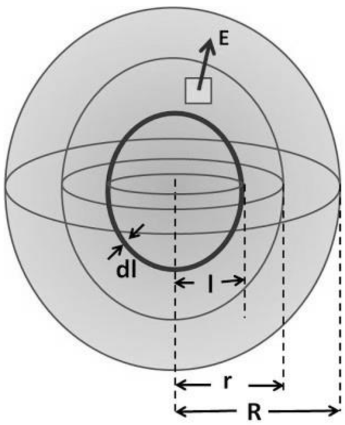
1. Assume a spherical ball of radius R carries a volume charge distribution given by the relation,

$$\rho = \rho_0 (1 - r/R)$$

Find the distance from the center where the electric field will be maximum.

Answer.



The electric field vector will be radially outwards at any point within or outside the sphere because charge density is isotropic (does not depend on θ but is only a function of only r).

(1) When r<R

Let us consider a spherical Gaussian Surface that is concentric with the sphere and of radius r. First calculate the total charge contained within this Gaussian Surface. Consider an infinitesimally thin spherical shell of radius l and thickness dl. The volume of this infinitesimally thin sphere will be= $4\pi l^2 dl$ and so the charge contained in this volume will be = $\rho(4\pi l^2 dl)$. The charge enclosed by the Gaussian surface can be obtained by integrating the charge distribution between $l = 0$ and l = r,

$$q = \int_0^r \rho(l)(4\pi l^2 dl)$$

$$= \int_0^r \rho_0 [1 - l/R](4\pi l^2 dl)$$

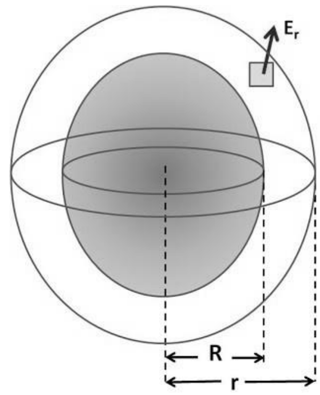
$$= \frac{4\rho_0 \pi r^3}{3}\left[1 - \frac{3r}{4R}\right] \quad \text{........1)}$$

From Gauss Law,

$$E(4\pi r^2) = q/\varepsilon_0$$

$$E = \frac{\rho_0 r}{3\varepsilon_0}\left[1 - \frac{3r}{4R}\right] \qquad r < R$$

(2) Whenr>R



The total charge contained in the Gaussian Surface is simply the total charge in the sphere is given by,

$$q = (4/3)\rho_0 \pi R^3 [1 - 3/4]$$
$$= \frac{\rho_0 \pi R^3}{3}$$

FromGaussLaw,

$$E(4\pi r^2) = \frac{\rho_0 \pi R^3}{3\varepsilon_0}$$
$$E = \frac{\rho_0 R^3}{12\varepsilon_0 r^2} \qquad r > R$$

The electric field is increasing inside the sphere but it's decreasing outside the sphere. So electric field will be maximum inside the sphere.

the electric field will be maximum where dE/dr=0. If we differentiate equation (1)

(1-3r/2R)=0

r=2R/3

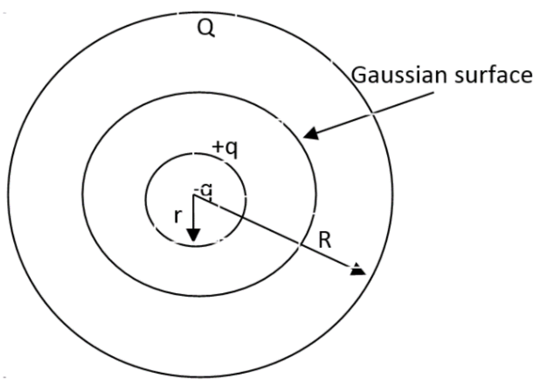
2. Consider a spherical conductor of radius R with a spherical cavity of radius r < R. Consider that the conductor is charged with a charge Q and in the center of the cavity there is an additional charge −q, such that |q| < |Q|. Find

(a) Charge in the inner surface of the conductor

(b) Charge on the outer surface of the conductor

(c) Electric field at a point *a* away from the surface for (i) a < r, (ii) R > a > r and (iii) a > R

Answer:



Since the electric field inside a conductor must be zero, the net charge enclosed by the Gaussian surface shown in the abovefigure must be zero. This implies that a charge +q must have been induced on the cavity surface. Since the conductor itself has a charge +Q, the amount of charge on the outer surface of the conductor must be= Q-q.

(i) *a*<r i.e inside the cavity where the charge is -q

Electric field from Gauss law due to this charge

$$\oint \overrightarrow{E_1}.\overrightarrow{dA} = \frac{-q}{\varepsilon_0}$$

$$E_1 4\pi r_1^2 = \frac{-q}{\varepsilon_0}$$

$$E_1 = -\frac{q}{4\pi\varepsilon_0 r_1^2}$$

(ii) R >*a*> r

In this region the total charge is zero. Thus the electric field will be zero. $E_2 = 0$

(iii) *a*> R

In this region the total charge is Q-q.

Thus electric field will be

$$E_3 = \frac{Q-q}{4\pi\varepsilon_0 r_3^2}$$

3. An infinite cylindrical conductor of radius R; carries a uniform current I. The current can be assumed to be uniformly distributed over the cross-sectional area of the conductor. Find the magnetic field as a function of the distance from the center of the cylindrical conductor.

Answer:

Because there is cylindrical symmetry in the current distribution, we might be able to use Ampere's Law to assist in determining the magnetic field. To use Ampere's Law, we need to find a path for which $\oint \vec{B}.d\vec{r}$ is easy to calculate. Because thewire is infinitely long and has axial symmetry about the axis, the strength of themagnetic field can only depend on r, the radial distance from the axis.

Case 1: outside the wire, r > R:

In this case all the current passes through the circular path.

$$\oint \vec{B}.d\vec{r} = \mu_0 I$$

$$2\pi r\left|\vec{B}\right| = \mu_0 I$$

$$\left|\vec{B}\right| = \frac{\mu_0 I}{2\pi r}$$

Case 2: inside the wire, r<R:

In this case only part of the current passes through the circular path. We need todetermine how much current passes through a circle of radius r centered on the axis.The amount of current is I times the ratio of the area of the circle to the crosssectional area of the wire: $I(\pi r^2)/(\pi R^2)$. Ampere's Law becomes

$$\oint \vec{B}.d\vec{r} = \mu_0 I$$

$$2\pi r\left|\vec{B}\right| = \mu_0 I\,\frac{\pi r^2}{\pi R^2}$$

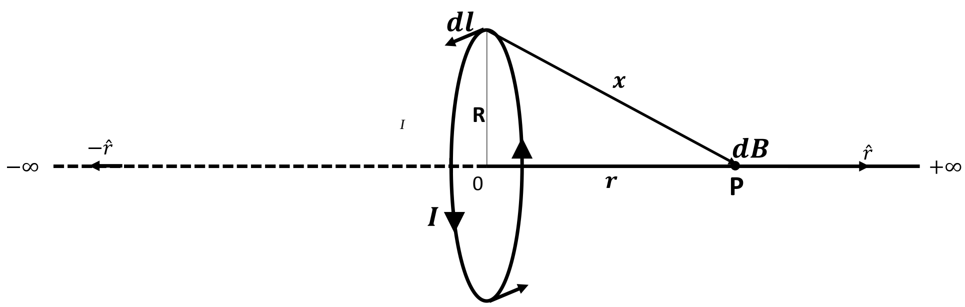
$$\left|\vec{B}\right| = \frac{\mu_0 I \pi r}{2\pi R^2}$$

So we see that inside the wire the magnet field strength increases linearly with distance from the axis until the wires edge. Outside the wire the magnetic field strength decreases inversely with distance (1/r).

4. A current I flows along a circular loop. Find the integral $\int_{-\infty}^{\infty} \vec{B}.d\vec{r}$ along the axis of the loop.

Answer:

Consider a conducting element d$\boldsymbol{l}$ of the loop as shown in figure. The magnitude dB of the magnetic field due to d$\boldsymbol{l}$ is given by the Biot-Savart law,



$$dB = \frac{\mu_0}{4\pi}\frac{I\left|dl \times x\right|}{x^3} \qquad \text{..........} (1) \quad (x^2 = r^2 + R^2)$$

Further, any element of the loop will be perpendicular to the displacement vector from the element to the axial point

Hence $\left|dl \times x\right| = xdl$

$$dB = \frac{\mu_0}{4\pi}\frac{Idl}{(r^2 + R^2)} \qquad \text{.............}(2)$$

The direction of dB is perpendicular to the plane formed by dl and x. It has a parallel and perpendicular component of dB to $\hat{r}$. When the components perpendicular to the $\hat{r}$ are summed over, they cancel out and we obtain a null result. The net contribution along $\hat{r}$-direction can be obtained by integrating $dB_r = dB\cos\theta$ over the loop.

$$\cos\theta = \frac{R}{(r^2 + R^2)^{1/2}}$$

$$dB_r = \frac{\mu_0}{4\pi}\frac{Idl\, R}{(r^2 + R^2)^{3/2}}$$

The summation of elements dl over the loop yields $2\pi R$, thecircumference of the loop. Thus, the magnetic field at P due to entirecircular loop is

$$B = \frac{\mu_0 I R^2}{2(r^2 + R^2)^{3/2}}\hat{r}$$

Now the integral $\int_{-\infty}^{\infty} \vec{B}.d\vec{r}$ can be written as

$$\int_{-\infty}^{\infty} \frac{\mu_0 I R^2 dr}{2(r^2+R^2)^{3/2}}$$

$$Let\ \frac{r}{R} = \tan\theta\ ,so\ '\theta'\ will\,be\ -\pi/2\ \ to\ \ \pi/2$$

$$\int_{-\pi/2}^{\pi/2} \frac{\mu_0 I R^3 \sec^2\theta d\theta}{2R^3\sec^3\theta}$$

$$\int_{-\pi/2}^{\pi/2} \frac{\mu_0 I \cos\theta d\theta}{2} \ = \ \mu_0 I$$

5. Consider circular wire loop that is shrinking and its circumference is decreasing at a rate of 12 cm/s. The loop is placed in constant uniform magnetic field of 0.5 T oriented perpendicular to the plane of the wire loop. Find the magnitude and direction of the induced emf at the instance when its circumference is 165 cm.

   Answer:

   Circumference C= 2πr

   Rate of change of circumference is

   $$\frac{dC}{dt} = 2\pi\frac{dr}{dt} = -0.12\ \ m/s\ \ (given)$$

   minus sign to denote the decrease in value

   Area of the loop is $\quad A = \pi r^2$

   The rate of change of area

   $$\frac{dA}{dt} = 2\pi r\frac{dr}{dt} = -0.12r\ \ m/s\ \ (replacing\ 2\pi\frac{dr}{dt}\ by -0.12)$$

   When the circumference is = 1.65 m

   The radius is 1.65/2π ≈ 0.26 m

   Now induced emf can be written as

$$e = -\frac{d}{dt}(BA\cos\theta)$$

$$e = -0.5\,\frac{dA}{dt} \qquad (\because\ \cos\theta = 1 and\ B = 0.5)$$

$$e = 0.5\times(0.12\times r) \qquad (Substituting\ \ for\ \frac{dA}{dt} = -0.12\times r)$$

$$e = 0.5\times 0.12\times 0.26 \ = \ 0.0156V$$

While the area of loop is decreasing, then the magnetic flux will also be decreasing. According to Lenz's law, the emf induced in the loop by this changing flux produces a current that sets up a field opposing the change. Thus the induced field will be in the same direction (i.e. out of page) and the current will be anticlockwise.