# Problems on Vector Bundles - MA556

Sankalp Gambhir

October 21, 2021

# Problem 3.6

**Problem.** Suppose $E$ is a vector bundle over $M$, and $U$ is an open set containing $p \in M$. Let $s$ be a smooth section of $E_U$. Show that there exists a smooth section $t$ of $E$ whose restriction to $U$ agrees with $s$ in a neighbourhood of $p$.

# Smooth Sections

A smooth section of a vector bundle $\pi : E \to M$ is a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}$.

The space of sections over a vector bundle $E$, $\Gamma(E)$ carries the structure of a vector space.

Given a smooth section $s$ of $E_U$, construct a smooth function $f$ over $M$ such that it is uniformly 1 in a neighbourhood of $p$ and 0 everywhere outside $U$.

Given a smooth section $s$ of $E_U$, construct a smooth function $f$ over $M$ such that it is uniformly 1 in a neighbourhood of $p$ and 0 everywhere outside $U$.
Note that $f \cdot s$ is a smooth section of $E_U$ as well. As it vanishes along the boundary, we can trivially extend it to a global smooth section

Given a smooth section $s$ of $E_U$, construct a smooth function $f$ over $M$ such that it is uniformly 1 in a neighbourhood of $p$ and 0 everywhere outside $U$.
Note that $f \cdot s$ is a smooth section of $E_U$ as well. As it vanishes along the boundary, we can trivially extend it to a global smooth section

$$t(x) = \begin{cases} f(x) \cdot s(x) \text{ if } x \in U \\ 0 \text{ otherwise.} \end{cases} \tag{1}$$

# Constructing the smooth function

Consider the following construction, beginning with the standard mollifier,

$$g(x) = e^{\frac{1}{1-x^2}} \ ,$$

$$f(x) = \begin{cases} 1 \text{ if } x \leq 0 \\ e^{1-\frac{1}{1-x^2}} \text{ if } 0 < x < 1 \\ 0 \text{ if } x \geq 1 \end{cases} \quad .$$

# Constructing the smooth function

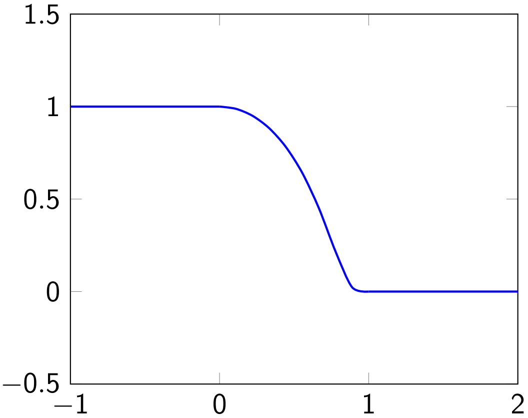
Consider the following construction, beginning with the standard mollifier,

$$g(x) = e^{\frac{1}{1-x^2}} \;,$$

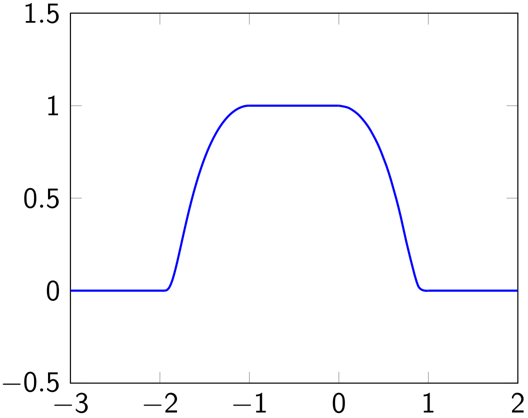
$$f(x) = \begin{cases} 1 \text{ if } x \leq 0 \\ e^{1-\frac{1}{1-x^2}} \text{ if } 0 < x < 1 \\ 0 \text{ if } x \geq 1 \end{cases} \quad .$$

Note that $f$ is indeed smooth. By transforming and taking products of functions of this kind, we can construct a required smooth function in $\mathbb{R}^n$ with $n$ being the dimension of our manifold.

Example choice in $\mathbb{R}$:



For an interval



By construction, it agrees with $s$ in a neighbourhood of $p$ when restricted. $t$ is the required section.

By construction, it agrees with $s$ in a neighbourhood of $p$ when restricted. $t$ is the required section.
This only matches on a neighbourhood, can we do better? Can we define a section on a slice of the space and extend it globally?

# Global Sections (lack thereof)

Consider a trivial bundle on $\mathbb{R}$ given by $\mathbb{R} \times \mathbb{R}$. Construct a section on $\mathbb{R} \setminus \{0\}$ given by $s(p) = (p, \frac{1}{p})$.

Can we extend this to a section on the whole space?

# Global Sections (lack thereof)

Consider a trivial bundle on $\mathbb{R}$ given by $\mathbb{R} \times \mathbb{R}$. Construct a section on $\mathbb{R} \setminus \{0\}$ given by $s(p) = (p, \frac{1}{p})$.

Can we extend this to a section on the whole space?

Note that we can still construct a global section that agrees with this section in a neighbourhood of any interior point of $\mathbb{R} \setminus \{0\}$.

# Problem 3.11

Let $M$ be a smooth manifold. Show that the tensor product over $C(M)$ of two projective $C(M)$-modules is again a projective $C(M)$-module.

# Reframing

Let $M$ be a smooth manifold. Show that the tensor product over $C(M)$ of two spaces of smooth sections over vector bundles is again a space of sections over a vector bundle.

# Reframing

Let $M$ be a smooth manifold. Show that the tensor product over $C(M)$ of two spaces of smooth sections over vector bundles is again a space of sections over a vector bundle.

For each pair of vector bundles $E_1, E_2$, $\exists E \in \mathrm{Bundle}_M$ such that

$$\Gamma(E_1) \otimes_{C(M)} \Gamma(E_2) \cong \Gamma(E) \ .$$

We propose that $E = E_1 \otimes E_2$.

We propose that $E = E_1 \otimes E_2$.
Suppose we have a map between the two spaces, $\chi$. We show that it is bijective.

We propose that $E = E_1 \otimes E_2$.
Suppose we have a map between the two spaces, $\chi$. We show that it is bijective.
Elements in the space $\Gamma(E_1) \otimes \Gamma(E_2)$ are tensor products of sections $s_1 \otimes s_2$, defined as mapping any point $p \in M$ to $s_1(p) \otimes s_2(p) \in E_{1p} \otimes E_{2p}$.

We propose that $E = E_1 \otimes E_2$.
Suppose we have a map between the two spaces, $\chi$. We show that it is bijective.
Elements in the space $\Gamma(E_1) \otimes \Gamma(E_2)$ are tensor products of sections $s_1 \otimes s_2$, defined as mapping any point $p \in M$ to $s_1(p) \otimes s_2(p) \in E_{1p} \otimes E_{2p}$.

$\chi$ maps this to itself in the RHS. $\chi$ is injective as an inclusion map.

# Surjectivity

If $E_1, E_2$ are trivial bundles of rank $k_1, k_2$ respectively, we can construct a linearly independent basis of sections for each $\Gamma(E_i)$. Let a choice of such bases be given by $\{a_i\}$ and $\{b_i\}$. These are frame fields for the individual bundles.

# Surjectivity

If $E_1, E_2$ are trivial bundles of rank $k_1, k_2$ respectively, we can construct a linearly independent basis of sections for each $\Gamma(E_i)$. Let a choice of such bases be given by $\{a_i\}$ and $\{b_i\}$. These are frame fields for the individual bundles.

The tensor product $\{a_i\} \otimes \{b_i\}$ then produces a frame field for $\Gamma(E)$. Since this field is trivial, its frame field has exactly the same dimension as its rank.

# Trivialising Bundles

**Theorem (Whitney Summands of Trivial Vector Bundles)**

*If $X$ is a paracompact Hausdorff space, and $E \to X$ is a topological vector bundle, then for every vector subbundle $E_1 \hookrightarrow E$, there exists a direct summand $E_2 \hookrightarrow E$ such that*

$$E_1 \oplus E_2 \cong E \ .$$

Consider trivialisations of the chosen bundles $E_1 \oplus E_1^\perp$ and $E_2 \oplus E_2^\perp$. We can then construct the diagram

$$\begin{tikzcd}
\Gamma(E_1) \otimes \Gamma(E_2) \arrow[d, "\chi"'] & \Gamma(E_1 \oplus E_1^\perp) \otimes \Gamma(E_2 \oplus E_2^\perp) \arrow[l, "\text{projection}"'] \arrow[d, "\chi^\perp"] \\
\Gamma(E_1 \otimes E_2) & \Gamma((E_1 \oplus E_1^\perp) \otimes (E_2 \oplus E_2^\perp)) \arrow[l, "\text{projection}"]
\end{tikzcd}$$

The diagram commutes. All the other arrows are known to be surjective, the surjectivity of $\chi$ follows.

The diagram for an object in each space

$$
\begin{tikzcd}
s_1 \otimes s_2 \arrow[d, "{\chi,\ \text{inclusion}}"'] & & (s_1 \oplus s_1^\perp) \otimes (s_2 \oplus s_2^\perp) \arrow[ll, "\text{projection}"'] \arrow[d, "{\chi^\perp,\ \text{inclusion}}"] \\
s_1 \otimes s_2 & & (s_1 \oplus s_1^\perp) \otimes (s_2 \oplus s_2^\perp) \arrow[ll, "\text{projection}"]
\end{tikzcd}
$$

Finally, since $\chi$ is injective and surjective, it gives an isomorphism between $\Gamma(E_1) \otimes \Gamma(E_2)$ and $\Gamma(E_1 \otimes E_2)$ as required.