Universal constructions on species

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In dealing with maps to and from direct sums of vector spaces, we will often follow Notation 1 below.

Notation 1. We will often need to consider linear maps of the form

$$f : \bigoplus_i V_i \to \bigoplus_j W_j,$$

where both sums are finite. Since direct sum over a finite set is the product as well as the coproduct for vector spaces, such a map is equivalent to a family of linear maps $f_{ij} : V_i \to W_j$. We refer to the f_{ij} as the matrix-components of f.

Important special cases arise as

$$f : \bigoplus_i V_i \to W \qquad \text{and} \qquad f : V \to \bigoplus_j W_j.$$

In these cases, we may write f as (f_i) and (f_j), respectively. We refer to the f_i or the f_j as the vector-components of f.

The dual of f can be written as

$$f^* : \bigoplus_j W_j^* \to \bigoplus_i V_i^*.$$

We note that $(f^*)_{ji} = (f_{ij})^*$.

1 Cauchy powers of a species

1.1 Cauchy powers of a species

Let q be a species. For $k \geq 1$, define the species q^k by

$$\mathrm{q}^k[A] := \bigoplus_{\substack{F\colon F \geq A, \\ \mathrm{rk}(F/A)=k-1}} \mathrm{q}[F]. \tag{1}$$

The sum is over all faces F greater than A whose rank is $k-1$ higher than the rank of A.

Suppose A and B have the same support. Then for every face F greater than A, there is a corresponding face $G := BF$ greater than B with the same support as F, and hence a linear map $\beta_{G,F}$: q$[F] \to$ q$[G]$. Following Notation 1, this yields a linear map $\beta_{B,A}$: q$^k[A] \to$ q$^k[B]$ whose (F, G)-component is $\beta_{G,F}$ when $G = BF$, and zero otherwise. This turns q^k into a species. We call q^k the k-th Cauchy power of the species q.

- If $k = 1$, then there is only one summand in (1), namely, $F = A$, and so q$^1 =$ q.
- Since the rank of $\mathcal{A}$ is finite, eventually all Cauchy powers are zero.

Suppose c is a comonoid. For $k \geq 1$, define

(2)

$$\Delta^{k-1} : \mathsf{c} \to \mathsf{c}^k, \qquad \Delta_A^{k-1} := (\Delta_A^F)_{\mathrm{rk}(F/A)=k-1}.$$

Note that Δ^0 is the identity map, while Δ^1 has vector-components Δ_A^F as F varies over all faces which cover A.

Similarly, for a monoid a, for $k \geq 1$, define

(3)

$$\mu^{k-1} : \mathsf{a}^k \to \mathsf{a}, \qquad \mu_A^{k-1} := (\mu_A^F)_{\mathrm{rk}(F/A)=k-1}.$$

Note that μ^0 is the identity map.

1.2 Commutative Cauchy powers of a species

The above discussion has a commutative counterpart. For this, it is convenient to work with the formulation of species in terms of flats.

Let q be a species. For $k \geq 1$, define the species $\mathrm{q}^{\underline{k}}$ by

$$\mathrm{q}^{\underline{k}}[\mathrm{Z}] := \bigoplus_{\substack{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}, \\ \mathrm{rk}(\mathrm{X}/\mathrm{Z}) = k-1}} \mathrm{q}[\mathrm{X}]. \tag{4}$$

We call $\mathrm{q}^{\underline{k}}$ the k-th commutative Cauchy power of the species q.

Compare and contrast with the definition of Cauchy powers. Now there are fewer summands since we are working with flats instead of faces.

Suppose c is a cocommutative comonoid. For $k \geq 1$, define

(5)

$$\Delta^{\underline{k-1}} : \mathsf{c} \to \mathsf{c}^{\underline{k}}, \qquad \Delta_{\mathrm{Z}}^{\underline{k-1}} := (\Delta_{\mathrm{Z}}^{\mathrm{X}})_{\mathrm{rk}(\mathrm{X}/\mathrm{Z})=k-1}.$$

Dually, for a commutative monoid a, for $k \geq 1$, define

(6)

$$\mu^{\underline{k-1}} : \mathsf{a}^{\underline{k}} \to \mathsf{a}, \qquad \mu_{\mathrm{Z}}^{\underline{k-1}} := (\mu_{\mathrm{Z}}^{\mathrm{X}})_{\mathrm{rk}(\mathrm{X}/\mathrm{Z})=k-1}.$$

Note that $\Delta^{\underline{0}}$ and $\mu^{\underline{0}}$ are the identity maps.

2 Primitive filtration and decomposable filtration

2.1 Primitive part of a comonoid

Let c be a comonoid. Define the subcomonoid $\mathcal{P}(\mathsf{c})$ by

$$\mathcal{P}(\mathsf{c})[A] := \bigcap_{F\colon F>A} \ker(\Delta_A^F : \mathsf{c}[A] \to \mathsf{c}[F]). \tag{7}$$

We refer to $\mathcal{P}(\mathsf{c})$ as the primitive part of c.

We will employ the term primitive element to refer to elements of the components $\mathcal{P}(\mathsf{c})[A]$.

Let $x \in \mathsf{c}[A]$. If $\Delta_A^F(x) = 0$ for all faces F which cover A, then by coassociativity, $\Delta_A^F(x) = 0$ for all faces F strictly greater than A. Thus,

$$\mathcal{P}(\mathsf{c})[A] = \bigcap_{\substack{F\colon F \geq A,\\ \mathrm{rk}(F/A)=1}} \ker(\Delta_A^F : \mathsf{c}[A] \to \mathsf{c}[F]).$$

Equivalently, in terms of (2),

$$\mathcal{P}(\mathsf{c}) = \ker \Delta^1.$$

2.2 Primitive filtration of a comonoid

More generally, for $k \geq 1$, define the subcomonoid $\mathcal{P}_k(\mathsf{c})$ by

(8)

$$\mathcal{P}_k(\mathsf{c})[A] := \bigcap_{\substack{F:\, F \geq A, \\ \mathrm{rk}(F/A) \geq k}} \ker(\Delta_A^F : \mathsf{c}[A] \to \mathsf{c}[F]).$$

Again by coassociativity, in (8), it suffices to intersect over faces F greater than A with $\mathrm{rk}(F/A) = k$. In other words,

$$\mathcal{P}_k(\mathsf{c}) = \ker \Delta^k.$$

This is the k-th term of the primitive filtration of c. The first term is the primitive part of c, that is, $\mathcal{P}_1(\mathsf{c}) = \mathcal{P}(\mathsf{c})$.

Observe that

(9)
$$\mathcal{P}_1(\mathsf{c}) \subseteq \mathcal{P}_2(\mathsf{c}) \subseteq \cdots \subseteq \mathsf{c} \quad \text{with} \quad \bigcup_{k \geq 1} \mathcal{P}_k(\mathsf{c}) = \mathsf{c}.$$

More precisely, $\mathcal{P}_k(\mathsf{c}) = \mathsf{c}$ as soon as k exceeds the rank of the arrangement. Further:

(10)
$$\Delta^{k-1}\big(\mathcal{P}_k(\mathsf{c})\big) \subseteq \mathcal{P}(\mathsf{c})^k.$$

A morphism $\mathsf{c} \to \mathsf{d}$ of comonoids induces a map of comonoids $\mathcal{P}_k(\mathsf{c}) \to \mathcal{P}_k(\mathsf{d})$. Thus, for each k, we have a functor

$$\text{(11)} \qquad \mathcal{P}_k : \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}) \to \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}).$$

We refer to $\mathcal{P} = \mathcal{P}_1$ as the primitive part functor. Depending on the context, we may want to view it as a functor from comonoids to species, or from bimonoids to species, and so on.

Proposition 1. *A morphism $f : \mathsf{c} \to \mathsf{d}$ of comonoids is injective iff the restriction $f : \mathcal{P}(\mathsf{c}) \to \mathcal{P}(\mathsf{d})$ is injective.*

2.3 Decomposable part of a monoid

Let a be a monoid. Define the submonoid $\mathcal{D}(\mathsf{a})$ by

(12)
$$\mathcal{D}(\mathsf{a})[A] := \sum_{F\colon F>A} \operatorname{image}(\mu_A^F : \mathsf{a}[F] \to \mathsf{a}[A]).$$

By associativity, it suffices to sum only over faces which cover A. In terms of (3),

$$\mathcal{D}(\mathsf{a}) = \operatorname{image} \mu_{\cdot}^{1}$$

We refer to $\mathcal{D}(\mathsf{a})$ as the decomposable part of a.

2.4 Decomposable filtration of a monoid

More generally, for $k \geq 1$, define the submonoid $\mathcal{D}_k(\mathsf{a})$ by

$$\mathcal{D}_k(\mathsf{a})[A] := \sum_{\substack{F\colon F \geq A \\ \mathrm{rk}(F/A) \geq k}} \mathrm{image}(\mu_A^F : \mathsf{a}[F] \to \mathsf{a}[A]).$$

This is the k-th term of the decomposable filtration of a. The first term is the decomposable part of a, that is, $\mathcal{D}_1 = \mathcal{D}$. Again by associativity, it suffices to sum over faces F greater than A with $\mathrm{rk}(F/A) = k$. In other words,

$$\mathcal{D}_k(\mathsf{a}) = \mathrm{image}\, \mu^k.$$

Observe that

$$\mathsf{a} \supseteq \mathcal{D}_1(\mathsf{a}) \supseteq \mathcal{D}_2(\mathsf{a}) \supseteq \dots \quad \text{with} \quad \bigcap_{k \geq 1} \mathcal{D}_k(\mathsf{a}) = 0.$$

2.5 Indecomposable part of a monoid

Let a be a monoid. The indecomposable part of a is the quotient species $\mathcal{Q}(\mathsf{a})$ defined by

$$\mathcal{Q}(\mathsf{a}) := \mathsf{a}/\mathcal{D}(\mathsf{a}).$$

This construction is functorial in a and yields the indecomposable part functor from the category of monoids to the category of species.

2.6 Duality

The notions of primitive part (primitive filtration) and decomposable part (decomposable filtration) are dual to each other. This is formalized below.

Proposition 2. *If* c *is a comonoid, then* $\mathcal{P}_k(\mathsf{c})$ *and* $\mathcal{D}_k(\mathsf{c}^*)$ *are orthogonal complements of each other under the canonical pairing between* c *and* c^**.*

Dually, if a *is a monoid, then* $\mathcal{D}_k(\mathsf{a})$ *and* $\mathcal{P}_k(\mathsf{a}^*)$ *are orthogonal complements of each other under the canonical pairing between* a *and* a^**.*

The duality between the primitive part and the indecomposable part can be phrased as follows.

$$\mathcal{P}(\mathsf{c})^* \cong \mathcal{Q}(\mathsf{c}^*) \quad \text{and} \quad \mathcal{Q}(\mathsf{a})^* \cong \mathcal{P}(\mathsf{a}^*).$$

This is a reformulation of Proposition 2 for $k = 1$.

2.7 Trivial (co)monoids

Every species can be turned into a monoid and a comonoid in a trivial way by letting

(13)

$$\mu_A^F = \begin{cases} 0 & \text{if } F > A, \\ \mathrm{id} & \text{if } F = A, \end{cases} \qquad \Delta_A^F = \begin{cases} 0 & \text{if } F > A, \\ \mathrm{id} & \text{if } F = A. \end{cases}$$

We refer to such a (co)monoid as a trivial (co)monoid. This defines functors

(14)

$$\mathcal{A}\text{-}\mathrm{Sp} \to \mathsf{Mon}(\mathcal{A}\text{-}\mathrm{Sp}) \quad \text{and} \quad \mathcal{A}\text{-}\mathrm{Sp} \to \mathsf{Comon}(\mathcal{A}\text{-}\mathrm{Sp}).$$

We call these the trivial monoid functor and the trivial comonoid functor, respectively. Observe that they map into the subcategories of commutative monoids and cocommutative comonoids, respectively.

Suppose c is a trivial comonoid. Then a morphism of comonoids $\mathrm{c} \to \mathrm{d}$ is the same as a map of species $\mathrm{c} \to \mathcal{P}(\mathrm{d})$, where $\mathcal{P}(\mathrm{d})$ is the primitive part of d. This is because $\mathcal{P}(\mathrm{d})$ is the largest trivial subcomonoid of d.

Dually, suppose a is a trivial monoid. Then a morphism of monoids $\mathrm{b} \to \mathrm{a}$ is the same as a map of species $\mathcal{Q}(\mathrm{b}) \to \mathrm{a}$, where $\mathcal{Q}(\mathrm{b})$ is the indecomposable part of b.

Thus:

Proposition 3. *The trivial comonoid functor is the left adjoint of the primitive part functor. Dually, the trivial monoid functor is the right adjoint of the indecomposable part functor.*

3 (Co)free (co)monoids on species

3.1 Free monoid on a species

For any species p, the species $\mathcal{T}(\mathsf{p})$ is defined by

$$\mathcal{T}(\mathsf{p})[A] := \bigoplus_{F\colon A \leq F} \mathsf{p}[F].$$

More precisely,

$$\mathcal{T}(\mathsf{p}) = \mathsf{p} + \mathsf{p}^2 + \mathsf{p}^3 + \dots,$$

the sum of the Cauchy powers of p.

The species $\mathcal{T}(\mathsf{p})$ carries the structure of a monoid: For $A \leq F$, define μ_A^F by

$$
\begin{tikzcd}
\mathcal{T}(\mathsf{p})[F] \arrow[r, dotted, "\mu_A^F"] & \mathcal{T}(\mathsf{p})[A] \\
\mathsf{p}[H] \arrow[u] \arrow[r, "\mathrm{id}"'] & \mathsf{p}[H] \arrow[u]
\end{tikzcd}
\tag{15}
$$

for each $F \leq H$. We refer to μ as the concatenation product.

A map of species $\mathsf{p} \to \mathsf{q}$ induces a morphism of monoids $\mathcal{T}(\mathsf{p}) \to \mathcal{T}(\mathsf{q})$. So we have a functor

$$\mathcal{T} : \mathcal{A}\text{-}\mathsf{Sp} \to \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp}).$$

Theorem 2. *The functor $\mathcal{T}$ is the left adjoint of the forgetful functor. Explicitly, for any species* p *and monoid* a*, there are natural bijections*

$$\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{T}(\mathsf{p}), \mathsf{a}) \xrightarrow{\cong} \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{p}, \mathsf{a}).$$

In other words, $\mathcal{T}(\mathsf{p})$ is the free monoid on the species p. This can be phrased as a universal property as follows.

Theorem 3. *Let* a *be a monoid,* p *a species, and* $\zeta : \mathsf{p} \to \mathsf{a}$ *a map of species. Then there exists a unique morphism of monoids* $\hat{\zeta} : \mathcal{T}(\mathsf{p}) \to \mathsf{a}$ *such that*

$$\begin{tikzcd} \mathcal{T}(\mathsf{p}) \arrow[r, dotted, "\hat{\zeta}"] & \mathsf{a} \\ \mathsf{p} \arrow[u] \arrow[ur, "\zeta"'] & \end{tikzcd}$$

commutes.

Explicitly, the map $\hat{\zeta}$ is as follows. Evaluating on the A-component, on the F-summand, the map is

$$\mathsf{p}[F] \xrightarrow{\zeta_F} \mathsf{a}[F] \xrightarrow{\mu_A^F} \mathsf{a}[A]. \tag{16}$$

3.2 Free q-bimonoid on a comonoid

Let c be a comonoid. Consider the monoid $\mathcal{T}(\mathsf{c})$ as discussed above.

Now, in addition, for each scalar q, the coproduct of c induces a coproduct on $\mathcal{T}(\mathsf{c})$ as follows. For $A \leq G$, define Δ_A^G by

(17)

$$\begin{tikzcd} \mathcal{T}(\mathsf{c})[A] \arrow[rr, dotted, "\Delta_A^G"] & & \mathcal{T}(\mathsf{c})[G] \\ \mathsf{c}[H] \arrow[u] \arrow[r, "\Delta_H^{HG}"'] & \mathsf{c}[HG] \arrow[r, "(\beta_q)_{GH,HG}"'] & \mathsf{c}[GH] \arrow[u] \end{tikzcd}$$

for each $A \leq H$.

We refer to Δ as the dequasishuffle coproduct. It is straightforward to check that the q-bimonoid axiom holds. So $\mathcal{T}(\mathsf{c})$ is a q-bimonoid.

This construction is clearly functorial in c. So we have a functor

$$\mathcal{T}_q : \mathsf{Comon}(\mathcal{A}\text{-}\mathtt{Sp}) \to q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathtt{Sp}). \tag{18}$$

We denote it by $\mathcal{T}_q$ to emphasize the dependence on q.

Theorem 4. *The functor $\mathcal{T}_q$ is the left adjoint of the forgetful functor. Explicitly, for any comonoid c and q-bimonoid h, there are natural bijections*

$$q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathtt{Sp})(\mathcal{T}_q(\mathsf{c}), \mathsf{h}) \xrightarrow{\cong} \mathsf{Comon}(\mathcal{A}\text{-}\mathtt{Sp})(\mathsf{c}, \mathsf{h}).$$

In other words, $\mathcal{T}_q(\mathsf{c})$ is the free q-bimonoid on the comonoid c. Its universal property is stated below.

Theorem 5. *Let* h *be a* q*-bimonoid,* c *a comonoid, and* $\zeta : \mathsf{c} \to \mathsf{h}$ *a morphism of comonoids. Then there exists a unique morphism of* q*-bimonoids* $\hat{\zeta} : \mathcal{T}_q(\mathsf{c}) \to \mathsf{h}$ *such that*

$$\begin{tikzcd} \mathcal{T}_q(\mathsf{c}) \arrow[r, dotted, "\hat{\zeta}"] & \mathsf{h} \\ \mathsf{c} \arrow[u] \arrow[ur, swap, "\zeta"] & \end{tikzcd}$$

commutes.

Explicitly, the morphism $\hat{\zeta}$ is given by (16) as before.

3.3 Cofree comonoid on a species

Dually, one can construct the cofree comonoid on a species p. It is denoted by $\mathcal{T}^\vee(\mathsf{p})$. As a species, it equals $\mathcal{T}(\mathsf{p})$.

The coproduct is defined as follows. For $A \leq F$ and $A \leq H$,

$$
\begin{tikzcd}
\mathcal{T}^\vee(\mathsf{p})[A] \arrow[r, dotted, "\Delta_A^F"] & \mathcal{T}^\vee(\mathsf{p})[F] \\
\mathsf{p}[H] \arrow[u] \arrow[r] & \begin{cases} \mathsf{p}[H] & \text{if } H \geq F, \\ 0 & \text{otherwise.} \end{cases} \arrow[u]
\end{tikzcd}
\tag{19}
$$

We refer to Δ as the deconcatenation coproduct.

3.4 Cofree q-bimonoid on a monoid

Now suppose a is a monoid. Then, for each scalar q, the comonoid $\mathcal{T}^\vee(\mathsf{a})$ discussed above carries a product: For $A \leq G \leq K$,

(20)

$$\begin{tikzcd}
\mathcal{T}^\vee(\mathsf{a})[G] \arrow[rrr, dotted, "\mu_A^G"] & & & \mathcal{T}^\vee(\mathsf{a})[A] \\
\mathsf{a}[K] \arrow[u] \arrow[r, "((\beta_q)_{HG,GH})"'] & \displaystyle\bigoplus_{\substack{H:\, A \leq H \\ GH = K}} \mathsf{a}[HG] \arrow[rr, "(\mu_H^{HG})"'] & & \displaystyle\bigoplus_{\substack{H:\, A \leq H \\ GH = K}} \mathsf{a}[H]. \arrow[u]
\end{tikzcd}$$

For the second map, the matrix-component is μ_H^{HG} on matching indices, and zero otherwise. We refer to μ as the quasishuffle product. One may check that $\mathcal{T}^\vee(\mathsf{a})$ is a q-bimonoid.

Thus, we have a functor

$$\mathcal{T}_q^{\vee} : \mathsf{Mon}(\mathcal{A}\text{-}\mathtt{Sp}) \to q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathtt{Sp}). \tag{21}$$

It is the right adjoint of the forgetful functor. In other words, $\mathcal{T}_q^{\vee}(\mathsf{a})$ is the cofree q-bimonoid on the monoid a.

3.5 Contragredience

The precise relation between the free and cofree constructions is as follows.

For any comonoid c and monoid a,

(22) $$\mathcal{T}_q^\vee(\mathsf{c}^*) = \mathcal{T}_q(\mathsf{c})^* \quad \text{and} \quad \mathcal{T}_q^\vee(\mathsf{a})^* = \mathcal{T}_q(\mathsf{a}^*).$$

This can be checked directly. Observe that the coproduct (17) sends one summand to exactly one summand. However, different summands can map to the same summand. Hence, when we pass to the dual, we see that the product (20) sends one summand to multiple summands in general.

In view of (22), we say that $\mathcal{T}_q$ and $\mathcal{T}_q^\vee$ are contragredients of each other.

3.6 Primitive filtration and decomposable filtration

The decomposable filtration of the free monoid, and the primitive filtration of the cofree comonoid on a species can be described in terms of the Cauchy powers of the species.

Proposition 4. *For any* $k \geq 1$*, for any species* p*,*

$$\mathcal{P}_k(\mathcal{T}^\vee(\mathsf{p})) = \mathop{+}_{1 \leq i \leq k} \mathsf{p}^i.$$

In particular, $\mathcal{P}(\mathcal{T}^\vee(\mathsf{p})) = \mathsf{p}$*. Explicitly,*

$$\mathcal{P}_k(\mathcal{T}^\vee(\mathsf{p}))[A] = \bigoplus_{\substack{F\colon F \geq A,\\ \mathrm{rk}(F/A) < k}} \mathsf{p}[F].$$

Proof. This follows from the coproduct formula (19). $\square$

Dually:

Proposition 5. *For any $k \geq 1$, for any species* p*,*

$$\mathcal{D}_k(\mathcal{T}(\mathsf{p})) = \underset{i>k}{+} \mathsf{p}^i.$$

Explicitly,

$$\mathcal{D}_k(\mathcal{T}(\mathsf{p}))[A] = \bigoplus_{\substack{F:\, F \geq A, \\ \mathrm{rk}(F/A) \geq k}} \mathsf{p}[F].$$

Proof. This follows from the product formula (15). $\square$

Note that Proposition 4 also applies to the cofree q-bimonoid $\mathcal{T}_q^{\vee}(\mathsf{a})$. The primitive filtration depends only on the coproduct, so the product of a and the parameter q play no role. Similar remarks apply to Proposition 5 and the free q-bimonoid $\mathcal{T}_q(\mathsf{c})$.

4 (Co)free (co)commutative (co)monoids on species

4.1 Free commutative monoid on a species

For any species p, the species $\mathcal{S}(\mathsf{p})$ is defined by

$$\mathcal{S}(\mathsf{p})[\mathrm{Z}] := \bigoplus_{\mathrm{X}\colon \mathrm{Z} \leq \mathrm{X}} \mathsf{p}[\mathrm{X}].$$

Equivalently,

$$\mathcal{S}(\mathsf{p}) = \mathsf{p} + \mathsf{p}^{\underline{2}} + \mathsf{p}^{\underline{3}} + \dots,$$

the sum of the commutative Cauchy powers of p.

The species $\mathcal{S}(\mathsf{p})$ carries the structure of a commutative monoid: For $\mathrm{Z} \leq \mathrm{X}$, define $\mu_{\mathrm{Z}}^{\mathrm{X}}$ by

(23)
$$\begin{tikzcd}
\mathcal{S}(\mathsf{p})[\mathrm{X}] \arrow[r, dotted, "\mu_{\mathrm{Z}}^{\mathrm{X}}"] & \mathcal{S}(\mathsf{p})[\mathrm{Z}] \\
\mathsf{p}[\mathrm{Y}] \arrow[u] \arrow[r, "\mathrm{id}"'] & \mathsf{p}[\mathrm{Y}] \arrow[u]
\end{tikzcd}$$

for each $\mathrm{X} \leq \mathrm{Y}$. A map of species $\mathsf{p} \to \mathsf{q}$ induces a morphism of monoids $\mathcal{S}(\mathsf{p}) \to \mathcal{S}(\mathsf{q})$. So we have a functor

$$\mathcal{S} : \mathcal{A}\text{-}\mathrm{Sp} \to \mathsf{Mon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathrm{Sp}).$$

It is the left adjoint of the forgetful functor. Explicitly, for any species p and commutative monoid a, there are natural bijections

$$\mathsf{Mon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathrm{Sp})(\mathcal{S}(\mathsf{p}), \mathsf{a}) \xrightarrow{\cong} \mathcal{A}\text{-}\mathrm{Sp}(\mathsf{p}, \mathsf{a}).$$

We say that $\mathcal{S}(\mathsf{p})$ is the free commutative monoid on the species p. Its universal property is given below.

Theorem 6. *Let* a *be a commutative monoid,* p *a species, and* $\zeta : \mathsf{p} \to \mathsf{a}$ *a map of species. Then there exists a unique morphism of monoids* $\hat{\zeta} : \mathcal{S}(\mathsf{p}) \to \mathsf{a}$ *such that the diagram*

$$\begin{tikzcd}
\mathcal{S}(\mathsf{p}) \arrow[r, dotted, "\hat{\zeta}"] & \mathsf{a} \\
\mathsf{p} \arrow[u] \arrow[ur, "\zeta"'] &
\end{tikzcd}$$

commutes.

Explicitly, the map $\hat{\zeta}$ is as follows. Evaluating on the Z-component, on the X-summand, the map is

$$\mathsf{p}[\mathrm{X}] \xrightarrow{\zeta_{\mathrm{X}}} \mathsf{a}[\mathrm{X}] \xrightarrow{\mu_{\mathrm{Z}}^{\mathrm{X}}} \mathsf{a}[\mathrm{Z}]. \tag{24}$$

4.2 Free bicommutative bimonoid on a cocommutative comonoid

Let c be a cocommutative comonoid. Consider the commutative monoid $\mathcal{S}(\mathsf{c})$ as discussed above. Now, in addition, the coproduct of c induces a cocommutative coproduct on $\mathcal{S}(\mathsf{c})$ as follows.

(25)
$$\begin{tikzcd} \mathcal{S}(\mathsf{c})[\mathrm{Z}] \arrow[r, dotted, "\Delta^{\mathrm{X}}_{\mathrm{Z}}"] & \mathcal{S}(\mathsf{c})[\mathrm{X}] \\ \mathsf{c}[\mathrm{Y}] \arrow[u] \arrow[r, "\Delta^{\mathrm{X}\vee\mathrm{Y}}_{\mathrm{Y}}"'] & \mathsf{c}[\mathrm{X}\vee\mathrm{Y}] \arrow[u] \end{tikzcd}$$

Thus $\mathcal{S}(\mathsf{c})$ is a bicommutative bimonoid.

So we have a functor

$$\mathcal{S} : {}^{\mathrm{co}}\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}) \to {}^{\mathrm{co}}\mathsf{Bimon}^{\mathrm{co}}(\mathcal{A}\text{-}\mathsf{Sp}). \tag{26}$$

It is the left adjoint of the forgetful functor. Explicitly, for any cocommutative comonoid c and bicommutative bimonoid h, there are natural bijections

$${}^{\mathrm{co}}\mathsf{Bimon}^{\mathrm{co}}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{S}(\mathsf{c}), \mathsf{h}) \xrightarrow{\cong} {}^{\mathrm{co}}\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c}, \mathsf{h}).$$

Thus, $\mathcal{S}(\mathsf{c})$ is the free bicommutative bimonoid on the cocommutative comonoid c.

We also have a functor

$$\mathcal{S} : \mathsf{Comon}(\mathcal{A}\text{-}\mathrm{Sp}) \to \mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathrm{Sp}). \tag{27}$$

which is the left adjoint to the forgetful functor.

Here, we start with any comonoid c not necessarily cocommutative. Then $\mathcal{S}(\mathsf{c})$ is a commutative bimonoid not necessarily cocommutative.

4.3 Cofree cocommutative comonoid on a species

Dually, one can construct the cofree cocommutative comonoid on a species p. It is denoted by $\mathcal{S}^\vee(\mathsf{p})$. As a species, it equals $\mathcal{S}(\mathsf{p})$. For a monoid a, $\mathcal{S}^\vee(\mathsf{a})$ carries the structure of a cocommutative bimonoid. Further, if a is commutative, then so is $\mathcal{S}^\vee(\mathsf{a})$. Explicitly, the coproduct and product in the latter case are as follows.

(28)

$$
\begin{tikzcd}
\mathcal{S}^\vee(\mathsf{a})[\mathrm{Z}] \arrow[r, dotted, "\Delta^{\mathrm{X}}_{\mathrm{Z}}"] & \mathcal{S}^\vee(\mathsf{a})[\mathrm{X}] \\
\mathsf{a}[\mathrm{Y}] \arrow[u] \arrow[r] & \begin{cases} \mathsf{a}[\mathrm{Y}] & \text{if } \mathrm{Y} \geq \mathrm{X}, \\ 0 & \text{otherwise.} \end{cases} \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{S}^\vee(\mathsf{a})[\mathrm{X}] \arrow[r, dotted, "\mu^{\mathrm{X}}_{\mathrm{Z}}"] & \mathcal{S}^\vee(\mathsf{a})[\mathrm{Z}] \\
\mathsf{a}[\mathrm{W}] \arrow[u] \arrow[r, "(\mu^{\mathrm{W}}_{\mathrm{Y}})"'] & \displaystyle\bigoplus_{\substack{\mathrm{Y}:\, \mathrm{Z} \leq \mathrm{Y} \\ \mathrm{X} \vee \mathrm{Y} = \mathrm{W}}} \mathsf{a}[\mathrm{Y}] \arrow[u]
\end{tikzcd}
$$

The functors

$$\mathcal{S}^{\vee} : \mathsf{Mon}(\mathcal{A}\text{-}\mathrm{Sp}) \to {}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathrm{Sp}), \tag{29}$$

and

$$\mathcal{S}^{\vee} : \mathsf{Mon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathrm{Sp}) \to {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathrm{Sp}) \tag{30}$$

are the right adjoints of the forgetful functors.

4.4 (Co)abelianization

For any comonoid c,

$$\mathcal{S}(\mathsf{c}) = \mathcal{T}(\mathsf{c})_{ab}, \tag{31}$$

where the latter is the abelianization of $\mathcal{T}(\mathsf{c})$. This follows by composing adjunctions, the first adjunction is between comonoids and bimonoids (Theorem 4), while the second is between bimonoids and commutative bimonoids.

Dually, for any monoid a,

$$\mathcal{S}^{\vee}(\mathsf{a}) = \mathcal{T}^{\vee}(\mathsf{a})^{coab}, \tag{32}$$

where the latter is the coabelianization of the bimonoid $\mathcal{T}^{\vee}(\mathsf{a})$.

4.5 Primitive filtration and decomposable filtration

The commutative analogues of Propositions 4 and 5 are given below.

Proposition 6. *For any* $k \geq 1$*, for any species* p*,*

$$\mathcal{P}_k(\mathcal{S}^\vee(\mathsf{p})) = \underset{1 \leq i \leq k}{+} \mathsf{p}^{\underline{i}}.$$

In particular, $\mathcal{P}(\mathcal{S}^\vee(\mathsf{p})) = \mathsf{p}$*. Explicitly,*

$$\mathcal{P}_k(\mathcal{S}^\vee(\mathsf{p}))[\mathrm{Z}] = \bigoplus_{\substack{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Z},\\ \mathrm{rk}(\mathrm{X}/\mathrm{Z}) < k}} \mathsf{p}[\mathrm{X}].$$

Proposition 7. *For any* $k \geq 1$*, for any species* p*,*

$$\mathcal{D}_k(\mathcal{S}(\mathsf{p})) = \underset{i>k}{+} \mathsf{p}^{\underline{i}}.$$

Explicitly,

$$\mathcal{D}_k(\mathcal{S}(\mathsf{p}))[\mathrm{Z}] = \bigoplus_{\substack{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z},\\ \mathrm{rk}(\mathrm{X}/\mathrm{Z}) \geq k}} \mathsf{p}[\mathrm{X}].$$

These results follow from the coproduct formula in (28) and the product formula in (23), respectively. Note very carefully that Cauchy powers have been replaced by commutative Cauchy powers.

5 (Co)free bimonoids arising from a species

5.1 From species to q-bimonoids

Recall from (14) the trivial (co)monoid functor, which views a species as a (co)monoid by letting all nontrivial (co)product components to be zero. By precomposing the functors $\mathcal{T}_q$ and $\mathcal{T}_q^\vee$ in (18) and (21) with the trivial (co)monoid functor, we obtain functors (denoted by the same symbols)

(33) $$\mathcal{T}_q, \mathcal{T}_q^\vee : \mathcal{A}\text{-}\mathsf{Sp} \to q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}).$$

In other words, for any species p, $\mathcal{T}_q(\mathrm{p})$ and $\mathcal{T}_q^\vee(\mathrm{p})$ carry canonical q-bimonoid structures. These are made explicit below.

The product and coproduct of $\mathcal{T}_q(\mathsf{p})$ are as follows.

(34)
$$\begin{tikzcd} \mathcal{T}_q(\mathsf{p})[F] \arrow[r, dotted, "\mu_A^F"] & \mathcal{T}_q(\mathsf{p})[A] \\ \mathsf{p}[H] \arrow[u] \arrow[r, "\mathrm{id}"'] & \mathsf{p}[H] \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{T}_q(\mathsf{p})[A] \arrow[r, dotted, "\Delta_A^G"] & \mathcal{T}_q(\mathsf{p})[G] \\ \mathsf{p}[H] \arrow[u] \arrow[r, "((\beta_q)_{GH,HG})"'] & \begin{cases} \mathsf{p}[GH] & \text{if } HG = H, \\ 0 & \text{otherwise.} \end{cases} \arrow[u] \end{tikzcd}$$

The product is concatenation as in (15), but note how the coproduct has simplified from (17). We refer to Δ as the deshuffle coproduct. It is cocommutative for $q = 1$.

Dually, the product and coproduct of $\mathcal{T}_q^\vee(\mathsf{p})$ are as follows.

(35)
$$\begin{tikzcd}
\mathcal{T}_q^\vee(\mathsf{p})[A] \arrow[r, dotted, "\Delta_A^F"] & \mathcal{T}_q^\vee(\mathsf{p})[F] \\
\mathsf{p}[H] \arrow[u] \arrow[r] & \begin{cases} \mathsf{p}[H] & \text{if } H \geq F, \\ 0 & \text{otherwise.} \end{cases} \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{T}_q^\vee(\mathsf{p})[G] \arrow[r, dotted, "\mu_A^G"] & \mathcal{T}_q^\vee(\mathsf{p})[A] \\
\mathsf{p}[K] \arrow[u] \arrow[r, "(\beta_q)_{HG,GH}"'] & \displaystyle\bigoplus_{\substack{H:\, GH=K,\\ HG=H,\\ A\leq H}} \mathsf{p}[H] \arrow[u]
\end{tikzcd}$$

The coproduct is deconcatenation as in (19), but the product has simplified from (20). We refer to μ as the shuffle product. It is commutative for $q = 1$.

By composing adjunctions, we deduce from Theorem 4 and its dual, and Proposition 3:

Theorem 7. *For any scalar q, the functor $\mathcal{T}_q$ is the left adjoint of $\mathcal{P}$, while $\mathcal{T}_q^\vee$ is the right adjoint of $\mathcal{Q}$. (The functors are between the categories of species and q-bimonoids.)*

The first adjunction is reformulated below as a universal property.

Theorem 8. *Let* h *be a* q*-bimonoid,* p *a species, and* $\zeta : \mathsf{p} \to \mathcal{P}(\mathsf{h})$ *a map of species. Then there exists a unique morphism of* q*-bimonoids* $\hat{\zeta} : \mathcal{T}_q(\mathsf{p}) \to \mathsf{h}$ *such that the diagram*

$$\begin{tikzcd} \mathcal{T}_q(\mathsf{p}) \arrow[r, dotted, "\hat{\zeta}"] & \mathsf{h} \\ \mathsf{p} \arrow[u, hook] \arrow[r, "\zeta"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd}$$

commutes.

Explicitly, the map $\hat{\zeta}$ is as follows. Evaluating on the A-component, on the F-summand, the map is

$$\mathsf{p}[F] \xrightarrow{\zeta_F} \mathcal{P}(\mathsf{h})[F] \hookrightarrow \mathsf{h}[F] \xrightarrow{\mu_A^F} \mathsf{h}[A]. \tag{36}$$

Note the similarity with (16).

5.2 From species to 0-bimonoids

We now specialize to $q = 0$. The product and coproduct of $\mathcal{T}_0(\mathsf{p})$ are given by concatenation and deconcatenation:

For $A \leq F$,

$$\Big(\mu_A^F : \mathsf{p}[H] \to \mathsf{p}[K]\Big) = \begin{cases} \mathrm{id} & \text{if } K = H, \\ 0 & \text{otherwise,} \end{cases} \tag{37}$$

for each $F \leq H$ and $A \leq K$.

For $A \leq G$,

$$\Big(\Delta_A^G : \mathsf{p}[H] \to \mathsf{p}[K]\Big) = \begin{cases} \mathrm{id} & \text{if } K = H, \\ 0 & \text{otherwise,} \end{cases} \tag{38}$$

for each $A \leq H$ and $G \leq K$.

Observe that this is also the product and coproduct of $\mathcal{T}_0^\vee(\mathsf{p})$. Thus,

$$\mathcal{T}_0 = \mathcal{T}_0^\vee \tag{39}$$

as functors from the category of species to the category of 0-bimonoids.

5.3 From species to bicommutative bimonoids

We now turn to the commutative aspects. By precomposing the functors $\mathcal{S}$ and $\mathcal{S}^\vee$ in (26) and (30) with the trivial (co)monoid functor, we obtain functors (denoted by the same symbols)

$$\mathcal{S}, \mathcal{S}^\vee : \mathcal{A}\text{-}\mathsf{Sp} \to {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}).$$

They go from the category of species to the category of bicommutative bimonoids. In fact, it turns out that

$$\mathcal{S} = \mathcal{S}^\vee. \tag{40}$$

This can be seen by specializing the formulas (23), (25) and (28).

Explicitly, for a species p, the product and coproduct of $\mathcal{S}(\mathsf{p})$ (and of $\mathcal{S}^\vee(\mathsf{p})$) are: For $\mathrm{Z} \leq \mathrm{X}$,

(41)

$$\begin{tikzcd} \mathcal{S}(\mathsf{p})[\mathrm{X}] \arrow[r, dotted, "\mu_{\mathrm{Z}}^{\mathrm{X}}"] & \mathcal{S}(\mathsf{p})[\mathrm{Z}] \\ \mathsf{p}[\mathrm{Y}] \arrow[u, hook] \arrow[r, "\mathrm{id}"'] & \mathsf{p}[\mathrm{Y}] \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S}(\mathsf{p})[\mathrm{Z}] \arrow[r, dotted, "\Delta_{\mathrm{Z}}^{\mathrm{X}}"] & \mathcal{S}(\mathsf{p})[\mathrm{X}] \\ \mathsf{p}[\mathrm{Y}] \arrow[u, hook] \arrow[r] & \begin{cases} \mathsf{p}[\mathrm{Y}] & \text{if } \mathrm{Y} \geq \mathrm{X}, \\ 0 & \text{otherwise.} \end{cases} \arrow[u, hook] \end{tikzcd}$$

By composing adjunctions, we deduce:

Theorem 9. *The functor $\mathcal{S}$ is the left adjoint of $\mathcal{P}$ and the right adjoint of $\mathcal{Q}$. (The functors are between the categories of species and bicommutative bimonoids.)*

For the adjunction between $\mathcal{S}$ and $\mathcal{P}$, we can take the functors to be between species and commutative bimonoids. Similarly, for the adjunction between $\mathcal{S}$ and $\mathcal{Q}$, we can take the functors to be between species and cocommutative bimonoids.

The first adjunction is reformulated below as a universal property.

Theorem 10. *Let* h *be a commutative bimonoid,* p *a species, and* $\zeta : \mathsf{p} \to \mathcal{P}(\mathsf{h})$ *a map of species. Then there exists a unique morphism of bimonoids* $\hat{\zeta} : \mathcal{S}(\mathsf{p}) \to \mathsf{h}$ *such that the diagram*

$$\begin{tikzcd} \mathcal{S}(\mathsf{p}) \arrow[r, dotted, "\hat{\zeta}"] & \mathsf{h} \\ \mathsf{p} \arrow[u, hook] \arrow[r, "\zeta"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd}$$

commutes.

The map $\hat{\zeta}$ is as follows. Evaluating on the Z-component, on the X-summand, the map is

$$\mathsf{p}[\mathrm{X}] \xrightarrow{\zeta_{\mathrm{X}}} \mathcal{P}(\mathsf{h})[\mathrm{X}] \hookrightarrow \mathsf{h}[\mathrm{X}] \xrightarrow{\mu_{\mathrm{Z}}^{\mathrm{X}}} \mathsf{h}[\mathrm{Z}]. \quad (42)$$

Note the similarity with (24).

6 Partial summary

Starting data	q-bimonoid	Product	Coproc
comonoid c	$\mathcal{T}_q(\mathsf{c})$	concatenation	dequasis
monoid a	$\mathcal{T}_q^{\vee}(\mathsf{a})$	quasishuffle	deconcate
species p	$\mathcal{T}_q(\mathsf{p})$	concatenation	deshu
species p	$\mathcal{T}_q^{\vee}(\mathsf{p})$	shuffle	deconcate
species p	$\mathcal{T}_0(\mathsf{p}) = \mathcal{T}_0^{\vee}(\mathsf{p})$	concatenation	deconcate