Diff Geo HW

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Metric Spaces

A *metric space* is a set X equipped with a map $d : X \times X \to \mathbb{R}$ such that :

1. $d(x,x) = 0$
2. $d(x,y) > 0$ if $x \neq y$
3. $d(x,y) = d(y,x)$
4. $d(x,z) \leq d(x,y) + d(y,z)$

for all $x, y, z \in X$

d: distance function or metric

Problem Set 3.1 - 1

Show that: For any points $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y}$ in a metric space X,
$|\boldsymbol{d}(\boldsymbol{a}, \boldsymbol{b}) - \boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y})| \leq \boldsymbol{d}(\boldsymbol{a}, \boldsymbol{x}) + \boldsymbol{d}(\boldsymbol{b}, \boldsymbol{y})$

Using the property of the absolute value function $|a| \leq b \iff -b \leq a \leq b$ we get:

$$-d(a, x) - d(b, y) \leq d(a, b) - d(x, y) \leq d(a, x) + d(b, y)$$

Now we focus on the a term $\alpha \triangleq d(a, x) + d(x, y) + d(y, b)$. Using property (4):

$$d(a, b) \leq d(a, x) + d(x, b) \leq d(a, x) + d(x, y) + d(y, b)$$

Rearranging terms and using property (3) we obtain:

$$d(a, b) - d(x, y) \leq d(a, x) + d(b, y)$$

Which is the RHS of the inequality. Similarly, the LHS can be proven*.
*Modify (4) into $d(x, z) - d(y, z) \leq d(x, y)$ and $\alpha = -d(a, x) - d(x, y) + d(y, b)$

Problem Set 3.1 - 4

Check that the diamond and square metrics on $\mathbb{R}^n$ are indeed metrics. Show that the euclidean metric on $\mathbb{R}^n$ is indeed a metric. (The triangle inequality in this context is equivalent to Minkowski's inequality.)

For any $x, y \in \mathbb{R}^n$ where $v = (v_1, v_2, ..., v_n)$:

1. Diamond metric: $d_1(x, y) = \sum_{i \in \mathcal{I}} |x_i - y_i|$
2. Euclidean metric: $d_2(x, y) = \sqrt{\sum_{i \in \mathcal{I}} |x_i - y_i|^2}$
3. Square metric: $d_\infty(x, y) = max_{i \in \mathcal{I}}\{|x_i - y_i|\}$

Now we just have to show that each of these metrics satisfy the four conditions that characterise metric spaces. We will look at each metric separately.

The Diamond metric

1. $d_1(x,x) = \sum_{i=1}^{n} |x_i - x_i| = 0$ for all $i \in \mathcal{I}$.
2. Suppose $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| = 0$. We know that that for $p \in \mathbb{R}, |p| \geq 0$ and $|p| = 0 \iff p = 0$ (positive definite). Therefore for the assumption to hold, $|x_i - y_i| = 0$ for all $i \in \mathcal{I}$. Hence $x_i = y_i$ for all $i \in \mathcal{I}$. This shows that for $x \neq y$, $d_1(x,y) > 0$.
3. We can rewrite $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |-(-x_i + y_i)| = \sum_{i=1}^{n} |(-x_i + y_i)| = \sum_{i=1}^{n} |y_i - x_i|$. Therefore, $d_1(x,y) = d_1(y,x)$.
4. Using the triangle inequality $|(x_i - y_i) + (y_i - z_i)| \leq |x_i - y_i| + |y_i - z_i| \rightarrow |x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$. Taking the summation, $\sum_{i=1}^{n} |x_i - z_i| \leq \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i|$. Hence, we have shown, $d_1(x,z) \leq d_1(x,y) + d_1(y,z)$.

The Euclidean metric

1. $d_2(x,x) = \sqrt{\sum_{i=1}^{n} |x_i - x_i|^2} = 0$ for all $i \in \mathcal{I}$.
2. Suppose $d_2(x,y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} = 0$. We know that that for $p \in \mathbb{R}, \sqrt{p} \geq 0$ and $\sqrt{p} = 0 \iff p = 0$ (positive definite). Therefore for the assumption to hold, $\sum_{i=1}^{n} |x_i - y_i|^2 = 0$ for all $i \in \mathcal{I}$. Further, $|x_i - y_i|^2 = 0$ for all $i \in \mathcal{I}$. Hence $x_i = y_i$ for all $i \in \mathcal{I}$. This shows that for $x \neq y$, $d_2(x,y) > 0$.
3. We can rewrite $d_2(x,y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} = \sqrt{\sum_{i=1}^{n} |-(-x_i + y_i)|^2} = \sqrt{\sum_{i=1}^{n} |(-x_i + y_i)|^2} = \sqrt{\sum_{i=1}^{n} |y_i - x_i|^2}$. Therefore, $d_2(x,y) = d_2(y,x)$.
4. Minkowski's inequality* says that $(\sum_{k=1}^{n} |\alpha_k + \beta_k|^p)^{\frac{1}{p}} \leq (\sum_{k=1}^{n} |\alpha_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{n} |\beta_k|^p)^{\frac{1}{p}}$. If we substitute $\alpha_k = x_k - y_k$ and $\beta_k = y_k - z_k$, we get the desired triangle inequality. $\sqrt{\sum_{i=1}^{n} |x_i - z_i|^2} \leq \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} + \sqrt{\sum_{i=1}^{n} |y_i - z_i|^2}$. Hence, we have shown, $d_2(x,z) \leq d_2(x,y) + d_2(y,z)$.

The Diamond metric

1. $d_1(x,x) = \sum_{i=1}^{n} |x_i - x_i| = 0$ for all $i \in \mathcal{I}$.
2. Suppose $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| = 0$. We know that that for $p \in \mathbb{R}, |p| \geq 0$ and $|p| = 0 \iff p = 0$ (positive definite). Therefore for the assumption to hold, $|x_i - y_i| = 0$ for all $i \in \mathcal{I}$. Hence $x_i = y_i$ for all $i \in \mathcal{I}$. This shows that for $x \neq y$, $d_1(x,y) > 0$.
3. We can rewrite $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |-(-x_i + y_i)| = \sum_{i=1}^{n} |(-x_i + y_i)| = \sum_{i=1}^{n} |y_i - x_i|$. Therefore, $d_1(x,y) = d_1(y,x)$.
4. Using the triangle inequality $|(x_i - y_i) + (y_i - z_i)| \leq |x_i - y_i| + |y_i - z_i| \rightarrow |x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$. Taking the summation, $\sum_{i=1}^{n} |x_i - z_i| \leq \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i|$. Hence, we have shown, $d_1(x,z) \leq d_1(x,y) + d_1(y,z)$.

Question 2 contd.

The Square metric

1. $d_\infty(x, x) = max_{i \in \mathcal{I}}\{|x_i - x_i|\} = max\{0, 0, .., 0\} = 0$ for all $i \in \mathcal{I}$.
2. Suppose $d_\infty(x, y) = max_{i \in \mathcal{I}}\{|x_i - y_i|\} = 0$ at $i = i*$. We know that that for $p \in \mathbb{R}, |p| \geq 0$ and $|p| = 0 \iff p = 0$ (positive definite). Therefore for the assumption to hold, $|x_i - y_i| \leq 0$ for all $i \in \mathcal{I}, i \neq i* \Rightarrow |x_i - y_i| = 0$. Hence $x_i = y_i$ for all $i \in \mathcal{I}$. This shows that for $x \neq y$, $d_\infty(x, y) > 0$.
3. We can rewrite $d_\infty(x, y) = max_{i \in \mathcal{I}}\{|x_i - y_i|\} = max_{i \in \mathcal{I}}\{|-(-x_i + y_i)|\} = max_{i \in \mathcal{I}}\{|(-x_i + y_i)|\} = max_{i \in \mathcal{I}}\{|y_i - x_i|\}$. Therefore, $d_\infty(x, y) = d_\infty(y, x)$.
4. Need help here*

Minkowski's Inequality

$$(\sum_{k=1}^{n} |\alpha_k + \beta_k|^p)^{\frac{1}{p}} \leq (\sum_{k=1}^{n} |\alpha_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{n} |\beta_k|^p)^{\frac{1}{p}}$$

$(\sum_{k=1}^{n} |\alpha_k|^p)^{\frac{1}{p}} = ||\alpha||_p$ is called the p-norm.The proof makes use of Holder's inequality. It is first shown that if α and β have a finite p-norm, so does $\alpha + \beta$ (convexity arguments).

$$||\alpha + \beta||_p^p = \int |\alpha + \beta|^p \, d\mu \leq \int (|\alpha| + |\beta|)|\alpha + \beta|^{p-1} \, d\mu$$

Now, applying Holder's inequality;

$$||\alpha + \beta||_p^p \leq (||\alpha||_p + ||\beta||_p) \frac{||\alpha + \beta||_p^p}{||\alpha + \beta||_p} \tag{1}$$

Rearranging this gives:

$$||\alpha + \beta||_p \leq ||\alpha||_p + ||\beta||_p \tag{2}$$

Minkowski's Inequality for the Euclidean Metric

For the case of the Euclidean metric, the inequality boils down to:

$$\sqrt{\sum_{i=1}^{n} |x_i - z_i|^2} \leq \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} + \sqrt{\sum_{i=1}^{n} |y_i - z_i|^2}$$

For $n = 1$, we retrieve the same inequality as that of the diamond metric, i.e.,
$\sqrt{|x_1 - z_1|^2} \leq \sqrt{|x_1 - y_1|^2} + \sqrt{|y_1 - z_1|^2} \Rightarrow |x_1 - z_1| \leq |x_1 - y_1| + |y_1 - z_1|$
Should I write something for $n = 2$?*

Ques 0

Test 1

Ques 0

Test 1

Problem Set 3.4 - 7 — Pushkar Mohile

Analysis Notes 3.4 Q.- Show that The discrete and indiscrete topologies on a set give rise to functors

$$\text{Set} \to \text{Top} \tag{3}$$

and these are the left and right adjoints, respectively, to the forgetful functor from Top to Set.

Solution :
We begin by recalling the definition of a functor. Given two Categories C amd D, A functor $\mathcal{F}$ assigns to every object of a C an object $\mathcal{F}(a)$ in D
For every morphism $f \in C(a, b)$ a corresponding morphism

$$\mathcal{F}(f) \in \mathrm{D}(\mathcal{F}(a), \mathcal{F}(b)) \tag{4}$$

which respects composition

$$\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g) \tag{5}$$

and maps id to id

Let us construct the functors corresponding to the discrete and indiscrete topologies on any set X given by $\tau_{disc} = 2^X$ and $\tau_{indisc} = \{\phi, X\}$. We will call them *disc*: Set $\to$ Top and *indisc*: Set $\to$ Top , defined in the following way

$$disc(X) \mapsto (X, \tau_{disc})$$
$$indisc(X) \mapsto (X, \tau_{indisc})$$

And for any function $f \in \mathrm{Set}(X, Y)$, $f \mapsto f$. The check we need to make here is that f is a continous function between sets with the discrete and indiscrete topology.

For the discrete topology, this is done by noting that

$$\forall U \in \tau_Y, f^-1(U) \subseteq X \in 2^X \tag{6}$$

and hence $f^{-1}(U)$ is open in τ_{disc}.
Similarly, for the indiscrete topology, the only open subsets of Y are ϕ, Y.
For $\phi \in \tau_Y$ we have $f^{-1}(\phi) = \phi \in \tau_X$ and $f^{-1}(Y) = X \cup \phi \in \tau_X$ and hence once again f is continous. The composition law is valid since composition of continous functions are continous. Hence the discrete and indiscrete topologies define the required functors.

For the second part, we have to show that these are left adjoint and right adjoints respectively to the forgetful functor defined as follows:

$$\begin{aligned} frg : \mathrm{Top} &\to \mathrm{Set} \\ (X, \tau_X) &\mapsto X \\ f \in \mathrm{Top}((X, \tau_X), (Y, tau_Y) &\mapsto f \in \mathrm{Set}(X, Y) \end{aligned}$$

ie we are forgetting the underlying topology and viewing the function f as a morphism between sets.

We recall the definitions of left and right adjoint functors. Given two categories C and D and functors $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ and $\mathcal{G} : \mathsf{D} to \mathsf{C}$, we say that $\mathcal{F}$ is a left adjoint and $\mathcal{G}$ is a right adjoint if for objects a in C and x in G , there is a bijection between the set of morphisms
$\mathsf{D}(\mathcal{F}(a), x) \xrightarrow{\cong} \mathsf{C}(a, \mathcal{G}(b))$
that is *natural* in a and x. The naturality condition is formally stated as follows: For any morphism $a \to a'$ in C and $x \to x'$ in D, we have the following commutative diagrams (Check Cat Theory lec. 2 or section 5 of the notes)

We now check the adjuction between *frg* and *disc* as defined previously. Let X be any set and (Y, τ_Y) be any topological space. We have to prove that

$$\mathrm{Top}(disc(X), (Y, \tau_Y)) \xrightarrow{\cong} \mathrm{Set}(X, Y) \tag{7}$$

This bijection is given by $f \mapsto f$ in both directions. We now have to simply check whether the two sets are the same. This is done as follows:

$$\mathrm{Top}(disc(X), (Y, \tau_Y)) \subset\xrightarrow{\cong} \mathrm{Set}(X, Y) \tag{8}$$

is obvious since continous functions are functions between the sets.

Next, note that

$$\text{Set}(X, Y) \subset \text{Top}(disc(X), (Y, \tau_Y)) \tag{9}$$

Proof: Let $f \in$ Set(X, Y), U_Y be any open set on τ_Y. $f^{-1}(U) \subseteq X \in 2^X$ and hence f is continous.

Thus we have proved that the two sets are equal. Finally we make note of the naturality condition. This holds because composition of functions and composition of continuos function commute with the functors.

This adjunction can be restated in terms of the following universal property of the discrete topology : The discrete topology on X is the topology such that every function from X to any topological space Y, τ_Y is continuos.

Finally we take a look at the right adjoint condition for the indiscrete topology. The conditions states that

$$\mathsf{Set}(Y, X) \xrightarrow{\cong} \mathsf{Top}((Y, \tau_Y), \mathit{indisc}(X)) \tag{10}$$

The checks are similar to the previously done checks. We mention the only nontrivial check:

$$\mathsf{Set}(Y, X) \subseteq \mathsf{Top}((Y, \tau_Y), \mathit{indisc}(X)) \tag{11}$$

For a given function $f \in \mathsf{Set}(Y, X)$, with the indiscrete topology on X, $f^{-1}(\phi) = \phi$ and $f^{-1}(X) = Y \cup \phi$, both of which are open wrt any topology on Y. Hence f is continous.

Topological Spaces

The following questions deal with the idea of Topological Spaces, so here's a quick recap on what exactly those are.

Topological Spaces: A *topological space* is a set X on which a *topology* τ is equipped. τ is a collection of subsets of X (or, τ is a subset of the power set 2^X of X) such that -

1. $\varnothing$ and X should belong to τ
2. the union of the elements in any subset of τ should belong to τ
3. the intersection of the elements in any finite subset of τ should belong to τ

Topological Spaces

The elements of τ are called *open sets*. Thus, a topological space is a pair (X, τ) consisting of a set and a topology on it.

We can reframe the axioms given on the previous slide in terms of open sets -

1. The empty and the full set are open.
2. Any arbitrary union of open sets is open.
3. Any finite intersection of open sets is open.

Question (8)

Show that: The euclidean, diamond, square metrics on $\mathbb{R}^2$ have the same underlying topology. (When we say continuous map from $\mathbb{R}^2$ to $\mathbb{R}$, it is w.r.t this topology.) Further, check that it coincides with the product topology on $\mathbb{R} \times \mathbb{R}$.

Question (4)

Ques - Show that: The underlying topology of the discrete metric is the discrete topology. If a set X has more than one element, then the indiscrete topology on X is not metrizable.

Both these subparts deal with one or the other extreme cases as far as topologies go. So let's look at them individually before solving the problem.

Discrete Topology: The textbook definition of a *discrete topology* is that it is a collection of all subsets of X, i.e, $\tau = 2^X$. There are a few interesting inferences to be drawn from this definition. Since every possible subset is an open subset in the discrete topology, in particular, every *singleton subset* is an open set in this topology.

Question (4)

Indiscrete Topology: The collection $\tau = \{\emptyset, X\}$ on X is the *indiscrete, or trivial topology* on X. A consequence of this collection is that all points in the set X cannot be distinguished from each other through topological means.

Now, let's look at the first part of the problem -
Show that the underlying topology of the discrete metric is the discrete topology

The discrete metric is as follows -

$$d_{\text{discrete}}(x, y) := \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{otherwise.} \end{cases}$$

Question (4)

Now, a metric d on a set X induces a topology τ by taking the idea of the open balls $B(x,r) = \{y : d(x,y) < r\}$ as basic open sets. We need to show that the d_{discrete} we are given produces the discrete topology $\tau = 2^X$.

Let $x \in X$ be an arbitrary element, and let $r \in (0,1)$; then by the definition of the discrete metric $B_d(x,r) = \{x\}$, so $\forall x \in X, \{x\}$ is an open set.

Now, by the axioms we discussed about topological spaces, any arbitrary union of open sets is open. Let $A \subseteq X$ be any arbitrary subset of X, then $A = \bigcup_{x \in A}\{x\}$, but we have shown that $\forall x \in X, \{x\}$ is an open set.

Question (4)

Since any arbitrary union of open sets is open, we can claim that A is an open set, as induced by the discrete metric. Since this claim holds for any $A \subseteq X$, we thus claim that every subset of X is open, i.e, $\forall A \subseteq X, A \in \tau$.

Since τ contains every possible subset of X, it is the power set 2^X of X. Thus, we have shown that the discrete metric induces a topology $\tau = 2^X$ on X. Since this is the definition of the discrete topology, we have shown that the underlying topology of the discrete metric is the discrete topology. $\square$

Question (4)

We now look at the next part of the problem -
Show that if a set X has more than one element, then the indiscrete topology on X is not metrizable.

We prove this by contradiction. Assume that there exists a metric d on the set X such that (X, d) is a metric space and that the topology induced by this metric on X is the indiscrete topology, $\tau = \{\emptyset, X\}$
X has at least 2 distinct elements x and y, i.e, $\exists x, y \in X$ s.t $x \neq y$.
$\implies d(x, y) = r > 0$
Now, consider the open ball $B(x, r/2)$. This open ball should be an open set in the topology that d induces.

Question (4)

But, $x \in B(x, r/2)$ and since $d(x, y) = r > r/2$, $y \notin B(x, r/2)$.

Thus, $B(x, r/2) \neq \emptyset$ and $B(x, r/2) \neq X$ (as there is at least one element $y \in X$ s.t $y \notin B(x, r/2)$).

Thus, the topology induced by the metric d cannot be the indiscrete topology, since $\tau_{\text{indiscrete}} = \{\emptyset, X\}$

Thus, we have shown that if a set X has more than one element, then the indiscrete topology on X is not metrizable. $\square$

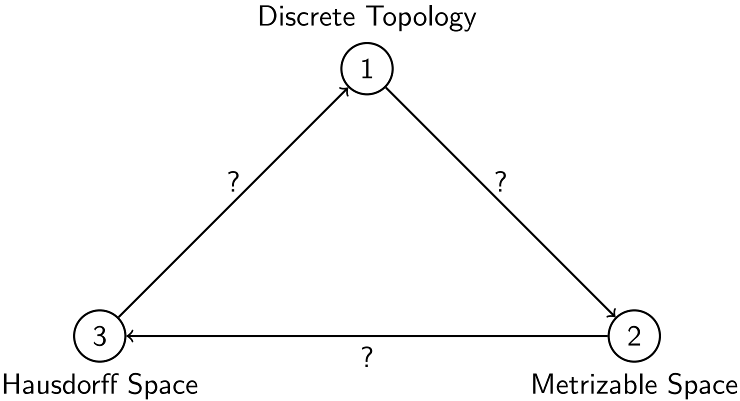
Exercise 3.4 - 15

Problem Statement

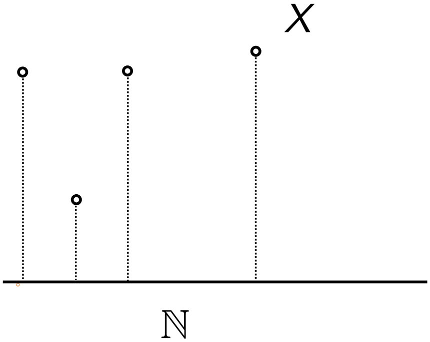
Let X be a *finite* topological space. Show that the following are equivalent:

1. X has the discrete topology.
2. X is metrizable.
3. X is Hausdorff.

Establishing Relationships



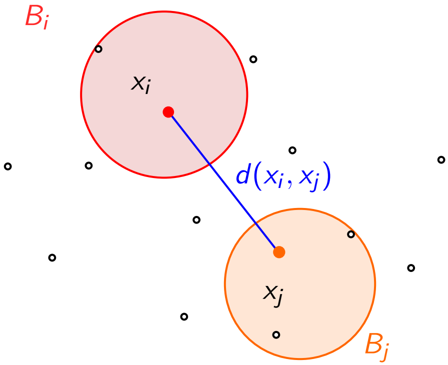
$1 \to 2$ — Discrete $\to$ Metrizable



This is equivalent to finding an order on the set and identifying it with $\mathbb{N}$. It only requires the Axiom of Choice!

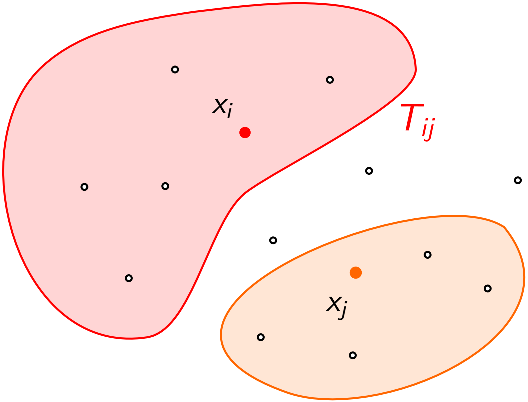
$2 \to 3$ — Metrizable $\to$ Hausdorff

Given that the space, say $(\mathcal{X}, \tau)$, is metrizable, there exists a metric $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ which induces the topology given by τ. Use this metric to define open balls B_i, B_j for any pair of points in $\mathcal{X}$, x_i, x_j. By shrinking these open balls, we can create non-intersecting open sets as required.



$3 \to 1$ — Hausdorff $\to$ Discrete

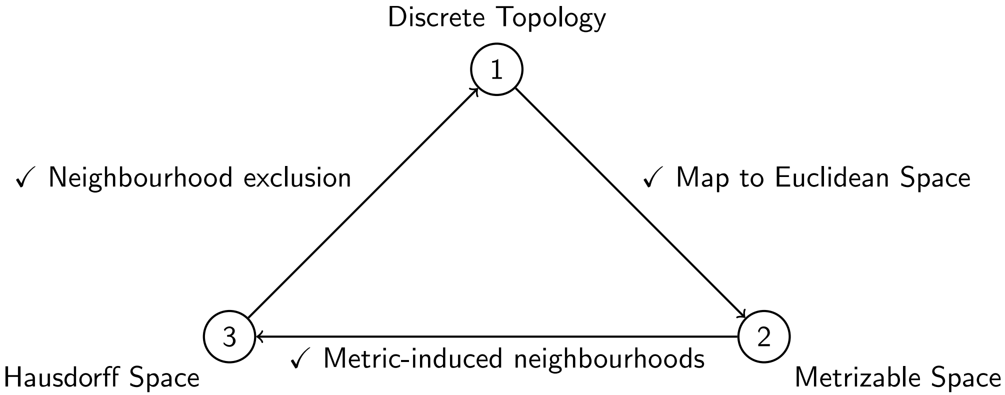
We know by the Hausdorff property that any two points are separable by neighbourhoods.



$$\forall x_i \bigcap_{x_j \in X} T_i(x_j) = x_i,$$

$$x_i \in \tau \ .$$

Establishing Relationships



Exercise 3.4 - 16

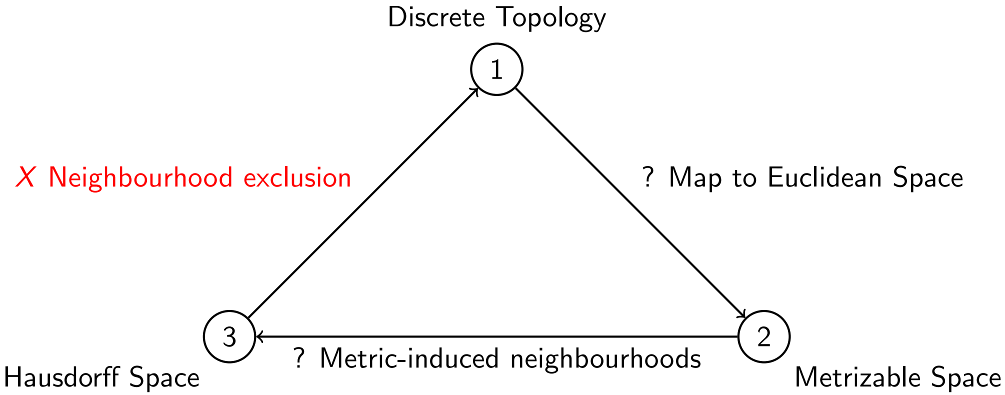
Problem Statement

Give an example of a topological space which is Hausdorff but not metrizable.

Examining the problem

We just proved Metrizable $\Leftrightarrow$ Hausdorff, so what gives? There is something quite important we used to establish all the ideas in that problem. *Finiteness.*

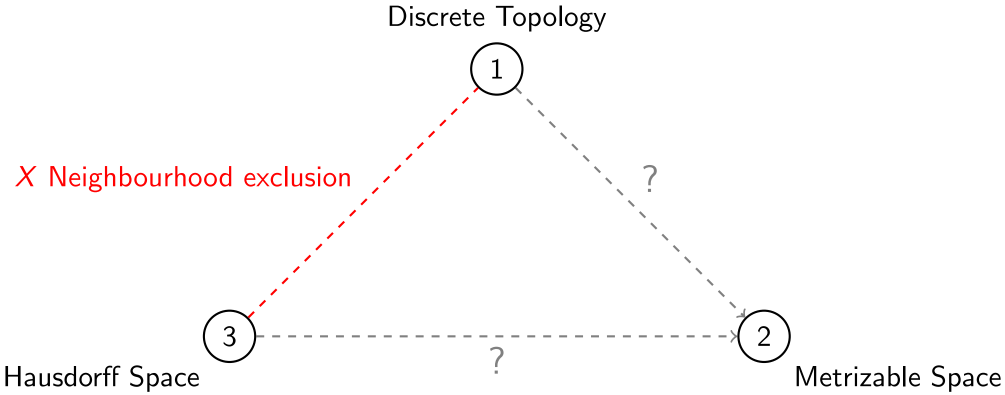
Morphing Relationships



Chasing Broken Bridges

We are now looking for a Hausdorff space that is not metrizable. It cannot be discrete, since we have the discrete metric for it, regardless of finiteness, i.e., the implication edge $1 \to 2$ holds without finiteness too.

Morphing Relationships



We have lost some information about the internal relationships while breaking one edge. If finiteness doesn't break things enough, what did we use that might?

A Chasm

We construct a topology from a metric space by constructing open balls around points. So, perhaps we can break metrizability by taking a Hausdorff space, and creating gaps in it that cannot be worked around with a metric.

Combining everything, consider the space $(\mathbb{R}, \tau)$ with τ being the usual topology on the real line. It is clearly Hausdorff, and metrizable.Construct from this a new space $*\mathbb{R}$ from $\mathbb{R}$ with an added point ω, a number larger than any finite real. Choose the set $\{\omega\}$ to additionally be open. We see that there are suddenly issues with defining a metric on this space.

Suppose we were able to metrize this space. That would imply we have defined a *real* distance between all points on the real line and ω , but this would imply that ω is contained within some finite open balls centered at said points. Intuitively, this does not make any sense per the definition of ω.

Non-metrizability of $*\mathbb{R}$.

If possible, suppose there exists a metric such that it induces the defined topology on $*R$, $d : *\mathbb{R} \times *\mathbb{R} \to \mathbb{R}$, defined as usual over the 'finite' numbers. If $\exists x \in \mathbb{R}, f : \mathbb{R} \to \mathbb{R}$ $d(x, \omega) = f(x)$, then pick any two points $x, y \neq \omega$, such that $d(x, y) = a$ for some real a.
Then by triangle inequality we must have

$$d(x, y) \leq d(x, \omega) + d(\omega, y) \text{ , so}$$
$$a \leq f(x) + f(y)$$

Since $a, x,$ and y were arbitrary, $f(x)$ must be unbounded $\forall x$, and thus d is not a proper metric. This is a contradiction. $\square$