Diff Geo HW

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August 22, 2021

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Metric Spaces

A *metric space* is a set X equipped with a map $d : X \times X \to \mathbb{R}$ such that :

1. $d(x, x) = 0$
2. $d(x, y) > 0$ if $x \neq y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$

for all $x, y, z \in X$

d: distance function or metric

Question 1

Show that: For any points $\boldsymbol{a, b, x, y}$ in a metric space X,
$\boldsymbol{|d(a,b) - d(x,y)| \leq d(a,x) + d(b,y)}$

Using the property of the absolute value function $|a| \leq b \iff -b \leq a \leq b$ we get:

$$-d(a,x) - d(b,y) \leq d(a,b) - d(x,y) \leq d(a,x) + d(b,y)$$

Now we focus on the a term $\alpha \triangleq d(a,x) + d(x,y) + d(y,b)$. Using property (4):

$$d(a,b) \leq d(a,x) + d(x,b) \leq d(a,x) + d(x,y) + d(y,b)$$

Rearranging terms and using property (3) we obtain:

$$d(a,b) - d(x,y) \leq d(a,x) + d(b,y)$$

Which is the RHS of the inequality. Similarly, the LHS can be proven*.
*Modify (4) into $d(x,z) - d(y,z) \leq d(x,y)$ and $\alpha = -d(a,x) - d(x,y) + d(y,b)$

Question 2

Check that the diamond and square metrics on $\mathbb{R}^n$ are indeed metrics. Show that the euclidean metric on $\mathbb{R}^n$ is indeed a metric. (The triangle inequality in this context is equivalent to Minkowski's inequality.)

For any $x, y \in \mathbb{R}^n$ where $v = (v_1, v_2, ..., v_n)$:

1. Diamond metric: $d_1(x, y) = \sum_{i \in \mathcal{I}} |x_i - y_i|$
2. Euclidean metric: $d_2(x, y) = \sqrt{\sum_{i \in \mathcal{I}} |x_i - y_i|^2}$
3. Square metric: $d_\infty(x, y) = max_{i \in \mathcal{I}}\{|x_i - y_i|\}$

Now we just have to show that each of these metrics satisfy the four conditions that characterise metric spaces. We will look at each metric separately.

Question 2 contd.

The Diamond metric

1. $d_1(x,x) = \sum_{i=1}^{n} |x_i - x_i| = 0$ for all $i \in \mathcal{I}$.
2. Suppose $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| = 0$. We know that that for $p \in \mathbb{R}$, $|p| \geq 0$ and $|p| = 0 \iff p = 0$ (positive definite). Therefore for the assumption to hold, $|x_i - y_i| = 0$ for all $i \in \mathcal{I}$. Hence $x_i = y_i$ for all $i \in \mathcal{I}$. This shows that for $x \neq y$, $d_1(x,y) > 0$.
3. We can rewrite $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |-(-x_i + y_i)| = \sum_{i=1}^{n} |(-x_i + y_i)| = \sum_{i=1}^{n} |y_i - x_i|$. Therefore, $d_1(x,y) = d_1(y,x)$.
4. Using the triangle inequality $|(x_i - y_i) + (y_i - z_i)| \leq |x_i - y_i| + |y_i - z_i| \rightarrow |x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$. Taking the summation, $\sum_{i=1}^{n} |x_i - z_i| \leq \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i|$. Hence, we have shown, $d_1(x,z) \leq d_1(x,y) + d_1(y,z)$.

Question 2 contd.

The Euclidean metric

1. $d_2(x,x) = \sqrt{\sum_{i=1}^{n} |x_i - x_i|^2} = 0$ for all $i \in \mathcal{I}$.
2. Suppose $d_2(x,y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} = 0$. We know that that for $p \in \mathbb{R}$, $\sqrt{p} \geq 0$ and $\sqrt{p} = 0 \iff p = 0$ (positive definite). Therefore for the assumption to hold, $\sum_{i=1}^{n} |x_i - y_i|^2 = 0$ for all $i \in \mathcal{I}$. Further, $|x_i - y_i|^2 = 0$ for all $i \in \mathcal{I}$. Hence $x_i = y_i$ for all $i \in \mathcal{I}$. This shows that for $x \neq y$, $d_2(x,y) > 0$.
3. We can rewrite $d_2(x,y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} = \sqrt{\sum_{i=1}^{n} |-(-x_i + y_i)|^2} = \sqrt{\sum_{i=1}^{n} |(-x_i + y_i)|^2} = \sqrt{\sum_{i=1}^{n} |y_i - x_i|^2}$. Therefore, $d_2(x,y) = d_2(y,x)$.
4. Minkowski's inequality* says that $\left(\sum_{k=1}^{n} |\alpha_k + \beta_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} |\alpha_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |\beta_k|^p\right)^{\frac{1}{p}}$. If we substitute $\alpha_k = x_k - y_k$ and $\beta_k = y_k - z_k$, we get the desired triangle inequality. $\sqrt{\sum_{i=1}^{n} |x_i - z_i|^2} \leq \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} + \sqrt{\sum_{i=1}^{n} |y_i - z_i|^2}$. Hence, we have shown, $d_2(x,z) \leq d_2(x,y) + d_2(y,z)$.

Question 2 contd.

The Diamond metric

1. $d_1(x,x) = \sum_{i=1}^{n} |x_i - x_i| = 0$ for all $i \in \mathcal{I}$.
2. Suppose $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| = 0$. We know that that for $p \in \mathbb{R}, |p| \geq 0$ and $|p| = 0 \iff p = 0$ (positive definite). Therefore for the assumption to hold, $|x_i - y_i| = 0$ for all $i \in \mathcal{I}$. Hence $x_i = y_i$ for all $i \in \mathcal{I}$. This shows that for $x \neq y$, $d_1(x,y) > 0$.
3. We can rewrite $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |-(-x_i + y_i)| = \sum_{i=1}^{n} |(-x_i + y_i)| = \sum_{i=1}^{n} |y_i - x_i|$. Therefore, $d_1(x,y) = d_1(y,x)$.
4. Using the triangle inequality $|(x_i - y_i) + (y_i - z_i)| \leq |x_i - y_i| + |y_i - z_i| \rightarrow |x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$. Taking the summation, $\sum_{i=1}^{n} |x_i - z_i| \leq \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i|$. Hence, we have shown, $d_1(x,z) \leq d_1(x,y) + d_1(y,z)$.

Question 2 contd.

The Square metric

1. $d_\infty(x,x) = max_{i\in\mathcal{I}}\{|x_i - x_i|\} = max\{0,0,..,0\} = 0$ for all $i \in \mathcal{I}$.
2. Suppose $d_\infty(x,y) = max_{i\in\mathcal{I}}\{|x_i - y_i|\} = 0$ at $i = i*$. We know that that for $p \in \mathbb{R}, |p| \geq 0$ and $|p| = 0 \iff p = 0$ (positive definite). Therefore for the assumption to hold, $|x_i - y_i| \leq 0$ for all $i \in \mathcal{I}, i \neq i* \Rightarrow |x_i - y_i| = 0$. Hence $x_i = y_i$ for all $i \in \mathcal{I}$. This shows that for $x \neq y$, $d_\infty(x,y) > 0$.
3. We can rewrite $d_\infty(x,y) = max_{i\in\mathcal{I}}\{|x_i - y_i|\} = max_{i\in\mathcal{I}}\{|-(-x_i + y_i)|\} = max_{i\in\mathcal{I}}\{|(-x_i + y_i)|\} = max_{i\in\mathcal{I}}\{|y_i - x_i|\}$. Therefore, $d_\infty(x,y) = d_\infty(y,x)$.
4. Need help here*

Minkowski's Inequality

$$(\sum_{k=1}^{n} |\alpha_k + \beta_k|^p)^{\frac{1}{p}} \leq (\sum_{k=1}^{n} |\alpha_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{n} |\beta_k|^p)^{\frac{1}{p}}$$

$(\sum_{k=1}^{n} |\alpha_k|^p)^{\frac{1}{p}} = ||\alpha||_p$ is called the p-norm.The proof makes use of Holder's inequality. It is first shown that if α and β have a finite p-norm, so does $\alpha + \beta$ (convexity arguments).

$$||\alpha + \beta||_p^p = \int |\alpha + \beta|^p \, d\mu \leq \int (|\alpha| + |\beta|)|\alpha + \beta|^{p-1} \, d\mu$$

Now, applying Holder's inequality;

$$||\alpha + \beta||_p^p \leq (||\alpha||_p + ||\beta||_p) \frac{||\alpha + \beta||_p^p}{||\alpha + \beta||_p} \tag{1}$$

Rearranging this gives:

$$||\alpha + \beta||_p \leq ||\alpha||_p + ||\beta||_p \tag{2}$$

Minkowski's Inequality for the Euclidean Metric

For the case of the Euclidean metric, the inequality boils down to:

$$\sqrt{\sum_{i=1}^{n} |x_i - z_i|^2} \leq \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} + \sqrt{\sum_{i=1}^{n} |y_i - z_i|^2}$$

For $n = 1$, we retrieve the same inequality as that of the diamond metric, i.e.,
$\sqrt{|x_1 - z_1|^2} \leq \sqrt{|x_1 - y_1|^2} + \sqrt{|y_1 - z_1|^2} \Rightarrow |x_1 - z_1| \leq |x_1 - y_1| + |y_1 - z_1|$
Should I write something for $n = 2$?*

Ques 0

Test 1

Ques 0

Test 1

Ques 0

Test 1

Topological Spaces

The following questions deal with the idea of Topological Spaces, so here's a quick recap on what exactly those are.

Topological Spaces: A *topological space* is a set X on which a *topology* τ is equipped. τ is a collection of subsets of X (or, τ is a subset of the power set 2^X of X) such that -

1. $\varnothing$ and X should belong to τ
2. the union of the elements in any subset of τ should belong to τ
3. the intersection of the elements in any finite subset of τ should belong to τ

Topological Spaces

The elements of τ are called *open sets*. Thus, a topological space is a pair (X, τ) consisting of a set and a topology on it.

We can reframe the axioms given on the previous slide in terms of open sets -

1. The empty and the full set are open.
2. Any arbitrary union of open sets is open.
3. Any finite intersection of open sets is open.

Question (8)

Show that: The euclidean, diamond, square metrics on $\mathbb{R}^2$ have the same underlying topology. (When we say continuous map from $\mathbb{R}^2$ to $\mathbb{R}$, it is w.r.t this topology.) Further, check that it coincides with the product topology on $\mathbb{R} \times \mathbb{R}$.

Question (4)

Ques - Show that: The underlying topology of the discrete metric is the discrete topology. If a set X has more than one element, then the indiscrete topology on X is not metrizable.

Both these subparts deal with one or the other extreme cases as far as topologies go. So let's look at them individually before solving the problem.

Discrete Topology: The textbook definition of a *discrete topology* is that it is a collection of all subsets of X, i.e, $\tau = 2^X$. There are a few interesting inferences to be drawn from this definition. Since every possible subset is an open subset in the discrete topology, in particular, every *singleton subset* is an open set in this topology.

Question (4)

Indiscrete Topology: The collection $\tau = \{\emptyset, X\}$ on X is the *indiscrete, or trivial topology* on X. A consequence of this collection is that all points in the set X cannot be distinguished from each other through topological means.

Now, let's look at the first part of the problem -
Show that the underlying topology of the discrete metric is the discrete topology

The discrete metric is as follows -

$$d_{\text{discrete}}(x, y) := \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{otherwise.} \end{cases}$$

Question (4)

Now, a metric d on a set X induces a topology τ by taking the idea of the open balls $B(x, r) = \{y : d(x, y) < r\}$ as basic open sets. We need to show that the d_{discrete} we are given produces the discrete topology $\tau = 2^X$.

Let $x \in X$ be an arbitrary element, and let $r \in (0, 1)$; then by the definition of the discrete metric $B_d(x, r) = \{x\}$, so $\forall x \in X, \{x\}$ is an open set.

Now, by the axioms we discussed about topological spaces, any arbitrary union of open sets is open. Let $A \subseteq X$ be any arbitrary subset of X, then $A = \bigcup_{x \in A}\{x\}$, but we have shown that $\forall x \in X, \{x\}$ is an open set.

Question (4)

Since any arbitrary union of open sets is open, we can claim that A is an open set, as induced by the discrete metric. Since this claim holds for any $A \subseteq X$, we thus claim that every subset of X is open, i.e, $\forall A \subseteq X, A \in \tau$.

Since τ contains every possible subset of X, it is the power set 2^X of X. Thus, we have shown that the discrete metric induces a topology $\tau = 2^X$ on X. Since this is the definition of the discrete topology, we have shown that the underlying topology of the discrete metric is the discrete topology. $\square$

Question (4)

We now look at the next part of the problem -
Show that if a set X has more than one element, then the indiscrete topology on X is not metrizable.

We prove this by contradiction. Assume that there exists a metric d on the set X such that (X, d) is a metric space and that the topology induced by this metric on X is the indiscrete topology, $\tau = \{\emptyset, X\}$
X has at least 2 distinct elements x and y, i.e, $\exists x, y \in X$ s.t $x \neq y$.
$\implies d(x, y) = r > 0$
Now, consider the open ball $B(x, r/2)$. This open ball should be an open set in the topology that d induces.

Question (4)

But, $x \in B(x, r/2)$ and since $d(x, y) = r > r/2$, $y \notin B(x, r/2)$.

Thus, $B(x, r/2) \neq \emptyset$ and $B(x, r/2) \neq X$ (as there is at least one element $y \in X$ s.t $y \notin B(x, r/2)$).

Thus, the topology induced by the metric d cannot be the indiscrete topology, since $\tau_{\text{indiscrete}} = \{\emptyset, X\}$

Thus, we have shown that if a set X has more than one element, then the indiscrete topology on X is not metrizable. $\square$

Ques 0

Test 1

Ques 0

Test 1