Are bananas a good source of recursion in your diet?

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Reference

Main reference:

Erik Meijer, Maarten Fokkinga, and Ross Paterson. **“Functional programming with bananas, lenses, envelopes and barbed wire”.** In: *Functional Programming Languages and Computer Architecture*. Ed. by John Hughes. Berlin, Heidelberg: Springer Berlin Heidelberg, 1991, pp. 124–144. ISBN: 978-3-540-47599-6

But also:

nLab authors. **recursion scheme.** `https://ncatlab.org/nlab/show/recursion+scheme`. Revision 5. Mar. 2025

Wikipedia contributors. **F-algebra — Wikipedia, The Free Encyclopedia.** [Online; accessed 25-March-2025]. 2024. URL: `https://en.wikipedia.org/w/index.php?title=F-algebra&oldid=1265837291`

Bartosz Milewski. **Category theory for programmers.** Bartosz Milewski, 2019

Back to basics

```
int fac(int target) {
  int i = 1;
  int n = 1;

  start:
    if (n > target) goto end;
    i *= n++;
    goto start;
  end:

  return i;
}

```

Abstracting out of shame

```
int facw(int target) {
  int i = 1;
  int n = 1;

  while (n <= target) i *= n++;

  return i;
}

```

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  if target <= 0 then
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  (1 to target).fold(1)(_ * _)
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- `goto` can express any computable function
- So can `while` loops
- What about recursion v our abstractions? Is there a subset that allows expressing a reasonably large set of functions? All the recursively computable functions?

Catamorphism

"Cata" - down/downwards

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def cataList[A, B](b: B)(*: (A, B) => B)(l: List[A]): B =
  l match {
    case Nil => b
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Given a `b : B`, a function `*: (A, B) => B`, $(\!|$`b, *`$|\!)$ is a catamorphism over lists of type `A`. The final type is `List[A] => B`.

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- stopping point / base case
- inductive case - a way to split a `B` into an `h` and a `t`. Sufficient to have an `h` and a `b1`.

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```
def anaList[A, B](p: B => Bool)(g: B => (A, B))(b: B): List[A] =
  if p(b) then
    Nil
  else
    val (h, b1) = g(b)
    h :: anaList(p)(f)(b1)

```

or an *'unfold'*. Written as concave lenses $[\!(g, p)\!]$.

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Look back at `facf`

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'Hylo' - 'dust/matter' (wood?). Originating from philosophical idea that form and matter are one. Written $[\![(b, *), (g, p)]\!]$.

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Hylomorphisms are more generally any functions whose call trees are "list-like". (I think these are exactly linear recursive functions?)

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Why? Why these abstractions?

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Given an endofunctor F such that the category of F-algebras has an initial object $(\mu F, in)$, the catamorphism for an F-algebra (A, φ) is the unique homomorphism from the initial F-algebra $(\mu F, in)$ to (A, φ). The unique morphism between the carriers is also denoted cata $\varphi : \mu F \to A$.

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Or in code:

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This one line describes *all* folds. Not over lists, not over trees, but *all possible* folds.

Just Enough Category Theory to be Dangerous

A *category* is a collection of objects (points) and a collection of morphisms (arrows) between them.

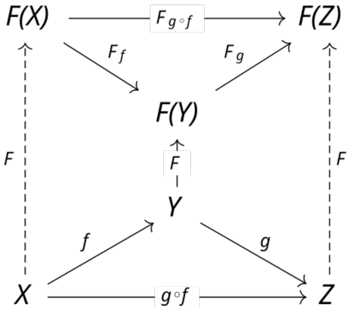
- For every object A, there is an identity morphism $id_A : A \to A$
- If $f : A \to B$ and $g : B \to C$, there must be a morphism $h : A \to C$ such that $g \circ f = h$.
- Composition is associative.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g \circ f"'] \arrow[loop left, "1_A"] & B \arrow[d, "g"] \arrow[loop right, "1_B"] \\
 & C \arrow[loop right, "1_C"]
\end{tikzcd}
$$

Just Enough Category Theory to be Dangerous

A *functor* is a mapping between categories that preserves the structure of the category.

- For every object A, there is a corresponding object $F(A)$.
- For every morphism $f : A \to B$, there is a morphism $F(f) : F(A) \to F(B)$.
- Identity is preserved: $F(id_A) = id_{F(A)}$.
- Composition is preserved: $F(g \circ f) = F(g) \circ F(f)$.



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For example, `List[_]` is a functor (`Type =>> Type`) that maps any type `A` to `List[A]`, and a function `A => B` to a function `List[A] => List[B]`.

Missing *initial*, *F-algebra*, and related terms. Hopefully understood better through code.

Implementation

Algebra

For an endofunctor F on a category C, an F-algebra is a pair (A, α) where A is an object in C and $\alpha : F(A) \to A$ is a morphism in C.

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We can check that these algebras form a category with algebra homomorphisms as their morphisms. This category has an *initial object*, which is a bit like the bottom element in a lattice, it has an arrow to every other object in the category.

So we are saying, an initial algebra can be 'embedded' (via a homomorphism) into all other algebras. It is a minimal F-algebra.

In particular, we used the fact that our language, the meta-logic, has a way to provide us least fixpoint semantics (via recursive type definitions), so the result is in fact the least fixpoint of the functor.

Why am I looking at recursion schemes?

- Well, it's interesting. What constructs do we finally really need to eliminate arbitrary recursion?
- What kind of analysis can we do on programs when we know they are composed of only these schemes?
- Existing work on decidable verification of programs with catamorphisms as theory extensions. Led to the questions "why catamorphisms?" and "what more is out there?"
- Once we (I) understand catamorphisms, the next thing to understand is why they lead to decidable theories.
- How do structured recursion schemes relate to other decidable fragments, e.g. sufficiently surjective functions, which (I think) subsume catamorphisms?

All good questions to hopefully answer in the next few lifetimes.

Thank you!