Diff Geo HW

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Test 0

Test 1

Test 0

Test 1

Test 2

Ques 0

Test 1

Ques 0

Test 1

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Ques 0

Test 1

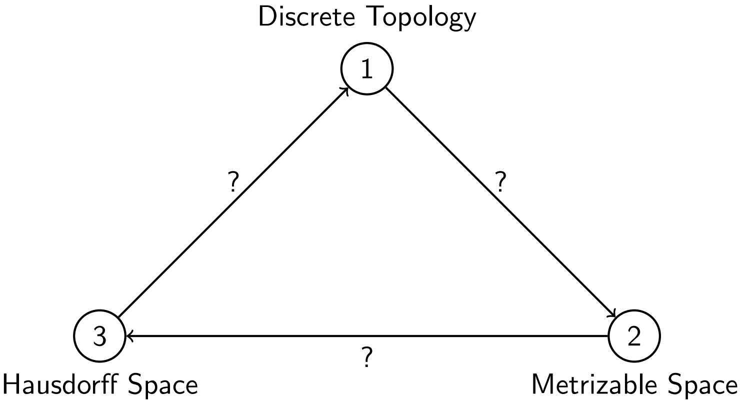
Exercise 3.4 - 15

Problem Statement

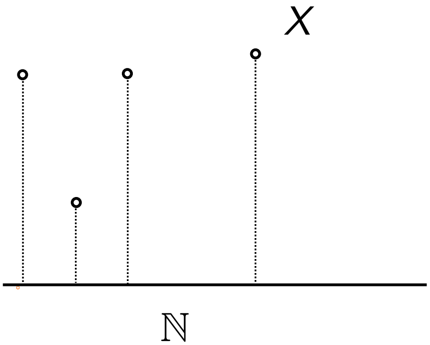
Let X be a *finite* topological space. Show that the following are equivalent:

1. X has the discrete topology.
2. X is metrizable.
3. X is Hausdorff.

Establishing Relationships



$1 \to 2$ — Discrete $\to$ Metrizable



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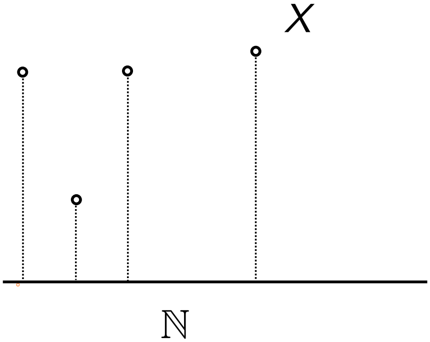

This is equivalent to finding an order on the set and identifying it with $\mathbb{N}$. It only requires the Axiom of Choice!

$2 \to 3$ — Metrizable $\to$ Hausdorff

Given that the space, say (X, τ), is metrizable, there exists a metric $d : X \times X \to \mathbb{R}$ which induces the topology given by τ.

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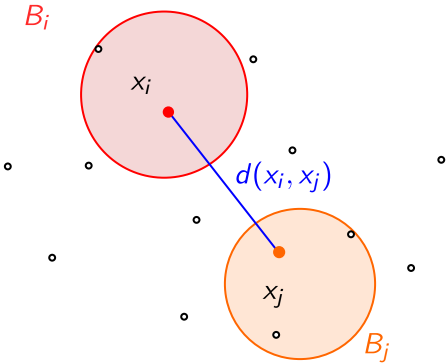

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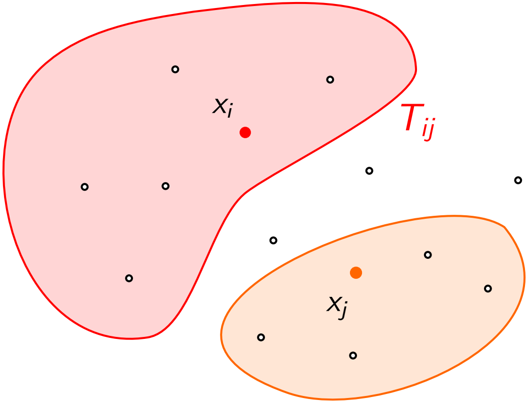

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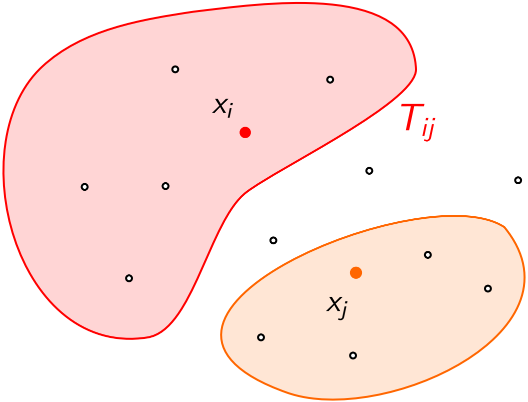

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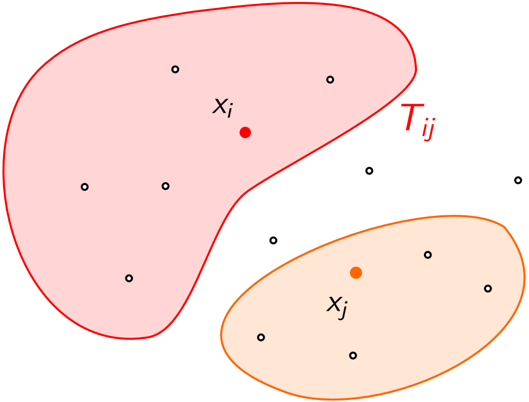

$$\forall x_i \bigcap_{x_j \in X} T_i(x_j) = x_i,$$

$$x_i \in \tau \; .$$

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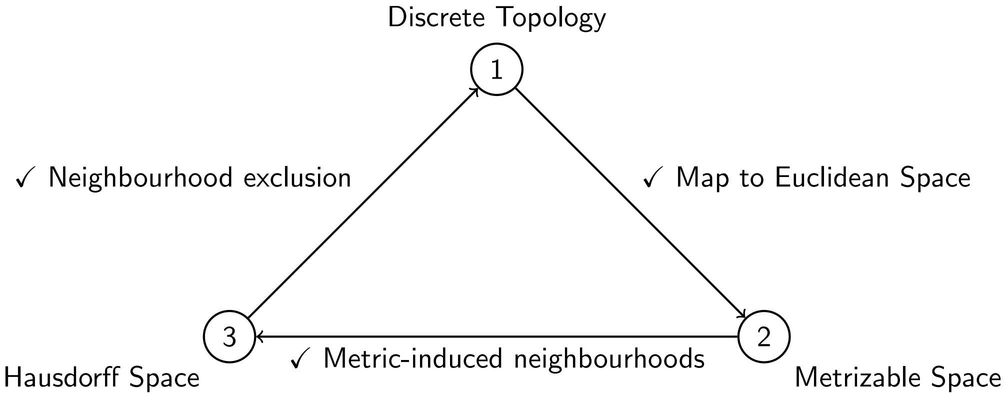

Exercise 3.4 - 16

Problem Statement

Give an example of a topological space which is Hausdorff but not metrizable.

Examining the problem

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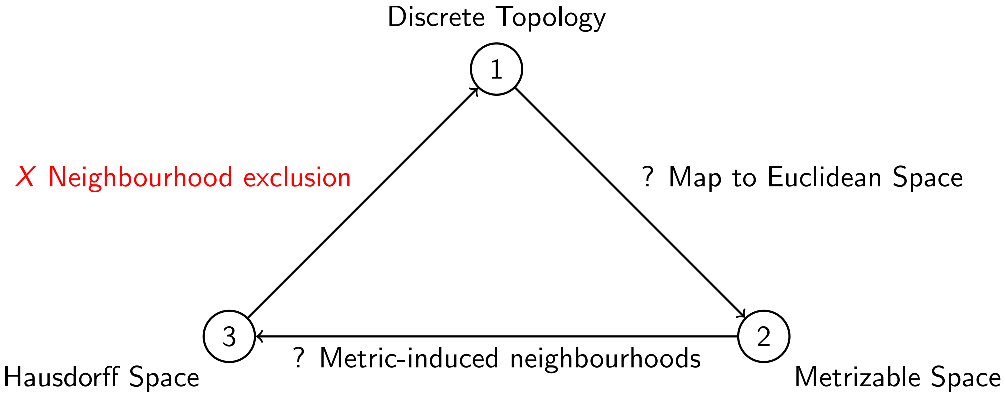
We just proved Metrizable $\Leftrightarrow$ Hausdorff, so what gives? There is something quite important we used to establish all the ideas in that problem. *Finiteness.*

Morphing Relationships



Chasing Broken Bridges

We are now looking for a Hausdorff space that is not metrizable. It cannot be discrete, since we have the discrete metric for it, regardless of finiteness, i.e., the implication edge $1 \to 2$ holds without finiteness too.

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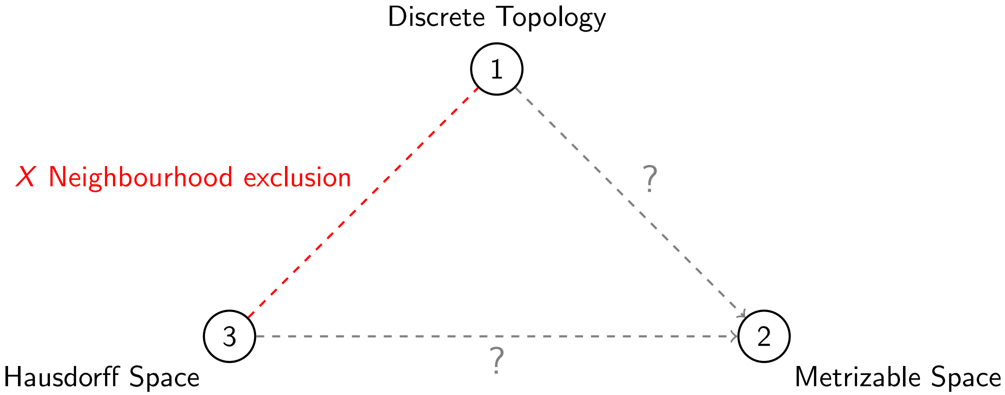

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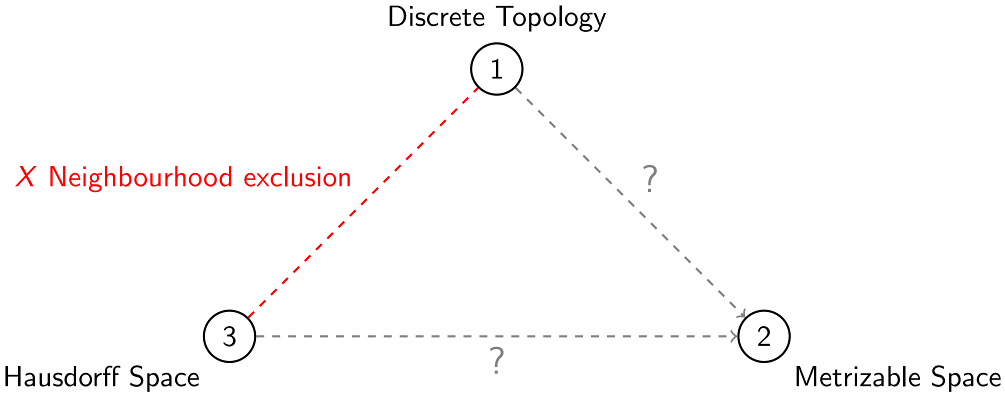

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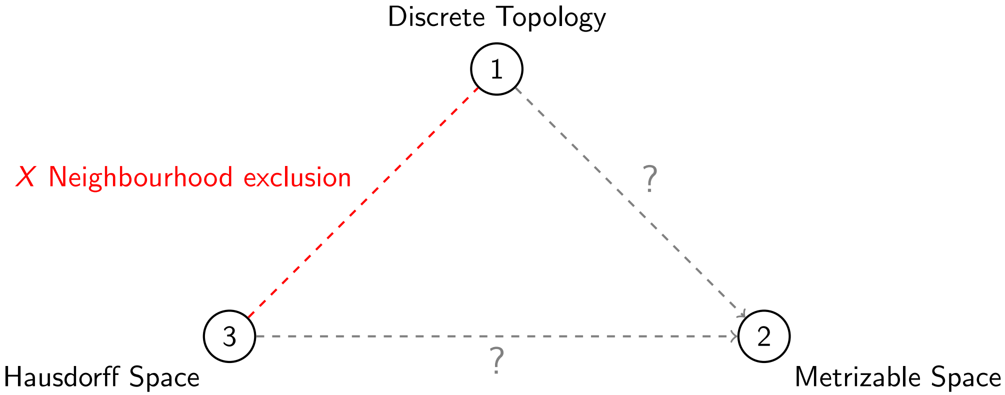

We have lost some information about the internal relationships while breaking one edge. If finiteness doesn't break things enough, what did we use that might?

A Chasm

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Non-metrizability of $*\mathbb{R}$.

If possible, suppose there exists a metric such that it induces the topology defined, $d : *\mathbb{R} \times *\mathbb{R} \to \mathbb{R}$, defined as usual over the 'finite' numbers.

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Then by triangle inequality we must have

$$d(x, y) \leq d(x, \omega) + d(\omega, y) \text{ , so}$$
$$a \leq f(x) + f(y)$$

Since $a, x,$ and y were arbitrary, $f(x)$ must be unbounded $\forall x$, and thus d is not a proper metric. This is a contradiction. $\square$