Exponential and logarithm

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1 Noncommutative exp-log correspondences

1.1 Action of the lune-incidence algebra

Recall the lune-incidence algebra $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$.

Let c be a comonoid and a be a monoid.

For any $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ and $f : \mathsf{c} \to \mathsf{a}$ a map of species, define $s \circ f : \mathsf{c} \to \mathsf{a}$ by

$$(s \circ f)_A := \sum_{F \colon F \geq A} s(A, F)\, \mu_A^F f_F \Delta_A^F. \tag{1}$$

This is a map of species.

To see this, fix A and B of the same support, and for any $F \geq A$, consider the commutative diagram below.

$$\begin{tikzcd}
\mathsf{c}[A] \arrow[r, "\Delta_A^F"] \arrow[d, "\beta_{B,A}"'] & \mathsf{c}[F] \arrow[r, "f_F"] \arrow[d, "\beta_{BF,F}"'] & \mathsf{a}[F] \arrow[r, "\mu_A^F"] \arrow[d, "\beta_{BF,F}"] & \mathsf{a}[A] \arrow[d, "\beta_{B,A}"] \\
\mathsf{c}[B] \arrow[r, "\Delta_B^{BF}"'] & \mathsf{c}[BF] \arrow[r, "f_{BF}"'] & \mathsf{a}[BF] \arrow[r, "\mu_B^{BF}"'] & \mathsf{a}[B]
\end{tikzcd}$$

The middle square commutes since f is a map of species,

while the side squares commute by naturality of the product and coproduct.

Moreover, $s(A, F) = s(B, BF)$.

Multiplying the above diagram by this scalar, summing over all $F \geq A$, and using the bijection between the stars of A and B, we see that $s \circ f$ is a map of species.

Moreover:

Lemma 1. *The assignment* $(s, f) \mapsto s \circ f$ *defines a left action of the lune-incidence algebra on* $\mathcal{A}\text{-}\mathrm{Sp}(\mathsf{c}, \mathsf{a})$.

Proof. This is checked below.

$$\begin{aligned}
(s \circ (t \circ f))_A &= \sum_{F \colon F \geq A} s(A, F)\, \mu_A^F (t \circ f)_F \Delta_A^F \\
&= \sum_{F \colon F \geq A} s(A, F)\, \mu_A^F \Big(\sum_{G \colon G \geq F} t(F, G)\, \mu_F^G f_G \Delta_F^G \Big) \Delta_A^F \\
&= \sum_{G \colon G \geq A} \Big(\sum_{F \colon G \geq F \geq A} s(A, F) t(F, G) \Big) \mu_A^G f_G \Delta_A^G \\
&= \sum_{G \colon G \geq A} (st)(A, G)\, \mu_A^G f_G \Delta_A^G \\
&= (st \circ f)_A.
\end{aligned}$$

Also, observe that $(\delta \circ f)_A = f_A$, where δ denotes the unit element of the lune-incidence algebra. $\square$

Thus, $\mathcal{A}\text{-}\mathrm{Sp}(\mathsf{c}, \mathsf{a})$ is a left module over the lune-incidence algebra.

Lemma 2. *Let $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ be such that $s(A, A) = 1$ for all A, and let $f : \mathsf{c} \to \mathsf{a}$ be a map of species from a comonoid* c *to a monoid* a.

Then $s \circ f$ and f agree when restricted to the primitive part $\mathcal{P}(\mathsf{c})$, and also when followed by the projection to the indecomposable part $\mathcal{Q}(\mathsf{a})$.

$$
\begin{tikzcd}
\mathsf{c} \arrow[r, shift left, "f"] \arrow[r, shift right, "s\circ f"'] & \mathsf{a} \\
\mathcal{P}(\mathsf{c}) \arrow[u, hook] \arrow[ur, bend right, "f=s\circ f"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{c} \arrow[r, shift left, "f"] \arrow[r, shift right, "s\circ f"'] \arrow[dr, bend right, "f=s\circ f"'] & \mathsf{a} \arrow[d, two heads] \\
& \mathcal{Q}(\mathsf{a})
\end{tikzcd}
$$

Proof. We explain the first statement, the second is similar.

Let us evaluate $(s \circ f)_A$ on $\mathcal{P}(\mathsf{c})[A]$.

In this case, $\Delta_A^F = 0$ for any $F > A$.

Hence, in the sum in (1), only the term corresponding to $F = A$ remains which is f_A.

Thus, $(s \circ f)_A = f_A$ on $\mathcal{P}(\mathsf{c})[A]$, as required. $\square$

Lemma 3. *Let* h *be a* q*-bimonoid and let* $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ *be such that* $s(A, A) = 1$ *for all* A*.*

If $f : \mathsf{h} \to \mathsf{h}$ *is a morphism of comonoids, then the first diagram below commutes.*

If $f : \mathsf{h} \to \mathsf{h}$ *is a morphism of monoids, then the second diagram below commutes.*

$$
\begin{tikzcd}
\mathsf{h} \arrow[r, shift left, "f"] \arrow[r, shift right, "s\circ f"'] & \mathsf{h} \\
\mathcal{P}(\mathsf{h}) \arrow[u, hook] \arrow[r, "f=s\circ f"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{h} \arrow[r, shift left, "f"] \arrow[r, shift right, "s\circ f"'] \arrow[d, two heads] & \mathsf{h} \arrow[d, two heads] \\
\mathcal{Q}(\mathsf{h}) \arrow[r, "f=s\circ f"'] & \mathcal{Q}(\mathsf{h})
\end{tikzcd}
$$

Proof. If $f : \mathsf{h} \to \mathsf{h}$ is a morphism of comonoids, then it preserves the primitive part, and we deduce from Lemma 2 that $s \circ f$ and f restrict to the same map on $\mathcal{P}(\mathsf{h})$.

The second part is similar. $\square$

1.2 Noncommutative exp-log correspondences

Fix a noncommutative zeta function $\boldsymbol{\zeta}$ and a noncommutative Möbius function $\boldsymbol{\mu}$ which are inverses of each other in the lune-incidence algebra.

For a map of species $f : \mathrm{c} \to \mathrm{a}$ from a comonoid c to a monoid a, we say that $\boldsymbol{\zeta} \circ f$ is an exponential of f and $\boldsymbol{\mu} \circ f$ is a logarithm of f.

In keeping with this terminology, we also use the notations $\mathbf{exp} \circ f$ and $\mathbf{log} \circ f$.

Explicitly, using formula (1),

(2a) $$\mathbf{exp}(f)_A = \sum_{F:\, F \geq A} \boldsymbol{\zeta}(A, F)\, \mu_A^F f_F \Delta_A^F,$$

(2b) $$\mathbf{log}(f)_A = \sum_{F:\, F \geq A} \boldsymbol{\mu}(A, F)\, \mu_A^F f_F \Delta_A^F.$$

Since these operations are inverse to each other, we obtain:

Proposition 1. *For a comonoid* $\mathtt{c}$ *and monoid* $\mathtt{a}$*, we have inverse bijections*

$$\mathcal{A}\text{-}\mathsf{Sp}(\mathtt{c}, \mathtt{a}) \underset{\mathbf{log}}{\overset{\mathbf{exp}}{\rightleftarrows}} \mathcal{A}\text{-}\mathsf{Sp}(\mathtt{c}, \mathtt{a}). \tag{3}$$

We refer to (3) as an exp-log correspondence.

Note very carefully that it depends on the particular $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ that we chose.

1.3 (Co)derivations and (co)monoid morphisms

Theorem 1. *For a cocommutative comonoid* c *and bimonoid* k*, we have inverse bijections*

(4)

$$\mathcal{A}\text{-}\mathsf{Sp}\big(\mathsf{c}, \mathcal{P}(\mathsf{k})\big) \underset{\mathbf{log}}{\overset{\mathbf{exp}}{\rightleftarrows}} \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c}, \mathsf{k}).$$

In other words:

If $f : \mathsf{c} \to \mathsf{k}$ is a coderivation, then $\mathbf{exp}(f) : \mathsf{c} \to \mathsf{k}$ is a morphism of comonoids.

If $g : \mathsf{c} \to \mathsf{k}$ is a morphism of comonoids, then $\mathbf{log}(g) : \mathsf{c} \to \mathsf{k}$ is a coderivation.

Proof. In view of (3), it suffices to show that $\mathbf{exp}$ and $\mathbf{log}$ map as stated.

We first show that if $f : \mathsf{c} \to \mathcal{P}(\mathsf{k})$ is a map of species, then $\boldsymbol{\zeta} \circ f : \mathsf{c} \to \mathsf{k}$ is a morphism of comonoids, that is,

$$\Delta_A^G (\boldsymbol{\zeta} \circ f)_A = (\boldsymbol{\zeta} \circ f)_G \Delta_A^G.$$

The calculation goes as follows.

$$
\begin{aligned}
\Delta_A^G(\boldsymbol{\zeta}\circ f)_A &= \sum_{F\colon F\geq A} \boldsymbol{\zeta}(A,F)\,\Delta_A^G \mu_A^F f_F \Delta_A^F \\
&= \sum_{F\colon F\geq A,\, FG=F} \boldsymbol{\zeta}(A,F)\,\mu_G^{GF}\beta_{GF,F} f_F \Delta_A^F \\
&= \sum_{F\colon F\geq A,\, FG=F} \boldsymbol{\zeta}(A,F)\,\mu_G^{GF} f_{GF}\beta_{GF,F} \Delta_A^F \\
&= \sum_{F\colon F\geq A,\, FG=F} \boldsymbol{\zeta}(A,F)\,\mu_G^{GF} f_{GF} \Delta_A^{GF} \\
&= \sum_{H\colon H\geq G} \Big(\sum_{F\colon F\geq A,\, FG=F,\, GF=H} \boldsymbol{\zeta}(A,F)\Big)\mu_G^H f_H \Delta_A^H \\
&= \Big(\sum_{H\colon H\geq G} \boldsymbol{\zeta}(G,H)\,\mu_G^H f_H \Delta_G^H \Big)\Delta_A^G \\
&= (\boldsymbol{\zeta}\circ f)_G \Delta_A^G.
\end{aligned}
$$

The first and last steps used definition (2a).

The second step made use of the bimonoid axiom for k and also the hypothesis that f maps into the primitive part of k.

The third step used that f is a map of species,

while the fourth step used cocommutativity of c.

In the next step, we introduced a new variable H for GF.

The sixth step used the lune-additivity formula and coassociativity of c.

In the other direction, we show that if $g : \mathsf{c} \to \mathsf{k}$ is a morphism of comonoids, then $\boldsymbol{\mu} \circ g$ maps c into $\mathcal{P}(\mathsf{k})$, that is,

$$\Delta_A^G(\boldsymbol{\mu} \circ g)_A = 0 \text{ for } G > A.$$

The calculation goes as follows.

$$
\begin{aligned}
\Delta_A^G(\boldsymbol{\mu} \circ g)_A &= \sum_{F\colon F \geq A} \boldsymbol{\mu}(A, F)\, \Delta_A^G \mu_A^F g_F \Delta_A^F \\
&= \sum_{F\colon F \geq A} \boldsymbol{\mu}(A, F)\, \Delta_A^G \mu_A^F \Delta_A^F g_A \\
&= \sum_{F\colon F \geq A} \boldsymbol{\mu}(A, F)\, \mu_G^{GF} \Delta_A^{GF} g_A \\
&= \sum_{H\colon H \geq G} \Big(\sum_{F\colon F \geq A,\, GF = H} \boldsymbol{\mu}(A, F) \Big) \mu_G^H \Delta_A^H g_A \\
&= 0.
\end{aligned}
$$

The first step used definition (2b).

The second step made use of the hypothesis that g is a morphism of comonoids.

The third step used the bimonoid axiom for k and the fact that g necessarily maps into the coabelianization of k.

In the next step, we introduced a new variable H for GF.

The last step used the noncommutative Weisner formula. $\square$

Remark 1. Though Theorem 1 is stated for an arbitrary bimonoid k, the exp-log correspondence (4) only depends on the coabelianization of k, so one may without loss of generality assume that k is cocommutative.

The dual of Theorem 1 is as follows.

Theorem 2. *For a bimonoid* h *and commutative monoid* a*, we have inverse bijections*

$$\mathcal{A}\text{-}\mathsf{Sp}\big(\mathcal{Q}(\mathsf{h}), \mathsf{a}\big) \underset{\mathbf{log}}{\overset{\mathbf{exp}}{\rightleftarrows}} \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h}, \mathsf{a}).$$

In other words:

If $f : \mathsf{h} \to \mathsf{a}$ is a derivation, then $\mathbf{exp}(f) : \mathsf{h} \to \mathsf{a}$ is a morphism of monoids.

If $g : \mathsf{h} \to \mathsf{a}$ is a morphism of monoids, then $\mathbf{log}(g) : \mathsf{h} \to \mathsf{a}$ is a derivation.

1.4 Logarithm of the identity map

Fix a noncommutative Möbius function $\boldsymbol{\mu}$.

For any bimonoid h, we have the operator
$\mathbf{log}(\mathrm{id}) : \mathsf{h} \to \mathsf{h}$.

This is a logarithm of the identity map on h.

Explicitly, using (2b), it is given by

$$\mathbf{log}(\mathrm{id})_A = \sum_{F\colon F \geq A} \boldsymbol{\mu}(A, F)\, \mu_A^F \Delta_A^F. \tag{5}$$

Proposition 2. *Let* h *be a bimonoid.*

If h *is cocommutative, then* $\mathbf{log}(\mathrm{id})$ *maps to* $\mathcal{P}(\mathsf{h})$ *and is in fact a projection from* h *onto* $\mathcal{P}(\mathsf{h})$*, or equivalently,* $\mathbf{log}(\mathrm{id})$ *is an idempotent operator on* h *whose image is* $\mathcal{P}(\mathsf{h})$*.*

If h *is commutative, then* $\mathbf{log}(\mathrm{id})$ *factors through* $\mathcal{Q}(\mathsf{h})$ *and splits the canonical projection* $\mathsf{h} \twoheadrightarrow \mathcal{Q}(\mathsf{h})$*, or equivalently,* $\mathbf{log}(\mathrm{id})$ *is an idempotent operator on* h *whose coimage is* $\mathcal{Q}(\mathsf{h})$*.*

Proof. Suppose h is cocommutative.

By taking $\mathsf{c} = \mathsf{k} := \mathsf{h}$ in Theorem 1, we see that $\mathbf{log}(\mathrm{id})$ maps to $\mathcal{P}(\mathsf{h})$.

Further, by Lemma 3, it is identity on $\mathcal{P}(\mathsf{h})$, and hence a projection.

The claim when h is commutative follows similarly by starting with $\mathsf{h} = \mathsf{a}$ in Theorem 2. $\square$

Corollary 1. *Let* h *be a bimonoid.*

If h *is cocommutative, then for any* $G > A$*,*

$$\sum_{F \colon F \geq A} \boldsymbol{\mu}(A, F)\, \Delta_A^G \mu_A^F \Delta_A^F = 0.$$

If h *is commutative, then for any* $G > A$*,*

$$\sum_{F \colon F \geq A} \boldsymbol{\mu}(A, F)\, \mu_A^F \Delta_A^F \mu_A^G = 0.$$

Proof. The first claim says that $\mathbf{log}(\mathrm{id})$ maps into $\mathcal{P}(\mathsf{h})$, while the second says that it factors through $\mathcal{Q}(\mathsf{h})$. $\square$

Example 1. Recall that the primitive part of the bimonoid of chambers Γ is the Lie species. Hence, by Proposition 2, $\mathbf{log}(\mathrm{id})$ yields an idempotent operator on Γ whose image is Lie.

Similarly, recall that the primitive part of the bimonoid of faces Σ is the Zie species. Hence, $\mathbf{log}(\mathrm{id})$ yields an idempotent operator on Σ whose image is Zie.

2 Commutative exp-log correspondence

2.1 Action of the flat-incidence algebra

Recall the flat-incidence algebra $\mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$.

Let c be a cocommutative comonoid and a be a commutative monoid.

For any $s \in \mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$ and f : c $\to$ a a map of species, define another map of species $s \circ f$: c $\to$ a by

$$(s \circ f)_{\mathrm{Z}} := \sum_{\mathrm{X} \colon \mathrm{X} \geq \mathrm{Z}} s(\mathrm{Z}, \mathrm{X})\, \mu^{\mathrm{X}}_{\mathrm{Z}} f_{\mathrm{X}} \Delta^{\mathrm{X}}_{\mathrm{Z}}. \tag{6}$$

This defines an action of the flat-incidence algebra on $\mathcal{A}$-$\mathrm{Sp}(\mathsf{c}, \mathsf{a})$. In other words, $\mathcal{A}$-$\mathrm{Sp}(\mathsf{c}, \mathsf{a})$ is a module over the flat-incidence algebra.

The check is similar to the one in Lemma 1 with faces replaced by flats.

Lemma 4. *Let* c *be a cocommutative comonoid and* a *be a commutative monoid.*

Then the action (1) *of the lune-incidence algebra on* $\mathcal{A}\text{-}\mathrm{Sp}(\mathsf{c}, \mathsf{a})$ *factors through the base-case map to yield an action of the flat-incidence algebra on* $\mathcal{A}\text{-}\mathrm{Sp}(\mathsf{c}, \mathsf{a})$ *which coincides with* (6)*.*

Proof. The calculation goes as follows. Let A be a face of support Z.

$$\begin{aligned}
(s \circ f)_A &= \sum_{F\colon F \geq A} s(A,F)\, \mu_A^F f_F \Delta_A^F \\
&= \sum_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Z}} \Big(\sum_{F\colon F\geq A,\, \mathrm{s}(F)=\mathrm{X}} s(A,F)\Big) \mu_{\mathrm{Z}}^{\mathrm{X}} f_{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}} \\
&= \sum_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Z}} \mathrm{bc}(s)(\mathrm{Z},\mathrm{X})\, \mu_{\mathrm{Z}}^{\mathrm{X}} f_{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}} \\
&= (\mathrm{bc}(s) \circ f)_{\mathrm{Z}}.
\end{aligned}$$

Recall that μ_A^F and $\mu_{\mathrm{Z}}^{\mathrm{X}}$ (and Δ_A^F and $\Delta_{\mathrm{Z}}^{\mathrm{X}}$) connect to each other by the maps $\beta_{\mathrm{Z},A}$ and $\beta_{\mathrm{X},F}$. For convenience, these maps have been suppressed in the above calculation. $\square$

2.2 Commutative exp-log correspondence

Let ζ and μ be the zeta function and Möbius function in the flat-incidence algebra.

For a map of species $f : \mathrm{c} \to \mathrm{a}$ from a cocommutative comonoid c to a commutative monoid a, we say that

$$\exp(f) := \zeta \circ f \quad \text{and} \quad \log(f) := \mu \circ f$$

are the exponential and logarithm of f, respectively.

Explicitly, using formula (6),

$$\exp(f)_{\mathrm{Z}} = \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mu_{\mathrm{Z}}^{\mathrm{X}} f_{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}, \tag{7a}$$

$$\log(f)_{\mathrm{Z}} = \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{X})\, \mu_{\mathrm{Z}}^{\mathrm{X}} f_{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}. \tag{7b}$$

In contrast to the noncommutative theory, these operations are now uniquely defined.

Since they are inverse to each other, we obtain inverse bijections

$$\mathcal{A}\text{-}\mathsf{Sp}\big(\mathsf{c},\mathsf{a}\big) \underset{\log}{\overset{\exp}{\rightleftarrows}} \mathcal{A}\text{-}\mathsf{Sp}\big(\mathsf{c},\mathsf{a}\big). \tag{8}$$

This is the commutative exp-log correspondence.

2.3 (Co)derivations and (co)monoid morphisms

Theorem 3. *For a cocommutative comonoid* c *and bicommutative bimonoid* k*, we have inverse bijections*

$$\mathcal{A}\text{-}\mathsf{Sp}\big(\mathsf{c}, \mathcal{P}(\mathsf{k})\big) \underset{\log}{\overset{\exp}{\rightleftarrows}} \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c}, \mathsf{k}).$$

In other words:

If $f : \mathsf{c} \to \mathsf{k}$ is a coderivation, then $\exp(f) : \mathsf{c} \to \mathsf{k}$ is a morphism of comonoids.

If $g : \mathsf{c} \to \mathsf{k}$ is a morphism of comonoids, then $\log(g) : \mathsf{c} \to \mathsf{k}$ is a coderivation.

Proof. The proof is similar to (and simpler than) that of Theorem 1; the two calculations are indicated below.

$$\begin{aligned}
\Delta^{\mathrm{Y}}_{\mathrm{Z}}(\zeta \circ f)_{\mathrm{Z}} &= \sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Z}} \Delta^{\mathrm{Y}}_{\mathrm{Z}} \mu^{\mathrm{X}}_{\mathrm{Z}} f_{\mathrm{X}} \Delta^{\mathrm{X}}_{\mathrm{Z}} \\
&= \sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Y}} \mu^{\mathrm{X}}_{\mathrm{Y}} f_{\mathrm{X}} \Delta^{\mathrm{X}}_{\mathrm{Z}} \\
&= \Big(\sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Y}} \mu^{\mathrm{X}}_{\mathrm{Y}} f_{\mathrm{X}} \Delta^{\mathrm{X}}_{\mathrm{Y}} \Big) \Delta^{\mathrm{Y}}_{\mathrm{Z}} \\
&= (\zeta \circ f)_{\mathrm{Y}} \Delta^{\mathrm{Y}}_{\mathrm{Z}}.
\end{aligned}$$

The first and last steps used definition (7a).

The second step made use of the bicommutative bimonoid axiom for k and also the hypothesis that f maps into the primitive part of k.

The third step used coassociativity of c.

$$
\begin{aligned}
\Delta^{\mathrm{Y}}_{\mathrm{Z}}(\mu \circ g)_{\mathrm{Z}} &= \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{X})\, \Delta^{\mathrm{Y}}_{\mathrm{Z}}\, \mu^{\mathrm{X}}_{\mathrm{Z}} g_{\mathrm{X}} \Delta^{\mathrm{X}}_{\mathrm{Z}} \\
&= \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{X})\, \Delta^{\mathrm{Y}}_{\mathrm{Z}}\, \mu^{\mathrm{X}}_{\mathrm{Z}} \Delta^{\mathrm{X}}_{\mathrm{Z}}\, g_{\mathrm{Z}} \\
&= \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{X})\, \mu^{\mathrm{Y} \vee \mathrm{X}}_{\mathrm{Y}} \Delta^{\mathrm{Y} \vee \mathrm{X}}_{\mathrm{Z}} g_{\mathrm{Z}} \\
&= \sum_{\mathrm{W}:\, \mathrm{W} \geq \mathrm{Y}} \Big(\sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z},\, \mathrm{Y} \vee \mathrm{X} = \mathrm{W}} \mu(\mathrm{Z}, \mathrm{X}) \Big) \mu^{\mathrm{W}}_{\mathrm{Y}}\, \Delta^{\mathrm{W}}_{\mathrm{Z}}\, g_{\mathrm{Z}} \\
&= 0.
\end{aligned}
$$

The first step used definition (7b).

The second step used the hypothesis that g is a morphism of comonoids.

The third step used the bicommutative bimonoid axiom and the coassociativity axiom for k.

The last step used the Weisner formula. $\square$

Dually:

Theorem 4. *For a bicommutative bimonoid* h *and commutative monoid* a*, we have inverse bijections*

$$\mathcal{A}\text{-}\mathsf{Sp}\big(\mathcal{Q}(\mathsf{h}), \mathsf{a}\big) \underset{\log}{\overset{\exp}{\rightleftarrows}} \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h}, \mathsf{a}).$$

In other words:

If $f : \mathsf{h} \to \mathsf{a}$ is a derivation, then $\exp(f) : \mathsf{h} \to \mathsf{a}$ is a morphism of monoids.

If $g : \mathsf{h} \to \mathsf{a}$ is a morphism of monoids, then $\log(g) : \mathsf{h} \to \mathsf{a}$ is a derivation.

2.4 Logarithm of the identity map

Note from (7b) that the logarithm of the identity map on a bicommutative bimonoid h is given by

(9) $$\log(\mathrm{id})_{\mathrm{Z}} := \sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{X})\, \mu_{\mathrm{Z}}^{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}.$$

Proposition 3. *Let* h *be a bicommutative bimonoid. Then* $\log(\mathrm{id})$ *is an idempotent operator on* h *whose image is* $\mathcal{P}(\mathsf{h})$ *and coimage is* $\mathcal{Q}(\mathsf{h})$ *yielding the commutative diagram of species*

(10)
$$\begin{tikzcd}
\mathsf{h} \arrow[r, "\log(\mathrm{id})"] \arrow[d, two heads] & \mathsf{h} \\
\mathcal{Q}(\mathsf{h}) \arrow[r, "\cong"'] & \mathcal{P}(\mathsf{h}). \arrow[u, hook]
\end{tikzcd}$$

In particular, $\mathcal{P}(\mathsf{h})$ *and* $\mathcal{Q}(\mathsf{h})$ *are isomorphic as species.*

Proof. One can imitate the proof of Proposition 2 by taking $\mathsf{c} = \mathsf{k} := \mathsf{h}$ in Theorem 3 and $\mathsf{h} = \mathsf{a}$ in Theorem 4 and so forth. $\square$

Corollary 2. *Let* $(\mathsf{h}, \mu, \Delta)$ *be a bicommutative bimonoid. Then for any* $\mathrm{Y} > \mathrm{Z}$*,*

$$\sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{X})\, \Delta^{\mathrm{Y}}_{\mathrm{Z}} \mu^{\mathrm{X}}_{\mathrm{Z}} \Delta^{\mathrm{X}}_{\mathrm{Z}} = 0 = \sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{X})\, \mu^{\mathrm{X}}_{\mathrm{Z}} \Delta^{\mathrm{X}}_{\mathrm{Z}} \mu^{\mathrm{Y}}_{\mathrm{Z}}\,.$$

Proof. These identities express the fact that the operator $\log(\mathrm{id})$ maps h into $\mathcal{P}(\mathsf{h})$ and factors through $\mathcal{Q}(\mathsf{h})$. $\square$

3 Primitive and group-like series of bimonoids

3.1 Primitive and group-like series of a comonoid

Let p be a species.

A series of p is a family of elements $v_F \in \mathrm{p}[F]$, one for each face F, such that

$$\beta_{G,F}(v_F) = v_G,$$

whenever F and G have the same support.

Let $\mathscr{S}(\mathrm{p})$ denote the space of series of p.

This construction is functorial in p, and defines a functor $\mathscr{S}$ from the category of species to the category of vector spaces.

Let (c, Δ) be a comonoid.

A series v of c is primitive if $\Delta_A^F(v_A) = 0$ for all $F > A$.

Let $\mathscr{P}(\mathsf{c})$ denote the set of all primitive series of c.

Observe that

$$\mathscr{P}(\mathsf{c}) = \mathscr{S}(\mathcal{P}(\mathsf{c})). \tag{11}$$

Similarly, a series v of c is group-like if $\Delta_A^F(v_A) = v_F$ for all $F \geq A$.

Let $\mathscr{G}(\mathsf{c})$ denote the set of all group-like series of c.

3.2 Action of the lune-incidence algebra

Recall the lune-incidence algebra $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$.

Let (a, μ) be a monoid.

For any $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ and a series v of a, define another series $s \circ v$ of a by

$$(s \circ v)_A := \sum_{F\colon F \geq A} s(A, F)\, \mu_A^F(v_F). \tag{12}$$

To see that this is indeed a series, we compute:

$$
\begin{aligned}
\beta_{B,A}((s \circ v)_A) &= \sum_{F\colon F \geq A} s(A,F)\, \beta_{B,A} \mu_A^F(v_F) \\
&= \sum_{F\colon F \geq A} s(A,F)\, \mu_B^{BF} \beta_{BF,F}(v_F) \\
&= \sum_{F\colon F \geq A} s(B,BF)\, \mu_B^{BF}(v_{BF}) \\
&= \sum_{G\colon G \geq B} s(B,G)\, \mu_B^G(v_G) \\
&= (s \circ v)_B.
\end{aligned}
$$

The second step used naturality of the product.

The third step used that v is a series. It also used that $s(A,F) = s(B,BF)$.

The fourth step used the bijection between the stars of A and B.

Lemma 5. *The assignment $(s, v) \mapsto s \circ v$ defines a left action of the lune-incidence algebra on $\mathscr{S}(\mathrm{a})$.*

Proof. This is checked below.

$$
\begin{aligned}
(s \circ (t \circ v))_A &= \sum_{F\colon F \geq A} s(A, F)\, \mu_A^F((t \circ v)_F) \\
&= \sum_{F\colon F \geq A} s(A, F)\, \mu_A^F\Big(\sum_{G\colon G \geq F} t(F, G)\, \mu_F^G(v_G)\Big) \\
&= \sum_{G\colon G \geq A} \Big(\sum_{F\colon G \geq F \geq A} s(A, F) t(F, G)\Big) \mu_A^G(v_G) \\
&= \sum_{G\colon G \geq A} (st)(A, G)\, \mu_A^G(v_G) \\
&= (st \circ v)_A.
\end{aligned}
$$

Also, observe that $(\delta \circ v)_A = v_A$, where δ denotes the unit element of the lune-incidence algebra. $\square$

Thus, $\mathscr{S}(\mathrm{a})$ for any monoid a is a left module over the lune-incidence algebra.

3.3 Exp-log correspondences

Fix a noncommutative zeta function $\boldsymbol{\zeta}$ and a noncommutative Möbius function $\boldsymbol{\mu}$ which are inverses of each other.

For a series v of a monoid a, define

$$\mathbf{exp}(v)_A := \sum_{F\colon F \geq A} \boldsymbol{\zeta}(A, F)\, \mu_A^F(v_F), \tag{13a}$$

$$\mathbf{log}(v)_A := \sum_{F\colon F \geq A} \boldsymbol{\mu}(A, F)\, \mu_A^F(v_F). \tag{13b}$$

Since these operations are inverse to each other, we obtain:

Proposition 4. *For any monoid* a*, we have inverse bijections*

$$\mathscr{S}(\mathsf{a}) \underset{\mathbf{log}}{\overset{\mathbf{exp}}{\rightleftarrows}} \mathscr{S}(\mathsf{a}). \tag{14}$$

We refer to (14) as an exp-log correspondence.

Note very carefully that it depends on the particular $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ that we chose.

When a carries the structure of a bimonoid, one can do more as follows.

Theorem 5. *For a bimonoid* h*, we have inverse bijections*

$$\mathscr{P}(\mathsf{h}) \underset{\mathbf{log}}{\overset{\mathbf{exp}}{\rightleftarrows}} \mathscr{G}(\mathsf{h}). \tag{15}$$

Proof. In view of (14), it suffices to show that $\mathbf{exp}$ and $\mathbf{log}$ map as stated.

Suppose v is a primitive series of h.

We check below that $\boldsymbol{\zeta} \circ v$ is a group-like series.

For $G \geq A$,

$$\begin{aligned}
\Delta_A^G((\boldsymbol{\zeta} \circ v)_A) &= \sum_{F\colon F \geq A} \boldsymbol{\zeta}(A, F)\, \Delta_A^G \mu_A^F(v_F) \\
&= \sum_{F\colon F \geq A} \boldsymbol{\zeta}(A, F)\, \mu_G^{GF} \beta_{GF,FG} \Delta_F^{FG}(v_F) \\
&= \sum_{F\colon F \geq A,\, FG = F} \boldsymbol{\zeta}(A, F)\, \mu_G^{GF} \beta_{GF,F}(v_F) \\
&= \sum_{F\colon F \geq A,\, FG = F} \boldsymbol{\zeta}(A, F)\, \mu_G^{GF}(v_{GF}) \\
&= \sum_{H\colon H \geq G} \Big(\sum_{F\colon F \geq A,\, FG = F,\, GF = H} \boldsymbol{\zeta}(A, F) \Big)\, \mu_G^H(v_H) \\
&= \sum_{H\colon H \geq G} \boldsymbol{\zeta}(G, H)\, \mu_G^H(v_H) \\
&= (\boldsymbol{\zeta} \circ v)_G.
\end{aligned}$$

The first and last steps used definition (12).

The second step used the bimonoid axiom.

Since v is a primitive series, $\Delta_F^{FG}(v_F)$ is zero unless $FG = F$. This was used in the third step.

In the fifth step, we introduced a new variable H for GF.

The sixth step used the lune-additivity formula.

Conversely, suppose v is a group-like series of h.

We check below that $\boldsymbol{\mu} \circ v$ is a primitive series. For $G > A$,

$$\begin{aligned}
\Delta_A^G((\boldsymbol{\mu} \circ v)_A) &= \sum_{F\colon F \geq A} \boldsymbol{\mu}(A, F)\, \Delta_A^G \mu_A^F(v_F) \\
&= \sum_{F\colon F \geq A} \boldsymbol{\mu}(A, F)\, \mu_G^{GF} \beta_{GF,FG} \Delta_F^{FG}(v_F) \\
&= \sum_{F\colon F \geq A} \boldsymbol{\mu}(A, F)\, \mu_G^{GF} \beta_{GF,FG}(v_{FG}) \\
&= \sum_{F\colon F \geq A} \boldsymbol{\mu}(A, F)\, \mu_G^{GF}(v_{GF}) \\
&= \sum_{H\colon H \geq G} \Big(\sum_{F\colon F \geq A,\, GF = H} \boldsymbol{\mu}(A, F) \Big)\, \mu_G^H(v_H) \\
&= 0.
\end{aligned}$$

The first step used definition (12).

The second step used the bimonoid axiom.

The third step used that v is a group-like series.

In the fifth step, we introduced a new variable H for GF.

The last step used the noncommutative Weisner formula. $\square$

Example 2. Recall the bimonoid of chambers Γ.

Consider its space of series $\mathscr{S}(\Gamma)$.

An element can be viewed as a family of scalars $(f(A, C))_{A \leq C}$ such that $f(A, C) = f(B, D)$ whenever A and B have the same support, and $AD = C$ and $BC = D$.

The identification is done via

$$(f(A, C))_{A \leq C} \longleftrightarrow \sum_{C:\, C \geq A} f(A, C)\, \mathtt{H}_{C/A} \in \Gamma[A] \text{ for each face } A.$$

We deduce that $\mathscr{S}(\Gamma)$ has a basis indexed by top-lunes.

Specializing (12) and using the product formula of Γ, we see that its module structure over the lune-incidence algebra is given by

$$(s \circ f)(A, C) = \sum_{F\colon A \leq F \leq C} s(A, F) f(F, C).$$

This is the same as the lune-incidence module considered in [**?**, Section 15.2.6].

The bijection (14) specializes to

$$g(F,C) = \sum_{G:\, F \leq G \leq C} \boldsymbol{\zeta}(F,G) f(G,C) \tag{16}$$

$$\iff f(F,C) = \sum_{G:\, F \leq G \leq C} \boldsymbol{\mu}(F,G) g(G,C).$$

This is the noncommutative Möbius inversion of [**?**, Section 15.4.1].

Recall that Lie is the primitive part of Γ, hence a primitive series of Γ is the same as a series of the Lie species.

Explicitly, a series f is primitive if it satisfies

$$\sum_{C:\, C \geq A,\, HC=D} f(A, C) = 0 \tag{17}$$

for all $A < H \leq D$.

Similarly, a series g is group-like if it satisfies

$$g(H, D) = \sum_{C:\, C \geq A,\, HC=D} g(A, C) \tag{18}$$

for all $A \leq H \leq D$.

These descriptions can be deduced from the coproduct formula of Γ.

The bijection (15) says that primitive series and group-like series of Γ correspond to each other under (16). This result was obtained in [**?**, Theorem 15.42] with the same proof as given here.

4 Primitive and group-like series of bicommutative bimonoids

4.1 Primitive and group-like series of cocommutative comonoids

Let p be a species. We work with the formulation in terms of flats.

A series of p is a family of elements $v_{\mathrm{X}} \in \mathsf{p}[\mathrm{X}]$, one for each flat X.

Let $\mathscr{S}(\mathsf{p})$ denote the space of series of p.

Let (c, Δ) be a cocommutative comonoid.

A series v of c is primitive if $\Delta_{\mathrm{X}}^{\mathrm{Y}}(v_{\mathrm{X}}) = 0$ for all $\mathrm{Y} > \mathrm{X}$.

Let $\mathscr{P}(\mathsf{c})$ denote the set of all primitive series of c.

Similarly, a series v of c is group-like if $\Delta_{\mathrm{X}}^{\mathrm{Y}}(v_{\mathrm{X}}) = v_{\mathrm{Y}}$ for all $\mathrm{Y} \geq \mathrm{X}$.

Let $\mathscr{G}(\mathsf{c})$ denote the set of all group-like series of c.

The above definitions are consistent with those in Section 3.1.

4.2 Action of the flat-incidence algebra

Recall the flat-incidence algebra $\mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$.

Let (a, μ) be a commutative monoid.

The flat-incidence algebra acts on $\mathscr{S}(\mathsf{a})$ by

$$(s \circ v)_{\mathrm{Z}} := \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} s(\mathrm{Z}, \mathrm{X})\, \mu_{\mathrm{Z}}^{\mathrm{X}}(v_{\mathrm{X}}). \tag{19}$$

Thus, $\mathscr{S}(\mathsf{a})$ is a module over the flat-incidence algebra.

4.3 Commutative exp-log correspondence

The exponential and logarithm of a series v are defined by the action of the zeta function ζ and Möbius function μ of the poset of flats, that is,

$$\exp(v)_{\mathrm{Z}} := \sum_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Z}} \mu_{\mathrm{Z}}^{\mathrm{X}}(v_{\mathrm{X}}), \tag{20a}$$

$$\log(v)_{\mathrm{Z}} := \sum_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Z}} \mu(\mathrm{Z},\mathrm{X})\, \mu_{\mathrm{Z}}^{\mathrm{X}}(v_{\mathrm{X}}). \tag{20b}$$

Since these operations are inverse to each other, we obtain inverse bijections

$$\mathscr{S}(\mathsf{a}) \underset{\log}{\overset{\exp}{\rightleftarrows}} \mathscr{S}(\mathsf{a}). \tag{21}$$

This is the commutative exp-log correspondence.

Theorem 6. *For a bicommutative bimonoid* h*, we have inverse bijections*

$$\mathscr{P}(\mathsf{h}) \underset{\log}{\overset{\exp}{\rightleftarrows}} \mathscr{G}(\mathsf{h}). \tag{22}$$

Proof. One can proceed as in the proof of Theorem 5.

Suppose v is a primitive series of h.

We check that $\zeta \circ v$ is a group-like series.

For $\mathrm{Y} \geq \mathrm{Z}$,

$$\begin{aligned}
\Delta_{\mathrm{Z}}^{\mathrm{Y}}((\zeta \circ v)_{\mathrm{Z}}) &= \sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Z}} \Delta_{\mathrm{Z}}^{\mathrm{Y}} \mu_{\mathrm{Z}}^{\mathrm{X}}(v_{\mathrm{X}}) \\
&= \sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Z}} \mu_{\mathrm{Y}}^{\mathrm{X} \vee \mathrm{Y}} \Delta_{\mathrm{X}}^{\mathrm{X} \vee \mathrm{Y}}(v_{\mathrm{X}}) \\
&= \sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Y}} \mu_{\mathrm{Y}}^{\mathrm{X}}(v_{\mathrm{X}}) \\
&= (\zeta \circ v)_{\mathrm{Y}}.
\end{aligned}$$

The first and last steps used definition (20a).

The second step used the bicommutative bimonoid axiom.

Since v is a primitive series, $\Delta_{\mathrm{X}}^{\mathrm{X} \vee \mathrm{Y}}(v_{\mathrm{X}})$ will be zero unless $\mathrm{Y} \leq \mathrm{X}$. This was used in the third step.

Conversely, suppose v is a group-like series of h.

We check that $\mu \circ v$ is a primitive series.

For $\mathrm{Y} > \mathrm{Z}$,

$$
\begin{aligned}
\Delta^{\mathrm{Y}}_{\mathrm{Z}}((\mu \circ v)_{\mathrm{Z}}) &= \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{X})\, \Delta^{\mathrm{Y}}_{\mathrm{Z}} \mu^{\mathrm{X}}_{\mathrm{Z}}(v_{\mathrm{X}}) \\
&= \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{X})\, \mu^{\mathrm{X} \vee \mathrm{Y}}_{\mathrm{Y}} \Delta^{\mathrm{X} \vee \mathrm{Y}}_{\mathrm{X}}(v_{\mathrm{X}}) \\
&= \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{X})\, \mu^{\mathrm{X} \vee \mathrm{Y}}_{\mathrm{Y}}(v_{\mathrm{X} \vee \mathrm{Y}}) \\
&= \sum_{\mathrm{W}:\, \mathrm{W} \geq \mathrm{Y}} \Big(\sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z},\, \mathrm{Y} \vee \mathrm{X} = \mathrm{W}} \mu(\mathrm{Z}, \mathrm{X}) \Big) \mu^{\mathrm{W}}_{\mathrm{Z}}(v_{\mathrm{W}}) \\
&= 0.
\end{aligned}
$$

The first step used definition (20b).

The second step used the bicommutative bimonoid axiom.

The third step used that v is a group-like series.

The last step used the Weisner formula. $\square$

Example 3. Consider the exponential bimonoid E. All its components equal the base field.

Thus, a series of E is a family of scalars $f(\mathrm{X})$, one for each flat X.

Specializing (19), we see that its module structure over the flat-incidence algebra is given by

$$(s \circ f)(\mathrm{X}) = \sum_{\mathrm{Y}:\, \mathrm{X} \leq \mathrm{Y}} s(\mathrm{X}, \mathrm{Y}) f(\mathrm{Y}).$$

The bijection (21) specializes to

(23)

$$g(\mathrm{X}) = \sum_{\mathrm{Y}:\, \mathrm{X} \leq \mathrm{Y}} f(\mathrm{Y}) \iff f(\mathrm{X}) = \sum_{\mathrm{Y}:\, \mathrm{X} \leq \mathrm{Y}} \mu(\mathrm{X}, \mathrm{Y}) g(\mathrm{Y}).$$

This is Möbius inversion in the poset of flats.

Observe that a series f of E is primitive if $f(\mathrm{X}) = 0$ for all $\mathrm{X} \neq \top$.

Similarly, a series g of E is group-like if $g(\mathrm{X}) = g(\mathrm{Y})$ for all X and Y.

For a primitive series f and group-like series g,

$$\exp(f)(\mathrm{X}) = f(\top) \quad \text{and} \quad \log(g)(\mathrm{X}) = \begin{cases} g(\top) & \text{if } \mathrm{X} = \top, \\ 0 & \text{otherwise.} \end{cases}$$

This gives a direct verification of (22) for h := E.

5 Problems I

1. Let h be an $\mathcal{A}$-bimonoid. Let F and G be faces both greater than A, and of the same support. Let K be a face greater than A such that $KF = KG = K$. For any $x \in \mathsf{h}[F]$, show that

$$\Delta^K_A(\mu^F_A(x) - \mu^G_A \beta_{G,F}(x)) = 0.$$

2. Use the bimonoid axiom and the noncommutative Weisner formula to directly check the identities in Corollary 1.

3. Show that: If $f : \mathsf{c} \to \mathsf{d}$ is a morphism of comonoids such that $\mathrm{im}(f) \cap \mathcal{P}(\mathsf{d}) = 0$, then $f = 0$.

4. Explain diagram (10) as explicitly as possible for $\mathsf{h} = \Pi$, the bimonoid of flats.

5. Let $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ be such that $s(A, A) = 1$ for all A. If for every bimonoid h and primitive series v of

h, $s \circ v$ is a group-like series of h, then is s necessarily a noncommutative zeta function?

If for every bimonoid h and group-like series v of h, $s \circ v$ is a primitive series of h, then is s necessarily a noncommutative Möbius function?

6 Problems II

1. Let F denote the species whose A-component $\mathsf{F}[A]$ is linearly spanned by faces with the same support as A. For A and B of the same support, let $\beta_{B,A}$ be the identity. We write

$$\beta_{B,A} : \mathsf{F}[A] \to \mathsf{F}[B], \qquad \mathtt{H}_{A'} \mapsto \mathtt{H}_{A'}.$$

   Check that: The species F carries the structure of a comonoid with coproduct defined by

$$\Delta_A^G : \mathsf{F}[A] \to \mathsf{F}[G], \qquad \mathtt{H}_{A'} \mapsto \mathtt{H}_{A'G}.$$

   Moreover, the map $\mathsf{F} \to \mathsf{E}$ defined on the A-component by

$$\mathsf{F}[A] \to \mathsf{E}[A], \qquad \mathtt{H}_{A'} \mapsto \mathtt{H}_{A}$$

   is a morphism of comonoids.

2. For a rank-one arrangement with chambers C and $\overline{C}$, noncommutative zeta functions are

characterized by a scalar p as follows.

$$\boldsymbol{\zeta}(O,O)=\boldsymbol{\zeta}(C,C)=\boldsymbol{\zeta}(\overline{C},\overline{C})=1,\ \boldsymbol{\zeta}(O,C)=p,\ \boldsymbol{\zeta}(O,\overline{C})=1-p.$$

Compute the corresponding noncommutative Möbius function. Check explicitly that for $\mathsf{h} := \Gamma$, $\mathbf{log}(\mathrm{id})$ is an idempotent operator on Γ whose image is Lie.

3. Use the bicommutative bimonoid axiom and the Weisner formula to directly check the identities in Corollary 2.

4. Check that the composite map

$$\mathcal{P}(\Gamma) \hookrightarrow \Gamma \twoheadrightarrow \mathcal{Q}(\Gamma)$$

is surjective.

5. For any comonoid c, check that primitive series and group-like series of c coincide with those of its coabelianization c^{coab}, that is,

$$\mathscr{P}(\mathsf{c}^{coab}) = \mathscr{P}(\mathsf{c}) \quad \text{and} \quad \mathscr{G}(\mathsf{c}^{coab}) = \mathscr{G}(\mathsf{c}).$$

7 Reading assignment

Read at least four/five sections from any part of the notes c.pdf starting with Chapter 2, and give a writeup on it.

Your writeup could include

- a brief summary of what you understood,
- a list of things you did not understand properly,
- overall suggestions for improving the exposition,
- additional questions/insights that you have,
- pointing out typos,

and so on.