CS738 Assignment 1

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1. Prove that the LTL formulae $\mathsf{F}\,\phi$ and $\neg\,\mathsf{G}\,(\neg\phi)$ are equivalent.

Let $M = (S, R, S_0, L)$ be an arbitrary model, and $\pi \in \mathsf{Paths}(M, s)$ an arbitrary path with $s \in S_0$. We prove that $\mathsf{F}\,\phi$ holds iff $\neg\,\mathsf{G}\,(\neg\phi)$ does.

($\Rightarrow$ direction). If $M, \pi \models \mathsf{F}\,\phi$, there exists an $i \in \mathbb{N}$ such that $M, \pi^i \models \phi$ per the definition of the operator F. Consider now the formula $\mathsf{G}\,(\neg\phi)$ — the semantics of the operator G imply that for this formula to hold, $\forall j \in \mathbb{N} : M, \pi^j \models \neg\phi$, but we know that this does not hold for $j = i$, as $M, \pi^k \models \neg\phi$ iff $M, \pi^k \not\models \phi$ for any k, as per the definition of $\neg$. Thus, $M, \pi \not\models \mathsf{G}\,(\neg\phi)$ and by extension $M, \pi \models \neg\,\mathsf{G}\,(\neg\phi)$.

($\Leftarrow$ direction). If $M, \pi \models \neg\,\mathsf{G}\,(\neg\phi)$, then, reducing using the definition of the operator $\neg$, $M, \pi \not\models \mathsf{G}\,(\neg\phi)$. Negating the definition of the operator G, we see that there must exist an $i \in \mathbb{N}$ such that the subformula does not hold, that is, $M, \pi^i \models \neg\neg\phi$. Reducing this using the double negation axiom, we get that $M, \pi^i \models \phi$. This is precisely the condition for $\mathsf{F}\,\phi$ to hold. So we get that $M, \pi \models \mathsf{F}\,\phi$.

Thus, the two formulae imply each other and are thus equivalent for the pair $\langle M, \pi \rangle$. But, since both were chosen arbitrarily from the respective domains, the equivalence holds true for all $\langle M, \pi \rangle$. Thus, the two formulae are equivalent.