Operads

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1 A little category theory

1.1 Monoidal categories

Informally, a monoidal category is a category C, equipped with a product which is "associative" and "unital". We write

$$A \bullet B$$

for the product of the objects A and B. Associativity means that

$$(A \bullet B) \bullet C \cong A \bullet (B \bullet C)$$

for any objects A, B and C. Unitality means that there is a distinguished object I such that

$$A \cong I \bullet A \quad \text{and} \quad A \cong A \bullet I$$

for any object A.

We denote a monoidal category by $(\mathsf{C}, \bullet, I)$. The category is C, the monoidal structure is $\bullet$, and the unit object is I. It is also convenient to write $(\mathsf{C}, \bullet)$, keeping the unit object implicit.

Example 1. The category of sets Set under Cartesian product $\times$ is a monoidal category. Any singleton set can serve as the unit object. Let us take $\{\emptyset\}$ for definiteness. We denote this monoidal category by $(\mathsf{Set}, \times, \{\emptyset\})$.

Example 2. The category of vector spaces Vec under tensor product $\otimes$ is a monoidal category, with the base field $\mathbb{k}$ as the unit object. Thus, we write $(\mathsf{Vec}, \otimes, \mathbb{k})$.

1.2 Monoids

A monoid in a monoidal category $(\mathsf{C}, \bullet)$ is a triple (A, μ, ι) where

$$\mu : A \bullet A \to A \quad \text{and} \quad \iota : I \to A$$

(the product and the unit) satisfy the associativity and unitality axioms, which state that the following diagrams commute.

$$\begin{tikzcd}
A \bullet A \bullet A \arrow[r, "\mathrm{id} \bullet \mu"] \arrow[d, "\mu \bullet \mathrm{id}"'] & A \bullet A \arrow[d, "\mu"] \\
A \bullet A \arrow[r, "\mu"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
I \bullet A \arrow[r, "\iota \bullet \mathrm{id}"] & A \bullet A \arrow[d, "\mu"'] & A \bullet I \arrow[l, "\mathrm{id} \bullet \iota"'] \\
& A \arrow[ul, "\cong"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

A morphism $(A, \mu, \iota) \to (A', \mu', \iota')$ of monoids is a map $A \to A'$ which commutes with μ and μ', and ι and ι'.

Example 3. A monoid in $(\mathsf{Set}, \times)$ is the same as a usual monoid. For instance, the Tits monoid is a monoid in $(\mathsf{Set}, \times)$.

A monoid in $(\mathsf{Vec}, \otimes)$ is an (associative) algebra. For instance, the algebra of matrices, the polynomial algebra, the Tits algebra are all monoids in $(\mathsf{Vec}, \otimes)$.

Dually, a comonoid in a monoidal category $(\mathsf{C}, \bullet)$ is a triple (C, Δ, ϵ) where

$$\Delta : C \to C \bullet C \quad \text{and} \quad \epsilon : C \to I$$

(the coproduct and the counit) satisfy the coassociativity and counitality axioms. These are obtained from the monoid axioms by replacing μ by Δ and ι by ϵ, and reversing the arrows with those labels. A morphism $(C, \Delta, \epsilon) \to (C', \Delta', \epsilon')$ of comonoids is a map $C \to C'$ which commutes with Δ and Δ', and ϵ and ϵ'.

2 Dispecies

2.1 Dispecies

Let $\mathcal{A}$-$\mathtt{dHyp}$ denote the discrete category whose objects are nested flats, that is, pairs (X, Y) of flats with $\mathrm{X} \leq \mathrm{Y}$. (The only morphisms are identities.)

An $\mathcal{A}$-dispecies is a functor

$$\mathbf{p} : \mathcal{A}\text{-}\mathtt{dHyp} \to \mathsf{Vec}.$$

A map of $\mathcal{A}$-dispecies $\mathbf{p} \to \mathbf{q}$ is a natural transformation.

This defines the category of $\mathcal{A}$-dispecies which we denote by $\mathcal{A}$-$\mathtt{dSp}$.

It is a functor category, and we also write

$$\mathcal{A}\text{-}\mathtt{dSp} = [\mathcal{A}\text{-}\mathtt{dHyp}, \mathsf{Vec}].$$

The value of an $\mathcal{A}$-dispecies $\mathbf{p}$ on an object (X, Y) will be denoted $\mathbf{p}[\mathrm{X}, \mathrm{Y}]$.

Using these components, one may say more directly:

An $\mathcal{A}$-dispecies $\mathbf{p}$ consists of a family of vector spaces $\mathbf{p}[\mathrm{X}, \mathrm{Y}]$, one for each $\mathrm{X} \leq \mathrm{Y}$.

A map of $\mathcal{A}$-dispecies $f : \mathbf{p} \to \mathbf{q}$ consists of a family of linear maps

$$f_{\mathrm{X},\mathrm{Y}} : \mathbf{p}[\mathrm{X}, \mathrm{Y}] \to \mathbf{q}[\mathrm{X}, \mathrm{Y}],$$

one for each $\mathrm{X} \leq \mathrm{Y}$.

An $\mathcal{A}$-dispecies $\mathbf{p}$ is finite-dimensional if the vector spaces $\mathbf{p}[\mathrm{X}, \mathrm{Y}]$ have finite dimension for all $\mathrm{X} \leq \mathrm{Y}$.

2.2 Dispecies and species

Dispecies and species are closely related.

Any $\mathcal{A}$-dispecies $\mathbf{p}$ gives rise to an $\mathcal{A}$-species p by fixing the second coordinate of the nested flat to be the maximum flat.

In other words, $\mathrm{p}[\mathrm{X}] := \mathbf{p}[\mathrm{X}, \top]$.

2.3 Product and coproduct

The zero dispecies $\mathbf{0}$ is the $\mathcal{A}$-dispecies all of whose components are zero, namely,

$$\mathbf{0}[\mathrm{X}, \mathrm{Y}] = 0. \tag{1}$$

This is the initial and terminal object in the category of $\mathcal{A}$-dispecies.

Given $\mathcal{A}$-dispecies $\mathbf{p}$ and $\mathbf{q}$, their direct sum $\mathbf{p} + \mathbf{q}$ is defined by

$$(\mathbf{p} + \mathbf{q})[\mathrm{X}, \mathrm{Y}] := \mathbf{p}[\mathrm{X}, \mathrm{Y}] \oplus \mathbf{q}[\mathrm{X}, \mathrm{Y}]. \tag{2}$$

This is the product and coproduct in the category of $\mathcal{A}$-dispecies.

It is clear that arbitrary (co)products also exist in this category.

3 Operads

3.1 Substitution product on dispecies

Let $\mathbf{p}$ and $\mathbf{q}$ be two $\mathcal{A}$-dispecies.

Define a new $\mathcal{A}$-dispecies $\mathbf{p} \circ \mathbf{q}$ by

(3) $$(\mathbf{p} \circ \mathbf{q})[\mathrm{X}, \mathrm{Z}] := \bigoplus_{\mathrm{Y}:\, \mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}} \mathbf{p}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{q}[\mathrm{Y}, \mathrm{Z}].$$

The sum is over all flats Y which lie in-between X and Z.

We refer to this operation as the substitution product of $\mathbf{p}$ and $\mathbf{q}$.

This yields a monoidal structure on $\mathcal{A}$-dSp.

The unit object is the dispecies $\mathbf{x}$ defined by

(4) $$\mathbf{x}[\mathrm{X}, \mathrm{Y}] = \begin{cases} \Bbbk & \text{if } \mathrm{X} = \mathrm{Y}, \\ 0 & \text{otherwise.} \end{cases}$$

The (unbracketed) substitution product of three dispecies $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ can be written as

$$(\mathbf{p}\circ\mathbf{q}\circ\mathbf{r})[\mathrm{X},\mathrm{W}] := \bigoplus_{\mathrm{X}\leq\mathrm{Y}\leq\mathrm{Z}\leq\mathrm{W}} \mathbf{p}[\mathrm{X},\mathrm{Y}]\otimes\mathbf{q}[\mathrm{Y},\mathrm{Z}]\otimes\mathbf{r}[\mathrm{Z},\mathrm{W}],$$

with the sum being over Y and Z.

This consideration readily extends to a finite number of dispecies. Let

$$\mathbf{p}^{\circ n} := \underbrace{\mathbf{p}\circ\mathbf{p}\circ\cdots\circ\mathbf{p}}_{n}$$

denote the n-fold substitution product of $\mathbf{p}$ with itself. By convention, $\mathbf{p}^{\circ 0} := \mathbf{x}$.

Observe that

$$(\mathbf{p}_1 + \mathbf{p}_2) \circ \mathbf{q} \cong \mathbf{p}_1 \circ \mathbf{q} + \mathbf{p}_2 \circ \mathbf{q}$$

and

$$\mathbf{q} \circ (\mathbf{p}_1 + \mathbf{p}_2) \cong \mathbf{q} \circ \mathbf{p}_1 + \mathbf{q} \circ \mathbf{p}_2.$$

3.2 Operads

An $\mathcal{A}$-operad is a monoid in the monoidal category of $\mathcal{A}$-dispecies $(\mathcal{A}\text{-}\mathrm{dSp}, \circ, \mathbf{x})$.

In other words, an $\mathcal{A}$-operad is an $\mathcal{A}$-dispecies $\mathbf{a}$ equipped with maps

$$\mathbf{a} \circ \mathbf{a} \to \mathbf{a} \quad \text{and} \quad \mathbf{x} \to \mathbf{a}$$

which satisfy the associativity and unitality axioms.

A morphism of $\mathcal{A}$-operads is a morphism of monoids.

We denote the category of $\mathcal{A}$-operads by $\mathcal{A}$-Op.

Explicitly, an $\mathcal{A}$-operad is an $\mathcal{A}$-dispecies $\mathbf{a}$ equipped with linear maps

(5)

$$\gamma : \mathbf{a}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y}, \mathrm{Z}] \to \mathbf{a}[\mathrm{X}, \mathrm{Z}] \quad \text{and} \quad \eta : \mathbb{k} \to \mathbf{a}[\mathrm{X}, \mathrm{X}],$$

the former for each $\mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}$ and the latter for each X, subject to the following conditions.

Associativity. *For any* $\mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z} \leq \mathrm{W}$*, the diagram*

(6a)

$$\begin{tikzcd}
\mathbf{a}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y},\mathrm{Z}] \otimes \mathbf{a}[\mathrm{Z},\mathrm{W}] \arrow[r, "\gamma\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\gamma"'] & \mathbf{a}[\mathrm{X},\mathrm{Z}] \otimes \mathbf{a}[\mathrm{Z},\mathrm{W}] \arrow[d, "\gamma"] \\
\mathbf{a}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y},\mathrm{W}] \arrow[r, "\gamma"'] & \mathbf{a}[\mathrm{X},\mathrm{W}]
\end{tikzcd}$$

commutes.

Unitality. *For any* $\mathrm{X} \leq \mathrm{Y}$*, the diagrams*

(6b)

$$\begin{tikzcd}
 & \mathbf{a}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y},\mathrm{Y}] \arrow[dr, "\gamma"] & \\
\mathbf{a}[\mathrm{X},\mathrm{Y}] \otimes \Bbbk \arrow[ur, "\mathrm{id}\otimes\eta"] \arrow[rr, "\cong"'] & & \mathbf{a}[\mathrm{X},\mathrm{Y}]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbf{a}[\mathrm{X},\mathrm{X}] \otimes \mathbf{a}[\mathrm{X},\mathrm{Y}] \arrow[dr, "\gamma"] & \\
\Bbbk \otimes \mathbf{a}[\mathrm{X},\mathrm{Y}] \arrow[ur, "\eta\otimes\mathrm{id}"] \arrow[rr, "\cong"'] & & \mathbf{a}[\mathrm{X},\mathrm{Y}]
\end{tikzcd}$$

commute.

We refer to γ as the substitution map of $\mathbf{a}$, and to η as the unit map of $\mathbf{a}$. We also refer to both of them as the structure maps of $\mathbf{a}$.

A morphism of $\mathcal{A}$-operads is a map $f : \mathbf{a} \to \mathbf{b}$ of $\mathcal{A}$-dispecies such that the diagrams

(7)

$$\begin{tikzcd}
\mathbf{a}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y},\mathrm{Z}] \arrow[r, "\gamma"] \arrow[d, "f_{\mathrm{X},\mathrm{Y}} \otimes f_{\mathrm{Y},\mathrm{Z}}"'] & \mathbf{a}[\mathrm{X},\mathrm{Z}] \arrow[d, "f_{\mathrm{X},\mathrm{Z}}"] \\
\mathbf{b}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{b}[\mathrm{Y},\mathrm{Z}] \arrow[r, "\gamma"'] & \mathbf{b}[\mathrm{X},\mathrm{Z}]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{a}[\mathrm{X},\mathrm{X}] \arrow[rr, "f_{\mathrm{X},\mathrm{X}}"] & & \mathbf{b}[\mathrm{X},\mathrm{X}] \\
& \Bbbk \arrow[ul, "\eta"] \arrow[ur, "\eta"'] &
\end{tikzcd}$$

commute.

Example 4. Let $\mathcal{A}$ be an arrangement of rank zero. There is only one flat in $\mathcal{A}$, namely $\perp$. The functor

$$(\mathcal{A}\text{-}\mathrm{dSp}, \circ) \to (\mathsf{Vec}, \otimes), \qquad \mathbf{p} \mapsto \mathbf{p}[\perp, \perp], \tag{8}$$

is an isomorphism of monoidal categories.

By passing to the categories of monoids, we deduce that the category of $\mathcal{A}$-operads is isomorphic to the category of (associative) algebras.

For an operad $\mathbf{a}$, the product and unit of the corresponding algebra are given by

$$\gamma : \mathbf{a}[\perp, \perp] \otimes \mathbf{a}[\perp, \perp] \to \mathbf{a}[\perp, \perp] \quad \text{and} \quad \eta : \mathbb{k} \to \mathbf{a}[\perp, \perp],$$

respectively.

3.3 Ideals and quotients

Subdispecies and quotient dispecies can be defined as in the case of species or vector spaces.

Similarly, for any map of dispecies $f : \mathbf{q} \to \mathbf{p}$, we have the notions of injectivity, surjectivity, kernel, cokernel, image and coimage.

Suppose $\mathbf{a}$ is an operad.

A subdispecies $\mathbf{q}$ of $\mathbf{a}$ is an ideal of $\mathbf{a}$ if it is preserved by the substitution map of $\mathbf{a}$, that is, for any $\mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}$, there are induced maps

$$\mathbf{a}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{q}[\mathrm{Y},\mathrm{Z}] \to \mathbf{q}[\mathrm{X},\mathrm{Z}] \quad \text{and} \quad \mathbf{q}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y},\mathrm{Z}] \to \mathbf{q}[\mathrm{X},\mathrm{Z}].$$

For any subdispecies $\mathbf{q}$, there is a smallest ideal of $\mathbf{a}$ which contains $\mathbf{q}$. It is obtained by intersecting all ideals of $\mathbf{a}$ which contain $\mathbf{q}$. We call this the ideal generated by $\mathbf{q}$.

Now suppose $\mathbf{q}$ is an ideal of $\mathbf{a}$. Then the quotient dispecies $\mathbf{a}/\mathbf{q}$ is an operad, and the quotient map $\mathbf{a} \to \mathbf{a}/\mathbf{q}$ is a morphism of operads:

The substitution map of $\mathbf{a}/\mathbf{q}$ is the dotted arrow in the diagram

$$\begin{tikzcd}
\mathbf{a}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y},\mathrm{Z}] \arrow[r] \arrow[d, two heads] & \mathbf{a}[\mathrm{X},\mathrm{Z}] \arrow[d, two heads] \\
(\mathbf{a}/\mathbf{q})[\mathrm{X},\mathrm{Y}] \otimes (\mathbf{a}/\mathbf{q})[\mathrm{Y},\mathrm{Z}] \arrow[r, dotted] & (\mathbf{a}/\mathbf{q})[\mathrm{X},\mathrm{Z}].
\end{tikzcd}$$

The kernel of the left-vertical map is

$$\mathbf{a}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{q}[\mathrm{Y},\mathrm{Z}] + \mathbf{q}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y},\mathrm{Z}].$$

Since $\mathbf{q}$ is an ideal, the top-horizontal map takes this subspace to $\mathbf{q}[\mathrm{X},\mathrm{Z}]$, which is the kernel of the right-vertical map. This yields the dotted arrow.

The unit map of $\mathbf{a}/\mathbf{q}$ is defined by the composite map

$$\Bbbk \to \mathbf{a}[\mathrm{X},\mathrm{X}] \twoheadrightarrow (\mathbf{a}/\mathbf{q})[\mathrm{X},\mathrm{X}].$$

3.4 Cooperads

Dually, an $\mathcal{A}$-cooperad is a comonoid in the monoidal category of $\mathcal{A}$-dispecies $(\mathcal{A}\text{-}\mathtt{dSp}, \circ, \mathbf{x})$.

A morphism of $\mathcal{A}$-cooperads is a morphism of comonoids.

Explicitly, an $\mathcal{A}$-cooperad consists of an $\mathcal{A}$-dispecies $\mathbf{c}$ equipped with linear maps

$$\mathbf{c}[\mathrm{X}, \mathrm{Z}] \to \mathbf{c}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{c}[\mathrm{Y}, \mathrm{Z}] \quad \text{and} \quad \mathbf{c}[\mathrm{X}, \mathrm{X}] \to \Bbbk$$

subject to the coassociativity and counitality axioms.

3.5 Duality between operads and cooperads

Every dispecies $\mathbf{p}$ has a dual. It is the dispecies $\mathbf{p}^*$ defined by

$$\mathbf{p}^*[\mathrm{X}, \mathrm{Y}] := \mathbf{p}[\mathrm{X}, \mathrm{Y}]^*.$$

The dual of a cooperad is an operad.

Conversely, the dual of a finite-dimensional operad is a cooperad.

4 Set-operads

One can also consider dispecies with values in the category of sets.

More formally, we replace Vec by Set in the preceding discussion.

The resulting notions are called set-dispecies and set-operads.

Tensor product and direct sum of vector spaces are replaced by cartesian product and disjoint union of sets.

4.1 Set-dispecies

An $\mathcal{A}$-set-dispecies is a functor

$$\mathrm{p} : \mathcal{A}\text{-}\mathtt{dHyp} \to \mathsf{Set}.$$

A map of $\mathcal{A}$-set-dispecies $\mathrm{p} \to \mathrm{q}$ is a natural transformation. We denote the category of $\mathcal{A}$-set-dispecies by

$$\mathcal{A}\text{-}\mathtt{SetdSp} = [\mathcal{A}\text{-}\mathtt{dHyp}, \mathsf{Set}].$$

Explicitly: An $\mathcal{A}$-set-dispecies p consists of a family of sets $\mathrm{p}[\mathrm{X}, \mathrm{Y}]$, one for each $\mathrm{X} \leq \mathrm{Y}$. A map of $\mathcal{A}$-set-dispecies $f : \mathrm{p} \to \mathrm{q}$ consists of a family of maps

$$f_{\mathrm{X},\mathrm{Y}} : \mathrm{p}[\mathrm{X}, \mathrm{Y}] \to \mathrm{q}[\mathrm{X}, \mathrm{Y}],$$

one for each $\mathrm{X} \leq \mathrm{Y}$.

4.2 Substitution product on set-dispecies

The substitution product of $\mathcal{A}$-set-dispecies p and q is defined by

$$(9) \quad (\mathrm{p} \circ \mathrm{q})[\mathrm{X}, \mathrm{Z}] := \bigsqcup_{\mathrm{Y}:\, \mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}} \mathrm{p}[\mathrm{X}, \mathrm{Y}] \times \mathrm{q}[\mathrm{Y}, \mathrm{Z}].$$

The sum is over all flats Y which lie in-between X and Z.

This yields a monoidal structure on $\mathcal{A}$-$\mathtt{SetdSp}$.

The unit object is the $\mathcal{A}$-set-dispecies x defined by

$$(10) \qquad \mathrm{x}[\mathrm{X}, \mathrm{Y}] = \begin{cases} \{\emptyset\} & \text{if } \mathrm{X} = \mathrm{Y}, \\ \emptyset & \text{otherwise.} \end{cases}$$

4.3 Set-operads

An $\mathcal{A}$-set-operad is a monoid in the monoidal category of $\mathcal{A}$-set-dispecies $(\mathcal{A}\text{-}\mathtt{SetdSp}, \circ, \mathrm{x})$. A morphism of $\mathcal{A}$-set-operads is a morphism of monoids.

Explicitly, an $\mathcal{A}$-set-operad is an $\mathcal{A}$-set-dispecies a equipped with maps

(11)

$$\gamma : \mathrm{a}[\mathrm{X}, \mathrm{Y}] \times \mathrm{a}[\mathrm{Y}, \mathrm{Z}] \to \mathrm{a}[\mathrm{X}, \mathrm{Z}] \quad \text{and} \quad \eta : \{\emptyset\} \to \mathrm{a}[\mathrm{X}, \mathrm{X}],$$

the former for each $\mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}$ and the latter for each X, subject to the associativity and unitality axioms.

4.4 Set-operad as a category

An $\mathcal{A}$-set-dispecies p determines a directed graph whose vertices are flats of $\mathcal{A}$, and elements of the set $\mathrm{p}[\mathrm{X}, \mathrm{Y}]$ are arrows from Y to X.

Given $\mathcal{A}$-set-dispecies p and q, the substitution product $\mathrm{p} \circ \mathrm{q}$ can be described in this language as follows.

An element of $(\mathrm{p} \circ \mathrm{q})[\mathrm{X}, \mathrm{Z}]$ is a pair (f, g) of arrows such that g starts from Z, f ends at X, and the endpoint of g equals the starting point of f.

Further:

Lemma 1. *An $\mathcal{A}$-set-operad determines a category whose objects are flats of $\mathcal{A}$. More precisely, for an $\mathcal{A}$-set-operad a, elements of the set $\mathrm{a}[\mathrm{X}, \mathrm{Y}]$ correspond to morphisms from Y to X.*

4.5 Linearization functor

Linearization of set-dispecies (set-operads) produces dispecies (operads).

We say an operad is linearized if it arises by linearizing a set-operad.

5 Commutative, associative and Lie operads

We now define the commutative, associative and Lie operads. We denote them by $\mathbf{Com}$, $\mathbf{As}$ and $\mathbf{Lie}$, respectively. These are connected operads.

An operad $\mathbf{a}$ is connected if the unit map η is an isomorphism for all X. This is equivalent to requiring that the vector spaces $\mathbf{a}[\mathrm{X}, \mathrm{X}]$ are one-dimensional for all X.

5.1 Commutative operad

The exponential dispecies $\mathbf{E}$ is defined by setting

$$\mathbf{E}[\mathrm{X}, \mathrm{Y}] := \Bbbk$$

for all $\mathrm{X} \leq \mathrm{Y}$.

The exponential dispecies carries the structure of an operad.

The substitution map is

(12) $\mathbf{E}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{E}[\mathrm{Y}, \mathrm{Z}] \to \mathbf{E}[\mathrm{X}, \mathrm{Z}], \quad \Bbbk \otimes \Bbbk \xrightarrow{\cong} \Bbbk.$

The unit map $\Bbbk \to \mathbf{E}[\mathrm{X}, \mathrm{X}]$ is the identity.

We call this the commutative operad and denote it by $\mathbf{Com}$.

It is a connected operad.

It is the linearization of the set-operad which is singleton in each component.

Since (12) is an isomorphism, by reversing arrows, we see that $\mathbf{E}$ also carries the structure of a cooperad.

We call this the commutative cooperad and denote it by $\mathbf{Com}^*$.

As suggested by the notation, it is the cooperad dual to $\mathbf{Com}$.

5.2 Associative operad

The dispecies of chambers $\mathbf{\Gamma}$ is defined by

$$\mathbf{\Gamma}[\mathrm{X}, \mathrm{Y}] := \Gamma[\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}],$$

where the rhs is the space of chambers of the arrangement $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$. Recall that the latter is the arrangement over X and under Y.

The dispecies of chambers carries the structure of an operad.

The substitution map

$$\mathbf{\Gamma}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{\Gamma}[\mathrm{Y}, \mathrm{Z}] \to \mathbf{\Gamma}[\mathrm{X}, \mathrm{Z}]$$

is defined by specializing the substitution map of chambers to the arrangement $\mathcal{A}_{\mathrm{X}}^{\mathrm{Z}}$.

(Note that the arrangements under and over the flat Y/X of $\mathcal{A}_{\mathrm{X}}^{\mathrm{Z}}$ are precisely $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$ and $\mathcal{A}_{\mathrm{Y}}^{\mathrm{Z}}$, respectively.)

The unit map $\mathbb{k} \to \mathbf{\Gamma}[\mathrm{X}, \mathrm{X}]$ is the identity.

We call this the associative operad and denote it by $\mathbf{As}$.

It is a connected operad.

Recall that lunes with base X and case Y correspond to chambers of $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$. The associative operad can thus also be understood in terms of lunes as follows.

Recall the category of lunes. In this category, morphisms only go from a bigger flat to a smaller flat. Hence, it arises from a set-operad as in Lemma 1. The linearization of this set-operad is precisely the associative operad.

The substitution map can equivalently be written as

(13)

$$\mathbf{As}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{As}[\mathrm{Y},\mathrm{Z}] \to \mathbf{As}[\mathrm{X},\mathrm{Z}], \quad \mathtt{H}_{\mathrm{L}} \otimes \mathtt{H}_{\mathrm{M}} \mapsto \mathtt{H}_{\mathrm{L}\circ\mathrm{M}}.$$

Alternatively, in term of nested faces, we may write

(14)

$$\mathbf{As}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{As}[\mathrm{Y},\mathrm{Z}] \to \mathbf{As}[\mathrm{X},\mathrm{Z}], \quad \mathtt{H}_{F/A} \otimes \mathtt{H}_{G/F} \mapsto \mathtt{H}_{G/A}$$

where A, F and G are faces with support X, Y and Z, respectively, and $A \leq F \leq G$. It is implicit that $\mathtt{H}_{F/A} = \mathtt{H}_{F'/A'}$ whenever $(A, F) \sim (A', F')$.

5.3 Lie operad

The Lie dispecies $\mathbf{Lie}$ is defined by

$$\mathbf{Lie}[\mathrm{X}, \mathrm{Y}] := \mathsf{Lie}[\mathcal{A}^{\mathrm{Y}}_{\mathrm{X}}],$$

where the rhs is the space of Lie elements of the arrangement $\mathcal{A}^{\mathrm{Y}}_{\mathrm{X}}$.

The Lie dispecies carries the structure of an operad given by the substitution map of Lie elements (in the same manner as discussed above for the dispecies of chambers).

This is the Lie operad, which we continue to denote by $\mathbf{Lie}$.

It is a connected suboperad of $\mathbf{As}$.

5.4 Morphisms

There are morphisms of operads

$$\mathbf{Lie} \to \mathbf{As} \to \mathbf{Com}. \tag{15}$$

The first morphism was mentioned above.

The second morphism is as follows. Evaluated on the (X, Y)-component, sends each basis chamber in $\mathbf{\Gamma}[\mathrm{X}, \mathrm{Y}]$ to $1 \in \Bbbk$.

6 Hadamard product

6.1 Hadamard product of dispecies

Let $\mathbf{p}$ and $\mathbf{q}$ be two dispecies. Define a new dispecies $\mathbf{p} \times \mathbf{q}$ by

(16) $$(\mathbf{p} \times \mathbf{q})[\mathrm{X}, \mathrm{Y}] := \mathbf{p}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{q}[\mathrm{X}, \mathrm{Y}].$$

This is the Hadamard product of $\mathbf{p}$ and $\mathbf{q}$.

This yields a monoidal structure on the category of dispecies $\mathcal{A}\text{-}\mathrm{dSp}$.

The unit object is the exponential dispecies $\mathbf{E}$.

By interchanging the tensor factors in (16), we see that there is an isomorphism of dispecies $\mathbf{p} \times \mathbf{q} \to \mathbf{q} \times \mathbf{p}$.

This defines a braiding (which is in fact a symmetry).

6.2 Hadamard product of (co)operads

Suppose $\mathbf{a}$ and $\mathbf{b}$ are operads. Then so is their Hadamard product $\mathbf{a} \times \mathbf{b}$.

Further, the symmetry $\mathbf{a} \times \mathbf{b} \to \mathbf{b} \times \mathbf{a}$ is a morphism of operads.

(The same is true for cooperads.)

The structure maps of $\mathbf{a} \times \mathbf{b}$ are obtained by tensoring those of $\mathbf{a}$ and $\mathbf{b}$. That is, the substitution map is

$$(\mathbf{a} \times \mathbf{b})[\mathrm{X}, \mathrm{Y}] \otimes (\mathbf{a} \times \mathbf{b})[\mathrm{Y}, \mathrm{Z}] \cong$$

$$(\mathbf{a}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y}, \mathrm{Z}]) \otimes (\mathbf{b}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{b}[\mathrm{Y}, \mathrm{Z}]) \xrightarrow{\gamma \otimes \gamma}$$
$$\mathbf{a}[\mathrm{X}, \mathrm{Z}] \otimes \mathbf{b}[\mathrm{X}, \mathrm{Z}] = (\mathbf{a} \times \mathbf{b})[\mathrm{X}, \mathrm{Z}],$$

while the unit map is

$$\Bbbk \to \Bbbk \otimes \Bbbk \xrightarrow{\eta \otimes \eta} \mathbf{a}[\mathrm{X}, \mathrm{X}] \otimes \mathbf{b}[\mathrm{X}, \mathrm{X}] = (\mathbf{a} \times \mathbf{b})[\mathrm{X}, \mathrm{X}].$$

6.3 An interchange law

There is an interchange law between the substitution and Hadamard products on dispecies.

This is briefly explained below.

Let $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ and s be dispecies. Then

$$((\mathbf{p} \times \mathbf{q}) \circ (\mathbf{r} \times \mathbf{s}))[\mathrm{X}, \mathrm{Z}] =$$

$$\bigoplus_{\mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}} (\mathbf{p}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{q}[\mathrm{X}, \mathrm{Y}]) \otimes (\mathbf{r}[\mathrm{Y}, \mathrm{Z}] \otimes \mathbf{s}[\mathrm{Y}, \mathrm{Z}])$$

and

$$((\mathbf{p} \circ \mathbf{r}) \times (\mathbf{q} \circ \mathbf{s}))[\mathrm{X}, \mathrm{Z}] =$$

$$\bigoplus_{\mathrm{X} \leq \mathrm{Y}, \mathrm{W} \leq \mathrm{Z}} (\mathbf{p}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{r}[\mathrm{Y}, \mathrm{Z}]) \otimes (\mathbf{q}[\mathrm{X}, \mathrm{W}] \otimes \mathbf{s}[\mathrm{W}, \mathrm{Z}]).$$

The first sum is over all Y in-between X and Z, while the second sum is over all Y and W in-between X and Z.

Each summand of the former also appears in the latter (for $\mathrm{Y} = \mathrm{W}$). Rearranging the middle two tensor factors yields a map of dispecies

(17a) $\zeta : (\mathbf{p} \times \mathbf{q}) \circ (\mathbf{r} \times \mathbf{s}) \to (\mathbf{p} \circ \mathbf{r}) \times (\mathbf{q} \circ \mathbf{s}).$

We also have maps

(17b)

$$\Delta_{\mathbf{x}} : \mathbf{x} \to \mathbf{x}\times\mathbf{x}, \quad \mu_{\mathbf{E}} : \mathbf{E}\circ\mathbf{E} \to \mathbf{E}, \quad \iota_{\mathbf{E}} = \epsilon_{\mathbf{x}} : \mathbf{x} \to \mathbf{E}.$$

The first map is defined to be the obvious isomorphism. The second and third maps are defined to be the structure maps of the commutative operad.

Proposition 1. *With the structure maps* (17a) *and* (17b)*,*

$$(\mathcal{A}\text{-dSp}, \circ, \mathbf{x}, \times, \mathbf{E})$$

is a 2-monoidal category. Moreover, it is $\times$-braided.

7 Orientation operad

7.1 Orientation operad

The dispecies $\mathbf{E^o}$ is defined by

$$\mathbf{E^o}[\mathrm{X}, \mathrm{Y}] := \mathsf{E^o}[\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}],$$

where the rhs is the orientation space of the arrangement $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$. Note that $\mathbf{E^o}[\mathrm{X}, \mathrm{X}] = \Bbbk$. The maps

$$\mathsf{E^o}[\mathcal{A}^{\mathrm{Y}}] \otimes \mathsf{E^o}[\mathcal{A}_{\mathrm{Y}}] \xrightarrow{\cong} \mathsf{E^o}[\mathcal{A}]$$

applied to $\mathcal{A}_{\mathrm{X}}^{\mathrm{Z}}$ turn $\mathbf{E^o}$ into a connected operad.

We call this the orientation operad and denote it by $\mathbf{Com^o}$.

7.2 Orientation functor

For any dispecies $\mathbf{p}$, define its oriented partner $\mathbf{p}^{\mathbf{o}}$ by

$$\mathbf{p}^{\mathbf{o}} := \mathbf{p} \times \mathbf{E}^{\mathbf{o}}. \tag{18}$$

The assignment $\mathbf{p} \mapsto \mathbf{p}^{\mathbf{o}}$ is functorial in $\mathbf{p}$.

This is the orientation functor on dispecies.

Note that $\mathbf{E}$ and $\mathbf{E}^{\mathbf{o}}$ are oriented partners of each other, that is, $\mathbf{E}^{\mathbf{o}} \times \mathbf{E}^{\mathbf{o}} \cong \mathbf{E}$.

This follows from the isomorphism

$$\mathsf{E}^{\mathbf{o}}[\mathcal{A}] \otimes \mathsf{E}^{\mathbf{o}}[\mathcal{A}] \xrightarrow{\cong} \mathbb{k}.$$

Hence, the orientation functor is an involution.

Further, for any dispecies $\mathbf{p}$ and $\mathbf{q}$, there are natural isomorphisms

(19) $$(\mathbf{p} \circ \mathbf{q})^{\mathbf{o}} \cong \mathbf{p}^{\mathbf{o}} \circ \mathbf{q}^{\mathbf{o}} \quad \text{and} \quad \mathbf{x}^{\mathbf{o}} \cong \mathbf{x}.$$

In other words, the orientation functor is a strong monoidal functor.

So, it preserves (co)operads.

This may be also be deduced as follows.

We know that the Hadamard product preserves (co)operads. Thus if $\mathbf{a}$ is an operad, then so is $\mathbf{a}^{\mathbf{o}}$ obtained by taking Hadamard product of $\mathbf{a}$ with $\mathbf{Com}^{\mathbf{o}}$.

Some notations and terminology

A dispecies $\mathbf{p}$ is positive if $\mathbf{p}[\mathrm{X}, \mathrm{X}] = 0$ for all X.

8 Operad presentations

8.1 Free operad

We begin with the construction of the free operad on a dispecies.

Let $\mathbf{e}$ be any dispecies.

Define the dispecies

$$\mathcal{F}_\circ(\mathbf{e}) := \underset{n \geq 0}{+} \mathbf{e}^{\circ n}. \tag{20}$$

It carries an operad structure.

The substitution map

$$\mathcal{F}_\circ(\mathbf{e}) \circ \mathcal{F}_\circ(\mathbf{e}) \to \mathcal{F}_\circ(\mathbf{e})$$

is defined by distributing the substitution product over the direct sum, and then using the identifications

$$\mathbf{e}^{\circ m} \circ \mathbf{e}^{\circ n} \xrightarrow{\cong} \mathbf{e}^{\circ (m+n)}.$$

The unit map $\mathbf{x} \to \mathcal{F}_\circ(\mathbf{e})$ identifies $\mathbf{x}$ with $\mathbf{e}^{\circ 0}$.

We say that $\mathcal{F}_\circ(\mathbf{e})$ is the operad freely generated by $\mathbf{e}$, or simply, the free operad on $\mathbf{e}$.

Let us understand this construction explicitly.

We have

$$\mathcal{F}_\circ(\mathbf{e})[\mathrm{X},\mathrm{Z}] = \bigoplus_{\mathrm{X}\leq\mathrm{Y}_1\leq\cdots\leq\mathrm{Y}_k\leq\mathrm{Z}} \mathbf{e}[\mathrm{X},\mathrm{Y}_1]\otimes\cdots\otimes\mathbf{e}[\mathrm{Y}_k,\mathrm{Z}],$$

where the sum is over all multichains in the interval $[\mathrm{X},\mathrm{Z}]$.

(A multichain is a chain in which elements are allowed to repeat.)

Now suppose Y is a flat in-between X and Z. Given a multichain in $[\mathrm{X}, \mathrm{Y}]$ and a multichain in $[\mathrm{Y}, \mathrm{Z}]$, one obtains a multichain in $[\mathrm{X}, \mathrm{Z}]$ by concatenation.

The product

$$\mathcal{F}_{\circ}(\mathbf{e})[\mathrm{X}, \mathrm{Y}] \otimes \mathcal{F}_{\circ}(\mathbf{e})[\mathrm{Y}, \mathrm{Z}] \to \mathcal{F}_{\circ}(\mathbf{e})[\mathrm{X}, \mathrm{Z}]$$

is defined by concatenating multichains, and tensoring the corresponding summands.

Observe that when $\mathbf{e}$ is positive,

$$\mathcal{F}_\circ(\mathbf{e})[\mathrm{X},\mathrm{Z}] = \bigoplus_{\mathrm{X}<\mathrm{Y}_1<\cdots<\mathrm{Y}_k<\mathrm{Z}} \mathbf{e}[\mathrm{X},\mathrm{Y}_1]\otimes\cdots\otimes\mathbf{e}[\mathrm{Y}_k,\mathrm{Z}],$$

the sum now being over all chains in the interval $[\mathrm{X},\mathrm{Z}]$.

The number of summands is finite in this case, and hence, if $\mathbf{e}$ is finite-dimensional, then so is $\mathcal{F}_\circ(\mathbf{e})$.

Example 5. Recall from Example 4 that for a rank-zero arrangement, a dispecies $\mathbf{e}$ is the same as a vector space $V := \mathbf{e}[\perp, \perp]$.

In this case, the free operad on $\mathbf{e}$ is the same as the tensor algebra of V.

8.2 Quadratic operads

We say an operad $\mathbf{a}$ has a presentation $\langle \mathbf{e} \,|\, \mathbf{r} \rangle$ if $\mathbf{a}$ is the quotient of the free operad $\mathcal{F}_\circ(\mathbf{e})$ by the ideal generated by $\mathbf{r}$.

(By assumption, $\mathbf{r}$ is a subdispecies of $\mathcal{F}_\circ(\mathbf{e})$.)

An operad $\mathbf{a}$ is quadratic if it has a presentation $\langle \mathbf{e} \,|\, \mathbf{r} \rangle$ in which $\mathbf{r}$ is a subdispecies of $\mathbf{e} \circ \mathbf{e}$.

In this case:

The ideal generated by $\mathbf{r}$ is spanned by $\mathbf{r}$, $\mathbf{e} \circ \mathbf{r}$, $\mathbf{r} \circ \mathbf{e}$, $\mathbf{e} \circ \mathbf{e} \circ \mathbf{r}$, and so on.

Thus, the canonical map $\mathbf{e} \to \mathbf{a}$ is injective, and $\mathbf{e}$ is a subdispecies of $\mathbf{a}$.

We say a dispecies $\mathbf{e}$ is concentrated in rank i if the component $\mathbf{e}[\mathrm{X}, \mathrm{Y}] = 0$ whenever $\mathrm{rk}(\mathrm{Y}/\mathrm{X}) \neq i$.

An operad $\mathbf{a}$ is binary quadratic if it has a presentation $\langle \mathbf{e} \,|\, \mathbf{r} \rangle$ in which $\mathbf{e}$ is concentrated in rank 1 and $\mathbf{r}$ is a subdispecies of $\mathbf{e} \circ \mathbf{e}$.

In this case, the subdispecies $\mathbf{r}$ is concentrated in rank 2.

A binary quadratic operad is necessarily connected.

Example 6. We now present the commutative operad as a binary quadratic operad.

Let $\mathbf{e}$ be the dispecies concentrated in rank 1 given by

$$\mathbf{e}[\mathrm{X}, \mathrm{Y}] := \mathbb{k}$$

for $\mathrm{rk}(\mathrm{Y}/\mathrm{X}) = 1$.

Hence, $\mathcal{F}_{\circ}(\mathbf{e})[\mathrm{X}, \mathrm{Z}]$ has a basis consisting of all maximal chains of flats in the interval $[\mathrm{X}, \mathrm{Z}]$.

In particular, for $\mathrm{rk}(\mathrm{Z}/\mathrm{X}) = 2$, $\mathcal{F}_{\circ}(\mathbf{e})[\mathrm{X}, \mathrm{Z}]$ has a basis consisting of flats Y strictly in-between X and Z.

Let $\mathbf{r}[\mathrm{X}, \mathrm{Z}]$ be its subspace spanned by elements of the form $\mathrm{Y} - \mathrm{Y}'$.

We claim that $\langle \mathbf{e} \,|\, \mathbf{r} \rangle$ is the commutative operad.

To see this, first consider two maximal chains in any interval $[\mathrm{X}, \mathrm{Z}]$ which differ in exactly one position as shown below.

$$
\begin{array}{ccccc}
 & & \mathrm{Y}_k & & \\
 & \diagup & & \diagdown & \\
\mathrm{X} - \mathrm{Y}_1 - \ldots - \mathrm{Y}_{k-1} & & & & \mathrm{Y}_{k+1} - \ldots - \mathrm{Z} \\
 & \diagdown & & \diagup & \\
 & & \mathrm{Y}'_k & &
\end{array}
$$

Since $\mathrm{Y}_k - \mathrm{Y}'_k$ belongs to $\mathbf{r}[\mathrm{Y}_{k-1}, \mathrm{Y}_{k+1}]$, the difference of the above two maximal chains belongs to the ideal generated by $\mathbf{r}$, and hence the two chains become equal in the quotient.

Now any two maximal chains in $[\mathrm{X}, \mathrm{Z}]$ are related to each other by a sequence in which two successive maximal chains differ in exactly one position.

It follows that any two maximal chains are equal in the quotient.

Example 7. We now present the associative operad as a binary quadratic operad.

Let $\mathbf{e}$ be the dispecies concentrated in rank 1 given by

$$\mathbf{e}[\mathrm{X}, \mathrm{Y}] := \mathbf{\Gamma}[\mathrm{X}, \mathrm{Y}]$$

for $\mathrm{rk}(\mathrm{Y}/\mathrm{X}) = 1$.

Since a rank-one arrangement has 2 chambers, these spaces are two-dimensional.

Let us denote the free operad $\mathcal{F}_{\circ}(\mathbf{e})$ by $\mathbf{mc}$.

Observe that $\mathbf{mc}[\mathrm{X}, \mathrm{Z}]$ has a basis indexed by all maximal chains of faces in $\mathcal{A}^{\mathrm{Z}}_{\mathrm{X}}$.

The substitution map is given by concatenation of maximal chains.

Now let $\mathrm{rk}(\mathrm{Z}/\mathrm{X}) = 2$. Fix any face A with support X. Then $\mathbf{mc}[\mathrm{X}, \mathrm{Z}]$ has a basis consisting of chains $(A \lessdot F \lessdot C)$ of faces, with C having support Z.

Now take $\mathbf{r}[\mathrm{X}, \mathrm{Z}]$ to be its subspace spanned by elements of the form

$$(A \lessdot F \lessdot C) - (A \lessdot G \lessdot C),$$

as C varies.

We claim that $\langle \mathbf{e} \,|\, \mathbf{r} \rangle$ is the associative operad.

We follow the argument in Example 6. Two maximal chains of faces are equal in the quotient iff they end in the same face.

Example 8. We now present the Lie operad as a binary quadratic operad.

Let $\mathbf{e}$ be the dispecies concentrated in rank 1 given by

$$\mathbf{e}[\mathrm{X}, \mathrm{Y}] := \mathbf{E^o}[\mathrm{X}, \mathrm{Y}]$$

for $\mathrm{rk}(\mathrm{Y}/\mathrm{X}) = 1$.

Now let $\mathrm{rk}(\mathrm{Z}/\mathrm{X}) = 2$ and $\mathrm{Y}_1, \ldots, \mathrm{Y}_n$ denote the flats strictly in-between X and Z.

(The number n depends on X and Z.)

Fix a face A with support X. Let $\mathbf{r}[\mathrm{X}, \mathrm{Z}]$ be the subspace spanned by

$$\sum_{i=1}^{n} [A \lessdot F_i] \otimes [F_i \lessdot C_i],$$

where F_i is any one of the two faces greater than A with support Y_i, and C_i is then chosen so that the orientation $[A \lessdot F_i \lessdot C_i]$ equals a fixed orientation of $\mathcal{A}_{\mathrm{X}}^{\mathrm{Z}}$ (independent of i).

The presentation $\langle \mathbf{e} \,|\, \mathbf{r} \rangle$ yields the Lie operad.

This is a nontrivial theorem.

Setting the above sum to 0 is the Jacobi identity.

8.3 Quadratic duals

All dispecies in this discussion are finite-dimensional. Let $(\mathbf{e}, \mathbf{r})$ be a quadratic data. Define $\mathbf{r}^{\perp}$ by the exact sequence

$$\mathbf{0} \to \mathbf{r}^{\perp} \hookrightarrow \mathbf{e}^* \circ \mathbf{e}^* = (\mathbf{e} \circ \mathbf{e})^* \twoheadrightarrow \mathbf{r}^* \to \mathbf{0}.$$

It is a subdispecies of $\mathbf{e}^* \circ \mathbf{e}^*$ obtained by taking the kernel of the dual of $\mathbf{r} \to \mathbf{e} \circ \mathbf{e}$. Equivalently, consider the pairing

$$(\mathbf{e}^* \circ \mathbf{e}^*) \times (\mathbf{e} \circ \mathbf{e}) = (\mathbf{e} \circ \mathbf{e})^* \times (\mathbf{e} \circ \mathbf{e}) \to \mathbf{E}.$$

Under this pairing, $\mathbf{r}^{\perp}$ is the subdispecies of $\mathbf{e}^* \circ \mathbf{e}^*$ which is orthogonal to $\mathbf{r}$.

Let us make the pairing more explicit using components. Observe that

$$(\mathbf{e}^* \circ \mathbf{e}^*)[\mathrm{X}, \mathrm{Z}] \otimes (\mathbf{e} \circ \mathbf{e})[\mathrm{X}, \mathrm{Z}] =$$

$$\bigoplus_{\mathrm{X} \leq \mathrm{Y}, \mathrm{Y}' \leq \mathrm{Z}} \mathbf{e}[\mathrm{X}, \mathrm{Y}]^* \otimes \mathbf{e}[\mathrm{Y}, \mathrm{Z}]^* \otimes \mathbf{e}[\mathrm{X}, \mathrm{Y}'] \otimes \mathbf{e}[\mathrm{Y}', \mathrm{Z}].$$

The sum is over flats Y and Y' which lie in-between X and Z. The map

$$(\mathbf{e}^* \circ \mathbf{e}^*)[\mathrm{X}, \mathrm{Z}] \otimes (\mathbf{e} \circ \mathbf{e})[\mathrm{X}, \mathrm{Z}] \longrightarrow \Bbbk$$

is as follows. The summands with $\mathrm{Y} \neq \mathrm{Y}'$ are sent to zero. For the summands with $\mathrm{Y} = \mathrm{Y}'$, we use the canonical pairing between a vector space and its dual. Then $\mathbf{r}^{\perp}[\mathrm{X}, \mathrm{Z}]$ is the subspace of $(\mathbf{e}^* \circ \mathbf{e}^*)[\mathrm{X}, \mathrm{Z}]$ which is orthogonal to $\mathbf{r}[\mathrm{X}, \mathrm{Z}]$ wrt the above pairing.

For any dispecies $\mathbf{p}$, define its oriented dual, denoted $\mathbf{p}^{\circledast}$, to be the oriented partner of $\mathbf{p}^*$. That is, $\mathbf{p}^{\circledast} = (\mathbf{p}^*)^{\mathbf{o}}$.

Let $\mathbf{a} = \langle \mathbf{e} \,|\, \mathbf{r} \rangle$ be a quadratic operad. The unoriented quadratic dual of $\mathbf{a}$ is defined to be

$$\mathbf{a}^{\color{blue}!} := \langle \mathbf{e}^* \,|\, \mathbf{r}^{\perp} \rangle.$$

The oriented quadratic dual of $\mathbf{a}$ is the oriented partner of $\mathbf{a}^{\color{blue}!}$, and we write it as

$$\mathbf{a}^{!} = \langle \mathbf{e}^* \,|\, \mathbf{r}^{\perp} \rangle^{\mathbf{o}} = \langle \mathbf{e}^{\circledast} \,|\, \mathbf{r}^{\ominus} \rangle,$$

with $\mathbf{r}^{\ominus} = (\mathbf{r}^{\perp})^{\mathbf{o}}$.

The basic properties of these two duals are stated below.

$$(\mathbf{a}^{\color{blue}!})^{\color{blue}!} \cong \mathbf{a}, \;\; (\mathbf{a}^{!})^{!} \cong \mathbf{a}, \;\; (\mathbf{a}^{\mathbf{o}})^{\color{blue}!} \cong (\mathbf{a}^{\color{blue}!})^{\mathbf{o}} \cong \mathbf{a}^{!}, \;\; (\mathbf{a}^{\mathbf{o}})^{!} \cong (\mathbf{a}^{!})^{\mathbf{o}} \cong \mathbf{a}^{\color{blue}!}.$$

Lemma 2. *The commutative operad and Lie operad are oriented quadratic duals of each other:*

$$\mathbf{Com}^! = \mathbf{Lie} \quad \textit{and} \quad \mathbf{Lie}^! = \mathbf{Com}.$$

The associative operad $\mathbf{As}$ *is isomorphic to its oriented quadratic dual:*

$$\mathbf{As}^! \cong \mathbf{As}.$$

Proof. The first statement follows from the observation that the relations defining $\mathbf{Com}$ are orthogonal to the Jacobi identities.

Simply put: In a vector space V with basis $e_1, \ldots, e_n$, the one-dimensional space spanned by the linear functional $e_1^* + \cdots + e_n^*$ is orthogonal to the subspace of V spanned by the vectors $e_i - e_j$, as i and j vary.

Let $\mathbf{As} = \langle \mathbf{e} \,|\, \mathbf{r} \rangle$. Observe that $\mathbf{r}^{\perp}[\mathrm{X}, \mathrm{Z}]$ is spanned by elements of the form

$$(A \lessdot F \lessdot C)^* + (A \lessdot G \lessdot C)^*,$$

since their pairing with elements of the form $(A \lessdot F \lessdot C) - (A \lessdot G \lessdot C)$ evaluates to 0. It follows that $\mathbf{r}^{\ominus}[\mathrm{X}, \mathrm{Z}]$ is spanned by elements of the form

$$(A{\lessdot}F{\lessdot}C)^*{\otimes}[A{\lessdot}F{\lessdot}C]{-}(A{\lessdot}G{\lessdot}C)^*{\otimes}[A{\lessdot}G{\lessdot}C].$$

Note that the sign between the two terms is now minus because $[A \lessdot F \lessdot C]$ and $[A \lessdot G \lessdot C]$ are opposite orientations.

To obtain self-duality of $\mathbf{As}$, we start with the isomorphism

$$\mathbf{e} \xrightarrow{\cong} \mathbf{e}^{\circledast}, \qquad (A \lessdot F) \mapsto (A \lessdot F)^* \otimes [A \lessdot F].$$

(Simply put, in a rank-one arrangement, a chamber determines an orientation.) Now observe that under the induced isomorphism $\mathbf{e} \circ \mathbf{e} \cong \mathbf{e}^{\circledast} \circ \mathbf{e}^{\circledast}$, the relations $\mathbf{r}$ correspond to $\mathbf{r}^{\ominus}$. Thus the presentations $\langle \mathbf{e} \,|\, \mathbf{r} \rangle$ and $\langle \mathbf{e}^{\circledast} \,|\, \mathbf{r}^{\ominus} \rangle$ are isomorphic, as required. $\square$

9 Operad modules

9.1 Species as a module over dispecies

Given a dispecies $\mathbf{p}$ and a species m, define the species $\mathbf{p} \circ \mathrm{m}$ by

$$(\mathbf{p} \circ \mathrm{m})[\mathrm{X}] := \bigoplus_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{X}} \mathbf{p}[\mathrm{X}, \mathrm{Y}] \otimes \mathrm{m}[\mathrm{Y}]. \tag{21}$$

We refer to $\mathbf{p} \circ \mathrm{m}$ as the substitution product of $\mathbf{p}$ and m.

Observe that there are natural isomorphisms

$$(\mathbf{p} \circ \mathbf{q}) \circ \mathrm{m} \cong \mathbf{p} \circ (\mathbf{q} \circ \mathrm{m}) \quad \text{and} \quad \mathbf{x} \circ \mathrm{m} \cong \mathrm{m}$$

for dispecies $\mathbf{p}$ and $\mathbf{q}$ and species m.

Thus, the category of species is a module over the monoidal category of dispecies.

9.2 Operad modules

The category of species is a module over the monoidal category of dispecies and an operad is a monoid in the latter.

Thus, associated to any operad $\mathbf{a}$, we have the category of $\mathbf{a}$-modules.

Let us spell out this construction.

Let $\mathbf{a}$ be an operad. An $\mathbf{a}$-module is a species m equipped with a map of species

$$\mathbf{a} \circ \mathsf{m} \to \mathsf{m}$$

satisfying the associativity and unitality axioms.

A map of $\mathbf{a}$-modules is a map of species $\mathsf{m} \to \mathsf{n}$ which commutes with the respective structure maps.

Explicitly, a module m over $\mathbf{a}$ is a species m equipped with linear maps

$$\gamma : \mathbf{a}[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{m}[\mathrm{Y}] \to \mathsf{m}[\mathrm{X}], \tag{22}$$

one for each $\mathrm{X} \leq \mathrm{Y}$, such that the following axioms hold.

Associativity. *For any $\mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}$, the diagram*

(23a)

$$\begin{tikzcd}
\mathbf{a}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y}, \mathrm{Z}] \otimes \mathsf{m}[\mathrm{Z}] \arrow[r, "\mathrm{id} \otimes \gamma"] \arrow[d, "\gamma \otimes \mathrm{id}"'] & \mathbf{a}[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{m}[\mathrm{Y}] \arrow[d, "\gamma"] \\
\mathbf{a}[\mathrm{X}, \mathrm{Z}] \otimes \mathsf{m}[\mathrm{Z}] \arrow[r, "\gamma"'] & \mathsf{m}[\mathrm{X}]
\end{tikzcd}$$

commutes.

Unitality. *For any X, the diagram*

(23b)

$$\begin{tikzcd}
& \mathbf{a}[\mathrm{X}, \mathrm{X}] \otimes \mathsf{m}[\mathrm{X}] \arrow[dr, "\gamma"] & \\
\Bbbk \otimes \mathsf{m}[\mathrm{X}] \arrow[ur, "\eta \otimes \mathrm{id}"] \arrow[rr, "\cong"'] & & \mathsf{m}[\mathrm{X}]
\end{tikzcd}$$

commutes.

A map of modules is a map f : m $\to$ n of species such that for any $\mathrm{X} \leq \mathrm{Y}$, the diagram

$$
\begin{array}{ccc}
\mathbf{a}[\mathrm{X},\mathrm{Y}] \otimes \mathsf{m}[\mathrm{Y}] & \xrightarrow{\gamma} & \mathsf{m}[\mathrm{X}] \\
{\scriptstyle \mathrm{id}\otimes f_{\mathrm{Y}}}\downarrow & & \downarrow{\scriptstyle f_{\mathrm{X}}} \\
\mathbf{a}[\mathrm{X},\mathrm{Y}] \otimes \mathsf{n}[\mathrm{Y}] & \xrightarrow[\gamma]{} & \mathsf{n}[\mathrm{X}]
\end{array}
\tag{24}
$$

commutes.

Proposition 2. *For any operad* $\mathbf{a}$*, the free* $\mathbf{a}$*-module over a species* n *is given by* $\mathbf{a} \circ \mathsf{n}$*, with the structure map given by the composite*

$$\mathbf{a} \circ (\mathbf{a} \circ \mathsf{n}) \xrightarrow{\cong} (\mathbf{a} \circ \mathbf{a}) \circ \mathsf{n} \xrightarrow{\gamma \circ \mathrm{id}} \mathbf{a} \circ \mathsf{n}.$$

Proof. This is an instance of the general result: For a monoid A in a monoidal category $(\mathsf{C}, \bullet)$ and an object M in a module category, the free A-module over M is given by $A \bullet M$. In our case, the monoidal category is $(\mathcal{A}\text{-}\mathtt{dSp}, \circ)$ and the module category is $\mathcal{A}\text{-}\mathtt{Sp}$. $\square$

Example 9. Let x be the species such that $\mathsf{x}[\mathrm{Y}]$ is $\Bbbk$ if $\mathrm{Y} = \top$, and 0 otherwise. For any dispecies $\mathbf{p}$, the species $\mathbf{p} \circ \mathsf{x}$ is given by

$$(\mathbf{p} \circ \mathsf{x})[\mathrm{X}] = \mathbf{p}[\mathrm{X}, \top].$$

Thus, any dispecies gives rise to a species by fixing the second coordinate of the nested flat to be the maximum flat. This was mentioned in Section 2.2.

Now suppose $\mathbf{a}$ is an operad. Then $\mathbf{a} \circ \mathsf{x}$ is the free $\mathbf{a}$-module over x. Its structure map is obtained by restricting the product of $\mathbf{a}$:

$$\mathbf{a}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y}, \top] \to \mathbf{a}[\mathrm{X}, \top].$$

Example 10. Let m be a module over $\mathbf{Com}$. Let us make this explicit. For each $\mathrm{X} \leq \mathrm{Y}$, we have a linear map

$$\mathbf{Com}[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{m}[\mathrm{Y}] \to \mathsf{m}[\mathrm{X}],$$

which is the same as a linear map

$$\mathsf{m}[\mathrm{Y}] \to \mathsf{m}[\mathrm{X}].$$

Let us denote this map by $\mu_{\mathrm{X}}^{\mathrm{Y}}$.

Associativity says that

$$\begin{tikzcd} & \mathsf{m}[\mathrm{Y}] \arrow[dr, "\mu_{\mathrm{X}}^{\mathrm{Y}}"] & \\ \mathsf{m}[\mathrm{Z}] \arrow[ur, "\mu_{\mathrm{Y}}^{\mathrm{Z}}"] \arrow[rr, "\mu_{\mathrm{X}}^{\mathrm{Z}}"'] & & \mathsf{m}[\mathrm{X}] \end{tikzcd}$$

commutes for all $\mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}$.

Unitality says that

$$(\mathsf{m}[\mathrm{X}] \xrightarrow{\mu_{\mathrm{X}}^{\mathrm{X}}} \mathsf{m}[\mathrm{X}]) = \mathrm{id}$$

for all X.

This is the same as a commutative monoid in species.

Example 11. Let m be a module over $\mathbf{As}$. Let us make this explicit. For each $\mathrm{X} \leq \mathrm{Y}$, we have a linear map

$$\mathbf{As}[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{m}[\mathrm{Y}] \to \mathsf{m}[\mathrm{X}].$$

Thus, for each $F \geq A$, with $\mathrm{s}(F) = \mathrm{Y}$ and $\mathrm{s}(A) = \mathrm{X}$, we have a map $\mathsf{m}[\mathrm{Y}] \to \mathsf{m}[\mathrm{X}]$.

For book-keeping purposes, for each flat X, we make copies of $\mathsf{m}[\mathrm{X}]$, one for each face A with support X, and denote that copy by $\mathsf{m}[A]$.

The above data can then be expressed as a family of linear maps

$$\mu_A^F : \mathsf{m}[F] \to \mathsf{m}[A],$$

one for each pair of faces $A \leq F$.

These are subject to the following conditions.

For A and B of the same support, and $A \leq F$,

$$\begin{tikzcd} \mathsf{m}[F] \arrow[r, "\mu_A^F"] \arrow[d, "\cong"'] & \mathsf{m}[A] \arrow[d, "\cong"] \\ \mathsf{m}[BF] \arrow[r, "\mu_B^{BF}"'] & \mathsf{m}[B], \end{tikzcd}$$

for every $A \leq F \leq G$,

$$\begin{tikzcd} & \mathsf{m}[F] \arrow[dr, "\mu_A^F"] & \\ \mathsf{m}[G] \arrow[ur, "\mu_F^G"] \arrow[rr, "\mu_A^G"'] & & \mathsf{m}[A], \end{tikzcd}$$

and

$$(\mathsf{m}[A] \xrightarrow{\mu_A^A} \mathsf{m}[A]) = \mathrm{id}\,.$$

This is the same as a monoid in species.

Example 12. We never defined a Lie monoid, but we can now define it as follows.

A Lie monoid is a module over the Lie operad.

Making this concept explicit is harder since it is not set-theoretic.

Summary:

Operad	Operad module
$\mathbf{As}$	monoid in species
$\mathbf{Com}$	commutative monoid in species
$\mathbf{Lie}$	Lie monoid in species

Dually, by reversing arrows, one can define a comodule over a cooperad. This is what happens in our familiar examples.

Cooperad	Cooperad comodule
$\mathbf{As}^*$	comonoid in species
$\mathbf{Com}^*$	cocommutative comonoid in species
$\mathbf{Lie}^*$	Lie comonoid in species

A bioperad is a triple $(\mathbf{a}, \mathbf{c}, \lambda)$ consisting of an operad $\mathbf{a}$, a cooperad $\mathbf{c}$, and a map of dispecies

$$\lambda : \mathbf{a} \circ \mathbf{c} \to \mathbf{c} \circ \mathbf{a}$$

such that the diagrams

$$\begin{tikzcd}
\mathbf{a} \circ \mathbf{c} \circ \mathbf{c} \arrow[r, "\lambda \circ \mathrm{id}"] & \mathbf{c} \circ \mathbf{a} \circ \mathbf{c} \arrow[r, "\mathrm{id} \circ \lambda"] & \mathbf{c} \circ \mathbf{c} \circ \mathbf{a} \\
\mathbf{a} \circ \mathbf{c} \arrow[u] \arrow[rr, "\lambda"'] & & \mathbf{c} \circ \mathbf{a} \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{a} \circ \mathbf{a} \circ \mathbf{c} \arrow[r, "\mathrm{id} \circ \lambda"] \arrow[d] & \mathbf{a} \circ \mathbf{c} \circ \mathbf{a} \arrow[r, "\lambda \circ \mathrm{id}"] & \mathbf{c} \circ \mathbf{a} \circ \mathbf{a} \arrow[d] \\
\mathbf{a} \circ \mathbf{c} \arrow[rr, "\lambda"'] & & \mathbf{c} \circ \mathbf{a}
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathbf{c} \arrow[dl] \arrow[dr] & \\
\mathbf{a} \circ \mathbf{c} \arrow[rr, "\lambda"'] & & \mathbf{c} \circ \mathbf{a}
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbf{a} & \\
\mathbf{a} \circ \mathbf{c} \arrow[ur] \arrow[rr, "\lambda"'] & & \mathbf{c} \circ \mathbf{a} \arrow[ul]
\end{tikzcd}$$

commute.

A $(\mathbf{a}, \mathbf{c}, \lambda)$-module is a species m which is a module over $\mathbf{a}$ and a comodule over $\mathbf{c}$ such that the diagram

$$\begin{tikzcd}
\mathbf{a} \circ \mathrm{m} \arrow[r] \arrow[d] & \mathrm{m} \arrow[r] & \mathbf{c} \circ \mathrm{m} \\
\mathbf{a} \circ \mathbf{c} \circ \mathrm{m} \arrow[rr, "\lambda \circ \mathrm{id}"'] & & \mathbf{c} \circ \mathbf{a} \circ \mathrm{m} \arrow[u]
\end{tikzcd}$$

commutes.

Examples of bioperads directly relevant for us are shown below.

Bioperad	Biperad module
$(\mathbf{As}, \mathbf{As}^*)$	bimonoid in species
$(\mathbf{Com}, \mathbf{As}^*)$	commutative bimonoid in species
$(\mathbf{As}, \mathbf{Com}^*)$	cocommutative bimonoid in species
$(\mathbf{Com}, \mathbf{Com}^*)$	bicommutative bimonoid in species

In each case, there is an appropriate map λ which we have kept hidden.

10 Series of a dispecies

10.1 Series of a dispecies

Let $\mathbf{p}$ be a dispecies. Define the space of series of $\mathbf{p}$ as

$$\mathscr{S}(\mathbf{p}) := \bigoplus_{\mathrm{X} \leq \mathrm{Y}} \mathbf{p}[\mathrm{X}, \mathrm{Y}]. \tag{25}$$

The sum is over both X and Y.

An element of $\mathscr{S}(\mathbf{p})$ is called a $\mathbf{p}$-series. It consists of a family of elements

$$s_{\mathrm{X},\mathrm{Y}} \in \mathbf{p}[\mathrm{X}, \mathrm{Y}],$$

one for each $\mathrm{X} \leq \mathrm{Y}$. We denote this family by

$$s = (s_{\mathrm{X},\mathrm{Y}}).$$

The vector space structure is given by

$$(s+t)_{\mathrm{X},\mathrm{Y}} = s_{\mathrm{X},\mathrm{Y}} + t_{\mathrm{X},\mathrm{Y}} \quad \text{and} \quad (c\, s)_{\mathrm{X},\mathrm{Y}} = c\, s_{\mathrm{X},\mathrm{Y}}$$

for $s, t \in \mathscr{S}(\mathbf{p})$ and $c \in \mathbb{k}$.

The construction of $\mathscr{S}(\mathbf{p})$ is functorial in $\mathbf{p}$. Thus, we have a functor

$$\mathscr{S} : \mathcal{A}\text{-}\mathtt{dSp} \to \mathsf{Vec}$$

from the category of dispecies to the category of vector spaces.

Proposition 3. *For an operad $\mathbf{a}$, its space of series $\mathscr{S}(\mathbf{a})$ carries the structure of an algebra.*

The product of $\mathscr{S}(\mathbf{a})$ can be expressed as follows. It sends $\mathbf{a}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y}', \mathrm{Z}]$ to $\mathbf{a}[\mathrm{X}, \mathrm{Z}]$ when $\mathrm{Y} = \mathrm{Y}'$ using the substitution map γ, and to zero otherwise.

Equivalently:

For $s, t \in \mathscr{S}(\mathbf{a})$, their product denoted $s \circ t$ is given by

$$(s \circ t)_{\mathrm{X},\mathrm{Z}} = \sum_{\mathrm{Y}\colon \mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}} \gamma(s_{\mathrm{X},\mathrm{Y}} \otimes t_{\mathrm{Y},\mathrm{Z}}), \tag{26}$$

where γ is the substitution map of $\mathbf{a}$.

We refer to $\mathscr{S}(\mathbf{a})$ as the algebra of series of $\mathbf{a}$.

Example 13. The algebra of series of the operad $\mathbf{x}$ is the Birkhoff algebra Π:

$$\mathscr{S}(\mathbf{x}) = \Pi, \qquad 1 \mapsto \mathtt{Q}_{\mathrm{X}} \text{ for } 1 \in \mathbf{x}[\mathrm{X}, \mathrm{X}].$$

10.2 Series of a species

Series of a species can be defined in the same way. The space of series of a species m is given by

$$\mathscr{S}(\mathsf{m}) := \bigoplus_{\mathrm{X}} \mathsf{m}[\mathrm{X}]. \tag{27}$$

The sum is over all flats X. Thus, a series of m is a family of elements

$$v_{\mathrm{X}} \in \mathsf{m}[\mathrm{X}],$$

one for each flat X.

Alternatively, it is a family of elements $v_F \in \mathsf{m}[F]$, one for each face F, such that

$$\beta_{G,F}(v_F) = v_G,$$

whenever F and G have the same support.

Proposition 4. *For a module* (m, γ) *over an operad* $(\mathbf{a}, \gamma, \eta)$*, its space of series* $\mathscr{S}(\mathsf{m})$ *carries the structure of a module over the algebra* $\mathscr{S}(\mathbf{a})$*.*

The module structure can be expressed as follows. It sends $\mathbf{a}[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{m}[\mathrm{Y}']$ to $\mathsf{m}[\mathrm{X}]$ when $\mathrm{Y} = \mathrm{Y}'$ using γ, and to zero otherwise.

Equivalently:

For $s \in \mathscr{S}(\mathbf{a})$ and $v \in \mathscr{S}(\mathsf{m})$, the action denoted $s \circ v$ is given by

$$(s \circ v)_{\mathrm{X}} = \sum_{\mathrm{Y}\colon \mathrm{X} \leq \mathrm{Y}} \gamma(s_{\mathrm{X},\mathrm{Y}} \otimes v_{\mathrm{Y}}).$$

11 Series of the commutative operad

11.1 Incidence algebra

Recall that the components of the commutative operad $\mathbf{Com}$ equal the base field $\Bbbk$ and the structure maps are the canonical identifications. Thus, we obtain:

Proposition 5. *A series of $\mathbf{Com}$ is a family of scalars*

$$(s_{\mathrm{Y}/\mathrm{X}})_{\mathrm{X}\leq\mathrm{Y}}.$$

The algebra structure on the space of series is given by

$$(s\circ t)_{\mathrm{Z}/\mathrm{X}} = \sum_{\mathrm{Y}:\,\mathrm{X}\leq\mathrm{Y}\leq\mathrm{Z}} s_{\mathrm{Y}/\mathrm{X}} t_{\mathrm{Z}/\mathrm{Y}}, \quad s,t \in \mathscr{S}(\mathbf{Com}).$$

The unit element is the family $(e_{\mathrm{Y}/\mathrm{X}})$ with $e_{\mathrm{X}/\mathrm{X}} = 1$ and $e_{\mathrm{Y}/\mathrm{X}} = 0$ for $\mathrm{Y} > \mathrm{X}$.

Observe that:

Proposition 6. *The algebra $\mathscr{S}(\mathbf{Com})$ coincides with the incidence algebra of the poset of flats $\Pi[\mathcal{A}]$.*

11.2 Modules

Let (a, μ) be a commutative monoid in species. Recall that this is the same as a module over $\mathbf{Com}$. Hence $\mathscr{S}(\mathsf{a})$ is a module over the algebra $\mathscr{S}(\mathbf{Com})$. Explicitly, the module structure is given by

$$(s \circ v)_{\mathrm{X}} = \sum_{\mathrm{Y}\colon \mathrm{X} \leq \mathrm{Y}} s_{\mathrm{Y}/\mathrm{X}} \mu_{\mathrm{X}}^{\mathrm{Y}}(v_{\mathrm{Y}}), \quad s \in \mathscr{S}(\mathbf{Com}),\, v \in \mathscr{S}(\mathsf{a}).$$

Example 14. Consider the exponential species E. All components equal the base field $\Bbbk$. Thus, a series of E is a choice of scalars v_{X} one for each flat X. Further, since E is a commutative monoid, $\mathscr{S}(\mathsf{E})$ is a module over $\mathscr{S}(\mathbf{Com})$. In fact, it is the incidence module of the poset of flats.

11.3 Exp and log series

Define the series $\exp$ and $\log$ by

$$\exp_{\mathrm{Y}/\mathrm{X}} := 1 \qquad \text{and} \qquad \log_{\mathrm{Y}/\mathrm{X}} := \mu(\mathrm{X}, \mathrm{Y}).$$

These correspond to the zeta and Möbius functions in the incidence algebra. They define operators on the series of a commutative monoid. Explicitly, for a series v of a commutative monoid (a, μ),

$$(\exp \circ v)_{\mathrm{X}} = \sum_{\mathrm{Y}\colon \mathrm{Y} \geq \mathrm{X}} \mu_{\mathrm{X}}^{\mathrm{Y}}(v_{\mathrm{Y}})$$

and

$$(\log \circ v)_{\mathrm{X}} = \sum_{\mathrm{Y}\colon \mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{X}, \mathrm{Y})\, \mu_{\mathrm{X}}^{\mathrm{Y}}(v_{\mathrm{Y}}).$$

Proposition 7. *For any commutative monoid* a*, the maps*

$$\mathscr{S}(\mathsf{a}) \underset{\log}{\overset{\exp}{\rightleftarrows}} \mathscr{S}(\mathsf{a}) \tag{28}$$

are inverse bijections.

Proof. This is clear since $\exp$ and $\log$ are inverses, that is,

$$\log \circ \exp = e = \exp \circ \log .$$

□

For the commutative monoid $\mathsf{a} = \mathsf{E}$, the correspondence (28) reduces to Möbius inversion.

11.4 Group-like and primitive series

Let (c, Δ) be a cocommutative comonoid in species. A series v of c is group-like if

$$\Delta_{\mathrm{X}}^{\mathrm{Y}}(v_{\mathrm{X}}) = v_{\mathrm{Y}}$$

for all $\mathrm{Y} \geq \mathrm{X}$.

Let $\mathscr{G}(\mathsf{c})$ denote the set of all group-like series of c. Note that a group-like series v is uniquely determined by its value $v_{\perp}$. Thus:

Lemma 3. *For a cocommutative comonoid* c*, we have* $\mathscr{G}(\mathsf{c}) = \mathsf{c}[\perp]$*.*

Let (c, Δ) be a cocommutative comonoid in species. A series v of c is primitive if

$$\Delta_{\mathrm{X}}^{\mathrm{Y}}(v_{\mathrm{X}}) = 0$$

for all $\mathrm{Y} > \mathrm{X}$.

Let $\mathscr{P}(\mathsf{c})$ denote the set of all primitive series of c. Observe that

$$\mathscr{P}(\mathsf{c}) = \mathscr{S}(\mathcal{P}(\mathsf{c})),$$

where $\mathcal{P}(\mathsf{c})$ is the primitive part of c.

Theorem 1. *If* h *is a bicommutative bimonoid, the maps in* (28) *restrict to inverse bijections*

$$\mathscr{P}(\mathsf{h}) \underset{\log}{\overset{\exp}{\rightleftarrows}} \mathscr{G}(\mathsf{h}). \tag{29}$$

The correspondences are natural in a and h.

Proof. We need to check that $\exp$ and $\log$ map as stated.

Suppose v is a primitive series of h. We need to show that $\exp \circ v$ is a group-like series. The calculation goes as follows. For $\mathrm{X} \leq \mathrm{Z}$,

$$\begin{aligned}
\Delta^{\mathrm{Z}}_{\mathrm{X}}((\exp \circ v)_{\mathrm{X}}) &= \sum_{\mathrm{Y} \colon \mathrm{Y} \geq \mathrm{X}} \Delta^{\mathrm{Z}}_{\mathrm{X}} \mu^{\mathrm{Y}}_{\mathrm{X}}(v_{\mathrm{Y}}) \\
&= \sum_{\mathrm{Y} \colon \mathrm{Y} \geq \mathrm{X}} \mu^{\mathrm{Y} \vee \mathrm{Z}}_{\mathrm{Z}} \Delta^{\mathrm{Y} \vee \mathrm{Z}}_{\mathrm{Y}}(v_{\mathrm{Y}}) \\
&= \sum_{\mathrm{Y} \colon \mathrm{Y} \geq \mathrm{Z}} \mu^{\mathrm{Y}}_{\mathrm{Z}}(v_{\mathrm{Y}}) \\
&= (\exp \circ v)_{\mathrm{Z}}.
\end{aligned}$$

The first and last steps used the definition. The second step used the bimonoid axiom. Since v is a primitive series, $\Delta^{\mathrm{Y} \vee \mathrm{Z}}_{\mathrm{Y}}(v_{\mathrm{Y}})$ will be zero unless $\mathrm{Z} \leq \mathrm{Y}$. This was used in the third step.

Conversely, suppose v is a group-like series of h. We need to show that $\log \circ v$ is a primitive series. The calculation goes as follows. For $\mathrm{X} < \mathrm{Z}$,

$$\begin{aligned}
\Delta^{\mathrm{Z}}_{\mathrm{X}}((\log \circ v)_{\mathrm{X}}) &= \sum_{\mathrm{Y}\colon \mathrm{Y} \geq \mathrm{X}} \log_{\mathrm{Y}/\mathrm{X}} \Delta^{\mathrm{Z}}_{\mathrm{X}} \mu^{\mathrm{Y}}_{\mathrm{X}}(v_{\mathrm{Y}}) \\
&= \sum_{\mathrm{Y}\colon \mathrm{Y} \geq \mathrm{X}} \log_{\mathrm{Y}/\mathrm{X}} \mu^{\mathrm{Y}\vee\mathrm{Z}}_{\mathrm{Z}} \Delta^{\mathrm{Y}\vee\mathrm{Z}}_{\mathrm{Y}}(v_{\mathrm{Y}}) \\
&= \sum_{\mathrm{Y}\colon \mathrm{Y} \geq \mathrm{X}} \log_{\mathrm{Y}/\mathrm{X}} \mu^{\mathrm{Y}\vee\mathrm{Z}}_{\mathrm{Z}}(v_{\mathrm{Y}\vee\mathrm{Z}}) \\
&= \sum_{\mathrm{W}\colon \mathrm{W} \geq \mathrm{Z}} \Bigg(\sum_{\substack{\mathrm{Y}\colon \mathrm{Y} \geq \mathrm{X}, \\ \mathrm{Y}\vee\mathrm{Z} = \mathrm{W}}} \log_{\mathrm{Y}/\mathrm{X}} \Bigg) \mu^{\mathrm{W}}_{\mathrm{X}}(v_{\mathrm{W}}) \\
&= 0.
\end{aligned}$$

The first step used the definition. The second step used the bimonoid axiom. The third step used that v is a group-like series. The last step used the Weisner formula. $\square$

Example 15. Take $\mathsf{h} = \mathsf{E}$, the exponential bimonoid. Recall that a series of E is a choice of scalars v_{X} one for each flat X.

A series v of E is primitive if $v_{\mathrm{X}} = 0$ for all $\mathrm{X} \neq \top$, and group-like if $v_{\mathrm{X}} = v_{\mathrm{Y}}$ for all X and Y.

For a primitive series v, its exponential is given by

$$(\exp \circ v)_{\mathrm{X}} = v_{\top},$$

and for a group-like series v, its logarithm is given by

$$(\log \circ v)_{\mathrm{X}} = \begin{cases} v_{\top} & \text{if } \mathrm{X} = \top, \\ 0 & \text{otherwise.} \end{cases}$$

This gives a direct verification of (29).

12 Series of the associative operad

12.1 Reduced incidence algebra

Consider the associative operad $\mathbf{As}$. Its space of series $\mathscr{S}(\mathbf{As})$ is a (noncommutative) algebra. It can be described as follows.

Proposition 8. *A series of $\mathbf{As}$ is a family of scalars*

$$(s_{F/A})_{A \leq F}$$

such that

$$s_{F/A} = s_{G/B}$$

whenever A and B have the same support, and $F = AG$ and $G = BF$.

The algebra structure on the space of series is given by

$$(s \circ t)_{G/A} = \sum_{F\colon A \leq F \leq G} s_{F/A} t_{G/F}, \quad s, t \in \mathscr{S}(\mathbf{As}).$$

The unit element is the family $(e_{F/A})$ with $e_{A/A} = 1$ and $e_{F/A} = 0$ for $F > A$.

Proof. Suppose s is a series of $\mathbf{As}$. Then for each $\mathrm{X} \leq \mathrm{Y}$, write

$$s_{\mathrm{X},\mathrm{Y}} = \sum_{F \colon F \geq A, \mathrm{s}(F) = \mathrm{Y}} s_{F/A} \mathtt{H}_{F/A} \ \in \ \mathbf{As}[\mathrm{X}, \mathrm{Y}],$$

where A is an arbitrary but fixed face of support X. This yields the family of scalars $(s_{F/A})$ with the stated properties. $\square$

Proposition 9. *The algebra $\mathscr{S}(\mathbf{As})$ is a subalgebra of the incidence algebra of the poset of faces $\Sigma[\mathcal{A}]$. More precisely, it is the reduced incidence algebra of the poset of faces under the equivalence relation*

$$(A, F) \sim (B, G) \iff \\ AB = A,\ BA = B,\ AG = F,\ BF = G.$$

In particular, it has a basis indexed by lunes.

Proof. The first part follows by comparing the product of series with the product in the incidence algebra. For the second part: The condition $s_{F/A} = s_{G/B}$ says that an element of the incidence algebra is a series of $\mathbf{As}$ precisely when it takes the same value on directed faces which are equivalent. $\square$

This is the same as the reduced incidence algebra $R(\Sigma)$ which we have considered earlier.

The map of operads $\mathbf{As} \to \mathbf{Com}$ induces an algebra homomorphism

$$\mathbf{s} : \mathscr{S}(\mathbf{As}) \to \mathscr{S}(\mathbf{Com}).$$

We call this the support mapsupport map. Explicitly, the support of a series t of $\mathbf{As}$ is given by

$$\mathbf{s}(t)_{\mathrm{Y}/\mathrm{X}} = \sum_{F \colon F \geq A, \mathrm{s}(F) = \mathrm{Y}} t_{F/A},$$

where A is a fixed face of support X.

This is the same as the map $R(\Sigma) \to I(\Pi)$ discussed earlier (after using the identifications of Propositions 6 and 9).

12.2 Modules

Let (a, μ) be a monoid in species. Recall that this is the same as a module over $\mathbf{As}$. Hence $\mathscr{S}(\mathsf{a})$ is a module over $\mathscr{S}(\mathbf{As})$. The module structure is given by

$$(s \circ v)_A = \sum_{F\colon A \leq F} s_{F/A} \mu_A^F(v_F), \quad s \in \mathscr{S}(\mathbf{As}),\, v \in \mathscr{S}(\mathsf{a}).$$

Example 16. Recall the monoid of chambers Γ. A series of Γ is a family of scalars $(v_{C/A})_{A \leq C}$ such that

$$v_{C/A} = v_{D/B}$$

whenever A and B have the same support, and $AD = C$ and $BC = D$. We deduce that $\mathscr{S}(\Gamma)$ has a basis indexed by top-lunes. Its module structure over $\mathscr{S}(\mathbf{As})$ is given by

$$(s \circ v)_{C/A} = \sum_{F\colon A \leq F \leq C} s_{F/A} v_{C/F}, \quad s \in \mathscr{S}(\mathbf{As}),\ v \in \mathscr{S}(\Gamma).$$

12.3 Exp and log series

Fix a noncommutative zeta function $\boldsymbol{\zeta}$ and let $\boldsymbol{\mu}$ be the noncommutative Möbius function inverse to it. Let $\mathbf{exp}$ and $\mathbf{log}$ denote the series

$$\mathbf{exp}_{G/F} := \boldsymbol{\zeta}(F,G) \quad \text{and} \quad \mathbf{log}_{G/F} := \boldsymbol{\mu}(F,G).$$

The support of $\mathbf{exp}$ is $\exp$ and the support of $\mathbf{log}$ is $\log$.

The exp-log correspondenceexp-log correspondence for commutative monoids generalizes to arbitrary monoids as follows.

Proposition 10. *For any monoid* a*, the maps*

$$\mathscr{S}(\mathsf{a}) \underset{\mathbf{log}}{\overset{\mathbf{exp}}{\rightleftarrows}} \mathscr{S}(\mathsf{a}) \tag{30}$$

are inverse bijections.

This is because $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ are inverses.

12.4 Group-like and primitive series

Let (c, Δ) be a comonoid in species. A series v of c is group-like if

$$\Delta_A^F(v_A) = v_F$$

for all $F \geq A$. Let $\mathscr{G}(\mathsf{c})$ denote the set of all group-like series of c.

Let (c, Δ) be a comonoid in species. A series v of c is primitive if

$$\Delta_A^F(v_A) = 0$$

for all $F > A$. Let $\mathscr{P}(\mathsf{c})$ denote the set of all primitive series of c. Observe that

$$\mathscr{P}(\mathsf{c}) = \mathscr{S}(\mathcal{P}(\mathsf{c})),$$

where $\mathcal{P}(\mathsf{c})$ is the primitive part of c.

Theorem 2. *If* h *is a bimonoid, the maps* (30) *restrict to inverse bijections*

$$\mathscr{P}(\mathsf{h}) \underset{\mathbf{log}}{\overset{\mathbf{exp}}{\rightleftarrows}} \mathscr{G}(\mathsf{h}). \tag{31}$$

The correspondences are natural in a and h.

Proof. We need to check that $\mathbf{exp}$ and $\mathbf{log}$ map as stated.

Suppose v is a primitive series of h. We need to check that $\mathbf{exp} \circ v$ is a group-like series. The calculation goes as follows. For $A \leq G$,

$$\begin{aligned}
\Delta_A^G((\mathbf{exp} \circ v)_A) &= \sum_{F\colon F \geq A} \mathbf{exp}_{F/A} \Delta_A^G \mu_A^F(v_F) \\
&= \sum_{F\colon F \geq A} \mathbf{exp}_{F/A} \mu_G^{GF} \beta_{GF,FG} \Delta_F^{FG}(v_F) \\
&= \sum_{F\colon F \geq A, FG=F} \mathbf{exp}_{F/A} \mu_G^{GF} \beta_{GF,F}(v_F) \\
&= \sum_{F\colon F \geq A, FG=F} \mathbf{exp}_{F/A} \mu_G^{GF}(v_{GF}) \\
&= \sum_{H\colon H \geq G} \Big(\sum_{\substack{F\colon F \geq A, GF=H \\ FG=F}} \mathbf{exp}_{F/A} \Big) \mu_G^H(v_H) \\
&= \sum_{H\colon H \geq G} \mathbf{exp}_{H/G} \mu_G^H(v_H) \\
&= (\mathbf{exp} \circ v)_G.
\end{aligned}$$

The first and last steps used the definition. The second step used the bimonoid axiom. Since v is a primitive series, $\Delta_F^{FG}(v_F)$ will be zero unless $FG = F$. This was used in the third step. In the fifth step, we introduced a new variable H for GF. The sixth step used the star-lune formula.

Suppose v is a group-like series of h. We need to check that $\mathbf{log} \circ v$ is a primitive series. The calculation goes as follows. For $A < G$,

$$\begin{aligned}
\Delta_A^G((\mathbf{log} \circ v)_A) &= \sum_{F\colon F \geq A} \mathbf{log}_{F/A} \Delta_A^G \mu_A^F(v_F) \\
&= \sum_{F\colon F \geq A} \mathbf{log}_{F/A} \mu_G^{GF} \beta_{GF,FG} \Delta_F^{FG}(v_F) \\
&= \sum_{F\colon F \geq A} \mathbf{log}_{F/A} \mu_G^{GF} \beta_{GF,FG}(v_{FG}) \\
&= \sum_{F\colon F \geq A} \mathbf{log}_{F/A} \mu_G^{GF}(v_{GF}) \\
&= \sum_{H\colon H \geq G} \Big(\sum_{F\colon F \geq A, GF = H} \mathbf{log}_{F/A} \Big) \, \mu_G^H(v_H) \\
&= 0.
\end{aligned}$$

The first step used the definition. The second step used the bimonoid axiom. The third step used that v is a group-like series. In the fifth step, we introduced a new variable H for GF. The last step used the noncommutative Weisner formula. $\square$