Problems

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1 Exam problems. Set I

Ritesh, Amit, Keshav, Neel, Vatsal, Yash, Sahas, Akash

1. Let $\Bbbk$ be a field. Describe the coproduct in the category of commutative $\Bbbk$-algebras. The coproduct of A and B is the tensor product $A \otimes_{\Bbbk} B$.

   Some students wrote $A \times B$. But then the universal map is bilinear as opposed to linear.

2. A directed graph G consists of a set of vertices V, a set of edges E, and two maps $s, t : E \to V$. It is reflexive if there is given in addition a map $i : V \to E$ such that $si = \mathrm{id}_V = ti$.

$$\begin{tikzcd} s(e) \arrow[r, "e"] & t(e) & & v \arrow[loop above, "i(v)"] \end{tikzcd}$$

   A morphism $f : G \to G'$ of reflexive directed graphs consists of a pair of maps $f_0 : V \to V'$

and $f_1 : E \to E'$ such that $sf_1(e) = f_0s(e)$, $tf_1(e) = f_0t(e)$, and $if_0(v) = f_1i(v)$ for all $e \in E$, $v \in V$. This defines the category of reflexive directed graphs.

Let S denote the Tits monoid of a rank-one arrangement. Show that the category of right S-modules is equivalent to the category of reflexive directed graphs.

The graph corresponding to a right S-set X has

$$V := \{x \in X \mid x{\cdot}C = x\} = \{x \in X \mid x{\cdot}\overline{C} = x\}$$

and

$$E := X,$$

s is the right action of C, t the right action of $\overline{C}$, and i is the inclusion of V into X. This leads to a functor in one direction. In the other direction, the right S-set corresponding to a reflexive graph G has $X := E$, with the right action of C given by is and of $\overline{C}$ given by it.

3. Let X be a set equipped with two monoid structures $\cdot$ and $\star$ with identity elements 1 and e, respectively, which satisfy the compatibility axiom

$$(x \cdot y) \star (z \cdot t) = (x \star z) \cdot (y \star t)$$

for all $x, y, z, t \in X$. Show that in fact the two monoid structures coincide, that is, $\cdot = \star$ and $1 = e$, and moreover, they are commutative. This is called the Eckmann-Hilton argument.

4. Compute all idempotents in the Tits algebra of a rank-one arrangement. Consider a general element $\alpha\,\mathtt{H}_O + \beta\,\mathtt{H}_C + \gamma\,\mathtt{H}_{\overline{C}}$. It is idempotent iff $\alpha^2 = \alpha$, $\beta(2\alpha + \beta + \gamma) = \beta$ and $\gamma(2\alpha + \beta + \gamma) = \gamma$. There are two cases. $\underline{\alpha = 0.}$ The equations become $\beta(\beta + \gamma - 1) = 0$ and $\gamma(\beta + \gamma - 1) = 0$. The solutions are $\beta = \gamma = 0$ and any β and γ with $\beta + \gamma = 1$. $\underline{\alpha = 1.}$ The equations become $\beta(\beta + \gamma + 1) = 0$ and $\gamma(\beta + \gamma + 1) = 0$. The

solutions are $\beta = \gamma = 0$ and any β and γ with $\beta + \gamma = -1$. In conclusion, the idempotents are

$$0, \quad \mathtt{H}_O, \quad \beta\,\mathtt{H}_C + (1-\beta)\,\mathtt{H}_{\overline{C}}, \quad \mathtt{H}_O + \beta\,\mathtt{H}_C + (-1-\beta)\,\mathtt{H}_{\overline{C}},$$

with β arbitrary. Use this to check that an element z of the Tits algebra is a special Zie element iff z is an idempotent and $\mathrm{s}(z) = \mathtt{Q}_\perp$. Apply the support map to the idempotents obtained above. The ones with support $\mathtt{Q}_\perp$ are precisely $\mathtt{H}_O + \beta\,\mathtt{H}_C + (-1-\beta)\,\mathtt{H}_{\overline{C}}$, with β arbitrary. This set of elements coincides with the set of special Zie elements.

5. Let $\Bbbk$ be a field. Consider the monoid with underlying set $\{1, e, f, ef, fe\}$. All elements are idempotent, and they are multiplied using the relations $efe = e$ and $fef = f$. Let A denote the linearization of this monoid over $\Bbbk$. It is 5 dimensional. Compute the radical of A. Is this algebra elementary?

The monoid is precisely the Janus monoid of a rank-one arrangement $\mathcal{A}$ with chambers C and $\overline{C}$:

$$\mathrm{id} \leftrightarrow (O,O),\ \ e \leftrightarrow (C,C),\ \ f \leftrightarrow (\overline{C},\overline{C}),$$

$$ef \leftrightarrow (C,\overline{C}),\ \ fe \leftrightarrow (\overline{C},C).$$

Its linearization is the Janus algebra. It is elementary with split-semisimple quotient the Birhoff algebra. The Tits algebra lies between the Janus and Birkhoff algebras. The radical of the Janus algebra is 3-dimensional and its nilpotency index is 3.

Students found $e - ef$ and $f - fe$ to be in the radical. This will lead up to the Tits algebra. While identifying the quotient, some students took basis 1 and $(e + f + ef + fe)$. The latter choice looks symmetric, but it is 0 in the quotient if $4 = 0$ in the field. Better instead to take 1 and e.

One student tried to express this algebra as a tensor product of Tits algebra but then did not pursue this approach.

Many students represented the multiplication tables of the monoid as a matrix, and were comfortable working with matrices.

2 Exam problems. Set II

Aryaman, Kartik, Manav, Pushkar, Sankalp, Som,
Apurva, Soumyadip, Uttam

1. Describe the coproduct in the category of monoids.

   Let X and Y be monoids. Their coproduct $X \sqcup Y$ is the quotient of the free monoid on the set

   $$(X \setminus \{1\}) \cup (Y \setminus \{1\})$$

   subject to the relations

   $$(x_1, x_2) = (x_1 x_2) \quad \text{and} \quad (y_1, y_2) = (y_1 y_2)$$

   with $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

   Explicitly, an element in $X \sqcup Y$ is a word in which nonidentity elements of X and Y appear alternately. The product is concatenation of words followed by (possible) cancellations at the point where the two words join.

   If you try something naive like $X \times Y$, then you will have difficulty showing the map in the universal property to be an algebra homomorphism. A couple of students tried this. One student tried disjoint union of X and Y and concocted a product on it, but then this runs into the same difficulty of the map in the universal property being an algebra homomorphism.

2. Let $\mathbb{k}$ be a field. Describe the category of left modules over the algebra $\mathbb{k}^n$.

   Let $\mathsf{Vec}_{\mathbb{k}}^n$ denote the n-fold cartesian product of the category of $\mathbb{k}$-vector spaces with itself. Explicitly, an object is an n-tuple $(V_1, \dots, V_n)$ of $\mathbb{k}$-vector spaces, and a morphism $(V_1, \dots, V_n) \to (W_1, \dots, W_n)$ is an n-tuple of linear maps $f_i : V_i \to W_i$.

   The category $\mathbb{k}^n$-Mod is equivalent to $\mathsf{Vec}_{\mathbb{k}}^n$. The functor

$$\mathcal{F} : \mathbb{k}^n\text{-Mod} \to \mathsf{Vec}_{\mathbb{k}}^n$$

   sends a module M to the n-tuple

$(e_1M, \ldots, e_nM)$. The key observation is that a map $f : M \to N$ of modules induces linear maps $f_i : e_iM \to e_iN$ for each n. Thus, $\mathcal{F}$ is a functor, and one can check that it determines an equivalence.

Some students constructed specific modules such as $\mathbb{k}^n \oplus \cdots \oplus \mathbb{k}^n$, or $\mathbb{k}^I$ with I being a subset of $[n]$, thinking that these are all the modules.

3. Find all complete systems of primitive orthogonal idempotents of the Tits algebra of a rank-one arrangement.

   We can use the calculation of idempotents in a previous problem. Since the Tits algebra has a one-dimensional radical, we deduce that each complete system has exactly two elements. One can check them to be

   $$\beta\,\mathtt{H}_C + (1-\beta)\,\mathtt{H}_{\overline{C}}, \quad \mathtt{H}_O - \beta\,\mathtt{H}_C + (-1+\beta)\,\mathtt{H}_{\overline{C}},$$

   with β arbitrary.

Some students used the isomorphism with upper triangular matrices of size 2. The complete systems take the form $(1, c)(0, 0)$ and $(0, -c)(0, 1)$. (These are matrices with first row followed by the second row.) But this argument probably requires char 2.

4. Show that the algebra of upper triangular n by n matrices for $n \geq 3$ cannot be isomorphic to the Tits algebra of any arrangement. Let A denote the algebra of upper triangular n by n matrices. The radical of A has nilpotency index n, and the quotient by the radical is $\mathbb{k}^n$. If it were the Tits algebra of $\mathcal{A}$, then $\mathcal{A}$ would be of rank $n-1$ and the number of flats would be n. This is possible for $n = 2$ (the case discussed above) but impossible for $n \geq 3$. For instance, the faces of a chamber have distinct supports, so this gives us at least $2^{\mathrm{rk}(\mathcal{A})}$ flats.

5. Let $\mathbb{k}$ be a field. Let $q \in \mathbb{k}$ be a scalar which is not

a root of unity. Let A_q be an associative $\mathbb{k}$-algebra with basis $\{1, e, f, ef, fe\}$ with unit element 1 and products

$$e \cdot e = e, \quad e \cdot f = q\,ef, \quad e \cdot ef = ef, \quad e \cdot fe = q\,e$$
$$f \cdot f = f, \quad f \cdot e = q\,fe, \quad f \cdot ef = q\,f, \quad f \cdot fe = fe$$
$$ef \cdot e = q\,e, \quad ef \cdot f = ef, \quad ef \cdot ef = q\,ef, \quad ef \cdot fe = e$$
$$fe \cdot e = fe, \quad fe \cdot f = q\,f, \quad fe \cdot ef = f, \quad fe \cdot fe = q\,fe.$$

Compute the radical of A_q.

This is the q-Janus algebra of a rank-one arrangement, as in the earlier related question. This algebra has radical 0, and in fact, is split-semisimple.

Some students successfully showed that there is no nilpotent ideal by brute force. One student almost got to the Q-basis. The idea is that you start with a nilpotent element. And multiply it on the left and right by the right elements (such as the Q-basis elements) to get multiples of pure basis elements such as e which are idempotent.

3 Possible topics for presentations

1. Operad structure of Lie (Section 10.6, b.pdf)
2. Radical and quiver of the incidence algebra (Section C.2, b.pdf)
3. Presentation of the category of lunes (Section 4.6, b.pdf). Connection with the lune-incidence algebra (Proposition 15.7 and Section C.1.13, b.pdf). Quiver of the lune-incidence algebra (Theorem 15.14, b.pdf)
4. The Saliola construction of the Eulerian idempotents of the Tits algebra (Section 11.2, b.pdf)
5. Dynkin elements, calculation of dimension of Lie, classical Lie elements (Sections 14.1, 14.2, 14.9, b.pdf)
6. Radical of the Tits algebra and Lie elements (Proposition 13.66, b.pdf). Radical series of the

module of chambers (Proposition 13.67, b.pdf)

7. Peirce decompositions of faces (Section 13.7, b.pdf)

8. Quiver of the Tits algebra (Section 13.10, b.pdf)