CS711 Assignment 3

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Question 1.

Proof. Given the set $S_{d,n}$ as defined, I induct on d and n for the polynomial $f(x_1, x_2, \ldots, x_n)$ of total degree atmost d.

Base case: $n = 1, d = 1$.

The relevant polynomial is $f(x_1) = a_1 x_1 + a_0$, where atleast one a_i is non-zero, since f is not identically zero. If $a_0 \neq 0$, set $x_1 = 0 < d$, else set $x_1 = 1 \leq d$. This assignment results in a non-vanishing evaluation and satisfies the constraints, i.e, it is in $S_{d,n}$.

Induction on n: Given a non-zero evaluating assignment can be found for each degree $c \leq d$, and number of variables $m < n$, I show that we can find one for degree d and n variables.

Consider a polynomial $f(x_1, x_2, \ldots, x_n)$. We can write,

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=0}^{d} x_1^i f_i(x_2, x_3, \ldots, x_n) \ ,$$

for polynomials $\{f_i\}$ in $n-1$ variables and degrees $\{d-i\}$ respectively. Choose an assignment for the largest $f_i \not\equiv 0$ such that it satisfies the constraints for $S_{d-i,n-1}$. After substitution, we obtain a univariate polynomial in x_1 of degree atmost d.

□

Question 2.

Proof. Given the set $T_{d,n} =$. TODO. □

Question 3.

Proof. Consider a multilinear polynomial $P \in \mathbb{F}[x_1, x_2, \ldots, x_n]$. Since it is multilinear, and thus linear in x_1, we can express it as:

$$P(x_1, x_2, \ldots, x_n) = x_1^0 P_1(x_2, x_3, \ldots, x_n) + x_1^1 P_1'(x_2, x_3, \ldots, x_n) \ .$$

As this decomposes into the $(n-1)$ problem, it is easy to see that if either of the polynomials P_1 or P_1' have non-zero assignments in $\{0,1\}^{n-1}$, then we can generate a non-zero assignment including $x_1 \in \{0,1\}$. The proof proceeds by induction.

Base case: $n = 1$.

P_1, P_1' in this case are constants, with atleast one of them non-zero. If P_1 is 0, choose $x_1 = 1$, else choose $x_1 = 1$. $P \not\equiv 0$ guarantees this assignment leads to a non-vanishing result.

Induction: Given that $\forall m \in \mathbb{N}, m < n$, every non-zero multilinear polynomial in m variables has a non-zero satisfying assignment, the same holds for non-zero multilinear polynomials in n variables.

As before, write

$$P(x_1, x_2, \ldots, x_n) = x_1^0 P_1(x_2, x_3, \ldots, x_n) + x_1^1 P_1'(x_2, x_3, \ldots, x_n) \ ,$$

where P_1, P_1' depend on atmost $n-1$ variables. If $P_1 \not\equiv 0$, by the induction hypothesis, it has an assignment such that it is non-vanishing. Append to such an assignment, $x_1 = 0$. Else, $P_1 \equiv 0$, so we must have $P_1' \not\equiv 0$ since P is given to be non-zero. Append $x_1 = 1$ to a non-vanishing assignment of P_1'.

This produces a satisfying assignment for any non-zero multilinear polynomial of degree n.

$\square$

Question 4.

Hello