Markowitz Portfolio Theory

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1 Introduction

Modern portfolio theory (MPT), or **mean-variance analysis**, is a mathematical framework for assembling a portfolio of assets such that the expected return is maximized for a given level of risk. It is a formalisation and extension of diversification in investing, the idea that owning different kinds of financial assets is less risky than owning only one type. Its key insight is that an asset's risk and return should not be assessed by itself, but by how it contributes to a portfolio's overall risk and return. The variance of return (or its transformation, the standard deviation) is used as a measure of risk, because it is tractable when assets are combined into portfolios. Often, the historical variance and covariance of returns is used as a proxy for the forward-looking versions of these quantities, but other, more sophisticated methods are available.

Economist **Harry Markowitz** introduced MPT in his 1952 paper **Portfolio Selection** for which he was later awarded a Nobel Memorial Prize in Economic Sciences. In 1940, **Bruno de Finetti** published the mean-variance analysis method, in the context of proportional reinsurance, under a stronger assumption. The paper was obscure and only became known to economists of the English-speaking world in 2006.

2 Preliminaries

In this section, we define the necessary terms and discuss other crucial concepts before our close look at the **Markowitz mean-variance portfolio theory**.

2.1 Quadratic Programming Problems

As the reader proceeds to the later sections, he may realise that portfolio theory/optimisation is just one large optimisation problem: a quadratic programming problem to be precise. But what is a quadratic program (QP)? To answer this question, we first define a ordering relation on $\mathbb{R}^{k\times 1}$ where $k \in \mathbb{N}$.

Definition 2.1. *The relation $\leq$ is defined on $\mathbb{R}^{k\times 1}$ as follows:*

$$x \leq y \text{ iff } x_i \leq y_i \; \forall \, i = 1, \ldots, k$$

where $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ *and* $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$.

Please note that this section introduces the reader to quadratic programming problems (in the most general form). However, no general methods for solving such a problem has been discussed, for this is beyond our scope. We do, however, discuss the solution of a very specific case. As the reader may guess, it is the very form our portfolio optimisation problem reduces to.

Definition 2.2 (Quadratic Program)**.** *The problem of finding the minimum value of*

$$f(x) = \frac{1}{2}x^tQx + c^tx$$

subject to

$$Ax \leq b, \quad Ex = d$$

where $x \in \mathbb{R}^{n\times 1}$ is the vector of variables that we wish to optimise, $Q \in \mathbb{R}^{n\times n}$ is a symmetric matrix, $c \in \mathbb{R}^{n\times 1}$, $A \in \mathbb{R}^{s\times n}, b \in \mathbb{R}^{s\times 1}, E \in \mathbb{R}^{t\times n}$ and $d \in \mathbb{R}^{t\times 1}$.

Associated with every QP is set of conditions called the **Karush-Kuhn-Tucker (KKT) conditions** which are used to solve it. They are defined as follows.

Definition 2.3 (KKT Conditions)**.** *Corresponding to the QP described in definition 2.2, the following conditions are called the KKT conditions.*

1. ***Stationarity.*** *Define the Lagrangian* $\mathcal{L}(x, \lambda, \mu) := \frac{1}{2}x^tQx + c^tx + \lambda^t(Ax - b) + \mu^t(Ex - d)$ *with* $\lambda \in \mathbb{R}^{s\times 1}$ *and* $\mu \in \mathbb{R}^{t\times 1}$*. The stationarity condition is given by*
$$\nabla_x \mathcal{L}(x, \lambda, \mu) = 0 \implies Qx + c + A^t\lambda + E^t\mu = 0$$
2. ***Primal Feasibility.*** *These are the constraints in the original QP, i.e.*
$$Ax \leq b, \quad Ex = d$$
3. ***Dual Feasibility.*** *This condition states that the Lagrange multipliers associated with the inequality constraints are non-ngative:*
$$\lambda \geq 0$$
4. ***Complementary Slackness.*** *This condition says*
$$\lambda_i(A_ix - b_i) = 0 \,\forall\, i = 1, \ldots, s$$
where $\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_s \end{pmatrix}, b = \begin{pmatrix} b_1 \\ \vdots \\ b_s \end{pmatrix}$ *and* $A = \begin{pmatrix} A_1 \\ \vdots \\ A_s \end{pmatrix}$ *where* $A_j \in \mathbb{R}^{1\times n} \,\forall\, j = 1, \ldots, s.$

But why do we even care about the KKT conditions of a QP? To answer this question, we first introduce a couple of terms.

Definition 2.4 (Positive Semi-Definite Matrix)**.** *A symmetric matrix* $A \in \mathbb{R}^{n\times n}$ *is said to be a positive semi-definite matrix if for each* $x \in \mathbb{R}^{n\times 1}\backslash\{0\}$, $x^tAx \geq 0$.

Definition 2.5 (Convex QP)**.** *The QP in definition 2.2 is said to be convex if* Q *is positive semi-definite.*

Now that we have covered our preliminaries, we state the following theorem (without proof) to illustrate how KKT conditions are utilised to solve QPs.

Theorem 2.1. *For a convex QP, the triple* (x^*, λ^*, μ^*) *satisfies the KKT conditions iff* x^* *is a global optimum of the QP.*

Therefore, to solve a convex QP, all one has to do is solve the corresponding KKT conditions. However, solving the KKT conditions is not an easy task. The complementary slackness condition introduces non-linearity to the system, and hence adds another extra layer of complexity.

Illustration 2.1. *Minimise* $f(x_1, x_2) = x_1^2 + x_2^2 - 2x_1 - x_2$ *given that* $x_1 + x_2 \leq 2$ *and* $x_1 - x_2 = 0$.

Solution. Observe that this QP may be rewritten as:
$$\text{Minimise} \quad \frac{1}{2}x^tQx + c^tx \quad \text{subject to} \quad Ax \leq b \quad \text{and} \quad Ex = d$$
where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, Q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, c = \begin{pmatrix} -2 \\ -1 \end{pmatrix}, A = \begin{pmatrix} 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 2 \end{pmatrix}, E = \begin{pmatrix} 1 & -1 \end{pmatrix}$ and $d = (0)$. Let $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^{2\times 1}$. Then $y^tQy = 2y_1^2 + 2y_2^2 \geq 0$. Hence, Q is positive semi-definite, and therefore the QP is convex. Let $\lambda, \mu \in \mathbb{R}$. Now, the KKT conditions are as follows:

1. *Stationarity.* $Qx + c + A^t\lambda + E^t\mu = 0 \implies \begin{pmatrix} 2x_1 - 2 + \lambda + \mu \\ 2x_2 - 1 + \lambda - \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
2. *Primal Feasibility.* $Ax \leq b, Ex = d \implies x_1 + x_2 \leq 2, x_1 - x_2 = 0$
3. *Dual Feasibility.* $\lambda \geq 0$
4. *Complementary Slackness.* $\lambda(x_1 + x_2 - 2) = 0$

Since the QP is convex, solving conditions 1-4 will give the optimal value for x. We have the following mutually exclusive and exhaustive cases:

Case 01. $(\lambda = 0)$. Then the KKT conditions reduce to:
$$2x_1 - 2 + \mu = 0$$

$$2x_2 - 1 - \mu = 0$$
$$x_1 - x_2 = 0$$
$$x_1 + x_2 \leq 2$$

Solving the equations, i.e. the first three equations here yields $(x_1, x_2, \mu) = (\frac{3}{4}, \frac{3}{4}, \frac{1}{2})$. These values also satisfy the fourth condition, i.e. the inequality.

Case 02. $(\lambda > 0)$. Now the KKT conditions reduce to:

$$2x_1 - 2 + \lambda + \mu = 0$$
$$2x_2 - 1 + \lambda - \mu = 0$$
$$x_1 - x_2 = 0$$
$$x_1 + x_2 - 2 = 0$$
$$x_1 + x_2 \leq 2$$
$$\lambda > 0$$

Upon solving the equations only, we obtain $(x_1, x_2, \lambda, \mu) = (1, 1, -\frac{1}{2}, \frac{1}{2})$. But these values do not satisfy the inequalities.

Hence, the solution to the system of KKT conditions is

$$(x_1, x_2, \lambda, \mu) = \left(\frac{3}{4}, \frac{3}{4}, 0, \frac{1}{2}\right)$$

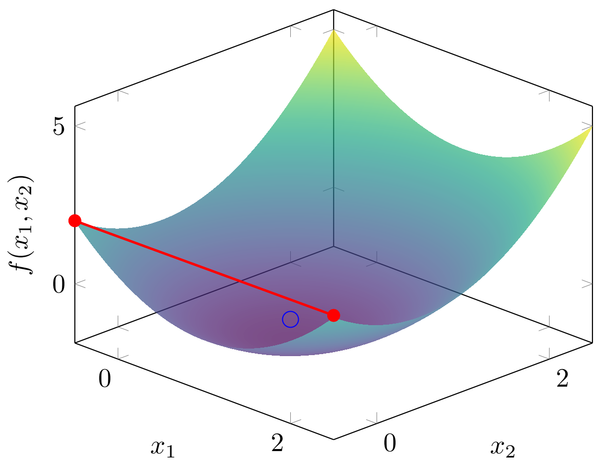
and consequently, the optimal value of x (say x^*) is given by

$$x^* = \begin{pmatrix} 3/4 \\ 3/4 \end{pmatrix}$$

Hence, the minimum value of f with the given constraints is

$$f\left(\frac{3}{4}, \frac{3}{4}\right) = -\frac{9}{8}$$

The 3D Plot of $f(x_1, x_2) = x_1^2 + x_2^2 - 2x_1 - x_2$



2.2 Basic Definitions

Definition 2.6 (Asset)**.** *An asset is something valuable that an individual, company or country owns or controls, with the expectation that it will provide a future economic benefit.*

Remark. Examples of assets include cash, land, gold etc.

Definition 2.7 (Financial Assets)**.** *A financial asset is a specific type of asset that derives its value from a contractual claim to future cash flows or ownership in another entity.*

Remark. Unlike physical assets (like a building or a piece of machinery), financial assets often do not have an inherent physical form; their value is based on a legal claim or the market's perception of their future earning potential. Options, futures and stocks are common examples of financial assets.

Definition 2.8 (Portfolio). *A portfolio refers to a collection of financial assets or investments held by an individual, an institution, or a fund.*

Definition 2.9 (Return on an Asset). *Suppose we purchase an asset for x_0 monetary units and later sell it for x_1. Then the return on that asset is defined as:*

$$R := \frac{x_0}{x_1}$$

Definition 2.10 (Rate of Return on an Asset). *Consider the asset described in definition 2.9. The rate of return on this asset is given by:*

$$r := R - 1 = \frac{x_1 - x_0}{x_0}$$

2.3 Short Selling

It is not always necessary that one owns the assets he/she sells in a market. This is known as **short selling**. It works somewhat as follows:
You ask a broker if their firm is holding a particular stock (say ABC) in the total pool owned by their customers. If so, you ask them to sell some or all of it on your behalf. Consequently, this becomes your debt. Suppose that you receive x_0 monetary units after the stocks have been sold. Eventually, you ask the brokerage to buy it back. Suppose x_1 monetary units were needed for the purchase. If $x_0 < x_1$, you end up making a profit; otherwise, a loss. In this case,

$$R = \frac{(-x_0)}{(-x_1)} = \frac{x_0}{x_1} \quad \text{and} \quad r = R - 1 = \frac{x_1 - x_0}{x_0} = \frac{(-x_1) - (-x_0)}{(-x_0)}$$

2.4 Returns from a Portfolio

Suppose we wish to construct a portfolio consisting of n number of assets. Assume that our initial budget is x_0 monetary units. For a particular asset $A_i, i = 1, 2, \ldots, n$, we allocate x_{0_i} monetary units from the total budget. Then, $\exists w_i \in (0, 1)$ s.t. $x_{0_i} = w_i x_0$. It is evident that $x_{0_1} + \cdots + x_{0_n} = x_0 \implies w_1 x_0 + \cdots w_n x_0 = x_0 \implies w_1 + \cdots + w_n = 1$. To incorporate short selling of assets in our portfolio, we allow these weights (w_i's) to be negative. However, the budget constraint still holds, i.e.

$$\sum_{i=1}^{n} w_i = 1$$

Let R_i denote the return on the asset $A_i, i = 1, \ldots, n$. This implies that asset A_i gets sold-off for $x_{1_i} = R_i x_{0_1} = R_i w_i x_0$ monetary units. Therefore, the total receipts from the portfolio is

$$x_1 = \sum_{i=1}^{n} x_{1_i} = \sum_{i=1}^{n} R_i w_i x_0 = x_0 \sum_{i=1}^{n} R_i w_i$$

Hence, the total return from the portfolio is

$$R = \frac{x_1}{x_0} = \sum_{i=1}^{n} R_i w_i$$

Additionally, suppose the rate of return from assets A_i is $r_i, i = 1, \ldots, n$. So, the total rate of return on the portfolio is:

$$r = R - 1 = \left(\sum_{i=1}^{n} R_i w_i \right) - \left(\sum_{i=1}^{n} w_i \right) = \sum_{i=1}^{n} (R_i - 1) w_i = \sum_{i=1}^{n} r_i w_i$$

3 Markowitz Mean-Variance Portfolio Theory

The main aim of the Markowitz porfolio theory is to find the weights for the portfolio described in section 2.4 such that risk is minimised and the expected returns is maximised given market and physical constraint(s). This may sound like a straightforward optimization problem, which it precisely is!

3.1 The Set-Up

Note that we remarked that our goals are to minimise risk and maximise expected returns. Therefore, it becomes necessary to define these terms in a more mathematical sense.

Again, consider the portfolio in subsection 2.4. Now these r_i's ($i = 1, \dots, n$) are assumed to be random variables. Subsequently, we define the random vector z as $z := \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$ Moreover, define

$$\mu_i := \mathbb{E}[r_i], i = 1, \dots, n$$

$$m := \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix}$$

$$w := \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

$$\Sigma := \mathrm{Cov}(z) = (\sigma_{ij})_{n \times n}$$

where $\sigma_{ij} = \mathrm{cov}(r_i, r_j), i, j = 1, \dots, n$. Note that Σ is a symmetric matrix.

3.1.1 What defines expected returns?

We have already seen that $r = r_1 w_1 + \cdots + r_n w_n$. So, r is also a random variable. r is the rate of return of our portfolio. Naturally, we expect the expectation value of r to be the expected return that we wish to maximise.

$$\mathbb{E}[r] = \mathbb{E}\left[\sum_{i=1}^{n} r_i w_i\right] = \sum_{i=1}^{n} w_i \mathbb{E}[r_i] = \sum_{i=1}^{n} w_i \mu_i = m^t w$$

3.1.2 What defines risk?

In the Markowitz theory, risk is modelled by the variance of the random variable r.

$$\mathrm{Var}(r) = \mathbb{E}[(r - \mathbb{E}[r])^2] = \mathbb{E}\left[\left(\sum_{i=1}^{n} r_i w_i - \sum_{i=1}^{n} w_i \mu_i\right)^2\right] = \mathbb{E}\left[\left(\sum_{i=1}^{n} w_i (r_i - \mu_i)\right)^2\right]$$

Let $x_i = r_i - \mu_i$. Then

$$\mathrm{Var}(r) = \mathbb{E}\left[\left(\sum_{i=1}^{n} w_i x_i\right)^2\right] = \mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j x_i x_j\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \mathbb{E}[x_i x_j]$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \mathbb{E}[(r_i - \mu_i)(r_j - \mu_j)] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \mathrm{cov}(r_i, r_j) = w^t \Sigma w$$

3.2 Minimising Risk for an Acceptable Baseline Return

Suppose $\mu_b \in \mathbb{R}$ is the acceptable baseline expected rate of return, i.e. $\mathbb{E}[r] = m^t w \geq \mu_b$. Then our optimisation problem becomes the following:

$$\text{Minimise} \quad \frac{1}{2} w^t \Sigma w \quad \text{subject to} \quad m^t w \geq \mu_b, \; e^t w = 1$$

where $e = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$. Here the variance is scaled by a factor of $\frac{1}{2}$ for the problem to become a standard **quadratic programming problem** (QP). Moreover, $e^t w = 1$ simplifies to $w_1 + \cdots + w_n = 1$ which is the budget constraint. The **Karush-Kuhn-Tucker** (KKT) conditions for this QP are as follows:

$$\Sigma w - \lambda m - \gamma e = 0 \text{ (stationarity condition)}$$

$$\mu_b \leq m^t w, e^t w = 1 \text{ (primal feasibility)}$$

$$\lambda \geq 0 \text{ (dual feasibility)}$$

$$\lambda^t(m^t w - \mu_b) = 0 \text{ (complementary slackness)}$$

Here $\lambda, \gamma \in \mathbb{R}$. By definition, Σ is symmetric and positive definite. Therefore, any triple (w, λ, γ) satisfying the KKT conditions is a solution to the QP. Assume Σ is invertible. Let $\bar{w}$ be an optimal value of w.

Case 3.2.1. ($\lambda = 0$, Inactive). In this case, the KKT conditions reduce to the following:

$$\Sigma\bar{w} - \gamma e = 0$$

$$e^t\bar{w} = 0$$

Solving this is an elementary task and involves some matrix algebra. Finally, we obtain:

$$\gamma = (e^t\Sigma^{-1}e)^{-1}$$

$$\bar{w} = (e^t\Sigma^{-1}e)^{-1}\Sigma^{-1}e$$

Note. The obtained value of w, i.e. $\bar{w}$ also solves the problem

$$\text{Minimise} \quad \frac{1}{2}w^t\Sigma w \quad \text{subject to} \quad e^t w = 1$$

Therefore, $\bar{w}$ gives the smallest possible variance (or risk) over all possible portfolios (since no return constraint is applicable). Consequently, in the following sections, we shall refer to this value of w as $w_{\text{min}-\text{var}}$, i.e.

$$w_{\text{min}-\text{var}} = (e^t\Sigma^{-1}e)^{-1}\Sigma^{-1}e = \frac{1}{e^t\Sigma^{-1}e}\Sigma^{-1}e$$

and the associated return by

$$\mu_{\text{min}-\text{var}} = m^t w_{\text{min}-\text{var}} = \frac{1}{e^t\Sigma^{-1}e}m^t\Sigma^{-1}e$$

Now we check if the minimum variance weights $w_{\text{min}-\text{var}}$ are feasible:

1. If $m^t w_{\text{min}-\text{var}} \geq \mu_b$, then $w_{\text{min}-\text{var}}$ is the solution of the QP.
2. If $m^t w_{\text{min}-\text{var}} < \mu_b$, we analyse the active condition.

Case 3.2.2. ($\lambda > 0$, Active). Now that $\lambda > 0$, the complementary slackness condition implies that $m^t\bar{w} = \mu_b$. Now the KKT conditions reduce to:

$$\Sigma\bar{w} - \lambda m - \gamma e = 0$$

$$\mu_b = m^t\bar{w}, e^t\bar{w} = 1$$

$$\lambda > 0$$

Eliminating $\bar{w}$ from this system, we get:

$$\mu_b = \lambda m^t\Sigma^{-1}m + \gamma m^t\Sigma^{-1}e$$

$$1 = \lambda m^t\Sigma^{-1}e + \gamma e^t\Sigma^{-1}e$$

or equivalently,

$$\begin{pmatrix} m^t\Sigma^{-1}m & m^t\Sigma^{-1}e \\ m^t\Sigma^{-1}e & e^t\Sigma^{-1}e \end{pmatrix}\begin{pmatrix} \lambda \\ \gamma \end{pmatrix} = \begin{pmatrix} \mu_b \\ 1 \end{pmatrix}$$

Let

$$T = \begin{pmatrix} m^t\Sigma^{-1}m & m^t\Sigma^{-1}e \\ m^t\Sigma^{-1}e & e^t\Sigma^{-1}e \end{pmatrix}$$

Let $\delta = \det(T) = (m^t\Sigma^{-1}m) \cdot (e^t\Sigma^{-1}e) - (m^t\Sigma^{-1}e)^2$. If $\delta = 0$, then $\exists\, \tau \in \mathbb{R}$ s.t. $m = \tau e$. (Follows from the Cauchy-Schwarz inequality.) If $\mu_b \neq \tau$, then the QP is infeasible. However, if $\mu_b = \tau$, then $w_{\text{min-var}}$ solves the QP.

Otherwise, if $\delta > 0$, the system can be solved. On solving, we obtain the optimal solution:

$$\bar{w} = (1 - \alpha)w_{\text{min}-\text{var}} + \alpha w_{\text{mk}}$$

where

$$w_{\text{mk}} = \frac{1}{e^t\Sigma^{-1}m}\Sigma^{-1}m$$

$$\alpha = \frac{\mu_b(m^t\Sigma^{-1}e)(e^t\Sigma^{-1}e) - (m^t\Sigma^{-1}e)^2}{\delta}$$

Remark. Observe that the optimal set of weights is a linear combination of the two sets of weights $w_{\text{min}-\text{var}}$ and w_{mk}, both of which satisfy the constraint $e^t w = 1$. We have labelled weights w_{mk} as the market weights since they incorporate all of the market information on the assets under consideration.

3.3 The Efficient Frontier

Ideally, one would want to study the relationship between risk and return simultaneously rather than consider a baseline return. This motivates us to define an optimisation problem similar to the QP in section 3.2 without considering the constraint $m^t w \geq \mu_b$, and also at the same time, devise an objective function to minimise risk and maximise return simultaneously. The following is one such attempt:

$$\text{Minimise} \quad \frac{1}{2}w^t\Sigma w - \lambda m^t w \quad \text{subject to} \quad e^t w = 1$$

where $\lambda > 0$. Note that this is also a QP with the corresponding KKT conditions being:

$$\Sigma w - \lambda m - \gamma e = 0, \gamma \in \mathbb{R}$$

$$e^t w = 1$$

Note that the unknowns here are w and γ, and this QP is inactive. Assume Σ is invertible. Then, solving this system yields:

$$\gamma = \frac{1 - \lambda m^t \Sigma^{-1} e}{e^t \Sigma^{-1} e}$$

$$w = (1 - \alpha_\lambda) w_{\text{min}-\text{var}} + \alpha_\lambda w_{\text{mk}}$$

where $\alpha_\lambda = \lambda(m^t\Sigma^{-1}e)$ and $w_{\text{min}-\text{var}}$ and w_{mk} are the same as in section 3.2. From here on, we shall refer to this optimum w as w_λ.

Now, the optimum return simplifies to:

$$\mu_b = m^t w_\lambda = \mu_{\text{min}-\text{var}} + \lambda \frac{\delta}{e^t \Sigma^{-1} e}$$

where δ has been defined in section 3.2

Remark. Notice that for $\lambda = 0$, $\mu_b = \mu_{\text{min}-\text{var}}$ (as expected from case 3.2.1). Also, if $\delta > 0$, as $\lambda \to \infty$, $\mu_b \to \infty$, i.e. an infinite risk implies infinite expected returns.

It is evident that the solution to this QP traces out all the solutions to the one in 3.2 for all possible values of μ_b as λ goes from 0 to ∞. Fix some $\lambda \geq 0$. Define the point (in the risk-return space)

$$G_\lambda := \left(\sqrt{\text{var}(r_\lambda)}, \mathbb{E}[r_\lambda]\right)$$

where $r_\lambda = w_\lambda^t z$ is the rate of return for weights w_λ. Therfore, $\mathbb{E}[r_\lambda] = w_\lambda^t m$ and $\text{Var}(r_\lambda) = w_\lambda^t \Sigma w_\lambda$. Note that this point specifies the least risk associated with some return and vice-versa. Then the graph

$$G = \{G_\lambda : \lambda \in \mathbb{R}_{\geq 0}\}$$

traces out the set of all effient portfolios. This upper part of this curve is known as the **efficient frontier**.

3.3.1 What does this curve look like?

Let $y^2 = \text{Var}(r_\lambda)$ and $x = \mathbb{E}[r_\lambda]$. Then

$$x = m^t w_\lambda$$

$$y^2 = w_\lambda^t \Sigma w_\lambda$$

From the KKT conditions, we have:

$$\Sigma w_\lambda - \lambda m - \gamma e = 0 \implies w_\lambda = \Sigma^{-1}(\lambda m + \gamma e)$$

Again,

$$x = m^t w_\lambda = m^t \Sigma^{-1}(\lambda m + \gamma e) = (m^t\Sigma^{-1}m)\lambda + (m^t\Sigma^{-1}e)\gamma$$

The budget constraint yields:

$$1 = e^t w_\lambda = e^t(\Sigma^{-1}(\lambda m + \gamma e)) = (e^t\Sigma^{-1}m)\lambda + (e^t\Sigma^{-1}e)\gamma$$

Let $A = e^t\Sigma^{-1}e, C = m^t\Sigma^{-1}m, B = m^t\Sigma^{-1}e = e^t\Sigma^{-1}m$ (since Σ is symmetric) and $D = AC - B^2$. Then the previous two equations become:

$$C\lambda + B\gamma = x$$

$$B\lambda + A\gamma = 1$$

which upon solving for λ and γ give

$$\lambda = \frac{Ax - B}{D}, \gamma = \frac{C - Bx}{D}$$

Again,

$$y^2 = w_\lambda^t \Sigma w_\lambda = (\Sigma^{-1}(\lambda m + \gamma e))^t \Sigma^{-1}(\Sigma^{-1}(\lambda m + \gamma e))$$

As Σ is symmetric,

$$y^2 = (\lambda m + \gamma e)^t \Sigma^{-1}(\lambda m + \gamma e) = \left(\frac{Ax - B}{D} m + \frac{C - Bx}{D} e\right)^t \Sigma^{-1} \left(\frac{Ax - B}{D} m + \frac{C - Bx}{D} e\right) =$$

Simplifying this further gives

$$y^2 = \frac{Ax^2 - 2Bx + C}{D}$$

Therefore,

$$\frac{\left(x - \frac{B}{A}\right)^2}{\frac{D}{A^2}} + \frac{y^2}{\frac{1}{A}} = -1$$

Hence, the efficient frontier is a **hyperbola**.

3.4 Effect of a Risk-Free Asset

Definition 3.1 (Risk-Free Asset)**.** *A risk-free asset is a theoretical or idealized financial asset that offers a guaranteed rate of return with no risk of financial loss. In other words, its future returns are known with certainty, and there is no possibility of defaulting on the principal or interest payments.*

Remark. Obviously, in the real world, one would expect some risk to be associated with any asset. However, risk-free assets are used to model assets with very little risk, e.g. treasury bills.

Definition 3.2 (Risk-Free Rate)**.** *The fixed, guaranteed rate of return offered by a risk-free asset is known as the risk-free rate.*

Consider the same portfolio as before, but this time include a risk-free asset f with risk-free rate r_f. Define $\hat{x} := \begin{pmatrix} r_f \\ r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$.

Clearly, $\hat{x}$ is a random vector. Then $\hat{\Sigma} = \mathrm{Cov}(\hat{x}) = \begin{pmatrix} 0 & 0 \\ 0 & \Sigma \end{pmatrix}$ where $\Sigma = \mathrm{Cov}(z)$ and $z = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$. Consequently, we have the following Markowitz QP:

$$\text{Minimise } \ \frac{1}{2} w^t \Sigma w \quad \text{subject to} \quad r_f w_0 + m^t w \geq \mu_b \text{ and } w_0 + e^t w = 1$$

Here w_0 is the weight assigned to the risk-free asset f. Since we can always (at least) achieve a rate of return r_f, $\mu_b \geq r_f$. Also assume that Σ is non-singular. Note that $w_0 > 0$ implies that we are investing at the risk-free rate. If $w_0 < 0$, it means that we are borrowing at the risk-free rate. The KKT conditions for this QP is as follows:

$$\lambda r_f + \gamma = 0$$

$$\Sigma w - \lambda m - \gamma e = 0$$

$$\lambda(r_f w_0 + m^t w - \mu_b) = 0$$

$$w_0 + e^t w = 1$$

$$r_f w_0 + m^t w \geq \mu_b$$

$$\lambda \geq 0$$

Case 3.4.1. ($\lambda = 0$, Inactive). The KKT conditions reduce to:

$$\lambda r_f + \gamma = 0$$

$$\Sigma w - \lambda m - \gamma e = 0$$

$$w_0 + e^t w = 1$$

The first of these implies that $\gamma = 0$. Consequently, it follows from the second of these equations that $w = 0$. Therefore, by the budget constraint, $w_0 = 1$. We check if $\mu_b \leq r_f$. If yes, this is the optimal solution. If not, we proceed to the active case.

Case 3.4.2 ($\lambda > 0$, Active). From the budget constraint, $1 - w_0 = e^t w = \lambda e^t \Sigma^{-1}(m - r_f e)$. Also, $\lambda \neq 0 \implies \mu_b - r_f w_0 = m^t w = \lambda m^t \Sigma^{-1}(m - r_f e)$. These last two equations may be compactly written as:

$$\begin{pmatrix} 1 & e^t\Sigma^{-1}(m - r_f e) \\ r_f & m^t\Sigma^{-1}(m - r_f e) \end{pmatrix} \begin{pmatrix} w_0 \\ \lambda \end{pmatrix} = \begin{pmatrix} 1 \\ r_f \end{pmatrix}$$

Solving this system yields

$$w_0 = 1 - (\mu_b - r_f)\frac{e^t\Sigma^{-1}(m - r_f e)}{(m - r_f e)^t\Sigma^{-1}(m - r_f e)}, \lambda = \frac{\mu_b - r_f}{(m - r_f e)^t\Sigma^{-1}(m - r_f e)}$$

if $m - r_f e \neq 0$. However, if $m - r_f e = 0$, then $w_0 = 1$ and hence $w = 0$. Now, the stationarity condition implies that

$$w = \lambda\Sigma(m - r_f e) = \frac{\mu_b - r_f}{(m - r_f e)^t\Sigma^{-1}(m - r_f e)}\Sigma(m - r_f e)$$

Finally, we have:

$$\begin{pmatrix} w_0 \\ w \end{pmatrix} = (1 - \alpha)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha\begin{pmatrix} 1 - e^t\Sigma^{-1}(m - r_f e) \\ \Sigma^{-1}(m - r_f e) \end{pmatrix}$$

where

$$\alpha = \frac{\mu_b - r_f}{(m - r_f e)^t\Sigma^{-1}(m - r_f e)}$$

3.4.1 Capital Asset Pricing Model (CAPM)

Note. Observe that the optimal weights are a linear combination of $w_f = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $w_M = \begin{pmatrix} 1 - e^t\Sigma^{-1}(m - r_f e) \\ \Sigma^{-1}(m - r_f e) \end{pmatrix}$. This observation yields the following result.

Theorem 3.1 (One Fund Theorem)**.** *If the selection of assets for investment includes a risk-free asset, then there exists a single fund F of risky assets such that every efficient portfolio can be constructed as a linear combination of the risk-free asset and the fund F.*

The question we now seek the answer of is: *What does the efficient frontier look like when one of the assets in the portfolio is risk-free?* Start by defining the **market portfolio**: the market portfolio has the weights

$$w_M = \begin{pmatrix} 1 - e^t\Sigma^{-1}(m - r_f e) \\ \Sigma^{-1}(m - r_f e) \end{pmatrix}$$

Therefore, the rate of return on this portfolio (a random variable) is given by:

$$r_M = w_M^t \hat{x} = w_M^t \begin{pmatrix} r_f \\ z \end{pmatrix} = \begin{pmatrix} r_f \\ z \end{pmatrix}^t \begin{pmatrix} 1 - e^t\Sigma^{-1}(m - r_f e) \\ \Sigma^{-1}(m - r_f e) \end{pmatrix}$$

Now we define an **efficient portfolio**. The weights of this portfolio are the same as the ones obtained after solving the previous QP. Using the **two fund theorem**, the rate of return on this portfolio is given by:

$$w_\lambda = (1 - \alpha)w_f + \alpha w_M \implies r = (1 - \alpha)r_f + \alpha r_M$$

Consequently,

$$\mathbb{E}[r] = \mathbb{E}[(1 - \alpha)r_f + \alpha r_M] = (1 - \alpha)r_f + \alpha\mathbb{E}[r_M] = r_f - \alpha r_f + \alpha\mathbb{E}[r_M] = r_f + \alpha(\mathbb{E}[r_M] - r_f)$$

Now,

$$\mathrm{Var}(r) = \mathrm{Var}((1 - \alpha)r_f + \alpha r_M) = \alpha^2 \cdot \mathrm{Var}(r_M) \implies \alpha = \frac{\sqrt{\mathrm{Var}(r)}}{\sqrt{\mathrm{Var}(r_M)}}$$

Combining the two equations above, we get:

$$\mathbb{E}[r] = r_f + \frac{\mathbb{E}[r_M] - r_f}{\sqrt{\mathrm{Var}(r_M)}}\sqrt{\mathrm{Var}(r)}$$

Quite clearly, the efficient front in this case is a **straight line**. This line is also called the **capital market line**, and the slope called the **price of risk** or the **Sharpe ratio**. Let $\mu_M = \mathbb{E}[r_M], \mu_r = \mathbb{E}[r], \sigma_M^2 = \mathrm{Var}(r_M)$ and $\sigma_r^2 = \mathrm{Var}(r)$. So, we have:

$$\mu_r = r_f + \frac{\mu_M - r_f}{\sigma_M}\sigma_r$$

Moreover,

$$\begin{aligned}\mu_M = \mathbb{E}[r_M] = \mathbb{E}\left[\begin{pmatrix} r_f \\ z \end{pmatrix}^t \begin{pmatrix} 1 - e^t\Sigma^{-1}(m - r_f e) \\ \Sigma^{-1}(m - r_f e) \end{pmatrix}\right] &= \mathbb{E}[(1 - e^t\Sigma^{-1}(m - r_f e))r_f + (\Sigma^{-1}(m - r_f e))^t z] \\ = (1 - e^t\Sigma^{-1}(m - r_f e))r_f + (\Sigma^{-1}(m - r_f e))^t\mathbb{E}[z] &= (1 - e^t\Sigma^{-1}(m - r_f e))r_f + (\Sigma^{-1}(m - r_f e))^t m \\ = r_f - r_f e^t\Sigma^{-1}(m - r_f e) + m^t\Sigma^{-1}(m - r_f e) &= r_f + (m^t\Sigma^{-1}(m - r_f e) - r_f e^t\Sigma^{-1}(m - r_f e)) \\ &= r_f + (m - r_f e)^t\Sigma^{-1}(m - r_f e)\end{aligned}$$

and similarly

$$\sigma_M^2 = \mathrm{Var}(r_M) = \mathrm{Var}((\Sigma^{-1}(m - r_f e))^t z) = (m - r_f e)^t\Sigma^{-1}(m - r_f e)$$

Therefore,

$$\text{Sharpe Ratio} = \frac{\mu_M - r_f}{\sigma_M} = \sqrt{(m - r_f e)^t\Sigma^{-1}(m - r_f e)}$$

Having seen the effects of risk-free assets, we would now like to carry forward the results obtained thus far to price assets. For this, we make a key observation:

Let i be any asset and consider a portfolio consisting of i and the market portfolio with rate of return r_M. The efficient frontier for the market portfolio and this asset i must lie entirely below the capital market line (the efficient frontier obtained when i is assumed to be a risk-free asset) since the addition of a risk-free asset constitutes some sort of certainty in the returns. (This can also be seen by sketching the two curves: we do have the closed form expressions of both which we have derived in the previous sections.) However, the key observation is that the capital market line is tangent to the efficient frontier of the market portfolio and the asset i.

Theorem 3.2 (CAPM)**.** *The expected return on any asset i satisfies*

$$\mathbb{E}[r_i] = \mu_i = r_f + \beta_i(\mu_M - r_f)$$

where

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

is called the asset-beta, $\mu_M = \mathbb{E}[r_M], \sigma_M^2 = \mathrm{Var}(r_M)$ and $\sigma_{iM} = \mathrm{cov}(r_i, r_M)$ and r_M is the rate of return on the market portfolio.

Proof. Suppose r_i is the rate of return on asset i. Consider a portfolio with rate of return $r_\alpha = \alpha r_i + (1 - \alpha)r_M$. The expected return $\mathbb{E}[r_\alpha] = \mu_\alpha = \alpha\mu_i + (1 - \alpha)\mu_M$ and variance $\sigma_\alpha^2 = \alpha^2\sigma_i^2 + 2\alpha(1 - \alpha)\sigma_{iM} + (1 - \alpha)^2\sigma_M^2$ where $\sigma_i^2 = \mathrm{Var}(r_i)$. Observe

$$\frac{dr_\alpha}{d\alpha} = r_i - r_M$$

$$\frac{d\sigma_\alpha}{d\alpha} = \frac{1}{\sigma_\alpha}(\alpha\sigma_i^2 + (1 - 2\alpha)\sigma_{iM} + (\alpha - 1)\sigma_M^2)$$

Therefore,

$$\frac{dr_\alpha}{d\sigma_\alpha} = \frac{\frac{dr_\alpha}{d\alpha}}{\frac{d\sigma_\alpha}{d\alpha}} = \frac{\sigma_\alpha(r_i - r_M)}{\alpha\sigma_i^2 + (1 - 2\alpha)\sigma_{iM} + (\alpha - 1)\sigma_M^2}$$

Now, the capital market line is tangent to the efficient frontier for the market portfolio and asset i. Therefore,

$$\frac{\mu_M - r_f}{\sigma_M} = \left(\frac{dr_\alpha}{d\sigma_\alpha}\right)_{\alpha=0} = \frac{\sigma_M(r_i - r_M)}{\sigma_{iM} - \sigma_M^2}$$

Solving for μ_M gives

$$\mu_i = r_f + \beta_i(\mu_M - r_f)$$

$\square$

Now we extend CAPM for pricing assets. Consider an asset purchased at price P and later sold at price Q. Therefore, the rate of return on this asset is

$$r = \frac{Q-P}{P}$$

and the expected rate of return is

$$\mu_r = \mathbb{E}[r] = \mathbb{E}\left[\frac{Q-P}{P}\right] = \frac{\mu_Q - P}{P}, \mu_Q = \mathbb{E}[Q]$$

The CAPM gives

$$\frac{\mu_Q - P}{P} = \mu_r = r_f + \beta_r(\mu_M - r_f) \implies P = \frac{\mu_Q}{1 + r_f + \beta_r(\mu_M - r_f)}$$

This formula is known as the **CAPM pricing formula**. How do we compute β_r?

$$\beta_r = \frac{\sigma_{rM}}{\sigma_M^2} = \frac{\text{cov}(r, r_M)}{\text{Var}(r_M)} = \frac{\text{cov}\left(\frac{Q-P}{P}, r_M\right)}{\sigma_M^2} = \frac{\text{cov}\left(\frac{Q}{P} - 1, r_M\right)}{\sigma_M^2} = \frac{\text{cov}\left(Q, r_M\right)}{P\sigma_M^2}$$

Putting this value of β_r in the CAPM pricing formula, we get

$$P = \frac{1}{1+r_f}\left(\mu_Q - \frac{\text{cov}(Q, r_M)(\mu_M - r_f)}{\sigma_M^2}\right)$$

Thi is known as the **certainty equivalent pricing formula**.

4 A Real-Life Problem

Suppose we want to create a portfolio consisting of the stocks of **Samsung Electronics**, **Apple Inc**, **Lenovo Group Ltd** and **Dell Inc**. All we have is the historical data of the four stock prices, and we wish to find the efficient frontier for our portfolio. A rough sketch of how to deal with this problem has been discussed below:

1. In essence, we want to plot the the graph of

$$y = \sqrt{\frac{Ax^2 - 2Bx + C}{D}}$$

where $A = e^t\Sigma^{-1}e, B = m^t\Sigma^{-1}e, C = m^t\Sigma^{-1}m$ and $D = AC - B^2$.

2. The two things we do not know are the matrices m and Σ. m is the column matrix of the expected rates of return of the four stocks and Σ is the corresponding covariance matrix.

3. Σ can be determined using statistical methods since we have the historical data of the stock prices.

4. The matrix m, or equivalently the expected rate of return for each stock, can be evaluated using the CAPM theorem.

4.1 Computation of Σ

1. Firstly, using the **yfinance** library in **Python**, we gather the historical data, i.e. the *close, high, low, open* and *volume* for Samsung Electronics, Apple Inc., Lenovo Group Ltd and Dell Inc. from 15 June 2020 to 25 June 2025 at 00:00 each day.

```
# Importing and installing the necessary libraries
!pip install yfinance
import yfinance as yf
import pandas as pd


start_date = '2020-06-15' # Setting the starting date
end_date = '2025-06-15' # Setting the ending date


# Storing the historical stock data for Samsung in a Pandas dataframe
ticker_symbol = '005930.KS'
samsung_data = yf.download(ticker_symbol, start=start_date, end=end_date)
samsung_df = pd.DataFrame(samsung_data)


```

```
# Doing the same for Apple
ticker_symbol = 'AAPL'
apple_data = yf.download(ticker_symbol, start=start_date, end=end_date)
apple_df = pd.DataFrame(apple_data)


# The Same for Lenovo
ticker_symbol = '0992.HK'
lenovo_data = yf.download(ticker_symbol, start=start_date, end=end_date)
lenovo_df = pd.DataFrame(lenovo_data)


# The same for Dell
ticker_symbol = 'DELL'
dell_data = yf.download(ticker_symbol, start=start_date, end=end_date)
dell_df = pd.DataFrame(dell_data)
```

The following is the obtained data. Please note that we are displaying just the first five entries of each dataframe.

Table 1: Samsung Stock Data (005930.KS)

Date	Price Close	High	Low	Open	Volume
2020-06-15	43901.445312	45749.001127	43901.445312	45221.128037	28772921
2020-06-16	45836.972656	45836.972656	44517.290142	45045.163148	21808375
2020-06-17	45924.957031	46540.808945	45133.147427	45836.978186	26672595
2020-06-18	46012.937500	46012.937500	45397.085564	45924.958652	15982926
2020-06-19	46540.816406	46540.816406	45397.091239	46276.879829	18157985

Table 2: Apple Stock Data (AAPL)

Date	Price Close	High	Low	Open	Volume
2020-06-15	83.352501	84.006218	80.822690	80.985515	138808800
2020-06-16	85.561523	85.833709	83.772920	85.410854	165428800
2020-06-17	85.442444	86.368339	85.320935	86.307585	114406400
2020-06-18	85.476486	85.894476	84.866510	85.398719	96820400
2020-06-19	84.988014	86.650252	83.877423	86.183663	264476000

Table 3: Lenovo Stock Data (0992.HK)

Date	Price Close	High	Low	Open	Volume
2020-06-15	3.235703	3.338549	3.219880	3.314815	40967948
2020-06-16	3.306904	3.322726	3.267348	3.283170	3585603
2020-06-17	3.362283	3.378105	3.306904	3.322726	24512233
2020-06-18	3.378104	3.401838	3.330637	3.338548	20244291
2020-06-19	3.386016	3.409750	3.354371	3.362282	30707019

Table 4: Dell Stock Data (DELL)

Date	Price Close	High	Low	Open	Volume
2020-06-15	22.033239	22.197108	21.359039	21.443315	4336259
2020-06-16	22.323524	22.801083	22.066018	22.716807	3159957
2020-06-17	22.248611	22.763625	22.225201	22.365660	4104235
2020-06-18	22.201788	22.257970	21.878733	22.117512	4487391
2020-06-19	22.904078	23.184993	22.501430	22.632523	6059478

2. For our analysis, we shall use the opening prices of the four stocks. The common dates in each of the four dataframes are identified, and a new dataframe is created consisting of the common dates and the respective opening prices of the four stocks.

```
# Finding common dates using the index (which is the date)

common_dates = samsung_df.index.intersection(apple_df.index).intersection(lenovo_df.
    index).intersection(dell_df.index)
```

```python



# Filtering the dataframes to keep only the common dates the 'Open' prices (without dropping any levels from the column index)

samsung_common = samsung_df.loc[common_dates].xs('Open', level='Price', axis=1)
apple_common = apple_df.loc[common_dates].xs('Open', level='Price', axis=1)
lenovo_common = lenovo_df.loc[common_dates].xs('Open', level='Price', axis=1)
dell_common = dell_df.loc[common_dates].xs('Open', level='Price', axis=1)



# Creating the new dataframe with common dates and 'Open' prices

common_opens_df = pd.DataFrame({
    'Samsung_Open': samsung_common['005930.KS'],
    'Apple_Open': apple_common['AAPL'],
    'Lenovo_Open': lenovo_common['0992.HK'],
    'Dell_Open': dell_common['DELL']
})


# Display the new dataframe

common_opens_df
```

Here is the ouput: Again, only the first five outputs are shown here. (There are 1,143 rows in total.)

Table 5: The Common Opens Dataframe

Date	Samsung_Open	Apple_Open	Lenovo_Open	Dell_Open
2020-06-15	45221.128037	80.985515	3.314815	21.443315
2020-06-16	45045.163148	85.410854	3.283170	22.716807
2020-06-17	45836.978186	86.307585	3.322726	22.365660
2020-06-18	45924.958652	85.398719	3.338548	22.117512
2020-06-19	46276.879829	86.183663	3.362282	22.632523

3. Now that we have the open prices of each of the four stocks at T distinct times, we may calcuate the rates of return for any particular stock as:

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}}, \quad t = 1, 2, \ldots, T$$

where P_t is the observed open price on the t-th point of observation and r_t is the corresponding rate of return. (Note that in this case, $T = 1143$.)

```python
# Compute the rate of return for each stock

samsung_returns = (common_opens_df['Samsung_Open'].shift(-1) -common_opens_df['Samsung_Open']) / common_opens_df['Samsung_Open']
apple_returns = (common_opens_df['Apple_Open'].shift(-1) - common_opens_df['Apple_Open']) / common_opens_df['Apple_Open']
lenovo_returns = (common_opens_df['Lenovo_Open'].shift(-1) - common_opens_df['Lenovo_Open']) / common_opens_df['Lenovo_Open']
dell_returns = (common_opens_df['Dell_Open'].shift(-1) - common_opens_df['Dell_Open']) / common_opens_df['Dell_Open']



# Create a new dataframe for the rates of return
returns_df = pd.DataFrame({
    'Samsung_Return': samsung_returns,
    'Apple_Return': apple_returns,
    'Lenovo_Return': lenovo_returns,
    'Dell_Return': dell_returns
})

```

```


# Display the returns dataframe

returns_df
```

The first five entries of the output are as follows:

Table 6: The Returns Dataframe

Date	Samsung_Return	Apple_Return	Lenovo_Return	Dell_Return
2020-06-15	-0.003891	0.054644	-9.546512e-03	0.059389
2020-06-16	0.017578	0.010499	1.204812e-02	-0.015458
2020-06-17	0.001919	-0.010531	4.761663e-03	-0.011095
2020-06-18	0.007663	0.009192	7.109200e-03	0.023285
2020-06-19	-0.011407	-0.009305	-2.645124e-08	0.00558

4. Having ascertained the historical rates of return for each of the four stocks at the same timestamps, we now proceed to computing the covariance matrix

$$\Sigma = \begin{pmatrix} \text{cov}(r_{sam}, r_{sam}) & \text{cov}(r_{sam}, r_{app}) & \text{cov}(r_{sam}, r_{len}) & \text{cov}(r_{sam}, r_{dell}) \\ \text{cov}(r_{app}, r_{sam}) & \text{cov}(r_{app}, r_{app}) & \text{cov}(r_{app}, r_{len}) & \text{cov}(r_{app}, r_{dell}) \\ \text{cov}(r_{len}, r_{sam}) & \text{cov}(r_{len}, r_{app}) & \text{cov}(r_{len}, r_{len}) & \text{cov}(r_{len}, r_{dell}) \\ \text{cov}(r_{dell}, r_{sam}) & \text{cov}(r_{dell}, r_{app}) & \text{cov}(r_{dell}, r_{len}) & \text{cov}(r_{dell}, r_{dell}) \end{pmatrix}$$

But how does one compute these covariances? For random variables r_i and r_j whose historical data are known to us at T distinct timestamps, we can approximate

$$\text{cov}(r_i, r_j) = \frac{1}{T-1} \sum_{t=1}^{T} (r_{i_t} - \bar{r_i})(r_{j_t} - \bar{r_j})$$

where r_{i_t} and r_{j_t} are the observed reurns values of r_i and r_j at timestamp t respectively. Moreover, $\bar{r_i}$ and $\bar{r_j}$ are the means of the historical values of r_i and r_j respectively, i.e.

$$\bar{r_i} = \frac{1}{T} \sum_{t=1}^{T} R_{i_t}, \quad \bar{r_i} = \frac{1}{T} \sum_{t=1}^{T} R_{j_t}$$

The Pandas library has a `cov.()` function that does this.

```
# Removing the last row as it contains NaN values due to the shift operation
returns_df_cleaned = returns_df.dropna()


# Calculating the covariance matrix using the .cov() function
covariance_matrix = returns_df_cleaned.cov()



#Displaying the covariance matrix
covariance_matrix
```

This computation yields:

$$\Sigma = \begin{pmatrix} 0.000315 & 0.000079 & 0.000124 & 0.000157 \\ 0.000079 & 0.000421 & 0.000107 & 0.000231 \\ 0.000124 & 0.000107 & 0.000885 & 0.000221 \\ 0.000157 & 0.000231 & 0.000221 & 0.000979 \end{pmatrix}$$

4.2 Computation of m