Discrete Mathematics and Logic (UE16CS205)

Unit 3 - Sets, Functions and Relations

Channa Bankapur (channabankapur@pes.edu),
Nirupama MP, Ramadevi, Sangeetha V, Chandana M

Sets

- A **set** is an unordered collection of objects.
- Sets are discrete structures used to group objects together, often the objects having similar properties.

- The objects in a set are called the **elements** or **members** of the set.
- A set is said to **contain** its elements.
- An element is said to **belong** to the set.

- $\mathbf{a \in S}$ denotes that "a" is an element of the set S.
- $\mathbf{a \notin S}$ denotes that "a" is not an element of the set S.

Set roster form: All the members of the set are listed separated by commas enclosed in curly braces.

Eg: A set of natural numbers from 1 to 5.

$S = \{1, 2, 3, 4, 5\}$

Eg: the set of positive integers less than hundred

$S = \{1, 2, 3, \ldots, 99\}$

Set builder form: Elements in the set are characterized by stating the property or properties they must have to be members.

Eg: the set of positive integers less than hundred

$A = \{ x \in \mathbf{Z}^+ \mid x < 100\}$

Eg: set of rational numbers.

$B = \{ p/q \mid p \in \mathbf{Z}; q \in \mathbf{Z}, \text{ and } q \neq 0\}$

Well known number sets:

- **N** = {0, 1, 2, 3, ...}, the set of natural numbers
- **Z** = {..., -2, -1, 0, 1, 2, ...}, the set of integers
- $\mathbf{Z}^+$ = {1, 2, 3, 4, ...}, set of positive integers
- **Q** = {p/q | p $\in$ **Z ;** q $\in$ **Z** and q 0} the set of rational numbers
- **R** the set of real numbers
- **C** the set of complex numbers

Equality of two sets

Two sets are equal if and only if they have the same elements.

i.e., if A and B are sets, then A and B are equal if and only if $\forall x\, (x \in A \leftrightarrow x \in B)$.

if A and B are equal sets, it can be denoted as **A = B**.

Eg: $\{a, b\} = \{a, b\}$

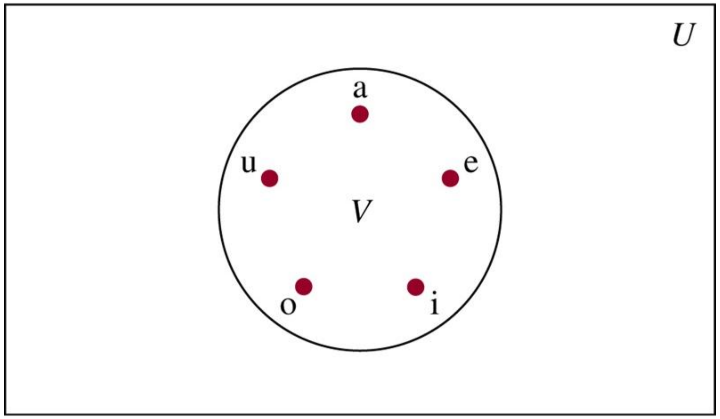
Eg: $\{1,3,5\} = \{3,1,5\}$

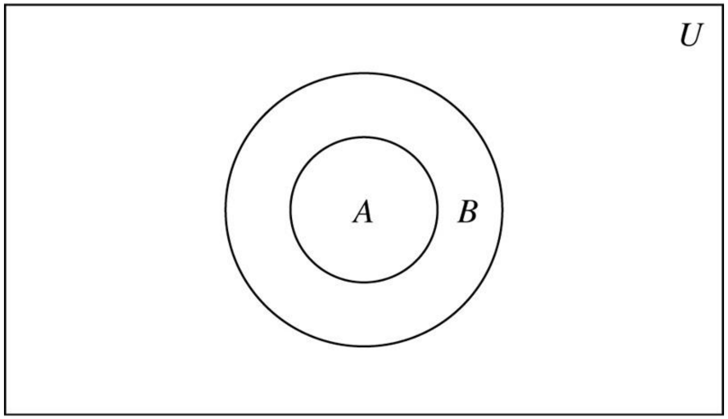
Eg: $\{1,3,3,5,5,5,5\} = \{1,3,5\}$

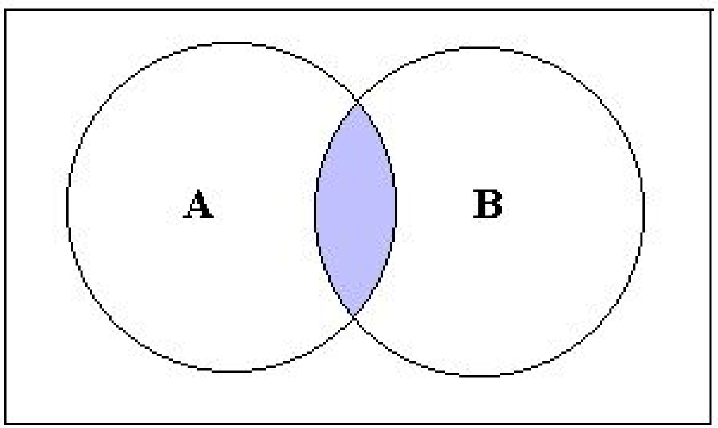
Eg: $\{1\} \neq \{\{1\}\}$

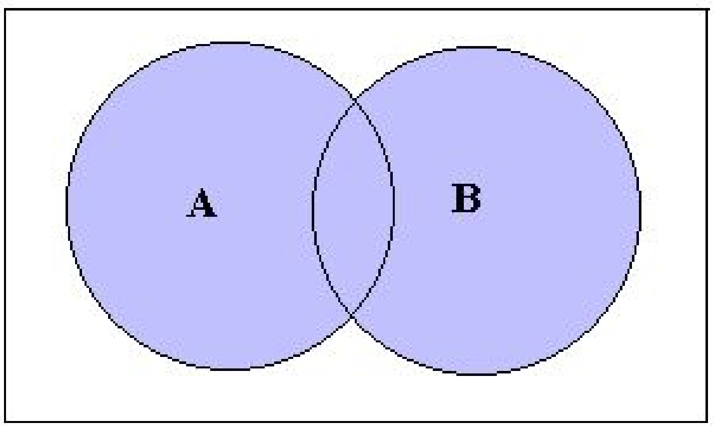
Venn Diagram

Sets can be represented graphically using Venn diagrams.









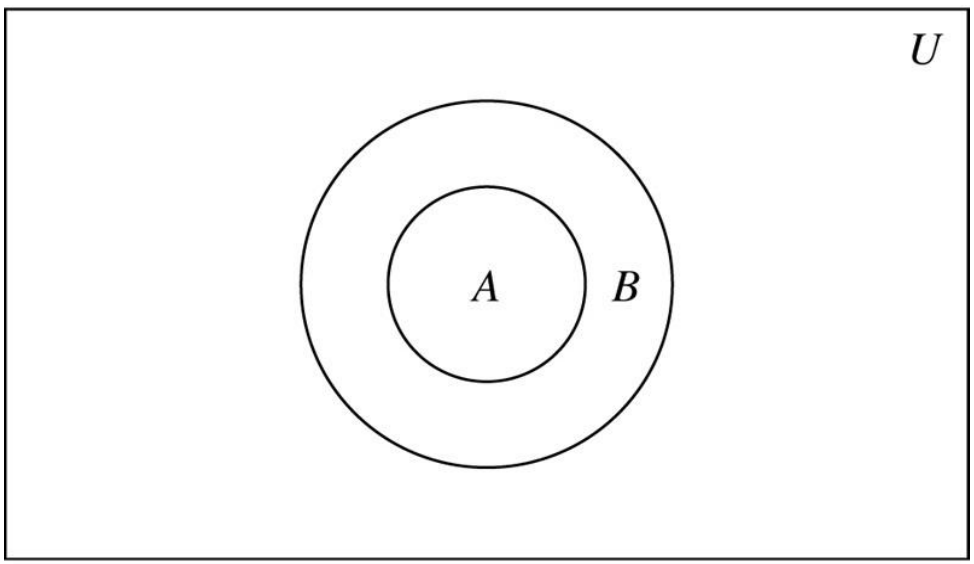
- Empty set / Null set
  - $\{\ \} = \Phi$

- Singleton set
  - $\{ a \}$, $\{ Z^+ \}$

- $\Phi \neq \{\Phi\}$

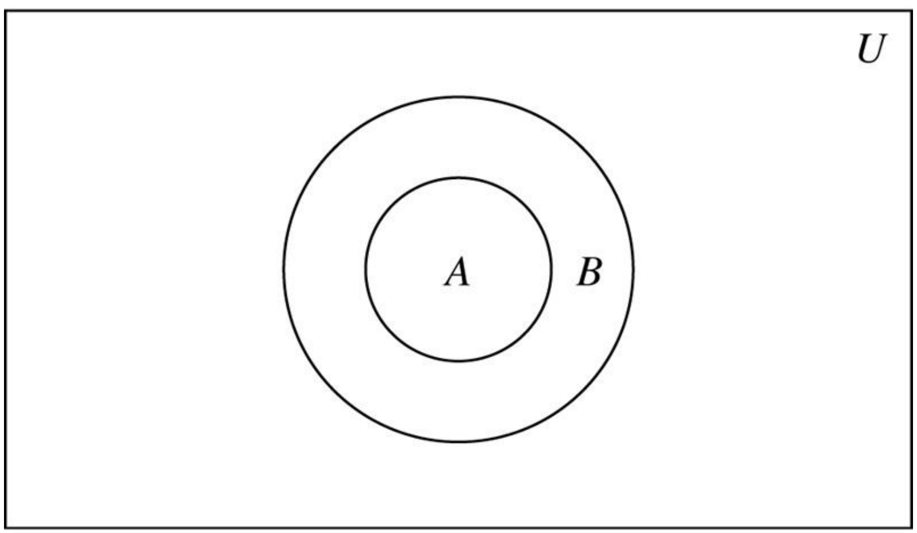
Subsets:

- The set A is called a subset of set B if and only if every element of A is also an element of B.
- We use the notation A ⊆ B to indicate that A is a subset of set B.
- A ⊆ B if and only if **$\forall x\ (x \in A \rightarrow x \in B)$** is true.



Proper Subset:

- When set A is a subset of set B but $A \neq B$, we write $A \subset B$ and say that A is a **proper subset** of B.
- For $A \subset B$ to be true it must be the case that $A \subseteq B$ and there must exist an element x of B that is not an element of A.
- That is, A is a proper subset of B if
  $\forall x\ (x \in A \rightarrow x \in B) \land \exists x\ (x \in B \land x \notin A)$ is true.



Theorem: For every set S,

1. $\varnothing \subseteq S$ 2. $S \subseteq S$

Proof of $\varnothing \subseteq S$:

Let S be a set.

$\varnothing \subseteq S$ iff $\forall x(x \in \varnothing \rightarrow x \in S)$.

Because the empty set contains no elements, it follows that $x \in \varnothing$ is always false.

It follows that the conditional statement $x \in \varnothing \rightarrow x \in S$ is always true, because the hypothesis is always false and a conditional statement with a false hypothesis is true.

That is $\forall x(x \in \varnothing \rightarrow x \in S)$ is true.

$\therefore \varnothing \subseteq S$

It's a vacuous proof.

Cardinality: If there are exactly **n** distinct elements in S where **n** is a non-negative integer, we say that S is a **finite set** and that **n** is the **cardinality** of S. The cardinality of S is denoted by **|S|**.

Eg: Set S of letters in the English alphabet. Then |S| = 26.

Eg: $|\varnothing| = 0$.

A set is said to be **infinite,** if it is not finite.

Eg: The set of positive integers {1, 2, 3, …} is infinite.

Power Set of S, P(S), is the set of all subsets of the set S.

If a set has **n** elements, then its power set has $\mathbf{2^n}$ elements.

Eg: $P(\{0,1,2\}) = \{\varnothing, \{0\}, \{1\}, \{2\}, \{0,1\}, \{1,2\}, \{0,2\}, \{0,1,2\})$

The **Ordered n-tuple** $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, …, a_n as its n^{th} element.

Cartesian product of A and B is $A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$

Eg: $A = \{a, b\}$ and $B = \{1, 2, 3\}$
$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$
$B \times A = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

$A \times B \neq B \times A$

$|A \times B| = |A| * |B|$

A subset R of AXB is called a **relation** from set A to set B.

The **cartesian product** of the sets A_1, A_2, …, A_n, denoted by $A_1 X A_2 X \cdots X A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for i = 1, 2, …, n.

That is, $A_1 X A_2 X \cdots X A_n = \{(a_1, a_2, ..., a_n) \mid a_i \in A_i$ for $i = 1,2,...,n\}$.

Truth Sets

Given Predicate P, and domain D, the truth set of P is the set of elements x in D for which P(x) is true.

The truth set of P(x) is denoted by $\{x \in D \mid P(x)\}$.

Q: What are the truth sets of the predicates P(x), Q(x), and R(x), where the domain is the set of integers and
P(x) is "$|x| = 1$",
Q(x) is "$x^2 = 2$", and
R(x) is "$|x| = x$" ?

$\forall_{x \in S}$ (P(x)) denotes the universal quantification of P(x) over all elements in the set S.

i.e., $\boldsymbol{\forall_{x \in S}}$ **(P(x))** is shorthand for $\boldsymbol{\forall x\ (x \in S \rightarrow P(x))}$.

$\exists_{x \in S}$ (P(x)) denotes the existential quantification of P(x) over all elements in S.

i.e., $\boldsymbol{\exists_{x \in S}}$ **(P(x))** is shorthand for $\boldsymbol{\exists x\ (x \in S \land P(x))}$.

Russell's Paradox

Let the domain be the set of all sets

$S = \{x \mid x \notin x\}$

Is S a member of itself?

Suppose, $S \in S$. Then, the predicate $x \notin x$ is false. Hence, S should not belong to S. It's a contradiction.

Suppose, $S \notin S$. Then, the predicate $x \notin x$ is true. Hence, S should belong to S. It's a contradiction.

Therefore, it is a paradox.

Analogy:

Predicate: I help people who can't help themselves.

Suppose I'm one of those people.

When I'm sick (i.e. I can't help myself), I'm in a paradox.

According to the predicate, I should help "me", but I can't do that because I'm sick. If I don't help myself, I'm violating the predicate.

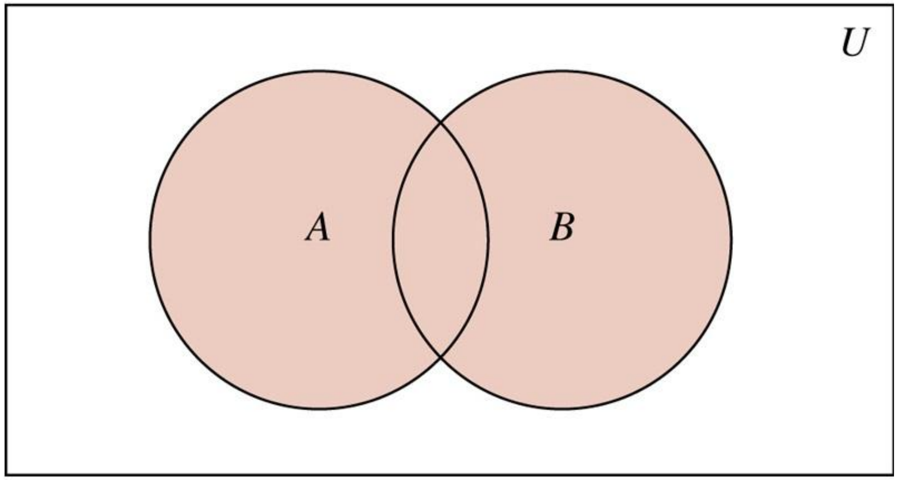
Russell’s Paradox

The Big Bang Theory Scene: Leonard’s car. “Play that funky music, white boy” is playing on the stereo.

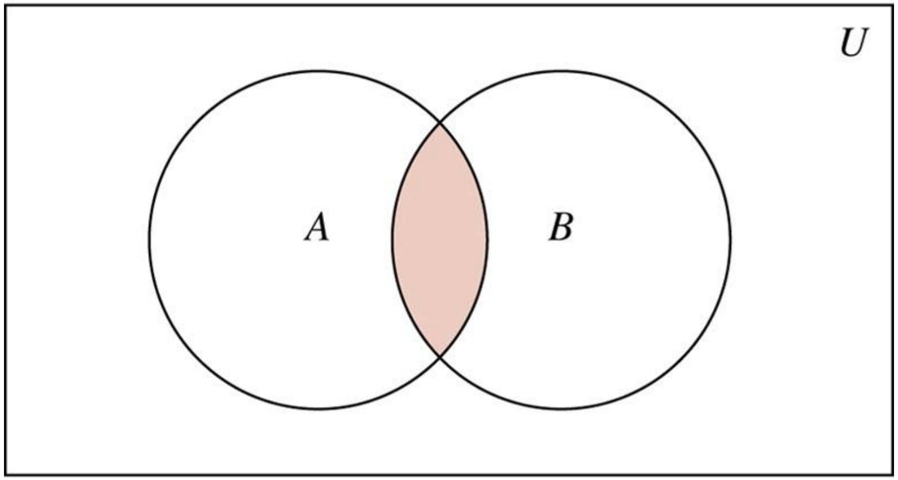
- Sheldon: So they’re requesting that the white boy play the funky music, yes?
  - Leonard: Yes.
- Sheldon: And this music we’re listening to right now is funky as well?
  - Leonard: Sure.
- Sheldon: Let me ask you this. Do you think this song is the music the white boy ultimately plays?
  - Leonard: It could be.
- Sheldon: So it’s like the musical equivalent of Russell’s Paradox, the question of whether the set of all sets that don’t contain themselves as members contains itself?
  - Leonard: Exactly.
- Sheldon: Well then I hate it. Music should just be fun.

Set Operations

- Union of sets A and B contains those elements in A, B or both.
    - $\mathbf{A \cup B = \{ x \mid x \in A \lor x \in B\}}$
- Intersection of sets A and B contains those elements in both A and B.
    - $\mathbf{A \cap B = \{ x \mid x \in A \land x \in B\}}$



$A \cup B$ is shaded.



$A \cap B$ is shaded.

Two sets are **disjoint** when their intersection is empty.

$A \cap B = \varnothing$

Eg: $\{1, 2\} \cap \{3, 4\} = \varnothing$

Q: What are the resulting sets of the following.

1. $\{1, 2, 3\} \cap \{1, 2\} = \{1, 2\}$
2. $\{1, 2, 3\} \cap \{R, G, B\} = \varnothing$
3. $\{1, 2, 3\} \cap \varnothing = \varnothing$
4. $\{1, 2, 3\} \cup \{1, 4\} = \{1, 2, 3, 4\}$
5. $\{1, 2, 3\} \cup \{R, G, B\} = \{1, 2, 3, R, G, B\}$
6. $\{1, 2, 3\} \cup \varnothing = \{1, 2, 3\}$
7. $\{1, 2, 3\} \cup \{\ \} = \{1, 2, 3\}$

Cardinality of union:

If sets A and B are disjoint,

$|A \cup B| = |A| + |B|$

In general,

$|A \cup B| = |A| + |B| - |A \cap B|$

Generalization of this result (of n sets) is called the **principle of inclusion-exclusion**.

Eg: $A = \{1, 2, 3\}$, $B = \{3, 4\}$

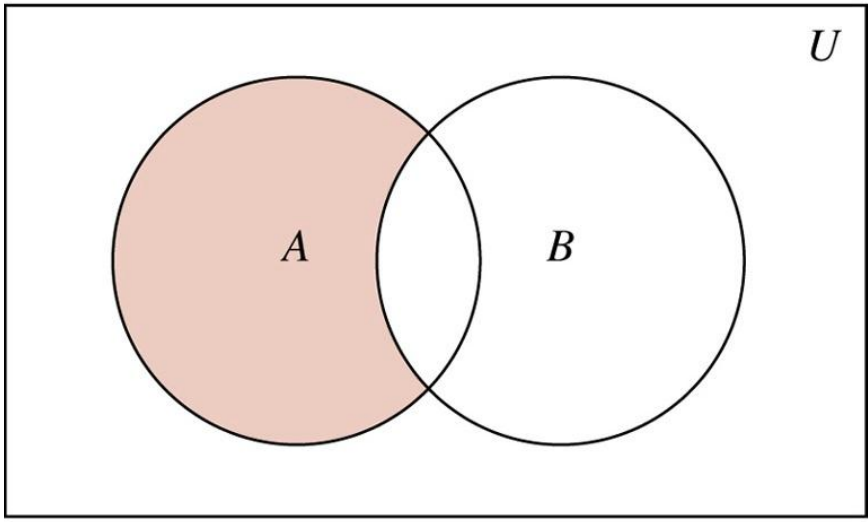
$|A \cup B| = |A| + |B| - |A \cap B| = 3+2-1 = 4$

Eg: $| \{1, 2, 3\} \cup \{ 2, 3, 4 \} | = 3+3-2 = 4$

Difference of two sets:

Difference of two sets, A - B, is set containing elements in A but not in B.

$A - B = \{ x \mid x \in A \land x \notin B\}$



$A - B$ is shaded.

Eg: {1, 2 } - {3, 4} = {1, 2}

Eg: {1, 2, 3 } - {3, 4} = {1, 2}

Eg: {1, 2, 3} - {1, 2, 3} = ∅

Eg: {1, 2, 3} - { 1, 2 } = {3}

Eg: {1, 2, 3} - { 1, 2, 3 } = { }

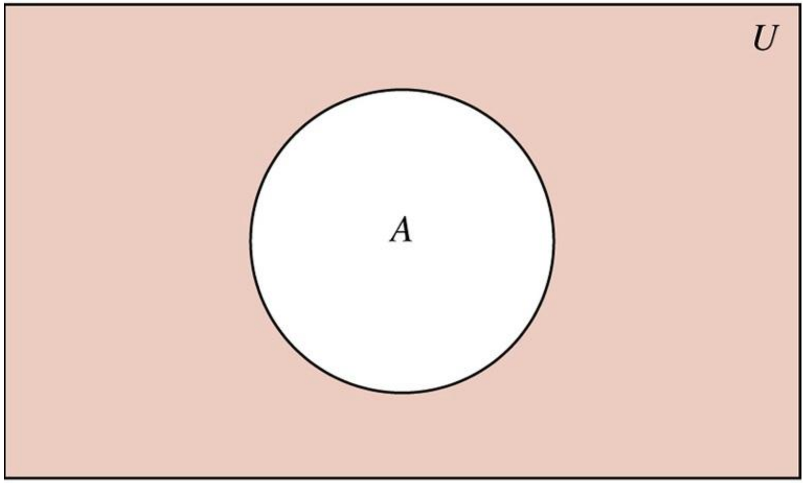
Eg: {1, 2, 3} - ∅ = {1, 2, 3}

Complement of a set:

Complement, $\bar{\mathbf{A}}$ (A bar) or A`, is the complement with respect to the universal set, U. That is, the difference U - A is the complement of A.

$\bar{A} = \{ x \mid x \in U \land x \notin A\}$

$\bar{A} = \{ x \mid x \notin A\}$



$\bar{A}$ is shaded.

Eg: U = { 1, 2, 3, 4, 5, 6 }

A = { 1, 2, 3, 4 }

$\bar{A}$ = { 5, 6}

Eg: U = {a, e, i, o, u }

A = {a, e, o }

$\bar{A}$ = {i, u}

TABLE 1 **Set Identities.**

Identity	*Name*
$A \cup \varnothing = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \varnothing$	Complement laws

Membership Tables
(Observe the similarities with the Truth Tables)

Eg: Use a **membership table** to show
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

TABLE 2 **A Membership Table for the Distributive Property.**

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Membership Table

Prove De Morgan’s law (A U B)` = A` ∩ B` using membership table.

A	B	$A \cup B$	$\overline{A \cup B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	1	1	0	1	0	0
0	0	0	1	1	1	1

Q: Prove De Morgan’s law (A ∩ B)` = A` U B` **without** using membership table.

Q: Use set builder notation and logical equivalences to establish the De Morgan's law $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Soln:

$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$	By definition of complement
$= \{x \mid \neg(x \in (A \cap B))\}$	By definition of $\notin$ symbol
$= \{x \mid \neg((x \in A) \wedge (x \in B))\}$	By definition of intersection
$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$	By De Morgan's law of logic
$= \{x \mid x \notin A \vee x \notin B$	By definition of $\notin$ symbol
$= \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$	By definition of complement
$= \{x \mid x \in \overline{A} \cup \overline{B}\}$	By definition of union
$= \overline{A} \cup \overline{B}$	By meaning of set builder notation

Therefore, $\overline{A \cap B} = \overline{A} \cup \overline{B}$

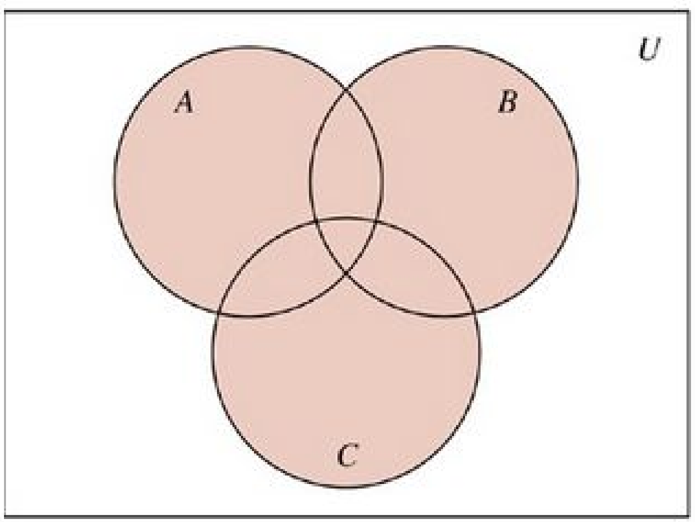
Generalized Unions and Intersections

Union:

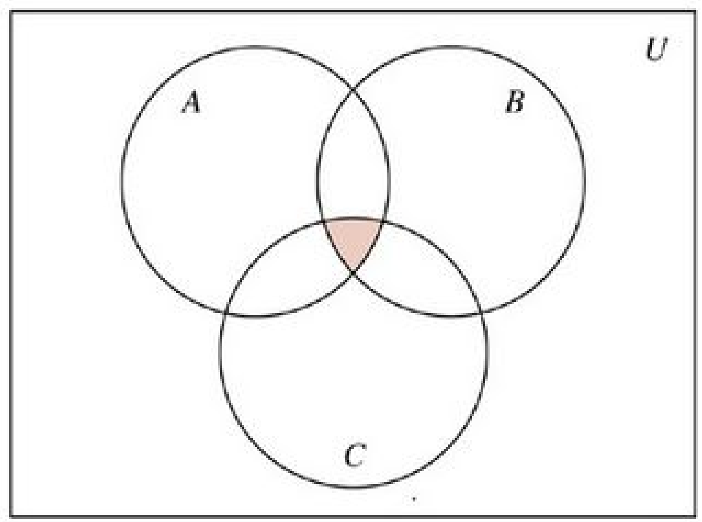
Union of a collection of sets contains elements that are members of at least one set in the collection. $A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$

Intersection

Intersection of a collection of sets contains elements that are members of all sets in the collection. $A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=0}^{n} A_i$



(a) $A \cup B \cup C$ is shaded.



(b) $A \cap B \cap C$ is shaded.

Computer Representation of Sets:

U = { 5, 4, 3, 2, 1, 0 } = 111111

A = { 2, 1, 0 } = 000111

B = { 3, 2 } = 001100

$A \cup B = A \vee B$

$\{2, 1, 0\} \cup \{3, 2\} = \{3, 2, 1, 0\}$

$$\begin{array}{rl} & 000111 \\ \vee & \underline{001100} \\ & 001111 \end{array} \qquad \begin{array}{rl} & \{2, 1, 0\} \\ \cup & \underline{\{3, 2\}} \\ & \{3, 2, 1, 0\} \end{array}$$

$A \cap B = A \wedge B$

$\{2, 1, 0\} \cap \{3, 2\} = \{2\}$

$$\begin{array}{rl} & 000111 \\ \wedge & \underline{001100} \\ & 000100 \end{array} \qquad \begin{array}{rl} & \{2, 1, 0\} \\ \cap & \underline{\{3, 2\}} \\ & \{2\} \end{array}$$

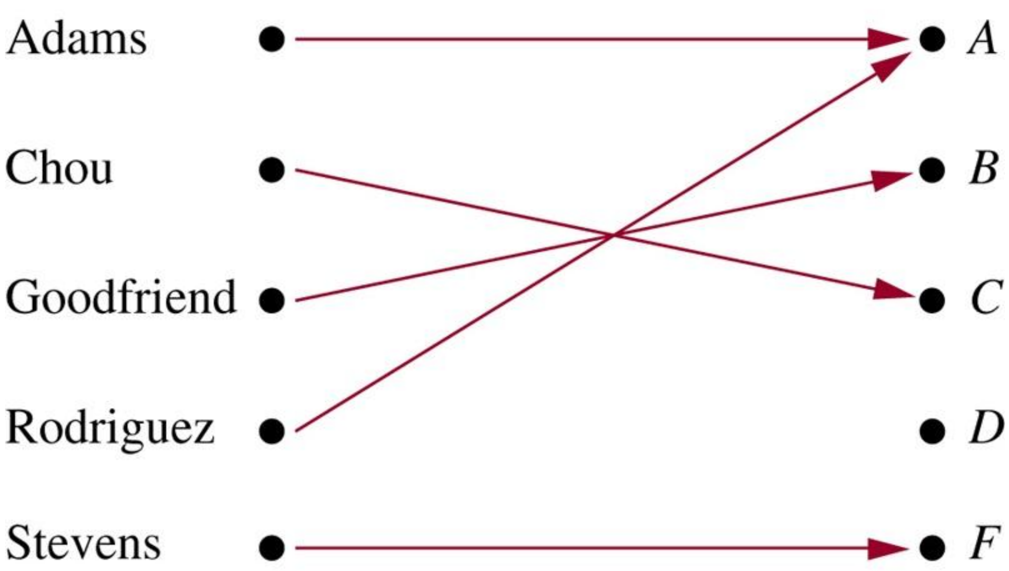
Functions

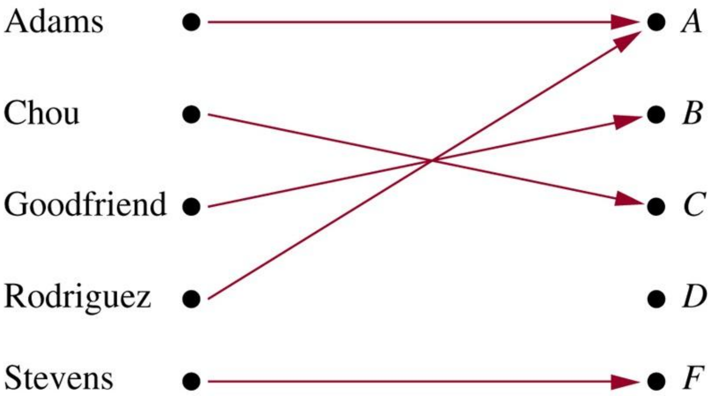
Function f from A to B is assignment of **exactly one element of B** to **each element of A**. f(a) = b where ‘b’ is an element of B assigned by ‘f’ to the element ‘a’ of A.

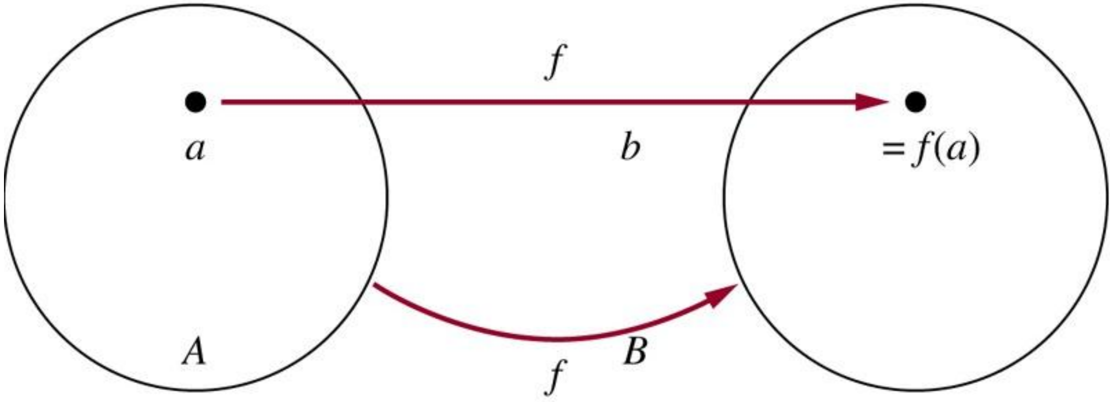
$f : A \rightarrow B$

Functions also called **mappings** or **transformations**.

Eg: **g : Students → Grades**







Domain of the function ‘g’ is the set of Students.

Co-domain of the function ‘g’ is the set of Grades.

Image of g(Goodfriend) is B.

Range of the function ‘g’ is {A, B, C, F}

Domain of $f : A \rightarrow B$ is A

Co-domain of $f : A \rightarrow B$ is B

Image $f(a) = b$ is b

Preimage $f(a) = b$ is a

Range of $f : A \rightarrow B$ is set of all images of elements of A

Two real-valued functions with the same domain can be added and multiplied.

Let $f_1 : A \rightarrow R$ and $f_2 : A \rightarrow R$

Then, $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

$(f_1 \bullet f_2)(x) = f_1(x) \bullet f_2(x)$

Eg:

$f_1 : R \rightarrow R$

$f_2 : R \rightarrow R$

$f_1(x) = x^2$

$f_2(x) = x - x^2$

$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + x - x^2 = x$

$(f_1\ f_2)(x) = f_1(x) \bullet f_2(x) = x^2 \bullet (x - x^2) = x^3 - x^4$

Image of a subset of the domain:

$f : A \to B$

$C \subseteq A$

Image of C under f is a subset of B.

$f(C) \subseteq B$

$f(C) = \{ x \mid \exists c \in C (x = f(c)) \}$

Types of functions

- One-to-One (Injective) function
- Onto (Surjective) function
- One-to-One Correspondence (Bijective) function

One-to-One (Injective) functions:

A function is said to be One-to-one or injective, if and only if f(a)=f(b) implies that a=b for all a and b in the domain of f.

That is, **f is one-to-one iff $\forall a \forall b\ (f(a) = f(b) \rightarrow a = b)$**

That is, **f is one-to-one iff $\forall a \forall b\ (a \neq b \rightarrow f(a) \neq f(b))$**

Eg: A = { a, b, c, d },

B = { 1, 2, 3, 4, 5 }

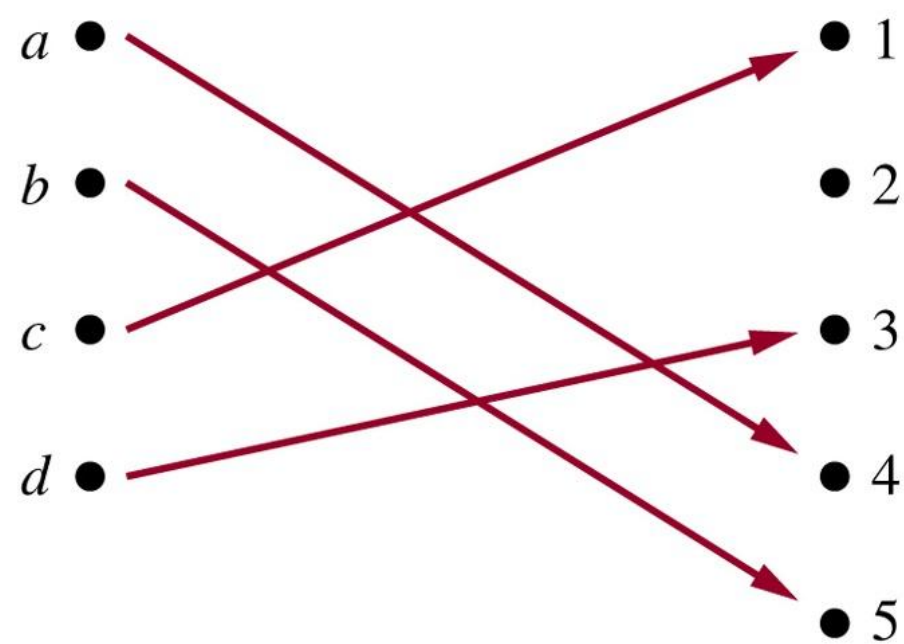
$g : A \rightarrow B$

g(a) = 4,

g(b) = 5,

g(c) = 1,

g(d) = 3



Let f be a function. Which of the following defines one-to-one function f?

1. $\forall a \forall b \, (f(a) = f(b) \leftrightarrow a = b \,)$
2. $\forall a \forall b \, (a = b \rightarrow f(a) = f(b) \,)$
3. $\forall a \forall b \, (f(a) = f(b) \rightarrow a = b \,)$
4. $\forall a \forall b \, (a \neq b \rightarrow f(a) \neq f(b) \,)$

Increasing/Decreasing functions:

Consider a function f whose domain and codomain are subsets of the set of real numbers.

Function f is **increasing** if $f(x) \leq f(y)$ for real $x < y$

That is, $\forall x \forall y(x < y \rightarrow f(x) \leq f(y))$

Function f is **strictly increasing** if $f(x) < f(y)$ for real $x < y$.

That is, $\forall x \forall y(x < y \rightarrow f(x) < f(y))$

Strictly increasing functions must be one-to-one.

Function f is **decreasing** if $f(x) \geq f(y)$ for real $x < y$.

That is, $\forall x \forall y(x < y \rightarrow f(x) \geq f(y))$

Function f is **strictly decreasing** if $f(x) > f(y)$ for real $x < y$.

That is, $\forall x \forall y(x < y \rightarrow f(x) > f(y))$

Strictly decreasing functions must be one-to-one.

Onto (Surjective) functions:

A function f from A to B is called onto or surjective, if and only if every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

i.e., f is Onto function iff every $b \in B$ has $a \in A$ with $f(a) = b$.

In short, **$\forall y \exists x\ (f(x) = y)$**

Eg: A = { a, b, c, d },

B = { 1, 2, 3 }

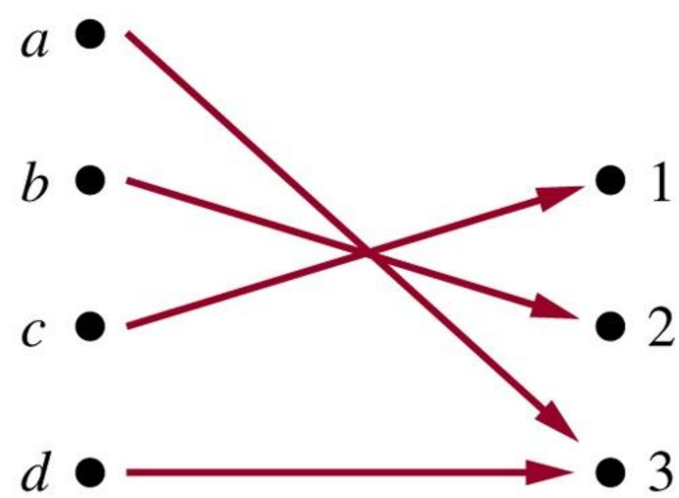
$G : A \rightarrow B$

G(a) = 3,

G(b) = 2,

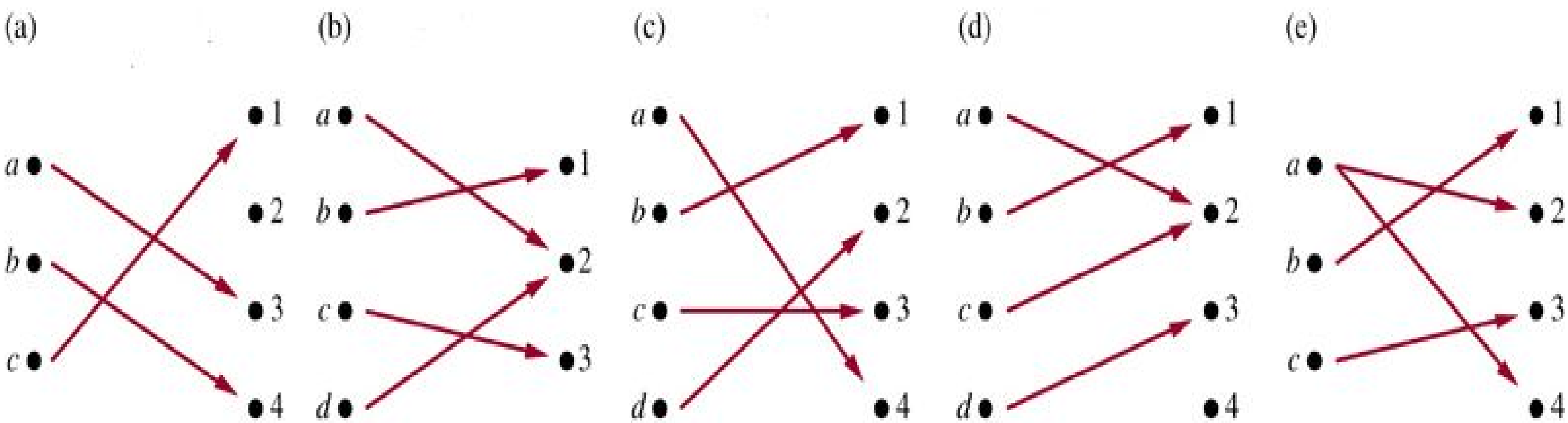
G(c) = 1,

G(d) = 3



One-to-One Correspondence (Bijection) functions:
A function is One-to-one correspondence or bijective function if it is one-to-one and onto.

Eg: What kind relations are these (one-to-one function, onto function, one-to-one correspondence)?



Q: Diagram the following functions and mention whether they are one-to-one, onto or one-to-one correspondence:

1. $f : \{a, b, c, d\} \rightarrow \{1, 2, 3, 4\}$

   $f(a) = 1$

   $f(b) = 2$

   $f(c) = 3$

   $f(d) = 4$

2. $g : \{a, b, c, d\} \rightarrow \{1, 2, 3, 4\}$

   $g(a) = 1$

   $g(b) = 1$

   $g(c) = 4$

   $g(d) = 4$

Inverse Function:

Inverse of a function **f** from A to B such that

$\mathbf{f}^{-1}(b) = a$ when $f(a) = b$.

Function f is **invertible** when **f** is one-to-one correspondence (i.e. one-to-one and onto), otherwise inverse function of **f** does not exists.

Eg: f : $\mathbf{R}^+ \rightarrow \mathbf{R}^+$

$f(x) = x^2$

$f^{-1}(y) = y^{1/2}$

$f(3) = 3^2 = 9$

$f^{-1}(9) = 9^{1/2} = 3$

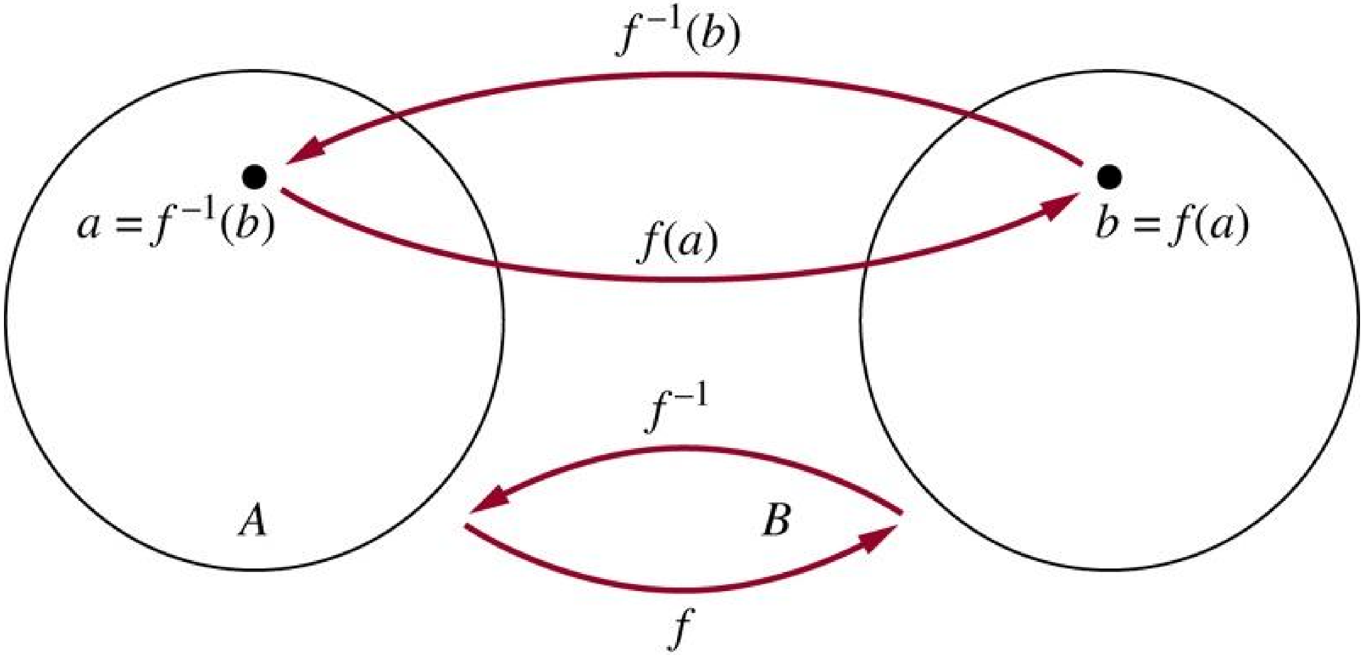
Eg: f : $\mathbf{Z} \rightarrow \mathbf{Z}$

$f(x) = x + 3$

$f^{-1}(y) = y - 3$

$f(20) = 23$

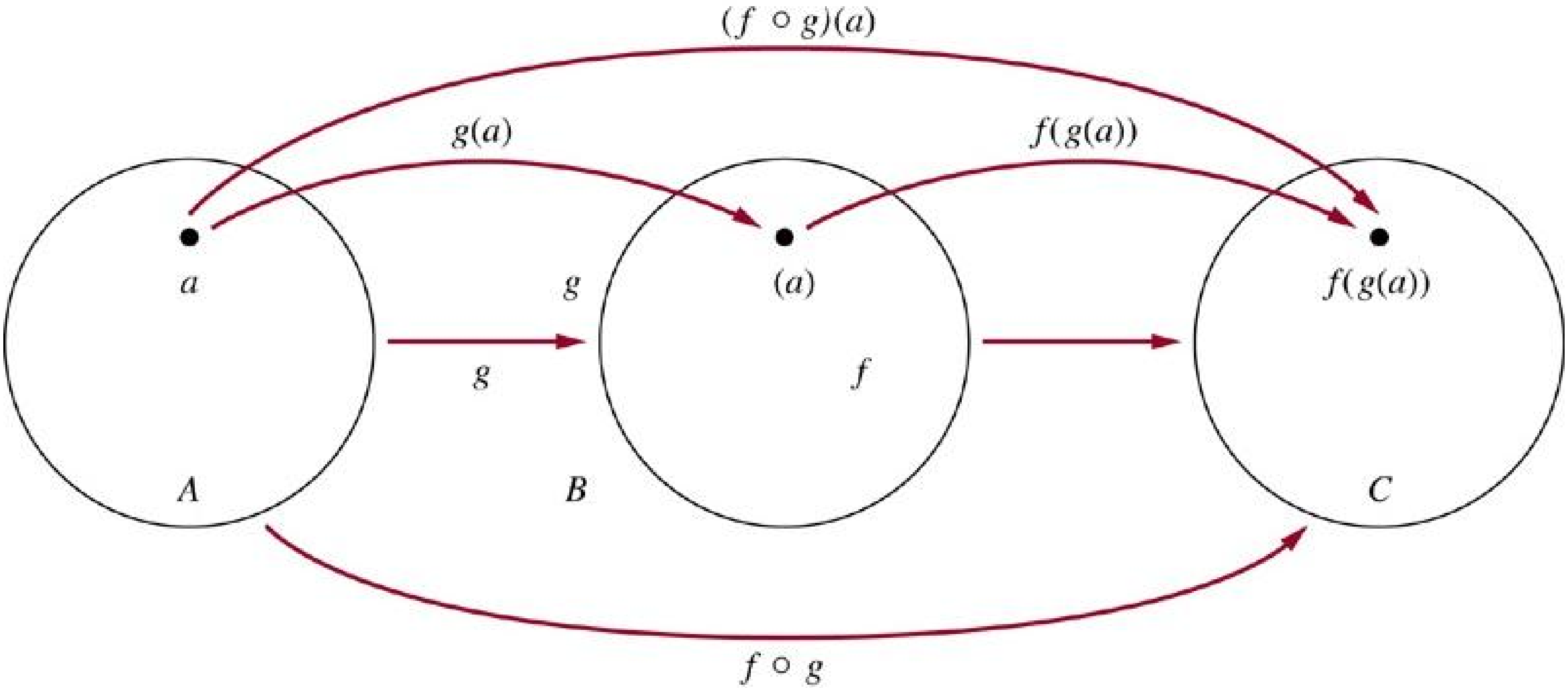
$f^{-1}(23) = 20$



Composition of Functions:

Composition of function g from A to B and function f from B to C.

$(f \circ g)(a) = f(g(a))$



$g: \{a, b, c\} \to \{X, Y, Z\}$

$g(a) = X$

$g(b) = Y$

$g(c) = Z$

$f: \{X, Y, Z\} \to \{1, 2, 3\}$

$f(X) = 1$

$f(Y) = 2$

$f(Z) = 3$

$(f \circ g)(a) = f(g(a)) = f(X) = 1$

$(f \circ g)(b) = f(g(b)) = f(Y) = 2$

$(f \circ g)(c) = f(g(c)) = f(Z) = 3$

$$\begin{tikzcd} a \arrow[r, "g"] & X \arrow[r, "f"] & 1 \\ b \arrow[r] & Y \arrow[r] & 2 \\ c \arrow[r] & Z \arrow[r] & 3 \end{tikzcd}$$

$g \circ f$ is not defined because *f* range, $\{1, 2, 3\}$, is not a subset of *g* domain, $\{a, b, c\}$

$(g \circ f)(X) = g(f(X)) = g(1)$ is not defined.

Eg:

$f(x) = 5x + 7$

$g(x) = 3x + 2$

$(f \circ g)(x) = f(g(x)) = f(3x+2) = 5(3x+2) + 7 = 15x + 17$

$(g \circ f)(x) = g(f(x)) = g(5x+7) = 3(5x+7) + 2 = 15x + 23$

$(f \circ g)(x) \neq (g \circ f)(x)$

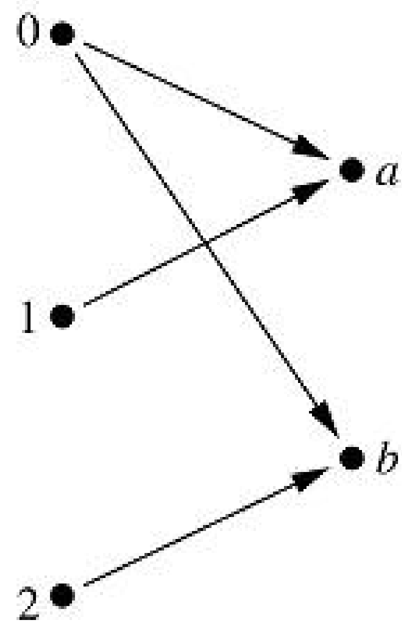
Relations

A **binary relation** from A to B is a subset of AXB.

Element **a** is related to **b** by **R** is denoted by **aRb**.

aRb denotes (a, b) ∈ R

$a \not R b$ denotes (a, b) ∉ R



R	a	b
0	×	×
1	×	
2		×

Eg:

A = {0, 1, 2}

B = {a, b}

A x B = { (0, a), (1, a), (0, b), (1, b), (2, a), (2, b) }

R = { (0, a), (1, a), (0, b), (2, b) } ⊆ A x B

Matrix Representation of Relations:

2-dimensional 0-1 matrix is used for binary relations.

One row for each element of A

One column for each element of B

$$m_{ij} = \begin{cases} 1 \; if \; (a_i, b_j) \in R \\ 0 \; if \; (a_i, b_j) \notin R \end{cases}$$

Directed Graph (Digraph) Representation of Relations:

Set V of vertices (nodes) representing elements of the sets.

Set E ordered pairs of elements of V called edges (arcs).

Vertex a is initial vertex and vertex b is terminal vertex of edge (a, b).

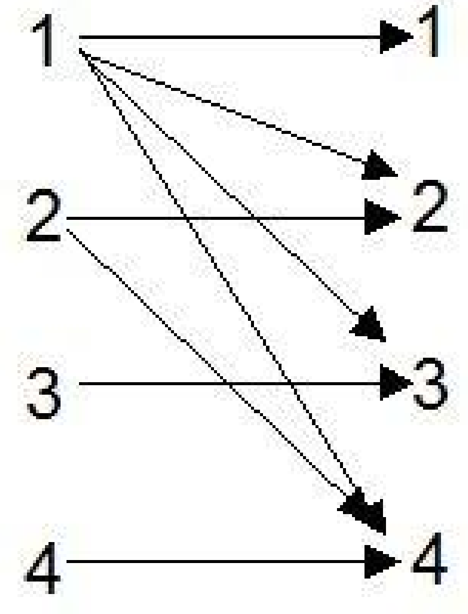
Edge (a, a) is a loop.

Relations on a Set:

Relation on the set A is a relation from A to A.

Eg: A = {1,2,3,4}

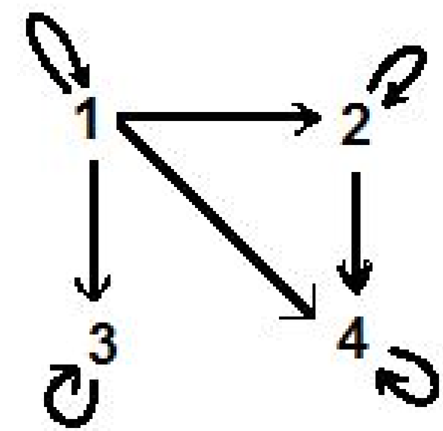
A x A = { (1,1), (1,2), (1,3), (1,4),
(2,1), (2,2), (2,3), (2,4),
(3,1), (3,2), (3,3), (3,4),
(4,1), (4,2), (4,3), (4,4) }



R = { (a, b) | a divides b}
= {(1,1), (1,2), (1,3), (1,4),
(2,2), (2,4), (3,3), (4,4)}

R1 = {(a, b) | a >= b}

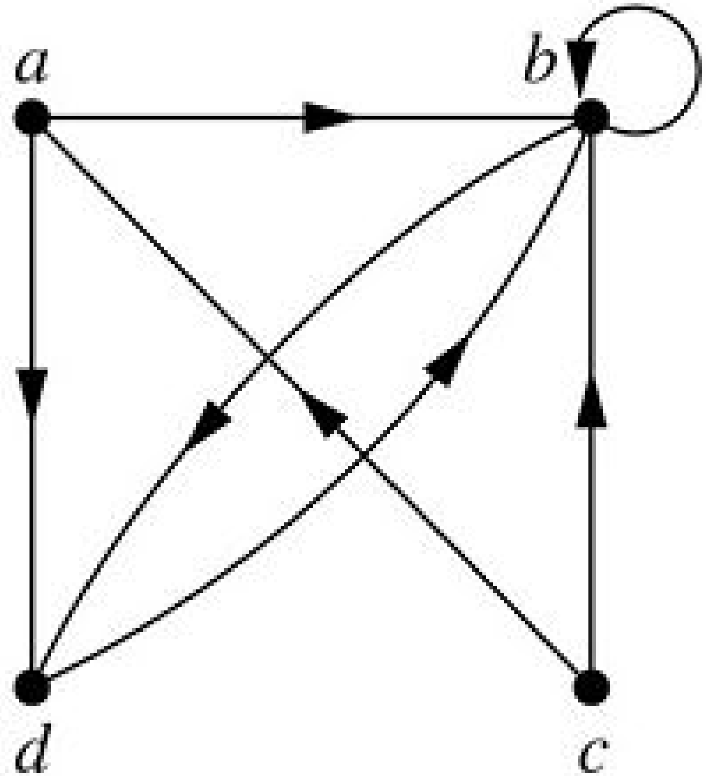
R2 = {(a, b) | a = b}



Eg: Set A = {a, b, c, d}

Relation R = {(a, d), (a, b), (b, b), (b, d), (c, b), (c, a), (d, b)}

	a	b	c	d
a	0	1	0	1
b	0	1	0	1
c	1	1	0	0
d	0	1	0	0



Eg: Set A = {1, 2, 3, 4}

Show the matrix and digraph representation of the relation

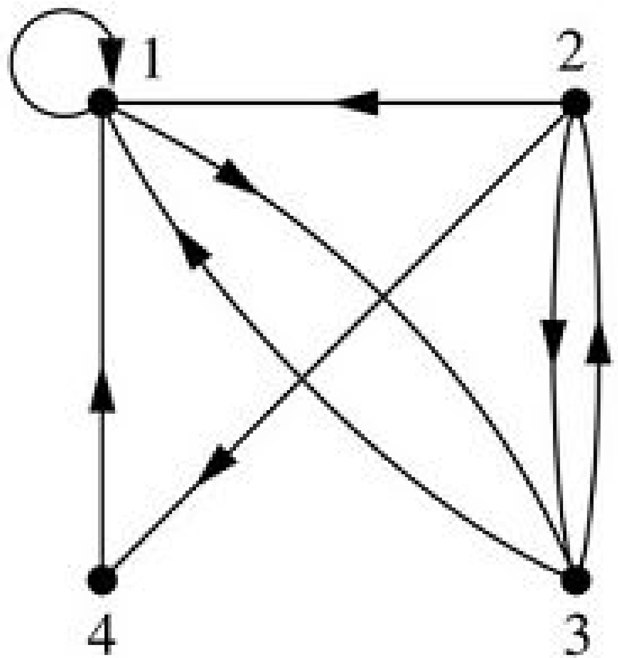
R = {(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)}

Eg: Set A = {1, 2, 3, 4}

Show the matrix and digraph representation of the relation

R = {(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)}

	1	2	3	4
1	1	0	1	0
2	1	0	1	1
3	1	1	0	0
4	1	0	0	0



Example relations:

- Let $A = \{1, 2, 3, 4\}$. Relation R on set A be $\{(1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (3,3), (4,4)\}$.
- Let $S = \{a,b,c,d,e,f\}$. Relation R on set S be $\{(a,a),(b,b),(b,c),(c,b),(c,c),(d,d),(d,e),(d,f),(e,d),(e,e),(e,f),(f,d),(f,e),(f,f)\}$
- Relation R on the set of integers such that aRb if and only if $a = b$ or $a = -b$.
- Relation R on the set of real numbers such that aRb if and only if $a - b$ is an integer.
- Relation $R = \{(a, b) \mid a \equiv b \pmod{10}\}$.
- Relation $R = \{(a, b) \mid a \equiv b \pmod{m}\}$, where $m \in Z^+ \land m>1$
- Relation R on the set of strings of English letters such that aRb if and only if Length(a) = Length(b).

Properties of Relations on a set:

1. Reflexive
2. Symmetric
3. Antisymmetric
4. Transitive

Reflexive Relations:

A relation R on a set A is reflexive iff

- $(a,a) \in R$ for every element $a \in A$.
- **$\forall a \in A\ ((a, a) \in R)$.**

Eg: A = {1, 2, 3, 4}

R = {(1, 1), (2, 2), (3, 3), (4, 4), (1, 2)} is reflexive

	1	**2**	**3**	**4**
1	1	1	0	0
2	0	1	0	0
3	0	0	1	0
4	0	0	0	1

Symmetric Relations:

A relation R on a set A is symmetric iff

- $(b,a) \in R$ whenever $(a,b) \in R$, for all $a,b \in A$.
- $\forall a \in A\ \forall b \in A\ ((a, b) \in R \rightarrow (b, a) \in R)$
- $\forall a \in A\ \forall b \in A\ ((a, b) \notin R \rightarrow (b, a) \notin R)$

Eg: Let set A = {1, 2, 3, 4}

R = {(1, 2), (2, 1), (1, 4), (4, 1), (3, 3)} symmetric

	1	**2**	**3**	**4**
1	0	1	0	1
2	1	0	0	0
3	0	0	1	0
4	1	0	0	0

Antisymmetric Relations:

A relation R on a set A is antisymmetric iff

- $a=b$ whenever $(a,b)\in R$ and $(b,a)\in R$, for all $a,b\in R$.
- $\forall a\in A\ \forall b\in A\ ((a,b)\in R \land (b,a)\in R \rightarrow a=b)$
- $\forall a\in A\ \forall b\in A\ ((a \neq b) \rightarrow (a, b)\notin R \lor (b, a)\notin R)$

Eg: Let set A = {1, 2, 3, 4}

R = {(1, 2), (3, 3), (4, 1)} antisymmetric

	1	**2**	**3**	**4**
1	0	1	0	0
2	0	0	0	0
3	0	0	1	0
4	1	0	0	0

Symmetric? Antisymmetric?

	1	2	3
1	0	1	1
2	1	0	0
3	1	0	1

	1	2	3
1	0	1	1
2	0	0	0
3	0	1	1

	1	2	3
1	0	1	0
2	0	0	0
3	0	0	1

	1	2	3
1	0	1	1
2	0	0	0
3	1	0	1

	1	2	3
1	0	0	0
2	0	0	0
3	0	0	1

	1	2	3
1	0	0	0
2	0	0	0
3	0	0	0

Symmetric? Antisymmetric?

	1	**2**	**3**		
1	0	1	1		
2	1	0	0		
3	1	0	1	**Y**	**N**

	1	**2**	**3**		
1	0	1	1		
2	0	0	0		
3	0	1	1	**N**	**Y**

	1	**2**	**3**		
1	0	1	0		
2	0	0	0		
3	0	0	1	**N**	**Y**

	1	**2**	**3**		
1	0	1	1		
2	0	0	0		
3	1	0	1	**N**	**N**

	1	**2**	**3**		
1	0	0	0		
2	0	0	0		
3	0	0	1	**Y**	**Y**

	1	**2**	**3**		
1	0	0	0		
2	0	0	0		
3	0	0	0	**Y**	**Y**

Transitive Relations:

A relation R on a set A is transitive iff

- (a,c)∈R whenever (a,b)∈R and (b,c)∈R, for all a,b,c∈R.
- **$\forall a \in A\ \forall b \in A\ \forall c \in A\ ((a,b) \in R \land (b,c) \in R \rightarrow (a,c) \in R)$**

	1	**2**	**3**	**4**
1	0	1	1	1
2	0	0	1	1
3	0	0	1	0
4	0	0	1	0

	1	**2**	**3**	**4**
1	1	0	1	0
2	0	0	0	0
3	1	0	1	0
4	0	0	0	0

Reflexive relation R on A, if $\forall a \in A, (a, a) \in R$.

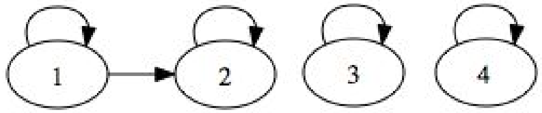
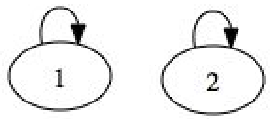
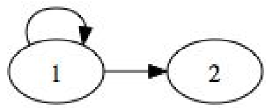
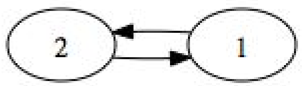
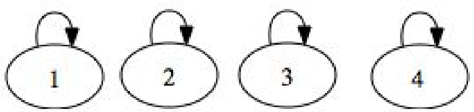
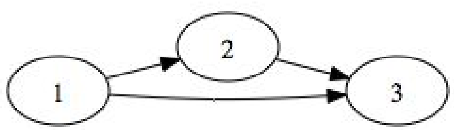
Symmetric relation R on A, if $\forall a \forall b \in A, (a, b) \in R \rightarrow (b, a) \in R$.

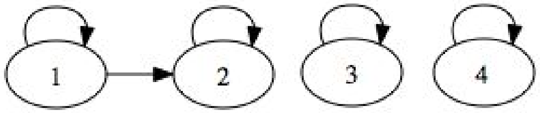
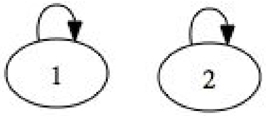
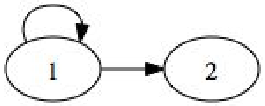
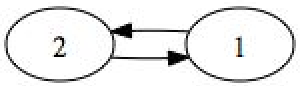
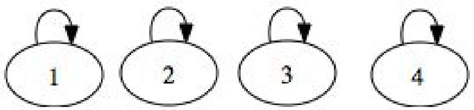
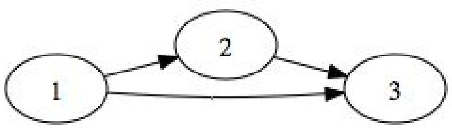
Antisymmetric relation R on A, if $\forall a \forall b \in A, (a, b) \in R \wedge (b, a) \in R \rightarrow a=b$.

Transitive relation R on A, if $\forall a \forall b \forall c \in A, (a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R$.

Relations on the set $A = \{1, 2, 3\}$

	reflexive	*symmetric*	*antisymmetric*	*transitive*
$R_0 = \{(1,1), (2,2), (3,3)\}$	Yes	Yes	Yes	Yes
$R_1 = \{(2,2), (2,3), (3,2)\}$	No	Yes	No (2,3) (3,2)	No (3,3)
$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$	Yes	Yes	No (1,2) (2,1)	Yes
$R_3 = \{(2,3), (3,2)\}$	No	Yes	No (2,3) (3,2)	No (2,2) (3,3)
$R_4 = \{(1,2), (2,3), (1,3)\}$	No	No	Yes	Yes

Relation	Reflexive	Symmetric	Antisymmetric	Transitive
				
				
				
				
				
				
				
				

Relation	Reflexive	Symmetric	Antisymmetric	Transitive
	Y	N	Y	Y
	Y	Y	Y	Y
	N	N	Y	Y
	N	Y	N	N
	Y	Y	Y	Y
	N	N	Y	Y
	N	N	Y	N
	N	N	N	N

Q: Mention the properties (reflexive, symmetric, antisymmetric and transitive properties) of the relations.

1. Relation of {a, b, c},
   R = {(a, a), (b, b), (c, c), (b, c), (c, b)}
   Ans: …

2. Relation of {a, b, c, d},
   S = {(a, c), (c, a), (a, d), (d, a), (d, d), (a, b), (b, a), (b, b)}
   Ans: …

Q: Mention the properties (reflexive, symmetric, antisymmetric and transitive properties) of the relations.

1. Relation of {a, b, c},
   R = {(a, a), (b, b), (c, c), (b, c), (c, b)}
   Ans: Y, Y, N, Y

2. Relation of {a, b, c, d},
   S = {(a, c), (c, a), (a, d), (d, a), (d, d), (a, b), (b, a), (b, b)}
   Ans: N, Y, N, N

Combining Relations:

Relations from A to B are subsets of A x B and can be combined in any way two sets can be combined.

Eg: $A = \{0, 1, 2\}$, $B = \{a, b\}$

$A \times B = \{ (0, a), (1, a), (0, b), (1, b), (2, a), (2, b) \}$

$R1 = \{(0, a), (0, b)\}$, $R2 = \{(0, b), (1, a), (1, b)\}$

$R1 \cap R2 = \{(0, b)\}$

$R1 \cup R2 = \{(0, a), (0, b), (1, a), (1, b)\}$

$R1 - R2 = \{(0, a)\}$

$R2 - R1 = \{(1, a), (1, b)\}$

$$\begin{aligned} R1 \oplus R2 &= R1 \cup R2 - R1 \cap R2 \\ &= \{(0, a), (0, b), (1, a), (1, b)\} - \{(0,b)\} \\ &= \{(0, a), (1, a), (1, b)\} \end{aligned}$$

Closure of Relations:

Let R be a relation on a set A. R may or may not have some property P, such as reflexivity, symmetry, or transitivity. If there is a relation S with property P containing R such that S is a subset of every relation with property P containing R, then S is called the closure of R with respect to P.

Reflexive Closure of a relation:

We see that the reflexive closure of R equals $R \cup \boldsymbol{\Delta}$, where $\boldsymbol{\Delta} = \{(a, a) \mid a \in A\}$ is the diagonal relation on A.

The relation R = {(1, 1), (1, 2), (2, 1), (3, 2)} on the set A = {1, 2, 3} is not reflexive. How can we produce a reflexive relation containing R that is as small as possible? This can be done by adding (2, 2) and (3, 3) to R, because these are the only pairs of the form (a, a) that are not in R. Clearly, this new relation contains R. Furthermore, any reflexive relation that contains R must also contain (2, 2) and (3, 3). Because this relation contains R, is reflexive, and is contained within every reflexive relation that contains R, it is called the reflexive closure of R.

Symmetric Closure of a relation:

The symmetric closure of R equals $R \cup R^{-1}$ is the symmetric closure of R, where $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.

The relation {(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 2)} on {1, 2, 3} is not symmetric. How can we produce a symmetric relation that is as small as possible and contains R? To do this, we need only add (2, 1) and (1, 3), because these are the only pairs of the form (b, a) with $(a, b) \in R$ that are not in R. This new relation is symmetric and contains R. Furthermore, any symmetric relation that contains R must contain this new relation, because a symmetric relation that contains R must contain (2, 1) and (1, 3). Consequently, this new relation is called the symmetric closure of R.

Transitive Closure of a relation:
Suppose that a relation R is not transitive. Can the transitive closure of a relation R be produced by adding all the pairs of the form (a, c), where (a, b) and (b, c) are already in the relation? Consider the relation R = {(1, 3), (1, 4), (2, 1), (3, 2)} on the set {1, 2, 3, 4}. This relation is not transitive because it does not contain all pairs of the form (a, c) where (a, b) and (b, c) are in R. The pairs of this form not in R are (1, 2), (2, 3), (2, 4), and (3, 1). Adding these pairs does not produce a transitive relation, because the resulting relation contains (3, 1) and (1, 4) but does not contain (3, 4). This shows that constructing the transitive closure of a relation is more complicated than constructing reflexive and symmetric closures.

A **path** from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ in G, where n is a nonnegative integer, and $x_0 = a$ and $x_n = b$, that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path. This path is denoted by $\mathbf{x_0, x_1, x_2, \ldots, x_{n-1}, x_n}$ and has length n. We view the empty set of edges as a path of length zero from a to a. A path of length $n \geq 1$ that begins and ends at the same vertex is called a **circuit** or cycle.

The term path also applies to relations. Carrying over the definition from directed graphs to relations, there is a path from a to b in R if there is a sequence of elements $\mathbf{a, x_1, x_2, \ldots, x_{n-1}, b}$ with $(a, \mathbf{x_1}) \in R, (\mathbf{x_1}, \mathbf{x_2}) \in R, \ldots,$ and $(\mathbf{x_{n-1}}, b) \in R$.

Composite Relations: $S \circ R$

if **(a, b)$\in$R** $\wedge$ **(b, c)$\in$S**, then **(a, c) $\in$ S $\circ$ R**

Eg:

A = {0, 1, 2}, B = {a, b}, C = {X, Y, Z}

Relation from A to B: R = {(0, a), (1, b), (2, b)}

Relation from B to C: S = {(a, X), (b, Y), (a, Z)}

$S \circ R$ = {(0,X), (0,Z), (1,Y), (2,Y)}

$R \circ S$ does not exists because co-domain of S is different from domain of R.

Powers of a relation R on a set: R^n

The powers of relation R for n=1, 2, 3, ... are defined by:

$R^1 = R$

$R^{n+1} = R^n \circ R$, where $n > 0$

Eg:

$R = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$

$R^2 = R \circ R$

$R^2 = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$

$R^3 = R^2 \circ R$

$R^3 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$

$R^4 = R^3 \circ R$

$R^4 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$

Theorem: Let R be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b if and only if $(a, b) \in R^n$.

Proof: We will use mathematical induction. By definition, there is a path from a to b of length one if and only if $(a, b) \in R$, so the theorem is true when $n = 1$. Assume that the theorem is true for the positive integer n. This is the inductive hypothesis. There is a path of length n+1 from a to b if and only if there is an element $c \in A$ such that there is a path of length one from a to c, so $(a, c) \in R$, and a path of length n from c to b, that is, $(c, b) \in R^n$. Consequently, by the inductive hypothesis, there is a path of length n+1 from a to b if and only if there is an element c with $(a, c) \in R$ and $(c, b) \in R^n$. But there is such an element if and only if $(a, b) \in R^{n+1}$. Therefore, there is a path of length n+1 from a to b if and only if $(a, b) \in R^{n+1}$. This completes the proof.

Transitive Closure of a relation:

Let R be a relation on a set A. The **connectivity relation R^*** consists of the pairs (a, b) such that there is a path of length at least one from a to b in R.

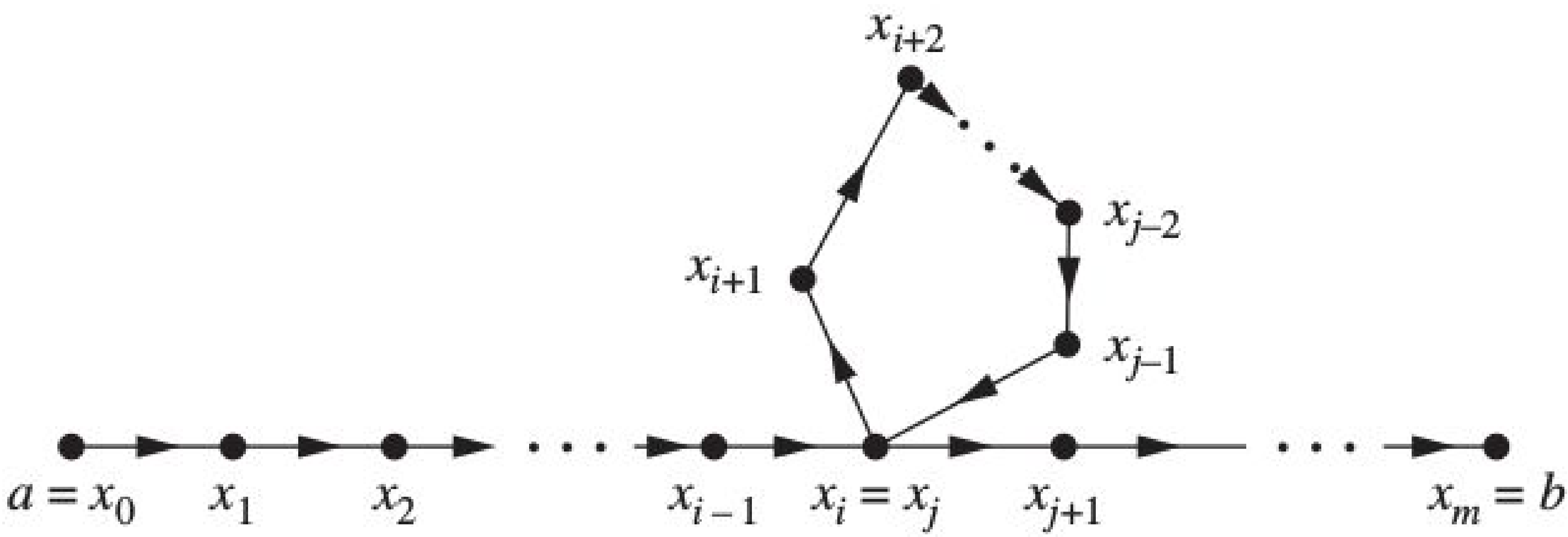
Because R^n consists of the pairs (a, b) such that there is a path of length n from a to b, it follows that R^* is the union of all the sets R^n. In other words,

R^* = Union of (R^n), for n=1 to ∞.

The **transitive closure** of a relation R equals the connectivity relation R^*.

Transitive Closure of a relation:

Let A be a set with n elements, and let R be a relation on A. If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n. Moreover, when a is not equal to b, if there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n−1.



Transitive Closure of a relation:

The **transitive closure of R** is the union of R, R^2, R^3, …, R^n.
This follows because there is a path in R^* between two vertices if and only if there is a path between these vertices in R^i, for some positive integer i with $i \le n$.
Therefore,
Transitive closure of R is $R^* = R \cup R^2 \cup R^3 \cup \cdots \cup R^n$

Transitive Closure of a relation:

Let M_R be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure R^* is

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \vee \mathbf{M}_R^{[n]}$$

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad \mathbf{M}_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad \mathbf{M}_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{M}_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Transitive Closure of a relation:

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \vee \mathbf{M}_R^{[n]}$$

A Procedure for Computing the Transitive Closure.

procedure *transitive closure* ($\mathbf{M}_R$: zero–one $n \times n$ matrix)
$\mathbf{A} := \mathbf{M}_R$
$\mathbf{B} := \mathbf{A}$
for $i := 2$ **to** n
    $\mathbf{A} := \mathbf{A} \odot \mathbf{M}_R$
    $\mathbf{B} := \mathbf{B} \vee \mathbf{A}$
return $\mathbf{B}$\{$\mathbf{B}$ is the zero–one matrix for R^*\}

Transitive Closure of a relation:

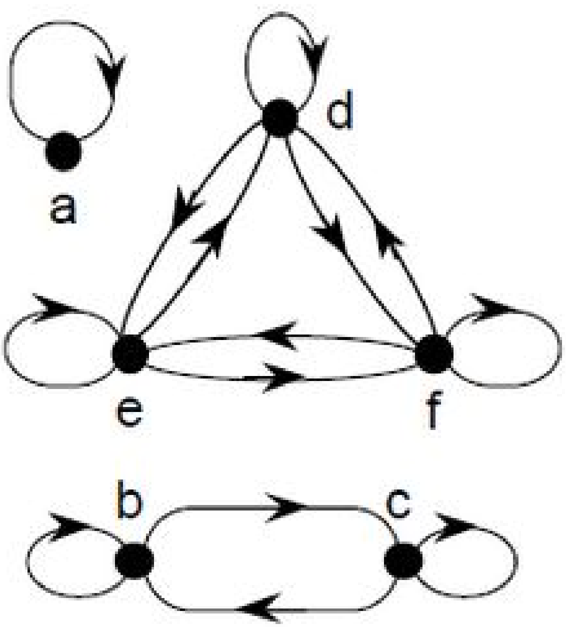
We know that to multiply two nxn matrices, it needs n^3 element-level multiplications. The procedure to find transitive closure needs n-1 such matrix multiplications. That implies, the running time of the procedure (aka algorithm) is in $O(n^4)$.

Warshall's algorithm finds transitive closure of a nxn matrix in $O(n^3)$ time.

Equivalence Relations:

A relation R on a set A is called an equivalence relation if it is **reflexive**, **symmetric**, and **transitive**.

Eg: Let S = {a,b,c,d,e,f}. Relation R on set S be {(a,a),(b,b),(b,c),(c,b),(c,c),(d,d),(d,e),(d,f),(e,d),(e,e),(e,f),(f,d),(f,e),(f,f)}



	a	b	c	d	e	f
a	1					
b		1	1			
c		1	1			
d				1	1	1
e				1	1	1
f				1	1	1

Q: Let $A = \{1, 2, 3, 4\}$ and

$R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (3,3), (4,4)\}$

be a relation on A. Verify that R is an equivalence relation.

Soln:

R is reflexive since it contains (1,1), (2,2), (3,3) and (4,4).

That is, $\forall x\ (x,x) \in R$

R is symmetric since it contains (1,2), (2,1), (3,4), (4,3) and no (a,b) where (b,a) is not in R.

That is, $\forall x \forall y\ ((x,y) \in R \rightarrow (y,x) \in R)$

R is transitive since for every pair of (x,y) and (y,z), there is (x,z) in R.

That is, $\forall x \forall y \forall z\ ((x,y) \in R \land (y,z) \in R \rightarrow (x,z) \in R)$

Therefore, R is an equivalence relation.

Q: Let R be a relation on the set of real numbers such that aRb iff a-b is an integer. Prove whether R is an equivalence relation.

Soln: …

Q: Let R be a relation on the set of real numbers such that aRb iff a-b is an integer. Prove whether R is an equivalence relation.

Soln: $a-a=0$ and $0 \in Z$

That is, $\forall a\ (aRa)$. $\therefore$ R is reflexive.

Let $a-b = k$ be an integer.

Then, $b-a = -k$, which is also an integer.

That is, if aRb, then bRa. $\therefore$ R is symmetric.

Let $a-b=k$ and $b-c=m$ where k and m are integers.

Then, $a-c = (a-b)-(c-b) = k-(-m)$, which is an integer.

That is, if aRb and bRc, then aRc. $\therefore$ R is transitive.

Because R is reflexive, symmetric and transitive,

R is an equivalence relation.

Q: Let ‘a’, ‘b’ and ‘m’ are integers with $m > 1$. Show that the relation $R = \{ (a, b) \mid a \equiv b \pmod{m} \}$ is an equivalence relation on the set of integers.

Soln:

If $a \equiv b \pmod{m}$, then “$m \mid (a - b)$” …… (read: m divides a-b)

…

Q: Let ‘a’, ‘b’ and ‘m’ are integers with $m > 1$. Show that the relation $R = \{ (a, b) \mid a \equiv b \pmod{m} \}$ is an equivalence relation on the set of integers.

Soln:

If $a \equiv b \pmod{m}$, then “$m \mid (a - b)$” …… (read: m divides a-b)

$a \equiv a \pmod{m}$ because $m|a-a$, which is same as $m|0$.

$\therefore$R is reflexive.

Let $a \equiv b \pmod{m}$

i.e., $m \mid (a-b)$

$mk = a-b$, where k is an integer

$m(-k) = b-a$

i.e., $b \equiv a \pmod{m}$ because $-k$ is also an integer

$\therefore$ R is symmetric.

Q: Let ‘a’, ‘b’ and ‘m’ are integers with $m > 1$. Show that the relation $R = \{ (a, b) \mid a \equiv b \pmod{m} \}$ is an equivalence relation on the set of integers.

Soln:

…

Let $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$

i.e., $mk = a-b$ and $ml = b-c$, where k and l are integers.

$mk+ml = a-b+b-c$

$m(k+l) = a-c$

i.e., $a \equiv c \pmod{m}$

$\therefore$ R is transitive.

R is reflexive, symmetric and transitive.

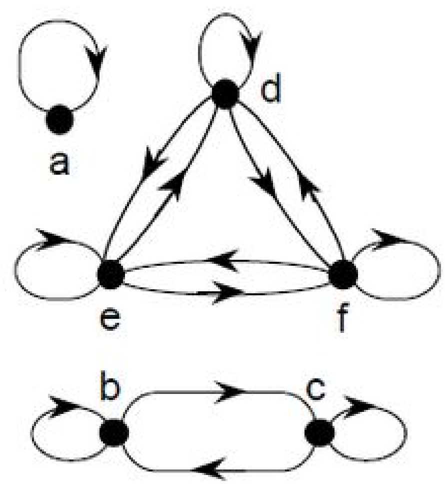
$\therefore$ R is an equivalence relation.

Equivalence Classes:

Let R be an equivalence relation on a set A. The set of all the elements that are related to an element ‘a’ of A is called equivalence class of ‘a’ denoted by $[a]_R$ or $[a]$ when the relation is implicit.

i.e. $[a]_R = \{s \mid (a,s) \in R\}$

Elements of $[a]_R$ are also known as representatives of $[a]_R$.



	a	b	c	d	e	f
a	1					
b		1	1			
c		1	1			
d				1	1	1
e				1	1	1
f				1	1	1

Q: What are the equivalence classes of 0 and 1 for the congruence modulo 10?

Soln:

The equivalence class of 0 contains all integers ‘a’ such that $a \equiv 0 \pmod{10}$.

$[0] = \{ \ldots, -20, -10, 0, 10, 20, \ldots \}$

Similarly, the equivalence class of 1 contains all integers ‘a’ such that $a \equiv 1 \pmod{10}$.

$[1] = \{ \ldots, -19, -9, 1, 11, 21, \ldots \}$

Congruence classes modulo m: are the equivalence classes of the relation congruence modulo m.

$[a]_m = \{ \ldots, a-2m, a-m, \mathbf{a}, a+m, a+2m, \ldots \}$

Q: What is the equivalence class of an integer ‘a’ for the equivalence relation R defined by
aRb iff a = b or a = -b?

Soln:
aRb iff a = b or a = -b means a related itself and negative of itself.

That is, $[a]_R = \{a, -a\}$

For example, $[10]_R = \{10, -10\}$
$[-100]_R = \{100, -100\}$
$[0]_R = \{0\}$

Theorem: Let R be an equivalence relation on a set A.
These statements for elements a and b of A are equivalent:
(i) aRb (ii) $[a] = [b]$ (iii) $[a] \cap [b] \neq \varnothing$

Note:
1. Two equivalence classes are either disjoint or identical.

2. Let R be an equivalence relation on a set A and let $a,b \in A$.
   If $[a] \neq [b]$ then $[a] \cap [b] = \varnothing$.

3. For $a,b \in A$, if $b \in [a]$ then $[a] = [b]$.

Partition of a Set:

Let A be a nonempty set. Let P be a set of nonempty subsets $A_1, A_2, \ldots, A_n$ of the set A such that

$A_i \cap A_j = \varnothing$ for $i \neq j$ … Mutually Exclusive

$A_1 \cup A_2 \cup \ldots \cup A_n = A$ … Collectively exhaustive

The set $P = \{A_1, A_2, \ldots, A_n\}$ is called the partition of A.

Partial Orderings

Partial Order:

A relation R on the set S is called a partial order/ordering if it is **reflexive**, **antisymmetric** and **transitive**.

Poset (S, R):

Relation R is a partial ordering on set S.

Eg: $(Z, \leq)$ is a Poset.

Eg: $(Z^{+}, |)$ is a Poset.

$(S, \preccurlyeq)$:

$(S, \preccurlyeq)$ is notation for poset where relation $\preccurlyeq$ is a partial ordering on set S.

Q: Show that (Z^+, |) is a Poset. (It's the divisibility relation)

Soln: a|a for every integer a.

∴ | is Reflexive

Whenever a≠b, at least one of a|b or b|a is false.

∴ | is antisymmetric.

Whenever a|b and b|c, a|c.

∴ | is transitive.

∴ (Z^+, |) is a Poset.

Q: Show that $(P(S), \subseteq)$ is a Poset.

Soln: …

Comparable and Incomparable:
Elements a and b are incomparable when they are elements of a poset $(S, \leqslant)$ such that neither $a \leqslant b$ nor $b \leqslant a$.

Eg: In poset $(Z^+, \leq)$, for every pair (a,b) either aRb or bRa. For instance, $10 \leq 20$. That is, every pair (a,b) is comparable.

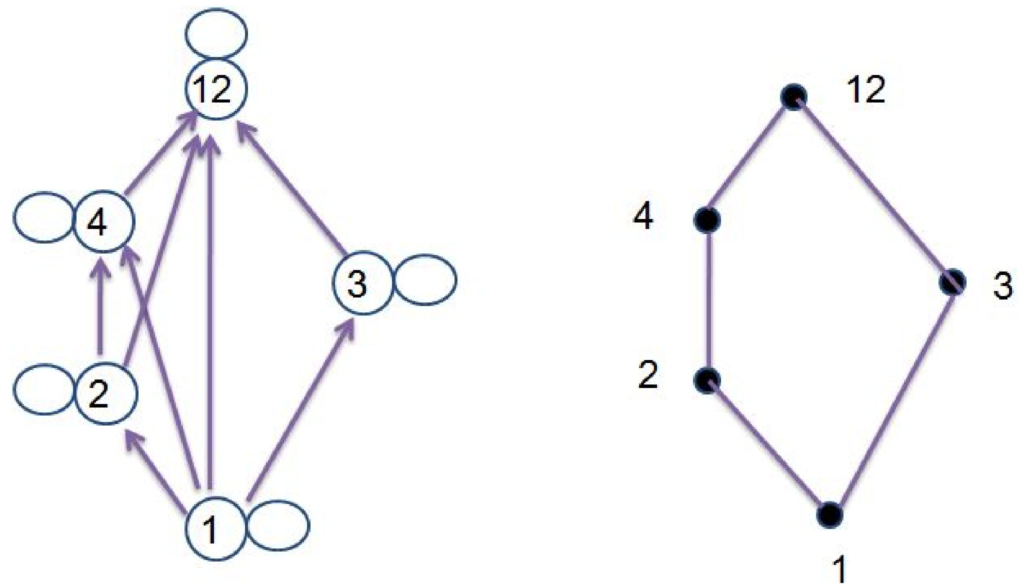
Eg: In poset $(Z^+, |)$, 5 and 7 are incomparable because $5|7$ is false and $7|5$ is false. Whereas 6 and 18 are comparable because 6 divides 18.

Total Ordering (Linear Ordering):
Every pair of elements in S are comparable.
Eg: $(Z, \leq)$

Hasse Diagram:

In a digraph of a Partial Order,

1. **remove self-loops** because we know the partial order is **reflexive**.



2. **remove direction marks** from edges because we know that the edges always point upwards as the relation is **antisymmetric**.

3. **remove transitive edges** because we know the relation is **transitive**. If there are edges (a,b) and (b,c), remove (a,c).

Eg: Course prerequisites

a: IntroCS

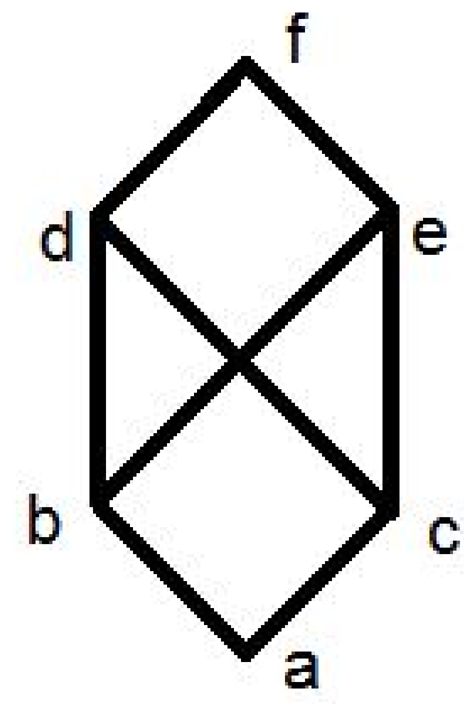
b: DML (requires IntroCS)

c: DS (requires IntroCS)

d: DAA (requires DS and DML)

e: DBMS (requires DS and DML)

f: AppDev (requires DAA and DBMS)



Maximal and Minimal elements:

'a' is a maximal in the poset $(S, \leqslant)$ if there is no $b \in S$ such that $a < b$.

'a' is a minimal in the poset $(S, \leqslant)$ if there is no $b \in S$ such that $b < a$.

Greatest and Least elements:

'a' is the greatest element of the poset $(S, \leqslant)$ if $b \leqslant a$ for all $b \in S$.

'a' is the least element of the poset $(S, \leqslant)$ if $a \leqslant b$ for all $b \in S$.

Upper Bound and Lower Bound elements:

If 'u' is an element of S such that $a \leqslant u$ for all elements $a \in A$, then u is an upper bound of A.

If 'l' is an element of S such that $l \leqslant a$ for all elements $a \in A$, then l is a lower bound of A.

Least Upper Bound and Greatest Lower Bound elements:

'x' is an upper bound that is less than every other upper bound of A.

'l' is a lower bound that is greater than every other lower bound of A.

Maximal elements:12

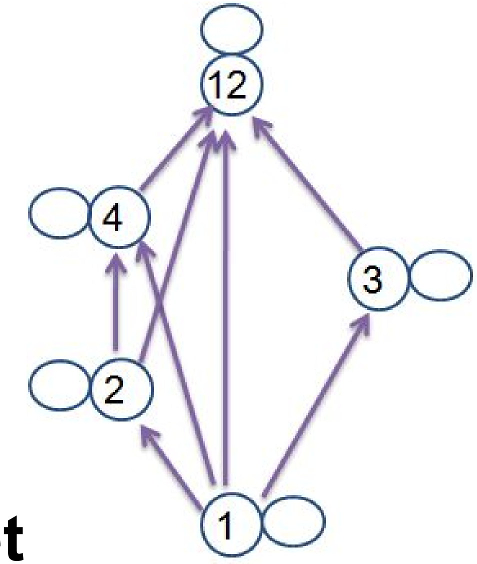
Minimal elements: 1

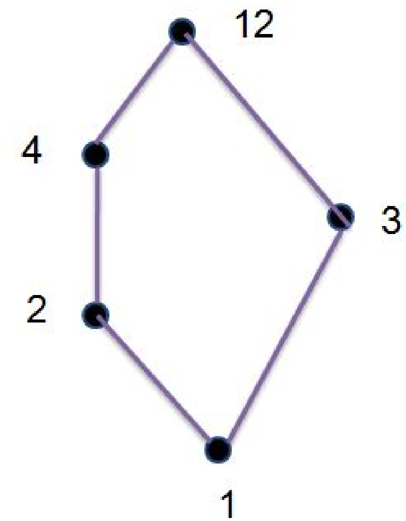
Greatest element: 12

Least element: 1

These above ones

are defined on the poset





The following are defined on a subset of the poset.

Let the subset be A= {2,3}

Upper bound of A: 12

Lower bound of A: 1

Least Upper bound of A: 12

Greatest Lower bound of A: 1

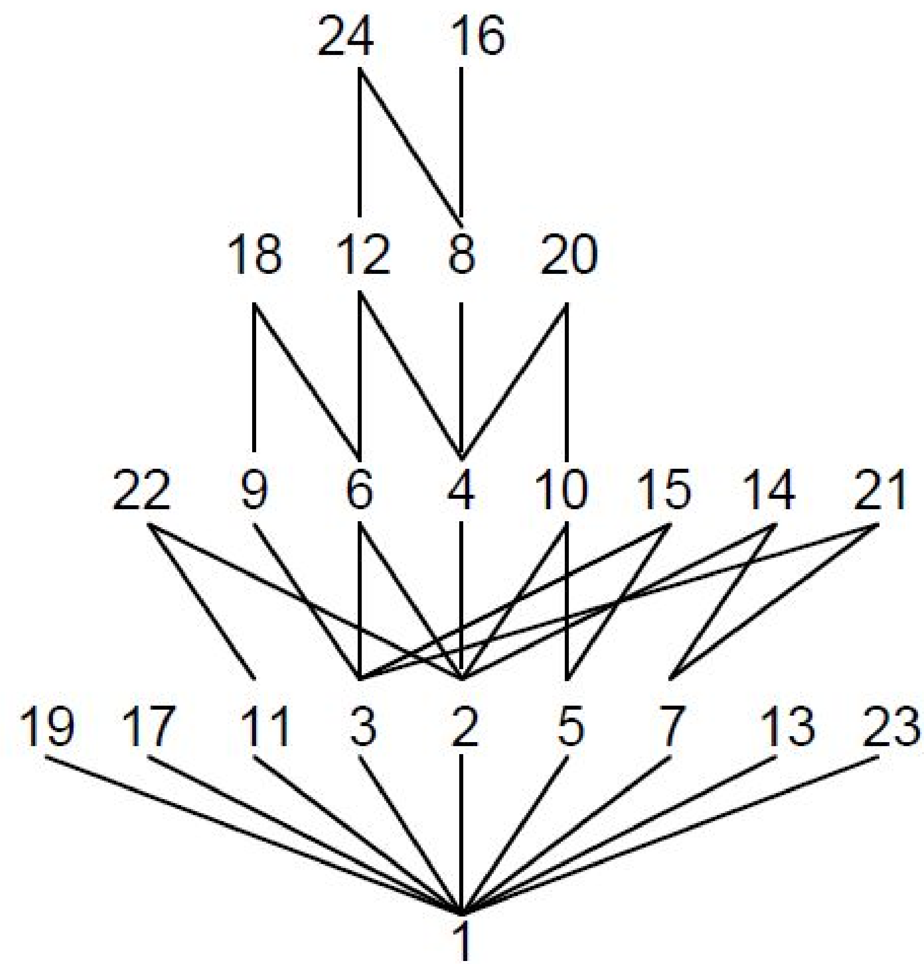
Eg: Poset ({1,2,...,24}, |). It's a "divides" relation.

Maximal elements:

Minimal elements:

Greatest element:

Least element:



For the subset A = {2,3},

Upper bound of A:

Lower bound of A:

Least Upper bound of A:

Greatest Lower bound of A:

For the subset A = {6,10}

Upper bound of A:

Lower bound of A:

Least Upper bound of A:

Greatest Lower bound of A:

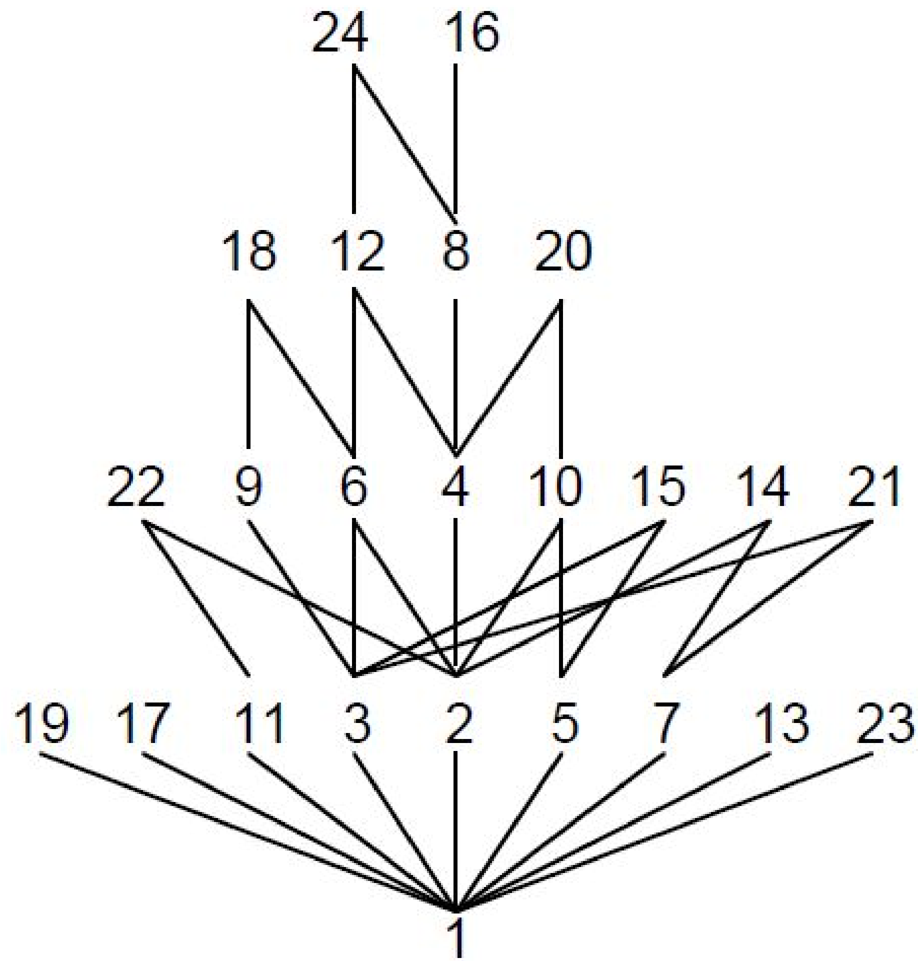
Eg: Poset ({1,2,...,24}, |). It's a "divides" relation.

Maximal elements: **24, 16, 18, 20, 22, 15, 14, 21, 19, 17, 13, 23**

Minimal elements: **1**

Greatest element: **None**

Least element: **1**



For the subset A = {2,3},

Upper bound of A: **6, 12, 18, 24**

Lower bound of A: **1**

Least Upper bound of A: **6**

Greatest Lower bound of A: **1**

For the subset A = {6,10}

Upper bound of A: **None**

Lower bound of A: **2, 1**

Least Upper bound of A: **None**

Greatest Lower bound of A: **2**

Eg: Let S = Power set of {a,b,c}. Poset (S, $\subseteq$).

Maximal elements:

Minimal elements:

Greatest element:

Least element:

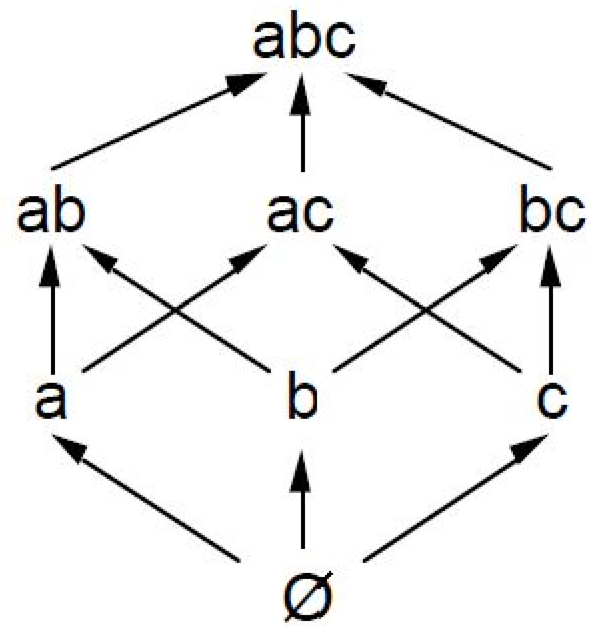
For the subset A = **{ab,b},**

Upper bound of A:

Lower bound of A:

Least Upper bound of A:

Greatest Lower bound of A:



Eg: Let S = Power set of {a,b,c}. Poset (S, $\subseteq$).

Maximal elements: **abc**

Minimal elements: **ϕ**

Greatest element: **abc**

Least element: **ϕ**

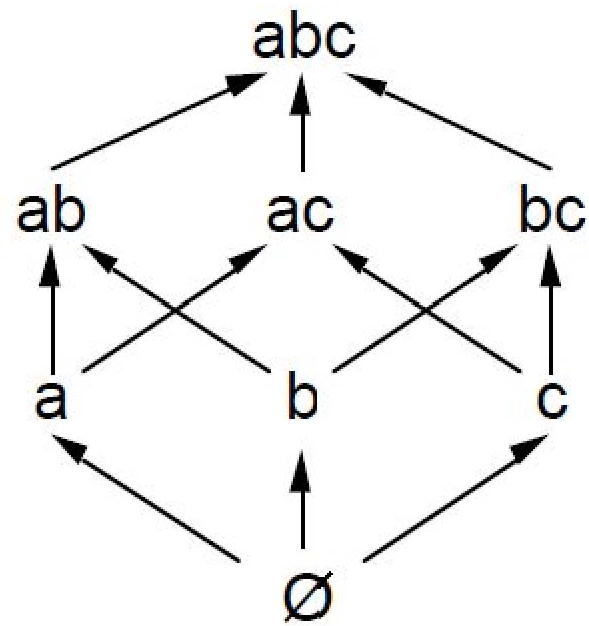
For the subset A = **{ab,b},**

Upper bound of A: **ab, abc**

Lower bound of A: **b, ϕ**

Least Upper bound of A: **ab**

Greatest Lower bound of A: **b**



Maximal elements:

Minimal elements:

Greatest element:

Least element:

For the subset A={d,e,f}

Upper bound of A:

Lower bound of A:

Least Upper bound of A:

Greatest Lower bound of A:

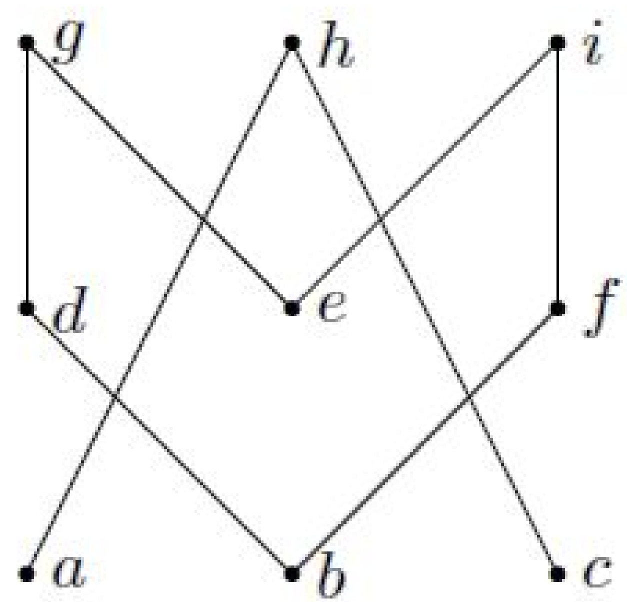
For the subset A={b,d}

Upper bound of A:

Lower bound of A:

Least Upper bound of A:

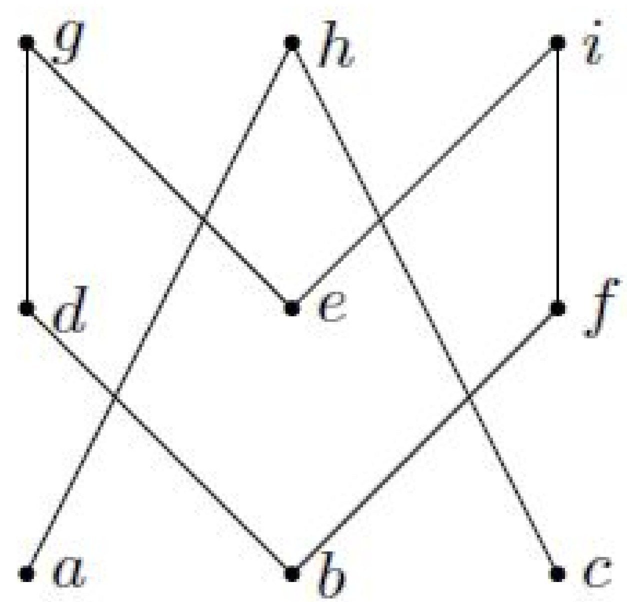
Greatest Lower bound of A:



Maximal elements: **g,h,i**
Minimal elements: **a,b,c,e**
Greatest element: **None**
Least element: **None**

For the subset A={d,e,f}
Upper bound of A: **None**
Lower bound of A: **None**
Least Upper bound of A: **None**
Greatest Lower bound of A: **None**

For the subset A={b,d}
Upper bound of A: **d,g**
Lower bound of A: **b**
Least Upper bound of A: **d**
Greatest Lower bound of A: **b**



Maximal elements:

Minimal elements:

Greatest element:

Least element:

For the subset A={c,e}

Upper bound of A:

Lower bound of A:

Least Upper bound of A:

Greatest Lower bound of A:

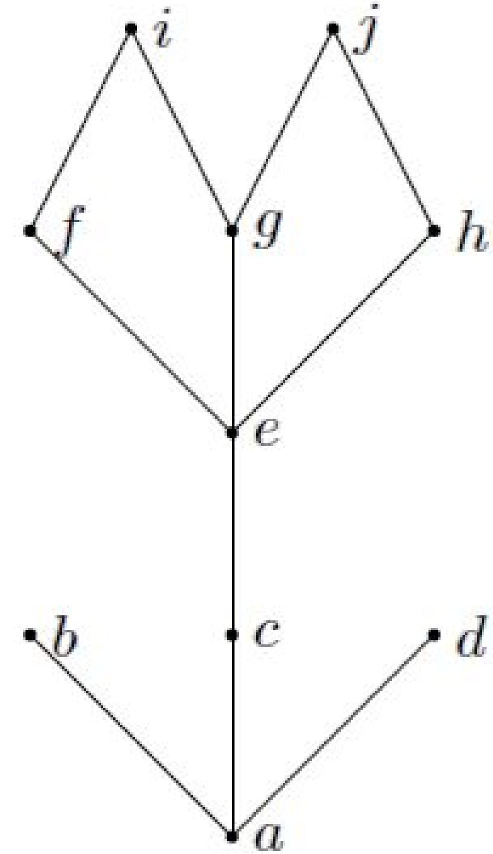
For the subset be A={b,i}

Upper bound of A:

Lower bound of A:

Least Upper bound of A:

Greatest Lower bound of A:



Maximal elements: **b,d,i,j**

Minimal elements: **a**

Greatest element: **None**

Least element: **a**

For the subset A={c,e}

Upper bound of A: **e,f,g,h,i,j**

Lower bound of A: **c,a**

Least Upper bound of A: **e**

Greatest Lower bound of A: **c**

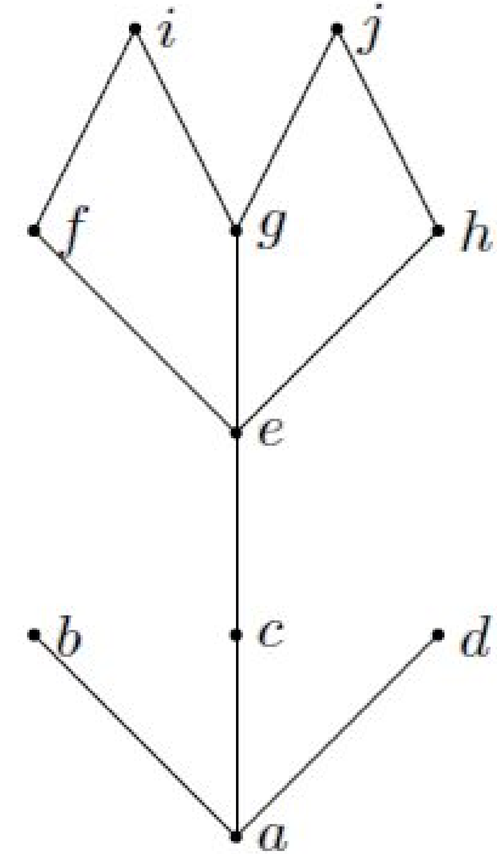
For the subset be A={b,i}

Upper bound of A: **None**

Lower bound of A: **a**

Least Upper bound of A: **None**

Greatest Lower bound of A: **a**

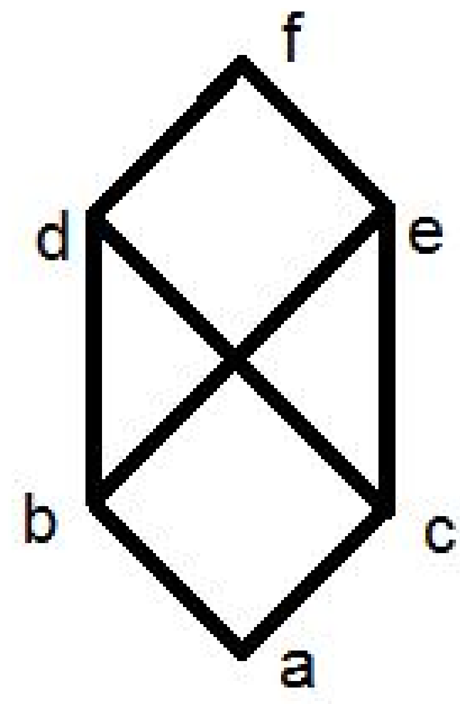


Maximal elements: **f**

Minimal elements: **a**

Greatest element: **f**

Least element: **a**



For the subset A={b, c}

Upper bound of A: **d,e,f**

Lower bound of A: **a**

Least Upper bound of A: **None**

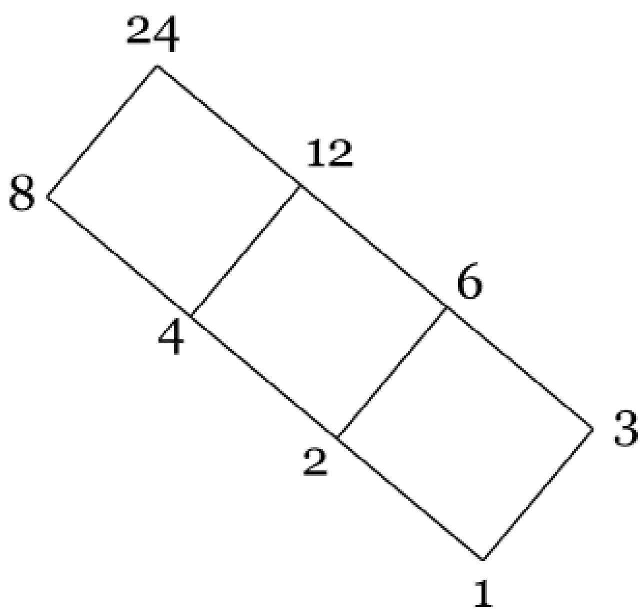
Greatest Lower bound of A: **a**

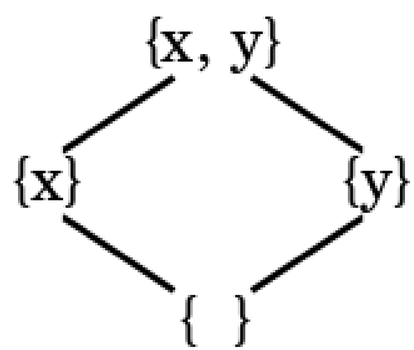
Note: The above example demonstrates, having multiple upper bounds for a pair of elements doesn’t guarantee to have a least upper bound.

Lattice:

A partially ordered set in which every pair of elements has both a **least upper bound** and **greatest lower bound** is called a **lattice**.

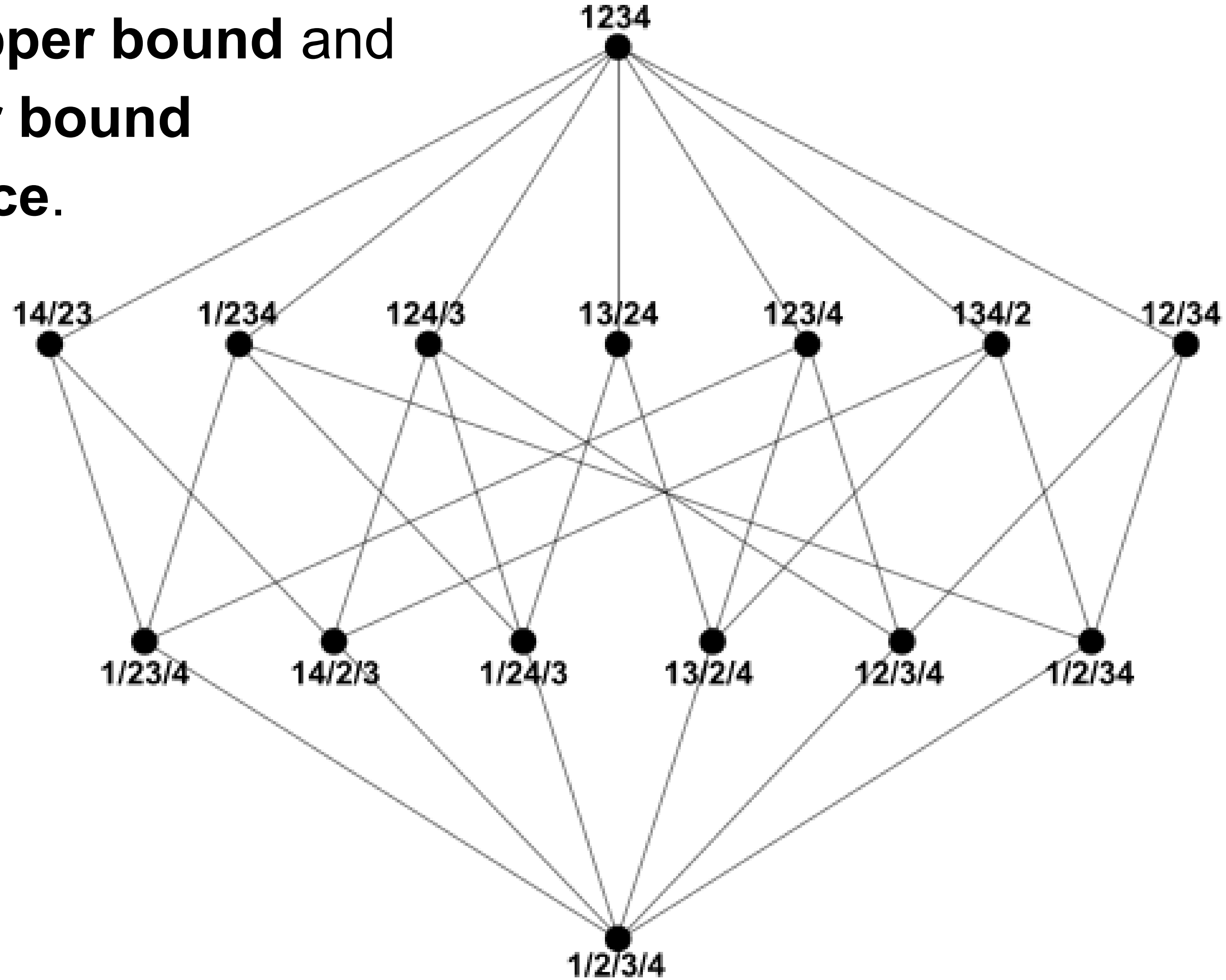
Eg: Poset (Z^+, |)





Lattice:

A partially ordered set in which every pair of elements has both a **least upper bound** and **greatest lower bound** is called a **lattice**.

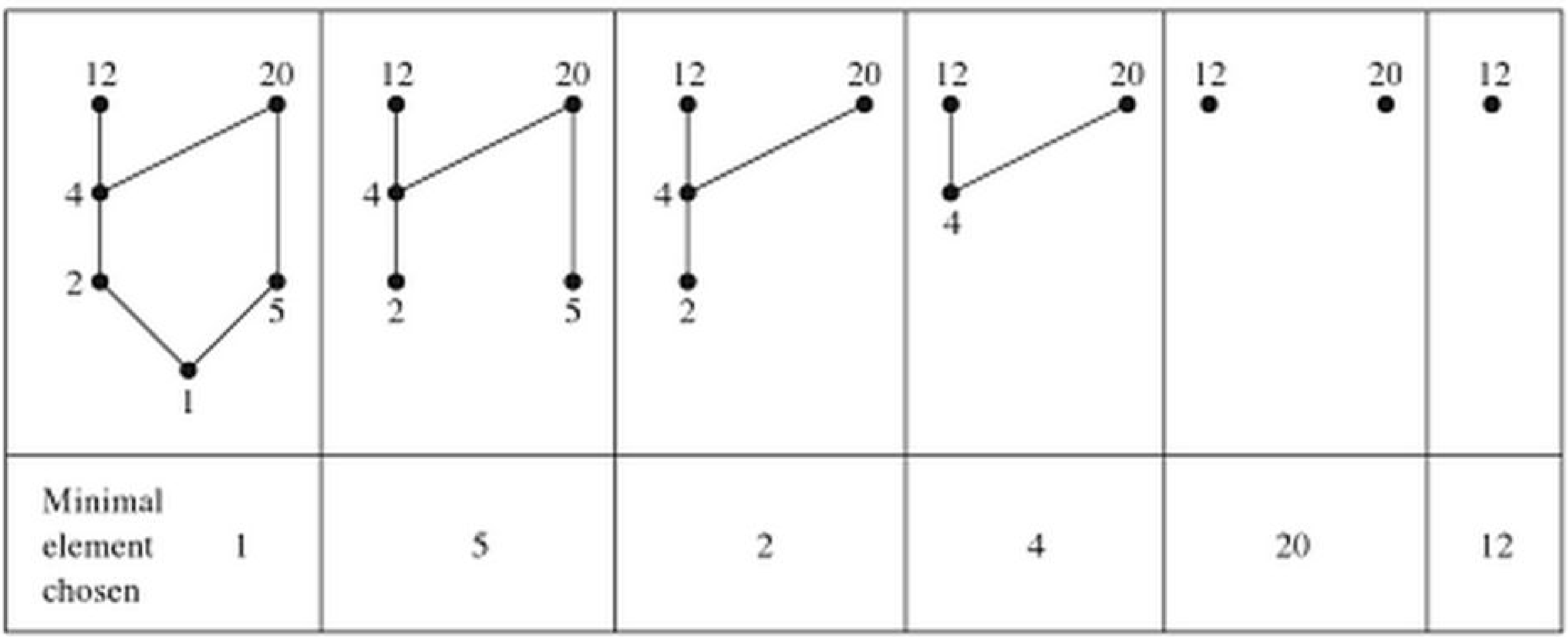


Topological Sorting: Constructing a **compatible total ordering** from a partial ordering is called **topological sorting**.

What makes the total ordering compatible with a partial ordering?

If (a,b)∈R, then a⩽b is in the total ordering.

If a and b are not comparable, then it is either a⩽b or b⩽a in the total ordering.



Lemma: Every finite nonempty poset $(S, \preccurlyeq)$ has at least one minimal element.

Topological Sorting Algorithm:

Algorithm 1 Topological Sorting

---

procedure *topological sort* $((S, \preccurlyeq)$: finite poset)
$k := 1$
while $S \neq \emptyset$
begin
  $a_k :=$ a minimal element of S {such an element exists by Lemma 1}
  $S := S - \{a_k\}$
  $k := k + 1$
end $\{a_1, a_2, \ldots, a_n$ is a compatible total ordering of $S\}$

Q: Find a compatible total ordering for the poset $(\{1, 2, 4, 5, 12, 20\}, |)$.

Soln: …

Q: Let S be Power set of {a,b,c}.
Find a compatible total ordering for the poset $(S, \subseteq)$.

Soln: …

Eg: How many functions are there from a set with **m** elements to a set with **n** elements?

Ans: n * n * … n (m times) = $\mathbf{n^m}$

Eg: How many **one-to-one** functions are there from a set with **m** elements to a set with **n** elements?

Ans: n * (n - 1) * (n - 2) * … * (n - m + 1),
where m ≤ n
= $\mathbf{{}^nP_m}$

Eg: How many **onto** functions … (**n ≤ m**)
Ans: …

<End of Set Theory />