Bimonoids as Functor Categories

1 Bicommutative Bimonoids

1.1 Base category for bicommutative bimonoids

We now define the category $\mathcal{A}$-$\mathtt{Hyp}^{\mathtt{e}}_{\mathtt{r}}$.

- Objects are flats of $\mathcal{A}$.
- A morphism $\mathrm{X} \to \mathrm{Y}$ is a flat X' greater than both X and Y. We represent this morphism as a triple $(\mathrm{X}, \mathrm{X}', \mathrm{Y})$, or more vividly as

(1)
$$\begin{array}{ccc} & \mathrm{X}' & \\ \nearrow & & \searrow \\ \mathrm{X} & & \mathrm{Y}. \end{array}$$

- The identity morphisms are $(\mathrm{X}, \mathrm{X}, \mathrm{X})$.

- Composition of morphisms is defined by

  (2) $(\mathrm{Y}, \mathrm{Y}', \mathrm{Z}) \circ (\mathrm{X}, \mathrm{X}', \mathrm{Y}) := (\mathrm{X}, \mathrm{X}' \vee \mathrm{Y}', \mathrm{Z})$.

  This is illustrated below.

$$\begin{tikzcd}[column sep=small]
 & & \mathrm{X}' \vee \mathrm{Y}' \arrow[dr, hook] & & \\
 & \mathrm{X}' \arrow[ur, hook] \arrow[dr, hook] & & \mathrm{Y}' \arrow[dr, hook] & \\
\mathrm{X} \arrow[ur, hook] & & \mathrm{Y} \arrow[ur, hook] & & \mathrm{Z}.
\end{tikzcd}$$

Whenever $\mathrm{X} \leq \mathrm{X}'$ and $\mathrm{Y} \leq \mathrm{X}'$, we have the morphisms $(\mathrm{X}, \mathrm{X}', \mathrm{X}')$ and $(\mathrm{X}', \mathrm{X}', \mathrm{Y})$. The first is from X to X', while the second is from X' to Y. To simplify notation, we denote them by $\mathrm{X} \xrightarrow{\Delta} \mathrm{X}'$ and $\mathrm{X}' \xrightarrow{\mu} \mathrm{Y}$. In other words,

$$
\begin{tikzcd}[column sep=small]
 & \mathrm{X}' \\
\mathrm{X} \arrow[ur, "\Delta"] &
\end{tikzcd}
:=
\begin{tikzcd}[column sep=small]
 & \mathrm{X}' \arrow[dr] & \\
\mathrm{X} \arrow[ur] & & \mathrm{X}'
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
\mathrm{X}' \arrow[dr, "\mu"] & \\
 & \mathrm{Y}
\end{tikzcd}
:=
\begin{tikzcd}[column sep=small]
 & \mathrm{X}' \arrow[dr] & \\
\mathrm{X}' \arrow[ur] & & \mathrm{Y}.
\end{tikzcd}
$$

What is the composite of these two morphisms?

Proposition 1. *The category $\mathcal{A}$-$\mathtt{Hyp}^{\mathtt{e}}_{\mathtt{r}}$ has a presentation given by generators*

$$\mathrm{X} \xrightarrow{\Delta} \mathrm{Y}, \quad \mathrm{Y} \xrightarrow{\mu} \mathrm{X},$$

and relations

(3a)
$$\begin{tikzcd} & \mathrm{X} \arrow[dr, "\Delta"] & \\ \mathrm{Z} \arrow[ur, "\Delta"] \arrow[rr, "\Delta"'] & & \mathrm{Y} \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{X} \arrow[dr, "\mu"] & \\ \mathrm{Y} \arrow[ur, "\mu"] \arrow[rr, "\mu"'] & & \mathrm{Z} \end{tikzcd}$$

(3b)
$$\begin{tikzcd} & \mathrm{X} \vee \mathrm{Y} \arrow[dr, "\mu"] & \\ \mathrm{X} \arrow[ur, "\Delta"] \arrow[dr, "\mu"'] & & \mathrm{Y} \\ & \mathrm{Z} \arrow[ur, "\Delta"'] & \end{tikzcd}$$

(3c)
$$(\mathrm{X} \xrightarrow{\Delta} \mathrm{X}) = \mathrm{id} = (\mathrm{X} \xrightarrow{\mu} \mathrm{X}).$$

By convention, Δ goes from a smaller flat to a bigger flat, and μ from a bigger flat to a smaller flat.

Proof Outline:

- We need to show that any morphism in C from X to Y can be uniquely written as a composite

$$\mathrm{X} \xrightarrow{\Delta} \mathrm{X}' \xrightarrow{\mu} \mathrm{Y}.$$

  - Use the diagrams in (3a) and (3b), to show that any morphism from X to Y can be reduced to one like the above.

- Also, check that morphisms compose by the rule (2). Use the diamond rule(3b).

Proposition 2. *The category of bicommutative bimonoids is equivalent to the functor category* $[\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{e}}_{\mathtt{r}}, \mathrm{Vec}]$.