Unit-III Temporal Logic

October 30, 2015

Introduction

- Temporal logic is a formal system for reasoning about time
- Temporal logic has found extensive application in computer science, because the behavior of both hardware and software is a function of time
- Some examples of specifications that use temporal concepts
  1. If a request is made to print a file, eventually the file will be printed
  2. The operating system will never deadlock
- The temporal aspects of these specification can be expressed in first-order logic using quantified variables for points in time:
  1. $\forall t_1 \ (\mathrm{RequestPrint}(t_1) \rightarrow \exists t_2 \ (t_2 \geq t_1 \wedge \mathrm{PrintedAt}(t_2)))$
  2. $\forall t \neg \ \mathrm{deadlocked}(t)$

Modal logics

- Modal logics express the distinction between what is necessarily true and what is possibly true
- For example, the statement '7 is a prime number' is necessarily true because-given the definitions of the concepts in the statement-the statement is true always and everywhere
- The statement the head of state of this country is a king is possibly true, because its truth changes from place to place and from time to time
- Temporal logic and modal logic are related because 'always' is similar to 'necessarily' and 'eventually' to 'possibly'

Small History about Temporal Logic

- Temporal and modal logics first appeared in Greek philosophy, their vague concepts proved difficult to formalize and an acceptable formal semantics for modal logic was first given by Saul Kripke in 1959
- In 1977, Amir Pnueli showed that temporal logic can specify properties of concurrent programs and that Kripkes semantics could be adapted to develop a formal theory of the temporal logic of programs.

Syntax and Semantics

- Two additional unary operators used in Propostional Temporal Logic are:
  1. $\Box$, read always
  2. $\Diamond$, read eventually
- The two unary temporal logic operators have the same precedence as negation

Examples for Syntactically correct formulas in PTL

- $p \wedge q$
- $\Box p$
- $\Diamond(p \wedge q) \rightarrow \Diamond p$
- $\Box\Box p \leftrightarrow \Box p$
- $\Diamond\Box p \leftrightarrow \Box\Diamond p$,
- $\neg \Diamond p \wedge \Box \neg q$

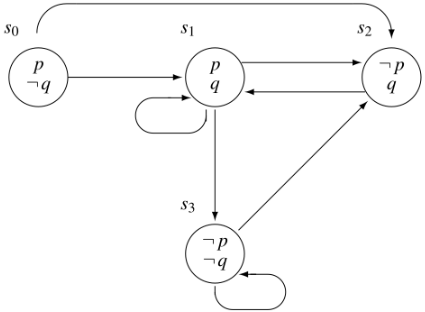
Semantics

- Informally, $\Box$ is a universal operator meaning 'for any time t in the future', while $\Diamond$ is an existential operator meaning 'for some time t in the future'

Definition (State Transition Diagram)

A state transition diagram is a directed graph. The nodes are states and the edges are transitions. Each state is labeled with a set of propositional literals such that clashing literals do not appear in any state

State Transition Diagram Example



Interpretation in PTL

Definition

An interpretation I for a formula A in PTL is a pair (S, ρ), where S $= \{s_1, \ldots, s_n\}$ is a set of states each of which is an assignment of truth values to the atomic propositions in A, $s_i : P \rightarrow \{T, F\}$, and ρ is a binary relation on the states, $\rho \subseteq S \times S$

Note

1. When displaying an interpretation graphically, the states are usually labeled only with the atomic propositions that are assigned T
2. If an atom is not shown in the label of a state, it is assumed to be assigned F
3. A binary relation can be considered to be a mapping from a state to a set of states $\rho : S \rightarrow 2^S$, so the relational notation $(s_1 , s_2) \in \rho$ will usually be written functionally as $s_2 \in \rho(s_1)$

Example

- In the previous figure, $s_0(p)$=T, $s_0(q)$=F, $s_1(p)$=T, $s_1(q)$=T,$s_2(p)$=F, $s_2(q)$=T, $s_3(p)$=F, $s_3(q)$=F
- $\rho(s_0)=\{s_1,s_2\}$, $\rho(s_1)=\{s_1,s_2,s_3\}$, $\rho(s_2)=\{s_1\}$, $\rho(s_3)=\{s_2,s_3\}$

Truth Value of A in S

Definition

Let A be a formula in PTL. $v_{I,s}$(A), the truth value of A in s, is defined by structural induction as follows:

- If A is p $\in$ P, then $v_{I,s}$(A) = s(p)
- If A is ¬A' then $v_{I,s}$(A) = T iff $v_{I,s}$(A') = F
- If A is A' $\vee$ A" then vI ,s(A) = T iff $v_{I,s(A')}$ = T or $v_{I,s(A")}$ = T, and similarly for the other Boolean operators.
- If A is $\Box$A' then $v_{I,s(A)}$ = T iff $v_{I,s'(A')}$ = T for all states s' $\in$ ρ(s).
- If A is $\Diamond$A' then $v_{I,s(A)}$ = T iff $v_{I,s'}$(A') = T for some state s'$\in$ ρ(s)

Note

- Note: The notation s $\models_I$ A is used for $v_{I,s(A)} = T$
- When I is clear from the context, it can be omitted s $\models$ A iff $v_s(A) = T$

Example

- Let us compute the truth value of the formula $\Box p \vee \Box q$ for each state s in the previous diagram.
- $\rho(s_0) = \{s_1, s_2\}$. Since $s_1 \models q$ and $s_2 \models q$, it follows that $s_0 \models \Box q$. By the semantics of $\vee$, $s_0 \models \Box p \vee \Box q$
- $s_3 \in \rho(s_1)$, but $s_3 \mid \neq p$ and $s_3 \mid \neq q$, so, $s_1 \mid \neq \Box p$ and $s_1 \mid \neq \Box q$. Therefore, $s1 \mid \neq \Box p \vee \Box q$
- $\rho(s_2) = s_1$. Since $s_1 \models p$, we have $s_2 \models \Box p$ and $s_2 \models \Box p \vee \Box q$
- $s_3 \in \rho(s_3)$. $s_3 \mid \neq \Box p \vee \Box q$ by the same argument used for s_1

Satisfiability and Validity

Definition

Let A be a formula in PTL.

- A is satisfiable iff there is an interpretation $I = (S, \rho)$ for A and a state $s \in S$ such that $s \models_I A$.
- A is valid iff for all interpretations $I = (S, \rho)$ for A and for all states $s \in S$, $s \models_I A$. Notation: $\models A$

Example

- The analysis we did for the formula $A = \Box p \vee \Box q$ in the previous example shows that A is satisfiable because $s_0 \models_I A$ or because $s\Box \models_I A$. The formulas A is not valid because $s_1 \models_I A$ or because $s_3 \models_I A$

Duality Theorem

Theorem

$\models \Box p \leftrightarrow \neg \Diamond \neg p$

Proof.

Let I = (S,ρ) be an arbitrary interpretation for the formula and let s be an arbitrary state in S .

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Assume that s $\models \Box$p, and suppose that s $\models \Diamond\neg$ p

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Proof.

Let I $=$ (S,ρ) be an arbitrary interpretation for the formula and let s be an arbitrary state in S .

Assume that s $\models \Box$p, and suppose that s $\models \Diamond\neg$ p

Then there exists a state s'$\in \rho$(s) such that s'$\models \neg$ p

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Proof.

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Assume that s $\models \Box$p, and suppose that s $\models \Diamond\neg$ p
Then there exists a state s'$\in \rho$(s) such that s'$\models \neg$ p
Since s $\models \Box$p, for all states t $\in \rho$(s), t$\models$p, in particular, s'$\models$p, contradicting s' $\models \neg$ p

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Since s $\models \Box$p, for all states t $\in \rho$(s), t$\models$p, in particular, s'$\models$p, contradicting s' $\models \neg$ p
$\therefore$ s $\models \neg \Diamond\neg$ p

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Assume that s $\models \Box$p, and suppose that s $\models \Diamond\neg$ p
Then there exists a state s'$\in \rho$(s) such that s'$\models \neg$ p
Since s $\models \Box$p, for all states t $\in \rho$(s), t$\models$p, in particular, s'$\models$p, contradicting s' $\models \neg$ p
$\therefore$ s $\models \neg \Diamond\neg$ p
Since I and s were arbitrary we have proved that $\models \Box p \rightarrow \neg\Diamond\neg$ p.
Similarly we can prove the converse. $\Box$

Models of Time

Definition (Reflexivity)

An interpretation I = (S,ρ) is reflexive iff ρ is a reflexive relation: for all s $\in$ S , (s,s) $\in \rho$, or s $\in \rho$(s) in functional notation

Definition (Transitivity)

An interpretation I = (S,ρ) is transitive iff ρ is a transitive relation: for all s_1 , s_2 , $s_3 \in$ S , $s_2 \in \rho(s_1) \vee s_3 \in \rho(s_2) \rightarrow s_3 \in \rho(s_1)$

Definition (Linearity)

An interpretation I = (S, ρ) is linear if ρ is a function, that is, for all s $\in$ S, there is at most one s' $\in$ S such that s'$\in \rho$(s)

Linear Temporal Logic(LTL)

Definition

Linear temporal logic (LTL) is propositional temporal logic whose interpretations are limited to transitions which are discrete, reflexive, transitive, linear and total

Definition (Interpretation for an LTL formula)

An interpretation for an LTL formula A is a path of states: $\sigma = s_0$, s_1 , s_2 , . . . ,

Satisfiable and Validity

Definition

Let A be a formula in LTL. A is satisfiable iff there is an interpretation σ for A such that $\sigma \models$ A. A is valid iff for all interpretations σ for A, $\sigma \models$ A. Notation: $\models$ A

Definition

A formula of the form $\circ$A or $\neg \circ$A is a next formula. A formula of the form $\Diamond$A or $\neg \Box$A is a future formula

Equivalent Formulas in LTL

Theorem

$\models \Box p \rightarrow \circ p$, $\models \circ p \rightarrow \Diamond p$, $\models \Box p \rightarrow \Diamond p$, $\models \circ p \leftrightarrow \neg \circ \neg\ p$