Discrete Mathematics and Logic (UE16CS205)

Unit 4 - Induction, Recursion and Recurrence Relations

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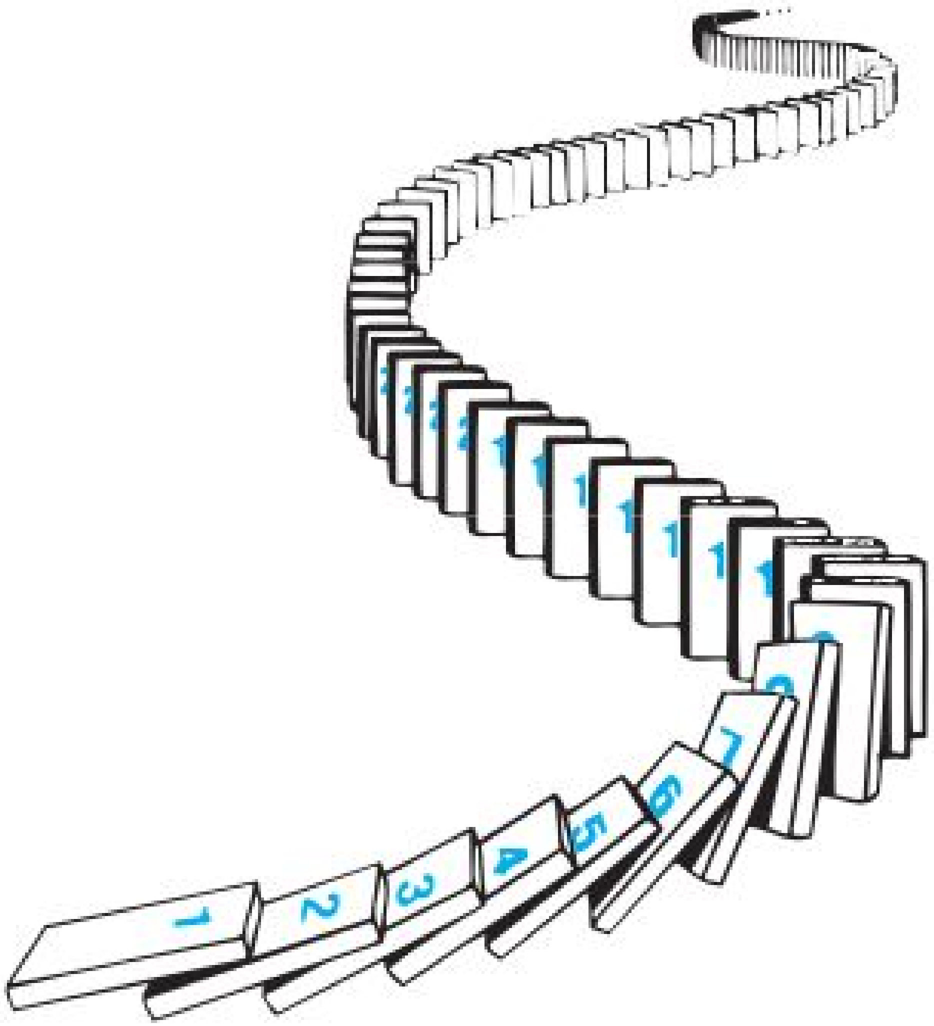
Mathematical Induction



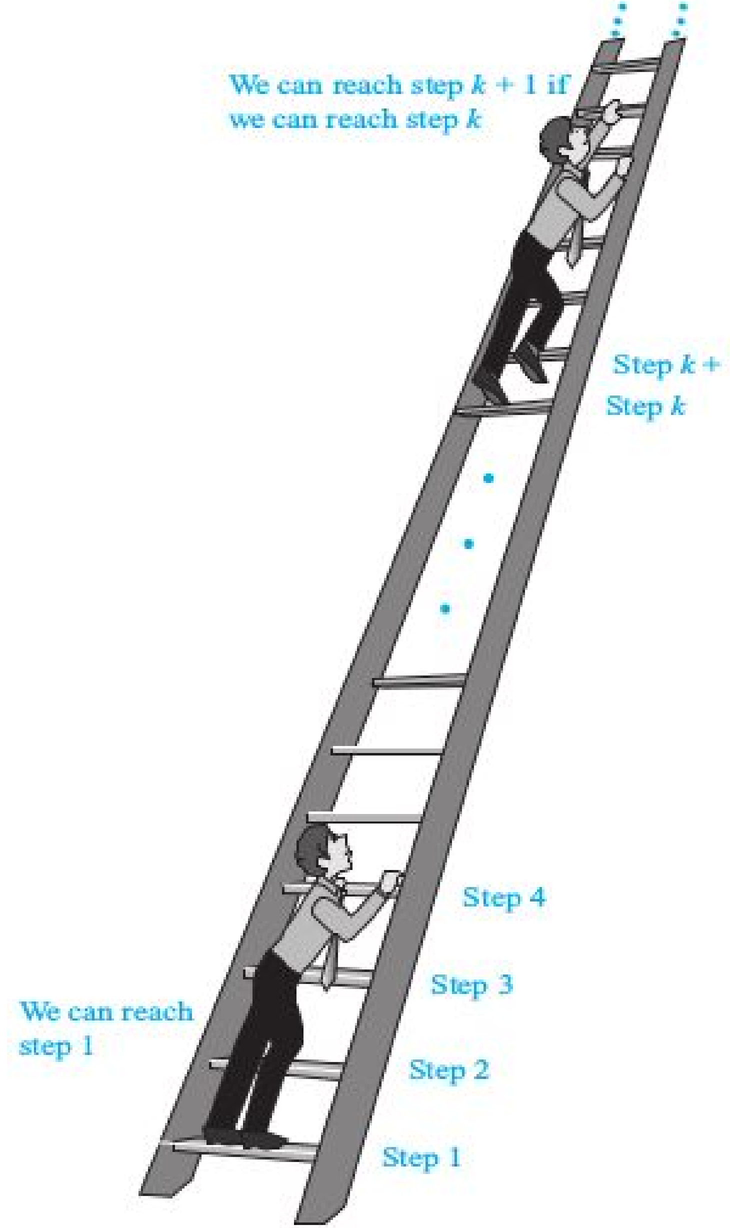
Illustrating How Mathematical Induction Works Using Dominoes.

Mathematical Induction

Suppose that we have an infinite ladder and we want to know whether we can reach every step on this ladder. We know two things:

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

Can we conclude that we can reach every rung?



Climbing an Infinite Ladder.

Mathematical Induction

By (1), we know that we can reach the first rung of the ladder. By (2), we can also reach the second rung because we can reach the first rung. Applying (2) again, we can also reach the third rung because we can reach the second rung. Continuing in this way, we can show that we can reach the fourth rung, the fifth rung, and so on. But can we conclude that we are able to reach every rung of this infinite ladder? The answer is yes, something we can verify using an important proof technique called **mathematical induction**.

That is, we can show that P(n) is true for every positive integer n, where P(n) is the statement that we can reach the n^{th} rung of the ladder.

Principle of Mathematical Induction:
To prove that P(n) is true for all positive integers n, where P(n) is a propositional function, we complete two steps:
Basis Step: We verify that P (1) is true.
Inductive Step: We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.

Expressed as a rule of inference, this proof technique can be stated as
$\mathbf{(P(1) \land \forall k\, (P(k) \to P(k+1))) \to \forall n\, P(n)}$,
when the domain is the set of positive integers.

Why is mathematical induction a valid proof technique? The well-ordering property as an axiom for the set of +ve integers states that every nonempty subset of the set of +ve integers has a least element. So, suppose we know that P(1) is true and P(k) → P(k+1) is true for all +ve integers k. To show that P(n) must be true for all +ve integers n, assume that there is at least one +ve integer for which P(n) is false. Then the set S of +ve integers for which P(n) is false is nonempty. Thus, by the well-ordering property, S has a least element, which will be denoted by m. We know that m cannot be 1, because P(1) is true. Because m is +ve and greater than 1, m−1 is a +ve integer. Furthermore, because m−1 is less than m, it is not in S, so P (m−1) must be true. Because the conditional statement P(m−1) → P(m) is also true, it must be the case that P(m) is true. This contradicts the choice of m. Hence, P(n) must be true for every positive integer n.

Eg: Show that if n is a positive integer, then
$\mathbf{1 + 2 + \cdots + n = n(n+1)/2}$

Let P(n) be the proposition that the sum of the first n positive integers, $1 + 2 + \cdot \cdot \cdot n = n(n+1)/2$.
We must do two things to prove that P(n) is true. Namely, we must show that P(1) is true and that the conditional statement P(k) implies P(k+1) is true for k = 1, 2, 3, … .

BASIS STEP: P (1) is true, because $1 = 1(1+1)/2$.

INDUCTIVE STEP: For the inductive hypothesis we assume that P(k) holds for an arbitrary positive integer k. That is, we assume that $1 + 2 + \cdots + k = k(k+1)/2$.

Eg: Show that if n is a positive integer, then

$1 + 2 + \cdots + n = n(n+1)/2$

…

Under this assumption, it must be shown that P(k+1) is true, namely, that

$1 + 2 + \cdot \cdot \cdot + k + (k + 1) = (k + 1)(k + 2)/2$ is true.

When we add k + 1 to both sides of the equation in P(k), we obtain

$1 + 2 + \cdot \cdot \cdot + k + (k + 1) = k(k+1)/2 + (k + 1)$

$= (k(k+1)+(2k+2)) / 2$

$= (k+1)(k+2)/2$

It shows that P(k+1) is true under the assumption that P(k) is true. This completes the inductive step.

So by mathematical induction we know that P(n) is true for all positive integers n.

Eg: Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathematical induction.

Conjecture:
Sum of the first n positive odd integers is n^2.
$1 + 3 + 5 + \ldots + (2n-1) = n^2$

Eg: Use mathematical induction to show that
$1 + 2 + 2^2 + \cdot \cdot \cdot + 2^n = 2^{n+1} - 1$
for all nonnegative integers n.

Eg: Sums of Geometric Progressions Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression with initial term a and common ratio r:

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1} \qquad \text{when } r \neq 1,$$

where n is a nonnegative integer.

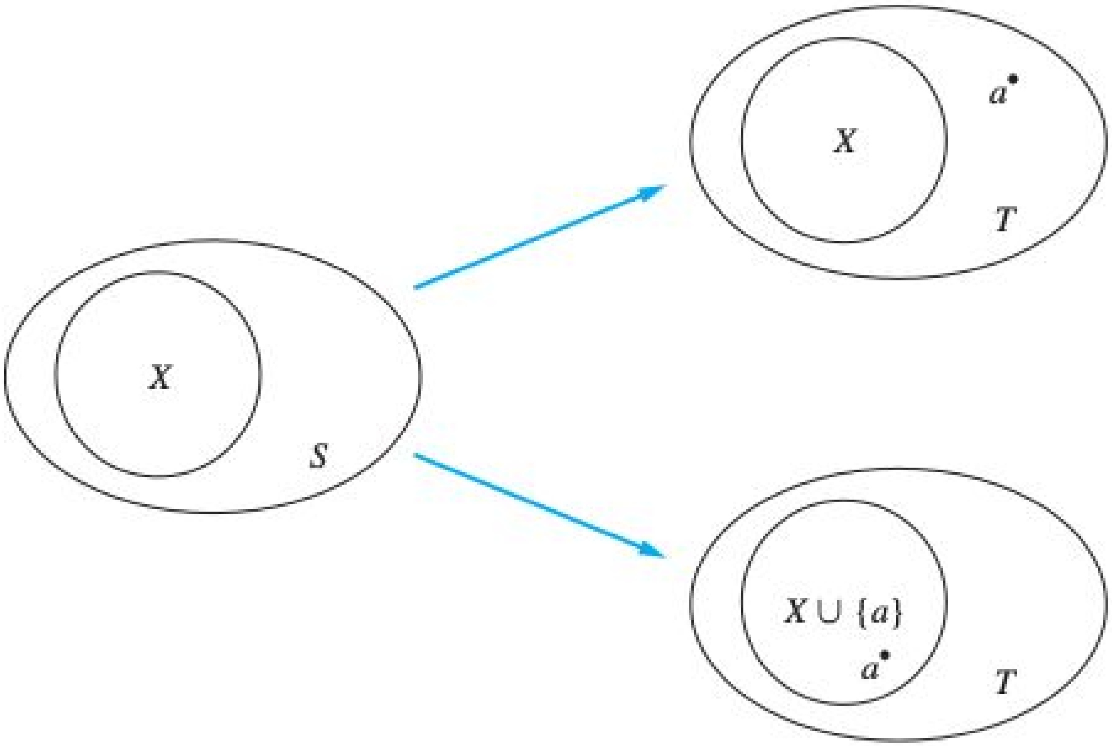
Eg: Use mathematical induction to prove the inequality $n < 2^n$ for all positive integers n.

Eg: Use mathematical induction to prove that $2^n < n!$ for every integer n with $n \geq 4$. (Note that this inequality is false for $n = 1, 2$, and 3.)

Eg: Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

Eg: Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer n.

Eg: The Number of Subsets of a Finite Set Use mathematical induction to show that if S is a finite set with n elements, where n is a nonnegative integer, then S has 2^n subsets.



Generating Subsets of a Set with $k + 1$ Elements. Here $T = S \cup \{a\}$.

Eg: Use mathematical induction to prove the following generalization of one of De Morgan's laws:

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

whenever A_1 , A_2 , … , A_n are subsets of a universal set U and n ≥ 2.

Template for Proofs by Mathematical Induction

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.
2. Write out the words "Basis Step." Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words "Inductive Step."
4. State, and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.
6. Prove the statement $P(k+1)$ making use the assumption $P(k)$. Be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.
7. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."
8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers n with $n \geq b$.

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[OPTIONAL] Eg: The input to a scheduling algorithm is a group of m proposed talks with preset starting and ending times. The goal is to schedule as many of these lectures as possible in the main lecture hall so that no two talks overlap. Suppose that talk t_j begins at time s_j and ends at time e_j. (No two lectures can proceed in the main lecture hall at the same time, but a lecture in this hall can begin at the same time another one ends.)

Eg: … scheduling algorithm …

Sort the talks in nondecreasing order of their ending times: $e_1 \leq e_2 \leq \ldots \leq e_m$.

The greedy algorithm starts by selecting a talk with the earliest end time and then proceeds by selecting at each stage a talk with the earliest ending time among all those talks that begin no sooner than when the last talk scheduled in the main lecture hall has ended. Using mathematical induction, show that this greedy algorithm is optimal in the sense that it always schedules the most talks possible in the main lecture hall.

Eg: … scheduling algorithm …

Let P(n) be the proposition that if the greedy algorithm schedules n talks in the main lecture hall, then it is not possible to schedule more than n talks
in this hall.

BASIS STEP: Suppose that the greedy algorithm managed to schedule just one talk, t_1 , in the main lecture hall. Every pair of talks are overlapping and hence only one talk can be scheduled.

INDUCTIVE STEP: The inductive hypothesis is that P(k) is true that the greedy algorithm always schedules the most possible talks when it selects k talks given any set of talks.

Eg: … scheduling algorithm …

We must show that P(k+1) follows from the assumption that P(k) is true, that is, we must show that under the assumption of P(k), the greedy algorithm always schedules the most possible talks when it selects k+1 talks.

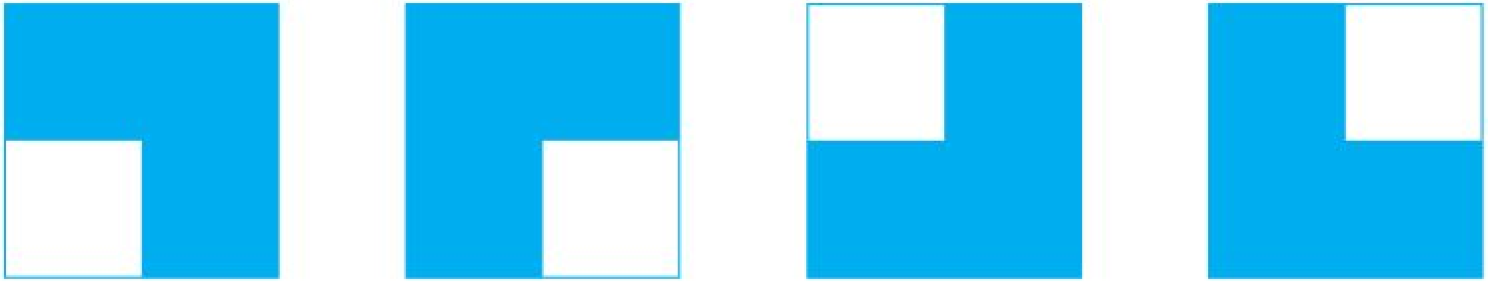
Our first step in completing the inductive step is to show there is a schedule including the most talks possible that contains talk t1, a talk with the earliest end time. It's easier to show t1 is in at least one of the schedules having k+1 talks, and it's the first task in the schedule. A schedule that begins with the talk ti in the list, where i>1, can be changed so that talk t1 replaces talk ti. To see this, note that because $e1 \leq ei$, all talks that were scheduled to follow talk ti can still be scheduled.

Eg: … scheduling algorithm …

Once we included talk t1, scheduling the talks so that as many as possible are scheduled is reduced to scheduling as many talks as possible that begin at or after time e1. Because the greedy algorithm schedules k talks when it creates this schedule, we can apply the inductive hypothesis to conclude that it has scheduled the most possible talks. So P(k+1) is true.

So, by mathematical induction we know that P(n) is true for all positive integers n. That is, we have proved that when the greedy algorithm schedules n talks, when n is a positive integer, then it is not possible to schedule more than n talks.

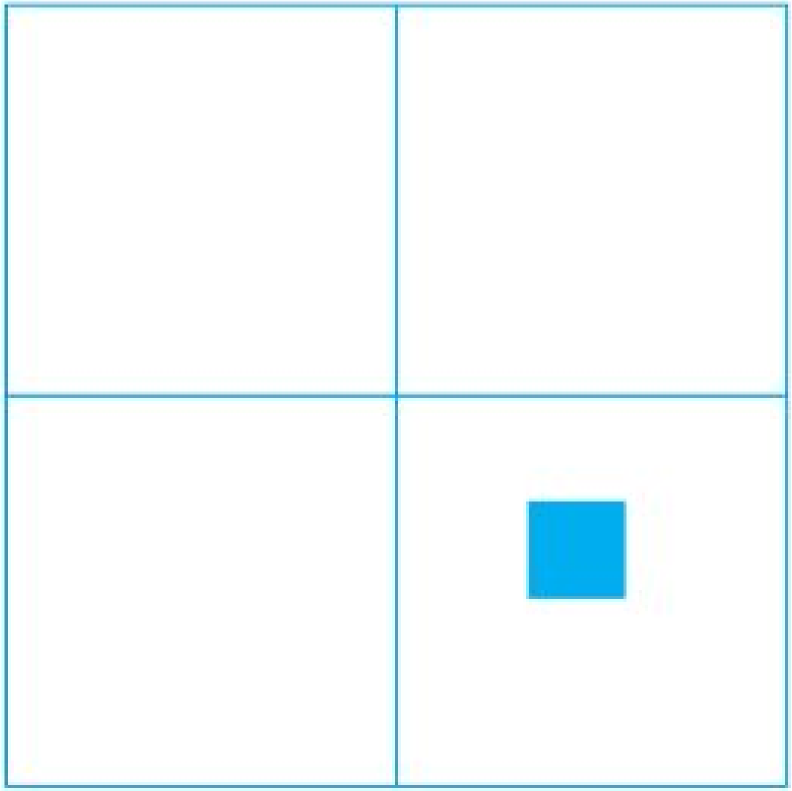
[OPTIONAL] Eg: Let n be a positive integer. Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes, where these pieces cover three squares at a time, as shown in the figure.



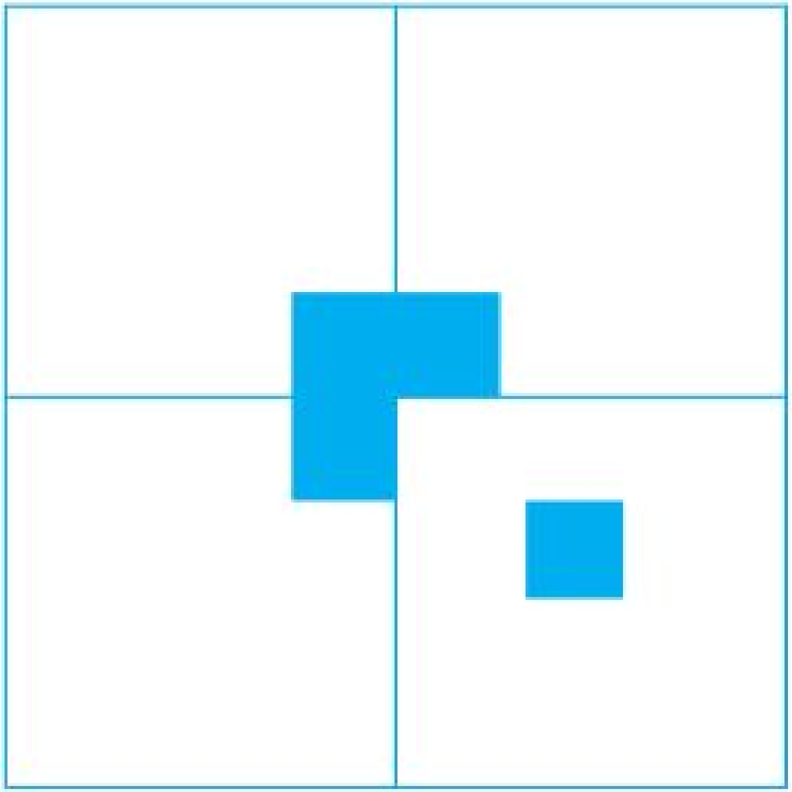
Tiling 2 × 2 Checkerboards with One Square Removed.

Solution: Let P(n) be the proposition that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes.

BASIS STEP: P(1) is true, because each of the four 2×2 checkerboards with one square removed can be tiled using one right triomino, as shown in the figure.



Dividing a $2^{k+1} \times 2^{k+1}$ Checkerboard into Four $2^k \times 2^k$ Checkerboards.



Tiling the $2^{k+1} \times 2^{k+1}$ Checkerboard with One Square Removed.

Strong Induction:

To prove that P(n) is true for all positive integers n, where P(n) is a propositional function, we complete two steps:

BASIS STEP:
We verify that the proposition P(1) is true.

INDUCTIVE STEP:
We show that the conditional statement
$$[P(1) \land P(2) \land \cdots \land P(k)] \rightarrow P(k+1)$$
is true for all positive integers k.

Note that when we use strong induction to prove that $P(n)$ is true for all positive integers n, our inductive hypothesis is the assumption that $P(j)$ is true for $j = 1, 2, \ldots, k$. That is, the inductive hypothesis includes all k statements $P(1), P(2), \ldots, P(k)$. Because we can use all k statements $P(1), P(2), \ldots, P(k)$ to prove $P(k+1)$, rather than just the statement $P(k)$ as in a proof by mathematical induction, strong induction is a more flexible proof technique.

Mathematical induction and strong induction are equivalent. That is, each can be shown to be a valid proof technique assuming that the other is valid. In particular, any proof using mathematical induction can also be considered to be a proof by strong induction.

Eg: Suppose we can reach the first and second rungs of an infinite ladder, and we know that if we can reach a rung, then we can reach two rungs higher. Can we prove that we can reach every rung using strong induction?

BASIS STEP: It simply verifies that we can reach the first two rungs.
INDUCTIVE STEP: The inductive hypothesis states that we can reach each of the first k rungs. We can complete the inductive step by noting that as long as $k \geq 2$, we can reach the $(k+1)^{st}$ rung from the $(k-1)^{st}$ rung because we know we can climb two rungs from a rung we can already reach, and because $k-1 \leq k$, by the inductive hypothesis we can reach the $(k-1)^{st}$ rung. This completes the inductive step and finishes the proof by strong induction.

Eg: Show that if n is an integer greater than 1, then n can be written as the product of primes.

Let P(n) be the proposition that n can be written as the product of primes.

BASIS STEP: P(2) is true, because 2 can be written as the product of one prime, itself.

INDUCTIVE STEP: The inductive hypothesis is the assumption that P(j) is true for all integers j with $2 \le j \le k$.
To complete the inductive step, it must be shown that P(k+1) is true under this assumption.
There are two cases to consider, namely, when k+1 is prime and when k+1 is composite.

If k+1 is prime, we immediately see that P(k+1) is true. Otherwise, k+1 is composite and can be written as the product of two positive integers a and b with $2 \le a \le b < k+1$. Because both a and b are integers at least 2 and not exceeding k, we can use the inductive hypothesis to write both a and b as the product of primes. Thus, if k+1 is composite, it can be written as the product of primes, namely, those primes in the factorization of a and those in the factorization of b.

Eg: Consider a game in which two players take turns removing any positive number of matches they want from one of two piles of matches. The player who removes the last match wins the game. Show that if the two piles contain the same number of matches initially, the second player can always guarantee a win.

Solution: Let n be the number of matches in each pile. Let P(n), the statement that the second player can win when there are initially n matches in each pile.

BASIS STEP: …

INDUCTIVE STEP: …

Eg: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution: Let $P(n)$ be the statement that postage of n cents can be formed using 4-cent and 5-cent stamps.

BASIS STEP: …

INDUCTIVE STEP: …

Recursion is the process of defining an object in terms of itself.

Recursively Defined Functions
We use two steps to define a function with the set of nonnegative integers as its domain:

BASIS STEP: Specify the value of the function at zero.

RECURSIVE STEP: Give a rule for finding
its value at an integer **from**
its values at smaller integers.

Eg: Recursive definition of sum of first n natural numbers.
BASIS STEP: $f(0) = 0$.
RECURSIVE STEP: $f(n) = f(n-1) + n$.

Eg: Fibonacci numbers, f_0, f_1, f_2, . . . , are defined by

BASIS STEP: $f_0 = 0$, $f_1 = 1$.

RECURSIVE STEP: $f_n = f_{n-1} + f_{n-2}$.

Eg: Give a recursive definition of $\mathbf{a^n}$, where **a** is a nonzero real number and **n** is a nonnegative integer.

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BASIS STEP: $a^0 = 1$.

RECURSIVE STEP: $a^n = a * a^{n-1}$.

Eg: Give a recursive definition of

$$\sum_{k=0}^{n} a_k .$$

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$$\sum_{k=0}^{n} a_k.$$

BASIS STEP:

$$\sum_{k=0}^{0} a_k = a_0.$$

RECURSIVE STEP:

$$\sum_{k=0}^{n+1} a_k = \left(\sum_{k=0}^{n} a_k\right) + a_{n+1}.$$

Recursive algorithms
An algorithm is called recursive if it solves a problem by reducing it to an instance of the same problem with smaller input.

Eg: Give a recursive definition and recursive algorithm for computing n!, where n is a nonnegative integer.

Recursive definition: $0! = 1$ and $n! = n * (n-1)!$

```
procedure factorial(n: nonnegative integer)
if n = 0 then return 1
else return n * factorial(n−1)
```

Eg: Give a recursive definition and a recursive algorithm for computing the greatest common divisor of two nonnegative integers a and b with b < a using Euclid's method.

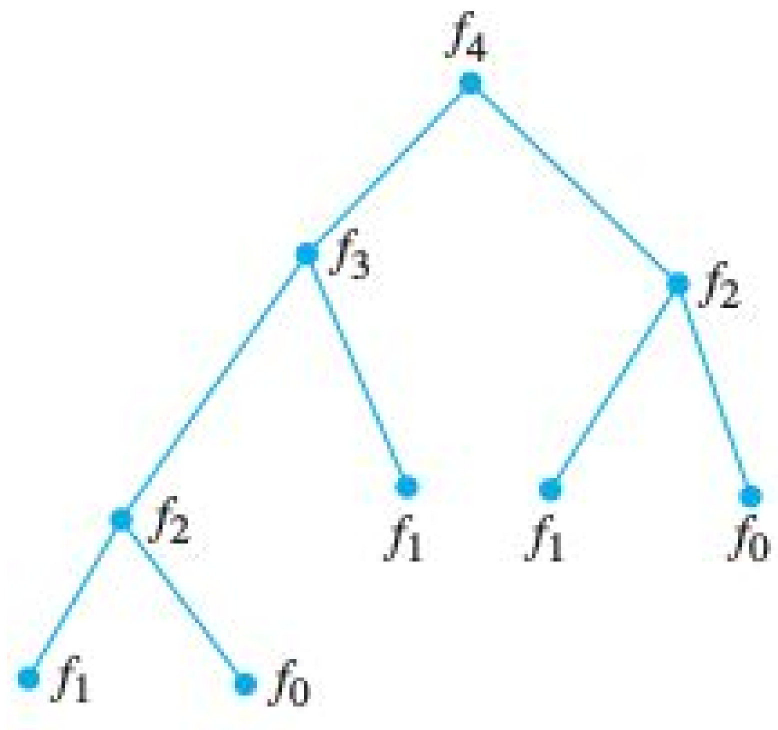
Recursive definition:
gcd(a, 0) = a, and
gcd(a, b) = gcd(b, a mod b)

```
procedure gcd(a, b: nonnegative integers
                    with b < a)
if b = 0 then return a
else return gcd(b, a mod b)
```

Eg: Give a recursive definition and a recursive algorithm for finding nth number in the fibonacci sequence.

Recursive definition:
f(0) = 0, f(1) = 1, and
f(n) = f(n-1) + f(n-2).

```
procedure f(n: nonnegative
               integer)
if n = 0 then return 0
else if n = 1 then return 1
else return f(n-1) + f(n-2)
```



Recurrence Relations

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, a_{n0}, a_{n1}, …, a_{n-1}, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

Eg: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, \ldots$, and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_5?

Solution:

$a_2 = 5 - 3 = 2$

$a_3 = 2 - 5 = -3$

$a_4 = -3 - 2 = -5$

$a_5 = -5 - (-3) = -2$

Eg: Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for n = 2, 3, 4, ... Answer the same question where

(b) $a_n = 2^n$ and

(c) $a_n = 5$.

Solution: True because $2(3(n-1))-3(n-2) = 3n$

(b) False because $2(2^{n-1})-2^{n-2} = 2^n$ is false.

(c) True because $2(5) - 5 = 5$.

Fibonacci numbers:

$f_n = f_{n-1} + f_{n-2}$, where $f_0 = 0$ and $f_1 = 1$.

Eg: Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive 0s. How many such bit strings are there of length eight.

Solution: …

Eg: Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive 0s. How many such bit strings are there of length eight.

Solution: $a_n = a_{n-1} + a_{n-2}$, where $a_1 = 2$ and $a_2 = 3$.

2, 3, 5, 8, 13, 21, 34, 55.

55 bit strings of length 8 don't have 2 consecutive 0s.

The Tower of Hanoi puzzle:

Move n disks from peg A to peg B using peg C.

Move top n-1 disk from peg A to C.

Move disk# n from peg A to B.

Move n-1 disks from peg C to B

# of moves, $H_n=2H_{n-1}+1$ with the initial condition $H_1=1$.

Power(x, y) = **x * Power(x, y-1)**, where $x \in R$, $y \in Z^+$ with the initial condition Power(x, 0) = 1.

Factorial(n) = **n * Factorial(n-1)**, where $n \in Z^+$ and Factorial(0) = 1.

Eg: Suppose that a person deposits Rs. 10,000 in a savings account at a bank yielding 11% per annum with interest compounded annually. How much will be in the account after 30 years?

$P_n = 1.11 * P_{n-1}$, P_0 = Rs. 10,000

P_{30} = Rs. 228,922

Solving Recurrence Relations

Finding solution to a recurrence relation means finding "**closed form expressions**" equivalent to the recurrence. That is as good as generating the sequence.

Q: Solve $a_n = a_{n-1} + C$, where a_0 is a constant
Solution: $\mathbf{a_n = C*n + a_0}$

Q: Solve $a_n = Ca_{n-1}$, where a_0 is a constant
Solution: $\mathbf{a_n = C^n * a_0}$

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

Linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where $c_1, c_2, c_3, \ldots, c_k$ are reals and $c_k \neq 0$.

Linear Homogeneous Recurrence Relations with Constant Coefficients

$\mathbf{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}}$

- $\mathbf{a_n = 2a_{n-1} + 3}$ is not homogeneous (not all terms has a factor a_j)
- $\mathbf{a_n = na_{n-1}}$ coefficient not constant
- $\mathbf{a_n = 3a^2_{n-1}}$ is not linear
- $\mathbf{a_n = a_{n-1} + a_{n-2}}$ is of degree 2
- $\mathbf{a_n = 4a_{n-2}}$ is of degree 2
- $\mathbf{a_n = 3a_{n-5} - 4a_{n-2}}$ is of degree 5

Linear Homogenous Recurrence Relation with constant coefficients occur often in modeling problems and can be systematically solved.

Theorem 1 - **For degree 2 and distinct roots r_1 and r_2**

$\{a_n\}$ is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

iff $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots,$

where $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2,

with constants α_1 and α_2.

$\{a_n\}$ is a solution of $a_n = c_1a_{n-1} + c_2a_{n-2}$

iff $a_n = \alpha_1 r_1^{\ n} + \alpha_2 r_2^{\ n}$ for n = 0, 1, 2, ...,

where $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2,

with constants α_1 and α_2

Recurrence: $a_n = c_1a_{n-1} + c_2a_{n-2}$

Characteristic equation: $r^2 - c_1 r - c_2 = 0$

Characteristic roots: r_1 and r_2 (distinct roots)

Solution for the recurrence: $a_n = \alpha_1 r_1^{\ n} + \alpha_2 r_2^{\ n}$ for n = 0,1,2,...,

Constants: α_1 and α_1 are constants, which can be

found using initial conditions a_0 and a_1.

Q: What is the solution (closed form) of
$a_n = a_{n-1} + 2a_{n-2}$ where $a_0 = 2$, $a_1 = 7$

Soln: $c_1 = 1$, $c_2 = 2$

$\mathbf{r^2 - c_1 r - c_2 = 0}$ is the characteristic equation

$r^2 - r - 2 = (r+1)(r-2) = 0$
$\mathbf{r_1 = 2, r_2 = -1}$ are the characteristic roots

Using initial conditions and roots, solve
$a_n = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n}$ for α_1 and α_2

$a_n = \alpha_1 2^n + \alpha_2(-1)^n$

$a_n = \alpha_1 2^n + \alpha_2(-1)^n$
for constants α_1 and α_2

$a_0 = 2$

$= \alpha_1 r_1^{\,0} + \alpha_2 r_2^{\,0}$

$= \alpha_1 2^0 + \alpha_2(-1)^0$

$= \alpha_1 + \alpha_2$

$\boldsymbol{\alpha_1 + \alpha_2 = 2}$

$a_1 = 7$

$= \alpha_1 r_1^{\,1} + \alpha_2 r_2^{\,1}$

$= \alpha_1 2^1 + \alpha_2(-1)^1$

$= 2\alpha_1 - \alpha_2$

$\boldsymbol{2\alpha_1 - \alpha_2 = 7}$

Solving simultaneous equations

$\alpha_1 + \alpha_2 = 2$

$2\alpha_1 - \alpha_2 = 7$

$\alpha_1 = 3$, $\alpha_2 = -1$

$a_n = \alpha_1 2^n + \alpha_2(-1)^n$

$\mathbf{a_n = 3*2^n + (-1)^{n+1}}$

is the solution (closed form) for

$a_n = a_{n-1} + 2a_{n-2}$ where $a_0 = 2$, $a_1 = 7$

Q: Find the solution (closed form) of $a_n = 7a_{n-1} - 10a_{n-2}$ where $a_0 = 2$, $a_1 = 1$

Soln: ...

Q: Find the solution (closed form) of $f_n = f_{n-1} + f_{n-2}$ where $f_0 = 0$, $f_1 = 1$

Soln: ...

Q: Find the solution (closed form) of $a_n = 2a_{n-1}$ where $a_0 = 3$

Soln: …

(Hint: it’s of degree 1, and it’s of a known form)

Q: Find the solution (closed form) of $a_n = 2a_{n-1}$ where $a_0 = 3$

Soln: This is a known form.

The solution of $a_n = Ca_{n-1}$ would be $\mathbf{a_0 * C^n}$

Therefore, solution of $a_n = 2a_{n-1}$ is $\mathbf{3 * 2^n}$

Alternate Soln: By substitution.

Q: Find the solution (closed form) of $a_n = 2a_{n-1}$ where $a_0 = 3$

Soln: By substitution

$a_n = 2a_{n-1}$ where $a_0 = 3$

$a_n = 2a_{n-1}$

$= 2(2a_{n-2}) = 2^2 a_{n-2}$

$= 2^2(2a_{n-3}) = 2^3 a_{n-3}$

$= 2^i a_{n-i}$

When n-i=0, i=n

$= 2^n a_0$

$= 3*2^n$

Q: What is the solution (closed form) of

$a_n = 6a_{n-1} - 9a_{n-2}$ where $a_0 = 1$, $a_1 = 6$

Soln: $c_1 = 6$, $c_2 = -9$

$r^2 - 6r + 9 = (r-3)(r-3) = 0$

$r_1 = 3$, $r_2 = 3$

but the roots are not distinct…

Theorem 2 - For degree 2 and 1 root (multiplicity 2)

$\{a_n\}$ is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

iff $a_n = \alpha_1 r_0^n + \alpha_2 \mathbf{n} r_0^n$ for $n=0,1,2,...,$

where $r^2 - c_1 r - c_2 = 0$ has one root r_0,

with constants α_1 and α_2.

Q: What is the solution (closed form) of

$a_n = 6a_{n-1} - 9a_{n-2}$ where $a_0 = 1$, $a_1 = 6$

Soln: $c_1 = 6$, $c_2 = -9$

$r^2 - 6r + 9 = (r-3)(r-3) = 0$

$r_0 = 3$

$a_n = \alpha_1 r_0^{\,n} + \alpha_2 n r_0^{\,n} = \alpha_1 3^n + \alpha_2 n 3^n$

for constants α_1 and α_2

$a_n = \alpha_1 r_0^{\,n} + \alpha_2 n r_0^{\,n}$

$= \alpha_1 3^n + \alpha_2 n 3^n$

$a_n = 3^n + n3^n$

Theorem 3 - For degree k and distinct roots

$\{a_n\}$ is a solution of $a_n = c_1a_{n-1} + c_2a_{n-2} + .. + c_ka_{n-k}$

iff $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + .. + \alpha_k r_k^n$ for n=0, 1, 2, ...,

where $r^k - c_1 r^{k-1} - .. - c_k = 0$ has k distinct roots

$r_1, r_2, .. r_k$, with constants $\alpha_1, \alpha_2, .. , \alpha_k$

Q: What is the solution (closed form) of

$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ where $a_0 = 2$, $a_1 = 5$, $a_2 = 15$

Soln: …

Theorem 4 Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then a sequence $\{a_n\}$is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = {} & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Suppose that the roots of the characteristic equation of a linear homogeneous recurrence relation are 2, 2, 2, 5, 5, and 9. The form of the general solution would be:

$$
\begin{aligned}
a_n = {} & (\boldsymbol{\alpha}_{1,0} + \boldsymbol{\alpha}_{1,1}\, n + \boldsymbol{\alpha}_{1,2}\, n^2)\, 2^n + \\
& (\boldsymbol{\alpha}_{2,0} + \boldsymbol{\alpha}_{2,1}\, n)\, 5^n + \\
& \boldsymbol{\alpha}_{3,0}\, 9^n.
\end{aligned}
$$

Q: What is the solution (closed form) of

$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ where $a_0 = 1$, $a_1 = -2$, $a_2 = -1$

Soln: …

Eg: How many bit strings of length n are there that **do not contain two consecutive 0s**?

Eg: How many bit strings of length n are there that **contain two consecutive 0s**?

Eg: How many bit strings of length n are there that contain **three** consecutive 0s?

Eg: How many bit strings of length n are there that **do not contain two consecutive 0s**?

There are only two possibilities; n^{th} bit is 0 or 1.

f(n) = the case where n^{th} bit 1 + the case where n^{th} bit is 0

When n^{th} bit is 1, there are f(n-1) bit strings.

When n^{th} bit is 0, $(n-1)^{th}$ bit has to be 1, and hence there are f(n-2) bit strings.

∴ $f(n) = f(n-1) + f(n-2)$, where $f(1) = 2$ and $f(2) = 3$.

Q: Given infinite number of red and blue balls, how many ways are there to arrange n balls such that no two red balls are adjacent?
OR How many bit strings of length n are there that **do not contain two consecutive 0s**?

To find f(n).
n^{th} ball is blue or red.
If n^{th} ball is blue, the count is f(n-1).
If n^{th} ball is red, $(n-1)^{th}$ ball has to be blue, and hence the count is f(n-2).
$\therefore$ f(n) = f(n-1) + f(n-2), f(1) = 2, f(2) = 3.

Eg: How many bit strings of length n are there that **contain two consecutive 0s**?

To find f(n).

When last bit is 1, there are f(n-1) bit strings.

When last two bits are 01, there are f(n-2) bit strings.

When last two bits are 00, there are 2^{n-2} bit strings.

$\therefore$ $f(n)= f(n-1) + f(n-2) + 2^{n-2}$, where $f(1) = 0$ and $f(2) = 1$.

Also, $f(n) = 2^n - f(n-1) - f(n-2)$

$2^n - 2^{n-2} = 2f(n-1) + 2f(n-2)$

$(3/2)2^{n-2} = f(n-1) + f(n-2)$ = "do not contain 2 consec. 0s."

Eg: How many bit strings of length n are there that **contain three consecutive 0s**?

To find f(n).

When 1st bit is 1, there are f(n-1) bit strings.

When 1st two bits are 01, there are f(n-2) bit strings.

When 1st three bits are 001, there are f(n-3) bit strings.

When 1st three bits are 000, there are 2^{n-3} bit strings.

$\therefore f(n)=f(n-1)+f(n-2)+f(n-3)+ 2^{n-3}$, where $f(1)=f(2)=0$ & $f(3)=1$.

< End of
Mathematical Induction,
Recursion and Recurrence Relations
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