Week 3: Forced oscillations(continued), coupled oscillations

Let us continue with forced oscillations. Recall that the amplitude maximum occurs at $\Omega^2 = \Omega_{res}^2 = \omega_0^2 - 2\beta^2$. The behaviour of the amplitude as a function of Ω is shown in Fig.1. It is useful to calculate a quantity known as the Full Width at Half Maximum (FWHM). To do this, first find out the value of the amplitude at its maximum. Then take the half of this value and draw a horizontal line at this half value on your amplitude-forcing frequency curve. The difference between the two frequency values at which the horizontal line intersects the amplitude versus frequency curve is the FWHM. In this case, it is easy to show that for small β it is close to $2\sqrt{3}\beta$. One also notes that the amplitude shifts towards the origin with increasing β.

How does the phase ϕ behave as a function of Ω? This is shown in the Fig 2. We have plotted $-\phi$ as a function of Ω here. It turns out that with decreasing β the curve becomes sharper (see the blue one). Moreover, in general there is always a phase difference between the harmonic forcing and the late time free oscillation. It is clear that damping plays a major role in controlling the nature of the amplitude and phase.

The nature of the oscillation away from resonance is shown in Figs. 3 and 4. One notes that the amplitude of the late time steady state oscillation is not large when one is off-resonance.

We now turn to analyse energy and power in a forced oscillatory system. Recall that the general solution can be written as

$$x(t) = x_d(t) + x_f(t) \tag{1}$$

where

$$x_d(t) = e^{-\beta t}\left(A\cos\omega t + B\sin\omega t\right) \tag{2}$$

$$x_f(t) = A_f(\Omega)\cos\left(\Omega t + \phi(\Omega)\right) \tag{3}$$

where A, B (evaluated using initial conditions) and A_f and ϕ are, as specified earlier.

The time rate of change of energy can be written, using the equation of motion of the system. Specifically, this becomes

$$\frac{dE}{dt} = -2\beta m\dot{x}^2 + mf_0\dot{x}\cos\Omega t \tag{4}$$

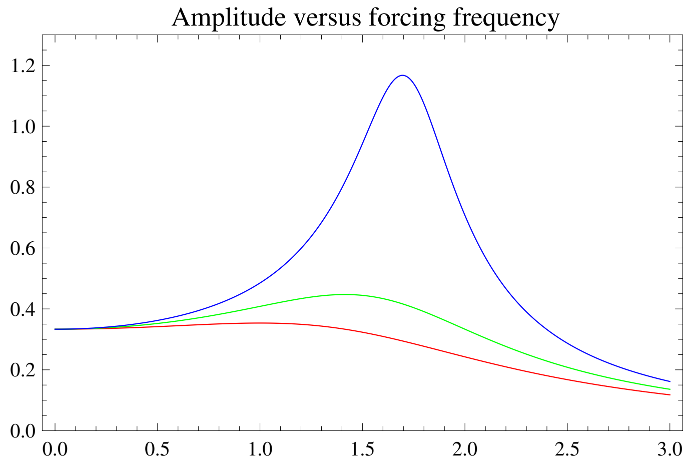

FIG. 1. Amplitude for $\beta = 1$ (red), $\beta = \frac{1}{\sqrt{2}}$ (green), $\beta = \frac{1}{4}$(blue)

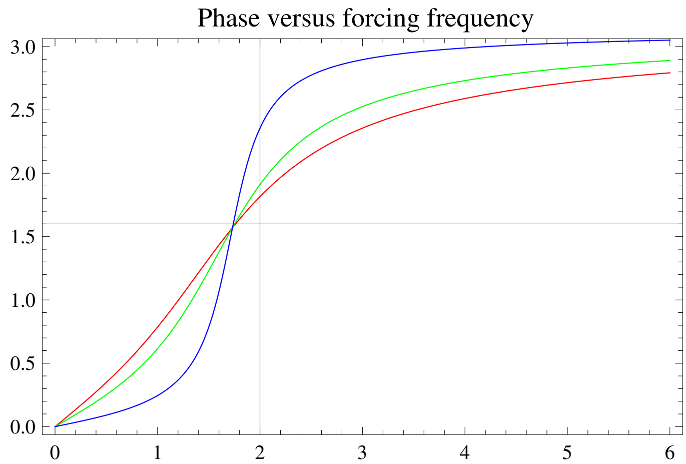

FIG. 2. (Negative of Phase for $\beta = 1$ (red), $\beta = \frac{1}{\sqrt{2}}$ (green), $\beta = \frac{1}{4}$(blue)

We will now need to use the solution $x(t)$ in the above expression of $\frac{dE}{dt}$. Notice that the first term (always negative) is the damping effect while the second one is the power drawn by the oscillator due to forcing. If we subsitute $x(t) = x_d(t) + x_f(t)$ we get the following:

$$\frac{dE}{dt} = -2\beta m \left(\dot{x}_d + \dot{x}_f\right)^2 + m f_0 \left(\dot{x}_d + \dot{x}_f\right) \cos \Omega t \tag{5}$$

At late times, we can afford to ignore the terms (like those having $\dot{x}_d^2$ and $\dot{x}_d$) which exponentially decay via overall factors of $e^{-2\beta t}$ or $e^{-\beta t}$. Thus, we get,

$$\left(\frac{dE}{dt}\right)_{approx} = -2\beta m \dot{x}_f^2 + m f_0 \dot{x}_f \cos \Omega t \tag{6}$$

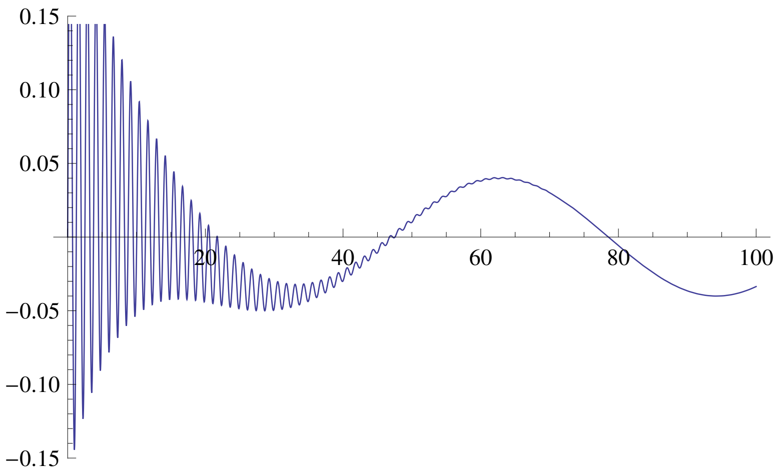

FIG. 3. $x(t)$ versus t when Ω much smaller than Ω_{res}

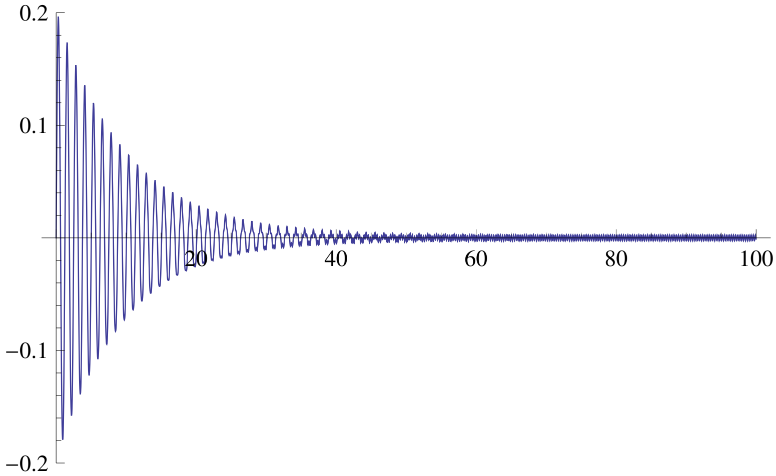

FIG. 4. $x(t)$ versus t when Ω much larger than Ω_{res}

Substituting x_f and its time derivative in the above and after some calculations, one can obtain the expression (not written here) for $\frac{dE}{dt}$. One can check from the expression that when $\Omega = \omega_0$, $\frac{dE}{dt}$ is zero. Also the time average, $\langle\frac{dE}{dt}\rangle$ is equal to zero, as it is for a free oscillation.

The average energy of the late time free oscillation can be calculated using the expression,

$$E^f_{avg} = \frac{1}{T}\int_0^T \left(\frac{1}{2}m\dot{x}_f^2 + \frac{1}{2}kx_f^2\right) dt \tag{7}$$

This yields, after some straightforward calculation, the following result,

$$E^f_{avg} = \frac{mf_0^2}{4}\frac{\Omega^2+\omega_0^2}{\left(\omega_0^2-\Omega^2\right)^2+4\beta^2\Omega^2} \tag{8}$$

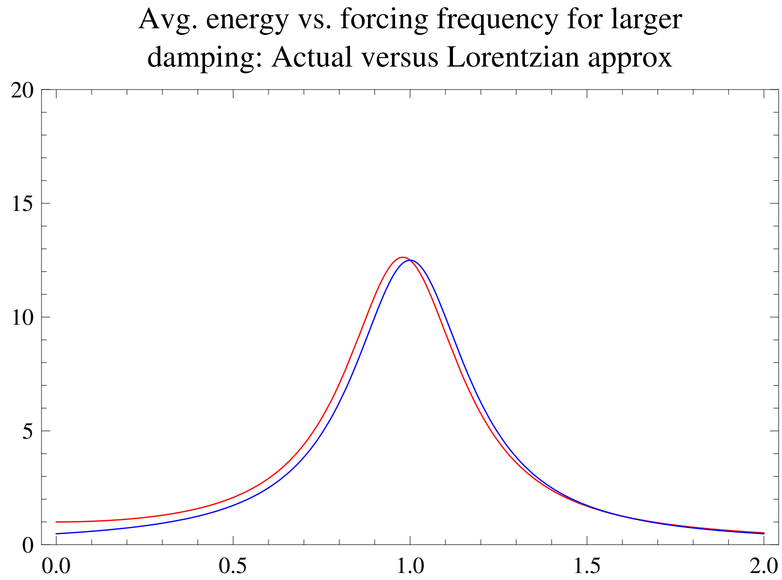

FIG. 5. Average energy vs forcing frequency

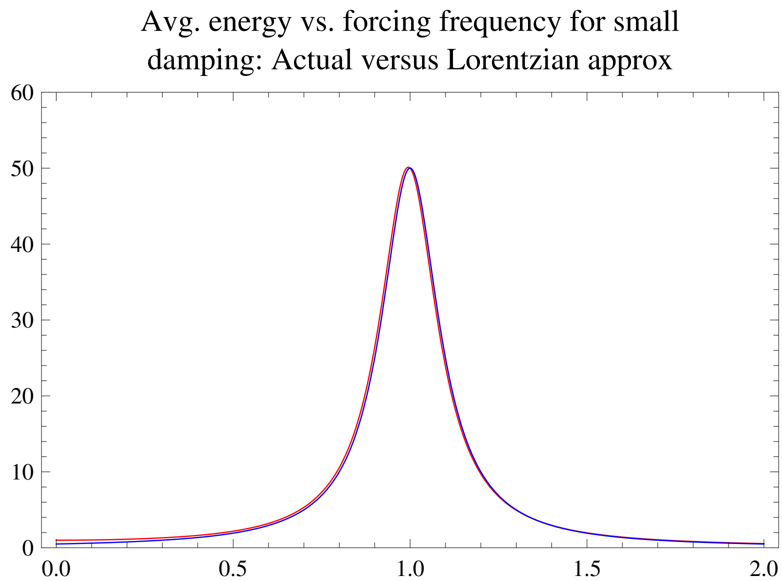

FIG. 6. Average energy vs forcing frequency

It can be checked that the above function has an exact maximum at $\Omega = \sqrt{\omega_0(2\omega - \omega_0)}$ which is close to ω_0 for small damping. The FWHM of this curve, for small damping is 2β. For small β it is possible to make an approximation as follows:

$$\left(\omega_0^2 - \Omega^2\right)^2 = (\omega_0 + \Omega)^2 (\omega_0 - \Omega)^2 \approx 4\omega_0^2 (\Omega - \omega_0)^2 \tag{9}$$

The average energy, under this approximation becomes

$$E^f_{avg} = \frac{mf_0^2}{8}\frac{1}{(\omega_0 - \Omega)^2 + \beta^2} \tag{10}$$

which is nothing but a **Lorentzian function** peaked at ω_0. The figures (Fig. 5 and 6) show how close the average energy is to the Lorentzian when we decrease the value of β.

In a similar manner, the average power drawn by the oscillator from the forcing agency can also be calculated as follows. The instantaneous power is given as:

$$P(t) = \frac{dE}{dt} = F(t)\dot{x} = -\Omega F(t) A_f \sin(\Omega t + \phi) \tag{11}$$

Using the expressions for A_f and ϕ one finds that the average power is

$$\langle P \rangle(\Omega) = -\frac{\Omega F_0 A_f \sin\phi}{2} = \frac{mf_0^2 \beta \Omega^2}{(\omega_0^2 - \Omega^2)^2 + 4\beta^2\Omega^2} \tag{12}$$

This function has an **exact** maximum at $\Omega = \omega_0$. For small β, we can, as before, make a Lorentzian approximation, which gives

$$\langle P \rangle(\Omega) = \frac{mf_0^2 \beta}{4\left[(\omega_0 - \Omega)^2 + \beta^2\right]} \tag{13}$$

This has a FWHM equal to 2β. The graphs are shown in Fig 7. Note the closeness to the Lorentzian.

Finally, let us look at the interesting situation where $\beta = 0$. The differential equation is

$$\ddot{x} + \omega_0^2 x = f_0 \cos\Omega t \tag{14}$$

We can solve this equation using standard methods (variation of parameters or complex method) to get a solution of the form

$$x = A\cos\omega_0 t + B\sin\omega_0 t + \frac{f_0}{\omega_0^2 - \Omega^2}\cos\Omega t \tag{15}$$

which is nothing but the $\beta = 0$ limit of the general solution quoted before. Note that there are no terms here which can become small at late times because there is no damping.

An important special case is that when $\Omega = \omega_0$. Once again, solving the equation using the method of variation of parameters one finds

$$x(t) = x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t + \frac{f_0}{2\omega_0} t \sin\omega_0 t \tag{16}$$

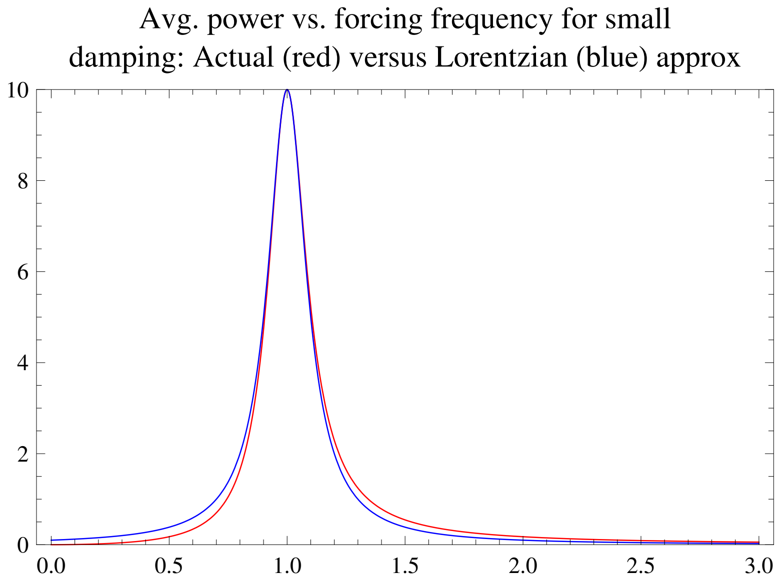

FIG. 7. Average power vs forcing frequency

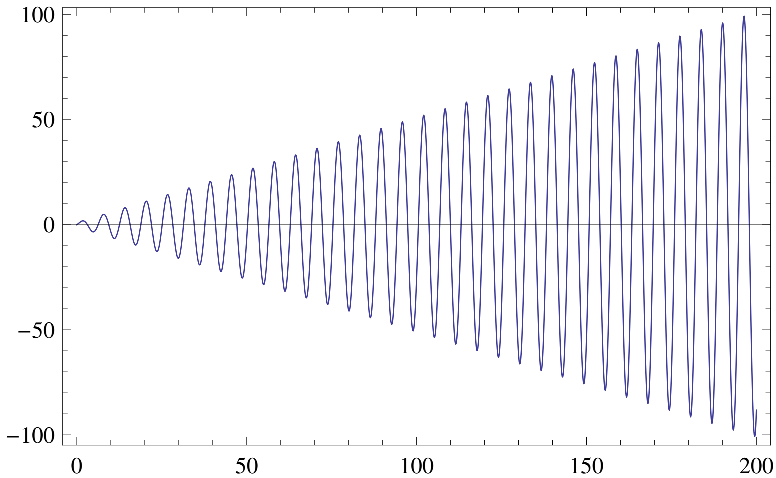

FIG. 8. $\beta = 0$, $\Omega = \omega_0$ case, continuous growth of amplitude

If we take $x_0 = 0$ then we get

$$x(t) = \left(\frac{v_0}{\omega_0} + \frac{f_0 t}{2\omega_0}\right)\sin\omega_0 t \tag{17}$$

Thus the oscillation amplitude **grows indefinitely** with time! This is shown in Fig. 8.

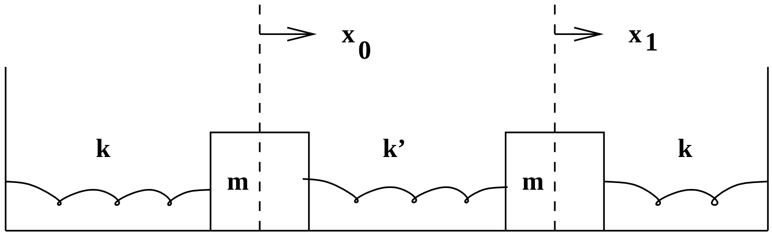

FIG. 9. Coupled mass-spring system

Coupled oscillations: We now turn to situations where there are more than one systems which are coupled to each other. This yields interesting novelties. The central question is related to –**what does the coupling do?**

The simplest such system is that of two mass-spring systems coupled to each other by a third spring. The configuration is shown in the Fig. 9. A similar set-up involves two coupled pendula –**you will work on this experiment in your lab class**.

The equations of motion for the two coordinates x_0 and x_1 are given as:

$$m\ddot{x}_0 = -kx_0 - k'\left(x_0 - x_1\right) \tag{18}$$

$$m\ddot{x}_1 = -kx_1 - k'\left(x_1 - x_0\right) \tag{19}$$

These two equations are a pair of **coupled, linear ordinary differential equations.** One can add and subtract these two equations to obtain two uncoupled equations for $q_0 = \frac{x_0+x_1}{2}$ and $q_1 = \frac{x_0-x_1}{2}$. The uncoupled equations are

$$\ddot{q}_0 = -\omega_0^2 q_0 \quad ; \quad \ddot{q}_1 = -\omega_1^2 q_1 \tag{20}$$

which can be solved easily to give

$$q_0 = A_0 \cos\left(\omega_0 t + \phi_0\right) \quad ; \quad q_1 = A_1 \cos\left(\omega_1 t + \phi_1\right) \tag{21}$$

where $\omega_0^2 = \frac{k}{m}$ and $\omega_1^2 = \frac{k+2k'}{m}$. Hence, finally we have

$$x_0 = A_0 \cos\left(\omega_0 t + \phi_0\right) + A_1 \cos\left(\omega_1 t + \phi_1\right) \tag{22}$$

$$x_1 = A_0 \cos\left(\omega_0 t + \phi_0\right) - A_1 \cos\left(\omega_1 t + \phi_1\right) \tag{23}$$

If we assume initial conditions $x_0(0) = a$, $x_1(0) = b$, $\dot{x}_0(0) = v_0$ and $\dot{x}_1(0) = v_1$, then we get the following relations between a, b, v_0, v_1 and A_0, A_1, ϕ_0, ϕ_1.

$$A_0 \cos\phi_0 + A_1 \cos\phi_1 = a$$

$$
\begin{aligned}
A_0 \cos\phi_0 - A_1 \cos\phi_1 &= b \\
\omega_0 A_0 \sin\phi_0 + \omega_1 A_1 \sin\phi_1 &= -v_0 \\
\omega_0 A_0 \sin\phi_0 - \omega_1 A_1 \sin\phi_1 &= -v_1
\end{aligned} \tag{24}
$$

This is the general solution.

We now turn towards a special class of solutions know as the **normal modes**. In such **normal modes** the coupled system oscillates at a **single frequency** as if it were a **single system**. This, of course, requires a specific choice of **initial conditions**. For example, if we want the system to oscillate at ω_0 then we must have $A_1 = 0$. This implies, from the above, the requirement that $x_0(0) = x_1(0)$ and $v_0(0) = v_1(0)$. Similarly, for the system to oscillate at ω_1 we need $A_0 = 0$ and therefore $x_0(0) = -x_1(0)$ and $v_0(0) = -v_1(0)$. Physically, in the ω_0 normal mode, the initial conditions are such that both the masses are equally displaced with equal velocities (along the same direction). In the ω_1 normal mode, the initial conditions are such that the masses are equally displaced with equal velocities (but along opposite directions). q_0 and q_1 are known as the **normal coordinates** of the system.

It is better to learn a general method for finding the normal modes of a given system. We illustrate it for ths system discussed above. One can easily generalise for more complicated systems. The general method involves assuming solutions for the 'complex' quantities z_0 and z_1 for the complexified equations of motion, in the form,

$$
z_0(t) = Ae^{i\omega t} \quad ; \quad z_1(t) = Be^{i\omega t} \tag{25}
$$

where the physical coordinate will be the real (or imaginary) part of the complex ones defined above. A, B are not functions of time. Let us now substitute the above into the equations of motion for z_0 and z_1. This gives, after some simple rearrangements,

$$
\begin{aligned}
\left(-m\omega^2 + k + k'\right) A - k'B &= 0 \\
-k'A + \left(-m\omega^2 + k + k'\right) B &= 0
\end{aligned} \tag{26}
$$

These are two algebraic equations in A and B. For a nontrivial solution to exist the coefficient determinant must vanish. Thus, we have

$$
\begin{vmatrix} -m\omega^2 + k + k' & -k' \\ -k' & -m\omega^2 + k + k' \end{vmatrix} = 0 \tag{27}
$$

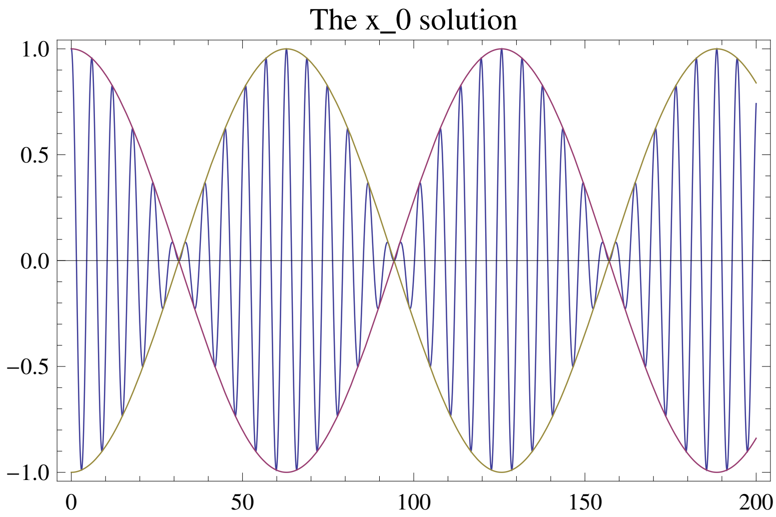

FIG. 10. The x_0 solution

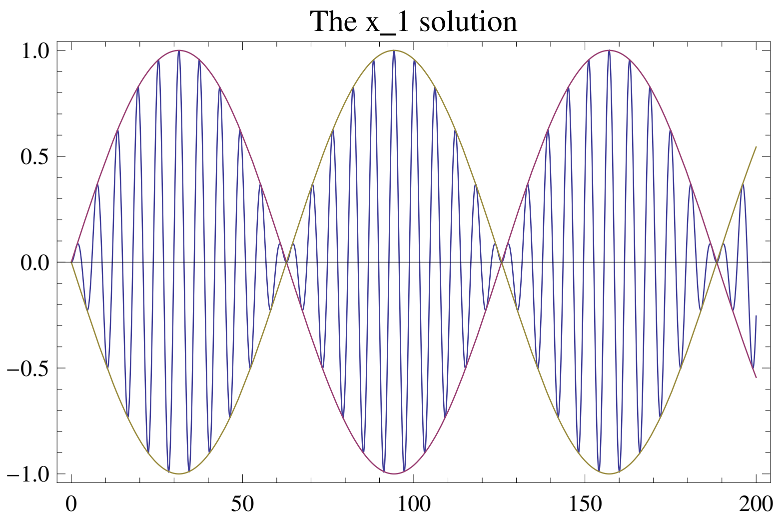

FIG. 11. The x_1 solution

which yields, on solving,

$$\omega_0^2 = \frac{k}{m} \quad ; \quad \omega_1^2 = \frac{k+2k'}{m} \tag{28}$$

The above method can be easily generalised to systems with more than two degrees of freedom like a system of say three masses and four springs, etc. The general procedure of finding the normal modes is a simple extrapolation of what we have done here.

Finally, let us look at a case when $\phi_0, \phi_1 = 0$ and $b = 0, v_1 = 0$. Physically this means that we have initially displaced one mass with a velocity v_0 and the other mass is left untouched.

The solutions now become,

$$x_0 = \frac{a}{2}\left(\cos\omega_0 t + \cos\omega_1 t\right) = a\cos\frac{\omega_0+\omega_1}{2}t\ \cos\frac{\omega_1-\omega_0}{2}\ t \tag{29}$$

$$x_1 = \frac{a}{2}\left(\cos\omega_0 t - \cos\omega_1 t\right) = a\sin\frac{\omega_0+\omega_1}{2}t\ \sin\frac{\omega_1-\omega_0}{2}\ t \tag{30}$$

It is easy to note that at $t = \frac{(2n+1)\pi}{\omega_0+\omega_1}$, x_0 becomes zero and x_1 is nonzero. The opposite happens when $t = \frac{2n\pi}{\omega_0+\omega_1}$. Thus there is an exchange of energy between the two masses. This is also known as **resonance**. Note that this effect is due to the coupling. The figures (Fig. 10 and 11) show the graphs of $x_0(t)$ and $x_1(t)$ for this case. If one takes the case when ω_0 and ω_1 are close, then we may write $\omega_0+\omega_1 \approx 2\omega_0$ and $\omega_1-\omega_0 = 2\Delta\omega$. Then, we can see that the oscillation with frequency ω_0 has an **amplitude modulation** at a lower frequency $\Delta\omega$, also known as the **beat** frequency.

Finally, let us write down what the equations would be if we had several masses connected by springs– a chain of masses and springs. If we assume that all masses and spring constants are all equal and the displacement of the i-th mass is $\xi_i(t)$ then its equation of motion is

$$m\ddot{\xi}_i(t) = -k\left(\xi_i - \xi_{i-1}(t)\right) - k\left(\xi_i(t) - \xi_{i+1}(t)\right) \tag{31}$$

We shall deal with this case again in the next lectures.