

PH423 Assignment 2

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Question 1.

- (a) Calculate the expectation values of \hat{J}_x , \hat{J}_y , \hat{J}_x^2 and \hat{J}_y^2 in the angular momentum states $|j, m\rangle$. Explain the result geometrically. (Using symmetry arguments may help).

We start with the expansion of the operators \hat{J}_x and \hat{J}_y in terms of the ladder operators

$$\hat{J}_x = \frac{1}{2} \cdot (\hat{J}_+ + \hat{J}_-) \quad (1)$$

and

$$\hat{J}_y = \frac{1}{2i} \cdot (\hat{J}_+ - \hat{J}_-) . \quad (2)$$

The application of the ladder operators on a state $|j, m\rangle$ changes it to a state of the form $c \cdot |j, m \pm 1\rangle$ for some $c \in \mathbb{C}$. So, given the orthogonality of the $|j, m\rangle$ states, we get that

$$\langle j, m | \hat{J}_x | j, m \rangle = \langle j, m | \hat{J}_y | j, m \rangle = 0 \quad \forall |j, m\rangle . \quad (3)$$

Squaring [Equation 1](#) and [2](#), we get the operators \hat{J}_x^2 and \hat{J}_y^2 in terms of the ladder operators. With the same argument as before, we see that only terms with equal powers of the two ladder operators will contribute, and using

$$\hat{J}_\pm |j, m\rangle = \hbar \sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1\rangle , \quad (4)$$

we get

$$\langle j, m | \hat{J}_y^2 | j, m \rangle = \langle j, m | \hat{J}_x^2 | j, m \rangle \quad (5)$$

$$= \langle j, m | \frac{1}{4} \cdot (\hat{J}_+^2 + \hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+ + \hat{J}_-^2) | j, m \rangle \quad (6)$$

$$= \langle j, m | \frac{1}{4} \cdot (\hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+) | j, m \rangle \quad (7)$$

$$= \langle j, m | \frac{\hbar^2}{4} \cdot \left(\sqrt{(j+m+1)(j-m)} \sqrt{(j-m)(j+m+1)} \right)$$

$$+ \sqrt{(j-m)(j+m+1)} \sqrt{(j+m+1)(j-m)} \Big) \cdot |j, m\rangle \quad (8)$$

$$= \frac{\hbar^2}{2} (j+m+1)(j-m) . \quad (9)$$

The values for x and y are not separately calculated as a trivial calculation shows they're equal. The same is easily argued using symmetry in the x-y plane. This symmetry also serves as an explanation for the expectation value, since there is similarly a reflection symmetry about either axis, the expectation cannot favor either $\pm x$ or $\pm y$.

- (b) Can the angular momentum $\hat{\mathbf{J}}$ be oriented entirely along the z (or x or y) axis? Give reasons in either case.

No. The operators do not commute. The momentum being completely along one axis would allow us to determine them simultaneously, violating the commutation condition.

2. Determine the eigenvalues and eigenvectors of the 2×2 matrix $\sigma \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector along the (θ, ϕ) direction and σ are the three Pauli matrices. This is basically the projection of the spin $1/2$ operator (apart from $\frac{\hbar}{2}$) along the direction of the unit vector $\hat{\mathbf{n}}$. Do this in two ways:

- (a) First by explicitly diagonalizing the matrix $\sigma \cdot \hat{\mathbf{n}}$.

The vector $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, where the σ_i matrices are -

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now we need to figure out what $\hat{\mathbf{n}}$ is. The unit vector points along the (θ, ϕ) direction. This is nothing but the unit vector $\hat{\mathbf{r}}$ in Polar co-ordinates.

$$\hat{\mathbf{n}} = \hat{\mathbf{r}} = \cos(\phi)\sin(\theta)\hat{\mathbf{i}} + \sin(\phi)\sin(\theta)\hat{\mathbf{j}} + \cos(\theta)\hat{\mathbf{k}}$$

Thus, $\hat{\mathbf{n}} = (\cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta), \cos(\theta))$. We know that $\mathbf{a} \cdot \mathbf{b} = a_i b_i$ (implicit summation over i)

Thus, $\sigma \cdot \hat{\mathbf{n}} = \sigma_i n_i$.

$$\begin{aligned} \sigma \cdot \hat{\mathbf{n}} &= \cos(\phi)\sin(\theta) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin(\phi)\sin(\theta) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos(\theta) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \therefore \sigma \cdot \hat{\mathbf{n}} &= \sin(\theta) \begin{pmatrix} 0 & \cos(\phi) - i \sin(\phi) \\ \cos(\phi) + i \sin(\phi) & 0 \end{pmatrix} + \cos(\theta) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \sin(\theta) \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix} + \cos(\theta) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos(\theta) \end{pmatrix} \end{aligned}$$

To find the eigenvalues and eigenvectors, we now need to diagonalize this matrix. Let the eigenvalues be represented by λ . The characteristic polynomial takes the following form.

$$(\cos(\theta) - \lambda)(-\cos(\theta) - \lambda) - \sin(\theta)e^{-i\phi} \sin(\theta)e^{i\phi} = 0$$

$$\begin{aligned}\therefore -\cos^2(\theta) + \lambda^2 - \sin^2(\theta) &= 0 \Rightarrow \lambda^2 - 1 = 0 \\ \therefore \lambda &= \pm 1\end{aligned}$$

for $\lambda = 1$, let the eigenvector be $\mathbf{v}_1 = (v_{1,1}, v_{1,2})$, thus

$$\begin{aligned}\begin{pmatrix} \cos(\theta) & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos(\theta) \end{pmatrix} \begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix} &= \begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix} \\ \therefore \cos(\theta) * v_{1,1} + \sin(\theta)e^{-i\phi} * v_{1,2} &= v_{1,1}, \quad \sin(\theta)e^{i\phi} * v_{1,1} - \cos(\theta) * v_{1,2} = v_{1,2} \\ v_{1,2} &= e^{i\phi} \frac{\sin(\theta)}{(\cos(\theta) + 1)} * v_{1,1}\end{aligned}$$

Thus, for eigenvalue $\lambda = 1$, the eigenvector $\mathbf{v}_1 = (v_{1,1}, e^{i\phi} \frac{\sin(\theta)}{(\cos(\theta)+1)} * v_{1,1})$

Likewise, for $\lambda = -1$, let the eigenvector be $\mathbf{v}_2 = (v_{2,1}, v_{2,2})$, thus

$$\begin{aligned}\begin{pmatrix} \cos(\theta) & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos(\theta) \end{pmatrix} \begin{pmatrix} v_{2,1} \\ v_{2,2} \end{pmatrix} &= \begin{pmatrix} -v_{2,1} \\ -v_{2,2} \end{pmatrix} \\ \therefore \cos(\theta) * v_{2,1} + \sin(\theta)e^{-i\phi} * v_{2,2} &= -v_{2,1}, \quad \sin(\theta)e^{i\phi} * v_{2,1} - \cos(\theta) * v_{2,2} = -v_{2,2} \\ v_{2,2} &= e^{i\phi} \frac{\sin(\theta)}{(1 - \cos(\theta))} * v_{2,1}\end{aligned}$$

Thus, for eigenvalue $\lambda = -1$, the eigenvector $\mathbf{v}_2 = (v_{2,1}, e^{i\phi} \frac{\sin(\theta)}{(1 - \cos(\theta))} * v_{2,1})$.

We thus have our two eigenvalues (± 1) and our two eigenvectors (\mathbf{v}_1 and \mathbf{v}_2)

- (b) By rotating the spinor pointing initially along the $+\hat{z}$ axis direction by appropriate angles, using the appropriate rotation operator. Convince yourself that one has to rotate by an angle θ counterclockwise around the y -axis and then by ϕ around the z -axis. Apart from overall phases, is the resultant spinor the same as the spin up eigenvector obtained in part (a)?

Let's start with the spinor pointing in the $+\hat{z}$ -direction.

$$\left| s_z = +\frac{\hbar}{2} \right\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{s.t. } S_z \left| s_z = +\frac{\hbar}{2} \right\rangle = +\frac{\hbar}{2} \left| s_z = +\frac{\hbar}{2} \right\rangle$$

If we apply consecutive rotation operators, we should be able to rotate this spinor into a general state, pointing in an arbitrary direction $\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ points in the (θ, ϕ) direction.

We first rotate this spinor by θ around the y -axis, and then by ϕ around the z -axis. The axis of spin now points in the direction $\hat{\mathbf{n}}$. Thus -

$$|\hat{n}+\rangle = U[R(\phi\hat{z})]U[R(\theta\hat{y})] \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

To find the explicit form of $|\hat{n}+\rangle$, we'll need the forms of the unitary matrices $U[R(\phi\hat{z})]$ and $U[R(\theta\hat{y})]$. We'll use the result given in Shankar -

$$U[R(\theta)] = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}(\hat{\theta} \cdot \boldsymbol{\sigma})$$

Looking at the particular case of rotation around y -axis by amount θ and then subsequently around z -axis by amount ϕ -

$$\begin{aligned} U[R(\theta\hat{y})] \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \left[\cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}\sigma_y \right] \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos\frac{\theta}{2} \\ 0 \end{bmatrix} - i\sin\frac{\theta}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix} \end{aligned}$$

Applying rotation around z -axis by amount ϕ now, we get

$$\begin{aligned} U[R(\phi\hat{z})] \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix} &= \left[\cos\frac{\phi}{2}I - i\sin\frac{\phi}{2}\sigma_z \right] \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix} \\ &= \begin{bmatrix} \cos\frac{\phi}{2}\cos\frac{\theta}{2} \\ \cos\frac{\phi}{2}\sin\frac{\theta}{2} \end{bmatrix} - i\sin\frac{\phi}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix} \\ &= \begin{bmatrix} \cos\frac{\theta}{2} \left(\cos\frac{\phi}{2} - i\sin\frac{\phi}{2} \right) \\ \sin\frac{\theta}{2} \left(\cos\frac{\phi}{2} + i\sin\frac{\phi}{2} \right) \end{bmatrix} \\ &= \begin{bmatrix} \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}} \end{bmatrix} \end{aligned}$$

This gives us a spinor $s_n = (s_{n1}, s_{n2}) = (\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}, \sin\frac{\theta}{2}e^{i\frac{\phi}{2}})$. If we recall our $\mathbf{v}_1 = (v_{1,1}, v_{1,2})$ from part (a), we recall the relation we obtained at the end.

$$v_{1,2} = e^{i\phi} \frac{\sin(\theta)}{(\cos(\theta) + 1)} * v_{1,1}$$

Substituting $v_{1,1} = s_{n1} = \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}$ (as our final spinor seems to suggest), we get -

$$\begin{aligned} v_{1,2} &= e^{i\phi} \frac{\sin(\theta)}{(\cos(\theta) + 1)} * v_{1,1} \\ &= e^{i\phi} \frac{\sin(\theta)}{(\cos(\theta) + 1)} * \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \end{aligned}$$

Recall $1 + \cos(A) = 2 * \cos^2(\frac{A}{2})$ and $\sin(A) = 2 * \sin(\frac{A}{2})\cos(\frac{A}{2})$

$$\begin{aligned}
e^{i\phi} \frac{\sin(\theta)}{(\cos(\theta) + 1)} * \cos \frac{\theta}{2} e^{-i\frac{\phi}{2}} &= e^{i\frac{\phi}{2}} \frac{\sin(\theta)}{2\cos^2(\frac{\theta}{2})} * \cos \frac{\theta}{2} \\
&= e^{i\frac{\phi}{2}} \frac{2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})}{2\cos^2(\frac{\theta}{2})} * \cos \frac{\theta}{2} \\
&= e^{i\frac{\phi}{2}} \sin(\frac{\theta}{2}) = s_{n2}
\end{aligned}$$

Therefore, apart from phase factors, the resultant spinor is the same as the spin up eigenvector we got in part (a).

Question 3.

- (a) Construct the matrices \hat{J}_x and \hat{J}_y for a particle with spin one, $j = 1$ (of course \hat{J}_z is already diagonal with eigenvalues $\hbar, 0, -\hbar$).

We can write the J_x operator as $\frac{J_+ + J_-}{2}$. We can write the matrix elements of this matrix in the $\langle j, m|$ basis as $\langle j, m' | \frac{J_+ + J_-}{2} | j, m \rangle$. Note that this matrix element will vanish if $m' = m$ or $|m' - m| > 1$. This gives us the following matrix for $\frac{J_+ + J_-}{2}$, when the basis elements are $|-1\rangle, |0\rangle, |1\rangle$, in that order.

$$\begin{bmatrix} 0 & a & 0 \\ b & 0 & c \\ 0 & d & 0 \end{bmatrix} \tag{10}$$

Now

$$\begin{aligned}
a &= \langle -1 | J_x | 0 \rangle \\
&= \langle -1 | \frac{J_-}{2} | 0 \rangle \\
&= \langle -1 | \frac{\hbar\sqrt{(1)(1+1) - (0)(0-1)}}{2} | -1 \rangle \\
&= \hbar \frac{\sqrt{2}}{2} \\
&= \frac{\hbar}{\sqrt{2}}
\end{aligned} \tag{11}$$

Now, since the matrix is hermitian, we have the following relation between a and b:

$$\begin{aligned}
b &= a^* \\
\Rightarrow b &= \frac{\hbar}{\sqrt{2}}
\end{aligned} \tag{12}$$

We can perform the same calculation for c :

$$\begin{aligned}
 c &= \langle 0 | J_x | 1 \rangle \\
 &= \langle 0 | \frac{J_-}{2} | 1 \rangle \\
 &= \langle 0 | \frac{\hbar \sqrt{(1)(1+1) - (1)(1-1)}}{2} | 1 \rangle \\
 &= \hbar \frac{\sqrt{2}}{2} \\
 &= \hbar \frac{1}{\sqrt{2}}
 \end{aligned} \tag{13}$$

Again, using the hermiticity argument, we get $d = c = \frac{\hbar}{\sqrt{2}}$. Therefore the final J_x matrix is:

$$\frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{14}$$

Now that we have J_x (and J_z is trivial), we can use the commutator relation to get J_y :

$$[J_x, J_z] = -i\hbar J_y \tag{15}$$

We write $[J_x, J_z]$ as

$$\frac{\hbar^2}{\sqrt{2}} \left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right) \tag{16}$$

With a little algebra we get

$$[J_x, J_z] = -i\hbar J_y = \frac{\hbar^2}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{17}$$

Finally we get

$$J_y = \frac{i\hbar}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{18}$$

- (b) An unpolarized beam of spin 1 particles enters a Stern-Gerlach filter that passes only particles with $S_z = \hbar$. After exiting this filter, the beam enters a second filter that passes particles with $S_x = \hbar$ and then finally it encounters a third filter that passes only particles with $S_z = -\hbar$. What fraction of the initial particles make it right through?

By computing the eigenvectors of the matrix J_y we get the results

$$|\langle S_x = i | S_z = j \rangle|^2 = \frac{1}{3} \quad (19)$$

for $i, j = -1, 0, 1$.

Since the beam is unpolarised, $1/3$ of the particles will pass through the first filter. Again, because of the above result, $1/3$ of the particles will pass through filter 2. Similarly, $1/3$ of these particles will then pass through filter 3. Finally we find that $1/27$ of the particles will pass through the whole set-up.

4. Consider the action of an infinitesimal rotation of magnitude ϵ about the \hat{n} axis of an angular momentum eigenstate $\psi_{l,m}(\theta, \phi)$ (or $|l, m\rangle$). Show that $U(R)\psi_{l,m} = \sum_m D_{m'm}\psi_{l,m'}$ and find the complex numbers $D_{m'm}$.

First, we write down $U(R(\epsilon\hat{n}))$ in terms of familiar operators assuming $\hat{n} = n_x\hat{x} + n_y\hat{y} + n_z\hat{z}$ to get

$$\begin{aligned} U(R(\epsilon\hat{n})) &= \exp\left(-\frac{i\epsilon}{\hbar} \cdot (\hat{\mathbf{J}} \cdot \hat{\mathbf{n}})\right) \\ &= \exp\left(-\frac{i\epsilon}{\hbar} \cdot (n_x\hat{\mathbf{J}}_x + n_y\hat{\mathbf{J}}_y + n_z\hat{\mathbf{J}}_z)\right) \end{aligned} \quad (20)$$

$$\begin{aligned} &= \hat{\mathbf{1}} - \frac{i\epsilon}{\hbar} \cdot (n_x\hat{\mathbf{J}}_x + n_y\hat{\mathbf{J}}_y + n_z\hat{\mathbf{J}}_z) + \mathcal{O}(\epsilon^2) \\ &\approx \hat{\mathbf{1}} - \frac{i\epsilon}{\hbar} \cdot (n_x\hat{\mathbf{J}}_x + n_y\hat{\mathbf{J}}_y + n_z\hat{\mathbf{J}}_z) \end{aligned} \quad (21)$$

Consider the action of this operator on an arbitrary state $|l, m\rangle$

$$U(R(\epsilon\hat{n}))|l, m\rangle = \hat{\mathbf{1}} - \frac{i\epsilon}{\hbar} \cdot (n_x\hat{\mathbf{J}}_x + n_y\hat{\mathbf{J}}_y + n_z\hat{\mathbf{J}}_z)|l, m\rangle. \quad (22)$$

Using the fact that the state is an eigenvector of the $\hat{\mathbf{J}}_z$ operator, and expanding $\hat{\mathbf{J}}_x, \hat{\mathbf{J}}_y$ as their respective forms in terms of the ladder operators, we get

$$\begin{aligned} U(R(\epsilon\hat{n}))|l, m\rangle &= \left(\hat{\mathbf{1}} - \frac{i\epsilon n_z}{\hbar} \cdot \hat{\mathbf{J}}_z\right)|l, m\rangle - \frac{i\epsilon}{\hbar} \cdot (n_x\hat{\mathbf{J}}_x + n_y\hat{\mathbf{J}}_y)|l, m\rangle \\ &= (1 - i\epsilon n_z \cdot m)|l, m\rangle - \frac{i\epsilon}{\hbar} \cdot \left(\frac{in_x + n_y}{2i}\hat{\mathbf{J}}_+ + \frac{in_x - n_y}{2i}\hat{\mathbf{J}}_-\right)|l, m\rangle \\ &= (1 - i\epsilon n_z \cdot m)|l, m\rangle - \frac{i\epsilon}{\hbar} \cdot \left(\frac{in_x + n_y}{2i} \cdot \hbar\sqrt{(l-m)(l+m+1)} \cdot |l, m+1\rangle\right. \\ &\quad \left.+ \frac{in_x - n_y}{2i} \cdot \hbar\sqrt{(l+m)(l-m-1)} \cdot |l, m-1\rangle\right) \end{aligned} \quad (23)$$

or writing it in the required format

$$U(R(\epsilon \hat{n})) |l, m\rangle = \sum_m' D_{m'm} |l, m'\rangle, \quad (24)$$

with

$$D_{m'm} = \begin{cases} 1 - i\epsilon n_z \cdot m & \text{if } m' = m \\ -\frac{\epsilon}{2} \cdot (in_x \pm n_y) \cdot \sqrt{(l \mp m)(l \pm m - 1)} & \text{if } m' = m \pm 1 \\ 0 & \text{otherwise.} \end{cases} \quad (25)$$

5. Prove that any function of the radial coordinate $f(r)$ where $r = |\mathbf{r}|$ and $\mathbf{X} \cdot \mathbf{P}$, where \mathbf{X} and \mathbf{P} are the position and momentum operators, are both scalar operators.

Under a symmetry operator U , operators change as $\mathcal{O}' = U^\dagger \mathcal{O} U$. A scalar operator being one which is invariant under rotations, i.e

$$S' = U^\dagger [R] S U [R] = S$$

where $U(R(\alpha)) = e^{-\frac{i}{\hbar} \alpha \mathbf{J}}$.

By considering infinitesimal rotations $\alpha = \epsilon$, we have

$$U[R(\alpha)] = \left(1 - \frac{i}{\hbar} \epsilon_i J_i\right)$$

Thus, our definition for a scalar operator becomes -

$$S' = \left(1 + \frac{i}{\hbar} \epsilon_i J_i\right) S \left(1 - \frac{i}{\hbar} \epsilon_i J_i\right) = S$$

which gives us $\frac{i}{\hbar} \epsilon_i [J_i, S] = 0$. Since ϵ was an arbitrary choice, we have

$$[J_i, S] = 0$$

as our definition of a scalar operator.

Considering $f(r)$, where $r = |\mathbf{r}|$ as our operator.

$$[J_i, f(r)] = [J_i, r] * f'(r)$$

$r = \sqrt{\sum_{i=1}^3 X_i^2}$, Thus

$$[J_i, r] = [J_i, X_1] * \frac{X_1}{r} + [J_i, X_2] * \frac{X_2}{r} + [J_i, X_3] * \frac{X_3}{r}$$

we know that $[J_i, X_j] = i\hbar\epsilon_{ijl}X_l$. Thus

$$[J_i, r] = [J_i, X_j] * \frac{X_j}{r} = \frac{1}{r} (i\hbar\epsilon_{ijl}X_lX_j)$$

$$\epsilon_{ijl}X_lX_j = [X_l, X_j] = 0 (l \neq j) \Rightarrow [J_i, r] = 0$$

Thus, since $[J_i, r] = 0$, we have $[J_i, f(r)] = [J_i, r] * f'(r) = 0 * f'(r) = 0$. Thus, $f(r)$ is a scalar operator.

Now considering $O = \mathbf{X} \cdot \mathbf{P}$ as our operator, we need to show $[J_i, O] = 0$

$$\mathbf{X} \cdot \mathbf{P} = X_i P_i \quad \text{implicit summation}$$

$$\begin{aligned} \therefore [J_i, O] &= [J_i, X_j P_j] \\ &= [J_i, X_j] P_j + X_j [J_i, P_j] \\ &= i\hbar\epsilon_{ijl}(X_l P_j + X_j P_l) \end{aligned}$$

Now, $\epsilon_{ijl}X_l P_j = [X_l, P_j]$ for $l \neq j$, but $[X_l, P_j] = 0, l \neq j$. Thus

$$i\hbar\epsilon_{ijl}(X_l P_j + X_j P_l) = 0 \Rightarrow [J_i, O] = 0$$

Since $[J_i, O] = 0$, we can say that the operator O is a scalar operator. Thus, $\mathbf{X} \cdot \mathbf{P}$ is a scalar operator

Question 6.

We know that the \mathbf{X}_i operators can be written in terms of the spherical tensor operators as follows: (notation is the same as that used in Shankar, Principles of Quantum Mechanics, 2ed, page 419)

$$\begin{aligned} V_1^{+1} &= \frac{iX_y - X_x}{\sqrt{2}} \\ V_1^0 &= X_z \\ V_1^{-1} &= -\frac{X_x + iX_y}{\sqrt{2}} \end{aligned} \tag{26}$$

Thus in general any linear combination of the \mathbf{X}_i s can be written in terms of the V_1^i s. Note that $\epsilon \cdot \mathbf{X}$ is exactly such a linear combination. Thus we may write

$$\hat{O} = \epsilon \cdot \mathbf{X} = \alpha_i V_1^i \quad (27)$$

Where the α_i are scalars, and summation over repeated values is implied.

Using this form we can write the transition probability for the Hydrogen atom as

$$|\langle n', l', m' | \alpha_i V_1^i | n, l, m \rangle| \quad (28)$$

Now since each V_1^i , acting on $|n, l, m\rangle$ can either:

- Increase the value of l by 1
- Decrease the value of l by 1
- Keep the value of l unchanged

Or give a superposition of the above. Since states of different l are orthogonal, $\alpha_i V_1^i |n, l, m\rangle$ and $\langle n', l', m' |$ won't have any common terms unless $|l - l'| = 1$ or $l = l'$.

Thus we get the relation

$$|\langle n', l', m' | \alpha_i V_1^i | n, l, m \rangle| = 0 \quad (29)$$

Unless $|l - l'| = 1$ or $l = l'$.

$$|\langle n', l, m' | \epsilon \cdot \mathbf{X} | n, l, m \rangle| = |\langle n', l', m' | P^\dagger \epsilon \cdot \mathbf{X} P | n, l, m \rangle| \quad (30)$$

Since $|n, l, m\rangle$ transforms as $|n, l, m\rangle \longrightarrow (-1)^l |n, l, m\rangle$ under parity,

$$(-1)^{l'+l} \langle n', l', m' | \epsilon \cdot \mathbf{X} | n, l, m \rangle = \langle n', l', m' | P^\dagger \epsilon \cdot \mathbf{X} P | n, l, m \rangle \quad (31)$$

Since \mathbf{X} transforms as $\mathbf{X} \longrightarrow -\mathbf{X}$ under parity, we get

$$(-1)^{l'+l} \langle n', l', m' | \epsilon \cdot \mathbf{X} | n, l, m \rangle = -\langle n', l', m' | \epsilon \cdot \mathbf{X} | n, l, m \rangle \quad (32)$$

Hence if $l + l'$ is even (i.e. when $l = l'$), we get

$$\langle n', l', m' | \epsilon \cdot \mathbf{X} | n, l, m \rangle = 0 \quad (33)$$