Tutorial 5: Solutions

1. The electric field of an electromagnetic wave is given as:

$$\vec{\mathbf{E}} = E_0 \hat{\mathbf{j}} \sin \frac{\pi z}{z_0} \cos(kx - \omega t) \tag{1}$$

(a) Describe the field given above. (b) Determine an expression for k in terms of other quantities. (c) Find the phase velocity of the wave.

(a) The given electric field of an electromagnetic wave is

$$\vec{\mathbf{E}} = E_0 \hat{\mathbf{j}} \sin \frac{\pi z}{z_0} \cos(kx - \omega t) \tag{2}$$

$$\vec{\mathbf{E}} = \frac{1}{2} E_0 \hat{\mathbf{j}} [\sin(\frac{\pi z}{z_0} + kx - \omega t) + \sin(\frac{\pi z}{z_0} - kx + \omega t)] \tag{3}$$

So, the given electric field is the superposition of two fields.

$$\vec{\mathbf{E}}_1 = \frac{1}{2} E_0 \hat{\mathbf{j}} \sin(kx + \frac{\pi z}{z_0} - \omega t) \tag{4}$$

$$\vec{\mathbf{E}}_2 = -\frac{1}{2} E_0 \hat{\mathbf{j}} \sin(kx - \frac{\pi z}{z_0} - \omega t) \tag{5}$$

$\vec{\mathbf{E}}_1$ is vibrating along positive y direction and $\vec{\mathbf{E}}_2$ is vibrating along negative y direction with amplitude $\frac{1}{2}E_0$ and the propagation directions $\vec{\mathbf{k}}_1 = k\hat{i} + \frac{\pi 1}{z_0}\hat{k}$ and $\vec{\mathbf{k}}_2 = k\hat{i} - \frac{\pi 1}{z_0}\hat{k}$. The resultant field is linearly polarised along y and varies sinusoidally with z (the $\sin \frac{\pi z}{z_0}$ factor).

(b) We know that electric field of the electromagnetic wave satisfies the following wave equation

$$\nabla^2 \vec{\mathbf{E}} = \frac{1}{c^2} \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} \tag{6}$$

$$\implies \frac{\partial^2 \vec{\mathbf{E}}}{\partial x^2} + \frac{\partial^2 \vec{\mathbf{E}}}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} \tag{7}$$

Now, after differentiating $\vec{\mathbf{E}} = E_0 \hat{\mathbf{j}} \sin \frac{\pi z}{z_0} \cos(kx - \omega t)$ twice with respect to x, z, and t respectively, we get from wave equation

$$k = \sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2}{z_0^2}} \tag{8}$$

(c) The phase velocity (v_p) is

$$v_p = \frac{\omega}{k} = \frac{\omega}{\sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2}{z_0^2}}} \tag{9}$$

2.(a) The electric field of a standing electromagnetic wave is given by $E = 2E_0 \sin kx \cos \omega t$. Derive an expression for $B(x,t)$. (b) A standing wave is given by $E = 100 \sin \frac{2\pi x}{3} \cos 5\pi t$. Determine the two waves that can be superimposed to generate it.

2.(a) From Faraday's law we have

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t} \tag{10}$$

Integrate to get

$$\begin{aligned} B(x,t) = -\int \frac{\partial E}{\partial x} dt &= -2E_0 k \cos kx \int \cos \omega t dt \\ &= -\frac{2E_0 k}{\omega} \cos kx \sin \omega t \end{aligned} \tag{11}$$

But $\frac{E_0 k}{\omega} = \frac{E_0}{c} = B_0$; thus

$$B(x,t) = -2B_0 \cos kx \sin \omega t \tag{12}$$

(b) The given standing wave is

$$E = 100 \sin \frac{2\pi x}{3} \cos 5\pi t \tag{13}$$

It can be rewritten as

$$E = 100 \left[\sin\left(\frac{2\pi x}{3} + 5\pi t\right) + \sin\left(\frac{2\pi x}{3} - 5\pi t\right)\right] \tag{14}$$

3.(a) Describe completely the state of polarization of each of the following waves:
$(i)\vec{\mathbf{E}} = \hat{\mathbf{i}} E_0 \cos(kz - \omega t) - \hat{\mathbf{j}} E_0 \cos(kz - \omega t)$
$(ii)\vec{\mathbf{E}} = \hat{\mathbf{i}} E_0 \sin(\omega t - kz) - \hat{\mathbf{j}} E_0 \cos(\omega t - kz - \frac{\pi}{4})$
$(iii)\vec{\mathbf{E}} = \hat{\mathbf{i}} E_0 \cos(\omega t - kz) - \hat{\mathbf{j}} E_0 \cos(\omega t - kz - \frac{\pi}{2})$
(b) Consider the disturbance given by $\vec{\mathbf{E}} = E_0[\hat{\mathbf{i}} \cos \omega t + \hat{\mathbf{j}} \cos(\omega t - \frac{\pi}{2})] \sin kz$. What kind of wave is it?

(a)(i) The given wave is

$$\vec{\mathbf{E}} = \hat{\mathbf{i}} E_0 \cos(kz - \omega t) - \hat{\mathbf{j}} E_0 \cos(kz - \omega t) = \hat{\mathbf{i}} E_0 \cos(kz - \omega t) + \hat{\mathbf{j}} E_0 \cos(kz - \omega t + \pi) \tag{15}$$

Thus, E_y lags e_x by π. We can also write it as,

$$\vec{\mathbf{E}} = (\hat{\mathbf{i}} - \hat{\mathbf{j}})E_0 \cos(kz - \omega t) \tag{16}$$

Now, $\tan\theta = \frac{E_0}{E_0} = 1$; So, $\theta = 45^o$.
Now, the direction of vibration

$$\hat{\mathbf{n}} = \hat{\mathbf{i}} \cos 45^o - \hat{\mathbf{j}} \sin 45^o = \frac{1}{\sqrt{2}}(\hat{\mathbf{i}} - \hat{\mathbf{j}}) \tag{17}$$

So,

$$\vec{\mathbf{E}} = \hat{\mathbf{n}}\sqrt{2}E_0 cos(kz - \omega t) \tag{18}$$

Since, the given wave has a fixed amplitude, it represents a linearly polarised wave with its electric vector making an angle 45^o with both x and y axis.
(ii) The given electric field vector is

$$\vec{\mathbf{E}} = \hat{\mathbf{i}}E_0 \sin(\omega t - kz) - \hat{\mathbf{j}}E_0 \cos(\omega t - kz - \frac{\pi}{4}) \tag{19}$$

$$\implies \vec{\mathbf{E}} = \hat{\mathbf{i}}E_0 \cos(\omega t - kz - \frac{\pi}{2}) - \hat{\mathbf{j}}E_0 \cos(\omega t - kz - \frac{\pi}{4}) \tag{20}$$

$$\implies \vec{\mathbf{E}} = \hat{\mathbf{i}}E_0 \cos(\omega t - kz - \frac{\pi}{4} - \frac{\pi}{4}) - \hat{\mathbf{j}}E_0 \cos(\omega t - kz - \frac{\pi}{4}) \tag{21}$$

$$\implies \vec{\mathbf{E}} = \hat{\mathbf{i}}E_0[\cos(\omega t - kz - \frac{\pi}{4})\frac{1}{\sqrt{2}} + \sin(\omega t - kz - \frac{\pi}{4})\frac{1}{\sqrt{2}}] - \hat{\mathbf{j}}E_0 \cos(\omega t - kz - \frac{\pi}{4}) \tag{22}$$

So, from the above equation,

$$E_x = \frac{E_0}{\sqrt{2}}[\cos(\omega t - kz - \frac{\pi}{4}) + \sin(\omega t - kz - \frac{\pi}{4})] \tag{23}$$

And,

$$E_y = E_0 \cos(\omega t - kz - \frac{\pi}{4}) \tag{24}$$

Now,

$$\sqrt{2}E_x - E_y = E_0 \sin(\omega t - kz - \frac{\pi}{4}) \tag{25}$$

And,

$$E_y = E_0 \cos(\omega t - kz - \frac{\pi}{4}) \tag{26}$$

Now squaring and adding these two equations we obtain

$$(\sqrt{2}E_x - E_y)^2 + E_y^2 = E_0^2 \tag{27}$$

$$\implies E_x^2 + \sqrt{2}E_xE_y + E_y^2 = \frac{E_0^2}{2} \tag{28}$$

This describe an ellipse. As there is a cross term in the above equation, there is a shift of major axis by an angle θ with respect to x-axis (or, E_x axis).

Let us give a rotation of coordinate system by an angle θ. The components of the electric field in the new coordinate system is,

$$E_x = E_x' \cos\theta - E_y' \sin\theta \tag{29}$$

And

$$E_y = E_x' \sin\theta + E_y' \cos\theta \tag{30}$$

So, in the new co-ordinate system, the equation is

$$E_x'^2 + E_y'^2 - \sqrt{2}(E_x'^2 \cos\theta \sin\theta - E_y'^2 \sin\theta\cos\theta) + \sqrt{2}E_x'E_y'(\cos^2\theta - \sin^2\theta) = \frac{E_0^2}{2} \tag{31}$$

So, in the new co-ordinate system the cross term should be zero. Hence,

$$\sqrt{2}E_x'E_y'(\cos^2\theta - \sin^2\theta) = 0 \tag{32}$$

So, $\cos 2\theta = 0$. $\implies \theta = \frac{\pi}{4}$.

Now putting $\theta = \frac{\pi}{4}$ in the equation (38) we obtain,

$$E_x'^2 + E_y'^2 - \sqrt{2}(\frac{E_x'^2}{2} - \frac{E_y'^2}{2}) = \frac{E_0^2}{2} \tag{33}$$

$$\implies \frac{E_x'^2}{\sqrt{2}+1} + \frac{E_y'^2}{\sqrt{2}-1} = \frac{E_0^2}{\sqrt{2}} \tag{34}$$

This is an equation of ellipse whose major and minor axes are shifted by an angle $\theta = \frac{\pi}{4}$

Further, at $z = 0$ plane,

At $t = 0$; $E_x = 0$; $E_y = \frac{E_0}{\sqrt{2}}$.

At some later time $t = \frac{\pi}{4\omega}$; $E_x = \frac{E_0}{\sqrt{2}}$; $E_y = 0$.

So, the given wave is a left handed elliptical polarised wave.

(iii) The given electric field is

$$\vec{\mathbf{E}} = \hat{\mathbf{i}}E_0\cos(\omega t - kz) - \hat{\mathbf{j}}E_0\cos(\omega t - kz - \frac{\pi}{2}) \tag{35}$$

$$\implies \vec{\mathbf{E}} = \hat{\mathbf{i}}E_0\cos(kz - \omega t) + \hat{\mathbf{j}}E_0\sin(kz - \omega t) \tag{36}$$

Now, from the above equation

$E_x = E_0 \cos(kz - \omega t)$.

$E_y = E_0 \sin(kz - \omega t)$.

Now, squaring and adding the above two equation, we get

$$E_x^2 + E_y^2 = E_0^2 \tag{37}$$

So, the above equation is an equation of a circle.

Further, at $z = 0$ plane

$E_x = E_0 \cos \omega t$, and

$E_y = -E_0 \sin \omega t$.

Now, at $t = 0$; $E_x = E_0$ and $E_y = 0$.

At a later time $t = \frac{\pi}{2\omega}$; $E_x = 0$ and $E_y = -E_0$.

So, the given wave equation is a right circularly polarised wave.

(b) The wave, $\vec{\mathbf{E}} = E_0[\hat{\mathbf{i}} \cos \omega t + \hat{\mathbf{j}} \cos(\omega t - \frac{\pi}{2})] \sin kz$ represents a left circularly polarised standing wave.

4.(a) Suppose that the intensity of the sunlight falling on the ground on a particular day is $140W/m^2$. What are the peak values of electric and magnetic fields associated with the incident radiation?

(b)A linearly polarised harmonic plane wave with a scalar amplitude of $10V/m$ is propagating along a line in the xy plane at 45 degrees to the x axis, with the xy plane as the plane of vibration. Write down a vector expression for the wave, assuming k_x and k_y are both positive. Calculate the flux density taking the wave to be in vacuum.

4.(a) The intensity of radiation (I) is defined as the magnitude of the time average of the Poynting vector.

$$I = |< \vec{\mathbf{S}} >| = \frac{E_0 B_0}{\mu_0} |< cos^2(kx - \omega t) >| \tag{38}$$

Since, the average value of $< cos^2(kx - \omega t) >$ is $\frac{1}{2}$ So,

$$I = \frac{E_0 B_0}{2\mu_0} = \frac{E_0^2}{2c\mu_0} = \frac{cB_0^2}{2\mu_0} \tag{39}$$

$$E_0 = \sqrt{2Ic\mu_0} \approx 324.9V/m \tag{40}$$

And,

$$B_0 = \sqrt{\frac{2I\mu_0}{c}} \approx 1 \times 10^{-6} Tesla. \tag{41}$$

(b) Since, the propagation is along a line in the xy plane at 45 degrees to the x-axis

$$\vec{\mathbf{K}} = k_x\hat{\mathbf{i}} + k_y\hat{\mathbf{j}} \tag{42}$$

$$k_x = k\cos 45 = \frac{k}{\sqrt{2}}; k_y = k\sin 45 = \frac{k}{\sqrt{2}} \tag{43}$$

So,

$$\mid k_x \mid = \mid k_y \mid = \frac{k}{\sqrt{2}} \tag{44}$$

Now the radiation is also in the xy plane.So, the direction of vibration of the harmonic wave

$$\hat{\mathbf{n}} = \frac{a\hat{\mathbf{i}} + b\hat{\mathbf{j}}}{\sqrt{a^2+b^2}} \tag{45}$$

Where a and b are any real quantity. Now the propagation direction and the direction of vibration should be perpendicular.

So,

$$\hat{\mathbf{n}}.\vec{\mathbf{K}} = 0 \tag{46}$$

$$\implies \frac{ak_x + bk_y}{\sqrt{a^2+b^2}} = 0 \tag{47}$$

$$\implies a = -b \tag{48}$$

Now the vibration is along $\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}(\hat{\mathbf{i}} - \hat{\mathbf{j}})$

So the vector expression of the wave is

$$\vec{\mathbf{E}} = \frac{10}{\sqrt{2}}(\hat{\mathbf{i}} - \hat{\mathbf{j}})\cos(\frac{kx}{\sqrt{2}} + \frac{ky}{\sqrt{2}} - \omega t) \tag{49}$$

Now the energy flux density is given as the magnitude of the time average of the Poynting's vector.

Energy flux density $= |< \vec{\mathbf{S}} >|$

So,

$$|< \vec{\mathbf{S}} >| = \frac{E_0 B_0}{\mu_0} |< cos^2(kx - \omega t) >| = \frac{E_0 B_0}{2\mu 0} = \frac{E_0^2}{2c\mu_0} \approx 63.3\, mJ\, m^{-2} s^{-1} \tag{50}$$

5.(a) A plane, harmonic linearly polarised light light wave has an electric field

$$E_z = E_0 \cos[\pi 10^{15}(t - \frac{x}{0.65c})] \tag{51}$$

Write down the frequency and wavelength of the light wave and obtain the index of refraction of the glass.

(b) Consider electromagnetic wave of wavelength $\lambda = 30cm$ in air. What is the frequency of such waves? If such waves waves pass from air into the block of quartz, for which the dielectric constant $K = 4.3$, what is their new speed, frequency, and wavelength?

5.(a)The given equation of the electric field is

$$E_z = E_0 \cos[\pi \times 10^{15}(t - \frac{x}{0.65c})] \tag{52}$$

From the above equation angular frequency $(\omega) = \pi \times 10^{15}$ rad/s.
So, the frequency $\nu = \frac{\omega}{2\pi} = 5 \times 10^{14}$ Hz.
Again from the given equation $k = \frac{\pi \times 10^{15}}{0.65c}\ m^{-1}$.
So, the wavelength $(\lambda) = \frac{2\pi}{k} \approx 390\,nm$.
Hence, the refractive index of glass is $(n_g) = \frac{c}{\nu\lambda} \approx 1.538$. ; Where c is the speed of light in vacuum.

(b) Given that the wavelength $(\lambda) = 30cm = 0.3m$.; and the refractive index of the quartz block $n_q = \sqrt{K} = \sqrt{4.3}$.
Now the frequency of the wave in air is $f = \frac{c}{n_a\lambda} \approx 10^9 Hz$. ; Where c is the speed of light in vacuum.
Now, the velocity of such waves in the quartz medium $v = \frac{c}{n_q} \approx 1.45 \times 10^8 m/s$.
Since, the frequency of the source is constant, the wavelength will change.
So, the new wavelength $(\lambda') = \frac{v}{n_q f} \approx 7 \times 10^{-4} m$.

6. Imagine that we have a non-absorbing glass plate of index n and thickness Δy, which stands between a source S and an observer P.

(i) If the unobstructed wave (without the plate present) is $E_u = E_0 e^{i\omega(t-\frac{y}{c})}$ show that with the plate in place the observer sees the wave given by,

$$E_p = E_0 e^{i\omega[t-(n-1)\frac{\Delta y}{c}-\frac{y}{c}]} \tag{53}$$

(ii) Show that if either $n \approx 1$ or Δy is small, then,

$$E_p = E_u + \frac{\omega(n-1)\Delta y}{c} E_u e^{-\frac{i\pi}{2}} \tag{54}$$

6.(i) The phase difference ($\Delta\phi$) due to the presence of the non-absorbing glass plate of thickness Δy is

$$\Delta\phi = \frac{2\pi}{\lambda}(n\Delta y - \Delta y) = \frac{2\pi c}{\lambda c}(n-1)\Delta y = \frac{2\pi f}{c}(n-1)\Delta y = \omega\frac{\Delta y}{c}(n-1) \tag{55}$$

Where n is the refractive index of the glass plate.
Due to this extra phase ($\Delta\phi$), the new wave will be

$$E_p = E_0 e^{i\omega(t-\frac{y}{c})} e^{-\Delta\phi} \tag{56}$$

$$\implies E_p = E_0 e^{i\omega[t-(n-1)\frac{\Delta y}{c}-\frac{y}{c}]} \tag{57}$$

(ii) Now,

$$E_p = E_u + \frac{\omega(n-1)\Delta y}{c} E_u e^{-\frac{i\pi}{2}} \tag{58}$$

$$\implies E_p = E_u e^{-i\omega(n-1)\frac{\Delta y}{c}} \tag{59}$$

Now if $n \approx 1$ or Δy is small then the exponent term is very small. Hence, expanding it in Taylor's series and neglecting the higher order terms, we obtain

$$E_p \approx E_u[1 - i\omega(n-1)\frac{\Delta y}{c}] \tag{60}$$

$$\implies E_p = E_u + \frac{\omega(n-1)\Delta y}{c} E_u e^{-\frac{i\pi}{2}} \tag{61}$$

7. (a) Draw a graph of θ_t versus θ_i for an air-glass boundary where the index of refraction of glass is $n_g = 1.5$.

(b) A glass prism whose cross section is an isosceles triangle stands with its (horizontal) base in water; the angles which its two equal sides make with the base are each equal to θ (see Figure 1.). An incident ray of light, above and parallel to the water surface and perpendicular to to the prism's axis, is internally reflected at the glass-water interface and subsequently re-emerges into the air. Taking the refractive indices of glass and water to be $\frac{3}{2}$ and $\frac{4}{3}$ show that θ must be atleast 25.9 degree.

7(a). From Snell's law

$$\theta_t = \sin^{-1}\left(\frac{\sin\theta_i}{n_g}\right)$$

The graph:

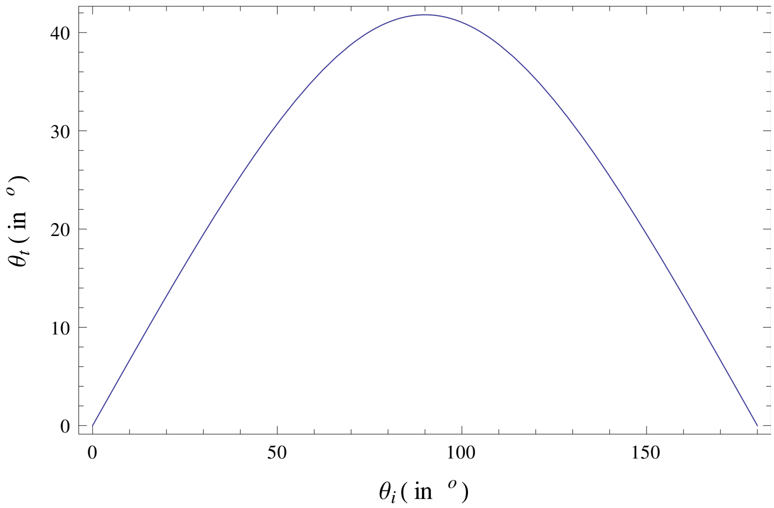

FIG. 1. θ_t versus θ_i

7(b). From the geometry of the prism, it can easily be seen that the angle of incidence at the first surface is $\theta_i = 90^o - \theta$. Using Snell's law, we have

$$n_g \sin\theta_t' = \sin(90^o - \theta)$$

$$\text{or}, n_g \sin\theta_t' = \cos\theta$$

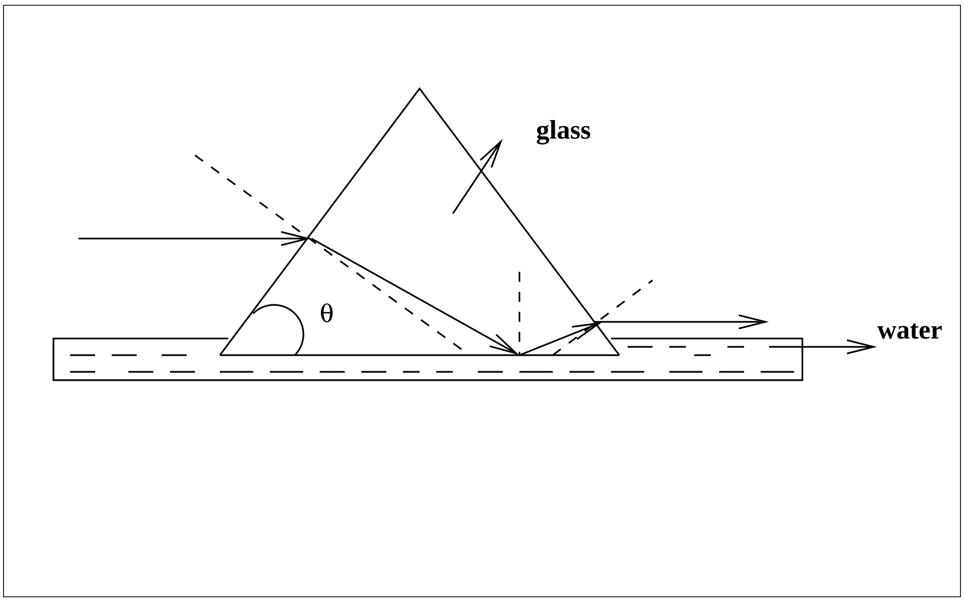

FIG. 2. Figure for Problem 7(b)

where, n_g(refractive index of the glass) $= \frac{3}{2}$ and θ_t' is the refraction angle at that surface. Suppose, θ_i' be the angle of incidence at the glass and water interface. Then, again from geometry of the prism, we have

$$\theta_i' = \theta_t' + \theta$$

Now, for internal reflaction at the glass-water interface, following inequality must be satisfied.

$$\sin\theta_i' \geq \frac{n_w}{n_g}$$

where, n_w(refractive index of water) $= \frac{4}{3}$. Using trigonometric identities and the previous two equations, we get the following inequality

$$\cos^2\theta + \sqrt{n_g^2 - \cos^2\theta}\,\sin\theta \geq n_w$$

Further simplifying the above inequality we get

$$\cos\theta \leq \sqrt{\frac{n_g^2 - n_w^2}{n_g^2 + 1 - 2n_w}}$$

$$\text{or,} \quad \theta \geq \cos^{-1}\left(\sqrt{\frac{n_g^2 - n_w^2}{n_g^2 + 1 - 2n_w}}\right)$$

Using the numerical values, finally we get $\theta_{min} \approx 25.9^o$.

** **8.** (a) Consider Snell's law of refraction. If the medium of incidence has an index $n_1 > 0$ and the other medium has an index $n_2 < 0$, then draw the refracted ray by assuming some angle of incidence θ_i. What difference do you notice in comparison to the case when $n_1, n_2 > 0$?

(b) Recall that the refractive index can be written as $n = \sqrt{\epsilon_r \mu_r}$ where $\epsilon_r = \frac{\epsilon}{\epsilon_0}$ and $\mu_r = \frac{\mu}{\mu_0}$. It is known that ϵ and μ are complex quantities and also functions of ω, with their real and imaginary parts related to physical quantities. Show, with an example, that using $\epsilon < 0$ and $\mu < 0$ it is possible to have the refractive index $n < 0$.

Note: Materials exhibiting negative index of refraction have actually been made in recent times though the original proposal was by the Russian physicist V. Veselago, in 1967. Such materials are known as metamaterials.

8.(a) Snell's law

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

$$\text{or}, \sin\theta_t = \frac{n_1}{n_2} \sin\theta_1$$

If $n_1 > 0$ and $n_2 < 0$, then $\sin\theta_t < 0$. Let us assume $n_1 = 1$, $n_2 = -1.5$, $\theta_i = 30$ degrees. Then we get $\sin\theta_t = -\frac{1}{3}$. Thus, $\theta_t = -19.27$ degrees. Note the minus sign. The refracted ray is now bent in a different way. Draw the incident and refracted rays, with the above values and you can see how different it is.

(b)The definition of refractive index is

$$n = \sqrt{\epsilon_r \mu_r} = \frac{\sqrt{\epsilon\mu}}{\sqrt{\epsilon_0 \mu_0}}$$

It is known that ϵ and μ are complex quantities. So, written in the polar form of a complex number, $\epsilon = |\epsilon| e^{i\phi_1}$ and $\mu = |\mu| e^{i\phi_2}$. So

$$n = \frac{\sqrt{|\epsilon||\mu|}}{\sqrt{\epsilon_0 \mu_0}} e^{i(\phi_1 + \phi_2)/2}$$

Now, if $\epsilon < 0$ and $\mu < 0$ then $\phi_1 = \phi_2 = \pi$. So,

$$n = \frac{\sqrt{|\epsilon||\mu|}}{\sqrt{\epsilon_0 \mu_0}} e^{i\pi}$$

$$\text{or,} \quad n = -\frac{\sqrt{|\epsilon||\mu|}}{\sqrt{\epsilon_0 \mu_0}} < 0$$