Zaslavsky formula

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1 Incidence algebras and Möbius functions

1.1 Incidence algebra of a poset

Let P be a finite poset.

A 1-chain in P is a pair $(x, y) \in P^2$ with $x \leq y$.

Fix a field $\mathbb{k}$.

An incidence function on P is a $\mathbb{k}$-valued function on the set of 1-chains in P

$$f : \{(x, y) \in P^2 \mid x \leq y\} \to \mathbb{k}.$$

Let $\mathrm{I}(P)$ denote the vector space of all incidence functions, with pointwise addition and scalar multiplication.

For $f, g \in \mathrm{I}(P)$, define a new function $fg \in \mathrm{I}(P)$ by

$$(fg)(x,z) = \sum_{y\colon x \le y \le z} f(x,y)g(y,z). \tag{1}$$

This turns $\mathrm{I}(P)$ into an algebra.

It is called the incidence algebra of P.

The unit element δ is given by

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In other words, $\delta f = f\delta = f$ for any $f \in \mathrm{I}(P)$.

Proposition 1. *Let $f \in \mathrm{I}(P)$ be such that $f(x,x) = 1$ for all $x \in P$. Then f is invertible in $\mathrm{I}(P)$ and*

(3)
$$f^{-1}(x,y) = \sum_{k \geq 0} (-1)^k \sum_{x = x_0 < x_1 < \cdots < x_k = y} f(x_0, x_1) \ldots f(x_{k-1}, x_k).$$

The summand corresponding to $k = 0$ is 0 unless $x = y$, in which case it is 1.

Proof. If r is the maximum length of a strict chain from x to y, then $(\delta - f)^k(x,y) = 0$ for all $k > r$. The result follows by expanding

$$f^{-1} = (\delta - (\delta - f))^{-1} = \sum_{k \geq 0} (\delta - f)^k. \qquad \square$$

More generally:

Proposition 2. *An incidence function f is invertible in $\mathrm{I}(P)$ iff $f(x,x) \neq 0$ for all $x \in P$.*

Example. Let $P = [n]$ under the usual order $1 < 2 < \cdots < n$.

Then $\mathrm{I}(P)$ is the algebra of upper triangular matrices of size n.

Proposition 2 says that an upper triangular matrix is invertible iff its diagonal entries are nonzero.

1.2 Zeta and Möbius functions of a poset

The zeta function $\zeta \in \mathrm{I}(P)$ is defined by

$$\zeta(x, y) = 1$$

for all $x \leq y$.

It is invertible.

Its inverse is the Möbius function $\mu \in \mathrm{I}(P)$. This may also be defined recursively as follows.

For any element x,

(4a) $$\mu(x, x) := 1$$

and for $x < y$,

$$\mu(x, y) := -\sum_{z\colon x \leq z < y} \mu(x, z) = -\sum_{z\colon x < z \leq y} \mu(z, y),$$

or equivalently,

(4b) $$\sum_{z\colon x \leq z \leq y} \mu(x, z) = \sum_{z\colon x \leq z \leq y} \mu(z, y) = 0.$$

Proposition 3 (Philip Hall formula)**.** *For any $x \leq y$ in P,*

(5) $$\mu(x, y) = \sum_{k \geq 0} (-1)^k c_k(x, y),$$

where $c_k(x, y)$ is the number of strict chains of length k from x to y.

Proof. This is a special case of (3). $\square$

1.3 Incidence module

Let $\mathrm{M}(P)$ denote the vector space of $\mathbb{k}$-valued functions on P.

The incidence algebra $\mathrm{I}(P)$ acts on $\mathrm{M}(P)$ on the left: For $f \in \mathrm{I}(P)$ and $g \in \mathrm{M}(P)$, define $fg \in \mathrm{M}(P)$ by

$$(fg)(x) = \sum_{y \colon x \leq y} f(x,y)g(y). \tag{6}$$

Thus, $\mathrm{M}(P)$ is a left module over $\mathrm{I}(P)$. We call it the incidence module of P.

For functions f and g on P, we have

(7)

$$g(x) = \sum_{y\colon x \le y} f(y) \iff f(x) = \sum_{y\colon x \le y} \mu(x,y) g(y).$$

This is equivalent to $g = \zeta f \iff f = \mu g$.

The passage from the first equation to the second is called Möbius inversion.

Similar to (6), there is also a right action of $\mathrm{I}(P)$ on $\mathrm{M}(P)$. Using it, we deduce

(8)

$$g(y) = \sum_{x\colon x \le y} f(x) \iff f(y) = \sum_{x\colon x \le y} g(x) \mu(x,y).$$

2 Euler characteristic

Let X be a finite cell complex.

Define the reduced Euler characteristic of X to be

(9) $$\chi(X) := -c_{-1} + c_0 - c_1 + c_2 - \ldots,$$

where c_i is the number of i-cells of X. By convention, 0-cells are vertices, 1-cells are edges, and so on.

The reduced Euler characteristic of a cell complex only depends on its underlying topology. Some well-known examples are recalled below.

The reduced Euler characteristic

- of a ball is 0,
- of the sphere of dimension n is $(-1)^n$, and, more generally,
- of the wedge of k spheres each of dimension n is $(-1)^n k$.

Recall that faces of an arrangement $\mathcal{A}$ are cells in a regular cellular decomposition of a sphere of dimension $\mathrm{rk}(\mathcal{A})-1$. Taking reduced Euler characteristics (9), we obtain

$$\sum_{F \in \Sigma[\mathcal{A}]} (-1)^{\mathrm{rk}(F)} = (-1)^{\mathrm{rk}(\mathcal{A})}. \tag{10}$$

For any flat X,

$$\sum_{\mathrm{Y}:\, \mathrm{Y} \leq \mathrm{X}} (-1)^{\mathrm{rk}(\mathrm{Y})} c^{\mathrm{Y}} = (-1)^{\mathrm{rk}(\mathrm{X})}, \tag{11}$$

where c^{Y} is the number of faces of support Y.

The two identities are equivalent. Applying the first to $\mathcal{A}^{\mathrm{X}}$ yields the second, while applying the second to $\mathrm{X} = \top$ yields the first.

3 Characteristic polynomial and Zaslavsky formula

3.1 Möbius number of an arrangement

For any arrangement $\mathcal{A}$, define

$$\mu(\mathcal{A}) := \mu(\perp, \top).$$

We refer to this as the Möbius number of $\mathcal{A}$.

It is the value of the Möbius function on the largest interval in the lattice of flats $\Pi[\mathcal{A}]$.

Proposition 4. *For any arrangement $\mathcal{A}$,*

$$(-1)^{\mathrm{rk}(\mathcal{A})}\mu(\mathcal{A}) = |\mu(\mathcal{A})|. \tag{12}$$

In other words, the sign of the Möbius number of an arrangement is the same as the parity of its rank.

We omit the proof.

3.2 Zaslavsky formula

The Zaslavsky formula counts the number of chambers in an arrangement in terms of the absolute values of the Möbius function of the lattice of flats.

It is given as follows.

Theorem 1. *For any arrangement $\mathcal{A}$,*

$$\sum_{\mathrm{X} \in \Pi[\mathcal{A}]} |\mu(\mathrm{X}, \top)| = \sum_{\mathrm{X} \in \Pi[\mathcal{A}]} |\mu(\mathcal{A}_{\mathrm{X}})| = c(\mathcal{A}), \tag{13}$$

where $c(\mathcal{A})$ is the number of chambers of $\mathcal{A}$.

Proof. For each flat X, put

$$f(\mathrm{X}) := (-1)^{\mathrm{rk}(\mathrm{X})} c^{\mathrm{X}},$$

where c^{X} is the number of faces of support X. Then, by (11),

$$g(\mathrm{Y}) := \sum_{\mathrm{X}\colon \mathrm{X} \leq \mathrm{Y}} f(\mathrm{X}) = \sum_{\mathrm{X}\colon \mathrm{X} \leq \mathrm{Y}} (-1)^{\mathrm{rk}(\mathrm{X})} c^{\mathrm{X}} = (-1)^{\mathrm{rk}(\mathrm{Y})}.$$

Now, by Möbius inversion (8),

$$f(\top) = \sum_{\mathrm{X}} g(\mathrm{X}) \mu(\mathrm{X}, \top).$$

The result now follows by applying Proposition 4 to each $\mathcal{A}_{\mathrm{X}}$. $\square$

There is a similar formula for face enumeration which is given below.

Corollary 1. *For any arrangement $\mathcal{A}$,*

$$\sum_{\mathrm{X} \le \mathrm{Y}} |\mu(\mathrm{X}, \mathrm{Y})| = \sum_{\mathrm{X} \le \mathrm{Y}} |\mu(\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}})| = d(\mathcal{A}), \tag{14}$$

where $d(\mathcal{A})$ is the number of faces of $\mathcal{A}$. (The sum is over both X and Y.)

Proof. Each face is a chamber of the arrangement under its support. So the result follows by applying the Zaslavsky formula (13) to $\mathcal{A}^{\mathrm{Y}}$ for each flat Y. □

3.3 Characteristic polynomial

For any arrangement $\mathcal{A}$, define a polynomial with integer coefficients in the variable t by

$$\chi(\mathcal{A},t) := \sum_{\mathrm{Y}} \mu(\mathrm{Y},\top)\, t^{\mathrm{rk}(\mathrm{Y})}. \tag{15}$$

This is the characteristic polynomial of $\mathcal{A}$. Its degree equals the rank of $\mathcal{A}$.

If $\mathcal{A}$ has rank 0, then $\chi(\mathcal{A},t) = 1$, independent of t.

Let us now consider the values $t = 0, 1, -1$.

For $t = 0$, only the summand for $\mathrm{Y} = \perp$ contributes to the rhs of (15). Thus,

(16a) $$\chi(\mathcal{A}, 0) = \mu(\mathcal{A}).$$

For $t = 1$, using (4a) and (4b),

(16b) $$\chi(\mathcal{A}, 1) = \begin{cases} 1 & \text{if } \mathcal{A} \text{ has rank } 0, \\ 0 & \text{otherwise.} \end{cases}$$

The case $t = -1$ is nontrivial. Using (12) and the Zaslavsky formula (13),

(16c) $$\chi(\mathcal{A}, -1) = (-1)^{\mathrm{rk}(\mathcal{A})} c(\mathcal{A}),$$

where $c(\mathcal{A})$ is the number of chambers in $\mathcal{A}$.

3.4 Examples

The Möbius number and characteristic polynomial of an arrangement only depend on its lattice of flats. Hence, cisomorphic arrangements have the same Möbius number and characteristic polynomial.

For the rank-one arrangement, we have

(17)
$$c(\mathcal{A}) = 2, \quad d(\mathcal{A}) = 3, \quad \mu(\mathcal{A}) = -1, \quad \chi(\mathcal{A}, t) = t - 1.$$

For the rank-two arrangement of n lines, with $n \geq 2$, we have

(18)
$$\begin{aligned} c(\mathcal{A}) &= 2n, \\ d(\mathcal{A}) &= 4n + 1, \\ \mu(\mathcal{A}) &= n - 1, \\ \chi(\mathcal{A}, t) &= t^2 - nt + n - 1. \end{aligned}$$