PH423 Assignment 2

Parth Sastry 180260026

Sahas Kamat 180260030

Sankalp Gambhir 180260032

September 22, 2020

1. Your question here.

[Sankalp: I got this one.]

We start with the expansion of the operators $\hat{\mathbf{J}}_x$ and $\hat{\mathbf{J}}_y$ in terms of the ladder operators

$$\hat{\mathbf{J}}_x = \frac{1}{2} \cdot (\hat{\mathbf{J}}_+ + \hat{\mathbf{J}}_-) \tag{1}$$

and

$$\hat{\mathbf{J}}_y = \frac{1}{2\imath} \cdot (\hat{\mathbf{J}}_+ - \hat{\mathbf{J}}_-) \ . \tag{2}$$

The application of the ladder operators on a state $|j, m\rangle$ changes it to a state of the form $c \cdot |j, m \pm 1\rangle$ for some $c \in \mathbb{C}$. So, given the orthogonality of the $|j, m\rangle$ states, we get that

$$\langle j, m|\hat{\mathbf{J}}_x|j, m\rangle = \langle j, m|\hat{\mathbf{J}}_y|j, m\rangle = 0 \qquad \forall\, |j, m\rangle \ . \tag{3}$$

Squaring Equation 1 and 2, we get the operators $\hat{\mathbf{J}}_x^2$ and $\hat{\mathbf{J}}_y^2$ in terms of the ladder operators. With the same argument as before, we see that only terms with equal powers of the two ladder operators will contribute, and using

$$\hat{\mathbf{J}}_\pm\, |j, m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)} \quad |j, m \pm 1\rangle \ , \tag{4}$$

we get

$$\begin{aligned}
\langle j, m|\hat{\mathbf{J}}_y^2|j, m\rangle &= \langle j, m|\hat{\mathbf{J}}_x^2|j, m\rangle &(5)\\
&= \langle j, m|\frac{1}{4} \cdot (\hat{\mathbf{J}}_+^2 + \hat{\mathbf{J}}_+\hat{\mathbf{J}}_- + \hat{\mathbf{J}}_-\hat{\mathbf{J}}_+ + \hat{\mathbf{J}}_-^2)|j, m\rangle &(6)\\
&= \langle j, m|\frac{1}{4} \cdot (\hat{\mathbf{J}}_+\hat{\mathbf{J}}_- + \hat{\mathbf{J}}_-\hat{\mathbf{J}}_+)|j, m\rangle &(7)\\
&= \langle j, m| \ \frac{1}{2} \cdot \Big(\sqrt{(j + m + 1)(j - m)}\sqrt{(j - m)(j + m + 1)} \\
&\quad + \sqrt{(j - m)(j + m + 1)}\sqrt{(j + m + 1)(j - m)} \ \Big) \cdot |j, m\rangle &(8)
\end{aligned}$$

$$= (j+m+1)(j-m) \tag{9}$$

The values for x and y are not separately calculated as a trivial calculation shows they're equal. The same is easily argued using symmetry in the x-y plane.

2. Determine the eigenvalues and eigenvectors of the 2 x 2 matrix $\boldsymbol{\sigma}.\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector along the (θ, ϕ) direction and $\boldsymbol{\sigma}$ are the three Pauli matrices. This is basically the projection of the spin 1/2 operator (apart from $\frac{\hbar}{2}$) along the direction of the unit vector $\hat{\mathbf{n}}$. Do this in two ways:

[Parth: Doing question 2, might have issues with part (b) make sure that it's correct]

(a) First by explicitly diagonalizing the matrix $\boldsymbol{\sigma}.\hat{\mathbf{n}}$.

The vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, where the σ_i matrices are -

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now we need to figure out what $\hat{\mathbf{n}}$ is. The unit vector points along the (θ, ϕ) direction. This is nothing but the unit vector $\hat{\mathbf{r}}$ in Polar co-ordinates.

$$\hat{\mathbf{n}} = \hat{\boldsymbol{r}} = \cos(\phi)\sin(\theta)\hat{\mathbf{i}} + \sin(\phi)\sin(\theta)\hat{\mathbf{j}} + \cos(\theta)\hat{\mathbf{k}}$$

Thus, $\hat{\mathbf{n}} = (\cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta), \cos(\theta))$. We know that $\mathbf{a}.\mathbf{b} = a_i b_i$ (implicit summation over i)

Thus, $\boldsymbol{\sigma}.\hat{\mathbf{n}} = \sigma_i n_i$.

$$\boldsymbol{\sigma}.\hat{\mathbf{n}} = \cos(\phi)\sin(\theta)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin(\phi)\sin(\theta)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos(\theta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\therefore \boldsymbol{\sigma}.\hat{\mathbf{n}} = \sin(\theta)\begin{pmatrix} 0 & \cos(\phi) - i * \sin(\phi) \\ \cos(\phi) + i * \sin(\phi) & 0 \end{pmatrix} + \cos(\theta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$= \sin(\theta)\begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix} + \cos(\theta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos(\theta) \end{pmatrix}$$

To find the eigenvalues and eigenvectors, we now need to diagonalize this matrix. Let the eigenvalues be represented by λ. The characteristic polynomial takes the following form.

$$(\cos(\theta) - \lambda)(-\cos(\theta) - \lambda) - \sin(\theta)e^{-i\phi} * \sin(\theta)e^{i\phi} = 0$$
$$\therefore -\cos^2(\theta) + \lambda^2 - \sin^2(\theta) = 0 \Rightarrow \lambda^2 - 1 = 0$$
$$\therefore \lambda = \pm 1$$

for $\lambda = 1$, let the eigenvector be $\mathbf{v_1} = (v_{1,1}, v_{1,2})$, thus

$$\begin{pmatrix} \cos(\theta) & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos(\theta) \end{pmatrix}\begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix} = \begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix}$$

$$\therefore \cos(\theta) * v_{1,1} + \sin(\theta)e^{-i\phi} * v_{1,2} = v_{1,1} \ , \ \sin(\theta)e^{i\phi} * v_{1,1} - \cos(\theta) * v_{1,2} = v_{1,2}$$
$$v_{1,2} = e^{i\phi}\frac{\sin(\theta)}{(\cos(\theta)+1)} * v_{1,1}$$

Thus, for eigenvalue $\lambda = 1$, the eigenvector $\mathbf{v_1} = (v_{1,1}, e^{i\phi}\frac{\sin(\theta)}{(\cos(\theta)+1)} * v_{1,1})$
Likewise, for $\lambda = -1$, let the eigenvector be $\mathbf{v_2} = (v_{2,1}, v_{2,2})$, thus

$$\begin{pmatrix} \cos(\theta) & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos(\theta) \end{pmatrix}\begin{pmatrix} v_{2,1} \\ v_{2,2} \end{pmatrix} = \begin{pmatrix} -v_{2,1} \\ -v_{2,2} \end{pmatrix}$$
$$\therefore \cos(\theta) * v_{2,1} + \sin(\theta)e^{-i\phi} * v_{2,2} = -v_{2,1}, \sin(\theta)e^{i\phi} * v_{2,1} - \cos(\theta) * v_{2,2} = -v_{2,2}$$
$$v_{2,2} = e^{i\phi}\frac{\sin(\theta)}{(1-\cos(\theta))} * v_{2,1}$$

Thus, for eigenvalue $\lambda = -1$, the eigenvector $\mathbf{v_2} = (v_{2,1}, e^{i\phi}\frac{\sin(\theta)}{(1-\cos(\theta))} * v_{2,1})$.
We thus have our two eigenvalues (± 1) and our two eigenvectors ($\mathbf{v_1}$ and $\mathbf{v_2}$)

(b) By rotating the spinor pointing initially along the $+\hat{\mathbf{z}}$ axis direction by appropriate angles, using the appropriate rotation operator. Convince yourself that one has to rotate by an angle θ counterclockwise around the y-axis and then by ϕ around the z-axis. Apart from overall phases, is the resultant spinor the same as the spin up eigenvector obtained in part **(a)**?

Let's start with the spinor pointing in the $+z$-direction.

$$\left|s_z = +\frac{\hbar}{2}\right\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ s.t \ S_z\left|s_z = +\frac{\hbar}{2}\right\rangle = +\frac{\hbar}{2}\left|s_z = +\frac{\hbar}{2}\right\rangle$$

If we apply consecutive rotation operators, we should be able to rotate this spinor into a general state, pointing in an arbitrary direction $\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ points in the (θ, ϕ) direction.
We first rotate this spinor by θ around the y-axis, and then by ϕ around the z-axis. The axis of spin now points in the direction $\hat{\mathbf{n}}$. Thus -

$$|\hat{n}+\rangle = U[R(\phi\hat{\mathbf{z}})]U[R(\theta\hat{\mathbf{y}})]\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

To find the explicit form of $|\hat{n}+\rangle$, we'll need the forms of the unitary matrices $U[R(\phi\hat{\mathbf{z}})]$ and $U[R(\theta\hat{\mathbf{y}})]$. We'll use the result given in Shankar -

$$U[R(\boldsymbol{\theta}] = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}(\hat{\theta}.\boldsymbol{\sigma})$$

Looking at the particular case of rotation around y-axis by amount θ and then subsequently around z-axis by amount ϕ -

$$
\begin{aligned}
U[R(\theta\hat{\mathbf{y}})]\begin{bmatrix}1\\0\end{bmatrix} &= \left[\cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}\sigma_y\right]\begin{bmatrix}1\\0\end{bmatrix}\\
&= \begin{bmatrix}\cos\frac{\theta}{2}\\0\end{bmatrix} - i\sin\frac{\theta}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\
&= \begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix}
\end{aligned}
$$

Applying rotation around z-axis by amount ϕ now, we get

$$
\begin{aligned}
U[R(\phi\hat{\mathbf{z}})]\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix} &= \left[\cos\frac{\phi}{2}I - i\sin\frac{\phi}{2}\sigma_z\right]\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix}\\
&= \begin{bmatrix}\cos\frac{\phi}{2}\cos\frac{\theta}{2}\\ \cos\frac{\phi}{2}\sin\frac{\theta}{2}\end{bmatrix} - i\sin\frac{\phi}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix}\\
&= \begin{bmatrix}\cos\frac{\theta}{2}\left(\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\right)\\ \sin\frac{\theta}{2}\left(\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right)\end{bmatrix}\\
&= \begin{bmatrix}\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}\\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{bmatrix}
\end{aligned}
$$

This gives us a spinor $s_n = (s_{n1}, s_{n2}) = (\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}, \sin\frac{\theta}{2}e^{i\frac{\phi}{2}})$. If we recall our $\mathbf{v_1} = (v_{1,1}, v_{1,2})$ from part **(a)**, we recall the relation we obtained at the end.

$$
v_{1,2} = e^{i\phi}\frac{\sin(\theta)}{(\cos(\theta)+1)} * v_{1,1}
$$

Substituting $v_{1,1} = s_{n1} = \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}$ (as our final spinor seems to suggest), we get -

$$
\begin{aligned}
v_{1,2} &= e^{i\phi}\frac{\sin(\theta)}{(\cos(\theta)+1)} * v_{1,1}\\
&= e^{i\phi}\frac{\sin(\theta)}{(\cos(\theta)+1)} * \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}
\end{aligned}
$$

Recall $1 + \cos(A) = 2 * \cos^2(\frac{A}{2})$ and $\sin(A) = 2 * \sin(\frac{A}{2})\cos(\frac{A}{2})$

$$
\begin{aligned}
e^{i\phi}\frac{\sin(\theta)}{(\cos(\theta)+1)} * \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} &= e^{i\frac{\phi}{2}}\frac{\sin(\theta)}{2cos^2(\frac{\theta}{2})} * \cos\frac{\theta}{2}\\
&= e^{i\frac{\phi}{2}}\frac{2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})}{2cos^2(\frac{\theta}{2})} * \cos\frac{\theta}{2}\\
&= e^{i\frac{\phi}{2}}\sin(\frac{\theta}{2}) = s_{n2}
\end{aligned}
$$

Therefore, apart from phase factors, the resultant spinor is the same as the spin up eigenvector we got in part **(a)**.

3. Your question here.

[Sahas: I got this one.]

4. Your question here.

[Sankalp: I got this one.]

5. Prove that any function of the radial coordinate $f(r)$ where $r = |\mathbf{r}|$ and $\mathbf{X} \cdot \mathbf{P}$, where $\mathbf{X}$ and $\mathbf{P}$ are the position and momentum operators, are both scalar operators.

[Parth: Doing question 5, I'm not spending as much time on this as question 2]

Under a symmetry operator U, operators change as $\mathbb{O}' = U^\dagger \mathbb{O} U$. A scalar operator being one which is invariant under rotations, i.e

$$S' = U^\dagger[R]SU[R] = S$$

where $U(R(\boldsymbol{\alpha}) = e^{-\frac{i}{\hbar}\boldsymbol{\alpha}\cdot\mathbf{J}})$.
By considering infinitesimal rotations $\boldsymbol{\alpha} = \boldsymbol{\epsilon}$, we have

$$U[R(\boldsymbol{\alpha})] = \left(1 - \frac{i}{\hbar}\epsilon_i J_i\right)$$

Thus, our definition for a scalar operator becomes -

$$S' = \left(1 + \frac{i}{\hbar}\epsilon_i J_i\right) S \left(1 - \frac{i}{\hbar}\epsilon_i J_i\right) = S$$

which gives us $\frac{i}{\hbar}\epsilon_i\,[J_i, S] = 0$. Since $\boldsymbol{\epsilon}$ was an arbitrary choice, we have

$$[J_i, S] = 0$$

as our definition of a scalar operator.

Considering $f(r)$, where $r = |\mathbf{r}|$ as our operator.

$$[J_i, f(r)] = [J_i, r] * f'(r)$$

$r = \sqrt{\sum_{i=1}^{3} X_i^2}$, Thus

$$[J_i, r] = [J_i, X_1] * \frac{X_1}{r} + [J_i, X_2] * \frac{X_2}{r} + [J_i, X_3] * \frac{X_3}{r}$$

we know that $\left[J_i, X_j\right] = i\hbar\epsilon_{ijl}X_l$. Thus

$$[J_i, r] = \left[J_i, X_j\right] * \frac{X_j}{r} = \frac{1}{r}\left(i\hbar\epsilon_{ijl}X_lX_j\right)$$

$$\epsilon_{ijl}X_lX_j = \left[X_l, X_j\right] = 0(l \neq j) \Rightarrow [J_i, r] = 0$$

Thus, since $[J_i, r] = 0$, we have $[J_i, f(r)] = [J_i, r] * f'(r) = 0 * f'(r) = 0$.
Thus, $f(r)$ is a scalar operator.

Now considering $O = \mathbf{X}\,.\,\mathbf{P}$ as our operator, we need to show $[J_i, O] = 0$

$$\begin{aligned}
\mathbf{X}.\mathbf{P} &= X_iP_i \qquad \text{implicit summation} \\
\therefore\ [J_i, O] &= \left[J_i, X_jP_j\right] \\
&= \left[J_i, X_j\right]P_j + X_j\left[J_i, P_j\right] \\
&= i\hbar\epsilon_{ijl}(X_lP_j + X_jP_l)
\end{aligned}$$

Now, $\epsilon_{ijl}X_lP_j = [X_l, P_j]$ for $l \neq j$, but $[X_l, P_j] = 0, l \neq j$. Thus

$$i\hbar\epsilon_{ijl}(X_lP_j + X_jP_l) = 0 \Rightarrow [J_i, O] = 0$$

Since $[J_i, O] = 0$, we can say that the operator O is a scalar operator.
Thus, **X.P** is a scalar operator

6. Your question here.

[Sahas: I got this one.]