Newman-Radford Rigidity and Noncommutative Möbius Theory

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Bimonoids for Hyperplane Arrangements
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1 Lune-incidence Algebra

1.1 Nested faces and lunes

A nested face is a pair of faces (A, F) with $F \geq A$.

Let A and B be faces having the same support. Recall that we have an isomorphism of posets:

$$\Sigma\left[\mathcal{A}\right]_A \xrightarrow{\cong} \Sigma\left[\mathcal{A}\right]_B \quad ; \quad F/A \mapsto BF/B$$

We define an equivalence relation $\sim$ on nested faces:

$$(A, F) \sim (B, G) \Leftrightarrow \mathrm{s}\,(A) = \mathrm{s}\,(B), G = BF, F = AG$$

The equivalence classes are called lunes.

1.2 Face-incidence algebra

As a set, $\mathrm{I}_{\text{face}}[\mathcal{A}]$ consists of $\mathbb{k}$-valued functions on nested faces:

$$f : \left\{ (A, F) \in \Sigma [\mathcal{A}]^2 \,|\, F \geq A \right\} \to \mathbb{k}$$

It forms a vector space under pointwise addition and scalar multiplication. Furthermore, for any $f, g \in \mathrm{I}_{\text{face}}[\mathcal{A}]$, we define $f \cdot g \in \mathrm{I}_{\text{face}}[\mathcal{A}]$ by:

$$(f \cdot g)(A, F) = \sum_{H : A \leq H \leq F} f(A, H)\, g(H, F)$$

Under this product, $\mathrm{I}_{\text{face}}[\mathcal{A}]$ forms an algebra with the multiplicative identity $\delta \in \mathrm{I}_{\text{face}}[\mathcal{A}]$ given by:

$$\delta(A, F) = \begin{cases} 1 & \text{if } A = F \\ 0 & \text{otherwise} \end{cases}$$

1.3 Lune-incidence algebra

Let $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ be the vector subspace of $\mathrm{I}_{\mathrm{face}}[\mathcal{A}]$ consisting of functions f such that $f(A, F) = f(B, G)$ whenever $(A, F) \sim (B, G)$.

Proposition. $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ *is a subalgebra of* $\mathrm{I}_{\mathrm{face}}[\mathcal{A}]$*. That is, it is closed under multiplication.*

We omit the proof.

We call $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ the lune-incidence algebra of $\mathcal{A}$.

Unlike $\mathrm{I}_{\mathrm{flat}}[\mathcal{A}]$ and $\mathrm{I}_{\mathrm{face}}[\mathcal{A}]$, $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ is not an incidence algebra of a poset.

1.4 Noncommutative zeta functions

A noncommutative zeta function is an element $\zeta \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ such that $\zeta(A, A) = 1$ for all faces A and

(1) $$\sum_{\substack{F : F \geq A, HF = G, \\ \mathrm{s}(F) = \mathrm{s}(G)}} \zeta(A, F) = \zeta(H, G)$$

for all $A \leq H \leq G$. We refer to (1) as the star-lune formula.

Setting $H = G$ and $X = \mathrm{s}(G)$ in (1), we get the following:

(2) $$\sum_{F : F \geq A, \mathrm{s}(F) = X} \zeta(A, F) = 1$$

for all $\mathrm{s}(A) \leq X$. We refer to (2) as the star-flat formula.

Lemma. *A noncommutative zeta function ζ is equivalent to a choice of scalars $\zeta\,(O,F)$, one for each face F, such that for each flat X,*

$$\sum_{F:s(F)=X} \zeta\,(O,F) = 1 \tag{3}$$

In particular, noncommutative zeta functions exist.

Proof sketch. If ζ is a noncommutative zeta function, then (3) is simply the star-flat formula with $A = O$.

Conversely, given scalars $\zeta\,(O,F)$ satisfying (3), we can recover $\zeta\,(A,F)$ for any $O \le A \le F$ using the star-lune formula:

$$\zeta\,(A,F) := \sum_{\substack{H:AH=F,\\ \mathrm{s}\,(H)=\mathrm{s}\,(F)}} \zeta\,(O,H)$$

$\square$

1.5 Noncommutative Möbius functions

A noncommutative Möbius function is an element $\boldsymbol{\mu} \in \mathrm{I}_{\text{lune}}[\mathcal{A}]$ such that $\boldsymbol{\mu}(A, A) = 1$ for all faces A and

(4) $$\sum_{F : F \geq A, HF = H} \boldsymbol{\mu}(A, F) = 0$$

for all $A < H \leq G$. We refer to (4) as the noncommutative Weisner formula.

Claim. *The inverse of a noncommutative zeta function* $\boldsymbol{\zeta}$ *in* $\mathrm{I}_{\text{lune}}[\mathcal{A}]$ *is a noncommutative Möbius function. In particular, noncommutative Möbius functions exist.*

We will prove this claim towards the end of this talk.

2 Newman-Radford for free bimonoid on a cocommutative comonoid

Let (c, Δ) be a comonoid.

We first recall the definition of $\mathcal{T}(\mathsf{c})$. As a species, it is given by:

$$\mathcal{T}(\mathsf{c})[A] = \bigoplus_{F : F \geq A} \mathsf{c}[F]$$

It is equipped with the concatenation product:

$$\begin{tikzcd}
\mathcal{T}(\mathsf{c})[H] \arrow[r, dotted, "\mu_A^H"] & \mathcal{T}(\mathsf{c})[A] \arrow[d, two heads] \\
\mathsf{c}[F] \arrow[u, hook] \arrow[r, "\mathrm{id}"] & \mathsf{c}[F]
\end{tikzcd}$$

and the dequasishuffle coproduct:

$$\begin{tikzcd}
\mathcal{T}(\mathsf{c})[A] \arrow[rr, dotted, "\Delta_A^H"] & & \mathcal{T}(\mathsf{c})[H] \arrow[d, two heads] \\
\mathsf{c}[F] \arrow[u, hook] \arrow[r, "\Delta_F^{FH}"] & \mathsf{c}[FH] \arrow[r, "\beta_{HF,FH}"] & \mathsf{c}[HF]
\end{tikzcd}$$

We now consider the bimonoid $\mathcal{T}(\mathsf{c}_t)$. As a species, it is given by:

$$\mathcal{T}(\mathsf{c}_t)[A] = \bigoplus_{F : F \geq A} \mathsf{c}[F]$$

It is equipped with the concatenation product:

$$\begin{tikzcd}
\mathcal{T}(\mathsf{c}_t)[H] \arrow[r, dotted, "\mu_A^H"] & \mathcal{T}(\mathsf{c}_t)[A] \arrow[d, two heads] \\
\mathsf{c}[F] \arrow[u, hook] \arrow[r, "\mathrm{id}"] & \mathsf{c}[F]
\end{tikzcd}$$

and the deshuffle coproduct:

$$\begin{tikzcd}
\mathcal{T}(\mathsf{c}_t)[A] \arrow[r, dotted, "\Delta_A^H"] & \mathcal{T}(\mathsf{c}_t)[H] \arrow[d, two heads] \\
\mathsf{c}[F] \arrow[u, hook] \arrow[r, "\beta_{HF,F}"] & \begin{cases} \mathsf{c}[HF] & \text{if } F = FH \\ 0 & \text{otherwise} \end{cases}
\end{tikzcd}$$

Observe that $\mathcal{T}(\mathsf{c})$ and $\mathcal{T}(\mathsf{c}_t)$ are identical as monoids. On the face of it, only their coproduct differs.

Question. Is the coproduct on $\mathcal{T}(\mathsf{c})$ genuinely different from the coproduct on $\mathcal{T}(\mathsf{c}_t)$?

The dequasishuffle coproduct on $\mathcal{T}(\mathsf{c})$ is cocommutative iff c is cocommutative. In contrast, $\mathcal{T}(\mathsf{c}_t)$ is always cocommutative.

Thus, if c is not cocommutative to begin with, then $\mathcal{T}(\mathsf{c})$ and $\mathcal{T}(\mathsf{c}_t)$ cannot possibly be isomorphic as bimonoids.

What happens when c is cocommutative?

What would an isomorphism $\mathcal{T}(\mathsf{c}) \to \mathcal{T}(\mathsf{c}_t)$ look like?

Proposition. *Fix a noncommutative zeta function* $\boldsymbol{\zeta} \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$*. For a cocommutative comonoid* c*, the map* $f : \mathcal{T}(\mathsf{c}) \to \mathcal{T}(\mathsf{c}_t)$ *given on the* A*-component by*

$$
\begin{tikzcd}
\mathcal{T}(\mathsf{c})[A] \arrow[r, dotted, "f_A"] & \mathcal{T}(\mathsf{c}_t)[A] \arrow[d, two heads] \\
\mathsf{c}[F] \arrow[u, hook] \arrow[r, "\boldsymbol{\zeta}(F,G)\Delta_F^G"] & \begin{cases} \mathsf{c}[G] & \text{if } G \geq F \\ 0 & \text{otherwise} \end{cases}
\end{tikzcd}
\tag{5}
$$

is an isomorphism of bimonoids.

Proof. Note that f is a well-defined morphism of species.

Since the matrix form of f is unitriangular, it is an isomorphism of species.

It remains to check that f is a morphism of bimonoids.

By the universal property of $\mathcal{T} : \mathsf{Comon}(\mathcal{A}\text{-}\mathtt{Sp}) \to \mathsf{Bimon}(\mathcal{A}\text{-}\mathtt{Sp})$, it suffices to check that $f : \mathsf{c} \to \mathcal{T}(\mathsf{c}_t)$ is a morphism of comonoids.

Thus, we need to verify that the following diagram commutes for all $H \geq A$:

$$\begin{tikzcd}
\mathsf{c}[H] \arrow[r, "f_H"] & \mathcal{T}(\mathsf{c}_t)[H] \\
\mathsf{c}[A] \arrow[u, "\Delta_A^H"] \arrow[r, "f_A"'] & \mathcal{T}(\mathsf{c}_t)[A] \arrow[u, "\Delta_A^H"']
\end{tikzcd}$$

$$
\begin{aligned}
\Delta_A^H \circ f_A &= \left(\bigoplus_{\substack{F:F\geq A,\\ F=\overline{F}H}} \beta_{HF,F} \right) \circ \left(\bigoplus_{K:K\geq A} \boldsymbol{\zeta}(A,K)\,\Delta_A^K \right) \\
&= \bigoplus_{\substack{F:F\geq A,\\ F=\overline{F}H}} \boldsymbol{\zeta}(A,F)\,\beta_{HF,F}\Delta_A^H \\
&= \bigoplus_{\substack{F:F\geq A,\\ F=\overline{F}H}} \boldsymbol{\zeta}(A,F)\,\Delta_A^{HF} \\
&= \bigoplus_{G:G\geq F} \left(\sum_{\substack{F:F\geq A, HF=G,\\ \overline{F}=FH}} \boldsymbol{\zeta}(A,F) \right) \Delta_A^G \\
&= \bigoplus_{G:G\geq H} \boldsymbol{\zeta}(H,G)\,\Delta_A^G \\
&= \left(\bigoplus_{G:G\geq H} \boldsymbol{\zeta}(H,G)\,\Delta_H^G \right) \circ \Delta_A^H \\
&= f_H \circ \Delta_A^H
\end{aligned}
$$

where we have used (1) in fifth step, with the $HF = G, F = FH$ replacing the conditions $HF = G, \mathrm{s}(F) = \mathrm{s}(G)$. □

Example. Taking c to be the exponential comonoid E, the above proposition gives us an isomorphim $\mathcal{T}(\mathsf{E}) \xrightarrow{\cong} \mathcal{T}(\mathsf{E}_t)$ that maps the $\mathtt{H}$-basis of Σ to a $\mathtt{Q}$-basis of Σ.

Remark. We have one such isomorphism for every choice of a noncommutative zeta function $\boldsymbol{\zeta} \in \mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$.

3 Zeta and Möbius as inverses

We now prove the claim we made earlier, viz. the inverse of a noncommutative zeta function $\boldsymbol{\zeta}$ in the lune-incidence algebra is a noncommutative Möbius function.

Note that a noncommutative zeta function $\boldsymbol{\zeta} \in \mathrm{I}_{\mathrm{lune}}\left[\mathcal{A}\right]$ is always invertible, since $\boldsymbol{\zeta}\left(A, A\right) = 1$ for all faces A.

In fact, if $\boldsymbol{\mu}$ denotes its inverse in the lune-incidence algebra, then we also have $\boldsymbol{\mu}\left(A, A\right) = 1$ for all faces A.

It remains to show that the inverse of $\boldsymbol{\zeta}$ satisfies (4).

Let $\boldsymbol{\zeta}$ be a noncommutative zeta function, and let $\boldsymbol{\mu}$ denote its inverse in the lune-incidence algebra.

Consider the isomorphism of bimonoids
$f : \mathcal{T}(\mathsf{c}) \xrightarrow{\cong} \mathcal{T}(\mathsf{c}_t)$, as given in (5).

The inverse map $g : \mathcal{T}(\mathsf{c}_t) \xrightarrow{\cong} \mathcal{T}(\mathsf{c})$ takes the form

(6)
$$\begin{tikzcd}
\mathcal{T}(\mathsf{c}_t)[A] \arrow[rr, dotted, "g_A"] & & \mathcal{T}(\mathsf{c})[A] \arrow[d, two heads] \\
\mathsf{c}[F] \arrow[u, hook] \arrow[rr, "\boldsymbol{\mu}(F,G)\Delta_F^G"] & & \begin{cases} \mathsf{c}[G] & \text{if } G \geq F \\ 0 & \text{otherwise} \end{cases}
\end{tikzcd}$$

By the universal property of $\mathcal{T} : \mathcal{A}\text{-}\mathsf{Sp} \to \mathsf{Bimon}(\mathcal{A}\text{-}\mathsf{Sp})$, g must restrict to a morphism of species
$g : \mathsf{c}_t \to \mathcal{P}(\mathcal{T}(\mathsf{c}))$.

That is, $\Delta_A^H \circ g_A = 0$ for all $H > A$.

$$
\begin{aligned}
&\Delta_A^H \circ g_A = 0 \\
\Rightarrow &\left(\bigoplus_{F:F\geq A} \beta_{HF,FH} \Delta_F^{FH} \right) \circ \left(\bigoplus_{K:K\geq A} \boldsymbol{\mu}\,(A,K)\,\Delta_A^K \right) = 0 \\
\Rightarrow &\bigoplus_{F:F\geq A} \boldsymbol{\mu}\,(A,F)\,\beta_{HF,FH}\Delta_F^{FH}\Delta_A^F = 0 \\
\Rightarrow &\bigoplus_{F:F\geq A} \boldsymbol{\mu}\,(A,F)\,\Delta_A^{HF} = 0 \\
\Rightarrow &\bigoplus_{G:G\geq A} \left(\sum_{F:F\geq A, HF=G} \boldsymbol{\mu}\,(A,F) \right) \Delta_A^G = 0
\end{aligned}
$$

Since this is true for any cocommutative comonoid c, we get the following:

$$
\sum_{F:F\geq A, HF=G} \boldsymbol{\mu}\,(A,F) = 0
$$

which is precisely the noncommutative Weisner formula (4).

Thus, we have proved the first half of the folllowing theorem:

Theorem. *In the lune-incidence algebra, the inverse of a noncommutative zeta function is a noncommutative Möbius function, and vice-versa*

Proof of the second half can be obtained by simply reversing all the implications.