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1 2-category

Recall: In a category, we have objects and morphisms between any two objects.

A 2-category has,

- Objects
- Morphisms between objects
- Morphisms between morphisms

2 Construction

2.1 Introduction

Let Cat denote the 2-category of all categories: 0-cells are categories, 1-cells are functors and 2-cells are natural transformations.

We will construct $m(\operatorname{Cat})$ such that 0-cells are monads, 1-cells are lax functors, 2-cells are morphisms between lax functors.

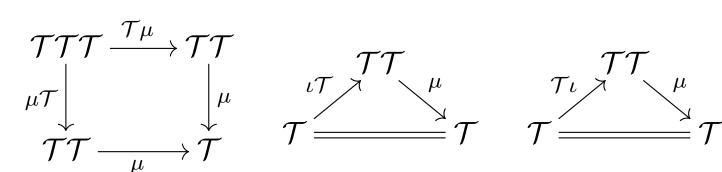
Table 1: 2-categories of higher monads.

2-category	0-cell	1-cell	2-cell
Cat	category	functor	nat. transf.
m(Cat)	monad	lax functor	Def. 2.2
c(Cat)	comonad	colax functor	Def. 2.3
mc(Cat)	bimonad	bilax functor	Def. 2.4

2.2 Construction of m(C)

A monad on a category C is a functor $\mathcal{T}:\mathsf{C}\to\mathsf{C}$ equipped with natural transformations $\mu:\mathcal{T}\mathcal{T}\to\mathcal{T}$ and $\iota:\mathrm{id}\to\mathcal{T}$ such that the diagrams

(1)



commute.

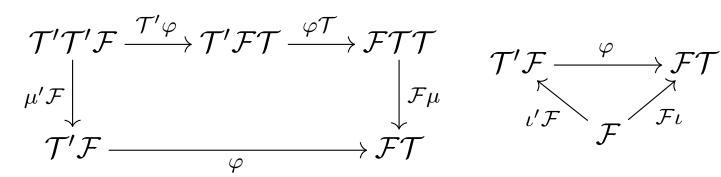
Suppose \mathcal{T} is a monad on C, and \mathcal{T}' is a monad on

C'. A lax functor from \mathcal{T} to \mathcal{T}' is a functor

 $\mathcal{F}:\mathsf{C}\to\mathsf{C}'$ equipped with a natural transformation

 $\varphi: \mathcal{T}'\mathcal{F} o \mathcal{F}\mathcal{T}$ such that the diagrams

(2)



commute. We use the notation $(\mathcal{F}, \varphi): \mathcal{T} \to \mathcal{T}'$.

Suppose (\mathcal{F},φ) and $(\tilde{\mathcal{F}},\tilde{\varphi})$ are lax functors from \mathcal{T} to \mathcal{T}' . Then a morphism $(\mathcal{F},\varphi) \to (\tilde{\mathcal{F}},\tilde{\varphi})$ is a natural transformation $\theta:\mathcal{F}\to\tilde{\mathcal{F}}$ such that the diagram

(3)
$$\begin{array}{ccc}
\mathcal{T}'\mathcal{F} & \xrightarrow{\varphi} \mathcal{F}\mathcal{T} \\
\mathcal{T}'\theta \downarrow & \downarrow \theta \mathcal{T} \\
\mathcal{T}'\tilde{\mathcal{F}} & \xrightarrow{\tilde{\varphi}} \tilde{\mathcal{F}}\mathcal{T}
\end{array}$$

commutes.

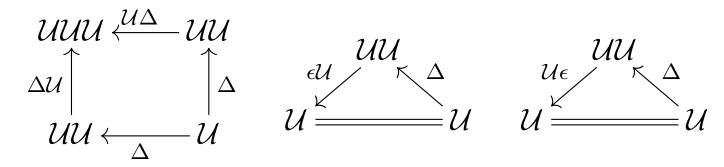
Proposition 1. Suppose $(\mathcal{F},\varphi):\mathcal{T}\to\mathcal{T}'$ and $(\mathcal{G},\gamma):\mathcal{T}'\to\mathcal{T}''$ are lax functors of monads. Then $(\mathcal{GF},\varphi\gamma):\mathcal{T}\to\mathcal{T}''$ is a lax functor of monads, where $\varphi\gamma$ is defined to be the composite

(4)
$$\mathcal{T}''\mathcal{GF} \xrightarrow{\gamma\mathcal{F}} \mathcal{GT}'\mathcal{F} \xrightarrow{\mathcal{G}\varphi} \mathcal{GFT}.$$

2.3 Construction of c(C)

A comonad on a category C is a functor $\mathcal{U}:\mathsf{C}\to\mathsf{C}$ equipped with natural transformations $\Delta:\mathcal{U}\to\mathcal{U}\mathcal{U}$ and $\epsilon:\mathcal{U}\to\mathrm{id}$ such that the diagrams

(5)



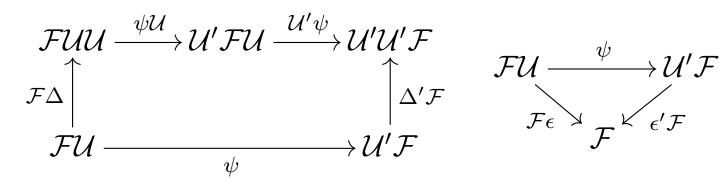
commute.

Suppose \mathcal{U} is a comonad on C, and \mathcal{U}' is a comonad on C'. A colax functor from \mathcal{U} to \mathcal{U}' is a functor

 $\mathcal{F}:\mathsf{C}\to\mathsf{C}'$ equipped with a natural transformation

 $\psi: \mathcal{F}\mathcal{U} o \mathcal{U}'\mathcal{F}$ such that the diagrams

(6)



commute. We use the notation $(\mathcal{F},\psi):\mathcal{U}\to\mathcal{U}'$.

Suppose (\mathcal{F},φ) and $(\tilde{\mathcal{F}},\tilde{\varphi})$ are colax functors from \mathcal{U} to \mathcal{U}' . Then a morphism $(\mathcal{F},\varphi) \to (\tilde{\mathcal{F}},\tilde{\varphi})$ is a natural transformation $\theta:\mathcal{F}\to\tilde{\mathcal{F}}$ such that the diagram

(7)
$$\begin{array}{ccc}
\mathcal{F}\mathcal{U} & \xrightarrow{\psi} \mathcal{U}'\mathcal{F} \\
\theta \mathcal{U} & & \downarrow \mathcal{U}'\theta \\
\tilde{\mathcal{F}}\mathcal{U} & \xrightarrow{\tilde{\psi}} \mathcal{U}'\tilde{\mathcal{F}}
\end{array}$$

commutes.

Proposition 2. If $(\mathcal{F}, \psi) : \mathcal{U} \to \mathcal{U}'$ and $(\mathcal{G}, \delta) : \mathcal{U}' \to \mathcal{U}''$ are colax functors of comonads, then so is $(\mathcal{G}\mathcal{F}, \delta\psi) : \mathcal{U} \to \mathcal{U}''$, where $\delta\psi$ is defined to be the composite

(8)
$$\mathcal{GFU} \xrightarrow{\mathcal{G}\psi} \mathcal{GU'F} \xrightarrow{\delta\mathcal{F}} \mathcal{U''\mathcal{GF}}.$$

2.4 Construction of mc(C)

A bimonad is a triple $(\mathcal{T},\mathcal{U},\lambda)$, where \mathcal{T} is a monad, \mathcal{U} is a comonad (both on the same category C), and $\lambda:\mathcal{T}\mathcal{U}\to\mathcal{U}\mathcal{T}$ is a natural transformation such that the diagrams

(9)
$$TUU \xrightarrow{\lambda U} UTU \xrightarrow{U\lambda} UUT$$

$$T\Delta \uparrow \qquad \qquad \uparrow \Delta \tau$$

$$TU \xrightarrow{\lambda} UTU \xrightarrow{\lambda} UT$$

(10)
$$\begin{array}{ccc}
\mathcal{T}\mathcal{T}\mathcal{U} & \xrightarrow{\mathcal{T}\lambda} \mathcal{T}\mathcal{U}\mathcal{T} & \xrightarrow{\lambda\mathcal{T}} \mathcal{U}\mathcal{T}\mathcal{T} \\
\mu\mathcal{U} & & \downarrow \mathcal{U}\mu \\
\mathcal{T}\mathcal{U} & \xrightarrow{\lambda} \mathcal{U}\mathcal{T}
\end{array}$$

commute. We refer to λ as a mixed distributive law linking ${\mathcal T}$ and ${\mathcal U}.$

Suppose $(\mathcal{T}, \mathcal{U}, \lambda)$ is a bimonad on C, and $(\mathcal{T}', \mathcal{U}', \lambda')$ is a bimonad on C'. A bilax functor

$$(\mathcal{T}, \mathcal{U}, \lambda) \to (\mathcal{T}', \mathcal{U}', \lambda')$$

is a triple $(\mathcal{F}, \varphi, \psi)$ such that $\mathcal{F}: \mathsf{C} \to \mathsf{C}'$ is a functor, $(\mathcal{F}, \varphi): \mathcal{T} \to \mathcal{T}'$ is a lax functor of monads, $(\mathcal{F}, \psi): \mathcal{U} \to \mathcal{U}'$ is a colax functor of comonads, and the diagram

(12)
$$T'\mathcal{F}\mathcal{U} \xrightarrow{\varphi\mathcal{U}} \mathcal{F}\mathcal{T}\mathcal{U} \xrightarrow{\mathcal{F}\lambda} \mathcal{F}\mathcal{U}\mathcal{T}$$

$$\uparrow \psi \downarrow \qquad \qquad \downarrow \psi \mathcal{T}$$

$$\mathcal{T}'\mathcal{U}'\mathcal{F} \xrightarrow{\lambda'\mathcal{F}} \mathcal{U}'\mathcal{T}'\mathcal{F} \xrightarrow{\mathcal{U}'\varphi} \mathcal{U}'\mathcal{F}\mathcal{T}$$

commutes.

Suppose $(\mathcal{F}, \varphi, \psi)$ and $(\tilde{\mathcal{F}}, \tilde{\varphi}, \tilde{\psi})$ are bilax functors $(\mathcal{T}, \mathcal{U}, \lambda) \to (\mathcal{T}', \mathcal{U}', \lambda')$. A morphism $(\mathcal{F}, \varphi, \psi) \to (\tilde{\mathcal{F}}, \tilde{\varphi}, \tilde{\psi})$ is a natural transformation $\theta: \mathcal{F} \to \tilde{\mathcal{F}}$ such that the diagrams (3) and (7) commute.

Proposition 3. If

 $(\mathcal{F}, \varphi, \psi): (\mathcal{T}, \mathcal{U}, \lambda) o (\mathcal{T}', \mathcal{U}', \lambda')$ and $(\mathcal{G}, \gamma, \delta): (\mathcal{T}', \mathcal{U}', \lambda') o (\mathcal{T}'', \mathcal{U}'', \lambda'')$ are bilax functors, then so is

$$(\mathcal{GF}, \varphi \gamma, \delta \psi) : (\mathcal{T}, \mathcal{U}, \lambda) \to (\mathcal{T}'', \mathcal{U}'', \lambda''),$$

where $\varphi\gamma$ and $\delta\psi$ are defined by (4) and (8).