Q1) If an iron ball of weight W = 98N stretches a spring by 1.09 m, how many cycles per minute will the mass-spring system execute? What will its motion be if we pull down the weight an additional 16 cm and let it start with zero initial velocity?

Solution:

(a) Let m be the mass of the ball, k is sprint constant. Then from the Hook's law

$$F = -k\Delta x$$

Given

$$F=98N,\ \Delta x= 1.09\text{ m}$$

Then $$k = \frac{98}{1.09}\text{ N/m} \qquad (1)$$

And

$$W=mg$$

$$m = \frac{98}{9.8}Kg = 10Kg \quad \text{(Use g=9.8 m/s}^2\text{)}$$

Hence the frequency of the oscillation

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{98}{1.09}\times\frac{1}{10}}$$

$$=2.99\text{ rad/sec} \approx 3\text{ rad/sec}$$

Cycle per minute

$$(f) = \frac{\omega}{2\pi}\times 60$$

$$= \frac{3}{2\pi}\times 60 = 28.65$$

$$\approx 29\text{ cycles/ min}$$

(b) If we pull down the weight an additional 16 cm and let it start with zero initial velocity, it will execute S.H.M. with the same frequency i.e. 29 cycle/ min.

We know that the equation of motion of S.H.M

$$\ddot{X} + \omega^2 X = 0$$

as calculated

$$\omega = 3$$

So, $$\ddot{X} + 9X = 0$$

Solution of the above equation is given by

$$X = A\cos 3t + B\sin 3t$$

using boundary condition

at $t=0,\ X=0.16$ m

and

$t=0,\ v=0$

gives

$$0.16 = A$$

$$0 = B$$

Then Solution of the equation of motion will be

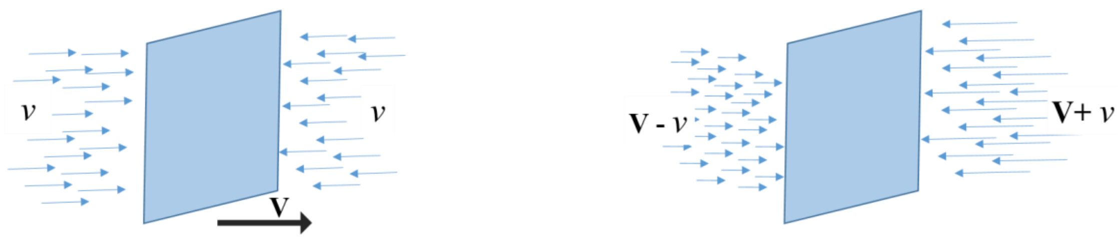
$$\boxed{X = A\cos 3t}$$

Q2) A flat plate moves at a velocity V along the direction normal to its planar surface, through a gas at low pressure. Show that the drag force on the plate is proportional to the speed (linear in speed damping).

Solution:

Assuming low pressure of the gas, the collision among the molecules are negligible compared to the collision of the gas molecule with the plate. The gas molecule has random velocity in all the direction, but for the purpose of calculating the force on the plate, only the velocity (or the component of the velocity) towards the plate is significant. For, simplification we consider this velocity of the molecules to be *v* on average.

Now, as the plate is moving with velocity **V** in the direction normal to its surface (say in the positive x direction), then the relative velocity (v_{rel}) of the molecule with respect to the plate will be **V + *v*** in one side of the plate and **V** - *v* on other side of the plate.



motions of plate and of the gas molecules towards plate

Relative velocity of the gas molecules with respect to plate

If F_1 and F_2 are the forces on the plate due to the collision with the gas molecules on two sides of the plate, then drag force on the plate is $F_{drag} = |F_1 - F_2|$.

Now force on plate F = rate of collision X change of momentum per collision

As all the particles within volume (A x v_{rel}.t) will collide with the plate in time t, so the rate of collision is proportional to the relative velocity (v_{rel}). Again the change in momentum per collision is also proportional to the relative velocity (v_{rel}). As a result, $F \propto {v_{rel}}^2$.

So $F_1 \propto (V+v)^2$ *and* $F_2 \propto (V-v)^2$ and the drag force on the plate is

$F_{drag} = |F_1 - F_2| \propto 4vV$ which is linearly proportional to the velocity of the plate.

Q3) A conductor in the shape of a square frame is suspended by an elastic thread and kept in a uniform magnetic field as shown below. Show that the system will exhibit oscillations and analyze the behavior of the oscillatory system.

Solution:

For a small displacement θ the two forces act on the loop

1- Restoring force due the twist in the thread (τ_R).
2- Torque due to the magnetic moment (τ_m) arises due to the induced e.m.f in the loop (arises due to the change in magnetic flux).

Then the total torque acting on the loop

$$\tau = \tau_R + \tau_m$$

1- Restoring force

$$\tau_R = -k\theta$$

2- Induced e.m.f

$$\begin{aligned}
\varepsilon &= -\frac{d\phi}{dt} \\
&= -\frac{d}{dt}(\vec{B}.\vec{A}) \\
&= -\frac{d}{dt}(Ba^2 \cos(90-\theta)) \\
&= -\frac{d}{dt}(Ba^2 \sin\theta) \\
&= -a^2 B\dot{\theta} \qquad \text{For small } \theta,\ \text{Sin}\theta=\theta
\end{aligned}$$

Let R be the resistance of the loop then

$$\varepsilon = iR = -a^2 B\dot{\theta}$$

$$i = -\frac{a^2 B\dot{\theta}}{R}$$

So the torque acting on the loop

$$\begin{aligned}
\tau_m &= \vec{m} \times \vec{B} \\
&= i\vec{A} \times \vec{B} \\
&= -\frac{a^2 B\dot{\theta}}{R}\vec{A} \times \vec{B} \\
&= -\frac{a^2 B\dot{\theta}}{R} a^2 B \sin(90-\theta) \\
&= -\frac{a^2 B\dot{\theta}}{R} a^2 B \cos\theta \\
&= -\frac{a^4 B^2\dot{\theta}}{R} \qquad \text{(For small } \theta\text{, } \cos\theta \approx 1\text{)}
\end{aligned}$$

So the equation of motion is

$$\tau = \tau_R + \tau_m$$

$$I\ddot{\theta} = -k\theta - \frac{a^4 B^2\dot{\theta}}{R}$$

(Let I be the moment of inertia of the system)

$$I\ddot{\theta} + \frac{a^4 B^2\dot{\theta}}{R} + k\theta = 0$$

$$\ddot{\theta} + \frac{a^4 B^2\dot{\theta}}{IR} + \frac{k}{I}\theta = 0$$

Let

$$2\gamma = \frac{a^4 B^2 \dot{\theta}}{IR}$$

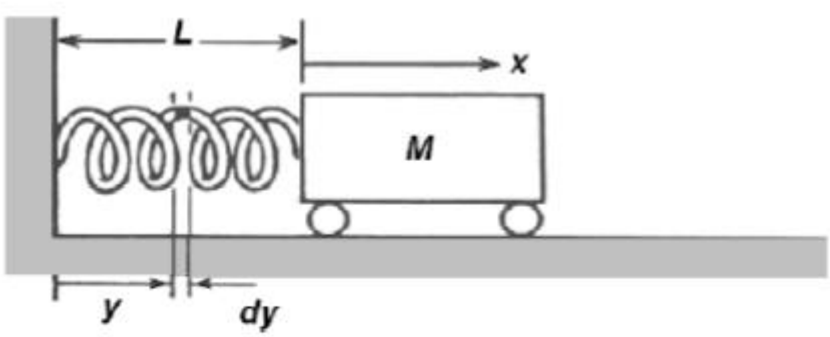
$$\omega^2 = \frac{k}{I}$$

Then

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega^2\theta = 0$$

Which, is a standard form of Damped harmonic motion. So, the system will behave as a Damped harmonic oscillator (Try to solve This equation!).

Q4) Let “m” be mass of a uniform spring with free-length “L”, and “k” be the stiffness constant of the spring. A mass “M” is attached to the spring as shown. Because of the way the spring is attached, the extension of the spring is linearly proportional to the distance from the fixed wall. Find the kinetic energy of the spring and using it determine the effective mass of spring. Thus, determine the natural frequency of oscillation of this system.



Solution:

Let us take a small element ‘dy’ of the spring at a distance y from the fixed end. Since the spring is of uniform density, mass of this element is $dm = \frac{m}{L}dy$; where m and L are the mass and free-length of the whole spring respectively.

Again as the displacement of the spring is linearly proportional to the distance from the fixed end and the displacement of the rightmost end of the spring being x (same as the displacement of the block)

$$\Delta y = \frac{y}{L} x$$

Assuming, dynamic shape of the spring is same as the static deformation, the velocity of the element at distance y is simply $u = \frac{y}{L} v$; where $v = \frac{dx}{dt}$ is the velocity of the block.

The kinetic energy of the element is $\frac{1}{2} u^2 \text{ dm}$ and so the total kinetic energy of the spring is

$$\int \frac{1}{2} u^2 \text{ dm} = \int_0^L \frac{1}{2} \left(\frac{yv}{L} \right)^2 \frac{m}{L} dy$$

$$= \frac{1}{2} \frac{m}{3} v^2$$

So the effective mass of the spring is m/3. In finding the kinetic energy of the system, the effective mass of m/3 of the spring has to be added to the mass of the block (M) to get the correct result of analyzing the oscillation of the system.

So the time period of oscillation of the system can be obtained from

$$\omega = \sqrt{\frac{k}{M + \frac{m}{3}}} \text{ and}$$

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{M + \frac{m}{3}}{k}}.$$