Construction of (Co)Monads and Bimonads

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1 2-category

Recall: In a category, we have objects and morphisms between any two objects.

A 2-category has,

- Objects
- Morphisms between objects
- Morphisms between morphisms

2 Construction

2.1 Introduction

Let Cat denote the 2-category of all categories: 0-cells are categories, 1-cells are functors and 2-cells are natural transformations.

We will construct $m(\mathsf{Cat})$ such that 0-cells are monads, 1-cells are lax functors, 2-cells are morphisms between lax functors.

Table 1: 2-categories of higher monads.

2-category	0-cell	1-cell	2-cell
Cat	category	functor	nat. transf.
$m(\mathsf{Cat})$	monad	lax functor	Def. 2.2
$c(\mathsf{Cat})$	comonad	colax functor	Def. 2.3
$mc(\mathsf{Cat})$	bimonad	bilax functor	Def. 2.4

2.2 Construction of $m(\mathsf{C})$

A monad on a category C is a functor $\mathcal{T} : \mathsf{C} \to \mathsf{C}$ equipped with natural transformations $\mu : \mathcal{T}\mathcal{T} \to \mathcal{T}$ and $\iota : \mathrm{id} \to \mathcal{T}$ such that the diagrams

(1)

$$
\begin{tikzcd}
\mathcal{T}\mathcal{T}\mathcal{T} \arrow[r, "\mathcal{T}\mu"] \arrow[d, "\mu\mathcal{T}"'] & \mathcal{T}\mathcal{T} \arrow[d, "\mu"] \\
\mathcal{T}\mathcal{T} \arrow[r, "\mu"'] & \mathcal{T}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{T}\mathcal{T} \arrow[dr, "\mu"] & \\
\mathcal{T} \arrow[ur, "\iota\mathcal{T}"] \arrow[rr, equal] & & \mathcal{T}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{T}\mathcal{T} \arrow[dr, "\mu"] & \\
\mathcal{T} \arrow[ur, "\mathcal{T}\iota"] \arrow[rr, equal] & & \mathcal{T}
\end{tikzcd}
$$

commute.

Suppose $\mathcal{T}$ is a monad on C, and $\mathcal{T}'$ is a monad on C'. A lax functor from $\mathcal{T}$ to $\mathcal{T}'$ is a functor $\mathcal{F} : \mathsf{C} \to \mathsf{C}'$ equipped with a natural transformation $\varphi : \mathcal{T}'\mathcal{F} \to \mathcal{F}\mathcal{T}$ such that the diagrams

(2)

$$\begin{tikzcd}
\mathcal{T}'\mathcal{T}'\mathcal{F} \arrow[r, "\mathcal{T}'\varphi"] \arrow[d, "\mu'\mathcal{F}"'] & \mathcal{T}'\mathcal{F}\mathcal{T} \arrow[r, "\varphi\mathcal{T}"] & \mathcal{F}\mathcal{T}\mathcal{T} \arrow[d, "\mathcal{F}\mu"] \\
\mathcal{T}'\mathcal{F} \arrow[rr, "\varphi"'] & & \mathcal{F}\mathcal{T}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{T}'\mathcal{F} \arrow[rr, "\varphi"] & & \mathcal{F}\mathcal{T} \\
& \mathcal{F} \arrow[ul, "\iota'\mathcal{F}"] \arrow[ur, "\mathcal{F}\iota"'] &
\end{tikzcd}$$

commute. We use the notation $(\mathcal{F}, \varphi) : \mathcal{T} \to \mathcal{T}'$.

Suppose $(\mathcal{F}, \varphi)$ and $(\tilde{\mathcal{F}}, \tilde{\varphi})$ are lax functors from $\mathcal{T}$ to $\mathcal{T}'$. Then a morphism $(\mathcal{F}, \varphi) \to (\tilde{\mathcal{F}}, \tilde{\varphi})$ is a natural transformation $\theta : \mathcal{F} \to \tilde{\mathcal{F}}$ such that the diagram

(3)
$$\begin{tikzcd}
\mathcal{T}'\mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\mathcal{T}'\theta"'] & \mathcal{F}\mathcal{T} \arrow[d, "\theta\mathcal{T}"] \\
\mathcal{T}'\tilde{\mathcal{F}} \arrow[r, "\tilde{\varphi}"'] & \tilde{\mathcal{F}}\mathcal{T}
\end{tikzcd}$$

commutes.

Proposition 1. *Suppose* $(\mathcal{F}, \varphi) : \mathcal{T} \to \mathcal{T}'$ *and* $(\mathcal{G}, \gamma) : \mathcal{T}' \to \mathcal{T}''$ *are lax functors of monads. Then* $(\mathcal{G}\mathcal{F}, \varphi\gamma) : \mathcal{T} \to \mathcal{T}''$ *is a lax functor of monads, where* $\varphi\gamma$ *is defined to be the composite*

$$\mathcal{T}''\mathcal{G}\mathcal{F} \xrightarrow{\gamma\mathcal{F}} \mathcal{G}\mathcal{T}'\mathcal{F} \xrightarrow{\mathcal{G}\varphi} \mathcal{G}\mathcal{F}\mathcal{T}. \tag{4}$$

2.3 Construction of $c(\mathsf{C})$

A comonad on a category C is a functor $\mathcal{U} : \mathsf{C} \to \mathsf{C}$ equipped with natural transformations $\Delta : \mathcal{U} \to \mathcal{U}\mathcal{U}$ and $\epsilon : \mathcal{U} \to \mathrm{id}$ such that the diagrams

(5)

$$
\begin{tikzcd}
\mathcal{U}\mathcal{U}\mathcal{U} & \mathcal{U}\mathcal{U} \arrow[l, "\mathcal{U}\Delta"'] \\
\mathcal{U}\mathcal{U} \arrow[u, "\Delta\mathcal{U}"] & \mathcal{U} \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{U}\mathcal{U} \arrow[dl, "\epsilon\mathcal{U}"'] & \\
\mathcal{U} \arrow[rr, equal] & & \mathcal{U} \arrow[ul, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{U}\mathcal{U} \arrow[dl, "\mathcal{U}\epsilon"'] & \\
\mathcal{U} \arrow[rr, equal] & & \mathcal{U} \arrow[ul, "\Delta"']
\end{tikzcd}
$$

commute.

Suppose $\mathcal{U}$ is a comonad on C, and $\mathcal{U}'$ is a comonad on C'. A colax functor from $\mathcal{U}$ to $\mathcal{U}'$ is a functor $\mathcal{F} : \mathsf{C} \to \mathsf{C}'$ equipped with a natural transformation $\psi : \mathcal{F}\mathcal{U} \to \mathcal{U}'\mathcal{F}$ such that the diagrams

(6)

$$
\begin{tikzcd}
\mathcal{F}\mathcal{U}\mathcal{U} \arrow[r, "\psi\mathcal{U}"] & \mathcal{U}'\mathcal{F}\mathcal{U} \arrow[r, "\mathcal{U}'\psi"] & \mathcal{U}'\mathcal{U}'\mathcal{F} \\
\mathcal{F}\mathcal{U} \arrow[u, "\mathcal{F}\Delta"] \arrow[rr, "\psi"'] & & \mathcal{U}'\mathcal{F} \arrow[u, "\Delta'\mathcal{F}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}\mathcal{U} \arrow[rr, "\psi"] \arrow[dr, "\mathcal{F}\epsilon"'] & & \mathcal{U}'\mathcal{F} \arrow[dl, "\epsilon'\mathcal{F}"] \\
& \mathcal{F} &
\end{tikzcd}
$$

commute. We use the notation $(\mathcal{F}, \psi) : \mathcal{U} \to \mathcal{U}'$.

Suppose $(\mathcal{F}, \varphi)$ and $(\tilde{\mathcal{F}}, \tilde{\varphi})$ are colax functors from $\mathcal{U}$ to $\mathcal{U}'$. Then a morphism $(\mathcal{F}, \varphi) \to (\tilde{\mathcal{F}}, \tilde{\varphi})$ is a natural transformation $\theta : \mathcal{F} \to \tilde{\mathcal{F}}$ such that the diagram

(7)
$$\begin{array}{ccc} \mathcal{F}\mathcal{U} & \xrightarrow{\psi} & \mathcal{U}'\mathcal{F} \\ {\scriptstyle \theta\mathcal{U}}\downarrow & & \downarrow{\scriptstyle \mathcal{U}'\theta} \\ \tilde{\mathcal{F}}\mathcal{U} & \xrightarrow[\tilde{\psi}]{} & \mathcal{U}'\tilde{\mathcal{F}} \end{array}$$

commutes.

Proposition 2. *If* $(\mathcal{F}, \psi) : \mathcal{U} \to \mathcal{U}'$ *and* $(\mathcal{G}, \delta) : \mathcal{U}' \to \mathcal{U}''$ *are colax functors of comonads, then so is* $(\mathcal{G}\mathcal{F}, \delta\psi) : \mathcal{U} \to \mathcal{U}''$*, where* $\delta\psi$ *is defined to be the composite*

$$\mathcal{G}\mathcal{F}\mathcal{U} \xrightarrow{\mathcal{G}\psi} \mathcal{G}\mathcal{U}'\mathcal{F} \xrightarrow{\delta\mathcal{F}} \mathcal{U}''\mathcal{G}\mathcal{F}. \tag{8}$$

2.4 Construction of $mc(\mathsf{C})$

A bimonad is a triple $(\mathcal{T}, \mathcal{U}, \lambda)$, where $\mathcal{T}$ is a monad, $\mathcal{U}$ is a comonad (both on the same category C), and $\lambda : \mathcal{T}\mathcal{U} \to \mathcal{U}\mathcal{T}$ is a natural transformation such that the diagrams

$$\begin{tikzcd}
\mathcal{T}\mathcal{U}\mathcal{U} \arrow[r, "\lambda\mathcal{U}"] & \mathcal{U}\mathcal{T}\mathcal{U} \arrow[r, "\mathcal{U}\lambda"] & \mathcal{U}\mathcal{U}\mathcal{T} \\
\mathcal{T}\mathcal{U} \arrow[u, "\mathcal{T}\Delta"] \arrow[rr, "\lambda"'] & & \mathcal{U}\mathcal{T} \arrow[u, "\Delta\mathcal{T}"']
\end{tikzcd} \tag{9}$$

$$\begin{tikzcd}
\mathcal{T}\mathcal{T}\mathcal{U} \arrow[r, "\mathcal{T}\lambda"] \arrow[d, "\mu\mathcal{U}"'] & \mathcal{T}\mathcal{U}\mathcal{T} \arrow[r, "\lambda\mathcal{T}"] & \mathcal{U}\mathcal{T}\mathcal{T} \arrow[d, "\mathcal{U}\mu"] \\
\mathcal{T}\mathcal{U} \arrow[rr, "\lambda"'] & & \mathcal{U}\mathcal{T}
\end{tikzcd} \tag{10}$$

$$\begin{tikzcd}
& \mathcal{T} & \\
\mathcal{T}\mathcal{U} \arrow[ur, "\mathcal{T}\epsilon"] \arrow[rr, "\lambda"'] & & \mathcal{U}\mathcal{T} \arrow[ul, "\epsilon\mathcal{T}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{U} \arrow[dl, "\iota\mathcal{U}"'] \arrow[dr, "\mathcal{U}\iota"] & \\
\mathcal{T}\mathcal{U} \arrow[rr, "\lambda"'] & & \mathcal{U}\mathcal{T}
\end{tikzcd} \tag{11}$$

commute. We refer to λ as a mixed distributive law linking $\mathcal{T}$ and $\mathcal{U}$.

Suppose $(\mathcal{T}, \mathcal{U}, \lambda)$ is a bimonad on C, and $(\mathcal{T}', \mathcal{U}', \lambda')$ is a bimonad on C'. A bilax functor

$$(\mathcal{T}, \mathcal{U}, \lambda) \to (\mathcal{T}', \mathcal{U}', \lambda')$$

is a triple $(\mathcal{F}, \varphi, \psi)$ such that $\mathcal{F} : \mathsf{C} \to \mathsf{C}'$ is a functor, $(\mathcal{F}, \varphi) : \mathcal{T} \to \mathcal{T}'$ is a lax functor of monads, $(\mathcal{F}, \psi) : \mathcal{U} \to \mathcal{U}'$ is a colax functor of comonads, and the diagram

$$\begin{tikzcd}
\mathcal{T}'\mathcal{F}\mathcal{U} \arrow[r, "\varphi\mathcal{U}"] \arrow[d, "\mathcal{T}'\psi"'] & \mathcal{F}\mathcal{T}\mathcal{U} \arrow[r, "\mathcal{F}\lambda"] & \mathcal{F}\mathcal{U}\mathcal{T} \arrow[d, "\psi\mathcal{T}"] \\
\mathcal{T}'\mathcal{U}'\mathcal{F} \arrow[r, "\lambda'\mathcal{F}"'] & \mathcal{U}'\mathcal{T}'\mathcal{F} \arrow[r, "\mathcal{U}'\varphi"'] & \mathcal{U}'\mathcal{F}\mathcal{T}
\end{tikzcd} \tag{12}$$

commutes.

Suppose $(\mathcal{F}, \varphi, \psi)$ and $(\tilde{\mathcal{F}}, \tilde{\varphi}, \tilde{\psi})$ are bilax functors $(\mathcal{T}, \mathcal{U}, \lambda) \to (\mathcal{T}', \mathcal{U}', \lambda')$. A morphism $(\mathcal{F}, \varphi, \psi) \to (\tilde{\mathcal{F}}, \tilde{\varphi}, \tilde{\psi})$ is a natural transformation $\theta : \mathcal{F} \to \tilde{\mathcal{F}}$ such that the diagrams (3) and (7) commute.

Proposition 3. *If* $(\mathcal{F}, \varphi, \psi) : (\mathcal{T}, \mathcal{U}, \lambda) \to (\mathcal{T}', \mathcal{U}', \lambda')$ *and* $(\mathcal{G}, \gamma, \delta) : (\mathcal{T}', \mathcal{U}', \lambda') \to (\mathcal{T}'', \mathcal{U}'', \lambda'')$ *are bilax functors, then so is*

$$(\mathcal{G}\mathcal{F}, \varphi\gamma, \delta\psi) : (\mathcal{T}, \mathcal{U}, \lambda) \to (\mathcal{T}'', \mathcal{U}'', \lambda''),$$

where $\varphi\gamma$ *and* $\delta\psi$ *are defined by* (4) *and* (8)*.*