

Diff Geo HW

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Test 0

Test 1

Test 0

Test 1

Test 2

Metric Spaces

A *metric space* is a set X equipped with a map $d : X \times X \rightarrow \mathbb{R}$ such that :

1. $d(x, x) = 0$
2. $d(x, y) > 0$ if $x \neq y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$

for all $x, y, z \in X$

d : distance function or metric

Question 1

Show that: For any points a, b, x, y in a metric space X ,
 $|d(a, b) - d(x, y)| \leq d(a, x) + d(b, y)$

Using the property of the absolute value function $|a| \leq b \iff -b \leq a \leq b$ we get:

$$-d(a, x) - d(b, y) \leq d(a, b) - d(x, y) \leq d(a, x) + d(b, y)$$

Now we focus on the a term $\alpha \triangleq d(a, x) + d(x, y) + d(y, b)$. Using property (4):

$$d(a, b) \leq d(a, x) + d(x, b) \leq d(a, x) + d(x, y) + d(y, b)$$

Rearranging terms and using property (3) we obtain:

$$d(a, b) - d(x, y) \leq d(a, x) + d(b, y)$$

Which is the RHS of the inequality. Similarly, the LHS can be proven*.

*Modify (4) into $d(x, z) - d(y, z) \leq d(x, y)$ and $\alpha = -d(a, x) - d(x, y) + d(y, b)$

Question 2

Check that the diamond and square metrics on \mathbb{R}^n are indeed metrics. Show that the euclidean metric on \mathbb{R}^n is indeed a metric. (The triangle inequality in this context is equivalent to Minkowski's inequality.)

For any $x, y \in \mathbb{R}^n$ where $v = (v_1, v_2, \dots, v_n)$:

1. Diamond metric: $d_1(x, y) = \sum_{i \in \mathcal{I}} |x_i - y_i|$
2. Euclidean metric: $d_2(x, y) = \sqrt{\sum_{i \in \mathcal{I}} |x_i - y_i|^2}$
3. Square metric: $d_\infty(x, y) = \max_{i \in \mathcal{I}} \{|x_i - y_i|\}$

Now we just have to show that each of these metrics satisfy the four conditions that characterise metric spaces. We will look at each metric separately.

Question 2 contd.

The Diamond metric

1. $d_1(x, x) = \sum_{i=1}^n |x_i - x_i| = 0$ for all $i \in \mathcal{I}$.
2. Suppose $d_1(x, y) = \sum_{i=1}^n |x_i - y_i| = 0$. We know that that for $p \in \mathbb{R}$, $|p| \geq 0$ and $|p| = 0 \iff p = 0$ (positive definite). Therefore for the assumption to hold, $|x_i - y_i| = 0$ for all $i \in \mathcal{I}$. Hence $x_i = y_i$ for all $i \in \mathcal{I}$. This shows that for $x \neq y$, $d_1(x, y) > 0$.
3. We can rewrite
$$d_1(x, y) = \sum_{i=1}^n |x_i - y_i| = \sum_{i=1}^n |-(-x_i + y_i)| = \sum_{i=1}^n |(-x_i + y_i)| = \sum_{i=1}^n |y_i - x_i|.$$
Therefore, $d_1(x, y) = d_1(y, x)$.
4. Using the triangle inequality
$$|(x_i - y_i) + (y_i - z_i)| \leq |x_i - y_i| + |y_i - z_i| \rightarrow |x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|.$$
Taking the summation, $\sum_{i=1}^n |x_i - z_i| \leq \sum_{i=1}^n |x_i - y_i| + \sum_{i=1}^n |y_i - z_i|$. Hence, we have shown, $d_1(x, z) \leq d_1(x, y) + d_1(y, z)$.

Question 2 contd.

The Euclidean metric

1. $d_2(x, x) = \sqrt{\sum_{i=1}^n |x_i - x_i|^2} = 0$ for all $i \in \mathcal{I}$.
2. Suppose $d_2(x, y) = \sqrt{\sum_{i=1}^n |x_i - y_i|^2} = 0$. We know that for $p \in \mathbb{R}$, $\sqrt{p} \geq 0$ and $\sqrt{p} = 0 \iff p = 0$ (positive definite). Therefore for the assumption to hold, $\sum_{i=1}^n |x_i - y_i|^2 = 0$ for all $i \in \mathcal{I}$. Further, $|x_i - y_i|^2 = 0$ for all $i \in \mathcal{I}$. Hence $x_i = y_i$ for all $i \in \mathcal{I}$. This shows that for $x \neq y$, $d_2(x, y) > 0$.
3. We can rewrite $d_2(x, y) = \sqrt{\sum_{i=1}^n |x_i - y_i|^2} = \sqrt{\sum_{i=1}^n | -(-x_i + y_i) |^2} = \sqrt{\sum_{i=1}^n |(-x_i + y_i)|^2} = \sqrt{\sum_{i=1}^n |y_i - x_i|^2}$. Therefore, $d_2(x, y) = d_2(y, x)$.
4. Minkowski's inequality* says that $(\sum_{k=1}^n |\alpha_k + \beta_k|^p)^{\frac{1}{p}} \leq (\sum_{k=1}^n |\alpha_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^n |\beta_k|^p)^{\frac{1}{p}}$. If we substitute $\alpha_k = x_k - y_k$ and $\beta_k = y_k - z_k$, we get the desired triangle inequality.
 $\sqrt{\sum_{i=1}^n |x_i - z_i|^2} \leq \sqrt{\sum_{i=1}^n |x_i - y_i|^2} + \sqrt{\sum_{i=1}^n |y_i - z_i|^2}$. Hence, we have shown, $d_2(x, z) \leq d_2(x, y) + d_2(y, z)$.

Question 2 contd.

The Diamond metric

1. $d_1(x, x) = \sum_{i=1}^n |x_i - x_i| = 0$ for all $i \in \mathcal{I}$.
2. Suppose $d_1(x, y) = \sum_{i=1}^n |x_i - y_i| = 0$. We know that that for $p \in \mathbb{R}$, $|p| \geq 0$ and $|p| = 0 \iff p = 0$ (positive definite). Therefore for the assumption to hold, $|x_i - y_i| = 0$ for all $i \in \mathcal{I}$. Hence $x_i = y_i$ for all $i \in \mathcal{I}$. This shows that for $x \neq y$, $d_1(x, y) > 0$.
3. We can rewrite
$$d_1(x, y) = \sum_{i=1}^n |x_i - y_i| = \sum_{i=1}^n |-(-x_i + y_i)| = \sum_{i=1}^n |(-x_i + y_i)| = \sum_{i=1}^n |y_i - x_i|.$$
Therefore, $d_1(x, y) = d_1(y, x)$.
4. Using the triangle inequality
$$|(x_i - y_i) + (y_i - z_i)| \leq |x_i - y_i| + |y_i - z_i| \rightarrow |x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|.$$
Taking the summation, $\sum_{i=1}^n |x_i - z_i| \leq \sum_{i=1}^n |x_i - y_i| + \sum_{i=1}^n |y_i - z_i|$. Hence, we have shown, $d_1(x, z) \leq d_1(x, y) + d_1(y, z)$.

Question 2 contd.

The Square metric

1. $d_{\infty}(x, x) = \max_{i \in \mathcal{I}} \{|x_i - x_i|\} = \max\{0, 0, \dots, 0\} = 0$ for all $i \in \mathcal{I}$.
2. Suppose $d_{\infty}(x, y) = \max_{i \in \mathcal{I}} \{|x_i - y_i|\} = 0$ at $i = i^*$. We know that for $p \in \mathbb{R}$, $|p| \geq 0$ and $|p| = 0 \iff p = 0$ (positive definite). Therefore for the assumption to hold, $|x_i - y_i| \leq 0$ for all $i \in \mathcal{I}, i \neq i^* \Rightarrow |x_i - y_i| = 0$. Hence $x_i = y_i$ for all $i \in \mathcal{I}$. This shows that for $x \neq y$, $d_{\infty}(x, y) > 0$.
3. We can rewrite $d_{\infty}(x, y) = \max_{i \in \mathcal{I}} \{|x_i - y_i|\} = \max_{i \in \mathcal{I}} \{| - (-x_i + y_i) |\} = \max_{i \in \mathcal{I}} \{|(-x_i + y_i)|\} = \max_{i \in \mathcal{I}} \{|y_i - x_i|\}$. Therefore, $d_{\infty}(x, y) = d_{\infty}(y, x)$.
4. Need help here*

Minkowski's Inequality

$$\left(\sum_{k=1}^n |\alpha_k + \beta_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^n |\alpha_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^n |\beta_k|^p\right)^{\frac{1}{p}}$$

$\left(\sum_{k=1}^n |\alpha_k|^p\right)^{\frac{1}{p}} = \|\alpha\|_p$ is called the p-norm. The proof makes use of Holder's inequality. It is first shown that if α and β have a finite p-norm, so does $\alpha + \beta$ (convexity arguments).

$$\|\alpha + \beta\|_p^p = \int |\alpha + \beta|^p d\mu \leq \int (|\alpha| + |\beta|)|\alpha + \beta|^{p-1} d\mu$$

Now, applying Holder's inequality;

$$\|\alpha + \beta\|_p^p \leq (\|\alpha\|_p + \|\beta\|_p) \frac{\|\alpha + \beta\|_p^p}{\|\alpha + \beta\|_p} \quad (1)$$

Rearranging this gives:

$$\|\alpha + \beta\|_p \leq \|\alpha\|_p + \|\beta\|_p \quad (2)$$

Minkowski's Inequality for the Euclidean Metric

For the case of the Euclidean metric, the inequality boils down to:

$$\sqrt{\sum_{i=1}^n |x_i - z_i|^2} \leq \sqrt{\sum_{i=1}^n |x_i - y_i|^2} + \sqrt{\sum_{i=1}^n |y_i - z_i|^2}$$

For $n = 1$, we retrieve the same inequality as that of the diamond metric, i.e.,

$$\sqrt{|x_1 - z_1|^2} \leq \sqrt{|x_1 - y_1|^2} + \sqrt{|y_1 - z_1|^2} \Rightarrow |x_1 - z_1| \leq |x_1 - y_1| + |y_1 - z_1|$$

Should I write something for $n = 2$?*

Ques 0

Test 1

Ques 0

Test 1

Ques 0

Test 1

Topological Spaces

The following questions deal with the idea of Topological Spaces, so here's a quick recap on what exactly those are.

Topological Spaces: A *topological space* is a set X on which a *topology* τ is equipped. τ is a collection of subsets of X (or, τ is a subset of the power set 2^X of X) such that -

1. \emptyset and X should belong to τ
2. the union of the elements in any subset of τ should belong to τ
3. the intersection of the elements in any finite subset of τ should belong to τ

The elements of τ are called *open sets*. Thus, a topological space is a pair (X, τ) consisting of a set and a topology on it.

We can reframe the axioms given on the previous slide in terms of open sets -

1. The empty and the full set are open.
2. Any arbitrary union of open sets is open.
3. Any finite intersection of open sets is open.

Problem Set 3.4 - 8

Show that: The euclidean, diamond, square metrics on \mathbb{R}^2 have the same underlying topology. (When we say continuous map from \mathbb{R}^2 to \mathbb{R} , it is w.r.t this topology.) Further, check that it coincides with the product topology on $\mathbb{R} \times \mathbb{R}$.

Before we jump into the proof for this, which is slightly convoluted, we need to talk about how to compare topologies. The set of all topologies on a set forms a partially ordered set with the binary relation \subseteq . With this relation, we can define a partial ordering that we use to compare topologies.

If there are two topologies τ and τ' on X such that $\tau \subseteq \tau'$, then τ is said to be a *coarser or weaker topology* than τ' and τ' is a *finer or stronger topology* than τ . An additional check is whether the two topologies are equal, if they aren't equal, then one can be called **strictly** finer or coarser than the other.

The three metrics in question were defined in the earlier slides, but for the sake of context, the diamond metric d_1 , the euclidean metric d_2 and the square metric d_∞ are defined over \mathbb{R}^2 as follows - (note that the notation used is $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$ are points in \mathbb{R}^2)

$$d_1(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|$$

$$d_2(\mathbf{x}, \mathbf{y}) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$$

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

Now, let us examine the relation between these three metrics.

$$d_{\infty}(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\} = \left(\max\{|x_1 - y_1|, |x_2 - y_2|\}^2\right)^{\frac{1}{2}}$$

We can also straightaway see that -

$$\max\{|x_1 - y_1|, |x_2 - y_2|\}^2 \leq |x_1 - y_1|^2 + |x_2 - y_2|^2$$

Note the fact that $f(x) = x^{\frac{1}{2}}$ is an increasing function for $x \geq 0$. Thus, we have -

$$\begin{aligned} \left(\max\{|x_1 - y_1|, |x_2 - y_2|\}^2\right)^{\frac{1}{2}} &\leq \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2} \\ \implies d_{\infty}(\mathbf{x}, \mathbf{y}) &\leq d_2(\mathbf{x}, \mathbf{y}) \end{aligned}$$

Now,

$$\begin{aligned}d_1(\mathbf{x}, \mathbf{y}) &= |x_1 - y_1| + |x_2 - y_2| = \sqrt{(|x_1 - y_1| + |x_2 - y_2|)^2} \\&= \sqrt{(|x_1 - y_1|^2 + |x_2 - y_2|^2 + 2 * |x_1 - y_1| * |x_2 - y_2|)} \geq \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2} \\&\implies d_1(\mathbf{x}, \mathbf{y}) \geq d_2(\mathbf{x}, \mathbf{y})\end{aligned}$$

(since $f(x) = \sqrt{x}$ is an increasing function, and $2 * |x_1 - y_1| * |x_2 - y_2| \geq 0$)

And since $d_\infty(\mathbf{x}, \mathbf{y}) \leq d_2(\mathbf{x}, \mathbf{y})$, we have the following ordering -

$$d_\infty(\mathbf{x}, \mathbf{y}) \leq d_2(\mathbf{x}, \mathbf{y}) \leq d_1(\mathbf{x}, \mathbf{y})$$

Now, consider the following -

$$d_{\infty}(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\} \geq |x_1 - y_1|$$

$$\text{Also, } \max\{|x_1 - y_1|, |x_2 - y_2|\} \geq |x_2 - y_2|$$

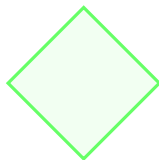
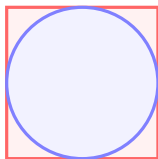
$$\implies 2 * \max\{|x_1 - y_1|, |x_2 - y_2|\} \geq |x_1 - y_1| + |x_2 - y_2|$$

$$\implies 2 * d_{\infty}(\mathbf{x}, \mathbf{y}) \geq d_1(\mathbf{x}, \mathbf{y})$$

Which implies, we have the following ordering -

$$d_{\infty}(\mathbf{x}, \mathbf{y}) \leq d_2(\mathbf{x}, \mathbf{y}) \leq d_1(\mathbf{x}, \mathbf{y}) \leq 2 * d_{\infty}(\mathbf{x}, \mathbf{y})$$

Now, we have 4 metrics $d_\infty(\mathbf{x}, \mathbf{y})$, $d_2(\mathbf{x}, \mathbf{y})$, $d_1(\mathbf{x}, \mathbf{y})$ and $2 * d_\infty(\mathbf{x}, \mathbf{y})$. (it is a trivial check to see that $2 * d_\infty(\mathbf{x}, \mathbf{y})$ also forms a metric). Before looking at the notion of open sets mathematically, we take a visual look at what open balls of the same radius look like, w.r.t each of these metrics.



Question (4)

Ques - Show that: The underlying topology of the discrete metric is the discrete topology. If a set X has more than one element, then the indiscrete topology on X is not metrizable.

Both these subparts deal with one or the other extreme cases as far as topologies go. So let's look at them individually before solving the problem.

Discrete Topology: The textbook definition of the *discrete topology* is that it is a collection of all subsets of X , i.e, $\tau = 2^X$. There are a few interesting inferences to be drawn from this definition. Since every possible subset is an open subset in the discrete topology, in particular, every *singleton subset* is an open set in this topology.

Question (4)

Indiscrete Topology: The collection $\tau = \{\emptyset, X\}$ on X is the *indiscrete, or trivial topology* on X . A consequence of this collection is that all points in the set X cannot be distinguished from each other through topological means.

Now, let's look at the first part of the problem -

Show that the underlying topology of the discrete metric is the discrete topology

The discrete metric is as follows -

$$d_{\text{discrete}}(x, y) := \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{otherwise.} \end{cases}$$

Question (4)

Now, a metric d on a set X induces a topology τ by taking the idea of the open balls $B(x, r) = \{y : d(x, y) < r\}$ as the bases of open sets. We need to show that the d_{discrete} we are given produces the discrete topology $\tau = 2^X$.

Let $x \in X$ be an arbitrary element, and let $r \in (0, 1]$; then by the definition of the discrete metric $B_d(x, r) = \{x\}$, so $\forall x \in X, \{x\}$ is an open set.

Now, by the axioms we discussed about topological spaces, any arbitrary union of open sets is open. Let $A \subseteq X$ be any arbitrary subset of X , then $A = \bigcup_{x \in A} \{x\}$, but we have shown that $\forall x \in X, \{x\}$ is an open set.

Question (4)

Since any arbitrary union of open sets is open, we can claim that A is an open set, as induced by the discrete metric. Since this claim holds for any $A \subseteq X$, we thus claim that every subset of X is open, i.e., $\forall A \subseteq X, A \in \tau$.

Since τ contains every possible subset of X , it is the power set 2^X of X . Thus, we have shown that the discrete metric induces a topology $\tau = 2^X$ on X . Since this is the definition of the discrete topology, we have shown that the underlying topology of the discrete metric is the discrete topology. \square

Question (4)

We now look at the next part of the problem -

Show that if a set X has more than one element, then the indiscrete topology on X is not metrizable.

We prove this by contradiction. Assume that there exists a metric d on the set X such that (X, d) is a metric space and that the topology induced by this metric on X is the indiscrete topology, $\tau = \{\emptyset, X\}$

X has at least 2 distinct elements x and y , i.e, $\exists x, y \in X$ s.t $x \neq y$.

$\implies d(x, y) = r > 0$

Now, consider the open ball $B(x, r/2)$. This open ball should be an open set in the topology that d induces.

Question (4)

But, $x \in B(x, r/2)$ and since $d(x, y) = r > r/2$, $y \notin B(x, r/2)$.

Thus, $B(x, r/2) \neq \emptyset$ and $B(x, r/2) \neq X$ (as there is at least one element $y \in X$ s.t. $y \notin B(x, r/2)$).

Thus, the topology induced by the metric d cannot be the indiscrete topology, since $\tau_{\text{indiscrete}} = \{\emptyset, X\}$

Thus, we have shown that if a set X has more than one element, then the indiscrete topology on X is not metrizable. \square

Ques 0

Test 1

Ques 0

Test 1