Lie and Zie elements

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1 Lie elements

1.1 Lie elements

Recall the left module of chambers $\Gamma[\mathcal{A}]$. We write a typical element as

$$z = \sum_C x^C \mathtt{H}_C.$$

An element $z \in \Gamma[\mathcal{A}]$ is a Lie element if

$$\sum_{C\colon HC=D} x^C = 0 \text{ for all } O < H \leq D. \tag{1}$$

This is a linear system in the variables x^C.

We denote the set of Lie elements by $\mathsf{Lie}[\mathcal{A}]$.

It is a subspace of $\Gamma[\mathcal{A}]$.

- Note very carefully that $H = O$ is excluded from (1): If not, then $z = 0$ would be the only solution since all its coefficients x^C would be forced to be zero.

- Since the condition (1) is in terms of the Tits product, cisomorphic arrangements have the "same" Lie elements. In particular, to understand $\mathsf{Lie}[\mathcal{A}]$, one may replace $\mathcal{A}$ by its essentialization.

Lemma 1. *If $\mathcal{A}$ has rank zero, then* $\mathsf{Lie}[\mathcal{A}] = \Gamma[\mathcal{A}] = \mathbb{k}$.

Proof. Suppose $\mathcal{A}$ has rank zero. Then, it has only one chamber which is the central face, so (1) is vacuously true. Hence $\mathsf{Lie}[\mathcal{A}] = \Gamma[\mathcal{A}]$, spanned by $\mathtt{H}_O$. □

Lemma 2. *If $\mathcal{A}$ has rank at least one, then the sum of the coefficients of any Lie element is zero. That is, $z \in \mathsf{Lie}[\mathcal{A}]$ implies*

$$\sum_C x^C = 0. \tag{2}$$

Proof. Let D be any chamber. Since $\mathcal{A}$ has rank at least one, $D > O$. So we may choose $H = D$ in (1). This yields (2). □

1.2 Friedrichs primitive part criterion

Recall that left modules over the Tits algebra have a primitive part.

Lemma 3. *The space of Lie elements is the primitive part of the left module of chambers:*

$$\mathcal{P}(\Gamma[\mathcal{A}]) = \mathsf{Lie}[\mathcal{A}].$$

Explicitly,

$$z \in \mathsf{Lie}[\mathcal{A}] \iff \mathtt{H}_H \cdot z = 0 \textit{ for all } H > O.$$

Proof. Let H be any face of $\mathcal{A}$. Then

$$\mathtt{H}_H \cdot \Big(\sum_C x^C \mathtt{H}_C\Big) = \sum_C x^C \mathtt{H}_{HC} = \sum_{D:\, H \leq D} \Big(\sum_{C:\, HC=D} x^C\Big) \mathtt{H}_D.$$

This equals 0 iff

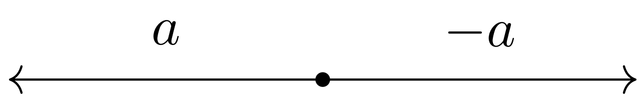
$$\sum_{C:\, HC=D} x^C = 0 \text{ for all } D \geq H.$$

The result follows from (1). $\square$

We refer to the characterization of Lie elements given by Lemma 3 as the Friedrichs criterion.

1.3 Rank-one and antisymmetry

Consider the rank-one arrangement in which the ambient space has dimension one, and there is only one hyperplane consisting of the origin.



In this case, $\mathsf{Lie}[\mathcal{A}]$ is one-dimensional.

The coefficients of the two chambers are a and $-a$.

The simplest choices are $a = 1$ and $a = -1$.

Either of them spans $\mathsf{Lie}[\mathcal{A}]$, and their sum is zero.

This can be shown as follows.

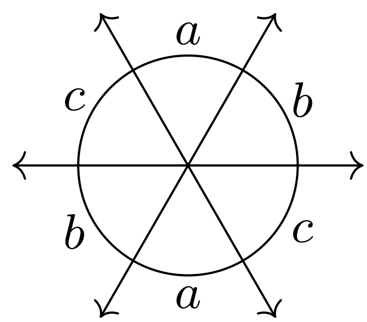
$$\left(\overset{1}{\bullet}\qquad\qquad \overset{\overline{1}}{\bullet}\right) + \left(\overset{\overline{1}}{\bullet}\qquad\qquad \overset{1}{\bullet}\right) = 0. \tag{3}$$

This is the antisymmetry relation.

(By convention, $\overline{1}$ denotes -1.)

1.4 Rank-two and Jacobi identity

Now consider the rank-two arrangement of 3 lines.



In this case, $\mathrm{Lie}[\mathcal{A}]$ is two-dimensional.

The coefficients of the chambers (read in clockwise cyclic order) are a, b, c, a, b and c subject to the condition $a+b+c=0$.

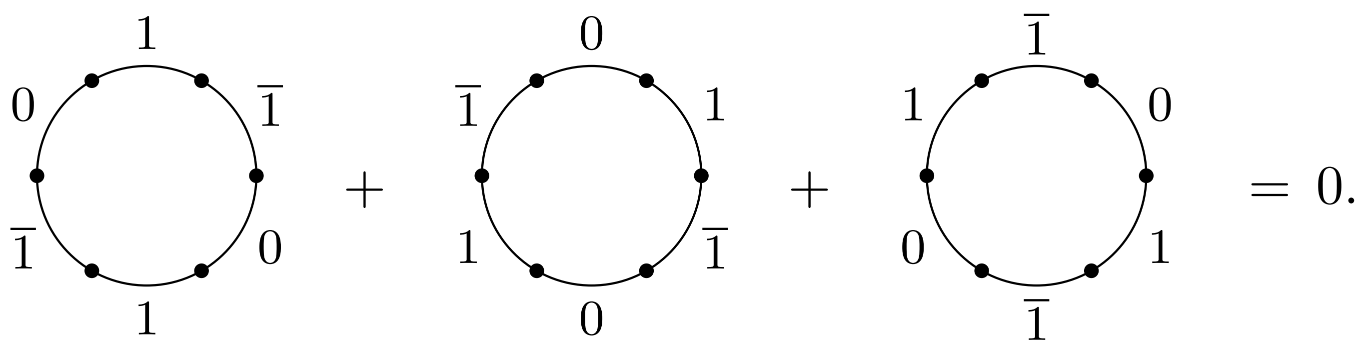
For example, one may take $a=1$, $b=-1$, and $c=0$.

Other similar choices are $a=0$, $b=1$, and $c=-1$, or $a=-1$, $b=0$, and $c=1$.

Any two of these yield a basis for $\mathrm{Lie}[\mathcal{A}]$, and the sum of all three is 0.

This can be shown as follows.

(4)



This is the Jacobi identity for the hexagon. (By convention, $\overline{1}$ denotes -1.)

The above analysis readily generalizes to the rank-two arrangement of n lines:

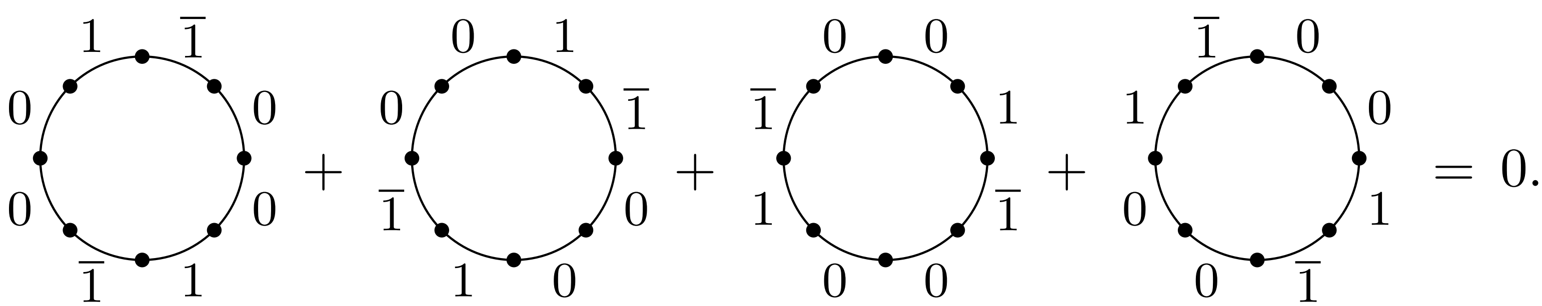
The hexagon gets replaced by a $2n$-gon, and $\mathrm{Lie}[\mathcal{A}]$ is $(n-1)$-dimensional.

The coefficients of the chambers (read in clockwise cyclic order) are $a_1, \ldots, a_n, a_1, \ldots, a_n$ subject to the condition $a_1 + \cdots + a_n = 0$.

Jacobi identity consists of n terms adding up to 0. Each term is a $2n$-gon whose two adjacent sides (and their opposites) have coefficients 1 and $\overline{1}$, and the remaining sides have coefficient 0.

For instance:

(5)



This is the Jacobi identity for the octagon.

2 Zie elements

2.1 Zie elements

Consider the Tits algebra $\Sigma[\mathcal{A}]$. Write a typical element as

$$z = \sum_F x^F \mathtt{H}_F.$$

An element $z \in \Sigma[\mathcal{A}]$ is a Zie element if

$$\sum_{F:\, HF=G} x^F = 0 \text{ for all } O < H \leq G. \tag{6}$$

This is a linear system in the variables x^F.

We denote the set of Zie elements by $\mathsf{Zie}[\mathcal{A}]$.

It is a subspace of $\Sigma[\mathcal{A}]$.

As for Lie elements, note that $H = O$ is excluded from the defining equations. Also, cisomorphic arrangements have the "same" Zie elements.

Lemma 4. *If* $\mathcal{A}$ *has rank zero, then* $\mathsf{Zie}[\mathcal{A}] = \Sigma[\mathcal{A}] = \Bbbk$*.*

Proof. Suppose $\mathcal{A}$ has rank zero. Then, it has only one face, namely, the central face, so (6) is vacuously true. Hence $\mathsf{Zie}[\mathcal{A}] = \Sigma[\mathcal{A}]$, spanned by $\mathtt{H}_O$. $\square$

Lemma 5. *Suppose z is a Zie element. Then*

$$\sum_{F:\, \mathrm{s}(F) \leq \mathrm{X}} x^F = 0 \text{ for all non-minimum flats } \mathrm{X}. \tag{7}$$

In particular, if $\mathcal{A}$ has rank at least one, then

$$\sum_{F} x^F = 0. \tag{8}$$

The sum is over all faces F.

Proof. Consider the special case of (6) in which $O < H = G$.

Let $\mathrm{X} := \mathrm{s}(H) = \mathrm{s}(G)$.

Recalling that $GF = G$ iff $\mathrm{s}(F) \leq \mathrm{X}$, we obtain (7).

Letting X be the maximum flat yields (8). $\square$

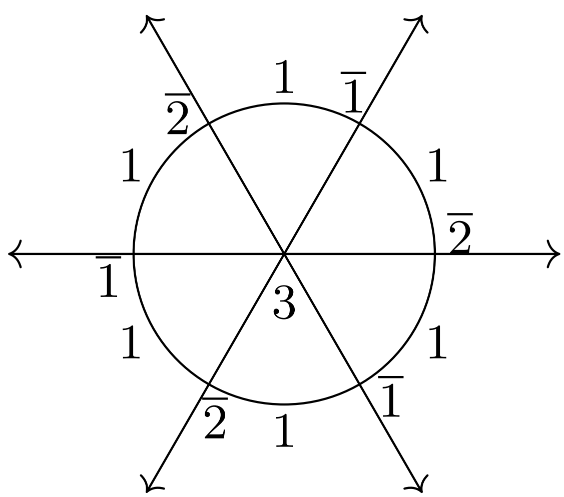
2.2 Zie in small ranks

Let $\mathcal{A}$ be the rank-one arrangement consisting of the central face, and chambers C and $\overline{C}$. In this case,

$$x^O \mathtt{H}_O + x^C \mathtt{H}_C + x^{\overline{C}} \mathtt{H}_{\overline{C}} \in \mathsf{Zie}[\mathcal{A}] \iff x^O + x^C + x^{\overline{C}} = 0.$$

Thus, $\mathsf{Zie}[\mathcal{A}]$ is two-dimensional.

A Zie element for the arrangement of 3 lines is shown in the diagram below.



2.3 Special Zie elements

A Zie element z is special if the coefficient in z of the central face is 1, that is, if $x^O = 1$.

Such elements do exist; examples will be given later.

Special Zie elements form an affine space of dimension one less than the dimension of $\mathrm{Zie}[\mathcal{A}]$.

Lemma 6. *For $z \in \Sigma[\mathcal{A}]$, the following conditions are equivalent.*

(9)

$$x^O = 1 \quad \textit{and} \quad \sum_{F:\, \mathrm{s}(F) \leq \mathrm{X}} x^F = 0 \text{ \textit{for all non-minimum flats} } \mathrm{X}.$$

(10) $$\sum_{F:\, \mathrm{s}(F) = \mathrm{X}} x^F = \mu(\perp, \mathrm{X}) \text{ \textit{for all flats} } \mathrm{X}.$$

(11)

$$\mathrm{s}(z) = \mathtt{Q}_{\perp}, \text{ \textit{the} } \mathtt{Q}\text{\textit{-basis element of the Birkhoff algebra}.}$$

When z is a special Zie element, all the above conditions hold.

Proof. For the equivalence between the first two conditions:

Denote the lhs of (10) by $f(\mathrm{X})$.

In (9), the condition $x^O = 1$ is the same as $f(\bot) = 1$, while the equations say: for any $\mathrm{Y} > \bot$,

$$\sum_{\mathrm{X}\colon \mathrm{X} \leq \mathrm{Y}} f(\mathrm{X}) = 0.$$

This linear system has a unique solution, namely, $f(\mathrm{X}) = \mu(\bot, \mathrm{X})$ for all X.

For the equivalence between the last two conditions: Note that

$$\mathrm{s}(z) = \sum_F x^F \mathtt{H}_{\mathrm{s}(F)} = \Big(\sum_{F\colon \mathrm{s}(F) = \mathrm{X}} x^F \Big) \mathtt{H}_{\mathrm{X}}.$$

This equals $\mathtt{Q}_\bot$ iff the term in parenthesis is $\mu(\bot, \mathrm{X})$.

By Lemma 5, a special Zie element satisfies condition (9), and hence the other two conditions as well. $\square$

2.4 Friedrichs primitive part criterion

The space of Zie elements is the primitive part of the Tits algebra (as a left module over itself). This is the Friedrichs criterion. It is elaborated below.

Lemma 7. *We have*

$$\mathcal{P}(\Sigma[\mathcal{A}]) = \mathsf{Zie}[\mathcal{A}].$$

Explicitly,

$$z \in \mathsf{Zie}[\mathcal{A}] \iff \mathtt{H}_H \cdot z = 0 \textit{ for all } H > O.$$

Proof. Let H be any face of $\mathcal{A}$. Then

$$\mathtt{H}_H \cdot \Big(\sum_F x^F \mathtt{H}_F\Big) = \sum_F x^F \mathtt{H}_{HF} = \sum_{G:H\leq G} \Big(\sum_{F:\, HF=G} x^F\Big) \mathtt{H}_G.$$

This equals 0 iff

$$\sum_{F:\, HF=G} x^F = 0 \text{ for all } G \geq H.$$

The result follows from (6). $\square$

We now discuss some consequences of the Friedrichs criterion.

Lemma 8. *The subspace* $\text{Zie}[\mathcal{A}]$ *is a right ideal of* $\Sigma[\mathcal{A}]$. *More precisely: If* z *is a special Zie element, then* $\text{Zie}[\mathcal{A}]$ *is the right ideal of* $\Sigma[\mathcal{A}]$ *generated by* z.

Proof. Let z be a special Zie element. For any element w of the Tits algebra, $z \cdot w$ is a Zie element since by Lemma 7,

$$\mathtt{H}_F \cdot (z \cdot w) = (\mathtt{H}_F \cdot z) \cdot w = 0$$

whenever $F > O$. Thus the right ideal generated by z is contained in $\text{Zie}[\mathcal{A}]$.

Equality holds since for any Zie element z',

$$z \cdot z' = \Big(\sum_F x^F \mathtt{H}_F\Big) \cdot z' = \sum_F x^F \mathtt{H}_F \cdot z' = x^O z' = z'$$

again using Lemma 7. $\square$

Lemma 9. *Any Zie element is a quasi-idempotent. More precisely, any Zie element z satisfies $z^2 = x^O z$.*

A nonzero Zie element is an idempotent iff it is special.

Proof. Let z be a Zie element. By Lemma 7,

$$z \cdot z = \Big(\sum_F x^F \mathtt{H}_F \Big) \cdot z = \sum_F x^F (\mathtt{H}_F \cdot z) = x^O z.$$

This proves the first claim.

Note that z is an idempotent iff $x^O z = z$. Assuming z to be nonzero, this happens precisely when $x^O = 1$, that is, when z is special. □

Lemma 10. *Conjugation of a special Zie element by an invertible element of the Tits algebra produces another special Zie element.*

Proof. Let u be an invertible element and z be a special Zie element.

We want to show that $u \cdot z \cdot u^{-1}$ is also a special Zie element.

We may assume that the coefficient of $\mathtt{H}_O$ in u is 1.

By Lemma 7, $u \cdot z = z$.

By Lemma 8, $z \cdot u^{-1}$ is a Zie element, and it is special because the coefficient of $\mathtt{H}_O$ in u^{-1} is also 1. $\square$

Lemma 11. *If z is a special Zie element then z is an idempotent and $\mathrm{s}(z) = \mathtt{Q}_\perp$.*

Proof. This follows from Lemmas 6 and 9. $\square$

2.5 Zie elements and primitive part of modules

Let z be an element of the Tits algebra Σ and h a left Σ-module. Recall that $\Psi_{\mathsf{h}}(z)$ denotes the linear operator on h given by left multiplication by z.

Proposition 1. *If z is a Zie element, then the image of $\Psi_{\mathsf{h}}(z)$ is contained in $\mathcal{P}(\mathsf{h})$.*

Moreover, $\Psi_{\mathsf{h}}(z)$ acts on $\mathcal{P}(\mathsf{h})$ by scalar multiplication by the coefficient of the central face in z.

If z is a special Zie element, then $\Psi_{\mathsf{h}}(z)$ projects h onto $\mathcal{P}(\mathsf{h})$.

Proof. Let $z = \sum_F x^F \mathtt{H}_F$. Let $h \in \mathsf{h}$. By Lemma 7,

$$\mathtt{H}_H \cdot (z \cdot h) = (\mathtt{H}_H \cdot z) \cdot h = 0$$

for all $H > O$. Thus $z \cdot h \in \mathcal{P}(\mathsf{h})$ as required. If h itself is primitive, then

$$z \cdot h = \sum_F x^F \mathtt{H}_F \cdot h = x^O h.$$

The claim about special Zie elements also follows. $\square$

Example. Consider the rank-one arrangement with chambers C and $\overline{C}$.

Observe that any special Zie element is of the form

$$\mathtt{H}_O - p\,\mathtt{H}_C - (1-p)\,\mathtt{H}_{\overline{C}},$$

where p is an arbitrary scalar.

Let us compute the action of this element on $\Gamma[\mathcal{A}]$.

$$\begin{aligned}(\mathtt{H}_O - p\,\mathtt{H}_C - (1-p)\,\mathtt{H}_{\overline{C}}) \cdot \mathtt{H}_C &= \mathtt{H}_C - p\,\mathtt{H}_C - (1-p)\,\mathtt{H}_{\overline{C}} \\ &= (1-p)\,\mathtt{H}_C - (1-p)\,\mathtt{H}_{\overline{C}},\end{aligned}$$

which is a Lie element.

Further,

$$(\mathtt{H}_O - p\,\mathtt{H}_C - (1-p)\,\mathtt{H}_{\overline{C}}) \cdot (\mathtt{H}_C - \mathtt{H}_{\overline{C}}) = \mathtt{H}_C - \mathtt{H}_{\overline{C}}.$$

So its action on a Lie element gives back the same Lie element.

This is consistent with Proposition 1.

2.6 Dimensions of Lie and Zie

Proposition 2. *For any finite-dimensional left* Σ*-module* h*,*

$$\dim(\mathcal{P}(\mathsf{h})) = \eta_{\perp}(\mathsf{h}) = \sum_{\mathrm{Y}} \mu(\perp, \mathrm{Y})\, \xi_{\mathrm{Y}}(\mathsf{h}), \tag{12}$$

with $\xi_{\mathrm{X}}(\mathsf{h})$ *and* $\eta_{\mathrm{X}}(\mathsf{h})$ *as before.*

Proof. We use that special Zie elements z do exist.

By Proposition 1, $\mathcal{P}(\mathsf{h}) = z \cdot \mathsf{h}$.

By Lemma 11, z is an idempotent which lifts $\mathtt{Q}_{\perp}$.

So $\dim(z \cdot \mathsf{h}) = \eta_{\perp}(\mathsf{h})$. □

Let us apply (12) to $\mathsf{h} = \Gamma$.

Combining the Friedrichs criterion (Lemmas 3) with the formula $\eta_\perp(\Gamma) = |\mu(\mathcal{A})|$, we obtain:

Theorem 1. *For any arrangement $\mathcal{A}$,*

$$\dim(\mathsf{Lie}[\mathcal{A}]) = \sum_{\mathrm{X}} \mu(\perp, \mathrm{X})\, c_{\mathrm{X}} = |\mu(\mathcal{A})| \tag{13}$$

where c_{X} is the number of chambers in $\mathcal{A}_{\mathrm{X}}$.

Similarly, we can also obtain

$$\dim(\mathsf{Zie}[\mathcal{A}]) = \sum_{\mathrm{X}} \mu(\perp, \mathrm{X})\, d_{\mathrm{X}} = \sum_{\mathrm{X}} |\mu(\mathcal{A}^{\mathrm{X}})|, \tag{14}$$

where d_{X} is the number of faces in $\mathcal{A}_{\mathrm{X}}$.