cos \omega t$$

FAMILIAR

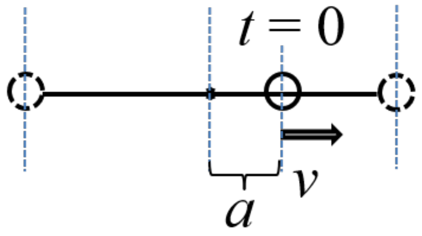
Impulsed at equilibrium



$$x(0) = 0; \qquad \dot{x}(0) = v_0$$

$$\Rightarrow x(t) = \frac{v_0}{\omega} \frac{1}{2i} \left(e^{i\omega t} - e^{-i\omega t}\right) = \frac{v_0}{\omega} \sin \omega t$$

Hit at intermediate position



$$x(0) = a; \qquad \dot{x}(0) = v$$

$$\Rightarrow x(t) = a \cos \omega t + \frac{v}{\omega} \sin \omega t = a_0 \cos(\omega t + \phi)$$

$$a_0 = \sqrt{a^2 + \left(\frac{v}{\omega}\right)^2}; \qquad \phi = tan^{-1}\left(-\frac{v}{a\omega}\right)$$

GENERAL

Q2

A simple harmonic oscillator with its angular frequency of 5 rad/s has a displacement of 1 cm and a velocity of 5 cm/s at the initial time, *t* = 0. Find

(a) total distance the oscillator moves during one cycle of its motion

(b) the complex amplitude and

(c) position of the oscillator at a later time *t* = π s.

SOLUTION

$$x(0) = a = 1\,cm; \qquad \dot{x}(0) = v = 5\,rad/s; \qquad \omega = 5\,rad/s$$

$$\Rightarrow x(t) = a_0 \cos(\omega t + \phi)$$

$$\Rightarrow a_0 = \sqrt{a^2 + \left(\frac{v}{\omega}\right)^2} = \sqrt{2}\,cm; \quad \phi = tan^{-1}\left(-\frac{v}{a\omega}\right) = -\frac{\pi}{4}$$

a) Total distance moved in one oscillation = $4a_0 = 4\sqrt{2}\,cm$

…Q2

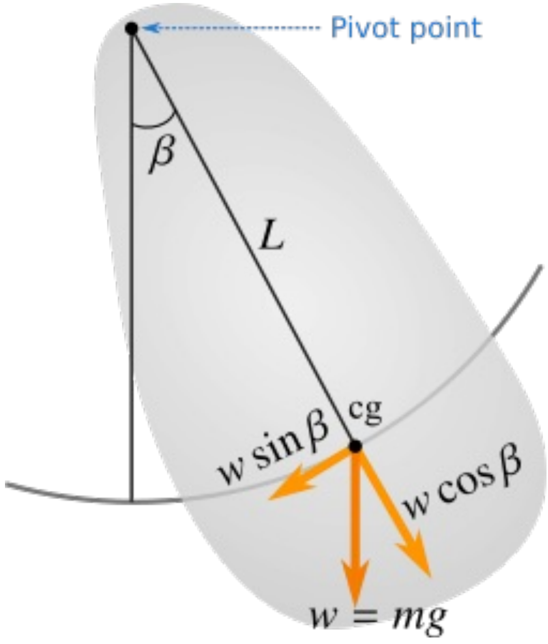
$$a_0 = \sqrt{a^2 + \left(\frac{v}{\omega}\right)^2} = \sqrt{2}\,cm; \quad \phi = tan^{-1}\left(-\frac{v}{\omega}\right) = -\frac{\pi}{4}$$

b) Complex amplitude = $a_0 e^{i\phi} = \sqrt{2}\,cm \times e^{-i\pi/4}$

c) $x(\pi\ s) = \sqrt{2}\cos\left(5\pi - \frac{\pi}{4}\right) cm = 1.37\ cm$

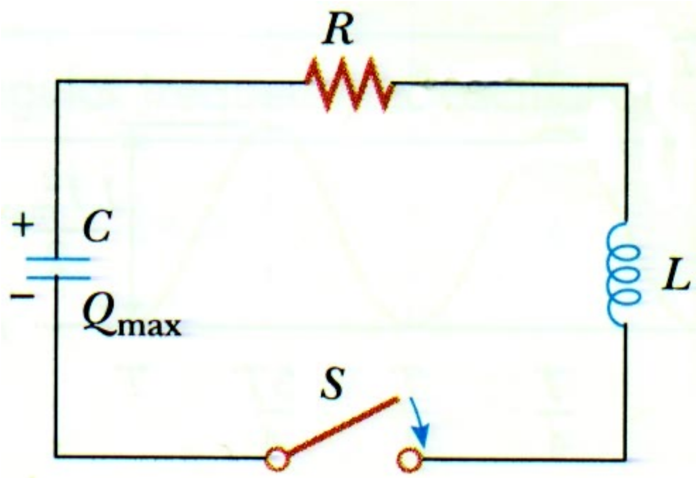
Oscillations: Few more examples

Physical Pendulum



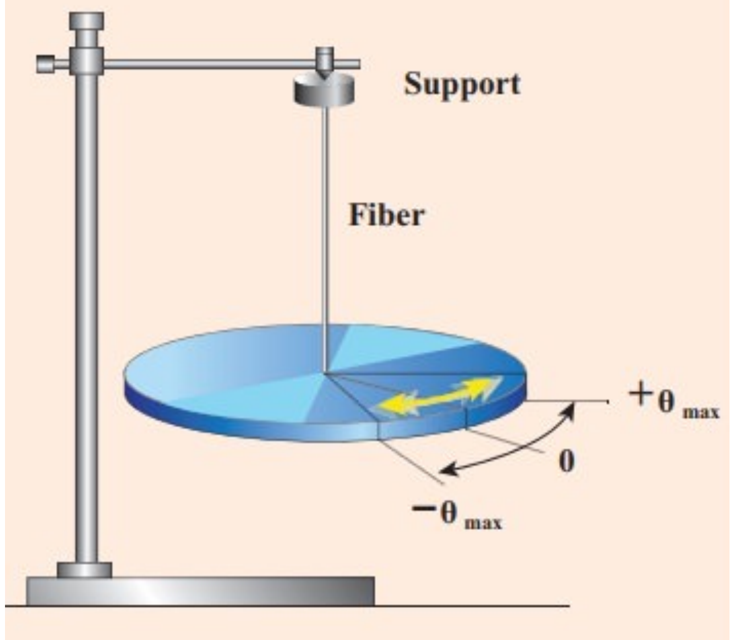
Simple and Physical Pendulums – Physics Key

Electrical Oscillator



EM Oscillations - LCR circuits - Physics 299 (louisville.edu)

Torsional Pendulum (Angular SHM)

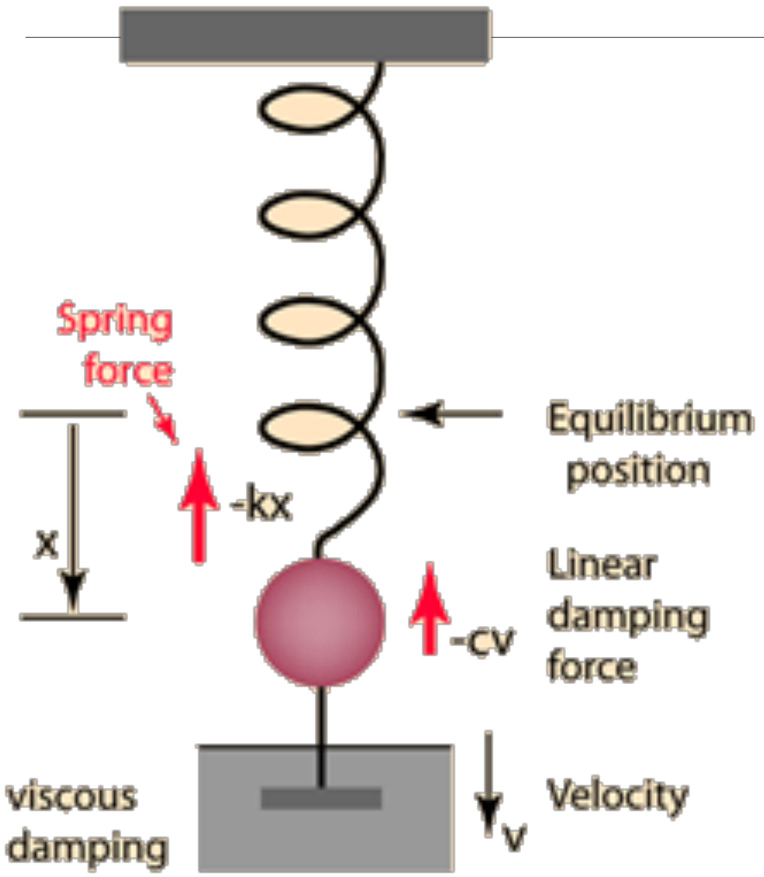


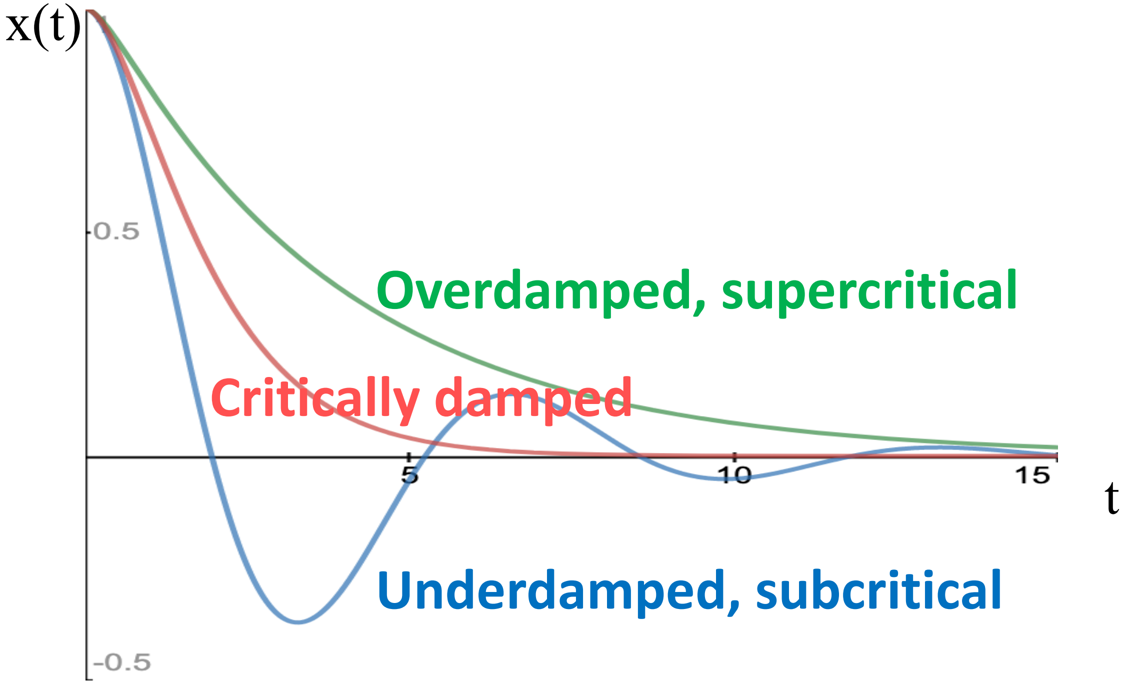
Time period and frequency of angular SHM(Simple Harmonic Motion) (brainkart.com)

OSCILLATIONS

1. Simple Harmonic (Free) Oscillations	-	Just studied
2. Damped Oscillations	-	Some familiarity
3. Forced Oscillations	-	Some familiarity

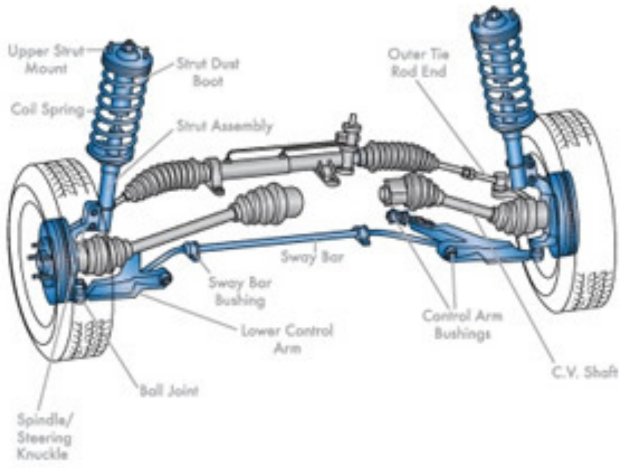
DAMPED OSCILLATIONS





Damped Harmonic Oscillator (gsu.edu)

Examples



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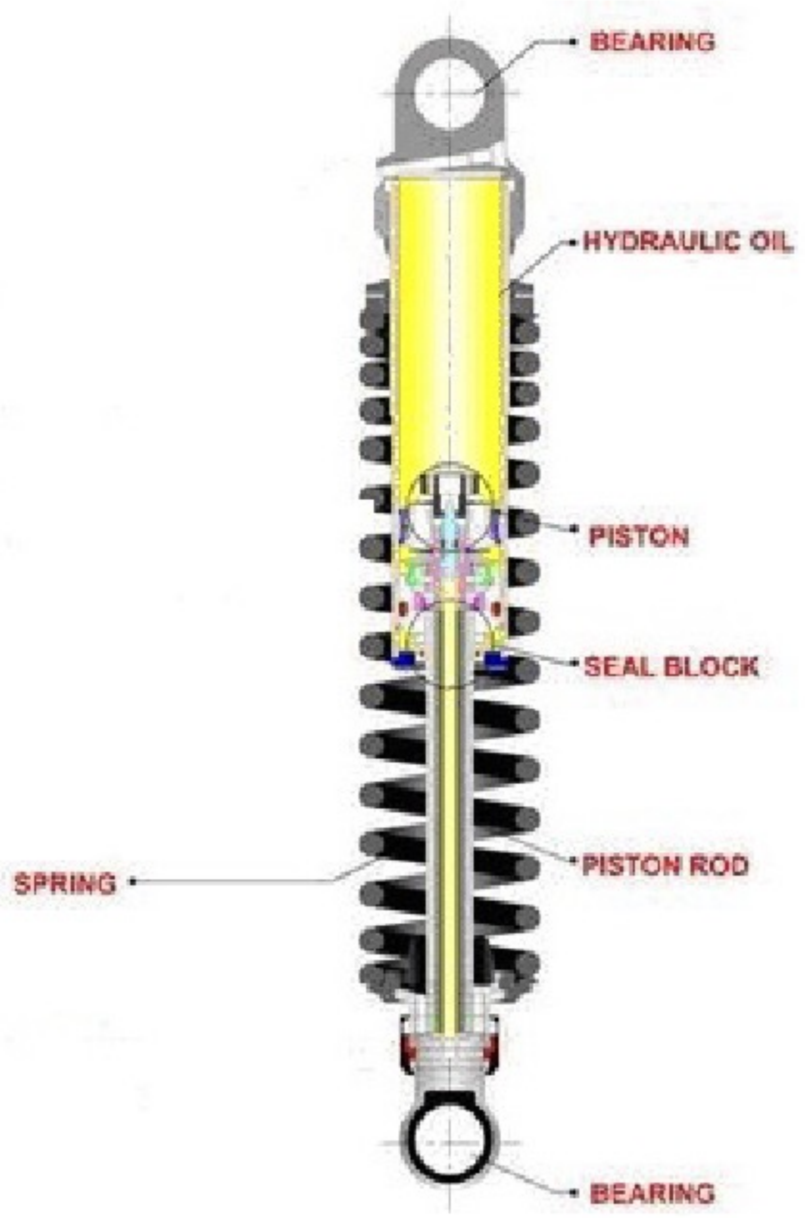
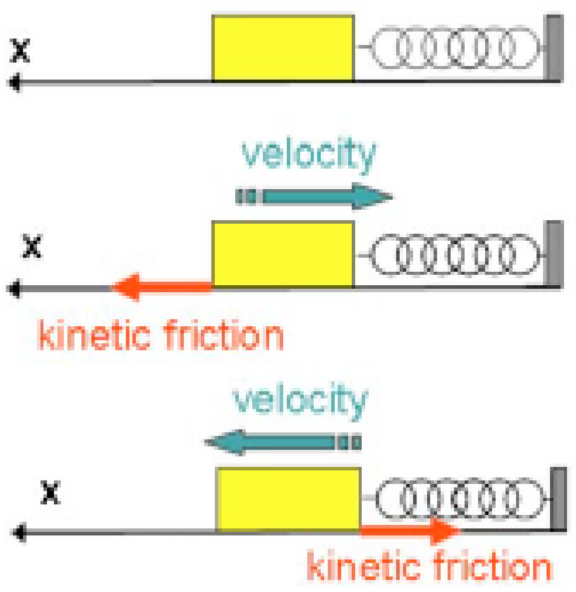


(2) Facebook

Examples



Damped oscillations and equilibrium in a mass-spring system subject to sliding friction forces: Integrating experimental and theoretical analyses: American Journal of Physics: Vol 78, No 11 (scitation.org)

Cute Boy Playing On Swing Stock Illustration - Download Image Now - Swing - Play Equipment, Swinging, Porch Swing - iStock (istockphoto.com)

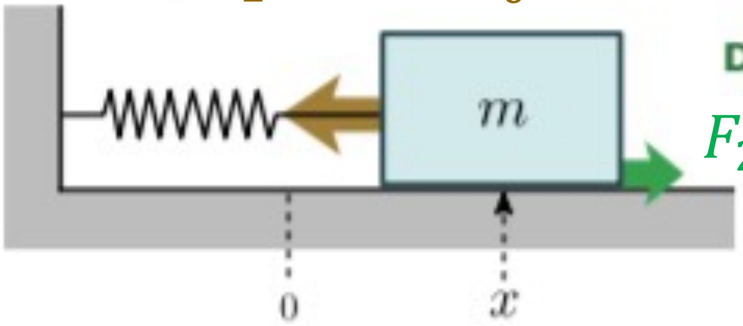
CUL Ul Door Closer For Fire Rated Doors Size 4 Size 5 Size 6 For Doors Up To 200kg (frd-doorcloser.com)

Damped Oscillations: The Force Equation and Analysis

5.1: The Damped Harmonic Oscillator - Physics LibreTexts

Spring force

$F_1 = -m\omega_0^2 x$



Damping force

$F_2 = -m\, 2\beta\, \dot{x}$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

Force equation: $m\ddot{x} = -m\omega_0^2 x - m\, 2\beta\, \dot{x}$

$$\boxed{\Rightarrow \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0}$$

- A 2nd order linear homogenous equation with constant coefficients

For solution, take $x = e^{qt}$

$$\Rightarrow q^2 + 2\beta q + \omega_0^2 = 0$$

$$\Rightarrow q = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

Let $\boxed{\omega^2 = \omega_0^2 - \beta^2}$ Altered frequency

Stiffness term Damping resistance term

$\Rightarrow q_1 = -\beta + i\omega,$ and $q_2 = -\beta - i\omega$

$$\Rightarrow x(t) = c_1 e^{(-\beta+i\omega)t} + c_2 e^{(-\beta-i\omega)t}$$

$$\boxed{\Rightarrow x(t) = \underbrace{\left(c_1 e^{i\omega t} + c_2 e^{-i\omega t}\right)}_{\text{Oscillatory?}} e^{-\beta t}}$$

Exponentially decaying

Damped Oscillations: General solution with initial conditions

$$\boxed{x(t) = \left(c_1 e^{i\omega t} + c_2 e^{-i\omega t}\right) e^{-\beta t}}$$

$$\dot{x}(t) = \left[(-\beta + i\omega)c_1 e^{i\omega t} + (-\beta - i\omega)c_2 e^{-i\omega t}\right] e^{-\beta t}$$

Then, at $t = 0$,

$c_1 + c_2 = x(0)$ and

$[(-\beta + i\omega)c_1 + (-\beta - i\omega)c_2] = \dot{x}(0)$

Solving for c_1 and c_2 gives

$$\boxed{c_1 = \frac{\dot{x}(0) + (\beta + i\omega)\, x(0)}{2i\omega}}$$ and $$\boxed{c_2 = \frac{-\dot{x}(0) - (\beta - i\omega)\, x(0)}{2i\omega}}$$

Damped Oscillations: General solution contd…

$$\boxed{x(t) = \left(c_1 e^{i\omega t} + c_2 e^{-i\omega t}\right) e^{-\beta t}}$$

$$\boxed{\omega^2 = \omega_0^2 - \beta^2}$$

$$\Rightarrow x(t) = \left(\frac{\dot{x}(0) + (\beta + i\omega)\, x(0)}{2i\omega} e^{i\omega t} + \frac{-\dot{x}(0) - (\beta - i\omega)\, x(0)}{2i\omega} e^{-i\omega t}\right) e^{-\beta t}$$

$$\boxed{\Rightarrow x(t) = \left(\underbrace{\left\{\frac{x(0)}{2} - i\,\frac{\dot{x}(0) + \beta\, x(0)}{2\omega}\right\}}_{\frac{\tilde{A}}{2} = \frac{a_0}{2} e^{i\alpha}} e^{i\omega t} + \underbrace{\left\{\frac{x(0)}{2} + i\,\frac{\dot{x}(0) + \beta\, x(0)}{2\omega}\right\}}_{\frac{\tilde{A}^*}{2} = \frac{a_0}{2} e^{-i\alpha}} e^{-i\omega t}\right) e^{-\beta t}}$$

considering $\boldsymbol{\omega}$ to be **real**

$$a_0 = \sqrt{\{x(0)\}^2 + \left\{\frac{\dot{x}(0) + \beta\, x(0)}{\omega}\right\}^2} \quad ; \quad \alpha = \tan^{-1}\left\{-\frac{\dot{x}(0) + \beta\, x(0)}{\omega x(0)}\right\}$$

Damped Oscillations: General solution contd...

$$\boxed{x(t) = \left(c_1 e^{i\omega t} + c_2 e^{-i\omega t}\right) e^{-\beta t}}$$

$$\boxed{\omega^2 = \omega_0^2 - \beta^2}$$

$$\boxed{\Rightarrow x(t) = \left(\frac{a_0}{2} e^{i\alpha} e^{i\omega t} + \frac{a_0}{2} e^{-i\alpha} e^{-i\omega t}\right) e^{-\beta t}}$$

$$= \frac{a_0}{2}\left\{e^{i(\alpha+\omega t)} + e^{-i(\alpha+\omega t)}\right\} e^{-\beta t}$$

$$= \frac{a_0}{2}\{\cos(\omega t + \alpha) + i \sin(\omega t + \alpha) + \cos(\omega t + \alpha) - i \sin(\omega t + \alpha)\} e^{-\beta t}$$

$$\boxed{\Rightarrow x(t) = a_0 \cos(\omega t + \alpha)\, e^{-\beta t}}$$

$\omega \equiv$ Real $\Rightarrow$ $\omega_0 > \beta$

$\Rightarrow$ Small damping

$\Rightarrow$ **Underdamped oscillations**

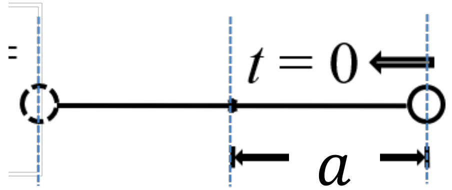
$$a_0 = \sqrt{\{x(0)\}^2 + \left\{\frac{\dot{x}(0) + \beta\, x(0)}{\omega}\right\}^2}$$

$$\alpha = \tan^{-1}\left\{-\frac{\dot{x}(0) + \beta\, x(0)}{\omega x(0)}\right\}$$

Underdamped Oscillations

$(\omega_0 > \beta)$ $\qquad \Rightarrow x(t) = a_0 \cos(\omega t + \alpha)\, e^{-\beta t}$

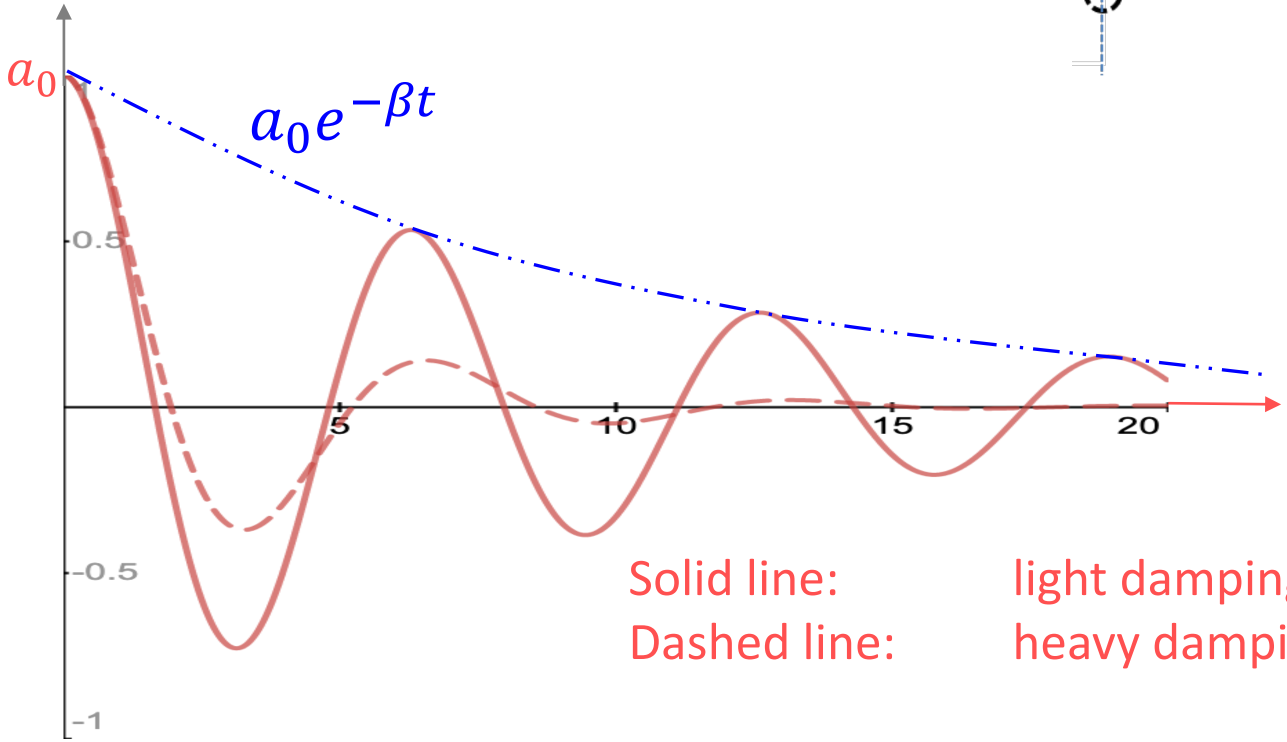
(a) The mass is pulled to one side and released from rest at t = 0



$x(0) = a; \qquad \dot{x}(0) = 0$

$$a_0 = \sqrt{a^2 + \left\{\frac{\beta\, a}{\omega}\right\}^2}$$

$$\alpha = \tan^{-1}\left\{-\frac{\beta}{\omega}\right\}$$



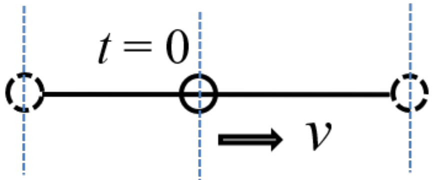
Solid line: light damping (lower β)

Dashed line: heavy damping (higher β)

Underdamped Oscillations

$(\omega_0 > \beta)$ $\Rightarrow x(t) \equiv a_0 \cos(\omega t + \alpha)\, e^{-\beta t}$

(b) The mass is hit and is given a speed v_0 at its equilibrium position at t = 0

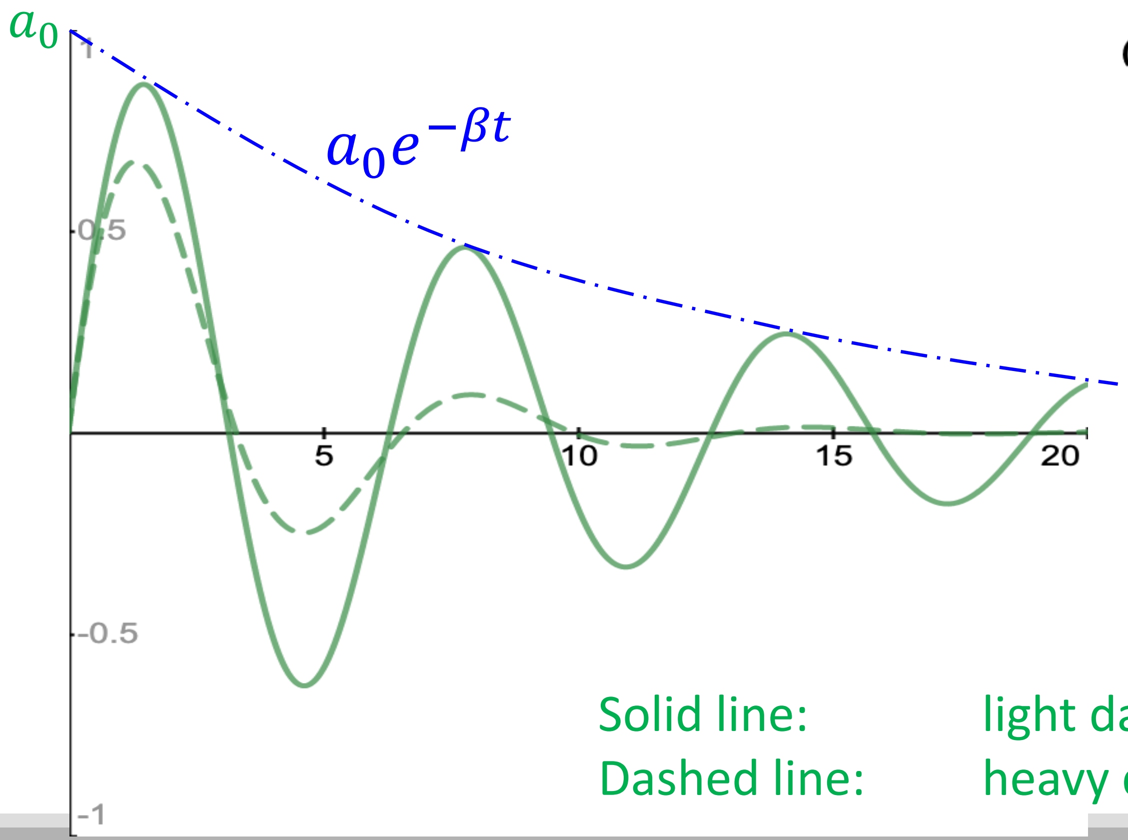


$x(0) = 0;\qquad \dot{x}(0) = v_0$

$$a_0 = \frac{v_0}{\omega}$$

$$\alpha = \tan^{-1}\{-\infty\} \Rightarrow \alpha = -\frac{\pi}{2}$$

$$\boxed{\Rightarrow x(t) = a_0 \sin(\omega t)\, e^{-\beta t}}$$



MODIFIED SLIDE

Damped Oscillations: Different cases

Case 2: Dominant damping (Overdamping) $(\omega_0 < \beta)$

Look back:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

$$x = e^{qt}$$

$$\Rightarrow q^2 + 2\beta q + \omega_0^2 = 0$$

$$\Rightarrow q = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

Let $\Omega^2 = \beta^2 - \omega_0^2$

$$\Rightarrow q = -\beta \pm \Omega$$

$$\Rightarrow \boxed{x(t) = \left(c_1 e^{\Omega t} + c_2 e^{-\Omega t}\right) e^{-\beta t}}$$

No oscillations, only decay

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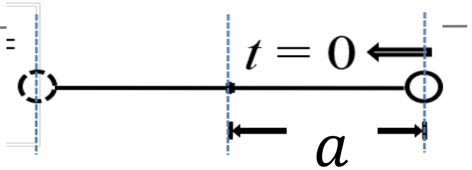
Overdamped Oscillations

$$\boxed{x(t) = \left(c_1 e^{\Omega t} + c_2 e^{-\Omega t}\right) e^{-\beta t}} \qquad \boxed{\Omega^2 = \beta^2 - \omega_0^2}$$

(a) The mass is pulled to one side and released from rest at t = 0



$$x(0) = a; \qquad \dot{x}(0) = 0$$

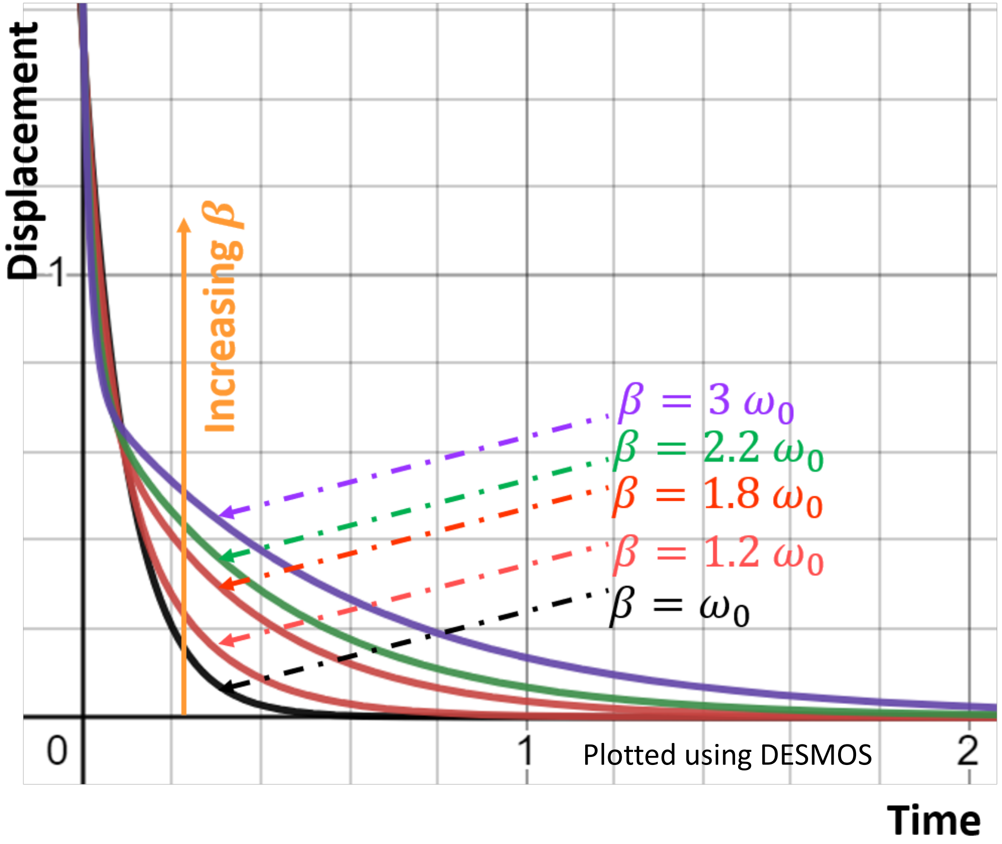
$$\Rightarrow c_1 + c_2 = a \text{ and } c_1 - c_2 = 0$$

$$\Rightarrow c_1 = c_2 = \frac{a}{2}$$

$$\Rightarrow x(t) = a\left(\frac{e^{\Omega t} + e^{-\Omega t}}{2}\right) e^{-\beta t}$$

$$\Rightarrow x(t) = a \cosh \Omega t \, e^{-\beta t}$$

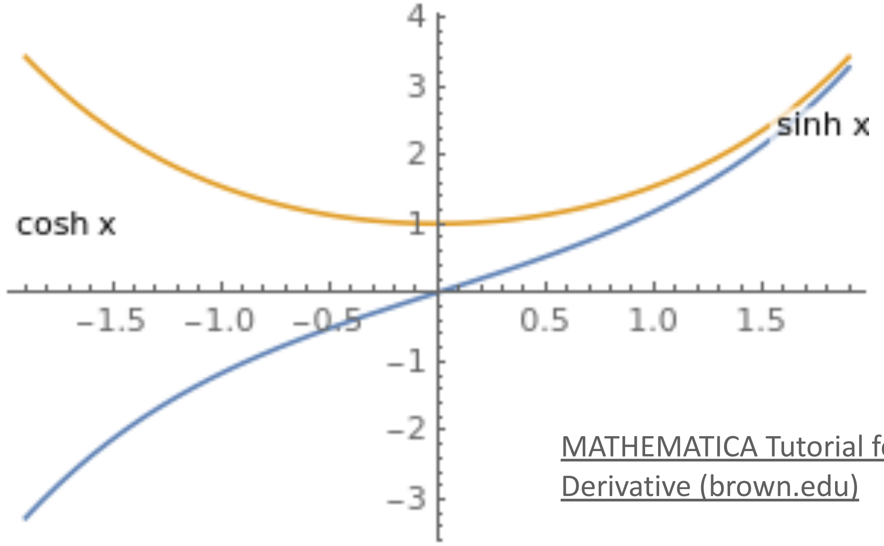
$$\Rightarrow x(t) = a \cosh\left\{\left(\sqrt{\beta^2 - \omega_0^2}\right) t\right\} e^{-\beta t}$$



Overdamped Oscillations

MODIFIED SLIDE (Informatory)

$$\boxed{x(t) = a \cosh \Omega t \, e^{-\beta t}}$$



[MATHEMATICA Tutorial for first course: Derivative (brown.edu)]

[hyperbolicfunctions.dvi (mathcentre.ac.uk)]

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\Rightarrow \cosh(ix) = \cos x$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\boxed{\Rightarrow \sinh(ix) = i \sin x}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cos^2 x + \sin^2 x = 1$$

[MATHEMATICA Tutorial for first course: Derivative (brown.edu)]

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\cos\,(x + y) = \cos x \cos y - \sin x \sin y$$

$$A \cosh x + B \sinh x = R \cosh(x + \delta)$$

$$R = \sqrt{B^2 - A^2}$$

$$\tanh \delta = \frac{B}{A}$$

Prove yourself.
Take help from:
[mc-TY-rcostheta-alpha-2009-1.dvi (mathcentre.ac.uk)]
[hyperbolicfunctions.dvi (mathcentre.ac.uk)]

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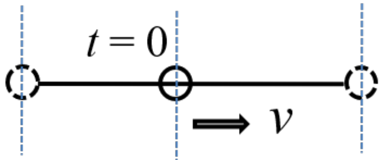
Overdamped Oscillations

$$\boxed{x(t) = \left(c_1 e^{\Omega t} + c_2 e^{-\Omega t}\right) e^{-\beta t}} \qquad \boxed{\Omega^2 = \beta^2 - \omega_0^2}$$

(a) Impulsed at equilibrium



$$x(0) = 0; \qquad \dot{x}(0) = v_0$$

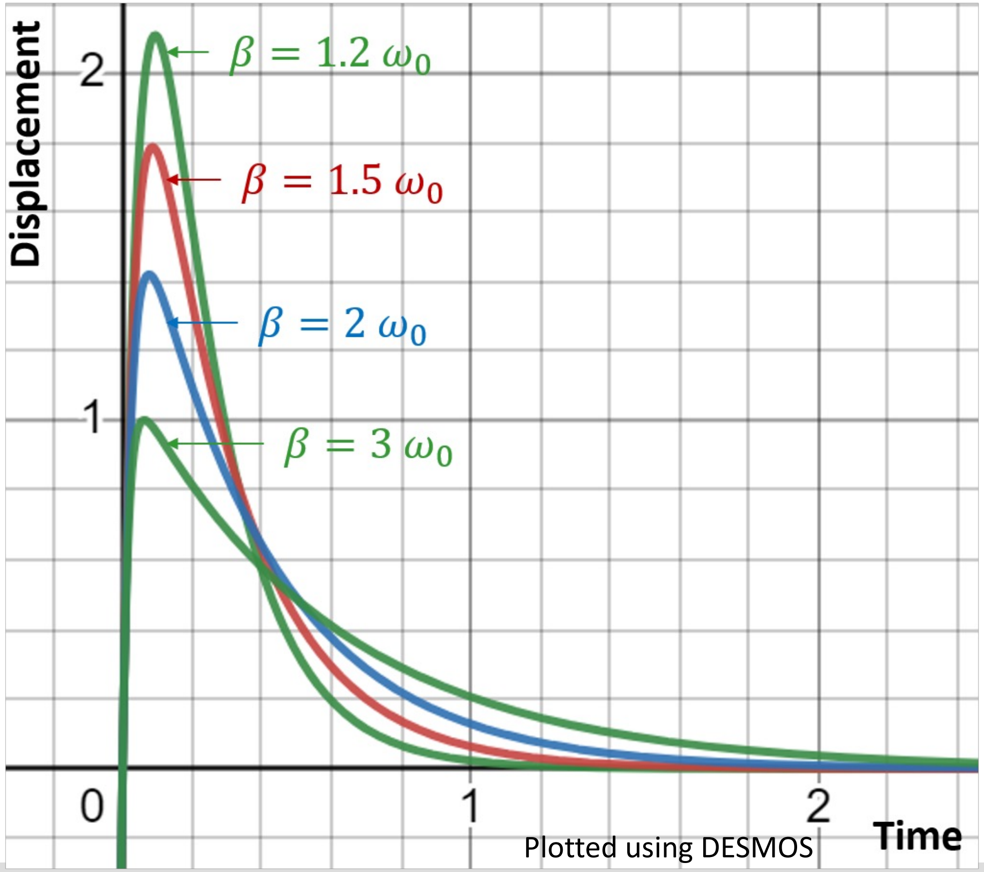
$$\Rightarrow c_1 + c_2 = 0 \text{ and } \quad \Omega(c_1 - c_2) = v_0$$

$$\Rightarrow c_1 = -c_2 = \frac{v_0}{2\Omega}$$

$$\Rightarrow x(t) = \frac{v_0}{\Omega}\left(\frac{e^{\Omega t} - e^{-\Omega t}}{2}\right) e^{-\beta t}$$

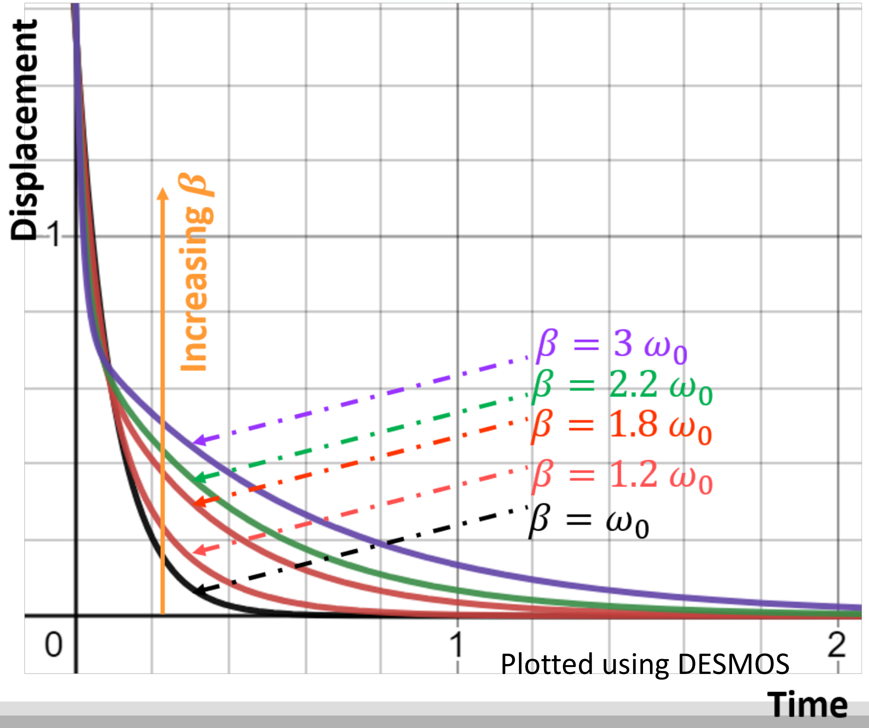
$$\Rightarrow x(t) = \frac{v_0}{\Omega} \sinh \Omega t \, e^{-\beta t}$$

$$\Rightarrow x(t) = \frac{v_0}{\sqrt{\beta^2 - \omega_0^2}} \sinh\left\{\left(\sqrt{\beta^2 - \omega_0^2}\right) t\right\} e^{-\beta t}$$



Overdamped Oscillators: Examples

- ➢ Push-button water faucet shut-off valves
- ➢ Door closer



Damped Oscillations: Different cases

Case 3: Equal stiffness and damping (Critical Damping) $(\omega_0 \approx \beta \Rightarrow \omega \to 0)$

Consider the general solution

$$x(t) = \left(\left\{\frac{x(0)}{2} - i\,\frac{\dot{x}(0) + \beta\, x(0)}{2\omega}\right\} e^{i\omega t} + \left\{\frac{x(0)}{2} + i\,\frac{\dot{x}(0) + \beta\, x(0)}{2\omega}\right\} e^{-i\omega t}\right) e^{-\beta t}$$

$$= \left\{x(0)\,\frac{e^{i\omega t} + e^{-i\omega t}}{2} + \frac{\dot{x}(0) + \beta\, x(0)}{\omega} \cdot \frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right\} e^{-\beta t}$$

$$= \left[x(0)\cos\omega t + \{\dot{x}(0) + \beta\, x(0)\}\,\frac{\sin\omega t}{\omega t}\,t\right] e^{-\beta t}$$

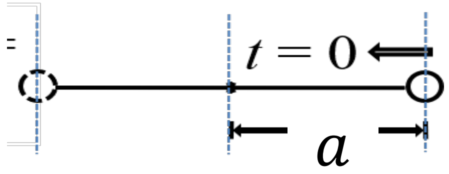
$$= [x(0) + \{\dot{x}(0) + \beta\, x(0)\}t]\; e^{-\beta t} \qquad \text{, for } \omega \to 0$$

Critically Damped Oscillations

$\omega_0 \approx \beta \Rightarrow \ \omega \to 0$

$$\boxed{x(t) = [x(0) + \{\dot{x}(0) + \beta\, x(0)\}t]\ e^{-\beta t}}$$

(a) The mass is pulled to one side and released from rest at t = 0

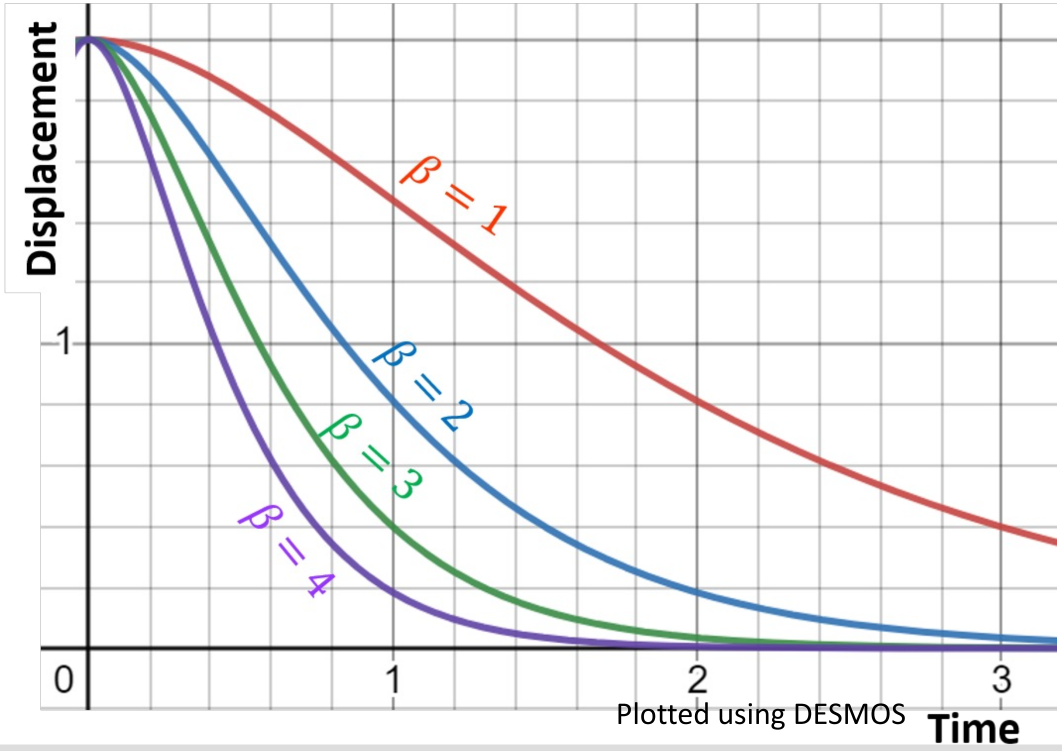


$x(0) = a; \qquad \dot{x}(0) = 0$

$$\boxed{\Rightarrow x(t) = a(1 + \beta t)e^{-\beta t}}$$

- a product of a linearly increasing term and an exponentially decreasing term

- The linear term dominates initially and is taken over later by the exponential decay term.

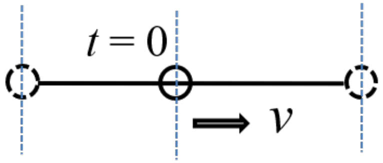


Plotted using DESMOS

Critically Damped Oscillations

$$\omega_0 \approx \beta \Rightarrow \ \omega \to 0$$

$$\boxed{x(t) = [x(0) + \{\dot{x}(0) + \beta\, x(0)\}t]\ e^{-\beta t}}$$



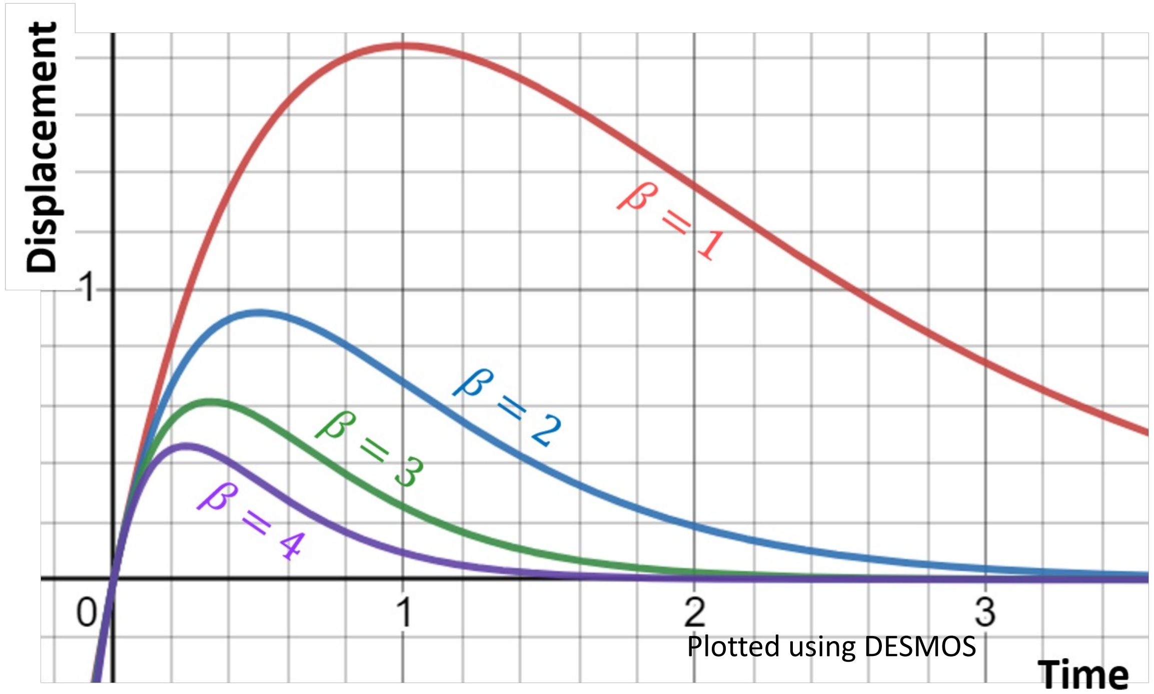
(b) Impulsed at equilibrium

$$x(0) = 0 \qquad\qquad \dot{x}(0) = v_0$$

$$\boxed{\Rightarrow x(t) = v_0 t\ e^{-\beta t}}$$

- a product of a linearly increasing term and an exponentially decreasing term

- The linear term dominates initially and is taken over later by the exponential decay term.

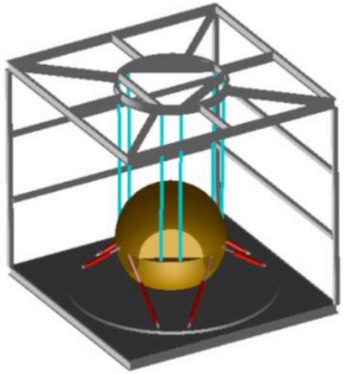


Critically Damped Oscillators: Examples

- Shock absorbers

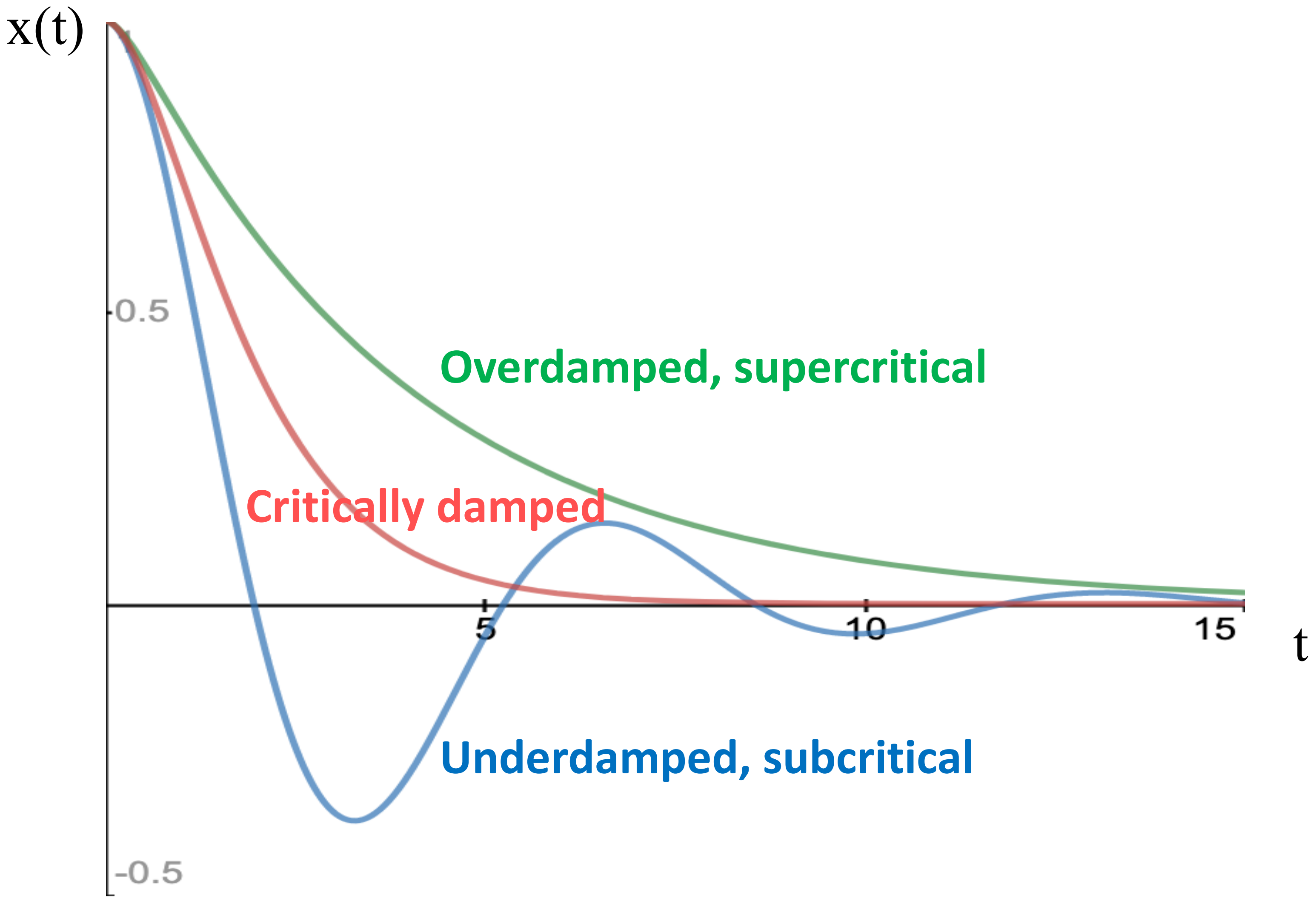
- Speedometer

- Wire bridges



- Skyscraper buildings

Damped Oscillations: Summary



Damped Oscillations: Further considerations

Energy in Underdamping

$$x(t) = a_0\cos(\omega t + \alpha)\, e^{-\beta t}$$

$$U = \frac{1}{2}k\, x^2(t) = \frac{1}{2}m\omega_0^2 a_0^2\, .\, e^{-2\beta t}.\cos^2(\omega t + \alpha) = \underbrace{\frac{1}{2}m\omega_0^2 a_0^2\, .\, e^{-2\beta t}}.\underbrace{\left[\frac{1 + \cos\{2(\omega t + \alpha)\}}{2}\right]}$$

Exponential decay in amplitude with decay constant -2β

Oscillations with frequency 2ω

$$\langle U\rangle = \frac{1}{2}m\omega_0^2 a_0^2\, .\, e^{-2\beta t}\left\langle\frac{1 + \cos\{2(\omega t + \alpha)\}}{2}\right\rangle = \frac{1}{4}m\omega_0^2 a_0^2\, .\, e^{-2\beta t}$$

DIY for SHM

Damped Oscillations: Further considerations

Energy in Underdamping

$$x(t) = a_0 \cos(\omega t + \alpha)\, e^{-\beta t}$$

$$\dot{x}(t) = -a_0 e^{-\beta t}[\beta \cos(\omega t + \alpha) + \omega \sin(\omega t + \alpha)]$$

$$= a_0 e^{-\beta t} \kappa \cos(\omega t + \alpha - \epsilon) \qquad \kappa = \sqrt{\omega^2 + \beta^2} = \omega_0; \;\; \tan\epsilon = -\omega/\beta$$

$$= a_0 \omega_0 e^{-\beta t} \cos(\omega t + \alpha - \epsilon)$$

$$KE = \frac{1}{2} m\, \dot{x}^2(t) = \underbrace{\frac{1}{2} m \omega_0^2 a_0^2 \,.\, e^{-2\beta t}}\,.\left[\underbrace{\frac{1 + \cos\{2(\omega t + \alpha - \epsilon)\}}{2}}\right]$$

$$\langle KE \rangle = \frac{1}{2} m \omega_0^2 a_0^2 \,.\, e^{-2\beta t} \left\langle \frac{1 + \cos\{2(\omega t + \alpha - \epsilon)\}}{2} \right\rangle = \frac{1}{4} m \omega_0^2 a_0^2 \,.\, e^{-2\beta t} = \langle U \rangle$$

DIY for SHM

Damped Oscillations: Further considerations

Energy in Underdamping

$\boxed{x(t) = a_0 \cos(\omega t + \alpha)\, e^{-\beta t}}$

$$U = \frac{1}{2} m\omega_0^2 a_0^2 \,.\, e^{-2\beta t}. \left[\frac{1 + \cos\{2(\omega t + \alpha)\}}{2}\right]$$

$$KE = \frac{1}{2} k\, \dot{x}^2(t) = \frac{1}{2} m\omega_0^2 a_0^2 \,.\, e^{-2\beta t}. \left[\frac{1+\cos\{2(\omega t+\alpha-\epsilon)\}}{2}\right];$$

$$\epsilon = \tan^{-1}\left(-\frac{\omega}{\beta}\right) = \tan^{-1}\left(-\frac{\sqrt{\omega_0^2 - \beta^2}}{\beta}\right)$$

$$\boxed{\frac{1}{2} m\omega_0^2 a_0^2 = E_0}$$

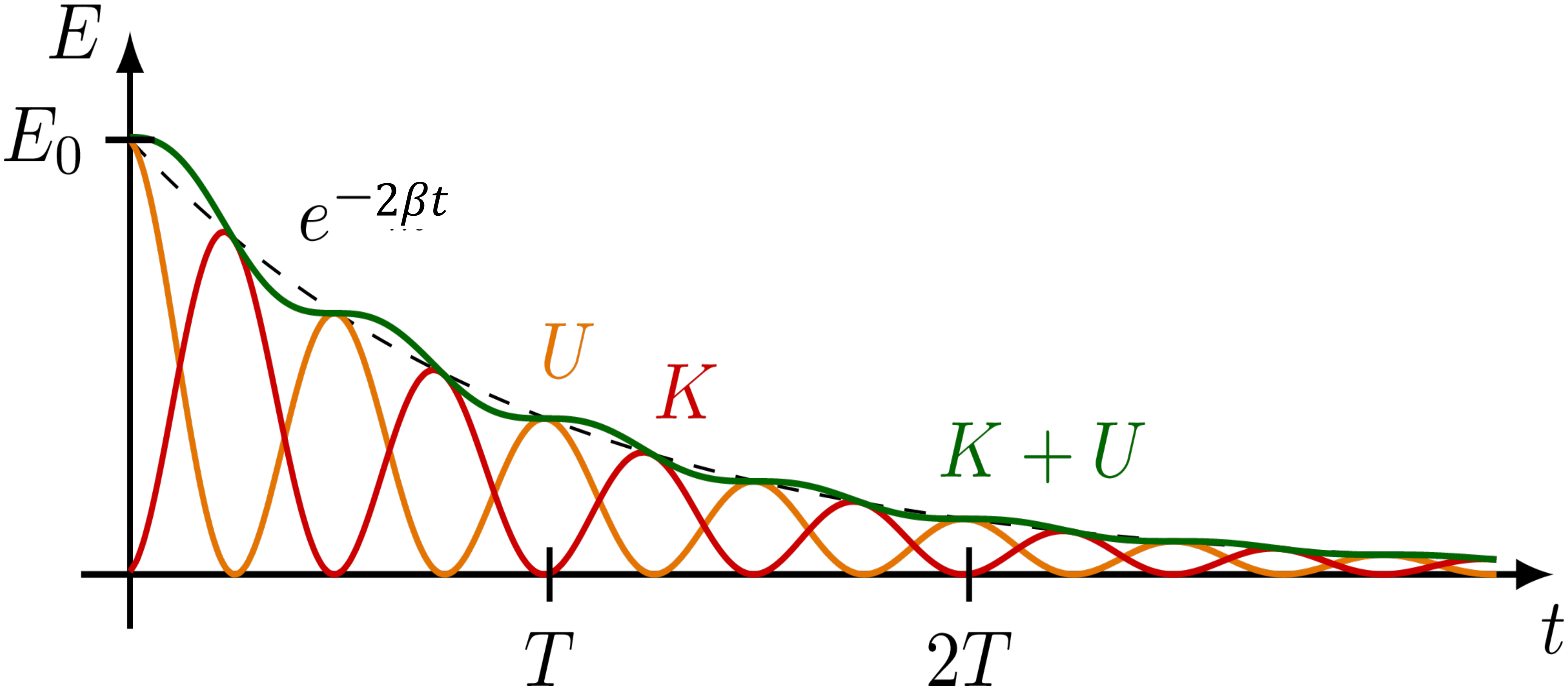
$$\Rightarrow E = U + KE = \frac{1}{2} m\omega_0^2 a_0^2 \,.\, e^{-2\beta t}[\underbrace{\cos\{2(\omega t + \alpha)\}} + \underbrace{\cos\{2(\omega t + \alpha - \epsilon)\}}]$$

$$\langle E \rangle = \langle U \rangle + \langle KE \rangle = \frac{1}{2} m\omega_0^2 a_0^2 \,.\, e^{-2\beta t} = E_0 e^{-2\beta t}$$

Damped Oscillations: Further considerations

Energy in Underdamping

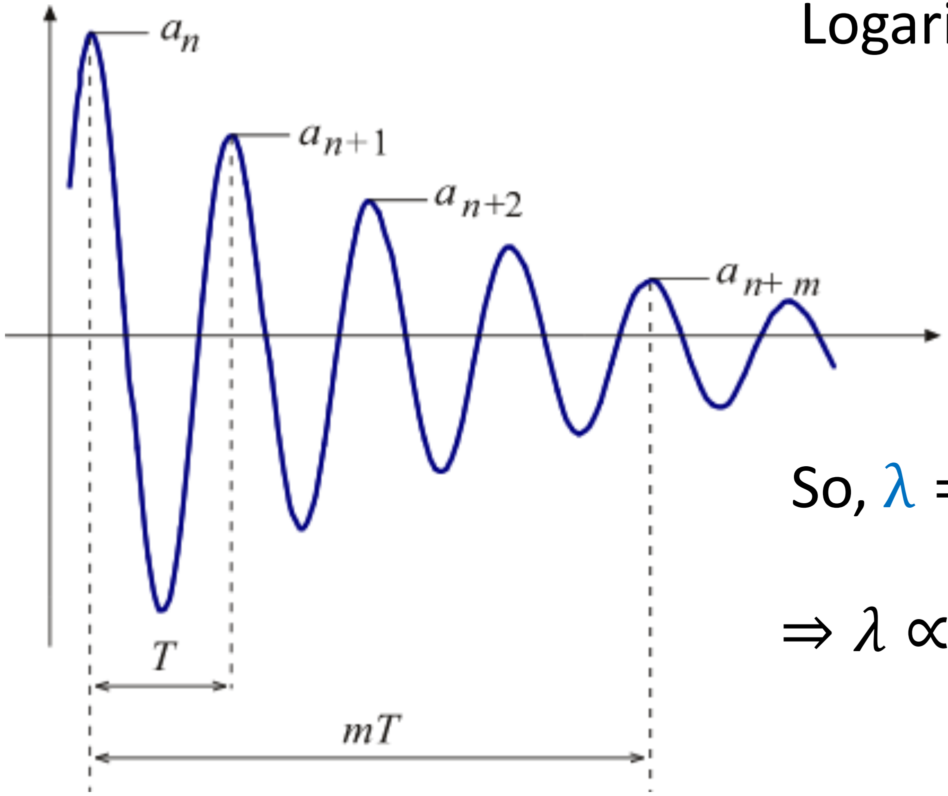
$$\boxed{x(t) = a_0 \cos(\omega t + \alpha)\, e^{-\beta t}}$$



Damped Oscillations: Further considerations

Logarithmic decrement (Underdamping)

$$x(t) = a_0 \cos(\omega t + \alpha)\, e^{-\beta t}$$



Logarithmic decrement $\lambda \equiv ln\left(\frac{a_n}{a_{n+1}}\right) = \frac{1}{m} ln\left(\frac{a_n}{a_{n+m}}\right)$

If $a_n = a_0 e^{-\beta t}$, then $a_{n+1} = a_0 e^{-\beta(t+T)}$ and

$$a_{n+m} = a_0 e^{-\beta(t+mT)}$$

So, $\lambda = ln\left(\frac{a_0 e^{-\beta t}}{a_0 e^{-\beta(t+T)}}\right) = \frac{1}{m} ln\left(\frac{a_0 e^{-\beta t}}{a_0 e^{-\beta(t+mT)}}\right) = \beta T$

$\Rightarrow \lambda \propto \beta$

- A measure of how fast the amplitude decreases.

Further, $\lambda = \beta T = \frac{2\pi\beta}{2\pi/T} = \frac{2\pi\beta}{\omega}$

Damped Oscillations: Further considerations

Quality Factor (Underdamping)

$\langle E(t)\rangle = E_0 e^{-2\beta t}$

Decay time τ = The time in which the total energy decays by *1/e* $= \dfrac{1}{2\beta}$

- The quality factor Q defined as

$$Q \equiv 2\pi\frac{\tau}{T} = \omega\tau = \frac{\omega}{2\beta} = \frac{\pi}{\lambda}$$

  is a convenient measure of how much longer the decay time is compared to the period. It is a measure of how many oscillations take place during the time the energy decays by the factor of 1/e.

- Informally, the quality factor represents the number of cycles completed by the oscillator before it "rings down" or "runs out of energy".
- Lightly damped oscillations are referred to as high Q, and heavier damped oscillations, as low Q.

Damped Oscillations: Further considerations

Quality Factor (Underdamping)

$$Q \equiv \frac{\omega}{2\beta}$$

In terms of energy (for high Q or low β),

$$Q = 2\pi \frac{\langle E(t)\rangle}{\langle E(t)\rangle - \langle E(t+T)\rangle} \equiv 2\pi \frac{\langle E(t)\rangle}{\Delta\langle E(t)\rangle} = 2\pi \frac{\text{Average energy stored}}{\text{Energy lost per cycle}}$$

$$\langle E(t)\rangle = E_0 e^{-2\beta t}$$

$$\langle \Delta E(t)\rangle = E_0 e^{-2\beta t} - E_0 e^{-2\beta(t+T)} = E_0 e^{-2\beta t}\left[1 - e^{-2\beta T}\right] \approx E_0 e^{-2\beta t}[1 - \{1 - 2\beta T + \cdots\}]$$

$$= 2\beta T E_0 e^{-2\beta t}$$

For small β

$$\Rightarrow 2\pi \frac{\langle E(t)\rangle}{\Delta\langle E(t)\rangle} = \frac{2\pi}{T} . \frac{1}{2\beta} = \frac{\omega}{2\beta} = \mathrm{Q}$$

Damped Oscillations: Further considerations

Quality Factor (Underdamping)

$$Q \equiv \frac{\omega}{2\beta}$$

In terms of displacement (for high Q or low β),

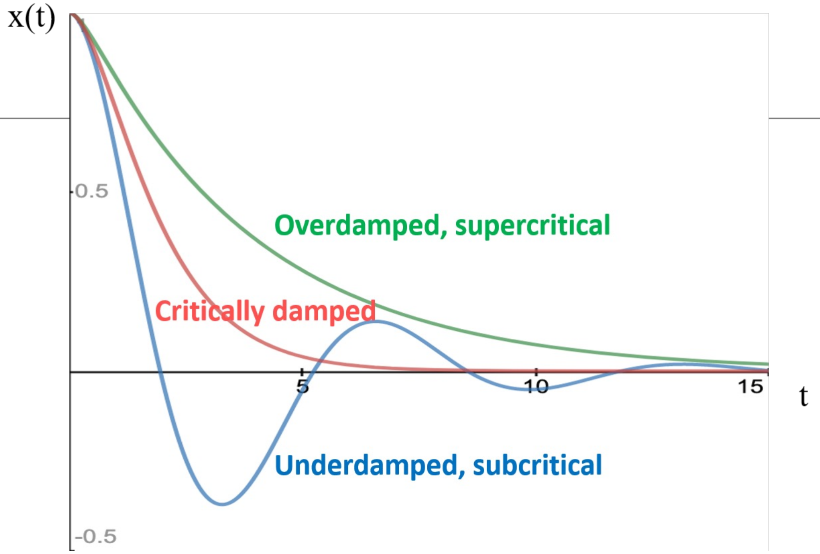
$$Q = \pi \frac{x(t)}{x(t) \; - x(t+T)} \equiv \pi \frac{x(t)}{\Delta x(t)}$$

$$= \pi \frac{\text{displacement at any instant}}{\text{reduction in the displacement in the next one cycle}}$$

- Verify yourself

Damped Oscillations: Further considerations

Phase diagram



Underdamped

$$x(t) = a_0 \cos(\omega t + \alpha)\, e^{-\beta t}$$

$$\Rightarrow \dot{x}(t) = -a_0 \omega_0 \cos(\omega t + \delta)\, e^{-\beta t}$$

Overdamped

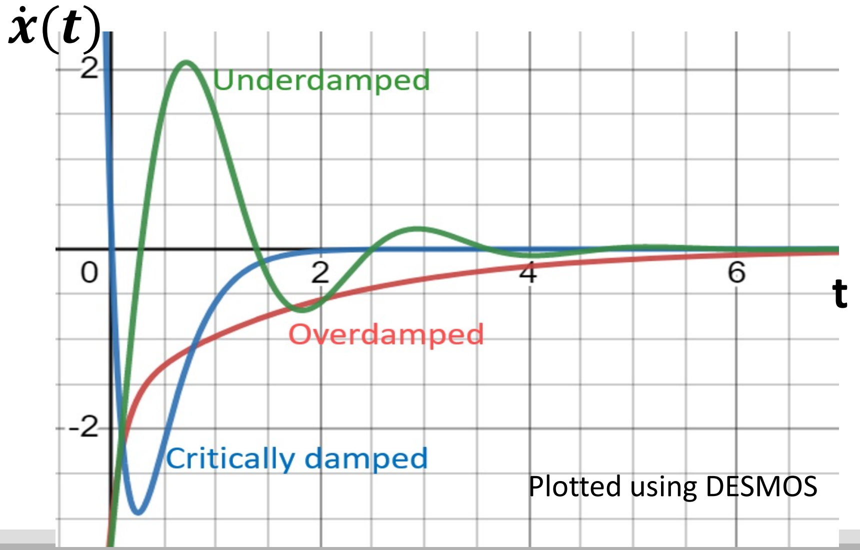
$$x(t) = a \cosh \Omega t\, e^{-\beta t}$$

$$\Rightarrow \dot{x}(t) = -\omega_0 a \cosh(\Omega t - \varepsilon)\, e^{-\beta t}$$

Critically damped

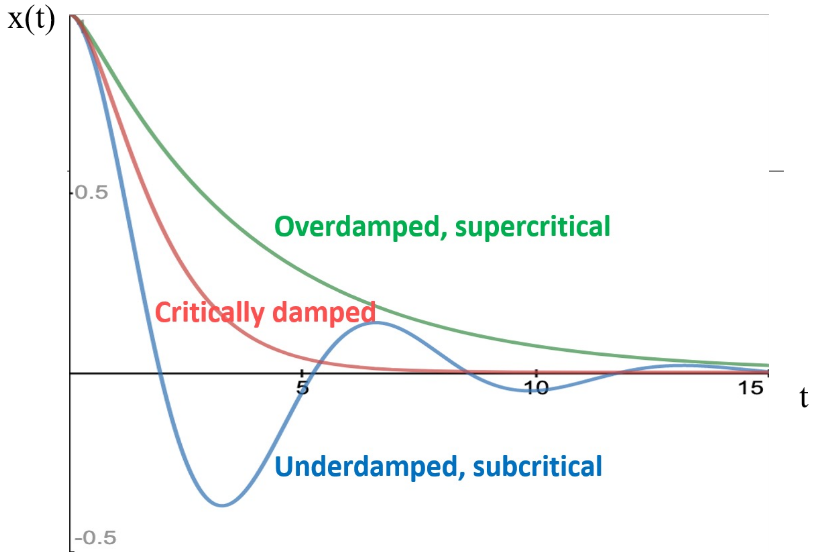
$$x(t) = a(1 + \beta t) e^{-\beta t}$$

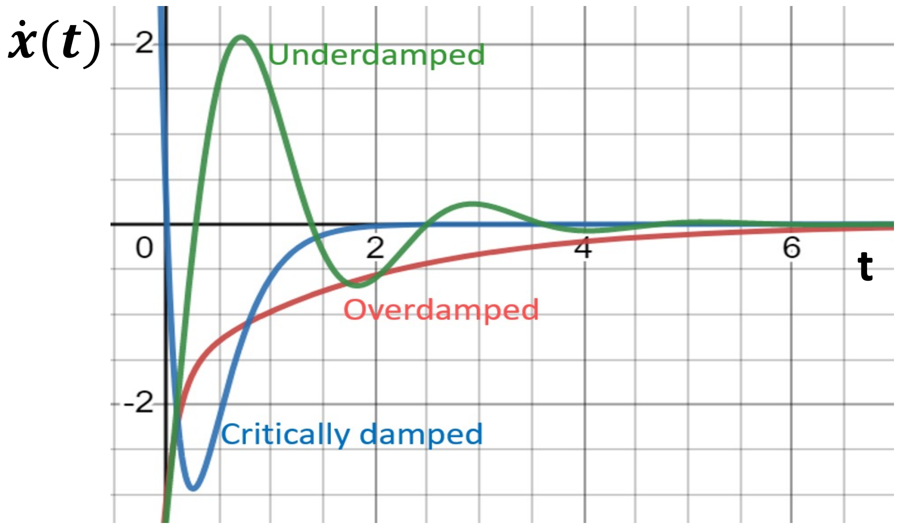
$$\Rightarrow \dot{x}(t) = -\beta^2 a t e^{-\beta t}$$

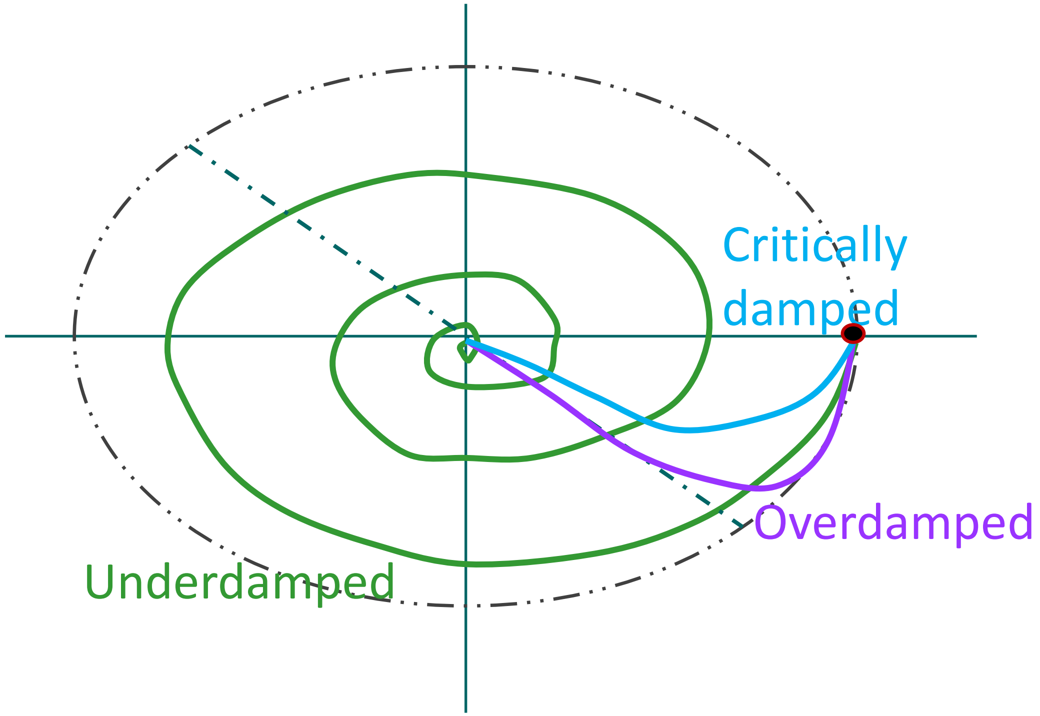


Damped Oscillations: Phase diagrams

MODIFIED SLIDE







Further reading (verification):
L06_321A.pdf (uvic.ca)

FORCED OSCILLATIONS

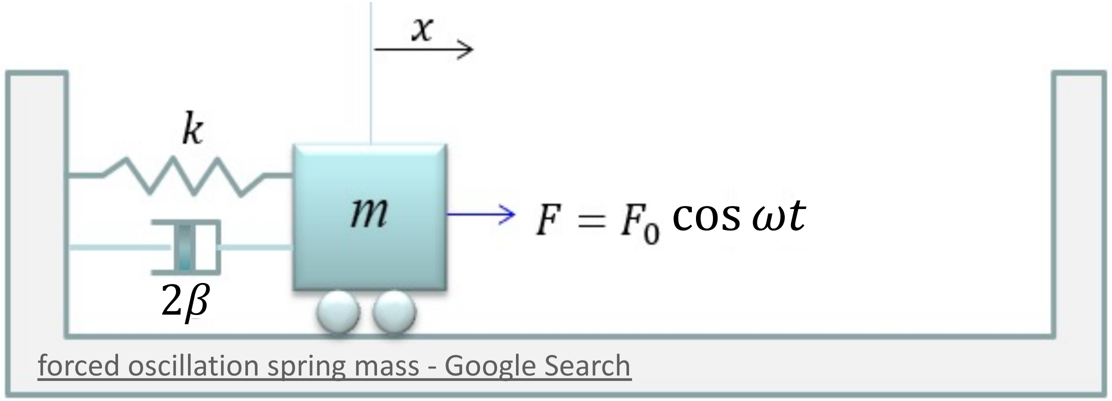
Occur when an oscillating system is driven by a periodic force that is external to the oscillating system.

In such a case, the oscillator is compelled to move at the frequency ω of the driving force.

The physically interesting aspect of a forced oscillator is its response—how much it moves due to the imposed driving force.

Forced Oscillation - an overview | ScienceDirect Topics

FORCED OSCILLATIONS



forced oscillation spring mass - Google Search

Two entities of importance:

1. The oscillator

   Frequency = ω_0 (without damping) $= \sqrt{\dfrac{k}{m}}$

2. The periodic force

   Frequency = ω (not the same as frequency with damping)

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t = f_0 \cos\omega t}$$

- An inhomogeneous differential equation.

Forced Oscillations: Solution

Microsoft Word - Notes-2nd order ODE pt2 (psu.edu)

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos\omega t}$$

Strategy

$$\boxed{x = x_C + x_P}$$

Complementary solution

- Solution of the corresponding homogeneous equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

- Never a solution of the full inhomogeneous equation

Dies out with time
(damped oscillations)
Insignificant at long times

Particular solution

- Necessarily a solution of the full inhomogeneous equation for some particular initial conditions

Survives at long times
Solution of interest

Forced Oscillations: The Complementary Solution

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

Recall:

Assume $x = e^{qt}$, that leads to $q^2 + 2\beta q + \omega_0^2 = 0 \Rightarrow q = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$

Say, $q_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}$, and $q_2 = -\beta + \sqrt{\beta^2 - \omega_0^2}$

CASE 1: Complex *q*'s

$q_1 = a + ib;$ $q_2 = a - ib = q_1^*$

The solution is:

$$x_C(t) = (C_1 \cos bt + C_2 \cos bt\,)e^{at}$$

$$\Rightarrow x_C(t) = C\cos(bt + \alpha)\,e^{at}$$

$\equiv$

UNDERDAMPING

$$q = -\beta \pm i\omega$$

$$x(t) = a_0 \cos(\omega t + \alpha)\,e^{-\beta t}$$

Forced Oscillations: The Complimentary Solution

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

Recall:

Assume $x = e^{qt}$, that leads to $q^2 + 2\beta q + \omega_0^2 = 0 \Rightarrow q = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$

Say, $q_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}$, and $q_2 = -\beta + \sqrt{\beta^2 - \omega_0^2}$

CASE 2: Real and distinct (unequal) q's

$q_1 = a + b;$ $q_2 = a - b$

The solution is:

$$x_C(t) = C_3 e^{q_1 t} + C_4 e^{q_2 t}$$

$$\equiv$$

OVERDAMPING

$$q = -\beta \pm \Omega$$

$$x(t) = \left(\left\{ \frac{x(0)}{2} - \frac{\dot{x}(0) + \beta\, x(0)}{2\Omega} \right\} e^{-\Omega t} + \left\{ \frac{x(0)}{2} + \frac{\dot{x}(0) + \beta\, x(0)}{2\Omega} \right\} e^{\Omega t} \right) e^{-\beta t}$$

Forced Oscillations: The Complimentary Solution

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

Recall:

Assume $x = e^{qt}$, that leads to $q^2 + 2\beta q + \omega_0^2 = 0 \Rightarrow q = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$

Say, $q_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}$, and $q_2 = -\beta + \sqrt{\beta^2 - \omega_0^2}$

CASE 3: Real and equal q's

$$q_1 = q_2 = a$$

The solution is:

$$x_C(t) = (C_5 + C_6 t)e^{qt}$$

$\equiv$

CRITICAL DAMPING

$$q \longrightarrow -\beta$$

$$x(t) = [x(0) + \{\dot{x}(0) + \beta\, x(0)\}t]\; e^{-\beta t}$$

Forced Oscillations: The Particular Solution

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos\omega t}$$

For the sake of mathematical simplicity, let us write the equation in its ‘complex’ form as:

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t} \qquad \text{..........(1)}$$

Observe the behavior of the function at RHS:

- It doesn’t change on taking first or second derivative, except for the constant coefficients.

Then, the solution can be taken as a general form of the function at RHS. Say,

$$z(t) = Ae^{i(\omega t - \phi)} \qquad \text{..........(2)}$$

The solution, then, will be:

$$x_P(t) = Re[z(t)] = A\cos(\omega t - \phi) \qquad \text{..........(3)}$$

Forced Oscillations: The Particular Solution

$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$ ....(1)

$z(t) = Ae^{i(\omega t - \phi)}$ ....(2)

Putting (2) in (1)

$$\Rightarrow \ (-\omega^2 + 2i\beta\omega + \omega_0^2)Ae^{i(\omega t - \phi)} = f_0 e^{i\omega t}$$

$$\Rightarrow \ (-\omega^2 + 2i\beta\omega + \omega_0^2)A = f_0 e^{i\phi} = f_0 \cos\phi + i f_0 \sin\phi$$

Equating real and imaginary parts:

$\Rightarrow (-\omega^2 + \omega_0^2)A = f_0 \cos\phi$ and $2\beta\omega A = f_0 \sin\phi$

$\Rightarrow A = \dfrac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}}$ and $\phi = \tan^{-1}\left(\dfrac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$

$x_P(t) = A\cos(\omega t - \phi)$ – an oscillations with the frequency of the periodic force

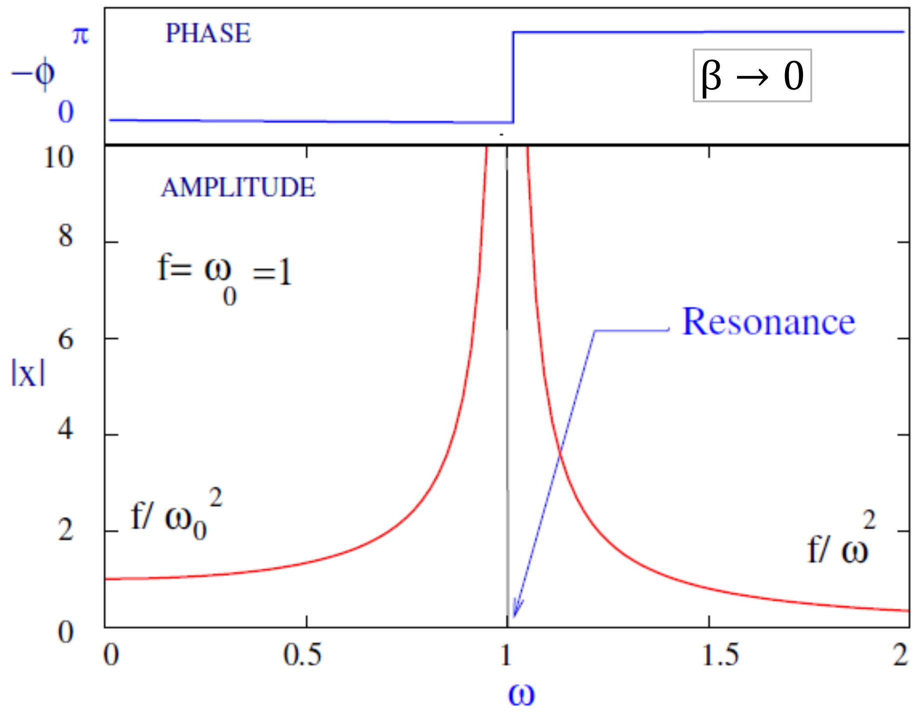
This is the solution which survives after some time.

Forced Oscillations: The Particular Solution

$$x_P(t) = A\cos(\omega t - \phi)$$

$$A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}}$$

$$\phi = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$



Specific points:

1. When $\omega \longrightarrow 0$, $A = \frac{F_0}{m\omega_0^2} = \frac{F_0}{k}$, $\phi = 0$

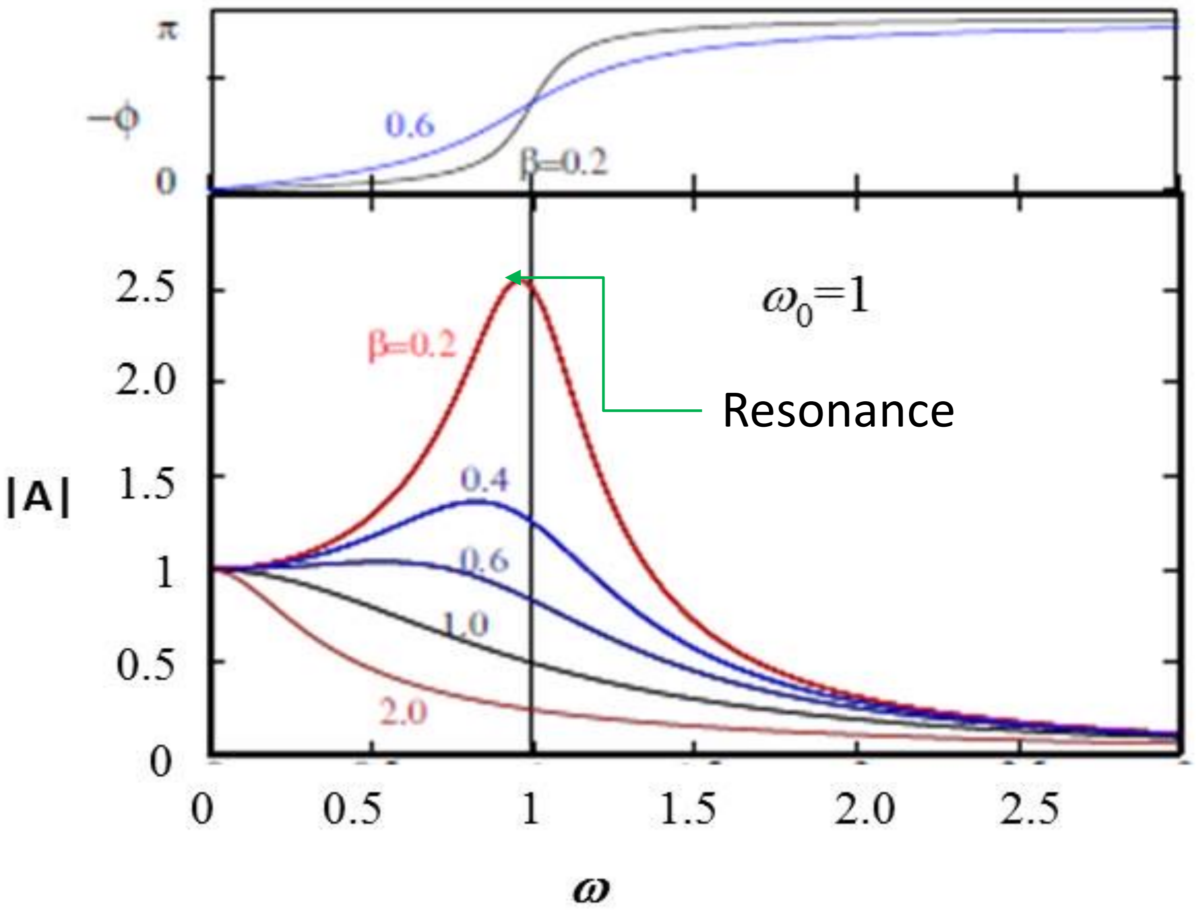
2. When $\omega \longrightarrow \omega_0$, $A = \frac{F_0}{m2\beta\omega_0} = \frac{F_0}{m\omega_0^2}\frac{\omega_0}{2\beta} = \frac{F_0}{k}Q$, $\phi = \frac{\pi}{2}$

- Remember $Q = \frac{\omega}{2\beta}$ for underdamped oscillations

3. When $\omega \longrightarrow \infty$, $A \longrightarrow 0$, $\phi = \pi$

Forced Oscillations: The Particular Solution

Finite β



$$A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}}$$

$$\phi = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

The damping ensures that the amplitude does not blow up at $\omega = \omega_0$ and it is finite for all values of ω.

The amplitude is maximum at

$$\omega = \sqrt{\omega_0^2 - 2\beta^2}$$

Verify yourself.

Forced Oscillations: The Full Solution

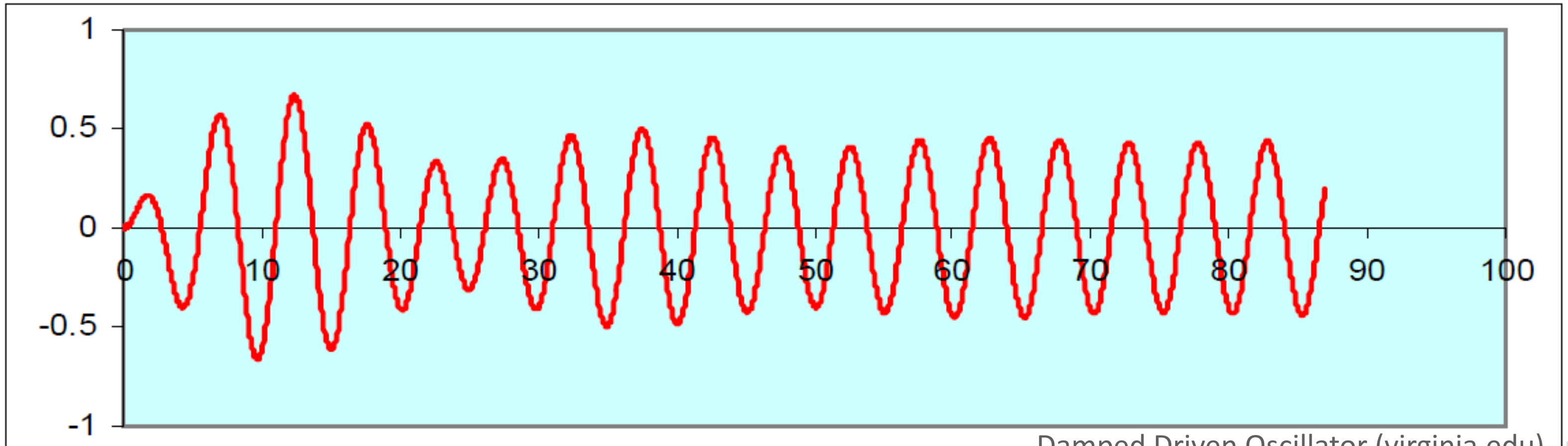
$$x = x_C + x_P$$

$$\Rightarrow x(t) = a_0\cos(\omega t + \alpha)\, e^{-\beta t} + A\cos(\omega t - \phi)$$

The frequency of the forced oscillator $\equiv \omega_F$

Corresponds to (under)damped oscillation $\equiv \omega_D$

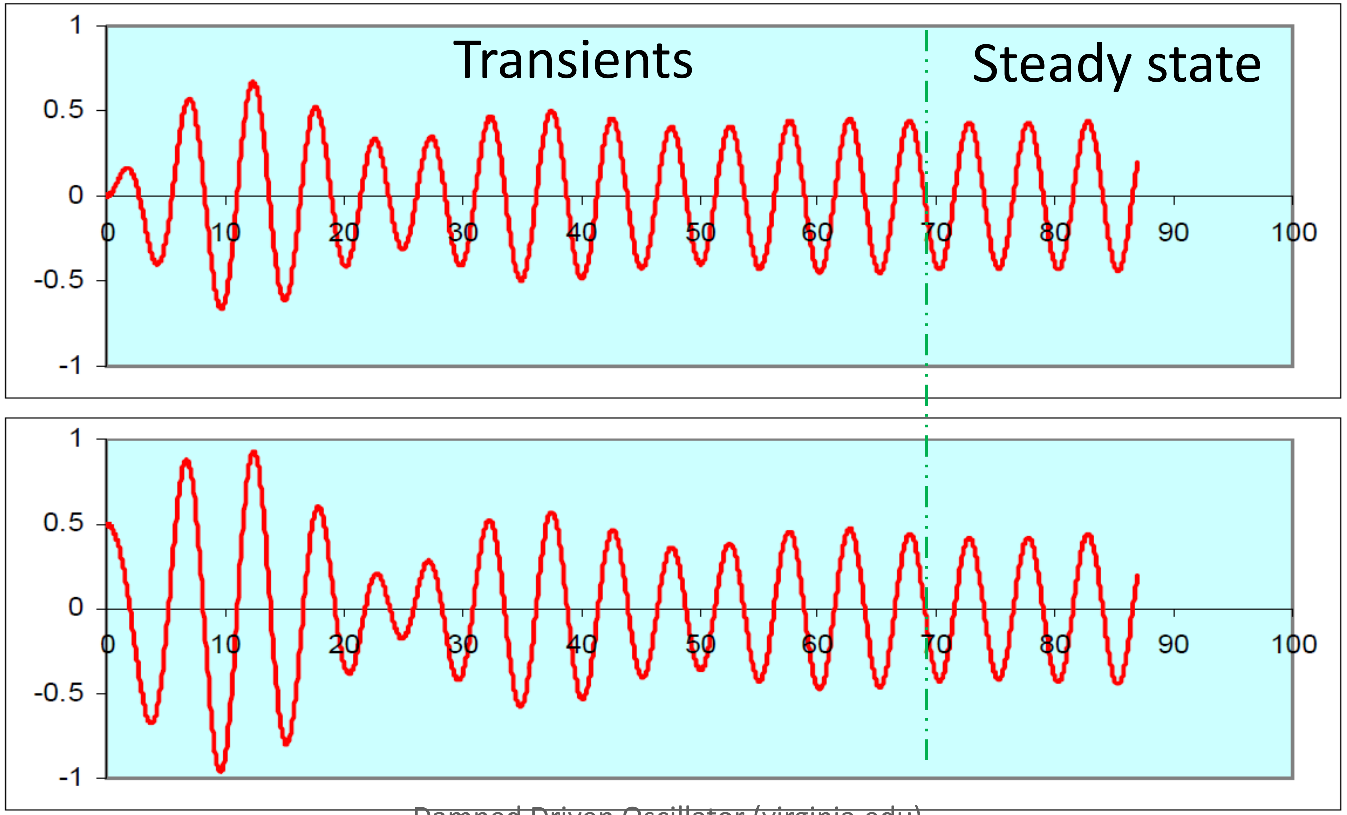
$$\Rightarrow x(t) \equiv \underbrace{a_0\cos(\omega_D t + \alpha_D)\, e^{-\beta t} + \underbrace{A\cos(\omega_F t - \phi_F)}_{\text{Oscillations later (steady state)}}}_{\text{Beats in the beginning (transients)}}$$



Damped Driven Oscillator (virginia.edu)

Forced Oscillations: The Full Solution

$$x(t) \equiv \underbrace{a_0\cos(\omega_D t + \alpha_D)\, e^{-\beta t} + \underbrace{A\cos(\omega_F t - \phi_F)}_{\text{Steady state}}}_{\text{Transients}}$$



Damped Driven Oscillator (virginia.edu)

Here's a pair of examples: the same driven damped oscillator, started with zero velocity, once from the origin and once from 0.5.

Notice that after about 70 seconds, the two curves are the same, both in amplitude and phase.

Forced Oscillations: The Quality Factor (Q-factor)

$$Q = \frac{\omega_0}{2\beta}$$

Remember

$Q = \frac{\omega}{2\beta}$ for underdamped oscillations

and

$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos \omega t}$

The Q factor is a measure of the 'quality' of an oscillator.

It is a measure of how many oscillations take place during the time the energy decays by the factor of 1/e.

Forced Oscillations: The Amplitude at Resonance

$$x_P(t) = A\cos(\omega t - \phi)$$

$$A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}}$$

The amplitude is maximum (A_{res}) at

$$\omega = \sqrt{\omega_0^2 - 2\beta^2}$$

$$\Rightarrow A_{res} = \frac{f_0}{\sqrt{(\omega_0^2 - \omega_0^2 + 2\beta^2)^2 + (2\beta)^2(\omega_0^2 - 2\beta^2)}}$$

$$\boxed{\Rightarrow A_{res} = \frac{f_0}{2\beta\sqrt{1 + \omega_0^2 - 2\beta^2}} \approx \frac{f_0}{2\beta\omega_0}}$$

Forced Oscillations: The Average Energy

$$x_P(t) = A\cos(\omega t - \phi)$$

$$U(\omega, t) = \frac{1}{2}kx^2(t)$$

$$= \frac{1}{2}m\omega_0^2 A^2 \cos^2(\omega t - \phi)$$

$$\Rightarrow \langle U(\omega)\rangle = \frac{1}{2}m\omega_0^2 A^2 \langle \cos^2(\omega t - \phi)\rangle$$

$$\Rightarrow \langle U(\omega)\rangle = \frac{1}{4}m\omega_0^2 A^2$$

$$\dot{x}(t) = -A\omega\sin(\omega t - \phi)$$

$$KE(\omega, t) = \frac{1}{2}m\dot{x}^2\,(t)$$

$$= \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t - \phi)$$

$$\Rightarrow \langle KE(\omega)\rangle = \frac{1}{2}m\omega^2 A^2 \langle \sin^2(\omega t - \phi)\rangle$$

$$\Rightarrow \langle KE(\omega)\rangle = \frac{1}{4}m\omega^2 A^2$$

Forced Oscillations: The Average Energy....

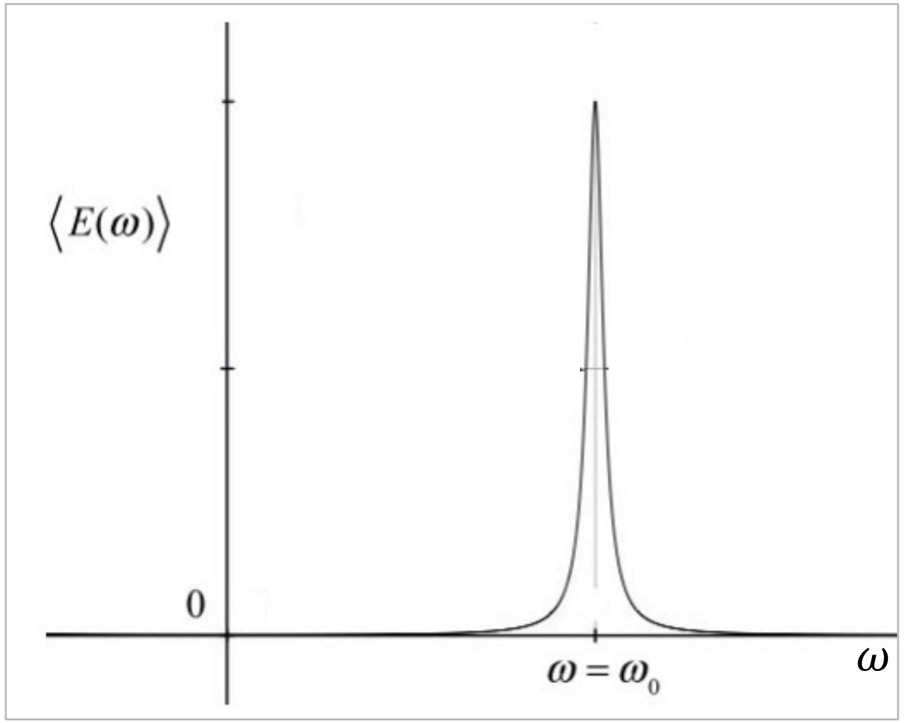
$$\langle U(\omega)\rangle = \frac{1}{4} m\omega_0^2 A^2$$

$$\langle KE(\omega)\rangle = \frac{1}{4} m\omega^2 A^2$$

$$\Rightarrow \langle E(\omega)\rangle = \langle U(\omega)\rangle + \langle KE(\omega)\rangle$$

$$= \frac{1}{4} m(\omega^2 + \omega_0^2) A^2$$

$$= \frac{m f_0^2}{4} \frac{(\omega^2 + \omega_0^2)}{[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]}$$



Forced Oscillations: Average Power put in the system by Driving Force

$$\dot{x}(t) = v = -A\omega \sin(\omega t - \phi)$$

Work done: $dW = \vec{F}.d\vec{x}$

$\Rightarrow$ Power: $$P(\omega, t) = \frac{dW}{dt} = \vec{F}.\vec{v} = Fv = -F_0\omega A \sin(\omega t - \phi) \cos \omega t$$

$$\Rightarrow \langle P(\omega) \rangle = -F_0\omega A \frac{1}{T} \int_0^T dt \sin(\omega t - \phi) \cos \omega t$$

$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

$$= F_0\omega A \frac{\sin \phi}{2}$$

$$= \frac{F_0^2 \omega \sin \phi}{2m\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + (2\beta\omega)^2}}$$

Forced Oscillations: **Average Power put in the system by Driving Force…**

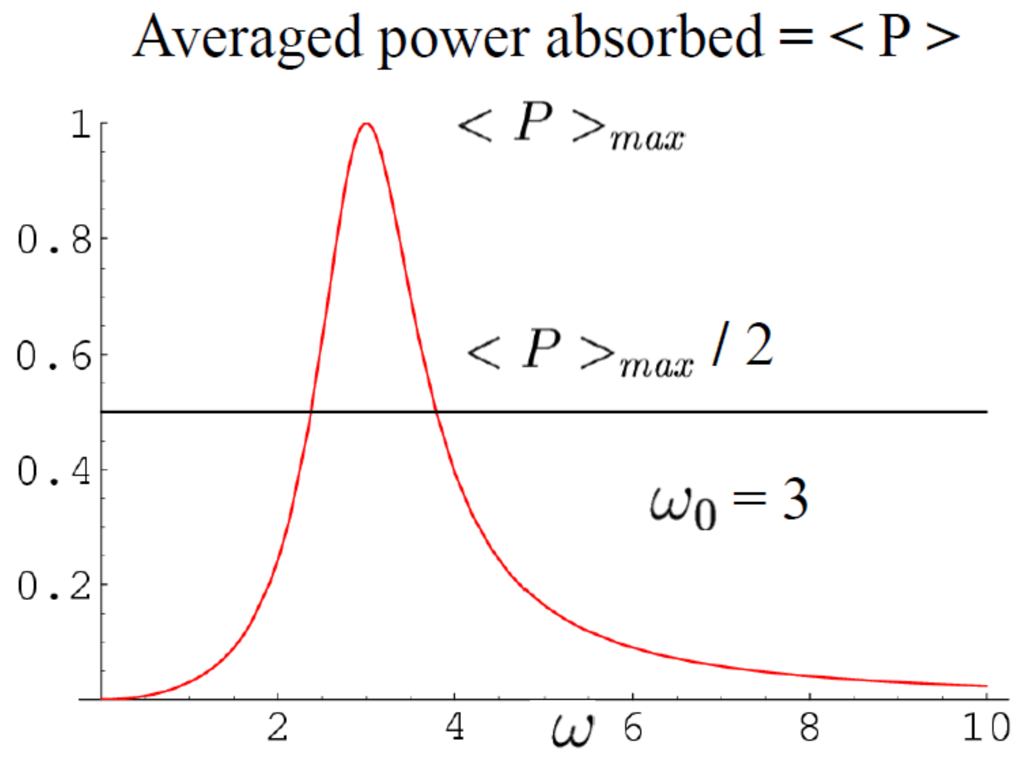
$$\langle P(\omega)\rangle = -\frac{F_0^2\omega\sin\phi}{2m\sqrt{(\omega_0^2-\omega^2)^2+(2\beta\omega)^2}}$$

Width at half-peak power $= 2\beta$

Verify yourself

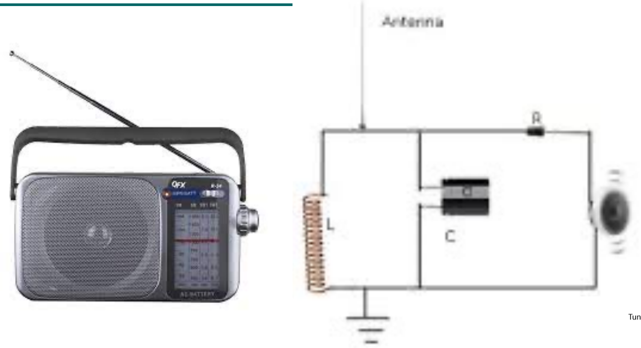
Remember

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0\cos\omega t}$$

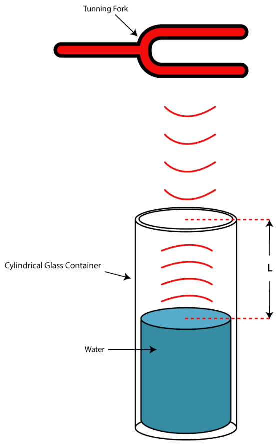


Forced Oscillations: **Examples**

1. Electrical resonance



When we turn the knob, the capacitance keeps on changing till the resonant frequency becomes equal to the frequency of the channel which we want to hear.



2. Acoustic resonance

3. Pohl's pendulum

4. Optical resonance

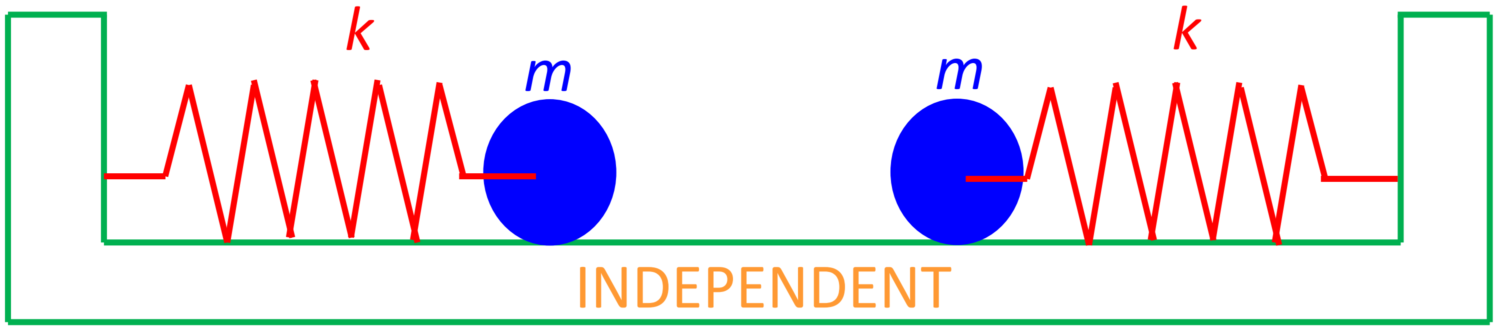
5. Nuclear magnetic resonance

....

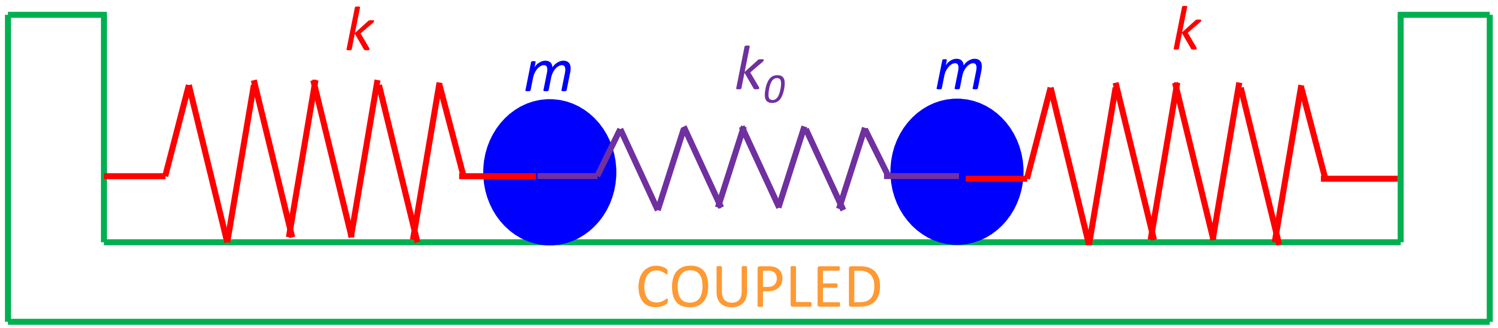
COUPLED OSCILLATIONS

Pushing Back and Forth: Coupled Oscillators | FOS Media Students' Blog (cmb.ac.lk)

COUPLED OSCILLATIONS



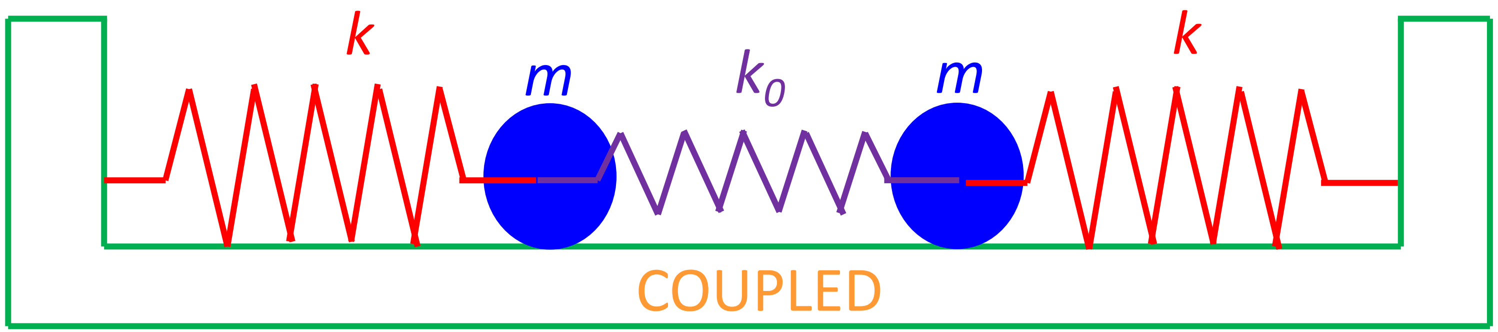
Each individual system (oscillator) executes SHM.



Oscillators get coupled. Motion of one mass affects the other.

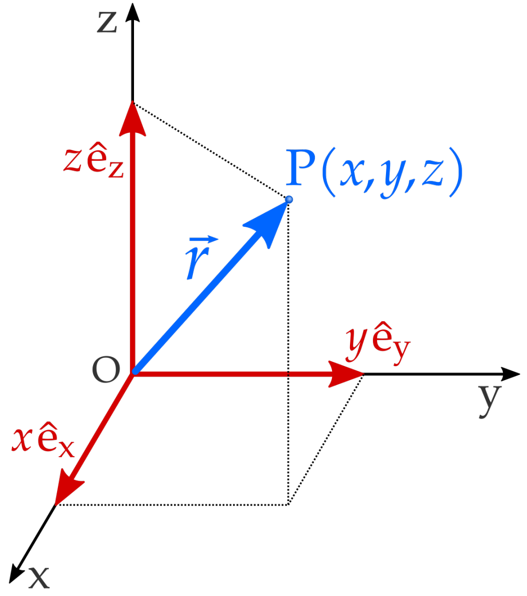
Chaotic motion of individual oscillators as well as of the combined system.

COUPLED OSCILLATIONS



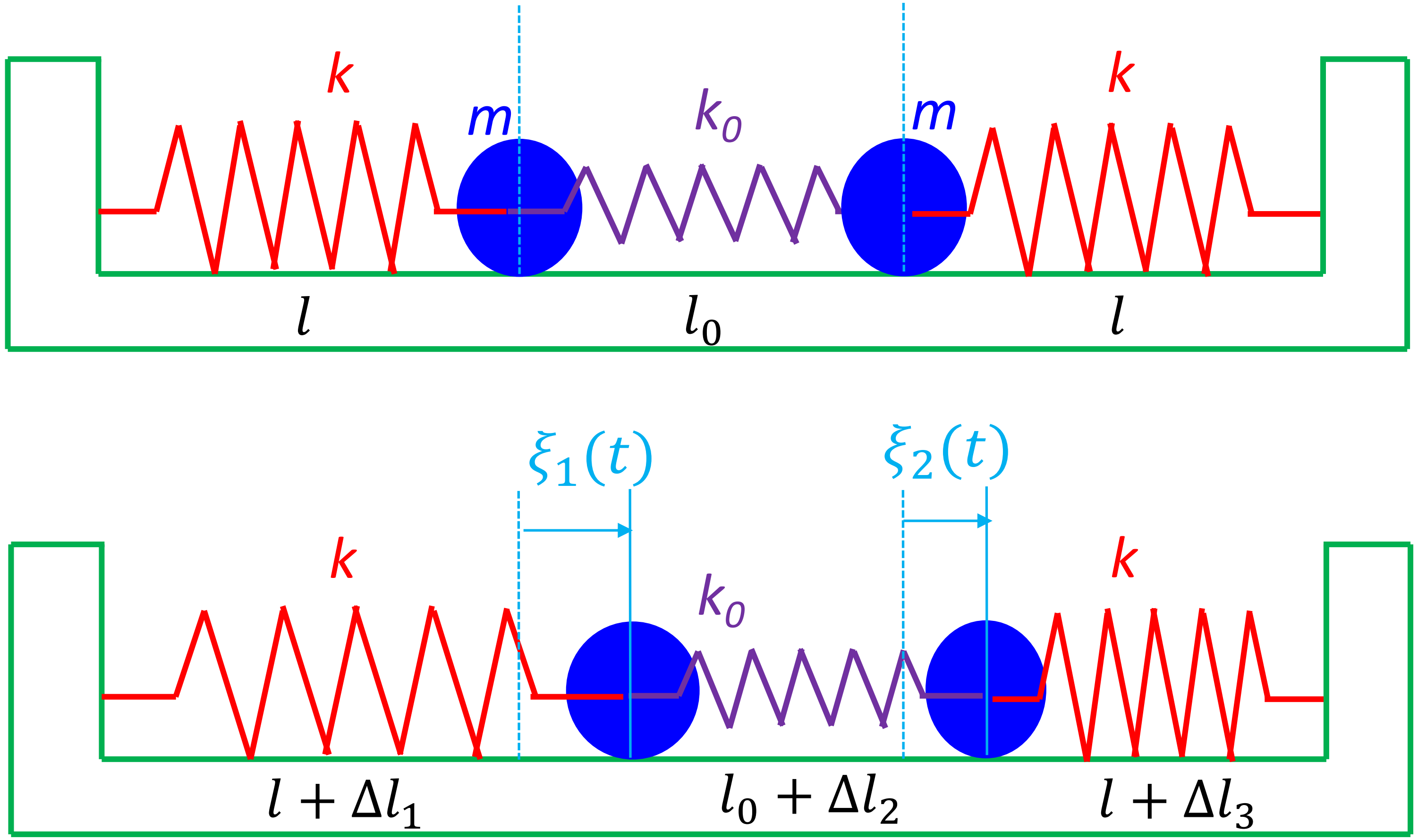
CHAOTIC MOTION: Can it be resolved into two (or more) simple, but mutually independent components (oscillations, modes)? Like vectors?

Need to analyse it mathematically.



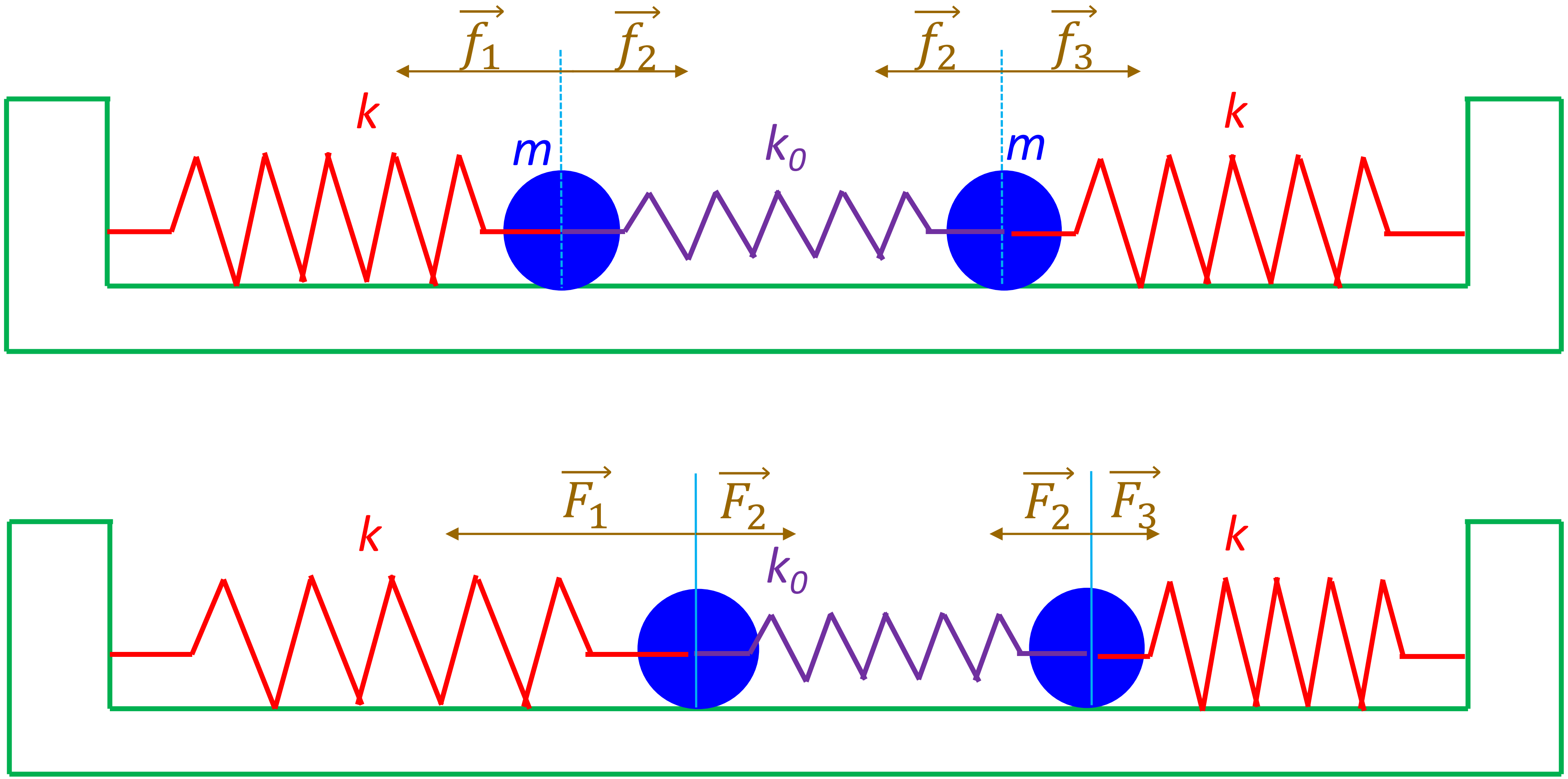
vectors - Google Search

The displacements



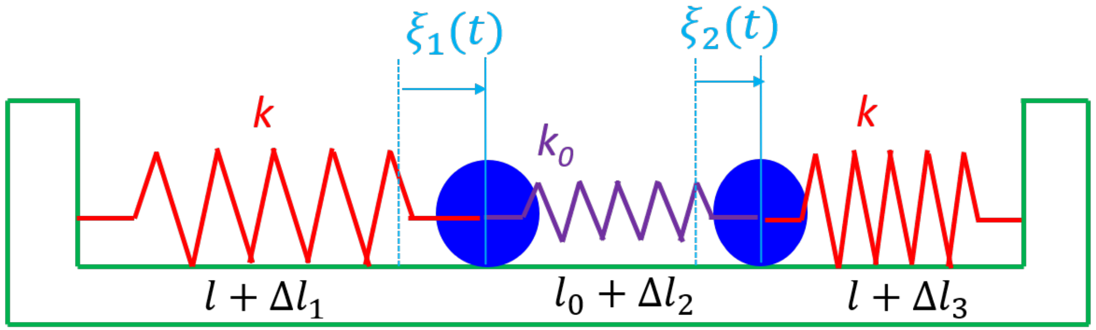
$$\Delta l_1 = \xi_1 \qquad \Delta l_2 = \xi_2 - \xi_1 \qquad \Delta l_3 = -\xi_2$$

The forces (consider stretched condition)

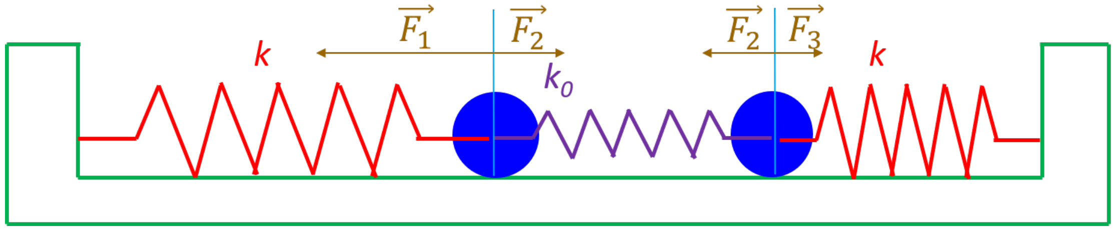


In non-equilibrium, the amounts of forces will change, but the directions of the forces will not change.

The force equations



$\Delta l_1 = \xi_1$ $\quad$ $\Delta l_2 = \xi_2 - \xi_1$ $\quad$ $\Delta l_3 = -\xi_2$

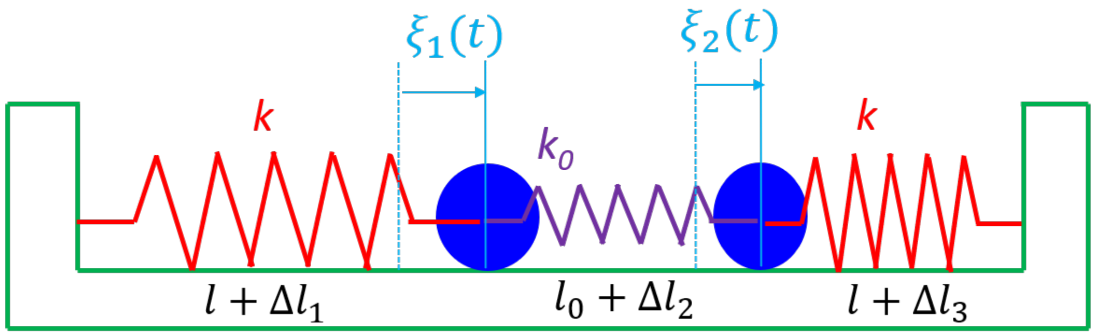


$$m\ddot{\xi}_1 = -F_1 + F_2$$
$$= -k\xi_1 + k_0(\xi_2 - \xi_1)$$
$$\boxed{\Rightarrow m\ddot{\xi}_1 + (k + k_0)\xi_1 - k_0\xi_2 = 0}$$
......(1)

$$m\ddot{\xi}_2 = -F_2 + F_3$$
$$= -k_0(\xi_2 - \xi_1) - k\xi_2$$
$$\boxed{\Rightarrow m\ddot{\xi}_2 + (k + k_0)\xi_2 - k_0\xi_1 = 0}$$
......(2)

(1) and (2): a coupled system of linear second order differential equations

The force equations: an alternative way



$\Delta l_1 = \xi_1$ $\quad$ $\Delta l_2 = \xi_2 - \xi_1$ $\quad$ $\Delta l_3 = -\xi_2$

$$U = \frac{1}{2}k(\Delta l_1)^2 + \frac{1}{2}k_0(\Delta l_2)^2 + \frac{1}{2}k(\Delta l_3)^2$$

$$= \frac{1}{2}k\xi_1^2 + \frac{1}{2}k_0(\xi_2 - \xi_1)^2 + \frac{1}{2}k(-\xi_2)^2$$

Then,

$$\text{Force on the 1}^{\text{st}}\text{ mass} = m\ddot{\xi}_1 = -\frac{\partial U}{\partial \xi_1}$$

$$= -k\xi_1 + k_0(\xi_2 - \xi_1)$$

$$\boxed{\Rightarrow m\ddot{\xi}_1 + (k + k_0)\xi_1 - k_0\xi_2 = 0} \quad \text{......(1)}$$

$$\text{Force on the 2}^{\text{nd}}\text{ mass} = m\ddot{\xi}_2 = -\frac{\partial U}{\partial \xi_2}$$

$$= -k_0(\xi_2 - \xi_1) - k\xi_2$$

$$\boxed{\Rightarrow m\ddot{\xi}_2 + (k + k_0)\xi_2 - k_0\xi_1 = 0} \quad \text{......(2)}$$

(1) and (2): a coupled system of linear second order differential equations

Solving the force equations

$$\boxed{m\ddot{\xi}_1 + (k+k_0)\xi_1 - k_0\xi_2 = 0} \quad ......(1)$$

$$\boxed{m\ddot{\xi}_2 + (k+k_0)\xi_2 - k_0\xi_1 = 0} \quad ......(2)$$

Method 1: ADD and SUBTRACT

(1) + (2) $\Rightarrow m\left(\ddot{\xi}_1 + \ddot{\xi}_2\right) + k(\xi_1 + \xi_2) = 0$ ......(3)

- a differential equation exclusively in a single variable $(\xi_1 + \xi_2) \equiv Y_1$

(1) - (2) $\Rightarrow m\left(\ddot{\xi}_1 - \ddot{\xi}_2\right) + (k + 2k_0)(\xi_1 - \xi_2) = 0$ ......(4)

- a differential equation exclusively in another single variable $(\xi_1 - \xi_2) \equiv Y_2$

Then, (3) $\Rightarrow m\dot{Y}_1 + kY_1 = 0 \Rightarrow Y_1 = Y_{10}\cos(\omega_1 t + \phi_1)$ - Mode 1

and (4) $\Rightarrow m\dot{Y}_2 + (k+2k_0)Y_2 = 0 \Rightarrow Y_2 = Y_{20}\cos(\omega_2 t + \phi_2)$ - Mode 2

SLOW

$$\omega_1 = \sqrt{\frac{k}{m}}$$

FAST

$$\omega_2 = \sqrt{\frac{(k+2k_0)}{m}}$$

$$\xi_1 = \frac{Y_1 + Y_2}{2}$$

- a linear combination

$$\xi_2 = \frac{Y_1 - Y_2}{2}$$

- a linear combination

Solving the force equations: More Appropriately

(1) + (2) $\Rightarrow m(\ddot{\xi}_1 + \ddot{\xi}_2) + k(\xi_1 + \xi_2) = 0$ ......(3)

- a differential equation exclusively in a single variable $(\xi_1 + \xi_2) \equiv \sqrt{2}X_1$

(1) - (2) $\Rightarrow m(\ddot{\xi}_1 - \ddot{\xi}_2) + (k + 2k_0)(\xi_1 - \xi_2) = 0$ ......(4)

- a differential equation exclusively in another single variable $(\xi_1 - \xi_2) \equiv \sqrt{2}X_2$

Then, (3) $\Rightarrow m\dot{X}_1 + kX_1 = 0 \Rightarrow X_1 = X_{10}\cos(\omega_1 t + \phi_1)$ - Mode 1

and (4) $\Rightarrow m\dot{X}_2 + (k + 2k_0)X_2 = 0 \Rightarrow X_2 = X_{20}\cos(\omega_2 t + \phi_2)$ - Mode 2

SLOW

$$\omega_1 = \sqrt{\frac{k}{m}}$$

FAST

$$\omega_2 = \sqrt{\frac{(k + 2k_0)}{m}}$$

$$\xi_1 = \frac{X_1 + X_2}{\sqrt{2}}$$

- a linear combination

$$\xi_2 = \frac{X_1 - X_2}{\sqrt{2}}$$

- a linear combination

Solving the force equations

$$m\ddot{\xi}_1 + (k + k_0)\xi_1 - k_0\xi_2 = 0 \quad \text{......(1)}$$

$$m\ddot{\xi}_2 + (k + k_0)\xi_2 - k_0\xi_1 = 0 \quad \text{......(2)}$$

Method 2: The MATRIX method

Assume $\xi_1(t) = A_1 e^{i\omega t}$ and $\xi_2(t) = A_2 e^{i\omega t}$

- same frequency to start with.
- Find A_1, A_2 and roots of ω.

Putting in (1) and (2):

$$-\omega^2 m A_1 + (k + k_0)A_1 - k_0 A_2 = 0$$

$$\Rightarrow \{-\omega^2 m + (k + k_0)\}A_1 + (-k_0)A_2 = 0 \quad \text{.......(5)}$$

and

$$-\omega^2 m A_2 + (k + k_0)A_2 - k_0 A_1 = 0$$

$$\Rightarrow (-k_0)A_1 + \{-\omega^2 m + (k + k_0)\}A_2 = 0 \quad \text{.......(6)}$$

Solving the force equations

$\{-\omega^2 m + (k + k_0)\}A_1 + (-k_0)A_2 = 0$ ....(5)

$(-k_0)A_1 + \{-\omega^2 m + (k + k_0)\}A_2 = 0$ ....(6)

Write in the matrix form

$$\begin{bmatrix} -\omega^2 m + (k + k_0) & -k_0 \\ -k_0 & -\omega^2 m + (k + k_0) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution exist only when the determinant of the coefficient matrix vanishes.

$$\Rightarrow \{-\omega^2 m + (k + k_0)\}^2 = k_0^2 \qquad \Rightarrow -\omega^2 m + (k + k_0) = \pm k_0$$

$$\boxed{\Rightarrow \omega_1 = \sqrt{\frac{k}{m}}, \qquad \text{and} \quad \omega_2 = \sqrt{\frac{(k + 2k_0)}{m}}}$$

Putting ω_1 in the matrix, we get $A_2 = A_1$, i. e., in-phase oscillations (mode 1).

Putting ω_2 in the matrix, we get $A_2 = -A_1$, i. e., out-of-phase oscillations (mode 2).

Solving the force equations

$$\boxed{\omega_1 = \sqrt{\frac{k}{m}}, \qquad \text{and} \quad \omega_2 = \sqrt{\frac{(k+2k_0)}{m}}}$$

Normal mode frequencies/
Characteristic frequencies/
Eigenfrequencies/
Frequency eigenvalues

In-phase mode

$$A_2 = A_1$$

Out-of-phase mode

$$A_2 = -A_1$$

Eigenvectors $\left(\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\right)$

Eigenvector of in-phase mode $\quad E_1 \equiv \begin{bmatrix} A_1 \\ A_1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (normalized)

Eigenvector of out-of-phase mode $\quad E_2 \equiv \begin{bmatrix} A_1 \\ -A_1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (normalized)

Solving the force equations

INNER PRODUCTS

$$\langle E_1, E_1\rangle = \frac{1}{\sqrt{2}}[1 \quad 1].\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = 1$$

$\Rightarrow E_1$ is normalized.

$$\langle E_2, E_2\rangle = \frac{1}{\sqrt{2}}[1 \quad -1].\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = 1$$

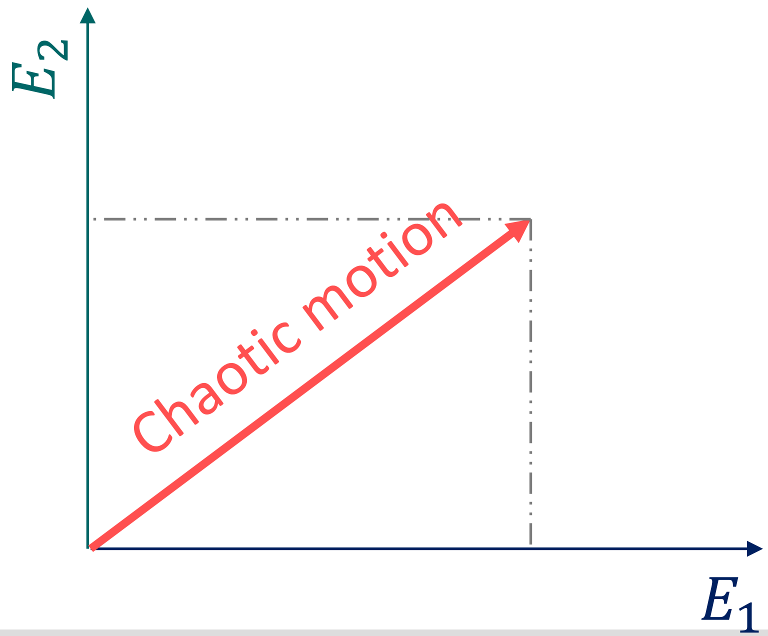
$\Rightarrow E_2$ is normalized.

$$\langle E_1, E_2\rangle = \frac{1}{\sqrt{2}}[1 \quad 1].\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = 0$$

$\Rightarrow E_1$ and E_2 are orthogonal.

This is the reason the two modes E_1 and E_2 are called 'NORMAL' modes.

- The differential equations are linear, so any linear combination of the two normal mode solutions is also a solution.
- In fact, the most general solution of the equations is an arbitrary linear combination of the two normal modes.



Detailed solutions for different cases

Remember

$$X_1 = X_{10}\cos(\omega_1 t + \phi_1)$$

$$X_2 = X_{20}\cos(\omega_2 t + \phi_2)$$

$$\xi_1 = \frac{X_1 + X_2}{\sqrt{2}} = \frac{1}{\sqrt{2}}[X_{10}\cos(\omega_1 t + \phi_1) + X_{20}\cos(\omega_2 t + \phi_2)]$$

$$\xi_2 = \frac{X_1 - X_2}{\sqrt{2}} = \frac{1}{\sqrt{2}}[X_{10}\cos(\omega_1 t + \phi_1) - X_{20}\cos(\omega_2 t + \phi_2)]$$

Detailed solutions for different cases

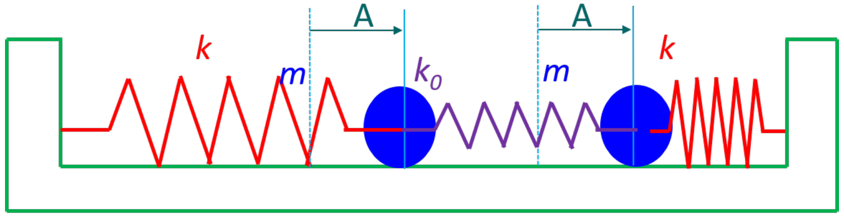
$$\xi_1 = \frac{1}{\sqrt{2}}[X_{10}\cos(\omega_1 t + \phi_1) + X_{20}\cos(\omega_2 t + \phi_2)]$$

$$\xi_2 = \frac{1}{\sqrt{2}}[X_{10}\cos(\omega_1 t + \phi_1) - X_{20}\cos(\omega_2 t + \phi_2)]$$

Case 1: Both the masses are displaced to one side by the same amount and released

$\Rightarrow \xi_1(0) = \xi_2(0) = A$ (say).....(7)

and $\dot{\xi}_1(0) = \dot{\xi}_2(0) = 0$ .....(8)



(7) $\Rightarrow A = \frac{1}{\sqrt{2}}[X_{10}\cos\phi_1 + X_{20}\cos\phi_2]$ ......(9) and

$A = \frac{1}{\sqrt{2}}[X_{10}\cos\phi_1 - X_{20}\cos\phi_2]$ ....(10)

(9)+(10) $\Rightarrow 2A = \frac{1}{\sqrt{2}}[2X_{10}\cos\phi_1] \Rightarrow X_{10}\cos\phi_1 = \sqrt{2}A$.........(11)

(9)-(10) $\Rightarrow X_{20}\cos\phi_2 = 0$.........(12)

Then,

$$\boxed{\xi_1 = \frac{X_{10}}{\sqrt{2}}\cos(\omega_1 t + \phi_1) = \xi_2}$$

Detailed solution: **Case 1**

$$\xi_1 = \frac{X_{10}}{\sqrt{2}} \cos(\omega_1 t + \phi_1) = \xi_2$$

$$\omega_1 = \sqrt{\frac{k}{m}}$$

$\dot{\xi}_1(0) = \dot{\xi}_2(0) = 0$ .....(8)

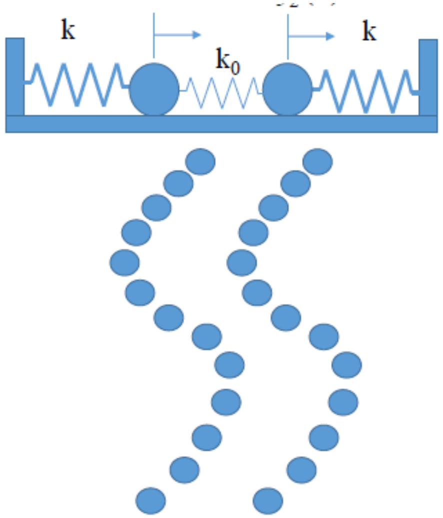
$X_{10} \cos \phi_1 = \sqrt{2} A$.........(11)

---

(8) $\Rightarrow -\frac{\omega_1 X_{10}}{\sqrt{2}} \sin \phi_1 = 0$ $\quad\quad \Rightarrow \phi_1 = 0$

$\Rightarrow X_{10} = \sqrt{2} A$ $\quad\quad$ using (11)

$\Rightarrow \xi_1 = A \cos \omega_1 t = \xi_2$ $\quad \equiv$ NORMAL MODE 1



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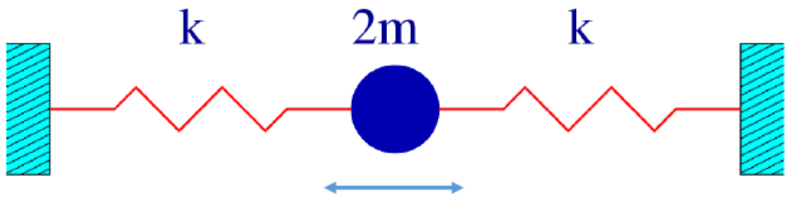


Oscillations of center of mass at frequency $\sqrt{\frac{2k}{2m}}$.

CENTRE OF MASS MODE

Detailed solutions for different cases

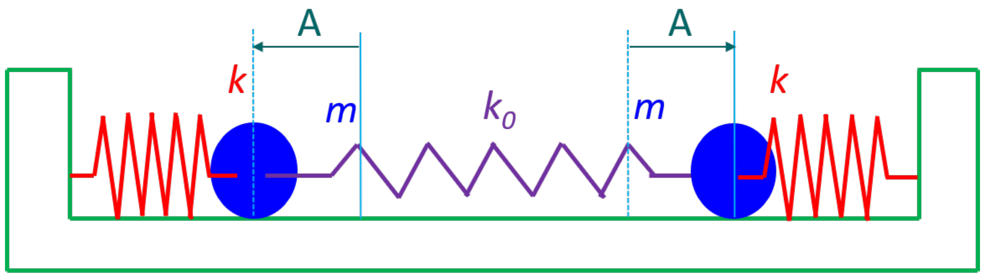
$$\xi_1 = \frac{1}{\sqrt{2}}[X_{10}\cos(\omega_1 t + \phi_1) + X_{20}\cos(\omega_2 t + \phi_2)]$$

$$\xi_2 = \frac{1}{\sqrt{2}}[X_{10}\cos(\omega_1 t + \phi_1) - X_{20}\cos(\omega_2 t + \phi_2)]$$

Case 2: The masses are displaced to either side by the same amount and released

$\Rightarrow \xi_1(0) = -\,\xi_2(0) = A$ .....(7')

and $\dot{\xi}_1(0) = \dot{\xi}_2(0) = 0$ ......(8')



(7') $\Rightarrow A = \frac{1}{\sqrt{2}}[X_{10}\cos\phi_1 + X_{20}\cos\ \phi_2]$ and

$$-A = \frac{1}{\sqrt{2}}[X_{10}\cos\phi_1 - X_{20}\cos\ \phi_2]$$

$$\Rightarrow 0 = \frac{1}{\sqrt{2}}[2X_{10}\cos\phi_1] \Rightarrow \cos\phi_1 = 0$$

$$\boxed{\Rightarrow \xi_1 = -\xi_2 = \frac{X_{20}}{\sqrt{2}}\cos(\omega_2 t + \phi_2)}$$

Next, $X_{20}\cos\phi_2 = \sqrt{2}A$

Detailed solution: **Case 2**

$$\xi_1 = -\xi_2 = \frac{X_{20}}{\sqrt{2}}\cos(\omega_2 t + \phi_2)$$

$$\omega_2 = \sqrt{\frac{(k + 2k_0)}{m}}$$

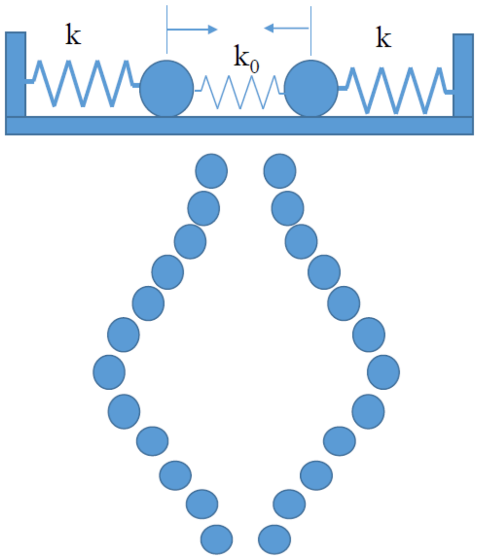
$\dot{\xi}_1(0) = \dot{\xi}_2(0) = 0$ .....(8')

$X_{20}\cos\phi_2 = \sqrt{2}A$

(8') $\Rightarrow -\frac{\omega_2 X_{20}}{\sqrt{2}}\sin\phi_2 = 0 \qquad \Rightarrow \phi_2 = 0$

$\Rightarrow X_{20} = \sqrt{2}A$ using (11)

$\Rightarrow \xi_1 = A\cos\omega_2 t = -\xi_2 \qquad \equiv$ NORMAL MODE 2



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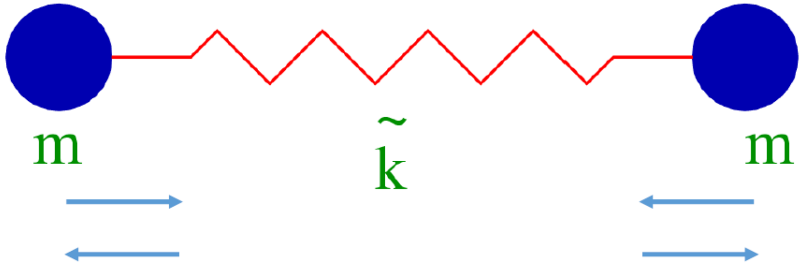


$$\tilde{k} = \frac{1}{2}(k + 2k_0)$$

Center of mass remains fixed.

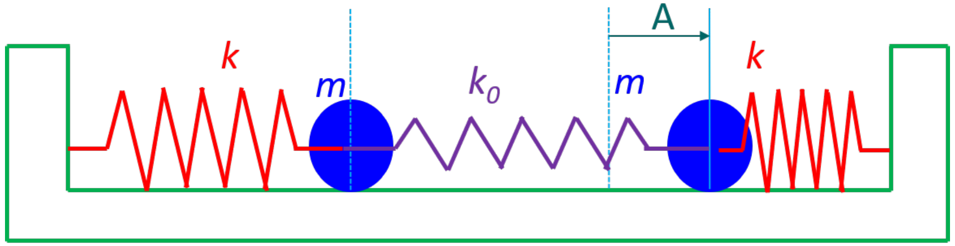
BREATHING MODE

Detailed solutions for different cases

$$\xi_1 = \frac{1}{\sqrt{2}}[X_{10}\cos(\omega_1 t + \phi_1) + X_{20}\cos(\omega_2 t + \phi_2)]$$

$$\xi_2 = \frac{1}{\sqrt{2}}[X_{10}\cos(\omega_1 t + \phi_1) - X_{20}\cos(\omega_2 t + \phi_2)]$$

Case 3: Pull one mass either left or right and other at zero displacement and release



$\Rightarrow \xi_1(0) = 0;\ \xi_2(0) = A$ …….(7”)

and $\dot{\xi}_1(0) = \dot{\xi}_2(0) = 0$ …..(8”)

(7”) $\Rightarrow X_{10}\cos\phi_1 = -X_{20}\cos\phi_2 = \frac{A}{\sqrt{2}}$ and

(8”)$\Rightarrow \phi_1 = \phi_2 = 0$

$$\Rightarrow X_{10} = -X_{20} = \frac{A}{\sqrt{2}}$$

$$\Rightarrow \xi_1 = \frac{A}{2}[\cos\omega_1 t + \cos\omega_2 t]; \qquad \xi_2 = \frac{A}{2}[\cos\omega_1 t - \cos\omega_2 t]$$

Verify yourself.

Detailed solution: **Case 3**

$$\xi_1 = \frac{A}{2}[\cos\omega_1 t + \cos\omega_2 t]; \qquad \xi_2 = \frac{A}{2}[\cos\omega_1 t - \cos\omega_2 t]$$

$$\Rightarrow \xi_1 = A\left[\cos\left(\frac{\omega_1+\omega_2}{2}\right)t\,.\cos\left(\frac{\omega_1-\omega_2}{2}\right)t\right]$$ - Beats

$$\xi_2 = A\left[\sin\left(\frac{\omega_1+\omega_2}{2}\right)t\,.\sin\left(\frac{\omega_2-\omega_1}{2}\right)t\right]$$ - Beats

Consider weak coupling: $k_0 \ll k$

$$\Rightarrow \omega_2 = \sqrt{\frac{k+2k_0}{m}} = \sqrt{\frac{k}{m}\left(1+2\frac{k_0}{k}\right)} \Rightarrow \omega_2 - \omega_1 \approx \frac{k_0}{\sqrt{km}};\, \omega_1+\omega_2 \approx 2\omega_1$$

(Taylor's expansion)

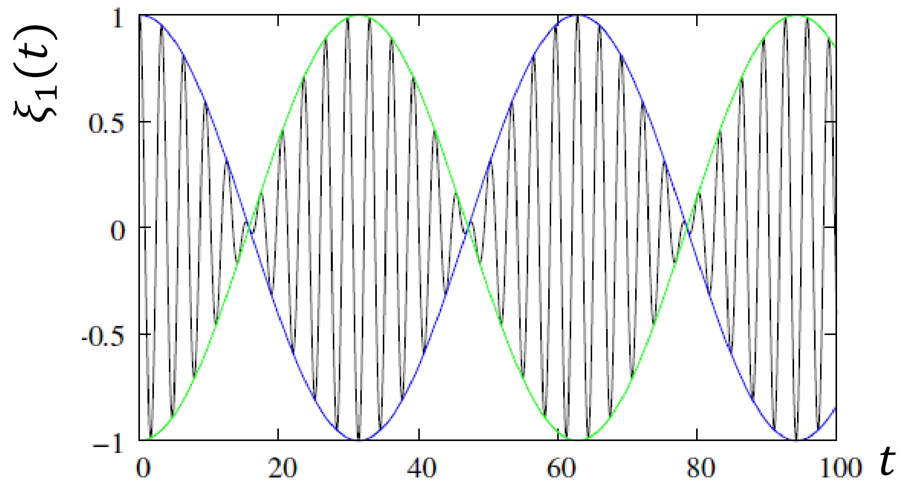
$$\Rightarrow \xi_1 = A\left[\cos\left(\frac{k_0}{2\sqrt{km}}\right)t\,.\cos\omega_1 t\right]$$

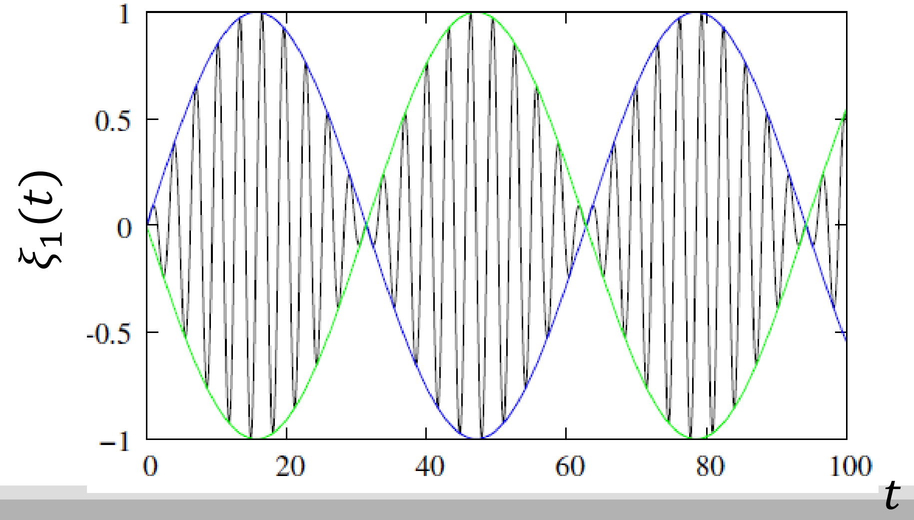
$$\xi_2 = A\left[\sin\left(\frac{k_0}{2\sqrt{km}}\right)t\,.\sin\omega_1 t\right]$$

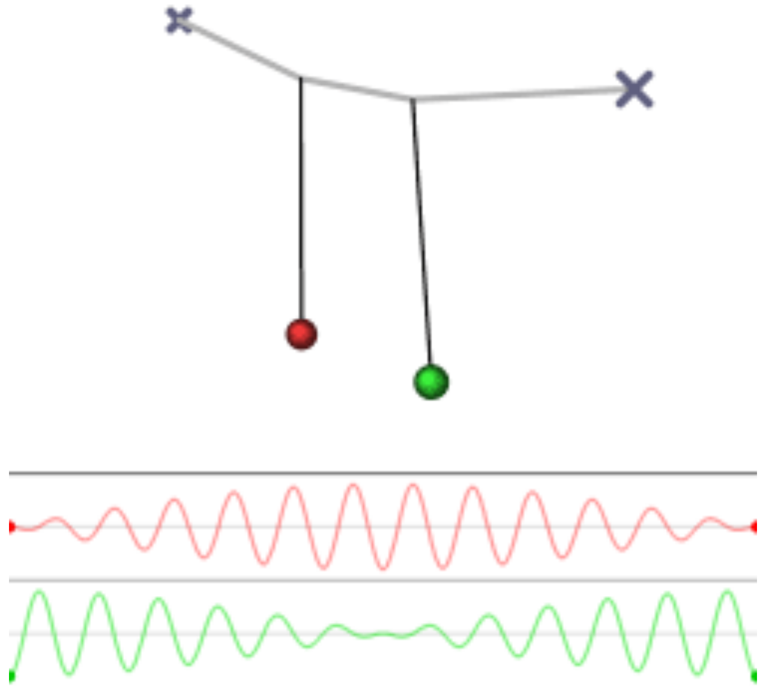
→ Resonance

Detailed solution: **Case 3 (RESONANCE)**

$$\xi_1 = A\left[\cos\left(\frac{k_0}{2\sqrt{km}}\right)t\,.\cos\omega_1\,t\right] \qquad \xi_2 = A\left[\sin\left(\frac{k_0}{2\sqrt{km}}\right)t\,.\sin\omega_1\,t\right]$$



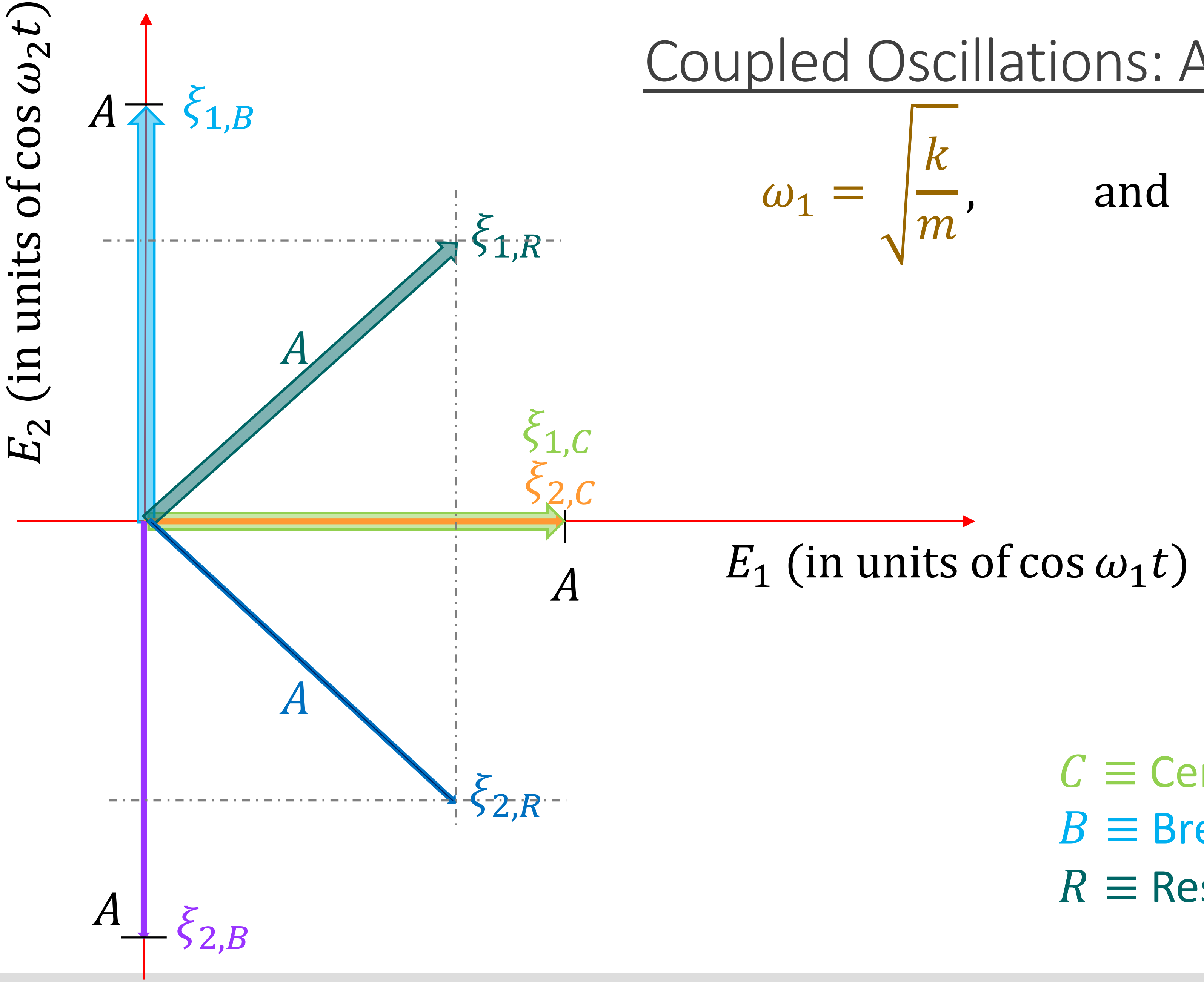




File:Coupled oscillators.gif - Wikipedia

Coupled Oscillations: A symbolic view

$$\omega_1 = \sqrt{\frac{k}{m}}, \qquad \text{and} \quad \omega_2 = \sqrt{\frac{(k+2k_0)}{m}}$$

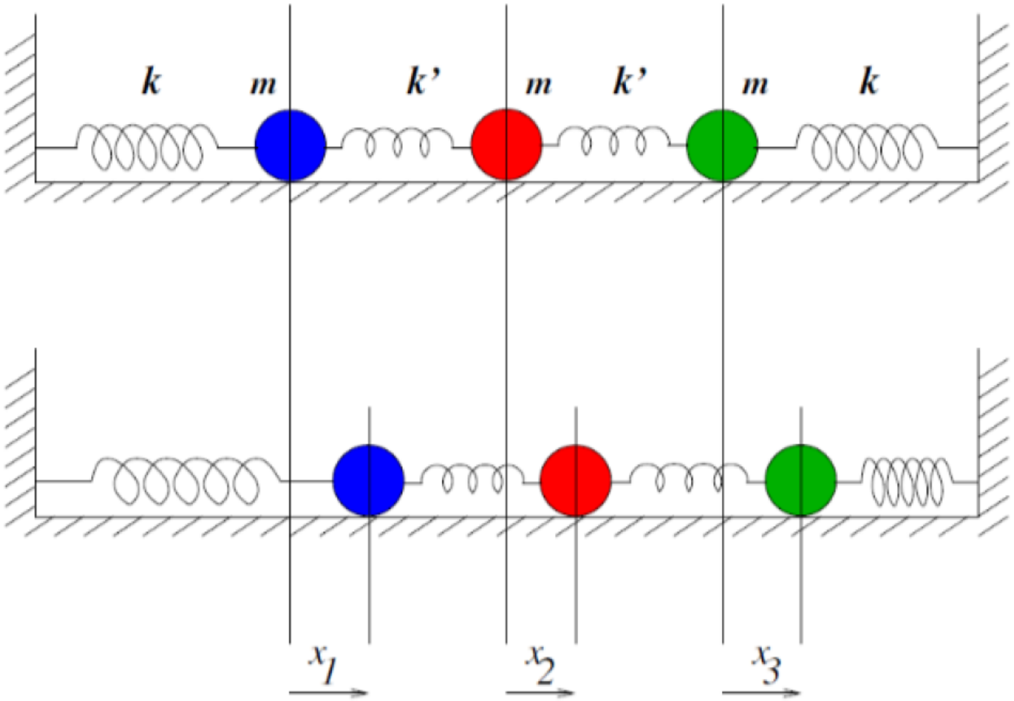


$C \equiv$ Centre of mass Mode

$B \equiv$ Breathing Mode

$R \equiv$ Resonance

3-Coupled Oscillators



$$m\ddot{x}_1 = -kx_1 - k'(x_1 - x_2)$$

$$m\ddot{x}_2 = -k'(x_2 - x_1) - k'(x_2 - x_3)$$

$$m\ddot{x}_3 = -k'(x_3 - x_2) - k\,x_3$$

Assume

$$x_1(t) = Ae^{i\omega t}$$

$$x_2(t) = Be^{i\omega t}$$

$$x_3(t) = Ce^{i\omega t}$$

3-Coupled Oscillators

The Matrix Equation:

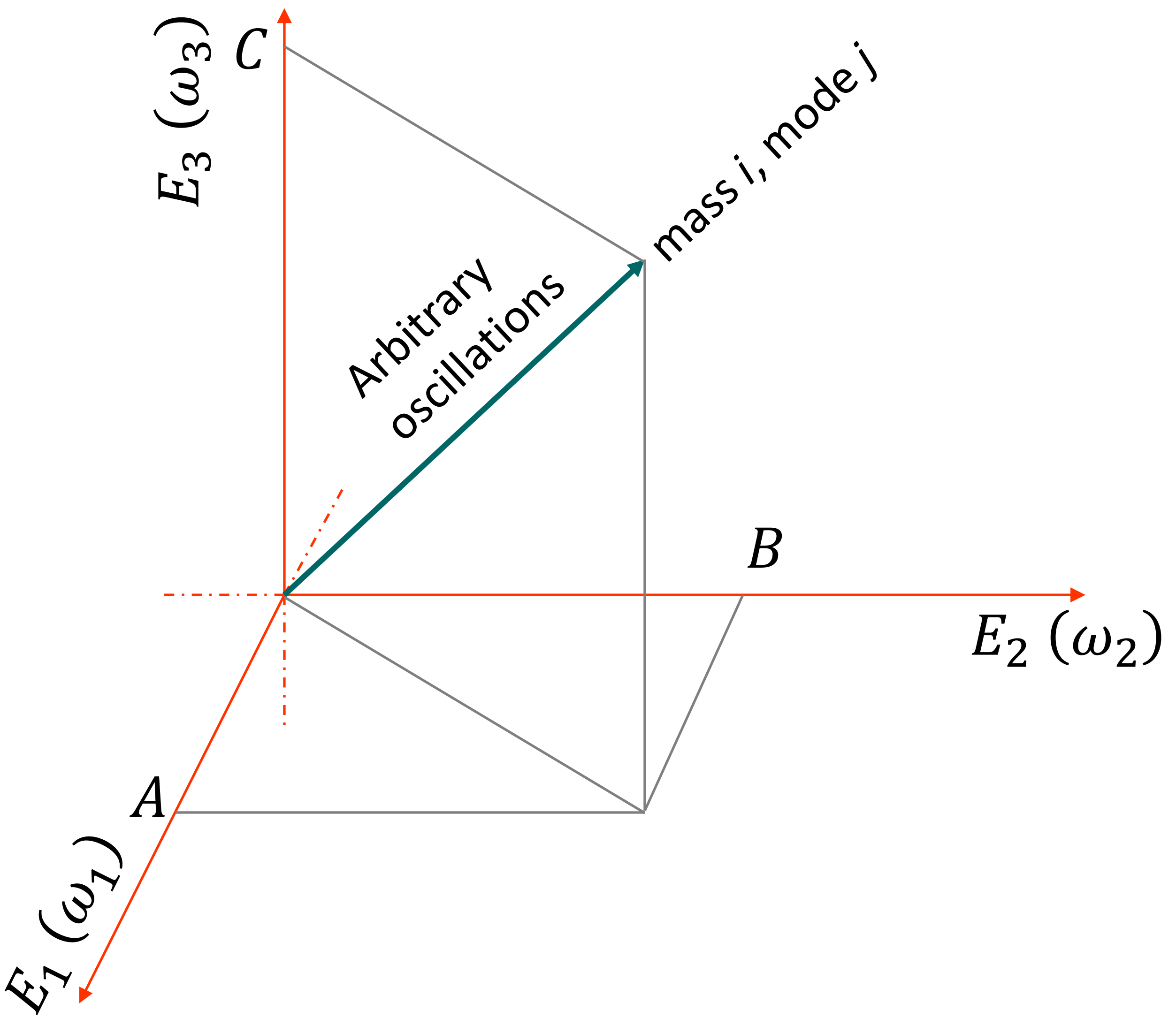
$$\begin{bmatrix} \omega^2 m - k - k' & k' & 0 \\ k' & \omega^2 m - 2k' & k' \\ 0 & k' & \omega^2 m - k - k' \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

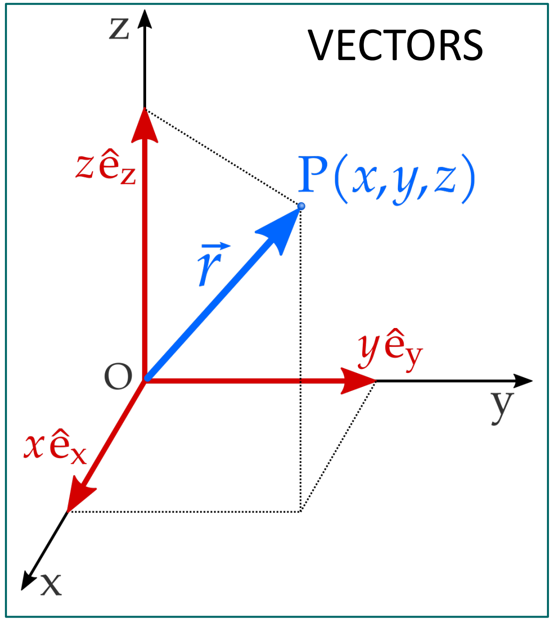
$\Rightarrow$ Three Eigenfrequencies

$$\omega_0^2 = \frac{k + k'}{m}$$

$$\omega_\pm^2 = \frac{1}{2m}\left[(k + 3k') \pm \sqrt{(k + 3k')^2 - 8kk'}\right]$$

3-Coupled Oscillators





N-coupled Oscillators

$\Rightarrow$ N Eigenfrequencies (N Modes)

$N \rightarrow \infty$

$\Rightarrow$ Waves