Joyal species

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1 Joyal species and Joyal bimonoids

1.1 Joyal species

Let $\mathtt{set}^{\times}$ denote the category whose objects are finite sets and whose morphisms are bijections.

A Joyal species is a functor

$$\mathrm{p} : \mathtt{set}^{\times} \to \mathsf{Vec}.$$

A map of Joyal species $\mathrm{p} \to \mathrm{q}$ is a natural transformation.

This defines the category of Joyal species which we denote by $\mathrm{J}\text{-}\mathsf{Sp}$.

It is a functor category, and we also write

$$\mathrm{J}\text{-}\mathsf{Sp} = [\mathtt{set}^{\times}, \mathsf{Vec}].$$

The value of a Joyal species p on a finite set J will be denoted $\mathrm{p}[J]$. We call it the J-component of p.

1.2 Joyal (co)monoids

The category of Joyal species $\mathsf{J}\text{-}\mathsf{Sp}$ is a monoidal category wrt the Cauchy product.

The Cauchy product of p and q is denoted $\mathsf{p} \cdot \mathsf{q}$ and its J-component is defined by

$$(\mathsf{p} \cdot \mathsf{q})[J] := \bigoplus_{J = S \sqcup T} \mathsf{p}[S] \otimes \mathsf{q}[T]. \tag{1}$$

We call $J = S \sqcup T$ a decomposition of J; it means that S and T are disjoint subsets whose union is J.

A monoid in $(\mathtt{J}\text{-}\mathtt{Sp}, \cdot)$ is called a Joyal monoid.

Thus, a Joyal monoid a entails linear maps

$$\mu_{S,T} : \mathsf{a}[S] \otimes \mathsf{a}[T] \to \mathsf{a}[J], \tag{2}$$

one for each decomposition $J = S \sqcup T$, subject to the naturality, associativity and unitality axioms.

Dually, a comonoid in $(\mathtt{J}\text{-}\mathtt{Sp}, \cdot)$ is called a Joyal comonoid.

Thus, a Joyal comonoid c entails linear maps

$$\Delta_{S,T} : \mathsf{c}[J] \to \mathsf{c}[S] \otimes \mathsf{c}[T], \tag{3}$$

one for each decomposition $J = S \sqcup T$, subject to the naturality, coassociativity and counitality axioms.

1.3 Joyal bimonoids and (co)commutative Joyal (co)monoids

The monoidal category of Joyal species $(\text{J-Sp}, \cdot)$ carries a family of braidings β_q, one for each scalar q.

The map $\beta_q : \mathsf{p} \cdot \mathsf{q} \to \mathsf{q} \cdot \mathsf{p}$, evaluated on the J-component, is the direct sum of the linear maps

(4)

$$(\beta_q)_{S,T} : \mathsf{p}[S] \otimes \mathsf{q}[T] \to \mathsf{q}[T] \otimes \mathsf{p}[S], \quad x \otimes y \mapsto q^{|S||T|} y \otimes x$$

over all decompositions $J = S \sqcup T$.

The notation $|S|$ stands for the cardinality of the set S.

A bimonoid in $(\text{J-}\mathtt{Sp}, \cdot, \beta_q)$ is called a Joyal q-bimonoid.

For $q = \pm 1$ and $q = 0$, we use the terms Joyal bimonoid, signed Joyal bimonoid and Joyal 0-bimonoid, respectively.

Explicitly, the q-bimonoid axiom says that for any two decompositions $J = S_1 \sqcup S_2 = T_1 \sqcup T_2$, the diagram

(5)

$$\begin{tikzcd}
\mathsf{h}[S_1]\otimes\mathsf{h}[S_2] \arrow[r, "\mu_{S_1,S_2}"] \arrow[d, "\Delta_{A,B}\otimes\Delta_{C,D}"'] & \mathsf{h}[J] \arrow[r, "\Delta_{T_1,T_2}"] & \mathsf{h}[T_1]\otimes\mathsf{h}[T_2] \\
\mathsf{h}[A]\otimes\mathsf{h}[B]\otimes\mathsf{h}[C]\otimes\mathsf{h}[D] \arrow[rr, "\mathrm{id}\otimes(\beta_q)_{B,C}\otimes\mathrm{id}"'] & & \mathsf{h}[A]\otimes\mathsf{h}[C]\otimes\mathsf{h}[B]\otimes\mathsf{h}[D] \arrow[u, "\mu_{A,C}\otimes\mu_{B,D}"']
\end{tikzcd}$$

commutes, where

$$A := S_1 \cap T_1,\ B := S_1 \cap T_2,\ C := S_2 \cap T_1,\ D := S_2 \cap T_2.$$

A commutative monoid in $(\mathtt{J\text{-}Sp}, \cdot, \beta_1)$ is called a commutative Joyal monoid, while a commutative monoid in $(\mathtt{J\text{-}Sp}, \cdot, \beta_{-1})$ is called a signed commutative Joyal monoid.

Explicitly, the commutativity axiom and signed commutativity axiom say that for any decomposition $J = S \sqcup T$, the diagrams

(6)

$$
\begin{tikzcd}
\mathsf{a}[S]\otimes\mathsf{a}[T] \arrow[rr, "\beta_{S,T}"] \arrow[dr, "\mu_{S,T}"'] & & \mathsf{a}[T]\otimes\mathsf{a}[S] \arrow[dl, "\mu_{T,S}"] \\
& \mathsf{a}[J] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{a}[S]\otimes\mathsf{a}[T] \arrow[rr, "(\beta_{-1})_{S,T}"] \arrow[dr, "\mu_{S,T}"'] & & \mathsf{a}[T]\otimes\mathsf{a}[S] \arrow[dl, "\mu_{T,S}"] \\
& \mathsf{a}[J], &
\end{tikzcd}
$$

respectively, commute.

Dually, we have (signed) cocommutative Joyal comonoids.

1.4 Cartesian extension functor

Fix a finite set I.

Let $\mathcal{A}$ be the braid arrangement on I.

Faces of $\mathcal{A}$ are compositions of I, while flats are partitions of I.

(Recall that a composition (partition) of I is a finite sequence (collection) of mutually disjoint nonempty subsets of I whose union is I. The subsets are called blocks.)

Any Joyal species p gives rise to an $\mathcal{A}$-species, denoted $\hat{\mathsf{p}}$, by setting its F-component to be

$$\hat{\mathsf{p}}[F] := \mathsf{p}[S_1] \otimes \cdots \otimes \mathsf{p}[S_k],$$

where $S_1, \ldots, S_k$ are the ordered blocks of F. The maps

$$\beta_{G,F} : \hat{\mathsf{p}}[F] \to \hat{\mathsf{p}}[G]$$

are defined by permuting tensor factors.

(Since F and G have the same support, they have the same blocks, but in a different order.)

Equivalently, one can work with set partitions and set

$$\hat{\mathsf{p}}[\mathrm{X}] := \bigotimes_{B \in \mathrm{X}} \mathsf{p}[B],$$

where the rhs is the unordered tensor product over blocks B of the set partition X.

This defines a functor

$$\text{J-Sp} \to \mathcal{A}\text{-Sp}$$

which we call the cartesian extension functor.

Remark 1. Note that $\mathsf{p}[\emptyset]$ played no role in this construction. So it is makes more sense to view this as a functor from connected Joyal species to $\mathcal{A}$-species.

Now suppose a is a Joyal monoid.

Then, for any subset J of I and composition F of J, iterating (2) yields a map $\hat{\mathrm{a}}[F] \to \mathrm{a}[J]$.

For $A \leq F$, both compositions of I, define

$$\mu_A^F : \hat{\mathrm{a}}[F] \to \hat{\mathrm{a}}[A]$$

by tensoring the maps $\hat{\mathrm{a}}[F^i] \to \mathrm{a}[J_i]$, where $A = (J_1, \ldots, J_k)$, and F^i is the part of F which refines J_i.

This turns $\hat{\mathrm{a}}$ into an $\mathcal{A}$-monoid.

In a similar manner, a Joyal comonoid yields an $\mathcal{A}$-comonoid.

Example. Let p be a Joyal species.

Take $I = \{a, b, c, d, e, f, g\}$ and $F = bde|af|cg$. Then

$$\mathsf{p}[bde|af|cg] = \mathsf{p}[bde] \otimes \mathsf{p}[af] \otimes \mathsf{p}[cg].$$

Consider the face $G = af|cg|bde$ which has the same support as F.

The map

$$\mathsf{p}[bde|af|cg] = \mathsf{p}[bde] \otimes \mathsf{p}[af] \otimes \mathsf{p}[cg] \xrightarrow{\beta_{G,F}} \mathsf{p}[af|cg|bde] = \mathsf{p}[af] \otimes \mathsf{p}[cg] \otimes \mathsf{p}[bde]$$

permutes the three tensor factors.

Now let a be a Joyal monoid.

For $H = bd|e|af|cg$ and $A = bde|afcg$,

$$\mathsf{a}[bd|e|af|cg] = \mathsf{a}[bd] \otimes \mathsf{a}[e] \otimes \mathsf{a}[af] \otimes \mathsf{a}[cg] \xrightarrow{\mu_A^H} \mathsf{a}[bde|afcg] = \mathsf{a}[bde] \otimes \mathsf{a}[afcg]$$

is given by $\mu_e^{bd} \otimes \mu_{cg}^{af}$.

This defines functors from the category of Joyal (co)monoids to the category of $\mathcal{A}$-(co)monoids.

We mention that these functors preserve (signed) (co)commutativity.

Further, they also induce a functor from the category of Joyal bimonoids to the category of $\mathcal{A}$-bimonoids.

2 Formal power series. Series of Joyal species

In this section, we assume that the field characteristic is 0.

2.1 Formal power series

Let $\mathscr{F}$ denote the space of formal power series in one variable χ whose constant term is 0.

A typical formal power series s is written as

$$s(\chi) = \sum_{n \geq 1} s_n \chi^n.$$

Recall that $\mathscr{F}$ is a monoid under substitution.

Explicitly, the substitution of t into s is given by

$$(s \circ t)_n := \sum_{(i_1, \ldots, i_k) \vDash n} s_k t_{i_1} \ldots t_{i_k}. \tag{7}$$

The sum is over all compositions of n.

Note very carefully that $\mathscr{F}$ is not an algebra because $s \circ (t + t') \neq s \circ t + s \circ t'$ in general.

Also note that the set of formal power series with $s_1 \neq 0$ constitutes a group under substitution.

2.2 From formal power series to lune-incidence algebras

Let s be a formal power series.

For any set compositions A and F with $A \leq F$, define

$$\hat{s}(A, F) := \prod_i s_{\deg(F/A)_i}, \tag{8}$$

where $\deg(F/A)_i$ is the number of blocks of F which refine the i-th block of A.

For example,

$$A = krish|na, \quad F = kr|i|sh|n|a, \quad \hat{s}(A, F) = s_3 s_2.$$

This is because $kr|i|sh$ which refines $krish$ has 3 blocks, while $n|a$ which refines na has 2 blocks.

In particular,

$$\hat{s}(O, F) = s_{\deg(F)},$$

where $\deg(F)$ denotes the number of blocks of F, and O is a one-block set composition.

Lemma 1. *For $J = S \sqcup T$, let A and F be compositions of S with $A \leq F$, and B and G be compositions of T with $B \leq G$.*

Then $A|B$ and $F|G$ are compositions of J with $A|B \leq F|G$, and

$$\hat{s}(A|B, F|G) = \hat{s}(A, F)\hat{s}(B, G).$$

Proof. This multiplicative property follows directly from (8). □

For any braid arrangement $\mathcal{A}$, observe that $\hat{s}$ defines an element of the lune-incidence algebra of $\mathcal{A}$.

In other words, we have a map

$$\mathscr{F} \to \mathrm{I}_{\mathrm{lune}}[\mathcal{A}], \qquad s \mapsto \hat{s} \tag{9}$$

for any such $\mathcal{A}$.

Lemma 2. *For any formal power series s and t, we have $\widehat{s \circ t} = \hat{s}\hat{t}$. In other words, the map (9) is a monoid homomorphism.*

Proof. We first do a special case. Using definition (8) and the product formulas (**??**) and (7),

$$\begin{aligned}
(\widehat{s \circ t})(O, G) &= (s \circ t)_{\deg(G)} \\
&= \sum_{(i_1,\ldots,i_k) \vDash \deg(G)} s_k t_{i_1} \ldots t_{i_k} \\
&= \sum_{F\colon F \leq G} s_{\deg(F)} \prod_i t_{\deg(G/F)_i} \\
&= \sum_{F\colon F \leq G} \hat{s}(O, F)\hat{t}(F, G) \\
&= (\hat{s}\hat{t})(O, G).
\end{aligned}$$

The general case can be deduced from Lemma 1. $\square$

2.3 Series of Joyal species

Let p be a Joyal species.

A series v of p is a family of elements $v_J \in \mathsf{p}[J]$, one for each nonempty finite set J, such that

$$\sigma(v_J) = v_{J'}$$

for each bijection $\sigma : J \to J'$.

Let $\mathscr{S}(\mathsf{p})$ denote the space of series of p.

This construction is functorial in p, and defines a functor $\mathscr{S}$ from the category of Joyal species to the category of vector spaces.

Let c be a Joyal comonoid.

A series v of c is primitive if $\Delta_{S,T}(v_J) = 0$ for each $J = S \sqcup T$, with S and T nonempty.

Similarly, a series v of c is group-like if $\Delta_{S,T}(v_J) = v_S \otimes v_T$ for each $J = S \sqcup T$, with S and T nonempty.

2.4 Cartesian extension functor

Recall the cartesian extension functor of Section 1.4 which associates an $\mathcal{A}$-species $\hat{\mathrm{p}}$ to a Joyal species p for any braid arrangement $\mathcal{A}$.

Let v be a series of a Joyal species p.

For any set composition $F = (S_1, \dots, S_k)$, put

$$\hat{v}_F := v_{S_1} \otimes \cdots \otimes v_{S_k} \quad \in \hat{\mathrm{p}}[F].$$

Then $\hat{v}$ defines a series of $\hat{\mathrm{p}}$.

Thus, we have a map

$$\mathscr{S}(\mathrm{p}) \to \mathscr{S}(\hat{\mathrm{p}}), \qquad v \mapsto \hat{v}. \tag{10}$$

Moreover, for a Joyal comonoid, if v is primitive or group-like, then so is $\hat{v}$.

2.5 Modules

Let a be a Joyal monoid and $\mathscr{S}(\mathsf{a})$ its space of series. The monoid $\mathscr{F}$ of formal power series acts on the left on $\mathscr{S}(\mathsf{a})$ by

(11)

$$(s \circ v)_J := \sum_{F \vDash J} s_{\deg(F)} \mu_O^F(\hat{v}_F) = \sum_{F \vDash J} \hat{s}(O, F) \mu_O^F(\hat{v}_F).$$

Here s is a formal power series and v is a series of a.

The sum is over all compositions of J.

Lemma 3. *For a formal power series s and series v of* a*, we have $\widehat{s \circ v} = \hat{s} \circ \hat{v}$. In other words, the map* (10) *is a module homomorphism.*

Proof. The identity is straightforward to verify using the multiplicative property in Lemma 1.

In fact, it can also be used to formally deduce that (11) defines a left action:

$$(s \circ (t \circ v))_J = (\widehat{s \circ (t \circ v)})_J = (\hat{s} \circ (\widehat{t \circ v}))_J = (\hat{s} \circ (\hat{t} \circ \hat{v}))$$

$$= ((\widehat{s \circ t}) \circ \hat{v})_J = (\widehat{(s \circ t) \circ v})_J = ((s \circ t) \circ v)_J.$$

The fifth step used Lemma 2. $\square$

2.6 Exp-log correspondence

Consider the formal power series

$$e^{x}-1 = \sum_{n\geq 1} \frac{x^n}{n!} \quad \text{and} \quad \log(1+x) = \sum_{n\geq 1} (-1)^{n-1}(n-1)!\frac{x^n}{n!}.$$

These are the exponential power series and logarithmic power series, respectively.

We also denote them by exp and log for short.

They are inverses of each other in $\mathscr{F}$.

Note very carefully that $(-1)^{n-1}(n-1)!$ is the Möbius number of the braid arrangement of rank $n-1$.

Lemma 4. *Under the map (9), we have*

$$\widehat{\exp}(A, F) = \prod_i \frac{1}{\deg(F/A)_i!}$$

and $$\widehat{\log}(A, F) = (-1)^{\mathrm{rk}(F/A)} \prod_i \frac{1}{\deg(F/A)_i}.$$

In other words, $\widehat{\exp}$ is the uniform noncommutative zeta function of the braid arrangement, and $\widehat{\log}$ is its inverse noncommutative Möbius function.

Proof. This is a specialization of (8). $\square$

For any Joyal monoid a, the action of the exponential power series and logarithmic power series sets up an exp-log correspondence on $\mathscr{S}(\mathsf{a})$.

It relates to the exp-log correspondence for the monoid $\hat{\mathsf{a}}$ via

$$(12) \quad \widehat{\exp} \circ \hat{v} = \widehat{\exp \circ v} \quad \text{and} \quad \widehat{\log} \circ \hat{v} = \widehat{\log \circ v},$$

with $\widehat{\exp}$ and $\widehat{\log}$ as in Lemma 4.

This is an instance of Lemma 3.

Example. Consider the exponential Joyal monoid E defined by $\mathsf{E}[J] = \Bbbk$ for all J, with all product components $\mu_{S,T}$ equal to the canonical identification $\Bbbk \otimes \Bbbk \xrightarrow{\cong} \Bbbk$.

A series v of E can be identified with a sequence of scalars $(v_n)_{n\geq 1}$ via $v_J = v_{|J|}$.

Observe that v is group-like if $v_n = \alpha^n$ for some scalar α, and primitive if $v_n = 0$ for all $n \geq 2$.

It is convenient to view a series v of E as the exponential generating function

(13)
$$v = \sum_{n\geq 1} v_n \frac{x^n}{n!}.$$

For any set composition $F = (S_1, \ldots, S_k)$, put $\hat{v}_F = v_{|S_1|} \ldots v_{|S_k|}$.

This is a product of scalars.

The action (11) of $\mathscr{F}$ on $\mathscr{S}(\mathrm{E})$ is given by

$$(s \circ v)_J = \sum_{F \vDash J} s_{\deg(F)} \hat{v}_F,$$

which can be rewritten as

$$(s \circ v)_n = \sum_{(i_1,\ldots,i_k) \vDash n} \binom{n}{i_1, \ldots, i_k} s_k v_{i_1} \ldots v_{i_k}.$$

This agrees with the action of $\mathscr{F}$ on exponential generating functions, that is, we substitute (13) in the power series s, and rewrite the resulting power series as an exponential generating function.

The identities (12) link the exponential and logarithmic power series to Möbius inversion in the poset of flats. In particular, the first identity recovers the result given in [**?**, Proposition 6.15].

Also note explicitly that the map (10) preserves primitive series and group-like series.

3 May operads

3.1 May operads

A positive Joyal species is a Joyal species p such that $\mathsf{p}[\emptyset] = 0$.

Let p and q be positive Joyal species.

Define a new positive Joyal species $\mathsf{p} \circ \mathsf{q}$ by

$$(\mathsf{p} \circ \mathsf{q})[J] := \bigoplus_{\mathrm{X} \vdash J} \mathsf{p}[\mathrm{X}] \otimes \left(\bigotimes_{S \in \mathrm{X}} \mathsf{q}[S] \right). \tag{14}$$

This is the substitution product of p and q.

The direct sum is over all partitions X of the set J, while the tensor product is over all blocks S of X.

This defines a monoidal structure on the category of positive Joyal species.

The unit object is the positive Joyal species x characteristic of singletons, namely,

$$\mathrm{x}[J] := \begin{cases} \Bbbk & \text{if } J \text{ is a singleton,} \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

A May operad is a monoid in this monoidal category.

In particular, a May operad p entails maps

$$\mathsf{p}[\mathrm{X}] \otimes \left(\bigotimes_{S \in \mathrm{X}} \mathsf{p}[S] \right) \to \mathsf{p}[J], \tag{16}$$

one for each partition X of J, subject to the associativity and unitality axioms.

3.2 From May operads to $\mathcal{A}$-operads

Let $\mathcal{A}$ denote any braid arrangement.

To a positive Joyal species p, we associate an $\mathcal{A}$-dispecies $\mathbf{p}$ as follows.

The (X, Y)-component of $\mathbf{p}$ is defined by

$$\mathbf{p}[\mathrm{X}, \mathrm{Y}] := \bigotimes_{S \in \mathrm{X}} \mathsf{p}[\mathrm{Y}_S],$$

where Y_S is the set whose elements are the blocks of Y which refine the block S of X.

This yields a functor from the category of positive Joyal species to the category of $\mathcal{A}$-dispecies.

Let us temporarily denote it by $\mathcal{F}$. For positive Joyal species p and q, there are natural isomorphisms

$$\mathcal{F}(\mathsf{p}) \circ \mathcal{F}(\mathsf{q}) \xrightarrow{\cong} \mathcal{F}(\mathsf{p} \circ \mathsf{q}) \quad \text{and} \quad \mathbf{x} \xrightarrow{\cong} \mathcal{F}(\mathsf{x}),$$

with products as in (**??**) and (14).

The second isomorphism is clear.

For the first one, we evaluate both sides on say the component (X, Z). In both cases, the sum splits over partitions Y that lie in-between X and Z, and corresponding summands can be identified by rearranging the tensor factors. We omit the details.

Thus, $\mathcal{F}$ is a strong monoidal functor.

Since strong functors preserve monoids, $\mathcal{F}$ sends a May operad to an $\mathcal{A}$-operad.

Remark 2. We do not formally discuss the classical associative, commutative and Lie operads here.

But we mention that the operads $\mathbf{As}$, $\mathbf{Com}$ and $\mathbf{Lie}$, when specialized to the braid arrangement, arise from their classical counterparts via the above construction.