Bimonoids for hyperplane arrangements

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Contents

Preface

Motivation

The geometry of braid arrangements is intimately related to the theory of connected graded Hopf algebras. Our path to this conclusion started with [17], where we noticed that the product and coproduct of certain combinatorial Hopf algebras are related to the geometric operations of join and link on the faces of a braid arrangement, while the compatibility axiom is related to the Tits product of those faces. This viewpoint was strengthened further when we studied Hopf monoids in Joyal species [18], [19], where the connection to the braid arrangement is more direct. This also suggested an extension of the theory to more general contexts involving real hyperplane arrangements or even certain semigroups replacing the Tits monoid of the braid arrangement. We mentioned this point for the first time in the introduction to [19] and then with more insistence in the end-of-chapter notes in [21].

Main players

The goal of this monograph is to embark on the theory of species and bimonoids for hyperplane arrangements. The main players that have emerged in this study are summarized in Table I below.

Table I. Coxeter bimonoids and Joyal bimonoids.

Starting data	Objects of interest	
hyperplane arrangement	species	
reflection arrangement	Coxeter species	Coxeter spaces
braid arrangement	Joyal species	graded vector spaces
hyperplane arrangement	bimonoids	
reflection arrangement	Coxeter bimonoids	Coxeter bialgebras
braid arrangement	Joyal bimonoids	graded bialgebras

Our hyperplane arrangements are assumed to be linear, that is, all hyperplanes pass through the origin. We will use the term 'classical' to refer to the mathematics of the braid arrangement. In most cases, classical objects will mean graded vector spaces and graded bialgebras, but they can also mean Joyal species and Joyal bimonoids.

In this book, we focus exclusively on species and bimonoids for which the starting data is a hyperplane arrangement. We also briefly indicate how they relate to Joyal species and Joyal bimonoids. Analogies with connected graded Hopf algebras are mentioned but not elaborated. In fact, these classical objects are better connected to Coxeter bimonoids and Coxeter bialgebras which are more structured and for which the starting data is a reflection arrangement. These Coxeter objects along with their relation to the classical theory will be explained in a separate work.

TABLE II. Coxeter operads and May operads.

Starting data	Objects of interest
Hyperplane arrangement	Operads
Reflection arrangement	Coxeter operads
Braid arrangement	May operads

Related objects are summarized in Table II. In this book, we briefly develop operads in the setting of hyperplane arrangements, and explain their connection to species and bimonoids. We also indicate how they relate to classical operads (which we call May operads). A proper understanding of this relationship requires consideration of Coxeter operads which will be treated in a separate work.

As a historical note, we mention that the picture in Tables I and II along with all the basic definitions emerged as [**18**] was nearing publication and became completely clear to us by the time [**19**] was published.

Synopsis

We begin by introducing the category of species for any hyperplane arrangement and the notions of monoid, comonoid, bimonoid therein. These may be viewed as an extension of corresponding notions in Joyal species from braid arrangements to an arbitrary arrangement. The main novelty is the usage of the Tits product on faces in the formulation of the bimonoid axiom. (We use the term 'bimonoid' rather than 'Hopf monoid' since we only treat the connected case.) A bimonoid can be commutative, cocommutative, both or neither. Illustrative examples include the exponential bimonoid, the bimonoid of chambers, the bimonoid of flats, the bimonoid of faces, the bimonoid of top-nested faces, the bimonoid of top-lunes, the bimonoid of bifaces, the bimonoid of chamber maps and the bimonoid of pairs of chambers. We also define signed bimonoids. More generally, for any scalar q, we introduce q-bimonoids, with $q = \pm 1$ specializing to bimonoids and signed bimonoids, respectively. This is done by deforming the bimonoid axiom using the distance function on faces of the arrangement. We also introduce the notion of a Lie monoid for any hyperplane arrangement.

We briefly consider operads in the setting of hyperplane arrangements. These may be seen as an extension of May operads from braid arrangements to an arbitrary arrangement. We define the commutative operad, associative operad, Lie operad for any hyperplane arrangement, and observe that left modules over these operads in the category of species are commutative monoids, monoids, Lie monoids, respectively. Any operad gives rise to a monad on species with operad-modules corresponding to monad-algebras. Thus, commutative monoids, monoids, Lie monoids can also be viewed as algebras over suitable monads. Moreover, we construct bimonads (mixed distributive laws) whose bialgebras are precisely bimonoids and their commutative and signed counterparts.

We lay out the basic theory of bimonoids for hyperplane arrangements. This includes a detailed discussion of primitive filtrations of comonoids and decomposable filtrations of monoids, the related Browder–Sweedler commutativity result and Milnor–Moore cocommutativity result, universal constructions of bimonoids, the Hadamard product of bimonoids and its freeness properties, the universal measuring comonoid and enrichment of the category of monoids over the category of comonoids, the antipode of a bimonoid and the Takeuchi formula. This is largely motivated by the classical theory of Hopf algebras [**867**] and the theory of Hopf monoids in Joyal species [**18**], [**19**]. The universal constructions, for instance, employ generalizations of the classical notions of (de)shuffles and (de)quasishuffles to arrangements which were given in [**17**]. We use (noncommutative) zeta and Möbius functions introduced in [**21**] to generalize the classical exponential and logarithm and obtain a family of exp-log correspondences between (co)derivations and (co)monoid morphisms, and between primitive and group-like series of a bimonoid. We consider (commutative, two-sided) characteristic operations on bimonoids and employ them to forge a precise connection of bimonoids and their commutative counterparts with the representation theory of the Birkhoff algebra, Tits algebra, Janus algebra. These algebras appear prominently in the recent semigroup literature; see for instance [**21**]. Characteristic operations by complete systems of primitive idempotents of the Tits algebra extend the classical theory of eulerian idempotents to arrangements.

We treat in detail many important structure results for bimonoids. These are analogues of well-known classical results for Hopf algebras. Their extension to arrangements appears here for the first time and contains many new ideas. These include the Loday–Ronco theorem for 0-bimonoids, the Leray–Samelson theorem for bicommutative bimonoids, the Borel–Hopf theorem for commutative bimonoids and for cocommutative bimonoids. We also generalize the Loday–Ronco theorem to q-bimonoids for q not a root of unity. This makes use of a classical factorization result of Varchenko on distance functions, and a q-analogue of zeta and Möbius functions and the resulting q-exp-log correspondence involving the q-exponential and q-logarithm. We treat the Poincaré–Birkhoff–Witt (PBW) and Cartier–Milnor–Moore (CMM) theorems relating Lie monoids and cocommutative bimonoids, as well as their dual versions relating Lie comonoids and commutative bimonoids, and highlight their connection with the Borel–Hopf theorem. These results come in two flavors, namely, unsigned and signed, with the two linked by the signature functor. We establish the Hoffman–Newman–Radford rigidity theorems which relate (de)shuffles and (de)quasishuffles in the setting of arrangements and are significant for the theory of zeta and Möbius functions. All our results are valid over a field of arbitrary characteristic.

Prerequisites

The prerequisites for reading this book pertain to three main areas: category theory, Hopf and Lie theory, and hyperplane arrangements. They are elaborated below.

Category theory. We assume familiarity with the basic language of category theory at the level of [**54**], [**591**], [**781**], [**785**]. Some concepts which are repeatedly used without explanation are functors, natural transformations, equivalences between categories, adjunctions, universal properties. Appendices are provided for more advanced concepts such as monads.

Hopf theory and Lie theory. While the entire theory here is developed from first principles, some exposure to classical Hopf theory at the level of [**867**] will be useful for motivational purposes. For Lie theory we require much less, basic familiarity with Lie algebras including the construction of the universal enveloping algebra is more than sufficient. We provide ample references to the classical literature. We point out that our monograph on species and Hopf algebras [**18**] is *not* a formal prerequisite for reading this work, though again some familiarity will be useful.

Hyperplane arrangements. General familiarity with hyperplane arrangements is sufficient to get started. The Tits monoid is a central object; familiarity with the Tits product and its basic properties suffices to understand large parts of the text. In certain places, we do need access to more specialized notions and results. These involve incidence algebras and noncommutative zeta and Möbius functions, the structure theory of the Tits algebra, distance functions, Lie and Zie elements, the descent, lune, Witt identities. To keep the book self-contained, this material is reviewed here in an introductory chapter. The reader interested in more details can consult [**21**]. This reference, however, is *not* a prerequisite for reading the text. In fact, many ideas there can be motivated and understood using the Hopf perspective developed in this book.

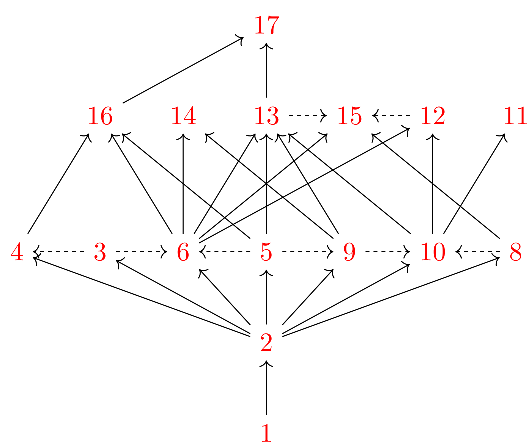
Readership

This book would be of interest to students and researchers working in the areas of hyperplane arrangements, semigroup theory, Joyal species, May operads, Hopf algebras, algebraic Lie theory, category theory. It is written with sufficient detail to make it accessible to well-prepared graduate students.

Organization

The text is organized in three parts. Part I introduces species and bimonoids, Part II develops their basic theory, and Part III establishes various structure results. A detailed summary of the contents is given in the main introduction. Each chapter in the text also has its own introduction. Further, there is a Notes section at the end of each chapter which provides historical commentary and detailed references to the literature. Appendices are provided for background material. Diagrams and pictures form an important component of our exposition. Numerous exercises (with generous hints) are interspersed throughout the book. We also list a few open problems. A list of notations, a list of tables, an author index, and a subject index are provided at the end of the book.

The diagram of interdependence of chapters is displayed below.



A dashed arrow indicates that the dependence is of a minimal nature. Chapter 7 is not shown in the above chart. It consists of examples and can be read in parallel to the theory developed in the other chapters. Some further guidelines on how to selectively read the book are given in the teaching section below.

Teaching

The book is suitable for a two-semester sequence with the first semester focusing on Parts I and II, and the second on Part III. With a careful selection of topics, it can also be used for courses of shorter duration or theme-based seminars. Details follow.

(0) First do the basic groundwork, namely: Review the Tits monoid and Birkhoff monoid from Chapter 1. Define monoids, comonoids, bimonoids in species (along with their commutative counterparts) from Chapter 2. Illustrate these notions with suitable examples from Chapter 7. This could then be followed by any of the routes given below.

(1) One plan could be to do the Leray–Samelson and Borel–Hopf theorems from Chapter 13. To state these results, first review the relevant universal constructions from Chapter 6. For the proofs, three entirely different paths can be taken. The first path goes through Chapter 5 on primitive and decomposable filtrations. The second path goes through Chapter 9 on the exponential and logarithm. The third path goes through Chapters 10 and 11 related to representation theory.
(2) The pattern in item (1) could be independently followed for the Loday–Ronco and the more general rigidity theorem for q-bimonoids (for q not a root of unity) from Chapter 13. This involves a lot of interesting q-calculus, with distance functions and the Varchenko factorization theorem from Chapter 1 playing a starring role.
(3) For those interested in zeta and Möbius functions, a nice option is to first do the universal constructions from Chapter 6 and then focus on the Hoffman–Newman–Radford rigidity theorems from Chapter 14. This entire chapter can be done independently of items (1) or (2), or relevant sections from it can also be done as a follow-up to either item (1) or item (2).
(4) A different plan could be to do Chapter 16 on Lie monoids. For this, first go over the commutative, associative, Lie operads and also the notion of operad modules from Chapter 4. The construction of the universal enveloping monoid requires some basic familiarity with Chapter 6. This can be followed with the Poincaré–Birkhoff–Witt theorem from Chapter 17. The final step would be to do the Cartier–Milnor–Moore theorem, but then this requires an exposure to the Borel–Hopf theorem from item (1).
(5) Independent of all the above, one can do Chapters 8 and 15 on the Hadamard product on species, OR bimonads from Chapter 3 and operads from Chapter 4 with applications to universal constructions, OR the antipode material from Chapter 12 which brings in Euler characteristics and the related descent, lune, Witt identities from Chapter 1.

Depending on interest, any of the above themes may be supplemented further in many different ways. For instance, one may discuss signed aspects of the theory, unifications via partial-flats, generalizations to left regular bands. One could also explore relevant historical facts using references in the Notes.

Comparison with previous work

In broad terms, the text builds on [**17**], [**18**, Part II], [**19**]. However, there are some technical differences as well as some new developments which we highlight below.

Differences. There are two main differences to be aware of. The theory presented here is local to a fixed hyperplane arrangement, while the theory presented in [**18**] applies to not just one braid arrangement but all braid arrangements taken together. This is a local-global issue. Secondly, the tensor product of vector spaces is central to the theory of Joyal bimonoids, but that is not the case for bimonoids for arrangements. This is a noncartesian-cartesian issue. In summary, what we have developed here is a local noncartesian theory of species. This has necessitated a technical change: monoidal categories are now replaced by monads, and bilax functors now go between bimonads as opposed to between braided monoidal categories. Thus, monoids in species are now algebras over a certain monad rather than monoids in a monoidal category, and so forth.

New developments. The extension to arrangements brings a completely new perspective to Hopf theory. Many aspects of the theory which were implicit before have now become more visible. In addition, several new aspects of a fundamental nature have also appeared. A summary is given below.

- clarity on the central role played by the Birkhoff monoid, Tits monoid, Janus monoid,
- formulation of bimonoids and Lie monoids using 'higher operations' involving faces of the arrangement as opposed to 'binary operations' involving vertices of the arrangement,
- formulation of the commutative aspects of the theory in terms of flats, and noncommutative aspects in terms of faces,
- interpretation of the categories of monoids, comonoids, bimonoids as functor categories just like the category of species,
- emergence of many interesting finite categories constructed from geometric objects such as faces, flats, lunes, bilunes,
- connection of the antipode of a bimonoid to the antipodal map on arrangements via the antipode opposition lemma and the op-cop constructions,
- emergence of the bimonoid of bifaces and related ideas such as two-sided characteristic operations,
- connection between representation theory of the Birkhoff algebra and bicommutative bimonoids, the Tits algebra and cocommutative bimonoids, the Janus algebra and arbitrary bimonoids, and more generally, between the q-Janus algebra and q-bimonoids,
- relevance of the Karoubi envelopes of the Birkhoff monoid, Tits monoid, Janus monoid to Hopf theory,
- systematic use of distance functions and the gate property to study deformations of bimonoids,
- emergence of the monoidal category of dispecies with the category of species as a left module category over it,
- connection between operads and incidence algebras, and in particular, between the commutative operad and the flat-incidence algebra, and the associative operad and the lune-incidence algebra,
- connection of the binary quadratic presentations of the commutative and associative operads to the strong connectivity property of the posets of flats and faces, respectively,
- emergence of the one-dimensional orientation and signature spaces of an arrangement to deal with signed aspects of the theory,

- emergence of noncommutative zeta and Möbius functions as a generalization of the classical exponential and logarithm, and their intimate connection to the Hoffman–Newman–Radford rigidity theorems as well as to the Borel–Hopf theorem and Poincaré–Birkhoff–Witt theorem,
- connection of the Borel–Hopf theorem to the Zaslavsky formula for enumeration of chambers and faces in a hyperplane arrangement,
- usage of (commutative, two-sided) characteristic operations and the structure theory of the Birkhoff algebra, Tits algebra, Janus algebra to give constructive proofs of the Leray–Samelson, Borel–Hopf, Loday–Ronco theorems, respectively.

Domain of validity. All our results are valid over a field of arbitrary characteristic. A key reason for this is the existence of noncommutative zeta and Möbius functions over any field.

Similarly, all our results (except those of an enumerative nature or pertaining to self-duality) are valid without any finite-dimensionality assumption on the species or on the monoids, comonoids, bimonoids involved. A key reason for this is that we are working in a noncartesian setting.

New results and topics. Some important new topics and results are listed below.

- We establish a noncommutative analogue of the Zaslavsky formula. It involves the antipodal map on arrangements, and has connections to the Witt identities via noncommutative Möbius inversion. These ideas are also intimately tied to the logarithm of the antipode map of bimonoids.
- We introduce partially commutative monoids as interpolating objects between monoids and commutative monoids. We formalize the close parallel between the Loday–Ronco and Leray–Samelson theorems using this approach. There is a similar parallel between Borel–Hopf and Leray–Samelson.
- We prove a rigidity theorem for q-bimonoids when q is not a root of unity. Setting $q = 0$ recovers the Loday–Ronco theorem. As a part of this story, we introduce the bilune-incidence algebra, define the two-sided q-zeta and q-Möbius functions therein, and use them to develop the q-exp-log correspondence. The two-sided q-zeta function is related to the inverse of the Varchenko matrices associated to the q-distance function on faces.
- We establish the Hoffman–Newman–Radford rigidity theorems. They come in different flavors; each flavor corresponds to a specific type of zeta and Möbius function. We use conjugation by the Hoffman–Newman–Radford isomorphisms to study the nondegeneracy of the mixed distributive law for bicommutative bimonoids and also for q-bimonoids for q not a root of unity.
- We study in depth the Hadamard product on species. This includes the construction of a variety of internal homs. We introduce the bimonoid of star families built from the internal hom for comonoids, and a similar bimonoid built from the universal measuring comonoid, and explain

their connection to exp-log correspondences. We also study freeness properties of bimonoids arising as Hadamard products of bimonoids.

- We introduce the Solomon operator on the free bimonoid on a species to give a constructive proof of the Poincaré–Birkhoff–Witt theorem. We also give a novel proof of the Cartier–Milnor–Moore theorem by linking the Tits algebra to the Lie-incidence algebra. The latter is an algebra associated to the Lie operad. There is a family of isomorphisms between the two algebras indexed by noncommutative zeta and Möbius functions.

In particular, this includes progress on some questions raised in our monograph on species and Hopf algebras [**18**, Questions 12.27, 12.39, 12.67].

Future directions

Coxeter species and Coxeter spaces. This monograph along with our previous work [**21**] gives a glimpse into how ideas from hyperplane arrangements and ideas from Hopf theory and algebraic Lie theory can interact and enrich each other. In a follow-up work, as mentioned in the paragraphs following Table I, we plan to:

- develop the notions of Coxeter bimonoids and Coxeter bialgebras, and the theory of Fock functors which relates the two notions,
- compare this picture with the classical picture of Joyal bimonoids, graded bialgebras and Fock functors, and
- in particular, explain how results about Coxeter bimonoids and Coxeter bialgebras can be used to deduce the corresponding results about Joyal bimonoids and graded bialgebras.

To deal with Coxeter species and Coxeter spaces, one needs to work with an invariant noncommutative zeta or Möbius function. This exists iff the field characteristic does not divide the order of the Coxeter group [**21**, Lemma 16.42]. This is how field characteristic issues eventually enter the picture.

The connection between Joyal species and species for arrangements is briefly indicated in Section 2.16 and Section 17.7. Similarly, the connection between formal power series and lune-incidence algebras is indicated in Section 9.8, see in particular Table 9.2.

Coxeter operads. In a similar vein, we plan to develop the notion of Coxeter operad mentioned in Table II. This will include aspects of homological algebra such as the Koszul theory of Coxeter operads, with the basic object being a differential graded Coxeter species. Given the wide applicability of May operads, this line of research looks very promising.

The connection between May operads and operads for arrangements is briefly indicated in Section 4.6.

Semigroup theory. Our notions of species and monoids, comonoids, bimonoids therein are defined for a fixed real hyperplane arrangement, with a central role played by the Tits monoid. These notions continue to make sense when the Tits monoid is replaced by an arbitrary left regular band (LRB) (Section 3.9) and large parts of the theory extend to this setting. For instance, see Section 4.14 for operads, Section 9.2.8 for the exp-log correspondence, Section 13.5.2 for the Leray–Samelson and Borel–Hopf theorems, Section 17.8 for the Poincaré–Birkhoff–Witt and Cartier–Milnor–Moore theorems. This line of study can lead to a better understanding of the interactions between Hopf theory and semigroup theory.

We point out a few situations where the generalization from arrangements to LRBs is not clear. There is no notion of opposite for an arbitrary LRB, so most of the descent, lune, Witt identities (Section 1.7) do not work as stated. This issue carries forward to the noncommutative Zaslavsky formula (Section 1.8) and to the antipode (Chapter 12). The orientation space and signature space do not work as stated. In particular, this affects all signed aspects of the theory such as signed commutative monoids (Section 2.5), the monad $\mathcal{E}$ (Section 3.2.6), signed Lie monoids (Section 16.7). In a similar vein, results such as rigidity of q-bimonoids for q not a root of unity (Theorem 13.77) rely on Theorem 1.10 on Varchenko matrices which is specific to arrangements. The presentation of the Lie operad given in Example 4.12 does not hold for an arbitrary LRB. So to define LRB Lie monoids (Section 17.8), one cannot directly use the Lie bracket (generator-relation) approach (Section 16.1.2); however, the operadic approach does work (Section 16.1.1). There are similar issues with the results in Section 2.12 which are linked to the presentations of the commutative and associative operads (Examples 4.9 and 4.10).

We mention that abstract distance functions on LRBs were introduced in [**21**, Appendix E] and deserve to be studied further.

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Introduction

We now describe the contents of the monograph in more detail along with pointers to important results. For organizational purposes, the text has been divided into three parts.

Part I

We begin Part I with a brief review of hyperplane arrangements. We then introduce the central objects, namely, species and monoids, comonoids and bimonoids for hyperplane arrangements, and initiate their basic study. We also briefly consider operads in the setting of hyperplane arrangements, and explain their connection to bimonoids. Monads and monoidal categories provide the categorical spine for these considerations.

Hyperplane arrangements. (Chapter 1.) A hyperplane arrangement $\mathcal{A}$ is a finite collection of hyperplanes in a real vector space. We assume that all hyperplanes pass through the origin. These hyperplanes break the space into subsets called faces. Let $\Sigma[\mathcal{A}]$ denote the set of faces. It is a graded poset under inclusion. We usually denote faces by the letters A, B, F, G, H, K. There is a unique minimum face called the central face. We denote it by O. Maximal faces are called chambers, and we denote them by the letters C, D, E. The set of faces $\Sigma[\mathcal{A}]$ is also a monoid. We call it the *Tits monoid*. The product of F and G is denoted FG and called the Tits product. Further, the product of a face and a chamber is a chamber, so the set of chambers $\Gamma[\mathcal{A}]$ is a left $\Sigma[\mathcal{A}]$-set.

Subspaces obtained by intersecting hyperplanes are called flats. Let $\Pi[\mathcal{A}]$ denote the set of flats. It is a graded lattice under inclusion. We usually denote flats by the letters X, Y, Z, W. The minimum and maximum flats are denoted $\perp$ and $\top$. The set of flats $\Pi[\mathcal{A}]$ is a commutative monoid under the join operation, that is, the product of X and Y is $\mathrm{X} \vee \mathrm{Y}$. We call this the *Birkhoff monoid*.

Every face has a support given by its linear span. It is a flat. We write $\mathrm{s}(F)$ for the support of F. The map $\mathrm{s} : \Sigma[\mathcal{A}] \to \Pi[\mathcal{A}]$ which sends F to $\mathrm{s}(F)$ is a morphism of monoids.

A biface is a pair (F, F') of faces such that F and F' have the same support. The *Janus monoid* $\mathrm{J}[\mathcal{A}]$ consists of bifaces (F, F') under the product $(F, F')(G, G') := (FG, G'F')$. It is the fiber product of the Tits monoid and its opposite monoid over the Birkhoff monoid.

The Tits algebra, Birkhoff algebra, Janus algebra are obtained from the corresponding monoids by linearization over a field $\Bbbk$. We denote them by $\Sigma[\mathcal{A}]$, $\Pi[\mathcal{A}]$, $\mathsf{J}[\mathcal{A}]$, respectively. Similarly, linearizing the set of chambers yields a left module $\Gamma[\mathcal{A}]$ over the Tits algebra. The space of chambers $\Gamma[\mathcal{A}]$ contains an important subspace, namely, the space of Lie elements which we denote by $\mathsf{Lie}[\mathcal{A}]$. Similarly, $\Sigma[\mathcal{A}]$ contains the space of Zie elements which we denote by $\mathsf{Zie}[\mathcal{A}]$.

For a flat X of $\mathcal{A}$, one can define the arrangement under X denoted $\mathcal{A}^{\mathrm{X}}$ and the arrangement over X denoted $\mathcal{A}_{\mathrm{X}}$. Further, for $\mathrm{X} \leq \mathrm{Y}$, we have the arrangement $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$ obtained by first going under Y and then over X, or equivalently, by first going over X and then under Y.

We make a note of some other important algebraic objects. The *flat-incidence algebra* is the incidence algebra of the lattice of flats. It contains the zeta function ζ and Möbius function μ. The *lune-incidence algebra* is a certain reduced incidence algebra of the poset of faces. It contains noncommutative zeta functions $\boldsymbol{\zeta}$ defined by the lune-additivity formula (1.42) and noncommutative Möbius functions $\boldsymbol{\mu}$ defined by the noncommutative Weisner formula (1.44). A related object that we introduce is the *bilune-incidence algebra*. For q not a root of unity, it contains the two-sided q-zeta function $\boldsymbol{\zeta}_q$ defined by the two-sided q-lune-additivity formula (1.66) and the two-sided q-Möbius function $\boldsymbol{\mu}_q$ defined by the two-sided q-Weisner formula (1.67).

The Zaslavsky formula for chamber enumeration is recalled in (1.84). We then establish noncommutative analogues of this formula, see (1.88) and (1.89). They involve noncommutative zeta and Möbius functions.

Species. (Chapter 2.) Fix a real hyperplane arrangement $\mathcal{A}$, and a field $\Bbbk$. A *species* p is a family of $\Bbbk$-vector spaces $\mathsf{p}[F]$, one for each face F of $\mathcal{A}$, together with linear maps

$$\beta_{G,F} : \mathsf{p}[F] \to \mathsf{p}[G],$$

whenever F and G have the same support, such that

$$\beta_{H,F} = \beta_{H,G}\beta_{G,F} \quad \text{and} \quad \beta_{F,F} = \mathrm{id},$$

the former whenever F, G, H have the same support, and the latter for every F. (The letter β suggests a connection to braiding in monoidal categories.) A map of species $f : \mathsf{p} \to \mathsf{q}$ is a family of linear maps

$$f_F : \mathsf{p}[F] \to \mathsf{q}[F],$$

one for each face F, such that $f_G \beta_{G,F} = \beta_{G,F} f_F$ whenever F and G have the same support. This defines the category of species.

Species can also be formulated using flats instead of faces as follows (Proposition 2.5). A *species* p is a family $\mathsf{p}[\mathrm{X}]$ of $\Bbbk$-vector spaces, one for each flat X of $\mathcal{A}$. A map of species $f : \mathsf{p} \to \mathsf{q}$ is a family of linear maps $f_{\mathrm{X}} : \mathsf{p}[\mathrm{X}] \to \mathsf{q}[\mathrm{X}]$, one for each flat X.

Either formulation can be used depending on convenience of the context.

Examples of species. (Chapter 7.) The *exponential species* E is one of the most basic and important examples of a species. It is defined by setting $\mathsf{E}[\mathrm{X}] := \Bbbk$ for all flats X. Alternatively, put $\mathsf{E}[A] := \Bbbk$ for all faces A and $\beta_{B,A} = \mathrm{id}$ for all faces A and B of the same support. We mention that E has a signed analogue E^- which we call the *signed exponential species*.

The *species of flats* Π is defined by setting the component $\Pi[\mathrm{X}]$ to be the linear span of flats greater than X. For clarity, the basis element of $\Pi[\mathrm{X}]$ indexed by the flat Y is denoted $\mathtt{H}_{\mathrm{Y}/\mathrm{X}}$.

The *species of chambers* Γ is defined by setting the component $\Gamma[A]$ to be the linear span of chambers greater than A. For clarity, the basis element of $\Gamma[A]$ indexed by the chamber C is denoted $\mathtt{H}_{C/A}$. For faces A and B of the same support,

$$\beta_{B,A} : \Gamma[A] \to \Gamma[B], \qquad \mathtt{H}_{C/A} \mapsto \mathtt{H}_{BC/B},$$

where BC denotes the Tits product of B and C.

The *species of faces* Σ is defined in a similar manner by replacing chambers by arbitrary faces. The inclusion map $\Gamma \hookrightarrow \Sigma$ is a map of species.

Many more examples are discussed in the text such as the species of charts, top-nested faces, top-lunes, bifaces, and so on.

Bimonoids. (Chapters 2 and 7.) A *monoid*, denoted (a, μ), is a species a equipped with linear maps

$$\mu_A^F : \mathsf{a}[F] \to \mathsf{a}[A],$$

one for each pair of faces $A \leq F$, such that

$$\mu_B^{BF} \beta_{BF,F} = \beta_{B,A} \mu_A^F, \quad \mu_A^G = \mu_A^F \mu_F^G, \quad \mu_A^A = \mathrm{id}\,.$$

These are the naturality, associativity, unitality axioms, respectively. In the naturality axiom, A and B have the same support and $A \leq F$, which implies B and BF have the same support and $B \leq BF$. In the associativity axiom, $A \leq F \leq G$. In the unitality axiom, A is an arbitrary face. We refer to μ as the *product* of a.

A *comonoid*, denoted (c, Δ), is defined dually using linear maps

$$\Delta_A^F : \mathsf{c}[A] \to \mathsf{a}[F]$$

for $A \leq F$. We refer to Δ as the *coproduct* of c.

A *bimonoid* is a triple $(\mathsf{h}, \mu, \Delta)$, where h is a species, (h, μ) is a monoid, (h, Δ) is a comonoid, and

$$\Delta_A^G \mu_A^F = \mu_G^{GF} \beta_{GF,FG} \Delta_F^{FG}$$

for any faces $A \leq F$ and $A \leq G$. This is the *bimonoid axiom.* Note very carefully how the product of the Tits monoid enters into the axiom. The idea is to change a product followed by a coproduct to a coproduct followed by a product. However, since the Tits monoid is not commutative, $FG \neq GF$ in general. Nonetheless, FG and GF have the same support, so they can be related by β, and this intervenes in the axiom. The axiom is shown in diagrammatic form below.

$$\begin{tikzcd}
\mathsf{h}[FG] \arrow[rr, "\beta_{GF,FG}"] & & \mathsf{h}[GF] \arrow[d, "\mu_G^{GF}"] \\
\mathsf{h}[F] \arrow[u, "\Delta_F^{FG}"] \arrow[dr, "\mu_A^F"'] & & \mathsf{h}[G] \\
& \mathsf{h}[A] \arrow[ur, "\Delta_A^G"'] &
\end{tikzcd}$$

More generally, for any scalar q, we define a q-bimonoid proceeding as above, but with the bimonoid axiom deformed to

$$\Delta_A^G \mu_A^F = q^{\operatorname{dist}(FG,GF)} \mu_G^{GF} \beta_{GF,FG} \Delta_F^{FG},$$

where $\operatorname{dist}(FG, GF)$ is the number of hyperplanes which separate the faces FG and GF. Setting $q = 1$ recovers the bimonoid axiom. Other parameter values of immediate interest are $q = -1$ and $q = 0$. We use the term *signed bimonoid* to refer to a (-1)-bimonoid.

The species of chambers Γ carries the structure of a bimonoid, with product and coproduct defined by

$$\begin{aligned} \mu_A^F : \Gamma[F] &\to \Gamma[A] \qquad\qquad & \Delta_A^F : \Gamma[A] &\to \Gamma[F] \\ \mathtt{H}_{C/F} &\mapsto \mathtt{H}_{C/A} & \mathtt{H}_{C/A} &\mapsto \mathtt{H}_{FC/F}. \end{aligned}$$

More generally, Γ becomes a q-bimonoid if the coproduct is deformed to

$$\mathtt{H}_{C/A} \mapsto q^{\operatorname{dist}(C,FC)}\, \mathtt{H}_{FC/F}.$$

To show dependence on q, we denote it by Γ_q. We call it the q-bimonoid of chambers.

The bimonoid of faces Σ and its q-analogue can be defined in a similar manner by replacing chambers by faces. The inclusion map $\Gamma \hookrightarrow \Sigma$ is a morphism of bimonoids.

We mention in passing that one can also define the q-bimonoid of top-nested faces and the q-bimonoid of bifaces.

(Co, bi)commutative bimonoids. (Chapters 2 and 7.) A monoid (a, μ) is *commutative* if

$$\mu_A^F = \mu_A^G \beta_{G,F}$$

whenever $A \leq F$ and $A \leq G$, and F and G have the same support. This is the *commutativity axiom.* Dually, a comonoid (c, Δ) is *cocommutative* if

$$\Delta_A^G = \beta_{G,F} \Delta_A^F$$

whenever $A \leq F$ and $A \leq G$, and F and G have the same support. This is the *cocommutativity axiom.* A bimonoid can be commutative, cocommutative, both or neither. If it is both, then we use the term *bicommutative.*

There is also a notion of a *signed commutative monoid*, and dually that of a *signed cocommutative comonoid.* Moreover, the two can be combined to yield the notion of a *signed bicommutative signed bimonoid.*

The bimonoid of chambers $\mathsf{\Gamma}$ is cocommutative but not commutative. Similarly, $\mathsf{\Gamma}_{-1}$ is signed cocommutative but not signed commutative. Similar remarks apply to the bimonoid of faces $\mathsf{\Sigma}$.

Commutativity can also be formulated directly in terms of flats as follows (Propositions 2.20, 2.21, 2.22).

A *commutative monoid*, denoted (a, μ), is a species a equipped with linear maps

$$\mu_{\mathrm{Z}}^{\mathrm{X}} : \mathsf{a}[\mathrm{X}] \to \mathsf{a}[\mathrm{Z}],$$

one for each pair of flats $\mathrm{Z} \leq \mathrm{X}$, such that

$$\mu_{\mathrm{Z}}^{\mathrm{X}} = \mu_{\mathrm{Z}}^{\mathrm{Y}} \mu_{\mathrm{Y}}^{\mathrm{X}} \quad \text{and} \quad \mu_{\mathrm{Z}}^{\mathrm{Z}} = \mathrm{id},$$

the former for every $\mathrm{Z} \leq \mathrm{Y} \leq \mathrm{X}$, and the latter for every Z. These are the associativity and unitality axioms, respectively. A morphism of commutative monoids $f : \mathsf{a} \to \mathsf{b}$ is a map of species f such that $f_{\mathrm{Z}} \mu_{\mathrm{Z}}^{\mathrm{X}} = \mu_{\mathrm{Z}}^{\mathrm{X}} f_{\mathrm{X}}$ for every $\mathrm{Z} \leq \mathrm{X}$. This defines the category of commutative monoids.

A *cocommutative comonoid*, denoted (c, Δ), is defined dually using linear maps

$$\Delta_{\mathrm{Z}}^{\mathrm{X}} : \mathsf{c}[\mathrm{Z}] \to \mathsf{c}[\mathrm{X}]$$

for $\mathrm{Z} \leq \mathrm{X}$.

A *bicommutative bimonoid* is a triple $(\mathsf{h}, \mu, \Delta)$, where h is a species, (h, μ) is a commutative monoid, (h, Δ) is a cocommutative comonoid, and

$$\Delta_{\mathrm{Z}}^{\mathrm{Y}} \mu_{\mathrm{Z}}^{\mathrm{X}} = \mu_{\mathrm{Y}}^{\mathrm{X} \vee \mathrm{Y}} \Delta_{\mathrm{X}}^{\mathrm{X} \vee \mathrm{Y}}$$

for any flats $\mathrm{Z} \leq \mathrm{X}$ and $\mathrm{Z} \leq \mathrm{Y}$. This is the *bicommutative bimonoid axiom*. It allows us to change a product followed by a coproduct to a coproduct followed by a product. Note very carefully how the product of the Birkhoff monoid (join operation on flats) has entered into the axiom. The axiom is shown in diagrammatic form below.

$$\begin{tikzcd}
& \mathsf{h}[\mathrm{X} \vee \mathrm{Y}] \arrow[dr, "\mu_{\mathrm{Y}}^{\mathrm{Y} \vee \mathrm{X}}"] & \\
\mathsf{h}[\mathrm{X}] \arrow[ur, "\Delta_{\mathrm{X}}^{\mathrm{X} \vee \mathrm{Y}}"] \arrow[dr, "\mu_{\mathrm{Z}}^{\mathrm{X}}"'] & & \mathsf{h}[\mathrm{Y}] \\
& \mathsf{h}[\mathrm{Z}] \arrow[ur, "\Delta_{\mathrm{Z}}^{\mathrm{Y}}"'] &
\end{tikzcd}$$

A morphism of bicommutative bimonoids is a map of the underlying species such that $f_{\mathrm{Z}} \mu_{\mathrm{Z}}^{\mathrm{X}} = \mu_{\mathrm{Z}}^{\mathrm{X}} f_{\mathrm{X}}$ and $f_{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}} = \Delta_{\mathrm{Z}}^{\mathrm{X}} f_{\mathrm{Z}}$ for $\mathrm{Z} \leq \mathrm{X}$.

The exponential species E is a bicommutative bimonoid with $\mu_{\mathrm{Z}}^{\mathrm{X}} = \mathrm{id}$ and $\Delta_{\mathrm{Z}}^{\mathrm{X}} = \mathrm{id}$. We mention that the signed exponential species E^{-} carries the structure of a signed bicommutative signed bimonoid.

The species of flats $\mathsf{\Pi}$ is a bicommutative bimonoid, with product and coproduct defined by

$$\begin{aligned} \mu_{\mathrm{Z}}^{\mathrm{Y}} : \mathsf{\Pi}[\mathrm{Y}] &\to \mathsf{\Pi}[\mathrm{Z}] \\ \mathtt{H}_{\mathrm{X}/\mathrm{Y}} &\mapsto \mathtt{H}_{\mathrm{X}/\mathrm{Z}} \end{aligned} \qquad\qquad \begin{aligned} \Delta_{\mathrm{Z}}^{\mathrm{Y}} : \mathsf{\Pi}[\mathrm{Z}] &\to \mathsf{\Pi}[\mathrm{Y}] \\ \mathtt{H}_{\mathrm{X}/\mathrm{Z}} &\mapsto \mathtt{H}_{\mathrm{X} \vee \mathrm{Y}/\mathrm{Y}}. \end{aligned}$$

Duality. (Chapter 2.) The duality operation on vector spaces extends to species. The dual p^* of a species p is defined by $\mathsf{p}^*[\mathrm{X}] := \mathsf{p}[\mathrm{X}]^*$. Duality interchanges (commutative) monoids and (cocommutative) comonoids, and preserves bimonoids. Thus, if h is a bimonoid, then so is h^*.

A bimonoid is *self-dual* if it is isomorphic to its own dual. For instance, E and $\mathsf{\Pi}$ are self-dual, but $\mathsf{\Gamma}$ and $\mathsf{\Sigma}$ are not (since $\mathsf{\Gamma}$ is cocommutative but not commutative, while $\mathsf{\Gamma}^*$ shows the opposite behavior).

Monads. (Chapter 3.) (Co, bi)monads on a category and (co, bi)lax functors linking them are reviewed in Appendix C. We recall from Definition C.4 that a bimonad is a triple $(\mathcal{V}, \mathcal{U}, \lambda)$ consisting of a monad $\mathcal{V}$, a comonad $\mathcal{U}$, and a mixed distributive law $\lambda : \mathcal{V}\mathcal{U} \to \mathcal{U}\mathcal{V}$ linking them. There are also notions of monad algebra, comonad coalgebra and bimonad bialgebra.

We construct a bimonad on species which we denote by $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$ (Theorem 3.4). A $\mathcal{T}$-algebra is the same as a monoid, a $\mathcal{T}^\vee$-coalgebra is the same as a comonoid, and a $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$-bialgebra is the same as a bimonoid (Proposition 3.5). Moreover, for any scalar q, one can deform the mixed distributive law λ to λ_q such that the resulting bialgebras are q-bimonoids (Theorem 3.6).

Similarly, we construct another bimonad on species denoted $(\mathcal{S}, \mathcal{S}^\vee, \lambda)$. A $\mathcal{S}$-algebra is the same as a commutative monoid, a $\mathcal{S}^\vee$-coalgebra is the same as a cocommutative comonoid, and a $(\mathcal{S}, \mathcal{S}^\vee, \lambda)$-bialgebra is the same as a bicommutative bimonoid. The precise connection between this bimonad and the previous one is summarized in Proposition 3.13.

Operads. (Chapter 4.) A *dispecies* $\mathbf{p}$ is a family $\mathbf{p}[\mathrm{X}, \mathrm{Y}]$ of $\Bbbk$-vector spaces, one for each pair (X, Y) of flats with $\mathrm{X} \leq \mathrm{Y}$. The category of dispecies carries a monoidal structure. For dispecies $\mathbf{p}$ and $\mathbf{q}$, define the dispecies $\mathbf{p} \circ \mathbf{q}$ by

$$(\mathbf{p} \circ \mathbf{q})[\mathrm{X}, \mathrm{Z}] := \bigoplus_{\mathrm{Y} :\, \mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}} \mathbf{p}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{q}[\mathrm{Y}, \mathrm{Z}].$$

We refer to this operation as the *substitution product* of $\mathbf{p}$ and $\mathbf{q}$. The unit object is the dispecies $\mathbf{x}$ defined by $\mathbf{x}[\mathrm{X}, \mathrm{Y}] = \Bbbk$ when $\mathrm{X} = \mathrm{Y}$, and 0 otherwise.

A monoid in this monoidal category is an *operad*. Explicitly, an operad is a dispecies $\mathbf{a}$ equipped with linear maps

$$\gamma : \mathbf{a}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y}, \mathrm{Z}] \to \mathbf{a}[\mathrm{X}, \mathrm{Z}] \quad \text{and} \quad \eta : \Bbbk \to \mathbf{a}[\mathrm{X}, \mathrm{X}],$$

the former for each $\mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}$ and the latter for each X, subject to associativity and unitality axioms.

The category of species is a left module category over the category of dispecies as follows. For a dispecies $\mathbf{p}$ and a species m, define the species $\mathbf{p} \circ \mathsf{m}$ by

$$(\mathbf{p} \circ \mathsf{m})[\mathrm{X}] := \bigoplus_{\mathrm{Y} :\, \mathrm{Y} \geq \mathrm{X}} \mathbf{p}[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{m}[\mathrm{Y}].$$

This yields the notion of a left $\mathbf{a}$-module for any operad $\mathbf{a}$. Explicitly, a left $\mathbf{a}$-module is a species m equipped with linear maps

$$\mathbf{a}[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{m}[\mathrm{Y}] \to \mathsf{m}[\mathrm{X}],$$

one for each $\mathrm{X} \leq \mathrm{Y}$, subject to associativity and unitality axioms. The free left $\mathbf{a}$-module over a species m is given by $\mathbf{a} \circ \mathsf{m}$ (Proposition 4.23).

A comonoid in the monoidal category of dispecies is a *cooperad*. It makes sense to talk of left comodules over a cooperad. Further, there is a duality functor on dispecies which interchanges operads and cooperads (and modules and comodules). A *bioperad* is a self-dual notion. It is a triple $(\mathbf{a}, \mathbf{c}, \lambda)$ consisting of an operad $\mathbf{a}$, a cooperad $\mathbf{c}$, and a mixed distributive law $\lambda : \mathbf{a} \circ \mathbf{c} \to \mathbf{c} \circ \mathbf{a}$ linking them. By combining operad-modules and cooperad-comodules, we obtain the notion of a bioperad-bimodule.

Commutative operad and associative operad. (Chapter 4.) The *commutative operad* $\mathbf{Com}$ is defined by $\mathbf{Com}[\mathrm{X}, \mathrm{Y}] := \Bbbk$ for all $\mathrm{X} \leq \mathrm{Y}$, with structure maps γ and η being identities. A left module over $\mathbf{Com}$ is precisely a commutative monoid. The commutative operad has a signed analogue denoted $\mathbf{Com}^{-}$. Left modules over it are signed commutative monoids.

The *associative operad* $\mathbf{As}$ is defined as follows. For any $\mathrm{X} \leq \mathrm{Y}$, set $\mathbf{As}[\mathrm{X}, \mathrm{Y}] := \Gamma[\mathcal{A}^{\mathrm{Y}}_{\mathrm{X}}]$, the space of chambers of the arrangement $\mathcal{A}^{\mathrm{Y}}_{\mathrm{X}}$. Equivalently, it is the linear span of symbols $\mathtt{H}_{F/A}$ with $A \leq F$, $\mathrm{s}(A) = \mathrm{X}$ and $\mathrm{s}(F) = \mathrm{Y}$, subject to the relation $\mathtt{H}_{F/A} = \mathtt{H}_{BF/B}$, whenever A and B have the same support. The structure map is given by

$$\gamma : \mathbf{As}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{As}[\mathrm{Y}, \mathrm{Z}] \to \mathbf{As}[\mathrm{X}, \mathrm{Z}], \qquad \mathtt{H}_{F/A} \otimes \mathtt{H}_{G/F} \mapsto \mathtt{H}_{G/A},$$

where A, F, G are faces with support X, Y, Z, respectively, and $A \leq F \leq G$. A left module over $\mathbf{As}$ is precisely a monoid, with $\mathtt{H}_{F/A}$ corresponding to the product component μ^F_A of the monoid.

Dualizing the commutative and associative operads yields the cooperads $\mathbf{Com}^*$ and $\mathbf{As}^*$. Left comodules over them are cocommutative comonoids and comonoids, respectively. Further, there is a mixed distributive law λ linking $\mathbf{As}$ and $\mathbf{As}^*$ such that left bimodules over the bioperad $(\mathbf{As}, \mathbf{As}^*, \lambda)$ are precisely bimonoids. Moreover, this law can be deformed by a scalar q such that the resulting left bimodules are q-bimonoids. Similarly, there is a bioperad $(\mathbf{Com}, \mathbf{Com}^*, \lambda)$ whose left bimodules are bicommutative bimonoids.

We now tie this with the earlier discussion on bimonads. A (co, bi)operad gives rise to a (co, bi)monad on species, and moreover, left (co, bi)modules over the (co, bi)operad are the same as (co, bi)algebras over the corresponding (co, bi)monad. The point is that the bioperad $(\mathbf{As}, \mathbf{As}^*, \lambda)$ yields the bimonad $(\mathcal{T}, \mathcal{T}^{\vee}, \lambda)$ (Theorem 4.33). Similarly, $(\mathbf{Com}, \mathbf{Com}^*, \lambda)$ yields the bimonad $(\mathcal{S}, \mathcal{S}^{\vee}, \lambda)$.

Lie operad. (Chapter 4.) The *Lie operad* $\mathbf{Lie}$ is defined as a suboperad of the associative operad $\mathbf{As}$ as follows. For any $\mathrm{X} \leq \mathrm{Y}$, set $\mathbf{Lie}[\mathrm{X}, \mathrm{Y}] := \mathsf{Lie}[\mathcal{A}^{\mathrm{Y}}_{\mathrm{X}}]$, the space of Lie elements of the arrangement $\mathcal{A}^{\mathrm{Y}}_{\mathrm{X}}$. Recall that this is a subspace of $\Gamma[\mathcal{A}^{\mathrm{Y}}_{\mathrm{X}}]$. The point is that the operad structure of $\mathbf{As}$ restricts to these subspaces and yields a suboperad.

Quadratic operads. (Chapter 4.) The *free operad* on a dispecies $\mathbf{e}$ is given by

$$\mathcal{F}_\circ(\mathbf{e}) := \underset{n\geq 0}{+} \mathbf{e}^{\circ n},$$

where $\mathbf{e}^{\circ n}$ is the n-fold substitution product of $\mathbf{e}$ with itself, and $+$ denotes the coproduct in the category of dispecies. Explicitly,

$$\mathcal{F}_\circ(\mathbf{e})[\mathrm{X},\mathrm{Z}] = \bigoplus_{\mathrm{X}\leq \mathrm{Y}_1\leq\cdots\leq \mathrm{Y}_k\leq \mathrm{Z}} \mathbf{e}[\mathrm{X},\mathrm{Y}_1]\otimes \mathbf{e}[\mathrm{Y}_1,\mathrm{Y}_2]\otimes\cdots\otimes \mathbf{e}[\mathrm{Y}_k,\mathrm{Z}],$$

where the sum is over all multichains in the interval $[\mathrm{X},\mathrm{Z}]$.

An operad $\mathbf{a}$ is *quadratic* if it can be written as a quotient of a free operad $\mathcal{F}_\circ(\mathbf{e})$ by an ideal generated by a subdispecies $\mathbf{r}$ of $\mathbf{e}\circ\mathbf{e}$. We denote this by $\mathbf{a} = \langle \mathbf{e} \mid \mathbf{r}\rangle$. We use the term *binary quadratic* if further $\mathbf{e}[\mathrm{X},\mathrm{Y}] = 0$ unless Y covers X in the lattice of flats. For any quadratic operad, one can talk about its oriented dual which is again a quadratic operad. The commutative, associative, Lie operads are binary quadratic. For the Lie operad, antisymmetry is incorporated in $\mathbf{e}$, while the Jacobi identities are in $\mathbf{r}$. The oriented quadratic dual of the associative operad is itself, while the commutative and Lie operads are oriented quadratic duals of each other (Proposition 4.14).

Part II

In Part II, we continue the development of the basic theory of bimonoids. We discuss the primitive filtration of a comonoid and dually the decomposable filtration of a monoid. We then discuss in detail various universal constructions starting with the free monoid and cofree comonoid on a species. We study the Hadamard functor and its specialization to the signature functor. The latter sets up an equivalence between the categories of bimonoids and signed bimonoids. We develop exp-log correspondences of a bimonoid by employing noncommutative zeta and Möbius functions. We forge a precise connection of bimonoids with modules over the Birkhoff algebra, Tits algebra, Janus algebra by considering characteristic operations on bimonoids. In our setting, the antipode of a bimonoid always exists and we study it using the Takeuchi formula.

Primitive filtrations and decomposable filtrations. (Chapters 5 and 7.) Every comonoid c has a *primitive part* $\mathcal{P}(\mathsf{c})$. It is a species whose A-component consists of those elements $x \in \mathsf{c}[A]$ such that $\Delta_A^F(x) = 0$ for all $F > A$. More generally, one can define a filtration of c whose first term is $\mathcal{P}(\mathsf{c})$. This is the *primitive filtration* of c. It turns c into a filtered comonoid. Dually, every monoid a has a *decomposable part* $\mathcal{D}(\mathsf{a})$, and more generally, a decomposable filtration which turns it into a filtered monoid. We refer to $\mathcal{Q}(\mathsf{a}) := \mathsf{a}/\mathcal{D}(\mathsf{a})$ as the *indecomposable part* of a.

Just like faces and chambers, Lie and Zie elements of an arrangement give rise to the Lie species and Zie species. The primitive part of the bimonoid of chambers Γ is the Lie species (Lemma 7.64), while that of the bimonoid of faces Σ is the Zie species (Lemma 7.69). We refer to these results as the *Friedrichs criteria* for Lie and Zie elements.

A map from a species to a comonoid is a *coderivation* if it maps into the primitive part of that comonoid. Dually, a map from a monoid to a species is a *derivation* if it factors through the indecomposable part of that monoid. A (co)derivation is the same as a (co)monoid morphism with the species viewed as a (co)monoid with all its nontrivial (co)product components being 0.

For a q-bimonoid, one can consider the primitive as well as the decomposable filtrations. Both of them turn it into a filtered q-bimonoid. Thus, for either filtration, one can consider the corresponding associated graded q-bimonoid. When $q = 1$, the associated graded bimonoid wrt the primitive filtration is commutative, and wrt the decomposable filtration is cocommutative (Propositions 5.62 and 5.65). These results have a signed analogue when $q = -1$. We call these the *Browder–Sweedler commutativity result* and the *Milnor–Moore cocommutativity result*, respectively.

For a bimonoid, there is a canonical map from its primitive part to its indecomposable part, see (5.50). This map is surjective iff the bimonoid is cocommutative, injective iff the bimonoid is commutative, and bijective iff the bimonoid is bicommutative (Proposition 5.56). In particular, for the bimonoid of faces Σ, this map is surjective. As an application, we deduce the existence of special Zie elements (Exercise 7.71).

Free monoid and free commutative monoid. (Chapters 6 and 7.) For a species p, define the species $\mathcal{S}(\mathsf{p})$ by

$$\mathcal{S}(\mathsf{p})[\mathrm{Z}] := \bigoplus_{\mathrm{X}\colon \mathrm{Z} \leq \mathrm{X}} \mathsf{p}[\mathrm{X}].$$

It carries the structure of a commutative monoid: For $\mathrm{Z} \leq \mathrm{Y}$, note that the summands in $\mathcal{S}(\mathsf{p})[\mathrm{Y}]$ are all contained in $\mathcal{S}(\mathsf{p})[\mathrm{Z}]$, and we define $\mu^{\mathrm{Y}}_{\mathrm{Z}}$ to be the canonical inclusion. In fact, $\mathcal{S}(\mathsf{p})$ is the *free commutative monoid* on p. In other words, $\mathcal{S}(\mathsf{p}) = \mathbf{Com} \circ \mathsf{p}$, where $\mathbf{Com}$ is the commutative operad.

Similarly, for a species p, define the species $\mathcal{T}(\mathsf{p})$ by

$$\mathcal{T}(\mathsf{p})[A] := \bigoplus_{F\colon A \leq F} \mathsf{p}[F].$$

The map $\beta_{B,A}$ is defined by summing the maps $\beta_{BF,F}$ of the species p over all $F \geq A$. Further, $\mathcal{T}(\mathsf{p})$ is a monoid with μ^F_A defined to be the canonical inclusion. This is the *free monoid* on p. In other words, $\mathcal{T}(\mathsf{p}) = \mathbf{As} \circ \mathsf{p}$, where $\mathbf{As}$ is the associative operad.

These constructions can be extended further. Let c be a comonoid. Then the coproduct of c induces coproducts on $\mathcal{S}(\mathsf{c})$ and $\mathcal{T}(\mathsf{c})$ turning them into bimonoids. Examples include the bimonoids that we have discussed earlier, namely,

$$\mathcal{S}(\mathsf{x}) = \mathsf{E}, \quad \mathcal{T}(\mathsf{x}) = \mathsf{\Gamma}, \quad \mathcal{S}(\mathsf{E}) = \mathsf{\Pi}, \quad \mathcal{T}(\mathsf{E}) = \mathsf{\Sigma}.$$

Here x denotes the species whose component $\mathsf{x}[\mathrm{Y}]$ is $\Bbbk$ if $\mathrm{Y} = \top$, and 0 otherwise. We view it as a comonoid in the only way possible with $\Delta^{\top}_{\top} = \mathrm{id}$ and the remaining coproduct components being zero.

In the case of $\mathcal{T}(\mathsf{c})$, one can do more. Its coproduct can be deformed using a scalar q such that it becomes a q-bimonoid. To show dependence on q, we write $\mathcal{T}_q(\mathsf{c})$. For instance, $\mathcal{T}_q(\mathsf{x}) = \Gamma_q$, the q-bimonoid of chambers. The universal property of $\mathcal{T}_q(\mathsf{c})$ is given in Theorem 6.6. The dual version is given in Theorem 6.13. These results can be seen as formal consequences of the existence of the bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda_q)$. Related universal properties involving the primitive part functor and indecomposable part functor are given in Theorems 6.31 and 6.34, respectively.

We mention that for a species p, one can also construct the free signed commutative monoid. We denote it by $\mathcal{E}(\mathsf{p})$. When $\mathsf{p} = \mathsf{x}$, we obtain the signed exponential species E^-.

Hadamard product. (Chapter 8.) For species p and q, their *Hadamard product* $\mathsf{p} \times \mathsf{q}$ is given by

$$(\mathsf{p} \times \mathsf{q})[F] := \mathsf{p}[F] \otimes \mathsf{q}[F].$$

This defines a symmetric monoidal structure on the category of species. Let $\hom^\times(\mathsf{p}, \mathsf{q})$ denote its internal hom. For a comonoid c and monoid a, the species $\hom^\times(\mathsf{c}, \mathsf{a})$ carries the structure of a monoid, while $\hom^\times(\mathsf{a}, \mathsf{c})$ carries the structure of a comonoid. We refer to them as the *convolution monoid* and *coconvolution comonoid*, respectively. Combining the two constructions, for bimonoids h and k, we obtain a bimonoid $\hom^\times(\mathsf{h}, \mathsf{k})$. This is the *biconvolution bimonoid*. When $\mathsf{h} = \mathsf{k}$, we write $\mathrm{end}^\times(\mathsf{h})$ for $\hom^\times(\mathsf{h}, \mathsf{h})$. A summary of these and related objects is given in Table 8.1.

The Hadamard product gives rise to a bilax functor between bimonads (Theorem 8.4). Hence, it preserves monoids, comonoids, bimonoids. Thus, we can consider its internal hom in these categories as well. Let $\mathcal{C}(\mathsf{c}, \mathsf{d})$ denote the internal hom in the category of comonoids. When c is a cocommutative comonoid, and k is a bimonoid, $\mathcal{C}(\mathsf{c}, \mathsf{k})$ carries the structure of a bimonoid. We refer to $\mathcal{C}(\mathsf{c}, \mathsf{k})$ as the *bimonoid of star families* (Section 8.4). If, in addition, c carries the structure of a bimonoid, then $\mathcal{C}(\mathsf{c}, \mathsf{k})$ can be realized as a subbimonoid of the biconvolution bimonoid $\hom^\times(\mathsf{c}, \mathsf{k})$ (Lemma 8.36). There is a similar bimonoid $\overline{\mathcal{C}}(\mathsf{h}, \mathsf{a})$ associated to a commutative monoid a and bimonoid h which is built out of the *universal measuring comonoid* (Section 8.6). The latter allows us to enrich the category of monoids over the category of comonoids. We describe the power and copower of this enriched category (Propositions 8.65 and 8.67).

For any species p, we let $\mathsf{p}^- := \mathsf{p} \times \mathsf{E}^-$, where E^- is the signed exponential species. We refer to p^- as the *signed partner* of p. This yields the *signature functor* which sends a species to its signed partner. It induces an adjoint equivalence between the categories of q-bimonoids and $(-q)$-bimonoids for any scalar q (Corollary 8.92).

Exp-log correspondences. (Chapter 9.) The lune-incidence algebra acts on the space of all maps from a comonoid c to a monoid a. We refer to the action of a noncommutative zeta function $\boldsymbol{\zeta}$ as an exponential, and to the action of a noncommutative Möbius functions $\boldsymbol{\mu}$ as a logarithm. This sets up exp-log correspondences on this space of maps (Proposition 9.9). Moreover, any logarithm of a comonoid morphism from a cocommutative comonoid to a bimonoid is a coderivation, and conversely, the exponential of a coderivation is a comonoid morphism (Theorem 9.11). In particular, any logarithm of the identity map on a cocommutative bimonoid maps into its primitive part (Proposition 9.17).

These results have commutative counterparts in which the (co, bi)monoids involved are (co, bi)commutative, and the lune-incidence algebra is replaced by the flat-incidence algebra (Theorem 9.40 and Proposition 9.47). The exponential and logarithm in this case are unique and defined by the action of the zeta function ζ and Möbius function μ of the poset of flats. Moreover, the logarithm of the identity map on a bicommutative bimonoid induces an isomorphism between its primitive and indecomposable parts. A general result of this nature is given in Proposition 9.55. There are also parallel results for q-bimonoids for q not a root of unity involving $\boldsymbol{\zeta}_q$ and $\boldsymbol{\mu}_q$ (Theorem 9.78, Proposition 9.84, Proposition 9.91), with $q = 0$ as an important special case (Theorem 9.103, Proposition 9.107, Proposition 9.111). We refer to the actions of $\boldsymbol{\zeta}_q$ and $\boldsymbol{\mu}_q$ as the q-exponential and q-logarithm, respectively.

A brief summary of these correspondences is given in Table 9.1.

An alternative equivalent approach to exp-log correspondences is through the notion of series of a species. A series v of a species p amounts to specifying elements $v_A \in \mathsf{p}[A]$ such that $\beta_{B,A}(v_A) = v_B$ whenever A and B have the same support. For a comonoid, one can further define primitive series and group-like series. The lune-incidence algebra acts on the space of series of a monoid, and the action of $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ sets up exp-log correspondences on this space (Proposition 9.116). For a commutative monoid, one can work instead with the flat-incidence algebra and the action of ζ and μ. For the exponential bimonoid E, this recovers Möbius inversion in the poset of flats (Example 9.127), while for the bimonoid of chambers Γ, this yields a noncommutative version of Möbius inversion (Example 9.118). Moreover, for any bimonoid, the exp-log correspondences set up a bijection between its primitive series and group-like series (Theorems 9.117 and 9.124).

The connection between the two approaches is made precise in Lemmas 9.130 and 9.133.

Modules over algebras and bimonoids in species. (Chapters 10 and 11.) The connection between the Tits algebra (and its relatives) and the theory of bimonoids can be strengthened further as follows. There are equivalences of categories:

(left) $\mathsf{J}[\mathcal{A}]$-modules	$\cong$	$\mathcal{A}$-bimonoids	Proposition 11.6
left $\Sigma[\mathcal{A}]$-modules	$\cong$	cocommutative $\mathcal{A}$-bimonoids	Proposition 11.1
right $\Sigma[\mathcal{A}]$-modules	$\cong$	commutative $\mathcal{A}$-bimonoids	Proposition 11.2
$\Pi[\mathcal{A}]$-modules	$\cong$	bicommutative $\mathcal{A}$-bimonoids	Proposition 11.5

In the lhs, $\mathsf{J}[\mathcal{A}]$ refers to the Janus algebra, $\Sigma[\mathcal{A}]$ to the Tits algebra, $\Pi[\mathcal{A}]$ to the Birkhoff algebra. A q-analogue of the first categorical equivalence is given in Proposition 11.7.

Given a bimonoid h, the component $\mathsf{h}[O]$ is a left $\mathsf{J}[\mathcal{A}]$-module with the action given by

$$\mathtt{H}_{(F,F')} \cdot x := \mu_O^F \beta_{F,F'} \Delta_O^{F'}(x).$$

We call this a *characteristic operation*. When h is either commutative or cocommutative, this simplifies to

$$\mathtt{H}_F \cdot x := \mu_O^F \Delta_O^F(x).$$

In this case, we obtain a right or left action, respectively, of $\Sigma[\mathcal{A}]$ on $\mathsf{h}[O]$. Further, when h is bicommutative, we may write

$$\mathtt{H}_{\mathrm{X}} \cdot x := \mu_\perp^{\mathrm{X}} \Delta_\perp^{\mathrm{X}}(x),$$

and we obtain an action of $\Pi[\mathcal{A}]$ on $\mathsf{h}[O]$.

Conversely, given a left $\mathsf{J}[\mathcal{A}]$-module M, we can define for each face F,

$$\mathsf{h}[F] := \mathtt{H}_{(F,F)} \cdot M.$$

This is the image of the idempotent operator defined by the action of $\mathtt{H}_{(F,F)}$. The structure map μ_A^F is defined by inclusion, while Δ_A^G and $\beta_{G,F}$ are defined by the action of $\mathtt{H}_{G,G}$. This turns h into a bimonoid. The remaining cases are similar.

The above categorical equivalences can also be derived by computing the Karoubi envelopes of the Birkhoff monoid, Tits monoid, Janus monoid, and using the interpretation of bimonoids as functor categories (Section 11.8).

Now let us see what these categorical equivalences signify for the bimonoids that we have discussed. For the Birkhoff algebra $\Pi[\mathcal{A}]$, the trivial representation yields the exponential bimonoid E, while the regular representation yields the bimonoid of flats Π, both of which are bicommutative. For the Tits algebra $\Sigma[\mathcal{A}]$, the left module of chambers yields the bimonoid of chambers Γ, while the left regular representation yields the bimonoid of faces Σ, both of which are cocommutative. For the Janus algebra $\mathsf{J}[\mathcal{A}]$, the left regular representation yields the bimonoid of bifaces J. A summary is given in Table 11.2.

These categorical equivalences can be used to relate concepts of Hopf theory with those of representation theory of monoid algebras. For instance: The Hadamard product of bimonoids corresponds to the tensor product of modules over monoid algebras (Exercise 11.9). Primitive and decomposable filtrations of bimonoids relate to primitive and decomposable series of modules over the Tits algebra. The Browder–Sweedler and Milnor–Moore (co)commutativity results relate to the fact that these series are Loewy (Proposition 11.20 and Exercise 11.21).

Antipode. (Chapter 12.) Define the *Takeuchi series*, denoted $\mathtt{Tak}$, of the bimonoid of faces Σ by

$$\mathtt{Tak}_A := \sum_{F:\, F \geq A} (-1)^{\dim(F)} \mathtt{H}_{F/A}.$$

It is group-like.

For any bimonoid h, there is a morphism of bimonoids from Σ to the biconvolution bimonoid $\mathrm{end}^{\times}(\mathsf{h})$ (Lemma 10.7). It is defined via characteristic operations. Under this morphism, the Takeuchi series yields a group-like series of $\mathrm{end}^{\times}(\mathsf{h})$. We call this the *antipode* of h and denote it by S_{h}. Explicitly,

$$(S_{\mathsf{h}})_A := \sum_{F\colon F \geq A} (-1)^{\dim(F)} \mu_A^F \Delta_A^F.$$

This is a linear map from $\mathsf{h}[A]$ to itself. We refer to the above formula as the *Takeuchi formula* (12.1). Up to signs, it equals the 0-logarithm of the identity map on h, see (12.5). The Takeuchi formula also has a commutative analogue (12.15).

Understanding the cancellations in the Takeuchi formula for a given bimonoid is a nontrivial problem. We solve this problem for (co)free bimonoids (Theorems 12.52 and 12.53) and their commutative analogues (Theorems 12.62 and 12.63). We also provide many illustrative examples. In a different direction, for any set-bimonoid, this problem can be reduced to an Euler characteristic computation of a "cell complex" (Lemma 12.86). In the cocommutative case, it involves descent identities, while in the commutative case, it involves lune identities.

The antipode is compatible with bimonoid morphisms (Lemma 12.2), the duality functor (Lemma 12.4), bimonoid filtrations (Lemma 12.5), the signature functor (Lemma 12.9). We formulate the antipode opposition Lemma 12.11 which explains the interaction of the antipode with op and cop constructions (Lemmas 12.12 and 12.15). We also compute the logarithm of the antipode map using the noncommutative Zaslavsky formula (Lemmas 12.24 and 12.26).

Part III

In Part III, we discuss various structure results on bimonoids, and their inter-relationships. We begin with the Leray–Samelson, Borel–Hopf, Loday–Ronco theorems. These deal with the categories of bicommutative bimonoids, (co)commutative bimonoids, 0-bimonoids, respectively. There is also a generalization of Loday–Ronco to the category of q-bimonoids for q not a root of unity. A summary is provided in Table 13.1. We present many different approaches to these results, each with its own flavor and advantage. Related summaries are provided in Tables 13.2 and 13.3.

The Hoffman–Newman–Radford rigidity theorems set up isomorphisms between bimonoids which arise out of similar universal constructions. The isomorphism in one direction involves a (noncommutative) zeta function, and in the other direction involves a (noncommutative) Möbius function. Moreover, the fact that these are morphisms of bimonoids is in principle equivalent to the fact that a zeta function satisfies the lune-additivity formula and a Möbius function satisfies the (noncommutative) Weisner formula. As an important consequence, the Hoffman–Newman–Radford isomorphisms can be used to establish the inverse relationship between zeta and Möbius functions.

Unlike Leray–Samelson or Loday–Ronco, Borel–Hopf does not provide an equivalence of categories. This problem is addressed by the more general Cartier–Milnor–Moore theorem. It involves the notion of Lie monoids and dually the notion of Lie comonoids in species. A related and important result used in its proof is the Poincaré–Birkhoff–Witt theorem. A summary is provided in Tables 17.1 and 17.2.

The Leray–Samelson and Loday–Ronco theorems. (Chapter 13.) Any bicommutative bimonoid h can be recovered from its primitive part as $\mathsf{h} \cong \mathcal{S}(\mathcal{P}(\mathsf{h}))$. This is the *Leray–Samelson theorem.* Similarly, a q-bimonoid can be recovered from its primitive part as $\mathsf{h} \cong \mathcal{T}_q(\mathcal{P}(\mathsf{h}))$ when q is not a root of unity. This includes the case $q = 0$ which we refer to as the *Loday–Ronco theorem.* To summarize, we have the following equivalences of categories.

$$\text{species} \;\cong\; \text{bicommutative bimonoids} \qquad \text{(Leray–Samelson Theorem 13.11)}$$

$$\text{species} \;\cong\; 0\text{-bimonoids} \qquad \text{(Loday–Ronco Theorem 13.2)}$$

$$\text{species} \;\cong\; q\text{-bimonoids for } q \text{ not a root of unity} \qquad \text{(Theorem 13.77)}$$

For Leray–Samelson, the functor in one direction is $\mathcal{S}$, and in the other direction is $\mathcal{P}$. For rigidity of q-bimonoids, the functor in one direction is $\mathcal{T}_q$, and in the other direction is $\mathcal{P}$. In fact, these functors determine adjoint equivalences.

A standard way to prove these rigidity theorems is by an induction on the primitive filtration of h. The analysis for Leray–Samelson and Loday–Ronco is straightforward. Further, there is a parallel between the two, with flats used for the former and faces for the latter. For q-bimonoids, the calculations get more interesting since distance functions also come into the picture.

Another way to prove Leray–Samelson is to employ the commutative exp-log correspondence of a bicommutative bimonoid. This approach is more explicit in the sense that the isomorphisms have explicit descriptions in terms of the zeta and Möbius functions. For q-bimonoids, we employ the two-sided q-zeta and q-Möbius functions.

A third way to prove Leray–Samelson is to use the categorical equivalence between bicommutative bimonoids and modules over the Birkhoff algebra, and the fact that the latter is split-semisimple with primitive orthogonal idempotents Q_{X}. More precisely, for each flat X, we identify $\mathcal{P}(\mathsf{h})[\mathrm{X}]$ with $\mathsf{Q}_{\mathrm{X}} \cdot \mathsf{h}[O]$. The latter is the characteristic operation by Q_{X} on $\mathsf{h}[O]$. These identifications give finer information on the Leray–Samelson isomorphism. Similar methods can also be used for Loday–Ronco, and more generally for rigidity of q-bimonoids.

The above rigidity theorems can also be written in dual form in terms of the indecomposable part functor $\mathcal{Q}$ instead of the primitive part functor $\mathcal{P}$. For the precise statements, see Theorems 13.21, 13.8, 13.93.

The Borel–Hopf theorem. (Chapter 13.) The *Borel–Hopf theorem* says that any cocommutative bimonoid h is cofree on its primitive part (Theorem 13.34). The isomorphism depends on the choice of a noncommutative zeta function. When the bimonoid is bicommutative, Borel–Hopf is contained in Leray–Samelson. Moreover, the latter can be used to deduce the former by passing to the associated graded wrt the primitive filtration (Exercise 13.44). This method is a minor variant on the standard argument by induction on the primitive filtration of h.

The exp-log correspondences can be used to give a more direct proof of Borel–Hopf. The point is to write down an explicit formula for the inverse isomorphism. As expected, it involves a noncommutative Möbius function (Theorem 13.38).

A third way to prove Borel–Hopf is to use the categorical equivalence between cocommutative bimonoids and left modules over the Tits algebra, and characteristic operations by the eulerian idempotents E_{X} in any fixed eulerian family E. More precisely, for each flat X, we identify $\mathcal{P}(\mathsf{h})[\mathrm{X}]$ with $\mathsf{E}_{\mathrm{X}} \cdot \mathsf{h}[O]$. (We mention that an eulerian family is a complete system of primitive orthogonal idempotents of the Tits algebra. Such eulerian families are in correspondence with noncommutative zeta and Möbius functions. This is in contrast to the Birkhoff algebra which has a unique complete system which corresponds to a unique zeta and Möbius function.)

The Borel–Hopf theorem also has a dual version. It says that any commutative bimonoid is free on its indecomposable part (Theorems 13.57 and 13.59). It can also be approached by any of the above methods.

The Hoffman–Newman–Radford rigidity theorems. (Chapter 14.) Every species can be viewed as a comonoid where all nontrivial coproduct components are zero. Thus, to every species p, we have the bimonoid $\mathcal{T}(\mathsf{p})$. Then, for any cocommutative comonoid c, the bimonoid $\mathcal{T}(\mathsf{c})$ is isomorphic to $\mathcal{T}(\mathsf{p})$, where p is the underlying species of c (Propositions 14.40 and 14.42). This result is intimately connected to noncommutative zeta and Möbius functions. It has a commutative analogue (Propositions 14.13 and 14.15) which involves the (commutative) zeta and Möbius functions. There is also a q-analogue for q not a root of unity (Propositions 14.63 and 14.65) in which the two-sided q-zeta and q-Möbius functions intervene.

We refer to these collectively as the *Hoffman–Newman–Radford rigidity theorems*. Interestingly, they can be used as a theoretical tool to establish the inverse relationship between zeta and Möbius functions (Theorem 14.89, Theorem 14.92, Example 14.33). As another application, we use conjugation by the Hoffman–Newman–Radford isomorphisms to study the nondegeneracy of the mixed distributive law for bicommutative bimonoids (Theorem 9.64 and Exercise 14.32) and also for q-bimonoids for q not a root of unity (Theorem 9.100 and Exercise 14.83). Interesting special cases are given in Theorem 7.10 and Theorem 7.51 which study certain self-duality morphisms of the bimonoid of flats and of the q-bimonoid of top-nested faces, respectively.

Freeness under Hadamard products. (Chapter 15.) We study further the Hadamard product on species which was introduced in Chapter 8. The

Hadamard product of two free monoids is again free (Lemma 15.1 and Proposition 15.2). Similarly, the Hadamard product of two free commutative monoids is again free commutative (Proposition 15.4). In either case, we give an explicit formula for a basis of the Hadamard product in terms of bases of its two factors, see (15.1) and (15.2). It involves the meet operation on faces and flats, respectively. We also show that the Hadamard product of bimonoids is free as a monoid if one of the two factors is free as a monoid (Theorem 15.34).

We study in detail the Hadamard product of a free bimonoid on a comonoid with a cofree bimonoid on a monoid. It is neither commutative nor cocommutative, so Borel–Hopf does not apply. This bimonoid is both free and cofree (Proposition 15.15). Interestingly, we prove this using Loday–Ronco (which is a theorem about 0-bimonoids). We give an explicit description of its primitive filtration by performing a change of basis (Proposition 15.23). We also give cancellation-free formulas for its antipode in either basis (Theorems 15.13 and 15.22). An illustrative example of this construction is the bimonoid of pairs of chambers (Section 15.5). We give a parallel discussion for a commutative counterpart of this construction (Section 15.3).

Lie monoids. (Chapters 16 and 17.) Left modules over the Lie operad are Lie monoids. In contrast to monoids and commutative monoids, these are harder to make explicit since the Lie operad is not set-theoretic. Using the presentation of the Lie operad, one can describe a Lie monoid in terms of a Lie bracket subject to antisymmetry and Jacobi identity.

To any Lie monoid g, one can associate its universal enveloping monoid $\mathcal{U}(\mathsf{g})$. The latter is a quotient of $\mathcal{T}(\mathsf{g})$ and carries the structure of a cocommutative bimonoid (Proposition 16.18). Further, the composite

$$\mathcal{S}(\mathsf{g}) \hookrightarrow \mathcal{T}(\mathsf{g}) \twoheadrightarrow \mathcal{U}(\mathsf{g})$$

is an isomorphism of comonoids. This is the *Poincaré–Birkhoff–Witt theorem*, or PBW for short (Theorem 17.9). The inclusion of $\mathcal{S}(\mathsf{g})$ into $\mathcal{T}(\mathsf{g})$ depends on the choice of a noncommutative zeta function. We give two proofs of PBW. One proof involves the explicit construction of an idempotent operator on $\mathcal{T}(\mathsf{g})$ whose image is $\mathcal{S}(\mathsf{g})$ and whose kernel coincides with the kernel of the projection $\mathcal{T}(\mathsf{g}) \twoheadrightarrow \mathcal{U}(\mathsf{g})$. We call this the *Solomon operator* (17.24).

The primitive part $\mathcal{P}(\mathsf{h})$ of a bimonoid h carries the structure of a Lie monoid (Proposition 16.2). The functors $\mathcal{U}$ and $\mathcal{P}$ define an adjoint equivalence between the category of Lie monoids and the category of cocommutative bimonoids. This is the *Cartier–Milnor–Moore theorem*, or CMM for short (Theorem 17.42). It can be formally deduced by combining PBW and Borel–Hopf.

Dually, left comodules over the Lie cooperad are Lie comonoids. To any Lie comonoid g, one can associate its universal coenveloping comonoid $\mathcal{U}^\vee(\mathsf{g})$. It is a subbimonoid of the commutative bimonoid $\mathcal{T}^\vee(\mathsf{g})$. The indecomposable part $\mathcal{Q}(\mathsf{h})$ of a bimonoid h carries the structure of a Lie comonoid. Dual PBW (Theorem 17.18) and dual CMM (Theorem 17.53) hold.

Part I

Species and operads

CHAPTER 1

Hyperplane arrangements

The goal of this preliminary chapter is to collect all the important notions and results on hyperplane arrangements that will be required in this text. For the most part, we only provide the statements; proofs can be found in [**21**], with additional information in the Notes section. The discussion includes

- geometric objects such as faces, flats, bifaces, partial-flats, nested faces, lunes, bilunes, cones,
- algebraic objects such as the Tits monoid, Birkhoff monoid, Janus monoid and their linearized algebras; Lie and Zie elements,
- combinatorial objects such as distance functions and Varchenko matrices; descent, lune, Witt identities; incidence algebras, zeta and Möbius functions along with their noncommutative and two-sided analogues, the Zaslavsky formula and its noncommutative analogue.

Readers with basic familiarity with hyperplane arrangements can directly start from Chapter 2 and return to this chapter when necessary.

1.1. Faces, bifaces, flats

Faces and flats are two important geometric objects associated to a hyperplane arrangement. The set of faces and the set of flats carry partial orders given by inclusion. Moreover, the set of faces carries the structure of a monoid called the Tits monoid, while the set of flats carries the structure of a commutative monoid called the Birkhoff monoid. The support map from faces to flats is order-preserving as well as a morphism of monoids.

We also review bifaces and the Janus monoid, cones, arrangements over and under flats.

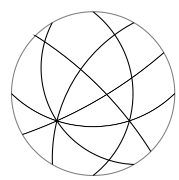
1.1.1. Hyperplane arrangements. A *hyperplane arrangement* $\mathcal{A}$ is a finite set of hyperplanes (codimension-one affine subspaces) in a fixed real vector space. The latter is called the *ambient space* of $\mathcal{A}$. The arrangement is *linear* if all its hyperplanes pass through the origin. Unless stated otherwise, all our arrangements are assumed to be linear.

Let O denote the intersection of all hyperplanes. We call it the *central face*. The *rank* of $\mathcal{A}$, denoted $\mathrm{rk}(\mathcal{A})$, is the difference between the dimensions of the ambient space and the central face. An arrangement has rank zero iff it has no hyperplanes. An arrangement has rank one iff it has exactly one hyperplane.

There are two standard ways to view a hyperplane arrangement, namely, the linear model and the spherical model. The latter is obtained from the former by first taking quotient by the central face and then cutting with the unit sphere. Some illustrative pictures are shown below.

The above is an arrangement of rank one with ambient space $\mathbb{R}$ and with one hyperplane, namely, the origin. The linear model is on the left and the spherical model on the right.

The above is the arrangement of three lines in the plane; the linear model is on the left and the spherical model on the right. More generally, one can consider the arrangement of n lines in the plane. This is an arrangement of rank two.



The above is a spherical model of six hyperplanes (great circles) in three-space. Only one half of the arrangement is visible in the picture, the other half being on the backside. This is an arrangement of rank three.

1.1.2. Faces and the Tits monoid. Each hyperplane has two associated half-spaces. A *face* of $\mathcal{A}$ is a subset of the ambient space obtained by intersecting half-spaces, with at least one half-space chosen for each hyperplane. In the spherical model, faces are precisely the vertices, edges, triangles, and so on. The arrangement is called *simplicial* if all faces are simplices.

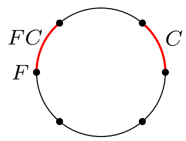
Let $\Sigma[\mathcal{A}]$ denote the set of faces of $\mathcal{A}$. It is a graded poset under inclusion. The minimum element is the central face. (It is not visible in the spherical model.) The rank of a face F is denoted $\mathrm{rk}(F)$. Note that F has a dimension in the ambient space, and

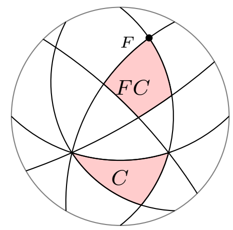
$$\mathrm{rk}(F) = \dim(F) - \dim(O).$$

A maximal face is called a *chamber*. We let $\Gamma[\mathcal{A}]$ denote the set of chambers. A corank-one face is called a *panel*; we also say a face F is a panel of a face G if $F \lessdot G$, that is, if G covers F in the poset of faces. Every face F has an *opposite face* $\overline{F}$ given by $\{-x \mid x \in F\}$. We usually denote faces by letters A, B, F, G, H, K, and chambers by letters C, D, E.

The intersection of two faces is a face, so meets exist in $\Sigma[\mathcal{A}]$. We denote the meet of F and G by $F \wedge G$. In contrast, joins may not exist.

The poset of faces $\Sigma[\mathcal{A}]$ carries a (noncommutative) monoid structure. We call this the *Tits monoid*, see [**21**, Section 1.4]. The central face O is the identity element. For faces F and G, we denote their Tits product by FG. Moreover, for F a face and C a chamber, FC is a chamber, thus, the set of chambers $\Gamma[\mathcal{A}]$ is a left $\Sigma[\mathcal{A}]$-set. This is illustrated below in ranks two and three, respectively.





Geometrically, among all chambers containing F, the chamber FC is closest to C. A more precise formulation of this property is given later in (1.22).

Some basic properties of the Tits product are listed below.

(1.1a) $$FF = F \text{ and } FGF = FG.$$

(1.1b) $$\text{If } G \leq H, \text{ then } FG \leq FH. \text{ In particular, } F \leq FG.$$

(1.1c) $$\text{If } FG = K \text{ and } F \leq H \leq K, \text{ then } HG = K.$$

(1.1d) $$HF = F \iff H \leq F.$$

(1.1e) $$H\overline{F} = F \iff H = F.$$

(1.1f) $$HF = G \iff HF\overline{G} = G.$$

A *left regular band*, or LRB for short, is a monoid in which the axiom $xyx = xy$ holds. By (1.1a), we see that the Tits monoid is a left regular band.

Example 1.1. Consider the monoid $S = \{-, 0, +\}$ with the following multiplication table. The entry in row a and column b represents the product ab.

	$-$	0	$+$
$-$	$-$	$-$	$-$
0	$-$	0	$+$
$+$	$+$	$+$	$+$

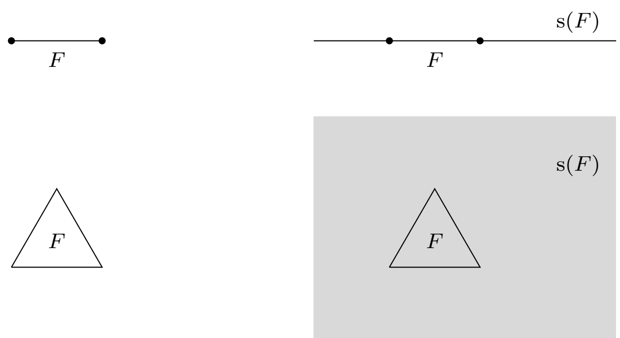
Observe that S is isomorphic to the Tits monoid of a rank-one arrangement.

1.1.3. Flats and the Birkhoff monoid. A *flat* of an arrangement $\mathcal{A}$ is a subspace of the ambient space obtained by intersecting an arbitrary subset of hyperplanes in $\mathcal{A}$. Let $\Pi[\mathcal{A}]$ denote the set of flats. It is a graded lattice under inclusion. We use the letters X, Y, Z, W to denote flats. The meet of X and Y is denoted by $\mathrm{X} \wedge \mathrm{Y}$, and join by $\mathrm{X} \vee \mathrm{Y}$. The minimum and maximum flats are denoted $\perp$ and $\top$, respectively. They coincide with the central face and ambient space, respectively. We view the lattice of flats $\Pi[\mathcal{A}]$ as a monoid with product given by the join. We call this the *Birkhoff monoid*, and refer to $\mathrm{X} \vee \mathrm{Y}$ as the Birkhoff product of X and Y. The minimum flat $\perp$ is the identity element. The Birkhoff monoid is a commutative left regular band.

1.1.4. Support map. The *support* of a face F, denoted $\mathrm{s}(F)$, is the smallest flat which contains F. It is the linear span of F. We say a flat X supports a face F if $\mathrm{s}(F) = \mathrm{X}$. The *support map*

$$\mathrm{s} : \Sigma[\mathcal{A}] \twoheadrightarrow \Pi[\mathcal{A}] \tag{1.2}$$

sends a face to its support. It is surjective and order-preserving. Illustrations of the support of a rank-two face and a rank-three face are shown below.



The support map is a morphism from the Tits monoid to the Birkhoff monoid, that is,

$$\mathrm{s}(FG) = \mathrm{s}(F) \vee \mathrm{s}(G). \tag{1.3}$$

Observe that FG and GF always have the same support. In particular, if $GF = G$, then FG and G have the same support. Similar useful observations are given below.

$$GF = G \iff \mathrm{s}(F) \leq \mathrm{s}(G). \tag{1.4}$$

$$FG = F \text{ and } GF = G \iff \mathrm{s}(F) = \mathrm{s}(G). \tag{1.5}$$

To summarize: The relation

$$F \sim G \iff FG = F \text{ and } GF = G \tag{1.6}$$

is an equivalence relation on the set of faces whose equivalence classes correspond to flats.

Exercise 1.2. Check that: For any faces F, G, H,

$$HF = G \text{ and } \mathrm{s}(F) = \mathrm{s}(G) \iff HF = G \text{ and } FH = F.$$

1.1.5. Bifaces and the Janus monoid. A *biface* is a pair (F, F') of faces such that F and F' have the same support. Let $\mathrm{J}[\mathcal{A}]$ denote the set of bifaces. The operation

$$(F, F')(G, G') := (FG, G'F')$$

turns $\mathrm{J}[\mathcal{A}]$ into a monoid. We call this the *Janus monoid*. The identity element is (O, O). The Janus monoid is canonically isomorphic to its opposite monoid via

$$\mathrm{J}[\mathcal{A}] \to \mathrm{J}[\mathcal{A}]^{\mathrm{op}}, \qquad (F, F') \mapsto (F', F).$$

Moreover, there is a commutative diagram of monoids

$$
\begin{tikzcd}
\mathrm{J}[\mathcal{A}] \arrow[r, two heads] \arrow[d, two heads] & \Sigma[\mathcal{A}]^{\mathrm{op}} \arrow[d, two heads, "\mathrm{s}"] \\
\Sigma[\mathcal{A}] \arrow[r, two heads, "\mathrm{s}"'] & \Pi[\mathcal{A}]
\end{tikzcd}
\qquad
\begin{tikzcd}
(F, F') \arrow[r, mapsto] \arrow[d, mapsto] & F' \arrow[d, mapsto] \\
F \arrow[r, mapsto] & \mathrm{s}(F) = \mathrm{s}(F')
\end{tikzcd}
\tag{1.7}
$$

with s being the support map, and the maps from $\mathrm{J}[\mathcal{A}]$ being the projections on the two coordinates, respectively.

The Janus monoid is a *regular band*, that is, an idempotent monoid in which the axiom $xyxzx = xyzx$ holds. See [**531**, Lemma 12] in this regard.

Example 1.3. Consider the monoid with underlying set $\{\mathrm{id}, e, f, ef, fe\}$. All elements are idempotent, and they are multiplied using the relations $efe = e$ and $fef = f$. This is precisely the Janus monoid of a rank-one arrangement $\mathcal{A}$ with chambers C and $\overline{C}$:

$$\mathrm{id} \leftrightarrow (O, O),\ e \leftrightarrow (C, C),\ f \leftrightarrow (\overline{C}, \overline{C}),\ ef \leftrightarrow (C, \overline{C}),\ fe \leftrightarrow (\overline{C}, C).$$

In this case, the Tits monoid is given by the set $\{\mathrm{id}, e, f\}$ of idempotents with $ef = e$ and $fe = f$. The opposite Tits monoid is the same set but with $ef = f$ and $fe = e$. The Birkhoff monoid consists of two idempotent elements, namely, id and $e = f$.

1.1.6. Cones. A *cone* of an arrangement $\mathcal{A}$ is a subset of the ambient space obtained by intersecting an arbitrary subset of half-spaces in $\mathcal{A}$. Let $\Omega[\mathcal{A}]$ denote the set of cones. It is a lattice under inclusion.

Faces and flats are cones. The inclusion $\Pi[\mathcal{A}] \hookrightarrow \Omega[\mathcal{A}]$ is a lattice morphism. Similarly, the inclusion $\Sigma[\mathcal{A}] \hookrightarrow \Omega[\mathcal{A}]$ preserves meets. It preserves joins whenever they exist in $\Sigma[\mathcal{A}]$.

Every cone V has an *opposite cone*, denoted $\overline{\mathrm{V}}$, obtained by intersecting the half-spaces opposite to those that define V. If V is a face, then $\overline{\mathrm{V}}$ is precisely its opposite face.

1.1.7. Strong connectivity. A finite poset P is *strongly connected* if given $x < y$ in P, one can transform any maximal chain from x to y to any other by successively changing one element of the chain to a different element.

Lemma 1.4. *Let $\mathcal{A}$ be any arrangement.*

(1) *The poset of faces $\Sigma[\mathcal{A}]$ with a top element adjoined is strongly connected. In particular, $\Sigma[\mathcal{A}]$ is also strongly connected.*
(2) *The lattice of flats $\Pi[\mathcal{A}]$ is strongly connected.*

Proof. See [**21**, Lemmas 1.34 and 1.35]. □

1.1.8. Category associated to a poset. To a finite poset P, one can associate the category whose objects are elements of P, with a unique morphism $x \to y$ whenever $x \leq y$ in P.

Proposition 1.5. *For a strongly connected poset P, the associated category has a presentation given by generators $\Delta : x \to y$, where y covers x, and relations*

$$\begin{tikzcd} x' \arrow[r, "\Delta"] & y \\ z \arrow[u, "\Delta"] \arrow[r, "\Delta"'] & x \arrow[u, "\Delta"'] \end{tikzcd}$$

whenever y covers both x and x', and they in turn cover z.

Proof. See [**21**, Proposition B.10]. □

1.1.9. Stars and top-stars. For a face F, let $\Sigma[\mathcal{A}]_F$ denote the set of faces of $\mathcal{A}$ which are greater than F. This is the *star* of F. It is a monoid under the Tits product with identity element F. For clarity, we denote elements of $\Sigma[\mathcal{A}]_F$ by K/F, where K is a face greater than F. Let $\Gamma[\mathcal{A}]_F$ denote the set of chambers of $\mathcal{A}$ which are greater than F. This is the *top-star* of F.

Lemma 1.6. *When F and G have the same support, we have an isomorphism*

$$\Sigma[\mathcal{A}]_F \xrightarrow{\cong} \Sigma[\mathcal{A}]_G, \qquad K/F \mapsto GK/G$$

of monoids, and hence of posets. The inverse is given by

$$\Sigma[\mathcal{A}]_G \xrightarrow{\cong} \Sigma[\mathcal{A}]_F, \qquad H/G \mapsto FH/F.$$

Further, it restricts to a bijection

$$\Gamma[\mathcal{A}]_F \xrightarrow{\cong} \Gamma[\mathcal{A}]_G, \qquad C/F \mapsto GC/G.$$

Proof. This is straightforward, see [**21**, Lemma 1.37]. □

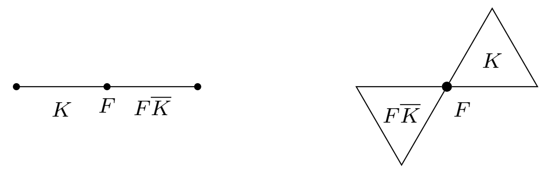
A rank-three illustration of the bijection between top-stars is shown below.

In the picture, faces F and G are of rank two and have the same support. The top-star of F consists of chambers C and D, which under the bijection correspond to chambers GC and GD in the top-star of G.

1.1.10. Arrangements under and over a flat. For any flat X, the arrangement under X is the arrangement $\mathcal{A}^{\mathrm{X}}$ whose ambient space is X and hyperplanes are codimension-one subspaces of X obtained by intersecting X with hyperplanes in $\mathcal{A}$ not containing X. Faces of $\mathcal{A}^{\mathrm{X}}$ can be canonically identified with faces of $\mathcal{A}$ with support smaller than X, and chambers can be identified with faces of $\mathcal{A}$ with support X.

For any flat Y, the arrangement over Y is the arrangement $\mathcal{A}_{\mathrm{Y}}$ consisting of those hyperplanes which contain Y. The ambient space remains the same. For any face F, let $\mathcal{A}_F := \mathcal{A}_{\mathrm{s}(F)}$. Thus, there is no distinction between $\mathcal{A}_F$

and $\mathcal{A}_G$ when F and G have the same support. However, for book-keeping purposes, we identify faces of $\mathcal{A}_F$ with the star of F and chambers with the top-star of F. Thus, K/F and C/F denote a face and chamber of $\mathcal{A}_F$, respectively. In this notation, the opposite of a face K/F of $\mathcal{A}_F$ is $F\overline{K}/F$. This is illustrated below.



Also note that $\mathrm{rk}(K/F) = \mathrm{rk}(K) - \mathrm{rk}(F)$.

The under and over constructions can be combined as follows. Let $\mathrm{Y} \leq \mathrm{X}$. Then one may first go under X and then over Y, or first go over Y and then under X. The resulting arrangements $(\mathcal{A}^{\mathrm{X}})_{\mathrm{Y}}$ and $(\mathcal{A}_{\mathrm{Y}})^{\mathrm{X}}$ are identical and we denote it by $\mathcal{A}^{\mathrm{X}}_{\mathrm{Y}}$.

1.1.11. Cartesian product of arrangements. For two arrangements $\mathcal{A}$ and $\mathcal{A}'$, one can form their *cartesian product* $\mathcal{A} \times \mathcal{A}'$. Its ambient space is $V \oplus V'$, where V and V' are the ambient spaces of $\mathcal{A}$ and $\mathcal{A}'$. Its hyperplanes are codimension-one subspaces of the form $\mathrm{H} \oplus V'$ and $V \oplus \mathrm{H}'$, where H and H' are hyperplanes of $\mathcal{A}$ and $\mathcal{A}'$.

1.2. Nested faces and lunes

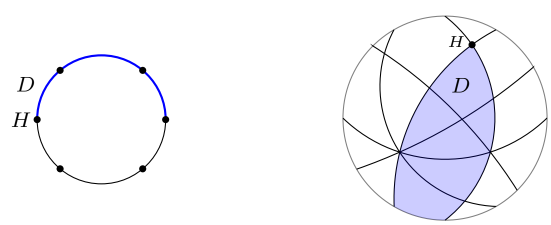
We review nested faces and lunes, and the support map relating them. This is analogous to the situation for faces and flats. The two pictures can be unified by consideration of h-faces and h-flats for any set h on which the Tits monoid acts on the left. We also recall briefly the category of lunes.

1.2.1. Top-nested faces and top-lunes. Let $\mathcal{A}$ be an arrangement. Let H be any face and D be a chamber greater than H. We refer to the pair (H, D) as a *top-nested face*. We define its *support* to be

$$\mathrm{s}(H, D) := \{C \mid HC = D\}. \tag{1.8}$$

This is a subset of $\Gamma[\mathcal{A}]$.

A *top-lune* is a subset of the set of chambers of the form $\mathrm{s}(H, D)$ for some top-nested face (H, D). Illustrations in ranks two and three are shown below.



The regions marked in blue are top-lunes. In the second picture, the top-lune is not fully visible; a small part is on the back side.

A top-lune corresponds to an equivalence class in the set of top-nested faces under the relation

$$(H,D) \sim (G,C) \iff HG = H,\ GH = G,\ HC = D,\ GD = C. \tag{1.9}$$

The class of (H,D) is the top-lune $\mathrm{s}(H,D)$. For the top-lune $\mathrm{s}(H,D)$, the flat $\mathrm{s}(H)$ is unique, and is called the *base* of that top-lune.

1.2.2. Nested faces and lunes. More generally: A *nested face* is a pair of faces (H,G) such that $H \le G$. We define its *support* to be

$$\begin{aligned} \mathrm{s}(H,G) &:= \{F \mid HF = G \text{ and } \mathrm{s}(F) = \mathrm{s}(G)\} \\ &= \{F \mid HF = G \text{ and } FH = F\}. \end{aligned} \tag{1.10}$$

This is a subset of $\Sigma[\mathcal{A}]$. The second equality above holds by Exercise 1.2.

A *lune* is a subset of the set of faces of the form $\mathrm{s}(H,G)$ for some nested face (H,G). It corresponds to an equivalence class in the set of nested faces under the relation

$$(H,G) \sim (K,F) \iff HK = H,\ KH = K,\ HF = G,\ KG = F. \tag{1.11}$$

The class of (H,G) is the lune $\mathrm{s}(H,G)$. For the lune $\mathrm{s}(H,G)$, the flats $\mathrm{s}(H)$ and $\mathrm{s}(G)$ are unique, and are, respectively, called the *base* and *case* of that lune. The case of a top-lune is the maximum flat.

Lunes of the form $\mathrm{s}(H,H)$ can be identified with the flat $\mathrm{s}(H)$, while lunes of the form $\mathrm{s}(O,H)$ can be identified with the face H. In this sense, every face and flat is a lune.

Lemma 1.7. *There are correspondences:*

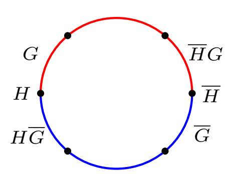
$$\begin{aligned} \textit{lunes of } \mathcal{A} \textit{ with base } \mathrm{X} &\longleftrightarrow \textit{faces of } \mathcal{A}_{\mathrm{X}}, \\ \textit{top-lunes of } \mathcal{A} \textit{ with base } \mathrm{X} &\longleftrightarrow \textit{chambers of } \mathcal{A}_{\mathrm{X}}, \\ \textit{lunes of } \mathcal{A} \textit{ with base } \mathrm{X} \textit{ and case } \mathrm{Y} &\longleftrightarrow \textit{chambers of } \mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}. \end{aligned}$$

Let $\Lambda[\mathcal{A}]$ denote the set of lunes, and $\widehat{\Lambda}[\mathcal{A}]$ denote the set of top-lunes. We view $\Sigma[\mathcal{A}]$ and $\Pi[\mathcal{A}]$ as subsets of $\Lambda[\mathcal{A}]$. Lunes are usually denoted by letters L, M, N. We denote the base and case of L by $\mathrm{b(L)}$ and $\mathrm{c(L)}$, respectively. The *slack* of L is defined by $\mathrm{sk(L)} := \mathrm{rk(c(L))} - \mathrm{rk(b(L))}$. Lunes of slack 0 are precisely flats. The slack of a face is its rank in the poset of faces.

Every lune L has an *opposite lune*, denoted $\overline{\mathrm{L}}$, defined as follows. The opposite of $\mathrm{s}(H,G)$ is

$$\overline{\mathrm{s}(H,G)} := \mathrm{s}(H, H\overline{G}) = \mathrm{s}(\overline{H}, \overline{G}). \tag{1.12}$$

In the picture below, the lunes in red and blue are opposite to each other.



Note that $\mathrm{b(L)} = \mathrm{b}(\overline{\mathrm{L}})$ and $\mathrm{c(L)} = \mathrm{c}(\overline{\mathrm{L}})$. If L is a face, then $\overline{\mathrm{L}}$ is precisely its opposite face. A lune equals its opposite iff it is a flat.

1.2.3. Category of lunes. We define the *category of lunes* of $\mathcal{A}$. Its objects are flats and morphisms are lunes. More precisely, a morphism from Y to X is a lune L whose base is X and case is Y. Identity morphisms are flats. Composition of morphisms is defined by

$$\mathrm{s}(F,G) \circ \mathrm{s}(G,H) = \mathrm{s}(F,H). \tag{1.13}$$

It is straightforward to check that this is well-defined. Moreover, we have $\overline{\mathrm{L} \circ \mathrm{M}} = \overline{\mathrm{L}} \circ \overline{\mathrm{M}}$ for composable lunes L and M.

Recall from [**21**, (4.3)] that the set of lunes $\Lambda[\mathcal{A}]$ carries a partial order denoted $\leq$. When $\mathrm{L} = \mathrm{M} \circ \mathrm{N}$, we say M is a left factor and N is a right factor of L. If a left factor M of L is given, then the corresponding right factor N is uniquely determined, and similarly, if a right factor N of L is given, then the corresponding left factor M is uniquely determined. Moreover, M is a left factor of L iff $\mathrm{b}(\mathrm{M}) = \mathrm{b}(\mathrm{L})$ and $\mathrm{M} \leq \mathrm{L}$. See [**21**, Lemma 3.16, Lemma 4.32, Exercise 4.36]. Similarly, N is a right factor of L implies $\mathrm{c}(\mathrm{N}) = \mathrm{c}(\mathrm{L})$ and $\mathrm{L} \leq \mathrm{N}$, but the converse does not hold in general.

1.2.4. h-faces and h-flats. Recall the Tits monoid $\Sigma[\mathcal{A}]$. Let h be a left $\Sigma[\mathcal{A}]$-set. We write $F \cdot x$ for the action of the face F on $x \in \mathrm{h}$. An h-*face* is a pair (F, x) such that $F \cdot x = x$. Define an equivalence relation on the set of h-faces by

$$(F,x) \sim (G,y) \iff FG = F,\ GF = G,\ F \cdot y = x,\ G \cdot x = y. \tag{1.14}$$

An equivalence class is an h-*flat*.

The *support* of an h-face is the h-flat to which it belongs. Let ${}^{\mathrm{h}}\Sigma[\mathcal{A}]$ denote the set of h-faces, and ${}^{\mathrm{h}}\Pi[\mathcal{A}]$ denote the set of h-flats. The *support map*

$$\mathrm{s} : {}^{\mathrm{h}}\Sigma[\mathcal{A}] \to {}^{\mathrm{h}}\Pi[\mathcal{A}]$$

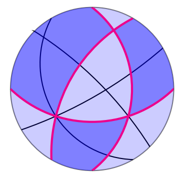
sends an h-face to its support.

If h is a singleton set, then an h-face is the same as a face, and an h-flat is the same as a flat. If h is the set of chambers $\Gamma[\mathcal{A}]$, then an h-face is the same as a top-nested face, and an h-flat is the same as a top-lune. Similarly, if h is the set of faces $\Sigma[\mathcal{A}]$, then an h-face is the same as a nested face, and an h-flat is the same as a lune. In all three cases, the support map specializes to the support map discussed in each context.

1.3. Partial-flats

Partial-flats interpolate between faces and flats. They arise from a suitable equivalence relation on faces. Subarrangements provide an important source of such equivalence relations.

1.3.1. Partial-support relations on chambers. A *partial-support relation on chambers* is an equivalence relation on the set of chambers $\Gamma[\mathcal{A}]$, denoted $\sim$, such that $C \sim D$ implies $FC \sim FD$. Equivalence classes of such a relation correspond to chambers of a subarrangement of $\mathcal{A}$. In fact, there is a bijection between subarrangements of $\mathcal{A}$ and partial-support relations on chambers of $\mathcal{A}$, see [**21**, Proposition 2.66]. This is illustrated below.



The subarrangement consists of the three hyperplanes shown as thick lines. These partition the set of chambers. The parts of this partition (highlighted in light and dark shades) are precisely the equivalence classes of the corresponding partial-support relation.

A useful property of a partial-support relation on chambers is noted below.

(1.15) Let $A \le C$, $A \le D$, $\mathrm{s}(A) = \mathrm{s}(B)$ and $C \sim BC$. Then $D \sim BD$.

1.3.2. Partial-support relations on faces. A *partial-support relation on faces* is an equivalence relation on the set of faces $\Sigma[\mathcal{A}]$, denoted $\sim$, which satisfies

$$F \sim G \implies \mathrm{s}(F) = \mathrm{s}(G), \tag{1.16a}$$

$$F \sim G \implies FH \sim GH, \tag{1.16b}$$

$$F \sim G \implies HF \sim HG. \tag{1.16c}$$

A partial-support relation is *geometric* if

$$\mathrm{s}(F) = \mathrm{s}(G) \text{ and } FH \sim GH \text{ for some } H \implies F \sim G. \tag{1.16d}$$

A geometric partial-support relation on faces restricts to a partial-support relation on chambers. Conversely, a partial-support relation on chambers extends to a geometric partial-support relation on faces: For $\mathrm{s}(F) = \mathrm{s}(G)$,

$$F \sim G :\iff FC \sim GC \text{ for some chamber } C.$$

Further, the two constructions are inverses of each other. So there is a bijection between partial-support relations on chambers, and geometric partial-support relations on faces.

1.3.3. Partial-flats. We refer to an equivalence class of a partial-support relation $\sim$ on faces as a *partial-flat*. We denote it by letters x, y, z, w. If $F \in \mathrm{x}$, then we say that x is the *partial-support* of F. By axiom (1.16a), each partial-flat has a well-defined support. We say a partial-flat is maximal if its support is the maximum flat.

Let $\Sigma_\sim[\mathcal{A}]$ denote the set of partial-flats. In fact, it is a left regular band (and hence also a poset). The product is defined such that the canonical maps

$$\Sigma[\mathcal{A}] \twoheadrightarrow \Sigma_\sim[\mathcal{A}] \twoheadrightarrow \Pi[\mathcal{A}]$$

are morphisms of monoids. We let xy denote the product of x and y. The partial order is given by $\mathrm{x} \le \mathrm{y}$ iff $\mathrm{xy} = \mathrm{y}$. We refer to $\Sigma_\sim[\mathcal{A}]$ as the *monoid of partial-flats*.

We say a partial-support relation $\sim$ is *finest* if $\Sigma_\sim[\mathcal{A}] = \Sigma[\mathcal{A}]$, and is *coarsest* if $\Sigma_\sim[\mathcal{A}] = \Pi[\mathcal{A}]$.

Lemma 1.8. *Let* x *and* x' *be equivalence classes of a geometric partial-support relation. Then* $\mathrm{x}\mathrm{x}' = \mathrm{x}'\mathrm{x}$ *iff* x *and* x' *have an upper bound iff* x *and* x' *have a join. In this situation,*

$$\mathrm{x} \vee \mathrm{x}' = \mathrm{x}\mathrm{x}' = \mathrm{x}'\mathrm{x}.$$

Proof. See [**21**, Lemma 2.74]. □

Lemma 1.9. *For a geometric partial-support relation* $\sim$*,* $\Sigma_\sim[\mathcal{A}]$ *is a subposet of the lattice of cones* $\Omega[\mathcal{A}]$ *closed under taking meets.*

Proof. See [**21**, Corollary 2.80]. □

1.3.4. Janus monoid for partial-flats. The Janus monoid can be generalized as follows. Fix two partial-support relations on faces, say $\sim$ and $\sim'$. A *partial-biflat* is a pair $(\mathrm{x}, \mathrm{x}')$ such that x is a partial-flat wrt $\sim$, x' is a partial-flat wrt $\sim'$, and x and x' have the same support. Let $\mathrm{J}_{\sim,\sim'}[\mathcal{A}]$ denote the set of partial-biflats. The operation

$$(\mathrm{x}, \mathrm{x}')(\mathrm{y}, \mathrm{y}') := (\mathrm{x}\mathrm{y}, \mathrm{y}'\mathrm{x}')$$

turns $\mathrm{J}_{\sim,\sim'}[\mathcal{A}]$ into a monoid. There is a commutative diagram of monoids

$$\begin{tikzcd} \mathrm{J}_{\sim,\sim'}[\mathcal{A}] \arrow[r] \arrow[d] & \Sigma_{\sim'}[\mathcal{A}]^{\mathrm{op}} \arrow[d] \\ \Sigma_\sim[\mathcal{A}] \arrow[r] & \Pi[\mathcal{A}]. \end{tikzcd} \tag{1.17}$$

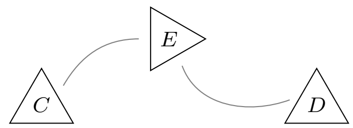
By specializing $\sim$ and $\sim'$ in various ways, the Janus monoid, Tits monoid, Birkhoff monoid, and also $\Sigma_\sim[\mathcal{A}]$ and $\Sigma_{\sim'}[\mathcal{A}]^{\mathrm{op}}$ are all special cases of $\mathrm{J}_{\sim,\sim'}[\mathcal{A}]$.

1.4. Minimal galleries, distance functions, Varchenko matrices

We review minimal galleries, gallery distance and the gate property. For any scalar q, we define the q-distance function on faces and summarize its properties. We briefly mention abstract distance functions, with the q-distance function as a motivating example. We also consider the family of Varchenko matrices, one for each flat, arising from the q-distance function.

1.4.1. Minimal galleries. Two chambers are *adjacent* if they are distinct and share a panel. A *gallery* is a sequence of chambers such that consecutive chambers are adjacent. It is well-known that the set of chambers is *gallery connected*, that is, there is always a gallery connecting any two chambers. Define the *gallery distance* $\mathrm{dist}(C, D)$ to be the minimum length of a gallery from C to D. Any gallery which achieves this minimum is a *minimal gallery* from C to D. The notion of adjacency and minimal galleries also make sense for faces with the same support (by working in the arrangement under that flat).

For chambers C, D, E, let $C \mathrel{\text{--}} E \mathrel{\text{--}} D$ mean that there is a minimal gallery from C to D passing through E. An illustration is shown below.

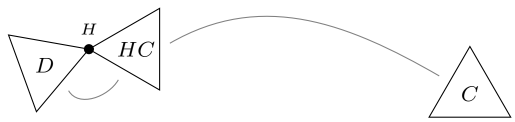


The notation extends to more than three chambers in the obvious way. Similarly, $F \mathrel{\text{--}} G \mathrel{\text{--}} H$ denotes a minimal gallery in $\mathcal{A}^{\mathrm{X}}$ for faces F, G, H with the same support X.

Some important facts are stated below.

(1.18) For any C and $H \leq D$, there is a minimal gallery $C \mathrel{\text{--}} HC \mathrel{\text{--}} D$.

This is the *gate property*. The chamber HC is the gate of the star of H wrt the chamber C.

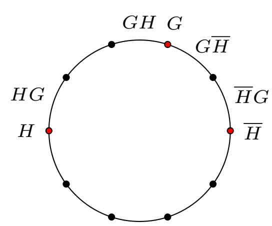


(1.19) For any H and C, there is a minimal gallery $HC \mathrel{\text{--}} C \mathrel{\text{--}} \overline{H}C$.

In particular, for any chambers C and D, we have $C \mathrel{\text{--}} D \mathrel{\text{--}} \overline{C}$.

(1.20) For any G and H, there is a minimal gallery $HG \mathrel{\text{--}} GH \mathrel{\text{--}} G\overline{H} \mathrel{\text{--}} \overline{H}G$.

This recovers (1.19) when G is a chamber. An illustration in rank two is shown below.



1.4.2. Gallery distance between chambers. A hyperplane *separates* two faces if they lie on opposite sides of that hyperplane. The gallery distance $\operatorname{dist}(C, D)$ is equal to the number of hyperplanes which separate C and D. It verifies the familiar properties of a metric:

(1.21a) $\operatorname{dist}(C, D) \geq 0$ with equality iff $C = D$,

(1.21b) $\operatorname{dist}(C, D) = \operatorname{dist}(D, C)$,

(1.21c) $\operatorname{dist}(C, E) \leq \operatorname{dist}(C, D) + \operatorname{dist}(D, E)$ with equality iff $C \mathrel{\text{--}} D \mathrel{\text{--}} E$.

The maximum gallery distance is $\operatorname{dist}(C, \overline{C})$. It is independent of C and equal to the number of hyperplanes in the arrangement.

For chambers C and D, and H any face of D,

$$\mathrm{dist}(C, D) = \mathrm{dist}(C, HC) + \mathrm{dist}(HC, D). \tag{1.22}$$

This is a reformulation of the gate property (1.18).

For faces A and B of the same support, and chambers $C, D \geq A$,

$$\mathrm{dist}(C, D) = \mathrm{dist}(BC, BD). \tag{1.23}$$

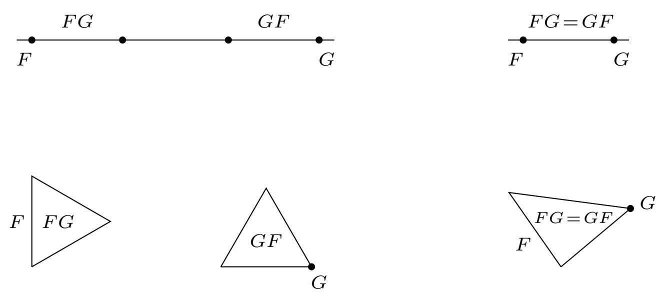
In other words, gallery distance is compatible with the bijection between top-stars given in Lemma 1.6.

1.4.3. Distance between faces. More generally: For any faces F and G, define $\mathrm{dist}(F, G)$ to be the number of hyperplanes which separate F and G. We have:

$$\mathrm{dist}(F, G) = \mathrm{dist}(FG, GF). \tag{1.24}$$

$$\mathrm{dist}(F, G) = 0 \iff FG = GF. \tag{1.25}$$

Illustrations in ranks two and three are shown below.



If F and G have the same support, and $F \leq C$, then

$$\mathrm{dist}(F, G) = \mathrm{dist}(C, GC). \tag{1.26}$$

Warning. Faces F and G with the same support, say X, correspond to chambers of $\mathcal{A}^{\mathrm{X}}$. However, $\mathrm{dist}(F, G)$ is in general larger than the gallery distance between F and G in $\mathcal{A}^{\mathrm{X}}$ (since there is more room to move in $\mathcal{A}$ than in $\mathcal{A}^{\mathrm{X}}$).

1.4.4. *q*-distance function on faces. For any scalar q, define a function υ_q on pairs of faces by

$$(\upsilon_q)_{F,G} = q^{\mathrm{dist}(F,G)}. \tag{1.27}$$

We call this the *q-distance function on faces*.

For $q = 1$, the function υ_1 is identically 1. For $q = -1$,

$$(\upsilon_{-1})_{F,G} = (-1)^{\mathrm{dist}(F,G)}. \tag{1.28}$$

We call this the *signed distance function on faces*. For $q = 0$,

$$(\upsilon_0)_{F,G} = \begin{cases} 1 & \text{if } FG = GF, \\ 0 & \text{otherwise.} \end{cases} \tag{1.29}$$

This is the 0-*distance function on faces*. In particular,

$$(\upsilon_0)_{C,D} = \begin{cases} 1 & \text{if } C = D, \\ 0 & \text{otherwise.} \end{cases}$$

The function υ_q satisfies the following properties, see [**21**, Propositions 8.3 and 8.4]. (We drop q from the notation for convenience.)

For any face F,

$$\upsilon_{F,F} = 1. \tag{1.30a}$$

If F and G have the same support, and $F \leq H$, then

$$\upsilon_{F,G} = \upsilon_{H,GH}. \tag{1.30b}$$

If F, G, H are faces with the same support and $F \text{ -- } G \text{ -- } H$, then

$$\upsilon_{H,G}\upsilon_{G,F} = \upsilon_{H,F}. \tag{1.30c}$$

For any faces F, H, K with $H \leq K$,

$$\upsilon_{K,F} = \upsilon_{K,HF}\upsilon_{HF,F}. \tag{1.30d}$$

For any faces F, H, K with $H \leq F$,

$$\upsilon_{K,F} = \upsilon_{K,HK}\upsilon_{HK,F}. \tag{1.30e}$$

If F and G have the same support, $F \leq H$ and $F \leq K$, then

$$\upsilon_{H,K} = \upsilon_{GH,GK}. \tag{1.30f}$$

Properties (1.30d) and (1.30e) can be deduced from the gate property (1.22).

For $q = \pm 1$, (1.30c) holds without the minimal gallery restriction. For $q = 1$, this is a triviality. For $q = -1$, it says: For faces F, G, H with the same support,

$$(\upsilon_{-1})_{H,G}(\upsilon_{-1})_{G,F} = (\upsilon_{-1})_{H,F}. \tag{1.31}$$

1.4.5. Abstract distance functions. A *distance function* on $\mathcal{A}$ is a function υ on bifaces (pairs of faces with the same support) which satisfies properties (1.30a), (1.30b), (1.30c), (1.30d), (1.30e). We mention that (1.30c) can be deduced from the remaining axioms, so it can removed from the definition. A distance function is completely determined by its values $\upsilon_{C,D}$ on chambers. This follows from (1.30b).

The motivating example of a distance function is the q-distance function. More generally, one can assign an arbitrary number (weight) $\mathrm{wt}(\mathrm{h})$ to each half-space h of $\mathcal{A}$, and define a distance function υ by setting

$$\upsilon_{C,D} := \prod \mathrm{wt}(\mathrm{h}), \tag{1.32}$$

where the product is over all half-spaces h which contain C but do not contain D. Setting all weights equal to q recovers the q-distance function.

A distance function is *log-antisymmetric* if (1.30c) holds without the minimal gallery restriction. In particular, the signed distance function is log-antisymmetric.

A *left distance function* is a function υ on bifaces which satisfies (1.30a), (1.30b), (1.30e). Similarly, a *right distance function* is a function on bifaces which satisfies (1.30a), (1.30b), (1.30d).

1.4.6. Abstract distance functions on LRBs. The above notion of (left, right) distance function makes sense for any LRB.

Now consider the monoid of partial-flats $\Sigma_{\sim}[\mathcal{A}]$, where $\sim$ is a partial-support relation on faces. Recall that it is a LRB. Define

$$\upsilon_{\mathrm{x},\mathrm{y}} := \begin{cases} 1 & \text{if x and y have an upper bound,} \\ 0 & \text{otherwise.} \end{cases} \tag{1.33}$$

This is a distance function on $\Sigma_{\sim}[\mathcal{A}]$. If $\sim$ is geometric, then by Lemma 1.8, this simplifies to

$$\upsilon_{\mathrm{x},\mathrm{y}} := \begin{cases} 1 & \text{if } \mathrm{x} = \mathrm{y}, \\ 0 & \text{otherwise.} \end{cases}$$

1.4.7. Varchenko matrices. Fix a flat X. For any scalar q, consider the matrix A indexed by faces of support X, whose entry in row F and column G is $q^{\mathrm{dist}(F,G)}$. We write it as

$$A = (q^{\mathrm{dist}(F,G)})_{\mathrm{s}(F)=\mathrm{s}(G)=\mathrm{X}}. \tag{1.34}$$

We call this the *Varchenko matrix* for the flat X. Note that the diagonal entries of A are all 1. When X is the maximum flat, we have $A = (q^{\mathrm{dist}(C,D)})$, as C and D vary over all chambers.

Theorem 1.10. *The determinant of the Varchenko matrix* (1.34) *factorizes as a product of terms of the form* $(1-q^n)$ *for varying* n. *In particular, the determinant of* (1.34) *is nonzero whenever* q *is not a root of unity.*

Proof. This is a nontrivial result, see for instance [**21**, Formulas (8.41) and (8.42)]. ☐

By Theorem 1.10, the Varchenko matrix (1.34) is invertible when q is not a root of unity. In this case, we write the inverse as

$$A^{-1} = (q^{F,G})_{\mathrm{s}(F)=\mathrm{s}(G)=\mathrm{X}}.$$

Explicitly, the scalars $q^{F,G}$ satisfy

$$\sum_{G:\,\mathrm{s}(G)=\mathrm{X}} q^{F,G} q^{\mathrm{dist}(G,K)} = \begin{cases} 1 & \text{if } F = K, \\ 0 & \text{if } F \neq K. \end{cases} \tag{1.35}$$

By Cramer's rule, each $q^{F,G}$ is a rational function in q with the denominator factorizing as a product of terms of the form $(1-q^n)$.

A *path* α in X is a finite sequence of faces $F_0 - F_1 - \cdots - F_n$ all of support X. We say that α is a path of length n from F_0 to F_n, and write $s(\alpha) = F_0$, $t(\alpha) = F_n$ and $l(\alpha) = n$. The path α is *non-stuttering* if any two consecutive faces in its sequence are distinct, that is, $F_i \neq F_{i+1}$ for all i. Note that a face by itself is a non-stuttering path of length 0.

For a non-stuttering path $\alpha = F_0 - \cdots - F_n$, define

$$q^\alpha := \prod_{i=1}^{n} q^{\mathrm{dist}(F_{i-1},F_i)} = q^{\sum_{i=1}^{n} \mathrm{dist}(F_{i-1},F_i)}.$$

Then observe that

$$q^{F,G} = \sum_{\alpha:\, s(\alpha)=F,\, t(\alpha)=G} (-1)^{l(\alpha)} q^\alpha. \tag{1.36}$$

The sum is over all non-stuttering paths α in X from F to G. The rhs makes sense because there are only finitely many non-stuttering paths which contribute to a given power of q.

1.5. Incidence algebras, and zeta and Möbius functions

We review the flat-incidence algebra, the lune-incidence algebra and the base-case map which relates the two. The flat-incidence algebra is the incidence algebra of the poset of flats. The lune-incidence algebra is the incidence algebra of the category of lunes. It is also a subalgebra of the incidence algebra of the poset of faces.

The flat-incidence algebra contains a unique zeta function ζ and Möbius function μ. In contrast, the lune-incidence algebra contains families of noncommutative zeta functions $\boldsymbol{\zeta}$ and noncommutative Möbius functions $\boldsymbol{\mu}$ which bear an inverse relationship to each other. They are characterized by the lune-additivity formula and the noncommutative Weisner formula, respectively. The base-case of any $\boldsymbol{\zeta}$ is ζ, and of any $\boldsymbol{\mu}$ is μ. We also introduce the noncommutative q-zeta function $\boldsymbol{\zeta}_q$ and noncommutative q-Möbius function $\boldsymbol{\mu}_q$ when q is not a root of unity. Applying base-case defines the q-zeta function ζ_q and q-Möbius function μ_q. The case $q = 0$ is of interest.

1.5.1. Flat-incidence algebra. Let $\mathcal{A}$ be an arrangement. A *nested flat* is a pair of flats (X, Y) such that $\mathrm{X} \leq \mathrm{Y}$. The *flat-incidence algebra*, denoted $\mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$, is the incidence algebra of the poset of flats. It consists of functions s on nested flats, with the product of s and t given by

$$(st)(\mathrm{X}, \mathrm{Z}) = \sum_{\mathrm{Y}:\, \mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}} s(\mathrm{X}, \mathrm{Y}) t(\mathrm{Y}, \mathrm{Z}). \tag{1.37}$$

1.5.2. Zeta and Möbius functions. The *zeta function* $\zeta \in \mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$ is defined by $\zeta(\mathrm{X}, \mathrm{Y}) = 1$ for all $\mathrm{X} \leq \mathrm{Y}$. It is invertible in the flat-incidence algebra and its inverse is the *Möbius function* $\mu \in \mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$. The latter is the unique element such that $\mu(\mathrm{X}, \mathrm{X}) = 1$ for all X and

$$\sum_{\mathrm{W}:\, \mathrm{W} \geq \mathrm{X},\, \mathrm{W} \vee \mathrm{Y} = \mathrm{Z}} \mu(\mathrm{X}, \mathrm{W}) = 0 \tag{1.38}$$

for all $\mathrm{X} < \mathrm{Y} \leq \mathrm{Z}$. This is the *Weisner formula*. Note very carefully that Y is strictly greater than X in this formula.

1.5.3. Face-incidence algebra and lune-incidence algebra. The *face-incidence algebra*, denoted $\mathrm{I}_{\mathrm{face}}[\mathcal{A}]$, is the incidence algebra of the poset of faces. It consists of functions s on nested faces, with the product of s and t given by

$$(st)(F,H) = \sum_{G\colon F \leq G \leq H} s(F,G)t(G,H). \tag{1.39}$$

The *lune-incidence algebra*, denoted $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$, is the subalgebra of $\mathrm{I}_{\mathrm{face}}[\mathcal{A}]$ consisting of those functions s such that

$$s(A,F) = s(B,G) \text{ whenever } (A,F) \sim (B,G), \tag{1.40}$$

with $\sim$ as in (1.11). It has a basis indexed by lunes.

1.5.4. Category of lunes. Recall the category of lunes from Section 1.2.3.

Proposition 1.11. *The lune-incidence algebra is the incidence algebra of the category of lunes. Explicitly, it consists of functions s on lunes, with the product of s and t given by*

$$(st)(\mathrm{N}) = \sum_{(\mathrm{L},\mathrm{M})\colon \mathrm{L}\circ\mathrm{M}=\mathrm{N}} s(\mathrm{L})t(\mathrm{M}). \tag{1.41}$$

The sum is over both L *and* M. *The unit element is the function which is* 1 *on lunes which are flats, and* 0 *otherwise.*

1.5.5. Projective lune-incidence algebra. For any $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$, define $\overline{s} \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ by $\overline{s}(A,F) := s(A,A\overline{F})$, or equivalently, by $\overline{s}(\mathrm{L}) := s(\overline{\mathrm{L}})$, see (1.12). We refer to $\overline{s}$ as the opposite of s. We say s is projective if $s = \overline{s}$. The map $s \mapsto \overline{s}$ is an involution of the lune-incidence algebra. Hence, projective functions in the lune-incidence algebra form a subalgebra. We call this the *projective lune-incidence algebra*.

1.5.6. Noncommutative zeta and Möbius functions. A *noncommutative zeta function* is an element $\boldsymbol{\zeta} \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ such that $\boldsymbol{\zeta}(A,A) = 1$ for all A and

$$\boldsymbol{\zeta}(H,G) = \sum_{\substack{F\colon F \geq A,\, HF=G \\ \mathrm{s}(F)=\mathrm{s}(G)}} \boldsymbol{\zeta}(A,F) \tag{1.42}$$

for all $A \leq H \leq G$. By Exercise 1.2, the condition $\mathrm{s}(F) = \mathrm{s}(G)$ in the above sum can be replaced by the condition $FH = F$. We refer to (1.42) as the *lune-additivity formula*.

In particular: For any flat X containing a face A,

$$\sum_{F\colon F \geq A,\, \mathrm{s}(F)=\mathrm{X}} \boldsymbol{\zeta}(A,F) = 1. \tag{1.43}$$

This arises by setting $H = G$ in (1.42), and letting X be the support of G. We refer to (1.43) as the *flat-additivity formula*.

Lemma 1.12. *A noncommutative zeta function $\boldsymbol{\zeta}$ is equivalent to a choice of scalars $\boldsymbol{\zeta}(O,F)$, one for each face F, such that for each flat X,*

$$\sum_{F:\, \mathrm{s}(F)=\mathrm{X}} \boldsymbol{\zeta}(O,F) = 1.$$

Proof. See [**21**, Lemma 15.18]. □

A *noncommutative Möbius function* is an element $\boldsymbol{\mu} \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ such that $\boldsymbol{\mu}(A,A) = 1$ for all A and

$$\sum_{F:\, F \geq A,\, HF=G} \boldsymbol{\mu}(A,F) = 0 \tag{1.44}$$

for all $A < H \leq G$. We refer to (1.44) as the *noncommutative Weisner formula*. Note very carefully that H is strictly greater than A in this formula.

Lemma 1.13. *A noncommutative Möbius function $\boldsymbol{\mu}$ is equivalent to a family of special Zie elements, one in each $\mathcal{A}_{\mathrm{X}}$, as X varies over all flats.*

Special Zie elements are defined in Section 1.12.5.

Proof. See [**21**, Lemma 15.24]. See also Exercise 9.120. □

Theorem 1.14. *In the lune-incidence algebra, the inverse of a noncommutative zeta function is a noncommutative Möbius function, and vice versa.*

Proof. This is a nontrivial result which was obtained in [**21**, Theorem 15.28]. This result is restated later in the text as Theorem 14.89 and a proof is given using methods developed in this monograph. □

Example 1.15. For a rank-one arrangement with chambers C and $\overline{C}$, noncommutative zeta and Möbius functions are characterized by a scalar p as follows.

$$\boldsymbol{\zeta}(O,O) = \boldsymbol{\zeta}(C,C) = \boldsymbol{\zeta}(\overline{C},\overline{C}) = 1, \quad \boldsymbol{\zeta}(O,C) = p, \quad \boldsymbol{\zeta}(O,\overline{C}) = 1-p,$$
$$\boldsymbol{\mu}(O,O) = \boldsymbol{\mu}(C,C) = \boldsymbol{\mu}(\overline{C},\overline{C}) = 1, \quad \boldsymbol{\mu}(O,C) = -p, \quad \boldsymbol{\mu}(O,\overline{C}) = p-1.$$

Exercise 1.16. Check that the involution $s \mapsto \overline{s}$ of the lune-incidence algebra preserves noncommutative zeta and Möbius functions. Deduce that if $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ are inverse, then so are $\overline{\boldsymbol{\zeta}}$ and $\overline{\boldsymbol{\mu}}$. As a consequence, the inverse of a projective noncommutative zeta function is a projective noncommutative Möbius function, and vice versa.

1.5.7. Set-theoretic and uniform noncommutative zeta functions. A noncommutative zeta function $\boldsymbol{\zeta}$ is called

- *set-theoretic* if the scalars $\boldsymbol{\zeta}(O,F)$ are either 0 or 1,
- *uniform* if $\boldsymbol{\zeta}(O,F) = \boldsymbol{\zeta}(O,G)$ whenever F and G have the same support.

In characteristic zero, the uniform noncommutative zeta function exists and is unique. We denote it by $\boldsymbol{\zeta}_u$, and its inverse by $\boldsymbol{\mu}_u$. Note that they are projective.

1.5.8. Base-case map. There is an algebra morphism

$$\mathrm{bc} : \mathrm{I}_{\mathrm{lune}}[\mathcal{A}] \to \mathrm{I}_{\mathrm{flat}}[\mathcal{A}] \tag{1.45}$$

defined by either of the two equivalent formulas

$$\mathrm{bc}(t)(\mathrm{X},\mathrm{Y}) := \sum_{F:\, F \geq A,\, \mathrm{s}(F)=\mathrm{Y}} t(A,F),$$

$$\mathrm{bc}(t)(\mathrm{X},\mathrm{Y}) := \sum_{\mathrm{L}:\, \mathrm{b}(\mathrm{L})=\mathrm{X},\, \mathrm{c}(\mathrm{L})=\mathrm{Y}} t(\mathrm{L}).$$

In the first sum, A is a fixed face of support X. The second sum is over all lunes L whose base is X and case is Y. We call bc the *base-case map*.

Lemma 1.17. *The base-case of a noncommutative zeta function* $\boldsymbol{\zeta} \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ *is the zeta function* $\zeta \in \mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$. *The base-case of a noncommutative Möbius function* $\boldsymbol{\mu} \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ *is the Möbius function* $\mu \in \mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$.

PROOF. The first statement follows from the flat-additivity formula (1.43). For the second statement, one checks that $\mathrm{bc}(\boldsymbol{\mu})$ satisfies the Weisner formula (1.38). For more details, see [**21**, Lemmas 15.17 and 15.23]. Also note that since bc is an algebra morphism, it preserves inverses, so the two statements imply each other in view of Theorem 1.14. □

Remark 1.18. The lune-incidence algebra and noncommutative zeta and Möbius functions therein can be defined for any LRB by using the product of the LRB in place of the Tits product. Theorem 1.14 continues to hold in this generality. Similarly, the flat-incidence algebra with its unique zeta and Möbius functions as well as the base-case map can all be constructed from the LRB by making use of its support lattice.

Now consider the special case when the LRB equals its support lattice. In this situation, the lune-incidence algebra and flat-incidence algebra coincide, and the base-case map is the identity. This happens, for instance, when the LRB is the Birkhoff monoid.

1.5.9. Noncommutative q-zeta and q-Möbius functions. Let q be any scalar which is not a root of unity. Let $\boldsymbol{\zeta}_q$ be an element of the lune-incidence algebra $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ such that $\boldsymbol{\zeta}_q(A,A) = 1$ for all A, and

$$\boldsymbol{\zeta}_q(H,G) = \sum_{\substack{F:\, F \geq A,\, HF=G \\ \mathrm{s}(F)=\mathrm{s}(G)}} \boldsymbol{\zeta}_q(A,F)\, q^{\mathrm{dist}(F,G)} \tag{1.46}$$

for all $A \leq H \leq G$. By Exercise 1.2, the condition $\mathrm{s}(F) = \mathrm{s}(G)$ in the above sum can be replaced by the condition $FH = F$. We refer to (1.46) as the *q-lune-additivity formula*.

In particular:

$$\sum_{\substack{F:\, F \geq A \\ \mathrm{s}(F)=\mathrm{s}(G)}} \boldsymbol{\zeta}_q(A,F)\, q^{\mathrm{dist}(F,G)} = 1 \tag{1.47}$$

for all $A \le G$. This arises by setting $H = G$ in (1.46). We refer to (1.47) as the *q-flat-additivity formula*.

Similarly, let $\boldsymbol{\mu}_q$ be an element of the lune-incidence algebra $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ such that $\boldsymbol{\mu}_q(A, A) = 1$ for all A, and

$$\sum_{F\colon F \ge A,\, HF = G} \boldsymbol{\mu}_q(A, F)\, q^{\mathrm{dist}(F,G)} = 0 \tag{1.48}$$

for all $A < H \le G$. We refer to (1.48) as the *noncommutative q-Weisner formula*.

Lemma 1.19. *In the lune-incidence algebra, the inverse of a $\boldsymbol{\zeta}_q$ is a $\boldsymbol{\mu}_q$, and vice versa.*

Proof. A proof is indicated later in the text in Exercise 14.93. ☐

A more precise result is given below.

Theorem 1.20. *The elements $\boldsymbol{\zeta}_q$ and $\boldsymbol{\mu}_q$ exist and are unique. Further, they are inverses of each other in the lune-incidence algebra.*

Proof. Existence and uniqueness of $\boldsymbol{\mu}_q$ follows from [**21**, Theorem 8.25] (applied to the star of A). The remaining claims follow from Lemma 1.19. ☐

We call $\boldsymbol{\zeta}_q$ the *noncommutative q-zeta function* and $\boldsymbol{\mu}_q$ the *noncommutative q-Möbius function*.

Example 1.21. For a rank-one arrangement with chambers C and $\overline{C}$,

$$\boldsymbol{\zeta}_q(O, C) = \boldsymbol{\zeta}_q(O, \overline{C}) = \frac{1}{1+q} \quad \text{and} \quad \boldsymbol{\mu}_q(O, C) = \boldsymbol{\mu}_q(O, \overline{C}) = \frac{-1}{1+q}.$$

For the rank-two arrangement of d lines in the plane, for a vertex F and chamber C,

$$\boldsymbol{\zeta}_q(O, F) = \frac{1}{1+q^{d-1}}, \quad \boldsymbol{\zeta}_q(F, C) = \frac{1}{1+q}, \quad \boldsymbol{\zeta}_q(O, C) = \frac{1-q}{(1-q^d)(1+q)},$$

$$\boldsymbol{\mu}_q(O, F) = \frac{-1}{1+q^{d-1}}, \quad \boldsymbol{\mu}_q(F, C) = \frac{-1}{1+q}, \quad \boldsymbol{\mu}_q(O, C) = \frac{1-q^{d-1}}{(1-q^d)(1+q^{d-1})}.$$

Remark 1.22. Setting $q = 1$ in (1.48) yields the noncommutative Weisner formula (1.44). The solution is no longer unique. The solutions are precisely noncommutative Möbius functions. Similarly, setting $q = 1$ in (1.46) yields the lune-additivity formula (1.42) whose solutions are noncommutative zeta functions.

Exercise 1.23. Check that: For arrangements $\mathcal{A}$ and $\mathcal{A}'$,

$$\mathrm{I}_{\mathrm{lune}}[\mathcal{A} \times \mathcal{A}'] = \mathrm{I}_{\mathrm{lune}}[\mathcal{A}] \otimes \mathrm{I}_{\mathrm{lune}}[\mathcal{A}']. \tag{1.49}$$

Under this identification, for q not a root of unity, $\boldsymbol{\zeta}_q \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A} \times \mathcal{A}']$ equals $\boldsymbol{\zeta}_q \otimes \boldsymbol{\zeta}_q \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}] \otimes \mathrm{I}_{\mathrm{lune}}[\mathcal{A}']$, and similarly, $\boldsymbol{\mu}_q \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A} \times \mathcal{A}']$ equals $\boldsymbol{\mu}_q \otimes \boldsymbol{\mu}_q \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}] \otimes \mathrm{I}_{\mathrm{lune}}[\mathcal{A}']$.

1.5.10. q-zeta and q-Möbius functions. Let q be any scalar which is not a root of unity. Define mutually inverse elements ζ_q and μ_q of the flat-incidence algebra by

$$\zeta_q := \mathrm{bc}(\boldsymbol{\zeta}_q) \quad \text{and} \quad \mu_q := \mathrm{bc}(\boldsymbol{\mu}_q), \tag{1.50}$$

where bc is the base-case map (1.45). We call ζ_q the *q-zeta function* and μ_q the *q-Möbius function*.

Example 1.24. For a rank-one arrangement,

$$\zeta_q(\bot, \top) = \frac{2}{1+q} \quad \text{and} \quad \mu_q(\bot, \top) = \frac{-2}{1+q}.$$

For the rank-two arrangement of d lines in the plane, for a rank-one flat X,

$$\zeta_q(\bot, \mathrm{X}) = \frac{2}{1+q^{d-1}}, \quad \zeta_q(\mathrm{X}, \top) = \frac{2}{1+q}, \quad \zeta_q(\bot, \top) = \frac{2d(1-q)}{(1-q^d)(1+q)},$$
$$\mu_q(\bot, \mathrm{X}) = \frac{-2}{1+q^{d-1}}, \quad \mu_q(\mathrm{X}, \top) = \frac{-2}{1+q}, \quad \mu_q(\bot, \top) = \frac{2d(1-q^{d-1})}{(1-q^d)(1+q^{d-1})}.$$

Exercise 1.25. Establish the analogue of Exercise 1.23 for ζ_q and μ_q instead of $\boldsymbol{\zeta}_q$ and $\boldsymbol{\mu}_q$.

1.5.11. Noncommutative 0-zeta and 0-Möbius functions. Let us now specialize Section 1.5.9 to $q = 0$. We call $\boldsymbol{\zeta}_0$ the *noncommutative* 0-*zeta function* and $\boldsymbol{\mu}_0$ the *noncommutative* 0-*Möbius function*. They have simple formulas, namely,

$$\boldsymbol{\zeta}_0(A, F) = 1 \quad \text{and} \quad \boldsymbol{\mu}_0(A, F) = (-1)^{\mathrm{rk}(F/A)}. \tag{1.51}$$

The fact that the above formula for $\boldsymbol{\zeta}_0$ satisfies (1.46) for $q = 0$ is the tautology $1 = 1$. The fact that the above formula for $\boldsymbol{\mu}_0$ satisfies (1.48) for $q = 0$ is equivalent to [**21**, Proposition 1.77].

Observe that $\boldsymbol{\zeta}_0$ and $\boldsymbol{\mu}_0$ are also the zeta and Möbius functions, respectively, of the poset of faces. The latter is the same as saying that the poset of faces is eulerian. See for instance, [**21**, Formula (1.43)].

Specializing (1.50) to $q = 0$ and using formulas (1.51), we obtain

$$\zeta_0(\mathrm{X}, \mathrm{Y}) = c_{\mathrm{X}}^{\mathrm{Y}} \quad \text{and} \quad \mu_0(\mathrm{X}, \mathrm{Y}) = (-1)^{\mathrm{rk}(\mathrm{Y}/\mathrm{X})}\, c_{\mathrm{X}}^{\mathrm{Y}}, \tag{1.52}$$

where $c_{\mathrm{X}}^{\mathrm{Y}}$ denotes the number of chambers in $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$ (see Notation 1.40). We call ζ_0 the 0-*zeta function* and μ_0 the 0-*Möbius function*.

1.5.12. Elements in the lune-incidence algebra. For any scalar α, define $r_\alpha, h_\alpha \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ by

$$r_\alpha(A, F) := \begin{cases} \alpha^{\mathrm{rk}(A)} & \text{if } F = A, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad h_\alpha(A, F) := \alpha^{\mathrm{rk}(F/A)}. \tag{1.53}$$

Some simple observations are listed below.

- $r_\alpha r_\beta = r_{\alpha\beta}$ for any α, β. Also r_1 is the identity. Thus, for $\alpha \neq 0$, r_α is invertible with inverse $r_{\alpha^{-1}}$. In particular, r_{-1} is an involution.
- $r_{-1} h_\alpha r_{-1} = h_{-\alpha}$ for any α.

- $h_\alpha h_\beta = h_{\alpha+\beta}$ for any α, β, provided $\mathcal{A}$ is simplicial. (This follows from the binomial theorem.) Also h_0 is the identity, and $h_1 = \boldsymbol{\zeta}_0$ and $h_{-1} = \boldsymbol{\mu}_0$, the noncommutative 0-zeta and 0-Möbius functions (Section 1.5.11).

Fix a pair of mutually inverse noncommutative zeta and Möbius functions $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$. For any scalar α, define $t_\alpha \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ by

$$t_\alpha := \boldsymbol{\zeta} r_\alpha \boldsymbol{\mu}. \tag{1.54}$$

We deduce that $t_\alpha t_\beta = t_{\alpha\beta}$ for any α, β. Also t_1 is the identity. Thus, for $\alpha \neq 0$, t_α is invertible with inverse $t_{\alpha^{-1}}$. In particular, t_{-1} is an involution.

1.6. Bilune-incidence algebra

We introduce the bilune-incidence algebra. It is the incidence algebra of the category of bilunes. There are algebra morphisms linking it to the lune-incidence algebra.

For q not a root of unity, we introduce the two-sided q-zeta function and two-sided q-Möbius function. These are elements of the bilune-incidence algebra, and satisfy identities which we call the two-sided q-lune-additivity formula and two-sided q-Weisner formula.

1.6.1. Local bifaces and bilunes. A *local biface* is a triple (A, F, F') such that $F, F' \geq A$ and $\mathrm{s}(F) = \mathrm{s}(F')$.

Define an equivalence relation on the set of local bifaces by

$$\begin{aligned}(A, F, F') \sim (B, G, G') \iff\ & AB = A,\ BA = B,\\ & AG = F,\ BF = G,\ AG' = F',\ BF' = G'.\end{aligned} \tag{1.55}$$

In other words, A and B have the same suport, and F and F' correspond to G and G', respectively, under the bijection between $\Sigma[\mathcal{A}]_A$ and $\Sigma[\mathcal{A}]_B$ in Lemma 1.6. This is illustrated below.

A *bilune* is a pair of lunes with the same base-case, that is, a pair $(\mathrm{L}, \mathrm{L}')$ such that $\mathrm{b}(\mathrm{L}) = \mathrm{b}(\mathrm{L}')$ and $\mathrm{c}(\mathrm{L}) = \mathrm{c}(\mathrm{L}')$.

Lemma 1.26. *Equivalence classes for the relation* (1.55) *are in one-to-one correspondence with bilunes. The class of* (A, F, F') *is* $(\mathrm{s}(A, F), \mathrm{s}(A, F'))$.

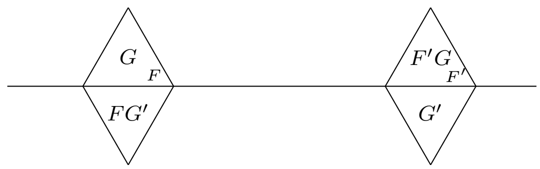
Exercise 1.27. Recall h-faces and h-flats from Section 1.2.4. Let h be the left $\Sigma[\mathcal{A}]$-set consisting of bifaces with action $A \cdot (F, F') = (AF, AF')$. Check that an h-face is the same as a local biface, and an h-flat is the same as a bilune.

1.6.2. Biface-incidence algebra. Let $\mathrm{I}_{\mathrm{biface}}[\mathcal{A}]$ denote the algebra consisting of functions s on local bifaces, with the product of s and t defined by

$$(st)(A,G,G') := \sum_{\substack{(F,F')\colon \mathrm{s}(F)=\mathrm{s}(F') \\ A\leq F\leq G \\ A\leq F'\leq G'}} s(A,F,F')t(F,G,FG'). \tag{1.56}$$

One may check that this product is associative. The unit element is the function which is 1 on local bifaces of the form (A,A,A) and 0 otherwise.

The two extreme choices for F, F' in the sum in (1.56) are $F = F' := A$, and $F := G$, $F' := G'$. Other choices lie somewhere in between. An illustration is shown below, with A as the central face.



We refer to $\mathrm{I}_{\mathrm{biface}}[\mathcal{A}]$ as the *biface-incidence algebra*.

Remark 1.28. We mention that there is another associative product on functions on local bifaces, namely,

$$(st)(A,G,G') := \sum_{\substack{(F,F')\colon \mathrm{s}(F)=\mathrm{s}(F') \\ A\leq F\leq G \\ A\leq F'\leq G'}} s(A,F,F')t(F',F'G,G'). \tag{1.57}$$

It is a canonical companion to (1.56). More precisely, the map $s \mapsto s'$, where $s'(A,F',F) := s(A,F,F')$ defines an isomorphism between the two products.

Exercise 1.29. For any scalar q, check that the map $s \mapsto s_q$, where

$$s_q(A,F,F') := q^{\mathrm{dist}(F,F')}\, s(A,F,F')$$

is an algebra morphism from $\mathrm{I}_{\mathrm{biface}}[\mathcal{A}]$ to itself. Moreover, it is an isomorphism when $q \neq 0$, with inverse $s \mapsto s_{q^{-1}}$.

1.6.3. Bilune-incidence algebra. Let $\mathrm{I}_{\mathrm{bilune}}[\mathcal{A}]$ denote the subspace of $\mathrm{I}_{\mathrm{biface}}[\mathcal{A}]$ consisting of those functions s on local bifaces such that

$$s(A,F,F') = s(B,G,G') \text{ whenever } (A,F,F') \sim (B,G,G'), \tag{1.58}$$

with $\sim$ as in (1.55). In fact:

Lemma 1.30. *The subspace* $\mathrm{I}_{\mathrm{bilune}}[\mathcal{A}]$ *is a subalgebra of* $\mathrm{I}_{\mathrm{biface}}[\mathcal{A}]$. *It has a basis indexed by bilunes.*

Proof. This is straightforward. □

We refer to $\mathrm{I}_{\mathrm{bilune}}[\mathcal{A}]$ as the *bilune-incidence algebra*.

Exercise 1.31. Check that the bilune-incidence algebra can also be obtained as the restriction of the product (1.57).

1.6.4. Category of bilunes. We define the *category of bilunes* of $\mathcal{A}$. Its objects are flats and morphisms are bilunes. More precisely, a morphism from Y to X is a bilune $(\mathrm{L}, \mathrm{L}')$ such that the base of both L and L′ is X and the case of both L and L′ is Y. Composition is done coordinatewise, that is,

$$(\mathrm{L}, \mathrm{L}') \circ (\mathrm{M}, \mathrm{M}') := (\mathrm{L} \circ \mathrm{M}, \mathrm{L}' \circ \mathrm{M}').$$

Identity morphisms are bilunes of the form (X, X).

Proposition 1.32. *The bilune-incidence algebra is the incidence algebra of the category of bilunes. Explicitly, it consists of functions s on bilunes, with the product of s and t given by*

$$(st)(\mathrm{N}, \mathrm{N}') = \sum_{(\mathrm{L},\mathrm{L}')\circ(\mathrm{M},\mathrm{M}')=(\mathrm{N},\mathrm{N}')} s(\mathrm{L}, \mathrm{L}')t(\mathrm{M}, \mathrm{M}'). \tag{1.59}$$

The sum is over both $(\mathrm{L}, \mathrm{L}')$ *and* $(\mathrm{M}, \mathrm{M}')$*. The unit element is the function which is* 1 *on bilunes of the form* (X, X)*, and* 0 *otherwise.*

The map $s \mapsto s'$ in Remark 1.28 induces an isomorphism on the bilune-incidence algebra. Explicitly, $s'(\mathrm{L}', \mathrm{L}) := s(\mathrm{L}, \mathrm{L}')$. We say a function s on bilunes is *symmetric* if $s(\mathrm{L}, \mathrm{L}') = s(\mathrm{L}', \mathrm{L})$, or equivalently, if $s(A, F, F') = s(A, F', F)$. The space of such functions is a subalgebra of the bilune-incidence algebra. We call it the *symmetric bilune-incidence algebra*.

1.6.5. Connection to the lune-incidence algebra. There are two algebra morphisms

$$p, q : \mathrm{I}_{\mathrm{bilune}}[\mathcal{A}] \to \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$$

defined by

$$\begin{aligned} p(s)(A, F) &:= \sum_{\substack{F' :\, F' \geq A \\ \mathrm{s}(F)=\mathrm{s}(F')}} s(A, F, F'), \\ q(s)(A, F') &:= \sum_{\substack{F :\, F \geq A \\ \mathrm{s}(F)=\mathrm{s}(F')}} s(A, F, F'), \end{aligned} \tag{1.60}$$

or equivalently,

$$\begin{aligned} p(s)(\mathrm{L}) &:= \sum_{\mathrm{L}' :\, \mathrm{bc}(\mathrm{L})=\mathrm{bc}(\mathrm{L}')} s(\mathrm{L}, \mathrm{L}'), \\ q(s)(\mathrm{L}') &:= \sum_{\mathrm{L} :\, \mathrm{bc}(\mathrm{L})=\mathrm{bc}(\mathrm{L}')} s(\mathrm{L}, \mathrm{L}'). \end{aligned} \tag{1.61}$$

There is also an algebra morphism

$$i : \mathrm{I}_{\mathrm{lune}}[\mathcal{A}] \to \mathrm{I}_{\mathrm{bilune}}[\mathcal{A}]$$

defined by

$$i(s)(A, F, F') := \begin{cases} s(A, F) & \text{if } F = F', \\ 0 & \text{otherwise,} \end{cases} \tag{1.62}$$

or equivalently,

$$i(s)(\mathrm{L},\mathrm{L}') := \begin{cases} s(\mathrm{L}) & \text{if } \mathrm{L}=\mathrm{L}', \\ 0 & \text{otherwise.} \end{cases} \tag{1.63}$$

The map i is a section to both p and q. Also note that the maps p and q coincide on the symmetric bilune-incidence algebra.

1.6.6. Two-sided q-zeta and q-Möbius functions. Let q be any scalar which is not a root of unity. Define an element ζ_q of the bilune-incidence algebra by

$$\zeta_q(A,F,F') := q^{F/A,F'/A}. \tag{1.64}$$

The latter are the scalars defined in (1.35) for the arrangement $\mathcal{A}_A$. By definition, for $A \le G, G'$ with $\mathrm{s}(G)=\mathrm{s}(G')$,

$$\sum_{\substack{F:\, F\geq A\\ \mathrm{s}(F)=\mathrm{s}(G)=\mathrm{s}(G')}} \zeta_q(A,G',F)\, q^{\mathrm{dist}(F,G)} = \begin{cases} 1 & \text{if } G=G', \\ 0 & \text{if } G\neq G'. \end{cases} \tag{1.65}$$

We call ζ_q the *two-sided q-zeta function*. Observe that $\zeta_q(A,F,F') = \zeta_q(A,F',F)$. Hence, ζ_q is symmetric, and belongs to the symmetric bilune-incidence algebra. We refer to (1.65) as the *two-sided q-flat-additivity formula*.

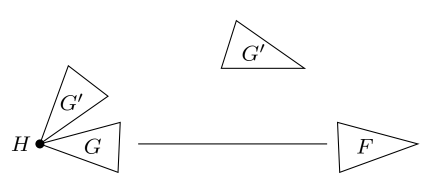
Lemma 1.33. *The element ζ_q is the unique element of the bilune-incidence algebra such that $\zeta_q(A,A,A)=1$ for all A, and*

$$\sum_{\substack{F:\, F\geq A,\, HF=G\\ \mathrm{s}(F)=\mathrm{s}(G)}} \zeta_q(A,F,G')\, q^{\mathrm{dist}(F,G)} = \begin{cases} \zeta_q(H,G,G') & \text{if } G'\geq H, \\ 0 & \text{otherwise,} \end{cases} \tag{1.66}$$

for all $A \le H \le G$ and $A \le G'$ with $\mathrm{s}(G)=\mathrm{s}(G')$.

We refer to (1.66) as the *two-sided q-lune-additivity formula*. In particular, setting $H=G$ in (1.66) recovers (1.65).

An illustration with $A=O$ is shown below. The two triangles labeled G' indicate the two alternatives in (1.66).



Proof. Uniqueness follows since (1.65) has a unique solution, namely, (1.64). We now check (1.66). The required calculation is shown below.

$$\begin{aligned}
\text{lhs} &= \sum_{\substack{F,K,K':\\ F\geq A,\ K,K'\geq H,\ HF=K\\ \mathrm{s}(F)=\mathrm{s}(G)=\mathrm{s}(K)=\mathrm{s}(K')}} \boldsymbol{\zeta}_q(A,F,G')\, q^{\mathrm{dist}(F,K)} q^{\mathrm{dist}(K,K')} \boldsymbol{\zeta}_q(H,G,K') \\
&= \sum_{\substack{F,K':\\ F\geq A,\ K'\geq H\\ \mathrm{s}(F)=\mathrm{s}(G)=\mathrm{s}(K')}} \boldsymbol{\zeta}_q(A,F,G')\, q^{\mathrm{dist}(F,K')} \boldsymbol{\zeta}_q(H,G,K') \\
&= \text{rhs}.
\end{aligned}$$

In the first step, we introduced two new variables K and K'. Summing over K' forces $K = G$ which is the lhs. This explains the first equality. The second step used property (1.30d) and eliminated K. In the last step, we summed over F which forced $K' = G'$. □

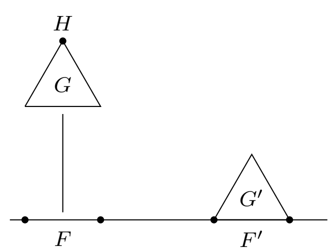
Let $\boldsymbol{\mu}_q$ denote the inverse of $\boldsymbol{\zeta}_q$ in the bilune-incidence algebra. We call $\boldsymbol{\mu}_q$ the *two-sided q-Möbius function*. By general principles [**21**, Lemma D.25, item (2)], it belongs to the symmetric bilune-incidence algebra.

Lemma 1.34. *The element $\boldsymbol{\mu}_q$ is the unique element of the bilune-incidence algebra such that $\boldsymbol{\mu}_q(A,A,A) = 1$ for all A, and*

$$\sum_{\substack{(F,F'):\ F,F'\geq A,\ \mathrm{s}(F)=\mathrm{s}(F')\\ HF=G,\ F'H=G'}} \boldsymbol{\mu}_q(A,F,F')\, q^{\mathrm{dist}(F,G)} = 0 \tag{1.67}$$

for all $A < H \leq G$ and $A < G'$ with $\mathrm{s}(G) = \mathrm{s}(G')$.

We refer to (1.67) as the *two-sided q-Weisner formula*. An illustration with $A = O$ is shown below.



Proof. This follows from Theorem 14.92 which is proved using ideas of this monograph. □

Setting $H = G$ in (1.67) yields:

$$\sum_{\substack{(F,F'):\ F,F'\geq A,\ \mathrm{s}(F)=\mathrm{s}(F')\\ F'G=G'}} \boldsymbol{\mu}_q(A,F,F')\, q^{\mathrm{dist}(F,G)} = 0 \tag{1.68}$$

for all $A < G, G'$ with $\mathrm{s}(G) = \mathrm{s}(G')$.

Exercise 1.35. Check that: With notation as in Exercise 1.29, we have $(\zeta_q)_{-1} = \zeta_{-q}$ and $(\mu_q)_{-1} = \mu_{-q}$. Explicitly,

$$\zeta_{-q}(A, F, F') = (-1)^{\mathrm{dist}(F,F')}\, \zeta_q(A, F, F'),$$
$$\mu_{-q}(A, F, F') = (-1)^{\mathrm{dist}(F,F')}\, \mu_q(A, F, F').$$

Example 1.36. For a rank-one arrangement with chambers C and $\overline{C}$,

$$\zeta_q(O, C, C) = \zeta_q(O, \overline{C}, \overline{C}) = \frac{1}{1-q^2}, \quad \zeta_q(O, C, \overline{C}) = \zeta_q(O, \overline{C}, C) = \frac{-q}{1-q^2},$$
$$\mu_q(O, C, C) = \mu_q(O, \overline{C}, \overline{C}) = \frac{-1}{1-q^2}, \quad \mu_q(O, C, \overline{C}) = \mu_q(O, \overline{C}, C) = \frac{q}{1-q^2}.$$

Recall that we have also defined elements $\boldsymbol{\zeta}_q$ and $\boldsymbol{\mu}_q$ of the lune-incidence algebra (Theorem 1.20). We now connect them to the present discussion.

Lemma 1.37. *We have*

$$p(\zeta_q) = q(\zeta_q) = \boldsymbol{\zeta}_q \quad \textit{and} \quad p(\mu_q) = q(\mu_q) = \boldsymbol{\mu}_q,$$

with the maps p and q as in (1.60).

Explicitly,

$$\boldsymbol{\zeta}_q(A, F') = \sum_{\substack{F:\, F \geq A \\ \mathrm{s}(F)=\mathrm{s}(F')}} \zeta_q(A, F, F'),$$

(1.69)

$$\boldsymbol{\mu}_q(A, F') = \sum_{\substack{F:\, F \geq A \\ \mathrm{s}(F)=\mathrm{s}(F')}} \mu_q(A, F, F').$$

Proof. We prove the claim about $\boldsymbol{\zeta}_q$. The claim about $\boldsymbol{\mu}_q$ then follows by taking inverses. Fix a face A and consider the q-flat-additivity formulas (1.47) as G varies over all faces of a given support X. Write this as a matrix equation $xD = y$, with $D = (q^{\mathrm{dist}(F,G)})$ with F, G greater than A and of support X. By definition, $D^{-1} := (q^{F/A, G/A})$. Now, in view of (1.64), observe that $x = yD^{-1}$ contains the first formula in (1.69). ☐

Exercise 1.38. Check that summing the two-sided q-lune-additivity formula (1.66) over all $G' \geq A$ yields the q-lune-additivity formula (1.46), while summing the two-sided q-Weisner formula (1.67) over all $G' > A$ yields the noncommutative q-Weisner formula (1.48).

Use this fact to give another proof of Lemma 1.37. The existence of $\boldsymbol{\zeta}_q$ and $\boldsymbol{\mu}_q$ is known from Theorem 1.20, but note that it also follows formally from the existence of ζ_q and μ_q using the above fact.

Exercise 1.39. Check that: For arrangements $\mathcal{A}$ and $\mathcal{A}'$,

(1.70)
$$\mathrm{I}_{\mathrm{bilune}}[\mathcal{A} \times \mathcal{A}'] = \mathrm{I}_{\mathrm{bilune}}[\mathcal{A}] \otimes \mathrm{I}_{\mathrm{bilune}}[\mathcal{A}'].$$

Under this identification, for q not a root of unity, $\zeta_q \in \mathrm{I}_{\mathrm{bilune}}[\mathcal{A} \times \mathcal{A}']$ equals $\zeta_q \otimes \zeta_q \in \mathrm{I}_{\mathrm{bilune}}[\mathcal{A}] \otimes \mathrm{I}_{\mathrm{bilune}}[\mathcal{A}']$, and $\mu_q \in \mathrm{I}_{\mathrm{bilune}}[\mathcal{A} \times \mathcal{A}']$ equals $\mu_q \otimes \mu_q \in \mathrm{I}_{\mathrm{bilune}}[\mathcal{A}] \otimes \mathrm{I}_{\mathrm{bilune}}[\mathcal{A}']$. (Compare with Exercise 1.23.)

1.6.7. Two-sided 0-zeta and 0-Möbius functions. Let us now specialize Section 1.6.6 to $q = 0$. We call $\boldsymbol{\zeta}_0$ the *two-sided 0-zeta function* and $\boldsymbol{\mu}_0$ the *two-sided 0-Möbius function*. They have simple formulas, namely,

$$\begin{aligned} \boldsymbol{\zeta}_0(A, F, F') &= \begin{cases} 1 & \text{if } F = F', \\ 0 & \text{otherwise,} \end{cases} \\ \boldsymbol{\mu}_0(A, F, F') &= \begin{cases} (-1)^{\mathrm{rk}(F/A)} & \text{if } F = F', \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{1.71}$$

In fact, these elements can be viewed inside the lune-incidence algebra. More precisely,

$$i(\boldsymbol{\zeta}_0) = \boldsymbol{\zeta}_0 \quad \text{and} \quad i(\boldsymbol{\mu}_0) = \boldsymbol{\mu}_0,$$

with the map i as in (1.62), and $\boldsymbol{\zeta}_0$ and $\boldsymbol{\mu}_0$ as in (1.51). (This phenomenon is specific to $q = 0$.)

1.7. Descent, lune, Witt identities

The regular cell complex defined by the spherical model of an arrangement has the topology of a sphere. More generally, any cone is either a sphere or a ball. Thus, one can evaluate expressions of the form $\sum_F (-1)^{\mathrm{rk}(F)}$ by relating them to the Euler characteristic of some cone. We collect together a number of identities of this nature.

Notation 1.40. For any arrangement $\mathcal{A}$, let $c(\mathcal{A})$ denote the number of chambers in $\mathcal{A}$. Recall that there are a number of arrangements associated to $\mathcal{A}$ such as $\mathcal{A}_F, \mathcal{A}^{\mathrm{X}}$, $\mathcal{A}^{\mathrm{X}}_{\mathrm{Y}}$. We usually write c_F instead of $c(\mathcal{A}_F)$, c^{X} instead of $c(\mathcal{A}^{\mathrm{X}})$, $c^{\mathrm{X}}_{\mathrm{Y}}$ instead of $c(\mathcal{A}^{\mathrm{X}}_{\mathrm{Y}})$.

1.7.1. Euler characteristic. In any arrangement $\mathcal{A}$,

$$\sum_F (-1)^{\mathrm{rk}(F)} = (-1)^{\mathrm{rk}(\mathcal{A})}. \tag{1.72}$$

The sum is over all faces. The lhs is the reduced Euler characteristic of the cell complex defined by the spherical model of $\mathcal{A}$.

Similarly, for any faces $A \leq G$,

$$\sum_{F\colon A \leq F \leq G} (-1)^{\mathrm{rk}(F/A)} = \begin{cases} 1 & \text{if } G = A, \\ 0 & \text{otherwise,} \end{cases} \tag{1.73}$$

and for any flats $\mathrm{Z} \leq \mathrm{X}$,

$$\sum_{\mathrm{Y}\colon \mathrm{Z} \leq \mathrm{Y} \leq \mathrm{X}} (-1)^{\mathrm{rk}(\mathrm{Y}/\mathrm{Z})} c^{\mathrm{Y}}_{\mathrm{Z}} = (-1)^{\mathrm{rk}(\mathrm{X}/\mathrm{Z})}. \tag{1.74}$$

Identity (1.73) states that the poset of faces of $\mathcal{A}$ is eulerian. We encountered this fact in Section 1.5.11.

1.7.2. Descent identities for faces. For any chambers C and D,

$$\sum_{H:\, HC=D} (-1)^{\mathrm{rk}(H)} = \begin{cases} (-1)^{\mathrm{rk}(D)} & \text{if } \overline{C} = D, \\ 0 & \text{otherwise.} \end{cases} \tag{1.75}$$

For any faces F and G,

$$\sum_{H:\, HF=G} (-1)^{\mathrm{rk}(H)} = \begin{cases} (-1)^{\mathrm{rk}(G)} & \text{if } \overline{F} \leq G, \\ 0 & \text{otherwise,} \end{cases} \tag{1.76a}$$

$$\sum_{H:\, HF\leq G} (-1)^{\mathrm{rk}(H)} = \begin{cases} (-1)^{\mathrm{rk}(G)} & \text{if } \overline{F} = G, \\ 0 & \text{otherwise.} \end{cases} \tag{1.76b}$$

We call these the *descent identities* for faces. See [**21**, Section 7.1.2].

Lemma 1.41. *Suppose $\mathcal{A}$ is simplicial. Then, for any faces H, H', K, K',*

$$\sum_{\substack{G:\, GH=H' \\ GK'=K}} (-1)^{\mathrm{rk}(G)} = \begin{cases} (-1)^{\mathrm{rk}(H'\wedge K)} & \textit{if the sum has exactly one term,} \\ 0 & \textit{otherwise.} \end{cases}$$

In the first alternative, the only choice is $G = H' \wedge K$.

Proof. Put $\mathrm{B} = \{G \mid GH = H',\, GK' = K\}$. Using (1.1c), (1.1f) and [**21**, Proposition 7.1], we observe:

- If $G \in \mathrm{B}$, then $G \leq H' \wedge K$,
- If $G \in \mathrm{B}$ and $G \leq F \leq H' \wedge K$, then $F \in \mathrm{B}$,
- If $G_1, G_2 \in \mathrm{B}$, then $G_1 \wedge G_2 \in \mathrm{B}$.

In particular, if B is nonempty, then $H' \wedge K \in \mathrm{B}$. Thus, B is a Boolean poset. So the alternating sum in the lhs above is zero unless B consists of the singleton element $H' \wedge K$. □

Exercise 1.42. In a simplicial arrangement, for faces F and G such that $GF = G$, let $\overline{\mathrm{Des}}(F, G)$ be the smallest face H of G such that $HF = G$, and $\mathrm{Des}(F, G)$ be the smallest face H of G such that $HF \leq G$. The point is that these smallest faces exist [**21**, Formulas (7.4a) and (7.4b)]. Observe that for chambers C and D, we have $\overline{\mathrm{Des}}(C, D) = \mathrm{Des}(C, D)$.

Show that in Lemma 1.41, in the first alternative, the only choice $G = H' \wedge K$ equals the join of $\overline{\mathrm{Des}}(H, H')$ and $\overline{\mathrm{Des}}(K', K)$ in the poset of faces. In the special case that H, H', K, K' are chambers C, C', D, D', the choice $G = C' \wedge D$ equals the join of $\mathrm{Des}(C, C')$ and $\mathrm{Des}(D', D)$.

1.7.3. Lune identities for faces. For any faces H and G,

$$\sum_{F:\, HF=G} (-1)^{\mathrm{rk}(F)} = \begin{cases} (-1)^{\mathrm{rk}(G)} & \text{if } H \leq G, \\ 0 & \text{otherwise,} \end{cases} \tag{1.77a}$$

$$\sum_{F:\, HF\leq G} (-1)^{\mathrm{rk}(F)} = \begin{cases} (-1)^{\mathrm{rk}(G)} & \text{if } H = G, \\ 0 & \text{otherwise.} \end{cases} \tag{1.77b}$$

These are the *lune identities* for faces. See [**21**, Section 7.2.2].

1.7.4. Descent-lune identities for flats. For any flats X and W,

$$\sum_{\mathrm{Y}\colon \mathrm{X}\vee\mathrm{Y}=\mathrm{W}} (-1)^{\mathrm{rk}(\mathrm{Y})} c^{\mathrm{Y}} = \begin{cases} (-1)^{\mathrm{rk}(\mathrm{W})} c_{\mathrm{X}}^{\mathrm{W}} & \text{if } \mathrm{X} \leq \mathrm{W}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.78a}$$

$$\sum_{\mathrm{Y}\colon \mathrm{X}\vee\mathrm{Y}\leq\mathrm{W}} (-1)^{\mathrm{rk}(\mathrm{Y})} c^{\mathrm{Y}} c_{\mathrm{X}\vee\mathrm{Y}}^{\mathrm{W}} = \begin{cases} (-1)^{\mathrm{rk}(\mathrm{W})} & \text{if } \mathrm{X} = \mathrm{W}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.78b}$$

These are the *descent-lune identities* for flats. See [**21**, Section 7.4.2].

Question 1.43. For flats X, Y, X′, Y′, give a cancellation-free formula for the expression

$$\sum_{\substack{\mathrm{W}\colon \mathrm{W}\vee\mathrm{Y}'=\mathrm{Y} \\ \mathrm{W}\vee\mathrm{X}=\mathrm{X}'}} (-1)^{\mathrm{rk}(\mathrm{W})} c^{\mathrm{W}}.$$

In the special case when X = X′ and Y = Y′, using identity (1.74), we see that the expression evaluates to $(-1)^{\mathrm{rk}(\mathrm{X}\wedge\mathrm{Y})}$.

1.7.5. Descent and lune identities for partial-flats. The identities for faces and flats can be unified using partial-flats. For partial-flats $\mathrm{x} \leq \mathrm{y}$, let $c_{\mathrm{x}}^{\mathrm{y}}$ denote the number of faces with partial-support y which are greater than some fixed face in x.

For partial-flats x and w,

$$\sum_{\mathrm{y}\colon \mathrm{yx}=\mathrm{w}} (-1)^{\mathrm{rk}(\mathrm{y})} c^{\mathrm{y}} = \begin{cases} (-1)^{\mathrm{rk}(\mathrm{w})} c_{\overline{\mathrm{x}}}^{\mathrm{w}} & \text{if } \overline{\mathrm{x}} \leq \mathrm{w}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.79a}$$

$$\sum_{\mathrm{y}\colon \mathrm{yx}\leq\mathrm{w}} (-1)^{\mathrm{rk}(\mathrm{y})} c^{\mathrm{y}} c_{\mathrm{yx}}^{\mathrm{w}} = \begin{cases} (-1)^{\mathrm{rk}(\mathrm{w})} & \text{if } \overline{\mathrm{x}} = \mathrm{w}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.79b}$$

These are the *descent identities* for partial-flats.

For partial-flats y and w,

$$\sum_{\mathrm{x}\colon \mathrm{yx}=\mathrm{w}} (-1)^{\mathrm{rk}(\mathrm{x})} c^{\mathrm{x}} = \begin{cases} (-1)^{\mathrm{rk}(\mathrm{w})} c_{\mathrm{y}}^{\mathrm{w}} & \text{if } \mathrm{y} \leq \mathrm{w}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.80a}$$

$$\sum_{\mathrm{x}\colon \mathrm{yx}\leq\mathrm{w}} (-1)^{\mathrm{rk}(\mathrm{x})} c^{\mathrm{x}} c_{\mathrm{yx}}^{\mathrm{w}} = \begin{cases} (-1)^{\mathrm{rk}(\mathrm{w})} & \text{if } \mathrm{y} = \mathrm{w}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.80b}$$

These are the *lune identities* for partial-flats.

See [**21**, Section 7.5.2].

1.7.6. Witt identities. For any top-nested face (A, D), and scalars x^C indexed by chambers C,

$$\sum_{H\colon A \leq H \leq D} (-1)^{\mathrm{rk}(H)} \Bigg(\sum_{C\colon HC = D} x^C \Bigg) = (-1)^{\mathrm{rk}(D)} \sum_{C\colon AC = A\overline{D}} x^C. \tag{1.81}$$

Setting A to be the central face, we obtain: For a fixed chamber D, and scalars x^C indexed by chambers C,

$$\sum_{H\colon H \leq D} (-1)^{\mathrm{rk}(H)} \Bigg(\sum_{C\colon HC = D} x^C \Bigg) = (-1)^{\mathrm{rk}(D)} x^{\overline{D}}. \tag{1.82}$$

This is the *Witt identity for chambers*. Similarly, for a fixed face G, and scalars x^F indexed by faces F,

$$\sum_{H\colon H \leq G} (-1)^{\mathrm{rk}(H)} \Bigg(\sum_{F\colon HF = G} x^F \Bigg) = (-1)^{\mathrm{rk}(G)} \sum_{F\colon \overline{F} \leq G} x^F. \tag{1.83}$$

This is the *Witt identity for faces*. See [**21**, Section 7.3].

1.8. Noncommutative Zaslavsky formula

We review the Zaslavsky formula for arrangements; it is given in terms of the Möbius function. We then establish a noncommutative analogue of this formula; it involves a mutually inverse noncommutative zeta and Möbius function. We call it the noncommutative Zaslavsky formula.

1.8.1. Möbius number and Zaslavsky formula. For any arrangement $\mathcal{A}$, define the *Möbius number* of $\mathcal{A}$ as

$$\mu(\mathcal{A}) := \mu(\perp, \top),$$

where the rhs is the Möbius function of the lattice of flats. The Möbius number can be either positive or negative; more precisely, the sign of $\mu(\mathcal{A})$ equals $(-1)^{\mathrm{rk}(\mathcal{A})}$.

For any arrangement $\mathcal{A}$,

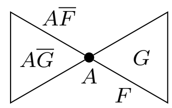
$$\sum_{\mathrm{X}} |\mu(\mathrm{X}, \top)| = c(\mathcal{A}), \tag{1.84}$$

where $c(\mathcal{A})$ is the number of chambers of $\mathcal{A}$. The sum is over all flats X. This is the *Zaslavsky formula*. Similarly,

$$\sum_{(\mathrm{X},\mathrm{Y})\colon \mathrm{X} \leq \mathrm{Y}} |\mu(\mathrm{X}, \mathrm{Y})| = f(\mathcal{A}), \tag{1.85}$$

where $f(\mathcal{A})$ is the number of faces of $\mathcal{A}$. The sum is over both X and Y. This formula follows by applying the Zaslavsky formula to $\mathcal{A}^{\mathrm{Y}}$ for each flat Y.

1.8.2. Noncommutative Zaslavsky formula. The following picture is useful for the discussion below.



Lemma 1.44. *Let $\boldsymbol{\zeta}$ be a noncommutative zeta function. Then, for any $A \leq G$,*

$$\boldsymbol{\zeta}(A, A\overline{G}) = \sum_{F\colon A \leq F \leq G} (-1)^{\mathrm{rk}(G/F)} \boldsymbol{\zeta}(F, G). \tag{1.86a}$$

In the special case that $\boldsymbol{\zeta}$ is projective,

$$\boldsymbol{\zeta}(A, G) = \sum_{F\colon A \leq F \leq G} (-1)^{\mathrm{rk}(G/F)} \boldsymbol{\zeta}(F, G). \tag{1.86b}$$

Proof. Identity (1.86a) can be deduced by combining the Witt identity for chambers (1.82) and the lune-additivity formula (1.42). Identity (1.86b) follows as a special case. □

Lemma 1.45. *Let $\boldsymbol{\mu}$ be a noncommutative Möbius function $\boldsymbol{\mu}$. Then, for any $A \leq G$,*

$$\boldsymbol{\mu}(A, A\overline{G}) = (-1)^{\mathrm{rk}(G/A)} \sum_{F\colon A \leq F \leq G} \boldsymbol{\mu}(A, F). \tag{1.87a}$$

In the special case that $\boldsymbol{\mu}$ is projective,

$$\boldsymbol{\mu}(A, G) = (-1)^{\mathrm{rk}(G/A)} \sum_{F\colon A \leq F \leq G} \boldsymbol{\mu}(A, F). \tag{1.87b}$$

Proof. Identity (1.87a) can be deduced by combining the Witt identity for faces (1.83) and the noncommutative Weisner formula (1.44). Identity (1.87b) follows as a special case. □

Lemma 1.46. *Let $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ be noncommutative zeta and Möbius functions which are inverse to each other. Then, for any $A \leq G$,*

$$\sum_{F\colon A \leq F \leq G} (-1)^{\mathrm{rk}(G/F)} \boldsymbol{\zeta}(A, A\overline{F})\, \boldsymbol{\mu}(F, G) = 1, \tag{1.88}$$

$$\sum_{F\colon A \leq F \leq G} (-1)^{\mathrm{rk}(G/F)} \boldsymbol{\zeta}(A, F)\, \boldsymbol{\mu}(F, F\overline{G}) = 1. \tag{1.89}$$

In the special case that $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ are projective,

$$\sum_{F\colon A \leq F \leq G} (-1)^{\mathrm{rk}(G/F)} \boldsymbol{\zeta}(A, F)\, \boldsymbol{\mu}(F, G) = 1. \tag{1.90}$$

Proof. Express (1.86a) in the form $t = s\boldsymbol{\zeta}$, where $s(A, F) = (-1)^{\mathrm{rk}(F)}$ and $t(A, F) = (-1)^{\mathrm{rk}(F)} \boldsymbol{\zeta}(A, A\overline{F})$. Hence, $t\boldsymbol{\mu} = s$ which yields (1.88). In a similar manner, (1.87a) yields (1.89). We also point out that (1.88) and (1.89) imply each other by replacing $\boldsymbol{\zeta}$ by $\overline{\boldsymbol{\zeta}}$, and $\boldsymbol{\mu}$ by $\overline{\boldsymbol{\mu}}$, or by replacing G by $A\overline{G}$. □

We say that (1.86a) and (1.88), and (1.87a) and (1.89) are related to each other by noncommutative Möbius inversion. We refer to either of (1.88), (1.89), (1.90) as the *noncommutative Zaslavsky formula*. A reformulation is given in the exercise below.

Exercise 1.47. We use the notations of Section 1.5.12. Check that: For any mutually inverse noncommutative zeta and Möbius functions $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$,

$$\overline{\boldsymbol{\zeta}}\, r_{-1}\, \boldsymbol{\mu}\, r_{-1} = h_1 = \boldsymbol{\zeta}\, r_{-1}\, \overline{\boldsymbol{\mu}}\, r_{-1}. \tag{1.91}$$

In addition, if $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ are projective, then

$$t_{-1} = h_1 r_{-1}. \tag{1.92}$$

Let us now recast the above results in the language of Proposition 1.11. Left and right factors of a lune are defined in Section 1.2.3.

Lemma 1.48. *Let $\boldsymbol{\zeta}$ be a noncommutative zeta function. Then, for any lune* L,

$$\boldsymbol{\zeta}(\overline{\mathrm{L}}) = \sum_{\substack{\mathrm{N}:\, \mathrm{N} \textit{ is a} \\ \textit{right factor of } \mathrm{L}}} (-1)^{\mathrm{sk}(\mathrm{N})} \boldsymbol{\zeta}(\mathrm{N}). \tag{1.93}$$

Let $\boldsymbol{\mu}$ be a noncommutative Möbius function $\boldsymbol{\mu}$. Then, for any lune L,

$$\boldsymbol{\mu}(\overline{\mathrm{L}}) = (-1)^{\mathrm{sk}(\mathrm{L})} \sum_{\substack{\mathrm{M}:\, \mathrm{M} \textit{ is a} \\ \textit{left factor of } \mathrm{L}}} \boldsymbol{\mu}(\mathrm{M}). \tag{1.94}$$

In the projective case, $\overline{\mathrm{L}}$ may be replaced by L.

These are reformulations of (1.86a) and (1.87a), respectively.

Lemma 1.49. *Let $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ be noncommutative zeta and Möbius functions which are inverse to each other. Then, for any lune* L,

$$\begin{aligned} \sum_{\mathrm{M},\mathrm{N}:\, \mathrm{L}=\mathrm{M}\circ\mathrm{N}} (-1)^{\mathrm{sk}(\mathrm{N})} \boldsymbol{\zeta}(\overline{\mathrm{M}})\, \boldsymbol{\mu}(\mathrm{N}) = 1, \\ \sum_{\mathrm{M},\mathrm{N}:\, \mathrm{L}=\mathrm{M}\circ\mathrm{N}} (-1)^{\mathrm{sk}(\mathrm{N})} \boldsymbol{\zeta}(\mathrm{M})\, \boldsymbol{\mu}(\overline{\mathrm{N}}) = 1. \end{aligned} \tag{1.95}$$

This is a reformulation of the noncommutative Zaslavsky formulas (1.88) and (1.89).

Exercise 1.50. Let $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ be projective noncommutative zeta and Möbius functions which are inverse to each other. Show that for any $A < G$,

$$\sum_{\substack{F:\, A \le F \le G \\ \mathrm{rk}(G/F) \text{ is even}}} \boldsymbol{\zeta}(A,F)\, \boldsymbol{\mu}(F,G) = 1/2 = \sum_{\substack{F:\, A \le F \le G \\ \mathrm{rk}(G/F) \text{ is odd}}} -\, \boldsymbol{\zeta}(A,F)\, \boldsymbol{\mu}(F,G).$$

Exercise 1.51. Use Lemma 1.17 to deduce the Zaslavsky formula (1.84) from either of the noncommutative Zaslavsky formulas (1.88) or (1.89).

Exercise 1.52. Use Lemma 1.17 to deduce the following identities from (1.86b) and (1.87b), respectively. For any $\mathrm{Z} \leq \mathrm{X}$,

$$\zeta(\mathrm{Z},\mathrm{X}) = \sum_{\mathrm{Y}:\, \mathrm{Z}\leq\mathrm{Y}\leq\mathrm{X}} (-1)^{\mathrm{rk}(\mathrm{X}/\mathrm{Y})} c_{\mathrm{Z}}^{\mathrm{Y}}\, \zeta(\mathrm{Y},\mathrm{X}), \tag{1.96}$$

$$\mu(\mathrm{Z},\mathrm{X}) = (-1)^{\mathrm{rk}(\mathrm{X}/\mathrm{Z})} \sum_{\mathrm{Y}:\, \mathrm{Z}\leq\mathrm{Y}\leq\mathrm{X}} \mu(\mathrm{Z},\mathrm{Y})\, c_{\mathrm{Y}}^{\mathrm{X}}. \tag{1.97}$$

The function ζ is identically 1, but for clarity, we have written it explicitly. Observe that (1.96) is the same as (1.74). We also mention that the Zaslavsky formula (1.84) is related to either (1.96) or (1.97) by Möbius inversion, see for instance [**21**, Proof of Theorem 1.81, Equations (9.5) and (9.8)].

1.8.3. Solomon coefficients. Let $\boldsymbol{\mu}$ be a noncommutative Möbius function. For faces G and H both greater than A such that $GH = G$, define

$$\mathrm{sln}_A^{G,H} := \sum_{F:\, F\geq A,\, FH=G} \boldsymbol{\mu}(A,F). \tag{1.98}$$

We refer to (1.98) as a *Solomon coefficient*.

One can reduce to the case when G and H have the same support. More precisely: Using (1.1f), we deduce that

$$\mathrm{sln}_A^{G,H} = \mathrm{sln}_A^{G,H\overline{G}}. \tag{1.99}$$

Note very carefully that G and $H\overline{G}$ have the same support.

For $A \leq G$, we have

$$\mathrm{sln}_A^{G,A\overline{G}} = \boldsymbol{\mu}(A,G) \quad \text{and} \quad \mathrm{sln}_A^{G,G} = (-1)^{\mathrm{rk}(G/A)}\boldsymbol{\mu}(A,A\overline{G}). \tag{1.100}$$

In the first case, there is only one term in the sum in (1.98) corresponding to $F = G$. In the second case, the sum is over $A \leq F \leq G$ which can be summed using (1.87a).

Example 1.53. For a rank-one arrangement $\mathcal{A}$ with chambers C and $\overline{C}$, in continuation of Example 1.15, we have

$$\mathrm{sln}_O^{O,O} = 1, \qquad \mathrm{sln}_O^{C,C} = 1-p, \qquad \mathrm{sln}_O^{\overline{C},\overline{C}} = p,$$
$$\mathrm{sln}_O^{C,\overline{C}} = \mathrm{sln}_O^{C,O} = -p, \qquad \mathrm{sln}_O^{\overline{C},C} = \mathrm{sln}_O^{\overline{C},O} = p-1.$$

An explicit evaluation of (1.98) for the braid arrangement is given later in Lemma 17.65.

Note that if $\boldsymbol{\mu}$ is replaced by the noncommutative 0-Möbius function $\boldsymbol{\mu}_0$ (Section 1.5.11), then the sum in (1.98) can be evaluated using the descent identity (1.76a).

1.9. Birkhoff algebra, Tits algebra, Janus algebra

The Birkhoff algebra is the linearization of the Birkhoff monoid; the canonical basis is denoted $\mathtt{H}$. It is a split-semisimple commutative algebra, and has a unique complete system of primitive orthogonal idempotents given by the $\mathtt{Q}$-basis.

The Tits algebra is the linearization of the Tits monoid; the canonical basis is denoted $\mathtt{H}$. It is an elementary algebra whose split-semisimple quotient is the Birkhoff algebra. Complete systems, called eulerian families and denoted $\mathtt{E}$, exist, but are no longer unique. They are parametrized by noncommutative zeta or Möbius functions. To any such function, one can associate a $\mathtt{Q}$-basis from which the eulerian family $\mathtt{E}$ can be uniquely determined.

The linearization of the Janus monoid is the Janus algebra. It is an elementary algebra whose split-semisimple quotient is the Birkhoff algebra. Interestingly, the Janus algebra admits a deformation by a scalar q, and this deformed algebra is split-semisimple when q is not a root of unity.

Some important elements in these algebras are summarized in Table 1.1.

TABLE 1.1. Birkhoff algebra, Tits algebra, q-Janus algebra.

Algebra	Elements
Birkhoff algebra	$\mathtt{H}_{\mathrm{X}}$, $\mathtt{Q}_{\mathrm{X}}$
Tits algebra	$\mathtt{H}_F$, $\mathtt{Q}_F$, $\mathtt{E}_{\mathrm{X}}$, $\mathtt{u}_{\mathrm{X}}$
q-Janus algebra for q not a root of unity	$\mathtt{H}_{(F,F')}$, $\mathtt{Q}'_{(F,F')}$, $\mathtt{Q}_{(F,F')}$, $\mathtt{u}_{(F,F')}$, $\mathtt{Q}^d_{(F,F')}$, $\mathtt{u}^d_{(F,F')}$

1.9.1. Birkhoff algebra. Recall the Birkhoff monoid $\Pi[\mathcal{A}]$ whose elements are flats of $\mathcal{A}$. Let $\mathsf{\Pi}[\mathcal{A}]$ denote its linearization over a field $\Bbbk$, with canonical basis $\mathtt{H}$. It is a commutative $\Bbbk$-algebra:

$$\mathtt{H}_{\mathrm{X}} \cdot \mathtt{H}_{\mathrm{Y}} := \mathtt{H}_{\mathrm{X} \vee \mathrm{Y}}. \tag{1.101}$$

We call this the *Birkhoff algebra*. Define the $\mathtt{Q}$-basis of $\mathsf{\Pi}[\mathcal{A}]$ by either of the two equivalent formulas

$$\begin{aligned} \mathtt{H}_{\mathrm{X}} &= \sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{X}} \mathtt{Q}_{\mathrm{Y}}, \\ \mathtt{Q}_{\mathrm{X}} &= \sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{X},\mathrm{Y})\, \mathtt{H}_{\mathrm{Y}}, \end{aligned} \tag{1.102}$$

where μ denotes the Möbius function of the poset of flats.

The Birkhoff algebra is a split-semisimple commutative algebra (of dimension equal to the number of flats in $\mathcal{A}$) and $\mathtt{Q}$ is its basis of primitive idempotents. In other words,

$$\mathtt{Q}_{\mathrm{X}} \cdot \mathtt{Q}_{\mathrm{Y}} = \begin{cases} \mathtt{Q}_{\mathrm{X}} & \text{if } \mathrm{X} = \mathrm{Y}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.103}$$

Moreover,

$$\mathtt{H}_{\mathrm{Y}} \cdot \mathtt{Q}_{\mathrm{X}} = \begin{cases} \mathtt{Q}_{\mathrm{X}} & \text{if } \mathrm{X} \geq \mathrm{Y}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.104}$$

1.9.2. Tits algebra. Recall the Tits monoid $\Sigma[\mathcal{A}]$ whose elements are faces of $\mathcal{A}$. Let $\mathsf{\Sigma}[\mathcal{A}]$ denote its linearization, with canonical basis $\mathtt{H}$. It is a $\Bbbk$-algebra:

$$\mathtt{H}_F \cdot \mathtt{H}_G := \mathtt{H}_{FG}. \tag{1.105}$$

We call this the *Tits algebra*.

Let $\mathsf{\Gamma}[\mathcal{A}]$ denote the linearization of the set of chambers $\Gamma[\mathcal{A}]$, with canonical basis $\mathtt{H}$. We call this the *space of chambers* of $\mathcal{A}$. It is a left module over the Tits algebra:

$$\mathtt{H}_F \cdot \mathtt{H}_C := \mathtt{H}_{FC}.$$

The linearization of the support map (1.2)

$$\mathrm{s} : \mathsf{\Sigma}[\mathcal{A}] \twoheadrightarrow \mathsf{\Pi}[\mathcal{A}] \tag{1.106}$$

is a morphism of algebras. One may check that the kernel of (1.106) is a nilpotent ideal, and hence it equals the radical of the Tits algebra. Thus, the Tits algebra is elementary, that is, the quotient by its radical is a split-semisimple commutative algebra.

Now fix a noncommutative zeta function $\boldsymbol{\zeta}$ and its inverse noncommutative Möbius function $\boldsymbol{\mu}$. Define a $\mathtt{Q}$-basis of $\mathsf{\Sigma}[\mathcal{A}]$ by either of the two equivalent formulas

$$\begin{aligned} \mathtt{H}_F &= \sum_{G:\, F \leq G} \boldsymbol{\zeta}(F,G)\, \mathtt{Q}_G, \\ \mathtt{Q}_F &= \sum_{G:\, F \leq G} \boldsymbol{\mu}(F,G)\, \mathtt{H}_G. \end{aligned} \tag{1.107}$$

Lemma 1.54. *For any faces F and G,*

$$\mathtt{H}_F \cdot \mathtt{Q}_G = \begin{cases} \mathtt{Q}_{FG} & \text{if } GF = G, \\ 0 & \text{if } GF > G. \end{cases} \tag{1.108}$$

In particular:

$$\mathtt{H}_F \cdot \mathtt{Q}_G = \mathtt{Q}_F \text{ if } F \text{ and } G \text{ have the same support,} \tag{1.109}$$

$$\mathtt{H}_F \cdot \mathtt{Q}_O = 0 \text{ if } F > O. \tag{1.110}$$

Proof. This can be derived using the noncommutative Weisner formula (1.44). We omit the details. $\square$

Exercise 1.55. Show that: For any faces F and G,

$$\mathtt{Q}_F \cdot \mathtt{Q}_G = \begin{cases} \sum_{\substack{K:\, K \geq F \\ \mathrm{s}(K)=\mathrm{s}(G)}} \mathrm{sln}_F^{K,FG}\, \mathtt{Q}_K & \text{if } GF = G, \\ 0 & \text{if } GF > G, \end{cases} \tag{1.111}$$

where $\mathrm{sln}_F^{K,FG}$ is the Solomon coefficient (1.98). (Express $\mathtt{Q}_F$ in the $\mathtt{H}$-basis and then employ formula (1.108).)

For each flat X, put

$$\mathtt{E}_{\mathrm{X}} := \sum_{F:\, \mathrm{s}(F)=\mathrm{X}} \boldsymbol{\zeta}(O,F)\, \mathtt{Q}_F. \tag{1.112}$$

For any flats X and Y,

$$\mathtt{E}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{Y}} = \begin{cases} \mathtt{E}_{\mathrm{X}} & \text{if } \mathrm{X} = \mathrm{Y}, \\ 0 & \text{if } \mathrm{X} \neq \mathrm{Y}. \end{cases} \tag{1.113}$$

This can be checked by direct calculation. We call $\mathtt{E}_{\mathrm{X}}$ an *eulerian idempotent* and $\mathtt{E}_{\perp}$ a *first eulerian idempotent*. The latter is given by

$$\mathtt{E}_{\perp} = \mathtt{Q}_O = \sum_G \boldsymbol{\mu}(O, G)\, \mathtt{H}_G. \tag{1.114}$$

The sum is over all faces G. Also observe that

$$\mathtt{H}_O = \sum_{\mathrm{X}} \mathtt{E}_{\mathrm{X}}.$$

The above observations on eulerian idempotents are a part of the following general result. For more details, see [**21**, Theorem 15.44].

Theorem 1.56. *The* $\mathtt{E}_{\mathrm{X}}$*, as* X *varies over all flats, is a complete system of primitive orthogonal idempotents of* $\Sigma[\mathcal{A}]$*. Moreover, all complete systems arise in this manner from a unique noncommutative zeta function* $\boldsymbol{\zeta}$*.*

We use the term *eulerian family* to refer to any complete system of the Tits algebra, and denote it by $\mathtt{E} := \{\mathtt{E}_{\mathrm{X}}\}$.

Exercise 1.57. For each flat X, put

$$\mathtt{u}_{\mathrm{X}} := \sum_{F:\, \mathrm{s}(F) = \mathrm{X}} \boldsymbol{\zeta}(O, F)\, \mathtt{H}_F. \tag{1.115}$$

Show that for any X,

$$\mathtt{u}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{X}} = \mathtt{E}_{\mathrm{X}}, \tag{1.116}$$

$$\mathtt{u}_{\mathrm{X}} \cdot \Big(\sum_{\mathrm{Y}: \mathrm{Y} \geq \mathrm{X}} \mathtt{E}_{\mathrm{Y}} \Big) = \mathtt{u}_{\mathrm{X}}, \tag{1.117}$$

$$\mathtt{E}_{\mathrm{X}} = \mathtt{u}_{\mathrm{X}} - \sum_{\mathrm{Y}: \mathrm{Y} > \mathrm{X}} \mathtt{u}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{Y}}. \tag{1.118}$$

The last formula may be viewed as a recursion to construct the eulerian idempotents. We call this the *Saliola construction*.

The connection of the $\mathtt{Q}$-basis of the Tits algebra with the $\mathtt{Q}$-basis of the Birkhoff algebra is as follows. For any face F and flat X,

$$\mathrm{s}(\mathtt{Q}_F) = \mathtt{Q}_{\mathrm{s}(F)} \quad \text{and} \quad \mathrm{s}(\mathtt{E}_{\mathrm{X}}) = \mathtt{Q}_{\mathrm{X}}. \tag{1.119}$$

The first formula follows from Lemma 1.17. The second can be deduced from the first using the flat-additivity formula (1.43). In particular,

$$\mathrm{s}(\mathtt{Q}_O) = \mathrm{s}(\mathtt{E}_{\perp}) = \mathtt{Q}_{\perp}. \tag{1.120}$$

1.9.3. Janus algebra. Recall the Janus monoid $\mathrm{J}[\mathcal{A}]$ whose elements are bifaces of $\mathcal{A}$. Let $\mathsf{J}[\mathcal{A}]$ denote its linearization, with canonical basis $\mathtt{H}$. It is a $\Bbbk$-algebra:

$$\mathtt{H}_{(F,F')} \cdot \mathtt{H}_{(G,G')} = \mathtt{H}_{(FG,G'F')}. \tag{1.121}$$

We call this the *Janus algebra*.

Linearizing diagram (1.7) yields the following commutative diagram of algebras.

$$\begin{tikzcd} \mathsf{J}[\mathcal{A}] \arrow[r, two heads] \arrow[d, two heads] & \Sigma[\mathcal{A}]^{\mathrm{op}} \arrow[d, two heads, "\mathrm{s}"] \\ \Sigma[\mathcal{A}] \arrow[r, two heads, "\mathrm{s}"'] & \Pi[\mathcal{A}]. \end{tikzcd} \tag{1.122}$$

One may check that the kernel of the composite map $\mathsf{J}[\mathcal{A}] \to \Pi[\mathcal{A}]$ is a nilpotent ideal, and hence it equals the radical of the Janus algebra. Thus, the Janus algebra is elementary.

1.9.4. *q*-Janus algebra. For any scalar q, the product

$$\mathtt{H}_{(F,F')} \cdot \mathtt{H}_{(G,G')} := q^{\operatorname{dist}(F',G)}\, \mathtt{H}_{(FG,G'F')} \tag{1.123}$$

is associative and defines an algebra. We call this the *q-Janus algebra* and denote it by $\mathsf{J}_q[\mathcal{A}]$. Setting $q = 1$ recovers the Janus algebra.

Observe that each $\mathtt{H}_{(F,F)}$ is an idempotent element.

Exercise 1.58. Check that: For any scalar q, the map

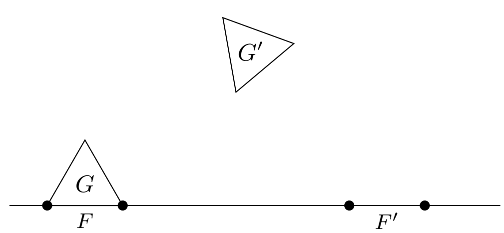
$$\mathsf{J}_q[\mathcal{A}] \to \mathsf{J}_{-q}[\mathcal{A}], \qquad \mathtt{H}_{(F,F')} \mapsto (-1)^{\operatorname{dist}(F,F')}\, \mathtt{H}_{(F,F')}$$

is an isomorphism of algebras. (Use (1.30b) and (1.31).) Thus, the q-Janus algebra and the $(-q)$-Janus algebra are isomorphic.

For q not a root of unity, define the $\mathtt{Q}$-basis of the q-Janus algebra $\mathsf{J}_q[\mathcal{A}]$ by the formula

$$\mathtt{H}_{(F,F')} = \sum_{G:\, G \geq F} \ \sum_{\substack{G':\, FG'=G \\ \mathrm{s}(G')=\mathrm{s}(G)}} q^{\operatorname{dist}(F',G')}\, \mathtt{Q}_{(G,G')}. \tag{1.124}$$

An illustration of how the bifaces (F, F') and (G, G') relate to each other is shown below.



The formula for a $\mathtt{Q}$-basis element in terms of $\mathtt{H}$-basis elements is given later in (1.128).

Lemma 1.59. *For q not a root of unity, the product in the* $\mathtt{Q}$*-basis of the q-Janus algebra* $\mathtt{J}_q[\mathcal{A}]$ *is given by*

$$\mathtt{Q}_{(F,F')} \cdot \mathtt{Q}_{(G,G')} = \begin{cases} \mathtt{Q}_{(F,G')} & \textit{if } F' = G, \\ 0 & \textit{otherwise.} \end{cases} \tag{1.125}$$

In particular, each $\mathtt{Q}_{(F,F)}$ *is an idempotent element.*

Proof. This is a direct calculation. See [**21**, Lemma 9.78] for details. Exercise 1.63 below suggests a slightly different way to do this calculation. ☐

Thus, for any flat X, the $\mathtt{Q}$-basis elements indexed by bifaces with support X form the basis of a matrix algebra. Further, $\mathtt{Q}$-basis elements for different flats are orthogonal, so $\mathtt{J}_q[\mathcal{A}]$ breaks as a product of these matrix algebras. This yields the following result stated in [**21**, Theorem 9.75].

Theorem 1.60. *Suppose q is not a root of unity. Then, the algebra* $\mathtt{J}_q[\mathcal{A}]$ *is split-semisimple, that is, isomorphic to a direct sum of matrix algebras over the base field. There is one matrix algebra for each flat* X*, with the size of the matrix being the number of faces with support* X.

We now turn to another basis intimately connected to the $\mathtt{Q}$-basis. For q not a root of unity, define the $\mathtt{Q}'$-basis of $\mathtt{J}_q[\mathcal{A}]$ by either of the two equivalent formulas

$$\begin{aligned} \mathtt{H}_{(F,F')} &= \sum_{\substack{G,G':\\ F\leq G,\, F'\leq G'\\ \mathrm{s}(G)=\mathrm{s}(G')}} \boldsymbol{\zeta}_q(F,G,FG')\, \mathtt{Q}'_{(G,G')}, \\ \mathtt{Q}'_{(F,F')} &= \sum_{\substack{G,G':\\ F\leq G,\, F'\leq G'\\ \mathrm{s}(G)=\mathrm{s}(G')}} \boldsymbol{\mu}_q(F,G,FG')\, \mathtt{H}_{(G,G')}, \end{aligned} \tag{1.126}$$

where $\boldsymbol{\zeta}_q$ and $\boldsymbol{\mu}_q$ are the two-sided q-zeta function and two-sided q-Möbius function $\boldsymbol{\mu}_q$ defined in Section 1.6.6.

Lemma 1.61. *The* $\mathtt{Q}$- *and* $\mathtt{Q}'$*-bases are related by*

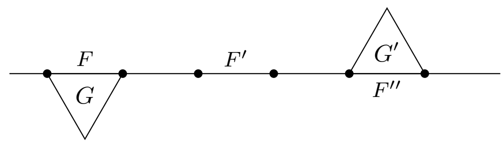
$$\begin{aligned} \mathtt{Q}'_{(F,F')} &= \sum_{\substack{F'':\\ \mathrm{s}(F'')=\mathrm{s}(F')=\mathrm{s}(F)}} q^{\mathrm{dist}(F',F'')}\, \mathtt{Q}_{(F,F'')}, \\ \mathtt{Q}_{(F,F')} &= \sum_{\substack{F'':\\ \mathrm{s}(F'')=\mathrm{s}(F')=\mathrm{s}(F)}} \boldsymbol{\zeta}_q(O,F',F'')\, \mathtt{Q}'_{(F,F'')}. \end{aligned} \tag{1.127}$$

Proof. The two formulas in (1.127) imply each other by (1.65). One may derive the first formula by substituting (1.124) into the second formula in (1.126), and simplifying using (1.68). We omit the details. ☐

Substituting the second formula in (1.126) into the second formula in (1.127) yields a formula for the $\mathtt{Q}$-basis in terms of the $\mathtt{H}$-basis, namely:

$$\mathtt{Q}_{(F,F')} = \sum_{\substack{F'':\\ \mathrm{s}(F'')=\mathrm{s}(F')}} \sum_{\substack{G,G':\\ F\leq G,\, F''\leq G'\\ \mathrm{s}(G)=\mathrm{s}(G')}} \boldsymbol{\zeta}_q(O,F',F'')\, \boldsymbol{\mu}_q(F,G,FG')\, \mathtt{H}_{(G,G')}. \tag{1.128}$$

This is the inverse of (1.124). An illustration of how the bifaces (F, F') and (G, G') relate to each other is shown below.



Lemma 1.62. *For q not a root of unity, the product in the $\mathtt{Q}'$-basis of the q-Janus algebra $\mathtt{J}_q[\mathcal{A}]$ is given by*

$$\mathtt{Q}'_{(F,F')} \cdot \mathtt{Q}'_{(G,G')} = \begin{cases} q^{\mathrm{dist}(F',G)}\, \mathtt{Q}'_{(F,G')} & \text{if } \mathrm{s}(F) = \mathrm{s}(G),\\ 0 & \text{otherwise.}\end{cases} \tag{1.129}$$

In particular, each $\mathtt{Q}'_{(F,F)}$ is an idempotent element.

Proof. This can be checked by a direct calculation which we omit. Alternatively, one can derive it by using product formula (1.125) in conjunction with the change of basis formula (1.127). □

Exercise 1.63. Derive (1.125) by using product formula (1.129) in conjunction with the change of basis formula (1.127).

Lemma 1.64. *For any scalar q, the linear map*

$$\mathtt{J}_q[\mathcal{A}] \to \mathtt{J}_q[\mathcal{A}]^{\mathrm{op}}, \qquad \mathtt{H}_{(F,F')} \mapsto \mathtt{H}_{(F',F)} \tag{1.130}$$

is an isomorphism of algebras. Thus, the q-Janus algebra is isomorphic to its opposite algebra.

Proof. Let us denote the above map by f. Then,

$$\begin{aligned} f(\mathtt{H}_{(F,F')} \cdot \mathtt{H}_{(G,G')}) &= f(q^{\mathrm{dist}(F',G)}\, \mathtt{H}_{(FG,G'F')}) = q^{\mathrm{dist}(F',G)}\, \mathtt{H}_{(G'F',FG)}\\ &= q^{\mathrm{dist}(G,F')}\, \mathtt{H}_{(G'F',FG)} = \mathtt{H}_{(G',G)} \cdot \mathtt{H}_{(F',F)} = f(\mathtt{H}_{(G,G')}) \cdot f(\mathtt{H}_{(F,F')}). \end{aligned}$$

Observe that this used the symmetry property (1.21b) of the distance function. □

For q not a root of unity, define the $\mathtt{Q}^d$-basis of the q-Janus algebra $\mathtt{J}_q[\mathcal{A}]$ as follows. Let $\mathtt{Q}^d_{(F',F)}$ be the element obtained by applying (1.130) to $\mathtt{Q}_{(F,F')}$. In other words, to get $\mathtt{Q}^d_{(F',F)}$, we first write $\mathtt{Q}_{(F,F')}$ in the $\mathtt{H}$-basis and then interchange the two coordinates in each term.

Lemma 1.65. *For q not a root of unity, the product in the $\mathtt{Q}^d$-basis of the q-Janus algebra $\mathsf{J}_q[\mathcal{A}]$ is given by*

$$\mathtt{Q}^d_{(G',G)} \cdot \mathtt{Q}^d_{(F',F)} = \begin{cases} \mathtt{Q}^d_{(G',F)} & \text{if } G = F', \\ 0 & \text{otherwise.} \end{cases} \tag{1.131}$$

In particular, each $\mathtt{Q}^d_{(F,F)}$ is an idempotent element.

PROOF. This follows from Lemmas 1.59 and 1.64. □

In contrast to the $\mathtt{Q}$-basis, observe that applying (1.130) to $\mathtt{Q}'_{(F,F')}$ indeed yields $\mathtt{Q}'_{(F',F)}$. This is a consequence of the symmetric roles played by the two coordinates in the change of basis formulas (1.126). We point out that

$$\mathtt{Q}_{(O,O)} = \mathtt{Q}'_{(O,O)} = \mathtt{Q}^d_{(O,O)} = \sum_{\substack{F,F':\\ \mathrm{s}(F)=\mathrm{s}(F')}} \boldsymbol{\mu}_q(O,F,F')\,\mathtt{H}_{(F,F')}. \tag{1.132}$$

However, the three bases are different in general. This is clear from Example 1.66 below. Also note that

$$\mathtt{H}_{(O,O)} = \sum_F \mathtt{Q}_{(F,F)} = \sum_F \mathtt{Q}^d_{(F,F)} = \sum_{\substack{F,F':\\ \mathrm{s}(F)=\mathrm{s}(F')}} \boldsymbol{\zeta}_q(O,F,F')\,\mathtt{Q}'_{(F,F')}. \tag{1.133}$$

The first two sums are over all faces F, while the last sum is over all bifaces (F,F').

Example 1.66. Let $\mathcal{A}$ be a rank-one arrangement with chambers C and $\overline{C}$. Suppose $q \neq \pm 1$. The $\mathtt{Q}$-basis of $\mathsf{J}_q[\mathcal{A}]$ is given by

$$\mathtt{Q}_{(C,C)} = \frac{1}{1-q^2}(\mathtt{H}_{(C,C)} - q\,\mathtt{H}_{(C,\overline{C})}), \quad \mathtt{Q}_{(C,\overline{C})} = \frac{1}{1-q^2}(\mathtt{H}_{(C,\overline{C})} - q\,\mathtt{H}_{(C,C)}),$$
$$\mathtt{Q}_{(\overline{C},C)} = \frac{1}{1-q^2}(\mathtt{H}_{(\overline{C},C)} - q\,\mathtt{H}_{(\overline{C},\overline{C})}), \quad \mathtt{Q}_{(\overline{C},\overline{C})} = \frac{1}{1-q^2}(\mathtt{H}_{(\overline{C},\overline{C})} - q\,\mathtt{H}_{(\overline{C},C)}),$$
$$\mathtt{Q}_{(O,O)} = \mathtt{H}_{(O,O)} - \frac{1}{1-q^2}(\mathtt{H}_{(C,C)} - q\,\mathtt{H}_{(C,\overline{C})} - q\,\mathtt{H}_{(\overline{C},C)} + \mathtt{H}_{(\overline{C},\overline{C})}).$$

The $\mathtt{Q}'$-basis of $\mathsf{J}_q[\mathcal{A}]$ is given by

$$\mathtt{Q}'_{(C,C)} = \mathtt{H}_{(C,C)}, \quad \mathtt{Q}'_{(C,\overline{C})} = \mathtt{H}_{(C,\overline{C})}, \quad \mathtt{Q}'_{(\overline{C},C)} = \mathtt{H}_{(\overline{C},C)}, \quad \mathtt{Q}'_{(\overline{C},\overline{C})} = \mathtt{H}_{(\overline{C},\overline{C})},$$
$$\mathtt{Q}'_{(O,O)} = \mathtt{H}_{(O,O)} - \frac{1}{1-q^2}(\mathtt{H}_{(C,C)} - q\,\mathtt{H}_{(C,\overline{C})} - q\,\mathtt{H}_{(\overline{C},C)} + \mathtt{H}_{(\overline{C},\overline{C})}).$$

The $\mathtt{Q}^d$-basis of $\mathsf{J}_q[\mathcal{A}]$ is given by

$$\mathtt{Q}^d_{(C,C)} = \frac{1}{1-q^2}(\mathtt{H}_{(C,C)} - q\,\mathtt{H}_{(\overline{C},C)}), \quad \mathtt{Q}^d_{(C,\overline{C})} = \frac{1}{1-q^2}(\mathtt{H}_{(C,\overline{C})} - q\,\mathtt{H}_{(\overline{C},\overline{C})}),$$
$$\mathtt{Q}^d_{(\overline{C},C)} = \frac{1}{1-q^2}(\mathtt{H}_{(\overline{C},C)} - q\,\mathtt{H}_{(C,C)}), \quad \mathtt{Q}^d_{(\overline{C},\overline{C})} = \frac{1}{1-q^2}(\mathtt{H}_{(\overline{C},\overline{C})} - q\,\mathtt{H}_{(C,\overline{C})}),$$
$$\mathtt{Q}^d_{(O,O)} = \mathtt{H}_{(O,O)} - \frac{1}{1-q^2}(\mathtt{H}_{(C,C)} - q\,\mathtt{H}_{(C,\overline{C})} - q\,\mathtt{H}_{(\overline{C},C)} + \mathtt{H}_{(\overline{C},\overline{C})}).$$

Exercise 1.67. Write down the relation between the $\mathtt{Q}^d$- and $\mathtt{Q}'$-bases by applying (1.130) to (1.127).

Exercise 1.68. For bifaces (F, F') and (G, G') of the same support,

$$\mathtt{Q}_{(F,F')} \cdot \mathtt{Q}'_{(G,G')} = \begin{cases} \mathtt{Q}'_{(F,G')} & \text{if } F' = G, \\ 0 & \text{otherwise,} \end{cases} \tag{1.134a}$$

$$\mathtt{Q}'_{(F,F')} \cdot \mathtt{Q}_{(G,G')} = q^{\operatorname{dist}(F',G)}\, \mathtt{Q}_{(F,G')}, \tag{1.134b}$$

$$\mathtt{Q}'_{(G',G)} \cdot \mathtt{Q}^d_{(F',F)} = \begin{cases} \mathtt{Q}'_{(G',F)} & \text{if } G = F', \\ 0 & \text{otherwise,} \end{cases} \tag{1.134c}$$

$$\mathtt{Q}^d_{(G',G)} \cdot \mathtt{Q}'_{(F',F)} = q^{\operatorname{dist}(F',G)}\, \mathtt{Q}^d_{(G',F)}. \tag{1.134d}$$

In particular, for any face F,

$$\begin{aligned} \mathtt{Q}_{(F,F)} \cdot \mathtt{Q}'_{(F,F)} &= \mathtt{Q}'_{(F,F)}, & \mathtt{Q}'_{(F,F)} \cdot \mathtt{Q}_{(F,F)} &= \mathtt{Q}_{(F,F)}, \\ \mathtt{Q}'_{(F,F)} \cdot \mathtt{Q}^d_{(F,F)} &= \mathtt{Q}'_{(F,F)}, & \mathtt{Q}^d_{(F,F)} \cdot \mathtt{Q}'_{(F,F)} &= \mathtt{Q}^d_{(F,F)}. \end{aligned} \tag{1.135}$$

The products in (1.134) are all zero when (F, F') and (G, G') do not have the same support.

Derive the formulas (1.134) using the change of basis (1.127) in conjunction with (1.125), (1.129), (1.131). There are two ways to do each calculation. In addition, the first two formulas imply the last two, and vice versa.

Exercise 1.69. For any bifaces (F, F') and (G, G'),

$$\mathtt{H}_{(F,F')} \cdot \mathtt{Q}_{(G,G')} = \begin{cases} q^{\operatorname{dist}(F',G)}\, \mathtt{Q}_{(FG,G')} & \text{if } \mathrm{s}(G) \geq \mathrm{s}(F), \\ 0 & \text{otherwise,} \end{cases} \tag{1.136a}$$

$$\mathtt{H}_{(F,F')} \cdot \mathtt{Q}'_{(G,G')} = \begin{cases} q^{\operatorname{dist}(F',G)}\, \mathtt{Q}'_{(FG,G')} & \text{if } \mathrm{s}(G) \geq \mathrm{s}(F), \\ 0 & \text{otherwise.} \end{cases} \tag{1.136b}$$

Derive the first formula by writing $\mathtt{H}_{(F,F')}$ in the $\mathtt{Q}$-basis and using product formula (1.125). For the second formula, either use the same strategy or deduce it from the first by using (1.127).

Similarly, check that: For bifaces (F, F') and (G, G') of the same support,

$$\mathtt{Q}_{(F,F')} \cdot \mathtt{H}_{(G,G')} = \begin{cases} \mathtt{Q}'_{(F,G')} & \text{if } F' = G, \\ 0 & \text{otherwise,} \end{cases} \tag{1.136c}$$

$$\mathtt{Q}'_{(F,F')} \cdot \mathtt{H}_{(G,G')} = q^{\operatorname{dist}(F',G)}\, \mathtt{Q}'_{(F,G')}. \tag{1.136d}$$

The above products are both zero when the support of (G, G') is strictly greater than that of (F, F').

By applying (1.130), write down similar formulas for $\mathtt{Q}^d$ instead of $\mathtt{Q}$.

We highlight some special cases.

- For any face F,

$$\begin{aligned} \mathtt{H}_{(F,F)} \cdot \mathtt{Q}_{(F,F)} &= \mathtt{Q}_{(F,F)}, & \mathtt{Q}_{(F,F)} \cdot \mathtt{H}_{(F,F)} &= \mathtt{Q}'_{(F,F)}, \\ \mathtt{H}_{(F,F)} \cdot \mathtt{Q}'_{(F,F)} &= \mathtt{Q}'_{(F,F)}, & \mathtt{Q}'_{(F,F)} \cdot \mathtt{H}_{(F,F)} &= \mathtt{Q}'_{(F,F)}, \\ \mathtt{H}_{(F,F)} \cdot \mathtt{Q}^d_{(F,F)} &= \mathtt{Q}'_{(F,F)}, & \mathtt{Q}^d_{(F,F)} \cdot \mathtt{H}_{(F,F)} &= \mathtt{Q}^d_{(F,F)}. \end{aligned} \tag{1.137}$$

- For any face $F > O$,

$$\mathtt{H}_{(F,F)} \cdot \mathtt{Q}_{(O,O)} = 0, \qquad \mathtt{Q}_{(O,O)} \cdot \mathtt{H}_{(F,F)} = 0. \tag{1.138}$$

Exercise 1.70. For q not a root of unity, for each biface (F, F'), define

$$\begin{aligned} \mathtt{u}_{(F,F')} &:= \sum_{\substack{F'': \\ \mathrm{s}(F'')=\mathrm{s}(F')=\mathrm{s}(F)}} \zeta_q(O, F', F'')\, \mathtt{H}_{(F,F'')}, \\ \mathtt{u}^d_{(F',F)} &:= \sum_{\substack{F'': \\ \mathrm{s}(F'')=\mathrm{s}(F')=\mathrm{s}(F)}} \zeta_q(O, F'', F')\, \mathtt{H}_{(F'',F)}. \end{aligned} \tag{1.139}$$

These are elements of the q-Janus algebra $\mathsf{J}_q[\mathcal{A}]$. Observe that $\mathtt{u}^d_{(F',F)}$ is the element obtained by applying (1.130) to $\mathtt{u}_{(F,F')}$.

Check that: For bifaces (F, F') and (G, G') of the same support,

$$\mathtt{H}_{(F,F')} \cdot \mathtt{u}_{(G,G')} = q^{\operatorname{dist}(F',G)}\, \mathtt{u}_{(F,G')}, \tag{1.140a}$$

$$\mathtt{u}_{(F,F')} \cdot \mathtt{H}_{(G,G')} = \begin{cases} \mathtt{H}_{(F,G')} & \text{if } F' = G, \\ 0 & \text{otherwise,} \end{cases} \tag{1.140b}$$

$$\mathtt{u}_{(F,F')} \cdot \mathtt{u}_{(G,G')} = \begin{cases} \mathtt{u}_{(F,G')} & \text{if } F' = G, \\ 0 & \text{otherwise.} \end{cases} \tag{1.140c}$$

In particular, each $\mathtt{u}_{(F,F)}$ is an idempotent element. Also note that

$$\mathtt{H}_{(F,F)} \cdot \mathtt{u}_{(F,F)} = \mathtt{u}_{(F,F)}, \qquad \mathtt{u}_{(F,F)} \cdot \mathtt{H}_{(F,F)} = \mathtt{H}_{(F,F)}. \tag{1.141}$$

Exercise 1.71. Check that: For bifaces (F, F') and (G, G') of the same support,

$$\mathtt{u}_{(F,F')} \cdot \mathtt{Q}_{(G,G')} = \begin{cases} \mathtt{Q}_{(F,G')} & \text{if } F' = G, \\ 0 & \text{otherwise,} \end{cases} \tag{1.142a}$$

$$\mathtt{u}_{(F,F')} \cdot \mathtt{Q}'_{(G,G')} = \begin{cases} \mathtt{Q}'_{(F,G')} & \text{if } F' = G, \\ 0 & \text{otherwise,} \end{cases} \tag{1.142b}$$

$$\mathtt{Q}_{(F,F')} \cdot \mathtt{u}_{(G,G')} = \begin{cases} \mathtt{Q}_{(F,G')} & \text{if } F' = G, \\ 0 & \text{otherwise,} \end{cases} \tag{1.142c}$$

$$\mathtt{Q}'_{(F,F')} \cdot \mathtt{u}_{(G,G')} = q^{\operatorname{dist}(F',G)}\, \mathtt{Q}_{(F,G')}. \tag{1.142d}$$

By applying (1.130), write down similar formulas for $\mathtt{u}^d$ instead of $\mathtt{u}$, and $\mathtt{Q}^d$ instead of $\mathtt{Q}$.

We highlight a particular case. For any face F,

$$\begin{aligned} \mathtt{u}_{(F,F)} \cdot \mathtt{Q}_{(F,F)} &= \mathtt{Q}_{(F,F)}, & \mathtt{Q}_{(F,F)} \cdot \mathtt{u}_{(F,F)} &= \mathtt{Q}_{(F,F)}, \\ \mathtt{u}_{(F,F)} \cdot \mathtt{Q}'_{(F,F)} &= \mathtt{Q}'_{(F,F)}, & \mathtt{Q}'_{(F,F)} \cdot \mathtt{u}_{(F,F)} &= \mathtt{Q}_{(F,F)}, \\ \mathtt{Q}^d_{(F,F)} \cdot \mathtt{u}^d_{(F,F)} &= \mathtt{Q}^d_{(F,F)}, & \mathtt{u}^d_{(F,F)} \cdot \mathtt{Q}^d_{(F,F)} &= \mathtt{Q}^d_{(F,F)}, \\ \mathtt{Q}'_{(F,F)} \cdot \mathtt{u}^d_{(F,F)} &= \mathtt{Q}'_{(F,F)}, & \mathtt{u}^d_{(F,F)} \cdot \mathtt{Q}'_{(F,F)} &= \mathtt{Q}^d_{(F,F)}. \end{aligned} \tag{1.143}$$

1.9.5. 0-Janus algebra. Let us now consider the case $q = 0$. The product of the 0-Janus algebra $\mathtt{J}_0[\mathcal{A}]$ is given by

$$\mathtt{H}_{(F,F')} \cdot \mathtt{H}_{(G,G')} = \begin{cases} \mathtt{H}_{(FG,G'F')} & \text{if } F'G = GF', \\ 0 & \text{otherwise.} \end{cases} \tag{1.144}$$

This is a specialization of (1.123) to $q = 0$. The $\mathtt{H}$- and $\mathtt{Q}$-bases of the 0-Janus algebra $\mathtt{J}_0[\mathcal{A}]$ are related by

$$\begin{aligned} \mathtt{H}_{(F,F')} &= \sum_{G:\, G \geq F} \mathtt{Q}_{(G,F'G)}, \\ \mathtt{Q}_{(F,F')} &= \sum_{G:\, G \geq F} (-1)^{\mathrm{rk}(G/F)}\, \mathtt{H}_{(G,F'G)}. \end{aligned} \tag{1.145}$$

The first formula is a specialization of (1.124). In particular,

$$\mathtt{Q}_{(O,O)} = \sum_F (-1)^{\mathrm{rk}(F)}\, \mathtt{H}_{(F,F)}. \tag{1.146}$$

The sum is over all faces F. This is a specialization of (1.132).

For $q = 0$, the $\mathtt{Q}$-basis, $\mathtt{Q}'$-basis, $\mathtt{Q}^d$-basis all coincide.

1.9.6. υ-Janus algebra. The q-Janus algebra can be generalized further as follows. Let υ be any distance function arising from a weight function as in (1.32). The product

$$\mathtt{H}_{(F,F')} \cdot \mathtt{H}_{(G,G')} := \upsilon_{F',G}\, \mathtt{H}_{(FG,G'F')} \tag{1.147}$$

is associative. The resulting algebra is the *υ-Janus algebra* which we denote by $\mathtt{J}_\upsilon[\mathcal{A}]$. Specializing υ to the q-distance function on faces recovers the q-Janus algebra.

1.10. Takeuchi element

We review the commutative Takeuchi element, the Takeuchi element, the two-sided Takeuchi element of an arrangement. They belong to the Birkhoff algebra, Tits algebra, Janus algebra, respectively.

1.10.1. Takeuchi element. Consider the element of the Tits algebra $\Sigma[\mathcal{A}]$ defined by

$$\mathtt{Tak}[\mathcal{A}] := \sum_F (-1)^{\mathrm{rk}(F)}\, \mathtt{H}_F. \tag{1.148}$$

The sum is over all faces F. This is the *Takeuchi element* of $\mathcal{A}$. It has order 2, that is,

$$\mathtt{Tak}[\mathcal{A}] \cdot \mathtt{Tak}[\mathcal{A}] = \mathtt{H}_O. \tag{1.149}$$

In particular, it is invertible, with its inverse being itself. Moreover, for all $z \in \Sigma[\mathcal{A}]$,

$$z \cdot \mathtt{Tak}[\mathcal{A}] = \mathtt{Tak}[\mathcal{A}] \cdot \overline{z}, \tag{1.150}$$

where

$$z = \sum_F x^F \mathtt{H}_F \quad \text{and} \quad \overline{z} = \sum_F x^F \mathtt{H}_{\overline{F}}.$$

Lemma 1.72. *For any noncommutative zeta function* $\boldsymbol{\zeta}$,

$$\mathtt{Tak}[\mathcal{A}] = \sum_F (-1)^{\mathrm{rk}(F)} \boldsymbol{\zeta}(O, \overline{F})\, \mathtt{Q}_F, \tag{1.151}$$

with $\mathtt{Q}$*-basis as defined in* (1.107). *In the special case that* $\boldsymbol{\zeta}$ *is projective,*

$$\mathtt{Tak}[\mathcal{A}] = \sum_F (-1)^{\mathrm{rk}(F)} \boldsymbol{\zeta}(O, F)\, \mathtt{Q}_F. \tag{1.152}$$

Proof. Formula (1.151) can be deduced using either identity (1.86a) and the first formula in (1.107), or the noncommutative Zaslavsky formula (1.88) and the second formula in (1.107). Formula (1.152) follows as a special case. □

1.10.2. Commutative Takeuchi element. Applying the support map s in (1.106) to the Takeuchi element (1.148), we obtain

$$\mathrm{s}(\mathtt{Tak}[\mathcal{A}]) = \sum_{\mathrm{X}} (-1)^{\mathrm{rk}(\mathrm{X})} c^{\mathrm{X}}\, \mathtt{H}_{\mathrm{X}}. \tag{1.153}$$

The sum is over all flats X and c^{X} is the number of faces with support X. This is the *commutative Takeuchi element* of $\mathcal{A}$. It is an element of the Birkhoff algebra. On the $\mathtt{Q}$-basis,

$$\mathrm{s}(\mathtt{Tak}[\mathcal{A}]) = \sum_{\mathrm{X}} (-1)^{\mathrm{rk}(\mathrm{X})}\, \mathtt{Q}_{\mathrm{X}}. \tag{1.154}$$

This follows from (1.152) and (1.119). Alternatively, it can be deduced from (1.153) using either identity (1.96) and the first formula in (1.102), or the Zaslavsky formula (1.84) and the second formula in (1.102).

1.10.3. Two-sided Takeuchi element. For any scalar q, consider the element of the q-Janus algebra $\mathtt{J}_q[\mathcal{A}]$ defined by

$$\mathbf{Tak}[\mathcal{A}] := \sum_F (-1)^{\mathrm{rk}(F)}\, \mathtt{H}_{(F,F)}. \tag{1.155}$$

The sum is over all faces F. This is the *two-sided Takeuchi element* of $\mathcal{A}$. For $q = 0$, it coincides with (1.146). The square of the two-sided Takeuchi element is given by

$$\mathbf{Tak}[\mathcal{A}] \cdot \mathbf{Tak}[\mathcal{A}] = \sum_{(K,G):\, K \le G} (-1)^{\mathrm{rk}(K)+\mathrm{rk}(G)} q^{\mathrm{dist}(G, \overline{K}G)}\, \mathtt{H}_{(G, \overline{K}G)}. \tag{1.156}$$

The sum is over both K and G. (The rhs is indeed cancellation-free.) Moreover, for all $z \in \mathsf{J}_q[\mathcal{A}]$,

$$z \cdot \mathbf{Tak}[\mathcal{A}] = \mathbf{Tak}[\mathcal{A}] \cdot \overline{z}, \tag{1.157}$$

where

$$z = \sum_{(F,F')} x^{F,F'} \mathtt{H}_{(F,F')} \quad \text{and} \quad \overline{z} = \sum_{(F,F')} x^{F,F'} \mathtt{H}_{(\overline{F},\overline{F'})}.$$

1.11. Orientation space and signature space

The orientation space and signature space are one-dimensional vector spaces associated to any arrangement. The former is constructed from maximal chains, and the latter from chambers. While the two constructions are quite different, they do share some common features. These ideas are developed more formally in Section 4.8 using the language of operads. We mention in passing that one can associate a signature space to any log-antisymmetric distance function, with the signed distance function recovering the notion discussed here.

1.11.1. Orientation space. For any arrangement $\mathcal{A}$, let $\mathsf{mc}[\mathcal{A}]$ denote the space spanned by maximal chains in the poset of faces $\Sigma[\mathcal{A}]$. For any flat Y, there is a map

$$\mathsf{mc}[\mathcal{A}^{\mathrm{Y}}] \otimes \mathsf{mc}[\mathcal{A}_{\mathrm{Y}}] \to \mathsf{mc}[\mathcal{A}] \tag{1.158}$$

obtained by concatenating maximal chains. More precisely, let f be a maximal chain of faces in $\mathcal{A}^{\mathrm{Y}}$. It defines a chain of faces f' in $\mathcal{A}$ which ends at a face F with support Y. Now let g be a maximal chain of faces in $\mathcal{A}_{\mathrm{Y}}$. Using the canonical identification $\Sigma[\mathcal{A}_{\mathrm{Y}}] \xrightarrow{\cong} \Sigma[\mathcal{A}_F]$, we obtain a chain of faces g' in $\mathcal{A}$ which starts at F. The concatenation of f' and g' is the required maximal chain in $\mathcal{A}$.

For any arrangement $\mathcal{A}$, let $\mathsf{E^o}[\mathcal{A}]$ denote the quotient of $\mathsf{mc}[\mathcal{A}]$ by the relation that two maximal chains which differ in exactly one position are negatives of each other. We call $\mathsf{E^o}[\mathcal{A}]$ the *orientation space* of $\mathcal{A}$. It is one-dimensional. (See Lemma 1.4, item (1) for a relevant result.) We denote the image of a maximal chain f in the orientation space by $[f]$. An *orientation* of $\mathcal{A}$ is an element of $\mathsf{E^o}[\mathcal{A}]$ of the form $[f]$ for some maximal chain f. Any arrangement (of rank at least one) has exactly two orientations which we may denote by σ and $-\sigma$.

Example 1.73. For a rank-one arrangement, with chambers C and $\overline{C}$, there are two maximal chains, namely, $O \lessdot C$ and $O \lessdot \overline{C}$. Since they differ in exactly one position, we have

$$[O \lessdot C] = -[O \lessdot \overline{C}].$$

For any rank-two arrangement: A maximal chain has the form $O \lessdot P \lessdot C$. The relations can be expressed as

$$[O \lessdot P \lessdot C] = -[O \lessdot Q \lessdot C] \quad \text{and} \quad [O \lessdot P \lessdot C] = -[O \lessdot P \lessdot D],$$

where in the former P and Q are the two vertices of C, and in the latter C and D are the two chambers greater than P. We can think of the two orientations as clockwise and anticlockwise. This is illustrated below.

The six maximal chains which give the anticlockwise orientation are shown on the left, while the six which give the clockwise orientation are shown on the right.

There is a canonical isomorphism

$$\mathsf{E^o}[\mathcal{A}] \otimes \mathsf{E^o}[\mathcal{A}] \xrightarrow{\cong} \Bbbk, \qquad \sigma \otimes \sigma \mapsto 1, \tag{1.159}$$

where σ is either of the two orientations of $\mathcal{A}$. Changing σ to $-\sigma$ incurs two minus signs, so the map is well-defined.

Moreover, for any flat Y, there is an isomorphism

$$\mathsf{E^o}[\mathcal{A}^{\mathrm{Y}}] \otimes \mathsf{E^o}[\mathcal{A}_{\mathrm{Y}}] \xrightarrow{\cong} \mathsf{E^o}[\mathcal{A}] \tag{1.160}$$

induced from the map (1.158). By construction, the diagram

$$\begin{tikzcd} \mathsf{mc}[\mathcal{A}^{\mathrm{Y}}] \otimes \mathsf{mc}[\mathcal{A}_{\mathrm{Y}}] \arrow[r] \arrow[d, two heads] & \mathsf{mc}[\mathcal{A}] \arrow[d, two heads] \\ \mathsf{E^o}[\mathcal{A}^{\mathrm{Y}}] \otimes \mathsf{E^o}[\mathcal{A}_{\mathrm{Y}}] \arrow[r] & \mathsf{E^o}[\mathcal{A}] \end{tikzcd} \tag{1.161}$$

commutes.

1.11.2. Signature space. We proceed to define the signature space of an arrangement $\mathcal{A}$. Start with the space of chambers $\Gamma[\mathcal{A}]$. Form the quotient space in which any two chambers C and D are related by

$$\mathtt{H}_C = (-1)^{\mathrm{dist}(C,D)} \mathtt{H}_D.$$

It follows from gallery connectedness and (1.31) that the quotient space is one-dimensional. We call it the *signature space* of $\mathcal{A}$ and denote it by $\mathsf{E}^-[\mathcal{A}]$.

More generally:

Definition 1.74. Fix $\mathrm{Z} \le \mathrm{X}$. Consider the vector space with basis consisting of symbols $\mathtt{H}_{F/A}$ with $A \le F$, $\mathrm{s}(A) = \mathrm{Z}$ and $\mathrm{s}(F) = \mathrm{X}$. Form the quotient space in which any two such symbols $\mathtt{H}_{F/A}$ and $\mathtt{H}_{F'/A'}$ are related by

$$\mathtt{H}_{F/A} = (-1)^{\mathrm{dist}(F',A'F)} \mathtt{H}_{F'/A'}.$$

This quotient space is one-dimensional. We denote it by $\mathbf{E}^-[\mathrm{Z},\mathrm{X}]$.

We write $\mathtt{H}_{[F/A]}$ for the class of $\mathtt{H}_{F/A}$. Any such element is a basis of $\mathbf{E}^-[\mathrm{Z},\mathrm{X}]$. Observe that $\mathbf{E}^-[\bot,\top]$ equals the signature space $\mathsf{E}^-[\mathcal{A}]$.

There are linear isomorphisms

$$\begin{aligned} \mathbf{E}^-[\mathrm{Z},\mathrm{X}] \otimes \mathbf{E}^-[\mathrm{X},\mathrm{Y}] &\xrightarrow{\cong} \mathbf{E}^-[\mathrm{Z},\mathrm{Y}] \qquad & \mathbf{E}^-[\mathrm{Z},\mathrm{Z}] &\xrightarrow{\cong} \Bbbk \\ \mathtt{H}_{[F/A]} \otimes \mathtt{H}_{[G/F]} &\mapsto \mathtt{H}_{[G/A]} & \mathtt{H}_{[A/A]} &\mapsto 1, \end{aligned} \tag{1.162}$$

the first for any $\mathrm{Z} \le \mathrm{X} \le \mathrm{Y}$ and the second for any Z. For the first map, we choose any $A \le F \le G$ such that A has support Z, F has support X, G has support Y. In the second map, A is any face with support Z.

For any $\mathrm{Z} \le \mathrm{X}$ and $\mathrm{Z} \le \mathrm{Y}$, there is a linear isomorphism

$$\begin{gathered} \mathbf{E}^-[\mathrm{Z},\mathrm{X}] \otimes \mathbf{E}^-[\mathrm{X},\mathrm{X}\vee\mathrm{Y}] \xrightarrow{\cong} \mathbf{E}^-[\mathrm{Z},\mathrm{Y}] \otimes \mathbf{E}^-[\mathrm{Y},\mathrm{X}\vee\mathrm{Y}] \\ \mathtt{H}_{[F/A]} \otimes \mathtt{H}_{[FK/F]} \mapsto (-1)^{\mathrm{dist}(F,K)} \mathtt{H}_{[K/A]} \otimes \mathtt{H}_{[KF/K]}. \end{gathered} \tag{1.163}$$

We choose here any $A \le F$ and $A \le K$ such that A has support Z, F has support X, and K has support Y.

Exercise 1.75. Check that: For any $\mathrm{Z} \le \mathrm{X}$ and $\mathrm{Z} \le \mathrm{Y}$, the diagram

$$\begin{tikzcd} \mathbf{E}^-[\mathrm{Z},\mathrm{X}] \otimes \mathbf{E}^-[\mathrm{X},\mathrm{X}\vee\mathrm{Y}] \arrow[rr, "\cong"] \arrow[dr, "\cong"'] & & \mathbf{E}^-[\mathrm{Z},\mathrm{Y}] \otimes \mathbf{E}^-[\mathrm{Y},\mathrm{X}\vee\mathrm{Y}] \arrow[dl, "\cong"] \\ & \mathbf{E}^-[\mathrm{Z},\mathrm{X}\vee\mathrm{Y}] & \end{tikzcd}$$

commutes. The special cases $\mathrm{X} \le \mathrm{Y}$ or $\mathrm{Y} \le \mathrm{X}$ are also of interest.

1.12. Lie and Zie elements

We review Lie elements, its substitution map and presentation involving antisymmetry in rank one and Jacobi identities in rank two. This will be formalized later as the Lie operad in Section 4.5. We also review the related notion of Zie elements.

1.12.1. Lie elements. Recall the space of chambers $\Gamma[\mathcal{A}]$. We write a typical element as

$$z = \sum_C x^C \mathtt{H}_C.$$

This is a *Lie element* if

$$\sum_{C:\, HC=D} x^C = 0 \quad \text{for all} \quad O < H \le D. \tag{1.164}$$

Note very carefully that H is strictly greater than O. We denote the set of Lie elements by $\mathsf{Lie}[\mathcal{A}]$. It is a subspace of $\Gamma[\mathcal{A}]$. Its dimension equals the absolute value of the Möbius number of the arrangement:

$$\dim(\mathsf{Lie}[\mathcal{A}]) = |\mu(\mathcal{A})|. \tag{1.165}$$

See [**21**, Formula (10.24)].

Lemma 1.76. *For any face F, view $\mathsf{Lie}[\mathcal{A}_F]$ as a subspace of $\Gamma[\mathcal{A}]$ via the composite of inclusion maps*

$$\mathsf{Lie}[\mathcal{A}_F] \hookrightarrow \Gamma[\mathcal{A}_F] \hookrightarrow \Gamma[\mathcal{A}].$$

Then

$$\sum_{F \text{ not a chamber}} \mathsf{Lie}[\mathcal{A}_F] = \big\{ \sum x^C \mathtt{H}_C \in \Gamma[\mathcal{A}] \mid \sum x^C = 0 \big\}.$$

The sum is over all faces F which are not chambers.

PROOF. See [21, Lemma 10.4 and Exercise 13.29]. $\square$

1.12.2. Substitution maps. For any flat Y, there is a map

$$\Gamma[\mathcal{A}^{\mathrm{Y}}] \otimes \Gamma[\mathcal{A}_{\mathrm{Y}}] \to \Gamma[\mathcal{A}]. \tag{1.166}$$

This is the *substitution map* of chambers. To define this map, pick any face F with support Y, consider the map

$$\Gamma[\mathcal{A}^{\mathrm{Y}}] \otimes \Gamma[\mathcal{A}_F] \to \Gamma[\mathcal{A}], \qquad \mathtt{H}_H \otimes \mathtt{H}_{C/F} \mapsto \mathtt{H}_{HC},$$

and identify $\Gamma[\mathcal{A}_F]$ with $\Gamma[\mathcal{A}_{\mathrm{Y}}]$. The result does not depend on the particular choice of F.

The map (1.166) restricts to the space of Lie elements; thus, for any flat Y, there is a map

$$\mathsf{Lie}[\mathcal{A}^{\mathrm{Y}}] \otimes \mathsf{Lie}[\mathcal{A}_{\mathrm{Y}}] \to \mathsf{Lie}[\mathcal{A}]. \tag{1.167}$$

This is the *substitution map* of Lie elements. By construction, the diagram

$$\begin{tikzcd} \Gamma[\mathcal{A}^{\mathrm{Y}}] \otimes \Gamma[\mathcal{A}_{\mathrm{Y}}] \arrow[r] & \Gamma[\mathcal{A}] \\ \mathsf{Lie}[\mathcal{A}^{\mathrm{Y}}] \otimes \mathsf{Lie}[\mathcal{A}_{\mathrm{Y}}] \arrow[r] \arrow[u, hook] & \mathsf{Lie}[\mathcal{A}] \arrow[u, hook] \end{tikzcd} \tag{1.168}$$

commutes. For more details, see [21, Proposition 10.42].

1.12.3. Antisymmetry and Jacobi identity. For $\mathcal{A}$ of rank one, $\mathsf{Lie}[\mathcal{A}]$ is one-dimensional. For a Lie element, the coefficients of the two chambers are negatives of each other. The simplest choices are 1 and -1. Either of them spans $\mathsf{Lie}[\mathcal{A}]$, and their sum is zero. This can be shown as follows.

$$\big(\overset{1}{\bullet} \qquad \overset{\overline{1}}{\bullet}\big) \;+\; \big(\overset{\overline{1}}{\bullet} \qquad \overset{1}{\bullet}\big) \;=\; 0. \tag{1.169}$$

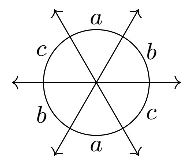
This is the *antisymmetry relation*. (By convention, $\overline{1}$ denotes -1.) Also note that

$$\mathsf{E}^{\mathsf{o}}[\mathcal{A}] \xrightarrow{\cong} \mathsf{Lie}[\mathcal{A}], \qquad [O \lessdot C] \mapsto \mathtt{H}_C - \mathtt{H}_{\overline{C}}. \tag{1.170}$$

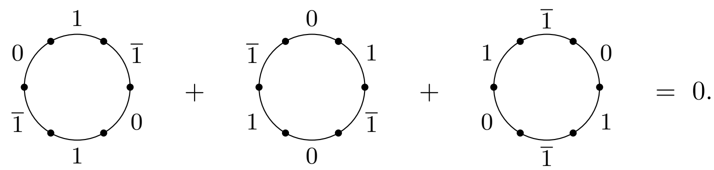
(Both spaces are 1-dimensional.)

For $\mathcal{A}$ the rank-two arrangement of n lines, $\mathsf{Lie}[\mathcal{A}]$ is $(n-1)$-dimensional. For a Lie element, the coefficients of the chambers (read in clockwise cyclic order) are $a_1, \dots, a_n, a_1, \dots, a_n$ subject to the condition $a_1 + \cdots + a_n = 0$.

For $n = 3$, a Lie element is

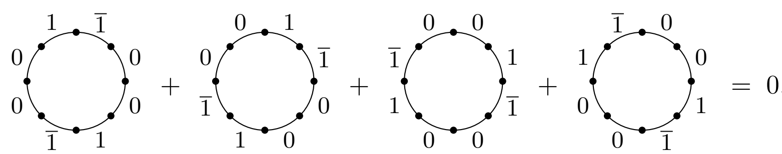


with $a + b + c = 0$. For example, one may take $a = 1$, $b = -1$, $c = 0$. Other similar choices are $a = 0$, $b = 1$, $c = -1$ and $a = -1$, $b = 0$, $c = 1$. Any two of these yield a basis for $\mathsf{Lie}[\mathcal{A}]$ and the sum of all three is 0. This can be shown as follows.

(1.171)


This is *Jacobi identity* for the hexagon.

In general, Jacobi identity consists of n terms adding up to 0. Each term is a $2n$-gon whose two adjacent sides (and their opposites) have coefficients 1 and $\bar{1}$, and the remaining sides have coefficient 0. For instance, for $n = 4$:

(1.172)


This is the *Jacobi identity* for the octagon.

The substitution map (1.167) combined with (1.170) yields

$$\bigoplus_{i=1}^{n} \mathsf{E}^{\mathbf{o}}[\mathcal{A}^{\mathrm{X}_i}] \otimes \mathsf{E}^{\mathbf{o}}[\mathcal{A}_{\mathrm{X}_i}] \to \mathsf{Lie}[\mathcal{A}],$$

where the X_i are the n lines (one-dimensional flats) of $\mathcal{A}$. This map is surjective. The lhs is n-dimensional, while the rhs is $(n-1)$-dimensional. The kernel is spanned by the element

$$\sum_{i=1}^{n} \tau^i \otimes \tau_i, \tag{1.173}$$

where τ^i and τ_i are orientations of $\mathcal{A}^{\mathrm{X}_i}$ and $\mathcal{A}_{\mathrm{X}_i}$ such that their concatenation (1.160) yields a fixed orientation of $\mathcal{A}$ (independent of i). This element corresponds to Jacobi identity.

1.12.4. Presentation. By iteration of (1.167), we see that for any maximal chain of flats $\bot \lessdot \mathrm{X}_1 \lessdot \cdots \lessdot \mathrm{X}_{r-1} \lessdot \top$, there is a map

$$\mathsf{Lie}[\mathcal{A}^{\mathrm{X}_1}] \otimes \mathsf{Lie}[\mathcal{A}^{\mathrm{X}_2}_{\mathrm{X}_1}] \otimes \cdots \otimes \mathsf{Lie}[\mathcal{A}_{\mathrm{X}_{r-1}}] \to \mathsf{Lie}[\mathcal{A}].$$

Note all arrangements involved in the lhs are of rank one. So, by identifying the tensor factors with orientation spaces, we obtain a map

$$\bigoplus_{z} \mathsf{E}^{\mathbf{o}}[\mathcal{A}^{\mathrm{X}_1}] \otimes \mathsf{E}^{\mathbf{o}}[\mathcal{A}^{\mathrm{X}_2}_{\mathrm{X}_1}] \otimes \cdots \otimes \mathsf{E}^{\mathbf{o}}[\mathcal{A}_{\mathrm{X}_{r-1}}] \to \mathsf{Lie}[\mathcal{A}], \tag{1.174}$$

where the sum is over all maximal chains of flats $z = (\bot \lessdot \mathrm{X}_1 \lessdot \cdots \lessdot \mathrm{X}_{r-1} \lessdot \top)$. The map (1.174) is surjective and its kernel is the subspace generated by the elements (1.173). We state this as follows.

Theorem 1.77. *The space* $\mathsf{Lie}[\mathcal{A}]$ *is freely generated by the orientation space in rank one subject to Jacobi identities in rank two.*

Proof. This is a nontrivial result which was proved in [**21**, Theorem 14.41]. For a more formal statement, see Example 4.12. ☐

1.12.5. Zie elements. Consider the Tits algebra $\Sigma[\mathcal{A}]$. We write a typical element as

$$z = \sum_F x^F \mathtt{H}_F.$$

This is a *Zie element* if

$$\sum_{F\colon HF=G} x^F = 0 \quad \text{for all} \quad O < H \leq G. \tag{1.175}$$

(Any Lie element is a Zie element.) We denote the set of Zie elements by $\mathsf{Zie}[\mathcal{A}]$. It is a subspace of $\Sigma[\mathcal{A}]$. Its dimension is given by the formula

$$\dim(\mathsf{Zie}[\mathcal{A}]) = \sum_{\mathrm{X}} |\mu(\mathcal{A}^{\mathrm{X}})|. \tag{1.176}$$

The sum is over all flats X. See [**21**, Formula (10.25)]. In fact, there is an isomorphism

$$\mathsf{Zie}[\mathcal{A}] \xrightarrow{\cong} \bigoplus_{\mathrm{X}} \mathsf{Lie}[\mathcal{A}^{\mathrm{X}}]. \tag{1.177}$$

See [**21**, Lemma 13.34]. The isomorphism (1.177) depends on the choice of $\mathtt{Q}$-basis of the Tits algebra. Explicitly, write the Zie element as $z = \sum_F x^F \mathtt{Q}_F$. Then, for each flat X, the element $\sum_{F\colon \mathrm{s}(F)=\mathrm{X}} x^F \mathtt{H}_F$ is a Lie element of $\mathcal{A}^{\mathrm{X}}$.

A Zie element z is *special* if the coefficient in z of the central face is 1, that is, if $x^O = 1$. Such elements do exist, a proof is given for instance in Exercise 7.71. A more precise result is stated below.

Lemma 1.78. *The first eulerian idempotent* $\mathtt{E}_\bot$ *of any eulerian family* $\mathtt{E}$ *of the Tits algebra is a special Zie element. Conversely, any special Zie element arises in this manner.*

Proof. This is contained in [**21**, Lemma 11.42]. ☐

In conjunction with formula (1.114), we see that special Zie elements are intimately connected to noncommutative Möbius functions. The precise relationship is given by Lemma 1.13.

1.13. Braid arrangement

We provide a brief recap on the combinatorics of the braid arrangement. For more details, see [**21**, Chapter 6, particularly Section 6.3].

1.13.1. Braid arrangement. Let J be a finite set. The *braid arrangement* on J consists of the hyperplanes

$$x_a = x_b$$

in $\mathbb{R}^J$, as a and b vary over elements of J with $a \neq b$. We denote it by $\mathcal{B}^J$.

The central face is the diagonal line where all the coordinates are equal to each other. The rank of $\mathcal{B}^J$ is $n-1$, where $n = |J|$.

1.13.2. Faces and compositions. A *composition* of the set J is a finite sequence $(S_1, \dots, S_k)$ of disjoint nonempty subsets of J such that

$$J = \bigsqcup_{i=1}^{k} S_i.$$

The subsets S_i are the blocks of the composition. Faces of $\mathcal{B}^J$ are in correspondence with compositions of J: Given a composition, the corresponding face is defined by the equations

$$\begin{cases} x_a = x_b & \text{if } a \text{ and } b \text{ lie in a common block,} \\ x_a \leq x_b & \text{if the block of } a \text{ precedes the block of } b. \end{cases}$$

We use F to denote either a composition or the corresponding face, and write $F \vDash J$.

A *linear order* on the set J is a composition of J into singleton blocks. Chambers of $\mathcal{B}^J$ are in correspondence with linear orders on J. The central face corresponds to the unique composition into one block. Inclusion of faces corresponds to refinement of compositions: $F \leq G$ if G is obtained by refining a number of blocks of F into smaller blocks. The opposite of $F = (S_1, \dots, S_k)$ is $\overline{F} = (S_k, \dots, S_1)$.

The Tits product of two faces $F = (S_1, \dots, S_p)$ and $G = (T_1, \dots, T_q)$ is the composition FG whose blocks are those pairwise intersections

$$A_{ij} := S_i \cap T_j$$

that are nonempty, ordered lexicographically in (i, j). For example,

$$(acde|bfg)(cdfg|b|ae) = (cd|ae|fg|b).$$

1.13.3. Flats and partitions. A *partition* of the set J is the set of blocks occurring in a composition of J, unprovided of any order. Flats of $\mathcal{B}^J$ are in correspondence with partitions of J. The flat corresponding to a partition is the subspace defined by the equations $x_a = x_b$ if a and b lie in a common block. We use X to denote either a partition or the corresponding flat, and write $\mathrm{X} \vdash J$.

Inclusion of flats corresponds to refinement of partitions. The maximum flat is the unique partition into singletons and the minimum flat the unique one-block partition. The support of a face $F = (S_1, \dots, S_k)$ is the flat $\mathrm{X} = \{S_1, \dots, S_k\}$.

1.13.4. Arrangements under and over a flat. The arrangement under a flat X of the braid arrangement $\mathcal{B}^J$ is combinatorially isomorphic to the braid arrangement on the set X, in which each block of X plays the role of one letter:

$$(\mathcal{B}^J)^{\mathrm{X}} \cong \mathcal{B}^{\mathrm{X}}.$$

The arrangement over X is combinatorially isomorphic to a cartesian product of braid arrangements: if $\mathrm{X} = \{S_1, \dots, S_k\}$, then

$$(\mathcal{B}^J)_{\mathrm{X}} \cong \mathcal{B}^{S_1} \times \cdots \times \mathcal{B}^{S_k}.$$

Notes

The main reference for this chapter is [**21**]. A few details along with some other historical references are given below.

Hyperplane arrangements. Most results of this chapter have appeared in the literature, though some of them are quite recent. We mention some important original sources. The Tits monoid arose in independent work of Tits [**883**, Section 2.30], [**884**, Proposition 1] and Bland [**119**, Section 5, page 62]. The gate property also originated in work of Tits [**883**, Section 3.19.6]. The (opposite) Tits algebra of the braid arrangement appeared in work of Epstein, Glaser, Stora [**291**, Section 1], [**292**, Section 4.1], see also [**290**, Definition 2.1]. Influential work on the Tits algebra was carried out by Bidigare, Hanlon, Rockmore [**112**], [**113**], Brown and Diaconis [**162**], [**163**], [**164**], Schocker [**815**], Saliola [**795**]. For the more general setting of left regular band algebras, see the papers by Margolis, Saliola, Steinberg [**651**], [**652**] and references therein. The split-semisimplicity of the Birkhoff algebra originated in work of Solomon [**838**, Theorem 1] and Greene [**372**, Section 1]. The support map from top-nested faces to top-lunes appeared in our monograph [**17**, Section 2.3]. The Zaslavsky formula (1.84) appeared in work of Zaslavsky [**927**, Theorem A] and of Las Vergnas in the more general context of oriented matroids [**568**, Proposition 8.1], [**569**, Theorem 3.1]. Computations of determinants of matrices associated to distance functions on hyperplane arrangements go back to Varchenko [**895**], in particular, Theorem 1.10 follows from his work. The distance function on faces (1.24) appeared in our monograph [**18**, Section 10.5.3]. Möbius functions of posets originated in work of Möbius [**701**] and their early theory was developed by Weisner [**906**], Ward [**902**], Hall [**410**] and Rota [**792**]. Lunes are considered by Bhaskaracharya for computing the surface area of a sphere [**110**, Goladhyaya, Bhuvanakosha, Verses 58 – 61]; the Sanskrit name for lune is 'vapraka'. To illustrate the decomposition of a sphere into lunes, Bhaskaracharya gives the example of the 'aamalaka' (gooseberry fruit).

Detailed historical references on all of the above topics can be found in our book [**21**]. Pointers to this book are as follows. The Birkhoff monoid and the Tits monoid are in [**21**, Chapter 1], cones are in [**21**, Chapter 2], lunes are in [**21**, Chapter 3], partial-flats are in [**21**, Section 2.8], distance functions and Varchenko matrices are in [**21**, Chapter 8], the category of lunes is in [**21**, Chapter 4], the lune-incidence algebra is in [**21**, Chapter 15], the Birkhoff algebra and the Tits algebra are in [**21**, Chapters 9 and 11], the Janus algebra is in [**21**, Section 9.9], the Takeuchi element and the two-sided Takeuchi element are in [**21**, Section 12.3], the orientation space is in [21, Section 14.4], Lie elements and Zie elements are in [21, Chapters 10 and 14], descent, lune, Witt identities are in [21, Chapter 7], gated sets are in [21, Appendix A]. We mention that partial-flats, h-faces and h-flats, abstract distance functions on LRBs, the category of lunes, the lune-incidence algebra, noncommutative zeta and Möbius functions, the Janus algebra, the Takeuchi element, Lie and Zie elements and the substitution maps, and most of the descent and lune identities appeared for the first time in [21]. Lie and Zie elements can be characterized is various ways such as the Friedrichs criterion, Ree criterion, Garsia criterion. We mention that some elements of Lie theory for arrangements were implicitly present in earlier work of Saliola [795].

The eulerian idempotents of the Tits algebra in Section 1.9.2 are directly constructed out of the noncommutative zeta and Möbius functions. This is different from the path taken in [21], where the eulerian idempotents are first constructed recursively taking (1.118) as the starting point, with the connection to zeta and Möbius functions brought out later. The flat-additivity formula (1.43) is closely connected to stationary distributions of random walks driven by elements of the Tits algebra [21, Theorem 12.17, Lemma 15.18].

Bilune-incidence algebra. The material in Section 1.6 which includes the bilune-incidence algebra, the category of bilunes, two-sided q-zeta and q-Möbius functions appears here for the first time. The closely related noncommutative q-zeta and q-Möbius functions in Section 1.5.9 have also not been explicitly considered before. However, the linear system (1.48) has indeed been formulated and studied in [21, (8.45) and Theorem 8.25].

Noncommutative Zaslavsky formula. Lemmas 1.44 and 1.45 are new, so are the noncommutative Zaslavsky formulas in Lemma 1.46. Identity (1.151) is new; the special case (1.152) is equivalent to [21, Identity (12.24)] in view of (1.112), but the argument given here is different. Identities (1.96) – (1.97) are equivalent to the Zaslavsky formula by Möbius inversion. They can also be deduced from a result of Kung [551, Theorem 4]. We mention that an interesting example of (1.87b), (1.90), Exercise 1.50 appears in work of McMullen [669] and Amelunxen and Lotz [27]. Details will appear elsewhere.

q-Janus algebra. Exercise 1.58 on the q-Janus algebra is given in [21, Exercise 9.72] only for $q = 1$. The material related to the $\mathtt{Q}'$-basis and the $\mathtt{Q}^d$-basis, and, in particular, formula (1.128) is new. The two-sided Takeuchi element in Section 1.10.3 is considered in [21, Section 12.3.6] only for $q = 1$.

General references. Some useful books on hyperplane arrangements are by De Concini and Procesi [236], Dimca [252], Orlik and Terao [727], [728], [729], Stanley [844]. Earlier sources are those of Grünbaum [381], Cartier [199], Jambu [482]. For emphasis on reflection arrangements and Coxeter theory, see for instance the books by Abramenko and Brown [2], Björner and Brenti [116], Borovik and Borovik [139], Borovik, Gelfand, White [140, Chapters 5 and 7], Bourbaki [148], Davis [229], Grove and Benson [380], Humphreys [474], Kane [509], Tits [883]. Short introductions can be found in [17, Chapter 1], [41, Section 6], [293, Section 3], [750, Chapter 11], [845, Section 3.11].

Oriented matroids and convex polytopes are two notions closely related to hyperplane arrangements. The standard reference for oriented matroids is the book by Björner, Las Vergnas, Sturmfels, White, Ziegler [117]. For convex polytopes, see the books by Ziegler [931], McMullen and Schulte [670], Grünbaum [382].

For semigroups, we mention the book by Steinberg [850].

CHAPTER 2

Species and bimonoids

Let $\mathcal{A}$ be an arbitrary but fixed hyperplane arrangement. We introduce the notion of an $\mathcal{A}$-species. Roughly speaking, an $\mathcal{A}$-species, denoted p, attaches a vector space $\mathsf{p}[F]$ to every face F of $\mathcal{A}$ along with linear isomorphisms

$$\beta_{G,F} : \mathsf{p}[F] \to \mathsf{p}[G]$$

whenever F and G have the same support. Next, we introduce the notion of an $\mathcal{A}$-monoid. This is an $\mathcal{A}$-species equipped with "product" maps

$$\mu_A^F : \mathsf{p}[F] \to \mathsf{p}[A],$$

one for each pair of faces $A \leq F$, subject to naturality, associativity, unitality axioms. For instance, associativity says that $\mu_A^F \mu_F^G = \mu_A^G$ whenever $A \leq F \leq G$. There is also a dual notion of an $\mathcal{A}$-comonoid defined using "coproduct" maps

$$\Delta_A^F : \mathsf{p}[A] \to \mathsf{p}[F],$$

and a mixed self-dual notion of an $\mathcal{A}$-bimonoid. Formally, there is a duality functor on the category of $\mathcal{A}$-species which interchanges $\mathcal{A}$-monoids and $\mathcal{A}$-comonoids, and preserves $\mathcal{A}$-bimonoids.

We also define commutativity for an $\mathcal{A}$-monoid, and dually cocommutativity for an $\mathcal{A}$-comonoid. An $\mathcal{A}$-bimonoid could be commutative, cocommutative, both or neither. A commutative $\mathcal{A}$-monoid can also be defined directly as an $\mathcal{A}$-species equipped with "product" maps

$$\mu_{\mathrm{Z}}^{\mathrm{X}} : \mathsf{p}[\mathrm{X}] \to \mathsf{p}[\mathrm{Z}],$$

one for each pair of flats $\mathrm{Z} \leq \mathrm{X}$, subject to associativity and unitality axioms. Note that in this formulation, an $\mathcal{A}$-species has a vector space $\mathsf{p}[\mathrm{X}]$ attached to each flat X of $\mathcal{A}$. This is possible since $\mathsf{p}[F]$ and $\mathsf{p}[G]$ are isomorphic whenever F and G have the same support X.

By associativity, an $\mathcal{A}$-monoid is completely determined by those μ_A^F in which F covers A. This allows us to formulate $\mathcal{A}$-monoids using such "covering" generators. Associativity among the covering generators can be expressed as $\mu_A^F \mu_F^G = \mu_A^{F'} \mu_{F'}^G$ whenever $\mathrm{rk}(G/A) = 2$, and F and F' are the two faces which lie strictly between A and G. The same can be done for $\mathcal{A}$-comonoids by replacing μ by Δ. For (co)commutative $\mathcal{A}$-(co)monoids, we use flats instead of faces.

In addition to the above, we discuss related objects such as q-bimonoids (which include bimonoids, signed bimonoids, 0-bimonoids), signed commutative monoids, and partially commutative monoids. The latter interpolate between monoids and commutative monoids. Similarly, the parallel between 0-bimonoids and bicommutative bimonoids is strengthened by introducing interpolating objects termed 0-$\sim$-bimonoids.

Species also have a set-theoretic analogue where vector spaces are replaced by sets. We refer to these as set-species. In the same vein, we have set-monoids, set-comonoids, set-bimonoids. However, one cannot define q-bimonoids or signed commutative monoids in this setting.

The notion of $\mathcal{A}$-species when specialized to the braid arrangements relates to the classical notion of Joyal species. Important basic examples of species and bimonoids are given in Chapter 7, and these could be read in parallel with the present and subsequent chapters.

Convention 2.1. Whenever convenient, we may drop $\mathcal{A}$ from the notation, and simply write species instead of $\mathcal{A}$-species, monoid instead of $\mathcal{A}$-monoid, and so on.

Convention 2.2. For objects a and b in a category C, we write $\mathsf{C}(a, b)$ for the set of morphisms from a to b. We denote an inclusion functor by *inc*, and forgetful functor by *frg*. An adjunction between categories C and D with $\mathcal{F}$ as the left adjoint and $\mathcal{G}$ as the right adjoint is denoted $\mathsf{C} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathsf{D}$.

2.1. Species

We introduce species. Let $\Bbbk$ be a field. This field will remain fixed throughout the text. Its characteristic will have no role to play in anything that we do. Let Vec denote the category of vector spaces over $\Bbbk$. The category of species is the category of functors from $\mathcal{A}$-$\mathtt{Hyp}$ to Vec, where $\mathcal{A}$-$\mathtt{Hyp}$ is a certain category (with a finite number of objects and morphisms) constructed from the given arrangement $\mathcal{A}$.

2.1.1. Base category. The category $\mathcal{A}$-$\mathtt{Hyp}$ is defined as follows. An object is a face of $\mathcal{A}$, and there is a unique morphism from one face to another face whenever the two faces have the same support.

When F and G have the same support, we write $\beta_{G,F} : F \to G$ for the unique morphism between them.

Proposition 2.3. *The category $\mathcal{A}$-$\mathtt{Hyp}$ has a presentation given by generators*

$$\beta_{G,F} : F \to G,$$

whenever F and G have the same support, and relations

$$\begin{tikzcd} & G \arrow[dr, "\beta_{H,G}"] & \\ F \arrow[ur, "\beta_{G,F}"] \arrow[rr, "\beta_{H,F}"'] & & H \end{tikzcd} \qquad\qquad (F \xrightarrow{\beta_{F,F}} F) = \mathrm{id},$$

the former whenever F, G, H have the same support, and the latter for any F.

It follows that $\beta_{G,F}$ and $\beta_{F,G}$ are inverse isomorphisms, that is, the diagrams

$$\begin{tikzcd} & G \arrow[dr, "\beta_{F,G}"] & \\ F \arrow[ur, "\beta_{G,F}"] \arrow[rr, "\mathrm{id}"'] & & F \end{tikzcd} \qquad \begin{tikzcd} & F \arrow[dr, "\beta_{G,F}"] & \\ G \arrow[ur, "\beta_{F,G}"] \arrow[rr, "\mathrm{id}"'] & & G \end{tikzcd}$$

commute, whenever F and G have the same support. We express this relation by writing $\beta^2 = \mathrm{id}$.

Since flats are gallery connected, the morphisms $\beta_{G,F}$ with F and G adjacent provide a smaller generating set. Composites of generators are now indexed by galleries, so the relations are: two galleries are the same if they have the same starting and ending face.

Proposition 2.4. *The category $\mathcal{A}$-$\mathtt{Hyp}$ is a disjoint union of indiscrete categories. In particular, it is a groupoid, that is, all morphisms are invertible. Further, it is equivalent to the discrete category whose objects are indexed by flats of $\mathcal{A}$.*

Proof. When F and G have different supports, there is no morphism between them. So $\mathcal{A}$-$\mathtt{Hyp}$ breaks as a union of connected pieces, one for each flat of $\mathcal{A}$. Further, each connected piece is an indiscrete category, meaning that there is exactly one morphism from one object to another. This proves the first statement. Since any indiscrete category is equivalent to the one-arrow category, the last statement follows. $\square$

Let us elaborate on the equivalence in Proposition 2.4. Let $\mathcal{A}$-$\mathtt{Hyp}'$ denote the discrete category on the set of flats of $\mathcal{A}$, that is, its objects are flats of $\mathcal{A}$, and the only morphisms are identities. There is a functor from $\mathcal{A}$-$\mathtt{Hyp}$ to $\mathcal{A}$-$\mathtt{Hyp}'$ which sends a face to its support. Conversely, for any flat X choose a face of support X, and define a functor from $\mathcal{A}$-$\mathtt{Hyp}'$ to $\mathcal{A}$-$\mathtt{Hyp}$ which sends a flat to its chosen face. The two functors define an equivalence between the categories $\mathcal{A}$-$\mathtt{Hyp}$ and $\mathcal{A}$-$\mathtt{Hyp}'$.

2.1.2. Species. An $\mathcal{A}$-*species* is a functor

$$\mathsf{p} : \mathcal{A}\text{-}\mathtt{Hyp} \to \mathsf{Vec}.$$

A map of $\mathcal{A}$-species $\mathsf{p} \to \mathsf{q}$ is a natural transformation. This defines the category of $\mathcal{A}$-species which we denote by $\mathcal{A}$-$\mathtt{Sp}$. It is a functor category, and we also write

$$\mathcal{A}\text{-}\mathtt{Sp} = [\mathcal{A}\text{-}\mathtt{Hyp}, \mathsf{Vec}].$$

For species p and q, let $\mathcal{A}$-$\mathtt{Sp}(\mathsf{p}, \mathsf{q})$ denote the space of all maps from p to q. This is consistent with Convention 2.2.

The value of an $\mathcal{A}$-species p on an object F will be denoted $\mathsf{p}[F]$. We call it the F-component of p. By Proposition 2.3, we obtain:

An $\mathcal{A}$-species p consists of a family of vector spaces $\mathsf{p}[F]$, one for each face F of $\mathcal{A}$, together with linear maps

$$\beta_{G,F} : \mathsf{p}[F] \to \mathsf{p}[G],$$

whenever F and G have the same support, such that the diagrams

$$\begin{tikzcd} & \mathsf{p}[G] \arrow[dr, "\beta_{H,G}"] & \\ \mathsf{p}[F] \arrow[ur, "\beta_{G,F}"] \arrow[rr, "\beta_{H,F}"'] & & \mathsf{p}[H] \end{tikzcd} \qquad (\mathsf{p}[F] \xrightarrow{\beta_{F,F}} \mathsf{p}[F]) = \mathrm{id} \tag{2.1}$$

commute, the former whenever F, G, H have the same support, and the latter for any F.

We refer to the maps $\beta_{G,F}$ as the structure maps of p. Observe that $\beta_{G,F}$ and $\beta_{F,G}$ are inverse linear isomorphisms, that is, the diagrams

$$\begin{tikzcd} & \mathsf{p}[G] \arrow[dr, "\beta_{F,G}"] & \\ \mathsf{p}[F] \arrow[ur, "\beta_{G,F}"] \arrow[rr, "\mathrm{id}"'] & & \mathsf{p}[F] \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{p}[F] \arrow[dr, "\beta_{G,F}"] & \\ \mathsf{p}[G] \arrow[ur, "\beta_{F,G}"] \arrow[rr, "\mathrm{id}"'] & & \mathsf{p}[G] \end{tikzcd} \tag{2.2}$$

commute, whenever F and G have the same support.

Similarly, a map of $\mathcal{A}$-species $f : \mathsf{p} \to \mathsf{q}$ consists of a family of linear maps

$$f_F : \mathsf{p}[F] \to \mathsf{q}[F],$$

one for each face F, such that whenever F and G have the same support, the diagram

$$\begin{tikzcd} \mathsf{p}[F] \arrow[r, "f_F"] \arrow[d, "\beta_{G,F}"'] & \mathsf{q}[F] \arrow[d, "\beta_{G,F}"] \\ \mathsf{p}[G] \arrow[r, "f_G"'] & \mathsf{q}[G] \end{tikzcd} \tag{2.3}$$

commutes. We refer to the map f_F as the F-component of f.

An $\mathcal{A}$-species p is *finite-dimensional* if its F-component $\mathsf{p}[F]$ has finite dimension for all faces F.

2.1.3. Product and coproduct. The *zero species* $\mathsf{0}$ is the $\mathcal{A}$-species all of whose components are zero:

$$\mathsf{0}[F] = 0. \tag{2.4}$$

This is the initial and terminal object in the category of $\mathcal{A}$-species.

For $\mathcal{A}$-species p and q, their *direct sum* $\mathsf{p} + \mathsf{q}$ is defined by

$$(\mathsf{p} + \mathsf{q})[A] := \mathsf{p}[A] \oplus \mathsf{q}[A], \tag{2.5}$$

with the linear maps $\beta_{G,F}$ of $\mathsf{p}+\mathsf{q}$ induced from those of p and q. This is the product and coproduct in the category of $\mathcal{A}$-species. It is clear that arbitrary products and coproducts also exist in this category.

2.1.4. Reformulation in terms of flats. Recall from the discussion after Proposition 2.4 that the category $\mathcal{A}$-$\mathtt{Hyp}$ is equivalent to $\mathcal{A}$-$\mathtt{Hyp}'$. As a consequence, the functor categories

$$[\mathcal{A}\text{-}\mathtt{Hyp}, \mathsf{Vec}] \quad \text{and} \quad [\mathcal{A}\text{-}\mathtt{Hyp}', \mathsf{Vec}]$$

are equivalent. The latter category yields the following reformulation of $\mathcal{A}$-species.

Proposition 2.5. *An $\mathcal{A}$-species p is a family $\mathsf{p}[\mathrm{X}]$ of vector spaces, one for each flat* X. *A map of $\mathcal{A}$-species $f : \mathsf{p} \to \mathsf{q}$ is a family of linear maps*

$$f_{\mathrm{X}} : \mathsf{p}[\mathrm{X}] \to \mathsf{q}[\mathrm{X}],$$

one for each flat X.

We elaborate on the connection between the two points of view. Starting with an $\mathcal{A}$-species p as above, one can set $\mathsf{p}[F] := \mathsf{p}[\mathrm{X}]$ for all F with support X, and $\beta_{G,F} = \mathrm{id}$ whenever F and G have the same support. This yields an $\mathcal{A}$-species in the first sense.

Conversely, given an $\mathcal{A}$-species p in the first sense, define $\mathsf{p}[\mathrm{X}] := \mathsf{p}[H]$ where H is an arbitrary but fixed face of support X, and we get an $\mathcal{A}$-species as above. Note that for any face F with support X, we have inverse isomorphisms

$$\beta_{F,\mathrm{X}} : \mathsf{p}[\mathrm{X}] \to \mathsf{p}[F] \quad \text{and} \quad \beta_{\mathrm{X},F} : \mathsf{p}[F] \to \mathsf{p}[\mathrm{X}] \tag{2.6}$$

and identities such as

$$\beta_{\mathrm{X},F} = \beta_{\mathrm{X},G}\beta_{G,F} \quad \text{and} \quad \beta_{G,F} = \beta_{G,\mathrm{X}}\beta_{\mathrm{X},F}$$

always hold. Also observe that for any flat X,

$$\mathsf{p}[\mathrm{X}] = \operatorname*{colim}_{F} \mathsf{p}[F] = \lim_{F} \mathsf{p}[F], \tag{2.7}$$

where the (co)limit is taken over morphisms $\beta_{G,F} : F \to G$, where F and G run over all faces with support X. (Since this is an indiscrete category, the (co)limit exists. It can be taken to be the value of p on any particular object. This is exactly how we defined $\mathsf{p}[\mathrm{X}]$.) Further, the maps (2.6) are then the canonical maps from the limit and to the colimit.

The point of view given by Proposition 2.5 is useful (and we will make use of it) while dealing with commutative aspects of the theory. However, this apparently simpler definition can be awkward in other situations.

2.2. Monoids, comonoids, bimonoids

One can define monoids, comonoids, and bimonoids in species. There is a notion of commutativity for monoids, and cocommutativity for comonoids. A bimonoid may be commutative, cocommutative, both, or neither. The resulting categories along with their notations are summarized in Table 2.1.

TABLE 2.1. Categories of (co, bi)monoids in species.

Category	Description	Category	Description
$\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})$	monoids	${}^{\mathsf{co}}\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})$	cocomm. comonoids
$\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})$	comonoids	$\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})$	comm. bimonoids
$\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$	bimonoids	${}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$	cocomm. bimonoids
$\mathsf{Mon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})$	comm. monoids	${}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})$	bicomm. bimonoids

For monoids a and b, let $\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{a}, \mathsf{b})$ denote the space of all monoid morphisms from a to b. Similar notations will be employed for the remaining categories in Table 2.1. This is consistent with Convention 2.2.

We first discuss monoids, comonoids, bimonoids. Commutativity aspects will be treated in Section 2.3.

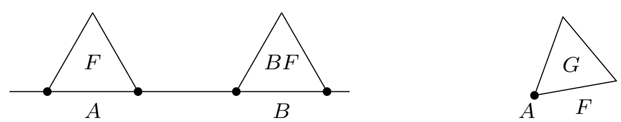
2.2.1. Monoids. An $\mathcal{A}$-*monoid* is an $\mathcal{A}$-species a equipped with linear maps

$$\mu_A^F : \mathsf{a}[F] \to \mathsf{a}[A],$$

one for each pair of faces $A \leq F$, such that the diagrams

$$\begin{tikzcd} \mathsf{a}[F] \arrow[r, "\beta_{BF,F}"] \arrow[d, "\mu_A^F"'] & \mathsf{a}[BF] \arrow[d, "\mu_B^{BF}"] \\ \mathsf{a}[A] \arrow[r, "\beta_{B,A}"'] & \mathsf{a}[B] \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{a}[F] \arrow[dr, "\mu_A^F"] & \\ \mathsf{a}[G] \arrow[ur, "\mu_F^G"] \arrow[rr, "\mu_A^G"'] & & \mathsf{a}[A] \end{tikzcd} \qquad (\mathsf{a}[A] \xrightarrow{\mu_A^A} \mathsf{a}[A]) = \mathrm{id} \tag{2.8}$$

commute. The first diagram is to be considered whenever A and B have the same support and $A \leq F$, the second diagram for every $A \leq F \leq G$, and the third diagram for every A. Illustrative pictures for the first two diagrams are shown below.



We refer to (2.8) as the *naturality*, *associativity*, *unitality* axioms, respectively. We denote an $\mathcal{A}$-monoid by a pair (a, μ), or simply by a with μ understood. We refer to the maps μ_A^F as the product components or structure maps of a.

A morphism $f : \mathsf{a} \to \mathsf{b}$ of $\mathcal{A}$-monoids is a map of $\mathcal{A}$-species such that for each $A \leq F$, the diagram

$$\begin{tikzcd} \mathsf{a}[F] \arrow[r, "f_F"] \arrow[d, "\mu_A^F"'] & \mathsf{b}[F] \arrow[d, "\mu_A^F"] \\ \mathsf{a}[A] \arrow[r, "f_A"'] & \mathsf{b}[A] \end{tikzcd} \tag{2.9}$$

commutes.

2.2.2. Comonoids. Dually, an $\mathcal{A}$-*comonoid* is an $\mathcal{A}$-species c equipped with linear maps

$$\Delta_A^F : \mathsf{c}[A] \to \mathsf{c}[F],$$

one for each $A \leq F$, such that the diagrams

$$\begin{tikzcd} \mathsf{c}[F] \arrow[r, "\beta_{BF,F}"] & \mathsf{c}[BF] \\ \mathsf{c}[A] \arrow[u, "\Delta_A^F"] \arrow[r, "\beta_{B,A}"'] & \mathsf{c}[B] \arrow[u, "\Delta_B^{BF}"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{c}[F] \arrow[dr, "\Delta_F^G"] & \\ \mathsf{c}[A] \arrow[ur, "\Delta_A^F"] \arrow[rr, "\Delta_A^G"'] & & \mathsf{c}[G] \end{tikzcd} \qquad (\mathsf{c}[A] \xrightarrow{\Delta_A^A} \mathsf{c}[A]) = \mathrm{id} \tag{2.10}$$

commute. The first diagram is to be considered whenever A and B have the same support and $A \leq F$, the second diagram for every $A \leq F \leq G$, and the third diagram for every A.

We refer to these as the *naturality*, *coassociativity*, *counitality axioms*, respectively. We denote an $\mathcal{A}$-comonoid by a pair (c, Δ), or simply by c with Δ understood. We refer to the maps Δ_A^F as the coproduct components or structure maps of c.

A morphism $f : \mathsf{c} \to \mathsf{d}$ of $\mathcal{A}$-comonoids is a map of $\mathcal{A}$-species such that for each $A \leq F$, the diagram

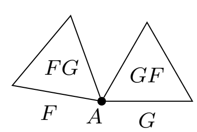
$$
\begin{tikzcd}
\mathsf{c}[F] \arrow[r, "f_F"] & \mathsf{d}[F] \\
\mathsf{c}[A] \arrow[u, "\Delta_A^F"] \arrow[r, "f_A"'] & \mathsf{d}[A] \arrow[u, "\Delta_A^F"']
\end{tikzcd}
\tag{2.11}
$$

commutes.

2.2.3. Bimonoids. An $\mathcal{A}$-*bimonoid* is a triple $(\mathsf{h}, \mu, \Delta)$, where h is an $\mathcal{A}$-species, (h, μ) is an $\mathcal{A}$-monoid, (h, Δ) is an $\mathcal{A}$-comonoid, and for any faces $A \leq F$ and $A \leq G$, the diagram

$$
\begin{tikzcd}
\mathsf{h}[F] \arrow[r, "\mu_A^F"] \arrow[d, "\Delta_F^{FG}"'] & \mathsf{h}[A] \arrow[r, "\Delta_A^G"] & \mathsf{h}[G] \\
\mathsf{h}[FG] \arrow[rr, "\beta_{GF,FG}"'] & & \mathsf{h}[GF] \arrow[u, "\mu_G^{GF}"']
\end{tikzcd}
\tag{2.12}
$$

commutes. An illustrative picture is shown below.



We refer to diagram (2.12) as the *bimonoid axiom*. As a shorthand, we write it as $\Delta\mu = \mu\beta\Delta$.

A morphism of $\mathcal{A}$-bimonoids is a map of the underlying $\mathcal{A}$-species which is a morphism of the underlying $\mathcal{A}$-monoids and $\mathcal{A}$-comonoids.

2.2.4. Degeneracies of the bimonoid axiom. We now consider special cases of the bimonoid axiom (2.12). We begin with the situation in which one of the five arrows becomes identity, so the pentagon reduces to a square.

Lemma 2.6. *Let $(\mathsf{h}, \mu, \Delta)$ be an $\mathcal{A}$-bimonoid. Then: For faces F and G both greater than A, the diagrams*

$$
\begin{tikzcd}
\mathsf{h}[F] \arrow[r, "\mu_A^F"] \arrow[d, "\beta_{GF,F}"'] & \mathsf{h}[A] \arrow[d, "\Delta_A^G"] \\
\mathsf{h}[GF] \arrow[r, "\mu_G^{GF}"'] & \mathsf{h}[G]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{h}[F] \arrow[r, "\mu_A^F"] \arrow[d, "\Delta_F^{FG}"'] & \mathsf{h}[A] \arrow[d, "\Delta_A^G"] \\
\mathsf{h}[FG] = \mathsf{h}[GF] \arrow[r, "\mu_G^{GF}"'] & \mathsf{h}[G]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{h}[F] \arrow[r, "\mu_A^F"] \arrow[d, "\Delta_F^{FG}"'] & \mathsf{h}[A] \arrow[d, "\Delta_A^G"] \\
\mathsf{h}[FG] \arrow[r, "\beta_{G,FG}"'] & \mathsf{h}[G]
\end{tikzcd}
$$

commute, respectively, when $FG = F$, when $FG = GF$, when $GF = G$.

We now look at a further degenerate situation in which two of the five arrows becomes identity, so the pentagon reduces to a triangle.

Lemma 2.7. *Let $(\mathsf{h}, \mu, \Delta)$ be an $\mathcal{A}$-bimonoid. Then: For faces F and G both greater than A and of the same support, the diagram*

$$\begin{tikzcd} \mathsf{h}[F] \arrow[rr, "\beta_{G,F}"] \arrow[dr, "\mu^F_A"'] & & \mathsf{h}[G] \\ & \mathsf{h}[A] \arrow[ur, "\Delta^G_A"'] & \end{tikzcd} \tag{2.13}$$

commutes.

For $A \leq F \leq G$, the diagrams

$$\begin{tikzcd} & \mathsf{h}[A] \arrow[dr, "\Delta^G_A"] & \\ \mathsf{h}[F] \arrow[ur, "\mu^F_A"] \arrow[rr, "\Delta^G_F"'] & & \mathsf{h}[G] \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{h}[A] \arrow[dr, "\Delta^F_A"] & \\ \mathsf{h}[G] \arrow[ur, "\mu^G_A"] \arrow[rr, "\mu^G_F"'] & & \mathsf{h}[F] \end{tikzcd} \tag{2.14}$$

commute.

Finally, we have:

Lemma 2.8. *For any $A \leq F$, the diagram*

$$\begin{tikzcd} \mathsf{h}[F] \arrow[rr, "\mathrm{id}"] \arrow[dr, "\mu^F_A"'] & & \mathsf{h}[F] \\ & \mathsf{h}[A] \arrow[ur, "\Delta^F_A"'] & \end{tikzcd} \tag{2.15}$$

commutes.

As a consequence of (2.15):

Corollary 2.9. *In any $\mathcal{A}$-bimonoid, the product components are injective, while the coproduct components are surjective.*

Exercise 2.10. Show by an example that the result of Corollary 2.9 does not hold for arbitrary $\mathcal{A}$-monoids and $\mathcal{A}$-comonoids.

Exercise 2.11. Derive (2.14) as a formal consequence of (2.15), associativity and coassociativity.

Exercise 2.12. What happens to the bimonoid axiom (2.12) if either $F = A$ or $G = A$?

Lemma 2.13. *The bimonoid axiom (2.12) is equivalent to the following two axioms (assuming (co)associativity and (co)unitality).*

$$\begin{tikzcd} \mathsf{h}[F] \arrow[r, "\mu^F_{F\wedge G}"] \arrow[d, "\Delta^{FG}_F"'] & \mathsf{h}[F \wedge G] \arrow[r, "\Delta^G_{F\wedge G}"] & \mathsf{h}[G] \\ \mathsf{h}[FG] \arrow[rr, "\beta_{GF,FG}"'] & & \mathsf{h}[GF] \arrow[u, "\mu^{GF}_G"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{h}[F] \arrow[rr, "\mathrm{id}"] \arrow[dr, "\mu^F_A"'] & & \mathsf{h}[F] \\ & \mathsf{h}[A] \arrow[ur, "\Delta^F_A"'] & \end{tikzcd} \tag{2.16}$$

Proof. It is clear that (2.12) implies (2.16). The first diagram is a special case of the bimonoid axiom, while the second diagram is the same as (2.15). For the converse, suppose $A \leq F \wedge G$. Then using the second diagram in (2.16) and (co)associativity,

$$\Delta^G_{F\wedge G}\mu^F_{F\wedge G} = \Delta^G_{F\wedge G}\Delta^{F\wedge G}_A \mu^{F\wedge G}_A \mu^F_{F\wedge G} = \Delta^G_A \mu^F_A.$$

Combining with the first diagram in (2.16), we obtain (2.12). In diagrammatic form,

$$\begin{tikzcd}
\mathsf{h}[F] \arrow[rr, "\mu^F_A"] \arrow[dr, "\mu^F_{F\wedge G}"] \arrow[dd, "\Delta^{FG}_F"'] & & \mathsf{h}[A] \arrow[rr, "\Delta^G_A"] \arrow[dr, "\Delta^{F\wedge G}_A"] & & \mathsf{h}[G] \\
& \mathsf{h}[F\wedge G] \arrow[ur, "\mu^{F\wedge G}_A"] \arrow[rr, "\mathrm{id}"'] & & \mathsf{h}[F\wedge G] \arrow[ur, "\Delta^G_{F\wedge G}"'] & \\
\mathsf{h}[FG] \arrow[rrrr, "\beta_{GF,FG}"'] & & & & \mathsf{h}[GF]. \arrow[uu, "\mu^{GF}_G"']
\end{tikzcd}$$

□

Exercise 2.14. Show that: For defining an $\mathcal{A}$-bimonoid, it suffices to require the bimonoid axiom (2.12) only for covers $A \lessdot F$ and $A \lessdot G$. (This fact is explained later in the text, see (2.85).)

Exercise 2.15. Show that: For defining a morphism between $\mathcal{A}$-monoids, it suffices to require diagram (2.9) only for covers $A \lessdot F$. Dually, for defining a morphism between $\mathcal{A}$-comonoids, it suffices to require diagram (2.11) only for covers $A \lessdot F$.

2.3. (Co)commutative (co)monoids

We now turn to the commutative aspects of the theory of bimonoids. Commutativity can be viewed as a property of a monoid. Alternatively, just as monoids are formulated in terms of faces, commutative monoids can be formulated in terms of flats. Flats are simpler objects than faces which is consistent with what we expect, namely, that the commutative theory be simpler than the general theory.

We also introduce double monoids and double comonoids. They are equivalent to commutative monoids and cocommutative comonoids, respectively, via an Eckmann–Hilton argument.

2.3.1. Commutative monoids. An $\mathcal{A}$-monoid (a, μ) is *commutative* if the diagram

(2.17)
$$\begin{tikzcd}
\mathsf{a}[F] \arrow[rr, "\beta_{G,F}"] \arrow[dr, "\mu^F_A"'] & & \mathsf{a}[G] \arrow[dl, "\mu^G_A"] \\
& \mathsf{a}[A] &
\end{tikzcd}$$

commutes, whenever $A \leq F$ and $A \leq G$, and F and G have the same support. Illustrative pictures are shown below.

We refer to diagram (2.17) as the *commutativity axiom*.

Lemma 2.16. *Let (a, μ) be an $\mathcal{A}$-monoid. Then the following are equivalent.*

(1) *(a, μ) is commutative.*
(2) *The diagram*

$$\begin{tikzcd}[column sep=small] \mathsf{a}[F] \arrow[rr, "\beta_{A\overline{F},F}"] \arrow[dr, "\mu_A^F"'] & & \mathsf{a}[A\overline{F}] \arrow[dl, "\mu_A^{A\overline{F}}"] \\ & \mathsf{a}[A] & \end{tikzcd} \tag{2.18}$$

commutes, whenever $A \leq F$.

(3) *Diagram (2.17) commutes, whenever F and G are adjacent with the same support and A is their common panel.*

Proof. Note that F and $A\overline{F}$ are opposite faces in the star of A. In particular, they have the same support. Further, if A has codimension one in F, then F and $A\overline{F}$ are adjacent with common panel A. This shows (1) implies (2), and (2) implies (3). It remains to show (3) implies (1). Since flats are gallery connected, to establish (2.17), we may assume that F and G are adjacent. Let H be their common panel. Diagram (2.17) can be filled in as follows.

$$\begin{tikzcd} \mathsf{a}[F] \arrow[rr, "\beta_{G,F}"] \arrow[dr, "\mu_H^F"] \arrow[ddr, bend right, "\mu_A^F"'] & & \mathsf{a}[G] \arrow[dl, "\mu_H^G"'] \arrow[ddl, bend left, "\mu_A^G"] \\ & \mathsf{a}[H] \arrow[d, "\mu_A^H"] & \\ & \mathsf{a}[A] & \end{tikzcd}$$

The top triangle commutes by hypothesis. The triangles on the two sides commute by associativity (2.8). ☐

Lemma 2.17. *Let (a, μ) be an $\mathcal{A}$-monoid. Then: It is commutative iff the diagram*

$$\begin{tikzcd} \mathsf{a}[F] \arrow[r, "\beta_{G,F}"] \arrow[d, "\mu_A^F"'] & \mathsf{a}[G] \arrow[d, "\mu_B^G"] \\ \mathsf{a}[A] \arrow[r, "\beta_{B,A}"'] & \mathsf{a}[B] \end{tikzcd} \tag{2.19}$$

commutes, whenever A and B have the same support, F and G have the same support, and $A \leq F$ and $B \leq G$.

Note very carefully that (2.19) is stronger than the naturality axiom in (2.8).

PROOF. For backward implication, take $A = B$. For forward implication, fill in diagram (2.19) as follows.

$$\begin{tikzcd} \mathsf{a}[F] \arrow[rr, "\beta"] \arrow[d, "\mu"'] \arrow[dr, "\beta"] & & \mathsf{a}[G] \arrow[d, "\mu"] \\ \mathsf{a}[A] \arrow[rr, "\beta"'] & \mathsf{a}[BF] \arrow[ur, "\beta"] \arrow[r, "\mu"] & \mathsf{a}[B]. \end{tikzcd}$$

The subscripts and superscripts on μ and β have been suppressed; they can be read off from the faces involved in their domain and codomain. For instance, the left-vertical μ is μ_A^F. The square commutes by naturality (2.8), the side triangle commutes by (2.17), the top triangle commutes by (2.1). $\square$

Exercise 2.18. Let a be a commutative $\mathcal{A}$-monoid. Show that: For $A \leq G$ and $\mathrm{s}(F) = \mathrm{s}(G)$, the diagram

$$\begin{tikzcd} \mathsf{a}[F] \arrow[r, "\beta_{G,F}"] \arrow[d, "\beta_{AF,F}"'] & \mathsf{a}[G] \arrow[d, "\mu_A^G"] \\ \mathsf{a}[AF] \arrow[r, "\mu_A^{AF}"'] & \mathsf{a}[A] \end{tikzcd} \tag{2.20}$$

commutes.

Exercise 2.19. Let h be an $\mathcal{A}$-bimonoid. Let F and G be faces both greater than A, and of the same support. Let K be a face greater than A such that $KF = KG = K$. For any $x \in \mathsf{h}[F]$, show that

$$\Delta_A^K(\mu_A^F(x) - \mu_A^G \beta_{G,F}(x)) = 0.$$

(Start with the third diagram in Lemma 2.6.)

2.3.2. Commutative monoids formulated in terms of flats. We now reformulate commutative monoids using the alternative definition of species given in Proposition 2.5.

Proposition 2.20. *A commutative $\mathcal{A}$-monoid is an $\mathcal{A}$-species a equipped with linear maps*

$$\mu_{\mathrm{Z}}^{\mathrm{X}} : \mathsf{a}[\mathrm{X}] \to \mathsf{a}[\mathrm{Z}],$$

one for each pair of flats $\mathrm{Z} \leq \mathrm{X}$, such that the diagrams

$$\begin{tikzcd} & \mathsf{a}[\mathrm{Y}] \arrow[dr, "\mu_{\mathrm{Z}}^{\mathrm{Y}}"] & \\ \mathsf{a}[\mathrm{X}] \arrow[ur, "\mu_{\mathrm{Y}}^{\mathrm{X}}"] \arrow[rr, "\mu_{\mathrm{Z}}^{\mathrm{X}}"'] & & \mathsf{a}[\mathrm{Z}] \end{tikzcd} \qquad (\mathsf{a}[\mathrm{Z}] \xrightarrow{\mu_{\mathrm{Z}}^{\mathrm{Z}}} \mathsf{a}[\mathrm{Z}]) = \mathrm{id} \tag{2.21}$$

commute, the first for every $\mathrm{Z} \leq \mathrm{Y} \leq \mathrm{X}$, and the second for every Z.

A morphism of commutative $\mathcal{A}$-monoids $f : \mathsf{a} \to \mathsf{b}$ is a family of linear maps
$$f_{\mathrm{X}} : \mathsf{a}[\mathrm{X}] \to \mathsf{b}[\mathrm{X}],$$
one for each flat X*, such that the diagram*

$$\begin{tikzcd} \mathsf{a}[\mathrm{X}] \arrow[r, "f_{\mathrm{X}}"] \arrow[d, "\mu_{\mathrm{Z}}^{\mathrm{X}}"'] & \mathsf{b}[\mathrm{X}] \arrow[d, "\mu_{\mathrm{Z}}^{\mathrm{X}}"] \\ \mathsf{a}[\mathrm{Z}] \arrow[r, "f_{\mathrm{Z}}"'] & \mathsf{b}[\mathrm{Z}] \end{tikzcd} \tag{2.22}$$

commutes, for every $\mathrm{Z} \leq \mathrm{X}$.

PROOF. We explain the first part. Suppose a is a commutative $\mathcal{A}$-monoid (in the usual sense). Then by Lemma 2.17, its product factors through the maps (2.6):

$$\begin{tikzcd} \mathsf{a}[F] \arrow[r, "\beta_{\mathrm{X},F}"] \arrow[d, "\mu_A^F"'] & \mathsf{a}[\mathrm{X}] \arrow[d, dashed, "\mu_{\mathrm{Z}}^{\mathrm{X}}"] \\ \mathsf{a}[A] \arrow[r, "\beta_{\mathrm{Z},A}"'] & \mathsf{a}[\mathrm{Z}]. \end{tikzcd}$$

Diagrams (2.21) of $\mu_{\mathrm{Z}}^{\mathrm{X}}$ follow from the corresponding diagrams (2.8) of μ_A^F.

Conversely, suppose we are given the maps $\mu_{\mathrm{Z}}^{\mathrm{X}}$. Then μ_A^F can be defined using the above diagram. It will satisfy the diagrams required of a commutative $\mathcal{A}$-monoid. $\square$

We refer to diagrams (2.21) as the *associativity* and *unitality axioms*, respectively.

2.3.3. Cocommutative comonoids. Dually, an $\mathcal{A}$-comonoid (c, Δ) is *cocommutative* if the diagram

$$\begin{tikzcd} \mathsf{c}[F] \arrow[rr, "\beta_{G,F}"] & & \mathsf{c}[G] \\ & \mathsf{c}[A] \arrow[ul, "\Delta_A^F"] \arrow[ur, "\Delta_A^G"'] & \end{tikzcd} \tag{2.23}$$

commutes, whenever $A \leq F$ and $A \leq G$, and F and G have the same support. We refer to diagram (2.23) as the *cocommutativity axiom*.

The entire discussion for commutative monoids holds for cocommutative comonoids (after reversing arrows labeled by μ and relabeling them by Δ). In particular:

Proposition 2.21. *A cocommutative $\mathcal{A}$-comonoid is an $\mathcal{A}$-species c equipped with linear maps*
$$\Delta_{\mathrm{Z}}^{\mathrm{X}} : \mathsf{c}[\mathrm{Z}] \to \mathsf{c}[\mathrm{X}],$$
one for each pair of flats $\mathrm{Z} \leq \mathrm{X}$*, such that the diagrams*

$$\begin{tikzcd} & \mathsf{c}[\mathrm{Y}] \arrow[dr, "\Delta_{\mathrm{Y}}^{\mathrm{X}}"] & \\ \mathsf{c}[\mathrm{Z}] \arrow[ur, "\Delta_{\mathrm{Z}}^{\mathrm{Y}}"] \arrow[rr, "\Delta_{\mathrm{Z}}^{\mathrm{X}}"'] & & \mathsf{c}[\mathrm{X}] \end{tikzcd} \qquad (\mathsf{c}[\mathrm{Z}] \xrightarrow{\Delta_{\mathrm{Z}}^{\mathrm{Z}}} \mathsf{c}[\mathrm{Z}]) = \mathrm{id} \tag{2.24}$$

commute, the first for every $\mathrm{Z} \leq \mathrm{Y} \leq \mathrm{X}$, *and the second for every* Z.

A morphism of cocommutative $\mathcal{A}$*-comonoids* $f : \mathsf{c} \to \mathsf{d}$ *is a family of linear maps*

$$f_{\mathrm{X}} : \mathsf{c}[\mathrm{X}] \to \mathsf{d}[\mathrm{X}],$$

one for each flat X, *such that the diagram*

$$\begin{tikzcd} \mathsf{c}[\mathrm{X}] \arrow[r, "f_{\mathrm{X}}"] & \mathsf{d}[\mathrm{X}] \\ \mathsf{c}[\mathrm{Z}] \arrow[u, "\Delta^{\mathrm{X}}_{\mathrm{Z}}"] \arrow[r, "f_{\mathrm{Z}}"'] & \mathsf{d}[\mathrm{Z}] \arrow[u, "\Delta^{\mathrm{X}}_{\mathrm{Z}}"'] \end{tikzcd} \tag{2.25}$$

commutes, for every $\mathrm{Z} \leq \mathrm{X}$.

We refer to diagrams (2.24) as the *coassociativity* and *counitality axioms*, respectively.

2.3.4. (Co, bi)commutative bimonoids. We say an $\mathcal{A}$-bimonoid is *commutative* if its underlying $\mathcal{A}$-monoid is commutative, and it is *cocommutative* if its underlying $\mathcal{A}$-comonoid is cocommutative. Similarly, an $\mathcal{A}$-bimonoid is *bicommutative* if it is both commutative and cocommutative.

Bicommutative bimonoids can be nicely reformulated in terms of the maps $\mu^{\mathrm{X}}_{\mathrm{Z}}$ and $\Delta^{\mathrm{Y}}_{\mathrm{Z}}$ as follows.

Proposition 2.22. *A bicommutative* $\mathcal{A}$*-bimonoid is a triple* $(\mathsf{h}, \mu, \Delta)$, *where* h *is an* $\mathcal{A}$*-species,* (h, μ) *is a commutative* $\mathcal{A}$*-monoid,* (h, Δ) *is a cocommutative* $\mathcal{A}$*-comonoid, and such that for any flats* $\mathrm{Z} \leq \mathrm{X}$ *and* $\mathrm{Z} \leq \mathrm{Y}$, *the diagram*

$$\begin{tikzcd} \mathsf{h}[\mathrm{X}] \arrow[r, "\mu^{\mathrm{X}}_{\mathrm{Z}}"] \arrow[d, "\Delta^{\mathrm{X}\vee\mathrm{Y}}_{\mathrm{X}}"'] & \mathsf{h}[\mathrm{Z}] \arrow[d, "\Delta^{\mathrm{Y}}_{\mathrm{Z}}"] \\ \mathsf{h}[\mathrm{X} \vee \mathrm{Y}] \arrow[r, "\mu^{\mathrm{X}\vee\mathrm{Y}}_{\mathrm{Y}}"'] & \mathsf{h}[\mathrm{Y}] \end{tikzcd} \tag{2.26}$$

commutes.

To obtain this, we only need to observe that the bimonoid axiom (2.12) simplifies to (2.26). We call this the *bicommutative bimonoid axiom*. As a shorthand, we write it as $\Delta\mu = \mu\Delta$.

Lemma 2.23. *The bicommutative bimonoid axiom* (2.26) *is equivalent to the following two axioms (assuming (co)associativity and (co)unitality).*

$$\begin{tikzcd} \mathsf{h}[\mathrm{X}] \arrow[r, "\mu^{\mathrm{X}}_{\mathrm{X}\wedge\mathrm{Y}}"] \arrow[d, "\Delta^{\mathrm{X}\vee\mathrm{Y}}_{\mathrm{X}}"'] & \mathsf{h}[\mathrm{X} \wedge \mathrm{Y}] \arrow[d, "\Delta^{\mathrm{Y}}_{\mathrm{X}\wedge\mathrm{Y}}"] \\ \mathsf{h}[\mathrm{X} \vee \mathrm{Y}] \arrow[r, "\mu^{\mathrm{X}\vee\mathrm{Y}}_{\mathrm{Y}}"'] & \mathsf{h}[\mathrm{Y}] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{h}[\mathrm{X}] \arrow[rr, "\mathrm{id}"] \arrow[dr, "\mu^{\mathrm{X}}_{\mathrm{Z}}"'] & & \mathsf{h}[\mathrm{X}] \\ & \mathsf{h}[\mathrm{Z}] \arrow[ur, "\Delta^{\mathrm{X}}_{\mathrm{Z}}"'] & \end{tikzcd} \tag{2.27}$$

Compare with Lemma 2.13.

Proof. Suppose (2.26) holds. Then setting $\mathrm{Z} := \mathrm{X} \wedge \mathrm{Y}$ yields the first diagram, while setting $\mathrm{X} = \mathrm{Y}$ yields the second diagram in (2.27) (since $\mathrm{X} \vee \mathrm{Y} = \mathrm{X} = \mathrm{Y}$ and $\mu_{\mathrm{Y}}^{\mathrm{X}\vee\mathrm{Y}} = \mathrm{id}$ and $\Delta_{\mathrm{X}}^{\mathrm{X}\vee\mathrm{Y}} = \mathrm{id}$ by unitality (2.21) and counitality).

Conversely, suppose (2.27) holds. Then fill in diagram (2.26) as follows.

$$
\begin{tikzcd}
\mathsf{h}[\mathrm{X}] \arrow[rrr, "\mu"] \arrow[dr, "\mu"'] \arrow[ddd, "\Delta"'] & & & \mathsf{h}[\mathrm{Z}] \arrow[dl, "\Delta"] \arrow[ddd, "\Delta"] \\
& \mathsf{h}[\mathrm{X} \wedge \mathrm{Y}] \arrow[urr, "\mu"] \arrow[dr, "\mathrm{id}"'] & & \\
& & \mathsf{h}[\mathrm{X} \wedge \mathrm{Y}] \arrow[dr, "\Delta"] & \\
\mathsf{h}[\mathrm{X} \vee \mathrm{Y}] \arrow[rrr, "\mu"'] & & & \mathsf{h}[\mathrm{Y}]
\end{tikzcd}
$$

The triangles on the two sides commute by (co)associativity (2.21) and its dual, while the square and the triangle in the middle commute by (2.27). $\square$

Lemma 2.24. *Let $(\mathsf{h}, \mu, \Delta)$ be an $\mathcal{A}$-bimonoid. Let F and G be faces both greater than A. Then the diagram*

$$
\begin{tikzcd}
\mathsf{h}[A] \arrow[r, "\Delta_A^F"] & \mathsf{h}[F] \arrow[r, "\mu_A^F"] & \mathsf{h}[A] \arrow[d, "\Delta_A^G"] \\
\mathsf{h}[G] \arrow[u, "\mu_A^G"] \arrow[r, "\Delta_G^{GF}"'] & \mathsf{h}[GF] \arrow[r, "\mu_G^{GF}"'] & \mathsf{h}[G]
\end{tikzcd}
\tag{2.28}
$$

commutes. If h is cocommutative, then the diagram

$$
\begin{tikzcd}
\mathsf{h}[A] \arrow[r, "\Delta_A^G"] \arrow[d, "\Delta_A^{FG}"'] & \mathsf{h}[G] \arrow[r, "\mu_A^G"] & \mathsf{h}[A] \arrow[d, "\Delta_A^F"] \\
\mathsf{h}[FG] \arrow[rr, "\mu_F^{FG}"'] & & \mathsf{h}[F]
\end{tikzcd}
\tag{2.29}
$$

commutes. If h is commutative, then the diagram

$$
\begin{tikzcd}
\mathsf{h}[F] \arrow[rr, "\Delta_F^{FG}"] \arrow[d, "\mu_A^F"'] & & \mathsf{h}[FG] \arrow[d, "\mu_A^{FG}"] \\
\mathsf{h}[A] \arrow[r, "\Delta_A^G"'] & \mathsf{h}[G] \arrow[r, "\mu_A^G"'] & \mathsf{h}[A]
\end{tikzcd}
\tag{2.30}
$$

commutes.

Proof. Diagram (2.28) can be filled in as follows.

$$
\begin{tikzcd}
\mathsf{h}[A] \arrow[rr, "\Delta"] & & \mathsf{h}[F] \arrow[rr, "\mu"] \arrow[dr, "\Delta"] & & \mathsf{h}[A] \arrow[dd, "\Delta"] \\
& \mathsf{h}[FG] \arrow[ur, "\mu"] \arrow[rr, "\mathrm{id}"] & & \mathsf{h}[FG] \arrow[dl, "\beta"] & \\
\mathsf{h}[G] \arrow[uu, "\mu"] \arrow[rr, "\Delta"'] & & \mathsf{h}[GF] \arrow[ul, "\beta"] \arrow[rr, "\mu"'] & & \mathsf{h}[G]
\end{tikzcd}
$$

The pentagons commute by the bimonoid axiom (2.12), and the triangles by (2.2) and (2.13).

Diagram (2.29) can be filled in as follows.

$$\begin{tikzcd}
\mathsf{h}[A] \arrow[r, "\Delta"] \arrow[dd, "\Delta"'] \arrow[dr, "\Delta"'] & \mathsf{h}[G] \arrow[r, "\mu"] \arrow[d, "\Delta"] & \mathsf{h}[A] \arrow[dd, "\Delta"] \\
 & \mathsf{h}[GF] \arrow[dl, "\beta"] & \\
\mathsf{h}[FG] \arrow[rr, "\mu"'] & & \mathsf{h}[F]
\end{tikzcd}$$

The pentagon commutes by the bimonoid axiom (2.12), and the triangles by coassociativity (2.10) and cocommutativity (2.23).

Diagram (2.30) can be checked similarly. □

Exercise 2.25. Let h be an $\mathcal{A}$-bimonoid, and p be a cocommutative subcomonoid of h. Check that (2.29) holds when restricted to the subspace $\mathsf{p}[A]$.

Exercise 2.26. Let h be an $\mathcal{A}$-bimonoid. Show that

(1) μ_O^F, as F varies, uniquely determine the remaining product components.
(2) h is commutative iff (2.17) holds for $A = O$. In this case, μ_O^F, as F varies, also uniquely detemine the $\beta_{G,F}$.
(3) Δ_O^F, as F varies, uniquely determine the remaining coproduct components.
(4) h is cocommutative iff (2.23) holds for $A = O$. In this case, Δ_O^F, as F varies, also uniquely detemine the $\beta_{G,F}$.

(Use Corollary 2.9.)

2.3.5. Double (co)monoids and Eckmann–Hilton. An $\mathcal{A}$-*double monoid* is a triple (a, μ, μ'), where a is an $\mathcal{A}$-species, (a, μ) and (a, μ') are $\mathcal{A}$-monoids, and for any faces $A \leq F$ and $A \leq G$, the diagram

(2.31)
$$\begin{tikzcd}
\mathsf{a}[F] \arrow[r, "\mu_A^F"] & \mathsf{a}[A] & \mathsf{a}[G] \arrow[l, "(\mu')_A^G"'] \\
\mathsf{a}[FG] \arrow[u, "(\mu')_F^{FG}"] \arrow[rr, "\beta_{GF,FG}"'] & & \mathsf{a}[GF] \arrow[u, "\mu_G^{GF}"']
\end{tikzcd}$$

commutes. We refer to diagram (2.31) as the *double monoid axiom*. A morphism of $\mathcal{A}$-double monoids is a map of the underlying $\mathcal{A}$-species which is a morphism of $\mathcal{A}$-monoids for both μ and μ'. This defines the category of $\mathcal{A}$-double monoids.

Dually, one can define the category of $\mathcal{A}$-double comonoids. An $\mathcal{A}$-*double comonoid* is a triple $(\mathsf{c}, \Delta, \Delta')$, where c is an $\mathcal{A}$-species, (c, Δ) and (c, Δ') are

$\mathcal{A}$-comonoids, and for any faces $A \leq F$ and $A \leq G$, the diagram

$$\begin{tikzcd} \mathsf{c}[F] \arrow[d, "(\Delta')^{FG}_{F}"'] & \mathsf{c}[A] \arrow[l, "\Delta^F_A"'] \arrow[r, "(\Delta')^G_A"] & \mathsf{c}[G] \arrow[d, "\Delta^{GF}_G"] \\ \mathsf{c}[FG] \arrow[rr, "\beta_{GF,FG}"'] & & \mathsf{c}[GF] \end{tikzcd} \tag{2.32}$$

commutes. We refer to diagram (2.32) as the *double comonoid axiom*. Morphisms between $\mathcal{A}$-double comonoids are defined similarly.

Proposition 2.27. *The category of $\mathcal{A}$-double monoids is equivalent to the category of commutative $\mathcal{A}$-monoids. Dually, the category of $\mathcal{A}$-double comonoids is equivalent to the category of cocommutative $\mathcal{A}$-comonoids.*

This is an avatar of the Eckmann–Hilton argument. The passage from commutative monoids to double monoids and vice versa is explained in the lemmas below. The argument in the comonoid case is similar.

Lemma 2.28. *Let (a, μ) be a commutative $\mathcal{A}$-monoid. Then (a, μ, μ) is an $\mathcal{A}$-double monoid.*

PROOF. Diagram (2.31) can be filled as follows.

$$\begin{tikzcd} \mathsf{a}[F] \arrow[r, "\mu^F_A"] & \mathsf{a}[A] & \mathsf{a}[G] \arrow[l, "\mu^G_A"'] \\ \mathsf{a}[FG] \arrow[u, "\mu^{FG}_F"] \arrow[ur, "\mu^{FG}_A"'] \arrow[rr, "\beta_{GF,FG}"'] & & \mathsf{a}[GF] \arrow[u, "\mu^{GF}_G"'] \arrow[ul, "\mu^{GF}_A"] \end{tikzcd}$$

We used associativity of μ (2.8) and the commutativity axiom (2.17). $\square$

Conversely:

Lemma 2.29. *Let (a, μ, μ') be an $\mathcal{A}$-double monoid. Then $\mu = \mu'$, and moreover, (a, μ) is a commutative $\mathcal{A}$-monoid.*

PROOF. In the double monoid axiom (2.31), first put $G = F$ to deduce $\mu = \mu'$, and then put $G = A\overline{F}$ to deduce (2.18). $\square$

Note the formal similarity of the double (co)monoid axiom with the bimonoid axiom. We will see a few instances where it is more natural to use the double (co)monoid axiom rather than the (co)commutativity axiom.

2.4. Deformed bimonoids and signed bimonoids

We now turn to deformed bimonoids. Distance functions on arrangements reviewed in Section 1.4 provide the theoretical framework for this discussion.

For any parameter q, by deforming the bimonoid axiom by the q-distance function on faces, one obtains the notion of a q-bimonoid. The value $q = 1$ recovers the usual notion of bimonoids. We use the term signed bimonoids for the value $q = -1$, and 0-bimonoids for the value $q = 0$. In these cases, the bimonoid axiom is deformed by the signed distance function on faces and the 0-distance function on faces, respectively. More generally, the bimonoid axiom can be deformed by any distance function v on the arrangement. This leads to the notion of v-bimonoids.

We denote the category of q-bimonoids by q-$\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$, and that of signed bimonoids by (-1)-$\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$.

2.4.1. *q*-bimonoids. Let q be any scalar. An *$\mathcal{A}$-q-bimonoid* is a triple $(\mathsf{h}, \mu, \Delta)$, where h is an $\mathcal{A}$-species, (h, μ) is an $\mathcal{A}$-monoid, (h, Δ) is an $\mathcal{A}$-comonoid, and for any faces $A \leq F$ and $A \leq G$, the diagram

$$\begin{tikzcd} \mathsf{h}[F] \arrow[r, "\mu_A^F"] \arrow[d, "\Delta_F^{FG}"'] & \mathsf{h}[A] \arrow[r, "\Delta_A^G"] & \mathsf{h}[G] \\ \mathsf{h}[FG] \arrow[rr, "(\beta_q)_{GF,FG}"'] & & \mathsf{h}[GF] \arrow[u, "\mu_G^{GF}"'] \end{tikzcd} \tag{2.33}$$

commutes, where

$$(\beta_q)_{GF,FG} := q^{\mathrm{dist}(GF,FG)} \beta_{GF,FG}, \tag{2.34}$$

with $\mathrm{dist}(GF, FG)$ being the number of hyperplanes which separate GF and FG. We refer to diagram (2.33) as the *q-bimonoid axiom*.

Since $\beta_1 = \beta$, axiom (2.33) reduces to the bimonoid axiom (2.12) when $q = 1$. In other words, an $\mathcal{A}$-1-bimonoid is the same as an $\mathcal{A}$-bimonoid. More formally, axiom (2.33) can be viewed as a deformation of (2.12) by the q-distance function on faces (1.27).

A morphism of $\mathcal{A}$-q-bimonoids is defined as for $\mathcal{A}$-bimonoids with q playing no role. In other words, a morphism of $\mathcal{A}$-q-bimonoids is a map of the underlying $\mathcal{A}$-species which is a morphism of the underlying $\mathcal{A}$-monoids and $\mathcal{A}$-comonoids.

Exercise 2.30. Check that the discussion in Section 2.2.4 continues to hold for $\mathcal{A}$-q-bimonoids with β replaced by $\beta_{q,\cdot}$. For instance: For faces F and G both greater than A and of the same support, the diagram

$$\begin{tikzcd} \mathsf{h}[F] \arrow[rr, "(\beta_q)_{G,F}"] \arrow[dr, "\mu_A^F"'] & & \mathsf{h}[G] \\ & \mathsf{h}[A] \arrow[ur, "\Delta_A^G"'] & \end{tikzcd} \tag{2.35}$$

commutes. In particular,

$$\Delta_A^F \mu_A^F = \mathrm{id}\,. \tag{2.36}$$

Deduce that the product components of any $\mathcal{A}$-q-bimonoid are injective, while the coproduct components are surjective.

Exercise 2.31. Show that: For any $\mathcal{A}$-q-bimonoid, $\mu_A^F \Delta_A^F$ is an idempotent operator on $\mathsf{h}[A]$ for any $A \leq F$. (Recall that an idempotent operator on a vector space V is a linear map $e : V \to V$ such that $e^2 = e$.)

Exercise 2.32. Extend the result of Exercise 2.14 to $\mathcal{A}$-q-bimonoids. (Use (1.30e).)

2.4.2. Signed bimonoids. The signed companion of $q = 1$ is the value $q = -1$. We use the term *signed $\mathcal{A}$-bimonoid* to refer to an $\mathcal{A}$-(-1)-bimonoid. In this case, for any faces $A \leq F$ and $A \leq G$, the diagram

$$\begin{tikzcd} \mathsf{h}[F] \arrow[r, "\mu_A^F"] \arrow[d, "\Delta_F^{FG}"'] & \mathsf{h}[A] \arrow[r, "\Delta_A^G"] & \mathsf{h}[G] \\ \mathsf{h}[FG] \arrow[rr, "(\beta_{-1})_{GF,FG}"'] & & \mathsf{h}[GF] \arrow[u, "\mu_G^{GF}"'] \end{tikzcd} \tag{2.37}$$

commutes, where

$$(\beta_{-1})_{GF,FG} := (-1)^{\mathrm{dist}(GF,FG)} \beta_{GF,FG}. \tag{2.38}$$

We refer to diagram (2.37) as the *signed bimonoid axiom*. It is a deformation of the bimonoid axiom (2.12) by the signed distance function on faces (1.28).

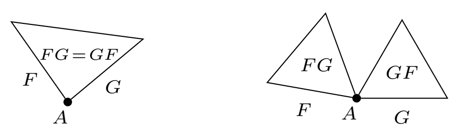
2.4.3. 0-bimonoids. We now focus on the value $q = 0$.

Lemma 2.33. *Let* h *be an* $\mathcal{A}$-0*-bimonoid. Then, for any faces* $A \leq F$ *and* $A \leq G$*, the first diagram below commutes if* $FG = GF$*, and the second diagram below commutes if* $FG \neq GF$*.*

$$\begin{tikzcd} \mathsf{h}[F] \arrow[r, "\mu_A^F"] \arrow[d, "\Delta_F^{FG}"'] & \mathsf{h}[A] \arrow[d, "\Delta_A^G"] \\ \mathsf{h}[FG] = \mathsf{h}[GF] \arrow[r, "\mu_G^{GF}"'] & \mathsf{h}[G] \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{h}[A] \arrow[dr, "\Delta_A^G"] & \\ \mathsf{h}[F] \arrow[ur, "\mu_A^F"] \arrow[rr, "0"'] & & \mathsf{h}[G] \end{tikzcd} \tag{2.39}$$

PROOF. This follows by setting $q = 0$ in the q-bimonoid axiom (2.33), and using the fact that $(\beta_0)_{GF,FG} = 0$ unless $FG = GF$. The latter follows from (1.25). $\square$

The two alternatives in (2.39) are illustrated below.



Thus, 0-bimonoids can be described as follows.

Lemma 2.34. *An* $\mathcal{A}$-0*-bimonoid is a triple* $(\mathsf{h}, \mu, \Delta)$*, where* h *is an* $\mathcal{A}$*-species,* (h, μ) *is an* $\mathcal{A}$*-monoid,* (h, Δ) *is an* $\mathcal{A}$*-comonoid such that*

$$\Delta_A^G \mu_A^F = \begin{cases} \mu_G^{GF} \Delta_F^{FG} & \text{if } FG = GF, \\ 0 & \text{otherwise,} \end{cases} \tag{2.40}$$

for any faces $A \leq F$ *and* $A \leq G$*.*

We refer to (2.40) as the 0*-bimonoid axiom*. Note its similarity with the bicommutative bimonoid axiom (2.26).

2.4.4. υ-bimonoids. The notion of q-bimonoids can be generalized further. Let υ be any distance function on $\mathcal{A}$ (Section 1.4.5). For any species p, let β_υ denote the family of maps

$$(\beta_\upsilon)_{G,F} : \mathsf{p}[F] \to \mathsf{p}[G], \qquad (\beta_\upsilon)_{G,F} := \upsilon_{G,F}\beta_{G,F}. \tag{2.41}$$

In other words, β_υ deforms β by the distance function υ.

An $\mathcal{A}$-υ-*bimonoid* is a triple $(\mathsf{h}, \mu, \Delta)$, where h is an $\mathcal{A}$-species, (h, μ) is an $\mathcal{A}$-monoid, (h, Δ) is an $\mathcal{A}$-comonoid, and for any faces $A \leq F$ and $A \leq G$, the diagram

$$\begin{tikzcd} \mathsf{h}[F] \arrow[r, "\mu_A^F"] \arrow[d, "\Delta_F^{FG}"'] & \mathsf{h}[A] \arrow[r, "\Delta_A^G"] & \mathsf{h}[G] \\ \mathsf{h}[FG] \arrow[rr, "(\beta_\upsilon)_{GF,FG}"'] & & \mathsf{h}[GF] \arrow[u, "\mu_G^{GF}"'] \end{tikzcd} \tag{2.42}$$

commutes. We refer to diagram (2.42) as the υ-*bimonoid axiom*.

We will also sometimes use the term $\mathcal{A}$-β_υ-bimonoid for an $\mathcal{A}$-υ-bimonoid. A morphism of $\mathcal{A}$-υ-bimonoids is as before with υ playing no role.

Note that for the distance function υ_q given by (1.27), an $\mathcal{A}$-υ_q-bimonoid is the same as an $\mathcal{A}$-q-bimonoid. In particular, an $\mathcal{A}$-υ_{-1}-bimonoid is the same as a signed $\mathcal{A}$-bimonoid.

2.4.5. Nowhere-zero distance functions. Let υ be any nowhere-zero distance function on the arrangement $\mathcal{A}$. For any species p, let β_υ denote the family of maps (2.41). By hypothesis, these maps are invertible. Let β_υ^{-1} denote the family obtained by inverting these maps, that is,

$$(\beta_\upsilon^{-1})_{G,F} := (\beta_\upsilon)_{F,G}^{-1}. \tag{2.43}$$

An important example is $\beta_\upsilon := \beta_q$ for $q \neq 0$. In this case, $\beta_q^{-1} = \beta_{q^{-1}}$. A further specialization is $q := 1$. By replacing β_υ by β_υ^{-1} in (2.42), we obtain the notion of a β_υ^{-1}-bimonoid.

2.5. Signed (co)commutative (co)monoids

The signed distance function υ_{-1} plays a defining role in the signed aspects of the theory of species. We saw this for signed bimonoids in Section 2.4. Next we would like to discuss signed commutativity. This is obtained by deforming the axioms (2.17) and (2.23) by υ_{-1}. Signed (co)commutative (co)monoids can also be formulated along the lines of Propositions 2.20 and 2.21. This requires the signature spaces defined in Section 1.11.2. They are also constructed out of υ_{-1}. We can also consider signed (co)commutativity for signed bimonoids.

To denote the categories associated to signed objects, we write a (-1) in front. For instance,

$$(-1)\text{-}\mathsf{Mon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})$$

denotes the category of signed commutative monoids. The same applies to all the categories in the right hand column of Table 2.1.

We mention in passing that the above ideas can be considered in the more general setting of log-antisymmetric distance functions (with υ_{-1} being a special case).

2.5.1. Signed commutative monoids. An $\mathcal{A}$-monoid (a, μ) is *signed commutative* if the diagram

$$
\begin{tikzcd}
\mathsf{a}[F] \arrow[rr, "(\beta_{-1})_{G,F}"] \arrow[dr, "\mu_A^F"'] & & \mathsf{a}[G] \arrow[dl, "\mu_A^G"] \\
& \mathsf{a}[A] &
\end{tikzcd}
\tag{2.44}
$$

commutes, whenever $A \leq F$ and $A \leq G$, and F and G have the same support. We refer to diagram (2.44) as the *signed commutativity axiom*. The notation $(\beta_{-1})_{G,F}$ is as in (2.38).

Compare and contrast (2.44) with (2.17). The parity of the distance between F and G now plays a role.

Exercise 2.35. Show that Lemma 2.17 has a signed analogue: replace β by β_{-1} in diagram (2.19). (The relevant property of distance functions is (1.30b).)

We now turn to the signed analogue of Proposition 2.20. This requires the signature spaces of Definition 1.74. Also, the following maps are useful.

For any species p and a basis element $\mathtt{H}_{[F/A]}$ of $\mathbf{E}^-[\mathrm{Z}, \mathrm{X}]$, define inverse isomorphisms

$$
\mathbf{E}^-[\mathrm{Z}, \mathrm{X}] \otimes \mathsf{p}[\mathrm{X}] \underset{[F/A] \otimes \beta_{\mathrm{X},F}}{\overset{[F/A]\, \beta_{F,\mathrm{X}}}{\rightleftarrows}} \mathsf{p}[F].
\tag{2.45}
$$

The forward map is $\mathtt{H}_{[F/A]} \otimes y \mapsto \beta_{F,\mathrm{X}}(y)$, while the backward map is $x \mapsto \mathtt{H}_{[F/A]} \otimes \beta_{\mathrm{X},F}(x)$.

Proposition 2.36. *A signed commutative $\mathcal{A}$-monoid is the same as an $\mathcal{A}$-species a equipped with linear maps*

$$\mu_{\mathrm{Z}}^{\mathrm{X}} : \mathbf{E}^-[\mathrm{Z}, \mathrm{X}] \otimes \mathsf{a}[\mathrm{X}] \to \mathsf{a}[\mathrm{Z}],$$

one for each pair of flats $\mathrm{Z} \leq \mathrm{X}$, such that the diagrams

$$
\begin{tikzcd}
\mathbf{E}^-[\mathrm{Z}, \mathrm{X}] \otimes \mathbf{E}^-[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{a}[\mathrm{Y}] \arrow[r, "\mathrm{id} \otimes \mu_{\mathrm{X}}^{\mathrm{Y}}"] \arrow[d] & \mathbf{E}^-[\mathrm{Z}, \mathrm{X}] \otimes \mathsf{a}[\mathrm{X}] \arrow[d, "\mu_{\mathrm{Z}}^{\mathrm{X}}"] \\
\mathbf{E}^-[\mathrm{Z}, \mathrm{Y}] \otimes \mathsf{a}[\mathrm{Y}] \arrow[r, "\mu_{\mathrm{Z}}^{\mathrm{Y}}"'] & \mathsf{a}[\mathrm{Z}]
\end{tikzcd}
$$

and

$$\big(\mathbf{E}^-[\mathrm{Z}, \mathrm{Z}] \otimes \mathsf{a}[\mathrm{Z}] \xrightarrow{\mu_{\mathrm{Z}}^{\mathrm{Z}}} \mathsf{a}[\mathrm{Z}]\big) = \mathrm{id}$$

commute, the first for every $\mathrm{Z} \leq \mathrm{X} \leq \mathrm{Y}$, and the second for every Z. (The diagrams employ the maps (1.162).)

The precise connection between the product components μ_A^F and μ_Z^X is shown below.

$$\begin{tikzcd} \mathsf{a}[F] \arrow[r, "\mu_A^F"] \arrow[d, "{[F/A]\otimes\beta_{X,F}}"'] & \mathsf{a}[A] \arrow[d, "\beta_{Z,A}"] \\ \mathbf{E}^-[Z,X]\otimes\mathsf{a}[X] \arrow[r, "\mu_Z^X"'] & \mathsf{a}[Z] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{a}[F] \arrow[r, "\mu_A^F"] & \mathsf{a}[A] \\ \mathbf{E}^-[Z,X]\otimes\mathsf{a}[X] \arrow[r, "\mu_Z^X"'] \arrow[u, "{[F/A]\,\beta_{F,X}}"] & \mathsf{a}[Z] \arrow[u, "\beta_{A,Z}"'] \end{tikzcd} \tag{2.46}$$

The left-vertical maps are as in (2.45).

2.5.2. Signed cocommutative comonoids. Dually, an $\mathcal{A}$-comonoid (c,Δ) is *signed cocommutative* if the diagram

$$\begin{tikzcd} \mathsf{c}[F] \arrow[rr, "(\beta_{-1})_{G,F}"] & & \mathsf{c}[G] \\ & \mathsf{c}[A] \arrow[ul, "\Delta_A^F"] \arrow[ur, "\Delta_A^G"'] & \end{tikzcd} \tag{2.47}$$

commutes, whenever $A \le F$ and $A \le G$, and F and G have the same support. We refer to diagram (2.47) as the *signed cocommutativity axiom*.

Reversing arrows in Proposition 2.36 yields an alternative description of a signed cocommutative $\mathcal{A}$-comonoid. It consists of linear maps

$$\Delta_Z^X : \mathsf{c}[Z] \to \mathbf{E}^-[Z,X]\otimes\mathsf{c}[X].$$

Note that we now need to use the inverse of the map (1.162) for the coassociativity axiom.

2.5.3. Signed (co)commutative signed bimonoids. One can also consider signed (co)commutativity for signed bimonoids:

We say a signed $\mathcal{A}$-bimonoid is *signed (co)commutative* if its underlying $\mathcal{A}$-(co)monoid is signed (co)commutative. Similarly, a signed $\mathcal{A}$-bimonoid is *signed bicommutative* if it is both signed commutative and signed cocommutative.

We have the following signed analogue of Proposition 2.22.

Proposition 2.37. *A signed bicommutative signed $\mathcal{A}$-bimonoid is the same as a triple (h,μ,Δ), where h is an $\mathcal{A}$-species, (h,μ) is a signed commutative $\mathcal{A}$-monoid, (h,Δ) is a signed cocommutative $\mathcal{A}$-comonoid, such that for any flats $Z \le X$ and $Z \le Y$, the diagram*

$$\begin{tikzcd} \mathbf{E}^-[Z,X]\otimes\mathsf{h}[X] \arrow[r, "\mu_Z^X"] \arrow[d, "\mathrm{id}\otimes\Delta_X^{X\vee Y}"'] & \mathsf{h}[Z] \arrow[r, "\Delta_Z^Y"] & \mathbf{E}^-[Z,Y]\otimes\mathsf{h}[Y] \\ \mathbf{E}^-[Z,X]\otimes\mathbf{E}^-[X,X\vee Y]\otimes\mathsf{h}[X\vee Y] \arrow[rr, "(-)\otimes\mathrm{id}"'] & & \mathbf{E}^-[Z,Y]\otimes\mathbf{E}^-[Y,X\vee Y]\otimes\mathsf{h}[X\vee Y] \arrow[u, "\mathrm{id}\otimes\mu_Y^{X\vee Y}"'] \end{tikzcd} \tag{2.48}$$

commutes, with the map $(-)$ given by (1.163).

We refer to diagram (2.48) as the *signed bicommutative signed bimonoid axiom.*

Exercise 2.38. Prove the signed analogue of Lemma 2.24, namely: Diagram (2.28) commutes for a signed $\mathcal{A}$-bimonoid h. If h is signed cocommutative, then (2.29) commutes. If h is signed commutative, then (2.30) commutes.

2.5.4. Unsigned and signed. The unsigned and signed worlds can be connected by a simple construction. It goes as follows.

For any species p with structure maps β, we define another species p_- whose F-component is $\mathsf{p}[F]$ and whose structure maps are β_{-1}. This uses (1.31). Further, a map $f : \mathsf{p} \to \mathsf{q}$ of species induces a map $f : \mathsf{p}_- \to \mathsf{q}_-$ of species. Thus, we have a functor

$$(-)_- : \mathcal{A}\text{-}\mathsf{Sp} \to \mathcal{A}\text{-}\mathsf{Sp}, \qquad \mathsf{p} \mapsto \mathsf{p}_-. \tag{2.49}$$

This is an isomorphism of categories (with its inverse being itself).

If a is a monoid, then so is a_- with product components μ_A^F the same as those of a. This uses (1.30b). Further, if a is commutative, then a_- is signed commutative, and vice versa. Dually, if c is a comonoid, then so is c_-. Further, if c is cocommutative, then c_- is signed cocommutative, and vice versa. Combining the two situations, if h is a q-bimonoid, then h_- is a $(-q)$-bimonoid. In particular, if h is a bimonoid, then h_- is a signed bimonoid, and vice versa.

This sets up an isomorphism between the categories of q-bimonoids and $(-q)$-bimonoids, and in particular, between bimonoids and signed bimonoids. Similarly, we have an isomorphism between the categories of (co)commutative (co)monoids and signed (co)commutative (co)monoids.

Closely related ideas are pursued in Section 8.10 where we study the signature functor. For the precise connection, see Section 8.10.7.

2.6. Subspecies and quotient species

The category of vector spaces is an abelian category (Appendix A.1), and hence so is the category of species. Thus, we can talk of subspecies and quotient species, kernels and cokernels, and so on. These notions can be defined explicitly by working componentwise and employing the corresponding notions for vector spaces on each component.

2.6.1. Subspecies and quotient species. For species p and q, we say p is a *subspecies* of q if each $\mathsf{p}[F]$ is a subspace of $\mathsf{q}[F]$, and the structure maps $\beta_{G,F}$ of p are obtained by restricting those of q. We write $\mathsf{p} \subseteq \mathsf{q}$. In this case, we have an inclusion map $\mathsf{p} \hookrightarrow \mathsf{q}$ of species. Further, we can form the *quotient species* q/p by taking quotients (of vector spaces) in each component. The structure maps of q/p are induced from those of q. Thus, we have the quotient map $\mathsf{q} \twoheadrightarrow \mathsf{q}/\mathsf{p}$ of species.

Let $f : \mathsf{p} \to \mathsf{q}$ be a map of species. Let $\ker(f)$ denote the subspecies of p whose F-component is the kernel of f_F. The species $\mathrm{im}(f)$, $\mathrm{coker}(f)$ and $\mathrm{coim}(f)$ are defined similarly. They are a subspecies of q, a quotient species of q, and a quotient species of p, respectively, and there are isomorphisms of species

$$\mathrm{im}(f) \cong \ker(\mathsf{q} \twoheadrightarrow \mathrm{coker}(f)) \quad \text{and} \quad \mathrm{coim}(f) \cong \mathrm{coker}(\ker(f) \hookrightarrow \mathsf{p}). \tag{2.50}$$

We have the following diagram of species.

$$
\begin{tikzcd}
0 \arrow[r] & \ker(f) \arrow[r, hook] & \mathsf{p} \arrow[r, "f"] \arrow[d, two heads] & \mathsf{q} \arrow[r, two heads] & \operatorname{coker}(f) \arrow[r] & 0 \\
& & \operatorname{coim}(f) \arrow[r, "\cong"'] & \operatorname{im}(f) \arrow[u, hook] & &
\end{tikzcd}
\tag{2.51}
$$

A map f of species is *injective* (*surjective*) if each component f_F is injective (surjective). Subspecies give rise to injective maps, and quotient species to surjective maps. A map of species is an isomorphism iff it is both injective and surjective.

2.6.2. Submonoids and quotient monoids. Suppose a is a subspecies of a monoid b. If $\mu_A^F : \mathsf{b}[F] \to \mathsf{b}[A]$ restricts to a map $\mathsf{a}[F] \to \mathsf{a}[A]$ for each $F \geq A$, we say a is a *submonoid* of b. This is equivalent to the inclusion $\mathsf{a} \hookrightarrow \mathsf{b}$ being a morphism of monoids. Suppose instead that b is a quotient species of a monoid a. If $\mu_A^F : \mathsf{a}[F] \to \mathsf{a}[A]$ factors through a map $\mathsf{b}[F] \to \mathsf{b}[A]$ for each $F \geq A$, we say b is a *quotient monoid* of a. This is equivalent to the projection $\mathsf{a} \twoheadrightarrow \mathsf{b}$ being a morphism of monoids. Similar terminology can be employed for comonoids and bimonoids.

Lemma 2.39. *Let $f : \mathsf{h} \to \mathsf{k}$ be a morphism of (co, bi)monoids. Then* $\ker(f)$ *is a sub(co, bi)monoid of* h*, while* $\operatorname{coker}(f)$ *is a quotient (co, bi)monoid of* k*. Hence, by* (2.50)*,* $\operatorname{im}(f)$*, or equivalently,* $\operatorname{coim}(f)$ *is also a (co, bi)monoid.*

PROOF. Suppose $f : \mathsf{a} \to \mathsf{b}$ is a morphism of monoids. Then the product of a restricts to $\ker(f)$, while the product of b projects onto $\operatorname{coker}(f)$:

$$
\begin{tikzcd}
\ker(f_F) \arrow[r, hook] \arrow[d, dashed] & \mathsf{a}[F] \arrow[r, "f_F"] \arrow[d, "\mu_A^F"'] & \mathsf{b}[F] \arrow[d, "\mu_A^F"] \\
\ker(f_A) \arrow[r, hook] & \mathsf{a}[A] \arrow[r, "f_A"'] & \mathsf{b}[A]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{a}[F] \arrow[r, "f_F"] \arrow[d, "\mu_A^F"'] & \mathsf{b}[F] \arrow[r, two heads] \arrow[d, "\mu_A^F"] & \operatorname{coker}(f_F) \arrow[d, dashed] \\
\mathsf{a}[A] \arrow[r, "f_A"'] & \mathsf{b}[A] \arrow[r, two heads] & \operatorname{coker}(f_A).
\end{tikzcd}
$$

The squares with solid edges coincide with (2.9). For the statement for comonoids, we reverse the vertical arrows and change μ to Δ. Combining the statements for monoids and comonoids yields the statement for bimonoids. $\square$

Note that any sub or quotient monoid of a commutative monoid is again commutative. Same remark applies to comonoids.

For subspecies p and q of a species r, their intersection $\mathsf{p} \cap \mathsf{q}$ (defined on each component as an intersection of subspaces) is also a subspecies of r. Similarly, the intersection of sub(co, bi)monoids of a (co, bi)monoid is also a sub(co, bi)monoid.

Let p be a subspecies of a comonoid c. We say c is cocommutative when restricted to p if the diagram

$$
\begin{tikzcd}
& \mathsf{p}[A] \arrow[dl, "\Delta_A^F"'] \arrow[dr, "\Delta_A^G"] & \\
\mathsf{c}[F] \arrow[rr, "\beta_{G,F}"'] & & \mathsf{c}[G]
\end{tikzcd}
$$

commutes, whenever $A \le F$ and $A \le G$, and F and G have the same support. A similar definition can be made for a subspecies of a monoid.

2.7. (Co)abelianizations of (co)monoids

We discuss the (co)abelianization construction which turns a (co)monoid into a (co)commutative (co)monoid. It also applies to bimonoids.

2.7.1. Abelianization of a monoid. We know that every commutative monoid is a monoid. Conversely, starting with a monoid a, the linear span of the elements

$$\mu_A^F(x) - \mu_A^G \beta_{G,F}(x) \ \in \mathsf{a}[A], \tag{2.52}$$

as x, F, G vary (with F and G both greater than A and of the same support), forms a submonoid of a: For any $B \le A$, applying μ_B^A to the above element, by associativity (2.8), yields

$$\mu_B^F(x) - \mu_B^G \beta_{G,F}(x),$$

which is again of the above form. Taking quotient of a by this submonoid yields a commutative monoid. This is the *abelianization* of a. We denote it by a_{ab}, and the quotient map $\mathsf{a} \twoheadrightarrow \mathsf{a}_{ab}$ by π_{a}. We refer to π_{a} as the *abelianization map*.

Lemma 2.40. *Any morphism of monoids f from a monoid a to a commutative monoid b factors through π_{a} yielding a commutative diagram*

$$\begin{tikzcd} \mathsf{a} \arrow[d, two heads, "\pi_{\mathsf{a}}"'] \arrow[dr, "f"] & \\ \mathsf{a}_{ab} \arrow[r, dashed] & \mathsf{b}. \end{tikzcd}$$

PROOF. We check below that the element (2.52) belongs to the kernel of f_A.

$$f_A \mu_A^G \beta_{G,F}(x) = \mu_A^G f_G \beta_{G,F}(x) = \mu_A^G \beta_{G,F} f_F(x) = \mu_A^F f_F(x) = f_A \mu_A^F(x).$$

(Note how the f moves from left to right and finally back to the left.) The third step used the commutativity axiom (2.17) for b. The remaining steps used that f is a map of species (2.3) and a morphism of monoids (2.9). $\square$

Thus, a_{ab} is the largest commutative quotient monoid of a. Equivalently, abelianization is the left adjoint of the inclusion functor. This is shown below.

$$\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{inc}{\overset{(-)_{ab}}{\rightleftarrows}} \mathsf{Mon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}). \tag{2.53}$$

Following Convention 2.2, we write the left adjoint above the right adjoint.

2.7.2. Coabelianization of a comonoid. Dually, every comonoid c has a largest cocommutative subcomonoid called the *coabelianization*. We denote it by c^{coab}, and the inclusion map $\mathsf{c}^{coab} \hookrightarrow \mathsf{c}$ by $\pi_{\mathsf{c}}^{\vee}$. We refer to $\pi_{\mathsf{c}}^{\vee}$ as the *coabelianization map*. Explicitly, $\mathsf{c}^{coab}[A]$ consists of those $x \in \mathsf{c}[A]$ for which

$$\beta_{G,F}\Delta_A^F(x) = \Delta_A^G(x) \tag{2.54}$$

for all faces F and G both greater than A and of the same support.

Lemma 2.41. *The image of any morphism of comonoids f from a cocommutative comonoid d to a comonoid c lies inside c^{coab} yielding a commutative diagram*

$$\begin{tikzcd} & \mathsf{c} \\ \mathsf{d} \arrow[ur, "f"] \arrow[r, dashed] & \mathsf{c}^{coab}. \arrow[u, hook, "\pi_{\mathsf{c}}^{\vee}"'] \end{tikzcd}$$

Proof. We check below condition (2.54) for $f_A(x) \in \mathsf{c}[A]$.

$$\beta_{G,F}\Delta_A^F f_A(x) = \beta_{G,F} f_F \Delta_A^F(x) = f_G \beta_{G,F}\Delta_A^F(x) = f_G \Delta_A^G(x) = \Delta_A^G f_A(x).$$

(Note how the f moves from right to left and finally back to the right.) The third step used the cocommutativity axiom (2.23) for d. The remaining steps used that f is a map of species (2.3) and a morphism of comonoids (2.11). $\square$

Thus, coabelianization is the right adjoint of the inclusion functor:

$$^{\mathsf{co}}\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{(-)^{coab}}{\overset{inc}{\rightleftarrows}} \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}). \tag{2.55}$$

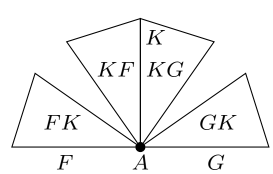
2.7.3. (Co)abelianization of a bimonoid. For a bimonoid h, one can consider both h_{ab} and h^{coab}. It turns out that both these are bimonoids.

Proposition 2.42. *Let h be a bimonoid. Then: h_{ab} is a commutative bimonoid, and further, it is bicommutative if h is cocommutative. Dually, h^{coab} is a cocommutative bimonoid, and further, it is bicommutative if h is commutative.*

Proof. We explain the first statement. We need to check that the linear span of the elements (2.52) is closed under the coproduct. This is done below.

$$\begin{aligned} &\Delta_A^K(\mu_A^F(x) - \mu_A^G\beta_{G,F}(x)) \\ &\qquad = \mu_K^{KF}\beta_{KF,FK}\Delta_F^{FK}(x) - \mu_K^{KG}\beta_{KG,GK}\Delta_G^{GK}\beta_{G,F}(x) \\ &\qquad\qquad = (\mu_K^{KF} - \mu_K^{KG}\beta_{KG,KF})(\beta_{KF,FK}\Delta_F^{FK}(x)). \end{aligned}$$

A way to visualize the faces involved in the calculation is shown below.



The calculation made use of (2.1), naturality of the coproduct (2.10) and the bimonoid axiom (2.12). $\square$

Thus, (2.53) extends to an adjunction between bimonoids and commutative bimonoids

$$\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{inc}{\overset{(-)_{ab}}{\rightleftarrows}} \mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}), \tag{2.56}$$

and also between cocommutative bimonoids and bicommutative bimonoids. Dually, (2.55) extends to an adjunction between bimonoids and cocommutative bimonoids

$$^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{(-)^{coab}}{\overset{inc}{\rightleftarrows}} \mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}), \tag{2.57}$$

and also between commutative bimonoids and bicommutative bimonoids.

For any cocommutative bimonoid h and commutative bimonoid k, we have a commutative diagram of bijections

$$\begin{tikzcd} \mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h},\mathsf{k}) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & {}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h},\mathsf{k}^{coab}) \arrow[d, leftrightarrow] \\ \mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h}_{ab},\mathsf{k}) \arrow[r, leftrightarrow] & {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h}_{ab},\mathsf{k}^{coab}). \end{tikzcd} \tag{2.58}$$

The bijections are encapsulated in the diagram of bimonoids below.

$$\begin{tikzcd} \mathsf{h} \arrow[r, dashed] \arrow[d, two heads] \arrow[dr, dashed] & \mathsf{k} \\ \mathsf{h}_{ab} \arrow[r, dashed] \arrow[ur, dashed] & \mathsf{k}^{coab} \arrow[u, hook] \end{tikzcd} \tag{2.59}$$

For a bimonoid h, the abelianization and coabelianization constructions can be iterated. Observe that there is a canonical morphism of bimonoids

$$(\mathsf{h}^{coab})_{ab} \to (\mathsf{h}_{ab})^{coab} \tag{2.60}$$

fitting into the commutative diagram

$$\begin{tikzcd} \mathsf{h}^{coab} \arrow[r, hook] \arrow[d, two heads] & \mathsf{h} \arrow[r, two heads] & \mathsf{h}_{ab} \\ (\mathsf{h}^{coab})_{ab} \arrow[rr] & & (\mathsf{h}_{ab})^{coab}. \arrow[u, hook] \end{tikzcd}$$

This is an instance of (2.59).

Exercise 2.43. Show that: For a comonoid $\mathsf{c} \neq \mathsf{0}$, the coabelianization of c cannot be $\mathsf{0}$. Similarly, for a monoid $\mathsf{a} \neq \mathsf{0}$, the abelianization of a cannot be $\mathsf{0}$.

Exercise 2.44. Show that: Every monoid a has a largest commutative submonoid. Explicitly, its A-component consists of those $x \in \mathsf{a}[A]$ for which

$$\mu_F^{F'} \beta_{F',A}(x) = \mu_F^{F''} \beta_{F'',A}(x)$$

for all $\mathrm{s}(A) = \mathrm{s}(F') = \mathrm{s}(F'')$ and $F \leq F', F''$. Deduce that the inclusion functor from commutative monoids to monoids also has a right adjoint. However, abelianization does not have a left adjoint (and dually coabelianization does not have a right adjoint) in general.

2.7.4. Signed (co)abelianization. By replacing β by β_{-1}, the above discussion can also be carried out in the signed setting. In this context, we use the terms *signed abelianization* and *signed coabelianization*.

The signed abelianization of a monoid a is obtained by taking quotient by the elements

$$\mu_A^F(x) - (-1)^{\mathrm{dist}(F,G)} \mu_A^G \beta_{G,F}(x) \ \in \mathsf{a}[A], \tag{2.61}$$

as x, F, G vary (with F and G both greater than A and of the same support). Similarly, the signed coabelianization of a comonoid c consists of those $x \in \mathsf{c}[A]$ for which

$$(-1)^{\mathrm{dist}(F,G)} \beta_{G,F} \Delta_A^F(x) = \Delta_A^G(x) \tag{2.62}$$

for all faces F and G both greater than A and of the same support.

Exercise 2.45. Prove the analogue of Proposition 2.42 for a signed bimonoid. (Use (1.31).)

2.8. Generating subspecies of monoids

Any subspecies p of a monoid a generates a submonoid which we denote by $\langle \mathsf{p} \rangle$. Moreover, for any subcomonoid c of a bimonoid h, the submonoid $\langle \mathsf{c} \rangle$ is in fact a subbimonoid of h and it is the smallest one containing c.

2.8.1. Generating subspecies of monoids. It makes sense to speak of the submonoid generated by a subspecies p of a monoid a. It is the smallest submonoid $\langle \mathsf{p} \rangle$ of a containing p and is the intersection of all submonoids of a containing p. Explicitly, for any face A,

$$\langle \mathsf{p} \rangle[A] = \sum_{F:\, F \geq A} \mu_A^F(\mathsf{p}[F]). \tag{2.63}$$

Each summand in the rhs is a subspace of $\mathsf{a}[A]$. It is clear that the rhs is contained in the lhs. For equality, we need to show that the rhs determines a submonoid of a. But this follows from associativity of μ.

We say a subspecies p *generates* a monoid a if $\langle \mathsf{p} \rangle = \mathsf{a}$. This property can also be phrased in terms of the free monoid on p, see Exercise 6.70.

Proposition 2.46. *Let $f : \mathsf{a} \to \mathsf{b}$ be a morphism of monoids, and let $\langle \mathsf{p} \rangle = \mathsf{a}$. Then f is completely determined by its restriction $\mathsf{p} \to \mathsf{b}$.*

Proof. This follows from (2.9) and (2.63). ☐

Proposition 2.47. *Let h be a bimonoid, and let $\langle \mathsf{p} \rangle = \mathsf{h}$ (as a monoid). Then: The coproduct of h is completely determined by its values on p. Further, if h is cocommutative when restricted to p, then h is cocommutative.*

Proof. The first claim follows from the bimonoid axiom (2.12). Suppose h is cocommutative on p. The diagram

$$\begin{tikzcd}
& \mathsf{h}[FG] \arrow[r, "\beta"] \arrow[dd, bend right, "\beta"'] & \mathsf{h}[GF] \arrow[r, "\mu"] \arrow[dd, bend right, "\beta"'] & \mathsf{h}[G] \arrow[dd, "\beta"] \\
\mathsf{p}[F] \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\mu"] & & \mathsf{h}[A] \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & \\
& \mathsf{h}[FH] \arrow[r, "\beta"'] & \mathsf{h}[HF] \arrow[r, "\mu"'] & \mathsf{h}[H]
\end{tikzcd}$$

shows that h is cocommutative on $\mu_A^F(\mathsf{p}[F])$. Now use (2.63). $\square$

Proposition 2.48. *Let* c *be a subcomonoid of a bimonoid* h*. Then* $\langle \mathsf{c} \rangle$ *is the smallest subbimonoid of* h *which contains* c*. Further, if* c *is cocommutative, then so is* $\langle \mathsf{c} \rangle$*, and in particular,* $\langle \mathsf{c} \rangle \subseteq \mathsf{h}^{coab}$.

Proof. The bimonoid axiom (2.12) implies

$$\Delta_A^G(\mu_A^F(\mathsf{c}[F])) \subseteq \mu_G^{GF}(\mathsf{c}[GF]).$$

In view of (2.63), we deduce that $\langle \mathsf{c} \rangle$ is a subcomonoid of h, proving the first claim. The second claim follows from Proposition 2.47. $\square$

Exercise 2.49. For a monoid a, consider the subspecies p consisting of elements of the form (2.52), where F and G are adjacent with the same support and A is their common panel. Show that $\mathsf{a}_{ab} = \mathsf{a}/\langle \mathsf{p} \rangle$, where $\langle \mathsf{p} \rangle$ is the submonoid of a generated by p.

Exercise 2.50. Let h be a bimonoid, and a be its submonoid generated by a subspecies p. Suppose the coproduct of h when applied to p lands in a. Then show that a is a subbimonoid of h.

2.9. Duality functor on species

Every vector space has a dual vector space, and every linear map has a dual linear map (Appendix A.2). This can be used to define duality in species. Duality interchanges (commutative) monoids and (cocommutative) comonoids. Moreover, abelianization and coabelianization are conjugates of each other wrt duality. Duality preserves bimonoids, and more generally, it also preserves q-bimonoids.

2.9.1. Duality functor. The dual of a species p is the species p^* defined as follows. For any face F, let

$$\mathsf{p}^*[F] := \mathsf{p}[F]^*,$$

where $\mathsf{p}[F]^*$ denotes the dual of the vector space $\mathsf{p}[F]$. For F and G of the same support, dualize $\beta_{F,G} : \mathsf{p}[G] \to \mathsf{p}[F]$ to obtain a linear map $\mathsf{p}^*[F] \to \mathsf{p}^*[G]$, and define this to be the structure map $\beta_{G,F}$ of p^*.

Moreover, a map $f : \mathsf{p} \to \mathsf{q}$ of species induces a map $f^* : \mathsf{q}^* \to \mathsf{p}^*$ of species. This defines a functor

$$(-)^* : \mathcal{A}\text{-}\mathsf{Sp} \to \mathcal{A}\text{-}\mathsf{Sp}^{\mathrm{op}}. \tag{2.64}$$

We call it the *duality functor* on species.

Let $f : \mathsf{p} \to \mathsf{q}$ be a map of species, and $f^* : \mathsf{q}^* \to \mathsf{p}^*$ its dual. Then

- $f = 0$ iff $f^* = 0$,
- f is injective iff f^* is surjective,
- f is surjective iff f^* is injective,
- f is bijective iff f^* is bijective.

This follows from the corresponding statements for linear maps.

2.9.2. Dual of a bimonoid. Let (a, μ) be a monoid in species. Its dual a^* is then a comonoid: The coproduct of a^*, denoted μ^*, is given by

$$\mathsf{a}^*[A] = \mathsf{a}[A]^* \xrightarrow{(\mu_A^F)^*} \mathsf{a}[F]^* = \mathsf{a}^*[F],$$

where $(\mu_A^F)^*$ is the linear dual of μ_A^F. The dual of the associativity axiom of a yields the coassociativity axiom of a^*. Further, if μ is commutative, then μ^* is cocommutative. The dual of the commutativity axiom of a yields the cocommutativity axiom of a^*.

Similarly, the dual c^* of a comonoid (c, Δ) is a monoid whose product Δ^* is given by

$$\mathsf{c}^*[F] = \mathsf{c}[F]^* \xrightarrow{(\Delta_A^F)^*} \mathsf{c}[A]^* = \mathsf{c}^*[A].$$

If Δ is cocommutative, then Δ^* is commutative.

These constructions do not require either a or c to be finite-dimensional. Combining the two, if $(\mathsf{h}, \mu, \Delta)$ is a bimonoid, then so is $(\mathsf{h}^*, \Delta^*, \mu^*)$. The dual of the bimonoid axiom of h yields the bimonoid axiom of h^*. More generally, if $(\mathsf{h}, \mu, \Delta)$ is a q-bimonoid, then so is $(\mathsf{h}^*, \Delta^*, \mu^*)$. For instance, the dual of a 0-bimonoid is a 0-bimonoid.

Observe that for a species p, monoid a, comonoid c, bimonoid h, we have natural maps

$$\mathsf{p} \hookrightarrow (\mathsf{p}^*)^*, \quad \mathsf{a} \hookrightarrow (\mathsf{a}^*)^*, \quad \mathsf{c} \hookrightarrow (\mathsf{c}^*)^*, \quad \mathsf{h} \hookrightarrow (\mathsf{h}^*)^* \tag{2.65}$$

of species, monoids, comonoids, bimonoids, respectively.

2.9.3. (Co)abelianization. For any monoid a and comonoid c,

$$(\mathsf{a}_{ab})^* = (\mathsf{a}^*)^{coab} \quad \text{and} \quad (\mathsf{c}^{coab})^* = (\mathsf{c}^*)_{ab}, \tag{2.66}$$

where we recall that a_{ab} denotes the abelianization of a, and c^{coab} the coabelianization of c. Hence, we say that abelianization and coabelianization are conjugates of each other wrt duality.

2.9.4. Self-duality. We now turn to the notion of self-duality. In this discussion, all species and (co, bi)monoids are assumed to be finite-dimensional.

Any species p is isomorphic to its dual p^*: Choose isomorphisms $\mathsf{p}[F] \cong \mathsf{p}^*[F]$, one for each face F, which are compatible with $\beta_{G,F}$. Equivalently, choose isomorphisms $\mathsf{p}[\mathrm{X}] \cong \mathsf{p}^*[\mathrm{X}]$, one for each flat X. Hence, we say that a species is self-dual.

We say a bimonoid h is *self-dual* if h and h^* are isomorphic as bimonoids. In contrast to species, a bimonoid may or may not be self-dual. For instance, if h is cocommutative but not commutative, then h^* will be commutative but not cocommutative. Hence, the two cannot be isomorphic.

In the finite-dimensional setting, the maps (2.65) become isomorphisms, and we treat them as equalities:

$$(\mathsf{p}^*)^* = \mathsf{p}, \quad (\mathsf{a}^*)^* = \mathsf{a}, \quad (\mathsf{c}^*)^* = \mathsf{c}, \quad (\mathsf{h}^*)^* = \mathsf{h}.$$

We say a morphism $f : \mathsf{p} \to \mathsf{p}^*$ of species is *self-dual* if $f = f^*$. The same definition can be made for bimonoids.

Proposition 2.51. *Let $f : \mathsf{h} \to \mathsf{h}^*$ be a self-dual morphism of bimonoids. Then* $\mathrm{im}(f)$*, or equivalently,* $\mathrm{coim}(f)$ *is a self-dual bimonoid.*

PROOF. For any morphism $f : \mathsf{p} \to \mathsf{q}$ of species and its dual $f^* : \mathsf{q}^* \to \mathsf{p}^*$,

$$\ker(f)^* = \mathrm{coker}(f^*) \quad \text{and} \quad \mathrm{coker}(f)^* = \ker(f^*).$$

Thus,

$$\begin{aligned} \mathrm{im}(f)^* = \ker(\mathsf{q} \to \mathrm{coker}(f))^* &= \mathrm{coker}((\mathsf{q} \to \mathrm{coker}(f))^*) \\ &= \mathrm{coker}(\ker(f^*) \to \mathsf{q}^*) = \mathrm{coim}(f^*). \end{aligned}$$

Now, let $f : \mathsf{h} \to \mathsf{h}^*$ be as in the proposition. By hypothesis, $f = f^*$. Since $\mathrm{im}(f)$ and $\mathrm{coim}(f)$ are canonically isomorphic, the result follows. □

Exercise 2.52. Let q be a subspecies of p. Check that if $f : \mathsf{p} \to \mathsf{p}^*$ is a self-dual map of species, then so is the composite $\mathsf{q} \hookrightarrow \mathsf{p} \xrightarrow{f} \mathsf{p}^* \twoheadrightarrow \mathsf{q}^*$.

2.10. Op and cop constructions

We discuss op and cop constructions on monoids and comonoids, and then extend them to bimonoids. The idea is to twist the (co)product of a given (co)monoid using the "braiding" β. We work in the more general setting of nowhere-zero distance functions.

2.10.1. b-species. The correct setting for op and cop constructions is that of b-species. These are more general than species, the key distinction is that for b-species, for faces F, G, H of the same support, the diagram

$$\begin{tikzcd} & \mathsf{p}[G] \arrow[dr, "\beta_{H,G}"] & \\ \mathsf{p}[F] \arrow[ur, "\beta_{G,F}"] \arrow[rr, "\beta_{H,F}"'] & & \mathsf{p}[H] \end{tikzcd} \tag{2.67}$$

commutes whenever there exists a minimal gallery $F \text{ -- } G \text{ -- } H$ (and not necessarily always). These are discussed in more detail later in Section 3.8. The discussion below is phrased for species, but it is also valid for b-species.

2.10.2. Opposite (co)monoids. Let v be any nowhere-zero distance function on $\mathcal{A}$. Recall the family β_v of maps defined in (2.41). It can be used to twist the product of a monoid: Suppose μ is a product on a species a. Then define the twisted product $\mu\beta_v$ as follows. For $F \geq A$,

$$(\mu\beta_v)^F_A : \mathsf{a}[F] \xrightarrow{(\beta_v)_{A\overline{F},F}} \mathsf{a}[A\overline{F}] \xrightarrow{\mu^{A\overline{F}}_A} \mathsf{a}[A].$$

Also recall the family β_v^{-1} of maps defined in (2.43). The twisted product $\mu\beta_v^{-1}$ is defined similarly using β_v^{-1} instead of β_v.

$$(\mu\beta_v^{-1})_A^F : \mathsf{a}[F] \xrightarrow{(\beta_v^{-1})_{A\overline{F},F}} \mathsf{a}[A\overline{F}] \xrightarrow{\mu_A^{A\overline{F}}} \mathsf{a}[A].$$

Twistings of the coproduct Δ, denoted $\beta_v\Delta$ and $\beta_v^{-1}\Delta$, are defined dually. For $F \geq A$,

$$(\beta_v\Delta)_A^F : \mathsf{c}[A] \xrightarrow{\Delta_A^{A\overline{F}}} \mathsf{c}[A\overline{F}] \xrightarrow{(\beta_v)_{F,A\overline{F}}} \mathsf{c}[F],$$

$$(\beta_v^{-1}\Delta)_A^F : \mathsf{c}[A] \xrightarrow{\Delta_A^{A\overline{F}}} \mathsf{c}[A\overline{F}] \xrightarrow{(\beta_v^{-1})_{F,A\overline{F}}} \mathsf{c}[F].$$

These twistings of the (co)product indeed yield (co)monoids:

Proposition 2.53. *If* $\mathsf{a} = (\mathsf{a}, \mu)$ *is a monoid, then so are*

$$\mathsf{a}^{\mathrm{op}} := (\mathsf{a}, \mu\beta_v) \quad \textit{and} \quad {}^{\mathrm{op}}\mathsf{a} := (\mathsf{a}, \mu\beta_v^{-1}).$$

Dually, if $\mathsf{c} = (\mathsf{c}, \Delta)$ *is a comonoid, then so are*

$$\mathsf{c}^{\mathrm{cop}} := (\mathsf{c}, \beta_v^{-1}\Delta) \quad \textit{and} \quad {}^{\mathrm{cop}}\mathsf{c} := (\mathsf{c}, \beta_v\Delta).$$

Proof. We show that a^{op} is a monoid (the remaining checks being similar). For that, we need to verify (2.8). Let A and B be faces with the same support, and $A \leq F$ and $G = BF$ (and hence $F = AG$). Then the naturality diagram can be filled in as follows.

$$\begin{tikzcd}
\mathsf{a}[F] \arrow[r, "(\beta_v)_{A\overline{F},F}"] \arrow[d, "\beta_{G,F}"'] & \mathsf{a}[A\overline{F}] \arrow[r, "\mu_A^{A\overline{F}}"] \arrow[d, "\beta_{B\overline{G},A\overline{F}}"] & \mathsf{a}[A] \arrow[d, "\beta_{B,A}"] \\
\mathsf{a}[G] \arrow[r, "(\beta_v)_{B\overline{G},G}"'] & \mathsf{a}[B\overline{G}] \arrow[r, "\mu_B^{B\overline{G}}"'] & \mathsf{a}[B]
\end{tikzcd}$$

The first square commutes because $v_{A\overline{F},F} = v_{B\overline{G},G}$ by (1.30f). The second square commutes by naturality since $BA\overline{F} = BA\overline{AG} = B\overline{G}$.

Let $A \leq F \leq G$. The associativity diagram can be filled in as follows.

$$\begin{tikzcd}
 & & \mathsf{a}[F] \arrow[dr, "(\beta_v)_{A\overline{F},F}"] & & \\
 & \mathsf{a}[F\overline{G}] \arrow[ur, "\mu_F^{F\overline{G}}"] \arrow[dr, "(\beta_v)_{A\overline{G},F\overline{G}}"] & & \mathsf{a}[A\overline{F}] \arrow[dr, "\mu_A^{A\overline{F}}"] & \\
\mathsf{a}[G] \arrow[ur, "(\beta_v)_{F\overline{G},G}"] \arrow[rr, "(\beta_v)_{A\overline{G},G}"'] & & \mathsf{a}[A\overline{G}] \arrow[ur, "\mu_{A\overline{F}}^{A\overline{G}}"] \arrow[rr, "\mu_A^{A\overline{G}}"'] & & \mathsf{a}[A]
\end{tikzcd}$$

By the gate property (1.18) (applied to the arrangement under the support of G), there is a minumum gallery $A\overline{G}$ -- $F\overline{G}$ -- G, so the first triangle commutes by (2.67) and by (1.30c). The second triangle commutes by associativity since $A \leq A\overline{F} \leq A\overline{G}$. The square commutes by naturality and the fact that $v_{A\overline{F},F} = v_{A\overline{G},F\overline{G}}$ by (1.30b).

Finally, the unitality axiom holds since $(\beta_v)_{A,A} = \mathrm{id} = \mu_A^A$. ∎

This construction of twisting the (co)product is functorial:

Lemma 2.54. *If* $\mathsf{a} \to \mathsf{b}$ *is a morphism of monoids, then so are* $\mathsf{a}^{\mathrm{op}} \to \mathsf{b}^{\mathrm{op}}$ *and* ${}^{\mathrm{op}}\mathsf{a} \to {}^{\mathrm{op}}\mathsf{b}$. *Dually, if* $\mathsf{c} \to \mathsf{d}$ *is a morphism of comonoids, then so are* $\mathsf{c}^{\mathrm{cop}} \to \mathsf{d}^{\mathrm{cop}}$ *and* ${}^{\mathrm{cop}}\mathsf{c} \to {}^{\mathrm{cop}}\mathsf{d}$.

Proof. We explain the first part of the first statement. Diagram (2.9) can be filled in as follows.

$$\begin{tikzcd}
\mathsf{a}[F] \arrow[r, "(\beta_v)_{A\overline{F},F}"] \arrow[d, "f_F"'] & \mathsf{a}[A\overline{F}] \arrow[r, "\mu_A^{A\overline{F}}"] \arrow[d, "f_{A\overline{F}}"] & \mathsf{a}[A] \arrow[d, "f_A"] \\
\mathsf{b}[F] \arrow[r, "(\beta_v)_{A\overline{F},F}"'] & \mathsf{b}[A\overline{F}] \arrow[r, "\mu_A^{A\overline{F}}"'] & \mathsf{b}[A]
\end{tikzcd}$$

The first square commutes by (2.3), while the second square commutes by (2.9). $\square$

Observe that for any monoid a and comonoid c,

$${}^{\mathrm{op}}(\mathsf{a}^{\mathrm{op}}) = \mathsf{a} = ({}^{\mathrm{op}}\mathsf{a})^{\mathrm{op}} \quad \text{and} \quad {}^{\mathrm{cop}}(\mathsf{c}^{\mathrm{cop}}) = \mathsf{c} = ({}^{\mathrm{cop}}\mathsf{c})^{\mathrm{cop}}$$

as monoids and comonoids, respectively.

Exercise 2.55. Show that $\mathsf{a} \to \mathsf{b}^{\mathrm{op}}$ is a morphism of monoids iff ${}^{\mathrm{op}}\mathsf{a} \to \mathsf{b}$ is a morphism of monoids. A similar statement applies to comonoids.

Exercise 2.56. Assume that the distance function $v \equiv 1$. Check that a monoid a is commutative iff $\mathsf{a} = \mathsf{a}^{\mathrm{op}} = {}^{\mathrm{op}}\mathsf{a}$, and dually, a comonoid c is cocommutative iff $\mathsf{c} = \mathsf{c}^{\mathrm{cop}} = {}^{\mathrm{cop}}\mathsf{c}$.

Exercise 2.57. Show that $(\mathsf{a}^{\mathrm{op}})^* = {}^{\mathrm{cop}}(\mathsf{a}^*)$ and $({}^{\mathrm{op}}\mathsf{a})^* = (\mathsf{a}^*)^{\mathrm{cop}}$ for any monoid a. Write down the dual statement for a comonoid c.

2.10.3. Opposite bimonoids. We now extend the above considerations to bimonoids.

Proposition 2.58. *Let* $\mathsf{h} = (\mathsf{h}, \mu, \Delta)$ *be a* β_v*-bimonoid. Then*

$$\mathsf{h}^{\mathrm{cop}} := (\mathsf{h}, \mu, \beta_v^{-1}\Delta) \quad \text{and} \quad {}^{\mathrm{op}}\mathsf{h} := (\mathsf{h}, \mu\beta_v^{-1}, \Delta)$$

are β_v^{-1}*-bimonoids, and*

$$\mathsf{h}^{\mathrm{op,cop}} := (\mathsf{h}, \mu\beta_v, \beta_v^{-1}\Delta) \quad \text{and} \quad {}^{\mathrm{op,cop}}\mathsf{h} := (\mathsf{h}, \mu\beta_v^{-1}, \beta_v\Delta)$$

are β_v*-bimonoids. In fact,*

$$\mathsf{h}^{\mathrm{op,cop}} = {}^{\mathrm{op}}(\mathsf{h}^{\mathrm{cop}}) \quad \text{and} \quad {}^{\mathrm{op,cop}}\mathsf{h} = ({}^{\mathrm{op}}\mathsf{h})^{\mathrm{cop}}. \tag{2.68}$$

Note very carefully that in (2.68), we use that $(\beta_v^{-1})^{-1} = \beta_v$.

A summary is given in Table 2.2. In the first row, a and c are the monoid and comonoid underlying h. The remaining rows are to be read similarly. We refer to these collectively as the *op and cop constructions*.

TABLE 2.2. Op and cop constructions on bimonoids.

β_v-bimonoid	h	a	c
β_v^{-1}-bimonoids	$\mathsf{h}^{\mathrm{cop}}$	a	$\mathsf{c}^{\mathrm{cop}}$
	${}^{\mathrm{op}}\mathsf{h}$	${}^{\mathrm{op}}\mathsf{a}$	c
β_v-bimonoids	$\mathsf{h}^{\mathrm{op},\mathrm{cop}}$	a^{op}	$\mathsf{c}^{\mathrm{cop}}$
	${}^{\mathrm{op},\mathrm{cop}}\mathsf{h}$	${}^{\mathrm{op}}\mathsf{a}$	${}^{\mathrm{cop}}\mathsf{c}$

PROOF. We check that $\mathsf{h}^{\mathrm{cop}}$ is a β_v^{-1}-bimonoid. The bimonoid diagram (2.42) can be filled in as follows.

$$\begin{tikzcd}
\mathsf{h}[F] \arrow[r, "\mu_A^F"] \arrow[d, "\Delta_F^{F\overline{G}}"'] & \mathsf{h}[A] \arrow[r, "\Delta_A^{A\overline{G}}"] & \mathsf{h}[A\overline{G}] \arrow[r, "(\beta_v^{-1})_{G,A\overline{G}}"] & \mathsf{h}[G] \\
\mathsf{h}[F\overline{G}] \arrow[rr, "(\beta_v)_{A\overline{G}F,F\overline{G}}"'] \arrow[d, "(\beta_v^{-1})_{FG,F\overline{G}}"'] & & \mathsf{h}[A\overline{G}F] \arrow[u, "\mu_{A\overline{G}}^{A\overline{G}F}"'] \arrow[dr, "(\beta_v^{-1})_{GF,A\overline{G}F}"'] & \\
\mathsf{h}[FG] \arrow[rrr, "(\beta_v^{-1})_{GF,FG}"'] & & & \mathsf{h}[GF] \arrow[uu, "\mu_G^{GF}"']
\end{tikzcd}$$

The pentagon commutes by (2.42). The square on the right commutes by naturality and the fact that $v_{A\overline{G},G} = v_{A\overline{G}F,GF}$ by (1.30b). By (1.20), there is a minimal gallery $A\overline{G}F \,\text{--}\, F\overline{G} \,\text{--}\, FG \,\text{--}\, GF$. Hence, the square on the bottom commutes by (2.67) and by (1.30c).

The check for ${}^{\mathrm{op}}\mathsf{h}$ is similar. By applying ${}^{\mathrm{op}}(-)$ to the β_v^{-1}-bimonoid $\mathsf{h}^{\mathrm{cop}}$ yields the β_v-bimonoid $\mathsf{h}^{\mathrm{op},\mathrm{cop}}$. A similar remark applies to ${}^{\mathrm{op},\mathrm{cop}}\mathsf{h}$. The remaining claims follow. $\square$

2.11. Monoids, comonoids, bimonoids as functor categories

Recall that the category of species $\mathcal{A}$-Sp is a functor category, namely, it is the category of functors from $\mathcal{A}$-Hyp to Vec. We now show that the categories of monoids and comonoids are also functor categories. More precisely, there is a canonical choice for C such that $[\mathsf{C}, \mathsf{Vec}]$ is the category of monoids, and dually $[\mathsf{C}^{\mathrm{op}}, \mathsf{Vec}]$ is the category of comonoids. In fact, the same is true of all categories in Table 2.1. Thus, the categories of commutative monoids, bimonoids, and so on are all functor categories. The notations that we employ for the base categories are summarized in Table 2.3.

The objects in these base categories are either faces or flats. The morphisms can be defined via a presentation using the same generators and relations that are used to define ((co, bi)commutative) (co, bi)monoids. In each case, the generators can be shuffled using the relations to obtain a normal form for the morphisms. These are summarized in Table 2.4. These yield nice direct descriptions of the base categories.

TABLE 2.3. (Co, bi)monoids in species as functor categories.

C	$[\mathsf{C}, \mathsf{Vec}]$	C	$[\mathsf{C}, \mathsf{Vec}]$
$\mathcal{A}\text{-}\mathtt{Hyp}^{\mathsf{d}}$	monoids	$\mathcal{A}\text{-}\mathtt{Hyp}^{\mathsf{d}}_{\mathtt{c}}$	bimonoids
$\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{c}}$	comonoids	$\mathcal{A}\text{-}\mathtt{Hyp}^{\mathsf{e}}_{\mathtt{c}}$	comm. bimonoids
$\mathcal{A}\text{-}\mathtt{Hyp}^{\mathsf{e}}$	comm. monoids	$\mathcal{A}\text{-}\mathtt{Hyp}^{\mathsf{d}}_{\mathtt{r}}$	cocomm. bimonoids
$\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}$	cocomm. comonoids	$\mathcal{A}\text{-}\mathtt{Hyp}^{\mathsf{e}}_{\mathtt{r}}$	bicomm. bimonoids

TABLE 2.4. Normal forms for morphisms.

Category	Normal form	Category	Normal form
$\mathcal{A}\text{-}\mathtt{Hyp}^{\mathsf{d}}$	$F \xrightarrow{\beta} G' \xrightarrow{\mu} G$	$\mathcal{A}\text{-}\mathtt{Hyp}^{\mathsf{d}}_{\mathtt{c}}$	$F \xrightarrow{\Delta} F' \xrightarrow{\beta} G'' \xrightarrow{\mu} G$
$\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{c}}$	$F \xrightarrow{\Delta} G' \xrightarrow{\beta} G$	$\mathcal{A}\text{-}\mathtt{Hyp}^{\mathsf{e}}_{\mathtt{c}}$	$F \xrightarrow{\Delta} H \xrightarrow{\beta} GH \xrightarrow{\mu} G$
$\mathcal{A}\text{-}\mathtt{Hyp}^{\mathsf{e}}$	$\mathrm{X} \xrightarrow{\mu} \mathrm{Y}$	$\mathcal{A}\text{-}\mathtt{Hyp}^{\mathsf{d}}_{\mathtt{r}}$	$F \xrightarrow{\Delta} FH \xrightarrow{\beta} H \xrightarrow{\mu} G$
$\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}$	$\mathrm{X} \xrightarrow{\Delta} \mathrm{Y}$	$\mathcal{A}\text{-}\mathtt{Hyp}^{\mathsf{e}}_{\mathtt{r}}$	$\mathrm{X} \xrightarrow{\Delta} \mathrm{X}' \xrightarrow{\mu} \mathrm{Y}$

2.11.1. Base category for (co)monoids. We define the category $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{c}}$. Objects are faces of $\mathcal{A}$. A morphism $F \to G$ is a face F' such that $F \leq F'$ and $\mathrm{s}(F') = \mathrm{s}(G)$. We represent this morphism as a triple (F, F', G), or more vividly as

$$\begin{array}{ll} F' & \to G. \\ \uparrow & \\ F & \end{array} \tag{2.69}$$

The arrow pointing up indicates going to a bigger face, while the horizontal arrow indicates going to a face of the same support. Composition of morphisms is defined by

$$(G, G', H) \circ (F, F', G) := (F, F'G', H). \tag{2.70}$$

This is illustrated below.

$$\begin{array}{ccccc} F'G' & \to & G' & \to & H. \\ \uparrow & & \uparrow & & \\ F' & \longrightarrow & G & & \\ \uparrow & & & & \\ F & & & & \end{array}$$

Since F' and G have the same support, and G' is greater than G, it follows that $F'G'$ is greater than F' and has the same support as G'.

The identity morphisms are (F, F, F). Whenever $F \leq F'$, we have the morphism (F, F', F'), and whenever F' and G have the same support, we have the morphism (F', F', G). To simplify notation, we denote them by $F \xrightarrow{\Delta} F'$

and $F' \xrightarrow{\beta} G$, respectively. In other words,

$$\begin{array}{c} F' \\ {\scriptstyle\Delta}\uparrow \\ F \end{array} \;:=\; \begin{array}{l} F' \to F' \\ \uparrow \\ F \end{array} \qquad \text{and} \qquad F' \xrightarrow{\beta} G \;:=\; \begin{array}{l} F' \to G. \\ \uparrow \\ F' \end{array}$$

The composite of these two morphisms is precisely (2.69).

Proposition 2.59. *The category $\mathcal{A}$-$\mathtt{Hyp_c}$ has a presentation given by generators*

$$F \xrightarrow{\Delta} G, \quad A \xrightarrow{\beta} B,$$

and relations

$$\begin{tikzcd} & G \arrow[dr, "\beta"] & \\ F \arrow[ur, "\beta"] \arrow[rr, "\beta"'] & & H \end{tikzcd} \qquad \begin{tikzcd} & AG \arrow[dr, "\beta"] & \\ A \arrow[ur, "\Delta"] \arrow[dr, "\beta"'] & & G \\ & B \arrow[ur, "\Delta"'] & \end{tikzcd} \qquad \begin{tikzcd} & G \arrow[dr, "\Delta"] & \\ F \arrow[ur, "\Delta"] \arrow[rr, "\Delta"'] & & H \end{tikzcd} \tag{2.71a}$$

$$(F \xrightarrow{\Delta} F) = \mathrm{id} = (F \xrightarrow{\beta} F). \tag{2.71b}$$

By convention, Δ goes from a smaller face to a bigger face, and β is between faces of the same support.

PROOF. Let C denote the category with the above presentation. Then any morphism in C from F to G can be uniquely written as a composite

$$F \xrightarrow{\Delta} F' \xrightarrow{\beta} G.$$

(This corresponds to (2.69).) To see this, let $F \to F_1 \to \cdots \to F_{n-1} \to G$ be a morphism from F to G. Put $F_0 = F$ and $F_n = G$ for uniformity of notation.

Using (2.71a), two consecutive Δ (resp. β) can be shortened to a single Δ (resp. β), and a β followed by a Δ can be replaced by a Δ followed by a β. So the given morphism can be reduced to a Δ followed by a β as claimed. Further observe that $F' = F_0 \dots F_n$, so uniqueness of F' also follows. To complete the argument, one checks that morphisms compose by the rule (2.70) which is straightforward. $\square$

Let $\mathcal{A}$-$\mathtt{Hyp^d}$ denote the opposite category of $\mathcal{A}$-$\mathtt{Hyp_c}$. A morphism $G \to F$ is a face F' such that $F \leq F'$ and $\mathrm{s}(F') = \mathrm{s}(G)$. We represent this morphism as a triple (G, F', F), or more vividly as

$$\begin{array}{l} G \to F' \\ \qquad \downarrow \\ \qquad F. \end{array}$$

It has a presentation similar to $\mathcal{A}$-$\mathtt{Hyp_c}$ with generators $G \xrightarrow{\mu} F$ and $A \xrightarrow{\beta} B$.

Proposition 2.60. *The category of comonoids in species is equivalent to the functor category $[\mathcal{A}\text{-}\mathtt{Hyp_c}, \mathsf{Vec}]$. Dually, the category of monoids in species is equivalent to $[\mathcal{A}\text{-}\mathtt{Hyp^d}, \mathsf{Vec}]$.*

Proof. The first statement follows from Proposition 2.59. More explicitly: in (2.71a), the first diagram translates to the first diagram in (2.1), the second and third diagrams to the first two diagrams in (2.8). The relation (2.71b) translates to the last diagrams in (2.1) and (2.8).

The second statement follows by dual considerations. □

2.11.2. Base category for bimonoids. We define the category $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{c}}^{\mathtt{d}}$. Objects are faces of $\mathcal{A}$. A morphism $F \to G$ is a pair of faces (F', G'') of the same support such that $F \leq F'$ and $G \leq G''$. We represent this morphism as a quadruple (F, F', G'', G), or more vividly as

$$\begin{array}{ccc} F' & \to & G'' \\ \uparrow & & \downarrow \\ F & & G. \end{array} \tag{2.72}$$

The up arrow is from a smaller face to a bigger face, the horizontal arrow is between faces of the same support, and the down arrow is from a bigger face to a smaller face. Composition of morphisms is defined by

$$(G, G', H'', H) \circ (F, F', G'', G) := (F, F'G', H''G'', H). \tag{2.73}$$

This is illustrated below.

$$\begin{tikzcd}[row sep=small] F'G' \arrow[r] & G''G' \arrow[rr] & & G'G'' \arrow[r] & H''G'' \arrow[d] \\ F' \arrow[u] \arrow[r] & G'' \arrow[u] \arrow[dr] & & G' \arrow[u, leftarrow] \arrow[r] & H'' \arrow[d] \\ F \arrow[u] & & G \arrow[ur] & & H \end{tikzcd} \tag{2.74}$$

Note that faces linked by horizontal arrows have the same support.

The identity morphisms are (F, F, F, F). Whenever $F \leq G$, we have the morphisms (F, G, G, G) and (G, G, G, F), and whenever A and B have the same support, we have the morphism (A, A, B, B). To simplify notation, we denote them by $F \xrightarrow{\Delta} G$, $G \xrightarrow{\mu} F$, $A \xrightarrow{\beta} B$, respectively. In other words,

$$\begin{array}{c} F' \\ {\scriptstyle\Delta}\!\uparrow \\ F \end{array} := \begin{array}{ccc} F' & \to & F' \\ \uparrow & & \downarrow \\ F & & F' \end{array} \qquad F' \xrightarrow{\beta} G'' := \begin{array}{ccc} F' & \to & G'' \\ \uparrow & & \downarrow \\ F' & & G'' \end{array} \qquad \begin{array}{c} G'' \\ {\scriptstyle\mu}\!\downarrow \\ G \end{array} := \begin{array}{ccc} G'' & \to & G'' \\ \uparrow & & \downarrow \\ G'' & & G. \end{array}$$

The composite of these three morphisms is precisely (2.72). This also makes it clear how both $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{c}}$ and $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}$ can be viewed as subcategories of $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{c}}^{\mathtt{d}}$.

Proposition 2.61. *The category $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{c}}^{\mathtt{d}}$ has a presentation given by generators*

$$F \xrightarrow{\Delta} G, \quad G \xrightarrow{\mu} F, \quad A \xrightarrow{\beta} B,$$

and relations

$$\begin{tikzcd}[column sep=small] & G \arrow[dr, "\mu"] & \\ H \arrow[ur, "\mu"] \arrow[rr, "\mu"'] & & F \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & G \arrow[dr, "\beta"] & \\ F \arrow[ur, "\beta"] \arrow[rr, "\beta"'] & & H \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & G \arrow[dr, "\Delta"] & \\ F \arrow[ur, "\Delta"] \arrow[rr, "\Delta"'] & & H \end{tikzcd} \tag{2.75a}$$

(2.75b)
$$
\begin{tikzcd}[column sep=small]
 & BF \arrow[dr, "\mu"] & \\
F \arrow[ur, "\beta"] \arrow[dr, "\mu"'] & & B \\
 & A \arrow[ur, "\beta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
FG \arrow[rr, "\beta"] & & GF \arrow[d, "\mu"] \\
F \arrow[u, "\Delta"] \arrow[dr, "\mu"'] & & G \\
 & A \arrow[ur, "\Delta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & AG \arrow[dr, "\beta"] & \\
A \arrow[ur, "\Delta"] \arrow[dr, "\beta"'] & & G \\
 & B \arrow[ur, "\Delta"'] &
\end{tikzcd}
$$

(2.75c)
$$(F \xrightarrow{\Delta} F) = (F \xrightarrow{\mu} F) = (F \xrightarrow{\beta} F) = \mathrm{id}\,.$$

By convention, Δ goes from a smaller face to a bigger face, μ from a bigger face to a smaller face, and β is between faces of the same support.

Proof. Let C denote the category with the above presentation. Then any morphism in C from F to G can be uniquely written as a composite

$$F \xrightarrow{\Delta} F' \xrightarrow{\beta} G'' \xrightarrow{\mu} G.$$

More precisely, an arbitrary morphism $F \to F_1 \to \cdots \to F_{n-1} \to G$ from F to G, by using the above relations, is equal to $F \to F_0 \dots F_n \to F_n \dots F_0 \to G$ (with $F_0 = F$ and $F_n = G$). We omit the details since they are similar to the proof of Proposition 2.59. The fact that the morphisms compose by the rule (2.73) is illustrated by the diagram (2.74): The pentagon and squares commute since they are relations. □

Proposition 2.62. *The category of bimonoids in species is equivalent to the functor category* $[\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{c}}, \mathsf{Vec}]$.

Proof. This follows from Proposition 2.61. The middle diagram in (2.75b) translates to the bimonoid axiom (2.12). The remaining translations are as in the proof of Proposition 2.60. □

2.11.3. Base category for (co)commutative (co)monoids. Let $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}$ denote the category whose objects are faces of $\mathcal{A}$, and there is a unique morphism $F \to G$ whenever $\mathrm{s}(F) \leq \mathrm{s}(G)$. Equivalently, we may take this to be the category whose objects are flats of $\mathcal{A}$, and there is a unique morphism $\mathrm{X} \to \mathrm{Y}$ whenever $\mathrm{X} \leq \mathrm{Y}$. In other words, this is the category associated to the poset of flats of $\mathcal{A}$ (in the sense of Section 1.1.8).

Proposition 2.63. *The category $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}$ has a presentation given by generators*

$$\mathrm{X} \xrightarrow{\Delta} \mathrm{Y}$$

with $\mathrm{X} \leq \mathrm{Y}$, and relations

(2.76)
$$
\begin{tikzcd}[column sep=small]
 & \mathrm{X} \arrow[dr, "\Delta"] & \\
\mathrm{Z} \arrow[ur, "\Delta"] \arrow[rr, "\Delta"'] & & \mathrm{Y}
\end{tikzcd}
\qquad\qquad
(\mathrm{Z} \xrightarrow{\Delta} \mathrm{Z}) = \mathrm{id},
$$

the first for every $\mathrm{Z} \leq \mathrm{X} \leq \mathrm{Y}$, and the second for every Z.

Proposition 2.64. *The category* $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}$ *has a presentation as for* $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{c}}$ *(Proposition 2.59) with the middle diagram in (2.71a) replaced by the following diagram.*

$$
\begin{tikzcd}
 & F \arrow[dr, "\beta"] & \\
A \arrow[ur, "\Delta"] \arrow[dr, "\beta"'] & & G \\
 & B \arrow[ur, "\Delta"'] &
\end{tikzcd}
\tag{2.77}
$$

Proof. We show that there is a unique morphism from $A \to G$ when $\mathrm{s}(A) \le \mathrm{s}(G)$ and no morphisms otherwise. The latter is clear. Using the relations (2.77), it is immediate that any morphism can be written as a Δ followed by a β. Further, the commutative diagram

$$
\begin{tikzcd}
 & F \arrow[dr, "\beta"] \arrow[dd, "\beta"] & \\
A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & G \\
 & AG \arrow[ur, "\beta"'] &
\end{tikzcd}
$$

shows that any such morphism from A to G is equal to the morphism with middle term AG. ☐

Let $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{e}}$ denote the opposite category of $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}$. It has a presentation as in Propositions 2.63 and 2.64, with the arrows labeled by Δ reversed (and relabeled by μ). In particular, the following relation holds.

$$
\begin{tikzcd}
 & G \arrow[dr, "\mu"] & \\
F \arrow[ur, "\beta"] \arrow[dr, "\mu"'] & & B \\
 & A \arrow[ur, "\beta"'] &
\end{tikzcd}
\tag{2.78}
$$

Proposition 2.65. *The category of cocommutative comonoids in species is equivalent to the functor category* $[\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}, \mathsf{Vec}]$*. Dually, the category of commutative monoids in species is equivalent to* $[\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{e}}, \mathsf{Vec}]$*.*

Proof. Let us consider the second statement. It follows from Proposition 2.20 and the dual of Proposition 2.63. Alternatively, compare Lemma 2.17 and the dual of Proposition 2.64. ☐

2.11.4. Base category for (co)commutative bimonoids. Define the category $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}^{\mathtt{d}}$ as follows. Objects are faces of $\mathcal{A}$. A morphism $F \to G$ is a face G' such that $\mathrm{s}(F) \le \mathrm{s}(G')$ and $G \le G'$. We represent this morphism as a triple (F, G', G), or more vividly as

$$
\begin{tikzcd}
 & G' \arrow[d] \\
F \arrow[ur] & G.
\end{tikzcd}
\tag{2.79}
$$

The arrow pointing up indicates going to a face of bigger support, while the arrow pointing down indicates going to a smaller face. Composition of morphisms is defined by

$$(G, H', H) \circ (F, G', G) := (F, H'G', H). \tag{2.80}$$

This is illustrated below.

$$\begin{tikzcd}[column sep=small, row sep=small] & & & H'G' \arrow[d] \\ & G' \arrow[d] \arrow[urr] & & H' \arrow[d] \\ F \arrow[ur] & G \arrow[urr] & & H \end{tikzcd}$$

Proposition 2.66. *The category $\mathcal{A}$-$\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{r}}$ has a presentation as for $\mathcal{A}$-$\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{c}}$ (Proposition 2.61) with the last relation in (2.75b) replaced by (2.77).*

Proof. Let C denote the category with the above presentation. As in the proof of Proposition 2.61, we can express any morphism in C from F to G in the form $F \xrightarrow{\Delta} F' \xrightarrow{\beta} G'' \xrightarrow{\mu} G$. Now as in the proof of Proposition 2.64, F' can be taken to be FG'', so we are only left with the requirement that $\mathrm{s}(G'') \geq \mathrm{s}(F)$. This is the normal form for the morphism. One then checks that morphisms compose by the rule (2.80). $\square$

Let $\mathcal{A}$-$\mathtt{Hyp}^{\mathtt{e}}_{\mathtt{c}}$ denote the opposite category of $\mathcal{A}$-$\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{r}}$. It has a presentation as for $\mathcal{A}$-$\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{c}}$ (Proposition 2.61) with the first relation in (2.75b) replaced by (2.78).

Proposition 2.67. *The category of cocommutative bimonoids in species is equivalent to the functor category $[\mathcal{A}$-$\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{r}}, \mathsf{Vec}]$. Dually, the category of commutative bimonoids in species is equivalent to $[\mathcal{A}$-$\mathtt{Hyp}^{\mathtt{e}}_{\mathtt{c}}, \mathsf{Vec}]$.*

2.11.5. Base category for bicommutative bimonoids. We define the category $\mathcal{A}$-$\mathtt{Hyp}^{\mathtt{e}}_{\mathtt{r}}$. Objects are flats of $\mathcal{A}$. A morphism $\mathrm{X} \to \mathrm{Y}$ is a flat X' greater than both X and Y. We represent this morphism as a triple $(\mathrm{X}, \mathrm{X}', \mathrm{Y})$, or more vividly as

$$\begin{tikzcd}[column sep=small, row sep=small] & \mathrm{X}' \arrow[dr] & \\ \mathrm{X} \arrow[ur] & & \mathrm{Y}. \end{tikzcd} \tag{2.81}$$

The arrow pointing up indicates going to a bigger flat, while the arrow pointing down indicates going to a smaller flat. Composition of morphisms is defined by

$$(\mathrm{Y}, \mathrm{Y}', \mathrm{Z}) \circ (\mathrm{X}, \mathrm{X}', \mathrm{Y}) := (\mathrm{X}, \mathrm{X}' \vee \mathrm{Y}', \mathrm{Z}). \tag{2.82}$$

This is illustrated below.

$$\begin{tikzcd}[column sep=small, row sep=small] & & \mathrm{X}' \vee \mathrm{Y}' \arrow[dr] & & \\ & \mathrm{X}' \arrow[ur] \arrow[dr] & & \mathrm{Y}' \arrow[dr] & \\ \mathrm{X} \arrow[ur] & & \mathrm{Y} \arrow[ur] & & \mathrm{Z}. \end{tikzcd}$$

The identity morphisms are $(\mathrm{X}, \mathrm{X}, \mathrm{X})$. Whenever $\mathrm{X} \leq \mathrm{Y}$, we have the morphisms $(\mathrm{X}, \mathrm{Y}, \mathrm{Y})$ and $(\mathrm{Y}, \mathrm{Y}, \mathrm{X})$. The first is from X to Y, while the second is from Y to X. To simplify notation, we denote them by $\mathrm{X} \xrightarrow{\Delta} \mathrm{Y}$ and $\mathrm{Y} \xrightarrow{\mu} \mathrm{X}$. In other words,

$$\begin{tikzcd}[column sep=small] & \mathrm{X}' \\ \mathrm{X} \arrow[ur, "\Delta"] \end{tikzcd} := \begin{tikzcd}[column sep=small] & \mathrm{X}' \arrow[dr] & \\ \mathrm{X} \arrow[ur] & & \mathrm{X}' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] \mathrm{X}' \arrow[dr, "\mu"] & \\ & \mathrm{Y} \end{tikzcd} := \begin{tikzcd}[column sep=small] & \mathrm{X}' \arrow[dr] & \\ \mathrm{X}' \arrow[ur] & & \mathrm{Y}. \end{tikzcd}$$

The composite of these two morphisms is precisely (2.81). This also makes it clear how both $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}$ and $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{e}}$ can be viewed as subcategories of $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{e}}_{\mathtt{r}}$.

Proposition 2.68. *The category $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{e}}_{\mathtt{r}}$ has a presentation given by generators*

$$\mathrm{X} \xrightarrow{\Delta} \mathrm{Y}, \quad \mathrm{Y} \xrightarrow{\mu} \mathrm{X},$$

and relations

$$\begin{tikzcd} & \mathrm{X} \arrow[dr, "\Delta"] & \\ \mathrm{Z} \arrow[ur, "\Delta"] \arrow[rr, "\Delta"'] & & \mathrm{Y} \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{X} \vee \mathrm{Y} \arrow[dr, "\mu"] & \\ \mathrm{X} \arrow[ur, "\Delta"] \arrow[dr, "\mu"'] & & \mathrm{Y} \\ & \mathrm{Z} \arrow[ur, "\Delta"'] & \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{X} \arrow[dr, "\mu"] & \\ \mathrm{Y} \arrow[ur, "\mu"] \arrow[rr, "\mu"'] & & \mathrm{Z} \end{tikzcd} \tag{2.83a}$$

$$(\mathrm{X} \xrightarrow{\Delta} \mathrm{X}) = \mathrm{id} = (\mathrm{X} \xrightarrow{\mu} \mathrm{X}). \tag{2.83b}$$

By convention, Δ goes from a smaller flat to a bigger flat, and μ from a bigger flat to a smaller flat.

Proof. Let C denote the category with the above presentation. We claim that any morphism in C from X to Y can be uniquely written as a composite

$$\mathrm{X} \xrightarrow{\Delta} \mathrm{X}' \xrightarrow{\mu} \mathrm{Y}.$$

(This corresponds to (2.81).) To see this, let $\mathrm{X} \to \mathrm{X}_1 \to \cdots \to \mathrm{X}_{n-1} \to \mathrm{Y}$ be a morphism from X to Y. Put $\mathrm{X}_0 = \mathrm{X}$ and $\mathrm{X}_n = \mathrm{Y}$ for uniformity of notation.

Two consecutive ups $\mathrm{X}_i \xrightarrow{\Delta} \mathrm{X}_{i+1} \xrightarrow{\Delta} \mathrm{X}_{i+2}$ can be shortened to a single up $\mathrm{X}_i \xrightarrow{\Delta} \mathrm{X}_{i+2}$, by using the first diagram in (2.83a). Similarly, two consecutive downs can be shortened to a single down. A down followed by an up can be replaced by an up followed by a down, by using the diamond in (2.83a).

By repeated application of these steps, the given morphism can be reduced to a single up followed by a single down as required. Further observe that the resulting X' is the join of all the X_i, so uniqueness also follows. It now remains to check that morphisms compose by the rule (2.82). But this is immediate from the diamond in (2.83a). ☐

Proposition 2.69. *The category $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{e}}_{\mathtt{r}}$ is equivalent to the category with presentation as for $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{c}}$ (Proposition 2.61) but with the first and last relations in (2.75b) replaced by (2.77) and (2.78).*

Proof. Let C denote the category with the above presentation. Arguing as in the proof of Proposition 2.66, we deduce that a morphism in C from F to G is a flat X which is greater than both $\mathrm{s}(F)$ and $\mathrm{s}(G)$, and composition of morphisms is given by the join of the indexing flats. Faces with the same support are all isomorphic to one another in C, so we may replace them by one object to get an equivalent category. ☐

Proposition 2.70. *The category of bicommutative bimonoids is equivalent to the functor category* $[\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{e}}_{\mathtt{r}}, \mathsf{Vec}]$.

Proof. This follows from Propositions 2.22 and 2.68. Alternatively, we may use Proposition 2.69 and the original definition of a bicommutative bimonoid which is given in terms of faces. ☐

2.11.6. (Co)completeness of the category of species. Consider the functor category $[\mathsf{C}, \mathsf{Vec}]$, where C is any category. Recall that the category Vec is (co)complete, that it, all (co)limits in Vec exist. Therefore, the functor category $[\mathsf{C}, \mathsf{Vec}]$ is also (co)complete, with (co)limits calculated pointwise.

The preceding discussion shows that the categories of (co, bi)monoids and their commutative counterparts are functor categories of the above kind, so they are (co)complete. For instance, the initial and terminal object in Vec is the zero vector space. Hence, the zero species (2.4) is the initial and terminal object in all categories listed in Table 2.1. In particular, the zero species is a (co)commutative (co, bi)monoid. Similarly, the product and coproduct in Vec is given by direct sum. Hence, the product and coproduct in the categories of Table 2.1 is given by direct sum (2.5).

2.12. Presentation for (co)monoids using covering generators

Recall from Section 2.11 that the categories of monoids, commutative monoids, and so on are all functor categories of the form $[\mathsf{C}, \mathsf{Vec}]$. The base categories C have presentations which mirror the axioms used to define these objects. We now observe that in each case, the set of generators can be made smaller by restricting to those μ and Δ which go between faces (or flats) whose ranks differ by 1. We call these the covering generators since in this case the larger face (or flat) covers the smaller face (or flat). The main effect on the set of relations is that the (co)associativity axiom instead of being a commutative triangle is now a commutative square. The relevant poset-theoretic property is strong connectivity (Section 1.1.7).

2.12.1. (Co)commutative (co)monoids. Recall the category $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}$ associated to the poset of flats of $\mathcal{A}$: objects are flats of $\mathcal{A}$, and there is a unique morphism $\mathrm{X} \to \mathrm{Y}$ whenever $\mathrm{X} \leq \mathrm{Y}$.

Proposition 2.71. *The category* $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}$ *has a presentation given by generators*

$$\Delta : \mathrm{X} \to \mathrm{Y},$$

where Y *covers* X*, and relations*

$$\begin{tikzcd} \mathrm{X}' \arrow[r, "\Delta"] & \mathrm{Y} \\ \mathrm{Z} \arrow[u, "\Delta"] \arrow[r, "\Delta"'] & \mathrm{X} \arrow[u, "\Delta"'] \end{tikzcd}$$

whenever Y *covers both* X *and* X'*, and they in turn cover* Z.

Proof. In view of Lemma 1.4, item (2), this is an instance of Proposition 1.5. For convenience, we spell out this argument.

Let C denote the category with the above presentation. For any $\mathrm{X} \leq \mathrm{Y}$, there is a morphism in C from X to Y: Pick a maximal chain of flats from X to Y. To finish the proof, we need to show that this morphism is unique. We proceed by induction on the length of the chain. Let $\mathrm{X}_0 \to \mathrm{X}_1 \to \cdots \to \mathrm{X}_n$ and $\mathrm{X}_0 \to \mathrm{X}'_1 \to \cdots \to \mathrm{X}_n$ be two maximal chains from X_0 to X_n.

$$\begin{tikzcd}
& \mathrm{X}_1 \arrow[r] & \mathrm{X}_2 \arrow[r] & \dots \arrow[r] & \mathrm{X}_{n-1} \arrow[dr] & \\
\mathrm{X}_0 \arrow[ur] \arrow[dr] \arrow[rrr, dashed] & & & \mathrm{Z} \arrow[ur] \arrow[dr] & & \mathrm{X}_n \\
& \mathrm{X}'_1 \arrow[r] & \mathrm{X}'_2 \arrow[r] & \dots \arrow[r] & \mathrm{X}'_{n-1} \arrow[ur] &
\end{tikzcd}$$

We may assume $\mathrm{X}_{n-1} \neq \mathrm{X}'_{n-1}$. Then they both cover $\mathrm{Z} := \mathrm{X}_{n-1} \wedge \mathrm{X}'_{n-1}$. Now pick any maximal chain from X_0 to Z. In the above illustration, the diamond commutes since it is a relation, while the parallelograms commute by induction. So the diagram commutes as required. $\square$

A similar result holds for the opposite category $\mathcal{A}$-$\mathtt{Hyp}^{\mathsf{e}}$ by reversing arrows. In conjunction with Proposition 2.65, we deduce:

Proposition 2.72. *A cocommutative comonoid is a species* c *equipped with linear maps*

$$\Delta_{\mathrm{X}}^{\mathrm{Y}} : \mathsf{c}[\mathrm{X}] \to \mathsf{c}[\mathrm{Y}],$$

whenever Y *covers* X*, such that the diagram*

$$\begin{tikzcd}
\mathsf{c}[\mathrm{X}'] \arrow[r, "\Delta_{\mathrm{X}'}^{\mathrm{Y}}"] & \mathsf{c}[\mathrm{Y}] \\
\mathsf{c}[\mathrm{Z}] \arrow[u, "\Delta_{\mathrm{Z}}^{\mathrm{X}'}"] \arrow[r, "\Delta_{\mathrm{Z}}^{\mathrm{X}}"'] & \mathsf{c}[\mathrm{X}] \arrow[u, "\Delta_{\mathrm{X}}^{\mathrm{Y}}"']
\end{tikzcd}$$

commutes, whenever Y *covers both* X *and* X'*, and they in turn cover* Z*.*

Proposition 2.73. *A commutative monoid is a species* a *equipped with linear maps*

$$\mu_{\mathrm{X}}^{\mathrm{Y}} : \mathsf{a}[\mathrm{Y}] \to \mathsf{a}[\mathrm{X}],$$

whenever Y *covers* X*, such that the diagram*

$$\begin{tikzcd}
\mathsf{a}[\mathrm{Y}] \arrow[r, "\mu_{\mathrm{X}'}^{\mathrm{Y}}"] \arrow[d, "\mu_{\mathrm{X}}^{\mathrm{Y}}"'] & \mathsf{a}[\mathrm{X}'] \arrow[d, "\mu_{\mathrm{Z}}^{\mathrm{X}'}"] \\
\mathsf{a}[\mathrm{X}] \arrow[r, "\mu_{\mathrm{Z}}^{\mathrm{X}}"'] & \mathsf{a}[\mathrm{Z}]
\end{tikzcd}$$

commutes, whenever Y *covers both* X *and* X'*, and they in turn cover* Z*.*

2.12.2. Bicommutative bimonoids. Recall the category $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}^{\mathtt{e}}$ which contains $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}$ and its opposite category $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{e}}$ as subcategories. A presentation for $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}^{\mathtt{e}}$ was given in Proposition 2.68. An alternative presentation using covering generators is given below.

Proposition 2.74. *The category $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}^{\mathtt{e}}$ has a presentation given by generators*

$$\Delta : \mathrm{X} \to \mathrm{Y}, \quad \mu : \mathrm{Y} \to \mathrm{X},$$

whenever Y *covers* X, *and relations*

$$\begin{tikzcd}
\mathrm{X}' \arrow[r, "\Delta"] & \mathrm{Y} \\
\mathrm{Z} \arrow[u, "\Delta"] \arrow[r, "\Delta"'] & \mathrm{X} \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathrm{X} \vee \mathrm{Y} \arrow[dr, dashed] & \\
\mathrm{X} \arrow[ur, dashed] \arrow[dr, "\mu"'] & & \mathrm{Y} \\
& \mathrm{Z} \arrow[ur, "\Delta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{X}' \arrow[d, "\mu"'] & \mathrm{Y} \arrow[l, "\mu"'] \arrow[d, "\mu"] \\
\mathrm{Z} & \mathrm{X}. \arrow[l, "\mu"]
\end{tikzcd}$$

In the diamond, X and Y both cover Z. But it does not imply that $\mathrm{X} \vee \mathrm{Y}$ covers X and Y. (If it covers one of them, then it will cover the other as well by dimension considerations.) The dotted arrows are the composites defined using any maximal chains of flats.

Proof. It is clear from the analysis done for $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}$ in Proposition 2.71 that the triangles in (2.83a) can be replaced by the two squares above. To see that the diamond in (2.83a) can be filled in using the above diamond, repeatedly use

$$\begin{tikzcd}
\mathrm{X} \arrow[r, "\Delta"] \arrow[d, "\mu"'] & \mathrm{X} \vee \mathrm{Y} \arrow[r, "\Delta"] \arrow[d, "\mu"] & \mathrm{X} \vee \mathrm{Y}' \arrow[d, "\mu"] \\
\mathrm{Z} \arrow[r, "\Delta"'] & \mathrm{Y} \arrow[r, "\Delta"'] & \mathrm{Y}'
\end{tikzcd}$$

and its dual diagram. Finally, (2.83b) must be removed. ☐

Exercise 2.75. Describe a bicommutative bimonoid using the above presentation of $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}^{\mathtt{e}}$. (The connection between the two is made by Proposition 2.70.)

2.12.3. (Co)monoids. Recall the category $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{c}}$ whose objects are faces of $\mathcal{A}$. A presentation for this category is given in Proposition 2.59. The result below gives a smaller presentation by restricting Δ to faces related by a cover relation.

Proposition 2.76. *The category $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{c}}$ has a presentation given by generators*

$$F \xrightarrow{\Delta} G, \quad A \xrightarrow{\beta} B,$$

and relations

$$\begin{tikzcd}
& G \arrow[dr, "\beta"] & \\
F \arrow[ur, "\beta"] \arrow[rr, "\beta"'] & & H
\end{tikzcd}
\qquad
\begin{tikzcd}
& AG \arrow[dr, "\beta"] & \\
A \arrow[ur, "\Delta"] \arrow[dr, "\beta"'] & & G \\
& B \arrow[ur, "\Delta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
F' \arrow[r, "\Delta"] & G \\
A \arrow[u, "\Delta"] \arrow[r, "\Delta"'] & F \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
(F \xrightarrow{\beta} F) = \mathrm{id}\,.$$

By convention, Δ goes from a face to a face covering it, and β is between faces of the same support.

Proof. By Lemma 1.4, item (1), the poset of faces is strongly connected. So by Proposition 1.5, the third relation in (2.71a) is equivalent to the third relation above.

Suppose A and B have the same support and $G \geq B$. Then by Tits projection, maximal chains from B to G correspond to maximal chains from A to AG (and corresponding faces have the same support). Using this fact, the second relation in (2.71a) can be seen to be equivalent to the second relation above. $\square$

We could further reduce the number of generators by restricting β to adjacent faces. A composite of such 'adjacent' generators amounts to a gallery. The first relation above would have to be replaced by: Galleries with the same starting and ending point define the same morphism.

A result similar to Lemma 2.76 holds for the opposite category $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathsf{d}}$ by reversing arrows. In conjunction with Proposition 2.60, we deduce:

Proposition 2.77. *A comonoid is a species* c *equipped with linear maps*

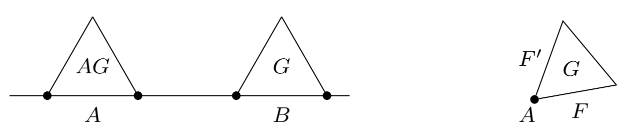
$$\Delta_F^G : \mathsf{c}[F] \to \mathsf{c}[G],$$

whenever G covers F, such that the diagrams

$$\begin{tikzcd} \mathsf{c}[AG] \arrow[r, "\beta_{G,AG}"] & \mathsf{c}[G] \\ \mathsf{c}[A] \arrow[u, "\Delta_A^{AG}"] \arrow[r, "\beta_{B,A}"'] & \mathsf{c}[B] \arrow[u, "\Delta_B^G"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{c}[F'] \arrow[r, "\Delta_{F'}^G"] & \mathsf{c}[G] \\ \mathsf{c}[A] \arrow[u, "\Delta_A^{F'}"] \arrow[r, "\Delta_A^F"'] & \mathsf{c}[F] \arrow[u, "\Delta_F^G"'] \end{tikzcd}$$

commute, where arrows labeled by Δ go from a face to a face covering it.

Illustrative pictures for the two diagrams are shown below.



Proposition 2.78. *A monoid is a species* a *equipped with linear maps*

$$\mu_F^G : \mathsf{a}[G] \to \mathsf{a}[F],$$

whenever G covers F, such that the diagrams

$$\begin{tikzcd} \mathsf{a}[AG] \arrow[r, "\beta_{G,AG}"] \arrow[d, "\mu_A^{AG}"'] & \mathsf{a}[G] \arrow[d, "\mu_B^G"] \\ \mathsf{a}[A] \arrow[r, "\beta_{B,A}"'] & \mathsf{a}[B] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{a}[G] \arrow[r, "\mu_{F'}^G"] \arrow[d, "\mu_F^G"'] & \mathsf{a}[F'] \arrow[d, "\mu_A^{F'}"] \\ \mathsf{a}[F] \arrow[r, "\mu_A^F"'] & \mathsf{a}[A] \end{tikzcd}$$

commute, where arrows labeled by μ go from a face to a face it covers.

2.12.4. Bimonoids. We now consider the category $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{c}}$. A presentation for it is given in Proposition 2.61. It can be shortened by restricting μ and Δ to faces related by a cover relation:

Proposition 2.79. *The category $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{c}}$ has a presentation given by generators*

$$F \xrightarrow{\Delta} G, \quad G \xrightarrow{\mu} F, \quad A \xrightarrow{\beta} B,$$

and relations

$$\begin{tikzcd} G \arrow[r, "\mu"] \arrow[d, "\mu"'] & F \arrow[d, "\mu"] \\ F' \arrow[r, "\mu"'] & A \end{tikzcd} \qquad \begin{tikzcd} & G \arrow[dr, "\beta"] & \\ F \arrow[ur, "\beta"] \arrow[rr, "\beta"'] & & H \end{tikzcd} \qquad \begin{tikzcd} F' \arrow[r, "\Delta"] & G \\ A \arrow[u, "\Delta"] \arrow[r, "\Delta"'] & F \arrow[u, "\Delta"'] \end{tikzcd} \tag{2.84a}$$

$$\begin{tikzcd} & BF \arrow[dr, "\mu"] & \\ F \arrow[ur, "\beta"] \arrow[dr, "\mu"'] & & B \\ & A \arrow[ur, "\beta"'] & \end{tikzcd} \qquad \begin{tikzcd} FG \arrow[rr, "\beta"] & & GF \arrow[d, dotted] \\ F \arrow[u, dotted] \arrow[dr, "\mu"'] & & G \\ & A \arrow[ur, "\Delta"'] & \end{tikzcd} \qquad \begin{tikzcd} & AG \arrow[dr, "\beta"] & \\ A \arrow[ur, "\Delta"] \arrow[dr, "\beta"'] & & G \\ & B \arrow[ur, "\Delta"'] & \end{tikzcd} \tag{2.84b}$$

$$(F \xrightarrow{\beta} F) = \mathrm{id}\,. \tag{2.84c}$$

By convention, Δ goes from a face to a face covering it, μ from a face to a face it covers, and β is between faces of the same support.

PROOF. Compare with Proposition 2.61. The relations involving β remain unchanged (except that μ and Δ now become covering generators). The triangles in (2.75a) involving μ and Δ change to squares. In (2.75c), the part involving μ and Δ must be removed. The middle diagram in (2.75b) changes to the middle diagram in (2.84b). Note that F and G both cover A, but FG may not cover F or GF may not cover G. The dotted arrows are the composites defined using maximal chains of faces. To see that the middle diagram in (2.75b) can be filled using diagrams (2.84b), repeatedly use

$$\begin{tikzcd} F \arrow[r, "\mu"] \arrow[d, "\Delta"'] & A \arrow[r, "\Delta"] & G \arrow[r, "\Delta"] & G' \\ FG \arrow[rr, "\beta"'] \arrow[d, "\Delta"'] & & GF \arrow[u, "\mu"'] \arrow[d, "\Delta"] & \\ FG' \arrow[rr, "\beta"'] & & GFG' \arrow[r, "\beta"'] & G'F \arrow[uu, "\mu"'] \end{tikzcd} \tag{2.85}$$

and its dual diagram. $\square$

Exercise 2.80. Describe a bimonoid using the above presentation of $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{c}}$. (The connection between the two is made by Proposition 2.62.)

Exercise 2.81. Give presentations for the categories $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{e}}_{\mathtt{c}}$ and $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{r}}$ as in Proposition 2.79 by modifying the first and third relations in (2.84b), respectively.

2.13. Partially commutative monoids

We introduce partially commutative monoids. This notion makes use of the concept of partial-flats discussed in Section 1.3. A partially commutative monoid is obtained by relaxing the commutativity axiom (2.17) to hold only when F and G belong to the same partial-flat. Since faces and flats are also instances of partial-flats, partially commutative monoids interpolate between monoids and commutative monoids: Faces go with monoids, flats with commutative monoids, and partial-flats with partially commutative monoids. Dually, one can define partially cocommutative comonoids, and the two constructions can be combined to define partially bicommutative bimonoids. They interpolate between bimonoids and bicommutative bimonoids. We also introduce 0-$\sim$-bicommutative bimonoids. These interpolate between 0-bimonoids and bicommutative bimonoids.

2.13.1. Species via partial-flats. For a partial-support relation $\sim$ on faces, let $\mathcal{A}\text{-}\mathtt{Hyp}_\sim$ denote the following category. An object is a partial-flat, and there is a unique morphism between partial-flats of the same support. When x and y have the same support, we use the notation $\beta_{\mathrm{y},\mathrm{x}} : \mathrm{x} \to \mathrm{y}$ for the unique morphism between them.

The category $\mathcal{A}\text{-}\mathtt{Hyp}_\sim$ interpolates between $\mathcal{A}\text{-}\mathtt{Hyp}$ and $\mathcal{A}\text{-}\mathtt{Hyp}'$, and is equivalent to both of them. Hence, the category of $\mathcal{A}$-species is equivalent to the functor category

$$[\mathcal{A}\text{-}\mathtt{Hyp}_\sim, \mathsf{Vec}].$$

In this language, an $\mathcal{A}$-species can be described as follows.

Proposition 2.82. *An $\mathcal{A}$-species is a family of vector spaces $\mathsf{p}[\mathrm{x}]$, one for each partial-flat x, together with linear maps*

$$\beta_{\mathrm{y},\mathrm{x}} : \mathsf{p}[\mathrm{x}] \to \mathsf{p}[\mathrm{y}],$$

whenever x and y have the same support, such that the diagrams

$$\begin{tikzcd} & \mathsf{p}[\mathrm{y}] \arrow[dr, "\beta_{\mathrm{z},\mathrm{y}}"] & \\ \mathsf{p}[\mathrm{x}] \arrow[ur, "\beta_{\mathrm{y},\mathrm{x}}"] \arrow[rr, "\beta_{\mathrm{z},\mathrm{x}}"'] & & \mathsf{p}[\mathrm{z}] \end{tikzcd} \qquad (\mathsf{p}[\mathrm{x}] \xrightarrow{\beta_{\mathrm{x},\mathrm{x}}} \mathsf{p}[\mathrm{x}]) = \mathrm{id}$$

commute, whenever x, y, z have the same support.

This interpolates between the standard definition of species and the one given in Proposition 2.5. These two extremes correspond to the two extreme choices for $\sim$.

2.13.2. Partially commutative monoids. Let $\sim$ be a partial-support relation on faces. An $\mathcal{A}$-monoid (a, μ) is *commutative wrt* $\sim$ if the diagram

$$\begin{tikzcd} \mathsf{a}[F] \arrow[rr, "\beta_{G,F}"] \arrow[dr, "\mu_A^F"'] & & \mathsf{a}[G] \arrow[dl, "\mu_A^G"] \\ & \mathsf{a}[A] & \end{tikzcd} \tag{2.86}$$

commutes, whenever $A \leq F$, $A \leq G$, $F \sim G$. We refer to diagram (2.86) as the *partial commutativity axiom*.

We employ the term $\sim$-commutative $\mathcal{A}$-monoid, or partially commutative $\mathcal{A}$-monoid (with $\sim$ kept implicit). This notion interpolates between monoids and commutative monoids, hence the terminology.

Partially cocommutative $\mathcal{A}$-comonoid is the dual notion.

Lemma 2.83. *Let* (a, μ) *be an* $\mathcal{A}$*-monoid. Then: It is partially commutative wrt* $\sim$ *iff the diagram*

$$\begin{tikzcd} \mathsf{a}[F] \arrow[r, "\beta_{G,F}"] \arrow[d, "\mu_A^F"'] & \mathsf{a}[G] \arrow[d, "\mu_B^G"] \\ \mathsf{a}[A] \arrow[r, "\beta_{B,A}"'] & \mathsf{a}[B] \end{tikzcd}$$

commutes, whenever $A \sim B$*,* $F \sim G$*,* $A \leq F$*,* $B \leq G$*.*

PROOF. The argument of Lemma 2.17 generalizes. ☐

Partially commutative monoids can also be formulated using the definition of species given by Proposition 2.82:

Proposition 2.84. *A* $\sim$*-commutative* $\mathcal{A}$*-monoid is an* $\mathcal{A}$*-species* a *equipped with linear maps*

$$\mu_{\mathrm{z}}^{\mathrm{x}} : \mathsf{a}[\mathrm{x}] \to \mathsf{a}[\mathrm{z}],$$

one for each $\mathrm{z} \leq \mathrm{x}$*, such that the diagrams*

$$(2.87)\qquad \begin{tikzcd} \mathsf{a}[\mathrm{x}] \arrow[r, "\beta_{\mathrm{wx},\mathrm{x}}"] \arrow[d, "\mu_{\mathrm{z}}^{\mathrm{x}}"'] & \mathsf{a}[\mathrm{wx}] \arrow[d, "\mu_{\mathrm{w}}^{\mathrm{wx}}"] \\ \mathsf{a}[\mathrm{z}] \arrow[r, "\beta_{\mathrm{w},\mathrm{z}}"'] & \mathsf{a}[\mathrm{w}] \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{a}[\mathrm{x}] \arrow[dr, "\mu_{\mathrm{z}}^{\mathrm{x}}"] & \\ \mathsf{a}[\mathrm{y}] \arrow[ur, "\mu_{\mathrm{x}}^{\mathrm{y}}"] \arrow[rr, "\mu_{\mathrm{z}}^{\mathrm{y}}"'] & & \mathsf{a}[\mathrm{z}] \end{tikzcd} \qquad (\mathsf{a}[\mathrm{z}] \xrightarrow{\mu_{\mathrm{z}}^{\mathrm{z}}} \mathsf{a}[\mathrm{z}]) = \mathrm{id}$$

commute. The first diagram is whenever z *and* w *have the same support and* $\mathrm{z} \leq \mathrm{x}$*, the second is for every* $\mathrm{z} \leq \mathrm{x} \leq \mathrm{y}$*, and the third is for every* z*.*

The argument is similar to the one given for Proposition 2.20. Note that the naturality axiom in (2.87) was not required in that case. When $\sim$ is finest, the axioms (2.87) specialize to (2.8).

2.13.3. Partially bicommutative bimonoids. For a partial-support relation $\sim$ on faces, an $\mathcal{A}$-bimonoid is $\sim$-(co)commutative if its underlying (co)monoid is $\sim$-(co)commutative.

For bicommutativity, we fix two partial-support relations on faces, say $\sim$ and $\sim'$, and consider bimonoids that are both $\sim$-commutative and $\sim'$-cocommutative. Note that by letting $\sim$ and $\sim'$ be the two extremes each, one recovers the four notions discussed earlier: bimonoids that are commutative, cocommutative, both, or neither. In the case when $\sim$ and $\sim'$ coincide, we use the term $\sim$-bicommutative. Such bimonoids can be nicely described using the formulation of species given by Proposition 2.82 and using Proposition 2.84 and its dual.

Proposition 2.85. *A $\sim$-bicommutative $\mathcal{A}$-bimonoid is the same as a triple $(\mathsf{h}, \mu, \Delta)$, where h is an $\mathcal{A}$-species, (h, μ) is a $\sim$-commutative $\mathcal{A}$-monoid, (h, Δ) is a $\sim$-cocommutative $\mathcal{A}$-comonoid, such that for any partial-flats $\mathrm{z} \leq \mathrm{x}$ and $\mathrm{z} \leq \mathrm{y}$, the diagram*

$$\begin{tikzcd} \mathsf{h}[\mathrm{x}] \arrow[r, "\mu_{\mathrm{z}}^{\mathrm{x}}"] \arrow[d, "\Delta_{\mathrm{x}}^{\mathrm{xy}}"'] & \mathsf{h}[\mathrm{z}] \arrow[r, "\Delta_{\mathrm{z}}^{\mathrm{y}}"] & \mathsf{h}[\mathrm{y}] \\ \mathsf{h}[\mathrm{xy}] \arrow[rr, "\beta_{\mathrm{yx},\mathrm{xy}}"'] & & \mathsf{h}[\mathrm{yx}] \arrow[u, "\mu_{\mathrm{y}}^{\mathrm{yx}}"'] \end{tikzcd} \tag{2.88}$$

commutes.

Observe how axiom (2.88) unifies the bimonoid axiom (2.12) and the bicommutative bimonoid axiom (2.26).

Exercise 2.86. Let $\sim$ be a partial-support relation on faces. By adding morphisms to $\mathcal{A}\text{-}\mathtt{Hyp}_{\sim}$, construct a category which interpolates between $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{c}}$ and $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}$, and serves as the base category for $\sim$-cocommutative comonoids. Do the same for $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathsf{d}}$ and $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathsf{e}}$, and for $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{c}}^{\mathsf{d}}$ and $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}^{\mathsf{e}}$. These will be the base categories for $\sim$-commutative monoids and for $\sim$-bicommutative bimonoids, respectively.

2.13.4. 0-$\sim$-bicommutative bimonoids. Let $\sim$ be a partial-support relation on faces. A 0-$\sim$-*bicommutative bimonoid* is a triple $(\mathsf{h}, \mu, \Delta)$ such that h is a species, (h, μ) is a $\sim$-commutative monoid, (h, Δ) is a $\sim$-cocommutative comonoid, and for any $\mathrm{z} \leq \mathrm{x}$ and $\mathrm{z} \leq \mathrm{y}$, the first diagram below commutes if x and y have an upper bound, and the second diagram below commutes if x and y do not have an upper bound.

$$\begin{tikzcd} \mathsf{h}[\mathrm{x}] \arrow[r, "\mu_{\mathrm{z}}^{\mathrm{x}}"] \arrow[d, "\Delta_{\mathrm{x}}^{\mathrm{xy}}"'] & \mathsf{h}[\mathrm{z}] \arrow[r, "\Delta_{\mathrm{z}}^{\mathrm{y}}"] & \mathsf{h}[\mathrm{y}] \\ \mathsf{h}[\mathrm{xy}] \arrow[rr, "\beta_{\mathrm{yx},\mathrm{xy}}"'] & & \mathsf{h}[\mathrm{yx}] \arrow[u, "\mu_{\mathrm{y}}^{\mathrm{yx}}"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{h}[\mathrm{z}] \arrow[dr, "\Delta_{\mathrm{z}}^{\mathrm{y}}"] & \\ \mathsf{h}[\mathrm{x}] \arrow[ur, "\mu_{\mathrm{z}}^{\mathrm{x}}"] \arrow[rr, "0"'] & & \mathsf{h}[\mathrm{y}] \end{tikzcd} \tag{2.89}$$

If $\sim$ is geometric, then by Lemma 1.8, the last condition can be rephrased as follows. For any $\mathrm{z} \leq \mathrm{x}$ and $\mathrm{z} \leq \mathrm{y}$, the first diagram below commutes if $\mathrm{xy} = \mathrm{yx}$, and the second diagram below commutes if $\mathrm{xy} \neq \mathrm{yx}$.

$$\begin{tikzcd} \mathsf{h}[\mathrm{x}] \arrow[r, "\mu_{\mathrm{z}}^{\mathrm{x}}"] \arrow[d, "\Delta_{\mathrm{x}}^{\mathrm{xy}}"'] & \mathsf{h}[\mathrm{z}] \arrow[d, "\Delta_{\mathrm{z}}^{\mathrm{y}}"] \\ \mathsf{h}[\mathrm{xy}] = \mathsf{h}[\mathrm{yx}] \arrow[r, "\mu_{\mathrm{y}}^{\mathrm{yx}}"'] & \mathsf{h}[\mathrm{y}] \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{h}[\mathrm{z}] \arrow[dr, "\Delta_{\mathrm{z}}^{\mathrm{y}}"] & \\ \mathsf{h}[\mathrm{x}] \arrow[ur, "\mu_{\mathrm{z}}^{\mathrm{x}}"] \arrow[rr, "0"'] & & \mathsf{h}[\mathrm{y}] \end{tikzcd} \tag{2.90}$$

This notion interpolates between 0-bimonoids and bicommutative bimonoids:

- $\sim$ is finest. In this case, 0-$\sim$-bicommutative bimonoids specialize to 0-bimonoids; diagrams (2.90) specialize to (2.39).

- $\sim$ is coarsest. In this case, 0-$\sim$-bicommutative bimonoids specialize to bicommutative bimonoids. Moreover, $\mathrm{xy} = \mathrm{yx}$ always holds, so the second diagram in (2.90) is irrelevant, while the first diagram in (2.90) specializes to (2.26).

2.13.5. Signed analogue. Let $\sim$ be a partial-support relation on faces. A monoid is *signed $\sim$-commutative* if diagram (2.44) commutes, whenever $A \leq F$ and $A \leq G$, and $F \sim G$. Signed $\sim$-cocommutative comonoids are defined dually. These notions can also be applied to signed bimonoids. We say a signed bimonoid is signed $\sim$-bicommutative if it is both signed $\sim$-commutative and signed $\sim$-cocommutative.

For partial-flats $\mathrm{z} \leq \mathrm{x}$, one can define one-dimensional vector spaces $\mathbf{E}^-[\mathrm{z}, \mathrm{x}]$ generalizing Definition 1.74 using the symbols $\mathtt{H}_{F/A}$ with $A \leq F$, A belonging to z and F belonging to x. The remaining discussion also generalizes. For instance, the map (1.163) takes the form

$$\mathbf{E}^-[\mathrm{z}, \mathrm{x}] \otimes \mathbf{E}^-[\mathrm{x}, \mathrm{xx}'] \to \mathbf{E}^-[\mathrm{z}, \mathrm{x}'] \otimes \mathbf{E}^-[\mathrm{x}', \mathrm{x}'\mathrm{x}]. \tag{2.91}$$

Using these spaces, one can also formulate signed $\sim$-commutative monoids along the lines of Proposition 2.36, signed $\sim$-cocommutative comonoids dually, and signed $\sim$-bicommutative signed bimonoids along the lines of Proposition 2.37.

When $\sim$ is finest, we recover the notion of signed bimonoids: The spaces $\mathbf{E}^-[A, F]$ are canonically isomorphic to $\Bbbk$ (with basis element $\mathtt{H}_{F/A}$). Note very carefully that the specialization of the map (2.91), namely,

$$\mathbf{E}^-[A, F] \otimes \mathbf{E}^-[F, FF'] \to \mathbf{E}^-[A, F'] \otimes \mathbf{E}^-[F', F'F]$$

is not the identity, but rather scalar multiplication by $(-1)^{\mathrm{dist}(F,F')}$. This is in agreement with the formulation of signed bimonoids.

2.14. Set-species and set-bimonoids

Let Set denote the category of sets. By replacing the codomain Vec by Set in the definition of species, one obtains the notion of a set-species. The theory of set-species proceeds in parallel to that of species, and the two are linked by the linearization functor. Certain aspects such as signed commutativity or duality are specific to the linear case.

2.14.1. Set-species. An $\mathcal{A}$-*set-species* is a functor

$$\mathrm{p} : \mathcal{A}\text{-}\mathtt{Hyp} \to \mathsf{Set}.$$

A map of $\mathcal{A}$-set-species $\mathrm{p} \to \mathrm{q}$ is a natural transformation. We denote the category of $\mathcal{A}$-set-species by

$$\mathcal{A}\text{-}\mathtt{SetSp} = [\mathcal{A}\text{-}\mathtt{Hyp}, \mathsf{Set}].$$

Explicitly: An $\mathcal{A}$-set-species consists of a family of sets $\mathrm{p}[F]$, one for each face F, together with bijections $\beta_{G,F}$ which satisfy (2.1). Similarly, a map of set-species $f : \mathrm{p} \to \mathrm{q}$ is a family of maps $f_F : \mathrm{p}[F] \to \mathrm{q}[F]$ which satisfy (2.3).

Equivalently: An $\mathcal{A}$-set-species is a family of sets $\mathrm{p}[\mathrm{X}]$, one for each flat X. A map of $\mathcal{A}$-set-species $f : \mathrm{p} \to \mathrm{q}$ is a family of maps $f_{\mathrm{X}} : \mathrm{p}[\mathrm{X}] \to \mathrm{q}[\mathrm{X}]$, one for each flat X.

2.14.2. Set-bimonoids. Monoids, comonoids, bimonoids in $\mathcal{A}$-set-species (as well as their commutative counterparts) are defined as for $\mathcal{A}$-species, with the understanding that all structure maps involved are now maps between sets (as opposed to linear maps). We refer to these as set-monoids, set-comonoids, set-bimonoids, and so on. For a set-bimonoid $(\mathrm{h}, \mu, \Delta)$, the *set-bimonoid axiom* says that for any faces $A \leq F$ and $A \leq G$, the diagram

$$
\begin{tikzcd}
\mathrm{h}[F] \arrow[r, "\mu_A^F"] \arrow[d, "\Delta_F^{FG}"'] & \mathrm{h}[A] \arrow[r, "\Delta_A^G"] & \mathrm{h}[G] \\
\mathrm{h}[FG] \arrow[rr, "\beta_{GF,FG}"'] & & \mathrm{h}[GF] \arrow[u, "\mu_G^{GF}"']
\end{tikzcd}
\tag{2.92}
$$

commutes.

The categories of double set-monoids and commutative set-monoids are equivalent, and dually, the categories of double set-comonoids and cocommutative set-comonoids are equivalent. This is the analogue of Proposition 2.27.

Note that there is no notion of a q-bimonoid or a signed commutative monoid in $\mathcal{A}$-set-species.

2.14.3. Exponential set-species. The *exponential set-species* E is defined by setting $\mathrm{E}[A] := \{*\}$ for any face A. In other words, each A-component is a singleton. For faces A and B of the same support, there is a unique map

$$\beta_{B,A} : \mathrm{E}[A] \to \mathrm{E}[B].$$

The exponential set-species E is the terminal object in the category of set-species.

Define maps

$$\mu_A^F : \mathrm{E}[F] \to \mathrm{E}[A] \quad \text{and} \quad \Delta_A^F : \mathrm{E}[A] \to \mathrm{E}[F], \tag{2.93}$$

to be identities for all $F \geq A$. They turn E into a set-bimonoid.

2.14.4. Linearization functor. Fix a field $\Bbbk$. Consider the functor

$$\Bbbk(-) : \mathsf{Set} \longrightarrow \mathsf{Vec},$$

which sends a set to the vector space with basis the given set. Composing a set-species p with this functor yields a species, which we denote by $\Bbbk\mathrm{p}$. Thus, we have

$$\Bbbk(-) : \mathcal{A}\text{-}\mathtt{SetSp} \longrightarrow \mathcal{A}\text{-}\mathtt{Sp}. \tag{2.94}$$

We call this the *linearization functor*.

Let a be a monoid in $\mathcal{A}$-set-species. Then $\Bbbk\mathrm{a}$ is a monoid in $\mathcal{A}$-species whose product components are obtained by linearizing the product components of a. Similarly, linearization preserves comonoids and bimonoids.

2.15. Bimonoids for a rank-one arrangement

In general for any arrangement, a ((co, bi)commutative) (co, bi)monoid consists of a bunch of vector spaces, and a bunch of linear maps between them satisfying some axioms. For instance, for the rank-zero arrangement, all notions coincide and amount to specifying a vector space. Let us now discuss the situation in rank one. Idempotent operators (Appendix A.5.1) play an important role in the discussion.

In this section, $\mathcal{A}$ denotes a rank-one arrangement.

2.15.1. Bicommutative bimonoids. A rank-one arrangement $\mathcal{A}$ has two flats, namely, $\perp$ and $\top$. In view of Proposition 2.5, an $\mathcal{A}$-species consists of two vector spaces V and U associated to $\perp$ and $\top$, respectively. We now build on this description.

Lemma 2.87. *A commutative $\mathcal{A}$-monoid is the same as a triple (U, μ, V), where U and V are arbitrary vector spaces, and $\mu : U \to V$ is any linear map.*

Cocommutative $\mathcal{A}$-comonoids have the same description.

PROOF. We employ Proposition 2.20. Given a commutative $\mathcal{A}$-monoid a, the corresponding linear algebra data is obtained by setting $V := \mathsf{a}[\perp]$ and $U := \mathsf{a}[\top]$, and $\mu := \mu_{\perp}^{\top}$. □

Since μ is arbitrary, we deduce that the structure maps of a comonoid need not be surjective, and of a monoid need not be injective. This fact was mentioned in Exercise 2.10.

Lemma 2.88. *A bicommutative $\mathcal{A}$-bimonoid is the same as a pair of vector spaces U and V, equipped with linear maps $\mu : U \to V$ and $\Delta : V \to U$ such that the composite $\Delta\mu$ is the identity.*

Equivalently, a bicommutative $\mathcal{A}$-bimonoid is a vector space V equipped with an idempotent operator e.

PROOF. We employ Lemma 2.23. Given a bicommutative $\mathcal{A}$-bimonoid h, put $V := \mathsf{h}[\perp]$, $U := \mathsf{h}[\top]$, $\mu := \mu_{\perp}^{\top}$ and $\Delta := \Delta_{\perp}^{\top}$. The first diagram in (2.27) is a tautology, while the second diagram yields the condition $\Delta\mu = \mathrm{id}$. This proves the first statement. The second statement follows from the first by Lemma A.1, with $e := \mu\Delta$ being the idempotent operator on V. □

The condition $\Delta\mu = \mathrm{id}$ implies that μ is injective and Δ is surjective. This is in agreement with the general observation in Corollary 2.9.

2.15.2. *q*-bimonoids. We now describe arbitrary $\mathcal{A}$-q-bimonoids in terms of idempotent operators extending the observations in Lemma 2.88.

Lemma 2.89. *An $\mathcal{A}$-q-bimonoid is the same as a vector space V equipped with idempotent operators e and f, and a linear isomorphism $\beta : e \cdot V \to f \cdot V$ such that the induced map $f : e \cdot V \to f \cdot V$ is $q\beta$, while the induced map $e : f \cdot V \to e \cdot V$ is $q\beta^{-1}$. In this situation, $efe = q^2 e$ and $fef = q^2 f$.*

PROOF. Let h be an $\mathcal{A}$-q-bimonoid. Put $V := \mathsf{h}[O]$. By Exercise 2.31, the linear maps

$$e := \mu_O^C \Delta_O^C \quad \text{and} \quad f := \mu_O^{\overline{C}} \Delta_O^{\overline{C}}$$

are idempotent operators on V. By Lemma A.1, the images of e and f, denoted $e \cdot V$ and $f \cdot V$, are isomorphic to $\mathsf{h}[C]$ and $\mathsf{h}[\overline{C}]$, respectively. Define $\beta : e \cdot V \to f \cdot V$ to be the map corresponding to $\beta_{\overline{C},C}$ via these isomorphisms. Then the induced map $f : e \cdot V \to f \cdot V$ is $q\beta$. This follows from (2.35) (using $\operatorname{dist}(C, \overline{C}) = 1$). Similarly, the induced map $e : f \cdot V \to e \cdot V$ is $q\beta^{-1}$.

Conversely, starting with the linear algebra data, one can define

$$\mathsf{h}[O] := V, \quad \mathsf{h}[C] := e \cdot V, \quad \mathsf{h}[\overline{C}] := f \cdot V, \quad \beta_{\overline{C},C} := \beta,$$

and μ and Δ to be the obvious inclusions and surjections. This is an $\mathcal{A}$-q-bimonoid. $\square$

Lemma 2.90. *An $\mathcal{A}$-bimonoid is the same as a vector space V equipped with idempotent operators e and f satisfying $efe = e$ and $fef = f$. Further, it is*

- *commutative iff $fe = e$ and $ef = f$,*
- *cocommutative iff $fe = f$ and $ef = e$,*
- *bicommutative iff $e = f$.*

Note that in the commutative case, $\operatorname{im}(e) = \operatorname{im}(f)$, while in the cocommutative case, $\ker(e) = \ker(f)$. Also observe that the claim about bicommutativity is in agreement with the result obtained in Lemma 2.88.

PROOF. Set $q = 1$ in Lemma 2.89. We see that β is now determined by e and f, and can be removed from the description by saying that the induced linear maps $f : e \cdot V \to f \cdot V$ and $e : f \cdot V \to e \cdot V$ are inverse isomorphisms. This translates to $efe = e$ and $fef = f$. For commutativity, $e \cdot V = f \cdot V$ with the induced linear maps being identity. This translates to $fe = e$ and $ef = f$. Cocommutativity is similar. Combining the two yields the claim about bicommutativity. $\square$

2.16. Joyal species and Joyal bimonoids

We briefly review the category of Joyal species and monoids, comonoids, bimonoids therein. We then explain how Joyal species give rise to $\mathcal{B}^J$-species, where $\mathcal{B}^J$ is the braid arrangement on a finite set J (Section 1.13). Importantly, this construction carries Joyal (co, bi)monoids to $\mathcal{A}$-(co, bi)monoids.

More details on Joyal species can be found in [**18**, Part II], [**19**].

2.16.1. Joyal species. Let $\mathtt{set}^\times$ denote the category whose objects are finite sets and whose morphisms are bijections. A *Joyal species* is a functor

$$\mathsf{p} : \mathtt{set}^\times \to \mathsf{Vec}.$$

A map of Joyal species $\mathsf{p} \to \mathsf{q}$ is a natural transformation. This defines the category of Joyal species which we denote by $\mathtt{JSp}$. It is a functor category, and we also write

$$\mathtt{JSp} = [\mathtt{set}^\times, \mathsf{Vec}].$$

The value of a Joyal species p on a finite set J will be denoted $\mathsf{p}[J]$. We call it the J-component of p.

2.16.2. Positive and connected Joyal species. A Joyal species p is *connected* if $\mathsf{p}[\emptyset] = \Bbbk$, and *positive* if $\mathsf{p}[\emptyset] = 0$. For a morphism $f : \mathsf{p} \to \mathsf{q}$ between connected Joyal species, we also require that $f_\emptyset = \mathrm{id}$.

A positive Joyal species p gives rise to a connected Joyal species by setting $\mathsf{p}[\emptyset] := \Bbbk$. Conversely, a connected Joyal species yields a positive Joyal species by forgetting $\mathsf{p}[\emptyset]$. Thus, they are equivalent notions.

2.16.3. Joyal (co)monoids. The category of Joyal species JSp is a monoidal category wrt the Cauchy product. The *Cauchy product* of p and q is denoted $\mathsf{p} \cdot \mathsf{q}$ and its J-component is defined by

$$(\mathsf{p} \cdot \mathsf{q})[J] := \bigoplus_{J = S \sqcup T} \mathsf{p}[S] \otimes \mathsf{q}[T]. \tag{2.95}$$

We call $J = S \sqcup T$ a decomposition of J, this means that S and T are disjoint subsets whose union is J. The subsets S and T may be empty.

A monoid in $(\mathsf{JSp}, \cdot)$ is called a *Joyal monoid*. Thus, a Joyal monoid a entails linear maps

$$\mu_{S,T} : \mathsf{a}[S] \otimes \mathsf{a}[T] \to \mathsf{a}[J], \tag{2.96}$$

one for each decomposition $J = S \sqcup T$, subject to the naturality, associativity, unitality axioms.

Dually, a comonoid in $(\mathsf{JSp}, \cdot)$ is called a *Joyal comonoid*. Thus, a Joyal comonoid c entails linear maps

$$\Delta_{S,T} : \mathsf{c}[J] \to \mathsf{c}[S] \otimes \mathsf{c}[T], \tag{2.97}$$

one for each decomposition $J = S \sqcup T$, subject to the naturality, coassociativity, counitality axioms.

2.16.4. Joyal bimonoids. The monoidal category of Joyal species $(\mathsf{JSp}, \cdot)$ carries a family of braidings β_q, one for each scalar q. The map $\beta_q : \mathsf{p} \cdot \mathsf{q} \to \mathsf{q} \cdot \mathsf{p}$, evaluated on the J-component, is the direct sum of the linear maps

$$(\beta_q)_{S,T} : \mathsf{p}[S] \otimes \mathsf{q}[T] \to \mathsf{q}[T] \otimes \mathsf{p}[S], \qquad x \otimes y \mapsto q^{|S||T|} y \otimes x \tag{2.98}$$

over all decompositions $J = S \sqcup T$. The notation $|S|$ stands for the cardinality of the set S. When $q = 1$, we write β instead of β_1.

A bimonoid in $(\mathsf{JSp}, \cdot, \beta_q)$ is called a *Joyal q-bimonoid*. For $q = \pm 1$ and $q = 0$, we use the terms *Joyal bimonoid*, *signed Joyal bimonoid* and *Joyal 0-bimonoid*, respectively. Explicitly, the *q-bimonoid axiom* says that for any two decompositions $J = S_1 \sqcup S_2 = T_1 \sqcup T_2$, the diagram

$$\begin{tikzcd}
\mathsf{h}[S_1] \otimes \mathsf{h}[S_2] \arrow[r, "\mu_{S_1,S_2}"] \arrow[d, "\Delta_{A,B} \otimes \Delta_{C,D}"'] & \mathsf{h}[J] \arrow[r, "\Delta_{T_1,T_2}"] & \mathsf{h}[T_1] \otimes \mathsf{h}[T_2] \\
\mathsf{h}[A] \otimes \mathsf{h}[B] \otimes \mathsf{h}[C] \otimes \mathsf{h}[D] \arrow[rr, "\mathrm{id} \otimes (\beta_q)_{B,C} \otimes \mathrm{id}"'] & & \mathsf{h}[A] \otimes \mathsf{h}[C] \otimes \mathsf{h}[B] \otimes \mathsf{h}[D] \arrow[u, "\mu_{A,C} \otimes \mu_{B,D}"']
\end{tikzcd} \tag{2.99}$$

commutes, where $A := S_1 \cap T_1,\ B := S_1 \cap T_2,\ C := S_2 \cap T_1,\ D := S_2 \cap T_2$.

2.16.5. (Co)commutative Joyal (co)monoids. The braiding β_q is a symmetry when $q = 1$ or -1. A commutative monoid in $(\mathsf{JSp}, \cdot, \beta)$ is called a *commutative Joyal monoid*, while a commutative monoid in $(\mathsf{JSp}, \cdot, \beta_{-1})$ is called a *signed commutative Joyal monoid*. Explicitly, the requirement is that for any decomposition $J = S \sqcup T$, the diagram

$$
\begin{tikzcd}
\mathsf{a}[S] \otimes \mathsf{a}[T] \arrow[rr] \arrow[dr, "\mu_{S,T}"'] & & \mathsf{a}[T] \otimes \mathsf{a}[S] \arrow[dl, "\mu_{T,S}"] \\
& \mathsf{a}[J] &
\end{tikzcd}
\tag{2.100}
$$

commutes, where the horizontal arrow is $\beta_{S,T}$ for the *commutativity axiom* and $(\beta_{-1})_{S,T}$ for the *signed commutativity axiom*. Dually, we have (signed) cocommutative Joyal comonoids.

2.16.6. From Joyal species to $\mathcal{B}^J$-species. Fix a finite set J and consider the braid arrangement $\mathcal{B}^J$. Recall from Section 1.13 that faces of $\mathcal{B}^J$ are identified with compositions of J, while flats with partitions of J.

Out of a Joyal species p we build a $\mathcal{B}^J$-species p^J, as follows. For a composition $F = (S_1, \dots, S_k)$ of J, we let the F-component of p^J be

$$\mathsf{p}^J[F] := \mathsf{p}[S_1] \otimes \cdots \otimes \mathsf{p}[S_k]. \tag{2.101}$$

The maps

$$\beta_{G,F} : \mathsf{p}^J[F] \to \mathsf{p}^J[G]$$

are defined by permuting tensor factors. (Since F and G have the same support, they consist of the same blocks, possibly listed in a different order.) Equivalently, one may work with set partitions and set

$$\mathsf{p}^J[\mathrm{X}] := \bigotimes_{B \in \mathrm{X}} \mathsf{p}[B],$$

where X is a partition of J, and the rhs is the unordered tensor product over blocks B of X.

For a morphism of Joyal species $f : \mathsf{p} \to \mathsf{q}$, we let $f^J : \mathsf{p}^J \to \mathsf{q}^J$ be the morphism of $\mathcal{B}^J$-species whose F-component is the tensor product of the maps $f_{S_i} : \mathsf{p}[S_i] \to \mathsf{q}[S_i]$. The assignments $\mathsf{p} \mapsto \mathsf{p}^J$, $f \mapsto f^J$ then define a functor

$$\mathsf{JSp} \to \mathcal{B}^J\text{-}\mathsf{Sp}. \tag{2.102}$$

Remark 2.91. Note that $\mathsf{p}[\emptyset]$ played no role in this construction. So the functor (2.102) factors through the category of positive Joyal species: we first forget the value on the empty set, then apply the preceding construction to obtain a $\mathcal{B}^J$-species. Equivalently, one may factor the functor through the category of connected Joyal species.

2.16.7. From Joyal (co, bi)monoids to $\mathcal{B}^J$-(co, bi)monoids. Now suppose a is a Joyal monoid. Then, for any subset S of J and composition F of S, iterating (2.96) yields a map $\mathsf{a}^J[F] \to \mathsf{a}[S]$. This is well-defined by associativity. For $A \leq F$, both compositions of J, define

$$\mu^F_A : \mathsf{a}^J[F] \to \mathsf{a}^J[A]$$

by tensoring the maps $\mathsf{a}^J[F^i] \to \mathsf{a}[S_i]$, where $A = (S_1, \dots, S_k)$, and F^i is the part of F which refines S_i. This turns a^J into an $\mathcal{B}^J$-monoid. In a similar manner, a Joyal comonoid yields a $\mathcal{B}^J$-comonoid. This defines functors from the category of Joyal (co)monoids to the category of $\mathcal{B}^J$-(co)monoids.

We mention that these functors preserve (signed) (co)commutativity. Further, they also induce a functor from the category of Joyal bimonoids to the category of $\mathcal{B}^J$-bimonoids.

Remark 2.92. Many examples of Joyal bimonoids are given in [**18**], [**19**]. These include the exponential Joyal bimonoid E, the bimonoid of linear orders $\mathsf{\Gamma}$, the Joyal bimonoid of set partitions $\mathsf{\Pi}$, the Joyal bimonoid of set compositions $\mathsf{\Sigma}$. Applying the above functor to them yields $\mathcal{B}^J$-bimonoids. We mention that these coincide with the bimonoids E, $\mathsf{\Gamma}$, $\mathsf{\Pi}$, $\mathsf{\Sigma}$ discussed in Chapter 7 when specialized to braid arrangements.

Remark 2.93. The reader interested in further understanding the connection between $\mathcal{B}^J$-bimonoids and Joyal bimonoids can consult [**19**, Section 10]. The bimonoid axiom (2.12) for Joyal bimonoids is given in [**19**, Diagram (127)], and a formulation of Joyal bimonoids similar to that of $\mathcal{B}^J$-bimonoids is stated in [**19**, Proposition 51]. For 0-bimonoids, compare the first diagram in (2.39) with [**18**, (9.6) and (9.7)], and the second diagram in (2.39) with [**18**, (9.5)].

These ideas will be developed in more detail in future work, also see Table I in the Preface in this regard.

Notes

As mentioned in the Preface, the theory of species and bimonoids for hyperplane arrangements is presented here for the first time. It is motivated by the classical theory of bialgebras, and that of bimonoids in Joyal species. References to this classical literature related to ideas developed in this chapter are given below. A similar remark applies to the Notes to subsequent chapters. In contrast to the classical case, all results in this book are valid over a field of arbitrary characteristic unless stated otherwise.

Bialgebras.
Early period. The origins of Hopf theory can be traced back to the paper of Hopf [**458**]. Algebras were standard objects by then, but Hopf seems to be the first one to consider the coproduct and make systematic use of it, see [**458**, Formula (4.6)]. Hopf's work was carried forward by Samelson [**799**], Leray [**595**], Borel [**135**]. More information about the content of their papers and related developments is given in the Notes to Chapters 5, 12, 13.

Cartan and Eilenberg in their book [**189**, Section XI.8] consider additional structure on an augmented algebra consisting of the coproduct and the antipode, thus anticipating the notion of a Hopf algebra. Earlier in Section XI.4, they do consider an abstract coproduct and explicitly define cocommutativity and coassociativity. The first formal definition of a coalgebra under that name appears in work of Cartier [**192**, Section 4.2] and Milnor and Moore [**695**, Definition 2.1], [**696**, Definition 1.2], [**706**, page 2]. A graded bialgebra is defined by Milnor and Moore under the name 'Hopf algebra' [**695**, Definition 4.1], [**696**, Definition 2.1], [**706**, page 4]. In Definition 4.16 of the first reference, they also consider a weaker notion (termed 'quasi Hopf algebra') where (co)associativity is not assumed. Hopf algebras also appear in Milnor's work on the Steenrod algebra [**694**, Theorem 1]. (For more in this direction, see [**849**, Chapters II and VI], [**709**, Chapter 6], [**821**, Section 14.2], [**419**, Sections 1.2.3 and 1.5.2].) An account of Milnor and Moore's work is given by Stasheff [**848**, pages 11 to 17].

A cocommutative bialgebra is defined by Cartier under the name 'hyperalgebra' [**192**, Section 2.1], [**194**, Section 2]. The same terminology is employed by Gabriel [**333**, Chapter I, Section 2, page 7]. This term was first used by Dieudonné in a sequence of papers on formal groups starting with [**248**]. In the fifth paper, Dieudonné arrives at an abstract definition of a hyperalgebra [**249**, page 208] which is a special case of the one given by Cartier. For more information on how Hopf theory fits into the theory of formal groups, see the books by Fröhlich [**330**, Section II.1], Dieudonné [**250**, Section I.2], Hazewinkel [**427**, Section 37]. The term 'bialgebra' seems to have been first used by Manin who says that it was suggested to him by Lazard in a private conversation [**640**, foootnote on page 14].

A bialgebra is called a 'hyperalgebra' by Halpern [**412**], [**413**] (with the term again borrowed from Dieudonné). In the papers [**412**], [**415**], he denotes a bialgebra by (H, ∇, Δ). (Originally, the choice of Δ for the coproduct came from the diagonal map $x \mapsto (x, x)$.) Halpern also uses the term 'Hopf algebra' for a connected graded commutative bialgebra and 'Pontryagin algebra' for a connected graded cocommutative bialgebra. Bourbaki uses the terms 'cogebra' and 'bigebra' [**149**, Section III.11]; this terminology is followed by Dieudonné [**250**] and Serre [**822**]. The term 'bicommutative' (for a bialgebra which is both commutative and cocommutative) is employed by Kane [**508**, page 11].

The classical analogue of (2.12) is the bialgebra axiom which is the compatibility between the product and coproduct. It is explicitly given by Halpern [**412**, Axiom (1.1)], [**413**, Axiom (1.1)], Milnor and Moore [**695**, Definition 4.1, axiom (3)], [**696**, Definition 2.1, axiom (3)], Zisman [**932**, page 5]. Milnor and Moore also mention the equivalent statement that the coproduct is a morphism of algebras. This condition is also given by Cartan and Eilenberg [**189**, Section XI.8, axiom (i)] and Cartier [**192**, Section 2.1, first line of axiom (HA_2)]. Other early references are by Araki [**39**, Section 1.1, axiom (iv)], Browder [**160**, bottom of page 155], Gugenheim [**390**, Definition 4.1, axiom (ii)]. Our analogue of this fact is given later in Lemma 6.65.

The fact that the algebra of polynomial functions on a linear algebraic group carries the structure of a commutative Hopf algebra appears implicitly in work of Cartier [**193**]. These ideas were carried forward by Hochschild and Mostow [**442**, page 27], [**443**, Section 3], [**444**, Chapters 2 and 3], [**445**, Chapter I], [**446**], [**447**]. The connection to affine group schemes was developed by Demazure, Gabriel, Grothendieck [**335**, Exposé VIIB, Section 2.2], [**242**, Proposition on page 146], [**243**, Proposition on page 181]. An early reference on the abelian case is [**333**, Chapter I, Sections 1 and 2]. Other references are by Dieudonné [**250**, Section I.1.6], Takeuchi [**872**], Yanagihara [**922**], Waterhouse [**903**, Chapter 1], Abe [**1**, Chapter 4], Cartier [**200**, Part I], Manin [**644**, Section 1.15]. For more recent references, see the books by Ferrer Santos and Rittatore [**304**, Chapter 5, Theorem 2.19], Jantzen [**484**, Part I, Sections 2.3 and 8.4], Milne [**693**, Corollary 3.7], Szamuely [**869**, Section 6.1], Connes and Marcolli [**221**, Chapter 1, Section 6.1], [**650**, Section 5.2], Elduque and Kochetov [**289**, Appendix A], Lorenz [**612**, Section 11.1].

A variant of the notion of Hopf algebra arose early on in work of Kac on von Neumann algebras [**504**], [**505**]. This work somewhat anticipated the emergence of quantum groups. For a recent reference, see the book by Neshveyev and Tuset [715].

Additional historical information on Hopf algebras can be found in the surveys by Cartier [202], Karaali [511], Andruskiewitsch and Ferrer Santos [35].

Later period. The first comprehensive work on Hopf algebras was carried out by Sweedler in his influential book [867]. Most of the terminology employed in his book is now standard. The lecture notes by Kaplansky [510] contain an extensive bibliography of the subject until 1975. On page 22, Kaplansky makes the following interesting remark.

> Examples of bialgebras are not easily constructed, and perhaps just about any example that a person constructs is worth exhibiting.

The survey paper by Artamonov [49] covers the period from 1970 to 1994.

Some books or monographs which study Hopf algebras in detail are by Abe [1], Dăscălescu, Năstăsescu, Raianu [228], Hazewinkel, Gubareni, Kirichenko [428], Montgomery [703], Radford [771], Underwood [888], Skowroński and Yamagata [832, Part VI], May and Ponto [663, Part 5], Lorenz [612, Part IV]. Shorter treatments can be found in our monograph [18, Chapter 2] and in the books by Bonfiglioli and Fulci [130, Section 3.6], Gracia-Bondía, Várilly, Figueroa [364, Section 1.B], [896, Section 1.2], Mac Lane [622, Section VI.9], Schneider [814], tom Dieck [885, Section 19.7], Khalkhali [526, Section 1.7]. See also the articles by Artamonov [50, Sections C.42 and C.43], Bergman [107], Hazewinkel [422]. A detailed treatment with emphasis on coalgebras and with plenty of historical information is given by Michaelis [688]. Coalgebras and bialgebras also find a mention in Jacobson's algebra textbook [481, Section 3.9].

For an analogue of Propositions 2.47 and 2.48, see [867, Proposition 3.2.5 and the exercise following it]. For an analogue of Propositions 2.53 and 2.58, see [228, Example 4.1.5, item (3)], [517, Proposition III.2.3], [535, page 12], [688, Remarks 2.37, items (b) and (c)], [703, Example 1.3.11], [771, Exercise 5.1.2]. For analogues of abelianization and coabelianization in Section 2.7, see [387, Sections 1.2 and 1.3].

Quantum groups. Hopf algebras rose in prominence with the advent of quantum groups in works of Drinfeld [262] and Jimbo [489], [490]. Some books on quantum groups with useful introductions to Hopf algebras are by Brown and Goodearl [161, Appendix I.9], Chari and Pressley [205, Chapter 4], Fuchs [331, Chapter 4], Joseph [497, Chapter 1], Kassel [517, Chapter III], Guichardet [391, Chapter II], Klimyk and Schmüdgen [535, Chapter 1], Majid [632, Chapter 1], Manin [642, Chapter 2], [645, Chapter 3], Shnider and Sternberg [825, Chapter 3]. Also see the expositions by De Concini and Procesi [235, Chapter 1] and Ram [773, Chapter I]. Later references are by Etingof and Schiffmann [296, Lecture 8], Deng, Du, Parshall, Wang [244, Chapter 5], Street [859, Chapters 6 to 9], Bulacu, Caenepeel, Panaite, Van Oystaeyen [175, Chapter 2].

Bialgebras in combinatorics and probability theory. The relevance of Hopf algebras to combinatorics was brought forth in work of Joni and Rota [496]. Additional early work in this direction was done by Nichols and Sweedler [721], Geissinger [344], Zelevinsky [929]. Some books or monographs in which Hopf algebras attached to combinatorial objects feature prominently are [17], [18], [377], [428], [673]. Connections to quantum groups are mentioned in [18, Chapter 20], [428, Chapter 8]. Additional references are given in the Notes to Chapter 7.

For bialgebras in probability theory, see for instance the books by Schürmann [**819**], Meyer [**683**, Chapter VII], Majid [**632**, Chapter 5], Franz and Schott [**315**, Chapter 3], and references therein.

(Signed) graded bialgebras. Graded bialgebras are built out of graded vector spaces. They come in two flavors, unsigned and signed. These are classical analogues of the bimonoids and signed bimonoids, respectively, in Sections 2.2.3 and 2.4.2. The classical analogue of signed commutativity (2.44) is the axiom $xy = (-1)^{|x||y|}yx$. Recall that this is the axiom satisfied by the cohomology cup product. Thus, interestingly, it was the signed case which appeared in Hopf's original work. This is also the case treated by Milnor and Moore, and which appears in textbooks on algebraic topology. A discussion of signed commutativity of the cup product from a purely categorical point of view is given in [**18**, Section 5.5.4]. Early references to signed commutative algebras are by Chevalley [**213**, Section V.5] and Moore [**706**, page 3].

More generally, one can consider a q-analogue of graded bialgebras, with $q = 1$ and $q = -1$ recovering the unsigned and signed cases, respectively. These are termed q-bialgebras in [**18**, Section 2.3], also see [**703**, Example 10.5.13]. They are classical analogues of the q-bimonoids in Section 2.4.1. In particular, 0-bialgebras are analogues of the 0-bimonoids in Section 2.4.3. For more on 0-bialgebras, see [**18**, Section 2.3.6]. A related usage is the term 'infinitesimal bialgebra' by Loday and Ronco [**610**, Proposition-Notation 2.3], [**608**, Section 4.2.1].

A notion related to signed graded bialgebras is that of superbialgebras which are built out of super vector spaces. Superalgebras are considered by Chevalley under the name 'semi-graded algebras' [**211**, page 5]. Early references to superbialgebras are by Kostant [**541**, page 221], Scheunert [**805**, Chapter I, Section 2.4], Boseck [**142**, page 58], [**143**, Section 1], Gould, Zhang, Bracken [**361**, Section 2], Majid [**632**, Definition 10.1.1]. For recent references, see [**657**, Section 2], [**659**, Section 3], [**34**, Section 1], [**26**, Section 1].

Some general references on signed aspects of graded vector spaces are [**75**, Chapter I], [**643**, Chapter 3], [**239**, Chapter 1], [**392**, Section 2.2], [**784**, Chapter 2], [**875**, Section 3.7], [**186**, Chapter 1]. Related notions of (signed) graded Lie algebras and Lie superalgebras are mentioned in the Notes to Chapter 16.

Color bialgebras. There is a more general notion of a color bialgebra where the space is graded by a commutative monoid. The trivial monoid yields bialgebras, $\mathbb{Z}_2$ yields superbialgebras, $\mathbb{N}$ yields graded bialgebras. Early references are by Mikhalev [**690**, Section 2], Bahturin, Mikhalev, Petrogradsky, Zaicev [**64**, Chapter 3, Section 2.9], Montgomery [**703**, Definition 10.5.11]. Recent references are by Kharchenko [**529**, Section 7.2, page 248], Andruskiewitsch, Angiono, Bagio [**33**]. A color bialgebra can be viewed as an analogue of a υ-bimonoid associated to a distance function υ in Section 2.4.4. Color bialgebras fit into the general framework of bimonoids in braided monoidal categories discussed below.

Connected bialgebras. There is a notion of connected bialgebras. In Sweedler's terminology, these are called (pointed) irreducible bialgebras [**867**, page 157]. In particular, this notion applies to graded bialgebras. However, when we use the term connected graded bialgebra, we mean something stronger, namely, a graded bialgebra whose degree zero component is the base field. This notion is equivalent to a positively graded bialgebra, see for instance [**18**, Section 2.3.5].

Bimonoids in braided monoidal categories. General references for monoidal categories are given in the Notes to Appendix B. Monoids and comonoids can be defined in any monoidal category. Early references are by Bénabou [**90**, Section 1], [**92**, Section 5.4.1], Mac Lane [**620**, Section 6, page 45], [**621**, Section 17], Jonah [**494**, Section 1], [**495**, Section 1]. The special case of categories with products and coproducts is considered by Eckmann and Hilton [**276**, Section 4]. Bimonoids can be defined in any braided monoidal category, and in particular, in any symmetric monoidal category. Early references for the symmetric case are by Mac Lane [**621**, Section 17], Hofmann [**452**, Part I], Saavedra Rivano [**793**, Section I.6], Larson [**566**, page 264], Pareigis [**736**, Section 2], Benson [**98**, Section 2.4], while the braided case is considered by Majid in a series of papers, see [**630**, Section 2] and references therein. Related historical information is given in Montgomery's book [**703**, Sections 10.4 and 10.5]. For later references, see [**873**], [**633**, Chapter 14], [**538**, Section 3.6.7], [**859**, Chapter 15], [**18**, Section 1.2], [**453**, Definition A3.64], [**758**], [**324**, Section 7.1], [**887**, Chapter 6], [**175**, Chapter 2]. For an analogue of Propositions 2.53 and 2.58 in this generality, see [**18**, Propositions 1.20 and 1.21], [**175**, Propositions 2.4 and 2.48].

A basic example of a braided monoidal category is the category of graded vector spaces equipped with the tensor product [**18**, Section 2.3]. It carries a family of braidings β_q, one for each scalar q. This yields the notion of q-bialgebras, with $q = 1$ and $q = -1$ recovering unsigned and signed graded bialgebras, respectively.

A closely related example is the category of Joyal species equipped with the Cauchy product (2.95). This product was studied by Joyal [**498**, page 9], [**500**, page 146]. For later references, see [**18**, Formula (8.6)], [**102**, Section 1.3, Formula (24)], [**675**, Formula (3.5)]. It also carries a family of braidings β_q, one for each scalar q, see (2.98). This yields the notion of q-bimonoids in Joyal species (Section 2.16). More historical information on Joyal species is given below.

A bimonoid can also be defined 'locally' using a Yang–Baxter operator on an object in a monoidal category. For the basic example of vector spaces, this idea is present in work of Gurevich [**399**, Definition in Section 6]. The term 'braided bialgebra' is used in this context, see the paper by Takeuchi [**874**, Definition 5.1]. (The algebra part of this structure is called a 'braided algebra', this notion is mentioned by Baez [**56**, page 886].) Some later references are [**18**, Section 19.9], [**529**, Chapter 6], [**46**, Definition 3.1].

Bimonoids in Joyal species.

Joyal species. Joyal's foundational work [**498**] introduced espèces de structures. We call them Joyal species. We quote from the introduction to [**498**]:

> La caractéristique principale de la théorie présentée ici est son degré de généralité et sa simplicité... L'accent est mis sur le transport des structures plutôt que sur leurs propriétés. Ce point de vue n'est pas sans évoquer celui d'Ehresmann et contraste avec celui de Bourbaki.

Bourbaki's point of view is exposed in [**145**], [**147**, Section IV.1]. Some later references are by Sonner [**840**] and Blanchard [**118**]. Ehresmann's ideas are exposed in his book [**282**, Chapter II], which builds on various earlier works of his including [**278**], [**279**], [**281**, Section I.2]. This line of thought is continued in Benabou's thesis [**93**] and in the book by Hasse and Michler [**420**, Chapter X]. Corry discusses Bourbaki's theory of structures in [**222**, Chapter 7] and also comments on Ehresmann's work on [**222**, pages 376–378]. Bourbaki's approach is criticized by Lambek in his review [**559**]. A definition closer to Joyal's is present in the book on category theory by Bucur and Deleanu [**173**, Section 4.4]. See also the notion of 'structured set' in the book by Arbib and Manes [**40**, Chapter 6] and of 'transportable construct' in Adámek's book [**3**, Definition on page 18].

Kelly [**524**] gave an early detailed discussion of the basic operations on the category of Joyal species. (This paper, though published in 2005, dates back to 1971.) A nice exposition on Joyal set-species is given in the book by Bergeron, Labelle, Leroux [**102**], see also the book by Méndez [**675**]. A generalization of Joyal species called 'stuff type' is considered by Baez and Dolan [**58**, Section 4]. For additional related references, see [**59**, Section 6, page 65].

Bimonoids in Joyal species. The study of algebraic structures (monoids, comonoids, bimonoids, Hopf monoids, Lie monoids) in Joyal's category of species was a main goal of our monographs [**18**, Parts II and III], [**19**]. Joyal initiated this study in [**498**, Section 7.1], [**500**, Section 4.2]. In the equivalent framework of symmetric group representations, monoids in Joyal species had appeared earlier in work of Barratt [**74**, Definition 3], under the name 'twisted algebras'. Twisted algebras and twisted Hopf algebras are the subject of work by Stover [**854**, Sections 3 and 4] and of Patras with Livernet, Reutenauer, Schocker [**604**], [**746**], [**747**], [**748**]. (More information on Stover's work is given in the Notes to Chapter 17.) They also enter in work of Fresse [**320**, Section 1.2.10] and Ginzburg and Schedler [**355**, Section 2.3]. Twisted commutative algebras (which are equivalent to commutative Joyal monoids) appear prominently in recent work of Sam and Snowden [**798**, Section 8.1.5], [**796**, Section 7.2], [**797**], see also the book by Bremner and Dotsenko [**156**, Definition 4.6.3.1].

In our monographs, bimonoids in Joyal species are treated in [**18**, Chapter 8], and q-bimonoids in [**18**, Chapter 9]. Many examples are given in [**18**, Chapters 12 and 13], [**19**, Section 9], see also the Notes to Chapter 7. The bimonoid axiom is given in [**18**, Diagram (8.18)], [**19**, Diagram (39)]. More information on how it relates to the bimonoid axiom (2.12) is given in Remark 2.93. The definition (2.101) is mentioned in [**18**, Notation 11.1], [**19**, Definition (118)]. The abelianization of a Joyal monoid and the coabelianization of a Joyal comonoid are mentioned in [**19**, Section 2.6].

For some recent work on Hopf monoids in Joyal species, see [**10**], [**12**], [**15**], [**31**], [**67**], [**94**], [**647**], [**648**], [**863**], [**876**], [**907**], [**908**].

Bimonoids for hyperplane arrangements. As stated in Table I in the Preface and briefly elaborated in Section 2.16, our bimonoids are a step towards generalizing connected bimonoids in Joyal species to hyperplane arrangements. The context for Joyal bimonoids is that of the braid arrangements. In more classical terms, our bimonoids relate to connected graded bialgebras. Such bialgebras always possess an antipode, and hence are Hopf algebras. The notion of antipode for our bimonoids is elaborated in Chapter 12. We also mention that our bimonoids are similar in spirit to braided bialgebras, where the braiding manifests locally in the form of a Yang–Baxter operator.

We recall some ideas from [**17**, Sections 6.3 – 6.8]. Starting with a family of posets satisfying certain (co)algebra axioms, we constructed graded (co)algebras using rank-one elements (vertices) in the posets to define the coproduct. The higher rank elements (faces) were linked to the iterated coproduct, with coassociativity related to rank-two faces (edges) and the two rank-one elements (vertices) it covered. This is similar to the considerations in Section 2.12, see in particular, Propositions 2.77 and 2.78. Also see Table 16.1 in this regard.

Eckmann–Hilton argument. In connection to Proposition 2.27, we mention that the classical Eckmann–Hilton argument originates in work of Eckmann and Hilton [**276**, Theorem 4.17], [**436**, Proposition 14.13 and Theorem 14.14]. It is discussed for instance by Schubert [**817**, Theorem 11.6.1], Gray [**368**, proof of Proposition 9.9], Switzer [868, Proposition 2.24], Baez and Dolan [57, page 6088], Bredon [154, Chapter VII, Theorem 3.3, item (3)], Aguilar, Gitler, Prieto [25, Lemma 2.10.10], Kock [538, Section 3.5, Exercise 2], Leinster [590, Lemma 1.2.4], tom Dieck [885, Proposition 4.3.1], Brown, Higgins, Sivera [165, Theorem 1.3.1], Simpson [826, Lemmas 2.1.1 and 2.1.2], Arkowitz [48, Proposition 2.2.12], Strom [862, Section 9.2.3].

A categorical version of this argument is given by Joyal and Street [501, Proposition 3], [503, Proposition 5.3]. It was originally suggested by Walters at the Sydney Category Seminar [901]. Related discussions and generalizations are given by Lack and Street [555, Remark 2.2], Balteanu, Fiedorowicz, Schwänzl, Vogt [69, Remarks 1.5 and 1.6], Kock [539, Proposition 3.2], and in our monograph [18, Propositions 6.11, 6.29, 6.59, Remark 6.30]. Generalizations to the context of operads are discussed by Batanin [76] and Schlank and Yanovski [806].

Monoids, comonoids, bimonoids as functor categories. For Joyal species, one may employ the alternative terminology in Table 2.5.

Table 2.5. Variants of Joyal species.

Joyal species with restrictions	$\longleftrightarrow$	cocommutative comonoids
Joyal species with extensions	$\longleftrightarrow$	commutative monoids
Joyal species with balanced operators	$\longleftrightarrow$	bicommutative bimonoids

Joyal species with restrictions were introduced by Schmitt [812]. The analogue of Proposition 2.65 is contained in [18, Proposition 8.29]. Joyal species with balanced operators appear in [18, Chapter 19]. These may also be called Joyal species with restrictions and extensions.

Presheaves on posets. Presheaves on posets and their homology are considered by Gabriel and Zisman [336, Appendix II] and Quillen [767, Section 7]. An $\mathcal{A}$-(co)monoid gives rise to a presheaf on the poset of faces of $\mathcal{A}$. A (co)commutative $\mathcal{A}$-(co)monoid is equivalent to a presheaf on the poset of flats. The resulting homology theory is related to that introduced in [18, Chapter 9] for Joyal comonoids. We plan to develop this aspect in future work. Another interesting point of contact between poset presheaves and hyperplane arrangements occurs in recent work of Everitt and Turner [297], [298].

Partially commutative monoids. Our notion of partially commutative monoids is motivated by ideas of Cartier and Foata [203]. They studied the monoid generated by an alphabet in which certain pairs of letters are allowed to commute with each other. There is now an immense literature on this subject; we mention [270], [664]. Schmitt [811] considered a bialgebra structure on the free partially commutative monoid of Cartier and Foata. Related discussion in our setting is given in Section 6.11.1.

Mackey functors and biset functors. Our notion of bimonoid for an arrangement resembles that of a Mackey functor for a finite group, see the paper by Webb [904], the recent book by Bouc [144] on the more general notion of a biset functor, and references therein.

CHAPTER 3

Bimonads on species

The goal of this chapter is to provide a categorical framework for the concepts introduced in Chapter 2. The usual categorical setting for monoids is a monoidal category. However, that is not the case here; the relevant concept is that of monads and algebras over monads. These notions are reviewed in Appendix C. The connection is as follows.

We construct a monad, denoted $\mathcal{T}$, on the category of $\mathcal{A}$-species, and observe that a $\mathcal{T}$-algebra is the same as an $\mathcal{A}$-monoid. Similarly, we construct a comonad, denoted $\mathcal{T}^\vee$, and a mixed distributive law λ between $\mathcal{T}$ and $\mathcal{T}^\vee$ yielding a bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$. A $\mathcal{T}^\vee$-coalgebra is the same as an $\mathcal{A}$-comonoid, and a $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$-bialgebra is the same as an $\mathcal{A}$-bimonoid. Moreover, λ can be deformed by a parameter q to yield a bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda_q)$ whose bialgebras are $\mathcal{A}$-q-bimonoids.

There is a commutative analogue of $\mathcal{T}$, which we denote by $\mathcal{S}$. Similarly, there is a comonad $\mathcal{S}^\vee$, and a bimonad $(\mathcal{S}, \mathcal{S}^\vee, \lambda)$. A $\mathcal{S}$-algebra is the same as a commutative $\mathcal{A}$-monoid, a $\mathcal{S}^\vee$-coalgebra is the same as a cocommutative $\mathcal{A}$-comonoid, a $(\mathcal{S}, \mathcal{S}^\vee, \lambda)$-bialgebra is the same as a bicommutative $\mathcal{A}$-bimonoid. The monads $\mathcal{T}$ and $\mathcal{S}$ are linked by abelianization, and $\mathcal{S}^\vee$ and $\mathcal{T}^\vee$ by coabelianization. There is also a signed analogue of $\mathcal{S}$, which we denote by $\mathcal{E}$, and so on. A summary is provided below.

TABLE 3.1. Bimonads on species and their bialgebras.

Bimonad	Bialgebras	Bimonad	Bialgebras
$(\mathcal{T}, \mathcal{T}^\vee, \lambda)$	bimonoids	$(\mathcal{T}, \mathcal{T}^\vee, \lambda_{-1})$	signed bimonoids
$(\mathcal{S}, \mathcal{T}^\vee, \lambda)$	com. bimonoids	$(\mathcal{E}, \mathcal{T}^\vee, \lambda_{-1})$	signed com. signed bimonoids
$(\mathcal{T}, \mathcal{S}^\vee, \lambda)$	cocom. bimonoids	$(\mathcal{T}, \mathcal{E}^\vee, \lambda_{-1})$	signed cocom. signed bimonoids
$(\mathcal{S}, \mathcal{S}^\vee, \lambda)$	bicom. bimonoids	$(\mathcal{E}, \mathcal{E}^\vee, \lambda_{-1})$	signed bicom. signed bimonoids

We also construct a bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$ on the category of set-species. Moreover, the linearization functor from set-species to species is bilax wrt the bimonads $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$. This formally explains why the linearization of a (co, bi)monoid in set-species yields a (co, bi)monoid in species.

We briefly discuss the Mesablishvili–Wisbauer rigidity theorem. It applies to the bimonads $(\mathcal{T}, \mathcal{T}^\vee, \lambda_0)$ and $(\mathcal{S}, \mathcal{S}^\vee, \lambda)$. As a consequence, the category of species is equivalent to the category of 0-bimonoids, as well as to the category of bicommutative bimonoids. These ideas are developed in more detail later in Chapter 13.

We extend the notion of species from hyperplane arrangements to the more general setting of left regular bands. Thus, for any LRB Σ, we have Σ-monoids, Σ-comonoids, Σ-bimonoids. They can also be understood formally in terms of the bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$. Moreover, λ can be deformed by any distance function υ on Σ. This gives rise to the notion of Σ-υ-bimonoids.

3.1. Bimonoids as bialgebras over a bimonad

We define a monad $\mathcal{T}$ and a comonad $\mathcal{T}^\vee$ on the category of $\mathcal{A}$-species. Further, we construct a mixed distributive law λ between $\mathcal{T}$ and $\mathcal{T}^\vee$, thus giving rise to a bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$. We also construct a deformation λ_q of λ for any scalar q. These constructions provide a categorical framework for $\mathcal{A}$-(co, bi)monoids in the following sense.

Proposition 3.1. *The following are equivalences of categories.*

$$\begin{aligned} \mathcal{T}\textit{-algebras} &\cong \mathcal{A}\textit{-monoids} \\ \mathcal{T}^\vee\textit{-coalgebras} &\cong \mathcal{A}\textit{-comonoids} \\ (\mathcal{T}, \mathcal{T}^\vee, \lambda)\textit{-bialgebras} &\cong \mathcal{A}\textit{-bimonoids} \\ (\mathcal{T}, \mathcal{T}^\vee, \lambda_q)\textit{-bialgebras} &\cong \mathcal{A}\textit{-}q\textit{-bimonoids} \end{aligned}$$

This result will be proved in the course of the discussion below.

3.1.1. Monad for monoids. Recall from Lemma 1.6 that there is a bijection between the stars of faces A and B whenever A and B have the same support. We now proceed to construct the monad

$$\mathcal{T} : \mathcal{A}\text{-}\mathsf{Sp} \to \mathcal{A}\text{-}\mathsf{Sp}.$$

For a species p, define

$$\mathcal{T}(\mathsf{p})[A] := \bigoplus_{F:\, A \leq F} \mathsf{p}[F]. \tag{3.1}$$

Suppose A and B have the same support. Then, for every face F greater than A, there is a corresponding face $G := BF$ greater than B with the same support as F, and hence a linear map $\beta_{G,F} : \mathsf{p}[F] \to \mathsf{p}[G]$. This yields a linear map

$$\beta_{B,A} : \mathcal{T}(\mathsf{p})[A] \to \mathcal{T}(\mathsf{p})[B]$$

whose (F, G)-component is $\beta_{G,F}$ when $G = BF$, and zero otherwise. (The notation is as in Appendix A.4.) This turns $\mathcal{T}(\mathsf{p})$ into a species. Further, if $f : \mathsf{p} \to \mathsf{q}$ is a map of species, then summing the components $f_F : \mathsf{p}[F] \to \mathsf{q}[F]$ yields a map of species $\mathcal{T}(f) : \mathcal{T}(\mathsf{p}) \to \mathcal{T}(\mathsf{q})$. Thus, $\mathcal{T}$ is a functor.

Observe that

$$\mathcal{T}\mathcal{T}(\mathsf{p})[A] = \bigoplus_{(F,G):\, A \leq F \leq G} \mathsf{p}[G].$$

The sum is over both F and G. Define a natural transformation

$$\mathcal{T}\mathcal{T} \to \mathcal{T}, \qquad \bigoplus_{(F,G)\colon A \leq F \leq G} \mathsf{p}[G] \to \bigoplus_{G\colon A \leq G} \mathsf{p}[G] \tag{3.2a}$$

by mapping each summand in the lhs identically to the matching summand in the rhs. In other words, for a given G, all summands labeled $\mathsf{p}[G]$ in the lhs map identically to the summand labeled $\mathsf{p}[G]$ in the rhs. There is also an obvious natural transformation

$$\mathrm{id} \to \mathcal{T}, \qquad \mathsf{p}[A] \to \bigoplus_{F\colon A \leq F} \mathsf{p}[F] \tag{3.2b}$$

given by inclusion. The maps (3.2a) and (3.2b) turn $\mathcal{T}$ into a monad. The associativity is illustrated below.

$$\begin{tikzcd} \mathcal{T}\mathcal{T}\mathcal{T} \arrow[r] \arrow[d] & \mathcal{T}\mathcal{T} \arrow[d] \\ \mathcal{T}\mathcal{T} \arrow[r] & \mathcal{T} \end{tikzcd} \qquad \begin{tikzcd} \displaystyle\bigoplus_{\substack{(F,G,H):\\ A \leq F \leq G \leq H}} \mathsf{p}[H] \arrow[r] \arrow[d] & \displaystyle\bigoplus_{\substack{(G,H):\\ A \leq G \leq H}} \mathsf{p}[H] \arrow[d] \\ \displaystyle\bigoplus_{\substack{(F,H):\\ A \leq F \leq H}} \mathsf{p}[H] \arrow[r] & \displaystyle\bigoplus_{\substack{H:\\ A \leq H}} \mathsf{p}[H]. \end{tikzcd}$$

Going down and across, or across and down, the map is the identity on each copy of $\mathsf{p}[H]$.

Proposition 3.2. *The category of algebras over the monad $\mathcal{T}$ is equivalent to the category of $\mathcal{A}$-monoids.*

Proof. Suppose p is a $\mathcal{T}$-algebra. This entails a map $\mathcal{T}(\mathsf{p}) \to \mathsf{p}$ of species subject to associativity and unitality axioms (C.19). On each A-component, we have a linear map

$$\bigoplus_{F\colon A \leq F} \mathsf{p}[F] \to \mathsf{p}[A].$$

This is equivalent to a family of linear maps $\mathsf{p}[F] \to \mathsf{p}[A]$, one for each $A \leq F$. Denote the linear map corresponding to $A \leq F$ by μ_A^F. The map $\mathcal{T}(\mathsf{p}) \to \mathsf{p}$ being a natural transformation imposes the naturality axiom in (2.8), while the associativity and unitality axioms impose the corresponding axioms in (2.8). Thus, a $\mathcal{T}$-algebra is the same as a monoid in species. Further, under this translation, a morphism of $\mathcal{T}$-algebras (C.20) is the same as a morphism of monoids. $\square$

This proves the first claim in Proposition 3.1.

3.1.2. Comonad for comonoids. Dually, we construct a comonad

$$\mathcal{T}^\vee : \mathcal{A}\text{-}\mathsf{Sp} \to \mathcal{A}\text{-}\mathsf{Sp}$$

as follows. As a functor, $\mathcal{T}^\vee := \mathcal{T}$. Thus, for a species p,

$$\mathcal{T}^\vee(\mathsf{p})[A] = \bigoplus_{F\colon A \leq F} \mathsf{p}[F].$$

The comonad structure on $\mathcal{T}^\vee$ is given by the natural transformation

$$\mathcal{T}^\vee \to \mathcal{T}^\vee\mathcal{T}^\vee, \qquad \bigoplus_{G:\, A\leq G} \mathsf{p}[G] \to \bigoplus_{(F,G):\, A\leq F\leq G} \mathsf{p}[G] \tag{3.3a}$$

which maps each summand in the lhs identically to all matching summands in the rhs, and the natural transformation

$$\mathcal{T}^\vee \to \mathrm{id}, \qquad \bigoplus_{F:\, A\leq F} \mathsf{p}[F] \to \mathsf{p}[A] \tag{3.3b}$$

which sends $\mathsf{p}[A]$ to itself, and all other summands to zero.

Analogous to the argument for $\mathcal{T}$, it is clear that the category of $\mathcal{T}^\vee$-coalgebras is equivalent to the category of $\mathcal{A}$-comonoids.

3.1.3. Bimonad for bimonoids. Mixed distributive law. We now define a natural transformation

$$\lambda : \mathcal{T}\mathcal{T}^\vee \to \mathcal{T}^\vee\mathcal{T}. \tag{3.4}$$

Evaluated on a species p, on the A-component, this entails a linear map

$$\bigoplus_{(F,G):\, A\leq F\leq G} \mathsf{p}[G] \to \bigoplus_{(F',G'):\, A\leq F'\leq G'} \mathsf{p}[G'].$$

The first sum is over F and G, while the second is over F' and G'. Following the notation in Appendix A.4, let $\lambda_{F,G,F',G'}$ denote the matrix-components of this map (with A kept suppressed). They are defined as follows.

$$\lambda_{F,G,F',G'} := \begin{cases} \mathsf{p}[G] \xrightarrow{\beta_{G',G}} \mathsf{p}[G'] & \text{if } FF' = G \text{ and } F'F = G', \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

In the first alternative, G and G' necessarily have the same support, so writing $\beta_{G',G}$ makes sense.

Exercise 3.3. Check that: For faces G, F', G' all greater than A,

$$\lambda_{A,G,F',G'} \neq 0 \iff (F',G') = (G,G).$$

By symmetry, there is a similar statement for $\lambda_{F,G,A,G'}$.

Theorem 3.4. *The triple $(\mathcal{T},\mathcal{T}^\vee,\lambda)$ is a bimonad on $\mathcal{A}$-species, or equivalently, λ is a mixed distributive law between $\mathcal{T}$ and $\mathcal{T}^\vee$.*

Proof. We need to check commutativity of the diagrams in (C.9a) and (C.9b). The diagrams in (C.9b) are easy to verify. (Exercise 3.3 is relevant here.) The left diagram in (C.9a) takes the following form.

$$\begin{tikzcd} \bigoplus\limits_{A\leq F\leq G\leq H} \mathsf{p}[H] \arrow[r, "\lambda"] & \bigoplus\limits_{A\leq F'\leq G'\leq H'} \mathsf{p}[H'] \arrow[r, "\lambda"] & \bigoplus\limits_{A\leq F'\leq G''\leq H''} \mathsf{p}[H''] \\ \bigoplus\limits_{A\leq F\leq H} \mathsf{p}[H] \arrow[u] \arrow[rr, "\lambda"'] & & \bigoplus\limits_{A\leq G''\leq H''} \mathsf{p}[H''] \arrow[u] \end{tikzcd} \tag{3.6}$$

The face A is fixed, the sums are over the remaining faces. To check that the diagram commutes, we need to check that the matrix-components of the composite map in the two directions coincide. Accordingly, fix $F \le H$ and $F' \le G'' \le H''$, and consider the matrix-component $\mathsf{p}[H] \to \mathsf{p}[H'']$.

Going up and across, we find that if the matrix-component is to be nonzero, the other indices are forced by (3.5): $G = FF'$, $G' = F'F$ and $H' = F'H$. This means that the matrix-component is the canonical identification $\beta_{H'',H} : \mathsf{p}[H] \to \mathsf{p}[H'']$ iff

$$FF' \le H, \quad F'FG'' = F'H, \quad G''F = H'';$$

else the matrix-component is zero. Now going across and up, we find that the matrix-component is the canonical identification iff

$$FG'' = H \quad \text{and} \quad G''F = H''.$$

We claim that the two sets of conditions are equivalent. The condition $G''F = H''$ is common to both, and

$$FG'' = H \iff FF' \le H \text{ and } F'FG'' = F'H.$$

For forward implication: Left multiplying $FG'' = H$ by F' yields $F'FG'' = F'H$. Also since $F' \le G''$, we have $FF' \le FG''$ by (1.1b). For backward implication: Left multiplying $F'FG'' = F'H$ by F yields $FF'FG'' = FF'H$. Now $FF'H = H$ by (1.1d), and $FF'FG'' = FF'G'' = FG''$ by (1.1d) and (1.1a). This proves the claim. It follows that diagram (3.6) commutes.

The right diagram in (C.9a) can be checked similarly. $\square$

Proposition 3.5. *The category of bialgebras over the bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$ is equivalent to the category of $\mathcal{A}$-bimonoids.*

Proof. Suppose h is a $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$-bialgebra, that is, h is a $\mathcal{T}$-algebra, a $\mathcal{T}^\vee$-coalgebra, and the diagram below on the left commutes. The diagram on the right is its evaluation on the A-component.

$$\begin{tikzcd}
\mathcal{T}(\mathsf{h}) \arrow[r] \arrow[d] & \mathsf{h} \arrow[r] & \mathcal{T}^\vee(\mathsf{h}) \\
\mathcal{T}(\mathcal{T}^\vee(\mathsf{h})) \arrow[rr] & & \mathcal{T}^\vee(\mathcal{T}(\mathsf{h})) \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\bigoplus_{A \le F} \mathsf{h}[F] \arrow[r] \arrow[d] & \mathsf{h}[A] \arrow[r] & \bigoplus_{A \le F'} \mathsf{h}[F'] \\
\bigoplus_{A \le F \le G} \mathsf{h}[G] \arrow[rr] & & \bigoplus_{A \le F' \le G'} \mathsf{h}[G'] \arrow[u]
\end{tikzcd}$$

Let us equate the matrix-components. Thus, each choice of faces $A \le F$ and $A \le F'$ yields a commutative diagram. Since the indices G and G' are forced by $G = FF'$ and $G' = F'F$, this diagram is precisely the bimonoid axiom (2.12) (with F' instead of G). Thus, a $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$-bialgebra is the same as a bimonoid in species. Further, morphisms of $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$-bialgebras correspond to morphisms of bimonoids. $\square$

3.1.4. Bimonad for q-bimonoids. For a scalar q, define

$$\lambda_q : \mathcal{T}\mathcal{T}^\vee \to \mathcal{T}^\vee\mathcal{T}, \qquad (\lambda_q)_{F,G,F',G'} := q^{\operatorname{dist}(G,G')}\lambda_{F,G,F',G'}. \tag{3.7}$$

The morphism λ_q may be viewed as a deformation of λ which one recovers by setting $q = 1$.

For $q = 0$, we have

$$(\lambda_0)_{F,G,F',G'} = \begin{cases} \mathrm{id} & \text{if } FF' = G = F'F = G', \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

A necessary and sufficient condition for the map (3.7) to be an isomorphism for q not a root of unity is given later in Theorem 9.100, with further considerations in Exercise 14.83.

Theorem 3.6. *For any scalar q, the triple $(\mathcal{T}, \mathcal{T}^\vee, \lambda_q)$ is a bimonad on $\mathcal{A}$-species. Further, the category of $(\mathcal{T}, \mathcal{T}^\vee, \lambda_q)$-bialgebras is equivalent to the category of $\mathcal{A}$-q-bimonoids.*

This generalizes Theorem 3.4 and Proposition 3.5. To prove it, one needs to understand in addition the role played by the parameter q. A careful analysis is done below for general distance functions.

3.1.5. Bimonad for υ-bimonoids. Let υ be any $\Bbbk$-valued function on the set of bifaces of $\mathcal{A}$. Define

$$\lambda_\upsilon : \mathcal{T}\mathcal{T}^\vee \to \mathcal{T}^\vee\mathcal{T}, \qquad (\lambda_\upsilon)_{F,G,F',G'} := \upsilon_{G,G'}\lambda_{F,G,F',G'}. \tag{3.9}$$

In other words, each matrix-component is multiplied by a scalar determined by the function υ.

Distance functions as defined in Section 1.4.5 provide a necessary and sufficient condition for (3.9) to be a mixed distributive law. This is explained below.

Theorem 3.7. *For υ a function on bifaces of $\mathcal{A}$, the triple $(\mathcal{T}, \mathcal{T}^\vee, \lambda_\upsilon)$ is a bimonad on $\mathcal{A}$-species iff υ is a distance function on $\mathcal{A}$.*

Proof. We need to run through the proof of Theorem 3.4. Diagram (C.9b) commutes iff $\upsilon_{G,G} = 1$ for all faces G. For the left diagram in (C.9a), consider diagram (3.6) with λ replaced by λ_υ. This diagram commutes iff

$$\upsilon_{G,G'}\upsilon_{H',H''} = \upsilon_{H,H''}. \tag{a}$$

The right diagram in (C.9a) gives a similar condition with the role of the two coordinates interchanged.

Suppose υ is a distance function. Then $\upsilon_{G,G} = 1$ by (1.30a), so (C.9b) commutes. Further,

$$\upsilon_{G,G'} = \upsilon_{H,H'} \tag{a1}$$

by (1.30b), and

$$\upsilon_{H,H'}\upsilon_{H',H''} = \upsilon_{H,H''}. \tag{a2}$$

by (1.30e). Combining (a1) and (a2) yields (a). So the left diagram in (C.9a) commutes. Similarly, by (1.30d), the right diagram in (C.9a) also commutes.

Conversely: Suppose (C.9a) and (C.9b) commute. Then $\upsilon_{G,G} = 1$ implies (1.30a). Further, choosing $F = G$, $F' = G'$, $H' = H'' = G''$ in (a) yields (a1) (after using $\upsilon_{H',H'} = 1$), while choosing $F = G = H$ and $H'' = G''$ yields (a2). Thus, (1.30b) and (1.30e) hold. Similarly, commutativity of the right diagram in (C.9a) implies (1.30d). We conclude that υ is a distance function. □

We point out that the left diagrams in (C.9a) and (C.9b) go with left distance functions, while the right diagrams go with right distance functions.

Similarly, by supplementing the argument for Proposition 3.5, we obtain:

Proposition 3.8. *For any distance function υ on $\mathcal{A}$, the category of bialgebras over the bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda_\upsilon)$ is equivalent to the category of $\mathcal{A}$-υ-bimonoids.*

Specializing the distance function to υ_q in the above results yields Theorem 3.6. Further specializing to $q = 1$ (in which case the distance function is identically 1) recovers Theorem 3.4 and Proposition 3.5. This completes the proof of Proposition 3.1.

Exercise 3.9. Observe that for finite sets A, A', B', C such that $A \subseteq C$ and $A' \subseteq B' \subseteq C$, the condition $A \cup B' = C$ is equivalent to the condition $B \cup B' = C$, where $B = A \cup A'$.

Now recall the distance function υ_0 from (1.29). Show that: For λ_{υ_0}, the commutativity of (C.9a) reduces to the above assertion about sets.

For a distance function υ, observe that the law λ_υ is defined by replacing β by β_υ in (3.5). Now let υ be nowhere-zero. In this case, we can define λ_υ^{-1} by replacing β by β_υ^{-1} in (3.5). Then $(\mathcal{T}, \mathcal{T}^\vee, \lambda_\upsilon^{-1})$ is a bimonad whose bialgebras are β_υ^{-1}-bimonoids.

3.2. Bicommutative bimonoids as bialgebras over a bimonad

We continue the discussion in Section 3.1 focussing now on the commutative aspects of the theory. We define a monad $\mathcal{S}$ whose algebras are commutative monoids. There is a natural transformation $\mathcal{T} \twoheadrightarrow \mathcal{S}$ which we call the abelianization. This induces a mixed distributive law between $\mathcal{S}$ and $\mathcal{T}^\vee$ whose bialgebras are commutative bimonoids. Dually, there is a comonad $\mathcal{S}^\vee$ whose coalgebras are cocommutative comonoids. It is related to $\mathcal{T}^\vee$ by the coabelianization $\mathcal{S}^\vee \hookrightarrow \mathcal{T}^\vee$. This induces a mixed distributive law between $\mathcal{T}$ and $\mathcal{S}^\vee$ whose bialgebras are cocommutative bimonoids. Finally, there is an induced mixed distributive law between $\mathcal{S}$ and $\mathcal{S}^\vee$ whose bialgebras are bicommutative bimonoids. This is summarized below.

Proposition 3.10. *The following are equivalences of categories.*

$$\begin{aligned}
\mathcal{S}\text{-}algebras &\cong commutative\ \mathcal{A}\text{-}monoids\\
\mathcal{S}^\vee\text{-}coalgebras &\cong cocommutative\ \mathcal{A}\text{-}comonoids\\
(\mathcal{S},\mathcal{T}^\vee,\lambda)\text{-}bialgebras &\cong commutative\ \mathcal{A}\text{-}bimonoids\\
(\mathcal{T},\mathcal{S}^\vee,\lambda)\text{-}bialgebras &\cong cocommutative\ \mathcal{A}\text{-}bimonoids\\
(\mathcal{S},\mathcal{S}^\vee,\lambda)\text{-}bialgebras &\cong bicommutative\ \mathcal{A}\text{-}bimonoids
\end{aligned}$$

Recall that (co)commutative (co)monoids have signed analogues. As expected, there is a monad $\mathcal{E}$ whose algebras are signed commutative monoids, and a comonad $\mathcal{E}^\vee$ whose coalgebras are signed cocommutative comonoids. The complete result is stated below.

Proposition 3.11. *The following are equivalences of categories.*

$$\begin{aligned}
\mathcal{E}\text{-}algebras &\cong signed\ commutative\ \mathcal{A}\text{-}monoids\\
\mathcal{E}^\vee\text{-}coalgebras &\cong signed\ cocommutative\ \mathcal{A}\text{-}comonoids\\
(\mathcal{E},\mathcal{T}^\vee,\lambda_{-1})\text{-}bialgebras &\cong signed\ commutative\ signed\ \mathcal{A}\text{-}bimonoids\\
(\mathcal{T},\mathcal{E}^\vee,\lambda_{-1})\text{-}bialgebras &\cong signed\ cocommutative\ signed\ \mathcal{A}\text{-}bimonoids\\
(\mathcal{E},\mathcal{E}^\vee,\lambda_{-1})\text{-}bialgebras &\cong signed\ bicommutative\ signed\ \mathcal{A}\text{-}bimonoids
\end{aligned}$$

3.2.1. Monad for commutative monoids. Define a functor

$$\mathcal{S} : \mathcal{A}\text{-}\mathsf{Sp} \to \mathcal{A}\text{-}\mathsf{Sp}$$

as follows. We use the formulation of species given in Proposition 2.5. For a species p, let

$$\mathcal{S}(\mathsf{p})[\mathrm{Z}] := \bigoplus_{\mathrm{X}\colon \mathrm{Z}\leq\mathrm{X}} \mathsf{p}[\mathrm{X}]. \tag{3.10}$$

Observe that

$$\mathcal{S}\mathcal{S}(\mathsf{p})[\mathrm{Z}] = \bigoplus_{(\mathrm{X},\mathrm{Y})\colon \mathrm{Z}\leq\mathrm{X}\leq\mathrm{Y}} \mathsf{p}[\mathrm{Y}].$$

The sum is over both X and Y. Define natural transformations $\mathcal{S}\mathcal{S} \to \mathcal{S}$ and $\mathrm{id} \to \mathcal{S}$ in a manner similar to $\mathcal{T}$ (with faces replaced by flats). This turns $\mathcal{S}$ into a monad.

Proposition 3.12. *The category of algebras over the monad $\mathcal{S}$ is equivalent to the category of commutative $\mathcal{A}$-monoids.*

PROOF. Suppose p is a $\mathcal{S}$-algebra. This entails a map $\mathcal{S}(\mathsf{p}) \to \mathsf{p}$ of species subject to associativity and unitality axioms (C.19). On each Z-component, we have a linear map

$$\bigoplus_{\mathrm{X}\colon \mathrm{Z}\leq\mathrm{X}} \mathsf{p}[\mathrm{X}] \to \mathsf{p}[\mathrm{Z}].$$

This is equivalent to a family of linear maps $\mathsf{p}[\mathrm{X}] \to \mathsf{p}[\mathrm{Z}]$, one for each $\mathrm{Z} \leq \mathrm{X}$. Denote the linear map corresponding to $\mathrm{Z} \leq \mathrm{X}$ by $\mu^{\mathrm{X}}_{\mathrm{Z}}$. The associativity and unitality axioms impose the corresponding axioms in (2.21). Thus, a

$\mathcal{S}$-algebra is the same as a commutative monoid in species (as formulated in Proposition 2.20). Further, under this translation, a morphism of $\mathcal{S}$-algebras (C.20) is the same as a morphism of commutative monoids. $\square$

In the usual formulation of species, we would write

$$\mathcal{S}(\mathsf{p})[A] := \bigoplus_{\mathrm{X}:\, \mathrm{s}(A) \leq \mathrm{X}} \mathsf{p}[\mathrm{X}], \tag{3.11}$$

where the sum is over all flats X greater than $\mathrm{s}(A)$, the support of A, and $\mathsf{p}[\mathrm{X}]$ is the colimit (2.7).

3.2.2. Abelianization. There is a natural transformation

$$\pi : \mathcal{T} \twoheadrightarrow \mathcal{S}. \tag{3.12}$$

Evaluated on a species p, on the A-component, on the F-summand, it is defined by $\beta_{\mathrm{s}(F),F}$, with the latter as in (2.6). We call (3.12) the *abelianization*. It is surjective in the sense that $\mathcal{T}(\mathsf{p}) \to \mathcal{S}(\mathsf{p})$ is surjective for all p. The diagrams

$$\begin{tikzcd} \mathcal{T}\mathcal{T} \arrow[r, "\pi\pi"] \arrow[d] & \mathcal{S}\mathcal{S} \arrow[d] \\ \mathcal{T} \arrow[r, "\pi"'] & \mathcal{S} \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathcal{T} \arrow[rr, "\pi"] & & \mathcal{S} \\ & \mathrm{id} \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{3.13}$$

commute. Formally, one can say that (id, π) is a lax functor of monads from $\mathcal{S}$ to $\mathcal{T}$ in the sense of (C.2). (By convention, the lax functor is from $\mathcal{S}$ to $\mathcal{T}$ and not the other way round.) This induces a functor from the category of $\mathcal{S}$-algebras (commutative monoids) to the category of $\mathcal{T}$-algebras (monoids). It has the familiar form: every commutative monoid is a monoid. This functor has a left adjoint, and this is called abelianization in Section 2.7.1.

3.2.3. Comonad for cocommutative comonoids. Dually, one can define a comonad

$$\mathcal{S}^\vee : \mathcal{A}\text{-}\mathsf{Sp} \to \mathcal{A}\text{-}\mathsf{Sp}.$$

As a functor, $\mathcal{S}^\vee = \mathcal{S}$, and the natural transformations $\mathcal{S}^\vee \to \mathcal{S}^\vee\mathcal{S}^\vee$ and $\mathcal{S}^\vee \to \mathrm{id}$ are defined in a manner similar to $\mathcal{T}^\vee$. Analogous to the argument for $\mathcal{S}$, it is clear that the category of $\mathcal{S}^\vee$-coalgebras is equivalent to the category of cocommutative $\mathcal{A}$-comonoids.

3.2.4. Coabelianization. There is a natural transformation

$$\pi^\vee : \mathcal{S}^\vee \hookrightarrow \mathcal{T}^\vee. \tag{3.14}$$

Evaluated on a species p, on the A-component, on the X-summand, it is defined by $\sum_F \beta_{F,\mathrm{X}}$, where F runs over faces greater than A which have support X. We call (3.14) the *coabelianization*. It is injective in the sense that $\mathcal{S}^\vee(\mathsf{p}) \to \mathcal{T}^\vee(\mathsf{p})$ is injective for all p. The functor $(\mathrm{id}, \pi^\vee)$ is now a colax functor of comonads from $\mathcal{S}^\vee$ to $\mathcal{T}^\vee$ in the sense of (C.6). That is, the diagrams dual to (3.13) commute. This induces a functor from the category of $\mathcal{S}^\vee$-coalgebras (cocommutative comonoids) to the category of $\mathcal{T}^\vee$-coalgebras (comonoids). It has the familiar form: every cocommutative comonoid is a comonoid. This functor has a right adjoint, and this is called coabelianization in Section 2.7.2.

3.2.5. Bimonads. Mixed distributive laws. Recall from Theorem 3.4 that there is a mixed distributive law λ between $\mathcal{T}$ and $\mathcal{T}^\vee$. This along with the (co)abelianizations induces the following commutative diagram.

$$
\begin{tikzcd}
& \mathcal{T}\mathcal{T}^\vee \arrow[rr, "\lambda"] \arrow[dd, two heads] & & \mathcal{T}^\vee\mathcal{T} \arrow[dd, two heads] \\
\mathcal{T}\mathcal{S}^\vee \arrow[ur, hook] \arrow[rr, dashed] \arrow[dd, two heads] & & \mathcal{S}^\vee\mathcal{T} \arrow[ur, hook] \arrow[dd, two heads] & \\
& \mathcal{S}\mathcal{T}^\vee \arrow[rr, dashed] & & \mathcal{T}^\vee\mathcal{S} \\
\mathcal{S}\mathcal{S}^\vee \arrow[ur, hook] \arrow[rr, dashed] & & \mathcal{S}^\vee\mathcal{S} \arrow[ur, hook] &
\end{tikzcd}
\tag{3.15}
$$

It then follows that there are three more mixed distributive laws: one between $\mathcal{S}$ and $\mathcal{T}^\vee$, one between $\mathcal{T}$ and $\mathcal{S}^\vee$, one between $\mathcal{S}$ and $\mathcal{S}^\vee$. We denote all of them by λ. The last one, namely,

$$
\lambda : \mathcal{S}\mathcal{S}^\vee \to \mathcal{S}^\vee\mathcal{S}
\tag{3.16}
$$

can be made explicit as follows. Evaluating on p, on the Z-component, we have

$$
\bigoplus_{(\mathrm{X},\mathrm{Y}):\,\mathrm{Z}\leq\mathrm{X}\leq\mathrm{Y}} \mathsf{p}[\mathrm{Y}] \to \bigoplus_{(\mathrm{X}',\mathrm{Y}'):\,\mathrm{Z}\leq\mathrm{X}'\leq\mathrm{Y}'} \mathsf{p}[\mathrm{Y}'].
$$

The first sum is over X and Y, while the second is over X' and Y'. The matrix-component for which $\mathrm{Y} = \mathrm{X} \vee \mathrm{X}' = \mathrm{Y}'$ is the identity map, while the remaining matrix-components are zero.

A necessary and sufficient condition for the map (3.16) to be an isomorphism is given later in Theorem 9.64, with an explicit diagonalization in Exercise 14.32.

Proposition 3.13. *We have bimonads*

$$
(\mathcal{T}, \mathcal{T}^\vee, \lambda), \quad (\mathcal{S}, \mathcal{T}^\vee, \lambda), \quad (\mathcal{T}, \mathcal{S}^\vee, \lambda), \quad (\mathcal{S}, \mathcal{S}^\vee, \lambda)
$$

on the category of $\mathcal{A}$-species. Further, there is a commutative diagram of bilax functors

$$
\begin{tikzcd}
(\mathcal{S}, \mathcal{S}^\vee, \lambda) \arrow[r] \arrow[d] & (\mathcal{T}, \mathcal{S}^\vee, \lambda) \arrow[d] \\
(\mathcal{S}, \mathcal{T}^\vee, \lambda) \arrow[r] & (\mathcal{T}, \mathcal{T}^\vee, \lambda).
\end{tikzcd}
\tag{3.17}
$$

The functors are bilax in the sense of (C.10) *with $\mathcal{F} := \mathrm{id}$ and the (co)lax structure given by (co)abelianization.*

Proof. Note that for each morphism in (3.17), the diagram (C.10) specializes to a square in (3.15). □

Exercise 3.14. Verify Proposition 3.10 that was stated at the beginning of the section. One can pass (3.17) to the categories of bialgebras using Proposition C.28. Check that this expresses the fact that a bicommutative bimonoid is a commutative bimonoid (as well as a cocommutative bimonoid), which in turn is a bimonoid.

3.2.6. (Co)monad for signed (co)commutative (co)monoids. Define a functor

$$\mathcal{E} : \mathcal{A}\text{-}\mathsf{Sp} \to \mathcal{A}\text{-}\mathsf{Sp}$$

as follows. We use the formulation of species given in Proposition 2.5. For a species p, let

$$\mathcal{E}(\mathsf{p})[\mathrm{Z}] := \bigoplus_{\mathrm{X}\colon \mathrm{Z}\leq \mathrm{X}} \mathbf{E}^{-}[\mathrm{Z},\mathrm{X}] \otimes \mathsf{p}[\mathrm{X}], \tag{3.18}$$

with $\mathbf{E}^{-}[\mathrm{Z},\mathrm{X}]$ as in Definition 1.74. Observe that

$$\mathcal{E}\mathcal{E}(\mathsf{p})[\mathrm{Z}] = \bigoplus_{(\mathrm{X},\mathrm{Y})\colon \mathrm{Z}\leq\mathrm{X}\leq\mathrm{Y}} \mathbf{E}^{-}[\mathrm{Z},\mathrm{X}] \otimes \mathbf{E}^{-}[\mathrm{X},\mathrm{Y}] \otimes \mathsf{p}[\mathrm{Y}].$$

The sum is over both X and Y. Define natural transformations $\mathcal{E}\mathcal{E} \to \mathcal{E}$ and $\mathrm{id} \to \mathcal{E}$ in a manner similar to $\mathcal{S}$. The additional ingredient is the usage of (1.162). This turns $\mathcal{E}$ into a monad.

Proposition 3.15. *The category of algebras over the monad $\mathcal{E}$ is equivalent to the category of signed commutative $\mathcal{A}$-monoids.*

Proof. This is similar to the argument for Proposition 3.12. We now make use of the formulation given in Proposition 2.36. □

There is a natural transformation

$$\pi_{-1} : \mathcal{T} \twoheadrightarrow \mathcal{E}. \tag{3.19}$$

Evaluated on a species p, on the A-component, on the F-summand, it is defined by

$$\mathsf{p}[F] \to \mathbf{E}^{-}[\mathrm{Z},\mathrm{X}] \otimes \mathsf{p}[\mathrm{X}], \qquad x \mapsto \mathtt{H}_{[F/A]} \otimes \beta_{\mathrm{X},F}(x).$$

The support of A is Z, and of F is X. We call (3.19) the *signed abelianization*. It is surjective. The diagrams (3.13) with $\mathcal{E}$ replacing $\mathcal{S}$ commute. Formally, (id, π_{-1}) is a lax functor of monads from $\mathcal{E}$ to $\mathcal{T}$ in the sense of (C.2). This induces a functor from the category of $\mathcal{E}$-algebras (signed commutative monoids) to the category of $\mathcal{T}$-algebras (monoids). It has the familiar form: every signed commutative monoid is a monoid.

Dually, there is a comonad

$$\mathcal{E}^{\vee} : \mathcal{A}\text{-}\mathsf{Sp} \to \mathcal{A}\text{-}\mathsf{Sp}.$$

As a functor, $\mathcal{E}^{\vee} = \mathcal{E}$, and the structure maps of $\mathcal{E}^{\vee}$ are defined in a manner similar to $\mathcal{S}^{\vee}$. We need to use the inverse of (1.162) here. The category of $\mathcal{E}^{\vee}$-coalgebras is equivalent to the category of signed cocommutative $\mathcal{A}$-comonoids.

Similarly, there is the *signed coabelianization*

$$\pi^{\vee}_{-1} : \mathcal{E}^{\vee} \hookrightarrow \mathcal{T}^{\vee}. \tag{3.20}$$

It is injective. The functor $(\mathrm{id}, \pi_{-1}^{\vee})$ now is a colax functor of comonads from $\mathcal{E}^{\vee}$ to $\mathcal{T}^{\vee}$ in the sense of (C.6). This induces a functor from the category of $\mathcal{E}^{\vee}$-coalgebras (signed cocommutative comonoids) to the category of $\mathcal{T}^{\vee}$-coalgebras (comonoids). It has the familiar form: every signed cocommutative comonoid is a comonoid.

The signed analogue of (3.15) is

$$
\begin{tikzcd}[row sep=small, column sep=small]
 & \mathcal{T}\mathcal{T}^{\vee} \arrow[rr, "\lambda_{-1}"] \arrow[dd, two heads] & & \mathcal{T}^{\vee}\mathcal{T} \arrow[dd, two heads] \\
\mathcal{T}\mathcal{E}^{\vee} \arrow[ur, hook] \arrow[rr, dashed] \arrow[dd, two heads] & & \mathcal{E}^{\vee}\mathcal{T} \arrow[ur, hook] \arrow[dd, two heads] & \\
 & \mathcal{E}\mathcal{T}^{\vee} \arrow[rr, dashed] & & \mathcal{T}^{\vee}\mathcal{E} \\
\mathcal{E}\mathcal{E}^{\vee} \arrow[ur, hook] \arrow[rr, dashed] & & \mathcal{E}^{\vee}\mathcal{E} \arrow[ur, hook] &
\end{tikzcd}
\tag{3.21}
$$

Note that this is induced from the mixed distributive law λ_{-1} between $\mathcal{T}$ and $\mathcal{T}^{\vee}$, and the signed (co)abelianizations. Thus, we obtain three more mixed distributive laws: one between $\mathcal{E}$ and $\mathcal{T}^{\vee}$, one between $\mathcal{T}$ and $\mathcal{E}^{\vee}$, one between $\mathcal{E}$ and $\mathcal{E}^{\vee}$. We denote all of them by λ_{-1}. The last one, namely,

$$\lambda_{-1} : \mathcal{E}\mathcal{E}^{\vee} \to \mathcal{E}^{\vee}\mathcal{E} \tag{3.22}$$

can be made explicit as follows. Evaluating on p, on the Z-component, we have

$$\bigoplus_{\substack{(\mathrm{X},\mathrm{Y}):\\ \mathrm{Z}\leq\mathrm{X}\leq\mathrm{Y}}} \mathbf{E}^{-}[\mathrm{Z},\mathrm{X}] \otimes \mathbf{E}^{-}[\mathrm{X},\mathrm{Y}] \otimes \mathsf{p}[\mathrm{Y}] \to \bigoplus_{\substack{(\mathrm{X}',\mathrm{Y}'):\\ \mathrm{Z}\leq\mathrm{X}'\leq\mathrm{Y}'}} \mathbf{E}^{-}[\mathrm{Z},\mathrm{X}'] \otimes \mathbf{E}^{-}[\mathrm{X}',\mathrm{Y}'] \otimes \mathsf{p}[\mathrm{Y}'].$$

The first sum is over X and Y, while the second is over X′ and Y′. The matrix-component for which $\mathrm{Y} = \mathrm{X} \vee \mathrm{X}' = \mathrm{Y}'$ is the map (1.163) tensor with the identity, while the remaining matrix-components are zero.

Proposition 3.16. *We have bimonads*

$$(\mathcal{T}, \mathcal{T}^{\vee}, \lambda_{-1}), \quad (\mathcal{E}, \mathcal{T}^{\vee}, \lambda_{-1}), \quad (\mathcal{T}, \mathcal{E}^{\vee}, \lambda_{-1}), \quad (\mathcal{E}, \mathcal{E}^{\vee}, \lambda_{-1})$$

on the category of $\mathcal{A}$-species. Further, there is a commutative diagram of bilax functors

$$
\begin{tikzcd}
(\mathcal{E}, \mathcal{E}^{\vee}, \lambda_{-1}) \arrow[r] \arrow[d] & (\mathcal{T}, \mathcal{E}^{\vee}, \lambda_{-1}) \arrow[d] \\
(\mathcal{E}, \mathcal{T}^{\vee}, \lambda_{-1}) \arrow[r] & (\mathcal{T}, \mathcal{T}^{\vee}, \lambda_{-1}).
\end{tikzcd}
\tag{3.23}
$$

The functors are bilax in the sense of (C.10) with $\mathcal{F} := \mathrm{id}$ and the (co)lax structure given by signed (co)abelianization.

3.3. Duality as a bilax functor

Recall from Section 2.9 that duality interchanges monoids and comonoids, and preserves bimonoids. These facts can be understood formally by the interaction of duality with the bimonad $(\mathcal{T}, \mathcal{T}^{\vee}, \lambda)$ on species. Details follow.

Recall from Theorem 3.6 that $(\mathcal{T}, \mathcal{T}^\vee, \lambda_q)$ is a bimonad on $\mathcal{A}\text{-}\mathsf{Sp}$. Now a monad on a category is a comonad on the opposite category, and vice versa. Thus, $(\mathcal{T}^\vee, \mathcal{T}, \lambda_q)$ is a bimonad on $\mathcal{A}\text{-}\mathsf{Sp}^{\mathrm{op}}$.

Since duality of vector spaces commutes with direct sums, there are natural isomorphisms

$$\mathcal{T}^\vee(\mathsf{p}^*) = \mathcal{T}(\mathsf{p})^* \quad \text{and} \quad \mathcal{T}^\vee(\mathsf{p})^* = \mathcal{T}(\mathsf{p}^*). \tag{3.24}$$

Recall that $\mathcal{T}$ and $\mathcal{T}^\vee$ are identical as functors. The compatibility of the above isomorphisms with their (co)monad structures is addressed below.

Proposition 3.17. *The duality functor* (2.64) *equipped with the structure maps* (3.24) *is bilax for*

$$(\mathcal{T}, \mathcal{T}^\vee, \lambda_q) \to (\mathcal{T}^\vee, \mathcal{T}, \lambda_q)$$

(in the sense of (C.10)*).*

Proof. Let p be any species. Then, under the canonical identification (3.24), the map $\mathcal{T}^\vee(\mathsf{p}^*) \to \mathcal{T}^\vee\mathcal{T}^\vee(\mathsf{p}^*)$ is the dual of the map $\mathcal{T}\mathcal{T}(\mathsf{p}) \to \mathcal{T}(\mathsf{p})$, and the map $\mathcal{T}^\vee(\mathsf{p}^*) \to \mathsf{p}^*$ is the dual of the map $\mathsf{p} \to \mathcal{T}(\mathsf{p})$. This shows that the duality functor is lax (C.2). By similar considerations, it is also colax (C.6). Next, up to canonical identifications, the map $\mathcal{T}\mathcal{T}^\vee(\mathsf{p}^*) \to \mathcal{T}^\vee\mathcal{T}(\mathsf{p}^*)$ is the dual of the map $\mathcal{T}\mathcal{T}^\vee(\mathsf{p}) \to \mathcal{T}^\vee\mathcal{T}(\mathsf{p})$. This shows that (C.10) holds. Hence, the duality functor is bilax. □

Hence, we say that the mixed distributive law λ_q is self-dual. For finite-dimensional species, this fact can be often used to reduce our work by half. For instance, in the proof of Theorem 3.4, it suffices to check the left diagrams in (C.9a) and (C.9b). The right diagrams then follow by duality.

Exercise 3.18. Let υ be a distance function, and let υ^t denote its transpose defined by $(\upsilon^t)_{F,G} := \upsilon_{G,F}$. Show that: The duality functor is bilax for

$$(\mathcal{T}, \mathcal{T}^\vee, \lambda_\upsilon) \to (\mathcal{T}^\vee, \mathcal{T}, \lambda_{\upsilon^t}).$$

Deduce that λ_υ is self-dual iff υ is symmetric, that is, $\upsilon = \upsilon^t$. (This is the situation of Proposition 3.17, where $\upsilon = \upsilon_q$, the q-distance function on faces.)

By Proposition C.28, a (co, bi)lax functor preserves (co, bi)algebras. Thus, one can deduce from Proposition 3.17 that the dual of a monoid is a comonoid and vice versa, while the dual of a bimonoid is a bimonoid (as explicitly discussed in Section 2.9).

In the above discussion, $\mathcal{T}$ can be replaced by $\mathcal{S}$, or $\mathcal{T}^\vee$ can be replaced by $\mathcal{S}^\vee$, or both. Using the natural isomorphisms

$$\mathcal{S}^\vee(\mathsf{p}^*) = \mathcal{S}(\mathsf{p})^* \quad \text{and} \quad \mathcal{S}(\mathsf{p}^*) = \mathcal{S}^\vee(\mathsf{p})^*, \tag{3.25}$$

one can see that the duality functor is bilax for

$$(\mathcal{S}, \mathcal{T}^\vee, \lambda) \to (\mathcal{S}^\vee, \mathcal{T}, \lambda), \quad (\mathcal{T}, \mathcal{S}^\vee, \lambda) \to (\mathcal{T}^\vee, \mathcal{S}, \lambda), \quad (\mathcal{S}, \mathcal{S}^\vee, \lambda) \to (\mathcal{S}^\vee, \mathcal{S}, \lambda).$$

As a consequence, the dual of a commutative monoid is a cocommutative comonoid, and so on. One may also replace $\mathcal{S}$ by $\mathcal{E}$, and $\mathcal{S}^\vee$ by $\mathcal{E}^\vee$. As a consequence, the dual of a signed commutative monoid is a signed cocommutative comonoid, and so on.

On the category of finite-dimensional species, the duality functor (2.64) is an equivalence of categories, and

$$\mathcal{T}^\vee(-) = \mathcal{T}((-)^*)^* \quad \text{and} \quad \mathcal{T}(-) = \mathcal{T}^\vee((-)^*)^*$$

determine each other. We say that $\mathcal{T}$ and $\mathcal{T}^\vee$ are conjugates of each other wrt duality. The first equality is as comonads, and the second is as monads. Similarly, $\mathcal{S}$ and $\mathcal{S}^\vee$ are conjugates of each other, and $\mathcal{E}$ and $\mathcal{E}^\vee$ are conjugates of each other.

3.4. Opposite transformation

We saw how bimonoids in species can be formally understood using bimonads. Similarly, op and cop constructions on bimonoids from Section 2.10 (and in particular the process of (co)abelianization) can be understood via bilax functors on bimonads. The key construction is that of the opposite transformation which links a face to its opposite face.

3.4.1. Opposite transformation. There is a natural transformation

$$\tau : \mathcal{T} \to \mathcal{T}. \tag{3.26}$$

Evaluated on a species p, the A-component of $\tau_{\mathsf{p}} : \mathcal{T}(\mathsf{p}) \to \mathcal{T}(\mathsf{p})$ is defined by summing the maps

$$\beta_{A\overline{F},F} : \mathsf{p}[F] \to \mathsf{p}[A\overline{F}]$$

over all $F \geq A$. For faces A and B of the same support, and $A \leq F$, the diagram

$$\begin{tikzcd} \mathsf{p}[F] \arrow[r, "\beta_{A\overline{F},F}"] \arrow[d, "\beta_{BF,F}"'] & \mathsf{p}[A\overline{F}] \arrow[d, "\beta_{B\overline{F},A\overline{F}}"] \\ \mathsf{p}[BF] \arrow[r, "\beta_{B\overline{F},BF}"'] & \mathsf{p}[B\overline{F}] \end{tikzcd} \tag{3.27}$$

commutes by (2.1). Since $B(A\overline{F}) = B\overline{F}$ and $B(\overline{BF}) = (B\overline{B})\overline{F} = B\overline{F}$, we deduce that τ_{p} is a map of species. Naturality in p follows from (2.3). Thus, τ is a natural transformation. We call it the *opposite transformation*. Observe that it is an involution. Since $\mathcal{T} = \mathcal{T}^\vee$ as functors, we also have $\tau : \mathcal{T}^\vee \to \mathcal{T}^\vee$.

Proposition 3.19. *The functor* $(\mathrm{id}, \tau) : \mathcal{T} \to \mathcal{T}$ *is lax, that is, the diagrams*

$$\begin{tikzcd} \mathcal{T}\mathcal{T} \arrow[r, "\tau\tau"] \arrow[d] & \mathcal{T}\mathcal{T} \arrow[d] \\ \mathcal{T} \arrow[r, "\tau"'] & \mathcal{T} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T} \arrow[rr, "\tau"] & & \mathcal{T} \\ & \mathrm{id} \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{3.28}$$

commute. Similarly, $(\mathrm{id}, \tau) : \mathcal{T}^\vee \to \mathcal{T}^\vee$ *is colax, that is, the diagrams*

$$\begin{tikzcd} \mathcal{T}^\vee\mathcal{T}^\vee \arrow[r, "\tau\tau"] & \mathcal{T}^\vee\mathcal{T}^\vee \\ \mathcal{T}^\vee \arrow[r, "\tau"'] \arrow[u] & \mathcal{T}^\vee \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}^\vee \arrow[rr, "\tau"] \arrow[dr] & & \mathcal{T}^\vee \arrow[dl] \\ & \mathrm{id} & \end{tikzcd} \tag{3.29}$$

commute.

Proof. We check the diagram on the left in (3.28). (The one on the right is easy to check.) Let us first look at the map

$$\tau_{\mathcal{T}(\mathsf{p})}\tau_{\mathsf{p}} : \mathcal{T}\mathcal{T}(\mathsf{p})[A] \to \mathcal{T}\mathcal{T}(\mathsf{p})[A].$$

This is a horizontal composite of two natural transformations. There are two equivalent ways to evaluate it. On the summand indexed by $A \leq F \leq G$, they are given by

$$\mathsf{p}[G] \to \mathsf{p}[F\overline{G}] \to \mathsf{p}[A\overline{G}] \qquad \text{and} \qquad \mathsf{p}[G] \to \mathsf{p}[A\overline{F}G] \to \mathsf{p}[A\overline{G}].$$

Both composites equal $\beta_{A\overline{G},G}$, and this is indeed the map

$$\tau_{\mathsf{p}} : \mathcal{T}(\mathsf{p})[A] \to \mathcal{T}(\mathsf{p})[A]$$

on the summand indexed by $A \leq G$. This shows that (id, τ) is lax. The colax check is similar. $\square$

Proposition 3.20. *The functors* $(\mathrm{id}, \tau, \mathrm{id})$ *and* $(\mathrm{id}, \mathrm{id}, \tau)$ *both from* $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$ *to itself are bilax. That is, the diagrams*

$$\begin{tikzcd} \mathcal{T}\mathcal{T}^\vee \arrow[r, "\tau\mathcal{T}^\vee"] \arrow[d, "\lambda"'] & \mathcal{T}\mathcal{T}^\vee \arrow[d, "\lambda"] \\ \mathcal{T}^\vee\mathcal{T} \arrow[r, "\mathcal{T}^\vee\tau"'] & \mathcal{T}^\vee\mathcal{T} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}\mathcal{T}^\vee \arrow[r, "\mathcal{T}\tau"] \arrow[d, "\lambda"'] & \mathcal{T}\mathcal{T}^\vee \arrow[d, "\lambda"] \\ \mathcal{T}^\vee\mathcal{T} \arrow[r, "\tau\mathcal{T}"'] & \mathcal{T}^\vee\mathcal{T} \end{tikzcd} \tag{3.30}$$

commute.

Proof. We check the first diagram. Evaluating on species p, starting in the summand $A \leq F \leq G$ and ending in the summand $A \leq F' \leq G'$, both directions force $G = FF'$ and $G' = F'\overline{F}$. In this case, the two composites are given by

$$\mathsf{p}[FF'] \to \mathsf{p}[A\overline{F}F'] \to \mathsf{p}[F'\overline{F}] \quad \text{and} \quad \mathsf{p}[FF'] \to \mathsf{p}[F'F] \to \mathsf{p}[F'\overline{F}].$$

Both composites equal $\beta_{F'\overline{F},FF'}$. $\square$

By composing bilax functors, we note that $(\mathrm{id}, \tau, \tau)$ is also bilax from $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$ to itself.

3.4.2. (Co)abelianization. Recall (co)abelianization π and $\pi^\vee$ from (3.12) and (3.14). The diagrams

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "\tau"] \arrow[dr, "\pi"'] & & \mathcal{T} \arrow[dl, "\pi"] \\ & \mathcal{S} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}^\vee \arrow[rr, "\tau"] & & \mathcal{T}^\vee \\ & \mathcal{S}^\vee \arrow[ul, "\pi^\vee"] \arrow[ur, "\pi^\vee"'] & \end{tikzcd} \tag{3.31}$$

commute. In fact, following the argument in the proof of Lemma 2.16, we deduce:

Lemma 3.21. $(\mathcal{S}, \pi)$ *is the coequalizer of* $\mathcal{T} \underset{\mathrm{id}}{\overset{\tau}{\rightrightarrows}} \mathcal{T}$ *in the category of monads on species. Dually,* $(\mathcal{S}^\vee, \pi^\vee)$ *is the equalizer of* $\mathcal{T}^\vee \underset{\mathrm{id}}{\overset{\tau}{\rightrightarrows}} \mathcal{T}^\vee$ *in the category of comonads on species.*

Similarly, diagram (3.17) arises by taking appropriate (co)equalizers in the category of bimonads.

3.4.3. Signed opposite transformation. The opposite transformation has a signed analogue

$$\tau_{-1} : \mathcal{T} \to \mathcal{T}.$$

It is defined similarly but with β replaced by β_{-1}. We call it the *signed opposite transformation*. It can also be viewed as $\tau_{-1} : \mathcal{T}^\vee \to \mathcal{T}^\vee$.

One can check that (id, τ_{-1}) is a lax functor of monads from $\mathcal{T}$ to $\mathcal{T}$ as well as a colax functor of comonads from $\mathcal{T}^\vee$ to $\mathcal{T}^\vee$. Further, the functors $(\mathrm{id}, \tau_{-1}, \mathrm{id})$, $(\mathrm{id}, \mathrm{id}, \tau_{-1})$, $(\mathrm{id}, \tau_{-1}, \tau_{-1})$ all from $(\mathcal{T}, \mathcal{T}^\vee, \lambda_{-1})$ to itself are bilax. In particular, diagrams (3.30) commute with λ replaced by λ_{-1} and τ replaced by τ_{-1}.

Now recall signed (co)abelianization from (3.19) and (3.20). The diagrams

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "\tau_{-1}"] \arrow[dr, "\pi_{-1}"'] & & \mathcal{T} \arrow[dl, "\pi_{-1}"] \\ & \mathcal{E} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}^\vee \arrow[rr, "\tau_{-1}"] & & \mathcal{T}^\vee \\ & \mathcal{E}^\vee \arrow[ul, "\pi^\vee_{-1}"] \arrow[ur, "\pi^\vee_{-1}"'] & \end{tikzcd}$$

commute, and $(\mathcal{E}, \pi_{-1})$ is the coequalizer of $\mathcal{T} \overset{\tau_{-1}}{\underset{\mathrm{id}}{\rightrightarrows}} \mathcal{T}$ in the category of monads on species, and $(\mathcal{E}^\vee, \pi^\vee_{-1})$ is the equalizer of $\mathcal{T}^\vee \overset{\tau_{-1}}{\underset{\mathrm{id}}{\rightrightarrows}} \mathcal{T}^\vee$ in the category of comonads on species. Similarly, diagram (3.23) arises by taking appropriate (co)equalizers in the category of bimonads.

3.4.4. Deformed opposite transformation. More generally, for any distance function υ, replacing β by β_υ yields a natural transformation

$$\tau_\upsilon : \mathcal{T} \to \mathcal{T}.$$

Further, if υ is nowhere-zero, then τ_υ is invertible. We denote its inverse by τ_υ^{-1}. It is obtained by replacing β by β_υ^{-1}.

Proposition 3.22. *The functor* $(\mathrm{id}, \tau_\upsilon) : \mathcal{T} \to \mathcal{T}$ *is lax, and* $(\mathrm{id}, \tau_\upsilon) : \mathcal{T}^\vee \to \mathcal{T}^\vee$ *is colax. The same holds with* τ_υ *replaced by* τ_υ^{-1}*.*

Proof. Repeating the proof of Proposition 3.19, we note by the gate property that $G \text{--} F\overline{G} \text{--} A\overline{G}$ and $G \text{--} A\overline{F}G \text{--} A\overline{G}$, and use (1.30c). □

Proposition 3.23. *The functors* $(\mathrm{id}, \tau_\upsilon^{-1}, \mathrm{id})$ *and* $(\mathrm{id}, \mathrm{id}, \tau_\upsilon^{-1})$ *both from*

$$(\mathcal{T}, \mathcal{T}^\vee, \lambda_\upsilon) \to (\mathcal{T}, \mathcal{T}^\vee, \lambda_\upsilon^{-1})$$

are bilax. Similarly, the functors $(\mathrm{id}, \tau_\upsilon, \mathrm{id})$ *and* $(\mathrm{id}, \mathrm{id}, \tau_\upsilon)$ *both from*

$$(\mathcal{T}, \mathcal{T}^\vee, \lambda_\upsilon^{-1}) \to (\mathcal{T}, \mathcal{T}^\vee, \lambda_\upsilon)$$

are bilax.

Explicitly, the first statement says that the diagrams

$$\begin{tikzcd} \mathcal{T}\mathcal{T}^\vee \arrow[r, "\tau_v^{-1}\mathcal{T}^\vee"] \arrow[d, "\lambda_v^{-1}"'] & \mathcal{T}\mathcal{T}^\vee \arrow[d, "\lambda_v"] \\ \mathcal{T}^\vee\mathcal{T} \arrow[r, "\mathcal{T}^\vee\tau_v^{-1}"'] & \mathcal{T}^\vee\mathcal{T} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}\mathcal{T}^\vee \arrow[r, "\mathcal{T}\tau_v^{-1}"] \arrow[d, "\lambda_v"'] & \mathcal{T}\mathcal{T}^\vee \arrow[d, "\lambda_v^{-1}"] \\ \mathcal{T}^\vee\mathcal{T} \arrow[r, "\tau_v^{-1}\mathcal{T}"'] & \mathcal{T}^\vee\mathcal{T} \end{tikzcd}$$

commute.

Proof. We check the first diagram. Repeating the proof of Proposition 3.20, the commutativity reduces to that of

$$\begin{tikzcd} \mathsf{p}[FF'] \arrow[r, "\beta_v^{-1}"] \arrow[d, "\beta_v^{-1}"'] & \mathsf{p}[F'F] \arrow[d, "\beta_v^{-1}"] \\ \mathsf{p}[A\overline{F}F'] \arrow[r, "\beta_v"'] & \mathsf{p}[F'\overline{F}]. \end{tikzcd}$$

By (1.20), there is a minimal gallery $A\overline{F}F' \text{ -- } F'\overline{F} \text{ -- } F'F \text{ -- } FF'$. Now apply (1.30c). ☐

We know that (co, bi)lax functors preserve (co, bi)monoids. The results in Section 2.10 on op and cop constructions can also be deduced by applying this principle to Propositions 3.22 and 3.23.

3.5. Lifting of monads to comonoids

The monad $\mathcal{T}$ on the category of species extends to the category of comonoids. Similarly, the monad $\mathcal{S}$ on the category of species extends to the category of cocommutative comonoids. Recall from Section 2.11 that the latter can be viewed as functor categories. We now explain the monads $\mathcal{T}$ and $\mathcal{S}$ from this perspective.

3.5.1. Monad on the category of comonoids. Recall the finite category $\mathcal{A}\text{-}\mathtt{Hyp}_\mathsf{c}$ from Section 2.11.1. It is generated by two kinds of morphisms, namely, Δ and β, see Proposition 2.59. The subcategory generated by the β is precisely $\mathcal{A}\text{-}\mathtt{Hyp}$. Recall that the category of species $\mathcal{A}\text{-}\mathtt{Sp}$ is the functor category $[\mathcal{A}\text{-}\mathtt{Hyp}, \mathsf{Vec}]$. We now extend the monad $\mathcal{T}$ from the category of species to the category $\mathcal{A}\text{-}\mathtt{Sp}_\mathsf{c} := [\mathcal{A}\text{-}\mathtt{Hyp}_\mathsf{c}, \mathsf{Vec}]$ as follows. Suppose p is a functor from $\mathcal{A}\text{-}\mathtt{Hyp}_\mathsf{c}$ to Vec. Then so is $\mathcal{T}(\mathsf{p})$: For $\Delta : A \to G$, the map

$$\mathcal{T}(\mathsf{p})(\Delta) : \mathcal{T}(\mathsf{p})[A] \to \mathcal{T}(\mathsf{p})[G],$$

on the F-summand, is

$$\mathsf{p}[F] \xrightarrow{\mathsf{p}(\Delta)} \mathsf{p}[FG] \xrightarrow{\mathsf{p}(\beta)} \mathsf{p}[GF].$$

The monad structure of $\mathcal{T}$ is as before.

Proposition 3.24. *The category of $\mathcal{T}$-algebras on $\mathcal{A}\text{-}\mathtt{Sp}_\mathsf{c}$ is equivalent to the category of bimonoids.*

PROOF. Suppose p is a $\mathcal{T}$-algebra. We explain how p is a bimonoid. The coproduct comes from $\mathsf{p}(\Delta)$. The product comes from $\mathcal{T}(\mathsf{p}) \to \mathsf{p}$. Naturality of this map wrt Δ yields the bimonoid axiom (2.12). $\square$

A conceptual understanding of this result is as follows. Recall the bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$ on the category of species. So a $\mathcal{T}$-algebra on the category of $\mathcal{T}^\vee$-coalgebras is the same as a $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$-bialgebra. By Proposition 3.1, this translates to: a $\mathcal{T}$-algebra on the category of comonoids is the same as a bimonoid. By Proposition 2.60, this further translates to Proposition 3.24.

3.5.2. Monad on the category of cocommutative comonoids. Recall the finite category $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}$ from Section 2.11.3. We now extend the monad $\mathcal{S}$ from the category of species to the category $\mathcal{A}\text{-}\mathtt{Sp}_{\mathtt{r}} := [\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}, \mathsf{Vec}]$ as follows. Suppose p is a functor from $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{r}}$ to Vec. Then so is $\mathcal{S}(\mathsf{p})$: For $\Delta : \mathrm{Z} \to \mathrm{Y}$, the map

$$\mathcal{S}(\mathsf{p})(\Delta) : \mathcal{S}(\mathsf{p})[\mathrm{Z}] \to \mathcal{S}(\mathsf{p})[\mathrm{Y}],$$

on the X-summand, is

$$\mathsf{p}[\mathrm{X}] \xrightarrow{\mathsf{p}(\Delta)} \mathsf{p}[\mathrm{X} \vee \mathrm{Y}].$$

The monad structure of $\mathcal{S}$ is as before.

Proposition 3.25. *The category of $\mathcal{S}$-algebras on $\mathcal{A}\text{-}\mathtt{Sp}_{\mathtt{r}}$ is equivalent to the category of bicommutative bimonoids.*

PROOF. The argument is similar to the one given for Proposition 3.24. If p is a $\mathcal{S}$-algebra, its coproduct comes from $\mathsf{p}(\Delta)$, the product comes from $\mathcal{S}(\mathsf{p}) \to \mathsf{p}$, and its naturality in Δ yields (2.26). $\square$

This result can be understood conceptually by considering the bimonad $(\mathcal{S}, \mathcal{S}^\vee, \lambda)$, and using Proposition 2.65.

Exercise 3.26. The category $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}$ is opposite to $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathtt{c}}$. Extend the comonad $\mathcal{T}^\vee$ on the category of species to the category $\mathcal{A}\text{-}\mathtt{Sp}^{\mathtt{d}} := [\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}, \mathsf{Vec}]$ such that the category of $\mathcal{T}^\vee$-coalgebras is equivalent to the category of bimonoids. What other results of this kind can we formulate?

3.6. Monad for partially commutative monoids

The monads $\mathcal{T}$ and $\mathcal{S}$ can be interpolated as follows. For any partial-support relation on faces $\sim$, there is a monad $\mathcal{T}_\sim$ whose algebras are $\sim$-commutative monoids. It specializes to $\mathcal{T}$ when $\sim$ is finest, and to $\mathcal{S}$ when $\sim$ is coarsest. Dually, one can construct a comonad $\mathcal{T}^\vee_\sim$. Moreover, the two constructions can be combined to yield a bimonad $(\mathcal{T}_\sim, \mathcal{T}^\vee_{\sim'})$, where $\sim$ and $\sim'$ are any two partial-support relations. Bialgebras over this bimonad are bimonoids that are $\sim$-commutative and $\sim'$-cocommutative. One can also approach 0-$\sim$-bicommutative bimonoids in this manner.

3.6.1. Monad for partially commutative monoids. Let $\sim$ be a partial-support relation on faces. We now construct a monad $\mathcal{T}_{\sim}$ whose algebras are $\sim$-commutative monoids. The easiest way to proceed is to use the formulation of species in terms of partial-flats given by Proposition 2.82: For a species p, let

$$\mathcal{T}_{\sim}(\mathsf{p})[\mathrm{z}] := \bigoplus_{\mathrm{x}:\, \mathrm{z} \leq \mathrm{x}} \mathsf{p}[\mathrm{x}], \tag{3.32}$$

and now proceed in analogy with (3.1).

The $\sim$-abelianization $\mathcal{T} \to \mathcal{T}_{\sim}$ is defined as follows. We work with the usual definition of species. For any partial-flat x, define

$$\mathsf{p}[\mathrm{x}] := \operatorname*{colim}_{F} \mathsf{p}[F],$$

the colimit being over all faces F which belong to x. The map $\mathcal{T} \to \mathcal{T}_{\sim}$ sends $\mathsf{p}[F]$ to $\mathsf{p}[\mathrm{x}]$, where x is the partial-flat to which F belongs. It is surjective and the usual abelianization factors through it:

$$\begin{tikzcd}
& \mathcal{T}_{\sim} \arrow[dr, two heads] & \\
\mathcal{T} \arrow[ur, two heads] \arrow[rr, two heads] & & \mathcal{S}.
\end{tikzcd}$$

Note that $\sim$-abelianization specializes to usual abelianization when $\sim$ is coarsest.

Dually, there is a comonad $\mathcal{T}^{\vee}_{\sim}$ whose coalgebras are $\sim$-cocommutative comonoids. Similarly, we have the $\sim$-coabelianization $\mathcal{T}^{\vee}_{\sim} \to \mathcal{T}^{\vee}$ which is injective, and which fits in the commutative diagram

$$\begin{tikzcd}
& \mathcal{T}^{\vee}_{\sim} \arrow[dr, hook] & \\
\mathcal{S}^{\vee} \arrow[ur, hook] \arrow[rr, hook] & & \mathcal{T}^{\vee}.
\end{tikzcd}$$

Now we combine the two constructions. For that purpose, we fix two partial-support relations $\sim$ and $\sim'$, and consider abelianization wrt $\sim$ and coabelianization wrt $\sim'$.

Proposition 3.27. *The mixed distributive law λ between $\mathcal{T}$ and $\mathcal{T}^{\vee}$ induces the following commutative diagram.*

$$\begin{tikzcd}
& \mathcal{T}\mathcal{T}^{\vee} \arrow[rr, "\lambda"] \arrow[dd, two heads] & & \mathcal{T}^{\vee}\mathcal{T} \arrow[dd, two heads] \\
\mathcal{T}\mathcal{T}^{\vee}_{\sim'} \arrow[ur, hook] \arrow[rr, dashed] \arrow[dd, two heads] & & \mathcal{T}^{\vee}_{\sim'}\mathcal{T} \arrow[ur, hook] \arrow[dd, two heads] & \\
& \mathcal{T}_{\sim}\mathcal{T}^{\vee} \arrow[rr, dashed] & & \mathcal{T}^{\vee}\mathcal{T}_{\sim} \\
\mathcal{T}_{\sim}\mathcal{T}^{\vee}_{\sim'} \arrow[ur, hook] \arrow[rr, dashed] & & \mathcal{T}^{\vee}_{\sim'}\mathcal{T}_{\sim} \arrow[ur, hook] &
\end{tikzcd}$$

(Compare with (3.15).)

PROOF. The two side faces clearly commute. Let us first understand the back face. Suppose (F_1, G_1) and (F_2, G_2) are such that $F_1 \sim F_2$, $F_2G_1 = G_2$ and $F_1G_2 = G_1$. The corresponding summands $\mathsf{p}[G_1]$ and $\mathsf{p}[G_2]$ of $\mathcal{T}\mathcal{T}^\vee(\mathsf{p})$ have the same image in $\mathcal{T}_\sim\mathcal{T}^\vee(\mathsf{p})$. Since F_1 and F_2 have the same support, observe that: The matrix-component $\lambda_{F_1,G_1,F',F'F_1}$ is nonzero iff the matrix-component $\lambda_{F_2,G_2,F',F'F_2}$ is nonzero. In this situation, by axiom (1.16c), we have $F'F_1 \sim F'F_2$. Thus, the corresponding summands $\mathsf{p}[F'F_1]$ and $\mathsf{p}[F'F_2]$ of $\mathcal{T}^\vee\mathcal{T}(\mathsf{p})$ have the same image in $\mathcal{T}^\vee\mathcal{T}_\sim(\mathsf{p})$. This shows that there is an induced map $\mathcal{T}_\sim\mathcal{T}^\vee(\mathsf{p}) \to \mathcal{T}^\vee\mathcal{T}_\sim(\mathsf{p})$.

One can argue dually for the top face. The existence of $\mathcal{T}_\sim\mathcal{T}^\vee_{\sim'}(\mathsf{p}) \to \mathcal{T}^\vee_{\sim'}\mathcal{T}_\sim(\mathsf{p})$ and the commutativity of the front face and the bottom face can be formally deduced using injectivity and surjectivity of the maps. $\square$

The above diagram yields a mixed distributive law λ between $\mathcal{T}_\sim$ and $\mathcal{T}^\vee_{\sim'}$. It also yields mixed distributive laws between $\mathcal{T}$ and $\mathcal{T}^\vee_{\sim'}$, and between $\mathcal{T}_\sim$ and $\mathcal{T}^\vee$. However, this does not need to be said separately, since these laws can be obtained by specializing $\sim$ or $\sim'$ appropriately. To summarize:

Theorem 3.28. *For any partial-support relations $\sim$ and $\sim'$ on faces, we have a bimonad $(\mathcal{T}_\sim, \mathcal{T}^\vee_{\sim'}, \lambda)$.*

If $\sim$ and $\sim'$ coincide, then the mixed distributive law between $\mathcal{T}_\sim$ and $\mathcal{T}^\vee_\sim$ can be described as follows. Evaluated on a species p, on the z-component, the linear map

$$\bigoplus_{(\mathrm{x},\mathrm{y}):\, \mathrm{z}\leq\mathrm{x}\leq\mathrm{y}} \mathsf{p}[\mathrm{y}] \to \bigoplus_{(\mathrm{x}',\mathrm{y}'):\, \mathrm{z}\leq\mathrm{x}'\leq\mathrm{y}'} \mathsf{p}[\mathrm{y}']$$

has matrix-components

$$\lambda_{\mathrm{x},\mathrm{y},\mathrm{x}',\mathrm{y}'} := \begin{cases} \mathsf{p}[\mathrm{y}] \xrightarrow{\beta_{\mathrm{y}',\mathrm{y}}} \mathsf{p}[\mathrm{y}'] & \text{if } \mathrm{x}\mathrm{x}' = \mathrm{y} \text{ and } \mathrm{x}'\mathrm{x} = \mathrm{y}', \\ 0 & \text{otherwise.} \end{cases} \tag{3.33}$$

(Compare with (3.5).)

Observe that bialgebras over $(\mathcal{T}_\sim, \mathcal{T}^\vee_{\sim'}, \lambda)$ are precisely bimonoids that are $\sim$-commutative and $\sim'$-cocommutative. When $\sim$ and $\sim'$ coincide, using (3.33), bialgebras over $(\mathcal{T}_\sim, \mathcal{T}^\vee_\sim, \lambda)$ can be phrased as in Proposition 2.85.

Exercise 3.29. Check directly that (3.33) is a mixed distributive law. This amounts to repeating the proof of Theorem 3.4 with faces replaced by partial-flats. The crucial fact used in the argument is that partial-flats, just like faces, form a LRB. The general result for LRBs is stated in Theorem 3.44.

3.6.2. Bimonad for 0-$\sim$-bicommutative bimonoids. Let $\sim$ be a partial-support relation on faces. We now define another mixed distributive law λ_0 between $\mathcal{T}_\sim$ and $\mathcal{T}^\vee_\sim$ as follows. Instead of (3.33), the matrix-components are given by

$$(\lambda_0)_{\mathrm{x},\mathrm{y},\mathrm{x}',\mathrm{y}'} := \begin{cases} \beta_{\mathrm{y}',\mathrm{y}} & \text{if } \mathrm{x}\mathrm{x}' = \mathrm{y},\ \mathrm{x}'\mathrm{x} = \mathrm{y}', \text{ and } \mathrm{y} \text{ and } \mathrm{y}' \text{ have an upper bound,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.34}$$

The upper bound is taken in the poset of partial-flats $\Sigma_\sim[\mathcal{A}]$. If the partial-support relation is geometric, then (3.34) takes the following simplified form in view of Lemma 1.8.

$$(\lambda_0)_{\mathrm{x},\mathrm{y},\mathrm{x}',\mathrm{y}'} = \begin{cases} \mathrm{id} & \text{if } \mathrm{xx}' = \mathrm{y} = \mathrm{x}'\mathrm{x} = \mathrm{y}', \\ 0 & \text{otherwise.} \end{cases} \tag{3.35}$$

Observe that when $\sim$ is finest, (3.35) reduces to (3.8), while when $\sim$ is coarsest, it reduces to (3.16).

Theorem 3.30. *The triple $(\mathcal{T}_\sim, \mathcal{T}_\sim^\vee, \lambda_0)$, with λ_0 given by (3.34), is a bimonad on $\mathcal{A}$-species. In particular, $(\mathcal{T}, \mathcal{T}^\vee, \lambda_0)$ with λ_0 as in (3.8), and $(\mathcal{S}, \mathcal{S}^\vee, \lambda)$ with λ as in (3.16) are bimonads on $\mathcal{A}$-species.*

Proof. This can be proved directly by repeating the proof of Theorem 3.4 with faces replaced by partial-flats. In addition, following the same notation, we need to show that H and H'' have an upper bound iff G and G' have an upper bound, and H' and H'' have an upper bound. This can be shown using (1.15).

A conceptual way to understand this is to go to the more general setting of LRB species in Section 3.9. We deform (3.33) by the distance function (1.33) and then apply Theorem 3.45 for LRBs. □

It is clear that bialgebras over this bimonad are 0-$\sim$-bicommutative bimonoids: x and x′ have an upper bound iff xx′ and x′x have an upper bound.

3.6.3. Signed analogue. There is a bimonad $(s\mathcal{T}_\sim, s\mathcal{T}_\sim^\vee, \lambda_{-1})$ whose bialgebras are signed $\sim$-bicommutative signed bimonoids. The construction parallels that of the bimonad $(\mathcal{E}, \mathcal{E}^\vee, \lambda_{-1})$, with flats replaced by partial-flats. One can then generalize (3.21) to obtain a signed analogue of the diagram in Proposition 3.27.

Exercise 3.31. Combine (3.34) with (2.91) to define another mixed distributive law for the monad $s\mathcal{T}_\sim$ and comonad $s\mathcal{T}_\sim^\vee$ such that bialgebras over this bimonad are *signed* 0-$\sim$-*bicommutative signed bimonoids*. Show that this notion interpolates between 0-bimonoids and signed bicommutative signed bimonoids.

3.7. Bimonad for set-species

Recall set-species from Section 2.14. We now construct a monad $\mathcal{T}$, a comonad $\mathcal{T}^\vee$ and a bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$ on the category of set-species. In the contrast to the linear case, the functors $\mathcal{T}$ and $\mathcal{T}^\vee$ are distinct in the set-theoretic case. This is because direct sum is both the product and the coproduct in Vec, while, in Set, the product and coproduct differ: product is cartesian product, while the coproduct is disjoint union.

The linearization functor from set-species to species is bilax wrt the bimonads $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$. This formally explains why the linearization of a (co, bi)monoid in set-species yields a (co, bi)monoid in species.

3.7.1. Monad for set-monoids. Define a functor

$$\mathcal{T} : \mathcal{A}\text{-}\mathtt{SetSp} \to \mathcal{A}\text{-}\mathtt{SetSp}$$

as follows. For a set-species p, let

$$\mathcal{T}(\mathrm{p})[A] := \bigsqcup_{F\colon A \leq F} \mathrm{p}[F]. \tag{3.36}$$

Note that

$$\mathcal{T}\mathcal{T}(\mathrm{p})[A] = \bigsqcup_{F\colon A\leq F} \ \bigsqcup_{G\colon F\leq G} \mathrm{p}[G] = \bigsqcup_{(F,G)\colon A\leq F\leq G} \mathrm{p}[G].$$

Define a natural transformation

$$\mathcal{T}\mathcal{T} \to \mathcal{T}, \qquad \bigsqcup_{(F,G)\colon A\leq F\leq G} \mathrm{p}[G] \to \bigsqcup_{G\colon A\leq G} \mathrm{p}[G]$$

as follows. To specify a map from a disjoint union, we need to specify it on each piece. Accordingly, the map on the (F, G) piece sends $\mathrm{p}[G]$ identically to the set $\mathrm{p}[G]$ appearing in the rhs. There is also an obvious natural transformation

$$\mathrm{id} \to \mathcal{T}, \qquad \mathrm{p}[A] \to \bigsqcup_{F\colon A\leq F} \mathrm{p}[F]$$

given by inclusion. The above maps turn $\mathcal{T}$ into a monad.

3.7.2. Comonad for set-comonoids. Similarly, define a functor

$$\mathcal{T}^\vee : \mathcal{A}\text{-}\mathtt{SetSp} \to \mathcal{A}\text{-}\mathtt{SetSp}$$

as follows. For a set-species p, let

$$\mathcal{T}^\vee(\mathrm{p})[A] := \bigtimes_{F\colon A\leq F} \mathrm{p}[F]. \tag{3.37}$$

We now define a comonad structure on $\mathcal{T}^\vee$. The natural transformation

$$\mathcal{T}^\vee \to \mathcal{T}^\vee\mathcal{T}^\vee, \qquad \bigtimes_{G\colon A\leq G} \mathrm{p}[G] \to \bigtimes_{(F,G)\colon A\leq F\leq G} \mathrm{p}[G]$$

is as follows. To specify a map to a cartesian product, we need to specify it to each of its factors. Accordingly, on the (F, G)-factor of the rhs, the map is the projection on the G-factor of the lhs. The natural transformation

$$\mathcal{T}^\vee \to \mathrm{id}, \qquad \bigtimes_{F\colon A\leq F} \mathrm{p}[F] \to \mathrm{p}[A]$$

projects on the A-factor of the lhs.

3.7.3. Bimonad for set-bimonoids. Mixed distributive law. We now proceed to define a natural transformation

$$\lambda : \mathcal{T}\mathcal{T}^{\vee} \to \mathcal{T}^{\vee}\mathcal{T}.$$

On a set-species p, on the A-component, this entails a map

$$\bigsqcup_{F:\, A \leq F} \Big(\bigtimes_{G:\, F \leq G} \mathrm{p}[G] \Big) \longrightarrow \bigtimes_{F':\, A \leq F'} \Big(\bigsqcup_{G':\, F' \leq G'} \mathrm{p}[G'] \Big). \tag{3.38}$$

Since the map is from a disjoint union to a cartesian product, it suffices to specify it on each component as follows.

$$\begin{tikzcd} \bigtimes\limits_{G:\, F \leq G} \mathrm{p}[G] \arrow[r, dashed] \arrow[d, two heads] & \bigsqcup\limits_{G':\, F' \leq G'} \mathrm{p}[G'] \\ \mathrm{p}[FF'] \arrow[r, "\beta_{F'F,FF'}"'] & \mathrm{p}[F'F], \arrow[u, hook] \end{tikzcd}$$

where the left vertical map is projection, while the right vertical map is inclusion.

Theorem 3.32. *The triple $(\mathcal{T}, \mathcal{T}^{\vee}, \lambda)$ is a bimonad on $\mathcal{A}$-set-species, or equivalently, λ is a mixed distributive law between $\mathcal{T}$ and $\mathcal{T}^{\vee}$.*

Proof. We need to check commutativity of the diagrams (C.9a) and (C.9b). The left diagram in (C.9a) takes the following form.

$$\begin{tikzcd} \bigsqcup\limits_{F} \bigtimes\limits_{G} \bigtimes\limits_{H} \mathrm{p}[H] \arrow[r, "\lambda"] & \bigtimes\limits_{F'} \bigsqcup\limits_{G'} \bigtimes\limits_{H'} \mathrm{p}[H'] \arrow[r, "\lambda"] & \bigtimes\limits_{F'} \bigtimes\limits_{G''} \bigsqcup\limits_{H''} \mathrm{p}[H''] \\ \bigsqcup\limits_{F} \bigtimes\limits_{H} \mathrm{p}[H] \arrow[u] \arrow[rr, "\lambda"'] & & \bigtimes\limits_{G''} \bigsqcup\limits_{H''} \mathrm{p}[H''] \arrow[u] \end{tikzcd}$$

The indexing convention is as follows. A face A is fixed. It is understood that the indexing faces increase from left to right. For instance, in the top-left term, the disjoint union is over all $F \geq A$, the first cartesian product is over all $G \geq F$, while the second cartesian product is over all $H \geq G$.

Observe that in the above diagram, the composite maps are from a disjoint union to a double cartesian product. So let us fix F and $F' \leq G''$, and look at the corresponding component. We claim that this map is given by

$$\begin{tikzcd} \bigtimes\limits_{H:\, F \leq H} \mathrm{p}[H] \arrow[r, dashed] \arrow[d, two heads] & \bigsqcup\limits_{H'':\, G'' \leq H''} \mathrm{p}[H''] \\ \mathrm{p}[FG''] \arrow[r, "\beta_{G''F,FG''}"'] & \mathrm{p}[G''F]. \arrow[u, hook] \end{tikzcd}$$

This is clear going across and up. While going up and across, the horizontal composite can be analyzed as follows.

$$
\begin{tikzcd}
\bigsqcup\limits_{A\leq F} \bigtimes\limits_{F\leq G} \bigtimes\limits_{G\leq H} \mathrm{p}[H] \arrow[r] & \bigtimes\limits_{A\leq F'} \bigsqcup\limits_{F'\leq G'} \bigtimes\limits_{G'\leq H'} \mathrm{p}[H'] \arrow[r] \arrow[d] & \bigtimes\limits_{A\leq F'} \bigtimes\limits_{F'\leq G''} \bigsqcup\limits_{G''\leq H''} \mathrm{p}[H''] \arrow[d] \\
\bigtimes\limits_{F\leq G} \bigtimes\limits_{G\leq H} \mathrm{p}[H] \arrow[r] \arrow[u] \arrow[d] & \bigsqcup\limits_{F'\leq G'} \bigtimes\limits_{G'\leq H'} \mathrm{p}[H'] \arrow[r] & \bigtimes\limits_{F'\leq G''} \bigsqcup\limits_{G''\leq H''} \mathrm{p}[H''] \arrow[d] \\
\bigtimes\limits_{FF'\leq H} \mathrm{p}[H] \arrow[r] \arrow[d] & \bigtimes\limits_{F'F\leq H'} \mathrm{p}[H'] \arrow[r] \arrow[u] \arrow[d] & \bigsqcup\limits_{G''\leq H''} \mathrm{p}[H''] \\
\mathrm{p}[FG''] \arrow[r] & \mathrm{p}[G'G''] \arrow[r] & \mathrm{p}[G''G] \arrow[u]
\end{tikzcd}
$$

The indexing convention is as follows. The disjoint sum or cartesian product is only over the second coordinate. So when we write $F \leq G$, the sum is only over G. In the first row, A is fixed, in the second row, F and F' are also fixed, in the third row, G'' is also fixed. In the passage from the second to the third row, we take $G = FF'$ and $G' = F'F$. The claim follows.

The right diagram in (C.9a) can be verified in a similar manner. The diagrams (C.9b) are straightforward to verify. $\square$

Exercise 3.33. Check that $\mathcal{T}$-algebras, $\mathcal{T}^\vee$-coalgebras, $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$-bialgebras are, respectively, monoids, comonoids, bimonoids in $\mathcal{A}$-set-species.

Exercise 3.34. Construct analogues of the monad $\mathcal{S}$ and the comonad $\mathcal{S}^\vee$ on $\mathcal{A}$-set-species, and show that (3.15) and Proposition 3.13 hold. Write down the analogue of (3.16). What about Theorem 3.28?

3.7.4. Linearization functor. The fact that the linearization functor (2.94) preserves (co, bi)monoids can be understood more formally as follows: For any set-species p, there are maps

$$\mathcal{T}(\Bbbk\mathrm{p}) \xrightarrow{\cong} \Bbbk\mathcal{T}(\mathrm{p}), \qquad \bigoplus_{F\colon A\leq F} \Bbbk\mathrm{p}[F] \to \Bbbk\Big(\bigsqcup_{F\colon A\leq F} \mathrm{p}[F]\Big), \tag{3.39a}$$

$$\Bbbk\mathcal{T}^\vee(\mathrm{p}) \to \mathcal{T}^\vee(\Bbbk\mathrm{p}), \qquad \Bbbk\Big(\bigtimes_{F\colon A\leq F} \mathrm{p}[F]\Big) \to \bigoplus_{F\colon A\leq F} \Bbbk\mathrm{p}[F] \tag{3.39b}$$

defined in the obvious manner. They are natural in p. Here $\mathcal{T}$ and $\mathcal{T}^\vee$ are the monads and comonads defined on both species and set-species.

Proposition 3.35. *The linearization functor equipped with structure maps* (3.39a) *and* (3.39b) *is bilax (in the sense of* (C.10)*).*

This is a routine check. By Proposition C.28, a (co, bi)lax functor preserves (co, bi)algebras. Thus, we may now formally deduce that linearization preserves (co, bi)monoids.

3.7.5. Abelianization. One cannot define (co)kernel of a map of set-species. However, one can define image of a map; (co)abelianization also makes sense. Further, linearization commutes with abelianization, that is, for any set-monoid a,

$$\Bbbk(\mathrm{a}_{ab}) = (\Bbbk\mathrm{a})_{ab}.$$

This follows from the fact that linearization is the left adjoint of the forgetful functor.

Exercise 3.36. Show that linearization does not commute with coabelianization in general.

3.8. Symmetries, braidings, lax braidings

Recall that the relation $\beta^2 = \mathrm{id}$ holds in the category $\mathcal{A}$-$\mathtt{Hyp}$. We now construct a category $\mathcal{A}$-$\mathtt{Hyp}(\mathsf{b})$, where this relation no longer holds but β is still invertible. Going a step further, one may also drop the invertibility requirement on β. This leads to an even more general category $\mathcal{A}$-$\mathtt{Hyp}(\mathsf{lb})$. Recall that functors on $\mathcal{A}$-$\mathtt{Hyp}$ are species. Similarly, we refer to functors on $\mathcal{A}$-$\mathtt{Hyp}(\mathsf{b})$ and $\mathcal{A}$-$\mathtt{Hyp}(\mathsf{lb})$ as b-species and lb-species. These three functor categories are analogous to monoidal categories equipped with a symmetry, braiding and a lax braiding, respectively. This is summarized below and motivates our terminology.

Table 3.2. Symmetries, braidings, lax braidings.

base category	functor category	analogous monoidal-type
$\mathcal{A}$-$\mathtt{Hyp}$	species	symmetry
$\mathcal{A}$-$\mathtt{Hyp}(\mathsf{b})$	b-species	braiding
$\mathcal{A}$-$\mathtt{Hyp}(\mathsf{lb})$	lb-species	lax braiding

Background information on braided monoidal categories can be found in [18, Chapter 1], see also Appendix B.

3.8.1. Base categories. The category $\mathcal{A}$-$\mathtt{Hyp}(\mathsf{lb})$ has a presentation given by generators

$$\beta : F \to G,$$

whenever F and G have the same support, and relations

$$\begin{tikzcd} & G \arrow[dr, "\beta"] & \\ F \arrow[ur, "\beta"] \arrow[rr, "\beta"'] & & H \end{tikzcd} \qquad\qquad (F \xrightarrow{\beta} F) = \mathrm{id},$$

the former whenever $F \,\text{--}\, G \,\text{--}\, H$, and the latter for any F.

Note very carefully the minimal gallery requirement for the first diagram above. Thus, $\beta^2 = (F \xrightarrow{\beta} G \xrightarrow{\beta} F) \neq \mathrm{id}$ in general. So, in contrast to $\mathcal{A}$-$\mathtt{Hyp}$, this category is not a groupoid. To get a groupoid, we need to add inverses. This construction is described below.

The category $\mathcal{A}\text{-}\mathtt{Hyp}(\mathsf{b})$ has a presentation given by generators

$$\beta : F \to G, \qquad \beta^{-1} : F \to G$$

whenever F and G have the same support, and relations

$$\begin{tikzcd} & G \arrow[dr, "\beta"] & \\ F \arrow[ur, "\beta"] \arrow[rr, "\beta"'] & & H \end{tikzcd} \qquad\qquad \begin{tikzcd} & G \arrow[dr, "\beta^{-1}"] & \\ F \arrow[ur, "\beta^{-1}"] \arrow[rr, "\beta^{-1}"'] & & H \end{tikzcd}$$

whenever $F\,\text{--}\,G\,\text{--}\,H$,

$$(F \xrightarrow{\beta} G \xrightarrow{\beta^{-1}} F) = \mathrm{id} = (F \xrightarrow{\beta^{-1}} G \xrightarrow{\beta} F)$$

for any F and G with the same support, and

$$(F \xrightarrow{\beta} F) = \mathrm{id} = (F \xrightarrow{\beta^{-1}} F)$$

for any F.

There are functors

$$\mathcal{A}\text{-}\mathtt{Hyp}(\mathsf{lb}) \to \mathcal{A}\text{-}\mathtt{Hyp}(\mathsf{b}) \to \mathcal{A}\text{-}\mathtt{Hyp} \tag{3.40}$$

which are the identity maps on the objects. The first functor sends β to β, while the second functor sends both β and β^{-1} to β.

The categories $\mathcal{A}\text{-}\mathtt{Hyp}(\mathsf{b})$ and $\mathcal{A}\text{-}\mathtt{Hyp}(\mathsf{lb})$ have a finite number of objects but infinitely many morphisms (assuming the rank of $\mathcal{A}$ to be at least one). For instance, for any G adjacent to F,

$$F \xrightarrow{\mathrm{id}} F, \quad F \xrightarrow{\beta} G \xrightarrow{\beta} F, \quad F \xrightarrow{\beta} G \xrightarrow{\beta} F \xrightarrow{\beta} G \xrightarrow{\beta} F, \quad \dots$$

are distinct morphisms from F to itself. In general, any morphism in $\mathcal{A}\text{-}\mathtt{Hyp}(\mathsf{lb})$ can be represented by a gallery in $\mathcal{A}^{\mathrm{X}}$, where X is the support of the faces in question. Further, the number of times a given hyperplane in $\mathcal{A}^{\mathrm{X}}$ is crossed does not depend on the choice of the gallery.

Exercise 3.37. Give an example of two morphisms in $\mathcal{A}\text{-}\mathtt{Hyp}(\mathsf{lb})$ between chambers say C and D which are distinct but which cross every hyperplane the same number of times.

3.8.2. b-species and lb-species. Recall that we defined species as functors from $\mathcal{A}\text{-}\mathtt{Hyp}$ to Vec. In a similar manner, let

$$\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{b}) := [\mathcal{A}\text{-}\mathtt{Hyp}(\mathsf{b}), \mathsf{Vec}] \quad \text{and} \quad \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{lb}) := [\mathcal{A}\text{-}\mathtt{Hyp}(\mathsf{lb}), \mathsf{Vec}].$$

These are the categories of b-species and lb-species, respectively. The functors (3.40) yield

$$\mathcal{A}\text{-}\mathsf{Sp} \hookrightarrow \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{b}) \hookrightarrow \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{lb}).$$

These are both inclusion functors. A lb-species p is a b-species if the maps $\mathsf{p}[\beta]$ are invertible for all β, and a b-species is a species if $\mathsf{p}[\beta]^2 = \mathrm{id}$ for all β.

The theory of b-species and lb-species is similar to that of species. We will briefly go over some of the basic constructions. One distinction to bear in mind is the following. Species can be defined in very general contexts such as LRBs, but the same is not true for b-species and lb-species since these concepts require the notion of minimal galleries.

3.8.3. Bimonads. The monad $\mathcal{T}$ and comonad $\mathcal{T}^\vee$ defined on the category of species extend to the categories of b-species and lb-species. Theorems 3.4 and 3.7 continue to hold: The main point is that only diagram (2.67) is required and not (2.1). The mixed distributive laws allow us to talk about (co, bi)monoids and also v-bimonoids in b-species and lb-species. The explicit definitions are the same as before.

The monad $\mathcal{S}$ and comonad $\mathcal{S}^\vee$ as functors are defined by

$$\mathcal{S}(\mathsf{p})[A] := \operatorname*{colim}_{F\colon A\leq F} \mathsf{p}[F] \qquad \text{and} \qquad \mathcal{S}^\vee(\mathsf{p})[A] := \lim_{F'\colon A\leq F'} \mathsf{p}[F']. \tag{3.41}$$

The (co)limit is taken over all morphisms involving faces greater than A. Since morphisms exist only between faces of the same support, the (co)limit can be expressed as a direct sum over all flats X which contain A. However, in contrast to (2.7), the limit and colimit can now differ, so we avoid using the notation $\mathsf{p}[\mathrm{X}]$. In any case, now the canonical maps $\beta_{F,\mathrm{X}}$ and $\beta_{\mathrm{X},F}$ in (2.6) will not be isomorphisms in general.

The monad structure on $\mathcal{S}$ entails linear maps

$$\operatorname*{colim}_{F\colon A\leq F}\ \operatorname*{colim}_{G\colon F\leq G} \mathsf{p}[G] \longrightarrow \operatorname*{colim}_{G\colon A\leq G} \mathsf{p}[G] \quad \text{and} \quad \mathsf{p}[A] \longrightarrow \operatorname*{colim}_{F\colon A\leq F} \mathsf{p}[F].$$

These are constructed from the universal property of the colimit. The comonad structure on $\mathcal{S}^\vee$ is defined dually by replacing colimit with limit and reversing arrows. The mixed distributive law between $\mathcal{S}$ and $\mathcal{S}^\vee$ is as follows.

$$\begin{tikzcd}
\operatorname*{colim}_{F\colon A\leq F}\ \lim_{G\colon F\leq G} \mathsf{p}[G] \arrow[rrr, dashed] & & & \lim_{F'\colon A\leq F'}\ \operatorname*{colim}_{G'\colon F'\leq G'} \mathsf{p}[G'] \arrow[d] \\
\lim_{G\colon F\leq G} \mathsf{p}[G] \arrow[u] \arrow[r] & \mathsf{p}[FF'] \arrow[r] & \mathsf{p}[F'F] \arrow[r] & \operatorname*{colim}_{G'\colon F'\leq G'} \mathsf{p}[G'].
\end{tikzcd} \tag{3.42}$$

Note that the functors $\mathcal{T}$ and $\mathcal{T}^\vee$ can also be defined as in (3.41) with the (co)limit taken only over the identity morphisms involving faces greater than A. So, in a way, diagram (3.42) illustrates all four mixed distributive laws of Proposition 3.13 with the (co)limits interpreted appropriately. Thus, they always arise by interchanging a limit and a colimit. In the law for set-species, a product and coproduct were interchanged, see (3.38). The same occurred for species but it was less evident since the product and coproduct for species are both given by direct sum.

The above discussion shows that it makes sense to talk of (co)commutative (co)monoids in b-species and lb-species. The definitions are as in (2.17) and (2.23). Lemmas 2.16 and 2.17 hold but not Proposition 2.20. In short, we have access to all previous results except those that refer to $\mathsf{p}[\mathrm{X}]$.

3.8.4. The inverse braiding. There is an isomorphism of categories

$$\mathcal{A}\text{-}\mathtt{Hyp}(\mathsf{b}) \to \mathcal{A}\text{-}\mathtt{Hyp}(\mathsf{b})$$

which interchanges β and β^{-1}. This induces an endofunctor on the category of b-species denoted

$$\mathcal{F}^{-1} : \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{b}) \to \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{b}), \qquad \mathsf{p} \mapsto \mathsf{p}^{-1}, \tag{3.43}$$

where

$$\mathsf{p}^{-1}[F] = \mathsf{p}[F], \quad \mathsf{p}^{-1}[\beta] = \mathsf{p}[\beta^{-1}], \quad \mathsf{p}^{-1}[\beta^{-1}] = \mathsf{p}[\beta].$$

Clearly, $\mathcal{F}^{-1}$ is invertible and its inverse is itself.

Recall the monad $\mathcal{T}$ and the comonad $\mathcal{T}^\vee$ on b-species. For any b-species p, there is a natural isomorphism

$$\mathcal{T}(\mathsf{p}^{-1}) \xrightarrow{\cong} \mathcal{T}(\mathsf{p})^{-1}$$

of b-species. Thus, we have natural transformations

$$\varphi : \mathcal{T}\mathcal{F}^{-1} \to \mathcal{F}^{-1}\mathcal{T} \qquad \text{and} \qquad \psi : \mathcal{F}^{-1}\mathcal{T}^\vee \to \mathcal{T}^\vee\mathcal{F}^{-1}.$$

They are inverses of each other.

Proposition 3.38. *The functor $(\mathcal{F}^{-1}, \varphi) : \mathcal{T} \to \mathcal{T}$ is lax, and $(\mathcal{F}^{-1}, \psi) : \mathcal{T}^\vee \to \mathcal{T}^\vee$ is colax. More generally, the functor*

$$(\mathcal{F}^{-1}, \varphi, \psi) : (\mathcal{T}, \mathcal{T}^\vee, \lambda) \to (\mathcal{T}, \mathcal{T}^\vee, \lambda^{-1})$$

is bilax. The same holds with positions of λ and λ^{-1} interchanged.

PROOF. The lax and colax checks are straightforward. The bilax axiom (C.10) is equivalent to the statement that λ^{-1} applied to the species p^{-1} equals λ applied to p, under the canonical identification of the components of p and p^{-1}. □

3.8.5. Deforming the braiding. Let υ be any nowhere-zero distance function on $\mathcal{A}$. For any b-species p, let β_υ and β_υ^{-1} denote the families of maps (2.41) and (2.43). Property (1.30c) implies that (2.67) holds with β replaced by β_υ. This yields a b-species p_υ whose F-component is $\mathsf{p}[F]$ and whose structure maps are β_υ and β_υ^{-1}. Thus, we have a functor

$$\mathcal{F}_\upsilon : \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{b}) \to \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{b}), \qquad \mathsf{p} \mapsto \mathsf{p}_\upsilon. \tag{3.44}$$

This is an isomorphism of categories with inverse given by $\mathcal{F}_{\upsilon^-}$, where υ^- is the distance function defined by $(\upsilon^-)_{G,F} := (\upsilon_{G,F})^{-1}$.

For any b-species p, there is a natural isomorphism

$$\mathcal{T}(\mathsf{p}_\upsilon) \xrightarrow{\cong} \mathcal{T}(\mathsf{p})_\upsilon$$

of b-species. This uses (1.30b). Thus, we have natural transformations

$$\varphi : \mathcal{T}\mathcal{F}_\upsilon \to \mathcal{F}_\upsilon\mathcal{T} \qquad \text{and} \qquad \psi : \mathcal{F}_\upsilon\mathcal{T}^\vee \to \mathcal{T}^\vee\mathcal{F}_\upsilon.$$

They are inverses of each other.

Proposition 3.39. *The functor $(\mathcal{F}_\upsilon, \varphi) : \mathcal{T} \to \mathcal{T}$ is lax, and $(\mathcal{F}_\upsilon, \psi) : \mathcal{T}^\vee \to \mathcal{T}^\vee$ is colax. More generally, for any distance function υ', the functor*

$$(\mathcal{F}_\upsilon, \varphi, \psi) : (\mathcal{T}, \mathcal{T}^\vee, \lambda_{\upsilon'}) \to (\mathcal{T}, \mathcal{T}^\vee, \lambda_{\upsilon' \times \upsilon^-})$$

is bilax. The same holds with λ^{-1} instead of λ.

PROOF. The lax and colax checks are straightforward. In addition, we need to check (C.10). The interesting stuff happens in the two λ arrows, the rest are canonical identifications. In the top arrow, we see υ', while in the bottom arrow, we see the Hadamard product of υ', υ and υ^- and the latter two cancel each other out. □

As a consequence: if a is a monoid, then so is a_υ with product components μ_A^F the same as those of a. Similarly, if c is a comonoid, then so is c_υ. Further, if h is a υ'-bimonoid, then h_υ is a $(\upsilon' \times \upsilon^-)$-bimonoid. In particular, if h is a bimonoid, then h_υ is a υ^--bimonoid, and if h is a υ-bimonoid, then h_υ is a bimonoid. These facts are also easy to verify directly.

Exercise 3.40. Check that: The functor $\mathcal{F}_\upsilon$ can be defined for species when υ is log-antisymmetric. Further, in this case, all the above statements hold. In the special case when υ is the signed distance function, we recover the functor $(-)_-$ in (2.49).

3.8.6. Opposite transformation on b-species. The opposite transformation $\tau : \mathcal{T} \to \mathcal{T}$ for b-species is defined as in (3.26). For (3.27), note that by the gate property $A \,\text{--}\, A\overline{F} \,\text{--}\, B\overline{F}$ and $F \,\text{--}\, BF \,\text{--}\, B\overline{F}$, hence the diagram commutes by (2.67). In contrast to species, $\tau^2 \neq \mathrm{id}$ (since β is not a symmetry). However, τ is invertible, its inverse denoted τ^{-1} is obtained by replacing β by β^{-1} in the definition.

More generally, for any nowhere-zero distance function υ, replacing β by β_υ yields a natural transformation $\tau_\upsilon : \mathcal{T} \to \mathcal{T}$. We denote its inverse by τ_υ^{-1}.

Exercise 3.41. Check that: τ^{-1} is the conjugate of τ by (3.43). Similarly, τ_υ is the conjugate of τ by (3.44).

Exercise 3.42. Check that Propositions 3.19 and 3.20 generalize to b-species. Write down the case $\upsilon \equiv 1$ explicitly. Use the previous exercise to deduce the general case from this special case.

Exercise 3.43. Check that: Lemma 2.16 holds for b-species. Deduce that the monad $\mathcal{S}$ and comonad $\mathcal{S}^\vee$ may also be constructed from $\mathcal{T}$ and $\mathcal{T}^\vee$ via the opposite transformation as in Lemma 3.21.

3.9. LRB species

The notion of species can be considered in more general settings than hyperplane arrangements such as left regular bands. We explain this briefly.

3.9.1. LRB species. Let Σ be a left regular band. Define the category Σ-LRB whose objects are elements of Σ, and there is a unique morphism between elements of the same support.

A Σ-*species* is a functor from Σ-LRB to Vec, and a map of Σ-species is a natural transformation between such functors. This defines the category of Σ-species which we denote by Σ-Sp. It is a functor category, and we also write

$$\Sigma\text{-}\mathsf{Sp} := [\Sigma\text{-}\mathsf{LRB}, \mathsf{Vec}].$$

3.9.2. LRB monoids, LRB comonoids, LRB bimonoids. For a left regular band Σ, the notions of monoid, comonoid, bimonoid for Σ-species can be defined using the same diagrams as for $\mathcal{A}$-species, namely, (2.8), (2.10), (2.12). We call them Σ-*monoid*, Σ-*comonoid*, Σ-*bimonoid*, respectively. A similar remark applies to their commutative counterparts. Moreover, by employing the support lattice of the LRB, we see that the alternative descriptions of (co)commutative (co)monoids given in Propositions 2.20 and 2.21 are also valid. One can also consider Σ-υ-bimonoids by taking υ to be a distance function on Σ.

Now consider the special case when Σ equals its support lattice. In this situation, there is no distinction between Σ-monoids and commutative Σ-monoids, and between Σ-comonoids and cocommutative Σ-comonoids. As a consequence, the notions of Σ-bimonoids, commutative Σ-bimonoids, cocommutative Σ-bimonoids, bicommutative Σ-bimonoids all coincide.

3.9.3. $\mathcal{A}$-species. Fix an arrangement $\mathcal{A}$. Let us clarify how the basic examples work.

(1). Let Σ be the Tits monoid of $\mathcal{A}$. Then the category Σ-$\mathtt{LRB}$ coincides with $\mathcal{A}$-$\mathtt{Hyp}$, and the category of Σ-species is the same as the category of $\mathcal{A}$-species. Further, Σ-monoids coincide with $\mathcal{A}$-monoids, Σ-comonoids with $\mathcal{A}$-comonoids, Σ-bimonoids with $\mathcal{A}$-bimonoids, and so on.

(2). Now let Σ be the Birkhoff monoid of $\mathcal{A}$. Then the category Σ-$\mathtt{LRB}$ coincides with $\mathcal{A}$-$\mathtt{Hyp}'$, and the category of Σ-species is equivalent to the category of $\mathcal{A}$-species. Further, Σ-monoids coincide with commutative $\mathcal{A}$-monoids, Σ-comonoids with cocommutative $\mathcal{A}$-comonoids, and Σ-bimonoids with bicommutative $\mathcal{A}$-bimonoids.

(3). This example unifies the previous two. Let $\sim$ be a partial-support relation on faces, and let Σ be the monoid of partial-flats. Then Σ-$\mathtt{LRB}$ coincides with $\mathcal{A}$-$\mathtt{Hyp}_\sim$, and the category of Σ-species is equivalent to the category of $\mathcal{A}$-species. Further, Σ-monoids coincide with $\sim$-commutative $\mathcal{A}$-monoids (Proposition 2.84), Σ-comonoids with $\sim$-cocommutative $\mathcal{A}$-comonoids, and Σ-bimonoids with $\sim$-bicommutative $\mathcal{A}$-bimonoids (Proposition 2.85).

3.9.4. Bimonad on LRB species. For a left regular band Σ, the monads $\mathcal{T}$ and $\mathcal{S}$, and their duals on the category of Σ-species can be constructed in the same way as for $\mathcal{A}$-species. Theorem 3.4 works in this generality and is stated below. The mixed distributive law is defined as in (3.5) with the Tits product replaced by the product in Σ.

Theorem 3.44. *For a left regular band Σ, the triple $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$ is a bimonad on Σ-species.*

When Σ is the Tits monoid, this recovers the bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$ in Theorem 3.4, when Σ is the Birkhoff monoid, this recovers the bimonad $(\mathcal{S}, \mathcal{S}^\vee, \lambda)$ with λ as in (3.16), when Σ is the monoid of partial-flats for a partial-support relation $\sim$, this recovers the bimonad $(\mathcal{T}_\sim, \mathcal{T}_\sim^\vee, \lambda)$ with λ as in (3.33).

Theorem 3.7 also works in the generality of Σ-species and is stated below. It requires the notion of a distance function on a LRB which was briefly mentioned in Section 1.4.6.

Theorem 3.45. *For υ a function on bifaces of a left regular band Σ, the triple $(\mathcal{T}, \mathcal{T}^\vee, \lambda_\upsilon)$ is a bimonad on Σ-species iff υ is a distance function on Σ.*

Apart from Theorem 3.7, another special case worth pointing out is Theorem 3.30. It arises from the distance function (1.33) on the monoid of partial-flats.

3.10. Mesablishvili–Wisbauer

We discuss MW-bimonads; these are bimonads on a category subject to certain additional axioms. The bimonads $(\mathcal{T}, \mathcal{T}^\vee, \lambda_0)$ and $(\mathcal{S}, \mathcal{S}^\vee, \lambda)$ on species are examples of MW-bimonads. By employing the Mesablishvili–Wisbauer ridigity theorem for MW-bimonads, we deduce that: The category of species is equivalent to the category of 0-bimonoids, as well as to the category of bicommutative bimonoids. These are the Loday–Ronco and Leray–Samelson theorems, respectively. These results are developed more carefully later in Chapter 13 (independent of the discussion here).

3.10.1. *MW*-bimonads. Let $\mathcal{V}$ be an endofunctor on a category C which carries the structure of a bimonad. Let us denote it by $(\mathcal{V}, \mu, \iota, \Delta, \epsilon, \lambda)$. We say that $\mathcal{V}$ is a *MW-bimonad* on C if the following diagrams commute.

(3.45a)
$$\begin{tikzcd} \mathcal{V}\mathcal{V} \arrow[r, "\mu"] \arrow[d, "\mathcal{V}\Delta"'] & \mathcal{V} \arrow[r, "\Delta"] & \mathcal{V}\mathcal{V} \\ \mathcal{V}\mathcal{V}\mathcal{V} \arrow[rr, "\lambda\mathcal{V}"'] & & \mathcal{V}\mathcal{V}\mathcal{V} \arrow[u, "\mathcal{V}\mu"'] \end{tikzcd}$$

(3.45b)
$$\begin{tikzcd} \mathcal{V}\mathcal{V} \arrow[r, "\mathcal{V}\epsilon"] \arrow[d, "\mu"'] & \mathcal{V} \arrow[d, "\epsilon"] \\ \mathcal{V} \arrow[r, "\epsilon"'] & \mathrm{id} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{id} \arrow[r, "\iota"] \arrow[d, "\iota"'] & \mathcal{V} \arrow[d, "\Delta"] \\ \mathcal{V} \arrow[r, "\mathcal{V}\iota"'] & \mathcal{V}\mathcal{V} \end{tikzcd}$$

(3.45c)
$$\begin{tikzcd} & \mathcal{V} \arrow[dr, "\epsilon"] & \\ \mathrm{id} \arrow[ur, "\iota"] \arrow[rr, equal] & & \mathrm{id} \end{tikzcd}$$

Example 3.46. Consider the bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda_0)$ on species with λ_0 as in (3.8). (Recall that $\mathcal{T} = \mathcal{T}^\vee$ as functors. Note very carefully that this is not true for set-species. It is true whenever we work with species with values in a category which has biproducts.) We claim that $(\mathcal{T}, \mathcal{T}^\vee, \lambda_0)$ is a MW-bimonad. The first diagram (3.45a) on a species p, evaluated on the A-component, takes the following form.

$$\begin{tikzcd} \bigoplus_{A\le F\le H} \mathsf{p}[H] \arrow[r, "\mu"] \arrow[d, "\mathcal{T}\Delta"'] & \bigoplus_{A\le H} \mathsf{p}[H] \arrow[r, "\Delta"] & \bigoplus_{A\le F'\le H} \mathsf{p}[H] \\ \bigoplus_{A\le F\le G\le H} \mathsf{p}[H] \arrow[rr, "\lambda_0\mathcal{T}"'] & & \bigoplus_{A\le F'\le G\le H} \mathsf{p}[H] \arrow[u, "\mathcal{T}\mu"'] \end{tikzcd}$$

In this diagram, the face A is fixed, and the sums are over the remaining faces. The top-horizontal composite map sends each summand in the lhs to all matching summands in the rhs. Going down, across and up results in the same map. This is because $G = FF' = F'F$ is determined by F and F' in view of (3.8). The remaining diagrams can be verified in a similar manner.

Now consider the bimonad $(\mathcal{S}, \mathcal{S}^\vee, \lambda)$ with λ as in (3.16). By an identical calculation to the one above with flats instead of faces, we see that diagrams (3.45a) – (3.45c) commute. Hence, $(\mathcal{S}, \mathcal{S}^\vee, \lambda)$ is a MW-bimonad.

More generally, for any geometric partial-support relation $\sim$, the bimonad $(\mathcal{T}_\sim, \mathcal{T}_\sim^\vee, \lambda_0)$ with λ_0 as in (3.35) is a MW-bimonad. This recovers the above two cases when $\sim$ is finest and coarsest, respectively.

Exercise 3.47. For a MW-bimonad $\mathcal{V}$, a natural transformation $S : \mathcal{V} \to \mathcal{V}$ is an *antipode* if the following diagrams commute.

$$\begin{tikzcd}
\mathcal{V}\mathcal{V} \arrow[rr, "S\mathcal{V}"] & & \mathcal{V}\mathcal{V} \arrow[d, "\mu"] \\
\mathcal{V} \arrow[u, "\Delta"] \arrow[r, "\epsilon"'] & \mathrm{id} \arrow[r, "\iota"'] & \mathcal{V}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{V}\mathcal{V} \arrow[rr, "\mathcal{V}S"] & & \mathcal{V}\mathcal{V} \arrow[d, "\mu"] \\
\mathcal{V} \arrow[u, "\Delta"] \arrow[r, "\epsilon"'] & \mathrm{id} \arrow[r, "\iota"'] & \mathcal{V}
\end{tikzcd}$$

Check that: The MW-bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda_0)$ has a unique antipode which evaluated on a species p, on the A-component, is given by

$$\bigoplus_{G\colon A \leq G} \mathsf{p}[G] \to \bigoplus_{G\colon A \leq G} \mathsf{p}[G], \qquad x \mapsto (-1)^{\mathrm{rk}(G/A)}\, x \tag{3.46}$$

for $x \in \mathsf{p}[G]$. Similarly, the MW-bimonad $(\mathcal{S}, \mathcal{S}^\vee, \lambda)$ has a unique antipode which evaluated on a species p, on the Z-component, is given by

$$\bigoplus_{\mathrm{X}\colon \mathrm{Z} \leq \mathrm{X}} \mathsf{p}[\mathrm{X}] \to \bigoplus_{\mathrm{X}\colon \mathrm{Z} \leq \mathrm{X}} \mathsf{p}[\mathrm{X}], \qquad x \mapsto \mu(\mathrm{Z}, \mathrm{X})\, x \tag{3.47}$$

for $x \in \mathsf{p}[\mathrm{X}]$. Unify these two formulas using the MW-bimonad $(\mathcal{T}_\sim, \mathcal{T}_\sim^\vee, \lambda_0)$.

Remark 3.48. Consider the bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda_q)$ on species with λ_q as in (3.7). This is not a MW-bimonad in general when $q \neq 0$. To see why, let us go back to the calculation in Example 3.46. The top-horizontal composite map remains the same, but the other composite map changes. This is because we no longer have the condition $FF' = F'F$, hence, along with F', we get $G' = F'F$ and $H' = F'H$ in the bottom-right indexing set. Thus, starting in the $\mathsf{p}[H]$-summand in the top-left, we end up in the $\mathsf{p}[H']$-summand in the top-right.

Exercise 3.49. For any MW-bimonad $\mathcal{V}$, the following diagrams commute.

$$\begin{tikzcd}
& \mathcal{V}\mathcal{V} \arrow[dr, "\lambda"] & \\
\mathcal{V} \arrow[ur, "\mathcal{V}\iota"] \arrow[rr, "\Delta"'] & & \mathcal{V}\mathcal{V}
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{V}\mathcal{V} \arrow[dr, "\mathcal{V}\epsilon"] & \\
\mathcal{V}\mathcal{V} \arrow[ur, "\lambda"] \arrow[rr, "\mu"'] & & \mathcal{V}
\end{tikzcd} \tag{3.48}$$

Deduce this fact using the following diagrams.

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "\mathcal{V}\iota"] \arrow[d, "\mathcal{V}\iota"'] & \mathcal{V}\mathcal{V} \arrow[r, "\mu"] \arrow[d, "\mathcal{V}\Delta"] & \mathcal{V} \arrow[r, "\Delta"] & \mathcal{V}\mathcal{V} \\
\mathcal{V}\mathcal{V} \arrow[r, "\mathcal{V}\mathcal{V}\iota"'] & \mathcal{V}\mathcal{V}\mathcal{V} \arrow[rr, "\lambda\mathcal{V}"'] & & \mathcal{V}\mathcal{V}\mathcal{V} \arrow[u, "\mathcal{V}\mu"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{V}\mathcal{V} \arrow[r, "\mu"] \arrow[d, "\mathcal{V}\Delta"'] & \mathcal{V} \arrow[r, "\Delta"] & \mathcal{V}\mathcal{V} \arrow[r, "\mathcal{V}\epsilon"] & \mathcal{V} \\
\mathcal{V}\mathcal{V}\mathcal{V} \arrow[rr, "\lambda\mathcal{V}"'] & & \mathcal{V}\mathcal{V}\mathcal{V} \arrow[u, "\mathcal{V}\mu"] \arrow[r, "\mathcal{V}\mathcal{V}\epsilon"'] & \mathcal{V}\mathcal{V} \arrow[u, "\mathcal{V}\epsilon"']
\end{tikzcd}$$

Check diagrams (3.48) directly for the MW-bimonads in Example 3.46.

3.10.2. Rigidity of bialgebras over MW-bimonads. Recall the following result from [**680**, Section 5.6].

Theorem 3.50. *Let $\mathcal{V}$ be a MW-bimonad on a category C. Assume that C admits limits or colimits and $\mathcal{V}$ preserves them. Then the following are equivalent.*

(1) *$\mathcal{V}$ has an antipode.*
(2) $(\mathrm{id}\,\mu)(\Delta\,\mathrm{id}) : \mathcal{V}\mathcal{V} \to \mathcal{V}\mathcal{V}$ *is an isomorphism.*
(3) $(\mu\,\mathrm{id})(\mathrm{id}\,\Delta) : \mathcal{V}\mathcal{V} \to \mathcal{V}\mathcal{V}$ *is an isomorphism.*
(4) *The functor $A \mapsto \mathcal{V}A$ is an equivalence from C to the category of $\mathcal{V}$-bialgebras over C.*

We call this the *Mesablishvili–Wisbauer rigidity theorem*.

Example 3.51. We continue the discussion in Example 3.46. Recall the MW-bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda_0)$. Let us first analyze the map $(\mathrm{id}\,\mu)(\Delta\,\mathrm{id})$. Evaluated on a species p, on the A-component, it is given by

$$\bigoplus_{A \leq G \leq H} \mathsf{p}[H] \to \bigoplus_{A \leq F \leq G \leq H} \mathsf{p}[H] \to \bigoplus_{A \leq F \leq H} \mathsf{p}[H],$$

with the matrix components equal to id when $F \leq G$, and zero otherwise. Thus, the map is unitriangular, and hence invertible. The companion map $(\mu\,\mathrm{id})(\mathrm{id}\,\Delta)$ is given by

$$\bigoplus_{A \leq F \leq H} \mathsf{p}[H] \to \bigoplus_{A \leq F \leq G \leq H} \mathsf{p}[H] \to \bigoplus_{A \leq G \leq H} \mathsf{p}[H],$$

with the matrix components equal to id when $F \leq G$, and zero otherwise. Thus, the map is unitriangular, and hence invertible. (Note very carefully the distinction from the previous map.) The inverse maps can be explicitly written in terms of the Möbius function of the poset of faces (which we recall only takes values ± 1). Note very carefully that it appears in the antipode formula (3.46).

For the same reason, the maps $(\mathrm{id}\,\mu)(\Delta\,\mathrm{id})$ and $(\mu\,\mathrm{id})(\mathrm{id}\,\Delta)$ are invertible for the MW-bimonad $(\mathcal{S}, \mathcal{S}^\vee, \lambda)$, and more generally for $(\mathcal{T}_\sim, \mathcal{T}_\sim^\vee, \lambda_0)$. For $(\mathcal{S}, \mathcal{S}^\vee, \lambda)$, the inverse maps can be written in terms of the Möbius function of the lattice of flats. Note very carefully that it appears in the antipode formula (3.47).

A related calculation is given in the exercise below.

Exercise 3.52. Let (a, μ) be a monoid, and $\iota_{\mathsf{a}} : \mathsf{a} \hookrightarrow \mathcal{T}^\vee(\mathsf{a})$ the canonical inclusion. Check that the composite

$$\mathcal{T}(\mathsf{a}) \xrightarrow{\mathcal{T}(\iota_{\mathsf{a}})} \mathcal{T}\mathcal{T}^\vee(\mathsf{a}) \xrightarrow{\lambda_0} \mathcal{T}^\vee\mathcal{T}(\mathsf{a}) \xrightarrow{\mathcal{T}^\vee(\mu)} \mathcal{T}^\vee(\mathsf{a})$$

evaluated on the A-component, on the G-summand, is given by the formula $\sum_{F:\, A \leq F \leq G} \mu_F^G$. In particular, deduce that it is an isomorphism.

Let (a, μ) be a commutative monoid, and $\iota_{\mathsf{a}} : \mathsf{a} \hookrightarrow \mathcal{S}^\vee(\mathsf{a})$ the canonical inclusion. Check that the composite

$$\mathcal{S}(\mathsf{a}) \xrightarrow{\mathcal{S}(\iota_{\mathsf{a}})} \mathcal{S}\mathcal{S}^\vee(\mathsf{a}) \xrightarrow{\lambda} \mathcal{S}^\vee\mathcal{S}(\mathsf{a}) \xrightarrow{\mathcal{S}^\vee(\mu)} \mathcal{S}^\vee(\mathsf{a})$$

evaluated on the Z-component, on the Y-summand, is given by the formula $\sum_{X:\, Z \le X \le Y} \mu_X^Y$. In particular, deduce that it is an isomorphism.

Theorem 3.50 when applied to Example 3.51 yields:

Theorem 3.53. *We have:*

(1) *The functor* $\mathcal{T}_0$ *is an equivalence between the categories of species and* 0*-bimonoids.*
(2) *The functor* $\mathcal{S}$ *is an equivalence between the categories of species and bicommutative bimonoids.*
(3) *For any geometric partial-support relation* $\sim$*, the functor* $\mathcal{T}_{0,\sim}$ *is an equivalence between the categories of species and* 0*-*$\sim$*-bicommutative bimonoids.*

The functor $\mathcal{T}_0$ is made explicit later in Section 6.4.3, the functor $\mathcal{S}$ in Section 6.5.1, the functor $\mathcal{T}_{0,\sim}$ in Section 6.11.2. (The notations $\mathcal{T}_0$ and $\mathcal{T}_{0,\sim}$ are used to emphasize the role of the parameter $q = 0$.) Item (1) is part of the Loday–Ronco theorem which is discussed in detail in Section 13.1, see Theorems 13.2 and 13.8. Similarly, item (2) is part of the Leray–Samelson theorem which is discussed in Section 13.2, see Theorems 13.11 and 13.21. Item (3) which unifies items (1) and (2) is part of Theorem 13.73.

Remark 3.54. It is true that for q not a root of unity, the categories of species and q-bimonoids are equivalent. This is part of the rigidity theorem for q-bimonoids which is discussed in Section 13.6. The case $q = 0$ recovers the Loday–Ronco theorem. However, in view of Remark 3.48, unlike Loday–Ronco, the rigidity theorem for q-bimonoids cannot be deduced from Theorem 3.50. It requires a more general approach, see for instance [**681**, Theorem 2.12].

Notes

Bimonads on species. The bimonads on species and the bilax functors between them which are constructed in this chapter are new. The notion of a distance function on a LRB was introduced in [**21**, Appendix E.2]. Their relevance to bimonads is brought forth by Theorem 3.45.

Mesablishvili–Wisbauer. Historical information about bimonads and mixed distributive laws is given in the Notes to Appendix C. The notion of MW-bimonads was introduced by Wisbauer [**912**, Section 5.13] under the name 'mixed bimonads'. It was developed further by Mesablishvili and Wisbauer [**680**, Definition 4.1], [**681**, Section 3]. In these references, MW-bimonads are simply called bimonads. The definition of the antipode in Exercise 3.47 is given in [**680**, Definition 5.2]. Theorem 3.50 is given in [**680**, Section 5.6], with a more general result in [**681**, Section 3.1]. Further results have been obtained by Livernet, Mesablishvili, Wisbauer [**603**]. For instance, Theorem 3.53 can also be deduced from [**603**, Theorem 5.8]. The related result in Exercise 3.49 is present in [**603**, Proposition 5.3 and Remark 5.4]. The result in Remark 3.54 fits into the framework of [**603**, Theorem 4.1]. In this reference, the term 'monad-comonad triple' is used for a bimonad whose monad and comonad have the same underlying functor, while MW-bimonads are called bimonads.

CHAPTER 4

Operads

This chapter assumes some basic familiarity with monoidal categories. An adequate reference is [**18**, Chapters 1 and 3], see also Appendix B. Monoids in a monoidal category are defined in [**18**, Section 1.2].

Let $\mathcal{A}$ be an arbitrary but fixed hyperplane arrangement. We introduce the notion of an $\mathcal{A}$-dispecies. The category of $\mathcal{A}$-dispecies carries a monoidal structure which we call the substitution product. Monoids wrt this product are called $\mathcal{A}$-operads. We describe the free $\mathcal{A}$-operad on an $\mathcal{A}$-dispecies. We then discuss operad presentations with an emphasis on binary quadratic operads. Apart from the substitution product, the category of $\mathcal{A}$-dispecies also carries the Hadamard product which turns it into a 2-monoidal category. Hopf operads are bimonoids in this 2-monoidal category. These ideas play a key role in the construction of the black and white circle products on binary quadratic operads.

We discuss three main examples of $\mathcal{A}$-operads, namely, commutative, associative, Lie. These are denoted **Com**, **As**, **Lie**, respectively. These are all binary quadratic. Further, under a suitable notion of quadratic duality, **Com** and **Lie** are dual to each other, while **As** is self-dual. These can be viewed as extensions of well-known facts from the classical theory of May operads.

The category of $\mathcal{A}$-species is a left module category over the monoidal category of $\mathcal{A}$-dispecies (under the substitution product). Thus, each $\mathcal{A}$-operad gives rise to a monad on $\mathcal{A}$-species. The associative operad **As** yields the monad $\mathcal{T}$, while the commutative operad **Com** yields the monad $\mathcal{S}$. Further, we have left modules over an $\mathcal{A}$-operad which are the same as algebras over the corresponding monad. Thus, a left **As**-module is the same as an $\mathcal{A}$-monoid, while a left **Com**-module is the same as a commutative $\mathcal{A}$-monoid. We mention that a left **Lie**-module is the same as an $\mathcal{A}$-Lie monoid in species. These objects are studied in detail in Chapter 16.

The notion dual to an $\mathcal{A}$-operad is that of an $\mathcal{A}$-cooperad. The latter is a comonoid wrt the substitution product. Moreover, one has the notion of an $\mathcal{A}$-bioperad which consists of an $\mathcal{A}$-operad, an $\mathcal{A}$-cooperad, and a mixed distributive law between them. Every $\mathcal{A}$-bioperad gives rise to a bimonad on $\mathcal{A}$-species. Moreover, left bimodules over an $\mathcal{A}$-bioperad are the same as bialgebras over the corresponding bimonad. The main example is the $\mathcal{A}$-bioperad consisting of the associative operad **As**, its dual associative cooperad $\mathbf{As}^*$, and a suitable mixed distributive law between them. This $\mathcal{A}$-bioperad yields the bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$ and its left bimodules are precisely $\mathcal{A}$-bimonoids.

We also introduce the signed commutative operad denoted $\mathbf{Com}^-$ which is constructed out of the signature space of an arrangement. This yields the monad $\mathcal{E}$, and thus a left $\mathbf{Com}^-$-module is the same as a signed commutative $\mathcal{A}$-monoid. In fact, every operad $\mathbf{a}$ has a signed partner denoted $\mathbf{a}^-$ obtained by taking Hadamard product of $\mathbf{a}$ with $\mathbf{Com}^-$. The operads $\mathbf{As}$ and $\mathbf{As}^-$ are canonically isomorphic which explains why we do not talk of signed $\mathcal{A}$-monoids. In the Lie case, the operad $\mathbf{Lie}^-$ gives rise to signed $\mathcal{A}$-Lie monoids.

Another important example of an operad is the orientation operad $\mathbf{Com}^{\mathbf{o}}$ which is constructed out of the orientation space of an arrangement. In fact, every operad $\mathbf{a}$ has a oriented partner denoted $\mathbf{a}^{\mathbf{o}}$ obtained by taking Hadamard product of $\mathbf{a}$ with $\mathbf{Com}^{\mathbf{o}}$.

To every operad, one can attach an (associative) algebra called its incidence algebra. The incidence algebra of the commutative operad is the flat-incidence algebra, of the associative operad is the lune-incidence algebra, and of the Lie operad is the Tits algebra. The incidence algebra of any connected quadratic operad is elementary and its quiver can be explicitly described.

Operads can also be defined in the more general setting of left regular bands. Interestingly, the commutative, associative, Lie operads extend to this setting.

4.1. Dispecies

We introduce dispecies. As suggested by the notation, they are similar to species. Recall that a species attaches a vector space to each flat. In comparison and contrast, a dispecies attaches a vector space to each pair of flats, one contained inside the other. Since this concept depends on an arrangement $\mathcal{A}$, we start off by writing $\mathcal{A}$-dispecies. However, later, for convenience, we will usually drop $\mathcal{A}$ from the notation.

4.1.1. Dispecies. Let $\mathcal{A}$-$\mathtt{dHyp}$ denote the discrete category whose objects are nested flats, that is, pairs (X, Y) of flats with $\mathrm{X} \leq \mathrm{Y}$. (The only morphisms are identities.)

An $\mathcal{A}$-*dispecies* is a functor

$$\mathbf{p} : \mathcal{A}\text{-}\mathtt{dHyp} \to \mathsf{Vec}.$$

A map of $\mathcal{A}$-dispecies $\mathbf{p} \to \mathbf{q}$ is a natural transformation. This defines the category of $\mathcal{A}$-dispecies which we denote by $\mathcal{A}$-$\mathtt{dSp}$. It is a functor category, and we also write

$$\mathcal{A}\text{-}\mathtt{dSp} = [\mathcal{A}\text{-}\mathtt{dHyp}, \mathsf{Vec}].$$

The value of an $\mathcal{A}$-dispecies $\mathbf{p}$ on an object (X, Y) will be denoted $\mathbf{p}[\mathrm{X}, \mathrm{Y}]$. Using these components, one may say more directly:

An $\mathcal{A}$-dispecies $\mathbf{p}$ consists of a family of vector spaces $\mathbf{p}[\mathrm{X}, \mathrm{Y}]$, one for each $\mathrm{X} \leq \mathrm{Y}$. A map of $\mathcal{A}$-dispecies $f : \mathbf{p} \to \mathbf{q}$ consists of a family of linear maps

$$f_{\mathrm{X},\mathrm{Y}} : \mathbf{p}[\mathrm{X}, \mathrm{Y}] \to \mathbf{q}[\mathrm{X}, \mathrm{Y}],$$

one for each $\mathrm{X} \leq \mathrm{Y}$.

An $\mathcal{A}$-dispecies $\mathbf{p}$ is *finite-dimensional* if the vector spaces $\mathbf{p}[\mathrm{X}, \mathrm{Y}]$ have finite dimension for all $\mathrm{X} \leq \mathrm{Y}$.

4.1.2. Dispecies and species. Dispecies and species are closely related. Compare and contrast the above discussion with the approach to species given by Proposition 2.5.

Any $\mathcal{A}$-dispecies $\mathbf{p}$ gives rise to an $\mathcal{A}$-species p by fixing the second coordinate of the nested flat to be the maximum flat. In other words, $\mathsf{p}[\mathrm{X}] := \mathbf{p}[\mathrm{X}, \top]$.

4.1.3. Product and coproduct. The *zero dispecies* $\mathbf{0}$ is the $\mathcal{A}$-dispecies all of whose components are zero, namely,

$$\mathbf{0}[\mathrm{X}, \mathrm{Y}] = 0. \tag{4.1}$$

This is the initial and terminal object in the category of $\mathcal{A}$-dispecies.

For $\mathcal{A}$-dispecies $\mathbf{p}$ and $\mathbf{q}$, their *direct sum* $\mathbf{p} + \mathbf{q}$ is defined by

$$(\mathbf{p} + \mathbf{q})[\mathrm{X}, \mathrm{Y}] := \mathbf{p}[\mathrm{X}, \mathrm{Y}] \oplus \mathbf{q}[\mathrm{X}, \mathrm{Y}]. \tag{4.2}$$

This is the product and coproduct in the category of $\mathcal{A}$-dispecies. It is clear that arbitrary products and coproducts also exist in this category.

4.2. Operads

We introduce the substitution product on dispecies. A monoid wrt this product is an operad. The notion dual to an operad is that of a cooperad. The latter is a comonoid wrt the substitution product. There is a duality functor on dispecies which interchanges operads and cooperads. Notions of subdispecies and quotient dispecies can be defined just as for species. Moreover, one can employ ideals of operads to form quotient operads.

4.2.1. Substitution product. Let $\mathbf{p}$ and $\mathbf{q}$ be two $\mathcal{A}$-dispecies. Define a new $\mathcal{A}$-dispecies $\mathbf{p} \circ \mathbf{q}$ by

$$(\mathbf{p} \circ \mathbf{q})[\mathrm{X}, \mathrm{Z}] := \bigoplus_{\mathrm{Y} \colon \mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}} \mathbf{p}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{q}[\mathrm{Y}, \mathrm{Z}]. \tag{4.3}$$

The sum is over all flats Y which lie between X and Z.

We refer to this operation as the *substitution product* of $\mathbf{p}$ and $\mathbf{q}$. This construction is natural in $\mathbf{p}$ and $\mathbf{q}$, that is, maps $\mathbf{p} \to \mathbf{p}'$ and $\mathbf{q} \to \mathbf{q}'$ induce a map $\mathbf{p} \circ \mathbf{q} \to \mathbf{p}' \circ \mathbf{q}'$. This yields a monoidal structure on $\mathcal{A}$-$\mathtt{dSp}$. The unit object is the $\mathcal{A}$-dispecies $\mathbf{x}$ defined by

$$\mathbf{x}[\mathrm{X}, \mathrm{Y}] = \begin{cases} \Bbbk & \text{if } \mathrm{X} = \mathrm{Y}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

The (unbracketed) substitution product of three dispecies $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ can be written as

$$(\mathbf{p} \circ \mathbf{q} \circ \mathbf{r})[\mathrm{X}, \mathrm{W}] := \bigoplus_{\mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z} \leq \mathrm{W}} \mathbf{p}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{q}[\mathrm{Y}, \mathrm{Z}] \otimes \mathbf{r}[\mathrm{Z}, \mathrm{W}],$$

with the sum being over Y and Z. This consideration readily extends to a finite number of dispecies. Let

$$\mathbf{p}^{\circ n} := \underbrace{\mathbf{p} \circ \mathbf{p} \circ \cdots \circ \mathbf{p}}_{n}$$

be the n-fold substitution product of $\mathbf{p}$ with itself. By convention, $\mathbf{p}^{\circ 0} := \mathbf{x}$.

Observe that

$$(\mathbf{p}_1 + \mathbf{p}_2) \circ \mathbf{q} \cong \mathbf{p}_1 \circ \mathbf{q} + \mathbf{p}_2 \circ \mathbf{q} \quad \text{and} \quad \mathbf{p} \circ (\mathbf{q}_1 + \mathbf{q}_2) \cong \mathbf{p} \circ \mathbf{q}_1 + \mathbf{p} \circ \mathbf{q}_2. \tag{4.5}$$

This is also true for arbitrary sums:

$$\big(\underset{i}{+} \mathbf{p}_i \big) \circ \mathbf{q} \cong \underset{i}{+} (\mathbf{p}_i \circ \mathbf{q}) \quad \text{and} \quad \mathbf{p} \circ \big(\underset{i}{+} \mathbf{q}_i \big) \cong \underset{i}{+} (\mathbf{p} \circ \mathbf{q}_i).$$

Thus, $(\mathcal{A}\text{-}\mathtt{dSp}, \circ)$ is an abelian monoidal category whose tensor product distributes over the coproduct on either side.

4.2.2. Operads. An $\mathcal{A}$-*operad* is a monoid in the monoidal category of $\mathcal{A}$-dispecies $(\mathcal{A}\text{-}\mathtt{dSp}, \circ, \mathbf{x})$. In other words, an $\mathcal{A}$-operad is an $\mathcal{A}$-dispecies $\mathbf{a}$ equipped with maps

$$\mathbf{a} \circ \mathbf{a} \to \mathbf{a} \quad \text{and} \quad \mathbf{x} \to \mathbf{a}$$

which satisfy the associativity and unitality axioms. A morphism of $\mathcal{A}$-operads is a morphism of monoids. We denote the category of $\mathcal{A}$-operads by $\mathcal{A}\text{-}\mathsf{Op}$.

Explicitly, an $\mathcal{A}$-operad is an $\mathcal{A}$-dispecies $\mathbf{a}$ equipped with linear maps

$$\gamma : \mathbf{a}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y}, \mathrm{Z}] \to \mathbf{a}[\mathrm{X}, \mathrm{Z}] \quad \text{and} \quad \eta : \Bbbk \to \mathbf{a}[\mathrm{X}, \mathrm{X}], \tag{4.6}$$

the former for each $\mathrm{X} \le \mathrm{Y} \le \mathrm{Z}$ and the latter for each X, subject to the following axioms.

Associativity. For any $\mathrm{X} \le \mathrm{Y} \le \mathrm{Z} \le \mathrm{W}$, the diagram

$$\begin{tikzcd}
\mathbf{a}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y},\mathrm{Z}] \otimes \mathbf{a}[\mathrm{Z},\mathrm{W}] \arrow[r, "\gamma\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\gamma"'] & \mathbf{a}[\mathrm{X},\mathrm{Z}] \otimes \mathbf{a}[\mathrm{Z},\mathrm{W}] \arrow[d, "\gamma"] \\
\mathbf{a}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y},\mathrm{W}] \arrow[r, "\gamma"'] & \mathbf{a}[\mathrm{X},\mathrm{W}]
\end{tikzcd} \tag{4.7a}$$

commutes.

Unitality. For any $\mathrm{X} \le \mathrm{Y}$, the diagrams

$$\begin{tikzcd}
& \mathbf{a}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y},\mathrm{Y}] \arrow[dr, "\gamma"] & \\
\mathbf{a}[\mathrm{X},\mathrm{Y}] \otimes \Bbbk \arrow[ur, "\mathrm{id}\otimes\eta"] \arrow[rr, "\cong"'] & & \mathbf{a}[\mathrm{X},\mathrm{Y}]
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbf{a}[\mathrm{X},\mathrm{X}] \otimes \mathbf{a}[\mathrm{X},\mathrm{Y}] \arrow[dr, "\gamma"] & \\
\Bbbk \otimes \mathbf{a}[\mathrm{X},\mathrm{Y}] \arrow[ur, "\eta\otimes\mathrm{id}"] \arrow[rr, "\cong"'] & & \mathbf{a}[\mathrm{X},\mathrm{Y}]
\end{tikzcd} \tag{4.7b}$$

commute.

We refer to γ as the *substitution map* of $\mathbf{a}$, and to η as the *unit map* of $\mathbf{a}$. We also refer to both of them as the structure maps of $\mathbf{a}$.

A morphism of $\mathcal{A}$-operads is a map $f : \mathbf{a} \to \mathbf{b}$ of $\mathcal{A}$-dispecies such that the diagrams

(4.8)
$$\begin{tikzcd} \mathbf{a}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y},\mathrm{Z}] \arrow[r, "\gamma"] \arrow[d, "f_{\mathrm{X},\mathrm{Y}} \otimes f_{\mathrm{Y},\mathrm{Z}}"'] & \mathbf{a}[\mathrm{X},\mathrm{Z}] \arrow[d, "f_{\mathrm{X},\mathrm{Z}}"] \\ \mathbf{b}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{b}[\mathrm{Y},\mathrm{Z}] \arrow[r, "\gamma"'] & \mathbf{b}[\mathrm{X},\mathrm{Z}] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{a}[\mathrm{X},\mathrm{X}] \arrow[rr, "f_{\mathrm{X},\mathrm{X}}"] & & \mathbf{b}[\mathrm{X},\mathrm{X}] \\ & \Bbbk \arrow[ul, "\eta"] \arrow[ur, "\eta"'] & \end{tikzcd}$$

commute.

Example 4.1. Let $\mathcal{A}$ be an arrangement of rank zero. There is only one flat in $\mathcal{A}$, namely, $\perp$. The functor

$$(4.9) \qquad (\mathcal{A}\text{-}\mathtt{dSp}, \circ) \to (\mathsf{Vec}, \otimes), \qquad \mathbf{p} \mapsto \mathbf{p}[\perp,\perp],$$

is an isomorphism of monoidal categories. By passing to the categories of monoids, we deduce that the category of $\mathcal{A}$-operads is isomorphic to the category of (associative) algebras. For an operad $\mathbf{a}$, the product and unit of the corresponding algebra are given by

$$\gamma : \mathbf{a}[\perp,\perp] \otimes \mathbf{a}[\perp,\perp] \to \mathbf{a}[\perp,\perp] \quad \text{and} \quad \eta : \Bbbk \to \mathbf{a}[\perp,\perp],$$

respectively.

4.2.3. Operad as a linear category. Recall that a *linear category* is a category enriched in the monoidal category of vector spaces (under tensor product).

Lemma 4.2. *An $\mathcal{A}$-operad determines a linear category whose objects are flats of $\mathcal{A}$. More precisely, for an $\mathcal{A}$-operad $\mathbf{a}$, the vector space $\mathbf{a}[\mathrm{X},\mathrm{Y}]$ is the space of morphisms from Y to X.*

Proof. The structure maps (4.6) define composition of morphisms and the identity morphisms in the category, and (4.7a) and (4.7b) are precisely the associativity and unitality axioms. $\square$

4.2.4. Ideals and quotients. Subdispecies and quotient dispecies can be defined as in the case of species (Section 2.6). Similarly, for any map of dispecies $f : \mathbf{q} \to \mathbf{p}$, we have the notions of injectivity, surjectivity, kernel, cokernel, image, coimage.

Suppose $\mathbf{a}$ is an operad. A subdispecies $\mathbf{q}$ of $\mathbf{a}$ is an *ideal* of $\mathbf{a}$ if it is preserved by the substitution map of $\mathbf{a}$, that is, for any $\mathrm{X} \le \mathrm{Y} \le \mathrm{Z}$, there are induced maps

$$\mathbf{a}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{q}[\mathrm{Y},\mathrm{Z}] \to \mathbf{q}[\mathrm{X},\mathrm{Z}] \quad \text{and} \quad \mathbf{q}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y},\mathrm{Z}] \to \mathbf{q}[\mathrm{X},\mathrm{Z}].$$

For any subdispecies $\mathbf{q}$, there is a smallest ideal of $\mathbf{a}$ which contains $\mathbf{q}$. It is obtained by intersecting all ideals of $\mathbf{a}$ which contain $\mathbf{q}$. We call this the ideal generated by $\mathbf{q}$.

Now suppose $\mathbf{q}$ is an ideal of $\mathbf{a}$. Then the quotient dispecies $\mathbf{a}/\mathbf{q}$ is an operad, and the quotient map $\mathbf{a} \to \mathbf{a}/\mathbf{q}$ is a morphism of operads: The substitution map of $\mathbf{a}/\mathbf{q}$ is the dotted arrow in the diagram

$$\begin{tikzcd}
\mathbf{a}[X,Y] \otimes \mathbf{a}[Y,Z] \arrow[r] \arrow[d, two heads] & \mathbf{a}[X,Z] \arrow[d, two heads] \\
(\mathbf{a}/\mathbf{q})[X,Y] \otimes (\mathbf{a}/\mathbf{q})[Y,Z] \arrow[r, dotted] & (\mathbf{a}/\mathbf{q})[X,Z].
\end{tikzcd}$$

The kernel of the left-vertical map is

$$\mathbf{a}[X,Y] \otimes \mathbf{q}[Y,Z] + \mathbf{q}[X,Y] \otimes \mathbf{a}[Y,Z].$$

Since $\mathbf{q}$ is an ideal, the top-horizontal map takes this subspace to $\mathbf{q}[X,Z]$, which is the kernel of the right-vertical map. This yields the dotted arrow.

The unit map of $\mathbf{a}/\mathbf{q}$ is defined by the composite map

$$\Bbbk \to \mathbf{a}[X,X] \twoheadrightarrow (\mathbf{a}/\mathbf{q})[X,X].$$

4.2.5. Cooperads. Dually, an $\mathcal{A}$-*cooperad* is a comonoid in the monoidal category of $\mathcal{A}$-dispecies $(\mathcal{A}\text{-}\mathtt{dSp}, \circ, \mathbf{x})$. A morphism of $\mathcal{A}$-cooperads is a morphism of comonoids.

Explicitly, an $\mathcal{A}$-cooperad consists of an $\mathcal{A}$-dispecies $\mathbf{c}$ equipped with linear maps

$$\mathbf{c}[X,Z] \to \mathbf{c}[X,Y] \otimes \mathbf{c}[Y,Z] \quad \text{and} \quad \mathbf{c}[X,X] \to \Bbbk$$

subject to the coassociativity and counitality axioms.

4.2.6. Duality between operads and cooperads. A duality functor can be defined on any functor category $[\mathsf{C}, \mathsf{Vec}]$ whenever C is a groupoid. This has been discussed before for the category of species where C was $\mathcal{A}\text{-}\mathtt{Hyp}$ (Section 2.9). Since $\mathcal{A}\text{-}\mathtt{dHyp}$ is a groupoid, we also have a *duality functor* on the category of $\mathcal{A}$-dispecies. Explicitly, the dual of $\mathbf{p}$, denoted $\mathbf{p}^*$, is defined by

$$\mathbf{p}^*[X,Y] := \mathbf{p}[X,Y]^*.$$

Let $\mathbf{p}$ and $\mathbf{q}$ be $\mathcal{A}$-dispecies. There is a canonical inclusion

$$\begin{aligned}
(\mathbf{p}^* \circ \mathbf{q}^*)[X,Z] = \bigoplus_{Y:\, X \leq Y \leq Z} & \mathbf{p}[X,Y]^* \otimes \mathbf{q}[Y,Z]^* \\
& \quad \big\downarrow \\
\Big(\bigoplus_{Y:\, X \leq Y \leq Z} & \mathbf{p}[X,Y] \otimes \mathbf{q}[Y,Z] \Big)^* = (\mathbf{p} \circ \mathbf{q})^*[X,Z]
\end{aligned}$$

and an identification $\mathbf{x}^* = \mathbf{x}$. These turn the duality functor into a lax monoidal functor

$$(\mathcal{A}\text{-}\mathtt{dSp}^{\mathrm{op}}, \circ) \to (\mathcal{A}\text{-}\mathtt{dSp}, \circ).$$

In particular, the dual of an $\mathcal{A}$-cooperad is an $\mathcal{A}$-operad. Restricted to the category of finite-dimensional $\mathcal{A}$-dispecies, duality is a strong monoidal functor, and an involution. In particular, the dual of a finite-dimensional $\mathcal{A}$-operad is an $\mathcal{A}$-cooperad.

4.3. Set-operads

Just as for species (Section 2.14), one can consider dispecies with values in the category of sets. More formally, we replace Vec by Set in the preceding discussion. The resulting notions are called set-dispecies and set-operads. Tensor product and direct sum of vector spaces are replaced by cartesian product and disjoint union of sets.

4.3.1. Set-dispecies. An $\mathcal{A}$-*set-dispecies* is a functor

$$\mathrm{p} : \mathcal{A}\text{-}\mathtt{dHyp} \to \mathsf{Set}.$$

A map of $\mathcal{A}$-set-dispecies $\mathrm{p} \to \mathrm{q}$ is a natural transformation. We denote the category of $\mathcal{A}$-set-dispecies by

$$\mathcal{A}\text{-}\mathtt{SetdSp} = [\mathcal{A}\text{-}\mathtt{dHyp}, \mathsf{Set}].$$

Explicitly: An $\mathcal{A}$-set-dispecies p consists of a family of sets $\mathrm{p}[\mathrm{X}, \mathrm{Y}]$, one for each $\mathrm{X} \leq \mathrm{Y}$. A map of $\mathcal{A}$-set-dispecies $f : \mathrm{p} \to \mathrm{q}$ consists of a family of maps

$$f_{\mathrm{X},\mathrm{Y}} : \mathrm{p}[\mathrm{X}, \mathrm{Y}] \to \mathrm{q}[\mathrm{X}, \mathrm{Y}],$$

one for each $\mathrm{X} \leq \mathrm{Y}$.

4.3.2. Substitution product on set-dispecies. The *substitution product* of $\mathcal{A}$-set-dispecies p and q is defined by

$$(\mathrm{p} \circ \mathrm{q})[\mathrm{X}, \mathrm{Z}] := \bigsqcup_{\mathrm{Y}:\, \mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}} \mathrm{p}[\mathrm{X}, \mathrm{Y}] \times \mathrm{q}[\mathrm{Y}, \mathrm{Z}]. \tag{4.10}$$

The sum is over all flats Y which lie between X and Z. This yields a monoidal structure on $\mathcal{A}\text{-}\mathtt{SetdSp}$. The unit object is the $\mathcal{A}$-set-dispecies x defined by

$$\mathrm{x}[\mathrm{X}, \mathrm{Y}] = \begin{cases} \{\emptyset\} & \text{if } \mathrm{X} = \mathrm{Y}, \\ \emptyset & \text{otherwise.} \end{cases} \tag{4.11}$$

4.3.3. Set-operads. An $\mathcal{A}$-*set-operad* is a monoid in the monoidal category of $\mathcal{A}$-set-dispecies $(\mathcal{A}\text{-}\mathtt{SetdSp}, \circ, \mathrm{x})$. A morphism of $\mathcal{A}$-set-operads is a morphism of monoids.

Explicitly, an $\mathcal{A}$-set-operad is an $\mathcal{A}$-set-dispecies a equipped with maps

$$\gamma : \mathrm{a}[\mathrm{X}, \mathrm{Y}] \times \mathrm{a}[\mathrm{Y}, \mathrm{Z}] \to \mathrm{a}[\mathrm{X}, \mathrm{Z}] \quad \text{and} \quad \eta : \{\emptyset\} \to \mathrm{a}[\mathrm{X}, \mathrm{X}], \tag{4.12}$$

the former for each $\mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}$ and the latter for each X, subject to the associativity and unitality axioms.

4.3.4. Set-operad as a category. An $\mathcal{A}$-set-dispecies p determines a directed graph whose vertices are flats of $\mathcal{A}$, and elements of the set $\mathrm{p}[\mathrm{X}, \mathrm{Y}]$ are arrows from Y to X. For $\mathcal{A}$-set-dispecies p and q, the substitution product $\mathrm{p} \circ \mathrm{q}$ can be described in this language as follows. An element of $(\mathrm{p} \circ \mathrm{q})[\mathrm{X}, \mathrm{Z}]$ is a pair (f, g) of arrows such that g starts from Z, f ends at X, and the endpoint of g equals the starting point of f. Further:

Lemma 4.3. *An $\mathcal{A}$-set-operad determines a category whose objects are flats of $\mathcal{A}$. More precisely, for an $\mathcal{A}$-set-operad* a*, elements of the set* $\mathrm{a}[\mathrm{X}, \mathrm{Y}]$ *correspond to morphisms from* Y *to* X*.*

This is the set-theoretic analogue of Lemma 4.2.

4.3.5. Linearization functor. Linearization of set-dispecies (set-operads) produces dispecies (operads). Compare with the linearization functor from set-species to species in (2.94).

We say an operad is *linearized* if it arises by linearizing a set-operad.

4.4. Connected and positive operads

We discuss connected operads and positive operads and explain how they are equivalent notions.

4.4.1. Connected operads. A *connected dispecies* is a dispecies $\mathbf{p}$ along with a specified morphism

$$\mathbf{x} \to \mathbf{p} \tag{4.13}$$

such that the components $\mathbf{x}[\mathrm{X},\mathrm{X}] \xrightarrow{\cong} \mathbf{p}[\mathrm{X},\mathrm{X}]$ are linear isomorphisms for all X. A map of connected dispecies is a map of dispecies $\mathbf{p} \to \mathbf{q}$ which commutes with the specified morphisms (4.13).

Let $\mathcal{A}\text{-}\mathtt{dSp}_0$ denote the category of connected dispecies. If $\mathbf{p}$ and $\mathbf{q}$ are connected, then so is $\mathbf{p} \circ \mathbf{q}$, with the specified morphism given by

$$\mathbf{x} \xrightarrow{\cong} \mathbf{x} \circ \mathbf{x} \to \mathbf{p} \circ \mathbf{q}.$$

Further, the dispecies $\mathbf{x}$ is connected with the specified morphism being the identity map. This yields a monoidal category $(\mathcal{A}\text{-}\mathtt{dSp}_0, \circ)$.

A *connected operad* is a monoid in $(\mathcal{A}\text{-}\mathtt{dSp}_0, \circ)$. Observe that for a connected operad, its unit map coincides with the specified morphism (4.13). Thus, equivalently, a connected operad is an operad $\mathbf{a}$ for which $\mathbf{a}[\mathrm{X},\mathrm{X}]$ is one-dimensional for all X.

Let $\mathcal{A}\text{-}\mathsf{Op}_0$ denote the category of connected operads. Important examples of connected operads are given in Section 4.5. For an operad which is not connected, see Example 4.1.

4.4.2. Positive operads. A *positive dispecies* is a dispecies $\mathbf{p}$ for which $\mathbf{p}[\mathrm{X},\mathrm{X}] = 0$ for all X. Let $\mathcal{A}\text{-}\mathtt{dSp}_+$ denote the category of positive dispecies. The substitution product of two positive dispecies is again positive; however, $\mathbf{x}$ is not a positive dispecies. Thus, $(\mathcal{A}\text{-}\mathtt{dSp}_+, \circ)$ is a nonunital monoidal category.

The following standard construction allows us to transform $(\mathcal{A}\text{-}\mathtt{dSp}_+, \circ)$ into a unital monoidal category. Define

$$\mathbf{p} \odot \mathbf{q} := \mathbf{p} \circ \mathbf{q} + \mathbf{p} + \mathbf{q}. \tag{4.14}$$

We call this the *modified substitution product*. Then $(\mathcal{A}\text{-}\mathtt{dSp}_+, \odot)$ is a monoidal category, with the zero dispecies as the unit object.

A *positive operad* is a nonunital monoid in $(\mathcal{A}\text{-}\mathtt{dSp}_+, \circ)$, or equivalently, a monoid in $(\mathcal{A}\text{-}\mathtt{dSp}_+, \odot)$. (The difference between an operad and a positive operad is that references to the unit map are dropped in the latter.)

Let $\mathcal{A}\text{-}\mathsf{Op}_+$ denote the category of positive operads.

4.4.3. Equivalence between connected and positive operads. Suppose $\mathbf{p}$ is a connected dispecies. We let $\mathbf{p}_+$ denote its positive part:

$$\mathbf{p}_+[\mathrm{X},\mathrm{Y}] = \begin{cases} \mathbf{p}[\mathrm{X},\mathrm{Y}] & \text{if } \mathrm{X} \neq \mathrm{Y}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.15}$$

Conversely, suppose $\mathbf{q}$ is a positive dispecies. Then

$$\mathbf{q}_0 := \mathbf{x} + \mathbf{q} \tag{4.16}$$

defines a connected dispecies. These constructions are functorial. Further, there are natural isomorphisms

$$(\mathbf{p} \circ \mathbf{q})_+ \cong \mathbf{p}_+ \odot \mathbf{q}_+ \quad \text{and} \quad (\mathbf{x} + \mathbf{p}) \circ (\mathbf{x} + \mathbf{q}) \cong \mathbf{x} + (\mathbf{p} \odot \mathbf{q}).$$

In the first isomorphism, $\mathbf{p}$ and $\mathbf{q}$ are connected, while in the second, they are positive. Thus, the functors

$$(-)_+ : (\mathcal{A}\text{-}\mathtt{dSp}_0, \circ) \to (\mathcal{A}\text{-}\mathtt{dSp}_+, \odot) \quad \text{and} \quad (-)_0 : (\mathcal{A}\text{-}\mathtt{dSp}_+, \odot) \to (\mathcal{A}\text{-}\mathtt{dSp}_0, \circ)$$

are strong monoidal functors. Moreover, they define an adjoint equivalence. As a consequence, the categories of connected operads and positive operads are equivalent.

4.5. Commutative, associative, Lie operads

We now define the commutative, associative, Lie operads. We denote them by **Com**, **As**, **Lie**, respectively. These are connected operads. The substitution map of chambers and of Lie elements from Section 1.12.2 plays an important role here.

4.5.1. Commutative operad. The *exponential dispecies* $\mathbf{E}$ is defined by setting

$$\mathbf{E}[\mathrm{X},\mathrm{Y}] := \Bbbk$$

for all $\mathrm{X} \leq \mathrm{Y}$.

The exponential dispecies carries the structure of an operad. The substitution map is

$$\mathbf{E}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{E}[\mathrm{Y},\mathrm{Z}] \to \mathbf{E}[\mathrm{X},\mathrm{Z}], \qquad \Bbbk \otimes \Bbbk \xrightarrow{\cong} \Bbbk. \tag{4.17}$$

The unit map $\Bbbk \to \mathbf{E}[\mathrm{X},\mathrm{X}]$ is the identity. We call this the *commutative operad* and denote it by **Com**. It is a connected operad. It is the linearization of the set-operad which is singleton in each component.

Since (4.17) is an isomorphism, by reversing arrows, we see that $\mathbf{E}$ also carries the structure of a cooperad. We call this the *commutative cooperad* and denote it by $\mathbf{Com}^*$. As suggested by the notation, it is the cooperad dual to **Com**.

4.5.2. Associative operad. The *dispecies of chambers* $\mathbf{\Gamma}$ is defined by

$$\mathbf{\Gamma}[\mathrm{X}, \mathrm{Y}] := \Gamma[\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}],$$

where the rhs is the space of chambers of the arrangement $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$. Recall that the latter is the arrangement over X and under Y.

The dispecies of chambers carries the structure of an operad. The substitution map

$$\mathbf{\Gamma}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{\Gamma}[\mathrm{Y}, \mathrm{Z}] \to \mathbf{\Gamma}[\mathrm{X}, \mathrm{Z}]$$

is defined by specializing (1.166) to the arrangement $\mathcal{A}_{\mathrm{X}}^{\mathrm{Z}}$. (Note that the arrangements under and over the flat Y/X of $\mathcal{A}_{\mathrm{X}}^{\mathrm{Z}}$ are precisely $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$ and $\mathcal{A}_{\mathrm{Y}}^{\mathrm{Z}}$, respectively.) The unit map $\Bbbk \to \mathbf{\Gamma}[\mathrm{X}, \mathrm{X}]$ is the identity. We call this the *associative operad* and denote it by **As**. It is a connected operad.

Recall from Lemma 1.7 that lunes with base X and case Y correspond to chambers of $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$. The associative operad can thus also be understood in terms of lunes as follows. Recall the category of lunes from Section 1.2.3. In this category, morphisms only go from a bigger flat to a smaller flat. Hence, it arises from a set-operad as in Lemma 4.3. The linearization of this set-operad is precisely the associative operad. The substitution map can equivalently be written as

$$\mathbf{As}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{As}[\mathrm{Y}, \mathrm{Z}] \to \mathbf{As}[\mathrm{X}, \mathrm{Z}], \qquad \mathtt{H}_{\mathrm{L}} \otimes \mathtt{H}_{\mathrm{M}} \mapsto \mathtt{H}_{\mathrm{L} \circ \mathrm{M}}. \tag{4.18}$$

Alternatively, in term of nested faces, using (1.13), we may write

$$\mathbf{As}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{As}[\mathrm{Y}, \mathrm{Z}] \to \mathbf{As}[\mathrm{X}, \mathrm{Z}], \qquad \mathtt{H}_{F/A} \otimes \mathtt{H}_{G/F} \mapsto \mathtt{H}_{G/A}, \tag{4.19}$$

where A, F, G are faces with support X, Y, Z, respectively, and $A \leq F \leq G$. It is implicit that $\mathtt{H}_{F/A} = \mathtt{H}_{F'/A'}$ whenever $(A, F) \sim (A', F')$ in the sense of (1.11).

The cooperad dual to **As** is $\mathbf{As}^*$. We call this the *associative cooperad*.

4.5.3. Lie operad. The *Lie dispecies* **Lie** is defined by

$$\mathbf{Lie}[\mathrm{X}, \mathrm{Y}] := \mathsf{Lie}[\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}],$$

where the rhs is the space of Lie elements of the arrangement $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$.

The Lie dispecies carries the structure of an operad given by the map (1.167) (in the same manner as discussed above for the dispecies of chambers). This is the *Lie operad*, which we continue to denote by **Lie**. It is a connected suboperad of **As** in view of (1.168).

The cooperad dual to **Lie** is $\mathbf{Lie}^*$. We call this the *Lie cooperad*.

4.5.4. Morphisms. There are morphisms of operads

$$\mathbf{Lie} \to \mathbf{As} \to \mathbf{Com}. \tag{4.20}$$

The first morphism was mentioned above. The second morphism is as follows. Evaluated on the (X, Y)-component, it sends each basis chamber in $\mathbf{\Gamma}[\mathrm{X}, \mathrm{Y}]$ to $1 \in \Bbbk$.

4.6. May operads

We recall May operads and explain how they give rise to $\mathcal{B}^J$-operads, where $\mathcal{B}^J$ is the braid arrangement on a finite set J (Section 1.13). This is done via a functor from Joyal species to $\mathcal{B}^J$-dispecies which is strong wrt the substitution products. Joyal species are reviewed in Section 2.16.

4.6.1. May operads. Recall from Section 2.16.2 that a Joyal species p is positive when $\mathsf{p}[\emptyset] = 0$. Let p and q be positive Joyal species. Define a new positive Joyal species $\mathsf{p} \circ \mathsf{q}$ by

$$(\mathsf{p} \circ \mathsf{q})[J] := \bigoplus_{\mathrm{X} \vdash J} \mathsf{p}[\mathrm{X}] \otimes \Big(\bigotimes_{S \in \mathrm{X}} \mathsf{q}[S] \Big). \tag{4.21}$$

This is the *substitution product* of p and q. The direct sum is over all partitions X of the set J, while the tensor product is over all blocks S of X. This defines a monoidal structure on the category of positive Joyal species. The unit object is the positive Joyal species x characteristic of singletons, namely,

$$\mathsf{x}[J] := \begin{cases} \Bbbk & \text{if } J \text{ is a singleton,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.22}$$

A *May operad* is a monoid in this monoidal category. In particular, a May operad p entails maps

$$\mathsf{p}[\mathrm{X}] \otimes \Big(\bigotimes_{S \in \mathrm{X}} \mathsf{p}[S] \Big) \to \mathsf{p}[J], \tag{4.23}$$

one for each partition X of J, subject to the associativity and unitality axioms.

4.6.2. From May operads to $\mathcal{B}^J$-operads. Fix a finite set J and consider the braid arrangement $\mathcal{B}^J$.

To a positive Joyal species p, we associate a $\mathcal{B}^J$-dispecies $\mathbf{p}$ as follows. The (X, Y)-component of $\mathbf{p}$ is defined by

$$\mathbf{p}[\mathrm{X}, \mathrm{Y}] := \bigotimes_{S \in \mathrm{X}} \mathsf{p}[\mathrm{Y}_S],$$

where Y_S is the set whose elements are the blocks of Y which refine the block S of X.

This yields a functor from the category of positive Joyal species to the category of $\mathcal{B}^J$-dispecies. Let us temporarily denote it by $\mathcal{F}$. For positive Joyal species p and q, there are natural isomorphisms

$$\mathcal{F}(\mathsf{p}) \circ \mathcal{F}(\mathsf{q}) \xrightarrow{\cong} \mathcal{F}(\mathsf{p} \circ \mathsf{q}) \quad \text{and} \quad \mathbf{x} \xrightarrow{\cong} \mathcal{F}(\mathsf{x}),$$

with the substitution products as in (4.3) and (4.21). The second isomorphism is clear. For the first one, we evaluate both sides on say the component (X, Z). In both cases, the sum splits over partitions Y that lie between X and Z, and corresponding summands can be identified by rearranging the tensor factors. We omit the details. Thus, $\mathcal{F}$ is a strong monoidal functor. Since strong monoidal functors preserve monoids, $\mathcal{F}$ sends a May operad to a $\mathcal{B}^J$-operad.

Remark 4.4. We do not formally discuss the classical associative, commutative, Lie operads here, see for instance [**18**, Appendix B.1.4]. But we mention that the operads **As**, **Com**, **Lie** defined in Section 4.5, when specialized to the braid arrangement, arise from their classical counterparts via the above construction.

4.7. Hadamard product. Hopf operads

The Hadamard product on dispecies is defined by taking componentwise tensor products. This product equips the category of dispecies with a monoidal structure which is different from the substitution product. Further, the two products are compatible in the sense that they turn the category of dispecies into a 2-monoidal category. Hopf operads are bimonoids in this 2-monoidal category. These are operads which have a compatible comonoid structure wrt the Hadamard product. We describe the internal hom for the Hadamard product. When applied to a cooperad and an operad, it produces an operad which we call the convolution operad.

This section assumes some familiarity with 2-monoidal categories. An adequate reference is [**18**, Chapter 6], also see the brief review in Appendix B.1. The main fact from here which we require later is that the Hadamard product preserves (co)operads. This statement can be understood directly, so we make it the starting point of our discussion.

4.7.1. Hadamard product of dispecies. Let $\mathbf{p}$ and $\mathbf{q}$ be two dispecies. Define a new dispecies $\mathbf{p} \times \mathbf{q}$ by

$$(\mathbf{p} \times \mathbf{q})[\mathrm{X}, \mathrm{Y}] := \mathbf{p}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{q}[\mathrm{X}, \mathrm{Y}]. \tag{4.24}$$

This is the *Hadamard product* of $\mathbf{p}$ and $\mathbf{q}$. This yields a monoidal structure on the category of dispecies $\mathcal{A}$-$\mathtt{dSp}$. The unit object is the exponential dispecies $\mathbf{E}$. By interchanging the tensor factors in (4.24), we see that there is an isomorphism of dispecies $\mathbf{p} \times \mathbf{q} \to \mathbf{q} \times \mathbf{p}$. This defines a braiding (which is in fact a symmetry).

4.7.2. Hadamard product of (co)operads. Suppose $\mathbf{a}$ and $\mathbf{b}$ are operads. Then so is their Hadamard product $\mathbf{a}\times\mathbf{b}$. Further, the symmetry $\mathbf{a}\times\mathbf{b} \to \mathbf{b}\times\mathbf{a}$ is a morphism of operads. (The same is true for cooperads.) The structure maps of $\mathbf{a} \times \mathbf{b}$ are obtained by tensoring those of $\mathbf{a}$ and $\mathbf{b}$. That is, the substitution map is

$$\begin{aligned}(\mathbf{a} \times \mathbf{b})[\mathrm{X}, \mathrm{Y}] \otimes (\mathbf{a} \times \mathbf{b})[\mathrm{Y}, \mathrm{Z}] \\ \cong (\mathbf{a}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y}, \mathrm{Z}]) \otimes (\mathbf{b}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{b}[\mathrm{Y}, \mathrm{Z}]) \\ \xrightarrow{\gamma\otimes\gamma} \mathbf{a}[\mathrm{X}, \mathrm{Z}] \otimes \mathbf{b}[\mathrm{X}, \mathrm{Z}] = (\mathbf{a} \times \mathbf{b})[\mathrm{X}, \mathrm{Z}],\end{aligned}$$

while the unit map is

$$\Bbbk \to \Bbbk \otimes \Bbbk \xrightarrow{\eta\otimes\eta} \mathbf{a}[\mathrm{X}, \mathrm{X}] \otimes \mathbf{b}[\mathrm{X}, \mathrm{X}] = (\mathbf{a} \times \mathbf{b})[\mathrm{X}, \mathrm{X}].$$

4.7.3. An interchange law. Recall the notion of interchange law from (B.2). There is an interchange law between the substitution and Hadamard products on dispecies. This is briefly explained below.

Let $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$ be dispecies. Then

$$((\mathbf{p} \times \mathbf{q}) \circ (\mathbf{r} \times \mathbf{s}))[\mathrm{X}, \mathrm{Z}] = \bigoplus_{\mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}} (\mathbf{p}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{q}[\mathrm{X}, \mathrm{Y}]) \otimes (\mathbf{r}[\mathrm{Y}, \mathrm{Z}] \otimes \mathbf{s}[\mathrm{Y}, \mathrm{Z}]),$$

$$((\mathbf{p} \circ \mathbf{r}) \times (\mathbf{q} \circ \mathbf{s}))[\mathrm{X}, \mathrm{Z}] = \bigoplus_{\mathrm{X} \leq \mathrm{Y}, \mathrm{W} \leq \mathrm{Z}} (\mathbf{p}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{r}[\mathrm{Y}, \mathrm{Z}]) \otimes (\mathbf{q}[\mathrm{X}, \mathrm{W}] \otimes \mathbf{s}[\mathrm{W}, \mathrm{Z}]).$$

The first sum is over all Y between X and Z, while the second sum is over all Y and W between X and Z. Each summand of the former also appears in the latter (for $\mathrm{Y} = \mathrm{W}$). Rearranging the middle two tensor factors yields a map of dispecies

$$\zeta : (\mathbf{p} \times \mathbf{q}) \circ (\mathbf{r} \times \mathbf{s}) \hookrightarrow (\mathbf{p} \circ \mathbf{r}) \times (\mathbf{q} \circ \mathbf{s}). \tag{4.25a}$$

We also have maps

$$\Delta_{\mathbf{x}} : \mathbf{x} \to \mathbf{x} \times \mathbf{x}, \qquad \mu_{\mathbf{E}} : \mathbf{E} \circ \mathbf{E} \to \mathbf{E}, \qquad \iota_{\mathbf{E}} = \epsilon_{\mathbf{x}} : \mathbf{x} \to \mathbf{E}. \tag{4.25b}$$

The first map is defined to be the obvious isomorphism. The second and third maps are defined to be the structure maps of the commutative operad.

Proposition 4.5. *With structure maps* (4.25a) *and* (4.25b),

$$(\mathcal{A}\text{-}\mathsf{dSp}, \circ, \mathbf{x}, \times, \mathbf{E})$$

is a 2*-monoidal category. Moreover, it is* $\times$*-braided.*

Proof. This is a routine verification of the axioms [18, Definition 6.1]. □

There also exist maps (4.25a) and (4.25b) with arrows reversed. This yields the 2-monoidal category $(\mathcal{A}\text{-}\mathsf{dSp}, \times, \circ)$. Note very carefully that the order of the monoidal structures has been reversed.

Let $\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{dSp}, \circ)$ and $\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{dSp}, \circ)$ denote the categories of operads and cooperads, respectively. Since $(\mathcal{A}\text{-}\mathsf{dSp}, \circ, \times)$ and $(\mathcal{A}\text{-}\mathsf{dSp}, \times, \circ)$ are 2-monoidal categories, it follows from [18, Proposition 6.35] that

$$(\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{dSp}, \circ), \times) \quad \text{and} \quad (\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{dSp}, \circ), \times)$$

are monoidal categories. (In fact, they are symmetric monoidal categories. This is the consequence of being $\times$-braided.) In particular, the Hadamard product of (co)operads is again a (co)operad. By unwinding the definitions, one can check that it coincides with the explicit description given earlier.

4.7.4. Hopf operads. Bimonoids in 2-monoidal categories are treated in [18, Section 6.5]. A *Hopf operad* is a bimonoid in $(\mathcal{A}\text{-}\mathsf{dSp}, \circ, \times)$. Explicitly, a Hopf operad is a dispecies $\mathbf{p}$ with maps

$$\mu : \mathbf{p} \circ \mathbf{p} \to \mathbf{p}, \qquad \iota : \mathbf{x} \to \mathbf{p}, \qquad \Delta : \mathbf{p} \to \mathbf{p} \times \mathbf{p}, \qquad \epsilon : \mathbf{p} \to \mathbf{E}$$

satisfying appropriate axioms [18, Definition 6.25]. Alternatively, in view of [18, Proposition 6.36], a Hopf operad is the same as a comonoid in the category of operads wrt the Hadamard product.

4.7.5. Internal hom for the Hadamard product. The notion of internal hom in a monoidal category is reviewed in Appendix B. We now proceed to construct the internal hom for the Hadamard product of dispecies.

For dispecies $\mathbf{p}$ and $\mathbf{q}$, let $\mathrm{hom}^{\times}(\mathbf{p},\mathbf{q})$ denote the dispecies defined by

$$\mathrm{hom}^{\times}(\mathbf{p},\mathbf{q})[\mathrm{X},\mathrm{Y}] := \mathrm{Hom}_{\Bbbk}(\mathbf{p}[\mathrm{X},\mathrm{Y}],\mathbf{q}[\mathrm{X},\mathrm{Y}]). \tag{4.26}$$

This gives rise to a functor

$$\mathcal{A}\text{-}\mathtt{dSp}^{\mathrm{op}} \times \mathcal{A}\text{-}\mathtt{dSp} \to \mathcal{A}\text{-}\mathtt{dSp}, \qquad (\mathbf{p},\mathbf{q}) \mapsto \mathrm{hom}^{\times}(\mathbf{p},\mathbf{q}).$$

It is the internal hom in the symmetric monoidal category $(\mathcal{A}\text{-}\mathtt{dSp},\times)$. That is, for any dispecies $\mathbf{p}$, $\mathbf{m}$, $\mathbf{n}$, there is a natural bijection

$$\mathcal{A}\text{-}\mathtt{dSp}(\mathbf{p}\times\mathbf{m},\mathbf{n}) \cong \mathcal{A}\text{-}\mathtt{dSp}\big(\mathbf{p},\mathrm{hom}^{\times}(\mathbf{m},\mathbf{n})\big). \tag{4.27}$$

There is a canonical map of dispecies

$$\mathbf{p}^{*}\times\mathbf{q} \to \mathrm{hom}^{\times}(\mathbf{p},\mathbf{q})$$

which is an isomorphism if either $\mathbf{p}$ or $\mathbf{q}$ is finite-dimensional. In particular,

$$\mathbf{p}^{*} \cong \mathrm{hom}^{\times}(\mathbf{p},\mathbf{E}).$$

4.7.6. Convolution operad. Let $\mathbf{c}$ be a cooperad and $\mathbf{a}$ an operad. Then the dispecies $\mathrm{hom}^{\times}(\mathbf{c},\mathbf{a})$ is an operad as follows. For maps $f:\mathbf{c}[\mathrm{X},\mathrm{Y}]\to\mathbf{a}[\mathrm{X},\mathrm{Y}]$ and $g:\mathbf{c}[\mathrm{Y},\mathrm{Z}]\to\mathbf{a}[\mathrm{Y},\mathrm{Z}]$, the substitution map sends $f\otimes g$ to the composite

$$\mathbf{c}[\mathrm{X},\mathrm{Z}] \to \mathbf{c}[\mathrm{X},\mathrm{Y}]\otimes\mathbf{c}[\mathrm{Y},\mathrm{Z}] \xrightarrow{f\otimes g} \mathbf{a}[\mathrm{X},\mathrm{Y}]\otimes\mathbf{a}[\mathrm{Y},\mathrm{Z}] \to \mathbf{a}[\mathrm{X},\mathrm{Z}].$$

We call this the *convolution operad*.

The map

$$\mathbf{c}^{*}\times\mathbf{a} \to \mathrm{hom}^{\times}(\mathbf{c},\mathbf{a})$$

is a morphism of operads, and

$$\mathbf{c}^{*} \cong \mathrm{hom}^{\times}(\mathbf{c},\mathbf{E})$$

is an isomorphism of operads.

The convolution operad arises from a lax monoidal structure on the internal hom of the Hadamard product, see Propositions B.6 and B.7.

4.8. Orientation functor and signature functor

We introduce two twisted versions of the commutative operad. We call them the orientation operad and the signed commutative operad. Taking Hadamard product with these operads gives rise to the orientation and signature functors on dispecies.

4.8.1. Orientation operad. The dispecies $\mathbf{E}^{\mathbf{o}}$ is defined by

$$\mathbf{E}^{\mathbf{o}}[\mathrm{X},\mathrm{Y}] := \mathsf{E}^{\mathsf{o}}[\mathcal{A}^{\mathrm{Y}}_{\mathrm{X}}],$$

where we recall from Section 1.11.1 that the rhs is the orientation space of the arrangement $\mathcal{A}^{\mathrm{Y}}_{\mathrm{X}}$. Note that $\mathbf{E}^{\mathbf{o}}[\mathrm{X},\mathrm{X}]=\Bbbk$. The maps (1.160) applied to $\mathcal{A}^{\mathrm{Z}}_{\mathrm{X}}$ turn $\mathbf{E}^{\mathbf{o}}$ into a connected operad. We call this the *orientation operad* and denote it by $\mathbf{Com}^{\mathbf{o}}$.

4.8.2. Orientation functor. For any dispecies $\mathbf{p}$, define its *oriented partner* $\mathbf{p}^{\mathbf{o}}$ by

$$\mathbf{p}^{\mathbf{o}} := \mathbf{p} \times \mathbf{E}^{\mathbf{o}}. \tag{4.28}$$

The assignment $\mathbf{p} \mapsto \mathbf{p}^{\mathbf{o}}$ is functorial in $\mathbf{p}$. This is the *orientation functor* on dispecies. Note that $\mathbf{E}$ and $\mathbf{E}^{\mathbf{o}}$ are oriented partners of each other, that is, $\mathbf{E}^{\mathbf{o}} \times \mathbf{E}^{\mathbf{o}} \cong \mathbf{E}$. This follows from (1.159). Hence, the orientation functor is an involution. Further, for any dispecies $\mathbf{p}$ and $\mathbf{q}$, there are natural isomorphisms

$$(\mathbf{p} \circ \mathbf{q})^{\mathbf{o}} \cong \mathbf{p}^{\mathbf{o}} \circ \mathbf{q}^{\mathbf{o}} \quad \text{and} \quad \mathbf{x}^{\mathbf{o}} \cong \mathbf{x}. \tag{4.29}$$

In other words, the orientation functor is a strong monoidal functor. So, it preserves (co)operads. This may be also be deduced as follows. We know that the Hadamard product preserves (co)operads. Thus, if $\mathbf{a}$ is an operad, then so is $\mathbf{a}^{\mathbf{o}}$ obtained by taking Hadamard product of $\mathbf{a}$ with $\mathbf{Com}^{\mathbf{o}}$.

4.8.3. Signed commutative operad. The dispecies $\mathbf{E}^-$ is defined by letting its components $\mathbf{E}^-[\mathrm{X}, \mathrm{Y}]$ be as in Definition 1.74. The maps (1.162) turn $\mathbf{E}^-$ into a connected operad. We call this the *signed commutative operad* and denote it by $\mathbf{Com}^-$.

4.8.4. Signature functor. For any dispecies $\mathbf{p}$, define its *signed partner* $\mathbf{p}^-$ by

$$\mathbf{p}^- := \mathbf{p} \times \mathbf{E}^-.$$

The assignment $\mathbf{p} \mapsto \mathbf{p}^-$ is functorial in $\mathbf{p}$. This is the *signature functor* on dispecies. Note that $\mathbf{E}$ and $\mathbf{E}^-$ are signed partners of each other. Further, for any dispecies $\mathbf{p}$ and $\mathbf{q}$, there are natural isomorphisms

$$(\mathbf{p} \circ \mathbf{q})^- \cong \mathbf{p}^- \circ \mathbf{q}^- \quad \text{and} \quad \mathbf{x}^- \cong \mathbf{x}.$$

Thus, the signature functor is strong, and it preserves (co)operads. So, if $\mathbf{a}$ is an operad, then so is $\mathbf{a}^-$, and if $\mathbf{c}$ is a cooperad, then so is $\mathbf{c}^-$.

Lemma 4.6. *The operads* $\mathbf{As}$ *and* $\mathbf{As}^-$ *are isomorphic. Explicitly, the map*

$$\mathbf{As}[\mathrm{X}, \mathrm{Y}] \to \mathbf{As}^-[\mathrm{X}, \mathrm{Y}], \qquad \mathtt{H}_{F/A} \mapsto \mathtt{H}_{F/A} \otimes \mathtt{H}_{[F/A]}$$

is an isomorphism of operads.

Proof. This follows from (1.162) and (4.19). $\square$

We refer to $\mathbf{Lie}^-$ as the *signed Lie operad*, and to its dual as the *signed Lie cooperad*. The latter is the same as $(\mathbf{Lie}^*)^-$ which is the signature functor applied to the Lie cooperad.

4.9. Operad presentations

We describe the free operad on a dispecies. We then discuss operad presentations with emphasis on quadratic operads and their duality. The commutative, associative, Lie operads are examples of quadratic operads. The associative operad is self-dual, while the commutative and Lie operads are duals of each other.

4.9.1. Free operad. We begin with the construction of the free operad on a dispecies. Let $\mathbf{e}$ be any dispecies. Define the dispecies

$$\mathcal{F}_\circ(\mathbf{e}) := \underset{n\geq 0}{+} \mathbf{e}^{\circ n}. \tag{4.30}$$

It carries an operad structure. The substitution map

$$\mathcal{F}_\circ(\mathbf{e}) \circ \mathcal{F}_\circ(\mathbf{e}) \to \mathcal{F}_\circ(\mathbf{e})$$

is defined by distributing the substitution product over the direct sum, and then using the identifications

$$\mathbf{e}^{\circ m} \circ \mathbf{e}^{\circ n} \xrightarrow{\cong} \mathbf{e}^{\circ(m+n)}.$$

The unit map $\mathbf{x} \to \mathcal{F}_\circ(\mathbf{e})$ identifies $\mathbf{x}$ with $\mathbf{e}^{\circ 0}$. The operad $\mathcal{F}_\circ(\mathbf{e})$ is the *free operad* on the dispecies $\mathbf{e}$. It satisfies the following universal property.

Theorem 4.7. *Let $\mathbf{a}$ be an operad, $\mathbf{e}$ a dispecies, $f : \mathbf{e} \to \mathbf{a}$ a map of dispecies. Then there exists a unique morphism of operads $\hat{f} : \mathcal{F}_\circ(\mathbf{e}) \to \mathbf{a}$ such that the diagram*

$$\begin{tikzcd}
\mathcal{F}_\circ(\mathbf{e}) \arrow[r, dashed, "\hat{f}"] & \mathbf{a} \\
\mathbf{e} \arrow[u, hook] \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes.

Let us understand the construction of $\mathcal{F}_\circ(\mathbf{e})$ explicitly. We have

$$\mathcal{F}_\circ(\mathbf{e})[\mathrm{X},\mathrm{Z}] = \bigoplus_{\mathrm{X}\leq \mathrm{Y}_1\leq\cdots\leq \mathrm{Y}_k\leq \mathrm{Z}} \mathbf{e}[\mathrm{X},\mathrm{Y}_1] \otimes \cdots \otimes \mathbf{e}[\mathrm{Y}_k,\mathrm{Z}],$$

where the sum is over all multichains in the interval $[\mathrm{X},\mathrm{Z}]$. (A multichain is a chain in which elements are allowed to repeat.) Now suppose Y is a flat between X and Z. Given a multichain in $[\mathrm{X},\mathrm{Y}]$ and a multichain in $[\mathrm{Y},\mathrm{Z}]$, one obtains a multichain in $[\mathrm{X},\mathrm{Z}]$ by concatenation. The product

$$\mathcal{F}_\circ(\mathbf{e})[\mathrm{X},\mathrm{Y}] \otimes \mathcal{F}_\circ(\mathbf{e})[\mathrm{Y},\mathrm{Z}] \to \mathcal{F}_\circ(\mathbf{e})[\mathrm{X},\mathrm{Z}]$$

is defined by concatenating multichains, and tensoring the corresponding summands.

Observe that when $\mathbf{e}$ is positive,

$$\mathcal{F}_\circ(\mathbf{e})[\mathrm{X},\mathrm{Z}] = \bigoplus_{\mathrm{X}< \mathrm{Y}_1<\cdots< \mathrm{Y}_k< \mathrm{Z}} \mathbf{e}[\mathrm{X},\mathrm{Y}_1] \otimes \cdots \otimes \mathbf{e}[\mathrm{Y}_k,\mathrm{Z}],$$

the sum now being over all chains in the interval $[\mathrm{X},\mathrm{Z}]$. The number of summands is finite in this case, and hence, if $\mathbf{e}$ is finite-dimensional, then so is $\mathcal{F}_\circ(\mathbf{e})$.

Example 4.8. Recall from Example 4.1 that for a rank-zero arrangement, a dispecies $\mathbf{e}$ is the same as a vector space $V := \mathbf{e}[\bot,\bot]$. In this case, the free operad on $\mathbf{e}$ is the same as the tensor algebra of V.

4.9.2. Quadratic operads. We say an operad $\mathbf{a}$ has a presentation $\langle \mathbf{e} \,|\, \mathbf{r} \rangle$ if $\mathbf{a}$ is the quotient of the free operad $\mathcal{F}_\circ(\mathbf{e})$ by the ideal generated by $\mathbf{r}$. (By assumption, $\mathbf{r}$ is a subdispecies of $\mathcal{F}_\circ(\mathbf{e})$.)

An operad $\mathbf{a}$ is *quadratic* if it has a presentation $\langle \mathbf{e} \,|\, \mathbf{r} \rangle$ in which $\mathbf{r}$ is a subdispecies of $\mathbf{e} \circ \mathbf{e}$. In this case: The ideal generated by $\mathbf{r}$ is spanned by $\mathbf{r}$, $\mathbf{e} \circ \mathbf{r}$, $\mathbf{r} \circ \mathbf{e}$, $\mathbf{e} \circ \mathbf{e} \circ \mathbf{r}$, and so on. Thus, the canonical map $\mathbf{e} \to \mathbf{a}$ is injective, and $\mathbf{e}$ is a subdispecies of $\mathbf{a}$.

We say a dispecies $\mathbf{e}$ is concentrated in rank i if the component $\mathbf{e}[\mathrm{X}, \mathrm{Y}] = 0$ whenever $\mathrm{rk}(\mathrm{Y}/\mathrm{X}) \neq i$. An operad $\mathbf{a}$ is *binary quadratic* if it has a presentation $\langle \mathbf{e} \,|\, \mathbf{r} \rangle$ in which $\mathbf{e}$ is concentrated in rank one and $\mathbf{r}$ is a subdispecies of $\mathbf{e} \circ \mathbf{e}$. In this case, the subdispecies $\mathbf{r}$ is concentrated in rank two. A binary quadratic operad is necessarily connected.

Example 4.9. We now present the commutative operad as a binary quadratic operad. Let $\mathbf{e}$ be the dispecies concentrated in rank one given by

$$\mathbf{e}[\mathrm{X}, \mathrm{Y}] := \Bbbk$$

for $\mathrm{rk}(\mathrm{Y}/\mathrm{X}) = 1$. Hence, $\mathcal{F}_\circ(\mathbf{e})[\mathrm{X}, \mathrm{Z}]$ has a basis consisting of all maximal chains of flats in the interval $[\mathrm{X}, \mathrm{Z}]$. In particular, for $\mathrm{rk}(\mathrm{Z}/\mathrm{X}) = 2$, $\mathcal{F}_\circ(\mathbf{e})[\mathrm{X}, \mathrm{Z}]$ has a basis consisting of flats Y strictly between X and Z. Let $\mathbf{r}[\mathrm{X}, \mathrm{Z}]$ be its subspace spanned by elements of the form $\mathrm{Y} - \mathrm{Y}'$. We claim that $\langle \mathbf{e} \,|\, \mathbf{r} \rangle$ is the commutative operad.

To see this, first consider two maximal chains in any interval $[\mathrm{X}, \mathrm{Z}]$ which differ in exactly one position as shown below.

$$\begin{array}{ccccc} & & \mathrm{Y}_k & & \\ & \diagup & & \diagdown & \\ \mathrm{X} - \mathrm{Y}_1 - \dots - \mathrm{Y}_{k-1} & & & & \mathrm{Y}_{k+1} - \dots - \mathrm{Z}. \\ & \diagdown & & \diagup & \\ & & \mathrm{Y}'_k & & \end{array}$$

Since $\mathrm{Y}_k - \mathrm{Y}'_k$ belongs to $\mathbf{r}[\mathrm{Y}_{k-1}, \mathrm{Y}_{k+1}]$, the difference of the above two maximal chains belongs to the ideal generated by $\mathbf{r}$, and hence the two chains become equal in the quotient. Now by Lemma 1.4, item (2), any two maximal chains in $[\mathrm{X}, \mathrm{Z}]$ are related to each other by a sequence in which two successive maximal chains differ in exactly one position. It follows that any two maximal chains are equal in the quotient.

Example 4.10. We now present the associative operad as a binary quadratic operad. Let $\mathbf{e}$ be the dispecies concentrated in rank one given by

$$\mathbf{e}[\mathrm{X}, \mathrm{Y}] := \mathbf{\Gamma}[\mathrm{X}, \mathrm{Y}]$$

for $\mathrm{rk}(\mathrm{Y}/\mathrm{X}) = 1$. Since a rank-one arrangement has two chambers, these spaces are two-dimensional. Let us denote the free operad $\mathcal{F}_\circ(\mathbf{e})$ by $\mathbf{mc}$. Observe that $\mathbf{mc}[\mathrm{X}, \mathrm{Z}]$ has a basis indexed by all maximal chains of faces in $\mathcal{A}^{\mathrm{Z}}_{\mathrm{X}}$. The substitution map is given by concatenation of maximal chains, see (1.158).

Now let $\mathrm{rk}(\mathrm{Z}/\mathrm{X}) = 2$. Fix any face A with support X. Then $\mathbf{mc}[\mathrm{X}, \mathrm{Z}]$ has a basis consisting of chains $(A \lessdot F \lessdot C)$ of faces, with C having support

Z. Now take $\mathbf{r}[\mathrm{X},\mathrm{Z}]$ to be its subspace spanned by elements of the form

$$(A \lessdot F \lessdot C) - (A \lessdot G \lessdot C),$$

as C varies. We claim that $\langle \mathbf{e} | \mathbf{r} \rangle$ is the associative operad.

We follow the argument in Example 4.9. Using Lemma 1.4, item (1), we see that two maximal chains of faces are equal in the quotient iff they end in the same face.

Remark 4.11. Let $\mathbf{s}$ be the subdispecies of $\mathbf{mc}$ spanned by elements which can be written as the difference of two maximal chains of faces differing in exactly one position. Then $\mathbf{mc}/\mathbf{s} = \mathbf{E}$. Instead, let $\mathbf{s}'$ be the subsdipecies spanned by elements which can be written as the sum of two maximal chains of faces differing in exactly one position. Then $\mathbf{mc}/\mathbf{s}' = \mathbf{E^o}$. Moreover, both $\mathbf{s}$ and $\mathbf{s}'$ are ideals. Thus, $\mathbf{E}$ and $\mathbf{E^o}$ carry operad structures, and these are precisely $\mathbf{Com}$ and $\mathbf{Com^o}$, respectively.

Example 4.12. We now present the Lie operad as a binary quadratic operad. Let $\mathbf{e}$ be the dispecies concentrated in rank one given by

$$\mathbf{e}[\mathrm{X},\mathrm{Y}] := \mathbf{E^o}[\mathrm{X},\mathrm{Y}]$$

for $\mathrm{rk}(\mathrm{Y}/\mathrm{X}) = 1$. Now let $\mathrm{rk}(\mathrm{Z}/\mathrm{X}) = 2$ and $\mathrm{Y}_1, \dots, \mathrm{Y}_n$ denote the flats strictly between X and Z. (The number n depends on X and Z.) Fix a face A with support X. Let $\mathbf{r}[\mathrm{X},\mathrm{Z}]$ be the subspace spanned by

$$\sum_{i=1}^{n} [A \lessdot F_i] \otimes [F_i \lessdot C_i],$$

where F_i is any one of the two faces greater than A with support Y_i, and C_i is then chosen so that the orientation $[A \lessdot F_i \lessdot C_i]$ equals a fixed orientation of $\mathcal{A}^{\mathrm{Z}}_{\mathrm{X}}$ (independent of i). The presentation $\langle \mathbf{e} | \mathbf{r} \rangle$ yields the Lie operad. This is equivalent to Theorem 1.77. Setting the above sum to 0 is the *Jacobi identity*.

4.9.3. Oriented partners. The orientation functor preserves binary quadratic operads. More precisely: Suppose $\mathbf{a}$ is a binary quadratic operad with presentation $\langle \mathbf{e} \,|\, \mathbf{r} \rangle$. Then its oriented partner $\mathbf{a^o}$ is also a binary quadratic operad with presentation $\langle \mathbf{e^o} \,|\, \mathbf{r^o} \rangle$. This can be checked using the key property (4.29).

The signature functor also preserves binary quadratic operads.

4.9.4. Quadratic duals. All dispecies in this discussion are assumed to be finite-dimensional. Let $\mathbf{e}$ be a dispecies, and $\mathbf{r}$ a subdispecies of $\mathbf{e} \circ \mathbf{e}$. Define $\mathbf{r}^{\perp}$ by the exact sequence

$$\mathbf{0} \to \mathbf{r}^{\perp} \hookrightarrow \mathbf{e}^* \circ \mathbf{e}^* = (\mathbf{e} \circ \mathbf{e})^* \twoheadrightarrow \mathbf{r}^* \to \mathbf{0}.$$

It is the subdispecies of $\mathbf{e}^* \circ \mathbf{e}^*$ obtained by taking the kernel of the dual of the inclusion $\mathbf{r} \hookrightarrow \mathbf{e} \circ \mathbf{e}$. Equivalently, consider the pairing

$$(\mathbf{e}^* \circ \mathbf{e}^*) \times (\mathbf{e} \circ \mathbf{e}) = (\mathbf{e} \circ \mathbf{e})^* \times (\mathbf{e} \circ \mathbf{e}) \to \mathbf{E}$$

defined on each component by the canonical pairing between a vector space and its dual. Under this pairing, $\mathbf{r}^{\perp}$ is the subdispecies of $\mathbf{e}^* \circ \mathbf{e}^*$ which is orthogonal to $\mathbf{r}$. This pairing may also be expressed as the composite

$$(\mathbf{e}^* \circ \mathbf{e}^*) \times (\mathbf{e} \circ \mathbf{e}) \to (\mathbf{e}^* \times \mathbf{e}) \circ (\mathbf{e}^* \times \mathbf{e}) \to \mathbf{E} \circ \mathbf{E} \ \to \mathbf{E},$$

with the first map coming from the interchange law between the substitution and Hadamard products, and the last one from the commutative operad structure.

Let us make the pairing more explicit using components. Observe that

$$(\mathbf{e}^* \circ \mathbf{e}^*)[\mathrm{X}, \mathrm{Z}] \otimes (\mathbf{e} \circ \mathbf{e})[\mathrm{X}, \mathrm{Z}] = \bigoplus_{\mathrm{X} \leq \mathrm{Y}, \mathrm{Y}' \leq \mathrm{Z}} \mathbf{e}[\mathrm{X}, \mathrm{Y}]^* \otimes \mathbf{e}[\mathrm{Y}, \mathrm{Z}]^* \otimes \mathbf{e}[\mathrm{X}, \mathrm{Y}'] \otimes \mathbf{e}[\mathrm{Y}', \mathrm{Z}].$$

The sum is over flats Y and Y' which lie between X and Z. The map

$$(\mathbf{e}^* \circ \mathbf{e}^*)[\mathrm{X}, \mathrm{Z}] \otimes (\mathbf{e} \circ \mathbf{e})[\mathrm{X}, \mathrm{Z}] \longrightarrow \Bbbk$$

is as follows. The summands with $\mathrm{Y} \neq \mathrm{Y}'$ are sent to zero. For the summands with $\mathrm{Y} = \mathrm{Y}'$, we use the canonical pairing between a vector space and its dual. Then $\mathbf{r}^{\perp}[\mathrm{X}, \mathrm{Z}]$ is the subspace of $(\mathbf{e}^* \circ \mathbf{e}^*)[\mathrm{X}, \mathrm{Z}]$ which is orthogonal to $\mathbf{r}[\mathrm{X}, \mathrm{Z}]$ wrt the above pairing.

Definition 4.13. Let $\mathbf{a} = \langle \mathbf{e} \mid \mathbf{r} \rangle$ be a quadratic operad. The *unoriented quadratic dual* of $\mathbf{a}$ is defined to be

$$\mathbf{a}_! := \langle \mathbf{e}^* \,|\, \mathbf{r}^{\perp} \rangle.$$

The *oriented quadratic dual* of $\mathbf{a}$ is the oriented partner of $\mathbf{a}_!$, and we write it as

$$\mathbf{a}^! = \langle \mathbf{e}^* \,|\, \mathbf{r}^{\perp} \rangle^{\mathbf{o}} = \langle \mathbf{e}^{\circledast} \,|\, \mathbf{r}^{\ominus} \rangle,$$

with $\mathbf{e}^{\circledast} = (\mathbf{e}^*)^{\mathbf{o}}$ and $\mathbf{r}^{\ominus} = (\mathbf{r}^{\perp})^{\mathbf{o}}$.

The basic properties of these two duals are stated below.

$$(\mathbf{a}_!)_! \cong \mathbf{a}, \qquad (\mathbf{a}^!)^! \cong \mathbf{a}, \qquad (\mathbf{a}^{\mathbf{o}})_! \cong (\mathbf{a}_!)^{\mathbf{o}} \cong \mathbf{a}^!, \qquad (\mathbf{a}^{\mathbf{o}})^! \cong (\mathbf{a}^!)^{\mathbf{o}} \cong \mathbf{a}_!.$$

Proposition 4.14. *The commutative operad and Lie operad are oriented quadratic duals of each other:*

$$\mathbf{Com}^! = \mathbf{Lie} \quad \textit{and} \quad \mathbf{Lie}^! = \mathbf{Com}.$$

The associative operad $\mathbf{As}$ *is isomorphic to its oriented quadratic dual:*

$$\mathbf{As}^! \cong \mathbf{As}.$$

PROOF. The first statement follows from the observation that the relations defining $\mathbf{Com}$ are orthogonal to the Jacobi identities. Simply put: In a vector space V with basis $e_1, \dots, e_n$, the one-dimensional space spanned by the linear functional $e_1^* + \cdots + e_n^*$ is orthogonal to the subspace of V spanned by the vectors $e_i - e_j$, as i and j vary.

Let $\mathbf{As} = \langle \mathbf{e} \,|\, \mathbf{r} \rangle$. Observe that $\mathbf{r}^{\perp}[\mathrm{X}, \mathrm{Z}]$ is spanned by elements of the form

$$(A \lessdot F \lessdot C)^* + (A \lessdot G \lessdot C)^*,$$

since their pairing with elements of the form $(A \lessdot F \lessdot C) - (A \lessdot G \lessdot C)$ evaluates to 0. It follows that $\mathbf{r}^{\ominus}[\mathrm{X}, \mathrm{Z}]$ is spanned by elements of the form

$$(A \lessdot F \lessdot C)^* \otimes [A \lessdot F \lessdot C] - (A \lessdot G \lessdot C)^* \otimes [A \lessdot G \lessdot C].$$

Note that the sign between the two terms is now minus because $[A \lessdot F \lessdot C]$ and $[A \lessdot G \lessdot C]$ are opposite orientations.

To obtain self-duality of $\mathbf{As}$, we start with the isomorphism

$$\mathbf{e} \xrightarrow{\cong} \mathbf{e}^{\circledast}, \qquad (A \lessdot F) \mapsto (A \lessdot F)^* \otimes [A \lessdot F].$$

(Simply put, in a rank-one arrangement, a chamber determines an orientation.) Now observe that under the induced isomorphism $\mathbf{e} \circ \mathbf{e} \cong \mathbf{e}^{\circledast} \circ \mathbf{e}^{\circledast}$, the relations $\mathbf{r}$ correspond to $\mathbf{r}^{\ominus}$. Thus, the presentations $\langle \mathbf{e} \,|\, \mathbf{r} \rangle$ and $\langle \mathbf{e}^{\circledast} \,|\, \mathbf{r}^{\ominus} \rangle$ are isomorphic, as required. $\square$

4.10. Black and white circle products

Let $\mathcal{A}$-bqOp denote the category whose objects are finite-dimensional binary quadratic operads, and a morphism from $\mathbf{a} = \langle \mathbf{e} \,|\, \mathbf{r} \rangle$ to $\mathbf{b} = \langle \mathbf{f} \,|\, \mathbf{s} \rangle$ is a map $f : \mathbf{e} \to \mathbf{f}$ of dispecies such that $f \circ f$ sends $\mathbf{r}$ to $\mathbf{s}$. We introduce the black and white circle products on this category. They are conjugates of each other wrt quadratic duality. There is also a colax monoidal functor linking the white circle product to the Hadamard product, and hence the internal cohom for the former carries the structure of a Hopf operad.

All dispecies in this section are assumed to be finite-dimensional.

4.10.1. White circle product. Let $\mathbf{a} = \langle \mathbf{e} \,|\, \mathbf{r} \rangle$ and $\mathbf{b} = \langle \mathbf{f} \,|\, \mathbf{s} \rangle$ be binary quadratic operads. Consider the inclusion

$$(\mathbf{e} \times \mathbf{f}) \circ (\mathbf{e} \times \mathbf{f}) \hookrightarrow (\mathbf{e} \circ \mathbf{e}) \times (\mathbf{f} \circ \mathbf{f})$$

obtained from the interchange law (4.25a) between the Hadamard and substitution products. Define a new binary quadratic operad by

$$\mathbf{a} \bigcirc \mathbf{b} := \langle \mathbf{e} \times \mathbf{f} \,|\, \mathbf{r}' \rangle, \tag{4.31}$$

where $\mathbf{r}'$ is the inverse image of $\mathbf{r} \times (\mathbf{f} \circ \mathbf{f}) \vee (\mathbf{e} \circ \mathbf{e}) \times \mathbf{s}$ under the above inclusion. (The symbol $\vee$ refers to the sum inside $(\mathbf{e} \circ \mathbf{e}) \times (\mathbf{f} \circ \mathbf{f})$. We do not use $+$ since that symbol is being used for direct sum.) This is the *white circle product* of $\mathbf{a}$ and $\mathbf{b}$. It defines a symmetric monoidal structure on $\mathcal{A}$-bqOp. The unit object is the commutative operad.

Consider the composite map of operads

$$\mathcal{F}_{\circ}(\mathbf{e} \times \mathbf{f}) \to \mathcal{F}_{\circ}(\mathbf{e}) \times \mathcal{F}_{\circ}(\mathbf{f}) \twoheadrightarrow \mathbf{a} \times \mathbf{b}.$$

Observe that $\mathbf{r}'$ maps to 0, so there is an induced map

$$\mathbf{a} \bigcirc \mathbf{b} \to \mathbf{a} \times \mathbf{b} \tag{4.32}$$

of operads.

4.10.2. Black circle product. Dually, consider the surjection

$$(\mathbf{e} \circ \mathbf{e}) \times (\mathbf{f} \circ \mathbf{f}) \twoheadrightarrow (\mathbf{e} \times \mathbf{f}) \circ (\mathbf{e} \times \mathbf{f}).$$

Define a new binary quadratic operad by

$$\mathbf{a} \mathbin{\color{gray}\bullet} \mathbf{b} := \langle \mathbf{e} \times \mathbf{f} \,|\, \mathbf{r}'' \rangle, \tag{4.33}$$

where $\mathbf{r}''$ is the image of $\mathbf{r} \times \mathbf{s}$ under the above surjection. This is the unoriented *black circle product* of $\mathbf{a}$ and $\mathbf{b}$.

Proposition 4.15. *The white circle product and the unoriented black circle product are conjugates of each other wrt the unoriented quadratic dual, that is,*

$$(\mathbf{a} \mathbin{\color{gray}\bullet} \mathbf{b})_! = \mathbf{a}_! \circ \mathbf{b}_!, \quad \textit{or equivalently,} \quad (\mathbf{a} \circ \mathbf{b})_! = \mathbf{a}_! \mathbin{\color{gray}\bullet} \mathbf{b}_!.$$

Proof. We explain the second statement. Recall that $\mathbf{r}^\perp$ and $\mathbf{s}^\perp$ denote the space of relations of $\mathbf{a}_!$ and $\mathbf{b}_!$, respectively. By definition, the dispecies $\mathbf{r}'$ for the white circle product is given by the pullback diagram

$$\begin{tikzcd}
(\mathbf{e} \times \mathbf{f}) \circ (\mathbf{e} \times \mathbf{f}) \arrow[r, hook] & (\mathbf{e} \circ \mathbf{e}) \times (\mathbf{f} \circ \mathbf{f}) \\
\mathbf{r}' \arrow[u, hook] \arrow[r, hook] & \mathbf{r} \times (\mathbf{f} \circ \mathbf{f}) \vee (\mathbf{e} \circ \mathbf{e}) \times \mathbf{s}. \arrow[u, hook]
\end{tikzcd}$$

Dualizing gives a pushout diagram. Hence, taking kernels gives a *surjective* map from the kernel of the right vertical map, namely, $\mathbf{r}^\perp \times \mathbf{s}^\perp$, to the kernel of the left vertical map, namely, the space of relations of $(\mathbf{a} \circ \mathbf{b})_!$. $\square$

As a consequence, the unoriented black circle product specifies another monoidal structure on $\mathcal{A}$-bqOp. The unit object is the unoriented quadratic dual of **Com**. This is the same as $\mathbf{Lie}^{\mathbf{o}}$, the oriented partner of the Lie operad. It is instructive to check this fact directly.

The oriented *black circle product*, denoted $\mathbf{a} \bullet \mathbf{b}$, is defined to be the conjugate of the unoriented black circle product wrt the orientation functor. That is,

$$(\mathbf{a} \bullet \mathbf{b})^{\mathbf{o}} = \mathbf{a}^{\mathbf{o}} \mathbin{\color{gray}\bullet} \mathbf{b}^{\mathbf{o}}, \quad \text{or equivalently,} \quad (\mathbf{a} \mathbin{\color{gray}\bullet} \mathbf{b})^{\mathbf{o}} = \mathbf{a}^{\mathbf{o}} \bullet \mathbf{b}^{\mathbf{o}}.$$

This is also a monoidal structure with unit object **Lie**. By composing two conjugations:

Proposition 4.16. *The white circle product and the oriented black circle product are conjugates of each other wrt the oriented quadratic dual, that is,*

$$(\mathbf{a} \bullet \mathbf{b})^! = \mathbf{a}^! \circ \mathbf{b}^!, \quad \textit{or equivalently,} \quad (\mathbf{a} \circ \mathbf{b})^! = \mathbf{a}^! \bullet \mathbf{b}^!.$$

Exercise 4.17. Show that $\mathbf{a}^{\mathbf{o}} \circ \mathbf{b} \cong (\mathbf{a} \circ \mathbf{b})^{\mathbf{o}} \cong \mathbf{a} \circ \mathbf{b}^{\mathbf{o}}$. The same holds for both black circle products as well.

4.10.3. Internal hom. The internal hom for the black circle products have a similar flavor to that for ordinary vector spaces:

Proposition 4.18. *There are natural bijections*

$$\mathcal{A}\text{-}\mathsf{bqOp}(\mathbf{a} \mathbin{\color{gray}\bullet} \mathbf{b}, \mathbf{c}) \cong \mathcal{A}\text{-}\mathsf{bqOp}(\mathbf{a}, \mathbf{b}_{!} \circ \mathbf{c}),$$

$$\mathcal{A}\text{-}\mathsf{bqOp}(\mathbf{a} \bullet \mathbf{b}, \mathbf{c}) \cong \mathcal{A}\text{-}\mathsf{bqOp}(\mathbf{a}, \mathbf{b}^{!} \circ \mathbf{c}).$$

Proof. We explain the first claim, the second is a formal consequence of the first. Write $\mathbf{a} = \langle \mathbf{e} \,|\, \mathbf{r} \rangle$, $\mathbf{b} = \langle \mathbf{f} \,|\, \mathbf{s} \rangle$, $\mathbf{c} = \langle \mathbf{g} \,|\, \mathbf{t} \rangle$. A map from $\mathbf{a} \mathbin{\color{gray}\bullet} \mathbf{b}$ to $\mathbf{c}$ is the same as a map $\mathbf{e} \times \mathbf{f} \to \mathbf{g}$ such that the composite

$$(\mathbf{e} \circ \mathbf{e}) \times (\mathbf{f} \circ \mathbf{f}) \twoheadrightarrow (\mathbf{e} \times \mathbf{f}) \circ (\mathbf{e} \times \mathbf{f}) \to \mathbf{g} \circ \mathbf{g}$$

takes $\mathbf{r} \times \mathbf{s}$ to $\mathbf{t}$. Similarly, a map from $\mathbf{a}$ to $\mathbf{b}_{!} \circ \mathbf{c}$ is the same as a map $\mathbf{e} \to \mathbf{f}^* \times \mathbf{g}$ such that the composite

$$\mathbf{e} \circ \mathbf{e} \to (\mathbf{f}^* \times \mathbf{g}) \circ (\mathbf{f}^* \times \mathbf{g}) \hookrightarrow (\mathbf{f}^* \circ \mathbf{f}^*) \times (\mathbf{g} \circ \mathbf{g})$$

takes $\mathbf{r}$ to $\mathbf{s}^{\perp} \times (\mathbf{g} \circ \mathbf{g}) \vee (\mathbf{f}^* \circ \mathbf{f}^*) \times \mathbf{t}$. The two statements are equivalent by elementary linear algebra. ☐

Thus,

$$\hom^{\color{gray}\bullet}(\mathbf{a}, \mathbf{b}) = \mathbf{a}_{!} \circ \mathbf{b} \quad \text{and} \quad \hom^{\bullet}(\mathbf{a}, \mathbf{b}) = \mathbf{a}^{!} \circ \mathbf{b}.$$

By Proposition B.4,

$$\operatorname{end}^{\color{gray}\bullet}(\mathbf{a}) = \mathbf{a}_{!} \circ \mathbf{a} \quad \text{and} \quad \operatorname{end}^{\bullet}(\mathbf{a}) = \mathbf{a}^{!} \circ \mathbf{a}$$

are monoids in the monoidal categories $(\mathcal{A}\text{-}\mathsf{bqOp}, \mathbin{\color{gray}\bullet})$ and $(\mathcal{A}\text{-}\mathsf{bqOp}, \bullet)$, respectively.

Dually, the internal cohom for the white circle product is given by $\mathbf{a} \mathbin{\color{gray}\bullet} \mathbf{b}_{!}$ which is the same as $\mathbf{a} \bullet \mathbf{b}^{!}$. As a consequence, $\mathbf{a} \bullet \mathbf{a}^{!}$ is a comonoid in the monoidal category $(\mathcal{A}\text{-}\mathsf{bqOp}, \circ)$. Now observe that the inclusion functor

$$(\mathcal{A}\text{-}\mathsf{bqOp}, \circ) \to (\mathcal{A}\text{-}\mathsf{Op}, \times)$$

is colax monoidal wrt the structure map (4.32). So it preserves comonoids. Hence, $\mathbf{a} \bullet \mathbf{a}^{!}$ is a comonoid wrt the Hadamard product on operads, which is the same as a Hopf operad. Thus:

Corollary 4.19. *For any finite-dimensional binary quadratic operad* $\mathbf{a}$*, the black circle product* $\mathbf{a} \bullet \mathbf{a}^{!}$ *is a Hopf operad.*

Corollary 4.20. *For any finite-dimensional binary quadratic operad* $\mathbf{a}$*, there exist morphisms of operads*

$$\mathbf{Lie}^{\mathbf{o}} \to \mathbf{a}_{!} \times \mathbf{a} \quad \textit{and} \quad \mathbf{Lie} \to \mathbf{a}^{!} \times \mathbf{a}. \tag{4.34}$$

They are obtained as the composites

$$\mathbf{Lie}^{\mathbf{o}} \to \mathbf{a}_{!} \circ \mathbf{a} \to \mathbf{a}_{!} \times \mathbf{a} \quad \text{and} \quad \mathbf{Lie} \to \mathbf{a}^{!} \circ \mathbf{a} \to \mathbf{a}^{!} \times \mathbf{a}.$$

For the first arrow, we used that $\mathbf{Lie}^{\mathbf{o}}$ and $\mathbf{Lie}$ are unit objects for the unoriented and oriented black circle products, respectively.

Note that the two maps can be obtained from each other by applying the orientation functor. Evaluated on (X, Y) with $\mathrm{rk}(\mathrm{Y}/\mathrm{X}) = 1$, the first map has the canonical form $\Bbbk \to V^* \otimes V$.

4.11. Left modules over operads

We now make a formal connection between species and dispecies. The category of species is a left module category over the monoidal category of dispecies. This allows us to consider left modules over operads in the category of species. Moreover, operads give rise to monads on species resulting in an equivalence between operad modules and monad algebras. We illustrate these results on the standard operads; notably **As**, **Com**, $\mathbf{Com}^-$, respectively, yield the monads $\mathcal{T}$, $\mathcal{S}$, $\mathcal{E}$ discussed in Chapter 3. Thus, a left **As**-module is the same as a monoid, a left **Com**-module is the same as a commutative monoid, a left $\mathbf{Com}^-$-module is the same as a signed commutative monoid.

Dually, we have the notion of left comodules over cooperads, cooperads giving rise to comonads on species, and so on. Since these notions and results are obtained from the above by reversing arrows in a formal manner, we do not discuss them explicitly.

4.11.1. Species as a left module over dispecies. For a dispecies $\mathbf{p}$ and a species m, define the species $\mathbf{p} \circ \mathsf{m}$ by

$$(\mathbf{p} \circ \mathsf{m})[\mathrm{X}] := \bigoplus_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{X}} \mathbf{p}[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{m}[\mathrm{Y}]. \tag{4.35}$$

We refer to $\mathbf{p} \circ \mathsf{m}$ as the substitution product of $\mathbf{p}$ and m. Observe that there are natural isomorphisms

$$(\mathbf{p} \circ \mathbf{q}) \circ \mathsf{m} \cong \mathbf{p} \circ (\mathbf{q} \circ \mathsf{m}) \quad \text{and} \quad \mathbf{x} \circ \mathsf{m} \cong \mathsf{m}$$

for dispecies $\mathbf{p}$ and $\mathbf{q}$ and species m. Thus, the category of species is a left module category over the monoidal category of dispecies (Appendix B.1.4). Moreover,

$$(\mathbf{p}_1 + \mathbf{p}_2) \circ \mathsf{m} \cong \mathbf{p}_1 \circ \mathsf{m} + \mathbf{p}_2 \circ \mathsf{m} \quad \text{and} \quad \mathbf{p} \circ (\mathsf{m}_1 + \mathsf{m}_2) \cong \mathbf{p} \circ \mathsf{m}_1 + \mathbf{p} \circ \mathsf{m}_2.$$

We now describe the enriched hom for species as a left module category over dispecies (Appendix B.2.4). For species m and n,

$$\mathrm{hom}^{\circ}(\mathsf{m}, \mathsf{n})[\mathrm{X}, \mathrm{Y}] := \mathrm{Hom}_{\Bbbk}(\mathsf{m}[\mathrm{Y}], \mathsf{n}[\mathrm{X}]).$$

By Proposition B.5, for any species m,

$$\mathrm{end}^{\circ}(\mathsf{m}) := \mathrm{hom}^{\circ}(\mathsf{m}, \mathsf{m})$$

is an operad. We call it the *endomorphism operad* of the species m. Explicitly, γ is given by composition of maps, while η sends $1 \in \Bbbk$ to the identity map.

Exercise 4.21. Check that: The substitution product of dispecies has an internal hom. Explicitly, for dispecies $\mathbf{p}$ and $\mathbf{q}$,

$$\mathrm{hom}^{\circ}(\mathbf{p}, \mathbf{q})[\mathrm{X}, \mathrm{Y}] := \bigoplus_{\mathrm{Z}:\, \mathrm{Z} \geq \mathrm{Y}} \mathrm{Hom}_{\Bbbk}(\mathbf{p}[\mathrm{Y}, \mathrm{Z}], \mathbf{q}[\mathrm{X}, \mathrm{Z}]).$$

Use the discussion in Appendix B.3.3 to deduce that the category of species is enriched over the monoidal category of dispecies and the copower is given by the substitution product (4.35).

4.11.2. Operad modules. The category of species is a left module category over the monoidal category of dispecies and an operad is a monoid in the latter. Thus, associated to any operad $\mathbf{a}$, we have the category of left $\mathbf{a}$-modules. Let us spell out this construction.

Let $\mathbf{a}$ be an operad. A left $\mathbf{a}$-module is a species m equipped with a map of species

$$\mathbf{a} \circ \mathsf{m} \to \mathsf{m}$$

satisfying the associativity and unitality axioms. A map of left $\mathbf{a}$-modules is a map of species $\mathsf{m} \to \mathsf{n}$ which commutes with the respective structure maps.

Explicitly, a left module m over $\mathbf{a}$ is a species m equipped with linear maps

$$\gamma : \mathbf{a}[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{m}[\mathrm{Y}] \to \mathsf{m}[\mathrm{X}], \tag{4.36}$$

one for each $\mathrm{X} \leq \mathrm{Y}$, such that the following axioms hold.

Associativity. For any $\mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}$, the diagram

$$\begin{tikzcd} \mathbf{a}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y}, \mathrm{Z}] \otimes \mathsf{m}[\mathrm{Z}] \arrow[r, "\mathrm{id} \otimes \gamma"] \arrow[d, "\gamma \otimes \mathrm{id}"'] & \mathbf{a}[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{m}[\mathrm{Y}] \arrow[d, "\gamma"] \\ \mathbf{a}[\mathrm{X}, \mathrm{Z}] \otimes \mathsf{m}[\mathrm{Z}] \arrow[r, "\gamma"'] & \mathsf{m}[\mathrm{X}] \end{tikzcd} \tag{4.37a}$$

commutes.

Unitality. For any X, the diagram

$$\begin{tikzcd} & \mathbf{a}[\mathrm{X}, \mathrm{X}] \otimes \mathsf{m}[\mathrm{X}] \arrow[dr, "\gamma"] & \\ \Bbbk \otimes \mathsf{m}[\mathrm{X}] \arrow[ur, "\eta \otimes \mathrm{id}"] \arrow[rr, "\cong"'] & & \mathsf{m}[\mathrm{X}] \end{tikzcd} \tag{4.37b}$$

commutes.

A map of left modules is a map $f : \mathsf{m} \to \mathsf{n}$ of species such that for any $\mathrm{X} \leq \mathrm{Y}$, the diagram

$$\begin{tikzcd} \mathbf{a}[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{m}[\mathrm{Y}] \arrow[r, "\gamma"] \arrow[d, "\mathrm{id} \otimes f_{\mathrm{Y}}"'] & \mathsf{m}[\mathrm{X}] \arrow[d, "f_{\mathrm{X}}"] \\ \mathbf{a}[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{n}[\mathrm{Y}] \arrow[r, "\gamma"'] & \mathsf{n}[\mathrm{X}] \end{tikzcd} \tag{4.38}$$

commutes.

Exercise 4.22. One can also consider left modules over set-operads. Recall from Lemma 4.3 that a set-operad can be viewed as a category. Show that a left module over a set-operad is the same as a functor from this category to the category of sets.

4.11.3. Free operad modules.

Proposition 4.23. *For any operad* $\mathbf{a}$*, the free left* $\mathbf{a}$*-module over a species* p *is given by* $\mathbf{a} \circ \mathsf{p}$*, with structure map given by the composite*

$$\mathbf{a} \circ (\mathbf{a} \circ \mathsf{p}) \xrightarrow{\cong} (\mathbf{a} \circ \mathbf{a}) \circ \mathsf{p} \xrightarrow{\gamma \circ \mathrm{id}} \mathbf{a} \circ \mathsf{p}.$$

Proof. This is an instance of the general result: For a monoid A in a monoidal category $(\mathsf{C}, \bullet)$ and an object M in a left module category over it, the free left A-module over M is given by $A \bullet M$. In our case, the monoidal category is $(\mathcal{A}\text{-}\mathsf{dSp}, \circ)$ and the left module category over it is $\mathcal{A}\text{-}\mathsf{Sp}$. □

The universal property of the free operad module is stated below.

Theorem 4.24. *Let* $\mathbf{a}$ *be an operad. Let* m *be a left* $\mathbf{a}$*-module,* p *a species,* $f : \mathsf{p} \to \mathsf{m}$ *a map of species. Then there exists a unique morphism of left* $\mathbf{a}$*-modules* $\hat{f} : \mathbf{a} \circ \mathsf{p} \to \mathsf{m}$ *such that the diagram*

$$\begin{tikzcd} \mathbf{a} \circ \mathsf{p} \arrow[r, dashed, "\hat{f}"] & \mathsf{m} \\ \mathsf{p} \arrow[u, hook] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes.

The vertical map is the composite $\mathsf{p} \cong \mathbf{x} \circ \mathsf{p} \hookrightarrow \mathbf{a} \circ \mathsf{p}$. Explicitly, the map $\hat{f}$ is

$$\mathbf{a} \circ \mathsf{p} \xrightarrow{\mathrm{id} \circ f} \mathbf{a} \circ \mathsf{m} \xrightarrow{\gamma} \mathsf{m}. \tag{4.39}$$

Example 4.25. Let x be the species defined by $\mathsf{x}[\mathrm{Y}]$ is $\Bbbk$ if $\mathrm{Y} = \top$, and 0 otherwise. (This is the same as the species characteristic of chambers considered later in Section 7.1.) For any dispecies $\mathbf{p}$, the species $\mathbf{p} \circ \mathsf{x}$ is given by

$$(\mathbf{p} \circ \mathsf{x})[\mathrm{X}] = \mathbf{p}[\mathrm{X}, \top].$$

Thus, any dispecies gives rise to a species by fixing the second coordinate of the nested flat to be the maximum flat. This was mentioned in Section 4.1.2.

Now suppose $\mathbf{a}$ is an operad. Then $\mathbf{a} \circ \mathsf{x}$ is the free left $\mathbf{a}$-module over x. Its structure map is obtained by restricting the product of $\mathbf{a}$:

$$\mathbf{a}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{a}[\mathrm{Y}, \top] \to \mathbf{a}[\mathrm{X}, \top].$$

4.11.4. Left modules over quadratic operads.

Proposition 4.26. *Let* $\mathbf{a} = \langle \mathbf{e} \mid \mathbf{r} \rangle$ *be a quadratic operad. Then a left* $\mathbf{a}$*-module is the same as a species* m *equipped with a map* $f : \mathbf{e} \circ \mathsf{m} \to \mathsf{m}$ *of species such that the composite*

$$\mathbf{r} \circ \mathsf{m} \to \mathbf{e} \circ \mathbf{e} \circ \mathsf{m} \xrightarrow{\mathrm{id} \circ f} \mathbf{e} \circ \mathsf{m} \xrightarrow{f} \mathsf{m}$$

is zero.

Proof. First note that a left module m over the free operad $\mathcal{F}_\circ(\mathbf{e})$ is the same as a map $\mathbf{e} \circ \mathsf{m} \to \mathsf{m}$ of species. Now a left $\mathbf{a}$-module is the same as a $\mathcal{F}_\circ(\mathbf{e})$-module in which the ideal generated by $\mathbf{r}$ acts by zero which is the same as $\mathbf{r}$ acting by zero. The result follows. □

4.11.5. Signed operad modules. Recall that for any operad $\mathbf{a}$, we have the operad $\mathbf{a}^-$ obtained by applying the signature functor on dispecies. We use the term *signed left* $\mathbf{a}$*-module* for a left $\mathbf{a}^-$-module.

For any species m, define another species m^- by

$$\mathsf{m}^-[\mathrm{X}] := \mathsf{m}[\mathrm{X}] \otimes \mathbf{E}^-[\mathrm{X}, \top]. \tag{4.40}$$

The assignment $\mathsf{m} \mapsto \mathsf{m}^-$ is functorial in m. This is the *signature functor* on species. It is studied further in Section 8.10.

Observe that for any species m,

$$(\mathbf{a} \circ \mathsf{m})^- = \mathbf{a}^- \circ \mathsf{m}^-. \tag{4.41}$$

We can deduce from here that m is a left $\mathbf{a}$-module iff m^- is a left $\mathbf{a}^-$-module. Thus, we obtain:

Proposition 4.27. *For any operad* $\mathbf{a}$*, there is an isomorphism between the category of left* $\mathbf{a}$*-modules and the category of left* $\mathbf{a}^-$*-modules.*

4.11.6. Operads and monads on species. Every operad $\mathbf{a}$ gives rise to a monad $\mathcal{V}_{\mathbf{a}}$ on species via

$$\mathcal{V}_{\mathbf{a}}(\mathsf{m}) := \mathbf{a} \circ \mathsf{m}. \tag{4.42}$$

A map between operads induces a structure on the identity functor on species which is lax wrt the corresponding monads.

Exercise 4.28. Use (4.41) to check that: For any operad $\mathbf{a}$, the signature functor on species is a (lax) isomorphism of monads $\mathcal{V}_{\mathbf{a}}$ and $\mathcal{V}_{\mathbf{a}^-}$.

We now turn to some examples. Recall the monads $\mathcal{T}$, $\mathcal{S}$, $\mathcal{E}$ from Sections 3.1.1, 3.2.1, 3.2.6, respectively.

Lemma 4.29. *The operads* $\mathbf{As}$, $\mathbf{Com}$, $\mathbf{Com}^-$ *yield the monads* $\mathcal{T}$, $\mathcal{S}$, $\mathcal{E}$, *respectively.*

Proof. The monad on species induced by $\mathbf{As}$ is $\mathcal{T}$: For any species m, the required isomorphism

$$\mathbf{As} \circ \mathsf{m} \to \mathcal{T}(\mathsf{m})$$

is as follows. Evaluating on the X-component, on the Y-summand for $\mathrm{Y} \geq \mathrm{X}$, it is given by

$$\begin{aligned} \mathbf{As}[\mathrm{X},\mathrm{Y}] \otimes \mathsf{m}[\mathrm{Y}] &\longrightarrow \bigoplus_{F\colon F \geq A,\, \mathrm{s}(F)=\mathrm{Y}} \mathsf{m}[F] \\ \Big(\sum_{F\colon F \geq A,\, \mathrm{s}(F)=\mathrm{Y}} x^{F/A} \mathtt{H}_{F/A}\Big) \otimes v &\longmapsto \sum_{F\colon F \geq A,\, \mathrm{s}(F)=\mathrm{Y}} x^{F/A} \beta_{F,\mathrm{Y}}(v). \end{aligned} \tag{4.43}$$

(Here A is an arbitrary but fixed face of support X.) It is now a routine check that the operad structure of $\mathbf{As}$ corresponds to the monad structure of $\mathcal{T}$.

The monad on species induced by $\mathbf{Com}$ is $\mathcal{S}$: For any species m, the required isomorphism

$$\mathbf{Com} \circ \mathsf{m} \to \mathcal{S}(\mathsf{m})$$

is as follows. Evaluating on the X-component, on the Y-summand for $\mathrm{Y} \geq \mathrm{X}$, it is the canonical identification

$$\mathbf{Com}[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{m}[\mathrm{Y}] \xrightarrow{\cong} \mathsf{m}[\mathrm{Y}]. \tag{4.44}$$

Similarly, the monad induced by $\mathbf{Com}^-$ is $\mathcal{E}$ via

$$\mathbf{Com}^-[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{m}[\mathrm{Y}] = \mathbf{E}^-[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{m}[\mathrm{Y}].$$

The details are straightforward. □

The map $\mathbf{As} \to \mathbf{Com}$ of operads in (4.20) yields the abelianization $\pi : \mathcal{T} \to \mathcal{S}$ defined in (3.12). Applying the signature functor to this map and using Lemma 4.6, we have a map $\mathbf{As} \to \mathbf{Com}^-$ of operads. This yields the signed abelianization $\pi_{-1} : \mathcal{T} \to \mathcal{E}$ defined in (3.19).

4.11.7. Operad modules and monad algebras. For any operad $\mathbf{a}$, the category of left $\mathbf{a}$-modules is isomorphic to the category of $\mathcal{V}_{\mathbf{a}}$-algebras, where $\mathcal{V}_{\mathbf{a}}$ is the monad defined by (4.42). This is a straightforward observation.

Lemma 4.30. *We have:*

- *The category of left* $\mathbf{As}$*-modules is isomorphic to the category of* $\mathcal{A}$*-monoids.*
- *The category of left* $\mathbf{Com}$*-modules is isomorphic to the category of commutative* $\mathcal{A}$*-monoids.*
- *The category of left* $\mathbf{Com}^-$*-modules is isomorphic to the category of signed commutative* $\mathcal{A}$*-monoids.*

Proof. This follows from Lemma 4.29 and Propositions 3.2, 3.12, 3.15. □

This result can also be understood directly as follows. Suppose a is a left module over $\mathbf{Com}$. Then, by (4.36), for each $\mathrm{X} \leq \mathrm{Y}$, we have a linear map

$$\mathbf{Com}[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{a}[\mathrm{Y}] \to \mathsf{a}[\mathrm{X}]$$

which we rewrite as

$$\mu_{\mathrm{X}}^{\mathrm{Y}} : \mathsf{a}[\mathrm{Y}] \to \mathsf{a}[\mathrm{X}].$$

Associativity (4.37a) and unitality (4.37b) say that diagrams (2.21) commute. This is the same as a commutative monoid in species as formulated in Proposition 2.20. The analysis for $\mathbf{Com}^-$ is similar, we now make use of Proposition 2.36.

Now suppose a is a left module over $\mathbf{As}$. Then, by (4.36), for each $\mathrm{X} \leq \mathrm{Y}$, we have a linear map

$$\mathbf{As}[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{a}[\mathrm{Y}] \to \mathsf{a}[\mathrm{X}].$$

Thus, for each $F \geq A$, with $\mathrm{s}(F) = \mathrm{Y}$ and $\mathrm{s}(A) = \mathrm{X}$, the basis element $\mathtt{H}_{F/A}$ yields a map $\mathsf{a}[\mathrm{Y}] \to \mathsf{a}[\mathrm{X}]$ which further yields

$$\mu_A^F : \mathsf{a}[F] \xrightarrow{\beta_{\mathrm{Y},F}} \mathsf{a}[\mathrm{Y}] \to \mathsf{a}[\mathrm{X}] \xrightarrow{\beta_{A,\mathrm{X}}} \mathsf{a}[A].$$

One may check that these maps, one for each $F \geq A$, are subject precisely to the axioms (2.21). Thus, a is the same as a monoid in species.

Remark 4.31. One may also formulate a monoid in species as follows. A monoid a consists of vector spaces $\mathsf{a}[\mathrm{X}]$, one for each flat X, with product components $\mu_{\mathrm{L}} : \mathsf{a}[\mathrm{Y}] \to \mathsf{a}[\mathrm{X}]$ indexed by lunes L with base X and case Y, subject to associativity and unitality axioms, namely, $\mu_{\mathrm{L}}\mu_{\mathrm{M}} = \mu_{\mathrm{L}\circ\mathrm{M}}$ whenever L and M are composable, and $\mu_{\mathrm{X}} = \mathrm{id}$ for any flat X.

Exercise 4.32. Note that Proposition 4.26 applied to the binary quadratic operad **As** yields a description of monoids whose structure maps only involve faces with a cover relation. Check that this is precisely Proposition 2.78. Similarly, check that for **Com**, the description of commutative monoids in terms of flats with a cover relation is precisely Proposition 2.73 .

TABLE 4.1. Operads and monads on species.

operad $\mathbf{a}$	monad $\mathcal{V}_{\mathbf{a}}$	operad module = monad algebra
As	$\mathcal{T}$	monoid
Com	$\mathcal{S}$	commutative monoid
$\mathbf{Com}^-$	$\mathcal{E}$	signed commutative monoid
Lie	$\mathcal{PT}$	Lie monoid
$\mathbf{Lie}^-$	$\mathcal{PT}_{-1}$	signed Lie monoid

Table 4.1 summarizes the above examples, and in addition says what happens with the Lie operad. The relevant monads are $\mathcal{PT}$ and $\mathcal{PT}_{-1}$. These are defined later in Section 13.7 and the connection to Lie is made in Proposition 16.6 and subsequent discussion.

The correspondence in Table 4.1 is further strengthened by the diagrams

$$\begin{array}{ccccc} \mathbf{Lie} & \longrightarrow & \mathbf{As} & \longrightarrow & \mathbf{Com} \\ \| & & \| & & \| \\ \mathcal{PT} & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{S} \end{array} \qquad \begin{array}{ccccc} \mathbf{Lie}^- & \longrightarrow & \mathbf{As} & \longrightarrow & \mathbf{Com}^- \\ \| & & \| & & \| \\ \mathcal{PT}_{-1} & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{E}. \end{array}$$

The morphisms on the left are as in (4.20) and (13.59), while those on the right are their signed counterparts.

4.11.8. Cooperad comodules. We mention that the cooperads dual to **As**, **Com**, $\mathbf{Com}^-$ give rise to the comonads $\mathcal{T}^\vee$, $\mathcal{S}^\vee$, $\mathcal{E}^\vee$, respectively. Thus,

- a left comodule over $\mathbf{As}^*$ is a comonoid in species,
- a left comodule over $\mathbf{Com}^*$ is a cocommutative comonoid in species,
- a left comodule over $(\mathbf{Com}^-)^*$ is a signed cocommutative comonoid in species.

This is summarized in Table 4.2. For entries related to (signed) Lie comonoids, see Section 16.8, particularly, Proposition 16.42.

TABLE 4.2. Cooperads and comonads on species.

cooperad $\mathbf{c}$	comonad $\mathcal{U}_{\mathbf{c}}$	cooperad comodule = comonad coalgebra
$\mathbf{As}^*$	$\mathcal{T}^\vee$	comonoid
$\mathbf{Com}^*$	$\mathcal{S}^\vee$	cocommutative comonoid
$(\mathbf{Com}^-)^*$	$\mathcal{E}^\vee$	signed cocommutative comonoid
$\mathbf{Lie}^*$	$\mathcal{Q}\mathcal{T}^\vee$	Lie comonoid
$(\mathbf{Lie}^-)^*$	$\mathcal{Q}\mathcal{T}^\vee_{-1}$	signed Lie comonoid

4.12. Bioperads. Mixed distributive laws

Mixed distributive laws is a general concept from category theory. A bioperad is a triple consisting of an operad, a cooperad, and a mixed distributive law linking them. Just as one defines left (co)modules over (co)operads, one can define left bimodules over bioperads. A bioperad gives rise to a bimonad on species with left bimodules over the bioperad corresponding to bialgebras over the corresponding bimonad.

We define a mixed distributive law between the associative operad and its dual cooperad. The corresponding bimonad on species in $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$ whose bialgebras are bimonoids in species. Similar considerations can be made for the commutative and signed cases.

4.12.1. Bioperads. Let $\mathbf{a}$ be an operad and $\mathbf{c}$ be a cooperad. A *mixed distributive law* between $\mathbf{a}$ and $\mathbf{c}$ is a map of dispecies

$$\lambda : \mathbf{a} \circ \mathbf{c} \to \mathbf{c} \circ \mathbf{a}$$

such that the diagrams

$$\begin{tikzcd} \mathbf{a}\circ\mathbf{c}\circ\mathbf{c} \arrow[r, "\lambda\circ\mathrm{id}"] & \mathbf{c}\circ\mathbf{a}\circ\mathbf{c} \arrow[r, "\mathrm{id}\circ\lambda"] & \mathbf{c}\circ\mathbf{c}\circ\mathbf{a} \\ \mathbf{a}\circ\mathbf{c} \arrow[u] \arrow[rr, "\lambda"'] & & \mathbf{c}\circ\mathbf{a} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{a}\circ\mathbf{a}\circ\mathbf{c} \arrow[r, "\mathrm{id}\circ\lambda"] \arrow[d] & \mathbf{a}\circ\mathbf{c}\circ\mathbf{a} \arrow[r, "\lambda\circ\mathrm{id}"] & \mathbf{c}\circ\mathbf{a}\circ\mathbf{a} \arrow[d] \\ \mathbf{a}\circ\mathbf{c} \arrow[rr, "\lambda"'] & & \mathbf{c}\circ\mathbf{a} \end{tikzcd}$$

$$\begin{tikzcd} & \mathbf{c} \arrow[dl] \arrow[dr] & \\ \mathbf{a}\circ\mathbf{c} \arrow[rr, "\lambda"'] & & \mathbf{c}\circ\mathbf{a} \end{tikzcd} \qquad \begin{tikzcd} & \mathbf{a} & \\ \mathbf{a}\circ\mathbf{c} \arrow[ur] \arrow[rr, "\lambda"'] & & \mathbf{c}\circ\mathbf{a} \arrow[ul] \end{tikzcd}$$

commute. The unlabeled arrows are induced from the (co)operad structures of $\mathbf{a}$ and $\mathbf{c}$. We refer to the triple $(\mathbf{a}, \mathbf{c}, \lambda)$ as a *bioperad*.

A morphism $(\mathbf{a}, \mathbf{c}, \lambda) \to (\mathbf{a}', \mathbf{c}', \lambda')$ of bioperads consists of a morphism of operads $\mathbf{a}' \to \mathbf{a}$ and a morphism of cooperads $\mathbf{c} \to \mathbf{c}'$ such that the diagram

$$\begin{tikzcd} \mathbf{a}'\circ\mathbf{c} \arrow[r] \arrow[d] & \mathbf{a}\circ\mathbf{c} \arrow[r, "\lambda"] & \mathbf{c}\circ\mathbf{a} \arrow[d] \\ \mathbf{a}'\circ\mathbf{c}' \arrow[r, "\lambda'"'] & \mathbf{c}'\circ\mathbf{a}' \arrow[r] & \mathbf{c}'\circ\mathbf{a} \end{tikzcd}$$

commutes.

4.12.2. Left bimodules over bioperads. A left $(\mathbf{a}, \mathbf{c}, \lambda)$-bimodule is a species m which is a left module over $\mathbf{a}$ and a left comodule over $\mathbf{c}$ such that the diagram

$$\begin{tikzcd} \mathbf{a} \circ \mathsf{m} \arrow[r] \arrow[d] & \mathsf{m} \arrow[r] & \mathbf{c} \circ \mathsf{m} \\ \mathbf{a} \circ \mathbf{c} \circ \mathsf{m} \arrow[rr, "\lambda \circ \mathrm{id}"'] & & \mathbf{c} \circ \mathbf{a} \circ \mathsf{m} \arrow[u] \end{tikzcd}$$

commutes. The unlabeled arrows are induced from the left (co)module structures of $\mathbf{a}$ and $\mathbf{c}$.

Note that: A morphism $(\mathbf{a}, \mathbf{c}, \lambda) \to (\mathbf{a}', \mathbf{c}', \lambda')$ of bioperads induces a functor from the category of left $(\mathbf{a}, \mathbf{c}, \lambda)$-bimodules to the category of left $(\mathbf{a}', \mathbf{c}', \lambda')$-bimodules.

4.12.3. Bioperads and bimonads. The connection between (co)operads and (co)monads in species discussed in Section 4.11 can be extended as follows.

Every bioperad $(\mathbf{a}, \mathbf{c}, \lambda)$ gives rise to a bimonad $(\mathcal{V}_{\mathbf{a}}, \mathcal{U}_{\mathbf{c}}, \lambda)$ on species via

$$\mathcal{V}_{\mathbf{a}}(\mathsf{m}) := \mathbf{a} \circ \mathsf{m}, \qquad \mathcal{U}_{\mathbf{c}}(\mathsf{m}) := \mathbf{c} \circ \mathsf{m},$$

with the mixed distributive law given by

$$\mathcal{V}_{\mathbf{a}}\mathcal{U}_{\mathbf{c}} \to \mathcal{U}_{\mathbf{c}}\mathcal{V}_{\mathbf{a}}, \qquad \mathbf{a} \circ \mathbf{c} \circ \mathsf{m} \xrightarrow{\lambda \circ \mathrm{id}} \mathbf{c} \circ \mathbf{a} \circ \mathsf{m}.$$

A morphism between bioperads induces a structure on the identity functor on species which is bilax wrt the corresponding bimonads. Further, the category of left $(\mathbf{a}, \mathbf{c}, \lambda)$-bimodules is isomorphic to the category of $(\mathcal{V}_{\mathbf{a}}, \mathcal{U}_{\mathbf{c}}, \lambda)$-bialgebras.

4.12.4. Mixed distributive law between As and As*. Consider the operad $\mathbf{As}$ and its dual cooperad $\mathbf{As}^*$. One can define a mixed distributive law λ between them as follows.

$$\begin{gathered} \bigoplus_{\mathrm{Y}\colon \mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}} \mathbf{As}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{As}^*[\mathrm{Y}, \mathrm{Z}] \longrightarrow \bigoplus_{\mathrm{Y}'\colon \mathrm{X} \leq \mathrm{Y}' \leq \mathrm{Z}} \mathbf{As}^*[\mathrm{X}, \mathrm{Y}'] \otimes \mathbf{As}[\mathrm{Y}', \mathrm{Z}] \\ \mathtt{H}_{F/A} \otimes \mathtt{M}_{G/F} \longmapsto \sum_{\substack{(F', G')\colon F' \geq A \\ FF' = G,\, F'F = G'}} \mathtt{M}_{F'/A} \otimes \mathtt{H}_{G'/F'} \end{gathered} \tag{4.45}$$

(We are writing $\mathtt{M}$ for the basis dual to $\mathtt{H}$.)

Theorem 4.33. *The triple $(\mathbf{As}, \mathbf{As}^*, \lambda)$ is a bioperad, and it yields the bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$ on species. Left bimodules over $(\mathbf{As}, \mathbf{As}^*, \lambda)$ are the same as bimonoids in species.*

PROOF. It is easy to see that (4.45) translates to (3.5). As expected, the argument that $(\mathbf{As}, \mathbf{As}^*, \lambda)$ is a bioperad is similar to the one in the proof of Theorem 3.4, so we omit it. The second part follows from Proposition 3.5. $\square$

More generally, for any distance function υ, one can deform λ by multiplying each summand in (4.45) by $\upsilon_{G,G'}$. This defines the bioperad $(\mathbf{As}, \mathbf{As}^*, \lambda_\upsilon)$ which then yields the bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda_\upsilon)$ given in Theorem 3.7. It follows from Proposition 3.8 that left bimodules over $(\mathbf{As}, \mathbf{As}^*, \lambda_\upsilon)$ are the same as υ-bimonoids. The case of q-bimonoids can be seen as a special case.

The mixed distributive law λ between $\mathbf{As}$ and $\mathbf{As}^*$ induces the following commutative diagram.

(4.46)
$$\begin{tikzcd}
& \mathbf{As} \circ \mathbf{As}^* \arrow[rr, "\lambda"] \arrow[dd, two heads] & & \mathbf{As}^* \circ \mathbf{As} \arrow[dd, two heads] \\
\mathbf{As} \circ \mathbf{Com}^* \arrow[ur, hook] \arrow[rr, dashed] \arrow[dd, two heads] & & \mathbf{Com}^* \circ \mathbf{As} \arrow[ur, hook] \arrow[dd, two heads] & \\
& \mathbf{Com} \circ \mathbf{As}^* \arrow[rr, dashed] & & \mathbf{As}^* \circ \mathbf{Com} \\
\mathbf{Com} \circ \mathbf{Com}^* \arrow[ur, hook] \arrow[rr, dashed] & & \mathbf{Com}^* \circ \mathbf{Com} \arrow[ur, hook] &
\end{tikzcd}$$

The dotted arrows are all mixed distributive laws. We denote all of them by λ. Explicitly, the mixed distributive law between $\mathbf{Com}$ and $\mathbf{Com}^*$ is given by

(4.47)
$$\bigoplus_{\mathrm{Y}\colon \mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}} \mathbf{Com}[\mathrm{X}, \mathrm{Y}] \otimes \mathbf{Com}^*[\mathrm{Y}, \mathrm{Z}] \longrightarrow \bigoplus_{\mathrm{Y}'\colon \mathrm{X} \leq \mathrm{Y}' \leq \mathrm{Z}} \mathbf{Com}^*[\mathrm{X}, \mathrm{Y}'] \otimes \mathbf{Com}[\mathrm{Y}', \mathrm{Z}]$$

with $1 \otimes 1$ from the Y-summand mapping to $\sum_{\mathrm{Y}' : \mathrm{Y} \vee \mathrm{Y}' = \mathrm{Z}} 1 \otimes 1$. Compare (4.47) with (3.16).

Proposition 4.34. *The following is a commutative diagram of bioperads.*

$$\begin{tikzcd}
(\mathbf{Com}, \mathbf{Com}^*, \lambda) \arrow[r] \arrow[d] & (\mathbf{As}, \mathbf{Com}^*, \lambda) \arrow[d] \\
(\mathbf{Com}, \mathbf{As}^*, \lambda) \arrow[r] & (\mathbf{As}, \mathbf{As}^*, \lambda)
\end{tikzcd}$$

Diagram (4.46) also has a signed analogue with $\mathbf{Com}$ replaced by $\mathbf{Com}^-$, and λ by λ_{-1}. Recall from Lemma 4.6 that $\mathbf{As}$ and $\mathbf{As}^-$ are canonically isomorphic.

Proposition 4.35. *The following is a commutative diagram of bioperads.*

$$\begin{tikzcd}
(\mathbf{Com}^-, (\mathbf{Com}^-)^*, \lambda_{-1}) \arrow[r] \arrow[d] & (\mathbf{As}, (\mathbf{Com}^-)^*, \lambda_{-1}) \arrow[d] \\
(\mathbf{Com}^-, \mathbf{As}^*, \lambda_{-1}) \arrow[r] & (\mathbf{As}, \mathbf{As}^*, \lambda_{-1})
\end{tikzcd}$$

The above results imply Propositions 3.13 and 3.16 which give the corresponding diagrams of bimonads.

Exercise 4.36. Use (1.163) to make the mixed distributive law between $\mathbf{Com}^-$ and its dual $(\mathbf{Com}^-)^*$ explicit. Compare with (3.22).

4.13. Incidence algebra of an operad

We introduce the incidence algebra of an operad. For a connected quadratic operad, this algebra is elementary with the Birkhoff algebra as its split-semisimple quotient. Moreover, the quiver of this algebra can be explicitly described. The incidence algebra of the commutative operad is the flat-incidence algebra, of the associative operad is the lune-incidence algebra, and of the Lie operad is the Tits algebra.

Our presentation here is very brief; more details will be given elsewhere. For background information on elementary algebras, see [**21**, Appendix D.8].

In this section, all operads are assumed to be finite-dimensional.

4.13.1. Operad incidence algebras. For any operad $\mathbf{a}$, we define the $\mathbf{a}$-incidence algebra by

$$\mathrm{I}(\mathbf{a}) := \bigoplus_{\mathrm{X} \leq \mathrm{Y}} \mathbf{a}[\mathrm{X}, \mathrm{Y}]. \tag{4.48}$$

The sum is over both X and Y. Elements in the (X, Y)-summand are multiplied with elements in the (Y, Z)-summand by the substitution map of $\mathbf{a}$; the remaining products are all zero. The unit element arises from the unit map of $\mathbf{a}$.

Observe that $\mathrm{I}(\mathbf{a})$ can also be viewed as the incidence algebra of the linear category determined by $\mathbf{a}$ in Lemma 4.2.

4.13.2. Elementary algebras. Let A be a finite-dimensional algebra. Its radical, denoted $\mathrm{rad}(A)$, is the largest nilpotent ideal of A. We say A is *elementary* if the quotient $A/\mathrm{rad}(A)$ is a split-semisimple commutative algebra.

Proposition 4.37. *Let* $\mathbf{a}$ *be a connected operad. The algebra* $\mathrm{I}(\mathbf{a})$ *is elementary with the Birkhoff algebra as its split-semisimple quotient. The radical of* $\mathrm{I}(\mathbf{a})$ *is given by*

$$\mathrm{rad}(\mathrm{I}(\mathbf{a})) = \bigoplus_{\mathrm{X} < \mathrm{Y}} \mathbf{a}[\mathrm{X}, \mathrm{Y}].$$

The sum is over both X *and* Y*.*

Here, the subspace $\bigoplus_{\mathrm{X}} \mathbf{a}[\mathrm{X}, \mathrm{X}]$ is identified with the Birkhoff algebra via its $\mathtt{Q}$-basis of primitive orthogonal idempotents.

Moreover:

Proposition 4.38. *For a connected quadratic operad* $\mathbf{a} = \langle \mathbf{e} \,|\, \mathbf{r} \rangle$*, the quiver of* $\mathrm{I}(\mathbf{a})$ *is as follows. The vertices are flats, and there are* $\dim \mathbf{e}[\mathrm{X}, \mathrm{Y}]$ *number of arrows from* Y *to* X *when* $\mathrm{X} < \mathrm{Y}$*, and no arrows otherwise.*

The proofs are straightforward.

4.13.3. Quadratic duality. For any connected quadratic operad $\mathbf{a}$, we have

$$\mathrm{I}(\mathbf{a}^!) \cong \mathrm{I}(\mathbf{a})^!. \tag{4.49}$$

The rhs refers to the quadratic dual of $\mathrm{I}(\mathbf{a})$ viewed as an algebra in the category of bimodules over the Birkhoff algebra.

4.13.4. Commutative, associative, Lie operads. Let us now look at some examples. Recall the flat-incidence algebra and lune-incidence algebra from Sections 1.5.1 and 1.5.3, respectively.

Proposition 4.39. *For the commutative operad, the* **Com***-incidence algebra coincides with the flat-incidence algebra, while for the associative operad, the* **As***-incidence algebra coincides with the lune-incidence algebra.*

These are straightforward observations.

Theorem 4.40. *For the Lie operad, the* **Lie***-incidence algebra is isomorphic to the Tits algebra.*

This is a nontrivial result which was obtained in [**21**, Theorem 13.54]. We elaborate briefly. Let $\mathtt{E}$ be an eulerian family of the Tits algebra, and let $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ be the corresponding noncommutative zeta function and noncommutative Möbius function. For any $\mathrm{X} \leq \mathrm{Y}$, there is a vector space isomorphism

$$\begin{aligned} \mathbf{Lie}[\mathrm{X},\mathrm{Y}] &\xrightarrow{\cong} \mathtt{E}_{\mathrm{X}} \cdot \Sigma[\mathcal{A}] \cdot \mathtt{E}_{\mathrm{Y}} \\ z &\longmapsto \sum_{F:\, \mathrm{s}(F)=\mathrm{X}} \boldsymbol{\zeta}(O,F)\, \beta_{F,\mathrm{X}}(z) \cdot \mathtt{E}_{\mathrm{Y}}. \end{aligned} \tag{4.50}$$

Each $\beta_{F,\mathrm{X}}(z)$ is a linear combination of faces greater than F which have support Y, and is obtained by viewing z in the star of F. Summing (4.50) over all $\mathrm{X} \leq \mathrm{Y}$ yields the algebra isomorphism in Theorem 4.40. Note very carefully that it depends on the choice of the eulerian family.

Some interesting consequences of Proposition 4.39 and Theorem 4.40 are given below.

Proposition 4.41. *The flat-incidence algebra and Tits algebra are quadratic duals of each other. The lune-incidence algebra is self-dual.*

Proof. This follows from Proposition 4.14 and (4.49). ☐

Proposition 4.42. *The quivers of the flat-incidence algebra and Tits algebra are identical: The vertices are flats, and there is one arrow from* Y *to* X *when* $\mathrm{X} \lessdot \mathrm{Y}$*, and no arrows otherwise. The quiver of the lune-incidence algebra has two arrows from* Y *to* X *when* $\mathrm{X} \lessdot \mathrm{Y}$*, and no arrows otherwise.*

Proof. This follows from Proposition 4.38 and the presentations of the commutative, associative, Lie operads given in Examples 4.9, 4.10, 4.12. ☐

4.13.5. Left modules over operad incidence algebras. Let m be a left module over an operad $\mathbf{a}$. Then

$$\mathrm{I}(\mathsf{m}) := \bigoplus_{\mathrm{Y}} \mathsf{m}[\mathrm{Y}] \tag{4.51}$$

is a left module over the algebra $\mathrm{I}(\mathbf{a})$. Elements in the (X,Y)-summand of $\mathrm{I}(\mathbf{a})$ act on elements in the Y-summand of $\mathrm{I}(\mathsf{m})$ by (4.36); the remaining actions are all zero.

Proposition 4.43. *For an operad* $\mathbf{a}$*, the category of left* $\mathbf{a}$*-modules is equivalent to the category of left* $\mathrm{I}(\mathbf{a})$*-modules. The equivalence sends a left* $\mathbf{a}$*-module* m *to the left* $\mathrm{I}(\mathbf{a})$*-module* $\mathrm{I}(\mathsf{m})$*.*

This is straightforward.

Example 4.44. Let (a, μ) be a commutative monoid in species. Recall that this is the same as a left module over **Com**. Hence, $\mathrm{I}(\mathsf{a})$ is a left module over $\mathrm{I}(\mathbf{Com})$ which is the flat-incidence algebra. Explicitly, the module structure is given by

$$(s \circ v)_{\mathrm{X}} = \sum_{\mathrm{Y}\colon \mathrm{X} \leq \mathrm{Y}} s(\mathrm{X}, \mathrm{Y})\, \mu_{\mathrm{X}}^{\mathrm{Y}}(v_{\mathrm{Y}}), \tag{4.52}$$

where $v = (v_{\mathrm{Y}}) \in \mathrm{I}(\mathsf{a})$.

Example 4.45. Let (a, μ) be a monoid in species. Recall that this is the same as a left module over **As**. Hence, $\mathrm{I}(\mathsf{a})$ is a left module over $\mathrm{I}(\mathbf{As})$ which is the lune-incidence algebra. Explicitly, the module structure is given by

$$(s \circ v)_A = \sum_{F\colon A \leq F} s(A, F)\, \mu_A^F(v_F), \tag{4.53}$$

where $v \in \mathrm{I}(\mathsf{a})$ is viewed as a family of elements $v_F \in \mathsf{a}[F]$, one for each face F, such that $\beta_{G,F}(v_F) = v_G$.

The ideas in Examples 4.44 and 4.45 are developed further in Sections 9.5 and 9.6.

4.13.6. Universal series. Let E denote the species defined by $\mathsf{E}[\mathrm{X}] := \Bbbk$ for all X. This is the exponential species which is studied in detail later in Section 7.2.

Let $\mathbf{a}$ be an operad. Then by Proposition 4.23, the species $\mathbf{a} \circ \mathsf{E}$ is a free left $\mathbf{a}$-module. Observe using (4.35) that its components are given by

$$(\mathbf{a} \circ \mathsf{E})[\mathrm{X}] = \bigoplus_{\mathrm{Y}\colon \mathrm{X} \leq \mathrm{Y}} \mathbf{a}[\mathrm{X}, \mathrm{Y}].$$

By summing over all X, we obtain an isomorphism of vector spaces

$$\mathrm{I}(\mathbf{a}) \xrightarrow{\cong} \mathrm{I}(\mathbf{a} \circ \mathsf{E}), \tag{4.54}$$

with the lhs defined by (4.48) and the rhs by (4.51). In fact, this is an isomorphism of left $\mathrm{I}(\mathbf{a})$-modules, with the lhs viewed as a left module over itself.

The unit element of the algebra $\mathrm{I}(\mathbf{a})$ yields a canonical element of $\mathrm{I}(\mathbf{a} \circ \mathsf{E})$. We call it the *universal series* of the species $\mathbf{a} \circ \mathsf{E}$. The notion of series of a species is developed later in Sections 9.5 and 9.6. Specializations of the universal series to the associative and commutative operads are given in Examples 9.119 and 9.128, respectively.

4.14. Operads for LRB species

We briefly indicate how operads can be defined in the context of LRB species (Section 3.9). Interestingly, the commutative, associative, Lie operads extend to this setting.

4.14.1. LRB dispecies. Let Σ be a left regular band. Let $\Sigma\text{-}\mathtt{dLRB}$ denote the discrete category whose objects are nested flats of Σ.

A Σ-*dispecies* is a functor from $\Sigma\text{-}\mathtt{dLRB}$ to Vec, and a map of Σ-dispecies is a natural transformation between such functors. This defines the category of Σ-dispecies which we denote by $\Sigma\text{-}\mathtt{dSp}$. It is a functor category and we also write

$$\Sigma\text{-}\mathtt{dSp} = [\Sigma\text{-}\mathtt{dLRB}, \mathsf{Vec}].$$

4.14.2. LRB operads. The *substitution product* of Σ-dispecies is defined as in (4.3). This turns $\Sigma\text{-}\mathtt{dSp}$ into a monoidal category.

A Σ-*operad* is a monoid in this monoidal category, and dually, a Σ-*cooperad* is a comonoid in this monoidal category.

Connected and positive Σ-operads are defined the same way as before. The free Σ-operad on a Σ-dispecies $\mathbf{e}$ is constructed as in (4.30). A quadratic Σ-operad is defined the same way as before.

4.14.3. Commutative, associative, Lie operads. The exponential dispecies, dispecies of chambers, Lie dispecies in Section 4.5 can be defined for any left regular band. Note very carefully that the linear system defining Lie elements (1.164) only involves the Tits product, and makes sense for any left regular band. Now going one step further yields the commutative Σ-operad, associative Σ-operad, Lie Σ-operad. We denote them by **Com**, **As**, **Lie** as before.

4.14.4. LRB operad modules. The substitution product of a Σ-dispecies and Σ-species is defined as in (4.35). This turns the category of Σ-species into a left module category over the monoidal category of Σ-dispecies. This yields the category of left $\mathbf{a}$-modules for any Σ-operad $\mathbf{a}$.

Every Σ-operad $\mathbf{a}$ gives rise to a monad $\mathcal{V}_{\mathbf{a}}$ on Σ-species via (4.42). The operad **As** yields the monad $\mathcal{T}$, while the operad **Com** yields the moand $\mathcal{S}$. As a consequence, left **As**-modules are the same as Σ-monoids, and left **Com**-modules are the same as commutative Σ-monoids. The operad **Lie** yields the monad $\mathcal{PT}$, and left **Lie**-modules are Σ-Lie monoids, see Section 17.8 for a brief elaboration.

4.14.5. LRB operad incidence algebras. For any finite-dimensional Σ-operad $\mathbf{a}$, we define the $\mathbf{a}$-incidence algebra $\mathrm{I}(\mathbf{a})$ by (4.48). Proposition 4.37 and Proposition 4.38 continue to hold. Similarly, for any left module m over $\mathbf{a}$, we define $\mathrm{I}(\mathsf{m})$ by (4.51). It is a left module over the algebra $\mathrm{I}(\mathbf{a})$. Proposition 4.43 continues to hold.

Notes

Operads for hyperplane arrangements are motivated by May operads. We provide below references to the literature on May operads along with other references relevant to the ideas of this chapter.

May operads. The term 'operad' and its first formal definition is due to May [**662**, Definition 1.1]. We have referred to these as May operads. This terminology is used by Markl [**653**, Definition 1], Yau and Johnson [**923**, Section 11.4]. The operad concept matured under different names in work of several authors such as Boardman and Vogt [**126**], Lazard [**581**], [**584**], Lambek [**560**], Stasheff [**846**]. Now there is a vast literature on operads. For book references, see those by Smirnov [**834**], Markl, Shnider, Stasheff [**654**], Leinster [**590**], Loday and Vallette [**611**], Bremner and Dotsenko [**156**], Fresse [**323**], [**324**], [**325**]. For related material, see [**401**], [**519**].

Substitution product. Kelly [**524**, Section 4] noted that May operads may be viewed as monoids under substitution; this is also done by Smirnov [**833**, Section 1, page 577], [**834**, Section 5.1, Definition on page 89], Joyal [**498**, Section 7, Examples 41, 44, 45]. Other references that treat this point are [**18**, Appendix B], [**321**, Part 1], [**590**, Appendix A.2], [**611**, Section 5.2], [**654**, Definition 1.67]. The form of the substitution operation as given in (4.21) is due to Joyal [**498**, Section 2.2, Definition 7]. He worked in the set-theoretic setting. The same is done by Bergeron, Labelle, Leroux [**102**, Section 1.4, Formula (2)], Méndez [**675**, Formula (3.11)]. The latter reference focuses on set-operads [**675**, Definition 3.1]. We mention that the substitution operation (4.21) distributes over the coproduct only on the left. This is in contrast to our situation, where it distributes on both sides, see (4.5).

PROPs. Recall that the connection between May operads and operads for hyperplane arrangements is via the construction in Section 4.6. This is similar to the construction of a PROP from a May operad [**6**, Section 2.3], see also [**653**, Example 60], [**611**, Section 5.4.1]. We mention that the notion of a PROP is due to Adams and Mac Lane [**621**, Section 24]. It is also discussed by Boardman and Vogt [**126**, Definition 2.44]. For a recent reference, see [**923**].

Hadamard product. Information on the Hadamard product of Joyal species is given in the Notes to Chapter 8.

Hopf operads. Hopf operads first appeared in work of Getzler and Jones [**351**]. For later references, see the books by Smirnov [**834**, Section 5.1, Definition on page 90], Fresse [**324**, Section 3.2], Loday and Vallette [**611**, Sections 5.1.12, 5.3.2, 5.3.3], Markl, Shnider, Stasheff [**654**, Definition 3.135]. Their connection with 2-monoidal categories is emphasized in our monograph [**18**, Appendix B.6], also see [**889**, Section 1.5]. We have followed the same approach here in Section 4.7 while extending them to arrangements. For an analogous example, see [**18**, Example 6.23], [**339**, Example 4.10]. The analogue of the interchange law (4.25a) is also mentioned in the early paper of Smirnov [**833**, Section 1, page 577].

The convolution operad appears in work of Berger and Moerdijk [**101**, Section 1], see also [**611**, Section 6.4.1]. This construction in our setting is given in Section 4.7.6.

Black and white circle products. The black and white circle products for quadratic algebras were introduced by Manin [**641**, Section 2], [**642**, Section 3], [**645**, Section 4]. The analogous constructions for binary quadratic operads are given by Ginzburg and Kapranov [**354**, Section 2.2]. For a more recent treatment, see [**889**, Section 3] and [**611**, Sections 4.5 and 8.8]. Quadratic algebras are briefly mentioned by Street [**859**, Example 12.5].

Operads for hyperplane arrangements. A notion of operads for hyperplane arrangements is proposed in our monograph [**21**, Section 15.9], a similar idea is present in work of Rains [**772**, page 794]. In the cited Section 15.9, we define the commutative, associative, Lie operads for arrangements, and also briefly mention operad incidence algebras. In this chapter, we have presented a local version of this theory in which a hyperplane arrangement has been fixed throughout.

We saw in Section 4.3 that a set-dispecies can be viewed as a directed graph on the set of flats. The substitution product of set-dispecies is the diamond product of directed graphs in [**18**, Example 6.17]. The connection of set-operads with categories in Lemma 4.3 is given in [**18**, Example 6.43]. The 2-monoidal structure on set-dispecies (defined as in Proposition 4.5) is compatible with the 2-monoidal structure on directed graphs given in [**18**, Example 6.17]. Substitution corresponds to arrows in series, and Hadamard to arrows in parallel. When the set-operad is connected, the category defined in Lemma 4.3 is an acyclic category in the sense of Kozlov [**544**, Chapter 10] and small category without loops in the sense of Bridson and Haefliger [**157**, Chapter III.$\mathcal{C}$, Section 1], see also [**768**, Definition 3.1], [**240**, Appendix A.1]. This includes the category of lunes. The presentation of the commutative and associative operads in Examples 4.9 and 4.10 is the linearization of [**21**, Propositions 4.41 and 4.42]. More precisely, the second result in this reference gives a presentation for the category of lunes.

S-algebras. Our notion of $\mathcal{A}$-operads is similar to the notion of S-algebras studied by Beilinson, Ginzburg, Schechtman [**88**, Section 3]. Here, S is a set equipped with a weight function $S \to \mathbb{Z}$. An S-algebra A consists of vector spaces A_{st} for $s, t \in S$ equipped with products $A_{st} \otimes A_{tu} \to A_{su}$ subject to associativity and unitality. They also study the class of S-algebras in which $A_{st} = 0$ whenever the weight of S is strictly greater than the weight of t. To get closer to our context, we can take S to be the set of flats, and for the weight function we can take rank of the flat.

Incidence algebras. Incidence algebras of categories appeared in work of Mitchell [**700**, Section 7] and Gabriel [**334**, Section II.1] (though not under that name). Proposition 4.43 is a special case of [**700**, Theorem 7.1] and [**334**, Proposition 2]. These include a similar result for quivers and quiver algebras [**53**, Chapter 3, Theorem 1.5], [**585**, Proposition 4.6].

Regarding Proposition 4.41: The duality between the flat-incidence algebra and the Tits algebra is due to Saliola [**795**, Proposition 9.6]. The self-duality of the lune-incidence algebra is mentioned in our monograph [**21**, Section 15.9]. Regarding Proposition 4.42: The quiver of the Tits algebra was computed by Saliola [**795**, Corollary 8.4]. The special case of the braid arrangement was done by Schocker [**815**, Theorem 8.1]. The quiver of the lune-incidence algebra was obtained in [**21**, Theorem 15.14]. For the quiver of the flat-incidence algebra, see for instance [**21**, Theorems 15.2 and C.14].

Future work. This chapter developed operads for hyperplane arrangements to the extent required in this book. To make full contact with May operads, one needs to work in a more structured setting, namely, that of reflection arrangements, see Table II in the Preface. The Coxeter group then plays the same role that the symmetric group plays in the classical theory. These ideas will be explained in a future work where we also plan to discuss Koszul theory and include more examples such as the Poisson operad for arrangements.

Part II

Basic theory of bimonoids

CHAPTER 5

Primitive filtrations and decomposable filtrations

Monoids, comonoids, bimonoids in species were introduced in Chapter 2. More generally, for any scalar q, we defined q-bimonoids. The case $q = 1$ recovered bimonoids, while $q = -1$ yielded the notion of signed bimonoids. We now proceed with the basic theory of these objects.

We discuss graded and filtered monoids, comonoids, bimonoids. Every comonoid has a primitive part and more generally a primitive filtration which turns it into a filtered comonoid. Dually, every monoid has a decomposable part and more generally a decomposable filtration which turns it into a filtered monoid. The indecomposable part of a monoid is the quotient by its decomposable part.

Every species can be turned into a (co)monoid by letting all its nontrivial (co)product components be 0. This defines a functor from the category of species to the category of (co)monoids which we call the trivial (co)monoid functor. The trivial comonoid functor is the left adjoint of the primitive part functor. Dually, the trivial monoid functor is the right adjoint of the indecomposable part functor. A map from a species to a comonoid is a coderivation if it maps into the primitive part of that comonoid. Dually, a map from a monoid to a species is a derivation if it factors through the indecomposable part of that monoid. A (co)derivation is the same as a (co)monoid morphism with the species viewed as a trivial (co)monoid.

We introduce Cauchy and commutative Cauchy powers of a species. The coproduct components of a comonoid can be encapsulated as maps from the comonoid to its Cauchy powers. The primitive filtration arises by taking kernels of these maps. Dually, the product components of a monoid can be encapsulated as maps to the monoid from its Cauchy powers. The decomposable filtration arises by taking images of these maps.

The primitive filtration provides a tool for establishing results on comonoids by doing an induction on the level of the filtration. For instance, one can show that a morphism of comonoids is injective iff it is injective on the primitive part. There is a dual result for monoids involving the indecomposable part.

We formulate some preliminary results on how the bimonoid axiom works on the primitive part of a bimonoid. We also give some related results on the primitive filtration. Each case is treated separately, namely, cocommutative bimonoids, bicommutative bimonoids, 0-bimonoids, and more generally,

q-bimonoids for q not a root of unity. These considerations will play an important role in inductive arguments for proving the rigidity theorems on bimonoids in Chapter 13.

For a q-bimonoid, one can also consider the filtration generated by its primitive part. For a q-bimonoid for q not a root of unity, this filtration coincides with the primitive filtration. This is also true for a (signed) bimonoid (which is the case $q = \pm 1$) provided it is (signed) cocommutative. In particular, q-bimonoids for q not a root of unity and (signed) cocommutative (signed) bimonoids are primitively generated.

For a q-bimonoid, there is a canonical map from its primitive part to its indecomposable part. For a q-bimonoid for q not a root of unity, this map is bijective. For a bimonoid, this map is surjective iff the bimonoid is cocommutative, injective iff the bimonoid is commutative, and bijective iff the bimonoid is bicommutative. Similar results hold for signed bimonoids.

For a q-bimonoid, both the primitive and the decomposable filtrations turn it into a filtered q-bimonoid. Thus, for either filtration, we can consider the corresponding associated graded q-bimonoid. For $q = \pm 1$, the associated graded (signed) bimonoid wrt the primitive filtration is (signed) commutative, and wrt the decomposable filtration is (signed) cocommutative. These are the Browder–Sweedler and Milnor–Moore (co)commutativity results.

5.1. Cauchy powers of a species

We introduce Cauchy and commutative Cauchy powers of a species, and discuss some maps involving them.

5.1.1. Cauchy powers of a species. Let p be a species. For $k \geq 1$, define the species p^k by

$$\mathsf{p}^k[A] := \bigoplus_{\substack{F\colon F \geq A \\ \mathrm{rk}(F/A) = k-1}} \mathsf{p}[F]. \tag{5.1}$$

The sum is over all faces F greater than A whose rank is $k-1$ higher than the rank of A. Suppose A and B have the same support. Then, for every face F greater than A, there is a corresponding face $G := BF$ greater than B with the same support as F, and hence a linear map $\beta_{G,F} : \mathsf{p}[F] \to \mathsf{p}[G]$. This yields a linear map

$$\beta_{B,A} : \mathsf{p}^k[A] \to \mathsf{p}^k[B]$$

whose (F, G)-component is $\beta_{G,F}$ when $G = BF$, and zero otherwise. (The notation is as in Appendix A.4.) This turns p^k into a species. We call p^k the k-th *Cauchy power* of the species p.

- If $k = 1$, then there is only one summand in (5.1), namely, $F = A$, and so $\mathsf{p}^1 = \mathsf{p}$.
- Since the rank of $\mathcal{A}$ is finite, eventually all Cauchy powers of any species are zero.

5.1.2. Maps involving Cauchy powers. Suppose c is a comonoid. For $k \geq 1$, define

$$\Delta^{k-1} : \mathsf{c} \to \mathsf{c}^k, \qquad \Delta^{k-1}_A := (\Delta^F_A)_{\mathrm{rk}(F/A)=k-1}. \tag{5.2}$$

Note that Δ^0 is the identity map, while Δ^1 has vector-components Δ^F_A as F varies over all faces which cover A.

Similarly, for a monoid a, for $k \geq 1$, define

$$\mu^{k-1} : \mathsf{a}^k \to \mathsf{a}, \qquad \mu^{k-1}_A := (\mu^F_A)_{\mathrm{rk}(F/A)=k-1}. \tag{5.3}$$

Note that μ^0 is the identity map.

Note that Δ^{k-1}_A goes from a vector space to a direct sum, while μ^{k-1}_A goes from a direct sum to a vector space. These maps have been defined by specifying their vector-components following Appendix A.4.

Exercise 5.1. Check that Δ^{k-1} and μ^{k-1} are maps of species. (Use naturality of the coproduct (2.10) and product (2.8).)

Suppose p is a species. Let q be any scalar. For $k \geq 1$, define

$$(\beta_q)^{k-1} : \mathsf{p}^k \to \mathsf{p}^k \tag{5.4}$$

as follows. Evaluated on the A-component,

$$(\beta_q)^{k-1}_A : \bigoplus_{\substack{F:\, F \geq A \\ \mathrm{rk}(F/A)=k-1}} \mathsf{p}[F] \to \bigoplus_{\substack{G:\, G \geq A \\ \mathrm{rk}(G/A)=k-1}} \mathsf{p}[G]$$

has matrix-components $q^{\mathrm{dist}(F,G)}\beta_{G,F}$ if F and G have the same support, and 0 otherwise. Note that $(\beta_q)^0$ is the identity map. Also, for $q = 0$, the map $(\beta_0)^{k-1}$ is the identity for any $k \geq 1$.

For q not a root of unity, define

$$(\beta^q)^{k-1} : \mathsf{p}^k \to \mathsf{p}^k \tag{5.5}$$

in a similar manner as above, with $q^{\mathrm{dist}(F,G)}$ replaced by $q^{F/A,G/A}$. The latter are the scalars defined in (1.35) for the arrangement $\mathcal{A}_A$.

Exercise 5.2. Check that $(\beta_q)^{k-1}$ and $(\beta^q)^{k-1}$ are maps of species (the latter when q is not a root of unity).

Lemma 5.3. *Let p be any species. Then, for q not a root of unity, and $k \geq 1$, the maps*

$$\begin{tikzcd} \mathsf{p}^k \arrow[r, shift left, "(\beta_q)^{k-1}"] & \mathsf{p}^k \arrow[l, shift left, "(\beta^q)^{k-1}"] \end{tikzcd}$$

are inverse isomorphisms.

Proof. Let us show that $(\beta_q)^{k-1}_A(\beta^q)^{k-1}_A = \mathrm{id}$ for any face A. That is, the composite map

$$\bigoplus_{\substack{F:\, F \geq A \\ \mathrm{rk}(F/A)=k-1}} \mathsf{p}[F] \xrightarrow{(\beta^q)^{k-1}_A} \bigoplus_{\substack{G:\, G \geq A \\ \mathrm{rk}(G/A)=k-1}} \mathsf{p}[G] \xrightarrow{(\beta_q)^{k-1}_A} \bigoplus_{\substack{K:\, K \geq A \\ \mathrm{rk}(K/A)=k-1}} \mathsf{p}[K]$$

is the identity. Let us consider the matrix-component starting in the summand $\mathsf{p}[F]$ and ending in the summand $\mathsf{p}[K]$. If F and K do not have the same support, then this map is zero. Else, the map is

$$\sum_{\substack{G:\, G \geq A \\ \mathrm{s}(F)=\mathrm{s}(G)=\mathrm{s}(K)}} q^{F/A,G/A} q^{\mathrm{dist}(G,K)}\, \beta_{K,G}\beta_{G,F}.$$

By (1.35) and (2.1), this equals identity if $F = K$, and 0 otherwise. This completes the check. Similarly, the composite in the other order is also identity. □

Recall the lune-incidence algebra $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ from Section 1.5.3. Fix an element $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$. Suppose p is a species. For $k \geq 1$, define

$$s^{(k-1)} : \mathsf{p}^k \to \mathsf{p}^k \tag{5.6}$$

as follows. Evaluated on the A-component,

$$s_A^{(k-1)} : \bigoplus_{\substack{F:\, F \geq A \\ \mathrm{rk}(F/A)=k-1}} \mathsf{p}[F] \to \bigoplus_{\substack{G:\, G \geq A \\ \mathrm{rk}(G/A)=k-1}} \mathsf{p}[G]$$

sends the summand $\mathsf{p}[F]$ to itself under scalar multiplication by $s(A, F)$. Note that $s^{(0)}$ is the identity map.

Exercise 5.4. For $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$, check that $s^{(k-1)}$ is a map of species. (Use (1.40).)

More generally: Recall the bilune-incidence algebra $\mathrm{I}_{\mathrm{bilune}}[\mathcal{A}]$ from Section 1.6.3. For $s \in \mathrm{I}_{\mathrm{bilune}}[\mathcal{A}]$ and p a species, and for $k \geq 1$, define

$$s^{(k-1)} : \mathsf{p}^k \to \mathsf{p}^k \tag{5.7}$$

as follows. Evaluated on the A-component, its matrix-component from $\mathsf{p}[F]$ to $\mathsf{p}[G]$ is $s(A, F, G)\beta_{G,F}$ if F and G have the same support, and 0 otherwise. We recover (5.6) by viewing the lune-incidence algebra inside the bilune-incidence algebra via the map (1.62).

Exercise 5.5. For $s \in \mathrm{I}_{\mathrm{bilune}}[\mathcal{A}]$, check that $s^{(k-1)}$ is a map of species. (Use (1.58).)

For $s(A, F, G) := q^{\mathrm{dist}(F,G)}$, the map (5.7) specializes to (5.4). Similarly, for q not a root of unity, and for $s(A, F, G) := q^{F/A,G/A}$, that is, $s = \boldsymbol{\zeta}_q$, the two-sided q-zeta function (1.64), the map (5.7) specializes to (5.5). In particular, both (5.4) and (5.5) are maps of species (as mentioned in Exercise 5.2).

5.1.3. Functoriality. Let $f : \mathsf{p} \to \mathsf{q}$ be a map of species. For each $k \geq 1$, we have an induced map of species

$$f_k : \mathsf{p}^k \to \mathsf{q}^k. \tag{5.8}$$

On the A-component, for any face F with $\mathrm{rk}(F/A) = k - 1$, the map f_k sends $\mathsf{p}[F]$ to $\mathsf{q}[F]$ via f_F. Note that $f_1 = f$.

Now suppose c is a comonoid, a is a monoid, and $f : \mathsf{c} \to \mathsf{a}$ is a map of species. Define

$$f^k : \mathsf{c} \to \mathsf{a}, \qquad f^k := \mu^{k-1} f_k \Delta^{k-1}. \tag{5.9}$$

Explicitly, on the A-component,

$$(f^k)_A = \sum_{F:\, \mathrm{rk}(F/A)=k-1} \mu_A^F \, f_F \, \Delta_A^F. \tag{5.10}$$

This is a map of species. Note that $f^1 = f$.

5.1.4. Commutative Cauchy powers of a species. The above discussion has a commutative counterpart which we touch upon briefly. For this, it is convenient to work with the formulation of species given in Proposition 2.5. Similarly, for (co)commutative (co)monoids, we work with the formulation given in Proposition 2.20 and its dual.

Let p be a species. For $k \geq 1$, define the species $\mathsf{p}^{\overline{k}}$ by

$$\mathsf{p}^{\overline{k}}[\mathrm{Z}] := \bigoplus_{\substack{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z} \\ \mathrm{rk}(\mathrm{X}/\mathrm{Z})=k-1}} \mathsf{p}[\mathrm{X}]. \tag{5.11}$$

We call $\mathsf{p}^{\overline{k}}$ the k-th *commutative Cauchy power* of the species p.

Compare and contrast with definition (5.1) of Cauchy powers. Now there are fewer summands since we are summing over flats instead of faces.

Suppose c is a cocommutative comonoid. For $k \geq 1$, define

$$\Delta^{\overline{k-1}} : \mathsf{c} \to \mathsf{c}^{\overline{k}}, \qquad \Delta_{\mathrm{Z}}^{\overline{k-1}} := (\Delta_{\mathrm{Z}}^{\mathrm{X}})_{\mathrm{rk}(\mathrm{X}/\mathrm{Z})=k-1}. \tag{5.12}$$

Dually, for a commutative monoid a, for $k \geq 1$, define

$$\mu^{\overline{k-1}} : \mathsf{a}^{\overline{k}} \to \mathsf{a}, \qquad \mu_{\mathrm{Z}}^{\overline{k-1}} := (\mu_{\mathrm{Z}}^{\mathrm{X}})_{\mathrm{rk}(\mathrm{X}/\mathrm{Z})=k-1}. \tag{5.13}$$

Note that $\Delta^{\overline{0}}$ and $\mu^{\overline{0}}$ are the identity maps.

Exercise 5.6. Recall the flat-incidence algebra $\mathrm{I}_{\mathsf{flat}}[\mathcal{A}]$ from Section 1.5.1. Formulate the analogue $s^{(\overline{k-1})}$ of (5.6) for $s \in \mathrm{I}_{\mathsf{flat}}[\mathcal{A}]$. Similarly, formulate analogues $f_{\overline{k}}$ and $f^{\overline{k}}$ of (5.8) and (5.9), respectively.

Exercise 5.7. Unify the notion of Cauchy powers and commutative Cauchy powers by formalizing the notion of $\sim$-commutative Cauchy powers for a partial-support relation $\sim$ on faces.

5.2. Graded and filtered bimonoids

One obtains the notion of graded species by replacing vector spaces by graded vector spaces in the definition of species. In a similar manner, we have graded (co, bi)monoids. A related notion is that of filtered species and filtered (co, bi)monoids. A filtered object can be turned into a graded one by taking quotients of successive components.

5.2.1. Graded bimonoids. A *grading* on a species p consists of subspecies p_i, one for each $i \geq 1$, such that

$$\mathsf{p} = \underset{i \geq 1}{+} \mathsf{p}_i.$$

We refer to p_i as the i-th graded component or the degree i component of p. We follow the convention that $\mathsf{p}_i = 0$ for $i \leq 0$.

A *graded species* is a species equipped with a grading. A map of graded species is a map of species which preserves the grading.

A *graded monoid* is a monoid a equipped with a grading (on its underlying species) such that each product component μ_A^F raises degree by $\mathrm{rk}(F/A)$. In other words, there are linear maps

$$\mu_A^F : \mathsf{a}_i[F] \to \mathsf{a}_{i+\mathrm{rk}(F/A)}[A].$$

In terms of (5.3), for $j \geq 0$, we have maps of species

$$\mu^j : \mathsf{a}_i^{j+1} \to \mathsf{a}_{i+j}.$$

Similarly, a *graded comonoid* is a comonoid c equipped with a grading such that each coproduct component Δ_A^F lowers degree by $\mathrm{rk}(F/A)$. In other words, there are linear maps

$$\Delta_A^F : \mathsf{c}_i[A] \to \mathsf{c}_{i-\mathrm{rk}(F/A)}[F].$$

In terms of (5.2), for $j \geq 0$, we have maps of species

$$\Delta^j : \mathsf{c}_i \to \mathsf{c}_{i-j}^{j+1}.$$

A morphism of graded (co)monoids is a morphism of (co)monoids which preserves the grading.

A *graded q-bimonoid* is a q-bimonoid equipped with a grading which turns it into a graded monoid and a graded comonoid. Thus, for a graded q-bimonoid h, we have linear maps

$$\mu_A^F : \mathsf{h}_i[F] \to \mathsf{h}_{i+\mathrm{rk}(F/A)}[A] \quad \text{and} \quad \Delta_A^F : \mathsf{h}_i[A] \to \mathsf{h}_{i-\mathrm{rk}(F/A)}[F].$$

In the q-bimonoid axiom (2.33), observe that if we start in the summand $\mathsf{h}_i[F]$, then we end in the summand $\mathsf{h}_{i+\mathrm{rk}(F)-\mathrm{rk}(G)}[G]$. A morphism of graded q-bimonoids is a morphism of q-bimonoids which preserves the grading.

Remark 5.8. Let h be a graded q-bimonoid. Suppose for each $i \geq 1$ and face A, we change the degree of $\mathsf{h}_i[A]$ from i to $i + \mathrm{rk}(A)$. Then, under this new grading, the maps μ_A^F and Δ_A^F indeed become degree-preserving. So one can equivalently work in a set-up where graded means that β, μ, Δ are all degree-preserving. Our convention works better in examples of interest, see for instance, the discussion on primitive filtrations (5.22), (5.52), decomposable filtrations (5.33), (5.54) and free graded monoids (6.82).

5.2.2. Filtered bimonoids. Filtrations come in two flavors, namely, ascending and descending. An *ascending filtration* of a species p consists of subspecies p_i, one for each $i \geq 1$, such that

$$\mathsf{p}_1 \subseteq \mathsf{p}_2 \subseteq \cdots \subseteq \mathsf{p}.$$

We follow the convention that $\mathsf{p}_i = 0$ for $i \leq 0$. Similarly, a *descending filtration* of p consists of subspecies p_i, one for each $i \geq 1$, such that

$$\mathsf{p} \supseteq \mathsf{p}_1 \supseteq \mathsf{p}_2 \supseteq \dots .$$

We follow the convention that $\mathsf{p}_i = \mathsf{p}$ for $i \leq 0$. The notions below apply to both ascending and descending filtrations. The formalism is similar to that of the graded case discussed above.

A *filtered species* is a species equipped with a filtration. A map of filtered species is a map of species which preserves the components of the filtration.

A *filtered monoid* is a monoid a equipped with a filtration (on its underlying species) such that each product component μ_A^F raises degree by $\mathrm{rk}(F/A)$. In other words, there are linear maps

$$\mu_A^F : \mathsf{a}_i[F] \to \mathsf{a}_{i+\mathrm{rk}(F/A)}[A].$$

Similarly, a *filtered comonoid* is a comonoid c equipped with a filtration such that each coproduct component Δ_A^F lowers degree by $\mathrm{rk}(F/A)$. In other words, there are linear maps

$$\Delta_A^F : \mathsf{c}_i[A] \to \mathsf{c}_{i-\mathrm{rk}(F/A)}[F].$$

A morphism of filtered (co)monoids is a morphism of (co)monoids which preserves the filtration.

A *filtered q-bimonoid* is a q-bimonoid equipped with a filtration which turns it into a filtered monoid and a filtered comonoid. Thus, for a filtered q-bimonoid h, we have linear maps

$$\mu_A^F : \mathsf{h}_i[F] \to \mathsf{h}_{i+\mathrm{rk}(F/A)}[A] \quad \text{and} \quad \Delta_A^F : \mathsf{h}_i[A] \to \mathsf{h}_{i-\mathrm{rk}(F/A)}[F]. \tag{5.14}$$

A morphism of filtered q-bimonoids is a morphism of q-bimonoids which preserves the filtration.

5.2.3. Associated graded bimonoids. Let p be a filtered species with ascending filtration. Consider the species obtained by taking sum of successive quotients

$$\mathrm{gr}(\mathsf{p}) := \mathop{+}_{j \geq 1} \mathsf{p}_j/\mathsf{p}_{j-1}. \tag{5.15}$$

This is a graded species, whose degree j component is the summand indexed by j. For a descending filtration, we put

$$\mathrm{gr}(\mathsf{p}) := \mathop{+}_{j \geq 1} \mathsf{p}_{j-1}/\mathsf{p}_j. \tag{5.16}$$

We refer to $\mathrm{gr}(\mathsf{p})$ as the *associated graded species* of p. This construction defines a functor from the category of filtered species to the category of graded species.

Observe that: If a is a filtered monoid, then $\mathrm{gr}(\mathsf{a})$ is a graded monoid. If c is a filtered comonoid, then $\mathrm{gr}(\mathsf{c})$ is a graded comonoid. If h is a filtered

q-bimonoid, then $\mathrm{gr}(\mathsf{h})$ is a graded q-bimonoid. The structure maps of $\mathrm{gr}(\mathsf{h})$ are obtained by applying gr to the structure maps of h. These constructions are all functorial.

Exercise 5.9. Let c be a filtered comonoid, a a filtered monoid, $f : \mathsf{c} \to \mathsf{a}$ a map of filtered species, and let f^k be as in (5.9). Check that: f^k is a map of filtered species, and moreover, $\mathrm{gr}(f^k) = \mathrm{gr}(f)^k$.

5.2.4. Filtration generated by subspecies of a monoid. Let a be a monoid and let p be a subspecies of a. For $k \geq 1$, define the subspecies $\mathcal{F}_k(\mathsf{a})$ of a by

$$\mathcal{F}_k(\mathsf{a})[A] := \sum_{F:\, \mathrm{rk}(F/A) \leq k-1} \mu_A^F(\mathsf{p}[F]).$$

Equivalently, in terms of (5.3),

$$\mathcal{F}_k(\mathsf{a}) = \sum_{1 \leq i \leq k} \mathrm{im}(\mu^{i-1}|_{\mathsf{p}^i}).$$

This is the k-th term of the filtration of a generated by p. The first term is $\mathcal{F}_1(\mathsf{a}) = \mathsf{p}$. In view of (2.63),

$$\mathcal{F}_1(\mathsf{a}) \subseteq \mathcal{F}_2(\mathsf{a}) \subseteq \cdots \subseteq \mathsf{a} \quad \text{with} \quad \bigcup_{k \geq 1} \mathcal{F}_k(\mathsf{a}) = \langle \mathsf{p} \rangle. \tag{5.17}$$

Observe that this turns a into a filtered monoid.

5.2.5. Commutative aspects. For (co)commutative (co)monoids, one may also formulate the above notions using flats instead of faces.

For a commutative monoid a which is either graded or filtered, each product component $\mu_{\mathrm{Z}}^{\mathrm{X}}$ raises degree by $\mathrm{rk}(\mathrm{X}/\mathrm{Z})$. In terms of (5.13), for $j \geq 0$, we have maps of species

$$\mu^{\overline{j}} : \mathsf{a}_i^{\overline{j+1}} \longrightarrow \mathsf{a}_{i+j}.$$

Similarly, for a cocommutative comonoid c which is either graded or filtered, each coproduct component $\Delta_{\mathrm{Z}}^{\mathrm{X}}$ lowers degree by $\mathrm{rk}(\mathrm{X}/\mathrm{Z})$. In terms of (5.12), for $j \geq 0$, we have maps of species

$$\Delta^{\overline{j}} : \mathsf{c}_i \longrightarrow \mathsf{c}_{i-j}^{\overline{j+1}}.$$

5.3. Primitive filtrations of comonoids

Any comonoid in species carries a filtration called the primitive filtration. The first term is the primitive part of the comonoid. A morphism of comonoids is injective iff it is injective on the primitive part.

5.3.1. Primitive part of a comonoid. Let c be a comonoid. Define the subcomonoid $\mathcal{P}(\mathsf{c})$ by

$$\mathcal{P}(\mathsf{c})[A] := \bigcap_{F:\, F > A} \ker(\Delta_A^F : \mathsf{c}[A] \to \mathsf{c}[F]). \tag{5.18}$$

We refer to $\mathcal{P}(\mathsf{c})$ as the *primitive part* of c. We will employ the term primitive element to refer to elements of the components $\mathcal{P}(\mathsf{c})[A]$.

Let $x \in \mathsf{c}[A]$. If $\Delta_A^F(x) = 0$ for all faces F which cover A, then by coassociativity (2.10), $\Delta_A^F(x) = 0$ for all faces F strictly greater than A. Thus,

$$\mathcal{P}(\mathsf{c})[A] = \bigcap_{\substack{F:\, F \geq A \\ \mathrm{rk}(F/A)=1}} \ker(\Delta_A^F : \mathsf{c}[A] \to \mathsf{c}[F]).$$

Equivalently, in terms of (5.2),

$$\mathcal{P}(\mathsf{c}) = \ker \Delta^1.$$

Note that $\mathcal{P}(\mathsf{c})[C] = \mathsf{c}[C]$ for any chamber C. If $\mathcal{A}$ has rank zero, then the only face is a chamber and hence $\mathcal{P}(\mathsf{c}) = \mathsf{c}$.

Exercise 5.10. For any comonoid c, check that $\mathcal{P}(\mathsf{c}^{coab}) = \mathcal{P}(\mathsf{c})$, where c^{coab} denotes the coabelianization of c.

Exercise 5.11. For any comonoid $\mathsf{c} \neq \mathsf{0}$, show that $\mathcal{P}(\mathsf{c}) \neq \mathsf{0}$. Use this fact to deduce the result of Exercise 2.43.

5.3.2. Primitive filtration of a comonoid. More generally, for $k \geq 1$, define the subcomonoid $\mathcal{P}_k(\mathsf{c})$ by

$$\mathcal{P}_k(\mathsf{c})[A] := \bigcap_{\substack{F:\, F \geq A \\ \mathrm{rk}(F/A)\geq k}} \ker(\Delta_A^F : \mathsf{c}[A] \to \mathsf{c}[F]). \tag{5.19}$$

Again by coassociativity, in (5.19), it suffices to intersect over faces F greater than A with $\mathrm{rk}(F/A) = k$. In other words,

$$\mathcal{P}_k(\mathsf{c}) = \ker \Delta^k.$$

This is the k-th term of the *primitive filtration* of c. The first term is the primitive part of c, that is, $\mathcal{P}_1(\mathsf{c}) = \mathcal{P}(\mathsf{c})$.

Observe that

$$\mathcal{P}_1(\mathsf{c}) \subseteq \mathcal{P}_2(\mathsf{c}) \subseteq \cdots \subseteq \mathsf{c} \quad \text{with} \quad \bigcup_{k\geq 1} \mathcal{P}_k(\mathsf{c}) = \mathsf{c}. \tag{5.20}$$

More precisely, $\mathcal{P}_k(\mathsf{c}) = \mathsf{c}$ as soon as k exceeds the rank of the arrangement.

Lemma 5.12. *A morphism $\mathsf{c} \to \mathsf{d}$ of comonoids induces a morphism of comonoids $\mathcal{P}_k(\mathsf{c}) \to \mathcal{P}_k(\mathsf{d})$.*

Proof. This follows from (2.11). ☐

Thus, for each k, we have a functor

$$\mathcal{P}_k : \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}) \to \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}). \tag{5.21}$$

We refer to $\mathcal{P} = \mathcal{P}_1$ as the *primitive part functor*. Depending on the context, we may want to view it as a functor from comonoids to species, or from bimonoids to species, and so on.

Exercise 5.13. For any comonoid c, check that the map $(\beta_q)^{k-1} : \mathsf{c}^k \to \mathsf{c}^k$ defined in (5.4) restricts to a map $\mathcal{P}(\mathsf{c})^k \to \mathcal{P}(\mathsf{c})^k$. A similar remark applies to the map $(\beta^q)^{k-1}$ defined in (5.5) (assuming that q is not a root of unity).

Exercise 5.14. Recall from Proposition 2.53 the comonoids $\mathsf{c}^{\mathrm{cop}}$ and ${}^{\mathrm{cop}}\mathsf{c}$ associated to a comonoid c. Check that the primitive filtrations of c, $\mathsf{c}^{\mathrm{cop}}$ and ${}^{\mathrm{cop}}\mathsf{c}$ coincide.

5.3.3. Filtered comonoid. Let c be a comonoid. Let us follow the convention that $\mathcal{P}_k(\mathsf{c}) = 0$ for $k \leq 0$. Observe that the coproduct components of c restrict to linear maps

$$\Delta_A^F : \mathcal{P}_k(\mathsf{c})[A] \to \mathcal{P}_{k-\mathrm{rk}(F/A)}(\mathsf{c})[F]. \tag{5.22}$$

Note very carefully that these maps decrease the level of the filtration. Thus, any comonoid is filtered by its primitive filtration in the sense of Section 5.2.2. Alternatively, (5.22) may be expressed as

$$\Delta^i\big(\mathcal{P}_k(\mathsf{c})\big) \subseteq \mathcal{P}_{k-i}(\mathsf{c})^{i+1} \tag{5.23}$$

for $i \geq 0$ (with no restriction on k). In particular, setting $i = k-1$,

$$\Delta^{k-1}\big(\mathcal{P}_k(\mathsf{c})\big) \subseteq \mathcal{P}(\mathsf{c})^k. \tag{5.24}$$

Exercise 5.15. For $0 \leq i \leq k$, show that

$$(\Delta^i)^{-1}(\mathcal{P}_{k-i}(\mathsf{c})^{i+1}) = \mathcal{P}_k(\mathsf{c}),$$

where the lhs refers to the inverse image of $\mathcal{P}_{k-i}(\mathsf{c})^{i+1}$ under the map Δ^i.

5.3.4. Associated graded comonoid. Now consider the graded species obtained by taking sum of successive quotients

$$\mathrm{gr}_{\mathcal{P}}(\mathsf{c}) := \mathop{+}_{j\geq 1} \mathcal{P}_j(\mathsf{c})/\mathcal{P}_{j-1}(\mathsf{c}) \tag{5.25}$$

as in (5.15). The first summand is the primitive part of c. The maps (5.22) induce a coproduct on $\mathrm{gr}_{\mathcal{P}}(\mathsf{c})$, and turn it into a graded comonoid. This defines a functor from the category of comonoids to the category of graded comonoids which sends c to $\mathrm{gr}_{\mathcal{P}}(\mathsf{c})$.

Exercise 5.16. Let c be a comonoid. Check that the primitive filtration of $\mathrm{gr}_{\mathcal{P}}(\mathsf{c})$ (viewed as a comonoid by forgetting its grading) is given by

$$\mathcal{P}_k(\mathrm{gr}_{\mathcal{P}}(\mathsf{c})) = \mathop{+}_{1\leq j\leq k} \mathcal{P}_j(\mathsf{c})/\mathcal{P}_{j-1}(\mathsf{c}).$$

In particular, its primitive part is given by

$$\mathcal{P}(\mathrm{gr}_{\mathcal{P}}(\mathsf{c})) = \mathcal{P}(\mathsf{c}). \tag{5.26}$$

Deduce that $\mathrm{gr}_{\mathcal{P}}(\mathrm{gr}_{\mathcal{P}}(\mathsf{c})) \cong \mathrm{gr}_{\mathcal{P}}(\mathsf{c})$ as graded comonoids.

Exercise 5.17. Let c be a graded comonoid. Check that $\mathcal{P}(\mathsf{c}) \supseteq \mathsf{c}_1$. Now suppose equality holds, that is, $\mathcal{P}(\mathsf{c}) = \mathsf{c}_1$. Show that

$$\mathcal{P}_k(\mathsf{c}) = \mathsf{c}_1 + \cdots + \mathsf{c}_k.$$

Deduce that $\mathrm{gr}_{\mathcal{P}}(\mathsf{c}) \cong \mathsf{c}$ as graded comonoids. The isomorphism identifies the degree k component $\mathcal{P}_k(\mathsf{c})/\mathcal{P}_{k-1}(\mathsf{c})$ with c_k.

As a special case, assume (5.26) and deduce the rest of Exercise 5.16.

5.3.5. Injectivity property.

Proposition 5.18. *A morphism $f : \mathsf{c} \to \mathsf{d}$ of comonoids is injective iff the restriction $f : \mathcal{P}(\mathsf{c}) \to \mathcal{P}(\mathsf{d})$ is injective.*

PROOF. Forward implication is clear. For backward implication, suppose $f : \mathcal{P}(\mathsf{c}) \to \mathcal{P}(\mathsf{d})$ is injective. We proceed by induction. Choose $z \in \mathsf{c}[A]$ which is in the kernel of f_A. By (5.20), there exists $k \geq 1$ such that $z \in \mathcal{P}_k(\mathsf{c})[A]$. If $k = 1$, then $z = 0$ by hypothesis. Suppose $k \geq 2$. Then by (5.22), for any F satisfying $\mathrm{rk}(F/A) = k - 1$, there is a commutative diagram

$$\begin{array}{ccc} \mathcal{P}_k(\mathsf{c})[A] & \xrightarrow{\Delta_A^F} & \mathcal{P}(\mathsf{c})[F] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{P}_k(\mathsf{d})[A] & \xrightarrow[\Delta_A^F]{} & \mathcal{P}(\mathsf{d})[F]. \end{array}$$

Since f is injective on the primitive part, it follows that $\Delta_A^F(z) = 0$, and hence $z \in \mathcal{P}_{k-1}(\mathsf{c})[A]$. By induction hypothesis, $z = 0$. □

Exercise 5.19. Show that:

(1) If c is a subcomonoid of d such that $\mathsf{c} \cap \mathcal{P}(\mathsf{d}) = 0$, then $\mathsf{c} = 0$. (Apply Proposition 5.18 to the morphism of comonoids $\mathsf{d} \to \mathsf{d}/\mathsf{c}$.) As a consequence, deduce the result of Exercise 5.11.

(2) If $f : \mathsf{c} \to \mathsf{d}$ is a morphism of comonoids such that $\mathrm{im}(f) \cap \mathcal{P}(\mathsf{d}) = 0$, then $f = 0$.

(3) Let c and d be graded comonoids such that $\mathcal{P}(\mathsf{d}) = \mathsf{d}_1$. If $f : \mathsf{c} \to \mathsf{d}$ is a morphism of graded comonoids such that $f(\mathsf{c}_1) = 0$, then $f = 0$. Equivalently: If $f, g : \mathsf{c} \to \mathsf{d}$ are morphisms of graded comonoids such that $f(\mathsf{c}_1) = g(\mathsf{c}_1)$, then $f = g$.

5.3.6. Commutative aspects. Let c be a cocommutative comonoid. In this case, the primitive filtration of c can be formulated in terms of flats instead of faces. The k-th term is given by

$$\mathcal{P}_k(\mathsf{c})[\mathrm{Z}] = \bigcap_{\substack{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z} \\ \mathrm{rk}(\mathrm{X}/\mathrm{Z}) \geq k}} \ker(\Delta_{\mathrm{Z}}^{\mathrm{X}} : \mathsf{c}[\mathrm{Z}] \to \mathsf{c}[\mathrm{X}]). \tag{5.27}$$

Equivalently,

$$\mathcal{P}_k(\mathsf{c}) = \ker \Delta^{\overline{k}},$$

with $\Delta^{\overline{k}}$ as in (5.12). Further,

$$\Delta^{\overline{i}}\big(\mathcal{P}_k(\mathsf{c})\big) \subseteq \mathcal{P}_{k-i}(\mathsf{c})^{\overline{i+1}} \tag{5.28}$$

for $i \geq 0$ (with no restriction on k). In particular, setting $i = k - 1$,

$$\Delta^{\overline{k-1}}\big(\mathcal{P}_k(\mathsf{c})\big) \subseteq \mathcal{P}(\mathsf{c})^{\overline{k}}. \tag{5.29}$$

5.4. Decomposable filtrations of monoids

We now look at the dual picture for monoids. Any monoid in species carries a filtration called the decomposable filtration. The first term is the decomposable part of the monoid. Moding it out yields the indecomposable part of the monoid. A morphism of monoids is surjective iff it is surjective on the indecomposable part.

5.4.1. Decomposable part of a monoid. Let a be a monoid. Define the submonoid $\mathcal{D}(\mathsf{a})$ by

$$\mathcal{D}(\mathsf{a})[A] := \sum_{F\colon F>A} \mathrm{im}(\mu_A^F : \mathsf{a}[F] \to \mathsf{a}[A]). \tag{5.30}$$

By associativity (2.8), it suffices to sum only over faces which cover A. In terms of (5.3),

$$\mathcal{D}(\mathsf{a}) = \mathrm{im}\,\mu^1.$$

We refer to $\mathcal{D}(\mathsf{a})$ as the *decomposable part* of a.

5.4.2. Decomposable filtration of a monoid. More generally, for $k \geq 1$, define the submonoid $\mathcal{D}_k(\mathsf{a})$ by

$$\mathcal{D}_k(\mathsf{a})[A] := \sum_{\substack{F\colon F\geq A\\ \mathrm{rk}(F/A)\geq k}} \mathrm{im}(\mu_A^F : \mathsf{a}[F] \to \mathsf{a}[A]).$$

Again by associativity, it suffices to sum over faces F greater than A with $\mathrm{rk}(F/A) = k$. In other words,

$$\mathcal{D}_k(\mathsf{a}) = \mathrm{im}\,\mu^k.$$

This is the k-th term of the *decomposable filtration* of a. The first term is the decomposable part of a, that is, $\mathcal{D}_1(\mathsf{a}) = \mathcal{D}(\mathsf{a})$.

Observe that

$$\mathsf{a} \supseteq \mathcal{D}_1(\mathsf{a}) \supseteq \mathcal{D}_2(\mathsf{a}) \supseteq \dots \quad \text{with} \quad \bigcap_{k\geq 1} \mathcal{D}_k(\mathsf{a}) = 0. \tag{5.31}$$

5.4.3. Indecomposable part of a monoid. Let a be a monoid. The *indecomposable part* of a is the quotient species $\mathcal{Q}(\mathsf{a})$ defined by

$$\mathcal{Q}(\mathsf{a}) := \mathsf{a}/\mathcal{D}(\mathsf{a}). \tag{5.32}$$

Equivalently,

$$\mathcal{Q}(\mathsf{a}) = \mathrm{coker}\,\mu^1.$$

This construction is functorial in a and yields the *indecomposable part functor* from the category of monoids to the category of species.

Exercise 5.20. For any monoid a, check that $\mathcal{Q}(\mathsf{a}_{ab}) = \mathcal{Q}(\mathsf{a})$, where a_{ab} denotes the abelianization of a.

5.4.4. Filtered monoid. Let a be a monoid. Let us follow the convention that $\mathcal{D}_k(\mathsf{a}) = \mathsf{a}$ for $k \leq 0$. Observe that the product components of a restrict to linear maps

$$\mu_A^F : \mathcal{D}_k(\mathsf{a})[F] \to \mathcal{D}_{k+\mathrm{rk}(F/A)}(\mathsf{a})[A]. \tag{5.33}$$

Note very carefully that these maps increase the level of the filtration. Thus, any monoid is filtered by its decomposable filtration in the sense of Section 5.2.2. Alternatively, (5.33) may be expressed as

$$\mu^i\big(\mathcal{D}_k(\mathsf{a})^{i+1}\big) \subseteq \mathcal{D}_{k+i}(\mathsf{a}) \tag{5.34}$$

for $i \geq 0$ (with no restriction on k). In particular, setting $k = 1$,

$$\mu^i\big(\mathcal{D}(\mathsf{a})^{i+1}\big) \subseteq \mathcal{D}_{i+1}(\mathsf{a}). \tag{5.35}$$

5.4.5. Associated graded monoid. Now consider the graded species obtained by taking sum of successive quotients

$$\mathrm{gr}_{\mathcal{D}}(\mathsf{a}) := \underset{j\geq 1}{+} \mathcal{D}_{j-1}(\mathsf{a})/\mathcal{D}_j(\mathsf{a}) \tag{5.36}$$

as in (5.16). The first summand is $\mathcal{Q}(\mathsf{a})$, the indecomposable part of a. The maps (5.33) induce a product on $\mathrm{gr}_{\mathcal{D}}(\mathsf{a})$, and turn it into a graded monoid. This defines a functor from the category of monoids to the category of graded monoids which sends a to $\mathrm{gr}_{\mathcal{D}}(\mathsf{a})$.

Exercise 5.21. Let a be a monoid. Check that the decomposable filtration of $\mathrm{gr}_{\mathcal{D}}(\mathsf{a})$ (viewed as a monoid by forgetting its grading) is given by

$$\mathcal{D}_k(\mathrm{gr}_{\mathcal{D}}(\mathsf{a})) = \underset{k\leq j-1}{+} \mathcal{D}_{j-1}(\mathsf{a})/\mathcal{D}_j(\mathsf{a}).$$

In particular, its indecomposable part is given by

$$\mathcal{Q}(\mathrm{gr}_{\mathcal{D}}(\mathsf{a})) = \mathcal{Q}(\mathsf{a}). \tag{5.37}$$

Deduce that $\mathrm{gr}_{\mathcal{D}}(\mathrm{gr}_{\mathcal{D}}(\mathsf{a})) \cong \mathrm{gr}_{\mathcal{D}}(\mathsf{a})$ as graded monoids.

Lemma 5.22. *For any monoid a and subspecies p, the following are equivalent.*

1. *The subspecies p and $\mathcal{D}(\mathsf{a})$ together linearly span a.*
2. *The monoid a is generated by p, that is, $\mathsf{a} = \langle \mathsf{p} \rangle$.*

Proof. (2) implies (1) is clear. For (1) implies (2), we check that (2.63) holds. We do this by a backward induction on the rank of A. When A is a chamber, this is trivial since $\mathcal{D}(\mathsf{a})[A] = 0$. This is the induction base. For the induction step: Given $x \in \mathsf{a}[A]$, by hypothesis, we may write $x = x_A + \sum_{F:\, F>A} \mu_A^F(x_F)$ for some $x_A \in \mathsf{p}[A]$, and $x_F \in \mathsf{a}[F]$. By induction hypothesis, each x_F is generated by p, and hence, so is x. This completes the induction step. □

Corollary 5.23. *For any monoid a, its indecomposable part $\mathcal{Q}(\mathsf{a})$ generates the monoid $\mathrm{gr}_{\mathcal{D}}(\mathsf{a})$.*

Proof. This follows from Lemma 5.22 and Exercise 5.21. □

5.4.6. Surjectivity property.

Proposition 5.24. *A morphism $f : \mathsf{a} \to \mathsf{b}$ of monoids is surjective iff the induced map $f : \mathcal{Q}(\mathsf{a}) \to \mathcal{Q}(\mathsf{b})$ is surjective.*

In view of (5.41) below, this result is dual to and can be formally deduced from Proposition 5.18. Alternatively, one can also proceed directly and dualize the argument given in the proof of Proposition 5.18. Combining Propositions 5.18 and 5.24, we deduce:

Proposition 5.25. *A morphism $f : \mathsf{h} \to \mathsf{k}$ of bimonoids is an isomorphism iff the induced maps $f : \mathcal{P}(\mathsf{h}) \to \mathcal{P}(\mathsf{k})$ and $f : \mathcal{Q}(\mathsf{h}) \to \mathcal{Q}(\mathsf{k})$ are isomorphisms.*

5.4.7. Commutative aspects. Let a be a commutative monoid. In this case, the decomposable filtration of a can be formulated in terms of flats instead of faces. The k-th term is given by

$$\mathcal{D}_k(\mathsf{a})[\mathrm{Z}] = \sum_{\substack{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z} \\ \mathrm{rk}(\mathrm{X}/\mathrm{Z}) \geq k}} \mathrm{im}(\mu_{\mathrm{Z}}^{\mathrm{X}} : \mathsf{a}[\mathrm{X}] \to \mathsf{a}[\mathrm{Z}]). \tag{5.38}$$

Equivalently,

$$\mathcal{D}_k(\mathsf{a}) = \mathrm{im}\, \mu^{\overline{k}},$$

with $\mu^{\overline{k}}$ as in (5.13). Further,

$$\mu^{\overline{i}}\big(\mathcal{D}_k(\mathsf{a})^{\overline{i+1}}\big) \subseteq \mathcal{D}_{k+i}(\mathsf{a}) \tag{5.39}$$

for $i \geq 0$ (with no restriction on k). In particular, setting $k = 1$,

$$\mu^{\overline{i}}\big(\mathcal{D}(\mathsf{a})^{\overline{i+1}}\big) \subseteq \mathcal{D}_{i+1}(\mathsf{a}). \tag{5.40}$$

5.4.8. Duality. The notions of primitive part (primitive filtration) and decomposable part (decomposable filtration) are dual to each other. This is formalized below.

Proposition 5.26. *If c is a comonoid, then $\mathcal{P}_k(\mathsf{c})$ and $\mathcal{D}_k(\mathsf{c}^*)$ are orthogonal complements of each other under the canonical pairing between c and c^*.*

Dually, if a is a monoid, then $\mathcal{D}_k(\mathsf{a})$ and $\mathcal{P}_k(\mathsf{a}^*)$ are orthogonal complements of each other under the canonical pairing between a and a^*.

The duality between the primitive part and the indecomposable part can be phrased as follows.

For any comonoid c and monoid a,

$$\mathcal{P}(\mathsf{c})^* \cong \mathcal{Q}(\mathsf{c}^*) \quad \text{and} \quad \mathcal{Q}(\mathsf{a})^* \cong \mathcal{P}(\mathsf{a}^*). \tag{5.41}$$

This is a reformulation of Proposition 5.26 for $k = 1$.

Exercise 5.27. Check that: For any comonoid c and monoid a,

$$\mathrm{gr}_{\mathcal{P}}(\mathsf{c})^* \cong \mathrm{gr}_{\mathcal{D}}(\mathsf{c}^*) \quad \text{and} \quad \mathrm{gr}_{\mathcal{D}}(\mathsf{a})^* \cong \mathrm{gr}_{\mathcal{P}}(\mathsf{a}^*).$$

The first isomorphism is of graded monoids, while the second is of graded comonoids.

5.5. Trivial (co)monoids and (co)derivations

We formulate trivial comonoids and relate them to the primitive part functor. Dually, we formulate trivial monoids and relate them to the indecomposable part functor. We also introduce the related notion of coderivations and derivations.

5.5.1. Trivial comonoids. Every species can be turned into a comonoid by defining

$$\Delta_A^F := \begin{cases} 0 & \text{if } F > A, \\ \mathrm{id} & \text{if } F = A. \end{cases} \tag{5.42}$$

We refer to such a comonoid as a *trivial comonoid*. This defines a functor

$$\mathit{trv} : \mathcal{A}\text{-}\mathsf{Sp} \to \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}). \tag{5.43}$$

We call this the *trivial comonoid functor*. Observe that it maps into the subcategory of cocommutative comonoids.

Suppose c is a trivial comonoid. Then a morphism of comonoids $\mathsf{c} \to \mathsf{d}$ is the same as a map of species $\mathsf{c} \to \mathcal{P}(\mathsf{d})$, where $\mathcal{P}(\mathsf{d})$ is the primitive part of d. This is because $\mathcal{P}(\mathsf{d})$ is the largest trivial subcomonoid of d. In other words:

Proposition 5.28. *The trivial comonoid functor is the left adjoint of the primitive part functor. (The functors are between the categories of species and comonoids.)*

Exercise 5.29. Check that: For the adjunction in Proposition 5.28, we can also take the functors to be between the categories of species and cocommutative comonoids. Deduce the result of Exercise 5.10 by composing the adjunctions

$$\mathcal{A}\text{-}\mathsf{Sp} \underset{\mathcal{P}}{\overset{\mathit{trv}}{\rightleftarrows}} {}^{\mathsf{co}}\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{(-)^{coab}}{\overset{\mathit{inc}}{\rightleftarrows}} \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}).$$

The second adjunction is given in (2.55).

5.5.2. Trivial monoids. Dually, every species can be turned into a monoid by defining

$$\mu_A^F := \begin{cases} 0 & \text{if } F > A, \\ \mathrm{id} & \text{if } F = A. \end{cases} \tag{5.44}$$

We refer to such a monoid as a *trivial monoid*. This defines a functor

$$\mathit{trv} : \mathcal{A}\text{-}\mathsf{Sp} \to \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp}). \tag{5.45}$$

We call this the *trivial monoid functor*. Observe that it maps into the subcategory of commutative monoids.

Suppose a is a trivial monoid. Then a morphism of monoids $\mathsf{b} \to \mathsf{a}$ is the same as a map of species $\mathcal{Q}(\mathsf{b}) \to \mathsf{a}$, where $\mathcal{Q}(\mathsf{b})$ is the indecomposable part of b. In other words:

Proposition 5.30. *The trivial monoid functor is the right adjoint of the indecomposable part functor. (The functors are between the categories of species and monoids.)*

Exercise 5.31. Deduce the result of Exercise 5.20 by formulating the dual of Exercise 5.29.

5.5.3. Derivations and coderivations. We now define (co, bi)derivations. For a species p and comonoid c, a map of species $f : \mathsf{p} \to \mathsf{c}$ is a *coderivation* if it maps into $\mathcal{P}(\mathsf{c})$. For a monoid a and species p, a map of species $f : \mathsf{a} \to \mathsf{p}$ is a *derivation* if it factors through $\mathcal{Q}(\mathsf{a})$. For a monoid a and comonoid c, a map of species $f : \mathsf{a} \to \mathsf{c}$ is a *biderivation* if it maps into $\mathcal{P}(\mathsf{c})$ and factors through $\mathcal{Q}(\mathsf{a})$. In other words, a map is a biderivation iff it is both a derivation and a coderivation.

This is illustrated below.

$$
\begin{tikzcd}
\mathsf{p} \arrow[r, "f"] \arrow[dr, dashed, "f"'] & \mathsf{c} \\
& \mathcal{P}(\mathsf{c}) \arrow[u, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{a} \arrow[r, "f"] \arrow[d, two heads] & \mathsf{p} \\
\mathcal{Q}(\mathsf{a}) \arrow[ur, dashed, "f"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{a} \arrow[r, "f"] \arrow[d, two heads] & \mathsf{c} \\
\mathcal{Q}(\mathsf{a}) \arrow[r, dashed, "f"'] & \mathcal{P}(\mathsf{c}) \arrow[u, hook]
\end{tikzcd}
$$

Observe that:

Lemma 5.32. *A map f is a coderivation iff $\Delta^F_A f_A = 0$ for $F > A$. Dually, f is a derivation iff $f_A \mu^F_A = 0$ for $F > A$.*

In other words:

Lemma 5.33. *A map of species $f : \mathsf{p} \to \mathsf{c}$ is a coderivation iff f is a morphism of comonoids (with the trivial coproduct on p). A map of species $f : \mathsf{a} \to \mathsf{p}$ is a derivation iff f is a morphism of monoids (with the trivial product on p).*

This result can also be seen as a consequence of Propositions 5.28 and 5.30.

Exercise 5.34. Check that: If $h : \mathsf{p} \to \mathsf{c}$ is a coderivation, $f : \mathsf{p}' \to \mathsf{p}$ a map of species, $g : \mathsf{c} \to \mathsf{c}'$ a comonoid morphism, then the composite $ghf : \mathsf{p}' \to \mathsf{c}'$ is a coderivation. Similarly, if $h : \mathsf{a} \to \mathsf{p}$ is a derivation, $f : \mathsf{a}' \to \mathsf{a}$ a monoid morphism, $g : \mathsf{p} \to \mathsf{p}'$ a map of species, then $ghf : \mathsf{a}' \to \mathsf{p}'$ is a derivation. Combining: If $h : \mathsf{a} \to \mathsf{c}$ is a biderivation, $f : \mathsf{a}' \to \mathsf{a}$ a monoid morphism, $g : \mathsf{c} \to \mathsf{c}'$ a comonoid morphism, then $ghf : \mathsf{a}' \to \mathsf{c}'$ is a biderivation.

Exercise 5.35. Check that: The dual of a derivation is a coderivation, and vice versa. The dual of a biderivation is again a biderivation.

5.6. Bimonoid axiom on the primitive part

We formulate some preliminary results on how the bimonoid axiom works on the primitive part of a bimonoid. We also give some related results on the primitive filtration. These will play an important role in inductive arguments for proving the rigidity theorems in Chapter 13.

5.6.1. 0-bimonoids. We begin with 0-bimonoids.

Lemma 5.36. *Let* h *be a* 0*-bimonoid. If* $x \in \mathcal{P}(\mathsf{h})[F]$*, then*

$$\Delta_A^G \mu_A^F(x) = \begin{cases} \mu_G^F(x) & \text{if } G \leq F, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Suppose $x \in \mathcal{P}(\mathsf{h})[F]$. Then $\Delta_F^{FG}(x) = 0$ unless $FG = F$. Applying the 0-bimonoid axiom (2.40), we see that the lhs equals 0 unless $FG = GF = F$, or equivalently, unless $G \leq F$. If this condition holds, then $\Delta_F^{FG} = \mathrm{id}$ and hence the lhs equals $\mu_G^F(x)$. □

Recall Cauchy powers of a species from Section 5.1.1.

Lemma 5.37. *Let* h *be a* 0*-bimonoid. Then* $\Delta^k \mu^k = \mathrm{id}$ *on* $\mathcal{P}(\mathsf{h})^{k+1}$ *for any* $k \geq 0$*.*

PROOF. We employ Lemma 5.36. For a face A, let us evaluate $\Delta_A^k \mu_A^k$ on an element x in $\mathcal{P}(\mathsf{h})^{k+1}[A]$. Suppose $x \in \mathcal{P}(\mathsf{h})[F]$. Then we need to sum $\Delta_A^G \mu_A^F(x)$ over all G greater than A which have the same rank as F. Only the term $G = F$ has a nonzero contribution (equal to x). □

Lemma 5.38. *Let* h *be a* 0*-bimonoid. Then, for* $k \geq 2$*,*

$$z \in \mathcal{P}_k(\mathsf{h})[A] \implies z - \mu_A^{k-1} \Delta_A^{k-1}(z) \in \mathcal{P}_{k-1}(\mathsf{h})[A].$$

PROOF. By (5.24), we have $\Delta_A^{k-1}(z) \in \mathcal{P}(\mathsf{h})^k[A]$. Now from Lemma 5.37,

$$\Delta_A^{k-1} \mu_A^{k-1} \Delta_A^{k-1}(z) = \Delta_A^{k-1}(z).$$

Hence, $z - \mu_A^{k-1} \Delta_A^{k-1}(z) \in \ker(\Delta_A^{k-1})$ as required. □

5.6.2. *q*-bimonoids. We now generalize the above results to q-bimonoids.

Lemma 5.39. *Let* h *be a* q*-bimonoid. If* $x \in \mathcal{P}(\mathsf{h})[F]$*, then*

$$\Delta_A^G \mu_A^F(x) = \begin{cases} \mu_G^{GF} (\beta_q)_{GF,F}(x) & \text{if } FG = F, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Suppose $x \in \mathcal{P}(\mathsf{h})[F]$. Then $\Delta_F^{FG}(x) = 0$ unless $FG = F$. Applying the q-bimonoid axiom (2.33), we see that the lhs equals 0 unless $FG = F$. If this condition holds, then $\Delta_F^{FG} = \mathrm{id}$ and hence the lhs equals $\mu_G^{GF}(\beta_q)_{GF,F}(x)$. □

Recall the maps $(\beta_q)^k$ from (5.4).

Lemma 5.40. *Let* h *be a* q*-bimonoid. Then* $\Delta^k \mu^k = (\beta_q)^k$ *on* $\mathcal{P}(\mathsf{h})^{k+1}$ *for any* $k \geq 0$*.*

PROOF. We employ Lemma 5.39. For a face A, let us evaluate $\Delta_A^k \mu_A^k$ on an element x in $\mathcal{P}(\mathsf{h})^{k+1}[A]$. Suppose $x \in \mathcal{P}(\mathsf{h})[F]$. Then we need to sum $\Delta_A^G \mu_A^F(x)$ over all G greater than A which have the same rank as F. Only those G with the same support as F have a nonzero contribution (equal to $(\beta_q)_{G,F}(x)$). □

Exercise 5.41. For any q-bimonoid h, check that $\Delta^j \mu^i = 0$ on $\mathcal{P}(\mathsf{h})^{i+1}$ for any $j > i$. (Use the second alternative in Lemma 5.39.)

Recall the maps $(\beta^q)^{k-1}$ from (5.5) (which are defined when q is not a root of unity).

Lemma 5.42. *Let h be a q-bimonoid for q not a root of unity. Then, for $k \geq 2$,*

$$z \in \mathcal{P}_k(\mathsf{h})[A] \implies z - \mu_A^{k-1}(\beta^q)_A^{k-1}\Delta_A^{k-1}(z) \in \mathcal{P}_{k-1}(\mathsf{h})[A].$$

Proof. Put $z' := z - \mu_A^{k-1}(\beta^q)_A^{k-1}\Delta_A^{k-1}(z)$. We calculate:

$$\begin{aligned}
\Delta_A^{k-1}(z') &= \Delta_A^{k-1}(z) - \Delta_A^{k-1}\mu_A^{k-1}(\beta^q)_A^{k-1}\Delta_A^{k-1}(z) \\
&= \Delta_A^{k-1}(z) - (\beta_q)_A^{k-1}(\beta^q)_A^{k-1}\Delta_A^{k-1}(z) \\
&= \Delta_A^{k-1}(z) - \Delta_A^{k-1}(z) \\
&= 0.
\end{aligned}$$

The second step used Lemma 5.40. We used here that $(\beta^q)_A^{k-1}\Delta_A^{k-1}(z) \in \mathcal{P}(\mathsf{h})^k[A]$. This follows by combining (5.24) and Exercise 5.13. The third step used Lemma 5.3. $\square$

The element z' in the above proof can be written explicitly as

$$z' = z - \sum_{\substack{(F,G):\, F,G \geq A, \\ \mathrm{rk}(F/A)=k-1, \\ \mathrm{s}(F)=\mathrm{s}(G)}} q^{F/A,G/A}\mu_A^G \beta_{G,F}\Delta_A^F(z). \tag{5.46}$$

The sum is over both F and G. We point out that by definition of the two-sided q-zeta function (1.64), the coefficient $q^{F/A,G/A}$ equals $\boldsymbol{\zeta}_q(A,F,G)$. The element z' satisfies the property that $\Delta_A^K(z') = 0$ for any face K greater than A such that $\mathrm{rk}(K/A) = k-1$.

We suggest here another way to understand the origin of the formula (5.46). First, approximate z' by

$$z - \sum_{\substack{F:\, F \geq A \\ \mathrm{rk}(F/A)=k-1}} \mu_A^F \Delta_A^F(z). \tag{5.47}$$

Applying Δ_A^K to this element does not give zero. So we add correction terms to get

$$z - \sum_{\substack{F:\, F \geq A \\ \mathrm{rk}(F/A)=k-1}} \mu_A^F \Delta_A^F(z) + \sum_{\substack{(F,G):\, F,G \geq A, \\ \mathrm{rk}(F/A)=k-1, \\ \mathrm{s}(F)=\mathrm{s}(G), F \neq G}} q^{\mathrm{dist}(F,G)}\mu_A^G \beta_{G,F}\Delta_A^F(z).$$

Continuing this procedure yields a series which can be written as

$$z - \sum_{\alpha} (-1)^{l(\alpha)} q^{\alpha}\, \mu_A^{t(\alpha)} \beta_{t(\alpha),s(\alpha)} \Delta_A^{s(\alpha)}(z),$$

where α varies over all non-stuttering paths in flats of rank $k-1$ in the star of A. By setting $F := s(\alpha)$ and $G := t(\alpha)$, we can rewrite this as a sum over non-stuttering paths from F to G (in the star of A), followed by a sum over (F, G). This yields z' in the form (5.46) and equals it in view of (1.36).

In the special case $q = 0$: The maps $(\beta_0)_A^{k-1}$ and $(\beta^0)_A^{k-1}$ are both identity, and Lemma 5.42 (and its proof) specializes to Lemma 5.38 (and its proof). Also observe that, the procedure given above for z' terminates after the first step itself, and z' equals (5.47).

5.6.3. Bicommutative bimonoids. We now turn to bicommutative bimonoids.

Lemma 5.43. *Let* h *be a bicommutative bimonoid. If* $x \in \mathcal{P}(\mathsf{h})[\mathrm{X}]$*, then*

$$\Delta_{\mathrm{Z}}^{\mathrm{Y}} \mu_{\mathrm{Z}}^{\mathrm{X}}(x) = \begin{cases} \mu_{\mathrm{Y}}^{\mathrm{X}}(x) & \text{if } \mathrm{Y} \leq \mathrm{X}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Apply the bicommutative bimonoid axiom (2.26). Since $x \in \mathcal{P}(\mathsf{h})[\mathrm{X}]$, $\Delta_{\mathrm{X}}^{\mathrm{X} \vee \mathrm{Y}}(x) = 0$ unless $\mathrm{X} \vee \mathrm{Y} = \mathrm{X}$, or equivalently, $\mathrm{Y} \leq \mathrm{X}$. ☐

Recall commutative Cauchy powers of a species from Section 5.1.4.

Lemma 5.44. *Let* h *be a bicommutative bimonoid. Then* $\Delta^{\overline{k}} \mu^{\overline{k}} = \mathrm{id}$ *on* $\mathcal{P}(\mathsf{h})^{\overline{k+1}}$ *for any* $k \geq 0$.

Proof. We employ Lemma 5.43. For any flat Z, let us evaluate $\Delta_{\mathrm{Z}}^{\overline{k}} \mu_{\mathrm{Z}}^{\overline{k}}$ on an element x in $\mathcal{P}(\mathsf{h})^{\overline{k+1}}[\mathrm{Z}]$. Suppose $x \in \mathcal{P}(\mathsf{h})[\mathrm{X}]$. Then we need to sum $\Delta_{\mathrm{Z}}^{\mathrm{Y}} \mu_{\mathrm{Z}}^{\mathrm{X}}(x)$ over all Y greater than Z which have the same rank as X. Only the term $\mathrm{Y} = \mathrm{X}$ has a nonzero contribution (equal to x). ☐

Lemma 5.45. *Let* h *be a bicommutative bimonoid. Then, for* $k \geq 2$,

$$z \in \mathcal{P}_k(\mathsf{h})[\mathrm{Z}] \implies z - \mu_{\mathrm{Z}}^{\overline{k-1}} \Delta_{\mathrm{Z}}^{\overline{k-1}}(z) \in \mathcal{P}_{k-1}(\mathsf{h})[\mathrm{Z}].$$

Proof. Firstly, $\Delta_{\mathrm{Z}}^{\overline{k-1}}(z) \in \mathcal{P}(\mathsf{h})^k[\mathrm{Z}]$. Now from Lemma 5.44,

$$\Delta_{\mathrm{Z}}^{\overline{k-1}} \mu_{\mathrm{Z}}^{\overline{k-1}} \Delta_{\mathrm{Z}}^{\overline{k-1}}(z) = \Delta_{\mathrm{Z}}^{\overline{k-1}}(z).$$

Hence, $z - \mu_{\mathrm{Z}}^{\overline{k-1}} \Delta_{\mathrm{Z}}^{\overline{k-1}}(z) \in \ker(\Delta_{\mathrm{Z}}^{\overline{k-1}})$ as required. ☐

Exercise 5.46. Let $\sim$ be a geometric partial-support relation on faces. Generalize Lemmas 5.36 and 5.43 to 0-$\sim$-bicommutative bimonoids. Do the same for Lemmas 5.37 and 5.44, and for Lemmas 5.38 and 5.45.

5.6.4. Cocommutative bimonoids. Fix a noncommutative zeta function $\boldsymbol{\zeta}$. By setting $s := \boldsymbol{\zeta}$ in (5.6), we obtain maps $\boldsymbol{\zeta}^{(k-1)}$ for any $k \geq 1$. Similarly, by setting $q = 1$ in (5.4), we have the maps $(\beta_1)^k$.

Lemma 5.47. *Let* h *be a cocommutative bimonoid. Then* $(\beta_1)^k \boldsymbol{\zeta}^{(k)} \Delta^k = \Delta^k$ *for any* $k \geq 0$.

PROOF. We compute on the A-component:

$$
\begin{aligned}
(\beta_1)^k_A \boldsymbol{\zeta}^{(k)}_A \Delta^k_A &= \sum_{\substack{F,G:\\ \mathrm{s}(F/A)=\mathrm{s}(G/A)\\ \mathrm{rk}(G/A)=\mathrm{rk}(F/A)=k}} \beta_{G,F}\, \boldsymbol{\zeta}(A,F)\, \Delta^F_A \\
&= \sum_{\substack{F,G:\\ \mathrm{s}(F/A)=\mathrm{s}(G/A)\\ \mathrm{rk}(G/A)=\mathrm{rk}(F/A)=k}} \boldsymbol{\zeta}(A,F)\, \Delta^G_A \\
&= \sum_{G:\, \mathrm{rk}(G/A)=k} \Bigg(\sum_{F:\, \mathrm{s}(F/A)=\mathrm{s}(G/A)} \boldsymbol{\zeta}(A,F) \Bigg) \Delta^G_A \\
&= \sum_{G:\, \mathrm{rk}(G/A)=k} \Delta^G_A \\
&= \Delta^k_A.
\end{aligned}
$$

The first step and last step used definitions. The second step used cocommutativity. The fourth step used the flat-additivity formula (1.43). ☐

Lemma 5.48. *Let* h *be a cocommutative bimonoid. Then, for* $k \geq 2$,

$$
z \in \mathcal{P}_k(\mathsf{h})[A] \implies z - \mu^{k-1}_A \boldsymbol{\zeta}^{(k-1)}_A \Delta^{k-1}_A(z) \in \mathcal{P}_{k-1}(\mathsf{h})[A].
$$

PROOF. Put $z' := z - \mu^{k-1}_A \boldsymbol{\zeta}^{(k-1)}_A \Delta^{k-1}_A(z)$. We calculate:

$$
\begin{aligned}
\Delta^{k-1}_A(z') &= \Delta^{k-1}_A(z) - \Delta^{k-1}_A \mu^{k-1}_A \boldsymbol{\zeta}^{(k-1)}_A \Delta^{k-1}_A(z) \\
&= \Delta^{k-1}_A(z) - (\beta_1)^{k-1}_A \boldsymbol{\zeta}^{(k-1)}_A \Delta^{k-1}_A(z) \\
&= \Delta^{k-1}_A(z) - \Delta^{k-1}_A(z) \\
&= 0.
\end{aligned}
$$

The second step used Lemma 5.40 for $q = 1$. The third step used Lemma 5.47. ☐

The element z' defined in the above proof can be explicitly written as

$$
z' = z - \sum_{F:\, \mathrm{rk}(F/A)=k-1} \boldsymbol{\zeta}(A,F)\, \mu^F_A \Delta^F_A(z). \tag{5.48}
$$

Compare and contrast with (5.46).

5.7. Primitively generated bimonoids and cocommutativity

We say a q-bimonoid is primitively generated if it is generated as a monoid by its primitive part. We show that a (signed) bimonoid h is primitively generated iff h is (signed) cocommutative. For this, we consider the filtration of h generated by its primitive part. This filtration is contained termwise in the primitive filtration of h. The two are equal iff h is (signed) cocommutative. A similar argument shows that q-bimonoids for q not a root of unity are primitively generated.

For a q-bimonoid, there is a canonical map from its primitive part to its indecomposable part. For a q-bimonoid for q not a root of unity, this map is bijective. For a bimonoid, this map is surjective iff the bimonoid is cocommutative, injective iff the bimonoid is commutative, and bijective iff the bimonoid is bicommutative. Similar results hold for signed bimonoids.

5.7.1. Primitively generated bimonoids. For any scalar q, we say a q-bimonoid h is *primitively generated* if $\mathsf{h} = \langle \mathcal{P}(\mathsf{h}) \rangle$, with notation as in (2.63). A related construction is given below.

Let h be a q-bimonoid. For $k \geq 1$, define the subspecies $\mathcal{F}_k(\mathsf{h})$ by

$$\mathcal{F}_k(\mathsf{h})[A] := \sum_{F:\, \mathrm{rk}(F/A) \leq k-1} \mu_A^F(\mathcal{P}(\mathsf{h})[F]).$$

This is the filtration of h generated by $\mathcal{P}(\mathsf{h})$ in the sense of Section 5.2.4. As a special case of (5.17),

$$\mathcal{F}_1(\mathsf{h}) \subseteq \mathcal{F}_2(\mathsf{h}) \subseteq \cdots \subseteq \mathsf{h} \quad \text{with} \quad \bigcup_{k \geq 1} \mathcal{F}_k(\mathsf{h}) = \langle \mathcal{P}(\mathsf{h}) \rangle. \tag{5.49}$$

Lemma 5.49. *If a bimonoid h is primitively generated, then h is cocommutative.*

PROOF. Since $\mathcal{P}(\mathsf{h})$ is cocommutative, by Proposition 2.48, we have $\langle \mathcal{P}(\mathsf{h}) \rangle \subseteq \mathsf{h}^{coab} \subseteq \mathsf{h}$. If h is primitively generated, then both inclusions become equalities, and h is cocommutative. $\square$

The converse also holds, see Proposition 5.51 below.

Lemma 5.50. *Let h be a bimonoid. For $k \geq 1$, we have $\mathcal{F}_k(\mathsf{h}) \subseteq \mathcal{P}_k(\mathsf{h})$. Equality holds iff h is cocommutative.*

PROOF. The inclusion follows from Exercise 5.41 for $q = 1$. We now prove the second part. Suppose equality holds. Then taking union over $k \geq 1$ on both sides, we obtain $\langle \mathcal{P}(\mathsf{h}) \rangle = \mathsf{h}$, which by Lemma 5.49 implies that h is cocommutative. Conversely, suppose h is cocommutative. To show equality, we do an induction on k. Clearly, $\mathcal{F}_1(\mathsf{h}) = \mathcal{P}_1(\mathsf{h})$. This is the induction base. For the induction step: Suppose $z \in \mathcal{P}_k(\mathsf{h})[A]$. Put $y := \mu_A^{k-1} \zeta_A^{(k-1)} \Delta_A^{k-1}(z)$. Clearly, y belongs to $\mathcal{F}_k(\mathsf{h})[A]$. Further, by Lemma 5.48, the element $z - y$ belongs to $\mathcal{P}_{k-1}(\mathsf{h})[A]$, which by induction hypothesis equals $\mathcal{F}_{k-1}(\mathsf{h})[A]$. Hence, $z = y + (z - y)$ belongs to $\mathcal{F}_k(\mathsf{h})[A]$. This completes the induction step. $\square$

Proposition 5.51. *A bimonoid h is primitively generated iff h is cocommutative.*

PROOF. We saw forward implication in Lemma 5.49. For backward implication: Suppose h is cocommutative. By Lemma 5.50, $\mathcal{F}_k(\mathsf{h}) = \mathcal{P}_k(\mathsf{h})$ for $k \geq 1$. Taking union of both sides over k and using (5.20) and (5.49) yields $\langle \mathcal{P}(\mathsf{h}) \rangle = \mathsf{h}$. $\square$

Another proof of backward implication in Proposition 5.51 is given later in Exercise 9.25.

Exercise 5.52. Let h be a bimonoid. Show that the filtration generated by the primitive part of h and h^{coab} coincide. (Use Exercise 5.10.) Deduce that the subbimonoid of h generated by $\mathcal{P}(\mathsf{h})$ equals the coabelianization of h, that is, $\langle \mathcal{P}(\mathsf{h}) \rangle = \mathsf{h}^{coab}$. (Apply Proposition 5.51 to h^{coab}.)

Exercise 5.53. Show that: A signed bimonoid h is primitively generated iff h is signed cocommutative. (This is the signed analogue of Proposition 5.51.)

Proposition 5.54. *Let h be a q-bimonoid for q not a root of unity. Then, for $k \geq 1$, we have $\mathcal{F}_k(\mathsf{h}) = \mathcal{P}_k(\mathsf{h})$. In particular, h is primitively generated.*

Proof. The proof is similar to that of Lemma 5.50. We use Exercise 5.41 as before, and Lemma 5.42 in place of Lemma 5.48. ☐

5.7.2. Canonical map between primitive part and indecomposable part. For any q-bimonoid h, there is a canonical map from the primitive part $\mathcal{P}(\mathsf{h})$ to the indecomposable part $\mathcal{Q}(\mathsf{h})$ given by the composite

$$\mathrm{pq}_{\mathsf{h}} : \mathcal{P}(\mathsf{h}) \hookrightarrow \mathsf{h} \twoheadrightarrow \mathcal{Q}(\mathsf{h}). \tag{5.50}$$

This map is clearly natural in h, that is, for any morphism $f : \mathsf{h} \to \mathsf{k}$ of q-bimonoids, the diagram

$$\begin{tikzcd} \mathcal{P}(\mathsf{h}) \arrow[r, "f"] \arrow[d, "\mathrm{pq}_{\mathsf{h}}"'] & \mathcal{P}(\mathsf{k}) \arrow[d, "\mathrm{pq}_{\mathsf{k}}"] \\ \mathcal{Q}(\mathsf{h}) \arrow[r, "f"'] & \mathcal{Q}(\mathsf{k}) \end{tikzcd} \tag{5.51}$$

commutes. This defines a natural transformation pq from $\mathcal{P}$ to $\mathcal{Q}$ (viewing both as functors from q-bimonoids to species).

The map (5.50) can be used to characterize (co)commutativity as follows.

Theorem 5.55. *For a bimonoid h, the following are equivalent.*

(1) *The map $\mathrm{pq}_{\mathsf{h}} : \mathcal{P}(\mathsf{h}) \to \mathcal{Q}(\mathsf{h})$ is surjective.*
(2) *The subspecies $\mathcal{P}(\mathsf{h})$ and $\mathcal{D}(\mathsf{h})$ together linearly span h.*
(3) *The bimonoid h is primitively generated, that is, $\mathsf{h} = \langle \mathcal{P}(\mathsf{h}) \rangle$.*
(4) *The filtration generated by the primitive part of h and the primitive filtration of h coincide, that is, $\mathcal{F}_k(\mathsf{h}) = \mathcal{P}_k(\mathsf{h})$ for $k \geq 1$.*
(5) *The bimonoid h is cocommutative.*

Proof. Items (1) and (2) are clearly equivalent. Items (2) and (3) are equivalent by Lemma 5.22. Items (3), (4), (5) are equivalent by Lemma 5.50 and Proposition 5.51. ☐

Proposition 5.56. *Let h be a bimonoid. Then*

(1) *h is cocommutative iff the map $\mathrm{pq}_{\mathsf{h}} : \mathcal{P}(\mathsf{h}) \to \mathcal{Q}(\mathsf{h})$ is surjective.*
(2) *h is commutative iff the map $\mathrm{pq}_{\mathsf{h}} : \mathcal{P}(\mathsf{h}) \to \mathcal{Q}(\mathsf{h})$ is injective.*
(3) *h is bicommutative iff the map $\mathrm{pq}_{\mathsf{h}} : \mathcal{P}(\mathsf{h}) \to \mathcal{Q}(\mathsf{h})$ is bijective.*

Proof. Item (1) is contained in Theorem 5.55. Item (2) follows from item (1) by duality. Item (3) follows by combining items (1) and (2). ☐

Another proof of forward implications in Proposition 5.56 is given later in Exercise 9.26. A result related to item (3) is given in Proposition 9.47.

Exercise 5.57. Formulate signed analogues of Theorem 5.55 and Proposition 5.56.

Proposition 5.58. *For any q-bimonoid h for q not a root of unity, the map $\mathrm{pq}_{\mathsf{h}} : \mathcal{P}(\mathsf{h}) \to \mathcal{Q}(\mathsf{h})$ is bijective.*

PROOF. In view of duality, it suffices to show that pq_{h} is surjective, or equivalently, that $\mathcal{P}(\mathsf{h})$ and $\mathcal{D}(\mathsf{h})$ together linearly span h. This follows from Proposition 5.54 and Lemma 5.22. $\square$

An improvement of this result is given later in Proposition 9.84.

5.8. Browder–Sweedler and Milnor–Moore

The primitive and decomposable filtrations turn any q-bimonoid into a filtered q-bimonoid. Thus, for either filtration, we can consider the corresponding associated graded q-bimonoid. The Browder–Sweedler commutativity result says that for $q = \pm 1$, the associated graded (signed) bimonoid wrt the primitive filtration is (signed) commutative. Dually, the Milnor–Moore cocommutativity result says that for $q = \pm 1$, the associated graded (signed) bimonoid wrt the decomposable filtration is (signed) cocommutative.

5.8.1. Associated graded bimonoid of the primitive filtration. Let h be a q-bimonoid. Consider the primitive filtration of h. The product components of h restrict to linear maps

$$\mu_A^F : \mathcal{P}_k(\mathsf{h})[F] \to \mathcal{P}_{k+\mathrm{rk}(F/A)}(\mathsf{h})[A]. \tag{5.52}$$

This is particularly easy to see when h is primitively generated. In general, this can be deduced from the q-bimonoid axiom (2.33). Alternatively, (5.52) may be expressed as

$$\mu^i\big(\mathcal{P}_k(\mathsf{h})^{i+1}\big) \subseteq \mathcal{P}_{k+i}(\mathsf{h}) \tag{5.53}$$

for $i \geq 0$ (with no restriction on k). Note very carefully that in contrast to the coproduct components (5.22), the product components increase the level of the filtration. We conclude that h is a filtered q-bimonoid.

Now consider the graded comonoid $\mathrm{gr}_{\mathcal{P}}(\mathsf{h})$ defined in (5.25). The maps (5.52) induce a product on $\mathrm{gr}_{\mathcal{P}}(\mathsf{h})$ which turn it into a graded q-bimonoid. This defines a functor from the category of q-bimonoids to the category of graded q-bimonoids which sends h to $\mathrm{gr}_{\mathcal{P}}(\mathsf{h})$.

Consider the special case $q = 1$. In other words, h is a bimonoid. Observe that if h is (co)commutative, then so is $\mathrm{gr}_{\mathcal{P}}(\mathsf{h})$. A similar remark applies to signed bimonoids which is the case $q = -1$.

Exercise 5.59. Show that: If k is a graded q-bimonoid such that $\mathcal{P}(\mathsf{k}) = \mathsf{k}_1$, then $\mathrm{gr}_{\mathcal{P}}(\mathsf{k}) \cong \mathsf{k}$ as graded q-bimonoids. Here $\mathrm{gr}_{\mathcal{P}}(\mathsf{k})$ is the graded q-bimonoid obtained by viewing k as a q-bimonoid by forgetting its grading. (This extends Exercise 5.17.)

5.8.2. Browder–Sweedler commutativity result. We begin with a result similar to Lemma 5.39. It will be useful in what follows.

Lemma 5.60. *Let* h *be a* q*-bimonoid. Let* F *and* K *be faces both greater than* A*, with* $\mathrm{rk}(K) \geq \mathrm{rk}(F)$*. Put* $k := \mathrm{rk}(K) - \mathrm{rk}(F)$*. If* $x \in \mathcal{P}_{k+1}(\mathsf{h})[F]$*, then*

$$\Delta_A^K \mu_A^F(x) = \begin{cases} (\beta_q)_{K,FK} \Delta_F^{FK}(x) & \textit{if } KF = K, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Applying the q-bimonoid axiom (2.33), we see that the lhs equals $\mu_K^{KF}(\beta_q)_{KF,FK}\Delta_F^{FK}(x)$. The first alternative follows. For the second alternative: Suppose $KF > K$. Then $\mathrm{rk}(FK/F) \geq k+1$ since $\mathrm{rk}(FK) = \mathrm{rk}(KF)$. Hence, by the hypothesis on x, we have $\Delta_F^{FK}(x) = 0$, and the lhs is zero. □

Lemma 5.61. *Let* h *be a bimonoid. Let* F *and* G *be faces both greater than* A*, and of the same support. If* $x \in \mathcal{P}_{k+1}(\mathsf{h})[F]$*, then*

$$\mu_A^F(x) - \mu_A^G \beta_{G,F}(x) \in \mathcal{P}_{k+\mathrm{rk}(F/A)}[A].$$

The point here is that the level of the primitive filtration only increases by $\mathrm{rk}(F/A) - 1$ (from $k+1$ to $k + \mathrm{rk}(F/A)$) which is one less than what one expects.

Proof. Let K be any face greater than A with $\mathrm{rk}(K/A) = k + \mathrm{rk}(F/A)$. We want to show that

$$\Delta_A^K(\mu_A^F(x) - \mu_A^G \beta_{G,F}(x)) = 0.$$

We consider two cases.

- Suppose $KF = KG = K$. Then we are done by Exercise 2.19. For convenience, let us spell this out. By the first alternative in Lemma 5.60 for $q = 1$,

$$\begin{aligned} \Delta_A^K(\mu_A^F(x) - \mu_A^G \beta_{G,F}(x)) &= \beta_{K,FK}\Delta_F^{FK}(x) - \beta_{K,GK}\Delta_G^{GK}\beta_{G,F}(x) \\ &= \beta_{K,FK}\Delta_F^{FK}(x) - \beta_{K,GK}\beta_{GK,FK}\Delta_F^{FK}(x) \\ &= \beta_{K,FK}\Delta_F^{FK}(x) - \beta_{K,FK}\Delta_F^{FK}(x) \\ &= 0. \end{aligned}$$

  The second step used (2.10), while the third step used (2.1).
- Suppose $KF > K$, or equivalently, $KG > K$. Then by the second alternative in Lemma 5.60, $\Delta_A^K \mu_A^F(x) = \Delta_A^K \mu_A^G \beta_{G,F}(x) = 0$.

□

Proposition 5.62. *For any bimonoid* h*, the bimonoid* $\mathrm{gr}_{\mathcal{P}}(\mathsf{h})$ *is commutative. Further, if* h *is cocommutative, then* $\mathrm{gr}_{\mathcal{P}}(\mathsf{h})$ *is bicommutative.*

Proof. The first part follows from Lemma 5.61. The second part is clear. □

We refer to this as the *Browder–Sweedler commutativity result*. It yields a functor from the category of bimonoids to the category of commutative bimonoids which sends h to $\mathrm{gr}_{\mathcal{P}}(\mathsf{h})$.

Exercise 5.63. Show that: If k is a graded bimonoid such that $\mathcal{P}(\mathsf{k}) = \mathsf{k}_1$, then k is commutative. (Use Exercise 5.59 and Proposition 5.62.) Conversely, use this fact to prove Proposition 5.62.

Exercise 5.64. Prove the signed analogue of Proposition 5.62, namely : For any signed bimonoid h, the signed bimonoid $\mathrm{gr}_{\mathcal{P}}(\mathsf{h})$ is signed commutative. Further, if h is signed cocommutative, then $\mathrm{gr}_{\mathcal{P}}(\mathsf{h})$ is signed bicommutative.

5.8.3. Associated graded bimonoid of the decomposable filtration.

Let h be a q-bimonoid. Consider the decomposable filtration of h. The coproduct components of h restrict to linear maps

$$\Delta_A^F : \mathcal{D}_k(\mathsf{h})[A] \to \mathcal{D}_{k-\mathrm{rk}(F/A)}(\mathsf{h})[F]. \tag{5.54}$$

This can be deduced from the q-bimonoid axiom (2.33). Alternatively, (5.54) may be expressed as

$$\Delta^i\big(\mathcal{D}_k(\mathsf{h})\big) \subseteq \mathcal{D}_{k-i}(\mathsf{h})^{i+1} \tag{5.55}$$

for $i \geq 0$ (with no restriction on k). Note very carefully that in contrast to the product components (5.33), the coproduct components decrease the level of the filtration. We conclude that h is a filtered q-bimonoid.

Now consider the graded monoid $\mathrm{gr}_{\mathcal{D}}(\mathsf{h})$ defined in (5.36). The maps (5.54) induce a coproduct on $\mathrm{gr}_{\mathcal{D}}(\mathsf{h})$ which turn it into a graded q-bimonoid. This defines a functor from the category of q-bimonoids to the category of graded q-bimonoids which sends h to $\mathrm{gr}_{\mathcal{D}}(\mathsf{h})$.

Consider the special case $q = 1$. In other words, h is a bimonoid. Observe that if h is (co)commutative, then so is $\mathrm{gr}_{\mathcal{D}}(\mathsf{h})$. A similar remark applies to signed bimonoids which is the case $q = -1$.

5.8.4. Milnor–Moore cocommutativity result.

Proposition 5.65. *For any bimonoid* h*, the bimonoid* $\mathrm{gr}_{\mathcal{D}}(\mathsf{h})$ *is cocommutative. Further, if* h *is commutative, then* $\mathrm{gr}_{\mathcal{D}}(\mathsf{h})$ *is bicommutative.*

We refer to this as the *Milnor–Moore cocommutativity result*. It is dual to Proposition 5.62 and can be deduced from it, see Exercise 5.67 below. A direct proof can also be given which we omit. Another proof is given in Exercise 5.66 below. This result also has a signed analogue as in Exercise 5.64.

Exercise 5.66. Show that: For any bimonoid h, the bimonoid $\mathrm{gr}_{\mathcal{D}}(\mathsf{h})$ is primitively generated. (Use Corollary 5.23.) Now apply Proposition 5.51 to deduce Proposition 5.65.

Exercise 5.67. This is a continuation of Exercise 5.27. Check that: For any bimonoid h,

$$\mathrm{gr}_{\mathcal{P}}(\mathsf{h})^* \cong \mathrm{gr}_{\mathcal{D}}(\mathsf{h}^*) \quad \text{and} \quad \mathrm{gr}_{\mathcal{D}}(\mathsf{h})^* \cong \mathrm{gr}_{\mathcal{P}}(\mathsf{h}^*)$$

as graded bimonoids. Use this to deduce Propositions 5.62 and 5.65 from each other.

Connections of the Browder–Sweedler commutativity and Milnor–Moore cocommutativity results to the representation theory of the Tits and Birkhoff algebras are developed later in Proposition 11.20 and Exercise 11.21.

Notes

References to the classical literature on Hopf algebras related to the primitive and decomposable filtrations (including their early history) are given below. This may be read in continuation to the Notes to Chapter 2.

Bialgebras. Classically, Cauchy powers of a species (5.1) correspond to tensor powers of a vector space. The maps Δ^k correspond to iterated coproducts of a coalgebra. Dually, the maps μ^k correspond to iterated products of an algebra.

Graded and filtered bialgebras. General references for graded and filtered bialgebras are the books by Abe [**1**, Chapter 1, Section 2.2 and Chapter 2, Section 4.1], Kharchenko [**529**, Section 1.6], Radford [**771**, Section 5.6], Sweedler [**867**, Sections 11.1 and 11.2]. For the analogue of Exercise 5.9, see for instance [**16**, Formula (5)].

Primitive elements and decomposable elements. Hopf [**458**, Section 14] uses the term 'maximal element' for any element not in the decomposable part of a Hopf algebra. In other words, his maximal element is a representative of an element in the indecomposable part. (We mention that Hopf is only working with a specific class of examples.) The terminology of maximal element is continued by Leray [**595**, Section 24]. Later, in [**595**, Section 25, Formula (15)], Leray introduces the notion of a primitive element under the name 'hypermaximal element'. This definition is again motivated by ideas of Hopf [**458**, Section 37] and Samelson [**799**]. The term 'primitive element' is used by Koszul [**542**, Section 10]. In fact, Koszul defines the primitive part as the orthogonal complement of the decomposable part, and later in [**542**, Lemma 10.1 and Formula (10.3)] derives the usual definition of a primitive element, namely, $\Delta(x) = 1 \otimes x + x \otimes 1$. The 'primitive' terminology is continued by Borel [**135**, Formula (20.1)], [**136**, Formula (2.3)], [**137**, page 404].

The analogue of the primitive-decomposable orthogonality in Proposition 5.26 is present in work of Koszul as mentioned above. The analogue of the equivalent primitive-indecomposable duality (5.41) is given by Milnor and Moore [**696**, Proposition 3.7], Zisman [**932**, Proposition 3,3], Bourgin [**151**, Chapter 12, Theorem 9.22], Wraith [**918**, Paragraph 3.7]. Some later references are by Kane [**508**, Section 1.5, Theorem A], Selick [**821**, Proposition 10.4.1], May and Ponto [**663**, Lemma 20.2.7].

Other early usages of primitive elements are by Cartier [**196**, Formula (17)], Halpern [**412**, page 128], Gabriel [**335**, Exposé VIIA, Section 3.2.3].

Primitive filtration. The coradical filtration of a connected graded coalgebra (without any specific name) appears in the paper by Browder [**160**, page 155]. Such a filtration is considered earlier by Cartier in the context of cocommutative bialgebras [**192**, Section 2.1, Formula (9)], [**194**, Section 2] and later by Dieudonné [**250**, Section II.2.2, Formula (5)]. For a bialgebra, Milnor and Moore consider the filtration generated by its primitive part [**695**, Definition 5.10].

The coradical filtration of a coalgebra is developed in detail in Sweedler's book [**867**, Section 9.1]. It is also treated by Grünenfelder [**383**, Section III.2], [**384**, Section 2]. The coradical filtration begins with the coradical of the coalgebra. For a connected graded bialgebra, the coradical equals its degree 0 part which is the base field. The next term is the direct sum of the coradical and the primitive part [**867**, Proposition 10.0.1]. In the equivalent setting of positively graded nonunital bialgebras, the coradical is not visible, and the first nontrivial term is the primitive part defined by the formula $\Delta_+(x) = 0$, where $\Delta_+(x) = \Delta(x) - 1 \otimes x - x \otimes 1$. This is analogous to our situation, and hence we prefer to use the term primitive filtration.

Some later references for the coradical filtration are [1, Chapter 2, Sections 3 and 4], [17, Section 3.1.2], [228, Section 3.1], [703, Chapter 5], [771, Chapter 4].

The analogue of Lemma 5.12 in the classical theory is [867, Theorem 9.1.4]. The analogue of Proposition 5.18 on injectivity is present in work of Milnor and Moore [695, Proposition 3.9], [696, Proposition 3.8, item (1)], Zisman [932, Lemma 3,2, item (a)], Quillen [766, Appendix B, Proposition 3.2], Heyneman and Sweedler [864, Lemma 4 on page 95], [865, Lemma 1], [867, Lemma 11.0.1], [432, Lemma 3.2.6], Grünenfelder [383, Corollary I.3.13], Kaplansky [510, Theorem 9]. See also [821, Proposition 10.4.2, item (b)], [663, Proposition 21.1.4]. An extension of this classical result to not necessarily connected bialgebras was given by Heyneman and Radford [431, Proposition 2.4.2]. For later references, see [703, Section 5.3], [771, Theorem 4.7.4], [295, Proposition 1.13.8], and for a further generalization to bimonoids in monoidal categories, see [42, Theorem 2.4].

The analogue of (5.26) is given by Browder [160, Proposition 1.2, item (iv)], see also [695, Proposition 5.11, item (4)]. The analogue of Exercises 5.16 and 5.17 is contained in Sweedler's book [867, Lemmas 11.2.1 and 11.2.3]. In his terminology, a graded comonoid c is called strictly graded if $\mathcal{P}(\mathsf{c}) = \mathsf{c}_1$. The analogue of Exercise 5.19 is given in [867, Corollaries 11.0.2 and 11.0.3 and Lemma 11.2.4].

Decomposable filtration. The decomposable filtration of an algebra is considered by Milnor and Moore [695, Definition 7.1] under the name 'augmentation filtration'. It is also present in Larson's thesis [563, Definition 13.1]. Its duality with the primitive filtration is given by Browder [160, page 155]. The dual of the primitive filtration is considered earlier by Cartier [192, bottom of page 2-05]. A more recent reference is [202, Section 2.4].

The analogue of Proposition 5.24 on surjectivity is given by Milnor and Moore [695, Proposition 3.8], [696, Proposition 3.8, item (2)], Zisman [932, Lemma 3,2, item (b)], see also [594, Lemma 1.0.3], [821, Proposition 10.4.2, item (a)], [663, Proposition 21.1.3]. The analogue of (5.37) and Corollary 5.23 is given by Browder [160, Proposition 1.1, item (iv)], see also [703, Lemma 5.6.6].

Primitive generation and cocommutativity. The analogue of Proposition 5.51 which connects primitive generation and cocommutativity is given by Milnor and Moore [695, Appendix], Larson [563, Theorem 15.2], [564, Proof of Theorem 3.5], Sweedler [864, page 96], [867, Corollary 13.0.3]. The special case that bicommutative bialgebras are primitively generated is treated by Halpern [412, Theorem 2.10], [415, (1.5)]. It is also given earlier by Moore [706, page 31] and Borel [136, Remark 2.7]. The original sources are the papers by Leray [595, Corollary 9 on page 133] and Samelson [799]. (We mention that Borel, Leray, Samelson are working with a specific class of examples.) Primitive generation of cocommutative bialgebras under more general hypothesis is given by Schmitt [813, Proposition 9.9].

The analogue of Proposition 5.56 is given by Milnor and Moore [695, Proposition 4.17 and Corollary 4.18], [696, Proposition 4.8], [708, Proposition 2]. Item (2) is given by Zisman [932, Proposition 3,5, and remark on page 8], Browder [160, Theorem 2.1], Bourgin [151, Chapter 12, Theorem 9.23]. Forward implication in item (3) for a special class of bicommutative bialgebras is present in earlier work of Leray [595, Theorems 9 and 10]. For later references, see [851, Proposition 3.1], [508, Section 1.5], [663, Corollary 22.3.3].

Browder–Sweedler and Milnor–Moore (co)commutativity results. The maps (5.22) and (5.52) show that any bimonoid is filtered wrt the primitive filtration. For classical analogues, see [867, Theorem 9.1.6 and Theorem 9.2.2, item (2)], [510, Theorem 17].

The analogue of Proposition 5.65 about cocommutativity of the associated graded wrt the decomposable filtration is implicitly present in work of Milnor and Moore [**695**]. They give the analogue of Exercise 5.66 in [**695**, Proposition 7.4].

The analogue of Proposition 5.62 about commutativity of the associated graded wrt the primitive filtration is deduced by Browder using duality as in Exercise 5.67 [**160**, Proposition 1.3, items (ii) and (iii)], also see his [**160**, Lemma 2.5]. The analogue of the closely related Exercise 5.63 is independently given by Sweedler [**867**, Theorem 11.2.5, item (a)]. For the analogue of Exercise 5.59 for $q = 1$, see his comments on [**867**, page 241]. For recent references, see [**22**, Proposition 1.6], [**16**, Lemma 1].

Derivations and coderivations. The classical analogues of the derivations and coderivations in Section 5.5.3 are the ϵ-derivations and ϵ-coderivations considered by Nichols [**719**, pages 5, 6, 10, 54, 56], [**720**, pages 66, 69, 70, 71, in particular, Proposition 6, item (i)]. An ϵ-derivation is called a differentiation by Hochschild [**445**, Section III.3, page 36]. It is also mentioned by Cartier [**202**, Formula (3.124)]. See the Notes to Chapter 9 for related information.

The analogue of Proposition 5.28 is given in Grünenfelder's thesis [**383**, Satz I.3.1].

Bimonoids in Joyal species. The k-th Cauchy power of a species (5.1) corresponds to the k-th power of a Joyal species under the Cauchy product (2.95).

Primitive filtration. The coradical filtration for positive comonoids in Joyal species is discussed in [**18**, Section 8.10]. The analogue of Proposition 5.18 is [**18**, Proposition 8.46]. The primitive part of a connected comonoid and the indecomposable part of a connected monoid are discussed in [**19**, Sections 5.5 and 5.6].

Derivations and coderivations. For a connected Joyal monoid a and Joyal species m viewed as an a-bimodule via the augmentation map of a, a map of Joyal species $f : \mathsf{a} \to \mathsf{m}$ is a derivation iff f factors through $\mathcal{Q}(\mathsf{a})$. There is a dual characterization of a coderivation from a Joyal species to a connected Joyal comonoid involving its primitive part. These facts motivate the definitions in Section 5.5.3.

Bimonoids for hyperplane arrangements. Primitive filtrations and decomposable filtrations for bimonoids for arrangements appear here for the first time.

CHAPTER 6

Universal constructions

We discuss the free monoid $\mathcal{T}(\mathsf{p})$ and the cofree comonoid $\mathcal{T}^{\vee}(\mathsf{p})$ on a species p. In addition, we discuss the free bimonoid $\mathcal{T}(\mathsf{c})$ on a comonoid c, and dually the cofree bimonoid $\mathcal{T}^{\vee}(\mathsf{a})$ on a monoid a. More generally, for any scalar q, we have the free q-bimonoid $\mathcal{T}_q(\mathsf{c})$ on c and the cofree q-bimonoid $\mathcal{T}_q^{\vee}(\mathsf{a})$ on a. Every species can be viewed as a (co)monoid where all nontrivial (co)product components are zero. Thus, to every species p, we have q-bimonoids $\mathcal{T}_q(\mathsf{p})$ and $\mathcal{T}_q^{\vee}(\mathsf{p})$.

TABLE 6.1. Universal bimonoids in species.

Starting data	q-bimonoid	Product	Coproduct
comonoid c	$\mathcal{T}_q(\mathsf{c})$	concatenation	q-dequasishuffle
monoid a	$\mathcal{T}_q^{\vee}(\mathsf{a})$	q-quasishuffle	deconcatenation
species p	$\mathcal{T}_q(\mathsf{p})$	concatenation	q-deshuffle
species p	$\mathcal{T}_q^{\vee}(\mathsf{p})$	q-shuffle	deconcatenation
species p	$\mathcal{T}_0(\mathsf{p}) = \mathcal{T}_0^{\vee}(\mathsf{p})$	concatenation	deconcatenation

The universal constructions are summarized in Table 6.1. We employ the terms concatenation and q-(quasi)shuffle for the products, and deconcatenation and q-de(quasi)shuffle for the coproducts. For $q = \pm 1$, the q-(quasi)shuffle product is (signed) commutative, while the q-de(quasi)shuffle coproduct is (signed) cocommutative. The concatenation product and deconcatenation coproduct do not depend on q, and do not satisfy any commutativity property. In terms of adjunctions, we have the following.

$$\text{species} \underset{\mathit{frg}}{\overset{\mathcal{T}}{\rightleftarrows}} \text{monoid} \qquad \text{comonoid} \underset{\mathcal{T}^{\vee}}{\overset{\mathit{frg}}{\rightleftarrows}} \text{species}$$

$$\text{comonoid} \underset{\mathit{frg}}{\overset{\mathcal{T}_q}{\rightleftarrows}} q\text{-bimonoid} \qquad q\text{-bimonoid} \underset{\mathcal{T}_q^{\vee}}{\overset{\mathit{frg}}{\rightleftarrows}} \text{monoid}$$

$$\text{species} \underset{\mathcal{P}}{\overset{\mathcal{T}_q}{\rightleftarrows}} q\text{-bimonoid} \qquad q\text{-bimonoid} \underset{\mathcal{T}_q^{\vee}}{\overset{\mathcal{Q}}{\rightleftarrows}} \text{species}$$

The functor $\mathcal{P}$ is the primitive part functor and $\mathcal{Q}$ the indecomposable part functor from Chapter 5.

In addition, we also discuss the free commutative monoid $\mathcal{S}(\mathsf{p})$ and the cofree cocommutative comonoid $\mathcal{S}^\vee(\mathsf{p})$ on a species p (and related constructions such as $\mathcal{S}(\mathsf{c})$ and $\mathcal{S}^\vee(\mathsf{a})$). These have signed analogues which we denote by $\mathcal{E}(\mathsf{p})$ and $\mathcal{E}^\vee(\mathsf{p})$.

The existence of the above universal constructions is a formal consequence of the existence of certain bimonads on species which we discussed in Chapter 3. For instance, since monoids in species are algebras over the monad $\mathcal{T}$, the free monoid on a species p is given by $\mathcal{T}(\mathsf{p})$, with the product defined using the monad structure of $\mathcal{T}$. Similarly, the free commutative monoid on a species p given by $\mathcal{S}(\mathsf{p})$ arises from the monad $\mathcal{S}$. The free (cofree) bimonoid on a comonoid (monoid) can be understood by employing the bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$. Its commutative analogues can be similarly understood using relevant bimonads involving $\mathcal{S}$ and $\mathcal{S}^\vee$. Readers unfamiliar with bimonads can understand the main ideas of this chapter by focusing on the explicit constructions of the universal objects.

The decomposable filtration of the free (commutative) monoid, and the primitive filtration of the cofree (cocommutative) comonoid on a species can be described in terms of the (commutative) Cauchy powers of the species. The abelianization of the free monoid is the free commutative monoid, while the coabelianization of the cofree comonoid is the cofree cocommutative comonoid. These considerations extend to (co)free bimonoids.

For any species p, there is a morphism of q-bimonoids between $\mathcal{T}_q(\mathsf{p})$ and $\mathcal{T}_q^\vee(\mathsf{p})$ which we call the q-norm map. It arises from freeness of $\mathcal{T}_q(\mathsf{p})$ and cofreeness of $\mathcal{T}_q^\vee(\mathsf{p})$. It is an isomorphism when q is not a root of unity. Invertibility of the Varchenko matrix associated to the q-distance function plays a critical role here. For $q = 0$, $\mathcal{T}_0(\mathsf{p}) = \mathcal{T}_0^\vee(\mathsf{p})$, and this isomorphism is the identity map.

We also discuss the (co)free graded (co)monoid on a graded species. Every species can be viewed as a graded species concentrated in degree 1. This is the situation that we focus on. The free graded monoid $\mathcal{T}(\mathsf{p})$ on a species p has a unique coproduct which turns it into a graded q-bimonoid. This is precisely the q-deshuffle coproduct. Dually, the q-shuffle product is the unique product which turns $\mathcal{T}^\vee(\mathsf{p})$ into a graded q-bimonoid.

Basic examples illustrating all of the above constructions are given in Chapter 7.

6.1. Free monoids on species

We construct the free monoid $\mathcal{T}(\mathsf{p})$ on a species p, and more generally, the free q-bimonoid $\mathcal{T}_q(\mathsf{c})$ on a comonoid c. It has the concatenation product and the q-dequasishuffle coproduct. We pay special attention to the cases $q = 0, \pm 1$. The existence of the universal object $\mathcal{T}_q(\mathsf{c})$ can be deduced from the fact that a q-bimonoid is the same as a bialgebra over the bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda_q)$ on the category of species (Section 3.1). This is in view of general results on bimonads given in Appendix C.2, in particular, Theorem C.32. We also apply this result to the bimonads $(\mathcal{T}, \mathcal{S}^\vee, \lambda)$ and $(\mathcal{T}, \mathcal{E}^\vee, \lambda_{-1})$ in Section 3.2

to describe the free cocommutative bimonoid on a cocommutative comonoid and its signed analogue.

6.1.1. Free monoid on a species. For any species p, recall from Section 3.1.1 the species $\mathcal{T}(\mathsf{p})$ defined by

$$\mathcal{T}(\mathsf{p})[A] := \bigoplus_{F\colon A \leq F} \mathsf{p}[F]. \tag{6.1}$$

Observe that

$$\mathcal{T}(\mathsf{p}) = \mathsf{p} + \mathsf{p}^2 + \mathsf{p}^3 + \dots, \tag{6.2}$$

the sum of the Cauchy powers of p defined in (5.1). It carries the structure of a monoid: For $A \leq F$, define μ_A^F by

$$\begin{tikzcd} \mathcal{T}(\mathsf{p})[F] \arrow[r, dashed, "\mu_A^F"] & \mathcal{T}(\mathsf{p})[A] \\ \mathsf{p}[H] \arrow[u, hook] \arrow[r, "\mathrm{id}"'] & \mathsf{p}[H] \arrow[u, hook] \end{tikzcd} \tag{6.3}$$

for each $F \leq H$. We refer to μ as the *concatenation product*. A map of species $\mathsf{p} \to \mathsf{q}$ induces a morphism of monoids $\mathcal{T}(\mathsf{p}) \to \mathcal{T}(\mathsf{q})$. So we have a functor

$$\mathcal{T} : \mathcal{A}\text{-}\mathsf{Sp} \to \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp}).$$

Theorem 6.1. *The functor $\mathcal{T}$ is the left adjoint of the forgetful functor. Explicitly, for any species p and monoid a, there is a natural bijection*

$$\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{T}(\mathsf{p}), \mathsf{a}) \xrightarrow{\cong} \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{p}, \mathsf{a}).$$

The unit of the adjunction is the inclusion $\mathsf{p} \hookrightarrow \mathcal{T}(\mathsf{p})$ arising from (6.2).

Proof. This is a specialization of Theorem C.32 to the monad $\mathcal{T}$ with $\mathsf{C} := \mathcal{A}\text{-}\mathsf{Sp}$. By general theory: For a species p, the $\mathcal{T}$-algebra structure of $\mathcal{T}(\mathsf{p})$ is given by the morphism

$$\mathcal{T}\mathcal{T}(\mathsf{p}) \to \mathcal{T}(\mathsf{p})$$

arising from the monad structure of $\mathcal{T}$ given in (3.2a) and (3.2b). Now, by Proposition 3.2, a $\mathcal{T}$-algebra is the same as a monoid. Under this translation, observe that the above map agrees with (6.3). $\square$

In other words, $\mathcal{T}(\mathsf{p})$ is the *free monoid* on the species p. This can be phrased as a universal property as follows.

Theorem 6.2. *Let a be a monoid, p a species, $f : \mathsf{p} \to \mathsf{a}$ a map of species. Then there exists a unique morphism of monoids $\hat{f} : \mathcal{T}(\mathsf{p}) \to \mathsf{a}$ such that the diagram*

$$\begin{tikzcd} \mathcal{T}(\mathsf{p}) \arrow[r, dashed, "\hat{f}"] & \mathsf{a} \\ \mathsf{p} \arrow[u, hook] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes.

Explicitly, the map $\hat{f}$ is as follows. Evaluating on the A-component, on the F-summand, the map is

$$\mathsf{p}[F] \xrightarrow{f_F} \mathsf{a}[F] \xrightarrow{\mu_A^F} \mathsf{a}[A]. \tag{6.4}$$

Definition 6.3. We say a monoid a is *free* on a species p if $\mathsf{a} \cong \mathcal{T}(\mathsf{p})$. Similarly, we say a q-bimonoid h is free on a species p if $\mathsf{h} \cong \mathcal{T}(\mathsf{p})$ as monoids.

Remark 6.4. By specializing Proposition 4.23 to $\mathbf{a} := \mathbf{As}$, the associative operad, and using Lemma 4.30, we obtain that the free monoid on a species p is given by $\mathbf{As} \circ \mathsf{p}$. In other words,

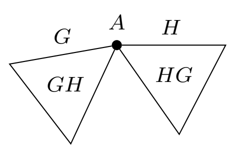
$$\mathcal{T}(\mathsf{p}) = \mathbf{As} \circ \mathsf{p}$$

as monoids. The two spaces are identified using (4.43). The universal property in Theorem 4.24 specializes to the one in Theorem 6.2, with the map (4.39) specializing to (6.4).

6.1.2. Free q-bimonoid on a comonoid. Let c be a comonoid. Consider the monoid $\mathcal{T}(\mathsf{c})$ discussed above. Now, in addition, for each scalar q, the coproduct of c induces a coproduct on $\mathcal{T}(\mathsf{c})$ as follows. For $A \leq G$, define Δ_A^G by

$$\begin{tikzcd}
\mathcal{T}(\mathsf{c})[A] \arrow[rr, dashed, "\Delta_A^G"] & & \mathcal{T}(\mathsf{c})[G] \\
\mathsf{c}[H] \arrow[u, hook] \arrow[r, "\Delta_H^{HG}"'] & \mathsf{c}[HG] \arrow[r, "(\beta_q)_{GH,HG}"'] & \mathsf{c}[GH] \arrow[u, hook]
\end{tikzcd} \tag{6.5}$$

for each $A \leq H$. The way the faces involved in the definition relate to each other is illustrated below.



We refer to Δ as the *q-dequasishuffle coproduct*. It is straightforward to check that the q-bimonoid axiom (2.33) holds. So $\mathcal{T}(\mathsf{c})$ is a q-bimonoid. This construction is clearly functorial in c. So we have a functor

$$\mathcal{T}_q : \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}) \to q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}). \tag{6.6}$$

We denote it by $\mathcal{T}_q$ to indicate the dependence on q.

Theorem 6.5. *The functor $\mathcal{T}_q$ is the left adjoint of the forgetful functor. Explicitly, for any comonoid c and q-bimonoid h, there is a natural bijection*

$$q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{T}_q(\mathsf{c}), \mathsf{h}) \xrightarrow{\cong} \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c}, \mathsf{h}).$$

Proof. This is a specialization of Theorem C.32 to the bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda_q)$ with $\mathsf{C} := \mathcal{A}\text{-}\mathsf{Sp}$. The general theory gives an explicit description of the product and coproduct of $\mathcal{T}_q(\mathsf{c})$. The way the product works was explained in Theorem 6.1. We now explain the coproduct part.

By the dual of Proposition 3.2, a $\mathcal{T}^\vee$-coalgebra is the same as a comonoid. If c is a $\mathcal{T}^\vee$-coalgebra, then using the given morphism $\mathsf{c} \to \mathcal{T}^\vee(\mathsf{c})$ and the mixed distributive law λ_q in (3.7), one can form the composite

$$\mathcal{T}(\mathsf{c}) \to \mathcal{T}\mathcal{T}^\vee(\mathsf{c}) \to \mathcal{T}^\vee\mathcal{T}(\mathsf{c}).$$

This turns $\mathcal{T}(\mathsf{c})$ into a $\mathcal{T}^\vee$-coalgebra. One can readily check that this map coincides with (6.5). $\square$

In other words, $\mathcal{T}_q(\mathsf{c})$ is the *free q-bimonoid* on the comonoid c. Its universal property is stated below.

Theorem 6.6. *Let* h *be a q-bimonoid,* c *a comonoid,* $f : \mathsf{c} \to \mathsf{h}$ *a morphism of comonoids. Then there exists a unique morphism of q-bimonoids* $\hat{f} : \mathcal{T}_q(\mathsf{c}) \to \mathsf{h}$ *such that the diagram*

$$\begin{tikzcd}
\mathcal{T}_q(\mathsf{c}) \arrow[r, dashed, "\hat{f}"] & \mathsf{h} \\
\mathsf{c} \arrow[u, hook] \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes.

Explicitly, the morphism $\hat{f}$ is given as before by (6.4).

Exercise 6.7. Check directly using (6.5) that the inclusion $\mathsf{c} \hookrightarrow \mathcal{T}_q(\mathsf{c})$ is a morphism of comonoids.

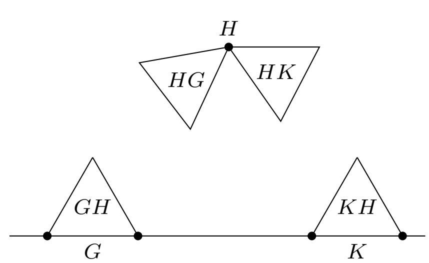
6.1.3. Free cocommutative bimonoid on a cocommutative comonoid. Put $q = 1$ in the preceding discussion. We have the free bimonoid $\mathcal{T}(\mathsf{c})$ on a comonoid c. In this case, we refer to the coproduct as the *dequasishuffle coproduct*. Further:

Lemma 6.8. *The bimonoid* $\mathcal{T}(\mathsf{c})$ *is cocommutative iff* c *is cocommutative.*

Proof. The cocommutativity axiom for $\mathcal{T}(\mathsf{c})$ is equivalent to the following. For faces H, G, K all greater than A, with G and K of the same support, the diagram

$$\begin{tikzcd}
 & \mathsf{c}[HG] \arrow[r, "\beta_{GH,HG}"] \arrow[d, dashed, "\beta_{HK,HG}"] & \mathsf{c}[GH] \arrow[d, "\beta_{KH,GH}"] \\
\mathsf{c}[H] \arrow[ur, "\Delta_H^{HG}"] \arrow[dr, "\Delta_H^{HK}"'] & & \\
 & \mathsf{c}[HK] \arrow[r, "\beta_{KH,HK}"'] & \mathsf{c}[KH]
\end{tikzcd}$$

commutes. An illustrative picture with $A = O$ is shown below.



If c is cocommutative, then by drawing in the dotted arrow, we see that the above diagram commutes, so $\mathcal{T}(\mathsf{c})$ is cocommutative. Conversely, if $\mathcal{T}(\mathsf{c})$ is cocommutative, then putting $H = A$ in the above diagram yields the cocommutativity axiom for c, so c is cocommutative. $\square$

So we have a functor

$$\mathcal{T} : {}^{\mathsf{co}}\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}) \to {}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}). \tag{6.7}$$

It is the left adjoint of the forgetful functor. This is again a special case of Theorem C.32 but now applied to the bimonad $(\mathcal{T}, \mathcal{S}^\vee, \lambda)$.

Now let $q = -1$. For a comonoid c, we have the free signed bimonoid $\mathcal{T}_{-1}(\mathsf{c})$. In this case, we refer to the coproduct as the *signed dequasishuffle coproduct*. Further, if c is signed cocommutative, then so is $\mathcal{T}_{-1}(\mathsf{c})$: Use the diagram in the proof of Lemma 6.8, with β_{-1} instead of β. Thus, we have a functor

$$\mathcal{T}_{-1} : (-1)\text{-}{}^{\mathsf{co}}\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}) \to (-1)\text{-}{}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}). \tag{6.8}$$

It is the left adjoint of the forgetful functor. This is again a special case of Theorem C.32 but now applied to the bimonad $(\mathcal{T}, \mathcal{E}^\vee, \lambda_{-1})$.

6.1.4. Free 0-bimonoid on a comonoid. We now specialize to $q = 0$. Suppose c is a comonoid. Specializing (6.3) and (6.5), the product and coproduct of $\mathcal{T}_0(\mathsf{c})$ are given below.

For $A \leq F$,

$$\big(\mu_A^F : \mathsf{c}[H] \to \mathsf{c}[K]\big) = \begin{cases} \mathrm{id} & \text{if } K = H, \\ 0 & \text{otherwise,} \end{cases} \tag{6.9}$$

for each $F \leq H$ and $A \leq K$.

For $A \leq G$,

$$\big(\Delta_A^G : \mathsf{c}[H] \to \mathsf{c}[K]\big) = \begin{cases} \Delta_H^K & \text{if } K = HG = GH, \\ 0 & \text{otherwise,} \end{cases} \tag{6.10}$$

for each $A \leq H$ and $G \leq K$.

6.2. Cofree comonoids on species

We construct the cofree comonoid $\mathcal{T}^\vee(\mathsf{p})$ on a species p, and more generally, the cofree q-bimonoid $\mathcal{T}_q^\vee(\mathsf{a})$ on a monoid a. It has the q-quasishuffle product and the deconcatenation coproduct. We also describe the cofree commutative bimonoid on a commutative monoid and its signed analogue. The discussion is dual to that in Section 6.1.

6.2.1. Cofree comonoid on a species. One can construct the *cofree comonoid* on a species p. We denote it by $\mathcal{T}^\vee(\mathsf{p})$. As a species, it equals $\mathcal{T}(\mathsf{p})$ and is given by (6.1). The coproduct is as follows. For $A \leq F$, define Δ_A^F by

$$
\begin{tikzcd}
\mathcal{T}^\vee(\mathsf{p})[A] \arrow[r, dashed, "\Delta_A^F"] & \mathcal{T}^\vee(\mathsf{p})[F] \\
\mathsf{p}[H] \arrow[u, hook] \arrow[r] & \begin{cases} \mathsf{p}[H] & \text{if } H \geq F, \\ 0 & \text{otherwise,} \end{cases} \arrow[u, hook]
\end{tikzcd}
\tag{6.11}
$$

for each $A \leq H$. We refer to Δ as the *deconcatenation coproduct*. A map of species $\mathsf{p} \to \mathsf{q}$ induces a morphism of comonoids $\mathcal{T}^\vee(\mathsf{p}) \to \mathcal{T}^\vee(\mathsf{q})$. So we have a functor

$$\mathcal{T}^\vee : \mathcal{A}\text{-}\mathsf{Sp} \to \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}).$$

Theorem 6.9. *The functor $\mathcal{T}^\vee$ is the right adjoint of the forgetful functor. Explicitly, for any species p and comonoid c, there is a natural bijection*

$$\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{p}) \xrightarrow{\cong} \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c}, \mathcal{T}^\vee(\mathsf{p})).$$

The counit of the adjunction is the projection $\mathcal{T}^\vee(\mathsf{p}) \twoheadrightarrow \mathsf{p}$ arising from (6.2). The universal property of $\mathcal{T}^\vee(\mathsf{p})$ can be phrased as follows.

Theorem 6.10. *Let c be a comonoid, p a species, $f : \mathsf{c} \to \mathsf{p}$ a map of species. Then there exists a unique morphism of comonoids $\hat{f} : \mathsf{c} \to \mathcal{T}^\vee(\mathsf{p})$ such that the diagram*

$$
\begin{tikzcd}
\mathsf{c} \arrow[r, dashed, "\hat{f}"] \arrow[dr, "f"'] & \mathcal{T}^\vee(\mathsf{p}) \arrow[d, two heads] \\
& \mathsf{p}
\end{tikzcd}
$$

commutes.

Explicitly, the map $\hat{f}$ is as follows. Evaluating on the A-component, into the F-summand, the map is

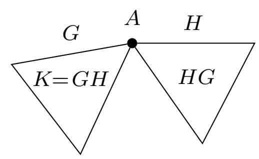
$$\mathsf{c}[A] \xrightarrow{\Delta_A^F} \mathsf{c}[F] \xrightarrow{f_F} \mathsf{p}[F]. \tag{6.12}$$

Definition 6.11. We say a comonoid c is *cofree* on a species p if $\mathsf{c} \cong \mathcal{T}^\vee(\mathsf{p})$. Similarly, we say a q-bimonoid h is cofree on a species p if $\mathsf{h} \cong \mathcal{T}^\vee(\mathsf{p})$ as comonoids.

6.2.2. Cofree *q*-bimonoid on a monoid. Now suppose a is a monoid. Then, for each scalar q, the comonoid $\mathcal{T}^\vee(\mathsf{a})$ discussed above carries a product: For $A \leq G$, define μ_A^G by

$$
\begin{tikzcd}
\mathcal{T}^\vee(\mathsf{a})[G] \arrow[rr, dashed, "\mu_A^G"] & & \mathcal{T}^\vee(\mathsf{a})[A] \\
\mathsf{a}[K] \arrow[u, hook] \arrow[r, "((\beta_q)_{HG,GH})"'] & \displaystyle\bigoplus_{\substack{H:\, A \leq H \\ GH = K}} \mathsf{a}[HG] \arrow[r, "(\mu_H^{HG})"'] & \displaystyle\bigoplus_{\substack{H:\, A \leq H \\ GH = K}} \mathsf{a}[H] \arrow[u, hook]
\end{tikzcd}
\tag{6.13}
$$

for each $G \leq K$. For the second map, the matrix-component is μ_H^{HG} on matching indices, and zero otherwise. The way the faces involved in the definition relate to each other is illustrated below.



Note very carefully that H is varying. We refer to μ as the *q-quasishuffle product*. One may check that $\mathcal{T}^\vee(\mathsf{a})$ is a q-bimonoid. This yields a functor

$$\mathcal{T}_q^\vee : \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp}) \to q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}). \tag{6.14}$$

Theorem 6.12. *The functor $\mathcal{T}_q^\vee$ is the right adjoint of the forgetful functor. Explicitly, for any monoid a and q-bimonoid h, there is a natural bijection*

$$\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h}, \mathsf{a}) \xrightarrow{\cong} q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h}, \mathcal{T}_q^\vee(\mathsf{a})).$$

Proof. This is a specialization of Theorem C.32 to the bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda_q)$ with $\mathsf{C} := \mathcal{A}\text{-}\mathsf{Sp}$. □

In other words, $\mathcal{T}_q^\vee(\mathsf{a})$ is the *cofree q-bimonoid* on the monoid a. Its universal property is stated below.

Theorem 6.13. *Let h be a q-bimonoid, a a monoid, $f : \mathsf{h} \to \mathsf{a}$ a morphism of monoids. Then there exists a unique morphism of q-bimonoids $\hat{f} : \mathsf{h} \to \mathcal{T}_q^\vee(\mathsf{a})$ such that the diagram*

$$\begin{tikzcd} \mathsf{h} \arrow[r, dashed, "\hat{f}"] \arrow[dr, "f"'] & \mathcal{T}_q^\vee(\mathsf{a}) \arrow[d, two heads] \\ & \mathsf{a} \end{tikzcd}$$

commutes.

Explicitly, the morphism $\hat{f}$ is given as before by (6.12).

Exercise 6.14. Check directly using (6.13) that the projection $\mathcal{T}_q^\vee(\mathsf{a}) \twoheadrightarrow \mathsf{a}$ is a morphism of monoids.

6.2.3. Cofree commutative bimonoid on a commutative monoid. Put $q = 1$ in the preceding discussion. We have the cofree bimonoid $\mathcal{T}^\vee(\mathsf{a})$ on a monoid a. In this case, we refer to the product as the *quasishuffle product*. Further:

Lemma 6.15. *The bimonoid $\mathcal{T}^\vee(\mathsf{a})$ is commutative iff a is commutative.*

So we have a functor

$$\mathcal{T}^\vee : \mathsf{Mon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \to \mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}).$$

It is the right adjoint of the forgetful functor. This is again a special case of Theorem C.32 but now applied to the bimonad $(\mathcal{S}, \mathcal{T}^\vee, \lambda)$.

Now let $q = -1$. For a monoid a, we have the cofree signed bimonoid $\mathcal{T}^{\vee}_{-1}(\mathsf{a})$. In this case, we refer to the product as the *signed quasishuffle product*. Further, if a is signed commutative, then so is $\mathcal{T}^{\vee}_{-1}(\mathsf{a})$. So we have a functor

$$\mathcal{T}^{\vee}_{-1} : (-1)\text{-}\mathsf{Mon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \to (-1)\text{-}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}).$$

It is the right adjoint of the forgetful functor. This is again a special case of Theorem C.32 but now applied to the bimonad $(\mathcal{E}, \mathcal{T}^{\vee}, \lambda_{-1})$.

6.2.4. Cofree 0-bimonoid on a monoid. We now specialize to $q = 0$. Suppose a is a monoid. Specializing (6.11) and (6.13), the product and coproduct of $\mathcal{T}^{\vee}_{0}(\mathsf{a})$ are given below.

For $A \leq G$,

$$\big(\mu^G_A : \mathsf{a}[K] \to \mathsf{a}[H]\big) = \begin{cases} \mu^K_H & \text{if } K = HG = GH, \\ 0 & \text{otherwise,} \end{cases} \tag{6.15}$$

for each $A \leq H$ and $G \leq K$.

For $A \leq F$,

$$\big(\Delta^F_A : \mathsf{a}[K] \to \mathsf{a}[H]\big) = \begin{cases} \mathrm{id} & \text{if } K = H, \\ 0 & \text{otherwise,} \end{cases} \tag{6.16}$$

for each $F \leq H$ and $A \leq K$.

6.2.5. Duality. The precise relation between the free and cofree constructions is as follows.

For any comonoid c and monoid a,

$$\mathcal{T}^{\vee}_{q}(\mathsf{c}^*) = \mathcal{T}_{q}(\mathsf{c})^* \quad \text{and} \quad \mathcal{T}^{\vee}_{q}(\mathsf{a})^* = \mathcal{T}_{q}(\mathsf{a}^*) \tag{6.17}$$

as q-bimonoids.

This can be checked directly or seen as a consequence of Proposition 3.17 and Proposition C.36. Observe that the coproduct (6.5) sends one summand to exactly one summand. However, different summands can map to the same summand. Hence, when we pass to the dual, we see that the product (6.13) sends one summand to multiple summands in general.

In view of (6.17), we say that $\mathcal{T}_q$ and $\mathcal{T}^{\vee}_q$ are conjugates of each other wrt duality.

6.3. (Co)free (co)commutative (co)monoids on species

We construct the free commutative monoid $\mathcal{S}(\mathsf{p})$ and the cofree cocommutative comonoid $\mathcal{S}^{\vee}(\mathsf{p})$ on a species p. We also make explicit the free bicommutative bimonoid on a (co)commutative (co)monoid. It arises from the bimonad $(\mathcal{S}, \mathcal{S}^{\vee}, \lambda)$ in Section 3.2. All these constructions have signed analogues.

6.3.1. Free commutative monoid on a species. For any species p, recall from Section 3.2.1 the species $\mathcal{S}(\mathsf{p})$ defined by

$$\mathcal{S}(\mathsf{p})[\mathrm{Z}] := \bigoplus_{\mathrm{X}:\, \mathrm{Z} \leq \mathrm{X}} \mathsf{p}[\mathrm{X}]. \tag{6.18}$$

Observe that

$$\mathcal{S}(\mathsf{p}) = \mathsf{p} + \mathsf{p}^{\overline{2}} + \mathsf{p}^{\overline{3}} + \dots, \tag{6.19}$$

the sum of the commutative Cauchy powers of p defined in (5.11). It carries the structure of a commutative monoid: For $\mathrm{Z} \leq \mathrm{X}$, define $\mu_{\mathrm{Z}}^{\mathrm{X}}$ by

$$\begin{tikzcd} \mathcal{S}(\mathsf{p})[\mathrm{X}] \arrow[r, dashed, "\mu_{\mathrm{Z}}^{\mathrm{X}}"] & \mathcal{S}(\mathsf{p})[\mathrm{Z}] \\ \mathsf{p}[\mathrm{Y}] \arrow[u, hook] \arrow[r, "\mathrm{id}"'] & \mathsf{p}[\mathrm{Y}] \arrow[u, hook] \end{tikzcd} \tag{6.20}$$

for each $\mathrm{X} \leq \mathrm{Y}$. A map of species $\mathsf{p} \to \mathsf{q}$ induces a morphism of monoids $\mathcal{S}(\mathsf{p}) \to \mathcal{S}(\mathsf{q})$. So we have a functor

$$\mathcal{S} : \mathcal{A}\text{-}\mathsf{Sp} \to \mathsf{Mon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}).$$

Theorem 6.16. *The functor $\mathcal{S}$ is the left adjoint of the forgetful functor. Explicitly, for any species p and commutative monoid a, there is a natural bijection*

$$\mathsf{Mon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{S}(\mathsf{p}), \mathsf{a}) \xrightarrow{\cong} \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{p}, \mathsf{a}).$$

The unit of the adjunction is the inclusion $\mathsf{p} \hookrightarrow \mathcal{S}(\mathsf{p})$ arising from (6.19).

Proof. This is a consequence of the fact that there is a monad on species, also called $\mathcal{S}$, whose algebras are commutative monoids. $\square$

We say that $\mathcal{S}(\mathsf{p})$ is the *free commutative monoid* on the species p. Its universal property is given below.

Theorem 6.17. *Let a be a commutative monoid, p a species, $f : \mathsf{p} \to \mathsf{a}$ a map of species. Then there exists a unique morphism of monoids $\hat{f} : \mathcal{S}(\mathsf{p}) \to \mathsf{a}$ such that the diagram*

$$\begin{tikzcd} \mathcal{S}(\mathsf{p}) \arrow[r, dashed, "\hat{f}"] & \mathsf{a} \\ \mathsf{p} \arrow[u, hook] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes.

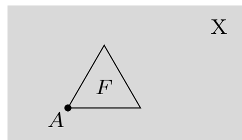
Explicitly, the map $\hat{f}$ is as follows. Evaluating on the Z-component, on the X-summand, the map is

$$\mathsf{p}[\mathrm{X}] \xrightarrow{f_{\mathrm{X}}} \mathsf{a}[\mathrm{X}] \xrightarrow{\mu_{\mathrm{Z}}^{\mathrm{X}}} \mathsf{a}[\mathrm{Z}]. \tag{6.21a}$$

Now suppose a and p are formulated in terms of faces (instead of flats), with $\mathcal{S}(\mathsf{p})$ as in (3.11). Then the map $\hat{f}$ is as follows. Evaluating on the A-component, on the X-summand for $\mathrm{X} \geq \mathrm{s}(A)$, the map is

$$\mathsf{p}[\mathrm{X}] \xrightarrow{\beta_{F,\mathrm{X}}} \mathsf{p}[F] \xrightarrow{f_F} \mathsf{a}[F] \xrightarrow{\mu_A^F} \mathsf{a}[A], \tag{6.21b}$$

where F is a fixed face of support X which is greater than A. An illustration is provided below.



The map (6.21b) does not depend on the specific choice of F.

Definition 6.18. We say a monoid a is *free commutative* on a species p if $\mathsf{a} \cong \mathcal{S}(\mathsf{p})$. Similarly, we say a bimonoid h is free commutative on a species p if $\mathsf{h} \cong \mathcal{S}(\mathsf{p})$ as monoids.

Remark 6.19. By specializing Proposition 4.23 to $\mathbf{a} := \mathbf{Com}$, the commutative operad, and using Lemma 4.30, we obtain that the free commutative monoid on a species p is given by $\mathbf{Com} \circ \mathsf{p}$. In other words,

$$\mathcal{S}(\mathsf{p}) = \mathbf{Com} \circ \mathsf{p}$$

as commutative monoids. The two spaces are identified using (4.44). The universal property in Theorem 4.24 specializes to the one in Theorem 6.17, with the map (4.39) specializing to (6.21a).

6.3.2. Free commutative bimonoid on a comonoid. Let c be a cocommutative comonoid. Consider the commutative monoid $\mathcal{S}(\mathsf{c})$ discussed above. Now, in addition, the coproduct of c induces a cocommutative coproduct on $\mathcal{S}(\mathsf{c})$ as follows. For $\mathrm{Z} \leq \mathrm{X}$,

$$\begin{tikzcd} \mathcal{S}(\mathsf{c})[\mathrm{Z}] \arrow[r, dashed, "\Delta_{\mathrm{Z}}^{\mathrm{X}}"] & \mathcal{S}(\mathsf{c})[\mathrm{X}] \\ \mathsf{c}[\mathrm{Y}] \arrow[u, hook] \arrow[r, "\Delta_{\mathrm{Y}}^{\mathrm{X}\vee\mathrm{Y}}"'] & \mathsf{c}[\mathrm{X}\vee\mathrm{Y}] \arrow[u, hook] \end{tikzcd} \tag{6.22}$$

for each $\mathrm{Y} \geq \mathrm{Z}$. Thus, $\mathcal{S}(\mathsf{c})$ is a bicommutative bimonoid. So we have a functor

$$\mathcal{S} : {}^{\mathsf{co}}\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}) \to {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}). \tag{6.23}$$

Theorem 6.20. *The functor $\mathcal{S}$ is the left adjoint of the forgetful functor. Explicitly, for any cocommutative comonoid c and bicommutative bimonoid h, there is a natural bijection*

$${}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{S}(\mathsf{c}), \mathsf{h}) \xrightarrow{\cong} {}^{\mathsf{co}}\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c}, \mathsf{h}).$$

Proof. This follows by applying Theorem C.32 to the bimonad $(\mathcal{S}, \mathcal{S}^\vee, \lambda)$. $\square$

In other words, $\mathcal{S}(\mathsf{c})$ is the *free bicommutative bimonoid* on the cocommutative comonoid c.

Similarly, the bimonad $(\mathcal{S}, \mathcal{T}^\vee, \lambda)$ yields the left adjoint

$$\mathcal{S} : \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}) \to \mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \tag{6.24}$$

of the forgetful functor. Here, we start with any comonoid c not necessarily cocommutative. Then $\mathcal{S}(\mathsf{c})$ is a commutative bimonoid not necessarily cocommutative. Its coproduct is given as follows. For $A \leq G$,

$$\begin{tikzcd}
\mathcal{S}(\mathsf{c})[A] \arrow[rrrr, dashed, "\Delta_A^G"] & & & & \mathcal{S}(\mathsf{c})[G] \\
\mathsf{c}[\mathrm{X}] \arrow[u, hook] \arrow[r, "\beta_{H,\mathrm{X}}"'] & \mathsf{c}[H] \arrow[r, "\Delta_H^{HG}"'] & \mathsf{c}[HG] \arrow[r, "\beta_{GH,HG}"'] & \mathsf{c}[GH] \arrow[r, "\beta_{\mathrm{s}(GH),GH}"'] & \mathsf{c}[\mathrm{s}(GH)] \arrow[u, hook]
\end{tikzcd} \tag{6.25}$$

for each $\mathrm{s}(A) \leq \mathrm{X}$. Here H is chosen to be any face with support X. The composite map does not depend on this choice.

The universal property of $\mathcal{S}(\mathsf{c})$ is stated below.

Theorem 6.21. *Let* h *be a commutative bimonoid,* c *a comonoid,* $f : \mathsf{c} \to \mathsf{h}$ *a morphism of comonoids. Then there exists a unique morphism of bimonoids* $\hat{f} : \mathcal{S}(\mathsf{c}) \to \mathsf{h}$ *such that the diagram*

$$\begin{tikzcd}
\mathcal{S}(\mathsf{c}) \arrow[r, dashed, "\hat{f}"] & \mathsf{h} \\
\mathsf{c} \arrow[u, hook] \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes.

Explicitly, the morphism $\hat{f}$ is given as before by (6.21a) or (6.21b).

6.3.3. Cofree cocommutative comonoid on a species. Dually, one can construct the cofree cocommutative comonoid on a species p. It is denoted by $\mathcal{S}^\vee(\mathsf{p})$. As a species, it equals $\mathcal{S}(\mathsf{p})$ and is given by (6.18). The coproduct is as follows. For $\mathrm{Z} \leq \mathrm{X}$,

$$\begin{tikzcd}
\mathcal{S}^\vee(\mathsf{p})[\mathrm{Z}] \arrow[r, dashed, "\Delta_{\mathrm{Z}}^{\mathrm{X}}"] & \mathcal{S}^\vee(\mathsf{p})[\mathrm{X}] \\
\mathsf{p}[\mathrm{Y}] \arrow[u, hook] \arrow[r] & \begin{cases} \mathsf{p}[\mathrm{Y}] & \text{if } \mathrm{Y} \geq \mathrm{X}, \\ 0 & \text{otherwise,} \end{cases} \arrow[u, hook]
\end{tikzcd} \tag{6.26}$$

for each $\mathrm{Y} \geq \mathrm{Z}$. This yields a functor

$$\mathcal{S}^\vee : \mathcal{A}\text{-}\mathsf{Sp} \to {}^{\mathsf{co}}\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}).$$

Theorem 6.22. *The functor* $\mathcal{S}^\vee$ *is the right adjoint of the forgetful functor. Explicitly, for any species* p *and cocommutative comonoid* c*, there is a natural bijection*

$$\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{p}) \xrightarrow{\cong} {}^{\mathsf{co}}\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c}, \mathcal{S}^\vee(\mathsf{p})).$$

The counit of the adjunction is the projection $\mathcal{S}^\vee(\mathsf{p}) \twoheadrightarrow \mathsf{p}$ arising from (6.19). The universal property of $\mathcal{S}^\vee(\mathsf{p})$ can be phrased as follows.

Theorem 6.23. *Let* c *be a cocommutative comonoid,* p *a species,* $f : \mathsf{c} \to \mathsf{p}$ *a map of species. Then there exists a unique morphism of comonoids* $\hat{f} : \mathsf{c} \to \mathcal{S}^\vee(\mathsf{p})$ *such that the diagram*

$$\begin{tikzcd} \mathsf{c} \arrow[r, dashed, "\hat{f}"] \arrow[dr, "f"'] & \mathcal{S}^\vee(\mathsf{p}) \arrow[d, two heads] \\ & \mathsf{p} \end{tikzcd}$$

commutes.

Explicitly, the map $\hat{f}$ is as follows. Evaluating on the Z-component, into the X-summand, the map is

$$\mathsf{c}[\mathrm{Z}] \xrightarrow{\Delta_{\mathrm{Z}}^{\mathrm{X}}} \mathsf{c}[\mathrm{X}] \xrightarrow{f_{\mathrm{X}}} \mathsf{p}[\mathrm{X}]. \tag{6.27a}$$

Now suppose c and p are formulated in terms of faces (instead of flats), with $\mathcal{S}^\vee(\mathsf{p})$ as in (3.11). Then the map $\hat{f}$ is as follows. Evaluating on the A-component, into the X-summand for $\mathrm{X} \geq \mathrm{s}(A)$, the map is

$$\mathsf{c}[A] \xrightarrow{\Delta_A^F} \mathsf{c}[F] \xrightarrow{f_F} \mathsf{p}[F] \xrightarrow{\beta_{\mathrm{X},F}} \mathsf{p}[\mathrm{X}], \tag{6.27b}$$

where F is a fixed face of support X which is greater than A. The map (6.27b) does not depend on the specific choice of F.

Definition 6.24. We say a comonoid c is *cofree cocommutative* on a species p if $\mathsf{c} \cong \mathcal{S}^\vee(\mathsf{p})$. Similarly, we say a bimonoid h is cofree cocommutative on a species p if $\mathsf{h} \cong \mathcal{S}^\vee(\mathsf{p})$ as comonoids.

6.3.4. Cofree cocommutative bimonoid on a monoid. For a monoid a, $\mathcal{S}^\vee(\mathsf{a})$ carries the structure of a cocommutative bimonoid. Further, if a is commutative, then so is $\mathcal{S}^\vee(\mathsf{a})$. In this case, the coproduct is as in (6.26), while the product is as follows. For $\mathrm{Z} \leq \mathrm{X}$,

$$\begin{tikzcd} \mathcal{S}^\vee(\mathsf{a})[\mathrm{X}] \arrow[r, dashed, "\mu_{\mathrm{Z}}^{\mathrm{X}}"] & \mathcal{S}^\vee(\mathsf{a})[\mathrm{Z}] \\ \mathsf{a}[\mathrm{W}] \arrow[u, hook] \arrow[r, "(\mu_{\mathrm{Y}}^{\mathrm{W}})"'] & \displaystyle\bigoplus_{\substack{\mathrm{Y}:\, \mathrm{Z} \leq \mathrm{Y} \\ \mathrm{X} \vee \mathrm{Y} = \mathrm{W}}} \mathsf{a}[\mathrm{Y}] \arrow[u, hook] \end{tikzcd} \tag{6.28}$$

for each $\mathrm{W} \geq \mathrm{X}$.

The functors

$$\mathcal{S}^\vee : \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp}) \to {}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}), \tag{6.29}$$

$$\mathcal{S}^\vee : \mathsf{Mon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \to {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \tag{6.30}$$

are the right adjoints of the forgetful functors. This follows by applying Theorem C.32 to the bimonads $(\mathcal{T}, \mathcal{S}^\vee, \lambda)$ and $(\mathcal{S}, \mathcal{S}^\vee, \lambda)$. The universal property of $\mathcal{S}^\vee(\mathsf{a})$ is stated below.

Theorem 6.25. *Let* h *be a cocommutative bimonoid,* a *a monoid,* $f : \mathsf{h} \to \mathsf{a}$ *a morphism of monoids. Then there exists a unique morphism of bimonoids* $\hat{f} : \mathsf{h} \to \mathcal{S}^\vee(\mathsf{a})$ *such that the diagram*

$$\begin{tikzcd} \mathsf{h} \arrow[r, dashed, "\hat{f}"] \arrow[dr, "f"'] & \mathcal{S}^\vee(\mathsf{a}) \arrow[d, two heads] \\ & \mathsf{a} \end{tikzcd}$$

commutes.

Explicitly, the morphism $\hat{f}$ is given as before by (6.27a) or (6.27b).

Exercise 6.26. Check using the (co)product formulas that for any cocommutative comonoid c and commutative monoid a,

$$\mathcal{S}^\vee(\mathsf{c}^*) = \mathcal{S}(\mathsf{c})^* \quad \text{and} \quad \mathcal{S}^\vee(\mathsf{a})^* = \mathcal{S}(\mathsf{a}^*) \tag{6.31}$$

as bicommutative bimonoids. This is the commutative analogue of (6.17). In view of (6.31), we say that $\mathcal{S}$ and $\mathcal{S}^\vee$ are conjugates of each other wrt duality.

6.3.5. Free signed commutative signed bimonoid. The preceding discussion has a signed analogue. For any species p, recall from Section 3.2.6 the species $\mathcal{E}(\mathsf{p})$ defined by

$$\mathcal{E}(\mathsf{p})[\mathrm{Z}] := \bigoplus_{\mathrm{X}:\, \mathrm{Z} \leq \mathrm{X}} \mathbf{E}^-[\mathrm{Z}, \mathrm{X}] \otimes \mathsf{p}[\mathrm{X}]. \tag{6.32}$$

The role of $\mathcal{S}$ is now played by $\mathcal{E}$, and of $\mathcal{S}^\vee$ by $\mathcal{E}^\vee$. We highlight one construction.

For a signed cocommutative comonoid c, we have the signed bicommutative signed bimonoid $\mathcal{E}(\mathsf{c})$ whose product and coproduct are given below.

For $\mathrm{Z} \leq \mathrm{X}$,

$$\begin{tikzcd} \mathbf{E}^-[\mathrm{Z}, \mathrm{X}] \otimes \mathcal{E}(\mathsf{c})[\mathrm{X}] \arrow[r, dashed, "\mu_{\mathrm{Z}}^{\mathrm{X}}"] & \mathcal{E}(\mathsf{c})[\mathrm{Z}] \\ \mathbf{E}^-[\mathrm{Z}, \mathrm{X}] \otimes \mathbf{E}^-[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{c}[\mathrm{Y}] \arrow[u, hook] \arrow[r, "(-)\otimes \mathrm{id}"'] & \mathbf{E}^-[\mathrm{Z}, \mathrm{Y}] \otimes \mathsf{c}[\mathrm{Y}], \arrow[u, hook] \end{tikzcd} \tag{6.33}$$

where the unnamed map $(-)$ is (1.162), and

$$\begin{tikzcd} \mathcal{E}(\mathsf{c})[\mathrm{Z}] \arrow[rr, dashed, "\Delta_{\mathrm{Z}}^{\mathrm{X}}"] & & \mathbf{E}^-[\mathrm{Z}, \mathrm{X}] \otimes \mathcal{E}(\mathsf{c})[\mathrm{X}] \\ \mathbf{E}^-[\mathrm{Z}, \mathrm{Y}] \otimes \mathsf{c}[\mathrm{Y}] \arrow[u, hook] \arrow[dr, "\mathrm{id} \otimes \Delta_{\mathrm{Y}}^{\mathrm{W}}"] & & \mathbf{E}^-[\mathrm{Z}, \mathrm{X}] \otimes \mathbf{E}^-[\mathrm{X}, \mathrm{W}] \otimes \mathsf{c}[\mathrm{W}] \arrow[u, hook] \\ & \mathbf{E}^-[\mathrm{Z}, \mathrm{Y}] \otimes \mathbf{E}^-[\mathrm{Y}, \mathrm{W}] \otimes \mathsf{c}[\mathrm{W}], \arrow[ur, "(-)\otimes \mathrm{id}"] & \end{tikzcd} \tag{6.34}$$

where $\mathrm{W} = \mathrm{X} \vee \mathrm{Y}$, and the unnamed map $(-)$ is (1.163).

These are the signed analogues of formulas (6.20) and (6.22), respectively.

Exercise 6.27. Formulate the signed analogue of Theorem 6.17. The signed analogues of (6.21a) and (6.21b) are

$$\mathbf{E}^{-}[\mathrm{Z},\mathrm{X}] \otimes \mathsf{p}[\mathrm{X}] \xrightarrow{\mathrm{id} \otimes f_{\mathrm{X}}} \mathbf{E}^{-}[\mathrm{Z},\mathrm{X}] \otimes \mathsf{a}[\mathrm{X}] \xrightarrow{\mu_{\mathrm{Z}}^{\mathrm{X}}} \mathsf{a}[\mathrm{Z}], \tag{6.35a}$$

$$\mathbf{E}^{-}[\mathrm{s}(A),\mathrm{X}] \otimes \mathsf{p}[\mathrm{X}] \xrightarrow{[F/A]\,\beta_{F,\mathrm{X}}} \mathsf{p}[F] \xrightarrow{f_F} \mathsf{a}[F] \xrightarrow{\mu_A^F} \mathsf{a}[A], \tag{6.35b}$$

with $[F/A]\,\beta_{F,\mathrm{X}}$ as in (2.45).

Exercise 6.28. Dually, formulate the signed analogue of Theorem 6.23. The signed analogues of (6.27a) and (6.27b) are

$$\mathsf{c}[\mathrm{Z}] \xrightarrow{\Delta_{\mathrm{Z}}^{\mathrm{X}}} \mathbf{E}^{-}[\mathrm{Z},\mathrm{X}] \otimes \mathsf{c}[\mathrm{X}] \xrightarrow{\mathrm{id} \otimes f_{\mathrm{X}}} \mathbf{E}^{-}[\mathrm{Z},\mathrm{X}] \otimes \mathsf{p}[\mathrm{X}], \tag{6.36a}$$

$$\mathsf{c}[A] \xrightarrow{\Delta_A^F} \mathsf{c}[F] \xrightarrow{f_F} \mathsf{p}[F] \xrightarrow{[F/A]\otimes\beta_{\mathrm{X},F}} \mathbf{E}^{-}[\mathrm{s}(A),\mathrm{X}] \otimes \mathsf{p}[\mathrm{X}], \tag{6.36b}$$

with $[F/A] \otimes \beta_{\mathrm{X},F}$ as in (2.45).

The construction of $\mathcal{E}(\mathsf{c})$ can be carried out more generally for any comonoid c, the result is now a signed commutative signed bimonoid.

Dually, for any monoid a, we have $\mathcal{E}^{\vee}(\mathsf{a})$ which is a signed cocommutative signed bimonoid.

6.3.6. Morphisms of bimonoids.

The universal constructions can be combined in various ways to obtain (signed) bimonoids and morphisms between them.

Theorem 6.29. *For any species* p*, the following are commutative diagrams of bimonoids and signed bimonoids, respectively.*

$$\begin{tikzcd}[column sep=small]
& \mathcal{T}\mathcal{T}^{\vee}(\mathsf{p}) \arrow[rr, "\lambda"] \arrow[dd, two heads] & & \mathcal{T}^{\vee}\mathcal{T}(\mathsf{p}) \arrow[dd, two heads] \\
\mathcal{T}\mathcal{S}^{\vee}(\mathsf{p}) \arrow[ur] \arrow[rr] \arrow[dd, two heads] & & \mathcal{S}^{\vee}\mathcal{T}(\mathsf{p}) \arrow[ur] \arrow[dd, two heads] & \\
& \mathcal{S}\mathcal{T}^{\vee}(\mathsf{p}) \arrow[rr] & & \mathcal{T}^{\vee}\mathcal{S}(\mathsf{p}) \\
\mathcal{S}\mathcal{S}^{\vee}(\mathsf{p}) \arrow[ur] \arrow[rr] & & \mathcal{S}^{\vee}\mathcal{S}(\mathsf{p}) \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& \mathcal{T}\mathcal{T}^{\vee}(\mathsf{p}) \arrow[rr, "\lambda_{-1}"] \arrow[dd, two heads] & & \mathcal{T}^{\vee}\mathcal{T}(\mathsf{p}) \arrow[dd, two heads] \\
\mathcal{T}\mathcal{E}^{\vee}(\mathsf{p}) \arrow[ur] \arrow[rr] \arrow[dd, two heads] & & \mathcal{E}^{\vee}\mathcal{T}(\mathsf{p}) \arrow[ur] \arrow[dd, two heads] & \\
& \mathcal{E}\mathcal{T}^{\vee}(\mathsf{p}) \arrow[rr] & & \mathcal{T}^{\vee}\mathcal{E}(\mathsf{p}) \\
\mathcal{E}\mathcal{E}^{\vee}(\mathsf{p}) \arrow[ur] \arrow[rr] & & \mathcal{E}^{\vee}\mathcal{E}(\mathsf{p}) \arrow[ur] &
\end{tikzcd} \tag{6.37}$$

Since p is a species, $\mathcal{T}^{\vee}(\mathsf{p})$ is a comonoid, and hence $\mathcal{T}\mathcal{T}^{\vee}(\mathsf{p})$ is a bimonoid (or a signed bimonoid if we use the law λ_{-1}). The remaining bimonoids and signed bimonoids are to be understood in a similar manner. The horizontal maps are the mixed distributive laws. For instance, $\lambda : \mathcal{T}\mathcal{T}^{\vee}(\mathsf{p}) \to \mathcal{T}^{\vee}\mathcal{T}(\mathsf{p})$ is given by (3.4), while $\mathcal{S}\mathcal{S}^{\vee}(\mathsf{p}) \to \mathcal{S}^{\vee}\mathcal{S}(\mathsf{p})$ is given by (3.16).

Proof. We discuss the first cube. Apply Proposition C.37 to each of the bilax functors in (3.17), diagram (C.24) specializes to one of the squares in (6.37). (Commutativity of the squares on the left- and right-vertical sides of the cube is clear.) $\square$

6.4. (Co)free bimonoids associated to species

To any species p, one can associate the q-bimonoid $\mathcal{T}_q(\mathsf{p})$ with the concatenation product and the q-deshuffle coproduct. This is the same as the free q-bimonoid on a trivial comonoid. This yields a functor from the category of species to the category of q-bimonoids. It is the left adjoint of the primitive part functor.

Dually, to any species p, one can associate the q-bimonoid $\mathcal{T}_q^\vee(\mathsf{p})$ with the q-shuffle product and deconcatenation coproduct. This is the same as the cofree q-bimonoid on a trivial monoid. This yields the right adjoint of the indecomposable part functor.

6.4.1. From species to q-bimonoids. Recall from Section 5.5 that we can view a species as a (co)monoid by letting all nontrivial (co)product components to be zero. By precomposing the functor $\mathcal{T}_q$ in (6.6) with the trivial comonoid functor (5.43), and $\mathcal{T}_q^\vee$ in (6.14) with the trivial monoid functor (5.45), we obtain functors (denoted by the same symbols)

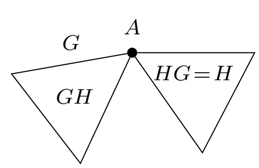
$$\mathcal{T}_q, \mathcal{T}_q^\vee : \mathcal{A}\text{-}\mathsf{Sp} \to q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}). \tag{6.38}$$

In other words, for any species p, $\mathcal{T}_q(\mathsf{p})$ and $\mathcal{T}_q^\vee(\mathsf{p})$ carry canonical q-bimonoid structures. These are made explicit below.

The product and coproduct of $\mathcal{T}_q(\mathsf{p})$ are as follows. In the first diagram, $H \geq F \geq A$, while in the second diagram, $H \geq A$, $G \geq A$.

$$\begin{tikzcd} \mathcal{T}_q(\mathsf{p})[F] \arrow[r, dashed, "\mu_A^F"] & \mathcal{T}_q(\mathsf{p})[A] \\ \mathsf{p}[H] \arrow[u, hook] \arrow[r, "\mathrm{id}"'] & \mathsf{p}[H] \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}_q(\mathsf{p})[A] \arrow[r, dashed, "\Delta_A^G"] & \mathcal{T}_q(\mathsf{p})[G] \\ \mathsf{p}[H] \arrow[u, hook] \arrow[r, "(\beta_q)_{GH,HG}"'] & \begin{cases} \mathsf{p}[GH] & \text{if } HG = H, \\ 0 & \text{otherwise.} \end{cases} \arrow[u, hook] \end{tikzcd} \tag{6.39}$$

The product is concatenation as in (6.3), but note how the coproduct has simplified from (6.5). The first alternative in the coproduct is illustrated below.

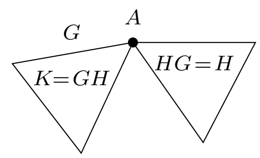


Compare and contrast with the picture after (6.5). We refer to Δ as the *q-deshuffle coproduct*.

Dually, the coproduct and product of $\mathcal{T}_q^\vee(\mathsf{p})$ are as follows. In the first diagram, $H \geq A$, $F \geq A$, while in the second diagram, $K \geq G \geq A$.

$$
\begin{tikzcd}
\mathcal{T}_q^\vee(\mathsf{p})[A] \arrow[r, dashed, "\Delta_A^F"] & \mathcal{T}_q^\vee(\mathsf{p})[F] \\
\mathsf{p}[H] \arrow[u, hook] \arrow[r] & \begin{cases} \mathsf{p}[H] & \text{if } H \geq F, \\ 0 & \text{otherwise.} \end{cases} \arrow[u, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{T}_q^\vee(\mathsf{p})[G] \arrow[r, dashed, "\mu_A^G"] & \mathcal{T}_q^\vee(\mathsf{p})[A] \\
\mathsf{p}[K] \arrow[u, hook] \arrow[r, "((\beta_q)_{HG,GH})"'] & \displaystyle\bigoplus_{\substack{H:\, GH=K \\ HG=H \\ A \leq H}} \mathsf{p}[H] \arrow[u, hook]
\end{tikzcd}
\tag{6.40}
$$

The coproduct is deconcatenation as in (6.11), but the product has simplified from (6.13). It is illustrated below.



Compare and contrast with the picture after (6.13). We refer to μ as the *q-shuffle product*.

Recall the adjunctions

$$
\mathcal{A}\text{-}\mathsf{Sp} \underset{\mathcal{P}}{\overset{trv}{\rightleftarrows}} \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{frg}{\overset{\mathcal{T}_q}{\rightleftarrows}} q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{\mathcal{T}_q^\vee}{\overset{frg}{\rightleftarrows}} \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{trv}{\overset{\mathcal{Q}}{\rightleftarrows}} \mathcal{A}\text{-}\mathsf{Sp}
$$

from Proposition 5.28, Theorem 6.5, Theorem 6.12, Proposition 5.30, respectively. By composing the first two and the last two, we deduce:

Theorem 6.30. *For any scalar q, the functor $\mathcal{T}_q$ is the left adjoint of the primitive part functor $\mathcal{P}$, while $\mathcal{T}_q^\vee$ is the right adjoint of the indecomposable part functor $\mathcal{Q}$. (The functors are between the categories of species and q-bimonoids.)*

These adjunctions are reformulated below as universal properties.

Theorem 6.31. *Let h be a q-bimonoid, p a species, $f : \mathsf{p} \to \mathcal{P}(\mathsf{h})$ a map of species. Then there exists a unique morphism of q-bimonoids $\hat{f} : \mathcal{T}_q(\mathsf{p}) \to \mathsf{h}$ such that the diagram*

$$
\begin{tikzcd}
\mathcal{T}_q(\mathsf{p}) \arrow[r, dashed, "\hat{f}"] & \mathsf{h} \\
\mathsf{p} \arrow[u, hook] \arrow[r, "f"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook]
\end{tikzcd}
$$

commutes.

Explicitly, the map $\hat{f}$ is as follows. Evaluating on the A-component, on the F-summand, the map is

$$
\mathsf{p}[F] \xrightarrow{f_F} \mathcal{P}(\mathsf{h})[F] \hookrightarrow \mathsf{h}[F] \xrightarrow{\mu_A^F} \mathsf{h}[A]. \tag{6.41}
$$

Note the similarity with (6.4).

Exercise 6.32. Check using formula (6.41) that $\hat{f}$ is a morphism of comonoids.

Exercise 6.33. Deduce from the coproduct formula in (6.39) that for any species p, we have $\mathsf{p} \subseteq \mathcal{P}(\mathcal{T}_q(\mathsf{p}))$. This inclusion is the unit of the adjunction between $\mathcal{T}_q$ and $\mathcal{P}$. It is strict in general. In contrast, $\mathsf{p} = \mathcal{P}(\mathcal{T}_q^\vee(\mathsf{p}))$, see the more general result given in Proposition 6.56.

Theorem 6.34. *Let* h *be a* q*-bimonoid,* p *a species,* $f : \mathcal{Q}(\mathsf{h}) \to \mathsf{p}$ *a map of species. Then there exists a unique morphism of* q*-bimonoids* $\hat{f} : \mathsf{h} \to \mathcal{T}_q^\vee(\mathsf{p})$ *such that the diagram*

$$\begin{tikzcd} \mathsf{h} \arrow[r, dashed, "\hat{f}"] \arrow[d, two heads] & \mathcal{T}_q^\vee(\mathsf{p}) \arrow[d, two heads] \\ \mathcal{Q}(\mathsf{h}) \arrow[r, "f"'] & \mathsf{p} \end{tikzcd}$$

commutes.

Explicitly, the map $\hat{f}$ is as follows. Evaluating on the A-component, into the F-summand, the map is

$$\mathsf{h}[A] \xrightarrow{\Delta_A^F} \mathsf{h}[F] \twoheadrightarrow \mathcal{Q}(\mathsf{h})[F] \xrightarrow{f_F} \mathsf{p}[F]. \tag{6.42}$$

Note the similarity with (6.12).

Exercise 6.35. Deduce from the product formula in (6.40) that for any species p, the projection $\mathcal{T}_q^\vee(\mathsf{p}) \twoheadrightarrow \mathsf{p}$ factors through $\mathcal{Q}\mathcal{T}_q^\vee(\mathsf{p})$. The induced map $\mathcal{Q}\mathcal{T}_q^\vee(\mathsf{p}) \twoheadrightarrow \mathsf{p}$ is the counit of the adjunction between $\mathcal{Q}$ and $\mathcal{T}_q^\vee$. This map is not the identity in general. In contrast, $\mathsf{p} = \mathcal{Q}(\mathcal{T}_q(\mathsf{p}))$, see the more general result given in Proposition 6.57.

6.4.2. From species to (co)commutative bimonoids. Put $q = \pm 1$ in the preceding discussion. The q-deshuffle coproduct is cocommutative for $q = 1$ and signed cocommutative for $q = -1$. (See Lemma 6.8.) In these cases, we use the terms *deshuffle coproduct* and *signed deshuffle coproduct*, respectively. Thus, to any species p, we have the cocommutative bimonoid $\mathcal{T}(\mathsf{p})$, and the signed cocommutative bimonoid $\mathcal{T}_{-1}(\mathsf{p})$.

Dually, the q-shuffle product is commutative for $q = 1$ and signed commutative for $q = -1$. (See Lemma 6.15.) In these cases, we use the terms *shuffle product* and *signed shuffle product*, respectively. Thus, to any species p, we have the commutative bimonoid $\mathcal{T}^\vee(\mathsf{p})$, and the signed commutative bimonoid $\mathcal{T}_{-1}^\vee(\mathsf{p})$.

Exercise 6.36. Show that: There are adjunctions between the categories of species and (co)commutative bimonoids:

$$\mathcal{A}\text{-}\mathsf{Sp} \underset{\mathcal{P}}{\overset{\mathcal{T}}{\rightleftarrows}} {}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \quad \text{and} \quad \mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{\mathcal{T}^\vee}{\overset{\mathcal{Q}}{\rightleftarrows}} \mathcal{A}\text{-}\mathsf{Sp}.$$

Similar adjunctions hold for the functors $\mathcal{T}_{-1}$ and $\mathcal{T}_{-1}^\vee$.

6.4.3. From species to 0-bimonoids. We now specialize to $q = 0$. The product and coproduct of $\mathcal{T}_0(\mathsf{p})$ are given by concatenation and deconcatenation:

For $A \leq F$,

$$\big(\mu_A^F : \mathsf{p}[H] \to \mathsf{p}[K]\big) = \begin{cases} \mathrm{id} & \text{if } K = H, \\ 0 & \text{otherwise,} \end{cases} \tag{6.43}$$

for each $F \leq H$ and $A \leq K$.

For $A \leq G$,

$$\big(\Delta_A^G : \mathsf{p}[H] \to \mathsf{p}[K]\big) = \begin{cases} \mathrm{id} & \text{if } K = H, \\ 0 & \text{otherwise,} \end{cases} \tag{6.44}$$

for each $A \leq H$ and $G \leq K$.

Observe that this is also the product and coproduct of $\mathcal{T}_0^\vee(\mathsf{p})$. Thus,

$$\mathcal{T}_0 = \mathcal{T}_0^\vee \tag{6.45}$$

as functors from the category of species to the category of 0-bimonoids.

Exercise 6.37. Check directly that $\mathcal{T}_0(\mathsf{p})$ is a 0-bimonoid by showing that the 0-bimonoid axiom (2.40) holds for (6.43) and (6.44).

Exercise 6.38. Check that the coproduct (6.10) specializes to (6.44) when c is the trivial comonoid, and the product (6.15) specializes to (6.43) when a is the trivial monoid.

6.4.4. Duality. The following diagrams of functors commute.

$$\begin{tikzcd} \mathcal{A}\text{-}\mathsf{Sp} \arrow[r, "\mathcal{T}_q"] \arrow[d, "(-)^*"'] & q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \arrow[d, "(-)^*"] \\ \mathcal{A}\text{-}\mathsf{Sp}^{\mathrm{op}} \arrow[r, "\mathcal{T}_q^\vee"'] & q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})^{\mathrm{op}} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}\text{-}\mathsf{Sp} & q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \arrow[l, "\mathcal{P}"'] \\ \mathcal{A}\text{-}\mathsf{Sp}^{\mathrm{op}} \arrow[u, "(-)^*"] & q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})^{\mathrm{op}} \arrow[l, "\mathcal{Q}"] \arrow[u, "(-)^*"'] \end{tikzcd}$$

This follows from (6.17) and (5.41). Moreover, the functors in the first diagram are the left adjoints of the functors in the right diagram. Note very carefully that for $\mathcal{T}_q^\vee$ and $\mathcal{Q}$, in the above diagrams, we have used the opposite categories, so $\mathcal{T}_q^\vee$ is the left adjoint and $\mathcal{Q}$ is the right adjoint.

Exercise 6.39. Show that: For any q-bimonoid h, the dual of the unit $\mathsf{h} \to \mathcal{T}_q^\vee\mathcal{Q}(\mathsf{h})$ can be identified with the counit $\mathcal{T}_q\mathcal{P}(\mathsf{h}^*) \to \mathsf{h}^*$ for the dual q-bimonoid h^*. (Use formulas (6.41) and (6.42).) Similarly, for any species p, the dual of the counit $\mathcal{Q}\mathcal{T}_q^\vee(\mathsf{p}) \to \mathsf{p}$ can be identified with the unit $\mathsf{p}^* \to \mathcal{P}\mathcal{T}_q(\mathsf{p}^*)$ for the dual species p^*. (See the descriptions given in Exercises 6.33 and 6.35.)

6.4.5. Interpolating the deshuffle and dequasishuffle coproducts. For any comonoid c, consider the q-bimonoids $\mathcal{T}_q(\mathsf{c})$ and $\mathcal{T}_q(\mathsf{c}_t)$, where c_t is c as a species but with the trivial coproduct. These q-bimonoids have the same product, namely, concatenation (6.3). However, their coproducts differ. The coproduct of the former is q-dequasishuffle (6.5), while that of the latter is q-deshuffle (6.39). The two can be interpolated as follows.

Let α be any scalar. For any comonoid (c, Δ), define a comonoid $(\mathsf{c}, \Delta_\alpha)$ with coproduct components

$$(\Delta_\alpha)^F_A := \alpha^{\mathrm{rk}(F/A)} \Delta^F_A. \tag{6.46}$$

Let us write $\mathsf{c}_\alpha := (\mathsf{c}, \Delta_\alpha)$. Observe that $\mathsf{c}_1 = \mathsf{c}$, while $\mathsf{c}_0 = \mathsf{c}_t$. Also if c is trivial to start with, then $\mathsf{c}_\alpha = \mathsf{c}$.

Now consider $\mathcal{T}_q(\mathsf{c}_\alpha)$, the free q-bimonoid on the comonoid c_α. It has product given by concatenation, and a coproduct which interpolates between the q-deshuffle and q-dequasishuffle coproducts. Explicitly, the Δ^{HG}_H in coproduct formula (6.5) gets replaced by $\alpha^{\mathrm{rk}(HG/H)} \Delta^{HG}_H$. When $\alpha = 0$, this becomes 0 unless $HG = H$.

Exercise 6.40. Check that: The map $\mathsf{c} \to \mathsf{c}_\alpha$ defined on the A-component by scalar multiplication by $\alpha^{\mathrm{rk}(A)}$ is a morphism of comonoids. Moreover, it is an isomorphism if $\alpha \neq 0$. This induces a morphism of q-bimonoids $\mathcal{T}_q(\mathsf{c}) \to \mathcal{T}_q(\mathsf{c}_\alpha)$ which is an isomorphism if $\alpha \neq 0$. Explicitly, on the A-component, it is given by

$$\bigoplus_{F:\, F \geq A} \mathsf{c}[F] \to \bigoplus_{H:\, H \geq A} \mathsf{c}[H], \qquad x \mapsto \alpha^{\mathrm{rk}(F)}\, x \tag{6.47}$$

for $x \in \mathsf{c}[F]$. The isomorphism issue when $\alpha = 0$ is addressed in Chapter 14, see in particular, Sections 14.4 and 14.5.

6.4.6. Interpolating the shuffle and quasishuffle products. Dually, for any monoid a, consider the q-bimonoids $\mathcal{T}^\vee_q(\mathsf{a})$ and $\mathcal{T}^\vee_q(\mathsf{a}_t)$, where a_t is a as a species but with the trivial product. These q-bimonoids have the same coproduct, namely, deconcatenation (6.11). However, their products differ. The product of the former is q-quasishuffle (6.13), while that of the latter is q-shuffle (6.40). The two can be interpolated as follows.

Let α be any scalar. For any monoid (a, μ), define a monoid (a, μ_α) with product components

$$(\mu_\alpha)^F_A := \alpha^{\mathrm{rk}(F/A)} \mu^F_A. \tag{6.48}$$

Let us write $\mathsf{a}_\alpha := (\mathsf{a}, \mu_\alpha)$. Observe that $\mathsf{a}_1 = \mathsf{a}$, while $\mathsf{a}_0 = \mathsf{a}_t$. Also if a is trivial to start with, then $\mathsf{a}_\alpha = \mathsf{a}$.

Now consider $\mathcal{T}^\vee_q(\mathsf{a}_\alpha)$, the cofree q-bimonoid on the monoid a_α. It has coproduct given by deconcatenation, and a product which interpolates between the q-shuffle and q-quasishuffle products. Explicitly, the μ^{HG}_H in product formula (6.13) gets replaced by $\alpha^{\mathrm{rk}(HG/H)} \mu^{HG}_H$. When $\alpha = 0$, this becomes 0 unless $HG = H$.

Exercise 6.41. Check that: The map $\mathsf{a}_\alpha \to \mathsf{a}$ defined on the A-component by scalar multiplication by $\alpha^{\mathrm{rk}(A)}$ is a morphism of monoids. Moreover, it is an isomorphism if $\alpha \neq 0$. This induces a morphism of q-bimonoids $\mathcal{T}^\vee_q(\mathsf{a}_\alpha) \to \mathcal{T}^\vee_q(\mathsf{a})$ which is an isomorphism if $\alpha \neq 0$. Explicitly, on the A-component, it is given by

$$\bigoplus_{F:\, F \geq A} \mathsf{a}[F] \to \bigoplus_{G:\, G \geq A} \mathsf{a}[G], \qquad x \mapsto \alpha^{\mathrm{rk}(F)}\, x \tag{6.49}$$

for $x \in \mathsf{a}[F]$. The isomorphism issue when $\alpha = 0$ is addressed in Chapter 14, see in particular, Sections 14.4 and 14.5.

6.5. (Co)free (co)commutative bimonoids associated to species

We now turn to the commutative aspects of the constructions in Section 6.4. We construct a functor from the category of species to the category of bicommutative bimonoids, and its signed analogue.

6.5.1. From species to bicommutative bimonoids. Let us view the trivial comonoid functor (5.43) as a functor from species to cocommutative comonoids, and similarly, the trivial monoid functor (5.45) as a functor from species to commutative monoids. By precomposing the functor $\mathcal{S}$ in (6.23) with the trivial comonoid functor, and $\mathcal{S}^\vee$ in (6.30) with the trivial monoid functor, we obtain functors (denoted by the same symbols)

$$\mathcal{S}, \mathcal{S}^\vee : \mathcal{A}\text{-}\mathsf{Sp} \to {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}).$$

They go from the category of species to the category of bicommutative bimonoids. In fact, it turns out that

$$\mathcal{S} = \mathcal{S}^\vee. \tag{6.50}$$

This can be seen by specializing formulas (6.20), (6.22), (6.26), (6.28). Explicitly, for a species p, the product and coproduct of $\mathcal{S}(\mathsf{p})$ (and of $\mathcal{S}^\vee(\mathsf{p})$) are as follows. In the first diagram, $\mathrm{Y} \geq \mathrm{X} \geq \mathrm{Z}$, while in the second diagram, $\mathrm{X} \geq \mathrm{Z}$, $\mathrm{Y} \geq \mathrm{Z}$.

$$\begin{tikzcd} \mathcal{S}(\mathsf{p})[\mathrm{X}] \arrow[r, dashed, "\mu_{\mathrm{Z}}^{\mathrm{X}}"] & \mathcal{S}(\mathsf{p})[\mathrm{Z}] \\ \mathsf{p}[\mathrm{Y}] \arrow[u, hook] \arrow[r, "\mathrm{id}"'] & \mathsf{p}[\mathrm{Y}] \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S}(\mathsf{p})[\mathrm{Z}] \arrow[r, dashed, "\Delta_{\mathrm{Z}}^{\mathrm{X}}"] & \mathcal{S}(\mathsf{p})[\mathrm{X}] \\ \mathsf{p}[\mathrm{Y}] \arrow[u, hook] \arrow[r] & \begin{cases} \mathsf{p}[\mathrm{Y}] & \text{if } \mathrm{Y} \geq \mathrm{X}, \\ 0 & \text{otherwise.} \end{cases} \arrow[u, hook] \end{tikzcd} \tag{6.51}$$

Exercise 6.42. Check directly that $\mathcal{S}(\mathsf{p})$ is a bicommutative bimonoid by showing that the bicommutative bimonoid axiom (2.26) holds for (6.51).

By composing adjunctions, we deduce:

Theorem 6.43. *The functor $\mathcal{S}$ is the left adjoint of $\mathcal{P}$ and the right adjoint of $\mathcal{Q}$. (The functors are between the categories of species and bicommutative bimonoids.)*

These adjunctions are reformulated below as universal properties.

Theorem 6.44. *Let h be a bicommutative bimonoid, p a species, $f : \mathsf{p} \to \mathcal{P}(\mathsf{h})$ a map of species. Then there exists a unique morphism of bimonoids*

$\hat{f} : \mathcal{S}(\mathsf{p}) \to \mathsf{h}$ *such that the diagram*

$$\begin{tikzcd} \mathcal{S}(\mathsf{p}) \arrow[r, dashed, "\hat{f}"] & \mathsf{h} \\ \mathsf{p} \arrow[u, hook] \arrow[r, "f"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd}$$

commutes.

The map $\hat{f}$ is as follows. Evaluating on the Z-component, on the X-summand, the map is

$$\mathsf{p}[\mathrm{X}] \xrightarrow{f_{\mathrm{X}}} \mathcal{P}(\mathsf{h})[\mathrm{X}] \hookrightarrow \mathsf{h}[\mathrm{X}] \xrightarrow{\mu^{\mathrm{X}}_{\mathrm{Z}}} \mathsf{h}[\mathrm{Z}]. \tag{6.52}$$

Note the similarity with (6.21a).

Theorem 6.45. *Let* h *be a bicommutative bimonoid,* p *a species,* $f : \mathcal{Q}(\mathsf{h}) \to \mathsf{p}$ *a map of species. Then there exists a unique morphism of bimonoids* $\hat{f} : \mathsf{h} \to \mathcal{S}(\mathsf{p})$ *such that the diagram*

$$\begin{tikzcd} \mathsf{h} \arrow[r, dashed, "\hat{f}"] \arrow[d, two heads] & \mathcal{S}(\mathsf{p}) \arrow[d, two heads] \\ \mathcal{Q}(\mathsf{h}) \arrow[r, "f"'] & \mathsf{p} \end{tikzcd}$$

commutes.

Explicitly, the map $\hat{f}$ is as follows. Evaluating on the Z-component, into the X-summand, the map is

$$\mathsf{h}[\mathrm{Z}] \xrightarrow{\Delta^{\mathrm{X}}_{\mathrm{Z}}} \mathsf{h}[\mathrm{X}] \twoheadrightarrow \mathcal{Q}(\mathsf{h})[\mathrm{X}] \xrightarrow{f_{\mathrm{X}}} \mathsf{p}[\mathrm{X}]. \tag{6.53}$$

Note the similarity with (6.27a).

Exercise 6.46. Deduce using (6.51) that $\mathcal{P}(\mathcal{S}(\mathsf{p})) = \mathsf{p}$ and $\mathcal{Q}(\mathcal{S}(\mathsf{p})) = \mathsf{p}$. Equivalently, the monads $\mathcal{PS}$ and $\mathcal{QS}$ on species (arising from the adjunction between $\mathcal{S}$ and $\mathcal{P}$, and between $\mathcal{Q}$ and $\mathcal{S}$) are both identity. For more general results, see Propositions 6.58 and 6.59 and the discussion in Section 13.2.

Remark 6.47. For the adjunction between $\mathcal{S}$ and $\mathcal{P}$ in Theorem 6.43, we can take the functors to be between the categories of species and commutative bimonoids. This follows by composing adjunctions:

$$\mathcal{A}\text{-}\mathsf{Sp} \underset{\mathcal{P}}{\overset{\mathcal{S}}{\rightleftarrows}} {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{(-)^{coab}}{\overset{inc}{\rightleftarrows}} \mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}).$$

The second adjunction is between inclusion and coabelianization, see comment after (2.57).

Further, for a commutative bimonoid h, the first diagram below induces the second diagram below.

$$\begin{tikzcd} \mathsf{p} \arrow[r, "f"] \arrow[dr, "f"'] & \mathcal{P}(\mathsf{h}^{coab}) \arrow[d, equal] \\ & \mathcal{P}(\mathsf{h}) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S}(\mathsf{p}) \arrow[r, dashed, "\hat{f}"] \arrow[dr, dashed, "\hat{f}"'] & \mathsf{h}^{coab} \arrow[d, hook] \\ & \mathsf{h} \end{tikzcd}$$

Similarly, for the adjunction between $\mathcal{S}$ and $\mathcal{Q}$, we can take the functors to be between the categories of species and cocommutative bimonoids. Further, for a cocommutative bimonoid h, the first diagram below induces the second diagram below.

$$\begin{tikzcd} \mathcal{Q}(\mathsf{h}) \arrow[r, "f"] \arrow[d, equal] & \mathsf{p} \\ \mathcal{Q}(\mathsf{h}_{ab}) \arrow[ur, "f"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{h} \arrow[r, dashed, "\hat{f}"] \arrow[d, two heads] & \mathcal{S}(\mathsf{p}) \\ \mathsf{h}_{ab} \arrow[ur, dashed, "\hat{f}"'] \end{tikzcd}$$

6.5.2. Duality. The following diagrams of functors commute.

$$\begin{tikzcd} \mathcal{A}\text{-}\mathsf{Sp} \arrow[r, "\mathcal{S}"] \arrow[d, "(-)^*"'] & {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \arrow[d, "(-)^*"] \\ \mathcal{A}\text{-}\mathsf{Sp}^{\mathrm{op}} \arrow[r, "\mathcal{S}"'] & {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})^{\mathrm{op}} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}\text{-}\mathsf{Sp} & {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \arrow[l, "\mathcal{P}"'] \\ \mathcal{A}\text{-}\mathsf{Sp}^{\mathrm{op}} \arrow[u, "(-)^*"] & {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})^{\mathrm{op}} \arrow[l, "\mathcal{Q}"] \arrow[u, "(-)^*"'] \end{tikzcd}$$

This follows from (6.31) and (5.41). Moreover, the functors in the first diagram are the left adjoints of the functors in the right diagram. Note very carefully that for $\mathcal{S}$ and $\mathcal{Q}$, in the above diagrams, we have used the opposite categories, so $\mathcal{S}$ is the left adjoint and $\mathcal{Q}$ is the right adjoint.

Exercise 6.48. Show that: For any bicommutative bimonoid h, the dual of the unit $\mathsf{h} \to \mathcal{S}\mathcal{Q}(\mathsf{h})$ can be identified with the counit $\mathcal{S}\mathcal{P}(\mathsf{h}^*) \to \mathsf{h}^*$ for the dual bicommutative bimonoid h^*. (Use formulas (6.52) and (6.53).) Similarly, for any species p, the dual of the counit $\mathcal{Q}\mathcal{S}(\mathsf{p}) \to \mathsf{p}$ can be identified with the unit $\mathsf{p}^* \to \mathcal{P}\mathcal{S}(\mathsf{p}^*)$ for the dual species p^*. In fact, both maps are identities. (See Exercise 6.46.)

6.5.3. From species to signed bicommutative signed bimonoids. The preceding discussion has a signed analogue which we briefly explain. By composing the trivial (co)monoid functors with the functors $\mathcal{E}$ and $\mathcal{E}^\vee$, we obtain functors

$$\mathcal{E}, \mathcal{E}^\vee : \mathcal{A}\text{-}\mathsf{Sp} \to (-1)\text{-}{}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}),$$

which are equal:

$$\mathcal{E} = \mathcal{E}^\vee. \tag{6.54}$$

This can be seen by specializing (6.33) and (6.34), and their duals. Explicitly, for a species p, the product and coproduct of $\mathcal{E}(\mathsf{p})$ (and of $\mathcal{E}^\vee(\mathsf{p})$) are as follows.

For $\mathrm{Z} \leq \mathrm{X}$,

$$\begin{tikzcd} \mathbf{E}^-[\mathrm{Z},\mathrm{X}] \otimes \mathcal{E}(\mathsf{p})[\mathrm{X}] \arrow[rr, dashed, "\mu_{\mathrm{Z}}^{\mathrm{X}}"] & & \mathcal{E}(\mathsf{p})[\mathrm{Z}] \\ \mathbf{E}^-[\mathrm{Z},\mathrm{X}] \otimes \mathbf{E}^-[\mathrm{X},\mathrm{Y}] \otimes \mathsf{p}[\mathrm{Y}] \arrow[u, hook] \arrow[rr, "(-)\otimes \mathrm{id}"'] & & \mathbf{E}^-[\mathrm{Z},\mathrm{Y}] \otimes \mathsf{p}[\mathrm{Y}], \arrow[u, hook] \end{tikzcd} \tag{6.55}$$

where the unnamed map $(-)$ is (1.162), and

$$\begin{tikzcd} \mathcal{E}(\mathsf{p})[\mathrm{Z}] \arrow[rr, dashed, "\Delta_{\mathrm{Z}}^{\mathrm{X}}"] & & \mathbf{E}^-[\mathrm{Z},\mathrm{X}] \otimes \mathcal{E}(\mathsf{p})[\mathrm{X}] \\ \mathbf{E}^-[\mathrm{Z},\mathrm{Y}] \otimes \mathsf{p}[\mathrm{Y}] \arrow[u, hook] \arrow[r] & \begin{cases} \mathbf{E}^-[\mathrm{Z},\mathrm{X}] \otimes \mathbf{E}^-[\mathrm{X},\mathrm{Y}] \otimes \mathsf{p}[\mathrm{Y}] & \text{if } \mathrm{Y} \geq \mathrm{X}, \\ 0 & \text{otherwise,} \end{cases} \arrow[ur, hook] \end{tikzcd} \tag{6.56}$$

where in the first alternative, the map is the inverse of (1.162) tensor with the identity. Use Exercise 1.75 to see why specializing $\mathrm{W} := \mathrm{Y}$ in (6.34) yields this formula.

By composing adjunctions, we deduce:

Theorem 6.49. *The functor $\mathcal{E}$ is the left adjoint of $\mathcal{P}$ and the right adjoint of $\mathcal{Q}$. (The functors are between the categories of species and signed bicommutative signed bimonoids.)*

The first adjunction is reformulated below as a universal property.

Theorem 6.50. *Let h be a signed commutative signed bimonoid, p a species, $f : \mathsf{p} \to \mathcal{P}(\mathsf{h})$ a map of species. Then there exists a unique morphism of signed bimonoids $\hat{f} : \mathcal{E}(\mathsf{p}) \to \mathsf{h}$ such that the diagram*

$$\begin{tikzcd} \mathcal{E}(\mathsf{p}) \arrow[r, dashed, "\hat{f}"] & \mathsf{h} \\ \mathsf{p} \arrow[u, hook] \arrow[r, "f"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd}$$

commutes.

The map $\hat{f}$ is as follows. Evaluating on the Z-component, on the X-summand, the map is

$$\mathbf{E}^-[\mathrm{Z},\mathrm{X}] \otimes \mathsf{p}[\mathrm{X}] \xrightarrow{\mathrm{id} \otimes f_{\mathrm{X}}} \mathbf{E}^-[\mathrm{Z},\mathrm{X}] \otimes \mathcal{P}(\mathsf{h})[\mathrm{X}] \hookrightarrow \mathbf{E}^-[\mathrm{Z},\mathrm{X}] \otimes \mathsf{h}[\mathrm{X}] \xrightarrow{\mu_{\mathrm{Z}}^{\mathrm{X}}} \mathsf{h}[\mathrm{Z}]. \tag{6.57}$$

Note the similarity with (6.35a).

The universal property of the adjunction between $\mathcal{E}$ and $\mathcal{Q}$ can be formulated similarly.

6.6. (Co)abelianizations of (co)free (co)monoids

The free monoid and the cofree comonoid on a species relate to the free commutative monoid and the cofree cocommutative comonoid via abelianization and coabelianization, respectively. More generally, the abelianization of the free bimonoid on a comonoid is the free commutative bimonoid on that comonoid, and dually, the coabelianization of the cofree bimonoid on a monoid is the cofree cocommutative bimonoid on that monoid.

6.6.1. Free monoid. Recall the abelianization of a bimonoid from Section 2.7. For any comonoid c,

$$\mathcal{S}(\mathsf{c}) = \mathcal{T}(\mathsf{c})_{ab}, \tag{6.58}$$

where the latter is the abelianization of the bimonoid $\mathcal{T}(\mathsf{c})$. This follows by composing the adjunctions in Theorem 6.5 for $q = 1$ and in (2.56):

$$\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{frg}{\overset{\mathcal{T}}{\rightleftarrows}} \mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{inc}{\overset{(-)^{ab}}{\rightleftarrows}} \mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}).$$

Explicitly, the abelianization map

$$\pi : \mathcal{T}(\mathsf{c}) \twoheadrightarrow \mathcal{S}(\mathsf{c}) \tag{6.59}$$

is as follows. Evaluating on the A-component, on the F-summand, it is given by $\beta_{\mathrm{s}(F),F}$.

Let us now specialize c to the trivial comonoid. Thus, for any species p,

$$\mathcal{S}(\mathsf{p}) = \mathcal{T}(\mathsf{p})_{ab}. \tag{6.60}$$

This can also be seen directly by composing the adjunctions:

$$\mathcal{A}\text{-}\mathsf{Sp} \underset{\mathcal{P}}{\overset{\mathcal{T}}{\rightleftarrows}} {}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{inc}{\overset{(-)^{ab}}{\rightleftarrows}} {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}).$$

The first adjunction is as in Exercise 6.36. The abelianization map

$$\pi : \mathcal{T}(\mathsf{p}) \twoheadrightarrow \mathcal{S}(\mathsf{p}) \tag{6.61}$$

is given by the same formula as in (6.59).

Exercise 6.51. Check directly using (co)product formulas (6.39) of $\mathcal{T}(\mathsf{p})$ and (6.51) of $\mathcal{S}(\mathsf{p})$ that (6.61) is a morphism of bimonoids.

Exercise 6.52. Check that: The morphism of bimonoids (6.59) arises from the universal property in Theorem 6.6 for $q = 1$ with $\mathsf{h} := \mathcal{S}(\mathsf{c})$ and the inclusion $f : \mathsf{c} \hookrightarrow \mathcal{S}(\mathsf{c})$ of comonoids. Similarly, the morphism (6.61) arises from the universal property in Theorem 6.31 for $q = 1$ with $\mathsf{h} := \mathcal{S}(\mathsf{p})$ and $f := \mathrm{id}$. (Recall from Exercise 6.46 that $\mathcal{P}(\mathcal{S}(\mathsf{p})) = \mathsf{p}$.)

Exercise 6.53. Let h be a commutative bimonoid. Check that: For a morphism of comonoids $f : \mathsf{c} \to \mathsf{h}$, the first diagram below commutes. The morphisms $\hat{f}$ are obtained from the universal properties in Theorem 6.6 for $q = 1$ and Theorem 6.21.

$$\begin{tikzcd} \mathcal{T}(\mathsf{c}) \arrow[r, dashed, "\hat{f}"] \arrow[d, "\pi"'] & \mathsf{h} \\ \mathcal{S}(\mathsf{c}) \arrow[ur, dashed, "\hat{f}"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}(\mathsf{p}) \arrow[r, dashed, "\hat{f}"] \arrow[d, "\pi"'] & \mathsf{h} \\ \mathcal{S}(\mathsf{p}) \arrow[ur, dashed, "\hat{f}"'] & \end{tikzcd}$$

Similarly, for a map of species $f : \mathsf{p} \to \mathcal{P}(\mathsf{h})$, the second diagram above commutes.

6.6.2. Cofree comonoid. Dually, for any monoid a,

$$\mathcal{S}^{\vee}(\mathsf{a}) = \mathcal{T}^{\vee}(\mathsf{a})^{coab}, \tag{6.62}$$

where the latter is the coabelianization of the bimonoid $\mathcal{T}^{\vee}(\mathsf{a})$. Explicitly, the coabelianization map

$$\pi^{\vee} : \mathcal{S}^{\vee}(\mathsf{a}) \hookrightarrow \mathcal{T}^{\vee}(\mathsf{a}) \tag{6.63}$$

is as follows. Evaluating on the A-component, on the X-summand, it is given by $\sum_F \beta_{F,\mathrm{X}}$, with the sum being over all F greater than A and of the same support as X.

Specializing to the trivial monoid: For any species p,

$$\mathcal{S}^{\vee}(\mathsf{p}) = \mathcal{T}^{\vee}(\mathsf{p})^{coab}, \tag{6.64}$$

with the coabelianization map

$$\pi^{\vee} : \mathcal{S}^{\vee}(\mathsf{p}) \hookrightarrow \mathcal{T}^{\vee}(\mathsf{p}) \tag{6.65}$$

given by the same formula as in (6.63).

6.6.3. Signed analogue. We now briefly deal with the signed case. For any comonoid c,

$$\mathcal{E}(\mathsf{c}) = \mathcal{T}_{-1}(\mathsf{c})_{ab}, \tag{6.66}$$

where the latter is the signed abelianization of the signed bimonoid $\mathcal{T}_{-1}(\mathsf{c})$. Explicitly, the signed abelianization map

$$\pi_{-1} : \mathcal{T}_{-1}(\mathsf{c}) \twoheadrightarrow \mathcal{E}(\mathsf{c}) \tag{6.67}$$

is as follows. Evaluating on the A-component, on the F-summand, it is given by $[F/A] \otimes \beta_{\mathrm{s}(F),F}$.

Specializing to the trivial comonoid: For any species p,

$$\mathcal{E}(\mathsf{p}) = \mathcal{T}_{-1}(\mathsf{p})_{ab}, \tag{6.68}$$

with the signed abelianization map

$$\pi_{-1} : \mathcal{T}_{-1}(\mathsf{p}) \twoheadrightarrow \mathcal{E}(\mathsf{p}) \tag{6.69}$$

given by the same formula as in (6.67).

Exercise 6.54. Check directly using (co)product formulas (6.39), (6.55), (6.56) that (6.69) is a morphism of signed bimonoids.

Dually, for any monoid a,

$$\mathcal{E}^\vee(\mathsf{a}) = \mathcal{T}^\vee_{-1}(\mathsf{a})^{coab}, \tag{6.70}$$

where the latter is the signed coabelianization of the signed bimonoid $\mathcal{T}^\vee_{-1}(\mathsf{a})$, and so on.

Question 6.55. For a comonoid c, describe the coabelianizations of $\mathcal{T}(\mathsf{c})$ and $\mathcal{S}(\mathsf{c})$ and the signed coabelianizations of $\mathcal{T}_{-1}(\mathsf{c})$ and $\mathcal{E}(\mathsf{c})$. Dually, for a monoid a, describe the abelianizations of $\mathcal{T}^\vee(\mathsf{a})$ and $\mathcal{S}^\vee(\mathsf{a})$, and the signed abelianizations of $\mathcal{T}^\vee_{-1}(\mathsf{a})$ and $\mathcal{E}^\vee(\mathsf{a})$.

6.7. Primitive filtrations and decomposable filtrations

The decomposable filtration of the free monoid, and the primitive filtration of the cofree comonoid on a species can be described in terms of Cauchy powers of the species. There are similar descriptions for (co)free (co)commutative (co)monoids in terms of commutative Cauchy powers. These results apply to any q-bimonoid which is either free as a (commutative) monoid or cofree as a (cocommutative) comonoid.

6.7.1. (Co)free (co)monoid. Let us begin with the free monoid and cofree comonoid.

Proposition 6.56. *For any $k \geq 1$, for any species p,*

$$\mathcal{P}_k(\mathcal{T}^\vee(\mathsf{p})) = \underset{1\leq i\leq k}{+} \mathsf{p}^i.$$

Explicitly, evaluating on the A-component,

$$\mathcal{P}_k(\mathcal{T}^\vee(\mathsf{p}))[A] = \bigoplus_{\substack{F\colon F\geq A\\ \mathrm{rk}(F/A)<k}} \mathsf{p}[F].$$

In particular, $\mathcal{P}(\mathcal{T}^\vee(\mathsf{p})) = \mathsf{p}$.

Dually:

Proposition 6.57. *For any $k \geq 1$, for any species p,*

$$\mathcal{D}_k(\mathcal{T}(\mathsf{p})) = \underset{i>k}{+} \mathsf{p}^i.$$

Explicitly, evaluating on the A-component,

$$\mathcal{D}_k(\mathcal{T}(\mathsf{p}))[A] = \bigoplus_{\substack{F\colon F\geq A\\ \mathrm{rk}(F/A)\geq k}} \mathsf{p}[F].$$

Thus, $\mathcal{Q}(\mathcal{T}(\mathsf{p})) = \mathsf{p}$.

The above results follow from coproduct formula (6.11) and product formula (6.3), respectively.

Note that Proposition 6.56 also applies to the cofree q-bimonoid $\mathcal{T}^\vee_q(\mathsf{a})$. The primitive filtration depends only on the coproduct, so the product of a and the parameter q play no role. Similar remarks apply to Proposition 6.57 and the free q-bimonoid $\mathcal{T}_q(\mathsf{c})$. For finer results, see Exercises 16.23 and 16.50.

6.7.2. (Co)commutative (co)free (co)monoids. The commutative analogues of Propositions 6.56 and 6.57 are given below. Note very carefully that Cauchy powers have been replaced by commutative Cauchy powers.

Proposition 6.58. *For any $k \geq 1$, for any species* p*,*

$$\mathcal{P}_k(\mathcal{S}^\vee(\mathsf{p})) = \underset{1\leq i \leq k}{+} \mathsf{p}^{\bar{i}}.$$

Explicitly, evaluating on the Z*-component,*

$$\mathcal{P}_k(\mathcal{S}^\vee(\mathsf{p}))[\mathrm{Z}] = \bigoplus_{\substack{\mathrm{X}:\, \mathrm{X}\geq \mathrm{Z} \\ \mathrm{rk}(\mathrm{X}/\mathrm{Z})<k}} \mathsf{p}[\mathrm{X}].$$

In particular, $\mathcal{P}(\mathcal{S}^\vee(\mathsf{p})) = \mathsf{p}$*.*

Proposition 6.59. *For any $k \geq 1$, for any species* p*,*

$$\mathcal{D}_k(\mathcal{S}(\mathsf{p})) = \underset{i>k}{+} \mathsf{p}^{\bar{i}}.$$

Explicitly, evaluating on the Z*-component,*

$$\mathcal{D}_k(\mathcal{S}(\mathsf{p}))[\mathrm{Z}] = \bigoplus_{\substack{\mathrm{X}:\, \mathrm{X}\geq \mathrm{Z} \\ \mathrm{rk}(\mathrm{X}/\mathrm{Z})\geq k}} \mathsf{p}[\mathrm{X}].$$

Thus, $\mathcal{Q}(\mathcal{S}(\mathsf{p})) = \mathsf{p}$*.*

The above results follow from coproduct formula (6.26) and product formula (6.20), respectively.

Exercise 6.60. Write down the signed analogues of the above results for $\mathcal{E}^\vee(\mathsf{p})$ and $\mathcal{E}(\mathsf{p})$.

Exercise 6.61. Show that: If a comonoid c is either cofree or cofree cocommutative on a species p, then $\mathcal{P}(\mathsf{c}) \cong \mathsf{p}$. Dually, if a monoid a is either free or free commutative on a species p, then $\mathcal{Q}(\mathsf{a}) \cong \mathsf{p}$.

Question 6.62. For a comonoid c, describe in explicit terms the primitive filtration and, in particular, the primitive part of $\mathcal{T}_q(\mathsf{c})$. Dually, for a monoid a, describe in explicit terms the decomposable filtration and, in particular, the decomposable part of $\mathcal{T}_q^\vee(\mathsf{a})$.

Related ideas are presented below.

Lemma 6.63. *Let q be any scalar. For any species* p*, the primitive part* $\mathcal{P}\mathcal{T}_q(\mathsf{p})[A]$ *consists of elements*

$$(x^F)_{F\geq A} \quad \textit{such that} \sum_{\substack{F:\, F\geq A \\ HF=G,\, \mathrm{s}(F)=\mathrm{s}(G)}} (\beta_q)_{G,F}(x^F) = 0 \quad \textit{for all } A < H \leq G. \tag{6.71}$$

Proof. By coproduct formula (6.39),

$$\Delta_A^H((x^F)_{F\geq A}) = \sum_{G:\, G\geq H} \; \sum_{\substack{F:\, F\geq A \\ HF=G,\, \mathrm{s}(F)=\mathrm{s}(G)}} (\beta_q)_{G,F}(x^F).$$

It follows that $\mathcal{P}\mathcal{T}_q(\mathsf{p})[A] = \bigcap_{H>A} \ker(\Delta_A^H)$ consists of the elements (6.71). $\square$

Exercise 6.64. Let q be any scalar. Use coproduct formula (6.5) to check that: For any comonoid c, the primitive part $\mathcal{P}\mathcal{T}_q(\mathsf{c})[A]$ consists of elements

$$(x^F)_{F\geq A} \text{ such that } \sum_{\substack{F:\, F\geq A \\ HF=G}} (\beta_q)_{G,FH} \Delta_F^{FH}(x^F) = 0 \text{ for all } A < H \leq G. \tag{6.72}$$

When c is trivial, this recovers (6.71).

The point is that solving (6.72) is a nontrivial problem, and this is only about the primitive part. We have not even considered the primitive filtration. Similar remarks apply to the dual problem on the decomposable filtration.

The answer to Question 6.62 is known in some special cases which we list below. To understand these properly, we need to refer to more advanced results proved later in the text.

- $q = 1$ and c is cocommutative. By the Borel–Hopf Theorem 13.34, the bimonoid $\mathcal{T}(\mathsf{c})$ is cofree cocommutative, so its primitive filtration is given by Proposition 6.58 with $\mathsf{p} := \mathcal{P}\mathcal{T}(\mathsf{c}) \cong \mathbf{Lie} \circ \mathsf{c}_t$, see Proposition 16.9 for more details. The special case when c is a trivial comonoid is dealt with in Corollary 16.7.
- $q = 1$ and a is commutative. By the Borel–Hopf Theorem 13.57, the bimonoid $\mathcal{T}^\vee(\mathsf{a})$ is free commutative, so its decomposable filtration is given by Proposition 6.59 with $\mathsf{p} := \mathcal{Q}\mathcal{T}^\vee(\mathsf{a}) \cong \mathbf{Lie}^* \circ \mathsf{a}_t$, see Proposition 16.44 for more details. The special case when a is a trivial monoid is dealt with in Corollary 16.43.
- q is not a root of unity. A q-bimonoid is both free and cofree (Section 13.6). So in principle, the primitive filtration of $\mathcal{T}_q(\mathsf{c})$ is given by Proposition 6.56, while the decomposable filtration of $\mathcal{T}^\vee(\mathsf{a})$ is given by Proposition 6.57. Explicitly: The case when c and a are trivial is given in Exercise 6.77. For the general case, one can use the HNR isomorphisms for q-bimonoids (Section 14.5).

Important special cases are discussed in Section 7.9.

6.8. Alternative descriptions of bimonoids

A bimonoid can be characterized by saying that its product is a morphism of comonoids, or dually, its coproduct is a morphism of monoids. These and related facts are explained below.

6.8.1. *q*-bimonoids. For any monoid (a, μ), by putting together the product components μ_A^F, we obtain a map of species $\mathcal{T}(\mathsf{a}) \to \mathsf{a}$. It makes sense to denote this map by μ itself. This is merely expressing the fact that a monoid is a $\mathcal{T}$-algebra, see Proposition 3.2 and its proof. Dually, for any comonoid (c, Δ), we have a map of species $\Delta : \mathsf{c} \to \mathcal{T}^\vee(\mathsf{c})$.

Lemma 6.65. *Let* h *be a species equipped with maps* μ *and* Δ *such that* (h, μ) *is a monoid and* (h, Δ) *is a comonoid. Then the following are equivalent.*

(1) $(\mathsf{h}, \mu, \Delta)$ *is a* q*-bimonoid.*
(2) $\mu : \mathcal{T}_q(\mathsf{h}) \to \mathsf{h}$ *is a morphism of comonoids.*
(3) $\Delta : \mathsf{h} \to \mathcal{T}_q^\vee(\mathsf{h})$ *is a morphism of monoids.*

The coproduct of $\mathcal{T}_q(\mathsf{h})$ is given by (6.5) with h viewed as a comonoid, while the product of $\mathcal{T}_q^\vee(\mathsf{h})$ is given by (6.13) with h viewed as a monoid.

Proof. Item (2) says that the following diagram commutes for any $A \leq G$.

$$\begin{tikzcd} \mathcal{T}_q(\mathsf{h})[A] \arrow[r, "\mu"] \arrow[d, "\Delta"'] & \mathsf{h}[A] \arrow[d, "\Delta"] \\ \mathcal{T}_q(\mathsf{h})[G] \arrow[r, "\mu"'] & \mathsf{h}[G] \end{tikzcd}$$

For any $F \geq A$, if we start in the summand $\mathsf{h}[F]$, then the above is precisely the q-bimonoid axiom (2.33). Thus, (2) is equivalent to (1). Similarly, (3) is also equivalent to (1). $\square$

Lemma 6.66. *Let* h *and* k *be* q*-bimonoids, and* $f : \mathsf{h} \to \mathsf{k}$ *a map of species. Then the following are equivalent.*

(1) f *is a morphism of* q*-bimonoids.*
(2) f *is a morphism of comonoids and the first diagram below commutes.*
(3) f *is a morphism of monoids and the second diagram below commutes.*

$$\begin{tikzcd} \mathcal{T}_q(\mathsf{h}) \arrow[r, "f"] \arrow[d, "\mu"'] & \mathcal{T}_q(\mathsf{k}) \arrow[d, "\mu"] \\ \mathsf{h} \arrow[r, "f"'] & \mathsf{k} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}_q^\vee(\mathsf{h}) \arrow[r, "f"] & \mathcal{T}_q^\vee(\mathsf{k}) \\ \mathsf{h} \arrow[r, "f"'] \arrow[u, "\Delta"] & \mathsf{k} \arrow[u, "\Delta"'] \end{tikzcd}$$

Proof. The first diagram says that f is a morphism of monoids, while the second says that it is a morphism of comonoids. $\square$

Exercise 6.67. Let $(\mathsf{h}, \mu, \Delta)$ be a q-bimonoid. Then $\mu : \mathcal{T}_q(\mathsf{h}) \twoheadrightarrow \mathsf{h}$ is a surjective morphism of q-bimonoids. Check this both directly and using the universal property given in Theorem 6.6. (The fact that μ is a morphism of comonoids is also contained in Lemma 6.65.) Dually, check that $\Delta : \mathsf{h} \hookrightarrow \mathcal{T}_q^\vee(\mathsf{h})$ is an injective morphism of q-bimonoids.

6.8.2. Commutative monoids. Recall the opposite transformation τ from Section 3.4. The following is a restatement of Lemma 3.21.

Lemma 6.68. *Let* (a, μ) *be an* $\mathcal{A}$*-monoid. Then: It is commutative iff the diagram*

$$\begin{tikzcd} \mathcal{T}(\mathsf{a}) \arrow[rr, "\tau_{\mathsf{a}}"] \arrow[dr, "\mu"'] & & \mathcal{T}(\mathsf{a}) \arrow[dl, "\mu"] \\ & \mathsf{a} & \end{tikzcd}$$

commutes.

There is a dual statement for cocommutative comonoids.

Lemma 6.69. *For any comonoid* c*, the map of species* $\tau_{\mathsf{c}} : \mathcal{T}(\mathsf{c}) \to \mathcal{T}(\mathsf{c})$ *is an isomorphism of comonoids. Dually, for any monoid* a*, the map of species* $\tau_{\mathsf{a}} : \mathcal{T}^{\vee}(\mathsf{a}) \to \mathcal{T}^{\vee}(\mathsf{a})$ *is an isomorphism of monoids.*

PROOF. We explain the first claim. Following the notation in (6.5) for $q = 1$, it boils down to the commutativity of the following diagram.

$$\begin{tikzcd}
\mathsf{c}[H] \arrow[r, "\Delta_H^{HG}"] \arrow[d, "\beta_{A\overline{H},H}"'] & \mathsf{c}[HG] \arrow[r, "\beta_{GH,HG}"] \arrow[d, "\beta_{A\overline{H}G,HG}"] & \mathsf{c}[GH] \arrow[d, "\beta_{G\overline{H},GH}"] \\
\mathsf{c}[A\overline{H}] \arrow[r, "\Delta_{A\overline{H}}^{A\overline{H}G}"'] & \mathsf{c}[A\overline{H}G] \arrow[r, "\beta_{G\overline{H},A\overline{H}G}"'] & \mathsf{c}[G\overline{H}]
\end{tikzcd}$$

The first square commutes by naturality of the coproduct (2.10), and the second by (2.1). Since τ_{c} is an involution, it is an isomorphism. □

Exercise 6.70. Check that: A subspecies p generates a monoid a iff the map $\mathcal{T}(\mathsf{p}) \to \mathsf{a}$ of monoids given in Theorem 6.2 is surjective. Use this and the preceding results to reprove Propositions 2.46 and 2.47.

6.8.3. Bicommutative bimonoids. For a commutative monoid (a, μ), we have a map of species $\mu : \mathcal{S}(\mathsf{a}) \to \mathsf{a}$, and dually, for a cocommutative comonoid (c, Δ), we have a map of species $\Delta : \mathsf{c} \to \mathcal{S}^{\vee}(\mathsf{c})$. One can use these to formulate commutative analogues of Lemma 6.65. For instance:

Lemma 6.71. *Let* h *be a species equipped with maps* μ *and* Δ *such that* (h, μ) *is a commutative monoid and* (h, Δ) *is a cocommutative comonoid. Then the following are equivalent.*

(1) $(\mathsf{h}, \mu, \Delta)$ *is a bicommutative bimonoid.*
(2) $\mu : \mathcal{S}(\mathsf{h}) \to \mathsf{h}$ *is a morphism of comonoids.*
(3) $\Delta : \mathsf{h} \to \mathcal{S}^{\vee}(\mathsf{h})$ *is a morphism of monoids.*

The coproduct of $\mathcal{S}(\mathsf{h})$ is given by (6.22) with h viewed as a cocommutative comonoid, while the product of $\mathcal{S}^{\vee}(\mathsf{h})$ is given by (6.28) with h viewed as a commutative monoid.

PROOF. Each of the three statements is equivalent to the bicommutative bimonoid axiom (2.26). □

Exercise 6.72. Formulate the analogue of Lemma 6.66 for bicommutative bimonoids.

6.9. Norm transformation

The q-norm transformation κ_q is a natural transformation between the functors $\mathcal{T}_q$ and $\mathcal{T}_q^{\vee}$ from the category of species to the category of q-bimonoids. Thus, for every species p, we have a q-norm map $(\kappa_q)_{\mathsf{p}} : \mathcal{T}_q(\mathsf{p}) \to \mathcal{T}_q^{\vee}(\mathsf{p})$ which is a morphism of q-bimonoids. It arises from the freeness of $\mathcal{T}_q$ and the cofreeness of $\mathcal{T}_q^{\vee}$. It is an isomorphism when q is not a root of unity. For $q = 0$, κ_0 is in fact the identity. It is of interest to understand the image of κ_q when q is a root of unity. The cases $q = \pm 1$ yield the functors $\mathcal{S}$ and $\mathcal{E}$, respectively.

More generally, we associate a map $\mathcal{T}_q(\mathsf{c}) \to \mathcal{T}_q^\vee(\mathsf{a})$ to a map of species $f : \mathsf{c} \to \mathsf{a}$ from a comonoid c to a monoid a. We recover the q-norm map $(\kappa_q)_\mathsf{p}$ by viewing p as a trivial (co)monoid and taking $f := \mathrm{id}$.

6.9.1. *q*-norm transformation. View $\mathcal{T}_q$ and $\mathcal{T}_q^\vee$ as functors from the category of species to the category of q-bimonoids as in (6.38). There is a natural transformation

$$\kappa_q : \mathcal{T}_q \to \mathcal{T}_q^\vee. \tag{6.73}$$

We call this the *q-norm transformation*. For a species p, the map

$$(\kappa_q)_\mathsf{p} : \mathcal{T}_q(\mathsf{p}) \to \mathcal{T}_q^\vee(\mathsf{p}), \tag{6.74}$$

evaluated on the A-component, on the F-summand, is defined by

$$\mathsf{p}[F] \to \bigoplus_{\substack{G:\, G \geq A \\ \mathrm{s}(G)=\mathrm{s}(F)}} \mathsf{p}[G], \qquad v \mapsto \sum_{\substack{G:\, G \geq A \\ \mathrm{s}(G)=\mathrm{s}(F)}} (\beta_q)_{G,F}(v).$$

We call this the *q-norm map*.

Lemma 6.73. *The q-norm map (6.74) is a morphism of q-bimonoids.*

Proof. This can be checked directly using formulas (6.39) and (6.40). Alternatively, we first note that $\mathcal{P}(\mathcal{T}_q^\vee(\mathsf{p})) = \mathsf{p}$ by Proposition 6.56, and then apply Theorem 6.31 to $f := \mathrm{id}$ to obtain a morphism of q-bimonoids from $\mathcal{T}_q(\mathsf{p})$ to $\mathcal{T}_q^\vee(\mathsf{p})$. Using formula (6.41), we see that this morphism indeed agrees with $(\kappa_q)_\mathsf{p}$. Dually, we note that $\mathcal{Q}(\mathcal{T}_q(\mathsf{p})) = \mathsf{p}$ by Proposition 6.57, then apply Theorem 6.34 to $f := \mathrm{id}$ to obtain a morphism from $\mathcal{T}_q(\mathsf{p})$ to $\mathcal{T}_q^\vee(\mathsf{p})$, and finally use formula (6.42) to check that it agrees with $(\kappa_q)_\mathsf{p}$. $\square$

Naturality of $(\kappa_q)_\mathsf{p}$ in p is clear.

Lemma 6.74. *The q-norm map (6.74) preserves Cauchy powers of p, and further, it is identity on the first Cauchy power $\mathsf{p}^1 = \mathsf{p}$. In general, its restriction $\mathsf{p}^k \to \mathsf{p}^k$ is given by the map $(\beta_q)^{k-1}$ as defined in (5.4).*

Proof. This follows from the definitions. $\square$

6.9.2. Specializations. For $q = \pm 1$, observe that κ_q factors as follows.

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\kappa_1"] \arrow[d, two heads] & \mathcal{T}^\vee \\ \mathcal{S} \arrow[r, "\cong"'] & \mathcal{S}^\vee \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}_{-1} \arrow[r, "\kappa_{-1}"] \arrow[d, two heads] & \mathcal{T}_{-1}^\vee \\ \mathcal{E} \arrow[r, "\cong"'] & \mathcal{E}^\vee \arrow[u, hook] \end{tikzcd} \tag{6.75}$$

The isomorphisms are as in (6.50) and (6.54), respectively. The vertical maps in the first diagram are abelianization (6.61) and coabelianization (6.65), while in the second diagram are their signed analogues, namely, (6.69) and its dual. The first diagram, when evaluated on a species, is an instance of (2.59).

For $q = 0$, observe that $\kappa_0 = \mathrm{id}$. This yields the natural isomorphism (6.45). More generally, for q not a root of unity (which includes $q = 0$), the q-norm map is an isomorphism:

Proposition 6.75. *Suppose q is not a root of unity. Then the q-norm transformation κ_q is an isomorphism. More precisely, for any species p, the component $(\kappa_q)_{\mathsf{p}} : \mathcal{T}_q(\mathsf{p}) \to \mathcal{T}_q^{\vee}(\mathsf{p})$ is an isomorphism of q-bimonoids, which moreover induces an isomorphism of species $\mathsf{p}^k \to \mathsf{p}^k$ for each $k \geq 1$.*

PROOF. This follows from Lemma 5.3 and Lemma 6.74. □

Exercise 6.76. Suppose q is not a root of unity, so the q-norm map (6.74) is an isomorphism.

(i) Use $\mathcal{P}(\mathcal{T}_q^{\vee}(\mathsf{p})) = \mathsf{p}$ to deduce that $\mathcal{P}(\mathcal{T}_q(\mathsf{p})) = \mathsf{p}$.
(ii) Use $\mathcal{Q}(\mathcal{T}_q(\mathsf{p})) = \mathsf{p}$ to deduce that $\mathcal{Q}(\mathcal{T}_q^{\vee}(\mathsf{p})) = \mathsf{p}$.

Exercise 6.77. Suppose q is not a root of unity. Show that: For any species p, the primitive filtration of $\mathcal{T}_q(\mathsf{p})$ is given by

$$\mathcal{P}_k(\mathcal{T}_q(\mathsf{p})) = \underset{1 \leq i \leq k}{+} \mathsf{p}^i.$$

(This result holds for $\mathcal{T}_q^{\vee}(\mathsf{p})$ by Proposition 6.56. Now apply Proposition 6.75. Alternatively, use Exercise 6.76, item (i) and Proposition 5.54 along with the observation that the rhs above is the filtration of $\mathcal{T}_q(\mathsf{p})$ generated by p.)

Dually, for any species p, the decomposable filtration of $\mathcal{T}_q^{\vee}(\mathsf{p})$ is given by

$$\mathcal{D}_k(\mathcal{T}_q^{\vee}(\mathsf{p})) = \underset{i > k}{+} \mathsf{p}^i.$$

(Now use Proposition 6.57 and Proposition 6.75.)

Exercise 6.78. Check that: For q not a root of unity, the inverse of the q-norm map (6.74)

$$\mathcal{T}_q^{\vee}(\mathsf{p}) \to \mathcal{T}_q(\mathsf{p}) \tag{6.76}$$

evaluated on the A-component, on the F-summand, can be written as

$$\mathsf{p}[F] \to \bigoplus_{\substack{G\colon G \geq A \\ \mathrm{s}(G) = \mathrm{s}(F)}} \mathsf{p}[G], \qquad v \mapsto \sum_{\substack{G\colon G \geq A \\ \mathrm{s}(G) = \mathrm{s}(F)}} \boldsymbol{\zeta}_q(A, F, G)\, \beta_{G,F}(v),$$

where $\boldsymbol{\zeta}_q$ is the two-sided q-zeta function (1.64).

A general context for Proposition 6.75 involving the q-exponential is given later in Proposition 9.91. The precise connection is explained in Example 9.92. This result is also a special case of the rigidity of q-bimonoids which we will discuss in Section 13.6. The precise connection is given in Example 13.80.

6.9.3. General q-norm transformation. The q-norm map can be put in a more general setting as follows. A general discussion is given in Proposition C.33.

For a comonoid c and monoid a, there is a natural bijection

$$\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a}) \xrightarrow{\cong} q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{T}_q(\mathsf{c}), \mathcal{T}_q^{\vee}(\mathsf{a})). \tag{6.77}$$

It is constructed as follows. Given a map $f : \mathsf{c} \to \mathsf{a}$ of species, first lift it to a morphism $\mathsf{c} \to \mathcal{T}_q^{\vee}(\mathsf{a})$ of comonoids, and then extend it to a morphism $g : \mathcal{T}_q(\mathsf{c}) \to \mathcal{T}_q^{\vee}(\mathsf{a})$ of q-bimonoids. Alternatively, first extend it to a morphism

$\mathcal{T}_q(\mathsf{c}) \to \mathsf{a}$ of monoids, and then lift it to a morphism $g : \mathcal{T}_q(\mathsf{c}) \to \mathcal{T}_q^\vee(\mathsf{a})$ of q-bimonoids. Explicitly, on the A-component, the (F, G)-matrix-component of g is given by the composite

$$\begin{tikzcd}[column sep=small] & & \mathsf{c}[GF] \arrow[dr, "f_{GF}"] & & \\ \mathsf{c}[F] \arrow[r, "\Delta_F^{FG}"] & \mathsf{c}[FG] \arrow[ur, "(\beta_q)_{GF,FG}"] \arrow[dr, "f_{FG}"'] & & \mathsf{a}[GF] \arrow[r, "\mu_G^{GF}"] & \mathsf{a}[G]. \\ & & \mathsf{a}[FG] \arrow[ur, "(\beta_q)_{GF,FG}"'] & & \end{tikzcd}$$

The diamond commutes, so the two ways to go from $\mathsf{c}[F]$ to $\mathsf{a}[G]$ match. We call g the q-norm map associated to f.

Similarly, for any cocommutative comonoid c and commutative monoid a, there is a natural bijection

$$\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a}) \xrightarrow{\cong} {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{S}(\mathsf{c}), \mathcal{S}^\vee(\mathsf{a})). \tag{6.78}$$

Explicitly, given a map $f : \mathsf{c} \to \mathsf{a}$ of species, the morphism $g : \mathcal{S}(\mathsf{c}) \to \mathcal{S}^\vee(\mathsf{a})$ of bimonoids is as follows. On the Z-component, the (X, Y)-matrix-component of g is given by the composite

$$\mathsf{c}[\mathrm{X}] \xrightarrow{\Delta_{\mathrm{X}}^{\mathrm{X}\vee\mathrm{Y}}} \mathsf{c}[\mathrm{X} \vee \mathrm{Y}] \xrightarrow{f_{\mathrm{X}\vee\mathrm{Y}}} \mathsf{a}[\mathrm{X} \vee \mathrm{Y}] \xrightarrow{\mu_{\mathrm{Y}}^{\mathrm{X}\vee\mathrm{Y}}} \mathsf{a}[\mathrm{Y}].$$

In this situation, we have the commutative diagram of bimonoids

$$\begin{tikzcd} \mathcal{T}(\mathsf{c}) \arrow[r] \arrow[d] & \mathcal{T}^\vee(\mathsf{a}) \\ \mathcal{S}(\mathsf{c}) \arrow[r] & \mathcal{S}^\vee(\mathsf{a}). \arrow[u] \end{tikzcd} \tag{6.79}$$

The top-horizontal map is the q-norm map for $q = 1$ associated to f, while the vertical maps are abelianization (6.59) and coabelianization (6.63). This is an instance of diagram (2.59).

There is a signed analogue which we omit.

Example 6.79. Let p be a species. View it as a trivial comonoid and trivial monoid, and apply the above discussion to $f := \mathrm{id}$. We obtain a morphism $g : \mathcal{T}_q(\mathsf{p}) \to \mathcal{T}_q^\vee(\mathsf{p})$ of q-bimonoids which coincides with the q-norm map (6.74). Only those (F, G)-matrix-components in which $\mathrm{s}(F) = \mathrm{s}(G)$ contribute.

Now let $q = 1$. Diagram (6.79) specializes to the left diagram in (6.75). Note that the morphism $\mathcal{S}(\mathsf{p}) \to \mathcal{S}^\vee(\mathsf{p})$ is the identity.

Exercise 6.80. Let p be a species. Put $\mathsf{c} := \mathcal{S}^\vee(\mathsf{p})$ and $\mathsf{a} := \mathcal{S}(\mathsf{p})$, the cofree cocommutative comonoid and the free commutative monoid on p, respectively. Check that:

(1) Applying (6.78) to the map $f : \mathcal{S}^\vee(\mathsf{p}) \twoheadrightarrow \mathsf{p} \hookrightarrow \mathcal{S}(\mathsf{p})$ yields the morphism $g : \mathcal{S}\mathcal{S}^\vee(\mathsf{p}) \to \mathcal{S}^\vee\mathcal{S}(\mathsf{p})$ in (6.37). An explicit formula is given in (3.16). (A more general fact is mentioned in the proof of Proposition C.34.)

(2) Applying (6.78) to the map $\mathrm{id} : \mathcal{S}^\vee(\mathsf{p}) \to \mathcal{S}(\mathsf{p})$ yields the morphism $\mathcal{S}\mathcal{S}^\vee(\mathsf{p}) \to \mathcal{S}^\vee\mathcal{S}(\mathsf{p})$ which evaluated on the Z-component

$$\bigoplus_{(\mathrm{X},\mathrm{Y}):\, \mathrm{Z}\le\mathrm{X}\le\mathrm{Y}} \mathsf{p}[\mathrm{Y}] \to \bigoplus_{(\mathrm{X}',\mathrm{Y}'):\, \mathrm{Z}\le\mathrm{X}'\le\mathrm{Y}'} \mathsf{p}[\mathrm{Y}']$$

sends a summand $\mathsf{p}[\mathrm{Y}]$ to all matching summands in the rhs.

Exercise 6.81. Check that the map (6.78) preserves isomorphisms if either c is a trivial comonoid or a is a trivial monoid. (Use a triangularity argument.) Use Exercise 6.80, item (2) to show that this is not true in general. A more precise result is given later in Exercise 9.60, see also Exercise 9.65.

Similar results can be obtained by replacing $\mathcal{S}$ by $\mathcal{T}_q$ as follows.

Exercise 6.82. Let p be a species. Put $\mathsf{c} := \mathcal{T}^\vee(\mathsf{p})$ and $\mathsf{a} := \mathcal{T}(\mathsf{p})$, the cofree comonoid and the free monoid on p, respectively. Check that:

(1) Applying (6.77) to the map $f : \mathcal{T}^\vee(\mathsf{p}) \twoheadrightarrow \mathsf{p} \hookrightarrow \mathcal{T}(\mathsf{p})$ yields the morphism $g : \mathcal{T}_q\mathcal{T}^\vee(\mathsf{p}) \to \mathcal{T}_q^\vee\mathcal{T}(\mathsf{p})$ in (3.7).

(2) Applying (6.77) to the map $\mathrm{id} : \mathcal{T}^\vee(\mathsf{p}) \to \mathcal{T}(\mathsf{p})$ yields the morphism $g : \mathcal{T}_q\mathcal{T}^\vee(\mathsf{p}) \to \mathcal{T}_q^\vee\mathcal{T}(\mathsf{p})$ which evaluated on the A-component

$$\bigoplus_{(F,G):\, A\le F\le G} \mathsf{p}[G] \to \bigoplus_{(F',G'):\, A\le F'\le G'} \mathsf{p}[G']$$

has matrix-components

$$g_{F,G,F',G'} := \begin{cases} \mathsf{p}[G] \xrightarrow{(\beta_q)_{G',G}} \mathsf{p}[G'] & \text{if } FG' = G \text{ and } F'G = G', \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 6.83. Suppose q is not a root of unity. Check that the map (6.77) preserves isomorphisms if either c is a trivial comonoid or a is a trivial monoid. (First do the case when both c and a are trivial following the proof of Proposition 6.75. This uses invertibility of the Varchenko matrix. Then deduce the general case using a triangularity argument.) A more general discussion is given later in Exercise 9.97.

For a comonoid c, let c_t denote the underlying species of c viewed as a trivial monoid. Similarly, for a monoid a, let a_t denote the underlying species of a viewed as a trivial comonoid. Specializing Exercise 6.83 first to $\mathrm{id} : \mathsf{c} \to \mathsf{c}_t$, and then to $\mathrm{id} : \mathsf{a}_t \to \mathsf{a}$, we obtain the following two results.

Proposition 6.84. *For q not a root of unity, and a comonoid c, the map of species $\mathcal{T}_q(\mathsf{c}) \to \mathcal{T}_q^\vee(\mathsf{c}_t)$ given on the A-component by*

$$\bigoplus_{F:\, F\ge A} \mathsf{c}[F] \to \bigoplus_{H:\, H\ge A} \mathsf{c}[H], \qquad x \mapsto \sum_{\substack{H\ge A\\ HF=H}} q^{\mathrm{dist}(F,H)}\, \beta_{H,FH} \Delta_F^{FH}(x) \tag{6.80}$$

for $x \in \mathsf{c}[F]$, is an isomorphism of q-bimonoids.

Dually:

Proposition 6.85. *For q not a root of unity, and a monoid a, the map of species $\mathcal{T}_q(\mathsf{a}_t) \to \mathcal{T}_q^\vee(\mathsf{a})$ given on the A-component by*

$$\bigoplus_{F:\, F\geq A} \mathsf{a}[F] \to \bigoplus_{G:\, G\geq A} \mathsf{a}[G], \qquad x \mapsto \sum_{\substack{G\geq A\\ FG=F}} q^{\operatorname{dist}(F,G)}\, \mu_G^{GF} \beta_{GF,F}(x) \tag{6.81}$$

for $x \in \mathsf{a}[F]$, is an isomorphism of q-bimonoids.

The map (6.80) is an instance of (6.12) applied to the canonical projection $f : \mathcal{T}_q(\mathsf{c}) \twoheadrightarrow \mathsf{c}_t$. Similarly, the map (6.81) is an instance of (6.4) applied to the canonical inclusion $f : \mathsf{a}_t \hookrightarrow \mathcal{T}_q^\vee(\mathsf{a})$.

A different proof of Propositions 6.84 and 6.85 is given in Exercise 9.94. Closely related results are given later in Section 14.5, see Exercise 14.80 in particular.

6.10. (Co)free graded (co)monoids on species

Recall from Section 5.2 the notions of graded and filtered q-bimonoids. We now study their relevance to universal constructions. Firstly, the free monoid and cofree comonoid on a species are graded via the Cauchy powers of that species. Further, they are graded as q-bimonoids under the q-deshuffle coproduct and q-shuffle product, respectively. The reason is that (de)concatenation and (de)shuffle send any summand such as $\mathsf{p}[F]$ to summands $\mathsf{p}[K]$, where K has the same rank as F.

The q-deshuffle coproduct has a uniqueness property in the graded setting. It is the unique coproduct on the free graded monoid on a species which yields a graded q-bimonoid. Dually, the q-shuffle product is the unique product on the cofree graded comonoid on a species which yields a graded q-bimonoid. As a consequence: The associated graded of any free q-bimonoid wrt the decomposable filtration has the q-deshuffle coproduct. Dually, the associated graded of any cofree q-bimonoid wrt the primitive filtration has the q-shuffle product. We also briefly mention commutative analogues of these results.

More generally, we consider the (co)free graded (co)monoid on a graded species. Further constructions such as the free graded q-bimonoid on a graded comonoid, and so on, can also be carried out in the same manner.

6.10.1. Free graded monoid on a species. Recall from (6.1) the species $\mathcal{T}(\mathsf{p})$ associated to a species p. Put

$$\mathcal{T}(\mathsf{p})_i := \mathsf{p}^i, \tag{6.82}$$

the i-th Cauchy power of p. This defines a grading on $\mathcal{T}(\mathsf{p})$ in view of (6.2). Moreover, it turns the free monoid $\mathcal{T}(\mathsf{p})$ into a graded monoid. In other words, the concatenation product component μ_A^F given in (6.3) sends $\mathsf{p}^i[F]$ to $\mathsf{p}^{i+\operatorname{rk}(F/A)}[A]$. Viewed as a graded monoid, $\mathcal{T}(\mathsf{p})$ satisfies the following universal property.

Theorem 6.86. *Let a be a graded monoid, p a species, $f : \mathsf{p} \to \mathsf{a}_1$ a map of species. Then there exists a unique morphism of graded monoids $\hat{f} : \mathcal{T}(\mathsf{p}) \to \mathsf{a}$*

such that the diagram

$$\begin{tikzcd} \mathcal{T}(\mathsf{p}) \arrow[r, dashed, "\hat{f}"] & \mathsf{a} \\ \mathsf{p} \arrow[u, hook] \arrow[r, "f"'] & \mathsf{a}_1 \arrow[u, hook] \end{tikzcd}$$

commutes.

The map $\hat{f}$ evaluated on the A-component, on the F-summand (which has degree $\mathrm{rk}(F/A)+1$), is given by

$$\mathsf{p}[F] \xrightarrow{f_F} \mathsf{a}_1[F] \xrightarrow{\mu_A^F} \mathsf{a}_{\mathrm{rk}(F/A)+1}[A]. \tag{6.83}$$

The map is degree-preserving, as required.

Proof. Apply Theorem 6.2 to get a unique morphism of monoids $\hat{f}$ given by (6.4). The extra step is to note that it is degree-preserving which was done above. $\square$

In a similar manner, for any scalar q, the q-bimonoid $\mathcal{T}_q(\mathsf{p})$ with concatenation product and q-deshuffle coproduct given in (6.39) is a graded q-bimonoid.

6.10.2. Cofree graded comonoid on a species. Dually, the cofree comonoid $\mathcal{T}^\vee(\mathsf{p})$ is a graded comonoid, with grading given by (6.82). The graded analogue of Theorem 6.10 is as follows.

Theorem 6.87. *Let* c *be a graded comonoid,* p *a species,* $f : \mathsf{c}_1 \to \mathsf{p}$ *a map of species. Then there exists a unique morphism of graded comonoids* $\hat{f} : \mathsf{c} \to \mathcal{T}^\vee(\mathsf{p})$ *such that the diagram*

$$\begin{tikzcd} \mathsf{c} \arrow[r, dashed, "\hat{f}"] \arrow[d, two heads] & \mathcal{T}^\vee(\mathsf{p}) \arrow[d, two heads] \\ \mathsf{c}_1 \arrow[r, "f"'] & \mathsf{p} \end{tikzcd}$$

commutes.

Explicitly, the map $\hat{f}$ is as follows. Evaluating on the A-component, into the F-summand, the map is

$$\mathsf{c}_{\mathrm{rk}(F/A)+1}[A] \xrightarrow{\Delta_A^F} \mathsf{c}_1[F] \xrightarrow{f_F} \mathsf{p}[F]. \tag{6.84}$$

The map is degree-preserving, as required.

Similarly, for any scalar q, the q-bimonoid $\mathcal{T}_q^\vee(\mathsf{p})$ with q-shuffle product and deconcatenation coproduct given in (6.40) is a graded q-bimonoid.

Exercise 6.88. Let c be a graded comonoid such that $\mathcal{P}(\mathsf{c}) = \mathsf{c}_1$. Deduce from Theorem 6.87 and Proposition 5.18 that there is an injective morphism of graded comonoids $\mathsf{c} \to \mathcal{T}^\vee(\mathsf{c}_1)$.

6.10.3. Uniqueness of the deshuffle coproduct. We now show that for any scalar q, if a graded q-bimonoid is free, then its coproduct is necessarily the q-deshuffle coproduct.

Proposition 6.89. *Consider the free graded monoid $\mathcal{T}(\mathsf{p})$ on a species p. Then for any scalar q, the q-deshuffle coproduct is the unique coproduct on $\mathcal{T}(\mathsf{p})$ which turns it into a graded q-bimonoid.*

PROOF. Suppose Δ is a coproduct which turns $\mathcal{T}(\mathsf{p})$ into a graded q-bimonoid. We want to show that Δ_A^G is given by (6.39). For $H \geq A$,

$$\Delta_A^G|_{\mathsf{p}[H]} = \Delta_A^G \mu_A^H|_{\mathsf{p}[H]} = \mu_G^{GH}(\beta_q)_{GH,HG}\Delta_H^{HG}|_{\mathsf{p}[H]}.$$

Since Δ is graded, $\Delta_H^{HG}|_{\mathsf{p}[H]} = 0$ unless $HG = H$. This yields the second alternative in (6.39). If $HG = H$, then $\Delta_H^{HG} = \mathrm{id}$, and the above calculation simplifies to the first alternative in (6.39). □

6.10.4. Uniqueness of the shuffle product. Dually, for any scalar q, if a graded q-bimonoid is cofree, then its product is necessarily the q-shuffle product:

Proposition 6.90. *Consider the cofree graded comonoid $\mathcal{T}^\vee(\mathsf{p})$ on a species p. Then for any scalar q, the q-shuffle product is the unique product on $\mathcal{T}^\vee(\mathsf{p})$ which turns it into a graded q-bimonoid.*

The proof is similar to that of Proposition 6.89, so we omit it. A more general result (with an independent proof) is given in Proposition 6.100 below.

6.10.5. Associated graded of a (co)free bimonoid. Consider the q-bimonoid $\mathcal{T}_q(\mathsf{p})$ with concatenation product and q-deshuffle coproduct given in (6.39). Its decomposable filtration is given in Proposition 6.57. Clearly, its associated graded wrt this filtration is isomorphic to itself, that is,

$$\mathrm{gr}_{\mathcal{D}}(\mathcal{T}_q(\mathsf{p})) \cong \mathcal{T}_q(\mathsf{p}) \tag{6.85}$$

as graded q-bimonoids.

Proposition 6.91. *Let h be any q-bimonoid which is isomorphic to $\mathcal{T}_q(\mathsf{p})$ as a monoid for some species p. Then $\mathrm{gr}_{\mathcal{D}}(\mathsf{h}) \cong \mathcal{T}_q(\mathsf{p})$ as graded q-bimonoids.*

PROOF. Use Proposition 6.89 and (6.85). □

In other words, for any free q-bimonoid h, its associated graded wrt the decomposable filtration is isomorphic to the free monoid on $\mathcal{Q}(\mathsf{h})$ equipped with the q-deshuffle coproduct. Recall that the deshuffle coproduct is cocommutative. Thus, for $q = 1$, the bimonoid $\mathrm{gr}_{\mathcal{D}}(\mathsf{h})$ is cocommutative. This is an illustration of the Milnor–Moore cocommutativity result (Proposition 5.65).

Dually, consider the q-bimonoid $\mathcal{T}_q^\vee(\mathsf{p})$ with q-shuffle product and deconcatenation coproduct given in (6.40). Its primitive filtration is given in Proposition 6.56. Clearly, its associated graded wrt this filtration is isomorphic to itself, that is,

$$\mathrm{gr}_{\mathcal{P}}(\mathcal{T}_q^\vee(\mathsf{p})) \cong \mathcal{T}_q^\vee(\mathsf{p}) \tag{6.86}$$

as graded q-bimonoids.

Proposition 6.92. *Let* h *be any q-bimonoid which is isomorphic to* $\mathcal{T}_q^\vee(\mathsf{p})$ *as a comonoid for some species* p*. Then* $\mathrm{gr}_{\mathcal{P}}(\mathsf{h}) \cong \mathcal{T}_q^\vee(\mathsf{p})$ *as graded q-bimonoids.*

PROOF. Use Proposition 6.90 and (6.86). □

In other words, for any cofree q-bimonoid h, its associated graded wrt the primitive filtration is isomorphic to the cofree comonoid on $\mathcal{P}(\mathsf{h})$ equipped with the q-shuffle product. Recall that the shuffle product is commutative. Thus, for $q = 1$, the bimonoid $\mathrm{gr}_{\mathcal{P}}(\mathsf{h})$ is commutative. This is an illustration of the Browder–Sweedler commutativity result (Proposition 5.62).

Exercise 6.93. Deduce that for any comonoid c and monoid a,

$$\mathrm{gr}_{\mathcal{D}}(\mathcal{T}_q(\mathsf{c})) \cong \mathcal{T}_q(\mathsf{c}_t) \quad \text{and} \quad \mathrm{gr}_{\mathcal{P}}(\mathcal{T}_q^\vee(\mathsf{a})) \cong \mathcal{T}_q^\vee(\mathsf{a}_t)$$

as graded q-bimonoids. Here c_t is c as a species but with the trivial coproduct, while a_t is a as a species but with the trivial product. Also check these facts directly.

6.10.6. Commutative aspects. Recall from (6.18) the species $\mathcal{S}(\mathsf{p})$ associated to a species p. Put

$$\mathcal{S}(\mathsf{p})_i := \mathsf{p}^{\bar{i}}, \tag{6.87}$$

the i-th commutative Cauchy power of p. This defines a grading on $\mathcal{S}(\mathsf{p})$ in view of (6.19). Moreover, with this grading, the bicommutative bimonoid $\mathcal{S}(\mathsf{p}) = \mathcal{S}^\vee(\mathsf{p})$ with product and coproduct given in (6.51) is graded. Let us temporarily denote the product and coproduct by μ and Δ, respectively.

Proposition 6.94. *Let* p *be a species. View* $\mathcal{S}(\mathsf{p})$ *as a graded monoid with product* μ*. Then* Δ *is the unique coproduct which turns it into a graded bicommutative bimonoid.*

Dually: view $\mathcal{S}(\mathsf{p})$ *as a graded comonoid with coproduct* Δ*. Then* μ *is the unique product which turns it into a graded bicommutative bimonoid.*

PROOF. This is the commutative analogue of Propositions 6.89 and 6.90, and can be proved in a similar manner by employing the bicommutative bimonoid axiom (2.26). □

Clearly, the associated graded of $\mathcal{S}(\mathsf{p})$ wrt either the decomposable or the primitive filtration is isomorphic to itself, that is,

$$\mathrm{gr}_{\mathcal{D}}(\mathcal{S}(\mathsf{p})) \cong \mathcal{S}(\mathsf{p}) \cong \mathrm{gr}_{\mathcal{P}}(\mathcal{S}(\mathsf{p})) \tag{6.88}$$

as graded bimonoids.

Proposition 6.95. *Let* h *be any bicommutative bimonoid. If* h *is isomorphic to* $\mathcal{S}(\mathsf{p})$ *as a monoid for some species* p*, then* $\mathrm{gr}_{\mathcal{D}}(\mathsf{h}) \cong \mathcal{S}(\mathsf{p})$ *as graded bimonoids. Dually, if* h *is isomorphic to* $\mathcal{S}(\mathsf{p})$ *as a comonoid for some species* p*, then* $\mathrm{gr}_{\mathcal{P}}(\mathsf{h}) \cong \mathcal{S}(\mathsf{p})$ *as graded bimonoids.*

PROOF. This is the commutative analogue of Propositions 6.91 and 6.92. It follows from Proposition 6.94 and (6.88). □

Exercise 6.96. Formulate signed analogues of the above results.

6.10.7. (Co)free graded (co)monoid on a graded species. Let p be a graded species. Then its k-th Cauchy power p^k is a graded species, with its degree i component defined by

$$\mathsf{p}^k_i[A] := \bigoplus_{\substack{F:\, F \geq A \\ \mathrm{rk}(F/A)=k-1}} \mathsf{p}_{i-k+1}[F]. \tag{6.89}$$

Note very carefully that the grading on p^k starts from k. This turns $\mathcal{T}(\mathsf{p})$ into a graded species in view of (6.2). The degree 1 component of $\mathcal{T}(\mathsf{p})$ is p_1. When p is concentrated in degree 1, p^k is concentrated in degree k. This was the situation considered in (6.82).

Recall that $\mathcal{T}(\mathsf{p})$ is a monoid under the concatenation product. The grading (6.89) turns it into a graded monoid. In fact, it is the *free graded monoid* on the graded species p. That is, any map of graded species from p to a graded monoid a extends uniquely to a morphism of graded monoids from $\mathcal{T}(\mathsf{p})$ to a. This is the graded analogue of Theorem 6.2. When p is concentrated in degree 1, we recover Theorem 6.86.

Similarly, for any graded comonoid c, the q-bimonoid $\mathcal{T}_q(\mathsf{c})$ with concatenation product and q-dequasishuffle coproduct given in (6.5) is a graded q-bimonoid. It satisfies the graded analogue of the universal property in Theorem 6.6.

Dually, for a graded species p, the comonoid $\mathcal{T}^\vee(\mathsf{p})$ under the deconcatenation coproduct is the *cofree graded comonoid* on p. That is, any map of graded species from a graded comonoid c to p lifts uniquely to a morphism of graded comonoids from c to $\mathcal{T}^\vee(\mathsf{p})$. This is the graded analogue of Theorem 6.10.

Similarly, for any graded monoid a, the q-bimonoid $\mathcal{T}^\vee_q(\mathsf{a})$ with deconcatenation coproduct and q-quasishuffle product given in (6.13) is a graded q-bimonoid. It satisfies the graded analogue of the universal property in Theorem 6.13.

Exercise 6.97. Check that: Lemmas 6.65 and 6.66 continue to hold for graded q-bimonoids.

Lemma 6.98. *Let h and k be graded q-bimonoids such that $\mathcal{P}(\mathsf{k}) = \mathsf{k}_1$. If $f : \mathsf{h} \to \mathsf{k}$ is a morphism of graded comonoids, then f is in fact a morphism of graded q-bimonoids.*

Proof. We check that f is a morphism of graded monoids. Equivalently, by Exercise 6.97, we check that the first diagram in Lemma 6.66 commutes. Both $f\mu$ and μf are morphisms of graded comonoids. Moreover, they are equal on the degree 1 component of $\mathcal{T}_q(\mathsf{h})$ which is h_1. So they are equal by Exercise 5.19, item (3). □

Exercise 6.99. We sketch a more direct proof of Lemma 6.98. Firstly, by Exercise 2.15, it suffices to check diagram (2.9) only for $A \lessdot F$. Moreover, in view of Exercise 5.17, it suffices to show that for any $i \geq 1$, $A \lessdot F$, $\mathrm{rk}(G/A) = i$, the diagram

$$\begin{tikzcd} \mathsf{h}_i[F] \arrow[r, "f_F"] \arrow[d, "\mu_A^F"'] & \mathsf{k}_i[F] \arrow[d, "\mu_A^F"] & \\ \mathsf{h}_{i+1}[A] \arrow[r, "f_A"'] & \mathsf{k}_{i+1}[A] \arrow[r, "\Delta_A^G"'] & \mathsf{k}_1[G] \end{tikzcd}$$

commutes. Check this diagram by considering two cases, namely, $GF = G$ and $GF > G$. In the latter case, use that $\mathrm{rk}(FG/F) \geq i$.

Proposition 6.100. *Let* c *be a graded comonoid such that* $\mathcal{P}(\mathsf{c}) = \mathsf{c}_1$. *If there exists a product on* c *which turns it into a graded* q*-bimonoid, then this product is unique.*

Proof. Suppose there are two such products on c. The identity map on c is a morphism of graded comonoids. Hence, it is also a morphism of graded q-bimonoids by Lemma 6.98. This implies that the two products are equal, as required. □

Observe that Proposition 6.90 and the second part of Proposition 6.94 follow as a consequence.

Exercise 6.101. Let k be a graded q-bimonoid such that $\mathcal{P}(\mathsf{k}) = \mathsf{k}_1$. Show that there is an injective morphism of graded q-bimonoids $\mathsf{k} \to \mathcal{T}_q^\vee(\mathsf{k}_1)$. (Use Exercise 6.88 and Lemma 6.98.)

6.11. Free partially bicommutative bimonoids

We briefly discuss universal constructions in the context of partial-support relations. One of them unifies $\mathcal{T}$ and $\mathcal{S}$, while another unifies $\mathcal{T}_0$ and $\mathcal{S}$.

6.11.1. $\sim$-bicommutative bimonoids. The noncommutative and commutative universal constructions can be understood in a unified manner using partial-support relations. Recall from Theorem 3.28 that for any partial-support relations $\sim$ and $\sim'$ on faces, we have a bimonad $(\mathcal{T}_\sim, \mathcal{T}^\vee_{\sim'}, \lambda)$. Applying Theorem C.32 yields two functors, one from $\sim'$-cocommutative comonoids and another from $\sim$-commutative monoids. They both go to $\sim$-commutative and $\sim'$-cocommutative bimonoids. We focus on the first functor.

Suppose first that $\sim'$ is finest. This yields the following. For any comonoid c, we have a $\sim$-commutative bimonoid $\mathcal{T}_\sim(\mathsf{c})$. Moreover, there are morphisms of bimonoids

$$\mathcal{T}(\mathsf{c}) \twoheadrightarrow \mathcal{T}_\sim(\mathsf{c}) \twoheadrightarrow \mathcal{S}(\mathsf{c}). \tag{6.90}$$

The first map is the $\sim$-abelianization map, while the composite is the abelianization map. The general case can be obtained from this one by noting that if c is $\sim'$-cocommutative, then so is $\mathcal{T}_\sim(\mathsf{c})$.

Now suppose $\sim$ and $\sim'$ coincide. For a $\sim$-cocommutative comonoid c, the product and coproduct of $\mathcal{T}_\sim(\mathsf{c})$ can be described as follows.

For $\mathrm{z} \leq \mathrm{x}$,

$$\begin{tikzcd}
\mathcal{T}_{\sim}(\mathsf{c})[\mathrm{x}] \arrow[r, dashed, "\mu_{\mathrm{z}}^{\mathrm{x}}"] & \mathcal{T}_{\sim}(\mathsf{c})[\mathrm{z}] \\
\mathsf{c}[\mathrm{y}] \arrow[u, hook] \arrow[r, "\mathrm{id}"'] & \mathsf{c}[\mathrm{y}] \arrow[u, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{T}_{\sim}(\mathsf{c})[\mathrm{z}] \arrow[rr, dashed, "\Delta_{\mathrm{z}}^{\mathrm{x}}"] & & \mathcal{T}_{\sim}(\mathsf{c})[\mathrm{x}] \\
\mathsf{c}[\mathrm{y}] \arrow[u, hook] \arrow[r, "\Delta_{\mathrm{y}}^{\mathrm{yx}}"'] & \mathsf{c}[\mathrm{yx}] \arrow[r, "\beta_{\mathrm{xy},\mathrm{yx}}"'] & \mathsf{c}[\mathrm{xy}]. \arrow[u, hook]
\end{tikzcd} \tag{6.91}$$

The first formula is clear. The second formula can be derived by following definitions and employing (3.33).

Observe that when $\sim$ is finest, (6.91) reduces to (6.3) and (6.5) for $q = 1$, while when $\sim$ is coarsest, it reduces to (6.20) and (6.22).

Exercise 6.102. For a $\sim$-commutative monoid a, explicitly describe the product and coproduct of $\mathcal{T}^{\vee}_{\sim}(\mathsf{a})$ along the lines of (6.91).

Exercise 6.103. For a partial-support relation $\sim$ on faces, describe the primitive filtration of $\mathcal{T}^{\vee}_{\sim}(\mathsf{p})$ and the decomposable filtration of $\mathcal{T}_{\sim}(\mathsf{p})$ using $\sim$-commutative Cauchy powers. This will unify Propositions 6.56 and 6.58, and Propositions 6.57 and 6.59.

6.11.2. 0-$\sim$-bicommutative bimonoids. Apply Theorem C.32 to the bimonad $(\mathcal{T}_{\sim}, \mathcal{T}^{\vee}_{\sim}, \lambda_0)$ given by Theorem 3.30. This yields a functor from the category of $\sim$-cocommutative comonoids to the category of 0-$\sim$-bicommutative bimonoids. It is the left adjoint of the forgetful functor. We denote it by $\mathcal{T}_{0,\sim}$ to avoid confusion with the functor $\mathcal{T}_{\sim}$ considered in Section 6.11.1.

Suppose $\sim$ is geometric. For a $\sim$-cocommutative comonoid c, the product and coproduct of $\mathcal{T}_{0,\sim}(\mathsf{c})$ are as follows.

For $\mathrm{z} \leq \mathrm{x}$,

$$\big(\mu_{\mathrm{z}}^{\mathrm{x}} : \mathsf{c}[\mathrm{w}] \to \mathsf{c}[\mathrm{w}']\big) = \begin{cases} \mathrm{id} & \text{if } \mathrm{w}' = \mathrm{w}, \\ 0 & \text{otherwise,} \end{cases} \tag{6.92}$$

for each $\mathrm{x} \leq \mathrm{w}$ and $\mathrm{z} \leq \mathrm{w}'$.

For $\mathrm{z} \leq \mathrm{y}$,

$$\big(\Delta_{\mathrm{z}}^{\mathrm{y}} : \mathsf{c}[\mathrm{w}] \to \mathsf{c}[\mathrm{w}']\big) = \begin{cases} \Delta_{\mathrm{w}}^{\mathrm{w}'} & \text{if } \mathrm{w}' = \mathrm{wy} = \mathrm{yw}, \\ 0 & \text{otherwise,} \end{cases} \tag{6.93}$$

for each $\mathrm{z} \leq \mathrm{w}$ and $\mathrm{y} \leq \mathrm{w}'$. In view of Lemma 1.8, the first alternative can be equivalently written as $\mathrm{w}' = \mathrm{wy}$ or $\mathrm{w}' = \mathrm{w} \vee \mathrm{y}$.

The first formula (6.92) is clear. The second formula (6.93) can be derived by following definitions and employing (3.35).

Observe that when $\sim$ is finest, (6.92) and (6.93) reduce to (6.9) and (6.10), while when $\sim$ is coarsest, they reduce to (6.20) and (6.22).

By precomposing $\mathcal{T}_{0,\sim}$ with the trivial comonoid functor, we obtain a functor from the category of species to the category of 0-$\sim$-bicommutative bimonoids. We continue to denote this by $\mathcal{T}_{0,\sim}$. It is now the left adjoint of the primitive part functor.

For a species p, the product and coproduct of $\mathcal{T}_{0,\sim}(\mathsf{p})$ are given by concatenation and deconcatenation:

For $\mathrm{z} \leq \mathrm{x}$,

$$\left(\mu_{\mathrm{z}}^{\mathrm{x}} : \mathsf{p}[\mathrm{w}] \to \mathsf{p}[\mathrm{w}']\right) = \begin{cases} \mathrm{id} & \text{if } \mathrm{w}' = \mathrm{w}, \\ 0 & \text{otherwise}, \end{cases} \tag{6.94}$$

for each $\mathrm{x} \leq \mathrm{w}$ and $\mathrm{z} \leq \mathrm{w}'$.

For $\mathrm{z} \leq \mathrm{y}$,

$$\left(\Delta_{\mathrm{z}}^{\mathrm{y}} : \mathsf{p}[\mathrm{w}] \to \mathsf{p}[\mathrm{w}']\right) = \begin{cases} \mathrm{id} & \text{if } \mathrm{w}' = \mathrm{w}, \\ 0 & \text{otherwise}, \end{cases} \tag{6.95}$$

for each $\mathrm{z} \leq \mathrm{w}$ and $\mathrm{y} \leq \mathrm{w}'$.

Exercise 6.104. Dually, construct the functor $\mathcal{T}^{\vee}_{0,\sim}$ from the category of species to the category of 0-$\sim$-bicommutative bimonoids. Show that $\mathcal{T}_{0,\sim} = \mathcal{T}^{\vee}_{0,\sim}$ provided $\sim$ is geometric. When $\sim$ is finest, it specializes to (6.45), and when $\sim$ is coarsest, it specializes to (6.50).

Notes

The constructions of this chapter are motivated by classical universal constructions on vector spaces and on Joyal species.

Bialgebras. To any vector space V, one can associate the Hopf algebras listed in Table 6.2. A summary of these Hopf algebras is given in [**18**, Section 2.6].

TABLE 6.2. Tensor algebra, shuffle algebra and their relatives.

Hopf algebra	product	coproduct
tensor algebra $\mathcal{T}(V)$	concatenation	deshuffle
shuffle algebra $\mathcal{T}^{\vee}(V)$	shuffle	deconcatenation
q-analogue of tensor algebra $\mathcal{T}_q(V)$	concatenation	q-deshuffle
q-analogue of shuffle algebra $\mathcal{T}^{\vee}_q(V)$	q-shuffle	deconcatenation
$\mathcal{T}_0(V) = \mathcal{T}^{\vee}_0(V)$	concatenation	deconcatenation
symmetric algebras $\mathcal{S}(V)$, $\mathcal{S}^{\vee}(V)$		
exterior algebras $\mathcal{E}(V)$, $\mathcal{E}^{\vee}(V)$		

All these Hopf algebras can be viewed as classical analogues of the objects discussed in Sections 6.4 and 6.5 with the species p playing the role of V, see also Table 6.1. The case when V is a positively graded vector space is closer to our setting. The tensor, symmetric, exterior algebras are discussed in many algebra textbooks (usually without any mention of their Hopf structures), see for instance [**925**].

More generally, to any positively graded algebra A and positively graded coalgebra C, one can associate the Hopf algebras listed in Table 6.3. When A has the trivial product and C the trivial coproduct, we recover the Hopf algebras in Table 6.2. These Hopf algebras can be viewed as classical analogues of the bimonoids

TABLE 6.3. Hopf algebras with (de)quasishuffle (co)product.

Hopf algebra	product	coproduct
$\mathcal{T}(C)$	concatenation	dequasishuffle
$\mathcal{T}^\vee(A)$	quasishuffle	deconcatenation
$\mathcal{T}_q(C)$	concatenation	q-dequasishuffle
$\mathcal{T}_q^\vee(A)$	q-quasishuffle	deconcatenation
$\mathcal{S}(C)$, $\mathcal{S}^\vee(A)$		
$\mathcal{E}(C)$, $\mathcal{E}^\vee(A)$		

$\mathcal{T}_q(\mathsf{c})$, $\mathcal{S}(\mathsf{c})$, $\mathcal{E}(\mathsf{c})$ in Sections 6.1.2, 6.3.2, 6.3.5, respectively, and of the bimonoids $\mathcal{T}_q^\vee(\mathsf{a})$, $\mathcal{S}^\vee(\mathsf{a})$, $\mathcal{E}^\vee(\mathsf{a})$ in Sections 6.2.2, 6.3.4, 6.3.5, respectively.

Tensor algebra. For the tensor Hopf algebra $\mathcal{T}(V)$, the product is concatenation and coproduct is deshuffle. The deshuffle coproduct of $\mathcal{T}(V)$ is given for instance by Schlessinger and Stasheff [**807**, page 315], Benson [**98**, Sections 2.2.4 and 2.4.1], Patras [**740**, Section 5], Loday and Vallette [**605**, Formula (2.1)], [**606**, Appendix A.5], [**611**, First statement in Proposition 1.3.2], Kassel, Rosso, Turaev [**517**, Theorem III.2.4], [**518**, Chapter 2, Example 1.5], Guichardet [**391**, Section II.1.7], Majid [**632**, Proposition 5.1.4], Klimyk and Schmüdgen [**535**, Section 1.2.6, Example 8], Fresse [**320**, Section 1.2.11], [**324**, page 241], Smirnov [**834**, Section 3.2, page 52], Michaelis [**688**, Proposition 3.31 and Remark 3.32]. It is implicit in [**149**, Exercise 3 on page 647], [**771**, Exercise 5.3.9].

Shuffle algebra. Dually, we have the shuffle Hopf algebra $\mathcal{T}^\vee(V)$ whose product is shuffle and coproduct is deconcatenation. Shuffle algebras came up in work of Cartan [**188**, Section 8.4, page 8-07] and Ree [**774**, Section 1]. The coalgebra structure is not considered in these references. Shuffle Hopf algebras occupy an entire chapter in Sweedler's book [**867**, Chapter XII]. Other early references are by Chen [**207**, Theorem 1.8], André [**29**, page 25], Nichols [**720**, Section 1], Radford [**770**, Section 1.1].

Shuffle Hopf algebras are treated in depth in Reutenauer's book [**777**, Sections 1.4, 1.5 and Chapter 6]. Some other references related to $\mathcal{T}^\vee(V)$ are by Schlessinger and Stasheff [**807**, pages 315 and 316], Block [**121**, page 282], Benson [**97**], [**98**, Section 2.4.2], Gerstenhaber and Schack [**349**, Section 2], Shnider and Sternberg [**825**, Section 3.8], Kassel [**517**, Exercise III.8.5, item (d)], Loday and Vallette [**606**, Appendix A.6], [**611**, Second statement in Proposition 1.3.2], Blute and Scott [**125**, Section 7], Smirnov [**834**, Section 3.2, page 52], Michaelis [**688**, page 724], Kharchenko [**527**, Section 4.1], Hazewinkel, Gubareni, Kirichenko [**428**, Example 3.4.6], Radford [**771**, Definition 7.11.5]. An example is considered by Connes and Marcolli [**221**, Definition 1.96]. See also the references below for the more general quasishuffle product.

The notion of a shuffle appears in early work of Eilenberg and Mac Lane [**285**, Section 5], [**619**, pages 10 and 11], [**622**, Chapter VIII, Theorem 8.8]. Other early reference to shuffles are by Chen, Fox, Lyndon [**617**, page 203], [**208**, pages 82 and 95], Schützenberger [**820**, Section IV]. See also [**353**, Section 2]. In the context of card shuffling, see the references in [**21**, Notes to Chapter 12].

The deconcatenation coproduct also appears in the bar construction, see for instance, the expositions by Stasheff [**847**, page 24], Hain [**404**, Definition 2.20], [**405**, Definition 5.13], [**406**, Section 1], Kane [**508**, Section 7.1, page 58], Félix, Halperin, Thomas [**302**, Example 4.6], [**303**, page 268], Loday and Vallette [**611**, Sections 1.2.6 and 2.2.1]. The term 'tensor coalgebra' is sometimes used for the deconcatenation coproduct.

q-analogues. The Hopf algebras $\mathcal{T}(V)$ and $\mathcal{T}^\vee(V)$ have q-analogues denoted $\mathcal{T}_q(V)$ and $\mathcal{T}_q^\vee(V)$. They are q-bialgebras in the sense of [**18**, Section 2.3]. The coproduct of $\mathcal{T}_q(V)$ is q-deshuffle, while the product of $\mathcal{T}_q^\vee(V)$ is q-shuffle. The q-shuffle product is considered by Duchamp, Klyachko, Krob, Thibon [**269**, Section 4.1]. For more general considerations, see the discussion under quantum shuffles below.

The classical analogues of Proposition 6.100 and Exercise 6.101 in the case $q = 1$ are contained in a result of Sweedler [**867**, Theorem 12.1.4]. The analogue of Exercise 6.88 is given in [**867**, Proposition 12.1.2], while the analogue of Lemma 6.98 is given in [**867**, Theorem 11.2.5, item (b)]. The analogues of the closely related Propositions 6.90 and 6.92 in the case $q = 1$ are given in [**22**, Propositions 1.4 and 1.5]. A more general context for the analogue of Exercise 6.101 is considered by Kharchenko [**528**, Proposition 3.3].

0-*analogue.* The q-bialgebras $\mathcal{T}_q(V)$ and $\mathcal{T}_q^\vee(V)$ coincide when $q = 0$. The product is concatenation, while the coproduct is deconcatenation. This is an example of a 0-bialgebra. It is explicitly considered by Loday and Ronco [**610**, Proposition-Notation 2.3], [**608**, Section 4.2.1]. It is also mentioned on [**18**, page 47].

The fact that concatenation and deconcatenation do not define a bialgebra structure is explicitly stated by Bourbaki [**149**, Remark on page 587]. Michaelis makes the interesting remark that this is in essence because concatenation and deconcatenation encode the same information [**688**, Note on page 724].

Symmetric algebra and exterior algebra. In connection to the q-analogues $\mathcal{T}_q(V)$ and $\mathcal{T}_q^\vee(V)$, one has the symmetric Hopf algebras $\mathcal{S}(V)$ and $\mathcal{S}^\vee(V)$ in the case $q = 1$, and the exterior Hopf algebras $\mathcal{E}(V)$ and $\mathcal{E}^\vee(V)$ in the case $q = -1$. The algebra $\mathcal{S}^\vee(V)$ along its divided power structure is considered by Cartan [**188**, Section 8.4, Proposition 4]. As a bialgebra, it is the same as the bialgebra of divided powers $\Gamma(M)$ considered by Cartier [**192**, page 3-12]; its cofreeness property is given later in [**192**, Proposition 7 on page 3-14]. See also the discussion in [**427**, Section E.6.4], [**682**, Section II.1]. The map $\mathcal{S}^\vee(V) \hookrightarrow \mathcal{T}^\vee(V)$ is given in [**766**, Appendix B, (3.4)]. The Hopf algebras $\mathcal{T}(V)$ and $\mathcal{S}(V)$ along with the map $\mathcal{T}(V) \twoheadrightarrow \mathcal{S}(V)$ are discussed in [**228**, Section 4.3]. The Hopf algebras $\mathcal{T}(V)$, $\mathcal{S}(V)$, $\mathcal{E}(V)$ are discussed in [**171**, Sections 15.12, 15.13, 15.14]. However, there is an error in this reference: the coproduct of $\mathcal{T}(V)$ is incorrectly given as deconcatenation instead of deshuffle. Symmetric and exterior algebras are reviewed in [**288**, Appendix 2].

Norm map. The q-norm map $\kappa_q : \mathcal{T}_q(V) \to \mathcal{T}_q^\vee(V)$ is given in our monograph [**18**, Formula (2.67)]. This is the analogue of (6.74). The case $q = 1$ is present in earlier work of Benson [**98**, Sections 2.3.4 and 2.4.5]. The analogues of diagrams (6.75) are given in [**18**, Diagrams (2.66) and (2.68)]. Diagram (2.66) in [**18**] is also discussed in some form in the book of Bonfiglioli and Fulci [**130**, Theorem 10.19]. The fact that κ_q is an isomorphism when q is not a root of unity is shown by Duchamp, Klyachko, Krob, Thibon [**269**, Proposition 4.5], this is the analogue of Proposition 6.75. For classical generalizations, see [**18**, Theorem 16.19 and Example 16.31].

Hopf algebras with dequasishuffle coproduct. We now turn to the entries in Table 6.3. To any positively graded coalgebra C, one can associate the Hopf algebra $\mathcal{T}(C)$

whose product is concatenation and coproduct is dequasishuffle. It has a commutative counterpart $\mathcal{S}(C)$. These objects appear in work of Moore [**707**, pages 1 and 2], and later in work of Newman [**716**, Remarks 2.3 and 2.5], [**717**, Sections 1.13 and 1.14]. Moore or Newman, however, do not explicitly work out the dequasishuffle coproduct. The analogue of Lemma 6.8 says that $\mathcal{T}(C)$ is cocommutative iff C is cocommutative; this is given in [**707**, Proposition 5, item (3)].

The special case $\mathcal{T}(\mathcal{T}^\vee(V)_+)$ is considered by Manchon [**636**, Section I.2], where $\mathcal{T}^\vee(V)_+$ is the positive part of $\mathcal{T}^\vee(V)$ with the deconcatenation coproduct (which is not cocommutative). He conjectures that the elements in [**636**, Theorem I.5] linearly span the primitive part of $\mathcal{T}(\mathcal{T}^\vee(V)_+)$; this is related to Question 6.62. A more recent reference for $\mathcal{T}(C)$ is [**18**, second construction in Section 2.6.5]. For related work, see [**159**].

The Hopf algebra $\mathcal{T}(C)$ is considered by Hoffman [**449**, Section 4] as the dual to the quasishuffle algebra $\mathcal{T}^\vee(A)$, see below. This point of view is further developed in [**174**, Theorem 2]. Condition (6.72) for $q = 1$ can be seen in analogy with [**174**, Lemma 2, item (1)], where it is referred to as the φ-extended Friedrichs criterion. An earlier reference is by Minh [**698**, Lemma 2, item (1)].

Hopf algebras with quasishuffle product. Dually, to any positively graded algebra A, one can associate the Hopf algebra $\mathcal{T}^\vee(A)$ whose product is quasishuffle and coproduct is deconcatenation. The term 'quasishuffle algebra' is often used for $\mathcal{T}^\vee(A)$. It has a cocommutative counterpart $\mathcal{S}^\vee(A)$. These objects also appear in Moore's work [**707**, page 3], but without explicit consideration of the quasishuffle product. The analogue of the bimonoid morphism $\mathcal{S}\mathcal{S}^\vee(\mathsf{p}) \to \mathcal{S}^\vee\mathcal{S}(\mathsf{p})$ in (6.37) is given on [**707**, page 5], see also Exercise 6.80, item (1).

The Hopf algebra $\mathcal{T}^\vee(A)$ is considered later by Newman and Radford [**718**, Section 1]; the quasishuffle product is explicitly described in their remark on [**718**, page 1032]. The analogue of Lemma 6.15 says that $\mathcal{T}^\vee(A)$ is commutative iff A is commutative; this is given in [**718**, Proposition 1.5, item (a)].

The quasishuffle product also appears in some form in independent work of Cartier on Baxter algebras [**198**, Formulas (7), (8), (9)]. He is in a setup analogous to Section 6.4.6 with $q = 1$, $\alpha = -1$. The exponent of α indicates the number of merges that take place in the quasishuffle. Let us write $\mathcal{T}_q^\vee(A_\alpha)$ for the analogue of the q-bimonoid $\mathcal{T}_q^\vee(\mathsf{a}_\alpha)$.

After Newman and Radford, the Hopf algebra $\mathcal{T}^\vee(A)$ is independently studied by Hoffman [**449**, Theorem 3.1]. (We mention that he assumes A to be commutative.) He also considers the deformation $\mathcal{T}_q^\vee(A)$ but only as a graded algebra [**449**, Theorem 5.1]. The analogue of (6.81) is his map Φ_q on [**449**, page 59]. He shows this to be an algebra isomorphism when q is not a root of unity in [**449**, Theorem 5.4]. This can be viewed as the analogue of Proposition 6.85 (without the part about coproducts). The construction of $\mathcal{T}^\vee(A)$ also appeared in work of Hazewinkel [**423**, Section 12] under the name 'generalized overlapping shuffle algebra'. Guo and Keigher consider the algebra $\mathcal{T}^\vee(A_\alpha)$ under the name 'mixable shuffle algebra' [**394**, Theorem 3.5].

Later references to $\mathcal{T}^\vee(A)$ are by Kreimer [**547**, Section 2], Ebrahimi-Fard and Guo [**272**, Section 2, Theorem 2.5], Loday and Ronco [**610**, Examples 1.7, item (b)], [**607**, Section 1], Manchon and Paycha [**639**, Section 4, page 4658], Radford [**771**, Section 7.11], Novelli, Patras, Thibon [**725**, Lemma 5.2] and in our monograph [**18**, Section 2.6.6]. Ihara, Kajikawa, Ohno, Okuda [**476**, page 189] consider both $\mathcal{T}^\vee(A)$ and $\mathcal{T}^\vee(A_{-1})$ but only as algebras. The analogue of isomorphism (6.49) for $\alpha = -1$ is given in [**476**, Proposition 2]. Hoffman and Ihara [**450**, Theorem 4.2] consider the Hopf algebras $\mathcal{T}^\vee(A)$ and $\mathcal{T}^\vee(A_{-1})$. The analogue of isomorphism (6.49) for $\alpha = -1$ is given in [**450**, Proposition 3.1]. The analogue of Lemma 6.69 pertaining to $\mathcal{T}^\vee(\mathsf{a})$ is given in [**450**, Proposition 4.2].

In the context of stochastic integrals, quasishuffles appeared in work of Gaines [**337**, Proposition 2.3], [**338**, Proposition 3], Li and Liu [**596**, Lemma 2.2], and independently in work of Hudson with Parthasarathy, Cohen, Eyre [**469**, Formula (4.3) and remarks after its proof], [**216**, Formula (1.3)], [**299**, Theorem 5.1]. The Hopf structure appears later in the paper of Hudson and Pulmannová [**471**, Section II], see also [**467**, Section 3], [**468**, Sections 6.2 and 6.3]. In these references, the terms 'Itô shuffle product' and 'sticky shuffle product' are used for the quasishuffle product. The deformation $\mathcal{T}^\vee(A_\alpha)$ is considered in [**471**, page 2091], [**467**, Section 4], while its signed analogue $\mathcal{T}^\vee_{-1}(A_\alpha)$ is considered in [**466**, Sections 2 and 3]. An earlier reference for $\mathcal{T}^\vee_{-1}(A)$ (as an algebra) is [**299**, Chapter 6]. For more in this direction, see [**226**], [**227**], [**273**], [**274**].

An important example of $\mathcal{T}^\vee(A)$ is the Hopf algebra of quasisymmetric functions which is discussed in the Notes to Chapter 7. For more examples, see for instance [**266**, Table 1]. More information on shuffles and quasishuffles can be found in the book by Guo [**393**, Section 3.1].

Quantum shuffles and quasishuffles. The notion of a q-shuffle generalizes further to that of a quantum shuffle. It was defined by Green [**371**, Section 4.3] and Rosso [**789**, Proposition 6], [**790**, Theorem and Definition 1], [**791**, Proposition 9]. For similar considerations, see our monograph [**18**, Examples 19.46, 19.47, 20.21] and works of Leclerc [**588**, Section 2.5], Lebed [**586**, Chapter 2], [**587**, Definition 2.8 and Theorem 1], Kharchenko [**529**, Sections 6.2 and 6.6], Terwilliger [**878**, Formulas (9) and (10)]. This notion extends to the quasishuffle setting. The main object here is the 'quantum quasishuffle algebra'. It appeared in works of Jian, Rosso, Zhang [**487**, Example 4.18], [**488**, Section 2]. It includes the 'q-quasishuffle algebra' $\mathcal{T}^\vee_q(A)$ considered by Hoffman as a special case. For later work, see [**485**, Section 2], [**301**, Section 2.3.1], [**300**, Section 4], [**486**, Section 3].

Shuffles, quasishuffles and Tits product. Recall that we defined $\mathcal{T}(\mathsf{c})$ and $\mathcal{T}^\vee(\mathsf{a})$ in terms of the Tits product. The precise connection of shuffles and quasishuffles to the Tits product is explained in our monographs [**17**, page 125, Formula (6.33) and Remark], [**18**, Section 10.11.3], [**21**, Sections 6.5.4 and 6.5.5]. The fact that q-deformations relate to the q-distance function on the braid arrangment was brought forth in [**18**]. This is manifest in the definitions of $\mathcal{T}_q(\mathsf{c})$ and $\mathcal{T}^\vee_q(\mathsf{a})$ given in the present work.

Freeness and cofreeness. The classical analogue of Theorem 6.10 is the cofreeness property of $\mathcal{T}^\vee(V)$. It is given in Sweedler's book [**867**, Theorem 12.0.2], see also [**718**, Theorem 1.1]. The classical analogue of Theorem 6.23 is the cofreeness property of $\mathcal{S}^\vee(V)$, see [**867**, Theorem 12.2.5], [**433**, Section 4.2], [**872**, Section 1.5.2]. For more information on cofree coalgebras, see [**17**, Section 3.1.1], [**18**, Section 2.6.4], [**36**, Theorems 2.2.2 and 5.2.10], [**71**, Section 4], [**124**], [**228**, Section 1.6], [**314**], [**424**], [**711**], [**766**, Appendix B, Proposition 4.1], [**771**, Sections 2.7 and 4.5], [**867**, Section 6.4]. For some classical generalizations, see [**37**], [**123**], [**373**, Chapter 1], [**374**], [**835**, Theorems 3.8 and 4.10].

The analogue of Theorem 6.6 for $q = 1$ and Theorem 6.21 is [**707**, Propositions 1 and 2], [**716**, Definition 2.2]. Dually, the analogue of Theorem 6.13 for $q = 1$ and Theorem 6.25 is [**707**, Propositions 3 and 4]. The former is also given in [**718**, Theorem 1.4].

For freeness of the shuffle and quasishuffle algebras, see the Notes to Chapter 16.

Nonconnected setting. The present monograph only pertains to the connected case. We mention that the classical free objects also exist in the nonconnected setting. Early references for the free bialgebra and the free Hopf algebra on an algebra or a coalgebra are by Stolberg [**853**, Theorem 4.4 on page 66], Sweedler [**867**, Proposition 3.2.4, and pages 134 and 135], Takeuchi [**871**, Sections 1 and 11]. More recent references are [**7**, Section 2], [**18**, first construction in Section 2.6.5], [**757**, Section 3.2], [**759**, Theorem 10 and Fact 11], [**771**, Sections 7.5 and 7.10]. A categorical setting is considered in [**18**, Section 6.10.2], [**758**, Section 3.4].

Bimonoids in Joyal species. Joyal described the free monoid and the free commutative monoid on a species [**498**, Section 7, Examples 42 and 43]. These and related constructions such as the free bimonoid on a comonoid are discussed in detail in our monograph [**18**, Chapter 11]. The free monoid $\mathcal{T}(\mathbf{X}_V)$ in Example 11.11 of this reference appears in earlier work of Barratt [**74**, Definition 3].

The exposition here may be viewed as a generalization of these constructions from the braid arrangement to arbitrary arrangements. The precise dictionary is as follows. Theorems 6.1 and 6.2 on the free monoid correspond to [**18**, Theorems 11.3 and 11.4]. Theorems 6.5 and 6.6 correspond to [**18**, Theorems 11.9 and 11.10] for $q = 1$ with the general case mentioned in [**18**, Section 11.7.1]. Theorem 6.10 on the cofree comonoid corresponds to [**18**, Theorem 11.19]. Theorem 6.13 corresponds to [**18**, Theorem 11.23] for $q = 1$ with the general case mentioned in [**18**, Section 11.7.2]. Theorems 6.17 and 6.21 on the free commutative monoid correspond to [**18**, Theorems 11.13 and 11.14]. Theorems 6.23 and 6.25 on the cofree cocommutative comonoid corresponds to [**18**, Theorems 11.26 and 11.27]. Section 6.3.5 on the signed analogues corresponds to [**18**, Section 11.7.3]. The adjunction between $\mathcal{T}_q$ and $\mathcal{P}$ for $q = 1$ in Theorem 6.30 corresponds to [**18**, first adjunction in Proposition 11.45]. Similarly, the adjunction between $\mathcal{S}$ and $\mathcal{P}$ in Theorem 6.43 corresponds to [**18**, second adjunction in Proposition 11.45]. Section 6.7 corresponds to [**18**, Section 11.9.4]. Section 6.6 on abelianization corresponds to [**18**, Section 11.6.2]. Section 6.9.1 on the q-norm transformation corresponds to [**18**, Section 11.7.5], Proposition 6.75 corresponds to [**18**, Theorem 11.35], diagrams (6.75) correspond to [**18**, Diagrams (11.26) and (11.31)]. The analogue of Section 6.9.3 on the generalized norm was not considered in [**18**].

Important examples of these constructions are discussed in [**18**, Chapter 12], both from a combinatorial and geometrical perspective. See the Notes to Chapter 7 for more information on this. We mention that the geometric perspective on these examples originates in [**17**, Sections 6.5 and 6.8].

The free monoid and free commutative monoid on Joyal species are also discussed in [**19**, Sections 6 and 7], [**675**, Example 3.22].

Bimonoids for hyperplane arrangements. Universal constructions for bimonoids for arrangements appear here for the first time.

CHAPTER 7

Examples of bimonoids

We discuss some basic and important examples of species. These include the exponential species E, species of chambers Γ, species of faces Σ, species of flats Π, species of top-nested faces $\widehat{\mathsf{Q}}$, species of top-lunes $\widehat{\Lambda}$, species of bifaces J. They are constructed from well-known objects associated to an arrangement such as faces, flats, chambers, top-nested faces, top-lunes, bifaces. The exponential species E is simpler and has all its components equal to the base field.

Each of these species carries the structure of a bimonoid. Some of them also admit q-analogues. These bimonoids along with their basic properties are summarized in Table 7.1 below. The table refers to two more species, namely, x and F. The species x is even simpler than E; we call it the species characteristic of chambers.

TABLE 7.1. Examples of bimonoids in species.

Bimonoid	Commutativity	Freeness
exponential bimonoid E	bicommutative	$\mathcal{S}(\mathsf{x})$
signed exponential bimonoid E^-	signed bicommutative	$\mathcal{E}(\mathsf{x})$
bimonoid of flats Π	bicommutative	$\mathcal{S}(\mathsf{E})$
bimonoid of top-lunes $\widehat{\Lambda}$	commutative	$\mathcal{S}(\Gamma^*)$
q-bimonoid of chambers Γ_q	cocommutative for $q = 1$ and signed cocommutative for $q = -1$	$\mathcal{T}_q(\mathsf{x})$
q-bimonoid of faces Σ_q	cocommutative for $q = 1$	$\mathcal{T}_q(\mathsf{E})$
q-bimonoid of top-nested faces $\widehat{\mathsf{Q}}_q$	–	$\mathcal{T}_q(\Gamma^*)$
q-bimonoid of bifaces J_q	–	$\mathcal{T}_q(\mathsf{F})$

By convention, $\Gamma = \Gamma_1$, $\Sigma = \Sigma_1$, $\widehat{\mathsf{Q}} = \widehat{\mathsf{Q}}_1$. The abelianization of Γ is E, of Σ is Π, of $\widehat{\mathsf{Q}}$ is $\widehat{\Lambda}$. The signed abelianization of Γ_{-1} is E^-. The latter is the signed exponential bimonoid which is constructed out of the signature space of the arrangement. We discuss the q-norm map from Γ_q to its dual. It is an isomorphism when q is a root of unity. The behavior at $q = 1$ is quite different. For $q = \pm 1$, the q-norm map factors through the (signed) abelianization map. Similar considerations apply to Σ_q.

The bimonoids E and Π are self-dual. The same is true of the q-bimonoids Γ_q and Σ_q when q is not a root of unity. In contrast, the bimonoids Γ and Σ are cocommutative but not commutative. So they cannot be self-dual. We also consider the q-bimonoid J_q which for $q = 1$ is neither commutative nor cocommutative. All these bimonoids arise from the universal constructions of Chapter 6.

We also introduce the Lie species and Zie species. These arise as the primitive part of Γ and Σ, respectively. As a consequence, they carry the structure of a Lie monoid. This last point will be properly addressed in Chapter 16.

Additional examples treated in later chapters are the bimonoid of chamber maps in Section 8.5 and the bimonoid of pairs of chambers in Section 15.5.

7.1. Species characteristic of chambers

We begin with the species characteristic of chambers. It plays an important role in the study of the exponential bimonoid and the bimonoid of chambers to be discussed in Sections 7.2 and 7.3.

7.1.1. Species characteristic of chambers. Define the species x by

$$\mathsf{x}[F] := \begin{cases} \Bbbk & \text{if } F \text{ is a chamber,} \\ 0 & \text{otherwise.} \end{cases} \tag{7.1}$$

The maps $\beta_{G,F}$ are defined to be the identity. This is the *species characteristic of chambers*.

The k-th Cauchy power (5.1) of x is given by

$$\mathsf{x}^k[F] := \begin{cases} \displaystyle\bigoplus_{C:\, C \geq F} \Bbbk & \text{if } F \text{ has corank } k-1, \\ 0 & \text{otherwise.} \end{cases} \tag{7.2}$$

In the first alternative, there is a copy of $\Bbbk$ for each chamber greater than F.

Equivalently, in the formulation given by Proposition 2.5, the species x can be defined by

$$\mathsf{x}[\mathrm{Y}] := \begin{cases} \Bbbk & \text{if } \mathrm{Y} \text{ is the maximum flat,} \\ 0 & \text{otherwise.} \end{cases} \tag{7.3}$$

The k-th commutative Cauchy power (5.11) of x is given by

$$\mathsf{x}^{\overline{k}}[\mathrm{Y}] := \begin{cases} \Bbbk & \text{if } \mathrm{Y} \text{ has corank } k-1, \\ 0 & \text{otherwise.} \end{cases} \tag{7.4}$$

7.1.2. Decorated version. More generally, for any vector space M, define the species x_M by

$$\mathsf{x}_M[F] := \begin{cases} M & \text{if } F \text{ is a chamber,} \\ 0 & \text{otherwise.} \end{cases} \tag{7.5}$$

Equivalently, $\mathsf{x}_M[\mathrm{Y}]$ is M if Y is the maximum flat, and 0 otherwise. When $M = \Bbbk$, we recover x.

We view x_M as a decorated version of x, with M as the space of decorations.

7.2. Exponential species

We start with one of the simplest bimonoids, namely, the exponential bimonoid. All its components equal the base field, and all its structure maps are the identity. It is bicommutative and self-dual. We discuss its primitive filtration and (co)freeness properties. We also introduce its signed analogue which is constructed out of the signature space of the arrangement.

The exponential bimonoid has a decorated version in which the base field is replaced by an arbitrary vector space. It also has a signed analogue.

7.2.1. Exponential species. The *exponential species* E is defined by setting $\mathsf{E}[A] := \Bbbk$ for any face A. For faces A and B of the same support, define

$$\beta_{B,A} : \mathsf{E}[A] \to \mathsf{E}[B] \tag{7.6}$$

to be the identity map $\Bbbk \to \Bbbk$. This is the linearization of the exponential set-species, that is, $\mathsf{E} = \Bbbk \mathrm{E}$, see Section 2.14.3. In the formulation given by Proposition 2.5, the exponential species E can be defined by $\mathsf{E}[\mathrm{X}] := \Bbbk$ for any flat X.

Observe from (7.4) that

$$\mathsf{E} = \mathsf{x} + \mathsf{x}^{\overline{2}} + \mathsf{x}^{\overline{3}} + \dots, \tag{7.7}$$

the sum of all commutative Cauchy powers of x.

7.2.2. Exponential bimonoid. For the exponential species E, define

$$\mu_A^F : \mathsf{E}[F] \to \mathsf{E}[A] \quad \text{and} \quad \Delta_A^F : \mathsf{E}[A] \to \mathsf{E}[F] \tag{7.8}$$

to be the identity maps $\Bbbk \to \Bbbk$ for all $F \geq A$. This turns E into a bimonoid. We call it the *exponential bimonoid*.

For clarity, let us write $\mathtt{H}_A$ for the basis element $1 \in \mathsf{E}[A]$. In this notation,

$$\beta_{B,A}(\mathtt{H}_A) = \mathtt{H}_B, \quad \mu_A^F(\mathtt{H}_F) = \mathtt{H}_A, \quad \Delta_A^F(\mathtt{H}_A) = \mathtt{H}_F.$$

The structure is set-theoretic, that is, the maps preserve basis elements.

Observe that E is bicommutative. So, in view of Proposition 2.22, one can express all the above in terms of flats instead of faces as follows. We let $\mathsf{E}[\mathrm{X}] := \Bbbk$ for any flat X, and

$$\mu_{\mathrm{Z}}^{\mathrm{X}} : \mathsf{E}[\mathrm{X}] \to \mathsf{E}[\mathrm{Z}] \quad \text{and} \quad \Delta_{\mathrm{Z}}^{\mathrm{X}} : \mathsf{E}[\mathrm{Z}] \to \mathsf{E}[\mathrm{X}] \tag{7.9}$$

be the identity maps for all $\mathrm{X} \geq \mathrm{Z}$. The basis element in $\mathsf{E}[\mathrm{Z}]$ may now be denoted $\mathtt{H}_{\mathrm{Z}}$.

The exponential bimonoid E is self-dual. The self-duality is via the canonical identification of $\Bbbk$ with $\Bbbk^*$.

7.2.3. Primitive part. The primitive part of E is given by

$$\mathcal{P}(\mathsf{E}) = \mathsf{x}.$$

Explicitly, the components $\mathsf{E}[C]$, as C varies over chambers, are primitive, while the remaining components do not contain any nonzero primitives.

Let us now consider the primitive filtration of E. The first term $\mathcal{P}_1(\mathsf{E})$ equals the primitive part $\mathcal{P}(\mathsf{E})$. The second term $\mathcal{P}_2(\mathsf{E})$ is the species, which is $\Bbbk$ on chambers and panels, and 0 otherwise. In general, $\mathcal{P}_k(\mathsf{E})$ is the species, which is $\Bbbk$ on faces with corank at most $k-1$, and 0 otherwise. Observe that this can be expressed as

$$\mathcal{P}_k(\mathsf{E}) = \mathsf{x} + \mathsf{x}^{\overline{2}} + \cdots + \mathsf{x}^{\overline{k}}, \tag{7.10}$$

the sum of the first k commutative Cauchy powers of x. This can be seen more directly by using the formulation of primitive filtration in terms of flats given in (5.27). Also note that (7.10) is the sequence of partial sums in the expression for E given in (7.7).

Exercise 7.1. Use (5.41) to deduce that the indecomposable part of E is given by

$$\mathcal{Q}(\mathsf{E}) = \mathsf{x}.$$

Also check this fact directly.

7.2.4. Freeness and cofreeness. The exponential bimonoid E is free as a commutative monoid and cofree as a cocommutative comonoid, both on the species x. Further, E is the free commutative bimonoid on x viewed as a trivial comonoid, and it is the cofree cocommutative bimonoid on x viewed as a trivial monoid. More precisely, consider the identification

$$\mathsf{E} \xrightarrow{\cong} \mathcal{S}(\mathsf{x}) = \mathcal{S}^{\vee}(\mathsf{x}) \tag{7.11}$$

obtained by combining (6.19) and (7.7). Explicitly, on the Z-component, it sends $\mathtt{H}_{\mathrm{Z}}$ to $1 \in \mathsf{x}[\top]$, the latter being a summand of $\mathcal{S}(\mathsf{x})[\mathrm{Z}]$. This is an isomorphism of bimonoids. This can be checked using (co)product formulas (6.51).

The formula for the primitive filtration of E given in (7.10) can now also be seen as a special case of Proposition 6.58.

Exercise 7.2. Let h be a bimonoid. Check that: If $f : \mathsf{E} \to \mathsf{h}$ is a morphism of monoids, then it is also a morphism of comonoids. Similarly, if $g : \mathsf{h} \to \mathsf{E}$ is a morphism of comonoids, then it is also a morphism of monoids. (See also Exercise 9.131.)

7.2.5. Signed exponential species. The exponential species has a signed analogue, which we denote by E^-. It is given by

$$\mathsf{E}^-[A] := \mathbf{E}^-[\mathrm{s}(A), \top],$$

where the latter is the signature space of Definition 1.74. Let us recall this construction.

First consider the vector space with basis consisting of symbols $\mathtt{H}_{C/A}$ with C a chamber, and $A \leq C$. Then $\mathsf{E}^-[A]$ is the quotient of this space by the relations

$$\mathtt{H}_{C/A} = (-1)^{\operatorname{dist}(C,D)}\, \mathtt{H}_{D/A}.$$

It is one-dimensional. We write $\mathtt{H}_{[C/A]}$ for the class of $\mathtt{H}_{C/A}$. Any such element is a basis of $\mathsf{E}^-[A]$.

For faces A and B of the same support, define

$$\beta_{B,A} : \mathsf{E}^-[A] \to \mathsf{E}^-[B], \qquad \mathtt{H}_{[C/A]} \mapsto \mathtt{H}_{[BC/B]}. \tag{7.12}$$

This is the species structure. Further, define

$$\begin{aligned} \mu_A^F : \mathsf{E}^-[F] &\to \mathsf{E}^-[A] & \Delta_A^F : \mathsf{E}^-[A] &\to \mathsf{E}^-[F] \\ \mathtt{H}_{[C/F]} &\mapsto \mathtt{H}_{[C/A]} & \mathtt{H}_{[C/A]} &\mapsto (-1)^{\operatorname{dist}(C,FC)}\, \mathtt{H}_{[FC/F]}. \end{aligned} \tag{7.13}$$

This turns E^- into a signed bimonoid. We call it the *signed exponential bimonoid*. It is signed bicommutative and self-dual.

Exercise 7.3. Show that $\mu_A^F \Delta_A^F = \mathrm{id}$ for the signed exponential bimonoid E^-. In fact, μ_A^F and Δ_A^F are inverse isomorphisms. (Check directly or use (2.36).)

We now reformulate E^- in terms of flats as in Proposition 2.37. Firstly,

$$\mathsf{E}^-[\mathrm{X}] = \mathbf{E}^-[\mathrm{X}, \top].$$

Further, the product and coproduct components

$$\begin{aligned} \mu_{\mathrm{Z}}^{\mathrm{X}} &: \mathbf{E}^-[\mathrm{Z},\mathrm{X}] \otimes \mathsf{E}^-[\mathrm{X}] \to \mathsf{E}^-[\mathrm{Z}] \\ \Delta_{\mathrm{Z}}^{\mathrm{X}} &: \mathsf{E}^-[\mathrm{Z}] \to \mathbf{E}^-[\mathrm{Z},\mathrm{X}] \otimes \mathsf{E}^-[\mathrm{X}] \end{aligned} \tag{7.14}$$

are inverse to each other and obtained by specializing (1.162) to $\mathrm{Y} := \top$. (To see this, use (2.46) and its dual.)

We deduce that the canonical identification

$$\mathsf{E}^- \xrightarrow{\cong} \mathcal{E}(\mathsf{x}) = \mathcal{E}^\vee(\mathsf{x}) \tag{7.15}$$

is an isomorphism of signed bimonoids, with the product and coproduct of the latter given by (6.55) and (6.56), respectively. This map is the signed analogue of (7.11).

7.2.6. Decorated exponential species. For any vector space M, one can construct a bicommutative bimonoid E_M by setting $\mathsf{E}_M[A] := M$ for any face A. The structure maps involving β, μ, Δ are defined to be identities. We refer to E_M as the *decorated exponential bimonoid*. When $M = \Bbbk$, we recover the exponential bimonoid E.

The decorated exponential bimonoid is free and cofree on the species x_M defined in (7.5), that is,

$$\mathsf{E}_M = \mathcal{S}(\mathsf{x}_M) = \mathcal{S}^\vee(\mathsf{x}_M).$$

The construction of E_M is functorial in M. That is, a linear map $M \to M'$ gives rise to a morphism $\mathsf{E}_M \to \mathsf{E}_{M'}$ of bimonoids. In other words, we have a functor from the category of vector spaces to the category of bicommutative bimonoids.

Similarly, given M, one can also construct a signed bimonoid E^-_M by setting

$$\mathsf{E}^-_M[A] := M \otimes \mathsf{E}^-[A]$$

for any face A. We refer to E^-_M as the *decorated signed exponential bimonoid*. When $M = \Bbbk$, we recover E^-. Further,

$$\mathsf{E}^-_M = \mathcal{E}(\mathsf{x}_M) = \mathcal{E}^\vee(\mathsf{x}_M).$$

7.2.7. Orientation species. We now introduce another important relative of the exponential species. Define the *orientation species* E^{o} by

$$\mathsf{E}^{\mathsf{o}}[F] := \mathsf{E}^{\mathsf{o}}[\mathcal{A}_F],$$

where the latter is the orientation space of the arrangement $\mathcal{A}_F$ as defined in Section 1.11.1. Explicitly, $\mathsf{E}^{\mathsf{o}}[F]$ is one-dimensional and spanned by any saturated chain of faces starting at F and ending at a chamber greater than F. Two such saturated chains which differ in only one position are negatives of each other.

7.3. Species of chambers

We introduce the bimonoid of chambers. It is free as a monoid, and may be viewed as the noncommutative analogue of the exponential bimonoid. Formally, the latter is the abelianization of the former. The bimonoid of chambers admits a deformation by a parameter q. When q is not a root of unity, it is isomorphic to its dual via the q-norm map. This includes the case $q = 0$.

One can interpolate between the exponential bimonoid and the bimonoid of chambers via geometric partial-support relations.

7.3.1. Species of chambers. For any face A, let $\Gamma[A]$ be the set of chambers greater than A. Whenever A and B have the same support, by Lemma 1.6, there is a bijection

$$\beta_{B,A} : \Gamma[A] \to \Gamma[B], \qquad C/A \mapsto BC/B.$$

Thus, Γ is a set-species.

Let $\mathsf{\Gamma} := \Bbbk\Gamma$ be the linearization of Γ. This is the *species of chambers*. Explicitly, $\mathsf{\Gamma}[A]$ is the linear span of the set of chambers greater than A. We use the letter $\mathtt{H}$ for the canonical basis of $\mathsf{\Gamma}[A]$. For faces A and B of the same support, we write

$$\beta_{B,A} : \mathsf{\Gamma}[A] \to \mathsf{\Gamma}[B], \qquad \mathtt{H}_{C/A} \mapsto \mathtt{H}_{BC/B}. \tag{7.16}$$

We claim that

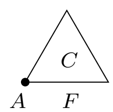
$$\mathsf{\Gamma} = \mathsf{x} + \mathsf{x}^2 + \mathsf{x}^3 + \dots, \tag{7.17}$$

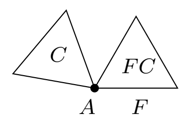
the sum of all Cauchy powers of x. By (7.2), the F-component of the rhs is a vector space with basis indexed by chambers C greater than F, and we identify this with the $\mathtt{H}$-basis of $\mathsf{\Gamma}[F]$. Compare and contrast (7.17) with (7.7).

7.3.2. Bimonoid of chambers. The species Γ carries the structure of a bimonoid. We call it the *bimonoid of chambers*. The product and coproduct are defined by

$$\begin{aligned} \mu_A^F : \Gamma[F] &\to \Gamma[A] \qquad\qquad & \Delta_A^F : \Gamma[A] &\to \Gamma[F] \\ \mathtt{H}_{C/F} &\mapsto \mathtt{H}_{C/A} & \mathtt{H}_{C/A} &\mapsto \mathtt{H}_{FC/F}. \end{aligned} \tag{7.18}$$

Since basis elements are preserved, this structure is set-theoretic. Illustrative pictures for the product and coproduct are shown below.

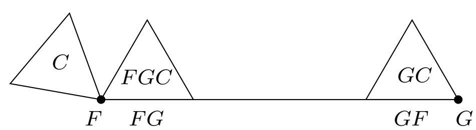




The bimonoid axiom (2.12) is checked below.

$$\begin{tikzcd} \mathtt{H}_{C/F} \arrow[r, mapsto] \arrow[d, mapsto] & \mathtt{H}_{C/A} \arrow[r, mapsto] & \mathtt{H}_{GC/G} \\ \mathtt{H}_{FGC/FG} \arrow[rr, mapsto] & & \mathtt{H}_{GC/GF} \arrow[u, mapsto] \end{tikzcd}$$

Here F and G are faces both greater than A, while C is a chamber greater than F. An illustrative picture is shown below, with A as the central face.



The bimonoid of chambers Γ is cocommutative but not commutative. Hence, it cannot be self-dual.

7.3.3. *q*-bimonoid of chambers. More generally, for any scalar q, the species of chambers carries the structure of a q-bimonoid which we denote by Γ_q. We call it the *q-bimonoid of chambers*. The product and coproduct are defined by

$$\begin{aligned} \mu_A^F : \Gamma_q[F] &\to \Gamma_q[A] \qquad\qquad & \Delta_A^F : \Gamma_q[A] &\to \Gamma_q[F] \\ \mathtt{H}_{C/F} &\mapsto \mathtt{H}_{C/A} & \mathtt{H}_{C/A} &\mapsto q^{\mathrm{dist}(C,FC)}\, \mathtt{H}_{FC/F}. \end{aligned} \tag{7.19}$$

Note that for $q = 1$, we have $\Gamma_1 = \Gamma$, the bimonoid of chambers. The q-bimonoid axiom (2.33) is checked below. It generalizes the previous calculation.

$$\begin{tikzcd} \mathtt{H}_{C/F} \arrow[r, mapsto] \arrow[d, mapsto] & \mathtt{H}_{C/A} \arrow[r, mapsto] & q^{\mathrm{dist}(C,GC)}\, \mathtt{H}_{GC/G} \\ q^{\mathrm{dist}(C,FGC)}\, \mathtt{H}_{FGC/FG} \arrow[rr, mapsto] & & q^{\mathrm{dist}(C,FGC)} q^{\mathrm{dist}(FGC,GC)}\, \mathtt{H}_{GC/GF} \arrow[u, mapsto] \end{tikzcd}$$

We used that $\mathrm{dist}(C, GC) = \mathrm{dist}(C, FGC) + \mathrm{dist}(FGC, GC)$. This follows from the gate property (1.22). Equivalently, one may directly apply property (1.30d).

7.3.4. Dual bimonoid. Let Γ^* denote the bimonoid dual to Γ. Let $\mathtt{M}$ denote the basis which is dual to the $\mathtt{H}$-basis. The product and coproduct of Γ^* are obtained by dualizing formulas (7.18). They are given by

$$\begin{aligned} &\mu_A^F : \Gamma^*[F] \to \Gamma^*[A] && \Delta_A^F : \Gamma^*[A] \to \Gamma^*[F] \\ &\mathtt{M}_{D/F} \mapsto \sum_{\substack{C\colon C \geq A \\ FC = D}} \mathtt{M}_{C/A} && \mathtt{M}_{C/A} \mapsto \begin{cases} \mathtt{M}_{C/F} & \text{if } F \leq C, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{7.20}$$

7.3.5. Dual q-bimonoid. More generally, for any scalar q, let Γ_q^* denote the q-bimonoid dual to Γ_q. Dualizing formulas (7.19), observe that its product and coproduct are given by

$$\begin{aligned} &\mu_A^F : \Gamma_q^*[F] \to \Gamma_q^*[A] && \Delta_A^F : \Gamma_q^*[A] \to \Gamma_q^*[F] \\ &\mathtt{M}_{D/F} \mapsto \sum_{\substack{C\colon C \geq A \\ FC = D}} q^{\operatorname{dist}(FC,C)}\, \mathtt{M}_{C/A} && \mathtt{M}_{C/A} \mapsto \begin{cases} \mathtt{M}_{C/F} & \text{if } F \leq C, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{7.21}$$

In contrast to Γ_q, the scalar q now appears in the product as opposed to the coproduct.

7.3.6. Primitive part. Observe from coproduct formula (7.21) that

$$\mathcal{P}(\Gamma_q^*) = \mathsf{x}. \tag{7.22}$$

Explicitly, the components $\Gamma_q^*[C]$, as C varies over chambers, are primitive, while the remaining components do not contain any nonzero primitives.

Let us now consider the primitive filtration of Γ_q^*. The first term $\mathcal{P}_1(\Gamma_q^*)$ equals the primitive part $\mathcal{P}(\Gamma_q^*)$. The second term $\mathcal{P}_2(\Gamma_q^*)$ is the species whose F-component is $\Gamma_q^*[F]$ if F is either a chamber or a panel, and 0 otherwise. In general, $\mathcal{P}_k(\Gamma_q^*)$ is the species whose F-component is $\Gamma_q^*[F]$ if F has corank at most $k-1$, and 0 otherwise. Observe that this can be expressed as

$$\mathcal{P}_k(\Gamma_q^*) = \mathsf{x} + \mathsf{x}^2 + \cdots + \mathsf{x}^k, \tag{7.23}$$

the sum of the first k Cauchy powers of x.

The primitive part of Γ_q is discussed separately later in Section 7.9.2.

7.3.7. Freeness and cofreeness. The q-bimonoid of chambers Γ_q is free as a monoid on the species x. Further, it is the free q-bimonoid on x viewed as a trivial comonoid. More precisely, the identification

$$\Gamma_q \xrightarrow{\cong} \mathcal{T}_q(\mathsf{x}), \qquad \mathtt{H}_{C/A} \mapsto 1 \in \mathsf{x}[C] \tag{7.24}$$

on the A-component, is an isomorphism of q-bimonoids. This can be checked using (6.39). The above map is the same as the one obtained by combining (6.2) and (7.17).

Dually, Γ_q^* is cofree as a comonoid on the species x. Further, it is the cofree q-bimonoid on x viewed as a trivial monoid. More precisely, the identification

$$\Gamma_q^* \xrightarrow{\cong} \mathcal{T}_q^\vee(\mathsf{x}), \qquad \mathtt{M}_{C/A} \mapsto 1 \in \mathsf{x}[C] \tag{7.25}$$

on the A-component, is an isomorphism of q-bimonoids. This can also be checked directly using (6.40).

The formula for the primitive filtration of Γ_q^* given in (7.23) can now also be seen as a special case of Proposition 6.56.

7.3.8. q-norm map and self-duality. Consider the map of species

$$\Gamma_q \to \Gamma_q^*, \qquad \mathtt{H}_{C/A} \mapsto \sum_{D:\, D\geq A} q^{\operatorname{dist}(C,D)}\, \mathtt{M}_{D/A} \tag{7.26}$$

on the A-component. We call it the *q-norm map* on chambers.

Lemma 7.4. *The q-norm map (7.26) is a self-dual morphism of q-bimonoids.*

PROOF. Let us write f for the map (7.26). Self-duality of f follows from property (1.21b). Lemma 1.6 and property (1.23) imply that f is a map of species. We claim that the following diagrams commute.

$$\begin{tikzcd} \Gamma_q[F] \arrow[r, "f_F"] \arrow[d, "\mu_A^F"'] & \Gamma_q^*[F] \arrow[d, "\mu_A^F"] \\ \Gamma_q[A] \arrow[r, "f_A"'] & \Gamma_q^*[A] \end{tikzcd} \qquad\qquad \begin{tikzcd} \Gamma_q^*[F] & \Gamma_q[F] \arrow[l, "f_F"'] \\ \Gamma_q^*[A] \arrow[u, "\Delta_A^F"] & \Gamma_q[A] \arrow[l, "f_A"] \arrow[u, "\Delta_A^F"'] \end{tikzcd}$$

They say that f is a morphism of monoids and of comonoids, respectively. Since they are duals of each other, it suffices to check that the first diagram commutes. We calculate:

$$\begin{aligned} \mu_A^F f_F(\mathtt{H}_{C/F}) &= \mu_A^F\Big(\sum_{E:\, E\geq F} q^{\operatorname{dist}(C,E)}\, \mathtt{M}_{E/F}\Big) \\ &= \sum_{E:\, E\geq F} \sum_{\substack{D:\, D\geq A \\ FD=E}} q^{\operatorname{dist}(C,E)} q^{\operatorname{dist}(E,D)}\, \mathtt{M}_{D/A} \\ &= \sum_{D:\, D\geq A} q^{\operatorname{dist}(C,D)}\, \mathtt{M}_{D/A} \\ &= f_A(\mathtt{H}_{C/A}) \\ &= f_A \mu_A^F(\mathtt{H}_{C/F}). \end{aligned}$$

The third step used the gate property (1.22).

Alternatively, one can avoid the above calculation by noting that the map (7.26) arises from the freeness of Γ_q and the cofreeness of Γ_q^* as follows. Using (7.22), apply Theorem 6.31 to $f := \mathrm{id}$ to obtain a morphism of q-bimonoids from $\mathcal{T}_q(\mathsf{x})$ to $\mathcal{T}_q^\vee(\mathsf{x})$. Using formula (6.41) and the identifications (7.24) and (7.25), we see that this morphism indeed agrees with (7.26). $\square$

Further, by Theorem 1.10 applied to the arrangement over the support of A, we deduce that:

Proposition 7.5. *Suppose q is not a root of unity. Then the q-norm map (7.26) is an isomorphism of q-bimonoids. In particular, the q-bimonoid Γ_q is self-dual.*

For instance, for a rank-one arrangement with chambers C and $\overline{C}$, the q-norm map on the O-component is

$$\mathtt{H}_C \mapsto \mathtt{H}_C + q\,\mathtt{H}_{\overline{C}}, \qquad \mathtt{H}_{\overline{C}} \mapsto q\,\mathtt{H}_C + \mathtt{H}_{\overline{C}},$$

which is invertible if $q \neq \pm 1$.

The above discussion can be seen as a special case of the discussion in Section 6.9. Observe that the q-norm map on chambers (7.26) is precisely the q-norm map (6.74) for $\mathtt{p} := \mathtt{x}$. Proposition 7.5 is then an instance of Proposition 6.75.

The cases $q = 0$ and $q = \pm 1$ of the q-norm map on chambers are discussed in more detail below.

7.3.9. 0-bimonoid of chambers. Let $q = 0$. Observe that the product and coproduct of Γ_0 are given by

$$\begin{aligned} \mu_A^F : \Gamma_0[F] &\to \Gamma_0[A] & \qquad \Delta_A^F : \Gamma_0[A] &\to \Gamma_0[F] \\ \mathtt{H}_{C/F} &\mapsto \mathtt{H}_{C/A} & \mathtt{H}_{C/A} &\mapsto \begin{cases} \mathtt{H}_{C/F} & \text{if } F \leq C, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{7.27}$$

This is the 0-*bimonoid of chambers*. It is free as a monoid and cofree as a comonoid, both on the species $\mathtt{x}$, which is its primitive part as well as its indecomposable part.

Also note that the product and coproduct of Γ_0^* are given by the same formulas (with $\mathtt{M}$ replacing $\mathtt{H}$). In other words, the map

$$\Gamma_0 \to \Gamma_0^*, \qquad \mathtt{H}_{C/A} \mapsto \mathtt{M}_{C/A}$$

is an isomorphism of 0-bimonoids, implying that Γ_0 is self-dual. This map is the case $q = 0$ of the map (7.26).

7.3.10. Abelianization. There is a close connection between the species of chambers and the exponential species as follows.

The map $\pi : \Gamma \to \mathsf{E}$ given by

$$\pi_A : \Gamma[A] \to \mathsf{E}[A], \qquad \mathtt{H}_{C/A} \mapsto \mathtt{H}_A \tag{7.28}$$

is a surjective morphism of bimonoids. This follows since the product and coproduct of Γ take a basis element to another basis element. The kernel of π_A is the subspace spanned by elements of the form

$$\mathtt{H}_{C/A} - \mathtt{H}_{D/A},$$

as C and D vary over chambers in A. These are precisely elements of the form (2.52), thus π is the abelianization map. The fact that E is the abelianization of Γ can also be seen from (6.60), (7.11) and (7.24) for $q = 1$.

Note that the kernel of π equals the kernel of the morphism of bimonoids

$$\Gamma \to \Gamma^*, \qquad \mathtt{H}_{C/A} \mapsto \sum_{D:\, D \geq A} \mathtt{M}_{D/A}. \tag{7.29}$$

This map is the case $q = 1$ of the map (7.26). Its image is one-dimensional. This yields the following commutative diagram of bimonoids.

$$\begin{tikzcd} \mathsf{\Gamma} \arrow[r] \arrow[d, two heads, "\pi"'] & \mathsf{\Gamma}^* \\ \mathsf{E} \arrow[r, "\cong"'] & \mathsf{E}^* \arrow[u, hook, "\pi^*"'] \end{tikzcd} \tag{7.30}$$

This is an instance of the first diagram in (6.75) applied to x.

Applying Proposition 2.51 to the self-dual map (7.29) recovers the fact that the bimonoid E is self-dual.

7.3.11. Signed abelianization. The preceding discussion has a signed analogue. It involves the signed exponential species.

Let $\mathsf{\Gamma}^- := \mathsf{\Gamma}_{-1}$. The signed abelianization map $\pi_{-1} : \mathsf{\Gamma}^- \to \mathsf{E}^-$ given by

$$(\pi_{-1})_A : \mathsf{\Gamma}[A] \to \mathsf{E}^-[A], \qquad \mathtt{H}_{C/A} \mapsto \mathtt{H}_{[C/A]} \tag{7.31}$$

is a surjective morphism of signed bimonoids. The kernel of $(\pi_{-1})_A$ is the subspace spanned by elements of the form

$$\mathtt{H}_{C/A} - (-1)^{\operatorname{dist}(C,D)} \mathtt{H}_{D/A},$$

as C and D vary over chambers greater than A. This is also the kernel of the morphism of signed bimonoids

$$\mathsf{\Gamma}^- \to (\mathsf{\Gamma}^-)^*, \qquad \mathtt{H}_{C/A} \mapsto \sum_{D:\, D \geq A} (-1)^{\operatorname{dist}(C,D)}\, \mathtt{M}_{D/A}.$$

This map is the case $q = -1$ of the map (7.26). Its image is one-dimensional. This yields the following commutative diagram of signed bimonoids.

$$\begin{tikzcd} \mathsf{\Gamma}^- \arrow[r] \arrow[d, two heads, "\pi_{-1}"'] & (\mathsf{\Gamma}^-)^* \\ \mathsf{E}^- \arrow[r, "\cong"'] & (\mathsf{E}^-)^* \arrow[u, hook, "\pi_{-1}^*"'] \end{tikzcd} \tag{7.32}$$

This is an instance of the second diagram in (6.75) applied to x.

7.3.12. Interpolating $\mathsf{\Gamma}$ and E using partial-flats. Let $\sim$ be a geometric partial-support relation on faces. We define the species $\mathsf{\Gamma}_\sim$ as follows. For any partial-flat x, let $\mathsf{\Gamma}_\sim[\mathrm{x}]$ be the linear span of all maximal partial-flats c greater than x. We denote these basis elements by $\mathtt{H}_{\mathrm{c}/\mathrm{x}}$. The species $\mathsf{\Gamma}_\sim$ carries the structure of a bimonoid with product and coproduct given by

$$\begin{aligned} \mu^{\mathrm{x}}_{\mathrm{z}} : \mathsf{\Gamma}_\sim[\mathrm{x}] &\to \mathsf{\Gamma}_\sim[\mathrm{z}] & \qquad \Delta^{\mathrm{x}}_{\mathrm{z}} : \mathsf{\Gamma}_\sim[\mathrm{z}] &\to \mathsf{\Gamma}_\sim[\mathrm{x}] \\ \mathtt{H}_{\mathrm{c}/\mathrm{x}} &\mapsto \mathtt{H}_{\mathrm{c}/\mathrm{z}} & \mathtt{H}_{\mathrm{c}/\mathrm{z}} &\mapsto \mathtt{H}_{\mathrm{xc}/\mathrm{x}}. \end{aligned} \tag{7.33}$$

It is cocommutative and $\sim$-commutative. This bimonoid interpolates between $\mathsf{\Gamma}$ and E. When $\sim$ is finest, $\mathsf{\Gamma}_\sim$ specializes to $\mathsf{\Gamma}$, while when $\sim$ is coarsest, $\mathsf{\Gamma}_\sim$ specializes to E. In fact,

$$\mathsf{\Gamma}_\sim = \mathcal{T}_\sim(\mathsf{x}), \tag{7.34}$$

with the latter as in (6.91). This interpolates between (7.24) for $q = 1$ and (7.11).

7.4. Species of flats

We discuss the bimonoid of flats. We consider two bases for it, the $\mathtt{H}$-basis and the $\mathtt{Q}$-basis. The motivation for the latter is that it simplifies the study of the bimonoid. The change of basis involves the zeta and Möbius functions of the poset of flats. The bimonoid of flats is bicommutative and self-dual. We provide explicit self-duality isomorphisms. Further, it is the free commutative and cofree cocommutative bimonoid on the exponential species, which is its primitive part.

The discussion here bears a resemblance to that in Section 1.9.1. The parallel will become clear when we discuss the equivalence between modules over the Birkhoff algebra and bicommutative bimonoids (Proposition 11.5).

7.4.1. Species of flats. Define a set-species Π as follows. For any flat X, let $\Pi[\mathrm{X}]$ be the set of all flats greater than X. Equivalently, it is the set of flats of $\mathcal{A}_{\mathrm{X}}$.

Let $\mathsf{\Pi} := \Bbbk\Pi$ be the linearization of Π. This is the *species of flats*. Let $\mathtt{H}$ denote its canonical basis.

We claim that

$$\mathsf{\Pi} = \mathsf{E} + \mathsf{E}^{\overline{2}} + \mathsf{E}^{\overline{3}} + \dots, \tag{7.35}$$

the sum of all commutative Cauchy powers of the exponential species E. Explicitly, the Z-component of the rhs is $\bigoplus_{\mathrm{X} \geq \mathrm{Z}} \mathsf{E}[\mathrm{X}]$. This is a vector space with basis indexed by flats X greater than Z, and we identify this with the $\mathtt{H}$-basis of $\mathsf{\Pi}[\mathrm{Z}]$.

7.4.2. Bimonoid of flats. The species of flats $\mathsf{\Pi}$ carries the structure of a bicommutative bimonoid. We call it the *bimonoid of flats*. The product and coproduct are defined by

$$\begin{aligned} \mu_{\mathrm{Z}}^{\mathrm{Y}} : \mathsf{\Pi}[\mathrm{Y}] &\to \mathsf{\Pi}[\mathrm{Z}] & \qquad \Delta_{\mathrm{Z}}^{\mathrm{Y}} : \mathsf{\Pi}[\mathrm{Z}] &\to \mathsf{\Pi}[\mathrm{Y}] \\ \mathtt{H}_{\mathrm{X}/\mathrm{Y}} &\mapsto \mathtt{H}_{\mathrm{X}/\mathrm{Z}} & \mathtt{H}_{\mathrm{X}/\mathrm{Z}} &\mapsto \mathtt{H}_{\mathrm{X}\vee\mathrm{Y}/\mathrm{Y}}. \end{aligned} \tag{7.36}$$

Since basis elements are preserved, this structure is set-theoretic. The bicommutative bimonoid axiom (2.26) is checked below.

$$\begin{tikzcd} \mathtt{H}_{\mathrm{W}/\mathrm{X}} \arrow[r, mapsto] \arrow[d, mapsto] & \mathtt{H}_{\mathrm{W}/\mathrm{Z}} \arrow[d, mapsto] \\ \mathtt{H}_{\mathrm{W}\vee(\mathrm{X}\vee\mathrm{Y})/\mathrm{X}\vee\mathrm{Y}} \arrow[r, mapsto] & \mathtt{H}_{\mathrm{W}\vee\mathrm{Y}/\mathrm{Y}} \end{tikzcd}$$

7.4.3. Birkhoff algebra. The bimonoid of flats $\mathsf{\Pi}$ also carries an internal structure. For each flat Z, the component $\mathsf{\Pi}[\mathrm{Z}]$ is an algebra with product in the $\mathtt{H}$-basis given by

$$\mathtt{H}_{\mathrm{X}/\mathrm{Z}} \cdot \mathtt{H}_{\mathrm{Y}/\mathrm{Z}} = \mathtt{H}_{\mathrm{X}\vee\mathrm{Y}/\mathrm{Z}}. \tag{7.37}$$

The unit element is $\mathtt{H}_{\mathrm{Z}/\mathrm{Z}}$. This algebra can be identified with the Birkhoff algebra of the arrangement $\mathcal{A}_{\mathrm{Z}}$: Compare (7.37) with (1.101).

7.4.4. $\mathtt{Q}$-basis. We now consider a second basis of the species of flats. Define the $\mathtt{Q}$-basis of Π, on the Z-component, by either of the two equivalent formulas

$$\begin{aligned} \mathtt{H}_{\mathrm{X}/\mathrm{Z}} &= \sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{X}} \mathtt{Q}_{\mathrm{Y}/\mathrm{Z}}, \\ \mathtt{Q}_{\mathrm{X}/\mathrm{Z}} &= \sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{X}, \mathrm{Y})\, \mathtt{H}_{\mathrm{Y}/\mathrm{Z}}. \end{aligned} \tag{7.38}$$

This is the same as applying definition (1.102) to the arrangement $\mathcal{A}_{\mathrm{Z}}$ for each flat Z. In particular,

$$\mathtt{Q}_{\mathrm{Z}/\mathrm{Z}} = \sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{Y})\, \mathtt{H}_{\mathrm{Y}/\mathrm{Z}}. \tag{7.39}$$

Proposition 7.6. *The product and coproduct of Π in the $\mathtt{Q}$-basis are given by*

$$\begin{array}{ll} \mu_{\mathrm{Z}}^{\mathrm{Y}} : \Pi[\mathrm{Y}] \to \Pi[\mathrm{Z}] & \qquad \Delta_{\mathrm{Z}}^{\mathrm{Y}} : \Pi[\mathrm{Z}] \to \Pi[\mathrm{Y}] \\ \mathtt{Q}_{\mathrm{X}/\mathrm{Y}} \mapsto \mathtt{Q}_{\mathrm{X}/\mathrm{Z}} & \qquad \mathtt{Q}_{\mathrm{X}/\mathrm{Z}} \mapsto \begin{cases} \mathtt{Q}_{\mathrm{X}/\mathrm{Y}} & \text{if } \mathrm{X} \geq \mathrm{Y}, \\ 0 & \text{otherwise.} \end{cases} \end{array} \tag{7.40}$$

The product works the same way as in the $\mathtt{H}$-basis, but the coproduct works differently.

PROOF. It is simpler to use the $\mathtt{Q}$-basis formulas to derive the $\mathtt{H}$-basis formulas. The product calculation is straightforward. The calculation for the coproduct goes as follows.

$$\Delta_{\mathrm{Z}}^{\mathrm{Y}}(\mathtt{H}_{\mathrm{X}/\mathrm{Z}}) = \Delta_{\mathrm{Z}}^{\mathrm{Y}}\Big(\sum_{\mathrm{W}:\, \mathrm{W} \geq \mathrm{X}} \mathtt{Q}_{\mathrm{W}/\mathrm{Z}} \Big) = \sum_{\mathrm{W}:\, \mathrm{W} \geq \mathrm{X} \vee \mathrm{Y}} \mathtt{Q}_{\mathrm{W}/\mathrm{Y}} = \mathtt{H}_{\mathrm{X} \vee \mathrm{Y}/\mathrm{Y}}.$$

This completes the proof. ∎

Exercise 7.7. Use the product and coproduct formulas in the $\mathtt{H}$-basis to derive the formulas in the $\mathtt{Q}$-basis. (Use the Weisner formula (1.38).)

The internal structure of Π in the $\mathtt{Q}$-basis, on the Z-component, is given by

$$\mathtt{Q}_{\mathrm{X}/\mathrm{Z}} \cdot \mathtt{Q}_{\mathrm{Y}/\mathrm{Z}} = \begin{cases} \mathtt{Q}_{\mathrm{X}/\mathrm{Z}} & \text{if } \mathrm{X} = \mathrm{Y}, \\ 0 & \text{otherwise.} \end{cases} \tag{7.41}$$

This follows from (1.103). The unit element in the $\mathtt{Q}$-basis is

$$\mathtt{H}_{\mathrm{Z}/\mathrm{Z}} = \sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{Z}} \mathtt{Q}_{\mathrm{Y}/\mathrm{Z}}. \tag{7.42}$$

Also, by (1.104),

$$\mathtt{H}_{\mathrm{Y}/\mathrm{Z}} \cdot \mathtt{Q}_{\mathrm{X}/\mathrm{Z}} = \begin{cases} \mathtt{Q}_{\mathrm{X}/\mathrm{Z}} & \text{if } \mathrm{X} \geq \mathrm{Y}, \\ 0 & \text{otherwise.} \end{cases} \tag{7.43}$$

7.4.5. Dual bimonoid. Let Π^* denote the bimonoid dual to Π. Let $\mathtt{M}$ be the basis which is dual to $\mathtt{H}$. The product and coproduct in the $\mathtt{M}$-basis are given by

$$\begin{array}{ll} \mu_Z^Y : \Pi^*[Y] \to \Pi^*[Z] & \Delta_Z^Y : \Pi^*[Z] \to \Pi^*[Y] \\ \mathtt{M}_{W/Y} \mapsto \displaystyle\sum_{\substack{X:\, X \geq Z \\ X \vee Y = W}} \mathtt{M}_{X/Z} & \mathtt{M}_{X/Z} \mapsto \begin{cases} \mathtt{M}_{X/Y} & \text{if } X \geq Y, \\ 0 & \text{otherwise.} \end{cases} \end{array} \tag{7.44}$$

Let $\mathtt{P}$ be the basis of Π^* dual to $\mathtt{Q}$. Observe that the product and coproduct formulas in the $\mathtt{P}$-basis are identical to those in the $\mathtt{Q}$-basis.

7.4.6. Primitive part. We work with the formulation provided by (5.27). Observe from coproduct formula (7.44) in the $\mathtt{M}$-basis that

$$\mathcal{P}(\Pi^*) = \mathsf{E}. \tag{7.45}$$

Each component $\mathcal{P}(\Pi^*)[Z]$ is one-dimensional, and is spanned by $\mathtt{M}_{Z/Z}$. (The same is true in the $\mathtt{P}$-basis.) More generally, the primitive filtration of Π^* can be expressed as

$$\mathcal{P}_k(\Pi^*) = \mathsf{E} + \mathsf{E}^{\overline{2}} + \cdots + \mathsf{E}^{\overline{k}},$$

the sum of the first k commutative Cauchy powers of E.

Similarly, coproduct formula (7.40) in the $\mathtt{Q}$-basis yields

$$\mathcal{P}(\Pi) = \mathsf{E}. \tag{7.46}$$

Each component $\mathcal{P}(\Pi)[Z]$ is one-dimensional, and is spanned by $\mathtt{Q}_{Z/Z}$.

7.4.7. Freeness and cofreeness. The bimonoid Π is the free commutative bimonoid on E viewed as a comonoid. Dually, Π^* is the cofree cocommutative bimonoid on E viewed as a monoid. More precisely:

Proposition 7.8. *There are isomorphisms of bimonoids*

$$\Pi \xrightarrow{\cong} \mathcal{S}(\mathsf{E}) \quad \textit{and} \quad \Pi^* \xrightarrow{\cong} \mathcal{S}^\vee(\mathsf{E}). \tag{7.47}$$

On the Z*-component, the first map sends* $\mathtt{H}_{X/Z}$ *to* $\mathtt{H}_X$ *in the* X*-summand, while the second map sends* $\mathtt{M}_{X/Z}$ *to* $\mathtt{H}_X$ *in the* X*-summand.*

(The (co)product of $\mathcal{S}(\mathsf{E})$ *is given by specializing* (6.20) *and* (6.22)*, while that of* $\mathcal{S}^\vee(\mathsf{E})$ *is given by specializing* (6.26) *and* (6.28)*.)*

The bimonoid Π is also the free commutative bimonoid on E viewed as a trivial comonoid. Dually, Π^* is the cofree cocommutative bimonoid on E viewed as a trivial monoid. More precisely:

Proposition 7.9. *There are isomorphisms of bimonoids*

$$\Pi \xrightarrow{\cong} \mathcal{S}(\mathsf{E}) = \mathcal{S}^\vee(\mathsf{E}) \xleftarrow{\cong} \Pi^*. \tag{7.48}$$

On the Z*-component, the first map sends* $\mathtt{Q}_{X/Z}$ *to* $\mathtt{H}_X$ *in the* X*-summand, while the second map sends* $\mathtt{P}_{X/Z}$ *to* $\mathtt{H}_X$ *in the* X*-summand.*

(The (co)product of $\mathcal{S}(\mathsf{E}) = \mathcal{S}^\vee(\mathsf{E})$ *is given by specializing* (6.51)*.)*

Note very carefully that the $\mathtt{Q}$-basis of Π and the $\mathtt{P}$-basis of Π^* are employed here.

7.4.8. Self-duality. The bimonoid Π is self-dual under the map $\mathtt{Q}_{X/Z} \mapsto \mathtt{P}_{X/Z}$. This result is contained in (7.48). One can modify this map by scaling in different components leading to other self-duality isomorphisms. This idea is pursued in more depth below.

Let (ξ_X) and (η_X) be two families of scalars indexed by flats, and related to each other by Möbius inversion:

$$\xi_X = \sum_{Y:\, Y \geq X} \eta_Y \quad \text{and} \quad \eta_X = \sum_{Y:\, Y \geq X} \mu(X, Y)\, \xi_Y. \tag{7.49}$$

We highlight two choices of ξ and η. The first choice is: For each flat X,

$$\xi_X = \begin{cases} 1 & \text{if } X = \top, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \eta_X = \mu(\mathcal{A}_X). \tag{7.50a}$$

The second choice is: For each flat X,

$$\xi_X = c_X \quad \text{and} \quad \eta_X = |\mu(\mathcal{A}_X)|. \tag{7.50b}$$

That this is a valid choice is equivalent to the Zaslavsky formula (1.84). (Here c_X is the number of chambers in $\mathcal{A}_X$.)

Theorem 7.10. *For any ξ and η related by (7.49), the map $\Pi \to \Pi^*$ given on the Z-component equivalently by*

$$\mathtt{H}_{X/Z} \mapsto \sum_{Y:\, Y \geq Z} \xi_{X \vee Y}\, \mathtt{M}_{Y/Z}, \qquad \text{or} \qquad \mathtt{Q}_{X/Z} \mapsto \eta_X\, \mathtt{P}_{X/Z} \tag{7.51}$$

is a morphism of bimonoids. In particular, this map is an isomorphism of bimonoids iff $\eta_X \neq 0$ for all X.

Proof. Recall that the product and coproduct formulas in the $\mathtt{P}$- and $\mathtt{Q}$-bases are identical, and are given by (7.40). One can then readily check that $\mathtt{Q}_{X/Z} \mapsto \eta_X\, \mathtt{P}_{X/Z}$ is a morphism of bimonoids. The formula for the map from the $\mathtt{H}$-basis to the $\mathtt{M}$-basis is a simple calculation:

$$\begin{aligned} \mathtt{H}_{X/Z} = \sum_{Y:\, Y \geq X} \mathtt{Q}_{Y/Z} \mapsto \sum_{Y:\, Y \geq X} \eta_Y \mathtt{P}_{Y/Z} \\ = \sum_{W:\, W \geq Z} \Big(\sum_{Y:\, Y \geq X \vee W} \eta_Y \Big) \mathtt{M}_{W/Z} = \sum_{W:\, W \geq Z} \xi_{X \vee W}\, \mathtt{M}_{W/Z}. \end{aligned}$$

□

The two choices (7.50a) and (7.50b) substituted in Theorem 7.10 yield:

Theorem 7.11. *Suppose $\mu(\mathcal{A}_X) \neq 0$ for all flats X. Then the maps*

$$\varphi, \psi : \Pi \to \Pi^*$$

defined by

$$\begin{aligned} \varphi(\mathtt{H}_{Y/Z}) &:= \sum_{X:\, X \geq Z,\, X \vee Y = \top} \mathtt{M}_{X/Z}, \\ \psi(\mathtt{H}_{Y/Z}) &:= \sum_{X:\, X \geq Z} c_{X \vee Y}\, \mathtt{M}_{X/Z} \end{aligned} \tag{7.52}$$

are isomorphisms of bimonoids.

We recall [**21**, Proposition 1.80] that the hypothesis $\mu(\mathcal{A}_{\mathrm{X}}) \neq 0$ holds in characteristic 0, and we indeed have isomorphisms (7.52).

Exercise 7.12. Check that: The map $\mathcal{S}\mathcal{S}^{\vee}(\mathsf{p}) \to \mathcal{S}^{\vee}\mathcal{S}(\mathsf{p})$ in (6.37) when specialized to $\mathsf{p} := \mathsf{x}$ yields the map φ in (7.52). (Use the explicit formula given in (3.16).)

Exercise 7.13. Consider the bijection (6.78). Put $\mathsf{c} := \mathsf{E}$ and $\mathsf{a} := \mathsf{E}$, and let $f : \mathsf{c} \to \mathsf{a}$ be the map which on the X-component is scalar multiplication by ξ_{X}, where (ξ_{X}) is any set of scalars indexed by flats. Check that, under the identification (7.47), the resulting morphism of bimonoids $g : \Pi \to \Pi^*$ matches the first formula in (7.51).

7.5. Species of charts and dicharts

We recall charts and dicharts for any arrangement, and then introduce bimonoids based on them. These bimonoids are bicommutative; their analysis parallels that of the bimonoid of flats from Section 7.4.

7.5.1. Charts and dicharts. A *chart* in $\mathcal{A}$ is a subset of the set of hyperplanes in $\mathcal{A}$. This is the same as a subarrangement of $\mathcal{A}$. We will use the letters g, h to denote charts. The *center* of a chart g, denoted $\mathrm{O}(g)$, is the flat obtained by intersecting all hyperplanes in g. A chart is *connected* if its center is the minimum flat. A chart is *coordinate* if it is connected and has size r, where $r := \mathrm{rk}(\mathcal{A})$.

For any chart g and flat X, let g_{X} denote the chart consisting of those hyperplanes in g which contain X, and let g^{X} denote the chart in $\mathcal{A}^{\mathrm{X}}$ obtained by intersecting the hyperplanes in g with X.

A *dichart* in $\mathcal{A}$ is a subset of the set of half-spaces in $\mathcal{A}$. We will use the letters r, s to denote dicharts. By $\mathrm{O}(s)$, we mean the flat obtained by intersecting the bounding hyperplanes of all half-spaces in s.

7.5.2. Bimonoid of charts. A chart in $\mathcal{A}_{\mathrm{X}}$ is the same as a chart in $\mathcal{A}$ whose center contains X. We denote it by g/X, where g is a chart in $\mathcal{A}$ with $\mathrm{O}(g) \geq \mathrm{X}$.

Now define a set-species G as follows. For any flat X, G[X] is the set of charts in $\mathcal{A}_{\mathrm{X}}$. Let $\mathsf{G} := \Bbbk\mathrm{G}$ be the linearization of G. This is the *species of charts*. Let $\mathtt{H}$ denote its canonical basis.

The species of charts carries the structure of a bicommutative bimonoid. We call it the *bimonoid of charts*. The product and coproduct are defined by

$$\begin{aligned} \mu_{\mathrm{Z}}^{\mathrm{Y}} : \mathsf{G}[\mathrm{Y}] &\to \mathsf{G}[\mathrm{Z}] \qquad\qquad & \Delta_{\mathrm{Z}}^{\mathrm{Y}} : \mathsf{G}[\mathrm{Z}] &\to \mathsf{G}[\mathrm{Y}] \\ \mathtt{H}_{g/\mathrm{Y}} &\mapsto \mathtt{H}_{g/\mathrm{Z}} & \mathtt{H}_{g/\mathrm{Z}} &\mapsto \mathtt{H}_{g_{\mathrm{Y}}/\mathrm{Y}}. \end{aligned} \tag{7.53}$$

This structure is set-theoretic.

7.5.3. $\mathtt{Q}$-basis. We now consider a second basis of the species of charts. Define the $\mathtt{Q}$-basis of G, on the X-component, by either of the two equivalent formulas

$$\begin{aligned} \mathtt{H}_{g/\mathrm{X}} &= \sum_{h\colon h\subseteq g} \mathtt{Q}_{h/\mathrm{X}}, \\ \mathtt{Q}_{g/\mathrm{X}} &= \sum_{h\colon h\subseteq g} (-1)^{|g\setminus h|}\, \mathtt{H}_{h/\mathrm{X}}. \end{aligned} \tag{7.54}$$

Note that if $\mathrm{X} \leq \mathrm{O}(g)$, then $\mathrm{X} \leq \mathrm{O}(h)$ for any $h \subseteq g$.

Proposition 7.14. *The product and coproduct of* G *in the* $\mathtt{Q}$*-basis are given by*

$$\begin{aligned} \mu_{\mathrm{Z}}^{\mathrm{Y}} : \mathsf{G}[\mathrm{Y}] &\to \mathsf{G}[\mathrm{Z}] \\ \mathtt{Q}_{g/\mathrm{Y}} &\mapsto \mathtt{Q}_{g/\mathrm{Z}} \end{aligned} \qquad \begin{aligned} \Delta_{\mathrm{Z}}^{\mathrm{Y}} : \mathsf{G}[\mathrm{Z}] &\to \mathsf{G}[\mathrm{Y}] \\ \mathtt{Q}_{g/\mathrm{Z}} &\mapsto \begin{cases} \mathtt{Q}_{g/\mathrm{Y}} & \textit{if } \mathrm{Y} \leq \mathrm{O}(g), \\ 0 & \textit{otherwise.} \end{cases} \end{aligned} \tag{7.55}$$

PROOF. It is easier to start with these formulas, and derive those in the $\mathtt{H}$-basis. The product formula is clear. The coproduct calculation is given below.

$$\Delta_{\mathrm{Z}}^{\mathrm{Y}}(\mathtt{H}_{g/\mathrm{Z}}) = \sum_{h\colon h\subseteq g} \Delta_{\mathrm{Z}}^{\mathrm{Y}}(\mathtt{Q}_{h/\mathrm{Z}}) = \sum_{h\colon h\subseteq g,\, \mathrm{Y}\leq \mathrm{O}(h)} \mathtt{Q}_{h/\mathrm{Y}} = \sum_{h\colon h\subseteq g_{\mathrm{Y}}} \mathtt{Q}_{h/\mathrm{Y}} = \mathtt{H}_{g_{\mathrm{Y}}/\mathrm{Y}}.$$

□

7.5.4. Dual bimonoid. Let G^* denote the bimonoid dual to G. Let $\mathtt{M}$ be the basis dual to $\mathtt{H}$. The product and coproduct in the $\mathtt{M}$-basis are given by

$$\begin{aligned} \mu_{\mathrm{Z}}^{\mathrm{Y}} : \mathsf{G}^*[\mathrm{Y}] &\to \mathsf{G}^*[\mathrm{Z}] \\ \mathtt{M}_{h/\mathrm{Y}} &\mapsto \sum_{g\colon g_{\mathrm{Y}}=h} \mathtt{M}_{g/\mathrm{Z}} \end{aligned} \qquad \begin{aligned} \Delta_{\mathrm{Z}}^{\mathrm{Y}} : \mathsf{G}^*[\mathrm{Z}] &\to \mathsf{G}^*[\mathrm{Y}] \\ \mathtt{M}_{h/\mathrm{Z}} &\mapsto \begin{cases} \mathtt{M}_{h/\mathrm{Y}} & \text{if } \mathrm{Y} \leq \mathrm{O}(h), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In the first formula, it is implicit that $\mathrm{O}(g) \geq \mathrm{Z}$.

Let $\mathtt{P}$ be the basis of G^* dual to $\mathtt{Q}$. Observe that the product and coproduct formulas in the $\mathtt{P}$-basis are identical to those in the $\mathtt{Q}$-basis.

7.5.5. Primitive part and (co)freeness. Recall that a chart g in $\mathcal{A}$ is connected if its center $\mathrm{O}(g)$ is the same as the central face of $\mathcal{A}$. Let cG denote the *species of connected charts*. Explicitly, $\mathsf{cG}[\mathrm{Z}]$ is spanned by charts in $\mathcal{A}$ whose center is Z.

The coproduct formula (7.55) in the $\mathtt{Q}$-basis shows that

$$\mathcal{P}(\mathsf{G}) \cong \mathsf{cG}. \tag{7.56}$$

The component $\mathcal{P}(\mathsf{G})[\mathrm{Z}]$ is spanned by $\mathtt{Q}_{g/\mathrm{Z}}$ such that $\mathrm{O}(g) = \mathrm{Z}$. The product and coproduct formulas (7.55) in the $\mathtt{Q}$-basis show that

$$\mathcal{S}(\mathsf{cG}) \cong \mathsf{G} \tag{7.57}$$

as bicommutative bimonoids. Thus, G is the free commutative and cofree cocommutative bimonoid on the species of connected charts.

Exercise 7.15. Give an example of a chart g and flat Y such that $\mathrm{O}(g) \leq \mathrm{Y}$ but $\mathrm{O}(g_{\mathrm{Y}}) \neq \mathrm{Y}$. Deduce that the subspecies of connected charts viewed in the $\mathtt{H}$-basis is *not* a subcomonoid of G in general.

7.5.6. Self-duality. Let (ξ_g) and (η_g) be two families of scalars indexed by charts such that

$$\xi_g = \sum_{h:\, h \subseteq g} \eta_h \quad \text{and} \quad \eta_g = \sum_{h:\, h \subseteq g} (-1)^{|g \setminus h|}\, \xi_h. \tag{7.58}$$

A nice example is

$$\xi_g = \begin{cases} 1 & \text{if } g \text{ has no hyperplanes,} \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \quad \eta_g := (-1)^{|g|}. \tag{7.59}$$

Theorem 7.16. *For any ξ and η related by (7.58), the map $\mathsf{G} \to \mathsf{G}^*$ given on the* Z*-component by*

$$\mathtt{H}_{g/\mathrm{Z}} \mapsto \sum_{h:\, \mathrm{O}(h) \geq \mathrm{Z}} \xi_{g \cap h}\, \mathtt{M}_{h/\mathrm{Z}}, \qquad \textit{or equivalently,} \qquad \mathtt{Q}_{g/\mathrm{Z}} \mapsto \eta_g\, \mathtt{P}_{g/\mathrm{Z}}$$

is a morphism of bimonoids. In particular, if $\eta_g \neq 0$ for all g, then this map is an isomorphism of bimonoids.

Specializing to (7.59), we obtain:

Theorem 7.17. *The map $\mathsf{G} \to \mathsf{G}^*$ given on the* Z*-component by*

$$\mathtt{H}_{g/\mathrm{Z}} \mapsto \sum_{h:\, g \cap h = \emptyset} \mathtt{M}_{h/\mathrm{Z}}, \qquad \textit{or equivalently,} \qquad \mathtt{Q}_{g/\mathrm{Z}} \mapsto (-1)^{|g|}\, \mathtt{P}_{g/\mathrm{Z}}$$

is an isomorphism of bimonoids. In the first formula from the $\mathtt{H}$-basis to the $\mathtt{M}$-basis, it is implicit that $\mathrm{O}(h) \geq \mathrm{Z}$.

7.5.7. Bimonoid of dicharts. Now define a set-species $\overrightarrow{\mathrm{G}}$ as follows. For any flat X, $\overrightarrow{\mathrm{G}}[\mathrm{X}]$ is the set of dicharts in $\mathcal{A}_{\mathrm{X}}$. We denote a typical element by r/X, where r is a dichart in $\mathcal{A}$ with $\mathrm{O}(r) \geq \mathrm{X}$. Let $\overrightarrow{\mathsf{G}} := \Bbbk\overrightarrow{\mathrm{G}}$ be the linearization of $\overrightarrow{\mathrm{G}}$. This is the *species of dicharts*. Let $\mathtt{H}$ denote its canonical basis.

The species of dicharts carries the structure of a bicommutative bimonoid. We call it the *bimonoid of dicharts*. The product and coproduct are defined by

$$\begin{aligned} \mu^{\mathrm{Y}}_{\mathrm{Z}} : \overrightarrow{\mathsf{G}}[\mathrm{Y}] &\to \overrightarrow{\mathsf{G}}[\mathrm{Z}] \\ \mathtt{H}_{r/\mathrm{Y}} &\mapsto \mathtt{H}_{r/\mathrm{Z}} \end{aligned} \qquad\qquad \begin{aligned} \Delta^{\mathrm{Y}}_{\mathrm{Z}} : \overrightarrow{\mathsf{G}}[\mathrm{Z}] &\to \overrightarrow{\mathsf{G}}[\mathrm{Y}] \\ \mathtt{H}_{r/\mathrm{Z}} &\mapsto \mathtt{H}_{r_{\mathrm{Y}}/\mathrm{Y}}, \end{aligned} \tag{7.60}$$

where r_{Y} is the dichart consisting of those half-spaces in r whose bounding hyperplane contains Y. This structure is set-theoretic.

Let $\overrightarrow{\mathsf{G}}^*$ denote the bimonoid dual to $\overrightarrow{\mathsf{G}}$. Let $\mathtt{M}$ be the basis dual to $\mathtt{H}$. The product and coproduct in the $\mathtt{M}$-basis are given by

$$\begin{aligned} \mu^{\mathrm{Y}}_{\mathrm{Z}} : \overrightarrow{\mathsf{G}}^*[\mathrm{Y}] &\to \overrightarrow{\mathsf{G}}^*[\mathrm{Z}] \\ \mathtt{M}_{s/\mathrm{Y}} &\mapsto \sum_{r:\, r_{\mathrm{Y}} = s} \mathtt{M}_{r/\mathrm{Z}} \end{aligned} \qquad\qquad \begin{aligned} \Delta^{\mathrm{Y}}_{\mathrm{Z}} : \overrightarrow{\mathsf{G}}^*[\mathrm{Z}] &\to \overrightarrow{\mathsf{G}}^*[\mathrm{Y}] \\ \mathtt{M}_{s/\mathrm{Z}} &\mapsto \begin{cases} \mathtt{M}_{s/\mathrm{Y}} & \text{if } \mathrm{Y} \leq \mathrm{O}(s), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In the first formula, it is implicit that $\mathrm{O}(r) \geq \mathrm{Z}$.

The analysis of the bimonoid of dicharts is similar to that of the bimonoid of charts. We elaborate briefly on self-duality.

Theorem 7.18. *The map*

$$\overrightarrow{\mathsf{G}} \to \overrightarrow{\mathsf{G}}^*, \qquad \mathtt{H}_{r/\mathrm{Z}} \mapsto \sum_{s:\, r \cap s = \emptyset} \mathtt{M}_{s/\mathrm{Z}}$$

on the Z *component, is an isomorphism of bimonoids. It is implicit that* $\mathrm{O}(s) \geq \mathrm{Z}$ *in the above sum.*

7.5.8. Morphisms of bimonoids. Flats, charts, dicharts are related by morphisms of bimonoids

$$\Pi \hookrightarrow \mathsf{G} \hookrightarrow \overrightarrow{\mathsf{G}}. \tag{7.61}$$

On the Z-component: The first map sends $\mathtt{H}_{\mathrm{X}/\mathrm{Z}}$ to $\mathtt{H}_{g/\mathrm{Z}}$, where g is the set of hyperplanes containing X. The second map sends $\mathtt{H}_{g/\mathrm{Z}}$ to $\mathtt{H}_{r/\mathrm{Z}}$, where r is the set of those half-spaces whose bounding hyperplane is in g.

7.6. Species of faces

We introduce the bimonoid of faces. It is cocommutative but not commutative. It contains the bimonoid of chambers as a subbimonoid. It is free as a monoid, and may be viewed as the noncommutative analogue of the bimonoid of flats. More precisely, the latter is the abelianization of the former. We consider two bases for the bimonoid of faces, the $\mathtt{H}$-basis and $\mathtt{Q}$-basis. Noncommutative zeta and Möbius functions intervene in the change of basis formulas.

Similar to the bimonoid of chambers, the bimonoid of faces admits a deformation by a parameter q. Now suppose q is not a root of unity. We define the $\mathtt{Q}$-basis in terms of noncommutative q-zeta and q-Möbius functions. We then give a necessary and sufficient condition for the q-norm map on faces (which is a self-dual morphism of q-bimonoids) to be an isomorphism. In particular, this yields explicit self-duality isomorphisms for the q-bimonoid of faces. This includes the case $q = 0$.

7.6.1. Species of faces. For any face A, let $\Sigma[A]$ denote the set of faces greater than A. For faces A and B with the same support, by Lemma 1.6, there is a bijection

$$\beta_{B,A} : \Sigma[A] \to \Sigma[B], \qquad F/A \mapsto BF/B.$$

Thus, Σ is a set-species.

Let $\boldsymbol{\Sigma} := \Bbbk\Sigma$ be the linearization of Σ. This is the *species of faces*. Explicitly, $\boldsymbol{\Sigma}[A]$ is the linear span of the set of faces greater than A. We use the letter $\mathtt{H}$ for the canonical basis of $\boldsymbol{\Sigma}[A]$. For faces A and B of the same support, we write

$$\beta_{B,A} : \boldsymbol{\Sigma}[A] \to \boldsymbol{\Sigma}[B], \qquad \mathtt{H}_{F/A} \mapsto \mathtt{H}_{BF/B}. \tag{7.62}$$

We claim that

$$\Sigma = \mathsf{E} + \mathsf{E}^2 + \mathsf{E}^3 + \dots, \tag{7.63}$$

the sum of all Cauchy powers of the exponential species E. Explicitly, the A-component of the rhs is $\bigoplus_{F \geq A} \mathsf{E}[F]$. This is a vector space with basis indexed by faces F greater than A, and we identify this with the $\mathtt{H}$-basis of $\Sigma[A]$.

7.6.2. Bimonoid of faces. The species Σ carries the structure of a bimonoid. We call it the *bimonoid of faces*. The product and coproduct are defined by

$$\begin{aligned} \mu_A^F : \Sigma[F] &\to \Sigma[A] \qquad & \Delta_A^G : \Sigma[A] &\to \Sigma[G] \\ \mathtt{H}_{K/F} &\mapsto \mathtt{H}_{K/A} & \mathtt{H}_{K/A} &\mapsto \mathtt{H}_{GK/G}. \end{aligned} \tag{7.64}$$

The structure is set-theoretic. The checking of the bimonoid axiom is similar to that for the bimonoid of chambers.

7.6.3. *q*-bimonoid of faces. More generally, for any scalar q, the species of faces carries the structure of a q-bimonoid which we denote by Σ_q. We call it the *q-bimonoid of faces*. The product and coproduct are defined by

$$\begin{aligned} \mu_A^F : \Sigma_q[F] &\to \Sigma_q[A] \qquad & \Delta_A^G : \Sigma_q[A] &\to \Sigma_q[G] \\ \mathtt{H}_{K/F} &\mapsto \mathtt{H}_{K/A} & \mathtt{H}_{K/A} &\mapsto q^{\mathrm{dist}(K,G)}\, \mathtt{H}_{GK/G}. \end{aligned} \tag{7.65}$$

Note that for $q = 1$, we have $\Sigma_1 = \Sigma$, the bimonoid of faces.

Exercise 7.19. Check that Σ_q satisfies the q-bimonoid axiom (2.33). (Use property (1.30d) of the q-distance function. Also see the similar calculation for the q-bimonoid of chambers Γ_q.)

Exercise 7.20. Recall that every chamber is a face. Check that the resulting injective map $\Gamma_q \hookrightarrow \Sigma_q$ is a morphism of q-bimonoids.

7.6.4. Tits algebra. The bimonoid of faces Σ also carries an internal structure. For each face A, the component $\Sigma[A]$ is an algebra with product in the $\mathtt{H}$-basis given by

$$\mathtt{H}_{F/A} \cdot \mathtt{H}_{G/A} = \mathtt{H}_{FG/A}. \tag{7.66}$$

The unit element is $\mathtt{H}_{A/A}$. This algebra can be identified with the Tits algebra of the arrangement $\mathcal{A}_A$: Compare (7.66) with (1.105).

Exercise 7.21. Check that: The maps $\beta_{B,A} : \Sigma[A] \to \Sigma[B]$ and $\Delta_A^G : \Sigma[A] \to \Sigma[G]$ are algebra morphisms. The map $\mu_A^F : \Sigma[F] \to \Sigma[A]$ preserves the Tits product but does not preserve the unit when $F > A$.

7.6.5. $\mathtt{Q}$-basis. Fix a noncommutative zeta function $\boldsymbol{\zeta}$ and its inverse noncommutative Möbius function $\boldsymbol{\mu}$. Define a $\mathtt{Q}$-basis of Σ, on the A-component, by either of the two equivalent formulas

$$\begin{aligned} \mathtt{H}_{F/A} &= \sum_{G:\, F \leq G} \boldsymbol{\zeta}(F,G)\, \mathtt{Q}_{G/A}, \\ \mathtt{Q}_{F/A} &= \sum_{G:\, F \leq G} \boldsymbol{\mu}(F,G)\, \mathtt{H}_{G/A}. \end{aligned} \tag{7.67}$$

This is obtained by applying (1.107) to the arrangement $\mathcal{A}_A$ for each face A. Similarly, from (1.110), we obtain: If $F > A$, then

$$\mathtt{H}_{F/A} \cdot \mathtt{Q}_{A/A} = 0. \tag{7.68}$$

Proposition 7.22. *The product and coproduct of Σ in the $\mathtt{Q}$-basis are given by*

$$\begin{gathered} \mu_A^F : \Sigma[F] \to \Sigma[A] \qquad\qquad \Delta_A^G : \Sigma[A] \to \Sigma[G] \\ \mathtt{Q}_{K/F} \mapsto \mathtt{Q}_{K/A} \qquad\qquad \mathtt{Q}_{K/A} \mapsto \begin{cases} \mathtt{Q}_{GK/G} & \text{if } KG = K, \\ 0 & \text{otherwise.} \end{cases} \end{gathered} \tag{7.69}$$

The product works the same way as in the $\mathtt{H}$-basis, but the coproduct works differently.

Proof. We employ (7.64). The product calculation is easy.

$$\mu_A^F(\mathtt{Q}_{K/F}) = \mu_A^F\big(\sum_{G:\, K \leq G} \boldsymbol{\mu}(K,G)\, \mathtt{H}_{G/F}\big) = \sum_{G:\, K \leq G} \boldsymbol{\mu}(K,G)\, \mathtt{H}_{G/A} = \mathtt{Q}_{K/A}.$$

The coproduct calculation is more interesting.

$$\begin{aligned} \Delta_A^G(\mathtt{Q}_{K/A}) &= \Delta_A^G\big(\sum_{F:\, K \leq F} \boldsymbol{\mu}(K,F)\, \mathtt{H}_{F/A}\big) \\ &= \sum_{F:\, K \leq F} \boldsymbol{\mu}(K,F)\, \mathtt{H}_{GF/G} \\ &= \sum_{H:\, H \geq GK} \big(\sum_{F:\, K \leq F,\, GF = H} \boldsymbol{\mu}(K,F)\big) \mathtt{H}_{H/G} \\ &= \sum_{H:\, H \geq GK} \big(\sum_{F:\, K \leq F,\, KGF = KH} \boldsymbol{\mu}(K,F)\big) \mathtt{H}_{H/G}. \end{aligned}$$

By the noncommutative Weisner formula (1.44), this is zero if $KG > K$. Assuming $KG = K$, the calculation continues as follows.

$$\begin{aligned} &= \sum_{H:\, H \geq GK} \boldsymbol{\mu}(K, KH)\, \mathtt{H}_{H/G} \\ &= \sum_{H:\, H \geq GK} \boldsymbol{\mu}(GK, H)\, \mathtt{H}_{H/G} \\ &= \mathtt{Q}_{GK/G}. \end{aligned}$$

The second-to-last step used (1.40) for $s := \boldsymbol{\mu}$.

Illustrative pictures for the cases $KG > K$ and $KG = K$, respectively, are shown below. The face A is taken to be the central face.

Alternatively: By definition, the coproduct component Δ_A^G is given by left multiplication by $\mathtt{H}_{G/A}$ in $\Sigma[A]$ followed by viewing the result inside $\Sigma[G]$. So the coproduct formula in the $\mathtt{Q}$-basis can be deduced from (1.108). $\square$

Exercise 7.23. For faces A and B of the same support, show that (7.62) can be written as

$$\beta_{B,A} : \Sigma[A] \to \Sigma[B], \qquad \mathtt{Q}_{F/A} \mapsto \mathtt{Q}_{BF/B}.$$

(Use Lemma 1.6 and (1.40) for $s := \boldsymbol{\mu}$.)

7.6.6. $\mathtt{Q}$-basis for q not a root of unity. Suppose q is not a root of unity. Recall the noncommutative q-zeta function $\boldsymbol{\zeta}_q$ and the noncommutative q-Möbius function $\boldsymbol{\mu}_q$ defined in Section 1.5.9. Define the $\mathtt{Q}$-basis of Σ_q, on the A-component, by either of the two equivalent formulas

$$\begin{aligned} \mathtt{H}_{F/A} &= \sum_{G:\, F \leq G} \boldsymbol{\zeta}_q(F,G)\, \mathtt{Q}_{G/A}, \\ \mathtt{Q}_{F/A} &= \sum_{G:\, F \leq G} \boldsymbol{\mu}_q(F,G)\, \mathtt{H}_{G/A}. \end{aligned} \tag{7.70}$$

This is the same as formulas (7.67) with $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ replaced by $\boldsymbol{\zeta}_q$ and $\boldsymbol{\mu}_q$, respectively. However, note very carefully that in contrast to that case, the $\mathtt{Q}$-basis here is unique.

Proposition 7.24. *For q not a root of unity, the product and coproduct of Σ_q in the $\mathtt{Q}$-basis are given by*

$$\begin{aligned} &\mu_A^F : \Sigma_q[F] \to \Sigma_q[A] &\qquad &\Delta_A^G : \Sigma_q[A] \to \Sigma_q[G] \\ &\mathtt{Q}_{K/F} \mapsto \mathtt{Q}_{K/A} & &\mathtt{Q}_{K/A} \mapsto \begin{cases} q^{\mathrm{dist}(K,G)}\, \mathtt{Q}_{GK/G} & \text{if } KG = K, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{7.71}$$

The product works the same way as in the $\mathtt{H}$-basis, but the coproduct works differently.

Proof. We calculate as in the proof of Proposition 7.22. In the coproduct calculation, we first use the identity

$$q^{\mathrm{dist}(F,G)} = q^{\mathrm{dist}(F,KG)} q^{\mathrm{dist}(KG,G)}$$

which holds by property (1.30d), and then use the noncommutative q-Weisner formula (1.48). $\square$

Exercise 7.25. Check that the result of Exercise 7.23 also holds for the $\mathtt{Q}$-basis of Σ_q for q not a root of unity.

7.6.7. Eulerian idempotents. We return to the bimonoid Σ. For any face A and flat $\mathrm{X} \geq \mathrm{s}(A)$, put

$$\mathtt{E}_{\mathrm{X}/A} = \sum_{\substack{F:\, F \geq A \\ \mathrm{s}(F) = \mathrm{X}}} \boldsymbol{\zeta}(A,F)\, \mathtt{Q}_{F/A}. \tag{7.72}$$

The $\mathtt{E}_{\mathrm{X}/A}$, as X varies, are eulerian idempotents of the Tits algebra $\Sigma[A]$, see (1.112). The first eulerian idempotent of $\Sigma[A]$ is

$$\mathtt{E}_{\mathrm{s}(A)/A} = \mathtt{Q}_{A/A} = \sum_{F:\, F \geq A} \boldsymbol{\mu}(A,F)\, \mathtt{H}_{F/A}. \tag{7.73}$$

We emphasize that it is not unique and depends on the choice of $\boldsymbol{\zeta}$ and $\boldsymbol{\mu} = \boldsymbol{\zeta}^{-1}$.

Exercise 7.26. Check that

$$\Delta_A^G(\mathtt{E}_{\mathrm{X}/A}) = \begin{cases} \mathtt{E}_{\mathrm{X}/G} & \text{if } \mathrm{X} \geq \mathrm{s}(G), \\ 0 & \text{otherwise.} \end{cases} \tag{7.74}$$

(Use definition (7.72), coproduct formula (7.69) and the lune-additivity formula (1.42).)

For any face A and flat $\mathrm{X} \geq \mathrm{s}(A)$, put

$$\mathtt{u}_{\mathrm{X}/A} = \sum_{\substack{F:\, F \geq A \\ \mathrm{s}(F) = \mathrm{X}}} \boldsymbol{\zeta}(A, F)\, \mathtt{H}_{F/A}. \tag{7.75}$$

In view of (1.116),

$$\mathtt{u}_{\mathrm{X}/A} \cdot \mathtt{E}_{\mathrm{X}/A} = \mathtt{E}_{\mathrm{X}/A}, \tag{7.76}$$

with the product in the lhs taken in the Tits algebra $\Sigma[A]$.

7.6.8. Dual bimonoid. Let Σ^* denote the bimonoid dual to Σ. Let $\mathtt{M}$ denote the basis which is dual to the $\mathtt{H}$-basis. The product and coproduct of Σ^* are obtained by dualizing formulas (7.64). They are given by

$$\begin{gathered} \mu_A^G : \Sigma^*[G] \to \Sigma^*[A] \qquad\qquad \Delta_A^F : \Sigma^*[A] \to \Sigma^*[F] \\ \mathtt{M}_{H/G} \mapsto \sum_{\substack{K:\, K \geq A \\ GK = H}} \mathtt{M}_{K/A} \qquad\qquad \mathtt{M}_{K/A} \mapsto \begin{cases} \mathtt{M}_{K/F} & \text{if } F \leq K, \\ 0 & \text{otherwise.} \end{cases} \end{gathered} \tag{7.77}$$

Let $\mathtt{P}$ be the basis of Σ^* dual to $\mathtt{Q}$. Its product and coproduct formulas can be written by dualizing (7.69). The coproduct formula works the same way as in the $\mathtt{M}$-basis.

7.6.9. Dual q-bimonoid. More generally, for any scalar q, let Σ_q^* denote the q-bimonoid dual to Σ_q. Dualizing formulas (7.65), its product and coproduct are given by

$$\begin{gathered} \mu_A^G : \Sigma_q^*[G] \to \Sigma_q^*[A] \qquad\qquad \Delta_A^F : \Sigma_q^*[A] \to \Sigma_q^*[F] \\ \mathtt{M}_{H/G} \mapsto \sum_{\substack{K:\, K \geq A \\ GK = H}} q^{\operatorname{dist}(H,K)}\, \mathtt{M}_{K/A} \qquad\qquad \mathtt{M}_{K/A} \mapsto \begin{cases} \mathtt{M}_{K/F} & \text{if } F \leq K, \\ 0 & \text{otherwise.} \end{cases} \end{gathered} \tag{7.78}$$

For q not a root of unity, let $\mathtt{P}$ be the basis of Σ_q^* dual to $\mathtt{Q}$. Its product and coproduct formulas can be written by dualizing (7.71). The coproduct formula works the same way as in the $\mathtt{M}$-basis.

7.6.10. Primitive part. We deduce from coproduct formula (7.78) in the $\mathtt{M}$-basis that

$$\mathcal{P}(\Sigma_q^*) = \mathsf{E}. \tag{7.79}$$

Each component $\mathcal{P}(\Sigma_q^*)[F]$ is one-dimensional, and is spanned by $\mathtt{M}_{F/F}$. (The same is true in the $\mathtt{P}$-basis of Σ^* or of Σ_q^* for q not a root of unity.) More generally, the primitive filtration of Σ_q^* can be expressed as

$$\mathcal{P}(\Sigma_q^*) = \mathsf{E} + \mathsf{E}^2 + \cdots + \mathsf{E}^k, \tag{7.80}$$

the sum of the first k Cauchy powers of E.

The primitive part of Σ_q is discussed separately later in Section 7.9.4.

Exercise 7.27. Use (5.41) to deduce that the indecomposable part of Σ is given by

$$\mathcal{Q}(\Sigma) = \mathsf{E}.$$

Also check this fact directly using product formula (7.64).

7.6.11. Freeness and cofreeness. We now state the (co)freeness properties of Σ_q and its dual. The proofs are straightforward, so we omit them.

The q-bimonoid Σ_q is the free q-bimonoid on E viewed as a comonoid. Dually, Σ_q^* is the cofree q-bimonoid on E viewed as a monoid. More precisely:

Proposition 7.28. *For any scalar q, there are isomorphisms of q-bimonoids*

$$\Sigma_q \xrightarrow{\cong} \mathcal{T}_q(\mathsf{E}) \quad \textit{and} \quad \Sigma_q^* \xrightarrow{\cong} \mathcal{T}_q^\vee(\mathsf{E}). \tag{7.81}$$

On the A-component, the first map sends $\mathtt{H}_{F/A}$ to $\mathtt{H}_F$ in the F-summand, while the second map sends $\mathtt{M}_{F/A}$ to $\mathtt{H}_F$ in the F-summand.

The (co)product of Σ_q in the $\mathtt{H}$-basis is given by (7.65), while that of Σ_q^* in the $\mathtt{M}$-basis is given by (7.78). The (co)product of $\mathcal{T}_q(\mathsf{E})$ is given by specializing (6.3) and (6.5), while that of $\mathcal{T}_q^\vee(\mathsf{E})$ is given by specializing (6.11) and (6.13).

Exercise 7.29. By specializing Theorem 6.6 and Theorem 6.13, check that Σ_q and Σ_q^* satisfy the following universal properties.

For h a q-bimonoid and $f : \mathsf{E} \to \mathsf{h}$ a morphism of comonoids, there exists a unique morphism of q-bimonoids $\hat{f} : \Sigma_q \to \mathsf{h}$ such that the diagram

$$\begin{tikzcd} \Sigma_q \arrow[r, dashed, "\hat{f}"] & \mathsf{h} \\ \mathsf{E} \arrow[u, hook] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes, where $\mathsf{E} \hookrightarrow \Sigma_q$ on the A-component sends $\mathtt{H}_A$ to $\mathtt{H}_{A/A}$.

For h a q-bimonoid and $f : \mathsf{h} \to \mathsf{E}$ a morphism of monoids, there exists a unique morphism of q-bimonoids $\hat{f} : \mathsf{h} \to \Sigma_q^*$ such that the diagram

$$\begin{tikzcd} \mathsf{h} \arrow[r, dashed, "\hat{f}"] \arrow[dr, "f"'] & \Sigma_q^* \arrow[d, two heads] \\ & \mathsf{E} \end{tikzcd}$$

commutes, where $\Sigma_q^* \twoheadrightarrow \mathsf{E}$ on the A-component sends $\mathtt{M}_{F/A}$ to $\mathtt{H}_A$ if $F = A$, and to 0 otherwise.

Let us now focus on $q = 1$. The bimonoid Σ is also the free bimonoid on E viewed as a trivial comonoid. Dually, Σ^* is the cofree bimonoid on E viewed as a trivial monoid. More precisely:

Proposition 7.30. *There are isomorphisms of bimonoids*

$$\Sigma \xrightarrow{\cong} \mathcal{T}(\mathsf{E}) \quad \textit{and} \quad \Sigma^* \xrightarrow{\cong} \mathcal{T}^\vee(\mathsf{E}). \tag{7.82}$$

On the A-component, the first map sends $\mathtt{Q}_{F/A}$ to $\mathtt{H}_F$ in the F-summand, while the second map sends $\mathtt{P}_{F/A}$ to $\mathtt{H}_F$ in the F-summand.

Note very carefully that the $\mathtt{Q}$-basis of Σ and the $\mathtt{P}$-basis of Σ^* are employed here. The relevant (co)product formulas are (7.69) and its dual. The (co)product of $\mathcal{T}(\mathsf{E})$ is given by specializing (6.39) and of $\mathcal{T}^\vee(\mathsf{E})$ is given by specializing (6.40) and also setting $q = 1$.

Similar considerations apply when q is not a root of unity. The q-bimonoid Σ_q is the free q-bimonoid on E viewed as a trivial comonoid. Dually, Σ_q^* is the cofree q-bimonoid on E viewed as a trivial monoid. More precisely:

Proposition 7.31. *For q not a root of unity, there are isomorphisms of q-bimonoids*

$$\Sigma_q \xrightarrow{\cong} \mathcal{T}_q(\mathsf{E}) \quad \textit{and} \quad \Sigma_q^* \xrightarrow{\cong} \mathcal{T}_q^\vee(\mathsf{E}). \tag{7.83}$$

On the A-component, the first map sends $\mathtt{Q}_{F/A}$ to $\mathtt{H}_F$ in the F-summand, while the second map sends $\mathtt{P}_{F/A}$ to $\mathtt{H}_F$ in the F-summand.

The (co)product of Σ_q in the $\mathtt{Q}$-basis is given by (7.71). The (co)product of $\mathcal{T}_q(\mathsf{E})$ is given by specializing (6.39) and of $\mathcal{T}_q^\vee(\mathsf{E})$ is given by specializing (6.40).

Remark 7.32. Note very carefully how the passage from the $\mathtt{H}$-basis to the $\mathtt{Q}$-basis of Σ (or of Σ_q when q is not a root of unity) allowed us to trivialize the (co)product of E. This phenomenon is studied in detail in Chapter 14, see in particular, Examples 14.51 and 14.76. Noncommutative zeta and Möbius functions play a central role in this story.

Exercise 7.33. Since $\Sigma = \mathcal{T}(\mathsf{E})$ and $\Pi = \mathcal{S}(\mathsf{E})$, we deduce from (6.58) that $\Pi = \Sigma_{ab}$, the abelianization of Σ. Explicitly, the abelianization map is given by

$$\pi : \Sigma \twoheadrightarrow \Pi, \qquad \mathtt{H}_{F/A} \mapsto \mathtt{H}_{\mathrm{s}(F/A)}. \tag{7.84}$$

We also call this the support map. The same formula holds on the $\mathtt{Q}$-basis as well. Dually, $\Pi^* = (\Sigma^*)^{coab}$. Check these facts directly.

7.6.12. *q*-norm map and self-duality. Let (ξ_X) be any set of scalars indexed by flats. For any scalar q, consider the map of species

$$\Sigma_q \to \Sigma_q^*, \qquad \mathtt{H}_{F/A} \mapsto \sum_{G:\, G \geq A} \xi_{\mathrm{s}(FG)}\, q^{\mathrm{dist}(F,G)}\, \mathtt{M}_{G/A} \tag{7.85}$$

on the A-component. We call it the *q-norm map* on faces.

Lemma 7.34. *The q-norm map on faces (7.85) is a self-dual morphism of q-bimonoids.*

PROOF. We first do the case $q = 1$. Let us temporarily write f for the map (7.85). Self-duality of f follows from the fact that $\mathrm{s}(FG) = \mathrm{s}(GF)$ which holds by (1.3). Lemma 1.6 and (1.3) imply that f is a map of species. We claim that the following diagrams commute.

$$\begin{tikzcd} \Sigma_q[F] \arrow[r, "f_F"] \arrow[d, "\mu_A^F"'] & \Sigma_q^*[F] \arrow[d, "\mu_A^F"] \\ \Sigma_q[A] \arrow[r, "f_A"'] & \Sigma_q^*[A] \end{tikzcd} \qquad \begin{tikzcd} \Sigma_q^*[F] & \Sigma_q[F] \arrow[l, "f_F"'] \\ \Sigma_q^*[A] \arrow[u, "\Delta_A^F"] & \Sigma_q[A] \arrow[l, "f_A"] \arrow[u, "\Delta_A^F"'] \end{tikzcd}$$

They say that f is a morphism of monoids and of comonoids, respectively. Since they are duals of each other, it suffices to check that the first diagram commutes. We calculate:

$$\begin{aligned} \mu_A^F f_F(\mathtt{H}_{K/F}) &= \mu_A^F\Big(\sum_{G:\, G\geq F} \xi_{\mathrm{s}(KG)}\, \mathtt{M}_{G/F}\Big) \\ &= \sum_{G:\, G\geq F}\ \sum_{\substack{H:\, H\geq A\\ FH=G}} \xi_{\mathrm{s}(KG)}\, \mathtt{M}_{H/A} \\ &= \sum_{H:\, H\geq A} \xi_{\mathrm{s}(KFH)}\, \mathtt{M}_{H/A} \\ &= \sum_{H:\, H\geq A} \xi_{\mathrm{s}(KH)}\, \mathtt{M}_{H/A} \\ &= f_A(\mathtt{H}_{K/A}) \\ &= f_A \mu_A^F(\mathtt{H}_{K/F}). \end{aligned}$$

The fourth step used (1.3).

For the case of general q, we use in addition properties of the q-distance function (1.30d), or equivalently, (1.30e), and also (1.30f). $\square$

The map (7.85) is an instance of the q-norm map in Section 6.9.3. This is elaborated in Exercise 7.37 below. It yields another proof of Lemma 7.34.

Exercise 7.35. Consider the self-dual morphism of q-bimonoids obtained as the composite

$$\Gamma_q \hookrightarrow \Sigma_q \to \Sigma_q^* \twoheadrightarrow \Gamma_q^*.$$

The middle map is (7.85), the first map is as in Exercise 7.20, while the last map is its dual. Check that this composite map coincides with (7.26) when $\xi_\top = 1$. Deduce Lemma 7.4 as a consequence.

Exercise 7.36. For $q = 1$, check that the map (7.85) factors through the (co)abelianization (7.84) to yield the following self-dual diagram of bimonoids.

$$\begin{tikzcd} \mathsf{\Sigma} \arrow[r] \arrow[d, two heads, "\pi"'] & \mathsf{\Sigma}^* \\ \mathsf{\Pi} \arrow[r, "\cong"'] & \mathsf{\Pi}^* \arrow[u, hook, "\pi^*"'] \end{tikzcd} \tag{7.86}$$

The bottom-horizontal map coincides with (7.51). Deduce that for $q = 1$, the map (7.85) is given by

$$\mathsf{\Sigma} \to \mathsf{\Sigma}^*, \qquad \mathtt{Q}_{F/A} \mapsto \eta_{\mathrm{s}(F)} \sum_{\substack{G:\, G \geq A \\ \mathrm{s}(G) = \mathrm{s}(F)}} \mathtt{P}_{G/A} \tag{7.87}$$

on the A-component.

Exercise 7.37. Consider the bijection (6.77). Put $\mathsf{c} := \mathsf{E}$ and $\mathsf{a} := \mathsf{E}$, and let $f : \mathsf{c} \to \mathsf{a}$ be the map which on the F-component is scalar multiplication by $\xi_{\mathrm{s}(F)}$. Check that, under the identification (7.81), the resulting morphism of q-bimonoids $g : \mathsf{\Sigma}_q \to \mathsf{\Sigma}_q^*$ matches (7.85). Further, for $q = 1$, diagram (6.79) specializes to (7.86). See Exercise 7.13 in this regard.

Fix a scalar q which is not a root of unity. Let (η_{X}) be the family of scalars which is related to (ξ_{X}) by

$$\xi_{\mathrm{X}} = \sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{X}} \zeta_q(\mathrm{X}, \mathrm{Y})\, \eta_{\mathrm{Y}} \quad \text{and} \quad \eta_{\mathrm{X}} = \sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{X}} \mu_q(\mathrm{X}, \mathrm{Y})\, \xi_{\mathrm{Y}}, \tag{7.88}$$

where ζ_q and μ_q are the q-zeta function and q-Möbius function in (1.50). (Note very carefully the difference with (7.49).)

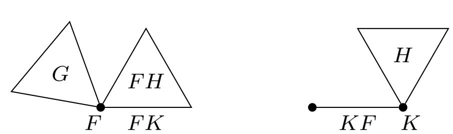
We now give a formula for the map (7.85) from the $\mathtt{Q}$-basis defined in (7.70) to its dual $\mathtt{P}$-basis.

Theorem 7.38. *Suppose q is not a root of unity. Then the q-norm map on faces (7.85) is given by*

$$\mathsf{\Sigma}_q \to \mathsf{\Sigma}_q^*, \qquad \mathtt{Q}_{F/A} \mapsto \eta_{\mathrm{s}(F)} \sum_{\substack{G:\, G \geq A \\ \mathrm{s}(G) = \mathrm{s}(F)}} q^{\mathrm{dist}(F,G)}\, \mathtt{P}_{G/A} \tag{7.89}$$

on the A-component. In particular, this map is an isomorphism of q-bimonoids iff $\eta_{\mathrm{X}} \neq 0$ for all X.

Proof. We start with formula (7.89) and derive (7.85). A useful illustration is shown below.



The computation goes as follows.

$$\begin{aligned}
\mathtt{H}_{F/A} &= \sum_{G:\, F\leq G} \zeta_q(F,G)\, \mathtt{Q}_{G/A} \\
&\mapsto \sum_{G:\, F\leq G} \zeta_q(F,G)\, \eta_{\mathrm{s}(G)} \sum_{\substack{H:\, H\geq A \\ \mathrm{s}(H)=\mathrm{s}(G)}} q^{\mathrm{dist}(G,H)}\, \mathtt{P}_{H/A} \\
&= \sum_{K:\, K\geq A} \Big(\sum_{\substack{G,H:\, F\leq G,\, K\leq H \\ \mathrm{s}(G)=\mathrm{s}(H)}} \zeta_q(F,G)\, q^{\mathrm{dist}(G,H)} \zeta_q(K,H)\, \eta_{\mathrm{s}(G)} \Big)\, \mathtt{M}_{K/A} \\
&= \sum_{K:\, K\geq A} \Big(\sum_{H:\, K\leq H,\, HF=H} \zeta_q(K,H)\, q^{\mathrm{dist}(KF,H)}\, \eta_{\mathrm{s}(H)} \Big)\, q^{\mathrm{dist}(F,K)}\, \mathtt{M}_{K/A} \\
&= \sum_{K:\, K\geq A} \Big(\sum_{H:\, KF\leq H} \zeta_q(KF,H)\, \eta_{\mathrm{s}(H)} \Big)\, q^{\mathrm{dist}(F,K)}\, \mathtt{M}_{K/A} \\
&= \sum_{K:\, K\geq A} \xi_{\mathrm{s}(KF)}\, q^{\mathrm{dist}(F,K)}\, \mathtt{M}_{K/A}.
\end{aligned}$$

The first step used (7.70), while the third step used its dual formula. The fourth step used the q-flat-additivity formula (1.47), while the fifth step used the q-lune-additivity formula (1.46). The last step used (1.50) and (7.88).

The second claim follows from the first and Theorem 1.10. $\square$

A different way of doing this calculation is suggested in the exercise below. A more conceptual proof of formula (7.89) using the q-exponential and q-logarithm is given later in Exercise 14.82. A proof of the last claim using similar ideas is given in Exercise 9.98.

Exercise 7.39. Check that: For q not a root of unity, the map (7.85) is given by

$$\Sigma_q \to \Sigma_q^*, \qquad \mathtt{H}_{F/A} \mapsto \sum_{\substack{G:\, G\geq A \\ GF=G}} q^{\mathrm{dist}(F,G)}\, \eta_{\mathrm{s}(G)}\, \mathtt{P}_{G/A} \tag{7.90}$$

on the A-component. (Use the noncommutative q-Weisner formula (1.48).) Prove formula (7.89) by showing that it also matches (7.90). (Use the q-flat-additivity formula (1.47).)

We highlight below two special cases of the last claim in Theorem 7.38.

Theorem 7.40. *Suppose q is not a root of unity. The map*

$$\Sigma_q \to \Sigma_q^*, \qquad \mathtt{H}_{F/A} \mapsto \sum_{G:\, G\geq A,\, \mathrm{s}(FG)=\top} q^{\mathrm{dist}(F,G)}\, \mathtt{M}_{G/A} \tag{7.91}$$

on the A-component, is an isomorphism of q-bimonoids iff $\mu_q(\mathrm{X},\top) \neq 0$ for any flat X.

Similarly, the map

$$\Sigma_q \to \Sigma_q^*, \qquad \mathtt{H}_{F/A} \mapsto \sum_{G:\, G\geq A} c_{\mathrm{s}(FG)}\, q^{\mathrm{dist}(F,G)}\, \mathtt{M}_{G/A} \tag{7.92}$$

on the A-component, is an isomorphism of q-bimonoids iff

$$\sum_{\mathrm{Y}\colon \mathrm{Y}\geq \mathrm{X}} \mu_q(\mathrm{X},\mathrm{Y})\, c_{\mathrm{Y}} \neq 0$$

for any flat X.

We recollect that these isomorphism criteria arise by writing the maps from the $\mathtt{Q}$-basis and into the $\mathtt{P}$-basis as in (7.89).

Example 7.41. Let $\mathcal{A}$ be a rank-one arrangement with chambers C and $\overline{C}$. Let us make explicit the map (7.91) on the O-component. On the $\mathtt{H}$-basis and into the $\mathtt{M}$-basis, it is given by

$$\mathtt{H}_O \mapsto \mathtt{M}_C + \mathtt{M}_{\overline{C}}, \qquad \mathtt{H}_C \mapsto \mathtt{M}_O + \mathtt{M}_C + q\,\mathtt{M}_{\overline{C}}, \qquad \mathtt{H}_{\overline{C}} \mapsto \mathtt{M}_O + q\,\mathtt{M}_C + \mathtt{M}_{\overline{C}}.$$

On the $\mathtt{Q}$-basis and into the $\mathtt{P}$-basis, it is given by

$$\mathtt{Q}_O \mapsto \frac{-2}{q+1}\,\mathtt{P}_O, \qquad \mathtt{Q}_C \mapsto \mathtt{P}_C + q\,\mathtt{P}_{\overline{C}}, \qquad \mathtt{Q}_{\overline{C}} \mapsto q\,\mathtt{P}_C + \mathtt{P}_{\overline{C}}.$$

We used that $\mu_q(\perp,\top) = \frac{-2}{q+1}$ from Example 1.24. In matrix form, the maps can be, respectively, written as

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & q \\ 1 & q & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{-2}{q+1} & 0 & 0 \\ 0 & 1 & q \\ 0 & q & 1 \end{pmatrix}.$$

Note very carefully the block-diagonal form of the second matrix. Since the change of basis is unitriangular, the determinants of the two matrices are equal. In this case, they both turn out to be $2(q-1)$.

The above discussion can be seen in analogy with the one in Section 7.4.8. Compare and contrast Theorem 7.38 with Theorem 7.10, and Theorem 7.40 with Theorem 7.11.

7.6.13. 0-bimonoid of faces. We now specialize to $q=0$. Observe that the product and coproduct of Σ_0 are given by

$$\begin{gathered} \mu_A^F : \Sigma_0[F] \to \Sigma_0[A] \qquad\qquad \Delta_A^G : \Sigma_0[A] \to \Sigma_0[G] \\ \mathtt{H}_{K/F} \mapsto \mathtt{H}_{K/A} \qquad\qquad \mathtt{H}_{K/A} \mapsto \begin{cases} \mathtt{H}_{GK/G} & \text{if } KG = GK, \\ 0 & \text{otherwise.} \end{cases} \end{gathered} \tag{7.93}$$

This is the 0-*bimonoid of faces*.

Define the $\mathtt{Q}$-basis of Σ_0, on the A-component, by either of the two equivalent formulas

$$\begin{aligned} \mathtt{H}_{F/A} &= \sum_{G\colon F\leq G} \mathtt{Q}_{G/A}, \\ \mathtt{Q}_{F/A} &= \sum_{G\colon F\leq G} (-1)^{\mathrm{rk}(G/F)}\, \mathtt{H}_{G/A}. \end{aligned} \tag{7.94}$$

This is the case $q=0$ of (7.70) in view of formulas (1.51). By specializing Proposition 7.24 to $q=0$, we obtain:

Proposition 7.42. *The product and coproduct of* Σ_0 *in the* $\mathtt{Q}$*-basis are given by*

$$\begin{array}{ccc} \mu_A^F : \Sigma_0[F] \to \Sigma_0[A] & & \Delta_A^G : \Sigma_0[A] \to \Sigma_0[G] \\ \mathtt{Q}_{K/F} \mapsto \mathtt{Q}_{K/A} & & \mathtt{Q}_{K/A} \mapsto \begin{cases} \mathtt{Q}_{K/G} & \textit{if } G \leq K, \\ 0 & \textit{otherwise.} \end{cases} \end{array} \tag{7.95}$$

Now consider the dual 0-bimonoid Σ_0^*. Its product and coproduct are given by

$$\begin{array}{ccc} \mu_A^G : \Sigma_0^*[G] \to \Sigma_0^*[A] & & \Delta_A^F : \Sigma_0^*[A] \to \Sigma_0^*[F] \\ \mathtt{M}_{H/G} \mapsto \displaystyle\sum_{\substack{K:\, K \geq A \\ KG = GK = H}} \mathtt{M}_{K/A} & & \mathtt{M}_{K/A} \mapsto \begin{cases} \mathtt{M}_{K/F} & \text{if } F \leq K, \\ 0 & \text{otherwise.} \end{cases} \end{array} \tag{7.96}$$

This follows by specializing (7.78) to $q = 0$, or by dualizing (7.93). Similarly, the product and coproduct formulas in the $\mathtt{P}$-basis can be written by dualizing (7.95).

7.7. Species of top-nested faces and top-lunes

We introduce the bimonoid of top-nested faces. It is neither commutative nor cocommutative. It is free as a monoid, and contains the bimonoid of faces as a subbimonoid. We also consider its abelianization which is the bimonoid of top-lunes. The latter contains the bimonoid of flats as a subbimonoid. The bimonoids of top-nested faces and top-lunes can thus be seen in analogy with the bimonoids of faces and flats. We consider two bases for the bimonoid of top-nested faces, the $\mathtt{H}$-basis and $\mathtt{K}$-basis.

The bimonoid of top-nested faces admits a deformation by a parameter q. Now suppose q is not a root of unity. We define the $\mathtt{Q}$-basis in terms of two-sided q-zeta and q-Möbius functions. We then give a necessary and sufficient condition for the q-norm map on top-nested faces (which is a self-dual morphism of q-bimonoids) to be an isomorphism.

The bimonoid of top-nested faces is studied further in Section 15.5.6 in connection to the bimonoid of pairs of chambers.

7.7.1. Species of top-nested faces. For any face A, let $\widehat{\mathsf{Q}}[A]$ be the linear span of the set of top-nested faces of the arrangement $\mathcal{A}_A$. This defines the species $\widehat{\mathsf{Q}}$. We call it the *species of top-nested faces*. We use the letter $\mathtt{H}$ for its canonical basis. Thus, a typical basis element of $\widehat{\mathsf{Q}}[A]$ is denoted $\mathtt{H}_{F/A,C/A}$, where F is a face and C is a chamber, both greater than A.

We claim that

$$\widehat{\mathsf{Q}} = \Gamma + \Gamma^2 + \Gamma^3 + \dots, \tag{7.97}$$

the sum of all Cauchy powers of the species of chambers Γ. Explicitly, the A-component of the rhs is $\bigoplus_{F \geq A} \Gamma[F]$. This is a vector space with basis indexed by top-nested faces (F, C), with F greater than A, and we identify this with the $\mathtt{H}$-basis of $\widehat{\mathsf{Q}}[A]$.

Define the $\mathtt{K}$-basis of $\widehat{\mathtt{Q}}$, on the A-component, by either of the two equivalent formulas

$$
\begin{aligned}
\mathtt{H}_{H/A,D/A} &= \sum_{G\colon A \leq G \leq H} \mathtt{K}_{G/A,D/A}, \\
\mathtt{K}_{H/A,D/A} &= \sum_{G\colon A \leq G \leq H} (-1)^{\mathrm{rk}(H/G)}\, \mathtt{H}_{G/A,D/A}.
\end{aligned}
\tag{7.98}
$$

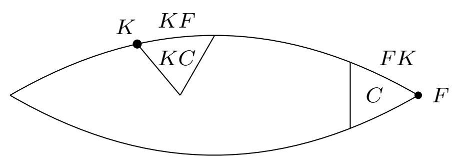
For the equivalence, we used (1.73).

7.7.2. *q*-bimonoid of top-nested faces. For any scalar q, the species of top-nested faces carries the structure of a q-bimonoid which we denote by $\widehat{\mathtt{Q}}_q$. We call it the *q-bimonoid of top-nested faces*. The product and coproduct are defined by

$$
\begin{gathered}
\mu_A^K : \widehat{\mathtt{Q}}_q[K] \to \widehat{\mathtt{Q}}_q[A] \\
\mathtt{H}_{F/K,C/K} \mapsto \mathtt{H}_{F/A,C/A},
\end{gathered}
\tag{7.99a}
$$

$$
\begin{gathered}
\Delta_A^K : \widehat{\mathtt{Q}}_q[A] \to \widehat{\mathtt{Q}}_q[K] \\
\mathtt{H}_{F/A,C/A} \mapsto \begin{cases} q^{\mathrm{dist}(F,K)}\, \mathtt{H}_{KF/K,KC/K} & \text{if } FK \leq C, \\ 0 & \text{otherwise.} \end{cases}
\end{gathered}
\tag{7.99b}
$$

An illustration for the coproduct with $A = O$ is shown below.



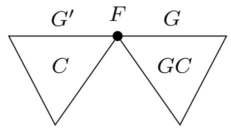
We write $\widehat{\mathtt{Q}}$ for $\widehat{\mathtt{Q}}_1$. This is the *bimonoid of top-nested faces*. It is neither commutative nor cocommutative. The product is the same as in (7.99a), while the coproduct (7.99b) simplifies with the power of q disappearing.

7.7.3. $\mathtt{Q}$-basis for *q* not a root of unity. Suppose q is not a root of unity. Recall the two-sided q-zeta function $\boldsymbol{\zeta}_q$ and the two-sided q-Möbius function $\boldsymbol{\mu}_q$ defined in Section 1.6.6. Define the $\mathtt{Q}$-basis of $\widehat{\mathtt{Q}}_q$, on the A-component, by either of the two equivalent formulas

$$
\begin{aligned}
\mathtt{H}_{F/A,C/A} &= \sum_{\substack{G'\colon F \leq G' \leq C \\ G\colon F \leq G,\, \mathrm{s}(G) = \mathrm{s}(G')}} \boldsymbol{\zeta}_q(F,G,G')\, \mathtt{Q}_{G/A,GC/A}, \\
\mathtt{Q}_{F/A,C/A} &= \sum_{\substack{G'\colon F \leq G' \leq C \\ G\colon F \leq G,\, \mathrm{s}(G) = \mathrm{s}(G')}} \boldsymbol{\mu}_q(F,G,G')\, \mathtt{H}_{G/A,GC/A}.
\end{aligned}
\tag{7.100}
$$

Observe that $\mathtt{H}_{C/A,C/A} = \mathtt{Q}_{C/A,C/A}$.

An illustration with $A = O$ is given below.



Proposition 7.43. *For q not a root of unity, the product and coproduct of $\widehat{\mathtt{Q}}_q$ in the $\mathtt{Q}$-basis are given by*

$$\begin{aligned} \mu_A^K : \widehat{\mathtt{Q}}_q[K] &\to \widehat{\mathtt{Q}}_q[A] \\ \mathtt{Q}_{F/K,C/K} &\mapsto \mathtt{Q}_{F/A,C/A}, \end{aligned} \tag{7.101a}$$

$$\begin{aligned} \Delta_A^K : \widehat{\mathtt{Q}}_q[A] &\to \widehat{\mathtt{Q}}_q[K] \\ \mathtt{Q}_{F/A,C/A} &\mapsto \begin{cases} q^{\operatorname{dist}(F,K)}\ \mathtt{Q}_{KF/K,KC/K} & \text{if } FK = F, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{7.101b}$$

The proof is similar to the one for Proposition 7.24, so we omit it. The coproduct calculation now requires the two-sided q-Weisner formula (1.67).

7.7.4. Dual q-bimonoid. For any scalar q, let $\widehat{\mathtt{Q}}_q^*$ denote the q-bimonoid dual to $\widehat{\mathtt{Q}}_q$. Let $\mathtt{M}$ denote the basis dual to $\mathtt{H}$, and $\mathtt{F}$ denote the basis dual to $\mathtt{K}$. By dualizing formulas (7.98), we obtain:

$$\begin{aligned} \mathtt{F}_{G/A,D/A} &= \sum_{H\colon G \leq H \leq D} \mathtt{M}_{H/A,D/A}, \\ \mathtt{M}_{G/A,D/A} &= \sum_{H\colon G \leq H \leq D} (-1)^{\operatorname{rk}(H/G)}\, \mathtt{F}_{H/A,D/A}. \end{aligned} \tag{7.102}$$

The product and coproduct of $\widehat{\mathtt{Q}}_q^*$ in the $\mathtt{M}$-basis are obtained by dualizing formulas (7.99). They are given by

$$\begin{aligned} \mu_A^K : \widehat{\mathtt{Q}}_q^*[K] &\to \widehat{\mathtt{Q}}_q^*[A] \\ \mathtt{M}_{G/K,D/K} &\mapsto \sum_{G'\colon G' \geq A,\, KG' = G} q^{\operatorname{dist}(K,G')}\, \mathtt{M}_{G'/A,G'D/A}, \end{aligned} \tag{7.103a}$$

$$\begin{aligned} \Delta_A^K : \widehat{\mathtt{Q}}_q^*[A] &\to \widehat{\mathtt{Q}}_q^*[K] \\ \mathtt{M}_{G/A,D/A} &\mapsto \begin{cases} \mathtt{M}_{G/K,D/K} & \text{if } K \leq G, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{7.103b}$$

Theorem 7.44. *Suppose $\mathcal{A}$ is simplicial. Then the coproduct and product of $\widehat{\mathtt{Q}}_q^*$ in the $\mathtt{F}$-basis are given by*

$$\begin{aligned} \Delta_A^K : \widehat{\mathtt{Q}}_q^*[A] &\to \widehat{\mathtt{Q}}_q^*[K] \\ \mathtt{F}_{G/A,D/A} &\mapsto \begin{cases} \mathtt{F}_{KG/K,D/K} & \textit{if } K \leq D, \\ 0 & \textit{otherwise,} \end{cases} \end{aligned}$$

$$\mu_A^K : \widehat{\mathsf{Q}}_q^*[K] \to \widehat{\mathsf{Q}}_q^*[A]$$
$$\mathtt{F}_{G/K,D/K} \mapsto \sum_{D' :\, D' \geq A,\, KD'=D} q^{\operatorname{dist}(K,G')}\, \mathtt{F}_{G'/A,D'/A},$$

where G' is the smallest face of D' greater than A such that either

$$G'K\overline{G} \leq D', \qquad \textit{or equivalently,} \qquad G \leq KG' \textit{ and } G'K \leq D'.$$

PROOF. The equivalence of the last two conditions follows from [**18**, Proposition 12.29]. The proof is similar to those of [**17**, Theorems 8.2.1 and 8.2.3], so we omit it. □

For q not a root of unity, let $\mathtt{P}$ be the basis of $\widehat{\mathsf{Q}}_q^*$ dual to $\mathtt{Q}$. Its product and coproduct formulas can be written by dualizing (7.101).

7.7.5. Primitive part. We deduce from coproduct formula (7.103b) in the $\mathtt{M}$-basis that

$$\mathcal{P}(\widehat{\mathsf{Q}}_q^*) = \mathsf{\Gamma}. \tag{7.104}$$

Each component $\mathcal{P}(\widehat{\mathsf{Q}}_q^*)[A]$ is spanned by the elements $\mathtt{M}_{A/A,D/A}$ for $D \geq A$. More generally, the primitive filtration of $\widehat{\mathsf{Q}}_q^*$ can be expressed as

$$\mathcal{P}(\widehat{\mathsf{Q}}_q^*) = \mathsf{\Gamma} + \mathsf{\Gamma}^2 + \cdots + \mathsf{\Gamma}^k, \tag{7.105}$$

the sum of the first k Cauchy powers of $\mathsf{\Gamma}$.

The primitive part of $\widehat{\mathsf{Q}}_q$ is discussed separately later in Section 7.9.5.

7.7.6. Freeness and cofreeness. We now state the (co)freeness properties of $\widehat{\mathsf{Q}}_q$ and its dual. They are similar to those of Σ_q with the role of the exponential species now played by the species of chambers.

The q-bimonoid $\widehat{\mathsf{Q}}_q$ is the free q-bimonoid on $\mathsf{\Gamma}^*$ viewed as a comonoid. Dually, $\widehat{\mathsf{Q}}_q^*$ is the cofree q-bimonoid on $\mathsf{\Gamma}$ viewed as a monoid. More precisely:

Proposition 7.45. *For any scalar q, there are isomorphisms of q-bimonoids*

$$\widehat{\mathsf{Q}}_q \xrightarrow{\cong} \mathcal{T}_q(\mathsf{\Gamma}^*) \quad \textit{and} \quad \widehat{\mathsf{Q}}_q^* \xrightarrow{\cong} \mathcal{T}_q^\vee(\mathsf{\Gamma}). \tag{7.106}$$

Evaluated on the A-component, the first map sends $\mathtt{H}_{F/A,C/A}$ to $\mathtt{M}_{C/F}$ in the F-summand, while the second map sends $\mathtt{M}_{G/A,D/A}$ to $\mathtt{H}_{D/G}$ in the G-summand.

The (co)product of $\widehat{\mathsf{Q}}_q$ in the $\mathtt{H}$-basis is given by (7.99), while that of $\widehat{\mathsf{Q}}_q^*$ in the $\mathtt{M}$-basis is given by (7.103). The (co)product of $\mathcal{T}_q(\mathsf{\Gamma}^*)$ is given by specializing (6.3) and (6.5), while that of $\mathcal{T}_q^\vee(\mathsf{\Gamma})$ is given by specializing (6.11) and (6.13).

Similar considerations apply when q is not a root of unity. The q-bimonoid $\widehat{\mathsf{Q}}_q$ is the free q-bimonoid on $\mathsf{\Gamma}^*$ viewed as a trivial comonoid. Dually, $\widehat{\mathsf{Q}}_q^*$ is the cofree q-bimonoid on $\mathsf{\Gamma}$ viewed as a trivial monoid. More precisely:

Proposition 7.46. *For q not a root of unity, there are isomorphisms of q-bimonoids*

$$\widehat{\mathsf{Q}}_q \xrightarrow{\cong} \mathcal{T}_q(\Gamma^*) \quad \text{and} \quad \widehat{\mathsf{Q}}^*_q \xrightarrow{\cong} \mathcal{T}^\vee_q(\Gamma). \tag{7.107}$$

Evaluated on the A-component, the first map sends $\mathtt{Q}_{F/A,C/A}$ to $\mathtt{M}_{C/F}$ in the F-summand, while the second map sends $\mathtt{P}_{G/A,D/A}$ to $\mathtt{H}_{D/G}$ in the G-summand.

The (co)product of $\widehat{\mathsf{Q}}_q$ in the $\mathtt{Q}$-basis is given by (7.101). The (co)product of $\mathcal{T}_q(\Gamma^*)$ is given by specializing (6.39) and of $\mathcal{T}^\vee_q(\Gamma)$ is given by specializing (6.40).

7.7.7. *q*-norm map and self-duality. For any scalar q, consider the map of species

$$\widehat{\mathsf{Q}}_q \to \widehat{\mathsf{Q}}^*_q, \qquad \mathtt{H}_{F/A,C/A} \mapsto \sum_{\substack{(G,D)\colon A\leq G\leq D\\ GF=D,\ FG=C}} q^{\operatorname{dist}(F,G)}\, \mathtt{M}_{G/A,D/A} \tag{7.108}$$

on the A-component. We call it the *q-norm map* on top-nested faces.

Lemma 7.47. *The q-norm map on top-nested faces (7.108) is a self-dual morphism of q-bimonoids.*

This can be checked directly. Another proof is given in the exercise below by viewing (7.108) as an instance of the q-norm map in Section 6.9.3.

Exercise 7.48. Consider the bijection (6.77). Put $\mathsf{c} := \Gamma^*$ and $\mathsf{a} := \Gamma$, and let $f : \mathsf{c} \to \mathsf{a}$ be the map which on the A-component sends $\mathtt{M}_{C/A}$ to $\mathtt{H}_{C/A}$ if $C = A$, and to 0 otherwise. Check that, under the identification (7.106), the resulting morphism of q-bimonoids $g : \widehat{\mathsf{Q}}_q \to \widehat{\mathsf{Q}}^*_q$ matches (7.108). This may also be viewed as a specialization of Exercise 6.82, item (1) to $\mathsf{p} := \mathsf{x}$.

For q not a root of unity, we now give a formula for the map (7.108) from the $\mathtt{Q}$-basis defined in (7.100) to its dual $\mathtt{P}$-basis.

Proposition 7.49. *Suppose q is not a root of unity. Then the q-norm map on top-nested faces (7.108) is given by*

$$\widehat{\mathsf{Q}}_q \to \widehat{\mathsf{Q}}^*_q, \quad \mathtt{Q}_{F/A,C/A} \mapsto \sum_{\substack{(G,D)\colon A\leq G\leq D\\ \mathrm{s}(G)=\mathrm{s}(F)}} q^{\operatorname{dist}(F,G)}\, \boldsymbol{\mu}_q(F,C,FD)\, \mathtt{P}_{G/A,D/A} \tag{7.109}$$

on the A-component.

This can be checked directly. A nice way to do this is indicated in the exercise below. A more conceptual proof using the q-exponential and q-logarithm is given later in Exercise 14.83.

Exercise 7.50. Check that: For q not a root of unity, the map (7.108) is given by

$$\widehat{\mathsf{Q}}_q \to \widehat{\mathsf{Q}}^*_q, \quad \mathtt{H}_{F/A,C/A} \mapsto \sum_{\substack{(G,D)\colon A\leq G\leq D\\ GF=G,\ FG\leq C}} q^{\operatorname{dist}(F,G)}\, \boldsymbol{\mu}_q(F,C,FD)\, \mathtt{P}_{G/A,D/A} \tag{7.110}$$

on the A-component. (Use the two-sided q-Weisner formula (1.67).) Prove formula (7.109) by showing that it also matches (7.110). (Use the two-sided q-flat-additivity formula (1.65).)

Theorem 7.51. *For q not a root of unity, the map (7.108) is an isomorphism of q-bimonoids iff*

$$\det\big(\boldsymbol{\mu}_q(A, C, D)_{C,D \geq A}\big) \neq 0$$

for all A. (In the above matrix, A is fixed, while C, D are the two varying indices.)

Proof. This can be deduced from Proposition 7.49 and Theorem 1.10 by expressing (7.110) as a direct sum over flats containing A with each summand written as a tensor product of two linear maps. $\square$

A conceptual proof using the q-exponential and q-logarithm is given in Theorem 9.100.

Exercise 7.52. Check that: For any scalar q, the map

$$\Sigma_q \hookrightarrow \widehat{\mathsf{Q}}_q, \qquad \mathtt{H}_{F/A} \mapsto \sum_{C\colon C \geq F} \mathtt{H}_{F/A, C/A} \tag{7.111}$$

is a morphism of q-bimonoids. For q not a root of unity, on the $\mathtt{Q}$-bases, it is given by the same formula:

$$\Sigma_q \hookrightarrow \widehat{\mathsf{Q}}_q, \qquad \mathtt{Q}_{F/A} \mapsto \sum_{C\colon C \geq F} \mathtt{Q}_{F/A, C/A}. \tag{7.112}$$

For any scalar q, the composite map

$$\Sigma_q \hookrightarrow \widehat{\mathsf{Q}}_q \to \widehat{\mathsf{Q}}_q^* \twoheadrightarrow \Sigma_q^*$$

is precisely (7.91). Here the first map is (7.111), the last map is its dual, and the middle map is (7.108). For q not a root of unity, it is given by

$$\Sigma_q \to \Sigma_q^*, \qquad \mathtt{Q}_{F/A} \mapsto \mu_q(\mathrm{s}(F), \top) \sum_{\substack{G\colon G \geq A \\ \mathrm{s}(G) = \mathrm{s}(F)}} q^{\operatorname{dist}(F,G)}\, \mathtt{P}_{G/A} \tag{7.113}$$

on the A-component. (Use (7.112), its dual, and (7.109).) This is a special case of (7.89).

Example 7.53. Let $\mathcal{A}$ be a rank-one arrangement with chambers C and $\overline{C}$. Let us make explicit the map (7.108) on the O-component. On the $\mathtt{H}$-basis and into the $\mathtt{M}$-basis, it is given by

$$\mathtt{H}_{O,C} \mapsto \mathtt{M}_{C,C}, \qquad \mathtt{H}_{C,C} \mapsto \mathtt{M}_{O,C} + \mathtt{M}_{C,C} + q\,\mathtt{M}_{\overline{C},\overline{C}},$$
$$\mathtt{H}_{O,\overline{C}} \mapsto \mathtt{M}_{\overline{C},\overline{C}}, \qquad \mathtt{H}_{\overline{C},\overline{C}} \mapsto \mathtt{M}_{O,\overline{C}} + q\,\mathtt{M}_{C,C} + \mathtt{M}_{\overline{C},\overline{C}}.$$

For q not a root of unity, on the $\mathtt{Q}$-basis and into the $\mathtt{P}$-basis, it is given by

$$\mathtt{Q}_{O,C} \mapsto \frac{-1}{1-q^2}\,\mathtt{P}_{O,C} + \frac{q}{1-q^2}\,\mathtt{P}_{O,\overline{C}}, \qquad \mathtt{Q}_{C,C} \mapsto \mathtt{P}_{C,C} + q\,\mathtt{P}_{\overline{C},\overline{C}},$$
$$\mathtt{Q}_{O,\overline{C}} \mapsto \frac{q}{1-q^2}\,\mathtt{P}_{O,C} + \frac{-1}{1-q^2}\,\mathtt{P}_{O,\overline{C}}, \qquad \mathtt{Q}_{\overline{C},\overline{C}} \mapsto q\,\mathtt{P}_{C,C} + \mathtt{P}_{\overline{C},\overline{C}}.$$

We used the values of $\boldsymbol{\mu}_q$ from Example 1.36. In matrix form, the maps can be, respectively, written as

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & q \\ 0 & 1 & q & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{-1}{1-q^2} & \frac{q}{1-q^2} & 0 & 0 \\ \frac{q}{1-q^2} & \frac{-1}{1-q^2} & 0 & 0 \\ 0 & 0 & 1 & q \\ 0 & 0 & q & 1 \end{pmatrix}.$$

Note very carefully the block-diagonal form of the second matrix. Since the change of basis is unitriangular, the determinants of the two matrices are equal. In this case, they both turn out to be 1.

In view of Exercise 7.52, the above maps can be used to recover the maps in Example 7.41.

7.7.8. 0-bimonoid of top-nested faces. Let us now specialize to $q = 0$. The change of basis formulas (7.100) simplify to

$$\begin{aligned} \mathtt{H}_{F/A,C/A} &= \sum_{G:\, F \le G \le C} \mathtt{Q}_{G/A,C/A}, \\ \mathtt{Q}_{F/A,C/A} &= \sum_{G:\, F \le G \le C} (-1)^{\mathrm{rk}(G/F)}\, \mathtt{H}_{G/A,C/A}. \end{aligned} \tag{7.114}$$

This follows from formulas (1.71).

For the 0-bimonoid $\widehat{\mathsf{Q}}_0$, the product formulas (7.99a) and (7.101a) remain as before, while the coproduct formulas (7.99b) and (7.101b) simplify.

The map (7.108) specializes to

$$\widehat{\mathsf{Q}}_0 \to \widehat{\mathsf{Q}}_0^*, \qquad \mathtt{H}_{F/A,C/A} \mapsto \sum_{\substack{G:\, A \le G \le C \\ GF = FG = C}} \mathtt{M}_{G/A,C/A} \tag{7.115}$$

on the A-component. Similarly, (7.109) specializes to

$$\widehat{\mathsf{Q}}_0 \to \widehat{\mathsf{Q}}_0^*, \qquad \mathtt{Q}_{F/A,C/A} \mapsto (-1)^{\mathrm{rk}(C/F)}\, \mathtt{P}_{F/A,C/A}. \tag{7.116}$$

on the A-component. In particular, the matrices appearing in Theorem 7.51 are diagonal with diagonal entries 1 or -1.

The map (7.113) specializes to

$$\Sigma_0 \to \Sigma_0^*, \qquad \mathtt{Q}_{F/A} \mapsto \mu_0(\mathrm{s}(F), \top)\, \mathtt{P}_{F/A} \tag{7.117}$$

on the A-component. (Recall that there is an explicit formula for μ_0 given by (1.52).)

7.7.9. Bimonoid of top-lunes. For any face A, let $\widehat{\Lambda}[A]$ be the linear span of the set of top-lunes of the arrangement $\mathcal{A}_A$. This defines the species $\widehat{\Lambda}$. We call it the *species of top-lunes*. We use the letter $\mathtt{H}$ for its canonical basis.

Recall that the support of a top-nested face is a top-lune, see (1.8). This yields a map of species

$$\pi : \widehat{\mathsf{Q}} \twoheadrightarrow \widehat{\Lambda}, \qquad \mathtt{H}_{F/A,C/A} \mapsto \mathtt{H}_{\mathrm{s}(F/A,C/A)}. \tag{7.118}$$

Moreover, the bimonoid structure of the former induces a bimonoid structure on the latter. We call it the *bimonoid of top-lunes*. Further, we have a commutative diagram of bimonoids

$$\begin{tikzcd} \widehat{\mathsf{Q}} \arrow[r, "\cong"] \arrow[d, two heads] & \mathcal{T}(\Gamma^*) \arrow[d, two heads] \\ \widehat{\Lambda} \arrow[r, "\cong"'] & \mathcal{S}(\Gamma^*). \end{tikzcd}$$

Thus, $\widehat{\Lambda}$ is the free commutative bimonoid on the comonoid Γ^*, and π is an instance of the abelianization map (6.59).

7.7.10. Morphisms of bimonoids. Consider the map $\pi^* : \mathsf{E}^* \hookrightarrow \Gamma^*$ in diagram (7.30), and view it as a morphism of comonoids. Applying the functors $\mathcal{T}$ and $\mathcal{S}$ to it, and combining with the abelianization (6.59), we obtain the commutative diagram of bimonoids

$$\begin{tikzcd} \mathcal{T}(\mathsf{E}^*) \arrow[r, hook] \arrow[d, two heads] & \mathcal{T}(\Gamma^*) \arrow[d, two heads] \\ \mathcal{S}(\mathsf{E}^*) \arrow[r, hook] & \mathcal{S}(\Gamma^*), \end{tikzcd} \quad \text{or equivalently,} \quad \begin{tikzcd} \Sigma \arrow[r, hook] \arrow[d, two heads] & \widehat{\mathsf{Q}} \arrow[d, two heads] \\ \Pi \arrow[r, hook] & \widehat{\Lambda}. \end{tikzcd} \tag{7.119}$$

Explicitly, on the A-component,

$$\Sigma \hookrightarrow \widehat{\mathsf{Q}}, \qquad \mathtt{H}_{F/A} \mapsto \sum_{C:\, C \geq F} \mathtt{H}_{F/A,C/A}. \tag{7.120}$$

This is the special case $q = 1$ of (7.120).

More generally: specializing the left diagram in (6.37) to $\mathsf{p} := \mathsf{x}$ yields the following self-dual commutative diagram of bimonoids.

$$\begin{tikzcd}[row sep=small, column sep=small] & \widehat{\mathsf{Q}} \arrow[rr] \arrow[dd, two heads] & & \widehat{\mathsf{Q}}^* \arrow[dd, two heads] \\ \Sigma \arrow[ur, hook] \arrow[rr] \arrow[dd, two heads] & & \widehat{\Lambda}^* \arrow[ur, hook] \arrow[dd, two heads] & \\ & \widehat{\Lambda} \arrow[rr] & & \Sigma^* \\ \Pi \arrow[ur, hook] \arrow[rr] & & \Pi^* \arrow[ur, hook] & \end{tikzcd} \tag{7.121}$$

Duality acts as 180 degree rotation around the line through the midpoint of the front bottom edge and the midpoint of the back top edge (and the center of the cube).

Explicitly, on the A-component,

$$\widehat{\mathsf{Q}} \to \widehat{\mathsf{Q}}^*, \qquad \mathtt{H}_{F/A,C/A} \mapsto \sum_{\substack{(G,D):\, A \leq G \leq D \\ GF = D,\, \overline{F}G = C}} \mathtt{M}_{G/A,D/A}. \tag{7.122}$$

This is the special case $q = 1$ of (7.108). It is not an isomorphism in general. The composite map

$$\Sigma \hookrightarrow \widehat{\mathsf{Q}} \to \widehat{\mathsf{Q}}^* \twoheadrightarrow \Sigma^*$$

is the special case of (7.91) with $q = 1$. The map $\mathsf{\Pi} \to \mathsf{\Pi}^*$ coincides with φ in (7.52) as mentioned in Exercise 7.12.

Similarly, on the A-component,

$$\widehat{\mathsf{\Lambda}} \to \mathsf{\Sigma}^*, \qquad \mathtt{H}_{\mathrm{L}} \mapsto \sum_{F/A \in \mathrm{L}^o} \mathtt{M}_{F/A}. \tag{7.123}$$

Here L is a top-lune of $\mathcal{A}_A$, and the sum is over all faces F/A in its interior. The interior of a lune is defined in [**21**, (3.11)].

Related morphisms of bimonoids are discussed later in Section 15.5.6.

7.8. Species of bifaces

We introduce the bimonoid of bifaces. It is neither commutative nor cocommutative. It is free as a monoid. We also consider its abelianization.

The bimonoid of bifaces admits a deformation by a parameter q. When q is not a root of unity, in addition to the canonical $\mathtt{H}$-basis, we also define the $\mathtt{Q}'$-basis in terms of two-sided q-zeta and q-Möbius functions.

7.8.1. The species F. Let F denote the species whose A-component $\mathsf{F}[A]$ is linearly spanned by faces with the same support as A. For A and B of the same support, let $\beta_{B,A}$ be the identity. We write

$$\beta_{B,A} : \mathsf{F}[A] \to \mathsf{F}[B], \qquad \mathtt{H}_{A'} \mapsto \mathtt{H}_{A'}.$$

The species F carries the structure of a comonoid with coproduct defined by

$$\Delta_A^G : \mathsf{F}[A] \to \mathsf{F}[G], \qquad \mathtt{H}_{A'} \mapsto \mathtt{H}_{A'G}.$$

It is not cocommutative.

Exercise 7.54. Check that: The species F is indeed a comonoid, that is, axioms (2.10) hold. Moreover, the map $\mathsf{F} \to \mathsf{E}$ defined on the A-component by

$$\mathsf{F}[A] \to \mathsf{E}[A], \qquad \mathtt{H}_{A'} \mapsto \mathtt{H}_A$$

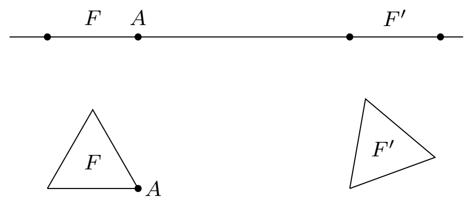
is a morphism of comonoids.

Dually, F^* is a monoid. Writing $\mathtt{M}$ for the basis dual to $\mathtt{H}$, the product is given by

$$\mu_A^F : \mathsf{F}^*[F] \to \mathsf{F}^*[A], \qquad \mathtt{M}_{F'} \mapsto \sum_{\substack{A' :\, A'F = F' \\ \mathrm{s}(A') = \mathrm{s}(A)}} \mathtt{M}_{A'}.$$

Note that since distinct faces of F' have distinct supports, there can be at most one choice for A'.

7.8.2. Species of bifaces. For any face A, let $\mathrm{J}[A]$ denote the set of bifaces (F, F') such that F is greater than A. Note very carefully that F' is *not* required to be greater than A. Illustrations in ranks two and three are shown below.



For A and B of the same support, there is a bijection

$$\beta_{B,A} : \mathrm{J}[A] \to \mathrm{J}[B], \qquad (F, F') \mapsto (BF, F').$$

Note very carefully that the second coordinate remains unchanged. This turns J into a set-species.

Let $\mathsf{J} := \Bbbk\mathrm{J}$ be the linearization of J. This is the *species of bifaces*. Let $\mathtt{H}$ denote its canonical basis. For A and B of the same support, we write

$$\beta_{B,A} : \mathsf{J}[A] \to \mathsf{J}[B], \qquad \mathtt{H}_{(F,F')} \mapsto \mathtt{H}_{(BF,F')}. \tag{7.124}$$

We claim that

$$\mathsf{J} = \mathsf{F} + \mathsf{F}^2 + \mathsf{F}^3 + \dots, \tag{7.125}$$

the sum of all Cauchy powers of the species F. Explicitly, the A-component of the rhs is $\bigoplus_{F \geq A} \mathsf{F}[F]$. This is a vector space with basis indexed by bifaces (F, F') with F greater than A, and we identify this with the $\mathtt{H}$-basis of $\mathsf{J}[A]$.

7.8.3. Bimonoid of bifaces. The species J carries the structure of a bimonoid. We call it the *bimonoid of bifaces*. The product and coproduct are defined by

$$\begin{aligned} \mu_A^F &: \mathsf{J}[F] \to \mathsf{J}[A] & \qquad \Delta_A^G &: \mathsf{J}[A] \to \mathsf{J}[G] \\ &\mathtt{H}_{(K,K')} \mapsto \mathtt{H}_{(K,K')} & &\mathtt{H}_{(K,K')} \mapsto \mathtt{H}_{(GK,K'G)}. \end{aligned} \tag{7.126}$$

The structure is set-theoretic. The bimonoid axiom (2.12) is checked below.

$$\begin{tikzcd} \mathtt{H}_{(K,K')} \arrow[r, mapsto] \arrow[d, mapsto] & \mathtt{H}_{(K,K')} \arrow[r, mapsto] & \mathtt{H}_{(GK,K'G)} \\ \mathtt{H}_{(FGK,K'FG)} \arrow[rr, mapsto] & & \mathtt{H}_{(GFK,K'FG)}. \arrow[u, mapsto] \end{tikzcd}$$

Here F and G are as in the bimonoid axiom, K is greater than F, and K' has the same support as K. Thus, $GFK = GK$ and $K'FG = K'G$, as required in the last step.

7.8.4. q-bimonoid of bifaces. More generally, for any scalar q, the species of bifaces carries the structure of a q-bimonoid which we denote by J_q. We call it the *q-bimonoid of bifaces*. The product and coproduct are defined by

$$\begin{aligned} &\mu_A^F : \mathsf{J}_q[F] \to \mathsf{J}_q[A] && \Delta_A^G : \mathsf{J}_q[A] \to \mathsf{J}_q[G] \\ &\mathtt{H}_{(K,K')} \mapsto \mathtt{H}_{(K,K')} && \mathtt{H}_{(K,K')} \mapsto q^{\mathrm{dist}(K,G)}\, \mathtt{H}_{(GK,K'G)}. \end{aligned} \tag{7.127}$$

Note that for $q = 1$, we have $\mathsf{J}_1 = \mathsf{J}$, the bimonoid of bifaces.

Exercise 7.55. Check that J_q satisfies the q-bimonoid axiom (2.33). (This is very similar to Exercise 7.19.)

Exercise 7.56. Check that the map $\mathsf{J}_q \to \mathsf{\Sigma}_q$ defined on the A-component by

$$\mathsf{J}_q[A] \to \mathsf{\Sigma}_q[A], \qquad \mathtt{H}_{(K,K')} \mapsto \mathtt{H}_{K/A}$$

is a morphism of q-bimonoids.

7.8.5. $\mathtt{Q}'$-basis for q not a root of unity. Suppose q is not a root of unity. Recall the two-sided q-zeta function $\boldsymbol{\zeta}_q$ and the two-sided q-Möbius function $\boldsymbol{\mu}_q$ defined in Section 1.6.6. Define the $\mathtt{Q}'$-basis of J_q, on the A-component, by either of the two equivalent formulas

$$\begin{aligned} \mathtt{H}_{(F,F')} &= \sum_{\substack{G,G':\\ F\leq G,\, F'\leq G'\\ \mathrm{s}(G)=\mathrm{s}(G')}} \boldsymbol{\zeta}_q(F,G,FG')\, \mathtt{Q}'_{(G,G')}, \\ \mathtt{Q}'_{(F,F')} &= \sum_{\substack{G,G':\\ F\leq G,\, F'\leq G'\\ \mathrm{s}(G)=\mathrm{s}(G')}} \boldsymbol{\mu}_q(F,G,FG')\, \mathtt{H}_{(G,G')}. \end{aligned} \tag{7.128}$$

The dependence on A lies in the fact that the first coordinate of each of these basis elements is greater than A. When $A = O$, this is precisely the $\mathtt{Q}'$-basis in (1.126).

Proposition 7.57. *For q not a root of unity, the product and coproduct of J_q in the $\mathtt{Q}'$-basis are given by*

$$\begin{aligned} &\mu_A^F : \mathsf{J}_q[F] \to \mathsf{J}_q[A] && \Delta_A^G : \mathsf{J}_q[A] \to \mathsf{J}_q[G] \\ &\mathtt{Q}'_{(K,K')} \mapsto \mathtt{Q}'_{(K,K')} && \mathtt{Q}'_{(K,K')} \mapsto \begin{cases} q^{\mathrm{dist}(K,G)}\, \mathtt{Q}'_{(GK,K')} & \textit{if } KG = K, \\ 0 & \textit{otherwise.} \end{cases} \end{aligned} \tag{7.129}$$

We omit the proof. It is similar to the proofs of Propositions 7.24 and 7.43. The coproduct calculation requires the two-sided q-Weisner formula (1.67).

7.8.6. Janus algebra. For any face A, let $\mathrm{J}^o[A]$ denote the set of bifaces (F, G) such that both F and G are greater than A. It is a subset of $\mathrm{J}[A]$. Denote the linearization of $\mathrm{J}^o[A]$ by $\mathsf{J}^o[A]$. It is an algebra with product given by

$$\mathtt{H}_{(F/A,F'/A)} \cdot \mathtt{H}_{(G/A,G'/A)} := \mathtt{H}_{(FG/A,G'F'/A)}. \tag{7.130}$$

The unit element is $\mathtt{H}_{(A/A,A/A)}$. This can be identified with the Janus algebra of the arrangement $\mathcal{A}_A$: Compare (7.130) with (1.121).

In general, $\mathsf{J}^o[A]$ is a proper subspace of $\mathsf{J}[A]$. The two coincide when A is the central face. The algebra structure of $\mathsf{J}^o[A]$ extends to $\mathsf{J}[A]$. This can be seen by viewing both as subalgebras of $\mathsf{J}[O]$. However, the species structure of J does not restrict to J^o in general.

7.8.7. *q*-Janus algebra. More generally, for any scalar q, and for any face A, we consider the algebra $\mathsf{J}^o_q[A]$. Its underlying space is the same as $\mathsf{J}^o[A]$ but the product is deformed as

$$\mathtt{H}_{(F/A,F'/A)} \cdot \mathtt{H}_{(G/A,G'/A)} := q^{\operatorname{dist}(F',G)}\, \mathtt{H}_{(FG/A,G'F'/A)}. \tag{7.131}$$

This can be identified with the q-Janus algebra of the arrangement $\mathcal{A}_A$: Compare (7.131) with (1.123). The algebra structure of $\mathsf{J}^o_q[A]$ extends to $\mathsf{J}_q[A]$.

Now suppose q is not a root of unity. In this situation, we have the $\mathtt{Q}$-basis, $\mathtt{Q}'$-basis, and $\mathtt{Q}^d$-basis of $\mathsf{J}^o_q[A]$. The $\mathtt{Q}'$-basis extends to $\mathsf{J}_q[A]$ as in (7.128). The product in the $\mathtt{Q}$-basis is given by

$$\mathtt{Q}_{(F/A,F'/A)} \cdot \mathtt{Q}_{(G/A,G'/A)} = \begin{cases} \mathtt{Q}_{(F/A,G'/A)} & \text{if } F' = G, \\ 0 & \text{otherwise.} \end{cases} \tag{7.132}$$

The same formula holds for the $\mathtt{Q}^d$-basis. Compare with (1.125) and (1.131). Some useful identities are recalled below.

$$\mathtt{Q}_{(A/A,A/A)} = \sum_{\substack{F,F':\, F,F' \geq A \\ \mathrm{s}(F)=\mathrm{s}(F')}} \boldsymbol{\mu}_q(A,F,F')\, \mathtt{H}_{(F/A,F'/A)}, \tag{7.133}$$

$$\mathtt{H}_{(A/A,A/A)} = \sum_{F:F\geq A} \mathtt{Q}_{(F/A,F/A)} = \sum_{F:F\geq A} \mathtt{Q}^d_{(F/A,F/A)}, \tag{7.134}$$

$$\begin{aligned} \mathtt{Q}_{(F/A,F/A)} \cdot \mathtt{Q}'_{(F/A,F/A)} &= \mathtt{Q}'_{(F/A,F/A)}, \\ \mathtt{Q}'_{(F/A,F/A)} \cdot \mathtt{Q}_{(F/A,F/A)} &= \mathtt{Q}_{(F/A,F/A)}, \end{aligned} \tag{7.135}$$

$$\mathtt{H}_{(F/A,F'/A)} \cdot \mathtt{Q}_{(G/A,G'/A)} = 0 \quad \text{if } \mathrm{s}(G) \ngeq \mathrm{s}(F), \tag{7.136}$$

$$\begin{aligned} \mathtt{H}_{(F/A,F/A)} \cdot \mathtt{Q}_{(F/A,F/A)} &= \mathtt{Q}_{(F/A,F/A)}, \\ \mathtt{H}_{(F/A,F/A)} \cdot \mathtt{Q}^d_{(F/A,F/A)} &= \mathtt{Q}'_{(F/A,F/A)}, \end{aligned} \tag{7.137}$$

for any $F > A$,

$$\begin{aligned} \mathtt{H}_{(F/A,F/A)} \cdot \mathtt{Q}_{(A/A,A/A)} &= 0, \\ \mathtt{Q}_{(A/A,A/A)} \cdot \mathtt{H}_{(F/A,F/A)} &= 0. \end{aligned} \tag{7.138}$$

Compare with (1.132), (1.133), (1.135), (1.136a), (1.137), (1.138).

For any biface (F, F') such that $F, F' \geq A$, define

$$\begin{aligned} \mathtt{u}_{(F/A,F'/A)} &:= \sum_{\substack{F''\colon F''\geq A \\ \mathrm{s}(F'')=\mathrm{s}(F')=\mathrm{s}(F)}} \zeta_q(A, F', F'')\, \mathtt{H}_{(F/A,F''/A)}, \\ \mathtt{u}^d_{(F'/A,F/A)} &:= \sum_{\substack{F''\colon F''\geq A \\ \mathrm{s}(F'')=\mathrm{s}(F')=\mathrm{s}(F)}} \zeta_q(A, F'', F')\, \mathtt{H}_{(F''/A,F/A)}. \end{aligned} \tag{7.139}$$

Compare with (1.139). For any $F \geq A$,

$$\mathtt{u}^d_{(F/A,F/A)} \cdot \mathtt{Q}'_{(F/A,F/A)} = \mathtt{Q}^d_{(F/A,F/A)}. \tag{7.140}$$

Compare with (1.143).

7.8.8. 0-Janus algebra. Let us now consider the special case $q = 0$. The product of the 0-Janus algebra $\mathtt{J}^o_0[A]$ is given by

$$\mathtt{H}_{(F/A,F'/A)} \cdot \mathtt{H}_{(G/A,G'/A)} = \begin{cases} \mathtt{H}_{(FG/A,G'F'/A)} & \text{if } F'G = GF', \\ 0 & \text{otherwise.} \end{cases} \tag{7.141}$$

Compare with (1.144). The $\mathtt{H}$- and $\mathtt{Q}$-bases are related by

$$\begin{aligned} \mathtt{H}_{(F/A,F'/A)} &= \sum_{G\colon G\geq F} \mathtt{Q}_{(G/A,F'G/A)}, \\ \mathtt{Q}_{(F/A,F'/A)} &= \sum_{G\colon G\geq F} (-1)^{\mathrm{rk}(G/F)}\, \mathtt{H}_{(G/A,F'G/A)}. \end{aligned} \tag{7.142}$$

Compare with (1.145). In particular,

$$\mathtt{Q}_{(A/A,A/A)} = \sum_{F\colon F\geq A} (-1)^{\mathrm{rk}(F/A)}\, \mathtt{H}_{(F/A,F/A)}. \tag{7.143}$$

Compare with (1.146).

7.8.9. Dual bimonoid. Let $\mathtt{J}^*$ denote the bimonoid dual to $\mathtt{J}$. Let $\mathtt{M}$ denote the basis which is dual to the $\mathtt{H}$-basis. The product and coproduct of $\mathtt{J}^*$ are obtained by dualizing formulas (7.126). They are given by

$$\begin{aligned} &\mu^G_A : \mathtt{J}^*[G] \to \mathtt{J}^*[A] && \Delta^F_A : \mathtt{J}^*[A] \to \mathtt{J}^*[F] \\ &\mathtt{M}_{(H,H')} \mapsto \sum_{\substack{K,K'\colon K\geq A \\ GK=H \\ K'G=H'}} \mathtt{M}_{(K,K')} && \mathtt{M}_{(K,K')} \mapsto \begin{cases} \mathtt{M}_{(K,K')} & \text{if } F \leq K, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{7.144}$$

7.8.10. Dual q-bimonoid. More generally, for any scalar q, let $\mathtt{J}^*_q$ denote the q-bimonoid dual to $\mathtt{J}_q$. Dualizing formulas (7.127), its product and coproduct are given by

$$\begin{aligned} &\mu^G_A : \mathtt{J}^*_q[G] \to \mathtt{J}^*_q[A] && \Delta^F_A : \mathtt{J}^*_q[A] \to \mathtt{J}^*_q[F] \\ &\mathtt{M}_{(H,H')} \mapsto \sum_{\substack{K,K'\colon K\geq A \\ GK=H \\ K'G=H'}} q^{\mathrm{dist}(H,K)}\, \mathtt{M}_{(K,K')} && \mathtt{M}_{(K,K')} \mapsto \begin{cases} \mathtt{M}_{(K,K')} & \text{if } F \leq K, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{7.145}$$

For q not a root of unity, let $\mathtt{P}'$ be the basis of J_q^* dual to $\mathtt{Q}'$. Its product and coproduct formulas can be written by dualizing (7.129).

7.8.11. Primitive part. We deduce from coproduct formula (7.145) in the $\mathtt{M}$-basis that

$$\mathcal{P}(\mathsf{J}_q^*) = \mathsf{F}^*. \tag{7.146}$$

Each component $\mathcal{P}(\mathsf{J}_q^*)[A]$ is spanned by the elements $\mathtt{M}_{A,A'}$ for A' with the same support as A. More generally, the primitive filtration of J_q^* can be expressed as

$$\mathcal{P}(\mathsf{J}_q^*) = \mathsf{F}^* + (\mathsf{F}^*)^2 + \cdots + (\mathsf{F}^*)^k, \tag{7.147}$$

the sum of the first k Cauchy powers of F^*.

7.8.12. Freeness and cofreeness. The q-bimonoid J_q is the free q-bimonoid on the comonoid F. Dually, J_q^* is the cofree q-bimonoid on the monoid F^*. More precisely:

Proposition 7.58. *For any scalar q, there are isomorphisms of q-bimonoids*

$$\mathsf{J}_q \xrightarrow{\cong} \mathcal{T}_q(\mathsf{F}) \quad \textit{and} \quad \mathsf{J}_q^* \xrightarrow{\cong} \mathcal{T}_q^\vee(\mathsf{F}^*). \tag{7.148}$$

On the A-component, the first map sends $\mathtt{H}_{(K,K')}$ to $\mathtt{H}_{K'}$ in the K-summand, while the second map sends $\mathtt{M}_{(K,K')}$ to $\mathtt{M}_{K'}$ in the K-summand.

The (co)product of $\mathcal{T}_q(\mathsf{F})$ is given by specializing (6.3) and (6.5), while that of $\mathcal{T}_q^\vee(\mathsf{F}^*)$ is given by specializing (6.11) and (6.13).

Exercise 7.59. The morphism of comonoids $\mathsf{F} \to \mathsf{E}$ from Exercise 7.54 extends to a morphism of q-bimonoids $\mathcal{T}_q(\mathsf{F}) \to \mathcal{T}_q(\mathsf{E})$. Check that this coincides with the morphism $\mathsf{J}_q \to \Sigma_q$ in Exercise 7.56 via the identifications (7.81) and (7.148).

Similar considerations apply when q is not a root of unity. The q-bimonoid J_q is the free q-bimonoid on F viewed as a trivial comonoid. Dually, J_q^* is the cofree q-bimonoid on F^* viewed as a trivial monoid. More precisely:

Proposition 7.60. *For q not a root of unity, there are isomorphisms of q-bimonoids*

$$\mathsf{J}_q \xrightarrow{\cong} \mathcal{T}_q(\mathsf{F}) \quad \textit{and} \quad \mathsf{J}_q^* \xrightarrow{\cong} \mathcal{T}_q^\vee(\mathsf{F}^*). \tag{7.149}$$

On the A-component, the first map sends $\mathtt{Q}'_{(K,K')}$ to $\mathtt{H}_{K'}$ in the K-summand, while the second map sends $\mathtt{P}'_{(K,K')}$ to $\mathtt{M}_{K'}$ in the K-summand.

The (co)product of J_q in the $\mathtt{Q}'$-basis is given by (7.129). The (co)product of $\mathcal{T}_q(\mathsf{F})$ is given by specializing (6.39) and of $\mathcal{T}_q^\vee(\mathsf{F}^*)$ is given by specializing (6.40).

7.8.13. (Co)abelianization. The abelianization of the bimonoid J is given by $\mathcal{S}(\mathsf{F})$, and dually, the coabelianization of J^* is given by $\mathcal{S}^\vee(\mathsf{F}^*)$:

$$\mathsf{J}_{ab} \xrightarrow{\cong} \mathcal{S}(\mathsf{F}) \qquad \text{and} \qquad (\mathsf{J}^*)^{coab} \xrightarrow{\cong} \mathcal{S}^\vee(\mathsf{F}^*). \tag{7.150}$$

This follows by combining (6.58), (6.62), (7.148). Explicitly, the abelianization of J can be described as follows. Its A-component is spanned by all faces K' such that $K'A = K'$. The β maps are all identities. The product and coproduct are given by

$$\begin{aligned} \mu_A^F : \mathsf{J}_{ab}[F] &\to \mathsf{J}_{ab}[A] \\ \mathtt{H}_{K'} &\mapsto \mathtt{H}_{K'} \end{aligned} \qquad\qquad \begin{aligned} \Delta_A^G : \mathsf{J}_{ab}[A] &\to \mathsf{J}_{ab}[G] \\ \mathtt{H}_{K'} &\mapsto \mathtt{H}_{K'G}. \end{aligned} \tag{7.151}$$

The canonical quotient map is given by

$$\mathsf{J} \to \mathsf{J}_{ab}, \qquad \mathtt{H}_{(K,K')} \mapsto \mathtt{H}_{K'}. \tag{7.152}$$

Exercise 7.61. Show that the coabelianization of J_{ab} can be described as follows. For any face A, the A-component is spanned by $\mathtt{H}_C$, as C varies over all chambers. All structure maps are identities. In other words, $(\mathsf{J}_{ab})^{coab}$ is the direct sum of copies of E, one copy for each chamber.

7.9. Lie and Zie species

Recall the notion of Lie and Zie elements from Section 1.12. Just as faces, flats, chambers have associated species, so do Lie and Zie elements. The Lie species is a subspecies of the species of chambers, in fact, it is the primitive part of the bimonoid of chambers. Similarly, the Zie species is the primitive part of the bimonoid of faces.

7.9.1. Lie species. Recall the species of chambers Γ from Section 7.3. We write

$$z = \sum_{C:\, C \geq A} x^{C/A}\, \mathtt{H}_{C/A}$$

for a typical element of $\Gamma[A]$. The letter $\mathtt{H}$ denotes the $\mathtt{H}$-basis of Γ.

Let Lie be the subspecies of Γ whose A-component is defined by

$$z \in \mathsf{Lie}[A] \iff \sum_{\substack{C:\, C \geq A \\ HC = D}} x^{C/A} = 0 \text{ for all } A < H \leq D. \tag{7.153}$$

The rhs is a linear system in the variables $x^{C/A}$.

For faces A and B with the same support, the bijection between the star of A and the star of B carries the equations defining $\mathsf{Lie}[A]$ to the equations defining $\mathsf{Lie}[B]$, so there is an induced map

$$\beta_{B,A} : \mathsf{Lie}[A] \to \mathsf{Lie}[B].$$

Thus, Lie is indeed a subspecies of Γ. We call it the *Lie species*.

Lemma 7.62. *For any chamber C, we have* $\mathsf{Lie}[C] = \Bbbk$.

Proof. There is no face strictly greater than any chamber, so the rhs of (7.153) is vacuously true for $A = C$. Hence, $\mathsf{Lie}[C] = \Gamma[C] = \Bbbk$, spanned by $\mathtt{H}_{C/C}$. ∎

Lemma 7.63. *Let $z \in \mathsf{Lie}[A]$. If A is not a chamber, then*

$$\sum_{C\colon C \geq A} x^{C/A} = 0,$$

that is, the sum of the coefficients of chambers greater than A is zero.

Proof. Let D be any chamber strictly greater than A. The equation for (D, D) in the rhs of (7.153) is precisely the one stated above. $\square$

7.9.2. Primitive part of the bimonoid of chambers. Recall that the species of chambers Γ is a bimonoid. In particular, it is a comonoid, and hence it has a primitive part (5.18).

Lemma 7.64. *The primitive part of the comonoid of chambers is the Lie species, that is,*

$$\mathcal{P}(\Gamma) = \mathsf{Lie}.$$

Explicitly, $z \in \mathsf{Lie}[A]$ iff $\Delta_A^H(z) = 0$ for all $H > A$.

Proof. Let $z \in \Gamma[A]$. For $H \geq A$, using coproduct formula (7.18),

$$\Delta_A^H(z) = \sum_{C\colon C \geq A} x^{C/A} \mathtt{H}_{HC/H} = \sum_{D\colon H \leq D} \Big(\sum_{\substack{C\colon C \geq A \\ HC = D}} x^{C/A} \Big) \mathtt{H}_{D/H},$$

and hence

$$z \in \ker(\Delta_A^H) \iff \sum_{\substack{C\colon C \geq A \\ HC = D}} x^{C/A} = 0 \text{ for all } D \text{ with } H \leq D.$$

Now z is primitive iff the lhs holds for all $H > A$, while z is a Lie element iff the rhs holds for all $H > A$ (via (7.153)). The result follows. $\square$

We refer to this characterization as the *Friedrichs criterion* for Lie elements. More generally, the primitive filtration of Γ can also be described in terms of Lie elements, see Exercise 13.52.

Exercise 7.65. Consider the specialization of the map (5.50) to $\mathsf{h} := \Gamma$. Check directly that this map $\mathrm{pq}_\Gamma : \mathcal{P}(\Gamma) \to \mathcal{Q}(\Gamma)$ is surjective. (Note that $\mathcal{Q}(\Gamma) = \mathsf{x}$, which is a lot smaller than $\mathcal{P}(\Gamma) = \mathsf{Lie}$.) Since Γ is cocommutative, this result is a special case of Proposition 5.56, item (1).

Exercise 7.66. Use coproduct formula (7.19) to check that: For any scalar q, for any face A,

$$z \in \mathcal{P}(\Gamma_q)[A] \iff \sum_{\substack{C\colon C \geq A \\ HC = D}} x^{C/A} q^{\mathrm{dist}(C,D)} = 0 \text{ for all } A < H \leq D. \tag{7.154}$$

Setting $q = 1$ recovers the Friedrichs criterion. Observe that (7.154) is a special case of (6.71) for $\mathsf{p} := \mathsf{x}$.

Show that: For q not a root of unity,

$$\mathcal{P}(\Gamma_q) = \mathsf{x}. \tag{7.155}$$

A direct way to proceed is as follows. For A not a chamber, consider the linear system in the rhs of (7.154) for $A < D$ with $H = D$, and apply Theorem 1.10 to the arrangement over the support of A to deduce that $z = 0$. Alternatively, (7.155) follows from (7.22) and Proposition 7.5. One may also apply Exercise 6.76, item (i), to $\mathsf{p} := \mathsf{x}$ and use (7.24).

More generally, for q not a root of unity, for $k \geq 1$,

$$\mathcal{P}_k(\Gamma_q) = \mathsf{x} + \mathsf{x}^2 + \cdots + \mathsf{x}^k. \tag{7.156}$$

(Use (7.23) and Proposition 7.5, or apply Exercise 6.77 to $\mathsf{p} := \mathsf{x}$ and use (7.24).)

7.9.3. Zie species. Recall the species of faces Σ from Section 7.6. We write

$$z = \sum_{F:\, F \geq A} x^{F/A}\, \mathtt{H}_{F/A}$$

for a typical element of $\Sigma[A]$. The letter $\mathtt{H}$ denotes the $\mathtt{H}$-basis of Σ.

Let Zie be the subspecies of Σ whose A-component is defined by

$$z \in \mathsf{Zie}[A] \iff \sum_{\substack{F:\, F \geq A \\ HF = G}} x^{F/A} = 0 \text{ for all } A < H \leq G. \tag{7.157}$$

The rhs is a linear system in the variables $x^{F/A}$.

For faces A and B with the same support, the bijection between the star of A and the star of B carries the equations defining $\mathsf{Zie}[A]$ to the equations defining $\mathsf{Zie}[B]$, so there is an induced map

$$\beta_{B,A} : \mathsf{Zie}[A] \to \mathsf{Zie}[B].$$

Thus, Zie is indeed a subspecies of Σ. We call it the *Zie species*.

Lemma 7.67. *We have*

$$z \in \mathsf{Zie}[A] \iff \sum_{\substack{F:\, F \geq A \\ HF \leq G}} x^{F/A} = 0 \textit{ for all } A < H \leq G. \tag{7.158}$$

PROOF. Fix $H > A$. Observe that the equation in the rhs of (7.158) for the pair (H, G) is the sum of the equations in the rhs of (7.157) for the pairs (H, G') with $H \leq G' \leq G$. Hence, the claim follows from a triangularity argument on the inclusion order on faces. □

Lemma 7.68. *For any chamber C, we have $\mathsf{Zie}[C] = \Bbbk$.*

PROOF. There is no face strictly greater than any chamber, so the rhs of (7.157) is vacuously true for $A = C$. Hence, $\mathsf{Zie}[C] = \Sigma[C] = \Bbbk$, spanned by $\mathtt{H}_{C/C}$. □

7.9.4. Primitive part of the bimonoid of faces. Recall that the species of faces Σ is a bimonoid. In particular, it is a comonoid.

Lemma 7.69. *The primitive part of the comonoid of faces is the Zie species, that is,*

$$\mathcal{P}(\Sigma) = \mathsf{Zie}.$$

Explicitly, $z \in \mathsf{Zie}[A]$ iff $\Delta_A^H(z) = 0$ for all $H > A$.

PROOF. Let $z \in \Sigma[A]$. For $H \geq A$, using coproduct formula (7.64),

$$\Delta_A^H(z) = \sum_{F\colon F \geq A} x^{F/A} \mathtt{H}_{HF/H} = \sum_{G: H \leq G} \Big(\sum_{\substack{F\colon F \geq A \\ HF=G}} x^{F/A} \Big) \mathtt{H}_{G/H},$$

and hence

$$z \in \ker(\Delta_A^H) \iff \sum_{\substack{F\colon F \geq A \\ HF=G}} x^{F/A} = 0 \text{ for all } G \text{ with } H \leq G.$$

Now z is primitive iff the lhs holds for all $H > A$, while z is a Zie element iff the rhs holds for all $H > A$ (via (7.157)). The result follows. $\square$

We refer to this characterization as the *Friedrichs criterion* for Zie elements. More generally, the primitive filtration of Σ can also be described in terms of Zie elements, see Exercise 13.52.

Exercise 7.70. Work out $\mathcal{P}(\Sigma)$ using the $\mathtt{Q}$-basis (7.67) by employing coproduct formula (7.69). Derive a description of Zie in terms of Lie elements in arrangements under flats. This is explained in detail later in Proposition 16.11, see also (1.177).

Exercise 7.71. Consider the specialization of the map (5.50) to $\mathsf{h} := \Sigma$. Since Σ is cocommutative, by Proposition 5.56, item (1), this map $\mathrm{pq}_\Sigma : \mathcal{P}(\Sigma) \to \mathcal{Q}(\Sigma)$ is surjective. Deduce that special Zie elements exist. (Use Exercise 7.27.)

Exercise 7.72. Use coproduct formula (7.65) to check that: For any scalar q, for any face A,

$$z \in \mathcal{P}(\Sigma_q)[A] \iff \sum_{\substack{F\colon F \geq A \\ HF=G}} x^{F/A} q^{\mathrm{dist}(F,G)} = 0 \text{ for all } A < H \leq G. \tag{7.159}$$

Setting $q = 1$ recovers the Friedrichs criterion. Observe that (7.159) is a special case of (6.72) for $\mathsf{c} := \mathsf{E}$.

Deduce that: For q not a root of unity,

$$\mathcal{P}(\Sigma_q) = \mathsf{E}. \tag{7.160}$$

Explicitly, $\mathcal{P}(\Sigma_q)[A]$ is spanned by the element

$$\mathtt{Q}_{A/A} := \sum_{F\colon F \geq A} \boldsymbol{\mu}_q(A, F)\, \mathtt{H}_{F/A},$$

where $\boldsymbol{\mu}_q$ is the noncommutative q-Möbius function defined by (1.48). We point out that the element $\mathtt{Q}_{A/A}$ is part of the $\mathtt{Q}$-basis defined in (7.70).

More generally, for q not a root of unity, for $k \geq 1$,

$$\mathcal{P}_k(\Sigma_q) = \mathsf{E} + \mathsf{E}^2 + \cdots + \mathsf{E}^k. \tag{7.161}$$

Explicitly, $\mathcal{P}_k(\Sigma_q)[A]$ is spanned by

$$\{\mathtt{Q}_{F/A} \mid \mathrm{rk}(F/A) < k\}.$$

(Use (7.80) and Theorem 7.38 with $\eta \equiv 1$, or apply Exercise 6.77 to $\mathsf{p} := \mathsf{E}$ and use (7.83).)

7.9.5. Primitive part of the bimonoid of top-nested faces. Recall the q-bimonoid of top-nested faces from Section 7.7.

Exercise 7.73. Show that: For q not a root of unity,

$$\mathcal{P}(\widehat{\mathsf{Q}}_q) = \Gamma^*. \tag{7.162}$$

Explicitly, $\mathcal{P}(\widehat{\mathsf{Q}}_q)[A]$ is spanned by the elements $\mathtt{Q}_{A/A,C/A}$ for $C \geq A$. These are the $\mathtt{Q}$-basis elements defined in (7.100).

More generally, for q not a root of unity, for $k \geq 1$,

$$\mathcal{P}_k(\widehat{\mathsf{Q}}_q) = \Gamma^* + (\Gamma^*)^2 + \cdots + (\Gamma^*)^k. \tag{7.163}$$

Explicitly, $\mathcal{P}_k(\widehat{\mathsf{Q}}_q)[A]$ is spanned by

$$\{\mathtt{Q}_{F/A,C/A} \mid C \geq A, \operatorname{rk}(F/A) < k\}.$$

(Apply Exercise 6.77 to $\mathsf{p} := \Gamma^*$ and use (7.107).) We point out that for $q = 0$, the $\mathtt{Q}$-basis elements have explicit formulas given by (7.114).

7.9.6. Primitive part of the bimonoid of bifaces. Recall the q-bimonoid of bifaces from Section 7.8.

Exercise 7.74. Show that: For q not a root of unity,

$$\mathcal{P}(\mathsf{J}_q) = \mathsf{F}. \tag{7.164}$$

Explicitly, $\mathcal{P}(\mathsf{J}_q)[A]$ is spanned by the elements $\mathtt{Q}'_{(A,A')}$ for A' with the same support as A. These are the $\mathtt{Q}'$-basis elements defined in (7.128).

More generally, for q not a root of unity, for $k \geq 1$,

$$\mathcal{P}_k(\mathsf{J}_q) = \mathsf{F} + \mathsf{F}^2 + \cdots + \mathsf{F}^k. \tag{7.165}$$

Explicitly, $\mathcal{P}_k(\mathsf{J}_q)[A]$ is spanned by

$$\{\mathtt{Q}'_{(F,F')} \mid \mathrm{s}(F') = \mathrm{s}(F), \operatorname{rk}(F/A) < k\}.$$

(Apply Exercise 6.77 to $\mathsf{p} := \mathsf{F}$ and use (7.149).)

Notes

The bimonoid of bifaces in Section 7.8 has a distinct local flavor and appears to be a completely new object. References to the literature for classical analogues of the remaining bimonoids in this chapter are given below. The geometric perspective on these objects was brought forth in [**17**, Chapter 6] in the context of Hopf algebras and in [**18**, Chapter 12] in the context of Hopf monoids in Joyal species. The translation between the geometry and combinatorics of the braid arrangement is recalled in Table 7.2; for more detail, see [**21**, Table 6.2], [**18**, Table 13.3]. The connection between Joyal species and species for arrangements was briefly explained in Section 2.16, see Remark 2.92 in particular.

TABLE 7.2. Geometry and combinatorics of the braid arrangement.

Species	Geometry	Combinatorics
Σ	face	set composition
Γ	chamber	linear order
Π	flat	set partition
$\widehat{Q}$	top-nested face	linear set composition
$\widehat{\Lambda}$	top-lune	linear set partition
G	chart	simple graph
$\overrightarrow{G}$	dichart	simple directed graph
$\Gamma\!\Gamma$	pair of chambers	pair of linear orders

Bimonoids in Joyal species. The Joyal monoid of linear orders first appeared in work of Barratt under the name 'permutation bi-ring' [**74**, page 10]. On page 11, he states that this is the free monoid on one generator. This is the analogue of (7.24) for $q = 1$.

Bergeron, Labelle, Leroux [**102**] give many examples of Joyal set-species. (They do not discuss (co, bi)monoids.) The Joyal species of set partitions and linear orders are given in [**102**, Section 1.1, page 7]. The classical analogue of the species characteristic of chambers defined in (7.1) is the Joyal species characteristic of singletons (4.22), see also [**102**, Section 1.1, (17)]. Monoids in Joyal set-species are discussed by Méndez [**675**, Definition 3.6], the exponential monoid and monoid of linear orders are given in [**675**, Example 3.22], the analogue of (7.17) is given in [**675**, Example 3.3].

The exponential Joyal bimonoid and the Joyal bimonoids of linear orders, (linear) set partitions, (linear) set compositions were studied in detail in [**18**]. The exponential Joyal bimonoid is discussed in [**18**, Section 8.5], the Joyal bimonoid of linear orders in [**18**, Sections 8.5 and 12.2], the Joyal bimonoid of set partitions in [**18**, Section 12.6], the Joyal bimonoid of set compositions in [**18**, Section 12.4], the Joyal bimonoid of linear set partitions in [**18**, Section 12.7], the Joyal bimonoid of linear set compositions in [**18**, Section 12.5]. In this reference, $\widehat{Q}$ and $\widehat{\Lambda}$ are denoted by $\overrightarrow{\Sigma}$ and $\overrightarrow{\Pi}$, respectively. The self-duality of the Joyal bimonoid of set partitions is described in [**18**, Proposition 12.48] using the isomorphism ψ in (7.52). The analogue of the second result in Theorem 7.40 is an improvement on [**18**, Proposition 12.26] (in the sense that it is very specific on the values of q which yield an isomorphism). For recent work on the Joyal bimonoid of set compositions and its primitive part (which is the Joyal species of Zie elements), see the papers by Liu, Norledge, Ocneanu [**601**], [**723**]. Interesting examples of Zie elements called 'Dynkin elements' were introduced in our monograph [**21**, Proposition 14.1]. They are associated to generic half-spaces of any hyperplane arrangement. For the braid arrangement, these elements are present in early work of Epstein, Glaser, Stora [**292**, Formula (1), page 26], [**290**, Formula (28)]. These formulas include the Dynkin elements; however not all elements in these formulas are Zie elements. We also point out that these formulas are phrased purely in combinatorial terms, with no mention of either generic half-spaces or Dynkin.

For the braid arrangement, charts correspond to simple graphs. The analogue of the bimonoid of charts is the Joyal bimonoid of simple graphs in [**18**, Section 13.2], [**19**, Section 9.4], [**10**].

The decorated exponential Joyal bimonoid is discussed in [**18**, Example 8.18]. It is referred to as the tensor algebra in [**746**, page 204], [**156**, Example 4.2.1.2]. It can be viewed as the analogue of E_M in Section 7.2.6.

For the Joyal bimonoid of pairs of linear orders, see the Notes to Chapter 15.

Bialgebras. The bimonoids in Joyal species mentioned above are closely related to many classical Hopf algebras, see Table 7.3. For details on the precise connection between the two, see [**18**, Chapter 17].

TABLE 7.3. Classical Hopf algebras.

Joyal bimonoids	Hopf algebras
E, $\mathsf{\Gamma}$	Hopf algebra of polynomials $\Bbbk[x]$, divided power Hopf algebra $\Bbbk\{x\}$
$\mathsf{\Sigma}$	Hopf algebra of noncommutative symmetric functions
$\mathsf{\Sigma}^*$	Hopf algebra of quasisymmetric functions
$\mathsf{\Pi}$	Hopf algebra of symmetric functions, and of symmetric functions in noncommuting variables
G	Hopf algebras of simple graphs
$\mathsf{I\Gamma}$	Hopf algebra of permutations, and of pairs of permutations

Hopf algebra of polynomials. The exponential Joyal bimonoid and the Joyal bimonoid of linear orders relate to the classical Hopf algebra of polynomials $\Bbbk[x]$ (in one variable x) as well as to the dual divided power Hopf algebra $\Bbbk\{x\}$. For references, see for instance those by Nichols and Sweedler [**721**, Sections III and XI], Abe [**1**, Example 2.6], Montgomery [**703**, Example 5.6.8], Kassel [**517**, Exercise III.8.2], Selick [**821**, Section 10.1, page 106], Hazewinkel, Gubareni, Kirichenko [**428**, Example 3.4.19], Underwood [**888**, Example 3.1.3]. For q-analogues, see the paper by Joni and Rota [**496**, Section VIII]. A summary is given in our monograph [**18**, Examples 2.3 and 2.9].

Hopf algebras of simple graphs. The Joyal bimonoid of simple graphs relates to Hopf algebras considered by Schmitt [**813**, Sections 12 and 13]. The precise connection is explained in our monograph [**18**, Section 17.5.3].

Hopf algebra of symmetric functions. The Joyal bimonoid of set partitions relates to the Hopf algebra of symmetric functions. It is most often viewed as a subalgebra of the algebra of formal power series in countably many variables, see for instance the books by Fulton [**332**], Macdonald [**625**], Sagan [**794**], Stanley [**843**]. The Hopf algebra viewpoint originated in work of Geissinger [**344**], Zelevinsky [**929**]. It is discussed in the book by Blessenohl and Schocker [**120**, Chapter 3]. More references are given below under noncommutative symmetric and quasisymmetric functions. As an abstract Hopf algebra, it had appeared earlier in work of Moore on universal constructions [**707**, page 6, Definitions and Notations] and of Cartier on formal groups [**197**, Corollary 2], see also [**427**, Theorem 36.1.11].

Hopf algebra of symmetric functions in noncommuting variables. The Joyal bimonoid of set partitions also relates to the Hopf algebra of symmetric functions in noncommuting variables. As an algebra, it appeared in work of Wolf [**916**] and was further studied by Gebhard, Rosas, Sagan [**343**], [**786**]. The Hopf algebra structure was pointed out in our monographs [**17**, Remark on page 84], [**18**, Section 17.4.1], and studied in [**103**], [**105**], [**106**], [**570**]. Related work includes [**9**], [**12**].

Hopf algebra of noncommutative symmetric functions and of quasisymmetric functions. The Joyal bimonoid of set compositions relates to the Hopf algebra of noncommutative symmetric functions introduced by Gelfand, Krob, Lascoux, Leclerc, Retakh, Thibon [**347**], and extensively studied further in a series of papers including [**548**], [**269**], [**549**], [**550**], [**268**]. This object had been considered earlier by Ditters in the context of formal groups [**253**, Section 1.2.7], [**254**, Section 1], [**255**, $Z(\infty, A)$ in Lemma 1.1], [**256**, Chapter I, Example 2.5, item (c)], see also his paper with Scholtens [**257**, Sentence after Theorem 1.2] and the book by Hazewinkel [**427**, Section 36.3.8].

The dual Joyal bimonoid of set compositions relates to the Hopf algebra of quasisymmetric functions. It was introduced by Gessel [**350**] as a subalgebra of the algebra of formal power series in countably many variables. It is discussed by Reutenauer [**777**, Section 9.4], Stanley [**843**, Section 7.19], see also [**109**]. The Hopf algebra structure was introduced by Malvenuto [**634**, Section 4.1]. It is an instance of the object $\mathcal{T}^\vee(A)$ discussed in the Notes to Chapter 6. The quasishuffle product of quasisymmetric functions is explicitly described by Ehrenborg [**277**, Lemma 3.3]. Quasisymmetric functions also appeared in work of Hoffman on multiple zeta functions [**448**, page 487], see also [**201**, Formula (94)], [**451**, Sections 4 and 5], [**294**, Section 2], [**477**, Section 1]. In this context, a quasi-shuffle is described by Bradley under the name 'stuffle' [**152**, Formula (2.1)], by Manchon and Paycha under the name 'mixable shuffle' [**638**, page 50]. It also appears in work of Goncharov [**359**, Section 7.1]. The universal property of the Hopf algebra of quasisymmetric functions [**11**, Theorem 4.1] can be viewed as the classical analogue of the universal property of $\mathbf{\Sigma}_q^*$ in Exercise 7.29 for $q = 1$. A topological point of view is given by Baker and Richter [**66**].

The Hopf algebras of symmetric functions, noncommutative symmetric functions, quasisymmetric functions are reviewed in our monograph [**17**, Sections 3.2 and 8.4] with further closely related examples in [**17**, Section 6.2]. Connections with Joyal species are brought forth in [**18**, Sections 17.3 and 17.4], see, in particular, the q-analogues in [**18**, Sections 17.3.4]. Surveys of these Hopf algebras are provided by Hazewinkel [**425**], [**426**] and Thibon [**879**]. More detailed discussions can be found for instance in the books by Hazewinkel, Gubareni, Kirichenko [**428**, Chapters 4 and 6], Méliot [**673**, Chapters 2 and 6], Grinberg and Reiner [**377**, Chapters 2 and 5]. A recent reference is the paper by Mason [**655**].

For the related Hopf algebra of permutations and of pairs of permutations, see the Notes to Chapter 15.

The generalizations of the above Hopf algebras to the setting of Coxeter bialgebras will be given in future work.

Bimonoids for hyperplane arrangements. The bimonoid of flats and the bimonoid of faces in the generality of arrangements are mentioned in [**21**, Sections 9.1.9 and 9.8.1], also see the remarks on [**21**, page 274]. The product and coproduct of the bimonoid of flats are given in [**21**, (9.17) and (9.15)], and the bicommutative bimonoid axiom in [**21**, Exercise 9.13]. The product and coproduct of the bimonoid of faces are given in [**21**, (9.74) and (9.72)], and the bimonoid axiom in [**21**, Exercise 9.62]. The facts in Exercise 7.21 are given in [**21**, Section 9.8.1]. Formulas (7.69) are equivalent to [**21**, Formulas (11.37)]. Formula (7.74) is related to [**21**, Formula (11.21)]. Lemmas 7.64 and 7.69 relate to [**21**, Lemmas 10.5 and 10.19], respectively.

Theorem 7.10 is equivalent to [**21**, Theorem 9.10] (in view of the categorical equivalence in Proposition 11.5). The result is stated more generally for lattices in [**21**, Theorem D.59]; the case of the map φ in (7.52) relates to a result of Dowling and Wilson [**261**, Lemma 1 on page 506], [**890**, Theorem 25.5]. The first result in Theorem 7.40 and the result in Theorem 7.51 can be viewed as noncommutative q-analogues of this result.

Charts and dicharts for arrangements are discussed in [**21**, Sections 2.6 and 9.2]. Theorem 7.16 is equivalent to [**21**, Theorem 9.15] (in view of the categorical equivalence in Proposition 11.5).

The material in Section 7.9.4 can be viewed as progress on [**18**, Question 12.27]. Similarly, Exercise 7.73 makes progress on [**18**, Question 12.39]. In this reference, only the $q = 0$ case for the primitive part was tackled.

Additional examples of bimonoids for arrangements are given in Sections 8.5 and 15.5. Many more examples built out of geometric data such as cones (preorders), top-cones (partial orders), gallery intervals (partial orders of order dimension 1 or 2), shards, deformations of the zonotope of the arrangement (generalized permutahedra), and so on will be given in future work. For cones and gallery intervals, the necessary ingredients for the construction are present in [**21**, Section 3.4]. For the bimonoid of top-lunes in Section 7.7.9, a relevant discussion is given in [**21**, Section 3.4.7].

CHAPTER 8

Hadamard product

This chapter assumes some basic familiarity with monoidal categories. An adequate reference is [**18**, Chapter 1], see also Appendix B.

We introduce the Hadamard product on the category of species. A key property of this product is that it preserves monoids, comonoids, bimonoids. In fact, for any scalars p and q, the Hadamard product of a p-bimonoid and a q-bimonoid is a pq-bimonoid. Similarly, the Hadamard product of (co)commutative (co)monoids is again (co)commutative. These facts can be seen as formal consequences of the bilax property of the Hadamard functor.

We construct the internal hom for the Hadamard product of species, and discuss its bilax property and the related constructions of the convolution monoid, coconvolution comonoid, biconvolution bimonoid. Moreover, we also construct the internal hom for the Hadamard product of monoids, comonoids and bimonoids, making critical use of the fact that these are functor categories just like the category of species. The internal hom for (co, bi)commutative bimonoids is intimately connected to the internal hom for the tensor product of modules over the Birkhoff algebra, Tits algebra, Janus algebra via certain categorical equivalences. The latter are developed later in Chapter 11.

We construct the universal measuring comonoid from one monoid to another monoid. It allows us to enrich the category of monoids over the category of comonoids. This enriched category possesses powers and copowers which we describe explicitly. The power is in fact the convolution monoid. The copower is a certain quotient of the free monoid on the Hadamard product of the given comonoid and monoid.

We introduce the bimonoid of star families. It is constructed out of a cocommutative comonoid and a bimonoid. It builds on the internal hom for the Hadamard product of comonoids. Moreover, it has a commutative counterpart which we call the bicommutative bimonoid of star families. This one builds on the internal hom for cocommutative comonoids. There is also an analogous construction starting with a bimonoid and a commutative monoid which builds on the universal measuring comonoid. These bimonoids play an important role in the study of the exp-log correspondences in Chapter 9.

A summary of the various constructions is provided in Table 8.1.

We introduce the signature functor on species. It is defined by taking Hadamard product with the signed exponential species. Recall that the latter carries the structure of a signed bimonoid. This sets up an equivalence between the categories of bimonoids and signed bimonoids.

Table 8.1. Convolution monoid and its relatives.

Starting data	Object	
comonoid c, monoid a	$\mathrm{hom}^{\times}(\mathsf{c},\mathsf{a})$	convolution monoid
	$\mathrm{hom}^{\times}(\mathsf{a},\mathsf{c})$	coconvolution comonoid
bimonoids h, k	$\mathrm{hom}^{\times}(\mathsf{h},\mathsf{k})$	biconvolution bimonoid
comonoids c, d	$\mathcal{C}(\mathsf{c},\mathsf{d})$	comonoid
cocomm. comonoid c, bimonoid k	$\mathcal{C}(\mathsf{c},\mathsf{k})$	bimonoid of star families
cocomm. comonoids c, d	${}^{\mathsf{co}}\mathcal{C}(\mathsf{c},\mathsf{d})$	cocomm. comonoid
cocomm. comonoid c, bicomm. bimonoid k	${}^{\mathsf{co}}\mathcal{C}(\mathsf{c},\mathsf{k})$	bicomm. bimonoid of star families
monoids a, b	$\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})$	universal measuring comonoid
comm. monoid a, bimonoid h	$\overline{\mathcal{C}}(\mathsf{h},\mathsf{a})$	bimonoid
comm. monoids a, b	${}^{\mathsf{co}}\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})$	universal measuring cocomm. comonoid
comm. monoid a, bicomm. bimonoid h	${}^{\mathsf{co}}\overline{\mathcal{C}}(\mathsf{h},\mathsf{a})$	bicomm. bimonoid
comonoid c, monoid a	$\mathsf{c} \triangleright \mathsf{a}$	monoid
monoids a, b	$\mathcal{M}(\mathsf{a},\mathsf{b})$	monoid
comm. monoids a, b	$\mathcal{M}^{\mathsf{co}}(\mathsf{a},\mathsf{b})$	comm. monoid

The Hadamard product is studied further in Chapter 15.

8.1. Hadamard functor

We define the Hadamard product of species. It defines a monoidal structure on the category of species, with the exponential species as the unit object. This product carries the structure of a bilax functor wrt the bimonad $(\mathcal{T}, \mathcal{T}^{\vee}, \lambda)$ on species, and hence it preserves bimonoids. In a related result, we show that the Hadamard product gives rise to a monad $\mathcal{L}$ on species which is compatible with the monad $\mathcal{T}$ and the comonad $\mathcal{T}^{\vee}$ in the sense that $(\mathcal{T}, \mathcal{L}, \mathcal{T}^{\vee})$ is a $(2,1)$-monad.

8.1.1. Hadamard functor. Let p and q be two species. Define a new species $\mathsf{p} \times \mathsf{q}$ by

$$(\mathsf{p} \times \mathsf{q})[F] := \mathsf{p}[F] \otimes \mathsf{q}[F]. \tag{8.1}$$

For any morphism $\beta_{G,F} : F \to G$, there is an induced linear map

$$\mathsf{p}[F] \otimes \mathsf{q}[F] \xrightarrow{\beta_{G,F} \otimes \beta_{G,F}} \mathsf{p}[G] \otimes \mathsf{q}[G],$$

so $\mathsf{p} \times \mathsf{q}$ is a species. This is the *Hadamard product* of p and q. This construction is natural in p and q, that is, maps $\mathsf{p} \to \mathsf{p}'$ and $\mathsf{q} \to \mathsf{q}'$ of species induce a map $\mathsf{p} \times \mathsf{q} \to \mathsf{p}' \times \mathsf{q}'$ of species. This yields a monoidal structure on the category of species $\mathcal{A}\text{-}\mathsf{Sp}$. The unit object is the exponential species E, which we recall from Section 7.2 is defined by $\mathsf{E}[A] = \Bbbk$ for all A. By interchanging the tensor factors in (8.1), we see that there is an isomorphism of species $\mathsf{p} \times \mathsf{q} \to \mathsf{q} \times \mathsf{p}$. This defines a braiding (which is in fact a symmetry).

We refer to

$$(- \times -) : \mathcal{A}\text{-}\mathsf{Sp} \times \mathcal{A}\text{-}\mathsf{Sp} \to \mathcal{A}\text{-}\mathsf{Sp}, \qquad (\mathsf{p}, \mathsf{q}) \mapsto \mathsf{p} \times \mathsf{q} \tag{8.2}$$

as the *Hadamard functor*.

8.1.2. Hadamard product of bimonoids. The Hadamard product preserves monoids in species: Suppose a and b are monoids. Then $\mathsf{a} \times \mathsf{b}$ is also a monoid with product components defined by

$$\begin{tikzcd}
(\mathsf{a} \times \mathsf{b})[F] \arrow[r, dashed, "\mu_A^F"] \arrow[d, equal] & (\mathsf{a} \times \mathsf{b})[A] \arrow[d, equal] \\
\mathsf{a}[F] \otimes \mathsf{b}[F] \arrow[r, "\mu_A^F \otimes \mu_A^F"'] & \mathsf{a}[A] \otimes \mathsf{b}[A].
\end{tikzcd} \tag{8.3}$$

(For instance, to check any diagram in (2.8) for $\mathsf{a} \times \mathsf{b}$, we take the corresponding diagram for a and for b, and take their tensor product.) In addition, if a and b are both commutative, then so is $\mathsf{a} \times \mathsf{b}$. In the formulation given by Proposition 2.20, the components of the structure maps of $\mathsf{a} \times \mathsf{b}$ can be written as

$$\mathsf{a}[\mathrm{X}] \otimes \mathsf{b}[\mathrm{X}] \xrightarrow{\mu_{\mathrm{Z}}^{\mathrm{X}} \otimes \mu_{\mathrm{Z}}^{\mathrm{X}}} \mathsf{a}[\mathrm{Z}] \otimes \mathsf{b}[\mathrm{Z}]. \tag{8.4}$$

Similarly, if $\mathsf{a} \to \mathsf{a}'$ and $\mathsf{b} \to \mathsf{b}'$ are morphisms of monoids, then so is $\mathsf{a} \times \mathsf{b} \to \mathsf{a}' \times \mathsf{b}'$.

Dually, if c and d are comonoids, then so is $\mathsf{c} \times \mathsf{d}$ with coproduct components defined by

$$\begin{tikzcd}
(\mathsf{c} \times \mathsf{d})[A] \arrow[r, dashed, "\Delta_A^F"] \arrow[d, equal] & (\mathsf{c} \times \mathsf{d})[F] \arrow[d, equal] \\
\mathsf{c}[A] \otimes \mathsf{d}[A] \arrow[r, "\Delta_A^F \otimes \Delta_A^F"'] & \mathsf{c}[F] \otimes \mathsf{d}[F].
\end{tikzcd} \tag{8.5}$$

In addition, if c and d are cocommutative, then so is $\mathsf{c} \times \mathsf{d}$. Similarly, if $\mathsf{c} \to \mathsf{c}'$ and $\mathsf{d} \to \mathsf{d}'$ are morphisms of comonoids, then so is $\mathsf{c} \times \mathsf{d} \to \mathsf{c}' \times \mathsf{d}'$.

Thus, the Hadamard product defines a monoidal structure on $\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})$, $\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})$, and their commutative counterparts. The unit object is E in all cases.

Lemma 8.1. *For any scalars p and q, if h is a p-bimonoid and k is a q-bimonoid, then $\mathsf{h} \times \mathsf{k}$ is a pq-bimonoid.*

PROOF. The tensor product of the p-bimonoid axiom of h and the q-bimonoid axiom of k yields the diagram

$$\begin{tikzcd}
\mathsf{h}[F] \otimes \mathsf{k}[F] \arrow[r, "\mu_A^F \otimes \mu_A^F"] \arrow[d, "\Delta_F^{FG} \otimes \Delta_F^{FG}"'] & \mathsf{h}[A] \otimes \mathsf{k}[A] \arrow[r, "\Delta_A^G \otimes \Delta_A^G"] & \mathsf{h}[G] \otimes \mathsf{k}[G] \\
\mathsf{h}[FG] \otimes \mathsf{k}[FG] \arrow[rr, "p^{\mathrm{dist}(F,G)}\beta_{GF,FG} \otimes q^{\mathrm{dist}(F,G)}\beta_{GF,FG}"'] & & \mathsf{h}[GF] \otimes \mathsf{k}[GF]. \arrow[u, "\mu_G^{GF} \otimes \mu_G^{GF}"']
\end{tikzcd}$$

Since $p^{\mathrm{dist}(F,G)} q^{\mathrm{dist}(F,G)} = (pq)^{\mathrm{dist}(F,G)}$, we see that the above is indeed the pq-bimonoid axiom of $\mathsf{h} \times \mathsf{k}$. $\square$

In particular:

- If h and k are bimonoids, then $\mathsf{h} \times \mathsf{k}$ is a bimonoid.
- If h is a signed bimonoid and k is a bimonoid or vice versa, then $\mathsf{h} \times \mathsf{k}$ is a signed bimonoid.
- If h and k are signed bimonoids, then $\mathsf{h} \times \mathsf{k}$ is a bimonoid.
- If h is a 0-bimonoid and k is a q-bimonoid, then $\mathsf{h} \times \mathsf{k}$ is a 0-bimonoid.

Since the Hadamard product preserves (co, bi)commutative bimonoids, it yields a monoidal structure on $\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$ and its commutative counterparts.

Exercise 8.2. Check that: The Hadamard product of a (co)commutative and a signed (co)commutative (co)monoid is signed (co)commutative. Similarly, the Hadamard product of two signed (co)commutative (co)monoids is (co)commutative.

Exercise 8.3. Let $\sim$ be a partial-support relation on faces. Suppose a and b are monoids and both are $\sim$-commutative. Show that $\mathsf{a} \times \mathsf{b}$ is also $\sim$-commutative. State the dual result for comonoids. Also formulate their signed analogues.

8.1.3. Hadamard functor as a bilax functor. We now try to understand the preceding observations in a more formal setting. Recall from Theorem 3.6 the family of bimonads $(\mathcal{T}, \mathcal{T}^\vee, \lambda_q)$ on the category of species, one for each scalar q.

Let us temporarily denote the Hadamard functor by $\mathcal{F}$. We now define natural transformations

$$\varphi : \mathcal{T}\mathcal{F} \to \mathcal{F}(\mathcal{T} \times \mathcal{T}) \quad \text{and} \quad \psi : \mathcal{F}(\mathcal{T}^\vee \times \mathcal{T}^\vee) \to \mathcal{T}^\vee \mathcal{F}. \tag{8.6}$$

Evaluated on a pair of species (p, q), on the A-component, the maps

$$\bigoplus_{A \leq F} \mathsf{p}[F] \otimes \mathsf{q}[F] \rightleftarrows \bigoplus_{A \leq F,\, A \leq G} \mathsf{p}[F] \otimes \mathsf{q}[G]$$

are the canonical inclusion for φ and the canonical projection for ψ. In the lhs, the sum is over F, while in the rhs, the sum is over F and G.

Theorem 8.4. *For any scalars p and q, the Hadamard functor*

$$(\times, \varphi, \psi) : (\mathcal{T}, \mathcal{T}^\vee, \lambda_p) \times (\mathcal{T}, \mathcal{T}^\vee, \lambda_q) \to (\mathcal{T}, \mathcal{T}^\vee, \lambda_{pq})$$

is bilax.

Proof. It is a routine check that $(\times, \varphi) : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is a lax functor of monads, and $(\times, \psi) : \mathcal{T}^\vee \times \mathcal{T}^\vee \to \mathcal{T}^\vee$ is a colax functor of comonads. This is commutativity of diagrams (C.2) and (C.6).

To show that $(\times, \varphi, \psi)$ is bilax, we need to further check commutativity of diagram (C.10). It takes the following form, evaluated on a pair of species (p, q).

$$\begin{tikzcd}
\mathcal{T}(\mathcal{T}^\vee(\mathsf{p}) \times \mathcal{T}^\vee(\mathsf{q})) \arrow[r, hook] \arrow[d, two heads] & \mathcal{T}\mathcal{T}^\vee(\mathsf{p}) \times \mathcal{T}\mathcal{T}^\vee(\mathsf{q}) \arrow[r] & \mathcal{T}^\vee\mathcal{T}(\mathsf{p}) \times \mathcal{T}^\vee\mathcal{T}(\mathsf{q}) \arrow[d, two heads] \\
\mathcal{T}\mathcal{T}^\vee(\mathsf{p} \times \mathsf{q}) \arrow[r] & \mathcal{T}^\vee\mathcal{T}(\mathsf{p} \times \mathsf{q}) \arrow[r, hook] & \mathcal{T}^\vee(\mathcal{T}(\mathsf{p}) \times \mathcal{T}(\mathsf{q})).
\end{tikzcd}$$

On the A-component, we get the following.

$$\begin{tikzcd}
\displaystyle\bigoplus_{\substack{A \le F \le G \\ A \le F \le H}} \mathsf{p}[G] \otimes \mathsf{q}[H] \arrow[r, hook, "F=K"] \arrow[d, two heads, "G=H"'] & \displaystyle\bigoplus_{\substack{A \le F \le G \\ A \le K \le H}} \mathsf{p}[G] \otimes \mathsf{q}[H] \arrow[r] & \displaystyle\bigoplus_{\substack{A \le F' \le G' \\ A \le K' \le H'}} \mathsf{p}[G'] \otimes \mathsf{q}[H'] \arrow[d, two heads, "F'=K'"] \\
\displaystyle\bigoplus_{A \le F \le G} \mathsf{p}[G] \otimes \mathsf{q}[G] \arrow[r] & \displaystyle\bigoplus_{A \le F' \le G'} \mathsf{p}[G'] \otimes \mathsf{q}[G'] \arrow[r, hook, "G'=H'"'] & \displaystyle\bigoplus_{\substack{A \le F' \le G' \\ A \le F' \le H'}} \mathsf{p}[G'] \otimes \mathsf{q}[H'].
\end{tikzcd}$$

The sums are over all the indices written under the $\bigoplus$ symbol, except the index A which is fixed. The labels $F = K$ and $G' = H'$ on the arrows indicate the summand in the target to which a given summand maps. Similarly, the labels $G = H$ and $F' = K'$ indicate the summands in the source which map canonically to a summand in the target, with the rest mapping to zero. We need to show that this diagram commutes.

Going across and down, the mixed distributive laws force $FF' = G$, $F'F = G'$, $KK' = H$ and $K'K = H'$, but since $F = K$ and $F' = K'$, we have $G = H$ and $G' = H'$. These are exactly the conditions while going down and across. Now assume that these equivalent conditions hold. Going across and down, the summand $\mathsf{p}[G] \otimes \mathsf{q}[H]$ maps to the summand $\mathsf{p}[G'] \otimes \mathsf{q}[H']$ by

$$p^{\mathrm{dist}(G,G')} q^{\mathrm{dist}(H,H')} \, \beta_{G',G} \otimes \beta_{H',H}.$$

The scalar comes from the laws λ_p and λ_q. Going down and across, the map is

$$(pq)^{\mathrm{dist}(G,G')} \, \beta_{G',G} \otimes \beta_{H',H},$$

with the scalar coming from the law λ_{pq}. Since $G = H$ and $G' = H'$, we see that the scalars match. So the diagram commutes, as required. $\square$

Exercise 8.5. Deduce the result of Lemma 8.1 as a formal consequence of Theorem 8.4. (Use Proposition C.28 which says that a bilax functor of bimonads preserves bialgebras.)

An in-between result is given below.

Theorem 8.6. *For any q-bimonoid* k*, the functor*

$$(-) \times \mathsf{k} : (\mathcal{T}, \mathcal{T}^\vee, \lambda_p) \to (\mathcal{T}, \mathcal{T}^\vee, \lambda_{pq})$$

is bilax.

PROOF. The above functor is the composite

$$\mathcal{A}\text{-}\mathsf{Sp} \to \mathcal{A}\text{-}\mathsf{Sp} \times \mathcal{A}\text{-}\mathsf{Sp} \to \mathcal{A}\text{-}\mathsf{Sp}, \qquad \mathsf{p} \mapsto (\mathsf{p}, \mathsf{k}) \mapsto \mathsf{p} \times \mathsf{k}.$$

Recall that a q-bimonoid k is the same as a bilax functor from the identity bimonad to $(\mathcal{T}, \mathcal{T}^\vee, \lambda_q)$. Since k is a q-bimonoid, we deduce that the first functor is bilax. The second functor is the Hadamard functor which is bilax by Theorem 8.4. Hence, their composite is bilax by Proposition C.5. $\square$

Explicitly, evaluated on a species p, on the A-component, the bilax structure maps

$$\bigoplus_{F\colon F \geq A} \mathsf{p}[F] \otimes \mathsf{k}[F] \rightleftarrows \Big(\bigoplus_{F\colon F \geq A} \mathsf{p}[F] \Big) \otimes \mathsf{k}[A] \tag{8.7}$$

are as follows. They preserve corresponding summands, and on a particular F-summand are given by $\mathrm{id} \otimes \mu_A^F$ and $\mathrm{id} \otimes \Delta_A^F$, respectively. Here μ_A^F and Δ_A^F denote the product and coproduct components of k.

We now turn to the bimonads in Propositions 3.13 and 3.16. These deal with the commutative aspects of the theory.

Exercise 8.7. Define analogues of the natural transformations (8.6) with $\mathcal{T}$ replaced by $\mathcal{S}$, and/or $\mathcal{T}^\vee$ replaced by $\mathcal{S}^\vee$.

- Prove the analogue of Theorem 8.4 for each of the bimonads $(\mathcal{S}, \mathcal{T}^\vee, \lambda)$, $(\mathcal{T}, \mathcal{S}^\vee, \lambda)$, $(\mathcal{S}, \mathcal{S}^\vee, \lambda)$.
- What would Exercise 8.5 say in these cases?
- Unify these results using the bimonad $(\mathcal{T}_\sim, \mathcal{T}^\vee_{\sim'})$ of Theorem 3.28.
- Formulate signed analogues of the above results.

In addition, one can combine unsigned and signed as follows. Let $\mathcal{F}$ temporarily denote the Hadamard functor. We define natural transformations

$$\varphi : \mathcal{E}\mathcal{F} \to \mathcal{F}(\mathcal{S} \times \mathcal{E}) \quad \text{and} \quad \psi : \mathcal{F}(\mathcal{S}^\vee \times \mathcal{E}^\vee) \to \mathcal{E}^\vee \mathcal{F}.$$

Evaluated on a pair of species (p, q), on the Z-component, the maps

$$\bigoplus_{\mathrm{Z} \leq \mathrm{X}} \mathsf{p}[\mathrm{X}] \otimes \mathsf{q}[\mathrm{X}] \otimes \mathbf{E}^-[\mathrm{Z}, \mathrm{X}] \rightleftarrows \bigoplus_{\mathrm{Z} \leq \mathrm{X},\, \mathrm{Z} \leq \mathrm{Y}} \mathsf{p}[\mathrm{X}] \otimes \mathsf{q}[\mathrm{Y}] \otimes \mathbf{E}^-[\mathrm{Z}, \mathrm{Y}]$$

are the canonical inclusion for φ and the canonical projection for ψ. In the lhs, the sum is over X, while in the rhs, the sum is over X and Y.

Theorem 8.8. *The Hadamard functor*

$$(\times, \varphi, \psi) : (\mathcal{S}, \mathcal{S}^\vee, \lambda) \times (\mathcal{E}, \mathcal{E}^\vee, \lambda_{-1}) \to (\mathcal{E}, \mathcal{E}^\vee, \lambda_{-1})$$

is bilax. (The mixed distributive laws λ and λ_{-1} are as in (3.16) *and* (3.22)*.)*

Theorem 8.9. *For any signed bicommutative signed bimonoid* k*, the functor*

$$(-) \times \mathsf{k} : (\mathcal{S}, \mathcal{S}^\vee, \lambda) \to (\mathcal{E}, \mathcal{E}^\vee, \lambda_{-1})$$

is bilax.

Explicitly, evaluated on a species p, on the Z-component, the bilax structure maps

$$\bigoplus_{\mathrm{X}:\,\mathrm{X}\geq \mathrm{Z}} \mathsf{p}[\mathrm{X}] \otimes \mathsf{k}[\mathrm{X}] \otimes \mathbf{E}^-[\mathrm{Z}, \mathrm{X}] \rightleftarrows \Bigl(\bigoplus_{\mathrm{X}:\,\mathrm{X}\geq \mathrm{Z}} \mathsf{p}[\mathrm{X}] \Bigr) \otimes \mathsf{k}[\mathrm{Z}] \tag{8.8}$$

are as follows. They preserve corresponding summands, and on a particular X-summand are given by $\mathrm{id} \otimes \mu_{\mathrm{Z}}^{\mathrm{X}}$ and $\mathrm{id} \otimes \Delta_{\mathrm{Z}}^{\mathrm{X}}$, respectively. Here $\mu_{\mathrm{Z}}^{\mathrm{X}}$ and $\Delta_{\mathrm{Z}}^{\mathrm{X}}$ denote the product and coproduct components of k.

The proofs are similar to those of Theorems 8.4 and 8.6, so we omit them.

Exercise 8.10. The Hadamard product preserves 0-$\sim$-bicommutative bimonoids. Deduce this by proving the analogue of Theorem 8.4 for the bimonad $(\mathcal{T}_\sim, \mathcal{T}_\sim^\vee, \lambda_0)$ of Theorem 3.30.

8.1.4. A higher monad. We now proceed to construct a $(2, 1)$-monad on species as in Definition C.13.

Define an endofunctor $\mathcal{L}$ on species by

$$\mathcal{L}(\mathsf{p}) := \mathop{+}_{n\geq 0} \mathsf{p}^{\times n}. \tag{8.9}$$

Explicitly, on a face A,

$$\mathcal{L}(\mathsf{p})[A] = \bigoplus_{n\geq 0} \mathsf{p}[A]^{\otimes n}.$$

Recall that a *weak composition* is a finite sequence of nonnegative integers and its degree is the sum of these integers. For a weak composition $\alpha = (a_1, \dots, a_k)$, let

$$\mathsf{p}[A]^{\otimes \alpha} := \mathsf{p}[A]^{\otimes a_1} \otimes \cdots \otimes \mathsf{p}[A]^{\otimes a_k}.$$

Observe that

$$\mathcal{L}\mathcal{L}(\mathsf{p})[A] = \bigoplus_{\alpha} \mathsf{p}[A]^{\otimes \alpha},$$

where the sum is over all weak compositions α. There is a natural transformation $\mathcal{L}\mathcal{L} \to \mathcal{L}$ which sends the summand $\mathsf{p}[A]^{\otimes \alpha}$ to $\mathsf{p}[A]^{\otimes \deg(\alpha)}$. There is also an obvious natural transformation $\mathrm{id} \to \mathcal{L}$. These turn $\mathcal{L}$ into a monad.

We now define natural transformations

$$\varphi : \mathcal{T}\mathcal{L} \to \mathcal{L}\mathcal{T} \quad \text{and} \quad \psi : \mathcal{L}\mathcal{T}^\vee \to \mathcal{T}^\vee\mathcal{L}. \tag{8.10}$$

Evaluated on a species p, on the A-component, the maps

$$\bigoplus_{A\leq F} \bigoplus_{n\geq 0} \mathsf{p}[F]^{\otimes n} \rightleftarrows \bigoplus_{n\geq 0} \bigoplus_{A\leq F_1,\dots,F_n} \mathsf{p}[F_1] \otimes \cdots \otimes \mathsf{p}[F_n]$$

are the canonical inclusion for φ and the canonical projection for ψ. In the lhs, the first sum is over F, and in the rhs, the second sum is over $F_1, \dots, F_n$.

Theorem 8.11. *The triple $(\mathcal{T}, \mathcal{L}, \mathcal{T}^\vee)$ is a $(2,1)$-monad on $\mathcal{A}$-species, with the three distributive laws given by (3.4) and (8.10). In particular, $(\mathcal{T}, \mathcal{L})$ is a double monad, and $(\mathcal{T}, \mathcal{T}^\vee)$ and $(\mathcal{L}, \mathcal{T}^\vee)$ are bimonads.*

Proof. The last claim is routine to verify. It then remains to check that diagram (C.17) commutes. It is copied below.

$$\begin{tikzcd}
\mathcal{T}\mathcal{L}\mathcal{T}^\vee \arrow[r, hook] \arrow[d, two heads] & \mathcal{L}\mathcal{T}\mathcal{T}^\vee \arrow[r] & \mathcal{L}\mathcal{T}^\vee\mathcal{T} \arrow[d, two heads] \\
\mathcal{T}\mathcal{T}^\vee\mathcal{L} \arrow[r] & \mathcal{T}^\vee\mathcal{T}\mathcal{L} \arrow[r, hook] & \mathcal{T}^\vee\mathcal{L}\mathcal{T}.
\end{tikzcd}$$

This is similar to the check performed in the proof of Theorem 8.4. Going across and down, or down and across, the composite map

$$\mathcal{T}\mathcal{L}\mathcal{T}^\vee(\mathsf{p})[A] = \bigoplus_{A \le F} \bigoplus_{m \ge 0} \bigoplus_{A \le G_1, \dots, G_m} \mathsf{p}[G_1] \otimes \cdots \otimes \mathsf{p}[G_m]$$
$$\longrightarrow \mathcal{T}^\vee\mathcal{L}\mathcal{T}(\mathsf{p})[A] = \bigoplus_{A \le F'} \bigoplus_{n \ge 0} \bigoplus_{A \le G'_1, \dots, G'_n} \mathsf{p}[G'_1] \otimes \cdots \otimes \mathsf{p}[G'_n]$$

has the following description. The matrix-components of this map are zero unless $m = n$, $G_1 = \cdots = G_m = G$ (say), $G'_1 = \cdots = G'_n = G'$ (say), $FF' = G$, and $F'F = G'$. If these conditions are met, then that matrix-component is the n-fold tensor power of the map $\beta_{G',G} : \mathsf{p}[G] \to \mathsf{p}[G']$. ☐

8.2. Internal hom for the Hadamard product

The internal hom for the tensor product on vector spaces is reviewed in Appendix A.3. Following this analogy, we construct the internal hom for the Hadamard product on species. Background information on internal hom in a monoidal category is given in Appendix B.

8.2.1. Internal hom for the Hadamard product. For species p and q, let $\mathrm{hom}^\times(\mathsf{p}, \mathsf{q})$ denote the species defined by

$$\mathrm{hom}^\times(\mathsf{p}, \mathsf{q})[A] := \mathrm{Hom}_{\Bbbk}(\mathsf{p}[A], \mathsf{q}[A]). \tag{8.11}$$

For a morphism $\beta_{B,A} : A \to B$, any linear map $f : \mathsf{p}[A] \to \mathsf{q}[A]$ induces a linear map

$$\begin{tikzcd}
\mathsf{p}[B] \arrow[r, dashed] \arrow[d, "\beta_{B,A}^{-1}"'] & \mathsf{q}[B] \\
\mathsf{p}[A] \arrow[r, "f"'] & \mathsf{q}[A]. \arrow[u, "\beta_{B,A}"']
\end{tikzcd} \tag{8.12}$$

Thus, $\mathrm{hom}^\times(\mathsf{p}, \mathsf{q})$ is a species. Further, this construction is natural in p and q, that is, maps $\mathsf{p}' \to \mathsf{p}$ and $\mathsf{q} \to \mathsf{q}'$ induce a map $\mathrm{hom}^\times(\mathsf{p}, \mathsf{q}) \to \mathrm{hom}^\times(\mathsf{p}', \mathsf{q}')$. This yields the functor

$$\mathrm{hom}^\times(-,-) : \mathcal{A}\text{-}\mathsf{Sp}^{\mathrm{op}} \times \mathcal{A}\text{-}\mathsf{Sp} \to \mathcal{A}\text{-}\mathsf{Sp}, \qquad (\mathsf{p}, \mathsf{q}) \mapsto \mathrm{hom}^\times(\mathsf{p}, \mathsf{q}).$$

It is straightforward to check that:

Proposition 8.12. *For any species* p*,* m*,* n*, there is a natural bijection*

$$\mathcal{A}\text{-}\mathtt{Sp}(\mathsf{p}\times\mathsf{m},\mathsf{n}) \cong \mathcal{A}\text{-}\mathtt{Sp}\big(\mathsf{p},\mathrm{hom}^{\times}(\mathsf{m},\mathsf{n})\big).$$

In other words, the functor $\mathrm{hom}^{\times}$ is the internal hom in the symmetric monoidal category $(\mathcal{A}\text{-}\mathtt{Sp},\times)$.

There is a canonical map of species

$$\mathsf{p}^{*}\times\mathsf{q}\to\mathrm{hom}^{\times}(\mathsf{p},\mathsf{q}) \tag{8.13}$$

which is an isomorphism if either p or q is finite-dimensional. In particular,

$$\mathsf{p}^{*}\cong\mathrm{hom}^{\times}(\mathsf{p},\mathsf{E}),$$

where E is the exponential species.

For any species p, let

$$\mathrm{end}^{\times}(\mathsf{p}) := \mathrm{hom}^{\times}(\mathsf{p},\mathsf{p}). \tag{8.14}$$

8.2.2. Monoids wrt the Hadamard product. Observe that a monoid wrt the Hadamard product is a species m such that each component $\mathsf{m}[A]$ is a $\Bbbk$-algebra, and $\beta_{B,A} : \mathsf{m}[A]\to\mathsf{m}[B]$ are $\Bbbk$-algebra morphisms. This is the same as a functor from $\mathcal{A}\text{-}\mathtt{Hyp}$ to the category of $\Bbbk$-algebras.

Example 8.13. For any species p, $\mathrm{end}^{\times}(\mathsf{p})$ is a monoid wrt the Hadamard product. Explicitly: For any face A, $\mathrm{end}^{\times}(\mathsf{p})[A]$ is the endomorphism algebra of the vector space $\mathsf{p}[A]$. Further, for any monoid m wrt the Hadamard product, a morphism of monoids $\mathsf{m}\to\mathrm{end}^{\times}(\mathsf{p})$ is the same as a (left) m-module structure on p. In this situation, we say that m acts on p.

For a more general discussion, see Proposition B.4.

Example 8.14. A monoid wrt the Hadamard product is the same as an algebra over the monad $\mathcal{L}$ from Section 8.1.4. The free monoid on a species p wrt the Hadamard product is given by $\mathcal{L}(\mathsf{p})$ as in (8.9). Explicitly: For any face A, the component $\mathcal{L}(\mathsf{p})[A]$ is the tensor algebra of the vector space $\mathsf{p}[A]$.

Exercise 8.15. Check that: The species of faces Σ equipped with the Tits product in each component is a monoid wrt the Hadamard product.

8.3. Biconvolution bimonoids

We construct the convolution monoid associated to a comonoid and a monoid. We also discuss the dual construction of the coconvolution comonoid. Further, these two constructions can be combined to form the biconvolution bimonoid. This object is associated to a pair of bimonoids.

The internal hom for Hadamard product plays a key role in these constructions. In fact, the (co, bi)convolution (co, bi)monoid arises from the (co, bi)lax property of the internal hom.

8.3.1. Convolution monoid. Let c be a comonoid and a a monoid. Then the species $\hom^{\times}(\mathsf{c},\mathsf{a})$ is a monoid as follows. The product component

$$\mu_A^F : \hom^{\times}(\mathsf{c},\mathsf{a})[F] \to \hom^{\times}(\mathsf{c},\mathsf{a})[A]$$

sends $f : \mathsf{c}[F] \to \mathsf{a}[F]$ to the composite

$$\mathsf{c}[A] \xrightarrow{\Delta_A^F} \mathsf{c}[F] \xrightarrow{f} \mathsf{a}[F] \xrightarrow{\mu_A^F} \mathsf{a}[A].$$

In other words,

$$\mu_A^F(f) = \mu_A^F f \Delta_A^F, \tag{8.15}$$

where in the rhs μ_A^F and Δ_A^F are the structure maps of a and c, respectively. Associativity for the product of $\hom^{\times}(\mathsf{c},\mathsf{a})$ follows at once from associativity of a and coassociativity of c. We refer to $\hom^{\times}(\mathsf{c},\mathsf{a})$ as the *convolution monoid*.

The map

$$\mathsf{c}^* \times \mathsf{a} \to \hom^{\times}(\mathsf{c},\mathsf{a}), \tag{8.16}$$

defined as in (8.13), is a morphism of monoids. It is an isomorphism if either c or a is finite-dimensional. In particular,

$$\mathsf{c}^* \cong \hom^{\times}(\mathsf{c},\mathsf{E}) \quad \text{and} \quad \mathsf{a} \cong \hom^{\times}(\mathsf{E},\mathsf{a}) \tag{8.17}$$

as monoids.

Exercise 8.16. Check that: The functor

$$\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})^{\mathrm{op}} \times \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp}) \to \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp}), \quad (\mathsf{c},\mathsf{a}) \mapsto \hom^{\times}(\mathsf{c},\mathsf{a})$$

turns $\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})$ into a left module category over the monoidal category $\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})^{\mathrm{op}}$ (Appendix B.1.4). Thus, for comonoids c and d and a monoid a, we have an isomorphism of monoids

$$\hom^{\times}(\mathsf{d} \times \mathsf{c},\mathsf{a}) \xrightarrow{\cong} \hom^{\times}(\mathsf{d},\hom^{\times}(\mathsf{c},\mathsf{a})).$$

Also relevant is the second isomorphism in (8.17).

Exercise 8.17. Recall the bimonoid of faces Σ from Section 7.6.2. Check that: The map $\Sigma \to \hom^{\times}(\mathsf{c},\mathsf{a})$, which on the A component sends $\mathtt{H}_{F/A}$ to $\mu_A^F \Delta_A^F : \mathsf{c}[A] \to \mathsf{a}[A]$, is a morphism of monoids.

8.3.2. Coconvolution comonoid. Let c be a comonoid and a a monoid. Then $\hom^{\times}(\mathsf{a},\mathsf{c})$ is a comonoid. The coproduct is given by

$$\Delta_A^F : \hom^{\times}(\mathsf{a},\mathsf{c})[A] \to \hom^{\times}(\mathsf{a},\mathsf{c})[F], \quad f \mapsto \Delta_A^F f \mu_A^F. \tag{8.18}$$

We refer to $\hom^{\times}(\mathsf{a},\mathsf{c})$ as the *coconvolution comonoid*.

The map

$$\mathsf{a}^* \times \mathsf{c} \to \hom^{\times}(\mathsf{a},\mathsf{c}), \tag{8.19}$$

defined as in (8.13), is a morphism of comonoids. It is an isomorphism if either c or a is finite-dimensional. In particular,

$$\mathsf{a}^* \cong \hom^{\times}(\mathsf{a},\mathsf{E}) \quad \text{and} \quad \mathsf{c} \cong \hom^{\times}(\mathsf{E},\mathsf{c}) \tag{8.20}$$

as comonoids.

8.3.3. Biconvolution bimonoid. Let h and k be bimonoids. Combining the preceding constructions, we obtain monoid and comonoid structures on $\mathrm{hom}^\times(\mathsf{h},\mathsf{k})$.

Proposition 8.18. *With these structures,* $\mathrm{hom}^\times(\mathsf{h},\mathsf{k})$ *is a bimonoid. In addition, the map*

$$\mathsf{h}^* \times \mathsf{k} \to \mathrm{hom}^\times(\mathsf{h},\mathsf{k}), \tag{8.21}$$

defined as in (8.13), is a morphism of bimonoids. It is an isomorphism if either h *or* k *is finite-dimensional.*

PROOF. The bimonoid axiom (2.12) is verified as follows. For any faces F and G both greater than A, and a linear map $f : \mathsf{h}[A] \to \mathsf{k}[A]$,

$$\begin{aligned}\Delta^G_A\mu^F_A(f) &= \Delta^G_A(\mu^F_A f \Delta^F_A) = \Delta^G_A\mu^F_A f \Delta^F_A\mu^G_A \\ &= \mu^{GF}_G \beta_{GF,FG}\Delta^{FG}_F f \mu^{FG}_F \beta_{FG,GF}\Delta^{GF}_G = \mu^{GF}_G\beta_{GF,FG}\Delta^{FG}_F(f).\end{aligned}$$

In the third equality, the bimonoid axioms for h and k were used. The following diagram is useful.

$$\begin{tikzcd}
\mathsf{h}[G] \arrow[r, "\mu^G_A"] \arrow[d, "\Delta^{GF}_G"'] & \mathsf{h}[A] \arrow[r, "\Delta^F_A"] & \mathsf{h}[F] \arrow[r, "f"] & \mathsf{k}[F] \arrow[r, "\mu^F_A"] \arrow[d, "\Delta^{FG}_F"] & \mathsf{k}[A] \arrow[r, "\Delta^G_A"] & \mathsf{k}[G] \\
\mathsf{h}[GF] \arrow[rr, "\beta_{FG,GF}"'] & & \mathsf{h}[FG] \arrow[u, "\mu^{FG}_F"] & \mathsf{k}[FG] \arrow[rr, "\beta_{GF,FG}"'] & & \mathsf{k}[GF] \arrow[u, "\mu^{GF}_G"']
\end{tikzcd}$$

The top row is the lhs. The rhs takes the longer route going down, across, up, across, down, across, up.

The map (8.21) is a morphism of bimonoids since it is a morphism of both monoids and comonoids. □

We refer to $\mathrm{hom}^\times(\mathsf{h},\mathsf{k})$ as the *biconvolution bimonoid*. In particular, for any bimonoid h, we obtain a bimonoid structure on $\mathrm{end}^\times(\mathsf{h})$. If h is finite-dimensional, the isomorphism

$$\mathsf{h}^* \times \mathsf{h} \to \mathrm{end}^\times(\mathsf{h}) \tag{8.22}$$

implies that the bimonoid $\mathrm{end}^\times(\mathsf{h})$ is self-dual. Further, in the linearized case $\mathsf{h} = \Bbbk\mathrm{h}$, we can use the pairs notation for $\mathrm{end}^\times(\mathsf{h})$: $\mathtt{K}_{y,x}$ as x and y vary over elements of $\mathrm{h}[A]$ form a linear basis for $\mathrm{end}^\times(\mathsf{h})[A]$. The product and coproduct of $\mathrm{end}^\times(\mathsf{h})$ in this notation are, respectively, given by

$$\mathrm{end}^\times(\mathsf{h})[F] \to \mathrm{end}^\times(\mathsf{h})[A], \qquad \mathtt{K}_{y,x} \to \sum_{z:\, \Delta^F_A(z)=y} \mathtt{K}_{z,\mu^F_A(x)},$$

$$\mathrm{end}^\times(\mathsf{h})[A] \to \mathrm{end}^\times(\mathsf{h})[F], \qquad \mathtt{K}_{y,x} \to \begin{cases}\mathtt{K}_{\Delta^F_A(y),\Delta^F_A(x)} & \text{if } \mu^F_A\Delta^F_A(y) = y,\\ 0 & \text{otherwise.}\end{cases}$$

Exercise 8.19. Generalize Proposition 8.18 as follows. Check that for any scalars p and q, if h is a p-bimonoid and k is a q-bimonoid, then $\mathrm{hom}^\times(\mathsf{h},\mathsf{k})$ is a pq-bimonoid.

8.3.4. Primitive part of the biconvolution bimonoid. We now discuss the primitive part of $\mathrm{hom}^{\times}(\mathsf{h},\mathsf{k})$ in relation to the primitive part of k and indecomposable part of h.

Proposition 8.20. *For bimonoids* h *and* k*,*

$$\begin{aligned} \mathrm{hom}^{\times}\big(\mathsf{h},\mathcal{P}(\mathsf{k})\big) &\subseteq \mathcal{P}\big(\mathrm{hom}^{\times}(\mathsf{h},\mathsf{k})\big),\\ \mathrm{hom}^{\times}\big(\mathcal{Q}(\mathsf{h}),\mathsf{k}\big) &\subseteq \mathcal{P}\big(\mathrm{hom}^{\times}(\mathsf{h},\mathsf{k})\big). \end{aligned} \tag{8.23}$$

Proof. Let $f \in \mathrm{hom}^{\times}\big(\mathsf{h},\mathcal{P}(\mathsf{k})\big)[A]$. That is, $f : \mathsf{h}[A] \to \mathcal{P}(\mathsf{k})[A]$. So $\Delta_A^F f = 0$ for every face $F > A$, where Δ_A^F denotes the coproduct of k. Hence, in the bimonoid $\mathrm{hom}^{\times}(\mathsf{h},\mathsf{k})$, the coproduct of f equals

$$\Delta_A^F f \mu_A^F = 0,$$

so f is primitive. The other inclusion is similar. □

8.3.5. Commutative aspects. Let c be a cocommutative comonoid and a a commutative monoid. Then $\mathrm{hom}^{\times}(\mathsf{c},\mathsf{a})$ is a commutative monoid. The proof is contained in the following diagram.

$$\begin{tikzcd} \mathsf{c}[A] \arrow[r, "\Delta_A^F"] \arrow[dr, "\Delta_A^G"'] & \mathsf{c}[F] \arrow[r, "f"] & \mathsf{a}[F] \arrow[r, "\mu_A^F"] \arrow[d, "\beta_{G,F}"'] & \mathsf{a}[A] \\ & \mathsf{c}[G] \arrow[u, "\beta_{F,G}"'] & \mathsf{a}[G] \arrow[ur, "\mu_A^G"'] & \end{tikzcd}$$

Alternatively, in this situation, one can proceed directly with the formulation given by Proposition 2.20, and define the product of $\mathrm{hom}^{\times}(\mathsf{c},\mathsf{a})$ as follows. The component

$$\mu_{\mathrm{Z}}^{\mathrm{X}} : \mathrm{hom}^{\times}(\mathsf{c},\mathsf{a})[\mathrm{X}] \to \mathrm{hom}^{\times}(\mathsf{c},\mathsf{a})[\mathrm{Z}]$$

sends $f : \mathsf{c}[\mathrm{X}] \to \mathsf{a}[\mathrm{X}]$ to the composite

$$\mathsf{c}[\mathrm{Z}] \xrightarrow{\Delta_{\mathrm{Z}}^{\mathrm{X}}} \mathsf{c}[\mathrm{X}] \xrightarrow{f} \mathsf{a}[\mathrm{X}] \xrightarrow{\mu_{\mathrm{Z}}^{\mathrm{X}}} \mathsf{a}[\mathrm{Z}].$$

In other words,

$$\mu_{\mathrm{Z}}^{\mathrm{X}}(f) = \mu_{\mathrm{Z}}^{\mathrm{X}} f \Delta_{\mathrm{Z}}^{\mathrm{X}}, \tag{8.24}$$

where in the rhs $\mu_{\mathrm{Z}}^{\mathrm{X}}$ and $\Delta_{\mathrm{Z}}^{\mathrm{X}}$ are the structure maps of a and c, respectively.

Dually, $\mathrm{hom}^{\times}(\mathsf{a},\mathsf{c})$ is a cocommutative comonoid. Combining the two constructions, if h and k are both bicommutative bimonoids, then so is $\mathrm{hom}^{\times}(\mathsf{h},\mathsf{k})$.

8.3.6. Internal hom as a bilax functor. We now try to understand the preceding constructions more formally. Since a monad on a category is a comonad on the opposite category and vice versa, the bimonad $(\mathcal{T},\mathcal{T}^{\vee},\lambda)$ on $\mathcal{A}$-Sp translates to the bimonad $(\mathcal{T}^{\vee},\mathcal{T})$ on $\mathcal{A}$-$\mathsf{Sp}^{\mathrm{op}}$.

We now define natural transformations

$$\varphi : \mathcal{T}\mathrm{hom}^{\times} \to \mathrm{hom}^{\times}(\mathcal{T}^{\vee}{\times}\mathcal{T}) \quad \text{and} \quad \psi : \mathrm{hom}^{\times}(\mathcal{T}{\times}\mathcal{T}^{\vee}) \to \mathcal{T}^{\vee}\mathrm{hom}^{\times}. \tag{8.25}$$

Evaluated on a pair of species (p,q), on the A-component, the maps

$$\bigoplus_{A\leq F} \mathrm{Hom}_{\Bbbk}(\mathsf{p}[F],\mathsf{q}[F]) \rightleftarrows \bigoplus_{A\leq F,\, A\leq G} \mathrm{Hom}_{\Bbbk}(\mathsf{p}[F],\mathsf{q}[G])$$

are the canonical inclusion for φ and the canonical projection for ψ. In the lhs, the sum is over F, and in the rhs, the sum is over F and G.

Theorem 8.21. *For any scalars p and q, the internal hom functor*

$$(\mathrm{hom}^{\times}, \varphi, \psi) : (\mathcal{T}^{\vee}, \mathcal{T}, \lambda_p) \times (\mathcal{T}, \mathcal{T}^{\vee}, \lambda_q) \to (\mathcal{T}, \mathcal{T}^{\vee}, \lambda_{pq})$$

is bilax.

PROOF. It is a routine check that $(\mathrm{hom}^{\times}, \varphi) : \mathcal{T}^{\vee} \times \mathcal{T} \to \mathcal{T}$ is a lax functor of monads, and $(\mathrm{hom}^{\times}, \psi) : \mathcal{T} \times \mathcal{T}^{\vee} \to \mathcal{T}^{\vee}$ is a colax functor of comonads. This is commutativity of diagrams (C.2) and (C.6).

To check further that $(\mathrm{hom}^{\times}, \varphi, \psi)$ is bilax, we need to check commutativity of diagram (C.10). It takes the following form, evaluated on a pair of species (p, q).

$$\begin{tikzcd}[column sep=small]
\mathcal{T}(\mathrm{hom}^{\times}(\mathcal{T}(\mathsf{p}),\mathcal{T}^{\vee}(\mathsf{q}))) \arrow[r, hook] \arrow[d, two heads] & \mathrm{hom}^{\times}(\mathcal{T}^{\vee}\mathcal{T}(\mathsf{p}),\mathcal{T}\mathcal{T}^{\vee}(\mathsf{q})) \arrow[r] & \mathrm{hom}^{\times}(\mathcal{T}\mathcal{T}^{\vee}(\mathsf{p}),\mathcal{T}^{\vee}\mathcal{T}(\mathsf{q})) \arrow[d, two heads] \\
\mathcal{T}\mathcal{T}^{\vee}(\mathrm{hom}^{\times}(\mathsf{p},\mathsf{q})) \arrow[r] & \mathcal{T}^{\vee}\mathcal{T}(\mathrm{hom}^{\times}(\mathsf{p},\mathsf{q})) \arrow[r, hook] & \mathcal{T}^{\vee}(\mathrm{hom}^{\times}(\mathcal{T}^{\vee}(\mathsf{p}),\mathcal{T}(\mathsf{q})))
\end{tikzcd}$$

The calculation is very similar to the one given in the proof of Theorem 8.4, one needs to replace the $\otimes$ symbol with the Hom symbol. So we omit the details. $\square$

Consider the lax functor of monads

$$(\mathrm{hom}^{\times}, \varphi) : \mathcal{T}^{\vee} \times \mathcal{T} \to \mathcal{T}.$$

An algebra wrt $\mathcal{T}^{\vee} \times \mathcal{T}$ is the same as a pair (c, a) consisting of a comonoid c and a monoid a. Since lax functors preserve algebras, we deduce that $\mathrm{hom}^{\times}(\mathsf{c}, \mathsf{a})$ is a monoid. This is precisely the convolution monoid. Similarly, the coconvolution comonoid and the biconvolution bimonoid arise from the colax and bilax property of the internal hom. We point out that the scalars p and q in Exercise 8.19 correspond to those in Theorem 8.21.

Exercise 8.22. Analogous to the Hadamard functor in Exercise 8.7, establish the various commutative analogues of Theorem 8.21. In particular, derive the commutative convolution monoid.

8.4. Internal hom for comonoids. Bimonoid of star families

We construct the internal hom for comonoids wrt the Hadamard product. We write $\mathcal{C}(\mathsf{c}, \mathsf{d})$ for the internal hom of c and d. Its A-component is defined using families of maps indexed by the star of A which are compatible with the coproducts of c and d. There is a similar construction for cocommutative comonoids which we denote by ${}^{\mathsf{co}}\mathcal{C}(\mathsf{c}, \mathsf{d})$. Thus, the symmetric monoidal categories of comonoids and cocommutative comonoids are closed.

For a cocommutative comonoid c and bimonoid k, the comonoid $\mathcal{C}(\mathsf{c}, \mathsf{k})$ carries the structure of a bimonoid. We refer to it as the bimonoid of star families. The coproduct comes from restricting the given family to a smaller family (which is routine), while the product comes from extending it to a larger family (which is interesting). If, in addition, c carries the structure of a bimonoid, then $\mathcal{C}(\mathsf{c},\mathsf{k})$ can be realized as a subbimonoid of the biconvolution bimonoid $\mathrm{hom}^{\times}(\mathsf{c},\mathsf{k})$. There is a commutative counterpart to $\mathcal{C}(\mathsf{c},\mathsf{k})$ when k is bicommutative. We denote it by ${}^{\mathsf{co}}\mathcal{C}(\mathsf{c},\mathsf{k})$, and refer to it as the bicommutative bimonoid of star families. It coincides with the coabelianization of $\mathcal{C}(\mathsf{c},\mathsf{k})$.

8.4.1. Internal hom for comonoids. Let c and d be comonoids. We proceed to construct a comonoid $\mathcal{C}(\mathsf{c},\mathsf{d})$. For a face A, an element of $\mathcal{C}(\mathsf{c},\mathsf{d})[A]$ is a family of linear maps

$$f_{F/A} : \mathsf{c}[F] \to \mathsf{d}[F],$$

one for each $A \leq F$, such that the diagram

$$\begin{array}{ccc} \mathsf{c}[F] & \xrightarrow{f_{F/A}} & \mathsf{d}[F] \\ {\scriptstyle \Delta_F^G}\big\downarrow & & \big\downarrow{\scriptstyle \Delta_F^G} \\ \mathsf{c}[G] & \xrightarrow[f_{G/A}]{} & \mathsf{d}[G] \end{array} \tag{8.26}$$

commutes for each $A \leq F \leq G$. We refer to $f_{A/A}$ as the base term, and to the remaining $f_{F/A}$ for $F > A$ as the higher terms.

Observe that $\mathcal{C}(\mathsf{c},\mathsf{d})[A]$ is a vector space with addition and scalar multiplication defined componentwise:

$$(f_{F/A}) + (g_{F/A}) := (f_{F/A} + g_{F/A}), \qquad c(f_{F/A}) := (cf_{F/A}).$$

Now suppose that we are given A and B of the same support, and an element $(f_{F/A})$ of $\mathcal{C}(\mathsf{c},\mathsf{d})[A]$. For each $F \geq A$, consider the composite

$$\mathsf{c}[G] \xrightarrow{\beta_{G,F}^{-1}} \mathsf{c}[F] \xrightarrow{f_{F/A}} \mathsf{d}[F] \xrightarrow{\beta_{G,F}} \mathsf{d}[G]$$

with $G := BF$. Denote this composite map by $g_{G/B}$. Naturality of the coproduct implies that the family $(g_{G/B})$ belongs to $\mathcal{C}(\mathsf{c},\mathsf{d})[B]$. The maps

$$\beta_{B,A} : \mathcal{C}(\mathsf{c},\mathsf{d})[A] \to \mathcal{C}(\mathsf{c},\mathsf{d})[B], \quad (f_{F/A}) \mapsto (g_{G/B})$$

turn $\mathcal{C}(\mathsf{c},\mathsf{d})$ into a species.

Further note that for each $A \leq G$, we have a linear map

$$\Delta_A^G : \mathcal{C}(\mathsf{c},\mathsf{d})[A] \to \mathcal{C}(\mathsf{c},\mathsf{d})[G]$$

obtained by restricting a family $(f_{F/A})$ to those faces F which are greater than G. It is a routine check that this turns $\mathcal{C}(\mathsf{c},\mathsf{d})$ into a comonoid. Further, this construction is natural in c and d, and yields a bifunctor. We claim that it is the internal hom in the monoidal category of comonoids wrt the Hadamard product. In other words:

Proposition 8.23. *For any comonoids c, m, n, there is a natural bijection*

$$\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c} \times \mathsf{m}, \mathsf{n}) \cong \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})\big(\mathsf{c}, \mathcal{C}(\mathsf{m},\mathsf{n})\big).$$

PROOF. We first go in the forward direction. Suppose we are given a morphism of comonoids $h : \mathsf{c} \times \mathsf{m} \to \mathsf{n}$. This entails a linear map

$$h_A : \mathsf{c}[A] \otimes \mathsf{m}[A] \to \mathsf{n}[A],$$

one for each face A, such that the diagrams

$$\begin{tikzcd} \mathsf{c}[F] \otimes \mathsf{m}[F] \arrow[r, "h_F"] \arrow[d, "\beta_{G,F} \otimes \beta_{G,F}"'] & \mathsf{n}[F] \arrow[d, "\beta_{G,F}"] \\ \mathsf{c}[G] \otimes \mathsf{m}[G] \arrow[r, "h_G"'] & \mathsf{n}[G] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{c}[F] \otimes \mathsf{m}[F] \arrow[r, "h_F"] \arrow[d, "\Delta_F^{F'} \otimes \Delta_F^{F'}"'] & \mathsf{n}[F] \arrow[d, "\Delta_F^{F'}"] \\ \mathsf{c}[F'] \otimes \mathsf{m}[F'] \arrow[r, "h_{F'}"'] & \mathsf{n}[F'] \end{tikzcd} \tag{8.27}$$

commute, the first whenever $\mathrm{s}(F) = \mathrm{s}(G)$, and the second whenever $F \leq F'$.

For each A and $x \in \mathsf{c}[A]$, define an element $(f_{F/A})$ of $\mathcal{C}(\mathsf{m}, \mathsf{n})[A]$ as follows. The linear map $f_{F/A}$ is the image of x under the composite

$$\mathsf{c}[A] \xrightarrow{\Delta_A^F} \mathsf{c}[F] \longrightarrow \mathrm{Hom}_{\Bbbk}(\mathsf{m}[F], \mathsf{n}[F]), \qquad x \mapsto f_{F/A}.$$

(The second map is induced by h_F.) The commutative diagram on the right implies that (8.26) commutes, so $(f_{F/A})$ is indeed an element of $\mathcal{C}(\mathsf{m}, \mathsf{n})[A]$. One can further check that this gives rise to a morphism of comonoids $\mathsf{c} \to \mathcal{C}(\mathsf{m}, \mathsf{n})$.

In the other direction, suppose we are given such a morphism of comonoids. Thus, for each A and $x \in \mathsf{c}[A]$, we have a family $(f_{F/A})$. Using the base term of this family, we define h_A as $h_A(x \otimes y) := f_{A/A}(y)$, and then check that the above two diagrams commute.

Finally, one may check that the two maps are inverse to each other. $\square$

Under an additional surjectivity hypothesis, $\mathcal{C}(\mathsf{c}, \mathsf{d})$ can be defined more simply as follows.

Lemma 8.24. *If the coproduct components of* c *are surjective, then* $\mathcal{C}(\mathsf{c}, \mathsf{d})[A]$ *is the same as the space of all linear maps*

$$f : \mathsf{c}[A] \to \mathsf{d}[A]$$

with the property that for each face $F \geq A$*, there exists a linear map* $\mathsf{c}[F] \to \mathsf{d}[F]$ *making the diagram*

$$\begin{tikzcd} \mathsf{c}[A] \arrow[r, "f"] \arrow[d, two heads, "\Delta_A^F"'] & \mathsf{d}[A] \arrow[d, "\Delta_A^F"] \\ \mathsf{c}[F] \arrow[r, dashed] & \mathsf{d}[F] \end{tikzcd} \tag{8.28}$$

commute. Moreover, the coproduct component Δ_A^F *of* $\mathcal{C}(\mathsf{c}, \mathsf{d})$ *sends* f *to the dotted arrow in* (8.28).

PROOF. The surjectivity hypothesis together with (8.26) shows that any family $(f_{F/A}) \in \mathcal{C}(\mathsf{c}, \mathsf{d})[A]$ is completely determined by its base term $f_{A/A}$ (that is, the dotted arrow is unique). $\square$

8.4.2. Primitive part. Recall that the A-component of $\hom^{\times}(\mathsf{c},\mathsf{d})$ consists of linear maps $\mathsf{c}[A] \to \mathsf{d}[A]$. Define a map of species

$$\mathcal{C}(\mathsf{c},\mathsf{d}) \to \hom^{\times}(\mathsf{c},\mathsf{d}), \tag{8.29}$$

which on the A-component, sends a family $(f_{F/A})$ to its base term $f_{A/A}$.

Lemma 8.25. *When restricted to the primitive part of $\mathcal{C}(\mathsf{c},\mathsf{d})$, the map (8.29) is injective, and in fact*

$$\mathcal{P}(\mathcal{C}(\mathsf{c},\mathsf{d})) = \hom^{\times}(\mathsf{c},\mathcal{P}(\mathsf{d})). \tag{8.30}$$

Proof. Suppose $f \in \mathcal{C}(\mathsf{c},\mathsf{d})[A]$ is primitive. Then, by definition of the coproduct, all higher terms of f must be zero, that is, $f_{F/A} = 0$ for $F > A$. Injectivity follows. Further, in this case, condition (8.26) reduces to $\Delta_A^G f_{A/A} = 0$ for all $G > A$, which is the same as saying that $f_{A/A}$ maps into $\mathcal{P}(\mathsf{d})[A]$. $\square$

Lemma 8.26. *If the coproduct components of c are surjective, then the map (8.29) is injective. For any bimonoid c, the map (8.29) is an injective morphism of comonoids, with the coconvolution coproduct on the latter.*

Proof. The first claim is clear from Lemma 8.24, the higher terms of a family $(f_{F/A})$ are determined by the base term $f_{A/A}$ via (8.28). Now suppose c is a bimonoid. Since $\Delta_A^F \mu_A^F = \mathrm{id}$ by (2.15), the structure maps Δ_A^F of c are necessarily surjective (as noted in Corollary 2.9), and further the dotted arrow in (8.28) is given by the map $\Delta_A^F f \mu_A^F$. Thus, the coproduct of $\mathcal{C}(\mathsf{c},\mathsf{d})$ coincides with the coconvolution coproduct of $\hom^{\times}(\mathsf{c},\mathsf{d})$. $\square$

When c is a bimonoid, by injectivity, the primitive part of $\mathcal{C}(\mathsf{c},\mathsf{d})$ is contained in the primitive part of $\hom^{\times}(\mathsf{c},\mathsf{d})$. In view of (8.30), we recover the first statement in (8.23).

Exercise 8.27. Check that: For any species p and comonoid c,

$$\mathcal{C}(\mathsf{c},\mathcal{T}^{\vee}(\mathsf{p})) \cong \mathcal{T}^{\vee}(\hom^{\times}(\mathsf{c},\mathsf{p})),$$

the cofree comonoid on the species $\hom^{\times}(\mathsf{c},\mathsf{p})$ (which is also the primitive part of the comonoid). This is consistent with formula (8.30).

8.4.3. Coabelianization. Let $\mathcal{C}(\mathsf{c},\mathsf{d})^{coab}$ denote the coabelianization of the comonoid $\mathcal{C}(\mathsf{c},\mathsf{d})$ as in Section 2.7.2. It is the largest cocommutative subcomonoid of $\mathcal{C}(\mathsf{c},\mathsf{d})$. Explicitly, its A-component consists of those elements $(f_{F/A})$ for which, in addition, the diagram

$$\begin{tikzcd} \mathsf{c}[F] \arrow[r, "f_{F/A}"] \arrow[d, "\beta_{G,F}"'] & \mathsf{d}[F] \arrow[d, "\beta_{G,F}"] \\ \mathsf{c}[G] \arrow[r, "f_{G/A}"'] & \mathsf{d}[G] \end{tikzcd} \tag{8.31}$$

commutes whenever F and G are faces greater than A with equal support.

Lemma 8.28. *Let c and d be cocommutative comonoids. If the coproduct components of c are surjective, then $\mathcal{C}(\mathsf{c},\mathsf{d})$ is cocommutative.*

Proof. We need to check that diagram (8.31) commutes. For that, consider the following diagram.

$$
\begin{tikzcd}
 & \mathsf{c}[F] \arrow[rr, "f_{F/A}"] \arrow[dd, "\beta_{G,F}"'] & & \mathsf{d}[F] \arrow[dd, "\beta_{G,F}"] \\
\mathsf{c}[A] \arrow[ur, two heads, "\Delta_A^F"] \arrow[rr, "f_{A/A}"] \arrow[dr, two heads, "\Delta_A^G"'] & & \mathsf{d}[A] \arrow[ur, "\Delta_A^F"] \arrow[dr, "\Delta_A^G"'] & \\
 & \mathsf{c}[G] \arrow[rr, "f_{G/A}"'] & & \mathsf{d}[G]
\end{tikzcd}
$$

The triangles commute by cocommutativity (2.23) and the parallelograms commute by (8.26). The surjectivity then forces the square to commute. □

8.4.4. Internal hom for cocommutative comonoids. Let c and d be cocommutative comonoids. We now construct a cocommutative comonoid ${}^{\mathsf{co}}\mathcal{C}(\mathsf{c},\mathsf{d})$. We work with the formulation provided by Proposition 2.21. For a flat Z, an element of ${}^{\mathsf{co}}\mathcal{C}(\mathsf{c},\mathsf{d})[\mathrm{Z}]$ is a family of linear maps

$$f_{\mathrm{X}/\mathrm{Z}} : \mathsf{c}[\mathrm{X}] \to \mathsf{d}[\mathrm{X}],$$

one for each $\mathrm{Z} \leq \mathrm{X}$, such that the diagram

$$
\begin{tikzcd}
\mathsf{c}[\mathrm{X}] \arrow[r, "f_{\mathrm{X}/\mathrm{Z}}"] \arrow[d, "\Delta_{\mathrm{X}}^{\mathrm{Y}}"'] & \mathsf{d}[\mathrm{X}] \arrow[d, "\Delta_{\mathrm{X}}^{\mathrm{Y}}"] \\
\mathsf{c}[\mathrm{Y}] \arrow[r, "f_{\mathrm{Y}/\mathrm{Z}}"'] & \mathsf{d}[\mathrm{Y}]
\end{tikzcd}
\tag{8.32}
$$

commutes for each $\mathrm{Z} \leq \mathrm{X} \leq \mathrm{Y}$. The rest of the construction is similar to that of $\mathcal{C}(\mathsf{c},\mathsf{d})$ with faces replaced by flats. The functor which sends (c,d) to ${}^{\mathsf{co}}\mathcal{C}(\mathsf{c},\mathsf{d})$ is the internal hom for the Hadamard product of cocommutative comonoids.

Exercise 8.29. Let c and d be cocommutative comonoids. Check directly that ${}^{\mathsf{co}}\mathcal{C}(\mathsf{c},\mathsf{d}) = \mathcal{C}(\mathsf{c},\mathsf{d})^{coab}$. The fact that the internal hom for cocommutative comonoids is the coabelianization of the internal hom for comonoids can be also seen as a formal consequence of (2.55).

Exercise 8.30. Let c and d be cocommutative comonoids. Formulate the analogue of Lemma 8.24 with flats insteads of faces. (In this case, under the surjectivity hypothesis, ${}^{\mathsf{co}}\mathcal{C}(\mathsf{c},\mathsf{d}) = \mathcal{C}(\mathsf{c},\mathsf{d})$ by Lemma 8.28.)

Exercise 8.31. Construct the internal hom in the category of partially cocommutative comonoids. (Use the formulation of species given by Proposition 2.82.)

8.4.5. Bimonoid of star families. Suppose c is a cocommutative comonoid, and k is a bimonoid. We proceed to turn $\mathcal{C}(\mathsf{c},\mathsf{k})$ into a bimonoid. Recall that it is a comonoid with coproduct defined by restricting the given family. For the product, we extend the family. This is done as follows. For $F \geq A$, define

$$\mu_A^F : \mathcal{C}(\mathsf{c},\mathsf{k})[F] \to \mathcal{C}(\mathsf{c},\mathsf{k})[A], \qquad f \mapsto \tilde{f}, \tag{8.33}$$

where for $K \geq A$, the map $\tilde{f}_{K/A}$ is given by the composite

$$\mathsf{c}[K] \xrightarrow{\Delta_K^{KF}} \mathsf{c}[KF] \xrightarrow{\beta_{FK,KF}} \mathsf{c}[FK] \xrightarrow{f_{FK/F}} \mathsf{k}[FK] \xrightarrow{\beta_{KF,FK}} \mathsf{k}[KF] \xrightarrow{\mu_K^{KF}} \mathsf{k}[K].$$

Observe that for $K \geq F$, we have $\tilde{f}_{K/A} = f_{K/F}$; thus $\tilde{f}$ extends f.

We now need to check that $\tilde{f}$ satisfies condition (8.26). For $A \leq K \leq G$, the diagram can be filled in as follows.

$$\begin{tikzcd}
\mathsf{c}[K] \arrow[r, "\Delta"] \arrow[dd, "\Delta"'] & \mathsf{c}[KF] \arrow[r, "\beta"] \arrow[d, "\Delta"'] & \mathsf{c}[FK] \arrow[r, "f"] \arrow[dd, "\Delta"'] & \mathsf{k}[FK] \arrow[r, "\beta"] \arrow[dd, "\Delta"'] & \mathsf{k}[KF] \arrow[r, "\mu"] \arrow[d, "\Delta"] & \mathsf{k}[K] \arrow[dd, "\Delta"] \\
& \mathsf{c}[KFG] \arrow[d, "\beta"'] \arrow[dr, "\beta"] & & & \mathsf{k}[KFG] \arrow[d, "\beta"] & \\
\mathsf{c}[G] \arrow[r, "\Delta"'] & \mathsf{c}[GF] \arrow[r, "\beta"'] & \mathsf{c}[FG] \arrow[r, "f"'] & \mathsf{k}[FG] \arrow[r, "\beta"'] \arrow[ur, "\beta"] & \mathsf{k}[GF] \arrow[r, "\mu"'] & \mathsf{k}[G]
\end{tikzcd}$$

The left pentagon is the double comonoid axiom of c (which is equivalent to c being cocommutative by Proposition 2.27), while the right pentagon is the bimonoid axiom (2.12) of k. The square in the middle commutes since f satisfies (8.26). Thus, indeed $\tilde{f} \in \mathcal{C}(\mathsf{c},\mathsf{k})[A]$, and μ_A^F is well-defined. It is now routine to verify that μ defines a product which is compatible with the coproduct.

We refer to $\mathcal{C}(\mathsf{c},\mathsf{k})$ as the *bimonoid of star families*.

Lemma 8.32. *If* k *is commutative, then so is* $\mathcal{C}(\mathsf{c},\mathsf{k})$.

PROOF. We need to check the commutativity axiom (2.17) for (8.33), that is,

$$\begin{tikzcd}
\mathcal{C}(\mathsf{c},\mathsf{k})[F] \arrow[rr, "\beta_{G,F}"] \arrow[dr, "\mu_A^F"'] & & \mathcal{C}(\mathsf{c},\mathsf{k})[G] \arrow[dl, "\mu_A^G"] \\
& \mathcal{C}(\mathsf{c},\mathsf{k})[A] &
\end{tikzcd}
\qquad
\begin{tikzcd}
f \arrow[rr, mapsto] \arrow[dr, mapsto] & & g \arrow[dl, mapsto] \\
& \tilde{f} = \tilde{g}. &
\end{tikzcd}$$

This is done below.

$$\begin{tikzcd}
& \mathsf{c}[KF] \arrow[r, "\beta_{FK,KF}"] \arrow[dd, "\beta_{KG,KF}"'] & \mathsf{c}[FK] \arrow[r, "f_{FK/F}"] \arrow[dd, "\beta_{GK,FK}"'] & \mathsf{k}[FK] \arrow[r, "\beta_{KF,FK}"] \arrow[dd, "\beta_{GK,FK}"] & \mathsf{k}[KF] \arrow[dd, "\beta_{KG,KF}"] \arrow[dr, "\mu_K^{KF}"] & \\
\mathsf{c}[K] \arrow[ur, "\Delta_K^{KF}"] \arrow[dr, "\Delta_K^{KG}"'] & & & & & \mathsf{k}[K] \\
& \mathsf{c}[KG] \arrow[r, "\beta_{GK,KG}"'] & \mathsf{c}[GK] \arrow[r, "g_{GK/G}"'] & \mathsf{k}[GK] \arrow[r, "\beta_{KG,GK}"'] & \mathsf{k}[KG] \arrow[ur, "\mu_K^{KG}"'] &
\end{tikzcd}$$

The triangle on the right commutes by commutativity of k. $\square$

Exercise 8.33. Let c be a cocommutative comonoid and k a bimonoid. Check that $\mathcal{C}(\mathsf{c},\mathsf{k})^{coab}$ is the subbimonoid of $\mathcal{C}(\mathsf{c},\mathsf{k})$ consisting of those families f which satisfy (8.31) with $\mathsf{d} := \mathsf{k}$. Moreover, $\mathcal{C}(\mathsf{c},\mathsf{k})^{coab} = \mathcal{C}(\mathsf{c},\mathsf{k}^{coab})^{coab}$ as bimonoids.

8.4.6. Bicommutative bimonoid of star families. Suppose c is a cocommutative comonoid, and k is a bicommutative bimonoid. We proceed to turn ${}^{\mathsf{co}}\mathcal{C}(\mathsf{c},\mathsf{k})$ into a bicommutative bimonoid by building on the discussion in Section 8.4.4. Recall that the coproduct is defined by restricting the given family. For the product, we extend the family as follows. For $\mathrm{X} \geq \mathrm{Z}$, define

$$\mu^{\mathrm{X}}_{\mathrm{Z}} : {}^{\mathsf{co}}\mathcal{C}(\mathsf{c},\mathsf{k})[\mathrm{X}] \to {}^{\mathsf{co}}\mathcal{C}(\mathsf{c},\mathsf{k})[\mathrm{Z}], \qquad f \mapsto \tilde{f}, \tag{8.34}$$

where for $\mathrm{Y} \geq \mathrm{Z}$, the map $\tilde{f}_{\mathrm{Y}/\mathrm{Z}}$ is given by the composite

$$\mathsf{c}[\mathrm{Y}] \xrightarrow{\Delta^{\mathrm{X}\vee\mathrm{Y}}_{\mathrm{Y}}} \mathsf{c}[\mathrm{X}\vee\mathrm{Y}] \xrightarrow{f_{\mathrm{X}\vee\mathrm{Y}/\mathrm{X}}} \mathsf{k}[\mathrm{X}\vee\mathrm{Y}] \xrightarrow{\mu^{\mathrm{X}\vee\mathrm{Y}}_{\mathrm{Y}}} \mathsf{k}[\mathrm{Y}].$$

We now need to check that $\tilde{f}$ satisfies (8.32). For $\mathrm{Z} \leq \mathrm{Y} \leq \mathrm{W}$, the diagram can be filled in as follows.

$$\begin{tikzcd}
\mathsf{c}[\mathrm{Y}] \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathsf{c}[\mathrm{X}\vee\mathrm{Y}] \arrow[r, "f"] \arrow[d, "\Delta"'] & \mathsf{k}[\mathrm{X}\vee\mathrm{Y}] \arrow[r, "\mu"] \arrow[d, "\Delta"] & \mathsf{k}[\mathrm{Y}] \arrow[d, "\Delta"] \\
\mathsf{c}[\mathrm{W}] \arrow[r, "\Delta"'] & \mathsf{c}[\mathrm{X}\vee\mathrm{W}] \arrow[r, "f"'] & \mathsf{k}[\mathrm{X}\vee\mathrm{W}] \arrow[r, "\mu"'] & \mathsf{k}[\mathrm{W}]
\end{tikzcd}$$

The left square commutes by coassociativity (2.24) of c, while the right square is the bicommutative bimonoid axiom (2.26) of k. The square in the middle commutes since f satisfies (8.32). Thus, indeed $\tilde{f} \in {}^{\mathsf{co}}\mathcal{C}(\mathsf{c},\mathsf{k})[\mathrm{Z}]$, and $\mu^{\mathrm{X}}_{\mathrm{Z}}$ is well-defined. It is now routine to check that μ defines a product which is compatible with the coproduct.

We refer to ${}^{\mathsf{co}}\mathcal{C}(\mathsf{c},\mathsf{k})$ as the *bicommutative bimonoid of star families*.

Exercise 8.34. Let c be a cocommutative comonoid and k a bicommutative bimonoid. Check that ${}^{\mathsf{co}}\mathcal{C}(\mathsf{c},\mathsf{k}) = \mathcal{C}(\mathsf{c},\mathsf{k})^{coab}$ as bimonoids. (The rhs is indeed a bicommutative bimonoid by Proposition 2.42 and Lemma 8.32.)

8.4.7. Connection to the biconvolution bimonoid. We now connect the bimonoid of star families to the biconvolution bimonoid when c carries the structure of a bimonoid.

Lemma 8.35. *Let c be a cocommutative comonoid, and d a bimonoid. Then the map (8.29) is a morphism of monoids, with the convolution product on the latter.*

Proof. Following the definition of μ^F_A given in (8.33), note that the base term of $\tilde{f}$ is determined by the base term of f via

$$\mathsf{c}[A] \xrightarrow{\Delta^F_A} \mathsf{c}[F] \xrightarrow{f_{F/F}} \mathsf{d}[F] \xrightarrow{\mu^F_A} \mathsf{d}[A].$$

This is precisely how the convolution product works. □

Lemma 8.36. *Let h and k be bimonoids, and h be cocommutative. Then $\mathcal{C}(\mathsf{h},\mathsf{k})$ is a subbimonoid of $\mathrm{hom}^{\times}(\mathsf{h},\mathsf{k})$ via the map (8.29). If, in addition, k is (co, bi)commutative, then so is $\mathcal{C}(\mathsf{h},\mathsf{k})$.*

Proof. The first claim follows from Lemmas 8.26 and 8.35. The second follows from Lemmas 8.28 and 8.32. □

It is possible to get hold of the above subbimonoid of $\hom^{\times}(\mathsf{h},\mathsf{k})$ directly without any explicit reference to $\mathcal{C}(\mathsf{c},\mathsf{k})$. This is explained below.

Lemma 8.37. *Let* h *and* k *be bimonoids, and* h *be cocommutative. Then the subspecies of* $\hom^{\times}(\mathsf{h},\mathsf{k})$ *consisting of those* f *which satisfy property* (8.28) *is a subbimonoid. It coincides with* $\mathcal{C}(\mathsf{h},\mathsf{k})$ *via the map* (8.29).

Proof. This is a restatement of Lemma 8.36 in view of Lemma 8.24, but the point here is to prove the first statement directly. This is done below.

For the coproduct:

$$\begin{tikzcd}
\mathsf{h}[F] \arrow[r, "\mu"] \arrow[d, "\Delta"'] & \mathsf{h}[A] \arrow[r, "f"] \arrow[dl, "\Delta"] & \mathsf{k}[A] \arrow[r, "\Delta"] \arrow[dr, "\Delta"'] & \mathsf{k}[F] \arrow[d, "\Delta"] \\
\mathsf{h}[G] \arrow[rrr, dashed] & & & \mathsf{k}[G]
\end{tikzcd}$$

The dotted arrow exists since f satisfies (8.28). The right triangle commutes by coassociativity, and the left triangle by (2.14). Thus, $\Delta_A^F f \mu_A^F$ satisfies (8.28).

For the product:

$$\begin{tikzcd}
\mathsf{h}[A] \arrow[r, "\Delta"] \arrow[dd, "\Delta"'] & \mathsf{h}[F] \arrow[r, "f"] \arrow[d, "\Delta"'] & \mathsf{k}[F] \arrow[r, "\mu"] \arrow[d, "\Delta"] & \mathsf{k}[A] \arrow[dd, "\Delta"] \\
 & \mathsf{h}[FG] \arrow[r, dashed] \arrow[d, "\beta"'] & \mathsf{k}[FG] \arrow[d, "\beta"] & \\
\mathsf{h}[G] \arrow[r, "\Delta"'] & \mathsf{h}[GF] \arrow[r, dashed] & \mathsf{k}[GF] \arrow[r, "\mu"'] & \mathsf{k}[G]
\end{tikzcd}$$

The middle dotted arrow exists since f satisfies (8.28). The bottom dotted arrow then exists since the β are isomorphisms. The left pentagon commutes by the double comonoid axiom of h, and the right pentagon by the bimonoid axiom of k. Thus, $\mu_A^F f \Delta_A^F$ satisfies (8.28). $\square$

Corollary 8.38. *For any bimonoid* k*, we have*

$$\mathcal{C}(\mathsf{E},\mathsf{k}) = \hom^{\times}(\mathsf{E},\mathsf{k}) = \mathsf{k} \tag{8.35}$$

as bimonoids.

Proof. Since the structure maps of E are all identities, condition (8.28) is vacuous in this case. $\square$

Lemma 8.39. *Let* h *be a cocommutative bimonoid. Then* $\mathcal{C}(\mathsf{h},\mathsf{h})$ *is a cocommutative subbimonoid of* $\operatorname{end}^{\times}(\mathsf{h})$*. Further, each component* $\operatorname{end}^{\times}(\mathsf{h})[A]$ *is an algebra under composition of linear maps, and this algebra structure restricts to* $\mathcal{C}(\mathsf{h},\mathsf{h})[A]$.

Proof. The first statement is a special case of Lemma 8.36. The second follows from (8.28). $\square$

For the result below, we make use of the Borel–Hopf theorem which we will prove later in Chapter 13.

Corollary 8.40. *For finite-dimensional cocommutative bimonoids* h *and* k,

$$\dim \mathcal{C}(\mathsf{h},\mathsf{k})[A] = \sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{s}(A)} (\dim \mathsf{h}[\mathrm{X}])(\dim \mathcal{P}(\mathsf{k})[\mathrm{X}]). \tag{8.36}$$

The sum is over flats X *containing the face* A.

Proof. From Lemma 8.36, $\mathcal{C}(\mathsf{h},\mathsf{k})$ is a cocommutative bimonoid. Now use (13.29) and (8.30) to deduce the result. □

8.5. Species of chamber maps

Recall from Section 7.3 the species of chambers Γ. We now consider the internal hom for comonoids $\mathcal{C}(\Gamma,\Gamma)$. It carries the structure of a cocommutative bimonoid. We call it the bimonoid of chamber maps. It is a specialization of the bimonoid of star families discussed in Section 8.4. To keep the discussion self-contained, we develop this example from first principles (with some details omitted).

8.5.1. Species of chamber maps. Recall that the species of chambers Γ carries the structure of a cocommutative bimonoid with product and coproduct given by (7.18). For any face A, let $\mathcal{C}(\Gamma,\Gamma)[A]$ denote the space of all linear maps $f : \Gamma[A] \to \Gamma[A]$ with the property that for each face $F \geq A$, there exists a linear map $\Gamma[F] \to \Gamma[F]$ making the diagram

$$\begin{tikzcd} \Gamma[A] \arrow[r, "f"] \arrow[d, two heads, "\Delta_A^F"'] & \Gamma[A] \arrow[d, two heads, "\Delta_A^F"] \\ \Gamma[F] \arrow[r, dashed] & \Gamma[F] \end{tikzcd} \tag{8.37}$$

commute.

It is clear that $\mathcal{C}(\Gamma,\Gamma)[A]$ is closed under addition and scalar multiplication under the inclusion

$$\mathcal{C}(\Gamma,\Gamma)[A] \to \mathrm{End}_{\Bbbk}(\Gamma[A]),$$

so it is a vector space. Further, given a morphism $\beta_{B,A} : A \to B$ and an element $f \in \mathcal{C}(\Gamma,\Gamma)[A]$, the composite

$$\Gamma[B] \xrightarrow{\beta_{B,A}^{-1}} \Gamma[A] \xrightarrow{f} \Gamma[A] \xrightarrow{\beta_{B,A}} \Gamma[B]$$

belongs to $\mathcal{C}(\Gamma,\Gamma)[B]$. This follows from naturality of the coproduct (2.10). This turns $\mathcal{C}(\Gamma,\Gamma)$ into a species. We call it the *species of chamber maps*. More information on its components is given below.

Lemma 8.41. *If* $f \in \mathcal{C}(\Gamma,\Gamma)[A]$, *then the dotted arrow in* (8.37) *is unique and given by* $\Delta_A^F f \mu_A^F$.

Proof. The uniqueness comes from the fact that the coproduct components Δ_A^F of Γ are surjective. The specified formula follows from (2.15). □

Lemma 8.42. *A linear map* $f : \Gamma[A] \to \Gamma[A]$ *belongs to* $\mathcal{C}(\Gamma,\Gamma)[A]$ *iff* (8.37) *holds for every face* F *which covers* A.

PROOF. We first make some preliminary definitions. We say a top-lune L is F-based if $\mathrm{L} = \mathrm{s}(F, E)$ for some chamber E greater than F. For any linear combination of chambers u, the content of u wrt a top-lune L is the sum of the coefficients in u of those chambers which belong to L.

Now we turn to the proof. We may assume without loss of generality that A is the central face. The condition (8.37) (with F fixed) can be reformulated as follows. For any adjacent chambers C and D which belong to the same F-based top-lune (that is, $FC = FD$), the contents of $f(\mathtt{H}_C)$ and $f(\mathtt{H}_D)$ are equal wrt any F-based top-lune. (Adjacent chambers suffice since top-lunes are gallery connected.)

Suppose the above property holds whenever F is a vertex. We want to then show that it holds for any face F. For the central face, there is nothing to do. So let F be any non-central face, and C and D be adjacent chambers such that $FC = FD$. By hypothesis, the common wall of C and D cannot contain F. So there exists a vertex H of F which is not on this common wall. This implies $HC = HD$. Since the property holds for vertices, the contents of $f(\mathtt{H}_C)$ and $f(\mathtt{H}_D)$ are equal wrt any H-based top-lune. But any F-based top-lune is the disjoint union of H-based top-lunes, see [**21**, Proposition 3.21]. So by adding these contents, we deduce that the contents of $f(\mathtt{H}_C)$ and $f(\mathtt{H}_D)$ are equal wrt any F-based top-lune.

(Note that each vertex of F provides a decomposition of a F-based top-lune, and different vertices give different decompositions. See for instance [**21**, Figure on page 87]. So to get a decomposition, we only need to choose one vertex, and that is what was done in the above argument.) □

Example 8.43. Let $\mathcal{A}$ be a rank-one arrangement consisting of two chambers C and $\overline{C}$ and the central face O. Then one may check that a linear map $\Gamma[O] \to \Gamma[O]$ given by (say)

$$\mathtt{H}_C \mapsto \alpha \mathtt{H}_C + \beta \mathtt{H}_{\overline{C}}, \qquad \mathtt{H}_{\overline{C}} \mapsto \gamma \mathtt{H}_C + \delta \mathtt{H}_{\overline{C}}$$

belongs to $\mathcal{C}(\Gamma, \Gamma)[O]$ iff $\alpha + \beta = \gamma + \delta$.

Exercise 8.44. Check that: For any arrangement, the identity and antipodal maps, namely,

$$\Gamma[A] \to \Gamma[A], \quad \mathtt{H}_{C/A} \mapsto \mathtt{H}_{C/A} \qquad \text{and} \qquad \Gamma[A] \to \Gamma[A], \quad \mathtt{H}_{C/A} \mapsto \mathtt{H}_{A\overline{C}/A}$$

are elements of $\mathcal{C}(\Gamma, \Gamma)[A]$.

8.5.2. Bimonoid of chamber maps. We now turn the species $\mathcal{C}(\Gamma, \Gamma)$ into a cocommutative bimonoid. The coproduct component Δ_A^F sends $f : \Gamma[A] \to \Gamma[A]$ to the dotted arrow in (8.37), or more explicitly to the map

$$\Gamma[F] \xrightarrow{\mu_A^F} \Gamma[A] \xrightarrow{f} \Gamma[A] \xrightarrow{\Delta_A^F} \Gamma[F].$$

The product component μ_A^F sends $f : \Gamma[F] \to \Gamma[F]$ to the composite

$$\Gamma[A] \xrightarrow{\Delta_A^F} \Gamma[F] \xrightarrow{f} \Gamma[F] \xrightarrow{\mu_A^F} \Gamma[A].$$

It is a routine check that this composite indeed belongs to $\mathcal{C}(\Gamma,\Gamma)[A]$. Observe that the product and coproduct is defined exactly as in (8.15) and (8.18). In other words, the injective map of species

$$\mathcal{C}(\Gamma,\Gamma) \hookrightarrow \mathrm{end}^{\times}(\Gamma)$$

turns $\mathcal{C}(\Gamma,\Gamma)$ into a subbimonoid of the biconvolution bimonoid $\mathrm{end}^{\times}(\Gamma)$. We refer to $\mathcal{C}(\Gamma,\Gamma)$ as the *bimonoid of chamber maps*.

8.5.3. Bimonoid of faces. We now relate the bimonoid of chamber maps $\mathcal{C}(\Gamma,\Gamma)$ to the bimonoid of faces Σ from Section 7.6.2. The map

$$\Sigma \hookrightarrow \mathrm{end}^{\times}(\Gamma), \tag{8.38}$$

which on the A component sends $\mathtt{H}_{K/A}$ to the linear map

$$\Gamma[A] \to \Gamma[A], \qquad \mathtt{H}_{C/A} \mapsto \mathtt{H}_{KC/A}$$

is a morphism of bimonoids. (See Exercise 8.17 in this regard.) In fact, this linear map belongs to $\mathcal{C}(\Gamma,\Gamma)[A]$: the dotted arrow in (8.37) sends $\mathtt{H}_{D/F}$ to $\mathtt{H}_{FKD/F}$. Thus, the morphism (8.38) factors through the bimonoid of chamber maps to yield the diagram of bimonoids

$$\begin{tikzcd} \Sigma \arrow[rr] \arrow[dr] & & \mathrm{end}^{\times}(\Gamma) \\ & \mathcal{C}(\Gamma,\Gamma). \arrow[ur, hook] & \end{tikzcd} \tag{8.39}$$

8.5.4. Primitive part. Now let us turn to the primitive part of the bimonoid of chamber maps. We have

$$\mathcal{P}(\mathcal{C}(\Gamma,\Gamma)) = \mathrm{hom}^{\times}(\Gamma,\mathcal{P}(\Gamma)) = \mathrm{hom}^{\times}(\Gamma,\mathsf{Lie}). \tag{8.40}$$

The first equality is an instance of (8.30), while the second one uses the Friedrichs criterion (Lemma 7.64).

Corollary 8.45. *We have*

$$\dim \mathcal{C}(\Gamma,\Gamma)[A] = \sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{s}(A)} (\dim \Gamma[\mathrm{X}])(\dim \mathsf{Lie}[\mathrm{X}]). \tag{8.41}$$

The sum is over flats X *containing the face* A.

Proof. This is an instance of formula (8.36). More explicitly, it follows from (8.40) and (13.29). A direct argument for this formula is sketched below.

Any element $f \in \mathcal{C}(\Gamma,\Gamma)[A]$ yields an element $\Delta_A^F(f) \in \mathcal{C}(\Gamma,\Gamma)[F]$ for each $F \geq A$. For simplicity, let us denote $\Delta_A^F(f)$ by $f_{F/A}$. These elements satisfy the following compatibilities.

- $\beta_{G,F}(f_{F/A}) = f_{G/A}$ whenever F and G have the same support, and
- $\Delta_F^G(f_{F/A}) = f_{G/A}$ whenever $G \geq F$.

Conversely: Suppose we are given elements $g_{F/A} \in \mathcal{C}(\Gamma,\Gamma)[F]$ for $F > A$ which are compatible in the above sense (with g instead of f). We want to find all $f \in \mathcal{C}(\Gamma,\Gamma)[A]$ such that $f_{F/A} = g_{F/A}$ for $F > A$. We claim that the solution set for f is an affine space of dimension

$$(\dim \Gamma[A])(\dim \mathsf{Lie}[A]).$$

To see this: First consider the case $g_{F/A} = 0$ for all $F > A$. Then the solution set is the linear space $\mathrm{Hom}_{\Bbbk}(\Gamma[A], \mathsf{Lie}[A])$ and the claim is clear. In the general case, note that the difference of two solutions belongs to this linear space. So all we need to do is to exhibit one solution. This is guaranteed by [**21**, Theorem 11.57] as follows. Fix a chamber $C \geq A$. This theorem implies that there exists a u (a linear combintation of chambers greater than A) such that $\Delta_A^F(u) = g_{F/A}(\mathtt{H}_{FC/F})$ for any $F > A$. Define $f(\mathtt{H}_{C/A}) := u$. Repeat this procedure for each $C \geq A$. This produces a solution f and the claim is proved. The formula now follows by induction on the corank of A. □

Example 8.46. For the braid arrangement on the set $[n]$, formula (8.41) applied to the central face is

$$\dim \mathcal{C}(\Gamma, \Gamma)[n] = \sum_{\mathrm{X}} \mathrm{X}!\, \mathrm{X}^{\mathrm{\Phi}}\!!,$$

where X runs over all partitions of the set $[n]$, and X! and $\mathrm{X}^{\mathrm{\Phi}}\!!$ are as defined in [**18**, Section 10.1.7]. For $n = 1, 2, 3$, this yields $1, 3, 19$. In the lhs above, to show the dependence on n, we have written $[n]$ instead of $[O]$.

8.6. Universal measuring comonoids

We construct the universal measuring comonoid $\overline{\mathcal{C}}(\mathsf{a}, \mathsf{b})$ from one monoid a to another monoid b. Its A-component is defined using families of maps indexed by the star of A which are compatible with the products of a and b. There is a similar construction for commutative monoids which we denote by ${}^{\mathsf{co}}\overline{\mathcal{C}}(\mathsf{a}, \mathsf{b})$. For a commutative monoid a and bimonoid h, the comonoid $\overline{\mathcal{C}}(\mathsf{h}, \mathsf{a})$ carries the structure of a bimonoid. If, in addition, a carries the structure of a bimonoid, then $\overline{\mathcal{C}}(\mathsf{h}, \mathsf{a})$ can be realized as a subbimonoid of the biconvolution bimonoid $\mathrm{hom}^{\times}(\mathsf{h}, \mathsf{a})$.

This section proceeds in parallel to Section 8.4. We only state the results. The constructions and proofs are similar, the key distinction being that certain arrows labeled Δ get reversed with label μ in the involved diagrams. As a result, in place of the bimonoid axiom, we need to employ the double monoid axiom, and in place of the double comonoid axiom, we need to employ the bimonoid axiom. The connection between the two situations is made precise using duality.

The above ideas are developed further in Section 8.7.

8.6.1. Universal measuring comonoid. Let a and b be monoids. We proceed to construct a comonoid $\overline{\mathcal{C}}(\mathsf{a}, \mathsf{b})$. For a face A, an element of $\overline{\mathcal{C}}(\mathsf{a}, \mathsf{b})[A]$ is a family of linear maps

$$f_{F/A} : \mathsf{a}[F] \to \mathsf{b}[F],$$

one for each $A \leq F$, such that the diagram

$$\begin{tikzcd} \mathsf{a}[G] \arrow[r, "f_{G/A}"] \arrow[d, "\mu_F^G"'] & \mathsf{b}[G] \arrow[d, "\mu_F^G"] \\ \mathsf{a}[F] \arrow[r, "f_{F/A}"'] & \mathsf{b}[F] \end{tikzcd} \tag{8.42}$$

commutes for each $A \leq F \leq G$. We refer to $f_{A/A}$ as the base term, and to the remaining $f_{F/A}$ for $F > A$ as the higher terms.

Clearly, $\overline{\mathcal{C}}(\mathsf{a}, \mathsf{b})[A]$ is a vector space with componentwise addition and scalar multiplication. Now suppose that we are given A and B of the same support, and an element $(f_{F/A})$ of $\overline{\mathcal{C}}(\mathsf{a}, \mathsf{b})[A]$. For each $F \geq A$, put $G := BF$ and let $g_{G/B}$ denote the composite

$$\mathsf{a}[G] \xrightarrow{\beta_{G,F}^{-1}} \mathsf{a}[F] \xrightarrow{f_{F/A}} \mathsf{b}[F] \xrightarrow{\beta_{G,F}} \mathsf{b}[G].$$

Naturality of the product implies that $(g_{G/B})$ belongs to $\overline{\mathcal{C}}(\mathsf{a}, \mathsf{b})[B]$. The maps

$$\beta_{B,A} : \overline{\mathcal{C}}(\mathsf{a}, \mathsf{b})[A] \to \overline{\mathcal{C}}(\mathsf{a}, \mathsf{b})[B], \quad (f_{F/A}) \mapsto (g_{G/B})$$

turn $\overline{\mathcal{C}}(\mathsf{a}, \mathsf{b})$ into a species. Further, for each $A \leq G$, the linear map

$$\Delta_A^G : \overline{\mathcal{C}}(\mathsf{a}, \mathsf{b})[A] \to \overline{\mathcal{C}}(\mathsf{a}, \mathsf{b})[G]$$

obtained by restricting $(f_{F/A})$ to faces F greater than G turns $\overline{\mathcal{C}}(\mathsf{a}, \mathsf{b})$ into a comonoid. We call it the *universal measuring comonoid*. This construction is natural in a and b, and yields a bifunctor. It satisfies the following universal property.

Proposition 8.47. *For a comonoid* c *and monoids* a *and* b*, there is a natural bijection*

$$\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})\big(\mathsf{a}, \hom^{\times}(\mathsf{c}, \mathsf{b})\big) \cong \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})\big(\mathsf{c}, \overline{\mathcal{C}}(\mathsf{a}, \mathsf{b})\big),$$

where $\hom^{\times}(\mathsf{c}, \mathsf{b})$ *is the convolution monoid.*

The proof is similar to that of Proposition 8.23. It is partly elaborated in the exercise below.

Exercise 8.48. Check that: For a comonoid c and monoids a and b, a morphism of monoids $\mathsf{a} \to \hom^{\times}(\mathsf{c}, \mathsf{b})$ is equivalent to a family of linear maps

$$h_A : \mathsf{c}[A] \otimes \mathsf{a}[A] \to \mathsf{b}[A],$$

one for each face A, such that

$$h_B(x \otimes \beta_{B,A}(y)) = \beta_{B,A} h_A(\beta_{A,B}(x) \otimes y) \tag{8.43a}$$

for A and B of the same support and $x \in \mathsf{c}[B]$ and $y \in \mathsf{a}[A]$, and

$$h_A(x \otimes \mu_A^F(y)) = \mu_A^F h_F(\Delta_A^F(x) \otimes y) \tag{8.43b}$$

for $F \geq A$ and $x \in \mathsf{c}[A]$ and $y \in \mathsf{a}[F]$. We refer to (8.43) as the *measuring conditions*. Compare and contrast with diagrams (8.27).

Lemma 8.49. *If the product components of* b *are injective, then* $\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})[A]$ *is the same as the space of all linear maps*

$$f : \mathsf{a}[A] \to \mathsf{b}[A]$$

with the property that for each face $F \geq A$*, there exists a linear map* $\mathsf{a}[F] \to \mathsf{b}[F]$ *making the diagram*

$$\begin{tikzcd} \mathsf{a}[F] \arrow[r, dotted] \arrow[d, "\mu_A^F"'] & \mathsf{b}[F] \arrow[d, hook, "\mu_A^F"] \\ \mathsf{a}[A] \arrow[r, "f"'] & \mathsf{b}[A] \end{tikzcd} \tag{8.44}$$

commute. Moreover, the coproduct component Δ_A^F *of* $\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})$ *sends* f *to the dotted arrow in* (8.44).

Compare and contrast with Lemma 8.24.

8.6.2. Primitive part. For monoids a and b, define a map of species

$$\overline{\mathcal{C}}(\mathsf{a},\mathsf{b}) \to \mathrm{hom}^{\times}(\mathsf{a},\mathsf{b}), \tag{8.45}$$

which on the A-component, sends a family $(f_{F/A})$ to its base term $f_{A/A}$.

Lemma 8.50. *When restricted to the primitive part of* $\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})$*, the map* (8.45) *is injective, and in fact*

$$\mathcal{P}(\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})) = \mathrm{hom}^{\times}(\mathcal{Q}(\mathsf{a}),\mathsf{b}). \tag{8.46}$$

Compare and contrast with (8.30).

Lemma 8.51. *If the product components of* b *are injective, then the map* (8.45) *is injective. For any bimonoid* b*, the map* (8.45) *is an injective morphism of comonoids, with the coconvolution coproduct on the latter.*

Compare and contrast with Lemma 8.26. When b is a bimonoid, by injectivity, the primitive part of $\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})$ is contained in the primitive part of $\mathrm{hom}^{\times}(\mathsf{a},\mathsf{b})$. In view of (8.46), we recover the second statement in (8.23).

Exercise 8.52. Check that: For any species p and monoid b,

$$\overline{\mathcal{C}}(\mathcal{T}(\mathsf{p}),\mathsf{b}) \cong \mathcal{T}^{\vee}(\mathrm{hom}^{\times}(\mathsf{p},\mathsf{b})),$$

the cofree comonoid on the species $\mathrm{hom}^{\times}(\mathsf{p},\mathsf{b})$ (which is also the primitive part of the comonoid). This is consistent with formula (8.46).

8.6.3. Coabelianization. Let $\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})^{coab}$ denote the coabelianization of the comonoid $\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})$ as in Section 2.7.2. Explicitly, its A-component consists of those elements $(f_{F/A})$ for which, in addition, the diagram

$$\begin{tikzcd} \mathsf{a}[F] \arrow[r, "f_{F/A}"] \arrow[d, "\beta_{G,F}"'] & \mathsf{b}[F] \arrow[d, "\beta_{G,F}"] \\ \mathsf{a}[G] \arrow[r, "f_{G/A}"'] & \mathsf{b}[G] \end{tikzcd} \tag{8.47}$$

commutes whenever F and G are faces greater than A with equal support.

Lemma 8.53. *Let* a *and* b *be commutative monoids. If the product components of* b *are injective, then* $\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})$ *is cocommutative.*

Compare and contrast with Lemma 8.28.

8.6.4. Universal measuring cocommutative comonoid. Let a and b be commutative monoids. We now construct a cocommutative comonoid ${}^{\mathsf{co}}\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})$. We work with the formulation provided by Proposition 2.21. For a flat Z, an element of ${}^{\mathsf{co}}\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})[\mathrm{Z}]$ is a family of linear maps

$$f_{\mathrm{X}/\mathrm{Z}} : \mathsf{a}[\mathrm{X}] \to \mathsf{b}[\mathrm{X}],$$

one for each $\mathrm{Z} \leq \mathrm{X}$, such that the diagram

$$\begin{tikzcd} \mathsf{a}[\mathrm{Y}] \arrow[r, "f_{\mathrm{Y}/\mathrm{Z}}"] \arrow[d, "\mu^{\mathrm{Y}}_{\mathrm{X}}"'] & \mathsf{b}[\mathrm{Y}] \arrow[d, "\mu^{\mathrm{Y}}_{\mathrm{X}}"] \\ \mathsf{a}[\mathrm{X}] \arrow[r, "f_{\mathrm{X}/\mathrm{Z}}"'] & \mathsf{b}[\mathrm{X}] \end{tikzcd} \tag{8.48}$$

commutes for each $\mathrm{Z} \leq \mathrm{X} \leq \mathrm{Y}$. The rest of the construction is similar to that of $\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})$ with faces replaced by flats. We call ${}^{\mathsf{co}}\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})$ the *universal measuring cocommutative comonoid*. It satisfies the following universal property.

Proposition 8.54. *For a cocommutative comonoid* c *and commutative monoids* a *and* b*, there is a natural bijection*

$$\mathsf{Mon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{a}, \hom^{\times}(\mathsf{c},\mathsf{b})) \cong {}^{\mathsf{co}}\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})\big(\mathsf{c}, {}^{\mathsf{co}}\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})\big),$$

where $\hom^{\times}(\mathsf{c},\mathsf{b})$ *is the convolution monoid.*

This is the commutative analogue of Proposition 8.47. It follows that for commutative monoids a and b, we have ${}^{\mathsf{co}}\overline{\mathcal{C}}(\mathsf{a},\mathsf{b}) = \overline{\mathcal{C}}(\mathsf{a},\mathsf{b})^{coab}$.

8.6.5. Bimonoid structure. Suppose h is a bimonoid, and a is a commutative monoid. We proceed to turn $\overline{\mathcal{C}}(\mathsf{h},\mathsf{a})$ into a bimonoid. Recall that it is a comonoid with coproduct defined by restricting the given family. For the product, we extend the family. This is done as follows. For $F \geq A$, define

$$\mu^F_A : \overline{\mathcal{C}}(\mathsf{h},\mathsf{a})[F] \to \overline{\mathcal{C}}(\mathsf{h},\mathsf{a})[A], \qquad f \mapsto \tilde{f}, \tag{8.49}$$

where for $K \geq A$, the map $\tilde{f}_{K/A}$ is given by the composite

$$\mathsf{h}[K] \xrightarrow{\Delta^{KF}_K} \mathsf{h}[KF] \xrightarrow{\beta_{FK,KF}} \mathsf{h}[FK] \xrightarrow{f_{FK/F}} \mathsf{a}[FK] \xrightarrow{\beta_{KF,FK}} \mathsf{a}[KF] \xrightarrow{\mu^{KF}_K} \mathsf{a}[K].$$

Observe that for $K \geq F$, we have $\tilde{f}_{K/A} = f_{K/F}$; thus $\tilde{f}$ extends f. One may check that $\tilde{f}$ indeed satisfies condition (8.42). Thus, μ^F_A is well-defined. It is routine to verify that μ defines a product which is compatible with the coproduct. This yields the bimonoid $\overline{\mathcal{C}}(\mathsf{h},\mathsf{a})$.

Lemma 8.55. *If* h *is cocommutative, then* $\overline{\mathcal{C}}(\mathsf{h},\mathsf{a})$ *is commutative.*

Compare and contrast with Lemma 8.32.

In a similar manner, for a bicommutative bimonoid h and a commutative monoid a, by building on the discussion in Section 8.6.4, we obtain a bicommutative bimonoid ${}^{\mathsf{co}}\overline{\mathcal{C}}(\mathsf{h},\mathsf{a})$.

8.6.6. Connection to the biconvolution bimonoid. We now state results analogous to those in Section 8.4.7.

Lemma 8.56. *Let* b *be a commutative monoid, and* a *a bimonoid. Then the map* (8.45) *is a morphism of monoids, with the convolution product on the latter.*

Lemma 8.57. *Let* h *and* k *be bimonoids, and* k *be commutative. Then* $\overline{\mathcal{C}}(\mathsf{h},\mathsf{k})$ *is a subbimonoid of* $\mathrm{hom}^{\times}(\mathsf{h},\mathsf{k})$ *via the map* (8.45)*. In addition, if* h *is commutative, then* $\overline{\mathcal{C}}(\mathsf{h},\mathsf{k})$ *is cocommutative, and if* h *is cocommutative, then* $\overline{\mathcal{C}}(\mathsf{h},\mathsf{k})$ *is commutative.*

Lemma 8.58. *Let* h *and* k *be bimonoids, and* k *be commutative. Then the subspecies of* $\mathrm{hom}^{\times}(\mathsf{h},\mathsf{k})$ *consisting of those* f *which satisfy property* (8.44) *is a subbimonoid. It coincides with* $\overline{\mathcal{C}}(\mathsf{h},\mathsf{k})$ *via the map* (8.45)*.*

Corollary 8.59. *For any bimonoid* h*, we have*

$$\overline{\mathcal{C}}(\mathsf{h},\mathsf{E}) = \mathrm{hom}^{\times}(\mathsf{h},\mathsf{E}) = \mathsf{h}^{*} \tag{8.50}$$

as bimonoids.

Lemma 8.60. *Let* h *be a commutative bimonoid. Then* $\overline{\mathcal{C}}(\mathsf{h},\mathsf{h})$ *is a cocommutative subbimonoid of* $\mathrm{end}^{\times}(\mathsf{h})$*. Further, each component* $\mathrm{end}^{\times}(\mathsf{h})[A]$ *is an algebra under composition of linear maps, and this algebra structure restricts to* $\overline{\mathcal{C}}(\mathsf{h},\mathsf{h})[A]$*.*

Corollary 8.61. *For finite-dimensional commutative bimonoids* h *and* k*,*

$$\dim \overline{\mathcal{C}}(\mathsf{h},\mathsf{k})[A] = \sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{s}(A)} (\dim \mathcal{Q}(\mathsf{h})[\mathrm{X}])(\dim \mathsf{k}[\mathrm{X}]). \tag{8.51}$$

The sum is over flats X *containing the face* A*.*

Exercise 8.62. Show that: For bimonoids h and k and a commutative monoid a, the following pieces of data are equivalent.

- A bimonoid morphism $\mathsf{h} \to \overline{\mathcal{C}}(\mathsf{k},\mathsf{a})$.
- A bimonoid morphism $\mathsf{k} \to \overline{\mathcal{C}}(\mathsf{h},\mathsf{a})$.
- A pair of comonoid morphisms $\mathsf{h} \to \overline{\mathcal{C}}(\mathsf{k},\mathsf{a})$ and $\mathsf{k} \to \overline{\mathcal{C}}(\mathsf{h},\mathsf{a})$.
- A pair of monoid morphisms $\mathsf{h} \to \mathrm{hom}^{\times}(\mathsf{k},\mathsf{a})$ and $\mathsf{k} \to \mathrm{hom}^{\times}(\mathsf{h},\mathsf{a})$.

The last item can be made explicit using Exercise 8.48. We refer to the resulting conditions as the *bimeasuring conditions*.

Use formula (8.50) to directly verify the special case $\mathsf{a} := \mathsf{E}$.

8.6.7. Duality. The internal hom for comonoids and the universal measuring comonoid are related by duality. Details follow.

For any comonoid c and monoid a, we have an isomorphism of comonoids

$$\mathcal{C}(\mathsf{c},\mathsf{a}^{*}) \xrightarrow{\cong} \overline{\mathcal{C}}(\mathsf{a},\mathsf{c}^{*}). \tag{8.52}$$

On each A-component, it sends a family $(f_{F/A})$ to the family $(\overline{f}_{F/A})$, with $\overline{f}_{F/A}$ defined by

$$\mathsf{a}[F] \hookrightarrow \mathsf{a}[F]^{**} \xrightarrow{f^{*}_{F/A}} \mathsf{c}[F]^{*}.$$

The map (8.52) fits into a commutative diagram of species

(8.53)
$$\begin{tikzcd} \mathcal{C}(\mathsf{c}, \mathsf{a}^*) \arrow[r, "\cong"] \arrow[d] & \overline{\mathcal{C}}(\mathsf{a}, \mathsf{c}^*) \arrow[d] \\ \hom^{\times}(\mathsf{c}, \mathsf{a}^*) \arrow[r, "\cong"'] & \hom^{\times}(\mathsf{a}, \mathsf{c}^*), \end{tikzcd}$$

with the vertical maps as in (8.29) and (8.45). The bottom-horizontal map is defined in a manner similar to the top-horizontal map.

For a cocommutative comonoid c and bimonoid h, the map (8.52) extends to an isomorphism of bimonoids

(8.54)
$$\mathcal{C}(\mathsf{c}, \mathsf{h}^*) \xrightarrow{\cong} \overline{\mathcal{C}}(\mathsf{h}, \mathsf{c}^*).$$

Similarly, for a cocommutative bimonoid k and bimonoid h, we have a commutative diagram of bimonoids

(8.55)
$$\begin{tikzcd} \mathcal{C}(\mathsf{k}, \mathsf{h}^*) \arrow[r, "\cong"] \arrow[d, hook] & \overline{\mathcal{C}}(\mathsf{h}, \mathsf{k}^*) \arrow[d, hook] \\ \hom^{\times}(\mathsf{k}, \mathsf{h}^*) \arrow[r, "\cong"'] & \hom^{\times}(\mathsf{h}, \mathsf{k}^*). \end{tikzcd}$$

The above story has a commutative analogue: For any cocommutative comonoid c and commutative monoid a, we have an isomorphism of comonoids

(8.56)
$${}^{\mathsf{co}}\mathcal{C}(\mathsf{c}, \mathsf{a}^*) \xrightarrow{\cong} {}^{\mathsf{co}}\overline{\mathcal{C}}(\mathsf{a}, \mathsf{c}^*),$$

and for a cocommutative comonoid c and bicommutative bimonoid h, we have an isomorphism of bimonoids

(8.57)
$${}^{\mathsf{co}}\mathcal{C}(\mathsf{c}, \mathsf{h}^*) \xrightarrow{\cong} {}^{\mathsf{co}}\overline{\mathcal{C}}(\mathsf{h}, \mathsf{c}^*).$$

In this situation, one can work with flats instead of faces.

8.7. Enrichment of the category of monoids over comonoids

The universal measuring comonoid constructed in Section 8.6 allows us to enrich the category of monoids over the category of comonoids. This enriched category possesses powers and copowers which we describe explicitly. The power is in fact the convolution monoid. The copower is a certain quotient of the free monoid on the Hadamard product of the given comonoid and monoid.

Enriched categories are recalled in Appendix B.1.5 and powers and copowers in Appendix B.3.

8.7.1. Enrichment of the category of monoids over comonoids. For any monoids m, n, p, we have a morphism of comonoids

$$\overline{\mathcal{C}}(\mathsf{n}, \mathsf{p}) \times \overline{\mathcal{C}}(\mathsf{m}, \mathsf{n}) \to \overline{\mathcal{C}}(\mathsf{m}, \mathsf{p}).$$

On the A-component, it sends a pair of families $((g_{F/A}), (f_{F/A}))$ to the family $((gf)_{F/A})$. Similarly, for any monoid a, we have a morphism of comonoids

$$\mathsf{E} \to \overline{\mathcal{C}}(\mathsf{a}, \mathsf{a})$$

which on the A-component sends the basis element of $\mathsf{E}[A]$ to the family consisting of identity maps. These morphisms satisfy appropriate associativity and unitality axioms. Thus, the category of monoids is enriched over the monoidal category of comonoids under the Hadamard product. In particular:

Lemma 8.63. *Let* a *be a monoid. Then* $\overline{\mathcal{C}}(\mathsf{a},\mathsf{a})$ *is a comonoid, and, in addition, it is a monoid wrt the Hadamard product on the category of comonoids. In other words,* $\overline{\mathcal{C}}(\mathsf{a},\mathsf{a})$ *is a bialgebra over the bimonad* $(\mathcal{L},\mathcal{T}^\vee)$.

The bimonad $(\mathcal{L},\mathcal{T}^\vee)$ is as in Theorem 8.11. Related information is given in Example 8.14.

Exercise 8.64. Use Exercise 8.16, Exercise B.3 and the discussion in Appendix B.3.3 to formally deduce that the category of monoids is enriched over the monoidal category of comonoids (under the Hadamard product).

8.7.2. Power. The power for the enriched category of monoids can be described using the convolution monoid.

Proposition 8.65. *For a comonoid* c *and monoids* a *and* b*, there is a natural identification of comonoids*

$$\overline{\mathcal{C}}(\mathsf{a}, \mathrm{hom}^\times(\mathsf{c},\mathsf{b})) \cong \mathcal{C}(\mathsf{c}, \overline{\mathcal{C}}(\mathsf{a},\mathsf{b})).$$

In other words, $\mathrm{hom}^\times(\mathsf{c},\mathsf{b})$ *is the power of* b *by* c *for the category of monoids enriched over the category of comonoids.*

The identification above is done in a manner similar to the bijection in Proposition 8.47. In fact, the latter can be deduced as a consequence of the above, see Exercise 9.137.

8.7.3. Copower. We now proceed to describe the copower for the enriched category of monoids. For any comonoid c and monoid a, consider the following monoid quotient of $\mathcal{T}(\mathsf{c}\times\mathsf{a})$. Its A-component is the quotient of $\mathcal{T}(\mathsf{c}\times\mathsf{a})[A]$ by the linear span of the elements given below.

For any $A \le F \le G$, and for any $x \in \mathsf{c}[F]$ and $y \in \mathsf{a}[G]$, consider

$$x \otimes \mu_F^G(y) - \Delta_F^G(x) \otimes y. \tag{8.58}$$

This connects an element of $\mathsf{c}[F] \otimes \mathsf{a}[F]$ to an element of $\mathsf{c}[G] \otimes \mathsf{a}[G]$.

Denote the quotient monoid by $\mathsf{c} \triangleright \mathsf{a}$. Under an additional surjectivity hypothesis, the relations (8.58) simplify as follows.

Lemma 8.66. *If the coproduct components of* c *are surjective, then the composite* $\mathsf{c}\times\mathsf{a} \hookrightarrow \mathcal{T}(\mathsf{c}\times\mathsf{a}) \twoheadrightarrow \mathsf{c}\triangleright\mathsf{a}$ *is surjective, and the kernel is the linear span of the elements*

$$x \otimes \mu_A^F(y), \tag{8.59}$$

where $y \in \mathsf{a}[F]$*, and* $x \in \mathsf{c}[A]$ *is such that* $\Delta_A^F(x) = 0$.

PROOF. Surjectivity is straightforward. To compute the kernel, the main observation is that (8.58) can be broken as a sum of two relations:

$$x \otimes \mu_F^G(y) - \Delta_F^G(x) \otimes y \\ = (-x' \otimes \mu_A^G(y) + x \otimes \mu_F^G(y)) + (x' \otimes \mu_A^G(y) - \Delta_F^G(x) \otimes y),$$

where $x' \in \mathsf{c}[A]$ is chosen such that $\Delta_A^F(x') = x$. It follows that if a linear combination of (8.58) belongs to $\mathsf{c}[A] \otimes \mathsf{a}[A]$, then it must be a linear combination of (8.59), as required. $\square$

Proposition 8.67. *For a comonoid* c *and monoids* a *and* b*, there is a natural identification of comonoids*

$$\overline{\mathcal{C}}(\mathsf{c} \triangleright \mathsf{a}, \mathsf{b}) \cong \mathcal{C}(\mathsf{c}, \overline{\mathcal{C}}(\mathsf{a}, \mathsf{b})).$$

In other words, $\mathsf{c} \triangleright \mathsf{a}$ *is the copower of* a *by* c *for the category of monoids enriched over the category of comonoids.*

The proof is similar to that of Proposition 8.65.

Proposition 8.68. *For a comonoid* c *and monoids* a *and* b*, there is a natural bijection*

$$\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c} \triangleright \mathsf{a}, \mathsf{b}) \cong \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})\big(\mathsf{c}, \overline{\mathcal{C}}(\mathsf{a}, \mathsf{b})\big).$$

This can be checked directly. It can also be deduced as a consequence of Proposition 8.67, see Exercise 9.137.

Example 8.69. For a monoid a and trivial comonoid c,

$$\mathsf{c} \triangleright \mathsf{a} = \mathcal{T}(\mathsf{c} \times \mathcal{Q}(\mathsf{a})),$$

the free monoid on the species $\mathsf{c} \times \mathcal{Q}(\mathsf{a})$. This follows by specializing the relations (8.58). The content of Propositions 8.47 and 8.68 in this case is elaborated below.

For monoids a and b and a trivial comonoid c, the following pieces of data are equivalent.

- A monoid morphism $\mathcal{T}(\mathsf{c} \times \mathcal{Q}(\mathsf{a})) \to \mathsf{b}$.
- A map of species $\mathsf{c} \times \mathcal{Q}(\mathsf{a}) \to \mathsf{b}$.
- A monoid morphism, or equivalently, a derivation $\mathsf{a} \to \mathrm{hom}^{\times}(\mathsf{c}, \mathsf{b})$.
- A comonoid morphism, or equivalently, a coderivation $\mathsf{c} \to \overline{\mathcal{C}}(\mathsf{a}, \mathsf{b})$.

For the third item, we note that $\mathrm{hom}^{\times}(\mathsf{c}, \mathsf{b})$ is a trivial monoid, so a monoid morphism into it is the same as a derivation by Lemma 5.33 and hence it factors through $\mathcal{Q}(\mathsf{a})$. A similar comment applies to the fourth item. Here $\mathcal{Q}(\mathsf{a})$ emerges when we consider the primitive part of $\overline{\mathcal{C}}(\mathsf{a}, \mathsf{b})$ and use (8.46).

Example 8.70. For a comonoid c and species p,

$$\mathsf{c} \triangleright \mathcal{T}(\mathsf{p}) = \mathcal{T}(\mathsf{c} \times \mathsf{p}),$$

the free monoid on the species $\mathsf{c} \times \mathsf{p}$. This can be checked directly. The content of Propositions 8.47 and 8.68 in this case is elaborated below.

For a comonoid c, monoid b, species p, the following pieces of data are equivalent.

- A monoid morphism $\mathcal{T}(\mathsf{c} \times \mathsf{p}) \to \mathsf{b}$.
- A map of species $\mathsf{c} \times \mathsf{p} \to \mathsf{b}$, or $\mathsf{p} \to \hom^{\times}(\mathsf{c}, \mathsf{b})$, or $\mathsf{c} \to \hom^{\times}(\mathsf{p}, \mathsf{b})$.
- A monoid morphism $\mathcal{T}(\mathsf{p}) \to \hom^{\times}(\mathsf{c}, \mathsf{b})$.
- A comonoid morphism $\mathsf{c} \to \mathcal{T}^{\vee}(\hom^{\times}(\mathsf{p}, \mathsf{b}))$.

For the last item, see Exercise 8.52.

8.7.4. A colax-lax adjunction. We now provide an alternative perspective on the copower $\mathsf{c} \triangleright \mathsf{a}$ of a monoid a by a comonoid c. It involves the theory of colax-lax adjunctions and mates from Appendix C.3.2.

Fix a comonoid c. Consider the functor

$$\mathcal{A}\text{-}\mathsf{Sp} \to \mathcal{A}\text{-}\mathsf{Sp}, \qquad \mathsf{p} \mapsto \mathsf{c} \times \mathsf{p}.$$

We denote it by $\mathsf{c} \times (-)$. It carries a colax structure wrt the monad $\mathcal{T}$ as follows. For any species p, on the A-component, the colax structure map

$$\mathsf{c}[A] \otimes \bigoplus_{F\colon F \geq A} \mathsf{p}[F] \to \bigoplus_{F\colon F \geq A} \mathsf{c}[F] \otimes \mathsf{p}[F],$$

on the F-summand, is given by $\Delta_A^F \otimes \mathrm{id}$. It is straightforward to check that diagrams (C.18) commute.

The existence of the internal hom for the Hadamard product (Proposition 8.12) says that $\mathsf{c} \times (-)$ has a right adjoint given by $\hom^{\times}(\mathsf{c}, -)$. It carries a lax structure wrt the monad $\mathcal{T}$ given by specializing (C.26). Explicitly: For any species p, on the A-component, the lax structure map

$$\bigoplus_{F\colon F \geq A} \mathrm{Hom}_{\Bbbk}(\mathsf{c}[F], \mathsf{p}[F]) \to \mathrm{Hom}_{\Bbbk}(\mathsf{c}[A], \bigoplus_{F\colon F \geq A} \mathsf{p}[F]),$$

on the F-summand, sends f to $f\Delta_A^F$.

To summarize:

Proposition 8.71. *For a comonoid* c*, the functor* $\mathsf{c} \times (-)$ *is colax wrt the monad* $\mathcal{T}$*, the functor* $\hom^{\times}(\mathsf{c}, -)$ *is lax wrt the monad* $\mathcal{T}$*, and the adjunction*

$$\mathcal{A}\text{-}\mathsf{Sp} \underset{\hom^{\times}(\mathsf{c},-)}{\overset{\mathsf{c}\times(-)}{\rightleftarrows}} \mathcal{A}\text{-}\mathsf{Sp}$$

is colax-lax.

One can lift this adjunction to the category of monoids. The key observation is that the coequalizer of the pair of maps (C.29) is precisely $\mathsf{c} \triangleright \mathsf{a}$. Note very carefully how the relations (8.58) emerge. As a consequence:

Proposition 8.72. *For a comonoid* c*, the functor* $\mathsf{c} \triangleright (-)$ *is the left adjoint of the functor* $\hom^{\times}(\mathsf{c}, -)$ *on the category of monoids, that is, for any monoids* a *and* b*, there is a natural bijection*

$$\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c} \triangleright \mathsf{a}, \mathsf{b}) \xrightarrow{\cong} \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{a}, \hom^{\times}(\mathsf{c}, \mathsf{b})).$$

This result also follows by combining Propositions 8.47 and 8.68.

Exercise 8.73. Check directly that diagrams (C.28) commute for $\mathcal{V} = \mathcal{V}' := \mathcal{T}$, $\mathcal{F} := \mathsf{c} \times (-)$, $\mathcal{G} := \hom^{\times}(\mathsf{c}, -)$.

The theory also yields the following. For any comonoid c, we have commutative diagrams of functors

$$
(8.60)\qquad
\begin{array}{ccc}
\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp}) & \xrightarrow{\mathsf{c}\,\triangleright(-)} & \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp}) \\
{\scriptstyle \mathcal{T}}\uparrow & & \uparrow{\scriptstyle \mathcal{T}} \\
\mathcal{A}\text{-}\mathsf{Sp} & \xrightarrow[\mathsf{c}\times(-)]{} & \mathcal{A}\text{-}\mathsf{Sp}
\end{array}
\qquad
\begin{array}{ccc}
\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp}) & \xrightarrow{\mathrm{hom}^{\times}(\mathsf{c},-)} & \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp}) \\
{\scriptstyle frg}\downarrow & & \downarrow{\scriptstyle frg} \\
\mathcal{A}\text{-}\mathsf{Sp} & \xrightarrow[\mathrm{hom}^{\times}(\mathsf{c},-)]{} & \mathcal{A}\text{-}\mathsf{Sp}.
\end{array}
$$

One can see this directly as follows. The diagram on the right clearly commutes, and hence so does the one on the left (by taking left adjoints). In particular, this recovers the result discussed in Example 8.70.

Exercise 8.74. Show that: For a finite-dimensional comonoid c and monoid a, we have an isomorphism of monoids

$$\mathsf{c} \triangleright \mathsf{a} \cong \mathcal{C}(\mathsf{c}, \mathsf{a}^*)^*.$$

8.8. Internal hom for monoids and bimonoids

We have constructed the internal hom for the Hadamard product on species, on comonoids, on cocommutative comonoids in Sections 8.2 and 8.4. The internal homs for the Hadamard product on monoids and bimonoids can be constructed using similar ideas. Moreover, these constructions can be understood in a unified manner by viewing the categories of (co, bi)monoids as functor categories as in Section 2.11. The abstract result for functor categories is stated in Theorem B.14 and its proof spells out the general construction.

8.8.1. Internal hom for commutative monoids. Let a and b be commutative monoids. Analogous to the cocommutative case in Section 8.4.4, we construct a commutative monoid $\mathcal{M}^{\mathsf{co}}(\mathsf{a}, \mathsf{b})$. We work with the formulation provided by Proposition 2.20. An element of $\mathcal{M}^{\mathsf{co}}(\mathsf{a}, \mathsf{b})[\mathrm{Z}]$ is a family of linear maps

$$f_{\mathrm{Z}/\mathrm{X}} : \mathsf{a}[\mathrm{X}] \to \mathsf{b}[\mathrm{X}],$$

one for each $\mathrm{X} \leq \mathrm{Z}$, such that the diagram

$$
(8.61)\qquad
\begin{array}{ccc}
\mathsf{a}[\mathrm{Y}] & \xrightarrow{f_{\mathrm{Z}/\mathrm{Y}}} & \mathsf{b}[\mathrm{Y}] \\
{\scriptstyle \mu^{\mathrm{Y}}_{\mathrm{X}}}\downarrow & & \downarrow{\scriptstyle \mu^{\mathrm{Y}}_{\mathrm{X}}} \\
\mathsf{a}[\mathrm{X}] & \xrightarrow[f_{\mathrm{Z}/\mathrm{X}}]{} & \mathsf{b}[\mathrm{X}]
\end{array}
$$

commutes for each $\mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}$. This defines the species $\mathcal{M}^{\mathsf{co}}(\mathsf{a}, \mathsf{b})$. It is a commutative monoid with structure maps given by restriction. The functor which sends (a, b) to $\mathcal{M}^{\mathsf{co}}(\mathsf{a}, \mathsf{b})$ is the internal hom for the Hadamard product of commutative monoids.

8.8.2. Internal hom for monoids. Contrary to what one may naively expect, the situation for monoids is more complicated than the one for comonoids in Section 8.4.1. To deal with monoids, we make use of the viewpoint provided by Proposition 2.60: A monoid is the same as a functor from the category $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathsf{d}}$ to the category Vec. Recall that a morphism in $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathsf{d}}$ is a triple (A, F', F) of faces, where $\mathrm{s}(A) = \mathrm{s}(F')$ and $F \leq F'$. (This particular morphism is from A to F.) In the present discussion, we will call this an m-triple.

Let a and b be monoids. We proceed to construct a monoid $\mathcal{M}(\mathsf{a}, \mathsf{b})$. An element of $\mathcal{M}(\mathsf{a}, \mathsf{b})[A]$ is a family of linear maps

$$f_{(A,F',F)} : \mathsf{a}[F] \to \mathsf{b}[F],$$

one for each m-triple (A, F', F), such that for any m-triples (A, F', F) and (F, G', G), the diagram

$$\begin{tikzcd} \mathsf{a}[F] \arrow[r, "f_{(A,F',F)}"] \arrow[d, "(F,G',G)"'] & \mathsf{b}[F] \arrow[d, "(F,G',G)"] \\ \mathsf{a}[G] \arrow[r, "f_{(A,G'F',G)}"'] & \mathsf{b}[G] \end{tikzcd} \tag{8.62}$$

commutes. (Note here that $(A, G'F', G)$ is the composite of (A, F', F) and (F, G', G) in $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathsf{d}}$.) Such families can be added and scalar-multiplied componentwise, so $\mathcal{M}(\mathsf{a}, \mathsf{b})[A]$ is a vector space. Further, to any m-triple (A, F', F), we can associate the linear map

$$\mathcal{M}(\mathsf{a}, \mathsf{b})[A] \xrightarrow{(A,F',F)} \mathcal{M}(\mathsf{a}, \mathsf{b})[F]$$

which sends a family f to the family g defined by

$$g_{(F,G',G)} := f_{(A,G'F',G)}. \tag{8.63}$$

This turns $\mathcal{M}(\mathsf{a}, \mathsf{b})$ into a monoid. This is the internal hom in the monoidal category of monoids wrt the Hadamard product. In other words:

Proposition 8.75. *For any monoids* a, m, n, *there is a natural bijection*

$$\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{a} \times \mathsf{m}, \mathsf{n}) \cong \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})\big(\mathsf{a}, \mathcal{M}(\mathsf{m}, \mathsf{n})\big).$$

Proof. We first go in the forward direction. Suppose we are given a morphism of monoids $h : \mathsf{a} \times \mathsf{m} \to \mathsf{n}$. This entails a linear map

$$h_A : \mathsf{a}[A] \otimes \mathsf{m}[A] \to \mathsf{n}[A],$$

one for each face A, such that for each m-triple (F, G', G) the diagram

$$\begin{tikzcd} \mathsf{a}[F] \otimes \mathsf{m}[F] \arrow[r, "h_F"] \arrow[d, "(F,G',G)\otimes(F,G',G)"'] & \mathsf{n}[F] \arrow[d, "(F,G',G)"] \\ \mathsf{a}[G] \otimes \mathsf{m}[G] \arrow[r, "h_G"'] & \mathsf{n}[G] \end{tikzcd}$$

commutes.

For each A and $x \in \mathsf{a}[A]$, define the element $(f_{(A,F',F)})$ of $\mathcal{M}(\mathsf{m},\mathsf{n})[A]$ as follows. The linear map $f_{(A,F',F)}$ is the image of x under the composite

$$\mathsf{a}[A] \xrightarrow{(A,F',F)} \mathsf{a}[F] \longrightarrow \mathrm{Hom}_{\Bbbk}(\mathsf{m}[F],\mathsf{n}[F]), \qquad x \mapsto f_{(A,F',F)}.$$

(The second map is induced by h_F.) The above commutative diagram implies that (8.62) commutes, so $(f_{(A,F',F)})$ is indeed an element of $\mathcal{M}(\mathsf{m},\mathsf{n})[A]$. One can further check that this gives rise to a morphism of monoids $\mathsf{a} \to \mathcal{M}(\mathsf{m},\mathsf{n})$.

In the other direction, suppose we are given such a morphism of monoids. Thus, for each A and $x \in \mathsf{a}[A]$, we have a family $(f_{(A,F',F)})$. We define h_A as $h_A(x \otimes y) := f_{(A,A,A)}(y)$, and then check that the above diagram commutes.

Finally, one may check that the two maps are inverse to each other. $\square$

Remark 8.76. The construction of the internal hom for monoids is more complicated than the one for comonoids. For monoids, defining the A-component to consist of families indexed by faces smaller than A, runs into an immediate difficulty. Namely, for faces A and B of the same support, there is no bijection in general between faces smaller than A and faces smaller than B; so it is not even clear how to turn this candidate into a species.

Exercise 8.77. Consider the largest commutative submonoid of $\mathcal{M}(\mathsf{a},\mathsf{b})$ as in Exercise 2.44. Check that its A-component consists of those families $(f_{(A,F',F)})$ indexed by m-triples (with first entry A) which satisfy $f_{(A,F',F)} = f_{(A,F'',F)}$, thus the families can be written more simply as $(f_{(A,F)})$ with $\mathrm{s}(F) \leq \mathrm{s}(A)$.

When a and b are commutative, this submonoid coincides with $\mathcal{M}^{\mathsf{co}}(\mathsf{a},\mathsf{b})$ by general principles. Check directly that the above description is equivalent to the one in Section 8.8.1.

8.8.3. Internal hom for cocommutative bimonoids. Observe that in the construction of the internal hom for the category of monoids, we made use of the fact that it was a functor category. This construction can be adapted to the other functor categories in Section 2.11, namely, replace m-triples by morphisms in the corresponding base category.

We elaborate on the case of cocommutative bimonoids. In this case, the relevant triples are (F,G',G) such that $\mathrm{s}(F) \leq \mathrm{s}(G')$ and $G \leq G'$. Let us call them cob-triples. Their composition rule is given in (2.80).

Let h and k be cocommutative bimonoids. Denote their internal hom by ${}^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k})$. An element of ${}^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k})[A]$ is a family of linear maps

$$f_{(A,F',F)} : \mathsf{h}[F] \to \mathsf{k}[F],$$

one for each cob-triple (A,F',F), such that for any cob-triples (A,F',F) and (F,G',G), the diagram

$$\begin{tikzcd} \mathsf{h}[F] \arrow[r, "f_{(A,F',F)}"] \arrow[d, "(F,G',G)"'] & \mathsf{k}[F] \arrow[d, "(F,G',G)"] \\ \mathsf{h}[G] \arrow[r, "f_{(A,G'F',G)}"'] & \mathsf{k}[G] \end{tikzcd} \tag{8.64}$$

commutes. To any cob-triple (A, F', F), we can associate the linear map

$$^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k})[A] \xrightarrow{(A,F',F)} {}^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k})[F]$$

which sends a family f to the family g defined by

$$g_{(F,G',G)} := f_{(A,G'F',G)}. \tag{8.65}$$

This turns $^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k})$ into a cocommutative bimonoid.

Exercise 8.78. Construct the internal hom for the category of comonoids by working with c-triples which are the morphisms in $\mathcal{A}\text{-}\mathtt{Hyp}_{\mathsf{c}}$. Why does this reduce to the previous construction in Section 8.4.1 (which was somewhat simpler)?

Exercise 8.79. Check that the internal hom for the category of cocommutative comonoids and commutative monoids constructed using the above method (the starting point being Proposition 2.65) coincides with the previous descriptions in Sections 8.4.4 and 8.8.1.

8.8.4. Comparison between internal homs. The inclusion and forgetful functors relating the various functor categories yield morphisms between the corresponding internal homs. For instance, the inclusion functor from cocommutative comonoids to comonoids yields a morphism of comonoids

$$^{\mathsf{co}}\mathcal{C}(\mathsf{c},\mathsf{d}) \to \mathcal{C}(\mathsf{c},\mathsf{d}).$$

(This is in fact the coabelianization map in view of Exercise 8.29.) Similarly, the forgetful functor from cocommutative bimonoids to monoids yields a morphism of monoids

$$^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k}) \to \mathcal{M}(\mathsf{h},\mathsf{k}), \tag{8.66}$$

and to comonoids yields a morphism of comonoids

$$^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k}) \to \mathcal{C}(\mathsf{h},\mathsf{k}), \tag{8.67}$$

and so on.

A more abstract explanation for the existence of these maps is as follows. The inclusion and forgetful functors relating the various functor categories are all lax monoidal functors wrt the Hadamard product. (In fact, they are strong monoidal functors.) So by (B.7), we obtain maps between the corresponding internal homs.

We now describe the map (8.66) explicitly. An element of $^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k})$ is a family indexed by cob-triples. Every m-triple is a cob-triple. So by restricting the indexing set to m-triples, we obtain an element of $\mathcal{M}(\mathsf{h},\mathsf{k})$. This is the required map.

For the map (8.67), a family $f \in {}^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k})[A]$ maps to $g \in \mathcal{C}(\mathsf{h},\mathsf{k})[A]$ with $g_{(A,F',F)} := f_{(A,F,F)}$. Alternatively, viewing $\mathcal{C}(\mathsf{h},\mathsf{k})$ as a subbimonoid of $\mathrm{hom}^{\times}(\mathsf{h},\mathsf{k})$ as in Lemma 8.36, $f \in {}^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k})[A]$ maps to $f_{(A,A,A)} \in \mathcal{C}(\mathsf{h},\mathsf{k})[A]$.

Exercise 8.80. Check that: An element $f \in {}^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k})[A]$ is primitive iff $f_{(A,F',F)} = 0$ for all $\mathrm{s}(F') > \mathrm{s}(A)$. Then restricting (8.66) to the primitive part yields an injective map of species

$$\mathcal{P}({}^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k})) \hookrightarrow \mathcal{M}(\mathsf{h},\mathsf{k}). \tag{8.68}$$

Its image on the A-component consists of those families $(f_{(A,F',F)})$ indexed by m-triples (with first entry A) for which

$$\begin{tikzcd} \mathsf{h}[F] \arrow[r, "f_{(A,F',F)}"] \arrow[d, "(F{,}G'{,}G)"'] & \mathsf{k}[F] \arrow[d, "(F{,}G'{,}G)"] \\ \mathsf{h}[G] \arrow[r, "g"'] & \mathsf{k}[G] \end{tikzcd} \tag{8.69}$$

commutes. Here (F, G', G) is any cob-triple, and $g = f_{(A,G'F',G)}$ if $\mathrm{s}(G'F') = \mathrm{s}(A)$ and 0 otherwise.

Exercise 8.81. Give a similar description for the primitive part of $\mathcal{B}^{\mathsf{co}}(\mathsf{h},\mathsf{k})$, the internal hom in the category of commutative bimonoids.

We now discuss some morphisms among the internal homs which do not arise from inclusion or forgetful functors.

Let h and k be bimonoids. Consider the evaluation map

$$\mathrm{hom}^{\times}(\mathsf{h},\mathsf{k}) \times \mathsf{h} \to \mathsf{k}, \qquad f \otimes x \mapsto f(x).$$

This is a morphism of monoids (but not of comonoids in general). This uses (2.15). The universal property of internal hom then yields a morphism of monoids

$$\mathrm{hom}^{\times}(\mathsf{h},\mathsf{k}) \to \mathcal{M}(\mathsf{h},\mathsf{k}). \tag{8.70}$$

Now let h and k be cocommutative bimonoids. View the bimonoid of star families $\mathcal{C}(\mathsf{h},\mathsf{k})$ as a subbimonoid of $\mathrm{hom}^{\times}(\mathsf{h},\mathsf{k})$ as in Lemma 8.36. By restricting the above evaluation map, we obtain a morphism of monoids

$$\mathcal{C}(\mathsf{h},\mathsf{k}) \times \mathsf{h} \to \mathsf{k}.$$

Since $\mathcal{C}(\mathsf{h},\mathsf{k})$ is the internal hom for comonoids, this map is in fact a morphism of bimonoids. Hence, we obtain a morphism of bimonoids

$$\mathcal{C}(\mathsf{h},\mathsf{k}) \hookrightarrow {}^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k}). \tag{8.71}$$

This map is evidently injective: Explicitly, the image of $f \in \mathcal{C}(\mathsf{h},\mathsf{k})[A]$ is the family

$$f_{(A,F',F)} := (A, F', F)(f) \ \in \mathcal{C}(\mathsf{h},\mathsf{k})[F].$$

In particular, $f_{(A,A,A)} = f$, that is, one can recover f from its image. One can check that (8.71) is a section to (8.67) implying that the latter is surjective.

8.8.5. Internal hom for the tensor product of modules. There is a general construction of the internal hom for the tensor product of modules over a monoid algebra. It is reviewed in Appendix B.5.

The discussion below is to be read in conjunction with Chapter 11 which establishes equivalences between categories of modules over certain monoid algebras and categories related to bimonoids. A summary is given in Table 11.1, see also Exercise 11.9. Thus, specializing the monoid X in Proposition B.19 to the Tits monoid $\Sigma[\mathcal{A}]$ yields the internal hom for cocommutative bimonoids. Similarly, specializing X to the opposite of $\Sigma[\mathcal{A}]$, to the Birkhoff monoid $\Pi[\mathcal{A}]$ and to the Janus monoid $\mathrm{J}[\mathcal{A}]$ yields the internal hom for commutative bimonoids, for bicommutative bimonoids and for bimonoids, respectively. It is instructive to check that in each case this approach yields the same answer as the one obtained by the approach in Section 8.8.3.

Let us elaborate on the case of the Tits monoid. Let h and k be cocommutative bimonoids. Then $\mathsf{h}[O]$, $\mathsf{k}[O]$ and ${}^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k})[O]$ are all $\Sigma[\mathcal{A}]$-modules with the face F acting by $\mu_O^F \Delta_O^F$ as in the proof of Proposition 11.1. We also have the $\Sigma[\mathcal{A}]$-module

$$\hom^{\otimes}(\mathsf{h}[O],\mathsf{k}[O])$$

constructed in Appendix B.5.3. Explicitly, an element $a \in \hom^{\otimes}(\mathsf{h}[O],\mathsf{k}[O])$ is a family of linear maps

$$a_K : \mathsf{h}[O] \to \mathsf{k}[O],$$

one for each face K, satisfying condition

$$a_{FK}(F \cdot m) = F \cdot a_K(m) \tag{8.72}$$

for any faces F, K, with the action of a face G on a given by

$$(G \cdot a)_K := a_{KG}. \tag{8.73}$$

See (B.27) and (B.28).

Lemma 8.82. *The map* ${}^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k})[O] \to \hom^{\otimes}(\mathsf{h}[O],\mathsf{k}[O])$ *which sends a family* f *to the family* a *defined by* $a_K := f_{O,K,O}$ *is an isomorphism of* $\Sigma[\mathcal{A}]$-*modules.*

Proof. We first check below that the map is well-defined, that is, the family a defined by $a_K := f_{O,K,O}$ satisfies condition (8.72).

$$\begin{tikzcd}
\mathsf{h}[O] \arrow[r, "\Delta_O^F=(O,F,F)"] \arrow[d, "a_K=f_{O,K,O}"'] & \mathsf{h}[F] \arrow[r, "\mu_O^F=(F,F,O)"] \arrow[d, "f_{O,FK,F}"] & \mathsf{h}[O] \arrow[d, "f_{O,FK,O}=a_{FK}"] \\
\mathsf{k}[O] \arrow[r, "\Delta_O^F=(O,F,F)"'] & \mathsf{k}[F] \arrow[r, "\mu_O^F=(F,F,O)"'] & \mathsf{k}[O]
\end{tikzcd}$$

Both squares commute since they are an instance of condition (8.64). Also, the horizontal composites $\mu_O^F \Delta_O^F = (O,F,O)$ are the action of F.

The fact that this is a module map follows from (8.65) applied to the cobtriple (O,G,O). To see that it is an isomorphism, we construct the inverse map. Given a family (a_K) satisfying condition (8.72), the outer rectangle in the diagram above commutes, and we define the map $f_{O,FK,F}$ as the projection of a_K and restriction of a_{FK}. We check below that f satisfies condition (8.64) with $A = O$.

$$\begin{tikzcd}
\mathsf{h}[O] \arrow[r, "\Delta_O^F=(O,F,F)"] \arrow[d, "a_{F'}=f_{O,F',O}"'] & \mathsf{h}[F] \arrow[r, "(F,G',G)"] \arrow[d, "f_{O,F',F}"] & \mathsf{h}[G] \arrow[r, "\mu_O^G=(G,G,O)"] \arrow[d, "f_{O,G'F',G}"] & \mathsf{h}[O] \arrow[d, "f_{O,G'F',O}=a_{G'F'}"] \\
\mathsf{k}[O] \arrow[r, "\Delta_O^F=(O,F,F)"'] & \mathsf{k}[F] \arrow[r, "(F,G',G)"'] & \mathsf{k}[G] \arrow[r, "\mu_O^G=(G,G,O)"'] & \mathsf{k}[O]
\end{tikzcd}$$

The outer rectangle commutes since it is an instance of condition (8.72), and the two side squares commute by definition of f, so the central square commutes (in view of surjectivity of Δ_O^F and injectivity of μ_O^F). $\square$

Lemma 8.82 can be viewed as giving a simplified description of the space ${}^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k})[O]$. A similar result related to Exercise 8.80 is given below.

Lemma 8.83. *An element of $\mathcal{P}({}^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k}))[A]$ is the same as a family of linear maps*

$$a_K : \mathsf{h}[O] \to \mathsf{k}[O],$$

one for each face K of the same support as A, such that

$$F \cdot a_K(m) = \begin{cases} a_{FK}(F \cdot m) & \text{if } \mathrm{s}(F) \leq \mathrm{s}(A), \\ 0 & \text{otherwise,} \end{cases} \tag{8.74}$$

for any face F and $m \in \mathsf{h}[O]$.

Explicitly, for an element $(f_{(A,F',F)})$ in the primitive part of ${}^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k})[A]$, we let $a_K := f_{(A,K,O)}$ for each face K of the same support as A.

Proof. We first check condition (8.74). The first alternative works as follows.

$$\begin{tikzcd}
\mathsf{h}[O] \arrow[r, "\Delta_O^F=(O,F,F)"] \arrow[d, "a_K=f_{A,K,O}"'] & \mathsf{h}[F] \arrow[r, "\mu_O^F=(F,F,O)"] \arrow[d, "f_{A,FK,F}"] & \mathsf{h}[O] \arrow[d, "f_{A,FK,O}=a_{FK}"] \\
\mathsf{k}[O] \arrow[r, "\Delta_O^F=(O,F,F)"'] & \mathsf{k}[F] \arrow[r, "\mu_O^F=(F,F,O)"'] & \mathsf{k}[O]
\end{tikzcd}$$

Both squares commute since they are an instance of condition (8.69). Also, the horizontal composites $\mu_O^F \Delta_O^F = (O,F,O)$ are the action of F. The second alternative works similarly, with the second and third vertical arrows being the zero map.

Conversely, suppose we are given a family (a_K) satisfying condition (8.74). Then we define $f_{A,FK,F}$ using the middle vertical arrow in the above diagram. It remains to check that f satisfies condition (8.69). The alternative $\mathrm{s}(G'F') =$

$\mathrm{s}(A)$ works as follows.

$$\begin{array}{ccccccc} \mathsf{h}[O] & \xrightarrow{\Delta_O^F=(O,F,F)} & \mathsf{h}[F] & \xrightarrow{(F,G',G)} & \mathsf{h}[G] & \xrightarrow{\mu_O^G=(G,G,O)} & \mathsf{h}[O] \\ {\scriptstyle a_{F'}=f_{A,F',O}}\big\downarrow & & \big\downarrow{\scriptstyle f_{A,F',F}} & & \big\downarrow{\scriptstyle f_{A,G'F',G}} & & \big\downarrow{\scriptstyle f_{A,G'F',O}=a_{G'F'}} \\ \mathsf{k}[O] & \xrightarrow[\Delta_O^F=(O,F,F)]{} & \mathsf{k}[F] & \xrightarrow[(F,G',G)]{} & \mathsf{k}[G] & \xrightarrow[\mu_O^G=(G,G,O)]{} & \mathsf{k}[O] \end{array}$$

The outer rectangle commutes since it is an instance of condition (8.74), and the two side squares commute by definition of f, so the central square commutes (in view of surjectivity of Δ_O^F and injectivity of μ_O^F). The alternative $\mathrm{s}(G'F') > \mathrm{s}(A)$ works similarly with the last two vertical arrows being the zero map. $\square$

The morphism (8.71) induces an injective map on the primitive part:

$$\mathcal{P}(\mathcal{C}(\mathsf{h},\mathsf{k})) \hookrightarrow \mathcal{P}({}^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k})). \tag{8.75}$$

A description of the former is given by (8.30), and of the latter by Lemma 8.83. The exercise below describes this map in these terms.

Exercise 8.84. Let $f : \mathsf{h}[A] \to \mathcal{P}(\mathsf{k})[A]$. This is an element of $\mathcal{P}(\mathcal{C}(\mathsf{h},\mathsf{k}))[A]$. For any K with the same support as A, define

$$a_K := \mu_O^K \beta_{K,A} f \Delta_O^A.$$

This is a family of maps from $\mathsf{h}[O]$ to $\mathsf{k}[O]$. Check that it is the image of f under (8.75) on the A-component. Also check directly that it satisfies condition (8.74).

Corollary 8.85. *For finite-dimensional cocommutative bimonoids* h *and* k,

$$\dim(\mathsf{h}[A])\dim(\mathcal{P}(\mathsf{k})[A])$$

is a lower bound on the dimensions of both $\mathcal{P}({}^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k}))[A]$ *and* $\mathcal{M}(\mathsf{h},\mathsf{k})[A]$.

Proof. The above formula is the dimension of $\mathcal{P}(\mathcal{C}(\mathsf{h},\mathsf{k}))[A]$. The first claim now follows from injectivity of (8.75) and the second from injectivity of (8.68). $\square$

Let ${}^{\mathsf{co}}\mathcal{B}^{\mathsf{co}}(\mathsf{h},\mathsf{k})$ denote the internal hom in the category of bicommutative bimonoids. The analogue of Lemma 8.83 is stated below.

Lemma 8.86. *An element of* $\mathcal{P}({}^{\mathsf{co}}\mathcal{B}^{\mathsf{co}}(\mathsf{h},\mathsf{k}))[\mathrm{Z}]$ *is the same as a linear map* $a : \mathsf{h}[\bot] \to \mathsf{k}[\bot]$ *such that*

$$\mathrm{Y} \cdot a(m) = \begin{cases} a(\mathrm{Y} \cdot m) & \text{if } \mathrm{Y} \leq \mathrm{Z}, \\ 0 & \text{otherwise,} \end{cases} \tag{8.76}$$

for any flat Y *and* $m \in \mathsf{h}[\bot]$.

Let us look at the extreme cases. When $\mathrm{Z} = \bot$, a primitive element in the Z-component is precisely a linear map $\mathsf{h}[\bot] \to \mathcal{P}(\mathsf{k})[\bot]$. When $\mathrm{Z} = \top$, a primitive element in the Z-component is precisely a map $\mathsf{h}[\bot] \to \mathsf{k}[\bot]$ of $\Pi[\mathcal{A}]$-modules.

When h and k are both bicommutative, so are $\mathrm{hom}^{\times}(\mathsf{h},\mathsf{k})$ and $\mathcal{C}(\mathsf{h},\mathsf{k})$, and the map (8.71) factors as follows.

$$\begin{tikzcd} \mathcal{C}(\mathsf{h},\mathsf{k}) \arrow[rr] \arrow[dr, hook] & & {}^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k}) \\ & {}^{\mathsf{co}}\mathcal{B}^{\mathsf{co}}(\mathsf{h},\mathsf{k}) \arrow[ur, hook] & \end{tikzcd}$$

As a consequence:

Corollary 8.87. *For finite-dimensional bicommutative bimonoids* h *and* k*,*

$$\dim(\mathsf{h}[\mathrm{Z}])\dim(\mathcal{P}(\mathsf{k})[\mathrm{Z}])$$

is a lower bound on the dimension of $\mathcal{P}({}^{\mathsf{co}}\mathcal{B}^{\mathsf{co}}(\mathsf{h},\mathsf{k}))[\mathrm{Z}]$.

8.9. Hadamard product of set-species

Just as for species, the Hadamard product can also be defined for set-species. We touch upon this briefly.

8.9.1. Hadamard product of set-species. The Hadamard product of two set-species p and q is the set-species $\mathrm{p}\times\mathrm{q}$ defined by

$$(\mathrm{p}\times\mathrm{q})[A] := \mathrm{p}[A]\times\mathrm{q}[A], \tag{8.77}$$

where in the rhs, $\times$ denotes the cartesian product of sets.

Set-monoids, set-comonoids, set-bimonoids are all preserved under the Hadamard product. The product and coproduct formulas are given by (8.3) and (8.5), with tensor product replaced by cartesian product.

Exercise 8.88. Define set-theoretic analogues of (8.6), with $\mathcal{T}$ and $\mathcal{T}^{\vee}$ given by (3.36) and (3.37), namely,

$$\varphi : \bigsqcup_{A\leq F} \big(\mathrm{p}[F]\times\mathrm{q}[F]\big) \longrightarrow \bigsqcup_{A\leq F,\, A\leq G} \mathrm{p}[F]\times\mathrm{q}[G],$$

$$\psi : \Big(\bigtimes_{A\leq F}\mathrm{p}[F]\Big)\times\Big(\bigtimes_{A\leq G}\mathrm{p}[G]\Big) \longrightarrow \bigtimes_{A\leq F}\big(\mathrm{p}[F]\times\mathrm{p}[F]\big).$$

The map φ is an inclusion, while ψ is a bijection. Prove the set-theoretic analogue of Theorem 8.4. (We need to assume $p = q = 1$, since one cannot talk of deformation in the context of set-species.) Deduce that set-(co, bi)monoids are preserved under the Hadamard product.

8.9.2. Biconvolution bimonoid. The internal hom for Hadamard product of two set-species p and q is the set-species $\mathrm{p}\times\mathrm{q}$ defined by

$$\mathrm{hom}^{\times}(\mathrm{p},\mathrm{q})[A] := \mathsf{Set}(\mathrm{p}[A],\mathrm{q}[A]), \tag{8.78}$$

where the rhs denotes the set of all functions from $\mathrm{p}[A]$ to $\mathrm{q}[A]$.

Exercise 8.89. Define set-theoretic analogues of (8.25), namely,

$$\varphi : \bigsqcup_{A\leq F}\mathsf{Set}(\mathrm{p}[F],\mathrm{q}[F]) \longrightarrow \mathsf{Set}\Big(\bigtimes_{A\leq F}\mathrm{p}[F],\ \bigsqcup_{A\leq G}\mathrm{q}[G]\Big),$$

$$\psi : \bigtimes_{A \leq F,\, A \leq G} \mathsf{Set}(\mathrm{p}[F], \mathrm{q}[G]) \longrightarrow \bigtimes_{A \leq F} \mathsf{Set}(\mathrm{p}[F], \mathrm{q}[F]),$$

and prove the set-theoretic analogue of Theorem 8.21. Deduce that there are set-theoretic analogues of the convolution monoid, coconvolution comonoid, biconvolution bimonoid. (The product and coproduct are defined in the same manner as before.)

8.9.3. Internal hom for comonoids and universal measuring comonoid. For set-comonoids c and d, the internal hom $\mathcal{C}(\mathrm{c}, \mathrm{d})$ can be constructed as in Section 8.4.1. Similarly, for set-monoids a and b, the universal measuring set-comonoid $\overline{\mathcal{C}}(\mathrm{a}, \mathrm{b})$ can be constructed as in Section 8.6.1. The category of set-monoids is enriched over the monoidal category of set-comonoids under the Hadamard product.

For c a cocommutative set-comonoid and k a set-bimonoid, $\mathcal{C}(\mathrm{c}, \mathrm{k})$ carries the structure of a set-bimonoid. Similarly, for a set-bimonoid h and commutative set-monoid a, $\overline{\mathcal{C}}(\mathrm{h}, \mathrm{a})$ carries the structure of a set-bimonoid.

8.10. Signature functor

The unsigned and signed worlds are linked by the signature functor. It is constructed by taking Hadamard product of the input species with the signed exponential species. The signature functor takes a bimonoid to a signed bimonoid, and vice versa, thus setting up an equivalence between the categories of bimonoids and signed bimonoids. More generally, it takes a q-bimonoid to a $(-q)$-bimonoid. The functors $\mathcal{T}_q$ and $\mathcal{T}_{-q}$ (from species to q-bimonoids and to $(-q)$-bimonoids) are conjugates of each other wrt the signature functor.

Similarly, the signature functor takes a (co)commutative (co)monoid to a signed (co)commutative (co)monoid, and vice versa, thus setting up an equivalence between the categories of (co)commutative (co)monoids and signed (co)commutative (co)monoids. The functors $\mathcal{S}$ and $\mathcal{E}$ (from species to bicommutative bimonoids and to signed bicommutative signed bimonoids) are conjugates of each other wrt the signature functor.

8.10.1. Signature functor. Recall the signed exponential species E^- from Section 7.2.5. It carries the structure of a signed bimonoid. For any species p, define

$$\mathsf{p}^- := \mathsf{p} \times \mathsf{E}^-.$$

Explicitly, on the F-component,

$$\mathsf{p}^-[F] = \mathsf{p}[F] \otimes \mathsf{E}^-[F]. \tag{8.79}$$

We refer to p^- as the *signed partner* of p. Observe that the signed partner of E is E^- as suggested by the notation. This yields a functor

$$(-)^- : \mathcal{A}\text{-}\mathsf{Sp} \to \mathcal{A}\text{-}\mathsf{Sp}, \qquad \mathsf{p} \mapsto \mathsf{p}^-. \tag{8.80}$$

We call this the *signature functor*. It coincides with the functor defined in (4.40).

The signature functor is involutive, that is,

$$(\mathsf{p}^-)^- \cong \mathsf{p}.$$

This is because $\mathsf{E}^- \times \mathsf{E}^- \cong \mathsf{E}$. In fact, we have an adjoint equivalence of categories

$$\mathcal{A}\text{-}\mathsf{Sp} \underset{(-)^-}{\overset{(-)^-}{\rightleftarrows}} \mathcal{A}\text{-}\mathsf{Sp}.$$

8.10.2. Bilax structure. Let us temporarily denote the signature functor by $\mathcal{F}$. We now define natural transformations

$$\varphi : \mathcal{T}\mathcal{F} \to \mathcal{F}\mathcal{T} \quad \text{and} \quad \psi : \mathcal{F}\mathcal{T}^\vee \to \mathcal{T}^\vee\mathcal{F}.$$

Evaluated on a species p, on the A-component, the maps

$$\bigoplus_{F:\, F \geq A} \mathsf{p}[F] \otimes \mathsf{E}^-[F] \rightleftarrows \bigoplus_{F:\, F \geq A} \mathsf{p}[F] \otimes \mathsf{E}^-[A] \tag{8.81}$$

are defined using the product μ_A^F of E^- for φ, and the coproduct Δ_A^F of E^- for ψ. Note that φ and ψ are inverse isomorphisms.

Theorem 8.90. *For any scalar q, the signature functor*

$$(-)^- : (\mathcal{T}, \mathcal{T}^\vee, \lambda_q) \to (\mathcal{T}, \mathcal{T}^\vee, \lambda_{-q}),$$

with structure maps (8.81), is a bilax isomorphism of bimonads (with its inverse being itself).

Proof. The fact that the signature functor is bilax is a special case of Theorem 8.6. Also note that the structure maps (8.7) specialize to (8.81). For the remaining claim, we check that the identification $(\mathsf{p}^-)^- \cong \mathsf{p}$ defines a bilax isomorphism $((-)^-)^- \to \mathrm{id}$. This is straightforward. $\square$

8.10.3. Interaction with bimonoids. Some consequences of Theorem 8.90 are stated below. These can also be checked directly.

Corollary 8.91. *If a is a monoid, then so is a^-. Similarly, if c is a comonoid, then so is c^-.*

Explicitly, the product components of a^- are given by

$$\mathsf{a}[F] \otimes \mathsf{E}^-[F] \xrightarrow{\mu_A^F \otimes \mu_A^F} \mathsf{a}[A] \otimes \mathsf{E}^-[A],$$

where the μ_A^F refer to the product components of a and E^-, respectively. This is a special case of (8.3). The coproduct components of c^- are given in a similar manner as a special case of (8.5).

Corollary 8.92. *Let q be any scalar. If h is a q-bimonoid, then h^- is a $(-q)$-bimonoid. Further,*

$$q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{(-)^-}{\overset{(-)^-}{\rightleftarrows}} (-q)\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$$

is an adjoint equivalence of categories.

In particular:

Corollary 8.93. *If* h *is a bimonoid, then* h^- *is a signed bimonoid, and vice versa. This defines an adjoint equivalence between the categories of bimonoids and signed bimonoids.*

Recall the free monoid $\mathcal{T}(\mathsf{p})$ from Section 6.1.1 and the cofree comonoid $\mathcal{T}^\vee(\mathsf{p})$ from Section 6.2.1.

Corollary 8.94. *For any species* p*, there are isomorphisms*

$$\mathcal{T}(\mathsf{p}^-) \xrightarrow{\cong} \mathcal{T}(\mathsf{p})^- \quad \text{and} \quad \mathcal{T}^\vee(\mathsf{p}^-) \xrightarrow{\cong} \mathcal{T}^\vee(\mathsf{p})^-$$

of monoids and comonoids, respectively.

PROOF. This follows from Proposition C.35. □

Recall the free q-bimonoid $\mathcal{T}_q(\mathsf{c})$ on a comonoid c from Section 6.1.2 and the cofree q-bimonoid $\mathcal{T}_q^\vee(\mathsf{a})$ on a monoid a from Section 6.2.2.

Corollary 8.95. *For any monoid* a *and comonoid* c*, there are isomorphisms of* q*-bimonoids*

$$\mathcal{T}_q(\mathsf{c}^-) \xrightarrow{\cong} \mathcal{T}_{-q}(\mathsf{c})^- \quad \text{and} \quad \mathcal{T}_q^\vee(\mathsf{a}^-) \xrightarrow{\cong} \mathcal{T}_{-q}^\vee(\mathsf{a})^-.$$

PROOF. This follows from Proposition C.36. □

Corollary 8.96. *For any species* p*, there are isomorphisms of* q*-bimonoids*

$$\mathcal{T}_q(\mathsf{p}^-) \xrightarrow{\cong} \mathcal{T}_{-q}(\mathsf{p})^- \quad \text{and} \quad \mathcal{T}_q^\vee(\mathsf{p}^-) \xrightarrow{\cong} \mathcal{T}_{-q}^\vee(\mathsf{p})^-.$$

PROOF. This follows from Corollary 8.95 by viewing p as a trivial (co)monoid. □

In other words, the conjugate of $\mathcal{T}_q$ (as a functor from species to q-bimonoids) wrt the signature functor is $\mathcal{T}_{-q}$.

Exercise 8.97. In Section 7.3, for any scalar q, we defined the q-bimonoid Γ_q of chambers. Check that

$$(\Gamma_q)^- = \Gamma_{-q}$$

as $(-q)$-bimonoids. This justifies our prior use of Γ^- to denote Γ_{-1}. Further check that diagrams (7.30) and (7.32) are related to each other by the signature functor.

8.10.4. Interaction with (co)commutative (co)monoids. Let us now consider the commutative aspects of the signature functor.

For any species p, there is a natural isomorphism

$$\mathcal{E}(\mathsf{p}^-) \xrightarrow{\cong} \mathcal{S}(\mathsf{p})^-, \qquad \text{or equivalently,} \qquad \mathcal{S}(\mathsf{p}^-) \xrightarrow{\cong} \mathcal{E}(\mathsf{p})^-. \tag{8.82}$$

Calculating using definitions,

$$\mathcal{E}(\mathsf{p}^-)[\mathrm{Z}] = \bigoplus_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Z}} \mathbf{E}^-[\mathrm{Z},\mathrm{X}] \otimes \mathsf{p}^-[\mathrm{X}] = \bigoplus_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Z}} \mathbf{E}^-[\mathrm{Z},\mathrm{X}] \otimes \mathbf{E}^-[\mathrm{X},\top] \otimes \mathsf{p}[\mathrm{X}],$$

and

$$\mathcal{S}(\mathsf{p})^-[\mathrm{Z}] = \bigoplus_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Z}} \mathbf{E}^-[\mathrm{Z},\top] \otimes \mathsf{p}[\mathrm{X}].$$

The natural isomorphism is defined using (1.162). We have the same isomorphisms linking $\mathcal{S}^\vee$ and $\mathcal{E}^\vee$:

$$\mathcal{E}^\vee(\mathsf{p}^-) \xrightarrow{\cong} \mathcal{S}^\vee(\mathsf{p})^-, \qquad \text{or equivalently,} \qquad \mathcal{S}^\vee(\mathsf{p}^-) \xrightarrow{\cong} \mathcal{E}^\vee(\mathsf{p})^-. \tag{8.83}$$

Theorem 8.98. *The signature functor*

$$(-)^- : (\mathcal{S}, \mathcal{S}^\vee, \lambda) \to (\mathcal{E}, \mathcal{E}^\vee, \lambda_{-1}),$$

with structure maps (8.82) and (8.83), is a bilax isomorphism of bimonads (with its inverse being itself).

Proof. The fact that the signature functor is bilax is a special case of Theorem 8.9 (since E^- is signed bicommutative). Also note that the structure maps (8.8) specialize to (8.82) and (8.83) in view of (7.14). The remaining check is straightforward. □

As a consequence:

Corollary 8.99. *If a is a commutative monoid, then a^- is a signed commutative monoid, and vice versa. If c is a cocommutative comonoid, then c^- is a signed cocommutative comonoid, and vice versa. If h is a bicommutative bimonoid, then h^- is a signed bicommutative signed bimonoid, and vice versa. Further,*

$$\mathsf{Mon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{(-)^-}{\overset{(-)^-}{\rightleftarrows}} (-1)\text{-}\mathsf{Mon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})$$

$$^{\mathsf{co}}\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{(-)^-}{\overset{(-)^-}{\rightleftarrows}} (-1)\text{-}^{\mathsf{co}}\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})$$

$$^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{(-)^-}{\overset{(-)^-}{\rightleftarrows}} (-1)\text{-}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})$$

are adjoint equivalences of categories.

Recall the bicommutative bimonoid $\mathcal{S}(\mathsf{p})$ from Section 6.5.1 and the signed bicommutative signed bimonoid $\mathcal{E}(\mathsf{p})$ from Section 6.5.3.

Corollary 8.100. *For any species p, there are isomorphisms*

$$\mathcal{S}(\mathsf{p}^-) \xrightarrow{\cong} \mathcal{E}(\mathsf{p})^- \quad \textit{and} \quad \mathcal{E}(\mathsf{p}^-) \xrightarrow{\cong} \mathcal{S}(\mathsf{p})^-$$

of bicommutative bimonoids and signed bicommutative signed bimonoids, respectively.

Proof. This can be proved along the lines of Corollary 8.96. □

In other words, $\mathcal{S}$ (as a functor from species to bicommutative bimonoids) and $\mathcal{E}$ (as a functor from species to signed bicommutative signed bimonoids) are conjugates of each other wrt the signature functor.

8.10.5. Interaction with primitive part and indecomposable part. Observe that: For any comonoid c, there is a natural isomorphism

$$\mathcal{P}(\mathsf{c}^-) \xrightarrow{\cong} \mathcal{P}(\mathsf{c})^- \tag{8.84}$$

of species. In other words, $\mathcal{P}$ is its own conjugate wrt the signature functor.

Similarly, for any monoid a, there is a natural isomorphism

$$\mathcal{Q}(\mathsf{a}^-) \xrightarrow{\cong} \mathcal{Q}(\mathsf{a})^- \tag{8.85}$$

of species. In other words, $\mathcal{Q}$ is its own conjugate wrt the signature functor.

Exercise 8.101. Check that: The conjugate of the adjunction $(\mathcal{T}_q, \mathcal{P})$ in Theorem 6.30 by the signature functor yields the adjunction $(\mathcal{T}_{-q}, \mathcal{P})$. In other words, the adjunction $(\mathcal{T}_{-q}, \mathcal{P})$ equals the composite

$$\mathcal{A}\text{-}\mathsf{Sp} \underset{(-)^-}{\overset{(-)^-}{\rightleftarrows}} \mathcal{A}\text{-}\mathsf{Sp} \underset{\mathcal{P}}{\overset{\mathcal{T}_q}{\rightleftarrows}} q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{(-)^-}{\overset{(-)^-}{\rightleftarrows}} (-q)\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}).$$

Similarly, the conjugate of the adjunction $(\mathcal{Q}, \mathcal{T}_q^\vee)$ by the signature functor yields the adjunction $(\mathcal{Q}, \mathcal{T}_{-q}^\vee)$. (Use (8.84), (8.85) and Corollary 8.96.)

Exercise 8.102. Check that: The conjugate of the adjunction $(\mathcal{S}, \mathcal{P})$ in Theorem 6.43 by the signature functor yields the adjunction $(\mathcal{E}, \mathcal{P})$ in Theorem 6.49. Similarly, the conjugate of the adjunction $(\mathcal{Q}, \mathcal{S})$ by the signature functor yields the adjunction $(\mathcal{Q}, \mathcal{E})$. (Use (8.84), (8.85) and Corollary 8.100.)

8.10.6. Interaction with duality. The signature functor and the duality functor viewed as bilax functors commute. (The bilax structures are given in Proposition 3.17 and Theorem 8.90.) Hence, we say that the signature functor is self-dual. In particular, for any q-bimonoid h, we have

$$(\mathsf{h}^-)^* = (\mathsf{h}^*)^-$$

as $(-q)$-bimonoids.

8.10.7. Comparison. Recall the functor $(-)_-$ introduced in Section 2.5.4. It has properties exactly like the signature functor. In fact, the two functors are isomorphic. To define a natural isomorphism $(-)_- \to (-)^-$, we proceed as follows. Fix a chamber C_0. For any species p, define the map $\mathsf{p}_- \to \mathsf{p}^-$, on the F-component, by

$$\mathsf{p}[F] \to \mathsf{p}[F] \otimes \mathsf{E}^-[F], \qquad x \mapsto x \otimes (-1)^{\operatorname{dist}(C_0, FC_0)}[FC_0/F].$$

This is an isomorphism of species. Diagram (2.3) is checked below.

$$\begin{array}{ccc}
\mathsf{p}[F] & \longrightarrow & \mathsf{p}[F] \otimes \mathsf{E}^-[F] \\
\big\downarrow & & \big\downarrow \\
\mathsf{p}[G] & \longrightarrow & \mathsf{p}[G] \otimes \mathsf{E}^-[G]
\end{array}$$

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & x \otimes (-1)^{\mathrm{dist}(C_0,FC_0)}[FC_0/F] \arrow[d, mapsto] \\
(-1)^{\mathrm{dist}(F,G)}\beta_{G,F}(x) \arrow[r, mapsto] & \beta_{G,F}(x) \otimes (-1)^{\mathrm{dist}(C_0,FC_0)}[GC_0/G].
\end{tikzcd}$$

This boils down to the identity

$$(-1)^{\mathrm{dist}(F,G)}(-1)^{\mathrm{dist}(C_0,GC_0)} = (-1)^{\mathrm{dist}(C_0,FC_0)}$$

which holds by (1.31).

Exercise 8.103. Check that: For a monoid a, the map $\mathsf{a}_- \to \mathsf{a}^-$ is an isomorphism of monoids. Dually, for a comonoid c, the map $\mathsf{c}_- \to \mathsf{c}^-$ is an isomorphism of comonoids. Combining, for a q-bimonoid h, the map $\mathsf{h}_- \to \mathsf{h}^-$ is an isomorphism of $(-q)$-bimonoids.

Notes

The Hadamard product on $\mathcal{A}$-species is motivated by the Hadamard products on graded vector spaces and on Joyal species. The terminology itself comes from the similarity with the Hadamard product of sequences and of matrices.

Hadamard product of sequences and of matrices. The termwise product of sequences originates in work of Hadamard [**402**, Formula (3)]. A product with a similar flavor is the entrywise product of matrices which appears in work of Schur [**818**, Section VII]. The sum of the entrywise products of two matrices had been considered earlier by Moutard [**710**, page 58]. An early usage of the name 'Hadamard product' is in the book by Halmos [**411**, page 174]. Information on the Hadamard product of matrices can be found in the book by Horn and Johnson [**462**, Chapter 5] with additional historical information in Horn's paper [**461**, page 92]. For connections of this product to Hopf theory, see [**18**, Chapter 20]. For the Hadamard product of sequences, see for instance [**102**, Section 2.1, Definition (21)], [**307**, page 303], [**845**, page 471], [**18**, page 23].

Bialgebras.

Hadamard product of graded vector spaces. The Hadamard product of graded vector spaces is stated in [**18**, Formula (2.3)], with its internal hom given in [**18**, Proposition 2.4]. This product is also considered in [**611**, Section 5.9.1], [**798**, Section 5.1.3].

Convolution algebra. Early references for the convolution algebra in Hopf theory are by Milnor and Moore [**695**, Proposition 8.2], [**696**, Proposition 6.3], Kostant [**540**, Section 1.1], Heyneman and Sweedler [**864**, page 14], [**867**, Section 4.0], [**432**, Section 1.5], Radford [**769**, Section 1.1], Hochschild [**444**, page 17], [**445**, page 5]. Some later references are by Bourbaki [**149**, Section III.11, Propositions 1, 2, 3], Caenepeel [**181**, page 177], Hazewinkel [**427**, Section 37.1.3], Kassel [**517**, Proposition III.3.1], Klimyk and Schmüdgen [**535**, pages 10 and 11], Lambe and Radford [**557**, Section 1.6.1], Montgomery [**703**, Section 1.4], Schürmann [**819**, page 27]. Some recent references are [**171**, Section 1.3], [**182**, page 5], [**244**, Proposition 5.4], [**228**, Section 4.2], [**364**, Proposition 1.22], [**611**, Section 1.6], [**612**, Section 9.1.5], [**663**, Construction 20.3.5], [**771**, Chapter 6], [**859**, pages 38 and 39], [**882**, Section 1.3.2].

For convolution monoids in the context of monoidal categories, see [**18**, Sections 1.2.4 and 3.4.5], [**475**, Lemma 2.3], [**898**, Section 2.2, Equation (5)]. Exercise 8.16

which talks of the action of comonoids on monoids may be compared with [**475**, Lemma 5.1], [**898**, Lemma 2.15 and Proposition 4.34].

Internal hom for coalgebras. The existence of the internal hom for cocommutative coalgebras is proved by Barr [**71**, Theorem 5.3]. The more general case of coalgebras appears in Porst's paper [**756**, Section 3.2]. These are classical analogues of ${}^{\mathrm{co}}\mathcal{C}(\mathsf{c},\mathsf{d})$ and $\mathcal{C}(\mathsf{c},\mathsf{d})$, respectively, from Section 8.4. The specific case of differential graded coalgebras is treated by Anel and Joyal [**36**, Theorem 2.5.1]. For later references, see [**475**, page 1123], [**899**, Proposition 2.1], [**897**, Proposition 3.3.6].

The analogue of the bimonoid of chamber maps $\mathcal{C}(\Gamma,\Gamma)$ in Section 8.5 is related to the Hopf algebra considered by Patras and Reutenauer [**745**, Theorem 9]. Their [**745**, Proposition-Definition 3] arises from (8.37). The analogue of the maps in diagram (8.39) are given in [**745**, Theorem 11]. The analogue of formula (8.40) is [**745**, Theorem 12]. They do not, however, mention any connection with the internal hom for coalgebras.

Universal measuring coalgebra. The universal measuring coalgebra (which is the classical analogue of $\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})$ from Section 8.6) was introduced by Sweedler [**866**, Section 1], [**867**, Section 7.0]; the analogue of the universal property in Proposition 8.47 is [**866**, Theorem 1.1], [**867**, Theorem 7.0.4], the analogue of Exercise 8.48 is [**867**, Proposition 7.0.1], the analogue of Lemma 8.63 is [**866**, page 267], [**867**, Exercise on page 146]. The latter is about a bialgebra structure on the universal measuring coalgebra. Interestingly, in our setting, the comonoid structure is wrt the comonad $\mathcal{T}^\vee$ on species, while the monoid structure is wrt the Hadamard product on species.

An early reference to measurings is by Barr [**71**, Sections 6 and 7]; analogues of Propositions 8.47 and 8.68 for c a cocommutative coalgebra are contained in [**71**, Propositions 7.4 and 7.8]. Barr also mentions that the category of algebras is enriched over the category of cocommutative coalgebras and uses the terms 'tensor' and 'cotensor' (which in our terminology are copower and power, respectively). These ideas are continued by Fox [**311**, Chapter III], [**312**, Sections 2 and 3], [**313**]. The analogue of relations (8.58) is given on [**313**, page 69]. These are also present in early work of Wraith [**919**, Relations J.1 on page 154]. Measurings also appear in later work of Grünenfelder and Paré [**389**, Section VII.3, Example 3.1.4], Batchelor [**78**], [**79**], [**80**]. The classical analogue of the bimonoid $\overline{\mathcal{C}}(\mathsf{h},\mathsf{a})$ in Section 8.6.5 is the bialgebra introduced by Mastnak [**656**, Section 3]. It is studied further in his papers with Grünenfelder [**387**, Proposition 2.3], [**388**]. The analogue of Lemma 8.55 is [**656**, Proposition 3.2], [**387**, Proposition 3.5]. The analogue of the first two items of Exercise 8.62 on bimeasurings is [**387**, Theorem 2.4]. The universal measuring cocommutative coalgebra is considered in [**387**, Proposition 1.4].

Anel and Joyal [**36**, Theorem 3.5.7] show that the category of differential graded algebras is enriched, tensored, cotensored over the monoidal category of differential graded coalgebras. This result is the analogue of Propositions 8.65 and 8.67. The analogue of relations (8.58) is given in [**36**, Proof of Theorem 3.4.1, item (m)]. Analogues of Exercise 8.27, Exercise 8.52, Example 8.70 are [**36**, Proposition 2.5.10, Proposition 3.5.13, Proposition 3.4.8]. The universal measuring comonoid in the general setting of monoidal categories is studied by Hyland, López Franco, Vasilakopoulou [**475**, Section 4], [**899**, Sections 3 and 4], [**897**, Section 6.1]. They credit Wraith for linking the universal measuring coalgebra to the enrichment of algebras over coalgebras. They formulate this in categorical terms in [**475**, Theorem 5.2], [**899**, Propositions 4.1 and 4.2], [**897**, Theorem 6.1.4], with a further generalization in [**898**, Theorems 2.18 and 4.37]. These include the result obtained by Anel and Joyal. Abstract results relevant to Exercise 8.64 are [**899**, Corollaries 2.2 and 2.3], [**897**, Corollary 4.3.4]. The bimonoid structure on the universal measuring comonoid is given in [**475**, Corollary 5.4]. This includes the bialgebra obtained by Mastnak. Other references to measurings are by Porst [**755**, Proposition 4], [**760**, Section 3.1.2], Lauve and Mastnak [**571**, Section 1]. For book references, see [**428**, Section 2.12], [**771**, Definition 15.1.1].

The analogue of the measuring condition (8.43b) also appears in the definition of module-algebras over a bialgebra. This is mentioned by Sweedler [**867**, page 153]. Some later references are by Abe [**1**, page 137], Montgomery [**703**, Definition 4.1.1], Kassel [**517**, Section V.6, Relations (6.1) and (6.2)], Majid [**632**, Relations (1.8)], [**633**, Definition 2.6], Chari and Pressley [**205**, page 109], Gracia-Bondía, Várilly, Figueroa [**364**, Definition 1.28], Brown and Goodearl [**161**, Definitions I.9.18], Giachetta, Mangiarotti, Sardanashvily [**352**, Section 10.2, page 551], Khalkhali [**526**, Example 1.7.6], Lorenz [**612**, Formulas (10.23)].

Bimonoids in Joyal species. The Hadamard product of Joyal set-species is given by Bergeron, Labelle, Leroux [**102**, Section 2.1, Formula (22)] under the name 'cartesian product'. Méndez uses the term Hadamard product [**675**, Section 3.2.2].

For Joyal species, an early reference for the Hadamard product is by Smirnov [**833**, Section 1, page 577]. It is stated in [**18**, Formula (8.7)]. The analogue of Theorem 8.4, namely, the bilax property of the Hadamard product, is given in [**18**, Proposition 9.5], with the $p = q = 1$ case treated earlier in [**18**, Proposition 8.58]. The analogue of Lemma 8.1 is given in [**18**, Corollary 9.6]. The analogue of Proposition 8.12 on the internal hom of the Hadamard product is given in [**18**, Proposition 8.63]. The analogue of Theorem 8.21, namely, the bilax property of the internal hom, in the case $p = q = 1$, is given in [**18**, Proposition 8.64]. The analogue of Proposition 8.71 is given in [**18**, Proposition 8.67]. The analogue of Theorem 8.11 is given in [**18**, Proposition 8.69]; the former talks about a $(2, 1)$-monad, the latter about a 3-monoidal category. More results on the Hadamard product can be found in [**20**], see the Notes to Chapter 15 for more details.

The Hadamard product of Joyal species is also considered in [**611**, Section 5.1.12], [**156**, Definition 6.3.1.3], [**798**, (5.1.4.1)].

Signature functor. The signature functor on Joyal species is explained in [**18**, Section 9.4.2]. The analogue of Theorem 8.90, namely, the bilax property of the signature functor is given in [**18**, Proposition 9.9]. The analogue of Corollary 8.92 is given in [**18**, Corollary 9.10]. It says that the categories of Joyal q-bimonoids and Joyal $(-q)$-bimonoids are isomorphic.

Bimonoids for hyperplane arrangements. The Hadamard product for bimonoids for arrangements is introduced here for the first time. The contents of this chapter are completely new from this point of view.

CHAPTER 9

Exponential and logarithm

Recall the flat-incidence, lune-incidence and bilune-incidence algebras from Sections 1.5 and 1.6. They are associated to a fixed hyperplane arrangement.

The lune-incidence algebra acts on the space of all maps from a comonoid c to a monoid a. In particular, any noncommutative zeta function $\boldsymbol{\zeta}$ defines an invertible operator on this space. We call such an operator an exponential. The inverse operator is given by a noncommutative Möbius function $\boldsymbol{\mu}$ and we call it a logarithm. For a cocommutative comonoid c and bimonoid k, any mutually inverse pair of exponential and logarithm sets up inverse bijections between coderivations from c to k and comonoid morphisms from c to k. Dually, for a bimonoid h and commutative monoid a, there are bijections between derivations and monoid morphisms from h to a. As a consequence, any logarithm of the identity map on a bimonoid h maps into $\mathcal{P}(\mathsf{h})$ when h is cocommutative, and factors through $\mathcal{Q}(\mathsf{h})$ when h is commutative.

Table 9.1. Exp-log correspondences.

Algebra	Starting data	Exp-log correspondence
lune-incidence algebra	cocommutative comonoid c, bimonoid k	(coderivations $\mathsf{c} \to \mathsf{k}$) $\longleftrightarrow$ (comonoid morphisms $\mathsf{c} \to \mathsf{k}$)
	bimonoid h, commutative monoid a	(derivations $\mathsf{h} \to \mathsf{a}$) $\longleftrightarrow$ (monoid morphisms $\mathsf{h} \to \mathsf{a}$)
flat-incidence algebra	cocommutative bimonoid h, commutative bimonoid k	(biderivations $\mathsf{h} \to \mathsf{k}$) $\longleftrightarrow$ (bimonoid morphisms $\mathsf{h} \to \mathsf{k}$)
bilune-incidence algebra	q-bimonoids h, k for q not a root of unity	(biderivations $\mathsf{h} \to \mathsf{k}$) $\longleftrightarrow$ (q-bimonoid morphisms $\mathsf{h} \to \mathsf{k}$)

This story has a commutative counterpart. It can either be deduced from the above via the base-case map or can also be developed independently. The flat-incidence algebra acts on the space of all maps from a cocommutative comonoid c to a commutative monoid a. The exponential and logarithm are now uniquely defined using the zeta function ζ and Möbius function μ of the poset of flats. For a cocommutative comonoid c and commutative bimonoid k, they set up a bijection between coderivations and comonoid morphisms from

c to k. Dually, for a cocommutative bimonoid h and commutative monoid a, there is a bijection between derivations and monoid morphisms from h to a. Combining, for a cocommutative bimonoid h and commutative bimonoid k, there is a bijection between biderivations and bimonoid morphisms from h to k. Moreover, the logarithm of the identity map on a bicommutative bimonoid h induces an isomorphism of species from $\mathcal{Q}(\mathsf{h})$ to $\mathcal{P}(\mathsf{h})$.

There is a parallel theory for q-bimonoids for q not a root of unity. It employs the two-sided q-zeta function $\boldsymbol{\zeta}_q$ and two-sided q-Möbius function $\boldsymbol{\mu}_q$. These are elements of the bilune-incidence algebra. Under certain commutativity assumptions, these can be replaced by the noncommutative q-zeta and q-Möbius functions from the lune-incidence algebra. In the special case $q = 0$, one can work inside the lune-incidence algebra without any loss of generality and all involved maps have explicit formulas. We refer to the actions of $\boldsymbol{\zeta}_q$ and $\boldsymbol{\mu}_q$ as the q-exponential and q-logarithm, respectively.

Exp-log correspondences can also be developed using the notion of series of a species. For a comonoid, one can further define primitive series and group-like series. The lune-incidence algebra acts on the space of series of a monoid, and any mutually inverse pair of a noncommutative zeta function $\boldsymbol{\zeta}$ and noncommutative Möbius function $\boldsymbol{\mu}$ defines inverse operations on this space. For the exponential bimonoid E, this recovers Möbius inversion in the poset of flats, while for the bimonoid of chambers Γ, this yields a noncommutative version of Möbius inversion. Moreover, for any bimonoid, the exp-log correspondences set up a bijection between its primitive series and group-like series. The passage from this approach to the previous approach involves the convolution monoid, the bimonoid of star families, and the universal measuring comonoid from Chapter 8.

The classical setting is as follows. The space of formal power series (viewed as a monoid under substitution) acts on the space of series of a Joyal monoid. The usual exponential and logarithmic power series yield the exp-log correspondence on it. The functor (2.102) relates this classical picture to the one for $\mathcal{A}$-monoids for any braid arrangement $\mathcal{A}$. More precisely, the action of the exponential and logarithmic power series corresponds to that of the uniform noncommutative zeta function and its inverse noncommutative Möbius function.

The above set of ideas can be brought to bear upon the Loday–Ronco, Leray–Samelson and Borel–Hopf theorems in Chapter 13. They are also intimately connected to the Hoffman–Newman–Radford rigidity theorems in Chapter 14.

Notation 9.1. For species p and q, recall that $\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{p},\mathsf{q})$ denotes the space of all maps from p to q. This space plays a prominent role in this chapter, particularly, $\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c},\mathsf{a})$ for c a comonoid and a a monoid. Further, recall that $\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{a},\mathsf{b})$ denotes the space of all monoid morphisms from a to b, and so on.

9.1. Exp-log correspondences

Recall the lune-incidence algebra $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ from Section 1.5.3. We define an action of the lune-incidence algebra on the space of all maps from a comonoid c to a monoid a. This can be used to formulate exp-log correspondences, one for each noncommutative zeta or Möbius function. We pay special attention to the logarithm of the identity map on a bimonoid. As an application, we obtain another proof of the fact that a cocommutative bimonoid is primitively generated.

9.1.1. Action of the lune-incidence algebra. Let c be a comonoid and a a monoid. For any $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ and $f : \mathsf{c} \to \mathsf{a}$ a map of species, define $s \circ f : \mathsf{c} \to \mathsf{a}$ by

$$(s \circ f)_A := \sum_{F:\, F \geq A} s(A, F)\, \mu_A^F f_F \Delta_A^F. \tag{9.1}$$

This is a map of species. To see this, fix A and B of the same support, and for any $F \geq A$, consider the commutative diagram below.

$$\begin{tikzcd}
\mathsf{c}[A] \arrow[r, "\Delta_A^F"] \arrow[d, "\beta_{B,A}"'] & \mathsf{c}[F] \arrow[r, "f_F"] \arrow[d, "\beta_{BF,F}"'] & \mathsf{a}[F] \arrow[r, "\mu_A^F"] \arrow[d, "\beta_{BF,F}"] & \mathsf{a}[A] \arrow[d, "\beta_{B,A}"] \\
\mathsf{c}[B] \arrow[r, "\Delta_B^{BF}"'] & \mathsf{c}[BF] \arrow[r, "f_{BF}"'] & \mathsf{a}[BF] \arrow[r, "\mu_B^{BF}"'] & \mathsf{a}[B]
\end{tikzcd}$$

The middle square commutes since f is a map of species (2.3), while the side squares commute by naturality of the product (2.8) and coproduct (2.10). Moreover, $s(A, F) = s(B, BF)$ by (1.40). Multiplying the above diagram by this scalar, summing over all $F \geq A$, and using the bijection between the stars of A and B as in Lemma 1.6, we see that $s \circ f$ is a map of species.

Alternatively, one can write

$$s \circ f = \sum_{k \geq 0} \mu^k f_{k+1} s^{(k)} \Delta^k = \sum_{k \geq 0} \mu^k s^{(k)} f_{k+1} \Delta^k,$$

with notations as in (5.2), (5.3), (5.6), (5.8).

Lemma 9.2. *For a comonoid c and monoid a, the assignment $(s, f) \mapsto s \circ f$ defines a left action of the lune-incidence algebra on $\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a})$. Thus, $\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a})$ is a left module over the lune-incidence algebra.*

Proof. This is checked below. For $s, t \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$,

$$\begin{aligned}
(s \circ (t \circ f))_A &= \sum_{F:\, F \geq A} s(A, F)\, \mu_A^F (t \circ f)_F \Delta_A^F \\
&= \sum_{F:\, F \geq A} s(A, F)\, \mu_A^F \Big(\sum_{G:\, G \geq F} t(F, G)\, \mu_F^G f_G \Delta_F^G \Big) \Delta_A^F \\
&= \sum_{G:\, G \geq A} \Big(\sum_{F:\, G \geq F \geq A} s(A, F) t(F, G) \Big) \mu_A^G f_G \Delta_A^G
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{G\colon G \geq A} (st)(A,G)\, \mu_A^G f_G \Delta_A^G \\
&= (st \circ f)_A.
\end{aligned}
$$

We used definitions (9.1) and (1.39), along with associativity of a and coassociativity of c.

Also, observe that $(\delta \circ f)_A = f_A$, where δ denotes the unit element of the lune-incidence algebra. $\square$

Lemma 9.3. *The action of the lune-incidence algebra on $\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c},\mathsf{a})$ is natural in c and a. More precisely: If $f : \mathsf{c}' \to \mathsf{c}$ is a morphism of comonoids, and $g : \mathsf{a} \to \mathsf{a}'$ is a morphism of monoids, then*

$$
\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c},\mathsf{a}) \to \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}',\mathsf{a}'), \qquad h \mapsto ghf \tag{9.2}
$$

is a map of left modules over the lune-incidence algebra. (The map ghf stands for the composite f followed by h followed by g.)

Proof. This is checked below. For $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$,

$$
\begin{aligned}
(g(s \circ h)f)_A &= g_A (s \circ h)_A f_A \\
&= \sum_{F\colon F \geq A} s(A,F)\, g_A \mu_A^F h_F \Delta_A^F f_A \\
&= \sum_{F\colon F \geq A} s(A,F)\, \mu_A^F g_F h_F f_F \Delta_A^F \\
&= \sum_{F\colon F \geq A} s(A,F)\, \mu_A^F (ghf)_F \Delta_A^F \\
&= (s \circ ghf)_A.
\end{aligned}
$$

We used (9.1), (2.9), (2.11). $\square$

Exercise 9.4. Check that: The action (9.1) is compatible with duality on species, that is, for any comonoid c and monoid a,

$$
\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c},\mathsf{a}) \to \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{a}^*,\mathsf{c}^*), \qquad f \mapsto f^*
$$

is a map of left modules over the lune-incidence algebra. In other words, $(s \circ f)^* = s \circ f^*$.

Lemma 9.5. *Let $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ be such that $s(A,A) = 1$ for all A, and $f : \mathsf{c} \to \mathsf{a}$ be a map of species from a comonoid c to a monoid a. Then $s \circ f$ and f agree when restricted to the primitive part $\mathcal{P}(\mathsf{c})$, and also when followed by the projection to the indecomposable part $\mathcal{Q}(\mathsf{a})$.*

$$
\begin{tikzcd}
\mathsf{c} \arrow[r, shift left, "f"] \arrow[r, shift right, "s\circ f"'] & \mathsf{a} \\
\mathcal{P}(\mathsf{c}) \arrow[u, hook] \arrow[ur, bend right, "f=s\circ f"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathsf{c} \arrow[r, shift left, "f"] \arrow[r, shift right, "s\circ f"'] \arrow[dr, bend right, "f=s\circ f"'] & \mathsf{a} \arrow[d, two heads] \\
& \mathcal{Q}(\mathsf{a})
\end{tikzcd}
$$

PROOF. We explain the first statement, the second is similar. Let us evaluate $(s \circ f)_A$ on $\mathcal{P}(\mathsf{c})[A]$. In this case, $\Delta_A^F = 0$ for any $F > A$. Hence, in the sum in (9.1), only the term corresponding to $F = A$ remains which is f_A. Thus, $(s \circ f)_A = f_A$ on $\mathcal{P}(\mathsf{c})[A]$, as required. □

Lemma 9.6. *Let* h *be a q-bimonoid, and* $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ *be such that* $s(A, A) = 1$ *for all* A. *If* $f : \mathsf{h} \to \mathsf{h}$ *is a morphism of comonoids, then the first diagram below commutes. If* $f : \mathsf{h} \to \mathsf{h}$ *is a morphism of monoids, then the second diagram below commutes.*

$$
\begin{tikzcd}
\mathsf{h} \arrow[r, shift left, "f"] \arrow[r, shift right, "s\circ f"'] & \mathsf{h} \\
\mathcal{P}(\mathsf{h}) \arrow[u, hook] \arrow[r, "f=s\circ f"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathsf{h} \arrow[r, shift left, "f"] \arrow[r, shift right, "s\circ f"'] \arrow[d, two heads] & \mathsf{h} \arrow[d, two heads] \\
\mathcal{Q}(\mathsf{h}) \arrow[r, "f=s\circ f"'] & \mathcal{Q}(\mathsf{h})
\end{tikzcd}
$$

PROOF. If $f : \mathsf{h} \to \mathsf{h}$ is a morphism of comonoids, then it preserves the primitive part, and we deduce from Lemma 9.5 that $s \circ f$ and f restrict to the same map on $\mathcal{P}(\mathsf{h})$. The second part is similar. □

Exercise 9.7. Let $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ be such that $s(A, A) = 1$ for all A, and $f : \mathsf{c} \to \mathsf{a}$ be a map of species from a comonoid c to a monoid a. Show that: If either c or a is trivial, then $s \circ f = f$.

Exercise 9.8. Recall opposite (co)monoids from Section 2.10.2. Let $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ and f a map of species from a comonoid c to a monoid a. Check that:

(1) $\overline{s} \circ f$ in $\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a}^{\mathrm{op}}) = s \circ f$ in $\mathcal{A}\text{-}\mathsf{Sp}({}^{\mathrm{cop}}\mathsf{c}, \mathsf{a})$.
(2) $\overline{s} \circ f$ in $\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, {}^{\mathrm{op}}\mathsf{a}) = s \circ f$ in $\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}^{\mathrm{cop}}, \mathsf{a})$.
(3) $\overline{s} \circ f$ in $\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a}) = s \circ f$ in either $\mathcal{A}\text{-}\mathsf{Sp}({}^{\mathrm{cop}}\mathsf{c}, {}^{\mathrm{op}}\mathsf{a})$ or $\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}^{\mathrm{cop}}, \mathsf{a}^{\mathrm{op}})$.

Here $\overline{s}$ denotes the opposite of s defined in Section 1.5.5.

9.1.2. Exp-log correspondences. Fix a noncommutative zeta function $\boldsymbol{\zeta}$ and a noncommutative Möbius function $\boldsymbol{\mu}$ which are inverses of each other in the lune-incidence algebra. For a map of species $f : \mathsf{c} \to \mathsf{a}$ from a comonoid c to a monoid a, we say that $\boldsymbol{\zeta} \circ f$ is an *exponential* of f and $\boldsymbol{\mu} \circ f$ is a *logarithm* of f. In keeping with this terminology, we also use the notations $\mathbf{exp}(f)$ and $\mathbf{log}(f)$. Explicitly, using formula (9.1),

$$\mathbf{exp}(f)_A = \sum_{F:\, F \geq A} \boldsymbol{\zeta}(A, F)\, \mu_A^F f_F \Delta_A^F, \tag{9.3a}$$

$$\mathbf{log}(f)_A = \sum_{F:\, F \geq A} \boldsymbol{\mu}(A, F)\, \mu_A^F f_F \Delta_A^F. \tag{9.3b}$$

Since $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ are inverse to each other, Lemma 9.2 implies the following result.

Proposition 9.9. *For a comonoid* c *and monoid* a, *we have inverse bijections*

$$\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a}) \underset{\mathbf{log}}{\overset{\mathbf{exp}}{\rightleftarrows}} \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a}). \tag{9.4}$$

We refer to (9.4) as an *exp-log correspondence*. Note very carefully that it depends on the particular $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ that we choose.

Lemma 9.10. *The exp-log correspondence (9.4) is natural in* c *and* a. *More precisely: If* $f : \mathsf{c}' \to \mathsf{c}$ *is a morphism of comonoids,* $h : \mathsf{c} \to \mathsf{a}$ *a map of species,* $g : \mathsf{a} \to \mathsf{a}'$ *a morphism of monoids, then*

$$\mathbf{exp}(ghf) = g\,\mathbf{exp}(h)f \quad \textit{and} \quad \mathbf{log}(ghf) = g\,\mathbf{log}(h)f.$$

Equivalently, the diagrams

$$\begin{tikzcd} \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c},\mathsf{a}) \arrow[r, "\mathbf{exp}"] \arrow[d] & \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c},\mathsf{a}) \arrow[d] \\ \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}',\mathsf{a}') \arrow[r, "\mathbf{exp}"'] & \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}',\mathsf{a}') \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c},\mathsf{a}) \arrow[d] & \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c},\mathsf{a}) \arrow[l, "\mathbf{log}"'] \arrow[d] \\ \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}',\mathsf{a}') & \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}',\mathsf{a}') \arrow[l, "\mathbf{log}"] \end{tikzcd}$$

commute.

This follows from Lemma 9.3.

9.1.3. (Co)derivations and (co)monoid morphisms. Recall the notions of derivation and coderivation from Section 5.5.3, see Lemma 5.32 in particular.

Theorem 9.11. *For a cocommutative comonoid* c *and bimonoid* k, *we have inverse bijections*

$$\mathcal{A}\text{-}\mathsf{Sp}\big(\mathsf{c},\mathcal{P}(\mathsf{k})\big) \underset{\mathbf{log}}{\overset{\mathbf{exp}}{\rightleftarrows}} \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c},\mathsf{k}). \tag{9.5}$$

In other words: If $f : \mathsf{c} \to \mathsf{k}$ is a coderivation, then $\mathbf{exp}(f) : \mathsf{c} \to \mathsf{k}$ is a morphism of comonoids. If $g : \mathsf{c} \to \mathsf{k}$ is a morphism of comonoids, then $\mathbf{log}(g) : \mathsf{c} \to \mathsf{k}$ is a coderivation.

Proof. In view of (9.4), it suffices to show that $\mathbf{exp}$ and $\mathbf{log}$ map as stated. We first show that if $f : \mathsf{c} \to \mathcal{P}(\mathsf{k})$ is a map of species, then $\boldsymbol{\zeta} \circ f : \mathsf{c} \to \mathsf{k}$ is a morphism of comonoids, that is,

$$\Delta_A^G(\boldsymbol{\zeta} \circ f)_A = (\boldsymbol{\zeta} \circ f)_G \Delta_A^G.$$

The calculation goes as follows.

$$\begin{aligned} \Delta_A^G(\boldsymbol{\zeta} \circ f)_A &= \sum_{F:\, F \geq A} \boldsymbol{\zeta}(A,F)\, \Delta_A^G \mu_A^F f_F \Delta_A^F \\ &= \sum_{F:\, F\geq A,\, FG=F} \boldsymbol{\zeta}(A,F)\, \mu_G^{GF} \beta_{GF,F} f_F \Delta_A^F \\ &= \sum_{F:\, F\geq A,\, FG=F} \boldsymbol{\zeta}(A,F)\, \mu_G^{GF} f_{GF} \beta_{GF,F} \Delta_A^F \\ &= \sum_{F:\, F\geq A,\, FG=F} \boldsymbol{\zeta}(A,F)\, \mu_G^{GF} f_{GF} \Delta_A^{GF} \\ &= \sum_{H:\, H\geq G} \Big(\sum_{F:\, F\geq A,\, FG=F,\, GF=H} \boldsymbol{\zeta}(A,F)\Big) \mu_G^H f_H \Delta_A^H \end{aligned}$$

$$= \Big(\sum_{H:\, H \geq G} \boldsymbol{\zeta}(G,H)\, \mu_G^H f_H \Delta_G^H \Big) \Delta_A^G$$
$$= (\boldsymbol{\zeta} \circ f)_G \Delta_A^G.$$

The first step and last step used definition (9.3a). The second step used Lemma 5.39 for $q = 1$. This made use of the bimonoid axiom (2.12) for k and also the hypothesis that f maps into the primitive part of k. The third step used that f is a map of species (2.3), while the fourth step used cocommutativity of c (2.23). In the next step, we introduced a new variable H for GF. The sixth step used the lune-additivity formula (1.42) and coassociativity of c (2.10).

In the other direction, we show that if $g : \mathsf{c} \to \mathsf{k}$ is a morphism of comonoids, then $\boldsymbol{\mu} \circ g$ maps c into $\mathcal{P}(\mathsf{k})$, that is,

$$\Delta_A^G (\boldsymbol{\mu} \circ g)_A = 0 \quad \text{for} \quad G > A.$$

The calculation goes as follows.

$$\begin{aligned}
\Delta_A^G (\boldsymbol{\mu} \circ g)_A &= \sum_{F:\, F \geq A} \boldsymbol{\mu}(A,F)\, \Delta_A^G \mu_A^F g_F \Delta_A^F \\
&= \sum_{F:\, F \geq A} \boldsymbol{\mu}(A,F)\, \Delta_A^G \mu_A^F \Delta_A^F g_A \\
&= \sum_{F:\, F \geq A} \boldsymbol{\mu}(A,F)\, \mu_G^{GF} \Delta_A^{GF} g_A \\
&= \sum_{H:\, H \geq G} \Big(\sum_{F:\, F \geq A,\, GF = H} \boldsymbol{\mu}(A,F) \Big) \mu_G^H \Delta_A^H g_A \\
&= 0.
\end{aligned}$$

The first step used definition (9.3b). The second step used (2.11). This made use of the hypothesis that g is a morphism of comonoids. The third step used diagram (2.29). This is justified since g necessarily maps into the coabelianization of k by Lemma 2.41. In the next step, we introduced a new variable H for GF. The last step used the noncommutative Weisner formula (1.44). $\square$

Dually:

Theorem 9.12. *For a bimonoid* h *and commutative monoid* a*, we have inverse bijections*

$$\mathcal{A}\text{-}\mathsf{Sp}\big(\mathcal{Q}(\mathsf{h}), \mathsf{a}\big) \underset{\mathbf{log}}{\overset{\mathbf{exp}}{\rightleftarrows}} \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h}, \mathsf{a}). \tag{9.6}$$

In other words: If $f : \mathsf{h} \to \mathsf{a}$ is a derivation, then $\mathbf{exp}(f) : \mathsf{h} \to \mathsf{a}$ is a morphism of monoids. If $g : \mathsf{h} \to \mathsf{a}$ is a morphism of monoids, then $\mathbf{log}(g) : \mathsf{h} \to \mathsf{a}$ is a derivation.

Proof. This can be deduced from Theorem 9.11 by duality. Alternatively, one can also proceed directly, and do the two calculations as in the proof of Theorem 9.11. $\square$

Remark 9.13. Though Theorem 9.11 is stated for an arbitrary bimonoid k, the exp-log correspondence (9.5) only depends on the coabelianization of k (see Exercise 5.10), so one may without loss of generality assume that k is cocommutative. Formally, we have a commutative diagram of bijections

$$\begin{tikzcd}
\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c},\mathsf{k}) \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c},\mathsf{k}^{coab}) \arrow[dl, leftrightarrow] \\
& \mathcal{A}\text{-}\mathsf{Sp}\big(\mathsf{c},\mathcal{P}(\mathsf{k})\big) &
\end{tikzcd}$$

with the oblique bijections being the exp-log correspondences. Dually, in Theorem 9.12, one may assume that h is commutative. This may be expressed formally as a commutative diagram of bijections similar to the one above.

In a manner similar to Lemma 9.10, the correspondences (9.5) and (9.6) are natural in c, k, and in h, a, respectively.

Exercise 9.14. Let $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ be such that $s(A,A) = 1$ for all A. Show that:

(1) If $f : \mathsf{c} \to \mathsf{k}$ a coderivation implies $s \circ f : \mathsf{c} \to \mathsf{k}$ a comonoid morphism for every cocommutative comonoid c and bimonoid k, then s is a noncommutative zeta function.

(2) If $g : \mathsf{c} \to \mathsf{k}$ a comonoid morphism implies $s \circ g : \mathsf{c} \to \mathsf{k}$ a coderivation for every cocommutative comonoid c and bimonoid k, then s is a noncommutative Möbius function.

A similar statement holds for derivations and monoid morphisms.

Lemma 9.15. *Let* c *be a comonoid,* h *a bimonoid,* a *a monoid. Suppose either* c *is cocommutative or* a *is commutative. Then:*

(1) *If* $f : \mathsf{c} \to \mathsf{h}$ *is a morphism of comonoids, and* $g : \mathsf{h} \to \mathsf{a}$ *is a morphism of monoids, then* $\mathbf{log}(gf) = \mathbf{log}(g)\,\mathbf{log}(f)$.

(2) *If* $f : \mathsf{c} \to \mathsf{h}$ *is a coderivation, and* $g : \mathsf{h} \to \mathsf{a}$ *is a derivation, then* $\mathbf{exp}(gf) = \mathbf{exp}(g)\,\mathbf{exp}(f)$.

The operation in $\mathbf{log}(g)\,\mathbf{log}(f)$ and $\mathbf{exp}(g)\,\mathbf{exp}(f)$ is composition of maps.

Proof. We explain item (1), item (2) is similar. Suppose c is cocommutative. Then:

$$\mathbf{log}(gf) = g\,\mathbf{log}(f) = \mathbf{log}(g)\,\mathbf{log}(f).$$

The first step used Lemma 9.10 for g a morphism of monoids. By (9.5), $\mathbf{log}(f)$ maps into $\mathcal{P}(\mathsf{h})$, and $\mathbf{log}(g) = g$ on $\mathcal{P}(\mathsf{h})$ by the first diagram in Lemma 9.5. This was used in the second step. See the first diagram below.

$$\begin{tikzcd}
\mathsf{c} \arrow[rr, "\mathbf{log}(gf)"] \arrow[d] \arrow[drr, "\mathbf{log}(f)"'] & & \mathsf{a} \\
\mathcal{P}(\mathsf{h}) \arrow[rr, hook] & & \mathsf{h} \arrow[u, shift left, "g"] \arrow[u, shift right, "\mathbf{log}(g)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{c} \arrow[rr, "\mathbf{log}(gf)"] \arrow[dr, shift left, "f"] \arrow[dr, shift right, "\mathbf{log}(f)"'] & & \mathsf{a} \\
& \mathsf{h} \arrow[ur, "\mathbf{log}(g)"] \arrow[r, two heads] & \mathcal{Q}(\mathsf{h}) \arrow[u]
\end{tikzcd}$$

Now suppose a is commutative. Then:

$$\mathbf{log}(gf) = \mathbf{log}(g)f = \mathbf{log}(g)\,\mathbf{log}(f).$$

The first step used Lemma 9.10 for f a morphism of comonoids. By (9.6), $\mathbf{log}(g)$ factors through $\mathcal{Q}(\mathsf{h})$, and $\mathbf{log}(f) = f$ when projected onto $\mathcal{Q}(\mathsf{h})$ by the second diagram in Lemma 9.5. This was used in the second step. See the second diagram above. □

Remark 9.16. In Lemma 9.15, suppose we work under the stronger hypotheses that c is cocommutative and a is commutative. Then both arguments in the proof work and the two diagrams above can be merged into one as follows.

$$\begin{tikzcd} \mathsf{c} \arrow[rr, "\mathbf{log}(gf)"] \arrow[d] \arrow[dr, "\mathbf{log}(f)"] & & \mathsf{a} \\ \mathcal{P}(\mathsf{h}) \arrow[r, hook] & \mathsf{h} \arrow[ur, "\mathbf{log}(g)"] \arrow[r, two heads] & \mathcal{Q}(\mathsf{h}) \arrow[u] \end{tikzcd}$$

Also in this case, items (1) and (2) formally imply each other.

9.1.4. Logarithm of the identity map. Fix a noncommutative Möbius function $\boldsymbol{\mu}$. For any q-bimonoid h, we have the operator $\mathbf{log}(\mathrm{id}) : \mathsf{h} \to \mathsf{h}$. This is a *logarithm of the identity map* on h. Explicitly, using (9.3b), it is given by

$$\mathbf{log}(\mathrm{id})_A = \sum_{F:\, F \geq A} \boldsymbol{\mu}(A, F)\, \mu_A^F \Delta_A^F. \tag{9.7}$$

To emphasize the dependence on h, we may also write $\mathbf{log}(\mathrm{id}_{\mathsf{h}})$.

Proposition 9.17. *Let* h *be a bimonoid. If* h *is cocommutative, then* $\mathbf{log}(\mathrm{id})$ *maps to* $\mathcal{P}(\mathsf{h})$ *and is in fact a projection from* h *onto* $\mathcal{P}(\mathsf{h})$*, or equivalently,* $\mathbf{log}(\mathrm{id})$ *is an idempotent operator on* h *whose image is* $\mathcal{P}(\mathsf{h})$*.*

If h *is commutative, then* $\mathbf{log}(\mathrm{id})$ *factors through* $\mathcal{Q}(\mathsf{h})$ *and splits the canonical projection* $\mathsf{h} \twoheadrightarrow \mathcal{Q}(\mathsf{h})$*, or equivalently,* $\mathbf{log}(\mathrm{id})$ *is an idempotent operator on* h *whose coimage is* $\mathcal{Q}(\mathsf{h})$*.*

The two situations when h is either cocommutative or commutative are, respectively, illustrated below.

$$\begin{tikzcd} \mathsf{h} \arrow[r, "\mathbf{log}(\mathrm{id})"] \arrow[dr, dashed, two heads] & \mathsf{h} \\ \mathcal{P}(\mathsf{h}) \arrow[u, hook] \arrow[r, "\mathbf{log}(\mathrm{id})=\mathrm{id}"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{h} \arrow[r, "\mathbf{log}(\mathrm{id})"] \arrow[d, two heads] & \mathsf{h} \arrow[d, two heads] \\ \mathcal{Q}(\mathsf{h}) \arrow[ur, dashed, hook] \arrow[r, "\mathbf{log}(\mathrm{id})=\mathrm{id}"'] & \mathcal{Q}(\mathsf{h}) \end{tikzcd} \tag{9.8}$$

Proof. Suppose h is cocommutative. By taking $\mathsf{c} = \mathsf{k} := \mathsf{h}$ in Theorem 9.11, we see that $\mathbf{log}(\mathrm{id})$ maps to $\mathcal{P}(\mathsf{h})$. Further, by Lemma 9.6, it is identity on $\mathcal{P}(\mathsf{h})$, and hence a projection. The claim when h is commutative follows similarly by starting with $\mathsf{h} = \mathsf{a}$ in Theorem 9.12. In either case, the fact that $\mathbf{log}(\mathrm{id})$ is an idempotent operator also follows from Lemma 9.15, item (1). □

Another proof of this result using characteristic operations is given later in Exercise 10.18.

Corollary 9.18. *Let* h *be a bimonoid. If* h *is cocommutative, then for any* $G > A$,

$$\sum_{F:\, F \geq A} \boldsymbol{\mu}(A, F)\, \Delta^G_A \mu^F_A \Delta^F_A = 0.$$

If h *is commutative, then for any* $G > A$,

$$\sum_{F:\, F \geq A} \boldsymbol{\mu}(A, F)\, \mu^F_A \Delta^F_A \mu^G_A = 0.$$

Proof. The first claim says that $\mathbf{log}(\mathrm{id})$ is a coderivation, that is, it maps into $\mathcal{P}(\mathsf{h})$. Similarly, the second claim says that $\mathbf{log}(\mathrm{id})$ is a derivation, that is, it factors through $\mathcal{Q}(\mathsf{h})$. ☐

Exercise 9.19. Use the bimonoid axiom (2.12) and noncommutative Weisner formula (1.44) to directly check the identities in Corollary 9.18.

Example 9.20. Recall from Lemma 7.64 that the primitive part of the bimonoid of chambers Γ is the Lie species. Hence, by Proposition 9.17, $\mathbf{log}(\mathrm{id})$ is an idempotent operator on Γ whose image is Lie. We emphasize that there is one such operator for each choice of $\boldsymbol{\mu}$. Explicitly, using (co)product formulas (7.18), we deduce that for any $C \geq A$,

$$\mathbf{log}(\mathrm{id})_A(\mathtt{H}_C) = \sum_{D:\, D \geq A} \mathrm{sln}^{D,C}_A\, \mathtt{H}_D, \tag{9.9}$$

with $\mathrm{sln}^{D,C}_A$ being the Solomon coefficient (1.98), is an element of $\mathsf{Lie}[A]$.

Similarly, recall from Lemma 7.69 that the primitive part of the bimonoid of faces Σ is the Zie species. Hence, $\mathbf{log}(\mathrm{id})$ is an idempotent operator on Σ whose image is Zie.

Example 9.21. For a rank-one arrangement $\mathcal{A}$ with chambers C and $\overline{C}$, and an $\mathcal{A}$-bimonoid h, in continuation of Example 1.15, we have

$$\mathbf{log}(\mathrm{id})_C = \mathrm{id} : \mathsf{h}[C] \to \mathsf{h}[C],$$
$$\mathbf{log}(\mathrm{id})_{\overline{C}} = \mathrm{id} : \mathsf{h}[\overline{C}] \to \mathsf{h}[\overline{C}],$$
$$\mathbf{log}(\mathrm{id})_O = \mathrm{id} - p\, \mu^C_O \Delta^C_O + (p-1)\, \mu^{\overline{C}}_O \Delta^{\overline{C}}_O : \mathsf{h}[O] \to \mathsf{h}[O].$$

Let us verify explicitly that for $\mathsf{h} := \Gamma$, the above is indeed an idempotent operator on Γ whose image is Lie. We are fine in the C- and $\overline{C}$-components since $\Gamma[C] = \mathsf{Lie}[C]$ and $\Gamma[\overline{C}] = \mathsf{Lie}[\overline{C}]$. In the O-component, we calculate using (co)product formulas (7.18):

$$\mathbf{log}(\mathrm{id})_O(\mathtt{H}_C) = (1-p)(\mathtt{H}_C - \mathtt{H}_{\overline{C}}) \quad \text{and} \quad \mathbf{log}(\mathrm{id})_O(\mathtt{H}_{\overline{C}}) = -p(\mathtt{H}_C - \mathtt{H}_{\overline{C}}),$$

which are both Lie elements. (They are specializations of (9.9).) Further, by subtracting, we obtain

$$\mathbf{log}(\mathrm{id})_O(\mathtt{H}_C - \mathtt{H}_{\overline{C}}) = \mathtt{H}_C - \mathtt{H}_{\overline{C}},$$

which completes the check.

Exercise 9.22. Use formula (9.9) to establish the following identities for the bimonoid of chambers Γ. For any $C \geq A$,

$$\sum_{K\colon K \geq A} \boldsymbol{\zeta}(A, K)\ \mathbf{log}(\mathrm{id})_K(\mathtt{H}_{KC}) = \mathtt{H}_C. \tag{9.10}$$

For any A,

$$\sum_{C\colon C \geq A} \boldsymbol{\zeta}(A, C)\ \mathbf{log}(\mathrm{id})_A(\mathtt{H}_C) = \begin{cases} \mathtt{H}_A & \text{if } A \text{ is a chamber,} \\ 0 & \text{otherwise.} \end{cases} \tag{9.11}$$

A more general context for the above discussion is provided later in Section 17.3.3. It involves replacing Γ by the bimonoid $\mathcal{T}(\mathsf{p})$.

Exercise 9.23. For a bimonoid h, let $f : \mathsf{h} \to \mathsf{h}$ be a morphism of monoids. Show that: For $H \geq A$ and $x \in \mathcal{P}(\mathsf{h})[H]$,

$$\mathbf{log}(f)_A \mu_A^H(x) = \sum_{\substack{G\colon G \geq A \\ \mathrm{s}(G) = \mathrm{s}(H)}} \mathrm{sln}_A^{G,H} f_A \mu_A^G \beta_{G,H}(x),$$

where $\mathrm{sln}_A^{G,H}$ is the Solomon coefficient (1.98). In addition, if $\mu_A^G \beta_{G,H}(x) = \mu_A^H(x)$ for all $G \geq A$ with $\mathrm{s}(G) = \mathrm{s}(H)$, then

$$\mathbf{log}(f)_A \mu_A^H(x) = \begin{cases} f_A(x) & \text{if } H = A, \\ 0 & \text{otherwise.} \end{cases}$$

(Use Lemma 5.39 and Lemma 9.10.)

Exercise 9.23 for $f := \mathrm{id}$ is related to the second calculation done later in the proof of Theorem 13.38. Also, the important special case $\mathsf{h} := \mathcal{T}(\mathsf{p})$ is considered later in Lemma 17.22, see Exercise 17.23.

Exercise 9.24. Let h be a bimonoid. Put

$$\mathsf{p} := \mathrm{im}(\mathbf{log}(\mathrm{id}_{\mathsf{h}})) \quad \text{and} \quad \mathsf{q} := \mathrm{coim}(\mathbf{log}(\mathrm{id}_{\mathsf{h}})),$$

the image and coimage, respectively, of the operator $\mathbf{log}(\mathrm{id}_{\mathsf{h}})$. We have the canonical inclusion $f : \mathsf{p} \hookrightarrow \mathsf{h}$ and projection $g : \mathsf{h} \twoheadrightarrow \mathsf{q}$. Consider the diagrams

$$\begin{tikzcd} \mathcal{T}(\mathsf{p}) \arrow[r, dashed, "\hat{f}"] & \mathsf{h} \\ \mathsf{p} \arrow[u, hook] \arrow[ur, hook, "f"'] & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathsf{h} \arrow[r, dashed, "\hat{g}"] \arrow[dr, two heads, "g"'] & \mathcal{T}^{\vee}(\mathsf{q}) \arrow[d, two heads] \\ & \mathsf{q} \end{tikzcd}$$

as in Theorems 6.2 and 6.10, respectively. Show that $\hat{f}$ is a surjective morphism of monoids, while $\hat{g}$ is an injective morphism of comonoids. (Use $\mathbf{exp}(\mathbf{log}(\mathrm{id})) = \mathrm{id}$.) We point out that when h is cocommutative, $\mathsf{p} = \mathcal{P}(\mathsf{h})$, while when h is commutative, $\mathsf{q} = \mathcal{Q}(\mathsf{h})$ by Proposition 9.17.

Exercise 9.25. Use Exercise 9.24 and Exercise 6.70 to deduce that any cocommutative bimonoid is primitively generated. This gives another proof of the nontrivial implication in Proposition 5.51.

Exercise 9.26. For a bimonoid h, recall the map $\mathrm{pq}_{\mathsf{h}} : \mathcal{P}(\mathsf{h}) \to \mathcal{Q}(\mathsf{h})$ from (5.50). Check that: If h is cocommutative, then the first diagram below commutes. If h is commutative, then the second diagram below commutes.

$$\begin{tikzcd} & \mathcal{P}(\mathsf{h}) \arrow[dr, two heads, "\mathrm{pq}_{\mathsf{h}}"] & \\ \mathsf{h} \arrow[ur, two heads, "\mathbf{log}(\mathrm{id}_{\mathsf{h}})"] \arrow[rr, two heads] & & \mathcal{Q}(\mathsf{h}) \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{Q}(\mathsf{h}) \arrow[dr, hook, "\mathbf{log}(\mathrm{id}_{\mathsf{h}})"] & \\ \mathcal{P}(\mathsf{h}) \arrow[ur, hook, "\mathrm{pq}_{\mathsf{h}}"] \arrow[rr, hook] & & \mathsf{h} \end{tikzcd}$$

The horizontal maps are the canonical maps. (Use Lemma 9.5 for $s := \boldsymbol{\mu}$, $\mathsf{c} = \mathsf{a} := \mathsf{h}$ and $f := \mathrm{id}$.) Deduce forward implications in Proposition 5.56 as a consequence. (The surjectivity and injectivity of the map $\mathrm{pq}_{\mathsf{h}} : \mathcal{P}(\mathsf{h}) \to \mathcal{Q}(\mathsf{h})$ is also indicated in the above diagrams.)

Exercise 9.27. Let $f : \mathsf{h} \to \mathsf{k}$ be a morphism of bimonoids. Then the diagram

$$\begin{tikzcd} \mathsf{h} \arrow[r, "\mathbf{log}(\mathrm{id}_{\mathsf{h}})"] \arrow[d, "f"'] & \mathsf{h} \arrow[d, "f"] \\ \mathsf{k} \arrow[r, "\mathbf{log}(\mathrm{id}_{\mathsf{k}})"'] & \mathsf{k} \end{tikzcd}$$

commutes. (Both directions evaluate to $\mathbf{log}(f)$ by Lemma 9.10.) Moreover, by Proposition 9.17, when h and k are either both cocommutative or commutative, we have the first or second diagram below, respectively.

$$\begin{tikzcd} \mathsf{h} \arrow[r, two heads, "\mathbf{log}(\mathrm{id}_{\mathsf{h}})"] \arrow[d, "f"'] & \mathcal{P}(\mathsf{h}) \arrow[r, hook] \arrow[d, "f"'] & \mathsf{h} \arrow[d, "f"] \\ \mathsf{k} \arrow[r, two heads, "\mathbf{log}(\mathrm{id}_{\mathsf{k}})"'] & \mathcal{P}(\mathsf{k}) \arrow[r, hook] & \mathsf{k} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{h} \arrow[r, two heads] \arrow[d, "f"'] & \mathcal{Q}(\mathsf{h}) \arrow[r, hook, "\mathbf{log}(\mathrm{id}_{\mathsf{h}})"] \arrow[d, "f"] & \mathsf{h} \arrow[d, "f"] \\ \mathsf{k} \arrow[r, two heads] & \mathcal{Q}(\mathsf{k}) \arrow[r, hook, "\mathbf{log}(\mathrm{id}_{\mathsf{k}})"'] & \mathsf{k} \end{tikzcd}$$

Show that: In the cocommutative case, $f : \mathsf{h} \to \mathsf{k}$ is surjective iff $f : \mathcal{P}(\mathsf{h}) \to \mathcal{P}(\mathsf{k})$ is surjective. Dually, in the commutative case, $f : \mathsf{h} \to \mathsf{k}$ is injective iff $f : \mathcal{Q}(\mathsf{h}) \to \mathcal{Q}(\mathsf{k})$ is injective.

9.1.5. Opposite exponential and opposite logarithm. For a map of species $f : \mathsf{c} \to \mathsf{a}$ from a comonoid c to a monoid a, let

$$\overline{\mathbf{exp}}(f)_A = \sum_{F:\, F \geq A} \boldsymbol{\zeta}(A, A\overline{F})\, \mu_A^F f_F \Delta_A^F, \tag{9.12a}$$

$$\overline{\mathbf{log}}(f)_A = \sum_{F:\, F \geq A} \boldsymbol{\mu}(A, A\overline{F})\, \mu_A^F f_F \Delta_A^F. \tag{9.12b}$$

In other words, $\overline{\mathbf{exp}}$ and $\overline{\mathbf{log}}$ are the exponential and logarithm associated to $\overline{\boldsymbol{\zeta}}$ and $\overline{\boldsymbol{\mu}}$, respectively. (Recall that $\overline{\boldsymbol{\zeta}}(A,F) := \boldsymbol{\zeta}(A, A\overline{F})$ and $\overline{\boldsymbol{\mu}}(A,F) := \boldsymbol{\mu}(A, A\overline{F})$.) We refer to $\overline{\mathbf{exp}}$ and $\overline{\mathbf{log}}$ as the opposite of $\mathbf{exp}$ and $\mathbf{log}$. In the special case that $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ are projective, we have $\overline{\mathbf{exp}} = \mathbf{exp}$ and $\overline{\mathbf{log}} = \mathbf{log}$.

In particular, for any q-bimonoid h, we have the operator $\overline{\mathbf{log}}(\mathrm{id}) : \mathsf{h} \to \mathsf{h}$ given by

$$\overline{\mathbf{log}}(\mathrm{id})_A = \sum_{F:\, F \geq A} \boldsymbol{\mu}(A, A\overline{F})\, \mu_A^F \Delta_A^F. \tag{9.13}$$

Exercise 9.28. Let h be a q-bimonoid for $q \neq 0$, and consider the q^{-1}-bimonoids ${}^{\mathrm{op}}\mathsf{h}$ and $\mathsf{h}^{\mathrm{cop}}$, and q-bimonoids $\mathsf{h}^{\mathrm{op,cop}}$ and ${}^{\mathrm{op,cop}}\mathsf{h}$ defined in Proposition 2.58. Use Exercise 9.8 to deduce that

$$\mathbf{log}(\mathrm{id}_{{}^{\mathrm{op}}\mathsf{h}}) = \overline{\mathbf{log}}(\mathrm{id}_{\mathsf{h}^{\mathrm{cop}}}) \quad \text{and} \quad \mathbf{log}(\mathrm{id}_{{}^{\mathrm{op,cop}}\mathsf{h}}) = \overline{\mathbf{log}}(\mathrm{id}_{\mathsf{h}}) = \mathbf{log}(\mathrm{id}_{\mathsf{h}^{\mathrm{op,cop}}}).$$

This holds more generally for any v-bimonoid h, where v is a nowhere-zero distance function.

Exercise 9.29. For a bimonoid h, deduce that

$$\mathbf{log}(\mathrm{id}_{{}^{\mathrm{op}}\mathsf{h}}) = \overline{\mathbf{log}}(\mathrm{id}_{\mathsf{h}}) \quad \text{and} \quad \mathbf{log}(\mathrm{id}_{\mathsf{h}^{\mathrm{cop}}}) = \overline{\mathbf{log}}(\mathrm{id}_{\mathsf{h}}),$$

the former when h is cocommutative and the latter when h is commutative.

9.1.6. Signed analogue. Theorems 9.11 and 9.12 have signed analogues which are given below.

Theorem 9.30. *For a signed cocommutative comonoid* c *and signed bimonoid* k*, we have inverse bijections*

$$\mathcal{A}\text{-}\mathsf{Sp}\big(\mathsf{c}, \mathcal{P}(\mathsf{k})\big) \underset{\mathbf{log}}{\overset{\mathbf{exp}}{\rightleftarrows}} \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c}, \mathsf{k}).$$

Proof. The calculations proceed as in the unsigned case, we now use instead the signed bimonoid axiom (2.37) and the signed cocommutativity axiom (2.47). For the second calculation, we further note that diagram (2.29) holds in the signed setting, see Exercise 2.38. □

Exercise 9.31. Check that: The action (9.1) is compatible with the functor $(-)_-$ defined in (2.49), that is, for any comonoid c and monoid a,

$$\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a}) \to \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}_-, \mathsf{a}_-), \qquad f \mapsto f$$

is a map of left modules over the lune-incidence algebra. Use this to deduce Theorem 9.30 from Theorem 9.11.

Dually:

Theorem 9.32. *For a signed bimonoid* h *and signed commutative monoid* a*, we have inverse bijections*

$$\mathcal{A}\text{-}\mathsf{Sp}\big(\mathcal{Q}(\mathsf{h}), \mathsf{a}\big) \underset{\mathbf{log}}{\overset{\mathbf{exp}}{\rightleftarrows}} \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h}, \mathsf{a}).$$

Exercise 9.33. Formulate the signed analogue of Proposition 9.17. (Replace (co)commutative bimonoid by signed (co)commutative signed bimonoid.)

9.2. Commutative exp-log correspondence

Recall the flat-incidence algebra $\mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$ from Section 1.5.1. We define an action of the flat-incidence algebra on the space of all maps from a cocommutative comonoid c to a commutative monoid a. This can be used to formulate the commutative exp-log correspondence. We develop this directly as well as explain how it can be deduced from the discussion in Section 9.1.

9.2.1. Action of the flat-incidence algebra. Let c be a cocommutative comonoid and a a commutative monoid. We work with the formulation given in Propositions 2.20 and 2.21. For any $s \in \mathrm{I}_{\text{flat}}[\mathcal{A}]$ and $f : \mathsf{c} \to \mathsf{a}$ a map of species, define another map of species $s \circ f : \mathsf{c} \to \mathsf{a}$ by

$$(s \circ f)_{\mathrm{Z}} := \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} s(\mathrm{Z}, \mathrm{X})\, \mu_{\mathrm{Z}}^{\mathrm{X}} f_{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}. \tag{9.14}$$

Alternatively, one can write

$$s \circ f = \sum_{k \geq 0} \mu^{\overline{k}} f_{\overline{k+1}} s^{(\overline{k})} \Delta^{\overline{k}} = \sum_{k \geq 0} \mu^{\overline{k}} s^{(\overline{k})} f_{\overline{k+1}} \Delta^{\overline{k}},$$

with notations as in (5.12), (5.13) and Exercise 5.6.

Lemma 9.34. *For a cocommutative comonoid* c *and commutative monoid* a*, the assignment* $(s, f) \mapsto s \circ f$ *defines an action of the flat-incidence algebra on* $\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a})$*. Thus,* $\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a})$ *is a module over the flat-incidence algebra.*

The check is similar to the one in the proof of Lemma 9.2 with faces replaced by flats.

Lemma 9.35. *For a cocommutative comonoid* c *and a commutative monoid* a*, the action* (9.1) *of the lune-incidence algebra on* $\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a})$ *factors through the base-case map* (1.45) *to yield an action of the flat-incidence algebra on* $\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a})$ *which coincides with* (9.14).

Proof. The calculation goes as follows. Let A be a face of support Z.

$$\begin{aligned}
(s \circ f)_A &= \sum_{F:\, F \geq A} s(A, F)\, \mu_A^F f_F \Delta_A^F \\
&= \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \Big(\sum_{F:\, F \geq A,\, \mathrm{s}(F) = \mathrm{X}} s(A, F) \Big) \mu_{\mathrm{Z}}^{\mathrm{X}} f_{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}} \\
&= \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mathrm{bc}(s)(\mathrm{Z}, \mathrm{X})\, \mu_{\mathrm{Z}}^{\mathrm{X}} f_{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}} \\
&= (\mathrm{bc}(s) \circ f)_{\mathrm{Z}}.
\end{aligned}$$

Recall that μ_A^F and $\mu_{\mathrm{Z}}^{\mathrm{X}}$ (and similarly Δ_A^F and $\Delta_{\mathrm{Z}}^{\mathrm{X}}$) connect to each other by the maps $\beta_{\mathrm{Z},A}$ and $\beta_{\mathrm{X},F}$ as in the diagram in the proof of Proposition 2.20. For convenience, these maps have been suppressed in the above calculation. □

Exercise 9.36. The action of the flat-incidence algebra on $\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a})$ is natural in c and a. Check this directly or deduce it from Lemma 9.3 and Lemma 9.35.

Exercise 9.37. Formulate the commutative analogues of Lemmas 9.5 and 9.6 and Exercise 9.7. (Start with $s \in \mathrm{I}_{\text{flat}}[\mathcal{A}]$ such that $s(\mathrm{Z}, \mathrm{Z}) = 1$ for all Z.)

9.2.2. Commutative exp-log correspondence. Let ζ and μ be the zeta function and Möbius function in the flat-incidence algebra. For a map of species $f : \mathsf{c} \to \mathsf{a}$ from a cocommutative comonoid c to a commutative monoid a, we say that

$$\exp(f) := \zeta \circ f \quad \text{and} \quad \log(f) := \mu \circ f$$

are the *exponential* and *logarithm* of f, respectively. Explicitly, using formula (9.14),

$$\exp(f)_{\mathrm{Z}} = \sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Z}} \mu^{\mathrm{X}}_{\mathrm{Z}} f_{\mathrm{X}} \Delta^{\mathrm{X}}_{\mathrm{Z}}, \tag{9.15a}$$

$$\log(f)_{\mathrm{Z}} = \sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{X})\, \mu^{\mathrm{X}}_{\mathrm{Z}} f_{\mathrm{X}} \Delta^{\mathrm{X}}_{\mathrm{Z}}. \tag{9.15b}$$

In contrast to the noncommutative theory, these operations are now uniquely defined. Since ζ and μ are inverse to each other, by Lemma 9.34, we obtain inverse bijections

$$\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a}) \underset{\log}{\overset{\exp}{\rightleftarrows}} \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a}). \tag{9.16}$$

This is the *commutative exp-log correspondence*.

Lemma 9.38. *When c is cocommutative and a is commutative, all exp-log correspondences (9.4) reduce to (9.16).*

PROOF. This follows from Lemma 1.17 and Lemma 9.35. □

Exercise 9.39. Deduce from Exercise 9.36 that: The commutative exp-log correspondence (9.16) is natural in c and a. This is the commutative analogue of Lemma 9.10. The formulation is similar: we now take c, c' to be cocommutative, a, a' to be commutative, and replace $\mathbf{exp}$, $\mathbf{log}$ by exp, log, respectively.

9.2.3. (Co, bi)derivations and (co, bi)monoid morphisms.

Theorem 9.40. *For a cocommutative comonoid c and bicommutative bimonoid k, we have inverse bijections*

$$\mathcal{A}\text{-}\mathsf{Sp}\big(\mathsf{c}, \mathcal{P}(\mathsf{k})\big) \underset{\log}{\overset{\exp}{\rightleftarrows}} \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c}, \mathsf{k}). \tag{9.17}$$

In other words: If $f : \mathsf{c} \to \mathsf{k}$ is a coderivation, then $\exp(f) : \mathsf{c} \to \mathsf{k}$ is a morphism of comonoids. If $g : \mathsf{c} \to \mathsf{k}$ is a morphism of comonoids, then $\log(g) : \mathsf{c} \to \mathsf{k}$ is a coderivation.

PROOF. This follows from Theorem 9.11 and Lemma 9.38. Alternatively, one can proceed directly. The proof is similar to (and simpler than) that of Theorem 9.11; the two calculations are indicated below.

For $\mathrm{Y} \geq \mathrm{Z}$,

$$\Delta^{\mathrm{Y}}_{\mathrm{Z}} (\zeta \circ f)_{\mathrm{Z}} = \sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Z}} \Delta^{\mathrm{Y}}_{\mathrm{Z}} \mu^{\mathrm{X}}_{\mathrm{Z}} f_{\mathrm{X}} \Delta^{\mathrm{X}}_{\mathrm{Z}}$$

$$
\begin{aligned}
&= \sum_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Y}} \mu^{\mathrm{X}}_{\mathrm{Y}} f_{\mathrm{X}} \Delta^{\mathrm{X}}_{\mathrm{Z}} \\
&= \Big(\sum_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Y}} \mu^{\mathrm{X}}_{\mathrm{Y}} f_{\mathrm{X}} \Delta^{\mathrm{X}}_{\mathrm{Y}} \Big) \Delta^{\mathrm{Y}}_{\mathrm{Z}} \\
&= (\zeta \circ f)_{\mathrm{Y}} \Delta^{\mathrm{Y}}_{\mathrm{Z}}.
\end{aligned}
$$

The first step and last step used definition (9.15a). The second step used Lemma 5.43. This made use of the bicommutative bimonoid axiom (2.26) for k and also the hypothesis that f maps into the primitive part of k. The third step used the coassociativity axiom (2.24) for c.

For $\mathrm{Y} > \mathrm{Z}$,

$$
\begin{aligned}
\Delta^{\mathrm{Y}}_{\mathrm{Z}} (\mu \circ g)_{\mathrm{Z}} &= \sum_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Z}} \mu(\mathrm{Z},\mathrm{X})\, \Delta^{\mathrm{Y}}_{\mathrm{Z}} \mu^{\mathrm{X}}_{\mathrm{Z}} g_{\mathrm{X}} \Delta^{\mathrm{X}}_{\mathrm{Z}} \\
&= \sum_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Z}} \mu(\mathrm{Z},\mathrm{X})\, \Delta^{\mathrm{Y}}_{\mathrm{Z}} \mu^{\mathrm{X}}_{\mathrm{Z}} \Delta^{\mathrm{X}}_{\mathrm{Z}} g_{\mathrm{Z}} \\
&= \sum_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Z}} \mu(\mathrm{Z},\mathrm{X})\, \mu^{\mathrm{Y}\vee\mathrm{X}}_{\mathrm{Y}} \Delta^{\mathrm{Y}\vee\mathrm{X}}_{\mathrm{Z}} g_{\mathrm{Z}} \\
&= \sum_{\mathrm{W}\colon \mathrm{W}\geq \mathrm{Y}} \Big(\sum_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Z},\, \mathrm{Y}\vee\mathrm{X}=\mathrm{W}} \mu(\mathrm{Z},\mathrm{X}) \Big) \mu^{\mathrm{W}}_{\mathrm{Y}} \Delta^{\mathrm{W}}_{\mathrm{Z}} g_{\mathrm{Z}} \\
&= 0.
\end{aligned}
$$

The first step used definition (9.15b). The second step used (2.25). This made use of the hypothesis that g is a morphism of comonoids. The third step used the bicommutative bimonoid axiom (2.26) and the coassociativity axiom (2.24) for k. The last step used the Weisner formula (1.38). ☐

Dually:

Theorem 9.41. *For a bicommutative bimonoid* h *and commutative monoid* a*, we have inverse bijections*

$$
\mathcal{A}\text{-}\mathsf{Sp}\big(\mathcal{Q}(\mathsf{h}),\mathsf{a}\big) \underset{\log}{\overset{\exp}{\rightleftarrows}} \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h},\mathsf{a}). \tag{9.18}
$$

In other words: If $f : \mathsf{h} \to \mathsf{a}$ is a derivation, then $\exp(f) : \mathsf{h} \to \mathsf{a}$ is a morphism of monoids. If $g : \mathsf{h} \to \mathsf{a}$ is a morphism of monoids, then $\log(g) : \mathsf{h} \to \mathsf{a}$ is a derivation.

Remark 9.42. We point out that Theorem 9.40 is valid for any commutative bimonoid k, and Theorem 9.41 is valid for any cocommutative bimonoid h. However, assuming the bimonoids to be bicommutative does not entail any loss of generality. See Remark 9.13 in this regard.

Combining Theorems 9.40 and 9.41 in the generality mentioned in Remark 9.42, we obtain:

Theorem 9.43. *For a cocommutative bimonoid* h *and commutative bimonoid* k*, we have inverse bijections*

$$\mathcal{A}\text{-}\mathsf{Sp}\big(\mathcal{Q}(\mathsf{h}), \mathcal{P}(\mathsf{k})\big) \underset{\log}{\overset{\exp}{\rightleftarrows}} \mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h}, \mathsf{k}). \tag{9.19}$$

In other words: If $f : \mathsf{h} \to \mathsf{k}$ is a biderivation, then $\exp(f) : \mathsf{h} \to \mathsf{k}$ is a morphism of bimonoids. If $g : \mathsf{h} \to \mathsf{k}$ is a morphism of bimonoids, then $\log(g) : \mathsf{h} \to \mathsf{k}$ is a biderivation. This can be shown as follows.

$$\begin{tikzcd} \mathsf{h} \arrow[r, "\log(g)"] \arrow[d, two heads] & \mathsf{k} \\ \mathcal{Q}(\mathsf{h}) \arrow[r, dashed] & \mathcal{P}(\mathsf{k}) \arrow[u, hook] \end{tikzcd} \tag{9.20}$$

In continuation of Remark 9.42, observe that we have a commutative diagram of bijections

$$\begin{tikzcd} \mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h}, \mathsf{k}) \arrow[rr, leftrightarrow] \arrow[dd, leftrightarrow] \arrow[dr, leftrightarrow] & & {}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h}, \mathsf{k}^{coab}) \arrow[dd, leftrightarrow] \arrow[dl, leftrightarrow] \\ & \mathcal{A}\text{-}\mathsf{Sp}\big(\mathcal{Q}(\mathsf{h}), \mathcal{P}(\mathsf{k})\big) \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h}_{ab}, \mathsf{k}) \arrow[rr, leftrightarrow] & & {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h}_{ab}, \mathsf{k}^{coab}) \end{tikzcd} \tag{9.21}$$

with the outside square as in (2.58), and the oblique bijections being the exp-log correspondences.

Example 9.44. For a bicommutative bimonoid h, a species p, a map of species $f : \mathsf{p} \to \mathcal{P}(\mathsf{h})$, the morphism of bimonoids $\hat{f} : \mathcal{S}(\mathsf{p}) \to \mathsf{h}$ in Theorem 6.44 is the exponential (9.15a) of the biderivation

$$\begin{tikzcd} \mathcal{S}(\mathsf{p}) \arrow[r, dashed] \arrow[d, two heads] & \mathsf{h} \\ \mathsf{p} \arrow[r, "f"'] & \mathcal{P}(\mathsf{h}). \arrow[u, hook] \end{tikzcd}$$

This can be checked directly: Let us continue to denote the above biderivation by f. We compute $\exp(f)_{\mathrm{Z}}$ on the summand $\mathsf{p}[\mathrm{Y}]$. Firstly, by definition of $\Delta^{\mathrm{X}}_{\mathrm{Z}}$ for $\mathcal{S}(\mathsf{p})$, only the terms $\mathrm{X} \leq \mathrm{Y}$ remain in the rhs of (9.15a). Moreover, since f_{X} involves projection of $\mathcal{S}(\mathsf{p})[\mathrm{X}]$ onto $\mathsf{p}[\mathrm{X}]$, only one term remains, namely, $\mathrm{X} = \mathrm{Y}$. We now see that $\exp(f)_{\mathrm{Z}}$ equals the formula for $\hat{f}$ given in (6.52), and the check is complete. Thus, $f \leftrightarrow \hat{f}$ is an instance of the bijection (9.19).

Similarly, the morphism of bimonoids $\hat{f} : \mathsf{h} \to \mathcal{S}(\mathsf{p})$ in Theorem 6.45 is the exponential (9.15a) of the biderivation

$$\begin{tikzcd} \mathsf{h} \arrow[r, dashed] \arrow[d, two heads] & \mathcal{S}(\mathsf{p}) \\ \mathcal{Q}(\mathsf{h}) \arrow[r, "f"'] & \mathsf{p}. \arrow[u, hook] \end{tikzcd}$$

This can be checked directly using the formula for $\hat{f}$ given in (6.53). Thus, $f \leftrightarrow \hat{f}$ is an instance of the bijection (9.19).

In view of Remark 6.47, the preceding two paragraphs work more generally when h is a commutative bimonoid and a cocommutative bimonoid, respectively. A further generalization is discussed in Exercise 9.59.

A special case is explained below.

Example 9.45. For any species p, recall the abelianization map $\pi : \mathcal{T}(\mathsf{p}) \to \mathcal{S}(\mathsf{p})$ given in (6.61), and the coabelianization map $\pi^\vee : \mathcal{S}^\vee(\mathsf{p}) \to \mathcal{T}^\vee(\mathsf{p})$ given in (6.65). Both are morphisms of bimonoids. They are the exponentials (9.15a), respectively, of the biderivations

$$\begin{tikzcd} \mathcal{T}(\mathsf{p}) \arrow[r, dashed] \arrow[d, two heads] & \mathcal{S}(\mathsf{p}) \\ \mathsf{p} \arrow[r, "\mathrm{id}"'] & \mathsf{p} \arrow[u, hook] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathcal{S}^\vee(\mathsf{p}) \arrow[r, dashed] \arrow[d, two heads] & \mathcal{T}^\vee(\mathsf{p}) \\ \mathsf{p} \arrow[r, "\mathrm{id}"'] & \mathsf{p}. \arrow[u, hook] \end{tikzcd}$$

Note very carefully that $\mathcal{T}(\mathsf{p})$ is cocommutative, $\mathcal{S}(\mathsf{p}) = \mathcal{S}^\vee(\mathsf{p})$ is bicommutative, $\mathcal{T}^\vee(\mathsf{p})$ is commutative.

9.2.4. Logarithm of the identity map. Note from (9.15b) that the *logarithm of the identity map* on a bicommutative bimonoid h is given by

$$\log(\mathrm{id})_{\mathrm{Z}} = \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{X})\, \mu_{\mathrm{Z}}^{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}. \tag{9.22}$$

Lemma 9.46. *When h is bicommutative, any logarithm of the identity map (9.7) reduces to (9.22).*

PROOF. This follows from the second part of Lemma 1.17 and Lemma 9.35. $\square$

Proposition 9.47. *Let h be a bicommutative bimonoid. Then* $\log(\mathrm{id})$ *is an idempotent operator on h whose image is $\mathcal{P}(\mathsf{h})$ and coimage is $\mathcal{Q}(\mathsf{h})$ yielding the commutative diagram of species*

$$\begin{tikzcd} \mathsf{h} \arrow[r, "\log(\mathrm{id})"] \arrow[d, two heads] & \mathsf{h} \\ \mathcal{Q}(\mathsf{h}) \arrow[r, "\cong"'] & \mathcal{P}(\mathsf{h}). \arrow[u, hook] \end{tikzcd} \tag{9.23}$$

In particular, $\mathcal{P}(\mathsf{h})$ and $\mathcal{Q}(\mathsf{h})$ are isomorphic as species. Moreover, the inverse of $\mathcal{Q}(\mathsf{h}) \to \mathcal{P}(\mathsf{h})$ is the map pq_{h} in (5.50).

Diagram (9.23) is an instance of the square in diagram (2.51).

PROOF. This follows by combining Proposition 9.17 with Lemma 9.46. Alternatively, one can take $\mathsf{h} = \mathsf{k}$ in Theorem 9.43 and then use Exercise 9.37. $\square$

Observe that the last claim in Proposition 9.47 gives more information on forward implication in Proposition 5.56, item (3).

Corollary 9.48. *Let* h *be a bicommutative bimonoid. Then* $\log(\mathrm{id}) : \mathsf{h} \twoheadrightarrow \mathcal{P}(\mathsf{h})$ *is a morphism of monoids (with the trivial product on* $\mathcal{P}(\mathsf{h})$*), while* $\log(\mathrm{id}) : \mathcal{Q}(\mathsf{h}) \hookrightarrow \mathsf{h}$ *is a morphism of comonoids (with the trivial coproduct on* $\mathcal{Q}(\mathsf{h})$*).*

Proof. We explain the first statement, the second is similar. By diagram (9.23), $\log(\mathrm{id}) : \mathsf{h} \twoheadrightarrow \mathcal{P}(\mathsf{h})$ factors through $\mathcal{Q}(\mathsf{h})$, so it is a derivation. Therefore, by Lemma 5.33, it is a morphism of monoids with the trivial product on $\mathcal{P}(\mathsf{h})$. $\square$

Corollary 9.49. *Let* $(\mathsf{h}, \mu, \Delta)$ *be a bicommutative bimonoid. Then, for any* $\mathrm{Y} > \mathrm{Z}$*,*

$$\sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{X})\, \Delta^{\mathrm{Y}}_{\mathrm{Z}} \mu^{\mathrm{X}}_{\mathrm{Z}} \Delta^{\mathrm{X}}_{\mathrm{Z}} = 0,$$

$$\sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{X})\, \mu^{\mathrm{X}}_{\mathrm{Z}} \Delta^{\mathrm{X}}_{\mathrm{Z}} \mu^{\mathrm{Y}}_{\mathrm{Z}} = 0.$$

Proof. These identities express the fact that $\log(\mathrm{id})$ is a biderivation, that is, it maps into $\mathcal{P}(\mathsf{h})$ and factors through $\mathcal{Q}(\mathsf{h})$. $\square$

Exercise 9.50. Use the bicommutative bimonoid axiom (2.26) and Weisner formula (1.38) to directly check the identities in Corollary 9.49. Alternatively, deduce them from the identities in Corollary 9.18.

Example 9.51. For a rank-one arrangement $\mathcal{A}$, and a bicommutative $\mathcal{A}$-bimonoid h, we have

$$\log(\mathrm{id})_{\top} = \mathrm{id} : \mathsf{h}[\top] \to \mathsf{h}[\top],$$

$$\log(\mathrm{id})_{\perp} = \mathrm{id} - \mu^{\top}_{\perp} \Delta^{\top}_{\perp} : \mathsf{h}[\perp] \to \mathsf{h}[\perp].$$

Thus, $\log(\mathrm{id})_{\perp}$ and $\mu^{\top}_{\perp} \Delta^{\top}_{\perp}$ are orthogonal idempotent operators on $\mathsf{h}[\perp]$ which add up to the identity. So the kernel of one equals the image of the other and vice versa. This yields a direct verification of Proposition 9.47.

Example 9.52. Recall the bicommutative bimonoid $\mathcal{S}(\mathsf{p})$ from Section 6.5.1 with product and coproduct as in (6.51). Diagram (9.23) specializes to

$$\begin{array}{ccc} \mathcal{S}(\mathsf{p}) & \xrightarrow{\log(\mathrm{id})} & \mathcal{S}(\mathsf{p}) \\ \downarrow\!\!\downarrow & & \uparrow \\ \mathsf{p} & \xrightarrow[\mathrm{id}]{} & \mathsf{p}, \end{array}$$

with the vertical maps being the canonical projection and inclusion. Explicitly, the map $\log(\mathrm{id})_{\mathrm{Z}}$, on the Y-summand, is given by scalar multiplication by $\sum_{\mathrm{X}:\, \mathrm{Z} \leq \mathrm{X} \leq \mathrm{Y}} \mu(\mathrm{Z}, \mathrm{X})$. This is zero if $\mathrm{Z} < \mathrm{Y}$, and 1 if $\mathrm{Z} = \mathrm{Y}$.

The above analysis applies to the exponential bimonoid E in view of (7.11). This can also be seen directly using (co)product formulas (7.9).

Example 9.53. Let us now make diagram (9.23) explicit for $\mathsf{h} = \Pi$, the bimonoid of flats. Using (co)product formulas (7.36) and (7.40), we obtain

$$\log(\mathrm{id})_{\mathrm{Z}}(\mathtt{H}_{\mathrm{Y}/\mathrm{Z}}) = \log(\mathrm{id})_{\mathrm{Z}}(\mathtt{Q}_{\mathrm{Y}/\mathrm{Z}}) = \begin{cases} \mathtt{Q}_{\mathrm{Z}/\mathrm{Z}} & \text{if } \mathrm{Y} = \mathrm{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

So the (co)image of $\log(\mathrm{id})$ is one-dimensional in each component, and can be identified with $\mathcal{Q}(\Pi) = \mathcal{P}(\Pi) = \mathsf{E}$, the exponential species, see (7.46). This observation can also be viewed as the special case $\mathsf{p} := \mathsf{E}$ in Example 9.52 via the first isomorphism in (7.48). Note very carefully that it involves the $\mathtt{Q}$-basis of Π.

Lemma 9.54. *Let $f : \mathsf{h} \to \mathsf{k}$ and $g : \mathsf{k} \to \mathsf{l}$ be morphisms of bimonoids. Further, let h be cocommutative, k be bicommutative, l be commutative. Then the following diagram of species commutes.*

(9.24)
$$\begin{tikzcd}
\mathsf{h} \arrow[rrrr, "\log(gf)"] \arrow[rrd, "\log(f)"'] \arrow[ddd, two heads] & & & & \mathsf{l} \\
& & \mathsf{k} \arrow[urr, "\log(g)"'] \arrow[dr] & & \\
& \mathcal{P}(\mathsf{k}) \arrow[ur] \arrow[rr, "\mathrm{pq}_{\mathsf{k}}"'] & & \mathcal{Q}(\mathsf{k}) \arrow[dr, dashed] & \\
\mathcal{Q}(\mathsf{h}) \arrow[ur, dashed] \arrow[rrrr, dashed] & & & & \mathcal{P}(\mathsf{l}) \arrow[uuu, hook]
\end{tikzcd}$$

PROOF. The outside square and the inside left and right squares are instances of (9.20). The upper triangle commutes by Lemma 9.15, item (1). (Since k is bicommutative, we are justified in writing log instead of **log**.) The middle triangle commutes by definition of pq_{k}. Finally, the bottom square commutes by uniqueness of the dotted arrows. $\square$

Proposition 9.55. *Let h and k be bicommutative bimonoids. Then the bijection (9.19) preserves isomorphisms.*

In other words: If $f : \mathcal{Q}(\mathsf{h}) \to \mathcal{P}(\mathsf{k})$ is an isomorphism of species, then $\exp(f) : \mathsf{h} \to \mathsf{k}$ is an isomorphism of bimonoids. If $g : \mathsf{h} \to \mathsf{k}$ is an isomorphism of bimonoids, then $\log(g) : \mathcal{Q}(\mathsf{h}) \to \mathcal{P}(\mathsf{k})$ is an isomorphism of species.

PROOF. We explain the second statement, the first is similar. Suppose $g : \mathsf{h} \to \mathsf{k}$ and $g' : \mathsf{k} \to \mathsf{h}$ are inverse isomorphisms of bimonoids. Specialize diagram (9.24) to $\mathrm{id}_{\mathsf{k}} = gg'$ and to $\mathrm{id}_{\mathsf{h}} = g'g$, and compare each of them with diagram (9.23). This yields the following commutative diagrams of species.

$$\begin{tikzcd}
\mathcal{Q}(\mathsf{h}) \arrow[r, "\log(g)"] \arrow[d, "\log(\mathrm{id}_{\mathsf{h}})"', "\cong"] & \mathcal{P}(\mathsf{k}) \arrow[d, "\cong"', "\mathrm{pq}_{\mathsf{k}}"] \\
\mathcal{P}(\mathsf{h}) & \mathcal{Q}(\mathsf{k}) \arrow[l, "\log(g')"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{Q}(\mathsf{h}) \arrow[r, "\log(g)"] & \mathcal{P}(\mathsf{k}) \\
\mathcal{P}(\mathsf{h}) \arrow[u, "\mathrm{pq}_{\mathsf{h}}", "\cong"'] & \mathcal{Q}(\mathsf{k}) \arrow[l, "\log(g')"] \arrow[u, "\cong", "\log(\mathrm{id}_{\mathsf{k}})"']
\end{tikzcd}$$

The vertical maps are isomorphisms by Proposition 9.47. The first (resp. second) diagram implies that $\log(g)$ is injective (resp. surjective) and $\log(g')$

is surjective (resp. injective). Hence, $\log(g)$ and $\log(g')$ are both isomorphisms as required. $\square$

Another proof is given in the exercise below.

Exercise 9.56. This is a continuation of Exercise 9.27. Let $f : \mathsf{h} \to \mathsf{k}$ be a morphism of bicommutative bimonoids. Then the diagram

$$\begin{tikzcd} \mathsf{h} \arrow[r, two heads] \arrow[d, "f"'] & \mathcal{Q}(\mathsf{h}) \arrow[r, "\log(\mathrm{id}_{\mathsf{h}})", "\cong"'] \arrow[d, "f"] & \mathcal{P}(\mathsf{h}) \arrow[r, hook] \arrow[d, "f"'] & \mathsf{h} \arrow[d, "f"] \\ \mathsf{k} \arrow[r, two heads] & \mathcal{Q}(\mathsf{k}) \arrow[r, "\cong", "\log(\mathrm{id}_{\mathsf{k}})"'] & \mathcal{P}(\mathsf{k}) \arrow[r, hook] & \mathsf{k} \end{tikzcd}$$

commutes. Show that: $f : \mathsf{h} \to \mathsf{k}$ is injective (surjective) iff $f : \mathcal{P}(\mathsf{h}) \to \mathcal{P}(\mathsf{k})$ is injective (surjective) iff $f : \mathcal{Q}(\mathsf{h}) \to \mathcal{Q}(\mathsf{k})$ is injective (surjective). Deduce Proposition 9.55 as a consequence.

Example 9.57. For a commutative bimonoid h, a species p, a map of species $f : \mathsf{p} \to \mathcal{P}(\mathsf{h})$, the morphism of bimonoids $\hat{f} : \mathcal{T}(\mathsf{p}) \to \mathsf{h}$ in Theorem 6.31 for $q = 1$ is the exponential (9.15a) of the biderivation

$$\begin{tikzcd} \mathcal{T}(\mathsf{p}) \arrow[r, dashed] \arrow[d, two heads] & \mathsf{h} \\ \mathsf{p} \arrow[r, "f"'] & \mathcal{P}(\mathsf{h}). \arrow[u, hook] \end{tikzcd}$$

This follows by combining Examples 9.44 and 9.45, and using Lemma 9.54 and Exercise 6.53. It can also be checked directly as follows. Let us continue to denote the above biderivation by f. We compute $\mathbf{exp}(f)_A$ on the summand $\mathsf{p}[G]$ using formula (9.3a). Only faces F of the same support as G contribute. This follows from definition of Δ^F_A for $\mathcal{T}(\mathsf{p})$ and the fact that f_F involves projection of $\mathcal{T}(\mathsf{p})[F]$ onto $\mathsf{p}[F]$. The calculation continues as follows.

$$\begin{aligned} \mathbf{exp}(f)_A &= \sum_{\substack{F:\, F \geq A, \\ \mathrm{s}(F)=\mathrm{s}(G)}} \boldsymbol{\zeta}(A,F)\, \mu^F_A f_F \beta_{F,G} \\ &= \sum_{\substack{F:\, F \geq A \\ \mathrm{s}(F)=\mathrm{s}(G)}} \boldsymbol{\zeta}(A,F)\, \mu^F_A \beta_{F,G} f_G = \Big(\sum_{\substack{F:\, F \geq A \\ \mathrm{s}(F)=\mathrm{s}(G)}} \boldsymbol{\zeta}(A,F) \Big) \mu^G_A f_G = \mu^G_A f_G. \end{aligned}$$

The third step used commutativity of h, while the last step used the flat-additivity formula (1.43). Note very carefully how $\boldsymbol{\zeta}(A,F)$ disappeared from the picture. We now see that $\mathbf{exp}(f)_A$ equals the formula for $\hat{f}$ given in (6.41).

Similarly, for a cocommutative bimonoid h, the morphism of bimonoids $\hat{f} : \mathsf{h} \to \mathcal{T}^\vee(\mathsf{p})$ in Theorem 6.34 for $q = 1$ is the exponential (9.15a) of the biderivation

$$\begin{tikzcd} \mathsf{h} \arrow[r, dashed] \arrow[d, two heads] & \mathcal{T}^\vee(\mathsf{p}) \\ \mathcal{Q}(\mathsf{h}) \arrow[r, "f"'] & \mathsf{p}. \arrow[u, hook] \end{tikzcd}$$

The formula for $\hat{f}$ is given in (6.42).

Example 9.58. Let c be a cocommutative comonoid, a a commutative monoid, $f : \mathsf{c} \to \mathsf{a}$ a map of species. Then the morphism of monoids $\hat{f} : \mathcal{S}(\mathsf{c}) \to \mathsf{a}$ in Theorem 6.17 is the exponential (9.15a) of the derivation in the first diagram below.

$$\begin{tikzcd} \mathcal{S}(\mathsf{c}) \arrow[r, dashed] \arrow[d, two heads] & \mathsf{a} \\ \mathsf{c} \arrow[ur, "\log(f)"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathsf{c} \arrow[r, dashed] \arrow[dr, "\log(f)"'] & \mathcal{S}^\vee(\mathsf{a}) \\ & \mathsf{a} \arrow[u, hook] \end{tikzcd}$$

Dually, the morphism of comonoids $\hat{f} : \mathsf{c} \to \mathcal{S}^\vee(\mathsf{a})$ in Theorem 6.23 is the exponential (9.15a) of the coderivation in the second diagram above.

We explain the first part. One way to proceed is to exponentiate the derivation and obtain $\hat{f}$. Alternatively, one can take the logarithm of $\hat{f}$. Since it factors through $\mathcal{Q}(\mathcal{S}(\mathsf{c})) = \mathsf{c}$, it suffices to compute it on c. But c is a subcomonoid of $\mathcal{S}(\mathsf{c})$, so we just get $\log(f)$ as required.

We point out that the first part is valid for any comonoid c, and the second part for any monoid a with the caveat that we use instead the exponential (9.3a) and replace $\log(f)$ by $\mathbf{log}(f)$. Let us work in this more general setup. Applying abelianization (6.59) and coabelianization (6.63), we deduce: For $f : \mathsf{c} \to \mathsf{a}$ with a commutative, the morphism of monoids $\hat{f} : \mathcal{T}(\mathsf{c}) \to \mathsf{a}$ in Theorem 6.2 is the exponential (9.3a) of the derivation in the first diagram below.

$$\begin{tikzcd} \mathcal{T}(\mathsf{c}) \arrow[r, dashed] \arrow[d, two heads] & \mathsf{a} \\ \mathsf{c} \arrow[ur, "\mathbf{log}(f)"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathsf{c} \arrow[r, dashed] \arrow[dr, "\mathbf{log}(f)"'] & \mathcal{T}^\vee(\mathsf{a}) \\ & \mathsf{a} \arrow[u, hook] \end{tikzcd}$$

Dually, for $f : \mathsf{c} \to \mathsf{a}$ with c cocommutative, the morphism of comonoids $\hat{f} : \mathsf{c} \to \mathcal{T}^\vee(\mathsf{a})$ in Theorem 6.10 is the exponential (9.3a) of the coderivation in the second diagram above.

Exercise 9.59. Deduce that: For c a cocommutative comonoid, h a commutative bimonoid, $f : \mathsf{c} \to \mathsf{h}$ a morphism of comonoids, the morphism of bimonoids $\hat{f} : \mathcal{S}(\mathsf{c}) \to \mathsf{h}$ in Theorem 6.21 is the exponential (9.15a) of the biderivation in the first diagram below.

$$\begin{tikzcd} \mathcal{S}(\mathsf{c}) \arrow[r, dashed] \arrow[d, two heads] & \mathsf{h} \\ \mathsf{c} \arrow[r, "\log(f)"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{h} \arrow[r, dashed] \arrow[d, two heads] & \mathcal{S}^\vee(\mathsf{a}) \\ \mathcal{Q}(\mathsf{h}) \arrow[r, "\log(f)"'] & \mathsf{a} \arrow[u, hook] \end{tikzcd}$$

Dually, for h a cocommutative bimonoid, a a commutative monoid, $f : \mathsf{h} \to \mathsf{a}$ a morphism of monoids, the morphism of bimonoids $\hat{f} : \mathsf{h} \to \mathcal{S}^\vee(\mathsf{a})$ in Theorem 6.25 is the exponential (9.15a) of the biderivation in the second diagram above.

Letting c be a trivial comonoid and a a trivial monoid recovers Example 9.44.

9.2.5. General norm transformation. We now apply the above ideas to the considerations in Section 6.9.3.

Exercise 9.60. For c a cocommutative comonoid, a a commutative monoid, $f : \mathsf{c} \to \mathsf{a}$ a map of species, let $g : \mathcal{S}(\mathsf{c}) \to \mathcal{S}^\vee(\mathsf{a})$ be the morphism of bimonoids obtained by applying (6.78) to f. Deduce that g is the exponential (9.15a) of the biderivation

$$\begin{tikzcd} \mathcal{S}(\mathsf{c}) \arrow[r, dashed] \arrow[d, two heads] & \mathcal{S}^\vee(\mathsf{a}) \\ \mathsf{c} \arrow[r, "\log(f)"'] & \mathsf{a}. \arrow[u, hook] \end{tikzcd}$$

In other words, (6.78) is the composite of the bijections

$$\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c},\mathsf{a}) \xrightarrow{\log} \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c},\mathsf{a}) \xrightarrow{\exp} {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{S}(\mathsf{c}), \mathcal{S}^\vee(\mathsf{a})).$$

The first is (9.16), while the second is an instance of (9.19). Deduce from Proposition 9.55 that:

g is an isomorphism of bimonoids iff $\log(f)$ is an isomorphism of species.

(Note that $\log(f) = f$ if either c is a trivial comonoid or a a trivial monoid. This recovers the result of Exercise 6.81.)

The second bijection above is discussed later in Exercise 14.29, with a related discussion in Exercise 14.30.

Exercise 9.61. For any cocommutative comonoid c and commutative monoid a, we have a commutative diagram of bijections

$$\begin{tikzcd}[column sep=small] \mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{T}(\mathsf{c}), \mathcal{T}^\vee(\mathsf{a})) \arrow[rr, leftrightarrow] & & {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{S}(\mathsf{c}), \mathcal{S}^\vee(\mathsf{a})) \\ & \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c},\mathsf{a}) \arrow[ul, leftrightarrow] \arrow[ur, leftrightarrow] & \end{tikzcd}$$

with the oblique bijections being the exp-log correspondences (9.19). Deduce this as an instance of diagram (9.21). Recall here that the abelianization of $\mathcal{T}(\mathsf{c})$ is $\mathcal{S}(\mathsf{c})$, and the coabelianization of $\mathcal{T}^\vee(\mathsf{a})$ is $\mathcal{S}^\vee(\mathsf{a})$ (Section 6.6).

For c a cocommutative comonoid, a a commutative monoid, $f : \mathsf{c} \to \mathsf{a}$ a map of species, let $g : \mathcal{T}(\mathsf{c}) \to \mathcal{T}^\vee(\mathsf{a})$ be the morphism of bimonoids obtained by applying (6.77) to f with $q = 1$. Use (6.79) and Exercise 9.60 to deduce that g is the exponential (9.15a) of the biderivation

$$\begin{tikzcd} \mathcal{T}(\mathsf{c}) \arrow[r, dashed] \arrow[d, two heads] & \mathcal{T}^\vee(\mathsf{a}) \\ \mathsf{c} \arrow[r, "\log(f)"'] & \mathsf{a}. \arrow[u, hook] \end{tikzcd}$$

Exercise 9.62. Let (ξ_{X}) be any set of scalars indexed by flats. Consider the map of species $f : \mathsf{E} \to \mathsf{E}$ which on the X-component is scalar multiplication by ξ_{X}. Check that $\log(f)_{\mathrm{Z}}$ is scalar multiplication by $\sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z},\mathrm{X})\, \xi_{\mathrm{X}}$. This scalar is η_{Z} as defined in (7.49). Now use Exercise 7.13 and Exercise 9.60 to deduce the last claim in Theorem 7.10.

9.2.6. Mixed distributive law for bicommutative bimonoids. We now give a necessary and sufficient condition for the mixed distributive law (3.16) for bicommutative bimonoids to be an isomorphism.

Exercise 9.63. Let p be a species. For the map $f : \mathcal{S}^\vee(\mathsf{p}) \twoheadrightarrow \mathsf{p} \hookrightarrow \mathcal{S}(\mathsf{p})$, check that $\log(f)_{\mathrm{Z}}$, on the X-summand, is given by scalar multiplication by $\mu(\mathrm{Z}, \mathrm{X})$.

Now as a consequence of Exercise 6.80, item (1) and Exercise 9.60, we deduce that:

Theorem 9.64. *For a species* p*, the map* $\mathcal{S}\mathcal{S}^\vee(\mathsf{p}) \to \mathcal{S}^\vee\mathcal{S}(\mathsf{p})$ *in* (3.16) *is an isomorphism of bimonoids iff for each flat* X*, either* $\mathsf{p}[\mathrm{X}] = 0$ *or* $\mu(\mathrm{Z}, \mathrm{X}) \neq 0$ *for all* $\mathrm{Z} \leq \mathrm{X}$*.*

In particular, we deduce that the map $\mathcal{S}\mathcal{S}^\vee(\mathsf{x}) \to \mathcal{S}^\vee\mathcal{S}(\mathsf{x})$ is an isomorphism iff $\mu(\mathrm{Z}, \top) \neq 0$ for all Z. In view of Exercise 7.12, this recovers a part of Theorem 7.11.

An explicit diagonalization of (3.16) is obtained later in Exercise 14.32.

Exercise 9.65. Let p be a species. For the map $\mathrm{id} : \mathcal{S}^\vee(\mathsf{p}) \to \mathcal{S}(\mathsf{p})$, check that $\log(\mathrm{id})_{\mathrm{Z}}$, on the X-summand, is identity if $\mathrm{X} = \mathrm{Z}$, and 0 otherwise. Thus, it is far from being an isomorphism. This is consistent with the fact that the map $\mathcal{S}\mathcal{S}^\vee(\mathsf{p}) \to \mathcal{S}^\vee\mathcal{S}(\mathsf{p})$ in Exercise 6.80, item (2) is not an isomorphism.

9.2.7. Signed analogue. Let c be a signed cocommutative comonoid and a a signed commutative monoid. We work with the formulation given in Proposition 2.36 and its dual. For any $s \in \mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$ and $f : \mathsf{c} \to \mathsf{a}$ a map of species, define another map of species $s \circ f : \mathsf{c} \to \mathsf{a}$ by

$$(s \circ f)_{\mathrm{Z}} := \sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Z}} s(\mathrm{Z}, \mathrm{X})\, \mu^{\mathrm{X}}_{\mathrm{Z}} (\mathrm{id} \otimes f_{\mathrm{X}}) \Delta^{\mathrm{X}}_{\mathrm{Z}}, \tag{9.25}$$

where id refers to the identity map on $\mathbf{E}^-[\mathrm{Z}, \mathrm{X}]$. This defines an action of the flat-incidence algebra on $\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a})$.

Exercise 9.66. Let c be a signed cocommutative comonoid and a a signed commutative monoid. Check that the action (9.1) of the lune-incidence algebra on $\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a})$ factors through the base-case map (1.45) to yield an action of the flat-incidence algebra on $\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a})$ which coincides with (9.25).

In the signed setting, by specializing (9.25), the *exponential* and *logarithm* of $f : \mathsf{c} \to \mathsf{a}$ are given by

$$\exp(f)_{\mathrm{Z}} = \sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Z}} \mu^{\mathrm{X}}_{\mathrm{Z}} (\mathrm{id} \otimes f_{\mathrm{X}}) \Delta^{\mathrm{X}}_{\mathrm{Z}}, \tag{9.26a}$$

$$\log(f)_{\mathrm{Z}} = \sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{X})\, \mu^{\mathrm{X}}_{\mathrm{Z}} (\mathrm{id} \otimes f_{\mathrm{X}}) \Delta^{\mathrm{X}}_{\mathrm{Z}}. \tag{9.26b}$$

We have inverse bijections

$$\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a}) \underset{\log}{\overset{\exp}{\rightleftarrows}} \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a}). \tag{9.27}$$

This is the *signed commutative exp-log correspondence*.

Exercise 9.67. The signed analogue of Lemma 9.38 says that when c is signed cocommutative and a is signed commutative, all exp-log correspondences (9.4) reduce to (9.27). Formulate signed analogues of Theorems 9.40, 9.41, 9.43. Deduce them from Theorems 9.30 and 9.32. Similarly, formulate the signed analogue of Propositions 9.47 and 9.55. (See Exercise 9.31.)

9.2.8. LRB species. Theorems 9.11 and 9.12 are valid in the more general setting of LRB species (Section 3.9). The same proofs go through. Let us recall here that noncommutative zeta and Möbius functions make sense for any LRB (Remark 1.18). Similarly, Proposition 9.17 and Corollary 9.18 also generalize to LRBs.

In the same vein, Theorems 9.40, 9.41, 9.43, Proposition 9.47, Corollary 9.49, Proposition 9.55 all generalize to LRBs.

When the LRB is specialized to the Tits monoid, one recovers all the above results. Also note very carefully that when the LRB is specialized to the Birkhoff monoid, Theorem 9.11 and Theorem 9.40 coincide, Theorem 9.12 and Theorem 9.41 coincide, and so on.

9.3. Deformed exp-log correspondences

Recall the bilune-incidence algebra $\mathrm{I}_{\mathrm{bilune}}[\mathcal{A}]$ from Section 1.6.3. We define an action of the bilune-incidence algebra on the space of all maps from a comonoid c to a monoid a. This extends the action of the lune-incidence algebra defined in Section 9.1. Now suppose that q is not a root of unity. We formulate the q-exp-log correspondence via the actions of the two-sided q-zeta and q-Möbius functions. We also define the q-logarithm of the identity map on a q-bimonoid, and use it to deduce that the primitive part and indecomposable part of a q-bimonoid are isomorphic. The case $q = 0$ is developed separately in Section 9.4.

9.3.1. Action of the bilune-incidence algebra. Let c be a comonoid and a a monoid. For any $s \in \mathrm{I}_{\mathrm{bilune}}[\mathcal{A}]$ and $f : \mathsf{c} \to \mathsf{a}$ a map of species, define $s \circ f : \mathsf{c} \to \mathsf{a}$ by

$$(s \circ f)_A := \sum_{\substack{F, F' \geq A \\ \mathrm{s}(F) = \mathrm{s}(F')}} s(A, F, F')\, \mu_A^F \beta_{F,F'} f_{F'} \Delta_A^{F'}. \tag{9.28a}$$

The sum is over both F and F'. Equivalently, using (2.3), one can rewrite

$$(s \circ f)_A := \sum_{\substack{F, F' \geq A \\ \mathrm{s}(F) = \mathrm{s}(F')}} s(A, F, F')\, \mu_A^F f_F \beta_{F,F'} \Delta_A^{F'}. \tag{9.28b}$$

The map $s \circ f$ is a map of species. This can be deduced using (1.58), (2.3), naturality of the product (2.8) and coproduct (2.10), and the bijection in Lemma 1.6.

Lemma 9.68. *For a comonoid* c *and monoid* a*, the assignment* $(s, f) \mapsto s \circ f$ *defines a left action of the bilune-incidence algebra on* $\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a})$*. Thus,* $\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a})$ *is a left module over the bilune-incidence algebra.*

This can be checked directly as in the proof of Lemma 9.2.

Exercise 9.69. For a comonoid c and monoid a, the action (9.28a) restricts to an action of the lune-incidence algebra on $\mathcal{A}$-$\mathsf{Sp}(\mathsf{c}, \mathsf{a})$ via the inclusion map (1.62). Check that this action coincides with (9.1).

Lemma 9.70. *We have:*

- *When* c *is cocommutative, the action* (9.28b) *factors through the map* p *in* (1.60) *to induce the action* (9.1).
- *When* a *is commutative, the action* (9.28a) *factors through the map* q *in* (1.60) *to induce the action* (9.1).
- *When* c *is cocommutative and* a *is commutative, the action factors further through the base-case map to induce the action* (9.14).

PROOF. When c is cocommutative, the composite $\beta_{F,F'}\Delta_A^{F'}$ in (9.28b) can be replaced by Δ_A^F, and the scalars $s(A, F, F')$ can then be summed over F', which is precisely how the map p is defined. The situation when a is commutative can be dealt with similarly. The last claim follows from Lemma 9.35. $\square$

Lemma 9.3 extends as follows.

Lemma 9.71. *The action of the bilune-incidence algebra on* $\mathcal{A}$-$\mathsf{Sp}(\mathsf{c}, \mathsf{a})$ *is natural in* c *and* a. *More precisely: The map* (9.2) *is a map of left modules over the bilune-incidence algebra.*

PROOF. This is checked below. For $s \in \mathrm{I}_{\mathrm{bilune}}[\mathcal{A}]$,

$$\begin{aligned}
(g(s \circ h)f)_A &= \sum_{F,F' \geq A,\, \mathrm{s}(F)=\mathrm{s}(F')} s(A, F, F')\, g_A \mu_A^F \beta_{F,F'} h_{F'} \Delta_A^{F'} f_A \\
&= \sum_{F,F' \geq A,\, \mathrm{s}(F)=\mathrm{s}(F')} s(A, F, F')\, \mu_A^F \beta_{F,F'} g_{F'} h_{F'} f_{F'} \Delta_A^{F'} \\
&= (s \circ ghf)_A.
\end{aligned}$$

Compare with the calculation in the proof of Lemma 9.3. $\square$

Exercise 9.72. Extend Exercise 9.4 as follows. Let $f : \mathsf{c} \to \mathsf{a}$ be a map of species from a comonoid c to a monoid a, and let $f^* : \mathsf{a}^* \to \mathsf{c}^*$ denote its dual. Check that for any $s \in \mathrm{I}_{\mathrm{bilune}}[\mathcal{A}]$, we have $(s \circ f)^* = s' \circ f^*$, where $s'(A, F', F) := s(A, F, F')$.

Exercise 9.73. Check that: Lemmas 9.5 and 9.6 and Exercise 9.7 hold more generally for $s \in \mathrm{I}_{\mathrm{bilune}}[\mathcal{A}]$ with $s(A, A, A) = 1$ for all A, and with $s \circ f$ as in (9.28a).

Exercise 9.74. Formulate the signed analogue of Lemma 9.70. (Replace (co)commutative (co)monoid by signed (co)commutative (co)monoid, and p and q by the algebra morphisms $s \mapsto p(s_{-1})$ and $s \mapsto q(s_{-1})$, respectively, with s_{-1} as in Exercise 1.29.)

9.3.2. *q*-exp-log correspondence. Let q be any scalar which is not a root of unity. Recall the two-sided q-zeta function $\boldsymbol{\zeta}_q$ and two-sided q-Möbius function $\boldsymbol{\mu}_q$ defined in Section 1.6.6. These are elements of the symmetric bilune-incidence algebra.

For a map of species $f : \mathsf{c} \to \mathsf{a}$ from a comonoid c to a monoid a, we say that

$$\mathbf{exp}_q(f) := \boldsymbol{\zeta}_q \circ f \quad \text{and} \quad \mathbf{log}_q(f) := \boldsymbol{\mu}_q \circ f$$

are the *q-exponential* and *q-logarithm* of f, respectively. Explicitly, using formula (9.28a),

$$\mathbf{exp}_q(f)_A = \sum_{\substack{F,F' \geq A \\ \mathrm{s}(F)=\mathrm{s}(F')}} \boldsymbol{\zeta}_q(A,F,F')\, \mu_A^F \beta_{F,F'} f_{F'} \Delta_A^{F'}, \tag{9.29a}$$

$$\mathbf{log}_q(f)_A = \sum_{\substack{F,F' \geq A \\ \mathrm{s}(F)=\mathrm{s}(F')}} \boldsymbol{\mu}_q(A,F,F')\, \mu_A^F \beta_{F,F'} f_{F'} \Delta_A^{F'}. \tag{9.29b}$$

These operations are uniquely defined. Since $\boldsymbol{\zeta}_q$ and $\boldsymbol{\mu}_q$ are inverse to each other, by Lemma 9.68, we obtain inverse bijections

$$\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c},\mathsf{a}) \underset{\mathbf{log}_q}{\overset{\mathbf{exp}_q}{\rightleftarrows}} \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c},\mathsf{a}). \tag{9.30}$$

This is the *q-exp-log correspondence*.

Now recall the noncommutative q-zeta function $\boldsymbol{\zeta}_q$ and noncommutative q-Möbius function $\boldsymbol{\mu}_q$ defined in Section 1.5.9. These are elements of the lune-incidence algebra.

Lemma 9.75. *If either c is cocommutative or a is commutative, then*

$$\mathbf{exp}_q(f)_A = \sum_{F:\, F \geq A} \boldsymbol{\zeta}_q(A,F)\, \mu_A^F f_F \Delta_A^F, \tag{9.31a}$$

$$\mathbf{log}_q(f)_A = \sum_{F:\, F \geq A} \boldsymbol{\mu}_q(A,F)\, \mu_A^F f_F \Delta_A^F. \tag{9.31b}$$

If c is cocommutative and a is commutative, then

$$\mathbf{exp}_q(f)_{\mathrm{Z}} = \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \zeta_q(\mathrm{Z},\mathrm{X})\, \mu_{\mathrm{Z}}^{\mathrm{X}} f_{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}, \tag{9.31c}$$

$$\mathbf{log}_q(f)_{\mathrm{Z}} = \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mu_q(\mathrm{Z},\mathrm{X})\, \mu_{\mathrm{Z}}^{\mathrm{X}} f_{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}. \tag{9.31d}$$

PROOF. This follows from Lemma 9.70, Lemma 1.37, and (1.50). □

Exercise 9.76. Check that: If either c is signed cocommutative or a is signed commutative, then $\mathbf{exp}_q(f)$ and $\mathbf{log}_q(f)$ are given as in (9.31a) and (9.31b) with $\boldsymbol{\zeta}_q$ and $\boldsymbol{\mu}_q$ replaced by $\boldsymbol{\zeta}_{-q}$ and $\boldsymbol{\mu}_{-q}$, respectively. (Use Exercise 9.74, Exercise 1.35, Lemma 1.37.)

Exercise 9.77. Deduce from Lemma 9.71 that: The q-exp-log correspondence (9.30) is natural in c and a. (Compare with Lemma 9.10.)

9.3.3. (Co, bi)derivations and (co, bi)monoid morphisms.

Theorem 9.78. *For q not a root of unity, for a comonoid* c *and q-bimonoid* k*, we have inverse bijections*

$$\mathcal{A}\text{-}\mathsf{Sp}\big(\mathsf{c}, \mathcal{P}(\mathsf{k})\big) \underset{\mathbf{log}_q}{\overset{\mathbf{exp}_q}{\rightleftarrows}} \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c}, \mathsf{k}). \tag{9.32}$$

In other words: If $f : \mathsf{c} \to \mathsf{k}$ is a coderivation, then $\mathbf{exp}_q(f) : \mathsf{c} \to \mathsf{k}$ is a morphism of comonoids. If $g : \mathsf{c} \to \mathsf{k}$ is a morphism of comonoids, then $\mathbf{log}_q(g) : \mathsf{c} \to \mathsf{k}$ is a coderivation.

Proof. The proof is similar to that of Theorem 9.11; the two calculations are indicated below.

For $G \geq A$,

$$\begin{aligned}
\Delta^G_A(\boldsymbol{\zeta}_q \circ f)_A &= \sum_{\substack{F,F' \geq A \\ \mathrm{s}(F)=\mathrm{s}(F')}} \boldsymbol{\zeta}_q(A, F, F')\, \Delta^G_A \mu^F_A \beta_{F,F'} f_{F'} \Delta^{F'}_A \\
&= \sum_{\substack{F,F' \geq A \\ \mathrm{s}(F)=\mathrm{s}(F')\geq \mathrm{s}(G)}} \boldsymbol{\zeta}_q(A, F, F') q^{\mathrm{dist}(F,G)}\, \mu^{GF}_G \beta_{GF,F} \beta_{F,F'} f_{F'} \Delta^{F'}_A \\
&= \sum_{H \geq G} \sum_{\substack{F,F' \geq A,\, GF=H \\ \mathrm{s}(F)=\mathrm{s}(F')=\mathrm{s}(H)}} \boldsymbol{\zeta}_q(A, F, F') q^{\mathrm{dist}(F,G)}\, \mu^H_G \beta_{H,F'} f_{F'} \Delta^{F'}_A \\
&= \sum_{\substack{H \geq G,\, F' \geq A \\ \mathrm{s}(H)=\mathrm{s}(F')}} \Big(\sum_{\substack{F \geq A,\, GF=H \\ \mathrm{s}(F)=\mathrm{s}(F')}} \boldsymbol{\zeta}_q(A, F, F') q^{\mathrm{dist}(F,G)} \Big) \mu^H_G \beta_{H,F'} f_{F'} \Delta^{F'}_A \\
&= \Big(\sum_{\substack{H,F' \geq G \\ \mathrm{s}(H)=\mathrm{s}(F')}} \boldsymbol{\zeta}_q(G, H, F')\, \mu^H_G \beta_{H,F'} f_{F'} \Delta^{F'}_G \Big) \Delta^G_A \\
&= (\boldsymbol{\zeta}_q \circ f)_G \Delta^G_A.
\end{aligned}$$

Note the use of Lemma 5.39 and the two-sided q-lune-additivity formula (1.66).

For $G > A$,

$$\begin{aligned}
\Delta^G_A(\boldsymbol{\mu}_q \circ g)_A &= \sum_{\substack{F,F' \geq A \\ \mathrm{s}(F)=\mathrm{s}(F')}} \boldsymbol{\mu}_q(A, F, F')\, \Delta^G_A \mu^F_A \beta_{F,F'} g_{F'} \Delta^{F'}_A \\
&= \sum_{\substack{F,F' \geq A \\ \mathrm{s}(F)=\mathrm{s}(F')}} \boldsymbol{\mu}_q(A, F, F')\, \Delta^G_A \mu^F_A \beta_{F,F'} \Delta^{F'}_A g_A \\
&= \sum_{\substack{F,F' \geq A \\ \mathrm{s}(F)=\mathrm{s}(F')}} \boldsymbol{\mu}_q(A, F, F') q^{\mathrm{dist}(F,G)}\, \mu^{GF}_G \beta_{GF,FG} \Delta^{FG}_F \beta_{F,F'} \Delta^{F'}_A g_A \\
&= \sum_{\substack{F,F' \geq A \\ \mathrm{s}(F)=\mathrm{s}(F')}} \boldsymbol{\mu}_q(A, F, F') q^{\mathrm{dist}(F,G)}\, \mu^{GF}_G \beta_{GF,F'G} \Delta^{F'G}_A g_A
\end{aligned}$$

$$= \sum_{\substack{H \geq G,\, H' \geq A \\ \mathrm{s}(H)=\mathrm{s}(H')}} \Big(\sum_{\substack{F, F' \geq A \\ \mathrm{s}(F)=\mathrm{s}(F') \\ GF=H,\, F'G=H'}} \boldsymbol{\mu}_q(A, F, F')\, q^{\mathrm{dist}(F,G)} \Big) \mu_G^H \beta_{H,H'} \Delta_A^{H'} g_A$$
$$= 0.$$

Note the use of the two-sided q-Weisner formula (1.67). □

Dually:

Theorem 9.79. *For q not a root of unity, for a q-bimonoid h and monoid a, we have inverse bijections*

$$\mathcal{A}\text{-}\mathsf{Sp}\big(\mathcal{Q}(\mathsf{h}), \mathsf{a}\big) \underset{\mathbf{log}_q}{\overset{\mathbf{exp}_q}{\rightleftarrows}} \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h}, \mathsf{a}). \tag{9.33}$$

In other words: If $f : \mathsf{h} \to \mathsf{a}$ is a derivation, then $\mathbf{exp}_q(f) : \mathsf{h} \to \mathsf{a}$ is a morphism of monoids. If $g : \mathsf{h} \to \mathsf{a}$ is a morphism of monoids, then $\mathbf{log}_q(g) : \mathsf{h} \to \mathsf{a}$ is a derivation.

Exercise 9.80. Prove Theorem 9.78 in the special case when c is cocommutative by directly working with the simpler formulation provided by Lemma 9.75. In this case, the q-lune-additivity formula (1.46) and the noncommutative q-Weisner formula (1.48) intervene in the two calculations. Same comment applies to Theorem 9.79 in the special case when a is commutative.

Similarly, in the signed setting, one can work with the formulation in Exercise 9.76.

Combining Theorems 9.78 and 9.79, we obtain:

Theorem 9.81. *For q not a root of unity, for q-bimonoids h and k, we have inverse bijections*

$$\mathcal{A}\text{-}\mathsf{Sp}\big(\mathcal{Q}(\mathsf{h}), \mathcal{P}(\mathsf{k})\big) \underset{\mathbf{log}_q}{\overset{\mathbf{exp}_q}{\rightleftarrows}} q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h}, \mathsf{k}). \tag{9.34}$$

In other words: If $f : \mathsf{h} \to \mathsf{k}$ is a biderivation, then $\mathbf{exp}_q(f) : \mathsf{h} \to \mathsf{k}$ is a morphism of q-bimonoids. If $g : \mathsf{h} \to \mathsf{k}$ is a morphism of q-bimonoids, then $\mathbf{log}_q(g) : \mathsf{h} \to \mathsf{k}$ is a biderivation.

Example 9.82. Suppose q is not a root of unity. For a q-bimonoid h, a species p, a map of species $f : \mathsf{p} \to \mathcal{P}(\mathsf{h})$, the morphism of q-bimonoids $\hat{f} : \mathcal{T}_q(\mathsf{p}) \to \mathsf{h}$ in Theorem 6.31 is the q-exponential (9.29a) of the biderivation

$$\begin{tikzcd} \mathcal{T}_q(\mathsf{p}) \arrow[r, dashed] \arrow[d, two heads] & \mathsf{h} \\ \mathsf{p} \arrow[r, "f"'] & \mathcal{P}(\mathsf{h}). \arrow[u, hook] \end{tikzcd}$$

(Recall from Exercise 6.35 that $\mathcal{Q}(\mathcal{T}_q(\mathsf{p})) = \mathsf{p}$.) This can be checked directly: Let us continue to denote the above biderivation by f. We compute $\mathbf{exp}_q(f)_A$ on the summand $\mathsf{p}[G]$. Only faces F' of the same support as G contribute.

This follows from definition of $\Delta_A^{F'}$ for $\mathcal{T}_q(\mathsf{p})$ and the fact that $f_{F'}$ involves projection of $\mathcal{T}_q(\mathsf{p})[F']$ onto $\mathsf{p}[F']$. The calculation continues as follows.

$$\begin{aligned}
\mathbf{exp}_q(f)_A &= \sum_{\substack{F,F' \geq A,\\ \mathrm{s}(F)=\mathrm{s}(F')=\mathrm{s}(G)}} \boldsymbol{\zeta}_q(A,F,F')\, q^{\mathrm{dist}(F',G)}\, \mu_A^F \beta_{F,F'} f_{F'} \beta_{F',G} \\
&= \sum_{\substack{F,F' \geq A,\\ \mathrm{s}(F)=\mathrm{s}(F')=\mathrm{s}(G)}} \boldsymbol{\zeta}_q(A,F,F')\, q^{\mathrm{dist}(F',G)}\, \mu_A^F f_F \beta_{F,G} \\
&= \sum_{\substack{F:\, F\geq A\\ \mathrm{s}(F)=\mathrm{s}(G)}} \Big(\sum_{\substack{F':\, F'\geq A\\ \mathrm{s}(F')=\mathrm{s}(F)=\mathrm{s}(G)}} \boldsymbol{\zeta}_q(A,F,F')\, q^{\mathrm{dist}(F',G)} \Big) \mu_A^F f_F \beta_{F,G} \\
&= \mu_A^G f_G.
\end{aligned}$$

The last step used the two-sided q-flat-additivity formula (1.65). We now see that $\mathbf{exp}_q(f)_A$ equals the formula for $\hat{f}$ given in (6.41). Thus, $f \leftrightarrow \hat{f}$ is an instance of the bijection (9.34). Interestingly, the q in $\boldsymbol{\zeta}_q$ and in the coproduct of $\mathcal{T}_q(\mathsf{p})$ canceled out, and so no q is visible in the formula for $\hat{f}$.

Similarly, the morphism of q-bimonoids $\hat{f} : \mathsf{h} \to \mathcal{T}_q^\vee(\mathsf{p})$ in Theorem 6.34 is the q-exponential (9.29a) of the biderivation

$$\begin{tikzcd}
\mathsf{h} \arrow[r, dashed] \arrow[d, two heads] & \mathcal{T}_q^\vee(\mathsf{p}) \\
\mathcal{Q}(\mathsf{h}) \arrow[r, "f"'] & \mathsf{p}. \arrow[u, hook]
\end{tikzcd}$$

(Recall from Exercise 6.33 that $\mathcal{P}(\mathcal{T}_q^\vee(\mathsf{p})) = \mathsf{p}$.) Thus, $f \leftrightarrow \hat{f}$ is an instance of the bijection (9.34).

A generalization is discussed in Exercise 9.96.

Lemma 9.83. *Suppose q is not a root of unity. Let c be a comonoid, h a q-bimonoid, a a monoid. Then:*

(1) *If $f : \mathsf{c} \to \mathsf{h}$ is a morphism of comonoids, and $g : \mathsf{h} \to \mathsf{a}$ is a morphism of monoids, then $\mathbf{log}_q(gf) = \mathbf{log}_q(g)\,\mathbf{log}_q(f)$.*

(2) *If $f : \mathsf{c} \to \mathsf{h}$ is a coderivation, and $g : \mathsf{h} \to \mathsf{a}$ is a derivation, then $\mathbf{exp}_q(gf) = \mathbf{exp}_q(g)\,\mathbf{exp}_q(f)$.*

Proof. This is similar to the proof of Lemma 9.15. More precisely, the situation here is analogous to the one in Remark 9.16. We now employ Exercise 9.73, Exercise 9.77 and (9.34). □

9.3.4. *q*-logarithm of the identity map. Let q be any scalar which is not a root of unity. Let h be a p-bimonoid for a scalar p possibly different from q. Note from (9.29b) that the *q-logarithm of the identity map* on h is given by

$$\mathbf{log}_q(\mathrm{id})_A = \sum_{\substack{F,F' \geq A\\ \mathrm{s}(F)=\mathrm{s}(F')}} \boldsymbol{\mu}_q(A,F,F')\, \mu_A^F \beta_{F,F'} \Delta_A^{F'}. \tag{9.35}$$

Proposition 9.84. *Let* h *be a* q*-bimonoid for* q *not a root of unity. Then* $\mathbf{log}_q(\mathrm{id})$ *is an idempotent operator on* h *whose image is* $\mathcal{P}(\mathsf{h})$ *and coimage is* $\mathcal{Q}(\mathsf{h})$ *yielding the commutative diagram of species*

$$\begin{tikzcd} \mathsf{h} \arrow[r, "\mathbf{log}_q(\mathrm{id})"] \arrow[d, two heads] & \mathsf{h} \\ \mathcal{Q}(\mathsf{h}) \arrow[r, "\cong"'] & \mathcal{P}(\mathsf{h}). \arrow[u, hook] \end{tikzcd} \tag{9.36}$$

In particular, $\mathcal{P}(\mathsf{h})$ *and* $\mathcal{Q}(\mathsf{h})$ *are isomorphic as species. Moreover, the inverse of* $\mathcal{Q}(\mathsf{h}) \to \mathcal{P}(\mathsf{h})$ *is the map* pq_{h} *in* (5.50).

Diagram (9.36) is an instance of the square in diagram (2.51).

Proof. One can take $\mathsf{h} = \mathsf{k}$ in Theorem 9.81 and then use Exercise 9.73. □

Observe that Proposition 5.58 follows from the above result. Another proof of this result using characteristic operations is given later in Exercise 10.37.

Corollary 9.85. *Let* h *be a* q*-bimonoid for* q *not a root of unity. Then* $\mathbf{log}_q(\mathrm{id}) : \mathsf{h} \twoheadrightarrow \mathcal{P}(\mathsf{h})$ *is a morphism of monoids (with the trivial product on* $\mathcal{P}(\mathsf{h})$*), while* $\mathbf{log}_q(\mathrm{id}) : \mathcal{Q}(\mathsf{h}) \hookrightarrow \mathsf{h}$ *is a morphism of comonoids (with the trivial coproduct on* $\mathcal{Q}(\mathsf{h})$*).*

Proof. This follows from diagram (9.36) and Lemma 5.33. □

Corollary 9.86. *Let* $(\mathsf{h}, \mu, \Delta)$ *be a* q*-bimonoid for* q *not a root of unity. Then, for any* $G > A$,

$$\sum_{\substack{F, F' \geq A \\ \mathrm{s}(F) = \mathrm{s}(F')}} \boldsymbol{\mu}_q(A, F, F')\, \Delta^G_A \mu^F_A \beta_{F,F'} \Delta^{F'}_A = 0,$$

$$\sum_{\substack{F, F' \geq A \\ \mathrm{s}(F) = \mathrm{s}(F')}} \boldsymbol{\mu}_q(A, F, F')\, \mu^F_A \beta_{F,F'} \Delta^{F'}_A \mu^G_A = 0.$$

The sums are over both F *and* F'.

Proof. These identities express the fact that $\mathbf{log}_q(\mathrm{id})$ is a biderivation, that is, it maps into $\mathcal{P}(\mathsf{h})$ and factors through $\mathcal{Q}(\mathsf{h})$. □

Example 9.87. Suppose q is not a root of unity. For a rank-one arrangement $\mathcal{A}$ with chambers C and $\overline{C}$, and an $\mathcal{A}$-q-bimonoid h, in continuation of Example 1.36, we have

$$\mathbf{log}_q(\mathrm{id})_O = \mathrm{id} - \frac{1}{1-q^2}(\mu^C_O \Delta^C_O + \mu^{\overline{C}}_O \Delta^{\overline{C}}_O) + \frac{q}{1-q^2}(\mu^C_O \beta_{C,\overline{C}} \Delta^{\overline{C}}_O + \mu^{\overline{C}}_O \beta_{\overline{C},C} \Delta^C_O).$$

This is a linear operator on $\mathsf{h}[O]$ whose coimage is $\mathcal{Q}(\mathsf{h})[O]$ and image is $\mathcal{P}(\mathsf{h})[O]$.

For $\mathsf{h} := \Gamma_q$, the q-bimonoid of chambers, one may readily check using formulas (7.19) that the above is the zero operator on $\Gamma_q[O]$. For instance,

$$\mathbf{log}_q(\mathrm{id})_O(\mathtt{H}_C) = \mathtt{H}_C - \frac{1}{1-q^2}(\mathtt{H}_C + q\,\mathtt{H}_{\overline{C}}) + \frac{q}{1-q^2}(q\,\mathtt{H}_C + \mathtt{H}_{\overline{C}}) = 0.$$

This is consistent with (7.155) which says that $\mathcal{P}(\Gamma_q) = \mathsf{x}$.

Similarly, for $\mathsf{h} := \Sigma_q$, the q-bimonoid of faces, using formulas (7.65),

$$\begin{aligned}\mathbf{log}_q(\mathrm{id})_O(\mathtt{H}_O) = \mathtt{H}_O - \frac{1}{1-q^2}(\mathtt{H}_C + \mathtt{H}_{\overline{C}}) + \frac{q}{1-q^2}(\mathtt{H}_C + \mathtt{H}_{\overline{C}})& \\ = \mathtt{H}_O - \frac{1}{1+q}(\mathtt{H}_C + \mathtt{H}_{\overline{C}}) = \mathtt{Q}_O&.\end{aligned}$$

This is an element of the $\mathtt{Q}$-basis defined in (7.70). The last step used the calculation in Example 1.21. We deduce that the image of $\mathbf{log}_q(\mathrm{id})_O$ on $\Sigma_q[O]$ is one-dimensional and spanned by $\mathtt{Q}_O$. This is consistent with (7.160) which says that $\mathcal{P}(\Sigma_q) = \mathsf{E}$.

More general calculations are indicated in the exercise below.

Exercise 9.88. Suppose q is not a root of unity. For $\mathsf{h} := \Gamma_q$, the q-bimonoid of chambers, using formulas (7.19), check that

$$\mathbf{log}_q(\mathrm{id})_A(\mathtt{H}_{C/A}) = \begin{cases} \mathtt{H}_{A/A} & \text{if } C = A, \\ 0 & \text{otherwise.} \end{cases}$$

Deduce that $\mathcal{Q}(\Gamma_q) = \mathcal{P}(\Gamma_q) = \mathsf{x}$. Similarly, for $\mathsf{h} := \Sigma_q$, the q-bimonoid of faces, using formulas (7.65) and (7.71), check that

$$\mathbf{log}_q(\mathrm{id})_A(\mathtt{H}_{K/A}) = \mathbf{log}_q(\mathrm{id})_A(\mathtt{Q}_{K/A}) = \begin{cases} \mathtt{Q}_{A/A} & \text{if } K = A, \\ 0 & \text{otherwise.} \end{cases}$$

Deduce that $\mathcal{Q}(\Sigma_q) = \mathcal{P}(\Sigma_q) = \mathsf{E}$.

A more general result is given below.

Example 9.89. Suppose q is not a root of unity. Recall the q-bimonoids $\mathcal{T}_q(\mathsf{p})$ and $\mathcal{T}_q^\vee(\mathsf{p})$ from Section 6.4.1. Diagram (9.36) specializes to

$$\begin{tikzcd} \mathcal{T}_q(\mathsf{p}) \arrow[r, "\mathbf{log}_q(\mathrm{id})"] \arrow[d, two heads] & \mathcal{T}_q(\mathsf{p}) \\ \mathsf{p} \arrow[r, "\mathrm{id}"'] & \mathsf{p} \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}_q^\vee(\mathsf{p}) \arrow[r, "\mathbf{log}_q(\mathrm{id})"] \arrow[d, two heads] & \mathcal{T}_q^\vee(\mathsf{p}) \\ \mathsf{p} \arrow[r, "\mathrm{id}"'] & \mathsf{p}, \arrow[u, hook] \end{tikzcd}$$

with the vertical maps being the canonical projection and inclusion. This can be checked directly using the special case (1.68) of the two-sided q-Weisner formula. In particular, we deduce that $\mathcal{P}(\mathcal{T}_q(\mathsf{p})) = \mathsf{p}$ and $\mathcal{Q}(\mathcal{T}_q^\vee(\mathsf{p})) = \mathsf{p}$. This result was also obtained in Exercise 6.76.

Lemma 9.90. *For q not a root of unity, for morphisms $f : \mathsf{h} \to \mathsf{k}$ and $g : \mathsf{k} \to \mathsf{l}$ of q-bimonoids, the following diagram of species commutes.*

(9.37)
$$
\begin{tikzcd}
\mathsf{h} \arrow[rrrr, "\mathbf{log}_q(gf)"] \arrow[rrd, "\mathbf{log}_q(f)"'] \arrow[ddd, two heads] & & & & \mathsf{l} \\
& & \mathsf{k} \arrow[rru, "\mathbf{log}_q(g)"'] \arrow[rd, two heads] & & \\
& \mathcal{P}(\mathsf{k}) \arrow[ru, hook] \arrow[rr, "\mathrm{pq}_{\mathsf{k}}"'] & & \mathcal{Q}(\mathsf{k}) \arrow[rd, dashed] & \\
\mathcal{Q}(\mathsf{h}) \arrow[ru, dashed] \arrow[rrrr, dashed] & & & & \mathcal{P}(\mathsf{l}) \arrow[uuu, hook]
\end{tikzcd}
$$

PROOF. This is similar to the proof of Lemma 9.54. $\square$

Proposition 9.91. *For q not a root of unity, for q-bimonoids h and k, the bijection (9.34) preserves isomorphisms.*

PROOF. This is similar to the proof of Proposition 9.55. We now employ diagrams (9.36) and (9.37). In one direction, if $g : \mathsf{h} \to \mathsf{k}$ and $g' : \mathsf{k} \to \mathsf{h}$ are inverse isomorphisms of q-bimonoids, then $\mathbf{log}_q(g)$ and $\mathbf{log}_q(g')$ are isomorphisms of species linked by the following diagrams.

$$
\begin{tikzcd}
\mathcal{Q}(\mathsf{h}) \arrow[r, "\mathbf{log}_q(g)"] \arrow[d, "\mathbf{log}_q(\mathrm{id}_{\mathsf{h}})"', "\cong"] & \mathcal{P}(\mathsf{k}) \arrow[d, "\cong"', "\mathrm{pq}_{\mathsf{k}}"] \\
\mathcal{P}(\mathsf{h}) & \mathcal{Q}(\mathsf{k}) \arrow[l, "\mathbf{log}_q(g')"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{Q}(\mathsf{h}) \arrow[r, "\mathbf{log}_q(g)"] & \mathcal{P}(\mathsf{k}) \\
\mathcal{P}(\mathsf{h}) \arrow[u, "\mathrm{pq}_{\mathsf{h}}", "\cong"'] & \mathcal{Q}(\mathsf{k}) \arrow[l, "\mathbf{log}_q(g')"] \arrow[u, "\cong", "\mathbf{log}_q(\mathrm{id}_{\mathsf{k}})"']
\end{tikzcd}
$$

The other direction is similar. $\square$

Example 9.92. Suppose q is not a root of unity. Recall from (6.74) the q-norm map $(\kappa_q)_{\mathsf{p}} : \mathcal{T}_q(\mathsf{p}) \to \mathcal{T}_q^{\vee}(\mathsf{p})$. As a special case of Example 9.82, observe that $(\kappa_q)_{\mathsf{p}}$ is the q-exponential of the biderivation

$$
\begin{tikzcd}
\mathcal{T}_q(\mathsf{p}) \arrow[r, dashed] \arrow[d, two heads] & \mathcal{T}_q^{\vee}(\mathsf{p}) \\
\mathsf{p} \arrow[r, "\mathrm{id}"'] & \mathsf{p}. \arrow[u, hook]
\end{tikzcd}
$$

By Proposition 9.91, we deduce that $(\kappa_q)_{\mathsf{p}}$ is an isomorphism of q-bimonoids. This result was obtained earlier in Proposition 6.75. The inverse isomorphism $\mathcal{T}_q^{\vee}(\mathsf{p}) \to \mathcal{T}_q(\mathsf{p})$ is the q-exponential of the biderivation

$$
\begin{tikzcd}
\mathcal{T}_q^{\vee}(\mathsf{p}) \arrow[r, dashed] \arrow[d, two heads] & \mathcal{T}_q(\mathsf{p}) \\
\mathsf{p} \arrow[r, "\mathrm{id}"'] & \mathsf{p}. \arrow[u, hook]
\end{tikzcd}
$$

This follows from the proof of Proposition 9.91 and Example 9.89. Alternatively, it can also be checked directly using formula (6.76).

Exercise 9.93. Suppose q is not a root of unity. Let $f : \mathsf{p} \to \mathsf{q}$ be a map of species. Check that: The q-exponential of the biderivation

$$\begin{tikzcd} \mathcal{T}_q(\mathsf{p}) \arrow[r, dashed] \arrow[d, two heads] & \mathcal{T}_q^\vee(\mathsf{q}) \\ \mathsf{p} \arrow[r, "f"'] & \mathsf{q}, \arrow[u, hook] \end{tikzcd}$$

evaluated on the A-component, on the F-summand, is given by

$$\sum_{\substack{F' \colon F' \geq A, \\ \mathrm{s}(F') = \mathrm{s}(F)}} (\beta_q)_{F',F}\, f_F, \quad \text{or equivalently,} \quad \sum_{\substack{F' \colon F' \geq A, \\ \mathrm{s}(F') = \mathrm{s}(F)}} f_{F'} (\beta_q)_{F',F}.$$

This map is an isomorphism of q-bimonoids iff f is an isomorphism of species. When $f = \mathrm{id}$, we recover the situation of Example 9.92.

Exercise 9.94. Suppose q is not a root of unity. Deduce the following as a special case of Example 9.82. For a comonoid c, the map (6.80) is the q-exponential of the biderivation in the first diagram below.

$$\begin{tikzcd} \mathcal{T}_q(\mathsf{c}) \arrow[r, dashed] \arrow[d, two heads] & \mathcal{T}_q^\vee(\mathsf{c}_t) \\ \mathsf{c} \arrow[r, "\mathrm{id}"'] & \mathsf{c}_t \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}_q(\mathsf{a}_t) \arrow[r, dashed] \arrow[d, two heads] & \mathcal{T}_q^\vee(\mathsf{a}) \\ \mathsf{a}_t \arrow[r, "\mathrm{id}"'] & \mathsf{a} \arrow[u, hook] \end{tikzcd}$$

Dually, for a monoid a, the map (6.81) is the q-exponential of the biderivation in the second diagram above.

Deduce Propositions 6.84 and 6.85 as a consequence of Proposition 9.91.

Example 9.95. Suppose q is not a root of unity. Let c be a comonoid, a a monoid, $f : \mathsf{c} \to \mathsf{a}$ a map of species. Then the morphism of monoids $\hat{f} : \mathcal{T}_q(\mathsf{c}) \to \mathsf{a}$ in Theorem 6.2 is the q-exponential (9.29a) of the derivation in the first diagram below.

$$\begin{tikzcd} \mathcal{T}_q(\mathsf{c}) \arrow[r, dashed] \arrow[d, two heads] & \mathsf{a} \\ \mathsf{c} \arrow[ur, "\mathbf{log}_q(f)"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathsf{c} \arrow[r, dashed] \arrow[dr, "\mathbf{log}_q(f)"'] & \mathcal{T}_q^\vee(\mathsf{a}) \\ & \mathsf{a} \arrow[u, hook] \end{tikzcd}$$

Dually, the morphism of comonoids $\hat{f} : \mathsf{c} \to \mathcal{T}_q^\vee(\mathsf{a})$ in Theorem 6.10 is the q-exponential (9.29a) of the coderivation in the second diagram above.

To see this, one may proceed as in Example 9.58. The analogues of Exercises 9.59, 9.60, 9.62 are given in the discussion below.

Exercise 9.96. Suppose q is not a root of unity. Deduce that: For c a comonoid, h a q-bimonoid, $f : \mathsf{c} \to \mathsf{h}$ a morphism of comonoids, the morphism of q-bimonoids $\hat{f} : \mathcal{T}_q(\mathsf{c}) \to \mathsf{h}$ in Theorem 6.6 is the q-exponential (9.29a) of the biderivation in the first diagram below.

$$\begin{tikzcd} \mathcal{T}_q(\mathsf{c}) \arrow[r, dashed] \arrow[d, two heads] & \mathsf{h} \\ \mathsf{c} \arrow[r, "\mathbf{log}_q(f)"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{h} \arrow[r, dashed] \arrow[d, two heads] & \mathcal{T}_q^\vee(\mathsf{a}) \\ \mathcal{Q}(\mathsf{h}) \arrow[r, "\mathbf{log}_q(f)"'] & \mathsf{a} \arrow[u, hook] \end{tikzcd}$$

Dually, for h a q-bimonoid, a a monoid, $f : \mathsf{h} \to \mathsf{a}$ a morphism of monoids, the morphism of q-bimonoids $\hat{f} : \mathsf{h} \to \mathcal{T}_q^\vee(\mathsf{a})$ in Theorem 6.13 is the q-exponential (9.29a) of the biderivation in the second diagram above.

Letting c be a trivial comonoid and a a trivial monoid recovers Example 9.82.

9.3.5. General q-norm transformation. We now apply the above ideas to the considerations in Section 6.9.3.

Exercise 9.97. Suppose q is not a root of unity. For c a comonoid, a a monoid, $f : \mathsf{c} \to \mathsf{a}$ a map of species, let $g : \mathcal{T}_q(\mathsf{c}) \to \mathcal{T}_q^\vee(\mathsf{a})$ be the morphism of q-bimonoids obtained by applying (6.77) to f. Deduce that g is the q-exponential (9.29a) of the biderivation

$$\begin{tikzcd} \mathcal{T}_q(\mathsf{c}) \arrow[r, dashed] \arrow[d, two heads] & \mathcal{T}_q^\vee(\mathsf{a}) \\ \mathsf{c} \arrow[r, "\mathbf{log}_q(f)"'] & \mathsf{a}. \arrow[u, hook] \end{tikzcd}$$

In other words, (6.77) is the composite of the bijections

$$\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c},\mathsf{a}) \xrightarrow{\mathbf{log}_q} \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c},\mathsf{a}) \xrightarrow{\mathbf{exp}_q} q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{T}_q(\mathsf{c}), \mathcal{T}_q^\vee(\mathsf{a})).$$

The first is (9.30), while the second is an instance of (9.34). Deduce from Proposition 9.91 that:

g is an isomorphism of q-bimonoids iff $\mathbf{log}_q(f)$ is an isomorphism of species.

(Note from Exercise 9.73 that $\mathbf{log}_q(f) = f$ if either c is a trivial comonoid or a is a trivial monoid. This recovers the result of Exercise 6.83.)

Exercise 9.98. Suppose q is not a root of unity. Let (ξ_{X}) be any set of scalars indexed by flats. Consider the map of species $f : \mathsf{E} \to \mathsf{E}$ which on the X-component is scalar multiplication by ξ_{X}. Check using (9.31d) that $\mathbf{log}_q(f)_{\mathrm{Z}}$ is scalar multiplication by $\sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mu_q(\mathrm{Z}, \mathrm{X})\, \xi_{\mathrm{X}}$. This scalar is η_{Z} as defined in (7.88). Now use Exercise 7.37 and Exercise 9.97 to deduce the last claim in Theorem 7.38.

9.3.6. Mixed distributive law for q-bimonoids. We now give a necessary and sufficient condition for the mixed distributive law (3.7) for q-bimonoids to be an isomorphism when q is not a root of unity.

Exercise 9.99. Suppose q is not a root of unity. Let p be a species. For the map $f : \mathcal{T}^\vee(\mathsf{p}) \twoheadrightarrow \mathsf{p} \hookrightarrow \mathcal{T}(\mathsf{p})$, check that $\mathbf{log}_q(f)_A$, on the H-summand, into the K-summand, is given by

$$\begin{cases} \boldsymbol{\mu}_q(A,K,H)\, \beta_{K,H} & \text{if } \mathrm{s}(H) = \mathrm{s}(K), \\ 0 & \text{otherwise.} \end{cases}$$

Now as a consequence of Exercise 6.82, item (1) and the isomorphism result in Exercise 9.97, we deduce that:

Theorem 9.100. *For q not a root of unity, and for a species p, the map $\mathcal{T}_q\mathcal{T}^\vee(\mathsf{p}) \to \mathcal{T}_q^\vee\mathcal{T}(\mathsf{p})$ in (3.7) is an isomorphism of q-bimonoids iff for each flat* X*, either $\mathsf{p}[\mathrm{X}] = 0$ or*

$$\det\left(\boldsymbol{\mu}_q(A,K,H)_{\substack{K,H\geq A\\ \mathrm{s}(K)=\mathrm{s}(H)=\mathrm{X}}}\right) \neq 0$$

for all A with $\mathrm{s}(A) \leq \mathrm{X}$. (In the above matrix, A and X *are fixed, while K, H are the two varying indices.)*

In particular, we deduce that the map $\mathcal{T}_q\mathcal{T}^\vee(\mathsf{x}) \to \mathcal{T}_q^\vee\mathcal{T}(\mathsf{x})$ is an isomorphism iff

$$\det\left(\boldsymbol{\mu}_q(A,C,D)_{C,D\geq A}\right) \neq 0$$

for all A. This recovers the result in Theorem 7.51.

More explicit considerations on Theorem 9.100 are given later in Exercise 14.83.

Exercise 9.101. Suppose q is not a root of unity. Let p be a species. For the map $\mathrm{id} : \mathcal{T}^\vee(\mathsf{p}) \to \mathcal{T}(\mathsf{p})$, check that $\mathbf{log}_q(\mathrm{id})_A$, on the H-summand, into the K-summand, is given by

$$\begin{cases} \left(\displaystyle\sum_{\substack{F'\colon A\leq F'\leq H\\ F: A\leq F\leq K\\ \mathrm{s}(F)=\mathrm{s}(F'),\, FH=K}} \boldsymbol{\mu}_q(A,F,F')\right)\beta_{K,H} & \text{if } \mathrm{s}(H)=\mathrm{s}(K),\\ 0 & \text{otherwise.}\end{cases}$$

Deduce that the map g in Exercise 6.82, item (2) is an isomorphism of q-bimonoids iff for each flat X, either $\mathsf{p}[\mathrm{X}] = 0$ or

$$\det\left(\left(\sum_{\substack{F'\colon A\leq F'\leq H\\ F: A\leq F\leq K\\ \mathrm{s}(F)=\mathrm{s}(F'),\, FH=K}} \boldsymbol{\mu}_q(A,F,F')\right)_{\substack{K,H\geq A\\ \mathrm{s}(K)=\mathrm{s}(H)=\mathrm{X}}}\right) \neq 0$$

for all A with $\mathrm{s}(A) \leq \mathrm{X}$. (In the above matrix, A and X are fixed, while K, H are the two varying indices. Each matrix entry is given by the sum over F, F'.)

The special case $\mathsf{p} := \mathsf{x}$ is given later in Theorem 15.47.

9.4. 0-exp-log correspondence

We now treat the 0-exp-log correspondence. We develop this directly as well as explain how it is a special case of the discussion in Section 9.3. Interestingly, it suffices to work here with the lune-incidence algebra.

9.4.1. 0-exp-log correspondence. Recall from Section 1.5.11 the noncommutative 0-zeta function $\boldsymbol{\zeta}_0$ and the noncommutative 0-Möbius function $\boldsymbol{\mu}_0$. These are elements of the lune-incidence algebra.

For a map of species $f : \mathsf{c} \to \mathsf{a}$ from a comonoid c to a monoid a, we say that

$$\mathbf{exp}_0(f) := \boldsymbol{\zeta}_0 \circ f \quad\text{and}\quad \mathbf{log}_0(f) := \boldsymbol{\mu}_0 \circ f$$

are the 0-*exponential* and 0-*logarithm* of f, respectively. Explicitly, using formula (9.1),

$$\mathbf{exp}_0(f)_A = \sum_{F\colon F \geq A} \mu^F_A f_F \Delta^F_A, \tag{9.38a}$$

$$\mathbf{log}_0(f)_A = \sum_{F\colon F \geq A} (-1)^{\mathrm{rk}(F/A)} \mu^F_A f_F \Delta^F_A. \tag{9.38b}$$

Since $\boldsymbol{\zeta}_0$ and $\boldsymbol{\mu}_0$ are inverse to each other, we obtain inverse bijections

$$\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c},\mathsf{a}) \underset{\mathbf{log}_0}{\overset{\mathbf{exp}_0}{\rightleftarrows}} \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c},\mathsf{a}). \tag{9.39}$$

This is the 0-*exp-log correspondence*.

A key observation is that for $q = 0$, the q-exponential (9.29a) and q-logarithm (9.29b) specialize to (9.38a) and (9.38b), respectively. This follows from formulas (1.71). Note very carefully how the bilune-incidence algebra disappears from the picture. In particular, the q-exp-log correspondence (9.30) specializes to (9.39).

Exercise 9.102. Similar to Lemma 9.10, deduce from Lemma 9.3 that: The 0-exp-log correspondence (9.39) is natural in c and a. This is also the special case $q = 0$ of Exercise 9.77.

9.4.2. (Co, bi)derivations and (co, bi)monoid morphisms.

Theorem 9.103. *For a comonoid* c *and* 0*-bimonoid* k*, we have inverse bijections*

$$\mathcal{A}\text{-}\mathsf{Sp}\big(\mathsf{c},\mathcal{P}(\mathsf{k})\big) \underset{\mathbf{log}_0}{\overset{\mathbf{exp}_0}{\rightleftarrows}} \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c},\mathsf{k}).$$

In other words: There is a bijection between coderivations from c to k and morphisms of comonoids from c to k.

Proof. This is the special case $q = 0$ of Theorem 9.78. One can also proceed directly. In that case, for the first calculation, we use Lemma 5.36, while for the second calculation, we use the 0-bimonoid axiom (2.40) and the fact that $\boldsymbol{\mu}_0$ satisfies the noncommutative 0-Weisner formula, that is, (1.48) for $q = 0$. □

Dually:

Theorem 9.104. *For a* 0*-bimonoid* h *and monoid* a*, we have inverse bijections*

$$\mathcal{A}\text{-}\mathsf{Sp}\big(\mathcal{Q}(\mathsf{h}),\mathsf{a}\big) \underset{\mathbf{log}_0}{\overset{\mathbf{exp}_0}{\rightleftarrows}} \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h},\mathsf{a}).$$

In other words: There is a bijection between derivations from h to a and morphisms of monoids from h to a.

Combining Theorems 9.103 and 9.104, we obtain:

Theorem 9.105. *For* 0*-bimonoids* h *and* k*, we have inverse bijections*

$$\mathcal{A}\text{-}\mathsf{Sp}\big(\mathcal{Q}(\mathsf{h}), \mathcal{P}(\mathsf{k})\big) \underset{\mathbf{log}_0}{\overset{\mathbf{exp}_0}{\rightleftarrows}} 0\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h}, \mathsf{k}). \tag{9.40}$$

In other words: There is a bijection between biderivations from h to k and morphisms of 0-bimonoids from h to k.

Exercise 9.106. Work out Example 9.82 directly for $q = 0$.

9.4.3. 0-logarithm of the identity map. Note from (9.38b) that the 0-*logarithm of the identity map* on a q-bimonoid h is given by

$$\mathbf{log}_0(\mathrm{id})_A = \sum_{F\colon F \geq A} (-1)^{\mathrm{rk}(F/A)}\, \mu_A^F \Delta_A^F. \tag{9.41}$$

This is closely related to the antipode of h studied in Chapter 12, see formula (12.5) in particular, and the calculations in Section 12.6.

By proceeding directly or as the special case $q = 0$ of Proposition 9.84, we get:

Proposition 9.107. *Let* h *be a* 0*-bimonoid. Then* $\mathbf{log}_0(\mathrm{id})$ *is an idempotent operator on* h *whose image is* $\mathcal{P}(\mathsf{h})$ *and coimage is* $\mathcal{Q}(\mathsf{h})$ *yielding the commutative diagram of species*

$$\begin{tikzcd} \mathsf{h} \arrow[r, "\mathbf{log}_0(\mathrm{id})"] \arrow[d, two heads] & \mathsf{h} \\ \mathcal{Q}(\mathsf{h}) \arrow[r, "\cong"'] & \mathcal{P}(\mathsf{h}). \arrow[u, hook] \end{tikzcd} \tag{9.42}$$

In particular, $\mathcal{P}(\mathsf{h})$ *and* $\mathcal{Q}(\mathsf{h})$ *are isomorphic as species. Moreover, the inverse of* $\mathcal{Q}(\mathsf{h}) \to \mathcal{P}(\mathsf{h})$ *is the map* pq_{h} *in* (5.50)*.*

Similarly, by proceeding directly or by specializing Corollaries 9.85 and 9.86, we get:

Corollary 9.108. *Let* h *be a* 0*-bimonoid. Then* $\mathbf{log}_0(\mathrm{id}) : \mathsf{h} \twoheadrightarrow \mathcal{P}(\mathsf{h})$ *is a morphism of monoids (with the trivial product on* $\mathcal{P}(\mathsf{h})$*), while* $\mathbf{log}_0(\mathrm{id}) : \mathcal{Q}(\mathsf{h}) \hookrightarrow \mathsf{h}$ *is a morphism of comonoids (with the trivial coproduct on* $\mathcal{Q}(\mathsf{h})$*).*

Corollary 9.109. *Let* $(\mathsf{h}, \mu, \Delta)$ *be a* 0*-bimonoid. Then, for any* $G > A$*,*

$$\sum_{F\colon F \geq A} (-1)^{\mathrm{rk}(F/A)}\, \Delta_A^G \mu_A^F \Delta_A^F = 0,$$

$$\sum_{F\colon F \geq A} (-1)^{\mathrm{rk}(F/A)}\, \mu_A^F \Delta_A^F \mu_A^G = 0.$$

Example 9.87 and Exercise 9.88 can be specialized to $q = 0$. This yields expressions for the 0-logarithm of the identity map on the 0-bimonoid of chambers Γ_0, and the 0-bimonoid of faces Σ_0. These calculations can also be done directly using (co)product formulas (7.27), (7.93), (7.95).

Example 9.110. For $q = 0$, the two diagrams in Example 9.89 coincide and equal

$$\begin{tikzcd} \mathcal{T}_0(\mathsf{p}) \arrow[r, "\mathbf{log}_0(\mathrm{id})"] \arrow[d, two heads] & \mathcal{T}_0(\mathsf{p}) \\ \mathsf{p} \arrow[r, "\mathrm{id}"'] & \mathsf{p}. \arrow[u, hook] \end{tikzcd}$$

This is also a specialization of diagram (9.42). Using the product (6.43) and coproduct (6.44), we also note that $\mathbf{log}_0(\mathrm{id})_A$, on the G-summand, is identity times the scalar $\sum_{F\colon A \le F \le G} (-1)^{\mathrm{rk}(F/A)}$. By (1.73), this is zero if $A < G$, and 1 if $A = G$.

Proposition 9.111. *For 0-bimonoids* h *and* k*, the bijection* (9.40) *preserves isomorphisms.*

This is the special case $q = 0$ of Proposition 9.91.

9.5. Primitive and group-like series of bimonoids

We define the space of series of a species. The lune-incidence algebra acts on the space of series of a monoid. This can be used to formulate exp-log correspondences, one for each noncommutative zeta or Möbius function. Each such correspondence restricts to a bijection between primitive series and group-like series of any bimonoid. For q not a root of unity, for any q-bimonoid, there is a q-exp-log correspondence defined using the noncommutative q-zeta and q-Möbius functions.

9.5.1. Space of series of a species. Let p be a species. A *series* of p is a family of elements $v_F \in \mathsf{p}[F]$, one for each face F, such that

$$\beta_{G,F}(v_F) = v_G,$$

whenever F and G have the same support. Let $\mathscr{S}(\mathsf{p})$ denote the space of series of p. This construction is functorial in p, and defines a functor $\mathscr{S}$ from the category of species to the category of vector spaces.

9.5.2. Primitive and group-like series of comonoids. Let (c, Δ) be a comonoid. A series v of c is *primitive* if $\Delta_A^F(v_A) = 0$ for all $F > A$. Let $\mathscr{P}(\mathsf{c})$ denote the space of all primitive series of c. Observe that

$$\mathscr{P}(\mathsf{c}) = \mathscr{S}(\mathcal{P}(\mathsf{c})). \tag{9.43}$$

Similarly, a series v of c is *group-like* if $\Delta_A^F(v_A) = v_F$ for all $F \geq A$. Let $\mathscr{G}(\mathsf{c})$ denote the space of all group-like series of c.

Exercise 9.112. For any comonoid c, check that primitive series and group-like series of c coincide with those of its coabelianization c^{coab}, that is,

$$\mathscr{P}(\mathsf{c}^{coab}) = \mathscr{P}(\mathsf{c}) \quad \text{and} \quad \mathscr{G}(\mathsf{c}^{coab}) = \mathscr{G}(\mathsf{c}).$$

(The first identity also follows formally from (9.43) and Exercise 5.10.)

Exercise 9.113. Check that primitive series and group-like series are preserved under morphisms of comonoids. In other words, a morphism $\mathsf{c} \to \mathsf{d}$ of comonoids induces maps $\mathscr{P}(\mathsf{c}) \to \mathscr{P}(\mathsf{d})$ and $\mathscr{G}(\mathsf{c}) \to \mathscr{G}(\mathsf{d})$.

9.5.3. Action of the lune-incidence algebra. Recall the lune-incidence algebra $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ from Section 1.5.3. Let (a, μ) be a monoid. For any $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ and a series v of a, define another series $s \circ v$ of a by

$$(s \circ v)_A := \sum_{F\colon F \geq A} s(A, F)\, \mu_A^F(v_F). \tag{9.44}$$

To see that this is indeed a series, we compute:

$$\begin{aligned}
\beta_{B,A}((s \circ v)_A) &= \sum_{F\colon F \geq A} s(A, F)\, \beta_{B,A} \mu_A^F(v_F) \\
&= \sum_{F\colon F \geq A} s(A, F)\, \mu_B^{BF} \beta_{BF,F}(v_F) \\
&= \sum_{F\colon F \geq A} s(B, BF)\, \mu_B^{BF}(v_{BF}) \\
&= \sum_{G\colon G \geq B} s(B, G)\, \mu_B^G(v_G) \\
&= (s \circ v)_B.
\end{aligned}$$

The second step used naturality of the product (2.8). The third step used that v is a series. It also used that $s(A, F) = s(B, BF)$ which holds by (1.40). The fourth step used the bijection between the stars of A and B as in Lemma 1.6.

Lemma 9.114. *The assignment $(s, v) \mapsto s \circ v$ defines a left action of the lune-incidence algebra on $\mathscr{S}(\mathsf{a})$.*

Proof. This is checked below.

$$\begin{aligned}
(s \circ (t \circ v))_A &= \sum_{F\colon F \geq A} s(A, F)\, \mu_A^F((t \circ v)_F) \\
&= \sum_{F\colon F \geq A} s(A, F)\, \mu_A^F\big(\sum_{G\colon G \geq F} t(F, G)\, \mu_F^G(v_G)\big) \\
&= \sum_{G\colon G \geq A} \big(\sum_{F\colon G \geq F \geq A} s(A, F) t(F, G)\big) \mu_A^G(v_G) \\
&= \sum_{G\colon G \geq A} (st)(A, G)\, \mu_A^G(v_G) \\
&= (st \circ v)_A.
\end{aligned}$$

Also, observe that $(\delta \circ v)_A = v_A$, where δ denotes the unit element of the lune-incidence algebra. $\square$

Thus, $\mathscr{S}(\mathsf{a})$ for any monoid a is a left module over the lune-incidence algebra. Observe that this structure coincides with the one discussed in Example 4.45.

Exercise 9.115. Check that: If $f : \mathsf{a} \to \mathsf{b}$ is a morphism of monoids, then $\mathscr{S}(f) : \mathscr{S}(\mathsf{a}) \to \mathscr{S}(\mathsf{b})$ is a map of left modules over the lune-incidence algebra.

9.5.4. Exp-log correspondences. Fix a noncommutative zeta function $\boldsymbol{\zeta}$ and a noncommutative Möbius function $\boldsymbol{\mu}$ which are inverses of each other. For a series v of a monoid a, define

$$\mathbf{exp}(v)_A := \sum_{F:\, F \geq A} \boldsymbol{\zeta}(A,F)\, \mu_A^F(v_F), \tag{9.45a}$$

$$\mathbf{log}(v)_A := \sum_{F:\, F \geq A} \boldsymbol{\mu}(A,F)\, \mu_A^F(v_F). \tag{9.45b}$$

Since $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ are inverse to each other, Lemma 9.114 implies the following result.

Proposition 9.116. *For any monoid* a*, we have inverse bijections*

$$\mathscr{S}(\mathsf{a}) \underset{\mathbf{log}}{\overset{\mathbf{exp}}{\rightleftarrows}} \mathscr{S}(\mathsf{a}). \tag{9.46}$$

We refer to (9.46) as an *exp-log correspondence*. Note very carefully that it depends on the particular $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ that we choose. When a carries the structure of a bimonoid, one can do more as follows.

Theorem 9.117. *For a bimonoid* h*, we have inverse bijections*

$$\mathscr{P}(\mathsf{h}) \underset{\mathbf{log}}{\overset{\mathbf{exp}}{\rightleftarrows}} \mathscr{G}(\mathsf{h}). \tag{9.47}$$

Proof. In view of (9.46), it suffices to show that **exp** and **log** map as stated. Suppose v is a primitive series of h. We check below that $\boldsymbol{\zeta} \circ v$ is a group-like series. For $G \geq A$,

$$\begin{aligned}
\Delta_A^G((\boldsymbol{\zeta} \circ v)_A) &= \sum_{F:\, F \geq A} \boldsymbol{\zeta}(A,F)\, \Delta_A^G \mu_A^F(v_F) \\
&= \sum_{F:\, F \geq A} \boldsymbol{\zeta}(A,F)\, \mu_G^{GF} \beta_{GF,FG} \Delta_F^{FG}(v_F) \\
&= \sum_{F:\, F \geq A,\, FG=F} \boldsymbol{\zeta}(A,F)\, \mu_G^{GF} \beta_{GF,F}(v_F) \\
&= \sum_{F:\, F \geq A,\, FG=F} \boldsymbol{\zeta}(A,F)\, \mu_G^{GF}(v_{GF}) \\
&= \sum_{H:\, H \geq G} \Big(\sum_{F:\, F \geq A,\, FG=F,\, GF=H} \boldsymbol{\zeta}(A,F) \Big)\, \mu_G^H(v_H) \\
&= \sum_{H:\, H \geq G} \boldsymbol{\zeta}(G,H)\, \mu_G^H(v_H) \\
&= (\boldsymbol{\zeta} \circ v)_G.
\end{aligned}$$

The first step and last step used definition (9.44). The second step used the bimonoid axiom (2.12). Since v is a primitive series, $\Delta_F^{FG}(v_F)$ is zero unless $FG = F$. This was used in the third step. In the fifth step, we introduced a new variable H for GF. The sixth step used the lune-additivity formula (1.42).

Conversely, suppose v is a group-like series of h. We check below that $\boldsymbol{\mu} \circ v$ is a primitive series. For $G > A$,

$$\begin{aligned}
\Delta_A^G((\boldsymbol{\mu} \circ v)_A) &= \sum_{F\colon F \geq A} \boldsymbol{\mu}(A,F)\, \Delta_A^G \mu_A^F(v_F) \\
&= \sum_{F\colon F \geq A} \boldsymbol{\mu}(A,F)\, \mu_G^{GF} \beta_{GF,FG} \Delta_F^{FG}(v_F) \\
&= \sum_{F\colon F \geq A} \boldsymbol{\mu}(A,F)\, \mu_G^{GF} \beta_{GF,FG}(v_{FG}) \\
&= \sum_{F\colon F \geq A} \boldsymbol{\mu}(A,F)\, \mu_G^{GF}(v_{GF}) \\
&= \sum_{H\colon H \geq G} \Big(\sum_{F\colon F \geq A,\, GF = H} \boldsymbol{\mu}(A,F) \Big)\, \mu_G^H(v_H) \\
&= 0.
\end{aligned}$$

The first step used definition (9.44). The second step used the bimonoid axiom (2.12). The third step used that v is a group-like series. In the fifth step, we introduced a new variable H for GF. The last step used the noncommutative Weisner formula (1.44). □

Example 9.118. Recall the bimonoid of chambers Γ (Section 7.3.2). Consider its space of series $\mathscr{S}(\Gamma)$. An element can be viewed as a family of scalars $(f(A,C))_{A \leq C}$ such that $f(A,C) = f(B,D)$ whenever A and B have the same support, and $AD = C$ and $BC = D$. The identification is done via

$$(f(A,C))_{A \leq C} \longleftrightarrow \sum_{C\colon C \geq A} f(A,C)\, \mathtt{H}_{C/A} \in \Gamma[A] \quad \text{for each face } A.$$

We deduce from (1.9) that $\mathscr{S}(\Gamma)$ has a basis indexed by top-lunes. Specializing (9.44) and using product formula (7.18), we see that its module structure over the lune-incidence algebra is given by

$$(s \circ f)(A,C) = \sum_{F\colon A \leq F \leq C} s(A,F) f(F,C).$$

This is the same as the lune-incidence module considered in [**21**, Section 15.2.6].

The bijection (9.46) specializes to

$$g(F,C) = \sum_{G\colon F \leq G \leq C} \boldsymbol{\zeta}(F,G) f(G,C) \iff f(F,C) = \sum_{G\colon F \leq G \leq C} \boldsymbol{\mu}(F,G) g(G,C). \tag{9.48}$$

This is the noncommutative Möbius inversion of [**21**, Section 15.4.1].

Recall from Lemma 7.64 that Lie is the primitive part of Γ, hence a primitive series of Γ is the same as a series of the Lie species. Explicitly, a series f is primitive if it satisfies

$$\sum_{C\colon C \geq A,\, HC = D} f(A,C) = 0 \tag{9.49}$$

for all $A < H \leq D$. Similarly, a series g is group-like if it satisfies

$$g(H, D) = \sum_{C:\, C \geq A,\, HC = D} g(A, C) \tag{9.50}$$

for all $A \leq H \leq D$. These descriptions can be deduced from coproduct formula (7.18).

The bijection (9.47) says that primitive series and group-like series of Γ correspond to each other under (9.48). This result was obtained in [**21**, Theorem 15.42] with the same proof as given here.

Example 9.119. Recall the bimonoid of faces Σ (Section 7.6.2). Consider its space of series $\mathscr{S}(\Sigma)$. An element can be viewed as a family of scalars $(f(A, F))_{A \leq F}$ such that $f(A, F) = f(B, G)$ whenever A and B have the same support, and $AG = F$ and $BF = G$. The identification is done via

$$(f(A, F))_{A \leq F} \longleftrightarrow \sum_{F:\, F \geq A} f(A, F)\, \mathtt{H}_{F/A} \in \Sigma[A] \quad \text{for each face } A. \tag{9.51}$$

Specializing (9.44) and using product formula (7.64), we see that $\mathscr{S}(\Sigma)$ in fact coincides with the lune-incidence algebra viewed as a left module over itself.

Recall from Lemma 7.69 that Zie is the primitive part of Σ, hence a primitive series of Σ is the same as a series of the Zie species. Explicitly, a series f is primitive if it satisfies

$$\sum_{F:\, F \geq A,\, HF = G} f(A, F) = 0 \tag{9.52}$$

for all $A < H \leq G$. Similarly, a series g is group-like if it satisfies

$$g(H, G) = \sum_{F:\, F \geq A,\, HF = G} g(A, F) \tag{9.53}$$

for all $A \leq H \leq G$. These descriptions can be deduced from coproduct formula (7.64). Note very carefully that (9.52) is the noncommutative Weisner formula (1.44), however, (9.53) differs from the lune-additivity formula (1.42).

There is a bijection between primitive series and group-like series of Σ given by (9.47). An interesting example of this correspondence is

$$\boldsymbol{\mu} \longleftrightarrow \delta, \tag{9.54}$$

where δ denotes the unit element of the lune-incidence algebra. To see this, note that $\mathbf{exp}(\boldsymbol{\mu}) = \boldsymbol{\zeta} \circ \boldsymbol{\mu} = \delta$ and $\mathbf{log}(\delta) = \boldsymbol{\mu} \circ \delta = \boldsymbol{\mu}$. It is also clear that $\boldsymbol{\mu}$ is primitive and δ is group-like. In terms of the identification (9.51),

$$\begin{aligned} \delta &\leftrightarrow \mathtt{H}_{A/A} \in \Sigma[A] \quad \text{for each } A, \\ \boldsymbol{\mu} &\leftrightarrow \mathtt{Q}_{A/A} \in \Sigma[A] \quad \text{for each } A, \end{aligned} \tag{9.55}$$

where $\mathtt{Q}_{A/A}$ is the $\mathtt{Q}$-basis element defined in (7.67). We refer to δ as the *universal series* of Σ. It is an instance of the universal series in Section 4.13.6 specialized to the associative operad.

Exercise 9.120. Use the analysis in Example 9.119 to deduce Lemma 1.13.

Exercise 9.121. Let $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ be such that $s(A, A) = 1$ for all A. Show that:

(1) If $v \in \mathscr{P}(\mathsf{h})$ implies $s \circ v \in \mathscr{G}(\mathsf{h})$ for every bimonoid h, then s is a noncommutative zeta function.
(2) If $v \in \mathscr{G}(\mathsf{h})$ implies $s \circ v \in \mathscr{P}(\mathsf{h})$ for every bimonoid h, then s is a noncommutative Möbius function.

9.5.5. *q*-exp-log correspondence. Let q be any scalar which is not a root of unity. Recall the noncommutative q-zeta function $\boldsymbol{\zeta}_q$ and noncommutative q-Möbius function $\boldsymbol{\mu}_q$ defined in Section 1.5.9.

For a series v of a monoid a, define

$$\mathbf{exp}_q(v)_A := \sum_{F\colon F \geq A} \boldsymbol{\zeta}_q(A, F)\, \mu_A^F(v_F), \tag{9.56a}$$

$$\mathbf{log}_q(v)_A := \sum_{F\colon F \geq A} \boldsymbol{\mu}_q(A, F)\, \mu_A^F(v_F). \tag{9.56b}$$

Since $\boldsymbol{\zeta}_q$ and $\boldsymbol{\mu}_q$ are inverse to each other, we obtain inverse bijections

$$\mathscr{S}(\mathsf{a}) \underset{\mathbf{log}_q}{\overset{\mathbf{exp}_q}{\rightleftarrows}} \mathscr{S}(\mathsf{a}). \tag{9.57}$$

We refer to (9.57) as the *q-exp-log correspondence*.

Theorem 9.122. *For a q-bimonoid* h *for* q *not a root of unity, we have inverse bijections*

$$\mathscr{P}(\mathsf{h}) \underset{\mathbf{log}_q}{\overset{\mathbf{exp}_q}{\rightleftarrows}} \mathscr{G}(\mathsf{h}). \tag{9.58}$$

Proof. The calculations are similar to those in Theorem 9.117. We now employ the q-bimonoid axiom (2.33), q-lune-additivity formula (1.46), noncommutative q-Weisner formula (1.48). $\square$

Exercise 9.123. Suppose q is not a root of unity. Let $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ be such that $s(A, A) = 1$ for all A. Show that:

(1) If $v \in \mathscr{P}(\mathsf{h})$ implies $s \circ v \in \mathscr{G}(\mathsf{h})$ for every q-bimonoid h, then s is the noncommutative q-zeta function $\boldsymbol{\zeta}_q$.
(2) If $v \in \mathscr{G}(\mathsf{h})$ implies $s \circ v \in \mathscr{P}(\mathsf{h})$ for every q-bimonoid h, then s is the noncommutative q-Möbius function $\boldsymbol{\mu}_q$.

9.6. Primitive and group-like series of bicomm. bimonoids

The story in Section 9.5 has a commutative counterpart. In this case, the flat-incidence algebra acts on the space of series of a commutative monoid. The exp-log correspondence is now uniquely defined using the zeta function and Möbius function of the poset of flats. This correspondence restricts to a bijection between primitive series and group-like series of any bicommutative bimonoid. We develop this directly as well as indicate how it can be deduced from the discussion in Section 9.5.

9.6.1. Space of series of a species. Let p be a species. We work with the formulation given in Proposition 2.5. A *series* of p is a family of elements $v_{\mathrm{X}} \in \mathsf{p}[\mathrm{X}]$, one for each flat X. Let $\mathscr{S}(\mathsf{p})$ denote the space of series of p. This definition is consistent with the one given in Section 9.5.1.

9.6.2. Primitive and group-like series of cocommutative comonoids. Let (c, Δ) be a cocommutative comonoid. A series v of c is *primitive* if $\Delta^{\mathrm{Y}}_{\mathrm{X}}(v_{\mathrm{X}}) = 0$ for all $\mathrm{Y} > \mathrm{X}$. Let $\mathscr{P}(\mathsf{c})$ denote the space of all primitive series of c. Similarly, a series v of c is *group-like* if $\Delta^{\mathrm{Y}}_{\mathrm{X}}(v_{\mathrm{X}}) = v_{\mathrm{Y}}$ for all $\mathrm{Y} \geq \mathrm{X}$. Let $\mathscr{G}(\mathsf{c})$ denote the space of all group-like series of c. These definitions are consistent with those in Section 9.5.2.

9.6.3. Action of the flat-incidence algebra. Recall the flat-incidence algebra $\mathrm{I}_{\mathsf{flat}}[\mathcal{A}]$ from Section 1.5.1. Let (a, μ) be a commutative monoid. The flat-incidence algebra acts on $\mathscr{S}(\mathsf{a})$ by

$$(s \circ v)_{\mathrm{Z}} := \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} s(\mathrm{Z}, \mathrm{X})\, \mu^{\mathrm{X}}_{\mathrm{Z}}(v_{\mathrm{X}}). \tag{9.59}$$

Thus, $\mathscr{S}(\mathsf{a})$ is a module over the flat-incidence algebra. This structure coincides with the one discussed in Example 4.44.

9.6.4. Commutative exp-log correspondence. The exponential and logarithm of a series v are defined by the action of the zeta function ζ and Möbius function μ of the poset of flats, that is,

$$\exp(v)_{\mathrm{Z}} := \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mu^{\mathrm{X}}_{\mathrm{Z}}(v_{\mathrm{X}}), \tag{9.60a}$$

$$\log(v)_{\mathrm{Z}} := \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{X})\, \mu^{\mathrm{X}}_{\mathrm{Z}}(v_{\mathrm{X}}). \tag{9.60b}$$

Since ζ and μ are inverse to each other, we obtain inverse bijections

$$\mathscr{S}(\mathsf{a}) \underset{\log}{\overset{\exp}{\rightleftarrows}} \mathscr{S}(\mathsf{a}). \tag{9.61}$$

This is the *commutative exp-log correspondence*.

Theorem 9.124. *For a bicommutative bimonoid* h*, we have inverse bijections*

$$\mathscr{P}(\mathsf{h}) \underset{\log}{\overset{\exp}{\rightleftarrows}} \mathscr{G}(\mathsf{h}). \tag{9.62}$$

PROOF. One can proceed directly as in the proof of Theorem 9.117. An alternative argument is given in Exercise 9.125 below.

Suppose v is a primitive series of h. We check that $\zeta \circ v$ is a group-like series. For $\mathrm{Y} \geq \mathrm{Z}$,

$$\begin{aligned}
\Delta^{\mathrm{Y}}_{\mathrm{Z}}((\zeta \circ v)_{\mathrm{Z}}) &= \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \Delta^{\mathrm{Y}}_{\mathrm{Z}} \mu^{\mathrm{X}}_{\mathrm{Z}}(v_{\mathrm{X}}) \\
&= \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mu^{\mathrm{X} \vee \mathrm{Y}}_{\mathrm{Y}} \Delta^{\mathrm{X} \vee \mathrm{Y}}_{\mathrm{X}}(v_{\mathrm{X}})
\end{aligned}$$

$$= \sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Y}} \mu_{\mathrm{Y}}^{\mathrm{X}}(v_{\mathrm{X}})$$
$$= (\zeta \circ v)_{\mathrm{Y}}.$$

The first step and last step used definition (9.60a). The second step used the bicommutative bimonoid axiom (2.26). Since v is a primitive series, $\Delta_{\mathrm{X}}^{\mathrm{X}\vee\mathrm{Y}}(v_{\mathrm{X}})$ will be zero unless $\mathrm{Y} \leq \mathrm{X}$. This was used in the third step.

Conversely, suppose v is a group-like series of h. We check that $\mu \circ v$ is a primitive series. For $\mathrm{Y} > \mathrm{Z}$,

$$\begin{aligned}
\Delta_{\mathrm{Z}}^{\mathrm{Y}}((\mu \circ v)_{\mathrm{Z}}) &= \sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{X})\, \Delta_{\mathrm{Z}}^{\mathrm{Y}} \mu_{\mathrm{Z}}^{\mathrm{X}}(v_{\mathrm{X}}) \\
&= \sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{X})\, \mu_{\mathrm{Y}}^{\mathrm{X}\vee\mathrm{Y}} \Delta_{\mathrm{X}}^{\mathrm{X}\vee\mathrm{Y}}(v_{\mathrm{X}}) \\
&= \sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{X})\, \mu_{\mathrm{Y}}^{\mathrm{X}\vee\mathrm{Y}}(v_{\mathrm{X}\vee\mathrm{Y}}) \\
&= \sum_{\mathrm{W}\colon \mathrm{W} \geq \mathrm{Y}} \Big(\sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Z},\, \mathrm{Y}\vee\mathrm{X} = \mathrm{W}} \mu(\mathrm{Z}, \mathrm{X}) \Big) \mu_{\mathrm{Z}}^{\mathrm{W}}(v_{\mathrm{W}}) \\
&= 0.
\end{aligned}$$

The first step used definition (9.60b). The second step used the bicommutative bimonoid axiom (2.26). The third step used that v is a group-like series. The last step used the Weisner formula (1.38). □

Exercise 9.125. Let a be a commutative monoid. Check that the action (9.44) of the lune-incidence algebra on $\mathscr{S}(\mathsf{a})$ factors through the base-case map (1.45) to yield an action of the flat-incidence algebra on $\mathscr{S}(\mathsf{a})$ which coincides with (9.59). Use Lemma 1.17 to deduce that all exp-log correspondences (9.46) reduce to (9.61). Deduce Theorem 9.124 from Theorem 9.117.

Exercise 9.126. Formulate the commutative analogue of Exercise 9.115. Either check it directly or deduce it from Exercise 9.125 and Exercise 9.115.

Example 9.127. Consider the exponential bimonoid E (Section 7.2). All its components equal the base field. Thus, a series of E is a family of scalars $f(\mathrm{X})$, one for each flat X. Specializing (9.59), we see that its module structure over the flat-incidence algebra is given by

$$(s \circ f)(\mathrm{X}) = \sum_{\mathrm{Y}\colon \mathrm{X} \leq \mathrm{Y}} s(\mathrm{X}, \mathrm{Y}) f(\mathrm{Y}).$$

The bijection (9.61) specializes to

$$g(\mathrm{X}) = \sum_{\mathrm{Y}\colon \mathrm{X} \leq \mathrm{Y}} f(\mathrm{Y}) \iff f(\mathrm{X}) = \sum_{\mathrm{Y}\colon \mathrm{X} \leq \mathrm{Y}} \mu(\mathrm{X}, \mathrm{Y}) g(\mathrm{Y}). \tag{9.63}$$

This is Möbius inversion in the poset of flats, see for instance [21, Equation (C.11)].

Observe that a series f of E is primitive if $f(\mathrm{X}) = 0$ for all $\mathrm{X} \neq \top$. Similarly, a series g of E is group-like if $g(\mathrm{X}) = g(\mathrm{Y})$ for all X and Y. For a primitive series f and group-like series g,

$$\exp(f)(\mathrm{X}) = f(\top) \quad \text{and} \quad \log(g)(\mathrm{X}) = \begin{cases} g(\top) & \text{if } \mathrm{X} = \top, \\ 0 & \text{otherwise.} \end{cases}$$

This gives a direct verification of (9.62) for $\mathsf{h} := \mathsf{E}$.

Example 9.128. Recall the bimonoid of flats Π (Section 7.4.2). Consider its space of series $\mathscr{S}(\Pi)$. An element can be viewed as a family of scalars $(f(\mathrm{Z},\mathrm{X}))_{\mathrm{Z}\leq\mathrm{X}}$ via the identification

$$(f(\mathrm{Z},\mathrm{X}))_{\mathrm{Z}\leq\mathrm{X}} \longleftrightarrow \sum_{\mathrm{X}:\, \mathrm{X}\geq\mathrm{Z}} f(\mathrm{Z},\mathrm{X})\, \mathtt{H}_{\mathrm{X}/\mathrm{Z}} \in \Pi[\mathrm{Z}] \quad \text{for each flat Z.} \tag{9.64}$$

Specializing (9.59) and using product formula (7.36), we see that $\mathscr{S}(\Pi)$ in fact coincides with the flat-incidence algebra viewed as a module over itself.

Observe that a series f of Π is primitive if it satisfies

$$\sum_{\mathrm{X}:\, \mathrm{X}\geq\mathrm{Z},\, \mathrm{Y}\vee\mathrm{X}=\mathrm{W}} f(\mathrm{Z},\mathrm{X}) = 0 \tag{9.65}$$

for all $\mathrm{Z} < \mathrm{Y} \leq \mathrm{W}$. Similarly, a series g of Π is group-like if it satisfies

$$g(\mathrm{Y},\mathrm{W}) = \sum_{\mathrm{X}:\, \mathrm{X}\geq\mathrm{Z},\, \mathrm{Y}\vee\mathrm{X}=\mathrm{W}} g(\mathrm{Z},\mathrm{X}) \tag{9.66}$$

for all $\mathrm{Z} \leq \mathrm{Y} \leq \mathrm{W}$. These descriptions can be deduced from coproduct formula (7.36). Note that (9.65) is the Weisner formula (1.38).

There is a bijection between primitive series and group-like series of Π given by (9.62). An interesting example of this correspondence is

$$\mu \longleftrightarrow \delta, \tag{9.67}$$

where δ denotes the unit element of the flat-incidence algebra. To see this, note that $\exp(\mu) = \zeta \circ \mu = \delta$ and $\log(\delta) = \mu \circ \delta = \mu$. It is also clear that μ is primitive and δ is group-like. In terms of the identification (9.64),

$$\begin{aligned} \delta &\leftrightarrow \mathtt{H}_{\mathrm{Z}/\mathrm{Z}} \in \Pi[\mathrm{Z}] \quad \text{for each Z,} \\ \mu &\leftrightarrow \mathtt{Q}_{\mathrm{Z}/\mathrm{Z}} \in \Pi[\mathrm{Z}] \quad \text{for each Z,} \end{aligned} \tag{9.68}$$

where $\mathtt{Q}_{\mathrm{Z}/\mathrm{Z}}$ is the $\mathtt{Q}$-basis element defined in (7.38). We refer to δ as the *universal series* of Π. It is an instance of the universal series in Section 4.13.6 specialized to the commutative operad.

Recall from (7.46) that E is the primitive part of Π, hence a primitive series of Π is the same as a series of E. More precisely, it is given by $(a_{\mathrm{Z}}\mathtt{Q}_{\mathrm{Z}/\mathrm{Z}})$, where a_{Z} are arbitrary scalars. The corresponding group-like series is given by $(\sum_{\mathrm{X}:\, \mathrm{X}\geq\mathrm{Z}} a_{\mathrm{X}}\mathtt{Q}_{\mathrm{X}/\mathrm{Z}})$.

9.7. Comparisons between exp-log correspondences

We presented two approaches to exp-log correspondences, the first one in Sections 9.1 and 9.2, and the second one in Sections 9.5 and 9.6. We now explain the precise connection between the two. The passage from the former to the latter is done by specializing the comonoid c to the exponential bimonoid E. The passage in the other direction is less trivial and involves the convolution monoid, the bimonoid of star families, and the universal measuring comonoid from Chapter 8.

9.7.1. Morphisms from the exponential bimonoid. Recall the exponential species E. For a species p, there is an isomorphism of vector spaces

$$\mathscr{S}(\mathsf{p}) = \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{E}, \mathsf{p}).$$

Explicitly, a series v of p corresponds to the map of species $\mathsf{E} \to \mathsf{p}$ whose F-component is

$$\mathsf{E}[F] \to \mathsf{p}[F], \qquad \mathtt{H}_F \mapsto v_F.$$

Moreover, for a comonoid c, observe that

$$\begin{aligned} \mathscr{P}(\mathsf{c}) &= \mathscr{S}(\mathcal{P}(\mathsf{c})) = \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{E}, \mathcal{P}(\mathsf{c})), \\ \mathscr{G}(\mathsf{c}) &= \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{E}, \mathsf{c}). \end{aligned} \tag{9.69}$$

The first is the space of all maps of species from E to $\mathcal{P}(\mathsf{c})$, or equivalently, the space of all coderivations from E to c. The second is the space of all morphisms of comonoids from E to c.

For a monoid a, observe that the action (9.44) is the special case $\mathsf{c} := \mathsf{E}$ of the action (9.1) under the identification $\mathscr{S}(\mathsf{a}) = \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{E}, \mathsf{a})$. Similarly, when a is commutative, the action (9.59) is the special case $\mathsf{c} := \mathsf{E}$ of the action (9.14). Hence:

Lemma 9.129. *The correspondence (9.46) is a specialization of (9.4), and (9.61) is a specialization of (9.16) to the comonoid* E.

Similarly, using the above identifications of the primitive series and group-like series:

Lemma 9.130. *Theorem 9.117 is a specialization of Theorem 9.11, and Theorem 9.124 is a specialization of Theorem 9.40 to the comonoid* E.

Exercise 9.131. Let (a, μ) be a monoid in species. A series v of a is *exponential* if $\mu_A^F(v_F) = v_A$ for all $F \geq A$. Let $\mathscr{E}(\mathsf{a})$ denote the space of all exponential series of a. Check that

$$\mathscr{E}(\mathsf{a}) = \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{E}, \mathsf{a}),$$

that is, exponential series of a correspond to monoid morphisms $\mathsf{E} \to \mathsf{a}$. Moreover, for a bimonoid h, any exponential series is group-like, that is, $\mathscr{E}(\mathsf{h}) \subseteq \mathscr{G}(\mathsf{h})$. This is equivalent to the first statement in Exercise 7.2.

9.7.2. Convolution monoid, internal hom for comonoids, and universal measuring comonoid. We now explain how to go in the other direction. For any species p and q, observe using (8.12) that

$$\mathscr{S}(\hom^{\times}(\mathsf{p},\mathsf{q})) = \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{p},\mathsf{q}). \tag{9.70}$$

For a comonoid c and monoid a, recall the convolution monoid $\hom^{\times}(\mathsf{c},\mathsf{a})$ from Section 8.3.1. We deduce from (8.15) that the action of the lune-incidence algebra given by (9.44) coincides with (9.1) via the identification (9.70) with $\mathsf{p} := \mathsf{c}$ and $\mathsf{q} := \mathsf{a}$. Similarly, when c is cocommutative and a is commutative, (8.24) says that the action of the flat-incidence algebra given by (9.59) coincides with (9.14). Hence:

Lemma 9.132. *The correspondence* (9.4) *is a specialization of* (9.46)*, and* (9.16) *is a specialization of* (9.61) *to the monoid* $\hom^{\times}(\mathsf{c},\mathsf{a})$.

Now recall from Section 8.4.1 the comonoid $\mathcal{C}(\mathsf{c},\mathsf{d})$ associated to comonoids c and d. We have

$$\begin{aligned} \mathscr{P}(\mathcal{C}(\mathsf{c},\mathsf{d})) &= \mathcal{A}\text{-}\mathsf{Sp}\big(\mathsf{c},\mathcal{P}(\mathsf{d})\big), \\ \mathscr{G}(\mathcal{C}(\mathsf{c},\mathsf{d})) &= \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c},\mathsf{d}). \end{aligned} \tag{9.71}$$

These can be directly established from the definitions. The key point is that if a family $(f_{F/A})$ is part of a primitive series or group-like series, then all its higher terms are determined by its base term. The first claim can also be deduced formally using (8.30) and (9.70):

$$\mathscr{P}(\mathcal{C}(\mathsf{c},\mathsf{d})) = \mathscr{S}(\mathcal{P}(\mathcal{C}(\mathsf{c},\mathsf{d}))) = \mathscr{S}(\hom^{\times}(\mathsf{c},\mathcal{P}(\mathsf{d}))) = \mathcal{A}\text{-}\mathsf{Sp}\big(\mathsf{c},\mathcal{P}(\mathsf{d})\big).$$

Similarly, the second claim may also be deduced by noting that a group-like series of $\mathcal{C}(\mathsf{c},\mathsf{d})$ is a morphism of comonoids from E to $\mathcal{C}(\mathsf{c},\mathsf{d})$, which by the internal hom property (Proposition 8.23) is the same as a morphism of comonoids from c to d.

Now consider $\mathcal{C}(\mathsf{c},\mathsf{k})$, where c is cocommutative and k is a bimonoid. It carries the structure of a bimonoid. This is the bimonoid of star families from Section 8.4.5. Similarly, when c is cocommutative and k is a bicommutative bimonoid, we have the bicommutative bimonoid of star families ${}^{\mathrm{co}}\mathcal{C}(\mathsf{c},\mathsf{k})$ from Section 8.4.6. By unwinding definitions:

Lemma 9.133. *Theorem 9.11 is a specialization of Theorem 9.117 to the bimonoid* $\mathcal{C}(\mathsf{c},\mathsf{k})$*, and Theorem 9.40 is a specialization of Theorem 9.124 to the bicommutative bimonoid* ${}^{\mathrm{co}}\mathcal{C}(\mathsf{c},\mathsf{k})$.

Exercise 9.134. Check that: The correspondence (9.5) is also the restriction of (9.46) applied to the convolution monoid $\hom^{\times}(\mathsf{c},\mathsf{k})$ via the diagram

$$\begin{tikzcd} \mathscr{P}(\mathcal{C}(\mathsf{c},\mathsf{k})) \arrow[r, hook] \arrow[d, equal] & \mathscr{S}(\mathcal{C}(\mathsf{c},\mathsf{k})) \arrow[d] & \mathscr{G}(\mathcal{C}(\mathsf{c},\mathsf{k})) \arrow[l, hook'] \arrow[d, equal] \\ \mathcal{A}\text{-}\mathsf{Sp}\big(\mathsf{c},\mathcal{P}(\mathsf{k})\big) \arrow[r, hook] & \mathscr{S}(\hom^{\times}(\mathsf{c},\mathsf{k})) & \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c},\mathsf{k}). \arrow[l, hook'] \end{tikzcd}$$

All horizontal maps are injections. The vertical map in the middle is induced from (8.29). Recall that the latter is a morphism of monoids by Lemma 8.35.

Now recall from Section 8.6.1 the universal measuring comonoid $\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})$ associated to monoids a and b. We have

$$
\begin{aligned}
\mathscr{P}(\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})) &= \mathcal{A}\text{-}\mathsf{Sp}\big(\mathcal{Q}(\mathsf{a}),\mathsf{b}\big),\\
\mathscr{G}(\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})) &= \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{a},\mathsf{b}).
\end{aligned}
\tag{9.72}
$$

The first claim can be deduced formally using (8.46) and (9.70):

$$\mathscr{P}(\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})) = \mathscr{S}(\mathcal{P}(\overline{\mathcal{C}}(\mathsf{a},\mathsf{b}))) = \mathscr{S}(\mathrm{hom}^{\times}(\mathcal{Q}(\mathsf{a}),\mathsf{b})) = \mathcal{A}\text{-}\mathsf{Sp}\big(\mathcal{Q}(\mathsf{a}),\mathsf{b}\big).$$

Similarly, the second claim may also be deduced by noting that a group-like series of $\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})$ is a morphism of comonoids from E to $\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})$, which by the universal property (Proposition 8.47) is the same as a morphism of monoids from a to b.

Now consider $\overline{\mathcal{C}}(\mathsf{h},\mathsf{a})$, where a is commutative and h is a bimonoid. It carries the structure of a bimonoid (Section 8.6.5). Similarly, when a is commutative and h is a bicommutative bimonoid, we have the bicommutative bimonoid ${}^{\mathsf{co}}\overline{\mathcal{C}}(\mathsf{h},\mathsf{a})$.

Lemma 9.135. *Theorem 9.12 is a specialization of Theorem 9.117 to the bimonoid $\overline{\mathcal{C}}(\mathsf{h},\mathsf{a})$, and Theorem 9.41 is a specialization of Theorem 9.124 to the bicommutative bimonoid ${}^{\mathsf{co}}\overline{\mathcal{C}}(\mathsf{h},\mathsf{a})$.*

Exercise 9.136. Check that: The correspondence (9.6) is also the restriction of (9.46) applied to the convolution monoid $\mathrm{hom}^{\times}(\mathsf{h},\mathsf{a})$ via the diagram

$$
\begin{array}{ccccc}
\mathscr{P}(\overline{\mathcal{C}}(\mathsf{h},\mathsf{a})) & \hookrightarrow & \mathscr{S}(\overline{\mathcal{C}}(\mathsf{h},\mathsf{a})) & \hookleftarrow & \mathscr{G}(\overline{\mathcal{C}}(\mathsf{h},\mathsf{a}))\\
\| & & \downarrow & & \|\\
\mathcal{A}\text{-}\mathsf{Sp}\big(\mathcal{Q}(\mathsf{h}),\mathsf{a}\big) & \hookrightarrow & \mathscr{S}(\mathrm{hom}^{\times}(\mathsf{h},\mathsf{a})) & \hookleftarrow & \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h},\mathsf{a}).
\end{array}
$$

All horizontal maps are injections. The vertical map in the middle is induced from (8.45). Recall that the latter is a morphism of monoids by Lemma 8.56.

Exercise 9.137. Check that applying $\mathscr{G}$ to the identification of comonoids in Proposition 8.65 yields the bijection in Proposition 8.47. In the same way, Proposition 8.67 implies Proposition 8.68. (Use (9.71) and (9.72).)

9.8. Formal power series. Series of Joyal species

Joyal species were discussed in Section 2.16. We now consider the space of series of a Joyal species. Formal power series (viewed as a monoid under substitution) operates on this space when the Joyal species carries the structure of a Joyal monoid. The action of the exponential and logarithmic power series yield the exp-log correspondence on the space of series.

We then relate these notions to parallel notions for $\mathcal{B}^J$-species, where $\mathcal{B}^J$ is the braid arrangement on a finite set J (Section 1.13). The role played by formal power series in the former context is played by the lune-incidence algebra in the latter. To begin with, out of a formal power series we build an element of the lune-incidence algebra of $\mathcal{B}^J$. This passage turns substitution of power series into multiplication in the lune-incidence algebra. The exponential power series yields the uniform noncommutative zeta function, while the logarithmic power series yields its inverse noncommutative Möbius function. Then, out of a series of a Joyal species p, we build a series of the $\mathcal{B}^J$-species p^J (which was constructed in Section 2.16.6). This passage turns the action of formal power series on the space of series of a Joyal monoid into the action of the lune-incidence algebra on the space of series of a p^J-monoid.

In this section, we assume that the field characteristic is 0.

9.8.1. Formal power series. Let $\mathscr{F}$ denote the space of *formal power series* in one variable χ whose constant term is 0. A typical formal power series s is written in functional notation as

$$s(\chi) = \sum_{n \geq 1} s_n \chi^n. \tag{9.73}$$

Recall that $\mathscr{F}$ is a monoid under substitution. Explicitly, the substitution of t into s is given by

$$(s \circ t)_n := \sum_{(i_1,\dots,i_k) \vDash n} s_k t_{i_1} \dots t_{i_k}. \tag{9.74}$$

The sum is over all compositions of n. In functional notation,

$$(s \circ t)(\chi) = s\big(t(x)\big).$$

Note very carefully that $\mathscr{F}$ is not an algebra because $s \circ (t+t') \neq s \circ t + s \circ t'$ in general. Also note that the set of formal power series with $s_1 \neq 0$ constitutes a group under substitution.

9.8.2. From formal power series to lune-incidence algebras. Let s be a formal power series. For any compositions A and F of J with $A \leq F$, define

$$s^J(A, F) := \prod_i s_{\deg(F/A)_i}, \tag{9.75}$$

where $\deg(F/A)_i$ is the number of blocks of F which refine the i-th block of A. For example,

$$A = krish|na, \quad F = kr|i|sh|n|a, \quad s^J(A, F) = s_3 s_2.$$

This is because $kr|i|sh$ which refines $krish$ has 3 blocks, while $n|a$ which refines na has 2 blocks. In particular,

$$s^J(O, F) = s_{\deg(F)}, \tag{9.76}$$

where $\deg(F)$ denotes the number of blocks of F, and O is the one-block set composition of J.

Lemma 9.138. *For $J = S \sqcup T$, let A and F be compositions of S with $A \leq F$, and B and G be compositions of T with $B \leq G$. Then $A|B$ and $F|G$ are compositions of J with $A|B \leq F|G$, and*

$$s^J(A|B, F|G) = s^S(A, F) s^T(B, G).$$

Proof. This multiplicative property follows directly from (9.75). ☐

Observe that s^J defines an element of the lune-incidence algebra of the braid arrangement $\mathcal{B}^J$. In other words, we have a map

$$\mathscr{F} \to \mathrm{I}_{\mathrm{lune}}[\mathcal{B}^J], \qquad s \mapsto s^J. \tag{9.77}$$

Note very carefully that this map does not preserve sums or products by scalars. On the other hand, it turns the substitution product of power series into the product of the lune-incidence algebra:

Lemma 9.139. *For any formal power series s and t, we have*

$$(s \circ t)^J = s^J t^J.$$

In other words, the map (9.77) is a morphism of monoids.

Proof. We first do a special case. Using definition (9.75) and product formulas (1.39) and (9.74),

$$\begin{aligned}
(s \circ t)^J(O, G) &= (s \circ t)_{\deg(G)} \\
&= \sum_{(i_1,\dots,i_k) \vDash \deg(G)} s_k t_{i_1} \dots t_{i_k} \\
&= \sum_{F\colon F \leq G} s_{\deg(F)} \prod_i t_{\deg(G/F)_i} \\
&= \sum_{F\colon F \leq G} s^J(O, F) t^J(F, G) \\
&= (s^J t^J)(O, G).
\end{aligned}$$

The general case can be deduced from Lemma 9.138. □

9.8.3. Examples. We now consider some basic examples of formal power series, and the corresponding elements of the lune-incidence algebra of the braid arrangement.

Example 9.140. Consider the formal power series

$$\begin{aligned}
e(\chi) &:= \exp(\chi) - 1 = \sum_{n \geq 1} \frac{\chi^n}{n!}, \\
l(\chi) &:= \log(1 + \chi) = \sum_{n \geq 1} (-1)^{n-1} (n-1)! \, \frac{\chi^n}{n!}.
\end{aligned}$$

These are the *exponential power series* and *logarithmic power series*, respectively. They are inverses of each other in the monoid $\mathscr{F}$. Note very carefully that $(-1)^{n-1}(n-1)!$ is the Möbius number of the braid arrangement of rank $n-1$.

We now make explicit the images of these formal power series under the map (9.77). Using (9.75), we deduce that

$$e^J(A, F) = \prod_i \frac{1}{\deg(F/A)_i!} \quad \text{and} \quad l^J(A, F) = (-1)^{\mathrm{rk}(F/A)} \prod_i \frac{1}{\deg(F/A)_i}.$$

In other words,

$$e^J = \boldsymbol{\zeta}_u \quad \text{and} \quad l^J = \boldsymbol{\mu}_u,$$

where $\boldsymbol{\zeta}_u$ is the uniform noncommutative zeta function of the braid arrangement $\mathcal{B}^J$, and $\boldsymbol{\mu}_u$ is its inverse (Section 1.5.7).

Exercise 9.141. Show that: If $s \in \mathscr{F}$ is such that $s^J \in \mathrm{I}_{\mathrm{lune}}[\mathcal{B}^J]$ is a noncommutative zeta function for every finite set J, then s is the exponential power series and s^J is the uniform noncommutative zeta function.

Example 9.142. Consider the formal power series

$$e_0(\chi) = \sum_{n \geq 1} \chi^n \quad \text{and} \quad l_0(\chi) = \sum_{n \geq 1} (-1)^{n-1} \chi^n.$$

Using (9.75), we deduce that

$$e_0^J(A, F) = 1 \quad \text{and} \quad l_0^J(A, F) = (-1)^{\mathrm{rk}(F/A)}.$$

In other words,

$$e_0^J = \boldsymbol{\zeta}_0 \quad \text{and} \quad l_0^J = \boldsymbol{\mu}_0,$$

the noncommutative 0-zeta function and noncommutative 0-Möbius function, respectively, of the braid arrangement (Section 1.5.11).

More generally, for any scalar α, define the formal power series

$$g_\alpha(\chi) := \sum_{n \geq 1} \alpha^{n-1} \chi^n.$$

Observe that $g_1(\chi) = e_0(\chi)$ and $g_{-1}(\chi) = l_0(\chi)$. Again using (9.75),

$$g_\alpha^J(A, F) = h_\alpha(A, F) = \alpha^{\mathrm{rk}(F/A)},$$

with h_α as in (1.53).

Example 9.143. For any scalar α, define the formal power series c_α by $c_\alpha(\chi) := \alpha\chi$. Using (9.75), we deduce that

$$c_\alpha^J(A, F) = \alpha r_\alpha(A, F) = \begin{cases} \alpha^{\deg(A)} & \text{if } F = A, \\ 0 & \text{otherwise,} \end{cases}$$

with r_α as in (1.53).

Similarly, for any scalar α, define the formal power series p_α by

$$p_\alpha(\chi) := (1 + x)^\alpha - 1 = (e \circ c_\alpha \circ l)(\chi).$$

In view of Lemma 9.139,

$$p_\alpha^J = \alpha \boldsymbol{\zeta}_u r_\alpha \boldsymbol{\mu}_u = \alpha t_\alpha,$$

with t_α as in (1.54).

More illustrations on how identities in the monoid of formal power series relate to identities in the lune-incidence algebra are given below.

Example 9.144. Observe that:

- c_{-1} is an involution, hence so is r_{-1}.
- $c_{-1} \circ e_0 \circ c_{-1} = l_0$, and hence $r_{-1}\, \boldsymbol{\zeta}_0\, r_{-1} = \boldsymbol{\mu}_0$.
- $e \circ c_{-1} \circ l \circ c_{-1} = e_0$, and hence $\boldsymbol{\zeta}_u\, r_{-1}\, \boldsymbol{\mu}_u\, r_{-1} = \boldsymbol{\zeta}_0$. The first identity says that $\exp(-\log(1-\chi)) - 1 = \chi/(1-\chi)$. The second is an instance of the noncommutative Zaslavsky formula (1.91).

Similarly, the identities

$$c_\alpha \circ c_\beta = c_{\alpha\beta}, \quad c_{-1} \circ g_\alpha \circ c_{-1} = g_{-\alpha}, \quad g_\alpha \circ g_\beta = g_{\alpha+\beta}, \quad p_\alpha \circ p_\beta = p_{\alpha\beta}$$

yield the identities listed in Section 1.5.12 (applied to the braid arrangement).

The above examples are summarized in Table 9.2.

TABLE 9.2. Formal power series and lune-incidence algebra.

formal power series		lune-incidence algebra elements	
$e(x)$	$\sum_{n\geq 1} \frac{x^n}{n!}$	$\boldsymbol{\zeta}_u$	$\prod_i \frac{1}{\deg(F/A)_i!}$
$l(x)$	$\sum_{n\geq 1} (-1)^{n-1} \frac{x^n}{n}$	$\boldsymbol{\mu}_u$	$(-1)^{\mathrm{rk}(F/A)} \prod_i \frac{1}{\deg(F/A)_i}$
$e_0(x)$	$\sum_{n\geq 1} x^n$	$\boldsymbol{\zeta}_0$	1
$l_0(x)$	$\sum_{n\geq 1} (-1)^{n-1} x^n$	$\boldsymbol{\mu}_0$	$(-1)^{\mathrm{rk}(F/A)}$
$g_\alpha(x)$	$\sum_{n\geq 1} \alpha^{n-1} x^n$	h_α	$\alpha^{\mathrm{rk}(F/A)}$
$c_\alpha(x)$	αx	αr_α	$\begin{cases} \alpha^{\deg(A)} & \text{if } F = A, \\ 0 & \text{otherwise} \end{cases}$
$p_\alpha(x)$	$(1+x)^\alpha - 1$	αt_α	$\alpha \boldsymbol{\zeta}_u r_\alpha \boldsymbol{\mu}_u$

9.8.4. Series of Joyal species. Let p be a Joyal species. A *series* v of p is a family of elements $v_J \in \mathsf{p}[J]$, one for each nonempty finite set J, such that

$$\sigma(v_J) = v_{J'}$$

for each bijection $\sigma : J \to J'$. Let $\mathscr{S}(\mathsf{p})$ denote the space of series of p. This construction is functorial in p, and defines a functor $\mathscr{S}$ from the category of Joyal species to the category of vector spaces.

Let c be a Joyal comonoid. A series v of c is *primitive* if $\Delta_{S,T}(v_J) = 0$ for each $J = S \sqcup T$, with S and T nonempty. Similarly, a series v of c is *group-like* if $\Delta_{S,T}(v_J) = v_S \otimes v_T$ for each $J = S \sqcup T$, with S and T nonempty.

9.8.5. From series of Joyal species to series of $\mathcal{B}^J$-species. Let p be a Joyal species. Recall the functor (2.102) which associates a $\mathcal{B}^J$-species p^J to p.

Let v be a series of p. For any composition $F = (S_1, \ldots, S_k)$ of J, put

$$v^J_F := v_{S_1} \otimes \cdots \otimes v_{S_k} \quad \in \ \mathsf{p}^J[F].$$

Then v^J defines a series of p^J. Thus, we have a map

$$\mathscr{S}(\mathsf{p}) \to \mathscr{S}(\mathsf{p}^J), \qquad v \mapsto v^J. \tag{9.78}$$

Observe that

$$v^J_O = v_J, \tag{9.79}$$

where O denotes the one-block composition of J.

Exercise 9.145. Let v be a series of a Joyal comonoid c. Check that: if v is primitive or group-like, then so is v^J.

9.8.6. Left modules. Let a be a Joyal monoid and $\mathscr{S}(\mathsf{a})$ its space of series. The monoid $\mathscr{F}$ of formal power series acts on the left on $\mathscr{S}(\mathsf{a})$ by

$$(s \circ v)_J := \sum_{F \vDash J} s_{\deg(F)} \mu_O^F(v_F^J) = \sum_{F \vDash J} s^J(O, F) \mu_O^F(v_F^J). \tag{9.80}$$

Here s is a formal power series and v is a series of a. The sum is over all compositions of J.

Lemma 9.146. *For a formal power series s and series v of a, we have*

$$(s \circ v)^J = s^J \circ v^J.$$

In other words, the map (9.78) is a morphism of left $\mathscr{F}$-modules, with $\mathscr{S}(\mathsf{a}^J)$ viewed as a left module over $\mathscr{F}$ via (9.77).

Proof. The identity is straightforward to verify using the multiplicative property in Lemma 9.138. □

In fact, the above identity can also be used to formally deduce that (9.80) defines a left action:

$$\begin{aligned}(s \circ (t \circ v))_J &= (s \circ (t \circ v))^J_O = (s^J \circ (t \circ v)^J)_O = (s^J \circ (t^J \circ v^J))_O \\ &= ((s^J t^J) \circ v^J)_O = ((s \circ t)^J \circ v^J)_O = ((s \circ t) \circ v)^J_O = ((s \circ t) \circ v)_J.\end{aligned}$$

The first step and last step used (9.79). The fifth step used Lemma 9.139.

9.8.7. Exp-log correspondence. Recall the exponential and logarithmic power series from Example 9.140. For any Joyal monoid a, the action of $e(\chi)$ and $l(\chi)$ via (9.80) sets up an *exp-log correspondence* on $\mathscr{S}(\mathsf{a})$. It relates to the exp-log correspondence (9.46) for the monoid a^J as follows. For v a series of a,

$$\boldsymbol{\zeta}_u \circ v^J = (e \circ v)^J \quad \text{and} \quad \boldsymbol{\mu}_u \circ v^J = (l \circ v)^J. \tag{9.81}$$

This is an instance of Lemma 9.146.

When the Joyal monoid a is commutative, the exp-log correspondence on $\mathscr{S}(\mathsf{a})$ relates to the exp-log correspondence (9.61) for the commutative monoid a^J as follows. For v a series of a,

$$\zeta \circ v^J = (e \circ v)^J \quad \text{and} \quad \mu \circ v^J = (l \circ v)^J, \tag{9.82}$$

where ζ and μ are zeta function and Möbius function of the poset of flats. This follows from (9.81) and Exercise 9.125.

Example 9.147. Consider the exponential Joyal monoid E defined by setting $\mathsf{E}[J] = \Bbbk$ for all J, with all product components $\mu_{S,T}$ equal to the canonical identification $\Bbbk \otimes \Bbbk \xrightarrow{\cong} \Bbbk$. A series v of E can be identified with a sequence of scalars $(v_n)_{n \geq 1}$ via $v_J = v_{|J|}$. Observe that v is group-like iff $v_n = \alpha^n$ for some scalar α, and primitive iff $v_n = 0$ for all $n \geq 2$. It is convenient to view a series v of E as the *exponential generating function*

$$v(x) = \sum_{n\geq 1} v_n \frac{\chi^n}{n!}. \tag{9.83}$$

In this notation, group-like series are those of the form $\exp(\alpha\chi)$, while primitive series are those of the form $\alpha\chi$.

Applying the functor (2.102) to E yields the $\mathcal{B}^J$-monoid E^J. Using the canonical identifications $\Bbbk^{\otimes n} \cong \Bbbk$, we see that E^J is the same as the exponential monoid E in Section 7.2 specialized to the arrangement $\mathcal{B}^J$. (Also see Remark 2.92 in this regard.) The space of series $\mathscr{S}(\mathsf{E}^J)$ is described explicitly in Example 9.127. It is easy to see directly that the map (9.78) preserves primitive series and group-like series.

Now let us turn to the action (9.80) of $\mathscr{F}$ on $\mathscr{S}(\mathsf{E})$. For any set composition $F = (S_1, \dots, S_k)$, we have $\mu_O^F(v_F^J) = v_{|S_1|} \dots v_{|S_k|}$. This is a product of scalars. The action (9.80) is given by

$$(s \circ v)_J = \sum_{F \vDash J} s_{\deg(F)} v_{|S_1|} \dots v_{|S_k|},$$

which may be rewritten as

$$(s \circ v)_n = \sum_{(i_1,\dots,i_k)\vDash n} \binom{n}{i_1, \dots, i_k} s_k v_{i_1} \dots v_{i_k}.$$

This agrees with the action of $\mathscr{F}$ on exponential generating functions: to compute $s \circ v$, we substitute (9.83) into (9.73), and rewrite the resulting power series as an exponential generating function. Thus, we may write $(s \circ v)(\chi) = s(v(\chi))$.

In view of Example 9.127, the identities (9.82) for the Joyal monoid E link the exponential and logarithmic power series to Möbius inversion in the poset of flats. In particular, the first identity becomes equivalent to the result given in [**21**, Proposition 6.15].

Notes

The classical exponential and logarithm functions and their inverse relationship are of prime importance in mathematics. Relevant references for ideas developed in this chapter are given below.

Bialgebras.
Primitives and group-likes. The exp-log correspondence between primitives and group-likes is considered by Gabriel [**335**, Exposé VIIB, Section 3.2]. In the setting of complete Hopf algebras, it is given by Quillen [**766**, Appendix A, Proposition 2.6]. It is also given by Fresse [**324**, Proposition 8.1.5], Chmutov, Duzhin, Mostovoy [**215**, Appendix A.2.12], and mentioned by Griffiths and Morgan [**376**, Section 17.5, page 175], Hain [**407**, Proposition 2.5.1]; see also the remarks by Menous [**677**, Section II], Foissy, Patras, Thibon [**309**, page 221], Ditters and Scholtens [**257**, Paragraph after Formula (6)]. This is the classical analogue of Theorem 9.117. For the special case of the completed tensor algebra (which can be seen in analogy with Example 9.118), the correspondence is considered by Ree [**774**, (2.1.1) and (2.2.2)]

without explicit mention of the coproduct. This case is also treated by Hochschild [**442**, pages 109 and 110], Bourbaki [**150**, Section II.6.2], Serre [**823**, Chapter IV, Corollary 7.3], Hazewinkel [**427**, Lemma 14.4.13], Reutenauer [**777**, Theorem 3.1, items (i) and (iii), Theorem 3.2, items (i) and (ii)], Shnider and Sternberg [**825**, Proposition 3.8.1], Bonfiglioli and Fulci [**130**, Theorem 3.24]. A related discussion is given by Michaelis [**688**, page 667]. For the completed tensor algebra with the dequasishuffle coproduct, see [**174**, Lemma 3]. An earlier reference is [**698**, Lemma 3].

(Co)derivations and (co)algebra morphisms. The classical analogue of the **exp** map in Theorems 9.11 and 9.12 is considered by Nichols [**719**, Chapter III, Theorem 6], [**720**, Lemma 7]. He shows that the exponential of an ϵ-derivation from a bialgebra to a commutative algebra is an algebra morphism, and dually, the exponential of an ϵ-coderivation from a cocommutative coalgebra to a bialgebra is a coalgebra morphism. The analogue of the **exp** map in Theorem 9.12 in the language of algebraic groups is considered by Demazure and Gabriel [**242**, Section II.6, 3.7 and Section IV.2, Proposition 4.1], [**243**, Section II.6, 3.7], Hochschild [**444**, page 69], [**445**, Section VIII.1], Cartier [**202**, Remark 3.9.1]. The classical analogue of Theorem 9.11 in a special case is also given by Helmstetter [**430**, Theorem 15].

Recent references on the classical analogue of Theorem 9.12 are by Manchon [**637**, Proposition 22, item (i)], Grinberg and Reiner [**377**, Section 1.7], see also [**615**, page 170]. A different classical analogue of Theorem 9.12 is given by Ebrahimi-Fard, Gracia-Bondía, Patras [**271**, Theorem 4.1], see also [**637**, Theorem 6], [**275**, Theorem 2.1], [**743**, Theorem 5.1], [**677**, Section III.C], [**678**, page 908], [**204**, Section 6.4]. It corresponds to the $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ of the braid arrangement which is constructed from Dynkin elements [**21**, Exercise 15.38 and Section 14.9.9]. (More references on Dynkin elements can be found in [**19**, Sections 14.5 and 14.6], [**21**, Notes to Chapter 14].)

Logarithm of the identity map. The classical analogue of the logarithm of the identity map (9.7) appears in work of Nichols [**719**, page 51], [**720**, page 67]. Its naturality is discussed in [**719**, Chapter III, Theorem 1], [**720**, Lemma 1]. This is a special case of the analogue of Lemma 9.10. The analogue of Proposition 9.17 is given in [**719**, Chapter III, Theorem 5]. Alternatively, the analogue of the second statement concerning $\mathcal{Q}(\mathsf{h})$ is given in [**720**, Proposition 6], while the analogue of the first statement concerning $\mathcal{P}(\mathsf{h})$ is stated in [**720**, Remark on page 71]. The analogue of Exercise 9.24 is given in [**719**, Chapter III, Theorem 3], [**720**, Proposition 3].

The classical analogue of the logarithm of the identity map (9.7) also appears in later works of Loday [**606**, Formula (4.5.2.5)], Patras [**740**, Theorem 3.5]. Helmstetter [**430**, page 183] considers it for universal enveloping algebras. Reutenauer [**776**, page 269], [**777**, pages 58 and 60], Garsia [**341**, page 362] consider it for the tensor algebra; see also earlier work of Bialynicki-Birula, Mielnik, Plebański [**111**, Formula (3.7)], [**689**, Formula (10.12)]. The logarithm of a coalgebra endomorphism of a cocommutative Hopf algebra is considered by Schmitt [**813**, Theorem 9.4]. He mentions the special case when the endomorphism is the identity on [**813**, page 318]. The analogue of Exercise 9.23 is given in [**813**, Theorem 9.5 and Corollary 9.6]. In the language of universal enveloping algebras, this result for $f = \mathrm{id}$ is also given by Helmstetter [**430**, Theorem 19]. The important case of the tensor algebra is considered by Reutenauer [**776**, Corollary 1.6], [**777**, Corollary 3.16]. Also see the Notes to Chapter 17 on the Solomon operator for related information.

Signed setting. The exponential map in a signed setting is considered by Ismail in his thesis [**478**, page 16]. In [**478**, Proposition on page 16], he considers a part of the analogue of Theorem 9.30 in the special case when $\mathsf{c} = \mathcal{E}(\mathsf{p})$ and $\mathsf{k} = \mathcal{E}(\mathsf{q})$.

0-*bialgebras.* The classical analogue of the 0-logarithm of the identity map (9.41) is the operator considered by Loday and Ronco [**609**, Section 3.3], [**610**, Proposition 2.5]. See Example 9.142 in this regard which connects the geometric series to the 0-exponential and 0-logarithm. The analogue of the diagram in Example 9.110 is given in [**610**, Proposition 2.5, item (d)].

The classical analogue of Theorem 9.103 in the special case $\mathsf{c} := \mathcal{T}_0(\mathsf{p}) =: \mathsf{k}$ is considered by Hoffman and Ihara [**450**, Proposition 4.1]. However, they do not interpret it as a 0-exp-log correspondence; they use the terms 'expansion' and 'contraction' instead of 0-exponential and 0-logarithm, respectively.

Internal hom for coalgebras and universal measuring coalgebra. Classical analogues of (9.71) and (9.72) are present in the paper by Anel and Joyal [**36**, Lemmas 2.5.8 and 3.5.11, Corollaries 2.7.26 and 3.9.30]. See also the comments by Fox [**311**, pages 22 and 23], [**312**, page 229]. Related information is given in the Notes to Chapter 8.

Joyal bimonoids. Series of Joyal species are discussed in detail in our monograph [**19**, Section 12]. The action (9.80) is given in [**19**, Formula (182)]. Group-like series and primitive series of a comonoid in Joyal species are defined in [**19**, Sections 12.3 and 12.4]. The analogue of Exercise 9.115 is [**19**, Proposition 67, item (v)]. The analogue of Proposition 9.116 and Theorem 9.117 for Joyal species is given in [**19**, Theorem 69]. The analogue of the universal series of $\mathbf{\Sigma}$ and its group-like property in Example 9.119 is given in [**19**, Formula (231) and Lemma 97]. The analogue of the corresponding primitive series $\boldsymbol{\mu}$ in (9.54) and (9.55) is given in [**19**, Formulas (239), (240), (241)]. The analogue of Theorems 9.11 and 9.12 is given in [**19**, Theorem 78]. The analogue of Proposition 9.17 is [**19**, Corollary 80]. The proofs here generalize the ones given in this reference. The analogue of (9.70) is given in [**19**, Formula (203)]. The analogue of Lemma 9.130 is mentioned in [**19**, proof of Theorem 69]. However, the analogue of Lemma 9.133 is new and not present in this reference.

Bimonoids for hyperplane arrangements. Exp-log correspondences in the context of species for arrangements using noncommutative zeta and Möbius functions are presented here for the first time. Noncommutative Möbius inversion was introduced in [**21**, Section 15.4]; Example 9.118 clarifies its connection to Hopf theory. Related to Example 9.119, a series of the Zie species has been equivalently called a Zie family in [**21**, Section 11.5.1], and also a Weisner function in [**21**, Section 15.7.2]. The connection of the exponential and logarithmic power series to zeta and Möbius functions was first brought forth in [**21**, Sections 6.6.5 and C.1.7]. Section 9.8.7 takes these ideas forward (as mentioned on [**21**, page 544] under future work). The subtle distinction between $\mathbf{exp}$ and $\overline{\mathbf{exp}}$ (or between $\mathbf{log}$ and $\overline{\mathbf{log}}$) in Section 9.1.5 is not visible in the classical theory since the uniform noncommutative zeta function is projective.

CHAPTER 10

Characteristic operations

Recall that the definition of a bimonoid in species makes use of the Tits monoid. The latter is a monoid structure on the set of faces. On the other hand, there is a bimonoid, namely, Σ, which is itself built out of faces. This double occurrence of faces acquires formal meaning now. Elements of Σ give rise to characteristic operations on any bimonoid h, see Table 10.1. This yields a morphism of bimonoids from Σ to the biconvolution bimonoid $\mathsf{end}^{\times}(\mathsf{h})$. In particular, for each face A, we have a linear map from $\Sigma[A]$ to the endomorphisms of $\mathsf{h}[A]$. Further, when h is either commutative or cocommutative, this map is an algebra antimorphism or algebra morphism, with the Tits product on the former and composition product on the latter, or equivalently, there is a right action or left action, respectively, of the Tits algebra $\Sigma[A]$ on $\mathsf{h}[A]$. When h is cocommutative, instead of $\mathsf{end}^{\times}(\mathsf{h})$, one can work with its subbimonoid of star families $\mathcal{C}(\mathsf{h}, \mathsf{h})$. Similarly, when h is commutative, one can work instead with the subbimonoid $\overline{\mathcal{C}}(\mathsf{h}, \mathsf{h})$.

TABLE 10.1. Characteristic operations by faces, flats, bifaces.

characteristic operation by a face	$\mathtt{H}_{F/A} \cdot h := \mu_A^F \Delta_A^F(h)$
comm. characteristic operation by a flat	$\mathtt{H}_{\mathrm{X}/\mathrm{Z}} \cdot h := \mu_{\mathrm{Z}}^{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}(h)$
two-sided characteristic operation by a biface	$\mathtt{H}_{(F/A, F'/A)} \cdot h := \mu_A^F \beta_{F,F'} \Delta_A^{F'}(h)$

The above story has a simpler commutative analogue. Recall that bicommutative bimonoids can be formulated using the Birkhoff monoid. The latter is a monoid structure on the set of flats. On the other hand, there is a bimonoid, namely, Π, which is itself built out of flats. Formally, elements of Π give rise to commutative characteristic operations on any bicommutative bimonoid h. This yields a morphism of bimonoids from Π to $\mathsf{end}^{\times}(\mathsf{h})$. Moreover, for any flat Z, the Birkhoff algebra $\Pi[\mathrm{Z}]$ acts on $\mathsf{h}[\mathrm{Z}]$. (Since the Birkhoff algebra is commutative, there is no distinction between left and right actions.)

There are more general operations one can consider on bimonoids by working with bifaces instead of faces. We call these the two-sided characteristic operations. The role of the Tits algebra is now played by the Janus algebra. More generally, one can also consider q-bimonoids whose components are acted upon by the q-Janus algebra.

Finally, we focus on certain idempotent operators on the components of bimonoids which arise by characteristic operations by idempotent elements in the Birkhoff algebra, Tits algebra, Janus algebra. For a bicommutative bimonoid, the components of its primitive and indecomposable parts arise as images of the action of the $\mathtt{Q}$-basis idempotents of the Birkhoff algebra. More generally, for a cocommutative bimonoid, the components of its primitive part arise as images of the action of the eulerian idempotents of the Tits algebra. The same is true for the indecomposable part of a commutative bimonoid. These results will play an important role in giving constructive proofs of the rigidity theorems in Chapter 13.

10.1. Characteristic operations

We begin with characteristic operations of the bimonoid of faces Σ on any bimonoid h. It gives rise to a morphism of bimonoids $\Sigma \to \mathrm{end}^{\times}(\mathsf{h})$. Further, when h is cocommutative, this morphism maps into $\mathcal{C}(\mathsf{h}, \mathsf{h})$, the subbimonoid of star families. Since this map preserves primitive parts, we obtain a map from the Zie species to $\mathrm{hom}^{\times}\big(\mathsf{h}, \mathcal{P}(\mathsf{h})\big)$. By passing to series, a family of special Zie elements, or equivalently, a noncommutative Möbius function gives rise to a map $\mathsf{h} \to \mathcal{P}(\mathsf{h})$ which is precisely $\mathbf{log}(\mathrm{id}_{\mathsf{h}})$. When h is commutative, we have a similar situation with $\mathrm{hom}^{\times}\big(\mathsf{h}, \mathcal{P}(\mathsf{h})\big)$ replaced by $\mathrm{hom}^{\times}\big(\mathcal{Q}(\mathsf{h}), \mathsf{h}\big)$.

10.1.1. Characteristic operations. Recall the species of faces Σ from Section 7.6. Let h be a bimonoid. For any face A, given $z \in \Sigma[A]$ and $h \in \mathsf{h}[A]$, define an element $z \cdot h \in \mathsf{h}[A]$ as follows. First, write

$$z = \sum_{F:\, F \geq A} a^{F/A}\, \mathtt{H}_{F/A}$$

for scalars $a^{F/A}$. Then set

$$z \cdot h := \sum_{F:\, F \geq A} a^{F/A}\, \mu_A^F \Delta_A^F(h). \tag{10.1}$$

In particular,

$$\mathtt{H}_{F/A} \cdot h := \mu_A^F \Delta_A^F(h). \tag{10.2}$$

We refer to this as a *characteristic operation*.

Recall from Section 7.6.4 that $\Sigma[A]$ is an algebra, which can be identified with the Tits algebra of the arrangement $\mathcal{A}_A$.

Lemma 10.1. *Let* h *be any bimonoid. Then:*

- *For any* $h \in \mathsf{h}[A]$,

$$\mathtt{H}_{A/A} \cdot h = h. \tag{10.3}$$

- *If* h *is cocommutative, then for any* $z, w \in \Sigma[A]$ *and* $h \in \mathsf{h}[A]$,

$$(z \cdot w) \cdot h = z \cdot (w \cdot h). \tag{10.4}$$

- *If* h *is commutative, then for any* $z, w \in \Sigma[A]$ *and* $h \in \mathsf{h}[A]$,

$$(z \cdot w) \cdot h = w \cdot (z \cdot h). \tag{10.5}$$

PROOF. Statement (10.3) follows from (co)unitality. The remaining statements are linear in z and w, so we assume them to be basis elements.

For (10.4), take $z = \mathtt{H}_{G/A}$ and $w = \mathtt{H}_{F/A}$. Then

$$\begin{aligned} z \cdot (w \cdot h) &= \mu_A^G \Delta_A^G \mu_A^F \Delta_A^F(h) \\ &= \mu_A^G \mu_G^{GF} \beta_{GF,FG} \Delta_F^{FG} \Delta_A^F(h) \\ &= \mu_A^{GF} \beta_{GF,FG} \Delta_A^{FG}(h) \\ &= \mu_A^{GF} \Delta_A^{GF}(h) \\ &= (z \cdot w) \cdot h. \end{aligned}$$

We used the bimonoid axiom (2.12), then associativity (2.8) and coassociativity (2.10), and finally cocommutativity (2.23).

The calculation for (10.5) proceeds similarly, except at the end, where β merges with μ instead of Δ. □

In other words: When h is cocommutative, for any face A, (10.1) defines a left action of the Tits algebra $\Sigma[A]$ on the space $\mathsf{h}[A]$. When h is commutative, there is a right action of $\Sigma[A]$ on $\mathsf{h}[A]$ given by

$$h \cdot z := z \cdot h. \tag{10.6}$$

Example 10.2. Recall that Σ is a cocommutative bimonoid with product and coproduct (7.64). Thus, we may take $\mathsf{h} = \Sigma$, resulting in a left action of the Tits algebra $\Sigma[A]$ on itself. This coincides with the usual action (7.66). Indeed, for faces F and G both greater than A,

$$\begin{aligned} \mathtt{H}_{F/A} \cdot \mathtt{H}_{G/A} &= \mu_A^F \Delta_A^F(\mathtt{H}_{G/A}) \\ &= \mu_A^F(\mathtt{H}_{FG/F}) \\ &= \mathtt{H}_{FG/A}. \end{aligned}$$

Now take $\mathsf{h} := \Gamma$. This is also a cocommutative bimonoid with product and coproduct (7.18). One may check that the action of $\mathtt{H}_{F/A}$ sends $\mathtt{H}_{C/A}$ to $\mathtt{H}_{FC/A}$. This is the usual left action of the Tits algebra on the space of chambers.

10.1.2. Review of bimonoid properties. We recall some related properties of the product and coproduct of any bimonoid h.

- For faces $A \leq F$,

$$\Delta_A^F \mu_A^F = \mathrm{id}. \tag{10.7}$$

- For any faces $A \leq F$ and $A \leq G$,

$$\Delta_A^F \mu_A^G \Delta_A^G \mu_A^F = \mu_F^{FG} \Delta_F^{FG}, \tag{10.8}$$

and, if h is commutative, then

$$\mu_A^G \Delta_A^G \mu_A^F = \mu_A^{FG} \Delta_F^{FG}, \tag{10.9}$$

and, if h is cocommutative, then

$$\Delta_A^F \mu_A^G \Delta_A^G = \mu_F^{FG} \Delta_A^{FG}. \tag{10.10}$$

- If $A \leq F \leq G$, then

$$\Delta_A^G \mu_A^F = \Delta_F^G \quad \text{and} \quad \Delta_A^F \mu_A^G = \mu_F^G. \tag{10.11}$$

These are restatements of (2.14), (2.15), (2.28), (2.29), (2.30), though not written in that order.

10.1.3. Interaction with the bimonoid structure. We use the above properties to study how characteristic operations interact with the product and coproduct of Σ and h. More precisely, we fix a pair of faces $F \geq A$. We take an element z in either $\Sigma[F]$ or $\Sigma[A]$, and an element h in either $\mathsf{h}[F]$ or $\mathsf{h}[A]$. There are two ways in which z and h can interact. For instance, if $z \in \Sigma[F]$ and $h \in \mathsf{h}[A]$, then we can consider $\mu_A^F(z) \cdot h$ and $z \cdot \Delta_A^F(h)$. The results below explain the relations between these two interactions.

Lemma 10.3. *Let h be a bimonoid. Let $A \leq F$. Then:*

- *For any $z \in \Sigma[F]$ and $h \in \mathsf{h}[A]$,*

$$\mu_A^F(z) \cdot h = \mu_A^F\big(z \cdot \Delta_A^F(h)\big). \tag{10.12}$$

- *For any $z \in \Sigma[A]$ and $h \in \mathsf{h}[F]$,*

$$\Delta_A^F(z) \cdot h = \Delta_A^F\big(z \cdot \mu_A^F(h)\big), \tag{10.13}$$

and if h is commutative, then

$$z \cdot \mu_A^F(h) = \mu_A^F\big(\Delta_A^F(z) \cdot h\big). \tag{10.14}$$

- *If h is cocommutative, then for any $z \in \Sigma[A]$ and $h \in \mathsf{h}[A]$,*

$$\Delta_A^F(z \cdot h) = \Delta_A^F(z) \cdot \Delta_A^F(h). \tag{10.15}$$

Proof. All statements are linear in z, so we may assume z is a basis element in each case. We go over them one by one.

- For (10.12): Take $z = \mathtt{H}_{G/F}$. Then

$$\mu_A^F\big(z \cdot \Delta_A^F(h)\big) = (\mu_A^F \mu_F^G \Delta_F^G \Delta_A^F)(h) = (\mu_A^G \Delta_A^G)(h) = \mathtt{H}_{G/A} \cdot h = \mu_A^F(z) \cdot h.$$

  We used associativity (2.8) and coassociativity (2.10).
- For (10.13): Take $z = \mathtt{H}_{G/A}$. Then

$$\Delta_A^F\big(z \cdot \mu_A^F(h)\big) = (\Delta_A^F \mu_A^G \Delta_A^G \mu_A^F)(h) = (\mu_F^{FG} \Delta_F^{FG})(h) = \mathtt{H}_{FG/F} \cdot h = \Delta_A^F(z) \cdot h.$$

  We used (10.8). In addition, if h is commutative,

$$\begin{aligned}\mu_A^F\big(\Delta_A^F(z) \cdot h\big) &= \mu_A^F\big(\mathtt{H}_{FG/F} \cdot h\big) = (\mu_A^F \mu_F^{FG} \Delta_F^{FG})(h) \\ &= (\mu_A^{FG} \Delta_F^{FG})(h) = (\mu_A^G \Delta_A^G \mu_A^F)(h) = \mathtt{H}_{G/A} \cdot \mu_A^F(h) = z \cdot \mu_A^F(h).\end{aligned}$$

  We used (10.9) and associativity (2.8).

- For (10.15): Take $z = \mathtt{H}_{G/A}$. Then

$$\begin{aligned}\Delta^F_A(z \cdot h) &= (\Delta^F_A \mu^G_A \Delta^G_A)(h) = (\mu^{FG}_F \Delta^{FG}_A)(h)\\ &= (\mu^{FG}_F \Delta^{FG}_F \Delta^F_A)(h) = \mathtt{H}_{FG/F} \cdot \Delta^F_A(h) = \Delta^F_A(z) \cdot \Delta^F_A(h).\end{aligned}$$

We used (10.10) and coassociativity (2.10).

□

The following properties complement (10.12)–(10.15) (and follow from the first of these).

Corollary 10.4. *Let* h *be a bimonoid. Let* $A \leq F$. *Then:*

- *For any* $z \in \Sigma[F]$ *and* $h \in \mathsf{h}[A]$,

$$z \cdot \Delta^F_A(h) = \Delta^F_A\big(\mu^F_A(z) \cdot h\big). \tag{10.16}$$

- *For any* $z \in \Sigma[F]$ *and* $h \in \mathsf{h}[F]$,

$$\mu^F_A(z \cdot h) = \mu^F_A(z) \cdot \mu^F_A(h). \tag{10.17}$$

Proof. Equation (10.16) follows by applying Δ^F_A to both sides of (10.12), in view of (10.7). Equation (10.17) follows by replacing h in (10.12) for $\mu^F_A(h)$ and employing (10.7). □

We consider some specializations, and related consequences.

Lemma 10.5. *Let* h *be a bimonoid. Let* $A \leq K \leq F$, *and* $x \in \mathsf{h}[K]$ *and* $y \in \mathsf{h}[A]$. *Then:*

$$\begin{aligned}\mu^K_A(\mathtt{H}_{F/K} \cdot x) &= \mathtt{H}_{F/A} \cdot \mu^K_A(x),\\ \mathtt{H}_{F/K} \cdot x &= \Delta^K_A(\mathtt{H}_{F/A} \cdot \mu^K_A(x)),\end{aligned} \tag{10.18}$$

$$\begin{aligned}\mathtt{H}_{F/K} \cdot \Delta^K_A(y) &= \Delta^K_A(\mathtt{H}_{F/A} \cdot y),\\ \mu^K_A(\mathtt{H}_{F/K} \cdot \Delta^K_A(y)) &= \mathtt{H}_{F/A} \cdot y,\end{aligned} \tag{10.19}$$

$$\mathtt{H}_{F/K} \cdot x = x \iff \mathtt{H}_{F/A} \cdot \mu^K_A(x) = \mu^K_A(x), \tag{10.20}$$

$$\mathtt{H}_{F/A} \cdot y = \mu^K_A(x) \iff \Delta^K_A(\mathtt{H}_{F/A} \cdot y) = x. \tag{10.21}$$

Proof. Let us prove the first identity in (10.18). By (10.17),

$$\mathtt{H}_{F/A} \cdot \mu^K_A(x) = \mu^K_A(\mathtt{H}_{F/K}) \cdot \mu^K_A(x) = \mu^K_A(\mathtt{H}_{F/K} \cdot x).$$

Alternatively, one may proceed directly and use (10.11):

$$\mathtt{H}_{F/A} \cdot \mu^K_A(x) = \mu^F_A \Delta^F_A \mu^K_A(x) = \mu^F_A \Delta^F_K(x) = \mu^K_A \mu^F_K \Delta^F_K(x) = \mu^K_A(\mathtt{H}_{F/K} \cdot x).$$

Applying Δ^K_A and using (10.7) yields the second identity in (10.18). Alternatively, one may directly use (10.13). The identities in (10.19) can be deduced from those in (10.18) by setting $x := \Delta^K_A(y)$. For (10.20), forward implication follows from the first identity in (10.18), while backward implication follows from the second identity in (10.19). Identity (10.21) can be deduced by applying Δ^K_A to the lhs, and μ^K_A to the rhs for the two implications, respectively. □

Lemma 10.6. *Let* h *be a cocommutative bimonoid. Let* $A \leq K$, $A \leq F$, *and* $x \in \mathsf{h}[A]$ *and* $y \in \mathsf{h}[K]$. *Then:*

$$\mathtt{H}_{KF/K} \cdot \Delta_A^K(x) = \Delta_A^K(\mathtt{H}_{F/A} \cdot x). \tag{10.22}$$

$$\mathtt{H}_{F/A} \cdot x = x \implies \mathtt{H}_{KF/K} \cdot \Delta_A^K(x) = \Delta_A^K(x). \tag{10.23}$$

$$x = \mathtt{H}_{F/A} \cdot \mu_A^K(y) \iff x = \mathtt{H}_{FK/A} \cdot x \text{ and } \Delta_A^K(x) = \mathtt{H}_{KF/K} \cdot y. \tag{10.24}$$

Proof. For (10.22): By (10.15),

$$\mathtt{H}_{KF/K} \cdot \Delta_A^K(x) = \Delta_A^K(\mathtt{H}_{F/A}) \cdot \Delta_A^K(x) = \Delta_A^K(\mathtt{H}_{F/A} \cdot x).$$

Alternatively, proceeding directly and using (10.10),

$$\begin{aligned}\mathtt{H}_{KF/K} \cdot \Delta_A^K(x) = \mu_K^{KF} \Delta_K^{KF} \Delta_A^K(x) = \mu_K^{KF} \Delta_A^{KF}(x) \\ = \Delta_A^K \mu_A^F \Delta_A^F(x) = \Delta_A^K(\mathtt{H}_{F/A} \cdot x).\end{aligned}$$

The implication (10.23) is immediate from (10.22).

For (10.24): Put $z = \mathtt{H}_{F/A} \cdot \mu_A^K(y)$. Then

$$\mathtt{H}_{KF/K} \cdot y = \Delta_A^K(\mathtt{H}_{F/A}) \cdot y = \Delta_A^K(\mathtt{H}_{F/A} \cdot \mu_A^K(y)) = \Delta_A^K(z).$$

Since $\mathtt{H}_{K/K} \cdot y = y$, we have $\mathtt{H}_{K/A} \cdot \mu_A^K(y) = \mu_A^K(y)$ by (10.20), and hence $\mathtt{H}_{FK/A} \cdot z = z$ by successive application of (10.23) and (10.20). This proves forward implication. For backward implication, we are given $\Delta_A^K(x) = \Delta_A^K(z)$. Compose both sides with $\mu_A^{FK} \beta_{FK,KF} \Delta_K^{KF}$ and use cocommutativity and $x = \mathtt{H}_{FK/A} \cdot x$ and $z = \mathtt{H}_{FK/A} \cdot z$ to deduce $x = z$. □

10.1.4. Canonical morphism from bimonoid of faces. Let h be a bimonoid. Recall the biconvolution bimonoid $\mathrm{end}^{\times}(\mathsf{h})$ from Section 8.3. Define a map of species

$$\Psi : \Sigma \to \mathrm{end}^{\times}(\mathsf{h}) \tag{10.25}$$

with A-component given by

$$\Psi_A : \Sigma[A] \to \mathrm{end}^{\times}(\mathsf{h})[A], \qquad \Psi_A(\mathtt{H}_{F/A}) := \mu_A^F \Delta_A^F : \mathsf{h}[A] \to \mathsf{h}[A].$$

Equivalently, by (10.1),

$$\Psi_A(z)(h) = z \cdot h$$

for all $z \in \Sigma[A]$ and $h \in \mathsf{h}[A]$.

Note that if h is cocommutative, then Ψ_A is an algebra morphism, with product on $\mathrm{end}^{\times}(\mathsf{h})$ given by composition.

Lemma 10.7. *The map* (10.25) *is a morphism of bimonoids.*

Proof. The fact that Ψ is a morphism of monoids is equivalent to (10.12), of comonoids to (10.13). □

A related result is given in Exercise 8.17.

Exercise 10.8. Show that if h is cocommutative, then Ψ is a morphism of monoids wrt the Hadamard product $\times$.

10.1.5. Internal hom for comonoids and universal measuring comonoid. For any cocommutative bimonoid h, recall the bimonoid $\mathcal{C}(\mathsf{h},\mathsf{h})$ of star families from Section 8.4. It is built from the internal hom for comonoids. It is a subbimonoid of $\mathrm{end}^{\times}(\mathsf{h})$ by Lemma 8.39. Similarly, for a commutative bimonoid h, we have the bimonoid $\overline{\mathcal{C}}(\mathsf{h},\mathsf{h})$ built from the universal measuring comonoid. It is a subbimonoid of $\mathrm{end}^{\times}(\mathsf{h})$ by Lemma 8.60.

Lemma 10.9. *For a cocommutative bimonoid* h*, the morphism of bimonoids* (10.25) *factors as*

$$\begin{tikzcd} \Sigma \arrow[rr] \arrow[dr] & & \mathrm{end}^{\times}(\mathsf{h}). \\ & \mathcal{C}(\mathsf{h},\mathsf{h}) \arrow[ur, hook] & \end{tikzcd} \tag{10.26}$$

Further, on each A-component, this is a diagram of algebras.

Proof. In view of Lemma 8.39, it remains to check that the image of Ψ belongs to $\mathcal{C}(\mathsf{h},\mathsf{h})$. For $f = \Psi_A(\mathtt{H}_{G/A})$, the dotted arrow in (8.28) is given by $\Psi_F(\mathtt{H}_{FG/F})$:

$$\begin{tikzcd} \mathsf{h}[A] \arrow[r, "\Delta"] \arrow[dd, "\Delta"'] & \mathsf{h}[G] \arrow[r, "\mu"] \arrow[d, "\Delta"] & \mathsf{h}[A] \arrow[dd, "\Delta"] \\ & \mathsf{h}[GF] \arrow[d, "\beta"] & \\ \mathsf{h}[F] \arrow[r, "\Delta"'] & \mathsf{h}[FG] \arrow[r, "\mu"'] & \mathsf{h}[F]. \end{tikzcd}$$

The pentagon on the left is the double comonoid axiom (2.32), while the one on the right is the bimonoid axiom (2.12). This completes the check. □

The special case $\mathsf{h} := \Gamma$, the bimonoid of chambers, of diagram (10.26) was encountered in (8.39).

Lemma 10.10. *For a commutative bimonoid* h*, the morphism of bimonoids* (10.25) *factors as*

$$\begin{tikzcd} \Sigma \arrow[rr] \arrow[dr] & & \mathrm{end}^{\times}(\mathsf{h}). \\ & \overline{\mathcal{C}}(\mathsf{h},\mathsf{h}) \arrow[ur, hook] & \end{tikzcd} \tag{10.27}$$

Further, on each A-component, this is a diagram of algebras.

Proof. We proceed as in the previous proof. In the diagram, the vertical arrows now go up instead of down with Δ replaced by μ. The pentagon on the left is the bimonoid axiom (2.12), while the one on the right is the double monoid axiom (2.31). □

10.1.6. Primitive part and indecomposable part. The map Ψ preserves the primitive part. This follows from Lemma 10.7. When h is either commutative or cocommutative, the following stronger statements hold. (They are stronger in view of (8.23).) A conceptual understanding via the bimonoid of star families and the universal measuring comonoid is given in Exercise 10.12 below.

Proposition 10.11. *Let h be a bimonoid.*

(i) *If h is cocommutative, then $\Psi\big(\mathcal{P}(\Sigma)\big) \subseteq \hom^{\times}\big(\mathsf{h}, \mathcal{P}(\mathsf{h})\big)$.*
(ii) *If h is commutative, then $\Psi\big(\mathcal{P}(\Sigma)\big) \subseteq \hom^{\times}\big(\mathcal{Q}(\mathsf{h}), \mathsf{h}\big)$.*

Proof. Consider (i). Let $z \in \mathcal{P}\big(\Sigma\big)[A]$ and $h \in \mathsf{h}[A]$. Then, for $F > A$, by (10.15),

$$\Delta^F_A(z \cdot h) = \Delta^F_A(z) \cdot \Delta^F_A(h) = 0.$$

Thus, $z \cdot h \in \mathcal{P}(\mathsf{h})[A]$.

The argument for (ii) is similar. Let $z \in \mathcal{P}\big(\Sigma\big)[A]$. Then, for $F > A$ and $h \in \mathsf{h}[F]$, by (10.14),

$$z \cdot \mu^F_A(h) = \mu^F_A\big(\Delta^F_A(z) \cdot h\big) = 0.$$

Thus, $z \cdot \mathcal{D}(\mathsf{h})[A] = 0$. □

Exercise 10.12. Deduce Proposition 10.11, item (i) from diagram (10.26) and formula (8.30), and item (ii) from diagram (10.27) and formula (8.46).

A more precise result is given below. Recall from Lemma 7.69 that $\mathcal{P}(\Sigma)$ is the same as the species of Zie elements.

Proposition 10.13. *Let h be a cocommutative bimonoid. Let A be any face. Then the left action of any Zie element of $\Sigma[A]$ sends $\mathsf{h}[A]$ to $\mathcal{P}(\mathsf{h})[A]$. Moreover, on $\mathcal{P}(\mathsf{h})[A]$ it acts by scalar multiplication by the coefficient of $\mathtt{H}_{A/A}$ in the Zie element.*

In particular, any special Zie element of $\Sigma[A]$ yields an idempotent operator on $\mathsf{h}[A]$ whose image is $\mathcal{P}(\mathsf{h})[A]$.

Proof. If $z \in \mathcal{P}\big(\Sigma\big)[A]$ and $h \in \mathsf{h}[A]$, then $z \cdot h \in \mathcal{P}(\mathsf{h})[A]$ (as in the proof of Proposition 10.11). Moreover, if $h \in \mathcal{P}(\mathsf{h})[A]$, then $\Delta^F_A(h) = 0$ for $F > A$, and by (10.1), $z \cdot h = a^{A/A} h$. □

Dually:

Proposition 10.14. *Let h be a commutative bimonoid. Let A be any face. Then the right action of any Zie element of $\Sigma[A]$ on $\mathsf{h}[A]$ factors through $\mathcal{Q}(\mathsf{h})[A]$. Moreover, on $\mathcal{Q}(\mathsf{h})[A]$ it acts by scalar multiplication by the coefficient of $\mathtt{H}_{A/A}$ in the Zie element.*

In particular, any special Zie element of $\Sigma[A]$ yields an idempotent operator on $\mathsf{h}[A]$ whose coimage is $\mathcal{Q}(\mathsf{h})[A]$.

The two situations when h is either cocommutative or commutative are, respectively, illustrated below.

$$
\begin{tikzcd}
\mathsf{h}[A] \arrow[r, "\text{left act. by Zie elt.}"] \arrow[dr, dashed, two heads] & \mathsf{h}[A] & & \mathsf{h}[A] \arrow[r, "\text{right act. by Zie elt.}"] \arrow[d, two heads] & \mathsf{h}[A] \arrow[d, two heads] \\
\mathcal{P}(\mathsf{h})[A] \arrow[u, hook] \arrow[r, "\text{mult. by scalar}"'] & \mathcal{P}(\mathsf{h})[A] \arrow[u, hook] & & \mathcal{Q}(\mathsf{h})[A] \arrow[r, "\text{mult. by scalar}"'] \arrow[ur, dashed, hook] & \mathcal{Q}(\mathsf{h})[A]
\end{tikzcd}
\tag{10.28}
$$

When the Zie element is special, the scalar multiplication map is the identity.

Exercise 10.15. Propositions 10.13 and 10.14 admit the following converses. Check that:

(i) For any face A, if an element of $\mathsf{\Sigma}[A]$ sends $\mathsf{h}[A]$ to $\mathcal{P}(\mathsf{h})[A]$ for every cocommutative bimonoid h, then it is a Zie element. (Take $\mathsf{h} := \mathsf{\Sigma}$.)

(ii) For any face A, if an element of $\mathsf{\Sigma}[A]$ sends $\mathsf{h}[A]$ through $\mathcal{Q}(\mathsf{h})[A]$ for every commutative bimonoid h, then it is a Zie element.

Lemma 10.16. *Let* h *be a bimonoid. Then, for* $h \in \mathsf{h}[A]$*,*

$$h \in \mathcal{P}(\mathsf{h})[A] \iff \mathtt{H}_{F/A} \cdot h = 0 \text{ for all } F > A, \tag{10.29}$$

$$h \in \mathcal{D}(\mathsf{h})[A] \iff h \in \sum_{F:\, F > A} \operatorname{im}(\mathtt{H}_{F/A} \cdot (-)). \tag{10.30}$$

Proof. For any $F > A$, since the map μ_A^F is injective, we have

$$\mathtt{H}_{F/A} \cdot h = 0 \iff \mu_A^F \Delta_A^F(h) = 0 \iff \Delta_A^F(h) = 0.$$

The first part follows. Similarly, the second part follows by using the fact that Δ_A^F is surjective. $\square$

Exercise 10.17. For a cocommutative bimonoid h, for any face A,

$$\text{for } h \in \mathsf{h}[A], \quad \mathtt{Q}_{A/A} \cdot h \in \mathcal{P}(\mathsf{h})[A],$$
$$\text{for } h \in \mathcal{P}(\mathsf{h})[A], \quad \mathtt{Q}_{A/A} \cdot h = h.$$

For a commutative bimonoid h, for any face A,

$$\text{for } h \in \mathsf{h}[A], \quad h \cdot \mathtt{Q}_{A/A} - h \in \mathcal{D}(\mathsf{h})[A],$$
$$\text{for } h \in \mathcal{D}(\mathsf{h})[A], \quad h \cdot \mathtt{Q}_{A/A} = 0.$$

Verify the first fact using (10.29) and (7.68), and the second using (10.30) and (7.68).

Now recall from Lemma 1.78 that every special Zie element of $\mathsf{\Sigma}[A]$ is of the form $\mathtt{Q}_{A/A}$ for some $\mathtt{Q}$-basis. Use this to reprove the last claim in Proposition 10.13 and in Proposition 10.14.

Exercise 10.18. Let h be a bimonoid. Use formula (7.73) to deduce that the characteristic operation of the first eulerian idempotent $\mathtt{Q}_{A/A}$ on $\mathsf{h}[A]$ is given by $\mathbf{log}(\mathrm{id})_A$, with the latter as defined in (9.7). Use this to reprove Proposition 9.17. Compare diagrams (10.28) with (9.8).

10.1.7. Universal series of the species of faces. Recall the space of series of a species from Section 9.5.1. Applying the series functor $\mathscr{S}$ to the morphism of bimonoids (10.25) yields a map of left modules

$$\mathscr{S}(\Psi) : \mathscr{S}(\Sigma) \to \mathscr{S}(\mathrm{end}^{\times}(\mathsf{h})) = \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{h}, \mathsf{h}) \tag{10.31}$$

over the lune-incidence algebra. This used Exercise 9.115. The equality is by (9.70). Further, this map preserves primitive and group-like series by Exercise 9.113.

Recall the universal series δ of Σ from Example 9.119. Observe that its image under the above map is the identity map on h, that is, $\mathscr{S}(\Psi)(\delta) = \mathrm{id}_{\mathsf{h}}$. This is a group-like series of $\mathrm{end}^{\times}(\mathsf{h})$. By (9.55), we deduce that its logarithm is the primitive series of $\mathrm{end}^{\times}(\mathsf{h})$ given by

$$\mathbf{log}(\mathrm{id}_{\mathsf{h}})_A : \mathsf{h}[A] \to \mathsf{h}[A], \qquad z \mapsto \mathsf{Q}_{A/A} \cdot z.$$

This is consistent with the claim in Exercise 10.18.

When h is cocommutative, the situation can be summarized as follows.

$$\begin{tikzcd} \mathscr{G}(\Sigma) \arrow[r] \arrow[d, "\mathbf{log}"'] & \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h}, \mathsf{h}) \arrow[d, "\mathbf{log}"] \\ \mathscr{P}(\Sigma) \arrow[r] & \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{h}, \mathcal{P}(\mathsf{h})) \end{tikzcd} \qquad \begin{tikzcd} \delta \arrow[r, mapsto] \arrow[d, mapsto] & \mathrm{id}_{\mathsf{h}} \arrow[d, mapsto] \\ \boldsymbol{\mu} \arrow[r, mapsto] & \mathbf{log}(\mathrm{id}_{\mathsf{h}}) \end{tikzcd} \tag{10.32}$$

For this, we made use of diagram (10.26) and descriptions of the primitve and group-like series of $\mathcal{C}(\mathsf{h}, \mathsf{h})$ given in (9.71).

When h is commutative, the situation can be summarized as follows.

$$\begin{tikzcd} \mathscr{G}(\Sigma) \arrow[r] \arrow[d, "\mathbf{log}"'] & \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h}, \mathsf{h}) \arrow[d, "\mathbf{log}"] \\ \mathscr{P}(\Sigma) \arrow[r] & \mathcal{A}\text{-}\mathsf{Sp}(\mathcal{Q}(\mathsf{h}), \mathsf{h}) \end{tikzcd} \qquad \begin{tikzcd} \delta \arrow[r, mapsto] \arrow[d, mapsto] & \mathrm{id}_{\mathsf{h}} \arrow[d, mapsto] \\ \boldsymbol{\mu} \arrow[r, mapsto] & \mathbf{log}(\mathrm{id}_{\mathsf{h}}) \end{tikzcd} \tag{10.33}$$

For this, we made use of diagram (10.27) and descriptions of the primitve and group-like series of $\overline{\mathcal{C}}(\mathsf{h}, \mathsf{h})$ given in (9.72).

10.1.8. Signed analogue. We mention that characteristic operations can also be defined for signed bimonoids exactly as in (10.1) and (10.2). We highlight one basic fact.

Let h be a signed bimonoid. When h is signed cocommutative, for any face A, (10.1) defines a left action of the Tits algebra $\Sigma[A]$ on $\mathsf{h}[A]$, that is, (10.4) holds. The calculation given in the proof of Lemma 10.1 gets modified as follows.

$$\begin{aligned} z \cdot (w \cdot h) &= \mu_A^G \Delta_A^G \mu_A^F \Delta_A^F(h) \\ &= \mu_A^G \mu_G^{GF} (\beta_{-1})_{GF,FG} \Delta_F^{FG} \Delta_A^F(h) \\ &= \mu_A^{GF} (\beta_{-1})_{GF,FG} \Delta_A^{FG}(h) \\ &= \mu_A^{GF} \Delta_A^{GF}(h) \\ &= (z \cdot w) \cdot h. \end{aligned}$$

We used the signed bimonoid axiom (2.37), then associativity (2.8) and coassociativity (2.10), and finally signed cocommutativity (2.47).

Dually, when h is signed commutative, for any face A, there is a right action of $\Sigma[A]$ on $\mathsf{h}[A]$.

10.2. Commutative characteristic operations

We now discuss the commutative analogue of the characteristic operations introduced in Section 10.1. The role of the bimonoid of faces is now played by the bimonoid of flats.

10.2.1. Commutative characteristic operations. Recall the species of flats Π from Section 7.4. Let h be a bicommutative bimonoid. For any flat Z, given $z \in \Pi[\mathrm{Z}]$ and $h \in \mathsf{h}[\mathrm{Z}]$, define an element $z \cdot h \in \mathsf{h}[\mathrm{Z}]$ as follows. First, write

$$z = \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} a^{\mathrm{X}/\mathrm{Z}}\, \mathtt{H}_{\mathrm{X}/\mathrm{Z}}$$

for scalars $a^{\mathrm{X}/\mathrm{Z}}$. Then set

$$z \cdot h := \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} a^{\mathrm{X}/\mathrm{Z}}\, \mu_{\mathrm{Z}}^{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}(h). \tag{10.34}$$

In particular,

$$\mathtt{H}_{\mathrm{X}/\mathrm{Z}} \cdot h := \mu_{\mathrm{Z}}^{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}(h). \tag{10.35}$$

We refer to this as a *commutative characteristic operation*.

Recall from Section 7.4.3 that $\Pi[\mathrm{Z}]$ is an algebra, which can be identified with the Birkhoff algebra of the arrangement $\mathcal{A}_{\mathrm{Z}}$.

Lemma 10.19. *For any $z, w \in \Pi[\mathrm{Z}]$ and $h \in \mathsf{h}[\mathrm{Z}]$,*

$$(z \cdot w) \cdot h = z \cdot (w \cdot h) \quad \textit{and} \quad \mathtt{H}_{\mathrm{Z}/\mathrm{Z}} \cdot h = h. \tag{10.36}$$

In other words, for any flat Z, (10.34) defines an action of the Birkhoff algebra $\Pi[\mathrm{Z}]$ on the space $\mathsf{h}[\mathrm{Z}]$. Since $\Pi[\mathrm{Z}]$ is a commutative algebra, there is no distinction between left and right actions.

Proof. The second statement follows from (co)unitality. It suffices to check the first statement on basis elements. Take $z = \mathtt{H}_{\mathrm{Y}/\mathrm{Z}}$ and $w = \mathtt{H}_{\mathrm{X}/\mathrm{Z}}$. Then

$$\begin{aligned} z \cdot (w \cdot h) &= \mu_{\mathrm{Z}}^{\mathrm{Y}} \Delta_{\mathrm{Z}}^{\mathrm{Y}} \mu_{\mathrm{Z}}^{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}(h) \\ &= \mu_{\mathrm{Z}}^{\mathrm{Y}} \mu_{\mathrm{Y}}^{\mathrm{Y}\vee\mathrm{X}} \Delta_{\mathrm{X}}^{\mathrm{X}\vee\mathrm{Y}} \Delta_{\mathrm{Z}}^{\mathrm{X}}(h) \\ &= \mu_{\mathrm{Z}}^{\mathrm{Y}\vee\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{Y}\vee\mathrm{X}}(h) \\ &= (z \cdot w) \cdot h. \end{aligned}$$

We used the bicommutative bimonoid axiom (2.26) followed by associativity (2.21) and coassociativity (2.24). □

Example 10.20. Recall that Π is a bicommutative bimonoid with product and coproduct (7.36). Thus, we may take $\mathsf{h} := \Pi$, resulting in an action of $\Pi[\mathrm{Z}]$ on itself. This coincides with the usual action (7.37). Indeed, for flats X and Y both greater than Z,

$$\begin{aligned}\mathtt{H}_{\mathrm{X}/\mathrm{Z}} \cdot \mathtt{H}_{\mathrm{Y}/\mathrm{Z}} &= \mu^{\mathrm{X}}_{\mathrm{Z}} \Delta^{\mathrm{X}}_{\mathrm{Z}}(\mathtt{H}_{\mathrm{Y}/\mathrm{Z}}) \\ &= \mu^{\mathrm{X}}_{\mathrm{Z}}(\mathtt{H}_{\mathrm{X}\vee\mathrm{Y}/\mathrm{X}}) \\ &= \mathtt{H}_{\mathrm{X}\vee\mathrm{Y}/\mathrm{Z}}.\end{aligned}$$

For $\mathsf{h} := \mathsf{E}$, the action of any flat is by the identity map.

10.2.2. Interaction with the bimonoid structure. For any bicommutative bimonoid h, for $\mathrm{Z} \leq \mathrm{X}$,

$$\Delta^{\mathrm{X}}_{\mathrm{Z}} \mu^{\mathrm{X}}_{\mathrm{Z}} = \mathrm{id}\,. \tag{10.37}$$

This is the second diagram in (2.27).

Lemma 10.21. *Let h be a bicommutative bimonoid. Let $\mathrm{Z} \leq \mathrm{X}$. Then:*

- *For any $z \in \Pi[\mathrm{X}]$ and $h \in \mathsf{h}[\mathrm{Z}]$,*

$$\mu^{\mathrm{X}}_{\mathrm{Z}}(z) \cdot h = \mu^{\mathrm{X}}_{\mathrm{Z}}\big(z \cdot \Delta^{\mathrm{X}}_{\mathrm{Z}}(h)\big), \tag{10.38}$$

  and

$$z \cdot \Delta^{\mathrm{X}}_{\mathrm{Z}}(h) = \Delta^{\mathrm{X}}_{\mathrm{Z}}(\mu^{\mathrm{X}}_{\mathrm{Z}}(z) \cdot h). \tag{10.39}$$

- *For any $z \in \Pi[\mathrm{Z}]$ and $h \in \mathsf{h}[\mathrm{X}]$,*

$$\Delta^{\mathrm{X}}_{\mathrm{Z}}(z) \cdot h = \Delta^{\mathrm{X}}_{\mathrm{Z}}\big(z \cdot \mu^{\mathrm{X}}_{\mathrm{Z}}(h)\big), \tag{10.40}$$

  and

$$z \cdot \mu^{\mathrm{X}}_{\mathrm{Z}}(h) = \mu^{\mathrm{X}}_{\mathrm{Z}}\big(\Delta^{\mathrm{X}}_{\mathrm{Z}}(z) \cdot h\big). \tag{10.41}$$

- *For any $z \in \Pi[\mathrm{Z}]$ and $h \in \mathsf{h}[\mathrm{Z}]$,*

$$\Delta^{\mathrm{X}}_{\mathrm{Z}}(z \cdot h) = \Delta^{\mathrm{X}}_{\mathrm{Z}}(z) \cdot \Delta^{\mathrm{X}}_{\mathrm{Z}}(h). \tag{10.42}$$

- *For any $z \in \Pi[\mathrm{X}]$ and $h \in \mathsf{h}[\mathrm{X}]$,*

$$\mu^{\mathrm{X}}_{\mathrm{Z}}(z \cdot h) = \mu^{\mathrm{X}}_{\mathrm{Z}}(z) \cdot \mu^{\mathrm{X}}_{\mathrm{Z}}(h). \tag{10.43}$$

Proof. We repeat the arguments in Lemma 10.3 and Corollary 10.4, with faces replaced by flats. ☐

10.2.3. Canonical morphism from bimonoid of flats. Let h be a bicommutative bimonoid. Define a map of species

$$\Psi : \Pi \to \mathrm{end}^{\times}(\mathsf{h}) \tag{10.44}$$

with Z-component given by

$$\Psi_{\mathrm{Z}} : \Pi[\mathrm{Z}] \to \mathrm{end}^{\times}(\mathsf{h})[\mathrm{Z}], \qquad \Psi_{\mathrm{Z}}(\mathtt{H}_{\mathrm{X}/\mathrm{Z}}) := \mu^{\mathrm{X}}_{\mathrm{Z}} \Delta^{\mathrm{X}}_{\mathrm{Z}} : \mathsf{h}[\mathrm{Z}] \to \mathsf{h}[\mathrm{Z}].$$

Equivalently by (10.34),

$$\Psi_{\mathrm{Z}}(z)(h) = z \cdot h$$

for all $z \in \Pi[\mathrm{Z}]$ and $h \in \mathsf{h}[\mathrm{Z}]$. Each Ψ_{Z} is an algebra morphism.

Lemma 10.22. *The map (10.44) is a morphism of bimonoids.*

Proof. The fact that Ψ is a morphism of monoids is equivalent to (10.38), of comonoids to (10.40). $\square$

Following the argument given in the proof of Lemmas 10.9 and 10.10, we obtain:

Lemma 10.23. *For a bicommutative bimonoid* h*, the morphism of bimonoids* (10.44) *factors as*

$$
\begin{tikzcd}
 & \mathcal{C}(\mathsf{h},\mathsf{h}) \arrow[dr, hook] & \\
\mathsf{\Pi} \arrow[ur] \arrow[rr] \arrow[dr] & & \mathrm{end}^{\times}(\mathsf{h}), \\
 & \overline{\mathcal{C}}(\mathsf{h},\mathsf{h}) \arrow[ur, hook] &
\end{tikzcd}
\tag{10.45}
$$

and on each Z*-component, this is a diagram of algebras.*

Example 10.24. Let $\mathcal{A}$ denote a rank-one arrangement. Let h be a bicommutative $\mathcal{A}$-bimonoid. By Lemma 2.88, this is the same as an idempotent operator e on $V = \mathsf{h}[\perp]$. In this language, $\mathrm{end}^{\times}(\mathsf{h})[\perp]$ is the space of all linear operators on V. As subspaces, $\mathcal{C}(\mathsf{h},\mathsf{h})[\perp]$ consists of those operators on V which preserve $\ker(e)$, while $\overline{\mathcal{C}}(\mathsf{h},\mathsf{h})[\perp]$ consists of those which preserve $\mathrm{im}(e)$. Finally, $\mathsf{\Pi}[\perp]$ is the two-dimensional subspace spanned by id and e.

10.2.4. Connection with characteristic operations. For a bicommutative bimonoid, characteristic operations and commutative characteristic operations are related as follows.

Lemma 10.25. *Let* h *be a bicommutative bimonoid. Then the map* (10.25) *factors through the support map* (7.84) *to yield a commutative diagram*

$$
\begin{tikzcd}
\mathsf{\Sigma} \arrow[d, two heads] \arrow[dr] & \\
\mathsf{\Pi} \arrow[r] & \mathrm{end}^{\times}(\mathsf{h})
\end{tikzcd}
\tag{10.46}
$$

of bimonoids. Further, the horizontal map is precisely (10.44).

Proof. Let F and G be faces both greater than A of the same support. Then $\mathtt{H}_{F/A}$ and $\mathtt{H}_{G/A}$ act the same way on $\mathsf{h}[A]$, that is, the diagram below commutes.

$$
\begin{tikzcd}
\mathsf{h}[A] \arrow[r, "\Delta_A^F"] \arrow[d, "\Delta_A^G"'] & \mathsf{h}[F] \arrow[d, "\mu_A^F"] \\
\mathsf{h}[G] \arrow[ur, "\beta_{F,G}"] \arrow[r, "\mu_A^G"'] & \mathsf{h}[A]
\end{tikzcd}
$$

The triangles commute by (co)commutativity. The result follows. $\square$

Diagram (10.46) can also be viewed as an instance of the universal property of the abelianization arising from its adjunction with the inclusion functor, see comment after (2.56).

10.2.5. Primitive part and indecomposable part. For a bicommutative bimonoid h,

$$\Psi\big(\mathcal{P}(\Pi)\big) \subseteq \hom^{\times}\big(\mathcal{Q}(\mathsf{h}), \mathcal{P}(\mathsf{h})\big),$$

with Ψ as in (10.44). This can be deduced from diagram (10.45) and combining formulas (8.30) and (8.46). Alternatively, we may combine both items of Proposition 10.11.

A more precise result is given below. Recall that $\mathtt{Q}_{Z/Z}$ is the unique element in $\mathcal{P}(\Pi)[Z]$ whose coefficient of $\mathtt{H}_{Z/Z}$ is 1. See (7.39) and (7.46).

Proposition 10.26. *Let h be a bicommutative bimonoid. Let Z be any flat. Then the action of any primitive element of $\Pi[Z]$ sends $\mathsf{h}[Z]$ to $\mathcal{P}(\mathsf{h})[Z]$ and through $\mathcal{Q}(\mathsf{h})[Z]$. Moreover, on $\mathcal{P}(\mathsf{h})[Z]$ and on $\mathcal{Q}(\mathsf{h})[Z]$, it acts by scalar multiplication by the coefficient of $\mathtt{H}_{Z/Z}$ in the primitive element of $\Pi[Z]$.*

In particular, the element $\mathtt{Q}_{Z/Z}$ yields an idempotent operator on $\mathsf{h}[Z]$ whose image is $\mathcal{P}(\mathsf{h})[Z]$ and coimage is $\mathcal{Q}(\mathsf{h})[Z]$.

Proof. Let $z \in \mathcal{P}\big(\Pi\big)[Z]$ and $h \in \mathsf{h}[Z]$. Then, for $X > Z$, by (10.42),

$$\Delta^{X}_{Z}(z \cdot h) = \Delta^{X}_{Z}(z) \cdot \Delta^{X}_{Z}(h) = 0.$$

Thus, $z \cdot h \in \mathcal{P}(\mathsf{h})[Z]$. Moreover, if $h \in \mathcal{P}(\mathsf{h})[Z]$, then $\Delta^{X}_{Z}(h) = 0$ for $X > Z$, and by (10.34), $z \cdot h = a^{Z/Z} h$. The argument about $\mathcal{Q}(\mathsf{h})[Z]$ is similar. $\square$

The situation is summarized below.

(10.47)
$$\begin{tikzcd}
& \mathsf{h}[Z] \arrow[r, "\text{act. by prim.}"] \arrow[dl, twoheadrightarrow] & \mathsf{h}[Z] \arrow[r, "\text{act. by prim.}"] \arrow[dr, twoheadrightarrow] & \mathsf{h}[Z] & \\
\mathcal{Q}(\mathsf{h})[Z] \arrow[r] & \mathcal{P}(\mathsf{h})[Z] \arrow[ur, hookrightarrow] & & \mathcal{Q}(\mathsf{h})[Z] \arrow[r] & \mathcal{P}(\mathsf{h})[Z] \arrow[ul, hookrightarrow]
\end{tikzcd}$$

Note in particular the induced map from $\mathcal{P}(\mathsf{h})[Z]$ to itself, and from $\mathcal{Q}(\mathsf{h})[Z]$ to itself. Both maps are given by multiplication by the same scalar.

Exercise 10.27. Show the following converse to Proposition 10.26. If an element of $\Pi[Z]$ sends $\mathsf{h}[Z]$ to $\mathcal{P}(\mathsf{h})[Z]$ for every bicommutative bimonoid h, then it is in $\mathcal{P}(\Pi)[Z]$. (Take $\mathsf{h} := \Pi$.) Similarly, if an element of $\Pi[Z]$ sends $\mathsf{h}[Z]$ through $\mathcal{Q}(\mathsf{h})[Z]$ for every bicommutative bimonoid h, then it is in $\mathcal{P}(\Pi)[Z]$.

Exercise 10.28. Deduce Proposition 10.26 from Propositions 10.13 and 10.14 using Lemma 10.25 and (1.120).

Exercise 10.29. Let h be a bicommutative bimonoid. Check that for $h \in \mathsf{h}[Z]$,

(10.48)
$$h \in \mathcal{P}(\mathsf{h})[Z] \iff \mathtt{H}_{X/Z} \cdot h = 0 \text{ for all } X > Z,$$

(10.49)
$$h \in \mathcal{D}(\mathsf{h})[Z] \iff h \in \sum_{X:\, X > Z} \operatorname{im}(\mathtt{H}_{X/Z} \cdot (-)).$$

Exercise 10.30. For a bicommutative bimonoid h, for any flat Z,

$$\begin{aligned}
&\text{for } h \in \mathsf{h}[\mathrm{Z}], \quad \mathtt{Q}_{\mathrm{Z}/\mathrm{Z}} \cdot h \in \mathcal{P}(\mathsf{h})[\mathrm{Z}] \text{ and } \mathtt{Q}_{\mathrm{Z}/\mathrm{Z}} \cdot h - h \in \mathcal{D}(\mathsf{h})[\mathrm{Z}],\\
&\text{for } h \in \mathcal{P}(\mathsf{h})[\mathrm{Z}], \quad \mathtt{Q}_{\mathrm{Z}/\mathrm{Z}} \cdot h = h,\\
&\text{for } h \in \mathcal{D}(\mathsf{h})[\mathrm{Z}], \quad \mathtt{Q}_{\mathrm{Z}/\mathrm{Z}} \cdot h = 0.
\end{aligned}$$

Verify these using (10.48), (10.49), and (7.43). Use this to reprove the last claim in Proposition 10.26.

Exercise 10.31. Let h be a bicommutative bimonoid. Use formula (7.39) to deduce that the characteristic operation of $\mathtt{Q}_{\mathrm{Z}/\mathrm{Z}}$ on $\mathsf{h}[\mathrm{Z}]$ is given by $\log(\mathrm{id})_{\mathrm{Z}}$, with the latter as defined in (9.22). Use this to reprove Proposition 9.47. Compare diagram (10.47) with (9.23).

10.2.6. Universal series of the species of flats. Recall the space of series of a species from Section 9.6.1. Applying the series functor $\mathscr{S}$ to the morphism of bimonoids (10.44) yields a map of modules

$$\mathscr{S}(\Psi) : \mathscr{S}(\Pi) \to \mathscr{S}(\mathrm{end}^{\times}(\mathsf{h})) = \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{h}, \mathsf{h}) \tag{10.50}$$

over the flat-incidence algebra. This used Exercise 9.126. The equality is by (9.70). Further, this map preserves primitive and group-like series.

Recall the universal series δ of Π from Example 9.128. Observe that its image under the above map is the identity map on h, that is, $\mathscr{S}(\Psi)(\delta) = \mathrm{id}_{\mathsf{h}}$. This is a group-like series of $\mathrm{end}^{\times}(\mathsf{h})$. By (9.68), we deduce that its logarithm is the primitive series of $\mathrm{end}^{\times}(\mathsf{h})$ given by

$$\log(\mathrm{id}_{\mathsf{h}})_{\mathrm{Z}} : \mathsf{h}[\mathrm{Z}] \to \mathsf{h}[\mathrm{Z}], \qquad z \mapsto \mathtt{Q}_{\mathrm{Z}/\mathrm{Z}} \cdot z.$$

This is consistent with the claim in Exercise 10.31.

The situation can be summarized as follows.

$$\begin{array}{ccc}
\mathscr{G}(\Pi) & \to & \mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h}, \mathsf{h})\\
{\scriptstyle \log}\downarrow & & \downarrow{\scriptstyle \log}\\
\mathscr{P}(\Pi) & \to & \mathcal{A}\text{-}\mathsf{Sp}(\mathcal{Q}(\mathsf{h}), \mathcal{P}(\mathsf{h}))
\end{array}
\qquad\qquad
\begin{array}{ccc}
\delta & \longmapsto & \mathrm{id}_{\mathsf{h}}\\
\downarrow & & \downarrow\\
\mu & \longmapsto & \log(\mathrm{id}_{\mathsf{h}})
\end{array} \tag{10.51}$$

For this, we made use of diagram (10.45) and descriptions of the primitve and group-like series of $\mathcal{C}(\mathsf{h}, \mathsf{h})$ and $\overline{\mathcal{C}}(\mathsf{h}, \mathsf{h})$ given in (9.71) and (9.72). One may also make use of diagrams (10.32) and (10.33).

10.2.7. Signed analogue. We mention that commutative characteristic operations can also be defined for signed bicommutative signed bimonoids exactly as in (10.34) and (10.35), with the components $\mu_{\mathrm{Z}}^{\mathrm{X}}$ and $\Delta_{\mathrm{Z}}^{\mathrm{X}}$ as in Proposition 2.37.

Lemma 10.19 continues to hold in this situation. The check now employs (1.162), (1.163) and Exercise 1.75. In other words, for any signed bicommutative signed bimonoid h, for any flat Z, the Birkhoff algebra $\Pi[\mathrm{Z}]$ acts on the space $\mathsf{h}[\mathrm{Z}]$.

10.3. Two-sided characteristic operations

We introduce two-sided characteristic operations on q-bimonoids by working with bifaces instead of faces.

10.3.1. Two-sided characteristic operations. For any scalar q, and any face A, recall the q-Janus algebra $\mathsf{J}_q^o[A]$ from Section 7.8.7.

Let h be a q-bimonoid. For any face A, given $z \in \mathsf{J}_q^o[A]$ and $h \in \mathsf{h}[A]$, define an element $z \cdot h \in \mathsf{h}[A]$ as follows. First, write

$$z = \sum_{\substack{(F,F')\colon F,F' \geq A \\ \mathrm{s}(F)=\mathrm{s}(F')}} a^{F/A,F'/A}\, \mathtt{H}_{(F/A,F'/A)}$$

for scalars $a^{F/A,F'/A}$. Then set

$$z \cdot h := \sum_{\substack{(F,F')\colon F,F' \geq A \\ \mathrm{s}(F)=\mathrm{s}(F')}} a^{F/A,F'/A}\, \mu_A^F \beta_{F,F'} \Delta_A^{F'}(h). \tag{10.52}$$

In particular,

$$\mathtt{H}_{(F/A,F'/A)} \cdot h := \mu_A^F \beta_{F,F'} \Delta_A^{F'}(h). \tag{10.53}$$

We refer to this as a *two-sided characteristic operation*.

Note very carefully that β and not β_q is employed in this definition.

Lemma 10.32. *Let h be any q-bimonoid. For any $z, w \in \mathsf{J}_q^o[A]$ and $h \in \mathsf{h}[A]$,*

$$(z \cdot w) \cdot h = z \cdot (w \cdot h) \quad \textit{and} \quad \mathtt{H}_{(A/A,A/A)} \cdot h = h. \tag{10.54}$$

In other words, for any face A, (10.52) defines a left action of the q-Janus algebra $\mathsf{J}_q^o[A]$ on the space $\mathsf{h}[A]$.

Proof. The second statement follows from (co)unitality. It suffices to check the first statement on basis elements. Accordingly, let $z = \mathtt{H}_{(G/A,G'/A)}$ and $w = \mathtt{H}_{(F/A,F'/A)}$. Let us first do the case $q = 1$. We calculate:

$$\begin{aligned}
z \cdot (w \cdot h) &= \mu_A^G \beta_{G,G'} \Delta_A^{G'} \mu_A^F \beta_{F,F'} \Delta_A^{F'}(h) \\
&= \mu_A^G \beta_{G,G'} \mu_{G'}^{G'F} \beta_{G'F,FG'} \Delta_F^{FG'} \beta_{F,F'} \Delta_A^{F'}(h) \\
&= \mu_A^G \mu_G^{GF} \beta_{GF,G'F} \beta_{G'F,FG'} \beta_{FG',F'G'} \Delta_{F'}^{F'G'} \Delta_A^{F'}(h) \\
&= \mu_A^{GF} \beta_{GF,F'G'} \Delta_A^{F'G'}(h) \\
&= (z \cdot w) \cdot h.
\end{aligned}$$

We used the bimonoid axiom (2.12) and then naturality and (co)associativity (2.8), (2.10).

For the general case, we use the q-bimonoid axiom (2.33) which yields an extra factor of $q^{\mathrm{dist}(G',F)}$. The same factor is also present in $z \cdot w$ in view of (7.131). This completes the check. $\square$

Remark 10.33. The above calculation justifies definition (10.52). Here is a simple way to understand its origin. For simplicity, take $q = 1$. Start with the special case

$$\mathtt{H}_{(F/A,F/A)} \cdot h := \mu_A^F \Delta_A^F(h)$$

which is very much like the formula for characteristic operations by faces, namely, (10.2). Now $\mathtt{H}_{(F/A,F'/A)} = \mathtt{H}_{(F/A,F/A)} \cdot \mathtt{H}_{(F'/A,F'/A)}$, and so

$$\mathtt{H}_{(F/A,F'/A)} \cdot h = \mu_A^F \Delta_A^F \mu_A^{F'} \Delta_A^{F'}(h) = \mu_A^F \beta_{F,F'} \Delta_A^{F'}(h).$$

The last step used (2.13). Note very carefully that the special case we started with did not have any β in it; the β emerged naturally when we composed and used (2.13).

Example 10.34. Recall the bimonoid of bifaces J with product and coproduct (7.126). Recall that for any face A, the space $\mathsf{J}^o[A]$ is contained inside $\mathsf{J}[A]$. Specializing $\mathsf{h} = \mathsf{J}$, we obtain a left action of the Janus algebra $\mathsf{J}^o[A]$ on $\mathsf{J}[A]$. This action extends the product (7.130) of $\mathsf{J}^o[A]$. Indeed,

$$\begin{aligned}
\mathtt{H}_{(F/A,F'/A)} \cdot \mathtt{H}_{(K,K')} &= \mu_A^F \beta_{F,F'} \Delta_A^{F'}(\mathtt{H}_{(K,K')}) \\
&= \mu_A^F \beta_{F,F'}(\mathtt{H}_{(F'K,K'F')}) \\
&= \mu_A^F(\mathtt{H}_{(FK,K'F')}) \\
&= \mathtt{H}_{(FK,K'F')}.
\end{aligned}$$

More generally, taking $\mathsf{h} = \mathsf{J}_q$ with product and coproduct (7.127), we obtain a left action of the q-Janus algebra $\mathsf{J}_q^o[A]$ on $\mathsf{J}_q[A]$ which extends the product (7.131).

10.3.2. Primitive part and indecomposable part. The following result gives a more general context to Lemma 10.16.

Lemma 10.35. *Let h be a q-bimonoid. Then, for $h \in \mathsf{h}[A]$,*

$$h \in \mathcal{P}(\mathsf{h})[A] \iff \mathtt{H}_{(F/A,F/A)} \cdot h = 0 \text{ for all } F > A. \tag{10.55}$$

$$h \in \mathcal{D}(\mathsf{h})[A] \iff h \in \sum_{F:\, F>A} \operatorname{im}(\mathtt{H}_{(F/A,F/A)} \cdot (-)). \tag{10.56}$$

Proof. For any $F > A$, since the map μ_A^F is injective, we have

$$\mathtt{H}_{(F/A,F/A)} \cdot h = 0 \iff \mu_A^F \Delta_A^F(h) = 0 \iff \Delta_A^F(h) = 0.$$

The first part follows. Similarly, the second part follows by using the fact that Δ_A^F is surjective. $\square$

Recall that for q not a root of unity, the q-Janus algebra has a $\mathtt{Q}$-basis. For the q-Janus algebra $\mathsf{J}_q^o[A]$, we denote the $\mathtt{Q}$-basis elements by $\mathtt{Q}_{(F/A,F'/A)}$.

Proposition 10.36. *Let h be a q-bimonoid for q not a root of unity. Let A be any face. Then the left action by the element $\mathtt{Q}_{(A/A,A/A)}$ of $\mathsf{J}_q^o[A]$ sends $\mathsf{h}[A]$ to $\mathcal{P}(\mathsf{h})[A]$ and through $\mathcal{Q}(\mathsf{h})[A]$, and moreover, it is identity on $\mathcal{P}(\mathsf{h})[A]$ and on $\mathcal{Q}(\mathsf{h})[A]$. Equivalently, $\mathtt{Q}_{(A/A,A/A)}$ yields an idempotent operator on $\mathsf{h}[A]$ whose image is $\mathcal{P}(\mathsf{h})[A]$ and coimage is $\mathcal{Q}(\mathsf{h})[A]$.*

PROOF. We first explain the claim related to the primitive part. Let $h \in \mathsf{h}[A]$. Then, for $F > A$, by the first formula in (7.138),

$$\mathtt{H}_{(F/A,F/A)} \cdot \mathtt{Q}_{(A/A,A/A)} \cdot h = 0.$$

Hence, by (10.55), $\mathtt{Q}_{(A/A,A/A)} \cdot h$ belongs to $\mathcal{P}(\mathsf{h})[A]$. Moreover, if $h \in \mathcal{P}(\mathsf{h})[A]$, then $\Delta_A^F(h) = 0$ for $F > A$, and by (10.52), $\mathtt{Q}_{(A/A,A/A)} \cdot h = h$. We also used here that the coefficient of $\mathtt{H}_{(A/A,A/A)}$ in $\mathtt{Q}_{(A/A,A/A)}$ is 1.

For the claim related to the indecomposable part, we make use of the second formula in (7.138) and (10.56). $\square$

Exercise 10.37. Let h be a q-bimonoid for q not a root of unity. Use (7.133) to deduce that the characteristic operation of $\mathtt{Q}_{(A/A,A/A)}$ on $\mathsf{h}[A]$ is given by $\mathbf{log}_q(\mathrm{id})_A$ as defined in (9.35). Use this to reprove Proposition 9.84.

10.4. Set-theoretic characteristic operations

Recall from Section 2.14 that one can define bimonoids in the category of set-species. These are set-bimonoids. Characteristic operations can also be considered in the set-theoretic setting.

10.4.1. Set-theoretic characteristic operations. For any face A, we have the Tits monoid $\Sigma[A]$ consisting of faces greater than A. Let h be a set-bimonoid. For any face A, the Tits monoid $\Sigma[A]$ operates on the set $\mathrm{h}[A]$ by

$$F/A \cdot x := \mu_A^F \Delta_A^F(x) \tag{10.57}$$

for $F/A \in \Sigma[A]$ and $x \in \mathrm{h}[A]$. This is a left action if h is cocommutative, and a right action if h is commutative. The linearization of this operation is (10.1).

For any flat Z, we have the Birkhoff monoid $\Pi[\mathrm{Z}]$ consisting of flats greater than Z. Let h be a bicommutative set-bimonoid. For any flat Z, the Birkhoff monoid $\Pi[\mathrm{Z}]$ operates on the set $\mathrm{h}[\mathrm{Z}]$ by

$$\mathrm{X}/\mathrm{Z} \cdot x := \mu_{\mathrm{Z}}^{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}(x) \tag{10.58}$$

for $\mathrm{X}/\mathrm{Z} \in \Pi[\mathrm{Z}]$ and $x \in \mathrm{h}[\mathrm{Z}]$. This is indeed an action. The linearization of this action is (10.34).

For any face A, we have the Janus monoid $\mathrm{J}^o[A]$ consisting of bifaces, with both coordinates greater than A. Let h be a set-bimonoid. For any face A, the Janus monoid $\mathrm{J}^o[A]$ operates on the set $\mathrm{h}[A]$ by

$$(F/A, F'/A) \cdot x := \mu_A^F \beta_{F,F'} \Delta_A^{F'}(x) \tag{10.59}$$

for $(F/A, F'/A) \in \mathrm{J}^o[A]$ and $x \in \mathrm{h}[A]$. This is a left action. The linearization of this action is (10.52).

Properties of characteristic operations that we established in the linear setting continue to hold.

10.4.2. Connection with primitive elements. Let h be a set-bimonoid and $\mathsf{h} = \Bbbk\mathrm{h}$ be its linearization. Let

$$h = \sum_{x \in \mathrm{h}[A]} a^x \, \mathtt{H}_x \in \mathsf{h}[A]$$

be an element, where a^x are scalars.

Lemma 10.38. *We have*

$$h \in \mathcal{P}(\mathsf{h})[A] \iff \sum_{x\colon F/A \cdot x = y} a^x = 0 \text{ for all } y \in \mathrm{h}[A] \text{ and all } F > A. \tag{10.60}$$

Proof. Observe that

$$\mathtt{H}_{F/A} \cdot h = \sum_{y \in \mathrm{h}[A]} \Big(\sum_{x\colon F/A \cdot x = y} a^x \Big) \mathtt{H}_y.$$

Now apply (10.29). ☐

10.5. Idempotent operators on bimonoid components

Idempotent elements in the Birkhoff algebra, Tits algebra, Janus algebra yield idempotent operators on the components of ((co, bi)commutative) bimonoids via characteristic operations. The main theme is to identify the images of these operators. We begin with idempotent operators on the components of bicommutative bimonoids that arise from the $\mathtt{H}$-basis and $\mathtt{Q}$-basis elements of the Birkhoff algebra. Further, we connect the $\mathtt{Q}$-basis idempotents to the components of the primitive and indecomposable parts of the bimonoid. We do a similar analysis for q-bimonoids for q not a root of unity, using $\mathtt{H}$-, $\mathtt{Q}$, $\mathtt{Q}^d$-bases elements of the q-Janus algebra. For (co)commutative bimonoids, we consider idempotents (1.115) and (1.112) constructed out of the $\mathtt{H}$- and $\mathtt{Q}$-bases elements of the Tits algebra using a noncommutative zeta function. The latter are the eulerian idempotents.

10.5.1. Birkhoff algebra.

Lemma 10.39. *Let* h *be a bicommutative bimonoid. Then, for any* $\mathrm{Z} \leq \mathrm{X}$*, the map* $\mu^{\mathrm{X}}_{\mathrm{Z}} \Delta^{\mathrm{X}}_{\mathrm{Z}}$ *is an idempotent operator on* $\mathsf{h}[\mathrm{Z}]$*, and there are inverse linear isomorphisms*

$$\mathsf{h}[\mathrm{X}] \underset{\Delta^{\mathrm{X}}_{\mathrm{Z}}}{\overset{\mu^{\mathrm{X}}_{\mathrm{Z}}}{\rightleftarrows}} \mathtt{H}_{\mathrm{X}/\mathrm{Z}} \cdot \mathsf{h}[\mathrm{Z}]. \tag{10.61}$$

Proof. Apply Lemma A.1 to $V = \mathsf{h}[\mathrm{Z}]$, $W = \mathsf{h}[\mathrm{X}]$, $p = \Delta^{\mathrm{X}}_{\mathrm{Z}}$, $i = \mu^{\mathrm{X}}_{\mathrm{Z}}$. The hypothesis $pi = \mathrm{id}_W$ holds by (10.37). Further, by (10.35), the linear operator $ip = \mu^{\mathrm{X}}_{\mathrm{Z}} \Delta^{\mathrm{X}}_{\mathrm{Z}}$ is the same as commutative characteristic operation by the element $\mathtt{H}_{\mathrm{X}/\mathrm{Z}}$. ☐

This result can be extended as follows.

Proposition 10.40. *Let* h *be a bicommutative bimonoid. For any flats* $\mathrm{X} \geq \mathrm{Z}$*, the space* $\mathtt{Q}_{\mathrm{X}/\mathrm{Z}} \cdot \mathsf{h}[\mathrm{Z}]$ *is a subspace of* $\mathtt{H}_{\mathrm{X}/\mathrm{Z}} \cdot \mathsf{h}[\mathrm{Z}]$*. Further, the isomorphisms* (10.61) *restrict to inverse isomorphisms*

$$\begin{tikzcd} \mathcal{P}(\mathsf{h})[\mathrm{X}] \arrow[r, dashed, shift left] \arrow[d, hook] & \mathtt{Q}_{\mathrm{X}/\mathrm{Z}} \cdot \mathsf{h}[\mathrm{Z}] \arrow[l, dashed, shift left] \arrow[d, hook] \\ \mathsf{h}[\mathrm{X}] \arrow[r, shift left, "\mu_{\mathrm{Z}}^{\mathrm{X}}"] & \mathtt{H}_{\mathrm{X}/\mathrm{Z}} \cdot \mathsf{h}[\mathrm{Z}]. \arrow[l, shift left, "\Delta_{\mathrm{Z}}^{\mathrm{X}}"] \end{tikzcd} \tag{10.62}$$

In particular,

$$\mathcal{P}(\mathsf{h})[\mathrm{Z}] = \mathtt{Q}_{\mathrm{Z}/\mathrm{Z}} \cdot \mathsf{h}[\mathrm{Z}].$$

Proof. By (7.43), $\mathtt{H}_{\mathrm{X}/\mathrm{Z}} \cdot \mathtt{Q}_{\mathrm{X}/\mathrm{Z}} = \mathtt{Q}_{\mathrm{X}/\mathrm{Z}}$, hence the first claim follows. For the second claim, we compute the image of $\mathtt{Q}_{\mathrm{X}/\mathrm{Z}} \cdot \mathsf{h}[\mathrm{Z}]$ under $\Delta_{\mathrm{Z}}^{\mathrm{X}}$ as follows.

Let Y be any flat. Recall from (7.46) that $\mathcal{P}(\Pi)[\mathrm{Y}]$ is one-dimensional and spanned by $\mathtt{Q}_{\mathrm{Y}/\mathrm{Y}}$. This along with Proposition 10.26 yields $\mathcal{P}(\mathsf{h})[\mathrm{Y}] = \mathtt{Q}_{\mathrm{Y}/\mathrm{Y}} \cdot \mathsf{h}[\mathrm{Y}]$. We now calculate:

$$\Delta_{\mathrm{Z}}^{\mathrm{X}}(\mathtt{Q}_{\mathrm{X}/\mathrm{Z}} \cdot \mathsf{h}[\mathrm{Z}]) = \Delta_{\mathrm{Z}}^{\mathrm{X}}(\mathtt{Q}_{\mathrm{X}/\mathrm{Z}}) \cdot \Delta_{\mathrm{Z}}^{\mathrm{X}}(\mathsf{h}[\mathrm{Z}]) = \mathtt{Q}_{\mathrm{X}/\mathrm{X}} \cdot \mathsf{h}[\mathrm{X}] = \mathcal{P}(\mathsf{h})[\mathrm{X}].$$

The first equality used (10.42). The second equality used coproduct formula (7.40) and the fact that $\Delta_{\mathrm{Z}}^{\mathrm{X}}$ maps $\mathsf{h}[\mathrm{Z}]$ onto $\mathsf{h}[\mathrm{X}]$. $\square$

The companion result with the primitive part of h replaced by the indecomposable part of h is stated below.

Proposition 10.41. *Let* h *be a bicommutative bimonoid. For any flats* $\mathrm{X} \geq \mathrm{Z}$*, the space* $\mathtt{Q}_{\mathrm{X}/\mathrm{Z}} \cdot \mathsf{h}[\mathrm{Z}]$ *is a quotient space of* $\mathtt{H}_{\mathrm{X}/\mathrm{Z}} \cdot \mathsf{h}[\mathrm{Z}]$*. (The quotient map is action by* $\mathtt{Q}_{\mathrm{X}/\mathrm{Z}}$*.) Further, the isomorphisms* (10.61) *induce inverse isomorphisms*

$$\begin{tikzcd} \mathtt{H}_{\mathrm{X}/\mathrm{Z}} \cdot \mathsf{h}[\mathrm{Z}] \arrow[r, shift left, "\Delta_{\mathrm{Z}}^{\mathrm{X}}"] \arrow[d, two heads] & \mathsf{h}[\mathrm{X}] \arrow[l, shift left, "\mu_{\mathrm{Z}}^{\mathrm{X}}"] \arrow[d, two heads] \\ \mathtt{Q}_{\mathrm{X}/\mathrm{Z}} \cdot \mathsf{h}[\mathrm{Z}] \arrow[r, dashed, shift left] & \mathcal{Q}(\mathsf{h})[\mathrm{X}]. \arrow[l, dashed, shift left] \end{tikzcd} \tag{10.63}$$

Exercise 10.42. Combine diagrams (10.62) and (10.63) and put $\mathrm{X} = \mathrm{Z}$ to deduce that: For a bicommutative bimonoid h, the canonical map $\mathrm{pq}_{\mathsf{h}} : \mathcal{P}(\mathsf{h}) \to \mathcal{Q}(\mathsf{h})$ is an isomorphism. (This result is contained in Proposition 5.56, item (3).)

10.5.2. Tits algebra. Fix a noncommutative zeta function $\boldsymbol{\zeta}$. For any flat X containing a face A, let $\mathtt{E}_{\mathrm{X}/A}$ and $\mathtt{u}_{\mathrm{X}/A}$ be as in (7.72) and (7.75), respectively. These are idempotent elements in $\Sigma[A]$. For any bimonoid h, observe that operation (10.1) by the element $\mathtt{u}_{\mathrm{X}/A}$ is the same as the linear operator

$$\sum_{F:\, F \geq A,\, \mathrm{s}(F) = \mathrm{X}} \boldsymbol{\zeta}(A, F)\, \mu_A^F \Delta_A^F. \tag{10.64}$$

Continuing in this setup:

Lemma 10.43. *Let* h *be a cocommutative bimonoid. Then, for any flat* X *containing a face* A, (10.64) *is an idempotent operator on* $\mathsf{h}[A]$*, and there are inverse linear isomorphisms*

$$\mathsf{h}[\mathrm{X}] \underset{\beta_{\mathrm{X},G}\Delta_A^G}{\overset{\sum_F \boldsymbol{\zeta}(A,F)\,\mu_A^F\beta_{F,\mathrm{X}}}{\rightleftarrows}} \mathsf{u}_{\mathrm{X}/A}\cdot\mathsf{h}[A], \tag{10.65}$$

where G *is any fixed face greater than* A *of support* X*, while the sum is over all faces* F *greater than* A *of support* X*.*

In the special case $\mathrm{X}=\mathrm{s}(A)$, the forward map is $\beta_{A,\mathrm{X}}$, the backward map is $\beta_{\mathrm{X},A}$, and the two are clearly inverses of each other.

Proof. Apply Lemma A.1 to $V=\mathsf{h}[A]$, $W=\mathsf{h}[\mathrm{X}]$, i equal to the forward map and p equal to the backward map in (10.65). Let us compute the composites in either direction.

$$\begin{aligned}
pi &= \sum_{F:\,F\geq A,\,\mathrm{s}(F)=\mathrm{X}} \beta_{\mathrm{X},G}\,\Delta_A^G\,\boldsymbol{\zeta}(A,F)\,\mu_A^F\,\beta_{F,\mathrm{X}}\\
&= \sum_{F:\,F\geq A,\,\mathrm{s}(F)=\mathrm{X}} \boldsymbol{\zeta}(A,F)\,\beta_{\mathrm{X},G}\,\beta_{G,F}\,\beta_{F,\mathrm{X}}\\
&= \Big(\sum_{F:\,F\geq A,\,\mathrm{s}(F)=\mathrm{X}} \boldsymbol{\zeta}(A,F)\Big)\ \mathrm{id}\\
&= \mathrm{id},
\end{aligned}$$

and

$$\begin{aligned}
ip &= \sum_{F:\,F\geq A,\,\mathrm{s}(F)=\mathrm{X}} \boldsymbol{\zeta}(A,F)\,\mu_A^F\,\beta_{F,\mathrm{X}}\,\beta_{\mathrm{X},G}\,\Delta_A^G\\
&= \sum_{F:\,F\geq A,\,\mathrm{s}(F)=\mathrm{X}} \boldsymbol{\zeta}(A,F)\,\mu_A^F\,\beta_{F,G}\,\Delta_A^G\\
&= \sum_{F:\,F\geq A,\,\mathrm{s}(F)=\mathrm{X}} \boldsymbol{\zeta}(A,F)\,\mu_A^F\,\Delta_A^F.
\end{aligned}$$

Note the use of the cocommutativity axiom (2.23) in the last step. The result follows. ∎

Lemma 10.44. *Let* h *be a cocommutative bimonoid. For any faces* $F\geq A$*, we have*

$$\mathcal{P}(\mathsf{h})[F]=\Delta_A^F(\mathtt{E}_{\mathrm{s}(F)/A}\cdot\mathsf{h}[A]). \tag{10.66}$$

In particular,

$$\mathcal{P}(\mathsf{h})[A]=\mathtt{E}_{\mathrm{s}(A)/A}\cdot\mathsf{h}[A]. \tag{10.67}$$

Proof. Since $\mathtt{E}_{\mathrm{s}(A)/A}$ is a first eulerian idempotent of $\Sigma[A]$, it is a special Zie element (Section 1.12.5). Formula (10.67) then follows from Proposition 10.13. To prove (10.66), we proceed as follows.

$$\mathcal{P}(\mathsf{h})[F]=\mathtt{E}_{\mathrm{s}(F)/F}\cdot\mathsf{h}[F]=\Delta_A^F(\mathtt{E}_{\mathrm{s}(F)/A})\cdot\Delta_A^F(\mathsf{h}[A])=\Delta_A^F(\mathtt{E}_{\mathrm{s}(F)/A}\cdot\mathsf{h}[A]).$$

The first equality holds by (10.67) which we just proved. The second step used (7.74) and the fact that Δ^F_A maps $\mathsf{h}[A]$ onto $\mathsf{h}[F]$. The last step used (10.15). $\square$

Proposition 10.45. *Let h be a cocommutative bimonoid. For any flat X containing a face A, the space $\mathtt{E}_{\mathrm{X}/A} \cdot \mathsf{h}[A]$ is a subspace of $\mathtt{u}_{\mathrm{X}/A} \cdot \mathsf{h}[A]$. Further, the isomorphisms (10.65) restrict to inverse isomorphisms*

$$\begin{tikzcd}
\mathcal{P}(\mathsf{h})[\mathrm{X}] \arrow[r, dashed, shift left] \arrow[d, hook] & \mathtt{E}_{\mathrm{X}/A} \cdot \mathsf{h}[A], \arrow[l, dashed, shift left] \arrow[d, hook] \\
\mathsf{h}[\mathrm{X}] \arrow[r, shift left, "\sum_F \boldsymbol{\zeta}(A,F)\, \mu^F_A\, \beta_{F,\mathrm{X}}"] & \mathtt{u}_{\mathrm{X}/A} \cdot \mathsf{h}[A], \arrow[l, shift left, "\beta_{\mathrm{X},G} \Delta^G_A"]
\end{tikzcd} \tag{10.68}$$

where G is any fixed face greater than A of support X, while the sum is over all faces F greater than A of support X.

Proof. By (7.76), $\mathtt{u}_{\mathrm{X}/A} \cdot \mathtt{E}_{\mathrm{X}/A} = \mathtt{E}_{\mathrm{X}/A}$, hence the first claim follows. Further, we can deduce from (10.66) that the dotted backward map above is an isomorphism. The second claim follows. $\square$

Companion results for a commutative bimonoid are given below.

Lemma 10.46. *Let h be a commutative bimonoid. Then, for any flat X containing a face A, (10.64) is an idempotent operator on $\mathsf{h}[A]$, and there are inverse linear isomorphisms*

$$\begin{tikzcd}
\mathsf{h}[\mathrm{X}] \arrow[r, shift left, "\mu^G_A \beta_{G,\mathrm{X}}"] & \mathsf{h}[A] \cdot \mathtt{u}_{\mathrm{X}/A}, \arrow[l, shift left, "\sum_F \boldsymbol{\zeta}(A,F)\, \beta_{\mathrm{X},F} \Delta^F_A"]
\end{tikzcd} \tag{10.69}$$

where G is any fixed face greater than A of support X, while the sum is over all faces F greater than A of support X.

Proof. The proof is similar to the one for Lemma 10.43. Now the commutativity axiom gets used in the last step instead of the cocommutativity axiom. $\square$

Proposition 10.47. *Let h be a commutative bimonoid. For any flat X containing a face A, the space $\mathsf{h}[A] \cdot \mathtt{E}_{\mathrm{X}/A}$ is a quotient space of $\mathsf{h}[A] \cdot \mathtt{u}_{\mathrm{X}/A}$. (The quotient map is right action by $\mathtt{E}_{\mathrm{X}/A}$.) Further, the isomorphisms (10.69) induce inverse isomorphisms*

$$\begin{tikzcd}
\mathsf{h}[\mathrm{X}] \arrow[r, shift left, "\mu^G_A \beta_{G,\mathrm{X}}"] \arrow[d, two heads] & \mathsf{h}[A] \cdot \mathtt{u}_{\mathrm{X}/A}, \arrow[l, shift left, "\sum_F \boldsymbol{\zeta}(A,F)\, \beta_{\mathrm{X},F} \Delta^F_A"] \arrow[d, two heads] \\
\mathcal{Q}(\mathsf{h})[\mathrm{X}] \arrow[r, dashed, shift left] & \mathsf{h}[A] \cdot \mathtt{E}_{\mathrm{X}/A}, \arrow[l, dashed, shift left]
\end{tikzcd} \tag{10.70}$$

where G is any fixed face greater than A of support X, while the sum is over all faces F greater than A of support X.

10.5.3. Janus algebra.

Lemma 10.48. *Let* h *be a bimonoid. Then, for any* $A \leq F$*, the map* $\mu_A^F \Delta_A^F$ *is an idempotent operator on* $\mathsf{h}[A]$*, and there are inverse linear isomorphisms*

$$\mathsf{h}[F] \underset{\Delta_A^F}{\overset{\mu_A^F}{\rightleftarrows}} \mathtt{H}_{F/A} \cdot \mathsf{h}[A]. \tag{10.71}$$

Proof. Apply Lemma A.1 to $V = \mathsf{h}[A]$, $W = \mathsf{h}[F]$, $p = \Delta_A^F$, $i = \mu_A^F$. The hypothesis $pi = \mathrm{id}_W$ holds by (10.7). Further, by (10.2), the linear operator $ip = \mu_A^F \Delta_A^F$ is the same as characteristic operation by the element $\mathtt{H}_{F/A}$. □

More generally:

Lemma 10.49. *Let* h *be a q-bimonoid. Then, for any* $A \leq F$*, the map* $\mu_A^F \Delta_A^F$ *is an idempotent operator on* $\mathsf{h}[A]$*, and there are inverse linear isomorphisms*

$$\mathsf{h}[F] \underset{\Delta_A^F}{\overset{\mu_A^F}{\rightleftarrows}} \mathtt{H}_{(F/A,F/A)} \cdot \mathsf{h}[A]. \tag{10.72}$$

Proof. Apply Lemma A.1 to $V = \mathsf{h}[A]$, $W = \mathsf{h}[F]$, $p = \Delta_A^F$, $i = \mu_A^F$. The hypothesis $pi = \mathrm{id}_W$ holds by (2.15). Further, by (10.53), the linear operator $ip = \mu_A^F \Delta_A^F$ is the same as two-sided characteristic operation by the element $\mathtt{H}_{(F/A,F/A)}$. □

This result can be extended as follows.

Proposition 10.50. *Let* h *be a q-bimonoid for q not a root of unity. Then, for any* $A \leq F$*, the space* $\mathtt{Q}_{(F/A,F/A)} \cdot \mathsf{h}[A]$ *is a subspace of* $\mathtt{H}_{(F/A,F/A)} \cdot \mathsf{h}[A]$*. Further, the isomorphisms* (10.72) *restrict to inverse isomorphisms*

$$\begin{tikzcd} \mathcal{P}(\mathsf{h})[F] \arrow[r, dashed, shift left] \arrow[d, hook] & \mathtt{Q}_{(F/A,F/A)} \cdot \mathsf{h}[A] \arrow[l, dashed, shift left] \arrow[d, hook] \\ \mathsf{h}[F] \arrow[r, shift left, "\mu_A^F"] & \mathtt{H}_{(F/A,F/A)} \cdot \mathsf{h}[A]. \arrow[l, shift left, "\Delta_A^F"] \end{tikzcd} \tag{10.73}$$

In particular,

$$\mathcal{P}(\mathsf{h})[A] = \mathtt{Q}_{(A/A,A/A)} \cdot \mathsf{h}[A].$$

Proof. The first claim follows from the first identity in (7.137). The second claim is verified below.

For any $G > F$,

$$\mu_A^G \Delta_A^G (\mathtt{Q}_{(F/A,F/A)} \cdot x) = \mathtt{H}_{(G/A,G/A)} \cdot \mathtt{Q}_{(F/A,F/A)} \cdot x = 0,$$

by (7.136). Since μ_A^G is injective, we deduce that for any $G > F$,

$$\Delta_F^G \Delta_A^F (\mathtt{Q}_{(F/A,F/A)} \cdot x) = \Delta_A^G (\mathtt{Q}_{(F/A,F/A)} \cdot x) = 0.$$

Hence, Δ_A^F maps $\mathtt{Q}_{(F/A,F/A)} \cdot \mathsf{h}[A]$ into $\mathcal{P}(\mathsf{h})[F]$.

Conversely: For simplicity, let us first work with the case $q = 0$. For $x \in \mathcal{P}(\mathsf{h})[F]$,

$$\begin{aligned} \mathtt{Q}_{(F/A,F/A)} \cdot \mu_A^F(x) &= \Big(\sum_{G:\, G \geq F} (-1)^{\mathrm{rk}(G/F)} \mathtt{H}_{(G/A,G/A)} \Big) \cdot \mu_A^F(x) \\ &= \sum_{G:\, G \geq F} (-1)^{\mathrm{rk}(G/F)} \mu_A^G \Delta_A^G \mu_A^F(x) \\ &= \mu_A^F(x). \end{aligned}$$

The first step used (7.142). The last step used Lemma 5.43, only the term $G = F$ contributed. This calculation shows that $\mu_A^F(x)$ belongs to $\mathtt{Q}_{(F/A,F/A)} \cdot \mathsf{h}[A]$ as required.

The calculation in the general case goes as follows.

$$\begin{aligned} \mathtt{Q}_{(F/A,F/A)} \cdot \mu_A^F(x) &= \Big(\sum_{\substack{G:\, G \geq A \\ \mathrm{s}(G) = \mathrm{s}(F)}} \zeta_q(A,F,G)\, \mathtt{H}_{(F/A,G/A)} + \text{higher terms} \Big) \cdot \mu_A^F(x) \\ &= \sum_{\substack{G:\, G \geq A \\ \mathrm{s}(G) = \mathrm{s}(F)}} \zeta_q(A,F,G)\, \mu_A^F \beta_{F,G} \Delta_A^G \mu_A^F(x) \\ &= \sum_{\substack{G:\, G \geq A \\ \mathrm{s}(G) = \mathrm{s}(F)}} \zeta_q(A,F,G)\, \mu_A^F \beta_{F,G} (\beta_q)_{G,F}(x) \\ &= \sum_{\substack{G:\, G \geq A \\ \mathrm{s}(G) = \mathrm{s}(F)}} \zeta_q(A,F,G)\, q^{\mathrm{dist}(G,F)}\, \mu_A^F(x) \\ &= \mu_A^F(x). \end{aligned}$$

The first step used formula (1.128) applied to $\mathcal{A}_A$. The higher terms refer to bifaces with support strictly greater than F. They do not contribute to the sum in view of the second alternative in Lemma 5.39. This was used in the second step. The third step used the first alternative in Lemma 5.39. The next step used (2.2). The last step used the two-sided q-flat-additivity formula (1.65). $\square$

The companion result with the primitive part of h replaced by the indecomposable part of h is stated below. Note very carefully that it employs the $\mathtt{Q}^d$-basis instead of the $\mathtt{Q}$-basis.

Proposition 10.51. *Let h be a q-bimonoid for q not a root of unity. Then, for any $A \leq F$, the space $\mathtt{Q}^d_{(F/A,F/A)} \cdot \mathsf{h}[A]$ is a quotient space of $\mathtt{H}_{(F/A,F/A)} \cdot \mathsf{h}[A]$. (The quotient map is left action by $\mathtt{Q}^d_{(F/A,F/A)} \cdot$.) Further, the isomorphisms* (10.72) *restrict to inverse isomorphisms*

$$\begin{tikzcd} \mathtt{H}_{(F/A,F/A)} \cdot \mathsf{h}[A] \arrow[r, shift left, "\Delta_A^F"] \arrow[d, two heads] & \mathsf{h}[F] \arrow[l, shift left, "\mu_A^F"] \arrow[d, two heads] \\ \mathtt{Q}^d_{(F/A,F/A)} \cdot \mathsf{h}[A] \arrow[r, shift left, dashed] & \mathcal{Q}(\mathsf{h})[F]. \arrow[l, shift left, dashed] \end{tikzcd} \tag{10.74}$$

Exercise 10.52. Combine diagrams (10.73) and (10.74) and put $F = A$ to deduce that: For a q-bimonoid h with q not a root of unity, the canonical map $\mathrm{pq}_{\mathsf{h}} : \mathcal{P}(\mathsf{h}) \to \mathcal{Q}(\mathsf{h})$ is an isomorphism. (This recovers Proposition 5.58.)

Exercise 10.53. Check Propositions 10.50 and 10.51 directly for a rank-one arrangement. (Use the formulas for the $\mathtt{Q}$- and $\mathtt{Q}^d$-bases elements provided in Example 1.66.)

Results very similar to those above can be obtained by replacing the $\mathtt{H}$-basis elements with a different set of idempotents. Details follow. For any faces $F, F' \geq A$, recall elements $\mathtt{u}_{(F/A,F'/A)}$ and $\mathtt{u}^d_{(F'/A,F/A)}$ from (7.139). For $F \geq A$, the operation (10.52) by the element $\mathtt{u}_{(F/A,F/A)}$ is the same as the linear operator

$$\sum_{\substack{F' \colon F' \geq A \\ \mathrm{s}(F')=\mathrm{s}(F)}} \boldsymbol{\zeta}_q(A,F,F')\, \mu_A^F \beta_{F,F'} \Delta_A^{F'}, \tag{10.75}$$

while for $F' \geq A$, the operation (10.52) by the element $\mathtt{u}^d_{(F'/A,F'/A)}$ is the same as the linear operator

$$\sum_{\substack{F \colon F \geq A \\ \mathrm{s}(F)=\mathrm{s}(F')}} \boldsymbol{\zeta}_q(A,F,F')\, \mu_A^F \beta_{F,F'} \Delta_A^{F'}. \tag{10.76}$$

In the first, the sum is over F', while in the second, the sum is over F.

Lemma 10.54. *Let h be a q-bimonoid for q not a root of unity. Then:*

- *For any $F' \geq A$, (10.76) is an idempotent operator on $\mathsf{h}[A]$, and there are inverse linear isomorphisms*

$$\mathsf{h}[F'] \underset{\Delta_A^{F'}}{\overset{\sum_F \boldsymbol{\zeta}_q(A,F,F')\,\mu_A^F \beta_{F,F'}}{\rightleftarrows}} \mathtt{u}^d_{(F'/A,F'/A)} \cdot \mathsf{h}[A]. \tag{10.77}$$

- *For any $F \geq A$, (10.75) is an idempotent operator on $\mathsf{h}[A]$, and there are inverse linear isomorphisms*

$$\mathsf{h}[F] \underset{\sum_{F'} \boldsymbol{\zeta}_q(A,F,F')\,\beta_{F,F'} \Delta_A^{F'}}{\overset{\mu_A^F}{\rightleftarrows}} \mathtt{u}_{(F/A,F/A)} \cdot \mathsf{h}[A]. \tag{10.78}$$

Proof. This follows from Lemma A.1. To check the hypothesis $pi = \mathrm{id}$ in either case, we use the two-sided q-flat-additivity formula (1.65). $\square$

This result can be extended as follows.

Proposition 10.55. *Let h be a q-bimonoid for q not a root of unity. Then:*

- *For any* $F' \geq A$, $\mathtt{Q}^d_{(F'/A,F'/A)} \cdot \mathsf{h}[A]$ *is a subspace of* $\mathtt{u}^d_{(F'/A,F'/A)} \cdot \mathsf{h}[A]$. *Further, the isomorphisms* (10.77) *restrict to inverse isomorphisms*

$$
\begin{tikzcd}
\mathcal{P}(\mathsf{h})[F'] \arrow[r, dashed, shift left] \arrow[d, hook] & \mathtt{Q}^d_{(F'/A,F'/A)} \cdot \mathsf{h}[A] \arrow[l, dashed, shift left] \arrow[d, hook] \\
\mathsf{h}[F'] \arrow[r, shift left, "\sum_F \zeta_q(A,F,F')\, \mu^F_A \beta_{F,F'}"] & \mathtt{u}^d_{(F'/A,F'/A)} \cdot \mathsf{h}[A]. \arrow[l, shift left, "\Delta^{F'}_A"]
\end{tikzcd}
\tag{10.79}
$$

- *For any* $F \geq A$, *the space* $\mathtt{Q}_{(F/A,F/A)} \cdot \mathsf{h}[A]$ *is a quotient space of* $\mathtt{u}_{(F/A,F/A)} \cdot \mathsf{h}[A]$. *(The quotient map is left action by* $\mathtt{Q}_{(F/A,F/A)}$*.) Further, the isomorphisms* (10.78) *restrict to inverse isomorphisms*

$$
\begin{tikzcd}
\mathsf{h}[F] \arrow[r, shift left, "\mu^F_A"] \arrow[d, two heads] & \mathtt{u}_{(F/A,F/A)} \cdot \mathsf{h}[A] \arrow[l, shift left, "\sum_{F'} \zeta_q(A,F,F')\, \beta_{F,F'} \Delta^{F'}_A"] \arrow[d, two heads] \\
\mathcal{Q}(\mathsf{h})[F] \arrow[r, dashed, shift left] & \mathtt{Q}_{(F/A,F/A)} \cdot \mathsf{h}[A]. \arrow[l, dashed, shift left]
\end{tikzcd}
\tag{10.80}
$$

These results are intimately connected to and can be deduced from Propositions 10.50 and 10.51. For instance, diagrams (10.73) and (10.79) can be deduced from each other by noting that for any $F' \geq A$, we have a commutative diagram

$$
\begin{tikzcd}
\mathtt{Q}_{(F'/A,F'/A)} \cdot \mathsf{h}[A] \arrow[r, dashed, shift left] \arrow[d, hook] & \mathtt{Q}^d_{(F'/A,F'/A)} \cdot \mathsf{h}[A] \arrow[l, dashed, shift left] \arrow[d, hook] \\
\mathtt{H}_{(F'/A,F'/A)} \cdot \mathsf{h}[A] \arrow[r, shift left] & \mathtt{u}^d_{(F'/A,F'/A)} \cdot \mathsf{h}[A]. \arrow[l, shift left]
\end{tikzcd}
$$

All spaces sit inside $\mathsf{h}[A]$. The bottom horizontal maps are given by left multiplication by $\mathtt{u}^d_{(F'/A,F'/A)}$ and by $\mathtt{H}_{(F'/A,F'/A)}$. To see that they induce the top horizontal maps, we first note from (7.135) that

$$\mathtt{Q}_{(F'/A,F'/A)} \cdot \mathsf{h}[A] = \mathtt{Q}'_{(F'/A,F'/A)} \cdot \mathsf{h}[A],$$

and then use the identities (7.137) and (7.140).

Exercise 10.56. Establish the first part of Proposition 10.55 directly by following the proof of Proposition 10.50.

Notes

The characteristic operations introduced in this chapter are motivated by the characteristic operations on graded bialgebras and on Joyal bimonoids.

Bialgebras.

Characteristic operations. Characteristic operations appeared in a topological setting in work of Leray under the heading 'puissances' [**595**, page 135]. He evaluates them on primitive elements which he calls 'hypermaximal elements', see [**595**, Formula (22) and Theorem 11].

The k-th characteristic operation on a bialgebra is given by $\mu^{(k)}\Delta^{(k)}$, where $\Delta^{(k)}$ and $\mu^{(k)}$ are, respectively, the iterated coproduct and product maps of the bialgebra. This can be viewed as the classical analogue of (10.2). In the context of group schemes (or commutative bialgebras), these operations appear in work of Tate and Oort [**877**, page 3] and implicitly in work of Gabriel [**335**, pages 272 and 273]. A purely algebraic setup is given by Gerstenhaber and Schack [**349**, Section 1], Patras [**740**, Definition 1.2]. The operator $\mu\Delta$ is studied by Shnider and Sternberg [**825**, Proposition 3.8.2]. It is also present in earlier work of Drinfeld [**263**, Proof of Proposition 3.7]. The p-th characteristic operation for a fixed prime p is considered by Hubbuck [**464**, Section 2].

Characteristic operations on bialgebras are called 'power maps' in [**514**], [**598**, page 348], 'Sweedler powers' in [**515**, Definition 2.1], 'Hopf powers' in [**562**, Section 1], [**704**, Section 3] and 'Adams operators' in [**202**, page 578], [**16**, Section 1.2]. They are considered by Diaconis, Pang, Ram in the context of Markov chains [**247**, page 540], [**733**]. They are also considered briefly in [**19**, Section 13.2] as special cases of characteristic operations on Joyal bimonoids, see below.

Characteristic operations on connected graded bialgebras by integer compositions are defined by Patras [**741**, Definition II,1]. This is the situation closest to our setting. (Recall from [**21**, Table 6.2] that integer compositions correspond to face-types of the braid arrangement.)

The classical analogue of two-sided characteristic operations (10.53) for $q = 1$ is considered by Kashina, Sommerhäuser, Zhu [**516**, Section 1.3]. The role of β is played by a permutation σ. They use the term σ-th Sweedler power for this operation. The proposition they state can be viewed as the analogue of (10.54) for $q = 1$.

Bimonoids in Joyal species. Characteristic operations on Joyal bimonoids indexed by set compositions are discussed in our monograph [**19**, Section 13]. The dictionary is as follows. The analogues of (10.1) and (10.2) are [**19**, (207) and (208)]. The analogue of Lemma 10.1 is [**19**, Theorem 82]. The analogues of Lemma 10.3 and Corollary 10.4 are [**19**, Theorem 83 and Proposition 87]. The analogue of Lemma 10.7 is [**19**, Proposition 88]. The analogue of Proposition 10.13 is [**19**, Theorem 89]. The analogue of Exercise 10.18 is contained in [**19**, Corollary 101].

Bimonoids for hyperplane arrangements. Characteristic operations on bimonoids for arrangements appear here for the first time.

CHAPTER 11

Modules over monoid algebras and bimonoids in species

We now forge a connection between bimonoids in species and representation theory of monoid algebras. More precisely, the category of cocommutative bimonoids is equivalent to the category of left modules over the Tits algebra. Similarly, commutative bimonoids relate to right modules over the Tits algebra, bicommutative bimonoids to modules over the Birkhoff algebra, arbitrary bimonoids to modules over the Janus algebra, and more generally, q-bimonoids to modules over the q-Janus algebra. Moreover, these equivalences are compatible with duality and base change and also have signed analogues.

Some illustrative examples are as follows. The (bicommutative) exponential bimonoid E corresponds to the trivial module over the Birkhoff algebra. Both are self-dual in an appropriate sense. The (bicommutative) bimonoid of flats Π corresponds to the Birkhoff algebra viewed as a module over itself. Similarly, the (cocommutative) bimonoid of chambers Γ corresponds to the left module of chambers over the Tits algebra, while the (cocommutative) bimonoid of faces Σ corresponds to the Tits algebra viewed as a left module over itself. The bimonoid of bifaces J corresponds to the Janus algebra viewed as a left module over itself.

We approach these results through characteristic operations on bimonoids introduced in Chapter 10 as follows. Recall that the bimonoid of faces Σ operates on any bimonoid h. Further, when h is either commutative or cocommutative, it defines, respectively, either a right or a left action of the Tits algebra $\Sigma[O]$ on $\mathsf{h}[O]$. Interestingly, information about this action suffices to recover the bimonoid h completely. As a consequence, the category of left $\Sigma[O]$-modules is equivalent to the category of cocommutative bimonoids, while the category of right $\Sigma[O]$-modules is equivalent to the category of commutative bimonoids. The other categorical equivalences are obtained in a similar manner by employing the commutative and two-sided characteristic operations.

These results can also be derived by computing the Karoubi envelopes of the Birkhoff monoid, Tits monoid, Janus monoid, and using the interpretation of bimonoids as functor categories from Section 2.11.

The categorical equivalences are also valid in the set-theoretic setting. Using these ideas, to any cocommutative set-bimonoid h, we associate the bimonoid of h-faces, and the bimonoid of h-flats. When h is the exponential set-bimonoid, this recovers the bimonoids of faces and flats, respectively.

11.1. Modules over the Tits algebra

Recall that the linearization of the Tits monoid is the Tits algebra. It is denoted by $\Sigma[\mathcal{A}]$. Note that

$$\Sigma[\mathcal{A}] = \Sigma[O], \qquad \mathtt{H}_F \leftrightarrow \mathtt{H}_{F/O},$$

where the latter refers to the O-component of the bimonoid Σ.

Proposition 11.1. *The category of left modules over the Tits algebra $\Sigma[\mathcal{A}]$ is equivalent to the category of cocommutative $\mathcal{A}$-bimonoids.*

Proof. We first construct a functor from the category of cocommutative $\mathcal{A}$-bimonoids to the category of left $\Sigma[\mathcal{A}]$-modules. Accordingly, suppose h is a cocommutative $\mathcal{A}$-bimonoid. Then $\mathsf{h}[O]$ is a left $\Sigma[\mathcal{A}]$-module, with the action of $\mathtt{H}_F$ on an element x given by the characteristic operation

$$\mathtt{H}_F \cdot x := \mu_O^F \Delta_O^F(x).$$

That this indeed defines an action is noted in (10.3) and (10.4). Further, if h and k are cocommutative $\mathcal{A}$-bimonoids and $f : \mathsf{h} \to \mathsf{k}$ is a morphism of $\mathcal{A}$-bimonoids, then the component $f_O : \mathsf{h}[O] \to \mathsf{k}[O]$ is a map of left $\Sigma[\mathcal{A}]$-modules as shown below.

$$\begin{tikzcd}
\mathsf{h}[O] \arrow[r, "\Delta_O^F"] \arrow[d, "f_O"'] & \mathsf{h}[F] \arrow[r, "\mu_O^F"] \arrow[d, "f_F"] & \mathsf{h}[O] \arrow[d, "f_O"] \\
\mathsf{k}[O] \arrow[r, "\Delta_O^F"'] & \mathsf{k}[F] \arrow[r, "\mu_O^F"'] & \mathsf{k}[O]
\end{tikzcd}$$

The squares commute by (2.9) and (2.11).

Now we construct a functor from the category of left $\Sigma[\mathcal{A}]$-modules to the category of cocommutative $\mathcal{A}$-bimonoids. Accordingly, suppose M is a left $\Sigma[\mathcal{A}]$-module. Then put

$$\mathsf{h}[F] := \mathtt{H}_F \cdot M.$$

This is the subspace of M onto which M projects by the action of the idempotent $\mathtt{H}_F$. Note that $\mathsf{h}[O] = M$. Whenever F and G have the same support, there is an isomorphism

$$\beta_{G,F} : \mathsf{h}[F] \to \mathsf{h}[G]$$

induced by the action of $\mathtt{H}_G$ (with the inverse induced by the action of $\mathtt{H}_F$). These clearly satisfy (2.1). Thus, h is an $\mathcal{A}$-species. Now let $A \leq F$. Then $AF = F$ and hence

$$\mathtt{H}_A \cdot (\mathtt{H}_F \cdot x) = (\mathtt{H}_A \cdot \mathtt{H}_F) \cdot x = \mathtt{H}_{AF} \cdot x = \mathtt{H}_F \cdot x,$$

so $\mathsf{h}[F]$ is a subspace of $\mathsf{h}[A]$. Define μ_A^F to be the inclusion map, and Δ_A^F to be the projection induced by the action of $\mathtt{H}_F$. This turns h into an $\mathcal{A}$-monoid and an $\mathcal{A}$-comonoid. The coproduct is cocommutative. The cocommutativity axiom (2.23) is checked below.

$$\mathtt{H}_G \cdot (\mathtt{H}_F \cdot x) = (\mathtt{H}_G \cdot \mathtt{H}_F) \cdot x = \mathtt{H}_{GF} \cdot x = \mathtt{H}_G \cdot x.$$

For the bimonoid axiom (2.12), we start with the element $\mathtt{H}_F \cdot x$, and the check reduces to

$$\mathtt{H}_G \cdot (\mathtt{H}_F \cdot x) = \mathtt{H}_{GF} \cdot \mathtt{H}_{FG} \cdot (\mathtt{H}_F \cdot x).$$

Thus, $(\mathsf{h}, \mu, \Delta)$ is indeed a cocommutative $\mathcal{A}$-bimonoid. Further, if M and N are left $\Sigma[\mathcal{A}]$-modules with h and k as the corresponding cocommutative $\mathcal{A}$-bimonoids, and $f : M \to N$ is a map of modules, then f restricts to linear maps

$$f_F : \mathsf{h}[F] \to \mathsf{k}[F],$$

one for each face F, and this family of maps constitutes a morphism $f : \mathsf{h} \to \mathsf{k}$ of $\mathcal{A}$-bimonoids.

Finally, we check that the functors we have constructed between modules and bimonoids define an equivalence. If we start from a module M, construct the bimonoid h, and then the corresponding module, we return to $\mathtt{H}_O \cdot M = M$. In the other direction, starting from a bimonoid h going to modules and back yields the bimonoid $\tilde{\mathsf{h}}$ with components

$$\tilde{\mathsf{h}}[F] = \mu_O^F \Delta_O^F \cdot \mathsf{h}[O].$$

Applying Lemma 10.48, we obtain a linear isomorphism $\mathsf{h}[F] \cong \tilde{\mathsf{h}}[F]$, for each face F. These constitute a natural isomorphism of $\mathcal{A}$-bimonoids $\mathsf{h} \cong \tilde{\mathsf{h}}$, in view of (A.4) and Exercise 2.26, items (3) and (4). $\square$

Proposition 11.2. *The category of right modules over the Tits algebra* $\Sigma[\mathcal{A}]$ *is equivalent to the category of commutative* $\mathcal{A}$*-bimonoids.*

Proof. The argument is similar to the one for Proposition 11.1, so we only briefly indicate how the functors work. If h is a commutative $\mathcal{A}$-bimonoid, then $\mathsf{h}[O]$ is a right $\Sigma[\mathcal{A}]$-module with action given by the characteristic operation

$$x \cdot \mathtt{H}_F := \mu_O^F \Delta_O^F(x).$$

Conversely, if M is a right $\Sigma[\mathcal{A}]$-module, then we set

$$\mathsf{h}[F] := M \cdot \mathtt{H}_F.$$

The product and coproduct are given by inclusion and projection induced by the right action. Moreover, $M \cdot \mathtt{H}_F$ and $M \cdot \mathtt{H}_G$ coincide whenever $\mathrm{s}(F) = \mathrm{s}(G)$, and $\beta_{G,F}$ is defined to be identity. So commutativity holds. The bimonoid axiom (2.12) reduces to

$$(x \cdot \mathtt{H}_F) \cdot \mathtt{H}_G = (x \cdot \mathtt{H}_F) \cdot \mathtt{H}_{FG}.$$

Note very carefully that cocommutativity requires $x \cdot \mathtt{H}_F = x \cdot \mathtt{H}_G$ whenever $\mathrm{s}(F) = \mathrm{s}(G)$, hence it does not hold in general. $\square$

Exercise 11.3. Describe the initial object, final object, product, coproduct in the category of left $\Sigma[\mathcal{A}]$-modules. Check that it matches the corresponding descriptions in the category of cocommutative bimonoids. Do the same for right $\Sigma[\mathcal{A}]$-modules.

Remark 11.4. Recall LRB species from Section 3.9. Propositions 11.1 and 11.2 generalize (with the same proof) to any LRB in place of the Tits monoid.

11.2. Modules over the Birkhoff algebra

Recall that the linearization of the Birkhoff monoid is the Birkhoff algebra. It is denoted by $\Pi[\mathcal{A}]$. Since $\Pi[\mathcal{A}]$ is commutative, there is no distinction between its left and right modules.

Proposition 11.5. *The category of modules over the Birkhoff algebra $\Pi[\mathcal{A}]$ is equivalent to the category of bicommutative $\mathcal{A}$-bimonoids.*

PROOF. The argument is similar to the one for Proposition 11.1, so we will be brief. Suppose h is a bicommutative $\mathcal{A}$-bimonoid. Then $\mathsf{h}[\perp]$ is a $\Pi[\mathcal{A}]$-module, with the action of $\mathtt{H}_{\mathrm{X}}$ on an element x given by the commutative characteristic operation

$$\mathtt{H}_{\mathrm{X}} \cdot x := \mu_{\perp}^{\mathrm{X}} \Delta_{\perp}^{\mathrm{X}}(x).$$

This defines an action as noted in (10.36). Further, if h and k are bicommutative $\mathcal{A}$-bimonoids and $f : \mathsf{h} \to \mathsf{k}$ is a morphism of $\mathcal{A}$-bimonoids, then the component $f_{\perp} : \mathsf{h}[\perp] \to \mathsf{k}[\perp]$ is a map of $\Pi[\mathcal{A}]$-modules as shown below.

$$\begin{tikzcd}
\mathsf{h}[\perp] \arrow[r, "\Delta_{\perp}^{\mathrm{X}}"] \arrow[d, "f_{\perp}"'] & \mathsf{h}[\mathrm{X}] \arrow[r, "\mu_{\perp}^{\mathrm{X}}"] \arrow[d, "f_{\mathrm{X}}"] & \mathsf{h}[\perp] \arrow[d, "f_{\perp}"] \\
\mathsf{k}[\perp] \arrow[r, "\Delta_{\perp}^{\mathrm{X}}"'] & \mathsf{k}[\mathrm{X}] \arrow[r, "\mu_{\perp}^{\mathrm{X}}"'] & \mathsf{k}[\perp]
\end{tikzcd}$$

The squares commute by (2.22) and (2.25).

Conversely: Suppose M is a $\Pi[\mathcal{A}]$-module. This defines an $\mathcal{A}$-species h whose X-component is given by

$$\mathsf{h}[\mathrm{X}] := \mathtt{H}_{\mathrm{X}} \cdot M.$$

For $\mathrm{Z} \leq \mathrm{X}$, we note that $\mathsf{h}[\mathrm{X}]$ is a subspace of $\mathsf{h}[\mathrm{Z}]$. Define $\mu_{\mathrm{Z}}^{\mathrm{X}}$ to be the inclusion map, and $\Delta_{\mathrm{Z}}^{\mathrm{X}}$ to be the projection induced by the action of $\mathtt{H}_{\mathrm{X}}$. The bicommutative bimonoid axiom (2.26) holds, and h is indeed a bicommutative $\mathcal{A}$-bimonoid. Further, if M and N are $\Pi[\mathcal{A}]$-modules with h and k as the corresponding bicommutative $\mathcal{A}$-bimonoids, and $f : M \to N$ is a map of modules, then f restricts to linear maps

$$f_{\mathrm{X}} : \mathsf{h}[\mathrm{X}] \to \mathsf{k}[\mathrm{X}],$$

one for each flat X, and this family of maps constitutes a morphism $f : \mathsf{h} \to \mathsf{k}$ of $\mathcal{A}$-bimonoids. $\square$

11.3. Modules over the Janus algebra

Recall that the linearization of the Janus monoid is the Janus algebra. It is denoted by $\mathsf{J}[\mathcal{A}]$. We write $\mathtt{H}_{(F,F')}$ for the element corresponding to (F, F'). Since the Janus monoid is isomorphic to its opposite monoid (via switching the two coordinates), the categories of left and right modules over the Janus algebra are isomorphic.

Proposition 11.6. *The category of (left) modules over the Janus algebra $\mathsf{J}[\mathcal{A}]$ is equivalent to the category of $\mathcal{A}$-bimonoids.*

PROOF. The argument is similar to the one for Proposition 11.1. We explain how the functors work.

Suppose h is an $\mathcal{A}$-bimonoid. Then $\mathsf{h}[O]$ is a left $\mathsf{J}[\mathcal{A}]$-module via the two-sided characteristic operation

$$\mathtt{H}_{(F,F')} \cdot x := \mu_O^F \beta_{F,F'} \Delta_O^{F'}(x).$$

This defines a left action as noted in (10.54) (for $q = 1$). Further, if h and k are $\mathcal{A}$-bimonoids and $f : \mathsf{h} \to \mathsf{k}$ is a morphism of $\mathcal{A}$-bimonoids, then the component $f_O : \mathsf{h}[O] \to \mathsf{k}[O]$ is a map of left $\mathsf{J}[\mathcal{A}]$-modules as shown below.

$$\begin{tikzcd}
\mathsf{h}[O] \arrow[r, "\Delta_O^{F'}"] \arrow[d, "f_O"'] & \mathsf{h}[F'] \arrow[r, "\beta_{F,F'}"] \arrow[d, "f_{F'}"] & \mathsf{h}[F] \arrow[r, "\mu_O^F"] \arrow[d, "f_F"] & \mathsf{h}[O] \arrow[d, "f_O"] \\
\mathsf{k}[O] \arrow[r, "\Delta_O^{F'}"'] & \mathsf{k}[F'] \arrow[r, "\beta_{F,F'}"'] & \mathsf{k}[F] \arrow[r, "\mu_O^F"'] & \mathsf{k}[O]
\end{tikzcd}$$

The squares commute by (2.3), (2.9), (2.11).

Conversely: Suppose M is a left $\mathsf{J}[\mathcal{A}]$-module. Then put

$$\mathsf{h}[F] := \mathtt{H}_{(F,F)} \cdot M.$$

This is the image of the idempotent $\mathtt{H}_{(F,F)}$ viewed as an operator on M. For F and G with the same support, the map $\beta_{G,F}$ is induced by the action of $\mathtt{H}_{(G,F)}$. These satisfy (2.1). Thus, h is an $\mathcal{A}$-species. Now let $A \leq F$. Then $AF = F$ and $FA = F$, and hence

$$\mathtt{H}_{(A,A)} \cdot (\mathtt{H}_{(F,F)} \cdot x) = (\mathtt{H}_{(A,A)} \cdot \mathtt{H}_{(F,F)}) \cdot x = \mathtt{H}_{(AF,FA)} \cdot x = \mathtt{H}_{(F,F)} \cdot x,$$

so $\mathsf{h}[F]$ is a subspace of $\mathsf{h}[A]$. Define μ_A^F to be the inclusion map, and Δ_A^F to be the projection induced by the action of $\mathtt{H}_{(F,F)}$. For the bimonoid axiom (2.12), we start with the element $\mathtt{H}_{(F,F)} \cdot x$, and the check reduces to

$$\mathtt{H}_{(G,G)} \cdot (\mathtt{H}_{(F,F)} \cdot x) = \mathtt{H}_{(GF,FG)} \cdot \mathtt{H}_{(FG,FG)} \cdot (\mathtt{H}_{(F,F)} \cdot x).$$

Thus, h is an $\mathcal{A}$-bimonoid. Further, if M and N are left $\mathsf{J}[\mathcal{A}]$-modules with h and k as the corresponding $\mathcal{A}$-bimonoids, and $f : M \to N$ is a map of modules, then f restricts to linear maps

$$f_F : \mathsf{h}[F] \to \mathsf{k}[F],$$

one for each face F, and this family of maps constitutes a morphism $f : \mathsf{h} \to \mathsf{k}$ of $\mathcal{A}$-bimonoids. □

Recall the q-Janus algebra $\mathsf{J}_q[\mathcal{A}]$ from Section 1.9.4. Proposition 11.6 can be generalized as follows.

Proposition 11.7. *For any scalar q, the category of (left) modules over the q-Janus algebra $\mathsf{J}_q[\mathcal{A}]$ is equivalent to the category of $\mathcal{A}$-q-bimonoids.*

PROOF. The same argument goes through, so we only point out the places where q intervenes. In the construction of the first functor, we use (10.54). In the construction of the second functor, the check of the q-bimonoid axiom (2.33) produces an extra factor of $q^{\operatorname{dist}(F,G)}$ on both sides. □

Exercise 11.8. Check that Proposition 11.7 (with the same proof) generalizes to the υ-Janus algebra, where υ is any distance function on $\mathcal{A}$ which arises from a weight function.

11.4. Examples

We now summarize the categorical equivalences between algebra-modules and bimonoids obtained in the preceding sections and then discuss some illustrative examples. For instance, the trivial module over the Birkhoff algebra corresponds to the (bicommutative) exponential bimonoid, while the left module of chambers over the Tits algebra corresponds to the (cocommutative) bimonoid of chambers.

11.4.1. Summary. For any algebra A, let $A\text{-}\mathsf{Mod}$ denote the category of left A-modules. The category of right A-modules is isomorphic to the category of left A^{op}-modules, where A^{op} denotes the algebra opposite to A.

A summary of the categorical equivalences obtained in the previous sections is given in Table 11.1.

TABLE 11.1. Categorical equivalences: algebra modules and bimonoids.

Modules over algebras		Bimonoids in species	
$\Sigma[\mathcal{A}]\text{-}\mathsf{Mod}$	left $\Sigma[\mathcal{A}]$-modules	${}^{\mathrm{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$	cocomm. bimonoids
$\Sigma[\mathcal{A}]^{\mathrm{op}}\text{-}\mathsf{Mod}$	right $\Sigma[\mathcal{A}]$-modules	$\mathsf{Bimon}^{\mathrm{co}}(\mathcal{A}\text{-}\mathsf{Sp})$	comm. bimonoids
$\Pi[\mathcal{A}]\text{-}\mathsf{Mod}$	$\Pi[\mathcal{A}]$-modules	${}^{\mathrm{co}}\mathsf{Bimon}^{\mathrm{co}}(\mathcal{A}\text{-}\mathsf{Sp})$	bicomm. bimonoids
$\mathsf{J}[\mathcal{A}]\text{-}\mathsf{Mod}$	(left) $\mathsf{J}[\mathcal{A}]$-modules	$\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$	bimonoids
$\mathsf{J}_q[\mathcal{A}]\text{-}\mathsf{Mod}$	(left) $\mathsf{J}_q[\mathcal{A}]$-modules	$q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$	q-bimonoids

Exercise 11.9. Recall from Appendix B.5 that the category of modules over a monoid algebra is a monoidal category under the tensor product. For the Birkhoff algebra, Tits algebra, Janus algebra, check that this monoidal structure corresponds to the Hadamard product of bicommutative bimonoids, cocommutative bimonoids, bimonoids, respectively. This fact was used in Section 8.8.5.

11.4.2. Examples. We now illustrate the equivalences in Table 11.1 on particular modules and bimonoids. Some basic examples are listed in Table 11.2. Using characteristic operations, the bimonoids listed in the second column yield the corresponding modules listed in the first column. These facts are contained in Examples 10.2, 10.20, 10.34.

More examples are given below.

Example 11.10. The left module over the q-Janus algebra which corresponds to the q-bimonoid of chambers Γ_q can be described as follows. It is linearly

TABLE 11.2. Examples of algebra modules and bimonoids.

Module	Bimonoid
trivial module over $\Pi[\mathcal{A}]$	exponential bimonoid E
left module $\Gamma[\mathcal{A}]$ over $\Sigma[\mathcal{A}]$	bimonoid of chambers Γ
$\Pi[\mathcal{A}]$ as a module over itself	bimonoid of flats Π
$\Sigma[\mathcal{A}]$ as a left module over itself	bimonoid of faces Σ
$\mathsf{J}[\mathcal{A}]$ as a left module over itself	bimonoid of bifaces J
$\mathsf{J}_q[\mathcal{A}]$ as a left module over itself	q-bimonoid of bifaces J_q

spanned by chambers. The left action of the q-Janus algebra is given by

$$\mathtt{H}_{(F,F')} \cdot \mathtt{H}_C = q^{\mathrm{dist}(C,F'C)}\, \mathtt{H}_{FC}.$$

This follows from formulas (7.16) and (7.19). Note very carefully the different roles played by F and F'.

When $q = 1$, the action factors through the quotient map $\mathsf{J}[\mathcal{A}] \to \Sigma[\mathcal{A}]$ to recover the left module of chambers $\Gamma[\mathcal{A}]$.

Example 11.11. The left module over the q-Janus algebra which corresponds to the q-bimonoid of faces Σ_q can be described as follows. It is linearly spanned by faces. The left action of the q-Janus algebra is given by

$$\mathtt{H}_{(F,F')} \cdot \mathtt{H}_G = q^{\mathrm{dist}(G,F')}\, \mathtt{H}_{FG}.$$

This follows from formulas (7.62) and (7.65).

Now suppose $q = 1$. The left action of the q-Janus algebra in the $\mathtt{Q}$-basis is given by

$$\mathtt{H}_{(F,F')} \cdot \mathtt{Q}_G = \begin{cases} \mathtt{Q}_{FG} & \text{if } \mathrm{s}(G) \geq \mathrm{s}(F) = \mathrm{s}(F'), \\ 0 & \text{otherwise.} \end{cases}$$

This follows from formula (7.69) and Exercise 7.23. In this case, the action factors through the quotient map $\mathsf{J}[\mathcal{A}] \to \Sigma[\mathcal{A}]$ to recover the Tits algebra $\Sigma[\mathcal{A}]$ viewed as a left module over itself. The above formula is consistent with formula (1.108).

Now suppose q is not a root of unity. The left action of the q-Janus algebra in the $\mathtt{Q}$-basis is given by

$$\mathtt{H}_{(F,F')} \cdot \mathtt{Q}_G = \begin{cases} q^{\mathrm{dist}(G,F')}\, \mathtt{Q}_{FG} & \text{if } \mathrm{s}(G) \geq \mathrm{s}(F) = \mathrm{s}(F'), \\ 0 & \text{otherwise.} \end{cases}$$

This follows from formula (7.71) and Exercise 7.25.

Exercise 11.12. The morphism $\mathsf{J}_q \to \Sigma_q$ of q-bimonoids from Exercise 7.56 corresponds to a map of modules over the q-Janus algebra. Make this explicit.

Example 11.13. Let c be any comonoid. Recall the q-bimonoid $\mathcal{T}_q(\mathsf{c})$ whose product and coproduct are given by (6.3) and (6.5), respectively. The corresponding left module over the q-Janus algebra can be described as follows.

Consider the vector space $\bigoplus_G \mathsf{c}[G]$. (The sum is over all faces.) The q-Janus algebra acts on it on the left by

$$\mathtt{H}_{(F,F')} \cdot x = q^{\mathrm{dist}(G,F')} \beta_{FG,GF'} \Delta_G^{GF'}(x)$$

for $x \in \mathsf{c}[G]$. When c is cocommutative and $q = 1$, the action factors through the quotient map $\mathsf{J}[\mathcal{A}] \to \Sigma[\mathcal{A}]$ to yield a left module over $\Sigma[\mathcal{A}]$:

$$\mathtt{H}_F \cdot x = \beta_{FG,GF} \Delta_G^{GF}(x).$$

Setting $\mathsf{c} := \mathsf{x}$ recovers the action in Example 11.10, while setting $\mathsf{c} := \mathsf{E}$ recovers the action in Example 11.11.

Exercise 11.14. Let c be any cocommutative comonoid. Consider the vector space $\bigoplus_{\mathrm{Y}} \mathsf{c}[\mathrm{Y}]$. (The sum is over all flats.) It is a module over the Birkhoff algebra with action given by

$$\mathtt{H}_{\mathrm{X}} \cdot x := \Delta_{\mathrm{Y}}^{\mathrm{X} \vee \mathrm{Y}}(x)$$

for $x \in \mathsf{c}[\mathrm{Y}]$. Check that this module corresponds to the bicommutative bimonoid $\mathcal{S}(\mathsf{c})$ whose product and coproduct are given by (6.20) and (6.22), respectively.

Exercise 11.15. Use Example 1.3 and Lemma 2.90 to deduce the results of Propositions 11.1, 11.2, 11.5, 11.6 for a rank-one arrangement.

11.5. Duality and base change

We connect the duality and base change constructions on algebra-modules to the duality and inclusion functors on bimonoids. As an application, we show that the associated graded of the primitive and decomposable series of modules over the Tits algebra factor through the support map to become modules over the Birkhoff algebra.

11.5.1. Duality. Let A be any algebra. The dual of a left A-module is a right A-module, or equivalently, a left A^{op}-module. This yields a (contravariant) *duality functor* from A-Mod to A^{op}-Mod.

The duality functor between module categories is compatible with the duality functor on the category of bimonoids (Section 2.9). In other words:

Lemma 11.16. *The following diagrams of functors commute.*

$$\begin{tikzcd} \Pi[\mathcal{A}]\text{-}\mathsf{Mod} \arrow[r, leftrightarrow] \arrow[d, "(-)^*"'] & {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \arrow[d, "(-)^*"] \\ \Pi[\mathcal{A}]\text{-}\mathsf{Mod} \arrow[r, leftrightarrow] & {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \end{tikzcd} \qquad \begin{tikzcd} \Sigma[\mathcal{A}]\text{-}\mathsf{Mod} \arrow[r, leftrightarrow] \arrow[d, "(-)^*"'] & {}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \arrow[d, "(-)^*"] \\ \Sigma[\mathcal{A}]^{\mathrm{op}}\text{-}\mathsf{Mod} \arrow[r, leftrightarrow] & \mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \end{tikzcd}$$

$$\begin{tikzcd}
\Sigma[\mathcal{A}]^{\mathrm{op}}\text{-}\mathsf{Mod} \arrow[r, leftrightarrow] \arrow[d, "(-)^*"'] & \mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \arrow[d, "(-)^*"] \\
\Sigma[\mathcal{A}]\text{-}\mathsf{Mod} \arrow[r, leftrightarrow] & {}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{J}_q[\mathcal{A}]\text{-}\mathsf{Mod} \arrow[r, leftrightarrow] \arrow[d, "(-)^*"'] & q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \arrow[d, "(-)^*"] \\
\mathsf{J}_q[\mathcal{A}]\text{-}\mathsf{Mod} \arrow[r, leftrightarrow] & q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})
\end{tikzcd}$$

The horizontal arrows are the categorical equivalences in Table 11.1. The duality functors here are contravariant by convention. (We recall from (1.130) that $\mathsf{J}_q[\mathcal{A}]$ is isomorphic to its opposite algebra.)

11.5.2. Base change. Let A and B be algebras and fix a morphism $f : A \to B$ of algebras. This gives rise to a faithful functor

$$\mathcal{F} : B\text{-}\mathsf{Mod} \to A\text{-}\mathsf{Mod}.$$

This functor views a B-module N as an A-module via $a \cdot n := f(a) \cdot n$ for $a \in A$ and $n \in N$. Further, if f is surjective, then $\mathcal{F}$ is also full, and in fact, B-Mod can be viewed as a full subcategory of A-Mod.

The functor $\mathcal{F}$ has a left adjoint which sends an A-module M to the B module $B \otimes_A M$. Thus, for an A-module M and a B-module N,

$$B\text{-}\mathsf{Mod}(B \otimes_A M, N) \cong A\text{-}\mathsf{Mod}(M, N).$$

The functor $\mathcal{F}$ also has a right adjoint which sends an A-module M to the B module $\mathrm{Hom}_A(B, M)$. Thus, for an A-module M and a B-module N,

$$A\text{-}\mathsf{Mod}(N, M) \cong B\text{-}\mathsf{Mod}(N, \mathrm{Hom}_A(B, M)).$$

Lemma 11.17. *The following diagram of functors commutes.*

$$\begin{tikzcd}
\Sigma[\mathcal{A}]\text{-}\mathsf{Mod} \arrow[rr, hook] \arrow[dr, leftrightarrow] & & \mathsf{J}[\mathcal{A}]\text{-}\mathsf{Mod} \arrow[dr, leftrightarrow] & \\
& {}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \arrow[rr, hook] & & \mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \\
\Pi[\mathcal{A}]\text{-}\mathsf{Mod} \arrow[uu, hook] \arrow[rr, hook] \arrow[dr, leftrightarrow] & & \Sigma[\mathcal{A}]^{\mathrm{op}}\text{-}\mathsf{Mod} \arrow[uu, hook] \arrow[dr, leftrightarrow] & \\
& {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \arrow[uu, hook] \arrow[rr, hook] & & \mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \arrow[uu, hook]
\end{tikzcd} \tag{11.1}$$

Proof. Consider the commutative diagram (1.122) of algebras. All morphisms are surjective. Hence, by passing to their module categories, we obtain the back face of the above cube. The front face simply says that every bicommutative bimonoid is a (co)commutative bimonoid which in turn is a bimonoid. The front and back faces are linked by the categorical equivalences in Table 11.1. It is a routine check that the squares linking the front and back faces commute. $\square$

Exercise 11.18. Check directly using Exercise 2.26, items (2) and (4), or deduce from diagram (11.1): For a cocommutative bimonoid k, if the left action of the Tits algebra on $\mathsf{k}[O]$ factors through the support map, then k is bicommutative. Dually, for a commutative bimonoid k, if the right action of the Tits algebra on $\mathsf{k}[O]$ factors through the support map, then k is bicommutative.

By taking left adjoints of the front and back faces in diagram (11.1), we see, in particular, that the abelianization functor (2.56) corresponds to

$$\mathsf{J}[\mathcal{A}]\text{-}\mathsf{Mod} \to \Sigma[\mathcal{A}]^{\mathrm{op}}\text{-}\mathsf{Mod}, \qquad M \mapsto \Sigma[\mathcal{A}]^{\mathrm{op}} \otimes_{\mathsf{J}[\mathcal{A}]} M. \tag{11.2}$$

Similarly, by taking right adjoints of the front and back faces in diagram (11.1), we see, in particular, that the coabelianization functor (2.57) corresponds to

$$\mathsf{J}[\mathcal{A}]\text{-}\mathsf{Mod} \to \Sigma[\mathcal{A}]\text{-}\mathsf{Mod}, \qquad M \mapsto \mathrm{Hom}_{\mathsf{J}[\mathcal{A}]}(\Sigma[\mathcal{A}], M). \tag{11.3}$$

Exercise 11.19. Observe that the functor (11.2) sends $\mathsf{J}[\mathcal{A}]$ to $\Sigma[\mathcal{A}]^{\mathrm{op}}$. The former corresponds to the bimonoid of bifaces J, and hence the latter must correspond to its abelianization J_{ab}. Use this to rederive the description of J_{ab} given in Section 7.8.13.

11.5.3. Primitive series and decomposable series of modules over the Tits algebra. For a cocommutative bimonoid h, consider the primitive filtration of h evaluated at the central face:

$$\mathcal{P}_1(\mathsf{h})[O] \subseteq \mathcal{P}_2(\mathsf{h})[O] \subseteq \cdots \subseteq \mathsf{h}[O].$$

This coincides with the primitive series of the left module $\mathsf{h}[O]$ over the Tits algebra, with primitive series as defined in [**21**, Section 13.1.1].

Dually, for a commutative bimonoid h, consider the decomposable filtration of h evaluated at the central face:

$$\mathsf{h}[O] \supseteq \mathcal{D}_1(\mathsf{h})[O] \supseteq \mathcal{D}_2(\mathsf{h})[O] \supseteq$$

This coincides with the decomposable series of the right module $\mathsf{h}[O]$ over the Tits algebra, with decomposable series as defined in [**21**, Section 13.1.2].

A module over the Tits algebra is *semisimple* if it is an arbitrary direct sum of simple modules. Equivalently, a module over the Tits algebra is semisimple if it arises from a module over the Birkhoff algebra via the support map. Recall that all simple modules are one-dimensional. Semisimple modules need not be finite-dimensional.

Proposition 11.20. *For a cocommutative bimonoid h, the primitive series of $\mathsf{h}[O]$ is a Loewy series, that is, each quotient*

$$\mathcal{P}_{j+1}(\mathsf{h})[O]/\mathcal{P}_j(\mathsf{h})[O]$$

is a semisimple module over the Tits algebra. Dually, for a commutative bimonoid h, the decomposable series of $\mathsf{h}[O]$ is a Loewy series, that is, each quotient

$$\mathcal{D}_j(\mathsf{h})[O]/\mathcal{D}_{j+1}(\mathsf{h})[O]$$

is a semisimple module over the Tits algebra.

PROOF. By the Browder–Sweedler commutativity result (Proposition 5.62), $\mathrm{gr}_{\mathcal{P}}(\mathsf{h})$ is bicommutative, and hence $\mathrm{gr}_{\mathcal{P}}(\mathsf{h})[O]$ is a module over the Birkhoff algebra. The first part follows. The second part follows similarly from the Milnor–Moore cocommutativity result (Proposition 5.65). ☐

Exercise 11.21. Conversely, use Proposition 11.20 and Exercise 11.18 to deduce that: For any cocommutative bimonoid h, the bimonoid $\mathrm{gr}_{\mathcal{P}}(\mathsf{h})$ is bicommutative, and for any commutative bimonoid h, the bimonoid $\mathrm{gr}_{\mathcal{D}}(\mathsf{h})$ is bicommutative.

Proposition 11.20 and Exercise 11.21 provide a nice illustration of how ideas from Hopf theory can be used to prove facts about representation theory of the Tits algebra, and vice versa.

11.6. Signed analogues

We now develop signed analogues of the categorical equivalences. They link modules over the Tits algebra to signed cocommutative signed bimonoids, and so on.

The signed analogues of Propositions 11.1 and 11.2 are as follows.

Proposition 11.22. *The category of left modules over the Tits algebra $\Sigma[\mathcal{A}]$ is equivalent to the category of signed cocommutative signed $\mathcal{A}$-bimonoids. Dually, the category of right modules over the Tits algebra $\Sigma[\mathcal{A}]$ is equivalent to the category of signed commutative signed $\mathcal{A}$-bimonoids.*

Proof. We focus on the first statement, the second being similar. Recall that the categories of cocommutative bimonoids and signed cocommutative signed bimonoids are isomorphic via the functor $(-)_{-}$ defined in (2.49). The first statement can now be seen as a consequence of Proposition 11.1. The equivalence in more direct terms works as follows.

Suppose h is a signed cocommutative signed $\mathcal{A}$-bimonoid. Then $\mathsf{h}[O]$ is a left $\Sigma[\mathcal{A}]$-module, with action defined by $\mathtt{H}_F \cdot x := \mu_O^F \Delta_O^F(x)$ as in the unsigned case. For more details, see Section 10.1.8. Conversely, suppose M is a left $\Sigma[\mathcal{A}]$-module. Put $\mathsf{h}[F] := \mathtt{H}_F \cdot M$, and define

$$\beta_{G,F} : \mathsf{h}[F] \to \mathsf{h}[G]$$

by the action of $\mathtt{H}_G$ multiplied by the scalar $(-1)^{\mathrm{dist}(F,G)}$. Define the product and coproduct as in the unsigned case. These turn h into a signed cocommutative signed $\mathcal{A}$-bimonoid. $\square$

This discussion can be summarized as follows.

Lemma 11.23. *The following diagrams of functors commute.*

(11.4)
$$\begin{tikzcd}[column sep=small] & {}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \arrow[dd, "(-)_-"] \\ \Sigma[\mathcal{A}]\text{-}\mathsf{Mod} \arrow[ur, leftrightarrow] \arrow[dr, leftrightarrow] & \\ & (-1)\text{-}{}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \arrow[dd, "(-)_-"] \\ \Sigma[\mathcal{A}]^{\mathrm{op}}\text{-}\mathsf{Mod} \arrow[ur, leftrightarrow] \arrow[dr, leftrightarrow] & \\ & (-1)\text{-}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \end{tikzcd}$$

The slanted arrows are the categorical equivalences in Propositions 11.1, 11.2 and 11.22. The functor $(-)_{-}$ is as in (2.49).

A related result is given below.

Lemma 11.24. *For any scalar q, the following diagram of functors commutes.*

$$\begin{tikzcd} \mathsf{J}_q[\mathcal{A}]\text{-}\mathsf{Mod} \arrow[r, leftrightarrow] \arrow[d] & q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \arrow[d, "(-)_-"] \\ \mathsf{J}_{-q}[\mathcal{A}]\text{-}\mathsf{Mod} \arrow[r, leftrightarrow] & (-q)\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \end{tikzcd} \tag{11.5}$$

The horizontal arrows are instances of the categorical equivalence in Proposition 11.7. The first vertical arrow is induced from the isomorphism in Exercise 1.58. In particular, note that for a q-bimonoid h, the component $\mathsf{h}[O]$ is a left $\mathsf{J}_{-q}[\mathcal{A}]$-module via the action

$$\mathtt{H}_{(F,F')} \cdot x := (-1)^{\operatorname{dist}(F,F')}\, \mu_O^F \beta_{F,F'} \Delta_O^{F'}(x).$$

Diagrams (11.4) can be viewed as restrictions of diagram (11.5) for $q = 1$ to bimonoids which are either commutative or cocommutative.

Exercise 11.25. Check that: The following diagram of functors commutes.

$$\begin{tikzcd} & {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \arrow[dd, "(-)_-"] \\ \Pi[\mathcal{A}]\text{-}\mathsf{Mod} \arrow[ur, leftrightarrow] \arrow[dr, leftrightarrow] & \\ & (-1)\text{-}{}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \end{tikzcd}$$

11.7. A unified viewpoint via partial-biflats

Recall the monoid $\mathrm{J}_{\sim,\sim'}[\mathcal{A}]$ from Section 1.3.4 associated to two fixed partial-support relations $\sim$ and $\sim'$ on faces. Let $\mathsf{J}_{\sim,\sim'}[\mathcal{A}]$ denote its linearization. We write $\mathtt{H}_{(\mathrm{x},\mathrm{x}')}$ for the element corresponding to the partial-biflat $(\mathrm{x}, \mathrm{x}')$. This algebra generalizes the Janus algebra, the Tits algebra, its opposite and the Birkhoff algebra. The results obtained earlier in this chapter can be unified as follows.

Proposition 11.26. *For partial-support relations $\sim$ and $\sim'$ on faces, the category of left modules over $\mathsf{J}_{\sim,\sim'}[\mathcal{A}]$ is equivalent to the category of $\sim$-commutative and $\sim'$-cocommutative $\mathcal{A}$-bimonoids.*

Proof. The argument generalizes the one for Proposition 11.6. Suppose h is an $\mathcal{A}$-bimonoid which is $\sim$-commutative and $\sim'$-cocommutative. Then $\mathsf{h}[O]$ is a left $\mathsf{J}_{\sim,\sim'}[\mathcal{A}]$-module via

$$\mathtt{H}_{(\mathrm{x},\mathrm{x}')} \cdot x := \mu_O^F \beta_{F,F'} \Delta_O^{F'}(x)$$

for some choice $F \in \mathrm{x}$ and $F' \in \mathrm{x}'$. This is well-defined since for some other $G \in \mathrm{x}$ and $G' \in \mathrm{x}'$,

$$\mu_O^G \beta_{G,G'} \Delta_O^{G'}(x) = \mu_O^F \beta_{F,G} \beta_{G,G'} \beta_{G',F'} \Delta_O^{F'}(x) = \mu_O^F \beta_{F,F'} \Delta_O^{F'}(x).$$

This used $\sim$-commutativity and $\sim'$-cocommutativity.

Conversely: Suppose M is a left $\mathsf{J}_{\sim,\sim'}[\mathcal{A}]$-module. Then put

$$\mathsf{h}[F] := \mathtt{H}_{(\mathrm{x},\mathrm{x}')} \cdot M,$$

where x is the partial-flat wrt $\sim$ which contains F, while x$'$ is the partial-flat wrt $\sim'$ which contains F'. The maps β, μ and Δ are defined as before, and we get an $\mathcal{A}$-bimonoid. We may further check that it is $\sim$-commutative and $\sim'$-cocommutative. $\square$

In a way, such a result is expected. This is because $\mathsf{J}_{\sim,\sim'}[\mathcal{A}]$ is a quotient of $\mathsf{J}[\mathcal{A}]$, so its category of left modules is a full subcategory of the category of left modules over $\mathsf{J}[\mathcal{A}]$. The latter is the category of $\mathcal{A}$-bimonoids by Proposition 11.6. The problem is then to describe this subcategory in a nice manner. This is precisely what was achieved in the above result.

For a partial-support relation $\sim$ on faces, let $\mathsf{\Sigma}_{\sim}[\mathcal{A}]$ denote the linearization of the monoid $\Sigma_{\sim}[\mathcal{A}]$. This is a specialization of $\mathsf{J}_{\sim,\sim'}[\mathcal{A}]$ in which $\sim'$ is coarsest. Proposition 11.26 yields the following.

Corollary 11.27. *Let $\sim$ be a partial-support relation on faces. The category of left modules over $\mathsf{\Sigma}_{\sim}[\mathcal{A}]$ is equivalent to the category of $\sim$-commutative and cocommutative bimonoids. Similarly, the category of right modules over $\mathsf{\Sigma}_{\sim}[\mathcal{A}]$ is equivalent to the category of commutative and $\sim$-cocommutative bimonoids.*

This result unifies Propositions 11.1, 11.2, 11.5. Instead of deducing it from Proposition 11.26, one can also deduce it by applying Propositions 11.1 and 11.2 to the LRB $\mathsf{\Sigma}_{\sim}[\mathcal{A}]$ (see Remark 11.4).

A summary of the equivalences is provided below.

TABLE 11.3. Algebra modules and partially (co)comm. bimonoids.

Modules over algebras	Bimonoids in species
left $\mathsf{J}_{\sim,\sim'}[\mathcal{A}]$-modules	$\sim$-comm. and $\sim'$-cocomm. bimonoids
left $\mathsf{\Sigma}_{\sim}[\mathcal{A}]$-modules	$\sim$-comm. and cocomm. bimonoids
right $\mathsf{\Sigma}_{\sim}[\mathcal{A}]$-modules	comm. and $\sim$-cocomm. bimonoids

11.8. Karoubi envelopes

We now present a more formal approach to understand the categorical equivalences. For that, we recall the notion of Karoubi envelope and Karoubi groupoid of a semigroup. We compute these categories for the Birkhoff monoid, Tits monoid, Janus monoid. They are summarized in Table 11.4.

For the Tits monoid, we obtain the categories $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{r}}$ and $\mathcal{A}\text{-}\mathtt{Hyp}$, respectively. These categories were discussed in Sections 2.1.1 and 2.11. Recall that these are the base categories for cocommutative bimonoids and species, respectively. Through a general result on Karoubi envelopes, we deduce that the category of cocommutative bimonoids is equivalent to the category of left modules over the Tits algebra. Similar phenomena are observed in the other cases as well.

TABLE 11.4. Karoubi envelopes of semigroups.

Semigroup	Karoubi envelope	Karoubi groupoid
Tits monoid $\Sigma[\mathcal{A}]$	$\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{r}}$	$\mathcal{A}\text{-}\mathtt{Hyp}$
$\Sigma[\mathcal{A}]^{\mathrm{op}}$	$\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{e}}_{\mathtt{c}}$	$\mathcal{A}\text{-}\mathtt{Hyp}$
Birkhoff monoid $\Pi[\mathcal{A}]$	$\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{e}}_{\mathtt{r}}$	$\mathcal{A}\text{-}\mathtt{Hyp}'$ (equiv. to $\mathcal{A}\text{-}\mathtt{Hyp}$)
Janus monoid $\mathrm{J}[\mathcal{A}]$	equiv. to $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{c}}$	equiv. to $\mathcal{A}\text{-}\mathtt{Hyp}$

11.8.1. Karoubi envelope of a semigroup. Let S be any semigroup. The *Karoubi envelope* $\mathsf{K}(S)$ of S is the following category associated to S.

- Objects are idempotents of S, and
- morphisms $x \to y$ are elements $f \in S$ satisfying $fx = f = yf$.

The identity morphism at x is x, while composition is product in S. More precisely, the composite of $x \xrightarrow{f} y$ and $y \xrightarrow{g} z$ is $x \xrightarrow{gf} z$.

Note that $\mathsf{K}(S^{\mathrm{op}}) = \mathsf{K}(S)^{\mathrm{op}}$, with S^{op} being the opposite semigroup, and $\mathsf{K}(S)^{\mathrm{op}}$ being the opposite category. Further, if x and y are idempotents satisfying $xyx = x$ and $yxy = y$, then x and y are isomorphic via $x \xrightarrow{yx} y$ and $y \xrightarrow{xy} x$. The subcategory of $\mathsf{K}(S)$ consisting of all isomorphisms is the *Karoubi groupoid* of S.

Now let S be a monoid. View S as a category with one object, denoted $*$, with morphisms indexed by elements of S. Observe that there is a full and faithful embedding $i : S \to \mathsf{K}(S)$ which sends the unique object $*$ in S to the identity element of S (which is an idempotent and hence an object of $\mathsf{K}(S)$).

Lemma 11.28. *For any monoid S, given a functor $\mathcal{F} : S \to \mathsf{Vec}$, there exists a functor $\hat{\mathcal{F}} : \mathsf{K}(S) \to \mathsf{Vec}$, unique up to a natural isomorphism, such that the diagram*

$$\begin{tikzcd} \mathsf{K}(S) \arrow[r, dashed, "\hat{\mathcal{F}}"] & \mathsf{Vec} \\ S \arrow[u, "i"] \arrow[ur, "\mathcal{F}"'] & \end{tikzcd}$$

commutes.

Explicitly: Put $\mathcal{F}(*) = V$. For any idempotent $x \in S$, define $\hat{\mathcal{F}}(x)$ to be the image of the linear map $\mathcal{F}(x) : V \to V$. For any morphism $f : x \to y$, define $\hat{\mathcal{F}}(f) : \hat{\mathcal{F}}(x) \to \hat{\mathcal{F}}(y)$ to be the restriction of $\mathcal{F}(f) : V \to V$. Any other choice for $\hat{\mathcal{F}}$ is naturally isomorphic to this one.

Theorem 11.29. *For any monoid S, the categories $[S, \mathsf{Vec}]$ and $[\mathsf{K}(S), \mathsf{Vec}]$ are equivalent.*

We point out that $[S, \mathsf{Vec}]$ is the same as the category of left S-modules.

PROOF. To get the equivalence: The functor in one direction is induced by the embedding $i : S \to \mathsf{K}(S)$. In the other direction, it arises from the universal property of $\mathsf{K}(S)$ given in Lemma 11.28. □

The following is a useful criterion to determine whether an embedding of a monoid yields its Karoubi envelope.

Lemma 11.30. *Let S be a monoid and $i : S \to \mathsf{C}$ a full and faithful embedding. Then C is equivalent to $\mathsf{K}(S)$ if $i(e)$ splits in C for every idempotent $e \in S$, and every object in C is a retract of $i(*)$.*

PROOF. See [**513**, Theorem I.6.12]. In this reference, the context is that of additive categories. □

11.8.2. Tits monoid. Now let us do the calculations. We begin with the Tits monoid. All elements of $\Sigma[\mathcal{A}]$ are idempotent. So objects of its Karoubi envelope $\mathsf{K}(\Sigma[\mathcal{A}])$ are faces.

Lemma 11.31. *The element H is a morphism from F to G in $\mathsf{K}(\Sigma[\mathcal{A}])$ iff $HF = H = GH$. Further, it is an isomorphism iff $H = G$ and $\mathrm{s}(H) = \mathrm{s}(F)$.*

PROOF. The first statement is immediate from the definition. For the second, suppose H is an isomorphism with inverse H'. This is equivalent to the equations

$$HF = H = GH,\ H'G = H' = FH',\ H'H = F,\ HH' = G.$$

The equation $H = GH$ implies $G \leq H$, while $HH' = G$ implies $H \leq G$. Thus, $H = G$. Similarly, $H' = F$. Substituting these back, we reduce to the equations $FG = F$ and $GF = G$. This is the same as $\mathrm{s}(F) = \mathrm{s}(G)$ by (1.5). □

Proposition 11.32. *For the Tits monoid $\Sigma[\mathcal{A}]$, the Karoubi envelope is $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{r}}$, while the Karoubi groupoid is $\mathcal{A}\text{-}\mathtt{Hyp}$. Dually, for $\Sigma[\mathcal{A}]^{\mathrm{op}}$, the Karoubi envelope is $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{e}}_{\mathtt{c}}$, while the Karoubi groupoid is $\mathcal{A}\text{-}\mathtt{Hyp}$.*

PROOF. The description of the morphisms in $\mathsf{K}(\Sigma[\mathcal{A}])$ given in Lemma 11.31 coincides with the description of the morphisms in the category $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{r}}$. (See definition prior to Proposition 2.66.) Moreover, it is easy to see that the composition laws match. Similarly, the description of the isomorphisms in $\mathsf{K}(\Sigma[\mathcal{A}])$ coincides with the description of the morphisms in the category $\mathcal{A}\text{-}\mathtt{Hyp}$. (See definition prior to Proposition 2.3.) This proves the claim about $\Sigma[\mathcal{A}]$.

The claim about $\Sigma[\mathcal{A}]^{\mathrm{op}}$ follows by taking opposites. □

Exercise 11.33. In the category $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{r}}$, by (2.80),

$$(F, F, O) \circ (O, F, F) = (O, F, O),$$
$$(O, F, F) \circ (F, F, O) = (F, F, F).$$

Use Lemma 11.30 to deduce that $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{r}}$ is equivalent to the Karoubi envelope of $\Sigma[\mathcal{A}]$.

11.8.3. Birkhoff monoid. Now let us look at the Birkhoff monoid $\Pi[\mathcal{A}]$. Again, all elements are idempotent. So objects of the Karoubi envelope $\mathsf{K}(\Pi[\mathcal{A}])$ are flats.

Lemma 11.34. *The element* Z *is a morphism from* X *to* Y *in* $\mathsf{K}(\Pi[\mathcal{A}])$ *iff* $\mathrm{X}, \mathrm{Y} \leq \mathrm{Z}$. *Further, it is an isomorphism iff* $\mathrm{X} = \mathrm{Z} = \mathrm{Y}$.

Proof. By definition, Z is a morphism from X to Y iff $\mathrm{X} \vee \mathrm{Z} = \mathrm{Z} = \mathrm{Y} \vee \mathrm{Z}$. This is the same as saying that Z is an upper bound for both X and Y. Further, if Z is an isomorphism with inverse Z', then $\mathrm{X} = \mathrm{Z} \vee \mathrm{Z}' = \mathrm{Y}$ which implies that $\mathrm{X} = \mathrm{Z} = \mathrm{Y}$. $\square$

Proposition 11.35. *For the Birkhoff monoid* $\Pi[\mathcal{A}]$*, the Karoubi envelope is* $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{e}}_{\mathtt{r}}$*, while the Karoubi groupoid is* $\mathcal{A}\text{-}\mathtt{Hyp}'$.

Proof. The description of the morphisms in $\mathsf{K}(\Pi[\mathcal{A}])$ in Lemma 11.34 coincides with the description of the morphisms in the category $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{e}}_{\mathtt{r}}$. (See definition prior to Proposition 2.68.) It is also clear that the composition laws match. Similarly, the description of the isomorphisms in $\mathsf{K}(\Pi[\mathcal{A}])$ coincides with the description of the morphisms in the category $\mathcal{A}\text{-}\mathtt{Hyp}'$. (See definition after Proposition 2.4.) $\square$

Exercise 11.36. In the category $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{e}}_{\mathtt{r}}$, by (2.82),

$$\begin{aligned}(\mathrm{X},\mathrm{X},\perp)\circ(\perp,\mathrm{X},\mathrm{X}) &= (\perp,\mathrm{X},\perp),\\ (\perp,\mathrm{X},\mathrm{X})\circ(\mathrm{X},\mathrm{X},\perp) &= (\mathrm{X},\mathrm{X},\mathrm{X}).\end{aligned}$$

Use Lemma 11.30 to deduce that $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{e}}_{\mathtt{r}}$ is equivalent to the Karoubi envelope of $\Pi[\mathcal{A}]$. (Compare with Exercise 11.33.)

11.8.4. Janus monoid. Now consider the Janus monoid $\mathrm{J}[\mathcal{A}]$. Again, all elements are idempotent. So objects of the Karoubi envelope $\mathsf{K}(\mathrm{J}[\mathcal{A}])$ are bifaces.

Lemma 11.37. *The element* (H, H') *is a morphism from* (F, F') *to* (G, G') *in* $\mathsf{K}(\mathrm{J}[\mathcal{A}])$ *iff* $G \leq H$ *and* $F' \leq H'$. *Further, it is an isomorphism iff* $H = G$ *and* $H' = F'$.

Proof. We prove the first statement. By definition, (H, H') is a morphism from (F, F') to (G, G') iff

$$(HF, F'H') = (H, H') = (GH, H'G').$$

The conditions $GH = H$ and $F'H' = H'$ are the same as $G \leq H$ and $F' \leq H'$. The remaining two conditions $HF = H$ and $H'G' = H'$ are implied by these two. For instance, $\mathrm{s}(H) = \mathrm{s}(H') \geq \mathrm{s}(F') = \mathrm{s}(F)$, and so $HF = H$. $\square$

A morphism in $\mathsf{K}(\mathrm{J}[\mathcal{A}])$ can be visualized as

$$\begin{array}{ccccc} & & H' & — & H \\ & & | & & | \\ F & — & F' & & G & — & G'. \end{array}$$

The horizontal lines mean same support, while the vertical lines mean containment. The composite of (H, H') and (K, K') can be shown as follows.

$$\begin{array}{ccccccccccc} & H' & - & H & & K' & - & K & \\ & | & & | & & | & & | & \\ F & - & F' & & G - G' & & N & - & N' \end{array} \quad = \quad \begin{array}{ccccc} & H'K' & - & KH & \\ & | & & | & \\ F & \text{---} & F' & & N \text{ --- } N'. \end{array}$$

Proposition 11.38. *For the Janus monoid* $\mathrm{J}[\mathcal{A}]$*, the Karoubi envelope is equivalent to* $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{c}}$*. More precisely,* $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{c}}$ *is the full subcategory of* $\mathsf{K}(\mathrm{J}[\mathcal{A}])$ *consisting of objects of the form* (F, F)*.*

PROOF. The second statement follows by comparing the composition law of $\mathsf{K}(\mathrm{J}[\mathcal{A}])$ described above with the composition law (2.73). The first statement then follows since objects of the form (F, F) cover all isomorphism classes in $\mathsf{K}(\mathrm{J}[\mathcal{A}])$. $\square$

Exercise 11.39. In the category $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{c}}$, by (2.73),

$$(F, F, F, O) \circ (O, F', F, F) = (O, F', F, O),$$
$$(O, F', F, F) \circ (F, F, F, O) = (F, F, F, F).$$

Use Lemma 11.30 to deduce that $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{c}}$ is equivalent to the Karoubi envelope of $\mathrm{J}[\mathcal{A}]$.

11.8.5. Modules over monoid algebras and bimonoids. Theorem 11.29 applied to the Tits monoid says that the category of left $\Sigma[\mathcal{A}]$-modules is equivalent to the functor category $[\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{r}}, \mathsf{Vec}]$. But by Proposition 2.67, the latter is the category of cocommutative bimonoids. Thus, we conclude that the categories of left $\Sigma[\mathcal{A}]$-modules and cocommutative bimonoids are equivalent. This is Proposition 11.1. Proposition 11.2 is obtained by working with the opposite of the Tits monoid. Similarly, specializing Theorem 11.29 to the Birkhoff monoid and using Proposition 2.70, we obtain Proposition 11.5. For the Janus monoid, we obtain Proposition 11.6. A point of distinction here is that the relevant category $\mathcal{A}\text{-}\mathtt{Hyp}^{\mathtt{d}}_{\mathtt{c}}$ is not equal to but is an equivalent full subcategory of the Karoubi envelope.

11.9. Monoid-sets and bimonoids in set-species

Recall from Section 2.14 that one can define bimonoids in the category of set-species. The categorical equivalences that we established for bimonoids continue to hold for set-bimonoids.

11.9.1. Categorical equivalences for set-bimonoids.

Proposition 11.40. *We have:*

- *The category of left* $\Sigma[\mathcal{A}]$*-sets is equivalent to the category of cocommutative* $\mathcal{A}$*-set-bimonoids.*
- *The category of right* $\Sigma[\mathcal{A}]$*-sets is equivalent to the category of commutative* $\mathcal{A}$*-set-bimonoids.*
- *The category of* $\Pi[\mathcal{A}]$*-sets is equivalent to the category of bicommutative* $\mathcal{A}$*-set-bimonoids.*

- *The category of left* $\mathrm{J}[\mathcal{A}]$*-sets is equivalent to the category of* $\mathcal{A}$*-set-bimonoids.*

This is the set-theoretic analogue of Propositions 11.1, 11.2, 11.5, 11.6. For proving it, the set-theoretic characteristic operations in Section 10.4 can be employed. Since the details are similar, we omit them.

11.9.2. Reflexive directed graphs. For a rank-one arrangement $\mathcal{A}$, right $\Sigma[\mathcal{A}]$-sets are related to directed graphs. Details follow.

A *directed graph* G consists of a set of vertices V, a set of edges E, and two maps $s, t : E \to V$. It is *reflexive* if there is given in addition a map $i : V \to E$ such that $si = \mathrm{id}_V = ti$.

$$s(e) \xrightarrow{e} t(e) \qquad \begin{array}{c} i(v) \\ \circlearrowright \\ v \end{array}$$

A morphism $f : G \to G'$ of reflexive directed graphs consists of a pair of maps $f_0 : V \to V'$ and $f_1 : E \to E'$ such that $sf_1(e) = f_0 s(e)$, $tf_1(e) = f_0 t(e)$, and $if_0(v) = f_1 i(v)$ for all $e \in E$, $v \in V$. This defines the category of reflexive directed graphs.

Exercise 11.41. Let F be the category with two objects 0 and 1 and generated by arrows $i : 0 \to 1$ and $s, t : 1 \to 0$ subject to $si = \mathrm{id}_0 = ti$.

$$\begin{tikzcd} 0 \arrow[r, "i" description] & 1 \arrow[l, bend right, "s"'] \arrow[l, bend left, "t"] \end{tikzcd}$$

Check that:

- F is equivalent to the Karoubi envelope of S^{op}, the opposite of the monoid S defined in Example 1.1. (Recall that S is isomorphic to the Tits monoid of a rank-one arrangement.)
- $[\mathsf{F}, \mathsf{Set}]$ is isomorphic to the category of reflexive directed graphs.

Deduce that the categories of right S-sets and that of reflexive directed graphs are equivalent. Explicitly, the graph corresponding to a right S-set X has

$$V := \{x \in X \mid x \cdot + = x\} = \{x \in X \mid x \cdot - = x\} \quad \text{and} \quad E := X,$$

s is the right action of $+$, t the right action of $-$, and i is the inclusion of V into X.

11.10. Bimonoids of h-faces and h-flats

To any cocommutative set-bimonoid h, we associate the bimonoid of h-faces, and the bimonoid of h-flats, and a morphism between the two. For the simplest case h = E, the bimonoids are Σ and Π, and the morphism is the support map. In the finite-dimensional setting, these constructions are isomorphic to the free bimonoid and the free commutative bimonoid, respectively, on the comonoid obtained by linearizing and dualizing h (viewed as a set-monoid).

11.10.1. Set-species of h-faces and h-flats. In view of Proposition 11.40, a cocommutative set-bimonoid is the same as a left $\Sigma[\mathcal{A}]$-set. We use this to translate the notion of h-faces and h-flats defined in Section 1.2.4 to the present setting.

Let h be a cocommutative set-bimonoid. We say that a pair $(F/A, x) \in \Sigma[A] \times \mathrm{h}[A]$ is an h-*face local to* A if

$$F/A \cdot x = x.$$

The set of h-faces local to A is

$${}^{\mathrm{h}}\Sigma[A] := \{(F/A, F/A \cdot x) \mid F \geq A,\ x \in \mathrm{h}[A]\}.$$

This follows from

$$F/A \cdot (F/A \cdot x) = F/A \cdot x.$$

We thus obtain a set-species ${}^{\mathrm{h}}\Sigma$. Its structure map is defined by

$$\beta_{G,F} : {}^{\mathrm{h}}\Sigma[A] \to {}^{\mathrm{h}}\Sigma[B], \qquad (F/A, x) \mapsto (BF/B, \beta_{G,F}(x)).$$

Similarly, by employing (1.14), one can define the set of h-flats local to A which we denote by ${}^{\mathrm{h}}\Pi[A]$. This yields the set-species ${}^{\mathrm{h}}\Pi$ and a canonical quotient map

$$ {}^{\mathrm{h}}\Sigma \twoheadrightarrow {}^{\mathrm{h}}\Pi. \tag{11.6}$$

We refer to ${}^{\mathrm{h}}\Sigma$ and ${}^{\mathrm{h}}\Pi$ as the set-species of h-faces and h-flats, respectively.

Some preliminary results on local h-faces are given below.

Lemma 11.42. *Let $A \leq G \leq F$. If $(F/A, x)$ is an* h-*face local to A, then so is $(G/A, x)$.*

Proof. Since $G \leq F$, we have $GF = F$. We are given that $x = F/A \cdot x$. Hence, by the set-analogue of (10.4),

$$G/A \cdot x = G/A \cdot (F/A \cdot x) = GF/A \cdot x = F/A \cdot x = x.$$

Thus, $(G/A, x)$ is an h-face. □

Lemma 11.43. *We have:*

(1) *Let $A \leq K \leq F$. Then $(F/K, x)$ is an* h-*face local to K iff $(F/A, \mu_A^K(x))$ is an* h-*face local to A.*
(2) *Let $A \leq K$ and $A \leq F$. If $(F/A, x)$ is an* h-*face local to A, then $(KF/K, \Delta_A^K(x))$ is an* h-*face local to K. In addition, if $K \leq F$, then $(F/K, \Delta_A^K(x))$ is an* h-*face local to K.*
(3) *Let $A \leq F$ and $A \leq H$. If $(H/A, x)$ is an* h-*face local to A, then $(FH/A, F/A \cdot x)$ is an* h-*face local to A.*

Proof. Item (1) follows from the set-analogue of (10.20), while item (2) from the set-analogue of (10.23). For item (3): Let $(H/A, x)$ be an h-face local to A. First by item (2), $(FH/F, \Delta_A^F(x))$ is an h-face local to F. Now applying item (1), $(FH/A, \mu_A^F \Delta_A^F(x))$ is an h-face local to A. □

11.10.2. Bimonoid of h-faces. Let ${}^{\mathrm{h}}\Sigma$ be the species obtained by linearizing the set-species of h-faces. Thus, ${}^{\mathrm{h}}\Sigma[A]$ is the vector space with basis $\mathtt{H}_{F/A,x}$ indexed by h-faces $(F/A, x)$ local to A.

We now turn ${}^{\mathrm{h}}\Sigma$ into a bimonoid. For $K \geq A$, define

$$\begin{aligned} \mu_A^K : {}^{\mathrm{h}}\Sigma[K] &\to {}^{\mathrm{h}}\Sigma[A] \\ \mathtt{H}_{F/K,x} &\mapsto \mathtt{H}_{F/A,\mu_A^K(x)}, \end{aligned} \tag{11.7}$$

$$\begin{aligned} \Delta_A^K : {}^{\mathrm{h}}\Sigma[A] &\to {}^{\mathrm{h}}\Sigma[K] \\ \mathtt{H}_{F/A,x} &\mapsto \begin{cases} \mathtt{H}_{KF/K,\Delta_A^K(x)} & \text{if } FK/A \cdot x = x, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{11.8}$$

These are well-defined by Lemma 11.43, items (1) and (2). (The alternative in the definition of Δ_A^K did not play a role here. It does in the proof below.)

Proposition 11.44. *For* h *a cocommutative set-bimonoid,* ${}^{\mathrm{h}}\Sigma$ *is a bimonoid with product* (11.7) *and coproduct* (11.8)*.*

PROOF. We verify the bimonoid axiom (2.12). Let J and K be faces greater than A and $(F/K, x)$ an h-face local to K. Then

$$\Delta_A^J \mu_A^K(\mathtt{H}_{F/K,x}) = \begin{cases} \mathtt{H}_{JF/J,\Delta_A^J\mu_A^K(x)} & \text{if } FJ/A \cdot \mu_A^K(x) = \mu_A^K(x), \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$\begin{aligned} &\mu_J^{JK}\beta_{JK,KJ}\Delta_K^{KJ}(\mathtt{H}_{F/K}, x) \\ &\qquad = \begin{cases} \mu_J^{JK}\beta_{JK,KJ}\Big(\mathtt{H}_{KJF/KJ,\Delta_K^{KJ}(x)}\Big) & \text{if } FKJ/K \cdot x = x, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We first check that the above alternatives coincide. Since $K \leq F$, we have $FKJ/K = FJ/K$. Further, by the set-analogue of (10.17), we have

$$FJ/A \cdot \mu_A^K(x) = \mu_A^K(FJ/K) \cdot \mu_A^K(x) = \mu_A^K(FJ/K \cdot x).$$

Now, since μ_A^K is injective (being split by Δ_A^K), we have

$$FJ \cdot \mu_A^K(x) = \mu_A^K(x) \iff FJ/K \cdot x = x,$$

and this shows that the alternatives coincide. Hence, working with the first alternative,

$$\begin{aligned} \mu_J^{JK}\beta_{JK,KJ}\Big(\mathtt{H}_{KJF/KJ,\Delta_K^{KJ}(x)}\Big) &= \mu_J^{JK}\Big(\mathtt{H}_{JF/JK,\beta_{JK,KJ}\Delta_K^{KJ}(x)}\Big) \\ &= \mathtt{H}_{JF/J,\mu_J^{JK}\beta_{JK,KJ}\Delta_K^{KJ}(x)} = \mathtt{H}_{JF/J,\Delta_A^J\mu_A^K(x)}, \end{aligned}$$

by the set-bimonoid axiom (2.92) for h. This completes the verification. $\square$

Let $f : \mathrm{h} \to \mathrm{k}$ be a morphism of cocommutative set-bimonoids. Define

$${}^{f}\Sigma : {}^{\mathrm{k}}\Sigma \to {}^{\mathrm{h}}\Sigma, \tag{11.9}$$

on the A-component, by

$$ {}^{\mathrm{k}}\Sigma[A] \to {}^{\mathrm{h}}\Sigma[A], \qquad \mathtt{H}_{F/A,y} \mapsto \sum_{\substack{x\in \mathrm{h}[A]:\\ f(x)=y,\, F/A\cdot x = x}} \mathtt{H}_{F/A,x}. $$

Proposition 11.45. *The map* (11.9) *is a morphism of bimonoids.*

PROOF. To show that ${}^{f}\Sigma$ preserves products, we need to verify that for any k-face $(F/K, y)$ local to K,

$$ \sum_{\substack{x\in \mathrm{h}[A]:\\ f(x)=\mu_A^K(y),\, F/A\cdot x = x}} \mathtt{H}_{F/A,x} = \sum_{\substack{z\in \mathrm{h}[K]:\\ f(z)=y,\, F/K\cdot z = z}} \mathtt{H}_{F/A,\mu_A^K(z)}. $$

For this, it suffices to check that the maps

$$ \begin{gathered} \{x \in \mathrm{h}[A] \mid f(x) = \mu_A^K(y),\ F/A\cdot x = x\} \leftrightarrow \{z \in \mathrm{h}[K] \mid f(z) = y,\ F/K\cdot z = z\} \\ x \mapsto \Delta_A^K(x) \\ \mu_A^K(z) \leftarrow\!\shortmid z \end{gathered} $$

are well-defined inverse bijections. They are inverse since $\Delta_A^K \mu_A^K(z) = z$ for any z, and $F/A \cdot x = x$ implies $\mu_A^K \Delta_A^K(x) = K/A \cdot x = x$ by Lemma 11.42.

We check they are well-defined. Let x be in the first set. Since f preserves coproducts, we have

$$ f\big(\Delta_A^K(x)\big) = \Delta_A^K\big(f(x)\big) = \Delta_A^K\big(\mu_A^K(y)\big) = y. $$

In addition, since $(F/A, x)$ is an h-face local to A, by Lemma 11.43, item (2), $(F/K, \Delta_A^K(x))$ is an h-face local to K. This shows that $\Delta_A^K(x)$ is in the second set.

Now take z in the second set. Since f preserves products,

$$ f\big(\mu_A^K(z)\big) = \mu_A^K\big(f(z)\big) = \mu_A^K(y). $$

In addition, since $(F/K, z)$ is an h-face local to K, by Lemma 11.43, item (1), $(F/A, \mu_A^K(z))$ is an h-face local to A. This shows that $\mu_A^K(z)$ is in the first set.

It remains to show that ${}^{f}\Sigma$ preserves coproducts. This boils down to the following equality, where $A \le K$ and $(F/A, y)$ a k-face local to A.

$$ \sum_{\substack{x\in \mathrm{h}[A]:\\ f(x)=y\\ FK/A\cdot x = x}} \mathtt{H}_{FK/K,\Delta_A^K(x)} = \begin{cases} \displaystyle\sum_{\substack{z\in \mathrm{h}[K]:\\ f(z)=\Delta_A^K(y)\\ KF/K\cdot z = z}} \mathtt{H}_{KF/K,z} & \text{if } FK/A\cdot y = y, \\ 0 & \text{otherwise.} \end{cases} $$

(The condition $F/A \cdot x = x$, which intervenes in $\Delta_A^K {}^{f}\Sigma(\mathtt{H}_{F/A,y})$, is implied by $FK/A \cdot x = x$ in view of Lemma 11.42.)

Consider the sets

$$ \{x \in \mathrm{h}[A] \mid f(x) = y,\ FK/A\cdot x = x\} $$

and

$$ \{z \in \mathrm{h}[K] \mid f(z) = \Delta_A^K(y),\ KF/K\cdot z = z\}. $$

If $FK/A \cdot y \neq y$, then the first set is empty. Indeed, if x belongs to this set, then

$$y = f(x) = f(FK/A \cdot x) = FK/A \cdot f(x) = FK/A \cdot y.$$

This proves the equality in the second alternative. To prove it in the first alternative, it suffices to check that the maps

$$\begin{aligned} x &\mapsto \Delta^K_A(x) \\ F/A \cdot \mu^K_A(z) &\leftarrow\!\shortmid z \end{aligned}$$

are well-defined inverse bijections between these sets, assuming $FK/A \cdot y = y$.

Starting from x in the first set, the composite yields

$$F/A \cdot \mu^K_A \Delta^K_A(x) = F/A \cdot K/A \cdot x = FK/A \cdot x = x.$$

The composite in the other direction sends z in the second set to

$$\Delta^K_A\big(F/A \cdot \mu^K_A(z)\big) = \Delta^K_A(F/A) \cdot z = KF/K \cdot z = z.$$

The second step used (10.13). Alternatively, using (10.8),

$$\Delta^K_A\big(F/A \cdot \mu^K_A(z)\big) = \Delta^K_A \mu^F_A \Delta^F_A \mu^K_A(z) = \mu^{KF}_K \Delta^{KF}_K(z) = KF/K \cdot z = z.$$

Thus, the maps are inverse.

We check they are well-defined. Let x be in the first set. Since f preserves coproducts, we have $f\big(\Delta^K_A(x)\big) = \Delta^K_A(y)$. This along with Lemma 11.43, item (2), shows that $\Delta^K_A(x)$ is in the second set. Now take z in the second set. Since f preserves (co)products, we have

$$\begin{aligned} f\big(F/A \cdot \mu^K_A(z)\big) = F/A \cdot \mu^K_A\big(f(z)\big) &= F/A \cdot \mu^K_A \Delta^K_A(y) \\ &= F/A \cdot K/A \cdot y = FK/A \cdot y = y. \end{aligned}$$

Also, by Lemma 11.43, item (1), $(KF/A, \mu^K_A(z))$ is an h-face local to A, and by item (3), $(FK/A, F/A \cdot \mu^K_A(z))$ is an h-face local to A. So $F/A \cdot \mu^K_A(z)$ belongs to the first set. This proves the remaining condition and completes the necessary verifications. $\square$

Thus:

Theorem 11.46. *There is a contravariant functor denoted* ${}^{-}\Sigma$ *from the category of cocommutative set-bimonoids to the category of bimonoids.*

11.10.3. Bimonoid of h-flats. Let ${}^{\mathrm{h}}\Pi$ denote the linearization of the set-species of h-flats. Consider the canonical quotient map ${}^{\mathrm{h}}\Sigma \twoheadrightarrow {}^{\mathrm{h}}\Pi$ obtained by linearizing (11.6). It is a routine check that the bimonoid structure of the former induces a bimonoid structure on the latter. Further, a morphism $f : \mathrm{h} \to \mathrm{k}$ of set-bimonoids induces a morphism of bimonoids ${}^{\mathrm{k}}\Pi[A] \to {}^{\mathrm{h}}\Pi[A]$ such that the diagram

$$\begin{tikzcd} {}^{\mathrm{k}}\Sigma[A] \arrow[r] \arrow[d, two heads] & {}^{\mathrm{h}}\Sigma[A] \arrow[d, two heads] \\ {}^{\mathrm{k}}\Pi[A] \arrow[r, dashed] & {}^{\mathrm{h}}\Pi[A] \end{tikzcd} \tag{11.10}$$

commutes. Thus:

Theorem 11.47. *There is a contravariant functor denoted* ${}^{-}\Pi$ *from the category of cocommutative set-bimonoids to the category of bimonoids, along with a natural transformation* ${}^{-}\Sigma \twoheadrightarrow {}^{-}\Pi$.

11.10.4. Connection with the biconvolution bimonoid. Recall from Section 8.3 that associated to any bimonoid h is the biconvolution bimonoid $\mathrm{end}^{\times}(\mathsf{h})$. Now let h be any cocommutative set-bimonoid, and $\mathsf{h} := \Bbbk\mathrm{h}$ denote its linearization. For an h-face $(F/A, x)$ local to A, let $f_{F/A,x}$ denote the linear endomorphism of $\mathsf{h}[A]$ defined on a basis element $y \in \mathrm{h}[A]$ by

$$f_{F/A,x}(y) = \begin{cases} x & \text{if } F/A \cdot y = x, \\ 0 & \text{otherwise.} \end{cases}$$

This gives rise to a map

$$ {}^{\mathrm{h}}\Sigma \to \mathrm{end}^{\times}(\mathsf{h}) \tag{11.11}$$

which on the A-component sends $\mathtt{H}_{F/A,x}$ to $f_{F/A,x}$.

Proposition 11.48. *The map* (11.11) *is a morphism of bimonoids.*

PROOF. Let us denote the map by φ. We check that φ preserves products. Let $(F/K, x)$ be an h-face local to K. We have

$$\mu_A^K \varphi(\mathtt{H}_{F/K,x}) = \mu_A^K(f_{F/K,x}) = \mu_A^K f_{F/K,x} \Delta_A^K,$$

where the latter μ_A^K and Δ_A^K denote the structure maps of h. On the other hand,

$$\varphi \mu_A^K(\mathtt{H}_{F/K,x}) = \varphi(\mathtt{H}_{F/A,\mu_A^K(x)}) = f_{F/A,\mu_A^K(x)}.$$

Let $y \in \mathrm{h}[A]$. We have

$$(\mu_A^K f_{F/K,x} \Delta_A^K)(y) = \begin{cases} \mu_A^K(x) & \text{if } F/K \cdot \Delta_A^K(y) = x, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f_{F/A,\mu_A^K(x)}(y) = \begin{cases} \mu_A^K(x) & \text{if } F/A \cdot y = \mu_A^K(x), \\ 0 & \text{otherwise.} \end{cases}$$

We only need to check that the alternatives coincide. This follows from the set-analogue of (10.21).

We check that φ preserves coproducts. Let $(F/A, x)$ be an h-face local to A and $y \in \mathrm{h}[K]$. We have

$$\Delta_A^K \varphi(\mathtt{H}_{F/A,x}) = \Delta_A^K(f_{F/A,x}) = \Delta_A^K f_{F/A,x} \mu_A^K,$$

and hence

$$\Delta_A^K \varphi(\mathtt{H}_{F/A,x})(y) = \begin{cases} \Delta_A^K(x) & \text{if } F/A \cdot \mu_A^K(y) = x, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, if $FK/A \cdot x = x$, then

$$\varphi \Delta_A^K(\mathtt{H}_{F/A,x}) = \varphi(\mathtt{H}_{KF/K,\Delta_A^K(x)}) = f_{KF/K,\Delta_A^K(x)},$$

and otherwise, $\varphi\Delta_A^K(\mathtt{H}_{F/A,x}) = 0$. Therefore,

$$\varphi\Delta_A^K(\mathtt{H}_{F/A,x})(y) = \begin{cases} \Delta_A^K(x) & \text{if } FK/A \cdot x = x \text{ and } KF/K \cdot y = \Delta_A^K(x), \\ 0 & \text{otherwise.} \end{cases}$$

The result now follows from the set-analogue of (10.24). □

11.10.5. Freeness of the bimonoids of h-faces and h-flats. Recall from Sections 6.1 and 6.3 the free bimonoid $\mathcal{T}(\mathsf{c})$ and the free commutative bimonoid $\mathcal{S}(\mathsf{c})$ on a comonoid c. Suppose h is a finite cocommutative set-bimonoid, with linearization $\mathsf{h} := \Bbbk\mathrm{h}$. Then, there is a natural isomorphism

$$^{\mathrm{h}}\Sigma \xrightarrow{\cong} \mathcal{T}(\mathsf{h}^*) \tag{11.12}$$

of bimonoids. On the A-component, it is defined by

$$^{\mathrm{h}}\Sigma[A] \xrightarrow{\cong} \bigoplus_{F:\, F \geq A} \mathsf{h}^*[F], \qquad \mathtt{H}_{F/A,x} \mapsto \mathtt{M}^*_{\Delta_A^F(x)}.$$

We are using $\mathtt{M}$ for the canonical basis of h, and $\mathtt{M}^*$ for the dual basis. Naturality means that for any morphism $f : \mathrm{h} \to \mathrm{k}$ of set-bimonoids, the diagram

$$\begin{tikzcd} ^{\mathrm{k}}\Sigma \arrow[r, "\cong"] \arrow[d] & \mathcal{T}(\mathsf{k}^*) \arrow[d] \\ ^{\mathrm{h}}\Sigma \arrow[r, "\cong"'] & \mathcal{T}(\mathsf{h}^*) \end{tikzcd} \tag{11.13}$$

commutes. Similarly, there is a natural isomorphism

$$^{\mathrm{h}}\Pi \xrightarrow{\cong} \mathcal{S}(\mathsf{h}^*) \tag{11.14}$$

of bimonoids such that the natural transformation $^{-}\Sigma \twoheadrightarrow {}^{-}\Pi$ corresponds to the abelianization $\mathcal{T} \to \mathcal{S}$ (Section 6.6).

These observations give alternative proofs of Propositions 11.44 and 11.45 in the finite-dimensional case.

Continuing further, observe that the map

$$\mathsf{h}^* \to \mathrm{end}^{\times}(\mathsf{h}), \qquad \mathtt{M}^*_x \mapsto \mathtt{K}_{x,x}$$

is a morphism of comonoids. Hence, by the universal property of $\mathcal{T}$, we obtain a morphism $\mathcal{T}(\mathsf{h}^*) \to \mathrm{end}^{\times}(\mathsf{h})$ of bimonoids. After identifying $\mathcal{T}(\mathsf{h}^*)$ with $^{\mathrm{h}}\Sigma$, this coincides with the morphism of Proposition 11.48 (and gives another proof for it).

11.10.6. Examples. When $\mathrm{h} = \mathrm{E}$ is the exponential set-species, there is a unique element in $\mathrm{E}[A]$, so a E-face local to A is the same as a face F greater than A. Similarly, a E-flat local to A is the same as a flat X containing A. One may check that

$$^{\mathrm{E}}\Sigma = \Sigma \quad \text{and} \quad ^{\mathrm{E}}\Pi = \Pi,$$

the bimonoids of faces and flats, respectively, and $^{\mathrm{E}}\Sigma \to {}^{\mathrm{E}}\Pi$ is the support map (7.84).

Now suppose $\mathrm{h} = \Gamma$ is the set-species of chambers. In this case,

$$F/A \cdot C/A = FC/A \quad \text{and} \quad FC = C \iff F \leq C.$$

Thus, a Γ-face local to A is a pair (F, C), where F is a face greater than A, C is a chamber, and $F \leq C$. This is the same as a top-nested face of $\mathcal{A}_A$. Similarly, a Γ-flat local to A is the same as a top-lune of $\mathcal{A}_A$. One may check that

$$^{\Gamma}\Sigma = \widehat{\mathsf{Q}} \quad \text{and} \quad ^{\Gamma}\Pi = \widehat{\Lambda},$$

the bimonoids of top-nested faces and top-lunes, respectively (Section 7.7), and $^{\Gamma}\Sigma \to {}^{\Gamma}\Pi$ is the map (7.118).

Now let h be any set-bimonoid. The unique map $\mathrm{h} \to \mathrm{E}$ is a morphism of set-bimonoids. If h is cocommutative, we therefore obtain a morphism of bimonoids

$$\Sigma = {}^{\mathrm{E}}\Sigma \to {}^{\mathrm{h}}\Sigma, \qquad \mathtt{H}_{F/A} \mapsto \sum_{x \in \mathrm{h}[A]:\, F/A \cdot x = x} \mathtt{H}_{F/A,x}. \tag{11.15}$$

The special case $\mathrm{h} = \Gamma$ yields the map $\Sigma \to \widehat{\mathsf{Q}}$ given in (7.120). Moreover, diagram (11.10) specializes to diagram (7.119).

Note that $\mathrm{end}^{\times}(\mathsf{E}) = \mathsf{E}$. Applying Proposition 11.48 to $\mathrm{h} = \mathrm{E}$, we obtain the canonical morphism $\Sigma \to \mathsf{E}$. Applying it to $\mathrm{h} = \Gamma$, we obtain a morphism $\widehat{\mathsf{Q}} \to \mathrm{end}^{\times}(\Gamma)$. It is discussed later in Section 15.5.6. More precisely, it is (15.42) for $q = 1$ (remembering that $\mathrm{end}^{\times}(\Gamma)$ is the same as $\mathsf{I}\Gamma$ by (15.39)).

Notes

Categorical equivalences between modules over monoid algebras and bimonoids in species. The categorical equivalences obtained in this chapter are all new. They make precise the remarks on [**21**, page 274]. These equivalences can be used to relate some of the results obtained here to those obtained in [**21**]. For instance: Proposition 10.13 and Exercise 10.15, item (i) on the action of Zie elements is equivalent to [**21**, Proposition 10.35]. Similarly, the part concerning $\mathcal{P}(\mathsf{h})$ of Proposition 10.26 is related to the first part of [**21**, Lemma D.54] specialized to the lattice of flats.

The left module over the q-Janus algebra discussed in Example 11.10 is defined in [**21**, Exercise 9.73].

Proposition 11.20 appears in [**21**, Propositions 13.4 and 13.6] (under the additional assumption that the modules are finite-dimensional). In this reference, the proof is phrased purely in the language of representation theory.

The result of Exercise 11.41 is at the basis of Lawvere's work on graphic toposes [**574**, Paragraph before Proposition 1], [**575**, First paragraph of Section 3]. An LRB is called a graphic monoid in these papers.

Karoubi envelopes. The Karoubi envelope construction appeared in an exercise in Freyd's book [**327**, page 61]. It is also present in his later book with Scedrov [**328**, Section 1.28]. In the context of additive categories, the construction is given in Karoubi's thesis [**512**, Lemma and Definition I.1.2.2] and later in his book [**513**, Proof of Theorem I.6.10]. Karoubi used it to give a proof of the Serre-Swan theorem [**512**, Section I.1.2, Example 1], [**513**, Theorem I.6.18]. The term 'Karoubi envelope' is used by Grothendieck and Verdier [**379**, Exposé I, Exercise 8.7.8 and Exposé IV, Exercise 7.5] and later by Lambek and Scott [**561**, Exercises on page 100]. Other terms in use are 'idempotent completion' and 'Cauchy completion'. A standard reference is Borceux's book [**132**, Chapter 6]. A brief discussion is given by Johnstone [**493**, pages 9 and 10]. It is also mentioned by Connes and Marcolli [**221**, Chapter 4, page 584]. For Cauchy completeness in the context of enriched categories, see the papers by Borceux and Dejean [**134**, Section 4], Carboni and Street [**185**, Section 3], Lawvere [**576**, Section 3]. The construction in the context of ∞-categories is given in Lurie's book [**616**, Section 5.1.4].

In the special case of semigroups considered in Section 11.8.1, the construction is implicit in Eilenberg's book [**283**, Exercise VIII.7.1] and Straubing's paper [**855**, Section 5.2]. It is discussed explicitly by Tilson [**881**, Section 16]. A more recent reference is the book by Rhodes and Steinberg [**779**, pages 279 and 280]. See also the paper by Costa and Steinberg [**224**, Section 3.2]. Theorem 11.29 (in the general categorical setting) is given in [**132**, Theorem 6.5.11], [**134**, Theorem 1, item (4)], [**493**, Corollary 1.1.9 on page 10]. In these references, the codomain is the category of sets. But it could be any category where idempotents split such as the category of vector spaces, see Lemma A.1.

The relevance of Karoubi envelopes to Hopf theory is pointed out in this work, possibly for the first time.

CHAPTER 12

Antipode

For any q-bimonoid h, the composite $\mu_A^F \Delta_A^F$ defines a map from $\mathsf{h}[A]$ to itself. By taking an alternating sum over all $F \geq A$, and doing this for each A yields a map of species from h to itself. We call this the antipode of h and refer to this formula as the Takeuchi formula. Up to signs, it equals the 0-logarithm of the identity map on h. There is also a commutative analogue of this formula for bicommutative bimonoids. We study interactions of the antipode with morphisms of bimonoids, the duality functor, bimonoid filtrations, the signature functor. The antipode is intimately related to the antipodal map on arrangements; this is brought forth by its interaction with op and cop constructions. We compute logarithm of the antipode map using the noncommutative Zaslavsky formula, and moreover relate it to logarithm of the identity map.

Understanding the cancellations in the Takeuchi formula for a given bimonoid is often a challenging combinatorial problem. We solve this problem for the bimonoids in Chapter 7. More generally, we provide cancellation-free formulas for the antipode of any bimonoid which arises from the universal constructions of Chapter 6. In a similar vein, for any set-bimonoid, this problem can be interpreted as the calculation of the Euler characteristic of a "cell complex". The descent and lune identities given in Section 1.7 provide motivating examples of such a calculation.

The antipode map is closely related to the Takeuchi elements associated to arrangements. More precisely, the A-component of the antipode map is the characteristic operation by the Takeuchi element of the arrangement over the support of A. This connection makes it possible to study the antipode using properties of the Takeuchi elements.

12.1. Takeuchi formula

We define the antipode of a q-bimonoid using the Takeuchi formula, and formulate the antipode problem. We also study how the antipode interacts with morphisms of bimonoids, the duality functor, bimonoid filtrations and the signature functor.

12.1.1. Antipode. Suppose $(\mathsf{h}, \mu, \Delta)$ is a q-bimonoid. We define its *antipode*

$$S : \mathsf{h} \to \mathsf{h}$$

as follows. (We will write S_{h} if we need to show the dependence on h.) The A-component is given by

$$S_A := \sum_{F:\, F \geq A} (-1)^{\dim(F)} \mu_A^F \Delta_A^F. \tag{12.1}$$

(This is a linear operator on $\mathsf{h}[A]$.)

We refer to (12.1) as the *Takeuchi formula*.

Lemma 12.1. *For any q-bimonoid h, the antipode $S : \mathsf{h} \to \mathsf{h}$ is a map of species. That is, the diagram*

$$\begin{tikzcd} \mathsf{h}[A] \arrow[r, "S_A"] \arrow[d, "\beta_{B,A}"'] & \mathsf{h}[A] \arrow[d, "\beta_{B,A}"] \\ \mathsf{h}[B] \arrow[r, "S_B"'] & \mathsf{h}[B] \end{tikzcd} \tag{12.2}$$

commutes.

Proof. For faces A and B of the same support, and $F \geq A$, the diagram

$$\begin{tikzcd} \mathsf{h}[A] \arrow[r, "\Delta_A^F"] \arrow[d, "\beta_{B,A}"'] & \mathsf{h}[F] \arrow[r, "\mu_A^F"] \arrow[d, "\beta_{G,F}"] & \mathsf{h}[A] \arrow[d, "\beta_{B,A}"] \\ \mathsf{h}[B] \arrow[r, "\Delta_B^G"'] & \mathsf{h}[G] \arrow[r, "\mu_B^G"'] & \mathsf{h}[B] \end{tikzcd}$$

commutes, with $G := BF$. This follows from naturality of the product (2.8) and coproduct (2.10). Multiplying by $(-1)^{\dim(F)}$, and summing over all $F \geq A$ yields (12.2). $\square$

An equivalent expression for the Takeuchi formula using Cauchy powers is given below.

$$S_A = (-1)^{\dim(A)} \sum_{k \geq 0} (-1)^k \mu_A^k \Delta_A^k, \tag{12.3}$$

with Δ^k and μ^k as in (5.2) and (5.3). In terms of the maps (5.9), this formula can be further rewritten as

$$S_A = (-1)^{\dim(A)} \sum_{k \geq 1} (-1)^{k-1} (\mathrm{id}^k)_A. \tag{12.4}$$

Another expression for the Takeuchi formula is

$$S_A = (-1)^{\dim(A)} \mathbf{log}_0(\mathrm{id})_A, \tag{12.5}$$

where the rhs refers to the 0-logarithm of the identity map (9.41).

The interactions of the antipode with the product and coproduct are studied in Section 12.2.

12.1.2. Antipode problem. In specific examples, cancellations frequently take place in the Takeuchi formula (12.1). Understanding these cancellations is often a challenging combinatorial problem. The antipode problem asks for an explicit, cancellation-free, formula for the antipode of a given q-bimonoid h. The problem may be formulated as follows: given a linear basis of the vector space $\mathsf{h}[A]$, we search for the structure constants of S_A on this basis. If h is the linearization of a set-bimonoid, then each component $\mathsf{h}[A]$ comes equipped with a canonical basis. In this or other cases, one may also be interested in other linear bases of $\mathsf{h}[A]$, and the corresponding structure constants.

12.1.3. Interaction with morphisms of bimonoids.

Lemma 12.2. *Let* $(\mathsf{h}, \mu, \Delta)$ *and* $(\mathsf{h}', \mu', \Delta')$ *be q-bimonoids with antipodes S and S', respectively, and $f : \mathsf{h} \to \mathsf{h}'$ be a morphism of q-bimonoids. Then the diagram*

$$
\begin{tikzcd}
\mathsf{h}[A] \arrow[r, "S_A"] \arrow[d, "f_A"'] & \mathsf{h}[A] \arrow[d, "f_A"] \\
\mathsf{h}'[A] \arrow[r, "S'_A"'] & \mathsf{h}'[A]
\end{tikzcd}
\tag{12.6}
$$

commutes. In other words, a morphism of q-bimonoids necessarily commutes with antipodes.

Proof. For any face $F \geq A$, since f is a morphism of monoids and comonoids, by (2.9) and (2.11), the diagram

$$
\begin{tikzcd}
\mathsf{h}[A] \arrow[r, "\Delta_A^F"] \arrow[d, "f_A"'] & \mathsf{h}[F] \arrow[r, "\mu_A^F"] \arrow[d, "f_F"] & \mathsf{h}[A] \arrow[d, "f_A"] \\
\mathsf{h}'[A] \arrow[r, "(\Delta')_A^F"'] & \mathsf{h}'[F] \arrow[r, "(\mu')_A^F"'] & \mathsf{h}'[A]
\end{tikzcd}
$$

commutes. Multiplying by $(-1)^{\dim(F)}$, and summing over all $F \geq A$ yields (12.6). $\square$

Exercise 12.3. Deduce Lemma 12.2 as a consequence of formula (12.5) and Exercise 9.102. More precisely, check that both directions in diagram (12.6) equal $(-1)^{\dim(A)} \mathbf{log}_0(f)_A$.

12.1.4. Interaction with the duality functor. Recall the duality functor from Section 2.9. For a q-bimonoid h, applying the duality functor to the antipode $S_{\mathsf{h}} : \mathsf{h} \to \mathsf{h}$ yields a map $\mathsf{h}^* \to \mathsf{h}^*$. We claim that this is in fact the antipode of h^*. In other words:

Lemma 12.4. *For a q-bimonoid* h*,*

$$S_{\mathsf{h}^*} = (S_{\mathsf{h}})^*. \tag{12.7}$$

Proof. Recall that the product and coproduct components of h^* are obtained by dualizing those of h. We calculate:

$$\begin{aligned}(S_{\mathsf{h}^*})_A &= \sum_{F\colon F\geq A} (-1)^{\dim(F)} (\Delta_A^F)^* (\mu_A^F)^* \\ &= \sum_{F\colon F\geq A} (-1)^{\dim(F)} (\mu_A^F \Delta_A^F)^* \\ &= \Big(\sum_{F\colon F\geq A} (-1)^{\dim(F)} \mu_A^F \Delta_A^F \Big)^* \\ &= (S_{\mathsf{h}})_A^*. \qquad \square\end{aligned}$$

12.1.5. Interaction with bimonoid filtrations. Recall from Section 5.2 the notions of a filtered q-bimonoid and its associated graded q-bimonoid.

Lemma 12.5. *Let h be a filtered q-bimonoid (with either ascending or descending filtration), and with antipode S_{h}. Then S_{h} preserves the filtration of h, and moreover,*

$$S_{\mathrm{gr}(\mathsf{h})} = \mathrm{gr}(S_{\mathsf{h}}),$$

that is, the antipode of $\mathrm{gr}(\mathsf{h})$ is gr applied to the antipode of h.

Proof. Let h_i denote the components of the filtration of h. By (5.14), we see that $\mu_A^F \Delta_A^F$ sends each h_i to itself. The Takeuchi formula (12.1) now implies that S_{h} indeed preserves the filtration. This proves the first claim. The second claim follows from the fact that the structure maps of $\mathrm{gr}(\mathsf{h})$ are obtained by applying gr to the structure maps of h. $\square$

Exercise 12.6. Deduce Lemma 12.5 by putting $f = \mathrm{id}$ in Exercise 5.9 and then using formula (12.4).

Recall from Section 5.8 that the primitive and decomposable filtrations of any q-bimonoid h turn it into a filtered q-bimonoid. Hence, by specializing Lemma 12.5, we obtain:

Lemma 12.7. *Let h be a q-bimonoid with antipode S. Then S preserves the primitive filtration and decomposable filtration of h, and moreover, the antipode of $\mathrm{gr}_{\mathcal{P}}(\mathsf{h})$ is given by $\mathrm{gr}_{\mathcal{P}}(S)$, and that of $\mathrm{gr}_{\mathcal{D}}(\mathsf{h})$ is given by $\mathrm{gr}_{\mathcal{D}}(S)$.*

Exercise 12.8. Let h be a q-bimonoid with antipode S. Check that: If $x \in \mathcal{P}(\mathsf{h})[A]$, then $S_A(x) = (-1)^{\dim(A)}x$. Dually, for $x \in \mathsf{h}[A]$, the elements $S_A(x)$ and $(-1)^{\dim(A)}x$ have the same image in the quotient $\mathcal{Q}(\mathsf{h})[A]$.

12.1.6. Interaction with the signature functor. Recall the signature functor on species from Section 8.10. For a q-bimonoid h, applying the signature functor to the antipode $S_{\mathsf{h}} : \mathsf{h} \to \mathsf{h}$ yields a map $\mathsf{h}^- \to \mathsf{h}^-$. We claim that this is in fact the antipode of h^-. In other words:

Lemma 12.9. *For a q-bimonoid h,*

$$S_{\mathsf{h}^-} = (S_{\mathsf{h}})^-. \tag{12.8}$$

PROOF. In the calculation below, there are two pairs of Δ_A^F and μ_A^F. One refers to the product and coproduct components of h, while the other refers to those of the signed exponential bimonoid E^- given in (7.13).

$$\begin{aligned}(S_{\mathsf{h}^-})_A &= \sum_{F\colon F\geq A} (-1)^{\dim(F)} (\mu_A^F \otimes \mu_A^F)(\Delta_A^F \otimes \Delta_A^F) \\ &= \sum_{F\colon F\geq A} (-1)^{\dim(F)} (\mu_A^F\Delta_A^F \otimes \mu_A^F\Delta_A^F) \\ &= \sum_{F\colon F\geq A} (-1)^{\dim(F)} (\mu_A^F\Delta_A^F \otimes \mathrm{id}) \\ &= \Big(\sum_{F\colon F\geq A} (-1)^{\dim(F)} \mu_A^F\Delta_A^F\Big) \otimes \mathrm{id} \\ &= (S_{\mathsf{h}})_A \otimes \mathrm{id}\,.\end{aligned}$$

The third step used Exercise 7.3. $\square$

12.2. Interaction with op and cop constructions

We now study the interaction of the antipode of a bimonoid with its product and coproduct. For a (co)commutative bimonoid, the antipode is a morphism of (co)monoids. This fact generalizes suitably to arbitrary bimonoids and further to q-bimonoids for $q \neq 0$; the op and cop constructions introduced in Section 2.10 intervene in this result. The antipode opposition lemma plays a key role in the proof. We also use these ideas to establish an involutive property and a group-like property of the antipode.

12.2.1. (Co)commutative bimonoids. We begin by showing that the antipode of a (co)commutative bimonoid is a morphism of (co)monoids.

Lemma 12.10. *For a commutative bimonoid* h*, the antipode* $S : \mathsf{h} \to \mathsf{h}$ *is a morphism of monoids. That is, for any faces* $F \geq A$*, the diagram*

(12.9)
$$\begin{tikzcd} \mathsf{h}[F] \arrow[r, "S_F"] \arrow[d, "\mu_A^F"'] & \mathsf{h}[F] \arrow[d, "\mu_A^F"] \\ \mathsf{h}[A] \arrow[r, "S_A"'] & \mathsf{h}[A] \end{tikzcd}$$

commutes. Dually, for a cocommutative bimonoid h*, the antipode* $S : \mathsf{h} \to \mathsf{h}$ *is a morphism of comonoids. That is, for any faces* $F \geq A$*, the diagram*

(12.10)
$$\begin{tikzcd} \mathsf{h}[F] \arrow[r, "S_F"] & \mathsf{h}[F] \\ \mathsf{h}[A] \arrow[r, "S_A"'] \arrow[u, "\Delta_A^F"] & \mathsf{h}[A] \arrow[u, "\Delta_A^F"'] \end{tikzcd}$$

commutes.

PROOF. We prove the first statement.

$$\begin{aligned}
S_A \mu_A^F &= \sum_{G\colon G \geq A} (-1)^{\dim(G)} \mu_A^G \Delta_A^G \mu_A^F \\
&= \sum_{G\colon G \geq A} (-1)^{\dim(G)} \mu_A^G \mu_G^{GF} \beta_{GF,FG} \Delta_F^{FG} \\
&= \sum_{G\colon G \geq A} (-1)^{\dim(G)} \mu_A^{FG} \Delta_F^{FG} \\
&= \mu_A^F \big(\sum_{G\colon G \geq A} (-1)^{\dim(G)} \mu_F^{FG} \Delta_F^{FG} \big) \\
&= \mu_A^F \Big(\sum_{K\colon K \geq F} \big(\sum_{\substack{G\colon G \geq A \\ FG = K}} (-1)^{\dim(G)} \big) \mu_F^K \Delta_F^K \Big) \\
&= \mu_A^F \big(\sum_{K\colon K \geq F} (-1)^{\dim(K)} \mu_F^K \Delta_F^K \big) \\
&= \mu_A^F S_F.
\end{aligned}$$

The first step and last step used (12.1). The second step used the bimonoid axiom (2.12). The third and fourth steps used associativity (2.8) and commutativity (2.17). The sixth step was crucial and used the lune identity (1.77a) for the arrangement $\mathcal{A}_A$. $\square$

12.2.2. Antipode opposition lemma. Recall that every face F has an opposite face $\overline{F}$. Locally, in the star of a face A, the opposite of F is given by $A\overline{F}$. The antipode is intimately connected to this notion of opposite.

Lemma 12.11. *Let* $(\mathsf{h}, \mu, \Delta)$ *be a q-bimonoid. Then, for any faces* $A \leq F$,

$$\sum_{G\colon G \geq A} (-1)^{\dim(G)} \mu_A^G \Delta_A^G \mu_A^F = \sum_{G\colon G \geq A\overline{F}} (-1)^{\dim(G)} \mu_A^G \Delta_A^G \mu_A^F, \tag{12.11}$$

and, dually,

$$\sum_{G\colon G \geq A} (-1)^{\dim(G)} \Delta_A^F \mu_A^G \Delta_A^G = \sum_{G\colon G \geq A\overline{F}} (-1)^{\dim(G)} \Delta_A^F \mu_A^G \Delta_A^G. \tag{12.12}$$

Observe that in both (12.11) and (12.12), there are fewer terms in the rhs than in the lhs. In the rhs, we only sum over those faces G which are greater than $A\overline{F}$.

PROOF. We prove (12.11). We calculate:

$$\begin{aligned}
\sum_{G\colon G \geq A} (-1)^{\dim(G)} \mu_A^G \Delta_A^G \mu_A^F &= \sum_{G\colon G \geq A} (-1)^{\dim(G)} \mu_A^{GF} (\beta_q)_{GF,FG} \Delta_F^{FG} \\
&= \sum_{K\colon K \geq A} \Big(\sum_{\substack{G\colon G \geq A \\ GF = K}} (-1)^{\dim(G)} \Big) \mu_A^K (\beta_q)_{K,FK} \Delta_F^{FK} \\
&= \sum_{K\colon K \geq A\overline{F}} (-1)^{\dim(K)} \mu_A^K (\beta_q)_{K,FK} \Delta_F^{FK}
\end{aligned}$$

$$= \sum_{K:\, K \geq A\overline{F}} (-1)^{\dim(K)} \mu_A^K \Delta_A^K \mu_A^F.$$

The crucial step was the evaluation of the sum in parenthesis. It used the descent identity (1.76a) for the arrangement $\mathcal{A}_A$. The remaining steps used associativity (2.8) and the q-bimonoid axiom (2.33). □

We refer to Lemma 12.11 as the *antipode opposition lemma*. A conceptual interpretation of this result is given later in Section 12.6.4.

12.2.3. q-bimonoids. We now generalize Lemma 12.10 to q-bimonoids for $q \neq 0$. This makes use of the op and cop constructions in Proposition 2.58. Recall that if h is a q-bimonoid, then ${}^{\mathrm{op}}\mathsf{h}$ and $\mathsf{h}^{\mathrm{cop}}$ are q^{-1}-bimonoids, and ${}^{\mathrm{op}}({}^{\mathrm{op}}\mathsf{h}) = \mathsf{h}$ and $(\mathsf{h}^{\mathrm{cop}})^{\mathrm{cop}} = \mathsf{h}$.

Lemma 12.12. *For $q \neq 0$, let h be a q-bimonoid with antipode S. Then:*

(1) *$S : {}^{\mathrm{op}}\mathsf{h} \to \mathsf{h}$ is a morphism of monoids. Equivalently, for any faces $F \geq A$, the diagram*

$$\begin{tikzcd} \mathsf{h}[F] \arrow[rr, "\mu_A^F"] \arrow[d, "S_F"'] & & \mathsf{h}[A] \arrow[d, "S_A"] \\ \mathsf{h}[F] \arrow[r, "(\beta_q)_{A\overline{F},F}"'] & \mathsf{h}[A\overline{F}] \arrow[r, "\mu_A^{A\overline{F}}"'] & \mathsf{h}[A] \end{tikzcd} \tag{12.13}$$

commutes.

(2) *$S : \mathsf{h} \to \mathsf{h}^{\mathrm{cop}}$ is a morphism of comonoids. Equivalently, for any faces $F \geq A$, the diagram*

$$\begin{tikzcd} \mathsf{h}[A] \arrow[rr, "\Delta_A^F"] \arrow[d, "S_A"'] & & \mathsf{h}[F] \arrow[d, "S_F"] \\ \mathsf{h}[A] \arrow[r, "\Delta_A^{A\overline{F}}"'] & \mathsf{h}[A\overline{F}] \arrow[r, "(\beta_q^{-1})_{F,A\overline{F}}"'] & \mathsf{h}[F] \end{tikzcd} \tag{12.14}$$

commutes.

The diagram encoding the fact that $S : {}^{\mathrm{op}}\mathsf{h} \to \mathsf{h}$ is a morphism of monoids is equivalent to (12.13) in view of Exercise 2.55.

Proof. We check that diagram (12.13) commutes. Going across and down in (12.13) is the lhs of (12.11). So we begin the calculation from the rhs of (12.11).

$$\begin{aligned} \sum_{G:\, G \geq A\overline{F}} (-1)^{\dim(G)} \mu_A^G \Delta_A^G \mu_A^F &= \sum_{G:\, G \geq A\overline{F}} (-1)^{\dim(G)} \mu_A^G \Delta_{A\overline{F}}^G \Delta_A^{A\overline{F}} \mu_A^F \\ &= \sum_{G:\, G \geq A\overline{F}} (-1)^{\dim(G)} \mu_A^G \Delta_{A\overline{F}}^G (\beta_q)_{A\overline{F},F} \\ &= \sum_{G:\, G \geq A\overline{F}} (-1)^{\dim(G)} \mu_A^{A\overline{F}} \mu_{A\overline{F}}^G \Delta_{A\overline{F}}^G (\beta_q)_{A\overline{F},F} \end{aligned}$$

$$= \sum_{H\colon H \geq F} (-1)^{\dim(H)} \mu_A^{A\overline{F}} (\beta_q)_{A\overline{F},F} \mu_F^H \Delta_F^H.$$

The first step used coassociativity of Δ, the second step used (2.35), the third step used associativity of μ, and the final step used naturality, first for Δ and then for μ. The faces $A\overline{F}$ and F have the same support, and the indices G and H correspond to each other under the bijection of Lemma 1.6. $\square$

Put $q = 1$ in Lemma 12.12. In addition, if h is commutative, then (12.13) specializes to (12.9). Similarly, if h is cocommutative, then (12.14) specializes to (12.10). Thus, we recover Lemma 12.10.

A conceptual approach to Lemma 12.12 is given later in Exercise 12.60.

Exercise 12.13. Use Exercise 2.57 to check that for finite-dimensional q-bimonoids, the two statements in Lemma 12.12 can be deduced from each other.

Exercise 12.14. Let h be a q-bimonoid for $q \neq 0$. Use Lemma 12.12 in conjunction with Lemma 5.12 and Exercise 5.14 to deduce that the antipode of h preserves the primitive filtration of h. Do the same for the decomposable filtration. (A more general result was obtained in Lemma 12.7.)

Lemma 12.15. *For $q \neq 0$, let h be a q-bimonoid. Then*

(1) *The antipodes of h, $\mathsf{h}^{\mathrm{op,cop}}$, ${}^{\mathrm{op,cop}}\mathsf{h}$ are all equal.*
(2) *The antipode is a morphism of q-bimonoids in two ways: $\mathsf{h} \to \mathsf{h}^{\mathrm{op,cop}}$ and ${}^{\mathrm{op,cop}}\mathsf{h} \to \mathsf{h}$.*

Proof. The first part follows directly from the Takeuchi formula (12.1). (This is tangible from the notations with β_q and β_q^{-1} canceling each other.) The second part follows from Lemma 12.12. $\square$

12.2.4. Involutive property.

Lemma 12.16. *For $q \neq 0$, let h be a q-bimonoid. Let S_{h} and $S_{{}^{\mathrm{op}}\mathsf{h}}$ denote the antipodes of h and ${}^{\mathrm{op}}\mathsf{h}$, respectively. Then S_{h} and $S_{{}^{\mathrm{op}}\mathsf{h}}$ are inverses under composition, that is,*

$$S_{\mathsf{h}} S_{{}^{\mathrm{op}}\mathsf{h}} = \mathrm{id} = S_{{}^{\mathrm{op}}\mathsf{h}} S_{\mathsf{h}}.$$

In particular, S_{h} is invertible. The same result holds replacing ${}^{\mathrm{op}}\mathsf{h}$ with $\mathsf{h}^{\mathrm{cop}}$.

Proof. Let us first prove $S_{\mathsf{h}} S_{{}^{\mathrm{op}}\mathsf{h}} = \mathrm{id}$. We compute:

$$\begin{aligned}
(S_{\mathsf{h}})_A (S_{{}^{\mathrm{op}}\mathsf{h}})_A &= \sum_{F \geq A,\, H \geq A} (-1)^{\dim(F)} (-1)^{\dim(H)} \mu_A^F \Delta_A^F \mu_A^{A\overline{H}} (\beta_q^{-1})_{A\overline{H},H} \Delta_A^H \\
&= \sum_{F \geq H \geq A} (-1)^{\dim(F)} (-1)^{\dim(H)} \mu_A^F \Delta_A^F \mu_A^{A\overline{H}} (\beta_q^{-1})_{A\overline{H},H} \Delta_A^H \\
&= \sum_{F \geq H \geq A} (-1)^{\dim(F)} (-1)^{\dim(H)} \mu_A^F \Delta_H^F \Delta_A^H \mu_A^{A\overline{H}} (\beta_q^{-1})_{A\overline{H},H} \Delta_A^H \\
&= \sum_{F \geq H \geq A} (-1)^{\dim(F)} (-1)^{\dim(H)} \mu_A^F \Delta_H^F (\beta_q)_{H,A\overline{H}} (\beta_q^{-1})_{A\overline{H},H} \Delta_A^H
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{F \geq H \geq A} (-1)^{\dim(F)} (-1)^{\dim(H)} \mu_A^F \Delta_A^F \\
&= \mathrm{id}\,.
\end{aligned}
$$

In the above calculation, the sums are over both F and H, while A remains fixed. The first step used the Takeuchi formula (12.1) both for S_{h} and $S_{^{\mathrm{op}}\mathsf{h}}$. The second step used (12.11) to reduce the indexing set. The third step used coassociativity, while the fourth used (2.35). In the next step, β_q and β_q^{-1} canceled each other out. In the last step, we first summed over H using (1.73) which forced $F = A$.

A similar calculation using (12.12) shows that $S_{^{\mathrm{op}}\mathsf{h}} S_{\mathsf{h}} = \mathrm{id}$. Alternatively, this can be deduced from $S_{\mathsf{h}} S_{^{\mathrm{op}}\mathsf{h}} = \mathrm{id}$ by symmetry since $^{\mathrm{op}}(^{\mathrm{op}}\mathsf{h}) = \mathsf{h}$.

The calculations for $\mathsf{h}^{\mathrm{cop}}$ are identical to those for $^{\mathrm{op}}\mathsf{h}$. ☐

Lemma 12.17. *Let h be a bimonoid which is either commutative or cocommutative. Then the antipode S of h is an involution under composition, that is, $SS = \mathrm{id}$.*

Proof. If h is commutative, then $^{\mathrm{op}}\mathsf{h} = \mathsf{h}$, while if h is cocommutative, then $\mathsf{h}^{\mathrm{cop}} = \mathsf{h}$. Now apply Lemma 12.16. ☐

12.2.5. Group-like property.

Lemma 12.18. *Let $(\mathsf{h}, \mu, \Delta)$ be a bimonoid with antipode S. Then, for any $A \leq F$, the diagram*

$$
\begin{tikzcd}
\mathsf{h}[F] \arrow[r, "S_F"] \arrow[d, "\mu_A^F"'] & \mathsf{h}[F] \\
\mathsf{h}[A] \arrow[r, "S_A"'] & \mathsf{h}[A] \arrow[u, "\Delta_A^F"']
\end{tikzcd}
$$

commutes.

Proof. We compose diagram (12.13) with Δ_A^F, and calculate:

$$
\Delta_A^F S_A \mu_A^F = \Delta_A^F \mu_A^{A\overline{F}} \beta_{A\overline{F},F} S_F = \beta_{F,A\overline{F}} \beta_{A\overline{F},F} S_F = S_F.
$$

The second step used (2.13). In the special case when h is commutative, one can use the simplified diagram (12.9) instead of (12.13). Similarly, when h is cocommutative, one can use diagram (12.10) instead of (12.13). ☐

12.3. Commutative Takeuchi formula

For a bicommutative bimonoid, the Takeuchi formula can be expressed using flats. We call this the commutative Takeuchi formula.

12.3.1. Antipode. Let h be a bicommutative bimonoid. Recall from Proposition 2.22 that h can be conveniently described using the structure maps $\mu_{\mathrm{Z}}^{\mathrm{X}}$ and $\Delta_{\mathrm{Z}}^{\mathrm{X}}$. In these terms:

Lemma 12.19. *The antipode S of a bicommutative bimonoid h is given by*

$$S_{\mathrm{Z}} = \sum_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Z}} (-1)^{\dim(\mathrm{X})} c_{\mathrm{Z}}^{\mathrm{X}}\, \mu_{\mathrm{Z}}^{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}, \tag{12.15}$$

where $c_{\mathrm{X}}^{\mathrm{Y}}$ is the number of chambers in the arrangement $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$.

Proof. This is a reformulation of (12.1): Fix a face A of support Z. Then each term $\mu_A^F \Delta_A^F$ in (12.1) translates to $\mu_{\mathrm{Z}}^{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}$, where $\mathrm{X} = \mathrm{s}(F)$. Further, the number of faces F greater than A and of support X is precisely $c_{\mathrm{Z}}^{\mathrm{X}}$. □

We refer to (12.15) as the *commutative Takeuchi formula*.

Lemma 12.20. *For a bicommutative bimonoid h, the antipode $S : \mathsf{h} \to \mathsf{h}$ is a morphism of bimonoids. That is, for any flats $\mathrm{X} \geq \mathrm{Z}$, the diagrams*

$$\begin{tikzcd} \mathsf{h}[\mathrm{X}] \arrow[r, "S_{\mathrm{X}}"] \arrow[d, "\mu_{\mathrm{Z}}^{\mathrm{X}}"'] & \mathsf{h}[\mathrm{X}] \arrow[d, "\mu_{\mathrm{Z}}^{\mathrm{X}}"] \\ \mathsf{h}[\mathrm{Z}] \arrow[r, "S_{\mathrm{Z}}"'] & \mathsf{h}[\mathrm{Z}] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{h}[\mathrm{X}] \arrow[r, "S_{\mathrm{X}}"] & \mathsf{h}[\mathrm{X}] \\ \mathsf{h}[\mathrm{Z}] \arrow[r, "S_{\mathrm{Z}}"'] \arrow[u, "\Delta_{\mathrm{Z}}^{\mathrm{X}}"] & \mathsf{h}[\mathrm{Z}] \arrow[u, "\Delta_{\mathrm{Z}}^{\mathrm{X}}"'] \end{tikzcd} \tag{12.16}$$

commute.

Proof. This follows from Lemma 12.10. Alternatively, one can proceed directly with (12.15), and use formula (1.78a). □

Lemma 12.21. *Let h be a bicommutative bimonoid with antipode S. Then, for any $\mathrm{X} \geq \mathrm{Z}$, the diagram*

$$\begin{tikzcd} \mathsf{h}[\mathrm{X}] \arrow[r, "S_{\mathrm{X}}"] \arrow[d, "\mu_{\mathrm{Z}}^{\mathrm{X}}"'] & \mathsf{h}[\mathrm{X}] \\ \mathsf{h}[\mathrm{Z}] \arrow[r, "S_{\mathrm{Z}}"'] & \mathsf{h}[\mathrm{Z}] \arrow[u, "\Delta_{\mathrm{Z}}^{\mathrm{X}}"'] \end{tikzcd}$$

commutes.

Proof. Post-compose the first diagram in (12.16) with $\Delta_{\mathrm{Z}}^{\mathrm{X}}$, or pre-compose the second diagram with $\mu_{\mathrm{Z}}^{\mathrm{X}}$, and use $\Delta_{\mathrm{Z}}^{\mathrm{X}} \mu_{\mathrm{Z}}^{\mathrm{X}} = \mathrm{id}$, see the second diagram in (2.27). □

12.3.2. Signed analogue. Now, let h be a signed bicommutative signed bimonoid. We work with the formulation given by Proposition 2.37.

Lemma 12.22. *The antipode S of a signed bicommutative signed bimonoid h is given by formula (12.15).*

The argument is similar to the proof of Lemma 12.19.

12.3.3. Unification via partial-support relations. Let $\sim$ be a partial-support relation on faces. Let h be a $\sim$-bicommutative bimonoid. We work with the formulation given by Proposition 2.85 in terms of partial-flats.

Lemma 12.23. *The antipode S of a $\sim$-bicommutative bimonoid h is given by*

$$S_{\mathrm{z}} = \sum_{\mathrm{x}\colon \mathrm{x} \geq \mathrm{z}} (-1)^{\dim(\mathrm{x})} c_{\mathrm{z}}^{\mathrm{x}} \mu_{\mathrm{z}}^{\mathrm{x}} \Delta_{\mathrm{z}}^{\mathrm{x}}, \tag{12.17}$$

where the coefficient $c_{\mathrm{z}}^{\mathrm{x}}$ is the number of faces in the partial-flat x *which are greater than a fixed face in* z*.*

Proof. The argument is the same as for Lemma 12.19. □

Observe that formula (12.17) specializes to the Takeuchi formula (12.1) when $\sim$ is finest, and to the commutative Takeuchi formula (12.15) when $\sim$ is coarsest.

12.4. Logarithm of the antipode

We compute any logarithm of the antipode of a q-bimonoid h. It agrees, up to signs, with the opposite logarithm of the identity map on h. The noncommutative Zaslavsky formula plays a key role in the calculation.

12.4.1. *q*-bimonoids. Recall from (9.3b) the logarithm of a map from a comonoid to a monoid. It depends on the choice of a noncommutative Möbius function $\boldsymbol{\mu}$. Any logarithm of the antipode of a q-bimonoid h can be expressed as follows.

Lemma 12.24. *For a q-bimonoid h,*

$$\mathbf{log}(S_{\mathsf{h}})_A = (-1)^{\dim(A)} \overline{\mathbf{log}}(\mathrm{id}_{\mathsf{h}})_A = (-1)^{\dim(A)} \sum_{F\colon F \geq A} \boldsymbol{\mu}(A, A\overline{F})\, \mu_A^F \Delta_A^F. \tag{12.18a}$$

In the special case that $\boldsymbol{\mu}$ is projective,

$$\mathbf{log}(S_{\mathsf{h}})_A = (-1)^{\dim(A)}\, \mathbf{log}(\mathrm{id}_{\mathsf{h}})_A = (-1)^{\dim(A)} \sum_{F\colon F \geq A} \boldsymbol{\mu}(A, F)\, \mu_A^F \Delta_A^F. \tag{12.18b}$$

Proof. We calculate:

$$\begin{aligned}
\mathbf{log}(S)_A &= \sum_{F\colon F \geq A} \boldsymbol{\mu}(A, F)\, \mu_A^F S_F \Delta_A^F \\
&= \sum_{F\colon F \geq A} \; \sum_{G\colon G \geq F} (-1)^{\dim(G)} \boldsymbol{\mu}(A, F)\, \mu_A^F \mu_F^G \Delta_F^G \Delta_A^F \\
&= \sum_{G\colon G \geq A} (-1)^{\dim(G)} \Big(\sum_{F\colon G \geq F \geq A} \boldsymbol{\mu}(A, F) \Big) \mu_A^G \Delta_A^G \\
&= (-1)^{\dim(A)} \sum_{G\colon G \geq A} \boldsymbol{\mu}(A, A\overline{G})\, \mu_A^G \Delta_A^G.
\end{aligned}$$

The first step used definition (9.3b). The second step used the Takeuchi formula (12.1). In the third step, we used (co)associativity, and then interchanged the sums. The last step used identity (1.87a). This proves (12.18a);

the second equality is by (9.13). Formula (12.18b) follows as a special case. (Note very carefully that the bimonoid axiom did not enter the above calculation, so it works for any q-bimonoid.)

Alternatively: Let s and t be elements of the lune-incidence algebra defined by $s(A,F) := 1$ and $t(A,F) := (-1)^{\mathrm{rk}(F/A)}\boldsymbol{\mu}(A, A\overline{F})$. The noncommutative Zaslavsky formula (1.89) implies $s = \boldsymbol{\zeta} t$, and hence $t = \boldsymbol{\mu} s$. Now let $f : \mathsf{h} \to \mathsf{h}$ be the map of species which on the F component is scalar multiplication by $(-1)^{\dim(F)}$. Observe from (12.1) that $S = s \circ f$, with the latter as in (9.1). Hence,

$$\mathbf{log}(S) = \mathbf{log}(s \circ f) = \boldsymbol{\mu} \circ (s \circ f) = (\boldsymbol{\mu} s) \circ f = t \circ f,$$

which equals the rhs of (12.18a). □

Exercise 12.25. Recall the op and cop constructions from Proposition 2.58. Use Exercise 9.8 to deduce that: For a q-bimonoid h for $q \neq 0$,

$$\mathbf{log}(S_{\mathrm{op}_{\mathsf{h}}}) = \overline{\mathbf{log}}(S_{\mathsf{h}^{\mathrm{cop}}}) \quad \text{and} \quad \mathbf{log}(S_{\mathrm{op,cop}_{\mathsf{h}}}) = \overline{\mathbf{log}}(S_{\mathsf{h}}) = \mathbf{log}(S_{\mathsf{h}^{\mathrm{op,cop}}}).$$

(Alternatively, one can also combine Lemma 12.24 with Exercise 9.28.)

12.4.2. Bicommutative bimonoids. Recall from (9.15b) the logarithm of a map from a cocommutative comonoid to a commutative monoid. The logarithm of the antipode of a bicommutative bimonoid h can be expressed as follows.

Lemma 12.26. *For a bicommutative bimonoid* h*,*

$$\log(S_{\mathsf{h}})_{\mathrm{Z}} = (-1)^{\dim(\mathrm{Z})}\log(\mathrm{id}_{\mathsf{h}})_{\mathrm{Z}} = (-1)^{\dim(\mathrm{Z})} \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z},\mathrm{X})\, \mu_{\mathrm{Z}}^{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}} = \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} (-1)^{\dim(\mathrm{X})} |\mu(\mathrm{Z},\mathrm{X})|\, \mu_{\mathrm{Z}}^{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}. \tag{12.19}$$

The second equality is by (9.22). The third equality holds since the sign of $\mu(\mathrm{Z},\mathrm{X})$ equals $(-1)^{\mathrm{rk}(\mathrm{X}/\mathrm{Z})}$.

Proof. This can be seen as a special case of Lemma 12.24. Alternatively, one can proceed directly using flats. We calculate:

$$\begin{aligned}
\log(S)_{\mathrm{Z}} &= \sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{Z}} \mu(\mathrm{Z},\mathrm{Y})\, \mu_{\mathrm{Z}}^{\mathrm{Y}} S_{\mathrm{Y}} \Delta_{\mathrm{Z}}^{\mathrm{Y}} \\
&= \sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{Z}} \; \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Y}} (-1)^{\dim(\mathrm{X})} \mu(\mathrm{Z},\mathrm{Y})\, c_{\mathrm{Y}}^{\mathrm{X}}\, \mu_{\mathrm{Z}}^{\mathrm{Y}} \mu_{\mathrm{Y}}^{\mathrm{X}} \Delta_{\mathrm{Y}}^{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{Y}} \\
&= \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} (-1)^{\dim(\mathrm{X})} \Big(\sum_{\mathrm{Y}:\, \mathrm{X} \geq \mathrm{Y} \geq \mathrm{Z}} \mu(\mathrm{Z},\mathrm{Y})\, c_{\mathrm{Y}}^{\mathrm{X}} \Big) \mu_{\mathrm{Z}}^{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}} \\
&= (-1)^{\dim(\mathrm{Z})} \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{Z},\mathrm{X})\, \mu_{\mathrm{Z}}^{\mathrm{X}} \Delta_{\mathrm{Z}}^{\mathrm{X}}.
\end{aligned}$$

The first step used definition (9.15b). The second step used the commutative Takeuchi formula (12.15). The last step used identity (1.97).

Alternatively: Let s and t be elements of the flat-incidence algebra defined by $s(\mathrm{Z},\mathrm{X}) := c_{\mathrm{Z}}^{\mathrm{X}}$ and $t(\mathrm{Z},\mathrm{X}) := |\mu(\mathrm{Z},\mathrm{X})|$. The Zaslavsky formula (1.84) implies $s = \zeta t$, and hence $t = \mu s$. Now let $f : \mathsf{h} \to \mathsf{h}$ be the map of species which on the X component is scalar multiplication by $(-1)^{\dim(\mathrm{X})}$. Observe from (12.15) that $S = s \circ f$, with the latter as in (9.14). Hence,

$$\log(S) = \log(s \circ f) = \mu \circ (s \circ f) = (\mu s) \circ f = t \circ f,$$

which equals the last expression in (12.19). □

Second proof. Another way to prove (12.19) is as follows.

$$\log(S)_{\mathrm{Z}} = S_{\mathrm{Z}} \log(\mathrm{id})_{\mathrm{Z}} = (-1)^{\dim(\mathrm{Z})} \log(\mathrm{id})_{\mathrm{Z}}.$$

The first step used Exercise 9.39. It applies since the antipode is a morphism of monoids by Lemma 12.20. The second step used Exercise 12.8. It applies since $\log(\mathrm{id})$ maps into the primitive part by Proposition 9.47. □

Exercise 12.27. Let h be a bimonoid which is either commutative or cocommutative. Prove formula (12.18a) along the lines of the second proof above. (Use Lemma 9.10, Lemma 12.12, Exercise 12.8, Proposition 9.17, Exercise 9.29.)

12.5. Examples

We calculate the antipodes of the bimonoids discussed in Chapter 7. The strategy is to substitute the product and coproduct formulas of the given bimonoid into the Takeuchi formula, and understand the resulting cancellations using the descent, lune and related identities from Section 1.7.

12.5.1. Exponential bimonoid.

We begin with the exponential bimonoid E from Section 7.2.

Proposition 12.28. *The antipode $S : \mathsf{E} \to \mathsf{E}$, on the A-component, is given by*

$$S_A(\mathtt{H}_A) = (-1)^{\dim(\mathcal{A})}\, \mathtt{H}_A. \tag{12.20}$$

Proof. Starting with the Takeuchi formula (12.1),

$$S_A(\mathtt{H}_A) = \Big(\sum_{F:\, F \geq A} (-1)^{\dim(F)} \Big) \mathtt{H}_A = (-1)^{\dim(\mathcal{A})} \mathtt{H}_A.$$

For the second equality, we used (1.72) on the arrangement $\mathcal{A}_A$. Alternatively, one may start with the commutative Takeuchi formula (12.15), and employ (1.74). □

We now turn to the signed exponential bimonoid E^- from Section 7.2.5.

Proposition 12.29. *The antipode $S : \mathsf{E}^- \to \mathsf{E}^-$, on the A-component, is given by*

$$S_A(\mathtt{H}_{[C/A]}) = (-1)^{\dim(\mathcal{A})}\, \mathtt{H}_{[C/A]}. \tag{12.21}$$

The calculation works exactly like the exponential bimonoid case in view of Exercise 7.3.

12.5.2. Bimonoid of chambers. Recall the q-bimonoid of chambers Γ_q from Section 7.3.

Proposition 12.30. *The antipode* $S : \Gamma_q \to \Gamma_q$*, on the* A*-component, is given by*

$$S_A(\mathtt{H}_{C/A}) = (-1)^{\dim(\mathcal{A})} q^{\operatorname{dist}(C, A\overline{C})}\, \mathtt{H}_{A\overline{C}/A}. \tag{12.22}$$

Proof. The calculation goes as follows. For $C \geq A$, by (12.1), using (7.19),

$$\begin{aligned} S_A(\mathtt{H}_{C/A}) &= \sum_{F:\, F \geq A} (-1)^{\dim(F)} q^{\operatorname{dist}(C, FC)} \mathtt{H}_{FC/A} \\ &= \sum_{D:\, D \geq A} \Big(\sum_{F:\, F \geq A, FC = D} (-1)^{\dim(F)} \Big) q^{\operatorname{dist}(C,D)} \mathtt{H}_{D/A}. \end{aligned}$$

Applying the descent identity (1.75) to the arrangement $\mathcal{A}_A$, we see that the sum in parenthesis above is zero unless $D = A\overline{C}$. In the latter case, the sum is $(-1)^{\dim(C)}$. ∎

Exercise 12.31. Put $q = \pm 1$ in formula (12.22). Use Lemma 12.2 and the abelianization map (7.28) and the signed abelianization map (7.31) to deduce the antipode formulas (12.20) and (12.21), respectively.

In view of Lemma 12.4, by dualizing formula (12.22), we obtain:

Proposition 12.32. *The antipode* $S : \Gamma_q^* \to \Gamma_q^*$*, on the* A*-component, is given by*

$$S_A(\mathtt{M}_{D/A}) = (-1)^{\dim(\mathcal{A})} q^{\operatorname{dist}(D, A\overline{D})}\, \mathtt{M}_{A\overline{D}/A}. \tag{12.23}$$

Exercise 12.33. By Lemma 12.2, the q-norm map on chambers (7.26) commutes with the antipodes of Γ_q and Γ_q^*. Check this fact directly using formulas (12.22) and (12.23).

For a geometric partial-support relation $\sim$ on faces, recall the bimonoid $\Gamma_\sim$ with product and coproduct given by (7.33).

Proposition 12.34. *The antipode* $S : \Gamma_\sim \to \Gamma_\sim$*, on the* z*-component, is given by*

$$S_{\mathrm{z}}(\mathtt{H}_{\mathrm{c}/\mathrm{z}}) = (-1)^{\dim(\mathcal{A})}\, \mathtt{H}_{\mathrm{z}\overline{\mathrm{c}}/\mathrm{z}}. \tag{12.24}$$

This can be deduced by a similar calculation using the descent identity (1.79a). Observe that formula (12.24) specializes to (12.22) for $q = 1$ when $\sim$ is finest, and to (12.20) when $\sim$ is coarsest.

12.5.3. Bimonoid of flats. Recall the bimonoid of flats Π from Section 7.4.

Proposition 12.35. *The antipode* $S : \Pi \to \Pi$*, on the* Z*-component, in the* $\mathtt{H}$*-basis, is given by*

$$S_{\mathrm{Z}}(\mathtt{H}_{\mathrm{X}/\mathrm{Z}}) = \sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{X}} (-1)^{\dim(\mathrm{Y})} c_{\mathrm{X}}^{\mathrm{Y}}\, \mathtt{H}_{\mathrm{Y}/\mathrm{Z}}. \tag{12.25}$$

PROOF. The calculation goes as follows. For $\mathrm{X} \geq \mathrm{Z}$, by the commutative Takeuchi formula (12.15), using (7.36),

$$\begin{aligned} \mathcal{S}_{\mathrm{Z}}(\mathtt{H}_{\mathrm{X}/\mathrm{Z}}) &= \sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{Z}} (-1)^{\dim(\mathrm{Y})} c_{\mathrm{Z}}^{\mathrm{Y}} \, \mathtt{H}_{\mathrm{Y} \vee \mathrm{X}/\mathrm{Z}} \\ &= \sum_{\mathrm{W}:\, \mathrm{W} \geq \mathrm{X}} \Big(\sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{Z},\, \mathrm{Y} \vee \mathrm{X} = \mathrm{W}} (-1)^{\dim(\mathrm{Y})} c_{\mathrm{Z}}^{\mathrm{Y}} \Big) \, \mathtt{H}_{\mathrm{W}/\mathrm{Z}} \\ &= \sum_{\mathrm{W}:\, \mathrm{W} \geq \mathrm{X}} (-1)^{\dim(\mathrm{W})} c_{\mathrm{X}}^{\mathrm{W}} \, \mathtt{H}_{\mathrm{W}/\mathrm{Z}}. \end{aligned}$$

In the last step, we applied the descent-lune identity (1.78a) to the arrangement $\mathcal{A}_{\mathrm{Z}}$. □

Proposition 12.36. *The antipode $\mathcal{S} : \Pi \to \Pi$, on the* Z*-component, in the* $\mathtt{Q}$*-basis, is given by*

$$\mathcal{S}_{\mathrm{Z}}(\mathtt{Q}_{\mathrm{X}/\mathrm{Z}}) = (-1)^{\dim(\mathrm{X})} \mathtt{Q}_{\mathrm{X}/\mathrm{Z}}. \tag{12.26}$$

PROOF. This is similar to the previous calculation. For $\mathrm{X} \geq \mathrm{Z}$, by (12.15), using (7.40),

$$\mathcal{S}_{\mathrm{Z}}(\mathtt{Q}_{\mathrm{X}/\mathrm{Z}}) = \Big(\sum_{\mathrm{Y}:\, \mathrm{X} \geq \mathrm{Y} \geq \mathrm{Z}} (-1)^{\dim(\mathrm{Y})} c_{\mathrm{Z}}^{\mathrm{Y}} \Big) \, \mathtt{Q}_{\mathrm{X}/\mathrm{Z}} = (-1)^{\dim(\mathrm{X})} \mathtt{Q}_{\mathrm{X}/\mathrm{Z}}.$$

The sum in parenthesis is evaluated using (1.74). □

Exercise 12.37. Combine (7.38) with (12.26) and write down an expression for the antipode of Π in the $\mathtt{H}$-basis. Now compare with (12.25) to deduce the following identity. For $\mathrm{X} \leq \mathrm{W}$,

$$\sum_{\mathrm{Y}:\, \mathrm{X} \leq \mathrm{Y} \leq \mathrm{W}} (-1)^{\dim(\mathrm{Y})} \mu(\mathrm{Y}, \mathrm{W}) = (-1)^{\dim(\mathrm{W})} c_{\mathrm{X}}^{\mathrm{W}}.$$

Deduce the Zaslavsky formula (1.84).

By dualizing formulas (12.25) and (12.26), we obtain:

Proposition 12.38. *The antipode $\mathcal{S} : \Pi^* \to \Pi^*$, on the* Z*-component, in the* $\mathtt{M}$*-basis, is given by*

$$\mathcal{S}_{\mathrm{Z}}(\mathtt{M}_{\mathrm{Y}/\mathrm{Z}}) = (-1)^{\dim(\mathrm{Y})} \sum_{\mathrm{X}:\, \mathrm{Z} \leq \mathrm{X} \leq \mathrm{Y}} c_{\mathrm{X}}^{\mathrm{Y}} \, \mathtt{M}_{\mathrm{X}/\mathrm{Z}}, \tag{12.27}$$

and in the $\mathtt{P}$*-basis, is given by*

$$\mathcal{S}_{\mathrm{Z}}(\mathtt{P}_{\mathrm{X}/\mathrm{Z}}) = (-1)^{\dim(\mathrm{X})} \mathtt{P}_{\mathrm{X}/\mathrm{Z}}. \tag{12.28}$$

12.5.4. Bimonoid of charts. Recall the bimonoid of charts G from Section 7.5. The antipode formula in the $\mathtt{H}$-basis is quite intricate, and makes use of the following result proved in [21, Proposition 7.99].

Lemma 12.39. *For any charts g, h in $\mathcal{A}$,*

$$\sum_{\mathrm{Y}:\, g_{\mathrm{Y}} = h} (-1)^{\mathrm{rk}(\mathrm{Y})} c^{\mathrm{Y}} = \begin{cases} (-1)^{\mathrm{rk}(\mathrm{O}(h))} c(g^{\mathrm{O}(h)}) & \textit{if } g_{\mathrm{O}(h)} = h, \\ 0 & \textit{otherwise,} \end{cases} \tag{12.29}$$

where $c(g^{\mathrm{O}(h)})$ is the number of chambers in the arrangement $g^{\mathrm{O}(h)}$.

Proposition 12.40. *The antipode $S : \mathtt{G} \to \mathtt{G}$, on the Z-component, in the $\mathtt{H}$-basis, is given by*

$$S_{\mathrm{Z}}(\mathtt{H}_{g/\mathrm{Z}}) = \sum_{\mathrm{X}\in\Pi[g]} (-1)^{\dim(\mathrm{X})}\, c(g^{\mathrm{X}})\, \mathtt{H}_{g_{\mathrm{X}}/\mathrm{Z}}, \tag{12.30}$$

where $\Pi[g]$ is the set of flats of g viewed as an arrangement, and $c(g^{\mathrm{X}})$ is the number of chambers in g^{X}.

Proof. This can be deduced from the commutative Takeuchi formula (12.15) and (12.29) applied to $\mathcal{A}_{\mathrm{Z}}$. □

Proposition 12.41. *The antipode $S : \mathtt{G} \to \mathtt{G}$, on the Z-component, in the $\mathtt{Q}$-basis, is given by*

$$S_{\mathrm{Z}}(\mathtt{Q}_{g/\mathrm{Z}}) = (-1)^{\dim(\mathrm{O}(g))}\, \mathtt{Q}_{g/\mathrm{Z}}. \tag{12.31}$$

The calculation is similar to the one in the proof of formula (12.26).

12.5.5. Bimonoid of faces. Recall the q-bimonoid of faces Σ_q from Section 7.6.

Proposition 12.42. *The antipode $S : \Sigma_q \to \Sigma_q$, on the A-component, in the $\mathtt{H}$-basis, is given by*

$$S_A(\mathtt{H}_{F/A}) = q^{\operatorname{dist}(F, A\overline{F})} \sum_{G:\, A\overline{F}\le G} (-1)^{\dim(G)}\, \mathtt{H}_{G/A}. \tag{12.32}$$

Proof. The calculation goes as follows. For $F \geq A$, by (12.1), using formulas (7.65),

$$\begin{aligned} S_A(\mathtt{H}_{F/A}) &= \sum_{H:\, H\geq A} (-1)^{\dim(H)} q^{\operatorname{dist}(F,HF)}\, \mathtt{H}_{HF/A} \\ &= \sum_{G:\, G\geq A} \Big(\sum_{H:\, H\geq A, HF=G} (-1)^{\dim(H)} \Big) q^{\operatorname{dist}(F,G)}\, \mathtt{H}_{G/A}. \end{aligned}$$

Applying the descent identity (1.76a) to the arrangement $\mathcal{A}_A$, we see that the sum in parenthesis above is zero unless $G \geq A\overline{F}$. In the latter case, the sum is $(-1)^{\dim(G)}$ and also $\operatorname{dist}(F, G) = \operatorname{dist}(F, A\overline{F})$. □

Now take $q = 1$.

Proposition 12.43. *The antipode $S : \Sigma \to \Sigma$, on the A-component, in the $\mathtt{Q}$-basis, is given by*

$$S_A(\mathtt{Q}_{F/A}) = (-1)^{\dim(F)}\, \mathtt{Q}_{A\overline{F}/A}. \tag{12.33}$$

Proof. We proceed as in the proof of (12.32) now using formulas (7.69). The coproduct formula in the $\mathtt{Q}$-basis is simpler. This has the effect that the G in the calculation has the same support as F, so it is forced to be $A\overline{F}$. □

Exercise 12.44. Use the support map (7.84) to deduce the antipode formulas of Π, namely, (12.25) and (12.26) from those of Σ, namely, (12.32) and (12.33).

Exercise 12.45. For q not a root of unity, use formulas (7.71) to show that the antipode $S : \Sigma_q \to \Sigma_q$, on the A-component, in the $\mathtt{Q}$-basis, is given by

$$S_A(\mathtt{Q}_{F/A}) = (-1)^{\dim(F)} q^{\mathrm{dist}(F, A\overline{F})}\, \mathtt{Q}_{A\overline{F}/A}. \tag{12.34}$$

By dualizing formula (12.32), we obtain:

Proposition 12.46. *The antipode $S : \Sigma_q^* \to \Sigma_q^*$, on the A-component, in the $\mathtt{M}$-basis, is given by*

$$S_A(\mathtt{M}_{G/A}) = (-1)^{\dim(G)} \sum_{F\colon A \leq F \leq A\overline{G}} q^{\mathrm{dist}(A\overline{F}, F)}\, \mathtt{M}_{F/A}. \tag{12.35}$$

Similarly, one can dualize formulas (12.33) and (12.34).

12.5.6. Bimonoid of top-nested faces. Recall the q-bimonoid of top-nested faces $\widehat{\mathsf{Q}}_q$ from Section 7.7.

Proposition 12.47. *The antipode $S : \widehat{\mathsf{Q}}_q \to \widehat{\mathsf{Q}}_q$, on the A-component, in the $\mathtt{H}$-basis, is given by*

$$S_A(\mathtt{H}_{F/A, C/A}) = q^{\mathrm{dist}(F, A\overline{F})} \sum_{G\colon A\overline{F} \leq G,\ FG \leq C} (-1)^{\dim(G)}\, \mathtt{H}_{G/A, GC/A}. \tag{12.36}$$

Proof. We proceed as in the proof of (12.32) using formulas (7.99). The cancellations occur in exactly the same manner. The extra condition $FG \leq C$ arises from the first alternative in the coproduct formula (7.99b). □

By dualizing (12.36), we obtain:

Proposition 12.48. *The antipode $S : \widehat{\mathsf{Q}}_q^* \to \widehat{\mathsf{Q}}_q^*$, on the A-component, in the $\mathtt{M}$-basis, is given by*

$$S_A(\mathtt{M}_{G/A, D/A}) = (-1)^{\dim(G)} \sum_{F\colon A \leq F \leq A\overline{G}} q^{\mathrm{dist}(A\overline{F}, F)}\, \mathtt{M}_{F/A, FD/A}. \tag{12.37}$$

12.5.7. Bimonoid of bifaces. Recall the q-bimonoid of bifaces J_q from Section 7.8.

Proposition 12.49. *The antipode $S : \mathsf{J}_q \to \mathsf{J}_q$, on the A-component, in the $\mathtt{H}$-basis, is given by*

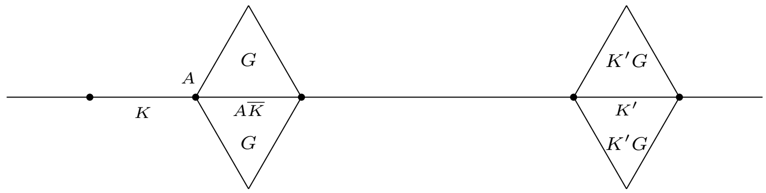
$$S_A(\mathtt{H}_{(K,K')}) = q^{\mathrm{dist}(K, A\overline{K})} \sum_{G\colon A\overline{K} \leq G} (-1)^{\dim(G)}\, \mathtt{H}_{(G, K'G)}. \tag{12.38}$$

Note that the term $\mathtt{H}_{(A\overline{K}, K')}$ appears in the above sum.

Proof. To prove this formula: Employing (12.1) and (7.127),

$$\begin{aligned} S_A(\mathtt{H}_{(K,K')}) &= \sum_{H\colon H \geq A} (-1)^{\dim(H)} q^{\mathrm{dist}(K, HK)}\, \mathtt{H}_{(HK, K'H)} \\ &= \sum_{G\colon G \geq A} \Big(\sum_{H\colon H \geq A, HK = G} (-1)^{\dim(H)} \Big) q^{\mathrm{dist}(K,G)}\, \mathtt{H}_{(G, K'G)}. \end{aligned}$$

Note that $K'H = K'HK = K'G$. The rest of the calculation is similar to the one for (12.32).



The picture illustrates the terms in formula (12.38) in rank three. Here A is a vertex, and K and K' are edges with the same support with K greater than A. There are three choices for G, one is the edge $A\overline{K}$, while the remaining two are the two triangles marked G. The corresponding choices for $K'G$ are K' itself and the two triangles marked $K'G$. $\square$

By dualizing formula (12.38), we obtain:

Proposition 12.50. *The antipode $S : \mathsf{J}_q^* \to \mathsf{J}_q^*$, on the A-component, in the* $\mathtt{M}$*-basis, is given by*

$$S_A(\mathtt{M}_{(G,G')}) = (-1)^{\dim(G)} \sum_{\substack{(K,K'):\\ A\leq K\leq A\overline{G}\\ K'G=G'}} q^{\operatorname{dist}(A\overline{K},K)}\, \mathtt{M}_{(K,K')}. \tag{12.39}$$

Note that the term $\mathtt{M}_{(A\overline{G},G')}$ appears in the above sum.

Exercise 12.51. Recall the commutative bimonoid J_{ab} whose product and coproduct are given by (7.151). Use these formulas along with the Takeuchi formula (12.1) to deduce that the antipode of J_{ab} is given by

$$S_A(\mathtt{H}_{K'}) = \sum_{G':\, K'\leq G'} (-1)^{\dim(G')}\, \mathtt{H}_{G'}.$$

(Use the lune identity (1.77a) and Lemma 1.6.) Alternatively, deduce this formula from (12.38) with $q = 1$ using the surjective map (7.152).

12.6. Antipodes of (co)free bimonoids

Recall from Section 6.1 the construction of the free q-bimonoid $\mathcal{T}_q(\mathsf{c})$ on a comonoid c, and dually from Section 6.2, the construction of the cofree q-bimonoid $\mathcal{T}_q^\vee(\mathsf{a})$ on a monoid a. We now write down cancellation-free formulas for their antipodes. In view of formula (12.5), this amounts to computing the 0-logarithm of their identity maps. We tie these calculations to the discussion in Section 12.2.

12.6.1. Free q-bimonoid on a comonoid.

Theorem 12.52. *For a comonoid c, the antipode of the q-bimonoid $\mathcal{T}_q(\mathsf{c})$ on the A-component $S_A : \mathcal{T}_q(\mathsf{c})[A] \to \mathcal{T}_q(\mathsf{c})[A]$, evaluated on the summand $\mathsf{c}[F]$,* *is given by*

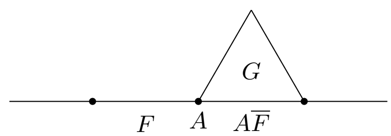
$$\mathsf{c}[F] \to \bigoplus_{G\colon G \geq A} \mathsf{c}[G]$$

$$S_A = q^{\operatorname{dist}(F, A\overline{F})} \sum_{G\colon A\overline{F} \leq G} (-1)^{\dim(G)} \Delta^{G}_{A\overline{F}} \beta_{A\overline{F}, F}. \tag{12.40}$$

In particular, if c is a trivial comonoid, then on the summand $\mathsf{c}[F]$,

$$S_A = (-1)^{\dim(F)} q^{\operatorname{dist}(F, A\overline{F})}\, \beta_{A\overline{F}, F}. \tag{12.41}$$

An illustration of how the faces A, F, G in formula (12.40) relate to one another is shown below.



Proof. From product and coproduct formulas (6.3) and (6.5) for $\mathcal{T}_q(\mathsf{c})$, we see that $\mu^H_A \Delta^H_A$ sends the component $\mathsf{c}[F]$ to $\mathsf{c}[HF]$. More precisely, it is the composite

$$\mathsf{c}[F] \xrightarrow{\Delta^{FH}_F} \mathsf{c}[FH] \xrightarrow{\beta_{HF,FH}} \mathsf{c}[HF],$$

multiplied by the coefficient $q^{\operatorname{dist}(F,H)}$. Applying the Takeuchi formula (12.1) to the summand $\mathsf{c}[F]$, we obtain

$$\begin{aligned}
S_A &= \sum_{H\colon H \geq A} (-1)^{\dim(H)} q^{\operatorname{dist}(F,H)} \beta_{HF,FH} \Delta^{FH}_F \\
&= \sum_{G\colon G \geq A} \Bigg(\sum_{\substack{H\colon H \geq A \\ HF = G}} (-1)^{\dim(H)} \Bigg) q^{\operatorname{dist}(F,G)} \beta_{G,FG} \Delta^{FG}_F \\
&= \sum_{G\colon A\overline{F} \leq G} (-1)^{\dim(G)} q^{\operatorname{dist}(F,G)} \beta_{G,FG} \Delta^{FG}_F \\
&= q^{\operatorname{dist}(F, A\overline{F})} \sum_{G : A\overline{F} \leq G} (-1)^{\dim(G)} \Delta^{G}_{A\overline{F}} \beta_{A\overline{F}, F}.
\end{aligned}$$

The main step, where the cancellations occur, is the third equality. It follows from the descent identity (1.76a) applied to the arrangement $\mathcal{A}_A$, see also Section 1.8.3. The last equality follows from naturality (2.10) and the fact that $\operatorname{dist}(F,G) = \operatorname{dist}(F, A\overline{F})$ if $A\overline{F} \leq G$. $\square$

12.6.2. Cofree q-bimonoid on a monoid. Dually:

Theorem 12.53. *For a monoid* a*, the antipode of the* q*-bimonoid* $\mathcal{T}^\vee_q(\mathsf{a})$ *on the* A*-component* $S_A : \mathcal{T}^\vee_q(\mathsf{a})[A] \to \mathcal{T}^\vee_q(\mathsf{a})[A]$*, evaluated on the summand* $\mathsf{a}[G]$*,* *is given by*

$$\mathsf{a}[G] \to \bigoplus_{F:\, F \geq A} \mathsf{a}[F]$$

$$S_A = (-1)^{\dim(G)} \sum_{F:\, A \leq F \leq A\overline{G}} q^{\operatorname{dist}(A\overline{F}, F)}\, \beta_{F, A\overline{F}} \mu^G_{A\overline{F}}. \tag{12.42}$$

In particular, if a is a trivial monoid, then on the summand $\mathsf{a}[G]$,

$$S_A = (-1)^{\dim(G)} q^{\operatorname{dist}(G, A\overline{G})}\, \beta_{A\overline{G}, G}. \tag{12.43}$$

Proof. Substitute the coproduct and product formulas (6.11) and (6.13) in the Takeuchi formula (12.1), and simplify as in the previous proof. Alternatively, if a is finite-dimensional, then we can deduce this result from (12.40) using (6.17) and (12.7). $\square$

12.6.3. Examples. We now return to the antipodes of the q-bimonoid of chambers and q-bimonoid of faces.

Example 12.54. Take $\mathsf{c} = \mathsf{x}$, the species characteristic of chambers, with the trivial coproduct. By (7.24), $\mathcal{T}_q(\mathsf{c}) = \Gamma_q$, the q-bimonoid of chambers, and formula (12.41) specializes to (12.22). Dually, take $\mathsf{a} = \mathsf{x}$, with the trivial product. By (7.25), $\mathcal{T}^\vee_q(\mathsf{a}) = \Gamma^*_q$, and formula (12.43) specializes to (12.23).

Example 12.55. Take $\mathsf{c} = \mathsf{E}$, the exponential bimonoid viewed as a comonoid, and $\mathsf{a} = \mathsf{E}$ viewed as a monoid. By (7.81), $\mathcal{T}_q(\mathsf{c}) = \Sigma_q$ and $\mathcal{T}^\vee_q(\mathsf{a}) = \Sigma^*_q$, and formulas (12.40) and (12.42) specialize to (12.32) and (12.35), respectively. We may also take $\mathsf{c} = \mathsf{E}$ with the trivial coproduct. In view of (7.82), formula (12.41) for $q = 1$ specializes to (12.33). Similarly, in view of (7.83), formula (12.41) specializes to (12.34).

Exercise 12.56. Take $\mathsf{c} = \Gamma^*$ and $\mathsf{a} = \Gamma$. By (7.106), $\mathcal{T}_q(\mathsf{c}) = \widehat{\mathsf{Q}}_q$ and $\mathcal{T}^\vee_q(\mathsf{a}) = \widehat{\mathsf{Q}}^*_q$. Check that formulas (12.40) and (12.42) specialize to (12.36) and (12.37), respectively.

Exercise 12.57. Take $\mathsf{c} = \mathsf{F}$ and $\mathsf{a} = \mathsf{F}^*$ as defined in Section 7.8.1. By (7.148), $\mathcal{T}_q(\mathsf{c}) = \mathsf{J}_q$ and $\mathcal{T}^\vee_q(\mathsf{a}) = \mathsf{J}^*_q$. Check that formulas (12.40) and (12.42) specialize to (12.38) and (12.39), respectively.

12.6.4. Op and cop constructions revisited. One can use Theorem 12.52 and Theorem 12.53 to give a more conceptual proof of the antipode opposition Lemma 12.11 as follows.

Lemma 12.58. *Let $(\mathsf{h}, \mu, \Delta)$ be a q-bimonoid. Then, for any faces $A \leq F$,*

$$\sum_{G:\, G \geq A} (-1)^{\dim(G)} \mu^G_A \Delta^G_A \mu^F_A = \sum_{G:\, G \geq A\overline{F}} (-1)^{\dim(G)} \mu^G_A \Delta^G_{A\overline{F}} (\beta_q)_{A\overline{F}, F}, \tag{12.44}$$

and, dually,

$$\sum_{G:\, G \geq A} (-1)^{\dim(G)} \Delta^F_A \mu^G_A \Delta^G_A = \sum_{G:\, G \geq A\overline{F}} (-1)^{\dim(G)} (\beta_q)_{F, A\overline{F}} \mu^G_{A\overline{F}} \Delta^G_A. \tag{12.45}$$

Proof. Consider the surjective morphism $\mu : \mathcal{T}_q(\mathsf{h}) \twoheadrightarrow \mathsf{h}$ of q-bimonoids given in Exercise 6.67. By Lemma 12.2, this morphism commutes with the antipodes. In view of Theorem 12.52, one can check that this statement is equivalent to (12.44). Note that the identity is a tautology for $F = A$. This reflects the fact that, for $F = A$, (12.40) composed with (μ_A^G) reduces to (12.1).

Dually, we start with the injective morphism of q-bimonoids $\Delta : \mathsf{h} \hookrightarrow \mathcal{T}_q^\vee(\mathsf{h})$, and use Theorem 12.53 to obtain (12.45). $\square$

By using (2.35), we see that Lemma 12.58 and Lemma 12.11 imply each other.

Exercise 12.59. Use Exercise 12.3 to deduce: For $\mu : \mathcal{T}_q(\mathsf{h}) \twoheadrightarrow \mathsf{h}$, the 0-logarithm $\mathbf{log}_0(\mu)$, evaluated on the A-component, on the F-summand, equals (12.44) times $(-1)^{\dim(A)}$. Dually, for $\Delta : \mathsf{h} \hookrightarrow \mathcal{T}_q^\vee(\mathsf{h})$, the 0-logarithm $\mathbf{log}_0(\Delta)$, evaluated on the A-component, into the F-summand, equals (12.45) times $(-1)^{\dim(A)}$.

An alternative way to obtain Lemma 12.12 is given in the exercise below.

Exercise 12.60. Let $(\mathsf{h}, \mu, \Delta)$ be a q-bimonoid. Use the explicit formula (12.40) to first show that $S : {}^{\mathrm{op}}\mathcal{T}_q(\mathsf{h}) \to \mathcal{T}_q(\mathsf{h})$ is a morphism of monoids. Next, use the commutative diagram

$$\begin{tikzcd} {}^{\mathrm{op}}\mathcal{T}_q(\mathsf{h}) \arrow[r, "S"] \arrow[d, two heads] & \mathcal{T}_q(\mathsf{h}) \arrow[d, two heads] \\ {}^{\mathrm{op}}\mathsf{h} \arrow[r, "S"'] & \mathsf{h} \end{tikzcd}$$

to deduce that $S : {}^{\mathrm{op}}\mathsf{h} \to \mathsf{h}$ is a morphism of monoids.

Repeat the above steps using $\mathcal{T}_q^\vee$ instead of $\mathcal{T}_q$ to deduce that $S : \mathsf{h} \to \mathsf{h}^{\mathrm{cop}}$ is a morphism of comonoids.

Conversely, one can use Lemma 12.12 to prove Theorem 12.52 and Theorem 12.53 as follows.

Exercise 12.61. For a comonoid c, first check directly that the antipode of $\mathcal{T}_q(\mathsf{c})$ on the A-component, on the summand $\mathsf{c}[A]$, is given by

$$\mathsf{c}[A] \to \bigoplus_{G:\, G \geq A} \mathsf{c}[G], \qquad \sum_{G:\, A \leq G} (-1)^{\dim(G)} \Delta_A^G, \tag{12.46}$$

and then use Lemma 12.12, item (1), to deduce the general formula (12.40).

Dually, for a monoid a, first check directly that the antipode of $\mathcal{T}_q^\vee(\mathsf{a})$ on the A-component, from the summand $\mathsf{a}[G]$ into the summand $\mathsf{a}[A]$, is given by $(-1)^{\dim(G)} \mu_A^G$, and then use Lemma 12.12, item (2), to deduce the general formula (12.42).

12.7. Antipodes of (co)free (co)commutative bimonoids

Recall the (co)free (co)commutative bimonoids, and their signed analogues defined in Section 6.3. We now write down cancellation-free formulas for their antipodes. We also generalize to partially bicommutative bimonoids which unifies all formulas.

12.7.1. Bicommutative bimonoids. Recall the bimonoid $\mathcal{S}(\mathsf{c})$ associated to a cocommutative comonoid c, and the bimonoid $\mathcal{S}^\vee(\mathsf{a})$ associated to a commutative monoid a. Both are bicommutative.

Theorem 12.62. *For a cocommutative comonoid* c*, the antipode of* $\mathcal{S}(\mathsf{c})$ *on the* Z *component* $S_{\mathrm{Z}} : \mathcal{S}(\mathsf{c})[\mathrm{Z}] \to \mathcal{S}(\mathsf{c})[\mathrm{Z}]$*, evaluated on the summand* $\mathsf{c}[\mathrm{X}]$*, is given by*

$$\mathsf{c}[\mathrm{X}] \to \bigoplus_{\mathrm{Y}\colon \mathrm{Y}\geq \mathrm{Z}} \mathsf{c}[\mathrm{Y}]$$

$$S_{\mathrm{Z}} = \sum_{\mathrm{Y}\colon \mathrm{X}\leq \mathrm{Y}} (-1)^{\dim(\mathrm{Y})} c^{\mathrm{Y}}_{\mathrm{X}} \Delta^{\mathrm{Y}}_{\mathrm{X}}. \tag{12.47}$$

In particular, if c *is a trivial comonoid, then on the summand* $\mathsf{c}[\mathrm{X}]$*,*

$$S_{\mathrm{Z}} = (-1)^{\dim(\mathrm{X})} \,\mathrm{id}\,. \tag{12.48}$$

PROOF. Apply Lemma 12.2 to the abelianization map $\mathcal{T}(\mathsf{c}) \twoheadrightarrow \mathcal{S}(\mathsf{c})$ (see (6.59)), and use (12.40) for $q = 1$ to deduce (12.47). Alternatively, one can compute directly with the commutative Takeuchi formula (12.15) and (co)product formulas (6.20) and (6.22), and use the descent-lune identity (1.78a). □

Dually:

Theorem 12.63. *For a commutative monoid* a*, the antipode of* $\mathcal{S}^\vee(\mathsf{a})$ *on the* Z *component* $S_{\mathrm{Z}} : \mathcal{S}^\vee(\mathsf{a})[\mathrm{Z}] \to \mathcal{S}^\vee(\mathsf{a})[\mathrm{Z}]$*, evaluated on the summand* $\mathsf{a}[\mathrm{Y}]$*, is given by*

$$\mathsf{a}[\mathrm{Y}] \to \bigoplus_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Z}} \mathsf{a}[\mathrm{X}]$$

$$S_{\mathrm{Z}} = (-1)^{\dim(\mathrm{Y})} \sum_{\mathrm{X}\colon \mathrm{Z}\leq \mathrm{X}\leq \mathrm{Y}} c^{\mathrm{Y}}_{\mathrm{X}} \mu^{\mathrm{Y}}_{\mathrm{X}}. \tag{12.49}$$

In particular, if a is a trivial monoid, then on the summand $\mathsf{a}[\mathrm{Y}]$,

$$S_{\mathrm{Z}} = (-1)^{\dim(\mathrm{Y})} \,\mathrm{id}\,. \tag{12.50}$$

PROOF. We can proceed as in the previous proof. Apply Lemma 12.2 to the coabelianization map $\mathcal{S}^\vee(\mathsf{a}) \hookrightarrow \mathcal{T}^\vee(\mathsf{a})$ (see (6.62)), and use (12.42) for $q = 1$ to deduce (12.49). Alternatively, substitute (co)product formulas (6.26) and (6.28) in the commutative Takeuchi formula (12.15) and simplify using the descent-lune identity (1.78a). □

12.7.2. Examples. We now return to the antipodes of the exponential bimonoid and bimonoid of flats.

Example 12.64. Take $\mathsf{c} = \mathsf{x}$, the species characteristic of chambers, with the trivial coproduct. By (7.11), $\mathcal{S}(\mathsf{c}) = \mathsf{E}$, the exponential bimonoid, and formula (12.48) specializes to (12.20). Dually, take $\mathsf{a} = \mathsf{x}$, with the trivial product. Again by (7.11), $\mathcal{S}^\vee(\mathsf{a}) = \mathsf{E}$, and formula (12.50) specializes to (12.20).

Example 12.65. Take $\mathsf{c} = \mathsf{E}$, the exponential bimonoid viewed as a comonoid, and $\mathsf{a} = \mathsf{E}$ viewed as a monoid. By (7.47), $\mathcal{S}(\mathsf{c}) = \Pi$ and $\mathcal{S}^\vee(\mathsf{a}) = \Pi^*$, and formulas (12.47) and (12.49) specialize to (12.25) and (12.27), respectively. We may also take $\mathsf{c} = \mathsf{E}$ with the trivial coproduct. In view of (7.48), formula (12.48) specializes to (12.26).

Exercise 12.66. Use (7.57) to deduce (12.31) as a special case of (12.48). Note very carefully that (12.30) is *not* a special case of (12.47), see Exercise 7.15 in this regard.

12.7.3. Signed bicommutative signed bimonoids. Recall the signed bimonoid $\mathcal{E}(\mathsf{c})$ associated to a signed cocommutative comonoid c, and the signed bimonoid $\mathcal{E}^\vee(\mathsf{a})$ associated to a signed commutative monoid a.

Theorem 12.67. *For a signed cocommutative comonoid* c*, the antipode of* $\mathcal{E}(\mathsf{c})$ *on the* Z*-component* $S_{\mathrm{Z}} : \mathcal{E}(\mathsf{c})[\mathrm{Z}] \to \mathcal{E}(\mathsf{c})[\mathrm{Z}]$*, evaluated on the summand indexed by* X*, is given by*

$$\mathbf{E}^-[\mathrm{Z},\mathrm{X}] \otimes \mathsf{c}[\mathrm{X}] \to \bigoplus_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{Z}} \mathbf{E}^-[\mathrm{Z},\mathrm{Y}] \otimes \mathsf{c}[\mathrm{Y}]$$

$$S_{\mathrm{Z}} = \sum_{\mathrm{Y}:\, \mathrm{X} \leq \mathrm{Y}} (-1)^{\dim(\mathrm{Y})} c^{\mathrm{Y}}_{\mathrm{X}} ((-) \otimes \mathrm{id})(\mathrm{id} \otimes \Delta^{\mathrm{Y}}_{\mathrm{X}}), \tag{12.51}$$

where $(-)$ *refers to the map* (1.162).

Proof. Apply Lemma 12.2 to the signed abelianization map $\mathcal{T}_{-1}(\mathsf{c}) \twoheadrightarrow \mathcal{E}(\mathsf{c})$, and use (12.40) for $q = -1$ to deduce (12.51). Alternatively, one can compute directly with (12.15) and formulas (6.33) and (6.34), and use the descent-lune identity (1.78a). □

Exercise 12.68. We now return to the antipode of the signed exponential bimonoid. Check that formula (12.51) specializes to (12.21) for $\mathsf{c} = \mathsf{x}$, with the trivial coproduct.

Dually:

Theorem 12.69. *For a signed commutative monoid* a*, the antipode of* $\mathcal{E}^\vee(\mathsf{a})$ *on the* Z*-component* $S_{\mathrm{Z}} : \mathcal{E}^\vee(\mathsf{a})[\mathrm{Z}] \to \mathcal{E}^\vee(\mathsf{a})[\mathrm{Z}]$*, evaluated on the summand indexed by* Y*, is given by*

$$\mathbf{E}^-[\mathrm{Z},\mathrm{Y}] \otimes \mathsf{a}[\mathrm{Y}] \to \bigoplus_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mathbf{E}^-[\mathrm{Z},\mathrm{X}] \otimes \mathsf{a}[\mathrm{X}]$$

$$S_{\mathrm{Z}} = (-1)^{\dim(\mathrm{Y})} \sum_{\mathrm{X}:\, \mathrm{Z} \leq \mathrm{X} \leq \mathrm{Y}} c^{\mathrm{Y}}_{\mathrm{X}} (\mathrm{id} \otimes \mu^{\mathrm{Y}}_{\mathrm{X}})((-) \otimes \mathrm{id}), \tag{12.52}$$

where $(-)$ *refers to the inverse of the map* (1.162).

The proof is similar to the one above.

12.7.4. Partially bicommutative bimonoids. Recall from Section 6.11.1 the bimonoid $\mathcal{T}_\sim(\mathsf{c})$ associated to a $\sim$-cocommutative comonoid c, and the bimonoid $\mathcal{T}^\vee_\sim(\mathsf{a})$ associated to a $\sim$-commutative monoid a. They are both $\sim$-bicommutative.

Theorem 12.70. *For a $\sim$-cocommutative comonoid c, the antipode of $\mathcal{T}_\sim(\mathsf{c})$ on the* z*-component $S_{\mathrm{z}} : \mathcal{T}_\sim(\mathsf{c})[\mathrm{z}] \to \mathcal{T}_\sim(\mathsf{c})[\mathrm{z}]$, evaluated on the summand $\mathsf{c}[\mathrm{x}]$, is given by*

$$\mathsf{c}[\mathrm{x}] \to \bigoplus_{\mathrm{y}:\, \mathrm{y} \geq \mathrm{z}} \mathsf{c}[\mathrm{y}]$$

$$S_{\mathrm{z}} = \sum_{\mathrm{y}:\, \mathrm{z}\overline{\mathrm{x}} \leq \mathrm{y}} (-1)^{\dim(\mathrm{y})} c^{\mathrm{y}}_{\mathrm{z}\overline{\mathrm{x}}} \Delta^{\mathrm{y}}_{\mathrm{z}\overline{\mathrm{x}}} \beta_{\mathrm{z}\overline{\mathrm{x}},\mathrm{x}}. \tag{12.53}$$

Proof. Apply Lemma 12.2 to the $\sim$-abelianization map $\mathcal{T}(\mathsf{c}) \twoheadrightarrow \mathcal{T}_\sim(\mathsf{c})$ in (6.90). Alternatively, one can compute directly with (12.17) and (co)product formulas (6.91), and use the descent identity (1.79a). □

Observe that formula (12.53) specializes to (12.40) when $\sim$ is finest, and to (12.47) when $\sim$ is coarsest.

Exercise 12.71. Deduce formula (12.24) as a special case of (12.53) for $\mathsf{c} = \mathsf{x}$.

Exercise 12.72. Write down antipode formulas for $\mathcal{T}^\vee_\sim(\mathsf{a})$, and also for the signed bimonoids $s\mathcal{T}_\sim(\mathsf{c})$ and $s\mathcal{T}^\vee_\sim(\mathsf{a})$.

12.8. Takeuchi element and characteristic operations

Recall from Section 1.10 the Takeuchi element of an arrangement. It is related via characteristic operations to the antipode of a bimonoid. More generally, the two-sided Takeuchi element is related to the antipode via two-sided characteristic operations. This provides another viewpoint on the antipode which can be used to establish some of its properties. For instance, the fact that the Takeuchi element is of order two translates to the fact that the square of the antipode of any (co)commutative bimonoid is the identity.

12.8.1. Takeuchi element. For each face A, define

$$\mathtt{Tak}_A := \sum_{F:\, F \geq A} (-1)^{\dim(F)}\, \mathtt{H}_{F/A}. \tag{12.54}$$

This is an element of the Tits algebra $\Sigma[A]$. It agrees up to sign with the Takeuchi element (1.148) of the arrangement $\mathcal{A}_A$. Thus, from (1.149), we have

$$\mathtt{Tak}_A \cdot \mathtt{Tak}_A = \mathtt{H}_{A/A}. \tag{12.55}$$

The elements $\mathtt{Tak}_A$, as A varies, are compatible with the structure of the bimonoid Σ in the following sense. For faces A and B of the same support, we have

$$\beta_{B,A}(\mathtt{Tak}_A) = \mathtt{Tak}_B. \tag{12.56}$$

For any faces $F \geq A$, we have

$$\Delta_A^F(\mathtt{Tak}_A) = \mathtt{Tak}_F. \tag{12.57}$$

This is equivalent to [**21**, First identity in (12.20)].

Lemma 12.73. *For any noncommutative zeta function* $\boldsymbol{\zeta}$,

$$\mathtt{Tak}_A = \sum_{F:\, F \geq A} (-1)^{\dim(F)} \boldsymbol{\zeta}(A, A\overline{F})\, \mathtt{Q}_{F/A}, \tag{12.58a}$$

with $\mathtt{Q}$*-basis as defined in* (7.67)*. In the special case that* $\boldsymbol{\zeta}$ *is projective,*

$$\mathtt{Tak}_A = \sum_{F:\, F \geq A} (-1)^{\dim(F)} \boldsymbol{\zeta}(A, F)\, \mathtt{Q}_{F/A}. \tag{12.58b}$$

Proof. This follows from Lemma 1.72 applied to the arrangement $\mathcal{A}_A$. More directly, we use either identity (1.86a) and the first formula in (7.67), or the noncommutative Zaslavsky formula (1.88) and the second formula in (7.67). ☐

12.8.2. Characteristic operations.

Now let h be any bimonoid. Observe that for $h \in \mathsf{h}[A]$,

$$S_A(h) = \mathtt{Tak}_A \cdot h, \tag{12.59}$$

where the rhs refers to the characteristic operation (10.1).

The following reformulates Lemma 12.10, and we also give another proof for it.

Lemma 12.74. *For a commutative bimonoid* h*, for any faces* $F \geq A$*, and* $h \in \mathsf{h}[F]$*, we have*

$$\mu_A^F(\mathtt{Tak}_F \cdot h) = \mathtt{Tak}_A \cdot \mu_A^F(h).$$

For a cocommutative bimonoid h*, for any faces* $F \geq A$*, and* $h \in \mathsf{h}[A]$*, we have*

$$\mathtt{Tak}_F \cdot \Delta_A^F(h) = \Delta_A^F(\mathtt{Tak}_A \cdot h).$$

Proof. In view of (12.57), these identities are equivalent to (10.14) and (10.15), respectively. ☐

The following is a restatement of Lemma 12.17, and we give another proof for it.

Lemma 12.75. *In any bimonoid which is either commutative or cocommutative, the square of its antipode is the identity.*

Proof. This follows from (12.55) and the fact that characteristic operations define an action of $\Sigma[A]$ on $\mathsf{h}[A]$ if the bimonoid h is either commutative or cocommutative. ☐

The following reformulates (1.150), and we give another proof for it.

Lemma 12.76. *For any faces* $F \geq A$*, we have*

$$\mathtt{H}_{A\overline{F}/A} \cdot \mathtt{Tak}_A = \mathtt{Tak}_A \cdot \mathtt{H}_{F/A}.$$

Proof. Let h be any cocommutative bimonoid. By combining (12.2), (12.10), (12.13), we have

$$\begin{tikzcd} \mathsf{h}[A] \arrow[r, "\Delta"] \arrow[d, "S_A"'] & \mathsf{h}[F] \arrow[rr, "\mu"] \arrow[d, "S_F"] & & \mathsf{h}[A] \arrow[d, "S_A"] \\ \mathsf{h}[A] \arrow[r, "\Delta"'] \arrow[rr, bend right, "\Delta"'] & \mathsf{h}[F] \arrow[r, "\beta"'] & \mathsf{h}[A\overline{F}] \arrow[r, "\mu"'] & \mathsf{h}[A]. \end{tikzcd}$$

We thus obtain

$$(\mu_A^{A\overline{F}} \Delta_A^{A\overline{F}}) S_A = S_A (\mu_A^F \Delta_A^F).$$

Translating these operators to elements of the Tits algebra $\Sigma[A]$, we obtain the desired identity. ☐

12.8.3. Takeuchi series. In the language of Section 9.5.1, the family of elements $\mathtt{Tak}_A$, one for each face A, defines a series of the species of faces Σ. This follows from (12.56). We call this the *Takeuchi series*, and denote it by $\mathtt{Tak}$. Moreover, formula (12.57) says that the Takeuchi series is group-like. Thus, by Theorem 9.117, any logarithm of $\mathtt{Tak}$ is a primitive series of Σ. It is computed below.

Lemma 12.77. *We have*

$$\mathbf{log}(\mathtt{Tak})_A = (-1)^{\dim(A)} \overline{\mathtt{Q}}_{A/A}, \tag{12.60a}$$

where the $\overline{\mathtt{Q}}$*-basis is defined from the* $\mathtt{H}$*-basis via* $\overline{\boldsymbol{\zeta}}$ *and* $\overline{\boldsymbol{\mu}}$*. In the special case that* $\boldsymbol{\mu}$ *is projective,*

$$\mathbf{log}(\mathtt{Tak})_A = (-1)^{\dim(A)} \mathtt{Q}_{A/A}. \tag{12.60b}$$

Proof. We calculate:

$$\begin{aligned} \mathbf{log}(\mathtt{Tak})_A &= \sum_{F:\, F\geq A} \boldsymbol{\mu}(A,F)\, \mu_A^F(\mathtt{Tak}_F) \\ &= \sum_{F:\, F\geq A} \sum_{G:\, G\geq F} (-1)^{\dim(G)} \boldsymbol{\mu}(A,F)\, \mu_A^F(\mathtt{H}_{G/F}) \\ &= \sum_{G:\, G\geq A} (-1)^{\dim(G)} \Big(\sum_{F:\, G\geq F\geq A} \boldsymbol{\mu}(A,F) \Big) \mathtt{H}_{G/A} \\ &= (-1)^{\dim(A)} \sum_{G:\, G\geq A} \boldsymbol{\mu}(A, A\overline{G})\, \mathtt{H}_{G/A} \\ &= (-1)^{\dim(A)} \overline{\mathtt{Q}}_{A/A}. \end{aligned}$$

The first two steps used definitions (9.45b) and (12.54). The third step used product formula (7.64), and then interchanged the sums. The fourth step used identity (1.87a). ☐

Exercise 12.78. Prove formula (12.60a) by applying product formula (7.69) on (12.58a) for the $\overline{\mathtt{Q}}$-basis, and using the fact that $\boldsymbol{\mu}$ and $\boldsymbol{\zeta}$ are inverses. One may also check instead that the exponential of (12.60a) is $\mathtt{Tak}$.

For any bimonoid $\mathtt{h}$, recall the map (10.31) which associates a series of $\mathrm{end}^{\times}(\mathtt{h})$ to any series of Σ. The image of the Takeuchi series $\mathtt{Tak}$ under this map is the antipode of $\mathtt{h}$, that is,

$$\mathscr{S}(\Psi)(\mathtt{Tak}) = S_{\mathtt{h}}.$$

This is a restatement of (12.59). It follows that $S_{\mathtt{h}}$ is a group-like series of $\mathrm{end}^{\times}(\mathtt{h})$. This is equivalent to Lemma 12.18. Further, if $\mathtt{h}$ is cocommutative, then by Lemma 10.9, $S_{\mathtt{h}}$ can be viewed as a group-like series of the bimonoid of star families $\mathcal{C}(\mathtt{h}, \mathtt{h})$. This is equivalent to saying that $S_{\mathtt{h}} : \mathtt{h} \to \mathtt{h}$ is a morphism of comonoids. This latter fact is given in Lemma 12.10.

Exercise 12.79. Deduce Lemma 12.24 from Lemma 12.77. (Note very carefully that the former deals with any q-bimonoid $\mathtt{h}$. Hence, use Exercise 8.17.) This explains the close similarity between the proofs of the two lemmas. Also note that

$$\mathbf{log}(\mathtt{Tak})_A = (-1)^{\dim(A)} \overline{\mathbf{log}}(\delta)_A, \tag{12.61}$$

where δ denotes the universal series of Σ, see Example 9.119. (Compare formulas (12.60a) and (9.55).) This is one way to understand why $\overline{\mathbf{log}}(\mathrm{id})$ appears in formula (12.18a).

12.8.4. Commutative Takeuchi element and commutative characteristic operations. For each flat Z, define

$$\mathrm{s}(\mathtt{Tak})_{\mathrm{Z}} := \sum_{\mathrm{X} \colon \mathrm{X} \geq \mathrm{Z}} (-1)^{\dim(\mathrm{X})} c^{\mathrm{X}}_{\mathrm{Z}} \, \mathtt{H}_{\mathrm{X}/\mathrm{Z}}. \tag{12.62}$$

This is an element of the Birkhoff algebra $\Pi[\mathrm{Z}]$. It agrees up to sign with the commutative Takeuchi element (1.153) of the arrangement $\mathcal{A}_{\mathrm{Z}}$. On the $\mathtt{Q}$-basis,

$$\mathrm{s}(\mathtt{Tak})_{\mathrm{Z}} = \sum_{\mathrm{X} \colon \mathrm{X} \geq \mathrm{Z}} (-1)^{\dim(\mathrm{X})} \, \mathtt{Q}_{\mathrm{X}/\mathrm{Z}}. \tag{12.63}$$

This follows from (1.154).

Now let $\mathtt{h}$ be any bicommutative bimonoid. Observe that for $h \in \mathtt{h}[\mathrm{Z}]$,

$$S_{\mathrm{Z}}(h) = \mathrm{s}(\mathtt{Tak})_{\mathrm{Z}} \cdot h, \tag{12.64}$$

where the rhs refers to the commutative characteristic operation (10.34).

12.8.5. Commutative Takeuchi series. In the language of Section 9.6.1, the family of elements $\mathrm{s}(\mathtt{Tak})_{\mathrm{Z}}$, one for each flat Z, defines a series of the species of flats Π. We call this the *commutative Takeuchi series*, and denote it by $\mathrm{s}(\mathtt{Tak})$. It is the image of the Takeuchi series under the support map (7.84). Since the latter is a morphism of bimonoids, the commutative Takeuchi series is also group-like. This can also be seen directly by applying coproduct formula (7.40) on (12.63). By Theorem 9.124, the logarithm of $\mathrm{s}(\mathtt{Tak})$ is a primitive series of Π. It is given as follows.

Lemma 12.80. *We have*

$$\log(\mathrm{s}(\mathtt{Tak}))_{\mathrm{X}} = (-1)^{\dim(\mathrm{X})} \mathtt{Q}_{\mathrm{X}/\mathrm{X}}. \tag{12.65}$$

PROOF. This can be deduced from formula (12.60a) and Exercise 9.125. Alternatively, one can proceed directly using definition (9.60b). Either apply product formula (7.36) on (12.62) and use identity (1.97), or apply product formula (7.40) on (12.63). One may also check instead that the exponential of (12.65) is $\mathrm{s}(\mathtt{Tak})$. $\square$

For any bicommutative bimonoid h, recall the map (10.50) which associates a series of $\mathrm{end}^{\times}(\mathsf{h})$ to any series of $\mathsf{\Pi}$. The image of the commutative Takeuchi series $\mathrm{s}(\mathtt{Tak})$ under this map is the antipode of h, that is, $\mathscr{S}(\Psi)(\mathrm{s}(\mathtt{Tak})) = S_{\mathsf{h}}$. This is a restatement of (12.64).

Exercise 12.81. Check that

$$\log(\mathrm{s}(\mathtt{Tak}))_{\mathrm{X}} = (-1)^{\dim(\mathrm{X})} \log(\delta)_{\mathrm{X}}, \tag{12.66}$$

where δ denotes the universal series of $\mathsf{\Pi}$, see Example 9.128. (Compare formulas (12.65) and (9.68).) Deduce formula (12.19) for the logarithm of the antipode of a bicommutative bimonoid.

12.8.6. Two-sided Takeuchi element and two-sided characteristic operations. For each face A, define

$$\mathbf{Tak}_A := \sum_{F:\, F \geq A} (-1)^{\dim(F)}\, \mathtt{H}_{(F,F)}. \tag{12.67}$$

This is an element of the q-Janus algebra $\mathsf{J}^o_q[A]$. It agrees up to sign with the two-sided Takeuchi element (1.155) of the arrangement $\mathcal{A}_A$.

Now let h be any q-bimonoid. Observe that for $h \in \mathsf{h}[A]$,

$$S_A(h) = \mathbf{Tak}_A \cdot h, \tag{12.68}$$

where the rhs refers to the two-sided characteristic operation (10.52).

The following reformulates Lemma 12.12.

Lemma 12.82. *Fix a q-bimonoid h, and faces $F \geq A$. For any $h \in \mathsf{h}[F]$, we have*

$$\mu_A^{A\overline{F}} (\beta_q)_{A\overline{F},F}(\mathbf{Tak}_F \cdot h) = \mathbf{Tak}_A \cdot \mu_A^F(h),$$

and for any $h \in \mathsf{h}[A]$, we have

$$\mathbf{Tak}_F \cdot \Delta_A^F(h) = (\beta_q^{-1})_{F,A\overline{F}} \Delta_A^{A\overline{F}}(\mathbf{Tak}_A \cdot h).$$

The following reformulates (1.157), and we give another proof for it.

Lemma 12.83. *For any A, and faces F and F' both greater than A, we have*

$$\mathtt{H}_{(A\overline{F}/A,A\overline{F'}/A)} \cdot \mathbf{Tak}_A = \mathbf{Tak}_A \cdot \mathtt{H}_{(F/A,F'/A)}.$$

Proof. Let h be any q-bimonoid. By combining (12.2), (12.13), (12.14), we have

$$\begin{tikzcd}
\mathsf{h}[A] \arrow[rr, "\Delta"] \arrow[d, "S_A"'] & & \mathsf{h}[F'] \arrow[r, "\beta"] \arrow[d, "S_{F'}"'] & \mathsf{h}[F] \arrow[rrr, "\mu"] \arrow[d, "S_F"] & & & \mathsf{h}[A] \arrow[d, "S_A"] \\
\mathsf{h}[A] \arrow[r, "\Delta"'] & \mathsf{h}[A\overline{F'}] \arrow[r, "\beta_q^{-1}"'] \arrow[rrr, bend right, "\beta"'] & \mathsf{h}[F'] \arrow[r, "\beta"'] & \mathsf{h}[F] \arrow[r, "\beta_q"'] & \mathsf{h}[A\overline{F}] \arrow[r, "\mu"'] & \mathsf{h}[A]. &
\end{tikzcd}$$

While combining the three β's into a single β, the q's cancel out because $\operatorname{dist}(A\overline{F'}, F') = \operatorname{dist}(F, A\overline{F})$. We thus obtain

$$(\mu_A^{A\overline{F}} \beta_{A\overline{F}, A\overline{F'}} \Delta_A^{A\overline{F'}}) S_A = S_A(\mu_A^F \beta_{F,F'} \Delta_A^{F'}).$$

Translating these operators to elements of the q-Janus algebra $\mathsf{J}_q^o[A]$, we obtain the desired identity. $\square$

Lemma 12.84. *In any q-bimonoid, the square of the antipode, on the A-component, is given by*

$$S_A^2 = \sum_{A \leq K \leq G} (-1)^{\dim(K)+\dim(G)} \mu_A^G (\beta_q)_{G, A\overline{K}G} \Delta_A^{A\overline{K}G}. \tag{12.69}$$

The sum is over K and G.

Proof. This follows from (1.156) applied to the arrangement $\mathcal{A}_A$ and the fact that two-sided characteristic operations define an action of $\mathsf{J}_q^o[A]$ on $\mathsf{h}[A]$ for any q-bimonoid h. $\square$

Exercise 12.85. Deduce Lemma 12.75 as a special case of Lemma 12.84.

12.9. Set-bimonoids

We now briefly comment on the antipode problem for bimonoids obtained by linearizing set-bimonoids. We make use of the notation and discussion in Section 10.4.

12.9.1. Bimonoids. Let h be a set-bimonoid, and let $\mathsf{h} := \Bbbk\mathrm{h}$ denote its linearization with canonical basis $\mathtt{H}$. For $x, y \in \mathrm{h}[A]$, define

$$\Sigma_{x,y} = \{F/A \in \Sigma[A] \mid F/A \cdot x = y\}.$$

For convenience, the dependence on A has been suppressed in the notation. The structure constants of the antipode of h can be determined from the sets $\Sigma_{x,y}$ as follows.

Lemma 12.86. *We have*

$$S_A(\mathtt{H}_x) = \sum_y \Big(\sum_{F/A \in \Sigma_{x,y}} (-1)^{\dim(F)} \Big) \mathtt{H}_y. \tag{12.70}$$

PROOF. By formula (12.59),

$$S_A(\mathtt{H}_x) = \sum_{F:\, F \geq A} (-1)^{\dim(F)} \mathtt{H}_{F/A \cdot x}$$

from which the result follows. □

Recall from Proposition 11.40 that a cocommutative set-bimonoid is the same as a left $\Sigma[\mathcal{A}]$-set. The sets $\Sigma_{x,y}$ in the context of left $\Sigma[\mathcal{A}]$-sets have been studied in [**21**, Section 7.7]. In particular, we introduced there a technique for evaluating the sum inside the parenthesis in (12.70) involving the Euler characteristic of a relative pair. See [**21**, Formula (7.32)]. The various descent identities are illustrations of this technique, and lead to a solution of the antipode problem. For instance, the bimonoids $\mathsf{\Gamma}$ and $\mathsf{\Sigma}$ are linearizations of cocommutative set-bimonoids. So the sum inside the parenthesis in (12.70) can be evaluated using the descent identities which is precisely what we did in the proofs of (12.22) and (12.32).

Dually, a commutative set-bimonoid is the same as a right $\Sigma[\mathcal{A}]$-set. In this case, it is better to write

$${}_{x,y}\Sigma = \{F/A \in \Sigma[A] \mid x \cdot F/A = y\}.$$

These sets in the context of right $\Sigma[\mathcal{A}]$-sets have been studied in [**21**, Section 7.9], with [**21**, Formula (7.41)] giving a possible method for evaluating the sum inside the parenthesis in (12.70). The various lune identities are illustrations of this technique, and lead to a solution of the antipode problem. For instance, this applies to the bimonoid J_{ab} which is the linearization of a commutative set-bimonoid. See Exercise 12.51.

Exercise 12.87. Let c be a set-comonoid, and let $\mathcal{T}(\mathrm{c})$ be the set-bimonoid which is free on c. Explicitly, the components of $\mathcal{T}(\mathrm{c})$ are given by (3.36), the product is inclusion, and the coproduct Δ_A^G sends the piece $\mathrm{c}[H]$ to the piece $\mathrm{c}[GH]$ via the composite map $\beta_{GH,HG}\Delta_H^{HG}$. Observe that

$$\Bbbk\mathcal{T}(\mathrm{c}) = \mathcal{T}(\Bbbk\mathrm{c}).$$

Compute the antipode of this bimonoid using (12.70). This is a special case of (12.40) for $q = 1$. (The sets $\Sigma_{x,y}$ are either empty or equal to $\Sigma_{F,G}$, where $x \in \mathrm{c}[F]$ and $y \in \mathrm{c}[G]$. So we can apply the descent identity (1.76a).)

12.9.2. Bicommutative bimonoids. We now look at the commutative analogue. Let h be a bicommutative set-bimonoid, and let $\mathsf{h} := \Bbbk\mathrm{h}$ denote its linearization with canonical basis $\mathtt{H}$. For $x, y \in \mathrm{h}[\mathrm{Z}]$, define

$$\Pi_{x,y} = \{\mathrm{X}/\mathrm{Z} \in \Pi[\mathrm{Z}] \mid \mathrm{X}/\mathrm{Z} \cdot x = y\}.$$

The structure constants of the antipode of h can be determined from these sets as follows.

Lemma 12.88. *We have*

$$S_{\mathrm{Z}}(\mathtt{H}_x) = \sum_y \Big(\sum_{\mathrm{X}/\mathrm{Z} \in \Pi_{x,y}} (-1)^{\dim(\mathrm{X})} c_{\mathrm{Z}}^{\mathrm{X}} \Big) \mathtt{H}_y. \tag{12.71}$$

PROOF. The proof is similar to that of (12.70). We now employ formula (12.64). □

Recall from Proposition 11.40 that a bicommutative set-bimonoid is the same as a $\Pi[\mathcal{A}]$-set. Information about the set $\Pi_{x,y}$ in the latter context is given in [**21**, Section 7.10], with [**21**, Formula (7.44)] giving a possible method to evaluate the sum inside the parenthesis in (12.71). The descent-lune identities are illustrations of this technique.

Notes

Hopf algebras.

Early history. The antipode map appears in the book of Cartan and Eilenberg [**189**, Section XI.8] under the name 'antipodism'. This term is used later by Dieudonné [**250**, Chapter II, Section 2.8], Cartier [**202**, pages 558 and 568], Gabriel [**335**, Exposé VIIB, page 511], Ditters [**254**, page 580], [**255**, page 1]. The antipode map for connected graded bialgebras is studied by Milnor and Moore under the name 'conjugation' [**695**, Proposition 8.2 and Definition 8.4], [**696**, Definition 6.9]. It is also present in the earlier notes by Moore [**706**, page 12, Theorem 5] but without any specific name. The 'conjugation' terminology is followed by Steenrod [**849**, Section II.4]. The term 'antipode' is used by Kostant [**540**, Section 1.1] and by Heyneman and Sweedler [**432**, page 204], [**864**, page 17], [**867**, page 71]; also see the later notes by Kaplansky [**510**, Section 6]. The antipode map is also mentioned by Hochschild [**442**, pages 27 and 28] under the name 'symmetry'. In later works, he switches to the term 'antipode' [**446**], [**444**, page 18], [**445**, page 5]. Similarly, Larson uses the term 'conjugation' in [**564**, Definition 5.4] and 'antipode' later in [**565**] and in the paper with Sweedler [**567**, page 77]. Bourbaki uses the term 'inversion' [**149**, Section III.11, Exercise 4]; this terminology is followed by Serre [**822**, Section 3.1]. Chase and Sweedler [**206**, page 54], Ditters [**253**, Definition 1.1.6], Milne [**693**, Definition 3.3] mention both 'antipode' and 'inverse'. The term 'involution' is used by Grünenfelder [**383**, Definition I.1.11], [**384**], Michaelis [**685**, page 34], Demazure and Gabriel [**242**, page 147], [**243**, page 181]. In [**241**, Section II.3, page 23], the term 'antipode' is mentioned along with 'involution'. The term 'coinverse' is used by Manin [**644**, Section 1.15.6], Bröcker and tom Dieck [**158**, Section III.7]. The term 'antipode' is now standard.

For a topological analogue of the antipode, see Whitehead's book [**909**, Chapter X.2.2]: if a connected CW-complex has an H-space structure, then it is group-like, that is, the multiplication admits a homotopy inverse.

Takeuchi formula. For a connected bialgebra, the antipode always exists. The Takeuchi formula for the antipode of a connected Hopf algebra appears in work of Sweedler [**867**, Proof of Lemma 9.2.3] and Takeuchi [**871**, Proof of Lemma 14]. For later references, see [**703**, Lemma 5.2.10], [**813**, page 314], [**17**, Formula (3.2)], [**18**, Formula (2.55)], [**637**, Formula (18)], [**771**, Proposition 6.2.2 and Lemma 7.6.2]. Compare with the formula given in (12.3).

For a connected bialgebra, the antipode can also be calculated recursively through formulas of Milnor and Moore [**695**, proof of Proposition 8.2]; see also [**849**, Section II.4], [**277**, Lemma 2.1], [**821**, Section 10.6], [**637**, Formulas (19) and (20)], [**18**, Formulas (2.56)].

Interaction with bimonoid filtrations. For Hopf algebras, a discussion analogous to the one in Section 12.1.5 is given by Radford [**771**, Section 7.9]. For the specific case of the primitive filtration, another reference is [**867**, Theorem 9.2.2, item (3)]. The proof given there corresponds to Exercise 12.14. For ideas related to the analogue of Exercise 12.6, see [**16**, Sections 1 and 2]. The classical analogue of Exercise 12.8 says that $S(x) = -x$ for x primitive. This simple fact is present in early work of Leray, it is the case $k = -1$ of [**595**, Theorem 11]. It also follows immediately from the recursive formula of Milnor and Moore. For later references, see for instance [**867**, pages 72 and 73], [**688**, Lemma 2.74, item (b)], [**637**, Proposition 16, item (i)], [**612**, Formula (9.32)].

Interaction with op and cop constructions. For Hopf algebras, analogues of the results in Lemma 12.12, Lemma 12.15, Lemma 12.16, Lemma 12.17 are discussed in many places. The original sources are those of Milnor and Moore [**695**, Propositions 8.6, 8.7, 8.8], [**696**, Propositions 6.14, 6.16, 6.17], [**706**, pages 12 to 18], Heyneman and Sweedler [**864**, page 18], [**867**, Proposition 4.0.1], [**432**, Proposition 1.5.2], Grünenfelder [**383**, Satz I.1.9]. Some later references are by Abe [**1**, Theorem 2.1.4], Caenepeel [**181**, Proposition 7.1.3], Hazewinkel [**427**, Proposition 37.1.8], Kassel [**517**, Section III.3], Klimyk and Schmüdgen [**535**, Section 1.2.4], Lambe and Radford [**557**, Propositions 1.6.1 and 1.6.2, Corollary 1.6.1], Majid [**632**, Proposition 1.3.1, Exercise 1.3.3], Montgomery [**703**, Proposition 1.5.10, Lemma 1.5.11, Corollary 1.5.12]. Some recent references are [**171**, Section 15.4], [**182**, Proposition 2], [**228**, Propositions 4.2.6 and 4.2.7, Corollary 4.2.8], [**428**, Theorems 3.3.9 and 3.3.10, Corollary 3.3.11], [**771**, Propositions 7.1.9 and 7.1.10, Corollary 7.1.11], [**832**, Proposition 2.8, Corollary 2.9, Corollary 2.12], [**859**, Proposition 9.1], [**882**, Section 1.3.3], [**612**, Section 9.3.5].

For the more general context of Hopf monoids in braided monoidal categories, see [**18**, Propositions 1.22 and 1.23, Corollary 1.24], [**324**, Propositions 7.1.10 and 7.1.12], [**175**, Proposition 2.65].

Our derivation of these results is based on the crucial Lemma 12.11 which we have called the antipode opposition lemma. This lemma makes a formal connection between the antipode and the antipodal map $F \mapsto \overline{F}$ on arrangements.

Antipode formulas. Recall the tensor and shuffle Hopf algebras from Tables 6.2 and 6.3. The antipodes of $\mathcal{T}(V)$ and $\mathcal{T}^\vee(V)$ are mentioned by Reutenauer [**777**, pages 19 and 35]. These are classical analogues of formulas (12.41) and (12.43) for $q = 1$. The antipode of $\mathcal{T}^\vee(A)$ (when A is commutative) is given by Hoffman [**449**, Theorem 3.2], see also his paper with Ihara [**450**, Theorem 4.2]. Other references are by Hudson [**467**, Theorem 1], [**468**, Formula (6.10)], see also [**227**, Definition 5.7, item (iii) and Remark 5.8]. This is the analogue of Theorem 12.53 for $q = 1$. Hoffman also states the dual result on page 58; this can be viewed as the analogue of Theorem 12.52 for $q = 1$. An analogue of formula (12.46) for the coalgebra $C := \mathcal{T}^\vee(V)_+$ is given by Manchon [**636**, Theorem I.1]. He then says that the antipode of the Hopf algebra $\mathcal{T}(\mathcal{T}^\vee(V)_+)$ can be computed from this formula using that the antipode is an antimorphism of algebras. This can be seen in analogy with Exercise 12.61.

Interest in the antipode from a combinatorial perspective originated in Schmitt's thesis [**809**], [**810**] and in work of Haiman and Schmitt [**403**], see also [**811**], [**813**]. We now consider specific examples, see Table 7.3. The classical analogue of (12.35) for $q = 1$ is the antipode formula for the Hopf algebra of quasisymmetric functions. It was obtained independently by Malvenuto and Reutenauer [**634**, Corollaire 4.20], [**635**, Corollary 2.3] and Ehrenborg [**277**, Proposition 3.4], see also [**23**, Formula (1.7)], [**11**, Formula (4.7)], [**17**, page 36], [**18**, page 573], [**673**, Lemma 6.16], [**377**, Theorem 5.1.11]. The q-analogue is given in [**18**, page 575]. The analogue of (12.32) for $q = 1$ is the antipode formula for the Hopf algebra of noncommutative symmetric functions in [**347**, Proposition 3.9], see also [**428**, Formulas (3.4.9) and (6.1.5)], [**673**, Proposition 6.4]. The analogues of (12.27) and (12.28) are the antipode formulas for the Hopf algebra of symmetric functions, see for instance [**17**, page 35], [**673**, page 72], [**377**, Section 2.4]. The analogue of (12.30) is the antipode formula for the Hopf algebra of simple graphs obtained by Humpert and Martin [**472**, Theorem 3.1]. For the Hopf algebra of permutations, see the Notes to Chapter 15. Explicit computations of antipode formulas for related Hopf algebras appear in [**13**], [**16**], [**18**, Chapter 17], [**24**, Section 6], [**95**], [**96**], [**104**], [**220**, page 219], [**287**], [**305**], [**364**, Section 14.1], [**545**], [**546**, Section 3.7.2], [**570**], [**676**], [**749**]. See also the paragraph on Hopf monoids below for additional related work.

Hopf monoids in Joyal species. The analogue of the Takeuchi formula (12.1) for connected Joyal q-bimonoids is discussed in our monographs [**18**, Proposition 8.13], [**19**, Section 5.3]. For the analogue of the formulas of Milnor and Moore, see [**18**, Proposition 8.14], [**19**, Section 5.2]. The analogue of diagram (12.13) is given in [**19**, Diagram (44)], followed by similar related facts. The analogue of Exercise 12.8 is given in [**19**, Proposition 12]. The analogue of Lemma 12.24 says that for a connected Joyal q-bimonoid h, $\log(S_{\mathsf{h}}) = -\log(\mathrm{id}_{\mathsf{h}})$. This follows from [**19**, Proposition 68].

Antipode formulas for (co)free Hopf monoids in Joyal species are obtained in [**18**, Section 11.8]. These follow by specializing the formulas in Sections 12.6 and 12.7 to the braid arrangements. The precise dictionary is as follows. The analogue of Theorem 12.52 is [**18**, Theorem 11.38], the analogue of Theorem 12.53 is [**18**, Theorem 11.39], the analogue of Theorem 12.62 is [**18**, Theorem 11.40], the analogue of Theorem 12.63 is [**18**, Theorem 11.41], the analogue of Theorem 12.67 is [**18**, Theorem 11.42], and the analogue of Theorem 12.69 is [**18**, Theorem 11.43]. We point out a conflict of notation: The term $c^{\mathrm{Y}}_{\mathrm{X}}\Delta^{\mathrm{Y}}_{\mathrm{X}}$ in this monograph is denoted $\Delta^{\mathrm{Y}}_{\mathrm{X}}$ in that reference. There is a similar conflict in the signed case.

The special case (12.41) of Theorem 12.52 is also given in [**19**, Theorem 18]. Similarly, the special case (12.48) of Theorem 12.62 is given in [**19**, Theorem 24].

We now consider specific examples. For the exponential species, the analogue of (12.20) is given in [**18**, Example 8.15], [**19**, Section 9.1], the analogue of (12.21) is given in [**18**, Section 9.3.2]. For the species of chambers, the analogue of (12.22) is given in [**18**, Example 8.16] for $q = 1$, and in [**18**, Propositions 9.14 and 12.3], [**19**, Formula (99)] for general q, the analogue of (12.23) is given in [**18**, Proposition 12.5]. For the species of flats, the analogue of (12.25) is given in [**18**, Theorem 12.47], [**19**, Theorem 33], the analogue of (12.26) is given in [**19**, Proposition 35], the analogue of (12.27) and (12.28) is [**18**, Theorem 12.44]. For the species of faces, the analogue of (12.32) is given in [**18**, Theorem 12.24], [**19**, Theorem 60], the analogue of (12.33) is given in [**19**, Proposition 59], the analogue of (12.35) is given in [**18**, Theorem 12.21]. For the species of top-nested faces, the analogue of (12.36) is given in [**18**, Theorem 12.36], the analogue of (12.37) is given in [**18**, Theorem 12.34]. For the species of pairs of chambers, see the Notes to Chapter 15.

Several explicit computations of antipode formulas are given in [**10**], together with combinatorial applications. The analogue of the antipode formula (12.30) is given in [**10**, Corollary 13.7], see also [**19**, Theorem 36]. For additional work on the antipode of Hopf monoids in Joyal species, see [**67**], [**94**]. See also the references above under the heading 'antipode formulas'.

Bimonoids for hyperplane arrangements. The notion of a bimonoid and its antipode which we have presented here can be viewed as progress on a question raised in our monograph [**18**, Question 12.67]. We recall that our bimonoids are to be seen in analogy with connected graded bialgebras which is the reason why we can talk of the antipode. Note very carefully that our entire discussion is built from the Takeuchi formula, which is very different from the way it is done in the classical theory (via the convolution algebra).

The minimum polynomial of the two-sided Takeuchi element was computed in [**21**, Theorem 12.52]. For any bimonoid h and each face A, this gives information about the minimum polynomial of S_A as a linear operator on $\mathsf{h}[A]$. The connection between the two-sided Takeuchi element and the antipode is via two-sided characteristic operations (12.68). It is mentioned on [**21**, page 364] that formula (1.157) is closely related to the fact that the antipode of a Hopf algebra reverses products and coproducts. The proof of Lemma 12.83 brings out this connection.

Part III

Structure theory for bimonoids

CHAPTER 13

Loday–Ronco, Leray–Samelson, Borel–Hopf

We discuss some important rigidity theorems related to the universal constructions in Chapter 6. They usually take the form of an adjoint equivalence between suitable categories. In particular, we improve upon the results mentioned in Section 3.10. A summary is provided in Table 13.1.

TABLE 13.1. Loday–Ronco, Leray–Samelson, Borel–Hopf.

Theorem	Starting data	Statement
Loday–Ronco	0-bimonoid h	$\mathcal{T}_0(\mathcal{P}(\mathsf{h})) \cong \mathsf{h}$ as 0-bimonoids $\mathsf{h} \cong \mathcal{T}_0(\mathcal{Q}(\mathsf{h}))$ as 0-bimonoids
	species p	$\mathsf{p} = \mathcal{P}(\mathcal{T}_0(\mathsf{p}))$ as species $\mathcal{Q}(\mathcal{T}_0(\mathsf{p})) = \mathsf{p}$ as species
q-analogue of Loday–Ronco	q-bimonoid h for q not a root of unity	$\mathcal{T}_q(\mathcal{P}(\mathsf{h})) \cong \mathsf{h}$ as q-bimonoids $\mathsf{h} \cong \mathcal{T}_q^\vee(\mathcal{Q}(\mathsf{h}))$ as q-bimonoids
	species p	$\mathsf{p} = \mathcal{P}(\mathcal{T}_q(\mathsf{p}))$ as species $\mathcal{Q}(\mathcal{T}_q^\vee(\mathsf{p})) = \mathsf{p}$ as species
Leray–Samelson	bicomm. bimonoid h	$\mathcal{S}(\mathcal{P}(\mathsf{h})) \cong \mathsf{h}$ as bimonoids $\mathsf{h} \cong \mathcal{S}(\mathcal{Q}(\mathsf{h}))$ as bimonoids
	species p	$\mathsf{p} = \mathcal{P}(\mathcal{S}(\mathsf{p}))$ as species $\mathcal{Q}(\mathcal{S}(\mathsf{p})) = \mathsf{p}$ as species
Borel–Hopf	cocomm. bimonoid h	$\mathcal{S}(\mathcal{P}(\mathsf{h})) \cong \mathsf{h}$ as comonoids
	comm. bimonoid h	$\mathsf{h} \cong \mathcal{S}(\mathcal{Q}(\mathsf{h}))$ as monoids

The Loday–Ronco theorem says that the category of 0-bimonoids is equivalent to the category of species. The two categories are linked by the functor $\mathcal{T}_0$ in one direction, and either the primitive part functor $\mathcal{P}$ or the indecomposable part functor $\mathcal{Q}$ in the other direction. In particular, 0-bimonoids are both free and cofree.

The Leray–Samelson theorem says that the category of bicommutative bimonoids is equivalent to the category of species. The two categories are now linked by the functor $\mathcal{S}$ in one direction, and either the primitive part

TABLE 13.2. Structure theorems via zeta and Möbius functions.

Data	Maps	Formulas
bicomm. bimonoid h	$\mathcal{S}(\mathcal{P}(\mathsf{h})) \to \mathsf{h}$	$\mathcal{P}(\mathsf{h})[\mathrm{X}] \xrightarrow{\mu_{\mathrm{Z}}^{\mathrm{X}}} \mathsf{h}[\mathrm{Z}]$
	$\mathsf{h} \to \mathcal{S}(\mathcal{P}(\mathsf{h}))$	$\mathsf{h}[\mathrm{Z}] \xrightarrow{\sum_{\mathrm{Y}} \mu(\mathrm{X},\mathrm{Y})\, \mu_{\mathrm{X}}^{\mathrm{Y}} \Delta_{\mathrm{Z}}^{\mathrm{Y}}} \mathcal{P}(\mathsf{h})[\mathrm{X}]$
	$\mathsf{h} \to \mathcal{S}(\mathcal{Q}(\mathsf{h}))$	$\mathsf{h}[\mathrm{Z}] \xrightarrow{\Delta_{\mathrm{Z}}^{\mathrm{X}}} \mathcal{Q}(\mathsf{h})[\mathrm{X}]$
	$\mathcal{S}(\mathcal{Q}(\mathsf{h})) \to \mathsf{h}$	$\mathcal{Q}(\mathsf{h})[\mathrm{X}] \xrightarrow{\sum_{\mathrm{Y}} \mu(\mathrm{X},\mathrm{Y})\, \mu_{\mathrm{Z}}^{\mathrm{Y}} \Delta_{\mathrm{X}}^{\mathrm{Y}}} \mathsf{h}[\mathrm{Z}]$
cocomm. bimonoid h	$\mathcal{S}(\mathcal{P}(\mathsf{h})) \to \mathsf{h}$	$\mathcal{P}(\mathsf{h})[\mathrm{X}] \xrightarrow{\sum_F \boldsymbol{\zeta}(A,F)\, \mu_A^F\, \beta_{F,\mathrm{X}}} \mathsf{h}[A]$
	$\mathsf{h} \to \mathcal{S}(\mathcal{P}(\mathsf{h}))$	$\mathsf{h}[A] \xrightarrow{\sum_G \boldsymbol{\mu}(K,G)\, \beta_{\mathrm{X},K}\, \mu_K^G \Delta_A^G} \mathcal{P}(\mathsf{h})[\mathrm{X}]$
comm. bimonoid h	$\mathsf{h} \to \mathcal{S}(\mathcal{Q}(\mathsf{h}))$	$\mathsf{h}[A] \xrightarrow{\sum_F \boldsymbol{\zeta}(A,F)\, \beta_{\mathrm{X},F} \Delta_A^F} \mathcal{Q}(\mathsf{h})[\mathrm{X}]$
	$\mathcal{S}(\mathcal{Q}(\mathsf{h})) \to \mathsf{h}$	$\mathcal{Q}(\mathsf{h})[\mathrm{X}] \xrightarrow{\sum_G \boldsymbol{\mu}(K,G)\, \mu_A^G \Delta_K^G \beta_{K,\mathrm{X}}} \mathsf{h}[A]$
0-bimonoid h	$\mathcal{T}_0(\mathcal{P}(\mathsf{h})) \to \mathsf{h}$	$\mathcal{P}(\mathsf{h})[F] \xrightarrow{\mu_A^F} \mathsf{h}[A]$
	$\mathsf{h} \to \mathcal{T}_0(\mathcal{P}(\mathsf{h}))$	$\mathsf{h}[A] \xrightarrow{\sum_G (-1)^{\mathrm{rk}(G/F)}\, \mu_F^G \Delta_A^G} \mathcal{P}(\mathsf{h})[F]$
	$\mathsf{h} \to \mathcal{T}_0(\mathcal{Q}(\mathsf{h}))$	$\mathsf{h}[A] \xrightarrow{\Delta_A^F} \mathcal{Q}(\mathsf{h})[F]$
	$\mathcal{T}_0(\mathcal{Q}(\mathsf{h})) \to \mathsf{h}$	$\mathcal{Q}(\mathsf{h})[F] \xrightarrow{\sum_G (-1)^{\mathrm{rk}(G/F)}\, \mu_A^G \Delta_F^G} \mathsf{h}[A]$
q-bimonoid h for q not a root of unity	$\mathcal{T}_q^{\vee}(\mathcal{P}(\mathsf{h})) \to \mathsf{h}$	$\mathcal{P}(\mathsf{h})[F'] \xrightarrow{\sum_F \boldsymbol{\zeta}_q(A,F,F')\, \mu_A^F \beta_{F,F'}} \mathsf{h}[A]$
	$\mathsf{h} \to \mathcal{T}_q^{\vee}(\mathcal{P}(\mathsf{h}))$	$\mathsf{h}[A] \xrightarrow{\sum_{G,G'} \boldsymbol{\mu}_q(F',G,G') \mu_{F'}^G \beta_{G,G'} \Delta_A^{G'}} \mathcal{P}(\mathsf{h})[F']$
	$\mathcal{T}_q(\mathcal{P}(\mathsf{h})) \to \mathsf{h}$	$\mathcal{P}(\mathsf{h})[F] \xrightarrow{\mu_A^F} \mathsf{h}[A]$
	$\mathsf{h} \to \mathcal{T}_q(\mathcal{P}(\mathsf{h}))$	$\mathsf{h}[A] \xrightarrow{\mathbf{log}_q(\mathrm{id})_F \sum_{F'} \boldsymbol{\zeta}_q(A,F,F') \beta_{F,F'} \Delta_A^{F'}} \mathcal{P}(\mathsf{h})[F]$
	$\mathsf{h} \to \mathcal{T}_q(\mathcal{Q}(\mathsf{h}))$	$\mathsf{h}[A] \xrightarrow{\sum_F \boldsymbol{\zeta}_q(A,F,F')\, \beta_{F',F} \Delta_A^F} \mathcal{Q}(\mathsf{h})[F']$
	$\mathcal{T}_q(\mathcal{Q}(\mathsf{h})) \to \mathsf{h}$	$\mathcal{Q}(\mathsf{h})[F'] \xrightarrow{\sum_{G,G'} \boldsymbol{\mu}_q(F',G,G') \mu_A^{G'} \beta_{G',G} \Delta_{F'}^G} \mathsf{h}[A]$
	$\mathsf{h} \to \mathcal{T}_q^{\vee}(\mathcal{Q}(\mathsf{h}))$	$\mathsf{h}[A] \xrightarrow{\Delta_A^F} \mathcal{Q}(\mathsf{h})[F]$
	$\mathcal{T}_q^{\vee}(\mathcal{Q}(\mathsf{h})) \to \mathsf{h}$	$\mathcal{Q}(\mathsf{h})[F] \xrightarrow{\sum_{F'} \boldsymbol{\zeta}_q(A,F,F') \mu_A^{F'} \beta_{F',F}\, \mathbf{log}_q(\mathrm{id})_F} \mathsf{h}[A]$

TABLE 13.3. Structure theorems via families of idempotent operators.

Theorem	Data	Inverse isomorphisms
Leray–Samelson	bicomm. bimonoid h	$\mathcal{P}(\mathsf{h})[\mathrm{X}] \underset{\Delta^{\mathrm{X}}_{\mathrm{Z}}}{\overset{\mu^{\mathrm{X}}_{\mathrm{Z}}}{\rightleftarrows}} \mathsf{Q}_{\mathrm{X/Z}} \cdot \mathsf{h}[\mathrm{Z}]$ $\mathsf{Q}_{\mathrm{X/Z}} \cdot \mathsf{h}[\mathrm{Z}] \underset{\mu^{\mathrm{X}}_{\mathrm{Z}}}{\overset{\Delta^{\mathrm{X}}_{\mathrm{Z}}}{\rightleftarrows}} \mathcal{Q}(\mathsf{h})[\mathrm{X}]$
Borel–Hopf	cocomm. bimonoid h	$\mathcal{P}(\mathsf{h})[\mathrm{X}] \underset{\beta_{\mathrm{X},G}\Delta^{G}_{A}}{\overset{\sum_F \boldsymbol{\zeta}(A,F)\, \mu^F_A\, \beta_{F,\mathrm{X}}}{\rightleftarrows}} \mathsf{E}_{\mathrm{X}/A} \cdot \mathsf{h}[A]$
	comm. bimonoid h	$\mathcal{Q}(\mathsf{h})[\mathrm{X}] \underset{\sum_F \boldsymbol{\zeta}(A,F)\, \beta_{\mathrm{X},F}\Delta^F_A}{\overset{\mu^G_A \beta_{G,\mathrm{X}}}{\rightleftarrows}} \mathsf{h}[A] \cdot \mathsf{E}_{\mathrm{X}/A}$
q-analogue of Loday–Ronco	q-bimonoid h for q not a root of unity	$\mathcal{P}(\mathsf{h})[F] \underset{\Delta^F_A}{\overset{\mu^F_A}{\rightleftarrows}} \mathsf{Q}_{(F/A,F/A)} \cdot \mathsf{h}[A]$ $\mathsf{Q}^d_{(F/A,F/A)} \cdot \mathsf{h}[A] \underset{\mu^F_A}{\overset{\Delta^F_A}{\rightleftarrows}} \mathcal{Q}(\mathsf{h})[F]$
		$\mathcal{P}(\mathsf{h})[F'] \underset{\Delta^{F'}_A}{\overset{\sum_F \boldsymbol{\zeta}_q(A,F,F')\mu^F_A\beta_{F,F'}}{\rightleftarrows}} \mathsf{Q}^d_{(F'/A,F'/A)} \cdot \mathsf{h}[A]$ $\mathcal{Q}(\mathsf{h})[F] \underset{\sum_{F'} \boldsymbol{\zeta}_q(A,F,F')\, \beta_{F,F'}\Delta^{F'}_A}{\overset{\mu^F_A}{\rightleftarrows}} \mathsf{Q}_{(F/A,F/A)} \cdot \mathsf{h}[A]$

functor $\mathcal{P}$ or the indecomposable part functor $\mathcal{Q}$ in the other direction. In particular, bicommutative bimonoids are both free commutative and cofree cocommutative. There is also a signed analogue of Leray–Samelson which applies to signed bicommutative signed bimonoids, where the role of the functor $\mathcal{S}$ is played by $\mathcal{E}$.

The Borel–Hopf theorem says that any cocommutative bimonoid is cofree on its primitive part, and dually, any commutative bimonoid is free on its indecomposable part. This result also has a signed analogue which applies to signed (co)commutative signed bimonoids. Observe that Borel–Hopf applies in a more general situation than Leray–Samelson but it does not provide an adjoint equivalence. This problem is addressed by the more general Cartier–Milnor–Moore theorem which we will prove in Chapter 17.

There are some interesting ways to interpolate the above rigidity theorems. Let $\sim$ be a partial-support relation on faces. Borel–Hopf and Leray–Samelson can be interpolated by working with $\sim$-bicommutative bimonoids. Similarly, Loday–Ronco and Leray–Samelson can be unified by working with 0-$\sim$-bicommutative bimonoids, where $\sim$ is in addition geometric.

The Loday–Ronco theorem is a special case of a more general result in which 0-bimonoids are replaced by q-bimonoids with q not a root of unity. The categories are now linked either by the functors $\mathcal{T}_q$ and $\mathcal{P}$, or by the functors $\mathcal{T}_q^\vee$ and $\mathcal{Q}$. We refer to this result as the rigidity of q-bimonoids. Invertibility of the Varchenko matrix associated to the q-distance function on faces plays a critical role here.

We present three broad approaches to the rigidity theorems in Table 13.1.

- The first approach is elementary and proceeds by an induction on the primitive filtration of the bimonoid. Here a key role is played by how the bimonoid axiom works on the primitive part. This was explained in Section 5.6.
- The second approach is more direct and proceeds by constructing an explicit inverse to the appropriate universal map. The universal map is defined using a zeta function and the inverse using a Möbius function. A compact summary is provided in Table 13.2. These maps have connections to the exponential and logarithm operators discussed in Chapter 9.
- The third approach is also constructive and employs (commutative, usual or two-sided) characteristic operations by suitable families of idempotents in the (Birkhoff, Tits or q-Janus) algebra, respectively, to decompose the given bimonoid. Relevant background results were discussed in Section 10.5. A compact summary is provided in Table 13.3.

All results are independent of the characteristic of the base field.

13.1. Loday–Ronco for 0-bimonoids

Every 0-bimonoid is the free monoid and the cofree comonoid on its primitive part. This shows that the categories of 0-bimonoids and species are equivalent. This is the Loday–Ronco theorem. It can also be phrased dually using the indecomposable part of the 0-bimonoid. (Recall that the primitive part and indecomposable part of a 0-bimonoid are naturally isomorphic.)

The main step in the proof is to show that for a 0-bimonoid h, the universal map into h from the free monoid on $\mathcal{P}(\mathsf{h})$ is an isomorphism. We give three arguments. The first one employs an induction based on the primitive filtration of h. The second one directly constructs the inverse isomorphism, and is linked to the inverse relationship between the 0-exponential and the 0-logarithm. The third one uses two-sided characteristic operations and is explained in a more general setup in Section 13.6.4.

13.1.1. Loday–Ronco. Freeness and cofreeness of 0-bimonoids. Recall from Section 6.4.3 the 0-bimonoid $\mathcal{T}_0(\mathsf{p})$ associated to a species p. The product is given by concatenation and the coproduct by deconcatenation. In view of (6.45), it does not matter whether we write $\mathcal{T}_0(\mathsf{p})$ or $\mathcal{T}_0^\vee(\mathsf{p})$.

Either from Proposition 6.56 or directly from (6.44), we note that

$$\mathcal{P}(\mathcal{T}_0(\mathsf{p})) = \mathsf{p}. \tag{13.1}$$

Thus, $\mathcal{P}\mathcal{T}_0 = \mathrm{id}$. We now show that composition of the functors in the other order, namely, $\mathcal{T}_0\mathcal{P}$, is also isomorphic to the identity.

Proposition 13.1. *For a* 0*-bimonoid* h*, there is a natural isomorphism*

$$\mathcal{T}_0(\mathcal{P}(\mathsf{h})) \xrightarrow{\cong} \mathsf{h} \tag{13.2}$$

of 0*-bimonoids. In particular, any* 0*-bimonoid is free as a monoid and cofree as a comonoid.*

The universal property of $\mathcal{T}_0$ stated in Theorem 6.31 (specialized to $q = 0$) applied to $f :=$ id on $\mathcal{P}(\mathsf{h})$ yields the morphism (13.2) of 0-bimonoids. Explicitly, using (6.41), the map on the A-component, on the F-summand, is

$$\mathcal{P}(\mathsf{h})[F] \hookrightarrow \mathsf{h}[F] \xrightarrow{\mu_A^F} \mathsf{h}[A]. \tag{13.3}$$

Equivalently, the map is μ^{k-1} on $\mathcal{P}(\mathsf{h})^k$ for each $k \geq 1$.

Proof. Let us denote the map (13.2) by f. We need to show that f is an isomorphism. In view of (13.1), f is injective (in fact, the identity) on the primitive part. Therefore, it is injective everywhere by Proposition 5.18. We now show it is surjective by an induction based on the primitive filtration of h. Take $z \in \mathsf{h}[A]$. By (5.20), there exists $k \geq 1$ such that $z \in \mathcal{P}_k(\mathsf{h})[A]$. If $k = 1$, then $z = f_A(z)$. Suppose $k \geq 2$. By (5.24), we have

$$\Delta_A^{k-1}(z) \in \mathcal{P}(\mathsf{h})^k[A]$$

and therefore

$$f_A\big(\Delta_A^{k-1}(z)\big) = \mu_A^{k-1}\Delta_A^{k-1}(z).$$

On the other hand, from Lemma 5.38, we have

$$z - \mu_A^{k-1}\Delta_A^{k-1}(z) \in \mathcal{P}_{k-1}(\mathsf{h})[A].$$

By induction hypothesis, this element belongs to the image of f_A, and we conclude that z is also in the image of f_A. □

Since the functor $\mathcal{T}_0$ and the primitive part functor $\mathcal{P}$ are adjoints (the case $q = 0$ of Theorem 6.30), and composing them either way yields natural isomorphisms with the identity, we obtain the following result.

Theorem 13.2. *The adjunction*

$$\mathcal{A}\text{-}\mathsf{Sp} \underset{\mathcal{P}}{\overset{\mathcal{T}_0}{\rightleftarrows}} 0\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$$

is an adjoint equivalence of categories.

This is the *Loday–Ronco theorem* for 0-bimonoids.

Corollary 13.3. *Any finite-dimensional* 0*-bimonoid is self-dual.*

Proof. Let h be a finite-dimensional 0-bimonoid. By Theorem 13.2, $\mathsf{h} = \mathcal{T}_0(\mathsf{p})$, where $\mathsf{p} = \mathcal{P}(\mathsf{h})$. Further, by (6.17) and (6.45), $\mathsf{h}^* = \mathcal{T}_0(\mathsf{p}^*)$. But we know that finite-dimensional species are self-dual, so $\mathsf{p} \cong \mathsf{p}^*$, and hence $\mathsf{h} \cong \mathsf{h}^*$, as required. □

Example 13.4. Recall from Section 7.3.9 the 0-bimonoid of chambers Γ_0. The fact that it is free and cofree on its primitive part can now be seen as a consequence of Proposition 13.1. Similarly, the self-duality of Γ_0 is a special case of Corollary 13.3.

13.1.2. The inverse isomorphism. We now give a more direct proof of Proposition 13.1 by explicitly describing the inverse of the map (13.2). We construct it using cofreeness of $\mathcal{T}_0(\mathcal{P}(\mathsf{h}))$ and the 0-logarithm of the identity map from Section 9.4.

For a 0-bimonoid h, consider the map of species $\mathbf{log}_0(\mathrm{id}) : \mathsf{h} \twoheadrightarrow \mathcal{P}(\mathsf{h})$ given in Proposition 9.107. Further, by Corollary 9.108, it is a morphism of monoids with the trivial product on $\mathcal{P}(\mathsf{h})$. Now apply Theorem 6.13 (specialized to $q = 0$) with $\mathsf{a} := \mathcal{P}(\mathsf{h})$ and $f := \mathbf{log}_0(\mathrm{id})$ to obtain a morphism $\hat{f}$ of 0-bimonoids

$$\mathsf{h} \to \mathcal{T}_0(\mathcal{P}(\mathsf{h})). \tag{13.4}$$

We can describe this explicitly using (6.12). Evaluating on the A-component, into the F-summand, the map is

$$\sum_{G\colon G \geq F} (-1)^{\mathrm{rk}(G/F)} \mu_F^G \Delta_A^G \,:\, \mathsf{h}[A] \to \mathcal{P}(\mathsf{h})[F]. \tag{13.5}$$

Proposition 13.5. *For a* 0*-bimonoid* h*, the map* (13.4) *is an isomorphism of* 0*-bimonoids. Moreover, it is the inverse of the map* (13.2)*.*

Proof. We directly check below that (13.2) and (13.4) are inverses. A conceptual explanation of this calculation is given in Section 13.1.3 below.

For any face A, the composite

$$\mathsf{h}[A] \to \bigoplus_{F\colon F \geq A} \mathcal{P}(\mathsf{h})[F] \to \mathsf{h}[A]$$

is given by

$$\begin{aligned}
\sum_{F\colon F \geq A} \sum_{G\colon G \geq F} (-1)^{\mathrm{rk}(G/F)} \mu_A^F \mu_F^G \Delta_A^G &= \sum_{F\colon F \geq A} \sum_{G\colon G \geq F} (-1)^{\mathrm{rk}(G/F)} \mu_A^G \Delta_A^G \\
&= \sum_{G\colon G \geq A} \Big(\sum_{F\colon G \geq F \geq A} (-1)^{\mathrm{rk}(G/F)} \Big) \mu_A^G \Delta_A^G \\
&= \mathrm{id}\,.
\end{aligned}$$

The first step used associativity of the product (2.8). For the last step, note that, by (1.73), the sum in parenthesis is zero unless $G = A$.

For faces F and G greater than a face A, the composite

$$\mathcal{P}(\mathsf{h})[F] \to \mathsf{h}[A] \to \mathcal{P}(\mathsf{h})[G]$$

is given by

$$\begin{aligned}
\sum_{H\colon H \geq G} (-1)^{\mathrm{rk}(H/G)} \mu_G^H \Delta_A^H \mu_A^F &= \sum_{H\colon F \geq H \geq G} (-1)^{\mathrm{rk}(H/G)} \mu_G^H \mu_H^F \\
&= \Big(\sum_{H\colon F \geq H \geq G} (-1)^{\mathrm{rk}(H/G)} \Big) \mu_G^F \\
&= \begin{cases} \mathrm{id} & \text{if } F = G, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

The first step used Lemma 5.36. The second step used associativity of the product (2.8). For the last step, by (1.73), the sum in parenthesis is zero unless $F = G$. $\square$

Note very carefully that we proved Proposition 13.5 from first principles, so this indeed gives another proof of Proposition 13.1. We refer to (13.2) and (13.4) as the *Loday–Ronco isomorphisms*.

13.1.3. 0-exponential and 0-logarithm. We now give a conceptual proof of Proposition 13.5 by deducing the inverse relationship between the Loday–Ronco isomorphisms from the inverse relationship between the 0-exponential and the 0-logarithm.

Exercise 13.6. For a 0-bimonoid h, consider the following biderivations f and g.

$$\begin{tikzcd} \mathcal{T}_0(\mathcal{P}(\mathsf{h})) \arrow[r, "f"] \arrow[d, two heads] & \mathsf{h} \\ \mathcal{P}(\mathsf{h}) \arrow[r, "\mathrm{id}"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{h} \arrow[r, "g"] \arrow[d, two heads] & \mathcal{T}_0(\mathcal{P}(\mathsf{h})) \\ \mathcal{Q}(\mathsf{h}) \arrow[r, "\mathbf{log}_0(\mathrm{id})"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd}$$

Check that: The maps (13.2) and (13.4) are the 0-exponentials (9.38a) of f and g, respectively. The claim about f is also a special case of Exercise 9.106.

Now deduce from Theorem 9.105 that $\mathbf{exp}_0(f)$ and $\mathbf{exp}_0(g)$ are morphisms of 0-bimonoids. Moreover, deduce from Proposition 9.111 and its proof that $\mathbf{exp}_0(f)$ and $\mathbf{exp}_0(g)$ are inverses of each other. More directly:

$$\begin{aligned} \mathbf{exp}_0(f)\,\mathbf{exp}_0(g) &= \mathbf{exp}_0(fg) = \mathbf{exp}_0(\mathbf{log}_0(\mathrm{id})) = \mathrm{id}, \\ \mathbf{exp}_0(g)\,\mathbf{exp}_0(f) &= \mathbf{exp}_0(gf) = \mathbf{exp}_0(\mathbf{log}_0(\mathrm{id})) = \mathrm{id}\,. \end{aligned}$$

In the first step of either calculation, use Lemma 9.83, item (2), for $q = 0$. To show $gf = \mathbf{log}_0(\mathrm{id})$ on $\mathcal{T}_0(\mathcal{P}(\mathsf{h}))$, use Example 9.110, and the fact that $\mathbf{log}_0(\mathrm{id}_{\mathsf{h}})$ and pq_{h} are inverses (Proposition 9.107).

The 0-exponential of a coderivation, and 0-logarithm of a comonoid morphism in Theorem 9.103 can be expressed in terms of the Loday–Ronco isomorphisms as follows.

Exercise 13.7. Let c be a comonoid and k a 0-bimonoid. Check that:

- For $f : \mathsf{c} \to \mathsf{k}$ a coderivation, its 0-exponential equals

$$\mathbf{exp}_0(f) : \mathsf{c} \to \mathcal{T}_0(\mathcal{P}(\mathsf{k})) \xrightarrow{\cong} \mathsf{k}.$$

  The first map arises from cofreeness of $\mathcal{T}_0(\mathcal{P}(\mathsf{k}))$ as in Theorem 6.10, while the second map is the isomorphism (13.2).
- For $g : \mathsf{c} \to \mathsf{k}$ a morphism of comonoids, its 0-logarithm equals

$$\mathbf{log}_0(g) : \mathsf{c} \to \mathsf{k} \xrightarrow{\cong} \mathcal{T}_0(\mathcal{P}(\mathsf{k})) \twoheadrightarrow \mathcal{P}(\mathsf{k}) \hookrightarrow \mathsf{k}.$$

  The isomorphism is as in (13.4).

13.1.4. Dual Loday–Ronco. The Loday–Ronco theorem in dual form is stated below. It is phrased in terms of the indecomposable part instead of the primitive part. The functor $\mathcal{T}_0$ is the right adjoint to the indecomposable part functor $\mathcal{Q}$.

Theorem 13.8. *The adjunction*

$$0\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{\mathcal{T}_0}{\overset{\mathcal{Q}}{\rightleftarrows}} \mathcal{A}\text{-}\mathsf{Sp}$$

is an adjoint equivalence of categories.

Proof. This can be deduced from Theorem 13.2 by duality. For the precise argument, see the proof of the more general Theorem 13.93 given later. □

Explicitly, for any 0-bimonoid h, the isomorphism

$$\mathsf{h} \to \mathcal{T}_0(\mathcal{Q}(\mathsf{h})) \tag{13.6}$$

on the A-component, into the F-summand, is given by

$$\mathsf{h}[A] \xrightarrow{\Delta_A^F} \mathsf{h}[F] \twoheadrightarrow \mathcal{Q}(\mathsf{h})[F].$$

The inverse isomorphism

$$\mathcal{T}_0(\mathcal{Q}(\mathsf{h})) \to \mathsf{h}, \tag{13.7}$$

on the A-component, on the F-summand, is given by

$$\sum_{G:\, G \geq F} (-1)^{\mathrm{rk}(G/F)}\, \mu_A^G \Delta_F^G : \mathcal{Q}(\mathsf{h})[F] \to \mathsf{h}[A].$$

The map as written is from $\mathsf{h}[F]$ to $\mathsf{h}[A]$, the point being that it factors through $\mathcal{Q}(\mathsf{h})[F]$.

13.2. Leray–Samelson for bicommutative bimonoids

Every bicommutative bimonoid is free commutative and cofree cocommutative on its primitive part. This shows that the categories of bicommutative bimonoids and species are equivalent. This is the Leray–Samelson theorem. It can also be phrased dually using the indecomposable part of the bimonoid. (Recall that the primitive part and indecomposable part of a bicommutative bimonoid are naturally isomorphic.) We also briefly discuss the signed analogue of the Leray–Samelson theorem.

Leray–Samelson bears a striking similarity to Loday–Ronco which we discussed in Section 13.1. This will be evident in the exposition below; the role of faces will now be played by flats. We present three arguments. The first is based on the primitive filtration, the second involves exponential and logarithm, and the third employs commutative characteristic operations by the complete system of primitive orthogonal idempotents of the Birkhoff algebra.

13.2.1. Leray–Samelson. Freeness and cofreeness of bicommutative bimonoids. Recall from Section 6.5.1 the bicommutative bimonoid $\mathcal{S}(\mathsf{p})$ associated to a species p. In view of (6.50), it does not matter whether we write $\mathcal{S}(\mathsf{p})$ or $\mathcal{S}^{\vee}(\mathsf{p})$.

Recall from Exercise 6.46 the simple fact that

$$\mathcal{P}(\mathcal{S}(\mathsf{p})) = \mathsf{p}. \tag{13.8}$$

Thus, $\mathcal{P}\mathcal{S} = \mathrm{id}$. We now show that composition of the functors in the other order, namely, $\mathcal{S}\mathcal{P}$, is also isomorphic to the identity.

Proposition 13.9. *For a bicommutative bimonoid* h*, there is a natural isomorphism*

$$\mathcal{S}(\mathcal{P}(\mathsf{h})) \xrightarrow{\cong} \mathsf{h} \tag{13.9}$$

of bimonoids. In particular, any bicommutative bimonoid is free as a commutative monoid and cofree as a cocommutative comonoid.

The universal property of $\mathcal{S}$ stated in Theorem 6.44 applied to $f := \mathrm{id}$ on $\mathcal{P}(\mathsf{h})$ yields the morphism (13.9) of bimonoids. Explicitly, using (6.52), the map on the Z-component, on the X-summand, is

$$\mathcal{P}(\mathsf{h})[\mathrm{X}] \hookrightarrow \mathsf{h}[\mathrm{X}] \xrightarrow{\mu_{\mathrm{Z}}^{\mathrm{X}}} \mathsf{h}[\mathrm{Z}]. \tag{13.10}$$

Equivalently, the map is $\mu^{\overline{k-1}}$ on $\mathcal{P}(\mathsf{h})^{\overline{k}}$ for each $k \geq 1$.

Proof. We need to show that the map constructed above is an isomorphism. In view of (13.8), the map is injective (in fact, the identity) on the primitive part. Therefore, it is injective everywhere by Proposition 5.18. We now show it is surjective by an induction based on the primitive filtration of h. Take $z \in \mathsf{h}[\mathrm{Z}]$. By (5.20), there exists $k \geq 1$ such that $z \in \mathcal{P}_k(\mathsf{h})[\mathrm{Z}]$. If $k = 1$, then $z = f_{\mathrm{Z}}(z)$. Suppose $k \geq 2$. By (5.29), we have

$$\Delta_{\mathrm{Z}}^{\overline{k-1}}(z) \in \mathcal{P}(\mathsf{h})^{\overline{k}}[\mathrm{Z}]$$

and therefore

$$f_{\mathrm{Z}}\big(\Delta_{\mathrm{Z}}^{\overline{k-1}}(z)\big) = \mu_{\mathrm{Z}}^{\overline{k-1}}\Delta_{\mathrm{Z}}^{\overline{k-1}}(z).$$

On the other hand, from Lemma 5.45, we have

$$z - \mu_{\mathrm{Z}}^{\overline{k-1}}\Delta_{\mathrm{Z}}^{\overline{k-1}}(z) \in \mathcal{P}_{k-1}(\mathsf{h})[\mathrm{Z}].$$

By induction hypothesis, this element belongs to the image of f_{Z}, and we conclude that z is also in the image of f_{Z}. $\square$

Remark 13.10. Compare and contrast the above proof with the proof of Proposition 13.1. Cauchy powers are replaced by commutative Cauchy powers, μ^k and Δ^k are replaced by $\mu^{\overline{k}}$ and $\Delta^{\overline{k}}$, and Lemma 5.38 is replaced by Lemma 5.45.

Recall from Theorem 6.43 that the functor $\mathcal{S}$ and the primitive part functor $\mathcal{P}$ are adjoints. Since composing them either way yields natural isomorphisms with the identity, we obtain the following result.

Theorem 13.11. *The adjunction*

$$\mathcal{A}\text{-}\mathsf{Sp} \underset{\mathcal{P}}{\overset{\mathcal{S}}{\rightleftarrows}} {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})$$

is an adjoint equivalence of categories.

This is the *Leray–Samelson theorem* for bicommutative bimonoids.

Corollary 13.12. *Any finite-dimensional bicommutative bimonoid is self-dual.*

Proof. Let h be a finite-dimensional bicommutative bimonoid. By Theorem 13.11, $\mathsf{h} = \mathcal{S}(\mathsf{p})$, where $\mathsf{p} = \mathcal{P}(\mathsf{h})$. Further, by (6.31) and (6.50), $\mathsf{h}^* = \mathcal{S}(\mathsf{p}^*)$. But we know that $\mathsf{p} \cong \mathsf{p}^*$, and hence $\mathsf{h} \cong \mathsf{h}^*$, as required. □

Exercise 13.13. Use Leray–Samelson to prove forward implication in Proposition 5.56, item (3). (First check it directly for $\mathcal{S}(\mathsf{p})$, and then use naturality (5.51).)

13.2.2. The inverse isomorphism. We now give a more direct proof of Proposition 13.9 by explicitly describing the inverse of the map (13.9). We construct it as follows using cofreeness of $\mathcal{S}^\vee(\mathcal{P}(\mathsf{h}))$ and the logarithm of the identity map from Section 9.2.

Let $\mu(\mathrm{X}, \mathrm{Y})$ denote the Möbius function of the poset of flats. For a bicommutative bimonoid h, consider the map of species $\log(\mathrm{id}) : \mathsf{h} \twoheadrightarrow \mathcal{P}(\mathsf{h})$ given in Proposition 9.47. Further, by Corollary 9.48, it is a morphism of monoids with the trivial product on $\mathcal{P}(\mathsf{h})$. Now apply Theorem 6.25 with $\mathsf{a} := \mathcal{P}(\mathsf{h})$ and $f := \log(\mathrm{id})$ to obtain a morphism $\hat{f}$ of bimonoids

$$\mathsf{h} \to \mathcal{S}(\mathcal{P}(\mathsf{h})). \tag{13.11}$$

We can describe this explicitly using (6.27a). Evaluating on the Z-component, into the X-summand, the map is

$$\sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{X}, \mathrm{Y})\, \mu_{\mathrm{X}}^{\mathrm{Y}} \Delta_{\mathrm{Z}}^{\mathrm{Y}} \;:\; \mathsf{h}[\mathrm{Z}] \to \mathcal{P}(\mathsf{h})[\mathrm{X}]. \tag{13.12}$$

Proposition 13.14. *For a bicommutative bimonoid* h*, the map* (13.11) *is an isomorphism of bimonoids. Moreover, it is the inverse of the map* (13.9).

Proof. We directly check below that (13.9) and (13.11) are inverses. A conceptual explanation of this calculation is given in Section 13.2.4 below.

For any flat Z, the composite

$$\mathsf{h}[\mathrm{Z}] \to \bigoplus_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mathcal{P}(\mathsf{h})[\mathrm{X}] \to \mathsf{h}[\mathrm{Z}]$$

is given by

$$\begin{aligned}
\sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \; \sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{X}, \mathrm{Y})\, \mu_{\mathrm{Z}}^{\mathrm{X}} \mu_{\mathrm{X}}^{\mathrm{Y}} \Delta_{\mathrm{Z}}^{\mathrm{Y}} &= \sum_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \; \sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{X}, \mathrm{Y})\, \mu_{\mathrm{Z}}^{\mathrm{Y}} \Delta_{\mathrm{Z}}^{\mathrm{Y}} \\
&= \sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{Z}} \Big(\sum_{\mathrm{X}:\, \mathrm{Y} \geq \mathrm{X} \geq \mathrm{Z}} \mu(\mathrm{X}, \mathrm{Y}) \Big) \mu_{\mathrm{Z}}^{\mathrm{Y}} \Delta_{\mathrm{Z}}^{\mathrm{Y}} \\
&= \mathrm{id}\,.
\end{aligned}$$

The first step used associativity of the product (2.8). For the last step, note that, by definition of the Möbius function, or equivalently, by the fact that ζ and μ are inverse, the sum in parenthesis is zero unless $\mathrm{Y} = \mathrm{Z}$.

For flats X and Y greater than a flat Z, the composite

$$\mathcal{P}(\mathsf{h})[\mathrm{X}] \to \mathsf{h}[\mathrm{Z}] \to \mathcal{P}(\mathsf{h})[\mathrm{Y}]$$

is given by

$$\begin{aligned}
\sum_{\mathrm{W}:\, \mathrm{W} \geq \mathrm{Y}} \mu(\mathrm{Y}, \mathrm{W})\, \mu_{\mathrm{Y}}^{\mathrm{W}} \Delta_{\mathrm{Z}}^{\mathrm{W}} \mu_{\mathrm{Z}}^{\mathrm{X}} &= \sum_{\mathrm{W}:\, \mathrm{X} \geq \mathrm{W} \geq \mathrm{Y}} \mu(\mathrm{Y}, \mathrm{W})\, \mu_{\mathrm{Y}}^{\mathrm{W}} \mu_{\mathrm{W}}^{\mathrm{X}} \\
&= \Big(\sum_{\mathrm{W}:\, \mathrm{X} \geq \mathrm{W} \geq \mathrm{Y}} \mu(\mathrm{Y}, \mathrm{W}) \Big) \mu_{\mathrm{Y}}^{\mathrm{X}} \\
&= \begin{cases} \mathrm{id} & \text{if } \mathrm{X} = \mathrm{Y}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

The first step used Lemma 5.43. The second step used associativity of the product (2.8). For the last step, note that the sum in parenthesis is zero unless $\mathrm{X} = \mathrm{Y}$. $\square$

Note very carefully that we proved Proposition 13.14 from first principles, so this indeed gives another proof of Proposition 13.9. We refer to (13.9) and (13.11) as the *Leray–Samelson isomorphisms*.

13.2.3. Examples.

Example 13.15. Take $\mathsf{h} = \mathsf{E}$, the exponential bimonoid (Section 7.2). The Leray–Samelson isomorphisms (13.9) and (13.11) specialize to (7.11). The self-duality of E can be seen as a special case of Corollary 13.12. The self-duality isomorphism of E that we have considered arises by using the canonical identification $\mathsf{x} \cong \mathsf{x}^*$ in the proof of Corollary 13.12.

Example 13.16. Take $\mathsf{h} = \Pi$, the bimonoid of flats (Section 7.4). The Leray–Samelson isomorphisms (13.9) and (13.11) specialize to the first map in (7.48). The self-duality isomorphism of Π given there arises from the canonical identification $\mathsf{E} \cong \mathsf{E}^*$ in the proof of Corollary 13.12. Other self-duality isomorphisms are explored in Section 7.4.8.

Example 13.17. Let $\mathcal{A}$ denote a rank-one arrangement. For a bicommutative $\mathcal{A}$-bimonoid h, the isomorphism (13.9), evaluated on the $\perp$-component, is given by

$$\mathcal{P}(\mathsf{h})[\perp] \oplus \mathsf{h}[\top] \xrightarrow{\cong} \mathsf{h}[\perp], \qquad (x^{\perp}, x^{\top}) \mapsto x^{\perp} + \mu_{\perp}^{\top}(x^{\top}).$$

The inverse isomorphism (13.11), evaluated on the $\perp$-component, is given by

$$\mathsf{h}[\perp] \xrightarrow{\cong} \mathcal{P}(\mathsf{h})[\perp] \oplus \mathsf{h}[\top], \qquad x \mapsto (x - \mu_{\perp}^{\top} \Delta_{\perp}^{\top}(x), \Delta_{\perp}^{\top}(x)).$$

The first coordinate is precisely $\log(\mathrm{id})_{\perp}(x)$, see also Example 9.51.

Now recall from Lemma 2.88 that a bicommutative $\mathcal{A}$-bimonoid h is equivalent to an idempotent operator e on $V = \mathsf{h}[\perp]$. The above isomorphism corresponds to the decomposition $V = \ker(e) \oplus \mathrm{im}(e)$.

13.2.4. Exponential and logarithm. We now give a conceptual proof of Proposition 13.14 by deducing the inverse relationship between the Leray–Samelson isomorphisms from the inverse relationship between the exponential and the logarithm.

Exercise 13.18. For a bicommutative bimonoid h, consider the following biderivations f and g.

$$\begin{tikzcd} \mathcal{S}(\mathcal{P}(\mathsf{h})) \arrow[r, "f"] \arrow[d, two heads] & \mathsf{h} \\ \mathcal{P}(\mathsf{h}) \arrow[r, "\mathrm{id}"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{h} \arrow[r, "g"] \arrow[d, two heads] & \mathcal{S}(\mathcal{P}(\mathsf{h})) \\ \mathcal{Q}(\mathsf{h}) \arrow[r, "\log(\mathrm{id})"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd}$$

Check that: The maps (13.9) and (13.11) are the exponentials (9.15a) of f and g, respectively. The claim about f is also a special case of Example 9.44.

Now deduce from Theorem 9.43 that $\exp(f)$ and $\exp(g)$ are morphisms of bimonoids. Moreover, deduce from Proposition 9.55 and its proof that $\exp(f)$ and $\exp(g)$ are inverses of each other. More directly:

$$\exp(f)\exp(g) = \exp(fg) = \exp(\log(\mathrm{id})) = \mathrm{id},$$
$$\exp(g)\exp(f) = \exp(gf) = \exp(\log(\mathrm{id})) = \mathrm{id}\,.$$

In the first step of either calculation, use Lemma 9.15, item (2). To show $gf = \log(\mathrm{id})$ on $\mathcal{S}(\mathcal{P}(\mathsf{h}))$, use Example 9.52, and the fact that $\log(\mathrm{id}_{\mathsf{h}})$ and pq_{h} are inverses (Proposition 9.47).

The exponential of a coderivation, and logarithm of a comonoid morphism in Theorem 9.40 can be expressed in terms of the Leray–Samelson isomorphisms as follows.

Exercise 13.19. Let c be a cocommutative comonoid and k a bicommutative bimonoid. Check that:

- For $f : \mathsf{c} \to \mathsf{k}$ a coderivation, its exponential equals
$$\exp(f) : \mathsf{c} \to \mathcal{S}(\mathcal{P}(\mathsf{k})) \xrightarrow{\cong} \mathsf{k}.$$
The first map arises from cofreeness of $\mathcal{S}(\mathcal{P}(\mathsf{k}))$ as in Theorem 6.23, while the second map is the isomorphism (13.9).
- For $g : \mathsf{c} \to \mathsf{k}$ a morphism of comonoids, its logarithm equals
$$\log(g) : \mathsf{c} \to \mathsf{k} \xrightarrow{\cong} \mathcal{S}(\mathcal{P}(\mathsf{k})) \twoheadrightarrow \mathcal{P}(\mathsf{k}) \hookrightarrow \mathsf{k}.$$
The isomorphism is as in (13.11).

13.2.5. Leray–Samelson via commutative characteristic operations. For any flat Z, let $\Pi[\mathrm{Z}]$ denote the Birkhoff algebra of $\mathcal{A}_{\mathrm{Z}}$ as in Section 7.4.3. Recall from Section 1.9.1 that the Birkhoff algebra is a split-semisimple commutative algebra. Now let h be a bicommutative bimonoid. Recall from Section 10.2 that for any flat Z, the Birkhoff algebra $\Pi[\mathrm{Z}]$ acts on $\mathsf{h}[\mathrm{Z}]$ via commutative characteristic operations. These facts can be combined to give another proof of the Leray–Samelson theorem as follows.

Lemma 13.20. *For a bicommutative bimonoid* h*, for any flat* Z*,*

$$\mathsf{h}[\mathrm{Z}] = \bigoplus_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Z}} \mathsf{Q}_{\mathrm{X}/\mathrm{Z}} \cdot \mathsf{h}[\mathrm{Z}].$$

PROOF. Formulas (7.41) and (7.42) yield the required decomposition. □

Now consider the induced isomorphisms in diagram (10.62). By summing these over all X greater than Z and using Lemma 13.20, we obtain a linear isomorphism

$$\mathcal{S}(\mathcal{P}(\mathsf{h}))[\mathrm{Z}] \xrightarrow{\cong} \mathsf{h}[\mathrm{Z}].$$

By construction, this coincides with the map (6.52) arising from the universal property of $\mathcal{S}$, hence it yields an isomorphism $\mathcal{S}(\mathcal{P}(\mathsf{h})) \xrightarrow{\cong} \mathsf{h}$ of bimonoids. Thus, we have reproved Proposition 13.9, which is the nontrivial part of the Leray–Samelson Theorem 13.11.

13.2.6. Dual Leray–Samelson. The Leray–Samelson theorem in dual form is stated below. It is phrased in terms of the indecomposable part instead of the primitive part. Recall from Theorem 6.43 that the functor $\mathcal{S}$ is the right adjoint to the indecomposable part functor $\mathcal{Q}$.

Theorem 13.21. *The adjunction*

$$^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{\mathcal{S}}{\overset{\mathcal{Q}}{\rightleftarrows}} \mathcal{A}\text{-}\mathsf{Sp}$$

is an adjoint equivalence of categories.

PROOF. It remains to check that the unit and counit of the adjunction are isomorphisms. This can be deduced by duality from Theorem 13.11 using Exercise 6.48. □

One may also proceed directly. The nontrivial part of the equivalence in Theorem 13.21 works as follows.

Proposition 13.22. *For a bicommutative bimonoid* h*, there is a natural isomorphism*

$$\mathsf{h} \xrightarrow{\cong} \mathcal{S}(\mathcal{Q}(\mathsf{h})) \tag{13.13}$$

of bimonoids.

The morphism (13.13) is obtained by applying the universal property of $\mathcal{S}$ stated in Theorem 6.45 to $f :=$ id on $\mathcal{Q}(\mathsf{h})$. Explicitly, using (6.53), the map on the Z-component, into the X-summand, is

$$\mathsf{h}[\mathrm{Z}] \xrightarrow{\Delta^{\mathrm{X}}_{\mathrm{Z}}} \mathsf{h}[\mathrm{X}] \twoheadrightarrow \mathcal{Q}(\mathsf{h})[\mathrm{X}]. \tag{13.14}$$

We also point out that in terms of characteristic operations, summing the induced isomorphisms in diagram (10.63) and using Lemma 13.20 yields the isomorphism (13.13).

Exercise 13.23. Applying the functor $\mathcal{P}$ to the map (13.13) and using (13.8) yields an isomorphism of species $\mathcal{P}(\mathsf{h}) \xrightarrow{\cong} \mathcal{Q}(\mathsf{h})$. Check that this coincides with the map (5.50). As a consequence, deduce forward implication in Proposition 5.56, item (3). See also Exercise 13.13.

Let h be a bicommutative bimonoid. Consider the map of species $\log(\mathrm{id}) : \mathcal{Q}(\mathsf{h}) \hookrightarrow \mathsf{h}$ given in Proposition 9.47. Further, by Corollary 9.48, it is a morphism of comonoids with the trivial coproduct on $\mathcal{Q}(\mathsf{h})$. Now apply Theorem 6.21 with $\mathsf{c} := \mathcal{Q}(\mathsf{h})$ and $f := \log(\mathrm{id})$ to obtain a morphism $\hat{f}$ of bimonoids

$$\mathcal{S}(\mathcal{Q}(\mathsf{h})) \to \mathsf{h}. \tag{13.15}$$

We can describe this explicitly using (6.21a). Evaluating on the Z-component, on the X-summand, the map is

$$\sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{X}, \mathrm{Y})\, \mu_{\mathrm{Z}}^{\mathrm{Y}} \Delta_{\mathrm{X}}^{\mathrm{Y}} \;:\; \mathcal{Q}(\mathsf{h})[\mathrm{X}] \to \mathsf{h}[\mathrm{Z}]. \tag{13.16}$$

The map as written is from $\mathsf{h}[\mathrm{X}]$ to $\mathsf{h}[\mathrm{Z}]$, the point being that it factors through $\mathcal{Q}(\mathsf{h})[\mathrm{X}]$.

Proposition 13.24. *For a bicommutative bimonoid* h*, the map* (13.15) *is an isomorphism of bimonoids. Moreover, it is the inverse of the map* (13.13)*.*

This can be checked directly as in the proof of Proposition 13.14 or along the lines of Exercise 13.18 as indicated below. We refer to (13.13) and (13.15) as the *dual Leray–Samelson isomorphisms*.

Exercise 13.25. For a bicommutative bimonoid h, consider the following biderivations f and g.

$$\begin{tikzcd} \mathsf{h} \arrow[r, "f"] \arrow[d, two heads] & \mathcal{S}(\mathcal{Q}(\mathsf{h})) \\ \mathcal{Q}(\mathsf{h}) \arrow[r, "\mathrm{id}"'] & \mathcal{Q}(\mathsf{h}) \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S}(\mathcal{Q}(\mathsf{h})) \arrow[r, "g"] \arrow[d, two heads] & \mathsf{h} \\ \mathcal{Q}(\mathsf{h}) \arrow[r, "\log(\mathrm{id})"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd}$$

Check that: The maps (13.13) and (13.15) are the exponentials (9.15a) of f and g, respectively. The claim about f is also a special case of Example 9.44. Now deduce from Proposition 9.55 and its proof that $\exp(f)$ and $\exp(g)$ are inverses of each other.

The exponential of a derivation, and logarithm of a monoid morphism in Theorem 9.41 can be expressed in terms of the Leray–Samelson isomorphisms as follows.

Exercise 13.26. Let h be a bicommutative bimonoid and a a commutative monoid. Check that:

- For $f : \mathsf{h} \to \mathsf{a}$ a derivation, its exponential equals

  $$\exp(f) : \mathsf{h} \xrightarrow{\cong} \mathcal{S}(\mathcal{Q}(\mathsf{h})) \to \mathsf{a}.$$

  The first map is the isomorphism (13.13), while the second map arises from freeness of $\mathcal{S}(\mathcal{Q}(\mathsf{h}))$ as in Theorem 6.17.

- For $g : \mathsf{h} \to \mathsf{a}$ a morphism of monoids, its logarithm equals

$$\log(g) : \mathsf{h} \twoheadrightarrow \mathcal{Q}(\mathsf{h}) \hookrightarrow \mathcal{S}(\mathcal{Q}(\mathsf{h})) \xrightarrow{\cong} \mathsf{h} \to \mathsf{a}.$$

The isomorphism is as in (13.15).

13.2.7. Signed analogue. The entire discussion above carries over to the signed setting. A very brief summary is given below. The role of the functor $\mathcal{S}$ is now played by $\mathcal{E}$, see Section 6.5.3.

Theorem 13.27. *The adjunction*

$$\mathcal{A}\text{-}\mathsf{Sp} \underset{\mathcal{P}}{\overset{\mathcal{E}}{\rightleftarrows}} (-1)\text{-}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})$$

is an adjoint equivalence of categories.

This is the *Leray–Samelson theorem* for signed bicommutative signed bimonoids.

PROOF. Theorems 13.11 and 13.27 (Leray–Samelson and its signed analogue) can readily be deduced from each other by conjugation by the signature functor. More precisely, we use Corollary 8.99 and Exercise 8.102. □

Let us elaborate further on the unit and counit of the adjunction in Theorem 13.27. We deduce from (6.56) that

$$\mathcal{P}(\mathcal{E}(\mathsf{p})) = \mathsf{p}. \tag{13.17}$$

Proposition 13.28. *For a signed bicommutative signed bimonoid* h*, there is a natural isomorphism*

$$\mathcal{E}(\mathcal{P}(\mathsf{h})) \xrightarrow{\cong} \mathsf{h} \tag{13.18}$$

of signed bimonoids.

The universal property of $\mathcal{E}$ stated in Theorem 6.50 applied to $f := \mathrm{id}$ on $\mathcal{P}(\mathsf{h})$ yields the morphism (13.18) of signed bimonoids. Explicitly, using (6.57), the map on the Z-component, on the X-summand, is

$$\mathbf{E}^-[\mathrm{Z},\mathrm{X}] \otimes \mathcal{P}(\mathsf{h})[\mathrm{X}] \hookrightarrow \mathbf{E}^-[\mathrm{Z},\mathrm{X}] \otimes \mathsf{h}[\mathrm{X}] \xrightarrow{\mu_{\mathrm{Z}}^{\mathrm{X}}} \mathsf{h}[\mathrm{Z}].$$

Exercise 13.29. Check that: For a signed bicommutative signed bimonoid h, the inverse of (13.18), namely,

$$\mathsf{h} \to \mathcal{E}(\mathcal{P}(\mathsf{h})) \tag{13.19}$$

is as follows. Evaluating on the Z-component, into the X-summand, the map is

$$\sum_{\mathrm{Y}:\,\mathrm{Y}\geq\mathrm{X}} \mu(\mathrm{X},\mathrm{Y})\,(\mathrm{id}\otimes\mu_{\mathrm{X}}^{\mathrm{Y}}\Delta_{\mathrm{X}}^{\mathrm{Y}})\Delta_{\mathrm{Z}}^{\mathrm{X}} \;:\; \mathsf{h}[\mathrm{Z}] \to \mathbf{E}^-[\mathrm{Z},\mathrm{X}] \otimes \mathcal{P}(\mathsf{h})[\mathrm{X}], \tag{13.20}$$

where id is the identity map on $\mathbf{E}^-[\mathrm{Z},\mathrm{X}]$.

We refer to (13.18) and (13.19) as the *signed Leray–Samelson isomorphisms*.

Exercise 13.30. Take $\mathsf{h} = \mathsf{E}^-$, the signed exponential bimonoid. Check that the signed Leray–Samelson isomorphisms (13.18) and (13.19) specialize to (7.15).

Exercise 13.31. For a signed bicommutative signed bimonoid h, consider the following biderivations f and g.

$$\begin{tikzcd} \mathcal{E}(\mathcal{P}(\mathsf{h})) \arrow[r, "f"] \arrow[d, two heads] & \mathsf{h} \\ \mathcal{P}(\mathsf{h}) \arrow[r, "\mathrm{id}"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{h} \arrow[r, "g"] \arrow[d, two heads] & \mathcal{E}(\mathcal{P}(\mathsf{h})) \\ \mathcal{Q}(\mathsf{h}) \arrow[r, "\log(\mathrm{id})"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd}$$

Check that: The maps (13.18) and (13.19) are the exponentials (9.26a) of f and g, respectively. Also, check that (13.19) is a special case of (6.36a) with $\mathsf{c} := \mathsf{h}$, $\mathsf{p} := \mathcal{P}(\mathsf{h})$, $f := \log(\mathrm{id})$.

The dual of Theorem 13.27 is stated below. It is also the signed analogue of Theorem 13.21.

Theorem 13.32. *The adjunction*

$$(-1)\text{-}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{\mathcal{E}}{\overset{\mathcal{Q}}{\rightleftarrows}} \mathcal{A}\text{-}\mathsf{Sp}$$

is an adjoint equivalence of categories.

13.3. Borel–Hopf for cocommutative bimonoids

Any cocommutative bimonoid is isomorphic as a comonoid to the cofree cocommutative comonoid on its primitive part. The isomorphism depends on the choice of a noncommutative zeta function. This is the Borel–Hopf theorem for cocommutative bimonoids. It also has a signed analogue. As a consequence of Borel–Hopf, we obtain a description of the primitive filtration of any cocommutative bimonoid. Also, specializing Borel–Hopf to the bimonoid of chambers, and taking dimensions of the spaces involved recovers the Zaslavsky formula for chamber enumeration.

We give three proofs of Borel–Hopf. The first proof is by an induction on the primitive filtration of the bimonoid. We also give a minor variant of this proof in which we pass to the associated graded bimonoid and then apply the Leray–Samelson theorem. The next proof proceeds by giving an explicit formula for the inverse isomorphism in terms of a noncommutative Möbius function, and is related to exponentials and logarithms. The last proof is more advanced and uses characteristic operations by eulerian idempotents of the Tits algebra.

The dual version of Borel–Hopf for commutative bimonoids is treated in Section 13.4.

13.3.1. Borel–Hopf. Cofreeness of cocommutative bimonoids. Recall the species $\mathcal{S}(\mathsf{p})$ from (6.18).

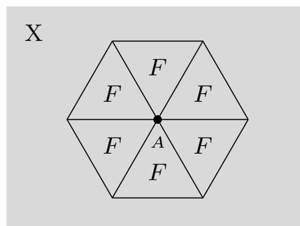
Fix a noncommutative zeta function $\boldsymbol{\zeta}$. For a bimonoid h, we define a natural map

$$\mathcal{S}(\mathcal{P}(\mathsf{h})) \to \mathsf{h} \tag{13.21}$$

as follows. Evaluating on the A-component, on the X-summand for $\mathrm{X} \geq \mathrm{s}(A)$, the map is

$$\sum_{F\colon F \geq A,\, \mathrm{s}(F)=\mathrm{X}} \boldsymbol{\zeta}(A, F)\, \mu_A^F \beta_{F,\mathrm{X}} \;:\; \mathcal{P}(\mathsf{h})[\mathrm{X}] \to \mathsf{h}[A]. \tag{13.22}$$

An illustration is provided below. The face A is the vertex in the center, and F runs over the 6 triangles shown.



More explicitly, (13.22) is the composite

$$\mathcal{P}(\mathsf{h})[\mathrm{X}] \hookrightarrow \mathsf{h}[\mathrm{X}] \to \bigoplus_{F\colon F \geq A,\, \mathrm{s}(F)=\mathrm{X}} \mathsf{h}[F] \to \mathsf{h}[A].$$

In the second map, the F-component $\mathsf{h}[\mathrm{X}] \to \mathsf{h}[F]$ is $\beta_{F,\mathrm{X}}$. The last map, on the F-component, is μ_A^F multiplied by the scalar $\boldsymbol{\zeta}(A, F)$.

The map (13.21) is a map of species. This uses that $\boldsymbol{\zeta}$ belongs to the lune-incidence algebra, that is, (1.40) holds for $s := \boldsymbol{\zeta}$. Further observe that when restricted to $\mathcal{P}(\mathsf{h})$, (13.21) is the inclusion of $\mathcal{P}(\mathsf{h})$ into h. Moreover:

Lemma 13.33. *For a bimonoid* h*, the map* (13.21) *is an injective morphism of comonoids.*

The product and coproduct of $\mathcal{S}(\mathcal{P}(\mathsf{h}))$ are given by (6.51) with $\mathsf{p} := \mathcal{P}(\mathsf{h})$. The above claim only concerns the comonoid structure. Another proof of this result is given later in Exercise 13.40. A related discussion is given in Exercise 17.3.

Proof. Let us denote the map (13.21) by f. We check below that f is a morphism of comonoids. Injectivity then follows from Proposition 5.18 (since the map is identity on the primitive part).

Fix a face A and a flat X containing A. Let H be any face greater than A. If $\mathrm{s}(H) \not\leq \mathrm{X}$, then Δ_A^H is zero on $\mathcal{P}(\mathsf{h})[\mathrm{X}]$ by (6.51) and it is also zero on the image of f_A by the second alternative in Lemma 5.39. So we may assume that $\mathrm{s}(H) \leq \mathrm{X}$. In this case, we need to check that the diagram

$$\begin{tikzcd} & \mathcal{P}(\mathsf{h})[\mathrm{X}] \arrow[dl, "f_A"'] \arrow[dr, "f_H"] & \\ \mathsf{h}[A] \arrow[rr, "\Delta_A^H"'] & & \mathsf{h}[H] \end{tikzcd}$$

commutes. Going left-down and across,

$$
\begin{aligned}
\sum_{\substack{F\colon F\geq A\\ \mathrm{s}(F)=\mathrm{X}}} \boldsymbol{\zeta}(A,F)\, \Delta_A^H \mu_A^F \beta_{F,\mathrm{X}} &= \sum_{\substack{F\colon F\geq A\\ \mathrm{s}(F)=\mathrm{X}}} \boldsymbol{\zeta}(A,F)\, \mu_H^{HF} \beta_{HF,F}\beta_{F,\mathrm{X}} \\
&= \sum_{\substack{F\colon F\geq A\\ \mathrm{s}(F)=\mathrm{X}}} \boldsymbol{\zeta}(A,F)\, \mu_H^{HF} \beta_{HF,\mathrm{X}} \\
&= \sum_{\substack{G\colon G\geq H\\ \mathrm{s}(G)=\mathrm{X}}} \Bigg(\sum_{\substack{F\colon F\geq A,\, HF=G\\ \mathrm{s}(F)=\mathrm{s}(G)}} \boldsymbol{\zeta}(A,F) \Bigg) \mu_H^G \beta_{G,\mathrm{X}} \\
&= \sum_{\substack{G\colon G\geq H\\ \mathrm{s}(G)=\mathrm{X}}} \boldsymbol{\zeta}(H,G)\, \mu_H^G \beta_{G,\mathrm{X}},
\end{aligned}
$$

which is the same as going right-down. In the first step, we used the bimonoid axiom (2.12) and counitality (since $FH = F$). In the last step, the sum in parenthesis was evaluated by the lune-additivity formula (1.42). This completes the check. □

Theorem 13.34. *For a cocommutative bimonoid* h*, the map* (13.21) *is an isomorphism of comonoids. In particular, any cocommutative bimonoid is isomorphic as a comonoid to the cofree cocommutative comonoid on its primitive part.*

This is the *Borel–Hopf theorem* for cocommutative bimonoids.

Proof. Let us denote the map (13.21) by f. It is injective by Lemma 13.33. We now show it is surjective by an induction based on the primitive filtration of h. Take $z \in \mathsf{h}[A]$. By (5.20), there exists $k \geq 1$ such that $z \in \mathcal{P}_k(\mathsf{h})[A]$. If $k = 1$, then $z = f_A(z)$. Suppose $k \geq 2$. Observe that the element

$$\sum_{F\colon \mathrm{rk}(F/A)=k-1} \boldsymbol{\zeta}(A,F)\, \mu_A^F \Delta_A^F(z)$$

belongs to the image of f_A. More precisely, it equals $f_A(\beta_{\mathrm{X},G}\Delta_A^G(z))$, where G is any face greater than A with support X. This used cocommutativity of h. Put z' equal to z minus the above element. Then by Lemma 5.48 and (5.48), $z' \in \mathcal{P}_{k-1}(\mathsf{h})[A]$, which by the induction hypothesis, is in the image of f_A. We conclude that z belongs to the image of f_A. This completes the induction step. □

Exercise 13.35. Deduce from Theorem 13.34 that any cocommutative bimonoid is primitively generated. (This recovers the nontrivial implication in Proposition 5.51.)

Exercise 13.36. Check that: For any bimonoid h, the map (13.21) induces an isomorphism $\mathcal{S}(\mathcal{P}(\mathsf{h})) \to \mathsf{h}^{coab}$ of comonoids.

Exercise 13.37. Use Borel–Hopf to deduce the result of Exercise 5.11 when c has the structure of a cocommutative bimonoid.

13.3.2. The inverse isomorphism. We now give a more direct proof of the Borel–Hopf Theorem 13.34 by explicitly describing the inverse of the map (13.21) in terms of $\boldsymbol{\zeta}^{-1}$. We construct it as follows using cofreeness of $\mathcal{S}(\mathcal{P}(\mathsf{h}))$ and a logarithm of the identity map from Section 9.1.

Fix a noncommutative Möbius function $\boldsymbol{\mu}$. Let h be a cocommutative bimonoid. Consider the map of species $\mathbf{log}(\mathrm{id}) : \mathsf{h} \twoheadrightarrow \mathcal{P}(\mathsf{h})$ given in Proposition 9.17. It is constructed out of $\boldsymbol{\mu}$. Now apply Theorem 6.23 with $\mathsf{c} := \mathsf{h}$, $\mathsf{p} := \mathcal{P}(\mathsf{h})$, $f := \mathbf{log}(\mathrm{id})$ to obtain a morphism $\hat{f}$ of comonoids

$$\mathsf{h} \to \mathcal{S}(\mathcal{P}(\mathsf{h})). \tag{13.23}$$

We can describe this explicitly using (6.27b). Evaluating on the A-component, into the X-summand for $\mathrm{X} \geq \mathrm{s}(A)$, the map is

$$\sum_{G\colon G \geq K} \boldsymbol{\mu}(K, G)\, \beta_{\mathrm{X},K} \mu_K^G \Delta_A^G \;:\; \mathsf{h}[A] \to \mathcal{P}(\mathsf{h})[\mathrm{X}], \tag{13.24}$$

where K is a fixed face of support X which is greater than A.

Theorem 13.38. *For a cocommutative bimonoid* h, *the map* (13.23) *is an isomorphism of comonoids. Moreover, it is the inverse of the map* (13.21) *when* $\boldsymbol{\mu}$ *and* $\boldsymbol{\zeta}$ *are inverse.*

Proof. We directly check below that (13.21) and (13.23) are inverses. A conceptual explanation of this calculation is given in Section 13.3.3 below.

For any face A, the composite

$$\mathsf{h}[A] \to \bigoplus_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{s}(A)} \mathcal{P}(\mathsf{h})[\mathrm{X}] \to \mathsf{h}[A]$$

is given by

$$\begin{aligned}
&\sum_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{s}(A)} \; \sum_{F\colon F \geq A,\, \mathrm{s}(F) = \mathrm{X}} \boldsymbol{\zeta}(A, F)\, \mu_A^F \beta_{F,\mathrm{X}} \sum_{G\colon G \geq F} \boldsymbol{\mu}(F, G)\, \beta_{\mathrm{X},F} \mu_F^G \Delta_A^G \\
&\quad = \sum_{F\colon F \geq A} \; \sum_{G\colon G \geq F} \boldsymbol{\zeta}(A, F) \boldsymbol{\mu}(F, G)\, \mu_A^F \mu_F^G \Delta_A^G \\
&\quad = \sum_{G\colon G \geq A} \Big(\sum_{F\colon G \geq F \geq A} \boldsymbol{\zeta}(A, F) \boldsymbol{\mu}(F, G) \Big) \mu_A^G \Delta_A^G \\
&\quad = \mu_A^A \Delta_A^A \\
&\quad = \mathrm{id}\,.
\end{aligned}$$

To simplify the calculation, note very carefully how at the beginning, we chose $K := F$ in (13.24). The second step used associativity of the product (2.8), while the third step used that $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ are inverse.

For any face A and flats X and Y greater than $\mathrm{s}(A)$, (fixing a face K greater than A of support Y), the composite

$$\mathcal{P}(\mathsf{h})[\mathrm{X}] \to \mathsf{h}[A] \to \mathcal{P}(\mathsf{h})[\mathrm{Y}]$$

is given by

$$\begin{aligned}
&\sum_{G\colon G\geq K}\ \sum_{F\colon F\geq A,\, \mathrm{s}(F)=\mathrm{X}} \boldsymbol{\mu}(K,G)\boldsymbol{\zeta}(A,F)\,\beta_{\mathrm{Y},K}\mu^G_K\Delta^G_A\mu^F_A\beta_{F,\mathrm{X}}\\
&\quad= \sum_{G\colon G\geq K}\ \sum_{F\colon F\geq A,\, FG=F,\, \mathrm{s}(F)=\mathrm{X}} \boldsymbol{\mu}(K,G)\boldsymbol{\zeta}(A,F)\,\beta_{\mathrm{Y},K}\mu^G_K\mu^{GF}_G\beta_{GF,F}\beta_{F,\mathrm{X}}\\
&\quad= \sum_{G\colon G\geq K,\, \mathrm{s}(G)\leq \mathrm{X}}\ \sum_{F\colon F\geq A,\, \mathrm{s}(F)=\mathrm{X}} \boldsymbol{\mu}(K,G)\boldsymbol{\zeta}(A,F)\,\beta_{\mathrm{Y},K}\mu^{GF}_K\beta_{GF,\mathrm{X}}\\
&\quad= \sum_{\substack{H,G\colon H\geq G\geq K\\ \mathrm{s}(H)=\mathrm{X}}} \Bigg(\sum_{\substack{F\colon F\geq A,\, GF=H\\ \mathrm{s}(F)=\mathrm{X}}} \boldsymbol{\zeta}(A,F)\Bigg)\boldsymbol{\mu}(K,G)\,\beta_{\mathrm{Y},K}\mu^H_K\beta_{H,\mathrm{X}}\\
&\quad= \sum_{H,G\colon H\geq G\geq K,\, \mathrm{s}(H)=\mathrm{X}} \boldsymbol{\mu}(K,G)\boldsymbol{\zeta}(G,H)\,\beta_{\mathrm{Y},K}\mu^H_K\beta_{H,\mathrm{X}}\\
&\quad= \begin{cases} \mathrm{id} & \text{if } \mathrm{X}=\mathrm{Y},\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

The first step used Lemma 5.39. The second step used associativity of the product (2.8). In the third step, we introduced a variable H for GF. The fourth step used the lune-additivity formula (1.42), while the last step used that $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ are inverse. □

Note very carefully that we proved Theorem 13.38 from first principles, so this indeed gives another proof of the Borel–Hopf Theorem 13.34. We refer to (13.21) and (13.23) as the *Borel–Hopf isomorphisms*.

Example 13.39. Let $\mathcal{A}$ denote a rank-one arrangement. Its noncommutative zeta and Möbius functions were characterized in Example 1.15. For a cocommutative $\mathcal{A}$-bimonoid h, the isomorphism (13.21), evaluated on the O-component, is given by

$$\begin{aligned}
&\mathcal{P}(\mathsf{h})[\bot]\oplus\mathsf{h}[\top] \xrightarrow{\cong} \mathsf{h}[O]\\
&\qquad (x^{\bot},x^{\top}) \longmapsto x^{\bot} + p\,\mu^C_O\beta_{C,\top}(x^{\top}) + (1-p)\,\mu^{\overline{C}}_O\beta_{\overline{C},\top}(x^{\top}).
\end{aligned}$$

The inverse isomorphism (13.23), evaluated on the O-component, is given by

$$\begin{aligned}
&\mathsf{h}[O] \xrightarrow{\cong} \mathcal{P}(\mathsf{h})[\bot]\oplus\mathsf{h}[\top]\\
&\qquad x \longmapsto (x - p\,\mu^C_O\Delta^C_O(x) + (p-1)\,\mu^{\overline{C}}_O\Delta^{\overline{C}}_O(x), \beta_{\top,C}\Delta^C_O(x)).
\end{aligned}$$

The first coordinate is precisely $\mathbf{log}(\mathrm{id})_O(x)$, see also Example 9.21. The second coordinate may also be written as $\beta_{\top,\overline{C}}\Delta^{\overline{C}}_O(x)$.

Now recall from Lemma 2.90 that a cocommutative $\mathcal{A}$-bimonoid h is equivalent to idempotent operators e and f on $V=\mathsf{h}[O]$ satisfying $ef=e$ and $fe=f$. The above isomorphism corresponds to a decomposition $V=U\oplus W$, where $U=\ker(e)=\ker(f)$ and $W=\{pe(x)+(1-p)f(x) \mid x\in V\}$. The latter is isomorphic to both $\mathrm{im}(e)$ and $\mathrm{im}(f)$.

13.3.3. Exponential and logarithm. We now give a conceptual proof of Theorem 13.38 by deducing the inverse relationship between the Borel–Hopf isomorphisms from the inverse relationship between exponentials and logarithms.

Exercise 13.40. For a cocommutative bimonoid h, consider the following coderivations f and g.

$$\begin{tikzcd} \mathcal{S}(\mathcal{P}(\mathsf{h})) \arrow[r, "f"] \arrow[dr, two heads] & \mathsf{h} \\ & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{h} \arrow[r, "g"] \arrow[dr, two heads, "\mathbf{log}(\mathrm{id})"'] & \mathcal{S}(\mathcal{P}(\mathsf{h})) \\ & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd}$$

(In fact, for any bimonoid h, f is a biderivation.) Check that: The maps (13.21) and (13.23) are exponentials (9.3a) of f and g, respectively. Deduce from Theorem 9.11 that $\mathbf{exp}(f)$ and $\mathbf{exp}(g)$ are morphisms of comonoids. This gives another proof of Lemma 13.33.

Exercise 13.41. We continue the notations in Exercise 13.40. Check that $\mathbf{exp}(f)$ and $\mathbf{exp}(g)$ are inverses as follows. First check:

$$\mathbf{exp}(f)\,\mathbf{exp}(g) = \mathbf{exp}(fg) = \mathbf{exp}(\mathbf{log}(\mathrm{id})) = \mathrm{id}\,.$$

Use Lemma 9.15, item (2) in the first step. Note very carefully that this requires f to be a derivation. Second check: Consider the diagram

$$\begin{tikzcd} \mathcal{S}(\mathcal{P}(\mathsf{h})) \arrow[r, "\mathbf{exp}(f)"] \arrow[dr, two heads, "p"'] & \mathsf{h} \arrow[r, "\mathbf{exp}(g)"] & \mathcal{S}(\mathcal{P}(\mathsf{h})) \arrow[dl, two heads, "p"] \\ & \mathcal{P}(\mathsf{h}) \arrow[r, hook, "i"'] & \mathsf{h}. \end{tikzcd}$$

Now combine

$$ip\,\mathbf{exp}(g)\,\mathbf{exp}(f) = \mathbf{log}(\mathrm{id})\,\mathbf{exp}(f) = \mathbf{log}(\mathbf{exp}(f)) = f$$

(with the second step using Lemma 9.10) and

$$ip\,\mathbf{exp}(g)\,\mathbf{exp}(f) = f \iff p\,\mathbf{exp}(g)\,\mathbf{exp}(f) = p \iff \mathbf{exp}(g)\,\mathbf{exp}(f) = \mathrm{id}\,.$$

An exponential of a coderivation, and logarithm of a comonoid morphism in Theorem 9.11 can be expressed in terms of the Borel–Hopf isomorphisms as follows.

Exercise 13.42. Let c be a cocommutative comonoid and k a cocommutative bimonoid. Check that:

- For $f : \mathsf{c} \to \mathsf{k}$ a coderivation, its exponential equals
$$\mathbf{exp}(f) : \mathsf{c} \to \mathcal{S}(\mathcal{P}(\mathsf{k})) \xrightarrow{\cong} \mathsf{k}.$$
The first map arises from cofreeness of $\mathcal{S}(\mathcal{P}(\mathsf{k}))$ as in Theorem 6.23, while the second map is the isomorphism (13.21).
- For $g : \mathsf{c} \to \mathsf{k}$ a morphism of comonoids, its logarithm equals
$$\mathbf{log}(g) : \mathsf{c} \to \mathsf{k} \xrightarrow{\cong} \mathcal{S}(\mathcal{P}(\mathsf{k})) \twoheadrightarrow \mathcal{P}(\mathsf{k}) \hookrightarrow \mathsf{k}.$$
The isomorphism is as in (13.23).

13.3.4. Connection with Leray–Samelson. The exercise below shows what happens to Borel–Hopf when the bimonoid h is bicommutative.

Exercise 13.43. For a bicommutative bimonoid h, the Borel–Hopf isomorphisms (13.21) and (13.23) reduce to the Leray–Samelson isomorphisms (13.9) and (13.11), respectively. Check this using Lemma 1.17.

Deduce the first part of Exercise 13.18 as a specialization of Exercise 13.40. Similarly, deduce Exercise 13.19 from Exercise 13.42.

In the next exercise, we indicate how Borel–Hopf can be deduced from Leray–Samelson using the construction in Section 5.8. The argument can be viewed as a minor variant of the one given in the proof of Theorem 13.34.

Exercise 13.44. Let h be a cocommutative bimonoid. Applying the functor $\mathrm{gr}_{\mathcal{P}}$ to (13.21) and using (6.88) and Lemma 13.33, we obtain a morphism

$$\mathcal{S}(\mathcal{P}(\mathsf{h})) \to \mathrm{gr}_{\mathcal{P}}(\mathsf{h}) \tag{13.25}$$

of graded comonoids. By the Browder–Sweedler commutativity result (Proposition 5.62), the bimonoid $\mathrm{gr}_{\mathcal{P}}(\mathsf{h})$ is bicommutative. Moreover, by (5.26), its primitive part is $\mathcal{P}(\mathsf{h})$. Check that (13.25) is an instance of the map (13.9) and hence an isomorphism of graded bimonoids. (Use Lemma 5.61 and the flat-additivity formula (1.43).) Deduce Theorem 13.34 as a consequence of this isomorphism.

Exercise 13.45. Let h be a cocommutative bimonoid. Show that $\mathrm{gr}_{\mathcal{P}}(\mathsf{h}) \cong \mathsf{h}$ as comonoids. More precisely, both are isomorphic to $\mathcal{S}(\mathcal{P}(\mathsf{h}))$ as comonoids.

The following exercise explains the precise relationship between the Borel–Hopf and Leray–Samelson isomorphisms.

Exercise 13.46. Show that: For a cocommutative bimonoid h, the diagram

$$\begin{tikzcd}
\mathsf{h} \arrow[r, "\cong"] \arrow[d, two heads, "\pi_{\mathsf{h}}"'] & \mathcal{S}(\mathcal{P}(\mathsf{h})) \arrow[d, two heads, "\mathcal{S}(\mathrm{pq}_{\mathsf{h}})"] \\
\mathsf{h}_{ab} \arrow[r, "\cong"'] & \mathcal{S}(\mathcal{Q}(\mathsf{h}))
\end{tikzcd}$$

commutes. The left-vertical map is the abelianization, while the right-vertical map is induced from (5.50). The horizontal maps are the isomorphisms (13.23) and (13.13). The latter applies since h_{ab} is bicommutative and $\mathcal{Q}(\mathsf{h}) = \mathcal{Q}(\mathsf{h}_{ab})$ by Exercise 5.20.

(First draw the diagonal arrow from h to $\mathcal{S}(\mathcal{Q}(\mathsf{h}))$. For the commutativity of the two triangles, use the second part of Remark 6.47 and the first part of Exercise 9.26, respectively.)

13.3.5. Borel–Hopf via characteristic operations. We now give another proof of the Borel–Hopf Theorem 13.34 which carries finer information. It makes use of characteristic operations by eulerian idempotents, and their close connection to primitive elements. We also describe the inverse of (13.21) in these terms. This is independent of the discussion in Section 13.3.2.

Let $\mathtt{E}$ be the eulerian family corresponding to a fixed noncommutative zeta function $\boldsymbol{\zeta}$. Let h be a cocommutative bimonoid. Recall from Section 10.1 that for any face A, the Tits algebra $\Sigma[A]$ acts on $\mathsf{h}[A]$ via characteristic operations.

Lemma 13.47. *We have*

$$\mathsf{h}[A] = \bigoplus_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{s}(A)} \mathtt{E}_{\mathrm{X}/A} \cdot \mathsf{h}[A].$$

The sum is over all flats X *containing* A.

Proof. This follows from (1.113) for the Tits algebra $\Sigma[A]$. □

Lemma 13.48. *For faces $F \geq A$, and a flat* X *containing F, the diagram*

$$\begin{tikzcd} \mathsf{h}[F] \arrow[r] & \mathtt{E}_{\mathrm{X}/F} \cdot \mathsf{h}[F] \\ \mathsf{h}[A] \arrow[r] \arrow[u, "\Delta_A^F"] & \mathtt{E}_{\mathrm{X}/A} \cdot \mathsf{h}[A] \arrow[u, "\Delta_A^F"'] \end{tikzcd} \tag{13.26}$$

commutes. The horizontal maps are left multiplication by $\mathtt{E}_{\mathrm{X}/F}$ and $\mathtt{E}_{\mathrm{X}/A}$, respectively. All maps are surjective.

Proof. The argument is similar to the one in the proof of Lemma 10.44, so we do not repeat it. □

Now consider the induced isomorphisms in diagram (10.68). By summing these over all flats X containing A, and using Lemma 13.47, we obtain the isomorphism (13.21). Moreover, its inverse

$$\mathsf{h} \to \mathcal{S}(\mathcal{P}(\mathsf{h})) \tag{13.27}$$

is as follows. Evaluating on the A-component, into the X-summand for $\mathrm{X} \geq \mathrm{s}(A)$, the map is given by the composite

$$\mathsf{h}[A] \xrightarrow{\mathtt{E}_{\mathrm{X}/A}\cdot(-)} \mathtt{E}_{\mathrm{X}/A} \cdot \mathsf{h}[A] \xrightarrow{\Delta_A^K} \mathcal{P}(\mathsf{h})[K] \xrightarrow{\beta_{\mathrm{X},K}} \mathcal{P}(\mathsf{h})[\mathrm{X}],$$

where K is any face greater than A of support X. Equivalently, it is given by

$$\mathsf{h}[A] \xrightarrow{\Delta_A^K} \mathsf{h}[K] \xrightarrow{\mathtt{E}_{\mathrm{X}/K}\cdot(-)} \mathcal{P}(\mathsf{h})[K] \xrightarrow{\beta_{\mathrm{X},K}} \mathcal{P}(\mathsf{h})[\mathrm{X}], \tag{13.28}$$

with K greater than A and of support X. This follows from (10.67) and (13.26).

To complete the proof of Theorem 13.34, it remains to check that (13.27) is a morphism of comonoids. For that, consider the map of species $f : \mathsf{h} \twoheadrightarrow \mathcal{P}(\mathsf{h})$ given by left multiplication by the first eulerian idempotent, that is, on the A-component, f_A sends x to $\mathtt{E}_{\mathrm{s}(A)/A} \cdot x$. Now apply Theorem 6.23 with $\mathsf{c} := \mathsf{h}$ and $\mathsf{p} := \mathcal{P}(\mathsf{h})$. Observe that (13.27) indeed coincides with the $\hat{f}$ given in (6.27b). Hence, it is a morphism of comonoids, as required.

Remark 13.49. We now reconcile the above discussion with the one in Section 13.3.2. The characteristic operation by the first eulerian idempotent (7.73) yields the operator (9.7). So the maps (13.24) and (13.28) indeed coincide, and hence so do (13.23) and (13.27). In both cases, we used the same argument, namely, cofreeness of $\mathcal{S}(\mathcal{P}(\mathsf{h}))$, to deduce that they are morphisms of comonoids.

13.3.6. Primitive filtration of a cocommutative bimonoid. The primitive filtration of $\mathcal{S}(\mathcal{P}(\mathsf{h}))$ is given by Proposition 6.58. (In this case, $\mathcal{S} = \mathcal{S}^\vee$.) Using the Borel–Hopf Theorem 13.34 and Proposition 10.45, we deduce:

Proposition 13.50. *The primitive filtration of a cocommutative bimonoid* h *is given by*

$$\mathcal{P}_k(\mathsf{h})[A] = \bigoplus_{\mathrm{X}:\, \mathrm{rk}(\mathrm{X}/A)<k} \mathsf{E}_{\mathrm{X}/A} \cdot \mathsf{h}[A].$$

For $A = O$, this is the primitive series of the left module $\mathsf{h}[O]$ over the Tits algebra, see Section 11.5.3. This expression for the primitive series is given in [**21**, Proposition 13.22] (with a discussion on a finer filtration indexed by flats).

13.3.7. Zaslavsky formula. As a consequence of the Borel–Hopf isomorphism (13.21), we obtain:

Corollary 13.51. *For a finite-dimensional cocommutative bimonoid* h*,*

$$\dim \mathsf{h}[\mathrm{X}] = \sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{X}} \dim \mathcal{P}(\mathsf{h})[\mathrm{Y}] \tag{13.29}$$

for any flat X.

Recall from Section 7.3 the bimonoid of chambers Γ. It is cocommutative. Combining the Borel–Hopf Theorem 13.34 and the Friedrichs criterion (Lemma 7.64), we obtain an isomorphism of comonoids

$$\mathcal{S}(\mathsf{Lie}) \cong \Gamma. \tag{13.30}$$

Setting $\mathsf{h} := \Gamma$, $\mathcal{P}(\mathsf{h}) := \mathsf{Lie}$, $\mathrm{X} := \perp$ in (13.29) and using (1.165) recovers the Zaslavsky formula (1.84) for chamber enumeration.

Similar considerations apply to faces. Recall from Section 7.6 the cocommutative bimonoid of faces Σ. Combining the Borel–Hopf Theorem 13.34 and the Friedrichs criterion (Lemma 7.69), we obtain an isomorphism of comonoids

$$\mathcal{S}(\mathsf{Zie}) \cong \Sigma. \tag{13.31}$$

Setting $\mathsf{h} := \Sigma$, $\mathcal{P}(\mathsf{h}) := \mathsf{Zie}$, $\mathrm{X} := \perp$ in (13.29) and using (1.176) recovers formula (1.85) for face enumeration.

Another application of formula (13.29) was given earlier in Corollary 8.40.

Exercise 13.52. Describe the primitive filtration of Γ by applying Proposition 6.58 to the isomorphism (13.30). Similarly, describe the primitive filtration of Σ using the isomorphism (13.31).

13.3.8. Signed analogue. Recall the species $\mathcal{E}(\mathsf{p})$ from (6.32). The Borel–Hopf theorem has a signed analogue with $\mathcal{S}$ replaced by $\mathcal{E}$. A brief summary is given below.

Fix a noncommutative zeta function $\boldsymbol{\zeta}$. For a signed bimonoid h, we define a natural map

$$\mathcal{E}(\mathcal{P}(\mathsf{h})) \to \mathsf{h} \tag{13.32}$$

as follows. Evaluating on the A-component, on the X-summand for $\mathrm{X} \geq \mathrm{s}(A)$, the map is

$$\sum_{\substack{F:\, F \geq A \\ \mathrm{s}(F)=\mathrm{X}}} \boldsymbol{\zeta}(A,F)\, \mu_A^F([F/A]\,\beta_{F,\mathrm{X}}) \;:\; \mathbf{E}^-[\mathrm{s}(A),\mathrm{X}] \otimes \mathcal{P}(\mathsf{h})[\mathrm{X}] \to \mathsf{h}[A], \tag{13.33}$$

with the map $[F/A]\,\beta_{F,\mathrm{X}}$ as in (2.45).

Theorem 13.53. *For any signed cocommutative signed bimonoid h, the map (13.32) is an isomorphism of comonoids. In particular, any signed cocommutative signed bimonoid is isomorphic as a comonoid to the cofree signed cocommutative comonoid on its primitive part.*

This is the *Borel–Hopf theorem* for signed cocommutative signed bimonoids.

Proof. The natural maps (13.21) and (13.32) are conjugates of each other wrt the signature functor. This can be deduced from (8.82) and (8.84). Thus, Theorems 13.34 and 13.53 (Borel–Hopf and its signed analogue) imply each other. □

Exercise 13.54. Check that: For a signed cocommutative signed bimonoid h, the inverse of (13.32)

$$\mathsf{h} \to \mathcal{E}(\mathcal{P}(\mathsf{h})) \tag{13.34}$$

is as follows. Evaluating on the A-component, into the X-summand for $\mathrm{X} \geq \mathrm{s}(A)$, the map is

$$\sum_{G:\, G \geq K} \boldsymbol{\mu}(K,G)\,([K/A] \otimes \beta_{\mathrm{X},K})\mu_K^G \Delta_A^G \;:\; \mathsf{h}[A] \to \mathbf{E}^-[\mathrm{s}(A),\mathrm{X}] \otimes \mathcal{P}(\mathsf{h})[\mathrm{X}], \tag{13.35}$$

where K is a fixed face of support X which is greater than A, and $[K/A] \otimes \beta_{\mathrm{X},K}$ is as in (2.45).

We refer to (13.32) and (13.34) as the *signed Borel–Hopf isomorphisms*.

Exercise 13.55. For a signed cocommutative signed bimonoid h, consider the following coderivations f and g.

$$\begin{tikzcd} \mathcal{E}(\mathcal{P}(\mathsf{h})) \arrow[r, "f"] \arrow[dr, two heads] & \mathsf{h} \\ & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{h} \arrow[r, "g"] \arrow[dr, two heads, "\mathbf{log}(\mathrm{id})"'] & \mathcal{E}(\mathcal{P}(\mathsf{h})) \\ & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd}$$

Check that: The maps (13.32) and (13.34) are exponentials (9.3a) of f and g, respectively. Deduce from Theorem 9.30 that $\mathbf{exp}(f)$ and $\mathbf{exp}(g)$ are morphisms of comonoids.

Also check that (13.34) is a special case of (6.36b) with $\mathsf{c} := \mathsf{h}$, $\mathsf{p} := \mathcal{P}(\mathsf{h})$, $f := \mathbf{log}(\mathrm{id})$.

13.4. Borel–Hopf for commutative bimonoids

We now consider the picture dual to the one in Section 13.3. Any commutative bimonoid is isomorphic as a monoid to the free commutative monoid on its indecomposable part. This is the Borel–Hopf theorem for commutative bimonoids. It also has a signed analogue. Borel–Hopf leads to a description of the decomposable filtration of any commutative bimonoid.

Since the discussion is formally similar to that in the previous section, we will be brief in our exposition.

13.4.1. Borel–Hopf. Freeness of commutative bimonoids. Fix a noncommutative zeta function $\boldsymbol{\zeta}$. Let h be a bimonoid. Define a natural map of species

$$\mathsf{h} \to \mathcal{S}(\mathcal{Q}(\mathsf{h})) \tag{13.36}$$

as follows. Evaluating on the A-component, into the X-summand for $\mathrm{X} \geq \mathrm{s}(A)$, the map is given by

$$\sum_{F:\, F \geq A,\, \mathrm{s}(F)=\mathrm{X}} \boldsymbol{\zeta}(A,F)\, \beta_{\mathrm{X},F} \Delta_A^F \,:\, \mathsf{h}[A] \to \mathcal{Q}(\mathsf{h})[\mathrm{X}]. \tag{13.37}$$

More explicitly, this is the composite

$$\mathsf{h}[A] \to \bigoplus_{F:\, F\geq A,\, \mathrm{s}(F)=\mathrm{X}} \mathsf{h}[F] \to \mathsf{h}[\mathrm{X}] \twoheadrightarrow \mathcal{Q}(\mathsf{h})[\mathrm{X}].$$

The first map, into the F-component, is Δ_A^F multiplied by the scalar $\boldsymbol{\zeta}(A,F)$. The second map, on the F-component, is $\beta_{\mathrm{X},F}$. The last map is the canonical projection which is suppressed in formula (13.37).

Lemma 13.56. *For a bimonoid* h*, the map* (13.36) *is a surjective morphism of monoids.*

The product and coproduct of $\mathcal{S}(\mathcal{Q}(\mathsf{h}))$ are given by (6.51) with $\mathsf{p} := \mathcal{Q}(\mathsf{h})$. The above result only concerns the monoid structure. Surjectivity of (13.36) can be deduced from Proposition 5.24.

Theorem 13.57. *For a commutative bimonoid* h*, the map* (13.36) *is an isomorphism of monoids. In particular, any commutative bimonoid is isomorphic as a monoid to the free commutative monoid on its indecomposable part.*

This is the *Borel–Hopf theorem* for commutative bimonoids. It is dual to Theorem 13.34 and can be proved in a similar manner.

Corollary 13.58. *For a finite-dimensional commutative bimonoid* h*,*

$$\dim \mathsf{h}[\mathrm{X}] = \sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{X}} \dim \mathcal{Q}(\mathsf{h})[\mathrm{Y}] \tag{13.38}$$

for any flat X*.*

13.4.2. The inverse isomorphism. We now proceed to describe the inverse of (13.36) using freeness of $\mathcal{S}(\mathcal{Q}(\mathsf{h}))$ and a logarithm of the identity map from Section 9.1.

Fix a noncommutative Möbius function $\boldsymbol{\mu}$. For a commutative bimonoid h, consider the map of species $\mathbf{log}(\mathrm{id}) : \mathcal{Q}(\mathsf{h}) \hookrightarrow \mathsf{h}$ given in Proposition 9.17. It is constructed out of $\boldsymbol{\mu}$. Now apply Theorem 6.17 with $\mathsf{a} := \mathsf{h}$, $\mathsf{p} := \mathcal{Q}(\mathsf{h})$, $f := \mathbf{log}(\mathrm{id})$ to obtain a morphism $\hat{f}$ of monoids

$$\mathcal{S}(\mathcal{Q}(\mathsf{h})) \to \mathsf{h}. \tag{13.39}$$

We can describe this explicitly using (6.21b). Evaluating on the A-component, on the X-summand for $\mathrm{X} \geq \mathrm{s}(A)$, the map is

$$\sum_{G:\, G \geq K} \boldsymbol{\mu}(K, G)\, \mu_A^G \Delta_K^G \beta_{K,\mathrm{X}} \;:\; \mathcal{Q}(\mathsf{h})[\mathrm{X}] \to \mathsf{h}[A], \tag{13.40}$$

where K is a fixed face of support X which is greater than A. The map as written is from $\mathsf{h}[\mathrm{X}]$ to $\mathsf{h}[A]$, the point being that it factors through $\mathcal{Q}(\mathsf{h})[\mathrm{X}]$.

Theorem 13.59. *For a commutative bimonoid* h*, the map* (13.39) *is an isomorphism of monoids. Moreover, it is the inverse of the map* (13.36) *when* $\boldsymbol{\mu}$ *and* $\boldsymbol{\zeta}$ *are inverse.*

Proof. It suffices to check that (13.36) and (13.39) are inverses. We omit this calculation. The dual calculation was done in detail in the proof of Theorem 13.38. □

We refer to (13.36) and (13.39) as the *dual Borel–Hopf isomorphisms*.

Exercise 13.60. Work out the dual Borel–Hopf isomorphisms for a rank-one arrangement along the lines of Example 13.39.

Exercise 13.61. For a commutative bimonoid h, consider the following derivations f and g.

$$\begin{tikzcd} \mathsf{h} \arrow[r, "f"] \arrow[d, two heads] & \mathcal{S}(\mathcal{Q}(\mathsf{h})) \\ \mathcal{Q}(\mathsf{h}) \arrow[ur, hook] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S}(\mathcal{Q}(\mathsf{h})) \arrow[r, "g"] \arrow[d, two heads] & \mathsf{h} \\ \mathcal{Q}(\mathsf{h}) \arrow[ur, hook, "\mathbf{log}(\mathrm{id})"'] & \end{tikzcd}$$

Check that: The maps (13.36) and (13.39) are exponentials (9.3a) of f and g, respectively. Deduce from Theorem 9.12 that $\mathbf{exp}(f)$ and $\mathbf{exp}(g)$ are morphisms of monoids.

An exponential of a derivation, and logarithm of a monoid morphism in Theorem 9.12 can be expressed in terms of the Borel–Hopf isomorphisms as follows.

Exercise 13.62. Let h be a commutative bimonoid and a a commutative monoid. Check that:

- For $f : \mathsf{h} \to \mathsf{a}$ a derivation, its exponential equals
$$\mathbf{exp}(f) : \mathsf{h} \xrightarrow{\cong} \mathcal{S}(\mathcal{Q}(\mathsf{h})) \to \mathsf{a}.$$
The first map is the isomorphism (13.36), while the second map arises from freeness of $\mathcal{S}(\mathcal{Q}(\mathsf{h}))$ as in Theorem 6.17.
- For $g : \mathsf{h} \to \mathsf{a}$ a morphism of monoids, its logarithm equals
$$\mathbf{log}(g) : \mathsf{h} \twoheadrightarrow \mathcal{Q}(\mathsf{h}) \hookrightarrow \mathcal{S}(\mathcal{Q}(\mathsf{h})) \xrightarrow{\cong} \mathsf{h} \to \mathsf{a}.$$
The isomorphism is as in (13.39).

13.4.3. Connection with Leray–Samelson.

Exercise 13.63. For a bicommutative bimonoid h, the dual Borel–Hopf isomorphisms (13.36) and (13.39) reduce to the dual Leray–Samelson isomorphisms (13.13) and (13.15), respectively. Check this using Lemma 1.17.

Deduce the first part of Exercise 13.25 as a specialization of Exercise 13.61.

Exercise 13.64. Let h be a commutative bimonoid. Applying the functor $\mathrm{gr}_{\mathcal{D}}$ to (13.36) and using (6.88) and Theorem 13.57, we obtain an isomorphism
$$\mathrm{gr}_{\mathcal{D}}(\mathsf{h}) \xrightarrow{\cong} \mathcal{S}(\mathcal{Q}(\mathsf{h}))$$
of graded monoids. By the Milnor–Moore cocommutativity result (Proposition 5.65), the bimonoid $\mathrm{gr}_{\mathcal{D}}(\mathsf{h})$ is bicommutative. Moreover, by (5.37), its indecomposable part is $\mathcal{Q}(\mathsf{h})$. Check that the above isomorphism is an instance of the map (13.13) and hence an isomorphism of graded bimonoids.

Exercise 13.65. Let h be a commutative bimonoid. Show that $\mathrm{gr}_{\mathcal{D}}(\mathsf{h}) \cong \mathsf{h}$ as monoids. More precisely, both are isomorphic to $\mathcal{S}(\mathcal{Q}(\mathsf{h}))$ as monoids.

Exercise 13.66. Show that: For a commutative bimonoid h, the diagram
$$\begin{tikzcd}
\mathcal{S}(\mathcal{P}(\mathsf{h})) \arrow[r, "\cong"] \arrow[d, hook, "\mathcal{S}(\mathrm{pq}_{\mathsf{h}})"'] & \mathsf{h}^{coab} \arrow[d, hook, "\pi_{\mathsf{h}}^{\vee}"] \\
\mathcal{S}(\mathcal{Q}(\mathsf{h})) \arrow[r, "\cong"'] & \mathsf{h}
\end{tikzcd}$$
commutes. The right-vertical map is the coabelianization, while the left-vertical map is induced from (5.50). The horizontal maps are the isomorphisms (13.39) and (13.9). The latter applies since h^{coab} is bicommutative and $\mathcal{P}(\mathsf{h}) = \mathcal{P}(\mathsf{h}^{coab})$ by Exercise 5.10.

(First draw the diagonal arrow from $\mathcal{S}(\mathcal{P}(\mathsf{h}))$ to h. For the commutativity of the two triangles, use the first part of Remark 6.47 and the second part of Exercise 9.26, respectively.)

13.4.4. Borel–Hopf via characteristic operations. We now explain the connection between the indecomposable part and characteristic operations by eulerian idempotents of the Tits algebra.

Let $\mathtt{E}$ be the eulerian family corresponding to a fixed noncommutative zeta function $\boldsymbol{\zeta}$. Let h be a commutative bimonoid.

Lemma 13.67. *We have*

$$\mathsf{h}[A] = \bigoplus_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{s}(A)} \mathsf{h}[A] \cdot \mathtt{E}_{\mathrm{X}/A}.$$

The sum is over all flats X *containing* A*.*

Now consider the induced isomorphisms in diagram (10.70). By summing these over all flats X containing A, and using Lemma 13.67, we obtain the isomorphism (13.36). Its inverse

$$\mathcal{S}(\mathcal{Q}(\mathsf{h})) \to \mathsf{h} \tag{13.41}$$

is as follows. Evaluating on the A-component, on the X-summand for $\mathrm{X} \geq \mathrm{s}(A)$, the map is

$$\mathcal{Q}(\mathsf{h})[\mathrm{X}] \xrightarrow{\beta_{K,\mathrm{X}}} \mathcal{Q}(\mathsf{h})[K] \xrightarrow{(-)\cdot \mathtt{E}_{\mathrm{X}/K}} \mathsf{h}[K] \xrightarrow{\mu_A^K} \mathsf{h}[A],$$

where K is any face greater than A, and of support X. The second map is induced by right multiplication by $\mathtt{E}_{\mathrm{X}/K}$ on $\mathsf{h}[K]$.

Comparing expressions, we see that (13.41) coincides with (13.39) as expected.

13.4.5. Decomposable filtration of a commutative bimonoid. The decomposable filtration of $\mathcal{S}(\mathcal{Q}(\mathsf{h}))$ is given by Proposition 6.59. Translating via Theorem 13.57, we deduce:

Proposition 13.68. *The decomposable filtration of a commutative bimonoid* h *is given by*

$$\mathcal{D}_k(\mathsf{h})[A] = \bigoplus_{\mathrm{X}:\, \mathrm{rk}(\mathrm{X}/A) \geq k} \mathsf{h}[A] \cdot \mathtt{E}_{\mathrm{X}/A}.$$

For $A = O$, this is the decomposable series of the right module $\mathsf{h}[O]$ over the Tits algebra, see Section 11.5.3. This expression for the decomposable series is given in [**21**, Proposition 13.24] (with a discussion on a finer filtration indexed by flats).

13.4.6. Signed analogue. Fix a noncommutative zeta function $\boldsymbol{\zeta}$. For a signed bimonoid h, we define a natural map

$$\mathsf{h} \to \mathcal{E}(\mathcal{Q}(\mathsf{h})) \tag{13.42}$$

as follows. Evaluating on the A-component, into the X-summand for $\mathrm{X} \geq \mathrm{s}(A)$, the map is

$$\sum_{\substack{F:\, F \geq A \\ \mathrm{s}(F) = \mathrm{X}}} \boldsymbol{\zeta}(A, F)\, ([F/A] \otimes \beta_{\mathrm{X},F})\Delta_A^F \,:\, \mathsf{h}[A] \to \mathbf{E}^-[\mathrm{s}(A), \mathrm{X}] \otimes \mathcal{Q}(\mathsf{h})[\mathrm{X}], \tag{13.43}$$

with $[F/A] \otimes \beta_{\mathrm{X},F}$ as in (2.45). The canonical projection from h to $\mathcal{Q}(\mathsf{h})$ is suppressed in notation (13.43).

Theorem 13.69. *For any signed commutative signed bimonoid* h*, the map* (13.42) *is an isomorphism of monoids. In particular, any signed commutative signed bimonoid is isomorphic as a monoid to the free signed commutative monoid on its indecomposable part.*

This is the *Borel–Hopf theorem* for signed commutative signed bimonoids.

Exercise 13.70. Check that: For a signed commutative signed bimonoid h, the inverse of (13.42)

$$\mathcal{E}(\mathcal{Q}(\mathsf{h})) \to \mathsf{h} \tag{13.44}$$

is as follows. Evaluating on the A-component, on the X-summand for $\mathrm{X} \geq \mathrm{s}(A)$, the map is

$$\sum_{G\colon G \geq K} \boldsymbol{\mu}(K,G)\, \mu_A^G \Delta_K^G([K/A]\, \beta_{K,\mathrm{X}}) \,:\, \mathbf{E}^-[\mathrm{s}(A), \mathrm{X}] \otimes \mathcal{Q}(\mathsf{h})[\mathrm{X}] \to \mathsf{h}[A], \tag{13.45}$$

where K is a fixed face of support X which is greater than A, and $[K/A]\, \beta_{K,\mathrm{X}}$ is as in (2.45).

We refer to (13.42) and (13.44) as the *signed dual Borel–Hopf isomorphisms*.

Exercise 13.71. For a signed commutative signed bimonoid h, consider the following derivations f and g.

$$\begin{tikzcd} \mathsf{h} \arrow[r, "f"] \arrow[d, two heads] & \mathcal{E}(\mathcal{Q}(\mathsf{h})) \\ \mathcal{Q}(\mathsf{h}) \arrow[ur, hook] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}(\mathcal{Q}(\mathsf{h})) \arrow[r, "g"] \arrow[d, two heads] & \mathsf{h} \\ \mathcal{Q}(\mathsf{h}) \arrow[ur, hook, "\mathbf{log}(\mathrm{id})"'] & \end{tikzcd}$$

Check that: The maps (13.42) and (13.44) are exponentials (9.3a) of f and g, respectively. Deduce from Theorem 9.32 that $\mathbf{exp}(f)$ and $\mathbf{exp}(g)$ are morphisms of monoids.

Also check that (13.44) is a special case of (6.35b) with $\mathsf{a} := \mathsf{h}$, $\mathsf{p} := \mathcal{Q}(\mathsf{h})$, $f := \mathbf{log}(\mathrm{id})$.

13.5. Unification using partially bicommutative bimonoids

We formalize the similarities between Loday–Ronco and Leray–Samelson, and between Borel–Hopf and Leray–Samelson by using the notion of partially bicommutative bimonoids.

13.5.1. Loday–Ronco and Leray–Samelson. Recall from Section 2.13.4 that 0-$\sim$-bicommutative bimonoids interpolate between 0-bimonoids and bicommutative bimonoids. They are associated to a partial-support relation $\sim$. If $\sim$ is geometric, then it arises from a subarrangement, say $\mathcal{A}_\sim$. A partial-flat in this case consists of faces which have the same support and which lie on the same side of every hyperplane in $\mathcal{A}_\sim$. The extreme cases are as follows. When $\sim$ is finest, $\mathcal{A}_\sim = \mathcal{A}$ and a 0-$\sim$-bicommutative bimonoid is a 0-bimonoid. When $\sim$ is coarsest, $\mathcal{A}_\sim$ is empty and a 0-$\sim$-bicommutative bimonoid is a bicommutative bimonoid.

We now establish rigidity of 0-$\sim$-bicommutative bimonoids in the case when $\sim$ is geometric, and unify Loday–Ronco and Leray–Samelson.

Recall from Section 6.11.2 the 0-$\sim$-bicommutative bimonoid $\mathcal{T}_{0,\sim}(\mathsf{p})$ associated to a species p. We note from (6.95) that

$$\mathcal{P}(\mathcal{T}_{0,\sim}(\mathsf{p})) = \mathsf{p}. \tag{13.46}$$

We now claim that composing the functors in the opposite order also yields an isomorphism with the identity.

Proposition 13.72. *For a geometric partial-support relation $\sim$ on faces, and a 0-$\sim$-bicommutative bimonoid h, there is a natural isomorphism*

$$\mathcal{T}_{0,\sim}(\mathcal{P}(\mathsf{h})) \xrightarrow{\cong} \mathsf{h}$$

of 0-$\sim$-bicommutative bimonoids. In particular, h is free as a $\sim$-commutative monoid and cofree as a $\sim$-cocommutative comonoid.

Proof. Repeat the proof of Proposition 13.1 with faces replaced by partial-flats. (We use Exercise 5.46.) Also compare with the proof of Proposition 13.9. $\square$

As a consequence:

Theorem 13.73. *For any geometric partial-support relation $\sim$ on faces, the functors $\mathcal{T}_{0,\sim}$ and $\mathcal{P}$ detemine an adjoint equivalence between the category of species and the category of 0-$\sim$-bicommutative bimonoids.*

This unifies Theorem 13.2 and Theorem 13.11. When $\sim$ is finest, we recover the former, while when $\sim$ is coarsest, we recover the latter.

Exercise 13.74. Show that any finite-dimensional 0-$\sim$-bicommutative bimonoid associated to a geometric partial-support relation $\sim$ on faces is self-dual.

13.5.2. LRB species. The Borel–Hopf Theorems 13.34 and 13.38 for cocommutative bimonoids, and the dual Theorems 13.57 and 13.59 for commutative bimonoids are valid in the more general setting of LRB species (Section 3.9). The Borel–Hopf isomorphisms (13.21) and (13.23) and the dual Borel–Hopf isomorphisms (13.36) and (13.39) are defined by the same formulas as before; recall here that noncommutative zeta and Möbius functions make sense for any LRB (Remark 1.18). In the same vein, the Leray–Samelson Theorems 13.11 and 13.21 generalize to any LRB.

When the LRB is specialized to the Tits monoid, one recovers all the above results.

13.5.3. Borel–Hopf and Leray–Samelson. Let $\sim$ be a partial-support relation on $\mathcal{A}$. A cocommutative bimonoid for the LRB $\Sigma_\sim[\mathcal{A}]$ is the same as an $\mathcal{A}$-bimonoid which is cocommutative and $\sim$-commutative. By the discussion in Section 13.5.2, Borel–Hopf holds for cocommutative bimonoids for $\Sigma_\sim[\mathcal{A}]$. When $\sim$ is finest, this result specializes to Borel–Hopf (Theorem 13.34). When

$\sim$ is coarsest, it specializes to a weaker form of Leray–Samelson (Proposition 13.9) which says that (13.9) is an isomorphism of comonoids.

Similarly, Borel–Hopf holds for commutative bimonoids for $\Sigma_\sim[\mathcal{A}]$. When $\sim$ is finest, this result specializes to Borel–Hopf (Theorem 13.57). When $\sim$ is coarsest, it specializes to a weaker form of Leray–Samelson (Proposition 13.22) which says that (13.13) is an isomorphism of monoids.

13.6. Rigidity of q-bimonoids for q not a root of unity

Recall from Section 13.1 the Loday–Ronco theorem which says that any 0-bimonoid is the free monoid and cofree comonoid on its primitive or indecomposable part. We now generalize this result to q-bimonoids for q not a root of unity. The methods of proof are the same as before. The added complication lies in the invertibility of the Varchenko matrix associated to the q-distance function on the arrangement. The q-exponential, q-logarithm, q-norm map, two-sided characteristic operations enter into the picture.

13.6.1. Primitive part of the q-bimonoid associated to a species. Recall from Exercise 6.33 that p is contained in the primitive part of the q-bimonoid $\mathcal{T}_q(\mathsf{p})$. Moreover, from Exercise 6.76, we know that equality holds when q is not a root of unity. The same result, along with a more direct proof, is given below.

Proposition 13.75. *Suppose q is not a root of unity. Then $\mathcal{P}\mathcal{T}_q = \mathrm{id}$, that is, for any species p,*

$$\mathcal{P}\mathcal{T}_q(\mathsf{p}) = \mathsf{p}.$$

Proof. Suppose $(x^F)_{F \geq A}$ is in the primitive part of $\mathcal{T}_q(\mathsf{p})[A]$ with $x^F \in \mathsf{p}[F]$. We need to show that $x^G = 0$ for all $A < G$. We employ Lemma 6.63. Let us only consider those equations in (6.71) where $H = G$ and further group them according to their support: For each flat $\mathrm{X} > \mathrm{s}(A)$, we have the set of equations

$$\sum_{\substack{F:\, F \geq A \\ \mathrm{s}(F) = \mathrm{s}(G)}} q^{\mathrm{dist}(F,G)} \beta_{G,F}(x^F) = 0$$

indexed by faces G of support X. By Theorem 1.10, the matrix $(q^{\mathrm{dist}(F,G)})$ is invertible if q is not a root of unity. So the only possible solutions in this case are $x^F = 0$. This completes the proof.

The above argument in more direct terms is as follows. The Witt identity (1.81) says that for any $A \leq G$,

$$\sum_{H: A \leq H \leq G} (-1)^{\mathrm{rk}(H)} \Bigg(\sum_{\substack{F:\, F \geq A \\ HF = G,\, \mathrm{s}(F) = \mathrm{s}(G)}} (\beta_q)_{G,F}(x^F) \Bigg) = (-1)^{\mathrm{rk}(G)} (\beta_q)_{G, A\overline{G}}(x^{A\overline{G}}).$$

This is an identity involving elements of the vector space $\mathsf{p}[G]$. By hypothesis, all terms in parenthesis are zero except the term corresponding to $H = A$ (and this term is x^G). This yields that for any $A \leq G$,

$$(-1)^{\mathrm{rk}(A)} x^G = (-1)^{\mathrm{rk}(G)} (\beta_q)_{G, A\overline{G}}(x^{A\overline{G}}).$$

Interchanging the roles of G and $A\overline{G}$, we get

$$(-1)^{\mathrm{rk}(A)} x^{A\overline{G}} = (-1)^{\mathrm{rk}(G)} (\beta_q)_{A\overline{G},G}(x^G).$$

Now combining the two identities, we deduce that

$$x^G = q^{2\,\mathrm{dist}(G,A\overline{G})} x^G.$$

If $A < G$, then G and $A\overline{G}$ are distinct, so $\mathrm{dist}(G, A\overline{G}) > 0$. In addition, if q is not a root of unity, then the coefficient of x^G in the rhs cannot be 1, which forces $x^G = 0$. $\square$

13.6.2. Adjoint equivalence between species and q-bimonoids. We now show that the composite in the other direction, namely, $\mathcal{T}_q\mathcal{P}$, is also the identity.

Proposition 13.76. *Suppose q is not a root of unity. For a q-bimonoid h, the natural map*

$$\mathcal{T}_q(\mathcal{P}(\mathsf{h})) \xrightarrow{\cong} \mathsf{h} \tag{13.47}$$

given on the A-component by

$$(\mu_A^F) : \bigoplus_{F:\, F \geq A} \mathcal{P}(\mathsf{h})[F] \longrightarrow \mathsf{h}[A],$$

is an isomorphism of q-bimonoids.

The universal property of $\mathcal{T}_q$ stated in Theorem 6.31 applied to $f := \mathrm{id}$ on $\mathcal{P}(\mathsf{h})$ yields the morphism (13.47) of q-bimonoids.

Proof. Let us denote the map (13.47) by f. By Proposition 13.75, the primitive part of $\mathcal{T}_q(\mathcal{P}(\mathsf{h}))$ is $\mathcal{P}(\mathsf{h})$. Thus, f is identity (and in particular injective) on the primitive part. Therefore, by Proposition 5.18, it is injective.

We now show it is surjective by an induction based on the primitive filtration of h. Take $z \in \mathsf{h}[A]$. By (5.20), there exists $k \geq 1$ such that $z \in \mathcal{P}_k(\mathsf{h})[A]$. If $k = 1$, then $z = f_A(z)$. Suppose $k \geq 2$. Recall the maps $(\beta^q)^{k-1}$ from (5.5) (which are defined when q is not a root of unity). Combining (5.24) and Exercise 5.13, we have

$$(\beta^q)_A^{k-1} \Delta_A^{k-1}(z) \in \mathcal{P}(\mathsf{h})^k[A]$$

and therefore

$$f_A\big((\beta^q)_A^{k-1} \Delta_A^{k-1}(z)\big) = \mu_A^{k-1} (\beta^q)_A^{k-1} \Delta_A^{k-1}(z).$$

Now consider the element

$$z' := z - \mu_A^{k-1} (\beta^q)_A^{k-1} \Delta_A^{k-1}(z).$$

By Lemma 5.42, z' belongs to $\mathcal{P}_{k-1}(\mathsf{h})[A]$, and hence by induction hypothesis, it belongs to the image of f_A. We conclude that

$$z = z' + \mu_A^{k-1} (\beta^q)_A^{k-1} \Delta_A^{k-1}(z) = z' + f_A\big((\beta^q)_A^{k-1} \Delta_A^{k-1}(z)\big)$$

is also in the image of f_A. $\square$

Since the functor $\mathcal{T}_q$ and the primitive part functor $\mathcal{P}$ are adjoints (Theorem 6.30), and composing them either way yields natural isomorphisms with the identity, we obtain the following result.

Theorem 13.77. *For q not a root of unity,*

$$\mathcal{A}\text{-}\mathsf{Sp} \underset{\mathcal{P}}{\overset{\mathcal{T}_q}{\rightleftarrows}} q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$$

is an adjoint equivalence of categories.

We call Theorem 13.77 the *rigidity theorem for q-bimonoids*. The $q = 0$ case recovers the Loday–Ronco Theorem 13.2.

Remark 13.78. More generally, the adjoint equivalence in Theorem 13.77 works for any q for which the Varchenko matrices (1.34) are invertible: The condition on q (not being a root of unity) came from Theorem 1.10 which was invoked in the proofs of Propositions 13.75 and 13.76. For instance, for a rank-one arrangement, the two propositions and the theorem work for $q \neq \pm 1$.

Example 13.79. Suppose q is not a root of unity. Let p be a species. Take $\mathsf{h} = \mathcal{T}_q(\mathsf{p})$. By Proposition 13.75, we have $\mathcal{P}(\mathsf{h}) = \mathsf{p}$, and it is easy to see that (13.47) is the identity map. We give two illustrations.

- Let $\mathsf{h} = \Gamma_q$, the q-bimonoid of chambers (Section 7.3). The map (13.47) coincides with the identification (7.24), see also (7.155).
- Let $\mathsf{h} = \Sigma_q$, the q-bimonoid of faces (Section 7.6). The map (13.47) coincides with the first identification in (7.83), see also (7.160).

Example 13.80. Suppose q is not a root of unity. Let p be a species. Take $\mathsf{h} = \mathcal{T}_q^{\vee}(\mathsf{p})$. Clearly, $\mathcal{P}(\mathsf{h}) = \mathsf{p}$, and one may check that (13.47) coincides with the q-norm map (6.74). As a consequence, we recover Proposition 6.75 which says that the q-norm map is an isomorphism when q is not a root of unity.

- Let $\mathsf{h} = \Gamma_q^*$. The map (13.47) coincides with the q-norm map on chambers (7.26) after the identifications (7.24) and (7.25).
- Let $\mathsf{h} = \Sigma_q^*$. The map (13.47) coincides with the q-norm map on faces (7.89) for $\eta \equiv 1$ after the identifications (7.83).

Corollary 13.81. *Any finite-dimensional q-bimonoid is self-dual for q not a root of unity.*

Proof. By Proposition 13.76, any q-bimonoid h is of the form $\mathcal{T}_q(\mathsf{p})$ for some species p. Hence,

$$\mathsf{h}^* \cong \mathcal{T}_q(\mathsf{p})^* \cong \mathcal{T}_q^{\vee}(\mathsf{p}^*) \cong \mathcal{T}_q(\mathsf{p}^*) \cong \mathcal{T}_q(\mathsf{p}) \cong \mathsf{h}.$$

The middle step used Proposition 6.75. The second-to-last step used finite-dimensionality of p. $\square$

This generalizes Corollary 13.3.

13.6.3. The inverse isomorphism. Let q be any scalar which is not a root of unity. Recall the two-sided q-zeta function $\boldsymbol{\zeta}_q$ and two-sided q-Möbius function $\boldsymbol{\mu}_q$ defined in Section 1.6.6. We now present an approach to the rigidity theorem which makes explicit use of these functions. In the process, we will also obtain a formula for the inverse of (13.47).

For a q-bimonoid h, consider the map

$$\mathcal{T}_q^\vee(\mathcal{P}(\mathsf{h})) \to \mathsf{h} \tag{13.48}$$

defined on the A-component, on the F'-summand, by

$$\mathcal{P}(\mathsf{h})[F'] \to \mathsf{h}[A], \qquad x \mapsto \sum_{\substack{F:\, F \geq A \\ \mathrm{s}(F)=\mathrm{s}(F')}} \boldsymbol{\zeta}_q(A, F, F')\, \mu_A^F \beta_{F,F'}(x).$$

In fact, this map is an isomorphism, as we will see below (in more than one way).

Exercise 13.82. Check that the map (13.48) equals the composite

$$\mathcal{T}_q^\vee(\mathcal{P}(\mathsf{h})) \xrightarrow{\cong} \mathcal{T}_q(\mathcal{P}(\mathsf{h})) \xrightarrow{\cong} \mathsf{h},$$

with the first map as in (6.76) for $\mathsf{p} := \mathcal{P}(\mathsf{h})$, and the second map as in (13.47). Deduce that it is an isomorphism of q-bimonoids.

Let p be a species. Use Examples 13.79 and 13.80 to deduce that: For $\mathsf{h} := \mathcal{T}_q^\vee(\mathsf{p})$, the map (13.48) is the identity map. For $\mathsf{h} := \mathcal{T}_q(\mathsf{p})$, the map (13.48) is the inverse of the q-norm map (6.76).

For a q-bimonoid h, consider the map

$$\mathsf{h} \to \mathcal{T}_q^\vee(\mathcal{P}(\mathsf{h})) \tag{13.49}$$

defined on the A-component, into the F'-summand, by

$$\mathsf{h}[A] \to \mathcal{P}(\mathsf{h})[F'], \qquad x \mapsto \sum_{\substack{G, G' \geq F' \\ \mathrm{s}(G)=\mathrm{s}(G')}} \boldsymbol{\mu}_q(F', G, G')\, \mu_{F'}^G \beta_{G,G'} \Delta_A^{G'}(x).$$

The sum is over both G and G'. This map arises from Theorem 6.13 with $\mathsf{a} := \mathcal{P}(\mathsf{h})$ (with the trivial product) and $f := \mathbf{log}_q(\mathrm{id})$, see Corollary 9.85 in this regard.

Proposition 13.83. *For q not a root of unity, for a q-bimonoid h, the maps (13.48) and (13.49) are inverse isomorphisms of q-bimonoids.*

PROOF. One can directly check that (13.48) and (13.49) are inverses. We leave these calculations as an exercise. A conceptual explanation is given in Exercise 13.85 below. $\square$

Note very carefully that the above argument gives another proof of Proposition 13.76. We refer to (13.48) and (13.49) as the *q-rigidity isomorphisms*.

Exercise 13.84. Check that: The inverse of (13.47), namely,

$$\mathsf{h} \to \mathcal{T}_q(\mathcal{P}(\mathsf{h})), \tag{13.50}$$

on the A-component, into the F-summand, is given by

$$\mathbf{log}_q(\mathrm{id})_F\Big(\sum_{\substack{F' \colon F' \geq A \\ \mathrm{s}(F') = \mathrm{s}(F)}} \boldsymbol{\zeta}_q(A, F, F')\, \beta_{F,F'} \Delta_A^{F'} \Big).$$

Since $\mathbf{log}_q(\mathrm{id})_F$ is defined using $\boldsymbol{\mu}_q$, this formula involves both $\boldsymbol{\zeta}_q$ and $\boldsymbol{\mu}_q$.

Exercise 13.85. For a q-bimonoid h for q not a root of unity, consider the following biderivations f and g.

$$\begin{tikzcd} \mathcal{T}_q^\vee(\mathcal{P}(\mathsf{h})) \arrow[r, "f"] \arrow[d, two heads] & \mathsf{h} \\ \mathcal{P}(\mathsf{h}) \arrow[r, "\mathrm{id}"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{h} \arrow[r, "g"] \arrow[d, two heads] & \mathcal{T}_q^\vee(\mathcal{P}(\mathsf{h})) \\ \mathcal{Q}(\mathsf{h}) \arrow[r, "\mathbf{log}_q(\mathrm{id})"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd}$$

Check that: The maps (13.48) and (13.49) are the q-exponentials (9.29a) of f and g, respectively. Compare the first claim with Example 9.82.

Now deduce from Theorem 9.81 that $\mathbf{exp}_q(f)$ and $\mathbf{exp}_q(g)$ are morphisms of q-bimonoids. Moreover, deduce from Proposition 9.91 and its proof that $\mathbf{exp}_q(f)$ and $\mathbf{exp}_q(g)$ are inverses of each other. This can also be done directly using Lemma 9.83, item (2) and Example 9.89. See Exercise 13.6 for the case $q = 0$.

Exercise 13.86. For a q-bimonoid h for q not a root of unity, consider the following biderivations f' and g'.

$$\begin{tikzcd} \mathcal{T}_q(\mathcal{P}(\mathsf{h})) \arrow[r, "f'"] \arrow[d, two heads] & \mathsf{h} \\ \mathcal{P}(\mathsf{h}) \arrow[r, "\mathrm{id}"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{h} \arrow[r, "g'"] \arrow[d, two heads] & \mathcal{T}_q(\mathcal{P}(\mathsf{h})) \\ \mathcal{Q}(\mathsf{h}) \arrow[r, "\mathbf{log}_q(\mathrm{id})"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd}$$

The maps (13.47) and (13.50) are the q-exponentials of f' and g', respectively. Check this directly or deduce it from Exercise 13.85 and Example 9.92.

The q-exponential of a coderivation, and q-logarithm of a comonoid morphism in Theorem 9.78 can be expressed in terms of the q-rigidity isomorphisms as follows.

Exercise 13.87. Suppose q is not a root of unity. Let c be a comonoid and k a q-bimonoid. Check that:

- For $f : \mathsf{c} \to \mathsf{k}$ a coderivation, its q-exponential equals
$$\mathbf{exp}_q(f) : \mathsf{c} \to \mathcal{T}_q^\vee(\mathcal{P}(\mathsf{k})) \xrightarrow{\cong} \mathsf{k}.$$
The first map arises from cofreeness of $\mathcal{T}_q^\vee(\mathcal{P}(\mathsf{k}))$ as in Theorem 6.10, while the second map is the isomorphism (13.48).
- For $g : \mathsf{c} \to \mathsf{k}$ a morphism of comonoids, its q-logarithm equals
$$\mathbf{log}_q(g) : \mathsf{c} \to \mathsf{k} \xrightarrow{\cong} \mathcal{T}_q^\vee(\mathcal{P}(\mathsf{k})) \twoheadrightarrow \mathcal{P}(\mathsf{k}) \hookrightarrow \mathsf{k}.$$
The isomorphism is as in (13.49).

See Exercise 13.7 for the case $q = 0$.

13.6.4. Rigidity via characteristic operations. The rigidity theorem for q-bimonoids can also be proved by employing the two-sided characteristic operations from Section 10.3. In this case, given a q-bimonoid h, for any face A, the q-Janus algebra $\mathsf{J}_q^o[A]$ acts on $\mathsf{h}[A]$, and the q-Janus algebra is also split-semisimple (though not commutative).

Lemma 13.88. *Let* h *be a* q*-bimonoid for* q *not a root of unity. Then, for any face* A,

$$\mathsf{h}[A] = \bigoplus_{F:\, F \geq A} \mathsf{Q}_{(F/A,F/A)} \cdot \mathsf{h}[A] = \bigoplus_{F':\, F' \geq A} \mathsf{Q}^d_{(F'/A,F'/A)} \cdot \mathsf{h}[A].$$

PROOF. Formulas (7.132) and (7.134) yield the two decompositions. (Note the remark that (7.132) holds for the Q^d-basis also.) $\square$

Now consider the induced isomorphisms in diagram (10.73). By summing these over all F greater than A and using Lemma 13.88, we obtain a linear isomorphism

$$\mathcal{T}_q(\mathcal{P}(\mathsf{h}))[A] \xrightarrow{\cong} \mathsf{h}[A].$$

By construction, this coincides with the map (6.41) arising from the universal property of $\mathcal{T}_q$, hence it yields an isomorphism $\mathcal{T}_q(\mathcal{P}(\mathsf{h})) \xrightarrow{\cong} \mathsf{h}$ of q-bimonoids. Thus, we have reproved Proposition 13.76, which is the nontrivial part of Theorem 13.77. We recall here that the proof of Proposition 10.50 is a lot simpler in the case $q = 0$.

Similarly, consider the induced isomorphisms in diagram (10.79). By summing these over all F' greater than A and using Lemma 13.88, we obtain a linear isomorphism

$$\mathcal{T}_q^{\vee}(\mathcal{P}(\mathsf{h}))[A] \xrightarrow{\cong} \mathsf{h}[A].$$

Observe that this is precisely the A-component of the isomorphism (13.48).

13.6.5. Rank-one arrangement. Let $\mathcal{A}$ denote a rank-one arrangement consisting of the central face O, and two chambers C and $\overline{C}$. The results below are special cases of Propositions 13.75 and 13.76, except that we are more precise on the values of q. See Remark 13.78. For illustration, we run through the arguments again.

Proposition 13.89. *Let* $\mathcal{A}$ *be a rank-one arrangement. If* $q \neq \pm 1$*, then* $\mathcal{P}\mathcal{T}_q = id$.

PROOF. For any $\mathcal{A}$-species p, note that

$$\mathcal{T}_q(\mathsf{p})[O] = \mathsf{p}[O] \oplus \mathsf{p}[C] \oplus \mathsf{p}[\overline{C}], \quad \mathcal{T}_q(\mathsf{p})[C] = \mathsf{p}[C], \quad \mathcal{T}_q(\mathsf{p})[\overline{C}] = \mathsf{p}[\overline{C}].$$

The nontrivial coproduct components of $\mathcal{T}_q(\mathsf{p})$ are given by

$$\Delta_O^C(x^O, x^C, x^{\overline{C}}) = x^C + q\, \beta_{C,\overline{C}}(x^{\overline{C}}),$$
$$\Delta_O^{\overline{C}}(x^O, x^C, x^{\overline{C}}) = x^{\overline{C}} + q\, \beta_{\overline{C},C}(x^C).$$

It is easy to see that for $q \neq \pm 1$,

$$\Delta_O^C(x^O, x^C, x^{\overline{C}}) = \Delta_O^{\overline{C}}(x^O, x^C, x^{\overline{C}}) = 0 \iff x^C = x^{\overline{C}} = 0$$

as required. □

Proposition 13.90. *Let $\mathcal{A}$ be a rank-one arrangement. If $q \neq \pm 1$, then $\mathcal{T}_q\mathcal{P} = id$. Explicitly, for any $\mathcal{A}$-q-bimonoid h with $q \neq \pm 1$, the map*

$$\mathrm{id} + \mu_O^C + \mu_O^{\overline{C}} : \mathcal{P}(\mathsf{h})[O] \oplus \mathsf{h}[C] \oplus \mathsf{h}[\overline{C}] \longrightarrow \mathsf{h}[O] \tag{13.51}$$

is an isomorphism.

We point out that since C and $\overline{C}$ are chambers,

$$\mathsf{h}[C] = \mathcal{P}(\mathsf{h})[C] \quad \text{and} \quad \mathsf{h}[\overline{C}] = \mathcal{P}(\mathsf{h})[\overline{C}].$$

PROOF. For injectivity, we use Proposition 13.89. For surjectivity, the expression for z' in (5.46) only involves $1-q^2$ in the denominator, so $q \neq \pm 1$ suffices. Let us see this explicitly. Let $z \in \mathsf{h}[O]$. Put

$$\begin{aligned} z' = z - \frac{1}{1-q^2}&(\mu_O^C \Delta_O^C(z) + \mu_O^{\overline{C}} \Delta_O^{\overline{C}}(z)) \\ &+ \frac{q}{1-q^2}(\mu_O^{\overline{C}} \beta_{\overline{C},C} \Delta_O^C(z) + \mu_O^C \beta_{C,\overline{C}} \Delta_O^{\overline{C}}(z)). \end{aligned}$$

Note that all terms in the rhs except z are in the image of either μ_O^C or $\mu_O^{\overline{C}}$. So it suffices to check that z' is primitive. Accordingly,

$$\begin{aligned} \Delta_O^C(z') = \Delta_O^C(z) - \frac{1}{1-q^2}\Delta_O^C(z) - \frac{q}{1-q^2}\beta_{C,\overline{C}}\Delta_O^{\overline{C}}(z) \\ + \frac{q^2}{1-q^2}\Delta_O^C(z) + \frac{q}{1-q^2}\beta_{C,\overline{C}}\Delta_O^{\overline{C}}(z) = 0. \end{aligned}$$

We used (2.2), (2.35), (2.36). By symmetry, $\Delta_O^{\overline{C}}(z') = 0$, and z' is primitive.

In this particular case, we point out that $z' = \mathbf{log}_q(\mathrm{id})_O(z)$, see Example 9.87. □

Exercise 13.91. Recall from Lemma 2.89 that an $\mathcal{A}$-q-bimonoid on a rank-one arrangement $\mathcal{A}$ can be described using idempotent operators on a vector space. Proposition 13.90 via this translation reduces to a statement of linear algebra. Check that this is precisely Lemma A.2.

Exercise 13.92. For a rank-one arrangement, the maps (13.48) and (13.49), on the O-component, are, respectively, given by

$$\mathcal{P}(\mathsf{h})[O] \oplus \mathsf{h}[C] \oplus \mathsf{h}[\overline{C}] \to \mathsf{h}[O]$$

$$(x, y, z) \mapsto x + \frac{1}{1-q^2}(\mu_O^C(y) + \mu_O^{\overline{C}}(z)) - \frac{q}{1-q^2}(\mu_O^{\overline{C}}\beta_{\overline{C},C}(y) + \mu_O^C\beta_{C,\overline{C}}(z)),$$

$$\mathsf{h}[O] \to \mathcal{P}(\mathsf{h})[O] \oplus \mathsf{h}[C] \oplus \mathsf{h}[\overline{C}]$$

$$x \mapsto \mathbf{log}_q(\mathrm{id})_O(x) + \Delta_O^C(x) + \Delta_O^{\overline{C}}(x).$$

Check directly that these maps are inverses of each other. (Use the formula for $\mathbf{log}_q(\mathrm{id})_O$ in Example 9.87.)

Similarly, the map (13.50), on the O-component, specializes to

$$\mathsf{h}[O] \to \mathcal{P}(\mathsf{h})[O] \oplus \mathsf{h}[C] \oplus \mathsf{h}[\overline{C}]$$

$$x \mapsto [\mathbf{log}_q(\mathrm{id})_O + \frac{1}{1-q^2}(\Delta_O^C + \Delta_O^{\overline{C}}) - \frac{q}{1-q^2}(\beta_{C,\overline{C}}\Delta_O^{\overline{C}} + \beta_{\overline{C},C}\Delta_O^C)](x).$$

Check directly that it is the inverse of (13.51).

13.6.6. Dual version. The dual version of Theorem 13.77 says that the indecomposable part functor $\mathcal{Q}$ and the functor $\mathcal{T}_q^\vee$ define an adjoint equivalence between the categories of species and q-bimonoids if q is not a root of unity.

Theorem 13.93. *For q not a root of unity,*

$$q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{\mathcal{T}_q^\vee}{\overset{\mathcal{Q}}{\rightleftarrows}} \mathcal{A}\text{-}\mathsf{Sp}$$

is an adjoint equivalence of categories.

Proof. We know from Theorem 6.30 that $\mathcal{Q}$ and $\mathcal{T}_q^\vee$ are adjoints. It remains to check that the unit and counit of this adjunction are isomorphisms. This can be deduced by duality from Theorem 13.77 using Exercise 6.39. □

One may also proceed directly. The nontrivial part of the equivalence in Theorem 13.93 works as follows.

Proposition 13.94. *For a q-bimonoid h for q not a root of unity, there is a natural isomorphism*

$$\mathsf{h} \xrightarrow{\cong} \mathcal{T}_q^\vee(\mathcal{Q}(\mathsf{h})) \tag{13.52}$$

of q-bimonoids.

The morphism (13.52) is obtained by applying the universal property of $\mathcal{T}_q^\vee$ stated in Theorem 6.34 to $f := \mathrm{id}$ on $\mathcal{Q}(\mathsf{h})$. Explicitly, using (6.42), the map on the A-component, into the F-summand, is

$$\mathsf{h}[A] \xrightarrow{\Delta_A^F} \mathsf{h}[F] \twoheadrightarrow \mathcal{Q}(\mathsf{h})[F]. \tag{13.53}$$

We also point out that in terms of characteristic operations, summing the induced isomorphisms in diagram (10.74) and using Lemma 13.88 (with Q^d instead of Q) yields the isomorphism (13.52).

Exercise 13.95. For q not a root of unity, show that for any q-bimonoid h, the map $\mathsf{h} \to \mathcal{T}_q^\vee(\mathcal{Q}(\mathsf{h}))$, namely, the counit of the adjunction between $\mathcal{Q}$ and $\mathcal{T}_q^\vee$, is an isomorphism as follows. When $\mathsf{h} = \mathcal{T}_q(\mathsf{p})$, the map specializes to the q-norm map (6.74) (see proof of Lemma 6.73) which we know is an isomorphism by Proposition 6.75. Moreover, every q-bimonoid is of this form by Proposition 13.76. This gives another way to derive Theorem 13.93.

For a q-bimonoid h for q not a root of unity, we have an isomorphism

$$\mathsf{h} \to \mathcal{T}_q(\mathcal{Q}(\mathsf{h})) \tag{13.54}$$

of q-bimonoids. (It connects to (13.52) by the q-norm map.) On the A-component, into the F'-summand, it is given by

$$\sum_{\substack{F\colon F\geq A \\ \mathrm{s}(F)=\mathrm{s}(F')}} \boldsymbol{\zeta}_q(A,F,F')\,\beta_{F',F}\Delta_A^F : \mathsf{h}[A] \to \mathcal{Q}(\mathsf{h})[F'].$$

The inverse isomorphism

$$\mathcal{T}_q(\mathcal{Q}(\mathsf{h})) \to \mathsf{h} \tag{13.55}$$

on the A-component, on the F'-summand, is given by

$$\sum_{\substack{G,G'\geq F' \\ \mathrm{s}(G)=\mathrm{s}(G')}} \boldsymbol{\mu}_q(F',G,G')\,\mu_A^{G'}\beta_{G',G}\Delta_{F'}^G : \mathcal{Q}(\mathsf{h})[F'] \to \mathsf{h}[A].$$

We refer to (13.54) and (13.55) also as the *q-rigidity isomorphisms*.

The inverse of (13.52), namely,

$$\mathcal{T}_q^{\vee}(\mathcal{Q}(\mathsf{h})) \to \mathsf{h}, \tag{13.56}$$

on the A-component, on the F-summand, is given by

$$\Big(\sum_{\substack{F'\colon F'\geq A \\ \mathrm{s}(F')=\mathrm{s}(F)}} \boldsymbol{\zeta}_q(A,F,F')\,\mu_A^{F'}\beta_{F',F}\Big)\,\mathbf{log}_q(\mathrm{id})_F.$$

The q-exponential of a derivation, and q-logarithm of a monoid morphism in Theorem 9.79 can be expressed in terms of the q-rigidity isomorphisms as follows.

Exercise 13.96. Let h be a q-bimonoid and a a monoid. Check that:

- For $f : \mathsf{h} \to \mathsf{a}$ a derivation, its q-exponential equals
$$\mathbf{exp}_q(f) : \mathsf{h} \xrightarrow{\cong} \mathcal{T}_q(\mathcal{Q}(\mathsf{h})) \to \mathsf{a}.$$
The first map is the isomorphism (13.54), while the second map arises from freeness of $\mathcal{T}_q(\mathcal{Q}(\mathsf{h}))$ as in Theorem 6.2.
- For $g : \mathsf{h} \to \mathsf{a}$ a morphism of monoids, its q-logarithm equals
$$\mathbf{log}_q(g) : \mathsf{h} \twoheadrightarrow \mathcal{Q}(\mathsf{h}) \hookrightarrow \mathcal{T}_q(\mathcal{Q}(\mathsf{h})) \xrightarrow{\cong} \mathsf{h} \to \mathsf{a}.$$
The isomorphism is as in (13.55).

Observe from Exercise 8.101 that conjugating the adjoint equivalence in either Theorem 13.77 or Theorem 13.93 by the signature functor has the effect of changing q to $-q$.

13.7. Monad for Lie monoids

Recall that any adjunction $(\mathcal{F},\mathcal{G})$ between categories C and D gives rise to a monad $\mathcal{G}\mathcal{F}$ on C and a comonad $\mathcal{F}\mathcal{G}$ on D. By Theorem 6.30, the functors $\mathcal{T}_q$ and $\mathcal{P}$ are adjoint between the category of species and the category of q-bimonoids. Thus, $\mathcal{P}\mathcal{T}_q$ is a monad on species, with structure maps

$$\mathcal{P}\mathcal{T}_q\mathcal{P}\mathcal{T}_q \to \mathcal{P}\mathcal{T}_q \quad \text{and} \quad \mathrm{id} \to \mathcal{P}\mathcal{T}_q$$

defined from the (co)unit of the adjunction.

If q is not a root of unity, then by Proposition 13.75, this monad is simply the identity. In contrast, the cases $q = \pm 1$ are of great interest as explained below.

There is an injective natural transformation

$$\iota : \mathcal{PT} \to \mathcal{T} \tag{13.57}$$

betweeen functors from the category of species to itself. Note very carefully that the two $\mathcal{T}$'s stand for different functors. The $\mathcal{T}$ on the left goes from species to bimonoids, while the $\mathcal{T}$ on the right goes from species to species by forgetting the bimonoid structure. The latter is also a monad with structure maps (3.2a) and (3.2b). Further, the diagrams

$$\begin{tikzcd} \mathcal{PTPT} \arrow[r, "\iota\iota"] \arrow[d] & \mathcal{TT} \arrow[d] \\ \mathcal{PT} \arrow[r, "\iota"'] & \mathcal{T} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{PT} \arrow[rr, "\iota"] & & \mathcal{T} \\ & \mathrm{id} \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{13.58}$$

commute: The first one can be deduced using the explicit description (6.41), while the second one is clear. Formally, (id, ι) is a lax functor of monads from $\mathcal{T}$ to $\mathcal{PT}$ in the sense of (C.2). Combining with the abelianization (3.12), we have morphisms of monads

$$\mathcal{PT} \xrightarrow{\iota} \mathcal{T} \xrightarrow{\pi} \mathcal{S}. \tag{13.59}$$

Recall that $\mathcal{T}$-algebras are monoids, and $\mathcal{S}$-algebras are commutative monoids. What then are $\mathcal{PT}$-algebras? These are Lie monoids which we will study in detail in Chapter 16.

Similarly, algebras over the monad $\mathcal{PT}_{-1}$ are signed Lie monoids, and we have morphisms of monads

$$\mathcal{PT}_{-1} \to \mathcal{T}_{-1} \to \mathcal{E}.$$

Notes

The results and terminology of this chapter are motivated by classical structure theorems in Hopf theory both in the context of vector spaces and of Joyal species.

Bialgebras. The classical Borel–Hopf, Leray–Samelson, Loday–Ronco theorems for Hopf algebras arose out of works of these authors. Details are given below.

Borel–Hopf for commutative bialgebras. The classical analogue of the Borel–Hopf Theorem 13.57 says that in characteristic zero, a connected (unsigned) graded commutative Hopf algebra is free over its indecomposable part. The analogue of Theorem 13.69 gives the same result for a connected signed graded commutative Hopf algebra.

The latter result originates in work of Hopf [**458**, Satz I], also see his announcement [**457**]. This was motivated by questions raised by Cartan in [**187**, Section VII]. Early exposition of Hopf's work is given by Samelson [**799**, Chapter I, Section 2], [**800**, Section 13] and Leray [**595**, Theorem 8]. It is presented in detail by Lefschetz in his algebraic topology book [**589**, Section VIII.9, Theorem 49.1]. Hopf's paper [**458**] is also reprinted in his selected works [**459**, pages 119–151] and collected works [**460**, pages 686–746]. The latter also contains an english translation by Charles Thomas. Brief historical summaries are provided by Hilton [**437**, pages 209 and 210], [**460**, Appendix 1, pages 1233–1235], Samelson [**801**, pages 134 and 135], Frei and Stammbach [**317**, page 1001]. Hopf, Samelson, Leray worked in a topological setting. Later, Borel provided a purely algebraic formulation of Hopf's result [**135**, Theorem 6.1]. Topological consequences are listed separately in [**135**, Section 7], also see his later paper [**136**, Section 1].

Early papers related to either Hopf's or Borel's work are those of Chevalley and Eilenberg [**214**, Theorem 18.1], Koszul [**542**, Theorem 10.2], Halpern [**412**, Theorem 2.1], [**414**, Section 2, Theorem A], Araki [**39**, page 406], May [**661**, Theorem 6], Nichols [**719**, Chapter II, Theorem 7]. See also the expositions by Lin [**597**, Section 1, Theorems 1.1 and 1.2], Michaelis [**688**, pages 593 to 596], Hofmann and Morris [**453**, Theorem A3.90].

The Borel–Hopf theorem is discussed in many places in the literature (in varying levels of generality). An elaborate account is given by Milnor and Moore [**695**, Theorems 7.5 and 7.11], [**696**, Theorem 4.6], [**706**, Theorem 5 on page 25]. See also the papers by Zisman [**932**, Theorem 4,2], Schoeller [**816**, Theorem 1, item (a)], Sjödin [**830**, Theorem 1, item (a)], Block [**122**, Lemma 1], Hain [**406**, Theorem on page 312]. A historical treatment is given by Dieudonné [**251**, Part 2, Section V1.2.A] and Cartier [**202**, Sections 2.3, 2.4, 2.5]. In the latter reference, Cartier also gives a modern formulation in [**202**, Theorems 3.8.3 and 3.9.1]. See also [**739**, Theorem I,6,4], [**740**, Proposition 4.3].

For book references, see for instance those by Bourgin [**151**, Chapter 12, Theorem 9.28], Dold [**260**, Proposition 10.16], Hatcher [**421**, Theorem 3C.4], Hazewinkel, Gubareni, Kirichenko [**428**, Theorems 3.8.17 and 3.13.9], Kane [**508**, Section 2.1, Theorem A], May and Ponto [**663**, Theorems 22.4.1 and 23.4.8], McCleary [**665**, Theorems 6.36 and 10.2], Mimura and Toda [**697**, Chapter VII, Theorem 1.3], Spanier [**841**, Section 5.8, Theorem 12], Whitehead [**909**, Theorem 8.11], Zabrodsky [**926**, Theorem 3.1.1]. Most of these are books on algebraic topology with the context for the theorem being the (co)homology of H-spaces. Warning: The term Hopf algebra is used with different meanings in these references. Also Borel–Hopf theorem is sometimes called Hopf–Leray theorem, see for instance [**688**, Remark 3.39].

The classical analogue of Theorem 13.59 is given by Nichols [**719**, Chapter III, Theorem 7, item (i)], [**720**, Theorem 8]. (Recall that this is an equivalent way of stating Borel–Hopf.) He uses the exponential map which corresponds to the uniform noncommutative zeta function of the braid arrangement, see Example 9.140. (The latter terminology was introduced in our monograph [**21**, Exercise 15.37].) The analogue of Theorem 13.57, in the language of algebraic groups, is given by Demazure and Gabriel [**242**, Section IV.2, Proposition 4.1] and Hochschild [**444**, Theorem 10.1], [**445**, Chapter VIII, Theorem 1.1]. The language translation involves the analogue of the first part of Exercise 13.62. A recent reference on the classical analogue of Theorem 13.59 is by Grinberg and Reiner [**377**, Section 1.7].

Borel–Hopf for cocommutative bialgebras. The classical analogue of the Borel–Hopf Theorem 13.34 says that in characteristic zero, a connected (unsigned) graded cocommutative Hopf algebra is cofree on its primitive part. The analogue of Theorem 13.53 gives the same result for a connected signed graded cocommutative Hopf algebra.

In the literature, this version of Borel–Hopf is usually stated for connected cocommutative Hopf algebras (without assuming graded). Also, it often appears implicitly inside the proof of CMM; references to CMM are given in the Notes to

Chapter 17. Early references for this version of Borel–Hopf are by Heyneman and Sweedler [**865**, Theorem 4 and second paragraph on page 516], [**867**, page 275], [**433**, Corollary 4.2.7]. It is developed further by Grünenfelder [**385**, Theorem on page 575]. In Montgomery's book [**703**, proof of Theorem 5.6.5], the proof is approached via Exercise 13.64; see also the dual result given in Exercise 13.44. A statement in the differential graded setting is given by Hain [**405**, Corollary 6.13]. The classical analogue of Theorem 13.38 is given by Nichols [**719**, Chapter III, Theorem 7, item (ii)], [**720**, Remark on page 71].

Borel–Hopf in the context of formal groups is given by Dieudonné [**250**, Section II.2.4, Proposition 1 and Section II.2.5].

Patras and Cartier [**739**, Lemma I,5,4 and Corollary I,5,5], [**741**, Proposition III,5], [**202**, proof of Theorem 3.8.1] prove Borel–Hopf using idempotents that arise from the diagonalization of the Adams elements. A similar treatment is given by Fresse [**324**, Theorem 7.2.16]; he works in a more general categorical setup. (Adams elements and their diagonalization are discussed in detail for instance in [**21**, Section 12.5.3]. The idempotents are denoted $\mathtt{E}_k$, while the Adams element of parameter n is denoted $\mathtt{Ads}_n$. Additional references are given in the notes to Chapter 12 of this book. See in particular [**19**, Theorem 106].) In the graded setting, the Garsia–Reutenauer idempotents indexed by partitions [**342**] yield finer information, see [**741**, Section V]. (More details on these idempotents can be found in [**21**, Section 16.11], where they are denoted $\mathtt{E}_\lambda$. These refine the idempotents $\mathtt{E}_k$ mentioned above, see formulas (16.78) and (16.79) in particular. Additional references are given in the notes to Chapter 16 of this book.) The proof of the Borel–Hopf Theorem 13.34 given in Section 13.3.5 employs eulerian idempotents of the Tits algebra in place of the classical Garsia–Reutenauer idempotents. See [**744**, Corollary 4.4] in this regard. We also mention that it is not clear how to generalize Adams elements from the braid arrangement to arbitrary arrangements.

The analogue of the decomposition in Lemma 13.47 for the case of the tensor algebra is given by Reutenauer [**777**, Theorem 3.7]. The classical analogues of Exercises 13.45 and 13.65 are given by Browder [**160**, Proposition 1.4].

Leray–Samelson for bicommutative bialgebras. The classical analogue of Proposition 13.9 originates in a conjecture of Hopf [**458**, Section 37] which was proved by Samelson [**799**, Chapter I, Section 3, Satz I and Satz I'] and later improved by Leray [**595**, Section 25, pages 132 to 135]. An algebraic treatment of this result was given by Milnor and Moore [**695**, Theorems 7.16 and 7.20 and Proposition 7.21], [**696**, Theorem 4.10 and Proposition 4.25], [**706**, Theorems 6 and 7 on pages 27 and 28], [**708**, Theorem 1]. See also the papers by Zisman [**932**, Theorem 5,1 and Remark 5,3], Wraith [**918**, Paragraphs 3.9 and 3.10], Steiner [**851**, Theorem 3.2], André [**30**, Theorems 3 and 4]. A historical treatment is given by Cartier [**202**, Section 2.5] and Dieudonné [**251**, Part 2, Section V1.2.B]. A discussion following Milnor and Moore is given by May and Ponto [**663**, Corollary 22.4.3, Theorem 22.4.4, Corollary 23.4.4]. Other book references are [**215**, Appendix A.2.11], [**453**, Corollary A3.84].

The papers by Leray and Samelson are among the first papers which initiated a study of the coproduct. See also the papers by Koszul [**542**, Theorem 10.2], Borel [**135**, Section 20, Remark (1)], [**136**, Section 2], [**137**, Sections 6 and 7], Halpern [**412**], [**413**], [**414**], [**415**], Browder [**160**, Corollary 2.3] in this regard along with the Notes to Chapter 5.

In the context of commutative algebraic groups: The analogue of Theorem 13.11 is given by Cartier [**196**, Section 6]. The connection between Borel–Hopf for commutative bialgebras and Leray–Samelson is mentioned by Demazure and Gabriel

[**242**, Section IV.2, last sentence of Proposition 4.1]. This can viewed as the classical analogue of Exercise 13.63. The analogues of Theorems 13.21 and 13.11 are items (a) and (b), respectively, of [**242**, Section IV.2, Proposition 4.2]. See also [**241**, last line on page 42].

In the context of commutative formal groups, the analogue of Theorem 13.21 is given by Fröhlich [**330**, Section II.2, Corollary 1 on page 55], Lazard [**583**, Paragraph II.3.2]. In the latter reference, the statements in Paragraphs II.3.4–II.3.6 can be seen as the analogue of Proposition 13.24. See also the paper by Honda [**455**, Theorem 1 and Proposition 1.6]. The one-dimensional case is present in the earlier papers [**582**, Proposition 4], [**454**, Proposition 2]. A recent reference is [**172**, Theorem E.1.1].

Loday–Ronco for 0*-bialgebras.* The classical analogue of Theorem 13.2 is due to Loday and Ronco [**610**, Theorem 2.6]. Their proof is by an explicit construction of the inverse as in Section 13.1.2; their map G corresponds to (13.4). The Loday–Ronco theorem is also formulated in [**18**, Theorem 2.13], [**602**, Theorem 4.2.1].

Classical generalizations. The classical Borel–Hopf theorem was extended to the setting of commutative braided bialgebras by Masuoka [**658**, Propositions 4.8 and 6.8]. He makes use of the exponential map following Nichols [**720**, Theorem 8] whose work was cited above. A generalization to the setting of operads was given by Fresse [**318**, Theorems 0.1, 0.2, 0.3], Oudom [**731**, Theorem 3.1], [**730**, Theorem 3.3], Patras [**742**, Theorem 2.4], with the commutative operad recovering the classical Borel–Hopf theorems. For earlier work related to the associative operad, see the paper by Berstein [**108**, Theorems 1.1 and 1.2].

The classical Loday–Ronco and Leray–Samelson theorems can be unified by working in the setting of generalized bialgebras. This was done by Loday [**608**, Theorem 2.3.7]. In his notation, the relevant triples of operads for the two theorems are (As, As, Vect) and (Com, Com, Vect), respectively. In this reference, no particular name is used for Loday–Ronco (Section 4.2.2), while the Leray–Samelson theorem is inappropriately called the Hopf–Borel theorem (Section 2.3 introduction and Section 4.1.8). Loday's result is also derived by Livernet, Mesablishvili, Wisbauer [**603**, Theorem 6.7].

Bimonoids in Joyal species. The Loday–Ronco Theorem 13.2 in the context of Joyal species first appeared in our monograph [**18**, Theorem 11.49], see also [**20**, Theorem 2.2]. The proof given there is by an induction on the coradical filtration as in Section 13.1.1. The Borel–Hopf Theorem 13.34 for Joyal species was proved by Stover [**854**, Proposition 12.2]. More details on his work are given in the Notes to Chapter 17.

Bimonoids for hyperplane arrangements. The rigidity theorems for arrangements presented in this chapter are new and appear here for the first time. Their connection with the Mesablishvili–Wisbauer rigidity theorem is explained in Section 3.10. Ideas closely related to the Borel–Hopf Theorem 13.34 in the special cases $\mathsf{h} = \Gamma$ and $\mathsf{h} = \Sigma$, namely, the bimonoids of chambers and faces, respectively, are given in [**21**, Section 13.5]. More precisely, the isomorphisms (13.30) and (13.31) are equivalent to [**21**, Decompositions (13.8) and (13.9)], respectively. We mention in passing that the rigidity theorem for q-bimonoids in Section 13.6 generalizes to v-bimonoids for any generic distance function v arising from weights on half-spaces (1.32). (Generic means that no nontrivial product of weights is 1.)

CHAPTER 14

Hoffman–Newman–Radford

We now discuss another class of results related to the universal constructions in Chapter 6 which we call the Hoffman–Newman–Radford (HNR) rigidity theorems. They come in different flavors which we explain one by one. In each case, the result provides explicit inverse isomorphisms between two universally constructed bimonoids. We call these the Hoffman–Newman–Radford (HNR) isomorphisms. A summary is provided in Table 14.1 below.

TABLE 14.1. Hoffman–Newman–Radford isomorphisms.

Starting data	HNR isomorphisms
cocommutative comonoid c	$\mathcal{T}(\mathsf{c}) \underset{\boldsymbol{\mu}}{\overset{\boldsymbol{\zeta}}{\rightleftarrows}} \mathcal{T}(\mathsf{c}_t)$
commutative monoid a	$\mathcal{T}^{\vee}(\mathsf{a}_t) \underset{\boldsymbol{\mu}}{\overset{\boldsymbol{\zeta}}{\rightleftarrows}} \mathcal{T}^{\vee}(\mathsf{a}_t)$
cocommutative comonoid c	$\mathcal{S}(\mathsf{c}) \underset{\mu}{\overset{\zeta}{\rightleftarrows}} \mathcal{S}(\mathsf{c}_t)$
commutative monoid a	$\mathcal{S}^{\vee}(\mathsf{a}_t) \underset{\mu}{\overset{\zeta}{\rightleftarrows}} \mathcal{S}^{\vee}(\mathsf{a}_t)$
comonoid c	$\mathcal{T}_q(\mathsf{c}) \underset{\boldsymbol{\mu}_q}{\overset{\boldsymbol{\zeta}_q}{\rightleftarrows}} \mathcal{T}_q(\mathsf{c}_t)$
monoid a	$\mathcal{T}_q^{\vee}(\mathsf{a}_t) \underset{\boldsymbol{\mu}_q}{\overset{\boldsymbol{\zeta}_q}{\rightleftarrows}} \mathcal{T}_q^{\vee}(\mathsf{a}_t)$

For a cocommutative comonoid c, the bimonoids $\mathcal{T}(\mathsf{c})$ and $\mathcal{T}(\mathsf{c}_t)$ are isomorphic, where c_t is the underlying species of c with the trivial coproduct. The product is concatenation in both, but the coproducts differ, it is dequasishuffle in the former and deshuffle in the latter (see Table 6.1). An explicit isomorphism can be constructed in either direction, one direction involves a noncommutative zeta function $\boldsymbol{\zeta}$, while the other direction involves a noncommutative Möbius function $\boldsymbol{\mu}$. These are the HNR isomorphisms. They are natural in c. Dually, for a commutative monoid a, the bimonoids $\mathcal{T}(\mathsf{a})$ and $\mathcal{T}(\mathsf{a}_t)$ are isomorphic. The coproduct is deconcatenation in both, but the products differ, it is quasishuffle in the former and shuffle in the latter. The HNR isomorphisms can be pictured as follows.

$$\text{dequasishuffle} \underset{\boldsymbol{\mu}}{\overset{\boldsymbol{\zeta}}{\rightleftarrows}} \text{deshuffle} \qquad\qquad \text{shuffle} \underset{\boldsymbol{\mu}}{\overset{\boldsymbol{\zeta}}{\rightleftarrows}} \text{quasishuffle}$$

Interestingly, these ideas can be used to prove that noncommutative zeta functions and noncommutative Möbius functions are inverse to each other in the lune-incidence algebra.

There is a commutative analogue of the above results which goes as follows. For a cocommutative comonoid c, the bicommutative bimonoids $\mathcal{S}(\mathsf{c})$ and $\mathcal{S}(\mathsf{c}_t)$ are isomorphic. Now the HNR isomorphisms are constructed using the zeta function and Möbius function of the poset of flats. Dually, for a commutative monoid a, the bicommutative bimonoids $\mathcal{S}^\vee(\mathsf{a})$ and $\mathcal{S}^\vee(\mathsf{a}_t)$ are isomorphic. As an application, we explain how the HNR isomorphisms can be used to diagonalize the mixed distributive law for bicommutative bimonoids.

There is also a q-analogue, for q not a root of unity, which says that for a comonoid c, the q-bimonoids $\mathcal{T}_q(\mathsf{c})$ and $\mathcal{T}_q(\mathsf{c}_t)$ are isomorphic, while for a monoid a, the q-bimonoids $\mathcal{T}_q^\vee(\mathsf{a})$ and $\mathcal{T}_q^\vee(\mathsf{a}_t)$ are isomorphic. The HNR isomorphisms involve the two-sided q-zeta and q-Möbius functions. If c is cocommutative or a is commutative, these can be replaced by the noncommutative q-zeta and q-Möbius functions. As an application, we explain how the HNR isomorphisms can be used to study the nondegeneracy of the mixed distributive law for q-bimonoids.

The case $q = 0$ is particularly nice to deal with and we make it the starting point of our discussion. Moreover, there is a striking parallel between this case and the commutative case. In fact, the two can be unified by working with 0-$\sim$-bicommutative bimonoids, where $\sim$ is a geometric partial-support relation on faces.

All results are independent of the characteristic of the base field.

14.1. Free 0-bimonoids on comonoids

Recall from Section 6.1.4 the free 0-bimonoid $\mathcal{T}_0(\mathsf{c})$ on a comonoid c. We show that it is naturally isomorphic to $\mathcal{T}_0(\mathsf{c}_t)$, where c_t is c as a species but with the trivial coproduct. In particular, we will see that the primitive part of $\mathcal{T}_0(\mathsf{c})$ is isomorphic to the underlying species of c. There is a dual result for the cofree 0-bimonoid $\mathcal{T}_0^\vee(\mathsf{a})$ on a monoid a.

Recall the Loday–Ronco Theorem 13.2 which says that every 0-bimonoid h is of the form $\mathcal{T}_0(\mathcal{P}(\mathsf{h}))$. The above result on $\mathcal{T}_0(\mathsf{c})$ can be viewed as an illustration of Loday–Ronco for $\mathsf{h} = \mathcal{T}_0(\mathsf{c})$. Similarly, the dual result on $\mathcal{T}_0^\vee(\mathsf{a})$ can be viewed as an illustration of the dual Loday–Ronco Theorem 13.8 for $\mathsf{h} = \mathcal{T}_0^\vee(\mathsf{a})$.

14.1.1. Free 0-bimonoid on a comonoid. For a comonoid c, let c_t denote the underlying species of c with the trivial coproduct. The product and coproduct of $\mathcal{T}_0(\mathsf{c})$ are given by (6.9) and (6.10), while that of $\mathcal{T}_0(\mathsf{c}_t)$ are given by (6.43) and (6.44) for $\mathsf{p} := \mathsf{c}_t$.

Lemma 14.1. *For a comonoid c, the map of species $\mathsf{c} \to \mathcal{T}_0(\mathsf{c}_t)$ given on the A-component by*

$$\mathsf{c}[A] \to \bigoplus_{H:\, H \geq A} \mathsf{c}[H], \qquad x \mapsto \sum_{H:\, H \geq A} \Delta_A^H(x) \tag{14.1}$$

is a morphism of comonoids.

First proof. Let f denote the above map. We need to check that the diagram

$$\begin{tikzcd} \mathsf{c}[A] \arrow[r, "\Delta_A^G"] \arrow[d, "f_A"'] \arrow[dr, dashed] & \mathsf{c}[G] \arrow[d, "f_G"] \\ \mathcal{T}_0(\mathsf{c}_t)[A] \arrow[r, "\Delta_A^G"'] & \mathcal{T}_0(\mathsf{c}_t)[G] \end{tikzcd}$$

commutes. This holds since the dotted arrow obtained by following both directions is given by

$$x \mapsto \sum_{H\colon H \geq G} \Delta_A^H(x).$$

Note that H only runs over faces greater than G. $\square$

Second proof. Observe that the map (14.1) is the 0-exponential (9.38a) of the canonical inclusion $\mathsf{c} \hookrightarrow \mathcal{T}_0(\mathsf{c}_t)$. Further, by (13.1), this inclusion maps into the primitive part of $\mathcal{T}_0(\mathsf{c}_t)$, so it is a coderivation. Hence, by Theorem 9.103, we deduce that (14.1) is a morphism of comonoids. $\square$

Proposition 14.2. *For a comonoid* c*, the map of species* $\mathcal{T}_0(\mathsf{c}) \to \mathcal{T}_0(\mathsf{c}_t)$ *given on the* A*-component by*

$$\bigoplus_{F\colon F \geq A} \mathsf{c}[F] \to \bigoplus_{H\colon H \geq A} \mathsf{c}[H], \qquad x \mapsto \sum_{H\colon H \geq F} \Delta_F^H(x) \tag{14.2}$$

for $x \in \mathsf{c}[F]$*, is an isomorphism of* 0*-bimonoids.*

Proof. By the universal property of $\mathcal{T}_0$ (Theorem 6.6 for $q = 0$), the morphism of comonoids $\mathsf{c} \to \mathcal{T}_0(\mathsf{c}_t)$ in Lemma 14.1 extends to a morphism of 0-bimonoids $\mathcal{T}_0(\mathsf{c}) \to \mathcal{T}_0(\mathsf{c}_t)$. By employing formula (6.4), we see that this indeed coincides with (14.2). Moreover, since the matrix form of (14.2) is unitriangular, it is an isomorphism. $\square$

Lemma 14.3. *For a comonoid* c*, the map of species* $\mathsf{c} \to \mathcal{T}_0(\mathsf{c})$ *given on the* A*-component by*

$$\mathsf{c}[A] \to \bigoplus_{H\colon H \geq A} \mathsf{c}[H], \qquad x \mapsto \sum_{H\colon H \geq A} (-1)^{\mathrm{rk}(H/A)} \Delta_A^H(x) \tag{14.3}$$

is a coderivation, that is, it maps into $\mathcal{P}(\mathcal{T}_0(\mathsf{c}))$.

First proof. The required calculation is done below. For $G > A$,

$$\begin{aligned} \Delta_A^G\Big(\sum_{H\colon H \geq A} (-1)^{\mathrm{rk}(H/A)} \Delta_A^H(x)\Big) &= \sum_{K} \sum_{\substack{H\colon H \geq A \\ HG = GH = K}} (-1)^{\mathrm{rk}(H/A)} \Delta_A^K(x) \\ &= \sum_{K\colon K \geq G} \Big(\sum_{\substack{H\colon H \geq A \\ HG = K}} (-1)^{\mathrm{rk}(H/A)} \Big) \Delta_A^K(x) \\ &= 0. \end{aligned}$$

In the first equality, we used coproduct formula (6.10) and coassociativity of c. For the second equality, we note that $GH = K$ implies $K \geq G$, and conversely, if $K \geq G$ and $HG = K$, then $GH = GHG = GK = K$. For the third equality, we used the descent identity (1.76a): since $A < G \leq K$, we cannot have $A\overline{G} \leq K$. $\square$

Second proof. Observe that the map (14.3) is the 0-logarithm (9.38b) of the canonical inclusion $\mathsf{c} \hookrightarrow \mathcal{T}_0(\mathsf{c})$. The latter is a morphism of comonoids by Exercise 6.7 for $q = 0$. Hence, by Theorem 9.103, we deduce that (14.3) is a coderivation. $\square$

Proposition 14.4. *For a comonoid* c*, the map of species* $\mathcal{T}_0(\mathsf{c}_t) \to \mathcal{T}_0(\mathsf{c})$ *given on the* A*-component by*

$$\bigoplus_{F:\, F \geq A} \mathsf{c}[F] \to \bigoplus_{H:\, H \geq A} \mathsf{c}[H], \qquad x \mapsto \sum_{H:\, H \geq F} (-1)^{\mathrm{rk}(H/F)} \Delta_F^H(x) \tag{14.4}$$

for $x \in \mathsf{c}[F]$*, is an isomorphism of* 0*-bimonoids.*

Proof. By the universal property of $\mathcal{T}_0$ (Theorem 6.31 for $q = 0$), the map $\mathsf{c} \to \mathcal{P}(\mathcal{T}_0(\mathsf{c}))$ in Lemma 14.3 extends to a morphism of 0-bimonoids $\mathcal{T}_0(\mathsf{c}_t) \to \mathcal{T}_0(\mathsf{c})$. By employing formula (6.41), we see that this indeed coincides with (14.4). Moreover, since the matrix form of (14.4) is unitriangular, it is an isomorphism. $\square$

Applying the primitive part functor to the map (14.4) and using (13.1) for $\mathsf{p} = \mathsf{c}_t$, we obtain:

Proposition 14.5. *For a comonoid* c*, the map* $\mathsf{c} \to \mathcal{P}(\mathcal{T}_0(\mathsf{c}))$ *in Lemma* 14.3 *is a natural isomorphism of species.*

Remark 14.6. The maps (14.2) and (14.4) are inverses of each other. This can be deduced from the fact that the poset of faces is eulerian (1.73). For the relevant formal setup, see Example 14.55. Hence, Propositions 14.2 and 14.4 imply each other. We did prove both of them separately though.

Exercise 14.7. For a comonoid c, consider the following biderivations f and g.

$$\begin{tikzcd} \mathcal{T}_0(\mathsf{c}) \arrow[r, "f"] \arrow[d, two heads] & \mathcal{T}_0(\mathsf{c}_t) \\ \mathsf{c} \arrow[r, "\cong", "\mathrm{id}"'] & \mathsf{c} \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}_0(\mathsf{c}_t) \arrow[r, "g"] \arrow[d, two heads] & \mathcal{T}_0(\mathsf{c}) \\ \mathsf{c} \arrow[r, "\cong"] & \mathcal{P}(\mathcal{T}_0(\mathsf{c})) \arrow[u, hook] \end{tikzcd}$$

The bottom horizontal map in the second diagram is as in (14.3). Check that: The maps (14.2) and (14.4) are the 0-exponentials (9.38a) of f and g, respectively. Now use Proposition 9.111 to reprove Propositions 14.2 and 14.4.

14.1.2. Cofree 0-bimonoid on a monoid. The dual results for the cofree 0-bimonoid on a monoid are stated below. Since the details are similar, we omit them. For a monoid a, let a_t denote the underlying species of a with the trivial product. The product and coproduct of $\mathcal{T}_0^\vee(\mathsf{a})$ are given by (6.15) and (6.16), while that of $\mathcal{T}_0^\vee(\mathsf{a}_t)$ are given by (6.43) and (6.44) for $\mathsf{p} := \mathsf{a}_t$.

Proposition 14.8. *For a monoid* a*, the map of species* $\mathcal{T}_0^\vee(\mathsf{a}_t) \to \mathcal{T}_0^\vee(\mathsf{a})$ *given on the* A*-component by*

$$\bigoplus_{H:\, H \geq A} \mathsf{a}[H] \to \bigoplus_{F:\, F \geq A} \mathsf{a}[F], \qquad x \mapsto \sum_{F:\, A \leq F \leq H} \mu_F^H(x) \tag{14.5}$$

for $x \in \mathsf{a}[H]$*, is an isomorphism of* 0*-bimonoids.*

Proposition 14.9. *For a monoid* a*, the map of species* $\mathcal{T}_0^\vee(\mathsf{a}) \to \mathcal{T}_0^\vee(\mathsf{a}_t)$ *given on the* A*-component by*

$$\bigoplus_{H:\, H \geq A} \mathsf{a}[H] \to \bigoplus_{F:\, F \geq A} \mathsf{a}[F], \qquad x \mapsto \sum_{F:\, A \leq F \leq H} (-1)^{\mathrm{rk}(H/F)} \mu_F^H(x) \tag{14.6}$$

for $x \in \mathsf{a}[H]$*, is an isomorphism of* 0*-bimonoids.*

Proposition 14.10. *For a monoid* a*, the map of species* $\mathcal{T}_0^\vee(\mathsf{a}) \to \mathsf{a}$ *given on the* A*-component by*

$$\bigoplus_{H:\, H \geq A} \mathsf{a}[H] \to \mathsf{a}[A], \qquad x \mapsto (-1)^{\mathrm{rk}(H/A)} \mu_A^H(x) \tag{14.7}$$

for $x \in \mathsf{a}[H]$*, induces a natural isomorphism* $\mathcal{Q}(\mathcal{T}_0^\vee(\mathsf{a})) \to \mathsf{a}$ *of species.*

Note that Propositions 14.8 and 14.9 imply each other, with (14.5) and (14.6) being inverses of each other. Also see Example 14.56.

Exercise 14.11. For a monoid a, consider the following biderivations f and g.

$$\begin{tikzcd} \mathcal{T}_0^\vee(\mathsf{a}_t) \arrow[r, "f"] \arrow[d, two heads] & \mathcal{T}_0^\vee(\mathsf{a}) \\ \mathsf{a} \arrow[r, "\cong", "\mathrm{id}"'] & \mathsf{a} \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}_0^\vee(\mathsf{a}) \arrow[r, "g"] \arrow[d, two heads] & \mathcal{T}_0^\vee(\mathsf{a}_t) \\ \mathcal{Q}(\mathcal{T}_0^\vee(\mathsf{a})) \arrow[r, "\cong"] & \mathsf{a} \arrow[u, hook] \end{tikzcd}$$

The bottom horizontal map in the second diagram is as in Proposition 14.10. Check that: The maps (14.5) and (14.6) are the 0-exponentials (9.38a) of f and g, respectively. Now use Proposition 9.111 to reprove Propositions 14.8 and 14.9.

14.1.3. HNR isomorphisms for 0-bimonoids. We collectively refer to (14.2), (14.4), and their duals (14.5), (14.6) as the *Hoffman–Newman–Radford isomorphisms* for 0-bimonoids. They are natural in c and a, that is, for any morphism of comonoids $f : \mathsf{c} \to \mathsf{c}'$ and morphism of monoids $g : \mathsf{a}' \to \mathsf{a}$, the following diagrams commute.

$$\begin{tikzcd} \mathcal{T}_0(\mathsf{c}) \arrow[r, shift left] \arrow[d, "\mathcal{T}_0(f)"'] & \mathcal{T}_0(\mathsf{c}_t) \arrow[l, shift left] \arrow[d, "\mathcal{T}_0(f)"] \\ \mathcal{T}_0(\mathsf{c}') \arrow[r, shift left] & \mathcal{T}_0(\mathsf{c}'_t) \arrow[l, shift left] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}_0^\vee(\mathsf{a}_t) \arrow[r, shift left] & \mathcal{T}_0^\vee(\mathsf{a}) \arrow[l, shift left] \\ \mathcal{T}_0^\vee(\mathsf{a}'_t) \arrow[r, shift left] \arrow[u, "\mathcal{T}_0^\vee(g)"] & \mathcal{T}_0^\vee(\mathsf{a}') \arrow[l, shift left] \arrow[u, "\mathcal{T}_0^\vee(g)"'] \end{tikzcd} \tag{14.8}$$

The horizontal maps are the HNR isomorphisms.

14.2. Free bicomm. bimonoids on cocomm. comonoids

Recall from Section 6.3 the free bicommutative bimonoid $\mathcal{S}(\mathsf{c})$ on a cocommutative comonoid c. We show that it is naturally isomorphic to $\mathcal{S}(\mathsf{c}_t)$, where c_t is c as a species but with the trivial coproduct. In particular, we will see that the primitive part of $\mathcal{S}(\mathsf{c})$ is isomorphic to the underlying species of c. There is a dual result for the cofree bicommutative bimonoid $\mathcal{S}^\vee(\mathsf{a})$ on a commutative monoid a. We also briefly deal with the signed analogues. We also explain how these ideas can be used to diagonalize the mixed distributive law for bicommutative bimonoids.

Recall the Leray–Samelson Theorem 13.11 which says that every bicommutative bimonoid h is of the form $\mathcal{S}(\mathcal{P}(\mathsf{h}))$. The above result on $\mathcal{S}(\mathsf{c})$ can be viewed as an illustration of Leray–Samelson for $\mathsf{h} = \mathcal{S}(\mathsf{c})$. Similarly, the dual result on $\mathcal{S}^\vee(\mathsf{a})$ can be viewed as an illustration of the dual Leray–Samelson Theorem 13.21 for $\mathsf{h} = \mathcal{S}^\vee(\mathsf{a})$.

The discussion parallels the one in Section 14.1, with flats replacing faces.

14.2.1. Free bicommutative bimonoid on a cocommutative comonoid. For a comonoid c, let c_t denote the underlying species of c with the trivial coproduct. The product and coproduct of $\mathcal{S}(\mathsf{c})$ are given by (6.20) and (6.22), while that of $\mathcal{S}(\mathsf{c}_t)$ are given by (6.51) for $\mathsf{p} := \mathsf{c}_t$.

Lemma 14.12. *For a cocommutative comonoid* c*, the map of species* $\mathsf{c} \to \mathcal{S}(\mathsf{c}_t)$ *given on the* Z*-component by*

$$\mathsf{c}[\mathrm{Z}] \to \bigoplus_{\mathrm{W}:\, \mathrm{W} \geq \mathrm{Z}} \mathsf{c}[\mathrm{W}], \qquad x \mapsto \sum_{\mathrm{W}:\, \mathrm{W} \geq \mathrm{Z}} \Delta^{\mathrm{W}}_{\mathrm{Z}}(x) \tag{14.9}$$

is a morphism of comonoids.

First proof. Let f denote the above map. We need to show that the diagram

$$\begin{tikzcd} \mathsf{c}[\mathrm{Z}] \arrow[r, "\Delta^{\mathrm{Y}}_{\mathrm{Z}}"] \arrow[d, "f_{\mathrm{Z}}"'] \arrow[dr, dotted] & \mathsf{c}[\mathrm{Y}] \arrow[d, "f_{\mathrm{Y}}"] \\ \mathcal{S}(\mathsf{c}_t)[\mathrm{Z}] \arrow[r, "\Delta^{\mathrm{Y}}_{\mathrm{Z}}"'] & \mathcal{S}(\mathsf{c}_t)[\mathrm{Y}] \end{tikzcd}$$

commutes. This holds since the dotted arrow obtained by following both directions is given by

$$x \mapsto \sum_{\mathrm{W}:\, \mathrm{W} \geq \mathrm{Y}} \Delta^{\mathrm{W}}_{\mathrm{Z}}(x).$$

Note that W only runs over flats greater than Y. $\square$

Second proof. Observe that f is the exponential (9.15a) of the canonical inclusion $\mathsf{c} \to \mathcal{S}(\mathsf{c}_t)$. Further, this inclusion maps into the primitive part of $\mathcal{S}(\mathsf{c}_t)$, so it is a coderivation. Hence, by Theorem 9.40, we deduce that f is a morphism of comonoids. $\square$

Proposition 14.13. *For a cocommutative comonoid* c*, the map of species* $\mathcal{S}(\mathsf{c}) \to \mathcal{S}(\mathsf{c}_t)$ *given on the* Z*-component by*

$$\bigoplus_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Z}} \mathsf{c}[\mathrm{X}] \to \bigoplus_{\mathrm{W}\colon \mathrm{W}\geq \mathrm{Z}} \mathsf{c}[\mathrm{W}], \qquad x \mapsto \sum_{\mathrm{W}\colon \mathrm{W}\geq \mathrm{X}} \Delta_{\mathrm{X}}^{\mathrm{W}}(x) \tag{14.10}$$

for $x \in \mathsf{c}[\mathrm{X}]$*, is an isomorphism of bimonoids.*

Proof. By the universal property of $\mathcal{S}$ given in Theorem 6.21, the morphism of comonoids $\mathsf{c} \to \mathcal{S}(\mathsf{c}_t)$ in Lemma 14.12 extends to a morphism of bimonoids $\mathcal{S}(\mathsf{c}) \to \mathcal{S}(\mathsf{c}_t)$. By employing formula (6.21a), we see that this indeed coincides with (14.10). Moreover, since the matrix form of (14.10) is unitriangular, it is an isomorphism. ☐

Let $\mu(\mathrm{X}, \mathrm{Y})$ denote the Möbius function of the poset of flats.

Lemma 14.14. *For a cocommutative comonoid* c*, the map of species* $\mathsf{c} \to \mathcal{S}(\mathsf{c})$ *given on the* Z*-component by*

$$\mathsf{c}[\mathrm{Z}] \to \bigoplus_{\mathrm{W}\colon \mathrm{W}\geq \mathrm{Z}} \mathsf{c}[\mathrm{W}], \qquad x \mapsto \sum_{\mathrm{W}\colon \mathrm{W}\geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{W})\Delta_{\mathrm{Z}}^{\mathrm{W}}(x) \tag{14.11}$$

is a coderivation, that is, it maps into $\mathcal{P}(\mathcal{S}(\mathsf{c}))$*.*

First proof. The required calculation is done below. For $\mathrm{X} > \mathrm{Z}$,

$$\Delta_{\mathrm{Z}}^{\mathrm{X}}\Big(\sum_{\mathrm{Y}\colon \mathrm{Y}\geq \mathrm{Z}} \mu(\mathrm{Z}, \mathrm{Y})\Delta_{\mathrm{Z}}^{\mathrm{Y}}(x)\Big) = \sum_{\mathrm{W}} \sum_{\substack{\mathrm{Y}\colon \mathrm{Y}\geq \mathrm{Z} \\ \mathrm{X}\vee\mathrm{Y}=\mathrm{W}}} \mu(\mathrm{Z}, \mathrm{Y})\Delta_{\mathrm{Z}}^{\mathrm{W}}(x) = 0.$$

The first equality used coproduct formula (6.22) and coassociativity of c, while the second used the Weisner formula (1.38). ☐

Second proof. Observe that the map (14.11) is the logarithm (9.15b) of the canonical inclusion $\mathsf{c} \to \mathcal{S}(\mathsf{c})$. The latter is a morphism of comonoids. Hence, by Theorem 9.40, we deduce that (14.11) is a coderivation. ☐

Proposition 14.15. *For a cocommutative comonoid* c*, the map of species* $\mathcal{S}(\mathsf{c}_t) \to \mathcal{S}(\mathsf{c})$ *given on the* Z*-component by*

$$\bigoplus_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Z}} \mathsf{c}[\mathrm{X}] \to \bigoplus_{\mathrm{W}\colon \mathrm{W}\geq \mathrm{Z}} \mathsf{c}[\mathrm{W}], \qquad x \mapsto \sum_{\mathrm{W}\colon \mathrm{W}\geq \mathrm{X}} \mu(\mathrm{X}, \mathrm{W})\Delta_{\mathrm{X}}^{\mathrm{W}}(x) \tag{14.12}$$

for $x \in \mathsf{c}[\mathrm{X}]$*, is an isomorphism of bimonoids.*

Proof. By the universal property of $\mathcal{S}$ given in Theorem 6.44, the map $\mathsf{c} \to \mathcal{P}(\mathcal{S}(\mathsf{c}))$ in Lemma 14.14 extends to a morphism of bimonoids $\mathcal{S}(\mathsf{c}_t) \to \mathcal{S}(\mathsf{c})$. By employing formula (6.52), we see that this indeed coincides with (14.12). Moreover, since the matrix form of (14.12) is unitriangular, it is an isomorphism. ☐

By applying the primitive part functor to the map (14.12), and using (13.8) for $\mathsf{p} = \mathsf{c}_t$, we obtain:

Proposition 14.16. *For a cocommutative comonoid* c*, the map* $\mathsf{c} \to \mathcal{P}(\mathcal{S}(\mathsf{c}))$ *in Lemma 14.14 is a natural isomorphism of species.*

Exercise 14.17. For a cocommutative comonoid c, consider the following biderivations f and g.

$$\begin{tikzcd} \mathcal{S}(\mathsf{c}) \arrow[r, "f"] \arrow[d, two heads] & \mathcal{S}(\mathsf{c}_t) \\ \mathsf{c} \arrow[r, "\cong", "\mathrm{id}"'] & \mathsf{c} \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S}(\mathsf{c}_t) \arrow[r, "g"] \arrow[d, two heads] & \mathcal{S}(\mathsf{c}) \\ \mathsf{c} \arrow[r, "\cong"] & \mathcal{P}(\mathcal{S}(\mathsf{c})) \arrow[u, hook] \end{tikzcd}$$

The bottom horizontal map in the second diagram is as in (14.11). Check that: The maps (14.10) and (14.12) are the exponentials (9.15a) of f and g, respectively. Now use Proposition 9.55 to reprove Propositions 14.13 and 14.15.

14.2.2. Cofree bicommutative bimonoid on a commutative monoid. The dual results for the cofree bicommutative bimonoid on a commutative monoid are given below. Since the details are similar, we only sketch the proofs. For a monoid a, let a_t denote the underlying species of a with the trivial product. The product and coproduct of $\mathcal{S}^\vee(\mathsf{a})$ are given by (6.26) and (6.28), while that of $\mathcal{S}^\vee(\mathsf{a}_t)$ are given by (6.51) for $\mathsf{p} := \mathsf{a}_t$.

Lemma 14.18. *For a commutative monoid* a*, the map of species* $\mathcal{S}^\vee(\mathsf{a}_t) \to \mathsf{a}$ *given on the* Z*-component by*

$$\bigoplus_{\mathrm{W}\colon \mathrm{W}\geq \mathrm{Z}} \mathsf{a}[\mathrm{W}] \to \mathsf{a}[\mathrm{Z}], \qquad x \mapsto \mu_{\mathrm{Z}}^{\mathrm{W}}(x) \tag{14.13}$$

for $x \in \mathsf{a}[\mathrm{W}]$*, is a morphism of monoids.*

Proof. This is easy to see directly. Alternatively, note that the above map is the exponential (9.15a) of the canonical projection $\mathcal{S}^\vee(\mathsf{a}_t) \to \mathsf{a}$ (which is a derivation) and then apply Theorem 9.41. □

Proposition 14.19. *For a commutative monoid* a*, the map* $\mathcal{S}^\vee(\mathsf{a}_t) \to \mathcal{S}^\vee(\mathsf{a})$ *of species given on the* Z*-component by*

$$\bigoplus_{\mathrm{W}\colon \mathrm{W}\geq \mathrm{Z}} \mathsf{a}[\mathrm{W}] \to \bigoplus_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Z}} \mathsf{a}[\mathrm{X}], \qquad x \mapsto \sum_{\mathrm{X}\colon \mathrm{Z}\leq \mathrm{X}\leq \mathrm{W}} \mu_{\mathrm{X}}^{\mathrm{W}}(x) \tag{14.14}$$

for $x \in \mathsf{a}[\mathrm{W}]$*, is an isomorphism of bimonoids.*

Proof. Apply the universal property of $\mathcal{S}^\vee$ given in Theorem 6.25 to the morphism of monoids in Lemma 14.18. □

Lemma 14.20. *For a commutative monoid* a*, the map of species* $\mathcal{S}^\vee(\mathsf{a}) \to \mathsf{a}$ *given on the* Z*-component by*

$$\bigoplus_{\mathrm{W}\colon \mathrm{W}\geq \mathrm{Z}} \mathsf{a}[\mathrm{W}] \to \mathsf{a}[\mathrm{Z}], \qquad x \mapsto \mu(\mathrm{Z}, \mathrm{W})\, \mu_{\mathrm{Z}}^{\mathrm{W}}(x) \tag{14.15}$$

for $x \in \mathsf{a}[\mathrm{W}]$*, is a derivation, that is, it factors through* $\mathcal{Q}(\mathcal{S}^\vee(\mathsf{a}))$*.*

Proof. One way is to proceed directly. Alternatively, note that the above map is the logarithm (9.15b) of the canonical projection $\mathcal{S}^\vee(\mathsf{a}) \twoheadrightarrow \mathsf{a}$ (which is a morphism of monoids) and then apply Theorem 9.41. □

Proposition 14.21. *For a commutative monoid* a*, the map* $\mathcal{S}^\vee(\mathsf{a}) \to \mathcal{S}^\vee(\mathsf{a}_t)$ *of species given on the* Z*-component by*

$$\bigoplus_{\mathrm{W}\colon \mathrm{W}\geq \mathrm{Z}} \mathsf{a}[\mathrm{W}] \to \bigoplus_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Z}} \mathsf{a}[\mathrm{X}], \qquad x \mapsto \sum_{\mathrm{X}\colon \mathrm{Z}\leq \mathrm{X}\leq \mathrm{W}} \mu(\mathrm{X},\mathrm{W})\, \mu_{\mathrm{X}}^{\mathrm{W}}(x) \tag{14.16}$$

for $x \in \mathsf{a}[\mathrm{W}]$*, is an isomorphism of bimonoids.*

Proof. Apply the universal property of $\mathcal{S}^\vee$ given in Theorem 6.45 to the map in Lemma 14.20. □

Proposition 14.22. *For a commutative monoid* a*, the map* $\mathcal{Q}(\mathcal{S}^\vee(\mathsf{a})) \to \mathsf{a}$ *in Lemma 14.20 is a natural isomorphism of species.*

Exercise 14.23. For a commutative monoid a, consider the following biderivations f and g.

$$\begin{tikzcd} \mathcal{S}^\vee(\mathsf{a}_t) \arrow[r, "f"] \arrow[d, two heads] & \mathcal{S}^\vee(\mathsf{a}) \\ \mathsf{a} \arrow[r, "\cong", "\mathrm{id}"'] & \mathsf{a} \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S}^\vee(\mathsf{a}) \arrow[r, "g"] \arrow[d, two heads] & \mathcal{S}^\vee(\mathsf{a}_t) \\ \mathcal{Q}(\mathcal{S}^\vee(\mathsf{a})) \arrow[r, "\cong"] & \mathsf{a} \arrow[u, hook] \end{tikzcd}$$

The bottom horizontal map in the second diagram is as in Lemma 14.20. Check that: The maps (14.14) and (14.16) are the exponentials (9.15a) of f and g, respectively. Now use Proposition 9.55 to reprove Propositions 14.19 and 14.21.

14.2.3. HNR isomorphisms for bicommutative bimonoids. We collectively refer to the maps (14.10), (14.12), and their duals (14.14), (14.16) as the *Hoffman–Newman–Radford isomorphisms* for bicommutative bimonoids. They are natural in c and a, that is, diagrams (14.8) commute with $\mathcal{S}$ in place of $\mathcal{T}_0$.

14.2.4. Signed analogue. The above discussion carries over to the signed setting. Recall from Section 6.3.5 the free signed bicommutative signed bimonoid $\mathcal{E}(\mathsf{c})$ on a signed cocommutative comonoid c.

Proposition 14.24. *For a signed cocommutative comonoid* c*, the map of species* $\mathcal{E}(\mathsf{c}) \to \mathcal{E}(\mathsf{c}_t)$ *given on the* Z*-component by*

$$\bigoplus_{\mathrm{X}\colon \mathrm{X}\geq \mathrm{Z}} \mathbf{E}^-[\mathrm{Z},\mathrm{X}] \otimes \mathsf{c}[\mathrm{X}] \to \bigoplus_{\mathrm{W}\colon \mathrm{W}\geq \mathrm{Z}} \mathbf{E}^-[\mathrm{Z},\mathrm{W}] \otimes \mathsf{c}[\mathrm{W}],$$
$$y \otimes x \mapsto \sum_{\mathrm{W}\colon \mathrm{W}\geq \mathrm{X}} ((-)\otimes \mathrm{id})(y \otimes \Delta_{\mathrm{X}}^{\mathrm{W}}(x))$$

for $x \in \mathsf{c}[\mathrm{X}]$*,* $y \in \mathbf{E}^-[\mathrm{Z},\mathrm{X}]$*, is an isomorphism of signed bimonoids. (The unnamed map* $(-)$ *is as in* (1.162)*.)*

Proposition 14.25. *For a signed cocommutative comonoid* c*, the map of species* $\mathcal{E}(\mathsf{c}_t) \to \mathcal{E}(\mathsf{c})$ *given on the* Z*-component by*

$$\bigoplus_{\mathrm{X}\colon \mathrm{X} \geq \mathrm{Z}} \mathbf{E}^-[\mathrm{Z},\mathrm{X}] \otimes \mathsf{c}[\mathrm{X}] \to \bigoplus_{\mathrm{W}\colon \mathrm{W} \geq \mathrm{Z}} \mathbf{E}^-[\mathrm{Z},\mathrm{W}] \otimes \mathsf{c}[\mathrm{W}],$$
$$y \otimes x \mapsto \sum_{\mathrm{W}\colon \mathrm{W} \geq \mathrm{X}} \mu(\mathrm{X},\mathrm{W})((-) \otimes \mathrm{id})(y \otimes \Delta^{\mathrm{W}}_{\mathrm{X}}(x))$$

for $x \in \mathsf{c}[\mathrm{X}]$*,* $y \in \mathbf{E}^-[\mathrm{Z},\mathrm{X}]$*, is an isomorphism of signed bimonoids. (The unnamed map* $(-)$ *is as in* (1.162)*.)*

Proposition 14.26. *For a signed cocommutative comonoid* c*, the map of species* $\mathsf{c} \to \mathcal{P}(\mathcal{E}(\mathsf{c}))$ *given on the* Z*-component by*

$$\mathsf{c}[\mathrm{Z}] \to \bigoplus_{\mathrm{Y}\colon \mathrm{Y} \geq \mathrm{Z}} \mathbf{E}^-[\mathrm{Z},\mathrm{Y}] \otimes \mathsf{c}[\mathrm{Y}], \qquad x \mapsto \sum_{\mathrm{Y}\colon \mathrm{Y} \geq \mathrm{Z}} \mu(\mathrm{Z},\mathrm{Y})\Delta^{\mathrm{Y}}_{\mathrm{Z}}(x) \tag{14.17}$$

is a natural isomorphism.

Exercise 14.27. Check directly that the image of (14.17) belongs to the primitive part. (Use Exercise 1.75.)

Dual results hold for $\mathcal{E}^\vee(\mathsf{a})$ which we omit. This is the cofree signed bicommutative signed bimonoid on a signed commutative monoid a.

14.2.5. Examples.

Example 14.28. Recall the bimonoid of flats Π from Section 7.4. View the exponential species E as a comonoid. Let E_t denote E with the trivial coproduct. The following is a commutative diagram of bimonoids.

$$\begin{tikzcd} \mathcal{S}(\mathsf{E}) \arrow[rr, "\cong"] & & \mathcal{S}(\mathsf{E}_t) \\ & \Pi \arrow[ul, "\cong"] \arrow[ur, "\cong"'] & \end{tikzcd}$$

The horizontal map is the HNR isomorphism (14.10) specialized to $\mathsf{c} := \mathsf{E}$. The map going up to the left is the first map in (7.47), while the one going up to the right is the first map in (7.48). They involve the $\mathtt{H}$-basis and $\mathtt{Q}$-basis, respectively, of Π, see (7.38).

Similarly, one can formulate the dual statement for Π^* by specializing (14.14) to $\mathsf{a} := \mathsf{E}$.

14.2.6. Conjugation by HNR isomorphisms. For any cocommutative comonoid c and commutative monoid a, suppose we are given a morphism of bimonoids from $\mathcal{S}(\mathsf{c})$ to $\mathcal{S}^\vee(\mathsf{a})$. It can be conjugated by the HNR isomorphisms to yield a morphism from $\mathcal{S}(\mathsf{c}_t)$ to $\mathcal{S}^\vee(\mathsf{a}_t)$ which are simpler objects. This can be viewed as a change of basis that simplifies the description of the given morphism. More precisely, we have bijections

$$^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{S}(\mathsf{c}),\mathcal{S}^\vee(\mathsf{a})) \rightleftarrows {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{S}(\mathsf{c}_t),\mathcal{S}^\vee(\mathsf{a}_t)) \tag{14.18}$$

given by

$$\begin{aligned} (\mathcal{S}(\mathsf{c}) \xrightarrow{g} \mathcal{S}^\vee(\mathsf{a})) &\longmapsto (\mathcal{S}(\mathsf{c}_t) \xrightarrow{\cong} \mathcal{S}(\mathsf{c}) \xrightarrow{g} \mathcal{S}^\vee(\mathsf{a}) \xrightarrow{\cong} \mathcal{S}^\vee(\mathsf{a}_t)) \\ (\mathcal{S}(\mathsf{c}_t) \xrightarrow{h} \mathcal{S}^\vee(\mathsf{a}_t)) &\longmapsto (\mathcal{S}(\mathsf{c}) \xrightarrow{\cong} \mathcal{S}(\mathsf{c}_t) \xrightarrow{h} \mathcal{S}^\vee(\mathsf{a}_t) \xrightarrow{\cong} \mathcal{S}^\vee(\mathsf{a})), \end{aligned}$$

with the isomorphisms being the HNR isomorphisms.

Exercise 14.29. Check that: For a cocommutative comonoid c and commutative monoid a, the following diagram of bijections commutes.

$$\begin{tikzcd} {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{S}(\mathsf{c}), \mathcal{S}^\vee(\mathsf{a})) \arrow[rr, shift left] \arrow[dr, Rightarrow, shift left, "\log"] & & {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{S}(\mathsf{c}_t), \mathcal{S}^\vee(\mathsf{a}_t)) \arrow[ll, shift left] \arrow[dl, Rightarrow, shift right, "\log"'] \\ & \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a}) \arrow[ul, Rightarrow, shift left, "\exp"] \arrow[ur, Rightarrow, shift right, "\exp"'] & \end{tikzcd}$$

The horizontal bijections are as in (14.18), while the exp-log correspondences are instances of (9.19). Let us make explicit the exp maps. Check that: The exponential of the first biderivation below is the functor $\mathcal{S} = \mathcal{S}^\vee$ applied to $f : \mathsf{c}_t \to \mathsf{a}_t$. Clearly, this is an isomorphism of bimonoids iff f is an isomorphism of species.

$$\begin{tikzcd} \mathcal{S}(\mathsf{c}_t) \arrow[r, dashed] \arrow[d, two heads] & \mathcal{S}^\vee(\mathsf{a}_t) \\ \mathsf{c} \arrow[r, "f"'] & \mathsf{a} \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S}(\mathsf{c}) \arrow[r, dashed] \arrow[d, two heads] & \mathcal{S}^\vee(\mathsf{a}) \\ \mathsf{c} \arrow[r, "f"'] & \mathsf{a} \arrow[u, hook] \end{tikzcd}$$

The exponential of the second biderivation above, on the Z-component, has (X, Y)-matrix-component given by

$$\sum_{\mathrm{W}:\, \mathrm{W} \geq \mathrm{X} \vee \mathrm{Y}} \mu_{\mathrm{Y}}^{\mathrm{W}} f_{\mathrm{W}} \Delta_{\mathrm{X}}^{\mathrm{W}},$$

where Δ is the coproduct of c, and μ the product of a. Deduce that: This is an isomorphism of bimonoids iff f is an isomorphism of species. This fact is an instance of Proposition 9.55.

14.2.7. Norm transformation. We now apply the above ideas to some of the considerations in Section 6.9.

Exercise 14.30. Let maps $f : \mathsf{c} \to \mathsf{a}$ and $g : \mathcal{S}(\mathsf{c}) \to \mathcal{S}^\vee(\mathsf{a})$ correspond to each other under the bijection (6.78). Use Exercise 14.29 and Exercise 9.60 to deduce that the composite map

$$\mathcal{S}(\mathsf{c}_t) \xrightarrow{\cong} \mathcal{S}(\mathsf{c}) \xrightarrow{g} \mathcal{S}^\vee(\mathsf{a}) \xrightarrow{\cong} \mathcal{S}^\vee(\mathsf{a}_t) \tag{14.19}$$

is the exponential (9.15a) of the biderivation

$$\begin{tikzcd} \mathcal{S}(\mathsf{c}_t) \arrow[r, dashed] \arrow[d, two heads] & \mathcal{S}^\vee(\mathsf{a}_t) \\ \mathsf{c} \arrow[r, "\log(f)"'] & \mathsf{a}. \arrow[u, hook] \end{tikzcd}$$

Exercise 14.31. Specialize Exercise 14.30 to the maps $f : \mathsf{E} \to \mathsf{E}$ and $g : \Pi \to \Pi^*$ in Exercise 7.13. Use Example 14.28 and Exercise 9.62 to deduce the second formula in (7.51) for the map $\Pi \to \Pi^*$.

14.2.8. Mixed distributive law for bicommutative bimonoids. We now rewrite the mixed distributive law (3.16) for bicommutative bimonoids by conjugating it with the HNR isomorphisms.

Exercise 14.32. Specialize Exercise 14.30 to the maps $f : \mathcal{S}^\vee(\mathsf{p}) \to \mathcal{S}(\mathsf{p})$ and $g : \mathcal{S}(\mathcal{S}^\vee(\mathsf{p})) \to \mathcal{S}^\vee(\mathcal{S}(\mathsf{p}))$ in Exercise 6.80, item (1). Use the formula for $\log(f)$ in Exercise 9.63 to show that the composite map (14.19)

$$\mathcal{S}(\mathcal{S}^\vee(\mathsf{p})_t) \to \mathcal{S}^\vee(\mathcal{S}(\mathsf{p})_t)$$

is diagonal: On the Z-component,

$$\bigoplus_{(\mathrm{X},\mathrm{Y}):\, \mathrm{Z} \leq \mathrm{X} \leq \mathrm{Y}} \mathsf{p}[\mathrm{Y}] \to \bigoplus_{(\mathrm{X}',\mathrm{Y}'):\, \mathrm{Z} \leq \mathrm{X}' \leq \mathrm{Y}'} \mathsf{p}[\mathrm{Y}']$$

is scalar multiplication by $\mu(\mathrm{X},\mathrm{Y})$ on the matrix-component for which $\mathrm{X} = \mathrm{X}'$ and $\mathrm{Y} = \mathrm{Y}'$, and 0 on the remaining matrix-components.

Use the above explicit description to recover the isomorphism criterion given in Theorem 9.64.

For $\mathsf{p} := \mathsf{x}$, the above map specializes to $\Pi \to \Pi^*$ given by the second formula in (7.51) for $\eta_{\mathrm{X}} = \mu(\mathrm{X}, \top)$.

14.2.9. Action of the flat-incidence algebra. For any cocommutative comonoid c, the flat-incidence algebra $\mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$ acts on the right on the species $\mathcal{S}(\mathsf{c})$ as follows. For each flat Z, the right action of $s \in \mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$ on the Z-component of $\mathcal{S}(\mathsf{c})$ is given by

$$\bigoplus_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mathsf{c}[\mathrm{X}] \to \bigoplus_{\mathrm{W}:\, \mathrm{W} \geq \mathrm{Z}} \mathsf{c}[\mathrm{W}], \qquad x \mapsto \sum_{\mathrm{W}:\, \mathrm{W} \geq \mathrm{X}} s(\mathrm{X},\mathrm{W})\, \Delta_{\mathrm{X}}^{\mathrm{W}}(x) \tag{14.20}$$

for $x \in \mathsf{c}[\mathrm{X}]$.

Example 14.33. The maps (14.10) and (14.12) are instances of (14.20) for $s := \zeta$ and $s := \mu$, respectively. It follows that they are inverses of each other. Hence, Propositions 14.13 and 14.15 imply each other. Recall that the Weisner formula (1.38) was used in the proof of the latter. Interestingly, one can also turn this picture around and say that the computation done in this proof yields a proof of the Weisner formula. This kind of observation will be seen in full force in the noncommutative and two-sided settings in Section 14.6.

Dually, for any commutative monoid a, the flat-incidence algebra $\mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$ acts on the left on the species $\mathcal{S}^\vee(\mathsf{a})$ as follows. For each flat Z, the left action of $s \in \mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$ on the Z-component of $\mathcal{S}^\vee(\mathsf{a})$ is given by

$$\bigoplus_{\mathrm{W}:\, \mathrm{W} \geq \mathrm{Z}} \mathsf{a}[\mathrm{W}] \to \bigoplus_{\mathrm{X}:\, \mathrm{X} \geq \mathrm{Z}} \mathsf{a}[\mathrm{X}], \qquad x \mapsto \sum_{\mathrm{X}:\, \mathrm{Z} \leq \mathrm{X} \leq \mathrm{W}} s(\mathrm{X},\mathrm{W})\, \mu_{\mathrm{X}}^{\mathrm{W}}(x) \tag{14.21}$$

for $x \in \mathsf{a}[\mathrm{W}]$.

Note that Propositions 14.19 and 14.21 imply each other, with (14.14) and (14.16) being inverses of each other. They are instances of (14.21).

14.3. Free 0-$\sim$-bicommutative bimonoids

We now unify the discussion in Sections 14.1 and 14.2. Let $\sim$ be a geometric partial-support relation on faces. Recall from Section 6.11.2 the free 0-$\sim$-bicommutative bimonoid $\mathcal{T}_{0,\sim}(\mathsf{c})$ on a $\sim$-cocommutative comonoid c. We now show that it is isomorphic to $\mathcal{T}_{0,\sim}(\mathsf{c}_t)$, where c_t is c as a species but with the trivial coproduct.

Proposition 14.34. *For a $\sim$-cocommutative comonoid c, the map of species $\mathcal{T}_{0,\sim}(\mathsf{c}) \to \mathcal{T}_{0,\sim}(\mathsf{c}_t)$ given on the z-component by*

$$\bigoplus_{\mathrm{x}:\, \mathrm{x} \geq \mathrm{z}} \mathsf{c}[\mathrm{x}] \to \bigoplus_{\mathrm{w}:\, \mathrm{w} \geq \mathrm{z}} \mathsf{c}[\mathrm{w}], \qquad x \mapsto \sum_{\mathrm{w}:\, \mathrm{w} \geq \mathrm{x}} \Delta^{\mathrm{w}}_{\mathrm{x}}(x) \tag{14.22}$$

for $x \in \mathsf{c}[\mathrm{x}]$, is an isomorphism of 0-$\sim$-bicommutative bimonoids.

Proof. This follows by generalizing the proofs of Propositions 14.2 and 14.13. $\square$

Let $\mu(\mathrm{z}, \mathrm{y})$ denote the Möbius function of the poset of partial-flats for the relation $\sim$.

Proposition 14.35. *For a $\sim$-cocommutative comonoid c, the map of species $\mathcal{T}_{0,\sim}(\mathsf{c}_t) \to \mathcal{T}_{0,\sim}(\mathsf{c})$ given on the z-component by*

$$\bigoplus_{\mathrm{x}:\, \mathrm{x} \geq \mathrm{z}} \mathsf{c}[\mathrm{x}] \to \bigoplus_{\mathrm{w}:\, \mathrm{w} \geq \mathrm{z}} \mathsf{c}[\mathrm{w}], \qquad x \mapsto \sum_{\mathrm{w}:\, \mathrm{w} \geq \mathrm{x}} \mu(\mathrm{x}, \mathrm{w}) \Delta^{\mathrm{w}}_{\mathrm{x}}(x) \tag{14.23}$$

for $x \in \mathsf{c}[\mathrm{x}]$, is an isomorphism of 0-$\sim$-bicommutative bimonoids.

Proof. This follows by inverting the map (14.22). $\square$

By applying the primitive part functor to (14.23) and using (13.46) for $\mathsf{p} = \mathsf{c}_t$, we obtain:

Proposition 14.36. *For a $\sim$-cocommutative comonoid c, the map of species $\mathsf{c} \to \mathcal{P}(\mathcal{T}_{0,\sim}(\mathsf{c}))$ given on the z-component by*

$$\mathsf{c}[\mathrm{z}] \to \bigoplus_{\mathrm{y}:\, \mathrm{y} \geq \mathrm{z}} \mathsf{c}[\mathrm{y}], \qquad x \mapsto \sum_{\mathrm{y}:\, \mathrm{y} \geq \mathrm{z}} \mu(\mathrm{z}, \mathrm{y}) \Delta^{\mathrm{y}}_{\mathrm{z}}(x) \tag{14.24}$$

is a natural isomorphism.

Exercise 14.37. Check directly that the image of (14.24) belongs to the primitive part: Start with coproduct formula (6.93) and then use the Weisner formula, namely, for partial-flats $\mathrm{z} < \mathrm{x} \leq \mathrm{w}$,

$$\sum_{\mathrm{y}:\, \mathrm{y} \geq \mathrm{z},\, \mathrm{x} \vee \mathrm{y} = \mathrm{w}} \mu(\mathrm{z}, \mathrm{y}) = 0.$$

This can then be used to give a direct proof of Proposition 14.35 (as done for Propositions 14.4 and 14.15).

Exercise 14.38. Write down the signed analogues of the above results, following the discussion of signed bicommutative signed bimonoids.

We mention that dual results hold for the cofree 0-$\sim$-bicommutative bimonoid $\mathcal{T}^{\vee}_{0,\sim}(\mathsf{a})$ on a $\sim$-commutative monoid a.

We refer to the maps (14.22), (14.23) and their duals as the *Hoffman–Newman–Radford isomorphisms* for 0-$\sim$-bicommutative bimonoids. They specialize to the HNR isomorphisms for 0-bimonoids when $\sim$ is finest, and to the HNR isomorphisms for bicommutative bimonoids when $\sim$ is coarsest.

14.4. Free bimonoids on cocommutative comonoids

Recall from Section 6.1 the free bimonoid $\mathcal{T}(\mathsf{c})$ on a comonoid c. The product is concatenation and the coproduct is dequasishuffle. If the coproduct of c is trivial, then the coproduct of $\mathcal{T}(\mathsf{c})$ simplifies to deshuffle (Section 6.4.1). If c is cocommutative, then the bimonoids $\mathcal{T}(\mathsf{c})$ and $\mathcal{T}(\mathsf{c}_t)$ are naturally isomorphic, where c_t denotes the underlying species of c with the trivial coproduct. The isomorphism in one direction involves a noncommutative zeta function, and in the other direction involves a noncommutative Möbius function. There is also a dual result connecting the shuffle and quasishuffle products of the cofree bimonoid on a commutative monoid. Propositions 14.13 and 14.19 from Section 14.2 can be viewed as commutative analogues of these results. We also briefly mention the signed analogues.

14.4.1. Deshuffle and dequasishuffle.

For a comonoid c, let c_t denote the underlying species of c with the trivial coproduct. The product and coproduct of $\mathcal{T}(\mathsf{c})$ are given by (6.3) and (6.5) for $q = 1$, while that of $\mathcal{T}(\mathsf{c}_t)$ are given by (6.39) for $\mathsf{p} := \mathsf{c}_t$ and $q = 1$.

Lemma 14.39. *Fix a noncommutative zeta function $\boldsymbol{\zeta}$. For a cocommutative comonoid c, the map of species $\mathsf{c} \to \mathcal{T}(\mathsf{c}_t)$ given on the A-component by*

$$\mathsf{c}[A] \to \bigoplus_{H\colon H \geq A} \mathsf{c}[H], \qquad x \mapsto \sum_{H\colon H\geq A} \boldsymbol{\zeta}(A,H)\, \Delta_A^H(x) \tag{14.25}$$

is a morphism of comonoids.

First proof. Let f denote the above map. We need to show that the diagram

$$\begin{tikzcd} \mathsf{c}[A] \arrow[r, "\Delta_A^G"] \arrow[d, "f_A"'] & \mathsf{c}[G] \arrow[d, "f_G"] \\ \mathcal{T}(\mathsf{c}_t)[A] \arrow[r, "\Delta_A^G"'] & \mathcal{T}(\mathsf{c}_t)[G] \end{tikzcd}$$

commutes. This is checked below. For $x \in \mathsf{c}[A]$,

$$\begin{aligned} \Delta_A^G f_A(x) &= \Delta_A^G\Big(\sum_{H\colon H\geq A} \boldsymbol{\zeta}(A,H)\, \Delta_A^H(x)\Big) \\ &= \sum_{H\colon H\geq A,\, HG=H} \boldsymbol{\zeta}(A,H)\, \beta_{GH,H} \Delta_A^H(x) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{H:\, H \geq A,\, HG=H} \boldsymbol{\zeta}(A,H)\, \Delta_A^{GH}(x) \\
&= \sum_{K:\, K \geq G} \Big(\sum_{\substack{H:\, H \geq A,\, GH=K \\ HG=H}} \boldsymbol{\zeta}(A,H) \Big) \Delta_A^{K}(x) \\
&= \sum_{K:\, K \geq G} \boldsymbol{\zeta}(G,K)\, \Delta_A^{K}(x) \\
&= f_G \Delta_A^G(x).
\end{aligned}
$$

The second step used coproduct formula (6.39) for $q = 1$ for $\mathcal{T}(\mathsf{c}_t)$. The third step used cocommutativity of c. The fifth step was critical and used the lune-additivity formula (1.42). $\square$

SECOND PROOF. Recall exponential of a map of species from (9.3a). Observe that the map (14.25) is an exponential of the canonical inclusion $\mathsf{c} \hookrightarrow \mathcal{T}(\mathsf{c}_t)$. Further, by Exercise 6.33 for $q = 1$, this inclusion maps into the primitive part of $\mathcal{T}(\mathsf{c}_t)$, so it is a coderivation. Hence, by Theorem 9.11, we deduce that (14.25) is a morphism of comonoids. $\square$

Proposition 14.40. *Fix a noncommutative zeta function $\boldsymbol{\zeta}$. For a cocommutative comonoid c, the map of species $\mathcal{T}(\mathsf{c}) \to \mathcal{T}(\mathsf{c}_t)$ given on the A-component by*

$$
\bigoplus_{F:\, F \geq A} \mathsf{c}[F] \to \bigoplus_{H:\, H \geq A} \mathsf{c}[H], \qquad x \mapsto \sum_{H:\, H \geq F} \boldsymbol{\zeta}(F,H)\, \Delta_F^H(x) \tag{14.26}
$$

for $x \in \mathsf{c}[F]$, is an isomorphism of bimonoids.

A quick justification for requiring c to be cocommutative is as follows. The bimonoid $\mathcal{T}(\mathsf{c}_t)$ is always cocommutative, while $\mathcal{T}(\mathsf{c})$ is cocommutative only if c is by Lemma 6.8.

PROOF. By the universal property of $\mathcal{T}$ (Theorem 6.6 for $q = 1$), the morphism of comonoids $\mathsf{c} \to \mathcal{T}(\mathsf{c}_t)$ in Lemma 14.39 extends to a morphism of bimonoids $\mathcal{T}(\mathsf{c}) \to \mathcal{T}(\mathsf{c}_t)$. By employing formula (6.4), we see that this indeed coincides with (14.26). Moreover, since the matrix form of (14.26) is unitriangular, it is an isomorphism. $\square$

Lemma 14.41. *Fix a noncommutative Möbius function $\boldsymbol{\mu}$. For a cocommutative comonoid c, the map of species $\mathsf{c} \to \mathcal{T}(\mathsf{c})$ given on the A-component by*

$$
\mathsf{c}[A] \to \bigoplus_{H:\, H \geq A} \mathsf{c}[H], \qquad x \mapsto \sum_{H:\, H \geq A} \boldsymbol{\mu}(A,H) \Delta_A^H(x) \tag{14.27}
$$

is a coderivation, that is, it maps into $\mathcal{P}(\mathcal{T}(\mathsf{c}))$.

First proof. The required calculation is done below. For $x \in \mathsf{c}[A]$ and $H > A$,

$$\begin{aligned}\Delta_A^H\Big(\sum_{F\colon F\geq A} \boldsymbol{\mu}(A,F)\Delta_A^F(x)\Big) &= \sum_{F\colon F\geq A} \boldsymbol{\mu}(A,F)\,\beta_{HF,FH}\Delta_F^{FH}\Delta_A^F(x)\\ &= \sum_{F\colon F\geq A} \boldsymbol{\mu}(A,F)\,\Delta_A^{HF}(x)\\ &= \sum_{G\colon G\geq H}\Big(\sum_{\substack{F\colon F\geq A\\ HF=G}} \boldsymbol{\mu}(A,F)\Big)\Delta_A^G(x)\\ &= 0.\end{aligned}$$

In the first step, we used coproduct formula (6.5) for $q = 1$ for $\mathcal{T}(\mathsf{c})$. In the second step, we used cocommutativity and coassociativity of c. The last step was critical and used the noncommutative Weisner formula (1.44). □

Second proof. Recall logarithm of a map of species from (9.3b). Observe that the map (14.27) is a logarithm of the canonical inclusion $\mathsf{c} \hookrightarrow \mathcal{T}(\mathsf{c})$. The latter is a morphism of comonoids by Exercise 6.7 for $q = 1$. Hence, by Theorem 9.11, we deduce that (14.27) is a coderivation. □

Proposition 14.42. *Fix a noncommutative Möbius function $\boldsymbol{\mu}$. For a cocommutative comonoid c, the map of species $\mathcal{T}(\mathsf{c}_t) \to \mathcal{T}(\mathsf{c})$ given on the A-component by*

$$\bigoplus_{F\colon F\geq A} \mathsf{c}[F] \to \bigoplus_{H\colon H\geq A} \mathsf{c}[H], \qquad x \mapsto \sum_{H\colon H\geq F} \boldsymbol{\mu}(F,H)\Delta_F^H(x) \tag{14.28}$$

for $x \in \mathsf{c}[F]$, is an isomorphism of bimonoids.

Proof. By the universal property of $\mathcal{T}$ (Theorem 6.31 for $q = 1$), the map $\mathsf{c} \to \mathcal{P}(\mathcal{T}(\mathsf{c}))$ in Lemma 14.41 extends to a morphism of bimonoids $\mathcal{T}(\mathsf{c}_t) \to \mathcal{T}(\mathsf{c})$. By employing formula (6.41), we see that this indeed coincides with (14.28). Moreover, since the matrix form of (14.28) is unitriangular, it is an isomorphism. □

14.4.2. Shuffle and quasishuffle. The dual results for the cofree bimonoid on a commutative monoid are given below. Since the details are similar, we only sketch the proofs. For a monoid a, let a_t denote the underlying species of a with the trivial product. The product and coproduct of $\mathcal{T}^\vee(\mathsf{a})$ are given by (6.11) and (6.13) for $q = 1$, while that of $\mathcal{T}^\vee(\mathsf{a}_t)$ are given by (6.40) for $\mathsf{p} := \mathsf{a}_t$ and $q = 1$.

Lemma 14.43. *Fix a noncommutative zeta function $\boldsymbol{\zeta}$. For a commutative monoid a, the map of species $\mathcal{T}^\vee(\mathsf{a}_t) \to \mathsf{a}$ given on the A-component by*

$$\bigoplus_{F\colon F\geq A} \mathsf{a}[F] \to \mathsf{a}[A], \qquad x \mapsto \boldsymbol{\zeta}(A,F)\,\mu_A^F(x)$$

for $x \in \mathsf{a}[F]$, is a morphism of monoids.

PROOF. One way is to proceed directly. Alternatively, note that the above map is an exponential of the canonical projection $\mathcal{T}^\vee(\mathsf{a}_t) \twoheadrightarrow \mathsf{a}$ (which is a derivation) and apply Theorem 9.12. $\square$

Proposition 14.44. *Fix a noncommutative zeta function* $\boldsymbol{\zeta}$*. For a commutative monoid* a*, the map of species* $\mathcal{T}^\vee(\mathsf{a}_t) \to \mathcal{T}^\vee(\mathsf{a})$ *given on the* A*-component by*

$$\bigoplus_{F:\, F \geq A} \mathsf{a}[F] \to \bigoplus_{G:\, G\geq A} \mathsf{a}[G], \qquad x \mapsto \sum_{G:\, A \leq G \leq F} \boldsymbol{\zeta}(G,F)\, \mu_G^F(x) \tag{14.29}$$

for $x \in \mathsf{a}[F]$*, is an isomorphism of bimonoids.*

PROOF. Apply the universal property of $\mathcal{T}^\vee$ (Theorem 6.13 for $q=1$) to the morphism of monoids in Lemma 14.43. $\square$

Lemma 14.45. *Fix a noncommutative Möbius function* $\boldsymbol{\mu}$*. For a commutative monoid* a*, the map of species* $\mathcal{T}^\vee(\mathsf{a}) \to \mathsf{a}$ *given on the* A*-component by*

$$\bigoplus_{F:\, F\geq A} \mathsf{a}[F] \to \mathsf{a}[A], \qquad x \mapsto \boldsymbol{\mu}(A,F)\, \mu_A^F(x)$$

for $x \in \mathsf{a}[F]$*, is a derivation, that is, it factors through* $\mathcal{Q}(\mathcal{T}^\vee(\mathsf{a}))$*.*

PROOF. One way is to proceed directly. Alternatively, note that the above map is a logarithm of the canonical projection $\mathcal{T}^\vee(\mathsf{a}) \twoheadrightarrow \mathsf{a}$ (which is a morphism of monoids by Exercise 6.14 for $q=1$) and apply Theorem 9.12. $\square$

Proposition 14.46. *Fix a noncommutative Möbius function* $\boldsymbol{\mu}$*. For a commutative monoid* a*, the map of species* $\mathcal{T}^\vee(\mathsf{a}) \to \mathcal{T}^\vee(\mathsf{a}_t)$ *given on the* A*-component by*

$$\bigoplus_{F:\, F \geq A} \mathsf{a}[F] \to \bigoplus_{G:\, G\geq A} \mathsf{a}[G], \qquad x \mapsto \sum_{G:\, A \leq G \leq F} \boldsymbol{\mu}(G,F)\, \mu_G^F(x) \tag{14.30}$$

for $x \in \mathsf{a}[F]$*, is an isomorphism of bimonoids.*

PROOF. Apply the universal property of $\mathcal{T}^\vee$ (Theorem 6.34 for $q=1$) to the map in Lemma 14.45. $\square$

14.4.3. HNR isomorphisms for (co)commutative bimonoids. We refer to (14.26) and (14.28) as the *Hoffman–Newman–Radford isomorphisms* for cocommutative bimonoids. Similarly, we refer to (14.29) and (14.30) as the *Hoffman–Newman–Radford isomorphisms* for commutative bimonoids. They are natural in c and a, that is, diagrams (14.8) commute with $\mathcal{T}$ in place of $\mathcal{T}_0$.

14.4.4. Signed analogue. The above discussion carries over to the signed setting. The main results are stated below. The formulas for the HNR isomorphisms remain the same as before.

In the comonoid case, we replace cocommutative by signed cocommutative, and $\mathcal{T}$ by $\mathcal{T}_{-1}$.

Proposition 14.47. *Fix a noncommutative zeta function* $\boldsymbol{\zeta}$*. For a signed cocommutative comonoid* c*, the map of species* $\mathcal{T}_{-1}(\mathsf{c}) \to \mathcal{T}_{-1}(\mathsf{c}_t)$ *given by formula* (14.26) *on the* A*-component is an isomorphism of signed bimonoids.*

Proposition 14.48. *Fix a noncommutative Möbius function* $\boldsymbol{\mu}$*. For a signed cocommutative comonoid* c*, the map of species* $\mathcal{T}_{-1}(\mathsf{c}_t) \to \mathcal{T}_{-1}(\mathsf{c})$ *given by formula* (14.28) *on the* A*-component is an isomorphism of signed bimonoids.*

In the monoid case, we replace commutative by signed commutative, and $\mathcal{T}^\vee$ by $\mathcal{T}^\vee_{-1}$.

Proposition 14.49. *Fix a noncommutative zeta function* $\boldsymbol{\zeta}$*. For a signed commutative monoid* a*, the map of species* $\mathcal{T}^\vee_{-1}(\mathsf{a}_t) \to \mathcal{T}^\vee_{-1}(\mathsf{a})$ *given by formula* (14.29) *on the* A*-component is an isomorphism of signed bimonoids.*

Proposition 14.50. *Fix a noncommutative Möbius function* $\boldsymbol{\mu}$*. For a signed commutative monoid* a*, the map of species* $\mathcal{T}^\vee_{-1}(\mathsf{a}) \to \mathcal{T}^\vee_{-1}(\mathsf{a}_t)$ *given by formula* (14.30) *on the* A*-component is an isomorphism of signed bimonoids.*

14.4.5. Compatibility between HNR isomorphisms. The HNR isomorphisms for (co)commutative bimonoids relate to the ones for bicommutative bimonoids via (co)abelianization. More precisely:

For a cocommutative comonoid c and commutative monoid a, the following diagrams commute.

$$\begin{tikzcd} \mathcal{T}(\mathsf{c}) \arrow[r, shift left] \arrow[d, "\pi"'] & \mathcal{T}(\mathsf{c}_t) \arrow[l, shift left] \arrow[d, "\pi"] \\ \mathcal{S}(\mathsf{c}) \arrow[r, shift left] & \mathcal{S}(\mathsf{c}_t) \arrow[l, shift left] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}^\vee(\mathsf{a}_t) \arrow[r, shift left] & \mathcal{T}^\vee(\mathsf{a}) \arrow[l, shift left] \\ \mathcal{S}^\vee(\mathsf{a}_t) \arrow[r, shift left] \arrow[u, "\pi^\vee"] & \mathcal{S}^\vee(\mathsf{a}) \arrow[l, shift left] \arrow[u, "\pi^\vee"'] \end{tikzcd} \tag{14.31}$$

The vertical maps are abelianization and coabelianization, respectively (Section 6.6). The horizontal maps are all HNR isomorphisms: In the first diagram, the top-horizontal maps are (14.26) and (14.28), while the bottom-horizontal maps are (14.10) and (14.12). In the second diagram, the top-horizontal maps are (14.29) and (14.30), while the bottom-horizontal maps are (14.14) and (14.16).

The verification is straightforward. Apart from definitions, one uses Lemma 1.17.

14.4.6. Examples.

Example 14.51. Recall the bimonoid of faces Σ from Section 7.6. View the exponential species E as a comonoid. Let E_t denote E with the trivial coproduct. The following is a commutative diagram of bimonoids.

$$\begin{tikzcd} \mathcal{T}(\mathsf{E}) \arrow[rr, "\cong"] & & \mathcal{T}(\mathsf{E}_t) \\ & \Sigma \arrow[ul, "\cong"] \arrow[ur, "\cong"'] & \end{tikzcd}$$

The horizontal map is the HNR isomorphism (14.26) specialized to $\mathsf{c} := \mathsf{E}$. The map going up to the left is the first map in (7.81) for $q = 1$, while the one going up to the right is the first map in (7.82). They involve the $\mathtt{H}$-basis and $\mathtt{Q}$-basis, respectively, of Σ, see (7.67).

Similarly, one can formulate the dual statement for Σ^* by specializing (14.29) to $\mathsf{a} := \mathsf{E}$.

How would diagrams (14.31) specialize for $\mathsf{c} := \mathsf{E}$ and $\mathsf{a} := \mathsf{E}$? See Example 14.28 and Exercise 7.33 in this regard.

Note very carefully that the above analysis cannot be carried out for the bimonoid of top-nested faces and its dual from Section 7.7 since these bimonoids are neither commutative nor cocommutative.

14.4.7. Conjugation by HNR isomorphisms. For any cocommutative comonoid c and commutative monoid a, we have bijections

$$\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{T}(\mathsf{c}), \mathcal{T}^\vee(\mathsf{a})) \rightleftarrows \mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{T}(\mathsf{c}_t), \mathcal{T}^\vee(\mathsf{a}_t)) \tag{14.32}$$

given by

$$\begin{aligned} (\mathcal{T}(\mathsf{c}) \xrightarrow{g} \mathcal{T}^\vee(\mathsf{a})) &\longmapsto (\mathcal{T}(\mathsf{c}_t) \xrightarrow{\cong} \mathcal{T}(\mathsf{c}) \xrightarrow{g} \mathcal{T}^\vee(\mathsf{a}) \xrightarrow{\cong} \mathcal{T}^\vee(\mathsf{a}_t)) \\ (\mathcal{T}(\mathsf{c}_t) \xrightarrow{h} \mathcal{T}^\vee(\mathsf{a}_t)) &\longmapsto (\mathcal{T}(\mathsf{c}) \xrightarrow{\cong} \mathcal{T}(\mathsf{c}_t) \xrightarrow{h} \mathcal{T}^\vee(\mathsf{a}_t) \xrightarrow{\cong} \mathcal{T}^\vee(\mathsf{a})), \end{aligned}$$

with the isomorphisms being the HNR isomorphisms. This change of basis in fact reduces to the one discussed in (14.18). More precisely:

Lemma 14.52. *For any cocommutative comonoid* c *and commutative monoid* a*, we have a commutative diagram of bijections*

$$\begin{tikzcd} \mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{T}(\mathsf{c}), \mathcal{T}^\vee(\mathsf{a})) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{T}(\mathsf{c}_t), \mathcal{T}^\vee(\mathsf{a}_t)) \arrow[d, leftrightarrow] \\ {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{S}(\mathsf{c}), \mathcal{S}^\vee(\mathsf{a})) \arrow[r, leftrightarrow] & {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{S}(\mathsf{c}_t), \mathcal{S}^\vee(\mathsf{a}_t)). \end{tikzcd}$$

The horizontal bijections are as in (14.32) and (14.18). The vertical bijections are instances of the composite bijection in (2.58) (between the top-left and bottom-right corners). Recall here that the abelianization of $\mathcal{T}(\mathsf{c})$ is $\mathcal{S}(\mathsf{c})$, and the coabelianization of $\mathcal{T}^\vee(\mathsf{a})$ is $\mathcal{S}^\vee(\mathsf{a})$ (Section 6.6).

PROOF. The bijections are encapsulated in the diagram below. The oblique maps are the HNR isomorphisms.

$$\begin{tikzcd} & \mathcal{T}(\mathsf{c}_t) \arrow[rr, dashed] \arrow[dd, two heads] & & \mathcal{T}^\vee(\mathsf{a}_t) \\ \mathcal{T}(\mathsf{c}) \arrow[ur, leftrightarrow] \arrow[rr, dashed] \arrow[dd, two heads] & & \mathcal{T}^\vee(\mathsf{a}) \arrow[ur, leftrightarrow] & \\ & \mathcal{S}(\mathsf{c}_t) \arrow[rr, dashed] & & \mathcal{S}^\vee(\mathsf{a}_t) \arrow[uu, hook] \\ \mathcal{S}(\mathsf{c}) \arrow[ur, leftrightarrow] \arrow[rr, dashed] & & \mathcal{S}^\vee(\mathsf{a}) \arrow[ur, leftrightarrow] \arrow[uu, hook] & \end{tikzcd} \tag{14.33}$$

The main step is to note that the two side squares commute by (14.31). $\square$

Exercise 14.53. Show that: For any cocommutative comonoid c and commutative monoid a, we have a commutative diagram of bijections

$$\begin{tikzcd}
\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{T}(\mathsf{c}), \mathcal{T}^{\vee}(\mathsf{a})) \arrow[rr, leftrightarrow] \arrow[dd, leftrightarrow] \arrow[dr, leftrightarrow] & & \mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{T}(\mathsf{c}_t), \mathcal{T}^{\vee}(\mathsf{a}_t)) \arrow[dd, leftrightarrow] \arrow[dl, leftrightarrow] \\
& \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a}) & \\
{}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{S}(\mathsf{c}), \mathcal{S}^{\vee}(\mathsf{a})) \arrow[rr, leftrightarrow] \arrow[ur, leftrightarrow] & & {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{S}(\mathsf{c}_t), \mathcal{S}^{\vee}(\mathsf{a}_t)) \arrow[ul, leftrightarrow]
\end{tikzcd}$$

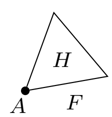
with the outside square as in Lemma 14.52, and the oblique bijections being the exp-log correspondences (9.19). (Use Exercise 14.29 for the bottom triangle and Exercise 9.61 for the two side triangles.)

Example 14.54. Specialize diagram (14.33) to $\mathsf{c} := \mathsf{E}$ and $\mathsf{a} := \mathsf{E}$. In view of Examples 14.28 and 14.51, this diagram connects the bimonoids Σ, Σ^*, Π, Π^*. This is the same as diagram (7.86) viewed both on the $\mathtt{H}$-basis and the $\mathtt{Q}$-basis.

14.4.8. Action of the lune-incidence algebra. For any comonoid c, the lune-incidence algebra $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ acts on the right on the species $\mathcal{T}(\mathsf{c})$ as follows. For each face A, the right action of $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ on the A-component of $\mathcal{T}(\mathsf{c})$ is given by

$$\bigoplus_{F:\, F \geq A} \mathsf{c}[F] \to \bigoplus_{H:\, H \geq A} \mathsf{c}[H], \qquad x \mapsto \sum_{H:\, H \geq F} s(F,H)\, \Delta^H_F(x) \tag{14.34}$$

for $x \in \mathsf{c}[F]$. An illustrative picture is shown below.



Example 14.55. Many maps which we have encountered earlier can now be seen as special cases of (14.34).

- For $s := \boldsymbol{\zeta}$, a noncommutative zeta function, we get (14.26), while for $s := \boldsymbol{\mu}$, a noncommutative Möbius function, we get (14.28). It follows that when $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ are inverse, the maps (14.26) and (14.28) are inverses of each other. Thus, Propositions 14.40 and 14.42 imply each other. Now interestingly, we gave independent proofs for both propositions, and this can be used to establish the inverse relationship between $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$, see Section 14.6 for details.
- For $s := \boldsymbol{\zeta}_0$, the noncommutative 0-zeta function, we get (14.2), while for $s := \boldsymbol{\mu}_0$, the noncommutative 0-Möbius function, we get (14.4). It follows that maps (14.2) and (14.4) are inverses of each other. See Remark 14.6.
- More generally, for $s := h_\alpha$ as in (1.53), we get the map

$$\bigoplus_{F:\, F \geq A} \mathsf{c}[F] \to \bigoplus_{H:\, H \geq A} \mathsf{c}[H], \qquad x \mapsto \sum_{H:\, H \geq F} \alpha^{\mathrm{rk}(H/F)}\, \Delta^H_F(x) \tag{14.35}$$

for $x \in \mathsf{c}[F]$.
- For $s := r_\alpha$ as in (1.53), we get (6.47).

Dually, for any monoid a, the lune-incidence algebra $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ acts on the left on the species $\mathcal{T}^\vee(\mathsf{a})$ as follows. For each face A, the left action of $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ on the A-component of $\mathcal{T}^\vee(\mathsf{a})$ is given by

$$\bigoplus_{F:\, F \geq A} \mathsf{a}[F] \to \bigoplus_{G:\, G \geq A} \mathsf{a}[G], \qquad x \mapsto \sum_{G:\, A \leq G \leq F} s(G,F)\, \mu_G^F(x) \tag{14.36}$$

for $x \in \mathsf{a}[F]$.

Example 14.56. Let us now specialize (14.36) as in Example 14.55.

- For $s := \boldsymbol{\zeta}$, we get (14.29), while for $s := \boldsymbol{\mu}$, we get (14.30). Note that Propositions 14.44 and 14.46 imply each other, with maps (14.29) and (14.30) being inverses of each other when $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ are inverse.
- For $s := \boldsymbol{\zeta}_0$, we get (14.5), while for $s := \boldsymbol{\mu}_0$, we get (14.6), which is its inverse.
- More generally, for $s := h_\alpha$, we get the map

$$\bigoplus_{F:\, F \geq A} \mathsf{a}[F] \to \bigoplus_{G:\, G \geq A} \mathsf{a}[G], \qquad x \mapsto \sum_{G:\, A \leq G \leq F} \alpha^{\mathrm{rk}(F/G)}\, \mu_G^F(x) \tag{14.37}$$

  for $x \in \mathsf{a}[F]$.
- For $s := r_\alpha$, we get (6.49).

Exercise 14.57. Recall $t_\alpha \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ from (1.54). Note very carefully that it depends on the choice of a $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$. Show that:

For a cocommutative comonoid c, the map of species $\mathcal{T}(\mathsf{c}) \to \mathcal{T}(\mathsf{c})$ given on the A-component by

$$\bigoplus_{F:\, F \geq A} \mathsf{c}[F] \to \bigoplus_{H:\, H \geq A} \mathsf{c}[H], \qquad x \mapsto \sum_{H:\, H \geq F} t_\alpha(F,H)\, \Delta_F^H(x) \tag{14.38}$$

for $x \in \mathsf{c}[F]$, is a morphism of bimonoids. It is an isomorphism if $\alpha \neq 0$. (Pre- and post-compose (6.47) by HNR isomorphisms.)

For a commutative monoid a, the map of species $\mathcal{T}^\vee(\mathsf{a}) \to \mathcal{T}^\vee(\mathsf{a})$ given on the A-component by

$$\bigoplus_{F:\, F \geq A} \mathsf{a}[F] \to \bigoplus_{G:\, G \geq A} \mathsf{a}[G], \qquad x \mapsto \sum_{G:\, A \leq G \leq F} t_\alpha(G,F)\, \mu_G^F(x) \tag{14.39}$$

for $x \in \mathsf{a}[F]$, is a morphism of bimonoids. It is an isomorphism if $\alpha \neq 0$. (Pre- and post-compose (6.49) by HNR isomorphisms.)

Exercise 14.58. For a comonoid c, let c_{-1} denote the comonoid obtained from c by multiplying Δ_A^F by $(-1)^{\mathrm{rk}(F/A)}$. Similarly, for a monoid a, we define a_{-1}. These are the specializations $\alpha = -1$ of (6.46) and (6.48). Use the noncommutative Zaslavsky formula (1.91) to show that:

For a cocommutative comonoid c, the map of species $\mathcal{T}(\mathsf{c}) \to \mathcal{T}(\mathsf{c}_{-1})$ given on the A-component by

$$\bigoplus_{F:\, F \geq A} \mathsf{c}[F] \to \bigoplus_{H:\, H \geq A} \mathsf{c}[H], \qquad x \mapsto \sum_{H:\, H \geq F} \Delta_F^H(x) \tag{14.40}$$

for $x \in \mathsf{c}[F]$, is an isomorphism of bimonoids.

For a commutative monoid a, the map of species $\mathcal{T}^\vee(\mathsf{a}_{-1}) \to \mathcal{T}^\vee(\mathsf{a})$ given on the A-component by

$$\bigoplus_{H\colon H \geq A} \mathsf{a}[H] \to \bigoplus_{F\colon F \geq A} \mathsf{a}[F], \qquad x \mapsto \sum_{F\colon A \leq F \leq H} \mu_F^H(x) \tag{14.41}$$

for $x \in \mathsf{a}[H]$, is an isomorphism of bimonoids.

Exercise 14.59. For any $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$, view (14.34) as an operator on the 0-bimonoid $\mathcal{T}_0(\mathsf{c}_t)$. Observe that it is a morphism of monoids. Check that its 0-logarithm is given on the A-component by

$$\bigoplus_{F\colon F \geq A} \mathsf{c}[F] \to \bigoplus_{H\colon H \geq A} \mathsf{c}[H], \quad x \mapsto \begin{cases} \sum_{H\colon H \geq A} s(A,H)\, \Delta_A^H(x) & \text{if } F = A, \\ 0 & \text{if } F > A, \end{cases}$$

for $x \in \mathsf{c}[F]$. Note that this map factors through the indecomposable part of $\mathcal{T}_0(\mathsf{c}_t)$ in agreement with Theorem 9.104.

For any $s \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$, view (14.36) as an operator on the 0-bimonoid $\mathcal{T}_0^\vee(\mathsf{a}_t)$. Observe that it is a morphism of comonoids. Check that its 0-logarithm is given on the A-component by

$$\bigoplus_{F\colon F \geq A} \mathsf{a}[F] \to \bigoplus_{G\colon G \geq A} \mathsf{a}[G], \qquad x \mapsto s(A,F)\, \mu_A^F(x)$$

for $x \in \mathsf{a}[F]$. Note that this maps into the primitive part of $\mathcal{T}_0^\vee(\mathsf{a}_t)$ in agreement with Theorem 9.103.

When $s = \delta$, the unit element of the lune-incidence algebra, both situations above specialize to Example 9.110.

Exercise 14.60. Recall the opposite transformation τ from (3.26). Show that: If s is projective and c is cocommutative, then the map (14.34) commutes with τ_{c}. Dually, if s is projective and a is commutative, then the map (14.36) commutes with τ_{a}.

14.4.9. Compatibility between the actions. Recall the base-case map (1.45). We now use it to relate the right actions (14.34) and (14.20), and dually, the left actions (14.36) and (14.21).

Lemma 14.61. *For any cocommutative comonoid* c*, the abelianization* $\pi : \mathcal{T}(\mathsf{c}) \to \mathcal{S}(\mathsf{c})$ *is a map of right* $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$*-modules, with* $\mathcal{S}(\mathsf{c})$ *viewed as a right* $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$*-module via the base-case map.*

For any commutative monoid a*, the coabelianization* $\pi^\vee : \mathcal{S}^\vee(\mathsf{a}) \to \mathcal{T}^\vee(\mathsf{a})$ *is a map of left* $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$*-modules, with* $\mathcal{S}^\vee(\mathsf{a})$ *viewed as a left* $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$*-module via the base-case map.*

The verification is straightforward. Observe that diagrams (14.31) follow as special cases, where the actions involved are HNR isomorphisms.

14.5. Free q-bimonoids on comonoids

Recall from Section 6.1 the free q-bimonoid $\mathcal{T}_q(\mathsf{c})$ on a comonoid c. The product is concatenation and the coproduct is q-dequasishuffle. If the coproduct of c is trivial, then the coproduct of $\mathcal{T}_q(\mathsf{c})$ simplifies to q-deshuffle (Section 6.4.1). For q not a root of unity, the q-bimonoids $\mathcal{T}_q(\mathsf{c})$ and $\mathcal{T}_q(\mathsf{c}_t)$ are naturally isomorphic, where c_t denotes the underlying species of c with the trivial coproduct. The isomorphism in one direction involves the two-sided q-zeta function, and in the other direction involves the two-sided q-Möbius function. We pay special attention to the case when c is cocommutative. There is a dual result for the cofree q-bimonoid $\mathcal{T}_q^\vee(\mathsf{a})$ on a monoid a. We also explain how these ideas can be used to study the nondegeneracy of the mixed distributive law for q-bimonoids.

Recall the rigidity Theorem 13.77 which says that for q not a root of unity, every q-bimonoid h is of the form $\mathcal{T}_q(\mathcal{P}(\mathsf{h}))$. The above result on $\mathcal{T}_q(\mathsf{c})$ can be viewed as an illustration for $\mathsf{h} = \mathcal{T}_q(\mathsf{c})$. Similarly, the dual result on $\mathcal{T}_q^\vee(\mathsf{a})$ can be viewed as an illustration of the dual Theorem 13.93 for $\mathsf{h} = \mathcal{T}_q^\vee(\mathsf{a})$.

The discussion closely follows the one in Section 14.4, with conceptual similarities with Section 14.2. Setting $q = 0$ recovers the discussion in Section 14.1.

14.5.1. Free q-bimonoid on a comonoid. For q not a root of unity, recall the two-sided q-zeta function $\boldsymbol{\zeta}_q$ and the two-sided q-Möbius function $\boldsymbol{\mu}_q$ defined in Section 1.6.6. They are elements of the bilune-incidence algebra.

Lemma 14.62. *For q not a root of unity, and a comonoid c, the map of species $\mathsf{c} \to \mathcal{T}_q(\mathsf{c}_t)$ given on the A-component by*

$$\mathsf{c}[A] \to \bigoplus_{H:\, H \geq A} \mathsf{c}[H], \qquad x \mapsto \sum_{\substack{H, H' \geq A \\ \mathrm{s}(H) = \mathrm{s}(H')}} \boldsymbol{\zeta}_q(A, H, H')\, \beta_{H,H'} \Delta_A^{H'}(x) \tag{14.42}$$

is a morphism of comonoids.

First proof. We proceed as in the first proof of Lemma 14.39. For $x \in \mathsf{c}[A]$,

$$\begin{aligned}
\Delta_A^G f_A(x) &= \Delta_A^G \Big(\sum_{\substack{H, H' \geq A \\ \mathrm{s}(H) = \mathrm{s}(H')}} \boldsymbol{\zeta}_q(A, H, H')\, \beta_{H,H'} \Delta_A^{H'}(x) \Big) \\
&= \sum_{\substack{H, H' \geq A \\ \mathrm{s}(H) = \mathrm{s}(H') \geq \mathrm{s}(G)}} \boldsymbol{\zeta}_q(A, H, H')\, q^{\mathrm{dist}(H,G)} \beta_{GH,H'} \Delta_A^{H'}(x) \\
&= \sum_{K \geq G} \sum_{\substack{H, H' \geq A,\, GH = K \\ \mathrm{s}(H) = \mathrm{s}(H') \geq \mathrm{s}(G)}} \boldsymbol{\zeta}_q(A, H, H')\, q^{\mathrm{dist}(H,G)} \beta_{K,H'} \Delta_A^{H'}(x) \\
&= \sum_{\substack{K \geq G,\, H' \geq A \\ \mathrm{s}(K) = \mathrm{s}(H')}} \Big(\sum_{\substack{H \geq A,\, GH = K \\ \mathrm{s}(H) = \mathrm{s}(K)}} \boldsymbol{\zeta}_q(A, H, H')\, q^{\mathrm{dist}(H,K)} \Big) \beta_{K,H'} \Delta_A^{H'}(x)
\end{aligned}$$

$$= \sum_{\substack{K,H' \geq G \\ \mathrm{s}(K)=\mathrm{s}(H')}} \boldsymbol{\zeta}_q(G,K,H')\, \beta_{K,H'} \Delta_A^{H'}(x)$$

$$= f_G \Delta_A^G(x).$$

Note the use of the two-sided q-lune-additivity formula (1.66). □

Second proof. Observe that the map (14.42) is the q-exponential (9.29a) of the canonical inclusion $\mathsf{c} \hookrightarrow \mathcal{T}_q(\mathsf{c}_t)$. Further, by Exercise 6.33, this inclusion maps into the primitive part of $\mathcal{T}_q(\mathsf{c}_t)$, so it is a coderivation. Hence, by Theorem 9.78, we deduce that (14.42) is a morphism of comonoids. □

Proposition 14.63. *For q not a root of unity, and a comonoid c, the map of species $\mathcal{T}_q(\mathsf{c}) \to \mathcal{T}_q(\mathsf{c}_t)$ given on the A-component by*

$$\bigoplus_{F\colon F \geq A} \mathsf{c}[F] \to \bigoplus_{H\colon H\geq A} \mathsf{c}[H], \qquad x \mapsto \sum_{\substack{H,H' \geq F \\ \mathrm{s}(H)=\mathrm{s}(H')}} \boldsymbol{\zeta}_q(F,H,H')\, \beta_{H,H'} \Delta_F^{H'}(x) \tag{14.43}$$

for $x \in \mathsf{c}[F]$, is an isomorphism of q-bimonoids.

Proof. By the universal property of $\mathcal{T}_q$ (Theorem 6.6), the morphism of comonoids $\mathsf{c} \to \mathcal{T}_q(\mathsf{c}_t)$ in Lemma 14.62 extends to a morphism of q-bimonoids $\mathcal{T}_q(\mathsf{c}) \to \mathcal{T}_q(\mathsf{c}_t)$. By employing formula (6.4), we see that this indeed coincides with (14.43). Moreover, since the matrix form of (14.43) is unitriangular, it is an isomorphism. □

Lemma 14.64. *For q not a root of unity, and a comonoid c, the map of species $\mathsf{c} \to \mathcal{T}_q(\mathsf{c})$ given on the A-component by*

$$\mathsf{c}[A] \to \bigoplus_{H\colon H\geq A} \mathsf{c}[H], \qquad x \mapsto \sum_{\substack{H,H' \geq A \\ \mathrm{s}(H)=\mathrm{s}(H')}} \boldsymbol{\mu}_q(A,H,H')\, \beta_{H,H'} \Delta_A^{H'}(x) \tag{14.44}$$

is a coderivation, that is, it maps into $\mathcal{P}(\mathcal{T}_q(\mathsf{c}))$.

First proof. We proceed as in the first proof of Lemma 14.41. For $x \in \mathsf{c}[A]$ and $H > A$,

$$\Delta_A^H\Big(\sum_{\substack{F,F' \geq A \\ \mathrm{s}(F)=\mathrm{s}(F')}} \boldsymbol{\mu}_q(A,F,F')\, \beta_{F,F'} \Delta_A^{F'}(x)\Big)$$

$$= \sum_{\substack{F,F' \geq A \\ \mathrm{s}(F)=\mathrm{s}(F')}} \boldsymbol{\mu}_q(A,F,F')\, q^{\mathrm{dist}(F,H)} \beta_{HF,F'H} \Delta_A^{F'H}(x)$$

$$= \sum_{\substack{G\geq H,\, G' \geq A \\ \mathrm{s}(G)=\mathrm{s}(G')}} \Big(\sum_{\substack{F,F' \geq A \\ \mathrm{s}(F)=\mathrm{s}(F') \\ HF=G,\, F'H=G'}} \boldsymbol{\mu}_q(A,F,F')\, q^{\mathrm{dist}(F,G)}\Big) \beta_{G,G'} \Delta_A^{G'}(x)$$

$$= 0.$$

Note the use of the two-sided q-Weisner formula (1.67). □

SECOND PROOF. Observe that the map (14.44) is the q-logarithm (9.29b) of the canonical inclusion $\mathsf{c} \hookrightarrow \mathcal{T}_q(\mathsf{c})$. The latter is a morphism of comonoids by Exercise 6.7. Hence, by Theorem 9.78, we deduce that (14.44) is a coderivation. $\square$

Proposition 14.65. *For q not a root of unity, and a comonoid c, the map of species $\mathcal{T}_q(\mathsf{c}_t) \to \mathcal{T}_q(\mathsf{c})$ given on the A-component by*

$$\bigoplus_{F\colon F\geq A} \mathsf{c}[F] \to \bigoplus_{H\colon H\geq A} \mathsf{c}[H], \qquad x \mapsto \sum_{\substack{H,H'\geq F\\ \mathrm{s}(H)=\mathrm{s}(H')}} \boldsymbol{\mu}_q(F,H,H')\, \beta_{H,H'} \Delta_F^{H'}(x) \tag{14.45}$$

for $x \in \mathsf{c}[F]$, is an isomorphism of q-bimonoids.

PROOF. By the universal property of $\mathcal{T}_q$ (Theorem 6.31), the map $\mathsf{c} \to \mathcal{P}(\mathcal{T}_q(\mathsf{c}))$ in Lemma 14.64 extends to a morphism of bimonoids $\mathcal{T}_q(\mathsf{c}_t) \to \mathcal{T}_q(\mathsf{c})$. By employing formula (6.41), we see that this indeed coincides with (14.45). Moreover, since the matrix form of (14.45) is unitriangular, it is an isomorphism. $\square$

Applying the primitive part functor to the map (14.45) and using Proposition 13.75 for $\mathsf{p} = \mathsf{c}_t$, we obtain:

Proposition 14.66. *For q not a root of unity, and a comonoid c, the map $\mathsf{c} \to \mathcal{P}(\mathcal{T}_q(\mathsf{c}))$ in Lemma 14.64 is a natural isomorphism of species.*

14.5.2. Cocommutative case. We now focus on the case when c is cocommutative. For q not a root of unity, recall the noncommutative q-zeta function $\boldsymbol{\zeta}_q$ and the noncommutative q-Möbius function $\boldsymbol{\mu}_q$ defined in Section 1.5.9. They are elements of the lune-incidence algebra.

Lemma 14.67. *For q not a root of unity, and a cocommutative comonoid c, the map of species $\mathsf{c} \to \mathcal{T}_q(\mathsf{c}_t)$ given on the A-component by*

$$\mathsf{c}[A] \to \bigoplus_{H\colon H\geq A} \mathsf{c}[H], \qquad x \mapsto \sum_{H\colon H\geq A} \boldsymbol{\zeta}_q(A,H) \Delta_A^H(x) \tag{14.46}$$

is a morphism of comonoids.

PROOF. This is a specialization of Lemma 14.62. When c is cocommutative, the map (14.42) simplifies to (14.46). This follows from (1.69). The first proof takes the following simpler form.

$$\begin{aligned}
\Delta_A^G f_A(x) &= \Delta_A^G\Big(\sum_{H\colon H\geq A} \boldsymbol{\zeta}_q(A,H)\, \Delta_A^H(x)\Big) \\
&= \sum_{H\colon H\geq A,\, HG=H} \boldsymbol{\zeta}_q(A,H)\, q^{\mathrm{dist}(H,G)} \beta_{GH,H} \Delta_A^H(x) \\
&= \sum_{H\colon H\geq A,\, HG=H} \boldsymbol{\zeta}_q(A,H)\, q^{\mathrm{dist}(H,G)} \Delta_A^{GH}(x) \\
&= \sum_{K\colon K\geq G} \Big(\sum_{\substack{H\colon H\geq A,\, GH=K\\ HG=H}} \boldsymbol{\zeta}_q(A,H)\, q^{\mathrm{dist}(H,K)}\Big) \Delta_A^K(x)
\end{aligned}$$

$$= \sum_{K\colon K \geq G} \boldsymbol{\zeta}_q(G,K)\, \Delta_A^K(x)$$
$$= f_G \Delta_A^G(x).$$

Note the use of the q-lune-additivity formula (1.46). $\square$

Proposition 14.68. *For q not a root of unity, and a cocommutative comonoid c, the map of species $\mathcal{T}_q(\mathsf{c}) \to \mathcal{T}_q(\mathsf{c}_t)$ given on the A-component by*

$$\bigoplus_{F\colon F \geq A} \mathsf{c}[F] \to \bigoplus_{H\colon H \geq A} \mathsf{c}[H], \qquad x \mapsto \sum_{H\colon H \geq F} \boldsymbol{\zeta}_q(F,H)\Delta_F^H(x) \tag{14.47}$$

for $x \in \mathsf{c}[F]$, is an isomorphism of q-bimonoids.

Proof. This is a specialization of Proposition 14.63. When c is cocommutative, the map (14.43) simplifies to (14.47). $\square$

Lemma 14.69. *For q not a root of unity, and a cocommutative comonoid c, the map of species $\mathsf{c} \to \mathcal{T}_q(\mathsf{c})$ given on the A-component by*

$$\mathsf{c}[A] \to \bigoplus_{H\colon H \geq A} \mathsf{c}[H], \qquad x \mapsto \sum_{H\colon H \geq A} \boldsymbol{\mu}_q(A,H)\Delta_A^H(x) \tag{14.48}$$

is a coderivation, that is, it maps into $\mathcal{P}(\mathcal{T}_q(\mathsf{c}))$.

Proof. This is a specialization of Lemma 14.64. When c is cocommutative, the map (14.44) simplifies to (14.48). This follows from (1.69). The first proof takes the following simpler form.

$$\begin{aligned}
\Delta_A^H\Big(\sum_{F\colon F \geq A} \boldsymbol{\mu}_q(A,F)\Delta_A^F(x)\Big) &= \sum_{F\colon F \geq A} \boldsymbol{\mu}_q(A,F)\, q^{\mathrm{dist}(F,H)} \beta_{HF,FH} \Delta_F^{FH} \Delta_A^F(x) \\
&= \sum_{F\colon F \geq A} \boldsymbol{\mu}_q(A,F)\, q^{\mathrm{dist}(F,H)} \Delta_A^{HF}(x) \\
&= \sum_{G\colon G \geq H} \Big(\sum_{\substack{F\colon F \geq A \\ HF = G}} \boldsymbol{\mu}_q(A,F)\, q^{\mathrm{dist}(F,G)} \Big) \Delta_A^G(x) \\
&= 0.
\end{aligned}$$

Note the use of the noncommutative q-Weisner formula (1.48). $\square$

Proposition 14.70. *For q not a root of unity, and a cocommutative comonoid c, the map of species $\mathcal{T}_q(\mathsf{c}_t) \to \mathcal{T}_q(\mathsf{c})$ given on the A-component by*

$$\bigoplus_{F\colon F \geq A} \mathsf{c}[F] \to \bigoplus_{H\colon H \geq A} \mathsf{c}[H], \qquad x \mapsto \sum_{H\colon H \geq F} \boldsymbol{\mu}_q(F,H)\Delta_F^H(x) \tag{14.49}$$

for $x \in \mathsf{c}[F]$, is an isomorphism of q-bimonoids.

Proof. This is a specialization of Proposition 14.65. The map (14.45) simplifies to (14.49). $\square$

The map (14.48) induces an isomorphism from c onto $\mathcal{P}(\mathcal{T}_q(\mathsf{c}))$. When specialized to $\mathsf{c} := \mathsf{x}$, it yields (7.155) in view of (7.24). When specialized to $\mathsf{c} := \mathsf{E}$, it yields (7.160) in view of (7.81).

Exercise 14.71. The signed case works as follows. Check that: For a signed cocommutative comonoid c, Lemma 14.67 and Proposition 14.68 hold with $\boldsymbol{\zeta}_q$ replaced by $\boldsymbol{\zeta}_{-q}$, and similarly, Lemma 14.69 and Proposition 14.70 hold with $\boldsymbol{\mu}_q$ replaced by $\boldsymbol{\mu}_{-q}$. (Use Exercise 1.35. Also see Exercise 9.76.)

14.5.3. Cofree q-bimonoid on a monoid. There are dual results for the cofree q-bimonoid $\mathcal{T}_q^\vee(\mathsf{a})$ on a monoid a.

Proposition 14.72. *For q not a root of unity, and a monoid a, the map of species $\mathcal{T}_q^\vee(\mathsf{a}_t) \to \mathcal{T}_q^\vee(\mathsf{a})$ given on the A-component by*

$$\bigoplus_{F:\, F\geq A} \mathsf{a}[F] \to \bigoplus_{G:\, G\geq A} \mathsf{a}[G], \quad x \mapsto \sum_{\substack{G:\, A\leq G\leq F \\ F':\, F'\geq G,\, \mathrm{s}(F')=\mathrm{s}(F)}} \boldsymbol{\zeta}_q(G,F',F)\, \mu_G^{F'} \beta_{F',F}(x) \tag{14.50}$$

for $x \in \mathsf{a}[F]$, is an isomorphism of q-bimonoids.

The inverse isomorphism is as follows.

Proposition 14.73. *For q not a root of unity, and a monoid a, the map of species $\mathcal{T}_q^\vee(\mathsf{a}) \to \mathcal{T}_q^\vee(\mathsf{a}_t)$ given on the A-component by*

$$\bigoplus_{F:\, F\geq A} \mathsf{a}[F] \to \bigoplus_{G:\, G\geq A} \mathsf{a}[G], \quad x \mapsto \sum_{\substack{G:\, A\leq G\leq F \\ F':\, F'\geq G,\, \mathrm{s}(F')=\mathrm{s}(F)}} \boldsymbol{\mu}_q(G,F',F)\, \mu_G^{F'} \beta_{F',F}(x) \tag{14.51}$$

for $x \in \mathsf{a}[F]$, is an isomorphism of q-bimonoids.

The maps simplify as follows when a is commutative.

Proposition 14.74. *For q not a root of unity, and a commutative monoid a, the map of species $\mathcal{T}_q^\vee(\mathsf{a}_t) \to \mathcal{T}_q^\vee(\mathsf{a})$ given on the A-component by*

$$\bigoplus_{F:\, F\geq A} \mathsf{a}[F] \to \bigoplus_{G:\, G\geq A} \mathsf{a}[G], \qquad x \mapsto \sum_{G:\, A\leq G\leq F} \boldsymbol{\zeta}_q(G,F)\, \mu_G^F(x) \tag{14.52}$$

for $x \in \mathsf{a}[F]$, is an isomorphism of q-bimonoids.

Proposition 14.75. *For q not a root of unity, and a commutative monoid a, the map of species $\mathcal{T}_q^\vee(\mathsf{a}) \to \mathcal{T}_q^\vee(\mathsf{a}_t)$ given on the A-component by*

$$\bigoplus_{F:\, F\geq A} \mathsf{a}[F] \to \bigoplus_{G:\, G\geq A} \mathsf{a}[G], \qquad x \mapsto \sum_{G:\, A\leq G\leq F} \boldsymbol{\mu}_q(G,F)\, \mu_G^F(x) \tag{14.53}$$

for $x \in \mathsf{a}[F]$, is an isomorphism of q-bimonoids.

The above results also hold for a signed commutative monoid a with $\boldsymbol{\zeta}_q$ replaced by $\boldsymbol{\zeta}_{-q}$, and $\boldsymbol{\mu}_q$ replaced by $\boldsymbol{\mu}_{-q}$.

14.5.4. HNR isomorphisms for q-bimonoids. We collectively refer to the maps (14.43), (14.45), and their duals (14.50), (14.51) as the *Hoffman–Newman–Radford isomorphisms* for q-bimonoids. In the commutative case, these specialize to (14.47), (14.49), and their duals (14.52), (14.53). They are natural in c and a, that is, diagrams (14.8) commute with $\mathcal{T}_q$ in place of $\mathcal{T}_0$.

14.5.5. Examples.

Example 14.76. Recall the q-bimonoid of faces Σ_q from Section 7.6. View the exponential species E as a comonoid. Let E_t denote E with the trivial coproduct. For q not a root of unity, the following is a commutative diagram of q-bimonoids.

$$\begin{tikzcd} \mathcal{T}_q(\mathsf{E}) \arrow[rr, "\cong"] & & \mathcal{T}_q(\mathsf{E}_t) \\ & \Sigma_q \arrow[ul, "\cong"] \arrow[ur, "\cong"'] & \end{tikzcd}$$

The horizontal map is the HNR isomorphism (14.47) specialized to $\mathsf{c} := \mathsf{E}$. The map going up to the left is the first map in (7.81), while the one going up to the right is the first map in (7.83). They involve the $\mathtt{H}$-basis and $\mathtt{Q}$-basis, respectively, of Σ_q, see (7.70).

Similarly, one can formulate the dual statement for Σ_q^* by specializing (14.52) to $\mathsf{a} := \mathsf{E}$.

Exercise 14.77. Recall the q-bimonoid of top-nested faces $\widehat{\mathsf{Q}}_q$ from Section 7.7. Formulate the analogue of Example 14.76 for $\widehat{\mathsf{Q}}_q$ for q not a root of unity. (Employ the isomorphisms (7.106) and (7.107).) Moreover, use naturality of the HNR isomorphisms to deduce the first part of Exercise 7.52.

Exercise 14.78. Recall the q-bimonoid of bifaces J_q from Section 7.8. Formulate the analogue of Example 14.76 for J_q for q not a root of unity. (Employ the isomorphisms (7.148) and (7.149).)

14.5.6. Conjugation by HNR isomorphisms. For q not a root of unity, for any comonoid c and monoid a, suppose we are given a morphism of q-bimonoids from $\mathcal{T}_q(\mathsf{c})$ to $\mathcal{T}_q^\vee(\mathsf{a})$. It can be conjugated by the HNR isomorphisms to yield a morphism from $\mathcal{T}_q(\mathsf{c}_t)$ to $\mathcal{T}_q^\vee(\mathsf{a}_t)$ which are simpler objects. This can be viewed as a change of basis that simplifies the description of the given morphism. More precisely, we have bijections

$$q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{T}_q(\mathsf{c}), \mathcal{T}_q^\vee(\mathsf{a})) \rightleftarrows q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{T}_q(\mathsf{c}_t), \mathcal{T}_q^\vee(\mathsf{a}_t)) \tag{14.54}$$

given by

$$\begin{aligned} (\mathcal{T}_q(\mathsf{c}) \xrightarrow{g} \mathcal{T}_q^\vee(\mathsf{a})) &\longmapsto (\mathcal{T}_q(\mathsf{c}_t) \xrightarrow{\cong} \mathcal{T}_q(\mathsf{c}) \xrightarrow{g} \mathcal{T}_q^\vee(\mathsf{a}) \xrightarrow{\cong} \mathcal{T}_q^\vee(\mathsf{a}_t)) \\ (\mathcal{T}_q(\mathsf{c}_t) \xrightarrow{h} \mathcal{T}_q^\vee(\mathsf{a}_t)) &\longmapsto (\mathcal{T}_q(\mathsf{c}) \xrightarrow{\cong} \mathcal{T}_q(\mathsf{c}_t) \xrightarrow{h} \mathcal{T}_q^\vee(\mathsf{a}_t) \xrightarrow{\cong} \mathcal{T}_q^\vee(\mathsf{a})), \end{aligned}$$

with the isomorphisms being the HNR isomorphisms.

Exercise 14.79. Suppose q is not a root of unity. Check that: For a comonoid c and monoid a, the following diagram of bijections commutes.

$$\begin{tikzcd} q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{T}_q(\mathsf{c}), \mathcal{T}_q^\vee(\mathsf{a})) \arrow[rr, shift left] \arrow[dr, shift left, "\mathbf{log}_q"] & & q\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{T}_q(\mathsf{c}_t), \mathcal{T}_q^\vee(\mathsf{a}_t)) \arrow[ll, shift left] \arrow[dl, shift right, "\mathbf{log}_q"'] \\ & \mathcal{A}\text{-}\mathsf{Sp}(\mathsf{c}, \mathsf{a}) \arrow[ul, shift left, "\mathbf{exp}_q"] \arrow[ur, shift right, "\mathbf{exp}_q"'] & \end{tikzcd}$$

The horizontal bijections are as in (14.54), while the q-exp-log correspondences are instances of (9.34).

14.5.7. *q*-norm transformation. We now apply the above ideas to the q-norm transformation in Section 6.9.

Exercise 14.80. Recall the q-norm map $(\kappa_q)_{\mathsf{p}}$ from (6.74). For a comonoid c, the map (6.80) equals the composite

$$\mathcal{T}_q(\mathsf{c}) \xrightarrow{\cong} \mathcal{T}_q(\mathsf{c}_t) \xrightarrow{(\kappa_q)_{\mathsf{c}_t}} \mathcal{T}_q^\vee(\mathsf{c}_t),$$

where the first map is the HNR isomorphism (14.43). Dually, for a monoid a, the map (6.81) equals the composite

$$\mathcal{T}_q(\mathsf{a}_t) \xrightarrow{(\kappa_q)_{\mathsf{a}_t}} \mathcal{T}_q^\vee(\mathsf{a}_t) \xrightarrow{\cong} \mathcal{T}_q^\vee(\mathsf{a}),$$

where the second map is the HNR isomorphism (14.50).

Check these facts either directly or by applying Exercise 14.79 to Example 9.92 and Exercise 9.94. Deduce Propositions 6.84 and 6.85 as a consequence.

Exercise 14.81. Suppose q is not a root of unity. Let maps $f : \mathsf{c} \to \mathsf{a}$ and $g : \mathcal{T}_q(\mathsf{c}) \to \mathcal{T}_q^\vee(\mathsf{a})$ correspond to each other under the bijection (6.77). Use Exercise 14.79 and Exercise 9.97 to deduce that the composite map

$$\mathcal{T}_q(\mathsf{c}_t) \xrightarrow{\cong} \mathcal{T}_q(\mathsf{c}) \xrightarrow{g} \mathcal{T}_q^\vee(\mathsf{a}) \xrightarrow{\cong} \mathcal{T}_q^\vee(\mathsf{a}_t) \tag{14.55}$$

is the q-exponential (9.29a) of the biderivation

$$\begin{tikzcd} \mathcal{T}_q(\mathsf{c}_t) \arrow[r, dashed] \arrow[d, two heads] & \mathcal{T}_q^\vee(\mathsf{a}_t) \\ \mathsf{c} \arrow[r, "\mathbf{log}_q(f)"'] & \mathsf{a}. \arrow[u, hook] \end{tikzcd}$$

If either c is a trivial comonoid or a is a trivial monoid, then $\mathbf{log}_q(f)$ can be replaced by f.

Exercise 14.82. Specialize Exercise 14.81 to the maps $f : \mathsf{E} \to \mathsf{E}$ and $g : \Sigma_q \to \Sigma_q^*$ in Exercise 7.37. Use Example 14.76 and Exercise 9.98 to deduce formula (7.89) for the map $\Sigma_q \to \Sigma_q^*$.

14.5.8. Mixed distributive law for *q*-bimonoids. For q not a root of unity, we now rewrite the mixed distributive law (3.7) for q-bimonoids by conjugating it with the HNR isomorphisms.

Exercise 14.83. Specialize Exercise 14.81 to the maps $f : \mathcal{T}^\vee(\mathsf{p}) \to \mathcal{T}(\mathsf{p})$ and $g : \mathcal{T}_q(\mathcal{T}^\vee(\mathsf{p})) \to \mathcal{T}_q^\vee(\mathcal{T}(\mathsf{p}))$ in Exercise 6.82, item (1). Use Exercise 9.93 and the formula for $\mathbf{log}_q(f)$ in Exercise 9.99 to deduce that the composite map (14.55)

$$\mathcal{T}_q(\mathcal{T}^\vee(\mathsf{p})_t) \to \mathcal{T}_q^\vee(\mathcal{T}(\mathsf{p})_t)$$

on the A-component,

$$\bigoplus_{(F,G):\, A \leq F \leq G} \mathsf{p}[G] \to \bigoplus_{(F',G'):\, A \leq F' \leq G'} \mathsf{p}[G']$$

is $q^{\mathrm{dist}(F,F')}\boldsymbol{\mu}_q(F,G,FG')\beta_{G',G}$ on the matrix-component for which $\mathrm{s}(F) = \mathrm{s}(F')$ and $\mathrm{s}(G) = \mathrm{s}(G')$, and 0 on the remaining matrix-components.

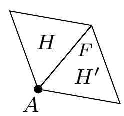
For $\mathsf{p} := \mathsf{x}$, the above map specializes to $\widehat{\mathsf{Q}}_q \to \widehat{\mathsf{Q}}_q^*$ given in (7.109).

Exercise 14.84. Write down the analogue of Exercise 14.83 for the maps in Exercise 6.82, item (2). (Use the formula for $\mathbf{log}_q(\mathrm{id})$ in Exercise 9.101.)

14.5.9. Action of the bilune-incidence algebra. For any comonoid c, the bilune-incidence algebra $\mathrm{I}_{\mathrm{bilune}}[\mathcal{A}]$ acts on the right on the species $\mathcal{T}(\mathsf{c})$ as follows. For each face A, the right action of $s \in \mathrm{I}_{\mathrm{bilune}}[\mathcal{A}]$ on the A-component of $\mathcal{T}(\mathsf{c})$ is given by

$$\bigoplus_{F\colon F \geq A} \mathsf{c}[F] \to \bigoplus_{H\colon H\geq A} \mathsf{c}[H], \qquad x \mapsto \sum_{\substack{H,H' \geq F\\ \mathrm{s}(H)=\mathrm{s}(H')}} s(F,H,H')\,\beta_{H,H'}\Delta_F^{H'}(x) \tag{14.56}$$

for $x \in \mathsf{c}[F]$. An illustrative picture is shown below.



Observe that under the algebra morphism i in (1.62), the induced action of the lune-incidence algebra on $\mathcal{T}(\mathsf{c})$ is precisely (14.34). Similarly, when c is cocommutative, the action (14.56) factors through the algebra morphism p in (1.60) to again yield the action (14.34).

Example 14.85. The maps (14.43) and (14.45) are instances of (14.56) for $s := \boldsymbol{\zeta}_q$ and $s := \boldsymbol{\mu}_q$, respectively. It follows that they are inverses of each other. Hence, Propositions 14.63 and 14.65 imply each other. In the case when c is cocommutative, the maps simplify to (14.47) and (14.49) and become instances of (14.34).

Now let us specialize to $q = 0$. Using formulas (1.71), we see that for $s := \boldsymbol{\zeta}_0$ and $s := \boldsymbol{\mu}_0$, (14.56) specializes to (14.2) and (14.4), respectively.

Dually, for any monoid a, the bilune-incidence algebra $\mathrm{I}_{\mathrm{bilune}}[\mathcal{A}]$ acts on the left on the species $\mathcal{T}^\vee(\mathsf{a})$ as follows. For each face A, the left action of $s \in \mathrm{I}_{\mathrm{bilune}}[\mathcal{A}]$ on the A-component of $\mathcal{T}^\vee(\mathsf{a})$ is given by

$$\bigoplus_{F\colon F\geq A} \mathsf{a}[F] \to \bigoplus_{G\colon G \geq A} \mathsf{a}[G], \qquad x \mapsto \sum_{\substack{G\colon A \leq G \leq F\\ F'\colon F' \geq G,\, \mathrm{s}(F')=\mathrm{s}(F)}} s(G,F',F)\,\mu_G^{F'}\beta_{F',F}(x) \tag{14.57}$$

for $x \in \mathsf{a}[F]$.

Observe that under the algebra morphism i in (1.62), the induced action of the lune-incidence algebra on $\mathcal{T}^\vee(\mathsf{a})$ is precisely (14.36). Similarly, when a is commutative, the action (14.57) factors through the algebra morphism q in (1.60) to again yield the action (14.36).

Example 14.86. The maps (14.50) and (14.51) are instances of (14.57) for $s := \boldsymbol{\zeta}_q$ and $s := \boldsymbol{\mu}_q$, respectively. In the case when a is commutative, the maps simplify to (14.52) and (14.53) and become instances of (14.36).

For $s := \boldsymbol{\zeta}_0$ and $s := \boldsymbol{\mu}_0$, (14.57) specializes to (14.5) and (14.6), respectively.

14.6. Zeta and Möbius as inverses

The Hoffman–Newman–Radford rigidity theorems can be used as a theoretical tool to establish the inverse relationship between zeta and Möbius functions, in the commutative, noncommutative as well as two-sided settings. The commutative case was indicated in Example 14.33. We focus on the remaining two cases below.

14.6.1. Noncommutative zeta and Möbius functions. Theorem 1.14 says that noncommutative zeta and Möbius functions are inverses of each other in the lune-incidence algebra. Interestingly, the ideas in Section 14.4 can be used to prove this result. As pointed out in Example 14.55, Propositions 14.40 and 14.42 imply each other if we assume this result. However, we gave independent proofs for both propositions, and this can be used to prove the result.

We begin with the converses to Lemma 14.39 and Lemma 14.41.

Lemma 14.87. *Let $\boldsymbol{\zeta} \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ be such that $\boldsymbol{\zeta}(A, A) = 1$ for all A. If for every cocommutative comonoid c, the map $\mathsf{c} \to \mathcal{T}(\mathsf{c}_t)$ given by (14.25) is a morphism of comonoids, then $\boldsymbol{\zeta}$ is a noncommutative zeta function.*

Proof. Following the calculation in the first proof of Lemma 14.39, we obtain: For any cocommutative comonoid c,

$$\Big(\sum_{\substack{H:\, H \geq A,\, GH = K \\ HG = H}} \boldsymbol{\zeta}(A, H) \Big) \Delta_A^K(x) = \boldsymbol{\zeta}(G, K)\, \Delta_A^K(x)$$

for $A \leq G \leq K$, and $x \in \mathsf{c}[A]$. By taking $\mathsf{c} := \mathsf{E}$ for which $\Delta_A^K = \mathrm{id}$, we deduce that $\boldsymbol{\zeta}$ satisfies the lune-additivity formula (1.42), and hence is a noncommutative zeta function. $\square$

Lemma 14.88. *Let $\boldsymbol{\mu} \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ be such that $\boldsymbol{\mu}(A, A) = 1$ for all A. If for every cocommutative comonoid c, the map $\mathsf{c} \to \mathcal{T}(\mathsf{c})$ given by (14.27) is a coderivation, then $\boldsymbol{\mu}$ is a noncommutative Möbius function.*

Proof. Following the calculation in the first proof of Lemma 14.41, we obtain: For any cocommutative comonoid c,

$$\Big(\sum_{\substack{F:\, F \geq A \\ HF = G}} \boldsymbol{\mu}(A, F) \Big) \Delta_A^G(x) = 0$$

for $A < H \leq G$, and $x \in \mathsf{c}[A]$. By taking $\mathsf{c} := \mathsf{E}$ for which $\Delta_A^G = \mathrm{id}$, we deduce that $\boldsymbol{\mu}$ satisfies the noncommutative Weisner formula (1.44), and hence is a noncommutative Möbius function. $\square$

Theorem 1.14 is restated below for convenience.

Theorem 14.89. *Let $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ be mutually inverse elements of the lune-incidence algebra. Then $\boldsymbol{\zeta}$ is a noncommutative zeta function iff $\boldsymbol{\mu}$ is a noncommutative Möbius function.*

Proof. Let $\boldsymbol{\zeta}$ be a noncommutative zeta function. For any cocommutative comonoid c, by Proposition 14.40, we have the isomorphism $\mathcal{T}(\mathsf{c}) \to \mathcal{T}(\mathsf{c}_t)$ of bimonoids given by (14.26). By inverting, we see that (14.28) is an isomorphism of bimonoids $\mathcal{T}(\mathsf{c}_t) \to \mathcal{T}(\mathsf{c})$ for $\boldsymbol{\mu} := \boldsymbol{\zeta}^{-1}$, and hence maps c into $\mathcal{P}(\mathcal{T}(\mathsf{c}))$. Lemma 14.88 now implies that $\boldsymbol{\mu}$ is a noncommutative Möbius function, as required.

Conversely, let $\boldsymbol{\mu}$ be a noncommutative Möbius function. For any cocommutative comonoid c, by Proposition 14.42, we have the isomorphism $\mathcal{T}(\mathsf{c}_t) \to \mathcal{T}(\mathsf{c})$ of bimonoids given by (14.28). By inverting, we see that (14.26) is an isomorphism of bimonoids $\mathcal{T}(\mathsf{c}) \to \mathcal{T}(\mathsf{c}_t)$ for $\boldsymbol{\zeta} := \boldsymbol{\mu}^{-1}$, and hence $\mathsf{c} \to \mathcal{T}(\mathsf{c}_t)$ is a morphism of comonoids. Lemma 14.87 now implies that $\boldsymbol{\zeta}$ is a noncommutative zeta function, as required. □

14.6.2. Two-sided q-zeta and q-Möbius functions.

A similar strategy can be used to prove the inverse relationship between the two-sided q-zeta and q-Möbius functions. We begin with the converses to Lemma 14.62 and Lemma 14.64.

Lemma 14.90. *For q not a root of unity, let $\boldsymbol{\zeta}_q \in \mathrm{I}_{\mathrm{bilune}}[\mathcal{A}]$ be such that $\boldsymbol{\zeta}_q(A,A,A) = 1$ for all A. If for every comonoid c, the map $\mathsf{c} \to \mathcal{T}_q(\mathsf{c}_t)$ given by (14.42) is a morphism of comonoids, then $\boldsymbol{\zeta}_q$ is the two-sided q-zeta function.*

Proof. Following the calculation in the first proof of Lemma 14.62, we obtain: For any comonoid c,

$$\sum_{\substack{H' \geq A \\ \mathrm{s}(K)=\mathrm{s}(H')}} \Big(\sum_{\substack{H \geq A,\, GH=K \\ \mathrm{s}(H)=\mathrm{s}(K)}} \boldsymbol{\zeta}_q(A,H,H')\, q^{\mathrm{dist}(H,K)} \Big) \beta_{K,H'} \Delta_A^{H'}(x) = \sum_{\substack{H' \geq G \\ \mathrm{s}(K)=\mathrm{s}(H')}} \boldsymbol{\zeta}_q(G,K,H')\, \beta_{K,H'} \Delta_A^{H'}(x)$$

for $A \leq G \leq K$, and $x \in \mathsf{c}[A]$. For any $H' \geq A$ of the same support as K, by choosing a comonoid c such that $\Delta_A^{H'} = \mathrm{id}$ while $\Delta_A^{H''} = 0$ for any $H'' > A$ with $H'' \neq H'$, we deduce that $\boldsymbol{\zeta}_q$ satisfies the two-sided q-lune-additivity formula (1.66), and hence is the two-sided q-zeta function. □

Lemma 14.91. *For q not a root of unity, let $\boldsymbol{\mu}_q \in \mathrm{I}_{\mathrm{bilune}}[\mathcal{A}]$ be such that $\boldsymbol{\mu}_q(A,A,A) = 1$ for all A. If for every comonoid c, the map $\mathsf{c} \to \mathcal{T}_q(\mathsf{c})$ given by (14.44) is a coderivation, then $\boldsymbol{\mu}_q$ is the two-sided q-Möbius function.*

Proof. Following the calculation in the first proof of Lemma 14.64, we obtain: For any comonoid c,

$$\sum_{\substack{G' \geq A \\ \mathrm{s}(G)=\mathrm{s}(G')}} \Big(\sum_{\substack{F,F' \geq A \\ \mathrm{s}(F)=\mathrm{s}(F') \\ HF=G,\, F'H=G'}} \boldsymbol{\mu}_q(A,F,F')\, q^{\mathrm{dist}(F,G)} \Big) \beta_{G,G'} \Delta_A^{G'}(x) = 0$$

for $A < H \leq G$, and $x \in \mathsf{c}[A]$. For any G' as above, by choosing a comonoid c such that $\Delta_A^{G'} = \mathrm{id}$ while $\Delta_A^{G''} = 0$ for $G'' > A$ with $G'' \neq G'$, we deduce that $\boldsymbol{\mu}_q$ satisfies the two-sided q-Weisner formula (1.67), and hence is the two-sided q-Möbius function. □

Theorem 14.92. *Let $\boldsymbol{\zeta}_q$ and $\boldsymbol{\mu}_q$ be mutually inverse elements of the bilune-incidence algebra. Then $\boldsymbol{\zeta}_q$ satisfies the conditions of Lemma 1.33 iff $\boldsymbol{\mu}_q$ satisfies the conditions of Lemma 1.34.*

Proof. Let $\boldsymbol{\zeta}_q$ satisfy the conditions of Lemma 1.33. Then Proposition 14.63 holds. By Example 14.85, Proposition 14.65 holds for $\boldsymbol{\mu}_q := \boldsymbol{\zeta}_q^{-1}$. Hence, by Lemma 14.91, $\boldsymbol{\mu}_q$ satisfies the conditions of Lemma 1.34, as required.

Conversely, let $\boldsymbol{\mu}_q$ satisfy the conditions of Lemma 1.34. Then Proposition 14.65 holds. By Example 14.85, Proposition 14.63 holds for $\boldsymbol{\zeta}_q := \boldsymbol{\mu}_q^{-1}$. Hence, by Lemma 14.90, $\boldsymbol{\zeta}_q$ satisfies the conditions of Lemma 1.33, as required. □

Exercise 14.93. Prove Lemma 1.19 using the same method as above by working with the case when c is cocommutative discussed in Section 14.5.2. (Begin with the converses to Lemma 14.67 and Lemma 14.69.)

Notes

The terminology of this chapter is motivated by classical results of Newman and Radford, and independently of Hoffman. More details are given below.

Bialgebras. The classical analogue of Proposition 14.13 pertaining to bicommutative bimonoids and of Proposition 14.40 pertaining to cocommutative bimonoids appears (somewhat implicitly) in Newman's thesis [**716**, Theorem 2.18], [**717**, Theorem 2.16]. In these results, Newman also deals with complications that arise in the case of positive characteristic. The analogue of the special case in Example 14.51 is also given by Ditters [**253**, Theorem 3.5.2, item (a)], [**255**, Theorem 3.3] with explicit use of the exponential map. Ditters' formula is equivalent, up to a normalization factor, to the change of basis formula relating the complete noncommutative symmetric functions and the power sums of the second kind considered in [**347**, Formula (25)]. Also see the discussion on [**21**, page 509].

The classical analogue of the dual Proposition 14.44 pertaining to commutative bimonoids appears explicitly in work of Newman and Radford [**718**, Theorem 1.12]. This result also appears independently in later work of Hoffman [**449**, Theorems 2.5 and 3.3]. Moreover, Hoffman makes explicit use of the classical exponential and logarithm (which is analogous to what we do). He also states the dual result in [**449**, Theorem 4.1] which can be viewed as the analogue of Propositions 14.40 and 14.42.

For later references, see [**639**, Theorem 3], [**273**, Definition 3.14], [**309**, Lemma 5.2], [**168**, Theorem 1], [**55**, Proposition 5.1], [**698**, Proposition 2], [**174**, Theorem 2, item (2)].

q-bialgebras. The HNR isomorphisms for q-bimonoids for q not a root of unity are new even from a classical perspective. However, as explained in the Notes to Chapter 6, Hoffman does consider the analogue of the isomorphism (6.81). Its connection to the HNR isomorphism is explained in Exercise 14.80.

Action of formal power series. Hoffman and Ihara define an action of the monoid of formal power series on the quasishuffle algebra $\mathcal{T}^\vee(A)$ when A is commutative, see [**450**, Theorem 3.1]. This is the classical analogue of the action (14.36) of the lune-incidence algebra on $\mathcal{T}^\vee(\mathsf{a})$ when a is commutative. (The connection between formal power series and the lune-incidence algebra is explained in Lemma 9.139.) In [**450**, Section 3.1], they consider many different linear operators arising from specific power series. The specializations (14.5) and (14.6) correspond to their maps Σ and Σ^{-1}, (14.29) and (14.30) correspond to their exp and log, (6.49) for $\alpha = -1$ corresponds to their T. The analogue of Exercise 14.57 pertaining to $\mathcal{T}^\vee(\mathsf{a})$ is [**450**, Theorem 3.2]. Similarly, the analogue of Exercise 14.58 pertaining to $\mathcal{T}^\vee(\mathsf{a})$ is [**450**, Corollary 3.3]. Their [**450**, Corollary 3.2] is a manifestation of the noncommutative Zaslavsky formula, also see Example 9.144 in this regard. The analogue of Exercise 14.59 pertaining to $\mathcal{T}_0^\vee(\mathsf{a}_t)$ is [**450**, Theorem 4.1], see also the Notes to Chapter 9 under 0-bialgebras. The analogue of Exercise 14.60 pertaining to monoids is [**450**, Proposition 4.3].

Some of the above operators had appeared in earlier work of Ihara, Kajikawa, Ohno, Okuda [**476**]. For instance, the analogue of Exercise 14.58 pertaining to $\mathcal{T}^\vee(\mathsf{a})$ is given in [**476**, Theorem 1] (as an algebra isomorphism). Similarly, the analogue of the map (14.37) was considered by Yamamoto [**921**, Proposition 3.3] with notation S^α. Hoffman and Ihara consider it with notation Σ^α.

For additional references on the action of formal power series, see [**920**], [**168**, Lemma 5].

Bimonoids for hyperplane arrangements. The Hoffman–Newman–Radford rigidity theorems for arrangements presented in this chapter are new and appear here for the first time. The proof of Theorem 14.89 (about the inverse relationship between zeta and Möbius) is closely related to the proof sketched in [**21**, Section 15.3.3]. The main difference is that the latter avoids explicit use of bimonoids. We mention in passing that the results of Section 14.5 can be extended to υ-bimonoids for any generic distance function υ arising from weights on half-spaces (1.32). (Generic means that no nontrivial product of weights is 1.)

CHAPTER 15

Freeness under Hadamard products

The Hadamard product on species was introduced in Chapter 8. We now study it further. The Hadamard product of two free monoids is again free. Similarly, the Hadamard product of two free commutative monoids is again free commutative. In either case, we give an explicit formula for a basis of the Hadamard product in terms of bases of its two factors. It involves the meet operation on faces and flats, respectively. We also show that the Hadamard product of bimonoids is free as a monoid if one of the two factors is free as a monoid.

We study in detail the Hadamard product of the free bimonoid on a comonoid with the cofree bimonoid on a monoid. It is neither commutative nor cocommutative, so Borel–Hopf does not apply. This bimonoid is both free and cofree. Interestingly, we prove this using Loday–Ronco (which is a theorem about 0-bimonoids). We also give a cancellation-free formula for its antipode. We give an explicit description of its primitive part, and more generally, its primitive filtration. An illustrative example of this construction is the bimonoid of pairs of chambers. We give a parallel discussion for a commutative counterpart where we take the Hadamard product of the free commutative bimonoid on a cocommutative comonoid with the cofree cocommutative bimonoid on a commutative monoid. (Since this bimonoid is bicommutative, it can also be tackled using Leray–Samelson.)

15.1. Freeness under Hadamard products

Recall from Section 8.1.2 that the Hadamard product on species preserves monoids. We now show that it also preserves free monoids. When there is given an isomorphism of monoids $\mathsf{a} \cong \mathcal{T}(\mathsf{q})$, we say that the species q is a *basis* of the free monoid a. For free monoids a and b, we provide an explicit formula for a basis of $\mathsf{a} \times \mathsf{b}$ in terms of the bases of a and b. There are similar results for free (signed) commutative monoids.

15.1.1. Hadamard product of free monoids. Let p and q be two species. Define a new species $\mathsf{p} \star \mathsf{q}$ by

$$(\mathsf{p} \star \mathsf{q})[A] := \bigoplus_{F \wedge G = A} \mathsf{p}[F] \otimes \mathsf{q}[G]. \tag{15.1}$$

The sum is over all pairs (F, G) of faces whose meet is the face A. This defines a nonunital symmetric monoidal structure on the category of species $\mathcal{A}\text{-}\mathsf{Sp}$.

Lemma 15.1. *For any species* p *and* q*, there is a natural isomorphism of monoids*

$$\mathcal{T}(\mathsf{p} \star \mathsf{q}) \cong \mathcal{T}(\mathsf{p}) \times \mathcal{T}(\mathsf{q}).$$

In particular, the Hadamard product of free monoids is again free.

PROOF. Using definitions (6.1) and (15.1),

$$\begin{aligned}
\mathcal{T}(\mathsf{p} \star \mathsf{q})[A] &= \bigoplus_{H \geq A} (\mathsf{p} \star \mathsf{q})[H] \\
&= \bigoplus_{H \geq A} \bigoplus_{F \wedge G = H} \mathsf{p}[F] \otimes \mathsf{q}[G] \\
&= \bigoplus_{F \geq A,\, G \geq A} \mathsf{p}[F] \otimes \mathsf{q}[G] \\
&= \mathcal{T}(\mathsf{p})[A] \otimes \mathcal{T}(\mathsf{q})[A] \\
&= \big(\mathcal{T}(\mathsf{p}) \times \mathcal{T}(\mathsf{q})\big)[A].
\end{aligned}$$

Moreover, it is clear from definitions (6.3) and (8.3) that this is an isomorphism of monoids. □

Recall from Section 6.4.3 that any free monoid $\mathcal{T}(\mathsf{p})$ equipped with the deconcatenation coproduct carries the structure of a 0-bimonoid and is denoted $\mathcal{T}_0(\mathsf{p})$. Further, recall from Lemma 8.1 that the Hadamard product preserves 0-bimonoids. Lemma 15.1 can be extended as follows.

Proposition 15.2. *For any species* p *and* q*, there is a natural isomorphism of* 0*-bimonoids*

$$\mathcal{T}_0(\mathsf{p} \star \mathsf{q}) \cong \mathcal{T}_0(\mathsf{p}) \times \mathcal{T}_0(\mathsf{q}).$$

PROOF. Using definitions (6.44) and (8.5), let us check that the above identification commutes with Δ_A^K for any $A \leq K$. Following the notations of the previous proof, we consider two cases.

- $\underline{H \not\geq K}$. In this case, Δ_A^K applied to $(\mathsf{p} \star \mathsf{q})[H]$ is zero. Also, since $F \wedge G = H$, both F and G cannot be greater than K. So Δ_A^K applied to either $\mathsf{p}[F]$ or $\mathsf{q}[G]$ is zero.
- $\underline{H \geq K}$. In this case, Δ_A^K identifies both $(\mathsf{p} \star \mathsf{q})[H]$ and its image in $\mathcal{T}(\mathsf{p})[A] \otimes \mathcal{T}(\mathsf{q})[A]$ to the corresponding summands in the K-component.

This completes the check. □

In view of the Loday–Ronco Theorem 13.2, Proposition 15.2 is equivalent to the following statement (for which we also provide a direct proof).

Proposition 15.3. *For any species* p *and* q*,*

$$\mathsf{p} \star \mathsf{q} = \mathcal{P}(\mathcal{T}_0(\mathsf{p}) \times \mathcal{T}_0(\mathsf{q})).$$

PROOF. The coproduct of $\mathcal{T}_0(\mathsf{p}) \times \mathcal{T}_0(\mathsf{q})$ is as follows. For $A \leq G$,

$$\big(\Delta_A^G : \mathsf{p}[H_1] \otimes \mathsf{q}[H_2] \to \mathsf{p}[K_1] \otimes \mathsf{q}[K_2]\big) = \begin{cases} \mathrm{id} & \text{if } K_1 = H_1 \text{ and } K_2 = H_2, \\ 0 & \text{otherwise,} \end{cases}$$

for each $A \le H_1, H_2$ and $G \le K_1, K_2$. The main observation is that if $H_1 \wedge H_2 = A$, then $\Delta_A^G = 0$ on the component $\mathsf{p}[H_1] \otimes \mathsf{q}[H_2]$ for all $G > A$. So such summands belong entirely to the primitive part. It is also easy to see that a primitive element cannot have a nonzero component in any of the remaining summands. $\square$

15.1.2. Hadamard product of free commutative monoids. The above discussion has a commutative counterpart. For this, it is convenient to work with the formulation of species given in Proposition 2.5.

Let p and q be two species. Define a new species $\mathsf{p} \diamond \mathsf{q}$ by

$$(\mathsf{p} \diamond \mathsf{q})[\mathrm{Z}] := \bigoplus_{\mathrm{X} \wedge \mathrm{Y} = \mathrm{Z}} \mathsf{p}[\mathrm{X}] \otimes \mathsf{q}[\mathrm{Y}]. \tag{15.2}$$

The sum is over all pairs (X, Y) of flats whose meet is Z. This defines a symmetric monoidal structure on the category of species $\mathcal{A}$-Sp. The unit object is x, the species characteristic of chambers defined by (7.3).

Recall that the free commutative monoid on any species p, denoted $\mathcal{S}(\mathsf{p})$, carries the structure of a bicommutative bimonoid. Explicitly, its product and coproduct are given by (6.51).

Proposition 15.4. *For any species p and q, there is a natural isomorphism of bicommutative bimonoids*

$$\mathcal{S}(\mathsf{p} \diamond \mathsf{q}) \cong \mathcal{S}(\mathsf{p}) \times \mathcal{S}(\mathsf{q}).$$

In particular, the Hadamard product of free commutative monoids is again free commutative.

Proof. Using definitions (6.18) and (15.2),

$$\begin{aligned}
\mathcal{S}(\mathsf{p} \diamond \mathsf{q})[\mathrm{Z}] &= \bigoplus_{\mathrm{Z}' \ge \mathrm{Z}} (\mathsf{p} \diamond \mathsf{q})[\mathrm{Z}'] \\
&= \bigoplus_{\mathrm{Z}' \ge \mathrm{Z}} \bigoplus_{\mathrm{X} \wedge \mathrm{Y} = \mathrm{Z}'} \mathsf{p}[\mathrm{X}] \otimes \mathsf{q}[\mathrm{Y}] \\
&= \bigoplus_{\mathrm{X} \ge \mathrm{Z},\, \mathrm{Y} \ge \mathrm{Z}} \mathsf{p}[\mathrm{X}] \otimes \mathsf{q}[\mathrm{Y}] \\
&= \mathcal{S}(\mathsf{p})[\mathrm{Z}] \otimes \mathcal{S}(\mathsf{q})[\mathrm{Z}] \\
&= \big(\mathcal{S}(\mathsf{p}) \times \mathcal{S}(\mathsf{q})\big)[\mathrm{Z}].
\end{aligned}$$

The fact that this is an isomorphism of monoids is clear. The argument for the coproduct is similar to the one given for $\mathcal{T}_0$ in the proof of Proposition 15.2. $\square$

In view of the Leray–Samelson Theorem 13.11, Proposition 15.4 is equivalent to the following statement.

Proposition 15.5. *For any species p and q,*

$$\mathsf{p} \diamond \mathsf{q} = \mathcal{P}(\mathcal{S}(\mathsf{p}) \times \mathcal{S}(\mathsf{q})).$$

This can also be seen directly. The argument is as in the proof of Proposition 15.3, with flats instead of faces.

There are three more companions of the $\diamond$ operation that emerge when we consider the signed setting, namely,

$$(\mathsf{p} \diamond_{-+} \mathsf{q})[\mathrm{Z}] := \bigoplus_{\mathrm{X} \wedge \mathrm{Y} = \mathrm{Z}} \mathbf{E}^-[\mathrm{Z}, \mathrm{X}] \otimes \mathsf{p}[\mathrm{X}] \otimes \mathsf{q}[\mathrm{Y}], \tag{15.3a}$$

$$(\mathsf{p} \diamond_{+-} \mathsf{q})[\mathrm{Z}] := \bigoplus_{\mathrm{X} \wedge \mathrm{Y} = \mathrm{Z}} \mathbf{E}^-[\mathrm{Z}, \mathrm{Y}] \otimes \mathsf{p}[\mathrm{X}] \otimes \mathsf{q}[\mathrm{Y}], \tag{15.3b}$$

$$(\mathsf{p} \diamond_{--} \mathsf{q})[\mathrm{Z}] := \bigoplus_{\mathrm{X} \wedge \mathrm{Y} = \mathrm{Z}} \mathbf{E}^-[\mathrm{Z}, \mathrm{X}] \otimes \mathbf{E}^-[\mathrm{Z}, \mathrm{Y}] \otimes \mathsf{p}[\mathrm{X}] \otimes \mathsf{q}[\mathrm{Y}]. \tag{15.3c}$$

The spaces $\mathbf{E}^-[\mathrm{Z}, \mathrm{X}]$ are as in Definition 1.74. In the above notation, $\mathsf{p} \diamond \mathsf{q}$ would be denoted $\mathsf{p} \diamond_{++} \mathsf{q}$.

Recall from Exercise 8.2 that the Hadamard product of a commutative monoid and a signed commutative monoid is signed commutative, while that of two signed commutative monoids is commutative. If both factors are free, then so is their Hadamard product. More precisely:

Proposition 15.6. *For any species* p *and* q*, there are natural isomorphisms of bicommutative bimonoids*

$$\mathcal{S}(\mathsf{p} \diamond_{++} \mathsf{q}) \cong \mathcal{S}(\mathsf{p}) \times \mathcal{S}(\mathsf{q}) \quad \textit{and} \quad \mathcal{S}(\mathsf{p} \diamond_{--} \mathsf{q}) \cong \mathcal{E}(\mathsf{p}) \times \mathcal{E}(\mathsf{q}),$$

and natural isomorphisms of signed bicommutative signed bimonoids

$$\mathcal{E}(\mathsf{p} \diamond_{-+} \mathsf{q}) \cong \mathcal{E}(\mathsf{p}) \times \mathcal{S}(\mathsf{q}) \quad \textit{and} \quad \mathcal{E}(\mathsf{p} \diamond_{+-} \mathsf{q}) \cong \mathcal{S}(\mathsf{p}) \times \mathcal{E}(\mathsf{q}).$$

Equivalently, by the Leray–Samelson Theorems 13.11 and 13.27, we have the following statement.

Proposition 15.7. *For any species* p *and* q*,*

$$\mathsf{p} \diamond_{++} \mathsf{q} = \mathcal{P}(\mathcal{S}(\mathsf{p}) \times \mathcal{S}(\mathsf{q})), \qquad \mathsf{p} \diamond_{--} \mathsf{q} = \mathcal{P}(\mathcal{E}(\mathsf{p}) \times \mathcal{E}(\mathsf{q})),$$
$$\mathsf{p} \diamond_{-+} \mathsf{q} = \mathcal{P}(\mathcal{E}(\mathsf{p}) \times \mathcal{S}(\mathsf{q})), \qquad \mathsf{p} \diamond_{+-} \mathsf{q} = \mathcal{P}(\mathcal{S}(\mathsf{p}) \times \mathcal{E}(\mathsf{q})).$$

The arguments are similar to the unsigned case, so we omit them.

15.1.3. Hadamard product of free partially commutative monoids. We now unify the discussion in Sections 15.1.1 and 15.1.2. Let $\sim$ be a geometric partial-support relation on faces. Let p and q be two species. Define a new species $\mathsf{p} \diamond_{\sim} \mathsf{q}$ by

$$(\mathsf{p} \diamond_{\sim} \mathsf{q})[\mathrm{z}] := \bigoplus_{\mathrm{x} \wedge \mathrm{y} = \mathrm{z}} \mathsf{p}[\mathrm{x}] \otimes \mathsf{q}[\mathrm{y}]. \tag{15.4}$$

The sum is over all pairs (x, y) of partial-flats whose meet is z. See Lemma 1.9 in this regard. The operation (15.4) specializes to (15.1) when $\sim$ is finest, and to (15.2) when $\sim$ is coarsest.

The free $\sim$-commutative monoid on any species p, denoted $\mathcal{T}_{0,\sim}(\mathsf{p})$, carries the structure of a 0-$\sim$-bicommutative bimonoid with product and coproduct given by (6.94) and (6.95).

Proposition 15.8. *For any species* p *and* q*, there is a natural isomorphism of* 0*-*$\sim$*-bicommutative bimonoids*

$$\mathcal{T}_{0,\sim}(\mathsf{p} \diamond_{\sim} \mathsf{q}) \cong \mathcal{T}_{0,\sim}(\mathsf{p}) \times \mathcal{T}_{0,\sim}(\mathsf{q}).$$

In particular, the Hadamard product of free $\sim$*-commutative monoids is again free* $\sim$*-commutative.*

Proof. This follows by generalizing the argument given in the proof of Proposition 15.2. □

In view of Theorem 13.73, Proposition 15.8 is equivalent to the following statement.

Proposition 15.9. *For any species* p *and* q*,*

$$\mathsf{p} \diamond_{\sim} \mathsf{q} = \mathcal{P}(\mathcal{T}_{0,\sim}(\mathsf{p}) \times \mathcal{T}_{0,\sim}(\mathsf{q})).$$

This can also be seen directly. The argument is as in the proof of Proposition 15.3, with partial-flats instead of faces.

Exercise 15.10. Generalize the operations $\diamond_{-+}$, $\diamond_{+-}$, $\diamond_{--}$ to partial-flats, and use them to generalize Propositions 15.6 and 15.7.

15.2. Product of free and cofree bimonoids

Recall from Sections 6.1.2 and 6.2.2 the free q-bimonoid $\mathcal{T}_q(\mathsf{c})$ on the comonoid c, and the cofree q-bimonoid $\mathcal{T}_q^\vee(\mathsf{a})$ on the monoid a. When $q = 1$, we write $\mathcal{T}(\mathsf{c})$ and $\mathcal{T}^\vee(\mathsf{a})$, respectively.

For a monoid a and comonoid c, we study the q-bimonoid

$$\mathcal{T\!T}_q(\mathsf{c}, \mathsf{a}) := \mathcal{T}(\mathsf{c}) \times \mathcal{T}_q^\vee(\mathsf{a}). \tag{15.5}$$

We describe its structure maps explicitly, and give a cancellation-free formula for its antipode. We employ the Loday–Ronco theorem to show that $\mathcal{T\!T}_q(\mathsf{c}, \mathsf{a})$ is free and cofree.

Notation 15.11. Let V be a $\Bbbk$-vector space. For book-keeping purposes, we denote an element $x \in V$ by the symbol $\mathtt{F}_x$. Thus, $\mathtt{F}_x + \mathtt{F}_y = \mathtt{F}_{x+y}$ and $c\mathtt{F}_x = \mathtt{F}_{cx}$ for $x, y \in V$ and $c \in \Bbbk$.

Now suppose $\varphi : V \to V$ is a linear isomorphism. Then define elements $\mathtt{M}_y$ of V by

$$\mathtt{F}_x = \mathtt{M}_{\varphi(x)}, \qquad \text{or equivalently,} \qquad \mathtt{F}_{\varphi^{-1}(y)} = \mathtt{M}_y.$$

It follows that $\mathtt{M}_x + \mathtt{M}_y = \mathtt{M}_{x+y}$ and $c\mathtt{M}_x = \mathtt{M}_{cx}$.

If $\{e_i\}$ is a basis for V, then so are $\{\mathtt{F}_{e_i}\}$ and $\{\mathtt{M}_{e_i}\}$. We call these the $\mathtt{F}$-basis and $\mathtt{M}$-basis, and use this terminology even when we have not picked a basis for V.

We introduce a partial order on pair of faces and use it to construct an isomorphism of species from $\mathcal{T\!T}_q(\mathsf{c}, \mathsf{a})$ to itself. In addition to the $\mathtt{F}$-basis, this defines the $\mathtt{M}$-basis, with the role of V played by the components $\mathcal{T\!T}_q(\mathsf{c}, \mathsf{a})[A]$. We describe the product and coproduct of $\mathcal{T\!T}_q(\mathsf{c}, \mathsf{a})$ in the $\mathtt{M}$-basis when c is cocommutative. The coproduct formula can be used to describe its primitive part and, more generally, its primitive filtration. We also give a cancellation-free formula for its antipode in the $\mathtt{M}$-basis.

Remark 15.12. We mention that under suitable assumptions such as commutativity of a or cocommutativity of c or q not a root of unity, one may use the HNR isomorphisms (Chapter 14) to simplify (15.5) to the case when the (co)product is trivial. However, even the study of such a special case, namely, $\mathbb{T}_q(\mathsf{p},\mathsf{q})$ for species p and q, is nontrivial.

15.2.1. $\mathtt{F}$-basis. We begin with the following general situation. Let p and q be any scalars. Let a be any monoid and c be any comonoid. Consider

$$\mathcal{T}_p(\mathsf{c}) \times \mathcal{T}_q^\vee(\mathsf{a}) \tag{15.6}$$

which is the Hadamard product of the free p-bimonoid on c with the cofree q-bimonoid on a. By Lemma 8.1, it is a pq-bimonoid. We now describe its structure maps explicitly by using the definitions for the (co)product of $\mathcal{T}_p(\mathsf{c})$, namely, (6.3), (6.5), of $\mathcal{T}_q^\vee(\mathsf{a})$, namely, (6.11), (6.13), and of the Hadamard product, namely, (8.3), (8.5).

The A-component is

$$(\mathcal{T}_p(\mathsf{c}) \times \mathcal{T}_q^\vee(\mathsf{a}))[A] = \bigoplus_{A \leq H,\, A \leq K} \mathsf{c}[H] \otimes \mathsf{a}[K].$$

The sum is over both H and K.

The map

$$\beta_{B,A} : \bigoplus_{A \leq H,\, A \leq K} \mathsf{c}[H] \otimes \mathsf{a}[K] \to \bigoplus_{B \leq H',\, B \leq K'} \mathsf{c}[H'] \otimes \mathsf{a}[K']$$

sends the (H,K)-summand to the (BH,BK)-summand via $\beta_{BH,H} \otimes \beta_{BK,K}$.

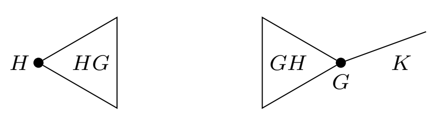
The coproduct

$$\Delta_A^G : \bigoplus_{A \leq H,\, A \leq K} \mathsf{c}[H] \otimes \mathsf{a}[K] \to \bigoplus_{G \leq H',\, G \leq K'} \mathsf{c}[H'] \otimes \mathsf{a}[K'], \tag{15.7}$$

on the (H,K)-summand, is given by

$$\mathsf{c}[H] \otimes \mathsf{a}[K] \xrightarrow{\Delta_H^{HG} \otimes \mathrm{id}} \mathsf{c}[HG] \otimes \mathsf{a}[K] \xrightarrow{(\beta_p)_{GH,HG} \otimes \mathrm{id}} \mathsf{c}[GH] \otimes \mathsf{a}[K]$$

if $G \leq K$, and zero otherwise. Thus, the (H,K)-summand maps to the (GH,K)-summand if $G \leq K$, and to zero otherwise. An illustration is provided below with $A = O$.



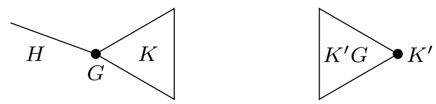
The product

$$\mu_A^G : \bigoplus_{G \leq H,\, G \leq K} \mathsf{c}[H] \otimes \mathsf{a}[K] \to \bigoplus_{A \leq H',\, A \leq K'} \mathsf{c}[H'] \otimes \mathsf{a}[K'] \tag{15.8}$$

sends the (H, K)-summand to a sum of those (H', K')-summands for which $H' = H$ and $GK' = K$. This map, projected on a particular (H', K')-summand, is given by

$$\mathsf{c}[H] \otimes \mathsf{a}[K] \xrightarrow{\mathrm{id} \otimes (\beta_q)_{K'G,GK'}} \mathsf{c}[H] \otimes \mathsf{a}[K'G] \xrightarrow{\mathrm{id} \otimes \mu_{K'}^{K'G}} \mathsf{c}[H] \otimes \mathsf{a}[K'].$$

An illustration is provided below with $A = O$.

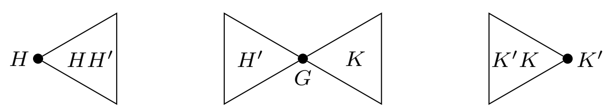


Theorem 15.13. *Suppose $\mathcal{A}$ is simplicial. The antipode $S : \mathcal{T}_p(\mathsf{c}) \times \mathcal{T}_q^{\vee}(\mathsf{a}) \to \mathcal{T}_p(\mathsf{c}) \times \mathcal{T}_q^{\vee}(\mathsf{a})$, on the A-component, on the (H, K)-summand, is given by*

$$\sum (-1)^{\dim(H' \wedge K)} (\beta_p)_{H',HH'} \Delta_H^{HH'} \otimes \mu_{K'}^{K'K} (\beta_q)_{K'K,K}. \tag{15.9}$$

The sum is into those (H', K')-summands for which there is a unique face G greater than A with $GH = H'$ and $GK' = K$. (In this case, the face G must equal $H' \wedge K$).

An illustration is provided below with $A = O$.



In particular, if c and a are both trivial, then the antipode, on the A-component, on the (H, K)-summand, is given by

$$\sum (-1)^{\dim(H' \wedge K)} (\beta_p)_{H',H} \otimes (\beta_q)_{K',K}. \tag{15.10}$$

The sum is now into those (H', K')-summands such that $\mathrm{s}(H') = \mathrm{s}(H)$, $\mathrm{s}(K') = \mathrm{s}(K)$, and there is a unique face G greater than A with $GH = H'$ and $GK' = K$.

Proof. Substituting (15.7) and (15.8) into the Takeuchi formula (12.1) and simplifying, the antipode S_A, on the (H, K)-summand, is given by

$$\sum_{H' \geq A,\, K' \geq A} \Big(\sum_{G \in \mathrm{B}} (-1)^{\dim(G)} \Big) (\beta_p)_{H',HH'} \Delta_H^{HH'} \otimes \mu_{K'}^{K'K} (\beta_q)_{K'K,K},$$

where

$$\mathrm{B} = \{G \mid G \geq A,\ GH = H',\ GK' = K\}.$$

Now apply Lemma 1.41 to the arrangement $\mathcal{A}_A$. $\square$

Setting $p = 1$ in (15.6) yields the q-bimonoid $\mathscr{T}_q(\mathsf{c}, \mathsf{a})$ in (15.5). The result below shows that specializing p in this manner entails very little loss of generality.

Lemma 15.14. *Let p be a nonzero scalar and q be any scalar. Let a be any monoid and c be any comonoid. Then the map*

$$\mathcal{T}_p(\mathsf{c}) \times \mathcal{T}_q^\vee(\mathsf{a}) \to \mathcal{T}(\mathsf{c}) \times \mathcal{T}_{pq}^\vee(\mathsf{a}),$$

which on the A-component

$$\bigoplus_{A \le H,\, A \le K} \mathsf{c}[H] \otimes \mathsf{a}[K] \to \bigoplus_{A \le H,\, A \le K} \mathsf{c}[H] \otimes \mathsf{a}[K]$$

sends the (H,K)-summand to itself by scalar multiplication by $p^{\mathrm{dist}(H,K)}$, is an isomorphism of pq-bimonoids.

Proof. Let us check that the map preserves coproducts. Accordingly, apply Δ_A^G to the (H,K)-summand. We may assume that $G \le K$ (else the summand maps to zero). The commutativity of

$$\begin{tikzcd}
\mathsf{c}[H] \otimes \mathsf{a}[K] \arrow[r, "\Delta_H^{HG} \otimes \mathrm{id}"] \arrow[d, "p^{\mathrm{dist}(H,K)}\,\mathrm{id}"'] & \mathsf{c}[HG] \otimes \mathsf{a}[K] \arrow[r, "(\beta_p)_{GH,HG} \otimes \mathrm{id}"] & \mathsf{c}[GH] \otimes \mathsf{a}[K] \arrow[d, "p^{\mathrm{dist}(GH,K)}\,\mathrm{id}"] \\
\mathsf{c}[H] \otimes \mathsf{a}[K] \arrow[r, "\Delta_H^{HG} \otimes \mathrm{id}"'] & \mathsf{c}[HG] \otimes \mathsf{a}[K] \arrow[r, "\beta_{GH,HG} \otimes \mathrm{id}"'] & \mathsf{c}[GH] \otimes \mathsf{a}[K]
\end{tikzcd}$$

reduces to the identity

$$p^{\mathrm{dist}(H,K)} = p^{\mathrm{dist}(H,GH)} p^{\mathrm{dist}(GH,K)},$$

which holds by property (1.30d) (or equivalently (1.30e)) applied to the distance function υ_p (1.27).

Similarly, for products, assuming $GK' = K$, the commutativity of

$$\begin{tikzcd}
\mathsf{c}[H] \otimes \mathsf{a}[K] \arrow[r, "\mathrm{id} \otimes (\beta_q)_{K'G,GK'}"] \arrow[d, "p^{\mathrm{dist}(H,K)}\,\mathrm{id}"'] & \mathsf{c}[H] \otimes \mathsf{a}[K'G] \arrow[r, "\mathrm{id} \otimes \mu_{K'}^{K'G}"] & \mathsf{c}[H] \otimes \mathsf{a}[K'] \arrow[d, "p^{\mathrm{dist}(H,K')}\,\mathrm{id}"] \\
\mathsf{c}[H] \otimes \mathsf{a}[K] \arrow[r, "\mathrm{id} \otimes (\beta_{pq})_{K'G,GK'}"'] & \mathsf{c}[H] \otimes \mathsf{a}[K'G] \arrow[r, "\mathrm{id} \otimes \mu_{K'}^{K'G}"'] & \mathsf{c}[H] \otimes \mathsf{a}[K']
\end{tikzcd}$$

reduces to the identity

$$q^{\mathrm{dist}(K,K')} p^{\mathrm{dist}(H,K')} = p^{\mathrm{dist}(H,K)} (pq)^{\mathrm{dist}(K,K')},$$

which holds by property (1.30d) (as above). $\square$

15.2.2. Freeness and cofreeness. We now employ the Loday–Ronco theorem to show that $\mathcal{T\!T}_q(\mathsf{c}, \mathsf{a})$ is both free and cofree.

Proposition 15.15. *For any scalars p and q, for any comonoid c and monoid a, the pq-bimonoid $\mathcal{T}_p(\mathsf{c}) \times \mathcal{T}_q^\vee(\mathsf{a})$ is free as a monoid and cofree as a comonoid. In particular, $\mathcal{T\!T}_q(\mathsf{c}, \mathsf{a})$ is both free and cofree.*

Proof. Observe that the coproduct does not involve the parameter q. Hence,

$$\mathcal{T}_p(\mathsf{c}) \times \mathcal{T}_q^\vee(\mathsf{a}) = \mathcal{T}_p(\mathsf{c}) \times \mathcal{T}_0^\vee(\mathsf{a})$$

as comonoids. Now the latter is a 0-bimonoid, hence by Proposition 13.1, it is cofree (on its primitive part). Similarly, by setting $p = 0$ instead, we deduce freeness. □

15.2.3. Self-duality.

Proposition 15.16. *Suppose $q \neq 0$. If a and c are finite-dimensional, then there is an isomorphism*

$$\mathcal{T\!T}_q(\mathsf{c}, \mathsf{a})^* \xrightarrow{\cong} \mathcal{T\!T}_q(\mathsf{a}^*, \mathsf{c}^*)$$

of q-bimonoids. In particular, if c is finite-dimensional, then $\mathcal{T\!T}_q(\mathsf{c}, \mathsf{c}^*)$ is self-dual.

Proof. The isomorphism is constructed as follows.

$$\begin{aligned}(\mathcal{T}(\mathsf{c}) \times \mathcal{T}_q^{\vee}(\mathsf{a}))^* \cong \mathcal{T}(\mathsf{c})^* \times \mathcal{T}_q^{\vee}(\mathsf{a})^* &\cong \mathcal{T}^{\vee}(\mathsf{c}^*) \times \mathcal{T}_q(\mathsf{a}^*) \\ &\cong \mathcal{T}_q(\mathsf{a}^*) \times \mathcal{T}^{\vee}(\mathsf{c}^*) \cong \mathcal{T}(\mathsf{a}^*) \times \mathcal{T}_q^{\vee}(\mathsf{c}^*).\end{aligned}$$

Finite-dimensionality justifies the manipulation of the duals. The last step used Lemma 15.14, which requires $q \neq 0$. □

For any finite-dimensional comonoid c, by specializing (8.22), we have an isomorphism of bimonoids

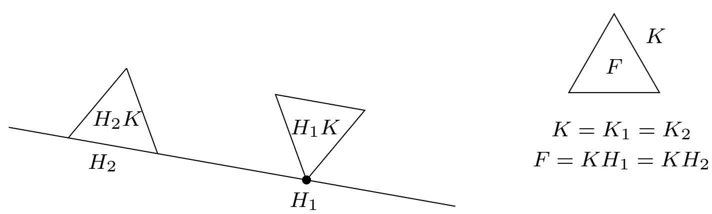
$$\mathcal{T\!T}(\mathsf{c}, \mathsf{c}^*) \xrightarrow{\cong} \mathrm{end}^{\times}(\mathcal{T}(\mathsf{c})).$$

Thus, $\mathcal{T\!T}(\mathsf{c}, \mathsf{c}^*)$ is an example of a biconvolution bimonoid. Alternatively, for any finite-dimensional monoid a, we have the biconvolution bimonoid $\mathcal{T\!T}(\mathsf{a}^*, \mathsf{a})$.

15.2.4. Partial order on pairs of faces. We now define a partial order on pairs of faces. We say that $(H_1, K_1) \leq (H_2, K_2)$ if

(i) $K_1 = K_2 = K$ (say),
(ii) $H_2 H_1 = H_2$ and $K H_1 = K H_2$,
(iii) $H_2 K \text{ -- } H_1 K \text{ -- } K H_2$.

It is clear from (ii) above that H_2K, H_1K, KH_2 all have the same support; so the gallery condition (iii) makes sense.



For any face A, let $\leq_A$ denote the restriction of the above partial order to faces greater than A. This agrees with the partial order on pairs of faces in the arrangement $\mathcal{A}_A$.

We recall the following results from [**21**, Lemmas 1.73 and 1.74].

Lemma 15.17. *For any faces G, H, K (all greater than A) with $G \le K$, there is a bijection*

$$\{H' \mid (H', K) \geq_A (H, K),\ A\overline{G} \le H'\} \longrightarrow \{H'' \mid (H'', K) \geq_G (GH, K)\}$$

which sends H' to GH'. The inverse sends H'' to $A\overline{G}H''$.

Lemma 15.18. *Let G, H, F, K be any faces with $G \le H$. Then*

$$(F, K) \geq (H, K) \iff \begin{matrix} (GF, GK) \geq (H, GK),\ FG = F, \\ FK \text{ -- } GFK \text{ -- } GKF \text{ -- } KF. \end{matrix}$$

We note that the above gallery condition can be equivalently written as either $FK \text{ -- } GKF \text{ -- } KF$ or $FK \text{ -- } GFK \text{ -- } KF$. In either case, the missing face can be recovered by applying the gate property to the star of G.

15.2.5. $\mathtt{M}$-basis. We use the above partial order on pairs of faces to define an isomorphism of species

$$\varphi : \mathcal{T}_q(\mathsf{c}, \mathsf{a}) \to \mathcal{T}_q(\mathsf{c}, \mathsf{a}). \tag{15.11}$$

On the A-component, on the (H, K)-summand, the map is given by

$$\begin{aligned} \mathsf{c}[H] \otimes \mathsf{a}[K] &\longrightarrow \bigoplus_{(H',K') \geq_A (H,K)} \mathsf{c}[H'] \otimes \mathsf{a}[K'], \\ x \otimes y &\longmapsto \sum_{(H',K) \geq_A (H,K)} \beta_{H',HH'} \Delta_H^{HH'}(x) \otimes y. \end{aligned}$$

This map is unitriangular for the partial order on pairs of faces, hence it is an isomorphism.

Following Notation 15.11, we denote $x \otimes y$ by the symbol $\mathtt{F}_{x\otimes y}$, and then define elements $\mathtt{M}_{x \otimes y}$ by

$$\mathtt{F}_{x\otimes y} = \sum_{(H',K) \geq_A (H,K)} \mathtt{M}_{\beta_{H',HH'} \Delta_H^{HH'}(x) \otimes y}.$$

This defines the $\mathtt{M}$-basis of $\mathcal{T}_q(\mathsf{c}, \mathsf{a})$.

Proposition 15.19. *Let c be cocommutative. The coproduct of $\mathcal{T}_q(\mathsf{c}, \mathsf{a})$, on the A-component, in the $\mathtt{M}$-basis*

$$\Delta_A^G : \bigoplus_{A \le H,\, A \le K} \mathsf{c}[H] \otimes \mathsf{a}[K] \to \bigoplus_{G \le H',\, G \le K'} \mathsf{c}[H'] \otimes \mathsf{a}[K'] \tag{15.12}$$

is as follows. On the (H, K)-summand, it is given by

$$\mathsf{c}[H] \otimes \mathsf{a}[K] \xrightarrow{\beta_{GH,H} \otimes \mathrm{id}} \mathsf{c}[GH] \otimes \mathsf{a}[K]$$

if $G \le K$ and $G \le A\overline{H}$, and zero otherwise.

Recall that $A\overline{H}$ is the opposite of H in the star of A. In the above setting, $A \le G$ and $A \le H$, so the condition $G \le A\overline{H}$ is equivalent to $A\overline{G} \le H$.

An illustration is provided below with $A = O$.

Proof. We check that the formulas in the $\mathtt{F}$-basis and in the $\mathtt{M}$-basis are consistent with each other. Since the change of basis formula expresses $\mathtt{F}$ in terms of the $\mathtt{M}$, it is convenient to compute the coproduct of an $\mathtt{F}$ element in two different ways and check that the results match. Accordingly, for $x \in \mathsf{c}[H]$, $y \in \mathsf{a}[K]$, and $G \leq K$, using (15.12),

$$\begin{aligned}\Delta^G_A(\mathtt{F}_{x\otimes y}) &= \Delta^G_A\Bigg(\sum_{(H',K)\geq_A (H,K)} \mathtt{M}_{\beta_{H',HH'}\Delta^{HH'}_H(x)\otimes y}\Bigg)\\ &= \sum_{\substack{(H',K)\geq_A (H,K)\\ A\overline{G}\leq H'}} \mathtt{M}_{\beta_{GH',H'}\beta_{H',HH'}\Delta^{HH'}_H(x)\otimes y}.\end{aligned}$$

Proceeding differently, using (15.7),

$$\begin{aligned}\Delta^G_A(\mathtt{F}_{x\otimes y}) &= \mathtt{F}_{\beta_{GH,HG}\Delta^{HG}_H(x)\otimes y}\\ &= \sum_{(H'',K)\geq_G (GH,K)} \mathtt{M}_{\beta_{H'',GHH''}\Delta^{GHH''}_{GH}\beta_{GH,HG}\Delta^{HG}_H(x)\otimes y}.\end{aligned}$$

By Lemma 15.17, there is a bijection between the indexing sets which sends H' to GH' in one direction, and H'' to $A\overline{G}H''$ in the other. The corresponding summands match because of the following commutative diagram.

$$\begin{tikzcd}
\mathsf{c}[H] \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "\Delta"] & \mathsf{c}[HG] \arrow[r, "\beta"] \arrow[d, "\Delta"] & \mathsf{c}[GH] \arrow[d, "\Delta"] \\
\mathsf{c}[HH'] \arrow[r, "\beta"'] \arrow[d, "\beta"'] & \mathsf{c}[HGH'] \arrow[r, "\beta"'] & \mathsf{c}[GHH'] \arrow[d, "\beta"] \\
\mathsf{c}[H'] \arrow[rr, "\beta"'] & & \mathsf{c}[GH']
\end{tikzcd}$$

The square commutes by naturality, one triangle commutes by coassociativity, and the other triangle commutes by cocommutativity. $\square$

Proposition 15.20. *The product of* $\mathcal{T}_q(\mathsf{c},\mathsf{a})$*, on the* A*-component, in the* $\mathtt{M}$*-basis*

$$\mu^G_A : \bigoplus_{G\leq H,\, G\leq K} \mathsf{c}[H]\otimes\mathsf{a}[K] \to \bigoplus_{A\leq H',\, A\leq K'} \mathsf{c}[H']\otimes\mathsf{a}[K'] \tag{15.13}$$

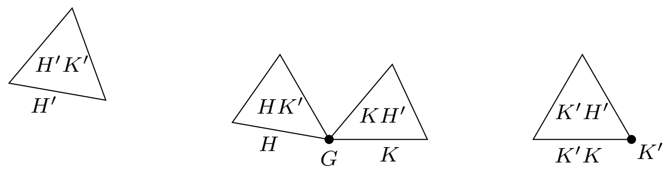
is as follows. It sends the (H,K)*-summand to a sum of those* (H',K')*-summands for which*

$$GK' = K, \quad GH' = H, \quad H'G = H', \quad H'K' \text{ -- } HK' \text{ -- } KH' \text{ -- } K'H'.$$

This map, projected on a particular (H', K')-summand, is given by the tensor product of the maps

$$\mathsf{c}[H] \xrightarrow{\beta_{H',H}} \mathsf{c}[H'] \quad \textit{and} \quad \mathsf{a}[K] \xrightarrow{(\beta_q)_{K'G,GK'}} \mathsf{a}[K'G] \xrightarrow{\mu^{K'G}_{K'}} \mathsf{a}[K'].$$

An illustration is provided below with $A = O$.



We note that the above gallery condition can be equivalently written as either $H'K' \text{--} KH' \text{--} K'H'$ or $H'K' \text{--} HK' \text{--} K'H'$. In either case, the missing face can be recovered by applying the gate property to the star of G.

Proof. We follow the proof method of Proposition 15.19. Accordingly, for $x \in \mathsf{c}[H]$, $y \in \mathsf{a}[K]$, and $G \leq H$, $G \leq K$, using (15.13),

$$\begin{aligned}\mu^G_A(\mathtt{F}_{x\otimes y}) &= \mu^G_A\Bigg(\sum_{(H',K)\geq_G (H,K)} \mathtt{M}_{\beta_{H',HH'}\Delta^{HH'}_H(x)\otimes y}\Bigg)\\ &= \sum_{(H'',K')}\ \sum_{(H',K)\geq_G (H,K)} \mathtt{M}_{\beta_{H'',H'}\beta_{H',HH'}\Delta^{HH'}_H(x)\otimes \mu^{K'G}_{K'}(\beta_q)_{K'G,GK'}(y)}.\end{aligned}$$

The first sum is over all (H'', K') such that $GK' = K$, $GH'' = H'$, $H''G = H''$, $H''K' \text{--} H'K' \text{--} KH'' \text{--} K'H''$.

Proceeding differently, using (15.8),

$$\begin{aligned}\mu^G_A(\mathtt{F}_{x\otimes y}) &= \sum_{K'\colon GK'=K} \mathtt{F}_{x\otimes \mu^{K'G}_{K'}(\beta_q)_{K'G,GK'}(y)}\\ &= \sum_{\substack{H''\colon\\ (H'',K')\geq_A (H,K')}}\ \sum_{K'\colon GK'=K} \mathtt{M}_{\beta_{H'',HH''}\Delta^{HH''}_H(x)\otimes \mu^{K'G}_{K'}(\beta_q)_{K'G,GK'}(y)}.\end{aligned}$$

By Lemma 15.18, there is a bijection between the indexing sets, and it is easy to see then that the corresponding summands match. □

Exercise 15.21. Check that: For $q = 0$, Proposition 15.20 specializes as follows. For the product component μ^G_A of $\mathcal{T}_0(\mathsf{c}, \mathsf{a})$, we sum over those (H', K') for which

$$GK' = K'G = K, \quad GH' = H, \quad H'G = H',$$

and the projection on a particular (H', K')-summand is the tensor product of the maps

$$\mathsf{c}[H] \xrightarrow{\beta_{H',H}} \mathsf{c}[H'] \quad \text{and} \quad \mathsf{a}[K] \xrightarrow{\mu^K_{K'}} \mathsf{a}[K'].$$

Further, check explicitly that the above product along with the coproduct (15.12) indeed satisfies the 0-bimonoid axiom (2.40).

Theorem 15.22. *Suppose $\mathcal{A}$ is simplicial. Let c be cocommutative. The antipode $S : \mathcal{T}_q(\mathsf{c}, \mathsf{a}) \to \mathcal{T}_q(\mathsf{c}, \mathsf{a})$, on the A-component, on the (H, K)-summand, in the* $\mathtt{M}$*-basis is given by*

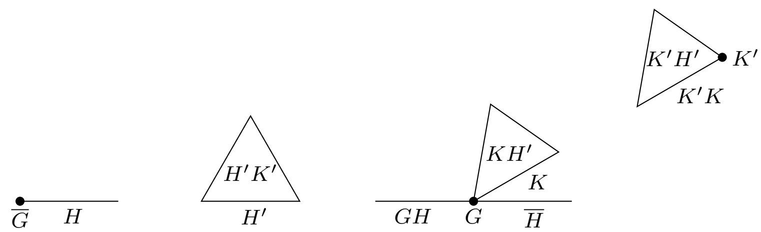
$$(-1)^{\dim(A\overline{H}\wedge K)} \sum \beta_{H',H} \otimes \mu_{K'}^{K'K}(\beta_q)_{K'K,K}. \tag{15.14}$$

The sum is into those (H', K')-summands such that

$$\mathrm{s}(H') = \mathrm{s}(H), \quad \mathrm{s}(K') \leq \mathrm{s}(K), \quad H'K' \text{ -- } KH' \text{ -- } K'H',$$

and for which there is a unique face G greater than A with $G\overline{H'} = A\overline{H}$ and $GK' = K$. (In this case, the face G must equal $A\overline{H} \wedge K$).

An illustration is provided below with $A = O$.



In particular, if a is trivial, formula (15.14) simplifies to

$$(-1)^{\dim(A\overline{H}\wedge K)} \sum \beta_{H',H} \otimes (\beta_q)_{K',K}, \tag{15.15}$$

with the condition $\mathrm{s}(K') \leq \mathrm{s}(K)$ replaced by $\mathrm{s}(K') = \mathrm{s}(K)$.

Proof. Substituting (15.12) and (15.13) into the Takeuchi formula (12.1) and simplifying, the antipode S_A, on the (H, K)-summand, is given by

$$\sum_{\substack{H' \geq A,\, K' \geq A \\ \mathrm{s}(H')=\mathrm{s}(H) \\ \mathrm{s}(K') \leq \mathrm{s}(K) \\ H'K' \text{ -- } KH' \text{ -- } K'H'}} \Big(\sum_{G \in \mathrm{B}} (-1)^{\dim(G)} \Big) \beta_{H',H} \otimes \mu_{K'}^{K'K}(\beta_q)_{K'K,K},$$

where

$$\mathrm{B} = \{G \mid G \geq A,\ G\overline{H'} = A\overline{H},\ GK' = K\}.$$

Now apply Lemma 1.41 to the arrangement $\mathcal{A}_A$. □

15.2.6. Primitive part and primitive filtration. We now discuss the primitive part of $\mathcal{T}(\mathsf{c}, \mathsf{a})$ when c is cocommutative. Since the coproduct of $\mathcal{T}_q(\mathsf{c}, \mathsf{a})$ does not depend on q, we drop it from the notation. We follow the notations of Proposition 15.19.

In the coproduct formula in the $\mathtt{M}$-basis (15.12), the map Δ_A^G either sends a summand to zero, or isomorphically onto a summand in the rhs, such that every summand in the rhs is the image of a unique summand. (This is because if $A\overline{G} \leq H$, then (H, K) can be recovered from (GH, K) by $H = A\overline{G}(GH)$.)

Thus, the kernel of Δ_A^G consists of precisely those summands which go to zero. In other words,

$$\ker(\Delta_A^G) = \bigoplus_{\substack{H \geq A,\, K \geq A \\ G \not\leq A\overline{H} \wedge K}} \mathsf{c}[H] \otimes \mathsf{a}[K].$$

As a consequence, we obtain the following description of the primitive part and primitive filtration.

Proposition 15.23. *Let* c *be cocommutative. In the* $\mathtt{M}$*-basis,*

$$\mathcal{P}(\mathcal{T\!\!T}(\mathsf{c}, \mathsf{a}))[A] = \bigoplus_{A\overline{H} \wedge K = A} \mathsf{c}[H] \otimes \mathsf{a}[K].$$

More generally, in the $\mathtt{M}$*-basis, for* $k \geq 1$*,*

$$\mathcal{P}_k(\mathcal{T\!\!T}(\mathsf{c}, \mathsf{a}))[A] = \bigoplus_{\mathrm{rk}((A\overline{H} \wedge K)/A) \leq k-1} \mathsf{c}[H] \otimes \mathsf{a}[K].$$

The sum in both formulas is over pairs of faces (H, K) with both H and K greater than A.

15.2.7. Cofreeness revisited. We continue to assume that c is cocommutative and work with the $\mathtt{M}$-basis. We now define an isomorphism

$$\mathcal{T\!\!T}(\mathsf{c}, \mathsf{a})[A] \to \bigoplus_{G:\, G \geq A} \mathcal{P}(\mathcal{T\!\!T}(\mathsf{c}, \mathsf{a}))[G]. \tag{15.16}$$

For any H and K both greater than A, observe that for $G := A\overline{H} \wedge K$,

$$\Delta_A^G(\mathsf{c}[H] \otimes \mathsf{a}[K]) \subseteq \mathcal{P}(\mathcal{T\!\!T}(\mathsf{c}, \mathsf{a}))[G].$$

This is how we define (15.16) on the (H, K)-summand. To see that it is an isomorphism, we describe its inverse. On the (H', K)-summand, with H' and K both greater than G, and $G\overline{H'} \wedge K = G$, the inverse map is

$$\mathsf{c}[H'] \otimes \mathsf{a}[K] \xrightarrow{\beta_{A\overline{G}H', H'} \otimes \mathrm{id}} \mathsf{c}[A\overline{G}H'] \otimes \mathsf{a}[K].$$

It is routine to check that this is indeed the inverse.

Let us now identify the rhs of (15.16) with $\mathcal{T}^\vee(\mathcal{P}(\mathcal{T\!\!T}(\mathsf{c}, \mathsf{a})))[A]$. This yields an isomorphism of species

$$\mathcal{T\!\!T}(\mathsf{c}, \mathsf{a}) \to \mathcal{T}^\vee(\mathcal{P}(\mathcal{T\!\!T}(\mathsf{c}, \mathsf{a}))). \tag{15.17}$$

Proposition 15.24. *Let* c *be cocommutative. The map* (15.17) *is an isomorphism of comonoids. In particular,* $\mathcal{T\!\!T}(\mathsf{c}, \mathsf{a})$ *is cofree on its primitive part.*

PROOF. Let us denote (15.17) by f. We check that f commutes with Δ_A^B for any $B \geq A$. Let us start with the summand $\mathsf{c}[H] \otimes \mathsf{a}[K]$. Put $G := A\overline{H} \wedge K$. By Proposition 15.19, Δ_A^B on this summand is zero unless $B \leq G$. The image of this summand under f belongs to $\mathcal{P}(\mathcal{T\!\!T}(\mathsf{c}, \mathsf{a}))[G]$, so Δ_A^B applied to it is also zero unless $B \le G$. So let us assume $B \le G$ and proceed. Note that since $A \le B \le A\overline{H}$, we have $B\overline{BH} \wedge K = A\overline{H} \wedge K = G$. So the check reduces to

$$\begin{tikzcd}[column sep=small] & \mathsf{c}[H] \otimes \mathsf{a}[K] \arrow[dl, "\Delta_A^B"'] \arrow[dr, "\Delta_A^G"] & \\ \mathsf{c}[BH] \otimes \mathsf{a}[K] \arrow[rr, "\Delta_B^G"'] & & \mathsf{c}[GH] \otimes \mathsf{a}[K] \end{tikzcd}$$

which holds by coassociativity. □

Another proof of Proposition 15.24 is given in the exercise below.

Exercise 15.25. Consider the map $f : \mathcal{T\!\!T}(\mathsf{c}, \mathsf{a}) \twoheadrightarrow \mathcal{P}(\mathcal{T\!\!T}(\mathsf{c}, \mathsf{a}))$ which sends the summands in the primitive part to themselves, and the rest to zero. Check that the morphism of comonoids $\hat{f}$ obtained by applying Theorem 6.10 coincides with (15.17).

Remark 15.26. We know from Proposition 15.15 that $\mathcal{T\!\!T}(\mathsf{c}, \mathsf{a})$ is cofree on its primitive part. Moreover, its proof also gives explicit inverse isomorphisms of comonoids

$$\mathcal{T}^\vee(\mathcal{P}(\mathcal{T\!\!T}(\mathsf{c}, \mathsf{a}))) \longleftrightarrow \mathcal{T\!\!T}(\mathsf{c}, \mathsf{a}).$$

These are specializations of the Loday–Ronco isomorphisms (13.2) and (13.4). They involve the product of $\mathcal{T\!\!T}_0(\mathsf{c}, \mathsf{a})$ which is described in Exercise 15.21. For instance, the forward map, on the G-summand of the A-component, is given by the product component μ_A^G of $\mathcal{T\!\!T}_0(\mathsf{c}, \mathsf{a})$.

Note very carefully that these inverse isomorphisms are different from the one in (15.17).

15.3. Product of free comm. and cofree cocomm. bimonoids

Recall from Sections 6.3.2 and 6.3.4 the free commutative bimonoid $\mathcal{S}(\mathsf{c})$ on the comonoid c, and the cofree cocommutative bimonoid $\mathcal{S}^\vee(\mathsf{a})$ on the monoid a.

For a commutative monoid a and cocommutative comonoid c, we study the bicommutative bimonoid

$$\boldsymbol{\mathcal{S}}(\mathsf{c}, \mathsf{a}) := \mathcal{S}(\mathsf{c}) \times \mathcal{S}^\vee(\mathsf{a}). \tag{15.18}$$

The discussion proceeds in analogy with the analysis of the bimonoid $\mathcal{T\!\!T}(\mathsf{c}, \mathsf{a})$ in Section 15.2. The partial order on pairs of faces is replaced by a partial order on pairs of flats. The latter is easier to handle since the subtleties with minimal galleries disappear, thus simplifying the analysis.

15.3.1. F-basis. Consider $\boldsymbol{\mathcal{S}}(\mathsf{c}, \mathsf{a})$ as defined in (15.18). Its Z-component is

$$\boldsymbol{\mathcal{S}}(\mathsf{c}, \mathsf{a})[\mathrm{Z}] = \bigoplus_{\mathrm{Z} \le \mathrm{X},\, \mathrm{Z} \le \mathrm{Y}} \mathsf{c}[\mathrm{X}] \otimes \mathsf{a}[\mathrm{Y}].$$

The sum is over both X and Y. We now make explicit the product and coproduct of $\boldsymbol{\mathcal{S}}(\mathsf{c}, \mathsf{a})$ in the $\mathtt{F}$-basis (following Notation 15.11).

The coproduct

$$\Delta_{\mathrm{Z}}^{\mathrm{W}} : \bigoplus_{\mathrm{Z} \leq \mathrm{X},\, \mathrm{Z} \leq \mathrm{Y}} \mathsf{c}[\mathrm{X}] \otimes \mathsf{a}[\mathrm{Y}] \to \bigoplus_{\mathrm{W} \leq \mathrm{X}',\, \mathrm{W} \leq \mathrm{Y}'} \mathsf{c}[\mathrm{X}'] \otimes \mathsf{a}[\mathrm{Y}']$$

is given by

$$\Delta_{\mathrm{Z}}^{\mathrm{W}}\big(\mathtt{F}_{x \otimes y}\big) = \begin{cases} \mathtt{F}_{\Delta_{\mathrm{X}}^{\mathrm{X} \vee \mathrm{W}}(x) \otimes y} & \text{if } \mathrm{Y} \geq \mathrm{W}, \\ 0 & \text{otherwise.} \end{cases} \tag{15.19}$$

This follows from definitions (6.22) and (6.26).

The product

$$\mu_{\mathrm{Z}}^{\mathrm{W}} : \bigoplus_{\mathrm{W} \leq \mathrm{X}',\, \mathrm{W} \leq \mathrm{Y}'} \mathsf{c}[\mathrm{X}'] \otimes \mathsf{a}[\mathrm{Y}'] \to \bigoplus_{\mathrm{Z} \leq \mathrm{X},\, \mathrm{Z} \leq \mathrm{Y}} \mathsf{c}[\mathrm{X}] \otimes \mathsf{a}[\mathrm{Y}]$$

is given by

$$\mu_{\mathrm{Z}}^{\mathrm{W}}\big(\mathtt{F}_{x \otimes y}\big) = \sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{Z},\, \mathrm{W} \vee \mathrm{Y} = \mathrm{Y}'} \mathtt{F}_{x \otimes \mu_{\mathrm{Y}}^{\mathrm{Y}'}(y)}. \tag{15.20}$$

This follows from definitions (6.20) and (6.28).

Exercise 15.27. Check that: The antipode $S : \boldsymbol{\mathcal{S}}(\mathsf{c}, \mathsf{a}) \to \boldsymbol{\mathcal{S}}(\mathsf{c}, \mathsf{a})$, on the Z-component, on the (X, Y)-summand, in the $\mathtt{F}$-basis is given by

$$S_{\mathrm{Z}}\big(\mathtt{F}_{x \otimes y}\big) = \sum_{\substack{\mathrm{X}':\, \mathrm{X}' \geq \mathrm{X} \\ \mathrm{Y}':\, \mathrm{Y} \geq \mathrm{Y}' \geq \mathrm{Z}}} \Bigg(\sum_{\substack{\mathrm{W}:\, \mathrm{W} \geq \mathrm{Z} \\ \mathrm{W} \vee \mathrm{Y}' = \mathrm{Y} \\ \mathrm{W} \vee \mathrm{X} = \mathrm{X}'}} (-1)^{\dim(\mathrm{W})} c_{\mathrm{Z}}^{\mathrm{W}} \Bigg) \mathtt{F}_{\Delta_{\mathrm{X}}^{\mathrm{X}'}(x) \otimes \mu_{\mathrm{Y}'}^{\mathrm{Y}}(y)}. \tag{15.21}$$

(Substitute the (co)product formulas (15.19) and (15.20) into the commutative Takeuchi formula (12.15).) Evaluating the sum in parenthesis is related to Question 1.43. Use identity (1.74) to deduce that, if c and a are both trivial, then

$$S_{\mathrm{Z}}\big(\mathtt{F}_{x \otimes y}\big) = (-1)^{\dim(\mathrm{X} \wedge \mathrm{Y})}\, \mathtt{F}_{x \otimes y}. \tag{15.22}$$

Use Proposition 15.4 to deduce formula (15.22) as a special case of formula (12.48) or (12.50).

15.3.2. Partial order on pairs of flats. We now define a partial order on pairs of flats. We say that $(\mathrm{X}_1, \mathrm{Y}_1) \leq (\mathrm{X}_2, \mathrm{Y}_2)$ if

- (i) $\mathrm{Y}_1 = \mathrm{Y}_2 = \mathrm{Y}$ (say),
- (ii) $\mathrm{X}_1 \leq \mathrm{X}_2$ and $\mathrm{Y} \vee \mathrm{X}_1 = \mathrm{Y} \vee \mathrm{X}_2$,
- (ii′) $\mathrm{X}_1 \leq \mathrm{X}_2 \leq \mathrm{Y} \vee \mathrm{X}_1$.

Conditions (ii) and (ii′) are equivalent.

For any flat Z, we use $\leq_{\mathrm{Z}}$ to denote the restriction of the above partial order to flats greater than Z.

15.3.3. $\mathtt{M}$-basis. The above partial order on pairs of flats yields an isomorphism of species

$$\varphi : \boldsymbol{\mathcal{S}}(\mathsf{c},\mathsf{a}) \to \boldsymbol{\mathcal{S}}(\mathsf{c},\mathsf{a}). \tag{15.23}$$

On the Z-component, on the (X, Y)-summand, the map is given by

$$\begin{aligned} \mathsf{c}[\mathrm{X}] \otimes \mathsf{a}[\mathrm{Y}] &\longrightarrow \bigoplus_{(\mathrm{X}',\mathrm{Y}') \geq_{\mathrm{Z}} (\mathrm{X},\mathrm{Y})} \mathsf{c}[\mathrm{X}'] \otimes \mathsf{a}[\mathrm{Y}'], \\ x \otimes y &\longmapsto \sum_{\mathrm{X}' :\, \mathrm{X} \leq \mathrm{X}' \leq \mathrm{X} \vee \mathrm{Y}} \Delta_{\mathrm{X}}^{\mathrm{X}'}(x) \otimes y. \end{aligned}$$

This map is unitriangular for the partial order on pairs of flats, hence it is an isomorphism.

Following Notation 15.11, we write $\mathtt{F}_{x\otimes y}$ for $x \otimes y$, and define elements $\mathtt{M}_{x\otimes y}$ by

$$\mathtt{F}_{x\otimes y} = \sum_{\mathrm{X}' :\, \mathrm{X} \leq \mathrm{X}' \leq \mathrm{X} \vee \mathrm{Y}} \mathtt{M}_{\Delta_{\mathrm{X}}^{\mathrm{X}'}(x) \otimes y}.$$

This defines the $\mathtt{M}$-basis of $\boldsymbol{\mathcal{S}}(\mathsf{c},\mathsf{a})$. We now describe the product and coproduct in the $\mathtt{M}$-basis.

Proposition 15.28. *The coproduct of $\boldsymbol{\mathcal{S}}(\mathsf{c},\mathsf{a})$ in the $\mathtt{M}$-basis is given by*

$$\Delta_{\mathrm{Z}}^{\mathrm{W}}(\mathtt{M}_{x\otimes y}) = \begin{cases} \mathtt{M}_{x\otimes y} & \text{if } \mathrm{Y} \geq \mathrm{W},\ \mathrm{X} \geq \mathrm{W}, \\ 0 & \text{otherwise,} \end{cases} \tag{15.24}$$

for $x \in \mathsf{c}[\mathrm{X}]$ and $y \in \mathsf{a}[\mathrm{Y}]$.

Note very carefully that this formula does not involve the coproduct of c, the latter enters into the change of basis formula.

Proof. We prove formula (15.24) by assuming it and using it to derive formula (15.19). We may assume $\mathrm{Y} \geq \mathrm{W}$, since otherwise we get zero.

$$\begin{aligned} \Delta_{\mathrm{Z}}^{\mathrm{W}}(\mathtt{F}_{x\otimes y}) &= \sum_{\mathrm{X}' :\, \mathrm{X} \leq \mathrm{X}' \leq \mathrm{X} \vee \mathrm{Y}} \Delta_{\mathrm{Z}}^{\mathrm{W}}(\mathtt{M}_{\Delta_{\mathrm{X}}^{\mathrm{X}'}(x) \otimes y}) \\ &= \sum_{\mathrm{X}' :\, \mathrm{X} \vee \mathrm{W} \leq \mathrm{X}' \leq \mathrm{X} \vee \mathrm{Y}} \mathtt{M}_{\Delta_{\mathrm{X}}^{\mathrm{X}'}(x) \otimes y} \\ &= \mathtt{F}_{\Delta_{\mathrm{X}}^{\mathrm{X}\vee\mathrm{W}}(x) \otimes y}. \end{aligned}$$

In the last step, note that $(\mathrm{X} \vee \mathrm{W}) \vee \mathrm{Y} = \mathrm{X} \vee \mathrm{Y}$ (since $\mathrm{W} \leq \mathrm{Y}$). We also used coassociativity of Δ. $\square$

Proposition 15.29. *The product of $\boldsymbol{\mathcal{S}}(\mathsf{c},\mathsf{a})$ in the $\mathtt{M}$-basis is given by*

$$\mu_{\mathrm{Z}}^{\mathrm{W}}(\mathtt{M}_{x\otimes y}) = \sum_{\mathrm{Y} :\, \mathrm{Y} \geq \mathrm{Z},\, \mathrm{W} \vee \mathrm{Y} = \mathrm{Y}'} \mathtt{M}_{x \otimes \mu_{\mathrm{Y}}^{\mathrm{Y}'}(y)}, \tag{15.25}$$

for $x \in \mathsf{c}[\mathrm{X}']$ and $y \in \mathsf{a}[\mathrm{Y}']$.

PROOF. We assume the formula in the $\mathtt{M}$-basis (15.25) and use it to derive the formula in the $\mathtt{F}$-basis (15.20). Note that the two formulas are identical.

$$\begin{aligned}
\mu_{\mathrm{Z}}^{\mathrm{W}}(\mathtt{F}_{x\otimes y}) &= \sum_{\mathrm{X}'':\, \mathrm{X}'\leq \mathrm{X}''\leq \mathrm{X}'\vee \mathrm{Y}'} \mu_{\mathrm{Z}}^{\mathrm{W}}\big(\mathtt{M}_{\Delta_{\mathrm{X}'}^{\mathrm{X}''}(x)\otimes y}\big)\\
&= \sum_{\mathrm{X}'':\, \mathrm{X}'\leq \mathrm{X}''\leq \mathrm{X}'\vee \mathrm{Y}'}\ \sum_{\mathrm{Y}:\, \mathrm{Y}\geq \mathrm{Z},\, \mathrm{W}\vee \mathrm{Y}=\mathrm{Y}'} \mathtt{M}_{\Delta_{\mathrm{X}'}^{\mathrm{X}''}(x)\otimes \mu_{\mathrm{Y}}^{\mathrm{Y}'}(y)}\\
&= \sum_{\mathrm{Y}:\, \mathrm{Y}\geq \mathrm{Z},\, \mathrm{W}\vee \mathrm{Y}=\mathrm{Y}'}\ \sum_{\mathrm{X}'':\, \mathrm{X}'\leq \mathrm{X}''\leq \mathrm{X}'\vee \mathrm{Y}'} \mathtt{M}_{\Delta_{\mathrm{X}'}^{\mathrm{X}''}(x)\otimes \mu_{\mathrm{Y}}^{\mathrm{Y}'}(y)}\\
&= \sum_{\mathrm{Y}:\, \mathrm{Y}\geq \mathrm{Z},\, \mathrm{W}\vee \mathrm{Y}=\mathrm{Y}'} \mathtt{F}_{x\otimes \mu_{\mathrm{Y}}^{\mathrm{Y}'}(y)}.
\end{aligned}$$

In the last step, note that $\mathrm{X}'\vee \mathrm{Y}' = \mathrm{X}'\vee \mathrm{Y}$ (since $\mathrm{W}\leq \mathrm{X}'$). □

Exercise 15.30. Check that: The antipode $S : \boldsymbol{\mathcal{S}}(\mathsf{c},\mathsf{a}) \to \boldsymbol{\mathcal{S}}(\mathsf{c},\mathsf{a})$, on the Z-component, on the (X, Y)-summand, in the $\mathtt{M}$-basis is given by

$$S_{\mathrm{Z}}(\mathtt{M}_{x\otimes y}) = \sum_{\mathrm{Y}':\, \mathrm{Y}\geq \mathrm{Y}'\geq \mathrm{Z}} \Bigg(\sum_{\substack{\mathrm{W}:\, \mathrm{X}\geq \mathrm{W}\geq \mathrm{Z}\\ \mathrm{W}\vee \mathrm{Y}'=\mathrm{Y}}} (-1)^{\dim(\mathrm{W})} c_{\mathrm{Z}}^{\mathrm{W}}\Bigg) \mathtt{M}_{x\otimes \mu_{\mathrm{Y}'}^{\mathrm{Y}}(y)}. \tag{15.26}$$

(Substitute the (co)product formulas (15.24) and (15.25) into the commutative Takeuchi formula (12.15).) Use identity (1.74) to deduce that, if a is trivial, then

$$S_{\mathrm{Z}}(\mathtt{M}_{x\otimes y}) = (-1)^{\dim(\mathrm{X}\wedge \mathrm{Y})}\, \mathtt{M}_{x\otimes y}. \tag{15.27}$$

15.3.4. Primitive part and primitive filtration. The primitive part and primitive filtration are easy to describe in the $\mathtt{M}$-basis. As a consequence of formula (15.24), we obtain:

Proposition 15.31. *In the* $\mathtt{M}$*-basis,*

$$\mathcal{P}(\boldsymbol{\mathcal{S}}(\mathsf{c},\mathsf{a}))[\mathrm{Z}] = \bigoplus_{\mathrm{X}\wedge \mathrm{Y}=\mathrm{Z}} \mathsf{c}[\mathrm{X}]\otimes \mathsf{a}[\mathrm{Y}] = (\mathsf{c}\diamond \mathsf{a})[\mathrm{Z}],$$

with the latter as in (15.2)*. More generally, in the* $\mathtt{M}$*-basis, for* $k\geq 1$*,*

$$\mathcal{P}_k(\boldsymbol{\mathcal{S}}(\mathsf{c},\mathsf{a}))[\mathrm{Z}] = \bigoplus_{\mathrm{rk}((\mathrm{X}\wedge \mathrm{Y})/\mathrm{Z})\leq k-1} \mathsf{c}[\mathrm{X}]\otimes \mathsf{a}[\mathrm{Y}].$$

The sum in both formulas is over pairs of flats (X, Y) with both X and Y greater than Z.

15.3.5. Freeness and cofreeness. Recall from Proposition 13.9 that every bicommutative bimonoid is free as a commutative monoid and cofree as a cocommutative comonoid. In particular, this applies to $\boldsymbol{\mathcal{S}}(\mathsf{c},\mathsf{a})$. We now show how cofreeness can also be deduced using the $\mathtt{M}$-basis. To that end, we define an isomorphism

$$\boldsymbol{\mathcal{S}}(\mathsf{c},\mathsf{a})[\mathrm{Z}] \to \bigoplus_{\mathrm{W}:\, \mathrm{W}\geq \mathrm{Z}} \mathcal{P}(\boldsymbol{\mathcal{S}}(\mathsf{c},\mathsf{a}))[\mathrm{W}]. \tag{15.28}$$

For any $W = X \wedge Y$, observe that

$$\Delta_Z^W(\mathsf{c}[X] \otimes \mathsf{a}[Y]) \subseteq \mathcal{P}(\boldsymbol{\mathcal{S}}(\mathsf{c}, \mathsf{a}))[W].$$

This is how we define (15.28) on the (X, Y)-summand. Now identify the rhs of (15.28) with $\mathcal{S}^\vee(\mathcal{P}(\boldsymbol{\mathcal{S}}(\mathsf{c}, \mathsf{a})))[Z]$. This yields an isomorphism of species

$$\boldsymbol{\mathcal{S}}(\mathsf{c}, \mathsf{a}) \to \mathcal{S}^\vee(\mathcal{P}(\boldsymbol{\mathcal{S}}(\mathsf{c}, \mathsf{a}))). \tag{15.29}$$

Proposition 15.32. *The map (15.29) is an isomorphism of comonoids. In particular, $\boldsymbol{\mathcal{S}}(\mathsf{c}, \mathsf{a})$ is cofree on its primitive part.*

Exercise 15.33. Suppose we modify φ in (15.23) by summing over all $X' \geq X$ (dropping the condition $X' \leq X \vee Y$). Show that the coproduct formula continues to be given by (15.24). Observe that the change of basis specified by the modified φ is precisely the Hadamard product of the HNR isomorphism (14.10) with $\mathcal{S}^\vee(\mathsf{a})$.

15.4. Product of bimonoids with one free factor

Recall from Section 15.1 that the Hadamard product preserves free monoids. We now prove a stronger result for the Hadamard product of bimonoids, namely, that if one of the factors is free as a monoid, then so is the Hadamard product. We discuss some examples which include the Hadamard product of a free monoid with a free commutative monoid.

15.4.1. Hadamard product of bimonoids with one free factor.

Theorem 15.34. *Let p and q be any scalars. Let h be a p-bimonoid. Let k be a q-bimonoid that is free as a monoid. Then $\mathsf{h} \times \mathsf{k}$ is a pq-bimonoid that is free as a monoid. More precisely, if q is a basis of k, then*

$$\mathsf{r} = \mathcal{P}(\mathsf{h} \times \mathcal{T}_0(\mathsf{q})) \tag{15.30}$$

is a basis of $\mathsf{h} \times \mathsf{k}$.

Proof. We know from Lemma 8.1 that $\mathsf{h} \times \mathsf{k}$ is a pq-bimonoid. Now, as monoids, we have

$$\mathsf{k} \cong \mathcal{T}_q(\mathsf{q}) = \mathcal{T}_0(\mathsf{q}).$$

Hence, as monoids,

$$\mathsf{h} \times \mathsf{k} \cong \mathsf{h} \times \mathcal{T}_0(\mathsf{q}).$$

But the latter is a 0-bimonoid by Lemma 8.1, and hence by Proposition 13.1 is free as a monoid on its primitive part. ☐

Corollary 15.35. *Let p and q be any scalars. Let h be a p-bimonoid. Then the pq-bimonoid $\mathsf{h} \times \Gamma_q$ is free as a monoid.*

Proof. This is a special case of Theorem 15.34, since Γ_q is free as a monoid on the species x, see (7.24). ☐

Let us understand (15.30) in explicit terms. Using (5.18) and (8.5), we calculate

$$\mathsf{r}[A] = \bigoplus_{G\colon G \geq A} \mathcal{P}'(\mathsf{h})[A,G] \otimes \mathsf{q}[G], \tag{15.31}$$

where

$$\mathcal{P}'(\mathsf{h})[A,G] := \bigcap_{K\colon A<K\leq G} \ker(\Delta_A^K : \mathsf{h}[A] \to \mathsf{h}[K]).$$

This is a subspace of $\mathsf{h}[A]$. The intersection is over all faces K strictly greater than A and less than G.

15.4.2. Examples. Suppose $\mathsf{h} = \mathcal{T}_0(\mathsf{p})$. Using coproduct formula (6.44),

$$\mathcal{P}'(\mathcal{T}_0(\mathsf{p}))[A,G] = \bigoplus_{F\colon F\wedge G=A} \mathsf{p}[F].$$

Substituting in (15.31), we see that $\mathsf{r} = \mathsf{p} \star \mathsf{q}$, and (15.30) specializes to the formula in Proposition 15.3.

Now suppose $\mathsf{h} = \mathcal{T}(\mathsf{p})$. Recall from (15.5) that $\mathsf{h} \times \mathcal{T}_0(\mathsf{q}) = \mathcal{T\!T}_0(\mathsf{p},\mathsf{q})$. An explicit description of its primitive part is given in Proposition 15.23. This required the introduction of the $\mathtt{M}$-basis, so it is much less straightforward. One can now deduce that

$$\mathcal{P}'(\mathcal{T}(\mathsf{p}))[A,G] = \bigoplus_{F\colon A\overline{F}\wedge G=A} \mathsf{p}[F],$$

where the rhs is taken in the $\mathtt{M}$-basis relative to G. We leave it as an exercise to interpret this statement correctly.

Now suppose $\mathsf{h} = \mathcal{T}_q(\mathsf{p})$ for $q \neq 0$. In this case, Lemma 15.14 gives an explicit isomorphism $\mathsf{h} \times \mathcal{T}_0(\mathsf{q}) \cong \mathcal{T\!T}_0(\mathsf{p},\mathsf{q})$, so we are back in the previous situation.

The example $\mathsf{h} = \mathcal{S}(\mathsf{p})$ is treated below.

15.4.3. Hadamard product of a free commutative monoid and a free monoid. Let p and q be two species. Define a new species $\mathsf{p} \diamond\!\star \mathsf{q}$ by

$$(\mathsf{p} \diamond\!\star \mathsf{q})[A] := \bigoplus_{\mathrm{X}\wedge G=A} \mathsf{p}[\mathrm{X}] \otimes \mathsf{q}[G]. \tag{15.32}$$

The sum is over all flats X and faces G whose meet (in the lattice of cones) is the face A, or equivalently, A is the largest face of G whose support is less than X.

Proposition 15.36. *For any species* p *and* q*, there is a natural isomorphism of species*

$$\mathsf{p} \diamond\!\star \mathsf{q} \cong \mathcal{P}(\mathcal{S}(\mathsf{p}) \times \mathcal{T}_0(\mathsf{q})),$$

and of 0*-bimonoids*

$$\mathcal{T}_0(\mathsf{p} \diamond\!\star \mathsf{q}) \cong \mathcal{S}(\mathsf{p}) \times \mathcal{T}_0(\mathsf{q}).$$

PROOF. By the Loday–Ronco Theorem 13.2, one can pass from one isomorphism to the other. So it suffices to establish the first isomorphism. Using the coproduct formula for $\mathcal{S}(\mathsf{p})$ given by (6.51), we calculate

$$\mathcal{P}'(\mathcal{S}(\mathsf{p}))[A,G] = \bigoplus_{\mathrm{X}\colon \mathrm{X}\wedge G = A} \mathsf{p}[\mathrm{X}].$$

Now substitute in (15.31), and formula (15.32) pops out. □

15.5. Species of pairs of chambers

Recall from Section 7.3 the bimonoid of chambers. The bimonoid of pairs of chambers is obtained by taking Hadamard product of the bimonoid of chambers with its dual. Thus, it is the simplest instance of the bimonoid studied in Section 15.2, namely, $\mathsf{c} = \mathsf{a} = \mathsf{x}$, and all the results proved there apply. To keep the discussion self-contained, we proceed from first principles, and state the main results (with most proofs omitted). In particular, we write down product, coproduct and antipode formulas in the $\mathtt{F}$-basis and $\mathtt{M}$-basis. We also briefly explain the connection with the bimonoid of top-nested faces.

15.5.1. Species of pairs of chambers. Let $\mathsf{\Gamma}[A]$ be the linear span of the set of pairs of chambers both greater than A. This defines a species $\mathsf{\Gamma}$. We refer to it as the *species of pairs of chambers*. We use the letter $\mathtt{K}$ for its canonical basis. Thus, a typical basis element will be denoted $\mathtt{K}_{C/A,D/A}$.

For the dual species $\mathsf{\Gamma}^*$, we use the letter $\mathtt{F}$ for the basis dual to $\mathtt{K}$. We also define a second basis called the $\mathtt{M}$-basis which is related to the $\mathtt{F}$-basis by

$$\mathtt{F}_{E/A,D/A} = \sum_{C\colon C\geq A,\, C \text{ -- } E \text{ -- } D} \mathtt{M}_{C/A,D/A}. \tag{15.33}$$

We use the letter $\mathtt{H}$ for the basis dual to $\mathtt{M}$. This is a second basis for $\mathsf{\Gamma}$. The relation between the $\mathtt{K}$-basis and $\mathtt{H}$-basis is obtained by dualizing (15.33).

15.5.2. q-bimonoid of pairs of chambers. For any scalar q, we now define a q-bimonoid structure on $\mathsf{\Gamma}^*$. To show the dependence on q, we write $\mathsf{\Gamma}^*_q$ from now on. The coproduct and product in the $\mathtt{F}$-basis are defined by

$$\begin{gathered} \Delta^G_A : \mathsf{\Gamma}^*_q[A] \to \mathsf{\Gamma}^*_q[G] \\ \mathtt{F}_{C/A,D/A} \mapsto \begin{cases} \mathtt{F}_{GC/G,D/G} & \text{if } G \leq D, \\ 0 & \text{otherwise,} \end{cases} \end{gathered} \tag{15.34a}$$

$$\begin{gathered} \mu^K_A : \mathsf{\Gamma}^*_q[K] \to \mathsf{\Gamma}^*_q[A] \\ \mathtt{F}_{C/K,D/K} \mapsto \sum_{D'\colon D'\geq A,\, KD'=D} q^{\operatorname{dist}(D,D')}\, \mathtt{F}_{C/A,D'/A}. \end{gathered} \tag{15.34b}$$

The q-bimonoid axiom may be checked directly. Alternatively, one can avoid this computation by making use of the isomorphisms (15.37) below.

Theorem 15.37. *The coproduct and product of* $\mathbb{\Gamma}^*_q$ *in the* $\mathtt{M}$*-basis are given by*

$$\Delta^G_A : \mathbb{\Gamma}^*_q[A] \to \mathbb{\Gamma}^*_q[G]$$
$$\mathtt{M}_{C/A,D/A} \mapsto \begin{cases} \mathtt{M}_{GC/G,D/G} & \textit{if } G \leq D \textit{ and } G \leq A\overline{C}, \\ 0 & \textit{otherwise,} \end{cases} \tag{15.35a}$$

$$\mu^K_A : \mathbb{\Gamma}^*_q[K] \to \mathbb{\Gamma}^*_q[A]$$
$$\mathtt{M}_{C/K,D/K} \mapsto \sum_{\substack{(C',D'):\\ C' \geq A,\, D' \geq A \\ KC'=C,\, KD'=D \\ C' \text{ -- } KC' \text{ -- } KD' \text{ -- } D'}} q^{\operatorname{dist}(D,D')}\, M_{C'/A,D'/A}. \tag{15.35b}$$

The product and coproduct formulas in the $\mathtt{K}$-basis and $\mathtt{H}$-basis can be written by dualizing these. For instance, the product and coproduct of $\mathbb{\Gamma}_q$ in the $\mathtt{K}$-basis are given by

$$\mu^G_A : \mathbb{\Gamma}_q[G] \to \mathbb{\Gamma}_q[A]$$
$$\mathtt{K}_{C/G,D/G} \mapsto \sum_{C' :\, C' \geq A,\, GC'=C} \mathtt{K}_{C'/A,D/A}, \tag{15.36a}$$

$$\Delta^K_A : \mathbb{\Gamma}_q[A] \to \mathbb{\Gamma}_q[K]$$
$$\mathtt{K}_{C/A,D/A} \mapsto \begin{cases} q^{\operatorname{dist}(KD,D)}\, \mathtt{K}_{C/K,KD/K} & \text{if } K \leq C, \\ 0 & \text{otherwise.} \end{cases} \tag{15.36b}$$

We refer to $\mathbb{\Gamma}_q$ as the *q-bimonoid of pairs of chambers*. We write $\mathbb{\Gamma}$ for $\mathbb{\Gamma}_1$. This is the *bimonoid of pairs of chambers*. It is neither commutative nor cocommutative. The product is the same as in (15.36a), while the coproduct (15.36b) simplifies with the power of q disappearing.

15.5.3. Hadamard products and self-duality. Using the (co)product formulas of the q-bimonoid of chambers and its dual, namely, (7.19) and (7.21), one may check that

$$\begin{aligned} &\mathbb{\Gamma}^*_q \xrightarrow{\cong} \Gamma \times \Gamma^*_q \\ &\mathtt{F}_{C/A,D/A} \mapsto \mathtt{H}_{C/A} \otimes \mathtt{M}_{D/A} \end{aligned} \quad \text{and} \quad \begin{aligned} &\mathbb{\Gamma}_q \xrightarrow{\cong} \Gamma^* \times \Gamma_q \\ &\mathtt{K}_{D/A,C/A} \mapsto \mathtt{M}_{D/A} \otimes \mathtt{H}_{C/A} \end{aligned} \tag{15.37}$$

as q-bimonoids. Observe that these two facts can be deduced from each other by duality.

Lemma 15.38. *For any scalar p and nonzero scalar q, the map*

$$\Gamma_q \times \Gamma^*_p \to \Gamma \times \Gamma^*_{pq}, \qquad \mathtt{H}_{C/A} \otimes \mathtt{M}_{D/A} \to q^{\operatorname{dist}(C,D)}\, \mathtt{H}_{C/A} \otimes \mathtt{M}_{D/A}$$

is an isomorphism of pq-bimonoids.

Setting $p = 1$ and using (15.37), we deduce:

Lemma 15.39. *For $q \neq 0$, the map*

$$s_q : \mathbb{\Gamma}_q \to \mathbb{\Gamma}_q^*, \qquad \mathtt{K}_{D/A,C/A} \mapsto q^{\mathrm{dist}(C,D)}\, \mathtt{F}_{C/A,D/A} \tag{15.38}$$

is an isomorphism of q-bimonoids. In particular, $\mathbb{\Gamma}_q$ is self-dual if $q \neq 0$.

Note that the self-duality of $\mathbb{\Gamma}$ (which is the case $q = 1$) is evident from (15.37) itself. Moreover, using (8.22) and (15.37), we have an isomorphism of bimonoids

$$\mathbb{\Gamma} \xrightarrow{\cong} \mathrm{end}^{\times}(\Gamma). \tag{15.39}$$

Thus, $\mathbb{\Gamma}$ is an example of a biconvolution bimonoid.

Exercise 15.40. Recall the signature functor on species from Section 8.10. Show that: For any scalar q, the signed partners of $\mathbb{\Gamma}_q$ and $\mathbb{\Gamma}_q^*$ are $\mathbb{\Gamma}_{-q}$ and $\mathbb{\Gamma}_{-q}^*$, respectively, that is,

$$(\mathbb{\Gamma}_q)^- \cong \mathbb{\Gamma}_{-q} \qquad \text{and} \qquad (\mathbb{\Gamma}_q^*)^- \cong \mathbb{\Gamma}_{-q}^*.$$

Exercise 15.41. Check using (15.37) that: Formulas (15.34) are specializations of (15.7) and (15.8). Similarly, formulas (15.35) are specializations of (15.12) and (15.13).

15.5.4. Primitive part and (co)freeness. The primitive part pertains to the coproduct, so the parameter q does not play any part. The primitive part of $\mathbb{\Gamma}^*$ can be read off from the coproduct formula (15.35a) in the $\mathtt{M}$-basis. Namely, for each A, the set

$$\{\mathtt{M}_{C/A,D/A} \mid D \wedge A\overline{C} = A\}$$

is a linear basis of $\mathcal{P}(\mathbb{\Gamma}^*)[A]$. More generally, the k-th term in the primitive filtration, on the A-component, has basis

$$\{\mathtt{M}_{C/A,D/A} \mid \mathrm{rk}((D \wedge A\overline{C})/A) = k - 1\}.$$

The coproduct formula also shows that $\mathbb{\Gamma}^*$ is cofree on its primitive part. More precisely, there is an isomorphism of comonoids

$$\mathbb{\Gamma}^* \xrightarrow{\cong} \mathcal{T}^{\vee}(\mathcal{P}(\mathbb{\Gamma}^*))$$

which on the A-component sends $\mathtt{M}_{C/A,D/A}$ to $\mathtt{M}_{KC/K,D/K}$ with $K := D \wedge A\overline{C}$. The latter belongs to $\mathcal{P}(\mathbb{\Gamma}^*)[K]$.

Finally, since we have cofreeness, one can use self-duality to deduce freeness as well.

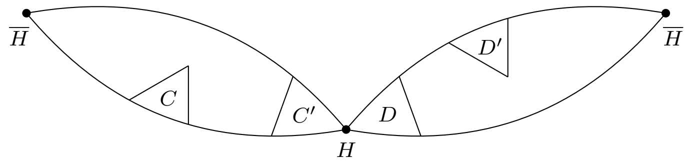
15.5.5. Antipode in the $\mathtt{F}$-basis and $\mathtt{M}$-basis. We give cancellation-free antipode formulas for $\mathbb{\Gamma}_q^*$ in the $\mathtt{F}$-basis and $\mathtt{M}$-basis.

Theorem 15.42. *Suppose $\mathcal{A}$ is simplicial. The antipode $S : \mathbb{\Gamma}_q^* \to \mathbb{\Gamma}_q^*$ in the $\mathtt{F}$-basis is given by*

$$\begin{gathered} \mathbb{\Gamma}_q^*[A] \to \mathbb{\Gamma}_q^*[A] \\ S_A(\mathtt{F}_{C/A,D/A}) = \sum (-1)^{\dim(C' \wedge D)} q^{\mathrm{dist}(D,D')}\, \mathtt{F}_{C'/A,D'/A}. \end{gathered} \tag{15.40}$$

The sum is over pairs $(C'/A, D'/A)$ *for which there is a unique face* H *greater than* A *with* $HC = C'$ *and* $HD' = D$*. (In this case, the face* H *must equal* $C' \wedge D$*).*

An illustration for $A = O$ is given below. In this figure, the two instances of $\overline{H}$ represent the same face.



PROOF. Substituting formulas (15.34) into the Takeuchi formula (12.1) and simplifying, we obtain:

$$S_A(\mathtt{F}_{C/A,D/A}) = \sum_{(C'/A,D'/A)} \Big(\sum_{H \in \mathrm{B}} (-1)^{\dim(H)} \Big) q^{\mathrm{dist}(D,D')} \, \mathtt{F}_{C'/A,D'/A},$$

where

$$\mathrm{B} = \{H \mid H \geq A,\ HC = C',\ HD' = D\}.$$

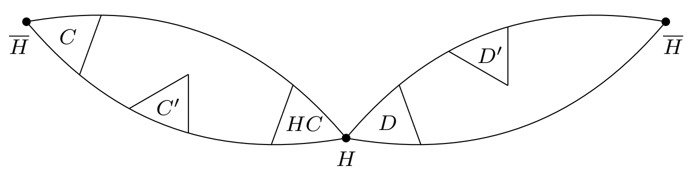
Now apply Lemma 1.41 to the arrangement $\mathcal{A}_A$. $\square$

Theorem 15.43. *Suppose* $\mathcal{A}$ *is simplicial. The antipode* $S : \mathbb{\Gamma}^*_q \to \mathbb{\Gamma}^*_q$ *in the* $\mathtt{M}$*-basis is given by*

$$\begin{gathered} \mathbb{\Gamma}^*_q[A] \to \mathbb{\Gamma}^*_q[A] \\ S_A(\mathtt{M}_{C/A,D/A}) = (-1)^{\dim(A\overline{C} \wedge D)} \sum q^{\mathrm{dist}(D,D')} \, \mathtt{M}_{C'/A,D'/A}. \end{gathered} \tag{15.41}$$

The sum is over pairs $(C'/A, D'/A)$ *such that* $C' - D - D'$ *and there is a unique face* H *greater than* A *with* $H\overline{C'} = A\overline{C}$ *and* $HD' = D$*. (In this case, the face* H *must equal* $A\overline{C} \wedge D$*.)*

An illustration for $A = O$ is given below.



PROOF. Substituting formulas (15.35) into the Takeuchi formula (12.1) and simplifying, we obtain:

$$S_A(\mathtt{M}_{C/A,D/A}) = \sum_{\substack{C',D' :\, C',D' \geq A \\ C' - D - D'}} \Big(\sum_{H \in \mathrm{B}} (-1)^{\dim(H)} \Big) q^{\mathrm{dist}(D,D')} \, \mathtt{M}_{C'/A,D'/A},$$

where

$$\mathrm{B} = \{H \mid H \geq A,\ H\overline{C'} = A\overline{C},\ HD' = D\}.$$

Now apply Lemma 1.41 to the arrangement $\mathcal{A}_A$. $\square$

Exercise 15.44. Check that: The antipode formulas (15.40) and (15.41) are specializations of formulas (15.10) and (15.15), respectively.

15.5.6. Bimonoid of top-nested faces. Recall the q-bimonoid of top-nested faces $\widehat{\mathsf{Q}}_q$ from Section 7.7. There is an injective morphism of q-bimonoids

$$\widehat{\mathsf{Q}}_q \hookrightarrow \mathsf{I\Gamma}_q. \tag{15.42}$$

On the A-component, it is equivalently given by

$$\mathtt{H}_{F/A,C/A} \mapsto \sum_{\substack{D:\, D\geq A\\ FD=C}} \mathtt{K}_{D/A,C/A}, \quad \text{or} \quad \mathtt{H}_{F/A,C/A} \mapsto \mathtt{H}_{A\overline{F}C/A,C/A}.$$

The equivalence of the two formulas can be deduced using [**21**, Proposition 3.4]. Note very carefully that the map itself does not depend on q. If $\mathcal{A}$ is simplicial, then the map may also be writtten as

$$\mathtt{K}_{F/A,C/A} \mapsto \sum_{\substack{D:\, D\geq A\\ \mathrm{Des}(D/A,C/A)=F/A}} \mathtt{K}_{D/A,C/A}.$$

The condition $\mathrm{Des}(D/A, C/A) = F/A$ means that F is the smallest face of C greater than A such that $FD = C$.

One may directly check that (15.42) is a morphism of q-bimonoids. Alternatively, it may be deduced from the universal property of $\widehat{\mathsf{Q}}_q$ as elaborated in the exercise below.

Exercise 15.45. Check that the map $\mathsf{\Gamma}^* \to \mathsf{I\Gamma}_q$ given on the A-component by $\mathtt{M}_{C/A} \mapsto \mathtt{K}_{C/A,C/A}$ is a morphism of comonoids. Applying Theorem 6.6 yields a morphism of q-bimonoids $\mathcal{T}_q(\mathsf{\Gamma}^*) \to \mathsf{I\Gamma}_q$. Use formula (6.4) to check that this morphism coincides with (15.42) (after the identification (7.106)).

By employing the self-dual map (15.38), one can form the composite

$$\widehat{\mathsf{Q}}_q \hookrightarrow \mathsf{I\Gamma}_q \xrightarrow{s_q} \mathsf{I\Gamma}_q^* \twoheadrightarrow \widehat{\mathsf{Q}}_q^*.$$

The last map is the dual of the first. Explicitly, the composite is given by

$$\widehat{\mathsf{Q}}_q \to \widehat{\mathsf{Q}}_q^*, \qquad \mathtt{H}_{F/A,C/A} \mapsto \sum_{\substack{(G,D):\, A\leq G\leq D\\ GC=D,\, \overline{F}D=C}} q^{\mathrm{dist}(F,G)}\, \mathtt{M}_{G/A,D/A}. \tag{15.43}$$

This is a self-dual morphism of q-bimonoids. Note very carefully that it is different from the one discussed in (7.108). The conditions $GC = D$ and $\overline{F}D = C$ are weaker than $GF = D$ and $FG = C$. It can also be viewed as a q-norm map on top-nested faces, as elaborated in the exercise below.

Exercise 15.46. Consider the bijection (6.77). Put $\mathsf{c} := \mathsf{\Gamma}^*$ and $\mathsf{a} := \mathsf{\Gamma}$, and let $f : \mathsf{c} \to \mathsf{a}$ be the map which on the A-component sends $\mathtt{M}_{C/A}$ to $\mathtt{H}_{C/A}$, that is, $f = \mathrm{id}$ as a map of species. Check that, under the identification (7.106), the resulting morphism of q-bimonoids $g : \widehat{\mathsf{Q}}_q \to \widehat{\mathsf{Q}}_q^*$ matches (15.43). This may also be viewed as a specialization of Exercise 6.82, item (2) to $\mathsf{p} := \mathsf{x}$.

Theorem 15.47. *For q not a root of unity, the map (15.43) is an isomorphism of q-bimonoids iff*

$$\det\Bigg(\Bigg(\sum_{\substack{F'\colon A\leq F'\leq C\\ F\colon A\leq F\leq D\\ \mathrm{s}(F)=\mathrm{s}(F'),\, FC=D}} \boldsymbol{\mu}_q(A,F,F')\Bigg)_{C,D\geq A}\Bigg)\neq 0$$

for all A. (In the above matrix, A is fixed, while C, D are the two varying indices. Each matrix entry is given by the sum over F, F'.)

Proof. This is the special case $\mathtt{p} := \mathtt{x}$ of the result in Exercise 9.101. Let us spell this out. Recall the q-logarithm (9.29b). For the map $\mathrm{id} : \Gamma^* \to \Gamma$,

$$\mathbf{log}_q(\mathrm{id})_A(\mathtt{M}_{C/A}) = \sum_{D\colon D\geq A} \alpha_A^{C,D}\, \mathtt{H}_{D/A},$$

where $\alpha_A^{C,D}$ is the sum over F, F' in parenthesis above. Now apply the isomorphism result in Exercise 9.97 and use Exercise 15.46. □

An expression for (15.43) from the $\mathtt{Q}$-basis and into the $\mathtt{P}$-basis can also be written. This would be the special case $\mathtt{p} := \mathtt{x}$ of Exercise 14.84.

Exercise 15.48. Check that: The composite map

$$\Sigma_q \hookrightarrow \widehat{\mathtt{Q}}_q \to \widehat{\mathtt{Q}}_q^* \twoheadrightarrow \Sigma_q^* \tag{15.44}$$

is the map (7.92). Here the first map is (7.111), the last map is its dual, and the middle map is (15.43).

Let us now specialize to $q = 0$. This is the 0-bimonoid of top-nested faces (Section 7.7.8).

The map (15.43) specializes to

$$\widehat{\mathtt{Q}}_0 \to \widehat{\mathtt{Q}}_0^*, \qquad \mathtt{H}_{F/A,C/A} \mapsto \sum_{G\colon A\leq G\leq C} \mathtt{M}_{G/A,C/A} \tag{15.45}$$

on the A-component. Equivalently,

$$\widehat{\mathtt{Q}}_0 \to \widehat{\mathtt{Q}}_0^*, \qquad \mathtt{Q}_{F/A,C/A} \mapsto \begin{cases} \mathtt{P}_{F/A,C/A} & \text{if } F = C,\\ 0 & \text{otherwise.}\end{cases} \tag{15.46}$$

on the A-component. In particular, the matrices appearing in Theorem 15.47 are diagonal with diagonal entries 1 or 0.

The map (15.44) specializes to

$$\Sigma_0 \to \Sigma_0^*, \qquad \mathtt{Q}_{F/A} \mapsto \begin{cases} \mathtt{P}_{F/A} & \text{if } F \text{ is a chamber},\\ 0 & \text{otherwise.}\end{cases} \tag{15.47}$$

on the A-component.

Now we consider the case $q = 1$. The map (15.43) for $q = 1$ leads to another cube of the form (7.121) with the same oblique and vertical maps, but with different horizontal maps.

$$
\begin{tikzcd}
& \widehat{\mathsf{Q}} \arrow[rr] \arrow[dd, two heads] & & \widehat{\mathsf{Q}}^* \arrow[dd, two heads] \\
\Sigma \arrow[ur] \arrow[rr] \arrow[dd, two heads] & & \widehat{\Lambda}^* \arrow[ur] \arrow[dd, two heads] & \\
& \widehat{\Lambda} \arrow[rr] & & \Sigma^* \\
\Pi \arrow[ur] \arrow[rr] & & \Pi^* \arrow[ur] &
\end{tikzcd}
\tag{15.48}
$$

Apart from (15.43) for $q = 1$, the remaining horizontal maps are as follows.

$$
\widehat{\Lambda} \to \Sigma^*, \qquad \mathtt{H}_{\mathrm{L}} \mapsto \sum_{F/A \in \mathrm{Cl}(\mathrm{L})} \mathtt{M}_{F/A}. \tag{15.49}
$$

The map is given on the A-component, with L a top-lune of $\mathcal{A}_A$. The sum is over all faces F/A in the closure of L (as opposed to its interior which was the case in (7.123)). The closure of a lune is defined in [**21**, (3.10)]. The map $\Sigma \to \widehat{\Lambda}^*$ is dual to this one. The map $\Pi \to \Pi^*$ coincides with the isomorphism ψ (and not φ) in (7.52).

We mention that the composite of the maps (7.120) and (15.42) for $q = 1$, namely,

$$
\Sigma \hookrightarrow \widehat{\mathsf{Q}} \hookrightarrow \mathsf{I\Gamma} \tag{15.50}
$$

is precisely (8.39) after the identification (15.39).

15.5.7. Bimonoid of pairs of faces. Recall that the bimonoid of pairs of chambers is obtained by taking Hadamard product of Γ with its dual. In a similar vein, one can define the *bimonoid of pairs of faces* by taking Hadamard product of Σ with its dual. One can also do something in-between by mixing faces and chambers. A summary is provided in Table 15.1. As indicated in the last column, these bimonoids are all instances of the bimonoid studied in Section 15.2. More generally, one may also consider q-analogues by combining Γ_q, Σ_q, and their duals in various ways.

TABLE 15.1. Hadamard products of faces and chambers.

bimonoid	indexing set	duality	$\mathcal{T}(\mathsf{c}) \times \mathcal{T}^\vee(\mathsf{a})$
$\Sigma \times \Sigma^*$	(face, face)	self-dual	$\mathsf{c} = \mathsf{a} = \mathsf{E}$
$\Sigma \times \Gamma^*$	(face, chamber)	duals of each other	$\mathsf{c} = \mathsf{E}$, $\mathsf{a} = \mathsf{x}$
$\Gamma \times \Sigma^*$	(chamber, face)		$\mathsf{c} = \mathsf{x}$, $\mathsf{a} = \mathsf{E}$
$\Gamma \times \Gamma^*$	(chamber, chamber)	self-dual	$\mathsf{c} = \mathsf{a} = \mathsf{x}$

Notes

The Hadamard product of graded vector spaces does not behave as nicely as the Hadamard product of Joyal species, see [**18**, Remark 8.65]. The results in this chapter are motivated by similar results in the context of Joyal species.

Bimonoids in Joyal species. The analogues of the results of Section 15.1 for Joyal species appear in [**20**, Section 3.4]. More precisely, the analogue of Proposition 15.2 is [**20**, Proposition 3.10]. The analogue of Theorem 15.34 and Corollary 15.35 is given in [**20**, Theorem 3.2 and Corollary 3.3].

The analogue of the bimonoid of pairs of chambers is the Joyal bimonoid of pairs of linear orders introduced in [**18**, Section 12.3]. The analogue of Theorem 15.37 is [**18**, Theorems 12.13 and 12.14]. The analogue of Lemma 15.39 is given in [**18**, Proposition 12.12]. The analogue of Section 15.5.4 on the primitive filtration is [**18**, Section 12.3.5]. The analogues of the antipode formulas in Theorems 15.42 and 15.43 are given in [**18**, Theorems 12.17 and 12.18].

The analogue of the cube (15.48) is a part of [**18**, Theorem 12.57]. The analogue of the result in Theorem 15.47 is an improvement on [**18**, Proposition 12.38] (in the sense that it is very specific on the values of q which yield an isomorphism). This can also be seen as progress on the question raised by Thibon and Ung [**880**, Section 3], see also [**425**, pages 80 and 81], [**18**, pages 361 and 575]. This will be clarified further in our future work on Coxeter bialgebras.

Apart from the bimonoid of pairs of chambers, analogues of the remaining three bimonoids in Table 15.1 were studied in unpublished work around the same time as [**17**], [**18**]. Relevant to the study of $\mathsf{\Sigma} \times \mathsf{\Gamma}^*$ and $\mathsf{\Gamma} \times \mathsf{\Sigma}^*$ are the partial orders involving faces and chambers defined in [**21**, Section 1.12.3].

Bialgebras. The Joyal bimonoid of pairs of linear orders relates to the Hopf algebra of permutations introduced by Malvenuto in her thesis [**634**], see Table 7.3. The precise connection is explained in our monograph [**18**, Section 17.2]. The Hopf algebra of permutations is also discussed in the books by Blessenohl and Schocker [**120**, Chapter 5], Hazewinkel, Gubareni, Kirichenko [**428**, Chapter 7], Grinberg and Reiner [**377**, Chapter 8]. It is called the 'Hopf algebra of free quasisymmetric functions' by Duchamp, Hivert, Novelli, Thibon [**268**, Section 3], [**267**]; the same terminology is continued in Méliot's book [**673**, Section 12.1].

The analogue of Lemma 15.39 for $q = 1$ appears in Malvenuto's thesis [**634**, Section 5.2] and in her paper with Reutenauer [**635**, Theorem 3.3]. The analogue of Theorem 15.37 for $q = 1$ is given by Aguiar and Sottile [**23**, Theorems 3.1 and 4.1], also see [**17**, Theorems 7.3.3 and 7.3.6]. The dual formulas are written down in [**17**, Theorems 7.5.2 and 7.5.5]. The analogue of Section 15.5.4 on the primitive filtration is [**23**, Section 6], see also [**18**, Section 17.2.5]. Results related to freeness and the primitive part were obtained earlier by Poirier and Reutenauer [**753**, Section 2] and by Duchamp, Hivert, Thibon [**268**, Sections 3.3 and 3.4]. The analogues of the antipode formulas in Theorems 15.42 and 15.43 for $q = 1$ are given in [**23**, Theorems 5.4 and 5.5], see also [**18**, Section 17.2.4]. The analogue of the deformation $\mathsf{I\Gamma}_q$ is considered by Foissy [**308**], see also [**18**, Section 17.2.7].

A closely related object is the Hopf algebra of pairs of permutations introduced in our monograph [**17**, Chapter 7]. In this setting, the analogue of Theorem 15.37 for $q = 1$ is [**17**, Theorems 7.3.1 and 7.3.4]. The primitive filtration of this Hopf algebra is computed in [**17**, Theorems 7.4.3]. For comparison with the discussion in Section 15.5.4, see the last two paragraphs in [**18**, Section 17.2.5].

Bimonoids for hyperplane arrangements. The results for the Hadamard product of bimonoids for arrangements presented in this chapter are new and appear here for the first time. The maps in diagram (15.48) (in the generality of left regular bands) first appeared in [**17**, Diagram (5.8)]. The maps $\mathsf{\Sigma} \to \widehat{\mathsf{Q}}$ in (7.120) and $\widehat{\mathsf{Q}} \to \mathsf{I\Gamma}$ in (15.42) for $q = 1$, and their composite $\mathsf{\Sigma} \to \mathsf{I\Gamma}$ in (15.50) are also discussed in some form in [**21**, Sections 9.4.6 and 9.4.7].

CHAPTER 16

Lie monoids

We introduce the notion of a Lie monoid. This goes hand-in-hand with the notions of (co)monoid, (co)commutative (co)monoid, bimonoid which we have studied in detail in earlier chapters. In contrast to monoids and commutative monoids, Lie monoids are considerably harder to formulate and study.

Operads were discussed in Chapter 4. Recall from Section 4.11 that left modules over the commutative and associative operads are commutative monoids and monoids, respectively. In a similar vein, left modules over the Lie operad are defined to be Lie monoids. One can also formulate Lie monoids in terms of a Lie bracket subject to antisymmetry and Jacobi identity. This arises from the presentation of the Lie operad. We work throughout with the definition of Lie monoids as left modules over the Lie operad; however, for additional clarity, we also illustrate many constructions and results using the Lie bracket formulation.

Every monoid carries the structure of a Lie monoid via the commutator bracket. For a bimonoid, the commutator bracket restricts to its primitive part. Thus, the primitive part $\mathcal{P}(\mathsf{h})$ of any bimonoid h carries the structure of a Lie monoid. In the other direction, to every Lie monoid g, one can associate its universal enveloping monoid $\mathcal{U}(\mathsf{g})$. The latter is a quotient of the cocommutative bimonoid $\mathcal{T}(\mathsf{g})$. These constructions yield the following adjunctions.

$$\text{Lie monoid} \underset{\text{commutator}}{\overset{\mathcal{U}}{\rightleftarrows}} \text{monoid}$$

$$\text{Lie monoid} \underset{\mathcal{P}}{\overset{\mathcal{U}}{\rightleftarrows}} \text{cocommutative bimonoid}$$

The free Lie monoid on a species can be described using the Lie operad. Recall that the functors $\mathcal{P}$ and $\mathcal{T}$ define an adjunction between the categories of species and bimonoids. The monad on species given by $\mathcal{P}\mathcal{T}$ coincides with the monad induced by the Lie operad. In particular, this explains why the primitive part of a bimonoid is a Lie monoid. It also shows that the primitive part of the bimonoid $\mathcal{T}(\mathsf{p})$ is the free Lie monoid on p. As a special case, the Zie species carries the structure of a Lie monoid and moreover, it is isomorphic to the free Lie monoid on the exponential species.

All the above considerations carry over to the signed setting. Thus, we have the notion of a signed Lie monoid which is a left module over the signed

Lie operad. A signed Lie monoid can also be formulated using signed antisymmetry and signed Jacobi identity which involve the signed distance function on faces.

We also briefly discuss the dual notion of Lie comonoids. They are left comodules over the Lie cooperad. They can also be formulated in terms of a Lie cobracket. Related notions are the cocommutator cobracket, cofree Lie comonoid, universal coenveloping comonoid $\mathcal{U}^\vee(\mathsf{g})$. The latter is a subbimonoid of the commutative bimonoid $\mathcal{T}^\vee(\mathsf{g})$. We have the following adjunctions.

$$\text{comonoid} \underset{\mathcal{U}^\vee}{\overset{\text{cocommutator}}{\rightleftarrows}} \text{Lie comonoid}$$

$$\text{commutative bimonoid} \underset{\mathcal{U}^\vee}{\overset{\mathcal{Q}}{\rightleftarrows}} \text{Lie comonoid}$$

The comonad $\mathcal{Q}\mathcal{T}^\vee$ on species arising from the adjunction between $\mathcal{Q}$ and $\mathcal{T}^\vee$ coincides with the comonad induced by the Lie cooperad.

16.1. Lie monoids

A Lie monoid is defined to be a left module over the Lie operad. Alternatively, it can also be formulated directly in terms of a Lie bracket subject to antisymmetry and Jacobi identity without making any explicit reference to the Lie operad. This parallels the classical definition of a Lie algebra. We present both approaches here. The fact that they are equivalent is in essence the presentation of the Lie operad.

16.1.1. Lie monoids as left modules. Recall the Lie operad **Lie** from Section 4.5. A *Lie monoid* is a left module over the Lie operad. A morphism of Lie monoids is a map of left modules. We denote the category of Lie monoids by $\mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp})$.

Explicitly: A Lie monoid is a species g equipped with linear maps

$$\gamma_{\mathrm{X}}^{\mathrm{Y}} : \mathbf{Lie}[\mathrm{X},\mathrm{Y}] \otimes \mathsf{g}[\mathrm{Y}] \to \mathsf{g}[\mathrm{X}],$$

one for each $\mathrm{X} \leq \mathrm{Y}$, which satisfy the following axioms.

Associativity. For any $\mathrm{X} \leq \mathrm{Y} \leq \mathrm{Z}$, the diagram

$$\begin{tikzcd} \mathbf{Lie}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{Lie}[\mathrm{Y},\mathrm{Z}] \otimes \mathsf{g}[\mathrm{Z}] \arrow[r, "\mathrm{id}\otimes\gamma_{\mathrm{Y}}^{\mathrm{Z}}"] \arrow[d, "\gamma\otimes\mathrm{id}"'] & \mathbf{Lie}[\mathrm{X},\mathrm{Y}] \otimes \mathsf{g}[\mathrm{Y}] \arrow[d, "\gamma_{\mathrm{X}}^{\mathrm{Y}}"] \\ \mathbf{Lie}[\mathrm{X},\mathrm{Z}] \otimes \mathsf{g}[\mathrm{Z}] \arrow[r, "\gamma_{\mathrm{X}}^{\mathrm{Z}}"'] & \mathsf{g}[\mathrm{X}] \end{tikzcd} \tag{16.1a}$$

commutes.

Unitality. For any X, the diagram

$$\begin{tikzcd} & \mathbf{Lie}[\mathrm{X},\mathrm{X}] \otimes \mathsf{g}[\mathrm{X}] \arrow[dr, "\gamma_{\mathrm{X}}^{\mathrm{X}}"] & \\ \Bbbk \otimes \mathsf{g}[\mathrm{X}] \arrow[ur, "\eta\otimes\mathrm{id}"] \arrow[rr, "\cong"'] & & \mathsf{g}[\mathrm{X}] \end{tikzcd} \tag{16.1b}$$

commutes. (Recall that $\mathbf{Lie}[\mathrm{X},\mathrm{X}] = \Bbbk$ and η is the identity map.)

A morphism $f : \mathsf{g} \to \mathsf{h}$ of Lie monoids is a map of species such that for any $\mathrm{X} \leq \mathrm{Y}$, the diagram

$$\begin{tikzcd} \mathbf{Lie}[\mathrm{X},\mathrm{Y}] \otimes \mathsf{g}[\mathrm{Y}] \arrow[r, "\gamma_{\mathrm{X}}^{\mathrm{Y}}"] \arrow[d, "\mathrm{id} \otimes f_{\mathrm{Y}}"'] & \mathsf{g}[\mathrm{X}] \arrow[d, "f_{\mathrm{X}}"] \\ \mathbf{Lie}[\mathrm{X},\mathrm{Y}] \otimes \mathsf{h}[\mathrm{Y}] \arrow[r, "\gamma_{\mathrm{X}}^{\mathrm{Y}}"'] & \mathsf{h}[\mathrm{X}] \end{tikzcd} \tag{16.2}$$

commutes.

These are specializations of (4.37a), (4.37b), (4.38) with $\mathbf{a} := \mathbf{Lie}$ and $\mathsf{m} := \mathsf{g}$.

16.1.2. Lie monoids by generators and relations. A *Lie monoid* is a species g equipped with linear maps

$$\nu_A^F : \mathsf{g}[F] \to \mathsf{g}[A],$$

one for each $A \lessdot F$, subject to the following axioms.

Naturality. For each morphism $\beta_{B,A} : A \to B$, and $A \lessdot F$, the diagram

$$\begin{tikzcd} \mathsf{g}[F] \arrow[r, "\beta_{BF,F}"] \arrow[d, "\nu_A^F"'] & \mathsf{g}[BF] \arrow[d, "\nu_B^{BF}"] \\ \mathsf{g}[A] \arrow[r, "\beta_{B,A}"'] & \mathsf{g}[B] \end{tikzcd} \tag{16.3a}$$

commutes. Note that $A \lessdot F$ implies $B \lessdot BF$.

Antisymmetry. For each $A \lessdot F$,

$$(\mathsf{g}[F] \xrightarrow{\nu_A^F} \mathsf{g}[A]) \ + \ (\mathsf{g}[F] \xrightarrow{\beta_{A\overline{F},F}} \mathsf{g}[A\overline{F}] \xrightarrow{\nu_A^{A\overline{F}}} \mathsf{g}[A]) \ = 0. \tag{16.3b}$$

Note that $A \lessdot F$ implies $A \lessdot A\overline{F}$.

Jacobi identity. For each $\mathrm{s}(A) \leq \mathrm{X}$ such that the face A has codimension two in the flat X,

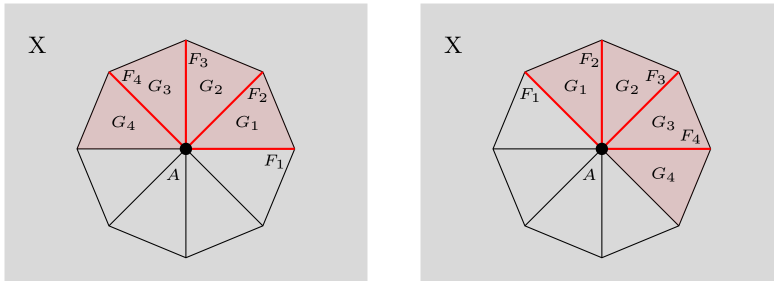
$$\sum_{i=1}^{n} (\mathsf{g}[G_1] \xrightarrow{\beta_{G_i,G_1}} \mathsf{g}[G_i] \xrightarrow{\nu_{F_i}^{G_i}} \mathsf{g}[F_i] \xrightarrow{\nu_A^{F_i}} \mathsf{g}[A]) \ = 0. \tag{16.3c}$$

The integer n and the faces F_i and G_i greater than A are obtained as follows. Since X/A is a rank-two flat of $\mathcal{A}_A$, its spherical model is a $2n$-gon for some $n \geq 2$. Pick any minimal gallery

$$G_1 \text{ -- } G_2 \text{ -- } \dots \text{ -- } G_n$$

consisting of distinct faces greater than A and of support X. Let F_1 be the outer panel of G_1, and F_i be the panel between G_{i-1} and G_i for $2 \leq i \leq n$.

Two illustrations for $n = 4$ are shown below.



Note very carefully that the G_i only sweep out half of the polygon.

Exercise 16.1. The formulation of Jacobi identity (16.3c), with A and X fixed, involves a choice. Show using antisymmetry (16.3b) that a different choice of minimal gallery leads to an equivalent condition.

We denote a Lie monoid by a pair (g, ν), or simply by g with ν understood. We refer to ν as the *Lie bracket*, and to the maps ν_A^F for $A \lessdot F$ as the Lie bracket components.

A morphism $f : \mathsf{g} \to \mathsf{h}$ of Lie monoids is a map of species such that for each $A \lessdot F$, the diagram

$$\begin{tikzcd} \mathsf{g}[F] \arrow[r, "f_F"] \arrow[d, "\nu_A^F"'] & \mathsf{h}[F] \arrow[d, "\nu_A^F"] \\ \mathsf{g}[A] \arrow[r, "f_A"'] & \mathsf{h}[A] \end{tikzcd} \tag{16.4}$$

commutes.

16.1.3. Back and forth. The two viewpoints on Lie monoids discussed above are equivalent. This follows from the presentation of the Lie operad given in Example 4.12. We now explain in explicit terms how to pass back and forth between the two viewpoints.

Suppose g is a Lie monoid defined as a left module with structure maps $\gamma_{\mathrm{X}}^{\mathrm{Y}}$ for $\mathrm{X} \leq \mathrm{Y}$. The Lie bracket component ν_A^F for $A \lessdot F$ is constructed as follows. Let X and Y be the supports of A and F, respectively. Observe that the element $\mathtt{H}_{F/A} - \mathtt{H}_{A\overline{F}/A}$ defines an element in $\mathbf{Lie}[\mathrm{X}, \mathrm{Y}]$. Now, define

$$\nu_A^F(v) := \beta_{A,\mathrm{X}} \gamma_{\mathrm{X}}^{\mathrm{Y}} \big((\mathtt{H}_{F/A} - \mathtt{H}_{A\overline{F}/A}) \otimes \beta_{\mathrm{Y},F}(v) \big), \quad v \in \mathsf{g}[F]. \tag{16.5}$$

Antisymmetry (16.3b) is verified below.

$$\begin{aligned} &\nu_A^F(v) + \nu_A^{A\overline{F}} \beta_{A\overline{F},F}(v) \\ &= \beta_{A,\mathrm{X}} \gamma_{\mathrm{X}}^{\mathrm{Y}} \big((\mathtt{H}_{F/A} - \mathtt{H}_{A\overline{F}/A}) \otimes \beta_{\mathrm{Y},F}(v) + (\mathtt{H}_{A\overline{F}/A} - \mathtt{H}_{F/A}) \otimes \beta_{\mathrm{Y},A\overline{F}} \beta_{A\overline{F},F}(v) \big) \\ &= \beta_{A,\mathrm{X}} \gamma_{\mathrm{X}}^{\mathrm{Y}} \big((\mathtt{H}_{F/A} - \mathtt{H}_{A\overline{F}/A}) \otimes \beta_{\mathrm{Y},F}(v) + (\mathtt{H}_{A\overline{F}/A} - \mathtt{H}_{F/A}) \otimes \beta_{\mathrm{Y},F}(v) \big) \\ &= 0. \end{aligned}$$

The main point here is that F and $A\overline{F}$ have the same support Y, and $\mathtt{H}_{F/A}$ and $\mathtt{H}_{A\overline{F}/A}$ appear with opposite signs in the two terms.

We now indicate why Jacobi identity (16.3c) holds. The map in each summand in (16.3c) involves four edges of the $2n$-gon, namely, two adjacent edges and their opposites; thus, each edge appears in exactly two summands with opposite signs which cancel.

Conversely, suppose g is a Lie monoid defined by Lie bracket components ν_A^F for $A \lessdot F$. The structure map $\gamma_{\mathrm{X}}^{\mathrm{Y}}$ for $\mathrm{X} \leq \mathrm{Y}$ is constructed as follows. First suppose $\mathrm{X} \lessdot \mathrm{Y}$. Let A and F be faces with supports X and Y such that $A \lessdot F$. Then $\mathbf{Lie}[\mathrm{X}, \mathrm{Y}]$ is one-dimensional and spanned by the element $\mathtt{H}_{F/A} - \mathtt{H}_{A\overline{F}/A}$. Define

$$\gamma_{\mathrm{X}}^{\mathrm{Y}}\big((\mathtt{H}_{F/A} - \mathtt{H}_{A\overline{F}/A}) \otimes v\big) := \beta_{\mathrm{X},A}\nu_A^F\beta_{F,\mathrm{Y}}(v), \quad v \in \mathsf{g}[\mathrm{Y}]. \tag{16.6}$$

This is well-defined by antisymmetry (16.3b). In the general case, pick a maximal chain of flats from X to Y, say $\mathrm{X} \lessdot \mathrm{X}_1 \lessdot \cdots \lessdot \mathrm{X}_{n-1} \lessdot \mathrm{Y}$. For

$$x_1 \otimes \cdots \otimes x_n \in \mathbf{Lie}[\mathrm{X}, \mathrm{X}_1] \otimes \cdots \otimes \mathbf{Lie}[\mathrm{X}_{n-1}, \mathrm{Y}],$$

let $\gamma(x_1 \otimes \cdots \otimes x_n)$ denote its image in $\mathbf{Lie}[\mathrm{X}, \mathrm{Y}]$ under iterated substitution. (Since $\mathbf{Lie}$ is a binary quadratic operad, every Lie element is expressible as a sum of such elements.) For $v \in \mathsf{g}[\mathrm{Y}]$, define

$$\gamma_{\mathrm{X}}^{\mathrm{Y}}\big(\gamma(x_1 \otimes \cdots \otimes x_n) \otimes v\big) := \gamma_{\mathrm{X}}^{\mathrm{X}_1}(x_1 \otimes \gamma_{\mathrm{X}_1}^{\mathrm{X}_2}(x_2 \otimes \cdots \otimes \gamma_{\mathrm{X}_{n-1}}^{\mathrm{Y}}(x_n \otimes v) \dots)), \tag{16.7}$$

with the maps in the rhs defined using (16.6). (This makes sense since $\mathrm{X}_i \lessdot \mathrm{X}_{i+1}$ for each i.) Antisymmetry (16.3b) and Jacobi identity (16.3c) ensure that $\gamma_{\mathrm{X}}^{\mathrm{Y}}$ is well-defined and satisfies the module axioms.

16.2. Commutator bracket and primitive part of bimonoids

Recall that every commutative monoid is in particular a monoid. In a similar vein, every monoid carries the structure of a Lie monoid via the commutator bracket. Moreover, for any bimonoid, the commutator bracket restricts to the primitive part of the bimonoid. This yields a functor from the category of bimonoids to the category of Lie monoids which we continue to call the primitive part functor.

16.2.1. Commutator bracket. Recall from Section 4.5 that the Lie operad $\mathbf{Lie}$ is a suboperad of the associative operad $\mathbf{As}$. Hence, there is an induced functor

$$\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp}) \to \mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp}). \tag{16.8}$$

We call this the *underlying Lie monoid functor*.

In particular, every monoid a carries the structure of a Lie monoid given by the composite

$$\mathbf{Lie} \circ \mathsf{a} \hookrightarrow \mathbf{As} \circ \mathsf{a} \to \mathsf{a}.$$

Explicitly: Let μ_A^F, one for each $A \leq F$, denote the product components of a. Then, for any $\mathrm{X} \leq \mathrm{Y}$, the Lie structure map $\gamma_{\mathrm{X}}^{\mathrm{Y}}$ of a is given by

$$\sum_{\substack{F:\, F \geq A \\ \mathrm{s}(F)=\mathrm{Y}}} x^{F/A}\, \mathtt{H}_{F/A} \otimes v \longmapsto \sum_{\substack{F:\, F \geq A \\ \mathrm{s}(F)=\mathrm{Y}}} x^{F/A}\, \beta_{\mathrm{X},A} \mu_A^F \beta_{F,\mathrm{Y}}(v), \quad v \in \mathsf{a}[\mathrm{Y}], \tag{16.9}$$

where A is any fixed face with $\mathrm{s}(A) = \mathrm{X}$. Since monoids are formulated using faces, it is also convenient to consider the map

$$\sum_{\substack{F:\, F \geq A \\ \mathrm{s}(F)=\mathrm{Y}}} x^{F/A}\, \mathtt{H}_{F/A} \otimes v \longmapsto \sum_{\substack{F:\, F \geq A \\ \mathrm{s}(F)=\mathrm{Y}}} x^{F/A}\, \mu_A^F \beta_{F,\mathrm{Y}}(v), \quad v \in \mathsf{a}[\mathrm{Y}], \tag{16.10}$$

whose image is in $\mathsf{a}[A]$ instead of $\mathsf{a}[\mathrm{X}]$. Let us denote this map by γ_A^{Y}.

Equivalently, one can work with the formulation of a monoid a given by Proposition 2.78. This only involves product components μ_A^F for $A \lessdot F$. In this situation, one can describe the Lie structure of a as follows. For $A \lessdot F$, the Lie bracket component ν_A^F of a is given by

$$\nu_A^F = \mu_A^F - \mu_A^{A\overline{F}} \beta_{A\overline{F},F}. \tag{16.11}$$

We refer to ν as the *commutator bracket*.

16.2.2. Primitive part of a bimonoid. Let h be a bimonoid. In particular, it is a monoid, and hence a Lie monoid via the functor (16.8). We now show that this Lie structure restricts to its primitive part $\mathcal{P}(\mathsf{h})$ as defined in (5.18).

Proposition 16.2. *If* h *is a bimonoid, then* $\mathcal{P}(\mathsf{h})$ *is a Lie submonoid of* h*.*

We give proofs using each of the two formulations of a Lie monoid.

First proof. For any $\mathrm{X} \leq \mathrm{Y}$, we need to show that the dotted arrow in the diagram

$$\begin{tikzcd}
\mathbf{Lie}[\mathrm{X},\mathrm{Y}] \otimes \mathcal{P}(\mathsf{h})[\mathrm{Y}] \arrow[r, hook] \arrow[d, dotted] & \mathbf{As}[\mathrm{X},\mathrm{Y}] \otimes \mathsf{h}[\mathrm{Y}] \arrow[d] \\
\mathcal{P}(\mathsf{h})[\mathrm{X}] \arrow[r, hook] & \mathsf{h}[\mathrm{X}]
\end{tikzcd}$$

exists. The map going across and down is described in (16.9). We need to show that its image belongs to $\mathcal{P}(\mathsf{h})[\mathrm{X}]$. Explicitly, we need to show that for any $G > A$, any Lie element

$$\sum_{F:\, F \geq A,\, \mathrm{s}(F)=\mathrm{Y}} x^{F/A}\, \mathtt{H}_{F/A}$$

(with A of support X) and $v \in \mathcal{P}(\mathsf{h})[\mathrm{Y}]$,

$$\Delta_A^G\Big(\sum_{F:\, F \geq A,\, \mathrm{s}(F)=\mathrm{Y}} x^{F/A} \mu_A^F(\beta_{F,\mathrm{Y}}(v))\Big) = 0.$$

Note very carefully that the expression in parenthesis is an element of $\mathsf{h}[A]$, so the term $\beta_{\mathrm{X},A}$ appearing in (16.9) is absent. In other words, we are working with the map (16.10). We now apply Lemma 5.39 for $q = 1$ to each summand in the lhs. For convenience, let us spell this out. By the bimonoid axiom (2.12), we need to show that for $G > A$,

$$\sum_{F\colon F\geq A,\, \mathrm{s}(F)=\mathrm{Y}} x^{F/A}\mu_G^{GF}\beta_{GF,FG}\Delta_F^{FG}\beta_{F,\mathrm{Y}}(v) = 0. \tag{16.12}$$

There are two cases.

- <u>$\mathrm{s}(G) \not\leq \mathrm{Y}$, or equivalently, $FG > F$ for all $F \geq A$ with $\mathrm{s}(F) = \mathrm{Y}$.</u>

  In this case, since $\beta_{F,\mathrm{Y}}(v)$ is a primitive element in $\mathsf{h}[F]$, applying Δ_F^{FG} yields 0.

- <u>$\mathrm{s}(G) \leq \mathrm{Y}$, or equivalently, $FG = F$ for all $F \geq A$ with $\mathrm{s}(F) = \mathrm{Y}$.</u>

  In this case, $\Delta_F^{FG} = \mathrm{id}$. Thus,

$$\begin{aligned}\text{lhs of (16.12)} &= \sum_{F\colon F\geq A,\, \mathrm{s}(F)=\mathrm{Y}} x^{F/A}\mu_G^{GF}\beta_{GF,\mathrm{Y}}(v)\\ &= \sum_{H\colon H\geq G,\, \mathrm{s}(H)=\mathrm{Y}} \Big(\sum_{\substack{F\colon F\geq A,\, \mathrm{s}(F)=\mathrm{Y}\\ GF=H}} x^{F/A}\Big)\mu_G^{H}\beta_{H,\mathrm{Y}}(v)\\ &= 0.\end{aligned}$$

  In the second step, we introduced a new variable $H = GF$. The sum in parenthesis is zero by the definition of a Lie element (1.164) applied to the arrangement $\mathcal{A}_A^{\mathrm{Y}}$.

This completes the proof. □

Second proof. For any $A \lessdot F$, let ν_A^F be the commutator bracket on h defined by (16.11). We need to check that it restricts to $\mathcal{P}(\mathsf{h})$, that is, the dotted arrow in the diagram

$$\begin{tikzcd} \mathcal{P}(\mathsf{h})[F] \arrow[r, hook] \arrow[d, dashed] & \mathsf{h}[F] \arrow[d, "\nu_A^F"] \\ \mathcal{P}(\mathsf{h})[A] \arrow[r, hook] & \mathsf{h}[A] \end{tikzcd}$$

exists. Accordingly, let $v \in \mathcal{P}(\mathsf{h})[F]$. We want to show that $\nu_A^F(v) \in \mathcal{P}(\mathsf{h})[A]$, that is, $\Delta_A^G(\nu_A^F(v)) = 0$, for any face $G \gtrdot A$. There are two cases.

- <u>$G \neq F$ and $G \neq A\overline{F}$.</u>

  In this case, since FG is strictly greater than F, $A\overline{F}G$ is strictly greater than $A\overline{F}$, and v is primitive, we deduce from the bimonoid axiom (2.12) that

$$\Delta_A^G(\mu_A^F(v)) = 0 = \Delta_A^G(\mu_A^{A\overline{F}}\beta_{A\overline{F},F}(v)).$$

  Equivalently, we can apply the second alternative in Lemma 5.39 for $q = 1$.

- <u>Either $G = F$ or $G = A\overline{F}$.</u>

  In this case, by (2.13),

$$\Delta_A^G(\mu_A^F(v)) = \beta_{G,F}(v) = \Delta_A^G(\mu_A^{A\overline{F}}\beta_{A\overline{F},F}(v)).$$

Thus, in both cases,

$$\Delta_A^G(\nu_A^F(v)) = \Delta_A^G(\mu_A^F(v)) - \Delta_A^G(\mu_A^{A\overline{F}}\beta_{A\overline{F},F}(v)) = 0$$

as required. □

Proposition 16.2 yields a functor

$$\mathcal{P} : \mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \to \mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp}) \tag{16.13}$$

from the category of bimonoids to the category of Lie monoids. We continue to call it the *primitive part functor*.

Example 16.3. For a cocommutative comonoid c and bimonoid k, recall from Section 8.4 the bimonoid of star families $\mathcal{C}(\mathsf{c},\mathsf{k})$. By formula (8.30), its primitive part is $\mathrm{hom}^{\times}(\mathsf{c},\mathcal{P}(\mathsf{k}))$. By Proposition 16.2, this carries the structure of a Lie monoid. Explicitly, the structure map

$$\mathbf{Lie}[\mathrm{X},\mathrm{Y}] \otimes \mathrm{hom}^{\times}(\mathsf{c},\mathcal{P}(\mathsf{k}))[\mathrm{Y}] \to \mathrm{hom}^{\times}(\mathsf{c},\mathcal{P}(\mathsf{k}))[\mathrm{X}]$$

can be specified as follows.

$$\begin{aligned}&\mathbf{Lie}[\mathrm{X},\mathrm{Y}] \otimes \mathrm{hom}^{\times}(\mathsf{c},\mathcal{P}(\mathsf{k}))[\mathrm{Y}] \otimes \mathsf{c}[\mathrm{X}]\\ &\qquad\xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\Delta_{\mathrm{X}}^{\mathrm{Y}}} \mathbf{Lie}[\mathrm{X},\mathrm{Y}] \otimes \mathrm{hom}^{\times}(\mathsf{c},\mathcal{P}(\mathsf{k}))[\mathrm{Y}] \otimes \mathsf{c}[\mathrm{Y}]\\ &\qquad\qquad\qquad\to \mathbf{Lie}[\mathrm{X},\mathrm{Y}] \otimes \mathcal{P}(\mathsf{k})[\mathrm{Y}] \to \mathcal{P}(\mathsf{k})[\mathrm{X}].\end{aligned}$$

The last map is the Lie structure map of $\mathcal{P}(\mathsf{k})$.

Equivalently, the Lie bracket

$$\nu_A^F : \mathrm{hom}^{\times}(\mathsf{c},\mathcal{P}(\mathsf{k}))[F] \to \mathrm{hom}^{\times}(\mathsf{c},\mathcal{P}(\mathsf{k}))[A]$$

is as follows. It sends the map f to the composite

$$\mathsf{c}[A] \xrightarrow{\Delta_A^F} \mathsf{c}[F] \xrightarrow{f} \mathcal{P}(\mathsf{k})[F] \xrightarrow{\nu_A^F} \mathcal{P}(\mathsf{k})[A].$$

The last map is the Lie bracket of $\mathcal{P}(\mathsf{k})$.

16.3. Free Lie monoids on species

The free Lie monoid can be described using the Lie operad. The monad on species arising from the Lie operad coincides with the monad $\mathcal{P}\mathcal{T}$ arising from the adjunction of Theorem 6.30 for $q = 1$. This yields another proof of the existence of the primitive part functor from bimonoids to Lie monoids. We also deduce that the primitive part of the free bimonoid is the free Lie monoid.

16.3.1. Free Lie monoid on a species.

Proposition 16.4. *The free Lie monoid on a species* p *is given by* $\mathbf{Lie} \circ \mathsf{p}$. *Explicitly,*

$$(\mathbf{Lie} \circ \mathsf{p})[\mathrm{X}] = \bigoplus_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{X}} \mathbf{Lie}[\mathrm{X},\mathrm{Y}] \otimes \mathsf{p}[\mathrm{Y}].$$

Proof. This is a specialization of Proposition 4.23 to the Lie operad. □

The universal property of the free Lie monoid is stated below. Let $\mathsf{p} \hookrightarrow \mathbf{Lie} \circ \mathsf{p}$ be the canonical inclusion.

Theorem 16.5. *Let g be a Lie monoid, p a species, $f : \mathsf{p} \to \mathsf{g}$ a map of species. Then there exists a unique morphism of Lie monoids $\hat{f} : \mathbf{Lie} \circ \mathsf{p} \to \mathsf{g}$ such that the diagram*

$$\begin{tikzcd} \mathbf{Lie} \circ \mathsf{p} \arrow[r, dashed, "\hat{f}"] & \mathsf{g} \\ \mathsf{p} \arrow[u, hook] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes.

Explicitly, the map $\hat{f}$ is given by

$$\mathbf{Lie} \circ \mathsf{p} \xrightarrow{\mathrm{id} \circ f} \mathbf{Lie} \circ \mathsf{g} \xrightarrow{\gamma} \mathsf{g}.$$

This is a specialization of Theorem 4.24 to the Lie operad.

For analogues of Proposition 16.4 for the free monoid and free commutative monoid, see Remarks 6.4 and 6.19, respectively. Similarly, analogues of Theorem 16.5 are given in Theorems 6.2 and 6.17, respectively.

16.3.2. Monad on species of the Lie operad. To any species p, one can associate the cocommutative bimonoid $\mathcal{T}(\mathsf{p})$. Let us recall this construction from Section 6.4. The A-component is defined by

$$\mathcal{T}(\mathsf{p})[A] := \bigoplus_{F\colon F \geq A} \mathsf{p}[F]. \tag{16.14}$$

The product component μ_A^G is given by

$$\bigoplus_{K\colon K \geq G} \mathsf{p}[K] \xrightarrow{\mu_A^G} \bigoplus_{F\colon F \geq A} \mathsf{p}[F], \qquad \mathsf{p}[K] \xrightarrow{\cong} \mathsf{p}[K], \tag{16.15}$$

while the coproduct component Δ_A^G is given by

$$\bigoplus_{F\colon F \geq A} \mathsf{p}[F] \xrightarrow{\Delta_A^G} \bigoplus_{K\colon K \geq G} \mathsf{p}[K], \qquad \mathsf{p}[F] \xrightarrow{\beta_{GF,F}} \begin{cases} \mathsf{p}[GF] & \text{if } FG = F, \\ 0 & \text{otherwise.} \end{cases} \tag{16.16}$$

Since $\mathcal{T}(\mathsf{p})$ is the free monoid on p, we identify it with $\mathbf{As} \circ \mathsf{p}$ using (4.43), see Remark 6.4. Moreover, recall from Theorem 6.30 for $q = 1$ that $\mathcal{T}$ viewed as a functor from species to bimonoids is the left adjoint of the primitive part functor $\mathcal{P}$. This yields the monad $\mathcal{P}\mathcal{T}$ on species as discussed in Section 13.7.

Proposition 16.6. *For any species p,*

$$\mathcal{P}\mathcal{T}(\mathsf{p}) = \mathbf{Lie} \circ \mathsf{p}. \tag{16.17}$$

Explicitly, $\mathcal{P}\mathcal{T}(\mathsf{p})[A]$ consists of elements

$$(v_F)_{F \geq A} \quad \textit{such that} \quad \sum_{\substack{F\colon F \geq A \\ GF=K,\, FG=F}} \beta_{K,F}(v_F) = 0 \quad \textit{for all } A < G \leq K. \tag{16.18}$$

Moreover, the monad structure of $\mathcal{P}\mathcal{T}$ corresponds to the operad structure of **Lie**. *In particular, a $\mathcal{P}\mathcal{T}$-algebra is the same as a Lie monoid.*

We point out that (16.18) is the special case $q = 1$ of (6.71).

PROOF. Let us compute the primitive part of $\mathcal{T}(\mathsf{p})[A]$. By coproduct formula (16.16),

$$\Delta_A^G\big((v_F)_{F\geq A}\big) = \sum_{K\colon K\geq G} \sum_{\substack{F\colon F\geq A \\ GF=K,\, \overline{F}G=F}} \beta_{K,F}(v_F).$$

It follows that the subspace $\mathcal{P}\mathcal{T}(\mathsf{p})[A] = \bigcap_{G>A} \ker(\Delta_A^G)$ consists of the elements (16.18). Further, from the identification (4.43) and the definition of a Lie element (1.164) applied to the arrangements $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$, we note that this subspace coincides with the image of the injective map

$$\bigoplus_{\mathrm{Y}\colon \mathrm{Y}\geq \mathrm{X}} \mathbf{Lie}[\mathrm{X},\mathrm{Y}] \otimes \mathsf{p}[\mathrm{Y}] \hookrightarrow \bigoplus_{\mathrm{Y}\colon \mathrm{Y}\geq \mathrm{X}} \mathbf{As}[\mathrm{X},\mathrm{Y}] \otimes \mathsf{p}[\mathrm{Y}] \xrightarrow{\cong} \bigoplus_{F\colon F\geq A} \mathsf{p}[F],$$

where $\mathrm{X} = \mathrm{s}(A)$. The first claim follows. The second claim is a formal consequence since **Lie** is a suboperad of **As**, and $\mathcal{P}\mathcal{T}$ is a submonad of $\mathcal{T}$. $\square$

Let h be a bimonoid. Then $\mathcal{P}(\mathsf{h})$ is a $\mathcal{P}\mathcal{T}$-algebra, or equivalently, a Lie monoid with the structure map

$$\mathcal{P}\mathcal{T}\mathcal{P}(\mathsf{h}) \to \mathcal{P}(\mathsf{h})$$

induced by the counit of the adjunction. This gives another derivation of Proposition 16.2.

Corollary 16.7. *For any species* p*, the primitive part of the bimonoid* $\mathcal{T}(\mathsf{p})$ *is the free Lie monoid on* p*. In particular, for a finite-dimensional species* p*,*

$$\dim \mathcal{P}(\mathcal{T}(\mathsf{p}))[\mathrm{X}] = \sum_{\mathrm{Y}\colon \mathrm{Y}\geq \mathrm{X}} |\mu(\mathrm{X},\mathrm{Y})| \dim \mathsf{p}[\mathrm{Y}], \tag{16.19}$$

where μ *denotes the Möbius function of the poset of flats.*

Thus, as a comonoid, $\mathcal{T}(\mathsf{p})$ *is isomorphic to the cofree cocommutative comonoid on the species* $\mathbf{Lie} \circ \mathsf{p}$*.*

The last claim follows from the Borel–Hopf Theorem 13.34.

Exercise 16.8. For a finite-dimensional species p, observe that

$$\dim \mathcal{T}(\mathsf{p})[\mathrm{X}] = \sum_{\mathrm{Y}\colon \mathrm{Y}\geq \mathrm{X}} c_{\mathrm{X}}^{\mathrm{Y}} \dim \mathsf{p}[\mathrm{Y}],$$

where $c_{\mathrm{X}}^{\mathrm{Y}}$ denotes the number of chambers in $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$. Deduce this result by combining dimension formulas (13.29), (16.19) with the Zaslavsky formula (1.84).

16.3.3. Primitive part of the free bimonoid on a comonoid. Recall the free bimonoid $\mathcal{T}(\mathsf{c})$ on a comonoid c from Section 6.1.2.

Proposition 16.9. *For any cocommutative comonoid* c*, there is an isomorphism of Lie monoids*

$$\mathcal{P}\mathcal{T}(\mathsf{c}) \cong \mathbf{Lie} \circ \mathsf{c}_t,$$

where c_t *is the underlying species of the comonoid* c*. The rhs is the free Lie monoid on* c_t*.*

Thus, as a comonoid, $\mathcal{T}(\mathsf{c})$ is isomorphic to the cofree cocommutative comonoid on the species $\mathbf{Lie} \circ \mathsf{c}_t$.

PROOF. By Proposition 14.40, $\mathcal{T}(\mathsf{c})$ is isomorphic to $\mathcal{T}(\mathsf{c}_t)$ as bimonoids via the HNR isomorphisms. The result follows by applying Corollary 16.7 to $\mathcal{T}(\mathsf{c}_t)$. □

16.4. Lie species and Zie species as Lie monoids

Lie species and Zie species are the primitive parts of the bimonoids of chambers and faces, respectively. Hence, they both carry the structure of a Lie monoid. In fact, they are the free Lie monoids on the species characteristic of chambers and on the exponential species, respectively.

16.4.1. Lie species. Recall the Lie species from Section 7.9.1. Observe that

$$\mathsf{Lie}[\mathrm{X}] = \mathbf{Lie}[\mathrm{X}, \top].$$

By restricting the product of the Lie operad, we have maps

$$\mathbf{Lie}[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{Lie}[\mathrm{Y}] \to \mathsf{Lie}[\mathrm{X}],$$

which turn the Lie species into a Lie monoid. In fact,

$$\mathsf{Lie} = \mathbf{Lie} \circ \mathsf{x},$$

where x is the species characteristic of chambers. Thus, Lie is the free Lie monoid on x, see Proposition 16.4. This can also be viewed as a special case of Example 4.25. It can also be deduced alternatively as follows.

Recall from Lemma 7.64 that the Lie species is the primitive part of the bimonoid of chambers Γ. Hence, by Proposition 16.2, it carries the structure of a Lie monoid. Recall from (7.24) for $q = 1$ that $\mathcal{T}(\mathsf{x}) = \Gamma$ as bimonoids. Hence, by applying Corollary 16.7, we deduce that Lie is the free Lie monoid on x.

Exercise 16.10. Since Lie is a Lie monoid, it can also be described by generators and relations. How does the Lie bracket work? Explicitly, verify Jacobi identity (16.3c) when the arrangement has rank two.

For any cocommutative comonoid c, the species $\hom^{\times}(\mathsf{c}, \mathsf{Lie})$ is a Lie monoid. This is the specialization of Example 16.3 to $\mathsf{k} := \Gamma$. When $\mathsf{c} := \mathsf{E}$, this recovers Lie. When $\mathsf{c} := \Gamma$, it is the primitive part of the bimonoid of chamber maps $\mathcal{C}(\Gamma, \Gamma)$ from Section 8.5.

16.4.2. Zie species. Now recall the Zie species from Section 7.9.3. It also carries the structure of a Lie monoid, but the situation is more complicated than that for the Lie species. We first recall from Lemma 7.69 that the Zie species is the primitive part of the bimonoid of faces Σ. Hence, by Proposition 16.2, it carries the structure of a Lie monoid. Recall from (7.82) that Σ is isomorphic to $\mathcal{T}(\mathsf{E})$ as bimonoids. This isomorphism depends on the choice of a Q-basis. Hence, by applying Corollary 16.7, we obtain:

Proposition 16.11. *There is an isomorphism of Lie monoids*

$$\mathsf{Zie} \cong \mathbf{Lie} \circ \mathsf{E}.$$

The rhs is the free Lie monoid on the exponential species E.

This isomorphism is consistent with the isomorphism described in (1.177) and mentioned in Exercise 7.70.

Alternatively, one can deduce the above result from Proposition 16.9 using the first isomorphism in (7.81) for $q = 1$. Note very carefully that E is viewed as a comonoid in this case.

For any cocommutative comonoid c, the species $\mathrm{hom}^{\times}(\mathsf{c}, \mathsf{Zie})$ is a Lie monoid. This is the specialization of Example 16.3 to $\mathsf{k} := \Sigma$. When $\mathsf{c} := \mathsf{E}$, this recovers the Lie monoid Zie.

16.5. Universal enveloping monoids

We associate to any Lie monoid g a monoid $\mathcal{U}(\mathsf{g})$ called the universal enveloping monoid of g. It is defined as the quotient of the free monoid $\mathcal{T}(\mathsf{g})$ by a submonoid denoted $\mathcal{I}(\mathsf{g})$. For each face A, we provide a linear spanning set of $\mathcal{I}(\mathsf{g})[A]$ using what we call H/A-relations. The construction of $\mathcal{U}(\mathsf{g})$ is functorial in g. In other words, $\mathcal{U}$ defines a functor from the category of Lie monoids to the category of monoids. It is the left adjoint of the underlying Lie monoid functor.

Moreover, the universal enveloping monoid $\mathcal{U}(\mathsf{g})$ carries the structure of a cocommutative bimonoid. It is induced from the bimonoid structure of $\mathcal{T}(\mathsf{g})$. Thus, $\mathcal{U}$ also defines a functor from the category of Lie monoids to the category of cocommutative bimonoids. It is the left adjoint of the primitive part functor $\mathcal{P}$.

16.5.1. *A/A*-relations. Suppose g is a Lie monoid. Consider the diagram

$$\begin{tikzcd} \mathbf{Lie} \circ \mathsf{g} \arrow[r, "\gamma"] \arrow[d, hook] & \mathsf{g} \arrow[d, hook] \\ \mathbf{As} \circ \mathsf{g} \arrow[r, "\cong"'] & \mathcal{T}(\mathsf{g}). \end{tikzcd} \tag{16.20}$$

The species $\mathcal{T}(\mathsf{g})$ is as in (16.14). The top-horizontal map is the Lie structure map of g. The vertical maps are the canonical inclusions. The bottom isomorphism is the identification (4.43). (This map is defined for any species g.)

The diagram (16.20) does not commute in general. Let us evaluate it on a flat X. Consider the summand of $(\mathbf{Lie} \circ \mathsf{g})[\mathrm{X}]$ corresponding to a flat $\mathrm{Y} \geq \mathrm{X}$. Pick any element

$$\sum_{F:\, F \geq A,\, \mathrm{s}(F) = \mathrm{Y}} x^{F/A}\, \mathtt{H}_{F/A} \otimes v \;\in\; \mathbf{Lie}[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{g}[\mathrm{Y}],$$

where A is a fixed face of support X. Apply the map going down and across to this element, and also the map going across and down, and take their difference. This equals

$$(16.21)\qquad \sum_{F:\, F\geq A,\, \mathrm{s}(F)=\mathrm{Y}} x^{F/A}\, \beta_{F,\mathrm{Y}}(v) - \beta_{A,\mathrm{X}}\gamma^{\mathrm{Y}}_{\mathrm{X}}\Big(\sum_{F:\, F\geq A,\, \mathrm{s}(F)=\mathrm{Y}} x^{F/A}\, \mathtt{H}_{F/A} \otimes v \Big).$$

This is an element of

$$(16.22)\qquad \Big(\bigoplus_{F:\, F\geq A,\, \mathrm{s}(F)=\mathrm{Y}} \mathsf{g}[F] \Big) + \mathsf{g}[A] \subseteq \mathcal{T}(\mathsf{g})[A].$$

We call this an A/A-relation and say that it is obtained from the flat Y/A. The sum in (16.22) is direct provided $\mathrm{Y} > \mathrm{s}(A)$. By definition, the quotient of $\mathcal{T}(\mathsf{g})[A]$ by the A/A-relations is the A-component of the coequalizer of $\mathbf{Lie} \circ \mathsf{g} \rightrightarrows \mathcal{T}(\mathsf{g})$, where the arrows are obtained by following the two directions in diagram (16.20).

16.5.2. *A/A*-relations from rank-zero and rank-one flats. Let us explicitly look at some A/A-relations.

Suppose the flat Y/A has rank zero, that is, $\mathrm{Y} = \mathrm{s}(A)$. Then the A/A-relation obtained from Y/A is 0: In this case, $\mathbf{Lie}[\mathrm{Y},\mathrm{Y}] = \Bbbk$, spanned by $\mathtt{H}_{A/A}$; so (16.21) becomes

$$\beta_{A,\mathrm{Y}}(v - \gamma^{\mathrm{Y}}_{\mathrm{Y}}(\mathtt{H}_{A/A} \otimes v)) = 0.$$

So nonzero A/A-relations arise only from flats of nonzero rank.

Suppose the flat Y/A has rank one, and it supports the vertices F/A and $A\overline{F}/A$. Then the A/A-relations obtained from Y/A are

$$(16.23)\qquad \beta_{F,\mathrm{Y}}(v) - \beta_{A\overline{F},\mathrm{Y}}(v) - \beta_{A,\mathrm{X}}\gamma^{\mathrm{Y}}_{\mathrm{X}}\big((\mathtt{H}_{F/A} - \mathtt{H}_{A\overline{F}/A}) \otimes v\big), \quad v \in \mathsf{g}[\mathrm{Y}]$$

with $\mathrm{X} := \mathrm{s}(A)$. These are elements of

$$\mathsf{g}[F] \oplus \mathsf{g}[A\overline{F}] \oplus \mathsf{g}[A] \subseteq \mathcal{T}(\mathsf{g})[A].$$

16.5.3. *H/A*-relations. For $H \geq A$, an H/A-relation is the image of an H/H-relation under the inclusion map

$$\mathcal{T}(\mathsf{g})[H] \hookrightarrow \mathcal{T}(\mathsf{g})[A].$$

If the relation is obtained from the flat Y/H, then it is an element of

$$(16.24)\qquad \Big(\bigoplus_{F:\, F\geq H,\, \mathrm{s}(F)=\mathrm{Y}} \mathsf{g}[F] \Big) + \mathsf{g}[H] \subseteq \mathcal{T}(\mathsf{g})[A].$$

For an H/A-relation to be nonzero, Y/H must have rank at least one, that is, $\mathrm{s}(H) < \mathrm{Y}$, in which case the sum in (16.24) is direct. An H/A-relation is given exactly as in (16.21), with H instead of A. Also note that if $\mathrm{s}(H) < \mathrm{Y}$, then the sum of the scalars $x^{F/H}$ appearing in the relation is zero by Lemma 7.63.

16.5.4. A submonoid of the free monoid. For a Lie monoid $\mathtt{g}$, define $\mathcal{I}(\mathtt{g})$ to be the smallest submonoid of $\mathcal{T}(\mathtt{g})$ such that its A-component contains all A/A-relations. Explicitly, $\mathcal{I}(\mathtt{g})[A]$ is the linear span of all H/A-relations as H varies over all faces greater than A. This is because, by (16.15), the product component μ_A^H of $\mathcal{T}(\mathtt{g})$ is precisely the inclusion of $\mathcal{T}(\mathtt{g})[H]$ in $\mathcal{T}(\mathtt{g})[A]$. We refer to $\mathcal{I}(\mathtt{g})$ as the monoid of relations.

The significance of the relations (16.23) is brought out by the following result.

Lemma 16.12. *For $\mathtt{g}$ a Lie monoid, $\mathcal{I}(\mathtt{g})$ is the smallest submonoid of $\mathcal{T}(\mathtt{g})$ such that its A-component contains all A/A-relations obtained from rank-one flats Y/A.*

Proof. The key fact is that **Lie** is a binary quadratic operad. We briefly sketch the argument. Consider the diagram

$$\begin{tikzcd}
\mathbf{Lie} \circ \mathbf{Lie} \circ \mathtt{g} \arrow[r, "(-)\circ \mathrm{id}"] \arrow[d, hook] & \mathbf{Lie} \circ \mathtt{g} \arrow[r, "\gamma"] \arrow[d, hook] \arrow[dr, phantom, "relation"] & \mathtt{g} \arrow[d, hook] \\
\mathbf{As} \circ \mathbf{As} \circ \mathtt{g} \arrow[r, "(-)\circ \mathrm{id}"'] & \mathbf{As} \circ \mathtt{g} \arrow[r, "\cong"'] & \mathcal{T}(\mathtt{g}).
\end{tikzcd}$$

The left square commutes (since **Lie** is a suboperad of **As**), while the right does not. Its lack of commutativity is yielding the relation. The following is a different filling of the same diagram.

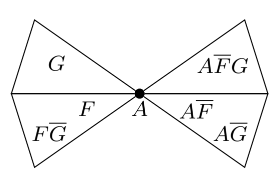
$$\begin{tikzcd}
\mathbf{Lie} \circ \mathbf{Lie} \circ \mathtt{g} \arrow[rr, "\mathrm{id} \circ \gamma"] \arrow[d, hook] \arrow[dr, phantom, "relation"] & & \mathbf{Lie} \circ \mathtt{g} \arrow[rr, "\gamma"] \arrow[dl, hook] \arrow[dr, hook] & & \mathtt{g} \arrow[d, hook] \arrow[dl, phantom, "relation"] \\
\mathbf{Lie} \circ \mathbf{As} \circ \mathtt{g} \arrow[r, "\cong"'] \arrow[d, hook] & \mathbf{Lie} \circ \mathcal{T}(\mathtt{g}) \arrow[dr, hook] & & \mathbf{As} \circ \mathtt{g} \arrow[r, "\cong"'] \arrow[dl, hook] & \mathcal{T}(\mathtt{g}) \\
\mathbf{As} \circ \mathbf{As} \circ \mathtt{g} \arrow[rr, "\cong"'] & & \mathbf{As} \circ \mathcal{T}(\mathtt{g}) \arrow[urr, bend right] & &
\end{tikzcd}$$

The bent arrow is the monoid structure of $\mathcal{T}(\mathtt{g})$. Now the top-left and top-right squares do not commute, while all the rest commute. Thus, we have broken one relation into two smaller relations. This also uses the existence of the map $\mathbf{As} \circ \mathcal{I}(\mathtt{g}) \to \mathcal{I}(\mathtt{g})$. Now apply induction. $\square$

We give a practical demonstration of the above argument. Suppose Z/A is a rank-two flat. Put $\mathrm{X} = \mathrm{s}(A)$. Consider the element

$$x := \mathtt{H}_{G/A} - \mathtt{H}_{F\overline{G}/A} - \mathtt{H}_{A\overline{F}G/A} + \mathtt{H}_{A\overline{G}/A} \in \mathbf{Lie}[\mathrm{X}, \mathrm{Z}],$$

where F/A is a vertex, and G/A is an edge greater than F/A with support Z/A.



Let $v \in \mathsf{g}[\mathrm{Z}]$. The A/A-relation obtained from Z/A using x and v then belongs to

$$\mathsf{g}[G] \oplus \mathsf{g}[F\overline{G}] \oplus \mathsf{g}[A\overline{F}G] \oplus \mathsf{g}[A\overline{G}] \oplus \mathsf{g}[A].$$

We explain how this element can be constructed from relations obtained from rank-one flats.

Let Y denote the support of F and $\overline{F}$. Observe that

$$\mathbf{Lie}[\mathrm{X},\mathrm{Y}] \otimes \mathbf{Lie}[\mathrm{Y},\mathrm{Z}] \to \mathbf{Lie}[\mathrm{X},\mathrm{Z}], \qquad y \otimes z \mapsto x$$

for $y := \mathtt{H}_{F/A} - \mathtt{H}_{A\overline{F}/A}$ and $z := \mathtt{H}_{G/F} - \mathtt{H}_{F\overline{G}/F}$. Put $w := \gamma^{\mathrm{Z}}_{\mathrm{Y}}(z \otimes v)$. The A/A-relation obtained from Y/A using y and w belongs to

$$\mathsf{g}[F] \oplus \mathsf{g}[A\overline{F}] \oplus \mathsf{g}[A].$$

Similarly, the F/F-relation (resp. $A\overline{F}/A\overline{F}$-relation) obtained from Y/F (resp. $\mathrm{Y}/A\overline{F}$) using z and v yields an element of

$$\mathsf{g}[G] \oplus \mathsf{g}[F\overline{G}] \oplus \mathsf{g}[F] \qquad (\text{resp. } \mathsf{g}[A\overline{F}G] \oplus \mathsf{g}[A\overline{G}] \oplus \mathsf{g}[A\overline{F}]).$$

An appropriate linear combination of these elements yields the A/A-relation obtained from Z/A.

Lemma 16.13. *Let (g, ν) be a Lie monoid. Then $\mathcal{I}(\mathsf{g})$ is the smallest submonoid of $\mathcal{T}(\mathsf{g})$ such that its A-component contains all elements of the form*

$$v - \beta_{A\overline{F},F}(v) - \nu^{F}_{A}(v), \quad v \in \mathsf{g}[F] \tag{16.25}$$

for $A \lessdot F$.

This is a reformulation of Lemma 16.12 in terms of the Lie bracket ν. The elements (16.25) are the same as those in (16.23) arising from rank-one flats Y/A.

16.5.5. Universal enveloping monoid. For any Lie monoid g, define $\mathcal{U}(\mathsf{g})$ to be the quotient of $\mathcal{T}(\mathsf{g})$ by the submonoid $\mathcal{I}(\mathsf{g})$. It follows that $\mathcal{U}(\mathsf{g})$ is a monoid. We call this the *universal enveloping monoid* of g. It is the largest monoid-quotient of $\mathcal{T}(\mathsf{g})$ such that the outside square in the diagram

$$\begin{tikzcd}
\mathbf{Lie}\circ\mathsf{g} \arrow[r, "\gamma"] \arrow[d, hook] & \mathsf{g} \arrow[d, hook] \arrow[ddr, bend left] & \\
\mathbf{As}\circ\mathsf{g} \arrow[r, "\cong"] \arrow[drr, bend right] & \mathcal{T}(\mathsf{g}) \arrow[dr, two heads] & \\
 & & \mathcal{U}(\mathsf{g})
\end{tikzcd} \tag{16.26}$$

commutes. (The bent maps are defined so that the triangles commute.) We emphasize that $\mathcal{U}(\mathsf{g})$ is *not* the coequalizer of the two arrows $\mathbf{Lie}\circ\mathsf{g} \rightrightarrows \mathcal{T}(\mathsf{g})$ obtained from the inside square. We need to take quotient by all H/A-relations since we want $\mathcal{U}(\mathsf{g})$ to be a monoid.

Equivalently, by Lemma 16.13, we have:

Lemma 16.14. *For a Lie monoid* (g, ν)*, the monoid* $\mathcal{U}(\mathsf{g})$ *is the quotient of* $\mathcal{T}(\mathsf{g})$ *by the submonoid of relations generated by*

$$v - \beta_{A\overline{F},F}(v) - \nu_A^F(v), \quad v \in \mathsf{g}[F] \tag{16.27}$$

for $A \lessdot F$.

The construction of $\mathcal{U}(\mathsf{g})$ is functorial in g: Any morphism of Lie monoids $\mathsf{g} \to \mathsf{g}'$ preserves H/A-relations, and hence induces a morphism of monoids $\mathcal{U}(\mathsf{g}) \to \mathcal{U}(\mathsf{g}')$. In other words, there is a commutative diagram of monoids

$$\begin{tikzcd} \mathcal{T}(\mathsf{g}) \arrow[r] \arrow[d, two heads] & \mathcal{T}(\mathsf{g}') \arrow[d, two heads] \\ \mathcal{U}(\mathsf{g}) \arrow[r] & \mathcal{U}(\mathsf{g}'). \end{tikzcd} \tag{16.28}$$

In particular, we have a functor

$$\mathcal{U} : \mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp}) \to \mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp}).$$

16.5.6. Adjunction with the underlying Lie monoid functor.

Theorem 16.15. *The functor* $\mathcal{U}$ *is the left adjoint of the underlying Lie monoid functor* (16.8)*. Explicitly, for any Lie monoid* g *and monoid* a*, there is a natural bijection*

$$\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{U}(\mathsf{g}), \mathsf{a}) \xrightarrow{\cong} \mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{g}, \mathsf{a}).$$

We explain using each of the two formulations of a Lie monoid.

First proof. Suppose $\mathsf{g} \to \mathsf{a}$ is a morphism of Lie monoids. Then we obtain a diagram

$$\begin{tikzcd} \mathbf{Lie} \circ \mathsf{g} \arrow[r, "\gamma"] \arrow[d, hook] & \mathsf{g} \arrow[d, hook] \arrow[ddr, bend left] & \\ \mathbf{As} \circ \mathsf{g} \arrow[r, "\cong"'] \arrow[d] & \mathcal{T}(\mathsf{g}) \arrow[dr] & \\ \mathbf{As} \circ \mathsf{a} \arrow[rr] & & \mathsf{a} \end{tikzcd} \tag{16.29}$$

with the outside pentagon commuting. The universal property of $\mathcal{T}$ (Theorem 6.2) gives the morphism of monoids $\mathcal{T}(\mathsf{g}) \to \mathsf{a}$. The manner in which it is constructed implies that the triangle and the bottom square commute, also see formula (6.4). So all A/A-relations are in the kernel of $\mathcal{T}(\mathsf{g}) \to \mathsf{a}$. Since this is a morphism of monoids, its kernel in fact contains all H/A-relations, and hence it induces a morphism of monoids $\mathcal{U}(\mathsf{g}) \to \mathsf{a}$.

In the other direction, suppose $\mathcal{U}(\mathsf{g}) \to \mathsf{a}$ is a morphism of monoids. Then the diagram

$$\begin{tikzcd} \mathsf{g} \arrow[r, hook] & \mathcal{T}(\mathsf{g}) \arrow[d, two heads] \arrow[dr] & \\ & \mathcal{U}(\mathsf{g}) \arrow[r] & \mathsf{a} \end{tikzcd}$$

yields a map of species $\mathsf{g} \to \mathsf{a}$. We again consider diagram (16.29). By construction, the triangle and the bottom square commute. Since the kernel of $\mathcal{T}(\mathsf{g}) \to \mathsf{a}$ includes A/A-relations, we deduce that the outside pentagon in (16.29) commutes. Hence, $\mathsf{g} \to \mathsf{a}$ is a morphism of Lie monoids as required.

It is clear that the two constructions are inverse to each other. $\square$

Second Proof. Let (g, ν) be a Lie monoid and (a, μ) be a monoid. Suppose $f : \mathsf{g} \to \mathsf{a}$ is a map of species. Then by (16.4) and (16.11), f is a morphism of Lie monoids iff

$$f_A \nu_A^F = (\mu_A^F - \mu_A^{A\overline{F}} \beta_{A\overline{F},F}) f_F \tag{16.30}$$

for $A \lessdot F$. By formula (6.4), the map f extends to a morphism of monoids $\mathcal{T}(\mathsf{g}) \to \mathsf{a}$ which on the A-component, on the F-summand, is

$$\mathsf{g}[F] \xrightarrow{f_F} \mathsf{a}[F] \xrightarrow{\mu_A^F} \mathsf{a}[A].$$

By Lemma 16.14, this map factors through $\mathcal{U}(\mathsf{g})$ iff the elements (16.27) map to zero. Writing this out yields the exact same condition (16.30). $\square$

The unit of the adjunction, namely,

$$i : \mathsf{g} \to \mathcal{U}(\mathsf{g}) \tag{16.31}$$

is the composite map $\mathsf{g} \hookrightarrow \mathcal{T}(\mathsf{g}) \twoheadrightarrow \mathcal{U}(\mathsf{g})$. It is a morphism of Lie monoids, with $\mathcal{U}(\mathsf{g})$ equipped with the commutator bracket.

Theorem 16.16. *Let a be a monoid, g a Lie monoid, $f : \mathsf{g} \to \mathsf{a}$ a morphism of Lie monoids. Then there exists a unique morphism of monoids $\hat{f} : \mathcal{U}(\mathsf{g}) \to \mathsf{a}$ such that the diagram*

$$\begin{tikzcd} \mathcal{U}(\mathsf{g}) \arrow[r, dashed, "\hat{f}"] & \mathsf{a} \\ \mathsf{g} \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes.

This reformulates the adjunction in Theorem 16.15 as a universal property. To recap: By Theorem 6.2, the map $f : \mathsf{g} \to \mathsf{a}$ extends uniquely to a morphism of monoids $\hat{f} : \mathcal{T}(\mathsf{g}) \to \mathsf{a}$. Since f is a morphism of Lie monoids, $\hat{f}$ factors through $\mathcal{U}(\mathsf{g})$ to yield $\hat{f} : \mathcal{U}(\mathsf{g}) \to \mathsf{a}$.

Exercise 16.17. Check that $\mathcal{U}(\mathsf{g})$ is the largest monoid-quotient of $\mathcal{T}(\mathsf{g})$ such that $\mathsf{g} \hookrightarrow \mathcal{T}(\mathsf{g}) \twoheadrightarrow \mathcal{U}(\mathsf{g})$ is a morphism of Lie monoids.

16.5.7. Bimonoid structure. First approach. So far $\mathcal{U}(\mathsf{g})$ has only been considered as a monoid. Now we equip it with a bimonoid structure.

Proposition 16.18. *There is a unique cocommutative bimonoid structure on $\mathcal{U}(\mathsf{g})$ such that the quotient map $\mathcal{T}(\mathsf{g}) \twoheadrightarrow \mathcal{U}(\mathsf{g})$ is a morphism of bimonoids.*

Proof. We need to show that the coproduct component $\Delta_A^G : \mathcal{T}(\mathsf{g})[A] \to \mathcal{T}(\mathsf{g})[G]$ given in (16.16) preserves relations. Accordingly, let u be a nonzero H/A-relation obtained from the flat Y/H. So it is an element of (16.24). We consider three cases, namely,

$$\mathrm{s}(G) \not\leq \mathrm{Y}, \qquad \mathrm{s}(G) \leq \mathrm{Y},\ HG = H, \qquad \mathrm{s}(G) \leq \mathrm{Y},\ HG > H.$$

In the first case, clearly $\Delta_A^G(u) = 0$. In the second case, Δ_A^G identifies $\mathsf{g}[H]$ with $\mathsf{g}[GH]$, and $\mathsf{g}[F]$ with $\mathsf{g}[GF]$. So $\Delta_A^G(u)$ belongs to

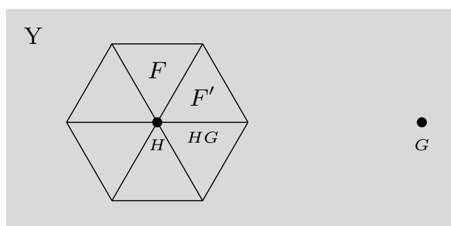
$$\Big(\bigoplus_{F\colon F \geq H,\, \mathrm{s}(F)=\mathrm{Y}} \mathsf{g}[GF] \Big) \oplus \mathsf{g}[GH] \subseteq \mathcal{T}(\mathsf{g})[G],$$

and it is easy to see that it is a GH/G-relation obtained from the flat Y/GH.

So suppose we are in the third case, that is, $\mathrm{s}(G) \leq \mathrm{Y}$ but $HG > H$. We claim that $\Delta_A^G(u) = 0$. To see this, we split the computation into two parts. Write

$$u = u_H + \sum_{F\colon F \geq H,\, \mathrm{s}(F)=\mathrm{Y}} u_F,$$

where $u_H \in \mathsf{g}[H]$ and $u_F \in \mathsf{g}[F]$ for each F. The summand $\mathsf{g}[H]$, and in particular u_H, maps to zero because $HG > H$. Thus, we need to show that the sum also maps to zero. An illustration is shown below.



In the figure, A is the central face (which is not visible), H and G are vertices, HG is an edge, and Y is the plane of the paper.

Write

$$\sum_{F\colon F \geq H,\, \mathrm{s}(F)=\mathrm{Y}} u_F = \sum_{\substack{F'\colon F' \geq H,\, \mathrm{s}(F')=\mathrm{Y} \\ F' \geq HG}} \Big(\sum_{\substack{F\colon F \geq H,\, \mathrm{s}(F)=\mathrm{Y} \\ HGF = F'}} u_F \Big).$$

Observe that $\Delta_A^G(u_F)$ can be written as the composite

$$\mathsf{g}[F] \xrightarrow{\beta_{HGF,F}} \mathsf{g}[HGF] = \mathsf{g}[F'] \xrightarrow{\beta_{GF,F'}} \mathsf{g}[GF].$$

Hence, by applying the definition of a Lie element (1.164) to the arrangement $\mathcal{A}_H^{\mathrm{Y}}$, we deduce that Δ_A^G of each inner sum above is 0. □

Remark 16.19. One can slightly simplify the proof of Proposition 16.18 as follows. In view of Exercise 2.50, it suffices to apply Δ_A^G only to A/A-relations. This more or less eliminates the second case. In the third case, we show as in the proof that for $G > A$, Δ_A^G applied to an A/A-relation yields zero.

Exercise 16.20. In view of Exercise 2.50, prove Proposition 16.18 by checking that for $G > A$, Δ_A^G applied to the relation (16.27) is zero.

Proposition 16.18 shows that diagram (16.28) is a commutative diagram of bimonoids. This yields a functor

$$\mathcal{U} : \mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp}) \to {}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}).$$

(Note that the same symbol as before is used to denote the new functor.)

Lemma 16.21. *The image of the map $i : \mathsf{g} \to \mathcal{U}(\mathsf{g})$ in (16.31) is a Lie submonoid of $\mathcal{U}(\mathsf{g})$ and is contained in $\mathcal{P}(\mathcal{U}(\mathsf{g}))$.*

Proof. We know that i is a morphism of Lie monoids, so the first assertion follows. By (16.17) or directly from coproduct formula (16.16), we deduce that g is in the primitive part of $\mathcal{T}(\mathsf{g})$. Since a morphism of comonoids preserves primitive parts, the second assertion follows from Proposition 16.18. $\square$

Note very carefully that we do not yet know whether i is injective. This is indeed true and is proved later in Corollary 17.13. Moreover, we will also see later in (17.33) that the image of i in fact equals $\mathcal{P}(\mathcal{U}(\mathsf{g}))$.

16.5.8. Bimonoid structure. Second approach. There is an alternative way to construct the coproduct of $\mathcal{U}(\mathsf{g})$ and prove Proposition 16.18. For that, recall from Section 6.2 the cofree bimonoid $\mathcal{T}^\vee(\mathsf{a})$ on the monoid a.

Lemma 16.22. *For any monoid a, the inclusion map $\mathsf{a} \hookrightarrow \mathcal{T}^\vee(\mathsf{a})$ is a morphism of Lie monoids.*

Here both a and $\mathcal{T}^\vee(\mathsf{a})$ are monoids, the product of the latter is given by formula (6.13) for $q = 1$. Note very carefully that the inclusion map is *not* a morphism of monoids, it is only a morphism of the underlying Lie monoids.

First proof. We verify that the diagram

$$\begin{tikzcd} \mathbf{Lie}[\mathrm{X},\mathrm{Y}] \otimes \mathsf{a}[\mathrm{Y}] \arrow[r, "\gamma_A^{\mathrm{Y}}"] \arrow[d, hook] & \mathsf{a}[A] \arrow[d, hook] \\ \mathbf{Lie}[\mathrm{X},\mathrm{Y}] \otimes \mathcal{T}^\vee(\mathsf{a})[\mathrm{Y}] \arrow[r, "\gamma_A^{\mathrm{Y}}"'] & \mathcal{T}^\vee(\mathsf{a})[A] \end{tikzcd}$$

commutes, with γ_A^{Y} as in (16.10). (This is the same as diagram (16.2) with X replaced by A in the right-vertical map via $\beta_{A,\mathrm{X}}$.) Going down and across,

$$\begin{aligned} \sum_{\substack{F:\, F \geq A \\ \mathrm{s}(F)=\mathrm{Y}}} x^{F/A}\, \mathtt{H}_{F/A} \otimes v \mapsto & \sum_{\substack{F:\, F\geq A \\ \mathrm{s}(F)=\mathrm{Y}}} \sum_{\substack{G:\, G\geq A \\ \mathrm{s}(G)\leq \mathrm{Y}}} x^{F/A} \mu_G^{GF} \beta_{GF,\mathrm{Y}}(v) \\ &= \sum_{\substack{(G,K):\, K\geq G\geq A \\ \mathrm{s}(K)=\mathrm{Y}}} \Bigg(\sum_{\substack{F:\, F\geq A \\ \mathrm{s}(F)=\mathrm{Y},\, GF=K}} x^{F/A} \Bigg) \mu_G^K \beta_{K,\mathrm{Y}}(v) \\ &= \sum_{\substack{F:\, F\geq A \\ \mathrm{s}(F)=\mathrm{Y}}} x^{F/A}\, \mu_A^F \beta_{F,\mathrm{Y}}(v), \end{aligned}$$

which is the same as going across and down. The first step used product formula (6.13). The second step introduced a new variable K for GF. By applying the definition of a Lie element (1.164) to the arrangement $\mathcal{A}_A^{\mathrm{Y}}$, the sum in parenthesis is 0 whenever $G > A$. $\square$

Second Proof. We check diagram (16.4), that is,

$$\begin{tikzcd} \mathsf{a}[F] \arrow[r, hook] \arrow[d, "\nu_A^F"'] & \mathcal{T}^\vee(\mathsf{a})[F] \arrow[d, "\nu_A^F"] \\ \mathsf{a}[A] \arrow[r, hook] & \mathcal{T}^\vee(\mathsf{a})[A]. \end{tikzcd}$$

Both ν_A^F are components of commutator brackets. We need to evaluate the second ν_A^F only on the summand $\mathsf{a}[F]$. It has the form

$$\mathsf{a}[F] \to \mathsf{a}[A] \oplus \mathsf{a}[F] \oplus \mathsf{a}[A\overline{F}].$$

The first component $\mathsf{a}[F] \to \mathsf{a}[A]$ of this map agrees with the commutator bracket component ν_A^F of a. The remaining two components can be checked to be

$$\mathsf{a}[F] \xrightarrow{\mathrm{id} - \beta_{F,A\overline{F}}\beta_{A\overline{F},F}} \mathsf{a}[F] \quad \text{and} \quad \mathsf{a}[F] \xrightarrow{\beta_{A\overline{F},F} - \beta_{A\overline{F},F}} \mathsf{a}[A\overline{F}]$$

and they are both zero. $\square$

Exercise 16.23. For a monoid a, deduce that $\mathcal{P}\mathcal{T}^\vee(\mathsf{a}) = \mathcal{P}\mathcal{S}^\vee(\mathsf{a}) = \mathsf{a}$ as Lie monoids. (Combine Lemma 16.22 with Propositions 6.56 and 6.58.) A related result is given later in Exercise 17.47.

As a consequence of Lemma 16.22, we obtain:

Lemma 16.24. *The composite map*

$$\mathsf{g} \xrightarrow{i} \mathcal{U}(\mathsf{g}) \hookrightarrow \mathcal{T}^\vee(\mathcal{U}(\mathsf{g})) \tag{16.32}$$

is a morphism of Lie monoids. The second map is the canonical inclusion.

Now apply Theorem 16.16 to the map (16.32) to obtain a morphism $\Delta : \mathcal{U}(\mathsf{g}) \to \mathcal{T}^\vee(\mathcal{U}(\mathsf{g}))$ of monoids fitting into the commutative diagram

$$\begin{tikzcd} \mathcal{U}(\mathsf{g}) \arrow[dr, dashed, "\Delta"] & \\ \mathsf{g} \arrow[u, "i"] \arrow[r] & \mathcal{T}^\vee(\mathcal{U}(\mathsf{g})). \end{tikzcd}$$

By Lemma 6.65, the map Δ is the same as a bimonoid structure on $\mathcal{U}(\mathsf{g})$. Moreover, by construction, the image of g under i is in the primitive part of $\mathcal{U}(\mathsf{g})$. So Theorem 6.31 (with $q = 1$) yields a morphism of bimonoids $\mathcal{T}(\mathsf{g}) \to \mathcal{U}(\mathsf{g})$. This map is clearly the usual one, so this proves Proposition 16.18.

16.5.9. Adjunction with the primitive part functor.

Theorem 16.25. *The functor $\mathcal{U}$ is the left adjoint of the primitive part functor $\mathcal{P}$. Explicitly, for any Lie monoid g and cocommutative bimonoid h, there is a natural bijection*

$$^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{U}(\mathsf{g}),\mathsf{h}) \xrightarrow{\cong} \mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{g},\mathcal{P}(\mathsf{h})).$$

We will see later in Theorem 17.42 that, in fact, this is an adjoint equivalence.

Proof. Suppose $\mathsf{g} \to \mathcal{P}(\mathsf{h})$ is a morphism of Lie monoids. Since $\mathcal{P}(\mathsf{h})$ is a Lie submonoid of h, we may apply Theorem 16.16 to obtain a commutative diagram:

$$\begin{tikzcd} \mathcal{U}(\mathsf{g}) \arrow[r, dashed] & \mathsf{h} \\ \mathsf{g} \arrow[u, "i"] \arrow[ur] \arrow[r] & \mathcal{P}(\mathsf{h}). \arrow[u, hook] \end{tikzcd}$$

The dotted arrow is a morphism of monoids. We need to show that it is a morphism of comonoids. For that, consider the diagram

$$\begin{tikzcd} \mathsf{g} \arrow[r, "i"] & \mathcal{U}(\mathsf{g}) \arrow[r] \arrow[d, "\Delta"'] & \mathsf{h} \arrow[d, "\Delta"] \\ & \mathcal{T}^\vee(\mathcal{U}(\mathsf{g})) \arrow[r] & \mathcal{T}^\vee(\mathsf{h}). \end{tikzcd}$$

All maps on the square are morphisms of monoids, see Lemma 6.65. Further, the image of g in $\mathcal{U}(\mathsf{g})$ is in the primitive part of $\mathcal{U}(\mathsf{g})$ by Lemma 16.21 and the image in h is in the primitive part of h by hypothesis. Thus, going along the two directions yields the same map $\mathsf{g} \to \mathcal{T}^\vee(\mathsf{h})$. So the square commutes by the uniqueness assertion in Theorem 16.16.

In the other direction, given a morphism $\mathcal{U}(\mathsf{g}) \to \mathsf{h}$ of bimonoids, we send it to the composite

$$\mathsf{g} \to \mathcal{P}\mathcal{U}(\mathsf{g}) \to \mathcal{P}(\mathsf{h}).$$

For the first map, use Lemma 16.21. $\square$

Remark 16.26. For the adjunction between $\mathcal{U}$ and $\mathcal{P}$ in Theorem 16.25, we can take the functors to be between the categories of Lie monoids and bimonoids. This follows by composing adjunctions:

$$\mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{\mathcal{P}}{\overset{\mathcal{U}}{\rightleftarrows}} {}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{(-)^{coab}}{\overset{inc}{\rightleftarrows}} \mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}).$$

The second adjunction is between inclusion and coabelianization, see (2.57). Compare and contrast with Remark 6.47.

The adjunction between $\mathcal{U}$ and $\mathcal{P}$ is phrased below as a universal property.

Theorem 16.27. *Let* h *be a bimonoid,* g *a Lie monoid,* $f : \mathsf{g} \to \mathcal{P}(\mathsf{h})$ *a morphism of Lie monoids. Then there exists a unique morphism of bimonoids* $\hat{f} : \mathcal{U}(\mathsf{g}) \to \mathsf{h}$ *such that the diagram*

$$\begin{tikzcd} \mathcal{U}(\mathsf{g}) \arrow[r, dashed, "\hat{f}"] & \mathsf{h} \\ \mathsf{g} \arrow[u, "i"] \arrow[r, "f"'] & \mathcal{P}(\mathsf{h}) \arrow[u, hook] \end{tikzcd}$$

commutes.

To recap: The map $\hat{f} : \mathcal{U}(\mathsf{g}) \to \mathsf{h}$ arises from the universal property of $\mathcal{U}(\mathsf{g})$ in Theorem 16.16. The extra step is that since f maps into $\mathcal{P}(\mathsf{h})$, the map $\hat{f}$ is in addition a morphism of comonoids.

16.5.10. Universal enveloping monoid of the free Lie monoid.

Proposition 16.28. *For any species* p*, we have*

$$\mathcal{U}(\mathbf{Lie} \circ \mathsf{p}) = \mathcal{T}(\mathsf{p}) \tag{16.33}$$

as bimonoids.

Proof. This follows by composing the adjunctions

$$\mathcal{A}\text{-}\mathsf{Sp} \underset{frg}{\overset{\mathbf{Lie}\circ(-)}{\rightleftarrows}} \mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{\mathcal{P}}{\overset{\mathcal{U}}{\rightleftarrows}} \mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}),$$

and comparing the result with the adjunction between $\mathcal{T}$ and $\mathcal{P}$ in Theorem 6.30 for $q = 1$. $\square$

16.6. Abelian Lie monoids

We consider abelian Lie monoids. They are Lie monoids whose Lie bracket is identically zero. Their universal enveloping monoids are free commutative monoids.

16.6.1. Abelian Lie monoid. We say that a Lie monoid is *abelian* if the structure maps $\gamma_{\mathrm{X}}^{\mathrm{Y}}$ are zero for all $\mathrm{X} < \mathrm{Y}$. Equivalently, a Lie monoid is abelian if all Lie bracket components ν_A^F are 0. Observe that every species is an abelian Lie monoid in a unique way. Thus, the category of species is isomorphic to the full subcategory of abelian Lie monoids.

16.6.2. Commutator bracket. Recall the commutator bracket of a monoid from Section 16.2. For a commutative monoid, the commutator bracket components (16.11) are all zero. This follows from the commutativity axiom (2.17). Thus, the functor (16.8) carries the subcategory of commutative monoids into the subcategory of abelian Lie monoids. Similarly, the primitive part functor (16.13) carries the subcategory of commutative bimonoids into the subcategory of abelian Lie monoids. Note that Proposition 16.2 is a triviality in this case.

16.6.3. Universal enveloping monoid. The monoid of relations $\mathcal{I}(\mathsf{g})$ of an abelian Lie monoid g can be explicitly described as follows.

Lemma 16.29. *For an abelian Lie monoid* g*, the submonoid* $\mathcal{I}(\mathsf{g})$ *of* $\mathcal{T}(\mathsf{g})$*, evaluated on the* A*-component, is linearly spanned by elements of the form* $z - \beta_{G,F}(z)$*, with* $z \in \mathsf{g}[F]$*, and* F *and* G *both greater than* A *and of the same support.*

Proof. First consider $z - \beta_{G,F}(z)$, where in addition, F and G are adjacent. Let H denote their common panel. Then $z - \beta_{G,F}(z)$ is an instance of the relation (16.23), with H instead of A, and G instead of $A\overline{F}$, so it belongs to $\mathcal{I}(\mathsf{g})[A]$. In general, by connecting F and G by a gallery, we deduce that $z - \beta_{G,F}(z)$ belongs to $\mathcal{I}(\mathsf{g})[A]$. For the reverse containment, we note that any nonzero H/A-relation arising from the flat Y/H has the form

$$\sum_{F\colon F \geq H,\, \mathrm{s}(F)=\mathrm{Y}} x^{F/H}\, \beta_{F,\mathrm{Y}}(v),$$

where $v \in \mathsf{g}[\mathrm{Y}]$ and the sum of the $x^{F/H}$ is zero. So it can be expressed as a linear combination of the given elements. This completes the argument. We mention that instead of using galleries, one can also proceed more abstractly by employing Lemma 1.76. $\square$

We now turn to the universal enveloping monoid of g. Recall from Section 6.3.1 the free commutative monoid $\mathcal{S}(\mathsf{p})$ on a species p. We deduce from Lemma 16.29 that:

Proposition 16.30. *For an abelian Lie monoid* g*, we have* $\mathcal{U}(\mathsf{g}) = \mathcal{S}(\mathsf{g})$*, the free commutative monoid on* g *(with* g *viewed as a species), with the canonical quotient* $\mathcal{T}(\mathsf{g}) \twoheadrightarrow \mathcal{U}(\mathsf{g})$ *being the abelianization map* (6.61)*.*

We know that $\mathcal{T}(\mathsf{g})$ induces a bimonoid structure on $\mathcal{U}(\mathsf{g})$. This is consistent with the fact that the abelianization map is a morphism of bimonoids. The coproduct of $\mathcal{S}(\mathsf{g})$ is as in (6.51).

16.7. Signed Lie monoids

We briefly discuss signed Lie monoids which are left modules over the signed Lie operad. (Recall that the definition of the latter involves the signed distance function υ_{-1}.) Signed Lie monoids can also be defined directly in terms of a Lie bracket subject to signed antisymmetry and signed Jacobi identity. Every monoid carries the structure of a signed Lie monoid via the signed commutator bracket. The primitive part of a signed bimonoid is a signed Lie monoid, and similarly the universal enveloping monoid of a signed Lie monoid is a signed bimonoid which is signed cocommutative.

We mention in passing that the above ideas can be considered in the more general setting of log-antisymmetric distance functions (with υ_{-1} being a special case).

16.7.1. Signed Lie monoids. Recall the signed Lie operad $\mathbf{Lie}^-$. A *signed Lie monoid* is a left module over the signed Lie operad. A morphism of signed Lie monoids is a map of left modules. We denote the category of signed Lie monoids by $(-1)\text{-}\mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp})$.

Explicitly: A signed Lie monoid is a species g equipped with linear maps

$$\gamma_{\mathrm{X}}^{\mathrm{Y}} : \mathbf{Lie}^-[\mathrm{X},\mathrm{Y}] \otimes \mathsf{g}[\mathrm{Y}] \to \mathsf{g}[\mathrm{X}],$$

one for each $\mathrm{X} \leq \mathrm{Y}$, satisfying axioms (16.1a) and (16.1b) with $\mathbf{Lie}^-$ instead of $\mathbf{Lie}$.

In terms of the Lie bracket, the definition is as follows. A signed Lie monoid is a species g equipped with linear maps

$$\nu_A^F : \mathsf{g}[F] \to \mathsf{g}[A],$$

one for each $A \lessdot F$, which satisfies the following axioms.

Naturality. For each morphism $\beta_{B,A} : A \to B$, and $A \lessdot F$, the diagram

$$\begin{tikzcd} \mathsf{g}[F] \arrow[r, "\beta_{BF,F}"] \arrow[d, "\nu_A^F"'] & \mathsf{g}[BF] \arrow[d, "\nu_B^{BF}"] \\ \mathsf{g}[A] \arrow[r, "\beta_{B,A}"'] & \mathsf{g}[B] \end{tikzcd} \tag{16.34a}$$

commutes.

Signed antisymmetry. For each $A \lessdot F$,

$$\Big(\mathsf{g}[F] \xrightarrow{\nu_A^F} \mathsf{g}[A]\Big) + (-1)^{\operatorname{dist}(A\overline{F},F)}\Big(\mathsf{g}[F] \xrightarrow{\beta_{A\overline{F},F}} \mathsf{g}[A\overline{F}] \xrightarrow{\nu_A^{A\overline{F}}} \mathsf{g}[A]\Big) \ = 0. \tag{16.34b}$$

Signed Jacobi identity. For each $\mathrm{s}(A) \leq \mathrm{X}$ such that the face A has codimension two in X,

$$\sum_{i=1}^{n} (-1)^{\operatorname{dist}(G_1,G_i)}\Big(\mathsf{g}[G_1] \xrightarrow{\beta_{G_i,G_1}} \mathsf{g}[G_i] \xrightarrow{\nu_{F_i}^{G_i}} \mathsf{g}[F_i] \xrightarrow{\nu_A^{F_i}} \mathsf{g}[A]\Big) \ = 0, \tag{16.34c}$$

with notation as in (16.3c).

Given a signed Lie monoid g in terms of structure maps $\gamma_{\mathrm{X}}^{\mathrm{Y}}$, its Lie bracket component ν_A^F is constructed by employing the element $\mathtt{H}_{[F/A]} \otimes (\mathtt{H}_{F/A} - \mathtt{H}_{A\overline{F}/A})$ of $\mathbf{Lie}^-[\mathrm{X},\mathrm{Y}]$. The notation and remaining details are as in the unsigned case, see (16.5).

16.7.2. Signed commutator bracket. Since the Lie operad $\mathbf{Lie}$ is a suboperad of $\mathbf{As}$, the signed Lie operad $\mathbf{Lie}^-$ is a suboperad of $\mathbf{As}^-$. By Lemma 4.6, the latter is isomorphic to $\mathbf{As}$. Hence, there is an induced functor

$$\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp}) \to (-1)\text{-}\mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp}). \tag{16.35}$$

We call this the *underlying signed Lie monoid functor*.

In particular, every monoid a carries the structure of a signed Lie monoid. For $A \lessdot F$, the Lie bracket component ν_A^F of a is given by

$$\nu_A^F = \mu_A^F - (-1)^{\operatorname{dist}(A\overline{F},F)} \mu_A^{A\overline{F}} \beta_{A\overline{F},F}. \tag{16.36}$$

We call ν the *signed commutator bracket*.

16.7.3. Free signed Lie monoid. The free signed Lie monoid on a species p is given by $\mathbf{Lie}^- \circ \mathsf{p}$. Explicitly,

$$(\mathbf{Lie}^- \circ \mathsf{p})[\mathrm{X}] = \bigoplus_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{X}} \mathbf{Lie}^-[\mathrm{X}, \mathrm{Y}] \otimes \mathsf{p}[\mathrm{Y}].$$

This is a specialization of Proposition 4.23 to the signed Lie operad.

16.7.4. Lie monoids and signed Lie monoids. Recall the signature functor (4.40) on species. For any species g,

$$(\mathbf{Lie} \circ \mathsf{g})^- = \mathbf{Lie}^- \circ \mathsf{g}^-. \tag{16.37}$$

This is a specialization of (4.41). Thus, g is a Lie monoid iff g^- is a signed Lie monoid.

Proposition 16.31. *The signature functor yields an isomorphism between the categories of Lie monoids and signed Lie monoids.*

This is a specialization of Proposition 4.27.

Exercise 16.32. Let us work with the formulation of the signature functor in (8.79). Let (g, ν) be a Lie monoid. Check that the Lie bracket component of g^- for $A \lessdot F$ is given by

$$\mathsf{g}[F] \otimes \mathsf{E}^-[F] \xrightarrow{\nu_A^F \otimes \mu_A^F} \mathsf{g}[A] \otimes \mathsf{E}^-[A],$$

where μ_A^F is the product component of E^- as defined in (7.13).

Exercise 16.33. Let a be a monoid, and let g denote its underlying Lie monoid. Check that the underlying signed Lie monoid of a^- coincides with g^-. Here a^- is the monoid given by Corollary 8.91. (Use (16.11), (16.36) and Exercise 16.32.)

16.7.5. Monad on species of the signed Lie operad. Recall from Theorem 6.30 for $q = -1$ that the functors $\mathcal{T}_{-1}$ and $\mathcal{P}$ are adjoint between the category of species and the category of signed bimonoids.

Lemma 16.34. *The monad $\mathcal{P}\mathcal{T}_{-1}$ is the conjugate of $\mathcal{P}\mathcal{T}$ wrt the signature functor, that is,*

$$\mathcal{P}\mathcal{T}_{-1}(\mathsf{m}) = (\mathcal{P}\mathcal{T}(\mathsf{m}^-))^-.$$

Proof. We have

$$\mathcal{P}\mathcal{T}_{-1}(\mathsf{m}) = \mathcal{P}(\mathcal{T}(\mathsf{m}^-)^-) = (\mathcal{P}\mathcal{T}(\mathsf{m}^-))^-.$$

The first step used Corollary 8.96, while the second step used (8.84). $\square$

Recall from (4.42) that every operad $\mathbf{a}$ gives rise to a monad $\mathcal{V}_{\mathbf{a}}$.

Lemma 16.35. *The monad $\mathcal{V}_{\mathbf{Lie}^-}$ is the conjugate of $\mathcal{V}_{\mathbf{Lie}}$ wrt the signature functor, that is,*

$$(\mathbf{Lie} \circ \mathsf{m}^-)^- = \mathbf{Lie}^- \circ \mathsf{m}.$$

This is a reformulation of (16.37).

Proposition 16.36. *For any species* p,

$$\mathcal{PT}_{-1}(\mathsf{p}) = \mathbf{Lie}^{-} \circ \mathsf{p}. \tag{16.38}$$

Moreover, the monad structure of $\mathcal{PT}_{-1}$ *corresponds to the operad structure of* $\mathbf{Lie}^{-}$. *In particular, a* $\mathcal{PT}_{-1}$*-algebra is the same as a signed Lie monoid.*

Proof. By Proposition 16.6, the monads $\mathcal{PT}$ and $\mathcal{V}_{\mathbf{Lie}}$ are isomorphic. Combining with Lemmas 16.34 and 16.35, we deduce that the monads $\mathcal{PT}_{-1}$ and $\mathcal{V}_{\mathbf{Lie}^{-}}$ are isomorphic. □

16.7.6. Primitive part of a signed bimonoid. It follows from Proposition 16.36 that the primitive part of a signed bimonoid is a signed Lie monoid. Thus, we have a functor

$$\mathcal{P} : (-1)\text{-}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \to (-1)\text{-}\mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp}) \tag{16.39}$$

from the category of signed bimonoids to the category of signed Lie monoids. We continue to call it the *primitive part functor*.

Further, the primitive part of the free signed bimonoid $\mathcal{T}_{-1}(\mathsf{p})$ is the free signed Lie monoid on p.

16.7.7. Universal enveloping monoid. For a signed Lie monoid g, define its *universal enveloping monoid*, denoted $\mathcal{U}_{-1}(\mathsf{g})$, by

$$\mathcal{U}_{-1}(\mathsf{g}) := \mathcal{U}(\mathsf{g}^{-})^{-}. \tag{16.40}$$

This yields a functor from the category of signed Lie monoids to the category of monoids.

Lemma 16.37. *For a signed Lie monoid* g*, the monoid* $\mathcal{U}_{-1}(\mathsf{g})$ *is the quotient of* $\mathcal{T}_{-1}(\mathsf{g})$ *by the submonoid of relations generated by*

$$v - (-1)^{\operatorname{dist}(A\overline{F},F)} \beta_{A\overline{F},F}(v) - \nu_A^F(v), \quad v \in \mathsf{g}[F] \tag{16.41}$$

for $A \lessdot F$.

This is the signed analogue of Lemma 16.14.

Theorem 16.38. *The functor* $\mathcal{U}_{-1}$ *is the left adjoint of the underlying signed Lie monoid functor* (16.35)*. Explicitly, for any signed Lie monoid* g *and monoid* a*, there is a natural bijection*

$$\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{U}_{-1}(\mathsf{g}), \mathsf{a}) \xrightarrow{\cong} (-1)\text{-}\mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{g}, \mathsf{a}).$$

This is the signed analogue of Theorem 16.15 and can be deduced from it using the signature functor. The unit of the adjunction, namely,

$$i : \mathsf{g} \to \mathcal{U}_{-1}(\mathsf{g}) \tag{16.42}$$

is the composite $\mathsf{g} \hookrightarrow \mathcal{T}_{-1}(\mathsf{g}) \twoheadrightarrow \mathcal{U}_{-1}(\mathsf{g})$. It is a morphism of signed Lie monoids, with $\mathcal{U}_{-1}(\mathsf{g})$ equipped with the signed commutator bracket.

Recall from Section 6.4 that $\mathcal{T}_{-1}(\mathsf{g})$ is a signed cocommutative signed bimonoid. It is identical to $\mathcal{T}(\mathsf{g})$ as a monoid, but the coproducts are different. The coproduct of $\mathcal{T}_{-1}(\mathsf{g})$ is obtained from that of $\mathcal{T}(\mathsf{g})$ given in (16.16) by replacing β by β_{-1}. The canonical quotient map $\mathcal{T}_{-1}(\mathsf{g}) \twoheadrightarrow \mathcal{U}_{-1}(\mathsf{g})$ induces a coproduct on the latter. This yields a functor

$$\mathcal{U}_{-1} : (-1)\text{-}\mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp}) \to (-1)\text{-}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$$

from the category of signed Lie monoids to the category of signed cocommutative signed bimonoids.

Theorem 16.39. *The functor $\mathcal{U}_{-1}$ is the left adjoint of the primitive part functor $\mathcal{P}$. Explicitly, for any signed Lie monoid g and signed cocommutative signed bimonoid h, there is a natural bijection*

$$(-1)\text{-}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{U}_{-1}(\mathsf{g}), \mathsf{h}) \xrightarrow{\cong} (-1)\text{-}\mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{g}, \mathcal{P}(\mathsf{h})).$$

This is the signed analogue of Theorem 16.25 and can be deduced from it using the signature functor.

16.8. Lie comonoids

We now turn to Lie comonoids. They are left comodules over the Lie cooperad. Every comonoid carries the structure of a Lie comonoid via the cocommutator cobracket. Moreover, for any bimonoid, the cocommutator cobracket descends to the indecomposable part of the bimonoid. We associate to any Lie comonoid k a comonoid $\mathcal{U}^\vee(\mathsf{k})$ called the universal coenveloping comonoid of k. It is a subcomonoid of the cofree comonoid $\mathcal{T}^\vee(\mathsf{k})$. This yields a functor $\mathcal{U}^\vee$ from Lie comonoids to comonoids which is the left adjoint of the underlying Lie comonoid functor. In fact, $\mathcal{U}^\vee(\mathsf{k})$ is a subbimonoid of $\mathcal{T}^\vee(\mathsf{k})$, so it carries the structure of a commutative bimonoid. Viewed as a functor from Lie comonoids to bimonoids, $\mathcal{U}^\vee$ is the left adjoint of the indecomposable part functor. Similar considerations carry over to the signed setting.

Since the discussion is formally similar to that for Lie monoids, we will be brief in our exposition.

16.8.1. Lie comonoids as left comodules. Recall the Lie cooperad. It is the cooperad dual to the Lie operad. A *Lie comonoid* is a left comodule over the Lie cooperad. A morphism of Lie comonoids is a map of left comodules. We denote the category of Lie comonoids by $\mathsf{LieComon}(\mathcal{A}\text{-}\mathsf{Sp})$.

Explicitly: A Lie comonoid is a species k equipped with linear maps

$$\delta_{\mathrm{X}}^{\mathrm{Y}} : \mathsf{k}[\mathrm{X}] \to \mathbf{Lie}[\mathrm{X}, \mathrm{Y}]^* \otimes \mathsf{k}[\mathrm{Y}],$$

one for each $\mathrm{X} \leq \mathrm{Y}$, subject to the comodule axioms. Thus, for any element of $\mathbf{Lie}[\mathrm{X}, \mathrm{Y}]$, we have a linear map $\mathsf{k}[\mathrm{X}] \to \mathsf{k}[\mathrm{Y}]$ obtained by evaluating $\delta_{\mathrm{X}}^{\mathrm{Y}}$ at that Lie element.

16.8.2. Lie comonoids by generators and relations. A *Lie comonoid* is a species k equipped with linear maps

$$\theta_A^F : \mathsf{k}[A] \to \mathsf{k}[F],$$

one for each $A \lessdot F$, subject to the following axioms.

Naturality. For each morphism $\beta_{B,A} : A \to B$, and $A \lessdot F$, the diagram

(16.43a)
$$\begin{tikzcd} \mathsf{k}[F] \arrow[r, "\beta_{BF,F}"] & \mathsf{k}[BF] \\ \mathsf{k}[A] \arrow[u, "\theta_A^F"] \arrow[r, "\beta_{B,A}"'] & \mathsf{k}[B] \arrow[u, "\theta_B^{BF}"'] \end{tikzcd}$$

commutes.

Antisymmetry. For each $A \lessdot F$,

(16.43b)
$$\big(\mathsf{k}[A] \xrightarrow{\theta_A^F} \mathsf{k}[F]\big) \;+\; \big(\mathsf{k}[A] \xrightarrow{\theta_A^{A\overline{F}}} \mathsf{k}[A\overline{F}] \xrightarrow{\beta_{F,A\overline{F}}} \mathsf{k}[F]\big) \;= 0.$$

Jacobi identity. For each $\mathrm{s}(A) \leq \mathrm{X}$ such that the face A has codimension two in the flat X,

(16.43c)
$$\sum_{i=1}^{n} \big(\mathsf{k}[A] \xrightarrow{\theta_A^{F_i}} \mathsf{k}[F_i] \xrightarrow{\theta_{F_i}^{G_i}} \mathsf{k}[G_i] \xrightarrow{\beta_{G_1,G_i}} \mathsf{k}[G_1]\big) \;= 0,$$

with notation as in (16.3c).

We denote a Lie comonoid by a pair (k, θ), or simply by k with θ understood. We refer to θ as the *Lie cobracket*, and to the maps θ_A^F for $A \lessdot F$ as the Lie cobracket components.

A morphism $f : \mathsf{k} \to \mathsf{h}$ of Lie comonoids is a map of species such that for each $A \lessdot F$, the diagram

(16.44)
$$\begin{tikzcd} \mathsf{k}[A] \arrow[r, "f_A"] \arrow[d, "\theta_A^F"'] & \mathsf{h}[A] \arrow[d, "\theta_A^F"] \\ \mathsf{k}[F] \arrow[r, "f_F"'] & \mathsf{h}[F] \end{tikzcd}$$

commutes.

16.8.3. Cocommutator cobracket. Dualizing the inclusion of the operad **Lie** in **As** yields a surjective morphism of cooperads $\mathbf{As}^* \twoheadrightarrow \mathbf{Lie}^*$. Hence, there is an induced functor

(16.45)
$$\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}) \to \mathsf{LieComon}(\mathcal{A}\text{-}\mathsf{Sp}).$$

We call this the *underlying Lie comonoid functor*.

In particular, every comonoid c carries the structure of a Lie comonoid given by the composite

$$\mathsf{c} \to \mathbf{As}^* \circ \mathsf{c} \to \mathbf{Lie}^* \circ \mathsf{c}.$$

Explicitly: Let Δ_A^F, one for each $A \leq F$, denote the coproduct components of c. Then, for any $\mathrm{X} \leq \mathrm{Y}$, the Lie structure map $\delta_{\mathrm{X}}^{\mathrm{Y}}$ of c, evaluated at the Lie element

$$\sum_{F:\, F \geq A,\, \mathrm{s}(F)=\mathrm{Y}} x^{F/A}\, \mathtt{H}_{F/A},$$

where A is any fixed face with $\mathrm{s}(A) = \mathrm{X}$, is given by

$$\mathsf{c}[\mathrm{X}] \to \mathsf{c}[\mathrm{Y}], \qquad v \longmapsto \sum_{F\colon F \geq A,\, \mathrm{s}(F)=\mathrm{Y}} x^{F/A}\, \beta_{\mathrm{Y},F} \Delta^F_A \beta_{A,\mathrm{X}}(v). \tag{16.46}$$

This is the dual of (16.9).

For c formulated as in Proposition 2.77 using coproduct components Δ^F_A for $A \lessdot F$, the Lie cobracket component θ^F_A is given by

$$\theta^F_A = \Delta^F_A - \beta_{F,A\overline{F}} \Delta^{A\overline{F}}_A. \tag{16.47}$$

We refer to θ as the *cocommutator cobracket*. This is the dual of (16.11).

16.8.4. Indecomposable part of a bimonoid. Let h be a bimonoid. In particular, it is a comonoid, and hence a Lie comonoid via the functor (16.45). The Lie structure descends to its indecomposable part $\mathcal{Q}(\mathsf{h})$ as defined in (5.32). This result is dual to the one in Proposition 16.2 and can be proved in the same manner. It yields a functor

$$\mathcal{Q} : \mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \to \mathsf{LieComon}(\mathcal{A}\text{-}\mathsf{Sp}) \tag{16.48}$$

from the category of bimonoids to the category of Lie comonoids. We continue to call it the *indecomposable part functor*.

Exercise 16.40. For a bimonoid h and commutative monoid a, recall from Section 8.6 the bimonoid $\overline{\mathcal{C}}(\mathsf{h}, \mathsf{a})$. By formula (8.46), its primitive part is $\mathrm{hom}^{\times}(\mathcal{Q}(\mathsf{h}), \mathsf{a})$. By Proposition 16.2, this carries the structure of a Lie monoid. Make this explicit in either formulation along the lines of Example 16.3. (Use that $\mathcal{Q}(\mathsf{h})$ is a Lie comonoid.)

16.8.5. Cofree Lie comonoid on a species. The cofree Lie comonoid on a species p is given by $\mathbf{Lie}^* \circ \mathsf{p}$. Its universal property is stated below. Let $\mathbf{Lie}^* \circ \mathsf{p} \twoheadrightarrow \mathsf{p}$ be the canonical projection.

Theorem 16.41. *Let k be a Lie comonoid, p a species, $f : \mathsf{k} \to \mathsf{p}$ a map of species. Then there exists a unique morphism of Lie comonoids $\hat{f} : \mathsf{k} \to \mathbf{Lie}^* \circ \mathsf{p}$ such that the diagram*

$$\begin{tikzcd} \mathsf{k} \arrow[r, dashed, "\hat{f}"] \arrow[dr, "f"'] & \mathbf{Lie}^* \circ \mathsf{p} \arrow[d, two heads] \\ & \mathsf{p} \end{tikzcd}$$

commutes.

For any species p, recall the commutative bimonoid $\mathcal{T}^{\vee}(\mathsf{p})$ from Section 6.4. Its coproduct and product are given by formulas (6.40) for $q = 1$. Moreover, the adjunction between $\mathcal{T}^{\vee}$ and $\mathcal{Q}$ in Theorem 6.30 for $q = 1$ yields the comonad $\mathcal{Q}\mathcal{T}^{\vee}$ on species.

Proposition 16.42. *For any species p,*

$$\mathcal{Q}\mathcal{T}^{\vee}(\mathsf{p}) = \mathbf{Lie}^* \circ \mathsf{p}. \tag{16.49}$$

Moreover, the comonad structure of $\mathcal{Q}\mathcal{T}^\vee$ corresponds to the cooperad structure of $\mathbf{Lie}^*$*. In particular, a $\mathcal{Q}\mathcal{T}^\vee$-coalgebra is the same as a Lie comonoid.*

This is dual to Proposition 16.6.

Corollary 16.43. *For any species p, the indecomposable part of the bimonoid $\mathcal{T}^\vee(\mathsf{p})$ is the cofree Lie comonoid on p. In particular, for a finite-dimensional species p,*

$$\dim \mathcal{Q}(\mathcal{T}^\vee(\mathsf{p}))[\mathrm{X}] = \sum_{\mathrm{Y}\colon \mathrm{Y}\geq \mathrm{X}} |\mu(\mathrm{X},\mathrm{Y})| \dim \mathsf{p}[\mathrm{Y}], \tag{16.50}$$

where μ denotes the Möbius function of the poset of flats.

Thus, as a monoid, $\mathcal{T}^\vee(\mathsf{p})$ is isomorphic to the free commutative monoid on the species $\mathbf{Lie}^* \circ \mathsf{p}$.

This is dual to Corollary 16.7. The last claim follows from the Borel–Hopf Theorem 13.57.

Recall the cofree bimonoid $\mathcal{T}^\vee(\mathsf{a})$ on a monoid a from Section 6.2.2.

Proposition 16.44. *For any commutative monoid a, there is an isomorphism of Lie comonoids*

$$\mathcal{Q}\mathcal{T}^\vee(\mathsf{a}) \cong \mathbf{Lie}^* \circ \mathsf{a}_t,$$

where a_t is the underlying species of the monoid a. The rhs is the cofree Lie comonoid on a_t.

Thus, as a monoid, $\mathcal{T}^\vee(\mathsf{a})$ is isomorphic to the free commutative monoid on the species $\mathbf{Lie}^* \circ \mathsf{a}_t$.

Proof. By Proposition 14.44, $\mathcal{T}^\vee(\mathsf{a})$ is isomorphic to $\mathcal{T}^\vee(\mathsf{a}_t)$ as bimonoids via the HNR isomorphisms. The result follows by applying Corollary 16.43 to $\mathcal{T}^\vee(\mathsf{a}_t)$. $\square$

This is dual to Proposition 16.9.

16.8.6. Universal coenveloping comonoid. For any Lie comonoid k, consider the diagram

$$\begin{array}{ccc} \mathcal{T}^\vee(\mathsf{k}) & \xrightarrow{\cong} & \mathbf{As}^* \circ \mathsf{k} \\ \downarrow & & \downarrow \\ \mathsf{k} & \xrightarrow[\delta]{} & \mathbf{Lie}^* \circ \mathsf{k}. \end{array}$$

The vertical maps are the canonical projections. The subspecies of $\mathcal{T}^\vee(\mathsf{k})$ on which the diagram commutes is as follows. Evaluated on the A-component, it consists of those $(v_F)_{F\geq A}$ such that for every Lie element

$$z = \sum_{F\colon F\geq A,\, \mathrm{s}(F)=\mathrm{Y}} x^{F/A}\, \mathtt{H}_{F/A} \;\in\; \mathbf{Lie}[\mathrm{X},\mathrm{Y}],$$

with $\mathrm{X} = \mathrm{s}(A)$,

$$\sum_{F\colon F\geq A,\, \mathrm{s}(F)=\mathrm{Y}} x^{F/A}\, \beta_{\mathrm{Y},F}(v_F) = \langle \delta_{\mathrm{X}}^{\mathrm{Y}} \beta_{\mathrm{X},A}(v_A), z\rangle. \tag{16.51}$$

The rhs is the evaluation of $\delta_{\mathrm{X}}^{\mathrm{Y}}$ on z. We call (16.51) the A/A-conditions.

For $H \geq A$, consider the map

$$\mathcal{T}^\vee(\mathsf{k})[A] \twoheadrightarrow \mathcal{T}^\vee(\mathsf{k})[H], \qquad (v_F)_{F\geq A} \mapsto (v_F)_{F\geq H}.$$

We say $(v_F)_{F\geq A}$ satisfies the H/A-conditions if its image $(v_F)_{F\geq H}$ satisfies the H/H-conditions.

Define $\mathcal{U}^\vee(\mathsf{k})$ to be the subspecies of $\mathcal{T}^\vee(\mathsf{k})$ whose A-component consists of those $(v_F)_{F\geq A}$ which satisfy the H/A-conditions for all $H \geq A$. In other words, $\mathcal{U}^\vee(\mathsf{k})$ is the largest subcomonoid of $\mathcal{T}^\vee(\mathsf{k})$ such that the outside square in the diagram

$$\begin{tikzcd}
\mathcal{U}^\vee(\mathsf{k}) \arrow[dr, hook] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathcal{T}^\vee(\mathsf{k}) \arrow[r, "\cong"'] \arrow[d, two heads] & \mathbf{As}^* \circ \mathsf{k} \arrow[d, two heads] \\
& \mathsf{k} \arrow[r, "\delta"'] & \mathbf{Lie}^* \circ \mathsf{k}
\end{tikzcd} \tag{16.52}$$

commutes. (The bent maps are defined so that the triangles commute.) We call $\mathcal{U}^\vee(\mathsf{k})$ the *universal coenveloping comonoid of a Lie comonoid* of k. It defines a functor

$$\mathcal{U}^\vee : \mathsf{LieComon}(\mathcal{A}\text{-}\mathsf{Sp}) \to \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp}).$$

Theorem 16.45. *The functor $\mathcal{U}^\vee$ is the right adjoint of the underlying Lie comonoid functor* (16.45)*. Explicitly, for any Lie comonoid k and comonoid c, there is a natural bijection*

$$\mathsf{LieComon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c}, \mathsf{k}) \xrightarrow{\cong} \mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c}, \mathcal{U}^\vee(\mathsf{k})).$$

The counit of the adjunction is

$$p : \mathcal{U}^\vee(\mathsf{k}) \to \mathsf{k}. \tag{16.53}$$

It equals the composite map $\mathcal{U}^\vee(\mathsf{k}) \hookrightarrow \mathcal{T}^\vee(\mathsf{k}) \twoheadrightarrow \mathsf{k}$. It is a morphism of Lie comonoids, with $\mathcal{U}^\vee(\mathsf{k})$ equipped with the cocommutator cobracket.

Theorem 16.46. *Let c be a comonoid, k a Lie comonoid, $f : \mathsf{c} \to \mathsf{k}$ a morphism of Lie comonoids. Then there exists a unique morphism of comonoids $\hat{f} : \mathsf{c} \to \mathcal{U}^\vee(\mathsf{k})$ such that the diagram*

$$\begin{tikzcd}
\mathsf{c} \arrow[r, dashed, "\hat{f}"] \arrow[dr, "f"'] & \mathcal{U}^\vee(\mathsf{k}) \arrow[d, "p"] \\
& \mathsf{k}
\end{tikzcd}$$

commutes.

This reformulates the adjunction in Theorem 16.45 as a universal property. By Theorem 6.10, the map $f : \mathsf{c} \to \mathsf{k}$ lifts uniquely to a morphism of comonoids $\hat{f} : \mathsf{c} \to \mathcal{T}^\vee(\mathsf{k})$ which then maps into $\mathcal{U}^\vee(\mathsf{k})$ since f is a morphism of Lie comonoids.

Exercise 16.47. Check that $\mathcal{U}^\vee(\mathsf{k})$ is the largest subcomonoid of $\mathcal{T}^\vee(\mathsf{k})$ such that $\mathcal{U}^\vee(\mathsf{k}) \hookrightarrow \mathcal{T}^\vee(\mathsf{k}) \twoheadrightarrow \mathsf{k}$ is a morphism of Lie comonoids.

Exercise 16.48. Check that: For a Lie comonoid k, and any $z \in \mathsf{k}[\top]$, the element $\sum_{C\colon C \geq A} \beta_{C,\top}(z)$ is an element of $\mathcal{U}^\vee(\mathsf{k})[A]$. The sum is over chambers C greater than A. (Use Lemma 7.63.)

So far $\mathcal{U}^\vee(\mathsf{k})$ has only been considered as a comonoid. In fact, it is a subbimonoid of $\mathcal{T}^\vee(\mathsf{k})$. This statement is dual to Proposition 16.18 and can be proved in a similar manner. Since $\mathcal{T}^\vee(\mathsf{k})$ is commutative, so is $\mathcal{U}^\vee(\mathsf{k})$. This yields a functor

$$\mathcal{U}^\vee : \mathsf{LieComon}(\mathcal{A}\text{-}\mathsf{Sp}) \to \mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}).$$

The duals of Lemma 16.22 and Exercise 16.23 are stated below.

Lemma 16.49. *For any comonoid* c*, the projection map* $\mathcal{T}(\mathsf{c}) \twoheadrightarrow \mathsf{c}$ *is a morphism of Lie comonoids.*

Exercise 16.50. For a comonoid c, deduce that $\mathcal{Q}\mathcal{T}(\mathsf{c}) = \mathcal{Q}\mathcal{S}(\mathsf{c}) = \mathsf{c}$ as Lie comonoids. (Combine Lemma 16.49 with Propositions 6.57 and 6.59.) A related result is given later in Exercise 17.55.

Theorem 16.51. *The functor* $\mathcal{U}^\vee$ *is the right adjoint of the indecomposable part functor* $\mathcal{Q}$*. Explicitly, for any Lie comonoid* k *and commutative bimonoid* h*, there is a natural bijection*

$$\mathsf{LieComon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathcal{Q}(\mathsf{h}), \mathsf{k}) \xrightarrow{\cong} \mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h}, \mathcal{U}^\vee(\mathsf{k})).$$

This can also be viewed as an adjunction between the categories of bimonoids and Lie comonoids. It is phrased below as a universal property.

Theorem 16.52. *Let* h *be a bimonoid,* k *a Lie comonoid,* $f : \mathcal{Q}(\mathsf{h}) \to \mathsf{k}$ *a morphism of Lie comonoids. Then there exists a unique morphism of bimonoids* $\hat{f} : \mathsf{h} \to \mathcal{U}^\vee(\mathsf{k})$ *such that the diagram*

$$\begin{tikzcd} \mathsf{h} \arrow[r, dashed, "\hat{f}"] \arrow[d, two heads] & \mathcal{U}^\vee(\mathsf{k}) \arrow[d, "p"] \\ \mathcal{Q}(\mathsf{h}) \arrow[r, "f"'] & \mathsf{k} \end{tikzcd}$$

commutes.

The map $\hat{f} : \mathsf{h} \to \mathcal{U}^\vee(\mathsf{k})$ arises from the universal property of $\mathcal{U}^\vee(\mathsf{k})$ in Theorem 16.46. The extra step is that since f factors through $\mathcal{Q}(\mathsf{h})$, the map $\hat{f}$ is in addition a morphism of monoids.

Proposition 16.53. *For any species* p*, we have*

$$\mathcal{U}^\vee(\mathbf{Lie}^* \circ \mathsf{p}) = \mathcal{T}^\vee(\mathsf{p}) \tag{16.54}$$

as bimonoids.

PROOF. This follows by composing the adjunctions

$$\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{\mathcal{U}^\vee}{\overset{\mathcal{Q}}{\rightleftarrows}} \mathsf{LieComon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{\mathbf{Lie}^*\circ(-)}{\overset{frg}{\rightleftarrows}} \mathcal{A}\text{-}\mathsf{Sp}$$

and comparing the result with the adjunction between $\mathcal{T}^\vee$ and $\mathcal{Q}$ in Theorem 6.30 for $q = 1$. □

16.8.7. Abelian Lie comonoids. We say that a Lie comonoid is *abelian* if the structure maps δ_X^Y are zero for all $X < Y$. Equivalently, a Lie comonoid is abelian if all Lie cobracket components θ_A^F are 0.

For a cocommutative comonoid, observe that the cocommutator cobracket components (16.47) are all zero by the cocommutativity axiom (2.23). Thus, the functor (16.45) carries the subcategory of cocommutative comonoids into the subcategory of abelian Lie comonoids. Similarly, the indecomposable part functor (16.48) carries the subcategory of cocommutative bimonoids into the subcategory of abelian Lie comonoids. The latter is isomorphic to the category of species.

The universal coenveloping comonoid of an abelian Lie comonoid can be explicitly described as follows.

Proposition 16.54. *For an abelian Lie comonoid* k*, we have* $\mathcal{U}^\vee(\mathsf{k}) = \mathcal{S}^\vee(\mathsf{k})$*, the free cocommutative comonoid on* k *(with* k *viewed as a species), with the canonical inclusion* $\mathcal{U}^\vee(\mathsf{k}) \hookrightarrow \mathcal{T}^\vee(\mathsf{k})$ *being the coabelianization map* (6.65).

This is the dual of Proposition 16.30, and can be proved in a similar manner.

16.8.8. Signed Lie comonoids. We briefly discuss the signed case. A *signed Lie comonoid* is a left comodule over the signed Lie cooperad. It can also be formulated using the Lie cobracket, with signed antisymmetry and signed Jacobi identity dual to (16.34b) and (16.34c).

Every comonoid carries the structure of a signed Lie comonoid given by the *signed cocommutator cobracket* dual to (16.36). This yields a functor from the category of comonoids to the category of signed Lie comonoids. We call this the *underlying signed Lie comonoid functor*.

For a signed bimonoid h, its Lie structure via the signed cocommutator cobracket descends to its indecomposable part $\mathcal{Q}(\mathsf{h})$. This defines a functor from the category of signed bimonoids to the category of signed Lie comonoids. We continue to call it the *indecomposable part functor*. When applied to the signed bimonoid $\mathcal{T}_{-1}^\vee(\mathsf{p})$, it yields the *cofree signed Lie comonoid* on the species p.

To any signed Lie comonoid k, one can associate a comonoid $\mathcal{U}_{-1}^\vee(\mathsf{k})$ called the *universal coenveloping comonoid of a signed Lie comonoid* of k. It is the conjugate of $\mathcal{U}^\vee(\mathsf{k})$ wrt the signature functor. This yields a functor which is the left adjoint of the underlying signed Lie comonoid functor. Moreover, $\mathcal{U}_{-1}^\vee(\mathsf{k})$ carries the structure of a signed commutative signed bimonoid. Viewed as a functor from signed Lie comonoids to signed bimonoids, $\mathcal{U}_{-1}^\vee$ is the left adjoint of the indecomposable part functor $\mathcal{Q}$.

Notes

The theory of Lie monoids for hyperplane arrangements is presented here for the first time. It is motivated by the classical theory of Lie algebras, and that of Lie monoids in Joyal species. Connections to this classical literature are given below.

Binary operations vs higher operations. In our exposition, we formulated monoids using product components μ_A^F, where F is any arbitrary face greater than A. These can be thought of as 'higher operations', with 'binary operations' being those components μ_A^F where F covers A. Using the presentation of the associative operad, monoids can also be formulated purely in terms of 'binary operations' (Proposition 2.78). This is closer to the classical approach.

Similarly, we formulated Lie monoids directly using the Lie operad. The structure maps can be thought of as 'higher bracket operations'. The key reason why it is possible to work effectively with these maps is the Tits product (which we recall was used to define the components of the Lie operad). Using the presentation of the Lie operad, we also formulated Lie monoids in terms of the Lie bracket which is a 'binary operation' subject to antisymmetry and Jacobi identity. For many proofs, we supplied an alternative argument using this second approach.

Table 16.1. Cover relations on faces and binary operations.

Hyperplane	Classical
monoid $\mu_A^F : \mathsf{a}[F] \to \mathsf{a}[A]$, $A \lessdot F$ (Proposition 2.78)	algebra $\mu : A \otimes A \to A$
comonoid $\Delta_A^F : \mathsf{c}[A] \to \mathsf{c}[F]$, $A \lessdot F$ (Proposition 2.77)	coalgebra $\Delta : C \to C \otimes C$
Lie monoid $\nu_A^F : \mathsf{g}[F] \to \mathsf{g}[A]$, $A \lessdot F$ (Section 16.1.2)	Lie algebra $[\ ,\ ] : \mathsf{g} \otimes \mathsf{g} \to \mathsf{g}$
Lie comonoid $\theta_A^F : \mathsf{k}[A] \to \mathsf{k}[F]$, $A \lessdot F$ (Section 16.8.2)	Lie coalgebra $\theta : \mathsf{k} \to \mathsf{k} \otimes \mathsf{k}$

Table 16.1 illustrates how binary operations appear in the theory developed here in comparison to the classical theory.

Lie algebras. The abstract definition of a Lie algebra appears in a paper by Jacobson [**479**, Section 1]. It is briefly mentioned by Birkhoff [**114**, Section 1]. It is also given by Chevalley in his book on Lie groups [**210**, Section IV.II, Definition 2]. Lie algebras were called infinitesimal groups in the older literature. Some historical information about Lie algebras is given by Bourbaki [**150**, Historical note, pages 410 to 429], Borel [**138**, Section I.2]. There are now many books on Lie theory, see for instance the book references for PBW given in the Notes to Chapter 17.

The connection of the theory developed here with classical Lie theory for some of the definitions and results that we have discussed is given in Table 16.2.

The commutator bracket $[x, y]$ is called an alternant by Birkhoff [**114**, Section 1]. An early source for the classical proof that $[x, y]$ is primitive when x and y are primitive is by Cartier [**192**, Section 3.5, Lemma 3]. It corresponds to the second proof of Proposition 16.2. Other early references are by Milnor and Moore [**695**,

TABLE 16.2. Hyperplane and classical Lie theory.

Hyperplane	Classical
Lie bracket component $\nu_A^F : \mathsf{g}[F] \to \mathsf{g}[A]$	$[\ ,\ ] : \mathsf{g} \otimes \mathsf{g} \to \mathsf{g}$
antisymmetry (16.3b)	$[x, y] + [y, x] = 0$
Jacobi identity (16.3c)	$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$
product component $\mu_A^F : \mathsf{a}[F] \to \mathsf{a}[A]$	$A \otimes A \to A, \ \ x \otimes y \mapsto xy$
commutator bracket component (16.11) $\nu_A^F = \mu_A^F - \mu_A^{A\overline{F}} \beta_{A\overline{F},F}$	$[x, y] = xy - yx$
Proposition 16.2	If x and y are primitive, then so is $[x, y]$
A/A-relation for a rank-one flat (16.23), (16.25)	$x \otimes y - y \otimes x - [x, y]$
A/A-relation for a rank-two flat	$x \otimes y \otimes z - x \otimes z \otimes y - y \otimes z \otimes x +$ $z \otimes y \otimes x - [x, [y, z]]$
H/A-relation	$x \otimes y \otimes z - y \otimes x \otimes z - [x, y] \otimes z$
Lemma 16.14	$U(\mathsf{g})$ is $T(\mathsf{g})$ mod the ideal generated by $x \otimes y - y \otimes x - [x, y]$
Lemma 16.22	$A \to A \otimes A, \ \ x \mapsto 1 \otimes x + x \otimes 1$ is a morphism of Lie algebras
map $\mathsf{g} \to \mathcal{T}^\vee(\mathcal{U}(\mathsf{g}))$ in (16.32)	coproduct $\mathsf{g} \to U(\mathsf{g}) \otimes U(\mathsf{g})$, $x \mapsto 1 \otimes x + x \otimes 1$

Proposition 5.8], [**696**, Proposition 5.6], Sweedler [**864**, page 94], [**867**, page 67]. Similarly, the classical proof that $A \to A \otimes A$, $\ x \mapsto 1 \otimes x + x \otimes 1$ is a morphism of Lie algebras corresponds to the second proof of Lemma 16.22.

We now elaborate further on the classical meaning of A/A-relations. In general, an A/A-relation is the difference between a Lie monomial and its expansion in tensors. Any such relation can be constructed from the basic relation $x \otimes y - y \otimes x - [x, y]$. For instance,

$$\begin{aligned}[x, [y, z]] &= x \otimes [y, z] - [y, z] \otimes x \\ &= x \otimes y \otimes z - x \otimes z \otimes y - y \otimes z \otimes x + z \otimes y \otimes x.\end{aligned}$$

This is the analogue of the demonstration given after the proof of Lemma 16.12.

Signed graded Lie algebras and Lie superalgebras. References for signed aspects of graded vector spaces are given in the Notes to Chapter 2. The classical analogues of the signed Lie monoids in Section 16.7 are signed graded Lie algebras. The latter are considered by Cartier [**191**, Section 1], Gerstenhaber [**348**, Section 2], Milnor and Moore [**695**, Proposition 5.2], [**696**, Lemma 5.3]. A closely related notion is that of Lie superalgebras. Early references are by Nijenhuis and Richardson [**722**, Section 1], Ross [787], [788, Section 2], Berezin and Kac [100, Formula (3.6)]. The first book on the subject is by Scheunert [805]. A survey of the subject till 1984 is given by Leĭtes [593].

Table 16.3. Hyperplane and classical signed Lie theory.

Hyperplane	Classical
Lie bracket component $\nu_A^F : \mathsf{g}[F] \to \mathsf{g}[A]$	$[\ ,\ ] : \mathsf{g} \otimes \mathsf{g} \to \mathsf{g}$
signed antisymmetry (16.34b)	$[x, y] + (-1)^{\lvert x \rvert \lvert y \rvert}[y, x] = 0$
signed Jacobi identity (16.34c)	$(-1)^{\lvert x \rvert\lvert z \rvert}[x, [y, z]] + (-1)^{\lvert y \rvert\lvert x \rvert}[y, [z, x]] +$ $(-1)^{\lvert z \rvert\lvert y \rvert}[z, [x, y]] = 0$
product component $\mu_A^F : \mathsf{a}[F] \to \mathsf{a}[A]$	$A \otimes A \to A, \quad x \otimes y \mapsto xy$
signed commutator bracket component (16.36) $\nu_A^F = \mu_A^F - (-1)^{\operatorname{dist}(A\overline{F},F)} \mu_A^{A\overline{F}} \beta_{A\overline{F},F}$	$[x, y] = xy - (-1)^{\lvert x \rvert\lvert y \rvert} yx$

The classical analogues of signed antisymmetry, signed Jacobi identity, signed commutator bracket are shown in Table 16.3. These are the signed analogues of those shown in Table 16.2.

Color Lie algebras. There is a more general notion of color Lie algebras where the space is graded by a commutative monoid. The trivial monoid yields Lie algebras, $\mathbb{Z}_2$ yields Lie superalgebras, $\mathbb{N}$ yields graded Lie algebras. Color Lie algebras were introduced by Ree [775, Section 1] under the name 'generalized Lie algebras'. They were rediscovered by Scheunert [804, Definition 2] building on ideas of Lukierski, Rittenberg, Wyler [614], [782], [783]. Early papers on the subject are by Agrawala [8], Green and Jarvis [370], Kleeman [533]. Color Lie algebras are also mentioned in Montgomery's book [703, Example 10.5.14]. Detailed treatments can be found in the books by Freund [326, Part I], Bahturin, Mikhalev, Petrogradsky, Zaicev [64, Chapters 1 and 2], Mikhalev and Zolotykh [692, Chapter 1], Kharchenko [529, Chapter 7], see also the book by Elduque and Kochetov [289]. Color Lie algebras fit into the general framework of Lie monoids in linear symmetric monoidal categories discussed below.

Universal enveloping algebras. The construction of the universal enveloping algebra of a Lie algebra (as a quotient of the tensor algebra) appeared implicitly in work of Poincaré [751, page 1066], [752, page 225]. It is also present in later works of Birkhoff [114] and Witt [913]. The same construction is given by Harish-Chandra [416, page 900] under the name 'general enveloping algebra'. In a later paper [417, page 50], he coined the name universal enveloping algebra which is now standard. Gelfand uses the term 'infinitesimal group ring' [345], [346]. The notion of an enveloping algebra of a Lie algebra is mentioned by Jacobson [479].

The classical analogue of the universal property in Theorem 16.16 is present in the papers of Birkhoff [114, Theorem 2] and Witt [913, Satz 1]. Some other early sources are by Cartier [192, Section 3.1], Milnor and Moore [695, Definition 5.3 and

Remarks 5.4], [**696**, Proposition 5.4], Jacobson [**480**, Section V.1, Definition 1 and Theorem 2]. The classical proof corresponds to the second proof of Theorem 16.15.

The coproduct of the universal enveloping algebra appears in the book by Cartan and Eilenberg [**189**, Section XIII.5]. They also mention properties of the coproduct which amount to saying that the universal enveloping algebra is a cocommutative Hopf algebra. This fact is said more explicitly later by Cartier [**192**, beginning of Section 3.4] and by Milnor and Moore [**695**, Definition 5.7], [**696**, Proposition 5.7]. The classical analogue of Theorem 16.25 on the adjunction between $\mathcal{U}$ and $\mathcal{P}$ is explicitly given by Grünenfelder [**383**, Theorem I.3.10], [**384**, Bemerkungen 2], see also [**687**, Appendix]. Related references are given in the Notes to Chapter 17 under Cartier–Milnor–Moore.

Most books on Lie theory discuss the universal enveloping algebra. This construction is also in Jacobson's algebra textbook [**481**, pages 142 and 145].

Free Lie algebras. Free objects for the associative operad, commutative operad, Lie operad are summarized in Table 16.4. The first column gives the objects discussed in the text, while the second column gives the corresponding classical objects. For more on the tensor algebra and symmetric algebra, see Table 6.2 in the Notes to Chapter 6 and the subsequent discussion.

TABLE 16.4. Types of free algebras.

Hyperplane	Classical
free monoid $\mathcal{T}(\mathsf{p}) = \mathbf{As} \circ \mathsf{p}$ (Section 6.1.1)	tensor algebra $\mathcal{T}(V)$
free commutative monoid $\mathcal{S}(\mathsf{p}) = \mathbf{Com} \circ \mathsf{p}$ (Section 6.3.1)	symmetric algebra $\mathcal{S}(V)$
free Lie monoid $\mathbf{Lie} \circ \mathsf{p}$ (Section 16.3.1)	free Lie algebra $Lie(V)$
$\mathcal{P}(\mathcal{T}(\mathsf{p})) = \mathbf{Lie} \circ \mathsf{p}$ (Proposition 16.6)	$\mathcal{P}(\mathcal{T}(V)) = Lie(V)$
$\mathcal{U}(\mathbf{Lie} \circ \mathsf{p}) = \mathcal{T}(\mathsf{p})$ (Proposition 16.28)	$\mathcal{U}(Lie(V)) = \mathcal{T}(V)$

Free Lie algebras originated in work of Hall [**409**], Magnus [**626**], [**627**], Witt [**913**]. They are also mentioned by Birkhoff [**114**, Theorem 3], They are briefly treated in the book by Cartan and Eilenberg [**189**, Chapter XIII, Exercises 5, 6, 7, 8]. There is a book by Reutenauer devoted to free Lie algebras [**777**], see also [**778**]. Shorter treatments are given by Bonfiglioli and Fulci [**130**, Section 2.2 and Chapter 8], Bourbaki [**150**, Chapter II], Bahturin [**61**, Chapters 2 and 3], Garsia [**341**], Serre [**823**, Chapter IV]. Free color Lie algebras were first considered by Ree [**775**, Section 2]. For a later reference, see [**64**, Chapter 2].

The classical analogue of Corollary 16.7 says that the primitive part of the tensor algebra is the free Lie algebra. This result originated in work of Friedrichs [**329**, footnote on page 19], and is usually called the Friedrichs criterion. Early papers related to this criterion are those of Cohn [**217**], Magnus [**628**, Theorem I], Lyndon [**618**], Finkelstein [**306**], Zassenhaus [**928**, Theorem 2], Ree [**774**, Theorem 2.1]. For later references, see for instance [**130**, Theorem 3.13], [**341**, Theorem 2.1, item (ii)], [**428**, Theorem 3.5.20], [**442**, Chapter X, Proposition 2.1], [**480**, Section V.4, Theorem 9], [**611**, Proposition 13.2.1, item (d)], [**613**, Theorem 5.3.13], [**629**, Section 5.6, page 336], [**777**, Theorem 1.4, items (i) and (iii)], [**823**, Chapter IV, Theorem 7.1], [900], [910]. The case of Lie superalgebras and color Lie algebras is due to Ree [775, Theorem 5.1], see also [64, Chapter 3, Theorem 2.10], [692, Theorems 19.21 and 19.22].

The classical analogue of formula (16.33) says that the universal enveloping algebra of the free Lie algebra is the tensor algebra. (This is an equality of bialgebras though it is usually stated only as an equality of algebras.) This result is due to Witt [913, Proof of Satz 2]. For later references, see for instance [150, Section II.3.1], [189, Chapter XIII, Exercise 7], [303, Section 21, item (c)], [480, Section V.4, Theorem 7], [613, Corollary 5.3.9], [777, Theorem 0.5], [823, Chapter IV, Theorem 4.2, item (1)]. The case of Lie superalgebras and color Lie algebras is due to Ree [775, Theorem 2.5], it is also given in Musson's book [712, Theorem 6.2.1].

Lie coalgebras. The classical analogues of the Lie comonoids in Section 16.8 are Lie coalgebras. Lie coalgebras appeared in work of André [28, page 359], [29, Section 3]. In the first paper, he defines a Lie coalgebra in terms of antisymmetry and Jacobi identity, while in the second paper, he defines a Lie coalgebra as a quotient of a coalgebra under the cocommutator cobracket. The latter is dual to the definition of Lie algebra employed by Milnor and Moore [695, Definitions 5.1]. Lie coalgebras were independently studied by Michaelis [684], [685, Section 1], [686, Definition 1.4], also see the historical note [688, Remark 3.52].

Early references dealing with Lie coalgebras are by Nichols [719, Chapter III, page 63], [720, Section 3], Hain [405, Definition 6.24], Block [121, Section 4], Griffing [373, Chapter 2], [375], Neisendorfer [713, page 432], Schlessinger and Stasheff [807, page 315].

A brief exposition of Lie coalgebras is given by Smirnov [834, Section 3.6] and Fresse [322, Section 4.2]. For a categorical context, see [362, Example 2.4], [363, Definition 3.1]. For some recent work related to cofree Lie coalgebras, see [827], [828]. For generalizations of coalgebras and Lie coalgebras, see [38]. In hindsight, these could be viewed in the framework of coalgebras over cooperads.

Lie coalgebras also appeared in work of Drinfeld [262, page 802] in his study of Lie bialgebras. For early references, see for instance [870], [632, Section 8.1].

Table 16.5. Lie coalgebras.

Hyperplane	Classical
Lie cobracket component $\theta_A^F : \mathsf{k}[A] \to \mathsf{k}[F]$	$\theta : \mathsf{k} \to \mathsf{k} \otimes \mathsf{k}, \ \ \theta(x) = \sum x_1 \otimes x_2$
antisymmetry (16.43b)	$\sum x_1 \otimes x_2 + x_2 \otimes x_1 = 0$
Jacobi identity (16.43c)	$\sum x_1 \otimes (x_2 \otimes x_3) + x_2 \otimes (x_3 \otimes x_1) + x_3 \otimes (x_1 \otimes x_2) = 0$
coproduct component $\Delta_A^F : \mathsf{c}[A] \to \mathsf{c}[F]$	$C \to C \otimes C, \ \ \Delta(x) = \sum x_1 \otimes x_2$
cocommutator cobracket component (16.47) $\theta_A^F = \Delta_A^F - \beta_{F,A\overline{F}} \Delta_A^{A\overline{F}}$	$\theta(x) = \sum x_1 \otimes x_2 - x_2 \otimes x_1$

A comparison of the hyperplane and classical setting is shown in Table 16.5. The cobracket of a Lie coalgebra and coproduct of a coalgebra have been written using the Sweedler notation [**867**, Section 1.2].

Universal coenveloping coalgebras. The universal coenveloping coalgebra of a Lie coalgebra is defined by André [**29**, Propositions 4, 5, 6] and Michaelis [**684**], [**685**, Section 3]. Other early references are by Nichols [**719**, Chapter III, page 64], [**720**, page 72], Block [**121**, Section 4]. It is also mentioned by Smirnov [**834**, bottom of page 70].

The dual of the universal enveloping algebra goes back to Cartier [**193**, Section 2]. It is studied further by Hochschild [**440**, page 500], [**441**, Part II, Section 2], [**443**, page 56], [**445**, pages 229–231], see also [**258**, pages 99 and 100]. Its precise connection with the universal coenveloping coalgebra is explained by Michaelis [**684**], [**685**, page 31], [**687**, Theorem 3.12], [**688**, Note on page 718].

Freeness of the shuffle and quasishuffle algebras. The classical analogue of Corollary 16.43 says in particular that the shuffle algebra $\mathcal{T}^\vee(V)$ in Table 6.2 is a free commutative algebra. (This can be seen as a special case of the Borel–Hopf theorem for commutative bialgebras, see the Notes to Chapter 13.) Freeness of the shuffle algebra was analyzed by Radford [**770**, Theorem 3.1.1, item (e)]. An early treatment is by Melançon and Reutenauer [**672**, Theorem on page 589], [**777**, Corollary 5.5 and Theorem 6.1], [**778**, Corollary 5]. They also mention an unpublished note by Perrin and Viennot from 1981. Some later references are by Lothaire [**613**, Exercise 5.3.6], Gaines [**337**, Theorem 3.2], [**338**, Theorem 4], Minh and Petitot [**699**, Theorem 2.1], Hoffman [**449**, Theorem 2.2], Hazewinkel [**426**, Theorem 5.5], Sklyar and Ignatovich [**831**, Corollary 3], Guo and Xie [**395**, Theorem 2.2], Foissy, Patras, Thibon [**309**, Section 6, page 227].

The classical analogue of Proposition 16.44 says in particular that the quasishuffle algebra $\mathcal{T}^\vee(A)$ in Table 6.3 is a free commutative algebra when A is commutative. This follows from freeness of the shuffle algebra via Hoffman–Newman–Radford rigidity, see the Notes to Chapter 14 for more on this. For results related to freeness of quasishuffle algebras, see for instance, [**337**, Proposition 3.4], [**338**, Proposition 6], [**596**, Theorem 2.7], [**174**, Theorem 3 and Theorem 5, item (4)], [**266**, Corollary 1], [**698**, Proposition 7, item (3) and Theorem 1, item (4)]. There is no mention of the cofree Lie coalgebra in any of these references.

The classical analogue of the dual Proposition 16.9 is related to the result stated by Hoffman [**449**, Theorem 4.2]. See also [**174**, Theorem 5, items (1) and (2)], [**698**, Theorem 1, items (1) and (2)].

Joyal Lie monoids. Lie monoids in Joyal species first appeared in work of Barratt [**74**, Definition 4] in the language of 'twisted Lie algebras'. In Definition 6, he considers a certain class of free Lie monoids and shows in Theorem 2 that these embed as submonoids of free monoids. Lie monoids were considered later by Joyal [**500**], Goerss [**358**], Stover [**854**], Aubry [**51**], [**52**]. A definition similar in spirit to the viewpoint of this text is given later in Section 17.7, see also [**18**, Section 11.9.1], [**19**, Section 2.9]. For more information on Joyal Lie monoids, see the Notes to Chapter 17.

Lie monoids in linear symmetric monoidal categories. Lie monoids can be defined in any linear symmetric monoidal category. An early reference is Manin's book [**642**, page 81], [**645**, Section 13.5.3]. For later references, see [**18**, Section 1.2.10], [**63**, Definition 1.5], [**239**, Section 1.3.1], [**318**, Appendix A.6], [**324**, Section 7.2.1], [**362**, Definition 2.1]. For the category of graded vector spaces, this yields (signed) graded Lie algebras, while for the category of Joyal species, this yields (signed) Joyal Lie monoids. More generally, for a commutative monoid A equipped with a skew-symmetric bicharacter, the category of A-graded vector spaces yields A-color Lie algebras.

A Lie monoid can also be defined 'locally' using an involutive Yang–Baxter operator on an object in a linear monoidal category. For the basic example of vector spaces, an early reference is by Gurevich [**396**, Definition in Section 5], [**397**, Definition in Section 2]. The term 'braided Lie algebra' is often used in this context. For later references, see [**529**, Section 7.3], [**530**, Section 4]. The categorical context is given in [**362**, Definition 2.3], [**363**, Definition 2.5], [**46**, Remark 5.2].

There exist extensions of the notion of Lie monoid to more general settings involving a nonsymmetric braiding or a noninvolutive Yang–Baxter operator. See the papers by Majid [**631**, Section 4], [**630**, Section 5.3], Pareigis [**737**, Section 4], [**738**, Sections 4 and 7], Ardizzoni [**43**, Definition 5.4], [**44**, Section 4].

For Friedrichs criterion for Lie algebras in a linear symmetric monoidal category, see [**324**, Proposition 7.2.14, item (b)]. For free braided Lie algebras, see [**529**, Section 7.4]. For Friedrichs criterion for braided Lie algebras, see [**530**, Theorem 5.3], [**529**, Theorems 7.6 and 7.7].

Lie monoids for hyperplane arrangements. Lie theory for hyperplane arrangements was initiated in our monograph [**21**, Chapters 10 and 14 and Section 15.9]. The substitution map of Lie (1.167) is given in [**21**, (10.28)] and the Lie operad is mentioned in [**21**, Section 15.9.2]. A result equivalent to Proposition 16.11 is mentioned in [**21**, Section 15.9.8]. The classical Dynkin basis and Lyndon basis for the space of Lie elements are extended to arrangements in [**21**, Chapter 14]; the connection between the hyperplane and classical theory is explained in [**21**, Section 14.9]. These bases are not discussed in the present text.

Lie monoids for arrangements are introduced here for the first time. In particular, this includes the geometric formulation of antisymmetry and Jacobi identity, and related constructions of the free Lie monoid and the universal enveloping monoid. Lie monoids for arrangements are a step towards generalizing positive Joyal Lie monoids and positively graded Lie algebras. They are also similar in spirit to braided Lie algebras, where the braiding manifests locally in the form of an involutive Yang–Baxter operator.

In a future work, we plan to explain Poisson monoids for arrangements. These are left modules over the Poisson operad. The latter is mentioned in the Notes to Chapter 4.

CHAPTER 17

Poincaré–Birkhoff–Witt and Cartier–Milnor–Moore

Recall from Chapter 16 that to every Lie monoid g, one can associate its universal enveloping monoid $\mathcal{U}(\mathsf{g})$. The latter carries the structure of a cocommutative bimonoid arising from the canonical quotient map $\mathcal{T}(\mathsf{g}) \twoheadrightarrow \mathcal{U}(\mathsf{g})$. The kernel of this map is the monoid of relations $\mathcal{I}(\mathsf{g})$ which is built out of the Lie structure of g.

The Poincaré–Birkhoff–Witt theorem (PBW) says that for any Lie monoid g, the universal enveloping monoid $\mathcal{U}(\mathsf{g})$ is isomorphic to $\mathcal{S}(\mathsf{g})$, the cofree cocommutative comonoid on g. The isomorphism is that of comonoids. It arises as the composite map $\mathcal{S}(\mathsf{g}) \hookrightarrow \mathcal{T}(\mathsf{g}) \twoheadrightarrow \mathcal{U}(\mathsf{g})$. The first map is a comonoid section to the abelianization map and depends on the choice of a noncommutative zeta function. The second map is the canonical quotient map.

Equivalently, PBW says that there is a decomposition $\mathcal{T}(\mathsf{g}) = \mathcal{S}(\mathsf{g}) \oplus \mathcal{I}(\mathsf{g})$. This defines an idempotent operator on $\mathcal{T}(\mathsf{g})$ whose image is $\mathcal{S}(\mathsf{g})$ and kernel is $\mathcal{I}(\mathsf{g})$, or equivalently, whose coimage is $\mathcal{U}(\mathsf{g})$. We call this the Solomon operator. It can be expressed as an exponential of the coderivation

$$\mathcal{T}(\mathsf{g}) \xrightarrow{\mathbf{log}(\mathrm{id})} \mathcal{P}\mathcal{T}(\mathsf{g}) \xrightarrow{\gamma} \mathsf{g} \hookrightarrow \mathcal{T}(\mathsf{g}).$$

The first map is a logarithm of the identity map, the middle map is the Lie structure map of g, the last map is the canonical inclusion. (Recall that exponential and logarithm depend on the choice of a noncommutative zeta and Möbius function.)

We give two proofs of PBW. The first one is elementary and inductively builds the Solomon operator. The second one starts with the above definition of the Solomon operator, and then establishes that it has the correct image and kernel.

The Cartier–Milnor–Moore theorem (CMM) says that the functors $\mathcal{U}$ and $\mathcal{P}$ determine an adjoint equivalence between the category of Lie monoids and the category of cocommutative bimonoids. It is a formal consequence of Borel–Hopf and PBW. A summary of these three results is provided in Table 17.1.

PBW and CMM also have dual versions. They go along with the Borel–Hopf theorem for commutative bimonoids. A summary is given in Table 17.2. Relevant notions are Lie comonoids and their universal coenveloping comonoids. PBW and CMM as well as their dual versions have signed analogues with $\mathcal{S}$ replaced by $\mathcal{E}$, with $\mathcal{U}$ replaced by $\mathcal{U}_{-1}$, and so on.

TABLE 17.1. CMM, Borel–Hopf, PBW.

Theorem	Starting data	Statement
CMM	cocommutative bimonoid h	$\mathcal{U}(\mathcal{P}(\mathsf{h})) \cong \mathsf{h}$ as bimonoids
	Lie monoid g	$\mathsf{g} \cong \mathcal{P}(\mathcal{U}(\mathsf{g}))$ as Lie monoids
Borel–Hopf	cocommutative bimonoid h	$\mathcal{S}(\mathcal{P}(\mathsf{h})) \cong \mathsf{h}$ as comonoids
PBW	Lie monoid g	$\mathcal{S}(\mathsf{g}) \cong \mathcal{U}(\mathsf{g})$ as comonoids

TABLE 17.2. Dual CMM, Borel–Hopf, dual PBW.

Theorem	Starting data	Statement
dual CMM	commutative bimonoid h	$\mathsf{h} \cong \mathcal{U}^{\vee}(\mathcal{Q}(\mathsf{h}))$ as bimonoids
	Lie comonoid k	$\mathcal{Q}(\mathcal{U}^{\vee}(\mathsf{k})) \cong \mathsf{k}$ as Lie comonoids
Borel–Hopf	commutative bimonoid h	$\mathsf{h} \cong \mathcal{S}^{\vee}(\mathcal{Q}(\mathsf{h}))$ as monoids
dual PBW	Lie comonoid k	$\mathcal{U}^{\vee}(\mathsf{k}) \cong \mathcal{S}^{\vee}(\mathsf{k})$ as monoids

The Leray–Samelson theorem for bicommutative bimonoids can be viewed as a restriction of the adjoint equivalence in CMM and also in dual CMM. This is illustrated below. (Recall that the category of species is isomorphic to the category of abelian Lie monoids and also to the category of abelian Lie comonoids.)

$$
\begin{tikzcd}
\text{Lie monoid} \arrow[r, leftrightarrow, "\text{CMM}"] & \text{cocommutative bimonoid} \\
\text{species} \arrow[u, hook] \arrow[d, hook] \arrow[r, leftrightarrow, "\text{LS}"] & \text{bicommutative bimonoid} \arrow[u, hook] \arrow[d, hook] \\
\text{Lie comonoid} \arrow[r, leftrightarrow, "\text{dual CMM}"'] & \text{commutative bimonoid}
\end{tikzcd}
$$

All results are independent of the characteristic of the base field.

17.1. Comonoid sections to the abelianization map

Recall the abelianization and coabelianization maps from Section 6.6. Both are morphisms of bimonoids. Noncommutative zeta functions can be used to construct comonoid sections to the abelianization map, and dually, monoid projections to the coabelianization map. Similar considerations apply to the signed abelianization and signed coabelianization maps.

17.1.1. Abelianization map. For any species p, recall the abelianization map $\mathcal{T}(\mathsf{p}) \twoheadrightarrow \mathcal{S}(\mathsf{p})$ given in (6.61). It is a morphism of bimonoids. We now construct comonoid sections to this map. For that, fix a noncommutative zeta function $\boldsymbol{\zeta}$. For a species p, define an injective map of species

$$\mathcal{S}(\mathsf{p}) \hookrightarrow \mathcal{T}(\mathsf{p}) \tag{17.1}$$

as follows. Evaluating on the A-component, on the X-summand for $\mathrm{X} \geq \mathrm{s}(A)$, the map is

$$\sum_{\substack{G\colon G \geq A \\ \mathrm{s}(G)=\mathrm{X}}} \boldsymbol{\zeta}(A,G)\, \beta_{G,\mathrm{X}} \,:\, \mathsf{p}[\mathrm{X}] \longrightarrow \bigoplus_{\substack{G\colon G \geq A \\ \mathrm{s}(G)=\mathrm{X}}} \mathsf{p}[G]. \tag{17.2}$$

In particular, the map (17.1) sends p to itself by the identity. This is the case when $\mathrm{X} = \mathrm{s}(A)$ in (17.2).

The map (17.1) is a section to the abelianization map. This follows from the flat-additivity formula (1.43).

Lemma 17.1. *The map* (17.1) *is an injective morphism of comonoids.*

Proof. Let us denote the map (17.1) by f. We check below that f is a morphism of comonoids. Fix a face A and a flat X containing A. Let $H \geq A$. If $\mathrm{s}(H) \not\leq \mathrm{X}$, then Δ^H_A is zero on $\mathsf{p}[\mathrm{X}]$ by the second alternative in (6.51) and it is also zero on the image of f_A by the second alternative in (6.39). So we may assume that $\mathrm{s}(H) \leq \mathrm{X}$. In this case, we need to check that the diagram

$$\begin{tikzcd}
& \mathsf{p}[\mathrm{X}] \arrow[dl, "f_A"'] \arrow[dr, "f_H"] & \\
\mathcal{T}(\mathsf{p})[A] \arrow[rr, "\Delta^H_A"'] & & \mathcal{T}(\mathsf{p})[H]
\end{tikzcd}$$

commutes. Going left-down and across,

$$\begin{aligned}
\sum_{F\colon F \geq A,\, \mathrm{s}(F)=\mathrm{X}} \boldsymbol{\zeta}(A,F)\, \Delta^H_A \beta_{F,\mathrm{X}} &= \sum_{F\colon F \geq A,\, \mathrm{s}(F)=\mathrm{X}} \boldsymbol{\zeta}(A,F)\, \beta_{HF,F} \beta_{F,\mathrm{X}} \\
&= \sum_{F\colon F \geq A,\, \mathrm{s}(F)=\mathrm{X}} \boldsymbol{\zeta}(A,F)\, \beta_{HF,\mathrm{X}} \\
&= \sum_{G\colon G \geq H,\, \mathrm{s}(G)=\mathrm{X}} \Bigg(\sum_{\substack{F\colon F \geq A,\, HF=G \\ \mathrm{s}(F)=\mathrm{s}(G)}} \boldsymbol{\zeta}(A,F) \Bigg) \beta_{G,\mathrm{X}} \\
&= \sum_{G\colon G \geq H,\, \mathrm{s}(G)=\mathrm{X}} \boldsymbol{\zeta}(H,G)\, \beta_{G,\mathrm{X}},
\end{aligned}$$

which is the same as going right-down. In the last step, the sum in parenthesis was evaluated by the lune-additivity formula (1.42). $\square$

Note the similarity of Lemma 17.1 with Lemma 13.33. The following two exercises explain how either result can be deduced from the other.

Exercise 17.2. Recall that $\mathcal{T}(\mathsf{p})$ is a cocommutative bimonoid and p is contained in its primitive part. The latter result is given in Exercise 6.33, and is also contained in (16.17). Now consider the maps

$$\mathcal{S}(\mathsf{p}) \to \mathcal{S}(\mathcal{P}(\mathcal{T}(\mathsf{p}))) \to \mathcal{T}(\mathsf{p}).$$

The first map is obtained by applying the functor $\mathcal{S}$ to the map $\mathsf{p} \hookrightarrow \mathcal{P}(\mathcal{T}(\mathsf{p}))$, while the second map is the map (13.21) applied to $\mathsf{h} := \mathcal{T}(\mathsf{p})$. Check that this composite is precisely (17.1). (Substitute product formula (6.3) in (13.22) and compare it with (17.2).) Now use Lemma 13.33 to deduce Lemma 17.1.

Exercise 17.3. For any bimonoid h, consider the maps

$$\mathcal{S}(\mathcal{P}(\mathsf{h})) \to \mathcal{T}(\mathcal{P}(\mathsf{h})) \to \mathsf{h}.$$

The first map is (17.1) applied to $\mathsf{p} := \mathcal{P}(\mathsf{h})$. The second map arises from Theorem 6.31 applied to $\mathsf{p} := \mathcal{P}(\mathsf{h})$ and $f := \mathrm{id}$. Check that the composite of these two maps is precisely (13.21). (Use formulas (17.2) and (6.41).) Now use Lemma 17.1 to deduce Lemma 13.33.

Exercise 17.4. Check that: The map (17.1) is an exponential (9.3a) of the coderivation

$$\mathcal{S}(\mathsf{p}) \twoheadrightarrow \mathsf{p} \hookrightarrow \mathcal{T}(\mathsf{p}).$$

The first map is the canonical projection, while the second is the canonical inclusion. Use Theorem 9.11 to deduce that (17.1) is a morphism of comonoids. This gives another proof of Lemma 17.1.

17.1.2. Coabelianization map. For any species p, recall the coabelianization map $\mathcal{S}^\vee(\mathsf{p}) \hookrightarrow \mathcal{T}^\vee(\mathsf{p})$ given in (6.65). It is a morphism of bimonoids. We now construct monoid projections to this map. For that, fix a noncommutative zeta function $\boldsymbol{\zeta}$. For a species p, define a surjective map of species

$$\mathcal{T}^\vee(\mathsf{p}) \twoheadrightarrow \mathcal{S}^\vee(\mathsf{p}) \tag{17.3}$$

as follows. Evaluating on the A-component, on the G-summand, the map is $\boldsymbol{\zeta}(A,G)\,\beta_{\mathrm{s}(G),G}$. In particular, the map (17.3) sends p to itself by the identity.

The map (17.3) is a projection to the coabelianization map. This follows from the flat-additivity formula (1.43).

Lemma 17.5. *The map* (17.3) *is a surjective morphism of monoids.*

This can be checked directly using product formulas (6.40) and (6.51).

Exercise 17.6. Note the similarity between Lemma 17.5 and Lemma 13.56. Explain how either result can be deduced from the other by formulating duals of Exercises 17.2 and 17.3.

Exercise 17.7. Check that: The map (17.3) is an exponential (9.3a) of the derivation

$$\mathcal{T}^\vee(\mathsf{p}) \twoheadrightarrow \mathsf{p} \hookrightarrow \mathcal{S}^\vee(\mathsf{p}).$$

The first map is the canonical projection, while the second is the canonical inclusion. Use Theorem 9.12 to deduce that (17.3) is a morphism of monoids.

17.1.3. Signed analogues. For any species p, recall the signed abelianization map $\mathcal{T}_{-1}(\mathsf{p}) \twoheadrightarrow \mathcal{E}(\mathsf{p})$ given in (6.69). Fix a noncommutative zeta function $\boldsymbol{\zeta}$. For any species p, define an injective morphism of comonoids

$$\mathcal{E}(\mathsf{p}) \hookrightarrow \mathcal{T}_{-1}(\mathsf{p}) \tag{17.4}$$

as follows. Evaluating on the A-component, on the X-summand for $\mathrm{X} \geq \mathrm{s}(A)$, the map is

$$\sum_{\substack{G:\, G \geq A \\ \mathrm{s}(G)=\mathrm{X}}} \boldsymbol{\zeta}(A,G)\,([G/A]\,\beta_{G,\mathrm{X}}) \,:\, \mathbf{E}^{-}[\mathrm{s}(A),\mathrm{X}] \otimes \mathsf{p}[\mathrm{X}] \longrightarrow \bigoplus_{\substack{G:\, G \geq A \\ \mathrm{s}(G)=\mathrm{X}}} \mathsf{p}[G], \tag{17.5}$$

with $[G/A]\,\beta_{G,\mathrm{X}}$ as in (2.45). This map is a section to the signed abelianization map.

Exercise 17.8. Check that: The map (17.4) is an exponential (9.3a) of the coderivation

$$\mathcal{E}(\mathsf{p}) \twoheadrightarrow \mathsf{p} \hookrightarrow \mathcal{T}_{-1}(\mathsf{p}).$$

Use Theorem 9.30 to deduce that (17.4) is a morphism of comonoids.

Dually, one can define monoid projections to the signed coabelianization map $\mathcal{E}^{\vee}(\mathsf{p}) \hookrightarrow \mathcal{T}^{\vee}_{-1}(\mathsf{p})$. We omit the details.

17.2. Poincaré–Birkhoff–Witt

For any species p, recall the bicommutative bimonoid $\mathcal{S}(\mathsf{p})$ from Section 6.5.1. Similarly, for any Lie monoid g, recall the universal enveloping monoid $\mathcal{U}(\mathsf{g})$ from Section 16.5. It carries the structure of a cocommutative bimonoid. The Poincaré–Birkhoff–Witt theorem says that for any Lie monoid g, the bimonoids $\mathcal{S}(\mathsf{g})$ and $\mathcal{U}(\mathsf{g})$ are isomorphic as comonoids. We write PBW for short. The isomorphism depends on the choice of a noncommutative zeta function and makes use of the comonoid section to the abelianization map from Section 17.1. PBW also has a signed analogue.

Dually, for any Lie comonoid k, the bimonoids $\mathcal{S}^{\vee}(\mathsf{k})$ and $\mathcal{U}^{\vee}(\mathsf{k})$ are isomorphic as monoids. We recall that $\mathcal{S}^{\vee}(\mathsf{k})$ is the same as $\mathcal{S}(\mathsf{k})$, while $\mathcal{U}^{\vee}(\mathsf{k})$ is the universal coenveloping comonoid of k.

17.2.1. Poincaré–Birkhoff–Witt. For a Lie monoid g, its universal enveloping monoid $\mathcal{U}(\mathsf{g})$ is a quotient of $\mathcal{T}(\mathsf{g})$ and moreover carries a bimonoid structure inherited from $\mathcal{T}(\mathsf{g})$. By composing the injective map (17.1) for $\mathsf{p} := \mathsf{g}$ with the canonical surjective map $\mathcal{T}(\mathsf{g}) \twoheadrightarrow \mathcal{U}(\mathsf{g})$, we obtain a morphism of comonoids

$$\mathcal{S}(\mathsf{g}) \to \mathcal{U}(\mathsf{g}). \tag{17.6}$$

Note very carefully that this map depends on the choice of a noncommutative zeta function $\boldsymbol{\zeta}$.

Theorem 17.9. *For any Lie monoid* g*, the map* (17.6) *is an isomorphism of comonoids.*

This is the *Poincaré–Birkhoff–Witt theorem*, or PBW for short. A proof is given in Section 17.2.3 below. Another proof using the Solomon operator is given later in Section 17.4, see Theorem 17.32. We refer to (17.6) as a *Poincaré–Birkhoff–Witt isomorphism*, or PBW isomorphism for short.

17.2.2. Borel–Hopf. For any cocommutative bimonoid h, consider the composite map

$$\mathcal{S}(\mathcal{P}(\mathsf{h})) \to \mathcal{U}(\mathcal{P}(\mathsf{h})) \to \mathsf{h}. \tag{17.7}$$

The first map arises from setting $\mathsf{g} := \mathcal{P}(\mathsf{h})$ in (17.6), while the second map is the counit of the adjunction between $\mathcal{U}$ and $\mathcal{P}$ from Theorem 16.25.

Lemma 17.10. *The composite map* (17.7) *coincides with the Borel–Hopf isomorphism* (13.21).

Proof. Consider the diagram

$$\begin{tikzcd}
 & \mathcal{T}(\mathcal{P}(\mathsf{h})) \arrow[d, two heads] \arrow[dr] & \\
\mathcal{S}(\mathcal{P}(\mathsf{h})) \arrow[ur, hook] \arrow[r] & \mathcal{U}(\mathcal{P}(\mathsf{h})) \arrow[r] & \mathsf{h}.
\end{tikzcd}$$

The left-up and right-down maps are as in Exercise 17.3. The triangles commute by definitions of the bottom-horizontal maps. So, by Exercise 17.3, their composite coincides with the map (13.21), which is an isomorphism by the Borel–Hopf Theorem 13.34. $\square$

This discussion is continued further in Section 17.5.

17.2.3. Proof of PBW. The surjectivity and injectivity of the map (17.6) are proved separately in Lemmas 17.11 and 17.12 below. Recall that the kernel of $\mathcal{T}(\mathsf{g}) \twoheadrightarrow \mathcal{U}(\mathsf{g})$ is the monoid of relations $\mathcal{I}(\mathsf{g})$ whose A-component is linearly spanned by H/A-relations, for $H \geq A$.

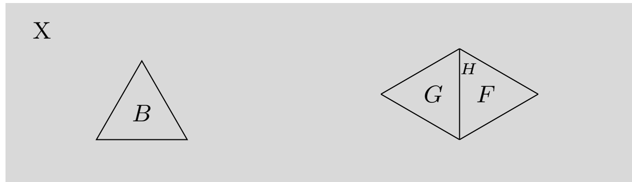
Lemma 17.11. *For any Lie monoid* g*, the map* (17.6) *is surjective.*

Proof. Let ρ denote the map $\mathcal{T}(\mathsf{g}) \twoheadrightarrow \mathcal{U}(\mathsf{g})$. We proceed to construct a surjective map σ fitting into a commutative diagram of species

$$\begin{tikzcd}
 & \mathcal{T}(\mathsf{g}) \arrow[dl, two heads, dashed, "\sigma"'] \arrow[dr, two heads, "\rho"] & \\
\mathcal{S}(\mathsf{g}) \arrow[rr] & & \mathcal{U}(\mathsf{g}).
\end{tikzcd}$$

In particular, this implies that (17.6) is surjective. (Here, we are only asserting existence of σ. It will also end up being unique.)

We focus on the case when $\boldsymbol{\zeta}$ is set-theoretic (since the gist of the argument is most transparent in this case). Let A be any face. We define σ_A on each summand $\mathsf{g}[F]$ of $\mathcal{T}(\mathsf{g})[A]$ by induction on the rank of F/A. Let X be the support of F, and let B be the unique face greater than A of support X such that $\boldsymbol{\zeta}(A, B) = 1$. We do another induction on the gallery distance of F from B. To start the induction, σ_A maps the summand $\mathsf{g}[B]$ to $\mathsf{g}[\mathrm{X}]$ via $\beta_{\mathrm{X},B}$. This also ensures that the σ_A we construct is surjective.



In the star of A, let G be a face adjacent to F which is closer to B, and let H be their common panel. (In the illustration above, A is taken to be the central face.) There could be more than one choice for G. Pick one. By induction hypothesis, σ_A has been defined on the summands $\mathtt{g}[G]$ and $\mathtt{g}[H]$. Now for any $v \in \mathtt{g}[\mathrm{X}]$, consider the H/A-relation

$$\beta_{F,\mathrm{X}}(v) - \beta_{G,\mathrm{X}}(v) - \beta_{H,\mathrm{Z}}\gamma^{\mathrm{X}}_{\mathrm{Z}}\big((\mathtt{H}_{F/H} - \mathtt{H}_{G/H}) \otimes v\big) \;\in\; \mathtt{g}[F] \oplus \mathtt{g}[G] \oplus \mathtt{g}[H]$$

(with $\mathrm{Z} := \mathrm{s}(H)$). This is an instance of (16.23) (with H instead of A, and G instead of $A\overline{F}$). Define σ_A on $\mathtt{g}[F]$ in such a way that it maps the above relation to zero (just like ρ_A). This completes the induction step, and finishes the proof when $\boldsymbol{\zeta}$ is set-theoretic.

Observe carefully that in the above argument, instead of doing a second induction on gallery distance, one can proceed more abstractly as follows. Use Lemma 1.76 to write

$$\mathtt{H}_{F/A} - \mathtt{H}_{B/A} = \sum_{H:\, H \geq A,\, \mathrm{s}(H) < \mathrm{X}} l_H,$$

where each l_H is a Lie element in the arrangement over the support of H and under X. (There may be more than one way of doing this. Pick one.) It follows that for any $v \in \mathtt{g}[\mathrm{X}]$,

$$\beta_{F,\mathrm{X}}(v) - \beta_{B,\mathrm{X}}(v) - \sum_{H} \beta_{H,\mathrm{s}(H)}\gamma^{\mathrm{X}}_{\mathrm{s}(H)}(l_H \otimes v) \;\in\; \mathtt{g}[F] \oplus \mathtt{g}[B] \oplus \big(\bigoplus_{H} \mathtt{g}[H]\big)$$

belongs to the monoid of relations, so ρ_A maps it to 0. By induction hypothesis, σ_A has been defined on $\mathtt{g}[B]$ and on the summands $\mathtt{g}[H]$. Define it on $\mathtt{g}[F]$ such that it maps the above relation to zero (just like ρ_A). The point is that this argument can be adapted to any noncommutative zeta function (not necessarily set-theoretic). $\square$

Lemma 17.12. *For any Lie monoid* $\mathtt{g}$*, the map* (17.6) *is injective.*

PROOF. Let the maps σ and ρ be as in the previous proof. To show that (17.6) is injective, we need to show that $\ker(\rho_A) \subseteq \ker(\sigma_A)$.

Suppose $\mathtt{g}' \twoheadrightarrow \mathtt{g}$ is a surjective morphism of Lie monoids, and suppose that the result holds for $\mathtt{g}'$. Then the result holds for $\mathtt{g}$ as well. To see this, consider the commutative diagram

$$\begin{tikzcd}
\ker(\rho_A) \arrow[r] \arrow[d, two heads] & \mathcal{T}(\mathtt{g}')[A] \arrow[r, "\sigma_A"] \arrow[d, two heads] & \mathcal{S}(\mathtt{g}')[A] \arrow[d, two heads] \\
\ker(\rho_A) \arrow[r] & \mathcal{T}(\mathtt{g})[A] \arrow[r, "\sigma_A"'] & \mathcal{S}(\mathtt{g})[A].
\end{tikzcd}$$

The key observation is that the first vertical map is surjective. To see this: Let u be an H/A-relation for g obtained from Y/H using $v \in \mathsf{g}[\mathrm{Y}]$. It is an element of (16.24). Since $\mathsf{g}' \twoheadrightarrow \mathsf{g}$ is surjective, choose an element $v' \in \mathsf{g}'[\mathrm{Y}]$ which maps to v. Let u' be the H/A-relation for g' obtained from Y/H using $v' \in \mathsf{g}'[\mathrm{Y}]$ and the same Lie element that was used to get u. Since $\mathsf{g}' \to \mathsf{g}$ is a morphism of Lie monoids, u' maps to u. This shows that the first vertical map is surjective. Hence, the top composite being zero implies that the bottom composite is also zero as required.

Next note that for any Lie monoid g, the structure map $\mathbf{Lie} \circ \mathsf{g} \twoheadrightarrow \mathsf{g}$ is a surjective morphism of Lie monoids. So it suffices to prove the result for the free Lie monoid $\mathsf{g}' := \mathbf{Lie} \circ \mathsf{g}$. Consider the commutative diagram:

$$\begin{tikzcd}
\mathcal{S}(\mathsf{g}') \arrow[r, "\cong"] \arrow[d] & \mathcal{S}(\mathcal{P}(\mathcal{U}(\mathsf{g}'))) \arrow[d] \arrow[dr, "\cong"] & \\
\mathcal{U}(\mathsf{g}') \arrow[r] & \mathcal{U}(\mathcal{P}(\mathcal{U}(\mathsf{g}'))) \arrow[r] & \mathcal{U}(\mathsf{g}').
\end{tikzcd}$$

The top-horizontal map is an isomorphism by Propositions 16.6 and 16.28. The right-down map is the isomorphism (17.7). The composite of the bottom-horizontal maps is the identity by the adjunction property. Hence, it follows that the vertical map on the left is an isomorphism, and in particular, injective. $\square$

This completes the proof of PBW.

17.2.4. Inclusion of a Lie monoid in its universal enveloping monoid.

An important consequence of PBW is the following.

Corollary 17.13. *For any Lie monoid g, the map $i : \mathsf{g} \to \mathcal{U}(\mathsf{g})$ defined in (16.31) is an injective morphism of Lie monoids.*

Proof. Since the map (17.1) is identity on p, we see that i is the restriction of (17.6) to g, and hence injective. $\square$

This result justifies the usage of the term enveloping that we have been using for $\mathcal{U}(\mathsf{g})$.

Exercise 17.14. Recall that $\mathsf{g} = \mathcal{P}\mathcal{S}(\mathsf{g})$. Deduce Lemma 17.12 as a consequence of Corollary 17.13 and Proposition 5.18.

Exercise 17.15. Check that: The map (17.6) is an exponential (9.3a) of the coderivation

$$\mathcal{S}(\mathsf{g}) \twoheadrightarrow \mathsf{g} \xrightarrow{i} \mathcal{U}(\mathsf{g}).$$

The first map is the canonical projection, while the second is the map (16.31). (Apply Lemma 9.10 to Exercise 17.4 and the morphism of monoids $\mathcal{T}(\mathsf{g}) \to \mathcal{U}(\mathsf{g})$.)

17.2.5. Signed analogue. PBW also works in the signed setting. We explain this briefly.

For any species p, recall the signed bicommutative signed bimonoid $\mathcal{E}(\mathsf{p})$ from Section 6.5.3. Similarly, for any signed Lie monoid g, recall the universal enveloping monoid $\mathcal{U}_{-1}(\mathsf{g})$ from Section 16.7.7. It is a quotient of $\mathcal{T}_{-1}(\mathsf{g})$ and moreover carries a signed bimonoid structure inherited from $\mathcal{T}_{-1}(\mathsf{g})$. By composing the injective map (17.4) for $\mathsf{p} := \mathsf{g}$ with the canonical surjective map $\mathcal{T}_{-1}(\mathsf{g}) \twoheadrightarrow \mathcal{U}_{-1}(\mathsf{g})$, we obtain a morphism of comonoids

$$\mathcal{E}(\mathsf{g}) \to \mathcal{U}_{-1}(\mathsf{g}). \tag{17.8}$$

Theorem 17.16. *For any signed Lie monoid g, the map (17.8) is an isomorphism of comonoids.*

This is the signed analogue of PBW and can be deduced from it using the signature functor. We call it the *signed Poincaré–Birkhoff–Witt theorem*, or signed PBW for short.

Exercise 17.17. Check that: The map (17.8) is an exponential (9.3a) of the coderivation

$$\mathcal{E}(\mathsf{g}) \twoheadrightarrow \mathsf{g} \xrightarrow{i} \mathcal{U}_{-1}(\mathsf{g}).$$

The first map is the canonical projection, while the second is the map (16.42).

17.2.6. Dual PBW. For a Lie comonoid k, recall its universal coenveloping comonoid of a Lie comonoid $\mathcal{U}^\vee(\mathsf{k})$ from Section 16.8.6. It is a subbimonoid of the commutative bimonoid $\mathcal{T}^\vee(\mathsf{k})$. By composing the canonical injective map $\mathcal{U}^\vee(\mathsf{k}) \hookrightarrow \mathcal{T}^\vee(\mathsf{k})$ with the surjective map (17.3) for $\mathsf{p} := \mathsf{k}$, we obtain a morphism of monoids

$$\mathcal{U}^\vee(\mathsf{k}) \to \mathcal{S}^\vee(\mathsf{k}). \tag{17.9}$$

Theorem 17.18. *For any Lie comonoid k, the map (17.9) is an isomorphism of monoids.*

This is the *dual Poincaré–Birkhoff–Witt theorem*, or dual PBW for short. The corresponding result involving the dual Solomon operator is given later in Theorem 17.40. We refer to (17.9) as a *dual Poincaré–Birkhoff–Witt isomorphism*, or dual PBW isomorphism for short.

Corollary 17.19. *For any Lie comonoid k, the map $p : \mathcal{U}^\vee(\mathsf{k}) \to \mathsf{k}$ defined in (16.53) is a surjective morphism of Lie comonoids.*

Exercise 17.20. Check that: The map (17.9) is an exponential (9.3a) of the derivation

$$\mathcal{U}^\vee(\mathsf{k}) \xrightarrow{p} \mathsf{k} \hookrightarrow \mathcal{S}^\vee(\mathsf{k}).$$

The first map is (16.53), while the second is the canonical inclusion. (Apply Lemma 9.10 to Exercise 17.7 and the morphism of comonoids $\mathcal{U}^\vee(\mathsf{k}) \to \mathcal{T}^\vee(\mathsf{k})$.)

Dual PBW also has a signed analogue which gives an isomorphism of monoids

$$\mathcal{U}^\vee_{-1}(\mathsf{k}) \to \mathcal{E}^\vee(\mathsf{k}). \tag{17.10}$$

It arises as the composite $\mathcal{U}^{\vee}_{-1}(\mathsf{k}) \hookrightarrow \mathcal{T}^{\vee}_{-1}(\mathsf{k}) \twoheadrightarrow \mathcal{E}^{\vee}(\mathsf{k})$. The latter is a monoid projection to the signed coabelianization map.

17.3. Projecting the free monoid onto the free Lie monoid

Recall logarithm of the identity map from Section 9.1.4. There is one such operator for each noncommutative Möbius function. For any species p, we have the free monoid $\mathcal{T}(\mathsf{p})$. It carries the structure of a cocommutative bimonoid. We now make explicit the operator $\mathbf{log}(\mathrm{id})$ on $\mathcal{T}(\mathsf{p})$. It is idempotent and its image is the primitive part $\mathcal{P}\mathcal{T}(\mathsf{p})$ which is the same as the free Lie monoid $\mathbf{Lie} \circ \mathsf{p}$. Dually, $\mathbf{log}(\mathrm{id})$ operates on $\mathcal{T}^{\vee}(\mathsf{p})$ with coimage equal to the indecomposable part $\mathcal{Q}\mathcal{T}^{\vee}(\mathsf{p})$ which is the same as the cofree Lie comonoid $\mathbf{Lie}^* \circ \mathsf{p}$.

Also recall that any logarithm of the identity map is the characteristic operation by a first eulerian idempotent. The above ideas can then also be expressed in this language.

17.3.1. Projection onto the free Lie monoid. Fix a noncommutative zeta function $\boldsymbol{\zeta}$ and its inverse noncommutative Möbius function $\boldsymbol{\mu}$. For any bimonoid h, recall the operator $\mathbf{log}(\mathrm{id})$ on h given by (9.7). When h is cocommutative, it is idempotent with image equal to $\mathcal{P}(\mathsf{h})$, see Proposition 9.17.

Now we specialize to $\mathsf{h} := \mathcal{T}(\mathsf{p})$. The latter is the cocommutative bimonoid associated to a species p with product (16.15) and coproduct (16.16). In conjunction with (16.17), we obtain:

Proposition 17.21. *For any species p, $\mathbf{log}(\mathrm{id})$ is an idempotent operator on the cocommutative bimonoid $\mathcal{T}(\mathsf{p})$ with image $\mathcal{P}\mathcal{T}(\mathsf{p}) = \mathbf{Lie} \circ \mathsf{p}$.*

For convenience, we put $\varphi := \mathbf{log}(\mathrm{id})$, thus, we have

$$\varphi : \mathcal{T}(\mathsf{p}) \twoheadrightarrow \mathcal{P}\mathcal{T}(\mathsf{p}) \hookrightarrow \mathcal{T}(\mathsf{p}). \tag{17.11}$$

We write φ^H_A for φ evaluated on the A-component, on the H-summand. In fact, φ^H_A only maps into those summands $G \geq A$ such that $\mathrm{s}(G) = \mathrm{s}(H)$, thus

$$\varphi^H_A : \mathsf{p}[H] \to \bigoplus_{\substack{G:\, G \geq A \\ \mathrm{s}(G)=\mathrm{s}(H)}} \mathsf{p}[G].$$

More precisely:

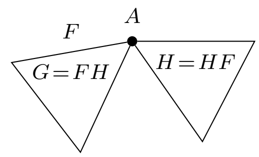
Lemma 17.22. *We have*

$$\varphi^H_A(z) = \sum_{G:\, G\geq A,\, \mathrm{s}(G)=\mathrm{s}(H)} \mathrm{sln}^{G,H}_A \beta_{G,H}(z), \tag{17.12a}$$

where $\mathrm{sln}^{G,H}_A$ is the Solomon coefficient (1.98). Equivalently,

$$\varphi^H_A(z) = \sum_{F:\, F\geq A,\, HF=H} \boldsymbol{\mu}(A,F)\, \beta_{FH,H}(z). \tag{17.12b}$$

Note that φ_A^A is the identity map on $\mathsf{p}[A]$ since $\boldsymbol{\mu}(A,A) = 1$. An illustration of how faces appearing in the above formulas relate to one another is shown below.



PROOF. We substitute formulas (16.15) and (16.16) in (9.7). For any $A \le F$, the operator $\mu_A^F \Delta_A^F$ on $\mathcal{T}(\mathsf{p})[A]$ is as follows. On the H-summand and into the G-summand, the map is $\beta_{G,H}$ if $H = HF$ and $G = FH$, and 0 otherwise. Formula (17.12a) follows by multiplying by $\boldsymbol{\mu}(A,F)$ and then summing over all F. $\square$

Exercise 17.23. Deduce Lemma 17.22 as a special case of Exercise 9.23. (Take $\mathsf{h} := \mathcal{T}(\mathsf{p})$ and $f := \mathrm{id}$.)

Since the primitive part of $\mathcal{T}(\mathsf{p})$ is the free Lie monoid $\mathbf{Lie} \circ \mathsf{p}$, there are Lie elements hidden in the map (17.11). More precisely: For $H \ge A$, define

$$\alpha_A^H := \sum_{G:\, G \ge A,\, \mathrm{s}(G)=\mathrm{s}(H)} \mathrm{sln}_A^{G,H}\, \mathtt{H}_{G/A}, \tag{17.13}$$

where $\mathrm{sln}_A^{G,H}$ is the Solomon coefficient (1.98).

Lemma 17.24. *For $H \ge A$, the element α_A^H is a Lie element of the arrangement $\mathcal{A}_A^{\mathrm{s}(H)}$.*

We refer to α_A^H as the Lie element associated to φ_A^H. When H is a chamber, α_A^H is an element of $\mathsf{Lie}[A]$.

17.3.2. Two identities. We record two identities involving the map (17.11) which will be used later.

Lemma 17.25. *For $H \ge A$ and $z \in \mathsf{p}[H]$, we have*

$$z = \sum_{K:\, K \ge A,\, HK=H} \boldsymbol{\zeta}(A,K)\, \varphi_K^{KH} \beta_{KH,H}(z). \tag{17.14}$$

FIRST PROOF. We calculate using (17.12b):

$$\begin{aligned}
\mathrm{rhs} &= \sum_{\substack{K:\, K \ge A \\ HK=H}} \boldsymbol{\zeta}(A,K) \varphi_K^{KH} \beta_{KH,H}(z) \\
&= \sum_{\substack{K:\, K \ge A \\ HK=H}} \boldsymbol{\zeta}(A,K) \sum_{\substack{F:\, F \ge K \\ \mathrm{s}(F) \le \mathrm{s}(H)}} \boldsymbol{\mu}(K,F) \beta_{FH,KH} \beta_{KH,H}(z) \\
&= \sum_{\substack{K:\, K \ge A \\ HK=H}} \boldsymbol{\zeta}(A,K) \sum_{\substack{F:\, F \ge K \\ \mathrm{s}(F) \le \mathrm{s}(H)}} \boldsymbol{\mu}(K,F) \beta_{FH,H}(z)
\end{aligned}$$

$$= \sum_{\substack{F\colon F\geq A\\ \mathrm{s}(F)\leq \mathrm{s}(H)}} \Big(\sum_{K\colon F\geq K\geq A} \boldsymbol{\zeta}(A,K)\boldsymbol{\mu}(K,F)\Big)\beta_{FH,H}(z)$$
$$= z$$
$$= \text{lhs}.$$

The fifth step used that $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ are inverse. □

Second proof. The identity (17.14) expresses the fact that the exponential and logarithm are inverse. More precisely, using the definition of exponential (9.3a), we note that the rhs of (17.14) equals

$$\mathbf{exp}(\varphi)_A(z) = \mathbf{exp}(\mathbf{log}(\mathrm{id}))_A(z) = \mathrm{id}_A(z) = z$$

which is the lhs of (17.14). □

Lemma 17.26. *For $H \geq A$ and $z \in \mathsf{p}[H]$, we have*

$$\sum_{G\colon G\geq A,\, \mathrm{s}(G)=\mathrm{s}(H)} \boldsymbol{\zeta}(A,G)\, \varphi_A^G \beta_{G,H}(z) = \begin{cases} z & \textit{if } H = A,\\ 0 & \textit{if } H > A. \end{cases} \tag{17.15}$$

First proof. We calculate using (17.12a):

$$\begin{aligned}
\text{lhs} &= \sum_{\substack{G\colon G\geq A\\ \mathrm{s}(G)=\mathrm{s}(H)}} \boldsymbol{\zeta}(A,G)\, \varphi_A^G \beta_{G,H}(z)\\
&= \sum_{\substack{G\colon G\geq A\\ \mathrm{s}(G)=\mathrm{s}(H)}} \boldsymbol{\zeta}(A,G) \sum_{\substack{K\colon K\geq A\\ \mathrm{s}(K)=\mathrm{s}(H)}} \mathrm{sln}_A^{K,G} \beta_{K,G}\beta_{G,H}(z)\\
&= \sum_{\substack{K\colon K\geq A\\ \mathrm{s}(K)=\mathrm{s}(H)}} \Big(\sum_{\substack{G\colon G\geq A\\ \mathrm{s}(G)=\mathrm{s}(H)}} \boldsymbol{\zeta}(A,G)\mathrm{sln}_A^{K,G}\Big) \beta_{K,H}(z)\\
&= \sum_{\substack{K\colon K\geq A\\ \mathrm{s}(K)=\mathrm{s}(H)}} \Big(\sum_{\substack{G\colon G\geq A\\ \mathrm{s}(G)=\mathrm{s}(H)}} \boldsymbol{\zeta}(A,G) \sum_{\substack{F\colon F\geq A\\ K=FG}} \boldsymbol{\mu}(A,F)\Big) \beta_{K,H}(z)\\
&= \sum_{\substack{K\colon K\geq A\\ \mathrm{s}(K)=\mathrm{s}(H)}} \Big(\sum_{F\colon K\geq F\geq A} \boldsymbol{\mu}(A,F) \sum_{\substack{G\colon G\geq A\\ FG=K\\ \mathrm{s}(G)=\mathrm{s}(K)}} \boldsymbol{\zeta}(A,G)\Big) \beta_{K,H}(z)\\
&= \sum_{\substack{K\colon K\geq A\\ \mathrm{s}(K)=\mathrm{s}(H)}} \Big(\sum_{F\colon K\geq F\geq A} \boldsymbol{\mu}(A,F)\boldsymbol{\zeta}(F,K)\Big) \beta_{K,H}(z).
\end{aligned}$$

The last step used the lune-additivity formula (1.42). Since $\boldsymbol{\mu}$ and $\boldsymbol{\zeta}$ are inverse, the sum above is zero if $H > A$, and z if $H = A$. □

Second proof. Let us denote the map (17.1) by f. The lhs of (17.15) equals

$$\varphi_A f_A \beta_{\mathrm{s}(H),H}(z) = \mathbf{log}(\mathrm{id})_A f_A \beta_{\mathrm{s}(H),H}(z) = \mathbf{log}(f)_A \beta_{\mathrm{s}(H),H}(z).$$

For the second step, we used Lemma 9.10 and Lemma 17.1. Now by Exercise 17.4, $\mathbf{log}(f)_A$ is the projection on the $\mathsf{p}[A]$ summand. Substituting this, we obtain the rhs of (17.15). □

17.3.3. Bimonoids of chambers and faces. Recall that the bimonoid of chambers Γ and the bimonoid of faces Σ are both cocommutative. Moreover, from (7.24) for $q = 1$, we have $\Gamma = \mathcal{T}(\mathsf{x})$. Similarly, from (7.82), we have $\Sigma = \mathcal{T}(\mathsf{E})$. Thus, specializing (17.11) to $\mathsf{p} := \mathsf{x}$ and to $\mathsf{p} := \mathsf{E}$ yields logarithm of the identity map on Γ and on Σ, respectively. This was considered in Example 9.20. Also see the related discussion on Lie and Zie species in Section 16.4.

Observe that the Lie element in formula (9.9) is precisely α_A^C as defined in (17.13). Moreover, for $\mathsf{h} := \Gamma$, identities (17.14) and (17.15) specialize to (9.10) and (9.11), respectively.

17.3.4. Characteristic operations by a first eulerian idempotent. Let us recast the preceding discussion in the language of characteristic operations (Section 10.1). Recall from Exercise 10.18 that $\mathbf{log}(\mathrm{id})$ is the same as the operator obtained by characteristic operation by the first eulerian idempotent $\mathtt{Q}_{A/A}$ in (7.73). We recall that $\mathtt{Q}_{A/A}$ is a special Zie element of $\Sigma[A]$. The fact that characteristic operation by a special Zie element projects a cocommutative bimonoid onto its primitive part is given in Proposition 10.13.

For any flat $\mathrm{X} \geq \mathrm{s}(A)$, put

$$(\mathtt{Q}_{A/A})^{\mathrm{X}} := \sum_{F\colon F \geq A,\, \mathrm{s}(F) \leq \mathrm{X}} \boldsymbol{\mu}(A, F)\, \mathtt{H}_{F/A}.$$

This is obtained from $\mathtt{Q}_{A/A}$ by removing terms involving faces whose support is not contained in X. It can be viewed as a first eulerian idempotent, and hence a special Zie element of the arrangement $\mathcal{A}_A^{\mathrm{X}}$. It is straightforward to see that for $H \geq A$,

$$\alpha_A^H = (\mathtt{Q}_{A/A})^{\mathrm{s}(H)} \cdot \mathtt{H}_{H/A}, \tag{17.16}$$

with α_A^H as defined in (17.13). Observe that Lemma 17.24 also follows from Proposition 10.13 applied to the bimonoid of chambers $\mathsf{h} := \Gamma$ in the arrangement under $\mathrm{s}(H)$, and the Friedrichs criterion (Lemma 7.64).

Lemma 17.25 expresses the fact that the sum of all eulerian idempotents (in any eulerian family) is the identity of the Tits algebra. More formally, we start with

$$\mathtt{H}_{A/A} = \sum_{K\colon A \leq K} \boldsymbol{\zeta}(A, K)\, \mathtt{Q}_{K/A}$$

which is a special case of the first formula in (7.67). By going under the flat $\mathrm{s}(H)$ and multiplying on the right by $\mathtt{H}_{H/A}$, we deduce

$$\mathtt{H}_{H/A} = \sum_{K\colon K \geq A,\, HK = H} \boldsymbol{\zeta}(A, K) \sum_{G\colon G \geq K,\, \mathrm{s}(G) = \mathrm{s}(H)} \mathrm{sln}_K^{G, KH}\, \mathtt{H}_{G/A},$$

with $\mathrm{sln}_K^{G,KH}$ as in (17.12a). Identity (17.14) can be deduced from this.

Similarly, Lemma 17.26 expresses the fact that in any eulerian family, the first eulerian idempotent is orthogonal to the eulerian idempotent indexed by the maximum flat when the arrangement has rank at least one.

17.3.5. Injection from the cofree Lie comonoid. We now briefly consider the dual situation. Recall from Proposition 9.17 that for any commutative bimonoid h, the operator $\mathbf{log}(\mathrm{id})$ on h is idempotent with coimage equal to $\mathcal{Q}(\mathsf{h})$. Now we specialize to $\mathsf{h} := \mathcal{T}^\vee(\mathsf{p})$. In conjunction with (16.49), we obtain:

Proposition 17.27. *For any species* p, $\mathbf{log}(\mathrm{id})$ *is an idempotent operator on the commutative bimonoid* $\mathcal{T}^\vee(\mathsf{p})$ *with coimage* $\mathcal{Q}\mathcal{T}(\mathsf{p}) = \mathbf{Lie}^* \circ \mathsf{p}$.

For convenience, we put $\varphi^\vee := \mathbf{log}(\mathrm{id})$, thus, we have

$$\varphi^\vee : \mathcal{T}^\vee(\mathsf{p}) \twoheadrightarrow \mathcal{Q}\mathcal{T}^\vee(\mathsf{p}) \hookrightarrow \mathcal{T}^\vee(\mathsf{p}). \tag{17.17}$$

We write $(\varphi^\vee)^H_A$ for $\varphi^\vee$ evaluated on the A-component, into the H-summand. Explicitly,

$$(\varphi^\vee)^H_A(z) = \sum_{G:\, G \geq A,\, \mathrm{s}(G)=\mathrm{s}(H)} \mathrm{sln}^{G,H}_A \beta_{H,G}(z), \tag{17.18}$$

where $\mathrm{sln}^{G,H}_A$ is the Solomon coefficient (1.98).

Recall from (16.49) that the indecomposable part of $\mathcal{T}^\vee(\mathsf{p})$ is the cofree Lie comonoid $\mathbf{Lie}^* \circ \mathsf{p}$. Thus, (17.17) yields a map $\mathbf{Lie}^* \circ \mathsf{p} \hookrightarrow \mathcal{T}^\vee(\mathsf{p})$. Evaluating on the A-component, from the $\mathrm{X} := \mathrm{s}(H)$-summand into the H-summand, it is given by evaluation on the Lie element α^H_A defined in (17.13).

17.3.6. (Co)free q-bimonoids. We briefly consider a more general situation. Recall from Section 6.1.2 the free q-bimonoid $\mathcal{T}_q(\mathsf{c})$ on the comonoid c. The logarithm of its identity map is as follows.

Lemma 17.28. *For a comonoid* (c, Δ),

$$\mathbf{log}(\mathrm{id}_{\mathcal{T}_q(\mathsf{c})})_A \text{ on the } H\text{-summand} = \sum_{\substack{G:\, G \geq A \\ GH=G}} \mathrm{sln}^{G,H}_A (\beta_q)_{G,HG} \Delta^{HG}_H, \tag{17.19}$$

where $\mathrm{sln}^{G,H}_A$ *is the Solomon coefficient* (1.98).

This is a routine calculation. When c is the trivial comonoid and $q = 1$, formula (17.19) specializes to (17.12a).

Dually, recall from Section 6.2.2 the cofree q-bimonoid $\mathcal{T}^\vee_q(\mathsf{a})$ on the monoid a. The logarithm of its identity map is as follows.

Lemma 17.29. *For a monoid* (a, μ),

$$\mathbf{log}(\mathrm{id}_{\mathcal{T}^\vee_q(\mathsf{a})})_A \text{ into the } H\text{-summand} = \sum_{\substack{G:\, G \geq A \\ GH=G}} \mathrm{sln}^{G,H}_A \mu^{HG}_H (\beta_q)_{HG,G}, \tag{17.20}$$

where $\mathrm{sln}^{G,H}_A$ *is the Solomon coefficient* (1.98).

When a is the trivial monoid and $q = 1$, formula (17.20) specializes to (17.18).

Let $(\mathsf{h}, \mu, \Delta)$ be a q-bimonoid. Consider the surjective morphism $\mu : \mathcal{T}_q(\mathsf{h}) \twoheadrightarrow \mathsf{h}$ of q-bimonoids given in Exercise 6.67. We deduce from (17.19) and Lemma 9.10 that its logarithm is as follows.

$$\mathbf{log}(\mu)_A \text{ on the } H\text{-summand } = \sum_{\substack{G:\, G \geq A \\ GH=G}} \mathrm{sln}_A^{G,H} \mu_A^G (\beta_q)_{G,HG} \Delta_H^{HG}. \tag{17.21}$$

A similar formula can be written down for the logarithm of the injective morphism $\Delta : \mathsf{h} \hookrightarrow \mathcal{T}_q^\vee(\mathsf{h})$.

$$\mathbf{log}(\Delta)_A \text{ into the } H\text{-summand } = \sum_{\substack{G:\, G \geq A \\ GH=G}} \mathrm{sln}_A^{G,H} \mu_H^{HG} (\beta_q)_{HG,G} \Delta_A^G. \tag{17.22}$$

For the 0-logarithm of the identity map on $\mathcal{T}_q(\mathsf{c})$ and $\mathcal{T}_q^\vee(\mathsf{a})$ (which, up to signs, is the same as the antipode), see Theorems 12.52 and 12.53. For the 0-logarithm of μ and Δ, see Exercise 12.59. Another calculation related to the 0-logarithm is given in Exercise 14.59.

Exercise 17.30. Use Propositions 16.9 and 16.44 to deduce the following generalizations of Propositions 17.21 and 17.27.

For any cocommutative comonoid c, $\mathbf{log}(\mathrm{id})$ is an idempotent operator on the cocommutative bimonoid $\mathcal{T}(\mathsf{c})$ with image $\mathcal{P}\mathcal{T}(\mathsf{c}) \cong \mathbf{Lie} \circ \mathsf{c}_t$. Dually, for any commutative monoid a, $\mathbf{log}(\mathrm{id})$ is an idempotent operator on the commutative bimonoid $\mathcal{T}^\vee(\mathsf{a})$ with coimage $\mathcal{Q}\mathcal{T}^\vee(\mathsf{a}) \cong \mathbf{Lie}^* \circ \mathsf{a}_t$.

17.4. Solomon operator

Let g be a Lie monoid. PBW implies the existence of an idempotent operator of comonoids on $\mathcal{T}(\mathsf{g})$ whose image is $\mathcal{S}(\mathsf{g})$ and coimage is $\mathcal{U}(\mathsf{g})$, or equivalently, whose kernel is $\mathcal{I}(\mathsf{g})$. (Recall from Section 16.5 that $\mathcal{I}(\mathsf{g})$ is the kernel of the canonical quotient $\mathcal{T}(\mathsf{g}) \twoheadrightarrow \mathcal{U}(\mathsf{g})$.) This is the Solomon operator associated to g. It is the morphism of comonoids arising as the composite

$$\mathcal{T}(\mathsf{g}) \twoheadrightarrow \mathcal{S}(\mathsf{g}) \hookrightarrow \mathcal{T}(\mathsf{g}).$$

The first map is the surjection σ constructed in the proof of Lemma 17.11, while the second map is the injection (17.1). One would like to have an explicit description of this operator. (Recall here that the map σ was defined inductively.) We now achieve this goal and in the process also give another proof of PBW. This is done as follows.

We start with a direct (non-inductive) definition of the Solomon operator. It is built out of the composite map which first projects $\mathcal{T}(\mathsf{g})$ onto its primitive part $\mathcal{P}\mathcal{T}(\mathsf{g})$ by a logarithm of the identity map, then identifies $\mathcal{P}\mathcal{T}(\mathsf{g})$ with $\mathbf{Lie} \circ \mathsf{g}$, and finally applies the Lie structure map of g. Using this direct definition, we establish that the Solomon operator is idempotent on $\mathcal{T}(\mathsf{g})$ with the correct image and kernel. PBW then follows as a consequence.

We also briefly discuss the dual Solomon operator of monoids on $\mathcal{T}^\vee(\mathsf{k})$ for a Lie comonoid k whose image is $\mathcal{U}^\vee(\mathsf{k})$ and coimage is $\mathcal{S}^\vee(\mathsf{k})$. It implies the dual PBW.

17.4.1. Solomon operator. Let g be a Lie monoid. Consider the composite map

$$\mathcal{T}(\mathsf{g}) \xrightarrow{\varphi} \mathcal{P}\mathcal{T}(\mathsf{g}) = \mathbf{Lie} \circ \mathsf{g} \xrightarrow{\gamma} \mathsf{g} \hookrightarrow \mathcal{T}(\mathsf{g}), \tag{17.23}$$

where φ is (17.11) with $\mathsf{p} := \mathsf{g}$, the equality is as in (16.17), γ is the Lie structure map of g, the last map is the canonical inclusion. The Solomon operator associated to g is built out of (17.23) as follows.

Define the *Solomon operator*

$$\psi : \mathcal{T}(\mathsf{g}) \to \mathcal{T}(\mathsf{g}) \tag{17.24}$$

as follows. Evaluating on the A-component, on the H-summand for $H \geq A$, it is given by

$$\sum_{K:\, K \geq A,\, HK=H} \boldsymbol{\zeta}(A,K)\, \gamma_K \varphi_K^{KH} \beta_{KH,H}, \tag{17.25}$$

where

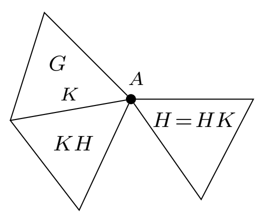
$$\gamma_K : (\mathbf{Lie} \circ \mathsf{g})[K] \to \mathsf{g}[K].$$

Equivalently, the (H,K)-matrix component of ψ_A (with H and K both greater than A) is given by the composite

$$\mathsf{g}[H] \xrightarrow{\beta_{KH,H}} \mathsf{g}[KH] \xrightarrow{\varphi_K^{KH}} \bigoplus_{\substack{G:\, G \geq K \\ \mathrm{s}(G)=\mathrm{s}(H)}} \mathsf{g}[G] \xrightarrow{\boldsymbol{\zeta}(A,K)\gamma_K} \mathsf{g}[K]$$

when $HK = H$, and zero otherwise. Observe that the scalar $\boldsymbol{\zeta}(A,K)$ is shown paired with γ_K, but it could equally well be paired with φ or β.

An illustration of how the different faces relate to one another is shown below.



Exercise 17.31. Check that: The Solomon operator (17.24) is an exponential (9.3a) of the coderivation (17.23). Deduce from Theorem 9.11 that (17.24) is a morphism of comonoids.

17.4.2. Image and coimage of the Solomon operator.

Theorem 17.32. *For a Lie monoid* g*, the Solomon operator* (17.24) *is an idempotent operator of comonoids. Further, its image is* $\mathcal{S}(\mathsf{g})$ *and its coimage is* $\mathcal{U}(\mathsf{g})$*, or equivalently, its kernel is* $\mathcal{I}(\mathsf{g})$*. In particular,* $\mathcal{T}(\mathsf{g}) = \mathcal{S}(\mathsf{g}) \oplus \mathcal{I}(\mathsf{g})$*.*

Here $\mathcal{S}(\mathsf{g})$ is viewed as a subspace of $\mathcal{T}(\mathsf{g})$ via the inclusion (17.1). Recall that $\mathcal{I}(\mathsf{g})$ is the kernel of the canonical quotient $\mathcal{T}(\mathsf{g}) \twoheadrightarrow \mathcal{U}(\mathsf{g})$. Its A-component is linearly spanned by H/A-relations for $H \geq A$.

The situation is summarized below.

$$\begin{tikzcd} 0 \arrow[r] & \ker(\psi) \arrow[r, hook] & \mathcal{T}(\mathsf{g}) \arrow[r, "\psi"] \arrow[d, two heads] & \mathcal{T}(\mathsf{g}) \arrow[r, two heads] & \operatorname{coker}(\psi) \arrow[r] & 0 \\ & & \mathcal{U}(\mathsf{g}) \arrow[r, "\cong"'] & \mathcal{S}(\mathsf{g}) \arrow[u, hook] & & \end{tikzcd} \tag{17.26}$$

This is an instance of diagram (2.51). Moreover, since ψ is idempotent, $\operatorname{coker}(\psi) \cong \ker(\psi) = \mathcal{I}(\mathsf{g})$, and the inverse of the above isomorphism $\mathcal{U}(\mathsf{g}) \to \mathcal{S}(\mathsf{g})$ is precisely (17.6). As a consequence, we obtain the PBW Theorem 17.9.

17.4.3. Second proof of PBW. We now prove Theorem 17.32. This gives a second proof of PBW. For convenience, the proof is divided into four steps.

(1) The image of $\mathrm{id} - \psi$ is contained in $\mathcal{I}(\mathsf{g})$.
(2) $\mathcal{I}(\mathsf{g})$ is in the kernel of ψ.
(3) The image of ψ is contained in $\mathcal{S}(\mathsf{g})$.
(4) ψ is identity on $\mathcal{S}(\mathsf{g})$.

The first two steps show that ψ is idempotent and its kernel is $\mathcal{I}(\mathsf{g})$. The next two steps identify the image of ψ to be $\mathcal{S}(\mathsf{g})$. These four steps are carried out in the four lemmas below.

Lemma 17.33. *The image of* $\mathrm{id} - \psi$ *is contained in* $\mathcal{I}(\mathsf{g})$.

Proof. This follows from identity (17.14) and definition (17.25): For $z \in \mathsf{g}[H]$,

$$z - \psi_A(z) = \sum_{K\colon K \geq A,\, HK = H} \boldsymbol{\zeta}(A, K)(\varphi_K^{KH} - \gamma_K \varphi_K^{KH})\beta_{KH,H}(z),$$

and the term in parenthesis is a K/A-relation, and hence belongs to $\mathcal{I}(\mathsf{g})[A]$. $\square$

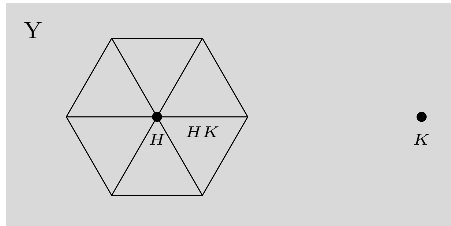
Lemma 17.34. $\mathcal{I}(\mathsf{g})$ *is in the kernel of* ψ.

Proof. Write $\psi_A = \sum_{K \geq A} \psi_A^K$, with ψ_A^K the part of ψ_A which maps into the summand $\mathsf{g}[K]$.

Consider an H/A-relation $z = z_1 - z_2$, where $z_2 \in \mathsf{g}[H]$ and $\gamma_H(z_1) = z_2$. Let Y denote the support of faces appearing in z_1. We need to show that $\psi_A^K(z_1) = \psi_A^K(z_2)$ for all $K \geq A$. We consider three cases.

- $\underline{\mathrm{s}(K) \not\leq \mathrm{Y}}$. In this case, both $\psi_A^K(z_1)$ and $\psi_A^K(z_2)$ are zero.
- $\underline{\mathrm{s}(K) \leq \mathrm{Y} \text{ and } HK > H}$. In this case, $\psi_A^K(z_2)$ is zero. To calculate $\psi_A^K(z_1)$, we may project first on HK. Since z_1 contains a Lie element in the arrangement $\mathcal{A}_H^{\mathrm{Y}}$, by applying the definition of a Lie element (1.164) to $\mathcal{A}_H^{\mathrm{Y}}$, this projection is zero. Hence, $\psi_A^K(z_1)$ is also zero. An

illustration is shown below.



In the figure, A is the central face (which is not visible), H and K are vertices, HK is an edge, and Y is the plane of the paper. The element z_1 involves the six triangles containing H.

- $\underline{\mathrm{s}(K) \leq \mathrm{Y} \text{ and } HK = H}$. By projecting on K, we may assume that $K \leq H$. Note that $\psi^K_A(z_2) = \boldsymbol{\zeta}(A, K)\, \gamma_K \varphi^H_K(z_2)$. On the other hand, $\psi^K_A(z_1)$ is obtained by combining the Lie element in z_1 with the Lie element α^K_H as defined in (17.13) using the substitution map of Lie elements as defined in (1.167), and then applying $\boldsymbol{\zeta}(A, K)\, \gamma_K$. (To get this description, we again used (1.164) as in the previous case.) Since $z_2 = \gamma_H(z_1)$, we conclude using the associativity axiom (16.1a) for the Lie monoid g that $\psi^K_A(z_1) = \psi^K_A(z_2)$.

This completes the argument. $\square$

A more formal approach to Lemma 17.34 is given in the following exercise.

Exercise 17.35. Show that: For any Lie monoid g, the image of the composite

$$\mathcal{I}(\mathsf{g}) \hookrightarrow \mathcal{T}(\mathsf{g}) \xrightarrow{\mathbf{log}(\mathrm{id})} \mathcal{PT}(\mathsf{g}) = \mathbf{Lie} \circ \mathsf{g}$$

is the subspecies of $\mathbf{Lie} \circ \mathsf{g}$ given by the image of $\gamma \circ \mathrm{id} - \mathrm{id} \circ \gamma$, where

$$\mathbf{Lie} \circ \mathbf{Lie} \circ \mathsf{g} \underset{\mathrm{id} \circ \gamma}{\overset{\gamma \circ \mathrm{id}}{\rightrightarrows}} \mathbf{Lie} \circ \mathsf{g}.$$

In the first map, γ is the structure map of the Lie operad, while in the second, it is the Lie structure map of g.

Deduce that the composite

$$\mathcal{I}(\mathsf{g}) \hookrightarrow \mathcal{T}(\mathsf{g}) \xrightarrow{\mathbf{log}(\mathrm{id})} \mathcal{PT}(\mathsf{g}) = \mathbf{Lie} \circ \mathsf{g} \xrightarrow{\gamma} \mathsf{g}$$

is zero. (This is essentially the content of the third case in the proof of Lemma 17.34.)

Now by Proposition 16.18, the inclusion $\mathcal{I}(\mathsf{g}) \hookrightarrow \mathcal{T}(\mathsf{g})$ is a morphism of comonoids. Use Exercise 17.31 and Lemma 9.10 to deduce Lemma 17.34. (The first two cases in the proof of this lemma only involved the coproduct of $\mathcal{T}(\mathsf{g})$ and not $\mathbf{log}(\mathrm{id})$, which explains the close similarity with the proof of Proposition 16.18.)

Lemma 17.36. *The image of ψ is contained in $\mathcal{S}(\mathsf{g})$.*

PROOF. Suppose K and K' (both greater than A) have the same support which is contained in $\mathrm{s}(H)$. Then

$$\beta_{K',K}\gamma_K\varphi_K^{KH}\beta_{KH,H} = \gamma_{K'}\varphi_{K'}^{K'H}\beta_{K'H,H}$$

on the H-summand. This follows since γ and φ are maps of species. One can use this identity to deduce that the image of ψ_A is contained in $\mathcal{S}(\mathsf{g})[A]$. $\square$

Lemma 17.37. *ψ is identity on $\mathcal{S}(\mathsf{g})$.*

PROOF. Write $\psi_A = \sum_{K\geq A}\psi_A^K$, with ψ_A^K the part of ψ_A which maps into the summand $\mathsf{g}[K]$.

Take $z \in \mathcal{S}(\mathsf{g})[A]$ and assume that it involves faces H greater than A of a fixed support, say X. In other words,

$$z = \sum_{H\colon H\geq A,\, \mathrm{s}(H)=\mathrm{X}} \boldsymbol{\zeta}(A,H)z_H,$$

with $\beta_{G,H}(z_H) = z_G$ for any H and G in the indexing set. Clearly, $\psi_A^K(z) = 0$ if $\mathrm{s}(K) \not\leq \mathrm{X}$. So suppose $\mathrm{s}(K) \leq \mathrm{X}$. Then

$$\begin{aligned}
\psi_A^K(z) &= \boldsymbol{\zeta}(A,K)\gamma_K\Big(\sum_{H\colon H\geq A,\, \mathrm{s}(H)=\mathrm{X}} \boldsymbol{\zeta}(A,H)\,\varphi_K^{KH}\beta_{KH,H}(z_H)\Big)\\
&= \boldsymbol{\zeta}(A,K)\gamma_K\Big(\sum_{\substack{H'\colon H'\geq K\\ \mathrm{s}(H')=\mathrm{X}}}\Big(\sum_{\substack{H\colon H\geq A,\, KH=H'\\ \mathrm{s}(H)=\mathrm{X}}} \boldsymbol{\zeta}(A,H)\Big)\varphi_K^{H'}\beta_{H',H}(z_H)\Big)\\
&= \boldsymbol{\zeta}(A,K)\gamma_K\Big(\sum_{H'\colon H'\geq K,\, \mathrm{s}(H')=\mathrm{X}} \boldsymbol{\zeta}(K,H')\,\varphi_K^{H'}(z_{H'})\Big)\\
&= \begin{cases} \boldsymbol{\zeta}(A,K)z_K & \mathrm{s}(K)=\mathrm{X},\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

The second step introduced a new variable $H' = KH$. The third step used the lune-additivity formula (1.42). The last step used identity (17.15) (with K and H' instead of A and H). Now by summing over all K with support X, we conclude that $\psi_A(z) = z$ as required. $\square$

17.4.4. Abelian Lie monoids. For an abelian Lie monoid g as in Section 16.6, the Solomon operator (17.24) on $\mathcal{T}(\mathsf{g})$ specializes as follows. Evaluating on the A-component, on the H-summand for $H \geq A$, it is given by

$$\mathsf{g}[H] \to \bigoplus_{K\colon K\geq A,\, \mathrm{s}(K)=\mathrm{s}(H)} \mathsf{g}[K], \qquad v \mapsto \sum_K \boldsymbol{\zeta}(A,K)\,\beta_{K,H}(v). \tag{17.27}$$

One can directly see from the flat-additivity formula (1.43) that this operator is idempotent. Its kernel and image can also be computed directly. Its image is $\mathcal{S}(\mathsf{g})$ viewed as a subcomonoid of $\mathcal{T}(\mathsf{g})$ via (17.1) for $\mathsf{p} := \mathsf{g}$. Its kernel, evaluated on the A-component, is linearly spanned by elements of the form $z - \beta_{G,F}(z)$, with $z \in \mathsf{g}[F]$, and F and G both greater than A and of the same support. This is the same as the kernel of the abelianization map $\mathcal{T}(\mathsf{g}) \twoheadrightarrow \mathcal{S}(\mathsf{g})$. By Theorem 17.32, this equals $\mathcal{I}(\mathsf{g})$ which recovers Proposition 16.30.

Exercise 17.38. Check that: For an abelian Lie monoid g, the Solomon operator is an exponential (9.3a) of the coderivation $\mathcal{T}(\mathsf{g}) \twoheadrightarrow \mathsf{g} \hookrightarrow \mathcal{T}(\mathsf{g})$. (Compare with Exercise 17.31.)

17.4.5. Dual Solomon operator. We now briefly consider the dual situation. Let k be a Lie comonoid. Consider the composite map

$$\mathcal{T}^\vee(\mathsf{k}) \twoheadrightarrow \mathsf{k} \xrightarrow{\delta} \mathbf{Lie}^* \circ \mathsf{k} = \mathcal{Q}\mathcal{T}^\vee(\mathsf{k}) \xrightarrow{\varphi^\vee} \mathcal{T}^\vee(\mathsf{k}), \tag{17.28}$$

where the first map is the canonical projection, δ is the Lie structure map of k, the equality is as in (16.49), $\varphi^\vee$ is (17.17) with $\mathsf{p} := \mathsf{k}$. The dual Solomon operator associated to k is built out of (17.28) as follows.

Define the *dual Solomon operator*

$$\psi^\vee : \mathcal{T}^\vee(\mathsf{k}) \to \mathcal{T}^\vee(\mathsf{k}) \tag{17.29}$$

as follows. Evaluating on the A-component, on the K-summand for $K \geq A$, it is given by

$$\boldsymbol{\zeta}(A, K) \sum_{H\colon H \geq A,\, HK = H} \beta_{H,KH} (\varphi^\vee)_K^{KH} \delta_K, \tag{17.30}$$

where

$$\delta_K : \mathsf{k}[K] \to (\mathbf{Lie}^* \circ \mathsf{k})[K].$$

Explicitly, $(\varphi^\vee)_K^{KH}$ evaluates the image of δ_K on the Lie element α_K^{KH} as defined in (17.13) to get an element of $\mathsf{k}[KH]$.

Exercise 17.39. Check that: The dual Solomon operator (17.29) is an exponential (9.3a) of the derivation (17.28). Deduce from Theorem 9.12 that (17.29) is a morphism of monoids.

Theorem 17.40. *For a Lie comonoid* k*, the dual Solomon operator* (17.29) *is an idempotent operator of monoids. Further, its image is* $\mathcal{U}^\vee(\mathsf{k})$ *and its coimage is* $\mathcal{S}^\vee(\mathsf{k})$*.*

This is dual to Theorem 17.32 and implies the dual PBW Theorem 17.18. The situation is summarized below.

$$\begin{tikzcd} 0 \arrow[r] & \ker(\psi^\vee) \arrow[r, hook] & \mathcal{T}^\vee(\mathsf{k}) \arrow[r, "\psi^\vee"] \arrow[d, two heads] & \mathcal{T}^\vee(\mathsf{k}) \arrow[r, two heads] & \operatorname{coker}(\psi^\vee) \arrow[r] & 0 \\ & & \mathcal{S}^\vee(\mathsf{k}) \arrow[r, "\cong"'] & \mathcal{U}^\vee(\mathsf{k}) \arrow[u, hook] & & \end{tikzcd} \tag{17.31}$$

Here $\mathcal{S}^\vee(\mathsf{k})$ is viewed as a quotient of $\mathcal{T}^\vee(\mathsf{k})$ via (17.3). The inverse of the above isomorphism $\mathcal{S}^\vee(\mathsf{k}) \to \mathcal{U}^\vee(\mathsf{k})$ is precisely (17.9).

The dual Solomon operator allows us to explicitly compute elements of $\mathcal{U}^\vee(\mathsf{k})$. For instance, for any $\mathrm{X} \geq \mathrm{s}(A)$, there are elements of $\mathcal{U}^\vee(\mathsf{k})[A]$ which only involve faces of support greater than X. The case when $\mathrm{X} = \top$ was dealt with in Exercise 16.48.

17.4.6. Abelian Lie comonoids. For an abelian Lie comonoid k as in Section 16.8.7, the dual Solomon operator (17.29) on $\mathcal{T}^\vee(\mathsf{k})$ specializes as follows. Evaluating on the A-component, on the K-summand for $K \geq A$, it is given by

$$\mathsf{k}[K] \to \bigoplus_{H\colon H \geq A,\, \mathrm{s}(H)=\mathrm{s}(K)} \mathsf{k}[H], \qquad v \mapsto \boldsymbol{\zeta}(A,K) \sum_H \beta_{H,K}(v). \tag{17.32}$$

In view of Theorem 17.40, this recovers Proposition 16.54.

Exercise 17.41. Check that: For an abelian Lie comonoid k, the dual Solomon operator is an exponential (9.3a) of the derivation $\mathcal{T}^\vee(\mathsf{k}) \twoheadrightarrow \mathsf{k} \hookrightarrow \mathcal{T}^\vee(\mathsf{k})$. (Compare with Exercise 17.39.)

17.5. Cartier–Milnor–Moore

The Cartier–Milnor–Moore theorem says that the categories of Lie monoids and cocommutative bimonoids are equivalent. We write CMM for short. The equivalence is given by the universal enveloping monoid functor in one direction and the primitive part functor in the other direction. We also explain connections of CMM with Leray–Samelson, and with PBW and Borel–Hopf.

One can also approach CMM via representation theory of the Tits algebra by employing Theorem 4.40 which we explain briefly.

17.5.1. Cartier–Milnor–Moore. Recall from Theorem 16.25 that the functors $\mathcal{U}$ and $\mathcal{P}$ are adjoints between the categories of Lie monoids and cocommutative bimonoids. In fact:

Theorem 17.42. *The adjunction*

$$\mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{\mathcal{P}}{\overset{\mathcal{U}}{\rightleftarrows}} {}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$$

is an adjoint equivalence of categories.

This is the *Cartier–Milnor–Moore theorem*, or CMM for short. Equivalently, it says that the counit and unit of the adjunction between $\mathcal{U}$ and $\mathcal{P}$ are natural isomorphisms:

$$\mathcal{U}(\mathcal{P}(\mathsf{h})) \xrightarrow{\cong} \mathsf{h} \quad \text{and} \quad \mathsf{g} \xrightarrow{\cong} \mathcal{P}(\mathcal{U}(\mathsf{g})). \tag{17.33}$$

Proof. The isomorphism of the unit follows from the PBW Theorem 17.9 by applying the functor $\mathcal{P}$ to (17.6) and using $\mathcal{P}(\mathcal{S}(\mathsf{g})) = \mathsf{g}$ from Exercise 6.46. The isomorphism of the counit follows from the Borel–Hopf isomorphism (17.7) and the PBW Theorem 17.9. This completes the proof.

Alternatively, for the counit: By Exercise 9.24, the map $\mathcal{T}(\mathcal{P}(\mathsf{h})) \to \mathsf{h}$ is surjective, which then implies that the counit is surjective. Equivalently, by Exercise 6.70, one can use the fact that h is primitively generated (Proposition 5.51). Injectivity of the counit can be deduced from Proposition 5.18. $\square$

Exercise 17.43. Show that: For any bimonoid h, we have an isomorphism of bimonoids $\mathcal{U}(\mathcal{P}(\mathsf{h})) \cong \mathsf{h}^{coab}$, the coabelianization of h. (Use Exercise 5.10.)

Exercise 17.44. Deduce that: For any Lie monoid g, the filtration of $\mathcal{U}(\mathsf{g})$ generated by the subspecies g (as defined in Section 5.2.4) coincides with the primitive filtration of $\mathcal{U}(\mathsf{g})$.

Exercise 17.45. Show that: For any Lie monoid g,

$$\operatorname{gr}_{\mathcal{P}}(\mathcal{U}(\mathsf{g})) \cong \mathcal{S}(\mathsf{g}) \tag{17.34}$$

as graded bimonoids. The grading on the latter is as in (6.87). Also,

$$\operatorname{gr}_{\mathcal{P}}(\mathcal{U}(\mathsf{g})) \cong \mathcal{U}(\mathsf{g})$$

as comonoids. (Use Exercises 13.44 and 13.45.)

Exercise 17.46. Let h and k be cocommutative bimonoids, and let $f : \mathcal{P}(\mathsf{h}) \to \mathcal{P}(\mathsf{k})$ be a morphism of Lie monoids. Show that: f extends uniquely to a morphism of bimonoids $\hat{f} : \mathsf{h} \to \mathsf{k}$. Further, $\hat{f}$ is also compatible with logarithm of the identity map, that is, diagram

$$\begin{tikzcd}
\mathsf{h} \arrow[r, two heads, "\mathbf{log}(\mathrm{id}_{\mathsf{h}})"] \arrow[d, dashed, "\hat{f}"'] & \mathcal{P}(\mathsf{h}) \arrow[r, hook] \arrow[d, "f"'] & \mathsf{h} \arrow[d, dashed, "\hat{f}"] \\
\mathsf{k} \arrow[r, two heads, "\mathbf{log}(\mathrm{id}_{\mathsf{k}})"'] & \mathcal{P}(\mathsf{k}) \arrow[r, hook] & \mathsf{k}
\end{tikzcd}$$

commutes. Moreover, if f is an isomorphism, then so is $\hat{f}$. (See Exercise 9.27.)

Exercise 17.47. For a monoid a, recall the cocommutative bimonoid $\mathcal{S}^\vee(\mathsf{a})$ from Section 6.3.4. Show that $\mathcal{S}^\vee(\mathsf{a}) \cong \mathcal{U}(\mathsf{a})$ as bimonoids, with a viewed as a Lie monoid while considering $\mathcal{U}(\mathsf{a})$. (Use Exercise 16.23.)

17.5.2. CMM, Borel–Hopf, PBW. These three theorems can be visualized as three vertices of a triangle with any two of them implying the third.

$$\begin{tikzcd}
 & \text{CMM} \arrow[dl, dash] \arrow[dr, dash] & \\
\text{BH} \arrow[rr, dash] & & \text{PBW}
\end{tikzcd}$$

We saw how Borel–Hopf and PBW imply CMM. The remaining two implications are explained below.

- CMM and PBW imply Borel–Hopf: For any cocommutative bimonoid h,
$$\mathcal{S}(\mathcal{P}(\mathsf{h})) \overset{\text{PBW}}{\cong} \mathcal{U}(\mathcal{P}(\mathsf{h})) \overset{\text{CMM}}{\cong} \mathsf{h}.$$
The first isomorphism is of comonoids, and the second of bimonoids.
- CMM and Borel–Hopf imply PBW: For any Lie monoid g,
$$\mathcal{S}(\mathsf{g}) \overset{\text{CMM}}{\cong} \mathcal{S}(\mathcal{P}(\mathcal{U}(\mathsf{g}))) \overset{\text{BH}}{\cong} \mathcal{U}(\mathsf{g}).$$
The first isomorphism is of bimonoids, and the second of comonoids.

Thus, if we assume CMM, then PBW and Borel–Hopf can be viewed as equivalent theorems. Note that both talk about an isomorphism of comonoids between a bicommutative bimonoid and a cocommutative bimonoid.

Exercise 17.48. Check that: For a Lie monoid g, the inverse of the PBW isomorphism (17.6) is an exponential (9.3a) of the coderivation

$$\mathcal{U}(\mathsf{g}) \xrightarrow{\mathbf{log}(\mathrm{id})} \mathsf{g} \hookrightarrow \mathcal{S}(\mathsf{g}).$$

(Put $\mathsf{h} := \mathcal{U}(\mathsf{g})$ in Exercises 13.40 and 13.41, and use $\mathcal{P}(\mathcal{U}(\mathsf{g})) = \mathsf{g}$ and Exercise 17.15.)

Exercise 17.49. Check that: For any Lie monoid g, the diagram

$$\begin{tikzcd} \mathcal{T}(\mathsf{g}) \arrow[r, two heads, "\mathbf{log}(\mathrm{id})"] \arrow[d, two heads] & \mathcal{P}\mathcal{T}(\mathsf{g}) \arrow[r, equal] \arrow[d, two heads] & \mathbf{Lie} \circ \mathsf{g} \arrow[r, hook] \arrow[d, two heads, "\gamma"] & \mathcal{T}(\mathsf{g}) \arrow[d, two heads] \\ \mathcal{U}(\mathsf{g}) \arrow[r, two heads, "\mathbf{log}(\mathrm{id})"'] & \mathcal{P}\mathcal{U}(\mathsf{g}) \arrow[r, equal] & \mathsf{g} \arrow[r, hook] & \mathcal{U}(\mathsf{g}) \end{tikzcd} \tag{17.35}$$

commutes. (See Exercise 9.27. The extra ingredient here is the map γ.)

Exercise 17.50. For a Lie monoid g, consider diagram (17.26). By Exercise 17.48 and Lemma 9.10, deduce that $\mathcal{T}(\mathsf{g}) \twoheadrightarrow \mathcal{S}(\mathsf{g})$ is an exponential of the composite

$$\mathcal{T}(\mathsf{g}) \twoheadrightarrow \mathcal{U}(\mathsf{g}) \xrightarrow{\mathbf{log}(\mathrm{id})} \mathsf{g} \hookrightarrow \mathcal{S}(\mathsf{g}).$$

Using Exercise 17.4, write $\mathcal{S}(\mathsf{g}) \hookrightarrow \mathcal{T}(\mathsf{g})$ as an exponential. Composing the two exponentials and using Lemma 9.15, item (2), express the Solomon operator ψ as an exponential. Using diagram (17.35), deduce that the result agrees with the one in Exercise 17.31.

17.5.3. Leray–Samelson. Recall abelian Lie monoids from Section 16.6. The universal enveloping monoid of an abelian Lie monoid is a bicommutative bimonoid, and conversely, the primitive part of a bicommutative bimonoid is an abelian Lie monoid. Thus, CMM restricts to an adjoint equivalence between the full subcategories of abelian Lie monoids and bicommutative bimonoids. Further, the category of abelian Lie monoids is isomorphic to the category of species. Thus, we deduce that the category of species and the category of bicommutative bimonoids are equivalent. This is precisely the Leray–Samelson Theorem 13.11 discussed earlier. This is summarized below.

$$\begin{tikzcd} \mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp}) \arrow[r, shift left, "\mathcal{U}"] & {}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp}) \arrow[l, shift left, "\mathcal{P}"] \\ \mathcal{A}\text{-}\mathsf{Sp} \arrow[u, hook] \arrow[r, shift left, "\mathcal{S}"] & {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \arrow[u, hook] \arrow[l, shift left, "\mathcal{P}"] \end{tikzcd}$$

Proposition 17.51. *For a Lie monoid* g*, the following are equivalent.*

(1) g *is abelian.*
(2) $\mathcal{U}(\mathsf{g})$ *is commutative.*
(3) *The map* (17.6) *is an isomorphism of bimonoids.*

Proof. This follows from the above discussion. For (2) implies (1), one may also directly use definition of $\mathcal{U}(\mathsf{g})$ and Corollary 17.13. $\square$

17.5.4. Signed analogue. CMM works in the signed setting. The *signed Cartier–Milnor–Moore theorem*, or signed CMM for short, says that the categories of signed Lie monoids and signed cocommutative signed bimonoids are equivalent. More precisely:

Theorem 17.52. *The adjunction*

$$(-1)\text{-}\mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{\mathcal{P}}{\overset{\mathcal{U}_{-1}}{\rightleftarrows}} (-1)\text{-}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$$

is an adjoint equivalence of categories.

Signed CMM follows from the usual CMM by composing the adjunction on either side by the adjunction determined by the signature functor and using (8.84) and (16.40).

17.5.5. Dual CMM. Recall from Theorem 16.51 that the functors $\mathcal{U}^\vee$ and $\mathcal{Q}$ are adjoints between the categories of Lie comonoids and commutative bimonoids. In fact:

Theorem 17.53. *The adjunction*

$$\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp}) \underset{\mathcal{U}^\vee}{\overset{\mathcal{Q}}{\rightleftarrows}} \mathsf{LieComon}(\mathcal{A}\text{-}\mathsf{Sp})$$

is an adjoint equivalence of categories.

This is the *dual Cartier–Milnor–Moore theorem*, or dual CMM for short. It can be deduced from the dual PBW Theorem 17.18 and the Borel–Hopf Theorem 13.57 for commutative bimonoids.

The discussion after CMM Theorem 17.42 can also be carried out in the dual setting. To avoid repetition, we only spell out some of the exercises.

Exercise 17.54. Let h and k be commutative bimonoids, and let $f : \mathcal{Q}(\mathsf{h}) \to \mathcal{Q}(\mathsf{k})$ be a morphism of Lie comonoids. Show that: f lifts uniquely to a morphism of bimonoids $\hat{f} : \mathsf{h} \to \mathsf{k}$. Further, $\hat{f}$ is also compatible with logarithm of the identity map, that is, diagram

$$\begin{tikzcd}
\mathsf{h} \arrow[r, two heads] \arrow[d, dashed, "\hat{f}"'] & \mathcal{Q}(\mathsf{h}) \arrow[r, hook, "\mathbf{log}(\mathrm{id}_\mathsf{h})"] \arrow[d, "f"] & \mathsf{h} \arrow[d, dashed, "\hat{f}"] \\
\mathsf{k} \arrow[r, two heads] & \mathcal{Q}(\mathsf{k}) \arrow[r, hook, "\mathbf{log}(\mathrm{id}_\mathsf{k})"'] & \mathsf{k}
\end{tikzcd}$$

commutes. Moreover, if f is an isomorphism, then so is $\hat{f}$. (See Exercise 9.27.)

Exercise 17.55. For a comonoid c, recall the commutative bimonoid $\mathcal{S}(\mathsf{c})$ from Section 6.3.2. Show that $\mathcal{S}(\mathsf{c}) \cong \mathcal{U}^\vee(\mathsf{c})$ as bimonoids, with c viewed as a Lie comonoid while considering $\mathcal{U}^\vee(\mathsf{c})$. (Use Exercise 16.50.)

Exercise 17.56. Check that: For a Lie comonoid k, the inverse of the dual PBW isomorphism (17.9) is an exponential (9.3a) of the derivation

$$\mathcal{S}^\vee(\mathsf{k}) \twoheadrightarrow \mathsf{k} \xhookrightarrow{\mathbf{log}(\mathrm{id})} \mathcal{U}^\vee(\mathsf{k}).$$

(Put $\mathsf{h} := \mathcal{U}^\vee(\mathsf{k})$ in Exercise 13.61, and use $\mathcal{Q}(\mathcal{U}^\vee(\mathsf{k})) = \mathsf{k}$ and Exercise 17.20.)

Dual CMM also has a signed analogue. It says that $\mathcal{Q}$ and $\mathcal{U}^\vee_{-1}$ determine an adjoint equivalence between the categories of signed Lie comonoids and signed commutative signed bimonoids.

17.5.6. CMM via representation theory of the Tits algebra. Recall the Tits algebra $\Sigma[\mathcal{A}]$. By Proposition 11.1, the category of left $\Sigma[\mathcal{A}]$-modules is equivalent to the category of cocommutative $\mathcal{A}$-bimonoids. By Theorem 4.40, the Tits algebra $\Sigma[\mathcal{A}]$ is isomorphic to the **Lie**-incidence algebra I(**Lie**). This isomorphism depends on the choice of an eulerian family $\mathtt{E}$. By Proposition 4.43, a left I(**Lie**)-module is the same as a left **Lie**-module, that is, a Lie monoid. Putting everything together:

$$\begin{tikzcd} \text{cocommutative bimonoids} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \text{Tits algebra modules} \arrow[d, leftrightarrow] \\ \text{Lie monoids} \arrow[r, leftrightarrow] & \textbf{Lie}\text{-incidence algebra modules.} \end{tikzcd} \tag{17.36}$$

Thus, the category of Lie monoids is equivalent to the category of cocommutative bimonoids, with one equivalence for each eulerian family $\mathtt{E}$. This is an avatar of the Cartier–Milnor–Moore theorem (Theorem 17.42). We now make the connection precise by showing that the functors involved in these equivalences are indeed isomorphic to $\mathcal{U}$ and $\mathcal{P}$.

Let h be a cocommutative bimonoid. Then $\mathsf{h}[O]$ is a left module over the Tits algebra. The components of the corresponding Lie monoid g are defined by

$$\mathsf{g}[\mathrm{X}] := \mathtt{E}_\mathrm{X} \cdot \mathsf{h}[O] \cong \mathcal{P}(\mathsf{h})[\mathrm{X}].$$

The last isomorphism is as in (10.68) with $A := O$.

Exercise 17.57. With notation as above, check that $\mathsf{g} \cong \mathcal{P}(\mathsf{h})$ as Lie monoids. (Use (4.50) and the diagram in the first proof of Proposition 16.2.)

Conversely, let g be a Lie monoid. Then

$$M := \bigoplus_{\mathrm{X}} \mathsf{g}[\mathrm{X}] = \mathcal{S}(\mathsf{g})[O]$$

is a left module over the Tits algebra. This yields a cocommutative bimonoid h with $\mathsf{h}[O] = M$. Now define $\mathsf{h}[O] \to \mathcal{U}(\mathsf{g})[O]$ as the O-component of the map (17.6).

Exercise 17.58. With notation as above, check that $\mathsf{h}[O] \to \mathcal{U}(\mathsf{g})[O]$ is an isomorphism of left modules over the Tits algebra. (First check that the map takes $\mathsf{g}[\mathrm{X}]$ to $\mathtt{E}_\mathrm{X} \cdot \mathcal{U}(\mathsf{g})[O]$. Then check that it is a map of left modules. Finally, use PBW to deduce that the map is an isomorphism.) Deduce that $\mathsf{h} \cong \mathcal{U}(\mathsf{g})$ as cocommutative bimonoids.

17.6. Lie monoids for a rank-one arrangement

In this section, $\mathcal{A}$ denotes a rank-one arrangement with chambers C and $\overline{C}$. In this situation, the notion of Lie monoids, related constructions such as the commutator bracket, the universal enveloping monoid, and results such as PBW and CMM can be understood in explicit terms. A related discussion on bimonoids is given in Section 2.15.

17.6.1. Species. We encode the linear algebra data in an $\mathcal{A}$-species p as follows.

$$\mathsf{p}[O] = V, \quad \mathsf{p}[C] = U, \quad \mathsf{p}[\overline{C}] = U' \quad \text{and} \quad \beta_{\overline{C},C} = \beta, \quad \beta_{C,\overline{C}} = \beta^{-1},$$
$$\mathsf{p}[\top] = W \quad \text{and} \quad \beta_{C,\top} = \alpha, \quad \beta_{\top,C} = \alpha^{-1}, \quad \beta_{\overline{C},\top} = \beta\alpha, \quad \beta_{\top,\overline{C}} = \alpha^{-1}\beta^{-1}.$$

Here V, U, U', W are vector spaces, and α, β are linear isomorphisms. We will employ these notations in the subsequent discussion.

17.6.2. Monoids and Lie monoids. We now formulate $\mathcal{A}$-monoids and $\mathcal{A}$-Lie monoids using linear algebra data.

Lemma 17.59. *An $\mathcal{A}$-monoid is the same as a diagram of vector spaces and linear maps*

$$\begin{tikzcd} U \arrow[rr, "\beta", "\cong"'] \arrow[dr, "\mu"'] & & U' \arrow[dl, "\mu'"] \\ & V. & \end{tikzcd}$$

Moreover, the $\mathcal{A}$-monoid is commutative iff the above diagram commutes.

Compare with Lemma 2.87.

Proof. For an $\mathcal{A}$-monoid, in addition to its data as an $\mathcal{A}$-species, we have linear maps $\mu_O^C = \mu$ and $\mu_O^{\overline{C}} = \mu'$. ☐

Lemma 17.60. *An $\mathcal{A}$-Lie monoid is the same as a diagram of vector spaces and linear maps*

$$\begin{tikzcd} U \arrow[rr, "\beta", "\cong"'] \arrow[dr, "\nu"'] & & U' \arrow[dl, "\nu'"] \\ & V & \end{tikzcd}$$

such that $\nu + \nu'\beta = 0$ (or equivalently, $\nu' + \nu\beta^{-1} = 0$). Moreover, the $\mathcal{A}$-Lie monoid is abelian iff $\nu = \nu' = 0$.

Proof. For an $\mathcal{A}$-Lie monoid, in addition to its data as an $\mathcal{A}$-species, we have linear maps $\nu_O^C = \nu$ and $\nu_O^{\overline{C}} = \nu'$. Antisymmetry (16.3b) yields the condition $\nu + \nu'\beta = 0$. Jacobi identity (16.3c) is vacuous since we are in rank one. ☐

Given linear algebra data as in Lemma 17.59, observe that setting

$$\nu := \mu - \mu'\beta \quad \text{and} \quad \nu' := \mu' - \mu\beta^{-1}$$

yields linear algebra data as in Lemma 17.60. This is the construction of the commutator bracket (16.11).

Exercise 17.61. Check that: A signed $\mathcal{A}$-Lie monoid can be described as in Lemma 17.60 with the condition $\nu + \nu'\beta = 0$ replaced by $\nu - \nu'\beta = 0$. The signed commutator bracket (16.36) translates to

$$\nu := \mu + \mu'\beta \quad \text{and} \quad \nu' := \mu' + \mu\beta^{-1}$$

(Use signed antisymmetry (16.34b) and the fact that $\operatorname{dist}(C, \overline{C}) = 1$.)

17.6.3. Free Lie monoid. The components of the free Lie monoid on an $\mathcal{A}$-species p are given by

$$(\mathbf{Lie} \circ \mathsf{p})[\top] = \mathsf{p}[\top] = W, \qquad (\mathbf{Lie} \circ \mathsf{p})[\perp] = V \oplus (\mathbf{Lie}[\perp, \top] \otimes W).$$

Now consider the cocommutative $\mathcal{A}$-bimonoid $\mathcal{T}(\mathsf{p})$. Its components are

$$\mathcal{T}(\mathsf{p})[C] = \mathsf{p}[C] = U, \quad \mathcal{T}(\mathsf{p})[\overline{C}] = \mathsf{p}[\overline{C}] = U', \quad \mathcal{T}(\mathsf{p})[O] = V \oplus U \oplus U',$$

and the coproduct is given by

$$\Delta_O^C : \mathcal{T}(\mathsf{p})[O] \to \mathcal{T}(\mathsf{p})[C], \qquad (v, u, u') \mapsto u + \beta^{-1}(u'),$$
$$\Delta_O^{\overline{C}} : \mathcal{T}(\mathsf{p})[O] \to \mathcal{T}(\mathsf{p})[\overline{C}], \qquad (v, u, u') \mapsto u' + \beta(u).$$

These are specializations of the coproduct formula (16.16). It follows that the primitive part $\mathcal{P}\mathcal{T}(\mathsf{p})[O]$ consists of elements of the form

$$(v, u, -\beta(u)), \quad u \in U, \ v \in V. \tag{17.37}$$

Thus, it can be identified with $(\mathbf{Lie} \circ \mathsf{p})[\perp]$. This gives a direct verification of formula (16.17).

17.6.4. Universal enveloping monoid. Let g be an $\mathcal{A}$-Lie monoid. Consider its universal enveloping $\mathcal{A}$-monoid $\mathcal{U}(\mathsf{g})$. It equals $\mathcal{T}(\mathsf{g})$ on the C- and $\overline{C}$-components, that is,

$$\mathcal{T}(\mathsf{g})[C] = \mathcal{U}(\mathsf{g})[C] = \mathsf{g}[C] = U \quad \text{and} \quad \mathcal{T}(\mathsf{g})[\overline{C}] = \mathcal{U}(\mathsf{g})[\overline{C}] = \mathsf{g}[\overline{C}] = U'.$$

However, $\mathcal{T}(\mathsf{g})$ and $\mathcal{U}(\mathsf{g})$ differ on the O-component. More precisely, $\mathcal{U}(\mathsf{g})[O]$ is the quotient of

$$\mathcal{T}(\mathsf{g})[O] = V \oplus U \oplus U'$$

by $\mathcal{I}(\mathsf{g})[O]$, the O-component of the monoid of relations, which consists of the elements

$$(-\nu(u), u, -\beta(u)), \quad u \in U. \tag{17.38}$$

These elements are instances of (16.25).

17.6.5. Poincaré–Birkhoff–Witt. Let g be an $\mathcal{A}$-Lie monoid. The components of $\mathcal{S}(\mathsf{g})$ are given by

$$\mathcal{S}(\mathsf{g})[\perp] = \mathsf{g}[\perp] \oplus \mathsf{g}[\top] = V \oplus W \quad \text{and} \quad \mathcal{S}(\mathsf{g})[\top] = \mathsf{g}[\top] = W.$$

The abelianization map $\mathcal{T}(\mathsf{g}) \twoheadrightarrow \mathcal{S}(\mathsf{g})$, evaluated on the O-component, is given by

$$V \oplus U \oplus U' \twoheadrightarrow V \oplus W, \qquad (v, u, u') \mapsto (v, \alpha^{-1}(u) + \alpha^{-1}\beta^{-1}(u')).$$

Now fix a noncommutative zeta function $\boldsymbol{\zeta}$. Following Example 1.15, we put

$$\boldsymbol{\zeta}(O, C) = p, \quad \boldsymbol{\zeta}(O, \overline{C}) = 1 - p, \quad \boldsymbol{\mu}(O, C) = -p, \quad \boldsymbol{\mu}(O, \overline{C}) = p - 1.$$

The map $\mathcal{S}(\mathsf{g}) \hookrightarrow \mathcal{T}(\mathsf{g})$ as in (17.1), evaluated on the O-component, is given by

$$V \oplus W \hookrightarrow V \oplus U \oplus U', \qquad (v, w) \mapsto (v, p\,\alpha(w), (1-p)\,\beta\alpha(w)). \tag{17.39}$$

It is a section to the abelianization map. We identify $\mathcal{S}(\mathsf{g})[O]$ with its image inside $\mathcal{T}(\mathsf{g})[O]$. Now observe that

$$\mathcal{T}(\mathsf{g})[O] = \mathcal{S}(\mathsf{g})[O] \oplus \mathcal{I}(\mathsf{g})[O].$$

The first summand consists of elements of the form $(v, p\,u, (1-p)\,\beta(u))$, while the second consists of elements of the form $(-\nu(u), u, -\beta(u))$. This in effect verifies the PBW Theorem 17.9.

The approach to PBW using the Solomon operator is sketched in the exercises below.

Exercise 17.62. Check that: The projection $\mathcal{T}(\mathsf{p}) \to \mathcal{PT}(\mathsf{p})$ as in (17.11), evaluated on the O-component, is given by

$$\begin{aligned} &V \oplus U \oplus U' \to V \oplus U \oplus U', \\ &\qquad (v, u, u') \mapsto (v, (1-p)u - p\beta^{-1}(u'), (p-1)\beta(u) + pu'). \end{aligned}$$

(Use formula (17.12b).) Note directly using (17.37) that the image indeed lies in the primitive part.

Exercise 17.63. Check that: The Solomon operator $\mathcal{T}(\mathsf{g}) \to \mathcal{T}(\mathsf{g})$ as in (17.24), evaluated on the O-component, is given by

$$\begin{aligned} &V \oplus U \oplus U' \to V \oplus U \oplus U', \\ &\quad (v, u, u') \mapsto (v + (1-p)\nu(u) + p\nu'(u'), p(u + \beta^{-1}(u')), (1-p)(\beta(u) + u')). \end{aligned}$$

Check that this linear operator is idempotent, its image is $\mathcal{S}(\mathsf{g})[O]$, its kernel is $\mathcal{I}(\mathsf{g})[O]$. This gives a direct verification of Theorem 17.32.

17.6.6. Cartier–Milnor–Moore. We now discuss the content of the CMM Theorem 17.42.

For any $\mathcal{A}$-Lie monoid g, we have $\mathcal{PU}(\mathsf{g})[O] = \mathsf{g}[O]$. This follows from PBW or can also be checked directly. Using relation (17.38), we see that

$$\mathcal{PT}(\mathsf{g})[O] \twoheadrightarrow \mathcal{PU}(\mathsf{g})[O], \qquad (v, u, -\beta(u)) \mapsto (v + \nu(u), 0, 0).$$

Let h be a cocommutative $\mathcal{A}$-bimonoid. First consider the map $\mathcal{TP}(\mathsf{h}) \to \mathsf{h}$, which on the O-component is given by

$$\mathrm{id} + \mu_O^C + \mu_O^{\overline{C}} : \mathcal{P}(\mathsf{h})[O] \oplus \mathsf{h}[C] \oplus \mathsf{h}[\overline{C}] \longrightarrow \mathsf{h}[O].$$

(This is the map in Proposition 13.90 but with $q = 1$. Precomposing with the section $\mathcal{SP}(\mathsf{h})[O] \hookrightarrow \mathcal{TP}(\mathsf{h})[O]$ in (17.39) for $\mathsf{g} = \mathcal{P}(\mathsf{h})$ yields the Borel–Hopf isomorphism given in Example 13.39.) The kernel of the map $\mathcal{TP}(\mathsf{h})[O] \to \mathsf{h}[O]$ consists of the elements

$$(-\mu_O^C(x) + \mu_O^{\overline{C}}\beta_{\overline{C},C}(x), x, -\beta_{\overline{C},C}(x)), \qquad x \in \mathsf{h}[C].$$

These coincide with the elements (17.38) for $\mathsf{g} = \mathcal{P}(\mathsf{h})$. Moreover, the map is surjective. Hence, it induces an isomorphism $\mathcal{UP}(\mathsf{h})[O] \to \mathsf{h}[O]$.

17.6.7. Lie comonoids. The above discussion can also be carried out in the dual setting. We give the formulation of $\mathcal{A}$-Lie comonoids below, and omit the rest.

Lemma 17.64. *An $\mathcal{A}$-Lie comonoid is the same as a diagram of vector spaces and linear maps*

$$\begin{tikzcd} U \arrow[rr, "\beta", "\cong"'] & & U' \\ & V \arrow[ul, "\theta"] \arrow[ur, "\theta'"'] & \end{tikzcd}$$

such that $\theta + \beta^{-1}\theta' = 0$ (or equivalently, $\theta' + \beta\theta = 0$). Moreover, the $\mathcal{A}$-Lie comonoid is abelian iff $\theta = \theta' = 0$.

Proof. For an $\mathcal{A}$-Lie comonoid, in addition to its data as an $\mathcal{A}$-species, we have linear maps $\theta^C_O = \theta$ and $\theta^{\overline{C}}_O = \theta'$. Antisymmetry (16.43b) yields the condition $\theta + \beta^{-1}\theta' = 0$. Jacobi identity (16.43c) is vacuous since we are in rank one. $\square$

17.7. Joyal Lie monoids

We extend the considerations of Section 2.16 to Lie monoids. We first recall the notion of Lie monoids for Joyal species and then explain how it gives rise to a $\mathcal{B}^J$-Lie monoid, where $\mathcal{B}^J$ is the braid arrangement on J.

17.7.1. Joyal Lie monoids. A *positive Joyal Lie monoid* is a positive Joyal species g equipped with linear maps

$$\mathsf{g}[S] \otimes \mathsf{g}[T] \to \mathsf{g}[S \sqcup T], \qquad x \otimes y \mapsto [x, y]_{S,T},$$

one for each pair of nonempty disjoint subsets S and T, which satisfy naturality, antisymmetry and Jacobi identity. We refer to $[x, y]_{S,T}$ as the *Lie bracket* of $x \in \mathsf{g}[S]$ and $y \in \mathsf{g}[T]$.

Naturality states that

$$(\sigma \sqcup \tau)([x, y]_{S,T}) = [\sigma(x), \tau(y)]_{S',T'} \tag{17.40a}$$

for any $x \in \mathsf{g}[S]$ and $y \in \mathsf{g}[T]$, with S and T disjoint, S' and T' disjoint, and bijections $\sigma : S \to S'$ and $\tau : T \to T'$.

Antisymmetry states that

$$[x, y]_{S,T} + [y, x]_{T,S} = 0 \tag{17.40b}$$

for any $x \in \mathsf{g}[S]$ and $y \in \mathsf{g}[T]$, with S and T disjoint.

Jacobi identity states that

$$[[x, y]_{R,S}, z]_{R\sqcup S,T} + [[z, x]_{T,R}, y]_{R\sqcup T,S} + [[y, z]_{S,T}, x]_{S\sqcup T,R} = 0 \tag{17.40c}$$

for any $x \in \mathsf{g}[R]$, $y \in \mathsf{g}[S]$, $z \in \mathsf{g}[T]$, with R, S, T disjoint. In the presence of antisymmetry (17.40b), Jacobi identity may be written in various equivalent forms, such as

$$[x, [y, z]_{S,T}]_{R,S\sqcup T} + [[y, x]_{S,R}, z]_{R\sqcup S,T} + [y, [z, x]_{T,R}]_{S,R\sqcup T} = 0. \tag{17.40d}$$

17.7.2. Joyal Lie comonoids. A *positive Joyal Lie comonoid* is a positive Joyal species k equipped with linear maps

$$\mathsf{k}[S \sqcup T] \to \mathsf{k}[S] \otimes \mathsf{k}[T], \qquad x \mapsto \theta_{S,T}(x),$$

subject to axioms dual to (17.40). For example, antisymmetry states that

$$\theta_{S,T}(x) + \beta_{T,S}(\theta_{T,S}(x)) = 0$$

for any $x \in \mathsf{c}[S \sqcup T]$.

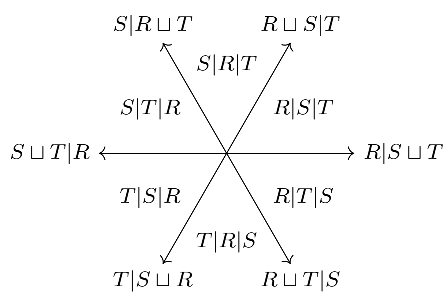
17.7.3. From Joyal Lie (co)monoids to $\mathcal{B}^J$-Lie (co)monoids. Recall from Section 2.16.6 that a positive Joyal species g gives rise to a $\mathcal{B}^J$-species g^J by setting

$$\mathsf{g}^J[F] := \mathsf{g}[S_1] \otimes \cdots \otimes \mathsf{g}[S_k]$$

for any set composition $F = (S_1, \dots, S_k)$. Now suppose g is a positive Joyal Lie monoid. We claim that g^J is a $\mathcal{B}^J$-Lie monoid. We work with the formulation in Section 16.1.2.

For $A \lessdot F$, note that F breaks some block of A into two blocks, say S and T. Define the Lie bracket ν_A^F of g^J to be $[\,,\,]_{S,T}$ on the $\mathsf{g}[S] \otimes \mathsf{g}[T]$ factor of $\mathsf{g}^J[F]$, and identity on the other tensor factors. The naturality axiom (16.3a) holds trivially. Antisymmetry (16.3b) follows from antisymmetry (17.40b). For Jacobi identity (16.3c), we fix a composition A and a partition X with $\mathrm{s}(A) \le \mathrm{X}$ and X having two more blocks than A. There are two cases to consider.

- X breaks one block of A into three blocks.
- X breaks two different blocks of A into two blocks each.



In the first case, we may write $A = F|R \sqcup S \sqcup T|G$, where F and G are set compositions and R, S, T are blocks of X. Jacobi identity (16.3c) for g^J requires that the sum of the following three terms

$$\mathsf{g}^J[F|R|S|T|G] \xrightarrow{\nu} \mathsf{g}^J[F|R|S \sqcup T|G] \xrightarrow{\nu} \mathsf{g}^J[F|R \sqcup S \sqcup T|G],$$

$$\mathsf{g}^J[F|R|S|T|G] \xrightarrow{\beta} \mathsf{g}^J[F|S|R|T|G] \xrightarrow{\nu} \mathsf{g}^J[F|R \sqcup S|T|G] \xrightarrow{\nu} \mathsf{g}^J[F|R \sqcup S \sqcup T|G],$$

$$\mathsf{g}^J[F|R|S|T|G] \xrightarrow{\beta} \mathsf{g}^J[F|S|T|R|G] \xrightarrow{\nu} \mathsf{g}^J[F|S|R \sqcup T|G] \xrightarrow{\nu} \mathsf{g}^J[F|R \sqcup S \sqcup T|G]$$

be 0. All these maps are identities on the outer tensor factors $\mathsf{g}[F]$ and $\mathsf{g}[G]$. Grouping the intermediate factors yields an instance of Jacobi identity (17.40d) for the Joyal Lie monoid g. Hence, the above sum is 0.

$$
\begin{array}{ccccc}
 & & R \sqcup S|T|U & & \\
 & & \uparrow & & \\
 & S|R|T|U & & R|S|T|U & \\
S|R|T \sqcup U & \longleftarrow & \cdot & \longrightarrow & R|S|T \sqcup U \\
 & S|R|U|T & & R|S|U|T & \\
 & & \downarrow & & \\
 & & R \sqcup S|U|T & &
\end{array}
$$

In the second case, we may write $A = F|R \sqcup S|T \sqcup U|G$, where R, S, T, U are blocks of X. Jacobi identity (16.3c) requires that the sum of the two terms

$$\mathsf{g}^J[F|R|S|T|U|G] \xrightarrow{\nu} \mathsf{g}^J[F|R|S|T \sqcup U|G] \xrightarrow{\nu} \mathsf{g}^J[F|R \sqcup S|T \sqcup U|G],$$

$$\mathsf{g}^J[F|R|S|T|U|G] \xrightarrow{\beta} \mathsf{g}^J[F|S|R|T|U|G] \xrightarrow{\nu} \mathsf{g}^J[F|R \sqcup S|T|U|G] \xrightarrow{\nu} \mathsf{g}^J[F|R \sqcup S|T \sqcup U|G],$$

be 0. As before, these maps are identities on the outer tensor factors $\mathsf{g}[F]$ and $\mathsf{g}[G]$. The sum of the intermediate factors yields an expression of the form

$$[x, y]_{R,S} \otimes [z, w]_{T,U} + [y, x]_{S,R} \otimes [z, w]_{T,U},$$

which is 0 by antisymmetry (17.40b) for the Joyal Lie monoid g.

A morphism of Joyal Lie monoids $\mathsf{f} \to \mathsf{g}$ yields a morphism of $\mathcal{B}^J$-Lie monoids $\mathsf{f}^J \to \mathsf{g}^J$. In this manner we obtain a functor from the category of positive Joyal Lie monoids to the category of $\mathcal{B}^J$-Lie monoids.

A similar construction can be carried out to turn Joyal Lie comonoids k into $\mathcal{B}^J$-Lie comonoids k^J.

17.7.4. Solomon coefficients. We now assume that the field characteristic is 0. Recall the Solomon coefficients from Section 1.8.3. For any composition H, let $\deg(H)$ denote the number of blocks of H.

Lemma 17.65. *For the braid arrangement, for* $\boldsymbol{\mu} := \boldsymbol{\mu}_u$,

$$\operatorname{sln}_O^{G,H} = \operatorname{sln}_O^{G,H\overline{G}} = (-1)^{s-1} \frac{1}{s\binom{d}{s}}, \tag{17.41}$$

where $d := \deg(G)$*, while* $s := \deg(K)$*, where* K *is the smallest face of* G *such that* $KH = G$.

In the notation of Exercise 1.42, we may write $K = \overline{\operatorname{Des}}(H, G)$. A combinatorial description can be given along the lines of [**18**, Definition 10.8].

Proof. This can be deduced for instance from [**21**, Formulas (12.43) and (12.51)] and the formula for $\boldsymbol{\mu}_u$ from Example 9.140. □

Formula (17.41) extends to $\operatorname{sln}_A^{G,H}$ by taking products of (17.41), one for each block of A.

17.7.5. Solomon operator. We continue to assume that the field characteristic is 0. Let g be a positive Joyal Lie monoid. For any composition G of J, define the *left bracketing operator*

$$L_J : \mathsf{g}^J[G] \to \mathsf{g}[J], \qquad x_1 \otimes x_2 \otimes \cdots \otimes x_d \mapsto [\ldots[x_1, x_2], \ldots, x_d].$$

Here $d := \deg(G)$. The rhs is computed using the Lie bracket of g.

Now for any composition H of J, define

$$\pi_J^H : \mathsf{g}^J[H] \to \mathsf{g}[J], \qquad \pi_J^H := \sum_{\substack{G \vDash J \\ \mathrm{s}(G)=\mathrm{s}(H)}} (-1)^{s-1} \frac{1}{s d\binom{d}{s}} L_J \beta_{G,H}. \tag{17.42}$$

The sum is over all compositions G of J of the same support as H. In other words, the blocks of G are obtained by rearranging the blocks of H. Further, $d := \deg(H)$ is fixed, while $s := \deg(K)$, where K is the smallest face of G such that $KH = G$. This is consistent with the notation in (17.41).

More generally, for any compositions $H \geq A$ of J, define

$$\pi_A^H := \pi_{S_1}^{H^1} \otimes \cdots \otimes \pi_{S_m}^{H^m},$$

where $A = (S_1, \ldots, S_m)$, and H^i is the part of H which refines S_i.

For any g, define the positive Joyal species $\mathcal{T}(\mathsf{g})$ by

$$\mathcal{T}(\mathsf{g})[J] := \bigoplus_{G \vDash J} \mathsf{g}^J[G].$$

Observe that $\mathcal{T}(\mathsf{g})^J = \mathcal{T}(\mathsf{g}^J)$ as $\mathcal{B}^J$-species.

Now define the Solomon operator ψ on $\mathcal{T}(\mathsf{g})$ as follows. The (H, K)-matrix component of ψ_J is a map from $\mathsf{g}^J[H]$ to $\mathsf{g}^J[K]$. If $HK > H$, it is defined to be zero, and if $HK = H$, it is defined by

$$\frac{1}{\deg(K)!} \pi_K^{KH} \beta_{KH,H}. \tag{17.43}$$

The condition $HK = H$ means that each block of K is obtained by merging some blocks of H. The map $\beta_{KH,H}$ rearranges the tensor factors appropriately.

Proposition 17.66. *For any positive Joyal Lie monoid* g*, the functor* (2.102) *carries the Solomon operator on* $\mathcal{T}(\mathsf{g})$ *to the Solomon operator on* $\mathcal{T}(\mathsf{g}^J)$*.*

The Solomon operator on $\mathcal{T}(\mathsf{g})$ is as defined above, while the one on $\mathcal{T}(\mathsf{g}^J)$ is (17.25) specialized to the braid arrangement $\mathcal{B}^J$ with $\boldsymbol{\zeta} := \boldsymbol{\zeta}_u$ and $\boldsymbol{\mu} := \boldsymbol{\mu}_u$.

Proof. The main part is to check that formula (17.25), when the latter is specialized as above and evaluated on the central face, equals (17.43). This is done in three steps. First, by substituting (17.41) in (17.12a), we obtain

$$\sum_G (-1)^{s-1} \frac{1}{s\binom{d}{s}} \beta_{G,H} \in (\mathbf{Lie} \circ \mathsf{g})[J].$$

Next, we employ the Dynkin-Specht-Wever theorem [**21**, Theorem 14.74]. (This result is stated for $\Gamma = \mathcal{T}(\mathsf{x})$ but it works more generally for $\mathcal{T}(\mathsf{g})$.) This allows us to insert the left bracketing operator at the expense of getting an extra factor of d in the denominator. Finally, recall from Example 9.140 that $\zeta_u(O, K) = \frac{1}{\deg(K)!}$. $\square$

17.8. Lie monoids in LRB species

Recall from Section 3.9 the notion of LRB species which generalizes the notion of $\mathcal{A}$-species. Also recall that LRB (co, bi)monoids generalize $\mathcal{A}$-(co, bi)monoids. We now consider LRB Lie monoids which generalize $\mathcal{A}$-Lie monoids, and indicate briefly how the theory generalizes to this setting.

17.8.1. LRB Lie monoids. Let Σ be a left regular band. A Σ-*Lie monoid* is defined as a left module over the Lie Σ-operad (Section 4.14).

A presentation of the Lie Σ-operad for arbitrary Σ is potentially complicated, so we do not consider the second approach to Lie monoids using the Lie bracket. (Rank-one flats supporting two vertices, and rank-two flats supporting an even-sided polygon are features which are specific to arrangements.) For the same reason, we also do not consider any definitions, results or proofs which involve the Lie bracket.

17.8.2. Primitive part of LRB bimonoids. The underlying LRB Lie monoid functor starting from the category of LRB monoids is obtained as in (16.8). Explicitly, it is given by formula (16.9). Proposition 16.2 holds for LRB bimonoids with the first proof going through.

17.8.3. Free LRB Lie monoids. The free LRB Lie monoid on a LRB species is given as in Proposition 16.4, with its universal property as in Theorem 16.5. Proposition 16.6 and Proposition 16.9 work as before. The general discussion on Lie and Zie species in Section 16.4 holds.

17.8.4. Universal enveloping LRB monoid. For a LRB Lie monoid g, the universal enveloping LRB monoid $\mathcal{U}(\mathsf{g})$ is defined as the quotient of $\mathcal{T}(\mathsf{g})$ by the H/A-relations. (The discussion on A/A- and H/A-relations goes through with the exception of (16.23) which involves rank-one flats.) Note very carefully that we do not consider Lemmas 16.13 and 16.14. Theorem 16.15 on the adjunction between $\mathcal{U}$ and the underlying LRB Lie monoid functor works with the first proof going through. Theorem 16.16 is a reformulation as a universal property.

For the bimonoid structure on $\mathcal{U}(\mathsf{g})$, the first approach using Proposition 16.18 and Lemma 16.21 works. The second approach with Lemma 16.22 (with its first proof) and Lemma 16.24 also works. The adjunction between $\mathcal{U}$ and $\mathcal{P}$ given in Theorem 16.25 holds for LRB Lie monoids and LRB bimonoids. Theorem 16.27 is a reformulation as a universal property. Proposition 16.28 holds for LRB species.

The situation for abelian LRB Lie monoids is as in Proposition 16.30. For the proof, we can use Lemma 1.76 which is true for any LRB.

17.8.5. LRB Lie comonoids. For any left regular band Σ, a Σ-*Lie comonoid* is defined as a left comodule over the Lie Σ-cooperad.

The underlying LRB Lie comonoid functor starting from the category of LRB comonoids is obtained as in (16.45). Similarly, the indecomposable part functor is as in (16.48). The cofree LRB Lie comonoid is defined in terms of the Lie LRB cooperad, and its universal property is as in Theorem 16.41. Proposition 16.42 works as before. The universal coenveloping LRB comonoid $\mathcal{U}^\vee(\mathsf{k})$ of a LRB Lie comonoid k is defined using the H/A-conditions (16.51). Theorems 16.45 and 16.46 on the adjunction between $\mathcal{U}^\vee$ and the underlying LRB Lie comonoid functor hold. Similarly, Theorems 16.51 and 16.52 on the adjunction between $\mathcal{U}^\vee$ and $\mathcal{Q}$ hold for LRB Lie comonoids and LRB bimonoids. Proposition 16.53 holds for LRB species. The situation for abelian LRB Lie comonoids is as in Proposition 16.54.

17.8.6. PBW and CMM. Recall that noncommutative zeta and Möbius functions make sense for any LRB. The PBW Theorem 17.9 holds for any LRB Lie monoid g. The PBW isomorphism is defined as in (17.6), with a section to the abelianization map as in (17.1). For the proof of Lemma 17.11, we can do the abstract argument using Lemma 1.76. The proof of Lemma 17.12 works as before. Similarly, the dual PBW Theorem 17.18 holds for any LRB Lie comonoid k, with the dual PBW isomorphism defined as in (17.9).

The discussion in Section 17.3 is valid for any LRB species p. The Solomon operator can be defined for any LRB Lie monoid g and Theorem 17.32 holds with the same proof. Similarly, the dual Solomon operator can be defined for any LRB Lie comonoid k and Theorem 17.40 holds.

The CMM Theorem 17.42 and dual CMM Theorem 17.53 generalize to any LRB species. For the related Borel–Hopf and Leray–Samelson theorems for LRB species, see Section 13.5.2.

Notes

The results and terminology of this chapter are motivated by the classical theorems of Poincaré–Birkhoff–Witt and Cartier–Milnor–Moore. Details on these are given below.

Lie algebras.

Poincaré–Birkhoff–Witt (PBW). The classical Poincaré–Birkhoff–Witt theorem, or PBW for short, originated in work of Poincaré [**751**, page 1066, second-to-last paragraph], [**752**, page 232, last sentence of Section III]. In our terminology, he showed that $\mathcal{T}(\mathsf{g}) = \mathcal{S}(\mathsf{g}) \oplus \mathcal{I}(\mathsf{g})$. A special case of PBW had appeared in earlier work of Capelli [**184**]. PBW was rediscovered later independently by Birkhoff [**114**] and Witt [**913**]. (Birkhoff and Witt do cite each other in their later works, see for instance [**115**], [**914**], [**915**].) The precise classical analogue of Theorem 17.9 was given later by Smoke [**836**, pages 11 and 12], [**837**, page 467] and Quillen [**766**, Appendix B, Theorem 2.3]. (The latter worked in the differential graded setting.) In PBW, proving surjectivity as in Lemma 17.11 is the easy part, while proving injectivity as in Lemma 17.12 is the hard part. As we saw, the geometric content of the surjectivity argument is that faces of a given support are gallery connected.

The name Poincaré–Birkhoff–Witt was promoted by Bourbaki [**146**, Section I.2.7] and it is now standard; other early usages are by Cartier [**190**], Gabriel [**335**, Exposé VIIB, Section 3.1]. PBW has also been called Birkhoff–Witt [**219**, Chapter VII, Theorem 6.3], [**438**, Chapter VII, Theorem 1.2], [**536**, Theorem 3.2], [**552**, Chapter 5, Section 9] and Poincaré–Witt [**189**, Chapter XIII, Section 3], [**192**, Section 3.1, Theorem 1 and Corollaries 1 and 2]. Early references for PBW over more general rings are [**195**], [**218**], [**434**], [**579**], [**580**], [**724**], [**829**]. More historical information on PBW can be found in the papers by Schmid [**808**], Ton-That and Tran [**886**], Grivel [**378**].

Symmetrization map. The classical analogue of the map (17.6) is usually called the symmetrization map in the literature. It corresponds to the uniform noncommutative zeta function of the braid arrangement. In this case, all faces of a given support are weighted equally. In general, a face F in the star of A is assigned a weight $\boldsymbol{\zeta}(A, F)$ such that the sum of the weights of faces of a given support is 1 (as in a probability distribution). This is the content of the flat-additivity formula (1.43).

The symmetrization map appeared implicitly in work of Poincaré cited above. He showed that any element of the universal enveloping algebra can be uniquely expressed using symmetric tensors. This fact is later mentioned by Gelfand [**345**, Section 2], [**346**, Section 2] and proved by Harish-Chandra [**416**, Theorem 1]. The symmetrization map appeared explicitly in a subsequent paper of Harish-Chandra [**418**, page 192]. The fact that $\mathcal{S}(\mathsf{g}) \cong \mathcal{U}(\mathsf{g})$ as coalgebras is implicit in Cartier's work [**192**, Section 3.4, Propositions 6 and 7], see also [**258**, Proposition 2.7.5]. Smoke [**836**, pages 11 and 12], [**837**, page 467] and Quillen [**766**, Appendix B, Theorem 2.3] explicitly say that the symmetrization map is a coalgebra isomorphism. Further, in [**766**, Appendix B, Section 3.9], Quillen connects the symmetrization map to the exponential map; this can be viewed as the analogue of Exercise 17.15.

Solomon coefficients. The coefficients $\mathrm{sln}_O^{G,H}$ in (17.41) (in the special case when G and H are linear orders) appeared in work of Solomon [**839**, Formula (2.3)] and Bialynicki-Birula, Mielnik, Plebański [**111**, Formula (3.7)], [**689**, Formula (10.12)]. Their definition was in terms of permutations. For later references, see for instance those by Reutenauer [**776**, Theorem 1.1 and Corollary 1.6], [**777**, Corollary 3.16], Strichartz [**861**, page 322], Helmstetter [**430**, Theorem 19], Schmitt [**813**, Theorem 9.5], Gelfand, Krob, Lascoux, Leclerc, Retakh, Thibon [**347**, Lemma 5.25]. See also [**21**, Formula (12.51)], [**81**, Theorem 1.1], [**82**, Theorem 1]. The case when G is a linear order and H a set composition is considered in [**725**, Proposition 7.5], [**273**, Theorem 4.8 and Corollary 5.11], [**274**, Theorems 2.1 and 7.2]. In the reference [**273**], a set composition is called a 'quasi-permutation'. Also, their notion of quasi-descent is subsumed by the considerations in [**21**, Section 7.1.1]. The formulation of Solomon coefficients in Section 1.8.3 in the generality of arrangements is new.

Logarithm of the identity on the tensor algebra. The classical analogue of the map (17.11) appears implicitly in work of Solomon [**839**, Formulas (2.1) and (2.3), Lemmas 5 and 6]. It is made explicit by Reutenauer [**776**, Corollary 1.6], [**777**, Corollary 3.16]. The analogue of the closely related bottom horizontal composite map in diagram (17.35) is given by Helmstetter [**430**, Theorem 19]. See the Notes to Chapter 9 for related information. The analogue of Proposition 17.21 says that the logarithm of the identity map on the tensor algebra $\mathcal{T}(V)$ has image the free Lie algebra $Lie(V)$. The analogue of Proposition 17.27 is the dual result for the logarithm of the identity map on the shuffle algebra $\mathcal{T}^\vee(V)$. The analogues of formulas (17.19) and (17.20) for $q = 1$ and Exercise 17.30 deal with the Hopf algebras $\mathcal{T}(C)$ and $\mathcal{T}^\vee(A)$ in Table 6.3; see [**309**, Section 6, page 227 and Proposition 6.1], [**725**, Proposition 7.5 and Theorem 9.2], [**273**, Corollaries 5.10 and 5.11], [**274**, Theorem 7.2], [**698**, Definition 1 and Proposition 1], [**174**, Definition 4, Lemma 8, Proposition 2, item (1)].

Recall from Section 17.3.3 that the map (9.9) on the bimonoid of chambers is a specialization of (17.11). For the braid arrangement, the map (9.9) for $A = O$, is given in [**21**, Formula (12.51)]. It is identical in form to the formula given by Reutenauer, with chambers (linear orders) instead of permutations. For the arrangement of type B, the map (9.9) for $A = O$, is given in [**21**, Formula (12.66)]. In this reference, the operator $\mathbf{log}(\mathrm{id})_O$ is viewed as the action of the first eulerian idempotent $\mathtt{E}_\perp$ of the Tits algebra.

Solomon operator. The analogues of the maps (17.42) and (17.43) are given by Solomon [**839**, Formulas (1.2) and (1.3)]. Solomon's formulas are also mentioned in Dixmier's book [**258**, Section 2.8.12, items (a) and (b)].

Book references for PBW. PBW is discussed or at least mentioned in many places in the literature. Early treatments are by Cartan and Eilenberg [**189**, Chapter XIII, Section 3], Chevalley [**212**, Chapter 5, Section 6, Propositions 1 and 2], Helgason [**429**, Chapter II, Proposition 1.9, and page 392], Hochschild [**441**, Part I, Section 1, Theorem 1], [**442**, Chapter X, Theorem 1.1], [**445**, Chapter XVI, Theorem 1.1], Jacobson [**480**, Section V.2], Serre [**823**, Chapter III, Theorem 4.3 and Lemma 4.5].

For later references, see those by Bahturin [**61**, Section 2.5.2], [**62**, Chapter IV, Theorem 2.3], Dixmier [**258**, Theorem 2.1.11 and Proposition 2.3.6], Humphreys [**473**, Chapter V, Section 17], Kassel [**517**, Theorem V.2.5], Loday [**606**, Section 3.3.4], Lothaire [**613**, Theorem 5.3.7], McConnell and Robson [**666**, Section 1.7.5], Moody and Pianzola [**705**, Section 1.8, Theorem 1], Onishchik and Vinberg [**726**, Section 3.1.2], Postnikov [**761**, Lecture 5, page 111], Reutenauer [**777**, Theorem 0.2], Varadarajan [**893**, Theorems 3.2.2 and 3.3.1], Weibel [**905**, Theorem 7.3.7], Želobenko [**930**, Chapter IX, Section 58, Theorem 1].

For more recent references, see [**130**, Theorem 2.94], [**156**, Theorem 2.5.3.1], [**215**, Appendix A.1.6], [**237**, Theorem 6.2.1], [**244**, Theorem 5.15], [**303**, Theorem 21.1], [**357**, Section 5.10, page 182], [**408**, Theorem 9.9], [**456**, Theorem 1.2.4], [**435**, Theorem 7.1.9], [**529**, Theorem 1.3], [**532**, Theorems 5.11 and 5.12], [**537**, Theorem 3.8 and Proposition 3.16], [**612**, Appendix D.5], [**671**, Section 5.2], [**765**, Section 7.1].

Warning: The precise statement of PBW varies from book to book. For instance, most references discuss the symmetrization map but only as a vector space isomorphism. Similarly, they mention the isomorphism (17.34) but only as algebras and not as bialgebras. In some references, the result of Corollary 17.13 is assumed. In this case, the proof of PBW simplifies since Lemma 17.12 immediately follows from Exercise 17.14. In some references, the result $\mathcal{P}(\mathcal{U}(\mathsf{g})) = \mathsf{g}$, which constitutes one part of CMM, is also proved as an application of PBW.

Signed PBW. In the context of Lie superalgebras or signed graded Lie algebras, PBW is discussed by Milnor and Moore [**695**, Theorems 5.15 and 5.16], [**696**, Theorem 5.21], Ross [**787**], [**788**, Theorem 2.1], Corwin, Ne'eman, Sternberg [**223**, Section IV]. It is mentioned by Kac [**506**, Section 1.1.3], Kostant [**541**, page 220] and later by Boseck [**141**, page 17], [**142**, pages 57 and 58]. It also appears in Scheunert's book [**805**, Section I.2.3, pages 26 and 27]. This is the classical analogue of Theorem 17.16.

The classical analogue of (17.8) is the signed symmetrization map. Ismail [**478**, Theorem and remarks on page 18] and Koszul [**543**, Lemma 4] explicitly say that the signed symmetrization map is an isomorphism of coalgebras. They also link this map to the exponential as done in Exercise 17.17. A later discussion is given by Helmstetter [**430**, Corollary 17].

For later references to signed PBW, see [**64**, Chapter 3, Theorems 2.2 and 2.5], [**186**, Theorem 1.6.5], [**239**, Section 1.3.7], [**663**, Sections 22.2 and 23.2], [**712**, Theorems 6.1.1, 6.1.2, 6.3.3], [**714**, Theorems 8.2.2 and 8.2.4], [**894**, Theorem 6.2.1]. Some books where signed PBW is mentioned are [**99**, page 279], [**209**, Section 1.4.1], [**259**, Section 1.1], [**299**, Theorem 3.6], [**316**, Sections 1.91 and 2.62], [**529**, Section 7.1], [**691**, Theorems 11.3.2 and 11.3.3], [**821**, Theorems 10.5.4 and 10.5.5], [**834**, Theorem on page 70], [**852**, Section 4.8].

PBW for color Lie algebras was obtained by Scheunert [**804**, Theorem 1]. See also [**649**, page 16], [**692**, Theorems 19.1, 19.2, 19.9], [**65**, Theorem 1.1], [**529**, Theorem 7.1].

Cartier–Milnor–Moore (CMM). The analogue of Theorem 17.42 is the classical Cartier–Milnor–Moore theorem for cocommutative bialgebras, or CMM for short. It first appeared in work of Cartier [**192**, Section 3.5, Propositions 6 and 8], [**194**, Theorem 1] and Milnor and Moore [**695**, Theorem 5.18], [**696**, Theorem 5.22]. The latter worked in the signed setting which is the analogue of Theorem 17.52. The analogue of Exercise 17.44 is given in [**192**, Section 3.4, Proposition 6], [**695**, Proposition 5.17].

Early references to CMM are by Quillen [**766**, Appendix B, Theorem 4.5] and Sweedler [**864**, Theorem 4 on page 103], [**867**, Theorem 13.0.1 and Proposition 13.0.2]. Quillen calls it the theorem of Cartier, Milnor, and Moore. Sweedler attributes this result to unpublished work of Kostant; also see [**541**, Proposition 3.2]. The crux of the proofs given by Quillen and Sweedler is to use Borel–Hopf and PBW (similar to what we do). They do not separately state Borel–Hopf; instead they develop it inside their proof to the extent required. CMM is also treated by Grünenfelder in his thesis [**383**, Theorems I.3.15 and III.3.3], [**384**, Theorems 1.3 and 3.2]. It is also mentioned in Larson's thesis [**563**, Theorem 3.3], [**564**, Theorem 3.5].

Later references to CMM are by Nichols [**719**, Chapter III, Theorem 9], Abe [**1**, Theorem 2.5.3], Hain [**404**, Theorem 5.4], [**405**, Theorem 6.12], Patras and Cartier [**739**, Theorem I,5,6], [**740**, Proposition 4.2], [**741**, Proposition III,1], [**200**, page 24], [**202**, Theorem 3.8.1], Montgomery [**703**, Theorem 5.6.5], Bourbaki [**150**, Section II.1.6], Gracia-Bondía, Várilly, Figueroa [**364**, Section 14.3], Connes and Marcolli [**221**, Theorem 1.22], May and Ponto [**663**, Theorem 22.3.1]. The last reference closely follows the original paper of Milnor and Moore. CMM is called the Theorem of Cartan–Milnor–Moore in Loday's book [**606**, Theorem A.10]. The attribution to Cartan is an error; Loday employs Cartier–Milnor–Moore in later work [**608**, Section 4.1]. The analogue of Exercise 17.46 is given by Nichols [**719**, Chapter III, Theorem 10], the claim about the isomorphism is given by Kaplansky [**510**, Theorem 18]. The analogue of Exercise 17.47 is given by Hudson [**467**, Section 7], [**468**, Section 6.4] and mentioned in his paper with Pulmannová [**471**, page 2091]. An earlier reference is his paper with Parthasarathy [**470**, Theorem 4.2].

CMM in the language of formal groups is given by Cartier [**196**, Theorem 3], Gabriel [**335**, Exposé VIIB, Corollary 3.3.2], Serre [**823**, Part II, Section V.6, Theorem 3], Fröhlich [**330**, Section II.2, Theorem 1], Demazure [**241**, page 42], Ditters [**256**, Theorem on page 28], Hazewinkel [**427**, Theorem 14.2.3].

The topological analogue of CMM is given by Milnor and Moore [**695**, Appendix], [**696**, Appendix]. It is explained in the book by Félix, Halperin, Thomas [**303**, Theorem 21.5], see also [**714**, Chapter 0, page 5], [**172**, Section B.1, page 424].

CMM in the pointed case. The present monograph only pertains to the connected case. We mention that the classical CMM generalizes to the wider class of pointed cocommutative Hopf algebras. Early references are by Larson [**563**, Theorem 16.4], [**564**, Theorem 5.3], Heyneman and Sweedler [**864**, page 67], [**867**, Theorem 8.1.5 and Section 13.1], [**432**, Theorem 3.5.8], Grünenfelder [**383**, Theorem III.3.10], [**384**, Theorem 3.13], Winter [**911**, Appendix B.3], Kostant [**541**, Theorem 3.3]. (The last reference is in the signed setting.) In the language of formal groups, it is given by Cartier [**196**, Theorems 2 and 3], Gabriel [**335**, Exposé VIIB, Section 2.5.2], Dieudonné [**250**, Section II.1, page 42]. Larson, Heyneman, Sweedler attribute this result to Kostant, while Dieudonné attributes it to Cartier and Gabriel.

For later references, see those by Montgomery [**703**, Corollary 5.6.4, item (3)], Andruskiewitsch [**32**, Theorem 1.1], Grünenfelder [**386**, Theorem 2.1], Cartier [**202**, Theorem 3.8.2], Khalkhali [**526**, Example 1.7.3], Etingof, Gelaki, Nikshych, Ostrik [**295**, Theorem 5.10.2], Lorenz [**612**, Section 10.4.3]. See also the historical note by Michaelis [**688**, Remark 3.42].

Dual PBW and dual CMM. There is a dual version of classical PBW and CMM involving Lie coalgebras and commutative bialgebras. These are analogues of Theorems 17.18 and 17.53, respectively. Early references to dual PBW are by André [**29**, Theorem 11] and Michaelis [**684**], [**685**, Theorem on page 52], [**686**, Theorem 3.8]. Early references to dual CMM are by André [**29**, Theorems 17 and 20], [**30**, Theorems 1 and 2], Sjödin [**830**, Theorem 2], Block [**121**, Theorem 4.9 and Corollary 5.4], [**122**, Theorems 1 and 2], Nichols [**719**, Chapter III, Theorems 12 and 14], [**720**, Theorems 11 and 14]. Dual CMM in the language of algebraic groups is given by Demazure and Gabriel [**242**, Section IV.2, Corollary 4.5] and Hochschild [**445**, Chapter XVI, Theorem 4.2]. The super case is given by Masuoka and Oka [**660**, Theorem 3.2]. Dual CMM is mentioned by Fresse [**322**, Section 4.2.6] and Loday [**608**, Section 3.6]. The analogue of Exercise 17.54 is given by Nichols [**719**, Chapter III, Theorem 13].

Joyal Lie monoids. PBW for Joyal species was proved by Joyal [**500**, Section 4.2, Theorem 2]. He deduces it from the classical PBW; he does not mention comonoids though. Lie monoids and Hopf monoids in Joyal species were studied by Stover [**854**] in the language of 'twisted Lie algebras' and 'twisted Hopf algebras'. The adjunction between $\mathcal{P}$ and $\mathcal{U}$ is given in [**854**, Proposition 7.10], while Borel–Hopf, PBW, CMM are given in [**854**, Proposition 12.2, Theorem 11.3, Theorem 8.4], respectively. These are analogues of Theorem 13.34, Theorem 17.9, Theorem 17.42, respectively. Interestingly, for Borel–Hopf and PBW, Stover implicitly works with set-theoretic noncommutative zeta functions. (He does not use this terminology, it was only introduced later in [**21**, Exercise 15.19].) In particular, he states Borel–Hopf and PBW only as an isomorphism of Joyal species (and not of Joyal comonoids).

A more recent exposition of Borel–Hopf, PBW, CMM for Joyal species is given in our monograph [**19**, Section 15, Theorems 118, 119, 120]. In this reference, the uniform noncommutative zeta function is employed (following the classical theory). Borel–Hopf is proved using characteristic operations by the corresponding eulerian idempotents (similar to what is done in Section 13.3.5). Information on these eulerian idempotents can be found in [**21**, Section 12.5.2]. The Solomon operator on $\mathcal{T}(\mathsf{g})$ for a positive Joyal Lie monoid g considered in Section 17.7.5 is new.

Work of Barratt, Joyal, Stover is mentioned in [**529**, page 273], [**530**, page 732]. More information on these references is given below.

Classical generalizations. PBW and CMM in any linear symmetric monoidal category is given by Fresse [**318**, Appendix], [**319**, Theorems 4.1.5 and 4.1.6], [**324**, Theorems 7.2.17 and 7.2.19]. The connection of the middle reference with Joyal species is elaborated in [**18**, Remark 11.47]. PBW in a linear symmetric monoidal category is also given by Deligne and Morgan [**239**, Section 1.3.7].

PBW for braided Lie algebras was obtained by Kharchenko [**528**, Theorems 5.2, 7.1, 7.2], [**529**, Theorem 7.4], [**530**, Theorems 4.2 and 4.3]. He also relates this result to PBW for quadratic algebras of Koszul type which is due to Braverman and Gaitsgory [**153**, Theorem 0.5], Polishchuk and Positselski [**754**, Chapter 5, Theorem 2.1]. An extension of Kharchenko's result to the nonsymmetric case is given by Ardizzoni [**44**, Theorem 5.4], [**45**, Theorems 3.11 and 5.2]. CMM for braided bialgebras is also present in work of Kharchenko [**528**, Theorem 6.1 and Lemma 6.2], see also the discussion by Ardizzoni and Menini [**46**, Theorem 8.1] (who work in a general categorical setup). For an extension to the nonsymmetric case, see [**47**, Theorem 5.5], [**44**, Corollary 5.5 and Theorem 5.7], [**43**, Theorems 6.8 and 6.9].

Dual CMM for braided bialgebras is given by Masuoka [**658**, Theorems 4.4 and 6.7] with related ideas in early work of Gurevich [**398**, Theorem 2].

A unification of the classical CMM, Loday–Ronco, Leray–Samelson theorems in the setting of generalized bialgebras is given by Loday [**608**, Theorems 2.3.7, 2.5.1, 2.6.3], see also the Notes to Chapter 13. In his notation, the triple relevant to CMM is (Com, As, Lie). Loday's result also includes other CMM type theorems present in the literature; a summary is given in [**608**, Section 6.3]. Livernet and Patras [**604**, Theorem 3.1.3] generalize [**854**, Proposition 7.10] of Stover.

A survey of some of the generalizations of PBW along with additional references is given by Grivel [**378**], Shepler and Witherspoon [**824**].

Lie monoids for hyperplane arrangements. PBW and CMM as well as the Solomon operator for arrangements are new and appear here for the first time. The approach to CMM in Section 17.5.6 via representation theory of the Tits algebra is also a novelty. It is an elaboration of the claim made in [**21**, Section 15.9.5].

How much of the classical generalizations of PBW and CMM mentioned above will go over to arrangements? This is a question for the next decade and beyond.

Appendices

APPENDIX A

Vector spaces

Let Vec denote the category of vector spaces over a field $\Bbbk$. It is an abelian category. We briefly review some standard material about this category which is relevant to us.

A.1. Kernel, cokernel, image, coimage

Let $f : V \to W$ be a linear map. Define

$$\ker(f) := \{v \in V \mid f(v) = 0\}.$$

This is a subspace of V. It is called the *kernel* of f. Similarly, define

$$\operatorname{im}(f) := \{w \in W \mid f(v) = w \text{ for some } v \in V\}.$$

This is a subspace of W. It is called the *image* of f. Dually, define

$$\operatorname{coker}(f) := W/\operatorname{im}(f).$$

This is a quotient space of W. It is called the *cokernel* of f. Similarly, define

$$\operatorname{coim}(f) := V/\ker(f).$$

This is a quotient space of V. It is called the *coimage* of f.

We have the following commutative diagram of vector spaces.

$$\begin{tikzcd} 0 \arrow[r] & \ker(f) \arrow[r, hook] & V \arrow[r, "f"] \arrow[d, two heads] & W \arrow[r, two heads] & \operatorname{coker}(f) \arrow[r] & 0 \\ & & \operatorname{coim}(f) \arrow[r, "\cong"'] & \operatorname{im}(f) \arrow[u, hook] & & \end{tikzcd} \tag{A.1}$$

A.2. Duality functor on vector spaces

For vector spaces V and W over a field $\Bbbk$, let $\operatorname{Hom}_{\Bbbk}(V, W)$ denote the space of all $\Bbbk$-linear maps from V to W. For a vector space V, define its *dual vector space* by

$$V^* := \operatorname{Hom}_{\Bbbk}(V, \Bbbk).$$

For any linear map $f : V \to W$, there is a dual linear map $f^* : W^* \to V^*$ which sends $g : W \to \Bbbk$ to $g \circ f : V \to \Bbbk$. This yields a functor

$$(-)^* : \mathsf{Vec} \to \mathsf{Vec}^{\mathrm{op}}, \tag{A.2}$$

which we call the *duality functor* on vector spaces.

Let $f : V \to W$ be a linear map and $f^* : W^* \to V^*$ its dual. Then

$$\operatorname{coker}(f^*) = \ker(f)^*, \quad \ker(f^*) = \operatorname{coker}(f)^*,$$
$$\operatorname{coim}(f^*) = \operatorname{im}(f)^*, \quad \operatorname{im}(f^*) = \operatorname{coim}(f)^*.$$

As a result, the dual of (A.1) is the same diagram for the dual map f^*. Also,

- $f = 0$ iff $f^* = 0$,
- f is injective iff f^* is surjective,
- f is surjective iff f^* is injective,
- f is bijective iff f^* is bijective.

The last property says that the duality functor (A.2) is conservative.

For any vector space W, there is a natural injective map $W \hookrightarrow W^{**}$. Moreover, for any vector spaces V and W, there is a natural bijection

$$\operatorname{Hom}_{\Bbbk}(V, W^*) \xrightarrow{\cong} \operatorname{Hom}_{\Bbbk}(W, V^*) \tag{A.3}$$

which sends $f : V \to W^*$ to the composite $W \hookrightarrow W^{**} \xrightarrow{f^*} V^*$. This says that the duality functor (A.2) is adjoint to itself.

A.3. Internal hom for the tensor product

Background information on internal hom is given in Appendix B.

For vector spaces V and W, let $V \otimes W$ denote their tensor product. For any vector spaces U, V, W, there is a natural bijection

$$\operatorname{Hom}_{\Bbbk}(U \otimes V, W) \cong \operatorname{Hom}_{\Bbbk}(U, \operatorname{Hom}_{\Bbbk}(V, W)).$$

Formally, one says that $\operatorname{Hom}_{\Bbbk}(V, W)$ is the internal hom for the tensor product on the category of vector spaces Vec. Since the internal hom exists, we say that the monoidal category $(\mathsf{Vec}, \otimes)$ is closed (Appendix B.2.3).

There is a canonical map of vector spaces

$$V^* \otimes W \to \operatorname{Hom}_{\Bbbk}(V, W)$$

which is an isomorphism if either V or W is finite-dimensional. We let

$$\operatorname{End}_{\Bbbk}(V) := \operatorname{Hom}_{\Bbbk}(V, V).$$

This is an algebra under composition. Further, for any algebra A, a morphism of algebras $A \to \operatorname{End}_{\Bbbk}(V)$ is the same as an A-module structure on V.

A.4. Linear maps between direct sums of vector spaces

We will often need to consider linear maps of the form

$$f : \bigoplus_i V_i \to \bigoplus_j W_j,$$

where both sums are finite. Since direct sum over a finite set is the product as well as the coproduct for vector spaces, such a map is equivalent to a family of linear maps $f_{ij} : V_i \to W_j$. We refer to the f_{ij} as the matrix-components of f. Important special cases arise as

$$f : \bigoplus_i V_i \to W \qquad \text{and} \qquad f : V \to \bigoplus_j W_j.$$

In these cases, we may write f as (f_i) and (f_j), respectively. We refer to the f_i or the f_j as the vector-components of f.

The matrix-components of a composite

$$\bigoplus_i U_i \xrightarrow{f} \bigoplus_j V_j \xrightarrow{g} \bigoplus_j W_k$$

are given by $(gf)_{ik} = \sum_j g_{jk} f_{ij}$.

The dual of f can be written as

$$f^* : \bigoplus_j W_j^* \to \bigoplus_i V_i^*.$$

We note that $(f^*)_{ji} = (f_{ij})^*$.

A.5. Idempotent operators

A.5.1. Idempotent operators. An *idempotent operator* on a vector space V is a linear map $e : V \to V$ such that $e^2 = e$. In this situation, we let $e \cdot V$ denote the image of e. We say a vector space W is a *retract* of $e : V \to V$ if there exist linear maps $p : V \to W$ and $i : W \to V$ such that $pi = \mathrm{id}_W$ and $ip = e$.

Lemma A.1. *Let V and W be vector spaces, and $p : V \to W$ and $i : W \to V$ be linear maps such that $pi = \mathrm{id}_W$. Put $e = ip : V \to V$. Then e is idempotent and there is an isomorphism $W \cong e \cdot V$ for which the following diagrams commute.*

$$\begin{tikzcd} & V \arrow[dl, "p"'] \arrow[dr, "e"] & \\ W \arrow[rr, "\cong"'] & & e \cdot V \end{tikzcd} \qquad\qquad \begin{tikzcd} & V & \\ W \arrow[ur, "i"] \arrow[rr, "\cong"'] & & e \cdot V \arrow[ul, hook'] \end{tikzcd} \tag{A.4}$$

Conversely, let $e : V \to V$ be idempotent. Put $W = e \cdot V$, and let $p : V \to W$ be induced by e and $i : W \to V$ be inclusion. Then $pi = \mathrm{id}_W$ and $ip = e$, that is, W is a retract of e.

Since every idempotent has a retract, we say that idempotents split in the category of vector spaces.

A.5.2. Rigidity of q-bimonoids for $q \neq \pm 1$. The following is an interesting fact from linear algebra which is relevant to the rigidity of $\mathcal{A}$-q-bimonoids in species for $q \neq \pm 1$, see Exercise 13.91.

Lemma A.2. *Suppose V is any vector space, and e and f are idempotent operators on V such that $efe = \alpha e$ and $fef = \alpha f$ for a scalar $\alpha \neq 1$. Then*

$$V = e \cdot V \oplus f \cdot V \oplus (\ker(e) \cap \ker(f)).$$

PROOF. Suppose $v \in e \cdot V \cap f \cdot V$. Then $e(v) = f(v) = v$. Hence, $v = efe(v) = \alpha e(v) = \alpha v$. Since $\alpha \neq 1$, we conclude that $v = 0$. Thus, $e \cdot V \cap f \cdot V = 0$. Now suppose

$$v \in (e \cdot V \oplus f \cdot V) \cap (\ker(e) \cap \ker(f)).$$

Write $v = y + z$ with $y \in e \cdot V$ and $z \in f \cdot V$. Applying e and f to v, we get $y + e(z) = 0 = f(y) + z$. Applying f to $y + e(z) = 0$, we obtain $f(y) + \alpha z = 0$.

Hence, $z = \alpha z$, and $z = 0$. By symmetry, we get $y = 0$. Thus, the sum of the subspaces $e \cdot V$, $f \cdot V$, $\ker(e) \cap \ker(f)$ is direct. We now need to show that it is all of V. Given $v \in V$, put

$$v' = v - \frac{1}{1-\alpha}(e(v) + f(v)) + \frac{1}{1-\alpha}(ef(v) + fe(v)).$$

Note that all terms in the rhs except v belong to $e \cdot V \oplus f \cdot V$. Now

$$e(v') = e(v) - \frac{1}{1-\alpha}(e(v) + ef(v)) + \frac{1}{1-\alpha}(ef(v) + \alpha e(v)) = 0.$$

Similarly, $f(v') = 0$. Thus, $v' \in \ker(e) \cap \ker(f)$. This completes the argument. $\square$

Remark A.3. How do we come up with the formula for v'? We can write $v' = v - v_e - v_f$ with $v_e \in e \cdot V$ and $v_f \in f \cdot V$. Applying e, f, ef and fe to the rhs yields 0. We can then solve for v_e and v_f.

Alternatively: We can approximate v' by

$$v - (e(v) + f(v)).$$

Applying e yields $-ef(v)$, while applying f yields $-fe(v)$. So we correct it to

$$v - (e(v) + f(v)) + (ef(v) + fe(v)).$$

Applying e yields $efe(v)$, while applying f yields $fef(v)$. So we correct it to

$$v - (e(v) + f(v)) + (ef(v) + fe(v)) - (efe(v) + fef(v)).$$

Continuing this procedure leads to an infinite series. Using the identities $efe = \alpha e$ and $fef = \alpha f$, we obtain

$$v - (1 + \alpha + \alpha^2 + \dots)(e(v) + f(v)) + (1 + \alpha + \alpha^2 + \dots)(ef(v) + fe(v)).$$

Compare this with v'.

APPENDIX B

Internal hom for monoidal categories

We review the notion of internal hom for a monoidal category. The discussion includes the endomorphism monoid, the convolution monoid, the internal hom for functor categories (which includes the category of modules over a monoid algebra). We also discuss the enriched counterpart of the tensor-hom adjunction, which gives rise to the notion of power and copower.

B.1. Monoidal and 2-monoidal categories

We set up the basic notation for monoidal categories, braided monoidal categories, 2-monoidal categories, left module categories over a monoidal category, and categories enriched over a monoidal category. For more information on monoidal categories and monoidal functors, see [**18**, Chapters 1 and 3], and on 2-monoidal categories, see [**18**, Chapter 6].

B.1.1. Monoidal category. A *monoidal category* is a triple $(\mathsf{C}, \bullet, I)$, where C is a category, $\bullet : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ is a functor, I is a distinguished object in C together with natural isomorphisms

$$(A \bullet B) \bullet C \to A \bullet (B \bullet C), \quad A \to I \bullet A, \quad A \to A \bullet I$$

which satisfy the pentagon and triangle axioms [**18**, Definition 1.1]. We refer to $\bullet$ as the monoidal structure or the tensor product, and to I as the unit object. It is also convenient to write $(\mathsf{C}, \bullet)$, keeping the unit object implicit.

The *reverse tensor product* $A \,\tilde{\bullet}\, B := B \bullet A$ endows C with another monoidal structure. The opposite category C^{op} is monoidal under either $\bullet$ or $\tilde{\bullet}$. The same object I serves as the unit for all four monoidal categories $(\mathsf{C}, \bullet)$, $(\mathsf{C}, \tilde{\bullet})$, $(\mathsf{C}^{\mathrm{op}}, \bullet)$, $(\mathsf{C}^{\mathrm{op}}, \tilde{\bullet})$.

Between monoidal categories, one can define *lax monoidal functor*s, *colax monoidal functor*s and *strong monoidal functor*s [**18**, Section 3.1].

B.1.2. Braided monoidal category. A *braided monoidal category* is a monoidal category $(\mathsf{C}, \bullet)$ together with a natural isomorphism

$$\beta_{A,B} : A \bullet B \to B \bullet A, \tag{B.1}$$

called the *braiding*, subject to certain axioms [**18**, Definition 1.2]. We write $(\mathsf{C}, \bullet, \beta)$ for a braided monoidal category. It is *symmetric* if β satisfies $\beta^2 = \mathrm{id}$. In this case, β is called a *symmetry*, and $(\mathsf{C}, \bullet, \beta)$ is called a *symmetric monoidal category*.

When $(\mathsf{C}, \bullet)$ is braided, $(\mathsf{C}, \bullet)$ and $(\mathsf{C}, \tilde{\bullet})$ are isomorphic monoidal categories.

We also mention a more general notion of a *lax braided monoidal category* in which (B.1) is not required to be an isomorphism [**18**, Definition 1.5]. In this case, β is called a *lax braiding*.

B.1.3. 2-monoidal category. A 2-*monoidal category* consists of a five-tuple $(\mathsf{C}, \diamond, I, \star, J)$, where $(\mathsf{C}, \diamond, I)$ and $(\mathsf{C}, \star, J)$ are monoidal categories with units I and J, respectively, along with a transformation called the *interchange law*

$$\zeta_{A,B,C,D} : (A \star B) \diamond (C \star D) \to (A \diamond C) \star (B \diamond D) \tag{B.2}$$

which is natural in A, B, C, D, and three morphisms

$$\Delta_I : I \to I \star I, \qquad \mu_J : J \diamond J \to J, \qquad \iota_J = \epsilon_I : I \to J, \tag{B.3}$$

satisfying appropriate associativity and unitality axioms [**18**, Definition 6.1]. Just as for a monoidal category, it is convenient to write $(\mathsf{C}, \diamond, \star)$, keeping the unit objects implicit.

Example B.1. Suppose $(\mathsf{C}, \bullet, \beta)$ is a braided monoidal category. This gives rise to a 2-monoidal category in which both monoidal structures equal $\bullet$. The interchange law (B.2) is constructed by applying the braiding β in (B.1) on the middle two tensor factors.

B.1.4. Module category over a monoidal category. Let $(\mathsf{C}, \bullet, I)$ be a monoidal category. A *left module category* over C is a category M together with a strong monoidal functor

$$\mathsf{C} \to \mathrm{End}(\mathsf{M}), \quad A \mapsto A \bullet (-).$$

Here $\mathrm{End}(\mathsf{M})$ denotes the monoidal category of endofunctors $\mathsf{M} \to \mathsf{M}$. The tensor product is composition $((\mathcal{F} \bullet \mathcal{G})(X) := \mathcal{F}(\mathcal{G}(X)))$ and the unit is the identity functor. This data is equivalent to that of a functor

$$\mathsf{C} \times \mathsf{M} \to \mathsf{M}, \qquad (A, X) \mapsto A \bullet X,$$

together with natural isomorphisms

$$(A \bullet B) \bullet X \to A \bullet (B \bullet X) \quad \text{and} \quad I \bullet X \to X$$

which satisfy the pentagon and triangle axioms as in the definition of a monoidal category. We refer to the above functor as the left action of C on M and denote the module category by $(\mathsf{M}, \bullet)$.

Exercise B.2. Let M be a category. Check that $\mathrm{End}(\mathsf{M})^{\mathrm{op}} = \mathrm{End}(\mathsf{M}^{\mathrm{op}})$ as monoidal categories. Deduce that a left action of $(\mathsf{C}^{\mathrm{op}}, \bullet)$ on M is the same as a left action of $(\mathsf{C}, \bullet)$ on M^{op}.

A *right module category* over $(\mathsf{C}, \bullet, I)$ is a left module category over $(\mathsf{C}, \tilde{\bullet}, I)$. This data is equivalent to that of a functor

$$\mathsf{M} \times \mathsf{C} \to \mathsf{M}, \qquad (X, A) \mapsto X \bullet A,$$

together with natural isomorphisms

$$X \bullet (A \bullet B) \to (X \bullet A) \bullet B \quad \text{and} \quad X \bullet I \to X,$$

satisfying appropriate axioms.

B.1.5. Enriched categories. Let $(\mathsf{C}, \bullet, I)$ be a monoidal category. A *category enriched over* C or more simply a C-*category*, denoted M, consists of the following data. A class of objects; for each pair of objects X, Y, an object $\mathsf{M}(X, Y)$ in C; for each triple of objects X, Y, Z, a morphism

$$\mathsf{M}(Y, Z) \bullet \mathsf{M}(X, Y) \to \mathsf{M}(X, Z)$$

in C; for each object X, a morphism $I \to \mathsf{M}(X, X)$ in C. These are subject to associativity and unitality axioms [**523**, Section 1.2]. We refer to the $\mathsf{M}(X, Y)$ as the hom-objects of the C-category.

The *underlying category* of a C-category M is the ordinary category that has the same objects as M, and for which the set of morphisms from X to Y is $\mathsf{C}(I, \mathsf{M}(X, Y))$. We do not distinguish notationally between M and its underlying category.

Exercise B.3. Suppose M is enriched over $(\mathsf{C}, \bullet, I)$. Check that the opposite category M^{op} is enriched over $(\mathsf{C}, \tilde{\bullet}, I)$.

B.2. Internal hom

We discuss internal hom for monoidal categories, and the related notions of endomorphism monoid and convolution monoid. The latter is also defined in the more general setting of 2-monoidal categories. We also consider enriched hom for left module categories.

B.2.1. Internal hom for a monoidal category. Let $(\mathsf{C}, \bullet)$ be a monoidal category. An *internal hom* for $(\mathsf{C}, \bullet)$ is a functor

$$\mathrm{hom}^{\bullet} : \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{C}$$

such that for any objects A, B, C in C, there is a natural bijection

$$\mathsf{C}(A \bullet B, C) \cong \mathsf{C}(A, \mathrm{hom}^{\bullet}(B, C)).$$

We call this the internal hom property.

By setting $A = \mathrm{hom}^{\bullet}(B, C)$, the identity morphism in the rhs yields a morphism

$$\mathrm{hom}^{\bullet}(A, B) \bullet A \to B. \tag{B.4}$$

Multiplying on the left by $\mathrm{hom}^{\bullet}(B, C)$ and composing by (B.4) (with B and C instead of A and B), we obtain

$$\mathrm{hom}^{\bullet}(B, C) \bullet \mathrm{hom}^{\bullet}(A, B) \bullet A \to C.$$

The internal hom property yields

$$\mathrm{hom}^{\bullet}(B, C) \bullet \mathrm{hom}^{\bullet}(A, B) \to \mathrm{hom}^{\bullet}(A, C). \tag{B.5}$$

Similarly, the unit constraint $I \bullet A \to A$ yields

$$I \to \mathrm{hom}^{\bullet}(A, A). \tag{B.6}$$

Proposition B.4. *For any object M in C,*

$$\mathrm{end}^{\bullet}(M) := \mathrm{hom}^{\bullet}(M, M)$$

is a monoid in $(\mathsf{C}, \bullet)$. If A is a monoid, then a left A-module structure on an object M is equivalent to a morphism of monoids

$$A \to \mathrm{end}^{\bullet}(M).$$

We call $\mathrm{end}^{\bullet}(M)$ the *endomorphism monoid*.

PROOF. By letting $A = B = C = M$ in (B.5) and (B.6), we obtain

$$\mathrm{end}^{\bullet}(M) \bullet \mathrm{end}^{\bullet}(M) \to \mathrm{end}^{\bullet}(M) \quad \text{and} \quad I \to \mathrm{end}^{\bullet}(M).$$

These turn $\mathrm{end}^{\bullet}(M)$ into a monoid. The internal hom property yields a correspondence

$$A \bullet M \to M \quad \longleftrightarrow \quad A \to \mathrm{end}^{\bullet}(M).$$

The second claim can be verified using this correspondence. □

We mention in passing that there is also a dual notion of an internal cohom which is defined by the property

$$\mathsf{C}(\mathrm{cohom}^{\bullet}(A, B), C) \cong \mathsf{C}(A, B \bullet C).$$

B.2.2. Comparison between internal homs. Suppose C and D are monoidal categories in which internal homs exist. If $\mathcal{F} : (\mathsf{C}, \bullet) \to (\mathsf{D}, \bullet)$ is a lax monoidal functor, then for any objects A and B in C, there is a morphism

$$\mathcal{F}(\mathrm{hom}^{\bullet}(A, B)) \to \mathrm{hom}^{\bullet}(\mathcal{F}(A), \mathcal{F}(B)). \tag{B.7}$$

To see this, consider the composite

$$\mathcal{F}(\mathrm{hom}^{\bullet}(A, B)) \bullet \mathcal{F}(A) \to \mathcal{F}(\mathrm{hom}^{\bullet}(A, B) \bullet A) \to \mathcal{F}(B).$$

The first map arises from the lax structure of $\mathcal{F}$, and the second from (B.4). Now apply the internal hom property in D to this composite to obtain (B.7).

B.2.3. Closed monoidal categories. A monoidal category $(\mathsf{C}, \bullet, I)$ is *left closed* if for each object B, the functor $(-) \bullet B$ has a right adjoint. This is equivalent to the existence of an internal hom functor, with the right adjoint being $\mathrm{hom}^{\bullet}(B, -)$. A monoidal category $(\mathsf{C}, \bullet, I)$ is *right closed* if each functor $B \bullet (-)$ has a right adjoint. A monoidal category is *biclosed* if it is both left and right closed. When distinction between the right adjoints of $(-) \bullet B$ and of $B \bullet (-)$ is warranted, one speaks of the left and right internal hom functors, respectively.

Note that $(\mathsf{C}, \bullet, I)$ is right closed iff $(\mathsf{C}, \tilde{\bullet}, I)$ is left closed, and the right internal hom for the former coincides with the left internal hom for the latter. This is denoted $\mathrm{hom}^{\tilde{\bullet}}$.

For a symmetric monoidal category, the properties of being left closed, right closed, and biclosed, are equivalent. If they hold, we say that the symmetric monoidal category is *closed*, and make no distinction between the left and right internal homs. In this case, we have

$$\mathrm{hom}^{\bullet}(A, \mathrm{hom}^{\bullet}(B, C)) \cong \mathrm{hom}^{\bullet}(A \bullet B, C) \cong \mathrm{hom}^{\bullet}(B, \mathrm{hom}^{\bullet}(A, C)). \tag{B.8}$$

When the monoidal category C is left closed, it is enriched over itself. More precisely, there is a C-category with the same objects as C and with hom-objects $\mathrm{hom}^\bullet(A, B)$. Since $\mathsf{C}(I, \mathrm{hom}^\bullet(A, B)) \cong \mathsf{C}(A, B)$, the underlying category is C itself.

When C is right closed, it is enriched over $(\mathsf{C}, \tilde{\bullet})$.

B.2.4. Enriched hom for a left module category. Let $(\mathsf{M}, \bullet)$ be a left module category over a monoidal category $(\mathsf{C}, \bullet, I)$. An *enriched hom* for M is a functor

$$\mathrm{hom}^\bullet : \mathsf{M}^{\mathrm{op}} \times \mathsf{M} \to \mathsf{C}$$

such that for any objects A in C, and X and Y in M, there is a natural bijection

$$\mathsf{M}(A \bullet X, Y) \cong \mathsf{C}(A, \mathrm{hom}^\bullet(X, Y)). \tag{B.9}$$

When the enriched hom exists, the category M is enriched over C. More precisely, there is a C-category with the same objects as M and with hom-objects $\mathrm{hom}^\bullet(X, Y)$. Since $\mathsf{C}(I, \mathrm{hom}^\bullet(X, Y)) \cong \mathsf{M}(X, Y)$, the underlying category is M itself.

Proposition B.4 generalizes to this setting. Thus:

Proposition B.5. *For any object M in M,*

$$\mathrm{end}^\bullet(M) := \mathrm{hom}^\bullet(M, M)$$

is a monoid in $(\mathsf{C}, \bullet)$. If A is a monoid, then a left A-module structure on an object M is equivalent to a morphism of monoids

$$A \to \mathrm{end}^\bullet(M).$$

B.2.5. Internal hom in a 2-monoidal category. Let $(\mathsf{C}, \diamond, I, \star, J)$ be a 2-monoidal category. Suppose that the internal hom for $\diamond$ exists. Applying the interchange law (B.2) and using (B.4) twice, we obtain the composite

$$\begin{aligned}&(\mathrm{hom}^\diamond(A, A') \star \mathrm{hom}^\diamond(B, B')) \diamond (A \star B)\\ &\qquad\qquad \to (\mathrm{hom}^\diamond(A, A') \diamond A) \star (\mathrm{hom}^\diamond(B, B') \diamond B) \to A' \star B'.\end{aligned}$$

The internal hom property yields

$$\mathrm{hom}^\diamond(A, A') \star \mathrm{hom}^\diamond(B, B') \to \mathrm{hom}^\diamond(A \star B, A' \star B'). \tag{B.10}$$

Similarly, the map $J \diamond J \to J$ yields

$$J \to \mathrm{hom}^\diamond(J, J). \tag{B.11}$$

Proposition B.6. *The internal hom for $\diamond$ is a lax monoidal functor wrt $\star$ with structure maps* (B.10) *and* (B.11)*.*

Recall that a lax monoidal functor preserves monoids. Moreover, a monoid in $\mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ wrt $\star$ is the same as a pair (C, A) consisting of a comonoid (C, Δ, ϵ) and a monoid (A, μ, ι) both wrt $\star$. As a consequence:

Proposition B.7. *Let $(\mathsf{C}, \diamond, I, \star, J)$ be a 2-monoidal category. For a comonoid C and a monoid A wrt $\star$, $\mathrm{hom}^\diamond(C, A)$ is a monoid wrt $\star$.*

This is the *convolution monoid*. The product is given by

$$\hom^{\diamond}(C,A)\star\hom^{\diamond}(C,A)\to\hom^{\diamond}(C\star C,A\star A)\xrightarrow{(\Delta,\mu)}\hom^{\diamond}(C,A) \tag{B.12}$$

and the unit by

$$J\to\hom^{\diamond}(J,J)\xrightarrow{(\epsilon,\iota)}\hom^{\diamond}(C,A). \tag{B.13}$$

Note how the lax structures get used in these descriptions.

Proposition B.8. *For a monoid A wrt $\star$, $\hom^{\diamond}(-,A)$ is a lax monoidal functor from C^{op} to C. Similarly, for a comonoid C wrt $\star$, $\hom^{\diamond}(C,-)$ is a lax monoidal functor from C to C.*

Proof. Since A is a monoid, the functor

$$\mathsf{C}^{\mathrm{op}}\to\mathsf{C}^{\mathrm{op}}\times\mathsf{C},\qquad B\mapsto(B,A)$$

is lax monoidal. Hence, composing with the internal hom yields a lax monoidal functor $\hom^{\diamond}(-,A)$. The second statement is proved similarly. $\square$

The existence of the convolution monoid can also be deduced from this result.

Let $(\mathsf{C},\bullet,\beta)$ be a braided monoidal category, and view it as a 2-monoidal category as in Example B.1. The above discussion then specializes as follows. Applying the braiding on the middle two factors and using (B.4) twice, we obtain

$$\begin{aligned}&\hom^{\bullet}(A,A')\bullet\hom^{\bullet}(B,B')\bullet A\bullet B\\&\qquad\to\hom^{\bullet}(A,A')\bullet A\bullet\hom^{\bullet}(B,B')\bullet B\to A'\bullet B'.\end{aligned}$$

The internal hom property yields

$$\hom^{\bullet}(A,A')\bullet\hom^{\bullet}(B,B')\to\hom^{\bullet}(A\bullet B,A'\bullet B'). \tag{B.14}$$

Specializing (B.6) yields $I\to\hom^{\bullet}(I,I)$. These equip the internal hom with a lax structure, and we deduce:

Proposition B.9. *Let $(\mathsf{C},\bullet,\beta)$ be a braided monoidal category. For a comonoid C and a monoid A, $\hom^{\bullet}(C,A)$ is a monoid.*

B.3. Powers and copowers

The tensor-hom adjunction in a closed monoidal category has an enriched counterpart, giving rise to the notion of power and copower. Given a category M enriched over a closed monoidal category C, an object X in M and an object A in C, the power $X\lhd A$ and the copower $A\rhd X$ are objects of M whose definitions are reviewed below.

B.3.1. Powers and copowers. Let $(\mathsf{C}, \bullet, I)$ be a monoidal category. Let M be a category enriched over C.

Assume first that C is left closed and let $\mathrm{hom}^{\bullet}$ be the left internal hom. For X an object in M and A an object in C, a *copower* of X by A is an object $A \triangleright X$ in M along with a natural isomorphism

$$\mathsf{M}(A \triangleright X, Y) \cong \mathrm{hom}^{\bullet}(A, \mathsf{M}(X, Y)) \tag{B.15}$$

in the category C (of the variable Y). When copowers exist for all X and A, we say that M is *copowered* by C.

Assume now that C is right closed and let $\mathrm{hom}^{\tilde{\bullet}}$ be the right internal hom. For Y an object in M and A an object in C, a *power* of Y by A is an object $Y \triangleleft A$ in M along with a natural isomorphism

$$\mathsf{M}(X, Y \triangleleft A) \cong \mathrm{hom}^{\tilde{\bullet}}(A, \mathsf{M}(X, Y)) \tag{B.16}$$

in the category C (of the variable X). When powers exist for all Y and A, we say that M is *powered* by C.

When C is symmetric and closed, powers and copowers satisfy

$$\mathsf{M}(A \triangleright X, Y) \cong \mathrm{hom}^{\bullet}(A, \mathsf{M}(X, Y)) \cong \mathsf{M}(X, Y \triangleleft A). \tag{B.17}$$

Example B.10. Let $(\mathsf{C}, \bullet)$ be a left closed monoidal category. View it as a category enriched over itself (via the left internal hom). Then C is copowered by itself: the copower of X by A is $A \bullet X$.

If in fact C is symmetric (and closed), then it is also powered by itself: the power of Y by A is $\mathrm{hom}^{\bullet}(A, Y)$. These assertions follow by comparing (B.8) and (B.17).

Exercise B.11. Let $(\mathsf{C}, \bullet, I)$ be a monoidal category and M a category enriched over it. Suppose $(\mathsf{C}, \bullet, I)$ is left closed. Then check that M is copowered by C iff M^{op} is powered by $(\mathsf{C}, \tilde{\bullet}, I)$. See Exercise B.3.

B.3.2. (Co)powers by sets. An ordinary category M is enriched over the symmetric monoidal category of sets under cartesian product. Let X and Y be objects of M and A a set. When M has products, the power of Y by A is Y^A, the product of A copies of Y. When M has coproducts, the copower of X by A is $A \cdot X$, the coproduct of A copies of X.

Example B.12. If M is itself the category of sets, then Y^A is the set of functions from A to Y, and $A \cdot X$ is the cartesian product $A \times X$.

For a different example, let M be the category of monoids. The power of Y by A is the monoid Y^A of functions from A to Y under pointwise product. The copower of X by A is the quotient of the free monoid on $A \times (X \setminus \{1\})$ by the relations

$$((a, x_1), \ldots, (a, x_n)) \equiv (a, x_1 \cdots x_n))$$

for $a \in A$ and $x_1, \ldots, x_n \in X \setminus \{1\}$.

B.3.3. Copowers and left actions. We extend part of Example B.10 to the setting of module categories.

Let C be a left closed monoidal category. Let M be a category enriched over C. Suppose M is copowered by C. Then M is a left module category over C under the action by copowers

$$\mathsf{C} \times \mathsf{M} \to \mathsf{M}, \qquad (A, X) \mapsto A \triangleright X.$$

Moreover, the enrichment over C defines an enriched hom for M, in the sense of Appendix B.2.4:

$$\mathrm{hom}^{\triangleright}(X, Y) := \mathsf{M}(X, Y).$$

The adjunction (B.9) follows by composing (B.15) with the functor $\mathsf{C}(I, -)$.

Conversely, let C be a monoidal category, and let M be a left module category over C with an enriched hom. Then M is enriched over C via the enriched hom. If in addition C is left closed, then M is copowered by C by means of the left action.

For details, see [**483**, Section 2], [**475**, Proposition 3.1], [**780**, Lemma 3.7.7 and Proposition 10.1.4]. A more general result is given in [**360**, Theorem 3.7].

Exercise B.13. Let $(\mathsf{C}, \bullet)$ be a right closed monoidal category. Let M be a category enriched over C and powered by C. Use Exercises B.2 and B.11 to show that there is a right action of $(\mathsf{C}^{\mathrm{op}}, \bullet)$ on M given by

$$\mathsf{M} \times \mathsf{C}^{\mathrm{op}} \to \mathsf{M}, \quad (X, A) \mapsto X \triangleleft A.$$

B.4. Internal hom for functor categories

Let $(\mathsf{D}, \bullet, I)$ be a monoidal category. Let $\mathcal{F}$ and $\mathcal{G}$ be two functors from C to D. Define a new functor $\mathcal{F} \bullet \mathcal{G}$ by

$$(\mathcal{F} \bullet \mathcal{G})(A) := \mathcal{F}(A) \bullet \mathcal{G}(A). \tag{B.18}$$

For any morphism $f : A \to B$, there is an induced morphism

$$\mathcal{F}(A) \bullet \mathcal{G}(A) \xrightarrow{f \bullet f} \mathcal{F}(B) \bullet \mathcal{G}(B),$$

so $\mathcal{F}\bullet\mathcal{G}$ is indeed a functor from C to D. Further, given natural transformations $\mathcal{F} \to \mathcal{F}'$ and $\mathcal{G} \to \mathcal{G}'$, there is an induced natural transformation $\mathcal{F} \bullet \mathcal{G} \to \mathcal{F}' \bullet \mathcal{G}'$. This yields a monoidal structure on $[\mathsf{C}, \mathsf{D}]$, the category of functors from C to D. The unit object is the functor which is I on all objects in C.

Any functor $\alpha : \mathsf{D} \to \mathsf{E}$ induces a functor

$$[\mathsf{C}, \mathsf{D}] \to [\mathsf{C}, \mathsf{E}], \qquad \mathcal{F} \mapsto \alpha(\mathcal{F}), \tag{B.19}$$

where $\alpha(\mathcal{F})$ denotes the composite of the functors α and $\mathcal{F}$. Further, if $\alpha : (\mathsf{D}, \bullet) \to (\mathsf{E}, \bullet)$ is a lax monoidal functor, then so is the induced functor.

Theorem B.14. *Let* C *be a small category and* $(\mathsf{D}, \bullet)$ *a complete left (right) closed monoidal category. Then, under the monoidal structure* (B.18)*, the functor category* $[\mathsf{C}, \mathsf{D}]$ *is left (right) closed as well.*

Proof. Let us work in the left closed setting. For objects B, C of D, let $\hom^\bullet(B, C)$ denote their left internal hom in D. Now let $\mathcal{F}$, $\mathcal{G}$ be any functors from C to D. We claim that their left internal hom in $[\mathsf{C}, \mathsf{D}]$ exists. Let us denote it by $\hom^\bullet(\mathcal{F}, \mathcal{G})$. Explicitly, for an object A of C, $\hom^\bullet(\mathcal{F}, \mathcal{G})(A)$ is defined to be the universal object in D with the following properties. For any morphism $f : A \to X$ in C, there is a morphism in D

$$\eta_f : \hom^\bullet(\mathcal{F}, \mathcal{G})(A) \to \hom^\bullet(\mathcal{F}(X), \mathcal{G}(X)) \tag{B.20}$$

such that for any $f : A \to X$ and $g : X \to Y$ in C, the diagram

$$\begin{tikzcd} \hom^\bullet(\mathcal{F}, \mathcal{G})(A) \arrow[r, "\eta_f"] \arrow[d, "\eta_{gf}"'] & \hom^\bullet(\mathcal{F}(X), \mathcal{G}(X)) \arrow[d] \\ \hom^\bullet(\mathcal{F}(Y), \mathcal{G}(Y)) \arrow[r] & \hom^\bullet(\mathcal{F}(X), \mathcal{G}(Y)) \end{tikzcd} \tag{B.21}$$

commutes. The unlabeled arrows above are induced by g. For any $A \to B$ in C, there is an induced morphism

$$\hom^\bullet(\mathcal{F}, \mathcal{G})(A) \to \hom^\bullet(\mathcal{F}, \mathcal{G})(B)$$

arising from the universal property of the latter.

Let us now verify the internal hom property, that is, for functors $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$,

$$[\mathsf{C}, \mathsf{D}](\mathcal{F} \bullet \mathcal{G}, \mathcal{H}) \cong [\mathsf{C}, \mathsf{D}](\mathcal{F}, \hom^\bullet(\mathcal{G}, \mathcal{H})).$$

Suppose we have a natural transformation from $\mathcal{F} \bullet \mathcal{G}$ to $\mathcal{H}$. Thus, for each object A in C, there is a morphism $h_A : \mathcal{F}(A) \bullet \mathcal{G}(A) \to \mathcal{H}(A)$ such that for any morphism $f : A \to B$ in C, the diagram

$$\begin{tikzcd} \mathcal{F}(A) \bullet \mathcal{G}(A) \arrow[r, "h_A"] \arrow[d, "f \bullet f"'] & \mathcal{H}(A) \arrow[d, "f"] \\ \mathcal{F}(B) \bullet \mathcal{G}(B) \arrow[r, "h_B"'] & \mathcal{H}(B) \end{tikzcd}$$

commutes.

This data may equivalently be expressed as follows. For each object A in C, there is a morphism $h_A : \mathcal{F}(A) \to \hom^\bullet(\mathcal{G}(A), \mathcal{H}(A))$ such that for any morphism $f : A \to B$ in C, the diagram

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, "h_A"] \arrow[dd, "\mathcal{F}(f)"'] & \hom^\bullet(\mathcal{G}(A), \mathcal{H}(A)) \arrow[dr] & \\ & & \hom^\bullet(\mathcal{G}(A), \mathcal{H}(B)) \\ \mathcal{F}(B) \arrow[r, "h_B"'] & \hom^\bullet(\mathcal{G}(B), \mathcal{H}(B)) \arrow[ur] & \end{tikzcd}$$

commutes. The unlabeled arrows above are induced by f.

Now suppose we have a natural transformation from $\mathcal{F}$ to $\hom^\bullet(\mathcal{G}, \mathcal{H})$. Thus, for each morphism $f : A \to X$ in C, there is a morphism

$$\alpha_f : \mathcal{F}(A) \to \hom^\bullet(\mathcal{G}(X), \mathcal{H}(X))$$

such that for any $f : A \to X$ and $g : X \to Y$ in C, the diagram

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\alpha_f"] \arrow[d, "\alpha_{gf}"'] & \hom^\bullet(\mathcal{G}(X), \mathcal{H}(X)) \arrow[d] \\
\hom^\bullet(\mathcal{G}(Y), \mathcal{H}(Y)) \arrow[r] & \hom^\bullet(\mathcal{G}(X), \mathcal{H}(Y))
\end{tikzcd}$$

commutes (with the unlabeled arrows induced from g), and for any $f : A \to B$ and $g : B \to X$, the diagram

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[dr, "\alpha_{gf}"] \arrow[dd, "\mathcal{F}(f)"'] & \\
& \hom^\bullet(\mathcal{G}(X), \mathcal{H}(X)) \\
\mathcal{F}(B) \arrow[ur, "\alpha_g"'] &
\end{tikzcd}$$

commutes. The second diagram shows that all the α_f are determined by α_{id}. By relabeling α_{id_A} by h_A, we see that this data is equivalent to the one obtained earlier. □

The internal hom in the functor category $[\mathsf{C}, \mathsf{D}]$ admits the following description as an end of a functor:

$$\hom^\bullet(\mathcal{F}, \mathcal{G})(A) = \int_B \hom^\bullet(\mathcal{F}(B), \mathcal{G}(B))^{\mathsf{C}(A,B)}. \tag{B.22}$$

The rhs involves the power of the object $\hom^\bullet(\mathcal{F}(B), \mathcal{G}(B))$ of D by the set $\mathsf{C}(A, B)$, and the end of the functor

$$\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{D}, \quad (X, Y) \mapsto \hom^\bullet(\mathcal{F}(X), \mathcal{G}(Y))^{\mathsf{C}(A,X)}. \tag{B.23}$$

Completeness of D guarantees the existence of powers and ends. For background on ends, see [**623**, Section IX.5].

We explain how formula (B.22) may be derived from the proof of Theorem B.14. For fixed A, $\mathcal{F}$, $\mathcal{G}$, the collection

$$\eta_f : \hom^\bullet(\mathcal{F}, \mathcal{G})(A) \to \hom^\bullet(\mathcal{F}(X), \mathcal{G}(X))$$

(indexed by $f : A \to X$ as in (B.20)) is equivalent to a collection

$$\hom^\bullet(\mathcal{F}, \mathcal{G})(A) \to \hom^\bullet(\mathcal{F}(X), \mathcal{G}(X))^{\mathsf{C}(A,X)},$$

indexed only by X, in view of the universal property of powers. Condition (B.21) in the proof corresponds to dinaturality of this collection. Hence, formula (B.23) holds.

The following exercise provides an alternative description for the internal hom in $[\mathsf{C}, \mathsf{D}]$. It requires additional hypotheses.

Exercise B.15. Assume that D is complete, cocomplete, and biclosed. Check that the internal hom in $[\mathsf{C}, \mathsf{D}]$ is also given by

$$\hom^\bullet(\mathcal{F}, \mathcal{G})(A) = \int_B \hom^\bullet(\mathsf{C}(A, B) \cdot \mathcal{F}(B), \mathcal{G}(B)), \tag{B.24}$$

where $S \cdot D$ denotes the copower of an object D of D by a set S. (Check that

$$\hom^\bullet(D, E^S) \cong \hom^\bullet(S \cdot D, E),$$

for any set S and objects D and E in D.)

Exercise B.16. Another description of the internal hom in $[\mathsf{C}, \mathsf{D}]$ is as follows. Fix an object A in C, and functors $\mathcal{F}, \mathcal{G} : \mathsf{C} \to \mathsf{D}$. Consider the comma category $A \uparrow \mathsf{C}$. An object is a morphism $f : A \to X$ in C. There is a functor

$$(A \uparrow \mathsf{C})^{\mathrm{op}} \times (A \uparrow \mathsf{C}) \to \mathsf{D}, \qquad (f_1, f_2) \mapsto \hom^\bullet(\mathcal{F}(X_1), \mathcal{G}(X_2)),$$

where $f_1 : A \to X_1$ and $f_2 : A \to X_2$. Show that $\hom^\bullet(\mathcal{F}, \mathcal{G})(A)$ is the end of this functor.

B.5. Modules over a monoid algebra

Fix a monoid X. Its linearization over a field $\Bbbk$ is an algebra denoted $\Bbbk X$. In fact, $\Bbbk X$ is a bialgebra whose coproduct is defined by linearizing the diagonal map $X \to X \times X$ which sends x to (x, x). Hence, the category of left $\Bbbk X$-modules is monoidal under the tensor product. We construct the internal hom for this monoidal structure beginning with the simpler case when X is a group, that is, when $\Bbbk X$ is a Hopf algebra. This can be viewed as a special case of the construction described in Section B.4.

B.5.1. Tensor product. Let X be a monoid. We use letters x, y, z, w to denote elements of X, and e to denote its identity element. The linearization of X, denoted $\Bbbk X$, is a $\Bbbk$-algebra.

Let $\Bbbk X\text{-}\mathsf{Mod}$ denote the category of left $\Bbbk X$-modules. Suppose M and N are left $\Bbbk X$-modules. Then their tensor product over $\Bbbk$, denoted $M \otimes N$, is a left $\Bbbk X$-module under the diagonal action:

$$x \cdot (m \otimes n) := (x \cdot m) \otimes (x \cdot n).$$

(The symbol $\cdot$ is used for the action on all three modules.) The tensor product is a monoidal structure for the category of left $\Bbbk X$-modules. The unit object is the base field $\Bbbk$ with the trivial action: $x \cdot 1 := 1$.

B.5.2. Internal hom for modules over a group algebra. Assume first that X is a group. Suppose M and N are left $\Bbbk X$-modules. Then $\mathrm{Hom}_\Bbbk(M, N)$ is a left $\Bbbk X$-module with the action given by

$$(x \cdot f)(m) := x \cdot f(x^{-1} \cdot m) \tag{B.25}$$

for $x \in X$, $f \in \mathrm{Hom}_\Bbbk(M, N)$, $m \in M$.

Proposition B.17. *For a group X and left $\Bbbk X$-modules M, N, P, there is a natural bijection*

$$\Bbbk X\text{-}\mathsf{Mod}(P \otimes M, N) \xrightarrow{\cong} \Bbbk X\text{-}\mathsf{Mod}(P, \mathrm{Hom}_\Bbbk(M, N)).$$

Proof. Recall: A linear map $\alpha : P \otimes M \to N$ corresponds to a linear map $\beta : P \to \mathrm{Hom}_\Bbbk(M, N)$. Now suppose that α is a map of left $\Bbbk X$-modules. Then check that the corresponding β satisfies the condition

$$\beta(x \cdot p)(x \cdot m) = x \cdot (\beta(p)(m))$$

for all $p \in P$, $m \in M$, $x \in X$. (Observe that all three actions are involved here.) This condition can be rephrased as

$$\beta(x \cdot p)(m) = x \cdot (\beta(p)(x^{-1} \cdot m)).$$

But this is the same as saying that β is a map of left $\Bbbk X$-modules. The result follows. $\square$

B.5.3. Internal hom for modules over a monoid algebra. Now we tackle the general case. So X is a monoid. Suppose M and N are left $\Bbbk X$-modules. Put

$$\hom^{\otimes}(M, N) := \mathrm{Hom}_{\Bbbk X}(M \otimes \Bbbk X, N). \tag{B.26}$$

In the rhs, $M \otimes \Bbbk X$ is a left $\Bbbk X$-module under the diagonal action. In fact, it is a $\Bbbk X$-bimodule, with the right action induced by the right action on the factor $\Bbbk X$. As a consequence, (B.26) is a left $\Bbbk X$-module.

Let us describe this module more explicitly. An element $a \in \hom^{\otimes}(M, N)$ is a family (a_w) indexed by elements w of X, where each $a_w : M \to N$ is a linear map and

$$a_{xw}(x \cdot m) = x \cdot a_w(m) \tag{B.27}$$

holds for any $x, w \in X$, and $m \in M$.

Next, $\hom^{\otimes}(M, N)$ is a vector space with addition and scalar multiplication given by

$$(a + b)_w := a_w + b_w \quad \text{and} \quad (ca)_w := ca_w$$

for $a, b \in \hom^{\otimes}(M, N)$ and $c \in \Bbbk$. For $a \in \hom^{\otimes}(M, N)$ and $y \in X$, define $y \cdot a \in \hom^{\otimes}(M, N)$ by

$$(y \cdot a)_w := a_{wy}. \tag{B.28}$$

To show that this indeed belongs to $\hom^{\otimes}(M, N)$, we need to check (B.27) with $y \cdot a$ replacing a, that is,

$$(y \cdot a)_{xw}(x \cdot m) = x \cdot (y \cdot a)_w(m).$$

But this is an instance of (B.27) with wy instead of w. Next we check that for any $z, y \in X$ and $a \in \hom^{\otimes}(M, N)$, we have $z \cdot (y \cdot a) = zy \cdot a$:

$$(z \cdot (y \cdot a))_w = (y \cdot a)_{wz} = a_{wzy} = (zy \cdot a)_w.$$

Thus, $\hom^{\otimes}(M, N)$ is a left $\Bbbk X$-module.

Example B.18. Suppose X is a group. For a left $\Bbbk X$-module M, let M_t denote the vector space M viewed as a trivial $\Bbbk X$-module. In this situation, the map

$$M \otimes \Bbbk X \to M_t \otimes \Bbbk X, \qquad m \otimes x \mapsto x^{-1} \cdot m \otimes x$$

is an isomorphism of $\Bbbk X$-modules, with inverse given by $m \otimes x \mapsto x \cdot m \otimes x$. We deduce from here that $\hom^{\otimes}(M, N) \cong \mathrm{Hom}_{\Bbbk}(M, N)$ as left $\Bbbk X$-modules, with action on the latter given by (B.25). Explicitly, the members a_x are determined by a_e via

$$a_x(m) := x \cdot a_e(x^{-1} \cdot m).$$

Proposition B.19. *For a monoid X and left $\Bbbk X$-modules M, N, P, there is a natural bijection*

$$\Bbbk X\text{-}\mathsf{Mod}(P \otimes M, N) \xrightarrow{\cong} \Bbbk X\text{-}\mathsf{Mod}(P, \mathrm{hom}^{\otimes}(M, N)).$$

Proof. This can be deduced from (B.26) using the standard Hom-tensor adjunction. Alternatively, one can argue directly using the explicit construction as follows.

As explained in the proof of Proposition B.17, an element of the lhs is the same as a linear map $\beta : P \to \mathrm{Hom}_{\Bbbk}(M, N)$ which satisfies the condition

$$\beta(x \cdot p)(x \cdot m) = x \cdot (\beta(p)(m)) \tag{B.29}$$

for all $p \in P$, $m \in M$, $x \in X$.

Given such a β, we define $\gamma : P \to \mathrm{hom}^{\otimes}(M, N)$ by

$$\gamma(p)_w := \beta(w \cdot p)$$

for $p \in P$ and $w \in X$. We need to check that $\gamma(p)$ belongs to $\mathrm{hom}^{\otimes}(M, N)$, that is, (B.27) holds with $\gamma(p)$ instead of a. This follows from (B.29) and definitions:

$$\gamma(p)_{xw}(x \cdot m) = \beta(xw \cdot p)(x \cdot m) = x \cdot (\beta(w \cdot p)(m)) = x \cdot (\gamma(p)_w(m)).$$

Further,

$$\gamma(x \cdot p)_w = \beta(w \cdot (x \cdot p)) = \beta(wx \cdot p) = \gamma(p)_{wx} = (x \cdot \gamma(p))_w.$$

Thus, $\gamma(x \cdot p) = x \cdot \gamma(p)$ and γ is a map of left $\Bbbk X$-modules.

Conversely, suppose we are given $\gamma : P \to \mathrm{hom}^{\otimes}(M, N)$ which is a map of left $\Bbbk X$-modules. Then define $\beta : P \to \mathrm{Hom}_{\Bbbk}(M, N)$ by

$$\beta(p) := \gamma(p)_e.$$

It satisfies condition (B.29):

$$\begin{aligned}\beta(x \cdot p)(x \cdot m) &= \gamma(x \cdot p)_e(x \cdot m) = (x \cdot \gamma(p))_e(x \cdot m)\\ &= \gamma(p)_x(x \cdot m) = x \cdot (\gamma(p)_e(m)) = x \cdot (\beta(p)(m)).\end{aligned}$$

It is routine to verify that the two constructions are inverse of each other. □

Exercise B.20. Deduce Proposition B.19 as a special case of Theorem B.14 and formula (B.24). (Take C to be the one-object category determined by the monoid X and D to be the category of $\Bbbk$-vector spaces.)

Notes

Monoidal categories. Monoidal categories appeared in work of Bénabou [**89**], [**90**] and Mac Lane [**620**, Section 2], [**621**, Section 15] under the name 'categories with multiplication'. Mac Lane also treated the case of symmetric monoidal categories. The fact that the pentagon and triangle axioms suffice for the definition was shown by Kelly [**520**, Theorem 3′]. The term 'monoidal category' is used by Eilenberg and Kelly [**284**, Chapters II and III].

Braided monoidal categories arose in the mid 1980's in work of Joyal and Street [**501**]. The concept is present in letters from Joyal to Grothendieck [**499**] and from

Breen to Deligne [**155**], in that decade. The work of Joyal and Street was continued in [**502**], [**503**].

Additional information on monoidal categories can be found in the books by Bulacu, Caenepeel, Panaite, Van Oystaeyen [**175**, Chapter 1], Kassel [**517**, Chapters XI and XIII], Kock [**538**, Chapter 3], Leinster [**590**, Chapter 3], Mac Lane [**623**, Chapters VII and XI], Melliès [**674**, Chapter 4], Street [**859**], Yetter [**924**, Chapters 3 and 5], and in our monograph [**18**, Chapters 1 and 3]. References related to bimonoids in braided monoidal categories are given in the Notes to Chapter 2.

2-monoidal categories. Categories with two compatible monoidal structures originated in work of Balteanu and Fiedorowicz [**68**]. The notion was suitably refined by Garner [**339**, Section 4.9] and, independently, in our monograph [**18**, Chapter 6], where the term 2-monoidal category was employed. Lamarche considered the same ingredients in [**556**, Sections 2.7, 2.8, 2.19]. Alternative formulations of a more restrictive nature were considered by Kock [**539**], Forcey, Siehler, Sowers [**310**], Vallette [**889**, Section 1.2], see [**18**, Remark 6.2] for more on this.

Example B.1 goes back to Joyal and Street [**503**, Propositions 5.2 and 5.3], see [**18**, Section 6.3]. The interchange law is called 'middle-four interchange' by Day [**233**]; for an earlier usage of this term, see for instance [**621**, page 77]. Bimonoids in the context of 2-monoidal categories appear in [**18**, Section 6.5], [**339**, Section 4.11], [**556**, Section 3]. 2-monoidal categories are studied further in [**77**], [**127**], [**128**], [**129**], [**131**], [**340**], [**555**], [**860**], often under the name 'duoidal categories'.

There is a more general notion of a higher monoidal category which specifies suitable compatibility conditions between any given number of monoidal structures on a category. It originated in works of Balteanu, Fiedorowicz, Schwänzl, Vogt [**68**], [**69**] and was refined in [**18**, Chapter 7]. Monads acting on higher monoidal categories are studied in [**14**].

Module categories. Module categories over monoidal categories are considered by Bénabou in his work on bicategories [**92**, Section 2.3]. Some later references are those by Pareigis [**735**], Hovey [**463**, Definition 4.1.6], McCrudden [**667**, Section 3], Janelidze and Kelly [**483**, Section 1, page 62], Schauenburg [**803**, Section 2.2], Leinster [**590**, Example 1.2.12]. For module categories with enriched hom, see the book by Riehl [**780**, Definition 10.1.3]. In the literature, a module category is sometimes called an 'actegory', while 'hommed category' is used for a module category with enriched homs.

Enriched categories. Enrichment over monoidal categories appeared in work of Bénabou [**91**], Eilenberg and Kelly [**284**, Section 6], Maranda [**646**, Section 4]. The special case of 2-categories is also mentioned in these papers, where they are called 2-categories, hypercategories, categories of the second type, respectively. (More references for 2-categories are given in the Notes to Appendix C.) Early references on enrichment are by Bunge [**176**, Section 1.2] and Dubuc [**265**]. Some later references are by Kelly [**523**], Borceux [**133**, Chapter 6], Johnstone [**493**, Section B2.1], Riehl [**780**, Chapter 3], May and Ponto [**663**, Section 16.3].

Closed categories and internal hom. Closed categories were introduced by Eilenberg and Kelly [**284**, Section I.2]. They discuss closed monoidal categories in [**284**, Section II.2] and cartesian closed categories in [**284**, Section IV.2]. The latter arose in work of Lawvere [**572**, Axiom 2], [**573**], [**577**]. Biclosed monoidal categories are considered by Lambek [**558**, Section 2], [**560**, Section 2]. Some book references on closed monoidal categories are those by Kelly [**523**, Section 1.5], Mac

Lane [**623**, Sections IV.6 and VII.7], Borceux [**133**, Section 6.1], Street [**859**, Chapter 12], see also [**60**], [**238**, Section 1], [**165**, Appendix C]. Examples pertaining to graded vector spaces and Joyal species can be found in our monograph [**18**, Section 1.3]. For emphasis on cartesian closed categories, see the books by Lawvere and Schanuel [**578**, Session 30], Lambek and Scott [**561**, Part I], McLarty [**668**, Chapter 6], Crole [**225**, Section 2.8], Johnstone [**493**, Section A1.5], Awodey [**54**, Chapter 6]. Our use of left closed agrees with that of Bruguières, Lack, Virelizier [**166**, Section 3.1], Hyland, López Franco, Vasilakopoulou [**475**, Section 2], but differs from that of Pareigis [**734**, page 199], Janelidze and Kelly [**483**, Section 2], Lurie [**616**, Definition A.1.3.4], Melliès [**674**, Section 4.5].

Powers and copowers. The idea of (co)powers is present in a paper of Beck in the specific context of 'topological categories' [**85**, page 139, item (2)]. A complete definition is given by Kelly [**521**, Section 4]. Information on powers and copowers for enriched categories can be found in the books by Dubuc [**265**, Section I.2], Borceux [**133**, Section 6.5], Kelly [**523**, Section 3.7], Riehl [**780**, Section 3.7], Kamps and Porter [**507**, Section III.4], Fresse [**325**, Sections 2.1.3 and 2.1.4]. In these references, the term 'tensor' is used in place of copower, and 'cotensor' in place of power. An example of powers and copowers appears in the text in Section 8.7. Related references are given in the Notes to Chapter 8. Another example of copowers is given in Exercise 4.21.

Internal hom for functor categories. When the codomain D is the category of sets, the result of Theorem B.14 is mentioned by Bunge [**176**, Section 1.12, item (iii)]. For book references, see for instance those by Lambek and Scott [**561**, Part II, Example 9.1], Mac Lane and Moerdijk [**624**, Section I.6, Proposition 1], Johnstone [**493**, Proposition 1.5.5 on page 48], Awodey [**54**, Theorem 8.14], Riehl [**780**, Example 1.5.6]. The last reference focuses on the category of simplicial sets. For Exercise B.16, see for instance Mac Lane and Moerdijk [**624**, Exercise 8, page 63], Simpson [**826**, Corollary 7.8.2].

Internal hom for functor categories is treated in greater generality by Day [**230**, Chapter 3], [**231**], [**232**], see also [**234**, Section 9]. Day deals with enriched categories and with monoidal structures on $[\mathsf{C}, \mathsf{D}]$ arising from a promonoidal structure on C and a monoidal structure on D. See [**231**, Theorem 3.3, Formulas (3.3) and (3.4)]; a formula closely related to (B.24) appears in [**230**, Section 3.3], [**231**, page 35]. These results are further extended to the setting of skew-monoidal categories by Campbell [**183**]; formula (B.22) is contained in his [**183**, Proposition 4.5]. Examples of Theorem B.14 appear in the text in Section 8.8.

Modules over monoid algebras. For modules over bialgebras (which include monoid algebras), the internal hom is constructed by Eilenberg and Kelly [**284**, Section IV.5], see also [**230**, Example 3.3.1], [**231**, Example 5.1]. A later reference is by Schauenburg [**802**, Proposition 3.3]. For the simpler case of modules over Hopf algebras (which include group algebras), see for instance [**859**, Example 12.2]. For group actions on sets, see [**133**, Example 6.1.9.e].

APPENDIX C

Higher monads

C.1. Higher monads

Let Cat denote the 2-category of all categories: 0-cells are categories, 1-cells are functors, 2-cells are natural transformations. We are mainly interested in higher monads in Cat (though these notions make sense in any 2-category). To each pair (p, q) of nonnegative integers, we associate a 2-category denoted $m^p c^q(\mathsf{Cat})$: 0-cells are (p, q)-monads, 1-cells are (p, q)-lax functors, 2-cells are morphisms between (p, q)-lax functors. For the initial values of p and q, we will also employ the terminology of Table C.1. Some of this is standard but some is not.

TABLE C.1. 2-categories of higher monads on levels 0, 1, 2.

		2-category	0-cell	1-cell
$(0, 0)$		Cat	category	functor
$(1, 0)$	Def. C.1	$m(\mathsf{Cat})$	monad	lax functor
$(0, 1)$	Def. C.3	$c(\mathsf{Cat})$	comonad	colax functor
$(2, 0)$	Def. C.6	$mm(\mathsf{Cat})$	double monad	double lax functor
$(1, 1)$	Def. C.4	$mc(\mathsf{Cat})$	bimonad	bilax functor
$(0, 2)$		$cc(\mathsf{Cat})$	double comonad	double colax functor

Thus, a $(0, 0)$-monad is simply a category, a $(1, 0)$-monad is a monad, a $(0, 1)$-monad is a comonad, a $(2, 0)$-monad is a double monad, a $(1, 1)$-monad is a bimonad and a $(0, 2)$-monad is a double comonad. Similarly, a $(0, 0)$-lax functor is simply a functor, a $(1, 0)$-lax functor is a lax functor, a $(0, 1)$-lax functor is a colax functor, and so forth.

We explicitly define these notions starting with (co)monads, (co)lax functors and morphisms between (co)lax functors. They are defined using commutative diagrams of natural transformations. There are three diagrams for (co)monads, two for (co)lax functors, and one for morphisms between (co)lax functors. These are objects on level one. In general, (p, q)-objects are on level $p + q$. The passage from level zero to level one, that is, the construction of $m(\mathsf{Cat})$ and $c(\mathsf{Cat})$ from Cat, is highly significant. We refer to these as the monad and comonad constructions. In general, starting with any 2-category

D, one can construct the 2-categories $m(\mathsf{D})$ and $c(\mathsf{D})$. So these constructions can be iterated. We show that $c(m(\mathsf{D})) = m(c(\mathsf{D}))$. Thus, the 2-categories that we obtain by successive iterations are indexed by pairs (p, q) of non-negative integers: $m^p c^q(\mathsf{D})$ is the 2-category obtained from D by p monad constructions and q comonad constructions.

$$
\begin{tikzcd}[column sep=tiny]
 & & & \mathsf{Cat} \arrow[dl, dash] \arrow[dr, dash] & & & \\
 & & m(\mathsf{Cat}) \arrow[dl, dash] \arrow[dr, dash] & & c(\mathsf{Cat}) \arrow[dl, dash] \arrow[dr, dash] & & \\
 & mm(\mathsf{Cat}) \arrow[dl, dash] \arrow[dr, dash] & & mc(\mathsf{Cat}) \arrow[dl, dash] \arrow[dr, dash] & & cc(\mathsf{Cat}) \arrow[dl, dash] \arrow[dr, dash] & \\
mmm(\mathsf{Cat}) \arrow[d, dotted, dash] & & mmc(\mathsf{Cat}) \arrow[d, dotted, dash] & & mcc(\mathsf{Cat}) \arrow[d, dotted, dash] & & ccc(\mathsf{Cat}) \arrow[d, dotted, dash] \\
{} & & {} & & {} & & {}
\end{tikzcd}
$$

FIGURE C.1. Monad and comonad constructions on Cat.

The structure of $m^p c^q(\mathsf{Cat})$ can be made explicit. On level two, the 0-cells, for instance bimonads, require four additional diagrams, while the 1-cells, for instance bilax functors, require one additional diagram. (The 2-cells do not require any additional diagrams.) On level three, the 0-cells, for instance triple monads, require one additional diagram. (The 1-cells and 2-cells do not require any additional diagrams.) The higher levels do not require any additional diagrams, thus they can be directly described using the notions defined in the first three levels.

C.1.1. Level one. We begin by discussing monads and comonads.

Definition C.1. A *monad* on a category C is a functor $\mathcal{V} : \mathsf{C} \to \mathsf{C}$ equipped with natural transformations $\mu : \mathcal{V}\mathcal{V} \to \mathcal{V}$ and $\iota : \mathrm{id} \to \mathcal{V}$ such that the diagrams

$$
\begin{tikzcd}
\mathcal{V}\mathcal{V}\mathcal{V} \arrow[r, "\mathcal{V}\mu"] \arrow[d, "\mu\mathcal{V}"'] & \mathcal{V}\mathcal{V} \arrow[d, "\mu"] \\
\mathcal{V}\mathcal{V} \arrow[r, "\mu"'] & \mathcal{V}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{V}\mathcal{V} \arrow[dr, "\mu"] & \\
\mathcal{V} \arrow[ur, "\iota\mathcal{V}"] \arrow[rr, equal] & & \mathcal{V}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{V}\mathcal{V} \arrow[dr, "\mu"] & \\
\mathcal{V} \arrow[ur, "\mathcal{V}\iota"] \arrow[rr, equal] & & \mathcal{V}
\end{tikzcd}
\tag{C.1}
$$

commute.

Let $\mathcal{V}$ be a monad on C, and $\mathcal{V}'$ a monad on C'. A *lax functor* from $\mathcal{V}$ to $\mathcal{V}'$ is a functor $\mathcal{F} : \mathsf{C} \to \mathsf{C}'$ equipped with a natural transformation $\varphi : \mathcal{V}'\mathcal{F} \to \mathcal{F}\mathcal{V}$ such that the diagrams

$$
\begin{tikzcd}
\mathcal{V}'\mathcal{V}'\mathcal{F} \arrow[r, "\mathcal{V}'\varphi"] \arrow[d, "\mu'\mathcal{F}"'] & \mathcal{V}'\mathcal{F}\mathcal{V} \arrow[r, "\varphi\mathcal{V}"] & \mathcal{F}\mathcal{V}\mathcal{V} \arrow[d, "\mathcal{F}\mu"] \\
\mathcal{V}'\mathcal{F} \arrow[rr, "\varphi"'] & & \mathcal{F}\mathcal{V}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{V}'\mathcal{F} \arrow[rr, "\varphi"] & & \mathcal{F}\mathcal{V} \\
 & \mathcal{F} \arrow[ul, "\iota'\mathcal{F}"] \arrow[ur, "\mathcal{F}\iota"'] &
\end{tikzcd}
\tag{C.2}
$$

commute. We use the notation $(\mathcal{F}, \varphi) : \mathcal{V} \to \mathcal{V}'$.

Let $(\mathcal{F},\varphi)$ and $(\tilde{\mathcal{F}},\tilde{\varphi})$ be lax functors from $\mathcal{V}$ to $\mathcal{V}'$. Then a morphism $(\mathcal{F},\varphi) \to (\tilde{\mathcal{F}},\tilde{\varphi})$ is a natural transformation $\theta : \mathcal{F} \to \tilde{\mathcal{F}}$ such that the diagram

$$\begin{tikzcd} \mathcal{V}'\mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\mathcal{V}'\theta"'] & \mathcal{F}\mathcal{V} \arrow[d, "\theta\mathcal{V}"] \\ \mathcal{V}'\tilde{\mathcal{F}} \arrow[r, "\tilde{\varphi}"'] & \tilde{\mathcal{F}}\mathcal{V} \end{tikzcd} \tag{C.3}$$

commutes.

This defines a 2-category, denoted $m(\mathsf{Cat})$, whose 0-cells are monads, 1-cells are lax functors of monads, 2-cells are morphisms of lax functors. The composition of 1-cells is explained below.

Proposition C.2. *Let $(\mathcal{F},\varphi) : \mathcal{V} \to \mathcal{V}'$ and $(\mathcal{G},\gamma) : \mathcal{V}' \to \mathcal{V}''$ be lax functors of monads. Then $(\mathcal{G}\mathcal{F},\varphi\gamma) : \mathcal{V} \to \mathcal{V}''$ is a lax functor of monads, where $\varphi\gamma$ is defined to be the composite*

$$\mathcal{V}''\mathcal{G}\mathcal{F} \xrightarrow{\gamma\mathcal{F}} \mathcal{G}\mathcal{V}'\mathcal{F} \xrightarrow{\mathcal{G}\varphi} \mathcal{G}\mathcal{F}\mathcal{V}. \tag{C.4}$$

PROOF. We need to check that the diagrams (C.2) commute for $(\mathcal{G}\mathcal{F},\varphi\gamma)$. This is illustrated below.

$$\begin{tikzcd}[column sep=small] \mathcal{V}''\mathcal{V}''\mathcal{G}\mathcal{F} \arrow[r, "\mathcal{V}''\gamma\mathcal{F}"] \arrow[dd, "\mu''\mathcal{G}\mathcal{F}"'] & \mathcal{V}''\mathcal{G}\mathcal{V}'\mathcal{F} \arrow[r, "\mathcal{V}''\mathcal{G}\varphi"] \arrow[dr, "\gamma\mathcal{V}'\mathcal{F}"'] & \mathcal{V}''\mathcal{G}\mathcal{F}\mathcal{V} \arrow[r, "\gamma\mathcal{F}\mathcal{V}"] & \mathcal{G}\mathcal{V}'\mathcal{F}\mathcal{V} \arrow[r, "\mathcal{G}\varphi\mathcal{V}"] & \mathcal{G}\mathcal{F}\mathcal{V}\mathcal{V} \arrow[dd, "\mathcal{G}\mathcal{F}\mu"] \\ & & \mathcal{G}\mathcal{V}'\mathcal{V}'\mathcal{F} \arrow[ur, "\mathcal{G}\mathcal{V}'\varphi"'] \arrow[d, "\mathcal{G}\mu'\mathcal{F}"] & & \\ \mathcal{V}''\mathcal{G}\mathcal{F} \arrow[rr, "\gamma\mathcal{F}"'] & & \mathcal{G}\mathcal{V}'\mathcal{F} \arrow[rr, "\mathcal{G}\varphi"'] & & \mathcal{G}\mathcal{F}\mathcal{V} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{V}''\mathcal{G}\mathcal{F} \arrow[r, "\gamma\mathcal{F}"] & \mathcal{G}\mathcal{V}'\mathcal{F} \arrow[r, "\mathcal{G}\varphi"] & \mathcal{G}\mathcal{F}\mathcal{V} \\ & \mathcal{G}\mathcal{F} \arrow[ul, "\iota''\mathcal{G}\mathcal{F}"] \arrow[u, "\mathcal{G}\iota'\mathcal{F}"] \arrow[ur, "\mathcal{G}\mathcal{F}\iota"'] & \end{tikzcd}$$

The pentagons and triangles commute by diagrams (C.2) for $(\mathcal{F},\varphi)$ and $(\mathcal{G},\gamma)$. $\square$

Comonads is the notion dual to a monad. To set up notation, let us briefly go through what we did for monads.

Definition C.3. A *comonad* on a category C is a functor $\mathcal{U} : \mathsf{C} \to \mathsf{C}$ equipped with natural transformations $\Delta : \mathcal{U} \to \mathcal{U}\mathcal{U}$ and $\epsilon : \mathcal{U} \to \mathrm{id}$ such that the diagrams

$$\begin{tikzcd} \mathcal{U}\mathcal{U}\mathcal{U} & \mathcal{U}\mathcal{U} \arrow[l, "\mathcal{U}\Delta"'] \\ \mathcal{U}\mathcal{U} \arrow[u, "\Delta\mathcal{U}"] & \mathcal{U} \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{U}\mathcal{U} \arrow[dl, "\epsilon\mathcal{U}"'] & \\ \mathcal{U} \arrow[rr, equal] & & \mathcal{U} \arrow[ul, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{U}\mathcal{U} \arrow[dl, "\mathcal{U}\epsilon"'] & \\ \mathcal{U} \arrow[rr, equal] & & \mathcal{U} \arrow[ul, "\Delta"'] \end{tikzcd} \tag{C.5}$$

commute.

Let $\mathcal{U}$ be a comonad on C, and $\mathcal{U}'$ a comonad on C'. A *colax functor* from $\mathcal{U}$ to $\mathcal{U}'$ is a functor $\mathcal{F}:\mathsf{C}\to\mathsf{C}'$ equipped with a natural transformation $\psi:\mathcal{F}\mathcal{U}\to\mathcal{U}'\mathcal{F}$ such that the diagrams

(C.6)
$$
\begin{tikzcd}
\mathcal{F}\mathcal{U}\mathcal{U} \arrow[r, "\psi\mathcal{U}"] & \mathcal{U}'\mathcal{F}\mathcal{U} \arrow[r, "\mathcal{U}'\psi"] & \mathcal{U}'\mathcal{U}'\mathcal{F} \\
\mathcal{F}\mathcal{U} \arrow[u, "\mathcal{F}\Delta"] \arrow[rr, "\psi"'] & & \mathcal{U}'\mathcal{F} \arrow[u, "\Delta'\mathcal{F}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}\mathcal{U} \arrow[rr, "\psi"] \arrow[dr, "\mathcal{F}\epsilon"'] & & \mathcal{U}'\mathcal{F} \arrow[dl, "\epsilon'\mathcal{F}"] \\
& \mathcal{F} &
\end{tikzcd}
$$

commute. We use the notation $(\mathcal{F},\psi):\mathcal{U}\to\mathcal{U}'$.

Let $(\mathcal{F},\varphi)$ and $(\tilde{\mathcal{F}},\tilde{\varphi})$ be colax functors from $\mathcal{U}$ to $\mathcal{U}'$. Then a morphism $(\mathcal{F},\varphi)\to(\tilde{\mathcal{F}},\tilde{\varphi})$ is a natural transformation $\theta:\mathcal{F}\to\tilde{\mathcal{F}}$ such that the diagram

(C.7)
$$
\begin{tikzcd}
\mathcal{F}\mathcal{U} \arrow[r, "\psi"] \arrow[d, "\theta\mathcal{U}"'] & \mathcal{U}'\mathcal{F} \arrow[d, "\mathcal{U}'\theta"] \\
\tilde{\mathcal{F}}\mathcal{U} \arrow[r, "\tilde{\psi}"'] & \mathcal{U}'\tilde{\mathcal{F}}
\end{tikzcd}
$$

commutes.

This defines a 2-category, denoted $c(\mathsf{Cat})$, whose 0-cells are comonads, 1-cells are colax functors of comonads, 2-cells are morphisms of colax functors. The composition of 1-cells works as follows. If $(\mathcal{F},\psi):\mathcal{U}\to\mathcal{U}'$ and $(\mathcal{G},\delta):\mathcal{U}'\to\mathcal{U}''$ are colax functors of comonads, then so is $(\mathcal{G}\mathcal{F},\delta\psi):\mathcal{U}\to\mathcal{U}''$, where $\delta\psi$ is defined to be the composite

(C.8)
$$\mathcal{G}\mathcal{F}\mathcal{U}\xrightarrow{\mathcal{G}\psi}\mathcal{G}\mathcal{U}'\mathcal{F}\xrightarrow{\delta\mathcal{F}}\mathcal{U}''\mathcal{G}\mathcal{F}.$$

C.1.2. Level two. We now discuss bimonads, double monads, double comonads.

Definition C.4. A *bimonad* on a category C is a triple $(\mathcal{V},\mathcal{U},\lambda)$, where $\mathcal{V}$ is a monad on C, $\mathcal{U}$ is a comonad on C, and $\lambda:\mathcal{V}\mathcal{U}\to\mathcal{U}\mathcal{V}$ is a natural transformation such that the diagrams

(C.9a)
$$
\begin{tikzcd}
\mathcal{V}\mathcal{U}\mathcal{U} \arrow[r, "\lambda\mathcal{U}"] & \mathcal{U}\mathcal{V}\mathcal{U} \arrow[r, "\mathcal{U}\lambda"] & \mathcal{U}\mathcal{U}\mathcal{V} \\
\mathcal{V}\mathcal{U} \arrow[u, "\mathcal{V}\Delta"] \arrow[rr, "\lambda"'] & & \mathcal{U}\mathcal{V} \arrow[u, "\Delta\mathcal{V}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{V}\mathcal{V}\mathcal{U} \arrow[r, "\mathcal{V}\lambda"] \arrow[d, "\mu\mathcal{U}"'] & \mathcal{V}\mathcal{U}\mathcal{V} \arrow[r, "\lambda\mathcal{V}"] & \mathcal{U}\mathcal{V}\mathcal{V} \arrow[d, "\mathcal{U}\mu"] \\
\mathcal{V}\mathcal{U} \arrow[rr, "\lambda"'] & & \mathcal{U}\mathcal{V}
\end{tikzcd}
$$

(C.9b)
$$
\begin{tikzcd}
& \mathcal{V} & \\
\mathcal{V}\mathcal{U} \arrow[ur, "\mathcal{V}\epsilon"] \arrow[rr, "\lambda"'] & & \mathcal{U}\mathcal{V} \arrow[ul, "\epsilon\mathcal{V}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{U} \arrow[dl, "\iota\mathcal{U}"'] \arrow[dr, "\mathcal{U}\iota"] & \\
\mathcal{V}\mathcal{U} \arrow[rr, "\lambda"'] & & \mathcal{U}\mathcal{V}
\end{tikzcd}
$$

commute. We refer to λ as a *mixed distributive law* linking $\mathcal{V}$ and $\mathcal{U}$.

Let $(\mathcal{V},\mathcal{U},\lambda)$ be a bimonad on C, and $(\mathcal{V}',\mathcal{U}',\lambda')$ a bimonad on C'. A *bilax functor*

$$(\mathcal{V},\mathcal{U},\lambda)\to(\mathcal{V}',\mathcal{U}',\lambda')$$

is a triple $(\mathcal{F},\varphi,\psi)$ such that $\mathcal{F}:\mathsf{C}\to\mathsf{C}'$ is a functor, $(\mathcal{F},\varphi):\mathcal{V}\to\mathcal{V}'$ is a lax functor of monads, $(\mathcal{F},\psi):\mathcal{U}\to\mathcal{U}'$ is a colax functor of comonads, and the diagram

$$\text{(C.10)}\qquad \begin{tikzcd} \mathcal{V}'\mathcal{F}\mathcal{U} \arrow[r, "\varphi\mathcal{U}"] \arrow[d, "\mathcal{V}'\psi"'] & \mathcal{F}\mathcal{V}\mathcal{U} \arrow[r, "\mathcal{F}\lambda"] & \mathcal{F}\mathcal{U}\mathcal{V} \arrow[d, "\psi\mathcal{V}"] \\ \mathcal{V}'\mathcal{U}'\mathcal{F} \arrow[r, "\lambda'\mathcal{F}"'] & \mathcal{U}'\mathcal{V}'\mathcal{F} \arrow[r, "\mathcal{U}'\varphi"'] & \mathcal{U}'\mathcal{F}\mathcal{V} \end{tikzcd}$$

commutes.

Let $(\mathcal{F},\varphi,\psi)$ and $(\tilde{\mathcal{F}},\tilde{\varphi},\tilde{\psi})$ be bilax functors from $(\mathcal{V},\mathcal{U},\lambda)$ to $(\mathcal{V}',\mathcal{U}',\lambda')$. A morphism $(\mathcal{F},\varphi,\psi)\to(\tilde{\mathcal{F}},\tilde{\varphi},\tilde{\psi})$ is a natural transformation $\theta:\mathcal{F}\to\tilde{\mathcal{F}}$ such that the diagrams (C.3) and (C.7) commute.

This defines a 2-category, denoted $b(\mathsf{Cat})$, whose 0-cells are bimonads, 1-cells are bilax functors, 2-cells are morphisms of bilax functors. The composition of 1-cells is explained below.

Proposition C.5. *If*

$$(\mathcal{F},\varphi,\psi):(\mathcal{V},\mathcal{U},\lambda)\to(\mathcal{V}',\mathcal{U}',\lambda') \quad \textit{and} \quad (\mathcal{G},\gamma,\delta):(\mathcal{V}',\mathcal{U}',\lambda')\to(\mathcal{V}'',\mathcal{U}'',\lambda'')$$

are bilax functors, then so is

$$(\mathcal{G}\mathcal{F},\varphi\gamma,\delta\psi):(\mathcal{V},\mathcal{U},\lambda)\to(\mathcal{V}'',\mathcal{U}'',\lambda''),$$

where $\varphi\gamma$ and $\delta\psi$ are defined by (C.4) *and* (C.8).

Proof. We know from Proposition C.2 that $\varphi\gamma$ is lax and dually that $\delta\psi$ is colax. We need to check that the diagram (C.10) commutes for $(\mathcal{G}\mathcal{F},\varphi\gamma,\delta\psi)$. This is done below.

$$\begin{tikzcd} \mathcal{V}''\mathcal{G}\mathcal{F}\mathcal{U} \arrow[r, "\gamma\mathcal{F}\mathcal{U}"] \arrow[d, "\mathcal{V}''\mathcal{G}\psi"'] & \mathcal{G}\mathcal{V}'\mathcal{F}\mathcal{U} \arrow[r, "\mathcal{G}\varphi\mathcal{U}"] \arrow[d, "\mathcal{G}\mathcal{V}'\psi"'] & \mathcal{G}\mathcal{F}\mathcal{V}\mathcal{U} \arrow[r, "\mathcal{G}\mathcal{F}\lambda"] & \mathcal{G}\mathcal{F}\mathcal{U}\mathcal{V} \arrow[d, "\mathcal{G}\psi\mathcal{V}"] \\ \mathcal{V}''\mathcal{G}\mathcal{U}'\mathcal{F} \arrow[r, "\gamma\mathcal{U}'\mathcal{F}"] \arrow[d, "\mathcal{V}''\delta\mathcal{F}"'] & \mathcal{G}\mathcal{V}'\mathcal{U}'\mathcal{F} \arrow[r, "\mathcal{G}\lambda'\mathcal{F}"] & \mathcal{G}\mathcal{U}'\mathcal{V}'\mathcal{F} \arrow[r, "\mathcal{G}\mathcal{U}'\varphi"] \arrow[d, "\delta\mathcal{V}'\mathcal{F}"] & \mathcal{G}\mathcal{U}'\mathcal{F}\mathcal{V} \arrow[d, "\delta\mathcal{F}\mathcal{V}"] \\ \mathcal{V}''\mathcal{U}''\mathcal{G}\mathcal{F} \arrow[r, "\lambda''\mathcal{G}\mathcal{F}"'] & \mathcal{U}''\mathcal{V}''\mathcal{G}\mathcal{F} \arrow[r, "\mathcal{U}''\gamma\mathcal{F}"'] & \mathcal{U}''\mathcal{G}\mathcal{V}'\mathcal{F} \arrow[r, "\mathcal{U}''\mathcal{G}\varphi"'] & \mathcal{U}''\mathcal{G}\mathcal{F}\mathcal{V} \end{tikzcd}$$

The hexagons commute by diagram (C.10) for $(\mathcal{F},\varphi,\psi)$ and $(\mathcal{G},\gamma,\delta)$. □

Definition C.6. A *double monad* on a category C is a triple $(\mathcal{V}_1,\mathcal{V}_2,\lambda)$, where $\mathcal{V}_1$ and $\mathcal{V}_2$ are monads on C, and $\lambda:\mathcal{V}_1\mathcal{V}_2\to\mathcal{V}_2\mathcal{V}_1$ is a natural transformation such that the diagrams

$$\text{(C.11a)}\qquad \begin{tikzcd} \mathcal{V}_1\mathcal{V}_2\mathcal{V}_2 \arrow[r, "\lambda\mathcal{V}_2"] \arrow[d, "\mathcal{V}_1\mu_2"] & \mathcal{V}_2\mathcal{V}_1\mathcal{V}_2 \arrow[r, "\mathcal{V}_2\lambda"] & \mathcal{V}_2\mathcal{V}_2\mathcal{V}_1 \arrow[d, "\mu_2\mathcal{V}_1"'] \\ \mathcal{V}_1\mathcal{V}_2 \arrow[rr, "\lambda"'] & & \mathcal{V}_2\mathcal{V}_1 \end{tikzcd} \qquad \begin{tikzcd} \mathcal{V}_1\mathcal{V}_1\mathcal{V}_2 \arrow[r, "\mathcal{V}_1\lambda"] \arrow[d, "\mu_1\mathcal{V}_2"] & \mathcal{V}_1\mathcal{V}_2\mathcal{V}_1 \arrow[r, "\lambda\mathcal{V}_1"] & \mathcal{V}_2\mathcal{V}_1\mathcal{V}_1 \arrow[d, "\mathcal{V}_2\mu_1"'] \\ \mathcal{V}_1\mathcal{V}_2 \arrow[rr, "\lambda"'] & & \mathcal{V}_2\mathcal{V}_1 \end{tikzcd}$$

(C.11b)
$$\begin{tikzcd} & \mathcal{V}_1 \arrow[dl, "\mathcal{V}_1\iota_2"'] \arrow[dr, "\iota_2\mathcal{V}_1"] & \\ \mathcal{V}_1\mathcal{V}_2 \arrow[rr, "\lambda"'] & & \mathcal{V}_2\mathcal{V}_1 \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{V}_2 \arrow[dl, "\iota_1\mathcal{V}_2"'] \arrow[dr, "\mathcal{V}_2\iota_1"] & \\ \mathcal{V}_1\mathcal{V}_2 \arrow[rr, "\lambda"'] & & \mathcal{V}_2\mathcal{V}_1 \end{tikzcd}$$

commute. We refer to λ as a *distributive law* linking $\mathcal{V}_1$ and $\mathcal{V}_2$.

Let $(\mathcal{V}_1, \mathcal{V}_2, \lambda)$ be a double monad on C, and $(\mathcal{V}_1', \mathcal{V}_2', \lambda')$ a double monad on C'. A *double lax functor*

$$(\mathcal{V}_1, \mathcal{V}_2, \lambda) \to (\mathcal{V}_1', \mathcal{V}_2', \lambda')$$

is a triple $(\mathcal{F}, \varphi_1, \varphi_2)$ such that $\mathcal{F} : \mathsf{C} \to \mathsf{C}'$ is a functor, $(\mathcal{F}, \varphi_1) : \mathcal{V}_1 \to \mathcal{V}_1'$ and $(\mathcal{F}, \varphi_2) : \mathcal{V}_2 \to \mathcal{V}_2'$ are lax functors of monads, and the diagram

(C.12)
$$\begin{tikzcd} \mathcal{V}_1'\mathcal{V}_2'\mathcal{F} \arrow[r, "\lambda'\mathcal{F}"] \arrow[d, "\mathcal{V}_1'\varphi_2"'] & \mathcal{V}_2'\mathcal{V}_1'\mathcal{F} \arrow[r, "\mathcal{V}_2'\varphi_1"] & \mathcal{V}_2'\mathcal{F}\mathcal{V}_1 \arrow[d, "\varphi_2\mathcal{V}_1"] \\ \mathcal{V}_1'\mathcal{F}\mathcal{V}_2 \arrow[r, "\varphi_1\mathcal{V}_2"'] & \mathcal{F}\mathcal{V}_1\mathcal{V}_2 \arrow[r, "\mathcal{F}\lambda"'] & \mathcal{F}\mathcal{V}_2\mathcal{V}_1 \end{tikzcd}$$

commutes.

Let $(\mathcal{F}, \varphi_1, \varphi_2)$ and $(\tilde{\mathcal{F}}, \tilde{\varphi}_1, \tilde{\varphi}_2)$ be double lax functors from $(\mathcal{V}_1, \mathcal{V}_2, \lambda)$ to $(\mathcal{V}_1', \mathcal{V}_2', \lambda')$. A morphism $(\mathcal{F}, \varphi_1, \varphi_2) \to (\tilde{\mathcal{F}}, \tilde{\varphi}_1, \tilde{\varphi}_2)$ is a natural transformation $\theta : \mathcal{F} \to \tilde{\mathcal{F}}$ such that the diagram (C.3) commutes for $\mathcal{V} := \mathcal{V}_1$ and $\varphi := \varphi_1$, and for $\mathcal{V} := \mathcal{V}_2$ and $\varphi := \varphi_2$.

This defines the 2-category of double monads. We denote it by $mm(\mathsf{Cat})$.

Proposition C.7. *Let $(\mathcal{V}_1, \mathcal{V}_2, \lambda)$ be a double monad. Then $\mathcal{V}_2\mathcal{V}_1$ is a monad with structure maps*

$$\mathcal{V}_2\mathcal{V}_1\mathcal{V}_2\mathcal{V}_1 \xrightarrow{\mathcal{V}_2\lambda\mathcal{V}_1} \mathcal{V}_2\mathcal{V}_2\mathcal{V}_1\mathcal{V}_1 \xrightarrow{\mu_2\mu_1} \mathcal{V}_2\mathcal{V}_1 \quad \textit{and} \quad \mathrm{id} = \mathrm{id}\,\mathrm{id} \xrightarrow{\iota_2\iota_1} \mathcal{V}_2\mathcal{V}_1.$$

Double comonads is the dual notion to double monads. The 2-category of double comonads is denoted by $cc(\mathsf{Cat})$.

C.1.3. Passage from level one to level two.

Proposition C.8. *Let $(\mathcal{V}, \mu, \iota)$ be a monad, and $(\mathcal{U}, \Delta, \epsilon)$ a comonad on a category C. Further, let $\lambda : \mathcal{V}\mathcal{U} \to \mathcal{U}\mathcal{V}$ be a natural transformation. Then the following are equivalent.*

1. *$(\mathcal{V}, \mathcal{U}, \lambda)$ is a bimonad.*
2. *$(\mathcal{U}, \lambda) : \mathcal{V} \to \mathcal{V}$ is a lax functor, and Δ and ϵ are morphisms between lax functors.*
3. *$(\mathcal{V}, \lambda) : \mathcal{U} \to \mathcal{U}$ is a colax functor, and μ and ι are morphisms between colax functors.*

Proof. Let us analyze statement (2).

- $(\mathcal{U}, \lambda) : \mathcal{V} \to \mathcal{V}$ is a lax functor is equivalent to the commutativity of the second diagrams in (C.9a) and (C.9b),
- Δ is a morphism between lax functors is equivalent the first diagram in (C.9a),

- ϵ is a morphism between lax functors is equivalent the first diagram in (C.9b).

Thus, statement (2) is equivalent to statement (1). The analysis for statement (3) is similar. □

Proposition C.9. *Let* $(\mathcal{V},\mathcal{U},\lambda)$ *be a bimonad on* C, *and* $(\mathcal{V}',\mathcal{U}',\lambda')$ *a bimonad on* C'. *If* $(\mathcal{F},\psi):\mathcal{U}\to\mathcal{U}'$ *is a colax functor of comonads, then so are* $\mathcal{V}'\mathcal{F}$ *and* $\mathcal{F}\mathcal{V}$ *with structure maps*

$$\mathcal{V}'\mathcal{F}\mathcal{U} \xrightarrow{\mathcal{V}'\psi} \mathcal{V}'\mathcal{U}'\mathcal{F} \xrightarrow{\lambda'\mathcal{F}} \mathcal{U}'\mathcal{V}'\mathcal{F} \quad \text{and} \quad \mathcal{F}\mathcal{V}\mathcal{U} \xrightarrow{\mathcal{F}\lambda} \mathcal{F}\mathcal{U}\mathcal{V} \xrightarrow{\psi\mathcal{V}} \mathcal{U}'\mathcal{F}\mathcal{V}. \tag{C.13}$$

Dually, if $(\mathcal{F},\varphi):\mathcal{V}\to\mathcal{V}'$ *is a lax functor of monads, then so are* $\mathcal{U}'\mathcal{F}$ *and* $\mathcal{F}\mathcal{U}$ *with structure maps*

$$\mathcal{V}'\mathcal{U}'\mathcal{F} \xrightarrow{\lambda'\mathcal{F}} \mathcal{U}'\mathcal{V}'\mathcal{F} \xrightarrow{\mathcal{U}'\varphi} \mathcal{U}'\mathcal{F}\mathcal{V} \quad \text{and} \quad \mathcal{V}'\mathcal{F}\mathcal{U} \xrightarrow{\varphi\mathcal{U}} \mathcal{F}\mathcal{V}\mathcal{U} \xrightarrow{\mathcal{F}\lambda} \mathcal{F}\mathcal{U}\mathcal{V}. \tag{C.14}$$

Proof. Let us check that $\mathcal{F}\mathcal{V}\mathcal{U}\to\mathcal{U}'\mathcal{F}\mathcal{V}$ is a colax functor of comonads. The first diagram in (C.6) is as follows.

$$\begin{tikzcd}
\mathcal{F}\mathcal{V}\mathcal{U}\mathcal{U} \arrow[r, "\mathcal{F}\lambda\mathcal{U}"] & \mathcal{F}\mathcal{U}\mathcal{V}\mathcal{U} \arrow[r, "\psi\mathcal{V}\mathcal{U}"] \arrow[dr, "\mathcal{F}\mathcal{U}\lambda"'] & \mathcal{U}'\mathcal{F}\mathcal{V}\mathcal{U} \arrow[r, "\mathcal{U}'\mathcal{F}\lambda"] & \mathcal{U}'\mathcal{F}\mathcal{U}\mathcal{V} \arrow[r, "\mathcal{U}'\psi\mathcal{V}"] & \mathcal{U}'\mathcal{U}'\mathcal{F}\mathcal{V} \\
& & \mathcal{F}\mathcal{U}\mathcal{U}\mathcal{V} \arrow[ur, "\psi\mathcal{U}\mathcal{V}"'] & & \\
\mathcal{F}\mathcal{V}\mathcal{U} \arrow[uu, "\mathcal{F}\mathcal{V}\Delta"] \arrow[rr, "\mathcal{F}\lambda"'] & & \mathcal{F}\mathcal{U}\mathcal{V} \arrow[u, "\mathcal{F}\Delta\mathcal{V}"'] \arrow[rr, "\psi\mathcal{V}"'] & & \mathcal{U}'\mathcal{F}\mathcal{V} \arrow[uu, "\Delta\mathcal{F}\mathcal{V}"']
\end{tikzcd}$$

The first pentagon commutes since $(\mathcal{V},\mathcal{U},\lambda)$ is a bimonad (C.9a), while the second pentagon commutes since $(\mathcal{F},\psi)$ is a colax functor of comonads (C.6).

The second diagram in (C.6) is as follows.

$$\begin{tikzcd}
\mathcal{F}\mathcal{V}\mathcal{U} \arrow[r, "\mathcal{F}\lambda"] \arrow[dr, "\mathcal{F}\mathcal{V}\epsilon"'] & \mathcal{F}\mathcal{U}\mathcal{V} \arrow[r, "\psi\mathcal{V}"] \arrow[d, "\mathcal{F}\epsilon\mathcal{V}"] & \mathcal{U}'\mathcal{F}\mathcal{V} \arrow[dl, "\epsilon'\mathcal{F}\mathcal{V}"] \\
& \mathcal{F}\mathcal{V} &
\end{tikzcd}$$

The triangles commute for the same reasons that the pentagons commuted above. □

Proposition C.10. *Let* $(\mathcal{V},\mathcal{U},\lambda)$ *be a bimonad on* C, *and* $(\mathcal{V}',\mathcal{U}',\lambda')$ *a bimonad on* C'. *Let* $\mathcal{F}:\mathsf{C}\to\mathsf{C}'$ *be a functor equipped with* φ *and* ψ *such that* $(\mathcal{F},\varphi):\mathcal{V}\to\mathcal{V}'$ *is a lax functor of monads, and* $(\mathcal{F},\psi):\mathcal{U}\to\mathcal{U}'$ *is a colax functor of comonads. Then the following are equivalent.*

(1) $(\mathcal{F},\varphi,\psi)$ *is a bilax functor.*
(2) $\varphi:\mathcal{V}'\mathcal{F}\to\mathcal{F}\mathcal{V}$ *is a morphism between colax functors.*
(3) $\psi:\mathcal{F}\mathcal{U}\to\mathcal{U}'\mathcal{F}$ *is a morphism between lax functors.*

Proof. Condition (2) is the commutativity of diagram (C.7) applied to the colax functors $\mathcal{V}'\mathcal{F}$ and $\mathcal{F}\mathcal{V}$ with structure maps (C.13). Observe that this diagram coincides with (C.10). Thus, conditions (1) and (2) are equivalent. Condition (3) is handled similarly. □

Analogous results hold for double (co)monads and double (co)lax functors.

C.1.4. Monad and comonad constructions. Monads, comonads, and so on can be defined in any 2-category: replace category by 0-cell, functor by 1-cell and natural transformation by 2-cell. With these changes all of the preceding discussion goes through.

Suppose D is a 2-category. Let $m(\mathsf{D})$ denote the 2-category whose 0-cells are monads in D, 1-cells are lax functors between monads, 2-cells are morphisms between lax functors. Similarly, let $c(\mathsf{D})$ (resp. $b(\mathsf{D})$) denote the 2-category whose 0-cells are (co, bi)monads in D, 1-cells are (co, bi)lax functors between (co, bi)monads, 2-cells are morphisms between (co, bi)lax functors.

Theorem C.11. *Let* D *be a* 2*-category. Then*

$$m(c(\mathsf{D})) = b(\mathsf{D}) = c(m(\mathsf{D}))$$

as 2*-categories.*

PROOF. By symmetry it suffices to show $m(c(\mathsf{D})) = b(\mathsf{D})$. A 0-cell in $m(c(\mathsf{D}))$ is a monad in $c(\mathsf{D})$. This entails:

- a comonad $\mathcal{U}$ in D. In particular, we have a 1-cell $\mathcal{U} : \mathsf{C} \to \mathsf{C}$ in D.
- a colax functor $\mathcal{U} \to \mathcal{U}$. Explicitly, we have a 1-cell $\mathcal{V} : \mathsf{C} \to \mathsf{C}$ and a 2-cell $\mathcal{V}\mathcal{U} \to \mathcal{U}\mathcal{V}$ in D such that diagrams (C.6) commute with $\mathcal{U}' = \mathcal{U}$ and $\mathcal{F} = \mathcal{V}$. These are the same as the first diagrams in (C.9a) and (C.9b).
- 2-cells $\mathcal{V}\mathcal{V} \to \mathcal{V}$ and $\mathrm{id} \to \mathcal{V}$ which satisfy (C.1) and which are morphisms between colax functors. The latter means that diagram (C.7) commutes with $\mathcal{F} := \mathcal{V}\mathcal{V}$ and $\tilde{\mathcal{F}} := \mathcal{V}$, and also with $\mathcal{F} := \mathrm{id}$ and $\tilde{\mathcal{F}} := \mathcal{V}$. These are the same as the second diagrams in (C.9a) and (C.9b).

Thus, a 0-cell in $m(c(\mathsf{D}))$ is the same as a bimonad in D. This analysis is identical to the one in Proposition C.8.

A 1-cell in $m(c(\mathsf{D}))$ is a lax functor of monads in $c(\mathsf{D})$. Suppose the monads in question are $(\mathcal{V},\mathcal{U})$ and $(\mathcal{V}',\mathcal{U}')$. This entails a colax functor $\mathcal{F} : \mathcal{U} \to \mathcal{U}'$, a lax functor $\mathcal{F} : \mathcal{V} \to \mathcal{V}'$ and a morphism between colax functors $\mathcal{V}'\mathcal{F} \to \mathcal{F}\mathcal{V}$. The latter is the same as the diagram (C.10). Thus, a 1-cell in $m(c(\mathsf{D}))$ is the same as a bilax functor in D. This analysis is identical to the one in Proposition C.10.

It is straightforward to see that a 2-cell in $m(c(\mathsf{D}))$ is a morphism between bilax functors in $c(\mathsf{D})$. $\square$

This result shows that starting with any 2-category, iterations of the monad and comonad constructions result in a triangle of 2-categories. This is illustrated for Cat in Figure C.1.

C.1.5. Levels three and higher. So far, we have explicitly described the 2-categories up to level two. An explicit description of $m^3(\mathsf{Cat})$ is given below; 0-cells are called triple monads.

Definition C.12. A *triple monad* on a category C is a triple $(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3)$ of monads on C equipped with natural transformations

$$\lambda_{12} : \mathcal{V}_1\mathcal{V}_2 \to \mathcal{V}_2\mathcal{V}_1, \quad \lambda_{13} : \mathcal{V}_1\mathcal{V}_3 \to \mathcal{V}_3\mathcal{V}_1, \quad \lambda_{23} : \mathcal{V}_2\mathcal{V}_3 \to \mathcal{V}_3\mathcal{V}_2,$$

such that $(\mathcal{V}_1, \mathcal{V}_2, \lambda_{12})$, $(\mathcal{V}_1, \mathcal{V}_3, \lambda_{13})$ and $(\mathcal{V}_2, \mathcal{V}_3, \lambda_{23})$ are double monads and the diagram

$$\begin{tikzcd}
\mathcal{V}_1\mathcal{V}_2\mathcal{V}_3 \arrow[r, "\lambda_{12}\mathcal{V}_3"] \arrow[d, "\mathcal{V}_1\lambda_{23}"'] & \mathcal{V}_2\mathcal{V}_1\mathcal{V}_3 \arrow[r, "\mathcal{V}_2\lambda_{13}"] & \mathcal{V}_2\mathcal{V}_3\mathcal{V}_1 \arrow[d, "\lambda_{23}\mathcal{V}_1"] \\
\mathcal{V}_1\mathcal{V}_3\mathcal{V}_2 \arrow[r, "\lambda_{13}\mathcal{V}_2"'] & \mathcal{V}_3\mathcal{V}_1\mathcal{V}_2 \arrow[r, "\mathcal{V}_3\lambda_{12}"'] & \mathcal{V}_3\mathcal{V}_2\mathcal{V}_1
\end{tikzcd} \tag{C.15}$$

commutes.

Let $(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3)$ and $(\mathcal{V}'_1, \mathcal{V}'_2, \mathcal{V}'_3)$ be triple monads. A triple lax functor between them is a quadruple $(\mathcal{F}, \varphi_1, \varphi_2, \varphi_3)$ such that $(\mathcal{F}, \varphi_1, \varphi_2)$, $(\mathcal{F}, \varphi_1, \varphi_3)$, $(\mathcal{F}, \varphi_2, \varphi_3)$ are double lax functors between the appropriate double monads.

Suppose $(\mathcal{F}, \varphi_1, \varphi_2, \varphi_3)$ and $(\tilde{\mathcal{F}}, \tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3)$ are triple lax functors from $(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3)$ to $(\mathcal{V}'_1, \mathcal{V}'_2, \mathcal{V}'_3)$. A morphism $(\mathcal{F}, \varphi_1, \varphi_2, \varphi_3) \to (\tilde{\mathcal{F}}, \tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3)$ is a natural transformation $\theta : \mathcal{F} \to \tilde{\mathcal{F}}$ such that (C.3) commutes for $\varphi := \varphi_1$, $\varphi := \varphi_2$ and $\varphi := \varphi_3$.

It is useful to employ the following notations for (p, q)-monads.

$$\begin{tikzcd}[column sep=small]
 & & & {} \arrow[dl, dash] \arrow[dr, dash] & & & \\
 & & \mathcal{V}_1 \arrow[dl, dash] \arrow[dr, dash] & & \mathcal{U}_1 \arrow[dl, dash] \arrow[dr, dash] & & \\
 & (\mathcal{V}_1, \mathcal{V}_2) \arrow[dl, dash] \arrow[dr, dash] & & (\mathcal{V}_1, \mathcal{U}_1) \arrow[dl, dash] \arrow[dr, dash] & & (\mathcal{U}_2, \mathcal{U}_1) \arrow[dl, dash] \arrow[dr, dash] & \\
(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3) & & (\mathcal{V}_1, \mathcal{V}_2, \mathcal{U}_1) & & (\mathcal{V}_1, \mathcal{U}_2, \mathcal{U}_1) & & (\mathcal{U}_3, \mathcal{U}_2, \mathcal{U}_1) \\
\vdots & & \vdots & & \vdots & & \vdots
\end{tikzcd}$$

For instance:

Definition C.13. A $(2,1)$-*monad* on a category C is a triple $(\mathcal{V}_1, \mathcal{V}_2, \mathcal{U}_1)$, where $\mathcal{V}_1$ and $\mathcal{V}_2$ are monads on C, and $\mathcal{U}_1$ is a comonad on C equipped with natural transformations

$$\mathcal{V}_1\mathcal{V}_2 \to \mathcal{V}_2\mathcal{V}_1, \quad \mathcal{V}_1\mathcal{U}_1 \to \mathcal{U}_1\mathcal{V}_1, \quad \mathcal{V}_2\mathcal{U}_1 \to \mathcal{U}_1\mathcal{V}_2, \tag{C.16}$$

such that $(\mathcal{V}_1, \mathcal{V}_2)$ is a double monad, $(\mathcal{V}_1, \mathcal{U}_1)$ and $(\mathcal{V}_2, \mathcal{U}_1)$ are bimonads and the diagram

$$\begin{tikzcd}
\mathcal{V}_1\mathcal{V}_2\mathcal{U}_1 \arrow[r] \arrow[d] & \mathcal{V}_2\mathcal{V}_1\mathcal{U}_1 \arrow[r] & \mathcal{V}_2\mathcal{U}_1\mathcal{V}_1 \arrow[d] \\
\mathcal{V}_1\mathcal{U}_1\mathcal{V}_2 \arrow[r] & \mathcal{U}_1\mathcal{V}_1\mathcal{V}_2 \arrow[r] & \mathcal{U}_1\mathcal{V}_2\mathcal{V}_1
\end{tikzcd} \tag{C.17}$$

commutes. (The arrows in this diagram are induced from those in (C.16).)

A $(1,2)$-monad and a $(0,3)$-monad on a category are defined along similar lines.

Proposition C.14. *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be monads, and $\mathcal{U}_1$ be a comonad equipped with* (C.16) *such that $(\mathcal{V}_1, \mathcal{V}_2)$ is a double monad, and $(\mathcal{V}_1, \mathcal{U}_1)$ and $(\mathcal{V}_2, \mathcal{U}_1)$ are bimonads. Then the following are equivalent.*

(1) *$(\mathcal{V}_1, \mathcal{V}_2, \mathcal{U}_1)$ is a $(2,1)$-monad.*
(2) *$\mathcal{U}_1 : (\mathcal{V}_1, \mathcal{V}_2) \to (\mathcal{V}_1, \mathcal{V}_2)$ is a double lax functor.*
(3) *$\mathcal{V}_2 : (\mathcal{V}_1, \mathcal{U}_1) \to (\mathcal{V}_1, \mathcal{U}_1)$ is a bilax functor.*
(4) *$\mathcal{V}_1 : (\mathcal{V}_2, \mathcal{U}_1) \to (\mathcal{V}_2, \mathcal{U}_1)$ is a double colax functor.*

Each statement boils down to commutativity of diagram (C.17). Statement (4) involves a double colax functor between bimonads, something that we have not discussed thus far. It is treated a little later.

The explicit description of the 2-category $m^p c^q(\mathsf{Cat})$ is given below.

Definition C.15. A *(p,q)-monad* is a $(p+q)$-tuple $(\mathcal{V}_1, \dots, \mathcal{V}_p, \mathcal{U}_q, \dots, \mathcal{U}_1)$ equipped with distributive laws, one for each subpair, such that every subtriple is a $(3,0)$- or a $(2,1)$- or a $(1,2)$- or a $(0,3)$-monad as may be the case.

A (p,q)-lax functor

$$(\mathcal{V}_1, \dots, \mathcal{V}_p, \mathcal{U}_q, \dots, \mathcal{U}_1) \to (\mathcal{V}'_1, \dots, \mathcal{V}'_p, \mathcal{U}'_q, \dots, \mathcal{U}'_1)$$

is a functor $\mathcal{F}$ equipped with a tuple $(\varphi_1, \dots, \varphi_p, \psi_q, \dots, \psi_1)$ of natural transformations such that $(\mathcal{F}, \varphi_i) : \mathcal{V}_i \to \mathcal{V}'_i$ are lax functors, $(\mathcal{F}, \psi_i) : \mathcal{U}_i \to \mathcal{U}'_i$ are colax functors, and any subpair is a double lax, or a bilax or a double colax functor as may be the case.

A morphism

$$(\mathcal{F}, \varphi_1, \dots, \varphi_p, \psi_q, \dots, \psi_1) \to (\tilde{\mathcal{F}}, \tilde{\varphi}_1, \dots, \tilde{\varphi}_p, \tilde{\psi}_q, \dots, \tilde{\psi}_1)$$

between (p,q)-lax functors is a natural transformation $\theta : \mathcal{F} \to \tilde{\mathcal{F}}$ such that diagram (C.3) commutes for all $\varphi := \varphi_i$ and diagram (C.7) commutes for all $\psi := \psi_i$.

C.1.6. Companion monad and comonad constructions. Analogous to $m(\mathsf{Cat})$ and $c(\mathsf{Cat})$, we now describe two more constructions denoted $m^*(\mathsf{Cat})$ and $c^*(\mathsf{Cat})$.

Definition C.16. Let $\mathcal{V}$ be a monad on C and $\mathcal{V}'$ a monad on C'. A *colax functor* of monads $\mathcal{V} \to \mathcal{V}'$ is a functor $\mathcal{F} : \mathsf{C} \to \mathsf{C}'$ equipped with a natural transformation $\psi : \mathcal{F}\mathcal{V} \to \mathcal{V}'\mathcal{F}$ such that the diagrams

(C.18)
$$\begin{tikzcd}
\mathcal{V}'\mathcal{V}'\mathcal{F} \arrow[d, "\mu'\mathcal{F}"'] & \mathcal{V}'\mathcal{F}\mathcal{V} \arrow[l, "\mathcal{V}'\psi"'] & \mathcal{F}\mathcal{V}\mathcal{V} \arrow[l, "\psi\mathcal{V}"'] \arrow[d, "\mathcal{F}\mu"] \\
\mathcal{V}'\mathcal{F} & & \mathcal{F}\mathcal{V} \arrow[ll, "\psi"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{V}'\mathcal{F} & & \mathcal{F}\mathcal{V} \arrow[ll, "\psi"'] \\
& \mathcal{F} \arrow[ul, "\iota'\mathcal{F}"] \arrow[ur, "\mathcal{F}\iota"'] &
\end{tikzcd}$$

commute. We use the notation $(\mathcal{F}, \psi) : \mathcal{V} \to \mathcal{V}'$. Note that diagrams (C.18) are obtained from (C.2) by reversing the horizontal arrows.

Similarly, a morphism $(\mathcal{F}, \psi) \to (\tilde{\mathcal{F}}, \tilde{\psi})$ between colax functors is a natural transformation $\theta : \mathcal{F} \to \tilde{\mathcal{F}}$ such that diagram (C.3) commutes with horizontal arrows reversed.

This defines a 2-category whose 0-cells are monads, 1-cells are colax functors of monads, 2-cells are morphisms of colax functors. We denote it by $m^*(\mathsf{Cat})$. Dually, we define the 2-category $c^*(\mathsf{Cat})$ whose 0-cells are comonads, 1-cells are lax functors of comonads, 2-cells are morphisms of lax functors.

In these constructions, Cat may be replaced by any 2-category D, and we denote the resulting 2-categories by $m^*(\mathsf{D})$ and $c^*(\mathsf{D})$. The relations among these constructions, and the earlier monad and comonad constructions are given below.

Theorem C.17. *Let* D *be a* 2*-category. Then*

$$m(c(\mathsf{D})) = c(m(\mathsf{D})), \quad m^*(c^*(\mathsf{D})) = c^*(m^*(\mathsf{D})),$$
$$m(m^*(\mathsf{D})) = m^*(m(\mathsf{D})), \quad c^*(c(\mathsf{D})) = c(c^*(\mathsf{D}))$$

as 2*-categories.*

This extends Theorem C.11 and the proof is a similar calculation.

TABLE C.2. 2-categories of higher monads on level 2.

2-category	0-cell	1-cell
$mm(\mathsf{Cat})$	double monad	double lax functor
$mm^*(\mathsf{Cat}) = m^*m(\mathsf{Cat})$	double monad	bilax functor
$m^*m^*(\mathsf{Cat})$	double monad	double colax functor
$c^*m(\mathsf{Cat})$	bimonad	double lax functor
$mc(\mathsf{Cat}) = cm(\mathsf{Cat})$	bimonad	bilax functor
$m^*c(\mathsf{Cat})$	bimonad	double colax functor
$mc^*(\mathsf{Cat})$	opp. bimonad	double lax functor
$m^*c^*(\mathsf{Cat}) = c^*m^*(\mathsf{Cat})$	opp. bimonad	bilax functor
$cm^*(\mathsf{Cat})$	opp. bimonad	double colax functor
$c^*c^*(\mathsf{Cat})$	double comonad	double lax functor
$cc^*(\mathsf{Cat}) = c^*c(\mathsf{Cat})$	double comonad	bilax functor
$cc(\mathsf{Cat})$	double comonad	double colax functor

The 2-categories that arise from Cat by a two-step process involving the monad and comonad constructions are shown in Table C.2. This supplements Table C.1. Among 0-cells, we have encountered double (co)monads and bimonads before. In addition, there is a notion of an opposite bimonad.

Definition C.18. An *opposite bimonad* on a category C is a triple $(\mathcal{U}, \mathcal{V}, \lambda)$, where $\mathcal{U}$ is a comonad on C, $\mathcal{V}$ is a monad on C, and $\lambda : \mathcal{U}\mathcal{V} \to \mathcal{V}\mathcal{U}$ is a natural transformation such that the diagrams (C.9a) and (C.9b) commute with the horizontal arrows reversed.

To unify notation, we denote a 0-cell in any of these 2-categories by $(\mathcal{B}_1, \mathcal{B}_2, \lambda)$. Each $\mathcal{B}_i$ is either a monad or a comonad, the distributive law has the form $\lambda : \mathcal{B}_1\mathcal{B}_2 \to \mathcal{B}_2\mathcal{B}_1$, and there are four commutative diagrams to be satisfied. Using this notation, the 1-cells can be described as follows.

Definition C.19. Let $(\mathcal{B}_1, \mathcal{B}_2, \lambda)$ and $(\mathcal{B}'_1, \mathcal{B}'_2, \lambda')$ be two 0-cells in the same 2-category.

- A double lax functor between them is a functor $\mathcal{F}$ equipped with natural transformations $\varphi_1 : \mathcal{B}'_1\mathcal{F} \to \mathcal{F}\mathcal{B}_1$ and $\varphi_2 : \mathcal{B}'_2\mathcal{F} \to \mathcal{F}\mathcal{B}_2$ such that $(\mathcal{F}, \varphi_1) : \mathcal{B}_1 \to \mathcal{B}'_1$ is lax, $(\mathcal{F}, \varphi_2) : \mathcal{B}_2 \to \mathcal{B}'_2$ is lax and diagram (C.12) commutes with $\mathcal{V} := \mathcal{B}$.
- A bilax functor between them is a functor $\mathcal{F}$ equipped with natural transformations $\varphi : \mathcal{B}'_1\mathcal{F} \to \mathcal{F}\mathcal{B}_1$ and $\psi : \mathcal{F}\mathcal{B}_2 \to \mathcal{B}'_2\mathcal{F}$ such that $(\mathcal{F}, \varphi) : \mathcal{B}_1 \to \mathcal{B}'_1$ is lax, $(\mathcal{F}, \psi) : \mathcal{B}_2 \to \mathcal{B}'_2$ is colax, and diagram (C.10) commutes with $\mathcal{V} := \mathcal{B}_1$ and $\mathcal{U} := \mathcal{B}_2$.
- Double colax functors are defined similarly.

(The reason this definition is more general than what was written before is that a (co)lax functor is defined for both monads and comonads.)

We now describe the 2-cells.

Definition C.20. Let $(\mathcal{F}, \varphi_1, \varphi_2)$ and $(\tilde{\mathcal{F}}, \tilde{\varphi}_1, \tilde{\varphi}_2)$ be double lax functors $(\mathcal{B}_1, \mathcal{B}_2, \lambda) \to (\mathcal{B}'_1, \mathcal{B}'_2, \lambda')$. A morphism $(\mathcal{F}, \varphi_1, \varphi_2) \to (\tilde{\mathcal{F}}, \tilde{\varphi}_1, \tilde{\varphi}_2)$ is a natural transformation $\theta : \mathcal{F} \to \tilde{\mathcal{F}}$ such that the diagram (C.3) commutes for $\mathcal{V} := \mathcal{B}_1$ and $\varphi := \varphi_1$, and for $\mathcal{V} := \mathcal{B}_2$ and $\varphi := \varphi_2$.

Morphisms between bilax functors and between double colax functors are defined in a similar manner.

This completes the description of all 2-categories which appear in Table C.2.

Fix a word α of length n in the free group on two letters u and v, and a pair (p, q) of nonnegative integers with $p+q = n$. We now define a 2-category $\alpha^{p,q}(\mathsf{Cat})$.

Definition C.21. A 0-cell is a tuple $(\mathcal{B}_1, \dots, \mathcal{B}_n)$ with each $\mathcal{B}_i$ either a monad or a comonad (on the same category C). It is a monad if the i-th letter in α is u and a comonad if the i-th letter in α is v. For each $1 \le i < j \le n$, there is a natural transformation

$$\lambda_{ij} : \mathcal{B}_i\mathcal{B}_j \to \mathcal{B}_j\mathcal{B}_i$$

such that $(\mathcal{B}_i, \mathcal{B}_j, \lambda_{ij})$ is a double monad, bimonad, opposite bimonad, or a double comonad, as may be the case.

A 1-cell between two such tuples $(\mathcal{B}_1, \dots, \mathcal{B}_n) \to (\mathcal{B}'_1, \dots, \mathcal{B}'_n)$ is a functor $\mathcal{F}$ equipped with a tuple $(\varphi_1, \dots, \varphi_n)$ such that $(\mathcal{F}, \varphi_i) : \mathcal{B}_i \to \mathcal{B}'_i$ are lax functors for $1 \le i \le p$, and colax functors for $p+1 \le i \le n$, and any subpair is a double lax, or a bilax or a double colax functor as may be the case.

A 2-cell $(\mathcal{F}, \varphi_1, \dots, \varphi_n) \to (\tilde{\mathcal{F}}, \tilde{\varphi}_1, \dots, \tilde{\varphi}_n)$ is a natural transformation $\theta : \mathcal{F} \to \tilde{\mathcal{F}}$ such that diagram (C.3) commutes for all $\varphi := \varphi_i$ with $1 \le i \le p$ and diagram (C.7) commutes for all $\psi := \varphi_i$ with $p+1 \le i \le n$.

The 2-categories which result from iterations of the monad and comonad constructions are precisely $\alpha^{p,q}(\mathsf{Cat})$ as α, p and q vary. The precise statement is given below, with Cat dropped from the notation for convenience. (In a way, this is apt since $\alpha^{p,q}(\mathsf{D})$ makes sense for any 2-category D.) If α has length two, then we recover the 2-categories in Table C.2.

Theorem C.22. *Let α be a word in the letters u and v. Write $\alpha = \alpha_1\alpha_2$, where α_1 is the initial segment consisting of the first p letters, and α_2 is the final segment consisting of the last q letters. Then*

$$\begin{aligned} m(\alpha^{p,q}) &= (\alpha_1 u \alpha_2)^{p+1,q}, \quad & c(\alpha^{p,q}) &= (\alpha_1 v \alpha_2)^{p,q+1}, \\ m^*(\alpha^{p,q}) &= (\alpha_1 u \alpha_2)^{p,q+1}, \quad & c^*(\alpha^{p,q}) &= (\alpha_1 v \alpha_2)^{p+1,q}. \end{aligned}$$

PROOF. This is a routine calculation, similar to the one in the proof of Theorem C.11. □

Theorem C.22 can be used to deduce Theorem C.17. For instance,

$$mc(\mathsf{D}) = m(v^{0,1}(\mathsf{D})) = (uv)^{0,1}(\mathsf{D}) = c(u^{1,0}(\mathsf{D})) = cm(\mathsf{D}).$$

We studied earlier the iteration of the m and c constructions on Cat. In our present notation,

$$m^p c^q(\mathsf{Cat}) = (u^p v^q)^{p,q}(\mathsf{Cat}).$$

(The word $\alpha = u^p v^q$ consists of p occurrences of u followed by q occurrences of v.)

$$\begin{tikzcd}[column sep=tiny]
 & & & \mathsf{Cat} \arrow[dl, dash] \arrow[dr, dash] & & & \\
 & & c^*(\mathsf{Cat}) \arrow[dl, dash] \arrow[dr, dash] & & m^*(\mathsf{Cat}) \arrow[dl, dash] \arrow[dr, dash] & & \\
 & c^*c^*(\mathsf{Cat}) \arrow[dl, dash] \arrow[dr, dash] & & c^*m^*(\mathsf{Cat}) \arrow[dl, dash] \arrow[dr, dash] & & m^*m^*(\mathsf{Cat}) \arrow[dl, dash] \arrow[dr, dash] & \\
c^*c^*c^*(\mathsf{Cat}) & & c^*c^*m^*(\mathsf{Cat}) & & c^*m^*m^*(\mathsf{Cat}) & & m^*m^*m^*(\mathsf{Cat}) \\
\vdots & & \vdots & & \vdots & & \vdots
\end{tikzcd}$$

FIGURE C.2.

Now let us consider iterations of the m^* and c^* constructions. They commute by Theorem C.17. Let $(c^*)^p(m^*)^q(\mathsf{Cat})$ denote the result of p applications of c^* and q applications of m^*. The first few iterates are shown in Figure C.2. Compare and contrast with Figure C.1. We have

$$(c^*)^p(m^*)^q(\mathsf{Cat}) = (v^p u^q)^{p,q}(\mathsf{Cat}).$$

Observe that a 0-cell is a n-tuple of which the first p are comonads and the last q are monads. So the mixed distributive laws are as for an opposite bimonad. For 1-cells, lax functors go with comonads, while colax functors go with monads.

$$
\begin{tikzcd}[column sep=tiny]
 & & & \mathsf{Cat} \arrow[dl, dash] \arrow[dr, dash] & & & \\
 & & m(\mathsf{Cat}) \arrow[dl, dash] \arrow[dr, dash] & & m^*(\mathsf{Cat}) \arrow[dl, dash] \arrow[dr, dash] & & \\
 & mm(\mathsf{Cat}) \arrow[dl, dash] \arrow[dr, dash] & & mm^*(\mathsf{Cat}) \arrow[dl, dash] \arrow[dr, dash] & & m^*m^*(\mathsf{Cat}) \arrow[dl, dash] \arrow[dr, dash] & \\
mmm(\mathsf{Cat}) & & mmm^*(\mathsf{Cat}) & & mm^*m^*(\mathsf{Cat}) & & m^*m^*m^*(\mathsf{Cat}) \\
\vdots & & \vdots & & \vdots & & \vdots
\end{tikzcd}
$$

FIGURE C.3.

Similarly, combining the m and m^* constructions, we obtain the diagram shown in Figure C.3. We have

$$m^p(m^*)^q(\mathsf{Cat}) = (u^n)^{p,q}(\mathsf{Cat}).$$

In all 2-categories at a fixed level, the 0-cells have the same description. At level n, a 0-cell is a n-tuple $(\mathcal{V}_1, \ldots, \mathcal{V}_n)$ of monads with a distributive law for each subpair such that any subtriple is a triple monad. The 2-categories differ in their 1-cells, the first p are related by lax functors, and the next $q = n - p$ are related by colax functors. All this is in principle identical to the lax-colax triangle for monoidal categories obtained in [**18**, Figure 7.1] (with monads playing the role of pseudomonoids).

Suppose we combine the m and c^* constructions. Then we obtain 2-categories

$$\mathsf{Cat},\ m(\mathsf{Cat}),\ c^*(\mathsf{Cat}),\ mc^*(\mathsf{Cat}),\ c^*m(\mathsf{Cat}),\ mm(\mathsf{Cat}),\ c^*c^*(\mathsf{Cat}), \ldots$$

indexed by the free group on two letters. These are 2-categories of the form $\alpha^{n,0}(\mathsf{Cat})$. A 0-cell in such a 2-category is a tuple obtained by writing a monad everytime an m occurs and a comonad everytime a c^* occurs (from left to right). For instance, a 0-cell in $c^*c^*m(\mathsf{Cat})$ is a triple $(\mathcal{V},\mathcal{U},\mathcal{U}')$ subject to further conditions.

A similar analysis applies if we combine the c and m^* constructions. We get 2-categories of the form $\alpha^{0,n}(\mathsf{Cat})$. Finally, one may combine all four constructions. We can separate out m and c^* (writing them first) from c and m^* (writing them later). So in general a 0-cell is a tuple consisting of monads and comonads in no particular order. Further, the tuple consists of two parts, with lax functors applying to the first part and colax functors applying to the second part. This was the general construction of $\alpha^{p,q}(\mathsf{Cat})$.

C.1.7. Duality. Let D be a 2-category. Let D^{op} denote the 2-category obtained from D by reversing 1-cells, and let D_{op} denote the 2-category obtained from D by reversing 2-cells. These two constructions commute with each other. Thus, $\mathsf{D}^{\mathrm{op}}_{\mathrm{op}}$ denotes the 2-category obtained from D by reversing both 1-cells and 2-cells.

The constructions $m(\mathsf{D})$, $c(\mathsf{D})$, $m^*(\mathsf{D})$, $c^*(\mathsf{D})$ relate to each other as follows.

Lemma C.23. *For any 2-category* D,

$$m^*(\mathsf{D}) = m(\mathsf{D}^{\mathrm{op}})^{\mathrm{op}}, \qquad c^*(\mathsf{D}) = c(\mathsf{D}^{\mathrm{op}})^{\mathrm{op}},$$
$$c(\mathsf{D}) = m(\mathsf{D}_{\mathrm{op}})_{\mathrm{op}}, \qquad c^*(\mathsf{D}) = m^*(\mathsf{D}_{\mathrm{op}})_{\mathrm{op}}$$

as 2-categories.

C.2. Higher monad algebras

Higher monad algebras are constructed in a straightforward manner from the 2-categories of higher monads. Here we concentrate on the case of algebras, coalgebras, bialgebras which arise from monads, comonads, bimonads, respectively.

C.2.1. Monad algebras. Let I denote the one-arrow category: I consists of one object and the identity morphism on that object. There is a unique functor $\mathrm{id} : \mathsf{I} \to \mathsf{I}$. This is a (co, bi)monad with structure maps uniquely defined. We call this the *identity (co, bi)monad*. More generally, it is a higher monad (in every sense of the term that we have discussed).

Now suppose $\mathcal{V}$ is a monad on a category C. Let $\mathcal{V}\text{-}\mathsf{Alg}(\mathsf{C})$ denote the category whose objects are lax functors from $\mathrm{id} \to \mathcal{V}$ and morphisms are morphisms between lax functors. We refer to this as the category of $\mathcal{V}$-algebras. An explicit description is given below. We will also denote the category of $\mathcal{V}$-algebras simply by $\mathcal{V}\text{-}\mathsf{Alg}$, with the category C implicit.

Definition C.24. Let $\mathcal{V}$ be a monad. A $\mathcal{V}$*-algebra* is an object A equipped with a morphism $\alpha : \mathcal{V}A \to A$ such that the diagrams

$$\begin{tikzcd} \mathcal{V}\mathcal{V}A \arrow[r, "\mu A"] \arrow[d, "\mathcal{V}\alpha"'] & \mathcal{V}A \arrow[d, "\alpha"] \\ \mathcal{V}A \arrow[r, "\alpha"'] & A \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{V}A \arrow[dr, "\alpha"] & \\ A \arrow[ur, "\iota A"] \arrow[rr, equal] & & A \end{tikzcd} \tag{C.19}$$

commute. A morphism $f : A \to B$ of $\mathcal{V}$-algebras is a map $A \to B$ such that the diagram

$$\begin{tikzcd} \mathcal{V}A \arrow[r, "\mathcal{V}f"] \arrow[d] & \mathcal{V}B \arrow[d] \\ A \arrow[r, "f"'] & B \end{tikzcd} \tag{C.20}$$

commutes.

By Proposition C.2, composite of lax functors is lax. Since a monad algebra is a lax functor from the identity monad, we deduce the following consequence.

Suppose $(\mathcal{F}, \varphi) : \mathcal{V} \to \mathcal{V}'$ is a lax functor of monads. If A is a $\mathcal{V}$-algebra, then $\mathcal{F}A$ is a $\mathcal{V}'$-algebra:

$$\mathcal{V}'\mathcal{F}A \to \mathcal{F}\mathcal{V}A \to \mathcal{F}A.$$

Further, if $A \to B$ is a morphism of $\mathcal{V}$-algebras, then $\mathcal{F}A \to \mathcal{F}B$ is a morphism of $\mathcal{V}'$-algebras. In other words, a lax functor of monads $(\mathcal{F}, \varphi) : \mathcal{V} \to \mathcal{V}'$ induces a functor

$$\mathcal{V}\text{-}\mathsf{Alg} \to \mathcal{V}'\text{-}\mathsf{Alg}.$$

Remark C.25. If we use colax functors instead of lax, then we obtain the following notion. A $\mathcal{V}$-coalgebra is an object A equipped with a morphism $\alpha : A \to \mathcal{V}A$ such that the diagrams

$$\begin{tikzcd} \mathcal{V}\mathcal{V}A \arrow[r, "\mu A"] & \mathcal{V}A \\ \mathcal{V}A \arrow[u, "\mathcal{V}\alpha"] & A \arrow[l, "\alpha"] \arrow[u, "\alpha"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{V}A & \\ A \arrow[ur, "\iota A"] \arrow[rr, equal] & & A \arrow[ul, "\alpha"'] \end{tikzcd}$$

commute. By the second diagram, $\alpha = \iota A$, and for this choice of α, the first diagram always commutes. So every object A is a $\mathcal{V}$-coalgebra in a unique manner.

Now we turn to the dual notion of comonad coalgebras. These are colax functors from the identity comonad. Explicitly:

Definition C.26. Let $\mathcal{U}$ be a comonad. A *$\mathcal{U}$-coalgebra* is an object C equipped with a morphism $C \to \mathcal{U}C$ such that the diagrams

$$\begin{tikzcd} \mathcal{U}\mathcal{U}C & \mathcal{U}C \arrow[l] \\ \mathcal{U}C \arrow[u] & C \arrow[l] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{U}C \arrow[dl] & \\ C \arrow[rr, equal] & & C \arrow[ul] \end{tikzcd} \tag{C.21}$$

commute. A morphism of $\mathcal{U}$-coalgebras is defined similarly.

Let $\mathcal{U}$-Coalg denote the category of $\mathcal{U}$-coalgebras. A colax functor of comonads $(\mathcal{F}, \psi) : \mathcal{U} \to \mathcal{U}'$ induces a functor

$$\mathcal{U}\text{-}\mathsf{Coalg} \to \mathcal{U}'\text{-}\mathsf{Coalg}.$$

Now combining monad algebras and comonad coalgebras, we have the notion of bimonad bialgebras. These are bilax functors from the identity bimonad. Explicitly:

Definition C.27. A *bialgebra over a bimonad* $(\mathcal{V}, \mathcal{U}, \lambda)$ is a triple (H, μ, Δ) such that (H, μ) is an algebra over $\mathcal{V}$, (H, Δ) is a coalgebra over $\mathcal{U}$, and the diagram

$$\begin{tikzcd} \mathcal{V}H \arrow[r] \arrow[d] & H \arrow[r] & \mathcal{U}H \\ \mathcal{V}\mathcal{U}H \arrow[rr] & & \mathcal{U}\mathcal{V}H \arrow[u] \end{tikzcd} \tag{C.22}$$

commutes. A morphism $H \to H'$ of bialgebras is a map from H to H' which is a morphism of the underlying algebras and coalgebras.

Let $(\mathcal{V}, \mathcal{U}, \lambda)$-$\mathsf{Bialg}$ denote the category of $(\mathcal{V}, \mathcal{U}, \lambda)$-bialgebras.

Proposition C.28. *A bilax functor of bimonads*
$$(\mathcal{F},\varphi,\psi):(\mathcal{V},\mathcal{U},\lambda)\to(\mathcal{V}',\mathcal{U}',\lambda')$$
induces a functor
$$(\mathcal{V},\mathcal{U},\lambda)\text{-}\mathsf{Bialg}\to(\mathcal{V}',\mathcal{U}',\lambda')\text{-}\mathsf{Bialg}.$$

Proof. By Proposition C.5, composite of bilax functors is bilax. Since a bimonad bialgebra is a bilax functor from the identity bimonad, the result follows. For convenience, the direct argument is also given below.

Let us show that if H is a bialgebra over $(\mathcal{V},\mathcal{U},\lambda)$, then $\mathcal{F}H$ is a bialgebra over $(\mathcal{V}',\mathcal{U}',\lambda')$.

We have seen before that $\mathcal{F}H$ is a $\mathcal{V}$-algebra and a $\mathcal{V}'$-coalgebra. It remains to check (C.22) for $\mathcal{F}H$. This follows from the commutativity of the following diagram.

$$\begin{tikzcd}
\mathcal{V}'\mathcal{F}H \arrow[r] \arrow[d] & \mathcal{F}\mathcal{V}H \arrow[r] \arrow[d] & \mathcal{F}H \arrow[r] & \mathcal{F}\mathcal{U}H \arrow[r] & \mathcal{U}'\mathcal{F}H \\
\mathcal{V}'\mathcal{F}\mathcal{U}H \arrow[r] \arrow[d] & \mathcal{F}\mathcal{V}\mathcal{U}H \arrow[rr] & & \mathcal{F}\mathcal{U}\mathcal{V}H \arrow[u] \arrow[dr] & \\
\mathcal{V}'\mathcal{U}'\mathcal{F}H \arrow[r] & \mathcal{U}'\mathcal{V}'\mathcal{F}H \arrow[rrr] & & & \mathcal{U}'\mathcal{F}\mathcal{V}H \arrow[uu]
\end{tikzcd}$$

The squares commute by naturality, the pentagon commutes by (C.22), and the hexagon commutes by (C.10). ☐

Algebras over a double (co)monad can be defined in a similar way. Note that: An algebra A over a double monad $(\mathcal{V}_1,\mathcal{V}_2,\lambda)$ is also an algebra over the monad $\mathcal{V}_2\mathcal{V}_1$ via
$$\mathcal{V}_2\mathcal{V}_1A\to\mathcal{V}_2A\to A.$$
Conversely, an algebra over $\mathcal{V}_2\mathcal{V}_1$ is an algebra over $(\mathcal{V}_1,\mathcal{V}_2,\lambda)$ via
$$\mathcal{V}_1A\to\mathcal{V}_2\mathcal{V}_1A\to A \quad\text{and}\quad \mathcal{V}_2A\to\mathcal{V}_2\mathcal{V}_1A\to A.$$
In fact, the category of $(\mathcal{V}_1,\mathcal{V}_2,\lambda)$-algebras is isomorphic to the category of $(\mathcal{V}_2\mathcal{V}_1)$-algebras.

C.2.2. Alternative descriptions of bialgebras. A bialgebra over a bimonad can be viewed as a coalgebra in a category of algebras, and vice versa. This is explained below.

Theorem C.29. *Let $(\mathcal{V},\mathcal{U},\lambda)$ be a bimonad. Then $\mathcal{U}$ induces a comonad on $\mathcal{V}$-Alg, and dually $\mathcal{V}$ induces a monad on $\mathcal{U}$-Coalg.*

Proof. By Theorem C.11, $\mathcal{U}$ is a comonad in $m(\mathsf{Cat})$ based at $\mathcal{V}$ which is a 0-cell in $m(\mathsf{Cat})$. So it induces a comonad on the category of lax functors from the identity monad to $\mathcal{V}$. Since we have not spoken in detail about 2-categories, the direct argument is also given below.

Suppose A is a $\mathcal{V}$-algebra. In particular, there is a morphism $\mathcal{V}A\to A$. Now define a morphism $\mathcal{V}(\mathcal{U}A)\to\mathcal{U}A$ as the composite

$$\mathcal{V}\mathcal{U}A\to\mathcal{U}\mathcal{V}A\to\mathcal{U}A. \tag{C.23}$$

The first map uses the mixed distributive law, while the second map uses the $\mathcal{V}$-algebra structure of A. This turns $\mathcal{U}A$ into a $\mathcal{V}$-algebra: The axioms (C.19) are checked below.

$$
\begin{tikzcd}
\mathcal{V}\mathcal{V}\mathcal{U}A \arrow[rr] \arrow[d] & & \mathcal{V}\mathcal{U}A \arrow[d] \\
\mathcal{V}\mathcal{U}\mathcal{V}A \arrow[r] \arrow[d] & \mathcal{U}\mathcal{V}\mathcal{V}A \arrow[r] \arrow[d] & \mathcal{U}\mathcal{V}A \arrow[d] \\
\mathcal{V}\mathcal{U}A \arrow[r] & \mathcal{U}\mathcal{V}A \arrow[r] & \mathcal{U}A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{V}\mathcal{U}A \arrow[r] & \mathcal{U}\mathcal{V}A \arrow[d] \\
\mathcal{U}A \arrow[u] \arrow[ur] \arrow[r] & \mathcal{U}A
\end{tikzcd}
$$

In the first diagram, the pentagon commutes by the second diagram in (C.9a), the first square commutes by naturality, and the second square commutes since A is a $\mathcal{V}$-algebra. In the second diagram, one triangle commutes by the first diagram in (C.9b) and the other triangle commutes since A is a $\mathcal{V}$-algebra. It is easy to see that if $A \to B$ is a morphism of $\mathcal{V}$-algebras, then so is $\mathcal{U}A \to \mathcal{U}B$. This yields a functor

$$\mathcal{U} : \mathcal{V}\text{-}\mathsf{Alg} \to \mathcal{V}\text{-}\mathsf{Alg}.$$

Since $\mathcal{U}$ is a comonad, there are natural transformations $\mathcal{U} \to \mathcal{U}\mathcal{U}$ and $\mathcal{U} \to \mathrm{id}$. Suppose A is a $\mathcal{V}$-algebra. Then $\mathcal{U}A \to \mathcal{U}\mathcal{U}A$ and $\mathcal{U}A \to A$ are morphisms of $\mathcal{V}$-algebras:

$$
\begin{tikzcd}
\mathcal{V}\mathcal{U}\mathcal{U}A \arrow[r] & \mathcal{U}\mathcal{V}\mathcal{U}A \arrow[r] & \mathcal{U}\mathcal{U}\mathcal{V}A \arrow[r] & \mathcal{U}\mathcal{U}A \\
\mathcal{V}\mathcal{U}A \arrow[u] \arrow[rr] & & \mathcal{U}\mathcal{V}A \arrow[u] \arrow[r] & \mathcal{U}A \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{V}A \arrow[rr] & & A \\
\mathcal{V}\mathcal{U}A \arrow[u] \arrow[r] & \mathcal{U}\mathcal{V}A \arrow[ul] \arrow[r] & \mathcal{U}A \arrow[u]
\end{tikzcd}
$$

In the first diagram, the pentagon commutes by the first diagram in (C.9a), and the square commutes by naturality. In the second diagram, the triangle commutes by the second diagram in (C.9b), and the square commutes by naturality. This shows that $\mathcal{U}$ induces a comonad on the category of $\mathcal{V}$-algebras. $\square$

Theorem C.30. *Let $(\mathcal{V}, \mathcal{U}, \lambda)$ be a bimonad. Then there are isomorphisms of categories*

$$\mathcal{U}\text{-}\mathsf{Coalg}(\mathcal{V}\text{-}\mathsf{Alg}(\mathsf{C})) = (\mathcal{V}, \mathcal{U}, \lambda)\text{-}\mathsf{Bialg}(\mathsf{C}) = \mathcal{V}\text{-}\mathsf{Alg}(\mathcal{U}\text{-}\mathsf{Coalg}(\mathsf{C})).$$

Proof. Suppose H is a $\mathcal{V}$-algebra and a $\mathcal{U}$-coalgebra. Then axiom (C.22) is equivalent to saying that $H \to \mathcal{U}H$ is a morphism of $\mathcal{V}$-algebras, with the $\mathcal{V}$-algebra structure of $\mathcal{U}H$ given by (C.23). The first equivalence follows. The second equivalence then follows by duality. $\square$

Corollary C.31. *Let $(\mathcal{V}, \mathcal{U}, \lambda)$ be a bimonad. If $A \to B$ is a morphism of $\mathcal{V}$-algebras, then $\mathcal{U}A \to \mathcal{U}B$ is a morphism of $(\mathcal{V}, \mathcal{U}, \lambda)$-bialgebras. Similarly, if $C \to D$ is a morphism of $\mathcal{U}$-coalgebras, then $\mathcal{V}C \to \mathcal{V}D$ is a morphism of $(\mathcal{V}, \mathcal{U}, \lambda)$-bialgebras.*

C.2.3. Free algebra over a monad. Suppose $\mathcal{V}$ is a monad over C and A is any object in C. Then $\mathcal{V}A$ is a $\mathcal{V}$-algebra whose structure map is defined by

$$\mathcal{V}\mathcal{V}A \xrightarrow{\mu A} \mathcal{V}A.$$

This construction defines a functor

$$\mathcal{V} : \mathsf{C} \to \mathcal{V}\text{-}\mathsf{Alg}(\mathsf{C}), \qquad A \mapsto \mathcal{V}A.$$

For any $\mathcal{V}$-algebra B and a morphism $f : A \to B$, there is a unique morphism $\mathcal{V}A \to B$ of $\mathcal{V}$-algebras such that the diagram

$$\begin{tikzcd} \mathcal{V}A \arrow[dr, dashed] & \\ A \arrow[u, "\iota A"] \arrow[r, "f"'] & B \end{tikzcd}$$

commutes. Explicitly, the morphism $\mathcal{V}A \to B$ is given by the composite

$$\mathcal{V}A \xrightarrow{\mathcal{V}f} \mathcal{V}B \to B.$$

This says that $\mathcal{V}A$ is the free $\mathcal{V}$-algebra on A.

Dually, $\mathcal{U}A$ carries the structure of a $\mathcal{U}$-coalgebra, and it is the cofree $\mathcal{U}$-coalgebra on A.

Theorem C.32. *For a monad $\mathcal{V}$ and a comonad $\mathcal{U}$, the functors*

$$\mathcal{V} : \mathsf{C} \to \mathcal{V}\text{-}\mathsf{Alg}(\mathsf{C}), \quad A \mapsto \mathcal{V}A \qquad \textit{and} \qquad \mathcal{U} : \mathsf{C} \to \mathcal{U}\text{-}\mathsf{Coalg}(\mathsf{C}), \quad A \mapsto \mathcal{U}A$$

are, respectively, the left and right adjoints of the forgetful functors.

For a bimonad $(\mathcal{V}, \mathcal{U}, \lambda)$, the functors

$$\mathcal{V} : \mathcal{U}\text{-}\mathsf{Coalg} \to (\mathcal{V}, \mathcal{U}, \lambda)\text{-}\mathsf{Bialg} \qquad \textit{and} \qquad \mathcal{U} : \mathcal{V}\text{-}\mathsf{Alg} \to (\mathcal{V}, \mathcal{U}, \lambda)\text{-}\mathsf{Bialg}$$

are, respectively, the left and right adjoints of the forgetful functors.

PROOF. The first statement follows from the preceding discussion. Now consider the second statement. For the claim regarding $\mathcal{V}$, apply the first statement to $\mathsf{C} := \mathcal{U}\text{-}\mathsf{Coalg}$: By Theorem C.29, $\mathcal{V}$ induces a monad on $\mathcal{U}\text{-}\mathsf{Coalg}$, and the category of $\mathcal{V}$-algebras on it is equivalent to the category of $(\mathcal{V}, \mathcal{U}, \lambda)$-bialgebras by Theorem C.30. The claim regarding $\mathcal{U}$ is similar. □

Proposition C.33. *Let $(\mathcal{V}, \mathcal{U}, \lambda)$ be a bimonad. Then, for any $\mathcal{U}$-coalgebra C and $\mathcal{V}$-algebra A, there is a natural bijection*

$$\mathsf{C}(C, A) \xrightarrow{\cong} (\mathcal{V}, \mathcal{U}, \lambda)\text{-}\mathsf{Bialg}(\mathcal{V}C, \mathcal{U}A).$$

PROOF. The adjunctions in Theorem C.32 yield bijections

$$\mathsf{C}(C, A) \cong \mathcal{U}\text{-}\mathsf{Coalg}(C, \mathcal{U}A) \cong (\mathcal{V}, \mathcal{U}, \lambda)\text{-}\mathsf{Bialg}(\mathcal{V}C, \mathcal{U}A)$$

and

$$\mathsf{C}(C, A) \cong \mathcal{V}\text{-}\mathsf{Alg}(\mathcal{V}C, A) \cong (\mathcal{V}, \mathcal{U}, \lambda)\text{-}\mathsf{Bialg}(\mathcal{V}C, \mathcal{U}A).$$

Both composites yield the same bijection and this is the bijection we want. Let us spell this out. Given a morphism $f : C \to A$ in C, first lift it to a morphism $C \to \mathcal{U}A$ of $\mathcal{U}$-coalgebras, and then extend it to a morphism $\mathcal{V}C \to \mathcal{U}A$ of $(\mathcal{V}, \mathcal{U}, \lambda)$-bialgebras. Alternatively, first extend to a morphism

$\mathcal{V}C \to A$ of $\mathcal{V}$-algebras, and then lift. Explicitly, the morphism is given by the composite

$$\begin{tikzcd}
 & & \mathcal{U}\mathcal{V}C \arrow[dr] & & \\
\mathcal{V}C \arrow[r] & \mathcal{V}\mathcal{U}C \arrow[ur] \arrow[rr] \arrow[dr] & & \mathcal{U}\mathcal{V}A \arrow[r] & \mathcal{U}A. \\
 & & \mathcal{V}\mathcal{U}A \arrow[ur] & &
\end{tikzcd}$$

Conversely, given a morphism $\mathcal{V}C \to \mathcal{U}A$ of $(\mathcal{V}, \mathcal{U}, \lambda)$-bialgebras, the composite

$$C \to \mathcal{V}C \to \mathcal{U}A \to A$$

yields a morphism in C. The two constructions are inverses of each other. $\square$

Proposition C.34. *Let $(\mathcal{V}, \mathcal{U}, \lambda)$ be a bimonad, and A be any object in C. Then $\lambda_A : \mathcal{V}\mathcal{U}A \to \mathcal{U}\mathcal{V}A$ is a morphism of $(\mathcal{V}, \mathcal{U}, \lambda)$-bialgebras.*

Proof. Note that $\mathcal{U}A$ is a $\mathcal{U}$-coalgebra, so $\mathcal{V}$ applied to it is a $(\mathcal{V}, \mathcal{U}, \lambda)$-bialgebra. Explicitly, the product and coproduct are

$$\mathcal{V}\mathcal{V}\mathcal{U}A \xrightarrow{\mu\mathcal{U}A} \mathcal{V}\mathcal{U}A \quad \text{and} \quad \mathcal{V}\mathcal{U}A \xrightarrow{\mathcal{V}\Delta A} \mathcal{V}\mathcal{U}\mathcal{U}A \xrightarrow{\lambda\mathcal{U}A} \mathcal{U}\mathcal{V}\mathcal{U}A.$$

The bialgebra structure of $\mathcal{U}\mathcal{V}A$ is similar. We need to check that $\mathcal{V}\mathcal{U}A \to \mathcal{U}\mathcal{V}A$ is a morphism of $\mathcal{U}$-coalgebras and of $\mathcal{V}$-algebras. The check reduces to the two diagrams in (C.9a), respectively.

Alternatively, we may apply Proposition C.33 to the composite $\mathcal{U}A \to A \to \mathcal{V}A$ and check that the resulting morphism of $(\mathcal{V}, \mathcal{U}, \lambda)$-bialgebras agrees with λ_A. $\square$

Proposition C.35. *Let $(\mathcal{F}, \varphi) : \mathcal{V} \to \mathcal{V}'$ be a lax functor of monads, and A be any object in C. Then $\mathcal{V}'\mathcal{F}A \to \mathcal{F}\mathcal{V}A$ is a morphism of $\mathcal{V}'$-algebras.*

Dually, let $(\mathcal{F}, \psi) : \mathcal{U} \to \mathcal{U}'$ be a colax functor of comonads, and A be any object in C. Then $\mathcal{F}\mathcal{U}A \to \mathcal{U}'\mathcal{F}A$ is a morphism of $\mathcal{U}'$-coalgebras.

Proof. We explain the first part, the second being similar. In the map $\mathcal{V}'\mathcal{F}A \to \mathcal{F}\mathcal{V}A$, the domain $\mathcal{V}'\mathcal{F}A$ is the free $\mathcal{V}'$-algebra on $\mathcal{F}A$, and the codomain $\mathcal{F}\mathcal{V}A$ is $\mathcal{F}$ applied to $\mathcal{V}A$ which is the free $\mathcal{V}$-algebra on A. The check that this map is a morphism of $\mathcal{V}'$-algebras reduces to the first diagram in (C.2). $\square$

Proposition C.36. *Let $(\mathcal{F}, \varphi, \psi) : (\mathcal{V}, \mathcal{U}, \lambda) \to (\mathcal{V}', \mathcal{U}', \lambda')$ be bilax.*

1. *If C is a $\mathcal{U}$-coalgebra, then $\mathcal{V}'\mathcal{F}C \to \mathcal{F}\mathcal{V}C$ is a morphism of $(\mathcal{V}', \mathcal{U}', \lambda')$-bialgebras.*
2. *If A is a $\mathcal{V}$-algebra, then $\mathcal{F}\mathcal{U}A \to \mathcal{U}'\mathcal{F}A$ is a morphism of $(\mathcal{V}', \mathcal{U}', \lambda')$-bialgebras.*

Proof. We explain the first part, the second being similar. By Proposition C.35, we know that $\mathcal{V}'\mathcal{F}C \to \mathcal{F}\mathcal{V}C$ is a morphism of $\mathcal{V}'$-algebras. The following diagram

$$\begin{tikzcd}
\mathcal{V}'\mathcal{F}C \arrow[r] \arrow[d] & \mathcal{V}'\mathcal{F}\mathcal{U}C \arrow[r] \arrow[d] & \mathcal{V}'\mathcal{U}'\mathcal{F}C \arrow[r] & \mathcal{U}'\mathcal{V}'\mathcal{F}C \arrow[d] \\
\mathcal{F}\mathcal{V}C \arrow[r] & \mathcal{F}\mathcal{V}\mathcal{U}C \arrow[r] & \mathcal{F}\mathcal{U}\mathcal{V}C \arrow[r] & \mathcal{U}'\mathcal{F}\mathcal{V}C
\end{tikzcd}$$

shows that it is a morphism of $\mathcal{U}'$-coalgebras. The square commutes by naturality, and the hexagon by (C.10). ☐

Proposition C.37. *Let $(\mathcal{F}, \varphi, \psi) : (\mathcal{V}, \mathcal{U}, \lambda) \to (\mathcal{V}', \mathcal{U}', \lambda')$ be bilax, and A be any object in C. Then the following diagram of $(\mathcal{V}', \mathcal{U}', \lambda')$-bialgebras*

$$\begin{tikzcd}
\mathcal{V}'\mathcal{F}\mathcal{U}A \arrow[r] \arrow[d] & \mathcal{F}\mathcal{V}\mathcal{U}A \arrow[r] & \mathcal{F}\mathcal{U}\mathcal{V}A \arrow[d] \\
\mathcal{V}'\mathcal{U}'\mathcal{F}A \arrow[r] & \mathcal{U}'\mathcal{V}'\mathcal{F}A \arrow[r] & \mathcal{U}'\mathcal{F}\mathcal{V}A
\end{tikzcd} \tag{C.24}$$

commutes.

Proof. The commutativity of the diagram follows from (C.10). It is straightforward to check directly that each map is a morphism of bialgebras. We indicate below how this may be deduced from the previous results.

By Proposition C.34, $\mathcal{V}\mathcal{U}A \to \mathcal{U}\mathcal{V}A$ is a morphism of bialgebras, and $\mathcal{F}$ preserves bialgebra morphisms by Proposition C.28. Hence, $\mathcal{F}\mathcal{V}\mathcal{U}A \to \mathcal{F}\mathcal{U}\mathcal{V}A$ is a morphism of bialgebras. Similarly, again by Proposition C.34, $\mathcal{V}'\mathcal{U}'\mathcal{F}A \to \mathcal{U}'\mathcal{V}'\mathcal{F}A$ is a morphism of bialgebras.

By Proposition C.35, $\mathcal{F}\mathcal{U}A \to \mathcal{U}'\mathcal{F}A$ is a morphism of $\mathcal{U}'$-coalgebras. So applying $\mathcal{V}'$ yields a morphism of $(\mathcal{V}', \mathcal{U}', \lambda')$-bialgebras by Corollary C.31. By a similar argument, $\mathcal{U}'\mathcal{V}'\mathcal{F}A \to \mathcal{U}'\mathcal{F}\mathcal{V}A$ is a morphism of bialgebras.

By Proposition C.36, $\mathcal{V}'\mathcal{F}\mathcal{U}A \to \mathcal{F}\mathcal{V}\mathcal{U}A$ and $\mathcal{F}\mathcal{U}\mathcal{V}A \to \mathcal{U}'\mathcal{F}\mathcal{V}A$ are morphism of bialgebras. ☐

C.3. Adjunctions

We briefly discuss lax-lax, colax-colax, colax-lax adjunctions.

C.3.1. Lax-lax and colax-colax adjunctions. A *lax-lax adjunction* is an adjunction in the 2-category $m(\mathsf{Cat})$. Dually, a *colax-colax adjunction* is an adjunction in the 2-category $c(\mathsf{Cat})$. Explicitly: Let C (resp. C') be a category equipped with a monad $\mathcal{V}$ (resp. $\mathcal{V}'$). Let $(\mathcal{F}, \varphi) : \mathcal{V} \to \mathcal{V}'$ and $(\mathcal{G}, \gamma) : \mathcal{V}' \to \mathcal{V}$ be lax functors. Suppose $(\mathcal{F}, \mathcal{G})$ is an adjunction between the categories C and C'. We say that this adjunction is *lax-lax* if the unit and counit of this adjunction are morphisms of lax functors, or equivalently, if the diagrams

$$\begin{tikzcd}
\mathcal{V} \arrow[r] \arrow[d] & \mathcal{G}\mathcal{F}\mathcal{V} \\
\mathcal{V}\mathcal{G}\mathcal{F} \arrow[r] & \mathcal{G}\mathcal{V}'\mathcal{F} \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}\mathcal{V}\mathcal{G} \arrow[d] & \mathcal{V}'\mathcal{F}\mathcal{G} \arrow[l] \arrow[d] \\
\mathcal{F}\mathcal{G}\mathcal{V}' \arrow[r] & \mathcal{V}'
\end{tikzcd} \tag{C.25}$$

commute. Dual diagrams can be drawn for colax-colax adjunctions using comonads $\mathcal{U}$ and $\mathcal{U}'$.

Proposition C.38. *A lax-lax adjunction between* C *and* C' *wrt monads* $\mathcal{V}$ *and* $\mathcal{V}'$ *induces an adjunction between* $\mathcal{V}$-Alg *and* $\mathcal{V}'$-Alg*. Dually, a colax-colax adjunction between* C *and* C' *wrt comonads* $\mathcal{U}$ *and* $\mathcal{U}'$ *induces an adjunction between* $\mathcal{U}$-Coalg *and* $\mathcal{U}'$-Coalg*.*

This follows from the definitions.

C.3.2. Colax-lax adjunctions and mates. Let C (resp. C') be a category equipped with a monad $\mathcal{V}$ (resp. $\mathcal{V}'$). Suppose $(\mathcal{F}, \mathcal{G})$ is an adjunction between the categories C and C'. Then colax structures on $\mathcal{F}$ correspond to lax structures on $\mathcal{G}$. Explicitly: Given a colax structure $\psi : \mathcal{F}\mathcal{V} \to \mathcal{V}'\mathcal{F}$, one can define the lax structure

$$\varphi : \mathcal{V}\mathcal{G} \to \mathcal{G}\mathcal{F}\mathcal{V}\mathcal{G} \to \mathcal{G}\mathcal{V}'\mathcal{F}\mathcal{G} \to \mathcal{G}\mathcal{V}'. \tag{C.26}$$

Conversely, given a lax structure $\varphi : \mathcal{V}\mathcal{G} \to \mathcal{G}\mathcal{V}'$, one can define the colax structure

$$\psi : \mathcal{F}\mathcal{V} \to \mathcal{F}\mathcal{V}\mathcal{G}\mathcal{F} \to \mathcal{F}\mathcal{G}\mathcal{V}'\mathcal{F} \to \mathcal{V}'\mathcal{F}. \tag{C.27}$$

The structures φ and ψ are called *mates* of each other. In this situation, we say that $(\mathcal{F}, \psi)$ and $(\mathcal{G}, \varphi)$ form a *colax-lax adjunction*.

Proposition C.39. *An adjunction is colax-lax iff either of the two diagrams*

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & \mathcal{G}\mathcal{F}\mathcal{V} \\ \downarrow & & \downarrow \\ \mathcal{V}\mathcal{G}\mathcal{F} & \longrightarrow & \mathcal{G}\mathcal{V}'\mathcal{F} \end{array} \qquad \begin{array}{ccc} \mathcal{F}\mathcal{V}\mathcal{G} & \longrightarrow & \mathcal{V}'\mathcal{F}\mathcal{G} \\ \downarrow & & \downarrow \\ \mathcal{F}\mathcal{G}\mathcal{V}' & \longrightarrow & \mathcal{V}' \end{array} \tag{C.28}$$

commute.

Proof. Suppose $(\mathcal{F}, \mathcal{G})$ is colax-lax. Then the commutativity of both diagrams can be checked by writing everything in terms of ψ (or φ). Conversely, suppose the first diagram commutes. Then applying $\mathcal{F}$ to it, and composing it with $\mathcal{F}\mathcal{G}\mathcal{V}'\mathcal{F} \to \mathcal{V}'\mathcal{F}$, we see that ψ is determined by φ exactly in the manner which says that ψ and φ are mates. Hence, $(\mathcal{F}, \mathcal{G})$ is colax-lax. $\square$

There is a way to obtain an adjunction between monad-algebras starting with a colax-lax adjunction. This is explained below.

Suppose $(\mathcal{F}, \psi)$ is a colax functor from $\mathcal{V}$ to $\mathcal{V}'$. Given a $\mathcal{V}$-algebra A with structure map $\alpha : \mathcal{V}A \to A$, consider the maps

$$\begin{gathered} \mathcal{V}'\mathcal{F}\mathcal{V}(A) \xrightarrow{\mathcal{V}'\psi} \mathcal{V}'\mathcal{V}'\mathcal{F}(A) \xrightarrow{\mu\mathcal{F}} \mathcal{V}'\mathcal{F}(A), \\ \mathcal{V}'\mathcal{F}\mathcal{V}(A) \xrightarrow{\mathcal{V}'\mathcal{F}(\alpha)} \mathcal{V}'\mathcal{F}(A). \end{gathered} \tag{C.29}$$

These maps form a reflexive pair in the category $\mathcal{V}'$-Alg. The joint splitting is $\mathcal{V}'\mathcal{F}(\iota)$. Assume that the coequalizer of such a pair exists, and call it $\mathcal{F}'(A)$. This then defines a functor

$$\mathcal{F}' : \mathcal{V}\text{-}\mathsf{Alg} \to \mathcal{V}'\text{-}\mathsf{Alg}. \tag{C.30}$$

One can check that the functors $\mathcal{F}$ and $\mathcal{F}'$ intertwine the free algebra functors.

Suppose $\mathcal{F}$ has a right adjoint $\mathcal{G}$. Let φ be the mate of ψ. Then $(\mathcal{G}, \varphi) : \mathcal{V}' \to \mathcal{V}$ is a lax functor of monads. Hence, there is an induced functor

$$\mathcal{G}' : \mathcal{V}'\text{-}\mathsf{Alg} \to \mathcal{V}\text{-}\mathsf{Alg}.$$

The functors $\mathcal{G}$ and $\mathcal{G}'$ intertwine the forgetful functors. Moreover, $\mathcal{G}'$ is the right adjoint of $\mathcal{F}'$ by the adjoint lifting theorem.

Notes

2-categories. 2-categories constitute a special case of two different notions: double categories and bicategories. The first concept to emerge was that of a double category, in the work of Ehresmann [**280**, Section 1], [**281**, Section II.4, page 389]. See also [**282**, Chapter I, Section A, Definition 30], and [**282**, page 324], where Ehresmann comments on the special case of 2-categories. Bicategories were defined by Bénabou [**92**]. Gabriel and Zisman employ 2-categories in their book [**336**, Chapter V, Section 1].

Accessible introductions are provided by Kelly and Street [**525**], [**857**], [**858**], Lack [**553**], Leinster [**592**], Power [**763**]. For book references, see those by Borceux [**132**, Chapter 7], Gray [**369**, Chapters I,2 and I,3], Mac Lane [**623**, Chapter XII], Johnstone [**493**, Section B1.1], Leinster [**590**, Chapter 1]. A 2-category is the same as a category enriched in Cat (viewed as a monoidal category under cartesian product). References on monoidal categories and enrichment are given in the Notes to Appendix B. Brief treatments of 2-categories, bicategories, enrichment can be found in [**18**, Appendix C], [**897**, Part I], [**367**, Chapter 2].

Monads. The notion of a monad (without any specific name) goes back to Godement [**356**, Appendix, Section 3, Axioms (A) and (B)]. Huber used the term 'standard construction' for comonad and 'dual standard construction' for monad [**465**]. This terminology was continued by Kleisli [**534**]. Monads and comonads were called triples and cotriples, respectively, by Eilenberg and Moore [**286**], and also by Beck [**83**], [**86**]. The term monad was first used by Bénabou who defined them in the general context of bicategories [**92**, Section 5.4]. Monads in a 2-category were studied by Street [**856**], see also his paper with Kelly [**525**, Section 3] and with Lack [**554**].

Monads also appear in May's theory of operads [**662**, Section 2]. His [**662**, Construction 2.4] of a monad from an operad can be seen in analogy with (4.42). This construction is also in Smirnov's book [**834**, page 12]. An example of a colax functor of monads (Definition C.16) is considered by Boardman and Vogt [**126**, Definition 2.27].

The monad terminology was promoted by Mac Lane [**623**, page 138] and is largely standard now. Monads are also discussed in the books by Schubert [**817**, Chapter 21], Arbib and Manes [**40**, Chapter 10], Lambek and Scott [**561**, Part 0, Section 6], Barr and Wells [**73**, Chapter 3], Adámek, Herrlich, Strecker [**4**, Chapter V]. For recent references, see the books by Awodey [**54**, Chapter 10], Grandis [**366**, Chapter 3], [**367**, Chapter 1], Melliès [**674**, Chapter 6], Adámek, Rosický, Vitale [**5**, Appendix A], Riehl [**781**, Chapter 5], Spivak [**842**, Section 7.3].

Distributive laws. Distributive laws for monads (Definition C.6) were introduced by Beck [**84**], [**87**]. On [**84**, page 133], he also writes a line about the mixed case. Distributive laws for comonads are defined in Barr's paper [**70**, Definition 2.1], [**72**, Definition 2.1 on page 251]. He credits Appelgate and Beck for this concept. Distributive and mixed distributive laws are both considered by Burroni [**178**], [**179**], [**180**], [**177**]. The definition of a bimonad and bialgebras over a bimonad is given by van Osdol [891, page 235], [892, top part of page 456], see also the paper by Wolff [917, Definition 2.1]. We are using the term 'bimonad' for a triple (monad, comonad, mixed distributive law). However, we warn that this term is used in a different sense by Moerdijk and others [702], [166], [167], [887], [127] and in yet another sense by Mesablishvili, Wisbauer, Livernet [680], [681], [603]. For the latter, see also the Notes to Chapter 3.

A related notion called entwined structures appeared independently later in work of Brzeziński and Majid [170]. Another early paper is that of Brzeziński and Hajac [169]. The connection between distributive laws and entwined structures is explained, for instance, in [439], [679].

Distributive laws are briefly discussed in the book by Barr and Wells [73, Section 9.2].

Duality. The constructions D^{op}, D_{op}, $\mathsf{D}^{\mathrm{op}}_{\mathrm{op}}$ in Appendix C.1.7 (which reverse 1-cells and 2-cells) appear in the paper by Street [856, Section 4]. Other early references are [525, page 82], [369, pages 4 and 5].

2-*category of monads.* Let D be any 2-category. Street [856, Sections 1 and 4] considers the 2-categories $m(\mathsf{D})$ and $m^*(\mathsf{D})$ under the notations $\mathrm{Mnd}(\mathsf{D})$ and $\mathrm{Mnd}(\mathsf{D}^*)^*$, respectively, and connects them by the duality construction, see Lemma C.23. His $\mathrm{Mnd}(\mathsf{D}^*_*)^*_*$ is the same as $c^*(\mathsf{D})$. In [856, Section 6], he considers $m(m(\mathsf{D}))$ and explains its connection to distributive laws. The fact $m(m^*(\mathsf{D})) = m^*(m(\mathsf{D}))$ in Theorem C.17 is stated in [856, Theorem 17]. Another early reference for the 2-categories $m^*(\mathsf{D})$ and $m(\mathsf{D})$ is by Burroni [177, pages 449 and 450]. She also considers comonads in these 2-categories to obtain the two mixed distributive laws in our Definitions C.18 and C.4, see [180, Definitions 2.1 and 2.2], [177, page 474].

Power and Watanabe consider the 2-categories $m(\mathsf{D})$ and $m^*(\mathsf{D})$ in [764, Definitions 2.5 and 2.8]. They also consider the 2-categories $c(\mathsf{D})$ and $c^*(\mathsf{D})$ in [764, Definition 5.2 and Proposition 5.4], respectively. The middle six entries in Table C.2 pertaining to bimonads and opposite bimonads are precisely the six entries in [764, Table 1]. See also [764, Proposition 6.3, Corollary 6.6, Corollary 7.1] which elaborate on these entries.

The 2-categories $m(\mathsf{Cat})$ and $m^*(\mathsf{Cat})$ are denoted $\mathbf{Mnd}_{\mathrm{lax}}$ and $\mathbf{Mnd}_{\mathrm{colax}}$, respectively, by Leinster [590, Section 6.1]. Our terminology of lax and colax functors between monads is the same as his.

Mates. The bijection between colax structures on the left adjoint and lax structures on the right adjoint in Appendix C.3.2 is given by Street [856, Theorem 9]. The general construction of mates is discussed by Kelly and Street [525, Propositions 2.1 and 2.2], [522, Section 1.1] and Gray [369, Corollary I,6.6], see also earlier papers by Linton [599, (1.10)], Kelly [521, Section 2.1], Palmquist [732, Proposition 3.8]. For later references, see those by Leinster [590, Section 6.1, pages 185 and 186], Lack [553, page 115], Grandis [365, Appendix A5], Gurski [400, Section 1.2].

Adjoint lifting theorem. Early references for the adjoint lifting theorem are by Diaconescu [245, Appendix], [246, Theorem 1.6], Johnstone [491, Theorem 2]. A special case was considered earlier by Linton [600, Corollary 1]. The result is stated in Johnstone's books [492, Theorem 0.15 on page 3], [493, Proposition 1.1.3 on page 5]. A detailed discussion is given by Borceux [133, Section 4.5, Theorem 4.5.6 and Exercise 4.8.6]. An example of the adjoint lifting theorem is given in diagrams (8.60).

For related and more general results, see [264, Theorem 1], [762], [73, Section 3.7].

Higher monoidal categories. Our motivation for considering higher monads comes from the parallel with higher monoidal categories [**18**, Chapter 7]. Compare Figures C.1, C.2, C.3 with [**18**, Figure 7.1]. In particular, compare Proposition C.10 with [**18**, Proposition 6.65, item (a)], Table C.1 with [**18**, Table 6.3], Proposition C.14 with [**18**, Proposition 7.3]. Also compare the lax-lax, colax-colax, colax-lax adjunctions in Appendix C.3 with those in [**18**, Section 3.9].

The main examples of bimonads in this text are given in Chapter 3. Bialgebras over these bimonads are various kinds of bimonoids in species (Propositions 3.1, 3.10, 3.11) which are the central objects of this book. Examples of bilax functors between bimonads are given in Propositions 3.13, 3.16, 3.17, 3.20, 3.23, 3.35. More examples are given in Theorems 8.4, 8.8, 8.21, 8.90, 8.98. Compare and contrast these with the bilax monoidal functors between braided monoidal categories in [**18**, Propositions 8.58, 8.64, 9.5, 9.9]. Do the same for the $(2,1)$-monad in Theorem 8.11 and the 3-monoidal category in [**18**, Proposition 8.69]. (See also the Notes to Chapter 8.)

Recall from [**18**, Tables 15.1 and 16.1] the various bilax Fock functors from the category of Joyal species to the category of graded vector spaces. In a follow-up work, we will construct bilax Fock functors between bimonads on the categories of Coxeter species and Coxeter spaces, see Table I in the Preface. Bialgebras over these bimonads will be termed Coxeter bimonoids and Coxeter bialgebras, respectively.

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List of Notations

A list of important notations used in the book is given below. They are grouped according to topic. We begin by listing some standard notations related to number systems, posets, vector spaces, and categories. These are followed by notations which are more specific to this book. Here we also indicate the chapters and sections where these notations can be located.

Abbreviations

iff	if and only if
lhs	left hand side
rhs	right hand side
wrt	with respect to

Number systems

$\mathbb{N}$	set of nonnegative integers $\{0, 1, 2, \dots\}$
$\mathbb{Z}$	set of integers
$\mathbb{Z}_2$	set of integers modulo 2
$\mathbb{R}$	set of real numbers
$\Bbbk$	field

Posets

$x \le y$	x is smaller than y
$x < y$	x is strictly smaller than y
$x \lessdot y$	x is covered by y, or y covers x
$\mathrm{rk}(x)$	rank of the element x in a graded poset
$\mathrm{rk}(P)$	rank of the poset P
$\wedge$, $\vee$	meet, join

Vector spaces

The following notations can be found in Appendix A.

U, V, W	vector spaces over a field $\Bbbk$
V^*	dual of the vector space V
$\mathrm{Hom}_{\Bbbk}(V, W)$	space of $\Bbbk$-linear maps from V to W
$\mathrm{End}_{\Bbbk}(V)$	space of $\Bbbk$-linear endomorphisms of V
$V \otimes W$	tensor product of V and W over $\Bbbk$
$\ker(f)$, $\mathrm{coker}(f)$	kernel, cokernel of a linear map f
$\mathrm{im}(f)$, $\mathrm{coim}(f)$	image, coimage of a linear map f

Categories

C, D	categories
C^{op}	category opposite to C
$\mathcal{F}$, $\mathcal{G}$	functors
inc	inclusion functor
frg	forgetful functor
$\mathsf{C}(a,b)$	set of morphisms from a to b in the category C
$\mathsf{C} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathsf{D}$	adjunction with left adjoint $\mathcal{F}$ and right adjoint $\mathcal{G}$
$(\mathsf{C}, \bullet)$	monoidal category
$(\mathsf{C}, \bullet, \beta)$	braided monoidal category
$\hom^{\bullet}(A,B)$	internal hom in $(\mathsf{C}, \bullet)$
$A \triangleright X$	copower of X by A
$Y \triangleleft A$	power of Y by A
Set	category of sets
Vec	category of vector spaces over a field $\Bbbk$
$\mathsf{K}(S)$	Karoubi envelope of the semigroup S

Hyperplane arrangements

Most of the notations below are introduced in Sections 1.1, 1.2, 1.3, 1.4, 1.8, 1.11.

Hyperplane arrangements.

$\mathcal{A}$	hyperplane arrangement
$\mathcal{A} \times \mathcal{A}'$	cartesian product of arrangements $\mathcal{A}$ and $\mathcal{A}'$
$\mathcal{A}^{\mathrm{X}}$	arrangement under the flat X of $\mathcal{A}$
$\mathcal{A}_{\mathrm{X}}$	arrangement over the flat X of $\mathcal{A}$
$\mathcal{A}_{\mathrm{Y}}^{\mathrm{X}}$	arrangement over Y and under X
$\mathrm{rk}(\mathcal{A})$	rank of the arrangement $\mathcal{A}$
$\mu(\mathcal{A})$	Möbius number of $\mathcal{A}$
$c(\mathcal{A})$	number of chambers in $\mathcal{A}$
$f(\mathcal{A})$	number of faces in $\mathcal{A}$
$\mathsf{E}^{\mathsf{o}}[\mathcal{A}]$	orientation space of $\mathcal{A}$
$\mathsf{E}^{-}[\mathcal{A}]$	signature space of $\mathcal{A}$

Faces, flats, lunes.

H	hyperplane
h	half-space
H^{+}, H^{-}	the two half-spaces bounded by the hyperplane H
A, B, F, G, H, K	faces
O	central face
$\overline{F}$	face opposite to F
$F \wedge G$	meet of faces F and G
FG	Tits product of faces F and G
K/F	face K viewed in the star of face F
$F\overline{K}/F$	face opposite to K/F in the star of face F
C, D, E	chambers
FC	Tits product of the face F and chamber C
$C \text{ -- } E \text{ -- } D$	minimal gallery from C to D passing through E

$F \,\text{--}\, G \,\text{--}\, H$	minimal gallery between faces F, G, H of the same support
$\mathrm{dist}(C, D)$	minimum length of a gallery connecting C and D
$\mathrm{dist}(F, G)$	distance between faces F and G
X, Y, Z, W	flats
$\mathrm{X} \wedge \mathrm{Y}$	meet of flats X and Y
$\mathrm{X} \vee \mathrm{Y}$	join/Birkhoff product of flats X and Y
$\perp$	minimum flat
$\top$	maximum flat
(H, G)	nested face
(H, D)	top-nested face
$\mathrm{s}(F)$	support of the face F
$\mathrm{s}(H, G)$	support of the nested face (H, G)
L, M, N	lunes
$\mathrm{b}(\mathrm{L})$	base of the lune L
$\mathrm{c}(\mathrm{L})$	case of the lune L
$\mathrm{sk}(\mathrm{L})$	slack of the lune L
$\mathrm{L} \circ \mathrm{M}$	composition of morphisms in the category of lunes
(F, F')	biface
(A, F, F')	local biface
$F \cdot x$, $x \cdot F$	left, right action of the face F on the element x
$\mathrm{X} \cdot x$	action of the flat X on the element x
$\mathrm{rk}(F)$	rank of the face F
$\mathrm{rk}(\mathrm{X})$	rank of the flat X

Sets and maps.

$\Sigma[\mathcal{A}]$	set of faces/Tits monoid
$\Gamma[\mathcal{A}]$	set of chambers
$\Pi[\mathcal{A}]$	set of flats/Birkhoff monoid
$\mathrm{J}[\mathcal{A}]$	set of bifaces/Janus monoid
$\Sigma[\mathcal{A}]_F$	star of F, that is, set of faces greater than F
$\Gamma[\mathcal{A}]_F$	top-star of F, that is, set of chambers greater than F
$\Omega[\mathcal{A}]$	set of cones
$\Lambda[\mathcal{A}]$	set of lunes
${}^{\mathrm{h}}\Sigma$	set of h-faces
${}^{\mathrm{h}}\Pi$	set of h-flats
s	support map
υ	abstract distance function
υ_q	q-distance function on faces

Birkhoff algebra, Tits algebra, Janus algebra

The Birkhoff algebra, Tits algebra, Janus algebra, q-Janus algebra (along with eulerian families and $\mathtt{Q}$-basis elements) are introduced in Sections 1.9. The (commutative, usual, two-sided) Takeuchi elements are defined in Section 1.10.

Birkhoff algebra, Tits algebra, Janus algebra.

$\mathsf{\Pi}[\mathcal{A}]$	Birkhoff algebra
$\mathsf{\Sigma}[\mathcal{A}]$	Tits algebra
$\mathsf{J}[\mathcal{A}]$	Janus algebra
$\mathsf{J}_q[\mathcal{A}]$	q-Janus algebra for a scalar q

Idempotents.

$\mathtt{Q}_{\mathrm{X}}$	$\mathtt{Q}$-basis element of the Birkhoff algebra
$\mathtt{E}$	eulerian family of the Tits algebra
$\mathtt{E}_{\mathrm{X}}$	eulerian idempotent of the Tits algebra
$\mathtt{E}_{\perp}$	first eulerian idempotent of the Tits algebra
$\mathtt{Q}_F$	$\mathtt{Q}$-basis element of the Tits algebra
$\mathtt{Q}_{(F,F')}$	$\mathtt{Q}$-basis element of the q-Janus algebra for q not a root of unity
$\mathtt{Q}'_{(F,F')}$	$\mathtt{Q}'$-basis element of the q-Janus algebra for q not a root of unity
$\mathtt{Q}^d_{(F,F')}$	$\mathtt{Q}^d$-basis element of the q-Janus algebra for q not a root of unity

The elements $\mathtt{Q}_{(F,F')}$, $\mathtt{Q}'_{(F,F')}$, $\mathtt{Q}^d_{(F,F')}$ are idempotent if $F = F'$.

Takeuchi elements.

$\mathtt{Tak}[\mathcal{A}]$	Takeuchi element of the Tits algebra
$\mathrm{s}(\mathtt{Tak}[\mathcal{A}])$	commutative Takeuchi element of the Birkhoff algebra
$\mathbf{Tak}[\mathcal{A}]$	two-sided Takeuchi element of the q-Janus algebra

Incidence algebras

Incidence algebras, zeta and Möbius functions and related notions are introduced in Sections 1.5 and 1.6.

Incidence algebras.

$\mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$	flat-incidence algebra
$\mathrm{I}_{\mathrm{face}}[\mathcal{A}]$	face-incidence algebra
$\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$	lune-incidence algebra
$\mathrm{I}_{\mathrm{biface}}[\mathcal{A}]$	biface-incidence algebra
$\mathrm{I}_{\mathrm{bilune}}[\mathcal{A}]$	bilune-incidence algebra
$\mathrm{I}_{\mathrm{Lie}}[\mathcal{A}]$	Lie-incidence algebra
bc	base-case map from lune- to flat-incidence algebra
p, q	maps from bilune- to lune-incidence algebra
i	map from lune- to bilune-incidence algebra
s, t	elements of any incidence algebra
$\overline{s}$	opposite of s in the lune-incidence algebra

Zeta and Möbius functions.

ζ	zeta function
μ	Möbius function
ζ_q	q-zeta function for q not a root of unity
μ_q	q-Möbius function for q not a root of unity
$\boldsymbol{\zeta}$	noncommutative zeta function
$\boldsymbol{\mu}$	noncommutative Möbius function
$\overline{\boldsymbol{\zeta}}$	opposite of $\boldsymbol{\zeta}$
$\overline{\boldsymbol{\mu}}$	opposite of $\boldsymbol{\mu}$
$\boldsymbol{\zeta}_u$	uniform noncommutative zeta function
$\boldsymbol{\mu}_u$	noncommutative Möbius function inverse to $\boldsymbol{\zeta}_u$
$\boldsymbol{\zeta}_q$	noncommutative q-zeta function for q not a root of unity
$\boldsymbol{\mu}_q$	noncommutative q-Möbius function for q not a root of unity
$\boldsymbol{\zeta}_q$	two-sided q-zeta function for q not a root of unity
$\boldsymbol{\mu}_q$	two-sided q-Möbius function for q not a root of unity
$\boldsymbol{\zeta}_0$	noncommutative 0-zeta function
$\boldsymbol{\mu}_0$	noncommutative 0-Möbius function

r_α, h_α, t_α	elements in the lune-incidence algebra for a scalar α
$\mathrm{sln}_A^{G,H}$	Solomon coefficient

The Solomon coefficients are introduced in Section 1.8.3. They depend on the choice of a noncommutative Möbius function $\boldsymbol{\mu}$.

Braid arrangements

Braid arrangements are a family of arrangements indexed by finite sets. They are briefly discussed in Section 1.13, see also Table 7.2.

$\mathcal{B}^J$	braid arrangement on the finite set J
$F \vDash J$	F is a composition of the set J
$\mathrm{X} \vdash J$	X is a partition of the set J

Species

Species and bimonoids are introduced in Chapter 2 with further theory developed in Parts II and III.

Species.

Species are introduced in Section 2.1, and monoids, comonoids, bimonoids in Section 2.2. Lie monoids and Lie comonoids are in Sections 16.1 and 16.8, respectively.

p, q, r	species
$\mathsf{0}$	zero species
$\mathsf{p}+\mathsf{q}$	direct sum of species p and q
q/p	quotient of q by the subspecies p
p^*	dual of the species p
a, b	monoids in species
c, d	comonoids in species
h, k	bimonoids in species
a_t	a viewed as a trivial monoid
c_t	c viewed as a trivial comonoid
a_α	monoid a whose product is deformed by a parameter α
c_α	comonoid c whose coproduct is deformed by a parameter α
g	Lie monoid in species
k	Lie comonoid in species
S_h	antipode of the bimonoid h

Structure maps.

$\beta_{G,F}$	structure map of a species for faces F, G of the same support
μ_A^F	product component of a monoid for faces $A \leq F$
Δ_A^F	coproduct component of a comonoid for faces $A \leq F$
$\mu_\mathrm{Z}^\mathrm{X}$	product component of a commutative monoid for flats $\mathrm{Z} \leq \mathrm{X}$
$\Delta_\mathrm{Z}^\mathrm{X}$	coproduct component of a cocommutative comonoid for flats $\mathrm{Z} \leq \mathrm{X}$
ν_A^F	Lie bracket component of a Lie monoid for faces $A \lessdot F$
θ_A^F	Lie cobracket component of a Lie comonoid for faces $A \lessdot F$

Operations.

The op and cop constructions on monoids, comonoids, bimonoids are defined in Section 2.10, see Table 2.2 in particular. Cauchy powers are introduced in Section 5.1.

a_{ab}	abelianization of the monoid a
c^{coab}	coabelianization of the comonoid c

a^{op}, ${}^{\mathrm{op}}\mathsf{a}$	opposite of the monoid a
$\mathsf{c}^{\mathrm{cop}}$, ${}^{\mathrm{cop}}\mathsf{c}$	opposite of the comonoid c
$\mathsf{h}^{\mathrm{cop}}$, ${}^{\mathrm{op}}\mathsf{h}$, $\mathsf{h}^{\mathrm{op,cop}}$, ${}^{\mathrm{op,cop}}\mathsf{h}$	op and cop constructions on the bimonoid h
p^{k}	k-th Cauchy power of the species p
$\mathsf{p}^{\overline{k}}$	k-th commutative Cauchy power of the species p

Primitive and decomposable filtrations.
These filtrations are introduced in Sections 5.3 and 5.4, respectively. The map pq_{h} is defined in (5.50).

$\mathcal{P}(\mathsf{c})$	primitive part of the comonoid c
$\mathcal{P}_k(\mathsf{c})$	k-th term of the primitive filtration of the comonoid c
$\mathcal{D}(\mathsf{a})$	decomposable part of the monoid a
$\mathcal{D}_k(\mathsf{a})$	k-th term of the decomposable filtration of the monoid a
$\mathcal{Q}(\mathsf{a})$	indecomposable part of the monoid a
pq_{h}	canonical map from $\mathcal{P}(\mathsf{h})$ to $\mathcal{Q}(\mathsf{h})$ of a q-bimonoid h

Universal constructions.
The constructions involving $\mathcal{T}$ and $\mathcal{S}$ are introduced in the first five sections of Chapter 6, see Table 6.1 for a summary. Those involving **Lie** and $\mathcal{U}$ can be found in Chapter 16.

$\mathcal{T}(\mathsf{p})$	free monoid on the species p
$\mathcal{T}_q(\mathsf{c})$	free q-bimonoid on the comonoid c
$\mathcal{T}^{\vee}(\mathsf{p})$	cofree comonoid on the species p
$\mathcal{T}_q^{\vee}(\mathsf{a})$	cofree q-bimonoid on the monoid a
$\mathcal{S}(\mathsf{p})$	free commutative monoid on the species p
$\mathcal{S}^{\vee}(\mathsf{p})$	cofree cocommutative comonoid on the species p
$\kappa_q : \mathcal{T}_q \to \mathcal{T}_q^{\vee}$	q-norm transformation
$\mathbf{Lie} \circ \mathsf{p}$	free Lie monoid on the species p
$\mathcal{U}(\mathsf{g})$	universal enveloping monoid of the Lie monoid g
$\mathbf{Lie}^{*} \circ \mathsf{p}$	cofree Lie comonoid on the species p
$\mathcal{U}^{\vee}(\mathsf{k})$	universal coenveloping comonoid of the Lie comonoid k

The classical analogues of these constructions include the tensor algebra, symmetric algebra, free Lie algebra, shuffle algebra, quasishuffle algebra, and so on. These are summarized in Tables 6.2, 6.3, 16.4.

Examples of species.
Most of the examples of species below are introduced in Chapter 7. Many of them admit the structure of a bimonoid and also have q-analogues. See Table 7.1 for a summary. The species of chamber maps is in Section 8.5, the species of pairs of chambers is in Section 15.5, and the species of h-faces and h-flats are in Section 11.10.

x	species characteristic of chambers
E	exponential species
E^{-}	signed exponential species
E_M	exponential species E decorated by the vector space M
E_M^{-}	signed exponential species E^{-} decorated by the vector space M
E^{o}	orientation species
$\mathsf{\Gamma}$	species of chambers
$\mathsf{\Pi}$	species of flats
$\mathsf{\Sigma}$	species of faces
J	species of bifaces

Lie	Lie species
Zie	Zie species
G	species of charts
cG	species of connected charts
$\overrightarrow{\mathsf{G}}$	species of dicharts
$\widehat{\mathsf{Q}}$	species of top-nested faces
$\widehat{\Lambda}$	species of top-lunes
$\mathsf{I\Gamma}$	species of pairs of chambers
$\mathcal{C}(\Gamma,\Gamma)$	species of chamber maps
${}^{\mathrm{h}}\Sigma$	species of h-faces
${}^{\mathrm{h}}\Pi$	species of h-flats

Hadamard product.
The Hadamard product of species (and all the notations below) are introduced in Chapter 8.

$\mathsf{p} \times \mathsf{q}$	Hadamard product of species p and q
$\mathsf{p}^{-} = \mathsf{p} \times \mathsf{E}^{-}$	signed partner of the species p
$\mathrm{hom}^{\times}(\mathsf{p},\mathsf{q})$	internal hom for Hadamard product of species p and q
$\mathrm{hom}^{\times}(\mathsf{c},\mathsf{a})$	convolution monoid
$\mathrm{hom}^{\times}(\mathsf{a},\mathsf{c})$	coconvolution comonoid
$\mathrm{hom}^{\times}(\mathsf{h},\mathsf{k})$	biconvolution bimonoid
$\mathrm{end}^{\times}(\mathsf{h})$	specialization of the biconvolution bimonoid
$\mathcal{C}(\mathsf{c},\mathsf{d})$	internal hom for Hadamard product of comonoids c and d
$\mathcal{C}(\mathsf{c},\mathsf{k})$	bimonoid of star families
${}^{\mathsf{co}}\mathcal{C}(\mathsf{c},\mathsf{k})$	bicommutative bimonoid of star families
$\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})$	universal measuring comonoid
${}^{\mathsf{co}}\overline{\mathcal{C}}(\mathsf{a},\mathsf{b})$	universal measuring cocommutative comonoid
$\mathsf{c} \triangleright \mathsf{a}$	copower in the category of monoids enriched over comonoids
$\mathcal{M}(\mathsf{a},\mathsf{b})$	internal hom for Hadamard product of monoids a and b
$\mathcal{M}^{\mathsf{co}}(\mathsf{a},\mathsf{b})$	internal hom for Hadamard product of com. monoids a and b
${}^{\mathsf{co}}\mathcal{B}(\mathsf{h},\mathsf{k})$	internal hom for Hadamard product of cocom. bimonoids h and k

The Hadamard product is further studied in Chapter 15. Some of the notations introduced there are given below.

$\mathsf{p} \star \mathsf{q}$	operation on p and q involving meet of faces
$\mathsf{p} \diamond \mathsf{q}$	operation on p and q involving meet of flats
$\mathsf{p} \diamond\!\star \mathsf{q}$	operation on p and q involving meet of a flat and a face
$\mathcal{T}\!\!\mathcal{T}_q(\mathsf{c},\mathsf{a})$	Hadamard product of $\mathcal{T}(\mathsf{c})$ and $\mathcal{T}_q^{\vee}(\mathsf{a})$
$\mathcal{S}(\mathsf{c},\mathsf{a})$	Hadamard product of $\mathcal{S}(\mathsf{c})$ and $\mathcal{S}^{\vee}(\mathsf{a})$

The operations $\mathsf{p} \star \mathsf{q}$, $\mathsf{p} \diamond \mathsf{q}$, $\mathsf{p} \diamond\!\star \mathsf{q}$ are defined in (15.1), (15.2), (15.32), respectively.

Exponential and logarithm.
The actions of incidence algebras on maps of species from a comonoid to a monoid are introduced in Chapter 9. This leads to various avatars of the exponential and logarithm. These are defined in the first four sections of this chapter.

$\mathbf{exp}(f)$	exponential of a map of species $f : \mathsf{c} \to \mathsf{a}$
$\mathbf{log}(f)$	logarithm of a map of species $f : \mathsf{c} \to \mathsf{a}$
$\mathbf{log}(\mathrm{id})$	logarithm of the identity map on a bimonoid h
$\overline{\mathbf{exp}}$	opposite of $\mathbf{exp}$
$\overline{\mathbf{log}}$	opposite of $\mathbf{log}$

$\exp(f)$	(commutative) exponential of f
$\log(f)$	(commutative) logarithm of f
$\log(\mathrm{id})$	logarithm of the identity map on a bicommutative bimonoid h
$\mathbf{exp}_q(f)$	q-exponential of f for q not a root of unity
$\mathbf{log}_q(f)$	q-logarithm of f for q not a root of unity
$\mathbf{log}_q(\mathrm{id})$	q-logarithm of the identity map on a q-bimonoid h
$\mathbf{exp}_0(f)$	0-exponential of f
$\mathbf{log}_0(f)$	0-logarithm of f
$\mathbf{log}_0(\mathrm{id})$	0-logarithm of the identity map on a 0-bimonoid h

Series.
Series of a species are introduced in Sections 9.5 and 9.6.

$\mathscr{S}(\mathsf{m})$	space of series of the species m
$\mathscr{E}(\mathsf{a})$	space of all exponential series of the monoid a
$\mathscr{P}(\mathsf{c})$	space of all primitive series of the comonoid c
$\mathscr{G}(\mathsf{c})$	space of all group-like series of the comonoid c
$\mathtt{Tak}$	Takeuchi series
$\mathrm{s}(\mathtt{Tak})$	commutative Takeuchi series

Formal power series.
These are discussed in Section 9.8. The series below are summarized in Table 9.2.

$e(x)$	exponential power series
$l(x)$	logarithmic power series
$e_0(x)$	0-analogue of the exponential power series
$l_0(x)$	0-analogue of the logarithmic power series
$g_\alpha(x)$	geometric series

Categories related to species

Base categories.
Most of the categories below are summarized in Table 2.3. For b-species and lb-species, see Table 3.2.

$\mathcal{A}$-$\mathtt{Hyp}$, $\mathcal{A}$-$\mathtt{Hyp}'$	base categories for species
$\mathcal{A}$-$\mathtt{Hyp}(\mathsf{b})$	base category for b-species
$\mathcal{A}$-$\mathtt{Hyp}(\mathsf{lb})$	base category for lb-species
$\mathcal{A}$-$\mathtt{Hyp}^{\mathsf{d}}$	base category for monoids
$\mathcal{A}$-$\mathtt{Hyp}_{\mathsf{c}}$	base category for comonoids
$\mathcal{A}$-$\mathtt{Hyp}^{\mathsf{e}}$	base category for commutative monoids
$\mathcal{A}$-$\mathtt{Hyp}_{\mathtt{r}}$	base category for cocommutative comonoids
$\mathcal{A}$-$\mathtt{Hyp}^{\mathsf{d}}_{\mathsf{c}}$	base category for bimonoids
$\mathcal{A}$-$\mathtt{Hyp}^{\mathsf{e}}_{\mathsf{c}}$	base category for commutative bimonoids
$\mathcal{A}$-$\mathtt{Hyp}^{\mathsf{d}}_{\mathtt{r}}$	base category for cocommutative bimonoids
$\mathcal{A}$-$\mathtt{Hyp}^{\mathsf{e}}_{\mathtt{r}}$	base category for bicommutative bimonoids

Species.
Many of the categories below are summarized in Table 2.2. Signed analogues follow the same notations with an additional (-1) in front.

$\mathcal{A}$-Sp	species
$\mathcal{A}$-SetSp	set-species
$\mathcal{A}$-$\mathsf{Sp}(\mathsf{b})$	b-species

$\mathcal{A}$-$\mathsf{Sp}(\mathsf{lb})$	lb-species
$\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})$	monoids in species
$\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})$	comonoids in species
$\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$	bimonoids in species
q-$\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$	q-bimonoids in species
0-$\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$	0-bimonoids in species
$\mathsf{Mon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})$	commutative monoids in species
${}^{\mathsf{co}}\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})$	cocommutative comonoids in species
$\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})$	commutative bimonoids in species
${}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$	cocommutative bimonoids in species
${}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})$	bicommutative bimonoids in species
$\mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp})$	Lie monoids in species
(-1)-$\mathsf{Mon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})$	signed commutative monoids in species
(-1)-${}^{\mathsf{co}}\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})$	signed cocommutative comonoids in species
(-1)-$\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$	signed bimonoids in species
(-1)-$\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})$	signed commutative signed bimonoids in species
(-1)-${}^{\mathsf{co}}\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$	signed cocommutative signed bimonoids in species
(-1)-${}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathcal{A}\text{-}\mathsf{Sp})$	signed bicommutative signed bimonoids in species
(-1)-$\mathsf{LieMon}(\mathcal{A}\text{-}\mathsf{Sp})$	signed Lie monoids in species

Spaces of maps.

$\mathcal{A}\text{-}\mathsf{Sp}(\mathsf{p},\mathsf{q})$	space of all maps of species from p to q
$\mathsf{Mon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{a},\mathsf{b})$	space of all monoid morphisms from a to b
$\mathsf{Comon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{c},\mathsf{d})$	space of all comonoid morphisms from c to d
q-$\mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})(\mathsf{h},\mathsf{k})$	space of all q-bimonoid morphisms from h to k

Functors.

$\Bbbk(-)$	linearization functor from set-species to species
trv	trivial (co)monoid functor from species to (co)monoids

These can be found in Section 2.14.4 and Section 5.5, respectively.

Monads on the category of species

The monads, comonads, bimonads on the category of species listed below are mostly introduced in Chapter 3, see also Section 4.11.

$\mathcal{T}$	monad whose algebras are monoids in species
$\mathcal{S}$	monad whose algebras are com. monoids in species
$\mathcal{T}_{\sim}$	monad whose algebras are partially com. monoids in species
$\mathcal{E}$	monad whose algebras are signed com. monoids in species
$\mathcal{PT}$	monad whose algebras are Lie monoids in species
$\mathcal{PT}_{-1}$	monad whose algebras are signed Lie monoids in species
$\mathcal{T}^{\vee}$	comonad whose coalgebras are comonoids in species
$\mathcal{S}^{\vee}$	comonad whose coalgebras are cocom. comonoids in species
$\mathcal{E}^{\vee}$	comonad whose coalgebras are signed cocom. comonoids in species
$(\mathcal{T},\mathcal{T}^{\vee},\lambda)$	bimonad whose bialgebras are bimonoids in species
$\tau:\mathcal{T}\to\mathcal{T}$	opposite transformation
$\tau_{-1}:\mathcal{T}\to\mathcal{T}$	signed opposite transformation

Dispecies and operads

Dispecies and operads (and all the notations below) are introduced in Chapter 4.

Dispecies.

$\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$	dispecies
$\mathbf{0}$	zero dispecies
$\mathbf{p}+\mathbf{q}$	direct sum of dispecies $\mathbf{p}$ and $\mathbf{q}$
$\mathbf{p}\circ\mathbf{q}$	substitution product of dispecies $\mathbf{p}$ and $\mathbf{q}$
$\mathbf{x}$	unit object for the substitution product
$\mathbf{p}^{\circ n}$	n-fold product of the dispecies $\mathbf{p}$ with itself
$\mathbf{p}\times\mathbf{q}$	Hadamard product of dispecies $\mathbf{p}$ and $\mathbf{q}$
$\mathbf{p}^{\mathbf{o}}$	oriented partner of the dispecies $\mathbf{p}$
$\mathbf{p}^{-}$	signed partner of the dispecies $\mathbf{p}$
$\mathbf{p}\circ\mathsf{m}$	substitution product of dispecies $\mathbf{p}$ and species m
$\mathsf{p}\circ\mathsf{q}$	substitution product of positive Joyal species p and q

Operads.

$\mathbf{a}$, $\mathbf{b}$	operads
$\mathbf{c}$, $\mathbf{d}$	cooperads
$(\mathbf{a},\mathbf{c},\lambda)$	bioperad
$\mathcal{F}_{\circ}(\mathbf{e})$	free operad on the dispecies $\mathbf{e}$
$\mathbf{mc}$	free operad indexed by maximal chains of faces
$\mathbf{a}\bigcirc\mathbf{b}$	white circle product of $\mathbf{a}$ and $\mathbf{b}$
$\mathbf{a}\circledcirc\mathbf{b}$, $\mathbf{a}\bullet\mathbf{b}$	black circle products of $\mathbf{a}$ and $\mathbf{b}$
$\mathrm{end}^{\circ}(\mathsf{m})$	endomorphism operad of the species m
$\langle\mathbf{e}\,\vert\,\mathbf{r}\rangle$	presentation of a quadratic operad
$\mathbf{a}_{!}=\langle\mathbf{e}^{*}\,\vert\,\mathbf{r}^{\perp}\rangle$	unoriented quadratic dual of $\mathbf{a}=\langle\mathbf{e}\,\vert\,\mathbf{r}\rangle$
$\mathbf{a}^{!}=\langle\mathbf{e}^{\circledast}\,\vert\,\mathbf{r}^{\ominus}\rangle$	oriented quadratic dual of $\mathbf{a}=\langle\mathbf{e}\,\vert\,\mathbf{r}\rangle$
$\mathcal{V}_{\mathbf{a}}$	monad on species induced by the operad $\mathbf{a}$
$\mathcal{U}_{\mathbf{c}}$	comonad on species induced by the cooperad $\mathbf{c}$
$(\mathcal{V}_{\mathbf{a}},\mathcal{U}_{\mathbf{c}},\lambda)$	bimonad on species induced by the bioperad $(\mathbf{a},\mathbf{c},\lambda)$

Examples of dispecies and operads.

The examples below are discussed in Sections 4.5 and 4.8.

$\mathbf{E}$	exponential dispecies
$\mathbf{Com}$	commutative operad
$\mathbf{Com}^{*}$	commutative cooperad
$\mathbf{\Gamma}$	dispecies of chambers
$\mathbf{As}$	associative operad
$\mathbf{Lie}$	Lie operad
$\mathbf{E}^{\mathbf{o}}$, $\mathbf{Com}^{\mathbf{o}}$	orientation operad
$\mathbf{E}^{-}$, $\mathbf{Com}^{-}$	signed commutative operad

Categories related to dispecies and operads.

$\mathcal{A}$-$\mathtt{dSp}$	dispecies
$\mathcal{A}$-$\mathtt{dSp}_{0}$	connected dispecies
$\mathcal{A}$-$\mathtt{dSp}_{+}$	positive dispecies
$\mathcal{A}$-Op	operads
$\mathcal{A}$-Op_{0}	connected operads
$\mathcal{A}$-Op_{+}	positive operads

List of Tables

The tables listed below provide compact summaries of important objects, constructions, examples and results relevant to the material presented in the book.

Author Index

The page numbers indicate the places where the author's work is cited.

Subject Index

The page numbers shown in bold indicate the place where the corresponding item is defined. Multiple bold numbers for the same item mean that the term is used in multiple related senses.