

Indian Institute of Technology Kharagpur
Department of Mathematics
MA11003 - Advanced Calculus
Problem Sheet - 4
Autumn 2022

1. Show that the function

$$f(x, y) = \begin{cases} \frac{x^2 + y^2}{|x| + |y|}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

is continuous at $(0, 0)$, but $f_x(0, 0)$ and $f_y(0, 0)$ do not exist.

2. Show that the following functions

$$(a) \quad f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

$$(b) \quad f(x, y) = \begin{cases} \frac{x^3 + y^3}{x - y}, & \text{if } x \neq y; \\ 0, & \text{if } x = y. \end{cases}$$

possess partial derivatives at $(0, 0)$, though it is not continuous at $(0, 0)$.

3. Find $f_x(x, y)$ and $f_y(x, y)$ using definition for the followings :

$$(a) \quad f(x, y) = x^2 + y^2,$$

$$(b) \quad f(x, y) = \sin(3x + 4y),$$

$$(c) \quad f(x, y) = ye^{-x} + xy.$$

4. Find $f_x(0, 0)$, $f_y(0, 0)$, $f_x(0, y)$ and $f_y(x, 0)$ for the followings :

$$(a) \quad f(x, y) = \begin{cases} \frac{xy}{x + y}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

$$(b) \quad f(x, y) = \log(1 + xy),$$

$$(c) \quad f(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 0, \\ 0, & \text{Otherwise} \end{cases}$$

(d) $f(x, y) = e^{x-y} - e^{y-x},$

(e) $f(x, y) = \begin{cases} \frac{x^3 - y^3}{x + y}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

5. Show that the following functions

(a) $f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2}, & \text{if } (x^2 + y^2) \neq 0; \\ 0, & \text{if } (x^2 + y^2) = 0. \end{cases}$

(b) $f(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2}, & \text{if } (x^2 + y^2) \neq 0; \\ 0, & \text{if } (x^2 + y^2) = 0. \end{cases}$

have first order partial derivatives at $(0, 0)$, and discuss the differentiability at $(0, 0)$.

6. Show that following function

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0), \\ 0 & (x, y) = (0, 0) \end{cases}$$

is continuous, possess first order partial derivatives but it is not differentiable at the origin.

7. Prove that the function $f(x, y) = \sqrt{|xy|}$ is not differentiable at $(0, 0)$, but that f_x and f_y both exists at origin and have the value 0. Show that f_x and f_y are continuous everywhere except at the origin.

8. Test the differentiability of the following functions at $(0, 0)$

(a) $f(x, y) = \begin{cases} (x^2 + y^2) \sin \frac{1}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

(b) $f(x, y) = \begin{cases} \frac{x^3 - 2y^3}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

9. Let $f(x, y) = \begin{cases} y \frac{x^2 - y^2}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

Find $f_{xx}(x, y)$, $f_{xy}(x, y)$, $f_{yx}(x, y)$ and $f_{yy}(x, y)$ at $(0, 0)$. Also check the differentiability of the function $f(x, y)$ at the origin.

10. For the function $f(x, y) = \begin{cases} \frac{x^2 y(x - y)}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

check that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$. Also check the differentiability of $f(x, y)$ at the origin

11. Find the total differential of the following functions

(a) $z = \tan^{-1}(x/y),$

(b) $u = e^{(x^2 + y^2 + z^2)},$

(c) $w = \sin(3x + 4y) + 5e^z$

(d) $w = z \ln y + y \ln z + xyz,$

(e) $u = \sqrt{x^2 + y^2 + z^2},$