

PH423 Assignment 2

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Question 1.

[Sankalp: I got this one.]

- (a) Calculate the expectation values of \hat{J}_x , \hat{J}_y , \hat{J}_x^2 and \hat{J}_y^2 in the angular momentum states $|j, m\rangle$. Explain the result geometrically. (Using symmetry arguments may help).

We start with the expansion of the operators \hat{J}_x and \hat{J}_y in terms of the ladder operators

$$\hat{J}_x = \frac{1}{2} \cdot (\hat{J}_+ + \hat{J}_-) \quad (1)$$

and

$$\hat{J}_y = \frac{1}{2i} \cdot (\hat{J}_+ - \hat{J}_-) . \quad (2)$$

The application of the ladder operators on a state $|j, m\rangle$ changes it to a state of the form $c \cdot |j, m \pm 1\rangle$ for some $c \in \mathbb{C}$. So, given the orthogonality of the $|j, m\rangle$ states, we get that

$$\langle j, m | \hat{J}_x | j, m \rangle = \langle j, m | \hat{J}_y | j, m \rangle = 0 \quad \forall |j, m\rangle . \quad (3)$$

Squaring [Equation 1](#) and [2](#), we get the operators \hat{J}_x^2 and \hat{J}_y^2 in terms of the ladder operators. With the same argument as before, we see that only terms with equal powers of the two ladder operators will contribute, and using

$$\hat{J}_{\pm} |j, m\rangle = \hbar \sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1\rangle , \quad (4)$$

we get

$$\langle j, m | \hat{J}_y^2 | j, m \rangle = \langle j, m | \hat{J}_x^2 | j, m \rangle \quad (5)$$

$$= \langle j, m | \frac{1}{4} \cdot (\hat{J}_+^2 + \hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+ + \hat{J}_-^2) | j, m \rangle \quad (6)$$

$$= \langle j, m | \frac{1}{4} \cdot (\hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+) | j, m \rangle \quad (7)$$

$$= \langle j, m | \frac{\hbar^2}{4} \cdot \left(\sqrt{(j+m+1)(j-m)} \sqrt{(j-m)(j+m+1)} + \sqrt{(j-m)(j+m+1)} \sqrt{(j+m+1)(j-m)} \right) \cdot |j, m\rangle \quad (8)$$

$$= \frac{\hbar^2}{2} (j+m+1)(j-m) . \quad (9)$$

16 The values for x and y are not separately calculated as a trivial calculation shows they're equal.
 17 The same is easily argued using symmetry in the x-y plane. This symmetry also serves as an
 18 explanation for the expectation value, since there is similarly a reflection symmetry about either
 19 axis, the expectation cannot favor either $\pm x$ or $\pm y$.

20 (b) Can the angular momentum $\hat{\mathbf{J}}$ be oriented entirely along the z(or x or y) axis? Give reasons in
 21 either case.

22 No. Kill me now.

23 **2. Determine the eigenvalues and eigenvectors of the 2×2 matrix $\sigma \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector along
 the (θ, ϕ) direction and σ are the three Pauli matrices. This is basically the projection of the spin
 1/2 operator (apart from $\frac{\hbar}{2}$) along the direction of the unit vector $\hat{\mathbf{n}}$. Do this in two ways:**

24

25 [Parth: Doing question 2, might have issues with part (b) make sure that it's correct]

26 (a) First by explicitly diagonalizing the matrix $\sigma \cdot \hat{\mathbf{n}}$.

27 The vector $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, where the σ_i matrices are -

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now we need to figure out what $\hat{\mathbf{n}}$ is. The unit vector points along the (θ, ϕ) direction. This is nothing but the unit vector $\hat{\mathbf{r}}$ in Polar co-ordinates.

$$\hat{\mathbf{n}} = \hat{\mathbf{r}} = \cos(\phi)\sin(\theta)\hat{\mathbf{i}} + \sin(\phi)\sin(\theta)\hat{\mathbf{j}} + \cos(\theta)\hat{\mathbf{k}}$$

Thus, $\hat{\mathbf{n}} = (\cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta), \cos(\theta))$. We know that $\mathbf{a} \cdot \mathbf{b} = a_i b_i$ (implicit summation over i)

Thus, $\sigma \cdot \hat{\mathbf{n}} = \sigma_i n_i$.

$$\begin{aligned} \sigma \cdot \hat{\mathbf{n}} &= \cos(\phi)\sin(\theta) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin(\phi)\sin(\theta) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos(\theta) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \therefore \sigma \cdot \hat{\mathbf{n}} &= \sin(\theta) \begin{pmatrix} 0 & \cos(\phi) - i \sin(\phi) \\ \cos(\phi) + i \sin(\phi) & 0 \end{pmatrix} + \cos(\theta) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \sin(\theta) \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix} + \cos(\theta) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos(\theta) \end{pmatrix} \end{aligned}$$

To find the eigenvalues and eigenvectors, we now need to diagonalize this matrix. Let the eigenvalues be represented by λ . The characteristic polynomial takes the following form.

$$\begin{aligned} (\cos(\theta) - \lambda)(-\cos(\theta) - \lambda) - \sin(\theta)e^{-i\phi} * \sin(\theta)e^{i\phi} &= 0 \\ \therefore -\cos^2(\theta) + \lambda^2 - \sin^2(\theta) &= 0 \Rightarrow \lambda^2 - 1 = 0 \\ \therefore \lambda &= \pm 1 \end{aligned}$$

28 for $\lambda = 1$, let the eigenvector be $\mathbf{v}_1 = (v_{1,1}, v_{1,2})$, thus

$$\begin{aligned} \begin{pmatrix} \cos(\theta) & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos(\theta) \end{pmatrix} \begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix} &= \begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix} \\ \therefore \cos(\theta) * v_{1,1} + \sin(\theta)e^{-i\phi} * v_{1,2} &= v_{1,1}, \quad \sin(\theta)e^{i\phi} * v_{1,1} - \cos(\theta) * v_{1,2} = v_{1,2} \\ v_{1,2} &= e^{i\phi} \frac{\sin(\theta)}{(\cos(\theta) + 1)} * v_{1,1} \end{aligned}$$

29 Thus, for eigenvalue $\lambda = 1$, the eigenvector $\mathbf{v}_1 = (v_{1,1}, e^{i\phi} \frac{\sin(\theta)}{(\cos(\theta)+1)} * v_{1,1})$

30 Likewise, for $\lambda = -1$, let the eigenvector be $\mathbf{v}_2 = (v_{2,1}, v_{2,2})$, thus

$$\begin{aligned} \begin{pmatrix} \cos(\theta) & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos(\theta) \end{pmatrix} \begin{pmatrix} v_{2,1} \\ v_{2,2} \end{pmatrix} &= \begin{pmatrix} -v_{2,1} \\ -v_{2,2} \end{pmatrix} \\ \therefore \cos(\theta) * v_{2,1} + \sin(\theta)e^{-i\phi} * v_{2,2} &= -v_{2,1}, \quad \sin(\theta)e^{i\phi} * v_{2,1} - \cos(\theta) * v_{2,2} = -v_{2,2} \\ v_{2,2} &= e^{i\phi} \frac{\sin(\theta)}{(1 - \cos(\theta))} * v_{2,1} \end{aligned}$$

31 Thus, for eigenvalue $\lambda = -1$, the eigenvector $\mathbf{v}_2 = (v_{2,1}, e^{i\phi} \frac{\sin(\theta)}{(1-\cos(\theta))} * v_{2,1})$.

32 We thus have our two eigenvalues (± 1) and our two eigenvectors (\mathbf{v}_1 and \mathbf{v}_2)

33
34 **(b)** By rotating the spinor pointing initially along the $+\hat{z}$ axis direction by appropriate angles, using the
35 appropriate rotation operator. Convince yourself that one has to rotate by an angle θ counterclock-
36 wise around the y -axis and then by ϕ around the z -axis. Apart from overall phases, is the resultant
37 spinor the same as the spin up eigenvector obtained in part **(a)**?

38 Let's start with the spinor pointing in the $+\hat{z}$ -direction.

$$\left| s_z = +\frac{\hbar}{2} \right\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad s.t. \quad S_z \left| s_z = +\frac{\hbar}{2} \right\rangle = +\frac{\hbar}{2} \left| s_z = +\frac{\hbar}{2} \right\rangle$$

39 If we apply consecutive rotation operators, we should be able to rotate this spinor into a general
40 state, pointing in an arbitrary direction $\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ points in the (θ, ϕ) direction.

41 We first rotate this spinor by θ around the y -axis, and then by ϕ around the z -axis. The axis of spin
42 now points in the direction $\hat{\mathbf{n}}$. Thus -

$$|\hat{n}+\rangle = U[R(\phi\hat{z})]U[R(\theta\hat{y})] \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

43 To find the explicit form of $|\hat{n}+\rangle$, we'll need the forms of the unitary matrices $U[R(\phi\hat{z})]$ and
44 $U[R(\theta\hat{y})]$. We'll use the result given in Shankar -

$$U[R(\theta)] = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}(\hat{\theta} \cdot \boldsymbol{\sigma})$$

45 Looking at the particular case of rotation around y -axis by amount θ and then subsequently around
46 z -axis by amount ϕ -

$$\begin{aligned} U[R(\theta\hat{y})] \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \left[\cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}\sigma_y \right] \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos\frac{\theta}{2} \\ 0 \end{bmatrix} - i\sin\frac{\theta}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix} \end{aligned}$$

Applying rotation around z -axis by amount ϕ now, we get

$$\begin{aligned} U[R(\phi\hat{z})] \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix} &= \left[\cos\frac{\phi}{2}I - i\sin\frac{\phi}{2}\sigma_z \right] \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix} \\ &= \begin{bmatrix} \cos\frac{\phi}{2}\cos\frac{\theta}{2} \\ \cos\frac{\phi}{2}\sin\frac{\theta}{2} \end{bmatrix} - i\sin\frac{\phi}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix} \\ &= \begin{bmatrix} \cos\frac{\theta}{2} \left(\cos\frac{\phi}{2} - i\sin\frac{\phi}{2} \right) \\ \sin\frac{\theta}{2} \left(\cos\frac{\phi}{2} + i\sin\frac{\phi}{2} \right) \end{bmatrix} \\ &= \begin{bmatrix} \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}} \end{bmatrix} \end{aligned}$$

47 This gives us a spinor $s_n = (s_{n1}, s_{n2}) = (\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}, \sin\frac{\theta}{2}e^{i\frac{\phi}{2}})$. If we recall our $\mathbf{v}_1 = (v_{1,1}, v_{1,2})$ from
48 part (a), we recall the relation we obtained at the end.

$$v_{1,2} = e^{i\phi} \frac{\sin(\theta)}{(\cos(\theta) + 1)} * v_{1,1}$$

49 Substituting $v_{1,1} = s_{n1} = \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}$ (as our final spinor seems to suggest), we get -

$$\begin{aligned}
v_{1,2} &= e^{i\phi} \frac{\sin(\theta)}{(\cos(\theta) + 1)} * v_{1,1} \\
&= e^{i\phi} \frac{\sin(\theta)}{(\cos(\theta) + 1)} * \cos \frac{\theta}{2} e^{-i\frac{\phi}{2}}
\end{aligned}$$

50 Recall $1 + \cos(\mathcal{A}) = 2 * \cos^2(\frac{\mathcal{A}}{2})$ and $\sin(\mathcal{A}) = 2 * \sin(\frac{\mathcal{A}}{2})\cos(\frac{\mathcal{A}}{2})$

$$\begin{aligned}
e^{i\phi} \frac{\sin(\theta)}{(\cos(\theta) + 1)} * \cos \frac{\theta}{2} e^{-i\frac{\phi}{2}} &= e^{i\frac{\phi}{2}} \frac{\sin(\theta)}{2\cos^2(\frac{\theta}{2})} * \cos \frac{\theta}{2} \\
&= e^{i\frac{\phi}{2}} \frac{2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})}{2\cos^2(\frac{\theta}{2})} * \cos \frac{\theta}{2} \\
&= e^{i\frac{\phi}{2}} \sin(\frac{\theta}{2}) = s_{n2}
\end{aligned}$$

51 Therefore, apart from phase factors, the resultant spinor is the same as the spin up eigenvector we
52 got in part (a).

53 Question 3.

54 [Sahas: I got this one.]

55 (a) Construct the matrices \hat{J}_x and \hat{J}_y for a particle with spin one, $j = 1$ (of course \hat{J}_z is already
56 diagonal with eigenvalues $\hbar, 0, -\hbar$).

57 We can write the J_x operator as $\frac{J_+ + J_-}{2}$. We can write the matrix elements of this matrix in the $\langle j, m|$
58 basis as $\langle j, m' | \frac{J_+ + J_-}{2} | j, m \rangle$. Note that this matrix element will vanish if $m' = m$ or $|m' - m| > 1$.
59 This gives us the following matrix for $\frac{J_+ + J_-}{2}$, when the basis elements are $|-1\rangle, |0\rangle, |1\rangle$, in that
60 order.

$$\begin{bmatrix} 0 & a & 0 \\ b & 0 & c \\ 0 & d & 0 \end{bmatrix} \tag{10}$$

61 Now

$$\begin{aligned}
a &= \langle -1 | J_x | 0 \rangle \\
&= \langle -1 | \frac{J_-}{2} | 0 \rangle \\
&= \langle -1 | \frac{\hbar \sqrt{(1)(1+1) - (0)(0-1)}}{2} | -1 \rangle \\
&= \hbar \frac{\sqrt{2}}{2} \\
&= \frac{\hbar}{\sqrt{2}}
\end{aligned} \tag{11}$$

62 Now, since the matrix is hermitian, we have the following relation between a and b:

$$\begin{aligned}
b &= a^* \\
\implies b &= \frac{\hbar}{\sqrt{2}}
\end{aligned} \tag{12}$$

63 We can perform the same calculation for c:

$$\begin{aligned}
c &= \langle 0 | J_x | 1 \rangle \\
&= \langle 0 | \frac{J_-}{2} | 1 \rangle \\
&= \langle 0 | \frac{\hbar \sqrt{(1)(1+1) - (1)(1-1)}}{2} | 1 \rangle \\
&= \hbar \frac{\sqrt{2}}{2} \\
&= \hbar \frac{1}{\sqrt{2}}
\end{aligned} \tag{13}$$

64 Again, using the hermiticity argument, we get $d = c = \frac{\hbar}{\sqrt{2}}$. Therefore the final J_x matrix is:

$$\frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{14}$$

65 Now that we have J_x (and J_z is trivial), we can use the commutator relation to get J_y :

$$[J_x, J_z] = -i\hbar J_y \tag{15}$$

66 We write $[J_x, J_z]$ as

$$\frac{\hbar^2}{\sqrt{2}} \left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right) \tag{16}$$

67 With a little algebra we get

$$[J_x, J_z] = -i\hbar J_y = \frac{\hbar^2}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad (17)$$

68 Finally we get

$$J_y = \frac{i\hbar}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad (18)$$

69 (b) An unpolarized beam of spin 1 particles enters a Stern-Gerlach filter that passes only particles
70 with $S_z = \hbar$. After exiting this filter, the beam enters a second filter that passes particles with
71 $S_x = \hbar$ and then finally it encounters a third filter that passes only particles with $S_z = -\hbar$. What
72 fraction of the initial particles make it right through?

73 By computing the eigenvectors of the matrix J_y we get the results

$$|\langle S_x = i | S_z = j \rangle|^2 = \frac{1}{3} \quad (19)$$

74 for $i, j = -1, 0, 1$.

75 Since the beam is unpolarised, $1/3$ of the particles will pass through the first filter. Again, because
76 of the above result, $1/3$ of the particles will pass through filter 2. Similarly, $1/3$ of these particles
77 will then pass through filter 3. Finally we find that $1/27$ of the particles will pass through the whole
78 set-up.

79 **4. Consider the action of an infinitesimal rotation of magnitude ϵ about the \hat{n} axis of an angular
momentum eigenstate $\psi_{l,m}(\theta, \phi)$ (or $|l, m\rangle$). Show that $U(R)\psi_{l,m} = \sum'_m D_{m'm}\psi_{l,m'}$ and find the complex
numbers $D_{m'm}$.**

80

81 [Sankalp: I got this one.]

82 First, we write down $U(R(\epsilon\hat{n}))$ in terms of familiar operators assuming $\hat{n} = n_x\hat{x} + n_y\hat{y} + n_z\hat{z}$ to get

$$\begin{aligned} U(R(\epsilon\hat{n})) &= \exp\left(-\frac{i\epsilon}{\hbar} \cdot (\hat{\mathbf{J}} \cdot \hat{\mathbf{n}})\right) \\ &= \exp\left(-\frac{i\epsilon}{\hbar} \cdot (n_x\hat{\mathbf{J}}_x + n_y\hat{\mathbf{J}}_y + n_z\hat{\mathbf{J}}_z)\right) \end{aligned} \quad (20)$$

$$\begin{aligned} &= \hat{\mathbf{1}} - \frac{i\epsilon}{\hbar} \cdot (n_x\hat{\mathbf{J}}_x + n_y\hat{\mathbf{J}}_y + n_z\hat{\mathbf{J}}_z) + \mathcal{O}(\epsilon^2) \\ &\approx \hat{\mathbf{1}} - \frac{i\epsilon}{\hbar} \cdot (n_x\hat{\mathbf{J}}_x + n_y\hat{\mathbf{J}}_y + n_z\hat{\mathbf{J}}_z) \end{aligned} \quad (21)$$

83 Consider the action of this operator on an arbitrary state $|l, m\rangle$

$$U(R(\epsilon\hat{\mathbf{n}}))|l, m\rangle = \left(1 - \frac{i\epsilon}{\hbar} \cdot (n_x \hat{J}_x + n_y \hat{J}_y + n_z \hat{J}_z)\right) |l, m\rangle. \quad (22)$$

84 Using the fact that the state is an eigenvector of the \hat{J}_z operator, and expanding \hat{J}_x, \hat{J}_y as their respective
85 forms in terms of the ladder operators, we get

$$\begin{aligned} U(R(\epsilon\hat{\mathbf{n}}))|l, m\rangle &= \left(1 - \frac{i\epsilon n_z}{\hbar} \cdot \hat{J}_z\right) |l, m\rangle - \frac{i\epsilon}{\hbar} \cdot (n_x \hat{J}_x + n_y \hat{J}_y) |l, m\rangle \\ &= (1 - i\epsilon n_z \cdot m) |l, m\rangle - \frac{i\epsilon}{\hbar} \cdot \left(\frac{in_x + n_y}{2i} \hat{J}_+ + \frac{in_x - n_y}{2i} \hat{J}_-\right) |l, m\rangle \\ &= (1 - i\epsilon n_z \cdot m) |l, m\rangle - \frac{i\epsilon}{\hbar} \cdot \left(\frac{in_x + n_y}{2i} \cdot \hbar\sqrt{(l-m)(l+m+1)} \cdot |l, m+1\rangle \right. \\ &\quad \left. + \frac{in_x - n_y}{2i} \cdot \hbar\sqrt{(l+m)(l-m-1)} \cdot |l, m-1\rangle\right) \end{aligned} \quad (23)$$

86 or writing it in the required format

$$U(R(\epsilon\hat{\mathbf{n}}))|l, m\rangle = \sum_m D_{m'm} |l, m'\rangle, \quad (24)$$

87 with

$$D_{m'm} = \begin{cases} 1 - i\epsilon n_z \cdot m & \text{if } m' = m \\ -\frac{\epsilon}{2} \cdot (in_x \pm n_y) \cdot \sqrt{(l \mp m)(l \pm m + 1)} & \text{if } m' = m \pm 1 \\ 0 & \text{otherwise.} \end{cases} \quad (25)$$

88 **5. Prove that any function of the radial coordinate $f(r)$ where $r = |\mathbf{r}|$ and $\mathbf{X} \cdot \mathbf{P}$, where \mathbf{X} and \mathbf{P} are
89 the position and momentum operators, are both scalar operators.**

90 [Parth: Doing question 5, I'm not spending as much time on this as question 2]

Under a symmetry operator U , operators change as $\mathcal{O}' = U^\dagger \mathcal{O} U$. A scalar operator being one which is invariant under rotations, i.e

$$S' = U^\dagger [R] S U [R] = S$$

91 where $U(R(\alpha)) = e^{-\frac{i}{\hbar} \alpha \cdot \mathbf{J}}$.

92 By considering infinitesimal rotations $\alpha = \epsilon$, we have

$$U[R(\alpha)] = \left(1 - \frac{i}{\hbar} \epsilon_i J_i\right)$$

93 Thus, our definition for a scalar operator becomes -

$$S' = \left(1 + \frac{i}{\hbar} \epsilon_i J_i\right) S \left(1 - \frac{i}{\hbar} \epsilon_i J_i\right) = S$$

94 which gives us $\frac{i}{\hbar} \epsilon_i [J_i, S] = 0$. Since ϵ was an arbitrary choice, we have

$$[J_i, S] = 0$$

95 as our definition of a scalar operator.

96 Considering $f(r)$, where $r = |\mathbf{r}|$ as our operator.

$$[J_i, f(r)] = [J_i, r] * f'(r)$$

97 $r = \sqrt{\sum_{i=1}^3 X_i^2}$, Thus

$$[J_i, r] = [J_i, X_1] * \frac{X_1}{r} + [J_i, X_2] * \frac{X_2}{r} + [J_i, X_3] * \frac{X_3}{r}$$

98 we know that $[J_i, X_j] = i\hbar \epsilon_{ijl} X_l$. Thus

$$[J_i, r] = [J_i, X_j] * \frac{X_j}{r} = \frac{1}{r} (i\hbar \epsilon_{ijl} X_l X_j)$$

$$\epsilon_{ijl} X_l X_j = [X_l, X_j] = 0 (l \neq j) \Rightarrow [J_i, r] = 0$$

99 Thus, since $[J_i, r] = 0$, we have $[J_i, f(r)] = [J_i, r] * f'(r) = 0 * f'(r) = 0$.

100 Thus, $f(r)$ is a scalar operator.

101

102 Now considering $O = \mathbf{X} \cdot \mathbf{P}$ as our operator, we need to show $[J_i, O] = 0$

$$\mathbf{X} \cdot \mathbf{P} = X_i P_i \quad \text{implicit summation}$$

$$\begin{aligned} \therefore [J_i, O] &= [J_i, X_j P_j] \\ &= [J_i, X_j] P_j + X_j [J_i, P_j] \\ &= i\hbar \epsilon_{ijl} (X_l P_j + X_j P_l) \end{aligned}$$

103 Now, $\epsilon_{ijl} X_l P_j = [X_l, P_j]$ for $l \neq j$, but $[X_l, P_j] = 0, l \neq j$. Thus

$$i\hbar \epsilon_{ijl} (X_l P_j + X_j P_l) = 0 \Rightarrow [J_i, O] = 0$$

104 Since $[J_i, O] = 0$, we can say that the operator O is a scalar operator.
 105 Thus, $\mathbf{X} \cdot \mathbf{P}$ is a scalar operator

106 Question 6.

107 [Sahas: I got this one.]

108 We know that the \mathbf{X}_i operators can be written in terms of the spherical tensor operators as follows:
 109 (notation is the same as that used in Shankar, Principles of Quantum Mechanics, 2ed, page 419)

$$\begin{aligned} V_1^{+1} &= \frac{i\mathbf{X}_y - \mathbf{X}_x}{\sqrt{2}} \\ V_1^0 &= \mathbf{X}_z \\ V_1^{-1} &= -\frac{\mathbf{X}_x + i\mathbf{X}_y}{\sqrt{2}} \end{aligned} \quad (26)$$

110 Thus in general any linear combination of the \mathbf{X}_i s can be written in terms of the V_1^i s. Note that $\epsilon \cdot \mathbf{X}$ is
 111 exactly such a linear combination. Thus we may write

$$\hat{O} = \epsilon \cdot \mathbf{X} = \alpha_i V_1^i \quad (27)$$

112 Where the α_i are scalars, and summation over repeated values is implied.

113 Using this form we can write the transition probability for the Hydrogen atom as

$$|\langle n', l', m' | \alpha_i V_1^i | n, l, m \rangle| \quad (28)$$

114 Now since each V_1^i , acting on $|n, l, m\rangle$ can either:

- 115 • Increase the value of l by 1
- 116 • Decrease the value of l by 1
- 117 • Keep the value of l unchanged

118 Or give a superposition of the above. Since states of different l are orthogonal, $\alpha_i V_1^i |n, l, m\rangle$ and
 119 $\langle n', l', m' |$ won't have any common terms unless $|l - l'| = 1$ or $l = l'$.

120 Thus we get the relation

$$|\langle n', l', m' | \alpha_i V_1^i | n, l, m \rangle| = 0 \quad (29)$$

121 Unless $|l - l'| = 1$ or $l = l'$.

122 Since EM theory is invariant under parity inversion, we must require that expectation values of the dipole
 123 moment be conserved under parity inversion.

$$|\langle n', l, m' | \epsilon \cdot \mathbf{X} | n, l, m \rangle| = |\langle n', l', m' | P^\dagger \epsilon \cdot \mathbf{X} P | n, l, m \rangle| \quad (30)$$

¹²⁴ Since $|n, l, m\rangle$ transforms as $|n, l, m\rangle \longrightarrow (-1)^l |n, l, m\rangle$ under parity,

$$(-1)^{l'+l} \langle n', l', m' | \epsilon \cdot \mathbf{X} | n, l, m \rangle = \langle n', l', m' | P^\dagger \epsilon \cdot \mathbf{X} P | n, l, m \rangle \quad (31)$$

¹²⁵ Since \mathbf{X} transforms as $\mathbf{X} \longrightarrow -\mathbf{X}$ under parity, we get

$$(-1)^{l'+l} \langle n', l', m' | \epsilon \cdot \mathbf{X} | n, l, m \rangle = -\langle n', l', m' | \epsilon \cdot \mathbf{X} | n, l, m \rangle \quad (32)$$

¹²⁶ Hence if $l + l'$ is even (i.e. when $l = l'$), we get

$$\langle n', l', m' | \epsilon \cdot \mathbf{X} | n, l, m \rangle = 0 \quad (33)$$