Category theory

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1 Categories

Mathematics speaks the language of category theory. Some basic categories which routinely appear in mathematics are listed in Table 1.

Table 1: Some basic categories of mathematics.

Category	Objects	Morphisms
Set	sets	functions
Poset	partially ordered sets	order-preserving maps
Monoid	monoids	monoid homomorphisms
Group	groups	group homomorphisms
Ring	rings	ring homomorphisms
Field	fields	field extensions
$\mathsf{Vec}_{\mathbb{k}}$	vector spaces over $\mathbb{k}$	$\mathbb{k}$-linear maps
Metric	metric spaces	continuous maps
Top	topological spaces	continuous maps
NLS	normed linear spaces	bounded linear maps

Informally, the category of sets consists of

- all sets $X, Y, \ldots$ (possibly infinite),
- all functions $X \to Y$ for any sets X and Y.

We denote this category by Set. Set theory is the study of this category.

The other entries in the table are to be interpreted in a similar manner.

For example, $\mathsf{Vec}_{\Bbbk}$ is the category whose objects are vector spaces over a field $\Bbbk$ and whose morphisms are $\Bbbk$-linear maps. Linear algebra is the study of this category.

These examples are discussed more formally in Section 2.

1.1 Categories

A category C consists of the following data.

- objects $a, b, \ldots$,
- for any two objects a and b, a set of morphisms $\mathsf{C}(a, b)$. We denote an element of $\mathsf{C}(a, b)$ by $f : a \to b$, and say that f is a morphism from a to b.
- for each object a, there is a distinguished element of $\mathsf{C}(a, a)$ denoted id_a.
- for any three objects a, b, c, a binary operation (function)

  $$\mathsf{C}(a, b) \times \mathsf{C}(b, c) \to \mathsf{C}(a, c), \qquad (f, g) \mapsto g \circ f,$$

  which is associative and unital:

  If $f : a \to b$, $g : b \to c$, $h : c \to d$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

  If $f : a \to b$, then $\mathrm{id}_b \circ f = f = f \circ \mathrm{id}_a$.

We refer to the binary operation above as the composition rule.

We say that $g \circ f$ is the composite of f and g, or is obtained by composing f with g.

The set $\mathsf{C}(a, b)$ could be empty, that is, there are no morphisms from a to b.

1.2 Picturing a category

A category can be pictured as a graph with objects as vertices, and morphism as arrows (directed edges). The identity morphisms are loops. The composition rule is illustrated by the diagram

$$\begin{tikzcd}
 & b \arrow[dr, "g"] & \\
a \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & c.
\end{tikzcd}$$

This is an instance of a ‘commutative’ diagram.

1.3 Finite categories

Here are some concrete ’tiny’ categories.

- the empty category with no objects and no morphisms. In a way, there is nothing to say.
- the category with one object a and one morphism id_a. There is only one way to define the composition rule. This is also called the one-arrow category.

$$\begin{tikzcd}
a \arrow[loop above, "\mathrm{id}_a"]
\end{tikzcd}$$

- the category with two objects a and b, exactly one morphism from a to b, and the identity morphisms id_a and id_b. This is also called the interval category.

$$\begin{tikzcd}
a \arrow[loop left, "\mathrm{id}_a"] \arrow[r] & b \arrow[loop right, "\mathrm{id}_b"]
\end{tikzcd}$$

Let us try to construct a category with one object a and two morphisms id_a and f.

$$\begin{tikzcd} a \arrow[loop left, "\mathrm{id}_a"'] \arrow[loop right, "f"] \end{tikzcd}$$

How do we define $f \circ f$?

1.4 Discrete categories

Consider the category on two objects a and b consisting of only the identity morphisms id_a and id_b.

$$\begin{tikzcd} a \arrow[loop left, "\mathrm{id}_a"] & & b \arrow[loop right, "\mathrm{id}_b"] \end{tikzcd}$$

This is the discrete category on two objects.

One can define the discrete category on any set of objects: the only morphisms are the identity morphisms.

Note: The one-arrow category is the discrete category on one object.

1.5 Indiscrete categories

One can also define the indiscrete category on any set of objects. In this case, there is exactly one morphism from one object to any other object.

Note: The one-arrow category is the indiscrete category on one object.

The indiscrete category on two objects is shown below.

$$\begin{tikzcd} a \arrow[loop left, "\mathrm{id}_a"] \arrow[r, shift left] & b \arrow[l, shift left] \arrow[loop right, "\mathrm{id}_b"] \end{tikzcd}$$

Can you picture the indiscrete category on three objects?

1.6 Isomorphisms in a category

We say $f : a \to b$ is an isomorphism in C if there exists a morphism $g : b \to a$ such that $f \circ g = \mathrm{id}_b$ and $g \circ f = \mathrm{id}_a$.

$$\begin{tikzcd} a \arrow[loop left, "\mathrm{id}_a"] \arrow[r, bend left, "f"] & b \arrow[l, bend left, "g"] \arrow[loop right, "\mathrm{id}_b"] \end{tikzcd}$$

The morphism g is called the inverse of f.

We say that the objects a and b are isomorphic if there exists an isomorphism $f : a \to b$. Two objects can be isomorphic via many different isomorphisms.

An isomorphism from an object to itself is usually called an automorphism.

1.7 Groupoid

A groupoid is a category in which every morphism is an isomorphism.

Any discrete category is a groupoid.

Any indiscrete category is a groupoid: Given any morphism $f : a \to b$, the unique morphism $g : b \to a$ is its inverse.

What is the precise relationship between groupoids and groups?

1.8 Subcategories

A subcategory of a category C is a category whose objects and morphisms are picked from the objects and morphisms of C, with the composition rule and identities inherited from C.

For instance, consider the category whose objects are finite sets, and morphisms are functions. This is a subcategory of the category of sets. Here we have restricted the class of objects and kept all morphisms between the allowed objects. This is called a full subcategory.

Other such examples are

- finite-dimensional vector spaces inside vector spaces,
- finite groups inside groups,
- abelian groups inside groups,
- linearly ordered sets inside posets,
- compact topological spaces inside topological spaces,

and so on.

In contrast, we could take the category whose objects are sets, but whose morphisms are injective maps.

This is also a subcategory of the category of sets.

Here we have kept all the objects but restricted the class of morphisms.

In general, in a subcategory, we restrict both the objects and the morphisms.

A subcategory D of C is called isomorphism-closed if the following condition holds.

If a is an object in D and $f : a \to b$ is an isomorphism in C, then b is an object is D and f is a morphism in D.

All examples of subcategories mentioned above are isomorphism-closed.

1.9 Opposite category

Every category C has an opposite category which we denote by C^{op}. It has the same objects as C, and

$$\mathsf{C}^{\mathrm{op}}(a,b) := \mathsf{C}(b,a).$$

The composition rule is

$$\mathsf{C}^{\mathrm{op}}(a,b)\times\mathsf{C}^{\mathrm{op}}(b,c) \to \mathsf{C}^{\mathrm{op}}(a,c), \qquad (f,g) \mapsto f\circ g,$$

where $f \circ g$ refers to the composition rule of C. Note very carefully that we write $f \circ g$ and not $g \circ f$.

2 Some well-known categories

2.1 Category of sets

We now define the category Set more formally.

Objects are sets, and morphisms are functions. More precisely, for any sets X and Y, $\mathsf{Set}(X, Y)$ is defined to be the set of all functions from X to Y.

For every set X, we have the identity function

$$\mathrm{id}_X : X \to X, \qquad \mathrm{id}_X(x) = x.$$

These are the identity morphisms.

Given a function $f : X \to Y$ and a function $g : Y \to Z$, there is a function $g \circ f : X \to Z$ defined by

$$(g \circ f)(x) = g(f(x)).$$

This is the composition rule for morphisms. It is associative and unital:

$$(h \circ (g \circ f))(x) = h(g(f(x))) = ((h \circ g) \circ f)(x)$$

and

$$(\mathrm{id}_b \circ f)(x) = f(x) = (f \circ \mathrm{id}_a)(x).$$

Thus we obtain a category.

- There is exactly one function from $\emptyset$ to $\emptyset$ which we denote by $\mathrm{id}_\emptyset$. This is consistent with the convention $0^0 = 1$.

- For any nonempty set X, there are no functions from X to $\emptyset$, that is, $\mathsf{Set}(X, \emptyset) = \emptyset$.

Note that $f : A \to B$ is a bijection iff there exists a function $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Thus a bijection is the same as an isomorphism in the category of sets.

Two sets are isomorphic iff there is a bijection between them.

Note that there could be many bijections between two given sets.

2.2 Category of monoids

A monoid is a set X with a distinguished element $e \in X$ (called the identity element) and a binary operation

$$X \times X \to X, \qquad (a, b) \mapsto a \cdot b,$$

which is associative and unital, that is,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c), \quad \text{and} \quad a \cdot e = e \cdot a = a,$$

for all $a, b, c \in X$.

A monoid homomorphism is a function $f : X \to Y$ between monoids such that $f(e) = e$ and $f(a \cdot b) = f(a) \cdot f(b)$ for all $a, b \in X$.

This yields a category Monoid whose objects are monoids, and morphisms are monoid homomorphisms. What is it that you need to check?

A monoid is called commutative if $a \cdot b = b \cdot a$ for all $a, b \in X$. This is a property of a monoid. Thus, we have the full subcategory of commutative monoids.

Example. The set $\mathbb{N}$ is a commutative monoid under addition (with 0 being the identity element).

The set $\mathbb{N}_+$ is a commutative monoid under multiplication (with 1 being the identity element).

Example. Let X be the set of all (finite) words in the alphabet $\{0, 1\}$, that is,

$$X = \{\ , 0, 1, 00, 01, 10, 11, 000, 001, \ldots\}.$$

The first element is the empty word. Then X is monoid under concatenation of words (writing words next to each other).

$$a = 001,\ b = 10, \quad a \cdot b = 00110.$$

The empty word is the identity.

Example. For any S, let X be the set of all functions from S to itself. Then X is a monoid under composition of functions, the identity element is the identity function.

In our notation, we can write $X = \mathsf{Set}(S, S)$.

In fact, in any category C and for any object a in C, the set $\mathsf{C}(a, a)$ is a monoid under composition of morphisms. Taking $\mathsf{C} := \mathsf{Set}$ recovers the above example.

2.3 Category of modules over a monoid

Fix a monoid X.

A left module over X or a left X-module is a set M equipped with a map

$$X \times M \to M, \qquad (a, m) \mapsto a \cdot m,$$

such that $a \cdot (b \cdot m) = (a \cdot b) \cdot m$ and $1 \cdot m = m$ for all $a, b \in X$ and $m \in M$.

In this case, we also say that X acts on M on the left, or there is a left action of X on M.

The term left X-set is also commonly used in place of left X-module.

Let M and N be left X-modules.

A module homomorphism from M to N is a map $f : M \to N$ such that $f(a \cdot m) = a \cdot f(m)$ for all $a \in X$ and $m \in M$.

This defines the category of left X-modules which we denote by X-Mod.

The category of right X-modules is defined similarly by using maps of the form

$$M \times X \to M, \qquad (m, a) \mapsto m \cdot a.$$

The category of right X-modules is isomorphic to the category of left X^{op}-modules, where X^{op} denote the monoid opposite to X.

For a commutative monoid X, there is no distinction between left and right modules.

A monoid is a group if every element in the monoid is invertible.

In particular, every group is a monoid.

Thus, it makes sense to talk of left/right modules over a group, left/right actions of a group on a set, and so on.

A commutative group is more commonly called an abelian group.

2.4 Category of vector spaces

Fix a field $\Bbbk$. A vector space over $\Bbbk$ is an abelian group (whose binary operation is denoted $(v, w) \mapsto v + w$) equipped with an operation

$$\Bbbk \times V \to V, \qquad (a, v) \mapsto av$$

called scalar multiplication such that

$$1v = v, \quad (ab)v = a(bv),$$

and

$$a(v + w) = av + aw, \quad (a + b)v = av + bv$$

for all $a, b \in \Bbbk$ and $v, w \in V$.

A $\Bbbk$-linear map between V and W is a function $f : V \to W$ such that

$$af(v) = f(av), \qquad f(v + w) = f(v) + f(w)$$

for all $a \in \Bbbk$ and $v, w \in V$. Equivalently, a $\Bbbk$-linear map f is a group homomorphism such that $af(v) = f(av)$.

This defines the category of vector spaces denoted $\mathsf{Vec}_{\Bbbk}$: objects are vector spaces over $\Bbbk$ and morphisms are $\Bbbk$-linear maps.

Isomorphisms in this category are invertible linear maps.

What is the monoid $\mathsf{Vec}_{\Bbbk}(V, V)$?

2.5 Category of graded vector spaces

Fix a field $\mathbb{k}$. A graded vector space over $\mathbb{k}$ is a sequence of vector spaces over $\mathbb{k}$.

We write $V = (V_n)_{n \geq 0}$. Here each V_n is a vector space over $\mathbb{k}$, and these together define the graded vector space V. We refer to V_n as the component of degree n of V.

Let V and W be graded vector spaces. A morphism $f : V \to W$ is a sequence of linear maps $f_n : V_n \to W_n$.

This defines the category of graded vector spaces.

2.6 Category of posets

A poset is a set P equipped with a relation $\leq$ subject to the axioms below. For all $x, y, z \in P$,

- $x \leq x$,
- $x \leq y$ and $y \leq x$ implies $x = y$,
- $x \leq y$ and $y \leq z$ implies $x \leq z$.

We read $x \leq y$ as “x is less than y”, or “y is greater than x”.

Poset is a shortform for partially ordered set. If $x \leq y$ but $x \neq y$, then it is customary to write $x < y$.

A poset P is linearly ordered or totally ordered if for any $x, y \in P$ either $x \leq y$ or $y \leq x$. (This is a property of a poset as opposed to imposing more structure.)

Suppose P and Q are posets. A function $f : P \to Q$ is order-preserving if $x \leq y$ implies $f(x) \leq f(y)$. This defines the category Poset whose objects are posets, and morphisms are order-preserving maps.

Let P be a poset.

- We say x is the bottom element of P if $x \leq y$ for all $y \in P$.
- Dually, we say x is the top element of P if $y \leq x$ for all $y \in P$.
- Given elements $x, y \in P$, the meet of x and y, denoted $x \wedge y$ is the largest element of P smaller than both x and y. That is, $x \wedge y \leq x$, $x \wedge y \leq y$, and ($z \leq x$ and $z \leq y$ implies $z \leq x \wedge y$).
- Dually, the join of x and y, denoted $x \vee y$ is the smallest element of P larger than both x and y.

Meets, joins, bottom and top elements may not exist, but they are unique whenever they exist.

A poset is called a lattice if meets and joins (of any two elements) exist. (This is a property of a poset.)

Example. The set of natural numbers $\mathbb{N}$ is a linearly ordered set. It has a bottom element, namely 0, but no top element. The meet of two numbers is the smaller of the two, while the join is the larger of the two.

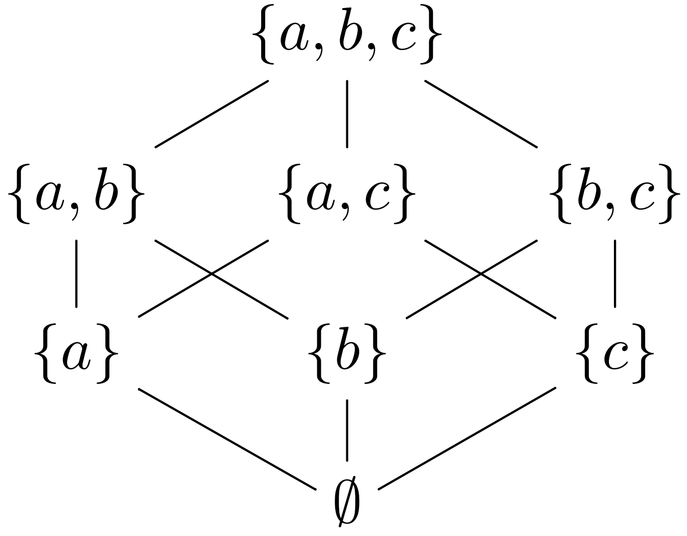
The set of integers $\mathbb{Z}$ is also a linearly ordered set. It has no bottom or top element, while meets and joins are as above.

The set of rational numbers $\mathbb{Q}$ is also a linearly ordered set.

Example. Let X be any set and 2^X denote its power set. An element of 2^X is a subset of X. This is a poset under inclusion, that is, we say $S \leq T$ if S is a subset of T. This is also called the Boolean poset of X.

The bottom element is the empty set, the top element is X, meet is given by intersection of sets, and join by union of sets.

The Hasse diagram of the Boolean poset of $\{a, b, c\}$ is shown below.



3 Functors

3.1 Functors

A functor is a way to compare categories.

Let C and D be two categories.

A functor $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ assigns to every object a in C, an object $\mathcal{F}(a)$ in D, and to every morphism $f : a \to b$ in C a morphism $\mathcal{F}(f) : \mathcal{F}(a) \to \mathcal{F}(b)$ in D such that

$$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f) \qquad \text{and} \qquad \mathcal{F}(\mathrm{id}) = \mathrm{id}\,.$$

See the illustration below.

$$\begin{tikzcd}
& b \arrow[dr, "g"] & & & & \mathcal{F}(b) \arrow[dr, "\mathcal{F}(g)"] & \\
a \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & c & \longmapsto & \mathcal{F}(a) \arrow[ur, "\mathcal{F}(f)"] \arrow[rr, "\mathcal{F}(g\circ f)"'] & & \mathcal{F}(c).
\end{tikzcd}$$

The diagram on the left is in the category C, while the one on the right is in the category D.

3.2 Forgetful functor

Observe that the objects in the categories listed in Table 1 are sets at the very least.

They are sets equipped with some extra structure. Similarly, the morphisms are functions at the very least.

This says that there is a functor from any of these categories to Set.

Such a functor is called a forgetful functor.

Example. Every poset is a set, and every order-preserving map is a function. Thus, there is a functor

$$\mathsf{Poset} \to \mathsf{Set}.$$

This is called a forgetful functor since it is constructed by forgetting the order relation of the poset.

Example. A ring has more structure than a monoid, in fact, there are two monoids present in a ring, so there are two forgetful functors $\mathsf{Ring} \to \mathsf{Monoid}$.

3.3 Inclusion functor

One must understand the difference between imposing structure and imposing conditions. The first gives rise to forgetful functors and the second to inclusion functors, that is, subcategories.

For instance, being a group is a property of a monoid. So $\mathsf{Group} \to \mathsf{Monoid}$ is an inclusion functor. Similarly, being a field is a property of a ring. So $\mathsf{Field} \to \mathsf{Ring}$ is also an inclusion functor.

3.4 Full and faithful functors

Suppose $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ is a functor. Then for any objects a and b of C, there is a function

$$\mathsf{C}(a, b) \to \mathsf{D}(\mathcal{F}(a), \mathcal{F}(b)), \qquad f \mapsto \mathcal{F}(f).$$

This is illustrated below.

$$\begin{tikzcd} a \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\vdots"] & b \arrow[r, phantom, "\longrightarrow"] & \mathcal{F}(a) \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\vdots"] & \mathcal{F}(b) \end{tikzcd}$$

We say that $\mathcal{F}$ is

- faithful if this function is injective,
- full if it is surjective,
- full and faithful if it is bijective,

for all a and b.

Observe that any inclusion functor (arising from a subcategory) is faithful. Further if the subcategory is full, then the inclusion functor is full and faithful.

3.5 Contravariant functors

There is a related notion of a contravariant functor $\mathcal{F} : \mathsf{C} \to \mathsf{D}$. It assigns to every object a in C, an object $\mathcal{F}(a)$ in D, and to every morphism $f : a \to b$ in C, a morphism $\mathcal{F}(f) : \mathcal{F}(b) \to \mathcal{F}(a)$ in D which respects composition.

Note the reversal of the arrow.

By making use of the notion of an opposite category, a contravariant functor can be interpreted as a usual functor (also sometimes called a covariant functor). This can be done in two ways, namely,

$$\mathcal{F} : \mathsf{C}^{\mathrm{op}} \to \mathsf{D} \qquad \text{and} \qquad \mathcal{F} : \mathsf{C} \to \mathsf{D}^{\mathrm{op}}.$$

4 Adjoint functors and universal properties

4.1 Linearization functor

Fix a field $\mathbb{k}$. To each set X, one can define a vector space, denoted $\mathbb{k}X$, whose elements are formal finite linear combinations

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n,$$

where $a_i \in \mathbb{k}$ and $x_i \in X$.

For addition, we add corresponding scalar coefficients. For scalar multiplication, we multiply each coefficient by the given scalar.

Observe that X is a basis of $\mathbb{k}X$.

This construction extends to a functor

$$\text{(1)} \qquad \mathsf{Set} \to \mathsf{Vec}_{\Bbbk}, \qquad X \mapsto \Bbbk X.$$

For this we need to specify how this construction works on morphisms.

Given a function $f : X \to Y$, the linear map $\Bbbk X \to \Bbbk Y$ is defined by replacing each x_i in a given linear combination with $f(x_i)$.

It is possible that some of the $f(x_i)$ coincide in which case we combine them into one term by adding the corresponding scalars.

We call (1) the linearization functor.

How does this relate to the forgetful functor from $\mathsf{Vec}_{\Bbbk}$ to Set?

4.2 A functor from sets to monoids

Let S be a set. Think of it as an alphabet.

Let $\mathcal{F}(S)$ be the set consisting of all words written using the alphabet S.

The operation of concatenation of words (writing one word after the other) turns S into a monoid.

The empty word is the identity element.

When $S = \{0, 1\}$, $\mathcal{F}(S)$ is the monoid discussed in Example 2.2.

Which familiar monoid do you get when S is a singleton set?

Now suppose S and T are two sets and $f : S \to T$ is any function.

This induces a function denoted $\mathcal{F}(f)$ from $\mathcal{F}(S)$ to $\mathcal{F}(T)$.

The function $\mathcal{F}(f)$ takes a word in S as input and produces a word in T as output by substituting letters in S with letters in T via the function f.

We claim that $\mathcal{F}(f)$ is a monoid homomorphism. This is because substitution of letters commutes with concatenation.

Further, $\mathcal{F}$ is compatible with composition since given $f : S \to T$ and $g : T \to U$, making a substitution via f and then via g is the same as making a substitution via $g \circ f$.

Thus, $\mathcal{F}$ defines a functor from Set to Monoid.

We also have the forgetful functor in the other direction from Monoid to Set. How do the two functors relate?

4.3 Adjoint functors

Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G}\colon \mathsf{D} \to \mathsf{C}$ be a pair of functors.

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "\mathcal{F}"] & \mathsf{D} \arrow[l, bend left, "\mathcal{G}"] \end{tikzcd}$$

We say that $\mathcal{F}$ is a left adjoint to $\mathcal{G}$ or that $\mathcal{G}$ is a right adjoint to $\mathcal{F}$ if for each object a in C and x in D, there exists a bijection

$$\mathsf{D}(\mathcal{F}(a), x) \xrightarrow{\cong} \mathsf{C}(a, \mathcal{G}(x)) \tag{2}$$

which is natural in a and x.

The term “natural” has a technical meaning which is given below for you to ponder.

The term “natural” means that for any morphism $a \to a'$ in C and $x \to x'$ in D, the diagrams

$$\begin{tikzcd}
\mathsf{D}(\mathcal{F}(a), x) \arrow[r] & \mathsf{C}(a, \mathcal{G}(x)) \\
\mathsf{D}(\mathcal{F}(a'), x) \arrow[r] \arrow[u] & \mathsf{C}(a', \mathcal{G}(x)) \arrow[u]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathsf{D}(\mathcal{F}(a), x) \arrow[r] \arrow[d] & \mathsf{C}(a, \mathcal{G}(x)) \arrow[d] \\
\mathsf{D}(\mathcal{F}(a), x') \arrow[r] & \mathsf{C}(a, \mathcal{G}(x'))
\end{tikzcd}$$

commute. The horizontal arrows are instances of (2). The vertical arrows are induced from the composition law and the functors.

Proposition 1. *The linearization functor is the left adjoint of the forgetful functor from* $\mathsf{Vec}_{\mathbb{k}}$ *to* Set. *That is, for each set* X *and vector space* V, *there exists a bijection*

$$\mathsf{Vec}_{\mathbb{k}}(\mathbb{k}X, V) \xrightarrow{\cong} \mathsf{Set}(X, V)$$

which is natural in X *and* V.

This encapsulates the property that a linear map is uniquely determined by what it does on the basis elements.

This property can also be formulated as follows.

Proposition 2. *Let V be a vector space, X a set, $f : X \to V$ a function. Then there exists a unique linear map $\hat{f} : \Bbbk X \to V$ such that the diagram*

$$\begin{tikzcd}
\Bbbk X \arrow[r, dashed, "\hat{f}"] & V \\
X \arrow[u, hook] \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes.

Proposition 3. *The functor $\mathcal{F}$ from* Set *to* Monoid *is the left adjoint of the forgetful functor from* Monoid *to* Set*. That is, for each set S and monoid A, there exists a bijection*

$$\mathsf{Monoid}(\mathcal{F}(S), A) \xrightarrow{\cong} \mathsf{Set}(S, A)$$

which is natural in S and A.

Due to this property, $\mathcal{F}(S)$ is also called the free monoid on S.

Equivalently, the free monoid satisfies the following universal property.

Proposition 4. *Let A be a monoid, S a set, $f : S \to A$ a function. Then there exists a unique monoid morphism $\hat{f} : \mathcal{F}(S) \to A$ such that the diagram*

$$\begin{tikzcd}
\mathcal{F}(S) \arrow[r, dashed, "\hat{f}"] & A \\
S \arrow[u, hook] \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes.

How would you construct the free commutative monoid on a set S?

This is the left adjoint of the forgetful functor from the category of commutative monoids to the category of sets.

Can you write down its universal property?

5 Natural transformation

5.1 Natural transformation

Suppose C and D are categories, and $\mathcal{F}$ and $\mathcal{G}$ are functors both from C to D.

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=L}] & \mathsf{D} \arrow[Rightarrow, from=U, to=L, "\eta"] \end{tikzcd}$$

A natural transformation $\eta : \mathcal{F} \Rightarrow \mathcal{G}$ assigns to each object a of C a morphism

$$\eta_a : \mathcal{F}(a) \to \mathcal{G}(a)$$

such that for any morphism $f : a \to b$ in C, the diagram

$$\begin{tikzcd} \mathcal{F}(a) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\eta_a"'] & \mathcal{F}(b) \arrow[d, "\eta_b"] \\ \mathcal{G}(a) \arrow[r, "\mathcal{G}(f)"'] & \mathcal{G}(b) \end{tikzcd}$$

commutes.

5.2 Equivalence of categories

Two categories C and D are called equivalent if there exist functors

$$\mathcal{F}\colon \mathsf{C} \to \mathsf{D} \quad \text{and} \quad \mathcal{G}\colon \mathsf{D} \to \mathsf{C}$$

such that $\mathcal{G}\mathcal{F}$ is naturally isomorphic to the identity functor on C, and $\mathcal{F}\mathcal{G}$ is naturally isomorphic to the identity functor on D.

In this situation, $\mathcal{F}$ (or $\mathcal{G}$) is said to be an equivalence between C and D.

Any (nonempty) indiscrete category is equivalent to the one-arrow category.

A full and faithful functor $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ induces an equivalence of categories between C and the full subcategory of D given by the image of $\mathcal{F}$.

6 Functor categories

6.1 Functor categories

Fix two categories C and D. Define the category $[\mathsf{C}, \mathsf{D}]$ as follows.

Objects of $[\mathsf{C}, \mathsf{D}]$ are functors from C to D. Morphisms are natural transformations between two such functors.

How would you compose natural transformations?

We refer to $[\mathsf{C}, \mathsf{D}]$ as the category of functors from C to D.

Any category which arises in this manner is called a functor category.

Example. What is $[\mathsf{C},\mathsf{D}]$ when C is the one-arrow category? This is D itself.

What is $[\mathsf{C},\mathsf{D}]$ when C is the discrete category on two objects?

6.2 Module categories

Example. Let X be a monoid. Let C_X denote the category with one object, whose morphisms are labeled by elements of X. For example, if $X = \{e, f\}$, then

$$\mathsf{C}_X = \boxed{\begin{tikzcd} \bullet \arrow[loop left, "e"] \arrow[loop right, "f"] \end{tikzcd}}\ .$$

Morphisms of C_X are composed using the product of X. The identity element e of X is the identity morphism.

The category X-Mod of left modules over X from Section 2.3 is the same as the functor category $[\mathsf{C}_X, \mathsf{Set}]$.

In general, one may view a functor from C to D as a ‘representation’ of C in the category D.

Going back to our example, we may also consider $[\mathsf{C}_X, \mathsf{Vec}]$. Functors from C_X to Vec are representations of the monoid X in the category of vector spaces (instead of sets). They are also called linear representations of X.

6.3 Graded vector spaces and Joyal species

Example. Let $\mathsf{C}_{\mathbb{N}}$ denote the discrete category on the set $\mathbb{N}$ of natural numbers.

$$\begin{tikzcd}
0 \arrow[loop above, "\mathrm{id}_0"] & 1 \arrow[loop above, "\mathrm{id}_1"] & 2 \arrow[loop above, "\mathrm{id}_2"] & \cdots
\end{tikzcd}$$

What is $[\mathsf{C}_{\mathbb{N}}, \mathsf{Vec}]$?

This is the category of graded vector spaces.

Example. For each natural number n, let $[n] := \{1, \ldots, n\}$. By convention, $[0]$ is the empty set.

Let S_n denote the set of bijections from $[n]$ to itself. This is also called the group of permutations of $[n]$. It has cardinality $n!$. The product in S_n is given by composing bijections.

Let $\hat{\mathsf{C}}_{\mathbb{N}}$ denote the category whose objects are the sets $[0]$, $[1]$, $[2]$, $\ldots$ and morphisms are bijections.

$$\begin{tikzcd}
{[0]} \arrow[loop above, "\mathrm{id}_{[0]}"] & {[1]} \arrow[loop above, "\mathrm{id}_{[1]}"] & {[2]} \arrow[loop above, "\mathrm{id}_{[2]}"] \arrow[loop below, "\sigma"] & \cdots
\end{tikzcd}$$

This is a disjoint union of one-object categories. It is a groupoid. The object $[n]$ has $n!$ loops, and there are no morphisms between $[m]$ and $[n]$ when $m \neq n$.

What is the functor category $[\hat{\mathsf{C}_{\mathbb{N}}}, \mathsf{Vec}]$?

An object p in this category is a sequence of vector spaces

$$\mathsf{p}[0], \mathsf{p}[1], \mathsf{p}[2], \ldots$$

such that for each n, the n-th component $\mathsf{p}[n]$ is a S_n-module.

A morphism from p to q is a family of maps $\mathsf{p}[n] \to \mathsf{q}[n]$ of S_n-modules, one for each n.

Example. Let $\mathsf{Set}^\times$ denote the category whose objects are finite sets and whose morphisms are bijections between finite sets. It is a groupoid.

The category is equivalent to the category $\hat{\mathsf{C}}_{\mathbb{N}}$. More precisely, the latter is the skeleton of the former.

The functor category $[\mathsf{Set}^\times, \mathsf{Vec}]$ is the category of Joyal species. It is equivalent to the category $[\hat{\mathsf{C}}_{\mathbb{N}}, \mathsf{Vec}]$.

Explicitly, a Joyal species p consists of a family of vector spaces p$[I]$, one for each finite set I, together with linear maps

$$\mathsf{p}[\sigma] : \mathsf{p}[I] \to \mathsf{p}[J],$$

one for each bijection $\sigma \colon I \to J$, such that

$$\mathsf{p}[\mathrm{id}_I] = \mathrm{id}_{\mathsf{p}[I]} \quad \text{and} \quad \mathsf{p}[\tau\sigma] = \mathsf{p}[\tau]\mathsf{p}[\sigma]$$

whenever $I \xrightarrow{\sigma} J \xrightarrow{\tau} K$ are composable bijections.

Similarly, a map of Joyal species f : p $\to$ q consists of a family of linear maps

$$f_I : \mathsf{p}[I] \to \mathsf{q}[I],$$

one for each finite set I, such that for each bijection $\sigma : I \to J$, the diagram

(3)
$$\begin{tikzcd}
\mathsf{p}[I] \arrow[r, "f_I"] \arrow[d, "\mathsf{p}[\sigma]"'] & \mathsf{q}[I] \arrow[d, "\mathsf{q}[\sigma]"] \\
\mathsf{p}[J] \arrow[r, "f_J"'] & \mathsf{q}[J]
\end{tikzcd}$$

commutes.

Joyal species are studied in detail in Part II of the book a.pdf where they are linked to Hopf algebras. See also Section 2.16 of c.pdf.

6.4 Functors between functor categories

Any functor α : $\mathsf{D} \to \mathsf{E}$ induces a functor

(4) $$[\mathsf{C}, \mathsf{D}] \to [\mathsf{C}, \mathsf{E}], \qquad \mathcal{F} \mapsto \alpha(\mathcal{F}),$$

where $\alpha(\mathcal{F})$ denotes the composite of the functors α and $\mathcal{F}$.

A common example of this is the functor

$$[\mathsf{C}, \mathsf{Set}] \to [\mathsf{C}, \mathsf{Vec}]$$

induced by the linearization functor (1).

Example. The category of simplicial sets (which you meet in algebraic topology) is an example of a functor category of the form $[\Delta, \mathsf{Set}]$ for an appropriate base category Δ (which encodes face and degeneracy maps).

Similarly, the category of simplicial spaces is the functor category $[\Delta, \mathsf{Vec}]$. The linearization of a simplicial set is a simplicial space.

For more detail, see for instance, Section 5.1 of the book a.pdf.

7 Limits and colimits

7.1 Initial and terminal objects

Fix a category C.

An object a in C is an initial object if for any object b, there is a unique morphism $a \to b$.

A category may not have any initial objects or it may have more than one initial object. All initial objects are isomorphic.

Dually, an object a in C is an terminal object if for any object b, there is a unique morphism $b \to a$.

Similar remarks apply.

The category of sets has a unique initial object which is the empty set $\emptyset$. Any singleton set is a terminal object. (Note that all singleton sets are isomorphic.)

What about the category of vector spaces?

What about the category of monoids?

7.2 Products and coproducts

Fix a category C.

Given objects a and b in C, their product is an object $a \times b$ in C equipped with morphisms $a \times b \to a$ and $a \times b \to b$ such that the following universal property holds.

For any object c and morphisms $f : c \to a$ and $g : c \to b$, there is a unique morphism $c \to a \times b$ such that the diagram

$$\begin{tikzcd}
 & c \arrow[dl, "f"'] \arrow[d, dashed] \arrow[dr, "g"] & \\
a & a \times b \arrow[l] \arrow[r] & b
\end{tikzcd}$$

commutes.

Dually, given objects a and b in C, their coproduct is an object $a \sqcup b$ in C equipped with morphisms $a \to a \sqcup b$ and $b \to a \sqcup b$ such that the following universal property holds.

For any object c and morphisms $f : a \to c$ and $g : b \to c$, there is a unique morphism $a \sqcup b \to c$ such that the diagram

$$\begin{tikzcd}
 & c & \\
a \arrow[r] \arrow[ur, "f"] & a \sqcup b \arrow[u, dashed] & b \arrow[l] \arrow[ul, "g"']
\end{tikzcd}$$

commutes.

The (co)product of a and b may not exist, but if it exists, then it is unique up to isomorphism.

Given sets X and Y, their cartesian product is defined by

$$X \times Y := \{(x, y) \mid x \in X, y \in Y\}.$$

Note that there are canonical projection maps

$$X \times Y \to X, \quad (x, y) \mapsto x,$$

and

$$X \times Y \to Y, \quad (x, y) \mapsto y.$$

Cartesian product satisfies the following universal property.

Given any set Z and functions $f : Z \to X$ and $g : Z \to Y$, there is a unique function $Z \to X \times Y$ denoted (f, g) such that the diagram

$$\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[d, dashed] \arrow[dr, "g"] & \\
X & X \times Y \arrow[l] \arrow[r] & Y
\end{tikzcd}$$

commutes. Explicitly, (f, g) is defined by

$$(f, g)(z) := (f(z), g(z)).$$

Conclusion: Cartesian product is the product in the category of sets.

Given sets X and Y, their disjoint union is defined by

$$X \sqcup Y := \{(x,0) \mid x \in X\} \cup \{(y,1) \mid y \in Y\}.$$

In the rhs, the first set is a copy of X, while the second is a copy of Y, but these copies have no common elements (though X and Y may have common elements).

Note that there are canonical inclusion maps

$$X \to X \sqcup Y, \quad x \mapsto (x,0),$$

and

$$Y \to X \sqcup Y, \quad y \mapsto (y,1).$$

The disjoint union satisfies the following universal property.

Given any set Z and functions $f : X \to Z$ and $g : Y \to Z$, there is a unique function $X \sqcup Y \to Z$

denoted $f \sqcup g$ such that the diagram

$$\begin{tikzcd}
 & Z & \\
X \arrow[r] \arrow[ur, "f"] & X \sqcup Y \arrow[u, dashed] & Y \arrow[l] \arrow[ul, "g"']
\end{tikzcd}$$

commutes. Conclusion: Disjoint union is the coproduct in the category of sets.

What is the product and coproduct in the category of vector spaces?

What about the category of monoids?

7.3 Limits and colimits of functors

Let $\mathcal{F}\colon \mathsf{D} \to \mathsf{C}$ be a functor. Consider an object V in C equipped with morphisms $\tau_Y\colon \mathcal{F}(Y) \to V$, one for each object Y in D, and such that for each morphism $f\colon Y \to Z$ in D the following diagram commutes.

$$\begin{tikzcd}
& V & \\
\mathcal{F}(Y) \arrow[ur, "\tau_Y"] \arrow[rr, bend right, "\mathcal{F}(f)"'] & & \mathcal{F}(Z) \arrow[ul, "\tau_Z"']
\end{tikzcd}$$

Such a structure is called a cone from the base $\mathcal{F}$ to the vertex V.

The colimit of a functor $\mathcal{F}\colon \mathsf{D} \to \mathsf{C}$ is an object of C, denoted

$$\operatorname{colim} \mathcal{F} \quad \text{or} \quad \operatorname*{colim}_{X} \mathcal{F}(X),$$

together with morphisms $\iota_Y : \mathcal{F}(Y) \to \operatorname{colim} \mathcal{F}$ for each object Y in D, satisfying the following properties.

- The maps ι_Y form a cone from the base $\mathcal{F}$ to the vertex $\operatorname{colim} \mathcal{F}$. In other words,

  $$\iota_Z \mathcal{F}(f) = \iota_Y$$

  for each morphism $f\colon Y \to Z$ in D.

- For any cone from $\mathcal{F}$ to a vertex V in C, there is a unique morphism $\operatorname{colim} \mathcal{F} \to V$, such that for each object Y in D the following diagram commutes, where τ_Y is the structure map of the cone to V.

$$\begin{tikzcd}
\mathcal{F}(Y) \arrow[r, "\iota_Y"] \arrow[dr, "\tau_Y"'] & \operatorname{colim} \mathcal{F} \arrow[d, dashed] \\
 & V
\end{tikzcd}$$

The colimit may not exist, but if it does, then it is unique up to isomorphism.

The notion of limit of a functor is defined dually.

Terminal objects and products are examples of limits.

Initial objects and coproducts are examples of colimits.

What about kernel and cokernel?

Theorem 1. *A right adjoint preserves all existing limits, and dually a left adjoint preserves all existing colimits.*

In particular, a right adjoint preserves terminal objects and products, and dually a left adjoint preserves initial objects and coproducts.

8 Operations on vector spaces

8.1 Subspaces

A subspace of a vector space V is a subset U of V which is closed under addition and scalar multiplication. Thus, there is a linear map

$$U \hookrightarrow V.$$

This is called the inclusion map. It is injective.

8.2 Quotient spaces

For any subspace U of V, one can form the quotient space V/U as follows.

We say two vectors v and w in V are equivalent if $v - w$ belongs to U. This defines an equivalence relation on V. Its equivalence classes are the elements of V/U, and we have a surjective map

$$V \twoheadrightarrow V/U.$$

This is called the quotient map.

Addition and scalar multiplication of V induces an addition and scalar multiplication on V/U. This turns V/U into a vector space.

Note that

$$\dim(V/U) = \dim V - \dim U.$$

By construction, the quotient map is a linear map. Its kernel is precisely the subspace U.

The inclusion map and the quotient map can be shown together as follows.

$$U \hookrightarrow V \twoheadrightarrow V/U.$$

8.3 Kernel, cokernel, image, coimage

Let $f : V \to W$ be a linear map.

Define

$$\ker(f) := \{v \in V \mid f(v) = 0\}.$$

This is a subspace of V. It is called the kernel of f.

Define

$$\mathrm{image}(f) := \{w \in W \mid f(v) = w \text{ for some } v\}.$$

This is a subspace of W. It is called the image of f.

Define

$$\operatorname{coker}(f) := W/\operatorname{image}(f).$$

This is a quotient space of W. It is called the cokernel of f.

Define

$$\operatorname{coimage}(f) := V/\ker(f).$$

This is a quotient space of V. It is called the coimage of f.

We have the following commutative diagram of vector spaces.

$$\begin{tikzcd}
0 \arrow[r] & \ker(f) \arrow[r, hook] & V \arrow[r, "f"] \arrow[d, two heads] & W \arrow[r, two heads] & \operatorname{coker}(f) \arrow[r] & 0 \\
& & \operatorname{coimage}(f) \arrow[r, "\cong"'] & \operatorname{image}(f) \arrow[u, hook] & &
\end{tikzcd}$$

One can think of $\ker(f)$ and $\operatorname{coker}(f)$ as a measure of the failure of f to be an isomorphism.

8.4 Dual of a vector space

Let V and W be two vector spaces over $\mathbb{k}$.

Let $\mathrm{Hom}_{\mathbb{k}}(V, W)$ denote the set of all $\mathbb{k}$-linear maps from V to W.

This set carries the structure of a vector space over $\mathbb{k}$ as follows.

Given linear maps f and g, define $f + g$ by

$$(f + g)(v) := f(v) + g(v).$$

Given a linear map f and a scalar c, define cf by

$$(cf)(v) := cf(v).$$

Given a vector space V, define its dual vector space denoted V^* by

$$V^* := \mathrm{Hom}_{\mathbb{k}}(V, \mathbb{k}).$$

Thus, a vector in V^* is a linear map from V to $\mathbb{k}$. It is also called a linear functional.

Now suppose $f : V \to W$ is any linear map. Then for any linear functional $g : W \to \mathbb{k}$, we have the linear functional $g \circ f : V \to \mathbb{k}$. This yields a map $W^* \to V^*$ which we denote by f^*. One can check that f^* is a linear map. It is called the dual linear map.

This yields a contravariant functor

$$(-)^* : \mathsf{Vec} \to \mathsf{Vec}.$$

On objects, it sends V to V^*, and on morphisms, it sends f to f^*. We call it the duality functor.

What happens if you dualize the diagram involving (co)kernel and (co)image of a linear map f?

Can you deduce row rank equals column rank of a matrix?

8.5 Direct sum

Fix two vector spaces V and W (over the same field).

Define their direct sum $V \oplus W$ as follows. Elements of $V \oplus W$ are pairs (v, w), with $v \in V$ and $w \in W$. Addition and scalar multiplication is define coordinate-wise. That is,

$$(v, w) + (v', w') := (v + v', w + w')$$

and

$$c(v, w) := (cv, cw).$$

Note that

$$\dim(V \oplus W) = \dim V + \dim W.$$

8.6 Tensor product

Fix two vector spaces V and W (over the same field $\Bbbk$).

Define their tensor product $V \otimes W$ as follows.

Elements of this space are formal sums

$$\sum_{i=1}^{n} a_i(v_i \otimes w_i), \qquad a_i \in \Bbbk, v_i \in V, w_i \in W$$

subject to the relations

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w,$$

$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2,$$

$$a(v \otimes w) = av \otimes w = v \otimes aw.$$

If V has basis $\{e_1, \ldots, e_m\}$ and W has basis $\{f_1, \ldots, f_n\}$, then $V \otimes W$ has basis $\{e_i \otimes f_j\}_{ij}$.

9 Algebras and modules

Algebras are linear analogues of monoids.

They are built out of vector spaces in a manner similar to how monoids are built out of sets.

There are two standard ways to deal with the binary operation in an algebra:

- use cartesian product of vector spaces and require bilinearity,
- use tensor product of vector spaces and require linearity.

9.1 Category of algebras

Fix a field $\mathbb{k}$.

An algebra over $\mathbb{k}$ is a vector space A over $\mathbb{k}$ equipped with a distinguished element 1 and a $\mathbb{k}$-bilinear map

$$A \times A \to A, \qquad (a,b) \mapsto a \cdot b$$

which is associative and unital. That is, for all $a, b, c \in A$,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \text{and} \quad 1 \cdot a = a \cdot 1 = a.$$

Alternatively, an algebra over $\mathbb{k}$ is a vector space A over $\mathbb{k}$ equipped with a distinguished element 1 and a $\mathbb{k}$-linear map

$$A \otimes A \to A, \qquad a \otimes b \mapsto a \cdot b$$

which is associative and unital.

Let A and B be algebras.

An algebra homomorphism from A to B is a linear map $f : A \to B$ such that $f(1) = 1$ and $f(a \cdot b) = f(a) \cdot f(b)$ for all $a, b \in A$.

This defines the category of $\mathbb{k}$-algebras which we denote by $\mathsf{Alg}_{\mathbb{k}}$.

An algebra A is commutative if $a \cdot b = b \cdot a$ for all $a, b \in A$. This defines the full subcategory of commutative $\mathbb{k}$-algebras which we denote by $\mathsf{Alg}_{\mathbb{k}}^{\mathrm{co}}$.

Since an algebra is a vector space, it has a dimension.

Some examples of $\mathbb{k}$-algebras to bear in mind are:

- $\mathbb{k}^n$,
- the algebra of polynomials $\mathbb{k}[x]$,
- $\mathbb{k}[x]/(x^n)$,
- algebra of square matrices of size n,
- algebra of upper-triangular matrices of size n.

$\mathbb{R}^2$ is a two-dimensional algebra over $\mathbb{R}$.

$\mathbb{C}$ is also a two-dimensional algebra over $\mathbb{R}$.

Are these two algebras isomorphic?

9.2 Tensor algebra and symmetric algebra

The forgetful functor from $\mathsf{Alg}_{\mathbb{k}}$ to $\mathsf{Vec}_{\mathbb{k}}$ has a left adjoint, which associates to a vector space V the tensor algebra $T(V)$ of V. This is the linear analogue of the free monoid on a set.

Similarly, the forgetful functor from $\mathsf{Alg}_{\mathbb{k}}^{\mathrm{co}}$ to $\mathsf{Vec}_{\mathbb{k}}$ has a left adjoint, which associates to a vector space V the symmetric algebra $S(V)$ of V. This is the linear analogue of the free commutative monoid on a set.

9.3 Linearization of a monoid

We have seen that the linearization of a set is a vector space. It is also possible to ‘linearize’ other set-theoretic objects.

For instance, the linearization of a monoid is an algebra. This yields a functor

$$\mathsf{Monoid} \to \mathsf{Alg}_{\mathbb{k}}.$$

For example, it sends the monoid $\mathbb{N}$ under addition to the algebra $\mathbb{k}[x]$.

9.4 Category of modules over an algebra

Fix an algebra A.

A left module over A or a left A-module is a vector space M over $\mathbb{k}$ equipped with a bilinear map

$$A \times M \to M, \qquad (a, m) \mapsto a \cdot m,$$

such that $a \cdot (b \cdot m) = (a \cdot b) \cdot m$ and $1 \cdot m = m$ for all $a, b \in A$ and $m \in M$.

In this case, we also say that A acts on M on the left, or there is a left action of A on M.

Let M and N be left A-modules.

A module homomorphism from M to N is a linear map $f : M \to N$ such that $f(a \cdot m) = a \cdot f(m)$ for all $a \in A$ and $m \in M$.

This defines the category of left A-modules which we denote by A-Mod.

The category of right A-modules is defined similarly by using maps of the form

$$M \times A \to M, \qquad (m, a) \mapsto m \cdot a.$$

The category of right A-modules is isomorphic to the category of left A^{op}-modules, where A^{op} denote the algebra opposite to A.

For a commutative algebra A, there is no distinction between left and right modules.

Some examples of modules to bear in mind are:

- A as a left and right module over itself.
- left module of column vectors (and right module of row vectors) over the algebra of square matrices.

9.5 Duality

Let A be any algebra.

Let M be a left A-module. Write M^* for the linear dual of M. Then M^* is a right A-module with the dual action: For $a \in A$ and $f \in M^*$,

$$(f \cdot a)(m) := f(a \cdot m).$$

Thus, the dual of a left A-module is a right A-module, or equivalently, a left A^{op}-module.

This yields a (contravariant) duality functor from A-Mod to A^{op}-Mod.

9.6 Base change

Let A and B be algebras and fix a morphism $f : A \to B$ of algebras.

This gives rise to a faithful functor

$$\mathcal{F} : B\text{-Mod} \to A\text{-Mod}.$$

This functor views a B-module N as a A-module via $a \cdot n := f(a) \cdot n$ for $a \in A$ and $n \in N$.

Further, if f is surjective, then $\mathcal{F}$ is also full, and in fact, B-Mod can be viewed as a full subcategory of A-Mod.

The functor $\mathcal{F}$ has a left adjoint which sends a A-module M to the B module $B \otimes_A M$. Thus, for a A-module M and a B-module N,

$$B\text{-Mod}(B \otimes_A M, N) \cong A\text{-Mod}(M, N).$$

The functor $\mathcal{F}$ also has a right adjoint which sends a A-module M to the B module $\mathrm{Hom}_A(B, M)$. Thus, for a A-module M and a B-module N,

$$A\text{-Mod}(N, M) \cong B\text{-Mod}(N, \mathrm{Hom}_A(B, M)).$$

10 Linear categories

Fix a field $\mathbb{k}$.

We say a category C is a $\mathbb{k}$-linear category if for any objects a and b, the set $\mathsf{C}(a,b)$ carries the structure of a $\mathbb{k}$-vector space. Moreover, these vector space structures must be compatible with composition of morphisms.

Thus, for any $f, g : a \to b$ and $\alpha \in \mathbb{k}$, we can talk of $f + g : a \to b$ and $\alpha f : a \to b$. The identities

$$(f+g)\circ h = f\circ h+g\circ h, \qquad h\circ(f+g) = h\circ f+h\circ g$$

$$(\alpha f)\circ g = \alpha(f\circ g), \qquad f\circ(\alpha g) = \alpha(f\circ g)$$

hold.

A $\mathbb{k}$-linear functor between two $\mathbb{k}$-linear categories can be defined along similar lines.

The category of $\mathbb{k}$-vector spaces is an example of a $\mathbb{k}$-linear category. Here $f + g$ is defined as the sum of the linear maps f and g, and αf as the scalar multiplication of the linear map f by α.

For any $\mathbb{k}$-algebra A, the category of (left) A-modules is a $\mathbb{k}$-linear category. For this, we check that if $f, g : M \to N$ are maps of A-modules, then so are $f + g$ and αf.

11 Exercises

1. Suppose C is a category with identity morphisms denoted id_a. Suppose for each object a, we are given a morphism id'_a which satisfies the unitality axiom. Then show that $\mathrm{id}_a = \mathrm{id}'_a$.

2. Show that any isomorphism $f : a \to b$ in a category C has a unique inverse.

3. Show that any two initial (terminal) objects in a category are isomorphic.

4. Let C be a category and a and b be any two objects. Show that the (co)product of a and b is unique up to isomorphism (assuming that it exists).

5. Suppose $f : a \to b$ is an isomorphism. Then for any object c, show that there is a bijection

$$\mathsf{C}(b, c) \to \mathsf{C}(a, c), \qquad h \mapsto h \circ f.$$

6. Show that: A functor preserves isomorphisms. That is, if $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ is a functor and $f : a \to b$ is an isomorphism in C, then $\mathcal{F}(f) : \mathcal{F}(a) \to \mathcal{F}(b)$ is an isomorphism in D.

7. Let C be a category. What are all functors from the one-arrow category to C? What are all functors from the interval category to C?

8. (Looping in a category)

   Suppose C is a category. Pick any object a in C. Consider the gadget with one object a, and set of morphisms $\mathsf{C}(a, a)$, with binary operation

   $$\mathsf{C}(a, a) \times \mathsf{C}(a, a) \to \mathsf{C}(a, a)$$

   induced from the composition rule of C. Does this define a subcategory of C? What are categories with exactly one object? What are functors between categories with exactly one object?

   What happens if instead of picking $\mathsf{C}(a, a)$ you only pick the subset of all automorphisms of a? Do you get a subcategory? What are categories with exactly one object in which every morphism is an isomorphism? What about functors between such categories?

9. (Looping in a linear category)

   Repeat the previous exercise with the category C replaced by a $\Bbbk$-linear category.

10. A groupoid is a category in which every morphism is an isomorphism. What is a groupoid with one object?

11. Show that the composite of two functors is a functor. For any category C, one always has the identity functor which takes every object to itself, and every morphism to itself. Does it make sense to define a category whose objects are categories, and morphisms are functors?

12 Exercises

1. Let $\mathbb{k}$ be a field and view it as an algebra over itself. Show that the category of left modules over $\mathbb{k}$ is the same as the category of $\mathbb{k}$-vector spaces.

   Can you generalize this and describe the category of left modules over the algebra $\mathbb{k}^n$? Since this algebra is commutative, there is no distinction between left and right modules. In particular, for modules M and N over $\mathbb{k}^n$, what are all module maps $M \to N$?

2. Consider the algebra A of square matrices of size n. (This algebra is not commutative for $n > 1$.) Show that the category of left A-modules is isomorphic to the category of right A-modules.

3. Let X be a monoid. Show that a left module M over X is the same as a monoid homomorphism from X to the monoid $\mathsf{Set}(M, M)$.

4. Let X be a monoid. Show that a left module over X is the same as a functor from X (viewed as a one-object category) to the category of sets.

5. Let A be an algebra. For a submodule N of an A-module M, check that the quotient space M/N inherits an A-module structure such that the quotient map $M \twoheadrightarrow M/N$ is a map of A-modules.

6. View $\mathbb{C}$ as a 2-dimensional algebra over $\mathbb{R}$. Describe the category of $\mathbb{C}$-modules. What are all the simple modules? What are their dimensions over $\mathbb{R}$.

7. Give an example of an algebra which is not the linearization of a monoid. What happens for $\Bbbk^n$?

13 References

Some nice books on category theory are listed below.

1. S. Mac Lane, Categories for the working mathematician, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.

2. T. Leinster, Basic category theory, Cambridge Studies in Advanced Mathematics, vol. 143, Cambridge University Press, Cambridge, 2014.

3. S. Awodey, Category theory, second ed., Oxford Logic Guides, vol. 52, Oxford University Press, Oxford, 2010.

4. E. Riehl, Category theory in context, Aurora, Dover Publications, 2016.

(More information/references on categories can be found in the Notes of the Appendices of c.pdf.)