Problems on Vector Bundles - MA556

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Problem 3.6

Problem. Suppose E is a vector bundle over M, and U is an open set containing $p \in M$. Let s be a smooth section of E_U. Show that there exists a smooth section t of E whose restriction to U agrees with s in a neighbourhood of p.

Smooth Sections

A smooth section of a vector bundle $\pi : E \to M$ is a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}$.

The space of sections over a vector bundle E, $\Gamma(E)$ carries the structure of a vector space.

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$$t(x) = \begin{cases} f(x) \cdot s(x) \text{ if } x \in U \\ 0 \text{ otherwise.} \end{cases} \tag{1}$$

Constructing the smooth function

Consider the following construction, beginning with the standard mollifier,

$$g(x) = e^{\frac{1}{1-x^2}} ,$$

$$f(x) = \begin{cases} 1 \text{ if } x \leq 0 \\ e^{1-\frac{1}{1-x^2}} \text{ if } 0 < x < 1 \\ 0 \text{ if } x \geq 1 \end{cases} .$$

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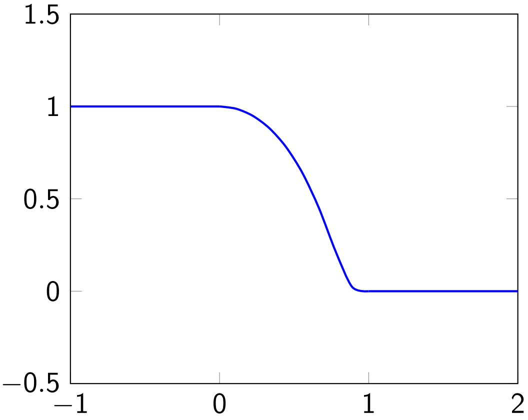
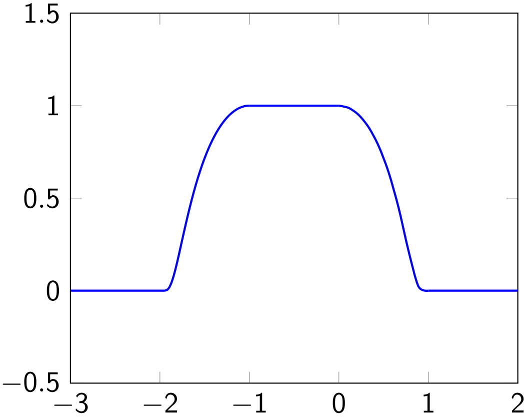
Note that f is indeed smooth. By transforming and taking products of functions of this kind, we can construct a required smooth function in $\mathbb{R}^n$ with n being the dimension of our manifold.

Example choice in $\mathbb{R}$:



For an interval



By construction, it agrees with s in a neighbourhood of p when restricted. t is the required section.

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This only matches on a neighbourhood, can we do better? Can we define a section on a slice of the space and extend it globally?

Global Sections (lack thereof)

Consider a trivial bundle on $\mathbb{R}$ given by $\mathbb{R} \times \mathbb{R}$. Construct a section on $\mathbb{R} \setminus \{0\}$ given by $s(p) = (p, \frac{1}{p})$.

Can we extend this to a section on the whole space?

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Note that we can still construct a global section that agrees with this section in a neighbourhood of any interior point of $\mathbb{R} \setminus \{0\}$.

Problem 3.11

Problem. Let M be a smooth manifold. Show that the tensor product over $C(M)$ of two projective $C(M)$-modules is again a projective $C(M)$-module.

Reframing

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For each pair of vector bundles E_1, E_2, $\exists E \in \mathrm{Bundle}_M$ such that

$$\Gamma(E_1) \otimes_{C(M)} \Gamma(E_2) \cong \Gamma(E) \ .$$

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If E_1, E_2 are trivial bundles of rank k_1, k_2 respectively, we can construct a linearly independent basis of sections for each $\Gamma(E_i)$. Let a choice of such bases be given by $\{a_i\}$ and $\{b_i\}$. These are frame fields for the individual bundles.

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Elements in the space $\Gamma(E_1) \otimes \Gamma(E_2)$ are tensor products of sections $s_1 \otimes s_2$, defined as mapping any point $p \in M$ to $s_1(p) \otimes s_2(p) \in E_{1p} \otimes E_{2p}$.

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χ maps this to itself in the RHS. χ is injective as an inclusion map.

Surjectivity

The tensor product $\{a_i\} \otimes \{b_i\}$ produces a frame field for $\Gamma(E)$. Since this field is trivial, its frame field has exactly the same dimension as its rank.

Trivialising Bundles

Theorem (Whitney Summands of Trivial Vector Bundles)

If X is a paracompact Hausdorff space, and $E \to X$ is a topological vector bundle, then for every vector subbundle $E_1 \hookrightarrow E$, there exists a direct summand $E_2 \hookrightarrow E$ such that

$$E_1 \oplus E_2 \cong E \ .$$

Consider trivialisations of the chosen bundles $E_1 \oplus E_1^{\perp}$ and $E_2 \oplus E_2^{\perp}$. We can then construct the diagram

$$\begin{tikzcd}
\Gamma(E_1) \otimes \Gamma(E_2) \arrow[d, "\chi"'] & & \Gamma(E_1 \oplus E_1^{\perp}) \otimes \Gamma(E_2 \oplus E_2^{\perp}) \arrow[ll, "\text{projection}"'] \arrow[d, "\chi^{\perp}"] \\
\Gamma(E_1 \otimes E_2) & & \Gamma((E_1 \oplus E_1^{\perp}) \otimes (E_2 \oplus E_2^{\perp})) \arrow[ll, "\text{projection}"]
\end{tikzcd}$$

The diagram commutes. All the other arrows are known to be surjective, the surjectivity of χ follows.

Similarly, for the injectivity, we have the diagram with flipped arrows

$$\begin{tikzcd}
\Gamma(E_1) \otimes \Gamma(E_2) \arrow[r, "\text{extension}"] \arrow[d, "\chi"'] & \Gamma(E_1 \oplus E_1^{\perp}) \otimes \Gamma(E_2 \oplus E_2^{\perp}) \arrow[d, "\chi^{\perp}"] \\
\Gamma(E_1 \otimes E_2) \arrow[r, "\text{extension}"'] & \Gamma((E_1 \oplus E_1^{\perp}) \otimes (E_2 \oplus E_2^{\perp}))
\end{tikzcd}$$

The extension maps take an element to its copy in the codomain with padded zeroes (when viewed as a vector).
The diagram commutes. All the other arrows are known to be surjective, the surjectivity of χ follows.

The diagram for an object in each space

$$\begin{tikzcd}
s_1 \otimes s_2 \arrow[d, "{\chi,\ \text{inclusion}}"'] & & (s_1 \oplus s_1^{\perp}) \otimes (s_2 \oplus s_2^{\perp}) \arrow[ll, "\text{projection}"'] \arrow[d, "{\chi^{\perp},\ \text{inclusion}}"] \\
s_1 \otimes s_2 & & (s_1 \oplus s_1^{\perp}) \otimes (s_2 \oplus s_2^{\perp}) \arrow[ll, "\text{projection}"]
\end{tikzcd}$$

Finally, since χ is injective and surjective, it gives an isomorphism between $\Gamma(E_1) \otimes \Gamma(E_2)$ and $\Gamma(E_1 \otimes E_2)$ as required.