Tutorial 1: Solutions

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1. Find the **frequency of oscillation** of the following simple systems (see the Fig 1.).

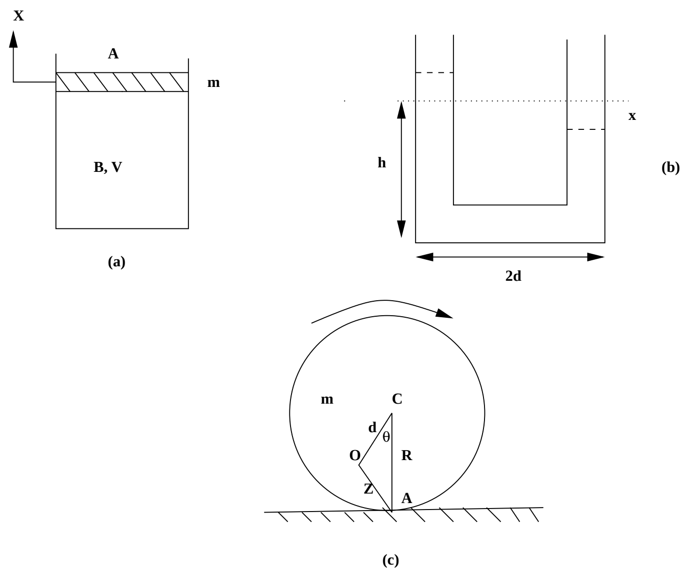

FIG. 1: Figure for Problem 1

* (a) A piston closes a cylinder containing gas. If B is the bulk modulus of elasticity of the gas, V is the volume of the gas and A is the cross sectional area of the piston, then find the frequency of longitudinal vibrations of the piston. Distinguish between the frequencies for isothermic or adiabatic processes.

(b) A liquid is kept in a U-shaped tube. If h is the equilibrium height of liquid in both arms and $2d$ is the separation between the two arms of the U tube then find the equation of motion for the vertical displacement x of the liquid surface. Also find the frequency of oscillation from the equation of motion.

(c) A non-uniform cylinder rolls on a rough surface. If I_O is the moment of inertia about O (centre of mass of cylinder) then find the frequency of small oscillations.

(a) The bulk modulus of a gas is defined as:

$$B = -\frac{\Delta P}{\Delta V/V} \tag{1}$$

where V is the equilibrium volume, ΔP is the change in pressure over the equilibrium pressure and ΔV is the corresponding change in volume. In our case $\Delta P = P - P_0$ is the acoustic pressure, i.e. the excess of the instantaneous pressure P over the equilibrium atmospheric pressure P_0. If A is the cross section of the piston and $x = x(t)$ its displacement from equilibrium, we have the equation of motion of the piston as:

$$m\ddot{x} = A\Delta P = -AB\frac{\Delta V}{V} = -\frac{BA^2}{V}x \tag{2}$$

since $\Delta V = Ax$. Therefore the frequency of oscillation is

$$\omega^2 = \frac{BA^2}{mV} \tag{3}$$

If the process is isothermic then PV =constant and $P_0\Delta V = V\Delta P = 0$. and $B = P_0$. Therefore,

$$\omega^2 = \frac{P_0A^2}{mV} \tag{4}$$

If the process is adiabatic then PV^γ =constant and $\gamma P_0 V^{\gamma-1}\Delta V + V^\gamma \Delta P = 0$, which leads to $B = \gamma P_0$ and a frequency given as

$$\omega^2 = \frac{\gamma P_0 A^2}{mV} \tag{5}$$

(b) At any cross-section of the tube, the excess force due to a vertical density of the displacement x of the surface of the liquid is $-2\rho Axg$ (ρ is the density, A is the cross sectional area). This excess force accelerates the whole mass of liquid, which is $2\rho A(h + d)$. Hence, the equation of motion is

$$-2\rho Agx = 2\rho A(h + d)\ddot{x} \tag{6}$$

Therefore, the frequency of oscillation is

$$\omega^2 = \frac{g}{h + d} = \frac{2g}{l} \tag{7}$$

where $l = 2(h + d)$ is the length of the column of liquid in the tube.

(c) Note the word *non-uniform* here. Taking moments about point A (see Figure 1), the instantaneous axis of rotation, we have by Newton's second law for rotational motion,

$$-mgd\sin\theta - I_A\ddot{\theta} \tag{8}$$

By the parallel axis theorem, $I_A = I_O + mz^2$, so

$$-mgd\sin\theta = \left(I_O + mz^2\right)\ddot{\theta} = \left[I_O + m(R^2 + d^2 - 2Rd\cos\theta)\right]\ddot{\theta} \tag{9}$$

For small angles, $cos\theta \sim 1$, $\sin\theta \sim \theta$ and thus

$$-mgd\theta = \left[I_O + m(R-d)^2\right]\ddot{\theta} \tag{10}$$

Hence, the frequency of oscillation is

$$\omega^2 = \frac{mgd}{I_O + m(R-d)^2} \tag{11}$$

2. (a) A simple harmonic oscillator has position x_0 and velocity v_0 at $t = 0$. Calculate the complex amplitude in terms of the initial conditions and use it to determine the particle's position $x(t)$ later.

* (b) For a simple harmonic motion find $\sqrt{\langle x^2 \rangle}$ and $\langle x^4 \rangle$.

* (c) A particle of mass $0.3kg$ moves in a potential $V(x) = 2e^{a^2x^2}$. It behaves like a simple harmonic oscillator for small displacements. Plot the potential qualitatively, in the domain $-\infty \leq x \leq \infty$, choosing $a = 1$ and $a = 2$. Find the period of small oscillations.

(a) For simple harmonic oscillator, having the angular frequency ω_0, the complex representation of its position, at any time t, is $\tilde{x}(t) = \tilde{A}e^{i\omega_0 t}$, where, $\tilde{A}$ is the complex amplitude and the actual expression of the position $x(t)$ is the real part of its complex counterpart, $i.e.$ $x(t) = Re\{\tilde{x}(t)\}$. Thus, its velocity is $v(t) = \dot{x}(t) = Re\{\dot{\tilde{x}}(t)\} = Re\{i\omega_0\tilde{A}e^{i\omega_0 t}\}$. Now $x(t = 0) = x_0$ and $v(t = 0) = v_0$. So $Re\{\tilde{A}\} = x_0$ and $Re\{i\omega_0\tilde{A}\} = v_0$. Assume $\tilde{A} = u + iv$, where u, v are real. Then $u = x_0$ and $-\omega_0 v = v_0$. So,

$$\tilde{A} = x_0 - i\frac{v_0}{\omega_0} \tag{12}$$

Therefore, the position of the particle, at later time, is

$$\begin{aligned} x(t) &= Re\{(x_0 - iv_0/\omega_0)e^{i\omega_0 t}\} \\ &= x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t \end{aligned} \tag{13}$$

(b) For simple harmonic motion (SHM),

$$x(t) = A\cos(\omega t + \phi)$$

where, A is the amplitude and ω is the angular frequency and ϕ is the phase. Now

$$< x^2(t) >= \frac{1}{T}\int_0^T x^2(t)dt \tag{14}$$

where, $T = 2\pi/\omega$ is the time period of the SHM. So

$$\begin{aligned} < x^2(t) > &= \frac{1}{T}\int_0^T A^2\cos^2(\omega t + \phi)dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} A^2\cos^2(\theta + \phi)d\theta \qquad ,(\theta = \omega t) \\ &= \frac{A^2}{4\pi}\int_0^{2\pi} [1 + \cos(2\theta + 2\phi)]d\theta \\ &= \frac{A^2}{2} \end{aligned}$$

So $\sqrt{< x^2 >} = A/\sqrt{2}$.

$$\begin{aligned} x^4 &= A^4\cos^4(\omega t + \phi) \\ &= \frac{A^4}{4}[1 + \cos(2\omega t + 2\phi)]^2 \\ &= \frac{A^4}{4}[1 + \cos^2(2\omega t + 2\phi) + 2\cos(2\omega t + 2\phi)] \end{aligned}$$

Now the first term in the square bracket is a constant, $< \cos^2(2\omega t + 2\phi) >= 1/2$ (from previous calculation) and $< \cos(2\omega t + 2\phi) >= 0$. So, $< x^4 >= A^4(1 + 1/2)/4 = 3A^4/8$.

(c) At $x = 0$, $V(x) = 2$ for both $a = 1$ and $a = 2$. For any a, $V(x)$ increases as x increases from 0, or x decreases from 0 and $V(x) = V(-x)$, i.e. $V(x)$ is symmetric about $x = 0$. For any particular value of x, $V(x)|_{a=2} > V(x)|_{a=1}$; so the curve $(V - x)$ for $a = 2$ is steeper than that for $a = 1$ and these two curves never intersect each other. Therefore, the qualitative plot of the curves look like Fig.2.

About $x_0 = 0$ (where the potential is minimum), for small x, $V(x) = 2e^{a^2x^2} \approx 2\,(1 + a^2x^2)$. So, the force $F = -\frac{dV}{dx} = -4a^2x$. Hence, the differential equation for the harmonic oscillator with a particle of mass m is

$$\ddot{x} + \left(\frac{4a^2}{m}\right)x = 0 \tag{15}$$

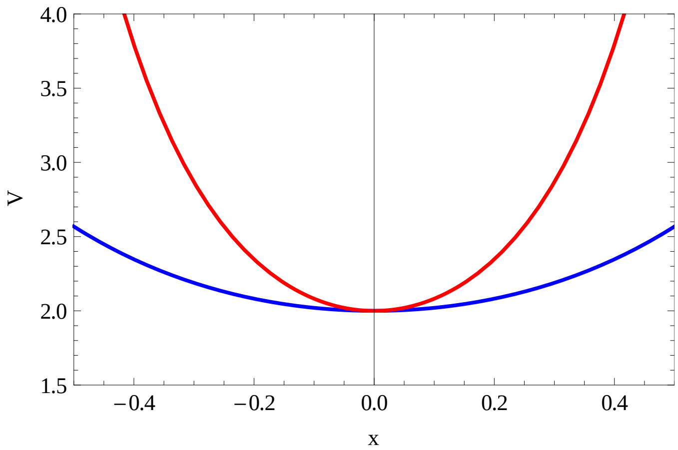

FIG. 2: Plot of the potential $V(x)$ in problem $2(c)$. Blue line for $a = 1$ and red line for $a = 2$.

and the period of oscillation

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{4a^2}} \tag{16}$$

For $a = 1$, $T = 1.72\,sec$ and for $a = 2$, $T = 0.86\,sec$.

3. * (a) A manufacturers specifications for the coil spring for the front suspension of a sports car requires a spring of 10 coils with a relaxed length of 0.316 m, and a length of 0.205 m when under a load of 399 kg. What is the **spring constant**? If the spring is cut into two equal pieces, what will be the spring constant of each piece? How does the spring constant vary with the number of coils n, in the spring ?
(b) A molecule of DNA is 2.17 micrometers long. The ends of the molecule become singly ionized – negative on one end, positive on the other. The helical molecule acts like a spring and compresses 1.00 % upon becoming charged. Determine the **effective spring constant** of the molecule.

(a)From Hooke's law we know

$$\Delta F = k\Delta x \tag{17}$$

Where ΔF is the change in the applied force, k is the spring constant and Δx is the change in length of the spring. Here $\Delta F = 399 \times 9.81N$ and $\Delta x = 0.111m$. So,

$$k = \frac{399 \times 9.81}{0.111} Nm^{-1} = 35.26 \times 10^3 Nm^{-1} \tag{18}$$

If the spring is cut into two equal pieces then suppose that k' is the spring constant of each of the two pieces. Then in Eq. (17), $\Delta x = 2\Delta x'$ where, $\Delta x'$ is change in the length of each spring for the same ΔF. Now $\Delta x = \Delta F/k$ and $\Delta x' = \Delta F/k'$. So $k' = 2k = 70.52 \times 10^3\, Nm^{-1}$

Suppose that each coil in the spring has the same spring constant k_0 and each one has change in length Δx_0 for the same ΔF in the Eq. (17). Then, for n coils in the spring, $\Delta x = n\Delta x_0 = n\Delta F/k_0$. But $\Delta x = \Delta F/k$. So, $k = k_0/n$. So the spring constant $k \propto 1/n$.

(**b**) As the helical molecule compresses 1.00% after becoming charged, so the final distance between them is

$$x_f = (2.17 - 0.0217) \times 10^{-6} = 2.1483 \times 10^{-6} m \tag{19}$$

Using Coulomb's law we can find the force between a positive and a negative charge when the separation between them is 2.1483 microns,

$$F_{coulomb} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{x_f^2} \tag{20}$$

where, e is the charge of the electron. In equilibrium, this force is balanced by the restoring force of the equivalent compressed spring. From Hooke's law we can find it as

$$F_{spring} = k_{eff}\Delta x \tag{21}$$

where $\Delta x = 0.0217 \times 10^{-6}\, m$. So the **effective spring constant** of the molecule is

$$\begin{aligned} k_{eff} &= \frac{1}{4\pi\epsilon_0}\frac{e^2}{x_f^2 \Delta x} \\ &\approx 2.3 \times 10^{-15}\, N/\mu m \end{aligned} \tag{22}$$

4.* A flat plate moves at a velocity V along the direction normal to its planar surface, through a gas at low pressure. Show that the drag force on the plate is proportional to the speed (**linear in speed** damping).

If the pressure of the gaseous medium, through which the flat plate is moving, is so low that the rate of collision of the gas molecules between themselves is much smaller than the rate of

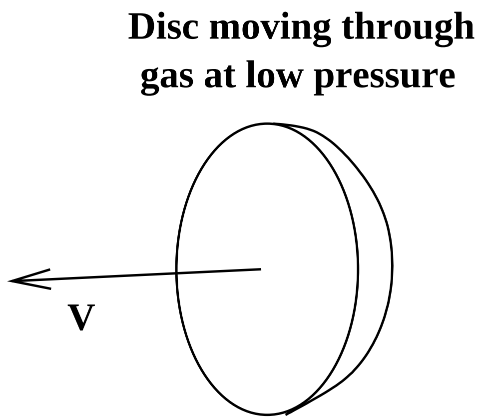

FIG. 3: Figure for Problem 4

collision of the gas molecules with the flat plate, then we may assume that the gas molecules only collide with the flat plate. For simplification, we further assume that the gas molecules are constrained to move in one direction, normal to the flat plate, with uniform speed (v) but randomly either in positive or negative direction. This means that the molecules are colliding with the plate on both sides of the plate. Thus, in one side, the gas molecules have the speed $v+V$ relative to the plate and on the other side, these have the relative speed $v-V$. The rate of collision is proportional to the relative speed. Again after the collision, the momentum transfer to the plate is also proportional to the relative speed. Therefore, the pressure on one side (on the left hand side of the figure), $P_1 \propto (v+V)^2$ and on the other side $P_2 \propto (v-V)^2$. These two pressure are acting in the opposite direction. So, the net pressure on the flat plate

$$P = P_1 - P_2 \propto 4vV$$

The direction of net pressure or net drag force is opposite to the motion of the flat plate. Thus, the drag force on the plate is proportional to the velocity V of the flat plate.

5. (a) If an iron ball of weight $W = 98N$ stretches a spring by 1.09 m, how many cycles per minute will the mass-spring system execute? What will its motion be if we pull down the weight an additional 16 cm and let it start with zero initial velocity?

(b) In the above problem, if we consider the effect of damping and change the damping constant β to one of the following three values with $y(0) = 0.16$ and $\dot{y}(0) = 0$, how does the motion change? (i) $\beta = 100$ kg/sec; (ii) $\beta = 60$ kg/sec (iii) $\beta = 10$ kg/sec.

(**a**) From Hooke's law we get

$$\Delta F = k\Delta x \tag{23}$$

where ΔF is the change in the applied force, k is the spring constant and Δx is the change in length of the spring. Here $\Delta F = 98N$ and $\Delta x = 1.09m$. Therefore, the spring constant,

$$k = \frac{98}{1.09} Nm^{-1} = 89.91 \simeq 90Nm^{-1} \tag{24}$$

Mass of the iron ball is $m = \frac{98}{9.81} = 9.99 \simeq 10kg$ Hence the frequency of the oscillation

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{90}{10}} = 3\, rad/sec \tag{25}$$

So, the number of cycles executed by the spring-mass system per minute will be $\frac{\omega}{2\pi} \times 60 = \frac{3\times 60}{2\pi} = 28.65 \approx 28$.

If we pull down the weight an additional $16cm$, and let it start with zero initial velocity, then it will execute simple harmonic oscillation with the angular frequency $\omega = 3rad/sec$. Its motion can be described by the second order differential equation,

$$\ddot{y} + 9y = 0 \tag{26}$$

It has the solution of the form

$$y = A\cos 3t + B\sin 3t \tag{27}$$

where A, B are constants. Using the initial conditions $y(0) = 0.16, \dot{y}(0) = 0$, we can find $A = 0.16, B = 0$. So the motion of the system is given by

$$y = 0.16\cos 3t \tag{28}$$

(**b**) Damped oscillation of a spring-mass system with spring constant k and damping constant β can be represented by the differential equation of the form

$$m\ddot{y} + \beta\dot{y} + ky = 0 \tag{29}$$

or

$$\ddot{y} + 2K\dot{y} + \omega_0^2 y = 0 \tag{30}$$

where $2K = \frac{\beta}{m}$ and $\omega_0 = \sqrt{\frac{k}{m}} = 3rad/sec$. We introduce a new variable $\xi(t)$ as

$$y(t) = \xi(t)e^{-Kt} \tag{31}$$

Using the above expression, we can get

$$\ddot{\xi} + \left(\omega_0^2 - K^2\right)\xi(t) = 0 \tag{32}$$

(i)For $\beta = 100kg/sec$,

$$K = \frac{\beta}{2m} = \frac{100}{2 \times 10} = 5sec^{-1}$$

So, $(\omega_0^2 - K^2) = 9 - 25 = -16$ and the differential equation takes the form

$$\ddot{\xi} - 16\xi(t) = 0 \tag{33}$$

It has the solution of the form

$$\xi(t) = Ae^{4t} + Be^{-4t} \tag{34}$$

Thus,

$$y(t) = e^{-5t}\left(Ae^{4t} + Be^{-4t}\right) \tag{35}$$

or

$$y(t) = Ae^{-t} + Be^{-9t} \tag{36}$$

where A, B are constants. Using the initial conditions $y(0) = 0.16, \dot{y}(0) = 0$, we can find $A = 0.18, B = -0.02$ .So,

$$y(t) = 0.18e^{-t} - 0.02e^{-9t} \tag{37}$$

The displacement decreases exponentially to zero and the motion is said to be overdamped.
(ii)For $\beta = 60kg/sec$,

$$K = \frac{\beta}{2m} = \frac{60}{2 \times 10} = 3sec^{-1}$$

So, $(\omega_0^2 - K^2) = 9 - 9 = 0$ and the differential equation takes the form

$$\ddot{\xi} = 0 \tag{38}$$

The solution is given by

$$\xi(t) = Ct + D \tag{39}$$

where C, D are constants.Thus,

$$y(t) = (Ct + D)\,e^{-3t} \tag{40}$$

Using the initial conditions $y(0) = 0.16, \dot{y}(0) = 0$, we can find $C = 0.48, D = 0.16$ Hence,

$$y(t) = (0.48t + 0.16)\, e^{-3t} \tag{41}$$

The motion is non-oscillatory and corresponds to critical damping.

(iii)For $\beta = 10kg/sec$,

$$K = \frac{\beta}{2m} = \frac{10}{2 \times 10} = 0.5sec^{-1}$$

So, $(\omega_0^2 - K^2) = 9 - 0.25 = 8.75$ and the differential equation takes the form

$$\ddot{\xi} + 8.75\xi(t) = 0 \tag{42}$$

The solution will be of the form

$$\xi(t) = P\cos\, 2.96t + Q\sin\, 2.96t \tag{43}$$

where P, Q are constants. So,

$$y(t) = (P\cos\, 2.96t + Q\sin\, 2.96t)\, e^{-0.5t} \tag{44}$$

Using the initial conditions $y(0) = 0.16, \dot{y}(0) = 0$, we can find $P = 0.16, Q = 0.03$ Hence,

$$y(t) = (0.16\cos\, 2.96t + 0.03\sin\, 2.96t)\, e^{-0.5t} \tag{45}$$

The amplitude of the oscillation decreases exponentially with time and the time period of the vibration is greater than that in the absence of damping.

6. You are given the following: (i) a conductor in the shape of a square frame (ii) an elastic thread (iii) a device that produces a uniform magnetic field.

(a) Make an arrangement which will exhibits oscillations.

(b) Analyse the behaviour of the oscillatory system you have constructed using your knowledge of oscillations.

(**a**) We can arrange the three components like the figure below. The square conducting loop will hang from a rigid roof by the elastic thread. Uniform magnetic field passes through the conducting square loop. In the absence of the magnetic field, the loop will perform torsional oscillation, due to the elasticity of the thread, about the axis of suspension. But in the presence of the magnetic field, there will be associated magnetic flux passing through the loop. As the loop oscillates, this magnetic flux will be changed and that will induce an electromotive force in the conducting loop and this will oppose the oscillatory motion of the loop. Hence the magnetic field will generate the damping effect in the torsional oscillation of the loop.

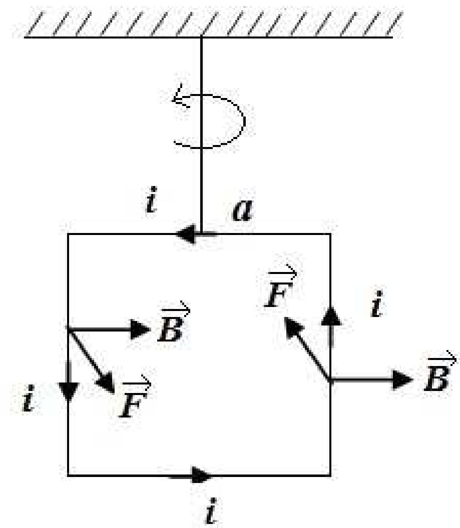

Fig. 2: Figure for problem 6

(**b**) Let us consider the area of the square loop is a^2, where a is the length of its one side and B is the magnitude of the uniform magnetic field. If κ is the torsional constant of the elastic thread, then the torque due to the torsion θ is

$$\tau_c = -\kappa\theta \tag{46}$$

where θ is angle of rotation of the loop about the axis of suspension. If I is the moment of inertia of the square conducting loop, then the equation of motion for torsional oscillation of the square loop is

$$I\ddot{\theta} = -\kappa\theta \tag{47}$$

Hence the angular frequency of the torsional oscillation is $\omega_0 = \sqrt{\kappa/I}$. Now the magnetic flux passing through the square loop is $\Phi = \vec{B} \cdot \vec{A}$, where $\vec{A}$ is an area vector which has the magnitude a^2 and the direction specified by the normal to the area of the loop. Since the angle of rotation is θ, then $\Phi = a^2 B \cos(90° - \theta) = a^2 B \sin\theta$. So the induced electromotive force $\varepsilon = -\frac{d\Phi}{dt} \approx -Ba^2\dot{\theta}$ (for small θ). Hence the induced current flowing in the loop is $i = -Ba^2\dot{\theta}/R$, where R is the resistance of the conducting loop. Now this conducting current will generate a magnetic moment $\vec{m} = i\vec{A}$. The torque acting on the square loop due to this induced magnetic moment is

$$\begin{aligned} \vec{\tau}_m &= \vec{m} \times \vec{B} \\ \text{so, } \tau_m &= -\frac{a^4B^2\dot{\theta}}{R}\cos\theta \\ &\approx -\frac{a^4B^2}{R}\dot{\theta}(\text{for small } \theta) \end{aligned} \tag{48}$$

So, in the presence of the uniform magnetic field, the equation of the oscillatory motion of the square conducting loop is

$$\ddot{\theta} + \frac{a^4B^2}{IR}\dot{\theta} + \omega_0^2\theta = 0 \tag{49}$$

This is the equation of a damped SHM.

7. A particle of mass m is free to move in the xy plane (**two dimensions**) under the action of a harmonic force $\vec{\mathbf{F}} = -k\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}\right) = -k\vec{\mathbf{r}}$.

* (a) Find the x, y equations of motion and solve them.

* (b) What are the conditions for motion in a circle and what is the period?

(c) What are the conditions for motion along the line at 45 degrees with the x axis and what is the period?

(**a**) If the particle has mass m, its equation of motion is

$$m\ddot{\vec{\mathbf{r}}} = \vec{\mathbf{F}} = -k\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}\right) = -k\vec{\mathbf{r}} \tag{50}$$

or since $\vec{\mathbf{r}} = \left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}\right)$, $\left(m\ddot{x}\hat{\mathbf{i}} + m\ddot{y}\hat{\mathbf{j}}\right) = \left(-kx\hat{\mathbf{i}} - ky\hat{\mathbf{j}}\right)$

Then

$$m\ddot{x} = -kx, m\ddot{y} = -ky \tag{51}$$

These equations have the solutions of the forms

$$x = A\cos\omega t + B\sin\omega t \tag{52}$$

$$y = C\cos\omega t + D\sin\omega t \tag{53}$$

where

$$\omega^2 = \sqrt{\frac{k}{m}} \tag{54}$$

Now we consider that, at $t = 0$, the particle is at the point $\vec{\mathbf{r}} = \left(a\hat{\mathbf{i}} + b\hat{\mathbf{j}}\right)$ and its velocity is $\dot{\vec{\mathbf{r}}} = \left(\upsilon_{\mathbf{1}}\hat{\mathbf{i}} + \upsilon_{\mathbf{2}}\hat{\mathbf{j}}\right)$. Using these conditions, we find $A = a, B = b, C = \upsilon_1\sqrt{\frac{m}{k}}$ and $D = \upsilon_2\sqrt{\frac{m}{k}}$. Therefore

$$x = a\cos\omega t + b\sin\omega t \tag{55}$$

$$y = c\cos\omega t + d\sin\omega t \tag{56}$$

where $C = c = \upsilon_1\sqrt{\frac{m}{k}}$, $D = d = \upsilon_2\sqrt{\frac{m}{k}}$.

(**b**) From the above equations we can find

$$\cos\omega t = \frac{dx - cy}{ad - bc}, \sin\omega t = \frac{ay - bx}{ad - bc} \tag{57}$$

Using the formula $\cos^2\omega t + \sin^2\omega t = 1$, we find

$$(dx - cy)^2 + (ay - bx)^2 = (ad - bc)^2 \tag{58}$$

or

$$(b^2 + d^2)x^2 - 2(cd + ab)xy + (a^2 + c^2)y^2 = (ad - bc)^2 \tag{59}$$

The above equation reduces to an equation of a circle if $(ad - bc) \neq 0$, $(b^2 + d^2) = (a^2 + c^2)$ and $(cd + ab) = 0$.

Only in the following situations, these conditions hold:

(i) $a = d = 0\,, b = c \neq 0$ **; or** $b = c = 0\,, a = -d \neq 0$ [**Right circularly (clockwise)**]

(ii) $a = d = 0\,, b = -c \neq 0$ **; or** $b = c = 0\,, a = d \neq 0$ [**Left circularly(anti-clockwise)**]

The period of oscillation is

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{60}$$

(**c**) For the motion along the line at 45 degrees with the x axis, the equation is $y = x$. So the conditions are $a = c \neq 0\,, b = d = 0$; or $a = c = 0\,, b = d \neq 0$. The period of oscillation will be same as before.

** **8.** A very simple model of price changes in time due to market demands (used in standard textbooks on economic dynamics) is constructed as follows. Let $P(t)$ be the price of a commodity at time t. The demanded quantity Q_d and the supplied quantity Q_s is given as follows, as a function of P, $\dot{P}$ and $\ddot{P}$ (all constants below, i.e. α_i, β_i are real and positive).

$$Q_d = \alpha_0 - \alpha_1 P + \alpha_2 \dot{P} - \alpha_3 \ddot{P} \tag{61}$$

$$Q_s = \beta_0 + \beta_1 P - \beta_2 \dot{P} + \beta_3 \ddot{P} \tag{62}$$

The expression for Q_d is explained as: demand falls when price goes up, demand grows when the market perceives that prices have an upward tendency and falls when this tendency is decreasing.

(a) Write down an explanation for the Q_s following the one for Q_d given above.

(b) The dynamics of price is give by the equation ($\lambda > 0$)

$$\dot{P} = \lambda \left(Q_d - Q_s\right) \tag{63}$$

Using the Q_d and Q_s above write down the differential equation for the price $P(t)$. Which equation does your equation resemble? When will the price show an oscillatory behaviour? *Optional: You may try to solve the equation and analyse the dynamics of price of a commodity in this simple model.*

(**a**) Q_s can be explained as the opposite of Q_d, i.e. supply increases with the increase of price, it decreases when the price has an upward tendency and increases when the tendency is downward.

(**b**)

$$\dot{P} = \lambda \left(Q_d - Q_s\right) \tag{64}$$

Now if we substitute the values of Q_d and Q_s in the above equation, we get

$$\dot{P} = \lambda \left(\alpha_0 - \alpha_1 P + \alpha_2 \dot{P} - \alpha_3 \ddot{P} - \beta_0 - \beta_1 P + \beta_2 \dot{P} - \beta_3 \ddot{P}\right) \tag{65}$$

or

$$\lambda \left(\alpha_3 + \beta_3\right) \ddot{P} + \left[1 - \lambda \left(\alpha_2 + \beta_2\right)\right] \dot{P} + \lambda \left(\alpha_1 + \beta_1\right) P - \lambda \left(\alpha_0 - \beta_0\right) = 0 \tag{66}$$

or

$$\ddot{P} + \frac{\left[1 - \lambda \left(\alpha_2 + \beta_2\right)\right]}{\lambda \left(\alpha_3 + \beta_3\right)} \dot{P} + \frac{\left(\alpha_1 + \beta_1\right)}{\left(\alpha_3 + \beta_3\right)} P - \frac{\left(\alpha_0 - \beta_0\right)}{\left(\alpha_3 + \beta_3\right)} = 0 \tag{67}$$

This equation resembles with the differential equation of the damped oscillation. So, the price will show oscillatory behaviour for

$$\frac{(\alpha_1 + \beta_1)}{(\alpha_3 + \beta_3)} > \left[\frac{[1 - \lambda\left(\alpha_2 + \beta_2\right)]}{2\lambda\left(\alpha_3 + \beta_3\right)} \right]^2 \tag{68}$$

For $\lambda(\alpha_2 + \beta_2) < 1$ the amplitude of the damped oscillation will decrease exponentially and after long time the price will reach a stable value $P_0 = \frac{\alpha_0 - \beta_0}{\alpha_1 + \beta_1}$, but if $\lambda(\alpha_2 + \beta_2) > 1$ then the amplitude of $P(t)$ will increase exponentially in time.

Tutorial 2: Solutions

Solutions prepared by Mr. Soumya Jana (soumyajana.physics@gmail.com)

1. A door shutter has a spring which, in the absence of damping, shuts the door in 0.5 seconds. The problem is that the door bangs shut with a speed 1 m/s. A damper with a damping coefficient β is introduced to ensure that the door shuts gradually. What are the times required for the door to shut and the velocities of the door at the instant it shuts, if $\beta = 0.5\pi$ and $\beta = 0.9\pi$. Note that the spring is unstretched when the door is shut.

The equation of displacement of the door shutter, in the absence of damping, is

$$x(t) = a\cos\omega_0 t + b\sin\omega_0 t$$

When the door is shut, then $x = 0$. The shutter shuts the door in 0.5 seconds; so $T_0/4 = 0.5\,sec$, where T_0 is the time period of oscillation of the shutter attached to the spring. Now imposing the conditions: $\dot{x}(t=0) = 0\,, \dot{x}(t = T_0/4) = -1\,m/s\,, T_0/4 = 2\pi/\omega_0 = 0.5\,sec$, we get $a = \frac{1}{\pi}\,m$ and $b = 0$.

Now, if a damper with a damping coefficient β is introduced, then the equation for $x(t)$ can be expressed as,

$$x(t) = e^{-\beta t}[c\cos\omega t + d\sin\omega t]$$

where $\omega = \sqrt{\omega_0^2 - \beta^2}$. Now imposing the conditions: $x(t=0) = a = \frac{1}{\pi}\,m\,, \dot{x}(t=0) = 0$, the equation becomes

$$x(t) = ae^{-\beta t}\left[\cos\omega t - \frac{\beta}{\omega}\sin\omega t\right]$$

Now, in the presence of damping, if the shutter shuts the door in time t_0, then $x(t_0) = 0$. Using the equation we have

$$\cos\omega t_0 = \frac{\beta}{\omega}\sin\omega t_0$$

$$\text{or,}\quad t_0 = \frac{1}{\omega}\tan^{-1}\left(\frac{\omega}{\beta}\right)$$

The velocity of the door at the instant it shuts is

$$\dot{x}(t = t_0) = -a\omega e^{-\beta t_0}\left(1 + \frac{\beta^2}{\omega^2}\right)\sin\omega t_0$$

For $\beta = 0.5\pi$, $t_0 = 0.38\,sec$, $\dot{x}(t_0) = -0.55\,m/s$ and for $\beta = 0.9\pi$, $t_0 = 0.33$ sec, $\dot{x}(t_0) = -0.39\,m/s$.

2. Practice with 2nd order differential equations:

* (a) Solve the initial value problem set up by the differential equation below:

$$\ddot{x} + 100x = 15\cos 5t + 20\sin 5t \tag{1}$$

where $x(t=0) = 25$ and $\dot{x}(t=0) = 0$.

(b) Find the particular solution $x_p(t)$ in the form $x_p(t) = C\cos(\omega t - \alpha)$ for the following ordinary dierential equation:

$$\ddot{x} + 3\dot{x} + 5x = -4\cos 5t. \tag{2}$$

(a) Note that Eq. (1) is the real part of the following complex equation:

$$\ddot{\tilde{x}} + 100\tilde{x} = 15e^{i5t} - 20ie^{i5t} \tag{3}$$

where x in Eq. (1) is real part of the complex function $\tilde{x}$ [i.e. $x(t) = Re\{\tilde{x}(t)\}$] . The general solution of the second-order differential equation, Eq. (3), is

$$\tilde{x}(t) = \tilde{x_c}(t) + \tilde{x_p}(t) \tag{4}$$

where $\tilde{x}_c(t)$ is *complementary solution* and $\tilde{x}_p(t)$ is *particular solution.* Complementary part $[\tilde{x}_c(t)]$ is the solution of the equation: $\ddot{\tilde{x}}_c + 100\tilde{x}_c = 0$. So the real part of $\tilde{x}_c$ is

$$x_c(t) \equiv Re\{\tilde{x}_c(t)\} = A\cos 10t + B\sin 10t \tag{5}$$

where A and B are two arbitrary real constants.

So, from Eq. (3) and Eq. (4), it follows that $\tilde{x}_p$ satisfies the following equation

$$\ddot{\tilde{x}}_p + 100\tilde{x}_p = 15e^{i5t} - 20ie^{i5t} \tag{6}$$

Assume a trial solution $\tilde{x}_p = \tilde{A}e^{i5t}$, where $\tilde{A}$ is time independent, and substitute it in Eq. (6). Then we have $\tilde{A} = \frac{1}{5} - \frac{4}{15}i$. So

$$\tilde{x}_p = \left(\frac{1}{5} - \frac{4}{15}i\right)e^{i5t} \tag{7}$$

$$x_p(t) \equiv Re\{\tilde{x}_p(t)\} = \frac{1}{5}\cos 5t + \frac{4}{15}\sin 5t \tag{8}$$

So the general solution for the Eq. (1) is

$$x(t) = A\cos\, 10t + B\sin\, 10t + \frac{1}{5}\cos\, 5t + \frac{4}{15}\sin\, 5t \tag{9}$$

Using the initial conditions $x(t=0) = 25$ and $\dot{x}(t=0) = 0$, we determine A and B: $A = \frac{124}{5}$ and $B = -\frac{2}{15}$.

(b) Writing the Eq. (2) in complex form

$$\ddot{\tilde{x}} + 3\dot{\tilde{x}} + 5\tilde{x} = -4e^{i5t} \tag{10}$$

and assuming the same trial solution for the complex particular solution $\tilde{x}_p(t) = \tilde{A}e^{i5t}$ and substituting it in the Eq. (10) we get

$$\begin{aligned} \tilde{A} &= \frac{4}{20-15i} = \frac{4(20+15i)}{20^2+15^2} \\ &= \frac{4}{25}e^{i\frac{3}{4}} \end{aligned} \tag{11}$$

So $\tilde{x}_p = \frac{4}{25}e^{i(5t+\frac{3}{4})}$. Hence

$$x_p = Re\{\tilde{x}_p\} = \frac{4}{25}\cos(5t + \frac{3}{4}) \tag{12}$$

So $C = \frac{4}{25}$, $\omega = 5$ and $\alpha = -\frac{3}{4}$.

3. Consider an undamped mass-spring system subject to a harmonic force $F = F_0 \cos \omega t$.

(a) Write down the equation of motion.

(b) Obtain the general solution for $x(t)$.

(c) If $\omega = \omega_0$ where ω_0 is the natural frequency of the un-forced system then how does $x(t)$ behave with time?

(d) If the external force is not harmonic but it is of the form $F(t) = \alpha t^n$ then will the motion $x(t)$ still be oscillatory? Solve for $\alpha = 1$, $n = 1$; $\alpha = -1$, $n = 2$ and demonstrate your answer.

(a) The equation of motion:

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\, \omega t \tag{13}$$

where $\omega_0^2 = k/m$.

(b) Writing the complex form of the eq. (13),

$$\ddot{\tilde{x}} + \omega_0^2 \tilde{x} = \frac{F_0}{m} e^{i\omega t}, \tag{14}$$

and substituting $\tilde{x} = \tilde{x}_p = \tilde{A}e^{i\omega t}$ in the eq. (14), we have $\tilde{A} = \frac{F_0}{m(\omega_0^2 - \omega^2)}$. So the genaral solution is

$$x(t) = A\cos\,\omega_0 t + B\sin\,\omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\,\omega t$$

(c) For $\omega = \omega_0$ the amplitude of the displacement diverges. This is called the resonance.

(d) For $\alpha = 1$, $n = 1$,

$$\begin{aligned}
&\ddot{x} + \omega_0^2 x = t \\
\Rightarrow\;& \ddot{x} + \omega_0^2 (x - t/\omega_0^2) = 0 \\
\Rightarrow\;& \ddot{y} + \omega_0^2 y = 0, \qquad (y = x - t/\omega_0^2) \\
\Rightarrow\;& y = C_1 \cos\,\omega_0 t + C_2 \sin\,\omega_0 t
\end{aligned}$$

So,

$$x(t) = t/\omega_0^2 + C_1 \cos\,\omega_0 t + C_2 \sin\,\omega_0 t \tag{15}$$

For $\alpha = -1$, $n = 2$,

$$\begin{aligned}
&\ddot{x} + \omega_0^2 x = -t^2 \\
\Rightarrow\;& \ddot{x} + \omega_0^2 (x + t^2/\omega_0^2) = 0 \\
\Rightarrow\;& \ddot{y} + \omega_0^2 \left(y - \frac{2}{\omega_0^4}\right) = 0, \qquad (y = x + t^2/\omega_0^2) \\
\Rightarrow\;& \ddot{z} + \omega_0^2 z = 0, \qquad (z = y - 2/\omega_0^4) \\
\Rightarrow\;& z = C_3 \cos\,\omega_0 t + C_4 \sin\,\omega_0 t
\end{aligned}$$

So the general solution is

$$x(t) = 2/\omega_0^4 - t^2/\omega_0^2 + C_3 \cos\,\omega_0 t + C_4 \sin\,\omega_0 t \tag{16}$$

Both of the solutions are non-oscillatory.

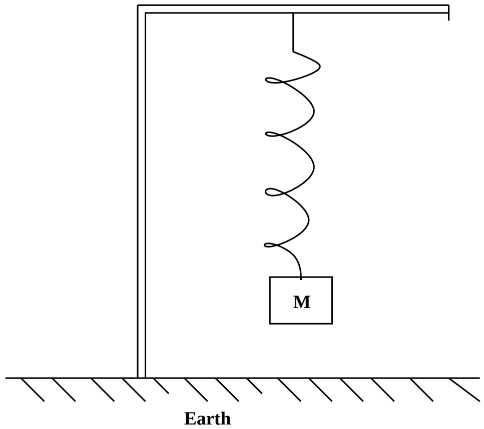

FIG. 1: Figure for Problem 4

*** 4. (Source: French's book/MIT OCW)** Imagine a simple seismograph consisting of a mass M hung from a spring on a rigid frame attached to the earth (see Figure 1). The spring force and the damping force depend on the displacement and velocity relative to the Earths surface, but the relevant acceleration (Newtons 2nd law) of M is relative to the fixed stars.

(a) Using y to denote the displacement of M relative to the earth and η to denote the displacement of the earths surface relative to the fixed stars, show that the equation of motion is

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega_0^2 y = -\frac{d^2\eta}{dt^2} \tag{17}$$

(b) Solve for y (steady-state) if $\eta = Ccos\omega t$.

(c) Sketch a graph of the amplitude A of the displacement y as a function of ω (supposing C the same for all ω).

(d) A typical long-period seismometer has a period of about 30 sec and a **quality factor** Q of about 2. As the result of a violent earthquake the Earths surface may oscillate with a period of about 20 minutes and with an amplitude such that the maximum acceleration is about 10^{-9} m/sec^2 . How small a value of A must be observable if this is to be detected?

(a) Suppose the equilbrium distance of the Earth's surface, relative to a fixed star, is r_0 and the equilibrium distance of M relative to the Earth's surface is y_0. The displacement of the Earth's surface relative to a fixed star is η and the displacement of the mass M relative to the Earth's surface is y. So, the displacement of M , relative to the fixed stars, is $\xi = (r_0 + \eta + y_0 + y) - (r_0 + y_0) = \eta + y$. Hence, the acceleration of M, relative to the fixed stars, is $a = \frac{d^2\xi}{dt^2} = \frac{d^2\eta}{dt^2} + \frac{d^2y}{dt^2}$. The force equation for M is

$$\begin{aligned} M\left(\frac{d^2\eta}{dt^2} + \frac{d^2y}{dt^2}\right) &= F_{spring} + F_{damping} \\ \text{where,} \quad F_{spring} &= -ky \\ F_{damping} &= -r\frac{dy}{dt} \end{aligned} \tag{18}$$

k is spring constant and r is another proportionality constant. Now define $\omega_0^2 = k/M$ and $\gamma = r/M$. Then Eq. (19) becomes

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega_0^2 y = -\frac{d^2\eta}{dt^2} \tag{19}$$

(b) If $\eta = C\cos\,\omega t$, then Eq. (19) becomes

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega_0^2 y = C\omega^2 \cos\,\omega t \tag{20}$$

The complementary solution will die out in time due to damping and, in steady state, only the particular solution survive. Writing the Eq. (20) in the complex form,

$$\frac{d^2\tilde{y}}{dt^2} + \gamma\frac{d\tilde{y}}{dt} + \omega_0^2 \tilde{y} = C\omega^2 e^{i\omega t} \tag{21}$$

we substitute $\tilde{y} = \tilde{A}e^{i\omega t}$ and find $\tilde{A}$ as,

$$\begin{aligned} \tilde{A} &= \frac{C\omega^2}{\omega_0^2 - \omega^2 + i\gamma\omega} \\ &= \frac{C\omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} e^{-i\phi} \end{aligned} \tag{22}$$

where $\phi = \tan^{-1}\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right)$. So the steady state solution is

$$\begin{aligned} y &= Re\{\tilde{y}\} \\ &= \frac{C\omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} \cos(\omega t - \phi) \end{aligned} \tag{23}$$

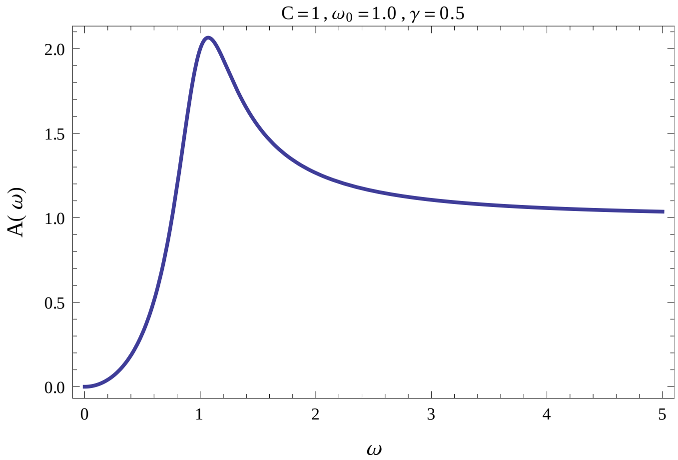

FIG. 2: Plot of $A(\omega)$, the amplitude of the displacement (y), as a function of ω

(c)

$$
\begin{aligned}
A(\omega) &= \frac{C\omega^2}{\sqrt{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2}} \\
\text{For } \omega \ll \omega_0\ , \quad A(\omega) &\approx C\frac{\omega^2}{\omega_0^2} \\
\text{For } \omega \gg \omega_0\ , \quad A(\omega) &= \frac{C}{\sqrt{\left(1-\frac{\omega_0^2}{\omega^2}\right)^2+\frac{\gamma^2}{\omega^2}}} \\
&\approx \frac{C}{\sqrt{1-2\frac{\omega_0^2}{\omega^2}+\frac{\gamma^2}{\omega^2}}} \\
&\approx C\left[1+\frac{(\omega_0^2-\gamma^2/2)}{\omega^2}\right]
\end{aligned}
$$

So for small ω, A increases with the increase of ω but for large ω, A decreases and for very large ω, A assymptotically reaches the value C. So there is a ω for which A is maximum and this is called the resonance frequency $\omega = \omega_{res}$. $\omega_{res} = \frac{\omega_0^2}{\sqrt{\omega_0^2-\gamma^2/2}} > \omega_0$. So the sketch of $A(\omega)$ will look like the plot in the Figure 2.

(d) Quality factor $Q = \frac{\omega_0}{2\beta} = \frac{2\pi}{\gamma T_0}$, for $\omega_0 = 2\pi/T_0$ and $\beta = \gamma/2$. Now it is given that, $Q = 2.0$, $T_0 = 30.0\, sec$. So, $\gamma = \pi/30\, sec^{-1}$.

Now time period of the oscillation of the Earth's surface is $T = 2\pi/\omega = 20$ minutes and its maximum acceleration $C\omega^2 = 10^{-9}\, m/sec^2$. So using the formula for $A(\omega)$, $A \approx 0.023\, \mu m$.

5. Consider a undamped harmonic oscillator of mass m and spring constant k subject to a constant force F. When the particle is in equilibrium the force is applied for τ seconds. Find the amplitude of oscillation after the force ceases to exist.

The equation of motion of the oscillator:

$$\begin{aligned} m\ddot{x} + kx &= F \quad , t \leq \tau \\ m\ddot{x} + kx &= 0 \quad , t \geq 0 \end{aligned}$$

For $t \leq \tau$, rewrite the equation as $\ddot{x} + \omega_0^2 (x - F/m\omega_0^2) = 0$. Define $y = x - F/m\omega_0^2$. Then $\ddot{y} + \omega_0^2 y = 0$. So, $y = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t \equiv A \cos(\omega_0 t + \phi)$. So $x(t \leq \tau) = F/m\omega_0^2 + A\cos(\omega_0 t + \phi)$. Now, $x(t=0) = \dot{x}(t=0) = 0$. So, $\phi = 0$ and $A = -F/m\omega_0^2$. So,

$$x(t \leq \tau) = \frac{F_0}{m\omega_0^2}(1 - \cos \omega_0 t) \tag{24}$$

Now $x(t > \tau) = A' \cos(\omega_0 t + \phi')$. From eq. (24), $x(t = \tau) = \frac{F_0}{m\omega_0^2}(1 - \cos \omega_0 \tau)$ and $\dot{x}(t = \tau) = \frac{F_0}{m\omega_0} \sin \omega_0 \tau$. So,

$$\begin{aligned} A' \cos(\omega_0 \tau + \phi) &= \frac{F_0}{m\omega_0^2}(1 - \cos \omega_0 \tau) \\ -A'\omega_0 \sin(\omega_0 \tau + \phi) &= \frac{F_0}{m\omega_0} \sin \omega_0 \tau \\ \text{so,} \qquad A' &= \sqrt{\frac{F_0^2}{m^2\omega_0^4}(1 - \cos \omega_0 \tau)^2 + \frac{F_0^2}{m^2\omega_0^4} \sin^2 \omega_0 \tau} \\ &= \frac{2F_0}{m\omega_0^2} |\sin(\omega_0 \tau/2)| \end{aligned}$$

*** 6.** A 50 gm mass is suspended by a spring of spring constant $k = 20N/m$. The mass performs steady state vertical oscillations of amplitude 1.3 cm due to an external harmonic force of frequency $\omega = 25s^{-1}$. The displacement lags behind the force by an angle $\frac{3\pi}{4}$. Find:
(a) The quality factor Q of the oscillator.
(b) The **work done by the external force** in one oscillation.

(a) Natural frequency of oscillation $\omega_0 = \sqrt{k/m} = 20\, sec^{-1}$. Phase angle, $\phi = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right) = \frac{3\pi}{4}$. $\omega = 25\, sec^{-1}$. Then $2\beta = \frac{\omega^2 - \omega_0^2}{\omega}$. So the quality factor Q of the oscillator:

$$
\begin{aligned}
Q &= \frac{\omega_0}{2\beta} \\
&= \frac{\omega\omega_0}{\omega^2 - \omega_0^2} \\
&\approx 2.22
\end{aligned}
$$

(b) For $F = F_0 \cos \omega t$, $x = \frac{F_0 \cos(\omega t - \phi)}{m\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}$ (in steady state). Then, work done by the external force in one oscillation:

$$
\begin{aligned}
W &= \int F dx = \int_0^T F \dot{x} dt \\
&= -\int_0^T \frac{F_0^2 \omega \cos \omega t \sin(\omega t - \phi)}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} dt \\
&= \frac{\pi F_0^2 \sin\phi}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \\
&= m\pi \left(\frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \right)^2 (\omega^2 - \omega_0^2) \sqrt{1 + \left(\frac{2\beta\omega}{\omega_0^2 - \omega^2} \right)^2} \sin\phi
\end{aligned}
$$

Now the amplitude of the oscillation: $x = \frac{F_0}{m\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}} = 1.3\,cm$, $\frac{2\beta\omega}{\omega_0^2-\omega^2} = \tan\phi = -1$, $\sin\phi = 1/\sqrt{2}$. So,

$$W \approx 5.97\,mJ$$

*** 7.** Two different forced oscillating systems are given to you. They have the same natural frequency ω_0. For one of them the **resonance** occurs at ω_{r1} and for the other at ω_{r2}. The damping coefficients are different: β_1 and β_2.

(a) Draw qualitatively the two resonance curves on the same graph paper, when $\beta_1 > \beta_2$. Can these two curves intersect?

(b) If $\beta_1 = 0.5$ units and $\beta_2 = 0.2$ units then find the expressions for ω_{r1}, ω_{r2} and the **FWHM** in each case.

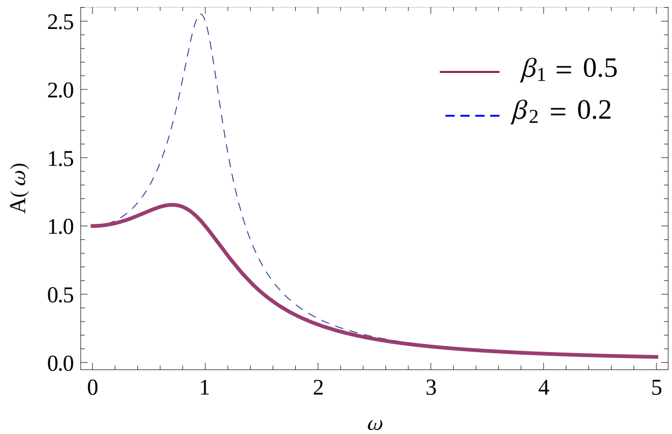

FIG. 3: Plot of resonance curves for two different β

(a) $\omega_{r1} = \sqrt{\omega_0^2 - 2\beta_1^2}$, $\quad \omega_{r2} = \sqrt{\omega_0^2 - 2\beta_2^2}$. Since $\beta_1 > \beta_2$, then $\omega_{r1} < \omega_{r2}$.

$$\begin{aligned} A(\omega_r) &= \frac{F_0}{m\sqrt{(\omega_0^2 - \omega_r^2)^2 + 4\beta^2\omega_r^2}} \\ &= \frac{F_0}{m\sqrt{(\omega_0^2 - \omega_r^2)^2 + 2(\omega_0^2 - \omega_r^2)\omega_r^2}} \\ &= \frac{F_0}{m\sqrt{\omega_0^4 - \omega_r^4}} \end{aligned}$$

So $A(\omega_{r1}) < A(\omega_{r2})$. The two resonance curves do not intersect due to the term $\beta^2\omega^2$ in the denominator of the expression of $A(\omega)$. So, the sketch of the two resonance curves on the same graph paper, will look like plot in Fig 3.

(b) The average energy of the oscillator:

$$E(\omega) = \frac{F_0^2}{4m} \frac{(\omega_0^2 + \omega^2)}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

For $E(\omega) = E_{\text{max}}/2 = E(\omega_0)/2$,

$$\begin{aligned} (\omega_0^2 - \omega^2)^2 &= 4\beta^2\omega_0^2 \\ \Rightarrow \Delta\omega &= \pm\frac{2\beta\omega_0}{\omega + \omega_0}, \qquad (\omega = \omega_0 + \Delta\omega) \\ &\approx \pm\beta, \qquad (\text{for}\beta \ll \omega_0) \end{aligned}$$

So, FWHM $= 2|\Delta\omega_0| = 2\beta$

**** 8. Curious oscillations!** Consider a candle of length L placed horizontally with its centre on a pivot. Light the candle at both the ends after you have sliced off the wax to expose the wick on the flat end. It is found that the candle oscillates as both the ends keep burning. How does one explain the oscillatory motion ?

Hint: Find out the rate of mass-loss at the both ends. Then use the relation Torque=Rate of change of angular momentum *to obtain the differential equation for θ. Try to solve this equation to demonstrate the oscilations.*

Let us assume that the rate of burning of the wax depends on the angle of the candle with the horizontal. Let us assume a linear relation. So the rate of mass loss at the each end is $\frac{dm}{dt} = \alpha - \beta\theta$. On the right hand side, $\dot{m}_R = \alpha - \beta\theta$ and, on the left hand side, $\dot{m}_L = \alpha + \beta\theta$. Assuming that the angle of rotation θ is small and the length of the candle L does not change appreciably in one oscillation, we find that the torque acting on the candle is $\tau = (m_R - m_g)gL/4$. So, $\dot{\tau} = -\beta gL\theta/2$. If M be the instantaneous total mass of the candle then the angular momentum of the candle $N \approx ML^2\dot{\theta}/12$. So $\tau = \dot{N}$ and $\dot{M} = \dot{m}_R + \dot{m}_l = 2\alpha$.

The detailed solution is difficult, though it has been done. See the detailed analysis in **S. Theodorakis and K. Paridi, American Journal of Physics, Vol. 77, Pg 1049 (2009)** if you want to learn more.

Tutorial 3: Solutions

Solutions prepared by Mr. Debabrata Paladhi (debabrata90@phy.iitkgp.ernet.in)

1. Two coupled pendula have the parameters: $m = 0.10$ kg, $l = 0.15$ m and $k = 5.0$ N/m. Initially, the left mass is held at $x_1 = 1.0$ cm and the right mass at $x_2 = 3.0$ cm. The masses are then released simultaneously. The masses are then released simultaneously. Find the **normal modes** of the system and the functional forms of $x_1(t)$ and $x_2(t)$ subject to the given initial conditions

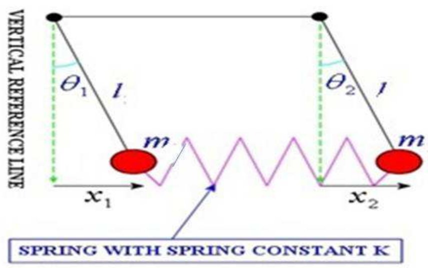

Figure 1: Figure for Problem 1

Figure 1 shows that two identical pendula, each having a mass m suspended by a rigid massless rod of length l. The masses are connected by a spring of spring constant k. If x_1 and x_2 are the displacements of the masses from their equilibrium position, then the equations of motion are

$$m\frac{d^2x_1}{dt^2} = -\frac{mgx_1}{l} - k(x_1 - x_2) \tag{1}$$

$$m\frac{d^2x_2}{dt^2} = -\frac{mgx_2}{l} - k(x_2 - x_1) \tag{2}$$

writing $\omega_0^2 = \frac{g}{l}$, where ω_0 is the natural frequency of oscillation of each pendulum, the equations become

$$\frac{d^2x_1}{dt^2} + (\omega_0^2 + \frac{k}{m})x_1 - \frac{k}{m}x_2 = 0 \tag{3}$$

$$\frac{d^2x_2}{dt^2} + (\omega_0^2 + \frac{k}{m})x_2 - \frac{k}{m}x_1 = 0 \tag{4}$$

To solve these equations we define two new co-ordinates $q_1 = x_1 + x_2,\ q_2 = x_1 - x_2$
Adding equation (3) and (4) gives

$$\frac{d^2q_1}{dt^2} + \omega_0^2 q_1 = 0 \tag{5}$$

and subtracting (4) from (3) gives

$$\frac{d^2q_2}{dt^2} + (\omega_0^2 + \frac{2k}{m})q_2 = 0 \tag{6}$$

So, the motion of the coupled system is described by two uncoupled equations of q_1 and q_2.
If $q_2 = 0$ then $x_1 = x_2$, so that the motion is described by the equation

$$\frac{d^2q_1}{dt^2} + \omega_0^2 q_1 = 0 \tag{7}$$

this is the in phase motion, where $\omega_0 = \sqrt{\frac{g}{l}}$
If $q_1 = 0$ then $x_1 = -x_2$, so that the motion is described by the equation

$$\frac{d^2q_2}{dt^2} + (\omega_0^2 + \frac{2k}{m})q_2 = 0 \tag{8}$$

So normal mode frequencies are $\omega_1 = \omega_0 = \sqrt{\frac{g}{l}}$ and $\omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}}$. For the given parameter values, $\omega_1 = 8.1\ rad/sec$ and $\omega_2 = 12.86\ rad/sec$. The normal amplitudes are $x_1(t=0) = x_2(t=0)$ for ω_1 and $x_1(t=0) = -x_2(t=0)$ for ω_2 if the pendulums are released from the rest.
For the given initial displacements, the system does not oscillate in either of the normal mode frequencies, the motion is a combination of them. Using the given initial conditions and after some straightforward calculations, we can easily see that

$$x_1(t) = 2\cos\omega_0 t - \cos\omega_1 t \tag{9}$$

and

$$x_2(t) = 2\cos\omega_0 t + \cos\omega_1 t \tag{10}$$

2. Consider the coupled LC circuits shown in the figure above. The circuits are driven with a voltage $V(t)$ at one end. Write the equations for the currents I_a and I_b. Find the 'normal coordinates' and the 'normal modes'. Explain how the above circuits can act as an **electrical low pass filter.**

The main governing differential equations for this circuit are the following

$$L\frac{dI_a}{dt} + \frac{Q_a}{C_0} + \frac{Q}{C} = V(t) \tag{11}$$

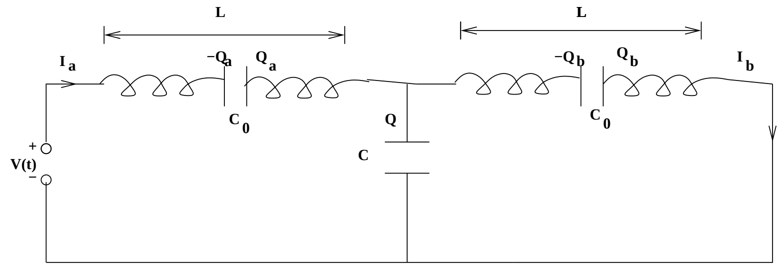

Figure 2: Figure for Problem 2

$$L\frac{dI_b}{dt} + \frac{Q_b}{C_0} = \frac{Q}{C} \tag{12}$$

By differentiating once again and using $\frac{dQ}{dt} = I_a - I_b$, $\frac{dQ_a}{dt} = I_a$ and $\frac{dQ_b}{dt} = I_b$ we can write

$$\frac{d^2 I_a}{dt^2} + (\frac{1}{LC_0} + \frac{1}{LC})I_a - \frac{1}{LC}I_b = \frac{1}{L}\frac{dV}{dt} \tag{13}$$

$$\frac{d^2 I_b}{dt^2} + (\frac{1}{LC_0} + \frac{1}{LC})I_b - \frac{1}{LC}I_a = 0 \tag{14}$$

Introducing new variables $I_1 = I_a - I_b$ and $I_2 = I_a - I_b$ we get

$$\frac{d^2 I_1}{dt^2} + \omega_1^2 I_1 = \frac{1}{L}\frac{dV}{dt} \tag{15}$$

$$\frac{d^2 I_2}{dt^2} + \omega_2^2 I_2 = \frac{1}{L}\frac{dV}{dt} \tag{16}$$

where $\omega_1 = \frac{1}{\sqrt{LC_0}}$ and $\omega_2 = \sqrt{\frac{1}{LC_0} + \frac{2}{LC}}$. These are the normal mode frequencies and corresponding normal mode coordinates are I_1 and I_2.

Assume that $V(t) = V_0 \sin \omega t$. Then solving eq. (13) and eq. (14), we get $I_1 = I_a + I_b = \frac{V_0 \omega}{L(\omega_1^2 - \omega^2)} \sin \omega t$ and $I_2 = I_a - I_b = \frac{V_0 \omega}{L(\omega_2^2 - \omega^2)} \sin \omega t$. So after some simplification we get

$$\frac{I_b}{I_a} = \frac{\omega_2^2 - \omega_1^2}{\omega_1^2 + \omega_2^2 - 2\omega^2} = \frac{1}{1 + 2\frac{\omega_1^2 - \omega^2}{\omega_2^2 - \omega_1^2}} = -\frac{1}{1 - 2\frac{\omega_2^2 - \omega^2}{\omega_2^2 - \omega_1^2}}$$

Note that $\left|\frac{I_b}{I_a}\right| > 1$, iff ω lies between ω_1 and ω_2. So this is an electrical band-pass filter. To behave like a low-pass filter $\omega_1 = 0$ or $C_0 \to \infty$, which means that one needs to short-circuit C_0.

3.

Three identical masses m are connected in series by four identical springs of spring constant k (each mass coupled to two strings, and the end springs to the walls).

(a) Calculate the normal modes of this system.
(b) What sets of initial displacements would you give to the three masses so that the system oscillates in each of these normal modes.

(a) If x_1, x_2 and x_3 are the displacements of the three masses from their equilibrium positions, then the three $F = ma$ equations are

$$m\frac{d^2x_1}{dt^2} = -kx_1 - k(x_1 - x_2) \tag{17}$$

$$m\frac{d^2x_2}{dt^2} = -k(x_2 - x_1) - k(x_2 - x_3) \tag{18}$$

$$m\frac{d^2x_3}{dt^2} = -k(x_3 - x_2) - kx_3 \tag{19}$$

It is not so obvious which combinations of these equations yield uncoupled equations. We will ,therefore, use the matrix method and guess a solution of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \exp(i\omega t) \tag{20}$$

Note that in normal mode oscillation all the masses have the same frequency. A_1, A_2 and A_3 are the amplitudes. Plugging this guess solution into Eq. (1) and collecting all the terms on the left hand side, and canceling the $\exp(i\omega t)$ factor, gives

$$\begin{pmatrix} -\omega^2 + 2\omega_0^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & -\omega^2 + 2\omega_0^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & -\omega^2 + 2\omega_0^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{21}$$

where $\omega_0^2 = \frac{k}{m}$. A nonzero (nontrivial) solution for (A_1, A_2, A_3) exists iff the determinant of this 3×3 matrix is zero. Setting it equal to zero and finding the roots of the equation, we get the normal mode frequencies

$$\omega_1 = \sqrt{2}\omega_0 \tag{22}$$

$$\omega_2 = \sqrt{(2 + \sqrt{2})\omega_0} \tag{23}$$

$$\omega_3 = \sqrt{(2 - \sqrt{2})\omega_0} \tag{24}$$

for $\omega_1 = \sqrt{2}\omega_0$ we get $A_2 = 0$ and $A_1 = -A_3$
for $\omega_2 = \sqrt{(2 + \sqrt{2})\omega_0}$ we get $A_1 = A_3 = -\frac{A_2}{\sqrt{2}}$
for $\omega_2 = \sqrt{(2 - \sqrt{2})\omega_0}$ we get $A_1 = A_3 = \frac{A_2}{\sqrt{2}}$

Normal mode co-ordinates are the following

$$X_1 = x_1 - x_3 \tag{25}$$
$$X_2 = x_1 - \sqrt{2}x_2 + x_3 \tag{26}$$
$$X_3 = x_1 + \sqrt{2}x_2 + x_3 \tag{27}$$

(b) The initials displacements are

$$x_1(t=0) = -x_3(t=0), \;\; x_2(t=0) = 0 \tag{28}$$
$$x_1(t=0) = x_3(t=0) = -\frac{x_2(t=0)}{\sqrt{2}} \tag{29}$$
$$x_1(t=0) = x_3(t=0) = \frac{x_2(t=0)}{\sqrt{2}} \tag{30}$$

assuming that all of the masses are released from rest.

4. Consider the system of two equal masses connected by three springs along a line. The springs which have their ends attached to the vertical support and one mass have spring constant k. The central spring whose ends are attached to the two masses has a spring constant k'. One of the masses is under a forcing $F(t) = F_0 \cos \omega t$. Include a damping term of the form $2\beta\times$ velocity.

(a) Write down the equations of motion for the two masses.
(b) Solve the equations by identifying the normal modes.
(c) Write down solutions for the two masses.
(d) Evaluate the square of the ratio of the amplitudes $\frac{|A_1|^2}{|A_2|^2}$ and plot it as a function of ω for negligible damping.
(e) Analyse your result in (d) and arrive at the result: **The unforced mass has large amplitude only when the forcing frequency lies between ω_1 and ω_2, otherwise it does not respond to forcing. The system acts like a mechanical band-pass filter.**

(a) If x_1 and x_2 are the displacements of the two masses from their equilibrium positions, then the two $F = ma$ equations are

$$m\frac{d^2x_1}{dt^2} = -kx_1 - k'(x_1 - x_2) - 2\beta\frac{dx_1}{dt} + F_0 \cos \omega t \tag{31}$$
$$m\frac{d^2x_2}{dt^2} = -kx_2 - k'(x_2 - x_1) - 2\beta\frac{dx_2}{dt} \tag{32}$$

which can be written as

$$\frac{d^2x_1}{dt^2} + \omega_1^2 x_1 + 2\beta_0\frac{dx_1}{dt} + \frac{k'}{m}(x_1 - x_2) = f_0 \cos \omega t \tag{33}$$
$$\frac{d^2x_2}{dt^2} + \omega_1^2 x_2 + 2\beta_0\frac{dx_2}{dt} + \frac{k'}{m}(x_2 - x_1) = 0 \tag{34}$$

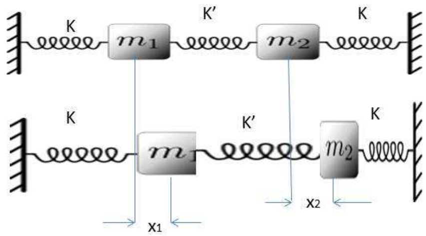

Figure 3: Figure for Problem 4

where $\omega_1^2 = \frac{k}{m}$ and $f_0 = F_0/m$.

(b) Now introducing two new variables $q_1 = x_1 + x_2$ and $q_2 = x_1 - x_2$ adding and subtracting subsequently the above two equations we get

$$\frac{d^2q_1}{dt^2} + 2\beta_0\frac{dq_1}{dt} + \omega_1^2 q_1 = f_0 \cos \omega t \tag{35}$$

$$\frac{d^2q_2}{dt^2} + 2\beta_0\frac{dq_2}{dt} + \omega_2^2 q_2 = f_0 \cos \omega t \tag{36}$$

where $\omega_2^2 = (\omega_1^2 + \frac{2k}{m})$ and $f_0 = \frac{F_0}{m}$

Let $\tilde{q}_1 = z_1 e^{i\omega t}$ and $\tilde{q}_2 = z_2 e^{i\omega t}$ be the complex solutions of the above two equations (in steady state). Putting these in the complex form of the equation (33) we get

$$z_1 = \frac{f_0}{(\omega_1^2 - \omega^2 + 2i\omega\beta_0)} = \frac{f_0 e^{i\phi_1}}{\sqrt{(\omega_1^2 - \omega^2)^2 + 4\omega^2\beta_0^2}} \tag{37}$$

where $\phi_1 = \tan^{-1}(-\frac{2\beta_0\omega}{(\omega_1^2 - \omega^2)})$. So the complex solution of equation (33) is

$$\tilde{q}_1 = \frac{f_0}{\sqrt{(\omega_1^2 - \omega^2)^2 + 4\omega^2\beta_0^2}} e^{i(\omega t + \phi_1)} \tag{38}$$

Similarly for the equation (34) we get

$$\tilde{q}_2 = \frac{f_0}{\sqrt{(\omega_2^2 - \omega^2)^2 + 4\omega^2\beta_0^2}} e^{i(\omega t + \phi_2)} \tag{39}$$

where $\phi_2 = \tan^{-1}(-\frac{2\beta_0\omega}{(\omega_2^2-\omega^2)})$.

(c) So the displacements of the two masses are the following

$$\tilde{x_1} = \frac{(\tilde{q_1}+\tilde{q_2})}{2}$$

$$\tilde{x_2} = \frac{(\tilde{q_1}-\tilde{q_2})}{2}$$

Putting the value of $\tilde{q_1}$ and $\tilde{q_2}$ we finally get the displacements of the two masses are

$$\tilde{x_1} = \frac{f_0}{2}\left(\frac{e^{i\phi_1}}{\sqrt{(\omega_1^2-\omega^2)^2+4\omega^2\beta_0^2}}+\frac{e^{i\phi_2}}{\sqrt{(\omega_2^2-\omega^2)^2+4\omega^2\beta_0^2}}\right)e^{i\omega t} \tag{40}$$

$$\tilde{x_2} = \frac{f_0}{2}\left(\frac{e^{i\phi_1}}{\sqrt{(\omega_1^2-\omega^2)^2+4\omega^2\beta_0^2}}-\frac{e^{i\phi_2}}{\sqrt{(\omega_2^2-\omega^2)^2+4\omega^2\beta_0^2}}\right)e^{i\omega t} \tag{41}$$

So finally

$$\tilde{x_1} = |A_1|e^{i(\omega t+\Phi_1)}, \;\; \tilde{x_2} = |A_2|e^{i(\omega t+\Phi_2)} \tag{42}$$

where $|A_1|^2 = \tilde{x_1}\tilde{x_1}^*$, $|A_2|^2 = \tilde{x_2}\tilde{x_2}^*$. So

$$|A_1^2| = \frac{f_0^2}{4}\left(\frac{e^{i\phi_1}}{\sqrt{(\omega_1^2-\omega^2)^2+4\omega^2\beta_0^2}}+\frac{e^{i\phi_2}}{\sqrt{(\omega_2^2-\omega^2)^2+4\omega^2\beta_0^2}}\right)$$
$$\left(\frac{e^{-i\phi_1}}{\sqrt{(\omega_1^2-\omega^2)^2+4\omega^2\beta_0^2}}+\frac{e^{-i\phi_2}}{\sqrt{(\omega_2^2-\omega^2)^2+4\omega^2\beta_0^2}}\right)$$

Using $\tan\phi_1 = -\frac{2\beta_0\omega}{(\omega_1^2-\omega^2)}$ and $\tan\phi_2 = -\frac{2\beta_0\omega}{(\omega_2^2-\omega^2)}$ we finally get

$$|A_1^2| = \frac{f_0^2}{4}\frac{(\omega_1^2+\omega_2^2-2\omega^2)^2+16\beta_0^2\omega^2}{((\omega_1^2-\omega^2)^2+2\beta_0^2\omega^2)((\omega_2^2-\omega^2)^2+2\beta_0^2\omega^2)} \tag{43}$$

Similarly for $|A_2|$ we can write

$$|A_2^2| = \frac{f_0^2}{4}\frac{(\omega_1^2-\omega_2^2)^2}{((\omega_1^2-\omega^2)^2+2\beta_0^2\omega^2)((\omega_2^2-\omega^2)^2+2\beta_0^2\omega^2)} \tag{44}$$

(d) So

$$\frac{|A_1|^2}{|A_2|^2} = \frac{(\omega_1^2+\omega_2^2-2\omega^2)^2+16\beta_0^2\omega^2}{(\omega_1^2-\omega_2^2)^2} \tag{45}$$

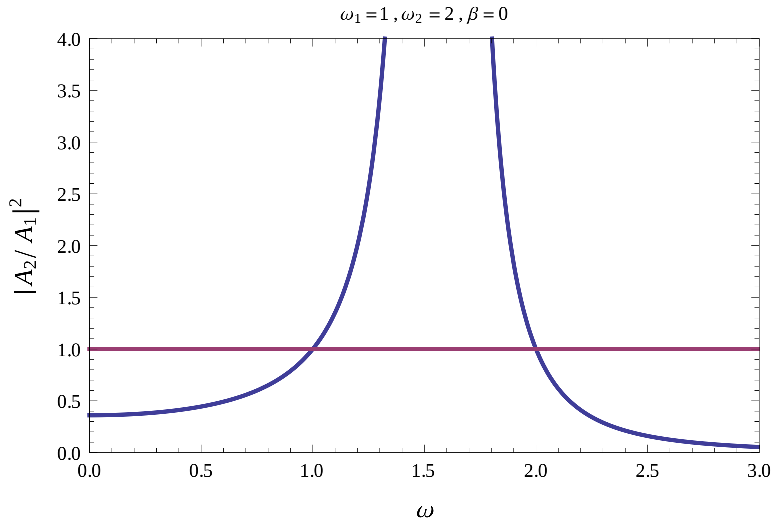

Figure 4: Plot of $|A_2/A_1|^2$ vs ω

(e) To analyse the above expression we rearrange as follows (taking $\beta_0 = 0$ for negligible damping)

$$\frac{|A_2|}{|A_1|} = \frac{\omega_2^2 - \omega_1^2}{\omega_1^2 + \omega_2^2 - 2\omega^2} = \frac{\omega_2^2 - \omega_1^2}{\omega_2^2 - \omega_1^2 + 2(\omega_1^2 - \omega^2)} = \frac{1}{1 + 2\frac{\omega_1^2 - \omega^2}{\omega_2^2 - \omega^2}}$$

So for $\omega < \omega_1$, $\frac{|A_2|}{|A_1|} < 1$.

5. Similarly for $\omega > \omega_2$ we get

$$\frac{|A_2|}{|A_1|} = \frac{\omega_2^2 - \omega_1^2}{\omega_1^2 + \omega_2^2 - 2\omega^2} = \frac{\omega_2^2 - \omega_1^2}{\omega_1^2 - \omega_2^2 + 2(\omega_2^2 - \omega^2)}$$
$$\frac{|A_2|}{|A_1|} = -\frac{\omega_2^2 - \omega_1^2}{\omega_2^2 - \omega_1^2 + 2(\omega^2 - \omega_2^2)} = -\frac{1}{1 + 2\frac{\omega^2 - \omega_2^2}{\omega_2^2 - \omega_1^2}}$$

So for $\omega > \omega_2$, $\frac{|A_2|}{|A_1|} < 1$. So the unforced mass has large amplitude only when the forcing frequency lies between ω_1 and ω_2, otherwise it does not respond to forcing.

(from MIT OCW) Two springs, each of constant k, support a rigid, massless platform to which a mass m is firmly attached. The position of this mass is $y_1(t)$. A second mass m hangs at the end of another spring (of constant k) from the center of the platform, as shown in the sketch. The position of this second mass is $y_2(t)$. Assume that the two longer springs move together with the same frequency and in the same plane.

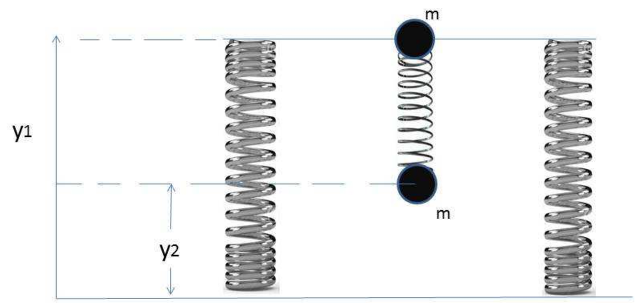

Figure 5: Figure for Problem 5

(a) Write the differential equations of motion for each of the masses.
(b) Solve the equations to nd the normal mode frequencies and nd suitable expressions for $y_1(t)$ and $y_2(t)$.
(c) Sketch the configuration of the system for each of the two normal modes. Label the sketches to indicate which configuration corresponds to the normal mode with low frequency ω_1 , and which conguration corresponds to the mode with high frequency ω_2
Left figure shows the system at rest and right figure shows it at any time t. Displacements from equilibrium are x_1 and x_2.Now

$$y_1 = d_1 + d_2 + x_1 \tag{46}$$
$$y_2 = d_2 + x_2 \tag{47}$$

The equations of motion are

$$m\ddot{x}_1 = -2kx_1 - k(x_1 - x_2)$$
$$\Rightarrow \ddot{x}_1 + 3\omega_0^2 x_1 - \omega_0^2 x_2 = 0 \tag{48}$$
$$m\ddot{x}_2 = -k(x_2 - x_1)$$
$$\Rightarrow \ddot{x}_2 + \omega_0^2 x_2 - \omega_0^2 x_1 \tag{49}$$

where $\omega_0^2 = \frac{k}{m}$.

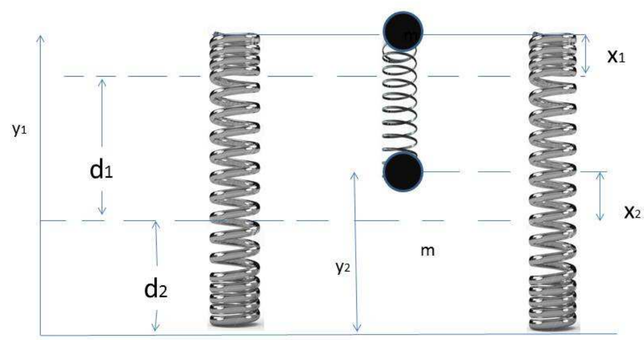

Figure 6:

(b) Substituting $x_1 = A\cos\omega t$ and $x_2 = B\cos\omega t$ in the equation of motion gives

$$
\begin{aligned}
A(3\omega_0^2 - \omega^2) = B\omega_0^2 \;\; and \;\; A\omega_0^2 &= B(\omega_0^2 - \omega^2) \\
\frac{A}{B} = \frac{\omega_0^2}{3\omega_0^2 - \omega^2} &= \frac{\omega_0^2 - \omega^2}{\omega_0^2} \\
So \;\; \omega_0^4 = 3\omega_0^4 - 4\omega^2\omega_0^2 + \omega^4 \;\Rightarrow\; \omega^4 - 4\omega^2\omega_0^2 + 2\omega_0^4 &= 0 \\
\omega_\pm^2 &= \omega_0^2(2 \pm \sqrt{2}) \\
So, \omega_1 = \omega_0(2-\sqrt{2})^{\frac{1}{2}}; \;\; \frac{B}{A} &= 1+\sqrt{2} \\
for, \omega_2 = \omega_0(2+\sqrt{2})^{\frac{1}{2}}; \;\; \frac{B}{A} &= 1-\sqrt{2}
\end{aligned}
$$

Hence the general solutions are

$$
\begin{aligned}
y_1(t) &= d_1 + d_2 + x_1(t) \\
&= d_1 + d_2 + A\cos(\omega_1 t + \alpha) + B\cos(\omega_2 t + \beta) \qquad (50) \\
y_2(t) &= d_2 + x_2(t) \\
&= d_2 + (1+\sqrt{2})A\cos(\omega_1 t + \alpha) + (1+\sqrt{2})B\cos(\omega_2 t + \beta) \qquad (51)
\end{aligned}
$$

The left figure shows the normal mode with higher frequency ω_2 such that $x_2(t) = (1-\sqrt{2})x_1(t)$. The right figure shows normal mode with lower frequency ω_1 such that $x_2(t) = (1+\sqrt{2})x_1(t)$.

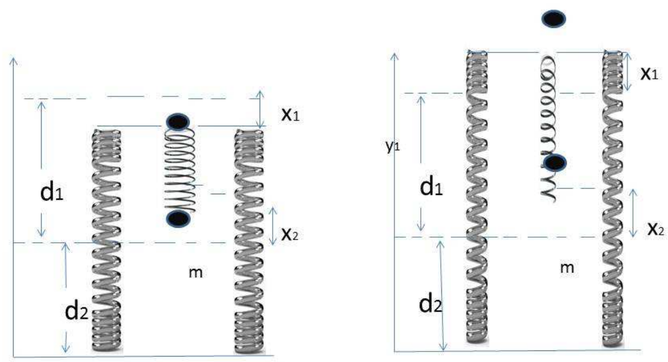

Figure 7: Sketch of the normal mode oscillations

6. A piece of delicate apparatus sits on a floor which has a vertical vibration of 20 cps. You wish to attenuate the jiggling by a factor of 100, so you set the apparatus on a cushion. About how far down should the top of the cushion sink when the apparatus is placed on it?

Let x_1 and x_2 are the displacements of the floor and apparatus respectively. Say, the masses of floor and apparatus are M and m. If k is the spring constant of the cushion then the equations of motion are

$$M\ddot{x}_1 + k(x_1 - x_2) = F_0 \cos \omega t \tag{52}$$

$$m\ddot{x}_2 + k(x_2 - x_1) = 0 \tag{53}$$

Introducing two new variables $X_1 = \frac{(Mx_1+mx_2)}{(M+m)}$, $X_2 = x_1 - x_2$, we get the uncoupled equations

$$\ddot{X}_1 = \frac{F_0}{(M+m)} \cos \omega t \tag{54}$$

$$\ddot{X}_2 + k(\frac{1}{m} + \frac{1}{M})X_2 = \frac{F_0}{M} \cos \omega t \tag{55}$$

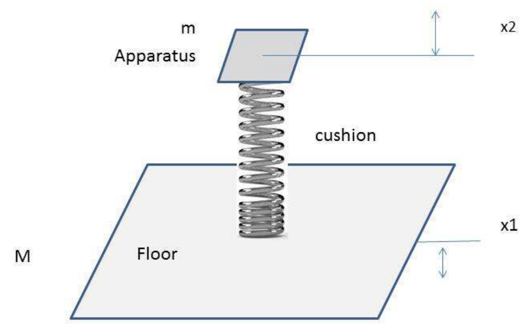

Figure 8: Figure for Problem 6

The solutions of the above equations are

$$X_1 = -\frac{F_0 \cos \omega t}{(M+m)\omega^2} \tag{56}$$

$$X_2 = -\frac{F_0 \cos \omega t}{M(k(\frac{1}{m}+\frac{1}{M})-\omega^2)} \tag{57}$$

$$\text{So,} \quad \frac{X_2}{X_1} = \frac{(x_1-x_2)(M+m)}{Mx_1+mx_2} = -\frac{(M+m)\omega^2}{M(k(\frac{1}{m}+\frac{1}{M})-\omega^2)}$$

$$\Rightarrow \frac{M(x_1-x_2)}{Mx_1+mx_2} = -\frac{\omega^2}{(k(\frac{1}{m}+\frac{1}{M})-\omega^2)} \tag{58}$$

After some calculation from the above equation we finally get the ratio of x_1 and x_2 and that is

$$\frac{x_2}{x_1} = \frac{\frac{k}{m}}{\frac{k}{m}-\omega^2} \tag{59}$$

From Hook's law can write $k\Delta x = mg$, $\Rightarrow \Delta x = \frac{mg}{k} = \frac{g}{\frac{k}{m}}$ Now using the above two equations we can easily find the value of Δx. Here $x_2/x_1 = -1/100$, $\omega = 40\pi\, rad/sec$. So $k/m = \omega^2/101 \approx 156\, Nm^{-1}kg^{-1}$. So, $\Delta x \approx 6\, cm$.

7. Consider two coupled pendula where pendulum 1 has mass m_1 and length l_1 and pendulum 2 has mass m_2 and length l_2. The coupling is through a horizontal spring of spring constant k.

* (a) Write down the equations of motion and solve the system of equations.

(b) Try to generalise using three pendula of various lengths and attached masses–you may also assume couplings which are different, i.e. with different spring constants.

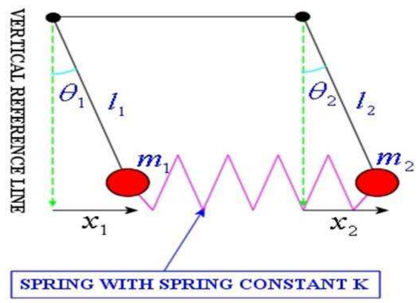

Figure 9: Figure for Problem 5

If x_1 and x_2 be the displacements of the masses m_1 and m_2 then the equations of motion are the following

$$m_1\ddot{x}_1 = -\frac{m_1 g}{l_1}x_1 - k(x_1 - x_2)$$

$$\Rightarrow \ddot{x}_1 + (\frac{g}{l_1} + \frac{k}{m_1})x_1 - \frac{k}{m_1}x_2 = 0 \tag{60}$$

$$and\ \ddot{x}_2 + (\frac{g}{l_2} + \frac{k}{m_2})x_2 - \frac{k}{m_2}x_1 = 0 \tag{61}$$

Now we guess a solution of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i\omega t} \tag{62}$$

with the goal of solving for ω, and also for the amplitudes A_1, A_2. Plugging this guess into the equations and canceling the $e^{i\omega t}$ factor gives

$$\begin{pmatrix} \frac{g}{l_1} + \frac{k}{m_1} - \omega^2 & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{g}{l_1} + \frac{k}{m_2} - \omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{63}$$

A nonzero solution for (A_1, A_2) exists only if the determinant of this matrix is zero. Setting it equal to zero gives

$$\omega_{1,2} = \sqrt{\frac{1}{2}(\frac{k}{m_1} + \frac{k}{m_2} + \frac{g}{l_1} + \frac{g}{l_2}) \pm \sqrt{(\frac{k}{m_1} + \frac{k}{m_2} + \frac{g}{l_1} + \frac{g}{l_2})^2} - 4((\frac{g}{l_1} + \frac{k}{m_1})(\frac{g}{l_2} + \frac{k}{m_2}) - \frac{k^2}{m_1 m_2})}$$

Tutorial 4: Solutions

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1.

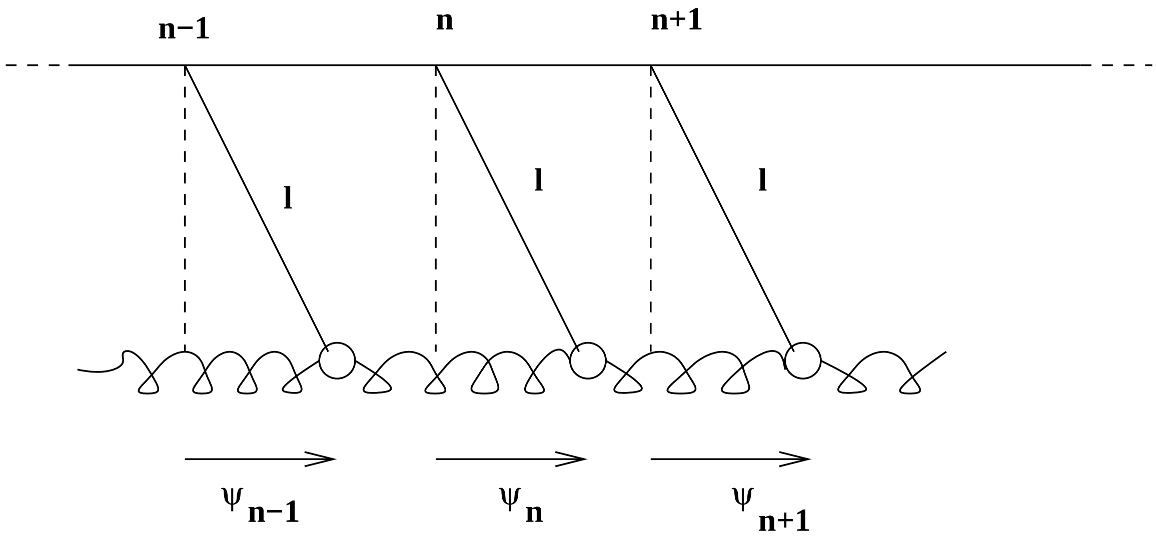

FIG. 1: Figure for Problem 1

Consider the system of coupled pendula shown in the figure above.

(a) Show that the equation of motion of the nth pendulum bob is given (for small oscillations) as:

$$\frac{d^2\Psi_n(t)}{dt^2} = -\frac{g}{l}\Psi_n(t) + \frac{aK}{m}\left(\frac{\Psi_{n+1}(t) - \Psi_n(t)}{a}\right) - \frac{aK}{m}\left(\frac{\Psi_n(t) - \Psi_{n-1}(t)}{a}\right) \tag{1}$$

(b) Take the **continuum limit** of the equation in (a) and obtain the equation below:

$$\frac{\partial^2\Psi}{\partial t^2} = -\omega_0^2\Psi + v_0^2\frac{\partial^2\Psi}{\partial z^2} \tag{2}$$

What are ω_0 and v_0?

(a) Net force on the n-th bob due to (n-1)th and (n+1)th bobs is: $-K(\Psi_n(t) - \Psi_{n-1}(t)) - K(\Psi_n(t) - \Psi_{n+1}(t))$. Therefore the equation of motion for the n-th bob is

$$m\frac{d^2\Psi_n(t)}{dt^2} = -mg\sin\theta_n - K(\Psi_n - \Psi_{n-1}) - K(\Psi_n - \Psi_{n+1}) \tag{3}$$

For small oscillation, $\sin\theta_n \approx \Psi_n/l$. So,

$$\frac{d^2\Psi_n(t)}{dt^2} = -\frac{g}{l}\Psi_n - \frac{K}{m}(\Psi_n - \Psi_{n-1}) + \frac{K}{m}(\Psi_{n+1} - \Psi_n) \tag{4}$$

or,

$$\frac{d^2\Psi_n(t)}{dt^2} = -\frac{g}{l}\Psi_n(t) + \frac{aK}{m}(\frac{\Psi_{n+1}(t) - \Psi_n(t)}{a}) - \frac{aK}{m}(\frac{\Psi_n(t) - \Psi_{n-1}(t)}{a}) \tag{5}$$

where, a is the equilibrium separation between the bobs.

(b) In case of continuum limit, n is very large ($n \to \infty$) and a is very small ($a \to 0$) but $na \to z_0$(a finite value). The equilibrium position of the nth bob is $z = na$. Therefore,in the equation,Ψ_n can be replaced by $\Psi(z,t)$, a function of both z and t.Hence the above equation can be written as,

$$\frac{\partial^2\Psi(z,t)}{\partial t^2} = -\frac{g}{l}\Psi(z,t) + \frac{aK}{m}\left(\frac{\Psi((n+1)a,t) - \Psi(na,t)}{a}\right) - \frac{aK}{m}\left(\frac{\Psi(na,t) - \Psi((n-1)a,t)}{a}\right) \tag{6}$$

or,

$$\frac{\partial^2\Psi(z,t)}{\partial t^2} = -\frac{g}{l}\Psi(z,t) + \frac{aK}{m}\frac{\partial\Psi(z,t)}{\partial z}\Big|_{z=na+\frac{a}{2}} - \frac{aK}{m}\frac{\partial\Psi(z,t)}{\partial z}\Big|_{z=na-\frac{a}{2}} \tag{7}$$

Using

$$\frac{\Psi((n+1)a,t) - \Psi(na,t)}{a} = \frac{\partial\Psi(z,t)}{\partial z}\Big|_{z=na+\frac{a}{2}} \tag{8}$$

$$\frac{\Psi(na,t) - \Psi((n-1)a,t)}{a} = \frac{\partial\Psi(z,t)}{\partial z}\Big|_{z=na-\frac{a}{2}} \tag{9}$$

$$\frac{\partial^2\Psi(z,t)}{\partial z^2} = \frac{1}{a}\left(\frac{\partial\Psi(z,t)}{\partial z}\Big|_{z=na+\frac{a}{2}} - \frac{\partial\Psi(z,t)}{\partial z}\Big|_{z=na-\frac{a}{2}}\right) \tag{10}$$

we have,

$$\frac{\partial^2\Psi(z,t)}{\partial t^2} = -\frac{g}{l}\Psi_n(z,t) + \frac{aK}{m}\left(a\frac{\partial^2\Psi_n(z,t)}{\partial z^2}\right) \tag{11}$$

or,

$$\frac{\partial^2\Psi(z,t)}{\partial t^2} = -\omega_0^2\Psi(z,t) + v_0^2\frac{\partial^2\Psi_n(z,t)}{\partial z^2} \tag{12}$$

Where

$$\omega_0 = \sqrt{\frac{g}{l}} \tag{13}$$

and

$$v_0^2 = \frac{Ka^2}{m} \tag{14}$$

If we write $K = \frac{k}{a}$ where k is generally the elastic modulas of the system under consideration, and $m = a\mu$ where μ is the mass per unit length then

$$v_0 = \sqrt{\frac{k}{\mu}} \tag{15}$$

2. Consider the wave equation given below.

$$\left[4\frac{\partial^2}{\partial x^2} + 9\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}\right]\Psi(\vec{r}, t) = 0 \tag{16}$$

(a) What is the speed of a **traveling wave** solution propagating along the x axis?

(b) What is the speed of a traveling wave solution along the y axis ?

(c) For what value of b is the traveling wave given belw a solution of the wave equation given above?

$$\Psi(\vec{r}, t) = e^{-[bx+y-5c_s t]^2} \tag{17}$$

(d) What is the speed of the traveling wave given above?

(a) For the travelling wave solution, propagating along x axis, the above equation reduces to,

$$4\frac{\partial^2\Psi(x,t)}{\partial x^2} = \frac{1}{c_s^2}\frac{\partial^2\Psi(x,t)}{\partial t^2} \tag{18}$$

or,

$$\frac{\partial^2\Psi(x,t)}{\partial x^2} = \frac{1}{4c_s^2}\frac{\partial^2\Psi(x,t)}{\partial t^2} \tag{19}$$

Comparing the above equation with the traveling wave equation propagating along x axis with velocity v i.e,

$$\frac{\partial^2\Psi(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\Psi(x,t)}{\partial t^2} \tag{20}$$

we get v=2c$_s$.

(b) Similarly for a wave traveling along the y axis the velocity is v=3c$_s$.

(c) $\Psi(r,t) = e^{-[bx+y-5c_s t]^2}$

Substituting

$$\frac{\partial^2\Psi(r,t)}{\partial x^2} = [-2b^2 + 4b^2(bx+y-5c_s t)^2]e^{-(bx+y-5c_s t)^2} \tag{21}$$

$$\frac{\partial^2\Psi(r,t)}{\partial y^2} = [-2 + 4(bx+y-5c_s t)^2]e^{-(bx+y-5c_s t)^2} \tag{22}$$

$$\frac{\partial^2\Psi(r,t)}{\partial t^2} = [-50c_s^2 + 100c_s^2(bx+y-5c_s t)^2]e^{-(bx+y-5c_s t)^2} \tag{23}$$

in the wave equation we get $b = \pm 2$.

(d) Substituting

$$\frac{\partial^2\Psi(r,t)}{\partial x^2} = [-2b^2 + 4b^2(bx + y - 5c_s t)^2]e^{-(bx+y-5c_s t)^2} \tag{24}$$

$$\frac{\partial^2\Psi(r,t)}{\partial y^2} = [-2 + 4(bx + y - 5c_s t)^2]e^{-(bx+y-5c_s t)^2} \tag{25}$$

$$\frac{\partial^2\Psi(r,t)}{\partial t^2} = [-\frac{50c_s^2}{v^2} + \frac{100c_s^2}{v^2}(bx + y - 5c_s t)^2]e^{-(bx+y-5c_s t)^2} \tag{26}$$

in the equation

$$\frac{\partial^2\Psi(r,t)}{\partial x^2} + \frac{\partial^2\Psi(r,t)}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2\Psi(r,t)}{\partial t^2} \tag{27}$$

we get

$$[-2b^2 - 2 + \frac{50c_s^2}{v^2}] + [4b^2 + 4 - \frac{100c_s^2}{v^2}] \times (bx + y - 5c_s t)^2 = 0 \tag{28}$$

or,

$$[-b^2 - 1 + \frac{25c_s^2}{v^2}][2 + 4(bx + y - 5c_s t)^2] = 0 \tag{29}$$

as the second term is always positive, therefore equating the first term to zero and putting $b = \pm 2$, we get $v = \sqrt{5}c_s$.

Alternatively,

$bx + y - 5c_s t = \vec{k}\cdot\vec{r} - \omega t$. So, $\vec{k} = b\hat{i} + \hat{j}$ and $\omega = 5c_s$. So, $v = \frac{\omega}{|\vec{k}|} = \frac{5c_s}{\sqrt{b^2+1}}$. $b = \pm 2$, so $v = \sqrt{5}c_s$.

3. Which of the folowing are traveling waves? If yes, what is the speed?

(i) $\Psi(x,t) = \sin^2\left[\pi\left(ax + bt\right)\right]$; (ii) $\Psi(x,t) = \sin^2\left[\pi\left(ax + bt\right)^2\right]$;
(iii) $\Psi(x,t) = \sin^2\left[\pi\left(ax^2 + bt\right)\right]$; (iv) $\Psi(x,t) = e^{\left[\pi\left(ax^2+bt^2+2\sqrt{ab}xt\right)\right]}$;
(v) $\Psi(x,y,t) = e^{\frac{1}{L^2}\left[\pi(ax+by-t)^2\right]}$

A traveling wave solution can always be represented in the form $\Psi(x,t) = f(x \pm vt)$. where v is the velocity of the wave.

(i)

$$\Psi(x,t) = \sin^2[\pi(ax + bt)] \tag{30}$$

or,

$$\Psi(x,t) = \sin^2[a\pi(x + \frac{b}{a}t)] \tag{31}$$

or,

$$\Psi(x,t) = \sin^2[a\pi(x+vt)] \tag{32}$$

where velocity v=$\frac{b}{a}$.

(ii)

A similar approach yields,

$$\Psi(x,t) = \sin^2[\pi(ax+bt)^2] \tag{33}$$

or,

$$\Psi(x,t) = \sin^2[a^2\pi(x+\frac{b}{a}t)^2] \tag{34}$$

or,

$$\Psi(x,t) = \sin^2[a^2\pi(x+vt)^2] \tag{35}$$

where v=$\frac{b}{a}$.

(iii)

$$\Psi(x,t) = \sin^2[\pi(ax^2+\frac{b}{t})] \tag{36}$$

is not a traveling wave.

(iv)

$$\Psi(x,t) = e^{[\pi(ax^2+bt^2+2\sqrt{ab}xt)]} \tag{37}$$

or,

$$\Psi(x,t) = e^{[a\pi(x+\sqrt{\frac{b}{a}}t)^2]} \tag{38}$$

or,

$$\Psi(x,t) = e^{[a\pi(x+vt)^2]} \tag{39}$$

where v=$\sqrt{\frac{b}{a}}$.

(v)

$$\Psi(x,y,t) = e^{\frac{\pi}{L^2}(ax+by-t)^2} \tag{40}$$

If the above one is a travling wave, it will satisfy the wave equation,

$$\frac{\partial^2\Psi(\vec{r},t)}{\partial x^2} + \frac{\partial^2\Psi(r,t)}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2\Psi(r,t)}{\partial t^2} \tag{41}$$

Substituting

$$\frac{\partial^2\Psi(\vec{r},t)}{\partial x^2} = \left[\frac{2\pi a^2}{L^2} + \frac{4\pi^2a^2}{L^2}(ax+by-t)^2\right] e^{\frac{\pi}{L^2}(ax+by-t)^2} \tag{42}$$

$$\frac{\partial^2\Psi(\vec{r},t)}{\partial y^2} = \left[\frac{2\pi b^2}{L^2} + \frac{4\pi^2 b^2}{L^2}(ax+by-t)^2\right] e^{\frac{\pi}{L^2}(ax+by-t)^2} \tag{43}$$

and,

$$\frac{\partial^2\Psi(\vec{r},t)}{\partial t^2} = \left[\frac{2\pi}{L^2} + \frac{4\pi^2}{L^2}(ax+by-t)^2\right] e^{\frac{\pi}{L^2}(ax+by-t)^2} \tag{44}$$

in the above equation we get,

$$\left(a^2+b^2-\frac{1}{v^2}\right)\left[\frac{2\pi}{L^2} + \frac{4\pi}{L^2}(ax+by-t)^2\right] = 0 \tag{45}$$

As $\left[\frac{2\pi}{L^2} + \frac{4\pi}{L^2}(ax+by-t)^2\right]$ is always a positive quantity and more importantly, above equation is to be satisfied for all points (x,y) in space. So equating $\left(a^2+b^2-\frac{1}{v^2}\right)$ to zero, we get v=$\frac{1}{\sqrt{a^2+b^2}}$.

Alternatively,

The most general form of a travelling wave in 2-D or, 3-D space is: $\Psi(\vec{r},t) \equiv f(\vec{k}\cdot\vec{r} \pm \omega t)$. The given function is of this form; hence it is a travelling wave. Here, $\vec{k} = a\hat{i} + b\hat{j}$ and $\omega = 1$. So, the speed of the wave propagation is $v = \frac{\omega}{|\vec{k}|} = \frac{1}{\sqrt{a^2+b^2}}$.

4. (a) The following is the expression for a **harmonic wave** which is a solution of the wave equation.

$$\Psi(\vec{r},t) = 20\cos\left(7x+6y-3z-10t+\frac{\pi}{3}\right) \tag{46}$$

(i) Find the equation for the **wavefronts** at time $t = 0$ s and $t = 0.5$ s. What is the geometrical surface representing the wavefront? Sketch it at these two times qualitatively.

(ii) What is the value of the phase at the point $(1,0,0)$ at time $t = 1$s.

(iii) What is the **phase velocity** of the wave?

(b) Two waves of the same frequency have the wave vectors $\vec{\mathbf{k_1}} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}\ m^{-1}$ and $\vec{\mathbf{k_2}} = 4\hat{\mathbf{i}} + 3\hat{\mathbf{j}}\ m^{-1}$. The two waves have the same phase at the point $(2,7,8)$ m. What is phase difference between the waves at the point $(3,5,8)$ m?

(a)(i) Wavefronts represent the geometrical surfaces on which at every point the phase is same, i.e. $\phi(x,y,z;t) = 7x+6y-3z-10t+\pi/3 = c$. So , at $t=0$, equation of the wavefront is $7x+6y-3z+\pi/3 = c$; here c represent the family of wavefronts each of which has the constant phase c everywhere on it. Similarly , at $t = 0.5\,s$ equation of the wavefront is $7x+6y-3z-5+\pi/3 = c$. Each of the wavefronts will now be shifted by distance

$\frac{\omega t}{|\vec{k}|} = \frac{5}{\sqrt{7^2+6^2+3^2}} = 0.5\,m$ along the direction of propagation. Geometrically, each of these equations represent a plane surface.

(ii) Putting $x = 1$, $y = 0$, $z = 0$, and $t = 1$ we get phase as $7 - 10 + \frac{\pi}{3} = \frac{\pi}{3} - 3$.
(iii) Phase velocity v=$\frac{\omega}{k}$. Here ω=10 and k=$\sqrt{7^2+6^2+(-3)^2}$=$\sqrt{94}$. Hence v=$\frac{10}{\sqrt{94}}\,m/s$.
(b) Suppose ϕ_1 and ϕ_2 represent the initial phases of the waves at the origin (0,0,0). As the waves have same frequency, so $\omega_1 = \omega_2 = \omega$ (say). Equating the phases for the point (2,7,8) we get,

$$\vec{k_1}.\vec{r} - \omega t + \phi_1 = \vec{k_2}.\vec{r} - \omega t + \phi_2 \tag{47}$$

i.e, $\phi_2 - \phi_1$=5.

At the point (3,5,8), phase difference is $[\vec{k_2}.\vec{r} - \omega t + \phi_2] - [\vec{k_1}.\vec{r} - \omega t + \phi_1]$=$[\vec{k_2}.\vec{r} - \vec{k_1}.\vec{r}] + [\phi_2 - \phi_1]$=[27-29]+5=3.

5. (a) A longitudinal **standing wave** $\Psi(\vec{r}, t) = A\cos kx \cos \omega t$ is maintained in a homogeneous medium of density ρ. Find expressions for the **potential energy density** and the **kinetic energy density**.
(b) Consider a **spherical wave** $\Psi(\vec{r}, t) = \frac{a}{r}\sin[k(r - c_s t)]$, with $k = 3m^{-1}$ and $c_s = 330ms^{-1}$.
(i) What is the frequency of the wave?
(ii) How much does the amplitude of the wave change over $\Delta r = \frac{2\pi}{k}$?
(iii) In which direction does this wave propagate?
(a) $\Psi(r, t) = A\cos(kx)\cos(\omega t)$.
Potential energy density: During the propagation of any longitudinal wave, a volume element of the medium is subjected to periodic compression and expansion. Suppose an increase in pressure from P_0 to $P_0 + p$ (p is acoustic pressure) changes the volume element by a factor $1 - s$, where s is the condensation. Now $s = \frac{\Psi(x)-\Psi(x+\delta x)}{\delta x} = -\frac{\partial \Psi}{\partial x}$ (see any standard text book like H.J. Pain). The work done in this process will be stored as the potential energy in the medium. So the potential energy density is $E_p = \int_0^s p ds$. Now from the definition of bulk modulus(B), we know $p = Bs$. So,

$$E_p = \frac{1}{2}Bs^2 = \frac{1}{2}B\left(\frac{\partial \Psi}{\partial x}\right)^2$$

$$
\begin{aligned}
&= \frac{1}{2} B A^2 k^2 \sin^2(kx)\cos^2(\omega t) \\
&= \frac{1}{2} \rho\omega^2 A^2 \sin^2(kx)\cos^2(\omega t)
\end{aligned}
$$

In the last equation we use the relation, $v = \omega/k = \sqrt{B/\rho}$.

Kinetic energy density: $E_k = \frac{1}{2}\rho(\frac{\partial\Psi(x,t)}{\partial t})^2 = \frac{\rho\omega^2 A^2}{2}\cos^2(kx)\sin^2(\omega t)$.

(b) $\Psi(r,t) = \frac{a}{r} sin[k(r - c_s t)]$

(i) Comparing the above equation with wave equation $\Psi(r,t) = \frac{a}{r} sin(k.r - \omega t)$ we get angular frequency $\omega = kc_s = 990 s^{-1}$.

(ii) $\Psi(r+\triangle r,t) = \frac{a}{r+\triangle r} sin[k(r+\triangle r - c_s t)] = \frac{a}{r+\triangle r} sin[k(r-c_s t)+2\pi] = \frac{a}{r+\triangle r} sin[k(r-c_s t)]$

New amplitude $A' = \frac{a}{r+\frac{2\pi}{k}} = \frac{3a}{2\pi+3r}$

putting the value of k we get $A' = \frac{3a}{2\pi+3r}$.

change in amplitude is then $\frac{A-A'}{A} = \frac{\frac{a}{r} - \frac{3a}{2\pi+3r}}{\frac{a}{r}} = \frac{2\pi}{2\pi+3r}$.

(iii) $\Psi(r,t) = \frac{a}{r} sin[k(r - c_s t)] = \frac{a}{r} sin[kc_s(\frac{r}{c_s} - t)] = \frac{1}{r} f(\frac{r}{c_s} - t)$

Therefore the wave is traveling radially in outward direction from the origin with velocity c_s.

6. (a) An **electromagnetic wave** is propagating along the direction $\hat{\mathbf{i}} + \hat{\mathbf{j}}$. What are the possible directions for the electric and magnetic fields of this wave?

(b) If the electric field of an electromagnetic wave is given as $\vec{\mathbf{E}} = E_0\left(3\vec{\mathbf{i}} - 2\vec{\mathbf{j}}\right)\cos\left(2x + 3y - \sqrt{3}z + 1.2\times 10^9 t\right)$. Find the direction of propagation, the magnetic field and the phase velocity of the wave.

(a) We use the relations: $\vec{E}\cdot\vec{k} = \vec{B}\cdot\vec{k} = \vec{E}\cdot\vec{B} = 0$. $\vec{k} = \hat{i} + \hat{j}$. So we have

$$
\begin{aligned}
E_x + E_y &= 0\,, \\
B_x + B_y &= 0\,, \\
E_x B_x + E_y B_y + E_z B_z &= 0
\end{aligned}
$$

So, $E_x = -E_y = E_0$, $B_x = -B_y = B_0$ and $E_z B_z = -2E_0 B_0$.

Special cases:

(i) $E_0 = 0$, $\vec{E} = E_z\hat{k}$, $\vec{B} = B_0(\hat{i} - \hat{j})$

(ii)$B_0 = 0$, $\vec{B} = B_z\hat{k}$, $\vec{E} = E_0(\hat{i} - \hat{j})$

(iii) $\vec{E} = E_0(\hat{i} - \hat{j}) + E_z\hat{k}$, $\vec{B} = B_0(\hat{i} - \hat{j}) - \frac{2E_0B_0}{E_z}\hat{k}$.

The magnitude and and sign of different variables , E_0, B_0, E_z, B_z are to be fixed by other relation $\hat{k} \times \vec{E} = c\vec{B}$ or, $\hat{k} \times \vec{B} = -\vec{E}/c$, where c is the speed of light.

(b) The direction of propagation is $2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \sqrt{3}\hat{\mathbf{k}}$. The magnetic field can be found from the relation $\vec{\mathbf{k}} \times \vec{\mathbf{E}} = \omega\vec{\mathbf{B}}$. Since $\vec{\mathbf{E}} = 3E_0\hat{\mathbf{i}} - 2E_0\hat{\mathbf{j}}$ we evaluate the curl $\vec{\mathbf{k}} \times \vec{\mathbf{E}}$ and divide it by ω to get the answer. Note $\hat{\mathbf{k}}$ is the unit vector along z while $\vec{\mathbf{k}}$ is the wave vector–do not mix up between them. The magnetic field is

$$\vec{\mathbf{B}} = \frac{E_0}{1.2 \times 10^9}\left[-2\sqrt{3}\hat{\mathbf{i}} - 3\sqrt{3}\hat{\mathbf{j}} - 13\hat{\mathbf{k}}\right] \tag{48}$$

the phase velocity is simply $\frac{\omega}{k}$. Here $k = |\vec{\mathbf{k}}| = 4m^{-1}$ and therefore $v_p = 3 \times 10^8 m.s^{-1} = c$.

7. The phase velocity v_p of ripples on a liquid surface is given as:

$$v_p = \sqrt{\frac{g\lambda}{2\pi} + \frac{2\pi S}{\lambda\rho}} \tag{49}$$

where g is the acceleration due to gravity, S is the surface tension and ρ is the density of the liquid.

(a) Write down the relation between ω and k, i.e. ω as a function of k. Plot, qualitatively, $\omega(k)$.

(b) Find an expression for the **group velocity**.

(c) When is the group velocity equal to the phase velocity?.

(d) If $g = 10 \quad \text{m}.s^{-2}$, $S = 4 \times 10^{-2} \quad \text{N}.m^{-1}$ and $\rho = 1000 \quad \text{kg}.m^{-3}$ then find v_g when $v_g = v_p$.

(a)

$$v_p = \sqrt{\frac{g\lambda}{2\pi} + \frac{2\pi S}{\lambda\rho}} \tag{50}$$

Substituting $v_p = \frac{\omega}{k}$ and $k = \frac{2\pi}{\lambda}$ and taking square of both sides of the above equation we get,

$$\omega^2 = gk + \frac{Sk^3}{\rho} \tag{51}$$

(b) Group velocity $v_g = \frac{d\omega}{dk}$

$$v_g = \frac{g + \frac{3Sk^2}{\rho}}{2\sqrt{gk + \frac{Sk^3}{\rho}}} \tag{52}$$

(c)

$v_g = v_p$ implies

$$\sqrt{\frac{g\lambda}{2\pi} + \frac{2\pi S}{\lambda\rho}} = \frac{g + \frac{3S^2k^2}{\rho}}{2\sqrt{gk + \frac{Sk^3}{\rho}}} \tag{53}$$

which gives $k = \frac{2\pi}{\lambda} = \sqrt{\frac{\rho g}{S}}$. Therefore $\lambda = 2\pi\sqrt{\frac{S}{\rho g}}$.

(d)

Substituting $k = \sqrt{\frac{\rho g}{S}}$ we get $v_g = \sqrt{2\sqrt{\frac{Sg}{\rho}}}$ and then using all the given values v_g comes out to be 0.2 m/s.

8. (a) Write down the **three dimensional** wave equation in (i) spherical and (ii) cylindrical coordinates. Obtain solutions of the form (i) $\Psi(r,t)$ and (ii) $\Psi(\rho,t)$. Explain physically the solutions you obtain.

(b) Let us assume a **change of coordinates** from (x,t) to (x',t') given by $x' = \gamma\,(x - vt)$, $t' = \gamma\left(t - \frac{vx}{c^2}\right)$. Show that the wave equation (one space and one time dimension) for $\Psi(x,t)$ remains unaltered when written in terms of x', t'. Here $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. The coordinate transformation is known as the **Lorentz transformation** which arises in **Einstein's special relativity**.

(a) The wave function Ψ in the wave equation

$$\nabla^2\Psi(\vec{r},t) = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2} \tag{54}$$

is a fuction of (r,θ,ϕ,t) in case of spherical coordinates. So the above equation becomes,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\,\theta}\frac{\partial}{\partial\theta}\left(\sin\,\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\,\theta}\frac{\partial^2\Psi}{\partial\phi^2} = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2} \tag{55}$$

For cylindriacal coordinates (ρ,φ,z), the wave equation(54) becomes:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Psi}{\partial\varphi^2} + \frac{\partial^2\Psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2} \tag{56}$$

(i) $\Psi(r,t)$**:** Eq. 55 becomes:

$$\frac{1}{r^2}\frac{\partial}{\partial r}[r^2\frac{\partial\Psi}{\partial r}] = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2} \tag{57}$$

Substituting

$$\Psi(r,t) = \frac{u(r,t)}{r} \tag{58}$$

in the above equation we get,

$$\frac{\partial^2 u}{\partial r^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{59}$$

which has the well known solution $u(r,t) = c_1 f(r-vt) + c_2 f(r+vt)$. Therefore, $\Psi(r,t) = \frac{C_1}{r} f(r-vt) + \frac{C_2}{r} f(r+vt)$. The first term represents a spherical wave propagating outward from the origin with a velocity v and the second term represent a similar wave propagating towards the origin.

(ii) $\Psi(\rho, t)$**:** the cylindrical wave equation(56) becomes,

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \frac{\partial \Psi}{\partial \rho}) = \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2} \tag{60}$$

Substituting $\Psi(\rho,t) = f_1(\rho) f_2(t)$ and assuming that $f_1(\rho) = \frac{u(\rho)}{\sqrt{\rho}}$ we get

$$\frac{d^2 u}{d\rho^2} + \left(k^2 + \frac{1}{4\rho^2}\right) u = 0 \tag{61}$$

where, k^2 is the constant of separation. and

$$\frac{d^2 f_2}{dt^2} + \omega^2 f_2 = 0 \tag{62}$$

where, $\omega^2 = v^2 k^2$.

For large value of ρ , the solution f can be written as, $\Psi(\rho, t) \approx \frac{c_1}{\sqrt{\rho}} e^{i(k\rho - \omega t)} + \frac{c_2}{\sqrt{\rho}} e^{i(k\rho + \omega t)}$. The first term represents an outgoing and the second term represent an incoming cylindrical wave respectively.

(b) Since $x' = \gamma(x - vt)$ and $t' = \gamma(t - \frac{vx}{c^2})$, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial t}$ can be written in terms of of $\frac{\partial}{\partial x'}$ and $\frac{\partial}{\partial t'}$. Obtaining this, work out what happens to the second derivatives. Substitute the second derivatives in the wave equation and note that it remains the same when re-written in terms of t', x'.