Modules over monoid algebras and bimonoids in species

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1 Characteristic operations

Recall that the definition of a bimonoid makes use of the Tits monoid.

On the other hand, there is a bimonoid, namely, Σ, which is itself built out of faces.

This double occurrence of faces acquires formal meaning now.

We show that elements of Σ give rise to characteristic operations on bimonoids.

1.1 Characteristic operations

Let h be a bimonoid.

Given $z \in \Sigma[A]$ and $h \in \mathsf{h}[A]$, define an element $z \cdot h \in \mathsf{h}[A]$ as follows.

First, write

$$z = \sum_{F\colon F \geq A} a^{F/A} \mathtt{H}_{F/A}$$

for scalars $a^{F/A}$.

Then set

$$z \cdot h := \sum_{F\colon F \geq A} a^{F/A} \mu_A^F \Delta_A^F(h). \tag{1}$$

In particular,

$$\mathtt{H}_{F/A} \cdot h := \mu_A^F \Delta_A^F(h).$$

We refer to these as characteristic operations.

Recall that for any face A, the component $\Sigma[A]$ is an algebra, which can be identified with the Tits algebra of the arrangement $\mathcal{A}_A$.

Lemma 1. *The following properties hold for any bimonoid* h*.*

- *For any* $h \in \mathsf{h}[A]$*,*

$$\mathtt{H}_{A/A} \cdot h = h. \tag{2}$$

- *If* h *is cocommutative, then for any* $z, w \in \Sigma[A]$ *and* $h \in \mathsf{h}[A]$*,*

$$(z \cdot w) \cdot h = z \cdot (w \cdot h). \tag{3}$$

- *If* h *is commutative, then for any* $z, w \in \Sigma[A]$ *and* $h \in \mathsf{h}[A]$*,*

$$(z \cdot w) \cdot h = w \cdot (z \cdot h). \tag{4}$$

In other words, when h is cocommutative, (1) defines a left action of the Tits algebra $\Sigma[A]$ on the space $\mathsf{h}[A]$.

When h is commutative, there is a right action of $\Sigma[A]$ on $\mathsf{h}[A]$ given by $h \cdot z := z \cdot h$.

Proof. Statement (2) follows from (co)unitality. The remaining statements are linear in z and w, so we assume them to be basis elements.

For (3), take $z = \mathtt{H}_{G/A}$ and $w = \mathtt{H}_{F/A}$. Then

$$\begin{aligned} z \cdot (w \cdot h) &= \mu_A^G \Delta_A^G \mu_A^F \Delta_A^F(h) \\ &= \mu_A^G \mu_G^{GF} \beta_{GF,FG} \Delta_F^{FG} \Delta_A^F(h) \\ &= \mu_A^{GF} \beta_{GF,FG} \Delta_A^{FG}(h) \\ &= \mu_A^{GF} \Delta_A^{GF}(h) \\ &= (z \cdot w) \cdot h. \end{aligned}$$

We used the bimonoid axiom, then (co)associativity, and finally cocommutativity.

The calculation for (4) proceeds similarly, except at the end, where β merges with μ instead of Δ. $\square$

Example. Recall that Σ is a cocommutative bimonoid.

Thus we may take $\mathrm{h} = \Sigma$, resulting in a left action of the Tits algebra $\Sigma[A]$ on itself.

This coincides with the usual action.

Indeed, for faces F and G greater than A,

$$\mathtt{H}_{F/A} \cdot \mathtt{H}_{G/A} = \mu_A^F \Delta_A^F(\mathtt{H}_{G/A}) = \mu_A^F(\mathtt{H}_{FG/F}) = \mathtt{H}_{FG/A}.$$

Now take $\mathsf{h} := \Gamma$.

This is also a cocommutative bimonoid.

One may check that the action of $\mathtt{H}_{F/A}$ sends $\mathtt{H}_{C/A}$ to $\mathtt{H}_{FC/A}$.

This is the usual left action of the Tits algebra on the module of chambers.

1.2 Review of bimonoid properties

Recall the following properties of a bimonoid h.

For any faces $A \leq F$ and $A \leq G$,

$$\Delta_A^F \mu_A^F = \mathrm{id}, \tag{5}$$

$$\Delta_A^F \mu_A^G \Delta_A^G \mu_A^F = \mu_F^{FG} \Delta_F^{FG}, \tag{6}$$

and, if h is commutative, then

$$\mu_A^G \Delta_A^G \mu_A^F = \mu_A^{FG} \Delta_F^{FG}, \tag{7}$$

and, if h is cocommutative, then

$$\Delta_A^F \mu_A^G \Delta_A^G = \mu_F^{FG} \Delta_A^{FG}. \tag{8}$$

If $A \leq F \leq G$, then

$$\Delta_A^G \mu_A^F = \Delta_F^G \quad \text{and} \quad \Delta_A^F \mu_A^G = \mu_F^G. \tag{9}$$

1.3 Interaction with the bimonoid structure

We use the above properties to study how characteristic operations interact with the product and coproduct of Σ and h.

More precisely, we fix a pair of faces $F \geq A$.

We take an element z in either $\Sigma[F]$ or $\Sigma[A]$, and an element h in either $\mathsf{h}[F]$ or $\mathsf{h}[A]$.

There are two ways in which z and h can interact.

For instance, if $z \in \Sigma[F]$ and $h \in \mathsf{h}[A]$, then we can consider $\mu_A^F(z) \cdot h$ and $z \cdot \Delta_A^F(h)$.

The results below explain the relations between these two interactions.

Proposition 1. *Let* h *be a bimonoid. Let* $A \leq F$*. Then:*

(i) For any $z \in \Sigma[F]$ *and* $h \in \mathsf{h}[A]$,

$$\mu_A^F(z) \cdot h = \mu_A^F\Big(z \cdot \Delta_A^F(h)\Big). \tag{10}$$

(ii) For any $z \in \Sigma[A]$ *and* $h \in \mathsf{h}[F]$,

$$\Delta_A^F(z) \cdot h = \Delta_A^F\Big(z \cdot \mu_A^F(h)\Big), \tag{11}$$

and if h *is commutative, then*

$$z \cdot \mu_A^F(h) = \mu_A^F\Big(\Delta_A^F(z) \cdot h\Big). \tag{12}$$

(iii) If h *is cocommutative, then for any* $z \in \Sigma[A]$ *and* $h \in \mathsf{h}[A]$,

$$\Delta_A^F(z \cdot h) = \Delta_A^F(z) \cdot \Delta_A^F(h). \tag{13}$$

Proof. All statements are linear in z, so we may assume z is a basis element in each case.

(i) We assume $z = \mathtt{H}_{G/F}$. We have

$$\begin{aligned} \mu_A^F\big(z \cdot \Delta_A^F(h)\big) &= (\mu_A^F \mu_F^G \Delta_F^G \Delta_A^F)(h) \\ &= (\mu_A^G \Delta_A^G)(h) \\ &= \mathtt{H}_{G/A} \cdot h \\ &= \mu_A^F(z) \cdot h. \end{aligned}$$

We used (co)associativity.

(ii) We assume $z = \mathtt{H}_{G/A}$. We have

$$\begin{aligned} \Delta_A^F\big(z \cdot \mu_A^F(h)\big) &= (\Delta_A^F \mu_A^G \Delta_A^G \mu_A^F)(h) \\ &= (\mu_F^{FG} \Delta_F^{FG})(h) \\ &= \mathtt{H}_{FG/F} \cdot h \\ &= \Delta_A^F(z) \cdot h. \end{aligned}$$

We used (6). In addition, if h is commutative,

$$\begin{aligned}
\mu_A^F\Big(\Delta_A^F(z)\cdot h\Big) &= \mu_A^F\Big(\mathtt{H}_{FG/F}\cdot h\Big)\\
&= (\mu_A^F\mu_F^{FG}\Delta_F^{FG})(h)\\
&= (\mu_A^{FG}\Delta_F^{FG})(h)\\
&= (\mu_A^G\Delta_A^G\mu_A^F)(h)\\
&= \mathtt{H}_{G/A}\cdot\mu_A^F(h)\\
&= z\cdot\mu_A^F(h).
\end{aligned}$$

We used (7) and associativity.

(iii) We again assume $z=\mathtt{H}_{G/A}$. We have

$$\begin{aligned}
\Delta_A^F(z\cdot h) &= (\Delta_A^F\mu_A^G\Delta_A^G)(h)\\
&= (\mu_F^{FG}\Delta_A^{FG})(h)\\
&= (\mu_F^{FG}\Delta_F^{FG}\Delta_A^F)(h)\\
&= \mathtt{H}_{FG/F}\cdot\Delta_A^F(h)\\
&= \Delta_A^F(z)\cdot\Delta_A^F(h).
\end{aligned}$$

We used (8) and coassociativity.

$\square$

The following properties complement (10)–(13) (and follow from the first of these).

Corollary 1. *Let* h *be a bimonoid. Let* $A \leq F$*. Then:*

(i) For any $z \in \Sigma[F]$ *and* $h \in \mathsf{h}[A]$*,*

$$z \cdot \Delta_A^F(h) = \Delta_A^F\big(\mu_A^F(z) \cdot h\big). \tag{14}$$

(ii) For any $z \in \Sigma[F]$ *and* $h \in \mathsf{h}[F]$*,*

$$\mu_A^F(z \cdot h) = \mu_A^F(z) \cdot \mu_A^F(h). \tag{15}$$

Proof. Equation (14) follows by applying Δ_A^F to both sides of (10), in view of (5). Equation (15) follows by replacing h in (10) for $\mu_A^F(h)$ and employing (5). $\square$

2 Commutative characteristic operations

Recall that bicommutative bimonoids can be formulated using the Birkhoff monoid.

On the other hand, there is a bimonoid, namely, Π, which is itself built out of flats.

Formally, elements of Π give rise to operations on bicommutative bimonoids.

This is the commutative analogue of the characteristic operations introduced in Section 1.

2.1 Commutative characteristic operations

Let h be a bicommutative bimonoid.

Given $z \in \Pi[\mathrm{Z}]$ and $h \in \mathsf{h}[\mathrm{Z}]$, define an element $z \cdot h \in \mathsf{h}[\mathrm{Z}]$ as follows.

First, write

$$z = \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} a^{\mathrm{X}/\mathrm{Z}} \mathtt{H}_{\mathrm{X}/\mathrm{Z}}$$

for scalars $a^{\mathrm{X}/\mathrm{Z}}$.

Then set

$$z \cdot h := \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} a^{\mathrm{X}/\mathrm{Z}} \mu_{\mathrm{Z}}^{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}(h). \tag{16}$$

In particular,

$$\mathtt{H}_{\mathrm{X}/\mathrm{Z}} \cdot h := \mu_{\mathrm{Z}}^{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}(h).$$

We refer to these as commutative characteristic operations.

Recall that for any flat Z, the component $\Pi[\mathrm{Z}]$ is an algebra, which can be identified with the Birkhoff algebra of the arrangement $\mathcal{A}_{\mathrm{Z}}$.

Lemma 2. *For any $z, w \in \Pi[\mathrm{Z}]$ and $h \in \mathrm{h}[\mathrm{Z}]$,*

(17) $(z \cdot w) \cdot h = z \cdot (w \cdot h)$ *and* $\mathtt{H}_{\mathrm{Z}/\mathrm{Z}} \cdot h = h.$

In other words, (16) defines an action of the Birkhoff algebra $\Pi[\mathrm{Z}]$ on the space $\mathrm{h}[\mathrm{Z}]$.

Since $\Pi[\mathrm{Z}]$ is a commutative algebra, there is no distinction between left and right actions.

Proof. The second statement follows from (co)unitality. It suffices to check the first statement on basis elements. Take $z = \mathtt{H}_{\mathrm{Y}/\mathrm{Z}}$ and $w = \mathtt{H}_{\mathrm{X}/\mathrm{Z}}$. Then

$$\begin{aligned} z \cdot (w \cdot h) &= \mu_{\mathrm{Z}}^{\mathrm{Y}} \Delta_{\mathrm{Z}}^{\mathrm{Y}} \mu_{\mathrm{Z}}^{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}(h) \\ &= \mu_{\mathrm{Z}}^{\mathrm{Y}} \mu_{\mathrm{Y}}^{\mathrm{Y}\vee\mathrm{X}} \Delta_{\mathrm{X}}^{\mathrm{X}\vee\mathrm{Y}} \Delta_{\mathrm{Z}}^{\mathrm{X}}(h) \\ &= \mu_{\mathrm{Z}}^{\mathrm{Y}\vee\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{Y}\vee\mathrm{X}}(h) \\ &= (z \cdot w) \cdot h. \end{aligned}$$

We used the bicommutative bimonoid axiom followed by (co)associativity. $\square$

Example. Recall that Π is a bicommutative bimonoid. Thus, we may take $\mathrm{h} := \Pi$, resulting in an action of $\Pi[\mathrm{Z}]$ on itself. This coincides with the usual action. Indeed, for flats X and Y greater than Z,

$$\mathtt{H}_{\mathrm{X}/\mathrm{Z}} \cdot \mathtt{H}_{\mathrm{Y}/\mathrm{Z}} = \mu_{\mathrm{Z}}^{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}(\mathtt{H}_{\mathrm{Y}/\mathrm{Z}}) = \mu_{\mathrm{Z}}^{\mathrm{X}}(\mathtt{H}_{\mathrm{X}\vee\mathrm{Y}/\mathrm{X}}) = \mathtt{H}_{\mathrm{X}\vee\mathrm{Y}/\mathrm{Z}}.$$

For $\mathrm{h} := \mathrm{E}$, the action of any flat is by the identity map.

2.2 Interaction with the bimonoid structure

Proposition 2. *Let* h *be a bicommutative bimonoid. Let* $\mathrm{Z} \leq \mathrm{X}$*. Then:*

- *For any* $z \in \Pi[\mathrm{X}]$ *and* $h \in \mathsf{h}[\mathrm{Z}]$*,*

(18) $$\mu_{\mathrm{Z}}^{\mathrm{X}}(z) \cdot h = \mu_{\mathrm{Z}}^{\mathrm{X}}\Big(z \cdot \Delta_{\mathrm{Z}}^{\mathrm{X}}(h)\Big),$$

  and

(19) $$z \cdot \Delta_{\mathrm{Z}}^{\mathrm{X}}(h) = \Delta_{\mathrm{Z}}^{\mathrm{X}}(\mu_{\mathrm{Z}}^{\mathrm{X}}(z) \cdot h).$$

- *For any* $z \in \Pi[\mathrm{Z}]$ *and* $h \in \mathsf{h}[\mathrm{X}]$*,*

(20) $$\Delta_{\mathrm{Z}}^{\mathrm{X}}(z) \cdot h = \Delta_{\mathrm{Z}}^{\mathrm{X}}\Big(z \cdot \mu_{\mathrm{Z}}^{\mathrm{X}}(h)\Big),$$

  and

(21) $$z \cdot \mu_{\mathrm{Z}}^{\mathrm{X}}(h) = \mu_{\mathrm{Z}}^{\mathrm{X}}\Big(\Delta_{\mathrm{Z}}^{\mathrm{X}}(z) \cdot h\Big).$$

- *For any* $z \in \Pi[\mathrm{Z}]$ *and* $h \in \mathsf{h}[\mathrm{Z}]$*,*

(22) $$\Delta_{\mathrm{Z}}^{\mathrm{X}}(z \cdot h) = \Delta_{\mathrm{Z}}^{\mathrm{X}}(z) \cdot \Delta_{\mathrm{Z}}^{\mathrm{X}}(h).$$

- *For any* $z \in \mathsf{\Pi}[\mathrm{X}]$ *and* $h \in \mathsf{h}[\mathrm{X}]$,

$$\mu_{\mathrm{Z}}^{\mathrm{X}}(z \cdot h) = \mu_{\mathrm{Z}}^{\mathrm{X}}(z) \cdot \mu_{\mathrm{Z}}^{\mathrm{X}}(h). \tag{23}$$

Proof. We essentially repeat the arguments in Proposition 1 and Corollary 1, with faces replaced by flats. □

3 Modules over algebras and bimonoids

3.1 Idempotent operators

Recall that an idempotent operator on a vector space V is a linear map $e : V \to V$ such that $e^2 = e$. In this situation, we let $e(V)$ denote the image of e.

Lemma 3. *Let V and W be vector spaces, and $p : V \to W$ and $i : W \to V$ linear maps such that $pi = \mathrm{id}_W$. Let $e = ip : V \to V$. Then e is idempotent and there is an isomorphism $W \cong e(V)$ for which the following diagrams commute.*

(24)
$$
\begin{tikzcd}
 & V \arrow[dl, "p"'] \arrow[dr, "e"] & \\
W \arrow[rr, "\cong"'] & & e(V)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & V & \\
W \arrow[ur, "i"] \arrow[rr, "\cong"'] & & e(V) \arrow[ul, hook']
\end{tikzcd}
$$

Proof. The maps $ei : W \to e(V)$ and $p|_{e(V)} : e(V) \to W$ are inverse. $\square$

3.2 Modules over the Tits algebra

Recall that the linearization of the Tits monoid is the Tits algebra. It is denoted by $\Sigma[\mathcal{A}]$. Note that

$$\Sigma[\mathcal{A}] = \Sigma[O], \qquad \mathtt{H}_F \leftrightarrow \mathtt{H}_{F/O},$$

where the latter refers to the O-component of the bimonoid Σ.

Proposition 3. *The category of left modules over $\Sigma[\mathcal{A}]$ is equivalent to the category of cocommutative $\mathcal{A}$-bimonoids.*

Proof. We first construct a functor from cocommutative $\mathcal{A}$-bimonoids to left $\Sigma[\mathcal{A}]$-modules.

Accordingly, suppose h is a cocommutative $\mathcal{A}$-bimonoid. Then $\mathrm{h}[O]$ is a left $\Sigma[\mathcal{A}]$-module, with the action of $\mathtt{H}_F$ on an element x given by

$$\mathtt{H}_F \cdot x := \mu_O^F \Delta_O^F(x).$$

Since h is cocommutative, this defines an action as noted in (2) and (3).

Further, if h and k are cocommutative $\mathcal{A}$-bimonoids and $f : \mathsf{h} \to \mathsf{k}$ is a morphism of $\mathcal{A}$-bimonoids, then the component $f_O : \mathsf{h}[O] \to \mathsf{k}[O]$ is a map of left $\Sigma[\mathcal{A}]$-modules as shown below.

$$\begin{tikzcd}
\mathsf{h}[O] \arrow[r, "\Delta_O^F"] \arrow[d, "f_O"'] & \mathsf{h}[F] \arrow[r, "\mu_O^F"] \arrow[d, "f_F"] & \mathsf{h}[O] \arrow[d, "f_O"] \\
\mathsf{k}[O] \arrow[r, "\Delta_O^F"'] & \mathsf{k}[F] \arrow[r, "\mu_O^F"'] & \mathsf{k}[O]
\end{tikzcd}$$

The squares commute since f is a morphism of comonoids and monoids.

Now we construct a functor from left $\Sigma[\mathcal{A}]$-modules to cocommutative $\mathcal{A}$-bimonoids.

Accordingly, suppose M is a left $\Sigma[\mathcal{A}]$-module. Then put

$$\mathsf{h}[F] := \mathtt{H}_F \cdot M.$$

This is the subspace of M onto which M projects by the action of the idempotent $\mathtt{H}_F$.

Note that $\mathsf{h}[O] = M$.

Whenever F and G have the same support, there is an isomorphism

$$\beta_{G,F} : \mathsf{h}[F] \to \mathsf{h}[G]$$

induced by the action of $\mathtt{H}_G$ (with the inverse induced by the action of $\mathtt{H}_F$). These turn h into an $\mathcal{A}$-species.

Now let $A \leq F$. Then $AF = F$ and hence

$$\mathtt{H}_A \cdot (\mathtt{H}_F \cdot x) = (\mathtt{H}_A \cdot \mathtt{H}_F) \cdot x = \mathtt{H}_{AF} \cdot x = \mathtt{H}_F \cdot x,$$

so $\mathsf{h}[F]$ is a subspace of $\mathsf{h}[A]$.

Define μ_A^F to be the inclusion map, and Δ_A^F to be the projection induced by the action of $\mathtt{H}_F$. This turns h into an $\mathcal{A}$-monoid and an $\mathcal{A}$-comonoid.

The coproduct is cocommutative. The cocommutativity axiom is checked below.

$$\mathtt{H}_G \cdot (\mathtt{H}_F \cdot x) = (\mathtt{H}_G \cdot \mathtt{H}_F) \cdot x = \mathtt{H}_{GF} \cdot x = \mathtt{H}_G \cdot x.$$

For the bimonoid axiom, we start with the element $\mathtt{H}_F \cdot x$, and the check reduces to

$$\mathtt{H}_G \cdot (\mathtt{H}_F \cdot x) = \mathtt{H}_{GF} \cdot \mathtt{H}_{FG} \cdot (\mathtt{H}_F \cdot x).$$

Thus, $(\mathsf{h}, \mu, \Delta)$ is indeed an $\mathcal{A}$-bimonoid.

Further, if M and N are left $\Sigma[\mathcal{A}]$-modules with h and k as the corresponding cocommutative $\mathcal{A}$-bimonoids, and $f : M \to N$ is a morphism of modules, then f restricts to linear maps

$$f_F : \mathsf{h}[F] \to \mathsf{k}[F],$$

one for each face F, and this family of maps constitutes a morphism $f : \mathsf{h} \to \mathsf{k}$ of $\mathcal{A}$-bimonoids.

Finally, we check that the functors we have constructed between modules and bimonoids define an equivalence.

If we start from a module M, construct the bimonoid h, and then the corresponding module, we return to $\mathtt{H}_O \cdot M = M$.

In the other direction, starting from a bimonoid h going to modules and back yields the bimonoid $\tilde{\mathsf{h}}$ with components

$$\tilde{\mathsf{h}}[F] = \mu_O^F \Delta_O^F(\mathsf{h}[O]).$$

Recall that $\Delta_O^F \mu_O^F = \mathrm{id}_{\mathsf{h}[F]}$. Applying Lemma 3 to this splitting, we obtain a linear isomorphism $\mathsf{h}[F] \cong \tilde{\mathsf{h}}[F]$, for each face F. These constitute a natural isomorphism of $\mathcal{A}$-bimonoids $\mathsf{h} \cong \tilde{\mathsf{h}}$, in view of (24). $\square$

Proposition 4. *The category of right modules over $\Sigma[\mathcal{A}]$ is equivalent to the category of commutative $\mathcal{A}$-bimonoids.*

Proof. The argument is similar to the one for Proposition 3, so we only briefly indicate how the functors work.

If h is a commutative $\mathcal{A}$-bimonoid, then $\mathsf{h}[O]$ is a right $\Sigma[\mathcal{A}]$-module with action given by

$$x \cdot \mathtt{H}_F := \mu_O^F \Delta_O^F(x).$$

Conversely, if M is a right $\Sigma[\mathcal{A}]$-module, then we set

$$\mathsf{h}[F] := M \cdot \mathtt{H}_F.$$

The product and coproduct are given by inclusion and projection induced by the right action. Observe that $M \cdot \mathtt{H}_F$ and $M \cdot \mathtt{H}_G$ coincide whenever $\mathrm{s}(F) = \mathrm{s}(G)$, and $\beta_{G,F}$ is defined to be identity. So commutativity holds. The bimonoid axiom reduces to

$$(x \cdot \mathtt{H}_F) \cdot \mathtt{H}_G = (x \cdot \mathtt{H}_F) \cdot \mathtt{H}_{FG}.$$

Note very carefully that cocommutativity requires $x \cdot \mathtt{H}_F = x \cdot \mathtt{H}_G$ whenever $\mathrm{s}(F) = \mathrm{s}(G)$, hence it does not hold in general. $\square$

3.3 Modules over the Birkhoff algebra

Recall that the linearization of the Birkhoff monoid is the Birkhoff algebra. It is denoted by $\Pi[\mathcal{A}]$. Since $\Pi[\mathcal{A}]$ is commutative, there is no distinction between its left and right modules.

Proposition 5. *The category of modules over* $\Pi[\mathcal{A}]$ *is equivalent to the category of bicommutative* $\mathcal{A}$*-bimonoids.*

Proof. The argument is similar to the one for Proposition 3, so we will be brief.

We make use of the commutative characteristic operation (16).

Suppose h is a bicommutative $\mathcal{A}$-bimonoid. Then h$[\perp]$ is a $\Pi[\mathcal{A}]$-module, with the action of $\mathtt{H}_{\mathrm{X}}$ on an element x given by

$$\mathtt{H}_{\mathrm{X}} \cdot x := \mu_{\perp}^{\mathrm{X}} \Delta_{\perp}^{\mathrm{X}}(x).$$

This defines an action as noted in (17).

Further, if h and k are bicommutative $\mathcal{A}$-bimonoids and $f : \mathsf{h} \to \mathsf{k}$ is a morphism of $\mathcal{A}$-bimonoids, then the component $f_\perp : \mathsf{h}[\perp] \to \mathsf{k}[\perp]$ is a map of $\Pi[\mathcal{A}]$-modules as shown below.

$$\begin{tikzcd}
\mathsf{h}[\perp] \arrow[r, "\Delta_\perp^{\mathrm{X}}"] \arrow[d, "f_\perp"'] & \mathsf{h}[\mathrm{X}] \arrow[r, "\mu_\perp^{\mathrm{X}}"] \arrow[d, "f_{\mathrm{X}}"] & \mathsf{h}[\perp] \arrow[d, "f_\perp"] \\
\mathsf{k}[\perp] \arrow[r, "\Delta_\perp^{\mathrm{X}}"'] & \mathsf{k}[\mathrm{X}] \arrow[r, "\mu_\perp^{\mathrm{X}}"'] & \mathsf{k}[\perp]
\end{tikzcd}$$

The squares commute since f is a morphism of comonoids and monoids.

Conversely: Suppose M is a $\Pi[\mathcal{A}]$-module. This defines an $\mathcal{A}$-species h whose X-component is given by

$$\mathsf{h}[\mathrm{X}] := \mathtt{H}_{\mathrm{X}} \cdot M.$$

For $\mathrm{Z} \leq \mathrm{X}$, we note that $\mathsf{h}[\mathrm{X}]$ is a subspace of $\mathsf{h}[\mathrm{Z}]$. Define $\mu_{\mathrm{Z}}^{\mathrm{X}}$ to be the inclusion map, and $\Delta_{\mathrm{Z}}^{\mathrm{X}}$ to be the projection induced by the action of $\mathtt{H}_{\mathrm{X}}$. The bicommutative bimonoid axiom holds, and h is indeed a bicommutative $\mathcal{A}$-bimonoid.

Further, if M and N are $\Pi[\mathcal{A}]$-modules with h and k as the corresponding bicommutative $\mathcal{A}$-bimonoids, and $f : M \to N$ is a morphism of modules, then f restricts to linear maps

$$f_{\mathrm{X}} : \mathsf{h}[\mathrm{X}] \to \mathsf{k}[\mathrm{X}],$$

one for each flat X, and this family of maps constitutes a morphism $f : \mathsf{h} \to \mathsf{k}$ of $\mathcal{A}$-bimonoids. $\square$

3.4 Summary

For any algebra A, let A-Mod denote the category of left A-modules. The category of right A-modules is isomorphic to the category of left A^{op}-modules, where A^{op} denote the algebra opposite to A.

A summary of the categorical equivalences obtained in the preceding discussion is given in Table 1.

Table 1:

Modules over algebras		Bimonoids in specie	
$\Sigma[\mathcal{A}]$-Mod	left $\Sigma[\mathcal{A}]$-modules	${}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$	cocom. b
$\Sigma[\mathcal{A}]^{\mathrm{op}}$-$\mathsf{Mod}$	right $\Sigma[\mathcal{A}]$-modules	$\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})$	com. bi
$\Pi[\mathcal{A}]$-Mod	$\Pi[\mathcal{A}]$-modules	${}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})$	bicom. bi

Illustrations of these equivalences on particular modules and bimonoids are given below.

Module	Bimonoid
trivial module over $\Pi[\mathcal{A}]$	Exponential bimonoid E
left module $\Gamma[\mathcal{A}]$ over $\Sigma[\mathcal{A}]$	Bimonoid of chambers Γ
$\Pi[\mathcal{A}]$ as a module over itself	Bimonoid of flats Π
$\Sigma[\mathcal{A}]$ as a left module over itself	Bimonoid of faces Σ

By using charactertistic operations, the bimonoids listed above yield the corresponding modules listed above. These facts are contained in Examples 1.1 and 2.1.

3.5 Janus monoid

A bi-face is a pair (F, F') of faces such that F and F' have the same support. Let $\mathrm{J}[\mathcal{A}]$ denote the set of bi-faces. The operation

$$(F, F')(G, G') := (FG, G'F')$$

turns $\mathrm{J}[\mathcal{A}]$ into a monoid. The unit element is (O, O). We call it the Janus monoid.

There is a commutative diagram of monoids

(25)
$$\begin{tikzcd}
\mathrm{J}[\mathcal{A}] \arrow[r, two heads] \arrow[d, two heads] & \Sigma[\mathcal{A}]^{\mathrm{op}} \arrow[d, two heads, "\mathrm{s}"] \\
\Sigma[\mathcal{A}] \arrow[r, two heads, "\mathrm{s}"'] & \Pi[\mathcal{A}]
\end{tikzcd}$$

with s being the support map, and the maps from J being the projections on the two coordinates, respectively.

3.6 Janus algebra

The linearization of the Janus monoid yields an algebra. We call this the Janus algebra, and denote it by $\mathsf{J}[\mathcal{A}]$. Using $\mathtt{H}$ for the canonical basis, we write

$$\mathtt{H}_{(F,F')} \cdot \mathtt{H}_{(G,G')} = \mathtt{H}_{(FG,G'F')}. \tag{26}$$

Linearizing diagram (25) yields the following commutative diagram of algebras.

$$\begin{tikzcd} \mathsf{J}[\mathcal{A}] \arrow[r, two heads] \arrow[d, two heads] & \Sigma[\mathcal{A}]^{\mathrm{op}} \arrow[d, two heads, "\mathrm{s}"] \\ \Sigma[\mathcal{A}] \arrow[r, two heads, "\mathrm{s}"'] & \Pi[\mathcal{A}]. \end{tikzcd} \tag{27}$$

For any face A, let $\mathsf{J}^o[A]$ denote the vector space linearly spanned by bi-faces (F, G) such that both F and G are greater than A.

It is an algebra with product given by

(28)

$$\mathtt{H}_{(F/A,F'/A)} \cdot \mathtt{H}_{(G/A,G'/A)} := \mathtt{H}_{(FG/A,G'F'/A)}.$$

The unit element is $\mathtt{H}_{(A/A,A/A)}$.

This can be identified with the Janus algebra of the arrangement $\mathcal{A}_A$.

3.7 Two-sided characteristic operations

Let h be a bimonoid.

Given $z \in \mathsf{J}^{o}[A]$ and $h \in \mathsf{h}[A]$, define an element $z \cdot h \in \mathsf{h}[A]$ as follows.

First, write

$$z = \sum_{\substack{(F,F')\colon F,F' \geq A \\ \mathrm{s}(F)=\mathrm{s}(F')}} a^{F/A,F'/A}\, \mathtt{H}_{(F/A,F'/A)}$$

for scalars $a^{F/A,F'/A}$.

Then set

$$z \cdot h := \sum_{\substack{(F,F')\colon F,F' \geq A \\ \mathrm{s}(F)=\mathrm{s}(F')}} a^{F/A,F'/A}\, \mu_A^F \beta_{F,F'} \Delta_A^{F'}(h). \tag{29}$$

In particular,

$$\mathtt{H}_{(F/A,F'/A)} \cdot h := \mu_A^F \beta_{F,F'} \Delta_A^{F'}(h).$$

We refer to these as two-sided characteristic operations.

Lemma 4. *The following holds for any bimonoid* h. *For any* $z, w \in \mathsf{J}^o[A]$ *and* $h \in \mathsf{h}[A]$,

(30)

$$(z \cdot w) \cdot h = z \cdot (w \cdot h) \quad \textit{and} \quad \mathtt{H}_{(A/A,A/A)} \cdot h = h.$$

In other words, (29) defines a left action of the Janus algebra $\mathsf{J}^o[A]$ on the space $\mathsf{h}[A]$.

Proof. The second statement follows from (co)unitality. It suffices to check the first statement on basis elements. Take $z = \mathtt{H}_{(G/A,G'/A)}$ and $w = \mathtt{H}_{(F/A,F'/A)}$. We calculate:

$$\begin{aligned} z \cdot (w \cdot h) &= \mu_A^G \beta_{G,G'} \Delta_A^{G'} \mu_A^F \beta_{F,F'} \Delta_A^{F'}(h) \\ &= \mu_A^G \beta_{G,G'} \mu_{G'}^{G'F} \beta_{G'F,FG'} \Delta_F^{FG'} \beta_{F,F'} \Delta_A^{F'}(h) \\ &= \mu_A^G \mu_G^{GF} \beta_{GF,G'F} \beta_{G'F,FG'} \beta_{FG',F'G'} \Delta_{F'}^{F'G'} \Delta_A^{F'}(h \\ &= \mu_A^{GF} \beta_{GF,F'G'} \Delta_A^{F'G'}(h) \\ &= (z \cdot w) \cdot h. \end{aligned}$$

We used the bimonoid axiom and then naturality and (co)associativity.

$\square$

3.8 Modules over the Janus algebra

Proposition 6. *The category of (left) modules over $\mathsf{J}[\mathcal{A}]$ is equivalent to the category of $\mathcal{A}$-bimonoids.*