Algebras of Charts, Dicharts and Cones

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1 Cones

1.1 The Lattice of Cones

Consider a hyperplane arrangement $\mathcal{A}$, we are familiar with the poset of faces $\Sigma[\mathcal{A}]$ and poset of flats $\Pi[\mathcal{A}]$.

Both of these can be embedded into a bigger poset, the poset of cones.

A cone of an arrangment $\mathcal{A}$ is a subset of the ambient space obtained by intersection of some subset of the half-spaces of the arrangement.

We let $\Omega[\mathcal{A}]$ denote the set of all cones, it is a poset under inclusion.

The intersection of all half-spaces, the center $\mathcal{O}$ is the minimum element.

The intersection of two cones is again a cone, so meets exist in $\Omega[\mathcal{A}]$.

Since the entire ambient space(the top flat $\top$) is a cone, joins also exist in $\Omega[\mathcal{A}]$.

Hence $\Omega[\mathcal{A}]$ is a lattice.

1.2 Inclusion of $\Sigma[\mathcal{A}]$ and $\Pi[\mathcal{A}]$ into $\Omega[\mathcal{A}]$

We note that a hyperplane is simply the intersection of the two half-spaces associated to it, so flats are cones.

Thus, we have an inclusion $\Pi[\mathcal{A}] \hookrightarrow \Omega[\mathcal{A}]$ which is also a lattice homomorphism.

Since a face is intersection of half-spaces, it is a cone and we have an inclusion $\Sigma[\mathcal{A}] \hookrightarrow \Omega[\mathcal{A}]$.

If V is a cone and F is a face such that $V \leq F$, then G is necessarily a face. Thus $\Sigma[\mathcal{A}]$ is a convex subposet and the inclusion map preserves meets.

The joins are preserved whenever they exist. Note that the join of two faces is in general a cone and and not a face.

1.3 Support and Base Maps

Firstly, note that we can extend the opposition map on faces to cones becauuse every half-space h has an opposite half-space $\bar{h}$.

The opposition map $\Omega[\mathcal{A}] \to \Omega[\mathcal{A}]$ is given by $V \mapsto \bar{V} = \{-x | x \in V\}$.

A cone V is a flat iff $V = \bar{V}$. In particular,$V \wedge \bar{V}$ and $V \vee \bar{V}$ are flats.

Now we extend the support map defined on faces to cones, this is done in an obvious manner: the support of a cone V is defined to be the smallest flat which contains V.

This gives a map s : $\Omega[\mathcal{A}] \twoheadrightarrow \Pi[\mathcal{A}]$. Note that its restriction to faces is indeed the previously defined support map on faces.

It is an order preserving map.

We have, for any cone V ,

$$s(V) = V \vee \bar{V}. \tag{1}$$

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The base of a cone V is the largest flat contained in V.

So we have a map $b : \Omega[\mathcal{A}] \to \Pi[\mathcal{A}]$ which maps $V \mapsto b(V)$. It is also an order preserving map.

For any cone V, we have

$$b(V) = V \wedge \bar{V}. \tag{2}$$

We make the following observations which will be important later: For $V \in \Omega[\mathcal{A}]$ and $X \in \Pi[\mathcal{A}]$,

$$s(V) \leq_{\Pi[\mathcal{A}]} X \iff V \leq_{\Omega[\mathcal{A}]} X \tag{3}$$

$$X \leq_{\Pi[\mathcal{A}]} b(V) \iff X \leq_{\Omega[\mathcal{A}]} V \tag{4}$$

2 Charts

A chart in $\mathcal{A}$ is a subset of the set of hyperplanes in $\mathcal{A}$.

We let $\mathrm{G}[\mathcal{A}]$ denote the set of charts in $\mathcal{A}$.

We partially order this set by reverse inclusion: $h \leq g$ iff $h \supseteq g$.

Since it is the power set of the set of hyperplanes, $\mathrm{G}[\mathcal{A}]$ is a Boolean poset.

The center of a chart g, denoted $O(g)$, is the intersection of all hyperplanes contained in g. It is a flat.

A chart is *connected* if its center is the minimum flat $\perp$.

3 Dicharts

A dichart in $\mathcal{A}$ is a subset of the set of half-spaces of in $\mathcal{A}$.

We denote the set of all dicharts in $\mathcal{A}$ by $\overrightarrow{\mathrm{G}}[\mathcal{A}]$.

Once again, we partially order this set by reverse inclusion, $r \leq s$ iff $r \supseteq s$.

Like $\mathrm{G}[\mathcal{A}]$, $\overrightarrow{\mathrm{G}}[\mathcal{A}]$ is also a Boolean poset.

4 Adjunctions Between Posets

Let P and Q be two finite posets.

Suppose

$$\lambda : P \to Q \quad \text{and} \quad \rho : Q \to P$$

are order-preserving maps, .ie., $p_1 \leq p_2 \implies \lambda(p_1) \leq \lambda(p_2)$ for any $p_1, p_2 \in P$ and similarly $q_1 \leq q_2 \implies \rho(q_1) \leq \rho(q_2)$ for any $q_1, q_2 \in Q$.

We say that (λ, ρ) is an adjunction if

$$\lambda(x) \leq y \iff x \leq \rho(y) \tag{5}$$

for all $x \in P$ and $y \in Q$.

We also say that ρ is the right adjoint of λ, and λ is the left adjoint of ρ.

Suppose (λ, ρ) is an adjunction. Then

$$\lambda(\rho(y)) \leq y \quad \text{and} \quad x \leq \rho(\lambda(x)). \tag{6}$$

Adjunctions can be composed. If (λ, ρ) is an adjunction between P and Q, and (μ, δ) is an adjunction between Q and R, then $(\mu\lambda, \rho\delta)$ is an adjunction between P and R.

Assume now that P and Q are finite lattices. Then, given λ, the right adjoint ρ exists iff λ preserves finite joins.

Dually, given ρ, the left adjoint λ exists iff ρ preserves finite meets.

5 Adjunctions between posets of flats, cones, charts and dicharts

There are two commutative diagrams of order-preserving maps, namely,

(7)
$$
\begin{tikzcd}
\mathrm{G}[\mathcal{A}] \arrow[r, "\lambda'"] & \overrightarrow{\mathrm{G}}[\mathcal{A}] \\
\Pi[\mathcal{A}] \arrow[u, "\lambda"] \arrow[r, "\mathrm{i}"'] & \Omega[\mathcal{A}] \arrow[u, "\vec{\lambda}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\overrightarrow{\mathrm{G}}[\mathcal{A}] \arrow[r, "\rho'"] \arrow[d, "\vec{\rho}"'] & \mathrm{G}[\mathcal{A}] \arrow[d, "\rho"] \\
\Omega[\mathcal{A}] \arrow[r, "\mathrm{b}"'] & \Pi[\mathcal{A}]
\end{tikzcd}
$$

We will define the maps appearing in this diagram in the following sections.

We will see that the maps in the first diagram are surjective while those in the second diagram are injective and that the corresponding pairs form adjunctions.

5.1 Flats and Cones

We have already seen the base map $b : \Omega[\mathcal{A}] \to \Pi[\mathcal{A}]$ and the inclusion map $i : \Pi[\mathcal{A}] \to \Omega[\mathcal{A}]$.

By (4), (i, b) is an adjunction with the base map a right adjoint of the inclusion map.

Although not shown in the diagram, (s, i), the pair consisting of the support map $s : \Omega[\mathcal{A}] \to \Pi[\mathcal{A}]$ and inclusion map $i : \Pi[\mathcal{A}] \to \Omega[\mathcal{A}]$ is also an adjunction by (3), with the support map being the left adjoint of the inclusion map.

It follows that the support map preserves joins and the base map preserves meets.

5.2 Flats and Charts

Flats and charts are related by order-preserving maps

(8) $$\lambda : \Pi[\mathcal{A}] \to \mathrm{G}[\mathcal{A}], \qquad \mathrm{X} \mapsto \{\mathrm{H} \mid \mathrm{H} \geq \mathrm{X}\},$$

and

(9) $$\rho : \mathrm{G}[\mathcal{A}] \to \Pi[\mathcal{A}], \qquad g \mapsto \bigcap_{\mathrm{H} \in g} \mathrm{H}.$$

Note that $\rho(g) = \mathrm{O}(g)$. It is easy to see that

(10) $$\lambda(\mathrm{X}) \leq g \iff \mathrm{X} \leq \rho(g).$$

So (λ, ρ) is an adjunction with λ as the left adjoint.

Hence, λ preserves joins while ρ preseves meets.

5.3 Charts and Dicharts

Define the map

$$\lambda' : \mathrm{G}[\mathcal{A}] \to \overrightarrow{\mathrm{G}}[\mathcal{A}] \tag{11}$$

which sends g to r, where $\mathrm{h} \in r$ if $\mathrm{b}(\mathrm{h}) \in g$.

Define the map

$$\rho' : \overrightarrow{\mathrm{G}}[\mathcal{A}] \to \mathrm{G}[\mathcal{A}] \tag{12}$$

which sends r to g, where $\mathrm{H} \in g$ if there exists a $\mathrm{h} \in r$ such that $\mathrm{H} = \mathrm{b}(\mathrm{h})$.

Then, (λ', ρ') is an adjunction between the posets $\mathrm{G}[\mathcal{A}]$ and $\overrightarrow{\mathrm{G}}[\mathcal{A}]$.

5.4 Cones and Dicharts

Define the maps

(13) $$\vec{\lambda} : \Omega[\mathcal{A}] \to \overrightarrow{\mathrm{G}}[\mathcal{A}], \qquad \mathrm{V} \mapsto \{\mathrm{h} \mid \mathrm{h} \geq \mathrm{V}\},$$

and

(14) $$\vec{\rho} : \overrightarrow{\mathrm{G}}[\mathcal{A}] \to \Omega[\mathcal{A}], \qquad r \mapsto \bigcap_{\mathrm{h} \in r} \mathrm{h}.$$

Then, $(\vec{\lambda}, \vec{\rho})$ is seen to be an adjunction between the posets $\Omega[\mathcal{A}]$ and $\overrightarrow{\mathrm{G}}[\mathcal{A}]$ since

$$\vec{\lambda}(\mathrm{V}) \leq r \iff \vec{\lambda}(\mathrm{V}) \supseteq r \iff$$

$$V \leq h \quad \forall \quad h \in r \iff \mathrm{V} \leq \bigcap_{\mathrm{h} \in r} \mathrm{h} = \vec{\rho}(r).$$

Note that the order relation being reverse inclusion played an important role in this pair being an adjunction.

6 Algebras of Charts, Dicharts and Cones

We already have the Birkhoff algebra, the algebra of flats with the operation $H_X.H_Y = H_{X \vee Y}$.

In an analogous manner, we can linearize the poset of cones, charts and dicharts to get their respective algebras with the operation induced by the join operation on the poset.

As usual, we will denote the algebras by $\Omega[\mathcal{A}]$, $\mathsf{G}[\mathcal{A}]$ and $\overrightarrow{\mathsf{G}}[\mathcal{A}]$ respectively.

For example, the product in $\mathsf{G}[\mathcal{A}]$ is given by

$$H_g := H_{h \cap g}. \tag{15}$$

Note that the join operation leads to intersection since the partial order is defined by reverse inclusion.

Since the above defined algebras are linearizations of finite lattices, they are split-semisimple commutative. So each algebra has associated with it a $\mathtt{Q}-$basis.

For example, for charts, explicitly we have,

$$\mathtt{H}_g = \sum_{h\colon h \subseteq g} \mathtt{Q}_h \quad \text{and} \quad \mathtt{Q}_g = \sum_{h\colon h \subseteq g} (-1)^{|g \setminus h|}\, \mathtt{H}_h.$$

where $g \setminus h$ denotes the set complement of h in g, i.e., set subtraction, which can easily seen to be the Möbius function for the poset of all subsets of a given set(in other words for the poset $(\mathcal{P}(S), \subseteq)$, where S is a finite set, we have $\mu(X, Y) = (-1)^{|Y \setminus X|}$ whenever $X \subseteq Y$).

For dicharts, the formula is similar. For cones, we have,

$$\mathtt{H}_{\mathrm{V}} = \sum_{\mathrm{W}\colon \mathrm{W} \geq \mathrm{V}} \mathtt{Q}_{\mathrm{W}} \quad \text{and} \quad \mathtt{Q}_{\mathrm{V}} = \sum_{\mathrm{W}\colon \mathrm{W} \geq \mathrm{V}} \mu(\mathrm{V}, \mathrm{W})\, \mathtt{H}_{\mathrm{W}},$$

where μ is the Möbius function of the lattice of cones.

Consider the first commuting diagram in the section on adjunctions. The maps are join preserving. So on linearization, we obtain a commuting diagram of algebras.

(16)
$$
\begin{tikzcd}
\mathsf{G}[\mathcal{A}] \arrow[r, "\lambda'"] & \overrightarrow{\mathsf{G}}[\mathcal{A}] \\
\Pi[\mathcal{A}] \arrow[u, "\lambda"] \arrow[r, "i"'] & \Omega[\mathcal{A}] \arrow[u, "\vec{\lambda}"']
\end{tikzcd}
$$

The maps in the $\mathtt{H}$-basis are straightforward. In the $\mathtt{Q}$-basis, they can be described by employing the lemma below.

Lemma 1. *Suppose $f : P \to Q$ is a join-preserving map between finite lattices. Consider its linearization $\mathbf{k}P \to \mathbf{k}Q$ with $H_x \mapsto H_{f(x)}$, it is an algebra homomorphism. On the $\mathtt{Q}-$basis of primitive idempotents, this map is given by*

$$\mathtt{Q}_z \mapsto \sum_{w:g(w)=z} \mathtt{Q}_w \tag{17}$$

where $g : Q \to P$ denotes the right adjoint of f.