Noncommutative Möbius theory

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1 Lattices (Commutative world)

1.1 Relations on a set

Let X be any set. A relation on X is a subset of $X \times X$.

Suppose R is a relation and $(x, y) \in R$. We read this as "x is related to y under R". One can visualize a relation as a graph whose vertices are the elements of X, and there is an arrow (edge) from x to y whenever $(x, y) \in R$.

A relation R is

- reflexive if $(x, x) \in R$ for all $x \in X$.
- symmetric if $(x, y) \in R$ implies $(y, x) \in R$ for all $x, y \in X$.
- antisymmetric if $(x, y) \in R$ and $(y, x) \in R$ implies $x = y$ for all $x, y \in X$.
- transitive if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$ for all $x, y, z \in X$.

Each of these is a property that a relation either has or does not have.

A preposet is a set equipped with a relation which is transitive and reflexive,

a poset is a set equipped with a relation which is transitive, reflexive, antisymmetric,

an equivalence relation is a relation which is transitive, symmetric, reflexive.

This is summarized below.

Name	Type of relation
preposet	transitive and reflexive
poset	transitive, antisymmetric, reflexive
equivalence relation	transitive, symmetric, reflexive

1.2 Equivalence relations

An equivalence relation on a set X is a relation which is transitive, symmetric, reflexive. In this context, it is customary to denote the relation by $\sim$ and write $x \sim y$ when x is related to y. We read this as "x is equivalent to y". In this notation, the definition of an equivalence relation takes the following form. For all $x, y, z \in X$,

- $x \sim x$,
- $x \sim y$ implies $y \sim x$,
- $x \sim y$ and $y \sim z$ implies $x \sim z$.

An equivalence relation allows us to write X as a disjoint union of equivalence classes. The equivalence class of an element x under $\sim$, denoted $[x]$, is defined as

$$[x] := \{y \in X \mid x \sim y\}.$$

1.3 Partially ordered sets

A poset is a set P equipped with a relation which is transitive, reflexive, antisymmetric.

Such a relation is usually denoted $\leq$.

We read $x \leq y$ as "x is less than y", or "y is greater than x".

In this notation, the poset axioms are as follows. For all $x, y, z \in P$,

- $x \leq x$,
- $x \leq y$ and $y \leq x$ implies $x = y$,
- $x \leq y$ and $y \leq z$ implies $x \leq z$.

Poset is a shortform for partially ordered set.

This term was coined by Birkhoff in the third edition of his famous book on Lattice Theory (1940).

If $x \leq y$ but $x \neq y$, then it is customary to write $x < y$.

A poset P is linearly ordered or totally ordered if for any $x, y \in P$ either $x \leq y$ or $y \leq x$.

1.4 Meets and joins

Let P be a poset.

- We say x is the minimum element of P if $x \leq y$ for all $y \in P$.
- Dually, we say x is the maximum element of P if $y \leq x$ for all $y \in P$.
- Given elements $x, y \in P$, the meet of x and y, denoted $x \wedge y$, is the largest element of P smaller than both x and y. That is, $x \wedge y \leq x$, $x \wedge y \leq y$, and ($z \leq x$ and $z \leq y$ implies $z \leq x \wedge y$).
- Dually, the join of x and y, denoted $x \vee y$, is the smallest element of P larger than both x and y.

Meets, joins, minimum and maximum elements may not exist, but they are unique whenever they exist.

A poset is called a lattice if meets and joins (of any two elements) exist.

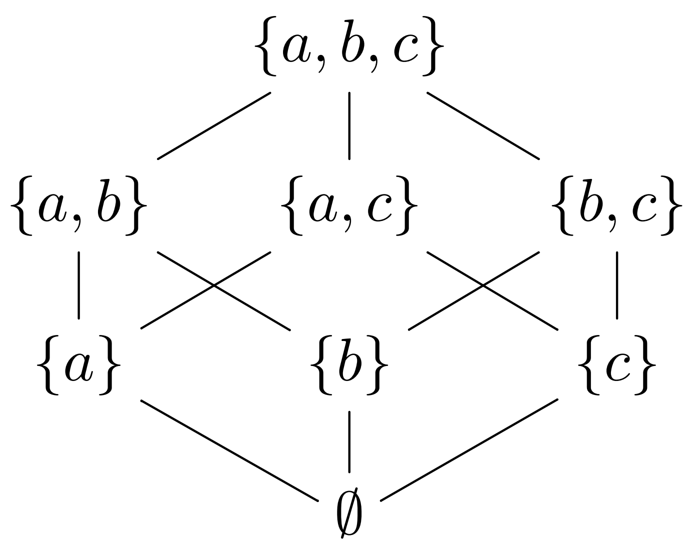
Example 1. Let X be any set and 2^X denote its power set. An element of 2^X is a subset of X.

This is a poset under inclusion, that is, we say $S \leq T$ if S is a subset of T. This is also called the Boolean poset of X.

The minimum element is the empty set, the maximum element is X, meet is given by intersection of sets, and join by union of sets.

We have already mentioned how to associate a graph with directed edges to any relation. The graph of a poset is called a Hasse diagram. It is customary not to draw the loops and the arrows implied by transitivity. Further, bigger elements are written vertically above the smaller elements, which makes the arrows on the edges redundant.

The Hasse diagram of the Boolean poset of $\{a, b, c\}$ is shown below.



Example 2. Let I be a finite set. A partition X of I is a collection X of disjoint nonempty subsets of I such that

$$I = \bigsqcup_{B \in \mathrm{X}} B.$$

The subsets B are the blocks of the partition.

For example, for $I = \{p, r, i, y, e, s, h\}$,

$$\mathrm{X} = \{\{i, e\}, \{p, r, y, s, h\}\}$$

is a partition with two blocks, namely $\{i, e\}$ and $\{p, r, y, s, h\}$.

Let $\Pi[I]$ denote the set of partitions of I. For partitions X and Y, we say that Y refines X if each block of X is a union of blocks of Y. In this case, we write $\mathrm{X} \leq \mathrm{Y}$. This defines a partial order on $\Pi[I]$ which is in fact a lattice.

The maximum element is the partition into singletons, and the minimum element is the partition whose only block is the whole set I.

The Hasse diagram of the set of partitions of $\{l, i, e\}$ is shown below.

$$
\begin{tikzcd}
& \{\{l\},\{i\},\{e\}\} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\{\{l\},\{i,e\}\} \arrow[dr, dash] & \{\{i\},\{l,e\}\} \arrow[d, dash] & \{\{e\},\{l,i\}\} \arrow[dl, dash] \\
& \{\{l,i,e\}\} &
\end{tikzcd}
$$

1.5 Incidence algebra of a poset

Fix a field $\mathbb{k}$. Let P be a finite poset. Let $I(P)$ denote the vector space of all $\mathbb{k}$-valued functions on the set

$$\{(x, y) \mid x \leq y\},$$

with addition and scalar multiplication defined pointwise.

For $f, g \in I(P)$, define a new function $fg \in I(P)$ by

$$(fg)(x, z) = \sum_{y \colon x \leq y \leq z} f(x, y) g(y, z). \tag{1}$$

This turns $I(P)$ into an algebra. It is called the incidence algebra of P. The unit element δ is given by

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

1.6 Zeta and Möbius functions of a poset

The zeta function $\zeta \in I(P)$ is defined by

$$\zeta(x, y) = 1$$

for all $x \leq y$.

It is invertible.

Its inverse is the Möbius function $\mu \in I(P)$.

This may also be defined recursively as follows.

For any element x,

(2a) $$\mu(x,x) := 1$$

and for $x < y$,

$$\mu(x,y) := -\sum_{z\colon x \le z < y} \mu(x,z) = -\sum_{z\colon x < z \le y} \mu(z,y),$$

or equivalently,

(2b) $$\sum_{z\colon x \le z \le y} \mu(x,z) = \sum_{z\colon x \le z \le y} \mu(z,y) = 0.$$

For more clarity, we may sometimes write μ_P instead of μ.

Example 3. Let $P = \{1, 2, \ldots, n\}$ with usual order. This is a linearly ordered set.

The incidence algebra $I(P)$ is the algebra of upper triangular square matrices of size n. The unit element δ is the identity matrix,

ζ is the upper triangular matrix with all entries 1, and μ is the upper triangular matrix with 1's on the diagonal, -1's on the slanted line above the diagonal and 0 elsewhere.

For example, for $n = 3$,

$$\zeta = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mu = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proposition 1 (Weisner formula)**.** *Let P be a finite lattice with $\perp$ and $\top$ as the minimum and maximum elements. Suppose $y > \perp$. Then for any element z,*

$$\sum_{x\colon y \vee x = z} \mu(\perp, x) = 0. \tag{3}$$

Proof. We prove (3) by induction on the partial order P. The base case is clear. We may assume $y \leq z$, otherwise the lhs is clearly 0.

$$\sum_{x\colon y \vee x = z} \mu(\perp, x) = \sum_{x\colon x \leq z} \mu(\perp, x) - \sum_{x\colon y \vee x < z} \mu(\perp, x).$$

The first term is zero by definition (2b), while the second term is zero by the induction hypothesis. This completes the induction step. □

Both the result and proof is due to Weisner (1935).

Proposition 2. *For any finite lattice P, consider the linear system*

$$\sum_{x\colon y\vee x=z} c_x = 0, \qquad \perp < y \leq z.$$

The solution space is one-dimensional and spanned by $(c_x = \mu(\perp, x))$.

Proof. It is clear from (3) that any scalar multiple of $(c_x = \mu(\perp, x))$ solves the above linear system. To see that these are the only solutions, restrict to the subsystem of equations satisfying $y = z$:

$$\sum_{x\colon x\leq z} c_x = 0, \qquad \perp < z.$$

Starting with an arbitrary value for $c_\perp$, we now see that $c_x = c_\perp \mu(\perp, x)$ is forced. □

1.7 Incidence module

Let $J(P)$ denote the vector space of $\mathbb{k}$-valued functions on a poset P.

The incidence algebra $I(P)$ acts on $J(P)$ on the left:

For $f \in I(P)$ and $g \in J(P)$, define $fg \in J(P)$ by

$$(fg)(x) = \sum_{y\colon x \leq y} f(x, y)g(y). \tag{4}$$

Thus, $J(P)$ is a left module over the incidence algebra $I(P)$. We call it the incidence module of P.

For functions f and g on P, we have

$$g(x) = \sum_{y\colon x \leq y} f(y) \iff f(x) = \sum_{y\colon x \leq y} \mu(x,y) g(y).$$

This is equivalent to $g = \zeta f \iff f = \mu g$.

The passage from the first equation to the second is called Möbius inversion.

In this situation, we say that g is the exponential of f, and f is the logarithm of g.

This is the simplest instance of the exponential map in Lie theory. Justification?

Example 4. Consider the Boolean poset on the set I.
For $S \subseteq T$, we have $\mu(S,T) = (-1)^{|T|-|S|}$.

So Möbius inversion takes the following form.

$$g(T) = \sum_{S \subseteq T} f(S) \iff f(T) = \sum_{S \subseteq T} (-1)^{|T|-|S|} g(S).$$

This is also called the inclusion–exclusion principle.

1.8 Exercises

1. For $f, g, h \in I(P)$, verify that $f(gh) = (fg)h$.

   For $f, g \in I(P)$ and $h \in J(P)$, verify that $f(gh) = (fg)h$.

2. Compute the Möbius function of the partition lattice $\Pi[I]$ of a finite set I.

3. Use the inclusion-exclusion principle to deduce that: For finite sets A, B, C, $\ldots$

$$|A \cup B| = |A| + |B| - |A \cap B|,$$

$$|A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap$$

   and so on.

4. Fix a parameter s. A Dirichlet series is an expression of the form

$$\hat{f}(s) = \sum_{n \geq 1} \frac{f(n)}{n^s},$$

with $f(n) \in \mathbb{k}$. The set of all Dirichlet series forms an algebra with componentwise sum and scalar multiplication. The product is defined by expanding and collecting like terms. Consider the Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

Show that this is an invertible element in the algebra of Dirichlet series and compute its inverse.

1.9 Review of finite-dimensional algebras

Fix a field $\mathbb{k}$.

Let A be a finite-dimensional algebra over $\mathbb{k}$.

Some examples to bear in mind are:

- $\mathbb{k}^n$,
- $\mathbb{k}[x]/(x^n)$,
- square matrices of size n,
- upper-triangular matrices of size n,
- incidence algebra of a poset P.

An element $x \in A$ is nilpotent if $x^n = 0$ for some positive integer n.

In the algebra of matrices, any strictly upper triangular matrix is nilpotent.

In the algebra $\mathbb{k}[x]/(x^n)$, the element x is nilpotent.

Lemma 1. *If x is nilpotent, then $1 + x$ is invertible with inverse*

$$(1 + x)^{-1} = 1 - x + x^2 - \dots .$$

Since x is nilpotent, the infinite sum in the rhs terminates.

In the incidence algebra $I(P)$ note that $\zeta - \delta$ is nilpotent. Hence, putting $x = \zeta - \delta$ in the above formula, we obtain

$$\mu = \delta - (\zeta - \delta) + (\zeta - \delta)^2 - \ldots .$$

Explicitly, this says

$$\mu(x, y) = c_0 - c_1 + c_2 - \ldots ,$$

where c_i is the number of chains from x to y of length i.

This is the Philip Hall formula for the Möbius function of a poset.

1.10 Complete systems of primitive orthogonal idempotents

An element $e \in A$ is an idempotent if $e^2 = e$.

Idempotents e and f are orthogonal if $ef = fe = 0$.

Note that for any idempotent e, the element $1 - e$ is also an idempotent which is orthogonal to e.

An idempotent e is primitive if it cannot be written as a sum of two orthogonal nonzero idempotents.

Lemma 2. *Every idempotent of A can be expressed as a sum of mutually orthogonal primitive idempotents.*

Proof. Let e be the given idempotent.

If e is primitive, then we are done.

If not, then write $e = f + g$, with both f and g nonzero orthogonal idempotents.

If f (or g) is not primitive, then write it as a sum of two orthogonal nonzero idempotents.

Continue.

If at some stage we have $e = e_1 + \cdots + e_k$, then $eA = e_1 A \oplus \cdots \oplus e_k A$, with each $e_i A \neq 0$.

So by finite-dimensionality of A, this process must end. $\square$

Applying this result to the unit element 1, we deduce:

There exists a family of mutually orthogonal primitive idempotents which sum up to 1.

Any such family is called a complete system of primitive orthogonal idempotents of A.

Complete refers to the fact that the idempotents sum up to 1.

Example 5. Consider the algebra of square matrices of size n. It acts on $\mathbb{k}^n$.

An idempotent is the same as a pair of complementary subspaces of $\mathbb{k}^n$, say (U, V). It acts by 0 on U and by 1 on V.

A nonzero idempotent is primitive precisely when V is one-dimensional. (If not, then it can be decomposed by breaking V.)

In particular:

A matrix with exactly one diagonal entry equal to 1 and all remaining entries 0, is a primitive idempotent.

Further, these matrices are orthogonal and their sum is the identity matrix, so they form a complete system of primitive orthogonal idempotents.

This is illustrated below for $n = 3$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Any other complete system is obtained by conjugating this system by an invertible matrix.

1.11 Split-semisimple commutative algebras

A commutative $\Bbbk$-algebra A is split-semisimple if it is isomorphic as an algebra to a product of copies of $\Bbbk$, that is, $A \cong \Bbbk^n$ for some n.

For $1 \leq i \leq n$, let e_i denote the element of A which corresponds to $(0, \ldots, 1, \ldots, 0) \in \Bbbk^n$ which is 1 in the i-th coordinate and zero elsewhere.

Observe that $f \in A$ is an idempotent iff f is a sum of some of the e_i.

In particular, the e_i are the only primitive idempotents of A.

These elements constitute a complete system of primitive orthogonal idempotents of A, and this system is unique.

The only algebra automorphisms of A are those obtained by permuting the e_i.

A split-semisimple commutative algebra does not contain any nonzero nilpotent elements. So an algebra such as $\mathbb{k}[x]/(x^n)$ for $n > 1$ cannot be split-semisimple.

1.12 Algebra of a lattice

Let P be a finite lattice with minimum element $\perp$ and maximum element $\top$.

Let $\Bbbk P$ denote the linearization of P over the field $\Bbbk$.

This is a commutative $\Bbbk$-algebra with product induced from the join operation in P.

Letting $\mathtt{H}$ denote the canonical basis,

$$\mathtt{H}_x \cdot \mathtt{H}_y := \mathtt{H}_{x \vee y}. \tag{5}$$

We use the symbol $\cdot$ to denote the product.

1.13 Semisimplicity and $\mathtt{Q}$-basis

Define the $\mathtt{Q}$-basis of $\Bbbk P$ by

$$\mathtt{H}_x = \sum_{y \colon y \geq x} \mathtt{Q}_y \qquad \text{or equivalently} \qquad \mathtt{Q}_x = \sum_{y \colon x \leq y} \mu(x, y)\, \mathtt{H}_y.$$

Here μ refers to the Möbius function of the lattice P. In particular,

$$\mathtt{H}_\perp = \sum_y \mathtt{Q}_y.$$

Theorem 1. *The linearization of a finite lattice is a split-semisimple commutative algebra. The primitive idempotents are given by elements of the* Q*-basis. In other words,*

$$\mathsf{Q}_x \cdot \mathsf{Q}_y = \begin{cases} \mathsf{Q}_x & \textit{if } x = y, \\ 0 & \textit{otherwise.} \end{cases} \tag{6}$$

Proof. An easy way to establish (6) is to assume it and deduce (5) from it. The required calculation is shown below.

$$\begin{aligned}
\mathtt{H}_x \cdot \mathtt{H}_y &= \Big(\sum_{z\colon z \geq x} \mathtt{Q}_z\Big) \cdot \Big(\sum_{w\colon w \geq y} \mathtt{Q}_w\Big) \\
&= \sum_{u\colon u \geq x \vee y} \mathtt{Q}_u \\
&= \mathtt{H}_{x \vee y}.
\end{aligned}$$

□

We also obtain

$$\mathtt{H}_y \cdot \mathtt{Q}_x = \begin{cases} \mathtt{Q}_x & \text{if } x \geq y, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

In particular, $\mathtt{H}_y \cdot \mathtt{Q}_\perp = 0$ for $y > \perp$.

Let us make this explicit.

$$\begin{aligned}
\mathtt{H}_y \cdot \mathtt{Q}_\perp &= \mathtt{H}_y \cdot \Big(\sum_w \mu(\perp, w)\,\mathtt{H}_w\Big) \\
&= \sum_w \mu(\perp, w)\,\mathtt{H}_{y\vee w} \\
&= \sum_{w'} \Big(\sum_{w\colon y\vee w = w'} \mu(\perp, w)\Big)\,\mathtt{H}_{w'}.
\end{aligned}$$

So each term in the parenthesis must be 0. This is exactly the Weisner formula (3) (and we have proved it again).

1.14 Summary

Given a finite lattice P:

How many zeta functions do we have?

$1.$

How many Möbius functions do we have?

$1.$

How many complete systems do we have?

$1.$

How many Q-bases do we have?

$1.$

1.15 Exercises

1. Let A be a $\mathbb{k}$-algebra. Let e be an element of A which satisfies $e^2 = \alpha e$ for some nonzero $\alpha \in \mathbb{k}$. (Such an e is called a quasi-idempotent.) Show that e/α is an idempotent element.

2. Let e be an idempotent in an algebra A. Suppose $e = f + g$ where f and g are orthogonal idempotents. Let M be any left A-module. Show that:

   - $eM = \{m \in M \mid e \cdot m = m\}$.
   - $ef = fe = f$ and $eg = ge = g$.
   - $eM = fM \oplus gM$.

   The above facts are relevant to Lemma 2 and its proof.

3. Show that any idempotent square matrix is diagonalizable. What are its eigenvalues and eigenspaces?

2 Left regular bands (Associative world)

2.1 Monoids

A monoid is a set Σ with a distinguished element $1 \in \Sigma$ (called the identity element) and a binary operation

$$\Sigma \times \Sigma \to \Sigma, \qquad (x, y) \mapsto xy,$$

which is associative and unital, that is,

$$(xy)z = x(yz), \quad \text{and} \quad x1 = 1x = x,$$

for all $x, y, z \in \Sigma$.

A monoid homomorphism is a function $f : \Sigma \to \Sigma'$ between monoids such that $f(1) = 1$ and $f(xy) = f(x)f(y)$ for all $x, y \in \Sigma$.

2.2 Left regular bands

A band or idempotent monoid is a monoid in which every element is idempotent: $x^2 = x$.

A left regular band is a monoid in which $xyx = xy$ for all x and y. By setting $y = 1$, we obtain $x^2 = x$. Thus, a left regular band is, in particular, a band.

2.3 Partial order

Let Σ be a finite left regular band (LRB). For $x, y \in \Sigma$, we say

$$x \leq y \text{ if } xy = y. \tag{8}$$

This defines a partial order on Σ. The necessary checks are done below. Note that $xy = y$ implies $yx = xyx = xy = y$.

- reflexivity, that is, $x \leq x$. This is because $x^2 = x$.
- transitivity, that is, $x \leq y$ and $y \leq z$ implies $x \leq z$. This is because $xz = x(yz) = (xy)z = yz = z$.
- antisymmetry, that is, $x \leq y$ and $y \leq x$ implies $x = y$. This is because $xy = yx = x = y$.

The unit element of Σ is the unique minimum element of this partial order. In general, there will be more than one maximal element. Note that

$$x \leq xy$$

for any $x, y \in \Sigma$.

Example 6. Any lattice P is a (commutative) monoid via

$$xy := x \vee y.$$

The unit element is $\perp$. Note that

$$xyx = x \vee y \vee x = x \vee y = xy.$$

Thus P is a LRB. The partial order (8) coincides with the partial order of P since

$$x \leq y \iff x \vee y = y.$$

For example, for the Boolean poset, the product is given by union, with the empty set as the unit element. The partial order is inclusion.

In view of this example: One may think of a LRB as a 'noncommutative lattice' in which the role of the join is played by the product operation.

Example 7. Fix a positive integer n. Let Σ denote the set of all n-tuples in which each entry is either 0, $+$ or $-$. Its cardinality is 3^n. For instance, for $n = 2$, the elements of Σ are

$$00,\ 0+,\ 0-,\ +0,\ -0,\ ++,\ +-,\ -+,\ --\,.$$

Let us denote the elements of Σ by capital letters such as F, G and H. Let $\epsilon_i(F)$ denotes the i-th entry in the tuple F. Given tuples F and G, define a tuple FG by the formula.

$$\epsilon_i(FG) := \begin{cases} \epsilon_i(F) & \text{if } \epsilon_i(F) \neq 0, \\ \epsilon_i(G) & \text{if } \epsilon_i(F) = 0. \end{cases} \tag{9}$$

For instance, for $n = 3$,

$$(0+0)(-00) = -+0,$$
$$(0+0)(+-0) = ++0.$$

This turns Σ into a LRB. The unit element is the tuples with all entries 0.

Observe that

(10)

$$F \leq G \iff \epsilon_i(F) = \epsilon_i(G) \text{ whenever } \epsilon_i(F) \neq 0.$$

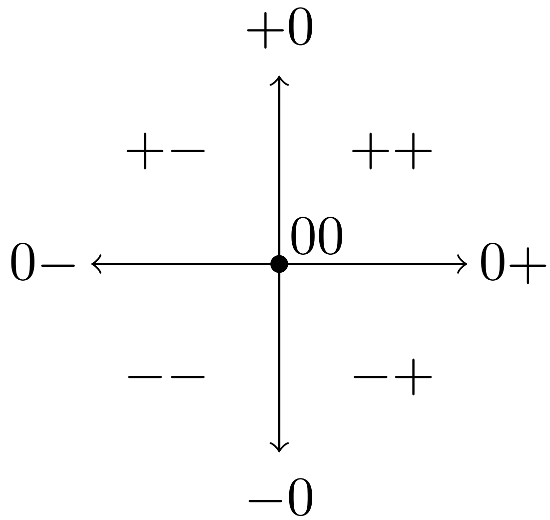
In other words, $F \leq G$ iff the tuple of F is obtained from that of G by replacing some $+$ and $-$ by 0. The maximal elements are those tuples which involve only $+$ and $-$ (and no 0).

For $n = 1$, the LRB Σ has three elements, namely, 1, $+$ and $-$, and the Hasse diagram is

$$\begin{tikzcd}[column sep=small] + & & - \\ & 0 \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

There is a nice way to visualize tuples in 0, $+$ or $-$. This is illustrated below for $n = 1$ and $n = 2$.

$$- \quad \longleftarrow\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\overset{0}{\bullet}-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!\longrightarrow \quad +$$



These picture can be vastly generalized as follows.

Example 8. Let $\mathcal{A}$ be a hyperplane arrangement, and let $\Sigma[\mathcal{A}]$ denote the set of faces. A face can be encoded as a tuple in 0, $+$ or $-$, with entries indexed by hyperplanes in $\mathcal{A}$. The formula (9) turns $\Sigma[\mathcal{A}]$ into a monoid. We call this the Tits monoid. Since the identity $FGF = FG$ holds, it is a left regular band. The partial order (8) is the usual inclusion of faces.

When $\mathcal{A}$ is the coordinate arrangement, we recover the previous example. Further, when $\mathcal{A}$ has rank 1, we recover the case $n = 1$.

Example 9. Fix a finite set S. We refer to elements of S as letters. Now define Σ as follows. Elements of Σ are words with no repetition of letters. Product xy of x and y is obtained by concatenating x and y, and removing those letters from y which have appeared in x. One can check that Σ is the free LRB on S.

Note that $x \leq y$ if x is an initial segment of y. The maximal elements are those words in which all letters appear exactly once.

For example, suppose $S = \{a, b\}$. The free LRB Σ on S equals $\{1, a, b, ab, ba\}$. The Hasse diagram of its partial order is given by

$$\begin{tikzcd}[arrows=dash]
ab & & ba \\
a \arrow[u] & & b \arrow[u] \\
& 1 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

2.4 Support map

Let Σ be a finite LRB. Define a relation $\preceq$ on Σ by

$$x \preceq y \text{ if } yx = y.$$

This is reflexive and transitive:

- $x \preceq x$. This is because $x^2 = x$.
- $x \preceq y$ and $y \preceq z$ implies $x \preceq z$. This is because $zx = (zy)x = z(yx) = zy = z$.

However, it is not necessarily antisymmetric. By identifying x and y whenever $x \preceq y$ and $y \preceq x$, we obtain a poset Π. We denote the quotient map by

$$\mathrm{s} : \Sigma \to \Pi.$$

We call it the support map. It is order-preserving. By construction,

$$\mathrm{s}(x) \leq \mathrm{s}(y) \iff yx = y$$

for any $x, y \in \Sigma$. It follows that

$$\mathrm{s}(x) = \mathrm{s}(y) \iff yx = y \text{ and } xy = x.$$

For any $x, y \in \Sigma$,

$$\mathrm{s}(xy) = \mathrm{s}(yx).$$

This is because $(xy)(yx) = xy$ and $(yx)(xy) = yx$.

Lemma 3. *The poset* Π *is a lattice, and*

$$\mathrm{s}(xy) = \mathrm{s}(x) \vee \mathrm{s}(y). \tag{11}$$

Proof. Since Π has a minimum element, it suffices to show that joins exist. Let $\mathrm{s}(x)$ and $\mathrm{s}(y)$ be any two elements in Π. Then

$$\mathrm{s}(x) \leq \mathrm{s}(xy) \quad \text{and} \quad \mathrm{s}(y) \leq \mathrm{s}(yx).$$

So $\mathrm{s}(xy)$ is an upper bound. Now suppose $\mathrm{s}(z)$ is also an upper bound. Then $zx = z$ and $zy = z$. Hence $z(xy) = (zx)y = zy = z$, so $\mathrm{s}(xy) \leq \mathrm{s}(z)$. Thus, $\mathrm{s}(xy)$ is the least upper bound as required. $\square$

We refer to Π as the support lattice of Σ.

Lemma 4. *An element $x \in \Sigma$ is maximal iff $\mathrm{s}(x)$ is maximal in Π.*

Proof. Exercise. $\square$

Example 10. When Σ is a lattice, with product given by the join, $\preceq$ is a partial order and coincides with the partial order of Σ. In this case, $\Pi = \Sigma$, and the support map is the identity.

For a hyperplane arrangement $\mathcal{A}$, the support lattice of the Tits monoid is the poset of flats of $\mathcal{A}$. You can check this for arrangement of 2 lines in the plane, and then for three lines in the plane.

What happens for the free LRB?

2.5 Notations and terminology

The following is motivated by the example of hyperplane arrangements.

Let Σ be a LRB with support lattice Π. We refer to an element of Σ as a face, and to an element of Π as a flat. We use capital letters such as A, F, G for faces, and X, Y, Z for flats. We may also refer to Σ as the poset of faces, and to Π as the poset of flats.

The minimum face is called the central face and denoted O. A maximal element of Σ is called a chamber. We denote a chamber by C, D or E. Let L denote the set of chambers. We use $\perp$ for the minimum flat, and $\top$ for the maximum flat. Thus, the support of O is $\perp$, while the support of any chamber is $\top$.

For a face F, let Σ_F denote the set of faces which are greater than F. This is the star of F. For clarity, we denote elements of Σ_F by K/F, where K is some face greater than F. Let L_F denote the set of chambers which are greater than F. This is the top-star of F.

Lemma 5. *Let F and G be faces with the same support, that is $FG = F$ and $GF = G$. Then we have bijections*

$$\Sigma_F \xrightarrow{\cong} \Sigma_G, \qquad K/F \mapsto GK/G$$

and

$$\mathrm{L}_F \xrightarrow{\cong} \mathrm{L}_G, \qquad C/F \mapsto GC/G.$$

2.6 Nested faces and lunes

A nested face is a pair (A, F) of faces such that $A \leq F$. Define an equivalence relation on the set of nested faces as follows.

$$(A, F) \sim (B, G) \iff AB = A,\ BA = B,\ AG = F,\ BF = G. \tag{12}$$

That is, A and B have the same support, and F/A and G/B correspond to each other under the bijection of the previous lemma.

An equivalence class is called a lune.

Lattice example: Nothing happens.

2.7 Reduced incidence algebra

Let P be a poset and $\sim$ be an equivalence relation on the set $\{(x, y) \mid x \leq y\}$. Let $I_\sim(P)$ denote the subspace of $I(P)$ consisting of those functions f such that $f(x, y) = f(u, v)$ whenever $(x, y) \sim (u, v)$. This subspace is not a subalgebra of $I(P)$ in general. If it is, then we say that the equivalence relation $\sim$ is order-compatible, and refer to $I_\sim(P)$ as the reduced incidence algebra associated to $\sim$.

2.8 Noncommutative theory begins

Let Σ be a LRB.

Lemma 6. *The equivalence relation* (12) *is order-compatible, and hence yields a reduced incidence algebra $I_\sim(\Sigma)$. It has a basis indexed by lunes.*

Proof. Let $f, g \in I_\sim(\Sigma)$. We show that $fg \in I_\sim(\Sigma)$. For that: Suppose $(A, F) \sim (B, G)$. Then

$$\begin{aligned}(fg)(A, F) &= \sum_{H\colon A \leq H \leq F} f(A, H) g(H, F) \\ &= \sum_{H'\colon B \leq H' \leq G} f(B, H') g(H', G) \\ &= (fg)(B, G).\end{aligned}$$

Thus, $fg \in I_\sim(\Sigma)$ as required. $\square$

Let $I(\Pi)$ denote the incidence algebra of the poset of flats. Consider the map

$$\mathbf{s} : I_{\sim}(\Sigma) \to I(\Pi)$$

given by

$$\mathbf{s}(f)(\mathrm{X}, \mathrm{Y}) = \sum_{F : \, \mathrm{s}(F/A) = \mathrm{Y}/\mathrm{X}} f(A, F), \tag{13}$$

where A is a fixed face of support X. The notation $\mathrm{s}(F/A) = \mathrm{Y}/\mathrm{X}$ means that $\mathrm{X} \leq \mathrm{Y}$, $A \leq F$, $\mathrm{s}(A) = \mathrm{X}$ and $\mathrm{s}(F) = \mathrm{Y}$.

We call $\mathbf{s}$ the support map. It is an algebra homomorphism, that is,

$$\mathbf{s}(fg) = \mathbf{s}(f)\,\mathbf{s}(g).$$

Lattice example: Nothing happens. Support map is the identity.

2.9 Noncommutative zeta functions

A noncommutative zeta function is an element $\zeta \in I_{\sim}(\Sigma)$ such that $\zeta(A, A) = 1$ for all A and

$$\zeta(H, G) = \sum_{\substack{F\colon F \geq A, HF = G,\\ \mathrm{s}(F) = \mathrm{s}(G)}} \zeta(A, F) \tag{14}$$

for all $A \leq H \leq G$.

We refer to (14) as the lune-additivity formula.

Putting $A = O$ in the lune-additivity formula, we obtain:

$$\zeta(H, G) = \sum_{\substack{F\colon HF = G,\\ \mathrm{s}(F) = \mathrm{s}(G)}} \zeta(O, F) \tag{15}$$

for all $H \leq G$. Thus, ζ is completely determined by the scalars $\zeta(O, F)$, as F varies over all faces. How arbitrary can these scalars be?

Putting $H = G$ in (15), we obtain:

$$1 = \zeta(G, G) = \sum_{F:\, \mathrm{s}(F)=\mathrm{s}(G)} \zeta(O, F).$$

Equivalently, for any flat X,

$$\sum_{F:\, \mathrm{s}(F)=\mathrm{X}} \zeta(O, F) = 1.$$

Lemma 7. *A noncommutative zeta function ζ is the same as a family of scalars $(\mathtt{u}^F)$ indexed by faces F which satisfy*

$$\sum_{F:\, \mathrm{s}(F)=\mathrm{X}} \mathtt{u}^F = 1 \tag{16}$$

for each flat X.

Proof. Given ζ, put $\mathtt{u}^F := \zeta(O, F)$. Conversely, given $(\mathtt{u}^F)$, define

$$\zeta(H, G) := \sum_{\substack{F:\, HF=G,\\ \mathrm{s}(F)=\mathrm{s}(G)}} \mathtt{u}^F.$$

We need to check that $\boldsymbol{\zeta}$ belongs to $I_{\sim}(\Sigma)$. That is, $(H,G) \sim (H',G')$ implies $\boldsymbol{\zeta}(H,G) = \boldsymbol{\zeta}(H',G')$, with $\sim$ as in (12), This holds because the indexing faces F are the same in both cases.

Let us now check the lune-additivity formula. Fix $A \leq H \leq G$.

$$
\begin{aligned}
rhs &= \sum_{\substack{F:\, F \geq A, HF=G,\\ \mathrm{s}(F)=\mathrm{s}(G)}} \boldsymbol{\zeta}(A,F) \\
&= \sum_{\substack{F:\, F \geq A, HF=G,\\ \mathrm{s}(F)=\mathrm{s}(G)}} \sum_{\substack{K:\, AK=F,\\ \mathrm{s}(K)=\mathrm{s}(F)}} \mathtt{u}^K \\
&= \sum_{\substack{K:\, HK=G,\\ \mathrm{s}(K)=\mathrm{s}(G)}} \mathtt{u}^K \\
&= \boldsymbol{\zeta}(H,G) \\
&= lhs. \qquad \square
\end{aligned}
$$

Lemma 8. *The support of any noncommutative zeta function $\boldsymbol{\zeta}$ is the zeta function $\zeta \in I(\Pi)$.*

Proof. Fix $\mathrm{X} \leq \mathrm{Y}$, and a face A of support X. Let G be any face greater than A with support Y. Then

$$\begin{aligned} \mathrm{s}(\boldsymbol{\zeta})(\mathrm{X}, \mathrm{Y}) &= \sum_{F \colon F \geq A, \mathrm{s}(F) = \mathrm{Y}} \boldsymbol{\zeta}(A, F) \\ &= \boldsymbol{\zeta}(G, G) \\ &= 1. \end{aligned}$$

For the second step, we put $H = G$ in (14). $\square$

2.10 Noncommutative Möbius functions

A noncommutative Möbius function is an element $\boldsymbol{\mu} \in I_{\sim}(\Sigma)$ such that $\boldsymbol{\mu}(A, A) = 1$ for all A and

(17) $$\sum_{F\colon F \geq A, HF=G} \boldsymbol{\mu}(A, F) = 0$$

for all $A < H \leq G$.

We refer to (17) as the noncommutative Weisner formula.

Lemma 9. *The support of any noncommutative Möbius function $\boldsymbol{\mu}$ is the Möbius function $\mu \in I(\Pi)$.*

Proof. We check that $\mathrm{s}(\boldsymbol{\mu})$ satisfies the Weisner formula (3), that is, for $\mathrm{Z} < \mathrm{Y} \leq \mathrm{W}$,

$$\sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Z}, \mathrm{Y} \vee \mathrm{X} = \mathrm{W}} \mathrm{s}(\boldsymbol{\mu})(\mathrm{Z}, \mathrm{X}) = 0.$$

Fix a face A of support Z, and face H greater than A

of support Y. Then

$$
\begin{aligned}
lhs &= \sum_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Z},\mathrm{Y}\vee\mathrm{X}=\mathrm{W}} \mathbf{s}(\boldsymbol{\mu})(\mathrm{Z},\mathrm{X}) \\
&= \sum_{F\colon F\geq A,\mathrm{Y}\vee\mathrm{s}(F)=\mathrm{W}} \boldsymbol{\mu}(A,F) \\
&= \sum_{F\colon F\geq A,\mathrm{s}(HF)=\mathrm{W}} \boldsymbol{\mu}(A,F) \\
&= \sum_{G\colon \mathrm{s}(G)=\mathrm{W}} \sum_{F\colon F\geq A,HF=G} \boldsymbol{\mu}(A,F) \\
&= 0 \\
&= rhs. \qquad \square
\end{aligned}
$$

2.11 The main result

Theorem 2. *Let Σ be a left regular band. Noncommutative zeta functions and noncommutative Möbius functions correspond to each other under taking inverses in the reduced incidence algebra $I_{\sim}(\Sigma)$.*

In other words, if $\boldsymbol{\zeta}$ is a noncommutative zeta function, then $\boldsymbol{\zeta}^{-1}$ is a noncommutative Möbius function. Conversely, if $\boldsymbol{\mu}$ is a noncommutative Möbius function, then $\boldsymbol{\mu}^{-1}$ is a noncommutative zeta function.

Explicitly, to say $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ are inverses means that for $H < G$,

$$\sum_{F\colon H \leq F \leq G} \boldsymbol{\zeta}(H,F)\boldsymbol{\mu}(F,G) = 0 = \sum_{F\colon H \leq F \leq G} \boldsymbol{\mu}(H,F)\boldsymbol{\zeta}(F,G).$$

In the lattice example, we have only one noncommutative zeta function, and one noncommutative Möbius function, and they are inverses.

Remark 1. Both the lune-additivity as well as the noncommutative Weisner formula refer to the monoid structure as well as the poset structure of Σ.

Example 11. Let Σ be the LRB of the rank-one arrangement. Its elements are O, C and $\overline{C}$. This is the same as the $n = 1$ case of the n-tuple example. In that notation, $O = 0$, $C = +$ and $\overline{C} = -$.

Fix an arbitrary scalar p. A noncommutative zeta function $\boldsymbol{\zeta}$ is characterized by

$$\boldsymbol{\zeta}(O, O) = \boldsymbol{\zeta}(C, C) = \boldsymbol{\zeta}(\overline{C}, \overline{C}) = 1,$$
$$\boldsymbol{\zeta}(O, C) = p, \quad \boldsymbol{\zeta}(O, \overline{C}) = 1 - p.$$

The lune-additivity formula says

$$\boldsymbol{\zeta}(O, C) + \boldsymbol{\zeta}(O, \overline{C}) = 1.$$

To find the inverse, we solve the equations

$$\boldsymbol{\zeta}(O, O)\boldsymbol{\mu}(O, C) + \boldsymbol{\zeta}(O, C)\boldsymbol{\mu}(C, C) = 0,$$
$$\boldsymbol{\zeta}(O, O)\boldsymbol{\mu}(O, \overline{C}) + \boldsymbol{\zeta}(O, \overline{C})\boldsymbol{\mu}(\overline{C}, \overline{C}) = 0.$$

We conclude that a noncommutative Möbius function

$\boldsymbol{\mu}$ is characterized by

$$\boldsymbol{\mu}(O,O) = \boldsymbol{\mu}(C,C) = \boldsymbol{\mu}(\overline{C},\overline{C}) = 1,$$

$$\boldsymbol{\mu}(O,C) = -p, \quad \boldsymbol{\mu}(O,\overline{C}) = p-1.$$

The noncommutative Weisner formula says

$$\boldsymbol{\mu}(O,O) + \boldsymbol{\mu}(O,C) + \boldsymbol{\mu}(O,\overline{C}) = 0.$$

2.12 Algebra of a left regular band

Let Σ be a LRB. Let $\mathsf{\Sigma}$ denote its linearization, with canonical basis $\mathtt{H}$. It is an algebra:

$$\mathtt{H}_F \cdot \mathtt{H}_G := \mathtt{H}_{FG}.$$

Let

$$\mathrm{s} : \mathsf{\Sigma} \twoheadrightarrow \mathsf{\Pi}$$

be the linearization of the support map. This is a morphism of algebras by (11), that is,

$$\mathrm{s}(x \cdot y) = \mathrm{s}(x) \cdot \mathrm{s}(y). \tag{18}$$

We know that $\mathsf{\Pi}$ is a split-semisimple commutative algebra. What about $\mathsf{\Sigma}$?

2.13 $\mathtt{Q}$-basis

We assume the main theorem and proceed.

Define the $\mathtt{Q}$-basis of Σ by

$$\mathtt{H}_F = \sum_{K\colon F \leq K} \boldsymbol{\zeta}(F,K)\, \mathtt{Q}_K$$

or equivalently,

$$\mathtt{Q}_F = \sum_{G\colon F \leq G} \boldsymbol{\mu}(F,G)\, \mathtt{H}_G.$$

Note that the $\mathtt{Q}$-basis depends on the choice of $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$.

Lemma 10. *We have*

$$\mathrm{s}(\mathtt{Q}_F) = \mathtt{Q}_{\mathrm{s}(F)},$$

where the latter is the $\mathtt{Q}$*-basis element of* Π*.*

Proof. This is a consequence of $\mathbf{s}(\boldsymbol{\mu}) = \mu$. The required calculation is done below.

$$\begin{aligned}
\mathrm{s}(\mathtt{Q}_F) &= \sum_{G\colon F \leq G} \boldsymbol{\mu}(F,G)\ \mathrm{s}(\mathtt{H}_G) \\
&= \sum_{G\colon F \leq G} \boldsymbol{\mu}(F,G)\, \mathtt{H}_{\mathrm{s}(G)} \\
&= \sum_{\mathrm{Y}} \sum_{G\colon F \leq G, \mathrm{s}(G) = \mathrm{Y}} \boldsymbol{\mu}(F,G)\, \mathtt{H}_{\mathrm{Y}} \\
&= \sum_{\mathrm{Y}} \mu(\mathrm{s}(F), \mathrm{Y})\, \mathtt{H}_{\mathrm{Y}} \\
&= \mathtt{Q}_{\mathrm{s}(F)}.
\end{aligned}$$

□

How do the Q-basis elements multiply? Are they mutually orthogonal primitive idempotents?

Example 12. Let us go back to the rank-one arrangement. There is one Q-basis for each scalar p. It is given by

$$\mathtt{Q}_C = \mathtt{H}_C, \quad \mathtt{Q}_{\overline{C}} = \mathtt{H}_{\overline{C}}, \quad \mathtt{Q}_O = \mathtt{H}_O - p\,\mathtt{H}_C - (1-p)\,\mathtt{H}_{\overline{C}}.$$

Let us understand orthogonality among these elements.

- $\mathtt{Q}_C$ and $\mathtt{Q}_{\overline{C}}$ are not orthogonal to each other.
- $\mathtt{Q}_C$ is orthogonal to $\mathtt{Q}_O$ when $p = 1$.
- $\mathtt{Q}_{\overline{C}}$ is orthogonal to $\mathtt{Q}_O$ when $p = 0$.

An appropriate linear combination of $\mathtt{Q}_C$ and $\mathtt{Q}_{\overline{C}}$ is orthogonal to $\mathtt{Q}_O$ for all p. Details below.

Define

$$\mathtt{E}_\top = p\,\mathtt{Q}_C + (1-p)\,\mathtt{Q}_{\overline{C}} \quad \text{and} \quad \mathtt{E}_\perp = \mathtt{Q}_O.$$

In the H-basis,

$$\mathtt{E}_\top = p\,\mathtt{H}_C + (1-p)\,\mathtt{H}_{\overline{C}} \quad \text{and} \quad \mathtt{E}_\perp = \mathtt{H}_O - p\,\mathtt{H}_C - (1-p)\,\mathtt{H}_{\overline{C}}.$$

Then $\mathtt{E} = \{\mathtt{E}_\perp, \mathtt{E}_\top\}$ is a complete system of primitive orthogonal idempotents of Σ. We call $\mathtt{E}$ an the Eulerian family. Note that it does not give a basis for Σ. This is expected since Σ is not commutative. The element $\mathtt{H}_C - \mathtt{H}_{\overline{C}}$ is nilpotent of order 2 and accounts for the missing dimension.

This algebra is isomorphic to the algebra of upper triangular 2 by 2 matrices (assuming $\mathbb{k}$ does not have characteristic 2). An explicit isomorphism is

$$\mathtt{H}_O \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathtt{H}_C \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \qquad \mathtt{H}_{\overline{C}} \mapsto \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$

Lemma 11. *When F and G have the same support, we have $\mathtt{H}_F \cdot \mathtt{Q}_G = \mathtt{Q}_F$.*

Proof. Use that $\boldsymbol{\mu} \in I_\sim(\Sigma)$ and hence $\boldsymbol{\mu}(G, K) = \boldsymbol{\mu}(F, FK)$ for any $K \geq G$. □

Lemma 12. *When $H > A$, we have $\mathtt{H}_H \cdot \mathtt{Q}_A = 0$.*

Proof.

$$\begin{aligned}
\mathtt{H}_H \cdot \mathtt{Q}_A &= \mathtt{H}_H \cdot \Big(\sum_{F \colon F \geq A} \boldsymbol{\mu}(A, F)\, \mathtt{H}_F \Big) \\
&= \sum_{F \colon F \geq A} \boldsymbol{\mu}(A, F)\, \mathtt{H}_{HF} \\
&= \sum_G \Big(\sum_{F \colon F \geq A, HF = G} \boldsymbol{\mu}(A, F) \Big) \mathtt{H}_G \\
&= 0.
\end{aligned}$$

The last step used the noncommutative Weisner formula. □

Lemma 13. *For any faces F and G,*

$$\mathtt{H}_F \cdot \mathtt{Q}_G = \begin{cases} \mathtt{Q}_{FG} & \text{if } GF = G, \\ 0 & \text{if } GF > G. \end{cases} \tag{19}$$

Proof. We employ the previous two lemmas and the following calculation.

$$\mathtt{H}_F \cdot \mathtt{Q}_G = \mathtt{H}_F \cdot \mathtt{H}_G \cdot \mathtt{Q}_G = \mathtt{H}_{FG} \cdot \mathtt{Q}_G = \mathtt{H}_{FG} \cdot \mathtt{H}_{GF} \cdot \mathtt{Q}_G.$$

$\square$

In particular, $\mathtt{H}_F \cdot \mathtt{Q}_O = 0$ for $F > O$.

Compare (19) with (7).

Proof of main theorem in one direction. Suppose $\boldsymbol{\mu}$ is a noncommutative Möbius function. Define the $\mathtt{Q}$-basis of Σ by

$$\mathtt{Q}_F := \sum_{G\colon F\leq G} \boldsymbol{\mu}(F,G)\,\mathtt{H}_G.$$

Then

$$\mathtt{H}_F \cdot \mathtt{Q}_G = \begin{cases} \mathtt{Q}_{FG} & \text{if } GF = G, \\ 0 & \text{if } GF > G. \end{cases}$$

The first case uses $\boldsymbol{\mu} \in I_{\sim}(\Sigma)$, while the second uses the noncommutative Weisner formula. Now consider

$$\mathtt{H}_A = \sum_{F\colon F\geq A} \boldsymbol{\mu}^{-1}(A,F)\mathtt{Q}_F.$$

Multiplying on the left by $\mathtt{H}_H$ for $H \geq A$, we obtain

$$\mathtt{H}_H = \sum_{F\colon F\geq A} \boldsymbol{\mu}^{-1}(A,F)\mathtt{H}_H \cdot \mathtt{Q}_F.$$

Let us manipulate the rhs.

$$
\begin{aligned}
\mathtt{H}_H &= \sum_{F\colon F \geq A, FH=F} \boldsymbol{\mu}^{-1}(A,F)\mathtt{Q}_{HF} \\
&= \sum_{G} \Big(\sum_{F\colon F \geq A, HF=G, \mathrm{s}(F)=\mathrm{s}(G)} \boldsymbol{\mu}^{-1}(A,F) \Big) \mathtt{Q}_G.
\end{aligned}
$$

Thus, we obtain

$$
\boldsymbol{\mu}^{-1}(H,G) = \sum_{F\colon F \geq A, HF=G, \mathrm{s}(F)=\mathrm{s}(G)} \boldsymbol{\mu}^{-1}(A,F).
$$

We conclude that $\boldsymbol{\mu}^{-1}$ is a noncommutative zeta function as required.

$\square$

Proof of main theorem in other direction. Suppose $\boldsymbol{\zeta}$ is a noncommutative zeta function. Define the $\mathtt{Q}$-basis of Σ by

$$\mathtt{H}_F = \sum_{K:\,F \le K} \boldsymbol{\zeta}(F,K)\,\mathtt{Q}_K.$$

We claim that

$$\mathtt{H}_F \cdot \mathtt{Q}_G = \begin{cases} \mathtt{Q}_{FG} & \text{if } GF = G, \\ 0 & \text{if } GF > G. \end{cases}$$

To prove this, we assume it to deduce $\mathtt{H}_F \cdot \mathtt{H}_G = \mathtt{H}_{FG}$.

$$\begin{aligned} \mathtt{H}_F \cdot \mathtt{H}_G &= \mathtt{H}_F \cdot \Big(\sum_{K:\,G \le K} \boldsymbol{\zeta}(G,K)\,\mathtt{Q}_K \Big) \\ &= \sum_{K:\,G \le K} \boldsymbol{\zeta}(G,K)\,\mathtt{H}_F \cdot \mathtt{Q}_K \\ &= \sum_{K:\,G \le K, KF = K} \boldsymbol{\zeta}(G,K)\,\mathtt{Q}_{FK} \\ &= \sum_{H:\,H \ge FG} \Big(\sum_{K:\,G \le K, KF=K, FK=H} \boldsymbol{\zeta}(G,K) \Big) \mathtt{Q}_H \end{aligned}$$

To finish the argument, we need to show that the sum in parenthesis is $\boldsymbol{\zeta}(FG, H)$. In the following calculation, F, G and H are fixed.

$$\sum_{\substack{K:\, G \le K, FK = H, \\ KF = K}} \boldsymbol{\zeta}(G, K) = \sum_{\substack{K:\, G \le K, GFK = GH, \\ \mathrm{s}(K) = \mathrm{s}(GH)}} \boldsymbol{\zeta}(G, K)$$
$$= \boldsymbol{\zeta}(GF, GH)$$
$$= \boldsymbol{\zeta}(FG, H).$$

The last equality uses $\boldsymbol{\zeta} \in I_{\sim}(\Sigma)$, while the second uses the lune-additivity formula.

This proves the claim. In particular, $\mathtt{H}_F \cdot \mathtt{Q}_G = 0$ for $F > G$. Now plug

$$\mathtt{Q}_G = \sum_{K:\, G \le K} \boldsymbol{\zeta}^{-1}(G, K)\, \mathtt{H}_K$$

into this formula to get that $\boldsymbol{\zeta}^{-1}$ satisfies the noncommutative Weisner formula. Thus, $\boldsymbol{\zeta}^{-1}$ is a noncommutative Möbius function. $\square$

2.14 Complete systems

Let Σ be a LRB and Σ be its linearization.

How many primitive orthogonal idempotents can we find in Σ?

Proposition 3. *The kernel of the support map $\Sigma \to \Pi$ is the largest niloptent ideal of Σ. It consists precisely of all nilpotent elements.*

By general theory of algebras, we conclude that any complete system of primitive orthogonal idempotents of Σ has cardinality equal to the number of flats.

Let ζ be a noncommutative zeta function. Define the $\mathtt{Q}$-basis of Σ by

$$\mathtt{H}_F = \sum_{K\colon F \le K} \zeta(F, K)\, \mathtt{Q}_K.$$

For each flat X, put

$$\mathtt{E}_{\mathrm{X}} := \sum_{F\colon \mathrm{s}(F) = \mathrm{X}} \zeta(O, F)\, \mathtt{Q}_F.$$

Theorem 3. *The family $\mathtt{E}$ is a complete system of primitive orthogonal idempotents of Σ.*

Conversely, every complete system arises from a unique noncommutative zeta function.

We omit the proofs.

The elements $\mathtt{E}_{\mathrm{X}}$ are called Eulerian idempotents. The idempotent corresponding to the minimum flat, namely $\mathtt{E}_{\perp}$ is called the first Eulerian idempotent. Note that

$$\mathtt{E}_{\perp} = \mathtt{Q}_O.$$

Summary:

Given a finite left regular band Σ:

How many zeta functions do we have?

How many Möbius functions do we have?

How many complete systems do we have?

How many Q-bases do we have?

Answer:

Such entities exist and they are equinumerous. More precisely, these sets can be viewed as an affine space of dimension equal to the number of faces minus the number of flats.

3 Problems

Exercise 3.1. *Starting with* (5)*, use the Weisner formula* (3) *to first prove* (7) *and then deduce* (6) *from it.*

Exercise 3.2. *Let* $a = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ *be a formal power series. Whenever* $a_0 = 0$*, let* e^a *denote the formal power series obtained by substituting* a *for* x *in*

$$e^x = \sum_{n\geq 0} \frac{x^n}{n!}.$$

Let $\Pi[I]$ *denote the poset of partitions of the set* I*. A formal power series* a *gives rise to a function* P_a *on* $\Pi[I]$ *for every finite set* I *via*

$$P_a(\mathrm{X}) := \prod_{B\in \mathrm{X}} a_{|B|}.$$

Show that P_{e^a} *is the exponential of* P_a*, that is,*

$$P_{e^a}(\mathrm{X}) = \sum_{\mathrm{Y}:\, \mathrm{Y}\geq \mathrm{X}} P_a(\mathrm{Y}).$$

(First do the case when X is the minimum set partition.)

Exercise 3.3. *Pick any noncommutative LRB. Describe its zeta functions, Möbius functions, complete systems and $\mathtt{Q}$-bases as explicitly as you can.*

Exercise 3.4. *Let Σ be the poset of faces of an arrangement, and let $\mathsf{\Sigma}$ denote its linearization. It is a commutative algebra under the product*

$$\mathtt{H}_F \cdot \mathtt{H}_G := \begin{cases} \mathtt{H}_{FG} & \textit{if } FG = GF, \\ 0 & \textit{otherwise.} \end{cases}$$

Is $\mathsf{\Sigma}$ split-semisimple?