# PH423 Assignment 2

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## Question 1.

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**(a) Calculate the expectation values of $\hat{\mathbf{J}}_x$, $\hat{\mathbf{J}}_y$, $\hat{\mathbf{J}}_x^2$ and $\hat{\mathbf{J}}_y^2$ in the angular momentum states $|j, m\rangle$. Explain the result geometrically. (Using symmetry arguments may help).**

We start with the expansion of the operators $\hat{\mathbf{J}}_x$ and $\hat{\mathbf{J}}_y$ in terms of the ladder operators

$$\hat{\mathbf{J}}_x = \frac{1}{2} \cdot (\hat{\mathbf{J}}_+ + \hat{\mathbf{J}}_-) \tag{1}$$

and

$$\hat{\mathbf{J}}_y = \frac{1}{2i} \cdot (\hat{\mathbf{J}}_+ - \hat{\mathbf{J}}_-) \ . \tag{2}$$

The application of the ladder operators on a state $|j, m\rangle$ changes it to a state of the form $c \cdot |j, m \pm 1\rangle$ for some $c \in \mathbb{C}$. So, given the orthogonality of the $|j, m\rangle$ states, we get that

$$\langle j, m|\hat{\mathbf{J}}_x|j, m\rangle = \langle j, m|\hat{\mathbf{J}}_y|j, m\rangle = 0 \qquad \forall\, |j, m\rangle \ . \tag{3}$$

Squaring Equation 1 and 2, we get the operators $\hat{\mathbf{J}}_x^2$ and $\hat{\mathbf{J}}_y^2$ in terms of the ladder operators. With the same argument as before, we see that only terms with equal powers of the two ladder operators will contribute, and using

$$\hat{\mathbf{J}}_\pm |j, m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)} \ \ |j, m \pm 1\rangle \ , \tag{4}$$

we get

$$\begin{aligned}
\langle j, m|\hat{\mathbf{J}}_y^2|j, m\rangle &= \langle j, m|\hat{\mathbf{J}}_x^2|j, m\rangle & (5)\\
&= \langle j, m|\frac{1}{4} \cdot (\hat{\mathbf{J}}_+^2 + \hat{\mathbf{J}}_+\hat{\mathbf{J}}_- + \hat{\mathbf{J}}_-\hat{\mathbf{J}}_+ + \hat{\mathbf{J}}_-^2)|j, m\rangle & (6)\\
&= \langle j, m|\frac{1}{4} \cdot (\hat{\mathbf{J}}_+\hat{\mathbf{J}}_- + \hat{\mathbf{J}}_-\hat{\mathbf{J}}_+)|j, m\rangle & (7)\\
&= \langle j, m| \ \frac{\hbar^2}{4} \cdot \Big(\sqrt{(j + m + 1)(j - m)}\sqrt{(j - m)(j + m + 1)}
\end{aligned}$$

$$+\sqrt{(j-m)(j+m+1)}\sqrt{(j+m+1)(j-m)}\ \Big) \cdot |j, m\rangle \tag{8}$$

$$= \frac{\hbar^2}{2}(j+m+1)(j-m)\ . \tag{9}$$

The values for x and y are not separately calculated as a trivial calculation shows they're equal. The same is easily argued using symmetry in the x-y plane. This symmetry also serves as an explanation for the expectation value, since there is similarly a reflection symmetry about either axis, the expectation cannot favor either ± x or ± y.

**(b) Can the angular momentum $\hat{\mathbf{J}}$ be oriented entirely along the z (or x or y) axis? Give reasons in either case.**

No. The operators do not commute. The momentum being completely along one axis would allow us to determine them simultaneously, violating the commutation condition.

**2. Determine the eigenvalues and eigenvectors of the 2 x 2 matrix $\boldsymbol{\sigma}.\hat{\mathbf{n}}$ , where $\hat{\mathbf{n}}$ is a unit vector along the $(\theta, \phi)$ direction and $\boldsymbol{\sigma}$ are the three Pauli matrices. This is basically the projection of the spin 1/2 operator (apart from $\frac{\hbar}{2}$) along the direction of the unit vector $\hat{\mathbf{n}}$. Do this in two ways:**

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**(a)** First by explicitly diagonalizing the matrix $\boldsymbol{\sigma}.\hat{\mathbf{n}}$.

The vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, where the $\sigma_i$ matrices are -

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now we need to figure out what $\hat{\mathbf{n}}$ is. The unit vector points along the $(\theta, \phi)$ direction. This is nothing but the unit vector $\hat{\mathbf{r}}$ in Polar co-ordinates.

$$\hat{\mathbf{n}} = \hat{\boldsymbol{r}} = \cos(\phi)\sin(\theta)\hat{\mathbf{i}} + \sin(\phi)\sin(\theta)\hat{\mathbf{j}} + \cos(\theta)\hat{\mathbf{k}}$$

Thus, $\hat{\mathbf{n}} = (\cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta), \cos(\theta))$. We know that $\mathbf{a}.\mathbf{b} = a_i b_i$ (implicit summation over i)
Thus, $\boldsymbol{\sigma}.\hat{\mathbf{n}} = \sigma_i n_i$.

$$\boldsymbol{\sigma}.\hat{\mathbf{n}} = \cos(\phi)\sin(\theta)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin(\phi)\sin(\theta)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos(\theta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\therefore\ \boldsymbol{\sigma}.\hat{\mathbf{n}} = \sin(\theta)\begin{pmatrix} 0 & \cos(\phi) - i * \sin(\phi) \\ \cos(\phi) + i * \sin(\phi) & 0 \end{pmatrix} + \cos(\theta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \sin(\theta)\begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix} + \cos(\theta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos(\theta) \end{pmatrix}$$

To find the eigenvalues and eigenvectors, we now need to diagonalize this matrix. Let the eigenvalues be represented by $\lambda$. The characteristic polynomial takes the following form.

$$(\cos(\theta) - \lambda)(-\cos(\theta) - \lambda) - \sin(\theta)e^{-i\phi} * \sin(\theta)e^{i\phi} = 0$$

$$\therefore -\cos^2(\theta) + \lambda^2 - \sin^2(\theta) = 0 \Rightarrow \lambda^2 - 1 = 0$$
$$\therefore \lambda = \pm 1$$

for $\lambda = 1$, let the eigenvector be $\mathbf{v_1} = (v_{1,1}, v_{1,2})$, thus

$$\begin{pmatrix} \cos(\theta) & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos(\theta) \end{pmatrix} \begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix} = \begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix}$$
$$\therefore \cos(\theta) * v_{1,1} + \sin(\theta)e^{-i\phi} * v_{1,2} = v_{1,1} \ , \ \sin(\theta)e^{i\phi} * v_{1,1} - \cos(\theta) * v_{1,2} = v_{1,2}$$
$$v_{1,2} = e^{i\phi}\frac{\sin(\theta)}{(\cos(\theta)+1)} * v_{1,1}$$

Thus, for eigenvalue $\lambda = 1$, the eigenvector $\mathbf{v_1} = (v_{1,1}, e^{i\phi}\frac{\sin(\theta)}{(\cos(\theta)+1)} * v_{1,1})$
Likewise, for $\lambda = -1$, let the eigenvector be $\mathbf{v_2} = (v_{2,1}, v_{2,2})$, thus

$$\begin{pmatrix} \cos(\theta) & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos(\theta) \end{pmatrix} \begin{pmatrix} v_{2,1} \\ v_{2,2} \end{pmatrix} = \begin{pmatrix} -v_{2,1} \\ -v_{2,2} \end{pmatrix}$$
$$\therefore \cos(\theta) * v_{2,1} + \sin(\theta)e^{-i\phi} * v_{2,2} = -v_{2,1}, \sin(\theta)e^{i\phi} * v_{2,1} - \cos(\theta) * v_{2,2} = -v_{2,2}$$
$$v_{2,2} = e^{i\phi}\frac{\sin(\theta)}{(1-\cos(\theta))} * v_{2,1}$$

Thus, for eigenvalue $\lambda = -1$, the eigenvector $\mathbf{v_2} = (v_{2,1}, e^{i\phi}\frac{\sin(\theta)}{(1-\cos(\theta))} * v_{2,1})$.
We thus have our two eigenvalues ($\pm 1$) and our two eigenvectors ($\mathbf{v_1}$ and $\mathbf{v_2}$)

**(b)** By rotating the spinor pointing initially along the $+\hat{\mathbf{z}}$ axis direction by appropriate angles, using the appropriate rotation operator. Convince yourself that one has to rotate by an angle $\theta$ counterclockwise around the $y$-axis and then by $\phi$ around the $z$-axis. Apart from overall phases, is the resultant spinor the same as the spin up eigenvector obtained in part **(a)**?

Let's start with the spinor pointing in the $+z$-direction.

$$\left| s_z = +\frac{\hbar}{2} \right\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ s.t\ S_z \left| s_z = +\frac{\hbar}{2} \right\rangle = +\frac{\hbar}{2} \left| s_z = +\frac{\hbar}{2} \right\rangle$$

If we apply consecutive rotation operators, we should be able to rotate this spinor into a general state, pointing in an arbitrary direction $\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ points in the $(\theta, \phi)$ direction.
We first rotate this spinor by $\theta$ around the $y$-axis, and then by $\phi$ around the $z$-axis. The axis of spin now points in the direction $\hat{\mathbf{n}}$. Thus -

$$|\hat{n}+\rangle = U[R(\phi\hat{\mathbf{z}})]U[R(\theta\hat{\mathbf{y}})] \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

To find the explicit form of $|\hat{n}+\rangle$, we'll need the forms of the unitary matrices $U[R(\phi\hat{\mathbf{z}})]$ and $U[R(\theta\hat{\mathbf{y}})]$. We'll use the result given in Shankar -

$$U[R(\boldsymbol{\theta}] = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}(\hat{\theta}.\boldsymbol{\sigma})$$

Looking at the particular case of rotation around $y$-axis by amount $\theta$ and then subsequently around $z$-axis by amount $\phi$ -

$$\begin{aligned}
U[R(\theta\hat{\mathbf{y}})]\begin{bmatrix}1\\0\end{bmatrix} &= \begin{bmatrix}\cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}\sigma_y\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\
&= \begin{bmatrix}\cos\frac{\theta}{2}\\0\end{bmatrix} - i\sin\frac{\theta}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\
&= \begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix}
\end{aligned}$$

Applying rotation around $z$-axis by amount $\phi$ now, we get

$$\begin{aligned}
U[R(\phi\hat{\mathbf{z}})]\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix} &= \begin{bmatrix}\cos\frac{\phi}{2}I - i\sin\frac{\phi}{2}\sigma_z\end{bmatrix}\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix}\\
&= \begin{bmatrix}\cos\frac{\phi}{2}\cos\frac{\theta}{2}\\ \cos\frac{\phi}{2}\sin\frac{\theta}{2}\end{bmatrix} - i\sin\frac{\phi}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix}\\
&= \begin{bmatrix}\cos\frac{\theta}{2}\left(\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\right)\\ \sin\frac{\theta}{2}\left(\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right)\end{bmatrix}\\
&= \begin{bmatrix}\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}\\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{bmatrix}
\end{aligned}$$

This gives us a spinor $s_n = (s_{n1}, s_{n2}) = (\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}, \sin\frac{\theta}{2}e^{i\frac{\phi}{2}})$. If we recall our $\mathbf{v_1} = (v_{1,1}, v_{1,2})$ from part **(a)**, we recall the relation we obtained at the end.

$$v_{1,2} = e^{i\phi}\frac{\sin(\theta)}{(\cos(\theta)+1)} * v_{1,1}$$

Substituting $v_{1,1} = s_{n1} = \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}$ (as our final spinor seems to suggest), we get -

$$\begin{aligned}
v_{1,2} &= e^{i\phi}\frac{\sin(\theta)}{(\cos(\theta)+1)} * v_{1,1}\\
&= e^{i\phi}\frac{\sin(\theta)}{(\cos(\theta)+1)} * \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}
\end{aligned}$$

Recall $1+\cos(A) = 2*\cos^2(\frac{A}{2})$ and $\sin(A) = 2*\sin(\frac{A}{2})\cos(\frac{A}{2})$

$$
\begin{aligned}
e^{i\phi}\frac{\sin(\theta)}{(\cos(\theta)+1)} * \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} &= e^{i\frac{\phi}{2}}\frac{\sin(\theta)}{2cos^2(\frac{\theta}{2})} * \cos\frac{\theta}{2} \\
&= e^{i\frac{\phi}{2}}\frac{2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})}{2cos^2(\frac{\theta}{2})} * \cos\frac{\theta}{2} \\
&= e^{i\frac{\phi}{2}}\sin(\frac{\theta}{2}) = s_{n2}
\end{aligned}
$$

Therefore, apart from phase factors, the resultant spinor is the same as the spin up eigenvector we got in part **(a)**.

## Question 3.

**(a) Construct the matrices $\hat{\mathrm{J}}_x$ and $\hat{\mathrm{J}}_y$ for a particle with spin one, $j = 1$ (of course $\hat{\mathrm{J}}_z$ is already diagonal with eigenvalues $\hbar, 0, -\hbar$).**

We can write the $J_x$ operator as $\frac{J_+ + J_-}{2}$. We can write the matrix elements of this matrix in the $\langle j, m|$ basis as $\langle j, m'| \frac{J_+ + J_-}{2} |j, m\rangle$. Note that this matrix element will vanish if $m' = m$ or $|m' - m| > 1$. This gives us the following matrix for $\frac{J_+ + J_-}{2}$, when the basis elements are $|-1\rangle, |0\rangle, |+1\rangle$, in that order.

$$
\begin{bmatrix} 0 & a & 0 \\ b & 0 & c \\ 0 & d & 0 \end{bmatrix} \tag{10}
$$

Now

$$
\begin{aligned}
a &= \langle -1| J_x |0\rangle \\
&= \langle -1| \frac{J_-}{2} |0\rangle \\
&= \langle -1| \frac{\hbar\sqrt{(1)(1+1)-(0)(0-1)}}{2} |-1\rangle \\
&= \hbar\frac{\sqrt{2}}{2} \\
&= \frac{\hbar}{\sqrt{2}}
\end{aligned} \tag{11}
$$

Now, since the matrix is hermitian, we have the following relation between a and b:

$$
\begin{aligned}
b &= a^* \\
\implies b &= \frac{\hbar}{\sqrt{2}}
\end{aligned} \tag{12}
$$

We can perform the same calculation for c:

$$\begin{aligned} c &= \langle 0| J_x |1\rangle \\ &= \langle 0| \frac{J_-}{2} |1\rangle \\ &= \langle 0| \frac{\hbar\sqrt{(1)(1+1)-(1)(1-1)}}{2} |1\rangle \\ &= \hbar\frac{\sqrt{2}}{2} \\ &= \hbar\frac{1}{\sqrt{2}} \end{aligned} \tag{13}$$

Again, using the hermiticity argument, we get $d = c = \frac{\hbar}{\sqrt{2}}$. Therefore the final $J_x$ matrix is:

$$\frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{14}$$

Now that we have $J_x$ (and $J_z$ is trivial), we can use the commutator relation to get $J_y$ :

$$[J_x, J_z] = -i\hbar J_y \tag{15}$$

We write $[J_x, J_z]$ as

$$\frac{\hbar^2}{\sqrt{2}}(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}) \tag{16}$$

With a little algebra we get

$$[J_x, J_z] = -i\hbar J_y = \frac{\hbar^2}{\sqrt{2}}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{17}$$

Finally we get

$$J_y = \frac{i\hbar}{\sqrt{2}}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{18}$$

**(b) An unpolarized beam of spin 1 particles enters a Stern-Gerlach filter that passes only particles with $S_z = \hbar$. After exiting this filter, the beam enters a second filter that passes particles with $S_x = \hbar$ and then finally it encounters a third filter that passes only particles with $S_z = -\hbar$. What fraction of the initial particles make it right through?**

By computing the eigenvectors of the matrix $J_y$ we get the results

$$|\langle S_x = i|S_z = j\rangle|^2 = \frac{1}{3} \tag{19}$$

for $i,j$ = -1,0,1.

Since the beam is unpolarised, 1/3 of the particles will pass through the first filter. Again, because of the above result, 1/3 of the particles will pass through filter 2. Similarly, 1/3 of these particles will then pass through filter 3. Finally we find that 1/27 of the particles will pass through the whole set-up.

**4. Consider the action of an infinitesimal rotation of magnitude $\epsilon$ about the $\hat{\mathbf{n}}$ axis of an angular momentum eigenstate $\psi_{l,m}(\theta, \phi)$ (or $|l, m\rangle$). Show that $U(R)\psi_{l,m} = \sum_m' D_{m'm}\psi_{l,m'}$ and find the complex numbers $D_{m'm}$.**

First, we write down $U\left(R\left(\epsilon\hat{\mathbf{n}}\right)\right)$ in terms of familiar operators assuming $\hat{\mathbf{n}} = n_x\hat{\mathbf{x}} + n_y\hat{\mathbf{y}} + n_z\hat{\mathbf{z}}$ to get

$$\begin{aligned} U\left(R\left(\epsilon\hat{\mathbf{n}}\right)\right) &= \exp\left(-\frac{i\epsilon}{\hbar}\cdot(\hat{\mathbf{J}}\cdot\hat{\mathbf{n}})\right) \\ &= \exp\left(-\frac{i\epsilon}{\hbar}\cdot(n_x\hat{\mathbf{J}}_x + n_y\hat{\mathbf{J}}_y + n_z\hat{\mathbf{J}}_z)\right) \qquad (20) \\ &= \hat{\mathbf{1}} - \frac{i\epsilon}{\hbar}\cdot(n_x\hat{\mathbf{J}}_x + n_y\hat{\mathbf{J}}_y + n_z\hat{\mathbf{J}}_z) + \mathcal{O}(\epsilon^2) \\ &\approx \hat{\mathbf{1}} - \frac{i\epsilon}{\hbar}\cdot(n_x\hat{\mathbf{J}}_x + n_y\hat{\mathbf{J}}_y + n_z\hat{\mathbf{J}}_z) \qquad (21) \end{aligned}$$

Consider the action of this operator on an arbitrary state $|l, m\rangle$

$$U\left(R\left(\epsilon\hat{\mathbf{n}}\right)\right)|l, m\rangle = \hat{\mathbf{1}} - \frac{i\epsilon}{\hbar}\cdot(n_x\hat{\mathbf{J}}_x + n_y\hat{\mathbf{J}}_y + n_z\hat{\mathbf{J}}_z)\,|l, m\rangle\ . \tag{22}$$

Using the fact that the state is an eigenvector of the $\hat{\mathbf{J}}_z$ operator, and expanding $\hat{\mathbf{J}}_x$, $\hat{\mathbf{J}}_y$ as their respective forms in terms of the ladder operators, we get

$$\begin{aligned} U\left(R\left(\epsilon\hat{\mathbf{n}}\right)\right)|l, m\rangle &= (\hat{\mathbf{1}} - \frac{i\epsilon n_z}{\hbar}\cdot\hat{\mathbf{J}}_z)\,|l, m\rangle - \frac{i\epsilon}{\hbar}\cdot(n_x\hat{\mathbf{J}}_x + n_y\hat{\mathbf{J}}_y)\,|l, m\rangle \\ &= (1 - i\epsilon n_z\cdot m)\,|l, m\rangle - \frac{i\epsilon}{\hbar}\cdot(\frac{in_x + n_y}{2i}\hat{\mathbf{J}}_+ + \frac{in_x - n_y}{2i}\hat{\mathbf{J}}_-)\,|l, m\rangle \\ &= (1 - i\epsilon n_z\cdot m)\,|l, m\rangle - \frac{i\epsilon}{\hbar}\cdot(\frac{in_x + n_y}{2i}\cdot\hbar\sqrt{(l-m)(l+m+1)}\cdot|l, m+1\rangle \\ &\quad + \frac{in_x - n_y}{2i}\cdot\hbar\sqrt{(l+m)(l-m-1)}\cdot|l, m-1\rangle) \qquad (23) \end{aligned}$$

or writing it in the required format

$$U\left(R\left(\epsilon\hat{\mathbf{n}}\right)\right)|l,m\rangle = \sum_{m}{}' D_{m'm}\,|l,m'\rangle \ , \tag{24}$$

with

$$D_{m'm} = \begin{cases} 1 - i\epsilon n_z \cdot m & \text{if } m' = m \\ -\frac{\epsilon}{2}\cdot(in_x \pm n_y)\cdot\sqrt{(l \mp m)(l \pm m - 1)} & \text{if } m' = m \pm 1 \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

**5. Prove that any function of the radial coordinate $f(r)$ where $r = |\mathbf{r}|$ and $\mathbf{X}\cdot\mathbf{P}$, where X and P are the position and momentum operators, are both scalar operators.**

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Under a symmetry operator $U$, operators change as $\mathbb{O}' = U^\dagger \mathbb{O} U$. A scalar operator being one which is invariant under rotations, i.e

$$S' = U^\dagger[R]SU[R] = S$$

where $U(R(\boldsymbol{\alpha}) = e^{-\frac{i}{\hbar}\alpha \mathrm{J}})$.
By considering infinitesimal rotations $\boldsymbol{\alpha} = \boldsymbol{\epsilon}$, we have

$$U[R(\boldsymbol{\alpha})] = \left(1 - \frac{i}{\hbar}\epsilon_i J_i\right)$$

Thus, our definition for a scalar operator becomes -

$$S' = \left(1 + \frac{i}{\hbar}\epsilon_i J_i\right) S \left(1 - \frac{i}{\hbar}\epsilon_i J_i\right) = S$$

which gives us $\frac{i}{\hbar}\epsilon_i\,[J_i, S] = 0$. Since $\boldsymbol{\epsilon}$ was an arbitrary choice, we have

$$[J_i, S] = 0$$

as our definition of a scalar operator.

Considering $f(r)$, where $r = |\mathbf{r}|$ as our operator.

$$[J_i, f(r)] = [J_i, r] * f'(r)$$

$r = \sqrt{\sum_{i=1}^{3} X_i^2}$, Thus

$$[J_i, r] = [J_i, X_1] * \frac{X_1}{r} + [J_i, X_2] * \frac{X_2}{r} + [J_i, X_3] * \frac{X_3}{r}$$

we know that $\left[J_i, X_j\right] = i\hbar\epsilon_{ijl}X_l$. Thus

$$[J_i, r] = \left[J_i, X_j\right] * \frac{X_j}{r} = \frac{1}{r}\left(i\hbar\epsilon_{ijl}X_lX_j\right)$$

$$\epsilon_{ijl}X_lX_j = \left[X_l, X_j\right] = 0(l \neq j) \Rightarrow [J_i, r] = 0$$

Thus, since $[J_i, r] = 0$, we have $[J_i, f(r)] = [J_i, r] * f'(r) = 0 * f'(r) = 0$.
Thus, $f(r)$ is a scalar operator.

Now considering $O = \mathbf{X}\ .\ \mathbf{P}$ as our operator, we need to show $[J_i, O] = 0$

$$\begin{aligned}
\mathbf{X.P} &= X_iP_i \qquad \text{implicit summation} \\
\therefore\ [J_i, O] &= \left[J_i, X_jP_j\right] \\
&= \left[J_i, X_j\right]P_j + X_j\left[J_i, P_j\right] \\
&= i\hbar\epsilon_{ijl}(X_lP_j + X_jP_l)
\end{aligned}$$

Now, $\epsilon_{ijl}X_lP_j = [X_l, P_j]$ for $l \neq j$, but $[X_l, P_j] = 0, l \neq j$. Thus

$$i\hbar\epsilon_{ijl}(X_lP_j + X_jP_l) = 0 \Rightarrow [J_i, O] = 0$$

Since $[J_i, O] = 0$, we can say that the operator $O$ is a scalar operator.
Thus, $\mathbf{X.P}$ is a scalar operator

## Question 6.

We know that the $\mathbf{X}_i$ operators can be written in terms of the spherical tensor operators as follows: (notation is the same as that used in Shankar, Principles of Quantum Mechanics, 2ed, page 419)

$$\begin{aligned}
V_1^{+1} &= \frac{i\mathbf{X}_y - \mathbf{X}_x}{\sqrt{2}} \\
V_1^{0} &= \mathbf{X}_z \\
V_1^{-1} &= -\frac{\mathbf{X}_x + i\mathbf{X}_y}{\sqrt{2}}
\end{aligned} \tag{26}$$

Thus in general any linear combination of the $\mathbf{X}_i$s can be written in terms of the $V_1^i$s. Note that $\epsilon \cdot \mathbf{X}$ is exactly such a linear combination. Thus we may write

$$\hat{O} = \epsilon \cdot \mathbf{X} = \alpha_i V_1^i \tag{27}$$

Where the $\alpha_i$ are scalars, and summation over repeated values is implied.

Using this form we can write the transition probability for the Hydrogen atom as

$$\left| \langle n', l', m' | \, \alpha_i V_1^i \, | n, l, m \rangle \right| \tag{28}$$

Now since each $V_1^i$, acting on $|n, l, m\rangle$ can either:

- Increase the value of $l$ by 1
- Decrease the value of $l$ by 1
- Keep the value of $l$ unchanged

Or give a superposition of the above. Since states of different $l$ are orthogonal, $\alpha_i V_1^i \, |n, l, m\rangle$ and $\langle n', l', m'|$ won't have any common terms unless $|l - l'| = 1$ or $l = l'$.

Thus we get the relation

$$\left| \langle n', l', m' | \, \alpha_i V_1^i \, | n, l, m \rangle \right| = 0 \tag{29}$$

Unless $|l - l'| = 1$ or $l = l'$.

$$|\langle n', l, m' | \, \epsilon \cdot \mathbf{X} \, | n, l, m \rangle| = \left| \langle n', l', m' | \, P^{\dagger} \epsilon \cdot \mathbf{X} P \, | n, l, m \rangle \right| \tag{30}$$

Since $|n, l, m\rangle$ transforms as $|n, l, m\rangle \longrightarrow (-1)^l \, |n, l, m\rangle$ under parity,

$$(-1)^{l'+l} \langle n', l', m' | \, \epsilon \cdot \mathbf{X} \, | n, l, m \rangle = \langle n', l', m' | \, P^{\dagger} \epsilon \cdot \mathbf{X} P \, | n, l, m \rangle \tag{31}$$

Since $\mathbf{X}$ transforms as $\mathbf{X} \longrightarrow -\mathbf{X}$ under parity, we get

$$(-1)^{l'+l} \langle n', l', m' | \, \epsilon \cdot \mathbf{X} \, | n, l, m \rangle = - \langle n', l', m' | \, \epsilon \cdot \mathbf{X} \, | n, l, m \rangle \tag{32}$$

Hence if $l + l'$ is even (i.e. when $l = l'$), we get

$$\langle n', l', m' | \, \epsilon \cdot \mathbf{X} \, | n, l, m \rangle = 0 \tag{33}$$