Lune-incidence Algebra and Non-commutative Möbius Theory

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Based on chapter 15

from Topics in Hyperplane Arrangements

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1 The main result

We will attempt to prove the following theorem:

Theorem 1. *The following pieces of data are equivalent:*

1. *a non-commutative zeta function* $\boldsymbol{\zeta}$ *of* $\mathcal{A}$
2. *a homogeneous section* $\mathtt{u}$ *of* $\mathcal{A}$
3. *an Eulerian family* $\mathtt{E}$ *of* $\mathcal{A}$
4. *a complete system of primitive orthogonal idempotents of* $\Sigma[\mathcal{A}]$
5. *an algebra section of the support map* $\mathrm{s} : \Sigma[\mathcal{A}] \to \Pi[\mathcal{A}]$
6. *a* $\mathtt{Q}$*-basis for* $\Sigma[\mathcal{A}]$
7. *a special Zie family* P *of* $\mathcal{A}$
8. *a non-commutative Möbius function* $\boldsymbol{\mu}$ *of* $\mathcal{A}$

We have seen that *2.* - *7.* are equivalent:

2. $\Rightarrow$ *3.* : Saliola construction

3. $\Rightarrow$ *2.* : base term

4. $\Leftrightarrow$ *5.* : general fact about elementary algebras

3. $\Rightarrow$ *5.* : $\mathtt{Q}_{\mathrm{X}} \mapsto \mathtt{E}_{\mathrm{X}}$

5. $\Rightarrow$ *3.* : image of $\mathtt{Q}_{\mathrm{X}}$ under algebra section

3. $\Rightarrow$ *6.* : $\mathtt{Q}_{\mathrm{F}} = \mathtt{H}_{\mathrm{F}} \cdot \mathtt{E}_{\mathrm{s\,(F)}}$

6. $\Rightarrow$ *3.* : $\mathtt{H}_{O} = \sum_{\mathrm{F}} \mathtt{u}^{\mathrm{F}} \mathtt{Q}_{\mathrm{F}}; \quad \mathtt{u}_{\mathrm{X}} = \sum_{\mathrm{F:s\,(F)=X}} \mathtt{u}^{\mathrm{F}} \mathtt{H}_{\mathrm{F}}$

6. $\Rightarrow$ *7.* : $\mathsf{P}_{\mathrm{X}} = (\beta_{\mathrm{X,F}} \circ \Delta_{\mathrm{F}})\,(\mathtt{Q}_{\mathrm{F}})$

7. $\Rightarrow$ *6.* : $\mathtt{Q}_{\mathrm{F}} = (\mu_{\mathrm{F}} \circ \beta_{\mathrm{F,X}})\,(\mathsf{P}_{\mathrm{X}})$

The theorem is proved by demonstrating the equivalences *1.* $\Leftrightarrow$ *2.* and *7.* $\Leftrightarrow$ *8.*

Objective: define non-commutative zeta functions for $\mathcal{A}$ and show that:

$$\textit{1.} \Leftrightarrow \textit{2.}$$

Due to the limited time, we will omit showing *7.* $\Leftrightarrow$ *8.*

The full details can be found in chapter 15 of notes.

We start off by recalling some classical Möbius theory.

2 Classical Möbius Theory

2.1 Incidence algebra of a poset

Recall that for any finite poset P, we defined its incidence algebra $\mathrm{I}\left(P\right)$ as the vector space of incidence functions:

$$f : \left\{(x, y) \in P^2 \mid x \leq y\right\} \to \mathbb{k}$$

and for any $f, g \in \mathrm{I}\left(P\right)$, we define $f \cdot g \in \mathrm{I}\left(P\right)$ by:

$$\left(f \cdot g\right)\left(x, z\right) = \sum_{y : x \leq y \leq z} f\left(x, y\right) g\left(y, z\right)$$

The (multiplicative) identity is given by $\delta \in \mathrm{I}\left(P\right)$, defined as follows:

$$\delta\left(x, y\right) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

The zeta function $\zeta \in \mathrm{I}[P]$ is defined by:

$$\zeta(x, y) = 1 \quad \text{for all } x \leq y$$

It is an invertible element of $\mathrm{I}[P]$.

Its inverse is the Möbius function $\mu \in \mathrm{I}[P]$, which is uniquely characterized by:

$$\mu(x, x) = 1 \quad \text{for all } x$$

$$\sum_{z: x \leq z \leq y} \mu(z, y) = 0 \quad \text{for all } x < y$$

2.2 Example: the flat-incidence algebra

Taking $P = \Pi[\mathcal{A}]$, the lattice of flats, we get the flat-incidence algebra $\mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$.

It consists of functions on nested flats (i.e. pairs (X, Y) s.t. $\mathrm{X} \le \mathrm{Y}$) with:

$$(f \cdot g)(\mathrm{X}, \mathrm{Y}) = \sum_{\mathrm{Y}:\mathrm{X} \le \mathrm{Y} \le \mathrm{Z}} f(\mathrm{X}, \mathrm{Y})\, g(\mathrm{Y}, \mathrm{Z})$$

We have the identity element, the zeta function, the Möbius function, the incidence module and Möbius inversion exactly as in the general case.

Since the underlying poset $\Pi\left[\mathcal{A}\right]$ is a lattice, $\mathrm{I}_{\mathrm{flat}}\left[\mathcal{A}\right]$ is very well-behaved as an algebra.

In fact, we have the following result:

Proposition 1. *The flat-incidence algebra* $\mathrm{I}_{\mathrm{flat}}\left[\mathcal{A}\right]$ *is elementary. Its split semi-simple quotient is the Birkhoff algebra* $\Pi\left[\mathcal{A}\right]$ *with the quotient map given by:*

$$\mathrm{I}_{\mathrm{flat}}\left[\mathcal{A}\right] \twoheadrightarrow \Pi\left[\mathcal{A}\right] \quad ; \quad f \mapsto \sum_{\mathrm{X}} f\left(\mathrm{X}, \mathrm{X}\right) \mathtt{Q}_{\mathrm{X}}$$

In particular, its radical consists of functions $f \in \mathrm{I}_{\mathrm{flat}}\left[\mathcal{A}\right]$ *with* $f\left(\mathrm{X}, \mathrm{X}\right) = 0$ *for all flats* X.

We omit the proof.

2.3 Example: the face-incidence algebra

Taking $P = \Sigma[\mathcal{A}]$, the poset of faces, we get the face-incidence algebra $\mathrm{I}_{\mathrm{face}}[\mathcal{A}]$.

It consists of functions on nested faces (i.e. pairs (F, G) s.t. $\mathrm{F} \leq \mathrm{G}$) with:

$$(f \cdot g)(\mathrm{F}, \mathrm{H}) = \sum_{\mathrm{G}:\mathrm{F} \leq \mathrm{G} \leq \mathrm{H}} f(\mathrm{F}, \mathrm{G})\, g(\mathrm{G}, \mathrm{H})$$

We have the identity element, the zeta function, the Möbius function, the incidence module and Möbius inversion exactly as in the general case.

$\mathrm{I}_{\mathrm{face}}[\mathcal{A}]$ is not quite as well-behaved as $\mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$.

We will focus instead on a subalgebra $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}] \subseteq \mathrm{I}_{\mathrm{face}}[\mathcal{A}]$ which, as we will see, is more well-behaved.

3 The Lune-incidence Algebra

3.1 Nested faces and lunes

Let A and B be faces of the same support. Recall that we have the isomorphism of posets:

$$\Sigma\left[\mathcal{A}_{\mathrm{A}}\right] \xrightarrow{\cong} \Sigma\left[\mathcal{A}_{\mathrm{B}}\right] \quad ; \quad \mathrm{F}/\mathrm{A} \mapsto \mathrm{BF}/\mathrm{B}$$

We define an equivalence relation $\sim$ on nested faces:

$$(\mathrm{A}, \mathrm{F}) \sim (\mathrm{B}, \mathrm{G}) \Leftrightarrow \mathrm{s}\,(\mathrm{A}) = \mathrm{s}\,(\mathrm{B}), \mathrm{G} = \mathrm{BF}, \mathrm{F} = \mathrm{AG}$$

or equivalently,

$$(\mathrm{A}, \mathrm{F}) \sim (\mathrm{B}, \mathrm{G}) \Leftrightarrow \mathrm{AB} = \mathrm{A}, \mathrm{BA} = \mathrm{BG} = \mathrm{BF}, \mathrm{F} = \mathrm{AG}$$

The equivalence classes are called lunes. We will typically label lunes as L, M, N.

Lemma 1. *Let* $(\mathrm{A}, \mathrm{F}) \sim (\mathrm{B}, \mathrm{G})$*. Then, for* H *such that* $\mathrm{A} \leq \mathrm{H} \leq \mathrm{F}$*, we have:*

$$(\mathrm{A}, \mathrm{H}) \sim (\mathrm{B}, \mathrm{BH}) \qquad (\mathrm{H}, \mathrm{F}) \sim (\mathrm{BH}, \mathrm{G})$$

Proof.

$\mathrm{ABH} = \mathrm{A} \cdot \mathrm{H} = \mathrm{H}$. This proves the first equivalence.

$$\mathrm{s}\,(\mathrm{BH}) = \mathrm{s}\,(\mathrm{B})\,\mathrm{s}\,(\mathrm{H}) = \mathrm{s}\,(\mathrm{A})\,\mathrm{s}\,(\mathrm{H}) = \mathrm{s}\,(\mathrm{AH}) = \mathrm{s}\,(\mathrm{H})$$

$$\mathrm{BHF} = \mathrm{BF} = \mathrm{G}$$

$\mathrm{AG} = \mathrm{F}$ and $\mathrm{A} \leq \mathrm{G} \leq \mathrm{F} \Rightarrow \mathrm{HG} = \mathrm{F}$. This proves the second equivalence. $\square$

3.2 Lune-incidence algebra

Let $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ be the vector subspace of $\mathrm{I}_{\mathrm{face}}[\mathcal{A}]$ consisting of functions f such that $f(\mathrm{A},\mathrm{F}) = f(\mathrm{B},\mathrm{G})$ whenever $(\mathrm{A},\mathrm{F}) \sim (\mathrm{B},\mathrm{G})$.

By the above lemma, for any $f, g \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$, if $(\mathrm{A},\mathrm{F}) \sim (\mathrm{B},\mathrm{G})$ then:

$$\begin{aligned}
(f \cdot g)(\mathrm{A},\mathrm{F}) &= \sum_{\mathrm{H}:\mathrm{A} \leq \mathrm{H} \leq \mathrm{G}} f(\mathrm{A},\mathrm{H})\, g(\mathrm{H},\mathrm{F}) \\
&= \sum_{\mathrm{H}:\mathrm{A} \leq \mathrm{H} \leq \mathrm{G}} f(\mathrm{B},\mathrm{BH})\, g(\mathrm{BH},\mathrm{G}) \\
&= \sum_{\mathrm{H}:\mathrm{B} \leq \mathrm{H} \leq \mathrm{G}} f(\mathrm{B},\mathrm{H})\, g(\mathrm{H},\mathrm{G}) \\
&= (f \cdot g)(\mathrm{B},\mathrm{G})
\end{aligned}$$

Thus, we have the result:

Proposition 2. $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ *is a subalgebra of the face-incidence algebra* $\mathrm{I}_{\mathrm{face}}[\mathcal{A}]$*.*
It consists of functions f which are constant on each lune L.

We call $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ the lune-incidence algebra of $\mathcal{A}$.

Warning. *Unlike the flat-incidence algebra and the face-incidence algebra, the lune-incidence algebra is not an incidence algebra of a poset.*

3.3 The base-support map

Given a nested face (F, G), there are two (possibly distinct) flats associated to it via the support map, viz. $\mathrm{s}\,(\mathrm{F})$ and $\mathrm{s}\,(\mathrm{G})$.

We call $\mathrm{s}\,(\mathrm{F})$, the base of (F, G), and $\mathrm{s}\,(\mathrm{G})$ the support of (F, G):

$$\mathrm{b}\,(\mathrm{F}, \mathrm{G}) = \mathrm{s}\,(\mathrm{F}) \qquad \mathrm{s}\,(\mathrm{F}, \mathrm{G}) = \mathrm{s}\,(\mathrm{G})$$

We make the following observations:

1. $\mathrm{b}\,(\mathrm{F}, \mathrm{G}) \leq \mathrm{s}\,(\mathrm{F}, \mathrm{G})$ for all nested faces (F, G).
2. If $(\mathrm{A}, \mathrm{F}) \sim (\mathrm{B}, \mathrm{G})$ then $\mathrm{b}\,(\mathrm{A}, \mathrm{F}) = \mathrm{b}\,(\mathrm{B}, \mathrm{G})$ and $\mathrm{s}\,(\mathrm{A}, \mathrm{F}) = \mathrm{s}\,(\mathrm{B}, \mathrm{G})$. Thus, the base and support maps are constant on lunes.

Hence, we have a map from lunes to nested flats given by $\mathrm{L} \mapsto (\mathrm{b}\,(\mathrm{L}), \mathrm{s}\,(\mathrm{L}))$.

Let (X, Y) be any nested flat and fix any face A with $\mathrm{s}(\mathrm{A}) = \mathrm{X}$.

Consider the map

$$\{\mathrm{F} \mid \mathrm{F} \geq \mathrm{A}, \mathrm{s}(\mathrm{F}) = \mathrm{Y}\} \to \{\mathrm{L} \mid \mathrm{b}(\mathrm{L}) = \mathrm{X}, \mathrm{s}(\mathrm{L}) = \mathrm{Y}\}$$
$$\mathrm{F} \mapsto (\mathrm{A}, \mathrm{F})_{\sim}$$

Since $(\mathrm{A}, \mathrm{F}) \sim (\mathrm{A}, \mathrm{G})$ implies $\mathrm{F} = \mathrm{G}$, the above map is injective.

Furthermore, since any $\mathrm{L} = (\mathrm{B}, \mathrm{G})_{\sim}$ with $\mathrm{s}(\mathrm{B}) = \mathrm{X}$ and $\mathrm{s}(\mathrm{G}) = \mathrm{Y}$ is the image of AG, the above map is also surjective.

Thus, the two sets are in bijection.

The base-support map $\mathrm{bs} : \mathrm{I}_{\mathrm{lune}}[\mathcal{A}] \to \mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$ takes any function $f \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ to the function $\mathrm{bs}(f) \in \mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$ given by:

$$\mathrm{bs}(f)(\mathrm{X}, \mathrm{Y}) = \sum_{L : \mathrm{b}(L) = \mathrm{X}, \mathrm{s}(L) = \mathrm{Y}} f(L)$$

By the discussion above, we have the alternative description:

$$\mathrm{bs}(f)(\mathrm{X}, \mathrm{Y}) = \sum_{\mathrm{F} : \mathrm{F} \geq \mathrm{A}, \mathrm{s}(\mathrm{F}) = \mathrm{Y}} f(\mathrm{A}, \mathrm{F})$$

for any face A of support X.

In particular, the definition is independent of choice of the face A.

Using the alternative description of the base-support map, it is easy to verify that

$$\mathrm{bs}\,(f \cdot g)\,(\mathrm{X}, \mathrm{Z}) = (\mathrm{bs}\,(f) \cdot \mathrm{bs}\,(g))\,(\mathrm{X}, \mathrm{Z})$$

for any $f, g \in \mathrm{I}_{\mathrm{lune}}\,[\mathcal{A}]$ and any nested flat (X, Z).

Proposition 3. *The base-support map* $\mathrm{bs} : \mathrm{I}_{\mathrm{lune}}\,[\mathcal{A}] \to \mathrm{I}_{\mathrm{flat}}\,[\mathcal{A}]$ *is an algebra homomorphism.*

We omit the proof.

3.4 Lune-incidence algebra is elementary

As promised, the lune-incidence algebra is well-behaved:

Proposition 4. *The lune-incidence algebra* $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ *is elementary. Its split semi-simple quotient is the Birkhoff algebra* $\Pi[\mathcal{A}]$ *with the quotient map given by:*

$$\mathrm{I}_{\mathrm{lune}}[\mathcal{A}] \twoheadrightarrow \Pi[\mathcal{A}] \quad ; \quad f \mapsto \sum_{\mathrm{X}} f(\mathrm{F}, \mathrm{F})\, \mathtt{Q}_{\mathrm{X}}$$

where, for each flat X*, we have fixed a face* F *of support* X*. In particular, its radical consists of functions* $f \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ *with* $f(\mathrm{F}, \mathrm{F}) = 0$ *for all faces* F*.*

We omit the proof.

3.5 Non-commutative zeta functions

A non-commutative zeta function is an element $\zeta \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ such that $\zeta(\mathrm{F},\mathrm{F}) = 1$ for all faces F and

$$(1) \qquad \sum_{\mathrm{F}:\mathrm{F}\geq \mathrm{A}, \mathrm{s}(\mathrm{F})=\mathrm{s}(\mathrm{G}), \mathrm{HF}=\mathrm{G},} \zeta(\mathrm{A},\mathrm{F}) = \zeta(\mathrm{H},\mathrm{G})$$

for all $\mathrm{A} \leq \mathrm{H} \leq \mathrm{G}$.

For $\mathrm{A} = \mathrm{H}$, we have $\mathrm{HF} = \mathrm{G} \Rightarrow \mathrm{F} = \mathrm{G}$ and so the second condition is automatically satisfied.

On the other hand, taking $\mathrm{H} = \mathrm{G}$, we get:

$$\sum_{\mathrm{F}:\mathrm{F}\geq \mathrm{A}, \mathrm{s}(\mathrm{F})=\mathrm{s}(\mathrm{G})} \zeta(\mathrm{A},\mathrm{F}) = \zeta(\mathrm{G},\mathrm{G}) = 1 \quad \text{for all } \mathrm{A} \leq \mathrm{G}$$

Thus, we have the following result:

Proposition 5. *The base-support map*

$$\mathrm{bs} : \mathrm{I}_{\mathrm{lune}} [\mathcal{A}] \to \mathrm{I}_{\mathrm{flat}} [\mathcal{A}]$$

maps any non-commutative zeta function $\boldsymbol{\zeta} \in \mathrm{I}_{\mathrm{lune}} [\mathcal{A}]$ *to the classical zeta function* $\zeta \in \mathrm{I}_{\mathrm{flat}} [\mathcal{A}]$*.*

Proof. For any nested flat (X, Y) we fix a face A of support X to obtain:

$$\begin{aligned} \mathrm{bs} (\boldsymbol{\zeta}) (\mathrm{X}, \mathrm{Y}) &= \sum_{\mathrm{F} : \mathrm{F} \geq \mathrm{A}, \mathrm{s}(\mathrm{F}) = \mathrm{Y}} \boldsymbol{\zeta} (\mathrm{A}, \mathrm{F}) \\ &= \boldsymbol{\zeta} (\mathrm{G}, \mathrm{G}) = 1 \end{aligned}$$

where $\mathrm{G} \geq A$ is any face of support Y. □

We have not yet made any comments about the existence or uniqueness of non-commutative zeta functions in $\mathrm{I}_{\mathrm{lune}} [\mathcal{A}]$. The following result should help.

3.6 Inching towards the main result

Lemma 2. *A non-commutative zeta function is the same as a homogeneous section.*

Proof. Given $\boldsymbol{\zeta}$, take $\mathtt{u}^{\mathrm{F}} = \boldsymbol{\zeta}\left(O, \mathrm{F}\right)$.

Taking $\mathrm{A} = O$ and $\mathrm{B} = \mathrm{G}$ in (1), we get:

$$\sum_{\mathrm{F}:\mathrm{s}\,(F)=\mathrm{X}} \mathtt{u}^{\mathrm{F}} = 1$$

for $\mathrm{X} = \mathrm{s}\left(\mathrm{B}\right)$, so that the scalars $\mathtt{u}^{\mathrm{F}}$ define a homogeneous section of $\mathcal{A}$.

Conversely, given a homogeneous section $\mathtt{u}$, take $\boldsymbol{\zeta}\left(\mathrm{A},\mathrm{F}\right)=\mathtt{u}_{\mathrm{A}}^{\mathrm{F}}$.

For $(\mathrm{A},\mathrm{F})\sim(\mathrm{B},\mathrm{G})$, we have:

$$\begin{aligned}\boldsymbol{\zeta}\left(\mathrm{A},\mathrm{F}\right)=\mathtt{u}_{\mathrm{A}}^{\mathrm{F}} &= \sum_{\mathrm{H}:\mathrm{AH}=\mathrm{F},\mathrm{s}\,(\mathrm{H})=\mathrm{s}\,(\mathrm{F})} \mathtt{u}^{\mathrm{H}} \\ &= \sum_{\mathrm{H}:\mathrm{BH}=\mathrm{G},\mathrm{s}\,(\mathrm{H})=\mathrm{s}\,(\mathrm{G})} \mathtt{u}^{\mathrm{H}}=\mathtt{u}_{\mathrm{B}}^{\mathrm{G}}=\boldsymbol{\zeta}\left(\mathrm{B},\mathrm{G}\right)\end{aligned}$$

so that $\boldsymbol{\zeta}\in\mathrm{I}_{\mathrm{lune}}\left[\mathcal{A}\right]$.

Since $\mathtt{u}_{\mathrm{F}}^{\mathrm{F}}=1$ for any face F, we obtain the first condition:

$$\boldsymbol{\zeta}\left(\mathrm{F},\mathrm{F}\right)=\mathtt{u}_{\mathrm{F}}^{\mathrm{F}}=1$$

Furthermore, for any $\mathrm{A} \leq \mathrm{H} \leq \mathrm{G}$, we have:

$$\sum_{\mathrm{F}:\mathrm{F}\geq\mathrm{A},\mathrm{s}\,(\mathrm{F})=\mathrm{s}\,(\mathrm{G}),\mathrm{HF}=\mathrm{G}} \mathtt{u}_{\mathrm{A}}^{\mathrm{F}}$$

$$=\sum_{\mathrm{F}:\mathrm{F}\geq\mathrm{A},\mathrm{s}\,(\mathrm{F})=\mathrm{s}\,(\mathrm{G}),\mathrm{HF}=\mathrm{G}} \left(\sum_{\mathrm{H}':\mathrm{AH}'=\mathrm{F},\mathrm{s}\,(\mathrm{H}')=\mathrm{s}\,(\mathrm{F})} \mathtt{u}^{\mathrm{H}'}\right)$$

$$=\sum_{\mathrm{H}':\mathrm{HAH}'=\mathrm{G},\mathrm{s}\,(\mathrm{H}')=\mathrm{s}\,(\mathrm{G})} \mathtt{u}^{\mathrm{H}'}$$

$$=\sum_{\mathrm{H}':\mathrm{HH}'=\mathrm{G},\mathrm{s}\,(\mathrm{H}')=\mathrm{s}\,(\mathrm{G})} \mathtt{u}^{\mathrm{H}'}$$

$$=\mathtt{u}_{\mathrm{H}}^{\mathrm{G}}$$

which gives us the second condition. $\square$