Hyperplane arrangements

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1 Faces and flats

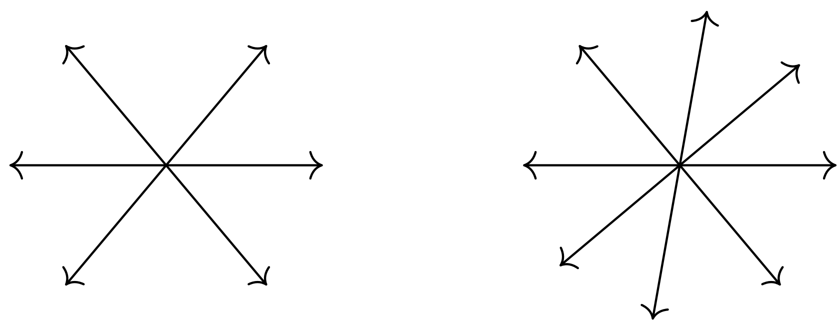
1.1 Hyperplane arrangements

A hyperplane arrangement $\mathcal{A}$ is a finite set of hyperplanes (codimension-one affine subspaces) in a fixed real vector space.

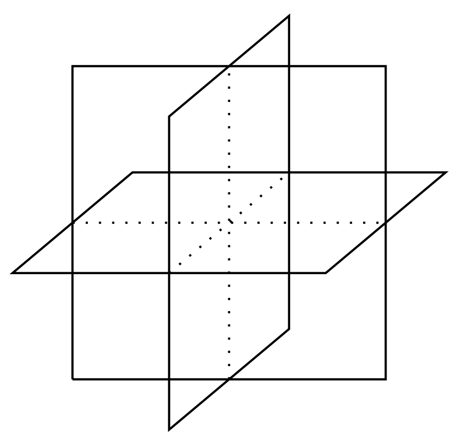
The latter is called the ambient space of $\mathcal{A}$.

The arrangement is linear if all its hyperplanes pass through the origin.

Unless stated otherwise, all our arrangements are assumed to be linear.



The arrangement on the left consists of three lines in $\mathbb{R}^2$, while the one on the right consists of four lines in $\mathbb{R}^2$.



Both the above arrangements consist of three planes in $\mathbb{R}^3$.

Let O denote the intersection of all hyperplanes. We call it the central face.

The rank of $\mathcal{A}$, denoted $\mathrm{rk}(\mathcal{A})$, is the difference between the dimensions of the ambient space and the central face.

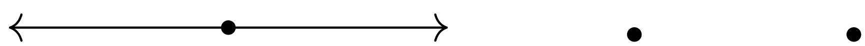
An arrangement has rank 0 iff it has no hyperplanes.

An arrangement has rank 1 iff it has exactly one hyperplane.

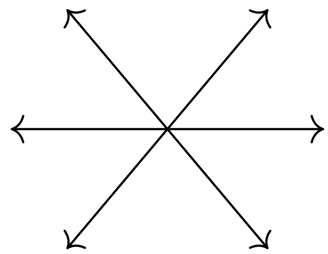
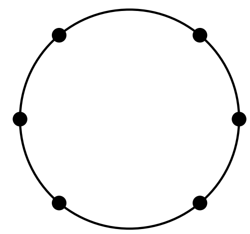
There are two standard ways to view a hyperplane arrangement, namely, the linear model and the spherical model.

The latter is obtained from the former by first taking quotient by the central face and then cutting with the unit sphere.

Some illustrative pictures are shown below.

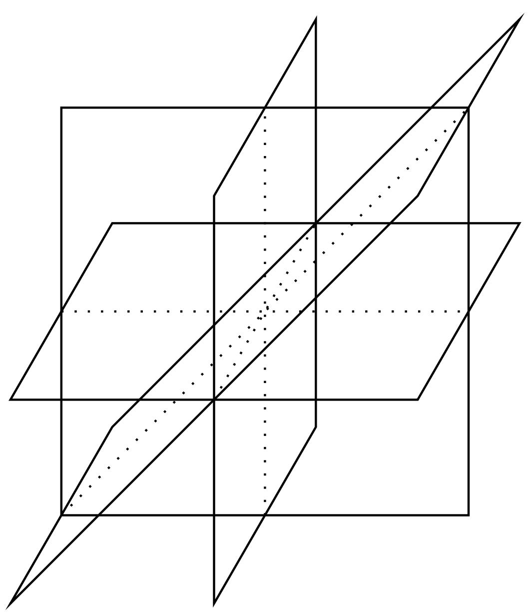
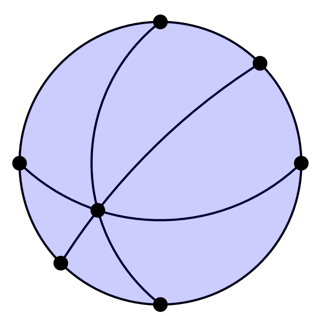


This is an arrangement of rank 1 with ambient space $\mathbb{R}$ and with one hyperplane, namely, the origin. The linear model is on the left and the spherical model on the right.

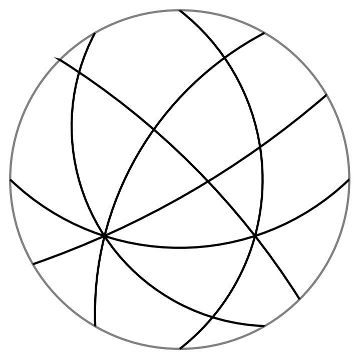
 

This is the arrangement of three lines in the plane; the linear model is on the left and the spherical model on the right.

More generally, one can consider the arrangement of n lines in the plane. This is an arrangement of rank two.

This is an arrangement of four planes in $\mathbb{R}^3$.



This is a spherical model of six hyperplanes (great circles) in three-space. Only one half of the arrangement is visible in the picture, the other half being on the backside.

This is an arrangement of rank three.

1.2 Faces and the Tits monoid

Each hyperplane has two associated half-spaces.

A face of $\mathcal{A}$ is a subset of the ambient space obtained by intersecting half-spaces, with at least one half-space chosen for each hyperplane.

In the spherical model, faces are precisely the vertices, edges, triangles, and so on.

The arrangement is called simplicial if all faces are simplices.

Let $\Sigma[\mathcal{A}]$ denote the set of faces of $\mathcal{A}$. It is a graded poset under inclusion. The minimum element is the central face. (It is not visible in the spherical model.)

The rank of a face F is denoted $\mathrm{rk}(F)$. Note that F has a dimension in the ambient space, and

$$\mathrm{rk}(F) = \dim(F) - \dim(O).$$

A maximal face is called a chamber. We let $\Gamma[\mathcal{A}]$ denote the set of chambers.

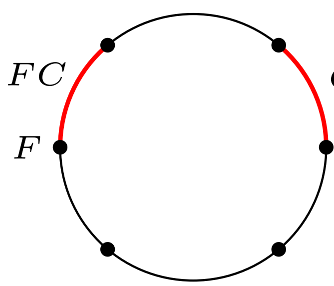
A corank-one face is called a panel; we also say a face F is a panel of a face G if $F \lessdot G$, that is, if G covers F in the poset of faces.

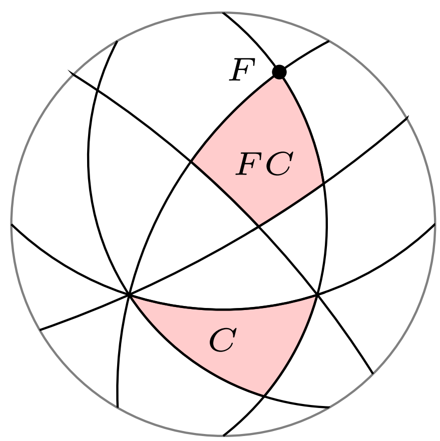
Every face F has an opposite face $\overline{F}$ given by $\{-x \mid x \in F\}$.

We usually denote faces by letters A, B, F, G, H, K, and chambers by letters C, D, E.

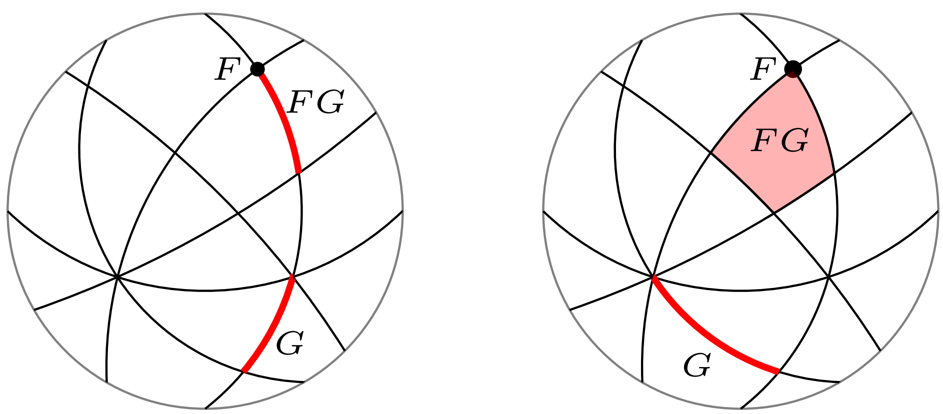
The poset of faces $\Sigma[\mathcal{A}]$ carries a (noncommutative) monoid structure. We call this the Tits monoid. The central face is the identity element. For faces F and G, we denote their Tits product by FG.

Moreover, for F a face and C a chamber, FC is a chamber, thus, the set of chambers $\Gamma[\mathcal{A}]$ is a left $\Sigma[\mathcal{A}]$-set. This is illustrated below.





Geometrically, among all chambers containing F, the chamber FC is closest to C.


F
FG
G
F
FG
G

Some basic properties of the Tits product are listed below.

(1) $$FF = F \text{ and } FGF = FG.$$

(2)

If $G \leq H$, then $FG \leq FH$. In particular, $F \leq FG$.

(3)

If $FG = K$ and $F \leq H \leq K$, then $HG = K$.

(4) $$HF = F \iff H \leq F,$$

(5) $$H\overline{F} = F \iff H = F.$$

A left regular band, or LRB for short, is a monoid in which the axiom $xyx = xy$ holds. By (1), we see that the Tits monoid is a left regular band.

1.3 Flats and the Birkhoff monoid

A flat of an arrangement $\mathcal{A}$ is a subspace of the ambient space obtained by intersecting an arbitrary subset of hyperplanes in $\mathcal{A}$.

Let $\Pi[\mathcal{A}]$ denote the set of flats. It is a graded lattice under inclusion.

We will use the letters X, Y, Z, W to denote flats. The minimum and maximum flats will be denoted $\bot$ and $\top$, respectively. They coincide with the central face and ambient space, respectively.

We view the lattice of flats $\Pi[\mathcal{A}]$ as a monoid with product given by the join. We call this the Birkhoff monoid, and refer to $\mathrm{X} \vee \mathrm{Y}$ as the Birkhoff product of X and Y. The minimum flat $\perp$ is the identity element.

The Birkhoff monoid is a commutative left regular band.

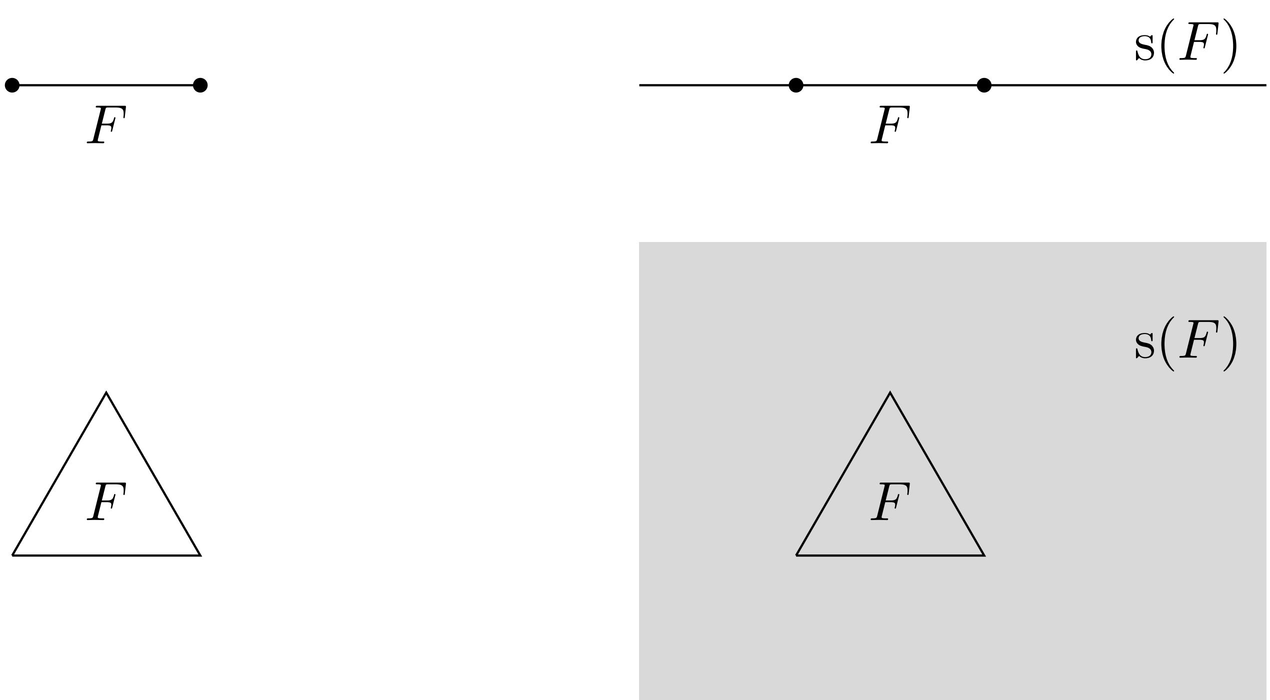
1.4 Support map

The support of a face F, denoted $\mathrm{s}(F)$, is the smallest flat which contains F. It is the linear span of F. We say a flat X supports a face F if $\mathrm{s}(F) = \mathrm{X}$.

The support map

$$\mathrm{s} : \Sigma[\mathcal{A}] \twoheadrightarrow \Pi[\mathcal{A}] \tag{6}$$

sends a face to its support. It is surjective and order-preserving. Illustrations of the support of a rank-two face and a rank-three face are shown below.



The support map is a homomorphism from the Tits monoid to the Birkhoff monoid, that is,

$$\mathrm{s}(FG) = \mathrm{s}(F) \vee \mathrm{s}(G). \tag{7}$$

Observe that FG and GF always have the same support. In particular, if $GF = G$, then FG and G have the same support. Similar useful observations are given below.

$$GF = G \iff \mathrm{s}(F) \leq \mathrm{s}(G). \tag{8}$$

$$FG = F \text{ and } GF = G \iff \mathrm{s}(F) = \mathrm{s}(G). \tag{9}$$

To summarize: The relation

$$F \sim G \iff FG = F \text{ and } GF = G \tag{10}$$

is an equivalence relation on the set of faces whose equivalence classes correspond to flats.

1.5 Bifaces and the Janus monoid

A biface is a pair (F, F') of faces such that F and F' have the same support.

Let $\mathrm{J}[\mathcal{A}]$ denote the set of bifaces. The operation

$$(F, F')(G, G') := (FG, G'F')$$

turns $\mathrm{J}[\mathcal{A}]$ into a monoid. We call this the Janus monoid. The identity element is (O, O).

The Janus monoid is canonically isomorphic to its opposite monoid via

$$\mathrm{J}[\mathcal{A}] \to \mathrm{J}[\mathcal{A}]^{\mathrm{op}}, \qquad (F, F') \mapsto (F', F).$$

Moreover, there is a commutative diagram of monoids

(11)
$$\begin{tikzcd}
\mathrm{J}[\mathcal{A}] \arrow[r, two heads] \arrow[d, two heads] & \Sigma[\mathcal{A}]^{\mathrm{op}} \arrow[d, two heads, "\mathrm{s}"] \\
\Sigma[\mathcal{A}] \arrow[r, two heads, "\mathrm{s}"'] & \Pi[\mathcal{A}]
\end{tikzcd}
\qquad
\begin{tikzcd}
(F, F') \arrow[r, mapsto] \arrow[d, mapsto] & F' \arrow[d, mapsto] \\
F \arrow[r, mapsto] & \mathrm{s}(F) = \mathrm{s}(F')
\end{tikzcd}$$

with s being the support map, and the maps from $\mathrm{J}[\mathcal{A}]$ being the projections on the two coordinates, respectively.

1.6 Stars and top-stars

For a face F, let $\Sigma[\mathcal{A}]_F$ denote the set of faces of $\mathcal{A}$ which are greater than F. This is the star of F.

It is a monoid under the Tits product with identity element F.

For clarity, we denote elements of $\Sigma[\mathcal{A}]_F$ by K/F, where K is a face greater than F.

Let $\Gamma[\mathcal{A}]_F$ denote the set of chambers of $\mathcal{A}$ which are greater than F. This is the top-star of F.

Lemma 1. *When F and G have the same support, we have an isomorphism*

$$\Sigma[\mathcal{A}]_F \xrightarrow{\cong} \Sigma[\mathcal{A}]_G, \qquad K/F \mapsto GK/G$$

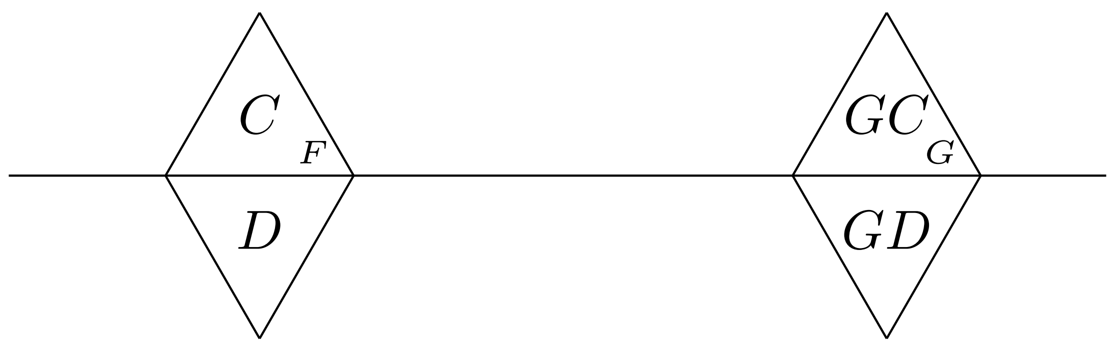
of monoids, and hence of posets. The inverse is given by

$$\Sigma[\mathcal{A}]_G \xrightarrow{\cong} \Sigma[\mathcal{A}]_F, \qquad H/G \mapsto FH/F.$$

Further, it restricts to a bijection

$$\Gamma[\mathcal{A}]_F \xrightarrow{\cong} \Gamma[\mathcal{A}]_G, \qquad C/F \mapsto GC/G.$$

A rank-three illustration of the last bijection is shown below.



In the picture, faces F and G are of rank two and have the same support. The top-star of F consists of chambers C and D, which under the bijection correspond to chambers GC and GD in the top-star of G.

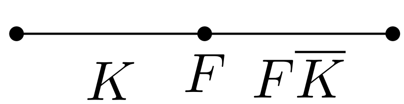
1.7 Arrangements under and over a flat

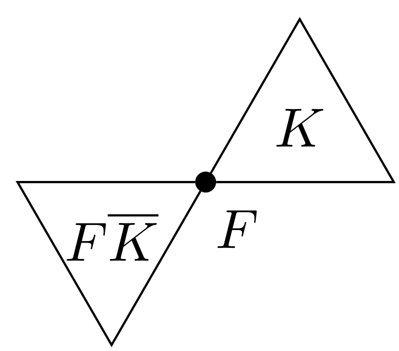
For any flat X, the arrangement under X is the arrangement $\mathcal{A}^{\mathrm{X}}$ whose ambient space is X and hyperplanes are codimension-one subspaces of X obtained by intersecting X with hyperplanes in $\mathcal{A}$ not containing X.

Faces of $\mathcal{A}^{\mathrm{X}}$ can be canonically identified with faces of $\mathcal{A}$ with support smaller than X, and chambers can be identified with faces of $\mathcal{A}$ with support X.

For any flat Y, the arrangement over Y is the arrangement $\mathcal{A}_{\mathrm{Y}}$ consisting of those hyperplanes which contain Y. The ambient space remains the same.

For any face F, let $\mathcal{A}_F := \mathcal{A}_{\mathrm{s}(F)}$. Thus, there is no distinction between $\mathcal{A}_F$ and $\mathcal{A}_G$ when F and G have the same support. However, for book-keeping purposes, we identify faces of $\mathcal{A}_F$ with the star of F, and chambers with the top-star of F. Thus, K/F and C/F denote a face and chamber of $\mathcal{A}_F$, respectively. In this notation, the opposite of a face K/F of $\mathcal{A}_F$ is $F\overline{K}/F$. This is illustrated below.





Also note that $\mathrm{rk}(K/F) = \mathrm{rk}(K) - \mathrm{rk}(F)$.

The under and over constructions can be combined together as follows.

Let $\mathrm{Y} \leq \mathrm{X}$. Then one may first go under X and then over Y, or first go over Y and then under X. The resulting arrangements $(\mathcal{A}^{\mathrm{X}})_{\mathrm{Y}}$ and $(\mathcal{A}_{\mathrm{Y}})^{\mathrm{X}}$ are identical and we denote it by $\mathcal{A}_{\mathrm{Y}}^{\mathrm{X}}$.

2 Nested faces and lunes

2.1 Top-nested faces and top-lunes

Let H be any face and D be a chamber greater than H. We refer to the pair (H, D) as a top-nested face.

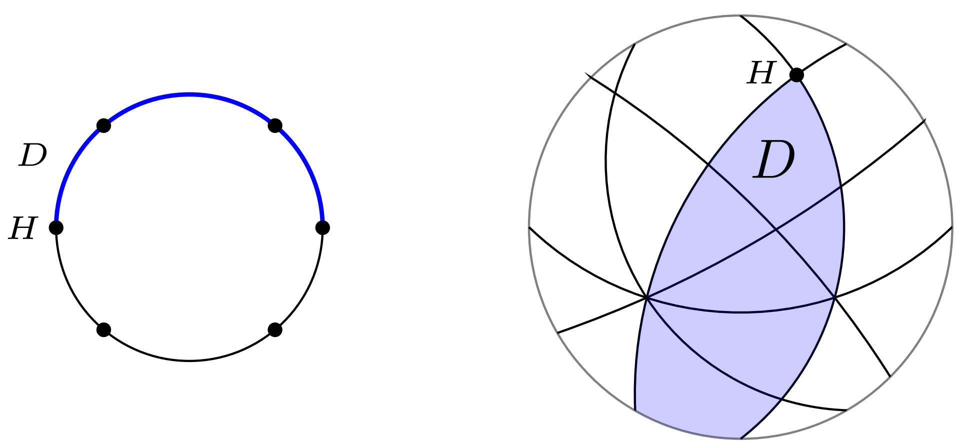
We define the support of such a top-nested face to be

(12) $$\mathrm{s}(H, D) := \{C \mid HC = D\}.$$

This is a subset of $\Gamma[\mathcal{A}]$.

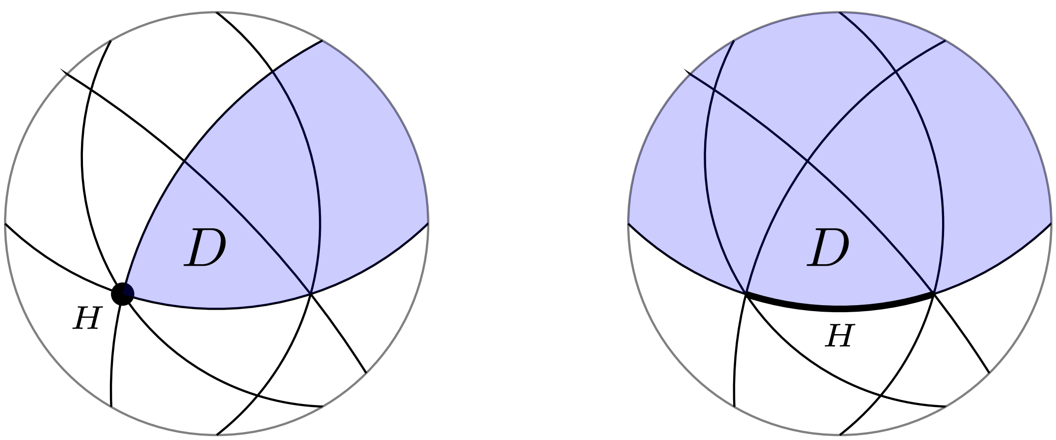
A top-lune is a subset of the set of chambers of the form $\mathrm{s}(H, D)$ for some top-nested face (H, D).

Illustrations in rank two and rank three are shown below.



The regions marked in blue are top-lunes. In the second picture, the top-lune is not fully visible; a small part is on the back side.

Two more illustrations in rank three are shown below.



A top-lune corresponds to an equivalence class in the set of top-nested faces under the relation

(13)

$$(H, D) \sim (G, C) \iff HG = H,\ GH = G,\ HC = D,\ GD = C.$$

The class of (H, D) is the top-lune $\mathrm{s}(H, D)$.

For the top-lune $\mathrm{s}(H, D)$, the flat $\mathrm{s}(H)$ is unique, and is called the base of that top-lune.

2.2 Nested faces and lunes

More generally:

A nested face is a pair of faces (H, G) such that $H \leq G$.

We define the support of such a nested face to be

(14)
$$\begin{aligned} \mathrm{s}(H, G) &:= \{F \mid HF = G \text{ and } \mathrm{s}(F) = \mathrm{s}(G)\} \\ &= \{F \mid HF = G \text{ and } FH = F\}. \end{aligned}$$

This is a subset of $\Sigma[\mathcal{A}]$.

A lune is a subset of the set of faces of the form $\mathrm{s}(H, G)$ for some nested face (H, G).

It corresponds to an equivalence class in the set of nested faces under the relation

(15)

$$(H, G) \sim (K, F) \Longleftrightarrow HK = H,\ KH = K,\ HF = G,\ KG = F.$$

The class of (H, G) is the lune $\mathrm{s}(H, G)$.

For the lune $\mathrm{s}(H, G)$, the flats $\mathrm{s}(H)$ and $\mathrm{s}(G)$ are unique, and are, respectively, called the base and case of that lune.

The case of a top-lune is the maximum flat.

Lunes of the form $\mathrm{s}(H, H)$ can be identified with the flat $\mathrm{s}(H)$, while lunes of the form $\mathrm{s}(O, H)$ can be identified with the face H.

Let $\Lambda[\mathcal{A}]$ denote the set of lunes, and $\widehat{\Lambda}[\mathcal{A}]$ denote the set of top-lunes.

Lunes will usually be denoted by letters L, M, N.

We denote the base and case of L by $\mathrm{b}(\mathrm{L})$ and $\mathrm{c}(\mathrm{L})$, respectively.

Lemma 2. *There are correspondences*

Lunes of $\mathcal{A}$ with base $\mathrm{X} \longleftrightarrow$ Faces of $\mathcal{A}_{\mathrm{X}}$,

Top-lunes of $\mathcal{A}$ with base $\mathrm{X} \longleftrightarrow$ Chambers of $\mathcal{A}_{\mathrm{X}}$,

Lunes of $\mathcal{A}$ with base X and case $\mathrm{Y} \longleftrightarrow$ Chambers of $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$.

2.3 Category of lunes

We define the category of lunes. Its objects are flats and morphisms are lunes.

More precisely, a morphism from Y to X is a lune L whose base is X and case is Y.

Identity morphisms are flats.

To define composition of morphisms, recall that every lune L can be written as the support of a nested face (F, G), that is, $\mathrm{L} = \mathrm{s}(F, G)$. Composition is then defined by

$$\mathrm{s}(F, G) \circ \mathrm{s}(G, H) = \mathrm{s}(F, H). \tag{16}$$

It is straightforward to check that this is well-defined.

3 Incidence algebras, and zeta and Möbius functions

3.1 Flat-incidence algebra

A nested flat is a pair of flats (X, Y) such that $\mathrm{X} \leq \mathrm{Y}$.

The flat-incidence algebra, denoted $\mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$, is the incidence algebra of the poset of flats.

It consists of functions s on nested flats, with the product of s and t given by

(17) $$(st)(\mathrm{X}, \mathrm{Z}) = \sum_{\mathrm{Y}:\, \mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}} s(\mathrm{X}, \mathrm{Y})t(\mathrm{Y}, \mathrm{Z}).$$

3.2 Zeta and Möbius functions

The zeta function $\zeta \in \mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$ is defined by $\zeta(\mathrm{X}, \mathrm{Y}) = 1$ for all $\mathrm{X} \leq \mathrm{Y}$.

It is invertible in the flat-incidence algebra and its inverse is the Möbius function $\mu \in \mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$.

The latter is the unique element such that $\mu(\mathrm{X}, \mathrm{X}) = 1$ for all X and

$$\sum_{\mathrm{W}\colon \mathrm{W} \geq \mathrm{X},\, \mathrm{W} \vee \mathrm{Y} = \mathrm{Z}} \mu(\mathrm{X}, \mathrm{W}) = 0 \tag{18}$$

for all $\mathrm{X} < \mathrm{Y} \leq \mathrm{Z}$.

This is the Weisner formula.

Note very carefully that Y is strictly greater than X in this formula.

3.3 Lune-incidence algebra

The face-incidence algebra, denoted $\mathrm{I}_{\mathrm{face}}[\mathcal{A}]$, is the incidence algebra of the poset of faces.

It consists of functions s on nested faces, with the product of s and t given by

$$(st)(F, H) = \sum_{G\colon F \leq G \leq H} s(F, G)t(G, H). \tag{19}$$

The lune-incidence algebra, denoted $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$, is the subalgebra of $\mathrm{I}_{\mathrm{face}}[\mathcal{A}]$ consisting of those functions s such that

(20)

$$s(A, F) = s(B, G) \text{ whenever } (A, F) \sim (B, G),$$

with $\sim$ as in (15).

It has a basis indexed by lunes.

3.4 Category of lunes

Recall the category of lunes from Section 2.3.

Proposition 1. *The lune-incidence algebra is the incidence algebra of the category of lunes. Explicitly, it consists of functions s on lunes, with the product of s and t given by*

$$(st)(\mathrm{N}) = \sum_{\mathrm{L} \circ \mathrm{M} = \mathrm{N}} s(\mathrm{L}) t(\mathrm{M}). \tag{21}$$

The sum is over both L and M. The unit element is the function which is 1 on lunes which are flats, and 0 otherwise.

3.5 Noncommutative zeta and Möbius functions

A noncommutative zeta function is an element $\zeta \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ such that $\zeta(A, A) = 1$ for all A and

$$\zeta(H, G) = \sum_{\substack{F:\, F \geq A,\, HF = G \\ \mathrm{s}(F) = \mathrm{s}(G)}} \zeta(A, F) \tag{22}$$

for all $A \leq H \leq G$.

The condition $\mathrm{s}(F) = \mathrm{s}(G)$ in the above sum can be replaced by the condition $FH = F$.

We refer to (22) as the lune-additivity formula.

In particular:

For any flat X containing a face A,

$$\sum_{F\colon F \geq A,\, \mathrm{s}(F)=\mathrm{X}} \boldsymbol{\zeta}(A,F) = 1. \tag{23}$$

This arises by setting $H = G$ in (22), and letting X be the support of G.

We refer to (23) as the flat-additivity formula.

Since $\boldsymbol{\zeta}$ belongs to the lune-incidence algebra, it satisfies

(24)
$$\boldsymbol{\zeta}(A, F) = \boldsymbol{\zeta}(B, G) \text{ whenever } (A, F) \sim (B, G),$$

with $\sim$ as in (15).

Lemma 3. *A noncommutative zeta function $\boldsymbol{\zeta}$ is equivalent to a choice of scalars $\boldsymbol{\zeta}(O, F)$, one for each face F, such that for each flat X,*

$$\sum_{F:\,\mathrm{s}(F)=\mathrm{X}} \boldsymbol{\zeta}(O, F) = 1.$$

Proof. See [**?**, Lemma 15.18]. $\square$

A noncommutative zeta function ζ is called

- set-theoretic if the scalars $\zeta(O, F)$ are either 0 or 1,
- projective if $\zeta(O, F) = \zeta(O, \overline{F})$ for all faces F, and
- uniform if $\zeta(O, F) = \zeta(O, G)$ whenever F and G have the same support.

A noncommutative Möbius function is an element $\boldsymbol{\mu} \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ such that $\boldsymbol{\mu}(A, A) = 1$ for all A and

(25) $$\sum_{F\colon F \geq A,\, HF=G} \boldsymbol{\mu}(A, F) = 0$$

for all $A < H \leq G$.

We refer to (25) as the noncommutative Weisner formula.

Note very carefully that H is strictly greater than A in this formula.

Since $\boldsymbol{\mu}$ belongs to the lune-incidence algebra, it satisfies

(26)

$$\boldsymbol{\mu}(A,F) = \boldsymbol{\mu}(B,G) \text{ whenever } (A,F) \sim (B,G),$$

with $\sim$ as in (15).

Lemma 4. *A noncommutative Möbius function $\boldsymbol{\mu}$ is equivalent to a family of special Zie elements, one in each $\mathcal{A}_{\mathrm{X}}$, as X varies over all flats.*

Proof. See [**?**, Lemma 15.24]. $\square$

Theorem 1. *In the lune-incidence algebra, the inverse of a noncommutative zeta function is a noncommutative Möbius function, and vice-versa.*

Proof. This is a nontrivial result which was obtained in [**?**, Theorem 15.28]. $\square$

3.6 Base-case map

There is an algebra homomorphism

$$\mathrm{bc} : \mathrm{I}_{\mathrm{lune}}[\mathcal{A}] \to \mathrm{I}_{\mathrm{flat}}[\mathcal{A}] \tag{27}$$

defined by

$$\mathrm{bc}(s)(\mathrm{X}, \mathrm{Y}) := \sum_{F:\, F \geq A,\, \mathrm{s}(F) = \mathrm{Y}} s(A, F) = \sum_{\mathrm{L}:\, \mathrm{b}(\mathrm{L}) = \mathrm{X},\, \mathrm{c}(\mathrm{L}) = \mathrm{Y}} s(\mathrm{L}).$$

In the first sum, A is a fixed face of support X. The second sum is over all lunes L whose base is X and case is Y.

We call bc the base-case map.

Lemma 5. *The base-case of any noncommutative zeta function* $\boldsymbol{\zeta} \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ *is the zeta function* $\zeta \in \mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$*.*

The base-case of any noncommutative Möbius function $\boldsymbol{\mu} \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ *is the Möbius function* $\mu \in \mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$*.*

Proof. The first statement follows from the flat-additivity formula (23).

For the second statement, one checks that $\mathrm{bc}(\boldsymbol{\mu})$ satisfies the Weisner formula (18).

For more details, see [**?**, Lemmas 15.17 and 15.23]. □

Remark 2. The converse to Lemma 5 is false in general. That is, there are functions other than $\boldsymbol{\zeta}$ or $\boldsymbol{\mu}$ whose base-case is ζ or μ.

4 Birkhoff algebra, Tits algebra and Janus algebra

4.1 Birkhoff algebra

Recall the Birkhoff monoid $\Pi[\mathcal{A}]$ whose elements are flats.

Let $\Pi[\mathcal{A}]$ denote its linearization over a field $\Bbbk$, with canonical basis $\mathtt{H}$.

It is a commutative $\Bbbk$-algebra:

$$\mathtt{H}_{\mathrm{X}} \cdot \mathtt{H}_{\mathrm{Y}} := \mathtt{H}_{\mathrm{X} \vee \mathrm{Y}}. \tag{28}$$

We call this the Birkhoff algebra.

4.2 Tits algebra

Recall the Tits monoid $\Sigma[\mathcal{A}]$ whose elements are faces.

Let $\mathrm{\Sigma}[\mathcal{A}]$ denote its linearization, with canonical basis $\mathtt{H}$.

It is a $\Bbbk$-algebra:

$$\mathtt{H}_F \cdot \mathtt{H}_G := \mathtt{H}_{FG}. \tag{29}$$

We call this the Tits algebra.

Let $\Gamma[\mathcal{A}]$ denote the linearization of the set of chambers $\Gamma[\mathcal{A}]$, with canonical basis $\mathtt{H}$. We call this the space of chambers of $\mathcal{A}$.

It is a left module over the Tits algebra:

$$\mathtt{H}_F \cdot \mathtt{H}_C := \mathtt{H}_{FC}.$$

The linearization of the support map (6)

$$\mathrm{s} : \Sigma[\mathcal{A}] \twoheadrightarrow \Pi[\mathcal{A}] \tag{30}$$

is a morphism of algebras.

4.3 Janus algebra

Recall the Janus monoid $\mathrm{J}[\mathcal{A}]$ whose elements are bifaces.

Let $\mathsf{J}[\mathcal{A}]$ denote its linearization, with canonical basis $\mathtt{H}$.

It is a $\Bbbk$-algebra:

$$\mathtt{H}_{(F,F')} \cdot \mathtt{H}_{(G,G')} = \mathtt{H}_{(FG,G'F')}. \tag{31}$$

We call this the Janus algebra.

Linearizing diagram (11) yields the following commutative diagram of algebras.

$$\begin{tikzcd} \mathsf{J}[\mathcal{A}] \arrow[r, two heads] \arrow[d, two heads] & \mathsf{\Sigma}[\mathcal{A}]^{\mathrm{op}} \arrow[d, two heads, "\mathrm{s}"] \\ \mathsf{\Sigma}[\mathcal{A}] \arrow[r, two heads, "\mathrm{s}"'] & \mathsf{\Pi}[\mathcal{A}]. \end{tikzcd} \tag{32}$$

5 Orientation space

For any arrangement $\mathcal{A}$, let $\mathsf{mc}[\mathcal{A}]$ denote the space spanned by maximal chains in the poset of faces $\Sigma[\mathcal{A}]$.

For any flat Y, there is a map

$$\mathsf{mc}[\mathcal{A}^{\mathrm{Y}}] \otimes \mathsf{mc}[\mathcal{A}_{\mathrm{Y}}] \to \mathsf{mc}[\mathcal{A}] \tag{33}$$

obtained by concatenating maximal chains.

More precisely, let f be a maximal chain of faces in $\mathcal{A}^{\mathrm{Y}}$. It defines a chain of faces f' in $\mathcal{A}$ which ends at a face F with support Y.

Now let g be a maximal chain of faces in $\mathcal{A}_{\mathrm{Y}}$. By using the canonical identification $\Sigma[\mathcal{A}_{\mathrm{Y}}] \xrightarrow{\cong} \Sigma[\mathcal{A}_F]$, we obtain a chain of faces g' in $\mathcal{A}$ which starts at F.

The concatenation of f' and g' is the required maximal chain in $\mathcal{A}$.

For any arrangement $\mathcal{A}$, let $\mathsf{E}^{\mathbf{o}}[\mathcal{A}]$ denote the quotient of $\mathsf{mc}[\mathcal{A}]$ by the relation that two maximal chains which differ in exactly one position are negatives of each other.

We call $\mathsf{E}^{\mathbf{o}}[\mathcal{A}]$ the orientation space of $\mathcal{A}$.

It is one-dimensional.

We denote the image of a maximal chain f in the orientation space by $[f]$. An orientation of $\mathcal{A}$ is an element of $\mathsf{E}^{\mathbf{o}}[\mathcal{A}]$ of the form $[f]$ for some maximal chain f.

Any arrangement (of rank at least one) has exactly two orientations which we may denote by σ and $-\sigma$.

Example 1. For the rank-one arrangement, with chambers C and $\overline{C}$, there are two maximal chains, namely, $O \lessdot C$ and $O \lessdot \overline{C}$.

Since they differ in exactly one position, we have

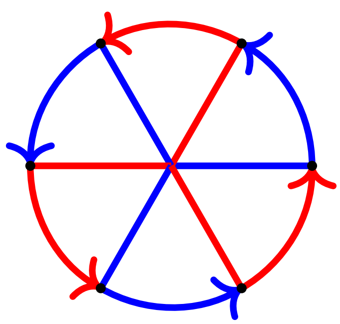
$$[O \lessdot C] = -[O \lessdot \overline{C}].$$

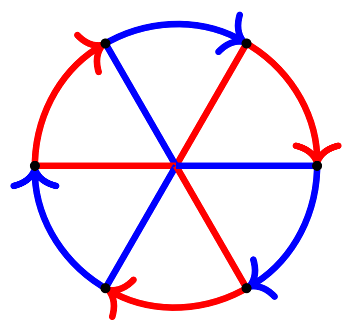
For any rank-two arrangement: A maximal chain has the form $O \lessdot P \lessdot C$. The relations can be expressed as

$$[O \lessdot P \lessdot C] = -[O \lessdot Q \lessdot C] \quad \text{and} \quad [O \lessdot P \lessdot C] = -[O \lessdot P \lessdot D],$$

where in the former P and Q are the two vertices of C, and in the latter C and D are the two chambers greater than P.

We can think of the two orientations as clockwise and anticlockwise. This is illustrated below.





The six maximal chains which give the anticlockwise orientation are shown on the left, while the six which give the clockwise orientation are shown on the right.

There is a canonical isomorphism

$$\mathsf{E}^{\mathbf{o}}[\mathcal{A}] \otimes \mathsf{E}^{\mathbf{o}}[\mathcal{A}] \xrightarrow{\cong} \mathbb{k}, \quad \sigma \otimes \sigma \mapsto 1, \tag{34}$$

where σ is either of the two orientations of $\mathcal{A}$. Changing σ to $-\sigma$ incurs two minus signs, so the map is well-defined.

Moreover, for any flat Y, there is an isomorphism

$$\mathsf{E}^{\mathbf{o}}[\mathcal{A}^{\mathrm{Y}}] \otimes \mathsf{E}^{\mathbf{o}}[\mathcal{A}_{\mathrm{Y}}] \xrightarrow{\cong} \mathsf{E}^{\mathbf{o}}[\mathcal{A}] \tag{35}$$

induced from the map (33). By construction, the diagram

$$\begin{tikzcd}
\mathsf{mc}[\mathcal{A}^{\mathrm{Y}}] \otimes \mathsf{mc}[\mathcal{A}_{\mathrm{Y}}] \arrow[r] \arrow[d] & \mathsf{mc}[\mathcal{A}] \arrow[d] \\
\mathsf{E}^{\mathbf{o}}[\mathcal{A}^{\mathrm{Y}}] \otimes \mathsf{E}^{\mathbf{o}}[\mathcal{A}_{\mathrm{Y}}] \arrow[r] & \mathsf{E}^{\mathbf{o}}[\mathcal{A}]
\end{tikzcd} \tag{36}$$

commutes.

6 Lie and Zie elements

6.1 Lie elements

Recall the space of chambers $\Gamma[\mathcal{A}]$. We write a typical element as

$$z = \sum_C x^C \mathtt{H}_C.$$

This is a Lie element if

(37) $$\sum_{C\colon HC=D} x^C = 0 \text{ for all } O < H \leq D.$$

We denote the set of Lie elements by $\mathsf{Lie}[\mathcal{A}]$.

It is a subspace of $\Gamma[\mathcal{A}]$. Its dimension equals the absolute value of the Möbius number of the arrangement:

(38) $$\dim(\mathsf{Lie}[\mathcal{A}]) = |\mu(\mathcal{A})|.$$

See [**?**, Formula (10.24)].

6.2 Substitution maps

For any flat Y, there is a map

$$\Gamma[\mathcal{A}^{\mathrm{Y}}] \otimes \Gamma[\mathcal{A}_{\mathrm{Y}}] \to \Gamma[\mathcal{A}]. \tag{39}$$

This is the substitution map of chambers.

To define this map, pick any face F with support Y, consider the map

$$\Gamma[\mathcal{A}^{\mathrm{Y}}] \otimes \Gamma[\mathcal{A}_F] \to \Gamma[\mathcal{A}], \qquad \mathtt{H}_H \otimes \mathtt{H}_{C/F} \mapsto \mathtt{H}_{HC},$$

and identify $\Gamma[\mathcal{A}_F]$ with $\Gamma[\mathcal{A}_{\mathrm{Y}}]$. The result does not depend on the particular choice of F.

The map (39) restricts to the space of Lie elements; thus, for any flat Y, there is a map

$$\mathsf{Lie}[\mathcal{A}^{\mathrm{Y}}] \otimes \mathsf{Lie}[\mathcal{A}_{\mathrm{Y}}] \to \mathsf{Lie}[\mathcal{A}]. \tag{40}$$

This is the substitution map of Lie elements.

By construction, the diagram

$$\begin{tikzcd} \Gamma[\mathcal{A}^{\mathrm{Y}}] \otimes \Gamma[\mathcal{A}_{\mathrm{Y}}] \arrow[r] & \Gamma[\mathcal{A}] \\ \mathsf{Lie}[\mathcal{A}^{\mathrm{Y}}] \otimes \mathsf{Lie}[\mathcal{A}_{\mathrm{Y}}] \arrow[r] \arrow[u] & \mathsf{Lie}[\mathcal{A}] \arrow[u] \end{tikzcd} \tag{41}$$

commutes. For more details, see [**?**, Proposition 10.42].

6.3 Antisymmetry and Jacobi identity

When $\mathcal{A}$ has rank one, $\mathsf{Lie}[\mathcal{A}]$ is one-dimensional.

For a Lie element, the coefficients of the two chambers are negatives of each other.

The simplest choices are 1 and -1. Either of them spans $\mathsf{Lie}[\mathcal{A}]$, and their sum is zero. This can be shown as follows.

$$(42)\qquad \Big(\overset{1}{\bullet}\qquad\qquad\overset{\textcolor{magenta}{1}}{\bullet}\Big) + \Big(\overset{\textcolor{magenta}{1}}{\bullet}\qquad\qquad\overset{1}{\bullet}\Big) \;=\; 0.$$

This is the antisymmetry relation.

(By convention, $\textcolor{magenta}{1}$ denotes -1.)

Also note that

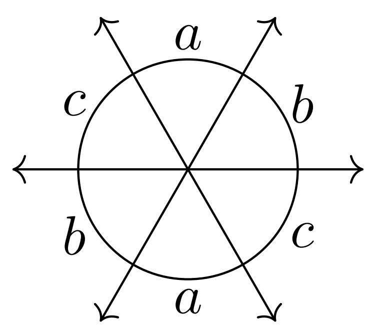
$$(43)\quad \mathsf{E}^{\mathbf{o}}[\mathcal{A}] \xrightarrow{\cong} \mathsf{Lie}[\mathcal{A}], \qquad [O \lessdot C] \mapsto \mathtt{H}_C - \mathtt{H}_{\overline{C}}.$$

(Both spaces are 1-dimensional.)

When $\mathcal{A}$ is the rank-two arrangement of n lines, Lie$[\mathcal{A}]$ is $(n-1)$-dimensional.

For a Lie element, the coefficients of the chambers (read in clockwise cyclic order) are $a_1, \ldots, a_n, a_1, \ldots, a_n$ subject to the condition $a_1 + \cdots + a_n = 0$.

For $n = 3$, a Lie element is

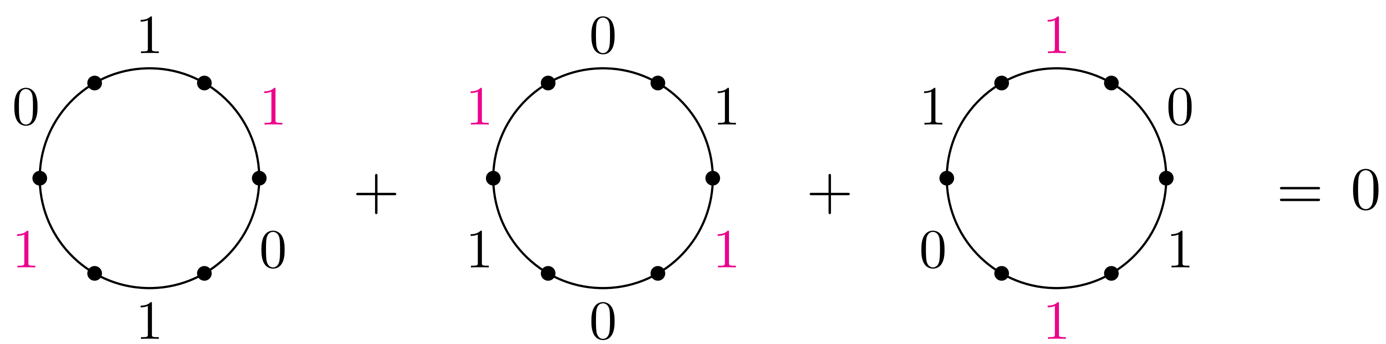


with $a + b + c = 0$.

For example, one may take $a = 1$, $b = -1$, $c = 0$. Other similar choices are $a = 0$, $b = 1$, $c = -1$ and $a = -1$, $b = 0$, $c = 1$.

Any two of these yield a basis for $\text{Lie}[\mathcal{A}]$ and the sum of all three is 0. This can be shown as follows.

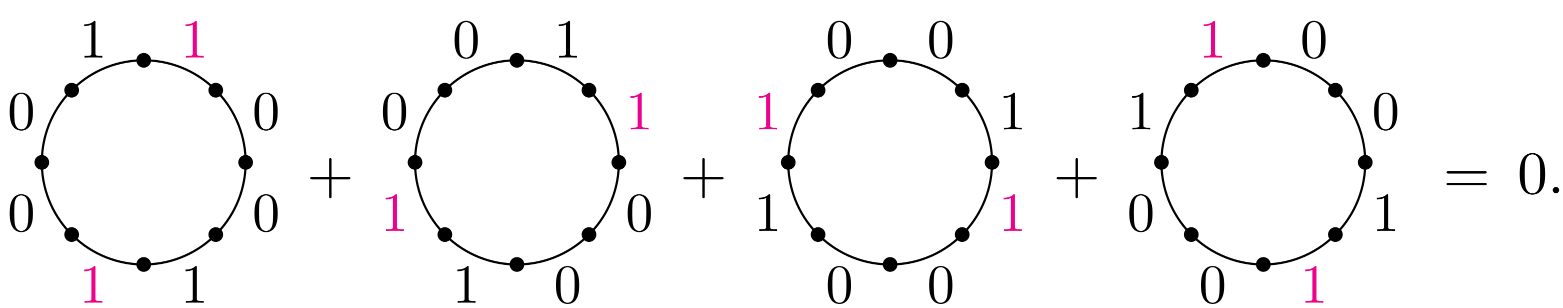
(44)



This is the Jacobi identity for the hexagon.

In general, the Jacobi identity consists of n terms adding up to 0. Each term is a $2n$-gon whose two adjacent sides (and their opposites) have coefficients 1 and 1, and the remaining sides have coefficient 0. For instance, for $n = 4$:

(45)



This is the Jacobi identity for the octagon.

The substitution map (40) combined with (43) yields

$$\bigoplus_{i=1}^{n} \mathsf{E}^{\mathbf{o}}[\mathcal{A}^{\mathrm{X}_i}] \otimes \mathsf{E}^{\mathbf{o}}[\mathcal{A}_{\mathrm{X}_i}] \to \mathsf{Lie}[\mathcal{A}],$$

where the X_i are the n lines (one-dimensional flats) of $\mathcal{A}$.

This map is surjective.

The lhs is n-dimensional while the rhs is $(n-1)$-dimensional.

The kernel is spanned by the element

$$\sum_{i=1}^{n} \tau^i \otimes \tau_i \tag{46}$$

where τ^i and τ_i are orientations of $\mathcal{A}^{\mathrm{X}_i}$ and $\mathcal{A}_{\mathrm{X}_i}$ such that their concatenation (35) yields a fixed orientation of $\mathcal{A}$ (independent of i).

This element corresponds to the Jacobi identity.

6.4 Presentation

By iteration of (40), we see that for any maximal chain of flats $\bot \lessdot \mathrm{X}_1 \lessdot \cdots \lessdot \mathrm{X}_{r-1} \lessdot \top$, there is a map

$$\mathsf{Lie}[\mathcal{A}^{\mathrm{X}_1}] \otimes \mathsf{Lie}[\mathcal{A}^{\mathrm{X}_2}_{\mathrm{X}_1}] \otimes \cdots \otimes \mathsf{Lie}[\mathcal{A}_{\mathrm{X}_{r-1}}] \to \mathsf{Lie}[\mathcal{A}].$$

Note all arrangements involved in the lhs are of rank one.

So, by identifying the tensor factors with orientation spaces, we obtain a map

(47)

$$\bigoplus_z \mathsf{E}^{\mathbf{o}}[\mathcal{A}^{\mathrm{X}_1}] \otimes \mathsf{E}^{\mathbf{o}}[\mathcal{A}^{\mathrm{X}_2}_{\mathrm{X}_1}] \otimes \cdots \otimes \mathsf{E}^{\mathbf{o}}[\mathcal{A}_{\mathrm{X}_{r-1}}] \to \mathsf{Lie}[\mathcal{A}],$$

where the sum is over all maximal chains of flats $z = (\bot \lessdot \mathrm{X}_1 \lessdot \cdots \lessdot \mathrm{X}_{r-1} \lessdot \top)$.

The map (47) is surjective and its kernel is the subspace generated by the elements (46).

We state this as follows.

Theorem 3. *The space* $\mathrm{Lie}[\mathcal{A}]$ *is freely generated by the orientation space in rank one subject to the Jacobi identities in rank two.*

Proof. This is a nontrivial result which was proved in [**?**, Theorem 14.41]. $\square$

6.5 Zie elements

Consider the Tits algebra $\Sigma[\mathcal{A}]$. We write a typical element as

$$z = \sum_F x^F \mathtt{H}_F.$$

This is a Zie element if

$$\sum_{F\colon HF=G} x^F = 0 \text{ for all } O < H \leq G. \tag{48}$$

(Any Lie element is a Zie element.) We denote the set of Zie elements by $\mathsf{Zie}[\mathcal{A}]$.

It is a subspace of $\Sigma[\mathcal{A}]$.

7 References

The following two books are the canonical references for this material.

1. M. Aguiar and S. Mahajan, Topics in hyperplane arrangements, Mathematical Surveys and Monographs, vol. 226, American Mathematical Society, Providence, RI, 2017.

2. M. Aguiar and S. Mahajan, Bimonoids for hyperplane arrangements, Encyclopedia of Mathematics and its Applications, vol. 173, Cambridge University Press, Cambridge, 2020.

They are available as the files b.pdf and c.pdf in our shared folder.