Lie monoids

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1 Lie monoids

1.1 Lie monoids as modules

Recall the Lie operad $\mathbf{Lie}$.

A Lie monoid is a module over the Lie operad.

A morphism of Lie monoids is a morphism of modules.

We denote the category of Lie monoids by $\mathrm{LieMon}(\mathcal{A}\text{-}\mathbb{S}\mathrm{p})$.

Explicitly: A Lie monoid is a species g equipped with linear maps

$$\gamma_{\mathrm{X}}^{\mathrm{Y}} : \mathbf{Lie}[\mathrm{X}, \mathrm{Y}] \otimes \mathrm{g}[\mathrm{Y}] \to \mathrm{g}[\mathrm{X}],$$

one for each $\mathrm{X} \leq \mathrm{Y}$, which satisfy the following axioms.

Associativity. *For any* $\mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}$*, the diagram*

(1a)

$$\begin{tikzcd}
\mathbf{Lie}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{Lie}[\mathrm{Y}, \mathrm{Z}] \otimes \mathrm{g}[\mathrm{Z}] \arrow[r, "\mathrm{id} \otimes \gamma"] \arrow[d, "\gamma \otimes \mathrm{id}"'] & \mathbf{Lie}[\mathrm{X}, \mathrm{Y}] \otimes \mathrm{g}[\mathrm{Y}] \arrow[d, "\gamma"] \\
\mathbf{Lie}[\mathrm{X}, \mathrm{Z}] \otimes \mathrm{g}[\mathrm{Z}] \arrow[r, "\gamma"'] & \mathrm{g}[\mathrm{X}]
\end{tikzcd}$$

commutes.

Unitality. *For any* X*, the diagram*

(1b)

$$\begin{tikzcd}
& \mathbf{Lie}[\mathrm{X}, \mathrm{X}] \otimes \mathrm{g}[\mathrm{X}] \arrow[dr, "\gamma"] & \\
\Bbbk \otimes \mathrm{g}[\mathrm{X}] \arrow[ur, "\eta \otimes \mathrm{id}"] \arrow[rr, "\cong"'] & & \mathrm{g}[\mathrm{X}]
\end{tikzcd}$$

commutes.

A morphism $f : \mathsf{g} \to \mathsf{h}$ of Lie monoids is a map of species such that for any $\mathrm{X} \leq \mathrm{Y}$, the diagram

$$
\begin{array}{ccc}
\mathbf{Lie}[\mathrm{X},\mathrm{Y}] \otimes \mathsf{g}[\mathrm{Y}] & \xrightarrow{\gamma} & \mathsf{g}[\mathrm{X}] \\
{\scriptstyle \mathrm{id} \otimes f_{\mathrm{Y}}} \downarrow & & \downarrow {\scriptstyle f_{\mathrm{X}}} \\
\mathbf{Lie}[\mathrm{X},\mathrm{Y}] \otimes \mathsf{h}[\mathrm{Y}] & \xrightarrow[\gamma]{} & \mathsf{h}[\mathrm{X}]
\end{array}
\tag{2}
$$

commutes.

1.2 Lie monoids by generators and relations

A Lie monoid is a species g equipped with the following structure.

For each $A \lessdot F$, there is a linear map

$$\nu_A^F : \mathrm{g}[F] \to \mathrm{g}[A].$$

We call this map the Lie bracket.

It is subject to the following axioms.

Naturality. *For each morphism $\beta_{B,A} : A \to B$, and $A \lessdot F$, the diagram*

(3a)
$$\begin{array}{ccc} \mathrm{g}[F] & \xrightarrow{\beta_{BF,F}} & \mathrm{g}[BF] \\ {\scriptstyle \nu_A^F}\big\downarrow & & \big\downarrow{\scriptstyle \nu_B^{BF}} \\ \mathrm{g}[A] & \xrightarrow[\beta_{B,A}]{} & \mathrm{g}[B] \end{array}$$

commutes.

Antisymmetry. *For each $A \lessdot F$,*

(3b)
$$(\mathrm{g}[F] \xrightarrow{\nu_A^F} \mathrm{g}[A]) + (\mathrm{g}[F] \xrightarrow{\beta_{A\overline{F},F}} \mathrm{g}[A\overline{F}] \xrightarrow{\nu_A^{A\overline{F}}} \mathrm{g}[A]) = 0.$$

Jacobi identity. *For each* $\mathrm{s}(A) \leq \mathrm{X}$ *such that the face* A *has codimension two in the flat* X*,*

(3c)

$$\sum_{i=1}^{n} \left(\mathrm{g}[G_1] \xrightarrow{\beta_{G_i,G_1}} \mathrm{g}[G_i] \xrightarrow{\nu_{F_i}^{G_i}} \mathrm{g}[F_i] \xrightarrow{\nu_{A}^{F_i}} \mathrm{g}[A]\right) = 0.$$

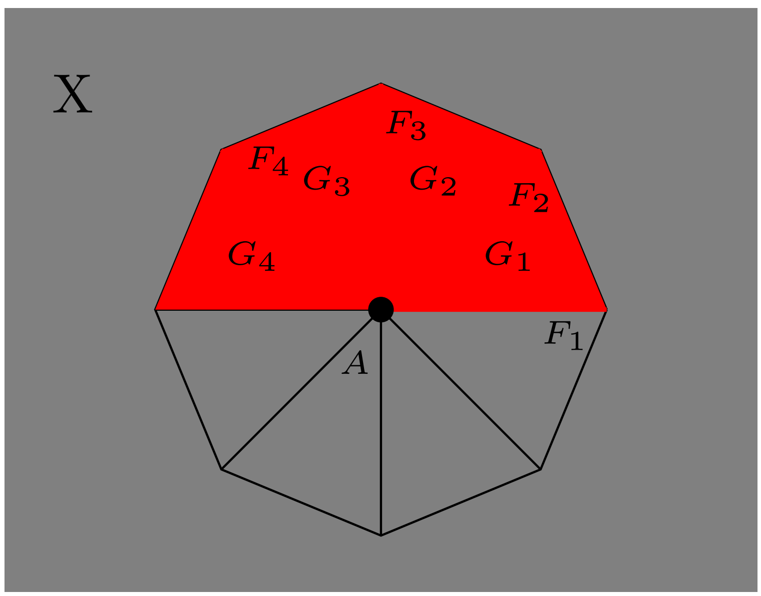
The integer n *and the faces* F_i *and* G_i *greater than* A *are obtained as follows.*

Since X/A *is a rank-two flat of* $\mathcal{A}_A$*, its spherical model is a* $2n$*-gon for some* n*. Pick any* $G_1 \gtrdot F_1 \gtrdot A$*. This then yields a unique gallery*

$$G_1 \text{ -- } G_2 \text{ -- } \dots \text{ -- } G_n$$

consisting of distinct faces greater than A *and of support* X *with* $F_2 \neq F_1$*, where* F_i *is the common panel of* G_{i-1} *and* G_i *for* $2 \leq i \leq n$*.*

An illustration for $n = 4$ is shown below.



Note very carefully that the G_i only sweep out half of the polygon.

A morphism $f : \mathsf{g} \to \mathsf{h}$ of Lie monoids is a map of species such that for each $A \lessdot F$, the diagram

$$
\begin{array}{ccc}
\mathsf{g}[F] & \xrightarrow{f_F} & \mathsf{h}[F] \\
{\scriptstyle \nu_A^F}\big\downarrow & & \big\downarrow{\scriptstyle \nu_A^F} \\
\mathsf{g}[A] & \xrightarrow[f_A]{} & \mathsf{h}[A]
\end{array}
\qquad (4)
$$

commutes.

1.3 Back and forth

The two viewpoints on Lie monoids discussed above are equivalent.

This follows from the presentation of the Lie operad.

We elaborate on the passage from the first to the second.

Suppose g is a Lie monoid defined as a module with structure maps $\gamma_{\mathrm{X}}^{\mathrm{Y}}$ for $\mathrm{X} \leq \mathrm{Y}$.

The Lie bracket ν_A^F for $A \lessdot F$ is constructed as follows.

Let X and Y be the supports of A and F, respectively. Observe that the element $\mathtt{H}_{F/A} - \mathtt{H}_{A\overline{F}/A}$ defines an element in $\mathbf{Lie}[\mathrm{X}, \mathrm{Y}]$.

Now, define

(5)

$$\nu_A^F(v) := \beta_{A,\mathrm{X}} \gamma_{\mathrm{X}}^{\mathrm{Y}} \big((\mathtt{H}_{F/A} - \mathtt{H}_{A\overline{F}/A}) \otimes \beta_{\mathrm{Y},F}(v) \big), \quad v \in \mathtt{g}[F]$$

2 Commutator bracket and primitive part of bimonoids

2.1 Commutator bracket

Recall that the Lie operad $\mathbf{Lie}$ is a suboperad of the associative operad $\mathbf{As}$.

Hence, there is an induced functor

$$\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp}) \to \mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp}). \tag{6}$$

We call this the underlying Lie monoid functor.

In particular, every monoid a carries the structure of a Lie monoid given by the composite

$$\mathbf{Lie} \circ \mathsf{a} \to \mathbf{As} \circ \mathsf{a} \to \mathsf{a}.$$

Explicitly:

Let μ_A^F, one for each $A \leq F$, denote the product components of a.

Then for any $\mathrm{X} \leq \mathrm{Y}$, the Lie structure map $\gamma_{\mathrm{X}}^{\mathrm{Y}}$ of a is given by

(7)

$$\sum_{\substack{F\colon F \geq A \\ \mathrm{s}(F)=\mathrm{Y}}} x^{F/A}\, \mathtt{H}_{F/A} \otimes v \longmapsto \sum_{\substack{F\colon F \geq A \\ \mathrm{s}(F)=\mathrm{Y}}} x^{F/A}\, \beta_{\mathrm{X},A} \mu_A^F \beta_{F,\mathrm{Y}}(v),$$

where A is any fixed face with $\mathrm{s}(A) = \mathrm{X}$.

Equivalently, one can work with the formulation of a monoid a given by covering generators.

This only involves product components μ_A^F for $A \lessdot F$.

In this situation, one can describe the Lie structure of a by specifying its Lie bracket as follows.

For $A \lessdot F$, the Lie bracket ν_A^F of a is given by

$$\nu_A^F = \mu_A^F - \mu_A^{A\overline{F}} \beta_{A\overline{F},F}. \tag{8}$$

We refer to (8) as the commutator bracket.

2.2 Abelian Lie monoids

We say that a Lie monoid is abelian if the structure maps $\gamma_{\mathrm{X}}^{\mathrm{Y}}$ are zero for all $\mathrm{X} < \mathrm{Y}$.

Equivalently, a Lie monoid is abelian if all Lie brackets ν_A^F are 0.

For a commutative monoid, the commutator bracket (8) is identically zero.

This follows from the commutativity axiom.

Hence the underlying Lie structure of a commutative monoid is abelian.

2.3 Primitive part of a bimonoid

Let h be a bimonoid.

In particular, it is a monoid, and hence a Lie monoid via the functor (6).

We now show that this Lie structure restricts to its primitive part $\mathcal{P}(\mathsf{h})$.

Proposition 1. *If* h *is a bimonoid, then* $\mathcal{P}(\mathsf{h})$ *is a Lie submonoid of* h*.*

We give proofs using each of the two formulations of a Lie monoid.

First proof. For any $\mathrm{X} \leq \mathrm{Y}$, we need to show that the dotted arrow in the diagram

$$\begin{tikzcd}
\mathbf{Lie}[\mathrm{X},\mathrm{Y}] \otimes \mathcal{P}(\mathsf{h})[\mathrm{Y}] \arrow[r, hook] \arrow[d, dashed] & \mathbf{As}[\mathrm{X},\mathrm{Y}] \otimes \mathsf{h}[\mathrm{Y}] \arrow[d] \\
\mathcal{P}(\mathsf{h})[\mathrm{X}] \arrow[r, hook] & \mathsf{h}[\mathrm{X}]
\end{tikzcd}$$

exists.

The map going across and down is described in (7).

We need to show that its image belongs to $\mathcal{P}(\mathsf{h})[\mathrm{X}]$.

Explicitly, we need to show that for any $G > A$, any Lie element

$$\sum_{F\colon F \geq A,\, \mathrm{s}(F)=\mathrm{Y}} x^{F/A}\, \mathtt{H}_{F/A}$$

(with A of support X) and $v \in \mathcal{P}(\mathsf{h})[\mathrm{Y}]$,

$$\Delta_A^G\Big(\sum_{F\colon F \geq A,\, \mathrm{s}(F)=\mathrm{Y}} x^{F/A} \mu_A^F(\beta_{F,\mathrm{Y}}(v))\Big) = 0.$$

By the bimonoid axiom, we need to show that

(9)
$$\sum_{F\colon F \geq A,\, \mathrm{s}(F)=\mathrm{Y}} x^{F/A} \mu_G^{GF} \beta_{GF,FG} \Delta_F^{FG} \beta_{F,\mathrm{Y}}(v) = 0.$$

There are two cases.

- $\mathrm{s}(G) \not\leq \mathrm{Y}$, or equivalently, $FG > F$ for all $F \geq A$ with $\mathrm{s}(F) = \mathrm{Y}$.

  In this case, since $\beta_{F,\mathrm{Y}}(v)$ is a primitive element in h$[F]$, applying Δ_F^{FG} yields 0.

- $\mathrm{s}(G) \leq \mathrm{Y}$, or equivalently, $FG = F$ for all $F \geq A$ with $\mathrm{s}(F) = \mathrm{Y}$.

  In this case, $\Delta_F^{FG} = \mathrm{id}$.

Thus,

$$\begin{aligned}
\text{lhs of (9)} &= \sum_{F\colon F \geq A,\, \mathrm{s}(F)=\mathrm{Y}} x^{F/A} \mu_G^{GF} \beta_{GF,\mathrm{Y}}(v) \\
&= \sum_{H\colon H \geq G,\, \mathrm{s}(H)=\mathrm{Y}} \Bigg(\sum_{\substack{F\colon F \geq A,\, \mathrm{s}(F)=\mathrm{Y} \\ GF=H}} x^{F/A} \Bigg) \mu_G^H \beta_{H,\mathrm{Y}}(v) \\
&= 0.
\end{aligned}$$

In the second step, we introduced a new variable $H = GF$. The sum in parenthesis is zero by the definition of a Lie element applied to the arrangement $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$.

This completes the proof. $\square$

Second proof. For any $A \lessdot F$, let ν_A^F be the commutator bracket on h defined by (8).

We need to check that it restricts to $\mathcal{P}(\mathsf{h})$, that is, the dotted arrow in the diagram

$$\begin{tikzcd}
\mathcal{P}(\mathsf{h})[F] \arrow[r, hook] \arrow[d, dashed] & \mathsf{h}[F] \arrow[d, "\nu_A^F"] \\
\mathcal{P}(\mathsf{h})[A] \arrow[r, hook] & \mathsf{h}[A]
\end{tikzcd}$$

exists.

Accordingly, let $v \in \mathcal{P}(\mathsf{h})[F]$.

We want to show that $\nu_A^F(v) \in \mathcal{P}(\mathsf{h})[A]$, that is, $\Delta_A^G(\nu_A^F(v)) = 0$, for any face $G \gtrdot A$.

There are two cases.

- $G \neq F$ and $G \neq A\overline{F}$.

  In this case, since FG is strictly greater than F, $A\overline{F}G$ is strictly greater than $A\overline{F}$, and v is primitive, we deduce from the bimonoid axiom that

$$\Delta_A^G(\mu_A^F(v)) = 0 = \Delta_A^G(\mu_A^{A\overline{F}}\beta_{A\overline{F},F}(v)).$$

- Either $G = F$ or $G = A\overline{F}$.

  In this case,

$$\Delta_A^G(\mu_A^F(v)) = \beta_{G,F}(v) = \Delta_A^G(\mu_A^{A\overline{F}}\beta_{A\overline{F},F}(v)).$$

Thus, in both cases,

$$\Delta_A^G(\nu_A^F(v)) = \Delta_A^G(\mu_A^F(v)) - \Delta_A^G(\mu_A^{A\overline{F}}\beta_{A\overline{F},F}(v)) = 0$$

as required. $\square$

Proposition 1 yields a functor

$$\mathcal{P} : \mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \to \mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp}) \tag{10}$$

from the category of bimonoids to the category of Lie monoids.

We continue to call it the primitive part functor.

3 Free Lie monoids on species

3.1 Free Lie monoid on a species

Proposition 2. *The free Lie monoid on a species* p *is given by* $\mathbf{Lie} \circ \mathsf{p}$*. Explicitly,*

$$(\mathbf{Lie} \circ \mathsf{p})[\mathrm{X}] = \bigoplus_{\mathrm{Y}\colon \mathrm{Y} \geq \mathrm{X}} \mathbf{Lie}[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{p}[\mathrm{Y}].$$

Proof. The free module over any operad $\mathbf{a}$ on a species p is given by $\mathbf{a} \circ \mathsf{p}$.

Now specialize $\mathbf{a}$ to the Lie operad. $\square$

The universal property of the free Lie monoid is stated below.

Let $\mathrm{p} \hookrightarrow \mathbf{Lie} \circ \mathrm{p}$ be the canonical inclusion.

Theorem 1. *Let* g *be a Lie monoid,* p *a species,* $f : \mathrm{p} \to \mathrm{g}$ *a map of species. Then there exists a unique morphism of Lie monoids* $\hat{f} : \mathbf{Lie} \circ \mathrm{p} \to \mathrm{g}$ *such that the diagram*

$$\begin{tikzcd} \mathbf{Lie} \circ \mathrm{p} \arrow[r, dashed, "\hat{f}"] & \mathrm{g} \\ \mathrm{p} \arrow[u, hook] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes.

Explicitly, the map $\hat{f}$ is given by

$$\mathbf{Lie} \circ \mathrm{p} \xrightarrow{\mathrm{id} \circ f} \mathbf{Lie} \circ \mathrm{g} \xrightarrow{\gamma} \mathrm{g}.$$

3.2 Primitive part of $\mathcal{T}(\mathsf{p})$

To any species p, one can associate the cocommutative bimonoid $\mathcal{T}(\mathsf{p})$. Let us recall this construction.

The A-component is defined by

$$\mathcal{T}(\mathsf{p})[A] := \bigoplus_{F\colon F \geq A} \mathsf{p}[F]. \tag{11}$$

The product component μ_A^G is given by

(12)

$$\bigoplus_{K\colon K \geq G} \mathsf{p}[K] \xrightarrow{\mu_A^G} \bigoplus_{F\colon F \geq A} \mathsf{p}[F], \qquad \mathsf{p}[K] \xrightarrow{\cong} \mathsf{p}[K],$$

while the coproduct component Δ_A^G is given by

(13)

$$\bigoplus_{F\colon F \geq A} \mathsf{p}[F] \xrightarrow{\Delta_A^G} \bigoplus_{K\colon K \geq G} \mathsf{p}[K], \quad \mathsf{p}[F] \xrightarrow{\beta_{GF,F}} \begin{cases} \mathsf{p}[GF] & \text{if } FG = F, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathcal{T}(\mathsf{p})$ is the free monoid on p, we can identify it with $\mathbf{As} \circ \mathsf{p}$ as follows.

$$\mathbf{As} \circ \mathsf{p} \xrightarrow{\cong} \mathcal{T}(\mathsf{p})$$

Evaluating on the X-component, on the Y-summand for $\mathrm{Y} \geq \mathrm{X}$, the map is given by

(14)

$$\mathbf{As}[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{p}[\mathrm{Y}] \longrightarrow \bigoplus_{\substack{F\colon F \geq A \\ \mathrm{s}(F) = \mathrm{Y}}} \mathsf{p}[F]$$

$$\Big(\sum_{\substack{F\colon F \geq A \\ \mathrm{s}(F) = \mathrm{Y}}} x^{F/A} \mathtt{H}_{F/A}\Big) \otimes v \longmapsto \sum_{\substack{F\colon F \geq A \\ \mathrm{s}(F) = \mathrm{Y}}} x^{F/A} \beta_{F,\mathrm{Y}}(v).$$

(Here A is an arbitrary but fixed face of support X.)

Proposition 3. *For any species* p*,*

$$\mathcal{PT}(\mathsf{p}) = \mathbf{Lie} \circ \mathsf{p}. \tag{15}$$

Proof. Let us compute the primitive part of $\mathcal{T}(\mathsf{p})[A]$.

Rewriting the coproduct formula (13),

$$\Delta_A^G\big((v_F)_{F\geq A}\big) = \sum_{K\colon K\geq G} \sum_{\substack{F\colon F\geq A \\ GF=K,\, \overline{F}G=F}} \beta_{K,F}(v_F).$$

It follows that the subspace

$\mathcal{PT}(\mathsf{p})[A] = \bigcap_{G>A} \ker(\Delta_A^G)$ consists of elements

$(v_F)_{F\geq A}$ such that $\displaystyle\sum_{\substack{F\colon F\geq A \\ GF=K,\, \overline{F}G=F}} \beta_{K,F}(v_F) = 0$ for all $K \geq G > A$.

Further, from the identification (14) and the definition of a Lie element applied to the arrangements $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$, we note that this subspace coincides with the image of the injective map

$$\bigoplus_{\mathrm{Y}\colon \mathrm{Y}\geq \mathrm{X}} \mathbf{Lie}[\mathrm{X},\mathrm{Y}]\otimes \mathsf{p}[\mathrm{Y}] \hookrightarrow \bigoplus_{\mathrm{Y}\colon \mathrm{Y}\geq \mathrm{X}} \mathbf{As}[\mathrm{X},\mathrm{Y}]\otimes \mathsf{p}[\mathrm{Y}] \xrightarrow{\cong} \bigoplus_{F\colon F\geq A} \mathsf{p}[F$$

where $\mathrm{X} = \mathrm{s}(A)$. $\square$

4 Universal enveloping monoids

4.1 A/A-relations

Suppose g is a Lie monoid. Consider the diagram

$$
\begin{tikzcd}
\mathbf{Lie} \circ \mathrm{g} \arrow[r] \arrow[d, hook] & \mathrm{g} \arrow[d, hook] \\
\mathbf{As} \circ \mathrm{g} \arrow[r, "\cong"'] & \mathcal{T}(\mathrm{g}).
\end{tikzcd}
\tag{16}
$$

The species $\mathcal{T}(\mathrm{g})$ is as in (11).

The top-horizontal map is the Lie structure map of g.

The vertical maps are the canonical inclusions.

The bottom isomorphism is the identification (14). (This map is defined for any species g.)

The diagram (16) does not commute in general.

Let us evaluate it on a flat X.

Consider the summand of $(\mathbf{Lie} \circ \mathrm{g})[\mathrm{X}]$ corresponding to a flat $\mathrm{Y} \geq \mathrm{X}$.

Pick any element

$$\sum_{F:\, F \geq A,\, \mathrm{s}(F)=\mathrm{Y}} x^{F/A}\, \mathtt{H}_{F/A} \otimes v \ \in \ \mathbf{Lie}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{g}[\mathrm{Y}],$$

where A is a fixed face of support X.

Apply the map going down and across to this element, and also the map going across and down, and take their difference. This equals

(17)
$$\sum_{\substack{F\colon F\geq A,\\ \mathrm{s}(F)=\mathrm{Y}}} x^{F/A}\,\beta_{F,\mathrm{Y}}(v)-\beta_{A,\mathrm{X}}\gamma_{\mathrm{X}}^{\mathrm{Y}}\Big(\sum_{\substack{F\colon F\geq A,\\ \mathrm{s}(F)=\mathrm{Y}}} x^{F/A}\,\mathtt{H}_{F/A}\otimes v\Big).$$

This is an element of

$$\Big(\bigoplus_{F\colon F\geq A,\,\mathrm{s}(F)=\mathrm{Y}} \mathrm{g}[F]\Big)+\mathrm{g}[A]\subseteq \mathcal{T}(\mathrm{g})[A]. \tag{18}$$

We call this a A/A-relation and say that it is obtained from the flat Y/A.

By definition, the quotient of $\mathcal{T}(\mathrm{g})[A]$ by the A/A-relations is the A-component of the coequalizer of $\mathbf{Lie}\circ\mathrm{g}\rightrightarrows\mathcal{T}(\mathrm{g})$, where the arrows are obtained by following the two directions in diagram (16).

4.2 A/A-relations from rank-zero and rank-one flats

Let us explicitly look at some A/A-relations.

Suppose the flat Y/A has rank zero, that is, $\mathrm{Y} = \mathrm{s}(A)$.

Then the A/A-relation obtained from Y/A is 0:

In this case, $\mathbf{Lie}[\mathrm{Y}, \mathrm{Y}] = \Bbbk$, spanned by $\mathtt{H}_{A/A}$; so (17) becomes

$$\beta_{A,\mathrm{Y}}(v - \gamma^{\mathrm{Y}}_{\mathrm{Y}}(\mathtt{H}_{A/A} \otimes v)) = 0.$$

So nonzero A/A-relations arise only from flats of nonzero rank, in which case the sum in (18) is direct.

Suppose the flat Y/A has rank one, and it supports the vertices F/A and $A\overline{F}/A$.

Then the A/A-relations obtained from Y/A are

(19)

$$\beta_{F,\mathrm{Y}}(v) - \beta_{A\overline{F},\mathrm{Y}}(v) - \beta_{A,\mathrm{X}}\gamma_{\mathrm{X}}^{\mathrm{Y}}\big((\mathtt{H}_{F/A} - \mathtt{H}_{A\overline{F}/A}) \otimes v\big), \quad v \in \mathrm{g}[\mathrm{Y}]$$

with $\mathrm{X} := \mathrm{s}(A)$.

These are elements of

$$\mathrm{g}[F] \oplus \mathrm{g}[A\overline{F}] \oplus \mathrm{g}[A] \subseteq \mathcal{T}(\mathrm{g})[A].$$

4.3 H/A-relations

For $H \geq A$, a H/A-relation is the image of a H/H-relation under the inclusion map

$$\mathcal{T}(\mathrm{g})[H] \hookrightarrow \mathcal{T}(\mathrm{g})[A].$$

If the relation is obtained from the flat Y/H, then it is an element of

$$\left(\bigoplus_{F\colon F \geq H,\, \mathrm{s}(F) = \mathrm{Y}} \mathrm{g}[F] \right) + \mathrm{g}[H] \subseteq \mathcal{T}(\mathrm{g})[A]. \tag{20}$$

For a H/A-relation to be nonzero, Y/H must have rank at least one, that is, $\mathrm{s}(H) < \mathrm{Y}$, in which case the sum in (20) is direct.

A H/A-relation is given exactly as in (17), with H instead of A.

Also note that if $\mathrm{s}(H) < \mathrm{Y}$, then the sum of the scalars $x^{F/H}$ appearing in the relation is zero.

4.4 A submonoid of the free monoid

For a Lie monoid g, define $\mathcal{I}(\mathrm{g})$ to be the smallest submonoid of $\mathcal{T}(\mathrm{g})$ such that its A-component contains all A/A-relations.

Explicitly, $\mathcal{I}(\mathrm{g})[A]$ is the linear span of all H/A-relations as H varies over all faces greater than A.

This is because, by (12), the product component μ_A^H of $\mathcal{T}(\mathrm{g})$ is precisely the inclusion of $\mathcal{T}(\mathrm{g})[H]$ in $\mathcal{T}(\mathrm{g})[A]$.

The significance of the relations (19) is brought out by the following result.

Lemma 1. *For* g *a Lie monoid,* $\mathcal{I}(\mathrm{g})$ *is the smallest submonoid of* $\mathcal{T}(\mathrm{g})$ *such that its* A*-component contains all* A/A*-relations obtained from rank-one flats* Y/A*.*

The key fact is that $\mathbf{Lie}$ is a binary quadratic operad.

We omit the details.

Lemma 2. *Let* g *be a Lie monoid. Then* $\mathcal{I}(\mathrm{g})$ *is the smallest submonoid of* $\mathcal{T}(\mathrm{g})$ *such that its* A*-component contains all elements of the form*

$$v - \beta_{A\overline{F},F}(v) - \nu_A^F(v), \quad v \in \mathrm{g}[F] \tag{21}$$

for $A \lessdot F$*.*

This is a reformulation of Lemma 1 in terms of the Lie bracket.

The elements (21) are the same as those in (19) arising from rank-one flats Y/A.

4.5 Universal enveloping monoid

Define $\mathcal{U}(\mathrm{g})$ to be the quotient of $\mathcal{T}(\mathrm{g})$ by the submonoid $\mathcal{I}(\mathrm{g})$. It follows that $\mathcal{U}(\mathrm{g})$ is a monoid.

We call this the universal enveloping monoid of the Lie monoid g.

It is the largest monoid-quotient of $\mathcal{T}(\mathrm{g})$ such that the outside square in the diagram

$$\begin{tikzcd} \mathbf{Lie} \circ \mathrm{g} \arrow[r] \arrow[d, hook] & \mathrm{g} \arrow[d, hook] \arrow[ddr, bend left] & \\ \mathbf{As} \circ \mathrm{g} \arrow[r] \arrow[dr, bend right] & \mathcal{T}(\mathrm{g}) \arrow[dr, two heads] & \\ & & \mathcal{U}(\mathrm{g}) \end{tikzcd} \tag{22}$$

commutes. We emphasize that $\mathcal{U}(\mathrm{g})$ is not the coequalizer of the two arrows $\mathbf{Lie} \circ \mathrm{g} \rightrightarrows \mathcal{T}(\mathrm{g})$ obtained from the inside square. We need to take quotient by all H/A-relations since we want $\mathcal{U}(\mathrm{g})$ to be a monoid.

Equivalently, in terms of the Lie bracket of g, by Lemma 2, we have:

Lemma 3. *For a Lie monoid* g*, the monoid* $\mathcal{U}(\mathrm{g})$ *is the quotient of* $\mathcal{T}(\mathrm{g})$ *by the submonoid of relations generated by*

$$v - \beta_{A\overline{F},F}(v) - \nu_A^F(v), \quad v \in \mathrm{g}[F] \tag{23}$$

for $A \lessdot F$*.*

The construction of $\mathcal{U}(\mathrm{g})$ is functorial in g:

Any morphism of Lie monoids $\mathrm{g} \to \mathrm{g}'$ preserves H/A-relations, and hence induces a morphism of monoids $\mathcal{U}(\mathrm{g}) \to \mathcal{U}(\mathrm{g}')$.

In other words, there is a commutative diagram of monoids

$$
\begin{array}{ccc}
\mathcal{T}(\mathrm{g}) & \longrightarrow & \mathcal{T}(\mathrm{g}') \\
\big\downarrow & & \big\downarrow \\
\mathcal{U}(\mathrm{g}) & \longrightarrow & \mathcal{U}(\mathrm{g}').
\end{array}
\tag{24}
$$

In particular, we have a functor

$$\mathcal{U} : \mathsf{LieMon}(\mathcal{A}\text{-}\mathrm{Sp}) \to \mathsf{Mon}(\mathcal{A}\text{-}\mathrm{Sp}).$$

4.6 Adjunction with the underlying Lie monoid functor

Theorem 2. *The functor $\mathcal{U}$ is the left adjoint of the underlying Lie monoid functor* (6)*. Explicitly, for any Lie monoid* g *and monoid* a*, there is a natural bijection*

(25)
$$\mathsf{Mon}(\mathcal{A}\text{-}\mathrm{Sp})(\mathcal{U}(\mathrm{g}), \mathsf{a}) \xrightarrow{\cong} \mathsf{LieMon}(\mathcal{A}\text{-}\mathrm{Sp})(\mathrm{g}, \mathsf{a}).$$

The unit of the adjunction (25) is

(26)
$$i : \mathrm{g} \to \mathcal{U}(\mathrm{g}).$$

It is a morphism of Lie monoids. It equals the composite map $\mathrm{g} \hookrightarrow \mathcal{T}(\mathrm{g}) \twoheadrightarrow \mathcal{U}(\mathrm{g})$.

The adjunction in Theorem 2 is reformulated below as a universal property.

Theorem 3. *Let* a *be a monoid,* g *a Lie monoid,* $f : \mathrm{g} \to \mathsf{a}$ *a morphism of Lie monoids. Then there exists a unique morphism of monoids* $\hat{f} : \mathcal{U}(\mathrm{g}) \to \mathsf{a}$ *such that the diagram*

$$\begin{tikzcd} \mathcal{U}(\mathrm{g}) \arrow[r, dashed, "\hat{f}"] & \mathsf{a} \\ \mathrm{g} \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes.

4.7 Abelian Lie monoids

If a Lie monoid g is abelian, then the monoid $\mathcal{I}(\mathrm{g})$ can be explicitly described as follows.

Lemma 4. *For* g *an abelian Lie monoid, the submonoid* $\mathcal{I}(\mathrm{g})$ *of* $\mathcal{T}(\mathrm{g})$*, evaluated on the* A*-component, is linearly spanned by elements of the form* $z - \beta_{G,F}(z)$*, with* $z \in \mathrm{g}[F]$*, and* F *and* G *both greater than* A *and of the same support.*

Proof. First consider $z - \beta_{G,F}(z)$, where in addition, F and G are adjacent. Let H denote their common panel.

Then $z - \beta_{G,F}(z)$ is an instance of the relation (19), with H instead of A, and G instead of $A\overline{F}$, so it belongs to $\mathcal{I}(\mathrm{g})[A]$.

In general, by connecting F and G by a gallery, we deduce that $z - \beta_{G,F}(z)$ belongs to $\mathcal{I}(\mathrm{g})[A]$.

For the reverse containment, we note that any nonzero H/A-relation arising from the flat Y/H has the form

$$\sum_{F\colon F \geq H,\, \mathrm{s}(F)=\mathrm{Y}} x^{F/H}\, \beta_{F,\mathrm{Y}}(v),$$

where $v \in \mathrm{g}[\mathrm{Y}]$ and the sum of the $x^{F/H}$ is zero.

So it can be expressed as a linear combination of the given elements.

This completes the argument. $\square$

We deduce from Lemma 4 that:

Proposition 4. *For* g *an abelian Lie monoid, we have* $\mathcal{U}(\mathrm{g}) = \mathcal{S}(\mathrm{g})$*, the free commutative monoid on* g *(with* g *viewed as a species), with the canonical quotient* $\mathcal{T}(\mathrm{g}) \twoheadrightarrow \mathcal{U}(\mathrm{g})$ *being the abelianization map.*

4.8 Bimonoid structure

So far $\mathcal{U}(\mathrm{g})$ has only been considered as a monoid. Now we equip it with a bimonoid structure.

Proposition 5. *There is a unique cocommutative bimonoid structure on* $\mathcal{U}(\mathrm{g})$ *such that the quotient map* $\mathcal{T}(\mathrm{g}) \twoheadrightarrow \mathcal{U}(\mathrm{g})$ *is a morphism of cocommutative bimonoids.*

Proof. We need to show that the coproduct component $\Delta_A^G : \mathcal{T}(\mathrm{g})[A] \to \mathcal{T}(\mathrm{g})[G]$ given in (13) preserves relations.

Accordingly, let u be a nonzero H/A-relation obtained from the flat Y/H.

So it is an element of (20).

There are three cases, namely,

$$\mathrm{s}(G) \not\leq \mathrm{Y}, \quad \mathrm{s}(G) \leq \mathrm{Y},\, HG = H, \quad \mathrm{s}(G) \leq \mathrm{Y},\, HG \neq H.$$

In the first case, clearly $\Delta_A^G(u) = 0$.

In the second case, Δ_A^G identifies g$[H]$ with g$[GH]$, and g$[F]$ with g$[GF]$.

So $\Delta_A^G(u)$ belongs to

$$\left(\bigoplus_{F\colon F\geq H,\, \mathrm{s}(F)=\mathrm{Y}} \mathrm{g}[GF]\right) \oplus \mathrm{g}[GH] \subseteq \mathcal{T}(\mathrm{g})[G],$$

and it is easy to see that it is a GH/G-relation obtained from the flat Y/GH.

So suppose we are in the third case, that is, $\mathrm{s}(G) \leq \mathrm{Y}$ but $HG \neq H$. We claim that $\Delta_A^G(u) = 0$.

To see this, we split the computation into two parts. Write

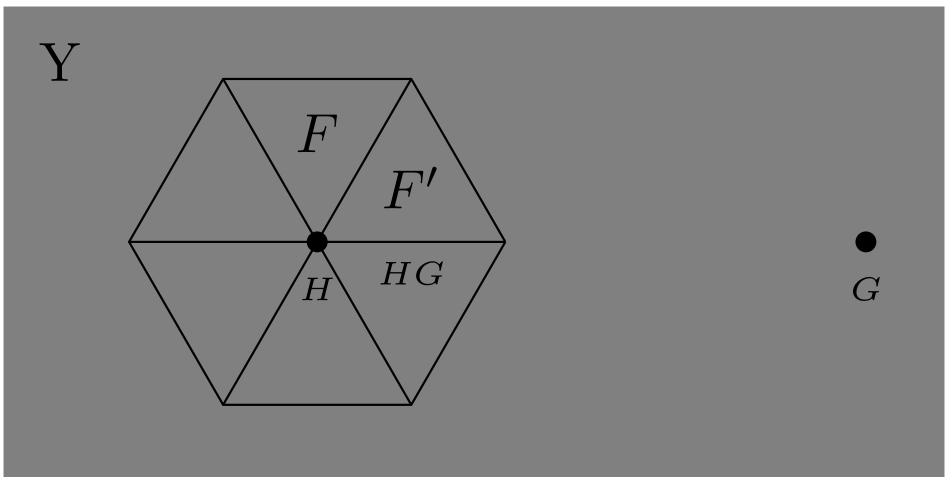
$$u = u_H + \sum_{F\colon F \geq H,\, \mathrm{s}(F) = \mathrm{Y}} u_F,$$

where $u_H \in \mathrm{g}[H]$ and $u_F \in \mathrm{g}[F]$ for each F.

The summand $\mathrm{g}[H]$, and in particular u_H, maps to zero because $HG \neq H$.

Thus we need to show that the sum also maps to zero.

An illustration is shown below.



In the figure, A is the central face (which is not visible), H and G are vertices, HG is an edge, and Y is the plane of the paper.

Write

$$\sum_{F\colon F\geq H,\, \mathrm{s}(F)=\mathrm{Y}} u_F = \sum_{\substack{F'\colon F'\geq H,\, \mathrm{s}(F')=\mathrm{Y} \\ F'\geq HG}} \sum_{\substack{F\colon F\geq H,\, \mathrm{s}(F)=\mathrm{Y} \\ HGF=F'}} u_F.$$

Observe that $\Delta_A^G(u_F)$ can be written as the composite

$$\mathrm{g}[F] \to \mathrm{g}[HGF] = \mathrm{g}[F'] \to \mathrm{g}[GF].$$

Hence, by the definition of a Lie element applied to the arrangement $\mathcal{A}_A^{\mathrm{Y}}$, we deduce that Δ_A^G of each inner sum above is 0. $\square$

Proposition 5 shows that diagram (24) is a commutative diagram of bimonoids. This yields a functor

$$\mathcal{U} : \mathsf{LieMon}(\mathcal{A}\text{-}\mathtt{Sp}) \to {}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathtt{Sp}).$$

(Note that the same symbol as before is used to denote the new functor.)

Lemma 5. *The image of the map* $i : \mathrm{g} \to \mathcal{U}(\mathrm{g})$ *in* (26) *is a Lie submonoid of* $\mathcal{U}(\mathrm{g})$ *and is contained in* $\mathcal{P}(\mathcal{U}(\mathrm{g}))$*.*

Proof. We know that i is a morphism of Lie monoids, so the first assertion follows.

By (15) or directly from the coproduct formula (13), we deduce that g is in the primitive part of $\mathcal{T}(\mathrm{g})$. Since a morphism of comonoids preserves primitive parts, the second assertion follows from Proposition 5. $\square$

4.9 Adjunction with the primitive part functor

Theorem 4. *The functor $\mathcal{U}$ is the left adjoint of the primitive part functorprimitive part functor $\mathcal{P}$. Explicitly, for any Lie monoid* g *and cocommutative bimonoid* h*, there is a natural bijection*

$$^{\mathrm{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{U}(\mathrm{g}), \mathrm{h}) \xrightarrow{\cong} \mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathrm{g}, \mathcal{P}(\mathrm{h})).$$

The adjunction between $\mathcal{U}$ and $\mathcal{P}$ is rephrased below as a universal property.

Theorem 5. *Let* h *be a bimonoid,* g *a Lie monoid,* $f : \mathrm{g} \to \mathcal{P}(\mathsf{h})$ *a morphism of Lie monoids. Then there exists a unique morphism of bimonoids* $\hat{f} : \mathcal{U}(\mathrm{g}) \to \mathsf{h}$ *such that the diagram*

$$\begin{tikzcd} \mathcal{U}(\mathrm{g}) \arrow[r, dashed, "\hat{f}"] & \mathsf{h} \\ \mathrm{g} \arrow[u, "i"] \arrow[r, "f"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd}$$

commutes.

5 Poincaré–Birkhoff–Witt

5.1 Borel-Hopf. Cofreeness of cocommutative bimonoids

Fix a noncommutative zeta function $\boldsymbol{\zeta}$.

For a bimonoid h, we define a natural map

(27) $$\mathcal{S}(\mathcal{P}(\mathsf{h})) \to \mathsf{h}$$

as follows.

Evaluating on the A-component, on the X-summand for $\mathrm{X} \geq \mathrm{s}(A)$, the map is

(28)
$$\sum_{F\colon F \geq A,\, \mathrm{s}(F)=\mathrm{X}} \boldsymbol{\zeta}(A, F)\, \mu_A^F \beta_{F,\mathrm{X}} \;:\; \mathcal{P}(\mathsf{h})[\mathrm{X}] \to \mathsf{h}[A].$$

Theorem 6. *For a cocommutative bimonoid* h*, the map* (27) *is an isomorphism of comonoids. In particular, any cocommutative bimonoid is isomorphic as a comonoid to the cofree cocommutative comonoid on its primitive part.*

This is the Borel-Hopf theorem for cocommutative bimonoids.

5.2 A comonoid section to the abelianization map

Fix a noncommutative zeta function ζ.

For any species p, we define an injective map of species

$$\mathcal{S}(\mathsf{p}) \hookrightarrow \mathcal{T}(\mathsf{p}) \tag{29}$$

as follows.

Evaluating on the A-component, on the X-summand for $\mathrm{X} \geq \mathrm{s}(A)$, the map is

(30)

$$\sum_{\substack{G:\, G \geq A \\ \mathrm{s}(G) = \mathrm{X}}} \zeta(A, G)\, \beta_{G,\mathrm{X}} \;:\; \mathsf{p}[\mathrm{X}] \longrightarrow \bigoplus_{\substack{G:\, G \geq A \\ \mathrm{s}(G) = \mathrm{X}}} \mathsf{p}[G].$$

In particular, the map (29) sends p to itself by the identity. This is the case when $\mathrm{X} = \mathrm{s}(A)$ in (30).

The map (29) is a section to the abelianization map $\mathcal{T}(\mathsf{p}) \twoheadrightarrow \mathcal{S}(\mathsf{p})$. This follows from the flat-additivity formula.

Lemma 6. *The map* (29) *is an injective morphism of comonoids.*

Proof. Recall that $\mathcal{T}(\mathsf{p})$ is a cocommutative bimonoid and p is contained in its primitive part.

The latter result is contained in (15).

Now consider the maps

$$\mathcal{S}(\mathsf{p}) \to \mathcal{S}(\mathcal{P}(\mathcal{T}(\mathsf{p}))) \to \mathcal{T}(\mathsf{p}).$$

The first map is the morphism of bimonoids obtained by applying the functor $\mathcal{S}$ to the map $\mathsf{p} \to \mathcal{P}(\mathcal{T}(\mathsf{p}))$, while the second map is the isomorphism of comonoids (27) applied to $\mathsf{h} := \mathcal{T}(\mathsf{p})$.

In particular, their composite is a morphism of comonoids.

This composite is precisely (29).

This can be seen by substituting the product formula (12) in (28) and comparing it with (30). $\square$

5.3 Poincaré–Birkhoff–Witt

Now let g be a Lie monoid.

Recall the universal enveloping monoid $\mathcal{U}(\mathrm{g})$.

It is defined as a quotient of $\mathcal{T}(\mathrm{g})$ and moreover carries a bimonoid structure inherited from $\mathcal{T}(\mathrm{g})$.

By composing the injective map (29) for p $:=$ g with the canonical surjective map $\mathcal{T}(\mathrm{g}) \twoheadrightarrow \mathcal{U}(\mathrm{g})$, we obtain a morphism

$$\mathcal{S}(\mathrm{g}) \to \mathcal{U}(\mathrm{g}) \tag{31}$$

of comonoids.

We emphasize that this map depends on the choice of a noncommutative zeta function ζ.

Theorem 7. *For any Lie monoid* $\mathfrak{g}$*, the map* (31) *is an isomorphism of comonoids.*

This is the Poincaré–Birkhoff–Witt theorem, or PBW for short.

An important consequence of PBW is the following.

Corollary 1. *For any Lie monoid* g*, the map* $i : \mathrm{g} \to \mathcal{U}(\mathrm{g})$ *defined in* (26) *is injective.*

Proof. Since the map (29) is identity on p, we see that i is the restriction of (31) to g, and hence injective. $\square$

This result justifies the usage of the term enveloping that is used for $\mathcal{U}(\mathrm{g})$.

5.4 Borel-Hopf

For any cocommutative bimonoid h, consider the composite map

$$\mathcal{S}(\mathcal{P}(\mathsf{h})) \to \mathcal{U}(\mathcal{P}(\mathsf{h})) \to \mathsf{h}. \tag{32}$$

The first map arises from setting $\mathsf{g} := \mathcal{P}(\mathsf{h})$ in (31), while the second map is the counit of the adjunction between $\mathcal{U}$ and $\mathcal{P}$ from Theorem 4.

Lemma 7. *The map* (32) *coincides with the Borel-Hopf isomorphism* (27)*.*

Proof. Consider the diagram

$$\begin{tikzcd}
 & \mathcal{T}(\mathcal{P}(\mathsf{h})) \arrow[d, two heads] \arrow[dr] & \\
\mathcal{S}(\mathcal{P}(\mathsf{h})) \arrow[ur, hook] \arrow[r] & \mathcal{U}(\mathcal{P}(\mathsf{h})) \arrow[r] & \mathsf{h}.
\end{tikzcd}$$

The left-oblique map is a section to the abelianization map, namely, (29) for p $:= \mathcal{P}(\mathsf{h})$.

The right-oblique map arises from the universal property of $\mathcal{T}$ applied to p $:= \mathcal{P}(\mathsf{h})$ and $f := \mathrm{id}$.

The triangles commute by definition of the bottom-horizontal maps.

Using (30) and the formula from the universal property, we see that the composite coincides with the map (27), and hence is an isomorphism by the Borel-Hopf Theorem 6. $\square$

6 Cartier–Milnor–Moore

6.1 Cartier–Milnor–Moore

Recall from Theorem 4 that the functors $\mathcal{U}$ and $\mathcal{P}$ are adjoints between the categories of Lie monoids and cocommutative bimonoids.

In fact:

Theorem 8. *The adjunction*

$$\mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{\mathcal{P}}{\overset{\mathcal{U}}{\rightleftarrows}} {}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$$

is an adjoint equivalence of categories.

This is the Cartier–Milnor–Moore theorem, or CMM for short.

Equivalently, it says that the counit and unit of the adjunction between $\mathcal{U}$ and $\mathcal{P}$ are natural isomorphisms:

$$\mathcal{U}(\mathcal{P}(\mathsf{h})) \xrightarrow{\cong} \mathsf{h} \quad \text{and} \quad \mathsf{g} \xrightarrow{\cong} \mathcal{P}(\mathcal{U}(\mathsf{g})). \tag{33}$$

Proof. The isomorphism of the unit follows from the PBW Theorem 7 by applying the functor $\mathcal{P}$ to (31) and using $\mathcal{P}(\mathcal{S}(\mathrm{g})) = \mathrm{g}$.

The isomorphism of the counit follows from the isomorphism (32) and the PBW Theorem 7. $\square$

6.2 Leray-Samelson

The universal enveloping monoid of an abelian Lie monoid is a bicommutative bimonoid, and conversely, the primitive part of a bicommutative bimonoid is an abelian Lie monoid.

Thus, CMM restricts to an adjoint equivalence between the full subcategories of abelian Lie monoids and bicommutative bimonoids.

Further, the category of abelian Lie monoids is isomorphic to the category of species.

Thus, we deduce that the category of species and the category of bicommutative bimonoids are equivalent.

This is precisely the Leray-Samelson Theorem.

6.3 CMM, Borel-Hopf, PBW

These three theorems can be visualized as the three vertices of a triangle with any two of them implying the third.

$$\begin{tikzcd}
& \text{CMM} \arrow[dl, dash] \arrow[dr, dash] & \\
\text{BH} \arrow[rr, dash] & & \text{PBW}
\end{tikzcd}$$

We saw how Borel-Hopf and PBW imply CMM.

The remaining two implications are explained below.

CMM and PBW imply Borel-Hopf:

For any cocommutative bimonoid h,

$$\mathsf{h} \overset{\text{CMM}}{\cong} \mathcal{U}(\mathcal{P}(\mathsf{h})) \overset{\text{PBW}}{\cong} \mathcal{S}(\mathcal{P}(\mathsf{h})).$$

The first isomorphism is of bimonoids, and the second of comonoids.

CMM and Borel-Hopf imply PBW:

For any Lie monoid g,

$$\mathcal{S}(\mathrm{g}) \overset{\mathrm{CMM}}{\cong} \mathcal{S}(\mathcal{P}(\mathcal{U}(\mathrm{g}))) \overset{\mathrm{BH}}{\cong} \mathcal{U}(\mathrm{g}).$$

The first isomorphism is of bimonoids, and the second of comonoids.

Thus, if we assume CMM, then PBW and Borel-Hopf can be viewed as equivalent theorems.

Note that both talk about an isomorphism of comonoids between a bicommutative bimonoid and a cocommutative bimonoid.