# Leray–Samelson Isomorphisms

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# 1 Introduction

Given a species $\mathrm{p}$, we can construct a bicommutative bimonoid $\mathcal{S}(\mathrm{p})$.

We have also seen that $\mathcal{P}(\mathcal{S}(\mathrm{p})) = \mathrm{p}$.

In the last talk, we saw that the composition of these two functors in the other order, that is, $\mathcal{S}(\mathcal{P}(\mathrm{h}))$ is isomorphic to $\mathrm{h}$, for any bicommutative bimonoid $\mathrm{h}$. These were the Leray–Samelson isomorphisms.

I aim to explain the relation of these isomorphisms to the exp-log correspondence.

## 2 Recall

The bicommutative bimonoind $\mathcal{S}(\mathsf{p})$ constructed from a species $\mathsf{p}$ is

(1) $$\mathcal{S}(\mathsf{p})[\mathrm{Z}] := \bigoplus_{\mathrm{X}:\,\mathrm{X}\geq\mathrm{Z}} \mathsf{p}[\mathrm{X}].$$

It is endowed with product and coproduct structure as,

$$
\begin{tikzcd}
\mathcal{S}(\mathsf{p})[\mathrm{X}] \arrow[r, dotted, "\mu_{\mathrm{Z}}^{\mathrm{X}}"] & \mathcal{S}(\mathsf{p})[\mathrm{Z}] \\
\mathsf{p}[\mathrm{Y}] \arrow[u] \arrow[r, "\mathrm{id}"'] & \mathsf{p}[\mathrm{Y}] \arrow[u]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{S}(\mathsf{p})[\mathrm{Z}] \arrow[r, dotted, "\Delta_{\mathrm{Z}}^{\mathrm{X}}"] & \mathcal{S}(\mathsf{p})[\mathrm{X}] \\
\mathsf{p}[\mathrm{Y}] \arrow[u] \arrow[r] & \begin{cases} \mathsf{p}[\mathrm{Y}] \text{ if } \mathrm{Y} \geq \mathrm{X}, \\ 0 \text{ otherwise.} \end{cases} \arrow[u]
\end{tikzcd}
$$

## 3 The Claim

Consider the two biderivations $f$ and $g$ as given below,

$$
\begin{tikzcd}
\mathcal{S}(\mathcal{P}(\mathsf{h})) \arrow[r, "f"] \arrow[d, two heads] & \mathsf{h} \\
\mathcal{P}(\mathsf{h}) \arrow[r, "\mathrm{id}"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathsf{h} \arrow[r, "g"] \arrow[d, two heads] & \mathcal{S}(\mathcal{P}(\mathsf{h})) \\
\mathcal{Q}(\mathsf{h}) \arrow[r, "\log(\mathrm{id})"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook]
\end{tikzcd}
$$

where $\mathsf{h}$ is a bicommutative bimonoid. Then $\exp(f)$ and $\exp(g)$ are the Leray–Samelson isomorphisms which we encountered before.

Remark: A biderivation factors through the indecomposable part of the bimonoid. In the first diagram, we have used the fact that $\mathcal{Q}(\mathcal{S}(\mathsf{p})) = \mathsf{p}$.

# 4 The Proofs

## 4.1 First Diagram

$$\begin{tikzcd}
\mathcal{S}(\mathcal{P}(\mathsf{h})) \arrow[r, "f"] \arrow[d, two heads] & \mathsf{h} \\
\mathcal{P}(\mathsf{h}) \arrow[r, "\mathrm{id}"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook]
\end{tikzcd}$$

On $\mathrm{Z}$-component, on $\mathrm{Y}$-summand with $x \in \mathcal{P}(\mathsf{h})[\mathrm{Y}]$, with $\mathrm{Y} \geq \mathrm{Z}$,

$$\begin{aligned}
\exp(f)_{\mathrm{Z}}(x) &= \sum_{\mathrm{X}:\mathrm{X}\geq\mathrm{Z}} \mu_{\mathrm{Z}}^{\mathrm{X}} f_{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}(x) \\
&= \sum_{\mathrm{X}:\mathrm{Y}\geq\mathrm{X}\geq\mathrm{Z}} \mu_{\mathrm{Z}}^{\mathrm{X}} f_{\mathrm{X}}(x), \quad x \in \mathcal{P}(\mathsf{h})[\mathrm{Y}] \\
&= \mu_{\mathrm{Z}}^{\mathrm{Y}}(x), \qquad\qquad\qquad\qquad x \in \mathsf{h}[\mathrm{Y}].
\end{aligned}$$

Compare with the map from $\mathcal{S}(\mathcal{P}(\mathsf{h})) \to \mathsf{h}$ given on the $\mathrm{Z}$-component, on the $\mathrm{Y}$-summand by

$$\mathcal{P}(\mathsf{h})[\mathrm{Y}] \hookrightarrow \mathsf{h}[\mathrm{Y}] \xrightarrow{\mu_{\mathrm{Z}}^{\mathrm{Y}}} \mathsf{h}[\mathrm{Z}].$$

### 4.2 Second Diagram

$$\begin{tikzcd}
\mathsf{h} \arrow[r, "g"] \arrow[d, two heads] & \mathcal{S}(\mathcal{P}(\mathsf{h})) \\
\mathcal{Q}(\mathsf{h}) \arrow[r, "\log(\mathrm{id})"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook]
\end{tikzcd}$$

On the $\mathrm{Z}$-component with $x \in \mathsf{h}[\mathrm{Z}]$

$$\begin{aligned}
\exp(g)_{\mathrm{Z}}(x) &= \sum_{\mathrm{X}:\mathrm{X}\geq\mathrm{Z}} \mu_{\mathrm{Z}}^{\mathrm{X}} g_{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}(x) \\
&= \sum_{\mathrm{X}:\mathrm{X}\geq\mathrm{Z}} g_{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}(x) \\
&= \sum_{\mathrm{X}:\mathrm{X}\geq\mathrm{Z}} \left( \sum_{\mathrm{Y}:\mathrm{Y}\geq\mathrm{X}} \mu(\mathrm{X},\mathrm{Y}) \mu_{\mathrm{X}}^{\mathrm{Y}} \Delta_{\mathrm{X}}^{\mathrm{Y}} \Delta_{\mathrm{Z}}^{\mathrm{X}}(x) \right) \\
&= \sum_{\mathrm{X}:\mathrm{X}\geq\mathrm{Z}} \left( \sum_{\mathrm{Y}:\mathrm{Y}\geq\mathrm{X}} \mu(\mathrm{X},\mathrm{Y}) \mu_{\mathrm{X}}^{\mathrm{Y}} \Delta_{\mathrm{Z}}^{\mathrm{Y}}(x) \right).
\end{aligned}$$

Restricting to the $\mathrm{X}$-summand, we get a map,

$$\sum_{\mathrm{Y}:\mathrm{Y}\geq\mathrm{X}} \mu(\mathrm{X},\mathrm{Y}) \mu_{\mathrm{X}}^{\mathrm{Y}} \Delta_{\mathrm{Z}}^{\mathrm{Y}} : \mathsf{h}[\mathrm{Z}] \to \mathcal{P}(\mathsf{h})[\mathrm{X}]$$

# 5 Putting it together

## 5.1 Morphism of Bimonoids

Recall that we proved for a cocommutative bimonoid $\mathsf{h}$ and commutative bimonoid $\mathsf{k}$, we have inverse bijections

$$\mathcal{A}\text{-}\mathsf{Sp}(\mathcal{Q}(\mathsf{h}), \mathcal{P}(\mathsf{k})) \underset{\log}{\overset{\exp}{\rightleftarrows}} \mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h}, \mathsf{k}) \ .$$

Hence, $\exp(f)$ and $\exp(g)$ are morphisms of bimonoids.

## 5.2 Inverse

Let us recall another lemma that if $f$ and $g$ are biderivations then

$$\exp(f)\exp(g) = \exp(fg)$$
$$\exp(g)\exp(f) = \exp(gf).$$

Now we will study the composite maps $fg$, and $gf$.

## 5.3 $fg$ map

$$
\begin{tikzcd}
\mathsf{h} \arrow[r, "g"] \arrow[d, two heads] & \mathcal{S}(\mathcal{P}(\mathsf{h})) \arrow[r, "\mathrm{id}"] & \mathcal{S}(\mathcal{P}(\mathsf{h})) \arrow[r, "f"] \arrow[d, two heads] & \mathsf{h} \\
\mathcal{Q}(\mathsf{h}) \arrow[r, "\log(\mathrm{id})"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] & \mathcal{P}(\mathsf{h}) \arrow[r, "\mathrm{id}"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook]
\end{tikzcd}
$$

It is equivalent to the diagram

$$
\begin{tikzcd}
\mathsf{h} \arrow[r, "fg"] \arrow[d, two heads] & \mathsf{h} \\
\mathcal{Q}(\mathsf{h}) \arrow[r, "\log(\mathrm{id})"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook]
\end{tikzcd}
$$

Hence, $fg = \log(\mathrm{id})$. And

$$\exp(f)\exp(g) = \exp(fg) = \exp(\log(\mathrm{id})) = \mathrm{id}\,.$$

## 5.4 $gf$ map

$$
\begin{tikzcd}
\mathcal{S}(\mathcal{P}(\mathsf{h})) \arrow[r, "f"] \arrow[d, two heads] & \mathsf{h} \arrow[r, "\mathrm{id}"] & \mathsf{h} \arrow[r, "g"] \arrow[d, two heads] & \mathcal{S}(\mathcal{P}(\mathsf{h})) \\
\mathcal{P}(\mathsf{h}) \arrow[r, "\mathrm{id}"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] & \mathcal{Q}(\mathsf{h}) \arrow[r, "\log(\mathrm{id})"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook]
\end{tikzcd}
$$

Let $x \in \mathcal{S}(\mathcal{P}(\mathsf{h}))[\mathrm{Z}]$. In particular, let $x \in \mathcal{P}(\mathsf{h})[\mathrm{Z}]$. Applying the map $g$ going into the $\mathrm{X}$-summand on $x$ gives,

$$
\sum_{\mathrm{X}:\mathrm{X}\geq\mathrm{Z}} \mu(\mathrm{Z},\mathrm{X})\mu_{\mathrm{Z}}^{\mathrm{X}}\Delta_{\mathrm{Z}}^{\mathrm{X}}(x) = \begin{cases} x & \text{if } \mathrm{X}=\mathrm{Z}, \\ 0 & \text{otherwise.} \end{cases}
$$

The above diagram is equivalent to the following diagram although it is not apparent.

$$
\begin{tikzcd}
\mathcal{S}(\mathcal{P}(\mathsf{h})) \arrow[r, "gf"] \arrow[d, two heads] & \mathcal{S}(\mathcal{P}(\mathsf{h})) \\
\mathcal{P}(\mathsf{h}) \arrow[r, "\log(\mathrm{id})"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook]
\end{tikzcd}
$$

Again let $x \in \mathcal{S}(\mathcal{P}(\mathsf{h}))[\mathrm{Z}]$. But this time, let $x \in \mathcal{P}(\mathsf{h})[\mathrm{Y}]$. Applying $\log(\mathrm{id})$ on $x$ with product and coproduct of $\mathcal{S}(\mathcal{P}(\mathsf{h}))$ gives

$$\sum_{\mathrm{X}:\mathrm{X}\geq\mathrm{Z}} \mu(\mathrm{Z},\mathrm{X})\mu_{\mathrm{Z}}^{\mathrm{X}}\Delta_{\mathrm{Z}}^{\mathrm{X}}(x) = \sum_{\mathrm{X}:\mathrm{Y}\geq\mathrm{X}\geq\mathrm{Z}} \mu(\mathrm{Z},\mathrm{X})\,\mathrm{id}(x)$$
$$= \begin{cases} x & \text{if } \mathrm{Y}=\mathrm{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $gf = \log(\mathrm{id})$. And

$$\exp(g)\exp(f) = \exp(gf) = \exp(\log(\mathrm{id})) = \mathrm{id}\,.$$

## 6 Facts

Let $\mathsf{c}$ be a cocommutative comonoid and $\mathsf{k}$ a bicommutative bimonoid. Then,

- For $f : \mathsf{c} \to \mathsf{k}$ a coderivation, its exponential equals

$$\exp(f) : \mathsf{c} \to \mathcal{S}(\mathcal{P}(\mathsf{k})) \xrightarrow{\cong} \mathsf{k}.$$

- For $g : \mathsf{c} \to \mathsf{k}$ a morphism of comonoids, its logarithm equals

$$\log(g) : \mathsf{c} \to \mathsf{k} \xrightarrow{\cong} \mathcal{S}(\mathcal{P}(\mathsf{k})) \twoheadrightarrow \mathcal{P}(\mathsf{k}) \hookrightarrow \mathsf{k}.$$

# 7 Conclusions

- Every bicommutative bimonoid is free as a commutative monoid or cofree as a cocommutative comonoid.
- The Leray–Samelson isomorphisms can be obtained by exponentiating biderivations.
- In turn, we can set up exp-log correspondence.