Braid arrangement and related examples

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1 Coordinate arrangement

1.1 Coordinate arrangement

The coordinate arrangement of rank n consists of the n hyperplanes

$$x_i = 0$$

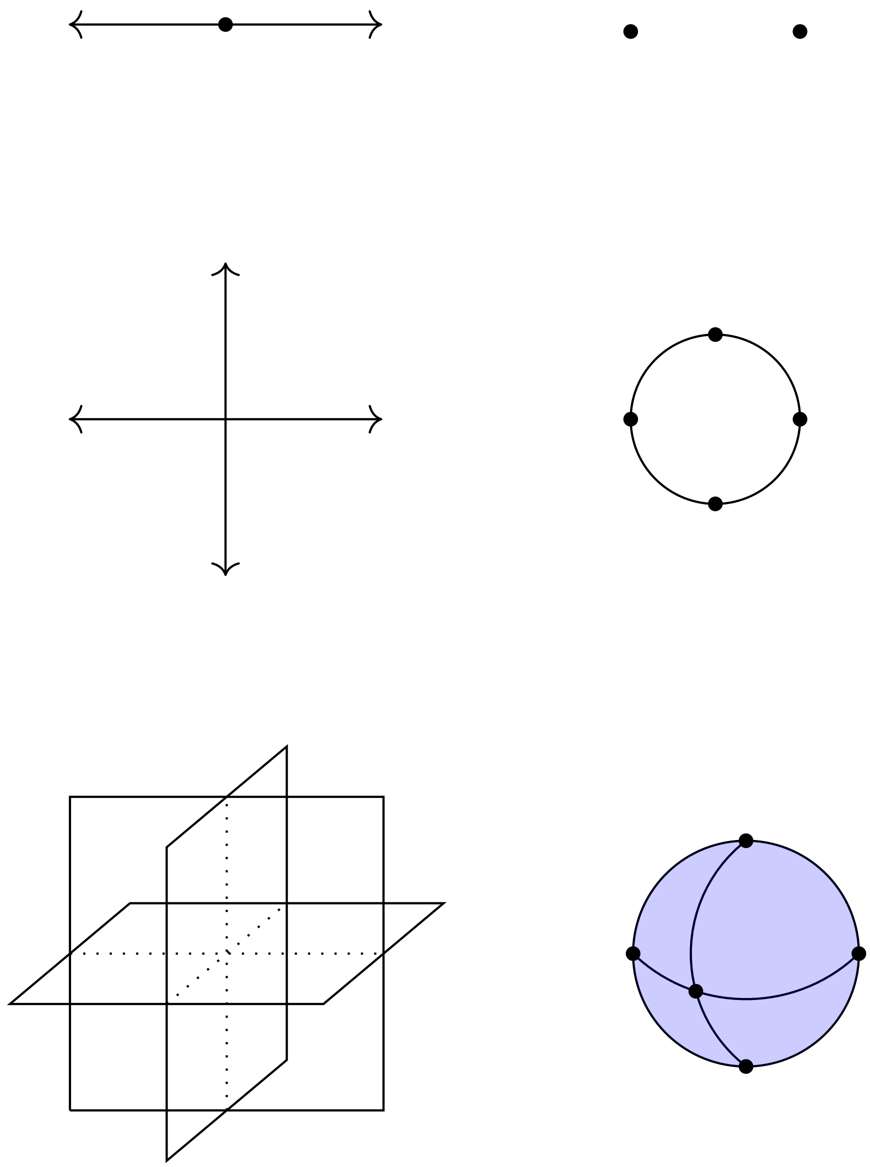
for $1 \leq i \leq n$.

It is the smallest arrangement of rank n in terms of number of hyperplanes.

It is the n-fold cartesian product of the arrangement of rank 1.

1.2 Small ranks

The linear and spherical models for $n = 1, 2, 3$ are shown below.



1.3 Faces and flats

Faces of $\mathcal{A}$ can be described as n-tuples in which each entry is either 0, $+$ or $-$. The Tits product on faces is given by

$$\epsilon_i(FG) := \begin{cases} \epsilon_i(F) & \text{if } \epsilon_i(F) \neq 0, \\ \epsilon_i(G) & \text{if } \epsilon_i(F) = 0, \end{cases} \tag{1}$$

where $\epsilon_i(F)$ denotes the i-th entry in the tuple representating F. (This is the same formula we had before.)

We have $F \lessdot G$ iff G is obtained from F by replacing exactly one 0 by either $+$ or $-$.

Chambers are n-tuples in which each entry is either $+$ or $-$.

For any flat, there is a unique set of hyperplanes whose intersection is that flat. Thus, flats can be identified with subsets of $[n]$.

The poset structure is given by reverse inclusion.

The support map sends a face to the subset consisting of those positions in its n-tuple which have a 0 entry.

The lattice of flats is a Boolean poset.

1.4 Arrangements under and over a flat. Cartesian product

Recall that a flat X of $\mathcal{A}$ is a subset of $[n]$.

The arrangement $\mathcal{A}^{\mathrm{X}}$ is the coordinate arrangement whose coordinates belong to X.

The arrangement $\mathcal{A}_{\mathrm{X}}$ is cisomorphic to the coordinate arrangement whose coordinates do not belong to X.

Similarly, the cartesian product of two coordinate arrangements is again a coordinate arrangement obtained by taking disjoint union of the two sets of coordinates.

To summarize: The family of all coordinate arrangements, as n varies, is closed under passage to arrangements under and over a flat, and under cartesian products.

2 Braid arrangement

2.1 Set compositions and set partitions

Let I be a finite set.

A composition of I is a finite sequence $(I_1, \dots, I_k)$ of disjoint nonempty subsets of I such that

$$I = \bigsqcup_{i=1}^{k} I_i.$$

The subsets I_i are the blocks of the composition.

We write $F \vDash I$ to indicate that $F = (I_1, \dots, I_k)$ is a composition of I.

When the blocks are singletons, a composition of I amounts to a linear order on I.

Let F and G be compositions of I.

We say G refines F if each block of F is a union of some contiguous set of blocks of G.

In this case, we write $F \leq G$. This defines a partial order on the set of compositions of I.

Maximal elements are linear orders. There is a unique minimum element given by the one-block composition of I.

A partition X of I is a collection X of disjoint nonempty subsets of I such that

$$I = \bigsqcup_{B \in \mathrm{X}} B.$$

The subsets B are the blocks of the partition.

We write $\mathrm{X} \vdash I$ to indicate that X is a partition of I.

Let X and Y be partitions of I.

We say that Y refines X if each block of X is a union of blocks of Y.

In this case, we write $\mathrm{X} \leq \mathrm{Y}$. This defines a partial order on the set of partitions of I which is in fact a lattice.

The top element is the partition into singletons and the bottom element is the partition whose only block is the whole set I.

2.2 Braid arrangement

The braid arrangement on n letters consists of the $\binom{n}{2}$ hyperplanes in $\mathbb{R}^n$ defined by

$$x_i = x_j$$

for $1 \leq i < j \leq n$.

This is also called the arrangement of type A_{n-1}.

It has rank $n - 1$.

It is not essential: The central face is one-dimensional and given by $x_1 = \cdots = x_n$.

The canonical linear order of the set $[n]$ is not relevant to the definition of the arrangement. So it is also useful to proceed as follows.

Let I be a finite set. The braid arrangement on I consists of the hyperplanes

$$x_a = x_b$$

in $\mathbb{R}^I$, as a and b vary over elements of I with $a \neq b$.

2.3 Faces and flats

Faces correspond to set compositions, and flats to set partitions.

A face is defined by a system of equalities and inequalities which may be encoded by a composition of I:

the equalities are used to define the blocks and the inequalities to order them.

For example, for $I = \{a, b, c, d, e\}$,

$$x_a = x_c \leq x_b = x_d \leq x_e \qquad \longleftrightarrow \qquad ac|bd|e.$$

(The blocks have been separated by vertical bars and ordered from left to right. There is no order within each block.)

Thus, faces correspond to compositions of the set I.

Under the above correspondence, chambers correspond to linear orders on I.

For example, for $I = \{a, b, c, d, e\}$,

$$x_a \leq x_c \leq x_b \leq x_d \leq x_e \qquad \longleftrightarrow \qquad a|c|b|d|e.$$

A flat is defined by a system of equalities which may be encoded by a partition of I:

the equalities are used to define the blocks.

For example, for $I = \{a, b, c, d, e\}$,

$$x_a = x_c,\ x_b = x_d,\ x_e \qquad \longleftrightarrow \qquad \{ac, bd, e\}.$$

(The blocks have been separated by commas. There is no order within each block or among the blocks.)

Thus, flats correspond to partitions of the set I.

2.4 Support map

The support map from faces to flats translates as follows.

The support of a composition F of I is the partition $\mathrm{s}(F)$ of I obtained by forgetting the order among the blocks:

if $F = (I_1, \ldots, I_k)$, then

$$\mathrm{s}(F) = \{I_1, \ldots, I_k\}.$$

2.5 Small ranks

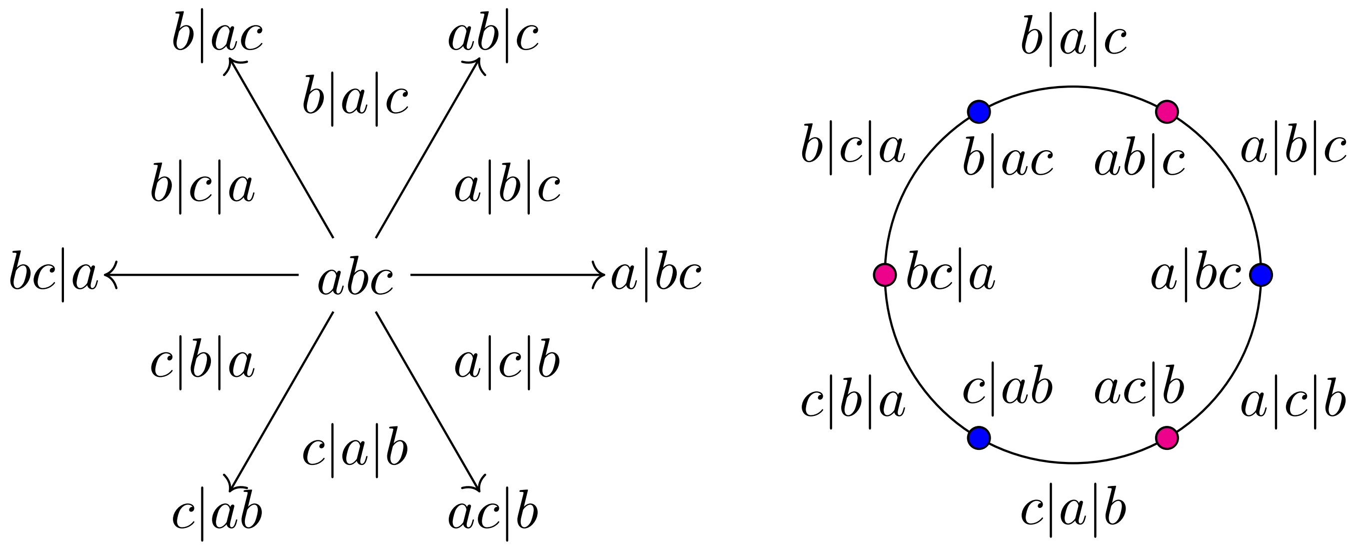
The braid arrangement on $I = \{a\}$ is the rank-zero arrangement containing no hyperplanes.

The braid arrangement on $I = \{a, b\}$ consists of one hyperplane $x_a = x_b$. It is cisomorphic to the rank-one arrangement whose ambient space is one-dimensional. The latter is shown below on the left with the spherical model on the right.

$$a|b \longleftarrow ab \longrightarrow b|a \qquad\qquad \underset{\bullet}{a|b} \quad \underset{\bullet}{b|a}$$

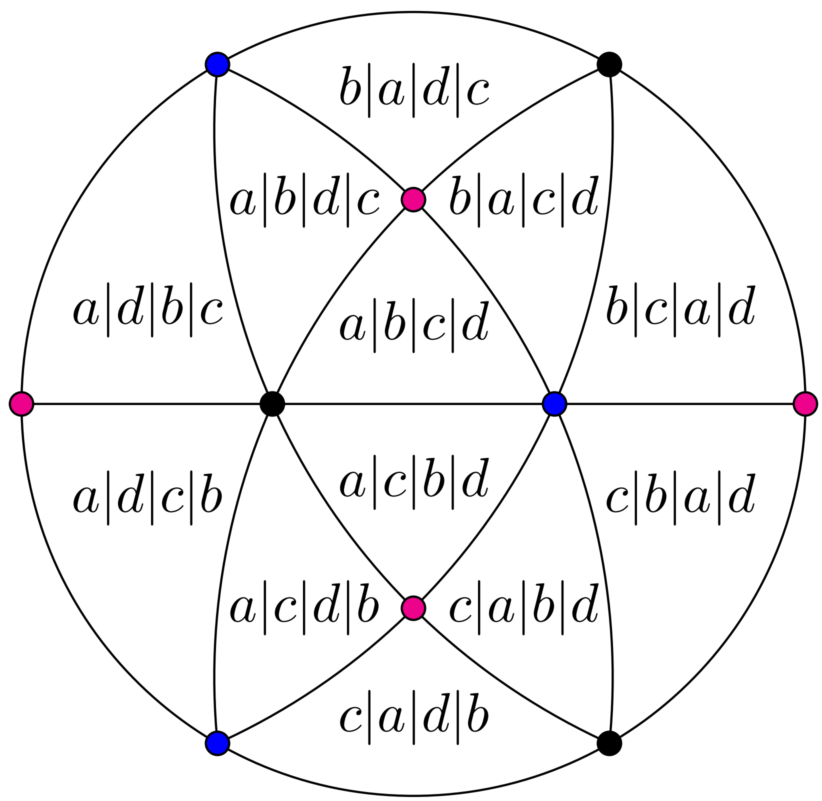
The central face corresponds to the one-block composition ab. It is not seen in the spherical model.

The braid arrangement on $I = \{a, b, c\}$ consists of the three hyperplanes $x_a = x_b$, $x_b = x_c$ and $x_a = x_c$. It is cisomorphic to the rank-two arrangement of three lines. The latter is shown below on the left with the spherical model on the right.

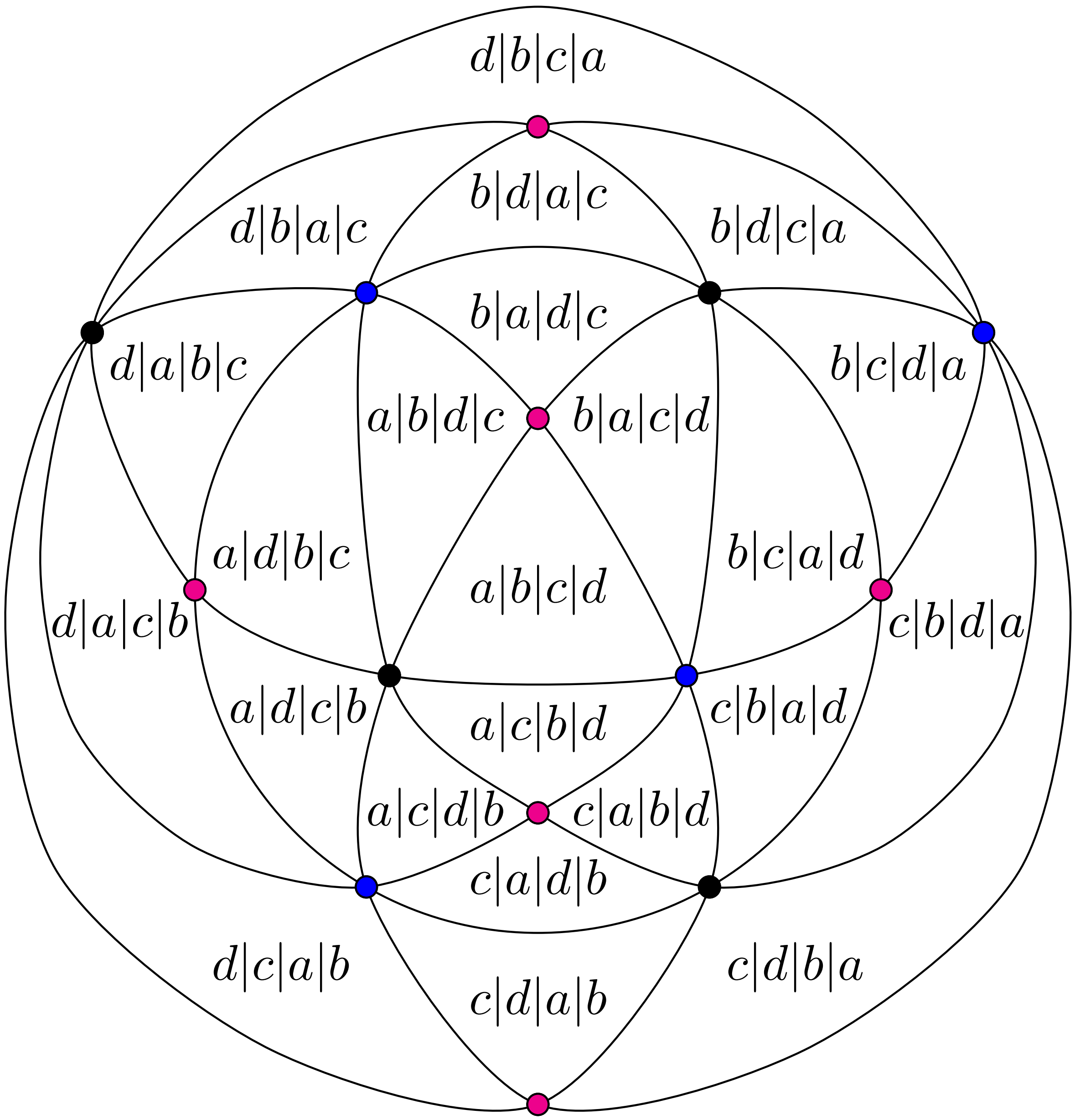


The faces are labeled by compositions of I. The central face which is not seen in the picture corresponds to the one-block composition abc. There are two types of vertices shown in blue and magenta, respectively.

The braid arrangement on $I = \{a, b, c, d\}$ consists of six hyperplanes. Its spherical model is shown below. The hyperplane $x_a = x_d$ is the outer circle, while $x_b = x_c$ is the horizontal line.



There are 24 triangles labeled by linear orders of which 12 are visible in the picture. The edges can be labeled by three-block compositions, and vertices by two-block compositions. There are three types of vertices shown in blue, magenta and black, respectively.



Here the spherical model has been flattened so that all triangles except $d|c|b|a$ are visible. The six hyperplanes can be seen in full as the six ovals.

2.6 Tits product

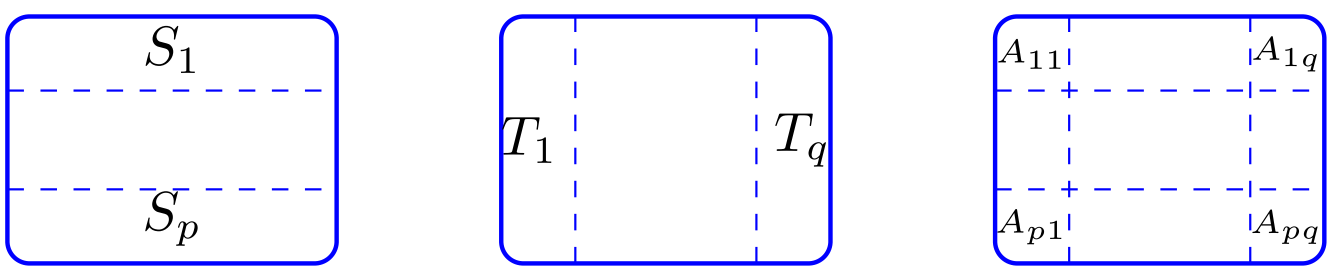
Let $F = (S_1, \ldots, S_p)$ and $G = (T_1, \ldots, T_q)$ be two compositions of I.

Consider the pairwise intersections

$$A_{ij} := S_i \cap T_j$$

for $1 \leq i \leq p$, $1 \leq j \leq q$.

A schematic picture is shown below.



The Tits product FG is the composition obtained by listing the nonempty intersections A_{ij} in lexicographic order of the indices (i, j):

(2) $$FG = (A_{11}, \ldots, A_{1q}, \ldots, A_{p1}, \ldots, A_{pq})\widehat{\ },$$

where the hat indicates that empty intersections are removed.

For example, to multiply $acde|bfg$ and $cdfg|b|ae$, we first compute the pairwise intersections.

$$\begin{bmatrix} acde \\ bfg \end{bmatrix} \qquad \begin{bmatrix} cdfg & b & ae \end{bmatrix} \qquad \begin{bmatrix} cd & \emptyset & ae \\ fg & b & \emptyset \end{bmatrix}$$

Now, we read the nonempty entries in the first row followed by those in the second to obtain:

$$(acde|bfg)(cdfg|b|ae) = (cd|ae|fg|b).$$

There is a similar operation on set partitions. To multiply X and Y, intersect the blocks of X with the blocks of Y and remove empty intersections. This operation is commutative, and in fact agrees with the join $\mathrm{X} \vee \mathrm{Y}$, which is the smallest common refinement of X and Y.

2.7 Degeneracies in the Tits product of two vertices

Let us look at the Tits product of two vertices in detail. A vertex is a set composition with two blocks. Suppose $F = (S, T)$ and $G = (S', T')$ are vertices. Put

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} := \begin{bmatrix} S \cap S' & S \cap T' \\ T \cap S' & T \cap T' \end{bmatrix}.$$

(Note that $FG = (A, B, C, D)^\frown$ and $GF = (A, C, B, D)^\frown$.)

Since S, T, S' and T' are nonempty, both entries in a row or column cannot be empty.

The remaining possibilities are listed below.

Combinatorics	Geometry
All entries are nonempty	FG and GF are triangles
One diagonal entry is empty and the rest are nonempty	FG and GF are distinct edges
One off-diagonal entry is empty and the rest are nonempty	$FG = GF$ is an edge
Diagonal entries are empty and the rest are nonempty	$F = \overline{G}$
Off-diagonal entries are empty and the rest are nonempty	$F = G$

The first row is the generic case.

The rest are degenerate cases of the generic case.

Observe how the combinatorial and geometric degeneracies go hand-in-hand.

2.8 Arrangements under and over a flat

Let $\mathcal{A}$ be any braid arrangement.

The arrangement under a flat X of the braid arrangement $\mathcal{A}$ is again cisomorphic to a braid arrangement.

More precisely, each block of X plays the role of one letter.

For example, for $\mathrm{X} = \{cdf, ae, bg\}$, the arrangement $\mathcal{A}^{\mathrm{X}}$ is cisomorphic to the braid arrangement on the three letters cdf, ae and bg.

The arrangement over a flat X is cisomorphic to a cartesian product of braid arrangements.

There is one braid arrangement for each block of X whose letters are the letters of that block.

For example, for $\mathrm{X} = \{cdf, ae, bg\}$, the arrangement $\mathcal{A}_{\mathrm{X}}$ is cisomorphic to the cartesian product of the three braid arrangements on $\{c, d, f\}$, $\{a, e\}$ and $\{b, g\}$, respectively.