Bimonads on Species

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Outline

Plan

The main aim is to provide an instance of a categorical framework for monoids, comonoids and bimonoids.

Instead of the usual monoidal category, the idea of monads and monad algebras is used.

Bimonoids as bialgebras over a bimonad

Proposition

The following are equivalences of categories.

$$\mathcal{T}\text{-}\mathit{algebras} \;\cong\; \mathcal{A}\text{-}\mathit{monoids}$$
$$\mathcal{T}^{\vee}\text{-}\mathit{coalgebras} \;\cong\; \mathcal{A}\text{-}\mathit{comonoids}$$
$$(\mathcal{T},\mathcal{T}^{\vee},\lambda)\text{-}\mathit{bialgebras} \;\cong\; \mathcal{A}\text{-}\mathit{bimonoids}$$

Plan

Construction of the $\mathcal{T}$ functor

Given a species p, define

$$\mathcal{T}(\mathrm{p})[A] := \bigoplus_{F\colon A \leq F} \mathrm{p}[F]. \tag{1}$$

Suppose A and B have the same support. We can define a linear map

$$(\beta_{B,A} : \mathcal{T}(\mathrm{p})[A] \to \mathcal{T}(\mathrm{p})[B])_{(F,G)} = \begin{cases} \beta_{G,F} & G = BF \\ 0 & \text{otherwise} \end{cases}$$

($_{(F,G)}$ stands for the (F, G) component of the map). This turns $\mathcal{T}(\mathrm{p})$ into a species.

Construction of the $\mathcal{T}$ functor

Further, if $f : \mathrm{p} \to \mathrm{q}$ is a map of species, then summing the components $f_F : \mathrm{p}[F] \to \mathrm{q}[F]$ yields a map of species $\mathcal{T}(f) : \mathcal{T}(\mathrm{p}) \to \mathcal{T}(\mathrm{q})$.

Thus $\mathcal{T}$ is a functor.

Definition of natural transformations μ and ι

Define a natural transformation

$$\mu : \mathcal{T}\mathcal{T} \to \mathcal{T}, \qquad \bigoplus_{A \le F \le G} \mathrm{p}[G] \to \bigoplus_{A \le G} \mathrm{p}[G] \tag{2}$$

by mapping each summand in the lhs identically to the matching summand in the rhs. In other words, for a given G, all summands labeled $\mathrm{p}[G]$ in the lhs map identically to the summand labeled $\mathrm{p}[G]$ in the rhs.

Definition of natural transformations μ and ι

There is also an obvious natural transformation

$$\iota : \mathrm{id} \to \mathcal{T}, \qquad \mathsf{p}[A] \to \bigoplus_{F \colon A \leq F} \mathsf{p}[F] \tag{3}$$

given by inclusion.

Monad construction

Recall the definition of a monad.

Definition

A monad on a category C is a functor $\mathcal{T} : \mathsf{C} \to \mathsf{C}$ equipped with natural transformations $\mu : \mathcal{T}\mathcal{T} \to \mathcal{T}$ and $\iota : \mathrm{id} \to \mathcal{T}$ such that the diagrams

$$\begin{tikzcd} \mathcal{T}\mathcal{T}\mathcal{T} \arrow[r, "\mathcal{T}\mu"] \arrow[d, "\mu\mathcal{T}"'] & \mathcal{T}\mathcal{T} \arrow[d, "\mu"] \\ \mathcal{T}\mathcal{T} \arrow[r, "\mu"'] & \mathcal{T} \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{T}\mathcal{T} \arrow[dr, "\mu"] & \\ \mathcal{T} \arrow[ur, "\iota\mathcal{T}"] \arrow[rr, equal] & & \mathcal{T} \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{T}\mathcal{T} \arrow[dr, "\mu"] & \\ \mathcal{T} \arrow[ur, "\mathcal{T}\iota"] \arrow[rr, equal] & & \mathcal{T} \end{tikzcd} \tag{4}$$

commute.

Monad construction

The maps (2) and (3) turn $\mathcal{T}$ into a monad. The diagrams commute simply by inclusion. For instance, the first diagram explicitly looks like:

$$\begin{tikzcd} \mathcal{T}\mathcal{T}\mathcal{T} \arrow[r] \arrow[d] & \mathcal{T}\mathcal{T} \arrow[d] \\ \mathcal{T}\mathcal{T} \arrow[r] & \mathcal{T} \end{tikzcd} \qquad \begin{tikzcd} \bigoplus_{A \le F \le G \le H} \mathrm{p}[H] \arrow[r] \arrow[d] & \bigoplus_{A \le G \le H} \mathrm{p}[H] \arrow[d] \\ \bigoplus_{A \le F \le H} \mathrm{p}[H] \arrow[r] & \bigoplus_{A \le H} \mathrm{p}[H] \end{tikzcd}$$

Monads and Monoids

Proposition

The category of algebras over the monad $\mathcal{T}$ is equivalent to the category of monoids in species.

Proof.

Suppose p is a $\mathcal{T}$-algebra. Evaluating on a face A,

$$\bigoplus_{F\colon A \leq F} \mathrm{p}[F] \to \mathrm{p}[A].$$

This is equivalent to a family of maps $\mathrm{p}[F] \to \mathrm{p}[A]$, one for each $A \leq F$.

Denote the map corresponding to $A \leq F$ by μ_A^F. One can check the naturality, associativity and unitality axioms required for a monoid. Thus a $\mathcal{T}$-algebra is the same as a monoid.

□

Plan

Construction of $\mathcal{T}^\vee$ functor

Dually, we construct a comonad

$$\mathcal{T}^\vee : \mathcal{A}\text{-}\mathsf{Sp} \to \mathcal{A}\text{-}\mathsf{Sp}$$

as follows.
As a functor, $\mathcal{T}^\vee := \mathcal{T}$. Thus, for a species p,

$$\mathcal{T}^\vee(\mathrm{p})[A] = \bigoplus_{F\colon A \leq F} \mathrm{p}[F].$$

Construction of natural transformations Δ and ϵ

The comonad structure on $\mathcal{T}^\vee$ is given by the natural transformation Δ

$$\Delta : \mathcal{T}^\vee \to \mathcal{T}^\vee\mathcal{T}^\vee, \qquad \bigoplus_{A\le G} \mathrm{p}[G] \to \bigoplus_{A\le F\le G} \mathrm{p}[G]$$

which maps each summand in the lhs identically to all matching summands in the rhs,

and the natural transformation ϵ

$$\epsilon : \mathcal{T}^\vee \to \mathrm{id}, \qquad \bigoplus_{F:\, A\le F} \mathrm{p}[F] \to \mathrm{p}[A]$$

which sends $\mathrm{p}[A]$ to itself, and all other summands to zero.

Comands and Comonoids

Recall that,

Definition

A comonad on a category C is a functor $\mathcal{U} : \mathsf{C} \to \mathsf{C}$ equipped with natural transformations $\Delta : \mathcal{U} \to \mathcal{U}\mathcal{U}$ and $\epsilon : \mathcal{U} \to \mathrm{id}$ such that the diagrams

$$
\begin{tikzcd}
\mathcal{U}\mathcal{U}\mathcal{U} & \mathcal{U}\mathcal{U} \arrow[l, "\mathcal{U}\Delta"'] \\
\mathcal{U}\mathcal{U} \arrow[u, "\Delta\mathcal{U}"] & \mathcal{U} \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{U}\mathcal{U} \arrow[dl, "\epsilon\mathcal{U}"'] & \\
\mathcal{U} \arrow[rr, equal] & & \mathcal{U} \arrow[ul, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{U}\mathcal{U} \arrow[dl, "\mathcal{U}\epsilon"'] & \\
\mathcal{U} \arrow[rr, equal] & & \mathcal{U} \arrow[ul, "\Delta"']
\end{tikzcd}
\tag{5}
$$

commute.

It is easy to check that $\mathcal{U} = \mathcal{T}^{\vee}$ on the category of species forms a comonad.

Comands and Comonoids

Analogous to the argument for $\mathcal{T}$, it is clear that the category of $\mathcal{T}^\vee$-coalgebras is equivalent to the category of comonoids in species.
Thus, we have shown,

$$\begin{aligned} \mathcal{T}\text{-algebras} &\cong \mathcal{A}\text{-monoids} \\ \mathcal{T}^\vee\text{-coalgebras} &\cong \mathcal{A}\text{-comonoids} \end{aligned}$$

Plan

We now define a natural transformation

$$\lambda : \mathcal{T}\mathcal{T}^{\vee} \to \mathcal{T}^{\vee}\mathcal{T}. \tag{6}$$

On a species p, on the A-component, this entails a linear map

$$\bigoplus_{A \leq F \leq G} \mathrm{p}[G] \to \bigoplus_{A \leq F' \leq G'} \mathrm{p}[G'].$$

Both spaces are indexed by pairs of faces (F, G) with $A \leq F \leq G$.

$$\lambda_{F,G,F',G'} := \begin{cases} \mathrm{p}[G] \xrightarrow{\beta_{G',G}} \mathrm{p}[G'] & \text{if } FF' = G \text{ and } F'F = G', \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Bimonad

Theorem

The triple $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$ is a bimonad, or equivalently, λ is a mixed distributive law between $\mathcal{T}$ and $\mathcal{T}^\vee$.

Bimonads and Bimonoids

Proposition

The category of bialgebras over the bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$ is equivalent to the category of bimonoids in species.

Proof.

Suppose h is a $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$-bialgebra, that is, h is a $\mathcal{T}$-algebra, a $\mathcal{T}^\vee$-coalgebra, and the following diagram commutes.

$$\begin{tikzcd}
\mathcal{T}(\mathrm{h}) \arrow[r] \arrow[d] & \mathrm{h} \arrow[r] & \mathcal{T}^\vee(\mathrm{h}) \\
\mathcal{T}(\mathcal{T}^\vee(\mathrm{h})) \arrow[rr] & & \mathcal{T}^\vee(\mathcal{T}(\mathrm{h})) \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\bigoplus_{A \leq F} \mathrm{h}[F] \arrow[r] \arrow[d] & \mathrm{h}[A] \arrow[r] & \bigoplus_{A \leq F'} \mathrm{h}[F'] \\
\bigoplus_{A \leq F \leq G} \mathrm{h}[G] \arrow[rr] & & \bigoplus_{A \leq F' \leq G'} \mathrm{h}[G'] \arrow[u]
\end{tikzcd}$$

$\square$

Bimonads and Bimonoids

Let us equate the matrix-components. Thus each choice of faces $A \leq F$ and $A \leq F'$ yields a commutative diagram. Since the indices G and G' are forced by $G = FF'$ and $G' = F'F$, this diagram is precisely the bimonoid axiom.

Thus a $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$-bialgebra is the same as a bimonoid in species.

Conclusion

Hence, we have shown that

$$\begin{aligned}
\mathcal{T}\text{-algebras} &\cong \mathcal{A}\text{-monoids} \\
\mathcal{T}^{\vee}\text{-coalgebras} &\cong \mathcal{A}\text{-comonoids} \\
(\mathcal{T}, \mathcal{T}^{\vee}, \lambda)\text{-bialgebras} &\cong \mathcal{A}\text{-bimonoids}
\end{aligned}$$

Plan

Pretty Pictures

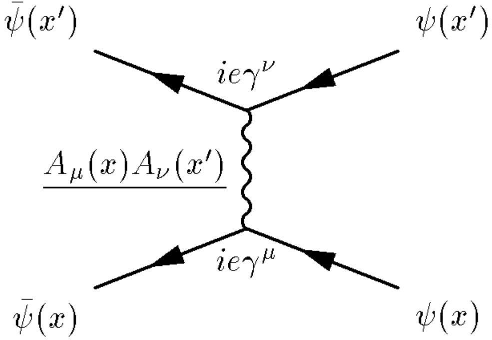

Figure: A Feynman Diagram

Is the resemblance to objects and morphisms only aesthetic?

[1] J. Baez, `http://math.ucr.edu/home/baez/rosetta.pdf`

Pretty Pictures

Category Theory	Physics	Topology	Logic	Computation
object	system	manifold	proposition	data type
morphism	process	cobordism	proof	program

Table 1: The Rosetta Stone (pocket version)

Figure: A bird's eye view [1]

Appreciate category theory due to its unifying power of mathematical structures and constructions.

[1] J. Baez, http://math.ucr.edu/home/baez/rosetta.pdf

Pretty Pictures

$$\frac{\text{“GOOD QM”}}{\texttt{von Neumann QM}} \simeq \frac{\text{HIGH-LEVEL language}}{\texttt{low-level language}}.$$

Figure: "Languages" [2]

Category theory offers a "high level language" to talk about quantum mechanics.

A higher level of sophistication in a theory allows us to ask the right questions.

[2] B. Coecke, `https://arxiv.org/pdf/quant-ph/0510032.pdf`

Thank You for your time!