Eulerian idempotents

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1 Homogeneous sections of the support map

1.1 Homogeneous sections

Recall the Birkhoff algebra $\Pi[\mathcal{A}]$, the Tits algebra $\Sigma[\mathcal{A}]$ and the support map relating them. Let

$$\mathtt{u} : \Pi[\mathcal{A}] \to \Sigma[\mathcal{A}]$$

be any section of the support map. (The section is only required to be a linear map, not an algebra map.)

For each flat X, let $\mathtt{u}_{\mathrm{X}} := \mathtt{u}(\mathtt{H}_{\mathrm{X}})$ denote the value of $\mathtt{u}$ on $\mathtt{H}_{\mathrm{X}}$. Thus

(1) $$\mathrm{s}(\mathtt{u}_{\mathrm{X}}) = \mathtt{H}_{\mathrm{X}}.$$

We say that a section $\mathtt{u}$ of the support map is homogeneous if each $\mathtt{u}_{\mathrm{X}}$ only involves faces of support X. That is,

$$\mathtt{u}_{\mathrm{X}} = \sum_{F\colon \mathrm{s}(F)=\mathrm{X}} \mathtt{u}^F \mathtt{H}_F \tag{2}$$

for scalars $\mathtt{u}^F$.

Applying the support map and using (1), we obtain

$$\sum_{F\colon \mathrm{s}(F)=\mathrm{X}} \mathtt{u}^F = 1. \tag{3}$$

Note that $\mathtt{u}_{\perp} = \mathtt{H}_O$.

Conversely, a choice of elements $\mathtt{u}_{\mathrm{X}}$ of the form (2) with property (3) determines a homogeneous section $\mathtt{u}$.

Lemma 1. *Let* X *be a flat, and* G *be a face with support* X*. Let* $\mathtt{u}_{\mathrm{X}}$ *be an element of* $\Sigma[\mathcal{A}]$ *of the form* (2)*. Then*

$$\mathtt{H}_G \cdot \mathtt{u}_{\mathrm{X}} = \mathtt{H}_G \iff \sum_{F:\, \mathrm{s}(F)=\mathrm{X}} \mathtt{u}^F = 1$$

$$\iff \mathtt{u}_{\mathrm{X}} \cdot \mathtt{u}_{\mathrm{X}} = \mathtt{u}_{\mathrm{X}} \text{ and } \mathtt{u}_{\mathrm{X}} \text{ is nonzero.}$$

Proof. Recall that $\mathrm{s}(F) = \mathrm{s}(G)$ implies $GF = G$. Hence,

$$\mathtt{H}_G \cdot \mathtt{u}_{\mathrm{X}} = \Big(\sum_{F:\, \mathrm{s}(F)=\mathrm{X}} \mathtt{u}^F \Big)\, \mathtt{H}_G$$

and

$$\mathtt{u}_{\mathrm{X}} \cdot \mathtt{u}_{\mathrm{X}} = \Big(\sum_{F:\, \mathrm{s}(F)=\mathrm{X}} \mathtt{u}^F \Big)\, \mathtt{u}_{\mathrm{X}}.$$

Both equivalences follow. Note the relevance of requiring $\mathtt{u}_{\mathrm{X}}$ to be nonzero. $\square$

The preceding discussion yields the following.

Lemma 2. *The following are equivalent.*

1. *A homogeneous section* $\mathtt{u}$ *of* $\mathcal{A}$*.*

2. *A family of scalars* $(\mathtt{u}^F)$ *indexed by faces* F*, which satisfy* (3) *for each flat* X*.*

3. *A family of nonzero elements* $\{\mathtt{u}_\mathrm{X}\}_{\mathrm{X}\in\Pi}$ *indexed by flats of the form* (2) *with*

$$\mathtt{u}_\mathrm{X} \cdot \mathtt{u}_\mathrm{X} = \mathtt{u}_\mathrm{X}, \tag{4}$$

that is, $\mathtt{u}_\mathrm{X}$ *is a nonzero idempotent of the Tits algebra.*

Lemma 3. *The dimension of the affine space of all homogeneous sections is equal to the number of faces minus the number of flats.*

Proof. Apply Lemma 2, item (2). For each flat we get the number of faces with that support minus one. $\square$

Suppose the base field is the real numbers and all scalars $\mathtt{u}^F$ are nonnegative. In this case, by Lemma 2, item (2), a homogeneous section constitutes a family of probability distributions: a distribution on the set of faces supported on X, one for each flat X.

1.2 Set-theoretic sections

Consider the (set-theoretic) support map relating the Birkhoff monoid $\Pi[\mathcal{A}]$ and the Tits monoid $\Sigma[\mathcal{A}]$. Let

$$\sec : \Pi[\mathcal{A}] \to \Sigma[\mathcal{A}]$$

be any section of the support map.

Note that $\sec(\top)$ is an arbitrarily chosen chamber.

Linearizing $\sec$ yields a homogeneous section $\mathtt{u}$. Explicitly, the scalars $\mathtt{u}^F$ are given by

$$\mathtt{u}^F := \begin{cases} 1 & \text{if } F \text{ is in the image of } \sec, \\ 0 & \text{otherwise.} \end{cases}$$

In this case, we say that the homogeneous section $\mathtt{u}$ is set-theoretic.

1.3 The uniform section

Suppose that the field characteristic is 0.

We say that a homogeneous section $\mathtt{u}$ is uniform if $\mathtt{u}^F = \mathtt{u}^G$ whenever F and G have the same support. Equivalently, $\mathtt{u}$ is uniform if

$$\mathtt{u}^F = \frac{1}{c^F},$$

where c^F is the number of faces with support $\mathrm{s}(F)$.

1.4 Projective sections

We say that a homogeneous section $\mathtt{u}$ is projective if $\mathtt{u}^F = \mathtt{u}^{\overline{F}}$ for all faces F.

Lemma 4. *Assume the rank of the arrangement to be at least one. Let $\Bbbk$ be any field. A projective section exists iff the characteristic of $\Bbbk$ is not 2.*

Proof. A projective section is the same as a choice of scalars $\mathtt{u}^{\{F,\overline{F}\}}$, one for each unordered pair of opposite faces $\{F, \overline{F}\}$, such that $\mathtt{u}^{\{O,O\}} = 1$, and for each non-minimum flat X,

$$2 \sum_{\{F,\overline{F}\}:\, \mathrm{s}(F)=\mathrm{X}} \mathtt{u}^{\{F,\overline{F}\}} = 1.$$

In particular, $2\,\mathtt{u}^{\{P,\overline{P}\}} = 1$ for any vertex P. Clearly, these equations can be solved iff the field characteristic is not 2. $\square$

The uniform section (assuming characteristic 0) is clearly projective. In contrast, a set-theoretic section of an arrangement of rank at least one can never be projective.

1.5 Example

Consider the rank-one arrangement with chambers C and $\overline{C}$. Homogeneous sections are characterized by an arbitrary scalar p via

$$\mathtt{u}^O = 1,\ \mathtt{u}^C = p,\ \mathtt{u}^{\overline{C}} = 1 - p.$$

There are two set-theoretic sections, namely,

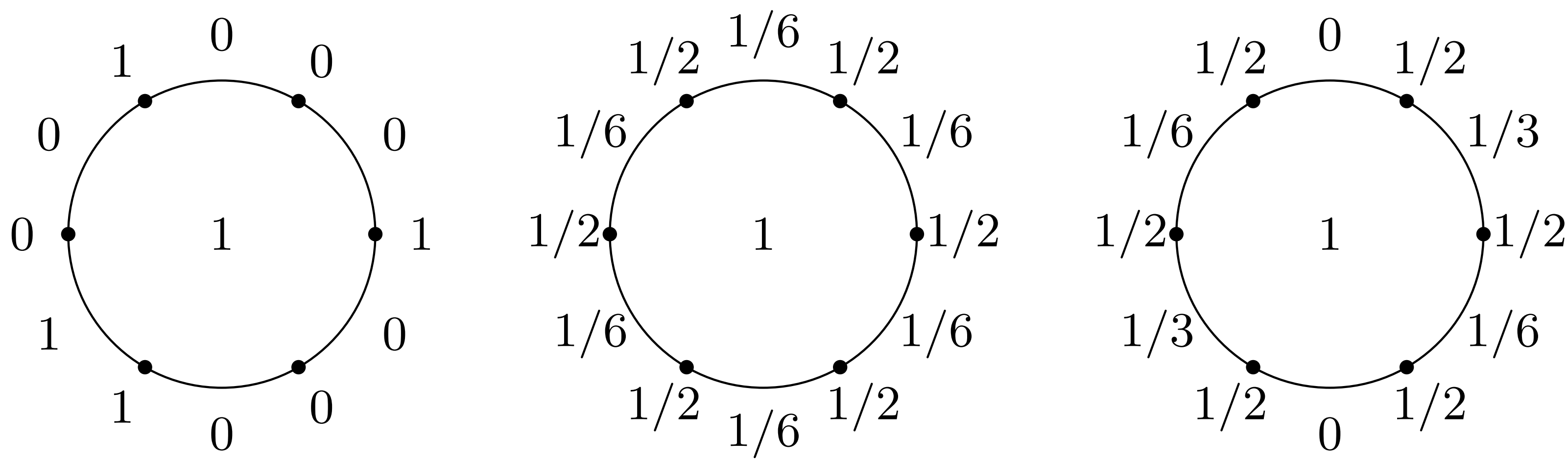
$$\mathtt{u}^O = 1,\ \mathtt{u}^C = 1,\ \mathtt{u}^{\overline{C}} = 0 \qquad \text{and} \qquad \mathtt{u}^O = 1,\ \mathtt{u}^C = 0,\ \mathtt{u}^{\overline{C}} = 1.$$

These are the cases $p = 1$ and $p = 0$, respectively. There is only one projective section and it is the uniform section. It is given by

$$\mathtt{u}^O = 1,\ \mathtt{u}^C = \mathtt{u}^{\overline{C}} = \frac{1}{2}.$$

This is the case $p = 1/2$.

Now consider the rank-two arrangement of 3 lines. The figure on the left shows a set-theoretic section, while the one on the right shows a projective section. The figure in the middle is the uniform section. The number written on the face F stands for the coefficient $\mathtt{u}^F$.



1.6 Induced section over a flat

Recall that for any face H of $\mathcal{A}$, the flats of $\mathcal{A}_H$ correspond to the flats of $\mathcal{A}$ which contain H. Whenever a flat X of $\mathcal{A}$ contains H, we write X/H for the corresponding flat of $\mathcal{A}_H$.

Suppose $\mathtt{u}$ is a homogeneous section of $\mathcal{A}$. For each $G \geq H$, define

$$\mathtt{u}_H^G := \sum_{\substack{F\colon HF=G,\\ \mathrm{s}(F)=\mathrm{s}(G)}} \mathtt{u}^F. \tag{5}$$

In particular, for any chamber $D \geq H$,

$$\mathtt{u}_H^D := \sum_{C\colon HC=D} \mathtt{u}^C. \tag{6}$$

Lemma 5. *A homogeneous section* $\mathtt{u}$ *of* $\mathcal{A}$ *induces a homogeneous section* $\mathtt{u}_H$ *of* $\mathcal{A}_H$*, with the scalar associated to the face* G/H *being* $\mathtt{u}_H^G$*.*

Proof. Let X be any flat containing H. Then

$$\begin{aligned}\sum_{G/H:\, \mathrm{s}(G/H)=\mathrm{X}/H} \mathtt{u}_H^G &= \sum_{\substack{G:\, G\geq H,\\ \mathrm{s}(G)=\mathrm{X}}} \sum_{\substack{F:\, HF=G,\\ \mathrm{s}(F)=\mathrm{s}(G)}} \mathtt{u}^F \\ &= \sum_{F:\, \mathrm{s}(F)=\mathrm{X}} \mathtt{u}^F \\ &= 1.\end{aligned}$$

The first step used (5), while the last step used (3). By Lemma 2, $\mathtt{u}_H$ is a homogeneous section. $\square$

Consistent with (2), for any face H, and flat X containing H, put

(7)
$$\mathtt{u}_{\mathrm{X}/H} = \sum_{G\colon G \geq H, \mathrm{s}(G)=\mathrm{X}} \mathtt{u}_H^G \, \mathtt{H}_{G/H}.$$

This is an element of $\Sigma[\mathcal{A}_H]$.

It is clear from (5) and (7) that

(8)
$$\mathtt{u}_{\mathrm{X}/H} = \Delta_H(\mathtt{u}_{\mathrm{X}}),$$

with Δ_H the map from $\Sigma[\mathcal{A}]$ to $\Sigma[\mathcal{A}_H]$.

Now let $\beta_{G,F}$ be the map from $\Sigma[\mathcal{A}_F]$ to $\Sigma[\mathcal{A}_G]$.

We deduce that

$\beta_{G,F}(\mathtt{u}_{\mathrm{X}/F}) = \mathtt{u}_{\mathrm{X}/G}$ for any F and G with the same support, and

$\Delta_{G/H}(\mathtt{u}_{\mathrm{X}/H}) = \mathtt{u}_{\mathrm{X}/G}$ for any $G \geq H$.

This can be expressed succintly as follows.

Lemma 6. *The homogeneous sections $\mathtt{u}_H$ of $\mathcal{A}_H$ induced from a homogeneous section $\mathtt{u}$ of $\mathcal{A}$ satisfy the following compatibility conditions.*

For any $G \geq H$,

$$(\mathtt{u}_H)_{G/H} = \mathtt{u}_G, \tag{9}$$

and for any F and G with the same support,

$$\beta_{G,F}(\mathtt{u}_F) = \mathtt{u}_G. \tag{10}$$

1.7 Induced section under a flat

Suppose $\mathtt{u}$ is a homogeneous section of $\mathcal{A}$. Then, for a fixed flat X, restricting $\mathtt{u}$ to flats $\mathrm{Y} \leq \mathrm{X}$ yields a homogeneous section of the arrangement $\mathcal{A}^{\mathrm{X}}$, which we denote by $\mathtt{u}^{\mathrm{X}}$.

1.8 Cartesian product

Suppose $\mathtt{u}$ is a homogeneous section of $\mathcal{A}$, and $\mathtt{u}'$ is a homogeneous section of $\mathcal{A}'$. Then we obtain an induced homogeneous section $\mathtt{u} \times \mathtt{u}'$ on $\mathcal{A} \times \mathcal{A}'$:

$$(\mathtt{u} \times \mathtt{u}')^{(F,F')} := \mathtt{u}^{F} \mathtt{u}'^{F'}.$$

Equivalently,

$$(\mathtt{u} \times \mathtt{u}')_{(\mathrm{X},\mathrm{X}')} = \mathtt{u}_{\mathrm{X}} \otimes \mathtt{u}'_{\mathrm{X}'}.$$

If $\mathtt{u}$ and $\mathtt{u}'$ are set-theoretic (uniform, projective), then so is $\mathtt{u} \times \mathtt{u}'$.

2 Eulerian idempotents

2.1 Eulerian idempotents

An Eulerian family of $\mathcal{A}$ is a set $\mathtt{E} := \{\mathtt{E}_{\mathrm{X}}\}_{\mathrm{X}\in\Pi}$ indexed by flats, where each $\mathtt{E}_{\mathrm{X}}$ is an element of the Tits algebra $\Sigma[\mathcal{A}]$ of the form

(11)
$$\mathtt{E}_{\mathrm{X}} = \sum_{F:\, \mathrm{s}(F)\geq \mathrm{X}} a^F \mathtt{H}_F$$

with a nonzero base term, that is, $a^G \neq 0$ for at least one face G of support X.

These elements are required to be idempotent and mutually orthogonal:

(12)
$$\mathtt{E}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{Y}} = \begin{cases} \mathtt{E}_{\mathrm{X}} & \text{if } \mathrm{X} = \mathrm{Y}, \\ 0 & \text{if } \mathrm{X} \neq \mathrm{Y}. \end{cases}$$

We refer to the $\mathtt{E}_{\mathrm{X}}$ as the Eulerian idempotents.

The element $\mathtt{E}_{\perp}$ associated to the minimum flat $\perp$ is the first Eulerian idempotent. The nonzero base term condition says that the coefficient of $\mathtt{H}_O$ in $\mathtt{E}_{\perp}$ is nonzero.

We will prove the following result.

Proposition 1. *Eulerian families of $\mathcal{A}$ are in correspondence with homogeneous sections of the support map of $\mathcal{A}$.*

In particular, such families always exist.

2.2 From a homogeneous section to an Eulerian family. Saliola construction

Suppose we are given a homogeneous section $\mathtt{u}$.

Define elements of $\Sigma[\mathcal{A}]$ indexed by flats recursively by the formula

$$\mathtt{E}_{\mathrm{X}} := \mathtt{u}_{\mathrm{X}} - \sum_{\mathrm{Y}:\mathrm{Y}>\mathrm{X}} \mathtt{u}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{Y}}, \tag{13}$$

beginning with the maximum flat and proceeding down.

Thus, $\mathtt{E}_{\top} = \mathtt{u}_{\top}$, and for a hyperplane X,

$$\mathtt{E}_{\mathrm{X}} = \mathtt{u}_{\mathrm{X}} - \mathtt{u}_{\mathrm{X}} \cdot \mathtt{E}_{\top} = \mathtt{u}_{\mathrm{X}} - \mathtt{u}_{\mathrm{X}} \cdot \mathtt{u}_{\top},$$

and so on till we reach $\mathtt{E}_{\perp}$ indexed by the minimum flat.

In $\mathtt{E}_{\mathrm{X}}$, the term $\mathtt{u}_{\mathrm{X}}$ involves faces with support X, while the remaining terms involve faces with support strictly greater than X.

Example. Suppose the homogeneous section $\mathtt{u}$ is set-theoretic arising from $\mathrm{sec} : \Pi[\mathcal{A}] \to \Sigma[\mathcal{A}]$.

Then formula (13) takes the simpler form:

$$\mathtt{E}_{\mathrm{X}} := \mathtt{H}_{\mathrm{sec}(\mathrm{X})} - \sum_{\mathrm{Y}:\mathrm{Y}>\mathrm{X}} \mathtt{H}_{\mathrm{sec}(\mathrm{X})} \cdot \mathtt{E}_{\mathrm{Y}}.$$

The first two steps are as follows.

- $\mathtt{E}_{\top} = \mathtt{H}_{\mathrm{sec}(\top)}$. This is a chamber. Call it $\mathtt{H}_C$.
- For any panel F in the image of sec, we have $\mathtt{E}_{\mathrm{s}(F)} = \mathtt{H}_F - \mathtt{H}_{FC}$.

In general, $\mathtt{E}_{\mathrm{X}}$ only contains faces in the star of $\mathrm{sec}(\mathrm{X})$.

Lemma 7. *We have*

$$\mathtt{u}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{X}} = \mathtt{E}_{\mathrm{X}}, \tag{14}$$

$$\mathtt{u}_{\mathrm{X}} \cdot \Big(\sum_{\mathrm{Y}:\mathrm{Y} \geq \mathrm{X}} \mathtt{E}_{\mathrm{Y}} \Big) = \mathtt{u}_{\mathrm{X}}, \tag{15}$$

$$\mathtt{H}_{F} \cdot \Big(\sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{s}(F)} \mathtt{E}_{\mathrm{Y}} \Big) = \mathtt{H}_{F}. \tag{16}$$

Proof. Formula (14) follows by premultiplying (13) with $\mathtt{u}_{\mathrm{X}}$ and using (4).

Substituting it in the lhs of (13) and rearranging terms yields (15).

Premultiplying this with $\mathtt{H}_{F}$ (where $\mathrm{X} = \mathrm{s}(F)$) and using Lemma 1 yields (16). □

In particular, by setting $\mathrm{X} = \perp$ in (15) and using $\mathtt{u}_{\perp} = \mathtt{H}_{O}$, or by setting $F = O$ in (16), we obtain:

$$\mathtt{H}_{O} = \sum_{\mathrm{X}} \mathtt{E}_{\mathrm{X}}. \tag{17}$$

Lemma 8. *For any face F and flat X, if $\mathrm{s}(F) \not\leq \mathrm{X}$, then $\mathtt{H}_F \cdot \mathtt{E}_{\mathrm{X}} = 0$.*

In particular, $\mathtt{H}_F \cdot \mathtt{E}_\perp = 0$ for $F > O$.

Proof. We do a backward induction on the rank of X.

If $\mathrm{X} = \top$, then the statement is vacuously true. This is the induction base.

The induction step is shown below. Put $\mathrm{Z} = \mathrm{s}(F)$.

(a)

$$\begin{aligned}
\mathtt{H}_F \cdot \mathtt{E}_{\mathrm{X}} &= \mathtt{H}_F \cdot \mathtt{u}_{\mathrm{X}} - \sum_{\mathrm{Y}:\mathrm{Y}>\mathrm{X}} \mathtt{H}_F \cdot \mathtt{u}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{Y}} \\
&= \mathtt{H}_F \cdot \mathtt{u}_{\mathrm{X}} - \sum_{\mathrm{Y}:\mathrm{Y}\geq \mathrm{Z}\vee\mathrm{X}} \mathtt{H}_F \cdot \mathtt{u}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{Y}} - \sum_{\mathrm{Y}:\mathrm{Y}>\mathrm{X},\, \mathrm{Z}\not\leq\mathrm{Y}} \mathtt{H}_F \cdot \mathtt{u}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{Y}}
\end{aligned}$$

Each term in $\mathtt{H}_F \cdot \mathtt{u}_{\mathrm{X}}$ has support $\mathrm{Z} \vee \mathrm{X}$.

It follows from Lemma 1 that

$$\mathtt{H}_F \cdot \mathtt{u}_{\mathrm{X}} = \mathtt{H}_F \cdot \mathtt{u}_{\mathrm{X}} \cdot \mathtt{u}_{\mathrm{Z}\vee\mathrm{X}}.$$

This along with the induction hypothesis implies that for any $\mathrm{Y} > \mathrm{X}$, $\mathrm{Z} \not\leq \mathrm{Y}$,

$$\mathtt{H}_F \cdot \mathtt{u}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{Y}} = \mathtt{H}_F \cdot \mathtt{u}_{\mathrm{X}} \cdot \mathtt{u}_{\mathrm{Z}\vee\mathrm{X}} \cdot \mathtt{E}_{\mathrm{Y}} = 0,$$

and hence the last term in (a) is zero.

The first two terms are manipulated as follows.

$$\begin{aligned} &\mathtt{H}_F \cdot \mathtt{u}_{\mathrm{X}} - \sum_{\mathrm{Y}:\mathrm{Y}\geq\mathrm{Z}\vee\mathrm{X}} \mathtt{H}_F \cdot \mathtt{u}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{Y}} \\ &\quad = \mathtt{H}_F \cdot \mathtt{u}_{\mathrm{X}} \cdot \Big(\mathtt{u}_{\mathrm{Z}\vee\mathrm{X}} - \sum_{\mathrm{Y}:\mathrm{Y}\geq\mathrm{Z}\vee\mathrm{X}} \mathtt{u}_{\mathrm{Z}\vee\mathrm{X}} \cdot \mathtt{E}_{\mathrm{Y}}\Big) = 0. \end{aligned}$$

In the last step, we used (15) to deduce that the term inside the parenthesis is zero. $\square$

We refer to Lemma 8 as the Saliola lemma.

Lemma 9. *Given a homogeneous section* $\mathtt{u}$*, the elements* $\mathrm{E_X}$ *defined by* (13) *yield an Eulerian family.*

Proof. It is clear that each $\mathrm{E_X}$ is of the form (11) and has a nonzero base term.

We need to check (12).

We do a backward induction on the rank of X.

By (4), $\mathrm{E}_\top = \mathtt{u}_\top$ is idempotent, and by the Saliola lemma, $\mathrm{E}_\top \cdot \mathrm{E_Y} = 0$ for any $\mathrm{Y} < \top$. This is the induction base.

The induction step is completed below.

- $\mathrm{X} \not\leq \mathrm{Y}$.

  In this case, $\mathrm{E_X} \cdot \mathrm{E_Y} = 0$ by the Saliola lemma.

- $\mathrm{X} < \mathrm{Y}$.

  In this case, by multiplying (13) on the right by $\mathrm{E_Y}$, we obtain

  $$\mathrm{E_X}\cdot\mathrm{E_Y} = \mathtt{u}_\mathrm{X}\cdot\mathrm{E_Y} - \sum_{\mathrm{Z}:\mathrm{Z}>\mathrm{X}} \mathtt{u}_\mathrm{X}\cdot\mathrm{E_Z}\cdot\mathrm{E_Y} = \mathtt{u}_\mathrm{X}\cdot\mathrm{E_Y} - \mathtt{u}_\mathrm{X}\cdot\mathrm{E_Y} = 0.$$

  In the second step, by the induction hypothesis, only the summand for $\mathrm{Z} = \mathrm{Y}$ contributed.

- $\mathrm{X} = \mathrm{Y}$.

  In this case, by multiplying (13) on the right by $\mathtt{E}_{\mathrm{X}}$, we obtain

  $$\mathtt{E}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{X}} = \mathtt{u}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{X}} - \sum_{\mathrm{Z}:\mathrm{Z}>\mathrm{X}} \mathtt{u}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{Z}} \cdot \mathtt{E}_{\mathrm{X}} = \mathtt{u}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{X}} = \mathtt{E}_{\mathrm{X}}.$$

  In the second step, by the induction hypothesis or by the Saliola lemma, the sum is zero. In the last step, we used (14).

$\square$

We refer to this construction of an Eulerian family starting from a homogeneous section as the Saliola construction.

2.3 From an Eulerian family to a homogeneous section. The base term

Suppose E is an Eulerian family. Using (11), write

$$\mathtt{E}_{\mathrm{X}} = \mathtt{u}_{\mathrm{X}} + \sum_{F\colon \mathrm{s}(F) > \mathrm{X}} a^F \mathtt{H}_F. \tag{18}$$

The element $\mathtt{u}_{\mathrm{X}}$ is the part of $\mathtt{E}_{\mathrm{X}}$ consisting of faces of support X. This is the base term of $\mathtt{E}_{\mathrm{X}}$. It is nonzero by hypothesis.

The remaining terms are the higher terms consisting of faces of support strictly greater than X.

Since $\mathtt{E}_{\mathrm{X}}$ is an idempotent,

$\mathtt{u}_{\mathrm{X}} +$ higher terms $= \mathtt{E}_{\mathrm{X}} = \mathtt{E}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{X}} = \mathtt{u}_{\mathrm{X}} \cdot \mathtt{u}_{\mathrm{X}} +$ higher terms.

Thus $\mathtt{u}_{\mathrm{X}} \cdot \mathtt{u}_{\mathrm{X}} = \mathtt{u}_{\mathrm{X}}$, and hence by Lemma 2 the $\mathtt{u}_{\mathrm{X}}$ yield a homogeneous section.

2.4 Equivalence between homogeneous sections and Eulerian families

We claim that the two constructions discussed above are inverse to each other.

One direction is clear.

The nontrivial direction is proved below.

Lemma 10. *Suppose* $\mathtt{E}$ *is an Eulerian family. Then* (13) *holds with the* $\mathtt{u}_{\mathrm{X}}$ *defined by* (18).

Proof. Write

$$\mathtt{E}_{\mathrm{X}} = \mathtt{u}_{\mathrm{X}} - \Big(\sum_{\mathrm{Y}:\mathrm{Y}>\mathrm{X}} \mathtt{u}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{Y}} \Big) + \mathrm{err}_{\mathrm{X}} \,.$$

We view $\mathrm{err}_{\mathrm{X}}$ as the error term and would like to show it to be 0 for all X.

We do this by a backward induction on the rank of X.

Clearly, $\mathtt{E}_{\top} = \mathtt{u}_{\top}$ and hence $\mathrm{err}_{\top} = 0$. This is the induction base.

For the induction step:

Suppose $\mathrm{err}_{\mathrm{Y}} = 0$ for all $\mathrm{Y} > \mathrm{X}$. In other words, suppose the E_{Y} for $\mathrm{Y} > \mathrm{X}$ are given by the Saliola construction (13).

For $\mathrm{Y} > \mathrm{X}$,

$$0 = (\mathtt{E}_{\mathrm{X}} - \mathrm{err}_{\mathrm{X}}) \cdot \mathtt{E}_{\mathrm{Y}} = \mathtt{E}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{Y}} - \mathrm{err}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{Y}} = -\,\mathrm{err}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{Y}}.$$

The first step used Lemma 9 while the last step used orthogonality of the Eulerian idempotents (12).

Hence, $\mathrm{err}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{Y}} = 0$ for all $\mathrm{Y} > \mathrm{X}$.

Also, by construction, $\mathrm{err}_{\mathrm{X}}$ only contains faces with support strictly greater than X.

Write $\mathrm{err}_{\mathrm{X}} = \sum x^F \mathtt{H}_F$.

Suppose $\mathrm{err}_{\mathrm{X}} \neq 0$.

Then there exists a face F such that $x^F \neq 0$ but $x^G = 0$ for all $G < F$.

In particular, $\mathrm{s}(F) > \mathrm{X}$.

Let us calculate the coefficient of $\mathtt{H}_F$ in $\mathrm{err}_\mathrm{X} \cdot \mathtt{E}_{\mathrm{s}(F)}$.

$$\begin{aligned}\langle \mathrm{err}_\mathrm{X} \cdot \mathtt{E}_{\mathrm{s}(F)}, \mathtt{H}_F \rangle = \langle x^F \mathtt{H}_F \cdot \mathtt{E}_{\mathrm{s}(F)}, \mathtt{H}_F \rangle = \langle x^F \mathtt{H}_F \cdot \mathtt{u}_{\mathrm{s}(F)}, \mathtt{H}_F \rangle \\ = \langle x^F \mathtt{H}_F, \mathtt{H}_F \rangle = x^F \neq 0.\end{aligned}$$

Thus $\mathrm{err}_\mathrm{X} \cdot \mathtt{E}_{\mathrm{s}(F)} \neq 0$, which is a contradiction.

Hence $\mathrm{err}_\mathrm{X} = 0$ as required. $\square$

2.5 Visualizing an Eulerian idempotent

One also deduces from (13) that Eulerian idempotents have a more rigid form than what is specified by (11), namely,

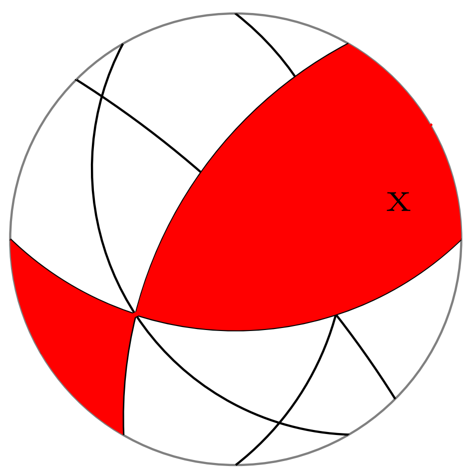
$$\mathtt{E}_{\mathrm{X}} = \sum_{F \colon \mathrm{s}(F) = \mathrm{X}} \ \sum_{G \colon G \geq F} a^G \mathtt{H}_G. \tag{19}$$

In other words, we need to sum only over those faces G which have a face F with support X.

Further, if F and F' both have support X, and $G \geq F$ and $G' \geq F'$ are such that $F'G = G'$, then
$a^G a^{F'} = a^{G'} a^F$.

In other words, the coefficients of faces in the star of F are in proportion to those in the star of F'.

An illustration in rank three is given below. For the flat X shown as the red line, the Eulerian idempotent $\mathrm{E_X}$ involves edges on that line and the chambers in the shaded region. (Only the front half is visible in the picture.)



2.6 Over and under a flat. Cartesian product

We now see how the correspondence between Eulerian families and homogeneous sections behaves under passage to arrangements over and under a flat, and with respect to cartesian product of arrangements.

Suppose $\mathtt{E}$ is an Eulerian family of $\mathcal{A}$. For any face H, and flat X containing H, define

$$\mathtt{E}_{\mathrm{X}/H} := \Delta_H(\mathtt{E}_{\mathrm{X}}), \tag{20}$$

with Δ_H the map from $\Sigma[\mathcal{A}]$ to $\Sigma[\mathcal{A}_H]$. Since it is an algebra homomorphism, it follows from (12) that

$$\mathtt{E}_H := \{\mathtt{E}_{\mathrm{X}/H}\}_{\mathrm{X}\geq \mathrm{s}(H)}$$

is a family of mutually orthogonal idempotents of $\Sigma[\mathcal{A}_H]$.

Lemma 11. *Let* $\mathtt{u}$ *be a homogeneous section of* $\mathcal{A}$*, and let* $\mathtt{E}$ *be its associated Eulerian family.*

Then for any face H*,* $\mathtt{E}_H$ *is the Eulerian family of* $\mathcal{A}_H$ *associated to the homogeneous section* $\mathtt{u}_H$*.*

In particular, $\mathtt{E}_{\mathrm{s}(H)/H}$ *is the first Eulerian idempotent of* $\mathcal{A}_H$*.*

Proof. Let $\mathtt{E}'_{\mathrm{X}/H}$ denote the Eulerian idempotents associated to $\mathtt{u}_H$.

We want to show $\mathtt{E}'_{\mathrm{X}/H} = \mathtt{E}_{\mathrm{X}/H}$.

We do a backward induction on the rank of X.

By (8), $\mathtt{u}_{\top/H} = \Delta_H(\mathtt{u}_\top)$, so the result holds for $\mathrm{X} = \top$. This is the induction base.

The induction step is as follows.

$$\begin{aligned}
\mathtt{E}'_{\mathrm{X}/H} &:= \mathtt{u}_{\mathrm{X}/H} - \sum_{\mathrm{Y}:\mathrm{Y}>\mathrm{X}} \mathtt{u}_{\mathrm{X}/H} \cdot \mathtt{E}'_{\mathrm{Y}/H} \\
&= \Delta_H(\mathtt{u}_{\mathrm{X}}) - \sum_{\mathrm{Y}:\mathrm{Y}>\mathrm{X}} \Delta_H(\mathtt{u}_{\mathrm{X}}) \cdot \Delta_H(\mathtt{E}_{\mathrm{Y}}) \\
&= \Delta_H(\mathtt{u}_{\mathrm{X}}) - \sum_{\mathrm{Y}:\mathrm{Y}>\mathrm{X}} \Delta_H(\mathtt{u}_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{Y}}) \\
&= \Delta_H(\mathtt{E}_{\mathrm{X}}).
\end{aligned}$$

The definition is used in the first step, and the induction hypothesis is used in the next step.

The remaining steps used (8) and the fact that Δ_H is an algebra homomorphism. $\square$

Suppose $\mathtt{E} := \{\mathtt{E}_{\mathrm{Y}}\}$ is an Eulerian family of $\mathcal{A}$.

For $\mathrm{Y} \leq \mathrm{X}$, let $\mathtt{E}^{\mathrm{X}}_{\mathrm{Y}}$ denote the image of $\mathtt{E}_{\mathrm{Y}}$ under the map $\Sigma[\mathcal{A}] \to \Sigma[\mathcal{A}^{\mathrm{X}}]$.

Since this map is an algebra homomorphism, it follows from (12) that

$$\mathtt{E}^{\mathrm{X}} := \{\mathtt{E}^{\mathrm{X}}_{\mathrm{Y}}\}_{\mathrm{Y} \leq \mathrm{X}}$$

is a family of mutually orthogonal idempotents of $\Sigma[\mathcal{A}^{\mathrm{X}}]$, that is,

$$\mathtt{E}^{\mathrm{X}}_{\mathrm{Y}} \cdot \mathtt{E}^{\mathrm{X}}_{\mathrm{Z}} = \begin{cases} \mathtt{E}^{\mathrm{X}}_{\mathrm{Y}} & \text{if } \mathrm{Y} = \mathrm{Z}, \\ 0 & \text{if } \mathrm{Y} \neq \mathrm{Z}. \end{cases} \tag{21}$$

Lemma 12. *Let $\mathtt{u}$ be a homogeneous section of $\mathcal{A}$, and let $\mathtt{E}$ be its associated Eulerian family. Then for any flat X, $\mathtt{E}^{\mathrm{X}}$ is the Eulerian family of $\mathcal{A}^{\mathrm{X}}$ associated to the homogeneous section $\mathtt{u}^{\mathrm{X}}$.*

Proof. This can be shown by induction on the rank of X similar to the proof of Lemma 11. □

Suppose $\mathtt{E} := \{\mathtt{E}_{\mathrm{X}}\}$ is an Eulerian family of $\mathcal{A}$, and $\mathtt{E}' := \{\mathtt{E}'_{\mathrm{X}'}\}$ is an Eulerian family of $\mathcal{A}'$. Then

$$\mathtt{E} \times \mathtt{E}' := \{\mathtt{E}_{\mathrm{X}} \otimes \mathtt{E}'_{\mathrm{X}'}\}_{(\mathrm{X},\mathrm{X}')}$$

is an Eulerian family of $\mathcal{A} \times \mathcal{A}'$. More precisely:

Lemma 13. *Let* $\mathtt{u}$ *and* $\mathtt{u}'$ *be homogeneous sections of* $\mathcal{A}$ *and* $\mathcal{A}'$*, and let* $\mathtt{E}$ *and* $\mathtt{E}'$ *be their associated Eulerian families. Then* $\mathtt{E} \times \mathtt{E}'$ *is the Eulerian family of* $\mathcal{A} \times \mathcal{A}'$ *associated to the homogeneous section* $\mathtt{u} \times \mathtt{u}'$*.*

3 Eulerian families, complete systems and algebra sections

We now show that any Eulerian family is a complete system of primitive orthogonal idempotents of the Tits algebra, and all complete systems are of this form.

Recall that for any elementary algebra, there is a correspondence between complete systems and algebra sections.

Hence, this result can be stated as follows.

Theorem 1. *The following pieces of data are equivalent.*

- *An Eulerian family* E *of* $\mathcal{A}$.
- *A complete system of primitive orthogonal idempotents of* $\Sigma[\mathcal{A}]$.
- *An algebra section* $\Pi[\mathcal{A}] \to \Sigma[\mathcal{A}]$ *of the support map.*

Let $\mathtt{E}$ be an Eulerian family associated to the homogeneous section $\mathtt{u}$.

Applying the support map to (13) and using (1) yields

$$\mathrm{s}(\mathtt{E}_{\mathrm{X}}) = \mathtt{H}_{\mathrm{X}} - \sum_{\mathrm{Y}:\mathrm{Y}>\mathrm{X}} \mathtt{H}_{\mathrm{X}} \cdot \mathrm{s}(\mathtt{E}_{\mathrm{Y}}).$$

Applying induction, we deduce that

$$\mathrm{s}(\mathtt{E}_{\mathrm{X}}) = \mathtt{Q}_{\mathrm{X}}. \tag{22}$$

Thus, the Eulerian idempotents map to the primitive idempotents of the Birkhoff algebra.

Further, the map

$$\Pi[\mathcal{A}] \hookrightarrow \Sigma[\mathcal{A}], \qquad \mathtt{Q}_{\mathrm{X}} \mapsto \mathtt{E}_{\mathrm{X}} \tag{23}$$

is an algebra section of the support map.

This is because the $\mathtt{E}_{\mathrm{X}}$ are idempotent, mutually orthogonal, and by (17), they add up to $\mathtt{H}_O$ (the unit element).

Since algebra sections correspond to complete systems, we have:

Theorem 2. *Any Eulerian family of $\mathcal{A}$ is a complete system of primitive orthogonal idempotents of $\Sigma[\mathcal{A}]$.*

For the converse:

Again as a general fact, any two algebra sections of the support map are conjugate by an element of $\mathtt{H}_O + \mathrm{rad}(\Sigma)$. (The latter is a subgroup of the group of invertible elements of the Tits algebra.)

Now use the result below.

Lemma 14. *Conjugation of any Eulerian family by an invertible element of the Tits algebra produces another Eulerian family.*

Proof. Let $\mathtt{E}$ be an Eulerian family, and v be an invertible element.

Put $\mathtt{E}'_{\mathrm{X}} := v \cdot \mathtt{E}_{\mathrm{X}} \cdot v^{-1}$.

Then clearly, the $\mathtt{E}'_{\mathrm{X}}$ are idempotent and mutually orthogonal, and each $\mathtt{E}'_{\mathrm{X}}$ only involves faces of support greater than X.

Further, $\mathrm{s}(\mathtt{E}'_{\mathrm{X}}) = \mathrm{s}(v) \cdot \mathrm{s}(\mathtt{E}_{\mathrm{X}}) \cdot \mathrm{s}(v^{-1}) = \mathrm{s}(\mathtt{E}_{\mathrm{X}}) = \mathtt{Q}_{\mathrm{X}}$, so $\mathtt{E}'_{\mathrm{X}}$ has a nonzero base term.

Thus, $\mathtt{E}'$ is an Eulerian family. $\square$

This completes the proof of Theorem 1.

4 $\mathtt{Q}$-bases of the Tits algebra

We introduce $\mathtt{Q}$-bases of the Tits algebra.

There is one such basis for every homogeneous section $\mathtt{u}$, or equivalently, one for every Eulerian family $\mathtt{E}$.

We compare the $\mathtt{Q}$-bases with the (unique) $\mathtt{Q}$-basis of the Birkhoff algebra.

4.1 $\mathtt{Q}$-basis of the Tits algebra

Let $\mathtt{E}$ be an Eulerian family.

For any face F, put

(24)
$$\mathtt{Q}_F := \mathtt{H}_F \cdot \mathtt{E}_{\mathrm{s}(F)}.$$

In particular, $\mathtt{Q}_O = \mathtt{E}_\perp$.

Lemma 15. *Each* $\mathtt{Q}_F$ *is a primitive idempotent, with*

$$\mathrm{s}(\mathtt{Q}_F) = \mathtt{Q}_{\mathrm{s}(F)}. \tag{25}$$

Further, the set $\{\mathtt{Q}_F\}$ *is a basis of the Tits algebra* $\Sigma[\mathcal{A}]$*.*

Proof. The following calculation shows that $\mathtt{Q}_F$ is an idempotent.

$$\mathtt{Q}_F \cdot \mathtt{Q}_F = \mathtt{H}_F \cdot \mathtt{E}_{\mathrm{s}(F)} \cdot \mathtt{H}_F \cdot \mathtt{E}_{\mathrm{s}(F)} = \mathtt{H}_F \cdot \mathtt{E}_{\mathrm{s}(F)} \cdot \mathtt{E}_{\mathrm{s}(F)} = \mathtt{H}_F \cdot \mathtt{E}_{\mathrm{s}(F)} = \mathtt{Q}_F.$$

Using (22), we deduce

$$\mathrm{s}(\mathtt{Q}_F) = \mathrm{s}(\mathtt{H}_F \cdot \mathtt{E}_{\mathrm{s}(F)}) = \mathrm{s}(\mathtt{H}_F) \cdot \mathrm{s}(\mathtt{E}_{\mathrm{s}(F)}) = \mathtt{H}_{\mathrm{s}(F)} \cdot \mathtt{Q}_{\mathrm{s}(F)} = \mathtt{Q}_{\mathrm{s}(F)}.$$

Thus, $\mathtt{Q}_F$ lifts the primitive idempotent $\mathtt{Q}_{\mathrm{s}(F)}$. So, it itself must be primitive.

The element $\mathtt{Q}_F$ written in the $\mathtt{H}$-basis only involves faces greater than F, and further by Lemma 1, $\mathtt{H}_F$ appears with coefficient 1. By triangularity, the set $\{\mathtt{Q}_F\}$ is a basis of $\Sigma[\mathcal{A}]$. $\square$

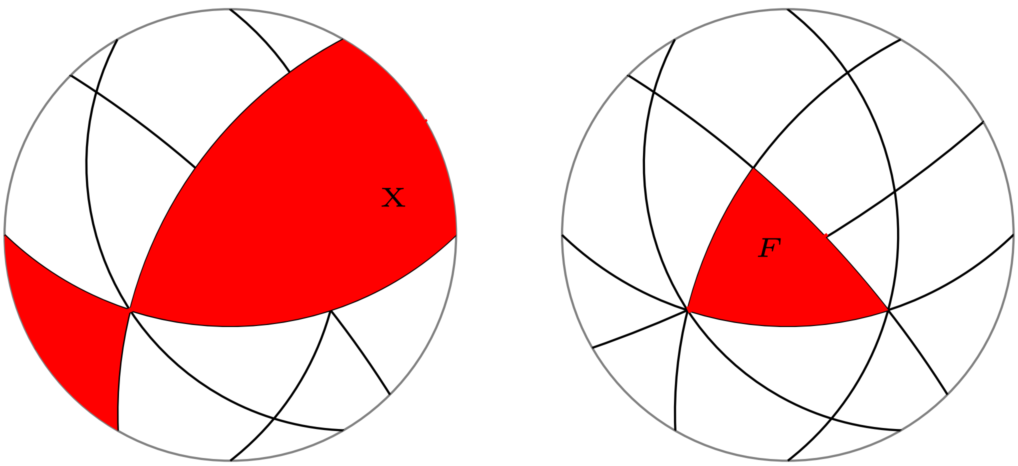
We refer to $\{\mathtt{Q}_F\}$ as the Q-basis. It follows from (2) and (14) that

$$\mathtt{E}_{\mathrm{X}} = \sum_{F\colon \mathrm{s}(F)=\mathrm{X}} \mathtt{u}^F \mathtt{Q}_F. \tag{26}$$

This is a more precise way of writing (19).

4.2 Visualizing a Q-basis element

An illustration of a Q-basis element is shown below.



On the left, we have redrawn the picture of the Eulerian idempotent E_{X} from Section 2.5.

The picture on the right shows Q_F with $\mathrm{s}(F) = \mathrm{X}$.

It involves the edge F and the two shaded chambers.

In a sense, E_{X} is local to X, while Q_F is local to F.

The passage between the two is governed by (24) and (26).

4.3 Rank-one

Consider the rank-one arrangement with chambers C and $\overline{C}$. Fix an arbitrary scalar p. Recall from Section 1.5 that any homogeneous section $\mathtt{u}$ is of the form $\mathtt{u}^O = 1$, $\mathtt{u}^C = p$, $\mathtt{u}^{\overline{C}} = 1 - p$. The associated Eulerian family $\mathtt{E}$ is

$$\mathtt{E}_\top = p\,\mathtt{H}_C + (1-p)\,\mathtt{H}_{\overline{C}} \quad \text{and} \quad \mathtt{E}_\perp = \mathtt{H}_O - p\,\mathtt{H}_C - (1-p)\,\mathtt{H}_{\overline{C}}.$$

The $\mathtt{Q}$-basis is

$$\mathtt{Q}_C = \mathtt{H}_C, \quad \mathtt{Q}_{\overline{C}} = \mathtt{H}_{\overline{C}}, \quad \mathtt{Q}_O = \mathtt{H}_O - p\,\mathtt{H}_C - (1-p)\,\mathtt{H}_{\overline{C}}.$$

Note very carefully that $\mathtt{Q}_O$ is not orthogonal to either $\mathtt{Q}_C$ or $\mathtt{Q}_{\overline{C}}$ in general.

4.4 Product of $\mathtt{H}$- and $\mathtt{Q}$-basis elements

Lemma 16. *For any faces F and G,*

(27)
$$\mathtt{H}_F \cdot \mathtt{Q}_G = \begin{cases} \mathtt{Q}_{FG} & \text{if } GF = G, \\ 0 & \text{if } GF > G. \end{cases}$$

In particular, if F and G have the same support, then

(28)
$$\mathtt{H}_F \cdot \mathtt{Q}_G = \mathtt{Q}_F.$$

Proof. For any F and G,

$$\mathtt{H}_F \cdot \mathtt{Q}_G = \mathtt{H}_F \cdot \mathtt{H}_G \cdot \mathtt{E}_{\mathrm{s}(G)} = \mathtt{H}_{FG} \cdot \mathtt{E}_{\mathrm{s}(G)}.$$

If $GF = G$, then $\mathrm{s}(FG) = \mathrm{s}(G)$, and the above quantity equals $\mathtt{Q}_{FG}$.

If $GF > G$, then $\mathrm{s}(FG) > \mathrm{s}(G)$, and the above quantity equals 0 by the Saliola lemma (Lemma 8). $\square$

4.5 Change of basis formulas

Consider the matrix $(\mathtt{u}_H^G)$ with entries defined by (5).

This matrix can be inverted. More precisely: For $F \leq G$, let $\mathtt{a}_F^G$ be the unique scalars which satisfy $\mathtt{a}_F^F = 1$ for all F, and

(29)
$$\sum_{K:\, F \leq K \leq G} \mathtt{u}_F^K \mathtt{a}_K^G = 0 = \sum_{K:\, F \leq K \leq G} \mathtt{a}_F^K \mathtt{u}_K^G$$

for all $F < G$.

Explicitly,

(30)
$$\mathtt{a}_F^G = -\mathtt{u}_F^G + \sum_{F<H<G} \mathtt{u}_F^H \mathtt{u}_H^G - \sum_{F<H<K<G} \mathtt{u}_F^H \mathtt{u}_H^K \mathtt{u}_K^G + \dots$$

for all $F < G$. The first sum is over H, the second sum is over H and K, and so on.

Note that some of the $\mathtt{a}_F^G$ could be negative even when all the $\mathtt{u}_F^G$ are nonnegative.

Lemma 17. *The* $\mathtt{H}$*- and* $\mathtt{Q}$*-bases of the Tits algebra are related by*

(31)
$$\mathtt{H}_F = \sum_{K:\, F \le K} \mathtt{u}_F^K\, \mathtt{Q}_K \quad \textit{and} \quad \mathtt{Q}_F = \sum_{G:\, F \le G} \mathtt{a}_F^G\, \mathtt{H}_G.$$

In particular,

$$\mathtt{H}_O = \sum_K \mathtt{u}^K \mathtt{Q}_K. \tag{32}$$

Proof. First note that (32) follows from (17) and (26).

Now multiply both sides of this identity on the left by $\mathtt{H}_F$, and then use (5) and (27).

This proves the first formula in (31), and the second formula then follows. □

We deduce from (26) and the second formula in (31) that for an Eulerian family $\mathtt{E}$ associated to the homogeneous section $\mathtt{u}$,

$$\mathtt{E}_{\mathrm{X}} = \sum_{F:\, \mathrm{s}(F) = \mathrm{X}} \ \sum_{G:\, F \le G} \mathtt{u}^F \mathtt{a}_F^G\, \mathtt{H}_G. \tag{33}$$

4.6 Over and under a flat. Cartesian product

The discussion of Section 2.6 implies:

Lemma 18. *A $\mathtt{Q}$-basis of $\Sigma[\mathcal{A}]$ induces a $\mathtt{Q}$-basis of $\Sigma[\mathcal{A}_H]$ for every face H and a $\mathtt{Q}$-basis of $\Sigma[\mathcal{A}^{\mathrm{X}}]$ for every flat X. Similarly, a $\mathtt{Q}$-basis of $\Sigma[\mathcal{A}]$ and a $\mathtt{Q}$-basis of $\Sigma[\mathcal{A}']$ induce a $\mathtt{Q}$-basis of $\Sigma[\mathcal{A} \times \mathcal{A}']$.*

Aspects of the passage to the arrangement over a flat are discussed in more detail below.

Lemma 19. *For $F \geq H$,*

(34)

$$\mathtt{H}_{F/H} = \sum_{K:\, F \leq K} \mathtt{u}_F^K\, \mathtt{Q}_{K/H} \quad \textit{and} \quad \mathtt{Q}_{F/H} = \sum_{G:\, F \leq G} \mathtt{a}_F^G\, \mathtt{H}_{G/H}.$$

These are the change of basis formulas for the $\mathtt{H}$- and $\mathtt{Q}$-bases of $\Sigma[\mathcal{A}_H]$.

Proof. By (9), the first formula is a restatement of the first formula in (31).

The second formula follows by inverting the matrix of coefficients $(\mathtt{u}_F^K)$ with $H \leq F \leq K$.

The coefficients of the inverse will match the $\mathtt{a}_F^G$. □

Lemma 20. *We have*

(35)

$$\Delta_G(\mathtt{Q}_K) = \begin{cases} \mathtt{Q}_{GK/G} & \text{if } KG = K, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \mu_F(\mathtt{Q}_{K/F}) = \mathtt{Q}_K.$$

In particular,

$$\mathtt{Q}_F = \mu_F(\mathtt{Q}_{F/F}). \tag{36}$$

Proof. For the first formula: By (24) and the fact that Δ_G is an algebra homomorphism,

$$\Delta_G(\mathtt{Q}_K) = \Delta_G(\mathtt{H}_K \cdot \mathtt{E}_{\mathrm{s}(K)}) = \Delta_G(\mathtt{H}_K) \cdot \Delta_G(\mathtt{E}_{\mathrm{s}(K)}).$$

If $KG \neq K$, then the rhs is zero by the Saliola lemma (Lemma 8).

If $KG = K$, then the calculation continues using (20) as follows.

$$\Delta_G(\mathtt{H}_K) \cdot \Delta_G(\mathtt{E}_{\mathrm{s}(K)}) = \mathtt{H}_{GK/G} \cdot \mathtt{E}_{\mathrm{s}(K)/G} = \mathtt{Q}_{GK/G}.$$

The last step again used (24) but for $\mathcal{A}_G$.

For the second formula:

$$\mu_F(\mathtt{Q}_{K/F}) = \mu_F(\Delta_F(\mathtt{Q}_K)) = \mathtt{H}_F \cdot \mathtt{Q}_K = \mathtt{Q}_K.$$

We used the first formula, $\mu_F \Delta_F(x) = \mathtt{H}_F \cdot x$ and (27) in that order. $\square$

Now $\mathtt{Q}_{F/F}$ is the first Eulerian idempotent of $\Sigma[\mathcal{A}_F]$ and μ_F is the inclusion map.

Thus, by (36), each $\mathtt{Q}$-basis element of $\mathcal{A}$ is interpretable as the first Eulerian idempotent of an arrangement over a flat of $\mathcal{A}$.

Lemma 21. *If F and G have the same support, then*

$$\beta_{G,F}(\mathsf{Q}_{K/F}) = \mathsf{Q}_{GK/G}. \tag{37}$$

In particular,

$$\beta_{G,F}(\mathsf{Q}_{F/F}) = \mathsf{Q}_{G/G}. \tag{38}$$

Proof. This follows from $\beta_{G,F}\Delta_F = \Delta_G$ and the first formula in (35). □

5 Families of Zie idempotents

Recall that a Zie element is special if its coefficient of the central face is 1.

We consider special Zie families made up of special Zie elements in arrangements over flats and extend Proposition 1 as follows.

Theorem 3. *The following pieces of data are equivalent.*

- *A homogeneous section* $\mathtt{u}$ *of* $\mathcal{A}$.
- *An Eulerian family* $\mathtt{E}$ *of* $\mathcal{A}$.
- *A special Zie family* P *of* $\mathcal{A}$.

Homogeneous sections are trivial to construct. All one needs to do is to assign scalars to each face such that the sum in each flat is 1.

In contrast, construction of Eulerian families or of special Zie elements is completely nontrivial.

5.1 Special Zie families

A special Zie family of $\mathcal{A}$ is a set $\mathsf{P} := \{\mathsf{P}_{\mathrm{X}}\}_{\mathrm{X}\in\Pi}$ indexed by flats, where each P_{X} is a special Zie element of the arrangement $\mathcal{A}_{\mathrm{X}}$ over X.

In particular, $\mathsf{P}_{\perp}$ is a special Zie element of $\mathcal{A}$.

5.2 From an Eulerian family to a special Zie family

Suppose $\mathtt{E}$ is an Eulerian family.

Recall: Using this family, one can define the $\mathtt{Q}$-basis of $\Sigma[\mathcal{A}]$, with basis element $\mathtt{Q}_F$ given by (24).

Written in the $\mathtt{H}$-basis, $\mathtt{Q}_F$ only involves faces greater than F.

The element $\mathtt{Q}_O$ equals the first Eulerian idempotent $\mathtt{E}_\perp$.

Similarly, the induced Eulerian family of $\mathcal{A}_F$ yields a $\mathtt{Q}$-basis of $\Sigma[\mathcal{A}_F]$ for any face F.

The element $\mathtt{Q}_{F/F}$ equals the first Eulerian idempotent $\mathtt{E}_{\mathrm{s}(F)/F}$.

Now, for each flat X, define an element of $\Sigma[\mathcal{A}_{\mathrm{X}}]$ by

$$\mathsf{P}_{\mathrm{X}} := \beta_{\mathrm{X},F}(\mathtt{Q}_{F/F}), \tag{39}$$

where F is any face with support X.

(It follows from (38) that the rhs does not depend on the specific choice of F.)

In particular,

$$\mathsf{P}_{\perp} = \mathtt{Q}_{O} = \mathtt{E}_{\perp}.$$

The Saliola lemma and the Friedrichs criterion imply that the first Eulerian idempotent $\mathtt{E}_{\perp}$ is a Zie element of $\mathcal{A}$. Also it is clearly special.

Similarly, for each flat X, P_{X} is a special Zie element of $\mathcal{A}_{\mathrm{X}}$ since $\mathtt{Q}_{F/F}$ is the first Eulerian idempotent of $\Sigma[\mathcal{A}_F]$ and $\beta_{\mathrm{X},F}$ is an algebra isomorphism.

Thus, we have constructed a special Zie family $\{\mathsf{P}_{\mathrm{X}}\}$.

5.3 From a special Zie family to an Eulerian family

We saw how to go from an Eulerian family to a special Zie family.

Now we show that this procedure can be reversed.

Accordingly:

- Suppose we are given a special Zie element P_{X} of $\mathcal{A}_{\mathrm{X}}$, one for each flat X of $\mathcal{A}$.

Starting from this data, we construct a homogeneous section $\mathtt{u}$, and its associated Eulerian family $\mathtt{E}$ and the $\mathtt{Q}$-basis.

In fact, we will first get hold of the $\mathtt{Q}$-basis, and use it to define $\mathtt{E}$ and $\mathtt{u}$.

Details follow.

For any face F with support X, define

$$\mathtt{Q}_{F/F} := \beta_{F,\mathrm{X}}(\mathsf{P}_\mathrm{X}) \quad \text{and} \quad \mathtt{Q}_F := \mu_F(\mathtt{Q}_{F/F}). \tag{40}$$

Since P_X is special, $\mathtt{Q}_F$ is of the form given in the second formula in (31).

By triangularity, the set $\{\mathtt{Q}_F\}$ as F varies is a basis of $\Sigma[\mathcal{A}]$.

Further, recall that the support of a special Zie element is $\mathtt{Q}_\perp$. Apply this fact to each P_X (which is a special Zie element in the arrangement over X) to deduce that

$$\mathrm{s}(\mathtt{Q}_F) = \mathtt{Q}_{\mathrm{s}(F)}. \tag{41}$$

Since $\mathtt{Q}$ is a basis, there exist unique scalars $\mathtt{u}^F$ such that

(42)
$$\mathtt{H}_O = \sum_F \mathtt{u}^F \mathtt{Q}_F.$$

Now set

(43)
$$\mathtt{E}_\mathrm{X} := \sum_{F:\, \mathrm{s}(F)=\mathrm{X}} \mathtt{u}^F \mathtt{Q}_F \quad \text{and} \quad \mathtt{u}_\mathrm{X} := \sum_{F:\, \mathrm{s}(F)=\mathrm{X}} \mathtt{u}^F \mathtt{H}_F.$$

By construction, (17) holds, that is, the sum of the $\mathtt{E}_\mathrm{X}$ is $\mathtt{H}_O$. We now claim that (3), (13) and (24) hold with $\mathtt{Q}$ as in (40) and $\mathtt{E}_\mathrm{X}$ and $\mathtt{u}_\mathrm{X}$ as in (43).

Applying the support map to (42) and using (41), we obtain

$$\mathtt{H}_\perp = \sum_F \mathtt{u}^F \mathtt{Q}_{\mathrm{s}(F)} = \sum_\mathrm{X} \Big(\sum_{F:\, \mathrm{s}(F)=\mathrm{X}} \mathtt{u}^F \Big) \mathtt{Q}_\mathrm{X}.$$

This shows that the sums in parenthesis are all 1. This proves (3).

Next:

- If F and G have the same support, then $\mathtt{H}_F \cdot \mathtt{Q}_G = \mathtt{Q}_F$.

  ($\mathtt{H}_F \cdot \mathtt{Q}_G = \mu_F \Delta_F \mu_G(\mathtt{Q}_{G/G}) = \mu_F \beta_{F,G} \Delta_G \mu_G(\mathtt{Q}_{G/G}) = \mu_F \beta_{F,G}(\mathtt{Q}_{G/G}) = \mu_F(\mathtt{Q}_{F/F}) = \mathtt{Q}_F$.)

- If $GF > G$, then $\mathtt{H}_F \cdot \mathtt{Q}_G = 0$.

  ($\mathtt{H}_F \cdot \mathtt{Q}_G = \mathtt{H}_F \cdot \mathtt{H}_G \cdot \mathtt{Q}_G = \mathtt{H}_{FG} \cdot \mathtt{Q}_G = \mathtt{H}_{FG} \cdot \mathtt{H}_{GF} \cdot \mathtt{Q}_G = \mathtt{H}_{FG} \cdot \mu_G(\mathtt{H}_{GF/G}) \cdot \mu_G(\mathtt{Q}_{G/G}) = \mathtt{H}_{FG} \cdot \mu_G(\mathtt{H}_{GF/G} \cdot \mathtt{Q}_{G/G}) = 0$ by the Friedrichs criterion since $\mathtt{Q}_{G/G}$ is a Zie element of $\Sigma[\mathcal{A}_G]$.)

The first statement along with (3) implies (24). The two statements together imply:

$$\mathtt{u}_{\mathrm{X}} \cdot \mathrm{E}_{\mathrm{X}} = \mathrm{E}_{\mathrm{X}} \qquad \text{and} \qquad \mathtt{u}_{\mathrm{X}} \cdot \mathrm{E}_{\mathrm{Y}} = 0 \text{ if } \mathrm{X} \not< \mathrm{Y}.$$

Now multiply both sides of (17) on the left by $\mathtt{u}_{\mathrm{X}}$ and use the above two identities to obtain

$$\mathtt{u}_{\mathrm{X}} = \mathrm{E}_{\mathrm{X}} + \sum_{\mathrm{Y}:\mathrm{X}<\mathrm{Y}} \mathtt{u}_{\mathrm{X}} \cdot \mathrm{E}_{\mathrm{Y}}.$$

This proves (13), and the claim is established.

Thus, we obtain an Eulerian family starting with a special Zie family. This construction is clearly inverse to the previous construction. This completes the proof of Theorem 3.

6 Decomposition of a module

6.1 Decomposition of a module

Proposition 2. *For any left* Σ*-module* h *and Eulerian family* $\mathtt{E}$*,*

(44) $$\mathsf{h} = \bigoplus_{\mathrm{X}} \mathtt{E}_{\mathrm{X}} \cdot \mathsf{h},$$

with

(45) $$\dim(\mathtt{E}_{\mathrm{X}} \cdot \mathsf{h}) = \eta_{\mathrm{X}}(\mathsf{h}).$$

Proof. This is a special case of a result for elementary algebras discussed earlier. □

6.2 Primitive part

Proposition 3. *For any left* Σ*-module* h *and Eulerian family* $\mathtt{E}$*,*

$$\mathcal{P}(\mathsf{h}) = \mathtt{E}_{\perp} \cdot \mathsf{h}. \tag{46}$$

Proof. Recall that for any special Zie element z, we have $z \cdot \mathsf{h} = \mathcal{P}(\mathsf{h})$. Now use the fact that $\mathtt{E}_{\perp}$ is a special Zie element. $\square$

7 Chamber element as a sum of Lie elements over flats

Recall that any Eulerian family yields a decomposition of a left module over the Tits algebra. Applied to the module of chambers, this yields an algebraic form of the Zaslavsky formula.

7.1 Algebraic form of Zaslavsky formula

Recall from Proposition 3 that for an Eulerian family $\mathtt{E}$ and a left module h, the summand $\mathtt{E}_\perp \cdot \mathsf{h}$ is the primitive part of h.

More generally, the summand $\mathtt{E}_\mathrm{X} \cdot \mathsf{h}$ is isomorphic to the primitive part of a certain module over $\Sigma[\mathcal{A}_\mathrm{X}]$.

We focus on the example of chambers.

For $\mathsf{h} = \Gamma$, the module in question is $\Gamma[\mathcal{A}_\mathrm{X}]$, and the decomposition (44) can thus be rephrased as

$$\Gamma[\mathcal{A}] \cong \bigoplus_{\mathrm{X}} \mathsf{Lie}[\mathcal{A}_\mathrm{X}]. \tag{47}$$

(Recall from the Friedrichs criterion that Lie is the primitive part of Γ.)

The isomorphism (47) may be viewed as an algebraic form of the Zaslavsky formula. It is developed in more detail below.

7.2 Chamber element as a sum of Lie elements over flats

Recall the maps $\beta_{F,\mathrm{X}}$, $\beta_{\mathrm{X},F}$, $\beta_{G,F}$, μ_F and Δ_F.

Lemma 22. *Fix a homogeneous section* $\mathtt{u}$ *with associated Eulerian family* $\mathtt{E}$.

For any flat X*, the linear map*

(48)
$$\mathsf{Lie}[\mathcal{A}_{\mathrm{X}}] \xrightarrow{\cong} \mathtt{E}_{\mathrm{X}} \cdot \Gamma[\mathcal{A}], \qquad z \mapsto \sum_{F:\, \mathrm{s}(F)=\mathrm{X}} \mathtt{u}^F \mu_F \beta_{F,\mathrm{X}}(z)$$

is an isomorphism. In particular,

(49)
$$\mathsf{Lie}[\mathcal{A}] = \mathtt{E}_{\perp} \cdot \Gamma[\mathcal{A}].$$

The inverse of (48) *is given by*

$$\mathtt{E}_{\mathrm{X}} \cdot \Gamma[\mathcal{A}] \to \mathsf{Lie}[\mathcal{A}_{\mathrm{X}}], \qquad z \mapsto \beta_{\mathrm{X},F} \Delta_F(z),$$

where F *is any face of support* X.

We elaborate on the isomorphism (48).

It says that $\mathrm{E}_{\mathrm{X}} \cdot \Gamma[\mathcal{A}]$ is the image of the composite map

$$\mathsf{Lie}[\mathcal{A}_{\mathrm{X}}] \hookrightarrow \Gamma[\mathcal{A}_{\mathrm{X}}] \to \bigoplus_{F\colon \mathrm{s}(F)=\mathrm{X}} \Gamma[\mathcal{A}_F] \to \Gamma[\mathcal{A}].$$

The second map projected on each F-component is $\beta_{F,\mathrm{X}}$, while the last map restricted to each F-component is μ_F multiplied by the scalar $\mathtt{u}^F$.

Informally, the composite map distributes a Lie element of $\mathcal{A}_{\mathrm{X}}$ over the stars of faces F with support X, with the star of F receiving weight $\mathtt{u}^F$.

Proof. First note that (49) is a special case of (46) in view of the Friedrichs criterion.

We now proceed to the general case. For a face F with support X,

$$\mathsf{Lie}[\mathcal{A}_F] = \mathtt{E}_{\mathrm{X}/F} \cdot \mathsf{\Gamma}[\mathcal{A}_F] = \Delta_F(\mathtt{E}_{\mathrm{X}}) \cdot \Delta_F(\mathsf{\Gamma}[\mathcal{A}]) = \Delta_F(\mathtt{E}_{\mathrm{X}} \cdot \mathsf{\Gamma}[\mathcal{A}]).$$

The first equality holds by (49) since $\mathtt{E}_{\mathrm{X}/F}$ is the first Eulerian idempotent of $\mathcal{A}_F$.

The second step used (20) and the fact that Δ_F maps $\mathsf{\Gamma}[\mathcal{A}]$ onto $\mathsf{\Gamma}[\mathcal{A}_F]$.

The last step used that Δ_F is an algebra homomorphism.

Let V be the image of the map

(a)

$$\mathsf{Lie}[\mathcal{A}_{\mathrm{X}}] \to \bigoplus_{F\colon \mathrm{s}(F)=\mathrm{X}} \mathsf{Lie}[\mathcal{A}_F], \qquad z \mapsto \sum_{F\colon \mathrm{s}(F)=\mathrm{X}} \beta_{F,\mathrm{X}}(z).$$

Explicitly, V consists of elements (z_F) such that $\beta_{G,F}(z_F) = z_G$. The map (a) is injective, so V is isomorphic to $\mathsf{Lie}[\mathcal{A}_{\mathrm{X}}]$.

By our first calculation and $\beta_{G,F}\Delta_F = \Delta_G$, we have a surjective map

(b) $$\mathtt{E}_{\mathrm{X}} \cdot \Gamma[\mathcal{A}] \to V, \qquad z \mapsto \sum_{F\colon \mathrm{s}(F)=\mathrm{X}} \Delta_F(z).$$

Moreover, for any $z \in \Gamma[\mathcal{A}]$,

$$\begin{aligned}\sum_{F\colon \mathrm{s}(F)=\mathrm{X}} \mathtt{u}^F \mu_F \Delta_F(\mathtt{E}_\mathrm{X} \cdot z) &= \sum_{F\colon \mathrm{s}(F)=\mathrm{X}} \mathtt{u}^F\, \mathtt{H}_F \cdot (\mathtt{E}_\mathrm{X} \cdot z) \\ &= \mathtt{u}_\mathrm{X} \cdot (\mathtt{E}_\mathrm{X} \cdot z) \\ &= (\mathtt{u}_\mathrm{X} \cdot \mathtt{E}_\mathrm{X}) \cdot z \\ &= \mathtt{E}_\mathrm{X} \cdot z.\end{aligned}$$

Thus, the map (b) is also injective, and its inverse is given by

(c) $V \to \mathtt{E}_\mathrm{X} \cdot \Gamma[\mathcal{A}], \qquad (z_F) \mapsto \displaystyle\sum_{F\colon \mathrm{s}(F)=\mathrm{X}} \mathtt{u}^F \mu_F(z_F).$

Composing (a) with (c) yields the isomorphism (48). $\square$

As a consequence:

Proposition 4. *For each homogeneous section* $\mathtt{u}$*, there is a linear isomorphism*

(50)
$$\bigoplus_{\mathrm{X}} \mathsf{Lie}[\mathcal{A}_{\mathrm{X}}] \cong \Gamma[\mathcal{A}], \qquad (z_{\mathrm{X}}) \mapsto \sum_{\mathrm{X}} \sum_{F\colon \mathrm{s}(F)=\mathrm{X}} \mathtt{u}^F \mu_F \beta_{F,\mathrm{X}}(z_{\mathrm{X}}).$$

The direct sum is over all flats.