Lie theory

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Terminology

$\mathcal{A}$ denotes a hyperplane arrangement which is fixed in the discussion.

$\Sigma[\mathcal{A}]$ denotes the poset of faces. It is a monoid under the Tits product. We call this the Tits monoid.

$\mathrm{L}[\mathcal{A}]$ denote the set of chambers. It is a left module over $\Sigma[\mathcal{A}]$.

$\Pi[\mathcal{A}]$ denotes the poset of flats. It is a monoid under the join operation. We call this the Birkhoff monoid.

The support map $\mathrm{s} : \Sigma[\mathcal{A}] \to \Pi[\mathcal{A}]$ is a morphism of monoids.

This picture can be linearized. Let $\mathsf{\Sigma}[\mathcal{A}]$ denote the Tits algebra, $\mathsf{L}[\mathcal{A}]$ denote the left module of chambers, and $\mathsf{\Pi}[\mathcal{A}]$ denote the Birkhoff algebra. In each case, we use the letter $\mathtt{H}$ for the canonical basis.

1 Lie elements

Recall the module of chambers $\mathsf{L}[\mathcal{A}]$. We write a typical element as

$$z = \sum_C x^C \mathtt{H}_C.$$

An element $z \in \mathsf{L}[\mathcal{A}]$ is a Lie element if

$$\sum_{C : HC = D} x^C = 0 \text{ for all } O < H \leq D. \tag{1}$$

This is a linear system in the variables x^C.

We denote the set of Lie elements by $\mathsf{Lie}[\mathcal{A}]$. It is a subspace of $\mathsf{L}[\mathcal{A}]$.

- Note that $H = O$ is excluded from (1): If not, then $z = 0$ would be the only solution since all its coefficients x^C would be forced to be zero.
- If $\mathcal{A}$ has rank zero, then $\mathsf{Lie}[\mathcal{A}] = \mathsf{L}[\mathcal{A}] = \Bbbk$, spanned by $\mathtt{H}_O$. This is because, in this case,

there is only one chamber namely the central face, so (1) is vacuously true.

Lemma. *Suppose $\mathcal{A}$ has rank at least one. Then the sum of the coefficients of any Lie element is zero, that is, $z \in$ Lie$[\mathcal{A}]$ implies*

$$\sum_C x^C = 0. \tag{2}$$

Proof. Let D be any chamber. Since $\mathcal{A}$ has rank at least one, $D > O$. So we may choose $H = D$ in (1). This yields $\sum_C x^C = 0$, as required. $\square$

1.1 Friedrichs primitive part criterion

For any left Σ-module h, let $\mathcal{P}(\mathsf{h})$ denote the subspace consisting of the elements z such that $\mathrm{H}_H \triangleright z = 0$ for all $H > O$. We refer to $\mathcal{P}(\mathsf{h})$ as the primitive part of h.

Lemma. *The space of Lie elements is the primitive* *part of the left module of chambers:*

$$\mathcal{P}(\mathsf{L}[\mathcal{A}]) = \mathsf{Lie}[\mathcal{A}].$$

Explicitly, $z \in \mathsf{Lie}[\mathcal{A}]$ *iff*

$$\mathtt{H}_H \triangleright z = 0$$

for all $H > O$*.*

Proof. Let H be any face of $\mathcal{A}$. Then

$$\begin{aligned} \mathtt{H}_H \triangleright \big(\sum_C x^C \mathtt{H}_C\big) &= \sum_C x^C \mathtt{H}_{HC} \\ &= \sum_{D\colon H \leq D} \Bigg(\sum_{C\colon HC=D} x^C\Bigg) \mathtt{H}_D. \end{aligned}$$

This equals 0 iff

$$\sum_{C\colon HC=D} x^C = 0 \text{ for all } D \geq H.$$

The result follows from (1). $\square$

We refer to this characterization of Lie elements as the Friedrichs criterion.

1.2 Ree top-lune criterion

Any top-directed face (H, D) gives rise to a top-lune

$$\mathrm{s}(H, D) := \{C \mid HC = D\}.$$

Note that D always belongs to this top-lune. Further, this top-lune is a singleton (consisting of D) iff $H = O$. The definition of a Lie element may now be rewritten as follows.

Lemma. *We have $z \in$ Lie$[\mathcal{A}]$ iff*

$$\sum_{C \in \mathrm{V}} x^C = 0 \tag{3}$$

for all non-singleton combinatorial top-lunes V in $\mathcal{A}$.

When V is the maximum flat, the above equation specializes to (2).

In fact, it suffices to consider only vertex-based top-lunes, since any non-singleton top-lune can be written as a disjoint union of vertex-based top-lunes.

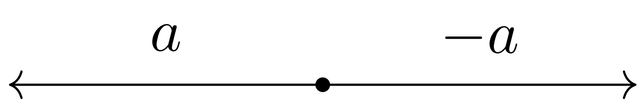
Lemma. *We have $z \in$ Lie$[\mathcal{A}]$ iff* (3) *holds for all* *vertex-based combinatorial top-lunes* V *in* $\mathcal{A}$*, or equivalently,* (1) *holds for all vertices* H*.*

We refer to this description of Lie elements as the Ree criterion. A Lie element may be visualized as a scalar assigned to each chamber such that the sum of the scalars in every vertex-based top-lune is zero. (The scalar assigned to C is x^C.)

1.3 Antisymmetry and Jacobi identity

Let us try to understand Lie elements of arrangements of small rank.

Consider the rank-one arrangement shown below.



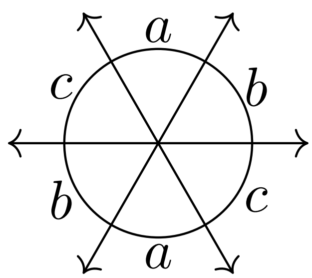
There is only one non-singleton top-lune consisting of the two chambers. It follows that $\mathsf{Lie}[\mathcal{A}]$ is one-dimensional. The coefficients of the two chambers are a and $-a$. The simplest choices are $a = 1$ and

$a = -1$. Either of them spans Lie$[\mathcal{A}]$, and their sum is zero. This can be shown as follows.

$$(4) \qquad \Big(\overset{1}{\bullet} \qquad\qquad \overset{\textcolor{magenta}{1}}{\bullet} \Big) + \Big(\overset{\textcolor{magenta}{1}}{\bullet} \qquad\qquad \overset{1}{\bullet} \Big) = 0.$$

This is the antisymmetry relation. (By convention, $\textcolor{magenta}{1}$ denotes -1.)

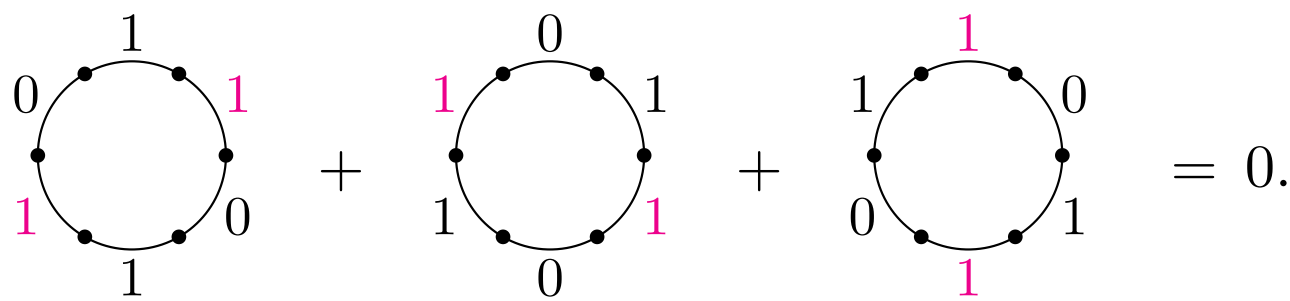
Now consider the dihedral arrangement of 3 lines.



There are six chambers. A non-singleton top-lune is either one of the six half-spaces or the full ambient space. It follows that Lie$[\mathcal{A}]$ is two-dimensional. The coefficients of the chambers (read in clockwise cyclic order) are a, b, c, a, b and c subject to the condition $a + b + c = 0$. For example, one may take $a = 1$, $b = -1$, and $c = 0$. Other similar choices are $a = 0$,

$b = 1$, and $c = -1$, or $a = -1$, $b = 0$, and $c = 1$. Any two of these yield a basis for Lie$[\mathcal{A}]$, and the sum of all three is 0. This can be shown as follows.

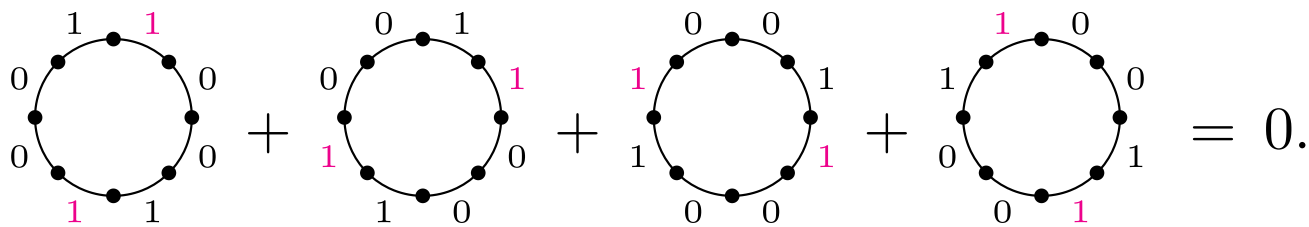
(5)



This is the Jacobi identity for the hexagon. (By convention, 1 denotes -1.)

The above analysis readily generalizes to the dihedral arrangement of n lines. The hexagon gets replaced by a $2n$-gon, and Lie$[\mathcal{A}]$ is $(n-1)$-dimensional. The coefficients of the chambers (read in clockwise cyclic order) are $a_1, \dots, a_n, a_1, \dots, a_n$ subject to the condition $a_1 + \cdots + a_n = 0$. Jacobi identity consists of n terms adding up to 0. Each term is a $2n$-gon whose two adjacent sides (and their opposites) have coefficients 1 and 1, and the remaining sides have coefficient 0. For instance:

(6)



This is the Jacobi identity for the octagon.

1.4 Lie elements and opposition map

Lemma. *If $z \in \mathsf{Lie}[\mathcal{A}]$, then x^D and $x^{\overline{D}}$ differ at most by a sign:*

(7)
$$x^D = (-1)^{\mathrm{rk}(\mathcal{A})} x^{\overline{D}}.$$

How will you prove this?

2 Zie elements

Consider the Tits algebra $\Sigma[\mathcal{A}]$. We write a typical element as

$$z = \sum_F x^F \mathtt{H}_F.$$

An element $z \in \Sigma[\mathcal{A}]$ is a Zie element if

$$\sum_{F\colon HF=G} x^F = 0 \text{ for all } O < H \leq G. \tag{8}$$

This is a linear system in the variables x^F.

We denote the set of Zie elements by $\mathsf{Zie}[\mathcal{A}]$. It is a subspace of $\Sigma[\mathcal{A}]$. Similar to Lie elements:

- Note that $H = O$ is excluded from the defining equations.
- If $\mathcal{A}$ has rank zero, then $\mathsf{Zie}[\mathcal{A}] = \Sigma[\mathcal{A}] = \Bbbk$. This is because, in this case, there is only one face, namely, the central face, so (8) is vacuously true. Hence $\mathsf{Zie}[\mathcal{A}] = \Sigma[\mathcal{A}]$, spanned by $\mathtt{H}_O$.

A Zie element z is special if the coefficient in z of the central face is 1, that is, if $x^O = 1$.

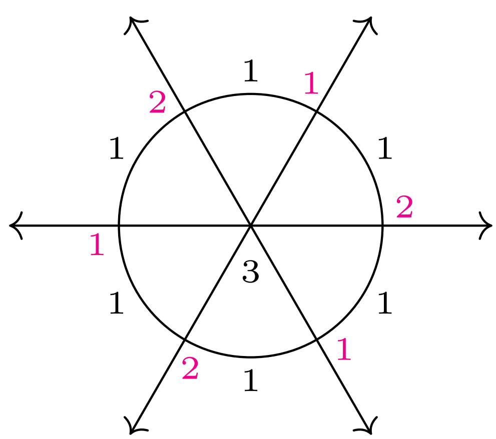
2.1 Zie elements in small ranks

Let $\mathcal{A}$ be the rank-one arrangement consisting of the central face, and chambers C and $\overline{C}$. Then, the ambient space is the only non-singleton lune in $\mathcal{A}$. Hence,

$$x^O \mathtt{H}_O + x^C \mathtt{H}_C + x^{\overline{C}} \mathtt{H}_{\overline{C}} \in \mathsf{Zie}[\mathcal{A}] \iff x^O + x^C + x^{\overline{C}} = 0.$$

Thus, $\mathsf{Zie}[\mathcal{A}]$ is two-dimensional.

Let $\mathcal{A}$ be the dihedral arrangement of 3 lines. A Zie element is shown in the diagram below.



The letters in magenta stand for negative numbers.

What is $\dim(\mathrm{Zie}[\mathcal{A}])$?

2.2 Flat equations and Möbius functions

Let us now concentrate on the equations indexed by $O < H \leq G$ with $H = G$. Put $\mathrm{X} := \mathrm{s}(H) = \mathrm{s}(G)$. Then X is a non-minimum flat, and

$$HF = G \iff \mathrm{s}(F) \leq \mathrm{X}.$$

This yields:

Lemma. *Suppose z is a Zie element. Then*

(9) $$\sum_{F:\,\mathrm{s}(F)\leq \mathrm{X}} x^F = 0 \text{ for all non-minimum flats } \mathrm{X}.$$

In particular, if $\mathcal{A}$ has rank at least one, then

(10) $$\sum_F x^F = 0.$$

Lemma. *Let $z \in \Sigma[\mathcal{A}]$. Then z satisfies (9) and $x^O = 1$ iff*

(11) $$\sum_{F \colon \mathrm{s}(F) = \mathrm{X}} x^F = \mu(\bot, \mathrm{X}) \text{ for all flats } \mathrm{X},$$

iff

$$\mathrm{s}(z) = \mathtt{Q}_\bot,$$

the $\mathtt{Q}$-basis element of the Birkhoff algebra.

In particular, if z is a special Zie element, then (11) holds.

Proof. For the first equivalence: Denote the lhs of (11) by $f(\mathrm{X})$. The condition $x^O = 1$ is the same as $f(\bot) = 1$, and the equations (9) are equivalent to saying: for any $\mathrm{Y} > \bot$,

$$\sum_{\mathrm{X} \colon \mathrm{X} \leq \mathrm{Y}} f(\mathrm{X}) = 0.$$

These together are equivalent to saying $f(\mathrm{X}) = \mu(\bot, \mathrm{X})$ for all X.

For the second equivalence, we only need to recall that

$$\mathtt{Q}_{\perp} = \sum_{\mathrm{X}} \mu(\perp, \mathrm{X})\, \mathtt{H}_{\mathrm{X}}. \qquad \square$$

2.3 Friedrichs primitive part criterion

The space of Zie elements is the primitive part of the Tits algebra (as a left module over itself). This is the Friedrichs criterion. It is elaborated below.

Lemma. *We have*

$$\mathcal{P}(\Sigma[\mathcal{A}]) = \mathsf{Zie}[\mathcal{A}].$$

Explicitly, $z \in \mathsf{Zie}[\mathcal{A}]$ *iff*

$$\mathtt{H}_H \triangleright z = 0$$

for all $H > O$*.*

Proof. Let H be any face of $\mathcal{A}$. Then

$$\begin{aligned}\mathtt{H}_H \triangleright \big(\sum_F x^F \mathtt{H}_F\big) &= \sum_F x^F \mathtt{H}_{HF} \\ &= \sum_{G:H\leq G} \Bigg(\sum_{F:\, HF=G} x^F \Bigg) \mathtt{H}_G.\end{aligned}$$

This equals 0 iff

$$\sum_{F:\, HF=G} x^F = 0 \text{ for all } G \geq H.$$

The result follows from (8). $\square$

Theorem. *The first Eulerian idempotent of any complete system of the Tits algebra is a special Zie element.*

Conversely, all special Zie elements arise in this manner.

Proof. Recall

$$\mathtt{Q}_O = \sum_G \boldsymbol{\mu}(O, G)\, \mathtt{H}_G.$$

By the noncommutative Weisner formula, $\mathtt{H}_F \triangleright \mathtt{Q}_O = 0$ for $F > O$. Hence, by Friedrichs criterion, $\mathtt{Q}_O$ is a special Zie element.

How will you prove the converse? $\square$

Lemma. *Any Zie element is a quasi-idempotent. More precisely, any Zie element z satisfies*

$$z^2 = x^O z.$$

A nonzero Zie element is an idempotent iff it is special.

Proof. Let z be a Zie element. By Friedrichs criterion,

$$z \triangleright z = \Big(\sum_F x^F \mathtt{H}_F\Big) \triangleright z = \sum_F x^F (\mathtt{H}_F \triangleright z) = x^O z.$$

This proves the first claim. Note that z is an idempotent iff $x^O z = z$. Assuming z to be nonzero, this happens precisely when $x^O = 1$, that is when z is special. $\square$

2.4 Ree lune criterion

Any directed face (H, G) gives rise to a lune

$$\mathrm{s}(H, G) = \{F \mid HF = G \text{ and } \mathrm{s}(F) = \mathrm{s}(G)\}.$$

Note that G always belongs to this lune. Further, this lune is a singleton (consisting of G) iff $H = O$. The closure, interior and boundary of $\mathrm{s}(H, G)$ are given by

$$\{F \mid HF \leq G\}, \quad \{F \mid HF = G\} \quad \text{and} \quad \{F \mid HF < G\}$$

respectively. This lune $\mathrm{s}(H, G)$ is a flat precisely when $H = G$, in which case its closure equals its interior.

For a lune V, let $\mathrm{Cl}(\mathrm{V})$, V^o and V^b denote its closure, interior and boundary.

Lemma. *We have* $z \in \mathsf{Zie}[\mathcal{A}]$ *iff*

(12)
$$\sum_{F \in \mathrm{V}^o} x^F = 0 \text{ for all non-singleton combinatorial lunes } \mathrm{V}.$$

When V runs over non-minimum flats, this statement specializes to (9).

The lemma also hold if V^o is replaced by $\mathrm{Cl}(\mathrm{V})$.

Why?

2.5 Image of the action of a Zie element

Let h be a left module over the Tits algebra. Let $\Psi(z)$ denotes the linear operator on h induced by the element $z \in \Sigma[\mathcal{A}]$. That is,

$$\Psi(z) : \mathsf{h} \to \mathsf{h}, \qquad \Psi(z)(h) := z \triangleright h.$$

Let $z(\mathsf{h})$ denote the image of this operator. In other words, $z(\mathsf{h})$ consists of all elements of the form $z \triangleright h$, as h varies over elements of h.

Example. Take h to be the module of chambers L on the rank-one arrangement. Let $z = \mathtt{H}_O + \mathtt{H}_C$. Then the linear operator $\Psi(z)$ is given by

$$\mathtt{H}_C \mapsto 2\mathtt{H}_C, \qquad \mathtt{H}_{\overline{C}} \mapsto \mathtt{H}_{\overline{C}} + \mathtt{H}_C.$$

Proposition. *Any Zie element sends* h *to* $\mathcal{P}(\mathsf{h})$*. Moreover, on* $\mathcal{P}(\mathsf{h})$ *it acts by scalar multiplication by its coefficient of the central face. In particular, any special Zie element projects* h *onto* $\mathcal{P}(\mathsf{h})$*.*

Proof. Let z be a Zie element and let $h \in \mathsf{h}$. Then

$$\mathtt{H}_H \triangleright (z \triangleright h) = (\mathtt{H}_H \triangleright z) \triangleright h = 0$$

for all $H > O$. Thus $z \triangleright h \in \mathcal{P}(\mathsf{h})$ as required. If h itself is primitive, then

$$z \triangleright h = \big(\sum_F x^F \mathtt{H}_F\big) \triangleright h = \sum_F x^F \mathtt{H}_F \triangleright h = x^O h.$$

$\square$

Example. Let us go back to the rank-one arrangement. A special Zie element is given by

$$\mathtt{H}_O - p\,\mathtt{H}_C - (1-p)\,\mathtt{H}_{\overline{C}},$$

where p is an arbitrary scalar. Let us compute the action of this element on L.

$$\begin{aligned}(\mathtt{H}_O - p\,\mathtt{H}_C - (1-p)\,\mathtt{H}_{\overline{C}}) \triangleright \mathtt{H}_C &= \mathtt{H}_C - p\,\mathtt{H}_C - (1-p)\,\mathtt{H}_{\overline{C}} \\ &= (1-p)\,\mathtt{H}_C - (1-p)\,\mathtt{H}_{\overline{C}},\end{aligned}$$

which is a Lie element. Further,

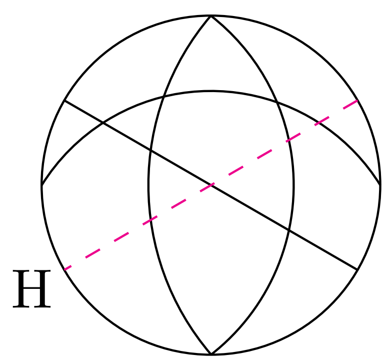
$$(\mathtt{H}_O - p\,\mathtt{H}_C - (1-p)\,\mathtt{H}_{\overline{C}}) \triangleright (\mathtt{H}_C - \mathtt{H}_{\overline{C}}) = \mathtt{H}_C - \mathtt{H}_{\overline{C}}.$$

So its action on a Lie element gives back the same Lie element.

3 Dynkin elements

3.1 Generic hyperplane

Let $\mathcal{A}$ be any arrangement of rank at least 1. A generic hyperplane wrt $\mathcal{A}$ is a codimension-one subspace of the ambient space which contains the central face O but does not contain any vertex of $\mathcal{A}$. For example:



Adding a generic hyperplane, say H, to $\mathcal{A}$ yields a new arrangement $\mathcal{A}'$. Let us compare the set of faces of $\mathcal{A}$ and $\mathcal{A}'$. A face of $\mathcal{A}$ which is not cut by H remains a face of $\mathcal{A}'$. In contrast, a face of $\mathcal{A}$ which is cut by H splits into three distinct faces of $\mathcal{A}'$: one face consists of those points which lie on H, while the remaining two consist of those points which lie on either side of H.

3.2 Dynkin element

Let H be a generic hyperplane wrt $\mathcal{A}$. There are two opposite half-spaces with base H. Fix one of them arbitrarily. Call it h. We say that h is generic wrt $\mathcal{A}$. Now define

$$(13) \qquad \theta_{\mathrm{h}} := \sum_{F\colon F \subseteq \mathrm{h}} (-1)^{\mathrm{rk}(F)} \, \mathtt{H}_F \quad \in \; \Sigma[\mathcal{A}].$$

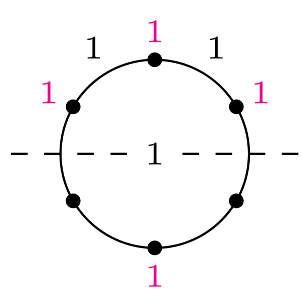
The sum is over all faces F of $\mathcal{A}$ which are contained in the fixed half-space h. These are precisely those faces of $\mathcal{A}$ which are not cut by H and which are on the h-side of H.

We refer to θ_{h} as the Dynkin element associated to the half-space h. The central face is contained in h and since its rank is zero, it appears in θ_{h} with coefficient 1.

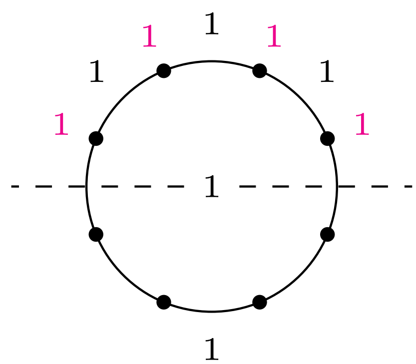
Example. Let $\mathcal{A}$ be the rank-one arrangement, with chambers C and $\overline{C}$. The origin is a generic hyperplane. In this case, $\mathcal{A} = \mathcal{A}'$. Thus, there are two generic half-spaces, and $\mathtt{H}_O - \mathtt{H}_C$ and $\mathtt{H}_O - \mathtt{H}_{\overline{C}}$ are the two Dynkin elements. Note that they are special Zie elements.

Example. Let $\mathcal{A}$ be the dihedral arrangement of n lines, with $n \geq 2$. The spherical model is the $2n$-gon. A line passing through the origin is generic wrt $\mathcal{A}$ if it cuts two opposite sides of the $2n$-gon. For definiteness, we demand that the lines bisect the two sides that they cut.

A Dynkin element for $n = 3$ is

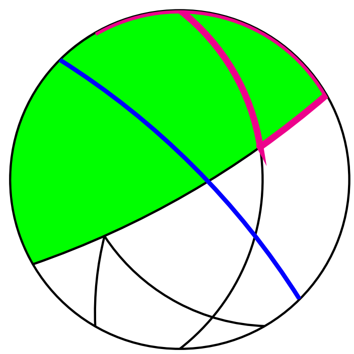


and for $n = 4$ is



Proposition. *For any generic half-space* h*, the Dynkin element* θ_{h} *is a special Zie element. In particular, it is an idempotent.*

This can be checked using the Ree criterion. We do not give a formal proof but illustrate it on an example.



The generic hyperplane is shown in blue, and the half-space h is the region to the right of it. The lune V is shown in green. It is bounded by two half-circles and is fully visible. The faces in $\mathrm{Cl}(\mathrm{V}) \cap \mathrm{h}$ are those on and inside the region defined by the magenta edges (consisting of a triangle and a rectangle). Topologically, this set is a ball with an edge hanging out.

For any arrangement $\mathcal{A}$, define

$$\mu(\mathcal{A}) := \mu(\perp, \top).$$

We refer to this as the Möbius number of $\mathcal{A}$. It is a particular value of the Möbius function of the lattice of flats $\Pi[\mathcal{A}]$.

Corollary. *The number of chambers contained in any generic half-space wrt $\mathcal{A}$ is given by $|\mu(\mathcal{A})|$.*

Proof. Apply (11) to the special Zie element θ_{h} for the flat $\mathrm{X} = \top$. $\square$

3.3 Action on chambers and Lie elements

Recall that the Tits algebra $\Sigma[\mathcal{A}]$ acts on the left on the module of chambers $\mathsf{L}[\mathcal{A}]$.

Proposition. *The Dynkin element θ_{h} is an idempotent operator which sends $\mathsf{L}[\mathcal{A}]$ onto $\mathsf{Lie}[\mathcal{A}]$.*

We now work towards a formula for the action of the Dynkin element on chambers. For a generic half-space h, and chambers C and D, put

$$\mathrm{A} = \{H \in \Sigma[\mathcal{A}] \mid HC = D\}$$

and

$$\mathrm{B} = \{H \in \Sigma[\mathcal{A}] \mid H \leq D, H \subseteq \mathrm{h}\}.$$

Both A and B consist of faces of D, with $D \in \mathrm{A}$ and $O \in \mathrm{B}$. Further,

$$\langle \theta_{\mathrm{h}} \triangleright \mathtt{H}_C, \mathtt{H}_D \rangle = \sum_{H \in \mathrm{A} \cap \mathrm{B}} (-1)^{\mathrm{rk}(H)}. \tag{14}$$

The lhs denotes the coefficient of $\mathtt{H}_D$ in $\theta_{\mathrm{h}} \triangleright \mathtt{H}_C$. We would like to understand the rhs.

For simplicity, we will assume the arrangement to be simplicial.

Let $\mathrm{Des}(C, D)$ denote the smallest face H of D such that $HC = D$. In other words,

$$HC = D \iff \mathrm{Des}(C, D) \leq H \leq D.$$

We say that $\mathrm{Des}(C, D)$ is the descent of D wrt C. Note that

$$\mathrm{Des}(C, D) = D \iff \overline{C} = D$$

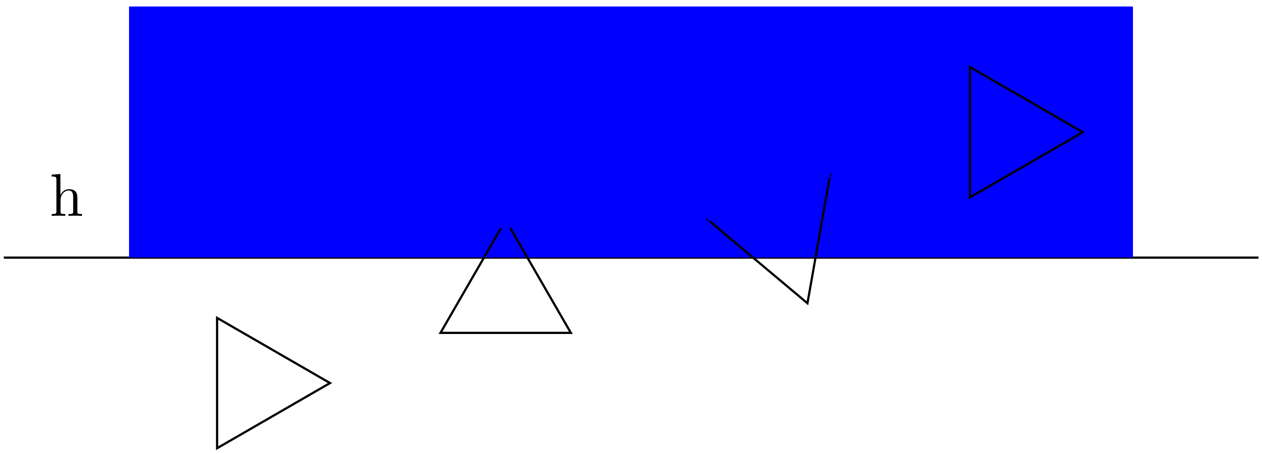
and

$$\mathrm{Des}(C, D) = O \iff C = D.$$

Thus,

$$\mathrm{A} = \{H \mid \mathrm{Des}(C, D) \leq H \leq D\}.$$

For a generic half-space h and a chamber D, let $\mathrm{h}(D)$ denote the largest face of D which is contained in h.



This is illustrated above in rank 3. The half-space h is shaded in light blue. Since the arrangement is simplicial, each chamber D is a triangle, and there are four possibilities for $\mathrm{h}(D)$ depending on how the vertices of D lie wrt h. Each case is shown separately with the face $\mathrm{h}(D)$ marked in dark blue.

Thus,

$$\mathrm{B} = \{H \mid H \leq \mathrm{h}(D)\}.$$

Lemma. *Let $\mathcal{A}$ be a simplicial arrangement. Then*

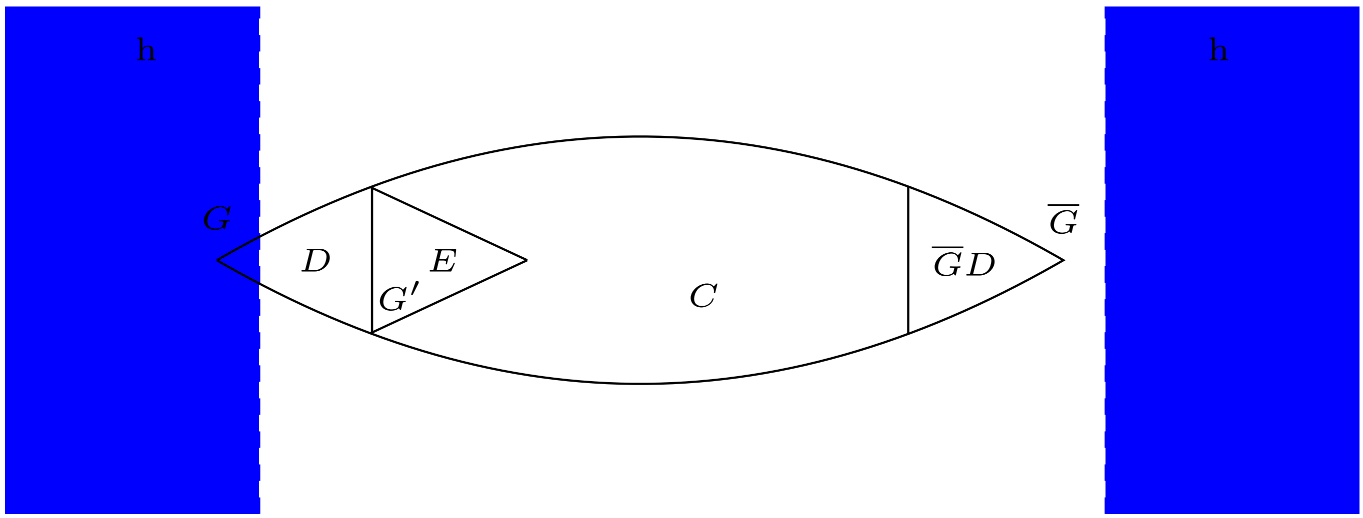
$$\theta_{\mathrm{h}} \triangleright \mathtt{H}_C = \sum_{D:\, \mathrm{Des}(C,D)=\mathrm{h}(D)} (-1)^{\mathrm{rk}(\mathrm{h}(D))}\, \mathtt{H}_D.$$

Proof. In the simplicial case, (14) simplifies to

$$\langle \theta_{\mathrm{h}} \triangleright \mathtt{H}_C, \mathtt{H}_D \rangle = \sum_{H:\, \mathrm{Des}(C,D)\leq H\leq \mathrm{h}(D)} (-1)^{\mathrm{rk}(H)}.$$

The indexing set (which could be empty) is a Boolean poset. So the sum will be zero unless the set is a singleton, that is, $\mathrm{Des}(C, D) = \mathrm{h}(D)$. $\square$

Lemma. *If* $\mathrm{Des}(C, D) = \mathrm{h}(D)$*, then* $\overline{C} \subseteq \mathrm{h}$*.*



Proof. For simplicity of notation, put $G := \mathrm{h}(D)$. Let G' be the face of D complementary to G. Then observe that

$$\mathrm{Des}(C, D) = \mathrm{h}(D) \iff C \text{ lies in the gallery interval } [E, \overline{G}D],$$

where E is the chamber opposite to D in the star of G'. But this entire gallery interval lies in the interior of $\overline{\mathrm{h}}$. (In the figure, the latter is the region between the two dotted lines.) So if $\mathrm{Des}(C, D) = \mathrm{h}(D)$, then $\overline{C}$ is contained in h. $\square$

Proposition. *Let $\mathcal{A}$ be a simplicial arrangement. Then*

(15)

$$\theta_{\mathrm{h}} \triangleright \mathtt{H}_C = \begin{cases} \mathtt{H}_C + (-1)^{\mathrm{rk}(\mathcal{A})} \mathtt{H}_{\overline{C}} + \sum_D \pm \mathtt{H}_D & \textit{if } \overline{C} \subseteq \mathrm{h}, \\ 0 & \textit{otherwise.} \end{cases}$$

The sum is over chambers D which are cut by the base of h (so that part of D lies in h and part in $\overline{\mathrm{h}}$) and which satisfy $\mathrm{Des}(C, D) = \mathrm{h}(D)$.

Proof. The second case follows from previous Lemmas. So suppose that $\overline{C} \subseteq \mathrm{h}$. Then

$$\mathrm{Des}(C, D) = \mathrm{h}(D) = O \iff D = C$$

and

$$\mathrm{Des}(C, D) = \mathrm{h}(D) = D \iff D = \overline{C}.$$

This yields the terms $\mathtt{H}_C$ and $(-1)^{\mathrm{rk}(\mathcal{A})} \mathtt{H}_{\overline{C}}$. In the remaining cases, $O < \mathrm{h}(D) < D$ and hence D is cut by the base of h. $\square$

3.4 Dynkin basis

Proposition. *Let $\mathcal{A}$ be any arrangement. For any generic half-space h wrt $\mathcal{A}$, the set*

$$\{\theta_{\mathrm{h}} \triangleright \mathtt{H}_C \mid \overline{C} \subseteq \mathrm{h}\} \tag{16}$$

is a basis of $\mathsf{Lie}[\mathcal{A}]$*.*

Proof. We prove this assuming that $\mathcal{A}$ is simplicial. For $\overline{C} \subseteq \mathrm{h}$, by the first case of formula (15), the term $\mathtt{H}_C$ only occurs in $\theta_{\mathrm{h}} \triangleright \mathtt{H}_C$, so these elements are linearly independent. Further, by Proposition 3.3, these elements span $\mathsf{Lie}[\mathcal{A}]$. Hence they form a basis. □

Corollary. *The dimension of* $\mathsf{Lie}[\mathcal{A}]$ *is* $|\mu(\mathcal{A})|$*.*

Proof. This follows from the previous result and Corollary 3.2. □

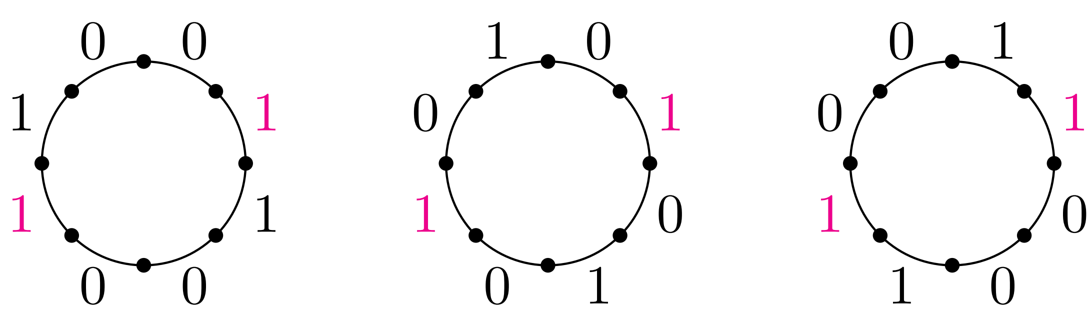
We call (16) the Dynkin basis associated to h.

Example. Consider the dihedral arrangement of n lines. In this case, we know independently that dimension of $\mathsf{Lie}[\mathcal{A}]$ and $|\mu(\mathcal{A})|$ are both $n-1$.

Now fix any two opposite chambers, say D and $\overline{D}$. Let h be one of the two half-spaces whose base bisects D and $\overline{D}$. Then the set

$$\{\mathtt{H}_C + \mathtt{H}_{\overline{C}} - \mathtt{H}_D - \mathtt{H}_{\overline{D}} \mid \overline{C} \subseteq \mathrm{h}\}$$

is a basis for $\mathsf{Lie}[\mathcal{A}]$. It has $n-1$ elements. This is precisely the Dynkin basis associated to h and also to $\overline{\mathrm{h}}$. For instance, for $n=4$, a choice for the Dynkin basis is shown below.



4 A little homological algebra

4.1 Chain complexes and homology

A chain complex is a sequence of vector spaces equipped with linear maps

$$\cdots \to \mathcal{C}_{k+1} \xrightarrow{\partial_{k+1}} \mathcal{C}_k \xrightarrow{\partial_k} \mathcal{C}_{k-1} \xrightarrow{\partial_{k-1}} \dots$$

such that the composite of two consecutive maps is 0, that is, $\partial_{k-1}\partial_k = 0$ for all k.

The $\mathcal{C}_k$ are called chain groups and ∂_k are called boundary maps.

Given a chain complex $\mathcal{C}$, for each k, define

$$\mathcal{H}_k(\mathcal{C}) := \frac{\ker \partial_k}{\operatorname{image} \partial_{k+1}}.$$

This is called the k-th homology group of $\mathcal{C}$.

Given a chain complex $\mathcal{C}$, one can dualize it to obtain

$$\cdots \leftarrow \mathcal{C}_{k+1}^* \xleftarrow{\partial_{k+1}^*} \mathcal{C}_k^* \xleftarrow{\partial_k^*} \mathcal{C}_{k-1}^* \xleftarrow{\partial_{k-1}^*} \cdots.$$

Let us write $\mathcal{C}^k$ instead of $\mathcal{C}_k^*$ and δ_k instead of ∂_k^*. That is,

$$\cdots \leftarrow \mathcal{C}^{k+1} \xleftarrow{\delta_{k+1}} \mathcal{C}^k \xleftarrow{\delta_k} \mathcal{C}^{k-1} \xleftarrow{\delta_{k-1}} \cdots.$$

This is called a cochain complex. The $\mathcal{C}^k$ are called cochain groups and δ_k are called coboundary maps.

The identity $\delta_k \delta_{k-1} = 0$ holds. We define

$$\mathcal{H}^k(\mathcal{C}) := \frac{\ker \delta_k}{\operatorname{image} \delta_{k-1}}.$$

This is called the k-th cohomology group of $\mathcal{C}$. It is isomorphic to the dual of $\mathcal{H}_k(\mathcal{C})$, with the duality induced by the duality between the chain and cochain complexes.

4.2 (Co)homology of the lattice of flats

Fix an arrangement $\mathcal{A}$. Put $r := \mathrm{rk}(\mathcal{A})$. We now associate a chain complex to the lattice of flats $\Pi[\mathcal{A}]$.

For $-1 \leq k \leq r-2$, the chain group $\mathcal{C}_k(\Pi[\mathcal{A}])$ is the vector space over $\mathbb{k}$ with basis consisting of chains

$$\perp < \mathrm{X}_1 < \cdots < \mathrm{X}_{k+1} < \top.$$

The remaining chain groups are 0. Note that

- $\mathcal{C}_{r-2}(\Pi[\mathcal{A}])$ has a basis of maximal chains, while
- $\mathcal{C}_{-1}(\Pi[\mathcal{A}])$ is one-dimensional and spanned by the chain $\perp < \top$.

The boundary operator

$$\partial_k : \mathcal{C}_k(\Pi[\mathcal{A}]) \rightarrow \mathcal{C}_{k-1}(\Pi[\mathcal{A}])$$

is given by

$$\partial_k(\perp < \mathrm{X}_1 < \cdots < \mathrm{X}_{k+1} < \top)$$

$$= \sum_{i=1}^{k+1}(-1)^i(\perp < \mathrm{X}_1 < \cdots < \hat{\mathrm{X}}_i < \cdots < \mathrm{X}_{k+1} < \top),$$

where by standard convention, $\hat{\mathrm{X}}_i$ means that X_i has been deleted from the chain.

One can readily check that $\partial_{k-1}\partial_k = 0$.

We write $\mathcal{H}_k(\Pi[\mathcal{A}])$ for the homology group in position k.

The cochain complex is obtained by dualizing the chain complex. We denote the cochain groups by $\mathcal{C}^k(\Pi[\mathcal{A}])$ and the coboundary operators by

$$\delta_k : \mathcal{C}^k(\Pi[\mathcal{A}]) \to \mathcal{C}^{k+1}(\Pi[\mathcal{A}]).$$

Explicitly,

$$(17) \quad \delta_k(\perp < \mathrm{X}_1 < \cdots < \mathrm{X}_{k+1} < \top)^* =$$

$$\sum_{i=1}^{k+2} (-1)^i \sum_{\mathrm{X}_{i-1} < \mathrm{X} < \mathrm{X}_i} (\perp < \mathrm{X}_1 < \ldots < \mathrm{X}_{i-1} < \mathrm{X} < \mathrm{X}_i < \ldots < \mathrm{X}_{k+1} < \top)^*,$$

with the convention that $\mathrm{X}_0 = \perp$ and $\mathrm{X}_{k+2} = \top$.

The superscript $*$ stands for the dual basis.

We write $\mathcal{H}^k(\Pi[\mathcal{A}])$ for the cohomology group in position k.

Proposition. *For any arrangement $\mathcal{A}$, the lattice of flats $\Pi[\mathcal{A}]$ has (co)homology only in position $r-2$, and further*

$$\dim \mathcal{H}_{r-2}(\Pi[\mathcal{A}]) = \dim \mathcal{H}^{r-2}(\Pi[\mathcal{A}]) = |\mu(\mathcal{A})|.$$

This is a result of Folkman. We omit the proof. We only mention that the second claim can be deduced from the first using the fact that $\mu(\mathcal{A})$ is the Euler characteristic of the (co)chain complex of $\Pi[\mathcal{A}]$.

The top homology $\mathcal{H}_{r-2}(\Pi[\mathcal{A}])$ is a well-studied object. Björner and Wachs have constructed a basis for it starting with a generic half-space h.

5 Orientation space

We discuss the notion of orientation for any arrangement $\mathcal{A}$. Let $\mathsf{E}^{\mathbf{o}}[\mathcal{A}]$ denote the space spanned by maximal chains in the poset of faces $\Sigma[\mathcal{A}]$ subject to the relations: If two maximal chains differ in exactly one position, then they are negatives of each other.

We call $\mathsf{E}^{\mathbf{o}}[\mathcal{A}]$ the orientation space of $\mathcal{A}$. We denote the image of a maximal chain f in the orientation space by $[f]$.

An orientation of $\mathcal{A}$ is an element of $\mathsf{E}^{\mathbf{o}}[\mathcal{A}]$ of the form $[f]$ for some maximal chain of faces f.

Example. Let $\mathcal{A}$ be the rank-one arrangement, with chambers C and $\overline{C}$. There are two maximal chains, namely $O \lessdot C$ and $O \lessdot \overline{C}$. Since they differ in exactly one position, we write

$$[O \lessdot C] = -[O \lessdot \overline{C}].$$

So $\mathsf{E}^{\mathbf{o}}[\mathcal{A}]$ is one-dimensional. It has two orientations, namely, $[O \lessdot C]$ which we call the right orientation, and $[O \lessdot \overline{C}]$ which we call the left orientation.

Example. Let $\mathcal{A}$ be the dihedral arrangement of n lines. A maximal chain has the form $O \lessdot F \lessdot C$. There are $4n$ maximal chains. The relations can be expressed as

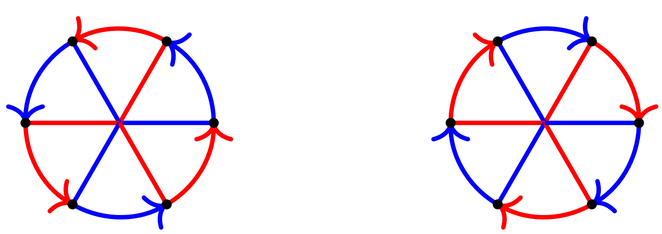
$$[O \lessdot F \lessdot C] = -[O \lessdot G \lessdot C],$$

where F and G are the two vertices of C, and

$$[O \lessdot F \lessdot C] = -[O \lessdot F \lessdot D],$$

where C and D are the two chambers greater than F.

Again we note that $\mathsf{E}^{\mathbf{o}}[\mathcal{A}]$ is one-dimensional. There are two orientations, which we can think of as clockwise and anticlockwise. This is illustrated below for $n = 3$.



The six maximal chains which give the anticlockwise orientation are shown on the left, while the six which give the clockwise orientation are shown on the right.

Lemma. *For any arrangement, the orientation space is one-dimensional. Any arrangement has two orientations.*

The relevant fact is that

- one can pass from one maximal chain to another by a sequence of maximal chains in which two consecutive maximal chains differ in exactly one position, and
- any such journey from a maximal chain back to itself takes an even number of steps.

We will use the letter σ to denote an orientation; the opposite orientation will be $-\sigma$.

There is a canonical isomorphism

$$\mathsf{E}^{\mathbf{o}}[\mathcal{A}] \otimes \mathsf{E}^{\mathbf{o}}[\mathcal{A}] \xrightarrow{\cong} \mathbb{k}, \qquad \sigma \otimes \sigma \mapsto 1, \tag{18}$$

where σ is either of the two orientations of $\mathcal{A}$. Changing σ to $-\sigma$ incurs two minus signs, so the map is well-defined.

For any flat X, there is an isomorphism

$$\mathsf{E}^{\mathbf{o}}[\mathcal{A}^{\mathrm{X}}] \otimes \mathsf{E}^{\mathbf{o}}[\mathcal{A}_{\mathrm{X}}] \xrightarrow{\cong} \mathsf{E}^{\mathbf{o}}[\mathcal{A}], \quad \sigma_1 \otimes \sigma_2 \mapsto \tau, \tag{19}$$

where τ is obtained by "concatenating" σ_1 and σ_2: Suppose c_1 is a maximal chain of faces in $\mathcal{A}^{\mathrm{X}}$ which represents σ_1. Let c_1' denote the corresponding chain in $\mathcal{A}$. It ends at a face with support X. Call that face F. Similarly, let c_2 be a maximal chain of faces in $\mathcal{A}_{\mathrm{X}}$ which represents σ_2. It corresponds to a chain c_2' in $\mathcal{A}$ which starts at F. The concatenation of c_1' and c_2' is a maximal chain in $\mathcal{A}$. Its class is the required τ.

Iterating this procedure yields, for any chain of flats

$(\perp < \mathrm{X}_1 < \cdots < \mathrm{X}_k < \top)$, an isomorphism

(20)

$$\mathsf{E}^{\mathbf{O}}[\mathcal{A}^{\mathrm{X}_1}] \otimes \mathsf{E}^{\mathbf{O}}[\mathcal{A}^{\mathrm{X}_2}_{\mathrm{X}_1}] \otimes \cdots \otimes \mathsf{E}^{\mathbf{O}}[\mathcal{A}_{\mathrm{X}_k}] \xrightarrow{\cong} \mathsf{E}^{\mathbf{O}}[\mathcal{A}].$$

6 Joyal-Klyachko-Stanley

Fix an arrangement $\mathcal{A}$. Put $r := \mathrm{rk}(\mathcal{A})$. We set up some terminology.

For any chain of faces $f = (F_1 < \cdots < F_k)$, define its support by

$$\mathrm{s}(f) := (\mathrm{s}(F_1) < \cdots < \mathrm{s}(F_k)).$$

This is a chain of flats.

For a maximal chain of faces f, let $\mathrm{last}(f)$ denote the last face in the chain f (which is necessarily a chamber).

For any orientation σ and a maximal chain of faces f,

$$(\sigma : f) := \begin{cases} 1 & \text{if } \sigma = [f], \\ -1 & \text{if } \sigma = -[f]. \end{cases}$$

Recall the cochain group $\mathcal{C}^{r-2}(\Pi[\mathcal{A}])$ with a basis consisting of maximal chains of flats. We now define a linear map

$$\mathcal{C}^{r-2}(\Pi[\mathcal{A}]) \otimes \mathsf{E}^{\mathbf{o}}[\mathcal{A}] \to \mathsf{L}[\mathcal{A}]. \tag{21}$$

We provide a number of equivalent definitions.

The map (21) is given by

$$z^* \otimes \sigma \mapsto \sum_{f:\, \mathrm{s}(f)=z} (\sigma : f)\, \mathtt{H}_{\mathrm{last}(f)}, \tag{22a}$$

where z is any maximal chain of flats and σ is an orientation.

The map (21) is given by

$$z^* \otimes \sigma \mapsto \sum_{D} \pm \mathtt{H}_D, \tag{22b}$$

where z is any maximal chain of flats and σ is an orientation. The sum is over those chambers D for which there exists a maximal chain of faces f with $\mathrm{last}(f) = D$ and $\mathrm{s}(f) = z$. The coefficient of $\mathtt{H}_D$ is

$+1$ if $[f] = \sigma$ and -1 if $[f] = -\sigma$.

The map (21) is given by

(22c) $\quad \mathrm{s}(f)^* \otimes [f] \mapsto (\mathtt{H}_{F_1} - \mathtt{H}_{G_1}) \triangleright \ldots \triangleright (\mathtt{H}_{F_r} - \mathtt{H}_{G_r}),$

where $f = (O \lessdot F_1 \lessdot \cdots \lessdot F_r)$ is a maximal chain of faces, and for $1 \leq i \leq r$, G_i is the face opposite to F_i in the star of F_{i-1} (with the convention $F_0 = O$).

Lemma. *The image of* (21) *belongs to* $\mathsf{Lie}[\mathcal{A}]$*.*

Proof. This can be proved from (22c) by using the Friedrichs criterion. For instance, in rank-two, we need to show for $F > O$,

$$\mathtt{H}_F \triangleright (\mathtt{H}_{F_1} - \mathtt{H}_{G_1}) \triangleright (\mathtt{H}_{F_2} - \mathtt{H}_{G_2}) = 0.$$

In the case when F is either F_1 or G_1, $\mathtt{H}_F \triangleright (\mathtt{H}_{F_1} - \mathtt{H}_{G_1})$ equals 0. If not, then this is some linear combination of chambers, so multiplying it with $(\mathtt{H}_{F_2} - \mathtt{H}_{G_2})$ produces 0, and we are done. This method generalizes to any r. $\square$

As a consequence, (21) induces a map

$$\mathcal{C}^{r-2}(\Pi[\mathcal{A}]) \otimes \mathsf{E}^{\mathbf{o}}[\mathcal{A}] \to \mathsf{Lie}[\mathcal{A}]. \tag{23}$$

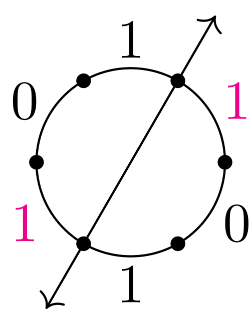
We refer to the image of $z^* \otimes \sigma$ under (22b) as the unbracketing of z wrt σ. Thus, the unbracketing of a maximal chain of flats (or any linear combination of them) determines, up to sign, a Lie element. The sign can be fixed by choosing an orientation.

Example. For the rank-one arrangement $\mathcal{A}$ with chambers C and $\overline{C}$, we have

$$\mathcal{C}^{-1}(\Pi[\mathcal{A}]) \otimes \mathsf{E}^{\mathbf{o}}[\mathcal{A}] \xrightarrow{\cong} \mathsf{Lie}[\mathcal{A}],$$
$$(\perp < \top)^* \otimes [O \lessdot C] \mapsto \mathtt{H}_C - \mathtt{H}_{\overline{C}}.$$

(Both sides are 1-dimensional.)

Example. Let $\mathcal{A}$ be the dihedral arrangement of n lines. Any maximal chain of flats has the form $\perp \lessdot \mathrm{H} \lessdot \top$. So maximal chains correspond to hyperplanes. Unbracketing $\perp \lessdot \mathrm{H} \lessdot \top$ yields a Lie element of the form $\mathtt{H}_C + \mathtt{H}_{\overline{C}} - \mathtt{H}_D - \mathtt{H}_{\overline{D}}$, where C and D are adjacent, and their common panel has support H. An example is shown below.



The unbracketing is done wrt the anticlockwise direction.

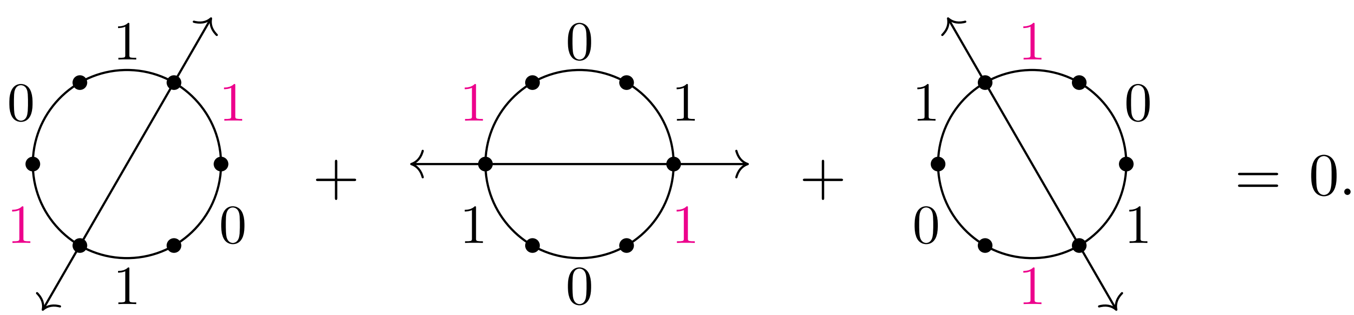
Let us go back to the general case.

A coboundary relation is an element of $\mathcal{C}^{r-2}(\Pi[\mathcal{A}])$ of the form $\delta_{r-3}(z^*)$ for some chain of flats z. (The coboundary map is given in (17).) Note that the top cohomology group $\mathcal{H}^{r-2}(\Pi[\mathcal{A}])$ is the quotient of $\mathcal{C}^{r-2}(\Pi[\mathcal{A}])$ by the subspace spanned by the coboundary relations.

Example. Let us return to the rank-two example. There is only one coboundary relation, namely

$$\sum_{\mathrm{H}} (\perp \lessdot \mathrm{H} \lessdot \top)^*.$$

The sum is over all hyperplanes. Note that this element after unbracketing produces the Jaocbi identity. This is illustrated below for $n = 3$.



In the figure, the orientation chosen for unbracketing is the anticlockwise direction. Thus, in this case, we have an induced isomorphism

$$\mathcal{H}^0(\Pi[\mathcal{A}]) \otimes \mathsf{E}^{\mathbf{o}}[\mathcal{A}] \to \mathsf{Lie}[\mathcal{A}].$$

Let us go back to the general case.

Lemma. *The map* (23) *sends any coboundary relation (tensored with an orientation) to zero.*

Proof. Let
$z = (\bot \lessdot \mathrm{X}_1 \lessdot \cdots \lessdot \mathrm{X}_{i-1} < \mathrm{X}_i \lessdot \cdots \lessdot \mathrm{X}_{k+1} \lessdot \top)$
be an element of $\mathcal{C}_{r-3}(\Pi)$, with
$\mathrm{rk}(\mathrm{X}_i) - \mathrm{rk}(\mathrm{X}_{i-1}) = 2$. Then

$$\delta_{r-3}(z^*) = (-1)^i \sum_w w^*,$$

where w runs over all maximal chains obtained from z by inserting a flat between X_{i-1} and X_i. Put
$\alpha = \delta_{r-3}(z^*) \otimes \sigma$ for some orientation σ. We need to show that (21) sends α to 0. Let us use (22b). If there is no chain of faces ending in D with support z, then the coefficient is clearly zero. So we may assume that
$f = (O \lessdot F_1 \lessdot \cdots \lessdot F_{i-1} < F_i \lessdot \cdots \lessdot F_{k+1} \lessdot D)$
is a chain of faces with support z. In this case, there

are exactly two faces that can be inserted between F_{i-1} and F_i. The resulting maximal chains have opposite orientations, so their contributions cancel. $\square$

Thus (23) induces a map

$$\mathcal{H}^{r-2}(\Pi[\mathcal{A}]) \otimes \mathsf{E}^{\mathbf{o}}[\mathcal{A}] \to \mathsf{Lie}[\mathcal{A}]. \tag{24}$$

By tensoring both sides by $\mathsf{E}^{\mathbf{o}}[\mathcal{A}]$ and using (18), it can be expressed in the equivalent form:

$$\mathcal{H}^{r-2}(\Pi[\mathcal{A}]) \to \mathsf{Lie}[\mathcal{A}] \otimes \mathsf{E}^{\mathbf{o}}[\mathcal{A}]. \tag{25}$$

Theorem. *The maps* (24) *and* (25) *are isomorphisms.*

We call this the Joyal-Klyachko-Stanley theorem, or JKS for short. We refer to (24) as the JKS isomorphism. Under this isomorphism the dual of the Björner-Wachs basis corresponds to the Dynkin basis. In fact, these two bases can be used to prove JKS.

7 Presentation for Lie

Let us slightly alter our viewpoint on the JKS isomorphism (24).

For the rank-one arrangement $\mathcal{A}$ with chambers C and $\overline{C}$, we have

$$\mathsf{E}^{\mathbf{o}}[\mathcal{A}] \xrightarrow{\cong} \mathsf{Lie}[\mathcal{A}], \qquad [O \lessdot C] \mapsto \mathtt{H}_C - \mathtt{H}_{\overline{C}}.$$

(Both spaces are 1-dimensional.)

Now suppose $\mathcal{A}$ is the dihedral arrangement of n lines. Then (23) along with the identification (19) can be rewritten as

$$\bigoplus_{i=1}^{n} \mathsf{E}^{\mathbf{o}}[\mathcal{A}^{\mathrm{X}_i}] \otimes \mathsf{E}^{\mathbf{o}}[\mathcal{A}_{\mathrm{X}_i}] \to \mathsf{Lie}[\mathcal{A}],$$

where the X_i are the n lines (one-dimensional flats) of $\mathcal{A}$. This map is surjective. The lhs is n-dimensional while the rhs is $(n-1)$-dimensional. The kernel is spanned by the element

$$\sum_{i=1}^{n} \tau^i \otimes \tau_i \tag{26}$$

where τ^i and τ_i are orientations of $\mathcal{A}^{\mathrm{X}_i}$ and $\mathcal{A}_{\mathrm{X}_i}$ such that their concatenation is (say) the anticlockwise orientation of $\mathcal{A}$. This element corresponds to the Jacobi identity.

Now let $\mathcal{A}$ be arbitrary. The map (23) can be rewritten as

$$\bigoplus_{z} \mathsf{E}^{\mathbf{o}}[\mathcal{A}^{\mathrm{X}_1}] \otimes \mathsf{E}^{\mathbf{o}}[\mathcal{A}^{\mathrm{X}_2}_{\mathrm{X}_1}] \otimes \cdots \otimes \mathsf{E}^{\mathbf{o}}[\mathcal{A}_{\mathrm{X}_{r-1}}] \to \mathsf{Lie}[\mathcal{A}], \tag{27}$$

where the sum is over all maximal chains of flats $z = (\bot \lessdot \mathrm{X}_1 \lessdot \cdots \lessdot \mathrm{X}_{r-1} \lessdot \top)$. (In this rewriting, the second tensor factor $\mathsf{E}^{\mathbf{o}}[\mathcal{A}]$ in the lhs of (23) is identified with the summands in the lhs above via (20).) Theorem 6 says that the kernel of (27) is the subspace generated by (26). (The latter corresponds to the coboundary relations.) To summarize:

Theorem. *The space* $\mathrm{Lie}[\mathcal{A}]$ *is freely generated by the orientation space in rank* 1 *subject to the Jacobi identities (in rank two).*

8 Substitution product of Lie

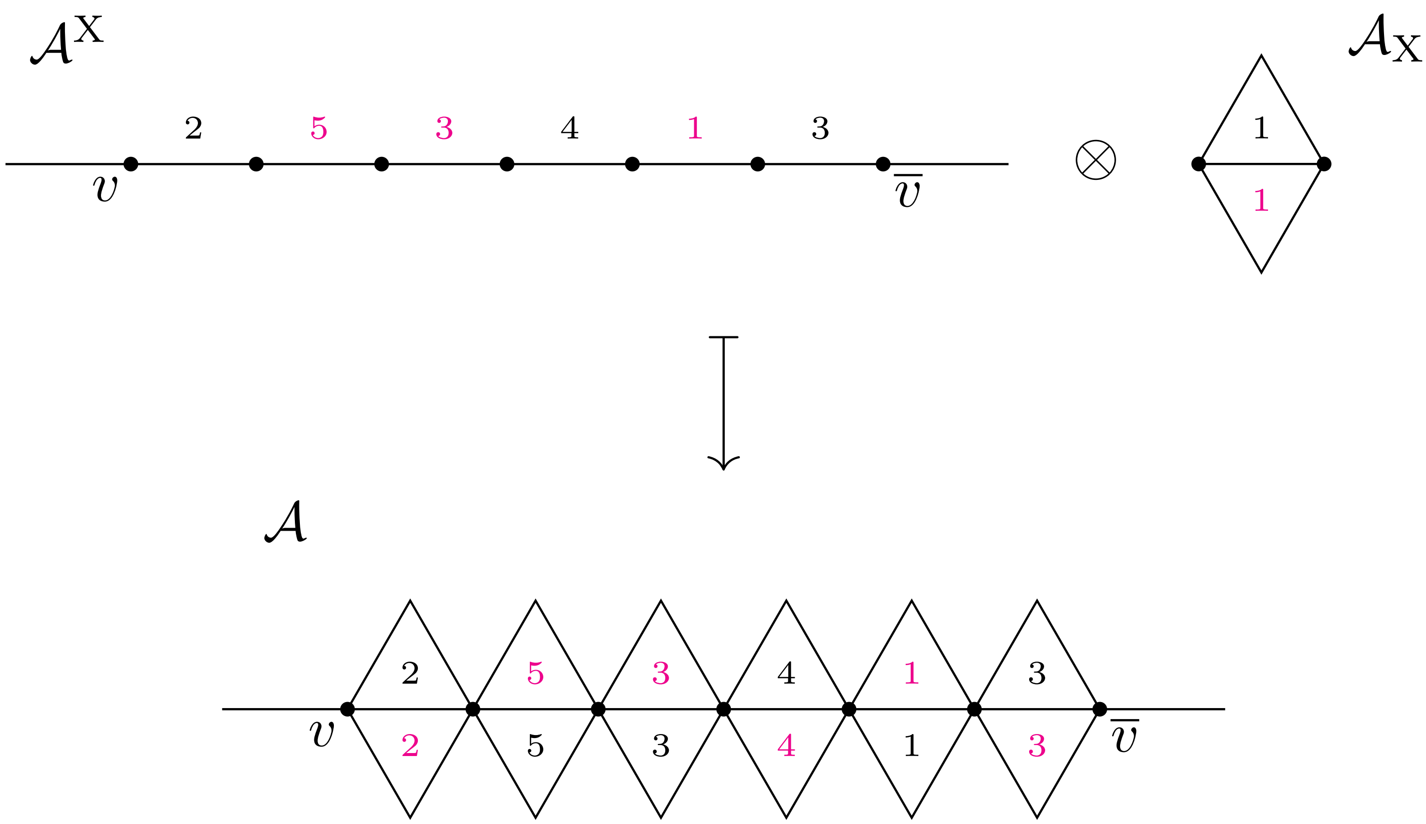
It is convenient to write $\mathcal{H}^{\mathsf{top}}(\Pi[\mathcal{A}])$ for the top-dimensional cohomology of the lattice of flats of $\mathcal{A}$. For any flat X, there is a map

$$\mathcal{H}^{\mathsf{top}}(\Pi[\mathcal{A}^{\mathrm{X}}]) \otimes \mathcal{H}^{\mathsf{top}}(\Pi[\mathcal{A}_{\mathrm{X}}]) \to \mathcal{H}^{\mathsf{top}}(\Pi[\mathcal{A}]) \tag{28}$$

obtained by concatenating: A maximal chain of flats in $\mathcal{A}^{\mathrm{X}}$ can be identified with a chain of flats in $\mathcal{A}$ ending at X, while a maximal chain of flats in $\mathcal{A}_{\mathrm{X}}$ can be identified with a chain of flats in $\mathcal{A}$ starting at X. So, concatenating the two yields a maximal chain of flats in $\mathcal{A}$. The map (28) is obtained by passing to the homology classes. Combining with (19) and using the JKS isomorphism (24) we obtain: For any flat X, there is a map

$$\mathsf{Lie}[\mathcal{A}^{\mathrm{X}}] \otimes \mathsf{Lie}[\mathcal{A}_{\mathrm{X}}] \to \mathsf{Lie}[\mathcal{A}]. \tag{29}$$

We call this the substitution product of Lie. How do we think of this map? An illustration is given below.



9 Classical Lie elements

For any letters a and b, define

$$[a, b] := a|b - b|a,$$

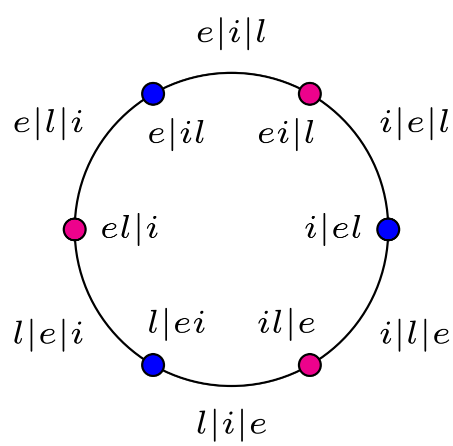
Note that

$$[a, b] = -[b, a].$$

Let us iterate this operation. Thus,

$$[[l, i], e] = [l|i - i|l, e] = l|i|e - i|l|e - e|l|i + e|i|l.$$

Let us visualize this element in the braid arrangement on $I = \{l, i, e\}$.



Thus, what we have is the Lie element obtained by unbracketing the maximal chain

$$\{l, i, e\} \lessdot \{\{l, i\}, \{e\}\} \lessdot \{\{l\}, \{i\}, \{e\}\}.$$

Note that

$$[[l, i], e] + [[e, l], i] + [[i, e], l] = 0,$$

and this is precisely our Jacobi identity.

Now it will also be clear why we used the term "unbracketing".

10 Problems

Exercise. *Prove formula* (7).

Exercise. *Let $\mathcal{A}$ be the dihedral arrangement of n lines. Show that the set of all Dynkin elements form a basis of $\mathrm{Zie}[\mathcal{A}]$.*

Exercise. *What is the number of generic half-spaces in the rank 3 braid arrangement? (Two generic half-spaces are to be regarded the same if they break the set of chambers in the same way.) Describe these half-spaces as well as you can. Are the corresponding Dynkin elements linearly independent?*

Exercise. *Let X and Y be complements in the lattice of flats, that is, $\mathrm{X} \wedge \mathrm{Y} = \bot$ and $\mathrm{X} \vee \mathrm{Y} = \top$. Each chamber F in $\mathcal{A}^{\mathrm{X}}$ is contained in a unique chamber of $\mathcal{A}_{\mathrm{Y}}$, which we denote by $\mathrm{Y}F$. Prove or disprove.*

If $\sum_F x^F \mathtt{H}_F$ is a Lie element of $\mathcal{A}^{\mathrm{X}}$, then $\sum_F x^F \mathtt{H}_{\mathrm{Y}F}$ is a Lie element of $\mathcal{A}_{\mathrm{Y}}$.

Exercise. *In the braid arrangement on the letters a, b, c, d, write down the Lie elements obtained by unbracketing the maximal chains*

$$(\{abcd\} \lessdot \{ab, cd\} \lessdot \{a, b, cd\} \lessdot \{a, b, c, d\})$$

and

$$(\{abcd\} \lessdot \{a, bcd\} \lessdot \{a, bc, d\} \lessdot \{a, b, c, d\}).$$

Also represent them in the bracket notation.