Examples of bimonoids in species

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1 Species characteristic of chambers

Define the species x by

$$\text{x}[F] := \begin{cases} \Bbbk & \text{if } F \text{ is a chamber,} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The maps $\beta_{G,F}$ are defined to be the identity. This is the species characteristic of chambers.

The k-th Cauchy power of x is given by

$$\text{x}^k[F] := \begin{cases} \bigoplus_{C:\, C \geq F} \Bbbk & \text{if } F \text{ has corank } k-1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In the first alternative, there is one copy of $\Bbbk$ for each chamber greater than F.

Equivalently, in terms of flats, the species x can be defined by

$$\mathsf{x}[\mathrm{Y}] := \begin{cases} \Bbbk & \text{if } \mathrm{Y} \text{ is the maximum flat,} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The k-th commutative Cauchy power of x is given by

$$\mathsf{x}^{\underline{k}}[\mathrm{Y}] := \begin{cases} \Bbbk & \text{if } \mathrm{Y} \text{ has corank } k-1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

2 Exponential species

2.1 Exponential set-species

The exponential set-species E is defined by

$$\mathrm{E}[A] := \{*\}$$

for any face A. In other words, each A-component is a singleton. For faces A and B of the same support, there is a unique map

$$\beta_{B,A} : \mathrm{E}[A] \to \mathrm{E}[B].$$

The exponential set-species E is the terminal object in the category of set-species.

2.2 Exponential species

The exponential species E is defined by

$$\mathsf{E}[A] := \Bbbk$$

for any face A. For faces A and B of the same support, define

$$\beta_{B,A} : \mathsf{E}[A] \to \mathsf{E}[B]$$

to be the identity map $\Bbbk \to \Bbbk$.

This is the linearization of the exponential set-species.

In terms of flats, the exponential species E can be defined by

$$\mathsf{E}[\mathrm{X}] := \Bbbk$$

for any flat X.

Observe from (4) that

(5) $$E = x + x^{\underline{2}} + x^{\underline{3}} + \dots,$$

the sum of all commutative Cauchy powers of x.

2.3 Exponential bimonoid

For the exponential species E, define

$$\mu_A^F : \mathsf{E}[F] \to \mathsf{E}[A] \quad \text{and} \quad \Delta_A^F : \mathsf{E}[A] \to \mathsf{E}[F]$$

to be the identity maps $\mathbb{k} \to \mathbb{k}$ for all $F \geq A$.

This turns E into a bimonoid.

We call it the exponential bimonoid.

For clarity, let us write $\mathtt{H}_A$ for the basis element $1 \in \mathsf{E}[A]$. By this convention,

$$\beta_{B,A}(\mathtt{H}_A) = \mathtt{H}_B, \quad \mu_A^F(\mathtt{H}_F) = \mathtt{H}_A \quad \text{and} \quad \Delta_A^F(\mathtt{H}_A) = \mathtt{H}_F.$$

Observe that E is bicommutative. So one can express all the above using flats instead of faces as follows. We let $\mathsf{E}[\mathrm{X}] := \mathbb{k}$ for any flat X, and

$$\mu_{\mathrm{Z}}^{\mathrm{X}} : \mathsf{E}[\mathrm{X}] \to \mathsf{E}[\mathrm{Z}] \quad \text{and} \quad \Delta_{\mathrm{Z}}^{\mathrm{X}} : \mathsf{E}[\mathrm{Z}] \to \mathsf{E}[\mathrm{X}]$$

be the identity maps for all $\mathrm{X} \geq \mathrm{Z}$. The basis element in $\mathsf{E}[\mathrm{Z}]$ may now be denoted $\mathtt{H}_{\mathrm{Z}}$.

The exponential bimonoid E is self-dual. The self-duality is via the canonical identification of $\mathbb{k}$ with $\mathbb{k}^*$.

2.4 Primitive part

The primitive part of E is given by

$$\mathcal{P}(\mathsf{E}) = \mathsf{x}.$$

Explicitly, the components $\mathsf{E}[C]$, as C varies over chambers, are primitive, while the remaining components do not contain any nonzero primitives.

Let us now consider the primitive filtration of E.

The first term $\mathcal{P}_1(\mathsf{E})$ equals the primitive part $\mathcal{P}(\mathsf{E})$.

The second term $\mathcal{P}_2(\mathsf{E})$ is the species, which is $\Bbbk$ on chambers and panels, and 0 otherwise.

In general, $\mathcal{P}_k(\mathsf{E})$ is the species, which is $\Bbbk$ on faces with corank at most $k-1$, and 0 otherwise.

Observe that this can be expressed as

$$\mathcal{P}_k(\mathsf{E}) = \mathsf{x} + \mathsf{x}^{\underline{2}} + \cdots + \mathsf{x}^{\underline{k}},$$

the sum of the first k commutative Cauchy powers of x.

2.5 (Co)freeness

The exponential bimonoid E is free as a commutative monoid and cofree as a cocommutative comonoid, both on the species x.

Further, E is the free commutative bimonoid on x viewed as a trivial comonoid, and it is the cofree cocommutative bimonoid on x viewed as a trivial monoid.

In other words, there is an isomorphism of bimonoids

$$\mathsf{E} \xrightarrow{\cong} \mathcal{S}(\mathsf{x}) = \mathcal{S}^{\vee}(\mathsf{x})$$

which on the Z-component, send $\mathtt{H}_{\mathrm{Z}}$ to $1 \in \mathsf{x}[\top]$.

The formula for the primitive filtration of E given above can now also be seen as a consequence of cofreeness.

3 Species of chambers

3.1 Species of chambers

The set-species of chambers Γ is defined by setting $\Gamma[A]$ to be the set of chambers greater than A. For clarity, we denote an element of $\Gamma[A]$ by C/A instead of just C. For A and B of the same support, define

$$\beta_{B,A} : \Gamma[A] \to \Gamma[B], \quad C/A \mapsto BC/B.$$

The species of chambers $\mathsf{\Gamma}$ is obtained by linearizing the set-species of chambers. Explicitly, $\mathsf{\Gamma}[A]$ is the linear span of chambers greater than A. We use the letter $\mathtt{H}$ for the canonical basis of $\mathsf{\Gamma}[A]$. For A and B of the same support, we write

$$\beta_{B,A} : \mathsf{\Gamma}[A] \to \mathsf{\Gamma}[B], \quad \mathtt{H}_{C/A} \mapsto \mathtt{H}_{BC/B}.$$

We claim that

$$\Gamma = \mathsf{x} + \mathsf{x}^2 + \mathsf{x}^3 + \dots, \tag{6}$$

the sum of all Cauchy powers of x.

By (2), the F component of the rhs is a vector space with basis indexed by chambers C greater than F, and we identify this with the $\mathtt{H}$-basis of $\Gamma[F]$.

Compare and contrast (6) with (5).

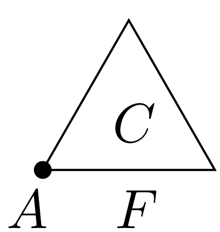
3.2 Bimonoid of chambers

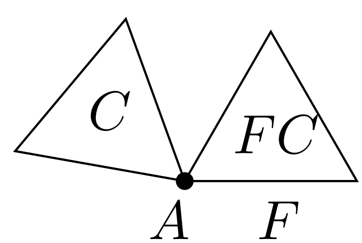
The species Γ carries the structure of a bimonoid.

The product and coproduct are defined by

$$
\begin{aligned}
\mu_A^F : \Gamma[F] &\to \Gamma[A] & \qquad \Delta_A^F : \Gamma[A] &\to \Gamma[F] \\
\mathtt{H}_{C/F} &\mapsto \mathtt{H}_{C/A} & \qquad \mathtt{H}_{C/A} &\mapsto \mathtt{H}_{FC/F}.
\end{aligned}
\tag{7}
$$

Illustrative pictures for the product and coproduct are shown below.



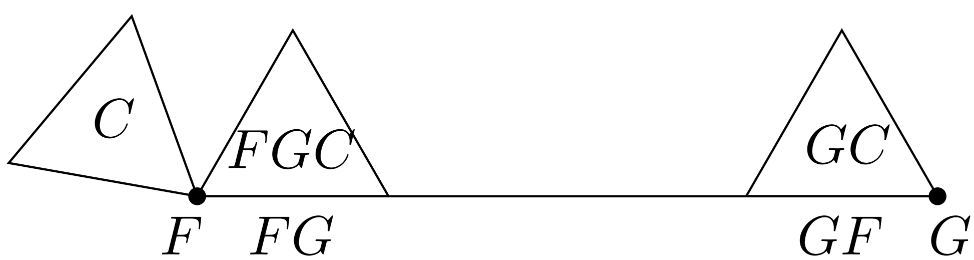


The bimonoid axiom is checked below.

$$\begin{tikzcd}
\mathtt{H}_{C/F} \arrow[r, mapsto] \arrow[d, mapsto] & \mathtt{H}_{C/A} \arrow[r, mapsto] & \mathtt{H}_{GC/G}. \\
\mathtt{H}_{FGC/FG} \arrow[rr, mapsto] & & \mathtt{H}_{GC/GF} \arrow[u, mapsto]
\end{tikzcd}$$

Here F and G are faces both greater than A, while C is a chamber greater than F.

An illustrative picture is shown below, with A as the central face.



The bimonoid Γ is cocommutative but not commutative. Hence it cannot be self-dual.

3.3 q-bimonoid of chambers

More generally, for any scalar q, the species of chambers carries the structure of a q-bimonoid which we denote by Γ_q.

The product and coproduct are defined by

(8)

$$\mu_A^F : \Gamma_q[F] \to \Gamma_q[A] \qquad \Delta_A^F : \Gamma_q[A] \to \Gamma_q[F]$$

$$\mathtt{H}_{C/F} \mapsto \mathtt{H}_{C/A} \qquad\qquad \mathtt{H}_{C/A} \mapsto q^{\mathrm{dist}(C,FC)}\mathtt{H}_{FC/F}.$$

Note that for $q = 1$, we have $\Gamma_1 = \Gamma$, the bimonoid of chambers.

The q-bimonoid axiom is checked below. It generalizes the previous calculation.

$$
\begin{tikzcd}
\mathtt{H}_{C/F} \arrow[r, mapsto] \arrow[d, mapsto] & \mathtt{H}_{C/A} \arrow[r, mapsto] & q^{\operatorname{dist}(C,GC)}\mathtt{H}_{GC/G}. \\
q^{\operatorname{dist}(C,FGC)}\mathtt{H}_{FGC/FG} \arrow[r, mapsto] & q^{\operatorname{dist}(C,FGC)}q^{\operatorname{dist}(FGC,GC)}\mathtt{H}_{GC/GF} \arrow[ur, mapsto] &
\end{tikzcd}
$$

We used that $\operatorname{dist}(C, GC) = \operatorname{dist}(C, FGC) + \operatorname{dist}(FGC, GC)$.

3.4 Dual bimonoid

Let Γ^* denote the bimonoid dual to Γ.

Let $\mathtt{M}$ denote the basis which is dual to the $\mathtt{H}$-basis.

The product and coproduct of Γ^* are obtained by dualizing formulas (7).

They are given by

(9)

$$\mu_A^F : \Gamma^*[F] \to \Gamma^*[A] \qquad \Delta_A^F : \Gamma^*[A] \to \Gamma^*[F]$$

$$\mathtt{M}_{D/F} \mapsto \sum_{\substack{C:\, C \geq A \\ FC = D}} \mathtt{M}_{C/A} \qquad \mathtt{M}_{C/A} \mapsto \begin{cases} \mathtt{M}_{C/F} & \text{if } F \leq C, \\ 0 & \text{otherwise.} \end{cases}$$

3.5 Dual q-bimonoid

More generally, let Γ_q^* denote the q-bimonoid dual to Γ_q.

Dualizing formulas (8), observe that its product and coproduct are given by

(10)

$$\mu_A^F : \Gamma_q^*[F] \to \Gamma_q^*[A] \qquad\qquad \Delta_A^F : \Gamma_q^*[A] \to \Gamma_q^*[F]$$

$$\mathtt{M}_{D/F} \mapsto \sum_{\substack{C:\, C \geq A \\ FC = D}} q^{\mathrm{dist}(FC,C)} \mathtt{M}_{C/A} \qquad\qquad \mathtt{M}_{C/A} \mapsto \begin{cases} \mathtt{M}_{C/F} \\ 0 \end{cases}$$

In contrast to Γ_q, the scalar q now appears in the product as opposed to the coproduct.

3.6 Primitive part

Observe from the coproduct formula (10) that

$$\mathcal{P}(\Gamma_q^*) = \mathrm{x}.$$

Explicitly, the components $\Gamma_q^*[C]$, as C varies over chambers, are primitive, while the remaining components do not contain any nonzero primitives.

Let us now consider the primitive filtration of Γ_q^*.

The first term $\mathcal{P}_1(\Gamma_q^*)$ equals the primitive part $\mathcal{P}(\Gamma_q^*)$.

The second term $\mathcal{P}_2(\Gamma_q^*)$ is the species whose F-component is $\Gamma_q^*[F]$ if F is either a chamber or a panel, and 0 otherwise.

In general, $\mathcal{P}_k(\Gamma_q^*)$ is the species whose F-component is $\Gamma_q^*[F]$ if F has corank at most $k-1$, and 0 otherwise.

Observe that this can be expressed as

$$\mathcal{P}_k(\Gamma_q^*) = \mathsf{x} + \mathsf{x}^2 + \cdots + \mathsf{x}^k,$$

the sum of the first k Cauchy powers of x.

Mention:

The primitive part of Γ is the Lie species.

For q not a root of unity,

$$\mathcal{P}(\Gamma_q) = \mathsf{x}.$$

3.7 (Co)freeness

The q-bimonoid of chambers Γ_q is free as a monoid on the species x. Further, it is the free q-bimonoid on x viewed as a trivial comonoid.

More precisely, there is an isomorphism of q-bimonoids

$$\Gamma_q \xrightarrow{\cong} \mathcal{T}_q(\mathsf{x}), \qquad \mathtt{H}_{C/A} \mapsto 1 \in \mathsf{x}[C].$$

This can be checked using the (co)product formulas of $\mathcal{T}_q(\mathsf{x})$.

Dually, $\mathsf{\Gamma}^*_q$ is cofree as a comonoid on the species x. Further, it is the cofree q-bimonoid on x viewed as a trivial monoid.

More precisely, there is an isomorphism of q-bimonoids

$$\mathsf{\Gamma}^*_q \xrightarrow{\cong} \mathcal{T}_q^\vee(\mathsf{x}), \qquad \mathtt{M}_{C/A} \mapsto 1 \in \mathsf{x}[C].$$

This can also be checked directly using the (co)product formulas of $\mathcal{T}_q^\vee(\mathsf{x})$.

The formula for the primitive filtration of $\mathsf{\Gamma}^*_q$ given above can also be seen as a consequence of cofreeness.

3.8 q-norm

Consider the map

(11)

$$\Gamma_q \to \Gamma_q^*, \qquad \mathtt{H}_{C/A} \mapsto \sum_{D:\, D \geq A} q^{\mathrm{dist}(C,D)}\, \mathtt{M}_{D/A}.$$

It is a self-dual morphism of q-bimonoids. It arises from the freeness of Γ_q and the cofreeness of Γ_q^*.

Nontrivial fact: This map is an isomorphism whenever q is not a root of unity, and in this case, the bimonoid Γ_q is self-dual.

Can you check this fact for the rank-one arrangement?

The cases $q = 0$ and $q = 1$ of (11) are discussed in more detail below.

3.9 0-bimonoid

Let $q = 0$. Observe that the product and coproduct of Γ_0 are given by:

$$\mu_A^F : \Gamma_0[F] \to \Gamma_0[A] \quad \Delta_A^F : \Gamma_0[A] \to \Gamma_0[F]$$

$$\mathtt{H}_{C/F} \mapsto \mathtt{H}_{C/A} \qquad\qquad \mathtt{H}_{C/A} \mapsto \begin{cases} \mathtt{H}_{C/F} & \text{if } F \leq C, \\ 0 & \text{otherwise.} \end{cases}$$

This is free as a monoid and cofree as a comonoid, both on the species x, which is its primitive part.

Also note that the product and coproduct of Γ_0^* are given by the same formulas (with $\mathtt{M}$ replacing $\mathtt{H}$). In other words, the map

$$\Gamma_0 \to \Gamma_0^*, \qquad \mathtt{H}_{C/A} \mapsto \mathtt{M}_{C/A}$$

is a morphism of 0-bimonoids, implying that Γ_0 is self-dual. The above map is the case $q = 0$ of the map (11).

3.10 Abelianization

There is a close connection between the species of chambers and the exponential species as follows.

The map $\pi : \mathsf{\Gamma} \to \mathsf{E}$ given by

$$\pi_A : \mathsf{\Gamma}[A] \to \mathsf{E}[A], \qquad \mathtt{H}_{C/A} \mapsto \mathtt{H}_A$$

is a surjective morphism of bimonoids.

This follows since the product and coproduct of $\mathsf{\Gamma}$ take a basis element to another basis element (rather than a sum as is the case for $\mathsf{\Gamma}^*$).

The kernel of π_A is the subspace spanned by elements of the form

$$\mathtt{H}_{C/A} - \mathtt{H}_{D/A},$$

as C and D vary over chambers in A. These are precisely elements of the form (**??**), thus π is the abelianization map.

Note that the kernel of π equals the kernel of the morphism of bimonoids

$$\mathsf{\Gamma} \to \mathsf{\Gamma}^*, \qquad \mathtt{H}_{C/A} \mapsto \sum_{D\colon D \geq A} \mathtt{M}_{D/A}.$$

This map is the case $q = 1$ of the map (11). Its image is one-dimensional. This yields the following commutative diagram of bimonoids.

(12)
$$\begin{tikzcd} \mathsf{\Gamma} \arrow[r] \arrow[d, "\pi"'] & \mathsf{\Gamma}^* \\ \mathsf{E} \arrow[r, "\cong"'] & \mathsf{E}^* \arrow[u, "\pi^*"'] \end{tikzcd}$$

4 Species of flats

4.1 Species of flats

Define a set-species Π as follows. For any flat X, let $\Pi[\mathrm{X}]$ be the set of all flats greater than X. Equivalently, it is the set of flats of $\mathcal{A}_{\mathrm{X}}$.

We denote the linearization of Π by $\mathsf{\Pi}$. This is the species of flats. Let $\mathtt{H}$ denote its canonical basis.

We claim that

$$\Pi = \mathsf{E} + \mathsf{E}^{\underline{2}} + \mathsf{E}^{\underline{3}} + \dots, \tag{13}$$

the sum of all commutative Cauchy powers of the exponential species E.

Explicitly, the Z-component of the rhs is $\bigoplus_{\mathrm{X} \geq \mathrm{Z}} \mathsf{E}[\mathrm{X}]$. This is a vector space with basis indexed by flats X greater than Z, and we identify this with the $\mathtt{H}$-basis of $\Pi[\mathrm{Z}]$.

4.2 Bimonoid of flats

The species of flats Π carries the structure of a bicommutative bimonoid, with the product and coproduct given by

(14)

$$\begin{aligned} \mu_{\mathrm{Z}}^{\mathrm{Y}} : \Pi[\mathrm{Y}] &\to \Pi[\mathrm{Z}] & \qquad \Delta_{\mathrm{Z}}^{\mathrm{Y}} : \Pi[\mathrm{Z}] &\to \Pi[\mathrm{Y}] \\ \mathtt{H}_{\mathrm{X}/\mathrm{Y}} &\mapsto \mathtt{H}_{\mathrm{X}/\mathrm{Z}} & \mathtt{H}_{\mathrm{X}/\mathrm{Z}} &\mapsto \mathtt{H}_{\mathrm{X}\vee\mathrm{Y}/\mathrm{Y}}. \end{aligned}$$

The bicommutative bimonoid axiom is checked below.

$$\begin{tikzcd} \mathtt{H}_{\mathrm{W}/\mathrm{X}} \arrow[r, mapsto] \arrow[d, mapsto] & \mathtt{H}_{\mathrm{W}/\mathrm{Z}} \arrow[d, mapsto] \\ \mathtt{H}_{\mathrm{W}\vee(\mathrm{X}\vee\mathrm{Y}))/\mathrm{X}\vee\mathrm{Y}} \arrow[r, mapsto] & \mathtt{H}_{\mathrm{W}\vee\mathrm{Y}/\mathrm{Y}} \end{tikzcd}$$

4.3 Birkhoff algebra

The bimonoid of flats Π also carries an internal structure.

For each flat Z, the component $\Pi[\mathrm{Z}]$ is an algebra with product in the $\mathtt{H}$-basis given by

$$\mathtt{H}_{\mathrm{X}/\mathrm{Z}} \cdot \mathtt{H}_{\mathrm{Y}/\mathrm{Z}} = \mathtt{H}_{\mathrm{X}\vee\mathrm{Y}/\mathrm{Z}}. \tag{15}$$

The unit element is $\mathtt{H}_{\mathrm{Z}/\mathrm{Z}}$.

This algebra can be identified with the Birkhoff algebra of the arrangement $\mathcal{A}_{\mathrm{Z}}$.

4.4 Dual bimonoid

Let Π^* denote the bimonoid dual to Π.

Let $\mathtt{M}$ be the basis which is dual to $\mathtt{H}$.

The product and coproduct in the $\mathtt{M}$-basis is given by

(16)

$$\mu_{\mathrm{Z}}^{\mathrm{Y}} : \Pi^*[\mathrm{Y}] \to \Pi^*[\mathrm{Z}] \qquad \Delta_{\mathrm{Z}}^{\mathrm{Y}} : \Pi^*[\mathrm{Z}] \to \Pi^*[\mathrm{Y}]$$

$$\mathtt{M}_{\mathrm{W/Y}} \mapsto \sum_{\substack{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}, \\ \mathrm{X} \vee \mathrm{Y} = \mathrm{W}}} \mathtt{M}_{\mathrm{X/Z}} \qquad \mathtt{M}_{\mathrm{X/Z}} \mapsto \begin{cases} \mathtt{M}_{\mathrm{X/Y}} & \text{if } \mathrm{X} \geq \mathrm{Y}, \\ 0 & \text{otherwise.} \end{cases}$$

4.5 Primitive part

Observe from the coproduct formula (16) in the $\mathtt{M}$-basis that

$$\mathcal{P}(\mathsf{\Pi}^*) = \mathsf{E}. \tag{17}$$

Each component $\mathcal{P}(\mathsf{\Pi}^*)[\mathrm{Z}]$ is one-dimensional, and is spanned by $\mathtt{M}_{\mathrm{Z}/\mathrm{Z}}$.

More generally, the primitive filtration of $\mathsf{\Pi}^*$ can be expressed as

$$\mathcal{P}_k(\mathsf{\Pi}^*) = \mathsf{E} + \mathsf{E}^{\underline{2}} + \cdots + \mathsf{E}^{\underline{k}},$$

the sum of the first k commutative Cauchy powers of E.

4.6 (Co)freeness

The bimonoid Π is the free commutative bimonoid on E, viewed as a comonoid.

Dually, Π^* is the cofree cocommutative bimonoid on E, viewed as a monoid.

More precisely, there are isomorphisms of bimonoids

$$\Pi \xrightarrow{\cong} \mathcal{S}(\mathrm{E}) \quad \text{and} \quad \Pi^* \xrightarrow{\cong} \mathcal{S}^{\vee}(\mathrm{E}). \tag{18}$$

On the Z-component, the first map sends $\mathtt{H}_{X/Z}$ to $\mathtt{H}_X$, while the second map sends $\mathtt{M}_{X/Z}$ to $\mathtt{H}_X$.

4.7 Self-duality

Is Π self-dual?

Yes.

In fact, every finite-dimensional bicommutative bimonoid is self-dual.

This is a consequence of the Borel-Hopf theorem.

See notes for more details.

5 Species of faces

5.1 Species of faces

For any face A, let $\Sigma[A]$ denote the set of faces greater than A. For faces A and B with the same support, there is a bijection

$$\beta_{B,A} : \Sigma[A] \to \Sigma[B], \qquad F/A \mapsto BF/B.$$

Thus, Σ is a set-species.

We denote the linearization of Σ by $\mathsf{\Sigma}$. This is the species of faces. Explicitly, $\mathsf{\Sigma}[A]$ is the linear span of the set of faces greater than A. We use the letter $\mathtt{H}$ for the canonical basis of $\mathsf{\Sigma}[A]$. For faces A and B of the same support, we write

$$\beta_{B,A} : \mathsf{\Sigma}[A] \to \mathsf{\Sigma}[B], \qquad \mathtt{H}_{F/A} \mapsto \mathtt{H}_{BF/B}.$$

We claim that

$$\Sigma = \mathsf{E} + \mathsf{E}^2 + \mathsf{E}^3 + \dots, \tag{19}$$

the sum of all Cauchy powers of the exponential species E.

Explicitly, the A-component of the rhs is $\bigoplus_{F \geq A} \mathsf{E}[F]$. This is a vector space with basis indexed by faces F greater than A, and we identify this with the $\mathtt{H}$-basis of $\Sigma[A]$.

5.2 Bimonoid of faces

The species Σ carries the structure of a bimonoid. The product and coproduct are defined by

(20)

$$\begin{aligned} \mu_A^F : \Sigma[F] &\to \Sigma[A] & \qquad \Delta_A^G : \Sigma[A] &\to \Sigma[G] \\ \mathtt{H}_{K/F} &\mapsto \mathtt{H}_{K/A} & \mathtt{H}_{K/A} &\mapsto \mathtt{H}_{GK/G}. \end{aligned}$$

5.3 q-bimonoid of faces

More generally, for any scalar q, the species of faces carries the structure of a q-bimonoid which we denote by Σ_q. The product and coproduct are defined by

(21)

$$\begin{aligned} \mu_A^F : \Sigma_q[F] &\to \Sigma_q[A] & \Delta_A^G : \Sigma_q[A] &\to \Sigma_q[G] \\ \mathtt{H}_{K/F} &\mapsto \mathtt{H}_{K/A} & \mathtt{H}_{K/A} &\mapsto q^{\mathrm{dist}(K,G)} \mathtt{H}_{GK/G} \end{aligned}$$

Note that for $q = 1$, we have $\Sigma_1 = \Sigma$, the bimonoid of faces.

5.4 Tits algebra

The bimonoid of faces Σ also carries an internal structure.

For each face A, the component $\Sigma[A]$ is an algebra with product in the $\mathtt{H}$-basis given by

$$\mathtt{H}_{F/A} \cdot \mathtt{H}_{G/A} = \mathtt{H}_{FG/A}. \tag{22}$$

The unit element is $\mathtt{H}_{A/A}$.

This algebra can be identified with the Tits algebra of the arrangement $\mathcal{A}_A$.

5.5 Dual bimonoid

Let Σ^* denote the bimonoid dual to Σ.

Let $\mathtt{M}$ denote the basis which is dual to the $\mathtt{H}$-basis.

The product and coproduct of Σ^* are obtained by dualizing formulas (20). They are given by

(23)

$$\mu_A^G : \Sigma^*[G] \to \Sigma^*[A] \qquad \Delta_A^F : \Sigma^*[A] \to \Sigma^*[F]$$

$$\mathtt{M}_{H/G} \mapsto \sum_{\substack{K:\, K \geq A \\ GK = H}} \mathtt{M}_{K/A} \qquad \mathtt{M}_{K/A} \mapsto \begin{cases} \mathtt{M}_{K/F} & \text{if } F \leq K \\ 0 & \text{otherwise.} \end{cases}$$

5.6 Dual q-bimonoid

Let Σ_q^* denote the q-bimonoid dual to Σ_q.

Dualizing formulas (21), its product and coproduct are given by

(24)

$$\mu_A^G : \Sigma_q^*[G] \to \Sigma_q^*[A] \qquad\qquad \Delta_A^F : \Sigma_q^*[A] \to \Sigma_q^*[F]$$

$$\mathtt{M}_{H/G} \mapsto \sum_{\substack{K:\, K \geq A \\ GK = H}} q^{\mathrm{dist}(H,K)} \mathtt{M}_{K/A} \qquad \mathtt{M}_{K/A} \mapsto \begin{cases} \mathtt{M}_{K/F} & \text{if} \\ 0 & \text{o} \end{cases}$$

5.7 Primitive part

We deduce from the coproduct formula (24) in the $\mathtt{M}$-basis that

$$\mathcal{P}(\Sigma_q^*) = \mathsf{E}.$$

Each component $\mathcal{P}(\Sigma_q^*)[F]$ is one-dimensional, and is spanned by $\mathtt{M}_{F/F}$. More generally, the primitive filtration of Σ_q^* can be expressed as

$$\mathcal{P}(\Sigma_q^*) = \mathsf{E} + \mathsf{E}^2 + \cdots + \mathsf{E}^k,$$

the sum of the first k Cauchy powers of E.

Mention:

The primitive part of Σ is the Zie species.

For q not a root of unity,

$$\mathcal{P}(\Sigma_q) = \mathsf{E}.$$

5.8 (Co)freeness

The q-bimonoid Σ_q is the free q-bimonoid on E, viewed as a comonoid.

Dually, Σ_q^* is the cofree q-bimonoid on E, viewed as a monoid.

More precisely, there are isomorphisms of q-bimonoids

(25) $$\Sigma_q \xrightarrow{\cong} \mathcal{T}_q(\mathsf{E}) \quad \text{and} \quad \Sigma_q^* \xrightarrow{\cong} \mathcal{T}_q^\vee(\mathsf{E}).$$

On the A-component, the first map sends $\mathtt{H}_{F/A}$ to $\mathtt{H}_F$, while the second map sends $\mathtt{M}_{F/A}$ to $\mathtt{H}_F$.