

# **Bimonoids as Functor Categories**

# 1 Bicommutative Bimonoids

## 1.1 Base category for bicommutative bimonoids

We now define the category  $\mathcal{A}\text{-Hyp}_r^e$ .

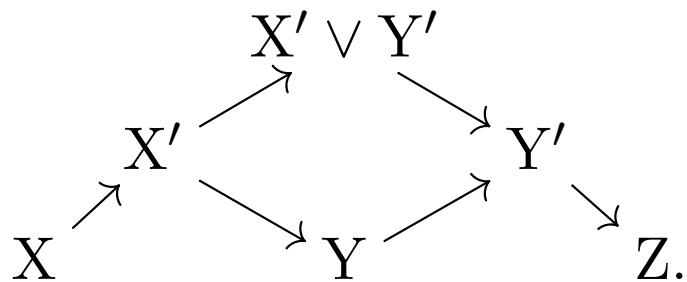
- Objects are flats of  $\mathcal{A}$ .
- A morphism  $X \rightarrow Y$  is a flat  $X'$  greater than both  $X$  and  $Y$ . We represent this morphism as a triple  $(X, X', Y)$ , or more vividly as

(1) 
$$\begin{array}{ccc} & X' & \\ X \nearrow & & \searrow Y \end{array}$$

- The identity morphisms are  $(X, X, X)$ .
- Composition of morphisms is defined by

$$(2) \quad (Y, Y', Z) \circ (X, X', Y) := (X, X' \vee Y', Z).$$

This is illustrated below.



Whenever  $X \leq X'$  and  $Y \leq X'$ , we have the morphisms  $(X, X', X')$  and  $(X', X', Y)$ . The first is from  $X$  to  $X'$ , while the second is from  $X'$  to  $Y$ . To simplify notation, we denote them by  $X \xrightarrow{\Delta} X'$  and  $X' \xrightarrow{\mu} Y$ . In other words,

$$\begin{array}{c} \Delta \\ \nearrow \\ X \end{array} \begin{array}{c} X' \\ \searrow \\ X' \end{array} := \begin{array}{c} X' \\ \nearrow \searrow \\ X \quad X' \end{array}$$

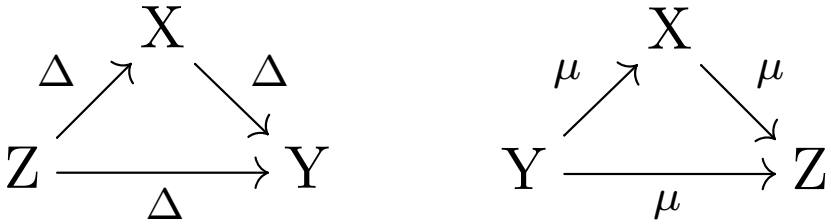
$$\begin{array}{c} X' \\ \searrow \\ Y \end{array} \begin{array}{c} X' \\ \nearrow \\ X' \end{array} := \begin{array}{c} X' \\ \nearrow \searrow \\ X' \quad Y \end{array}$$

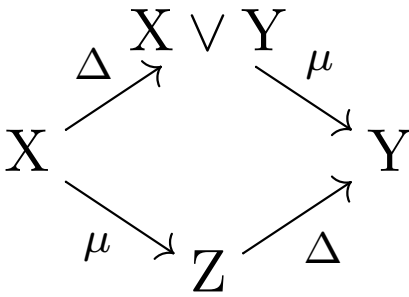
What is the composite of these two morphisms?

**Proposition 1.** *The category  $\mathcal{A}\text{-Hyp}_r^e$  has a presentation given by generators*

$$X \xrightarrow{\Delta} Y, \quad Y \xrightarrow{\mu} X,$$

*and relations*

(3a) 

(3b) 

(3c)  $(X \xrightarrow{\Delta} X) = \text{id} = (X \xrightarrow{\mu} X).$

*By convention,  $\Delta$  goes from a smaller flat to a bigger flat, and  $\mu$  from a bigger flat to a smaller flat.*

## Proof Outline:

- We need to show that any morphism in  $\mathcal{C}$  from  $X$  to  $Y$  can be uniquely written as a composite

$$X \xrightarrow{\Delta} X' \xrightarrow{\mu} Y.$$

- Use the diagrams in (3a) and (3b), to show that any morphism from  $X$  to  $Y$  can be reduced to one like the above.
- Also, check that morphisms compose by the rule (2). Use the diamond rule(3b).

**Proposition 2.** *The category of bicommutative bimonoids is equivalent to the functor category  $[\mathcal{A}\text{-Hyp}_r^e, \text{Vec}]$ .*