Diff Geo HW

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Metric Spaces

A *metric space* is a set X equipped with a map $d : X \times X \to \mathbb{R}$ such that :

1. $d(x, x) = 0$
2. $d(x, y) > 0$ if $x \neq y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$

for all $x, y, z \in X$

d: distance function or metric

Problem Set 3.1 - 1

Show that: For any points $\boldsymbol{a, b, x, y}$ in a metric space X,
$\boldsymbol{|d(a,b) - d(x,y)| \leq d(a,x) + d(b,y)}$

Using the property of the absolute value function $|p| \leq q \iff -q \leq p \leq q$ we get:

$$-d(a,x) - d(b,y) \leq d(a,b) - d(x,y) \leq d(a,x) + d(b,y)$$

Now we focus on the term $\alpha \triangleq d(a,x) + d(x,y) + d(y,b)$. Using the triangle inequality:

$$d(a,b) \leq d(a,x) + d(x,b) \leq d(a,x) + d(x,y) + d(y,b)$$

Rearranging terms and using symmetry we obtain:

$$d(a,b) - d(x,y) \leq d(a,x) + d(b,y)$$

Which is the RHS of the inequality. Similarly, the LHS can be proven:
$d(x,z) - d(y,z) \leq d(x,y)$ and $\alpha = -d(a,x) - d(x,y) + d(y,b)$

Problem Set 3.1 - 4

Check that the diamond and square metrics on $\mathbb{R}^n$ are indeed metrics. Show that the euclidean metric on $\mathbb{R}^n$ is indeed a metric. (The triangle inequality in this context is equivalent to Minkowski's inequality.)

For any $x, y \in \mathbb{R}^n$ where $x = (x_1, x_2, \ldots, x_n)$:

1. Diamond metric: $d_1(x, y) = \sum_{i \in \mathcal{I}} |x_i - y_i|$
2. Euclidean metric: $d_2(x, y) = \sqrt{\sum_{i \in \mathcal{I}} |x_i - y_i|^2}$
3. Square metric: $d_\infty(x, y) = max_{i \in \mathcal{I}}\{|x_i - y_i|\}$

Now we just have to show that each of these metrics satisfy the four conditions that characterise metric spaces. We will look at each metric separately.

The Diamond metric

- $d_1(x, x) = \sum_{i=1}^{n} |x_i - x_i| = 0.$

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- $d_1(x, x) = \sum_{i=1}^{n} |x_i - x_i| = 0$.
- We know that that for $p \in \mathbb{R}$, $|p| \geq 0$ and $|p| = 0 \iff p = 0$ (positive definite).

  Let us assume that there exists distinct x, y such that $d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| = 0$.

  However for the assumption to hold, $|x_i - y_i| = 0$ for all $i \in \mathcal{I}$. Hence $x_i = y_i$. This shows that for $x \neq y$, $d_1(x, y) > 0$.

- We can rewrite $d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} | - (-x_i + y_i)| = \sum_{i=1}^{n} |y_i - x_i|$. Therefore, $d_1(x, y) = d_1(y, x)$.

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- We use the inequality
  $|(x_i - y_i) + (y_i - z_i)| \leq |x_i - y_i| + |y_i - z_i| \Rightarrow |x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$.

  Taking the summation, $\sum_{i=1}^{n} |x_i - z_i| \leq \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i|$. Hence, we have shown, $d_1(x, z) \leq d_1(x, y) + d_1(y, z)$.

The Euclidean metric

- $d_2(x,x) = \sqrt{\sum_{i=1}^{n} |x_i - x_i|^2} = 0.$
- We know that that for $p \in \mathbb{R}$, $\sqrt{p} \geq 0$ and $\sqrt{p} = 0 \iff p = 0$ (positive definite).

  Suppose $d_2(x,y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} = 0.$

  Therefore for the assumption to hold, $\sum_{i=1}^{n} |x_i - y_i|^2 = 0$. Further, $|x_i - y_i|^2 = 0$ for all $i \in \mathcal{I}$. Hence $x_i = y_i$. This shows that for $x \neq y$, $d_2(x,y) > 0$.

The Euclidean metric

- We can rewrite $d_2(x,y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} = \sqrt{\sum_{i=1}^{n} |-(-x_i + y_i)|^2} = \sqrt{\sum_{i=1}^{n} |y_i - x_i|^2}$. Therefore, $d_2(x,y) = d_2(y,x)$.

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- We can rewrite
$d_2(x,y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} = \sqrt{\sum_{i=1}^{n} |-(-x_i + y_i)|^2} = \sqrt{\sum_{i=1}^{n} |y_i - x_i|^2}$.
Therefore, $d_2(x,y) = d_2(y,x)$.

- Minkowski's inequality* says that
$(\sum_{k=1}^{n} |\alpha_k + \beta_k|^p)^{\frac{1}{p}} \leq (\sum_{k=1}^{n} |\alpha_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{n} |\beta_k|^p)^{\frac{1}{p}}$.

  If we substitute $\alpha_k = x_k - y_k$, $\beta_k = y_k - z_k$ and $p = 2$, we get the desired triangle inequality.
$\sqrt{\sum_{i=1}^{n} |x_i - z_i|^2} \leq \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} + \sqrt{\sum_{i=1}^{n} |y_i - z_i|^2}$.
Hence, we have shown, $d_2(x,z) \leq d_2(x,y) + d_2(y,z)$.

The Square metric

- $d_\infty(x,x) = max_{i\in\mathcal{I}}\{|x_i - x_i|\} = max\{0,0,..,0\} = 0$

- We know that that for $p \in \mathbb{R}, |p| \geq 0$ and $|p| = 0 \iff p = 0$ (positive definite).

  Let us assume $d_\infty(x,y) = max_{i\in\mathcal{I}}\{|x_i - y_i|\} = 0$ at $i = i*$.

  Therefore for the assumption to hold, $|x_i - y_i| \leq 0$ for all $i \in \mathcal{I}, i \neq i*$
  $\Rightarrow |x_i - y_i| = 0$. Hence $x_i = y_i$. This shows that for $x \neq y$, $d_\infty(x,y) > 0$.

The Square metric

- We can rewrite $d_\infty(x,y) = max_{i\in\mathcal{I}}\{|x_i - y_i|\} = max_{i\in\mathcal{I}}\{|-(-x_i + y_i)|\} = max_{i\in\mathcal{I}}\{|y_i - x_i|\}$. Therefore, $d_\infty(x,y) = d_\infty(y,x)$.

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- For $x, y, z \in \mathbb{R}$ we have proven the triangle inequality $|x_1 - z_1| \leq |x_1 - y_1| + |y_1 - z_1|$.

  Now, for $x, y, z \in \mathbb{R}^n$, let $k \in \mathcal{I}$ correspond to $max_{i\in\mathcal{I}}\{|x_i - z_i|\} = |x_k - z_k|$.

  $|x_k - y_k| \leq max_{i\in\mathcal{I}}\{|x_i - y_i|\} = d_\infty(x,y)$
  $|y_k - z_k| \leq max_{i\in\mathcal{I}}\{|y_i - z_i|\} = d_\infty(y,z)$

  Hence, $max_{i\in\mathcal{I}}\{|x_i - z_i|\} \leq max_{i\in\mathcal{I}}\{|x_i - y_i|\} + max_{i\in\mathcal{I}}\{|y_i - z_i|\} \Rightarrow d_\infty(x,z) \leq d_\infty(x,y) + d_\infty(y,z)$.

Minkowski's Inequality

$$(\sum_{k=1}^{n} |\alpha_k + \beta_k|^p)^{\frac{1}{p}} \leq (\sum_{k=1}^{n} |\alpha_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{n} |\beta_k|^p)^{\frac{1}{p}}$$

$(\sum_{k=1}^{n} |\alpha_k|^p)^{\frac{1}{p}} = ||\alpha||_p$ is called the p-norm.The proof makes use of Hölder's inequality. It is first shown that if α and β have a finite p-norm, so does $\alpha + \beta$ (convexity arguments).

$$||\alpha + \beta||_p^p = \int |\alpha + \beta|^p \, d\mu \leq \int (|\alpha| + |\beta|)|\alpha + \beta|^{p-1} \, d\mu$$

Hölder's inequality on the L_k norm $||rs||_1 \leq ||r||_p ||s||_q, \frac{1}{p} + \frac{1}{q} = 1$.
Now, applying Hölder's inequality (split product $\frac{1}{p} = p, \frac{1}{q} = 1 - \frac{1}{p}$:

$$||\alpha + \beta||_p^p \leq (||\alpha||_p + ||\beta||_p) \frac{||\alpha + \beta||_p^p}{||\alpha + \beta||_p}$$

Rearranging this gives:

$$||\alpha + \beta||_p \leq ||\alpha||_p + ||\beta||_p$$

Minkowski's Inequality for the Euclidean Metric

For the case of the Euclidean metric, the inequality boils down to:

$$\sqrt{\sum_{i=1}^{n} |x_i - z_i|^2} \leq \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} + \sqrt{\sum_{i=1}^{n} |y_i - z_i|^2}$$

For $n = 1$, we retrieve the well-known inequality

$$\sqrt{|x_1 - z_1|^2} \leq \sqrt{|x_1 - y_1|^2} + \sqrt{|y_1 - z_1|^2} \Rightarrow |x_1 - z_1| \leq |x_1 - y_1| + |y_1 - z_1|$$

Problem Set 3.1 - 5 — Karthik Dasigi

Problem Statement For any metric spaces X and Y, put three metrics on $X \times Y$ by analogy with the Euclidean, diamon, and square metrics on $\mathbb{R}^2$. Show that: For any metrix space X, the distance function $d : X \times X \to \mathbb{R}$ is continuous (wrt either of the three metrics on $X \times X$).

Metric analogues

Suppose d_x is the distance function on X, and d_y is the distance function on Y. Analogous to the three metrics on $\mathbb{R}^2$, we can create three metrics on $X \times Y$ that describe the distance between the points (x_1, y_1) and (x_2, y_2):

- Euclidean$_{X \times Y}$: $\sqrt{d_x(x_1, x_2)^2 + d_y(y_1, y_2)^2}$
- Square$_{X \times Y}$: $\mathbf{max}(d_x(x_1, x_2), d_y(y_1, y_2))$
- Diamond$_{X \times Y}$: $d_x(x_1, x_2) + d_y(y_1, y_2)$

Continuity of the distance function

We need to now show that the metric on a metric space X, when viewed as a function from the larger space $X \times X$ to $\mathbb{R}$ is continuous wrt to the metrics defined here.

Continuity

Definition (Continuity)

Suppose X and Y are metric spaces. A function $f : X \to Y$ is continuous if for any point $x_0 \in X$, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x, x_0) < \delta \text{ implies } d(f(x), f(x_0)) < \epsilon \tag{1}$$

Continuity — Diamond metric

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For the function $d : X \times X \to \mathbb{R}$ we need to show that, for some point $(x_0, y_0) \in X$, given an $\epsilon > 0$, there exists a δ such that

$$\Delta((x, y), (x_0, y_0)) < \delta \Rightarrow |d(x, y) - d(x_0, y_0)| < \epsilon \ . \tag{2}$$

Note that the diamond metric $\Delta((x, y), (x_0, y_0)) = d(x, x_0) + d(y, y_0)$ and the function d is the metric on X.

We begin with the given bound ϵ such that

$$\begin{aligned} |d(x,y) - d(x_0,y_0)| &< \epsilon \ , \\ d(x,y) - d(x_0,y_0) &< \epsilon \ , \\ d(x,y) + d(x_0,y_0) &< \epsilon + 2d(x_0,y_0) \ . \end{aligned} \tag{3}$$

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Adding $2d(y,x_0)$ to both sides and using the triangle inequality, we obtain

$$\boxed{\begin{aligned} T = d(x,x_0) + d(y,y_0) &\leq (d(x,y) + d(y,x_0)) + (d(y,x_0) + d(x_0,y_0)) \\ &< \epsilon + 2d(x_0,y_0) + 2d(y,x_0) \ . \end{aligned} \tag{4}}$$

Peeking along a side

To resolve the terms dependent on only y, let us consider the dimensionally reduced problem, or proving continuity along the y-axis inside the ϵ-ball around the point (x_0, y_0), i.e.

$$\text{Given } \epsilon \in \mathbb{R} > 0 \text{ as before, with}$$
$$|d(x_0, y) - d(x_0, y_0)| < \epsilon \ .$$

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It is easy to now see the reduction

$$\begin{aligned} d(x_0, y) - d(x_0, y_0) &< \epsilon \ , \\ d(x_0, y) &< \epsilon + d(x_0, y_0) \ . \end{aligned} \tag{5}$$

We see that this is a bound on the y dependant term that is constant for a given (x_0, y_0).

Collecting results

From the results for T and y in 4 and 5, we obtain the bound on T

$$T < \epsilon + 2d(x_0, y_0) + 2d(y, x_0) < 3\epsilon + 4d(x_0, y_0) = \delta(\epsilon) \tag{6}$$

as required. Using the other half of the absolute value inequality, $d(x, y) - d(x_0, y_0) > -\epsilon$, we similarly get the bound

$$T < 5\epsilon + 4d(x_0, y_0) \ .$$

So inequality 6 still gives the tightest bound.

Continuity for other metrics

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To expand the proof, we make use of the following property: for positive x and y,

$$x + y \geq (x^p + y^p)^{1/p} \text{ for } p \geq 1 \tag{7}$$

(This is can be viewed as the *Minkowski* inequality applied to a 1-dimensional vector)

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(This is can be viewed as the *Minkowski* inequality applied to a 1-dimensional vector)
This means that the same $\delta(\epsilon)$ can be used for the proof of continuity of the other two metrics.

$$(d(x, x_0)^p + d(y, y_0)^p)^{1/p} \leq d(x, x_0) + d(y, y_0) < 3\epsilon + 4d(x_0, y_0) = \delta(\epsilon) \tag{8}$$

Putting $p = 2$ proves continuity for the Euclidean metric, and $p = \infty$ proves continuity for the Square metric

Problem Set 3.1 - 9 — Karthik Dasigi

Problem Statement A function $X \to Y$ between metric spaces is an *isometry* if it preserves distances, that is, $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. For instance, the map

$$\mathbb{R}^2 \to \mathbb{R}, \;\; t \mapsto (\frac{3}{5}t + 1, \frac{4}{5}t - 5) \tag{9}$$

is an isometry, with the image being the line $4x = 3y + 19$. In which of the metric categories is bijective isometry the notion of isomorphism.

Isomorphisms

Definition (Isomorphism)

We say a morphism $f : a \to b$ is an isomorphism in C if there exists a morphism $g : b \to a$ such that $f \circ g = id_b$ and $g \circ f = id_a$. The morphism g is called the inverse of f. The objects a and b are said to be isomorphic if there exists an isomorphism $f : a \to b$.

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For two isomorphic metric spaces X and Y, if $f : a \to b$ is an isomorphism and g its inverse, then for any $x \in X$

$$g(f(x)) = id_x(x) = x \tag{10}$$

Isomorphisms in Metric_{wc}

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Definition (Weak contraction)

A function $f : X \to Y$ between metric spaces is a weak contraction if

$$d(f(x), f(y)) \leq (x, y) \tag{11}$$

for all $x, y \in X$

Suppose $f : X \to Y$ is an isomorphism between $X, Y \in \text{Metric}_{wc}$, and g is it's inverse. For $x_1, x_2 \in X$, the isomorphisms f and g must satisfy (from 11)

$$d(f(x_1), f(x_2)) \leq d(x_1, x_2) \tag{12}$$

$$d(g(f(x_1)), g(f(x_2))) \leq d(f(x_1), f(x_2)) \tag{13}$$

From 10, we see that

$$\begin{gathered} d(g(f(x_1)), g(f(x_2))) = d(x_1, x_2) \leq d(f(x_1), f(x_2)) \\ \implies d(x_1, x_2) = d(f((x_1), f(x_2))) \end{gathered}$$

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Therefore, isomorphisms in Metric_{wc} are isometries.

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Definition (Lipschitz continuity)

A function $f : X \to Y$ between metric spaces is Lipschitz continuous if there exists a $K > 0$ such that

$$d(f(x), f(y)) \leq Kd(x, y) \tag{14}$$

for all $x, y \in X$

Suppose $f : X \to Y$ is an isomorphism between $X, Y \in \text{Metric}_L$, and g is it's inverse. For $x_1, x_2 \in X$, the isomorphisms f and g must satisfy (from 14)

$$\begin{aligned}
d(f(x_1), f(x_2)) &\leq Kd(x_1, x_2) && (15)\\
d(g(f(x_1)), g(f(x_2))) &\leq K'd(f(x_1), f(x_2)) && (16)\\
&\text{Again, by using 14 we get} && (17)\\
d(f(x_1), f(x_2)) &\leq Kd(x_1, x_2) && (18)\\
d(x_1, x_2) &\leq K'd(f(x_1), f(x_2)) && (19)\\
\implies KK' &\geq 1 && (20)
\end{aligned}$$

Suppose $f : X \to Y$ is an isomorphism between $X, Y \in \text{Metric}_L$, and g is it's inverse. For $x_1, x_2 \in X$, the isomorphisms f and g must satisfy (from 14)

$$d(f(x_1), f(x_2)) \leq Kd(x_1, x_2) \tag{15}$$

$$d(g(f(x_1)), g(f(x_2))) \leq K' d(f(x_1), f(x_2)) \tag{16}$$

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$$\implies KK' \geq 1 \tag{20}$$

This means that for any isomorphisms in Metric_L, the coefficients $K, K' \neq 1$. Thus, not all isomorphisms in Metric_L are isometries.

Detour to Problem Set 3.1 - 8

In question 8 of the problem set we are asked to prove the following inclusion functors exist:

$$\mathrm{Metric}_{wc} \to \mathrm{Metric}_L \to \mathrm{Metric}_u \to \mathrm{Metric}$$

and that the inclusions are not proper.

Proofs:

- Proof that weak contractions are Lipschitz continuous:
  if $f : X \to Y$ is a weak contraction then,

  $$d(f(x_1), f(x_2)) \leq d(x_1, x_2)$$

  thus, $f : X \to Y$ is Lipschitz continuous with $K = 1$

- Proof that Lipschitz continuous functions are uniformly continuous: if $f : X \to Y$ is Lipschitz continuous then,

$$d(f(x_1), f(x_2)) \leq Kd(x_1, x_2) \text{ for some } K \geq 0$$

  if we take $\delta = \epsilon / K$, for some $\epsilon > 0$, then ,

$$d(x_1, d_2) < \delta \implies d(f(x_1), f(x_2)) < \epsilon$$

  thus, $f : X \to Y$ is uniformly continuous
- Proof that uniform continuous functions are continuous: if $f : X \to Y$ is uniformly continuous then, if given $\epsilon > 0$, there exists $\delta > 0$ such that:

$$d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon$$

  Clearly, a uniform continuous function is continuous.

Examples where the inclusions are not proper:

- The function $f(x) = 2x$ on $\mathbb{R}$ is Lipschitz continuous but not a weak contraction
- The function $f(x) = \sqrt{x}$ on $[0, \infty]$ is uniformly continuous but not Lipschitz continuous
- The function $f(x) = 1/x$ on $[0, 1]$ is continuous but not uniformly continuous

Wrapping it up

Now that we have shown that we have functors:

$$\mathrm{Metric}_{wc} \to \mathrm{Metric}_L \to \mathrm{Metric}_u \to \mathrm{Metric}$$

From this we can say that since not all isomorphisms in Metric_L aren't isometries,

Wrapping it up

Now that we have shown that we have functors:

$$\mathsf{Metric}_{wc} \to \mathsf{Metric}_L \to \mathsf{Metric}_u \to \mathsf{Metric}$$

From this we can say that since not all isomorphisms in Metric_L aren't isometries, it means that not all isomorphisms in Metric_u and Metric are isometries.
Thus only for the category $\mathsf{Metric}_w c$, bijective isometry gives the notion of isomorphism.

Topological Spaces

The following questions deal with the idea of Topological Spaces, so here's a quick recap on what exactly those are.

Topological Spaces: A *topological space* is a set X on which a *topology* τ is equipped. τ is a collection of subsets of X (or, τ is a subset of the power set 2^X of X) such that -

1. $\varnothing$ and X should belong to τ
2. the union of the elements in any subset of τ should belong to τ
3. the intersection of the elements in any finite subset of τ should belong to τ

The elements of τ are called *open sets*. Thus, a topological space is a pair (X, τ) consisting of a set and a topology on it.

We can reframe the axioms given on the previous slide in terms of open sets -

1. The empty and the full set are open.
2. Any arbitrary union of open sets is open.
3. Any finite intersection of open sets is open.

Problem Set 3.4 - 8

Show that: The euclidean, diamond, square metrics on $\mathbb{R}^2$ have the same underlying topology. (When we say continuous map from $\mathbb{R}^2$ to $\mathbb{R}$, it is w.r.t this topology.) Further, check that it coincides with the product topology on $\mathbb{R} \times \mathbb{R}$.

Before we jump into the proof for this, we need to talk about how to compare topologies. The set of all topologies on a set forms a partially ordered set with the binary relation $\subseteq$. With this relation, we can define a partial ordering that we use to compare topologies.

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Show that: The euclidean, diamond, square metrics on $\mathbb{R}^2$ have the same underlying topology. (When we say continuous map from $\mathbb{R}^2$ to $\mathbb{R}$, it is w.r.t this topology.) Further, check that it coincides with the product topology on $\mathbb{R} \times \mathbb{R}$.

Before we jump into the proof for this, we need to talk about how to compare topologies. The set of all topologies on a set forms a partially ordered set with the binary relation $\subseteq$. With this relation, we can define a partial ordering that we use to compare topologies.

If there are two topologies τ and τ' on X such that $\tau \subseteq \tau'$, then τ is said to be a *coarser or weaker topology* than τ' and τ' is a *finer or stronger topology* than τ. An additional check is whether the two topologies are equal, if they aren't equal, then one can be called **strictly** finer or coarser than the other.

Lemma 13.3 from Munkres' Toplogy

Let $\mathcal{B}$ and $\mathcal{B}'$ be the bases for the topologies τ and τ' on X. Then the following are equivalent -

1. τ' is finer than τ
2. for each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$

The Metrics

The three metrics in question were defined in the earlier slides, but for the sake of context, the diamond metric d_1, the euclidean metric d_2 and the square metric d_∞ are defined over $\mathbb{R}^2$ as follows - (note that the notation used is $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{y} = (y_1, y_2)$ are points in $\mathbb{R}^2$)

$$d_1(\boldsymbol{x}, \boldsymbol{y}) = |x_1 - y_1| + |x_2 - y_2|$$

$$d_2(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$$

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

Let τ_{square}, $\tau_{\text{euclidean}}$ and τ_{diamond} be the respective topologies generated by the square, euclidean and diamond metrics.

Problem Set 3.4 - 8

Now, let us examine the relation between these three metrics.

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\} = \left(\max\{|x_1 - y_1|, |x_2 - y_2|\}^2\right)^{\frac{1}{2}}$$

We can also straightaway see that -

$$\max\{|x_1 - y_1|, |x_2 - y_2|\}^2 \leq |x_1 - y_1|^2 + |x_2 - y_2|^2$$

Note the fact that $f(x) = x^{\frac{1}{2}}$ is an increasing function for $x \geq 0$. Thus, we have -

$$\left(\max\{|x_1 - y_1|, |x_2 - y_2|\}^2\right)^{\frac{1}{2}} \leq \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$$
$$\implies d_\infty(\boldsymbol{x}, \boldsymbol{y}) \leq d_2(\boldsymbol{x}, \boldsymbol{y})$$

Now,

$$d_1(\boldsymbol{x}, \boldsymbol{y}) = |x_1 - y_1| + |x_2 - y_2| = \sqrt{(|x_1 - y_1| + |x_2 - y_2|)^2}$$
$$= \sqrt{(|x_1 - y_1|^2 + |x_2 - y_2|^2 + 2 * |x_1 - y_1| * |x_2 - y_2|)} \geq \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$$
$$\implies d_1(\boldsymbol{x}, \boldsymbol{y}) \geq d_2(\boldsymbol{x}, \boldsymbol{y})$$

(since $f(x) = \sqrt{x}$ is an increasing function, and $2 * |x_1 - y_1| * |x_2 - y_2| \geq 0$

And since $d_\infty(\boldsymbol{x}, \boldsymbol{y}) \leq d_2(\boldsymbol{x}, \boldsymbol{y})$, we have the following ordering -

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) \leq d_2(\boldsymbol{x}, \boldsymbol{y}) \leq d_1(\boldsymbol{x}, \boldsymbol{y})$$

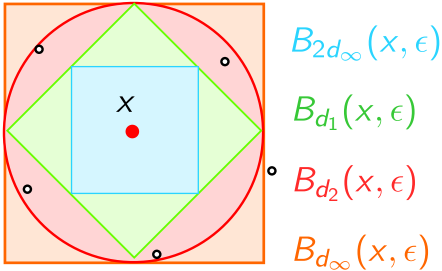
Now, consider the following -

$$
\begin{gathered}
d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\} \geq |x_1 - y_1| \\
\text{Also, } \max\{|x_1 - y_1|, |x_2 - y_2|\} \geq |x_2 - y_2| \\
\implies 2 * \max\{|x_1 - y_1|, |x_2 - y_2|\} \geq |x_1 - y_1| + |x_2 - y_2| \\
\implies 2 * d_\infty(\boldsymbol{x}, \boldsymbol{y}) \geq d_1(\boldsymbol{x}, \boldsymbol{y})
\end{gathered}
$$

Which implies, we have the following ordering -

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) \leq d_2(\boldsymbol{x}, \boldsymbol{y}) \leq d_1(\boldsymbol{x}, \boldsymbol{y}) \leq 2 * d_\infty(\boldsymbol{x}, \boldsymbol{y})$$

Now, we have 4 metrics $d_\infty(\boldsymbol{x}, \boldsymbol{y})$, $d_2(\boldsymbol{x}, \boldsymbol{y})$, $d_1(\boldsymbol{x}, \boldsymbol{y})$ and $2 * d_\infty(\boldsymbol{x}, \boldsymbol{y})$. (it is a trivial check to see that $2 * d_\infty(\boldsymbol{x}, \boldsymbol{y})$ also forms a metric). Before looking at the notion of open sets mathematically, we take a visual look at what open balls of the same radius look like, w.r.t each of these metrics.



Now we look at this mathematically. Consider the following ordering -

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) \leq d_2(\boldsymbol{x}, \boldsymbol{y}) \leq 2 * d_\infty(\boldsymbol{x}, \boldsymbol{y})$$

For all $\boldsymbol{x} \in \mathbb{R}^2$ and $\epsilon > 0$, since $d_2(\boldsymbol{x}, \boldsymbol{y}) \leq 2 * d_\infty(\boldsymbol{x}, \boldsymbol{y})$,

$$2 * d_\infty(\boldsymbol{x}, \boldsymbol{y}) < \epsilon \implies d_2(\boldsymbol{x}, \boldsymbol{y}) < \epsilon$$
$$\text{Thus, } d_\infty(\boldsymbol{x}, \boldsymbol{y}) < \frac{\epsilon}{2} \implies d_2(\boldsymbol{x}, \boldsymbol{y}) < \epsilon$$

This means that the open ball $B_{d_\infty}\left(\boldsymbol{x}, \frac{\epsilon}{2}\right)$ in the topology induced by d_∞ is contained within the open ball $B_{d_2}(\boldsymbol{x}, \epsilon)$ in the topology induced by d_2. Implying that the square metric topology is *finer* than the Euclidean metric topology.

(since you've shown that every possible open set in the topology induced by d_2 will be present in the topology induced by $d_\infty \implies \tau_{\text{euclidean}} \subseteq \tau_{\text{square}}$)

Similarly, since $d_\infty(\boldsymbol{x}, \boldsymbol{y}) \leq d_2(\boldsymbol{x}, \boldsymbol{y})$, we have -

$$d_2(\boldsymbol{x}, \boldsymbol{y}) < \epsilon \implies d_\infty(\boldsymbol{x}, \boldsymbol{y}) < \epsilon$$

This means that the open ball $B_{d_2}(\boldsymbol{x}, \epsilon)$ in the topology induced by d_2 is contained within the open ball $B_{d_\infty}(\boldsymbol{x}, \epsilon)$ in the topology induced by d_∞. Implying that the Euclidean metric topology is *finer* than the square metric topology.

(since you've shown that every possible open set in the topology induced by d_∞ will be present in the topology induced by $d_2 \implies \tau_{\text{square}} \subseteq \tau_{\text{euclidean}}$)

Combining the two results together, we have shown that

$$\tau_{\text{square}} \subseteq \tau_{\text{euclidean}} \subseteq \tau_{\text{square}} \implies \tau_{\text{square}} = \tau_{\text{euclidean}}$$

We perform the same process as earlier. We have the following ordering -

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) \leq d_1(\boldsymbol{x}, \boldsymbol{y}) \leq 2 * d_\infty(\boldsymbol{x}, \boldsymbol{y})$$

For all $\boldsymbol{x} \in \mathbb{R}^2$ and $\epsilon > 0$, since $d_1(\boldsymbol{x}, \boldsymbol{y}) \leq 2 * d_\infty(\boldsymbol{x}, \boldsymbol{y})$,

$$2 * d_\infty(\boldsymbol{x}, \boldsymbol{y}) < \epsilon \implies d_1(\boldsymbol{x}, \boldsymbol{y}) < \epsilon$$

$$\text{Thus, } d_\infty(\boldsymbol{x}, \boldsymbol{y}) < \frac{\epsilon}{2} \implies d_1(\boldsymbol{x}, \boldsymbol{y}) < \epsilon$$

This means that the open ball $B_{d_\infty}\left(\boldsymbol{x}, \frac{\epsilon}{2}\right)$ in the topology induced by d_∞ is contained within the open ball $B_{d_1}(\boldsymbol{x}, \epsilon)$ in the topology induced by d_1. Implying that the square metric topology is *finer* than the diamond metric topology.

(since you've shown that every possible open set in the topology induced by d_1 will be present in the topology induced by $d_\infty \implies \tau_{\text{diamond}} \subseteq \tau_{\text{square}}$)

Similarly, since $d_\infty(\boldsymbol{x}, \boldsymbol{y}) \leq d_1(\boldsymbol{x}, \boldsymbol{y})$, we have -

$$d_1(\boldsymbol{x}, \boldsymbol{y}) < \epsilon \implies d_\infty(\boldsymbol{x}, \boldsymbol{y}) < \epsilon$$

This means that the open ball $B_{d_1}(\boldsymbol{x}, \epsilon)$ in the topology induced by d_1 is contained within the open ball $B_{d_\infty}(\boldsymbol{x}, \epsilon)$ in the topology induced by d_∞. Implying that the diamond metric topology is *finer* than the square metric topology.

(since you've shown that every possible open set in the topology induced by d_∞ will be present in the topology induced by $d_1 \implies \tau_{\text{square}} \subseteq \tau_{\text{diamond}}$)

Combining the two results together, we have shown that

$$\tau_{\text{square}} \subseteq \tau_{\text{diamond}} \subseteq \tau_{\text{square}} \implies \tau_{\text{square}} = \tau_{\text{diamond}}$$

Thus, we have shown that
$\tau_{\text{square}} = \tau_{\text{diamond}}$ and $\tau_{\text{square}} = \tau_{\text{euclidean}} \implies \tau_{\text{square}} = \tau_{\text{euclidean}} = \tau_{\text{diamond}}$ $\square$

We now need to show that they coincide with the product topology on $\mathbb{R} \times \mathbb{R}$. We show this for the square metric as follows.

Let $\boldsymbol{B} = (a_1, b_1) \times (a_2, b_2)$ be a basis element for the product topology on $\mathbb{R} \times \mathbb{R}$, let $\boldsymbol{x} = (x_1, x_2)$ be some element of $\boldsymbol{B}$. Now, by the definition of open sets, there exist ϵ_1 and ϵ_2 such that -

$$(x_1 - \epsilon_1, x_1 + \epsilon_1) \subset (a_1, b_1) \text{ and } (x_2 - \epsilon_2, x_2 + \epsilon_2) \subset (a_2, b_2)$$

Choose $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then $B_{d_\infty}(\boldsymbol{x}, \epsilon) \subset \boldsymbol{B}$ (since
$B_{d_\infty}(\boldsymbol{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \subset (a_1, b_1) \times (a_2, b_2)$).
Hence, we have shown that each open set in the product topology will be present within the square metric topology. Hence, $\tau_{\text{product}} \subseteq \tau_{\text{square}}$

Now, let $B_{d_\infty}(\boldsymbol{x}, \epsilon)$ be an arbitrary open ball in $\mathbb{R}^2$ with the square metric topology τ_{square}. This open ball is a basis element for the square metric topology (since the basis elements of metric-induced topologies are the open balls generated by that metric). But -

$$B_{d_\infty}(\boldsymbol{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon)$$

is a basis element for the product topology as well! (since the basis elements of the product topology are arbitrary cartesian products of open sets in $\mathbb{R}$). This implies that every open set in the topology induced by the square metric will be present in the product topology. Hence $\tau_{\text{square}} \subseteq \tau_{\text{product}}$.

Combining the two results, $\tau_{\text{square}} \subseteq \tau_{\text{product}} \subseteq \tau_{\text{square}} \implies \tau_{\text{square}} = \tau_{\text{product}}$.

$$\tau_{\text{square}} = \tau_{\text{euclidean}} = \tau_{\text{diamond}} = \tau_{\text{product}} \quad \square$$

Problem Set 3.4 - 4

Ques - Show that: The underlying topology of the discrete metric is the discrete topology. If a set X has more than one element, then the indiscrete topology on X is not metrizable.

Both these subparts deal with one or the other extreme cases as far as topologies go. So let's look at them individually before solving the problem.

Discrete Topology: The textbook definition of the *discrete topology* is that it is a collection of all subsets of X, i.e, $\tau = 2^X$. There are a few interesting inferences to be drawn from this definition. Since every possible subset is an open subset in the discrete topology, in particular, every *singleton subset* is an open set in this topology.

Indiscrete Topology: The collection $\tau = \{\emptyset, X\}$ on X is the *indiscrete, or trivial topology* on X. A consequence of this collection is that all points in the set X cannot be distinguished from each other through topological means.

Now, let's look at the first part of the problem -

Show that the underlying topology of the discrete metric is the discrete topology

The discrete metric is as follows -

$$d_{\text{discrete}}(x, y) := \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{otherwise.} \end{cases}$$

Now, a metric d on a set X induces a topology τ by taking the idea of the open balls $B(x, r) = \{y : d(x, y) < r\}$ as the bases of open sets. We need to show that the d_{discrete} we are given produces the discrete topology $\tau = 2^X$.

Let $x \in X$ be an arbitrary element, and let $r \in (0, 1]$; then by the definition of the discrete metric $B_d(x, r) = \{x\}$, so $\forall x \in X, \{x\}$ is an open set.

Now, we know that any arbitrary union of open sets is open. Let $A \subseteq X$ be any arbitrary subset of X, then $A = \bigcup_{x \in A}\{x\}$, but we have shown that $\forall x \in X, \{x\}$ is an open set.

Since any arbitrary union of open sets is open, we can claim that A is an open set, as induced by the discrete metric. Since this claim holds for any $A \subseteq X$, we thus claim that every subset of X is open, i.e, $\forall A \subseteq X, A \in \tau$.

Since τ contains every possible subset of X, it is the power set 2^X of X. Thus, we have shown that the discrete metric induces a topology $\tau = 2^X$ on X. Since this is the definition of the discrete topology, we have shown that the underlying topology of the discrete metric is the discrete topology. $\square$

We now look at the next part of the problem -

Show that if a set X has more than one element, then the indiscrete topology on X is not metrizable.

We prove this by contradiction. Assume that there exists a metric d on the set X such that (X, d) is a metric space and that the topology induced by this metric on X is the indiscrete topology, $\tau = \{\emptyset, X\}$

X has at least 2 distinct elements x and y, i.e, $\exists x, y \in X$ s.t $x \neq y$.

$\implies d(x, y) = r > 0$

Now, consider the open ball $B(x, r/2)$. This open ball should be an open set in the topology that d induces.

But, $x \in B(x, r/2)$ and since $d(x, y) = r > r/2$, $y \notin B(x, r/2)$.

Thus, $B(x, r/2) \neq \emptyset$ and $B(x, r/2) \neq X$ (as there is at least one element $y \in X$ s.t $y \notin B(x, r/2)$).

Thus, the topology induced by the metric d cannot be the indiscrete topology, since $\tau_{\text{indiscrete}} = \{\emptyset, X\}$

Thus, we have shown that if a set X has more than one element, then the indiscrete topology on X is not metrizable. $\square$

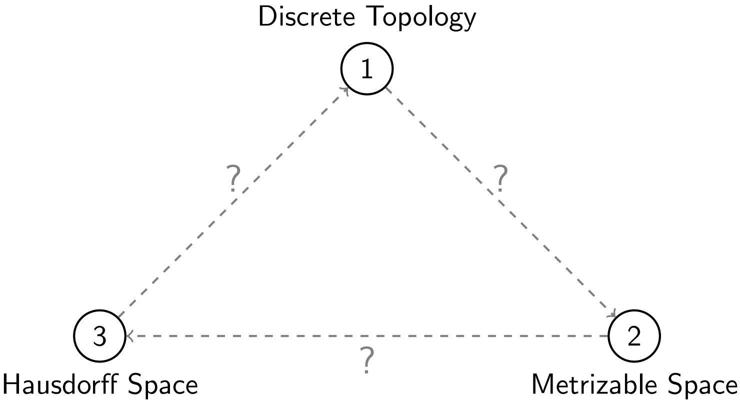
Exercise 3.4 - 15

Problem Statement

Let X be a *finite* topological space. Show that the following are equivalent:

1. X has the discrete topology.
2. X is metrizable.
3. X is Hausdorff.

Establishing Relationships



$1 \to 2$ — Discrete $\to$ Metrizable

We've already seen this done by Parth. We choose the discrete metric

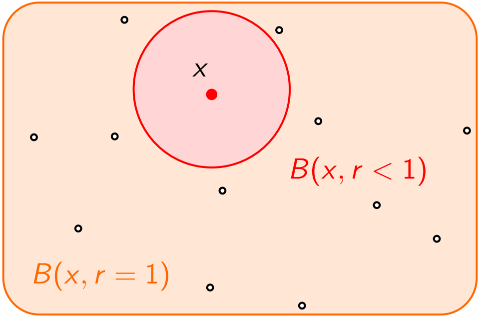
$$X \in \text{Metric}$$
$$d : X \times X \to \mathbb{R}$$

$$d(x,y) = \begin{cases} 1 \text{ if } x \neq y, \\ 0 \text{ otherwise.} \end{cases}$$

$1 \to 2$ — Discrete $\to$ Metrizable

We've already seen this done by Parth. We choose the discrete metric

$$X \in \text{Metric} \qquad d(x,y) = \begin{cases} 1 \text{ if } x \neq y, \\ 0 \text{ otherwise.} \end{cases}$$
$$d : X \times X \to \mathbb{R}$$



$2 \to 3$ — Metrizable $\to$ Hausdorff

Given that the space, say (X, τ), is metrizable, there exists a metric $d : X \times X \to \mathbb{R}$ which induces the topology given by τ.

$2 \to 3$ — Metrizable $\to$ Hausdorff

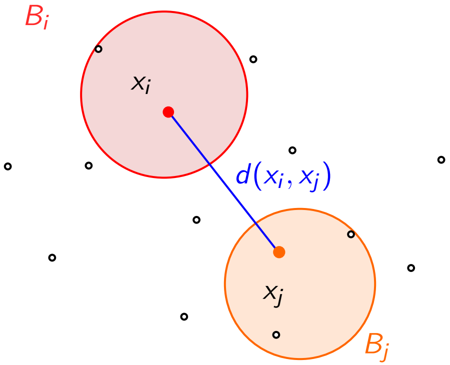
Given that the space, say (X, τ), is metrizable, there exists a metric $d : X \times X \to \mathbb{R}$ which induces the topology given by τ. Use this metric to define open balls B_i, B_j for any pair of points in X, x_i, x_j.

$2 \to 3$ — Metrizable $\to$ Hausdorff

Given that the space, say (X, τ), is metrizable, there exists a metric $d : X \times X \to \mathbb{R}$ which induces the topology given by τ. Use this metric to define open balls B_i, B_j for any pair of points in X, x_i, x_j. By shrinking these open balls, we can create non-intersecting open sets as required.

$2 \to 3$ — Metrizable $\to$ Hausdorff

Given that the space, say (X, τ), is metrizable, there exists a metric $d : X \times X \to \mathbb{R}$ which induces the topology given by τ. Use this metric to define open balls B_i, B_j for any pair of points in X, x_i, x_j. By shrinking these open balls, we can create non-intersecting open sets as required.



$3 \to 1$ — Hausdorff $\to$ Discrete

We know by the Hausdorff property that any two points are separable by neighbourhoods.

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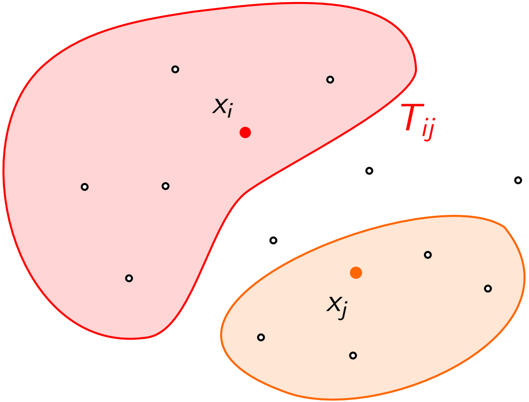

$3 \to 1$ — Hausdorff $\to$ Discrete

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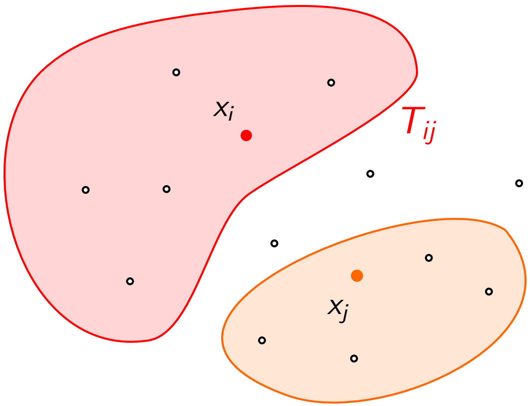
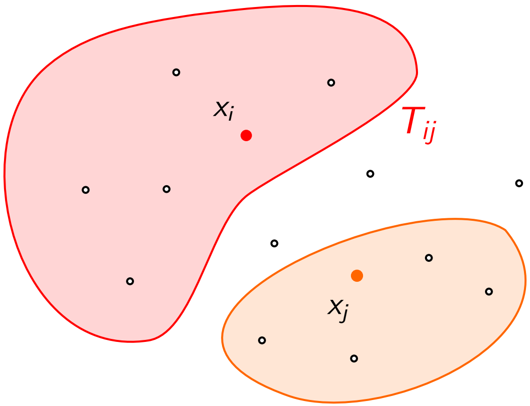


$$\forall x_i \bigcap_{x_j \in X} T_{ij} = \{x_i\},$$

$3 \to 1$ — Hausdorff $\to$ Discrete

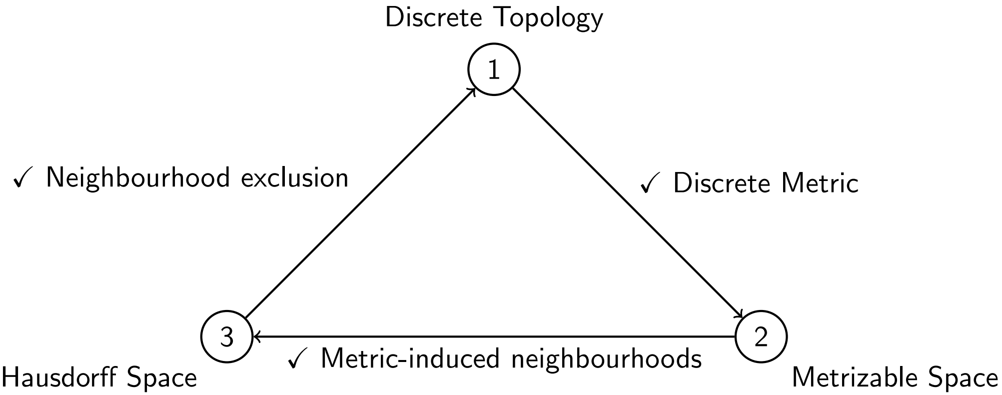
We know by the Hausdorff property that any two points are separable by neighbourhoods.



$$\forall x_i \bigcap_{x_j \in X} T_{ij} = \{x_i\},$$

$$\{x_i\} \in \tau \ .$$

Establishing Relationships



Exercise 3.4 - 16

Problem Statement
Give an example of a topological space which is Hausdorff but not metrizable.

Examining the problem

We just proved Metrizable $\Leftrightarrow$ Hausdorff, so what gives?

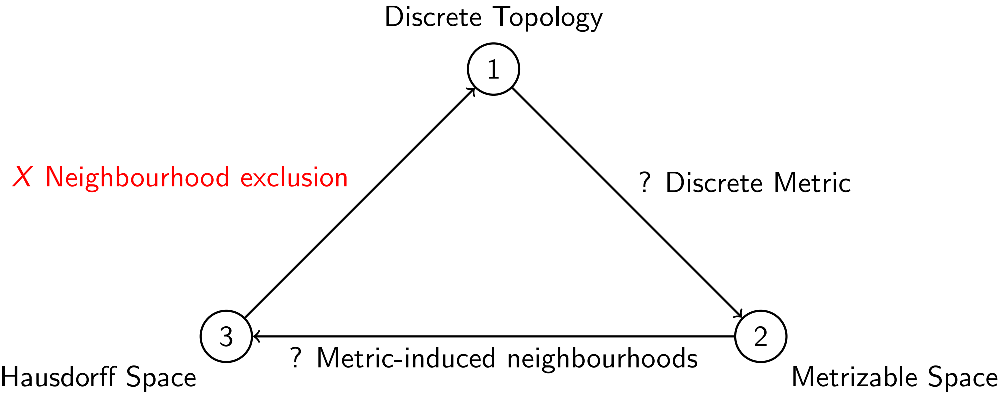
Examining the problem

We just proved Metrizable $\Leftrightarrow$ Hausdorff, so what gives? There is something quite important we used to establish all the ideas in that problem.

Examining the problem

We just proved Metrizable $\Leftrightarrow$ Hausdorff, so what gives? There is something quite important we used to establish all the ideas in that problem. *Finiteness.*

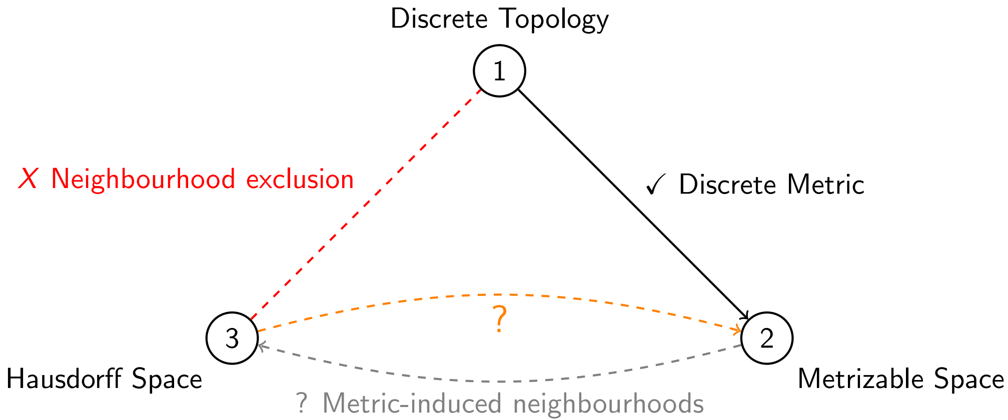
Morphing Relationships



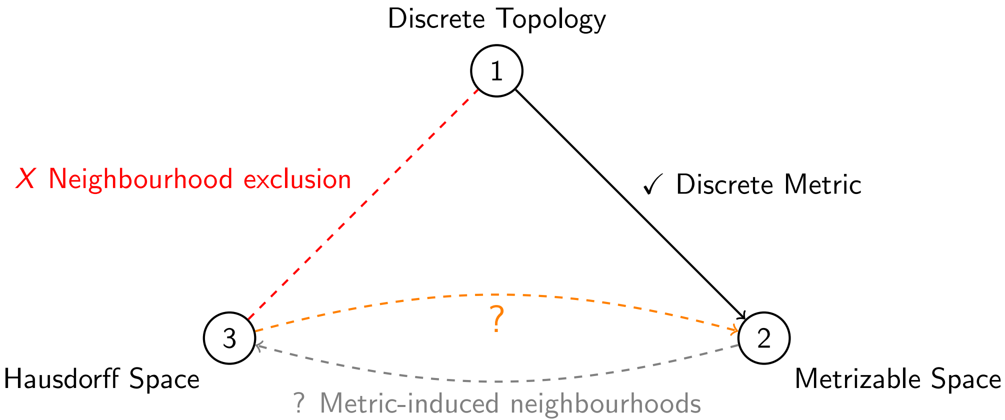
Chasing Broken Bridges

We are now looking for a Hausdorff space that is not metrizable. It cannot be discrete, since we have the discrete metric for it, regardless of finiteness, i.e., the implication edge $1 \to 2$ holds without finiteness too.

Morphing Relationships

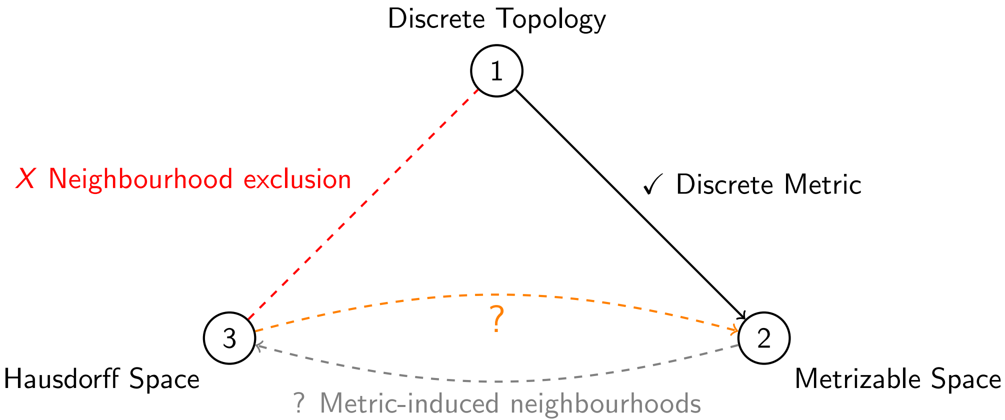


Morphing Relationships



We have lost some information about the internal relationships while breaking one edge.

Morphing Relationships



We have lost some information about the internal relationships while breaking one edge. If finiteness doesn't break things enough, what did we use that might?

A Chasm

We construct a topology from a metric space by constructing open balls around points.

A Chasm

We construct a topology from a metric space by constructing open balls around points. So, perhaps we can break metrizability by taking a Hausdorff space, and creating gaps in it that cannot be worked around with a metric.

Combining everything, consider the space $(\mathbb{R}, \tau)$ with τ being the usual topology on the real line. It is clearly Hausdorff, and metrizable.

Combining everything, consider the space $(\mathbb{R}, \tau)$ with τ being the usual topology on the real line. It is clearly Hausdorff, and metrizable. Construct from this a new space $*\mathbb{R}$ from $\mathbb{R}$ with an added point ω, a number larger than any finite real. Choose the set $\{\omega\}$ to additionally be open.

Combining everything, consider the space $(\mathbb{R}, \tau)$ with τ being the usual topology on the real line. It is clearly Hausdorff, and metrizable. Construct from this a new space $*\mathbb{R}$ from $\mathbb{R}$ with an added point ω, a number larger than any finite real. Choose the set $\{\omega\}$ to additionally be open. We see that there are suddenly issues with defining a metric on this space.

Suppose we were able to metrize this space. That would imply we have defined a *real* distance between all points on the real line and ω

Suppose we were able to metrize this space. That would imply we have defined a *real* distance between all points on the real line and ω , but this would imply that ω is contained within some finite open balls centered at said points. Intuitively, this does not make any sense per the definition of ω.

Non-metrizability of $*\mathbb{R}$.

If possible, suppose there exists a metric such that it induces the defined topology on $*\mathbb{R}$, $d : *\mathbb{R} \times *\mathbb{R} \to \mathbb{R}$, defined as usual over the 'finite' numbers.

Non-metrizability of $*\mathbb{R}$.

If possible, suppose there exists a metric such that it induces the defined topology on $*\mathbb{R}$, $d : *\mathbb{R} \times *\mathbb{R} \to \mathbb{R}$, defined as usual over the 'finite' numbers. So $\exists f : \mathbb{R} \to \mathbb{R},$ such that $\forall x \in \mathbb{R}\ d(x, \omega) = f(x)$. Now, pick any two points $x, y \neq \omega$, such that $d(x, y) = a$ for some real a.

Non-metrizability of $*\mathbb{R}$.

If possible, suppose there exists a metric such that it induces the defined topology on $*\mathbb{R}$, $d : *\mathbb{R} \times *\mathbb{R} \to \mathbb{R}$, defined as usual over the 'finite' numbers. So $\exists f : \mathbb{R} \to \mathbb{R},$ such that $\forall x \in \mathbb{R}\ d(x,\omega) = f(x)$. Now, pick any two points $x, y \neq \omega$, such that $d(x,y) = a$ for some real a.

Then by the triangle inequality, we must have

$$d(x,y) \leq d(x,\omega) + d(\omega,y) \text{ , so}$$
$$a \leq f(x) + f(y) \text{ .}$$

Since $a, x,$ and y were arbitrary, $f(x)$ must be unbounded $\forall x$, and thus d is not a proper metric. This happens because ω is a transfinite number. We have a contradiction. $\square$

Problem Set 3.4 - 7 — Pushkar Mohile

Analysis Notes 3.4 Q.- Show that the discrete and indiscrete topologies on a set give rise to functors

$$\text{Set} \to \text{Top} \tag{21}$$

and these are the left and right adjoints, respectively, to the forgetful functor from Top to Set.

Solution :
We begin by recalling the definition of a functor. Given two categories C and D, a functor $\mathcal{F}$ assigns to every object $a \in \mathrm{C}$ an object $\mathcal{F}(a) \in \mathrm{D}$, and for every morphism $f \in C(a, b)$ a corresponding morphism

$$\mathcal{F}(f) \in \mathrm{D}(\mathcal{F}(a), \mathcal{F}(b)) \tag{22}$$

which respects composition

$$\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g) \tag{23}$$

and maps id to id.

Let us construct the functors corresponding to the discrete and indiscrete topologies on any set X given by $\tau_{disc} = 2^X$ and $\tau_{indisc} = \{\emptyset, X\}$. We will call them *disc*: Set $\to$ Top and *indisc*: Set $\to$ Top , defined in the following way

$$disc(X) \mapsto (X, \tau_{disc})$$
$$indisc(X) \mapsto (X, \tau_{indisc})$$

And for any function $f \in \mathrm{Set}(X, Y)$, $f : X \to Y \mapsto f : (X, \tau_X) \to (Y, \tau_Y)$. The check we need to make here is that f is a continuous function between sets with the discrete and indiscrete topologies.

For the discrete topology, this is done by noting that

$$\forall \text{ open sets } U \in \tau_Y, f^{-1}(U) \subseteq X \in 2^X \tag{24}$$

and hence $f^{-1}(U)$ is open in τ_{disc}.
Similarly, for the indiscrete topology, the only open subsets of Y are $\emptyset$, Y.
For $\emptyset \in \tau_Y$ we have $f^{-1}(\emptyset) = \emptyset \in \tau_X$ and $f^{-1}(Y) = X \cup \emptyset \in \tau_X$ and hence once again f is continuous. The composition law is valid since composition of continuous functions are continuous. Hence the discrete and indiscrete topologies define the required functors.

For the second part, we have to show that these are left adjoint and right adjoints respectively to the forgetful functor defined as follows:

$$\begin{gathered} frg : \mathsf{Top} \to \mathsf{Set} \\ (X, \tau_X) \mapsto X \\ f \in \mathsf{Top}((X, \tau_X), (Y, \tau_Y)) \mapsto f \in \mathsf{Set}(X, Y) \end{gathered}$$

i.e. we are forgetting the underlying topology and viewing the function f as a morphism between sets.

We recall the definitions of left and right adjoint functors. Given two categories C and D and functors $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ and $\mathcal{G} : \mathsf{D} \to \mathsf{C}$, we say that $\mathcal{F}$ is a left adjoint and $\mathcal{G}$ is a right adjoint if for objects $a \in \mathsf{C}$ and $x \in \mathsf{D}$, there is a bijection between the sets of morphisms

$$\mathsf{D}(\mathcal{F}(a), x) \xrightarrow{\cong} \mathsf{C}(a, \mathcal{G}(x))$$

that is *natural* in a and x. The naturality condition is formally stated as follows: For any morphism $a \to a'$ in C and $x \to x'$ in D, we have the following commutative diagrams (Check Cat Theory lec. 2 or section 4.3 of the cat theory notes)

$$\begin{tikzcd}
\mathsf{D}(\mathcal{F}(a), x) \arrow[r] & \mathsf{C}(a, \mathcal{G}(x)) \\
\mathsf{D}(\mathcal{F}(a'), x) \arrow[r] \arrow[u] & \mathsf{C}(a', \mathcal{G}(x)) \arrow[u]
\end{tikzcd}$$

And, the naturality condition for $x \to x'$ gives us the following commutative diagram:

$$\begin{tikzcd}
\mathsf{D}(\mathcal{F}(a), x) \arrow[r] \arrow[d] & \mathsf{C}(a, \mathcal{G}(x)) \arrow[d] \\
\mathsf{D}(\mathcal{F}(a), x') \arrow[r] & \mathsf{C}(a', \mathcal{G}(x'))
\end{tikzcd}$$

We now check the adjuction between *frg* and *disc* as defined previously. Let X be any set and (Y, τ_Y) be any topological space. We have to prove that

$$\mathrm{Top}(disc(X), (Y, \tau_Y)) \xrightarrow{\cong} \mathrm{Set}(X, Y) \tag{25}$$

This bijection is given by $f \mapsto f$ in both directions. We now have to simply check whether the two sets are the same. This is done as follows:

$$\mathrm{Top}(disc(X), (Y, \tau_Y)) \subseteq \mathrm{Set}(X, Y) \tag{26}$$

is obvious since continuous functions are functions between the sets.

Next, note that

$$\mathrm{Set}(X, Y) \subseteq \mathrm{Top}(disc(X), (Y, \tau_Y)) \tag{27}$$

Proof.

Let $f \in \mathrm{Set}(X, Y)$, U_Y be any open set on τ_Y. $f^{-1}(U) \subseteq X \in 2^X$ and hence f is continuous.
Thus we have proved that the two sets are equal. Finally we make note of the naturality condition. This holds because composition of functions and composition of continuous function commute with the functors. $\square$

Given $h \in \mathit{Set}(X, X')$ and $j \in \mathit{Top}((Y, \tau_Y), (Y', \tau_{Y'}))$,

$$
\begin{tikzcd}
\mathrm{Top}(\mathit{disc}(X), (Y, \tau_Y)) \arrow[r, "\mathrm{id}"] & \mathrm{Set}(X, Y) \\
\mathrm{Top}(\mathit{disc}(X'), (Y, \tau_Y)) \arrow[r, "\mathrm{id}"'] \arrow[u, "(-) \circ h"] & \mathrm{Set}(X', Y) \arrow[u, "(-) \circ h"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{Top}(\mathit{disc}(X), (Y, \tau_Y)) \arrow[r, "\mathrm{id}"] \arrow[d, "j \circ (-)"'] & \mathrm{Set}(X, Y) \arrow[d, "j \circ (-)"] \\
\mathrm{Top}(\mathit{disc}(X), (Y', \tau_{Y'})) \arrow[r, "\mathrm{id}"'] & \mathrm{Set}(X, Y')
\end{tikzcd}
$$

This adjunction can be restated in terms of the following universal property of the discrete topology : The discrete topology on X is the topology such that every function from X to any topological space Y, τ_Y is continuous.

Finally we take a look at the right adjoint condition for the indiscrete topology. The conditions states that

$$\mathsf{Set}(Y, X) \xrightarrow{\cong} \mathsf{Top}((Y, \tau_Y), \mathit{indisc}(X)) \tag{28}$$

The checks are similar to the previously done checks. We mention the only nontrivial check:

$$\mathsf{Set}(Y, X) \subseteq \mathsf{Top}((Y, \tau_Y), \mathit{indisc}(X)) \tag{29}$$

For a given function $f \in \mathsf{Set}(Y, X)$, with the indiscrete topology on X, $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(X) = Y \cup \emptyset$, both of which are open wrt any topology on Y. Hence f is continuous.