Random walks

Swapneel Mahajan

http://www.math.iitb.ac.in/~swapneel

1 Stationary distribution

1.1 Top-eigenvectors

Fix an element $w = \sum_F w^F \mathtt{H}_F$ of the Tits algebra.

The content of w is defined to be the sum of the w^F.

For each flat X, put

$$\lambda_{\mathrm{X}} = \sum_{F\colon \mathrm{s}(F) \leq \mathrm{X}} w^F. \tag{1}$$

In particular, $\lambda_\perp = w^O$, and $\lambda_\top$ is the content of w.

Let $u = \sum_C u^C \mathtt{H}_C$ be a chamber element.

We say that u is a top-eigenvector for w if u is of content 1 and

$$w \cdot u = \lambda\, u \tag{2}$$

for some scalar λ, the eigenvalue.

Thus, the sum of the u^C is 1, and by taking the content in (2), we see that $\lambda = \lambda_\top$ necessarily.

Note in passing that an eigenvector u of w with eigenvalue different from $\lambda_\top$ must have content 0.

Lemma 1. *Let w be an element of the Tits algebra, and u be a chamber element of content 1.*

Then u is a top-eigenvector of w iff for each chamber C,

$$(\lambda_{\top} - \lambda_{\perp})\, u^C = \sum_{F\colon O<F\leq C} w^F\, u^C_F, \tag{3}$$

with

$$u^D_H := \sum_{C\colon HC=D} u^C. \tag{4}$$

Proof. This is a straightforward calculation.

$$\begin{aligned}
w \cdot u &= \big(\sum_{H} w^{H}\,\mathtt{H}_{H}\big) \cdot \big(\sum_{C} u^{C}\,\mathtt{H}_{C}\big) \\
&= \sum_{H,C} w^{H} u^{C}\,\mathtt{H}_{HC} \\
&= \sum_{D} \Big(\sum_{H\colon H \leq D} w^{H} \big(\sum_{C\colon HC=D} u^{C} \big) \Big)\,\mathtt{H}_{D} \\
&= \sum_{D} \big(\sum_{H\colon H \leq D} w^{H} u^{D}_{H} \big)\,\mathtt{H}_{D}
\end{aligned}$$

Comparing with the coefficient of $\mathtt{H}_D$ in $\lambda_\top u$, we obtain

$$\lambda_\top\, u^{D} = \sum_{H\colon H \leq D} w^{H} u^{D}_{H}.$$

The summand in the rhs corresponding to $H = O$ is $w^{O} u^{D}_{O} = \lambda_\perp u^{D}$.

Bringing it to the lhs yields (3). $\square$

Given w, for any face F, define $w_F := \Delta_F(w)$. It is an element of the Tits algebra of $\mathcal{A}_F$.

Explicitly,

(5)

$$w_F = \sum_{G\colon G \geq F} w_F^G \, \mathtt{H}_{G/F}, \quad \text{where} \quad w_F^G := \sum_{K\colon FK=G} w^K.$$

Similarly, given u, for any face F, define $u_F := \Delta_F(u)$. Explicitly,

$$u_F = \sum_{C\colon C \geq F} u_F^C \, \mathtt{H}_{C/F}.$$

Since Δ_F is an algebra homomorphism, we have the key fact:

If u is an eigenvector for w, then u_F is an eigenvector for w_F, with the same eigenvalue.

1.2 Brown-Diaconis formula

We say an element w of the Tits algebra is top-separating if $\lambda_{\mathrm{X}} \neq \lambda_{\top}$ for any $\mathrm{X} \neq \top$.

Our goal now is to show that a top-separating element has a unique top-eigenvector.

Lemma 2. *Let w be an element of the Tits algebra such that $\lambda_\top \neq \lambda_\perp$.*

Suppose for each face $H > O$, we are given a top-eigenvector v_H of w_H such that the v_H satisfy the following compatibility conditions. For any $G \geq H$,

$$(v_H)_{G/H} = v_G, \tag{6}$$

and for any F and G with the same support,

$$\beta_{G,F}(v_F) = v_G. \tag{7}$$

Then there is a unique top-eigenvector u of w satisfying

$$u_H = v_H \tag{8}$$

for each $H > O$.

Explicitly, for a chamber C, the scalar u^C is given by

$$u^C = \frac{1}{\lambda_\top - \lambda_\perp} \sum_{F\colon O<F\leq C} w^F v_F^C. \tag{9}$$

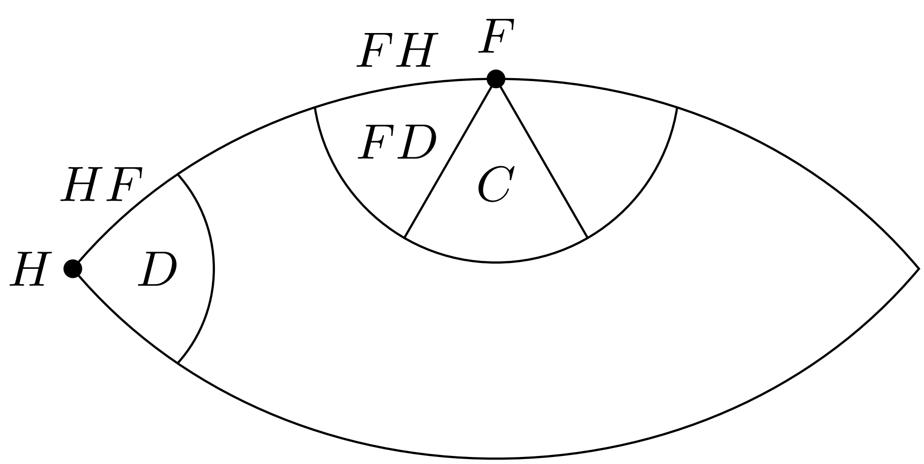
Proof. Suppose u is a top-eigenvector of w satisfying (8) for each $H > O$. Since $\lambda_\top \neq \lambda_\perp$, (3) can be rewritten as in (9). This proves uniqueness of u.

For existence of u, we check below that the u defined by (9) satisfies (8).

$$\begin{aligned} u_H^D &= \sum_{C\colon HC=D} u^C \\ &= \frac{1}{\lambda_\top - \lambda_\perp} \sum_{C\colon HC=D} \; \sum_{F\colon O<F\leq C} w^F \, v_F^C \\ &= \frac{1}{\lambda_\top - \lambda_\perp} \sum_{F\colon O<F,\, HF\leq D} w^F \sum_{C\colon F\leq C,\, HC=D} v_F^C . \end{aligned}$$

The condition $HC = D$ in the inside sum can be replaced by $FHC = FD$, so by (6), the inside sum equals v_{FH}^{FD}, and by (7), this further equals v_{HF}^{D}.

This is illustrated below.



Substituting, the calculation continues as follows.

$$
\begin{aligned}
u_H^D &= \frac{1}{\lambda_\top - \lambda_\perp} \sum_{F\colon O<F,\, HF\le D} w^F\, v_{HF}^D \\
&= \frac{1}{\lambda_\top - \lambda_\perp} \sum_{G\colon H\le G\le D} v_G^D \sum_{F\colon O<F,\, HF=G} w^F \\
&= \frac{1}{\lambda_\top - \lambda_\perp} \left((w_H^H - \lambda_\perp)\, v_H^D + \sum_{G\colon H<G\le D} w_H^G\, v_G^D \right) \\
&= \frac{1}{\lambda_\top - \lambda_\perp} \left((w_H^H - \lambda_\perp)\, v_H^D + (\lambda_\top - w_H^H)\, v_H^D \right) \\
&= v_H^D.
\end{aligned}
$$

In the third step, we broke the first sum depending on whether $G = H$ or $G > H$, and used (5). In the second-last step, we used (3) for the eigenvector v_H of w_H.

Finally, to see that u has content 1, we recall that u and u_H have the same content, and $u_H = v_H$ and v_H has content 1. □

Theorem 1. *Suppose w is a top-separating element of the Tits algebra.*

Then w has a unique top-eigenvector u.

Its eigenvalue is $\lambda_\top$.

Explicitly, in rank at least one, for a chamber C, the scalar u^C is given by

(10)

$$u^C = \frac{w^C}{\lambda_\top - \lambda_\perp} + \sum_{O<F<C} \frac{w^F w^C_F}{(\lambda_\top - \lambda_\perp)(\lambda_\top - \lambda_{\mathrm{s}(F)})} +$$

$$+ \sum_{O<F<G<C} \frac{w^F w^G_F w^C_G}{(\lambda_\top - \lambda_\perp)(\lambda_\top - \lambda_{\mathrm{s}(F)})(\lambda_\top - \lambda_{\mathrm{s}(G)})} + \dots.$$

The first sum is over F, the second sum is over F and G, and so on. (The top-separating condition ensures that the denominators are nonzero.)

We refer to (10) as the Brown-Diaconis formula.

Proof. We show that w has a unique top-eigenvector u by induction on the rank of $\mathcal{A}$.

For rank 0, $C = O$ and clearly $u^C = 1$ is the unique eigenvector. This is the induction base.

Since w is top-separating, so is w_H for any face H. Hence by the induction hypothesis, for each $H > O$, the element w_H has a unique top-eigenvector, say v_H.

By uniqueness, the v_H must satisfy the compatibility conditions (6) and (7).

Therefore by Lemma 2, w has a unique top-eigenvector u satisfying $u_H = v_H$. This proves both existence and uniqueness of u.

Formula (10) follows by inductively applying (9) to each u^C_F. $\square$

Example. Let $\mathcal{A}$ be the rank-one arrangement consisting of two chambers C and $\overline{C}$.

An element of the Tits algebra w is top-separating if $w^C + w^{\overline{C}} \neq 0$.

If this happens, then w has a unique top-eigenvector u whose coefficients are

$$u^C = \frac{w^C}{w^C + w^{\overline{C}}} \quad \text{and} \quad u^{\overline{C}} = \frac{w^{\overline{C}}}{w^C + w^{\overline{C}}}.$$

Only the first term in (10) contributed.

For any rank-two arrangement, the unique top-eigenvector u of a top-separating element w has coefficients

$$u^C = \frac{w^C}{\lambda_\top - \lambda_\perp} + \frac{w^P w^C_P}{(\lambda_\top - \lambda_\perp)(\lambda_\top - \lambda_{\mathrm{s}(P)})} + \frac{w^Q w^C_Q}{(\lambda_\top - \lambda_\perp)(\lambda_\top - \lambda_{\mathrm{s}(Q)})},$$

where P and Q are the two vertices of C.

1.3 Stationary distribution

Suppose the scalars w^F are nonnegative and add up to 1.

Then w can be interpreted as a probability distribution on the set of faces.

It induces a random walk on the set of chambers: Suppose we are currently in chamber C. Then pick a face F at random (with probability w^F) and move to FC.

With this interpretation, a top-eigenvector u for w is the same as a stationary distribution for this random walk (provided all coefficients of u are nonnegative).

Theorem 2. *Suppose w is a top-separating probability distribution on the set of faces. Then the associated random walk has a unique stationary distribution u given by the Brown-Diaconis formula* (10)*.*

This is essentially a restatement of Theorem 1 with a small additional observation.

We need to know that the coefficients of the eigenvector u are nonnegative, but this is clear from Brown-Diaconis formula.

2 Diagonalizability and eigensections

By definition, an element of the Tits algebra is diagonalizable if it can be expressed as a linear combination of mutually orthogonal idempotents.

We show that elements which satisfy a separating condition or a nonnegativity condition are diagonalizable.

The key idea is to choose an appropriate homogeneous section $\mathtt{u}$ so that the given element w can be expressed using the Eulerian family associated to $\mathtt{u}$.

We refer to such a $\mathtt{u}$ as an eigensection of w.

2.1 Eigensections

For any element of the Tits algebra $w = \sum_F w^F \mathtt{H}_F$, set

(11)

$$w^{\mathrm{X}} := \sum_{F\colon \mathrm{s}(F) \leq \mathrm{X}} w^F \mathtt{H}_F \quad \text{and} \quad w_{\mathrm{X}} := \sum_{F\colon \mathrm{s}(F) = \mathrm{X}} w^F \mathtt{H}_F.$$

By definition,

$$w^{\mathrm{X}} = \sum_{\mathrm{Y}\colon \mathrm{Y} \leq \mathrm{X}} w_{\mathrm{Y}}.$$

Definition 3. Let w be an element of the Tits algebra and $\mathtt{u}$ be a homogeneous section.

We say that $\mathtt{u}$ is an eigensection for w if there exist scalars $\lambda = (\lambda_{\mathrm{X}})$ indexed by flats X, such that for any flat X,

$$w^{\mathrm{X}} \cdot \mathtt{u}_{\mathrm{X}} = \lambda_{\mathrm{X}}\, \mathtt{u}_{\mathrm{X}}. \tag{12}$$

We refer to $\lambda = (\lambda_{\mathrm{X}})$ as the eigenvalues of w.

Observe that an eigensection of w is the same as a family $(\mathtt{u}_{\mathrm{X}})$, where each $\mathtt{u}_{\mathrm{X}}$ is a top-eigenvector of w^{X} in the arrangement $\mathcal{A}^{\mathrm{X}}$.

Since $\mathtt{u}_{\mathrm{X}}$ has content 1, taking the content of both sides of (12), we note that λ_{X} is given by (1). In particular, it depends only on w and not on the choice of $\mathtt{u}$.

Proposition 1. *Given a homogeneous section* $\mathtt{u}$ *and* $\lambda = (\lambda_{\mathrm{X}})$*, there exists a unique* w *with eigenvalues* λ *and eigensection* $\mathtt{u}$*.*

Proof. To construct w, we need to construct w_{X} for each flat X. We do that by induction on the rank of X. Setting $\mathrm{X} := \perp$ in (12) and using $\mathtt{u}_{\perp} = \mathtt{H}_O$ yields

$$w^{\perp} = w_{\perp} = \lambda_{\perp} \mathtt{H}_O.$$

Now suppose that w_{Y} are uniquely constructed for all $\mathrm{Y} < \mathrm{X}$. To construct w_{X}, we need to solve the equation

$$\Big(w_{\mathrm{X}} + \sum_{\mathrm{Y}:\, \mathrm{Y}<\mathrm{X}} w_{\mathrm{Y}}\Big) \cdot \mathtt{u}_{\mathrm{X}} = \lambda_{\mathrm{X}}\, \mathtt{u}_{\mathrm{X}}.$$

(This is a reformulation of (12).) We know that $w_{\mathrm{X}} \cdot \mathtt{u}_{\mathrm{X}} = w_{\mathrm{X}}$ always holds. Thus

$$w_{\mathrm{X}} := \lambda_{\mathrm{X}}\, \mathtt{u}_{\mathrm{X}} - \Big(\sum_{\mathrm{Y}:\, \mathrm{Y}<\mathrm{X}} w_{\mathrm{Y}}\Big) \cdot \mathtt{u}_{\mathrm{X}}$$

is the unique solution. This completes the induction step. $\square$

A more precise result is given below.

Proposition 2. *Given a triple* $(w, \lambda, \mathtt{u})$*,*

(13)

$$\mathtt{u} \textit{ is an eigensection of } w \textit{ with eigenvalues } \lambda \iff w = \sum_{\mathrm{X}} \lambda_{\mathrm{X}}\, \mathtt{E}_{\mathrm{X}},$$

where $\mathtt{E}$ *is the Eulerian family associated to* $\mathtt{u}$*.*

Proof. Forward implication. Since the sum of the $\mathtt{E}_{\mathrm{X}}$ is $\mathtt{H}_O$, it suffices to show that $w \cdot \mathtt{E}_{\mathrm{X}} = \lambda_{\mathrm{X}}\, \mathtt{E}_{\mathrm{X}}$. This follows from:

$$w \cdot \mathtt{E}_{\mathrm{X}} = w^{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{X}} = w^{\mathrm{X}} \cdot \mathtt{u}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{X}} = \lambda_{\mathrm{X}}\, \mathtt{u}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{X}} = \lambda_{\mathrm{X}}\, \mathtt{E}_{\mathrm{X}}.$$

The first equality used the Saliola lemma. The remaining ones used $\mathtt{u}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{X}} = \lambda_{\mathrm{X}}$ and (12).

Backward implication. We provide two arguments. Applying Proposition 1, let v be the unique element with eigenvalues λ and eigensection $\mathtt{u}$. Now apply the forward implication to $(v, \lambda, \mathtt{u})$ to obtain

$$v = \sum_{\mathrm{X}} \lambda_{\mathrm{X}}\, \mathtt{E}_{\mathrm{X}}.$$

Therefore $v = w$.

Alternatively: Restricting $w = \sum_{\mathrm{Y}} \lambda_{\mathrm{Y}}\, \mathtt{E}_{\mathrm{Y}}$ to faces of support smaller than X,

$$w^{\mathrm{X}} = \sum_{\mathrm{Y}\colon \mathrm{Y} \leq \mathrm{X}} \lambda_{\mathrm{Y}}\, \mathtt{E}^{\mathrm{X}}_{\mathrm{Y}},$$

where $\mathtt{E}^{\mathrm{X}}_{\mathrm{Y}}$ is the part of $\mathtt{E}_{\mathrm{Y}}$ consisting of faces of support smaller than X. In particular, $\mathtt{E}^{\mathrm{X}}_{\mathrm{X}} = \mathtt{u}_{\mathrm{X}}$. Hence

$$w^{\mathrm{X}} \cdot \mathtt{u}_{\mathrm{X}} = \sum_{\mathrm{Y}\colon \mathrm{Y} \leq \mathrm{X}} \lambda_{\mathrm{Y}}\, \mathtt{E}^{\mathrm{X}}_{\mathrm{Y}} \cdot \mathtt{E}^{\mathrm{X}}_{\mathrm{X}} = \lambda_{\mathrm{X}}\, \mathtt{E}^{\mathrm{X}}_{\mathrm{X}} = \lambda_{\mathrm{X}}\, \mathtt{u}_{\mathrm{X}}.$$

The third equality used orthogonality. $\square$

Since every complete system is an Eulerian family, every diagonalizable element can be diagonalized using an Eulerian family.

In conjunction with Proposition 2, we obtain:

Corollary 1. *An element of the Tits algebra is diagonalizable iff it has an eigensection.*

2.2 Diagonalizability for separating elements and the Brown formulas

We say an element w of the Tits algebra is separating if for any $\mathrm{X} < \mathrm{Y}$, we have $\lambda_{\mathrm{X}} \neq \lambda_{\mathrm{Y}}$.

This condition is stronger than the top-separating condition. More precisely, w is separating iff w^{X} (viewed as an element of the Tits algebra of $\mathcal{A}^{\mathrm{X}}$) is top-separating for each X.

Theorem 4. *Suppose w is a separating element of the Tits algebra.*

Then w has a unique eigensection $\mathtt{u}$.

Explicitly, $\mathtt{u}_F^F = 1$ and for $F < G$,

(14)

$$\mathtt{u}_F^G = \frac{w_F^G}{\lambda_{\mathrm{s}(G)} - \lambda_{\mathrm{s}(F)}} + \sum_{F<H<G} \frac{w_F^H w_H^G}{(\lambda_{\mathrm{s}(G)} - \lambda_{\mathrm{s}(F)})(\lambda_{\mathrm{s}(G)} - \lambda_{\mathrm{s}(H)})}$$

$$+ \sum_{F<H<K<G} \frac{w_F^H w_H^K w_K^G}{(\lambda_{\mathrm{s}(G)} - \lambda_{\mathrm{s}(F)})(\lambda_{\mathrm{s}(G)} - \lambda_{\mathrm{s}(H)})(\lambda_{\mathrm{s}(G)} - \lambda_{\mathrm{s}(K)})} + \dots$$

and $\mathtt{u}^G = \mathtt{u}_O^G$.

The first sum is over H, the second sum is over H and K, and so on. The scalars w_F^G are as in (5).

Further, w is diagonalizable, with $w = \sum_{\mathrm{X}} \lambda_{\mathrm{X}} \mathtt{E}_{\mathrm{X}}$ for a unique Eulerian family $\mathtt{E}$.

Proof. The last claim follows from the first by (13).

For the first claim: To construct $\mathtt{u}$, we need to construct each $\mathtt{u}_{\mathrm{X}}$.

This is a top-eigenvector of w^{X} in $\mathcal{A}^{\mathrm{X}}$, and we can apply Theorem 1. $\square$

The following identity is useful to invert the matrix $(\mathtt{u}_F^G)$.

Lemma 3. *Let $x_0, x_1, \ldots, x_n$ be distinct scalars. Then*

$$(-1)^n \prod_{i=1}^{n} \frac{1}{x_i - x_0}$$

$$= \sum_{(a_1,\ldots,a_k)\vDash n} (-1)^k \prod_{j=1}^{k} \frac{1}{(x_{b_j} - x_{b_j - 1}) \ldots (x_{b_j} - x_{b_{j-1}})},$$

where $b_j = a_1 + \cdots + a_j$ and $b_0 = 0$.

The sum is over all compositions $(a_1, \ldots, a_k)$ of n.

Proof. Note that $x_n - x_{n-1}$ appears in all terms in the rhs.

Split the rhs into two depending on whether $a_k = 1$ or $a_k > 1$.

Denoting the rhs by $f(x_0, \ldots, x_n)$, this yields the recursion

$$f(x_0, \ldots, x_n) = \frac{1}{x_n - x_{n-1}} \big(-f(x_0, \ldots, x_{n-1}) + f(x_0, \ldots, x_{n-2}, x_n) \big).$$

Note that in the second term, the variable x_{n-1} is absent.

Solving this recursion yields the result. $\square$

Theorem 5. *Let w be a separating element in the Tits algebra, and $\mathtt{u}$ be its unique eigensection. Let $\mathtt{E}$ be the associated Eulerian family, and $\mathtt{Q}$ the associated basis. Then*

(15)

$$\mathtt{E}_{\mathrm{X}} = \sum_{F\colon \mathrm{s}(F)=\mathrm{X}} \; \sum_{G\colon F \leq G} \mathtt{u}^F \mathtt{a}_F^G \, \mathtt{H}_G \quad \textit{and} \quad \mathtt{Q}_F = \sum_{G\colon F\leq G} \mathtt{a}_F^G \, \mathtt{H}_G,$$

where

(16)

$$\mathtt{a}_F^G = -\frac{w_F^G}{\lambda_{\mathrm{s}(G)} - \lambda_{\mathrm{s}(F)}} + \sum_{F<H<G} \frac{w_F^H w_H^G}{(\lambda_{\mathrm{s}(H)} - \lambda_{\mathrm{s}(F)})(\lambda_{\mathrm{s}(G)} - \lambda_{\mathrm{s}(F)})}$$
$$- \sum_{F<H<K<G} \frac{w_F^H w_H^K w_K^G}{(\lambda_{\mathrm{s}(H)} - \lambda_{\mathrm{s}(F)})(\lambda_{\mathrm{s}(K)} - \lambda_{\mathrm{s}(F)})(\lambda_{\mathrm{s}(G)} - \lambda_{\mathrm{s}(F)})} + \dots.$$

The first sum is over H, the second sum is over H and K, and so on.

Proof. Formulas (15) are general formulas we have seen before.

The nontrivial part is to obtain the formula for $\mathtt{a}_F^G$.

For this, substitute (14) in

$$\mathtt{a}_F^G = -\mathtt{u}_F^G + \sum_{F<H<G} \mathtt{u}_F^H \mathtt{u}_H^G - \sum_{F<H<K<G} \mathtt{u}_F^H \mathtt{u}_H^K \mathtt{u}_K^G + \dots,$$

collect together the terms involving w_F^G, $w_F^H w_H^G$, and so on, and simplify each coefficient using Lemma 3.

This yields (16). $\square$

We refer to (15), with $\mathtt{u}^F$ and $\mathtt{a}_F^G$ given by (14) and (16), as the Brown formulas for the Eulerian idempotents of a separating element.

3 Braid arrangement

Let $\mathcal{A}$ be the braid arrangement on p letters.

3.1 Riffle shuffle

A riffle shuffle is a common method used by people to shuffle a deck of cards.

It is described mathematically by the Gilbert-Shannon-Reeds model:

Cut the deck of cards into two heaps according to a binomial distribution, and then riffle them together such that cards drop from the left or right heaps with probability proportional to the number of cards in each heap.

The n-shuffle, for any integer $n \geq 2$, can be defined in a similar manner by cutting the deck of cards into n ordered heaps and riffling them together.

The 2-shuffle is the same as the riffle shuffle.

The inverse n-shuffle works as follows:

Label each card randomly with an integer from 1 to n.

Move all the cards labeled 1 to the bottom of the deck, preserving their relative order.

Next move all the cards labeled 2 above these again preserving their relative order and so on.

3.2 AS elements

For any set composition G, let $\deg(G)$ denote the number of blocks of G.

Similarly, for any set partition X, let $\deg(\mathrm{X})$ denotes the number of blocks of X.

Put

$$\mathsf{T}_k := \sum_{F\colon \deg(F)=k} \mathtt{H}_F.$$

For any integer n, define the AS element of parameter n to be

$$\mathrm{AS}_n := \sum_F \binom{n}{\deg(F)} \mathtt{H}_F = \sum_{k=1}^{p} \binom{n}{k} \mathsf{T}_k. \tag{17}$$

Up to normalization, the inverse n-shuffle is precisely the the linear operator resulting from the action of the element AS_n on the module of chambers.

Lemma 4. *For any integer m, and any set partition X,*

$$\sum_{F\colon \mathrm{s}(F)\leq \mathrm{X}} \binom{m}{\deg(F)} = m^{\deg(\mathrm{X})},$$

where $\deg(\mathrm{X})$ denotes the number of blocks of X.

Proof. First assume m is positive. Take m boxes labeled 1 to m. The rhs counts the number of ways of putting each block of X in one of the m boxes. Each such assignment yields a set composition F with $\mathrm{s}(F) \leq \mathrm{X}$: each box is a block of F with empty boxes deleted. The number of assignments which yield the same F is precisely $\binom{m}{\deg(F)}$. This is the lhs. This proves the identity for m positive.

For the general case, note that both sides are polynomials in m. Since they agree for infinitely many values of m, they must agree for all values of m. $\square$

By Lemma 4 and definition (1), we obtain:

Lemma 5. *The eigenvalues of* AS_n *are given by*

$$\lambda_{\mathrm{X}} = n^{\deg(\mathrm{X})}$$

for each set partition X*.*

Let E_{X} denote the Eulerian idempotents for the uniform section. Put

$$\mathtt{E}_k := \sum_{\mathrm{X}:\, \deg(\mathrm{X})=k} \mathtt{E}_{\mathrm{X}} \quad \text{for } 1 \leq k \leq p. \tag{18}$$

Proposition 3. *The* AS *elements diagonalize as follows.*

$$\mathrm{AS}_n = \sum_{\mathrm{X}} n^{\deg(\mathrm{X})}\, \mathtt{E}_{\mathrm{X}} = \sum_{k=1}^{p} n^k\, \mathtt{E}_k. \tag{19}$$

Proof. Put $w := \mathrm{AS}_n$.

Note that w^{X} is invariant under the action of the symmetric group on the blocks of X.

So the uniform section is an eigensection of w.

Hence (19) holds by Proposition 2 in view of the eigenvalue calculation done above. $\square$

As a consequence of (19):

Lemma 6. *For any integers m and n,*

$$\mathrm{AS}_m \cdot \mathrm{AS}_n = \mathrm{AS}_{mn}. \tag{20}$$

Similarly:

Theorem 6. *For the braid arrangement on p letters, the subalgebra of the Tits algebra generated by the elements AS_n is a split-semisimple commutative algebra, with primitive idempotents E_k, for $1 \leq k \leq p$.*

Let us now consider the parameter value $n = -1$. Using definition (17),

$$\mathrm{AS}_{-1} = \sum_{F} (-1)^{\deg(F)}\, \mathtt{H}_F.$$

It diagonalizes as

$$\mathrm{AS}_{-1} \;=\; \sum_{\mathrm{X}} (-1)^{\deg(\mathrm{X})}\, \mathtt{E}_{\mathrm{X}} \;=\; \sum_{k=1}^{p} (-1)^k\, \mathtt{E}_k.$$

3.3 Degrees and factorials

For set compositions $F \leq G$, let $(G/F)_i$ denote the set composition consisting of those contiguous blocks of G which refine the i-th block of F.

For any set composition G, recall that $\deg(G)$ denotes the number of blocks of G. More generally, for $F \leq G$, let

$$\deg(G/F) = \prod_i \deg(G/F)_i. \tag{21}$$

In particular, $\deg(G/O) = \deg(G)$.

For F, a set composition consisting of two blocks, $\deg(G/F)$ is the product of two numbers, one for each block of F, as in the following example.

$$F = krish|na, \quad G = kr|i|sh|n|a, \quad \deg(G/F) = 3.2 = 6.$$

Here $kr|i|sh$ which refines $krish$ has 3 blocks, while $n|a$ which refines na has 2 blocks.

3.4 Eulerian idempotents

Theorem 7. *The Eulerian idempotents for the uniform section are given by*

(22)

$$\mathtt{E}_{\mathrm{X}} = \frac{1}{\deg!(\mathrm{X})} \sum_{F \colon \mathrm{s}(F)=\mathrm{X}} \; \sum_{G \colon F \leq G} \frac{(-1)^{\mathrm{rk}(G/F)}}{\deg(G/F)} \, \mathtt{H}_G,$$

where $\deg!(\mathrm{X})$ *is the factorial of the number of blocks of* X*.*

In particular, the first Eulerian idempotent is

$$\mathtt{E}_{\perp} = \sum_{F} \frac{(-1)^{\mathrm{rk}(F)}}{\deg(F)} \, \mathtt{H}_F. \tag{23}$$

For $p = 2$, the two Eulerian idempotents are

$$\mathtt{E}_{\top} = \mathtt{E}_{1,2} = \frac{1}{2}(\mathtt{H}_{1|2}+\mathtt{H}_{2|1}) \quad \text{and} \quad \mathtt{E}_{\perp} = \mathtt{E}_{12} = \mathtt{H}_{12}-\frac{1}{2}(\mathtt{H}_{1|2}+\mathtt{H}_{2|1}).$$

For $p = 3$, the five Eulerian idempotents are

$$\mathtt{E}_{\top} = \mathtt{E}_{1,2,3} = \frac{1}{6}(\mathtt{H}_{1|2|3} + \mathtt{H}_{1|3|2} + \mathtt{H}_{2|3|1} + \mathtt{H}_{2|1|3} + \mathtt{H}_{3|1|2} + \mathtt{H}_{3|2|1}),$$

$$\mathtt{E}_{1,23} = \frac{1}{2}(\mathtt{H}_{1|23} + \mathtt{H}_{23|1}) - \frac{1}{4}(\mathtt{H}_{1|2|3} + \mathtt{H}_{1|3|2} + \mathtt{H}_{2|3|1} + \mathtt{H}_{3|2|1}),$$

$$\mathtt{E}_{2,13} = \frac{1}{2}(\mathtt{H}_{2|13} + \mathtt{H}_{13|2}) - \frac{1}{4}(\mathtt{H}_{2|1|3} + \mathtt{H}_{2|3|1} + \mathtt{H}_{1|3|2} + \mathtt{H}_{3|2|1}),$$

$$\mathtt{E}_{3,12} = \frac{1}{2}(\mathtt{H}_{3|12} + \mathtt{H}_{12|3}) - \frac{1}{4}(\mathtt{H}_{3|1|2} + \mathtt{H}_{3|2|1} + \mathtt{H}_{1|2|3} + \mathtt{H}_{2|1|3}),$$

$$\begin{aligned}\mathtt{E}_{\perp} = \mathtt{E}_{123} = \mathtt{H}_{123} &- \frac{1}{2}(\mathtt{H}_{1|23} + \mathtt{H}_{23|1} + \mathtt{H}_{2|13} + \mathtt{H}_{13|2} + \mathtt{H}_{3|12} + \mathtt{H}_{12|3}) \\ &+\frac{1}{3}(\mathtt{H}_{1|2|3} + \mathtt{H}_{1|3|2} + \mathtt{H}_{2|3|1} + \mathtt{H}_{2|1|3} + \mathtt{H}_{3|1|2} + \mathtt{H}_{3|2|1}).\end{aligned}$$