Birkhoff algebra and Tits algebra

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1 Birkhoff algebra and Tits algebra

1.1 Birkhoff algebra

Recall the set of flats $\Pi[\mathcal{A}]$.

Since it is a lattice under the partial order of inclusion, it carries a monoid structure given by the join operation.

Let $\Pi[\mathcal{A}]$ denote its linearization over a field $\Bbbk$, with canonical basis $\mathtt{H}$.

It is a commutative $\Bbbk$-algebra:

$$\mathtt{H}_{\mathrm{X}} \cdot \mathtt{H}_{\mathrm{Y}} := \mathtt{H}_{\mathrm{X} \vee \mathrm{Y}}.$$

We call this the Birkhoff algebra.

1.2 Tits algebra

Recall the set of faces $\Sigma[\mathcal{A}]$.

It carries the structure of a monoid under the Tits product. This is the Tits monoid.

Let $\mathsf{\Sigma}[\mathcal{A}]$ denote its linearization over a field $\Bbbk$, with canonical basis $\mathtt{H}$. It is an algebra:

$$\mathtt{H}_F \cdot \mathtt{H}_G := \mathtt{H}_{FG}.$$

We call this the Tits algebra.

Let

$$\mathrm{s} : \mathsf{\Sigma}[\mathcal{A}] \twoheadrightarrow \mathsf{\Pi}[\mathcal{A}] \tag{1}$$

be the linearization of the support map.

This is a surjective morphism of algebras, that is,

$$\mathrm{s}(x \cdot y) = \mathrm{s}(x) \cdot \mathrm{s}(y). \tag{2}$$

1.3 Left module of chambers

Let $\Gamma[\mathcal{A}]$ denote the linearization of the set of chambers $\Gamma[\mathcal{A}]$ over a field $\Bbbk$, with canonical basis $\mathtt{H}$.

It is a two-sided ideal in the Tits algebra.

In particular, it is a left module over the Tits algebra:

$$\mathtt{H}_F \cdot \mathtt{H}_C := \mathtt{H}_{FC}.$$

We call this the left module of chambers.

1.4 Examples of finite-dimensional algebras

Fix a field $\mathbb{k}$. Let A be a finite-dimensional algebra over $\mathbb{k}$. Some examples to bear in mind are:

- $\mathbb{k}^n$,
- $\mathbb{k}[x]/(x^n)$,
- algebra of square matrices of size n,
- incidence algebra of a finite poset P. This class of algebras includes the algebra of upper-triangular matrices of size n.

Let M be a finite-dimensional (left or right) module over A. Some examples to bear in mind are:

- A as a left and right module over itself.
- left module of column vectors (and right module of row vectors) over the algebra of square matrices.
- (left and right) incidence module of the incidence algebra of a poset P.

1.5 Modules and representations

Let M be a finite-dimensional left module over A. That is, M is a finite-dimensional vector space over $\Bbbk$ equipped with a bilinear map

$$A \times M \to M, \qquad (a, m) \mapsto am,$$

such that $a(bm) = (ab)m$ and $1m = m$ for all $a, b \in A$ and $m \in M$.

Any $w \in A$ gives rise to a linear operator

$$\Psi_M(w) : M \to M, \qquad m \mapsto wm$$

defined by left multiplication by w.

This gives rise to an algebra homomorphism

$$\Psi_M : A \to \mathrm{End}_{\Bbbk}(M).$$

The latter is the algebra of endomorphisms of M, where the product is composition:

$$(fg)(m) = f(g(m)).$$

We say that Ψ_M is the representation of A associated to the module M.

Similarly, a right A-module M is defined by a bilinear map

$$M \times A \to M, \qquad (m, a) \mapsto ma,$$

In this case, we let $\Psi_M(w)$ denote right multiplication by w. The resulting map Ψ_M is an algebra antimorphism.

Standard terms of linear algebra apply to $\Psi_M(w)$.

For instance, we say that the operator $\Psi_M(w)$ is diagonalizable if M can be expressed as a direct sum of subspaces such that $\Psi_M(w)$ acts on each subspace by multiplication by a scalar.

The scalars are the eigenvalues of $\Psi_M(w)$ and the subspaces the eigenspaces.

For a left module M, let wM denote the image of the linear operator $\Psi_M(w)$.

In other words, wM consists of all elements of the form wm, as m varies over elements of M.

For a right module M, we denote the image by Mw.

1.6 Faithful and simple modules

A left A-module M is faithful if the representation Ψ_M is injective.

The annihilator $\mathrm{ann}(M)$ of a left A-module M is the kernel of Ψ_M:

$$\mathrm{ann}(M) := \{a \in A \mid am = 0 \text{ for all } m \in M\}.$$

Thus, M is faithful iff $\mathrm{ann}(M) = 0$.

Similar considerations apply to right A-modules.

A (left or right) module over A is simple if it is nonzero and has no proper submodules.

Any one-dimensional A-module M is simple.

1.7 Characters

Let A be a finite-dimensional $\mathbb{k}$-algebra.

The character of a (left or right) A-module M is the linear functional

(3) $$\chi_M : A \to \mathbb{k}, \qquad \chi_M(w) = \mathrm{Tr}(\Psi_M(w)),$$

where $\mathrm{Tr}(\Psi_M(w))$ denotes trace of the linear operator $\Psi_M(w)$.

A linear functional on A is called a character of A if it is the character of some A-module M.

Isomorphic modules have equal characters, but non-isomorphic modules may give rise to the same character.

A multiplicative character of A is an algebra homomorphism

$$\chi : A \to \mathbb{k}.$$

Lemma 1. *If M is a one-dimensional A-module M, then χ_M is multiplicative. Conversely, given a multiplicative character χ, there exists a one-dimensional module M, unique up to isomorphism, such that $\chi_M = \chi$.*

Proof. If M is one-dimensional, then $\mathrm{Tr} : \mathrm{End}_{\mathbb{k}}(M) \to \mathbb{k}$ is an isomorphism of algebras, so χ_M is an algebra morphism. For the converse, $M \cong \mathbb{k}$ with $am = \chi(a)m$. □

For any A-module M,

$$\chi_M(1) = (\dim_{\mathbb{k}} M) \cdot 1,$$

where on the right 1 denotes the unit element of the ground field $\mathbb{k}$. It follows that if $\mathbb{k}$ is of characteristic 0,

$$\chi_M(1) = 1 \iff \dim_{\mathbb{k}} M = 1 \iff \chi_M \text{ is a multiplicative character.}$$

1.8 Endomorphism algebra of chambers

Recall from Section 1.5 that a left module M over an algebra A gives rise to an algebra homomorphism from A to the endomorphism algebra of M.

The Tits algebra $\Sigma[\mathcal{A}]$ acts on the left module of chambers $\Gamma[\mathcal{A}]$.

This gives rise to an algebra homomorphism

$$\Sigma[\mathcal{A}] \to \mathrm{End}_{\Bbbk}(\Gamma[\mathcal{A}]), \tag{4}$$

the latter being the algebra of endomorphisms of $\Gamma[\mathcal{A}]$.

Let $\mathcal{A}$ be the arrangement of rank one with chambers C and $\overline{C}$. Identifying the endomorphism algebra of chambers with 2 by 2 matrices, the map (4) is given by

$$\alpha\,\mathtt{H}_O + \beta\,\mathtt{H}_C + \gamma\,\mathtt{H}_{\overline{C}} \mapsto \begin{pmatrix} \alpha+\beta & \beta \\ \gamma & \alpha+\gamma \end{pmatrix}. \tag{5}$$

Observe directly that this map is injective. Its image consists of those matrices whose column sums are equal.

The matrix in (5) has eigenvalues α and $\alpha+\beta+\gamma$ with eigenvectors $\mathtt{H}_C - \mathtt{H}_{\overline{C}}$ and $\beta\,\mathtt{H}_C + \gamma\,\mathtt{H}_{\overline{C}}$, respectively. One can deduce from here that $\Gamma[\mathcal{A}]$ has a unique one-dimensional submodule, namely, the subspace spanned by $\mathtt{H}_C - \mathtt{H}_{\overline{C}}$. In particular, $\Gamma[\mathcal{A}]$ does not decompose as a direct sum of simple modules.

Following (3), taking trace of the matrix in (5), we see that the character of $\Gamma[\mathcal{A}]$ is the linear functional

$$\chi_{\Gamma[\mathcal{A}]} : \Sigma[\mathcal{A}] \to \Bbbk, \qquad \alpha\,\mathtt{H}_O+\beta\,\mathtt{H}_C+\gamma\,\mathtt{H}_{\overline{C}} \mapsto 2\alpha+\beta+\gamma. \tag{6}$$

1.9 Complete systems of primitive orthogonal idempotents

Let A be a finite-dimensional algebra.

An element $e \in A$ is an idempotent if $e^2 = e$.

Idempotents e and f are orthogonal if $ef = fe = 0$. In this case, $e + f$ is also an idempotent.

Note that for any idempotent e, $1 - e$ is also an idempotent and it is orthogonal to e.

A nonzero idempotent e is primitive if it cannot be written as a sum of two orthogonal nonzero idempotents.

Lemma 2. *Every nonzero idempotent of A can be expressed as a sum of mutually orthogonal primitive idempotents.*

Proof. Let e be the given idempotent.

If e is primitive, then we are done.

If not, then write $e = f + g$, with both f and g nonzero orthogonal idempotents.

If f (or g) is not primitive, then write it as a sum of two orthogonal nonzero idempotents.

Continue this procedure.

If at some stage we have $e = e_1 + \cdots + e_k$, then $eA = e_1A \oplus \cdots \oplus e_kA$, with each $e_iA \neq 0$.

So by finite-dimensionality of A, this procedure must terminate. $\square$

Applying this result to the unit element 1, we deduce that there exists a family of mutually orthogonal primitive idempotents which sum up to 1.

Any such family is called a complete system of primitive orthogonal idempotents of A.

Complete refers to the fact that the idempotents sum up to 1.

Thus:

Proposition 1. *Any finite-dimensional algebra has a complete system of primitive orthogonal idempotents.*

Let $e \in A$ be an idempotent.

Let M be a left A-module.

The linear operator $\Psi_M(e)$ is diagonalizable.

Its eigenvalues are 1 and 0 with eigenspaces eM and $(1-e)M$, respectively.

Its trace is the dimension of eM. Thus,

$$\chi_M(e) = \mathrm{Tr}(\Psi_M(e)) = \dim eM. \tag{7}$$

Example. Consider the algebra of square matrices of size n. It acts on the left on $\mathbb{k}^n$, with an n-tuple written as a column vector. This is a faithful module. An idempotent is the same as a pair of complementary subspaces of $\mathbb{k}^n$, say (U, V). It acts by 0 on U and by 1 on V. A nonzero idempotent is primitive precisely when V is one-dimensional. (If not, then it can be decomposed by breaking V.)

In particular: A matrix with exactly one diagonal entry equal to 1 and all remaining entries 0, is a primitive idempotent. Further, these matrices are orthogonal and their sum is the identity matrix, so they form a complete system of primitive orthogonal idempotents. This is illustrated below for $n = 3$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Any other complete system is obtained by conjugating this system by an invertible matrix.

1.10 Nilpotents

An element $a \in A$ is nilpotent if there exists an integer $k \geq 1$ such that $a^k = 0$.

For any nilpotent element $a \in A$ and left A-module M,

$$\chi_M(a) = \mathrm{Tr}(\Psi_M(a)) = 0. \tag{8}$$

This is because the trace of any nilpotent matrix is 0.

Note that there is only one element in A which is both idempotent and nilpotent, namely, 0.

2 Split-semisimple commutative algebras

2.1 Split-semisimple commutative algebras

A commutative $\Bbbk$-algebra A is split-semisimple if it is isomorphic as an algebra to a product of copies of $\Bbbk$, that is, $A \cong \Bbbk^n$ for some n.

For $1 \leq i \leq n$, let e_i denote the element of A which corresponds to $(0, \ldots, 1, \ldots, 0) \in \Bbbk^n$ which is 1 in the i-th coordinate and zero elsewhere.

Observe that $f \in A$ is an idempotent iff f is a sum of some of the e_i.

In particular, the e_i are the only primitive idempotents of A.

These elements constitute a complete system of primitive orthogonal idempotents of A, and this system is unique.

The only algebra automorphisms of A are those obtained by permuting the e_i.

A split-semisimple commutative algebra does not contain any nonzero nilpotent elements. So an algebra such as $\mathbb{k}[x]/(x^n)$ for $n > 1$ cannot be split-semisimple.

2.2 Modules

Suppose A is a split-semisimple commutative algebra, and M is an A-module.

Then each $e_i M$ is a submodule of M, and further

$$M = \bigoplus_{i=1}^{n} e_i M. \tag{9}$$

An element $z \in A$ acts on $e_i M$ by scalar multiplication by the coefficient of e_i in z.

Note that each $e_i A$ is one-dimensional.

For each $1 \leq i \leq n$, put

$$\eta_i(M) := \chi_M(e_i) = \dim e_i M. \tag{10}$$

The second equality can be seen directly, or as an instance of (7).

Some important consequences of the above discussion are given below.

Theorem 1. *A split-semisimple commutative algebra A of dimension n has n distinct simple modules (up to isomorphism).*

They are one-dimensional.

For $1 \leq i \leq n$, the i-th simple module is given by $e_i A$, or equivalently, by the multiplicative character

$$\chi_i : A \to \mathbb{k}, \qquad z \mapsto \langle z, e_i \rangle,$$

where $\langle z, e_i \rangle$ denotes the coefficient of e_i in z.

Theorem 2. *Let A be a split-semisimple commutative algebra. Each A-module M is a direct sum of simple modules with the multiplicity of the i-th simple module being $\eta_i(M)$. In particular, M is faithful iff $\eta_i(M) > 0$ for each i.*

By definition of χ_i,

$$z = \sum_i \chi_i(z)\, e_i.$$

Thus, for any $z \in A$,

$$\chi_M(z) = \sum_{i=1}^{n} \chi_i(z)\, \eta_i(M). \tag{11}$$

Theorem 3. *Let A be a split-semisimple commutative algebra.*

For any element $w \in A$, the linear operator $\Psi_M(w)$ is diagonalizable.

Writing $w = \sum_i \lambda_i e_i$, the operator $\Psi_M(w)$ has eigenvalues λ_i and the eigenspace of λ_i is $e_i M$.

In particular, the multiplicity of λ_i is $\eta_i(M)$.

It is possible that $e_i M$ is 0 for some i in which case the eigenvalue λ_i does not occur.

It may also happen that the λ_i are not distinct. In that case, the eigenspaces are obtained by lumping together the corresponding $e_i M$. For instance, if $w = e_i$, then the eigenvalues are 1 and 0. The eigenspace for 1 is $e_i M$ and the eigenspace for 0 is the sum of the remaining $e_j M$.

Proposition 2. *The characters of a split-semisimple commutative algebra A of dimension n correspond to families $(\eta_i)_{1\leq i\leq n}$ of nonnegative integers, with the multiplicative ones corresponding to those families in which exactly one η_i is 1 and the rest are 0.*

The character χ and the family (η_i) relate by $\chi(e_i) = \eta_i$. Note that the character determines the module M (up to isomorphism), with η_i being the number of times the i-th simple module occurs in M.

3 Algebra of a finite lattice

3.1 Algebra of a lattice

Let P be a finite lattice with minimum element $\perp$ and maximum element $\top$.

Let $\Bbbk P$ denote the linearization of P over the field $\Bbbk$.

This is a commutative $\Bbbk$-algebra with product induced from the join operation in P.

Letting $\mathtt{H}$ denote the canonical basis,

$$\mathtt{H}_x \cdot \mathtt{H}_y := \mathtt{H}_{x \vee y}. \tag{12}$$

3.2 $\mathtt{Q}$-basis and split-semisimplicity

Define the $\mathtt{Q}$-basis of $\Bbbk P$ by

(13)

$$\mathtt{H}_x = \sum_{y\colon y \geq x} \mathtt{Q}_y \qquad \text{or equivalently} \qquad \mathtt{Q}_x = \sum_{y\colon y \geq x} \mu(x,y)\, \mathtt{H}_y .$$

Here μ refers to the Möbius function of the lattice P. In particular,

$$\mathtt{H}_\perp = \sum_y \mathtt{Q}_y . \tag{14}$$

Theorem 4. *The linearization of a finite lattice is a split-semisimple commutative algebra.*

The unique complete system of primitive orthogonal idempotents is given by the $\mathtt{Q}$*-basis. In other words,*

$$\mathtt{Q}_x \cdot \mathtt{Q}_y = \begin{cases} \mathtt{Q}_x & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Proof. An easy way to establish (15) is to assume it and deduce (12) from it.

The required calculation is shown below.

$$\begin{aligned} \mathtt{H}_x \cdot \mathtt{H}_y &= \Big(\sum_{z:\, z \geq x} \mathtt{Q}_z \Big) \cdot \Big(\sum_{w:\, w \geq y} \mathtt{Q}_w \Big) \\ &= \sum_{u:\, u \geq x \vee y} \mathtt{Q}_u \\ &= \mathtt{H}_{x \vee y}. \end{aligned}$$

□

Also from (13) and (15), we obtain

$$\mathtt{H}_y \cdot \mathtt{Q}_x = \begin{cases} \mathtt{Q}_x & \text{if } x \geq y, \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

In particular,

$$\mathtt{H}_y \cdot \mathtt{Q}_\perp = 0 \text{ for } y > \perp.$$

3.3 Linear functionals

Suppose $f : \mathbb{k}P \to \mathbb{k}$ is a linear map.

Then define (set-theoretic) maps $\xi, \eta : P \to \mathbb{k}$ as follows.

For each $x \in P$, let

(17) $$\xi_x = f(\mathtt{H}_x) \qquad \text{and} \qquad \eta_x = f(\mathtt{Q}_x).$$

We deduce from (13) that

(18) $$\xi_x = \sum_{y\colon y \geq x} \eta_y \quad \text{and} \quad \eta_x = \sum_{y\colon y \geq x} \mu(x,y)\, \xi_y.$$

Further, linearizing ξ in the $\mathtt{H}$-basis or η in the $\mathtt{Q}$-basis recovers f.

Thus among f, ξ and η, knowing any one determines the remaining two.

Some interesting choices for ξ and η are given below.

Example. For $x \in P$, put

(19)

$$\xi_x = \begin{cases} 1 & \text{if } x = \top, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \eta_x = \mu_P(x, \top).$$

In general, η will take both positive and negative values.

Example. Let M be a finite-dimensional module over $\Bbbk P$.

For each element $x \in P$, define

(20)

$$\xi_x(M) := \dim(\mathtt{H}_x M) \quad \text{and} \quad \eta_x(M) := \dim(\mathtt{Q}_x M).$$

These scalars are always nonnegative integers, since they are dimensions of spaces.

Recall from (7) that for any idempotent operator, the dimension of its image is its trace.

Since $\mathtt{H}_x$ and $\mathtt{Q}_x$ are idempotents, the linear functional f associated to $\xi_x(M)$ (or to $\eta_x(M)$) is the character χ_M of M.

3.4 Simple modules and diagonalizability

Let $\eta_x(M)$ be as in (20).

Theorem 5. *The algebra $\Bbbk P$ has $|P|$ distinct simple modules (up to isomorphism).*

They are one-dimensional.

The simple module corresponding to $x \in P$ is defined by the multiplicative character

$$\chi_x : \Bbbk P \to \Bbbk, \qquad \sum_y b^y \, \mathtt{Q}_y \mapsto b^x .$$

On the $\mathtt{H}$-basis, the multiplicative character is given by

$$\chi_x : \Bbbk P \to \Bbbk, \qquad \sum_y a^y \, \mathtt{H}_y \mapsto \sum_{y \colon y \leq x} a^y .$$

Proof. The claim about the simple modules and the character formula on the $\mathtt{Q}$-basis follows from Theorems 1 and 4. The formula on the $\mathtt{H}$-basis can then be deduced as follows.

$$\sum_y a^y \, \mathtt{H}_y \mapsto \sum_y a^y \sum_{z \colon z \geq y} \mathtt{Q}_z \mapsto \sum_z \Big(\sum_{y \colon y \leq z} a^y \Big) \, \mathtt{Q}_z \mapsto \sum_{y \colon y \leq x} a^y .$$

□

Theorem 6. *Any module M over the algebra $\Bbbk P$ is a direct sum of simple modules with $\eta_x(M)$ being the multiplicity of the simple module corresponding to $x \in P$.*

In particular, M is faithful iff $\eta_x(M) > 0$ for each $x \in P$.

Proof. This follows from Theorems 2 and 4. $\square$

Theorem 7. *Let M be a module over $\Bbbk P$.*

For $\alpha = \sum_x a^x\, \mathtt{H}_x$, the linear operator $\Psi_M(\alpha)$ is diagonalizable.

It has an eigenvalue

$$\lambda_x(\alpha) = \sum_{y\colon y \le x} a^y \tag{21}$$

for each $x \in P$, with multiplicity $\eta_x(M)$.

Proof. This follows from Theorems 3 and 4 and the $\mathtt{H}$-basis formula in Theorem 5. $\square$

4 Birkhoff algebra

We return to the Birkhoff algebra $\Pi[\mathcal{A}]$.

4.1 $\mathtt{Q}$-basis and split-semisimplicity

Define the $\mathtt{Q}$-basis of $\Pi[\mathcal{A}]$ by

(22)
$$\mathtt{H}_{\mathrm{X}} = \sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{X}} \mathtt{Q}_{\mathrm{Y}} \qquad \text{or equivalently} \qquad \mathtt{Q}_{\mathrm{X}} = \sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{X}, \mathrm{Y})\, \mathtt{H}_{\mathrm{Y}}.$$

In particular, the unit element is

(23)
$$\mathtt{H}_{\perp} = \sum_{\mathrm{Y}} \mathtt{Q}_{\mathrm{Y}}.$$

Specializing Theorem 4, we obtain:

Theorem 8. *The Birkhoff algebra is a split-semisimple commutative algebra.*

Its dimension equals the number of flats in $\mathcal{A}$.

The unique complete system of primitive orthogonal idempotents is given by the $\mathtt{Q}$-basis:

$$\mathtt{Q}_{\mathrm{X}} \cdot \mathtt{Q}_{\mathrm{Y}} = \begin{cases} \mathtt{Q}_{\mathrm{X}} & \text{\textit{if} } \mathrm{X} = \mathrm{Y}, \\ 0 & \text{\textit{otherwise.}} \end{cases} \tag{24}$$

By (16), we have:

$$\mathtt{H}_{\mathrm{Y}} \cdot \mathtt{Q}_{\mathrm{X}} = \begin{cases} \mathtt{Q}_{\mathrm{X}} & \text{if } \mathrm{X} \geq \mathrm{Y}, \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

From now on, whenever convenient, we will abbreviate $\Pi[\mathcal{A}]$ to Π.

4.2 Rank-one

Let $\mathcal{A}$ be the arrangement of rank one.

It has two flats, namely, the minimum flat $\perp$ and the maximum flat $\top$.

The $\mathtt{Q}$-basis elements are given by

$$\mathtt{Q}_\perp = \mathtt{H}_\perp - \mathtt{H}_\top, \qquad \mathtt{Q}_\top = \mathtt{H}_\top.$$

One can readily check that they define a complete system.

4.3 Linear functionals

Let (ξ_{X}) and (η_{X}) be two families of scalars indexed by flats which are related by

(26)
$$\xi_{\mathrm{X}} = \sum_{\mathrm{Y}\colon \mathrm{Y}\geq\mathrm{X}} \eta_{\mathrm{Y}} \quad \text{and} \quad \eta_{\mathrm{X}} = \sum_{\mathrm{Y}\colon \mathrm{Y}\geq\mathrm{X}} \mu(\mathrm{X},\mathrm{Y})\, \xi_{\mathrm{Y}}.$$

They correspond to the linear functional f : Π $\to \Bbbk$ by

(27)
$$\xi_{\mathrm{X}} = f(\mathtt{H}_{\mathrm{X}}) \qquad \text{and} \qquad \eta_{\mathrm{X}} = f(\mathtt{Q}_{\mathrm{X}}).$$

See (17) and (18).

Some choices for these families are given below.

Example. For each flat X, put

(28) $$\xi_{\mathrm{X}} = \begin{cases} 1 & \text{if } \mathrm{X} = \top, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \eta_{\mathrm{X}} = \mu(\mathcal{A}_{\mathrm{X}}).$$

This choice is a specialization of (19).

Example. Let h be a finite-dimensional module over Π. For each flat X, put

(29)
$$\xi_{\mathrm{X}}(\mathsf{h}) := \dim(\mathtt{H}_{\mathrm{X}} \cdot \mathsf{h}) \quad \text{and} \quad \eta_{\mathrm{X}}(\mathsf{h}) := \dim(\mathtt{Q}_{\mathrm{X}} \cdot \mathsf{h}).$$

This choice is a specialization of (20).

Example. For each flat X, put

(30) $$\xi_{\mathrm{X}} = c_{\mathrm{X}} \quad \text{and} \quad \eta_{\mathrm{X}} = |\mu(\mathcal{A}_{\mathrm{X}})|,$$

where c_{X} is the number of chambers in $\mathcal{A}_{\mathrm{X}}$. That this is a valid choice is equivalent to the Zaslavsky formula.

4.4 Simple modules and diagonalizability

Let $\eta_{\mathrm{X}}(\mathsf{h})$ be as in (29).

Theorem 9. *The Birkhoff algebra* Π *has one simple module (up to isomorphism) for each flat* X*.*

It is one-dimensional and defined by the multiplicative character

$$\chi_{\mathrm{X}} : \Pi \to \Bbbk, \qquad \sum_{\mathrm{Y}} w^{\mathrm{Y}} \mathtt{H}_{\mathrm{Y}} \mapsto \sum_{\mathrm{Y}:\, \mathrm{Y} \leq \mathrm{X}} w^{\mathrm{Y}}.$$

Proof. This is a special case of Theorem 5. □

Theorem 10. *Any finite-dimensional module* h *is a direct sum of simple modules with* $\eta_{\mathrm{X}}(\mathsf{h})$ *being the multiplicity of the simple module corresponding to the flat* X*.*

Proof. This is a special case of Theorem 6. □

Theorem 11. *Let* h *be a finite-dimensional module over the Birkhoff algebra.*

For $w = \sum_{\mathrm{X}} w^{\mathrm{X}} \mathtt{H}_{\mathrm{X}}$*, the linear operator* $\Psi_{\mathsf{h}}(w)$ *is diagonalizable.*

It has an eigenvalue

$$\lambda_{\mathrm{X}}(w) = \sum_{\mathrm{Y}\colon \mathrm{Y} \leq \mathrm{X}} w^{\mathrm{Y}} \tag{31}$$

for each X*, with multiplicity* $\eta_{\mathrm{X}}(\mathsf{h})$*.*

Proof. This is a special case of Theorem 7. □

5 Elementary algebras

5.1 Radical of an algebra

Let A be an algebra.

An ideal N of A is nilpotent if there exists an integer $k \geq 1$ such that $N^k = 0$.

The smallest k for which this happens is the nilpotency index of N.

In other words: N has nilpotency index k iff the product of any k elements in N is zero, and there exist $k - 1$ elements whose product is nonzero.

The sum of all nilpotent ideals of A is again a nilpotent ideal. This ideal is defined to be the radical of A.

In other words, the radical of A is the largest nilpotent ideal of A.

We denote it by $\mathrm{rad}(A)$.

It is contained in the set of all nilpotent elements of A.

Notation 12. For any ideal I of A, we have the quotient map $A \twoheadrightarrow A/I$. Whenever such a map is under discussion, for $z \in A$, we will write $\bar{z}$ for its image in A/I.

Proposition 3. *Suppose N is a nilpotent ideal of an algebra A such that A/N is a split-semisimple commutative algebra.*

Then $N = \mathrm{rad}(A)$ and it consists precisely of the nilpotent elements of A.

Proof. Since N is nilpotent, it is contained in $\mathrm{rad}(A)$, which in turn is contained in the set of all nilpotent elements.

Suppose $z \in A$ is nilpotent.

Then, so is its image $\bar{z} \in A/N$.

However, since A/N is a split-semisimple commutative algebra, it has no nonzero nilpotent elements.

Hence, $\bar{z} = 0$, and $z \in N$.

Thus N consists precisely of the nilpotent elements of A, and equals $\mathrm{rad}(A)$. $\square$

5.2 Elementary algebras

An algebra A is elementary if the quotient $A/\operatorname{rad}(A)$ is a split-semisimple commutative algebra.

Let us denote this quotient by $\bar{A}$.

We assume that $\bar{A}$ has dimension n and denote its primitive idempotents by $e_1, \ldots, e_n$.

Also following standard notation, for $z \in A$, we write $\bar{z}$ for its image in $\bar{A}$.

5.3 Radical of the Tits algebra

Let N denote the kernel of the support map (1). We set out to prove that

$$\mathsf{N} = \mathrm{rad}(\Sigma[\mathcal{A}]),$$

the radical of the Tits algebra.

Since the support map is an algebra homomorphism, N is an ideal of the Tits algebra.

Let $z = \sum_F x^F \mathtt{H}_F$ be any element of $\Sigma[\mathcal{A}]$. Then observe that

(32) $$z \in \mathsf{N} \iff \sum_{F\colon \mathrm{s}(F)=\mathrm{X}} x^F = 0 \text{ for all flats } \mathrm{X}.$$

In particular, for this to occur, $x^O = 0$.

Note that for any faces F and G with the same support, $\mathtt{H}_F - \mathtt{H}_G$ belongs to N, and elements of this form linearly span N.

An element of the Tits algebra is homogeneous if it is a linear combination of faces with the same support.

For any such element x, let us denote this common support by $\mathbf{s}(x)$.

By convention, $\mathbf{s}(0) = \top$, the maximmum flat.

Note that an arbitrary element of the Tits algebra can be written as a linear combination of homogeneous elements.

The product of homogeneous elements is again homogeneous.

Further, $\mathbf{s}(x), \mathbf{s}(y) \leq \mathbf{s}(x \cdot y)$ for any homogeneous elements x and y.

Lemma 3. *If $x \in \mathsf{N}$ is homogeneous and F is a face such that $\mathbf{s}(x) \leq \mathrm{s}(F)$, then $\mathtt{H}_F \cdot x = 0$.*

More generally, if x and y are homogeneous, $x \in \mathsf{N}$ and $\mathbf{s}(x) \leq \mathbf{s}(y)$, then $y \cdot x = 0$.

Proof. The second statement follows from the first.

To prove the first: Write $x = \sum_{K:\, \mathrm{s}(K)=\mathrm{X}} a^K \mathtt{H}_K$, where $\mathrm{X} = \mathbf{s}(x)$.

By hypothesis, $FK = F$ for all K of support X.

Thus,

$$\mathtt{H}_F \cdot x = \sum_{K:\, \mathrm{s}(K)=\mathrm{X}} a^K \mathtt{H}_F \cdot \mathtt{H}_K = \Big(\sum_{K:\, \mathrm{s}(K)=\mathrm{X}} a^K \Big) \mathtt{H}_F = 0,$$

by (32). $\square$

Lemma 4. *For any nonnegative integer k, the ideal N^k only contains elements which are linear combinations of faces of rank at least k.*

Proof. Consider $x_1 \cdot x_2 \cdot \ldots \cdot x_k \in \mathsf{N}^k$, where each x_i is a homogeneous element of N. Then

$$\perp \leq \mathbf{s}(x_1) \leq \mathbf{s}(x_1 \cdot x_2) \leq \cdots \leq \mathbf{s}(x_1 \cdot \ldots \cdot x_k).$$

If equality holds in any place, say $\mathbf{s}(x_1 \cdot \ldots \cdot x_{i-1}) = \mathbf{s}(x_1 \cdot \ldots \cdot x_i)$, then $\mathbf{s}(x_i) \leq \mathbf{s}(x_1 \cdot \ldots \cdot x_{i-1})$, and hence $x_1 \cdot \ldots \cdot x_i = 0$ by Lemma 3.

Thus we may assume

$$\perp < \mathbf{s}(x_1) < \mathbf{s}(x_1 \cdot x_2) < \cdots < \mathbf{s}(x_1 \cdot \ldots \cdot x_k)$$

from which we deduce that $x_1 \cdot x_2 \cdot \ldots \cdot x_k$ is a linear combination of faces of rank at least k. $\square$

As a consequence:

Proposition 4. *The ideal* N *is nilpotent.*

Proposition 5. *The Tits algebra is elementary.*

Its split-semisimple quotient is the Birkhoff algebra, with the support map as the quotient map.

In particular, the radical of the Tits algebra is the kernel of the support map: $\mathrm{rad}(\Sigma[\mathcal{A}]) = \mathsf{N}$ *and it consists precisely of the nilpotent elements of the Tits algebra.*

Proof. Apply Proposition 3 to the nilpotent ideal N, and use Theorem 8. All claims follow. $\square$

5.4 Simple modules over an elementary algebra

Theorem 13. *Let A be elementary.*

Then A has n distinct simple modules (up to isomorphism).

They are one-dimensional.

For $1 \leq i \leq n$, the i-th simple module is defined by the multiplicative character

$$\chi_i : A \to \mathbb{k}, \qquad z \mapsto \langle \bar{z}, e_i \rangle.$$

In fact, there is a correspondence between simple modules over A and over $\bar{A}$.

Proof. Let M be a simple A-module.

Then JM is a submodule of M, where $J = \mathrm{rad}(A)$.

By simplicity of M, this submodule is either M or 0.

The nilpotency of J forces $JM = 0$.

So the action of A factors through the quotient map $A \twoheadrightarrow \bar{A}$, and M is a simple $\bar{A}$-module.

Conversely, any simple $\bar{A}$-module is a simple A-module.

So there is a correspondence between simple modules over A and over $\bar{A}$.

Now apply Theorem 1. $\square$

By definition of χ_i,

$$\bar{z} = \sum_i \chi_i(z)\, e_i. \tag{33}$$

It follows that $z \in \mathrm{rad}(A)$ iff $\chi_i(z) = 0$ for all i.

5.5 Simple modules over the Tits algebra

Theorem 14. *The simple modules over $\Sigma[\mathcal{A}]$ are one-dimensional and indexed by flats.*

Let χ_{X} denote the multiplicative character corresponding to the flat X. It is specified by

$$\mathrm{s}(z) = \sum_{\mathrm{X}} \chi_{\mathrm{X}}(z)\, \mathtt{Q}_{\mathrm{X}}. \tag{34}$$

On a $\mathtt{H}$-basis element, it is given by

$$\chi_{\mathrm{X}}(\mathtt{H}_F) = \begin{cases} 1 & \textit{if } \mathrm{s}(F) \leq \mathrm{X}, \\ 0 & \textit{otherwise.} \end{cases} \tag{35}$$

Equivalently, for $w = \sum_F w^F \mathtt{H}_F$,

$$\chi_{\mathrm{X}}(w) = \sum_{F:\, \mathrm{s}(F) \leq \mathrm{X}} w^F. \tag{36}$$

Proof. Apply Theorem 13. This yields the first two statements. In particular, $\chi_{\mathrm{X}}(\mathtt{H}_F)$ is the coefficient of $\mathtt{Q}_{\mathrm{X}}$ in $\mathtt{H}_{\mathrm{s}(F)}$. Now use (22) to first get (35) and then (36). $\square$

In particular, the multiplicative characters for the minimum and maximum flats are given by

(37)

$$\chi_\perp(\mathtt{H}_F) = \begin{cases} 1 & \text{if } F = O, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \chi_\top(\mathtt{H}_F) = 1 \text{ for all } F.$$

5.6 Modules

Let M be a (left or right) module over an elementary algebra A.

As a consequence of (8), the character of M factors through the quotient map $A \to \bar{A}$ yielding the commutative diagram

$$\begin{tikzcd} A \arrow[d, two heads] \arrow[dr, "\chi_M"] & \\ \bar{A} \arrow[r, dotted, "\chi_M"'] & \mathbb{k}. \end{tikzcd}$$

We continue to denote the induced linear functional on $\bar{A}$ by χ_M.

For $1 \le i \le n$, put

(38) $$\eta_i(M) := \chi_M(e_i).$$

Observe that for any $w \in A$,

(39) $$\chi_M(w) = \sum_{i=1}^{n} \chi_i(w)\, \eta_i(M).$$

Let $0 = M_0 \lessdot M_1 \lessdot \cdots \lessdot M_k = M$ be any composition series of M.

This is a filtration of M in which M_{j-1} is a maximal proper submodule of M_j.

Then each M_j/M_{j-1}, called a composition factor, is a simple module and hence one-dimensional by Theorem 13.

The associated graded module of the filtration, namely,

$$\bar{M} := \bigoplus_{j=1}^{k} M_j/M_{j-1}$$

is both an A-module and an $\bar{A}$-module.

Thus, for $w \in A$, the operators $\Psi_{\bar{M}}(w)$ and $\Psi_{\bar{M}}(\bar{w})$ coincide.

Further, we claim that the eigenvalues (and hence trace) of the operator $\Psi_M(w)$ coincide with those of the operator $\Psi_{\bar{M}}(\bar{w})$.

To see this, pick a basis of M by first picking a nonzero element from M_1, followed by an element of M_2 which is not in M_1, and so on. This basis does not depend on w. It induces a basis of $\bar{M}$. In these bases, $\Psi_{\bar{M}}(\bar{w})$ is a diagonal matrix, while $\Psi_M(w)$ is an upper triangular matrix whose diagonal part agrees with $\Psi_{\bar{M}}(\bar{w})$. This proves the claim.

In particular, the induced functional χ_M on $\bar{A}$ is the character $\chi_{\bar{M}}$ of the module $\bar{M}$.

Example. Let A be the algebra of upper triangular matrices of size n. It is elementary. The radical is the ideal of strictly upper triangular matrices. Elements of the quotient $\bar{A}$ can be identified with diagonal matrices.

Let M be the left A-module of column vectors. For $0 \leq i \leq n$, let M_i denote the submodule consisting of vectors whose last $n - i$ entries are zero. This defines a composition series of M. Let $\bar{M}$ denote its associated graded module. The action of any upper triangular matrix on $\bar{M}$ is via its diagonal part.

Some consequences of the above discussion are stated below.

Theorem 15. *Let A be elementary and M be an A-module.*

Then in any composition series of M, the number of times the simple module associated to χ_i appears as a composition factor is $\eta_i(M)$.

Proof. We have $\eta_i(M) = \chi_M(e_i) = \chi_{\bar{M}}(e_i)$. Now use Theorem 2. $\square$

Theorem 16. *Let A be elementary and M be an A-module.*

Then all elements of A are simultaneously triangularizable on M.

For $w \in A$, the eigenvalues of the linear operator $\Psi_M(w)$ are $\chi_i(w)$, and the multiplicity of $\chi_i(w)$ is $\eta_i(M)$.

Proof. For the second part, we can use Theorem 3 since $\Psi_M(w)$ and $\Psi_{\bar{M}}(\bar{w})$ have the same eigenvalues. $\square$

It is interesting that all eigenvalues of $\Psi_M(w)$ belong to the ground field $\mathbb{k}$.

The number $\eta_i(M)$ which is the multiplicity of $\chi_i(w)$ only depends on i and not on w. We call it the generic multiplicity associated to the index i.

It is possible that $\eta_i(M)$ is 0 for some i in which case the eigenvalue $\chi_i(w)$ does not occur.

It may also happen that $i \neq j$ but $\chi_i(w) = \chi_j(w) = \lambda$ (say). In this case, the multiplicity of λ will be the sum of $\eta_i(M)$ over those i for which $\chi_i(w) = \lambda$.

Note very carefully that Theorem 16 makes no claim about the diagonalizability of $\Psi_M(w)$.

Proposition 6. *For an elementary algebra A, there is a correspondence between (multiplicative) characters of A and (multiplicative) characters of $\bar{A}$.*

Thus, a character of A corresponds to a family $(\eta_i)_{1 \le i \le n}$ of nonnegative integers.

It is multiplicative if exactly one η_i is 1 and the rest are 0.

Proof. For the second part, we can use Proposition 2. □

Proposition 7. *Let A be elementary and M be an A-module. Let $\hat{e}_1, \ldots, \hat{e}_n$ be a complete system of primitive orthogonal idempotents of A such that $\hat{e}_i$ lifts e_i. Then*

$$M = \bigoplus_i \hat{e}_i M$$

and

$$\dim \hat{e}_i M = \eta_i(M).$$

Proof. The decomposition is clear. For the formula:

$$\dim \hat{e}_i M = \chi_M(\hat{e}_i) = \chi_M(e_i) = \eta_i(M).$$

We used (7) and (38). □

Note very carefully that Proposition 7 does not claim that the $\hat{e}_i M$ are submodules of M.

5.7 Modules over the Tits algebra

Let h be a finite-dimensional left module over the Tits algebra, and Ψ_{h} the associated representation.

For any element w of the Tits algebra, $\Psi_{\mathsf{h}}(w)$ denotes the linear operator on h given by multiplication by w, and $w \cdot \mathsf{h}$ denotes its image. Thus,

$$\Psi_{\mathsf{h}}(w) : \mathsf{h} \to \mathsf{h}, \qquad \Psi_{\mathsf{h}}(w)(h) := w \cdot h.$$

Following (3), the character of h is the linear functional

$$\chi_{\mathsf{h}} : \Sigma \to \mathbb{k}, \qquad \chi_{\mathsf{h}}(w) = \mathrm{Tr}(\Psi_{\mathsf{h}}(w)),$$

where Tr denotes trace.

Recall from Proposition 5 that the Tits algebra is elementary.

We now apply the general discussion in Section 5.6 to the module h.

The character χ_{h} factors through the support map yielding the commutative diagram

$$\begin{tikzcd}
\Sigma \arrow[d, "\mathrm{s}"'] \arrow[dr, "\chi_{\mathsf{h}}"] & \\
\Pi \arrow[r, dotted, "\chi_{\mathsf{h}}"'] & \Bbbk.
\end{tikzcd}$$

The induced linear functional on Π is also denoted χ_{h}.

Following (27), for each flat X, put

(40) $$\xi_{\mathrm{X}}(\mathsf{h}) = \chi_{\mathsf{h}}(\mathtt{H}_{\mathrm{X}}) \qquad \text{and} \qquad \eta_{\mathrm{X}}(\mathsf{h}) = \chi_{\mathsf{h}}(\mathtt{Q}_{\mathrm{X}}).$$

Thus,

(41)
$$\xi_{\mathrm{X}}(\mathsf{h}) = \sum_{\mathrm{Y}:\,\mathrm{Y}\geq\mathrm{X}} \eta_{\mathrm{Y}}(\mathsf{h}) \quad \text{or equivalently} \quad \eta_{\mathrm{X}}(\mathsf{h}) = \sum_{\mathrm{Y}:\,\mathrm{Y}\geq\mathrm{X}} \mu(\mathrm{X},\mathrm{Y})\,\xi_{\mathrm{Y}}(\mathsf{h}).$$

The integer $\eta_{\mathrm{X}}(\mathsf{h})$ agrees with (38).

It is the number of times the simple module associated to χ_{X} appears as a composition factor in a composition series of h.

By (39), for $w \in \Sigma$,

(42) $$\chi_{\mathsf{h}}(w) = \sum_{\mathrm{X}} \chi_{\mathrm{X}}(w)\,\eta_{\mathrm{X}}(\mathsf{h}).$$

Recall from (7) that the trace of an idempotent operator is the dimension of its image.

Applying this to the idempotent $\mathtt{H}_F$, we get

$$\xi_{\mathrm{X}}(\mathsf{h}) = \dim(\mathtt{H}_F \cdot \mathsf{h}), \tag{43}$$

where F is any face with support X.

The fact that this number does not depend on the particular choice of F can also be seen directly:

Lemma 5. *Let F and G be faces of the same support. For a left module h, there is an isomorphism*

$$\mathtt{H}_F \cdot \mathsf{h} \xrightarrow{\cong} \mathtt{H}_G \cdot \mathsf{h}$$

given by multiplication by $\mathtt{H}_G$, with inverse given by multiplication by $\mathtt{H}_F$.

Proof. This follows from the property $FGF = FG$ and $FG = F$ and $GF = G \iff \mathrm{s}(F) = \mathrm{s}(G)$. ∎

Similarly, we have

(44) $$\eta_{\mathrm{X}}(\mathsf{h}) = \dim(\mathsf{Q}_F \cdot \mathsf{h})$$

for any idempotent Q_F which lifts Q_{X}. Such idempotents will be constructed later.

If the action of Σ on h factors through the support map, then h becomes a module over Π, and $\xi_{\mathrm{X}}(\mathsf{h})$ and $\eta_{\mathrm{X}}(\mathsf{h})$ coincide with (29).

Example. For the left module of chambers Γ,

$$\xi_{\mathrm{X}}(\Gamma) = c_{\mathrm{X}} \quad \text{and} \quad \eta_{\mathrm{X}}(\Gamma) = |\mu(\mathcal{A}_{\mathrm{X}})|, \tag{45}$$

where c_{X} is the number of chambers in $\mathcal{A}_{\mathrm{X}}$.

This can be understood as follows.

The space $\mathtt{H}_F \cdot \Gamma$ has a basis consisting of all chambers greater than F, so its dimension is c_F.

This yields the formula for $\xi_{\mathrm{X}}(\Gamma)$.

The formula for $\eta_{\mathrm{X}}(\Gamma)$ then follows from (30) (in view of (26) and (41)).

The character of the left module of chambers Γ is given by

$$\chi_{\Gamma}(w) = \sum_{\mathrm{X}} \chi_{\mathrm{X}}(w)\, |\mu(\mathcal{A}_{\mathrm{X}})|. \tag{46}$$

This follows from (42) and (45).

Recall that a linear functional on Σ is called a character of Σ if it is the character of some Σ-module h.

Multiplicative characters are those which arise from one-dimensional modules.

Proposition 8. *The characters of the Tits algebra correspond to families (η_{X}) of nonnegative integers indexed by flats, with the multiplicative ones corresponding to those families in which exactly one η_{X} is 1 and the rest are 0.*

Proof. This follows from Proposition 6. $\square$

5.8 Eigenvalues and multiplicities

Theorem 16 gives the eigenvalues and multiplicities of the action of any element of an elementary algebra on a module.

Applying it to the Tits algebra and using (36), we obtain:

Theorem 17. *Let* h *be a finite-dimensional (left or right) module over the Tits algebra* Σ*.*

Then all elements of Σ *are simultaneously triangularizable on* h*.*

For $w = \sum_F w^F \mathtt{H}_F$*, the linear operator* $\Psi_{\mathsf{h}}(w)$ *has an eigenvalue*

$$\lambda_{\mathrm{X}}(w) := \chi_{\mathrm{X}}(w) = \sum_{F:\, \mathrm{s}(F) \leq \mathrm{X}} w^F \tag{47}$$

for each $\mathrm{X} \in \Pi$*, with multiplicity* $\eta_{\mathrm{X}}(\mathsf{h})$ *given by* (40)*.*

5.9 Bidigare-Hanlon-Rockmore

By specializing Theorem 17 to the left module of chambers $\mathsf{h} = \Gamma$ and using formula (45), we obtain:

Theorem 18. *For* $w = \sum_F w^F \mathtt{H}_F$*, the linear operator* $\Psi_\Gamma(w)$ *has an eigenvalue* $\lambda_{\mathrm{X}}(w)$ *defined by* (47) *for each* $\mathrm{X} \in \Pi$*, with multiplicity* $|\mu(\mathcal{A}_{\mathrm{X}})|$*.*

This is the Bidigare-Hanlon-Rockmore theorem, or BHR for short.

Note very carefully that this result makes no claim about the diagonalizability of $\Psi_\Gamma(w)$.

Example. Let $\mathcal{A}$ be the rank-one arrangement with chambers C and $\overline{C}$. It has two flats, namely, $\perp$ and $\top$.

Let $w = \alpha\,\mathtt{H}_O + \beta\,\mathtt{H}_C + \gamma\,\mathtt{H}_{\overline{C}}$. By BHR, the eigenvalues of $\Psi_{\Gamma}(w)$ are

$$\lambda_{\perp}(w) = \alpha \quad \text{and} \quad \lambda_{\top}(w) = \alpha + \beta + \gamma,$$

and both have multiplicity one. This is consistent with the explicit calculations done earlier.

Let k denote the submodule of Γ spanned by $\mathtt{H}_C - \mathtt{H}_{\overline{C}}$. Then $0 \lessdot \mathsf{k} \lessdot \Gamma$ is a composition series of Γ. The eigenvalue $\lambda_{\perp}$ corresponds to the composition factor k, while $\lambda_{\top}$ corresponds to the composition factor Γ/k. The calculation for the latter goes as follows.

$$(\alpha\,\mathtt{H}_O+\beta\,\mathtt{H}_C+\gamma\,\mathtt{H}_{\overline{C}})\cdot\mathtt{H}_C = \alpha\,\mathtt{H}_C+\beta\,\mathtt{H}_C+\gamma\,\mathtt{H}_{\overline{C}} = (\alpha+\beta+\gamma)\,\mathtt{H}_C$$

since $\mathtt{H}_C$ and $\mathtt{H}_{\overline{C}}$ represent the same element of Γ/k.

6 Primitive part of a left module

For a left Σ-module h, the primitive part of h is the subspace defined by

$$\mathcal{P}(\mathsf{h}) = \bigcap_{F>O} \ker(\Psi_{\mathsf{h}}(\mathtt{H}_F) : \mathsf{h} \to \mathsf{h}).$$

In other words,

$$z \in \mathcal{P}(\mathsf{h}) \iff \mathtt{H}_F \cdot z = 0 \text{ for all } F > O.$$

7 Over and under a flat. Cartesian product

We briefly discuss how the Tits algebra behaves under passage to arrangements over and under a flat, and with respect to cartesian product of arrangements.

7.1 Over a flat

For faces F and G with the same support, there is an algebra isomorphism

(48) $\beta_{G,F} : \Sigma[\mathcal{A}_F] \to \Sigma[\mathcal{A}_G], \qquad \mathtt{H}_{K/F} \mapsto \mathtt{H}_{GK/G}.$

Its inverse is $\beta_{F,G}$.

Similarly, for any face with support X, there are canonical inverse algebra isomorphisms

(49)

$\beta_{\mathrm{X},F} : \Sigma[\mathcal{A}_F] \to \Sigma[\mathcal{A}_\mathrm{X}]$ and $\beta_{F,\mathrm{X}} : \Sigma[\mathcal{A}_\mathrm{X}] \to \Sigma[\mathcal{A}_F]$.

Identities such as

$$\beta_{\mathrm{X},F} = \beta_{\mathrm{X},G}\beta_{G,F} \quad \text{and} \quad \beta_{G,F} = \beta_{G,\mathrm{X}}\beta_{\mathrm{X},F}$$

always hold.

For any face H of $\mathcal{A}$, the map

$$\Delta_H : \Sigma[\mathcal{A}] \to \Sigma[\mathcal{A}_H], \qquad \mathtt{H}_F \mapsto \mathtt{H}_{HF/H} \tag{50}$$

is an algebra homomorphism.

For faces F and G with the same support, the diagram

$$\begin{tikzcd} & \Sigma[\mathcal{A}] \arrow[dl, "\Delta_F"'] \arrow[dr, "\Delta_G"] & \\ \Sigma[\mathcal{A}_F] \arrow[rr, "\beta_{G,F}"'] & & \Sigma[\mathcal{A}_G] \end{tikzcd} \tag{51}$$

commutes.

For faces $F \leq G$, the diagram

$$\begin{tikzcd} & \Sigma[\mathcal{A}] \arrow[dl, "\Delta_F"'] \arrow[dr, "\Delta_G"] & \\ \Sigma[\mathcal{A}_F] \arrow[rr, "\Delta_{G/F}"'] & & \Sigma[\mathcal{A}_G] \end{tikzcd} \tag{52}$$

commutes, where $\Delta_{G/F}(\mathtt{H}_{K/F}) = \mathtt{H}_{GK/G}$.

Let

$$\mu_F : \Sigma[\mathcal{A}_F] \to \Sigma[\mathcal{A}], \qquad \mathtt{H}_{K/F} \mapsto \mathtt{H}_K. \tag{53}$$

This is a section of the map (50), that is, $\Delta_F \mu_F = \mathrm{id}$. Composing in the other direction yields

$$\mu_F \Delta_F(x) = \mathtt{H}_F \cdot x. \tag{54}$$

Note that μ_F preserves products, that is, $\mu_F(x \cdot y) = \mu_F(x) \cdot \mu_F(y)$, but it does not preserve the identity, so it is not an algebra homomorphism.

7.2 Under a flat

For any flat X, the linear map

(55)

$$\Sigma[\mathcal{A}] \to \Sigma[\mathcal{A}^{\mathrm{X}}], \qquad \sum_F x^F \mathtt{H}_F \mapsto \sum_{F\colon \mathrm{s}(F) \leq \mathrm{X}} x^F \mathtt{H}_F$$

is an algebra homomorphism.

7.3 Cartesian product

For any arrangements $\mathcal{A}$ and $\mathcal{A}'$, there is an algebra isomorphism

(56)

$$\Sigma[\mathcal{A} \times \mathcal{A}'] \to \Sigma[\mathcal{A}] \otimes \Sigma[\mathcal{A}'], \qquad \mathtt{H}_{(F,F')} \mapsto \mathtt{H}_F \otimes \mathtt{H}_{F'}.$$

Similarly, there is an isomorphism

(57)

$$\Gamma[\mathcal{A} \times \mathcal{A}'] \to \Gamma[\mathcal{A}] \otimes \Gamma[\mathcal{A}'], \qquad \mathtt{H}_{(C,C')} \mapsto \mathtt{H}_C \otimes \mathtt{H}_{C'}.$$

8 The Wedderburn theorem for semisimple algebras

8.1 Semisimple algebras

Let A be a $\mathbb{k}$-algebra.

We say A is semisimple if $\mathrm{rad}(A) = 0$, that is, if 0 is the only nilpotent ideal in A.

The Wedderburn structure theorem says the following.

Theorem 19. *A $\mathbb{k}$-algebra is semisimple iff it is isomorphic to a product of matrix algebras over division $\mathbb{k}$-algebras.*

Recall that a division $\mathbb{k}$-algebra is a nonzero $\mathbb{k}$-algebra in which every nonzero element is invertible.

If the division $\mathbb{k}$-algebras involved are all $\mathbb{k}$, then we say that the semisimple algebra is split.

In other words, A is split-semisimple iff it is isomorphic to a product of matrix algebras over $\mathbb{k}$.

Corollary 1. *A semisimple algebra is commutative iff it is isomorphic to a product of fields which are finite extensions of* $\mathbb{k}$*.*

Similarly, a split-semisimple algebra is commutative iff it is isomorphic to a product of copies of $\mathbb{k}$*.*

The latter notion was elaborated in Section 2.

8.2 The Schur lemma

Lemma 6. *Let A be a $\mathbb{k}$-algebra. Let $f : M \to N$ be a nonzero map of left A-modules. Then:*

1. *If M is simple, then f is injective.*
2. *If N is simple, then f is surjective.*

Proof. Since f is nonzero, $\ker(f) \neq M$ and $\operatorname{im}(f) \neq 0$.

Hence, M simple implies $\ker(f) = 0$, and N simple implies $\operatorname{im}(f) = N$. $\square$

This is called the Schur lemma.

Corollary 2. *Let A be a $\mathbb{k}$-algebra, and M and N be simple left A-modules.*

Then either $M \cong N$ (as left A-modules) or $\operatorname{Hom}_A(M, N) = 0$.

8.3 Semisimple modules

Let M be a left A-module.

We say M is semisimple if any of the following equivalent conditions hold.

- Every submodule of M is a direct summand of M (that is, has a complementary submodule).
- M is the direct sum of a family of simple modules.
- M is the sum of a family of simple modules.

8.4 Radical of a module

For a left A-module M, the radical of M is the intersection of all maximal submodules of M. We denote it by $\mathrm{rad}(M)$.

It is also given by

(58) $$\mathrm{rad}(M) = \mathrm{rad}(A)M.$$

Also,

(59) $$\mathrm{rad}(M) = 0 \iff M \text{ is semisimple.}$$

We omit the proofs.

Observe that A is semisimple iff A is a semisimple as a left module over itself.

8.5 Sketch of proof

We give a sketch of the forward implication of Theorem 19.

Suppose A is a semisimple algebra.

View A as a left module over itself.

Write

(60)

$$A = (M_{11} \oplus \cdots \oplus M_{1n_1}) \oplus (M_{21} \oplus \cdots \oplus M_{2n_2}) \\ \oplus \cdots \oplus (M_{m1} \oplus \cdots \oplus M_{mn_m}),$$

where each M_{ij} is a simple left A-module, and they have been grouped together according to their isomorphism class.

The equality in (60) is as objects in the category of left A-modules.

Now loop on both sides to get an equality of $\mathbb{k}$-algebras.

The loop object on the lhs is A^{op}, the opposite algebra of A.

The loop object on the rhs is a product of m matrix algebras over division $\mathbb{k}$-algebras of size $n_1, \ldots, n_m$. This can be deduced from the Schur lemma.

This completes the argument.

8.6 Examples

- $\mathbb{k}^n$. This is split-semisimple. There are n matrix algebras over $\mathbb{k}$ each of size 1.
- Algebra of square matrices. This is split-semisimple. There is one matrix algebra.
- $\mathbb{C}$ as a two-dimensional algebra over $\mathbb{R}$. This is semisimple but not split. There is one matrix algebra of size 1 over the division algebra $\mathbb{C}$.

In general, for any algebra A, $A/\operatorname{rad}(A)$ is semisimple.

As a first step towards understanding A, we try to understand where $A/\operatorname{rad}(A)$ fits in Theorem 19.

Subsequent steps involve lifting idempotents from $A/\operatorname{rad}(A)$ to A, understanding complete systems of A, etc.

We did some of this for the Tits algebra (which we recall is an elementary algebra).

8.7 Group algebras

Let $\mathbb{k}$ be any field.

Theorem 20. *Let G be a finite group. The group algebra $\mathbb{k}G$ is semisimple iff the characteristic of $\mathbb{k}$ does not divide the order of G.*

This is called the Maschke theorem.

Proof. We explain only the backward implication. Suppose the characteristic of $\Bbbk$ does not divide the order of G.

Let M be any submodule of $\Bbbk G$. We need to produce a complementary submodule N, that is $M \oplus N = \Bbbk G$.

For this, we pick any idempotent linear operator p on $\Bbbk G$ whose image is M. Define another operator e on $\Bbbk G$ by

$$e(x) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot p(g \cdot x)$$

for $x \in \Bbbk G$. We think of e as the average of p over G.

One can check that e is an idempotent operator on $\Bbbk G$ whose image is M, and moreover, it is a map of $\Bbbk G$-modules.

Now put $N = \ker(e)$. $\square$

9 Exercises

1. Show that the sum of two multiplicative characters of an algebra A may not be multiplicative.
2. What is the character of the algebra $\mathbb{k}^n$ viewed as a module over itself? Is it multiplicative?
3. Show that there are no nonzero nilpotent ideals in the algebra of square matrices of size n (for n fixed). Deduce that the radical of this algebra is the zero ideal.
4. What is the radical of the algebra $\mathbb{k}[x]/(x^n)$ (for n fixed)?
5. Show that for the algebra of square matrices of size n, the left module of column vectors is a simple module of dimension n.

10 Problems

1. Show that all arrangements of 3 lines in the plane (passing through the origin) are gisomorphic.

2. For any face F, describe the left ideal generated by F in the Tits monoid $\Sigma[\mathcal{A}]$. Show that it is two-sided, and in particular, contains the star of F. Say explicitly what happens when F is the central face and when F is a chamber.

3. Prove or disprove. For any faces F, G and H,
$G \leq H \implies GF \leq HF$.

4. Compute all idempotents in the Tits algebra of the rank-one arrangement.

5. Show that the algebra of upper triangular n by n matrices for $n \geq 3$ cannot be isomorphic to the Tits algebra of any arrangement.

11 Problems

1. For each flat X, let $\xi_{\mathrm{X}} = c_{\mathrm{X}}^2$, where c_{X} is the number of chambers in $\mathcal{A}_{\mathrm{X}}$. Define η_{X} via (26). Are the η_{X} nonnegative?

2. An element z of the Tits algebra is a special Zie element iff z is an idempotent and $\mathrm{s}(z) = \mathtt{Q}_{\perp}$. Verify this statement directly for the rank-one arrangement.

3. List all special Zie families of the rank-one arrangement. Compute the corresponding Eulerian families and check that we indeed get all of them.

4. Let $\mathtt{u}$ be a homogeneous section with associated Eulerian family $\mathtt{E}$ and $\mathtt{Q}$-basis. Fix a specific $\mathtt{Q}$-basis element, say $\mathtt{Q}_F$. Give an example of a homogeneous section $\mathtt{u}'$ whose associated Eulerian family $\mathtt{E}'$ satisfies $\mathtt{Q}_F = \mathtt{E}'_{\mathrm{s}(F)}$.

5. Check that for any faces F and G,

$$\Delta_G \mu_F = \mu_{GF/G} \beta_{GF,FG} \Delta_{FG/F}.$$

   We call this the bimonoid axiom for faces. It links the Tits algebras of $\mathcal{A}$, $\mathcal{A}_F$, $\mathcal{A}_G$, $\mathcal{A}_{FG}$ and $\mathcal{A}_{FG}$.

12 Reading assignment

Read at least one/two sections from any part of the notes b.pdf (including the appendices), and give a writeup on it.

Your writeup could include

- a brief summary of what you understood,
- a list of things you did not understand properly,
- overall suggestions for improving the exposition,
- additional questions/insights that you have,
- thoughts on the exercises listed,
- suggestions to improve some picture or draw more pictures,
- pointing out typos,

and so on.