Diff Geometry Presentation 2

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Problem 3.6

Problem. Suppose E is a vector bundle over M, and U is an open set containing $p \in M$. Let s be a smooth section of E_U. Show that there exists a smooth section t of E whose restriction to U agrees with s in a neighbourhood of p.

Given a smooth section s of E_U, construct a smooth function f over M such that it is uniformly 1 in a neighbourhood of p and 0 everywhere outside U.
Note that $f \cdot s$ is a smooth section of E_U as well. As it vanishes along the boundary, we can trivially extend it to a global smooth section

$$t(x) = \begin{cases} f(x) \cdot s(x) \text{ if } x \in U \\ 0 \text{ otherwise.} \end{cases} \tag{1}$$

By construction, it agrees with s in a neighbourhood of p when restricted. t is the required section.
This only matches on a neighbourhood, can we do better? Can we define a section on a slice of the space and extend it globally?

Global Sections (lack thereof)

Consider a trivial bundle on $\mathbb{R}$ given by $\mathbb{R} \times \mathbb{R}$. Construct a section on $\mathbb{R} \setminus \{0\}$ given by $s(p) = (p, \frac{1}{p})$.

Can we extend this to a section on the whole space?

Note that we can still construct a global section that agrees with this section in a neighbourhood of any interior point of $\mathbb{R} \setminus \{0\}$.

Problem 3.11

Let M be a smooth manifold. Show that the tensor product over $C(M)$ of two projective $C(M)$-modules is again a projective $C(M)$ module.

Reframing

Let M be a smooth manifold. Show that the tensor product over $C(M)$ of two spaces of smooth sections over vector bundles is again a space of sections over a vector bundle.

For each pair of vector bundles E_1, E_2, $\exists E \in \mathrm{Bundle}_M$ such that

$$\Gamma(E_1) \otimes_{C(M)} \Gamma(E_2) \cong \Gamma(E) \; .$$

We propose that $E = E_1 \otimes E_2$.
Suppose we have a map between the two spaces, χ. We show that it is bijective.
Elements in the space $\Gamma(E_1) \otimes \Gamma(E_2)$ are tensor products of sections $s_1 \otimes s_2$, defined as mapping any point $p \in M$ to $s_1(p) \otimes s_2(p)$.

χ maps this to itself in the RHS. χ is injective as an inclusion map.

Surjectivity

If E_1, E_2 are trivial bundles of rank k_1, k_2 respectively, we can construct a linearly independent basis of sections for each $\Gamma(E_i)$. Let a choice of such bases be given by $\{e_i\}$ and $\{f_i\}$.

The tensor product $\{e_i\} \otimes \{f_i\}$ then produces a linearly independent basis for $\Gamma(E)$.

Trivialising Bundles

Theorem (Whitney Summands of Trivial Vector Bundles)

If X is a paracompact Hausdorff space, and $E \to X$ is a topological vector bundle, then for every vector subbundle $E_1 \hookrightarrow E$, there exists a direct summand $E_2 \hookrightarrow E$ such that

$$E_1 \oplus E_2 \cong E \ .$$

Consider trivialisations of the chosen bundles $E_1 \oplus E_1^\perp$ and $E_2 \oplus E_2^\perp$. We can then construct the diagram

$$\begin{tikzcd}
\Gamma(E_1) \otimes \Gamma(E_2) \arrow[d, "\chi"'] & \Gamma(E_1 \oplus E_1^\perp) \otimes \Gamma(E_2 \oplus E_2^\perp) \arrow[l, "\text{projection}"'] \arrow[d, "\chi^\perp"] \\
\Gamma(E_1 \otimes E_2) & \Gamma((E_1 \oplus E_1^\perp) \otimes (E_2 \oplus E_2^\perp)) \arrow[l, "\text{projection}"]
\end{tikzcd}$$

All the other arrows are known to be surjective, the surjectivity of χ follows.

Finally, since χ is injective and surjective, it gives an isomorphism between $\Gamma(E_1) \otimes \Gamma(E_2)$ and $\Gamma(E_1 \otimes E_2)$ as required.