Species for hyperplane arrangements

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1 Species

1.1 Base category

The category $\mathcal{A}$-$\mathtt{Hyp}$ is defined as follows.

An object is a face of $\mathcal{A}$, and there is a unique morphism from one face to another face whenever the two faces have the same support.

When F and G have the same support, we write $\beta_{G,F} : F \to G$ for the unique morphism between them.

Proposition 1. *The category $\mathcal{A}$-$\mathtt{Hyp}$ has a presentation given by generators*

$$\beta_{G,F} : F \to G,$$

whenever F and G have the same support, and relations

$$\begin{tikzcd} & G \arrow[dr, "\beta_{H,G}"] & \\ F \arrow[ur, "\beta_{G,F}"] \arrow[rr, "\beta_{H,F}"'] & & H \end{tikzcd} \qquad (F \xrightarrow{\beta_{F,F}} F) = \mathrm{id}$$

whenever F, G and H have the same support.

It follows that $\beta_{G,F}$ and $\beta_{F,G}$ are inverse isomorphisms, that is, the diagrams

$$
\begin{tikzcd}
 & G \arrow[dr, "\beta_{F,G}"] & \\
F \arrow[ur, "\beta_{G,F}"] \arrow[rr, "\mathrm{id}"'] & & F
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F \arrow[dr, "\beta_{G,F}"] & \\
G \arrow[ur, "\beta_{F,G}"] \arrow[rr, "\mathrm{id}"'] & & G
\end{tikzcd}
$$

commute, whenever F and G have the same support.

We express this relation by writing $\beta^2 = \mathrm{id}$.

Proposition 2. *The category $\mathcal{A}$-$\mathtt{Hyp}$ is a disjoint union of indiscrete categories. In particular, it is a groupoid, that is, all morphisms are invertible. Further, it is equivalent to the discrete category whose objects are indexed by flats of $\mathcal{A}$.*

Proof. When F and G have different supports, there is no morphism between them. So $\mathcal{A}$-$\mathtt{Hyp}$ breaks as a union of connected pieces, one for each flat of $\mathcal{A}$. Further, each connected piece is an indiscrete category, meaning that there is exactly one morphism from one object to another. This proves the first statement. Since any indiscrete category is equivalent to the one-arrow category, the last statement follows. $\square$

Let us elaborate on the above equivalence.

Let $\mathcal{A}$-$\mathtt{Hyp}'$ denote the discrete category on the set of flats of $\mathcal{A}$, that is, its objects are flats of $\mathcal{A}$, and the only morphisms are identity.

There is a functor from $\mathcal{A}$-$\mathtt{Hyp}$ to $\mathcal{A}$-$\mathtt{Hyp}'$ which sends a face to its support.

Conversely, for any flat X choose a face of support X, and define a functor from $\mathcal{A}$-$\mathtt{Hyp}'$ to $\mathcal{A}$-$\mathtt{Hyp}$ which sends a flat to its chosen face.

The two functors define an equivalence between the categories $\mathcal{A}$-$\mathtt{Hyp}$ and $\mathcal{A}$-$\mathtt{Hyp}'$.

1.2 Species

An $\mathcal{A}$-species is a functor

$$\mathrm{p} : \mathcal{A}\text{-}\mathtt{Hyp} \to \mathsf{Vec}.$$

A map of $\mathcal{A}$-species $\mathrm{p} \to \mathrm{q}$ is a natural transformation.

This defines the category of $\mathcal{A}$-species which we denote by $\mathcal{A}$-$\mathtt{Sp}$.

It is a functor category, and we also write

$$\mathcal{A}\text{-}\mathtt{Sp} = [\mathcal{A}\text{-}\mathtt{Hyp}, \mathsf{Vec}].$$

The value of an $\mathcal{A}$-species p on an object F will be denoted $\mathrm{p}[F]$. We call it the F-component of p. By Proposition 1, we obtain:

An $\mathcal{A}$-species consists of a family of vector spaces $\mathsf{p}[F]$, one for each face F of $\mathcal{A}$, together with linear maps

$$\beta_{G,F} : \mathsf{p}[F] \to \mathsf{p}[G],$$

whenever F and G have the same support, such that

(1)

$$\begin{tikzcd} & \mathsf{p}[G] \arrow[dr, "\beta_{H,G}"] & \\ \mathsf{p}[F] \arrow[ur, "\beta_{G,F}"] \arrow[rr, "\beta_{H,F}"'] & & \mathsf{p}[H] \end{tikzcd} \qquad (\mathsf{p}[F] \xrightarrow{\beta_{F,F}} \mathsf{p}[F]) = \mathrm{id}$$

commute, whenever F, G and H have the same support.

It follows that $\beta_{G,F}$ and $\beta_{F,G}$ are inverse linear isomorphisms, that is, the diagrams

(2)

$$\begin{tikzcd} & \mathsf{p}[G] \arrow[dr, "\beta_{F,G}"] & \\ \mathsf{p}[F] \arrow[ur, "\beta_{G,F}"] \arrow[rr, "\mathrm{id}"'] & & \mathsf{p}[F] \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{p}[F] \arrow[dr, "\beta_{G,F}"] & \\ \mathsf{p}[G] \arrow[ur, "\beta_{F,G}"] \arrow[rr, "\mathrm{id}"'] & & \mathsf{p}[G] \end{tikzcd}$$

commute, whenever F and G have the same support.

Similarly, a map of $\mathcal{A}$-species $f : \mathsf{p} \to \mathsf{q}$ consists of a family of linear maps

$$f_F : \mathsf{p}[F] \to \mathsf{q}[F],$$

one for each face F, such that whenever F and G have the same support, the diagram

$$\begin{tikzcd} \mathsf{p}[F] \arrow[r, "f_F"] \arrow[d, "\beta_{G,F}"'] & \mathsf{q}[F] \arrow[d, "\beta_{G,F}"] \\ \mathsf{p}[G] \arrow[r, "f_G"'] & \mathsf{q}[G] \end{tikzcd} \tag{3}$$

commutes. We refer to f_F as the F-component of f.

An $\mathcal{A}$-species p is finite-dimensional if its F-component $\mathsf{p}[F]$ has finite dimension for all faces F.

1.3 Product and coproduct

The zero species $\mathsf{0}$ is the $\mathcal{A}$-species all of whose components are zero:

$$\mathsf{0}[F] = 0. \tag{4}$$

This is the initial and terminal object in the category of $\mathcal{A}$-species.

Given $\mathcal{A}$-species p and q, their direct sum $\mathsf{p} + \mathsf{q}$ is defined by

$$(\mathsf{p} + \mathsf{q})[A] := \mathsf{p}[A] \oplus \mathsf{q}[A], \tag{5}$$

with the linear maps $\beta_{G,F}$ of $\mathsf{p} + \mathsf{q}$ induced from those of p and q. This is the product and coproduct in the category of $\mathcal{A}$-species.

1.4 Reformulation using flats

Recall from the discussion after Proposition 2 that the category $\mathcal{A}$-$\mathtt{Hyp}$ is equivalent to $\mathcal{A}$-$\mathtt{Hyp}'$. As a consequence, the functor categories

$$[\mathcal{A}\text{-}\mathtt{Hyp}, \mathsf{Vec}] \quad \text{and} \quad [\mathcal{A}\text{-}\mathtt{Hyp}', \mathsf{Vec}]$$

are equivalent. The latter category yields the following reformulation of $\mathcal{A}$-species.

Proposition 3. *An $\mathcal{A}$-species p is a family $\mathsf{p}[\mathrm{X}]$ of vector spaces, one for each flat X. A map of $\mathcal{A}$-species $f : \mathsf{p} \to \mathsf{q}$ is a family of linear maps*

$$f_{\mathrm{X}} : \mathsf{p}[\mathrm{X}] \to \mathsf{q}[\mathrm{X}],$$

one for each flat X.

We elaborate on the connection between the two points of view. Starting with an $\mathcal{A}$-species p as above, one can set p$[F] :=$ p$[\mathrm{X}]$ for all F with support X, and $\beta_{G,F} = \mathrm{id}$ whenever F and G have the same support. This yields an $\mathcal{A}$-species in the first sense.

Conversely, given an $\mathcal{A}$-species in the first sense, define p$[\mathrm{X}] :=$ p$[H]$ where H is an arbitrary but fixed face of support X, and we get an $\mathcal{A}$-species as above. Note that for any face F with support X, we have inverse isomorphisms

(6)

$$\beta_{F,\mathrm{X}} : \mathsf{p}[\mathrm{X}] \to \mathsf{p}[F] \quad \text{and} \quad \beta_{\mathrm{X},F} : \mathsf{p}[F] \to \mathsf{p}[\mathrm{X}]$$

and identities such as

$$\beta_{\mathrm{X},F} = \beta_{\mathrm{X},G}\beta_{G,F} \quad \text{and} \quad \beta_{G,F} = \beta_{G,\mathrm{X}}\beta_{\mathrm{X},F}$$

always hold.

2 Monoids, comonoids and bimonoids

One can define monoids, comonoids, and bimonoids in species.

There is a notion of (co)commutativity for (co)monoids.

A bimonoid may be commutative, cocommutative, both, or neither.

The resulting categories along with their notations are summarized in Table 1.

Table 1: Categories of (co, bi)monoids in species.

Category	Description	Category	D
$\mathsf{Mon}(\mathcal{A}\text{-}\mathrm{Sp})$	monoids	${}^{\mathsf{co}}\mathsf{Comon}(\mathcal{A}\text{-}\mathrm{Sp})$	cocor
$\mathsf{Comon}(\mathcal{A}\text{-}\mathrm{Sp})$	comonoids	$\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathrm{Sp})$	com
$\mathsf{Bimon}(\mathcal{A}\text{-}\mathrm{Sp})$	bimonoids	${}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathrm{Sp})$	coco
$\mathsf{Mon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathrm{Sp})$	com. monoids	${}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathrm{Sp})$	bicor

We first dicuss monoids, comonoids and bimonoids. Commutativity aspects will be treated in the next section.

2.1 Monoids

An $\mathcal{A}$-monoid is an $\mathcal{A}$-species a equipped with linear maps

$$\mu_A^F : \mathsf{a}[F] \to \mathsf{a}[A],$$

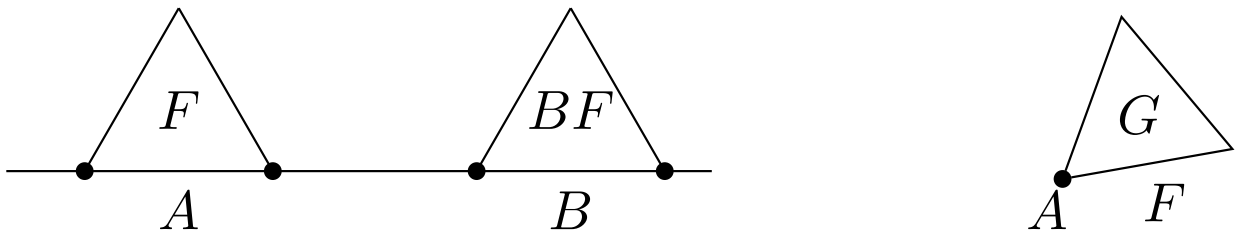
one for each pair of faces $A \leq F$, such that the following diagrams commute.

(7)

$$\begin{tikzcd} \mathsf{a}[F] \arrow[r, "\beta_{BF,F}"] \arrow[d, "\mu_A^F"'] & \mathsf{a}[BF] \arrow[d, "\mu_B^{BF}"] \\ \mathsf{a}[A] \arrow[r, "\beta_{B,A}"'] & \mathsf{a}[B] \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{a}[F] \arrow[dr, "\mu_A^F"] & \\ \mathsf{a}[G] \arrow[ur, "\mu_F^G"] \arrow[rr, "\mu_A^G"'] & & \mathsf{a}[A] \end{tikzcd} \qquad \left(\mathsf{a}[A] \xrightarrow{\mu_A^A} \mathsf{a}[A]\right) = \mathrm{id}$$

The first diagram is to be considered whenever A and B have the same support, and $A \leq F$, and the second diagram for every $A \leq F \leq G$.

Illustrative pictures are shown below.



We refer to (7) as the naturality, associativity and unitality axioms, respectively. We denote an $\mathcal{A}$-monoid by a pair (a, μ), or simply by a with μ understood. We refer to the maps μ_A^F as the product components or structure maps of a.

A morphism f : a $\rightarrow$ b of $\mathcal{A}$-monoids is a map of $\mathcal{A}$-species such that for each $A \leq F$, the diagram

(8)
$$\begin{tikzcd} \mathrm{a}[F] \arrow[r, "f_F"] \arrow[d, "\mu_A^F"'] & \mathrm{b}[F] \arrow[d, "\mu_A^F"] \\ \mathrm{a}[A] \arrow[r, "f_A"'] & \mathrm{b}[A] \end{tikzcd}$$

commutes.

2.2 Comonoids

The dual notion is that of $\mathcal{A}$-comonoids. An $\mathcal{A}$-comonoid is an $\mathcal{A}$-species c equipped with linear maps

$$\Delta_A^F : \mathsf{c}[A] \to \mathsf{c}[F],$$

one for each $A \leq F$, such that the following diagrams commute.

(9)

$$\begin{tikzcd} \mathsf{c}[A] \arrow[r, "\beta_{B,A}"] \arrow[d, "\Delta_A^F"'] & \mathsf{c}[B] \arrow[d, "\Delta_B^{BF}"] \\ \mathsf{c}[F] \arrow[r, "\beta_{BF,F}"'] & \mathsf{c}[BF] \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{c}[F] \arrow[dr, "\Delta_F^G"] & \\ \mathsf{c}[A] \arrow[ur, "\Delta_A^F"] \arrow[rr, "\Delta_A^G"'] & & \mathsf{c}[G] \end{tikzcd} \qquad \left(\mathsf{c}[A] \xrightarrow{\Delta_A^A} \mathsf{c}[A]\right) = \mathrm{id}$$

The first diagram is to be considered whenever A and B have the same support, and $A \leq F$, and the second diagram for every $A \leq F \leq G$.

We refer to (9) as the naturality, coassociativity and counitality axioms, respectively. We denote an $\mathcal{A}$-comonoid by a pair (c, Δ), or simply by c with Δ understood. We refer to the maps Δ_A^F as the coproduct components or structure maps of c.

A morphism $f : \mathsf{c} \to \mathsf{d}$ of $\mathcal{A}$-comonoids is a map of $\mathcal{A}$-species such that for each $A \leq F$, the diagram

(10)
$$\begin{tikzcd} \mathsf{c}[A] \arrow[r, "f_A"] \arrow[d, "\Delta_A^F"'] & \mathsf{d}[A] \arrow[d, "\Delta_A^F"] \\ \mathsf{c}[F] \arrow[r, "f_F"'] & \mathsf{d}[F] \end{tikzcd}$$

commutes.

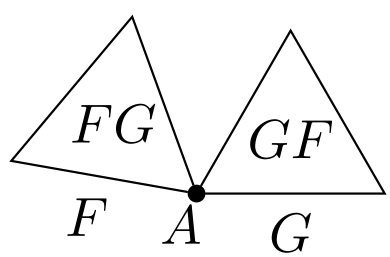
2.3 Bimonoids

An $\mathcal{A}$-bimonoid is a triple $(\mathsf{h}, \mu, \Delta)$, where h is an $\mathcal{A}$-species, (h, μ) is an $\mathcal{A}$-monoid, (h, Δ) is an $\mathcal{A}$-comonoid, and such that for any faces $A \leq F$ and $A \leq G$, the diagram

$$\begin{tikzcd} \mathsf{h}[F] \arrow[r, "\mu_A^F"] \arrow[d, "\Delta_F^{FG}"'] & \mathsf{h}[A] \arrow[r, "\Delta_A^G"] & \mathsf{h}[G] \\ \mathsf{h}[FG] \arrow[rr, "\beta_{GF,FG}"'] & & \mathsf{h}[GF] \arrow[u, "\mu_G^{GF}"'] \end{tikzcd} \tag{11}$$

commutes.

An illustrative picture is shown below.



We refer to diagram (11) as the bimonoid axiom. As a shorthand, we may write it as $\Delta\mu = \mu\beta\Delta$.

A morphism of $\mathcal{A}$-bimonoids $f : \mathsf{h} \to \mathsf{k}$ is a map of the underlying $\mathcal{A}$-species which is a morphism of the underlying $\mathcal{A}$-monoids and $\mathcal{A}$-comonoids.

Lemma 1. *Suppose* $(\mathsf{h}, \mu, \Delta)$ *is an* $\mathcal{A}$*-bimonoid. Then: For faces* F *and* G *both greater than* A*, and of the same support, the diagram*

(12)
$$\begin{tikzcd} \mathsf{h}[F] \arrow[rr, "\beta_{G,F}"] \arrow[dr, "\mu_A^F"'] & & \mathsf{h}[G] \\ & \mathsf{h}[A] \arrow[ur, "\Delta_A^G"'] & \end{tikzcd}$$

commutes.

For $A \leq F \leq G$*, the diagrams*

(13)
$$\begin{tikzcd} & \mathsf{h}[A] \arrow[dr, "\Delta_A^G"] & \\ \mathsf{h}[F] \arrow[ur, "\mu_A^F"] \arrow[rr, "\Delta_F^G"'] & & \mathsf{h}[G] \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{h}[A] \arrow[dr, "\Delta_A^F"] & \\ \mathsf{h}[G] \arrow[ur, "\mu_A^G"] \arrow[rr, "\mu_F^G"'] & & \mathsf{h}[F] \end{tikzcd}$$

commute.

In particular, for any $A \leq F$, the diagram

(14)
$$\begin{tikzcd} \mathsf{h}[F] \arrow[rr, "\mathrm{id}"] \arrow[dr, "\mu_A^F"'] & & \mathsf{h}[F] \\ & \mathsf{h}[A] \arrow[ur, "\Delta_A^F"'] & \end{tikzcd}$$

commutes.

Proof. All the above diagrams are special cases of the bimonoid axiom (11). The special nature of F and G forces two of the five arrows to become identity, so the pentagon reduces to triangles. For instance: Suppose F and G have the same support. Then $FG = F$ and $GF = G$, so by (co)unitality, $\Delta_F^{FG} = \mathrm{id}$ and $\mu_G^{GF} = \mathrm{id}$. Hence (12) follows. $\square$

As a consequence of (14):

In any $\mathcal{A}$-bimonoid, the product components are injective, and the coproduct components are surjective.

This is not true for arbitrary monoids and comonoids.

3 (Co)commutative (co)monoids

3.1 Commutative monoids

An $\mathcal{A}$-monoid (a, μ) is commutative if the diagram

$$
\begin{tikzcd}
\mathsf{a}[F] \arrow[rr, "\beta_{G,F}"] \arrow[dr, "\mu_A^F"'] & & \mathsf{a}[G] \arrow[dl, "\mu_A^G"] \\
& \mathsf{a}[A] &
\end{tikzcd}
\tag{15}
$$

commutes, whenever $A \leq F$ and $A \leq G$, and F and G have the same support.

Lemma 2. *Suppose (a, μ) is an $\mathcal{A}$-monoid. The following are equivalent.*

1. *(a, μ) is commutative.*

2. *The diagram*

$$\begin{tikzcd}
\mathsf{a}[F] \arrow[rr, "\beta_{A\overline{F},F}"] \arrow[dr, "\mu_A^F"'] & & \mathsf{a}[A\overline{F}] \arrow[dl, "\mu_A^{A\overline{F}}"] \\
& \mathsf{a}[A] &
\end{tikzcd} \tag{16}$$

   commutes, whenever $A \leq F$.

3. *Diagram (15) commutes, whenever F and G are adjacent with the same support and A is their common panel.*

Proof. Note that F and $A\overline{F}$ are opposite faces in the star of A. In particular, they have the same support. Further, if A has codimension one in F, then F and $A\overline{F}$ are adjacent with common panel A. This shows (1) implies (2), and (2) implies (3). It remains to show (3) implies (1).

Since flats are gallery connected, to establish (15), we may assume that F and G are adjacent. Let H be their common panel. Diagram (15) can be filled in as follows.

$$\begin{tikzcd}
\mathsf{a}[F] \arrow[rr, "\beta_{G,F}"] \arrow[dr, "\mu^F_H"] \arrow[ddr, bend right, "\mu^F_A"'] & & \mathsf{a}[G] \arrow[dl, "\mu^G_H"'] \arrow[ddl, bend left, "\mu^G_A"] \\
& \mathsf{a}[H] \arrow[d, "\mu^H_A"] & \\
& \mathsf{a}[A] &
\end{tikzcd}$$

The top triangle commutes by hypothesis. The triangles on the two sides commute by associativity (7). $\square$

Lemma 3. *Suppose* (a, μ) *is an* $\mathcal{A}$*-monoid. It is commutative iff the diagram*

$$\begin{tikzcd} \mathsf{a}[F] \arrow[r, "\mu_A^F"] \arrow[d, "\beta_{G,F}"'] & \mathsf{a}[A] \arrow[d, "\beta_{B,A}"] \\ \mathsf{a}[G] \arrow[r, "\mu_B^G"'] & \mathsf{a}[B] \end{tikzcd} \tag{17}$$

commutes, whenever A *and* B *have the same support,* F *and* G *have the same support, and* $A \leq F$ *and* $B \leq G$*.*

Note very carefully that (17) is stronger than the naturality axiom in (7).

Proof. For the backward implication, take $A = B$. For the forward implication, fill in (17) as follows.

$$\begin{tikzcd}
\mathsf{a}[F] \arrow[rr, "\mu"] \arrow[dd, "\beta"'] \arrow[dr, "\beta"] & & \mathsf{a}[A] \arrow[dd, "\beta"] \\
& \mathsf{a}[BF] \arrow[dl, "\beta"] \arrow[dr, "\mu"] & \\
\mathsf{a}[G] \arrow[rr, "\mu"'] & & \mathsf{a}[B].
\end{tikzcd}$$

The subscripts on μ and β have been suppressed, they can be read off from the faces involved in their domain and codomain. For instance, the top-horizontal μ is μ_A^F. The square commutes by naturality (7), the bottom triangle commutes by (15), and the side triangle commutes by (1). $\square$

We now reformulate commutative monoids using the alternative definition of species given in Proposition 3.

Proposition 4. *A commutative $\mathcal{A}$-monoid is an $\mathcal{A}$-species* a *equipped with linear maps*

$$\mu_{\mathrm{Z}}^{\mathrm{X}} : \mathsf{a}[\mathrm{X}] \to \mathsf{a}[\mathrm{Z}],$$

one for each pair of flats $\mathrm{Z} \leq \mathrm{X}$, such that the diagrams

(18)

$$\begin{tikzcd} & \mathsf{a}[\mathrm{Y}] \arrow[dr, "\mu_{\mathrm{Z}}^{\mathrm{Y}}"] & \\ \mathsf{a}[\mathrm{X}] \arrow[ur, "\mu_{\mathrm{Y}}^{\mathrm{X}}"] \arrow[rr, "\mu_{\mathrm{Z}}^{\mathrm{X}}"'] & & \mathsf{a}[\mathrm{Z}] \end{tikzcd} \qquad (\mathsf{a}[\mathrm{Z}] \xrightarrow{\mu_{\mathrm{Z}}^{\mathrm{Z}}} \mathsf{a}[\mathrm{Z}]) = \mathrm{id}$$

commute, the first for every $\mathrm{Z} \leq \mathrm{Y} \leq \mathrm{X}$, and the second for every Z.

A morphism of commutative $\mathcal{A}$-monoids f : $\mathsf{a} \to \mathsf{b}$ is a family of linear maps

$$f_{\mathrm{X}} : \mathsf{a}[\mathrm{X}] \to \mathsf{b}[\mathrm{X}],$$

one for each flat X, such that the diagram

$$\begin{tikzcd} \mathsf{a}[\mathrm{X}] \arrow[r, "f_{\mathrm{X}}"] \arrow[d, "\mu_{\mathrm{Z}}^{\mathrm{X}}"'] & \mathsf{b}[\mathrm{X}] \arrow[d, "\mu_{\mathrm{Z}}^{\mathrm{X}}"] \\ \mathsf{a}[\mathrm{Z}] \arrow[r, "f_{\mathrm{Z}}"'] & \mathsf{b}[\mathrm{Z}] \end{tikzcd} \tag{19}$$

commutes, for every $\mathrm{Z} \leq \mathrm{X}$.

Proof. We explain the first part. Suppose a is a commutative $\mathcal{A}$-monoid (in the usual sense). Then by Lemma 3, its product factors through the maps (6):

$$\begin{tikzcd}
\mathsf{a}[F] \arrow[r, "\mu_A^F"] \arrow[d, "\beta_{\mathrm{X},F}"'] & \mathsf{a}[A] \arrow[d, "\beta_{\mathrm{Z},A}"] \\
\mathsf{a}[\mathrm{X}] \arrow[r, dotted, "\mu_{\mathrm{Z}}^{\mathrm{X}}"'] & \mathsf{a}[\mathrm{Z}].
\end{tikzcd}$$

Diagrams (18) of $\mu_{\mathrm{Z}}^{\mathrm{X}}$ follow from the corresponding diagrams (7) of μ_A^F.

Conversely, suppose we are given the maps $\mu_{\mathrm{Z}}^{\mathrm{X}}$. Then μ_A^F can be defined using the above diagram. It will satisfy the diagrams required of a commutative $\mathcal{A}$-monoid. $\square$

We refer to (18) as the associativity and unitality axioms, respectively.

3.2 Cocommutative comonoids

Dually, an $\mathcal{A}$-comonoid (c, Δ) is cocommutative if the diagram

(20)
$$\begin{tikzcd} & \mathsf{c}[A] \arrow[dl, "\Delta_A^F"'] \arrow[dr, "\Delta_A^G"] & \\ \mathsf{c}[F] \arrow[rr, "\beta_{G,F}"'] & & \mathsf{c}[G] \end{tikzcd}$$

commutes, whenever $A \leq F$ and $A \leq G$, and F and G have the same support.

The entire discussion for commutative monoids holds for cocommutative comonoids (after reversing arrows labeled by μ and relabeling them by Δ). In particular, a cocommutative $\mathcal{A}$-comonoid can be described using maps

$$\Delta_{\mathrm{Z}}^{\mathrm{X}} : \mathsf{c}[\mathrm{Z}] \to \mathsf{c}[\mathrm{X}],$$

one for each pair of flats $\mathrm{Z} \leq \mathrm{X}$.

3.3 Bicommutative bimonoids

We say an $\mathcal{A}$-bimonoid is (co)commutative if its underlying $\mathcal{A}$-(co)monoid is (co)commutative.

Similarly, an $\mathcal{A}$-bimonoid is bicommutative if it is both commutative and cocommutative.

Bicommutative bimonoids can be nicely reformulated in terms of the maps $\mu_{\mathrm{Z}}^{\mathrm{X}}$ and $\Delta_{\mathrm{Z}}^{\mathrm{Y}}$:

Proposition 5. *A bicommutative $\mathcal{A}$-bimonoid is a triple $(\mathsf{h}, \mu, \Delta)$, where h is an $\mathcal{A}$-species, (h, μ) is a commutative $\mathcal{A}$-monoid, (h, Δ) is a cocommutative $\mathcal{A}$-comonoid, and such that for any flats $\mathrm{Z} \leq \mathrm{X}$ and $\mathrm{Z} \leq \mathrm{Y}$, the diagram*

$$
\begin{tikzcd}
\mathsf{h}[\mathrm{X}] \arrow[r, "\mu_{\mathrm{Z}}^{\mathrm{X}}"] \arrow[d, "\Delta_{\mathrm{X}}^{\mathrm{X}\vee\mathrm{Y}}"'] & \mathsf{h}[\mathrm{Z}] \arrow[d, "\Delta_{\mathrm{Z}}^{\mathrm{Y}}"] \\
\mathsf{h}[\mathrm{X} \vee \mathrm{Y}] \arrow[r, "\mu_{\mathrm{Y}}^{\mathrm{X}\vee\mathrm{Y}}"'] & \mathsf{h}[\mathrm{Y}]
\end{tikzcd}
\tag{21}
$$

commutes.

To obtain this, we only need to observe that the bimonoid axiom (11) simplifies to (21). We call this the bicommutative bimonoid axiom. As a shorthand, we may write it as $\Delta\mu = \mu\Delta$.

4 Deformed and signed bimonoids

4.1 q-bimonoids

Let q be any scalar.

An $\mathcal{A}$-q-bimonoid is a triple $(\mathsf{h}, \mu, \Delta)$, where h is an $\mathcal{A}$-species, (h, μ) is an $\mathcal{A}$-monoid, (h, Δ) is an $\mathcal{A}$-comonoid, and such that for any faces $A \leq F$ and $A \leq G$, the diagram

$$\begin{tikzcd} \mathsf{h}[F] \arrow[r, "\mu_A^F"] \arrow[d, "\Delta_F^{FG}"'] & \mathsf{h}[A] \arrow[r, "\Delta_A^G"] & \mathsf{h}[G] \\ \mathsf{h}[FG] \arrow[rr, "(\beta_q)_{GF,FG}"'] & & \mathsf{h}[GF] \arrow[u, "\mu_G^{GF}"'] \end{tikzcd} \tag{22}$$

commutes, where

$$(\beta_q)_{GF,FG} := q^{\operatorname{dist}(GF,FG)} \beta_{GF,FG}, \tag{23}$$

with $\operatorname{dist}(GF, FG)$ being the number of hyperplanes which separate GF and FG.

A morphism of $\mathcal{A}$-q-bimonoids is defined as before with q playing no role. In other words, a morphism of $\mathcal{A}$-q-bimonoids is a map of the underlying $\mathcal{A}$-species which is a morphism of the underlying $\mathcal{A}$-monoids and $\mathcal{A}$-comonoids.

Since $\beta_1 = \beta$, axiom (22) reduces to (11) when $q = 1$. In other words, an $\mathcal{A}$-1-bimonoid is the same as an $\mathcal{A}$-bimonoid. The signed companion of $q = 1$ is the value $q = -1$. We use the term signed $\mathcal{A}$-bimonoid to refer to an $\mathcal{A}$-(-1)-bimonoid.

We denote the category of q-bimonoids by q-Bimon$(\mathcal{A}\text{-}\mathtt{Sp})$, and that of signed bimonoids by (-1)-Bimon$(\mathcal{A}\text{-}\mathtt{Sp})$.

4.2 0-bimonoids

We now focus on the value $q = 0$. Let h be an $\mathcal{A}$-0-bimonoid. Then for any faces $A \leq F$ and $A \leq G$, the following diagrams commute. If $FG = GF$, then

(24a)
$$\begin{tikzcd} & \mathsf{h}[F] \arrow[r, "\mu_A^F"] \arrow[d, "\Delta_F^{FG}"'] & \mathsf{h}[A] \arrow[d, "\Delta_A^G"] \\ \mathsf{h}[FG] = & \mathsf{h}[GF] \arrow[r, "\mu_G^{GF}"'] & \mathsf{h}[G] \end{tikzcd}$$

and if $FG \neq GF$, then

(24b)
$$\begin{tikzcd} & \mathsf{h}[A] \arrow[dr, "\Delta_A^G"] & \\ \mathsf{h}[F] \arrow[ur, "\mu_A^F"] \arrow[rr, "0"'] & & \mathsf{h}[G]. \end{tikzcd}$$

This follows by setting $q = 0$ in axiom (22), and using the fact that $(\beta_0)_{GF,FG} = 0$ unless $FG = GF$.

To summarize:

An $\mathcal{A}$-0-bimonoid is the same as a triple $(\mathrm{h}, \mu, \Delta)$, where h is an $\mathcal{A}$-species, (h, μ) is an $\mathcal{A}$-monoid, (h, Δ) is an $\mathcal{A}$-comonoid such that

$$\Delta_A^G \mu_A^F = \begin{cases} \mu_G^{GF} \Delta_F^{FG} & \text{if } FG = GF, \\ 0 & \text{otherwise,} \end{cases} \tag{25}$$

for any faces $A \leq F$ and $A \leq G$.

Note the similarity of (25) with the bicommutative bimonoid axiom (21).

5 Signed commutative monoids

Commutative monoids have a signed analogue. An $\mathcal{A}$-monoid (a, μ) is signed commutative if the diagram

$$
\begin{tikzcd}
\mathsf{a}[F] \arrow[rr, "(\beta_{-1})_{G,F}"] \arrow[dr, "\mu_A^F"'] & & \mathsf{a}[G] \arrow[dl, "\mu_A^G"] \\
& \mathsf{a}[A] &
\end{tikzcd}
\tag{26}
$$

commutes, whenever $A \leq F$ and $A \leq G$, and F and G have the same support. Here

$$(\beta_{-1})_{G,F} := (-1)^{\mathrm{dist}(G,F)} \beta_{G,F}.$$

This is the special case $q = -1$ in (23).

Compare and contrast (26) with (15). The parity of the distance between F and G now plays a role.

6 Subspecies, quotient species and (co)abelianization

6.1 Subspecies and quotient species

Given species p and q, we say p is a subspecies of q if each $\mathrm{p}[F]$ is a subspace of $\mathrm{q}[F]$, and the structure maps $\beta_{G,F}$ of p are obtained by restricting those of q. We write $\mathrm{p} \subseteq \mathrm{q}$. In this case, we have an inclusion map $\mathrm{p} \hookrightarrow \mathrm{q}$ of species.

Further, we can form the quotient species q/p by taking quotients (of vector spaces) in each component. The structure maps of q/p are induced from those of q. Thus, we have the quotient map $\mathrm{q} \twoheadrightarrow \mathrm{q}/\mathrm{p}$ of species.

Let $f : \mathsf{p} \to \mathsf{q}$ be a map of species. Let $\ker(f)$ denote the subspecies of p whose F-component is the kernel of f_F. The species $\mathrm{image}(f)$, $\mathrm{coker}(f)$ and $\mathrm{coimage}(f)$ are defined similarly. They are a subspecies of q, a quotient species of q, and a quotient species of p, respectively, and there are isomorphisms of species

$$\text{(27)} \quad \mathrm{image}(f) \cong \ker(\mathsf{q} \twoheadrightarrow \mathrm{coker}(f)) \quad \text{and} \quad \mathrm{coimage}(f) \cong \mathrm{coker}(\ker(f) \hookrightarrow \mathsf{p}).$$

We have the following diagram of species.

$$\begin{tikzcd}
0 \arrow[r] & \ker(f) \arrow[r, hook] & \mathsf{p} \arrow[r, "f"] \arrow[d, two heads] & \mathsf{q} \arrow[r, two heads] & \mathrm{coker}(f) \arrow[r] & 0 \\
& & \mathrm{coimage}(f) \arrow[r, "\cong"'] & \mathrm{image}(f) \arrow[u, hook] & &
\end{tikzcd}$$

A map f of species is injective (surjective) if each component f_F is injective (surjective).

Subspecies give rise to injective maps, and quotient species to surjective maps.

An isomorphism of species is both injective and surjective.

6.2 Submonoids and quotient monoids

Suppose a is a subspecies of a monoid b. If μ_A^F : b$[F] \to$ b$[A]$ restricts to a map a$[F] \to$ a$[A]$ for each $F \geq A$, we say a is a submonoid of b. This is equivalent to the inclusion a $\hookrightarrow$ b being a morphism of monoids.

Suppose instead that b is a quotient species of a monoid a. If μ_A^F : a$[F] \to$ a$[A]$ factors through a map b$[F] \to$ b$[A]$ for each $F \geq A$, we say b is a quotient monoid of a. This is equivalent to the projection a $\twoheadrightarrow$ b being a morphism of monoids.

Similar terminology can be employed for comonoids and bimonoids.

Proposition 6. *Let $f : \mathsf{h} \to \mathsf{k}$ be a morphism of (co, bi)monoids. Then $\ker(f)$ is a sub(co, bi)monoid of h, while $\operatorname{coker}(f)$ is a quotient (co, bi)monoid of k. Hence, by* (27)*, $\operatorname{image}(f)$ (or equivalently $\operatorname{coimage}(f)$) is also a (co, bi)monoid.*

Proof. Suppose $f : \mathsf{a} \to \mathsf{b}$ is a morphism of monoids. Then the product of a restricts to $\ker(f)$, while the product of b projects onto $\operatorname{coker}(f)$:

$$\begin{tikzcd}
\ker(f_F) \arrow[r] \arrow[d, dotted] & \mathsf{a}[F] \arrow[r, "f_F"] \arrow[d, "\mu_A^F"'] & \mathsf{b}[F] \arrow[d, "\mu_A^F"] \\
\ker(f_A) \arrow[r] & \mathsf{a}[A] \arrow[r, "f_A"'] & \mathsf{b}[A]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{a}[F] \arrow[r, "f_F"] \arrow[d, "\mu_A^F"'] & \mathsf{b}[F] \arrow[r] \arrow[d, "\mu_A^F"] & \operatorname{coker}(f_F) \arrow[d, dotted] \\
\mathsf{a}[A] \arrow[r, "f_A"'] & \mathsf{b}[A] \arrow[r] & \operatorname{coker}(f_A).
\end{tikzcd}$$

The squares with solid edges coincide with (8). For the statement for comonoids, we reverse the vertical arrows and change μ to Δ. Combining the statements for monoids and comonoids yields the statement for bimonoids. $\square$

Note that any sub or quotient (co)monoid of a (co)commutative (co)monoid is again (co)commutative.

6.3 (Co)abelianization

Every commutative monoid is a monoid. Conversely, starting with a monoid a, the linear span of the elements

$$\mu_A^F(x) - \mu_A^G \beta_{G,F}(x) \ \in \mathsf{a}[A], \tag{28}$$

as x, F and G vary (with F and G both greater than A and of the same support), forms a submonoid of a: For any $B \leq A$, applying μ_B^A to the above element, by associativity (7), yields

$$\mu_B^F(x) - \mu_B^G \beta_{G,F}(x),$$

which is again of the above form. Taking quotient of a by this submonoid yields a commutative monoid. This is the abelianization of a. We denote it by a_{ab}, and the quotient map $\mathsf{a} \twoheadrightarrow \mathsf{a}_{ab}$ by π_{a}.

Any morphism of monoids from a to a commutative monoid b factors through π_{a} yielding a commutative diagram

$$\begin{tikzcd} \mathsf{a} \arrow[d, "\pi_{\mathsf{a}}"'] \arrow[dr] & \\ \mathsf{a}_{ab} \arrow[r, dotted] & \mathsf{b}. \end{tikzcd}$$

Thus, a_{ab} is the largest commutative quotient monoid of a. Equivalently, abelianization is the left adjoint of the inclusion functor. This is shown below.

(29) $$\mathsf{Mon}(\mathcal{A}\text{-}\mathrm{Sp}) \underset{}{\overset{(-)_{ab}}{\rightleftarrows}} \mathsf{Mon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathrm{Sp}).$$

By convention, we write the left adjoint above the right adjoint.

Dually, every comonoid c has a largest cocommutative subcomonoid called the coabelianization. We denote it by c^{coab}, and the inclusion map $\mathsf{c}^{coab} \hookrightarrow \mathsf{c}$ by $\pi_{\mathsf{c}}^{\vee}$. Explicitly, $\mathsf{c}^{coab}[A]$ consists of those $x \in \mathsf{c}[A]$ for which

$$\beta_{G,F}\Delta_A^F(x) = \Delta_A^G(x)$$

for all faces F and G both greater than A and of the same support. The image of any morphism of comonoids from a cocommutative comonoid d to a comonoid c lies inside c^{coab} yielding a commutative diagram

$$\begin{tikzcd} & \mathsf{c} \\ \mathsf{d} \arrow[ur] \arrow[r, dotted] & \mathsf{c}^{coab}. \arrow[u, "\pi_{\mathsf{c}}^{\vee}"'] \end{tikzcd}$$

Thus, coabelianization is the right adjoint of the inclusion functor.

Proposition 7. *Let* h *be a bimonoid. Then:* h_{ab} *is a commutative bimonoid, and further, it is bicommutative if* h *is cocommutative. Dually,* h^{coab} *is a cocommutative bimonoid, and further, it is bicommutative if* h *is commutative.*

Proof. We explain the first statement. We need to check that the linear span of the elements (28) is closed under the coproduct. This is checked below.

$$\Delta_A^K(\mu_A^F(x) - \mu_A^G \beta_{G,F}(x)) =$$
$$\mu_K^{KF} \beta_{KF,FK} \Delta_F^{FK}(x) - \mu_K^{KG} \beta_{KG,GK} \Delta_G^{GK} \beta_{G,F}(x) =$$
$$(\mu_K^{KF} - \mu_K^{KG} \beta_{KG,KF})(\beta_{KF,FK} \Delta_F^{FK}(x)).$$

We made use of (1), naturality (9) and the bimonoid axiom (11). $\square$

Thus, for a bimonoid h, the abelianization and coabelianization constructions can be iterated. Observe that there is a canonical morphism of bimonoids

$$(\mathsf{h}^{coab})_{ab} \to (\mathsf{h}_{ab})^{coab}. \tag{30}$$

7 Duality

7.1 Duality functor

The dual of a species p is the species p^* defined as follows. Let

$$\mathsf{p}^*[F] := \mathsf{p}[F]^*,$$

where $\mathsf{p}[F]^*$ denotes the dual of the vector space $\mathsf{p}[F]$.

For F and G of the same support, we have the invertible linear map $\beta_{G,F} : \mathsf{p}[F] \to \mathsf{p}[G]$. Dualize it and then take its inverse (or equivalently, take its inverse and then dualize) to obtain a linear map $\mathsf{p}^*[F] \to \mathsf{p}^*[G]$. We let this be the $\beta_{G,F}$ map for p^*.

Moreover a map f : p $\to$ q of species induces a map f^* : $\mathrm{q}^* \to \mathrm{p}^*$ of species.

This defines a functor

(31) $$(-)^* : \mathcal{A}\text{-Sp} \to \mathcal{A}\text{-Sp}^{\mathrm{op}}.$$

We call it the duality functor on species.

7.2 Dual of a bimonoid

Let (a, μ) be a monoid in species. Its dual a^* is then a comonoid:

The coproduct of a^*, denoted μ^*, is given by

$$\mathsf{a}^*[A] = \mathsf{a}[A]^* \xrightarrow{(\mu_A^F)^*} \mathsf{a}[F]^* = \mathsf{a}^*[F],$$

where $(\mu_A^F)^*$ is the linear dual of μ_A^F. The dual of the associativity axiom of a yields the coassociativity axiom of a^*.

Further, if μ is commutative, then μ^* is cocommutative. The dual of the commutativity axiom of a yields the cocommutativity axiom of a^*.

Similarly, the dual c^* of a comonoid (c, Δ) is a monoid whose product Δ^* is given by

$$\mathsf{c}^*[F] = \mathsf{c}[F]^* \xrightarrow{(\Delta_A^F)^*} \mathsf{c}[A]^* = \mathsf{c}^*[A].$$

If Δ is cocommutative, then Δ^* is commutative.

Note that these constructions do not require either a or c to be finite-dimensional.

Combining the two constructions, if $(\mathsf{h}, \mu, \Delta)$ is a bimonoid, then so is $(\mathsf{h}^*, \Delta^*, \mu^*)$. The dual of the bimonoid axiom of h yields the bimonoid axiom of h^*.

More generally, if $(\mathsf{h}, \mu, \Delta)$ is a q-bimonoid, then so is $(\mathsf{h}^*, \Delta^*, \mu^*)$. For instance, the dual of a 0-bimonoid is a 0-bimonoid.

7.3 (Co)abelianization

For any monoid a and comonoid c,

(32) $$(\mathsf{a}_{ab})^* = (\mathsf{a}^*)^{coab} \quad \text{and} \quad (\mathsf{c}^{coab})^* = (\mathsf{c}^*)_{ab},$$

where recall that a_{ab} denotes the abelianization of a, and c^{coab} the coabelianization of c.

Hence we say that abelianization and coabelianization are contragredients of each other.

7.4 Self-duality

We now turn to the notion of self-duality. In this discussion, all species and (co, bi)monoids are assumed to be finite-dimensional.

Any species p is isomorphic to its dual p^*: Choose isomorphisms $\mathrm{p}[F] \cong \mathrm{p}^*[F]$, one for each face F, which are compatible with $\beta_{G,F}$. Equivalently, choose isomorphisms $\mathrm{p}[\mathrm{X}] \cong \mathrm{p}^*[\mathrm{X}]$, one for each flat X. Hence, we say that a species is self-dual.

We say a bimonoid h is self-dual if h and h^* are isomorphic as bimonoids.

In contrast to species, a bimonoid may or may not be self-dual.

For instance, if h is cocommutative but not commutative, then h^* will be commutative but not cocommutative. Hence the two cannot be isomorphic.

Observe that for a species p, monoid a, comonoid c, and bimonoid h,

$$(\mathsf{p}^*)^* = \mathsf{p}, \quad (\mathsf{a}^*)^* = \mathsf{a}, \quad (\mathsf{c}^*)^* = \mathsf{c}, \quad (\mathsf{h}^*)^* = \mathsf{h}.$$

These identifications are canonical, so we show them as equalities.

We say a morphism $f : \mathsf{p} \to \mathsf{p}^*$ of species is self-dual if $f = f^*$. The same definition can be made for bimonoids.

Proposition 8. *Suppose $f : \mathsf{h} \to \mathsf{h}^*$ is a self-dual morphism of bimonoids. Then $\mathrm{image}(f)$ (or equivalently $\mathrm{coimage}(f)$) is a self-dual bimonoid.*

Proof. For any morphism $f : \mathsf{p} \to \mathsf{q}$ of species and its dual $f^* : \mathsf{q}^* \to \mathsf{p}^*$,

$$\ker(f)^* = \mathrm{coker}(f^*) \quad \text{and} \quad \mathrm{coker}(f)^* = \ker(f^*).$$

Thus,

$$\begin{aligned} \mathrm{image}(f)^* &= \ker(\mathsf{q} \to \mathrm{coker}(f))^* \\ &= \mathrm{coker}((\mathsf{q} \to \mathrm{coker}(f))^*) \\ &= \mathrm{coker}(\ker(f^*) \to \mathsf{q}^*) = \mathrm{coimage}(f^*). \end{aligned}$$

Now, let $f : \mathsf{h} \to \mathsf{h}^*$ be as in the proposition. By hypothesis, $f = f^*$. Since $\mathrm{image}(f)$ and $\mathrm{coimage}(f)$ are canonically isomorphic, the result follows. $\square$

8 Set-species

8.1 Set-species

An $\mathcal{A}$-set-species is a functor

$$\mathrm{p} : \mathcal{A}\text{-}\mathtt{Hyp} \to \mathsf{Set}.$$

A map of $\mathcal{A}$-set-species $\mathrm{p} \to \mathrm{q}$ is a natural transformation. We denote the category of $\mathcal{A}$-set-species by

$$\mathcal{A}\text{-}\mathtt{SetSp} = [\mathcal{A}\text{-}\mathtt{Hyp}, \mathsf{Set}].$$

Explicitly: An $\mathcal{A}$-set-species consists of a family of sets $\mathrm{p}[F]$, one for each face F, together with bijections $\beta_{G,F}$ which satisfy (1). Similarly, a map of set-species $f : \mathrm{p} \to \mathrm{q}$ is a family of maps $f_F : \mathrm{p}[F] \to \mathrm{q}[F]$ which satisfy (3).

Equivalently: An $\mathcal{A}$-set-species is a family of sets $\mathrm{p}[\mathrm{X}]$, one for each flat X. A map of $\mathcal{A}$-set-species $f : \mathrm{p} \to \mathrm{q}$ is a family of maps $f_{\mathrm{X}} : \mathrm{p}[\mathrm{X}] \to \mathrm{q}[\mathrm{X}]$, one for each flat X.

8.2 Bimonoids

Monoids, comonoids and bimonoids in set-species (as well as their commutative counterparts) are defined as for species, with the understanding that all structure maps involved are now maps between sets (as opposed to linear maps).

Note that there is no notion of a q-bimonoid or a signed commutative monoid in set-species.

8.3 Linearization functor

Fix a field $\Bbbk$. Consider the functor

$$\Bbbk(-) : \mathsf{Set} \longrightarrow \mathsf{Vec},$$

which sends a set to the vector space with basis the given set. Composing a set-species p with this functor yields a species, which we will denote by $\Bbbk\mathrm{p}$ or p. Thus, we have

$$\Bbbk(-) : \mathcal{A}\text{-}\mathtt{SetSp} \longrightarrow \mathcal{A}\text{-}\mathtt{Sp}. \tag{33}$$

We call this the linearization functor.

Let a be a monoid in set-species. Then $\Bbbk\mathrm{a}$ is a monoid in species whose product components are obtained by linearizing the product components of a. Similarly, linearization preserves comonoids and bimonoids.

9 Problems I

1. Show that the bimonoid axiom (11) is equivalent to the following two axioms.

(34)

$$
\begin{tikzcd}
\mathsf{h}[F] \arrow[r, "\mu^F_{F\wedge G}"] \arrow[d, "\Delta^{FG}_F"'] & \mathsf{h}[F\wedge G] \arrow[r, "\Delta^G_{F\wedge G}"] & \mathsf{h}[G] \\
\mathsf{h}[FG] \arrow[rr, "\beta_{GF,FG}"'] & & \mathsf{h}[GF] \arrow[u, "\mu^{GF}_G"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{h}[F] \arrow[rr, "\mathrm{id}"] \arrow[dr, "\mu^F_A"'] & & \mathsf{h}[F] \\
& \mathsf{h}[A] \arrow[ur, "\Delta^F_A"'] &
\end{tikzcd}
$$

2. Let a be a commutative $\mathcal{A}$-monoid. Show that: For $A \le G$ and $\mathrm{s}(F) = \mathrm{s}(G)$, the diagram

(35)
$$
\begin{tikzcd}
\mathsf{a}[F] \arrow[r, "\beta_{G,F}"] \arrow[d, "\beta_{AF,F}"'] & \mathsf{a}[G] \arrow[d, "\mu^G_A"] \\
\mathsf{a}[AF] \arrow[r, "\mu^{AF}_A"'] & \mathsf{a}[A]
\end{tikzcd}
$$

commutes.

3. Define a double monoid to be a triple (a, μ, μ'),

where a is a species, (a, μ) and (a, μ') are monoids, and for any faces $A \leq F$ and $A \leq G$, the diagram

$$\begin{tikzcd}
\mathsf{a}[F] \arrow[r, "\mu_A^F"] & \mathsf{a}[A] & \mathsf{a}[G] \arrow[l, "(\mu')_A^G"'] \\
\mathsf{a}[FG] \arrow[u, "(\mu')_F^{FG}"] \arrow[rr, "\beta_{GF,FG}"'] & & \mathsf{a}[GF] \arrow[u, "\mu_G^{GF}"']
\end{tikzcd}$$

commutes. Show that double monoids and commutative monoids are equivalent notions.

4. Show that: For any $\mathcal{A}$-q-bimonoid, $\mu_A^F \Delta_A^F$ is an idempotent operator on $\mathsf{h}[A]$ for any $A \leq F$. (Recall that an idempotent operator on a vector space V is a linear map $e : V \to V$ such that $e^2 = e$.)

5. Show that: For a comonoid $\mathsf{c} \neq 0$, the coabelianization of c cannot be 0. Similarly, for a monoid $\mathsf{a} \neq 0$, the abelianization of a cannot be 0.

10 Problems II

1. For any comonoid $\mathsf{c} \neq 0$, show that $\mathcal{P}(\mathsf{c}) \neq 0$.

2. Check directly that $\mathcal{S}(\mathsf{p})$ is a bicommutative bimonoid by showing that the bicommutative bimonoid axiom (21) holds for (**??**).

3. Check that: For a bimonoid h, if $f : \mathsf{E} \to \mathsf{h}$ is a morphism of monoids, then it is also a morphism of comonoids.

4. Let (ξ_{X}) be any set of scalars indexed by flats. Check that: The map

   (36)
   $$\Sigma \to \Sigma^*, \qquad \mathtt{H}_{F/A} \mapsto \sum_{G\colon G \geq A} \xi_{\mathrm{s}(FG)}\, \mathtt{M}_{G/A}$$

   on the A component, is a self-dual morphism of bimonoids.

5. For any species p, formulate the cofreeness of the comonoid $\mathcal{T}^{\vee}(\mathsf{p})$ as a universal property.