Hyperplane arrangements

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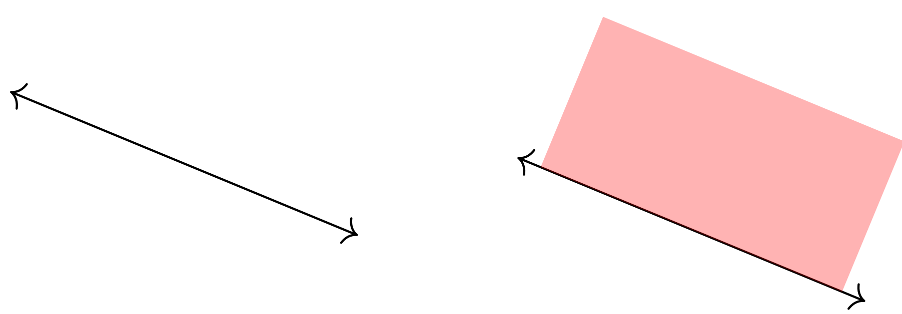
1 Hyperplane arrangements

1.1 Hyperplanes and half-spaces

Let V be a finite-dimensional vector space over $\mathbb{R}$.

A codimension-one affine subspace of V is called a hyperplane.

A half-space is a subset of V which lies on one side of some hyperplane.



The picture on the left shows a hyperplane in $\mathbb{R}^2$ which is the same as a line, while the one on the right shows a half-space.

The bounding hyperplane is the boundary of the half-space. By convention, a half-space is closed, that is, it contains its boundary.

The interior of the half-space is the half-space minus its boundary.

Each hyperplane has two associated half-spaces which lie on its two sides.

1.2 Hyperplane arrangements

A hyperplane arrangement $\mathcal{A}$ is a finite set of hyperplanes in V.

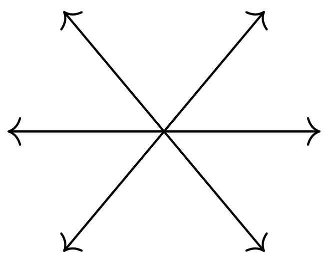
The space V is called the ambient space of $\mathcal{A}$.

The arrangement is linear if all its hyperplanes pass through the origin.

We assume from now on that $\mathcal{A}$ is linear. The center of $\mathcal{A}$ is the subspace obtained by intersecting all hyperplanes of $\mathcal{A}$.

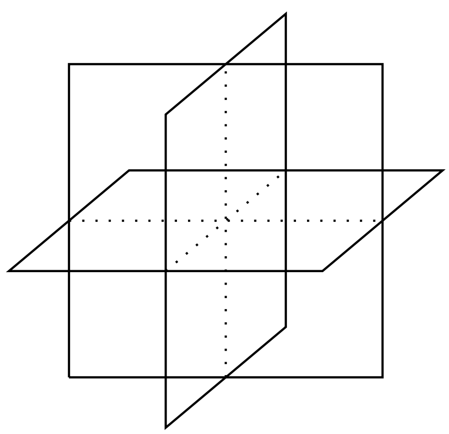
$\mathcal{A}$ is essential if its center is the zero subspace.

The rank of $\mathcal{A}$, denoted $\mathrm{rk}(\mathcal{A})$, is the difference between the dimensions of the ambient space and the center. In particular, the rank of an essential arrangement equals the dimension of the ambient space.



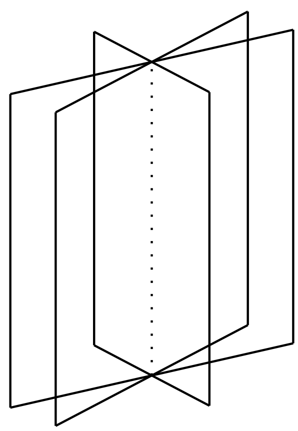
Three lines in $\mathbb{R}^2$

This arrangement is central and essential. It has rank 2.



Three planes in $\mathbb{R}^3$

This arrangement is central and essential. It has rank 3.



Three planes in $\mathbb{R}^3$

This arrangement is central but not essential. It has rank 2.

1.3 Faces

A half-space of an arrangement $\mathcal{A}$ is a half-space of the ambient space V whose bounding hyperplane belongs to $\mathcal{A}$.

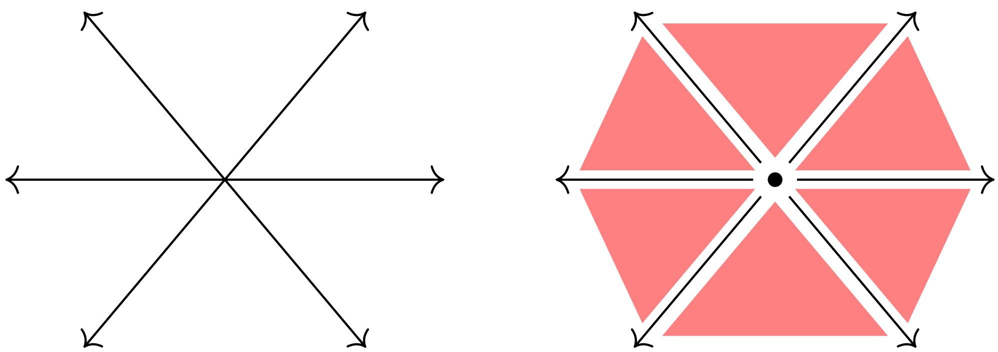
A face of $\mathcal{A}$ is a subset of V obtained by intersecting half-spaces of $\mathcal{A}$, with at least one associated half-space chosen for each hyperplane.

The interior of a face F is the subset of F obtained by intersecting F with the interiors of those half-spaces used to define F whose boundary does not contain F.

Every point in the ambient space belongs to the interior of a unique face. In other words, the interiors of all faces partition the ambient space.

The center of $\mathcal{A}$ is a face. We call it the central face and denote it by O.

Consider the arrangement of three lines (hyperplanes) in the plane passing through the origin.



The partition of the plane by face interiors is illustrated on the right. There are 13 faces.

Let $\Sigma[\mathcal{A}]$ denote the set of all faces.

It is a graded poset under inclusion, with the central face O as its minimum element.

Each face F has a dimension, and the rank of F is the dimension of F minus the dimension of O. We write this as

$$\mathrm{rk}(F) = \dim(F) - \dim(O).$$

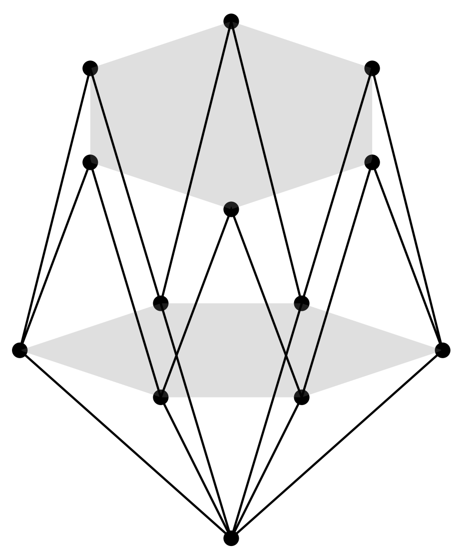
The rank of the poset $\Sigma[\mathcal{A}]$ equals the rank of $\mathcal{A}$.

A maximal face of $\Sigma[\mathcal{A}]$ is called a chamber. We denote the set of chambers by $\Gamma[\mathcal{A}]$.

A rank-one face is called a vertex, a rank-two face is called an edge, while a corank-one face is called a panel.

For the arrangement of three lines in the plane, there are six edges or chambers (sectors), six vertices or panels (rays) and the central face (the origin).

The Hasse diagram of the poset of faces is shown below. It has rank 2.



The minimum element is the origin, the rank-one elements are the rays, and the rank-two elements are the sectors.

Faces of an arrangement will generally be denoted by the letters F, G, H, K, and chambers by the letters C, D, E.

The intersection of two faces is a face, so meets exist in $\Sigma[\mathcal{A}]$.

We denote the meet of F and G by $F \wedge G$.

In contrast, joins may not exist.

We denote the join of F and G by $F \vee G$ (whenever it exists).

It exists precisely when F and G have a common upper bound.

In particular, the join of two distinct chambers cannot exist.

In summary, $\Sigma[\mathcal{A}]$ is a graded meet-semilattice; it is not a lattice unless the rank of $\mathcal{A}$ is 0.

1.4 Opposition map

Every face F has an opposite face, denoted $\overline{F}$, which is given by

$$\overline{F} := \{-x \mid x \in F\}.$$

The opposition map on faces

$$\Sigma[\mathcal{A}] \to \Sigma[\mathcal{A}], \qquad F \mapsto \overline{F} \tag{1}$$

sends every face to its opposite. It is an order-preserving involution. That is,

$$\overline{\overline{F}} = F \qquad \text{and} \qquad F \leq G \iff \overline{F} \leq \overline{G}.$$

Since chambers are maximal faces, the opposition map restricts to an involution on the set of chambers $\Gamma[\mathcal{A}]$. Thus, every chamber C has an opposite chamber $\overline{C}$.

1.5 Isomorphism of arrangements

We say two arrangements $\mathcal{A}$ and $\mathcal{A}'$ are geometrically isomorphic, or gisomorphic for short, if there is a linear isomorphism between their ambient spaces which induces a bijection between the two sets of hyperplanes. We refer to any such isomorphism as a gisomorphism.

We say two arrangements $\mathcal{A}$ and $\mathcal{A}'$ are combinatorially isomorphic, or cisomorphic for short, if the poset of faces $\Sigma[\mathcal{A}]$ and $\Sigma[\mathcal{A}']$ are isomorphic. We refer to any such isomorphism as a cisomorphism.

It is clear that gisomorphic implies cisomorphic. But the converse is not true.

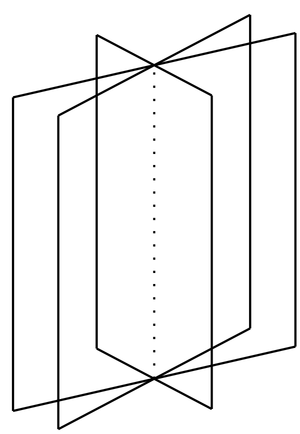
1.6 Essentialization

If an arrangement $\mathcal{A}$ is not essential, then we can make it essential by taking quotient of the ambient space by its center. This is the essentialization of $\mathcal{A}$.

This construction loses information about the dimension of the center, however it does not affect the poset of faces.

If an arrangement is not essential, then its essentialization is cisomorphic but not gisomorphic to the original arrangement (since the two ambient spaces have different dimensions).

Consider the arrangement of three planes in $\mathbb{R}^3$ shown below.



Its essentialization is gisomorphic to an arrangement of three lines in $\mathbb{R}^2$.

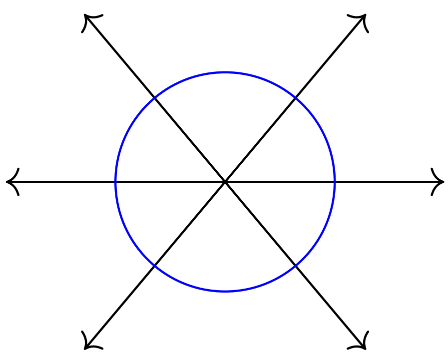
1.7 Cell complex of an arrangement

The poset of faces $\Sigma[\mathcal{A}]$ has the structure of a regular cell complex. The construction goes as follows.

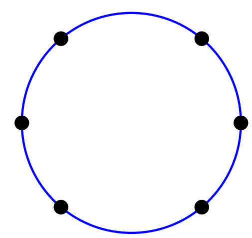
We assume that $\mathcal{A}$ is essential. (If not, we repeat the following on the essentialization of $\mathcal{A}$.)

Put a norm on the ambient space, cut the hyperplane arrangement by the unit sphere, and identify faces of the arrangement with cells on the sphere to obtain the cell complex.

This is illustrated below on the arrangement of 3 lines in the plane.

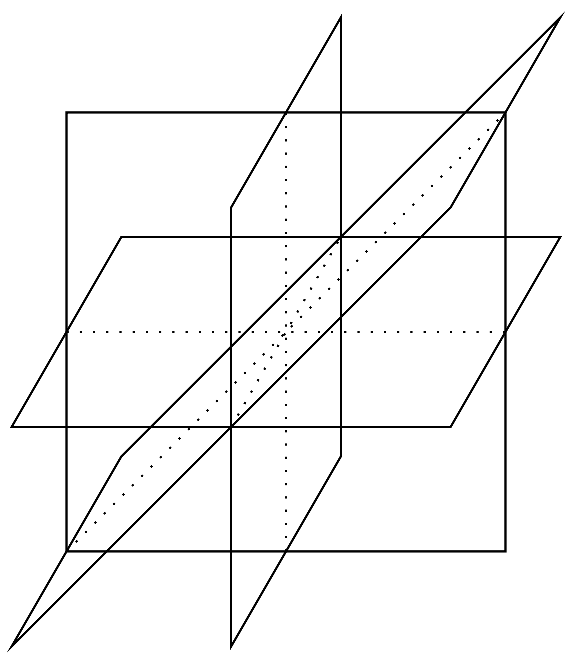


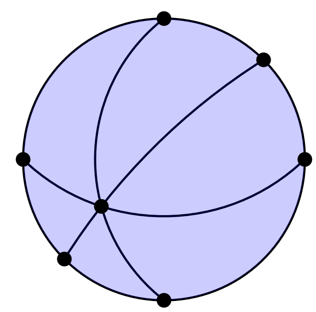
so we see



The central face $O = \{0\}$ is not visible in the spherical model; it corresponds to the unique face of rank 0 of the cell complex.

Another illustration for an arrangement of four planes in $\mathbb{R}^3$ is given below. The spherical model is shown on the right.





This construction explains the motivation for calling rank-one faces as vertices; they are indeed vertices of the associated cell complex, though they are rays in the original arrangement. Similar comment applies to edges.

1.8 Simplicial arrangements

An essential arrangement is simplicial if each chamber is a simplicial cone, that is, a cone over a simplex with the origin as the cone-point.

In general, an arrangement is said to be simplicial if its essentialization is simplicial.

Equivalently, an arrangement is simplicial iff its associated cell complex is a pure simplicial complex.

We refer to faces of a simplicial arrangement as simplices.

2 Arrangements of small rank

2.1 Rank 0

An arrangement has rank 0 iff it has no hyperplanes.

All these arrangements are clearly cisomorphic.

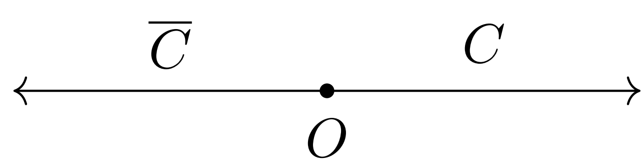
There is only one arrangement of rank 0 which is essential, namely, the arrangement whose ambient space is zero-dimensional.

2.2 Rank 1

An arrangement has rank 1 iff it has exactly one hyperplane.

All these arrangements are cisomorphic.

An arrangement of rank 1 is essential iff its ambient space is one-dimensional (with the origin as the unique hyperplane). This is illustrated below.



There are two chambers (rays) which we will usually denote by C and $\overline{C}$.

2.3 Rank 2

An arrangement has rank 2 iff it has at least two hyperplanes and they all pass through a codimension-two subspace of the ambient space.

An essential arrangement of rank 2 consists of n lines through the origin in a two-dimensional space, with $n \geq 2$.

The arrangement of three lines in the plane is an example with $n = 3$.

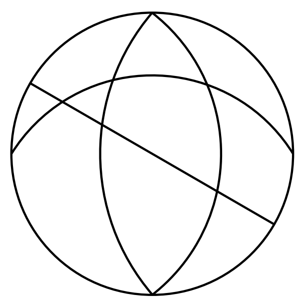
Any two rank-two arrangements with the same number of lines are cisomorphic.

When the lines are equally spaced, the arrangement is called dihedral.

Any two dihedral arrangements with the same number of lines are gisomorphic.

2.4 Rank 3

The figure below shows the spherical model of an arrangement of rank 3 consisting of five hyperplanes. A hyperplane in this case is the same as a great circle. Only one half of the arrangement is visible in the picture, the other half being on the backside.



The chambers are either triangles or quadrilaterals, so the arrangement is not simplicial. Three of the hyperplanes pass through the north and south poles. Note that the region between two adjacent longitudes contains either 3 or 4 chambers. The fact that these numbers can differ is of importance.

Rank-three arrangements can be visualized in this manner, and are very useful to develop a geometric feel for notions that we discuss. These arrangements abound; to classify even the simplicial ones is nontrivial.

3 Flats

3.1 Flats

A flat of an arrangement $\mathcal{A}$ is a subspace of the ambient space obtained by intersecting some subset of hyperplanes of $\mathcal{A}$.

In particular, a flat has a dimension.

Let $\Pi[\mathcal{A}]$ denote the set of flats.

It is a graded poset under inclusion, with the center as the minimum element, and the ambient space as the maximum element. (The center is the only subset which is both a face and a flat.)

The rank of a flat X is the dimension of X minus the dimension of the center.

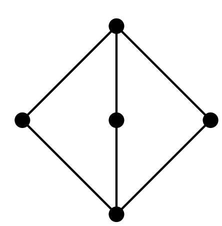
Intersection of two flats is a flat, so meets exist in the poset of flats.

Further, since there is a maximum element, joins exist as well.

Thus $\Pi[\mathcal{A}]$ is a lattice.

We will use the letters X, Y, Z, W to denote flats. The minimum and maximum flats will be denoted $\perp$ and $\top$, respectively. We denote the meet of X and Y by $\mathrm{X} \wedge \mathrm{Y}$ and the join by $\mathrm{X} \vee \mathrm{Y}$. We write $[\mathrm{X}, \mathrm{Z}]$ for the interval consisting of all flats which lie between X and Z.

The Hasse diagram of the poset of flats for the arrangement of 3 lines in the plane is shown below.



It has rank 2. It consists of the minimum flat $\perp$ (center), the three lines (hyperplanes), and the maximum flat $\top$ (ambient space).

3.2 Support map

The support of a face F is the smallest flat which contains F.

It is the intersection of all flats which contain F,

or equivalently, the intersection of all hyperplanes which contain F,

or equivalently, the linear span of F.

The support map

$$\mathrm{s} : \Sigma[\mathcal{A}] \twoheadrightarrow \Pi[\mathcal{A}] \tag{2}$$

sends a face to its support. It is surjective and order-preserving.

We say a flat X supports a face F if $\mathrm{s}(F) = \mathrm{X}$, that is, the support of F is X.

The minimum flat $\perp$ supports exactly one face, namely, the central face O.

Any rank-one flat supports two vertices, which are opposite to each other.

The maximum flat $\top$ supports chambers.

3.3 Combinatorial flats

A face F is a top-dimensional face of a flat X if X supports F.

Note that a flat is the union of its top-dimensional faces.

(For instance, a line through the origin is the union of its two opposite rays starting at the origin.)

This suggests the following alternative approach to flats.

A combinatorial flat is a subset of $\Sigma[\mathcal{A}]$ consisting of all faces with the same support.

In other words, a combinatorial flat precisely consists of the top-dimensional faces of some flat.

For a combinatorial flat X, define its closure to be

$$\mathrm{Cl}(\mathrm{X}) = \{F \in \Sigma[\mathcal{A}] \mid F \leq G \text{ for some } G \in \mathrm{X}\}.$$

This is the set of all faces contained in X viewed as a geometric flat (subset of the ambient space).

Equivalently, it is the set of faces whose support is smaller than X.

We have

$$\mathrm{X} \leq \mathrm{Y} \iff \mathrm{Cl}(\mathrm{X}) \subseteq \mathrm{Cl}(\mathrm{Y})$$

and

$$\mathrm{Cl}(\mathrm{X} \wedge \mathrm{Y}) = \mathrm{Cl}(\mathrm{X}) \cap \mathrm{Cl}(\mathrm{Y}).$$

Since a flat and a combinatorial flat are equivalent notions, we will usually just say a “flat” with the context determining which notion is being used.

4 Tits monoid and Birkhoff monoid

4.1 Sign sequences

For a hyperplane H, let us denote its two associated half-spaces by H^+ and H^-. The choice of $+$ and $-$ is arbitrary but fixed. It is convenient to let $\mathrm{H}^0 := \mathrm{H}$. Observe that $\mathrm{H}^0 = \mathrm{H}^+ \cap \mathrm{H}^-$.

In this notation, a face of $\mathcal{A} = \{\mathrm{H}_i\}_{i \in I}$ is a subset of the ambient space of the form

$$F = \bigcap_{i \in I} \mathrm{H}_i^{\epsilon_i},$$

where $\epsilon_i \in \{+, 0, -\}$. Different choices of ϵ_i can yield the same face.

However, there is a canonical way to write F in this form, namely,

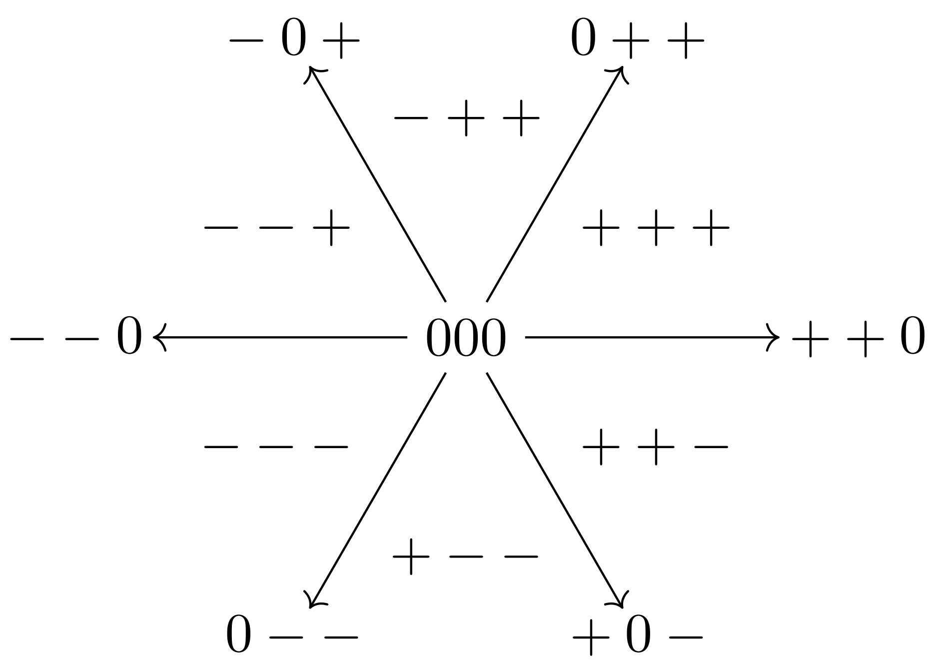
$$F = \bigcap_{i \in I} \mathrm{H}_i^{\epsilon_i(F)},$$

where $\epsilon_i(F)$ is 0 if F lies in H_i, it is $+$ if the interior of F lies in the interior of H_i^+, and it is $-$ if the interior of F lies in the interior of H_i^-. We refer to

$$(\epsilon_i(F))_{i \in I}$$

as the sign sequence of F.

A possible selection of sign sequences for the arrangement of three lines is shown below. Note very carefully that not all sign sequences occur. For instance, no chamber has sign sequence $+-+$.



The central face is the unique face F for which $\epsilon_i(F) = 0$ for each i, while a chamber is a face F for which $\epsilon_i(F) \neq 0$ for each i. The sign sequence of $\overline{F}$ is obtained by reversing each sign in the sign sequence of F:

(3) $$\epsilon_i(\overline{F}) = -\epsilon_i(F).$$

Observe that

(4)

$F \leq G \iff \epsilon_i(F) = \epsilon_i(G)$ whenever $\epsilon_i(F) \neq 0$.

In other words, $F \leq G$ iff the sign sequence of F is obtained from that of G by replacing some $+$ and $-$ by 0.

4.2 Tits monoid

For faces F and G, define the face FG by

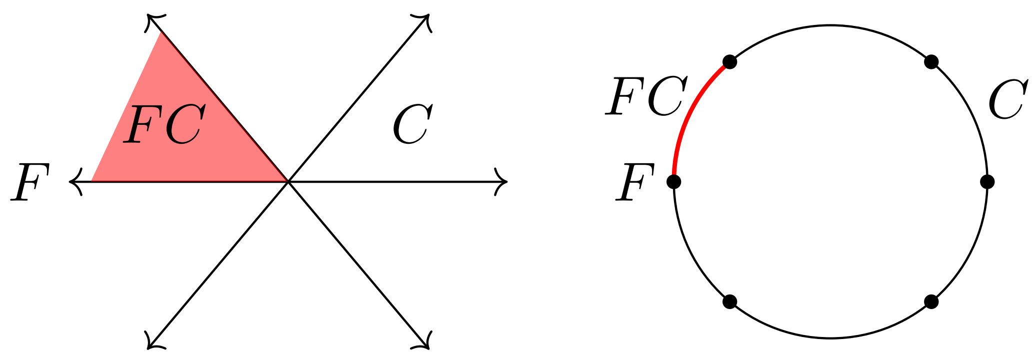
$$\epsilon_i(FG) := \begin{cases} \epsilon_i(F) & \text{if } \epsilon_i(F) \neq 0, \\ \epsilon_i(G) & \text{if } \epsilon_i(F) = 0. \end{cases} \tag{5}$$

We refer to FG as the Tits product of F and G.

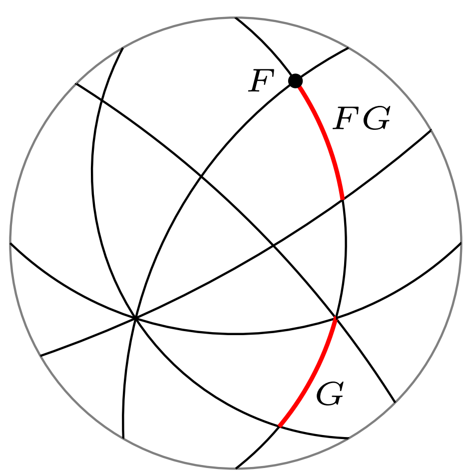
The product has a geometric meaning: if we move from an interior point of F to an interior point of G along a straight line then FG is the face that we are in after moving a small positive distance.

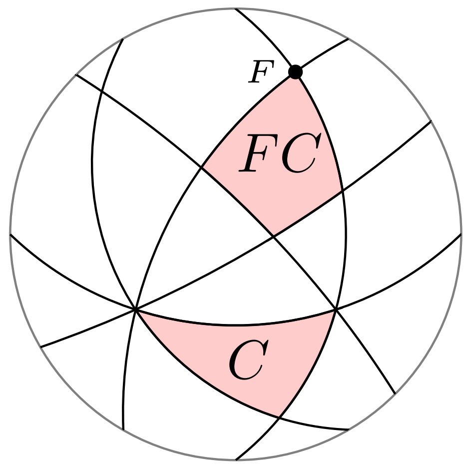
Hence, we also say that FG is the Tits projection of G on F.

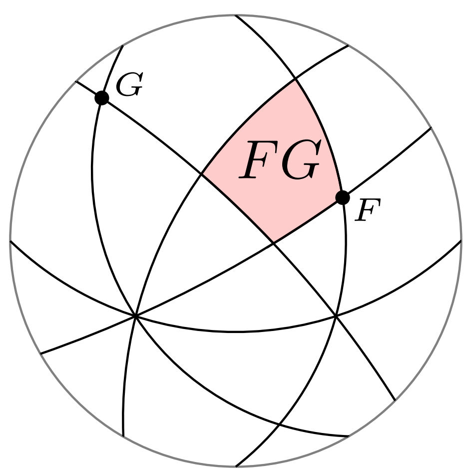
An example in the arrangement of three lines is shown below: F is a vertex (ray), C is an edge (sector) and FC is another edge (sector) which has F as a vertex.



Another illustration of the Tits product in the spherical model of a rank-three arrangement is shown below.




F
FC
C


G
FG
F

Observe that: For any $i \in I$,

(6)

$$\epsilon_i(FG) = 0 \iff \epsilon_i(F) = 0 \text{ and } \epsilon_i(G) = 0.$$

In other words, a hyperplane contains FG iff it contains both F and G.

The Tits product is associative, that is,

$$F(GH) = (FG)H$$

for any faces F, G, H.

Further, the central face is the identity element for the product.

Thus the set of faces $\Sigma[\mathcal{A}]$ is a monoid.

We call this the Tits monoid.

For any faces F and G, we have

$$FF = F \quad \text{and} \quad FGF = FG. \tag{7}$$

This follows from (5). The first identity is a special case of the second obtained by setting $G = O$.

A monoid is which every element is idempotent is called a band. Further, if the identity $xyx = xy$ holds for all elements x, y, then it is called a left regular band. The Tits monoid is an example of a left regular band.

For any faces F and G,

$$\overline{FG} = \overline{F}\,\overline{G}. \tag{8}$$

This follows from (3) and (5). Thus, the opposition map (1) is an automorphism of the Tits monoid.

Let C be a chamber. Then for any face F, the faces FC and CF are both chambers, and in fact $CF = C$. Thus the set of chambers $\Gamma[\mathcal{A}]$ is a two-sided ideal in $\Sigma[\mathcal{A}]$.

4.3 Tits monoid and poset of faces

The partial order on faces is completely determined by the Tits product: For any faces H and F,

(9a) $$HF = F \iff H \le F,$$

(9b) $$H\overline{F} = F \iff H = F.$$

This follows from (3), (4), (5). In particular, for any face F,

(10) $$F\overline{F} = F.$$

Some further interactions between the product and the partial order are summarized below. They can be verified in a similar manner.

Lemma 1. *The following properties hold for any faces* F, G, H, K.

1. *If* $G \leq H$, *then* $FG \leq FH$. *In particular,* $F \leq FG$.
2. *If* $FG = K$ *and* $F \leq H \leq K$, *then* $HG = K$.
3. $FG \wedge F\overline{G} = F$. *In particular,* $G \wedge \overline{G} = O$.
4. $F \wedge G = F\overline{G} \wedge G = F\overline{G} \wedge G\overline{F}$.
5. $F \leq H$ *and* $G \leq H$ *imply* $FG \leq H$.

4.4 Birkhoff monoid

Recall the lattice of flats $\Pi[\mathcal{A}]$.

It has the structure of a monoid: the product of X and Y is defined to be $\mathrm{X} \vee \mathrm{Y}$.

The unit element is the minimum flat $\perp$.

We call this the Birkhoff monoid.

It is commutative.

4.5 Support map

Recall the support map from faces to flats defined in (2). For any faces F and G,

$$\mathrm{s}(FG) = \mathrm{s}(F) \vee \mathrm{s}(G). \tag{11}$$

This follows from (6). Further, $\mathrm{s}(O) = \perp$. Thus, the support map is a homomorphism from the Tits monoid to the Birkhoff monoid.

Observe that FG and GF always have the same support. In particular, if $GF = G$, then FG and G have the same support. Similar useful observations are given below.

(12) $$GF = G \iff \mathrm{s}(F) \leq \mathrm{s}(G).$$

Either of these conditions is equivalent to the condition

$$\epsilon_i(G) = 0 \text{ implies } \epsilon_i(F) = 0 \text{ for all } i.$$

It follows that

(13) $$FG = F \text{ and } GF = G \iff \mathrm{s}(F) = \mathrm{s}(G).$$

Either of these conditions is equivalent to the condition

$$\epsilon_i(F) = 0 \text{ iff } \epsilon_i(G) = 0 \text{ for all } i.$$

To summarize: The relation

$$F \sim G \iff FG = F \text{ and } GF = G \tag{14}$$

is an equivalence relation on the set of faces whose equivalence classes correspond to flats. In fact, note that the equivalence classes are precisely combinatorial flats (Section 3.3).

4.6 Join of faces

We say that two faces F and G are joinable if their join exists in $\Sigma[\mathcal{A}]$, or equivalently, if there is a face greater than both F and G.

Proposition 1. *Distinct subfaces of a face have distinct supports.*

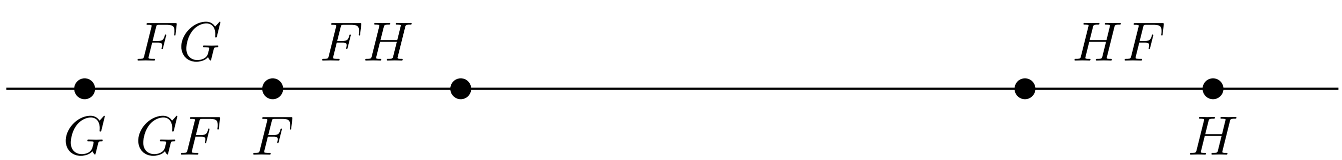
Proof. Suppose F and G are distinct subfaces of a face K. Then by (4), $\epsilon_i(F) = \epsilon_i(K)$ whenever $\epsilon_i(F) \neq 0$, and $\epsilon_i(G) = \epsilon_i(K)$ whenever $\epsilon_i(G) \neq 0$. Thus, the set of hyperplanes H_i for which $\epsilon_i(F) = 0$ differs from the one for which $\epsilon_i(G) = 0$. So by the sign sequence condition given after (13), the supports of F and G are distinct. $\square$

Proposition 2. *Two faces F and G are joinable iff $FG = GF$. In this situation,*

$$F \vee G = FG = GF.$$

In particular, faces with the same support are joinable iff they are equal.

Proof. Any face which contains F and G must contain FG and GF by Lemma 1, item (5). Since FG and GF have the same support, the only way a face can be greater than both of them is if $FG = GF$. All claims follow. $\square$



In the figure, the vertices F and G are joinable, the edge connecting them is their join. On the other hand, the vertices F and H are not joinable since the edges

FH and HF are distinct.

5 Arrangements under and over a flat

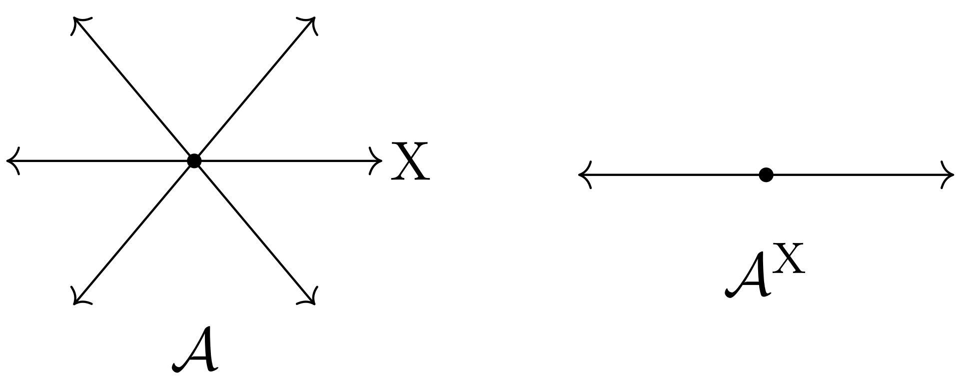
5.1 Under a flat

Let X be any fixed flat of $\mathcal{A}$.

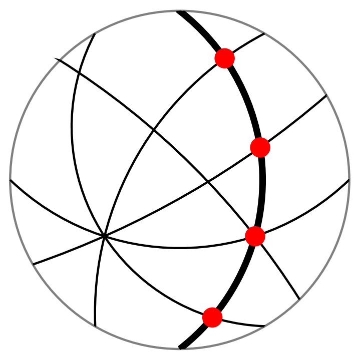
The arrangement under X is

$$\mathcal{A}^{\mathrm{X}} = \{\mathrm{H} \cap \mathrm{X} \mid \mathrm{H} \in \mathcal{A},\ \mathrm{X} \not\subseteq \mathrm{H}\}.$$

It is a hyperplane arrangement with ambient space X. Its hyperplanes are obtained by intersecting X with hyperplanes in $\mathcal{A}$ not containing it. Its center is the same as that of $\mathcal{A}$. An example follows.



The arrangement under a rank-two flat in a rank-three arrangement is illustrated below. It has eight vertices, four of which are visible in the picture.



The arrangement $\mathcal{A}^{\mathrm{X}}$ singles out the portion of $\mathcal{A}$ below X. Faces, chambers and flats of $\mathcal{A}^{\mathrm{X}}$ are as follows.

$$\Sigma[\mathcal{A}^{\mathrm{X}}] = \{F \in \Sigma[\mathcal{A}] \mid \mathrm{s}(F) \leq \mathrm{X}\},$$
$$\Gamma[\mathcal{A}^{\mathrm{X}}] = \{F \in \Sigma[\mathcal{A}] \mid \mathrm{s}(F) = \mathrm{X}\},$$
$$\Pi[\mathcal{A}^{\mathrm{X}}] = \{\mathrm{Y} \in \Pi[\mathcal{A}] \mid \mathrm{Y} \leq \mathrm{X}\}.$$

For any face K, let $\mathcal{A}^K := \mathcal{A}^{\mathrm{s}(K)}$.

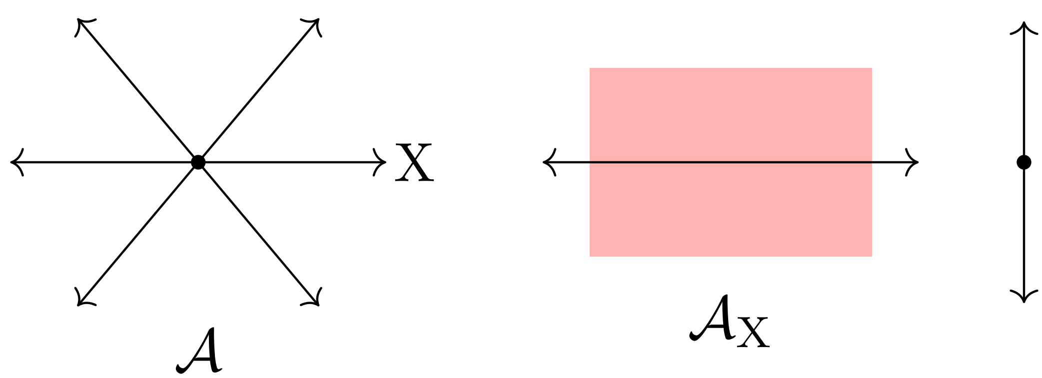
5.2 Over a flat

Let X be any fixed flat of $\mathcal{A}$.

The arrangement over X is

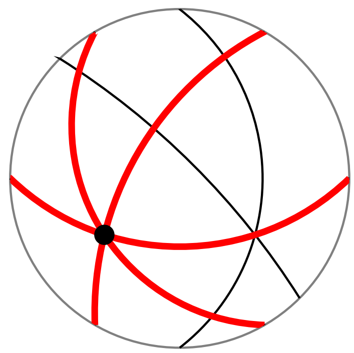
$$\mathcal{A}_{\mathrm{X}} = \{\mathrm{H} \in \mathcal{A} \mid \mathrm{X} \subseteq \mathrm{H}\}.$$

It is a hyperplane arrangement with the same ambient space as $\mathcal{A}$. It consists of the hyperplanes of $\mathcal{A}$ which contain X. The center is X. An example follows.



The essentialization of $\mathcal{A}_{\mathrm{X}}$ (up to gisomorphism) is shown on the far right.

The arrangement over a rank-one flat of a rank-three arrangement is illustrated below. It consists of the three red lines.



Roughly, the arrangement $\mathcal{A}_{\mathrm{X}}$ singles out the portion of $\mathcal{A}$ above X. Flats of $\mathcal{A}_{\mathrm{X}}$ are flats of $\mathcal{A}$ which contain X. Faces and chambers are in canonical correspondence with faces and chambers of $\mathcal{A}$ that contain any fixed face F of support X.

$$\Sigma[\mathcal{A}_{\mathrm{X}}] \cong \{G \in \Sigma[\mathcal{A}] \mid F \leq G\},$$

$$\Gamma[\mathcal{A}_{\mathrm{X}}] \cong \{C \in \Gamma[\mathcal{A}] \mid F \leq C\},$$

$$\Pi[\mathcal{A}_{\mathrm{X}}] = \{\mathrm{Y} \in \Pi[\mathcal{A}] \mid \mathrm{X} \leq \mathrm{Y}\}.$$

We elaborate on this below.

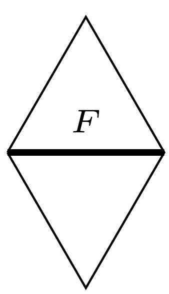
5.3 Stars and top-stars

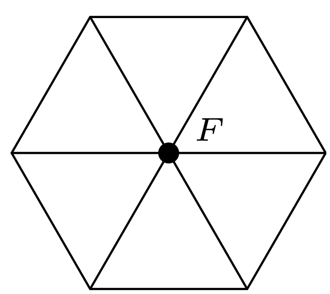
For a face F, let $\Sigma[\mathcal{A}]_F$ denote the set of faces of $\mathcal{A}$ which are greater than F. This is the star of F.

For clarity, we denote elements of $\Sigma[\mathcal{A}]_F$ by K/F, where K is a face greater than F.

The star of a chamber is a singleton consisting of the chamber itself, while the star of the central face is the set of all faces.

Let $\Gamma[\mathcal{A}]_F$ denote the set of chambers of $\mathcal{A}$ which are greater than F. This is the top-star of F.





The star of F is illustrated above in rank three. In the picture on the left, F is an edge and its top-star consists of two chambers, while in the picture on the right, F is a vertex and its top-star consists of six chambers.

Observe that the Tits product of $\Sigma[\mathcal{A}]$ restricts to $\Sigma[\mathcal{A}]_F$.

This turns the latter into a monoid, which is a left regular band.

However, note very carefully that the unit element is now F.

Lemma 2. *When F and G have the same support, we have an isomorphism*

$$\Sigma[\mathcal{A}]_F \xrightarrow{\cong} \Sigma[\mathcal{A}]_G, \qquad K/F \mapsto GK/G$$

of monoids and hence of posets. Further, it restricts to a bijection

$$\Gamma[\mathcal{A}]_F \xrightarrow{\cong} \Gamma[\mathcal{A}]_G, \qquad C/F \mapsto GC/G.$$

Proof. Using (7), one can check that the first map is a monoid homomorphism. Further it is an isomorphism with the map

$$\Sigma[\mathcal{A}]_G \xrightarrow{\cong} \Sigma[\mathcal{A}]_F, \qquad H/G \mapsto FH/F$$

as its inverse. This can be deduced from (9a) and (13). □

Lemma 3. *Let X be a flat. Then for any face F with support X, there is an isomorphism*

$$\Sigma[\mathcal{A}]_F \xrightarrow{\cong} \Sigma[\mathcal{A}_{\mathrm{X}}]$$

of monoids and hence of posets. Further, when F and G both have support X, the diagram

$$\begin{tikzcd} \Sigma[\mathcal{A}]_F \arrow[rr] \arrow[dr] & & \Sigma[\mathcal{A}]_G \arrow[dl] \\ & \Sigma[\mathcal{A}_{\mathrm{X}}] & \end{tikzcd}$$

commutes.

For any face F, let $\mathcal{A}_F := \mathcal{A}_{\mathrm{s}(F)}$.

Thus, there is no distinction between $\mathcal{A}_F$ and $\mathcal{A}_G$ when F and G have the same support.

However, for book-keeping purposes, we would like to keep them apart.

Hence, we identify faces of $\mathcal{A}_F$ with the star of F, and chambers with the top-star of F.

Thus, K/F and C/F denote a face and chamber of $\mathcal{A}_F$.

5.4 Between flats

The preceding constructions can be combined.

Let X be a flat contained in another flat Y, that is, $\mathrm{X} \leq \mathrm{Y}$.

Then X corresponds to a flat of $\mathcal{A}^{\mathrm{Y}}$, and Y to a flat of $\mathcal{A}_{\mathrm{X}}$.

Thus, one may first consider the arrangement under Y and within it the arrangement over X, or the other way around.

The resulting arrangements $(\mathcal{A}^{\mathrm{Y}})_{\mathrm{X}}$ and $(\mathcal{A}_{\mathrm{X}})^{\mathrm{Y}}$ are the same. We use $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$ to denote this arrangement.

Flats of $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$ correspond to flats of $\mathcal{A}$ which lie between X and Y. In other words,

$$\Pi[\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}] = [\mathrm{X}, \mathrm{Y}].$$

We will also use the notations $\mathcal{A}_F^{\mathrm{X}}$ (when support of F is smaller than X) and $\mathcal{A}_F^G$ (when $F \leq G$). We identify their faces with faces of $\mathcal{A}$ which are greater than F and of support smaller than X.

6 Cartesian product of arrangements

6.1 Cartesian product

Given two arrangements $\mathcal{A}$ and $\mathcal{A}'$, one can form their cartesian product $\mathcal{A} \times \mathcal{A}'$.

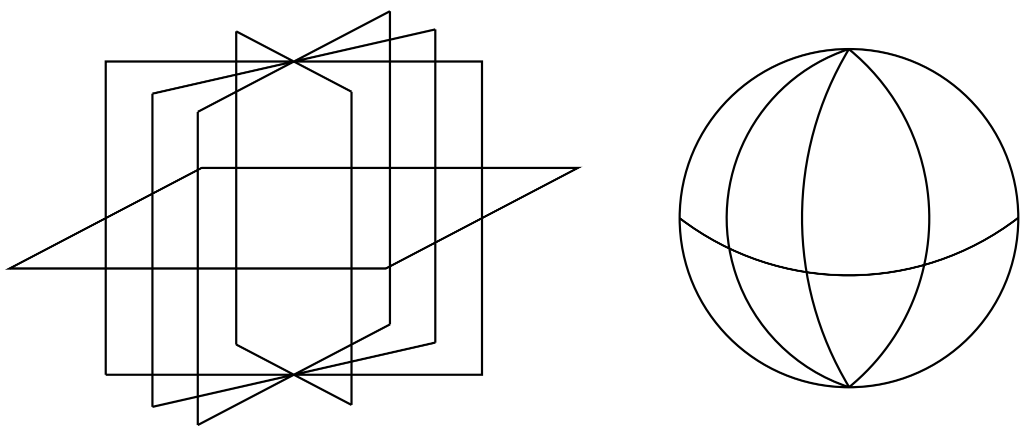
Its ambient space is $V \oplus V'$, where V and V' are the ambient spaces of $\mathcal{A}$ and $\mathcal{A}'$.

Its hyperplanes are codimension-one subspaces of the form $\mathrm{H} \oplus V'$ and $V \oplus \mathrm{H}'$, where H and H' are hyperplanes of $\mathcal{A}$ and $\mathcal{A}'$.

Observe that

$$\mathrm{rk}(\mathcal{A} \times \mathcal{A}') = \mathrm{rk}(\mathcal{A}) + \mathrm{rk}(\mathcal{A}').$$

The cartesian product of the rank-one arrangement and the rank-two arrangement of 4 lines is shown below.



In this arrangement, there exists a hyperplane which contains all vertices except two. The hyperplane is the equator and the two vertices are the north pole and south pole.

The operation of taking cartesian product is associative and commutative (up to gisomorphism). The essential arrangement of rank 0 serves as the unit. In general, taking cartesian product with a rank-zero arrangement has the effect of fattening up the center.

6.2 Faces and flats

A face of $\mathcal{A} \times \mathcal{A}'$ is the same as a pair (F, F'), where F is a face of $\mathcal{A}$ and F' is a face of $\mathcal{A}'$. In other words,

$$\Sigma[\mathcal{A} \times \mathcal{A}'] = \Sigma[\mathcal{A}] \times \Sigma[\mathcal{A}'].$$

This identification is an isomorphism of monoids, that is,

$$(F, F')(G, G') = (FG, F'G'). \tag{15}$$

One way to see this is to note that the sign sequence of (F, F') can be identified with the sign sequence of F followed by the sign sequence of F'. Either directly or as a formal consequence of (9a),

$$(F, F') \leq (G, G') \iff F \leq G \text{ and } F' \leq G'.$$

A chamber of $\mathcal{A} \times \mathcal{A}'$ is the same as a pair (C, C'), where C is a chamber of $\mathcal{A}$ and C' is a chamber of

$\mathcal{A}'$. Thus,

$$\Gamma[\mathcal{A} \times \mathcal{A}'] = \Gamma[\mathcal{A}] \times \Gamma[\mathcal{A}'].$$

A flat of $\mathcal{A} \times \mathcal{A}'$ is the same as a pair $(\mathrm{X}, \mathrm{X}')$, where X is a flat of $\mathcal{A}$ and X' is a flat of $\mathcal{A}'$. Thus,

$$\Pi[\mathcal{A} \times \mathcal{A}'] = \Pi[\mathcal{A}] \times \Pi[\mathcal{A}'].$$

This identification is an isomorphism of posets, that is,

$$(\mathrm{X}, \mathrm{X}') \leq (\mathrm{Y}, \mathrm{Y}') \iff \mathrm{X} \leq \mathrm{Y} \text{ and } \mathrm{X}' \leq \mathrm{Y}'.$$

Let $\bot'$ and $\top'$ denote the minimum and maximum flats of $\mathcal{A}'$. Under the above identification, a hyperplane of $\mathcal{A} \times \mathcal{A}'$ is either $(\mathrm{H}, \top')$ with H an hyperplane of $\mathcal{A}$ or $(\top, \mathrm{H}')$ with H' an hyperplane of $\mathcal{A}'$.

6.3 Under and over a flat of a product

For a flat $(\mathrm{X}, \mathrm{X}')$ of $\mathcal{A} \times \mathcal{A}'$,

$$(\mathcal{A}\times\mathcal{A}')^{(\mathrm{X},\mathrm{X}')} = \mathcal{A}^{\mathrm{X}}\times(\mathcal{A}')^{\mathrm{X}'} \quad \text{and} \quad (\mathcal{A}\times\mathcal{A}')_{(\mathrm{X},\mathrm{X}')} = \mathcal{A}_{\mathrm{X}}$$

Specializing to the flat $(\top, \bot')$ of $\mathcal{A} \times \mathcal{A}'$, we get

$$(\mathcal{A} \times \mathcal{A}')^{(\top,\bot')} \cong \mathcal{A} \quad \text{and} \quad (\mathcal{A} \times \mathcal{A}')_{(\top,\bot')} \cong \mathcal{A}'.$$

A similar remark applies to the flat $(\bot, \top')$. Thus, $\mathcal{A}$ and $\mathcal{A}'$ may both be seen as arrangements under and over a flat of $\mathcal{A} \times \mathcal{A}'$, up to cisomorphism.