

PH423 Assignment 1

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Question 1.

[Sankalp: I got this one.] We consider the action of $[\hat{X}, \hat{P}]$ on an arbitrary state ϕ ,

$$[\hat{X}, \hat{P}] \phi(x) . \quad (1)$$

Starting with equation 1, we first write it in bra-ket notation as

$$[\hat{X}, \hat{P}] \phi(x) = \langle x | [\hat{X}, \hat{P}] | \phi \rangle . \quad (2)$$

Since $[\hat{X}, \hat{P}] = i\hbar$, we get

$$\langle x | [\hat{X}, \hat{P}] | \phi \rangle = \langle x | i\hbar | \phi \rangle . \quad (3)$$

On either side, we can introduce the identity operator as $[\hat{X}, \hat{P}] \cdot \hat{1}$ and $i\hbar \cdot \hat{1}$, and use the position space completeness relation $\int dx' |x'\rangle \langle x'| = \hat{1}$ to obtain

$$\int dx' \langle x | [\hat{X}, \hat{P}] | x' \rangle \langle x' | \phi \rangle = \int dx' \langle x | i\hbar | x' \rangle \langle x' | \phi \rangle . \quad (4)$$

Expanding the commutator operation $\langle x | [\hat{X}, \hat{P}] | x' \rangle$ as

$$\langle x | [\hat{X}, \hat{P}] | x' \rangle = \langle x | \hat{X} \hat{P} | x' \rangle - \langle x | \hat{P} \hat{X} | x' \rangle$$

and applying the action of \hat{X} on $|x'\rangle$ and $\langle x|$, we obtain

$$\int dx' (x - x') \langle x | \hat{P} | x' \rangle \langle x' | \phi \rangle = \int dx' \langle x | i\hbar | x' \rangle \langle x' | \phi \rangle . \quad (5)$$

Since the state $\phi(x)$ chosen was arbitrary, we get the weak equivalence

$$(x - x') \langle x | \hat{P} | x' \rangle = i\hbar \langle x | x' \rangle . \quad (6)$$

We know $\langle x | x' \rangle = \delta(x - x')$, and using

$$\delta(x) = -x\delta'(x)$$

we get

$$(x - x') \langle x | \hat{\mathbf{P}} | x' \rangle = -i\hbar(x - x') \frac{d}{dx} \delta(x - x') . \quad (7)$$

And finally, dividing both sides by $(x - x')$, assuming $x \neq x'$, we get the required matrix element

$$\langle x | \hat{\mathbf{P}} | x' \rangle = -i\hbar \frac{d}{dx} \delta(x - x') . \quad (8)$$

□

Question 2.

[Parth: Question 2 is mine] Starting from the abstract ket form of the Schrodinger eqn. derive the Schrodinger eqn. in the co-ordinate basis

The Schrodinger eqn. in the abstract ket form is given as follows -

$$\hat{\mathbf{H}} |\psi\rangle = E |\psi\rangle \quad (9)$$

In the position basis, $\hat{\mathbf{H}}$ can be expressed in the form of a matrix, while $|\psi\rangle$ can be expressed in terms of components in the basis. In the position basis, the components of $|\psi\rangle$ form the wavefunction.

$$\langle x | \psi \rangle = \psi(x) . \quad (10)$$

The Hamiltonian becomes a matrix in the position basis, with components given by -

$$H_{xy} = \langle x | \hat{\mathbf{H}} | y \rangle . \quad (11)$$

Applying this to equation 9. We multiply by $\langle x |$ on the left side and insert an identity on the right hand side. The identity is given by -

$$\hat{\mathbf{1}} = \int dy |y\rangle \langle y| \quad (12)$$

So now we obtain -

$$\int dy \langle x | \hat{\mathbf{H}} | y \rangle \langle y | \psi \rangle = E \langle x | \psi \rangle . \quad (13)$$

simplifying to components, we get -

$$\int dy H_{xy} \psi(y) = E \psi(x) . \quad (14)$$

now our components are -

$$H_{xy} = -\frac{\hbar^2}{2m} \delta''(x - y) + V \delta(x - y) \quad (15)$$

29 we plug this in to equation 14. We get -

$$\int dy \left[-\frac{\hbar^2}{2m} \delta''(x-y) \psi(y) \right] + \int dy V \psi(y) \delta(x-y) = E \psi(x) \quad (16)$$

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$$-\frac{\hbar^2}{2m} \int dy \delta''(x-y) \psi(y) + V \psi(x) = E \psi(x) \quad (17)$$

31 Using Integration by Parts on the integral twice, we obtain -

$$-\frac{\hbar^2}{2m} \int dy \delta(x-y) \psi''(y) + V \psi(x) = E \psi(x) \quad (18)$$

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$$-\frac{\hbar^2}{2m} \psi''(x) + V \psi(x) = E \psi(x) \quad (19)$$

33 Generalising this a bit -

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi = E \psi \quad (20)$$

34 This is quite obviously, our Schrodinger's Equation in co-ordinate form.

35 Question 3.

36 [Sahas: This one's mine.]

37 We use the identity $\int_{-\infty}^{+\infty} |p\rangle \langle p| dp = \hat{\mathbf{1}}$ to get:

$$\langle x|U(t)|x'\rangle = \int_{-\infty}^{+\infty} \langle x|p\rangle \langle p|x'\rangle e^{-ip^2 t/2m\hbar} dp \quad (21)$$

38 We use the representation of momentum eigenvectors in the position basis to obtain:

$$\langle x|U(t)|x'\rangle = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} e^{ip(x-x')/\hbar} \times e^{-ip^2 t/2m\hbar} dp \quad (22)$$

39 Finally, we integrate the resulting Gaussian:

$$\langle x|U(t)|x'\rangle = \left(\frac{m}{2\pi\hbar it}\right)^{1/2} e^{im(x-x')^2/2\hbar t} \quad (23)$$