# QEEE DSA05
# DATA STRUCTURES AND ALGORITHMS

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LECTURE 7, 26 AUGUST 2014

# Recall that …

- BFS and DFS are two systematic ways to explore a graph
  - Both take time linear in the size of the graph with adjacency lists
- Recover paths by keeping parent information
- BFS can compute shortest paths, in terms of number of edges
- DFS numbering can reveal many interesting features

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  - Distance travelled between two stations
  - Typical time between two locations during peak hour traffic

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- **Shortest path** from $v_0$ to $v_n$ : minimum cost

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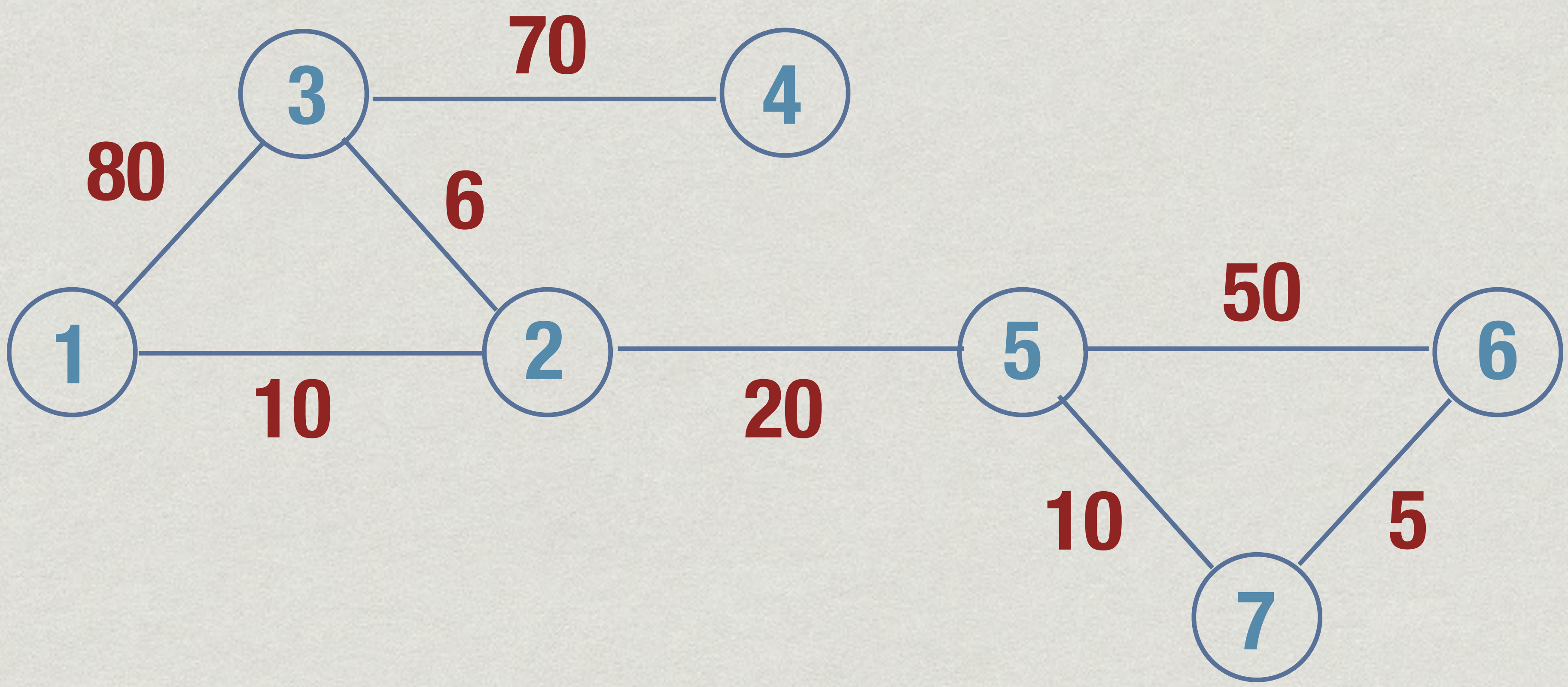

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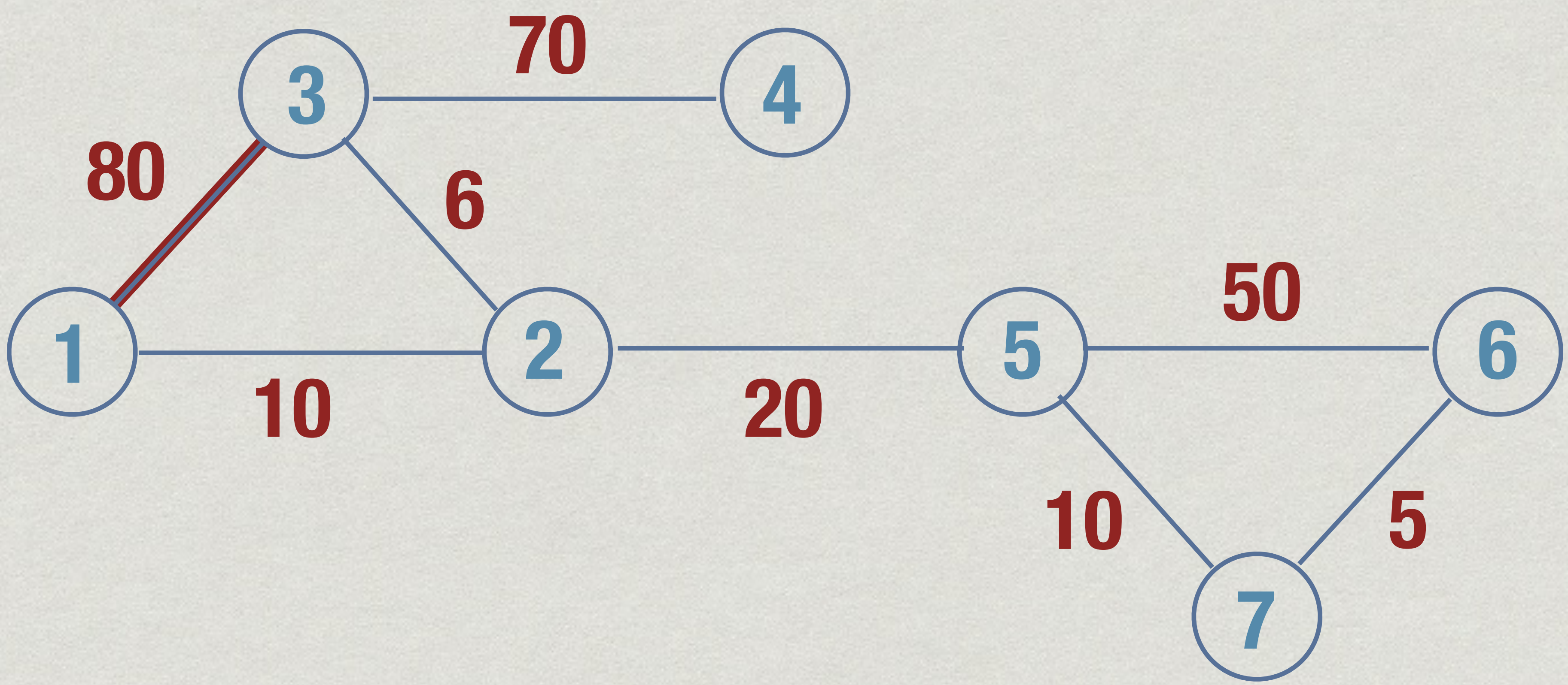

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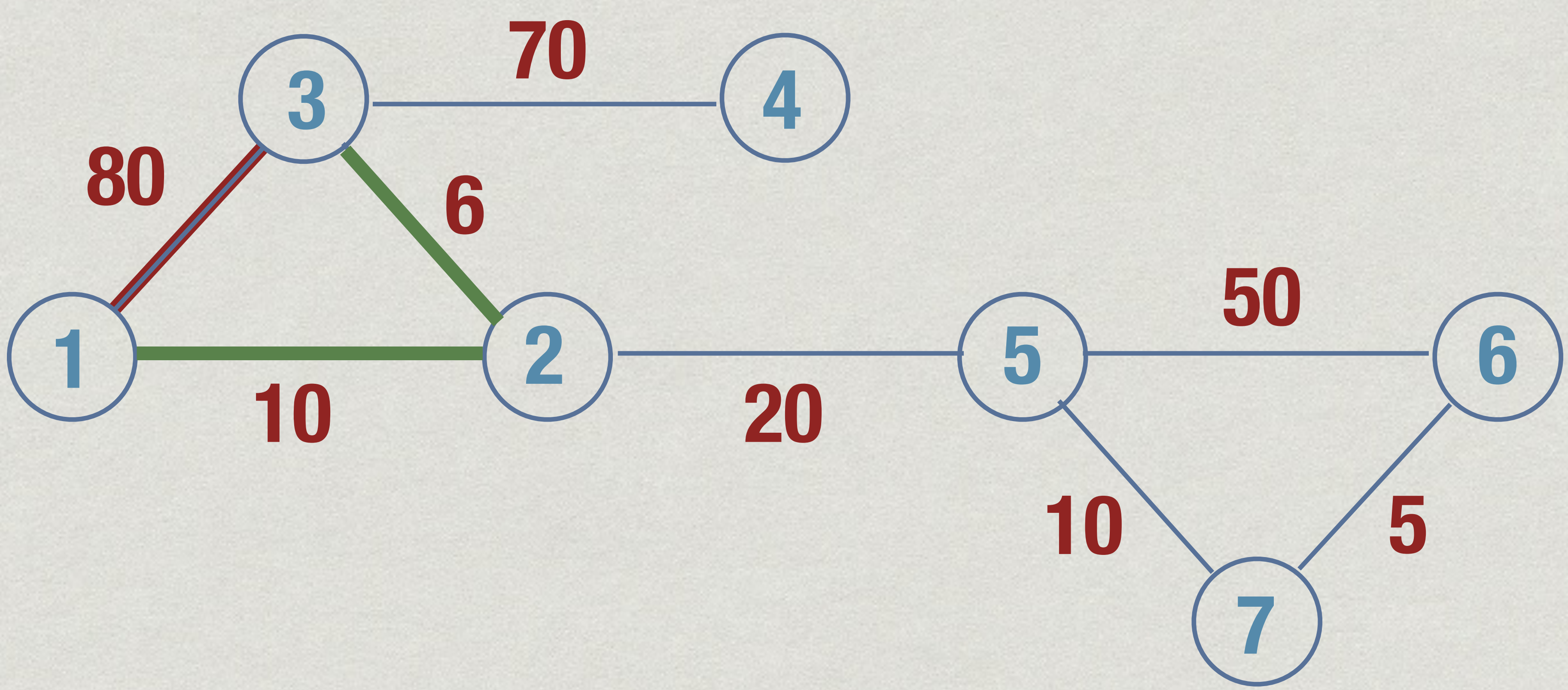

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  - Transport finished product from factory (single source) to all retail outlets
  - Courier company delivers items from distribution centre (single source) to addressees

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  - Railway routes, shortest way to travel between any pair of cities

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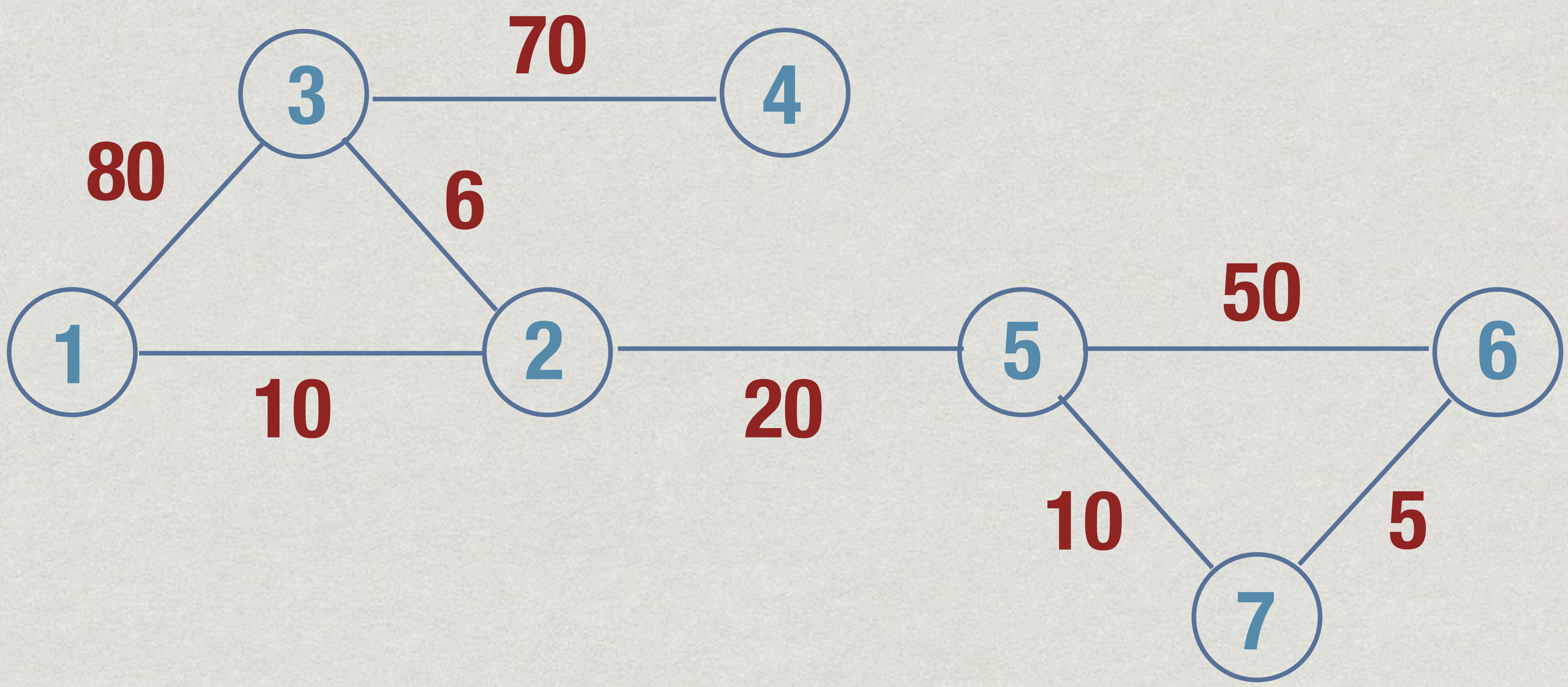
- For instance, shortest paths from 1 to 2,3,…,7



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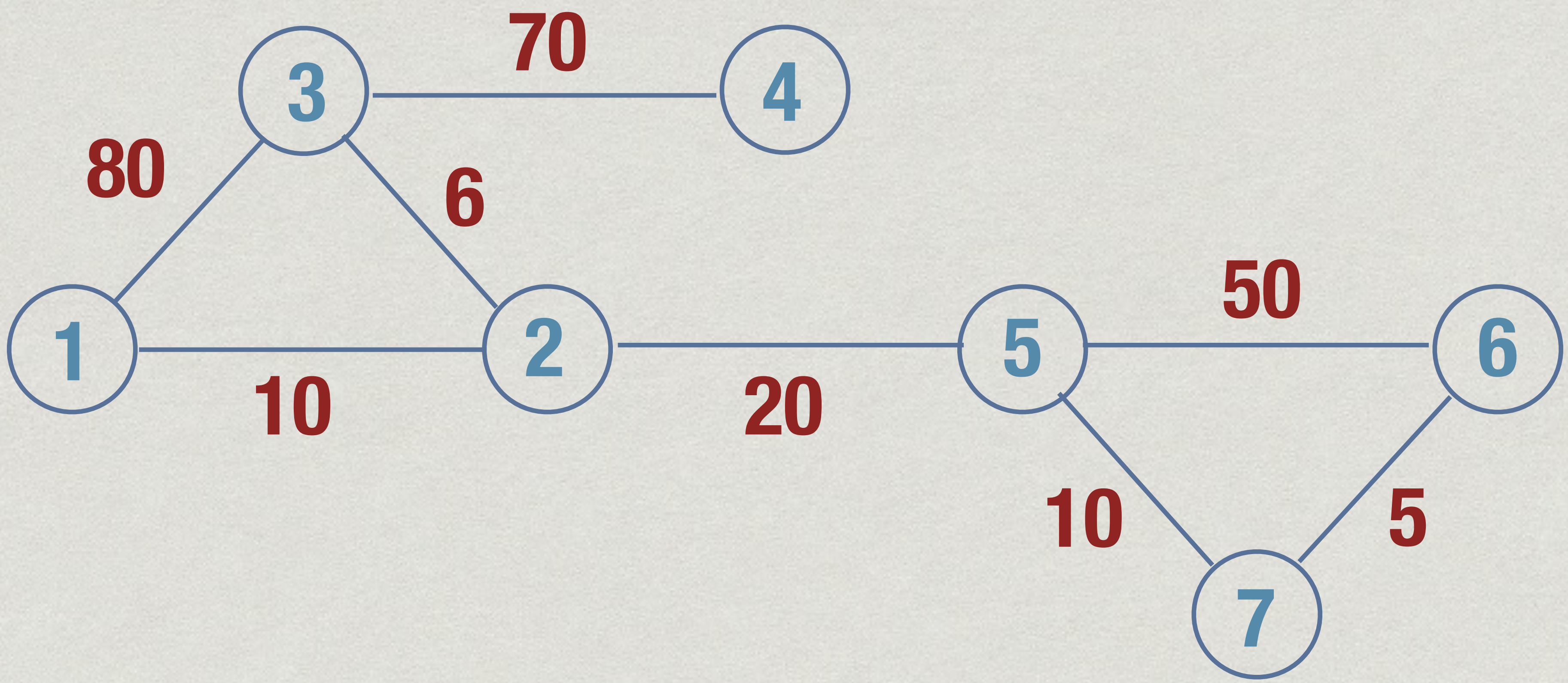

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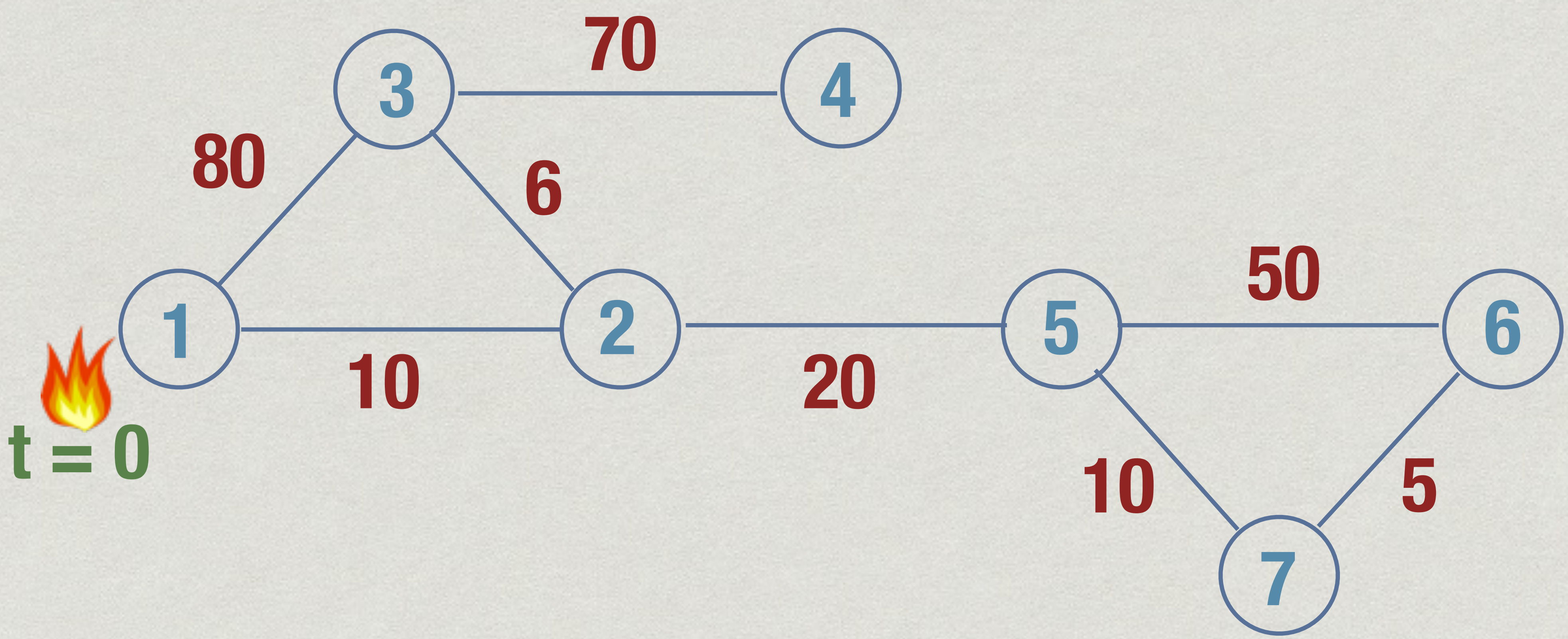

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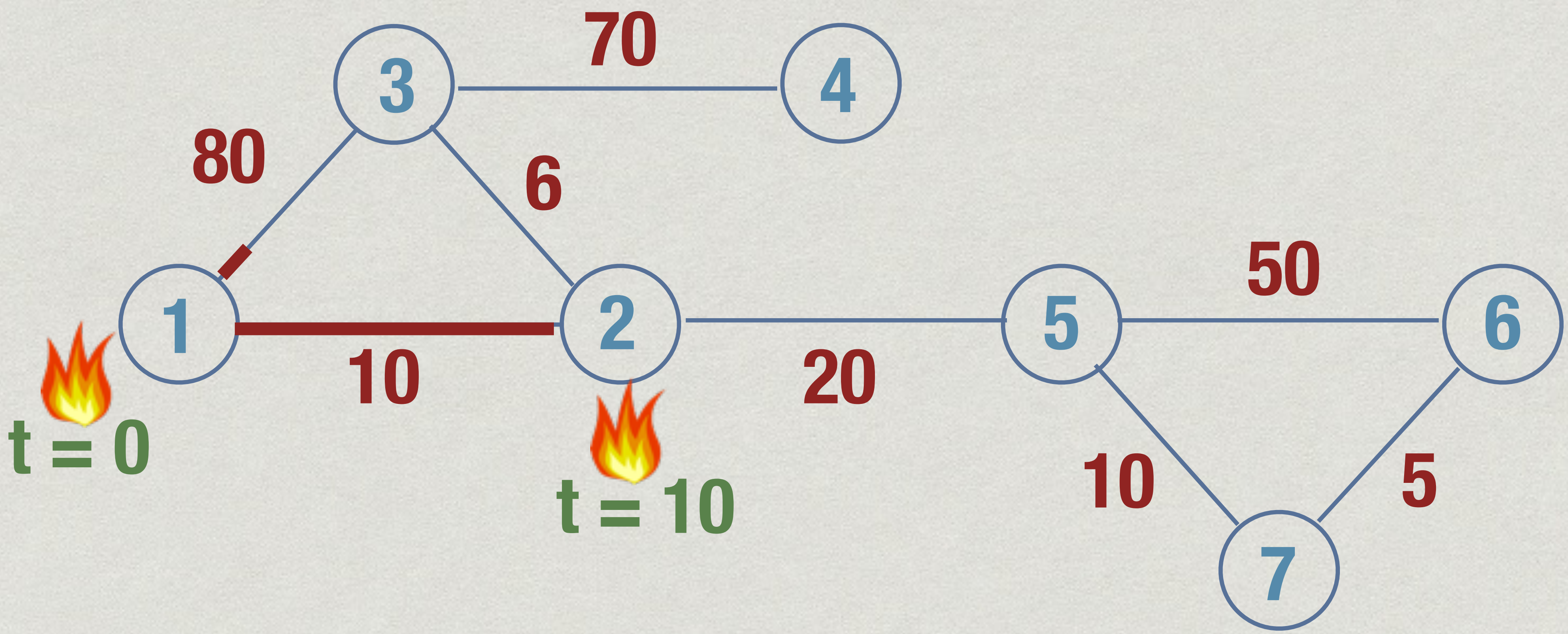

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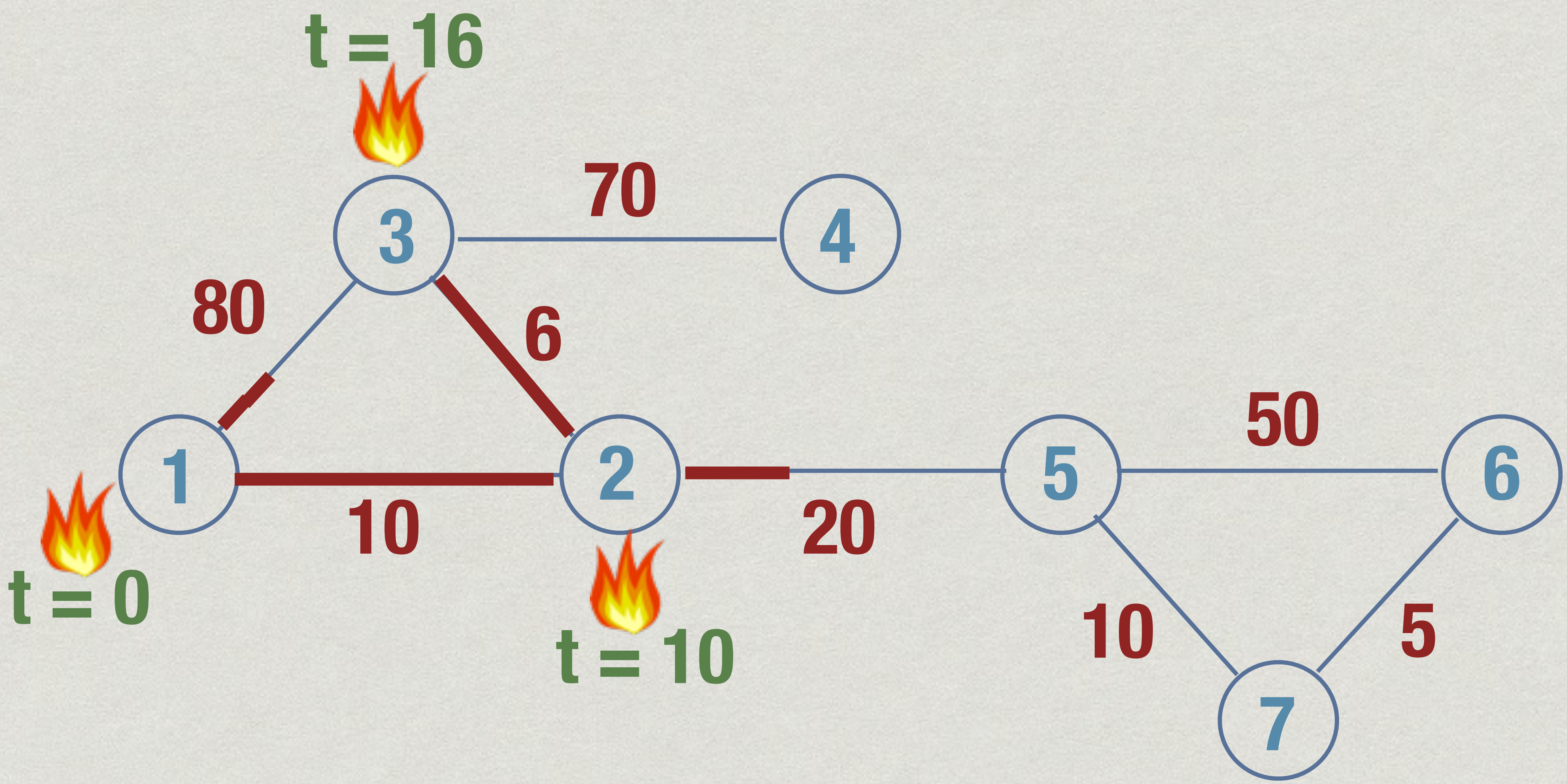

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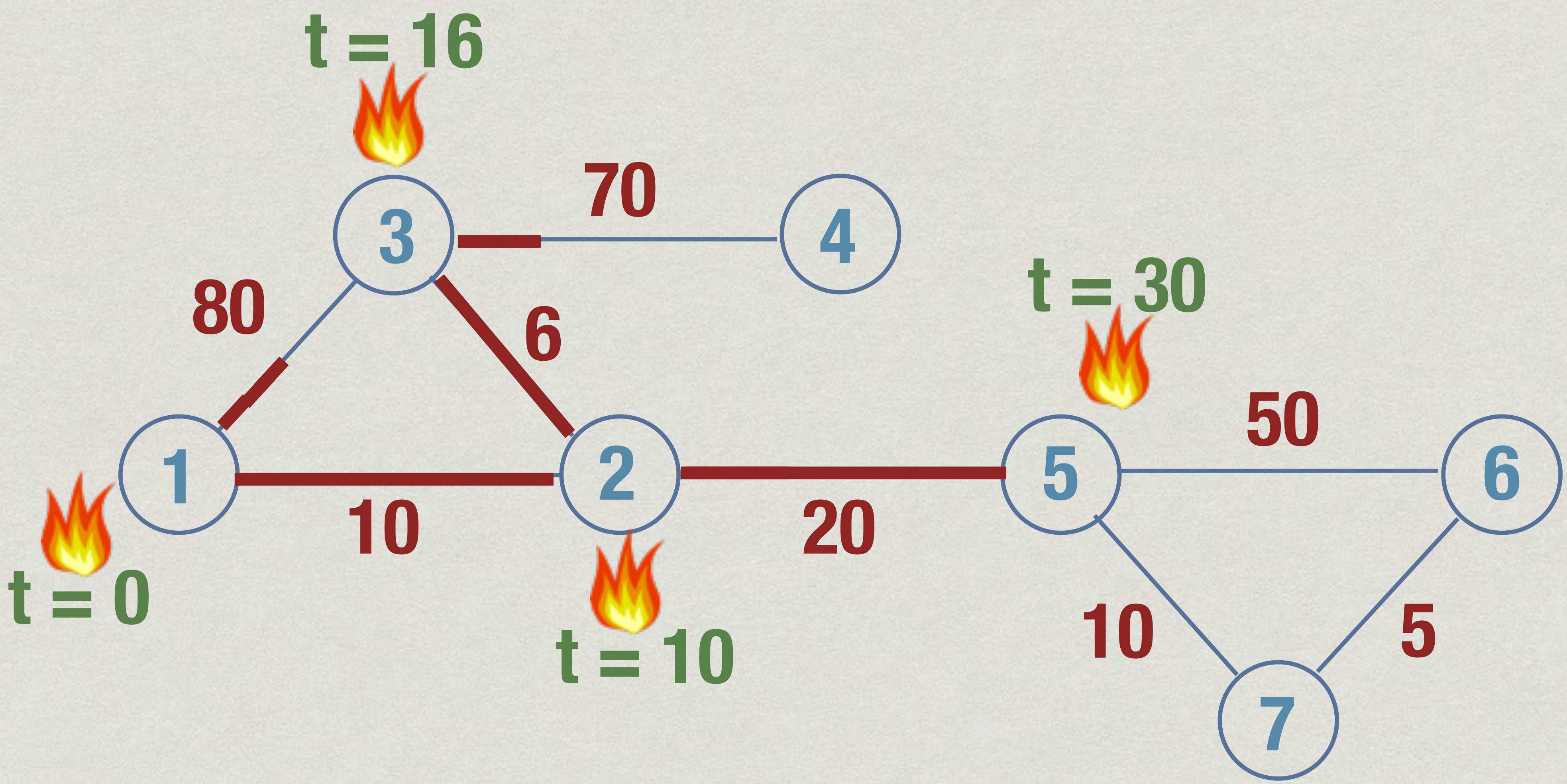

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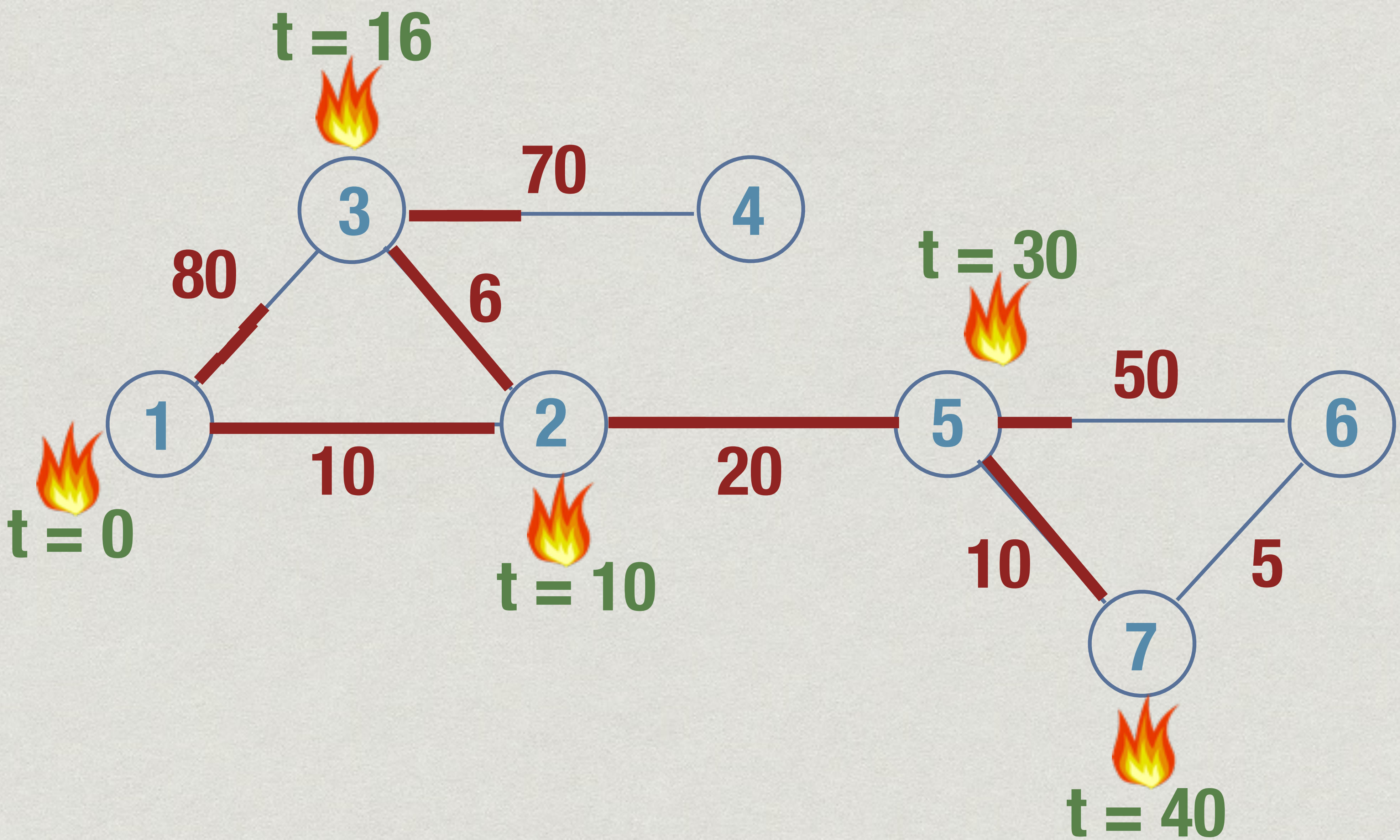

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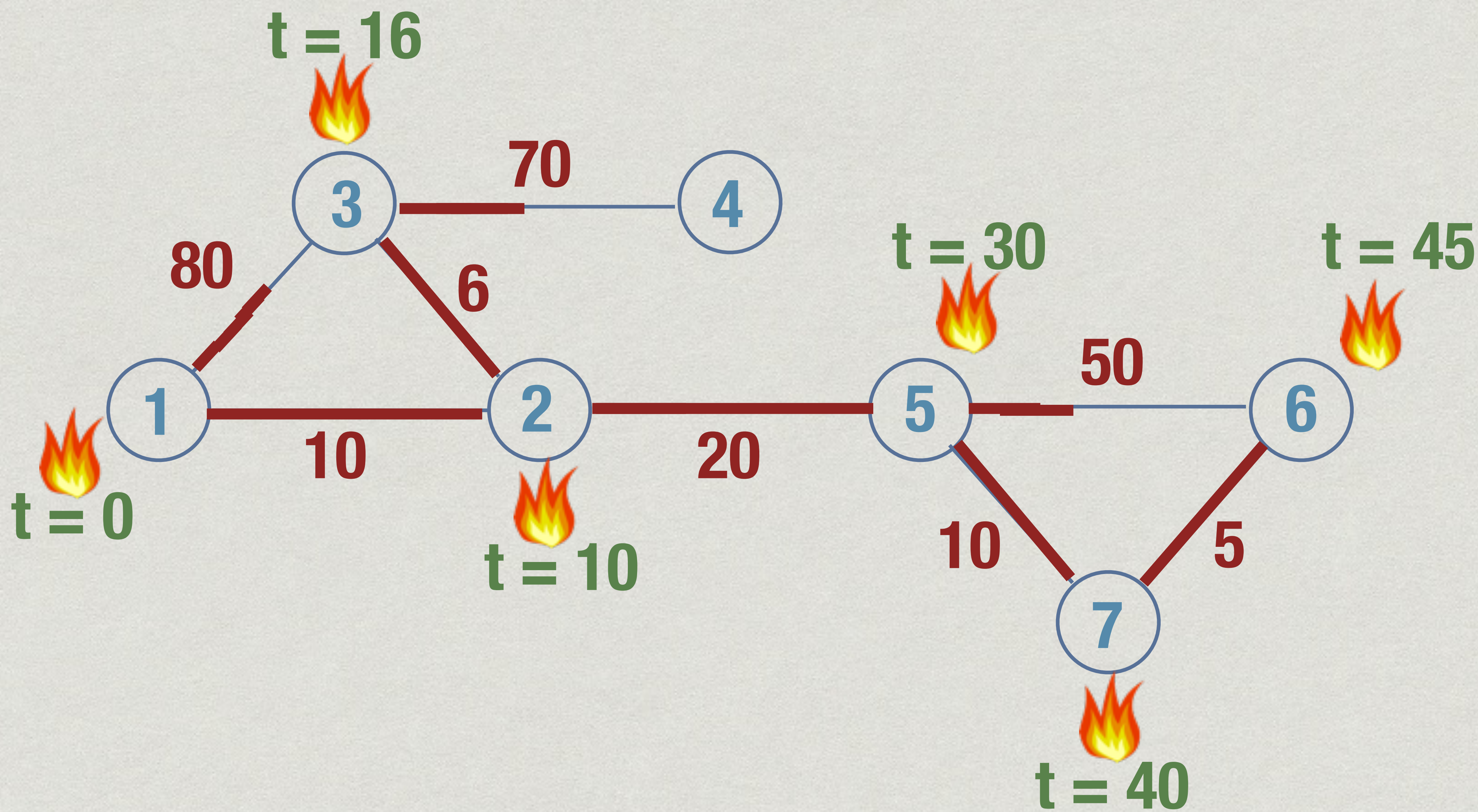

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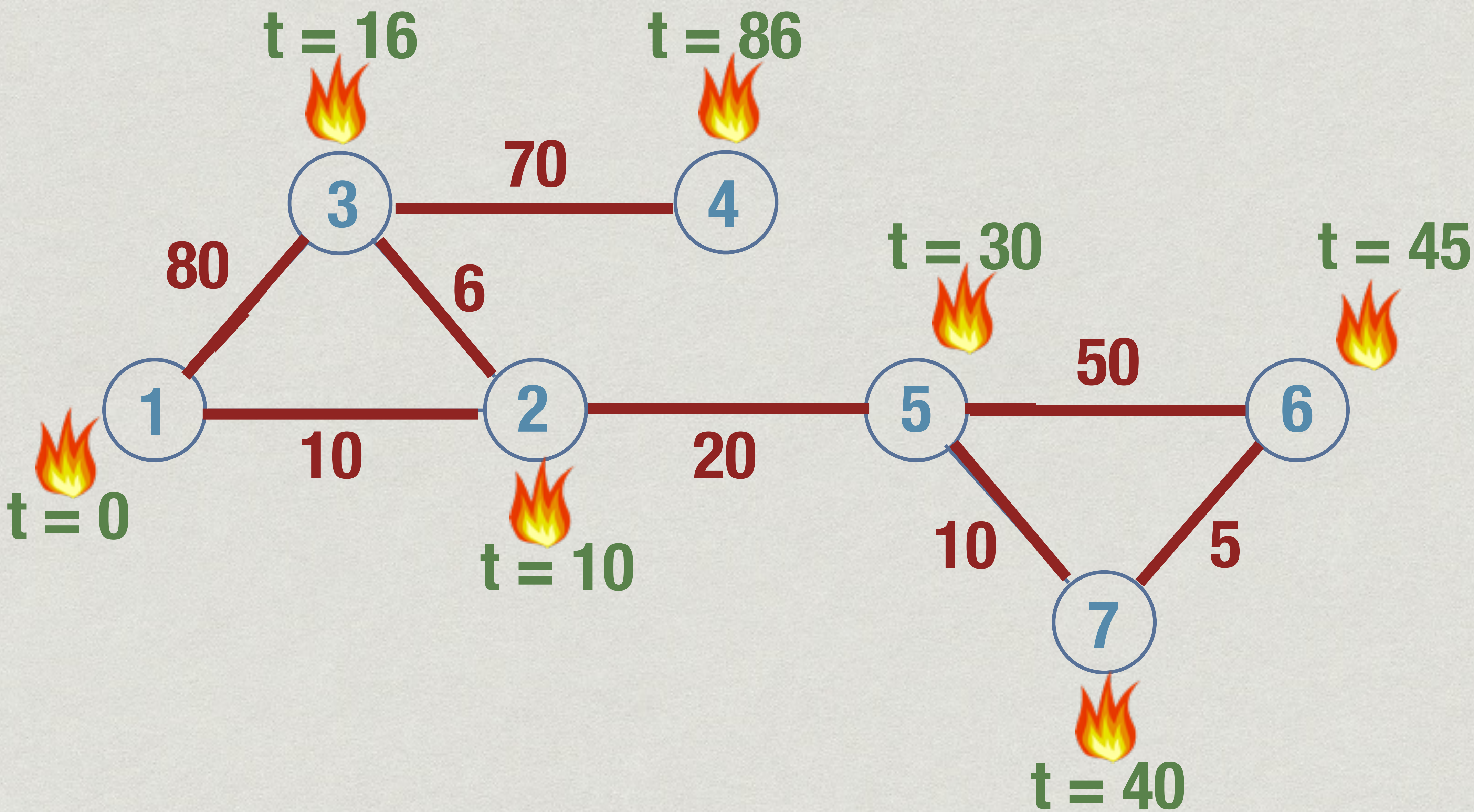

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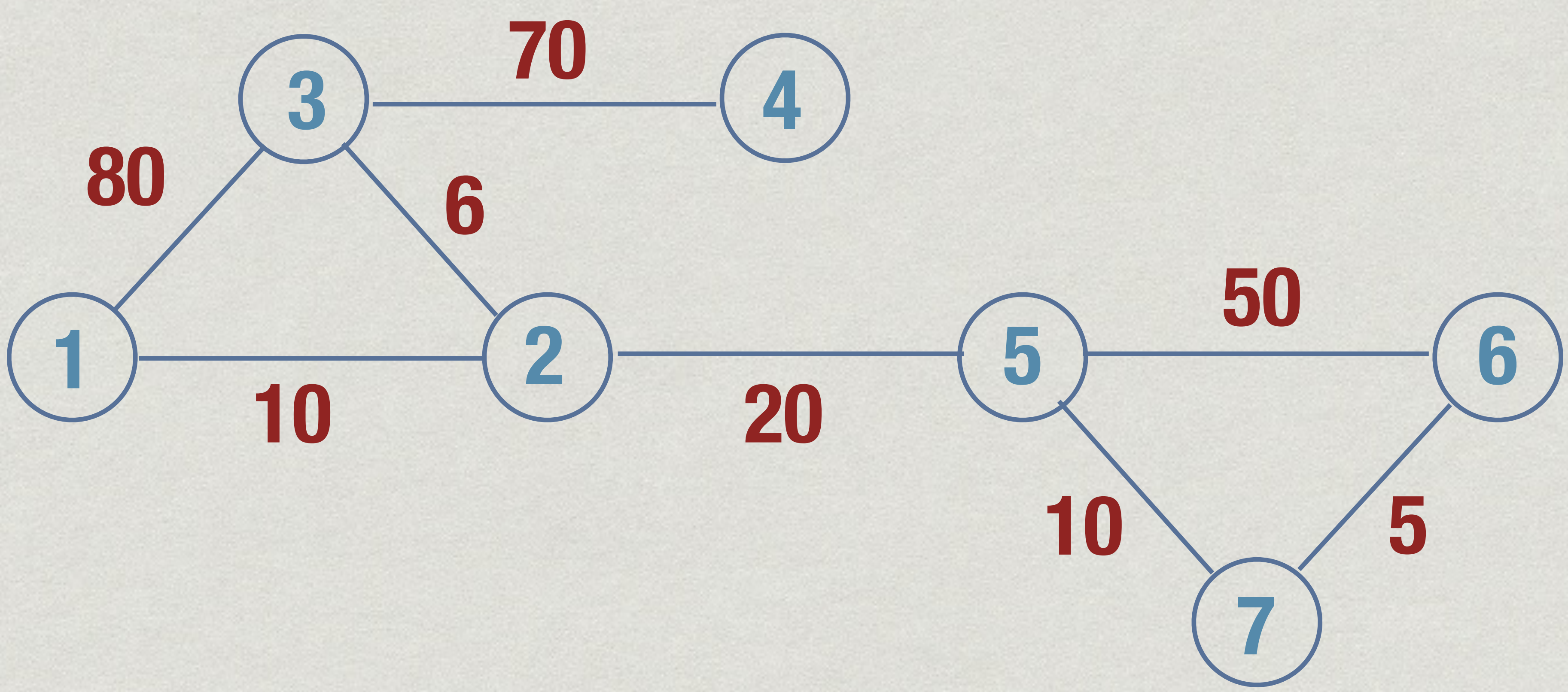

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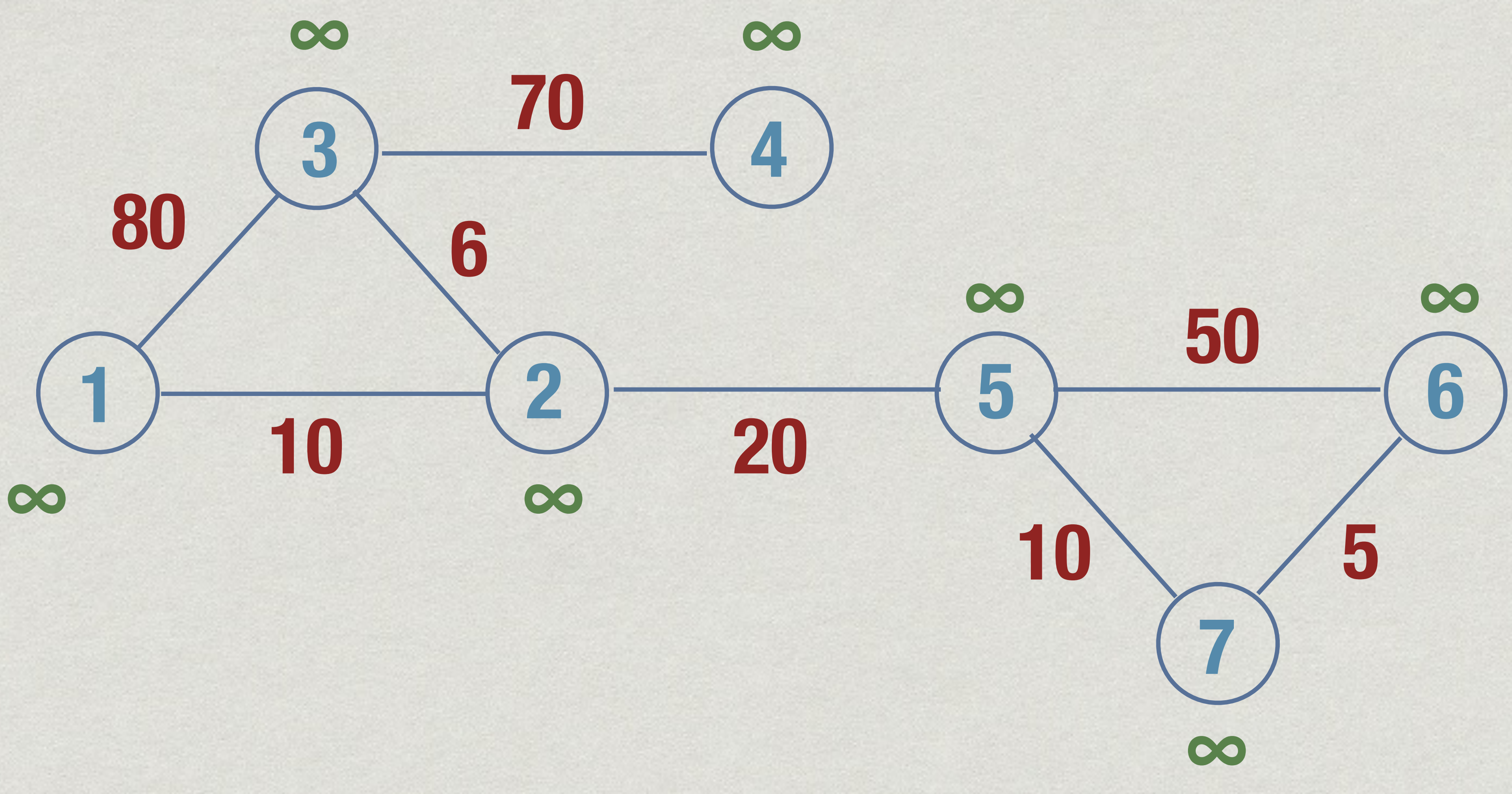

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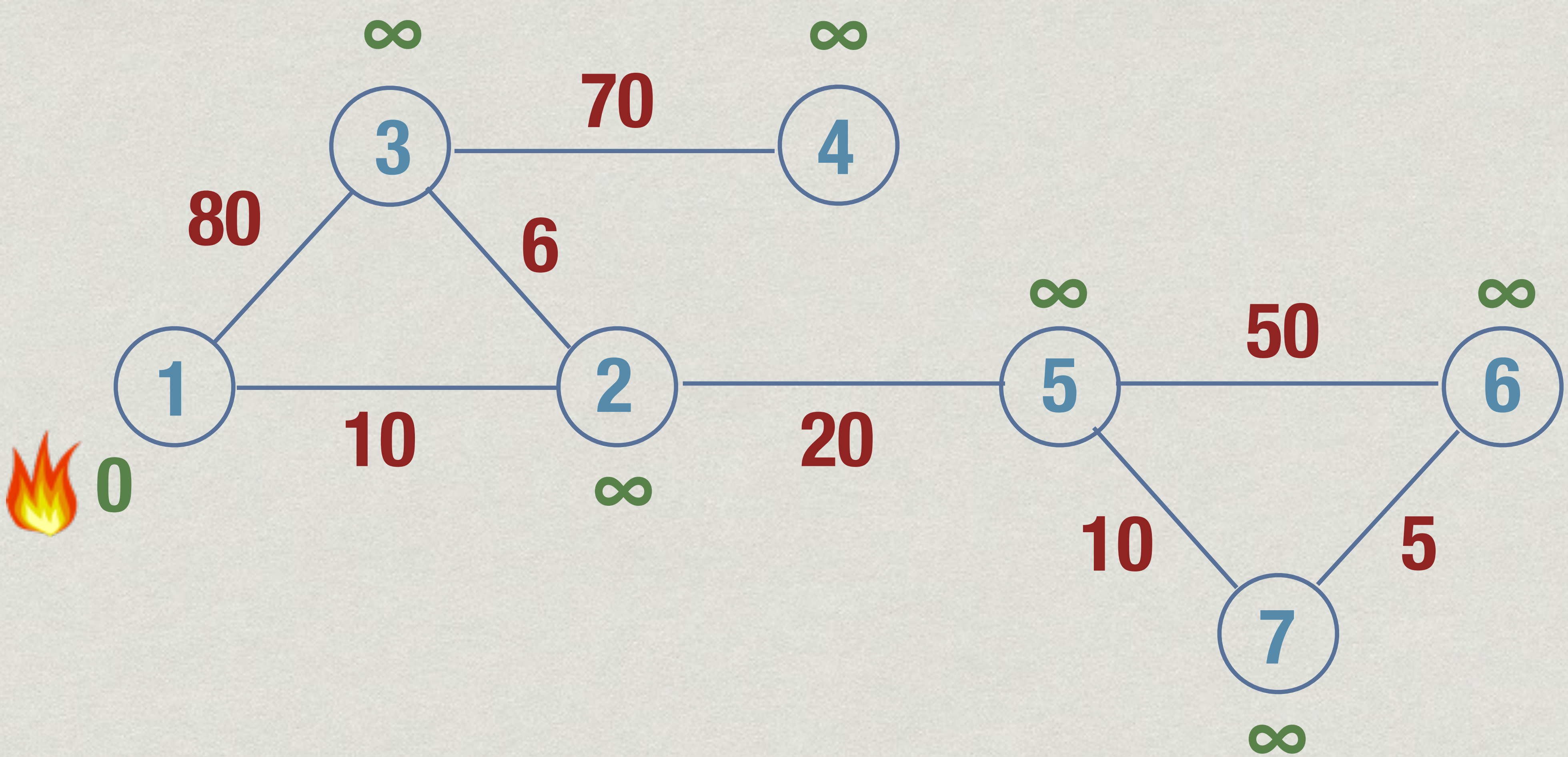

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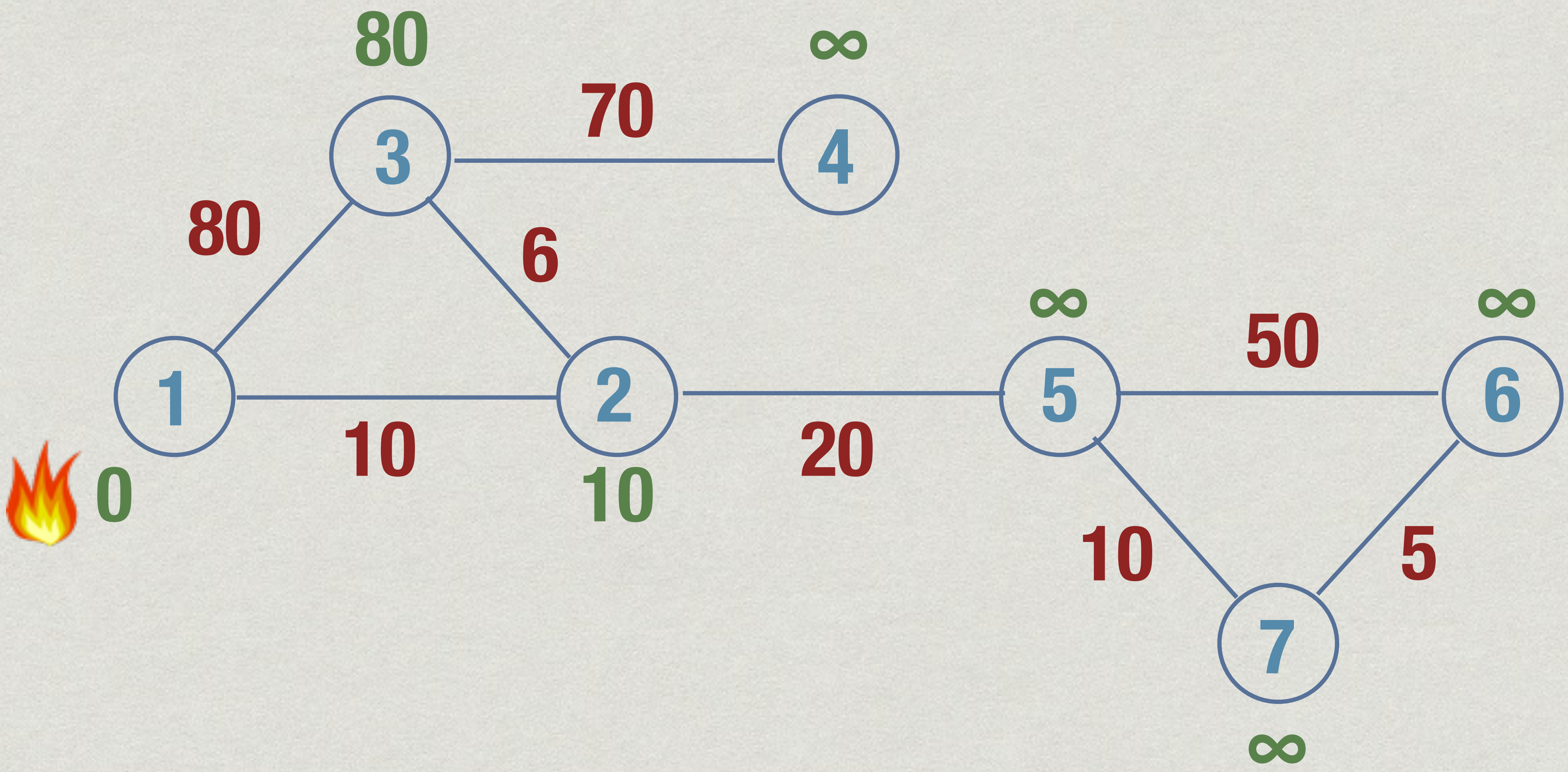

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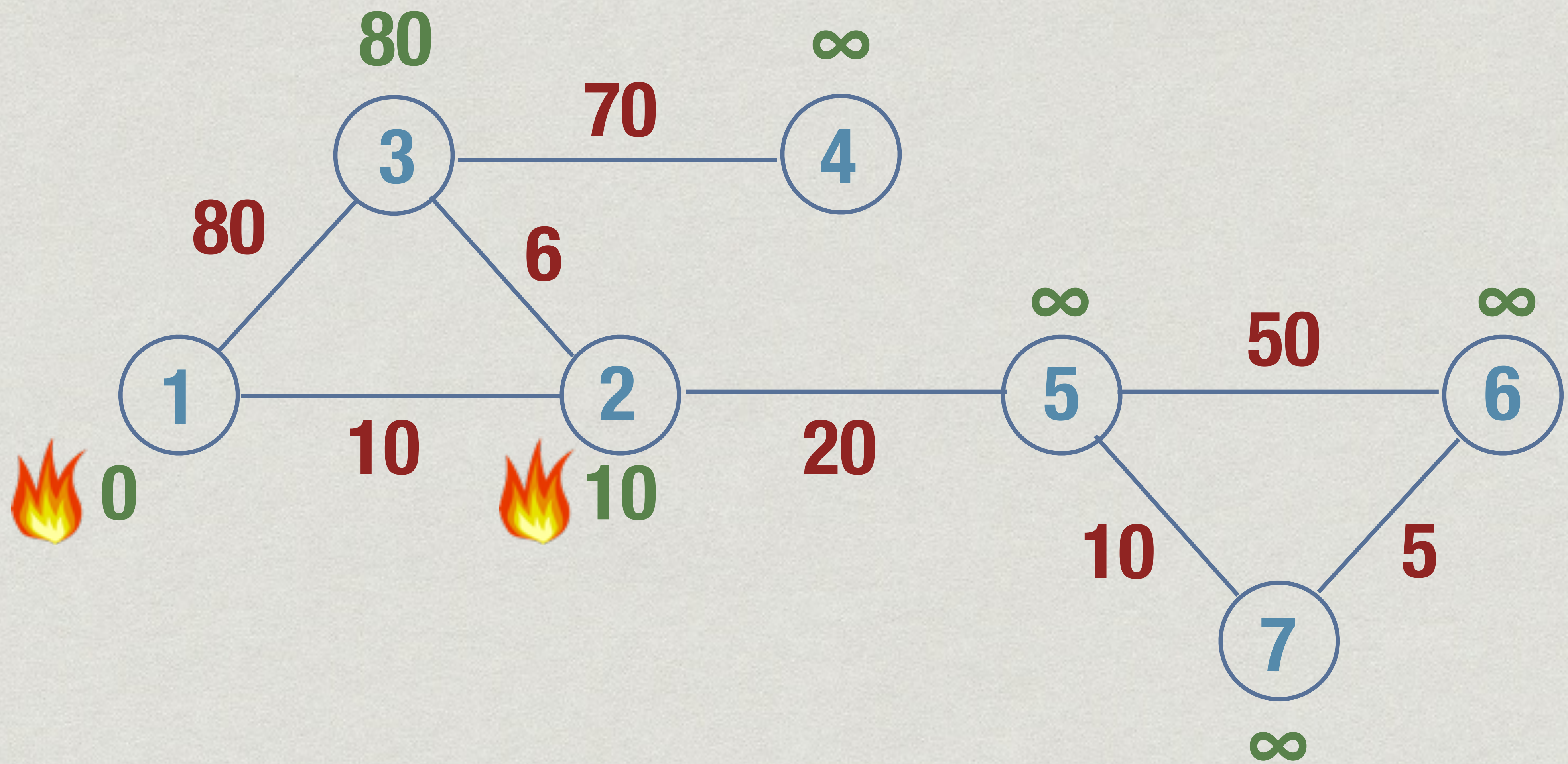

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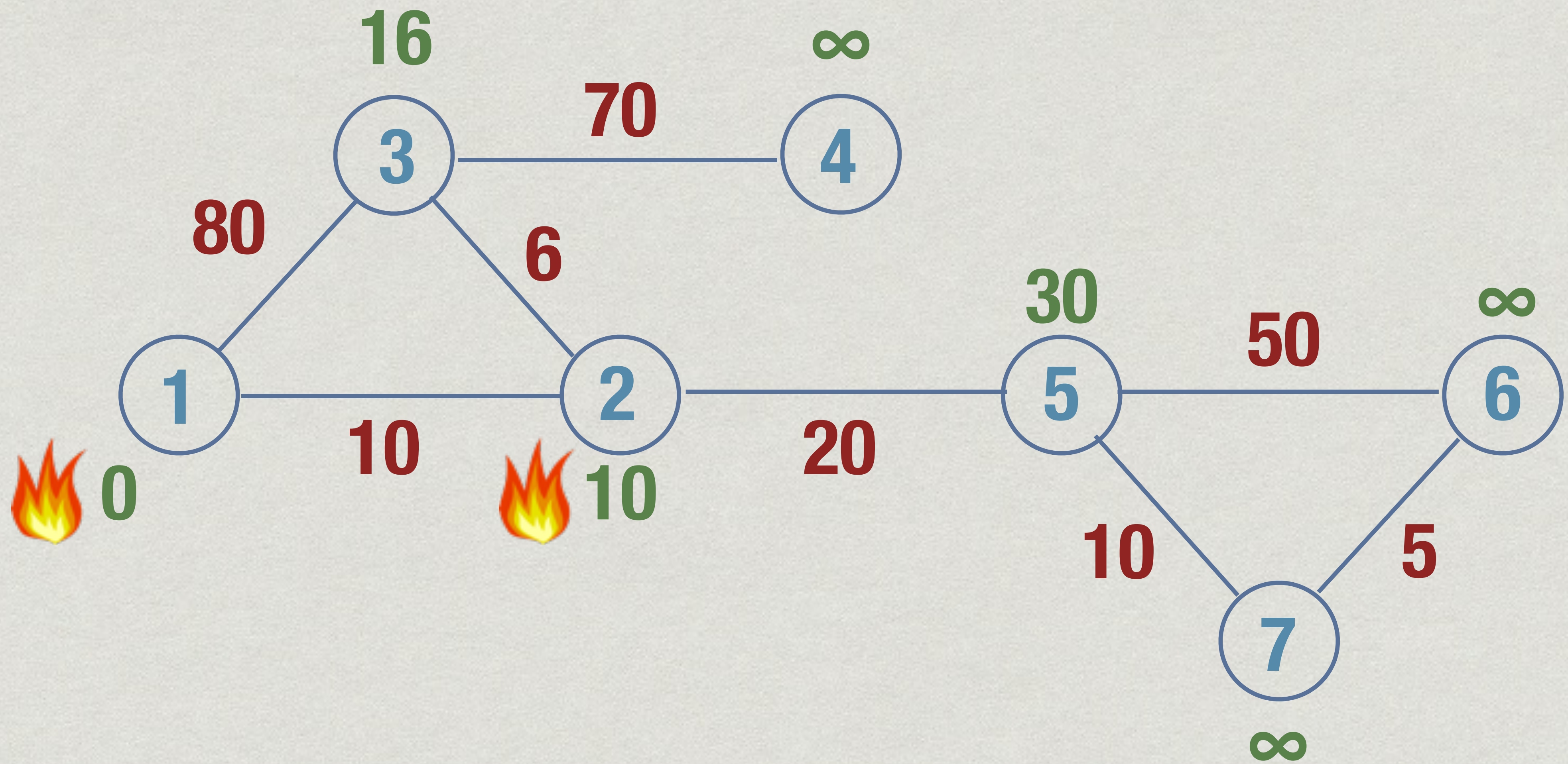

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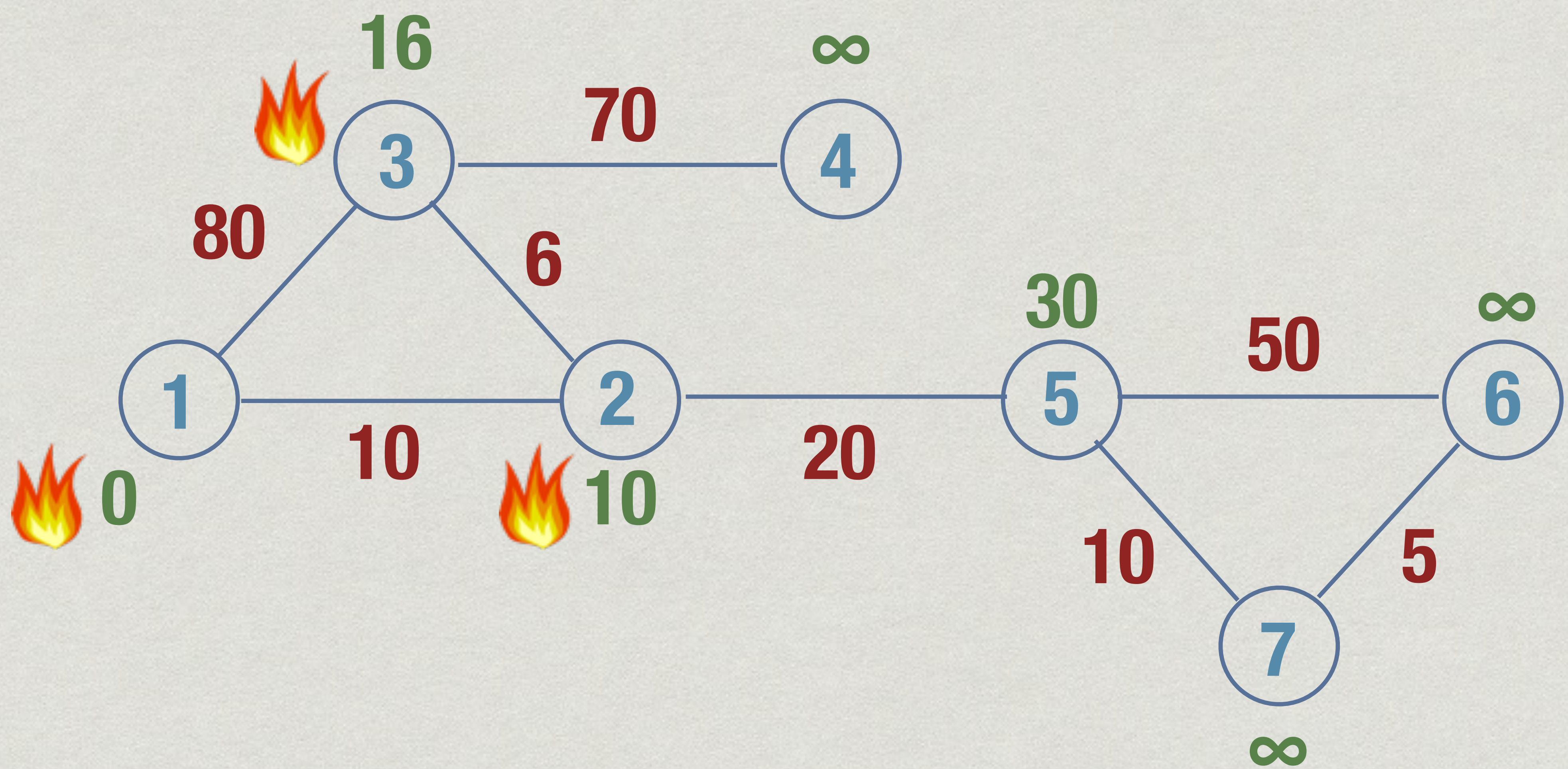

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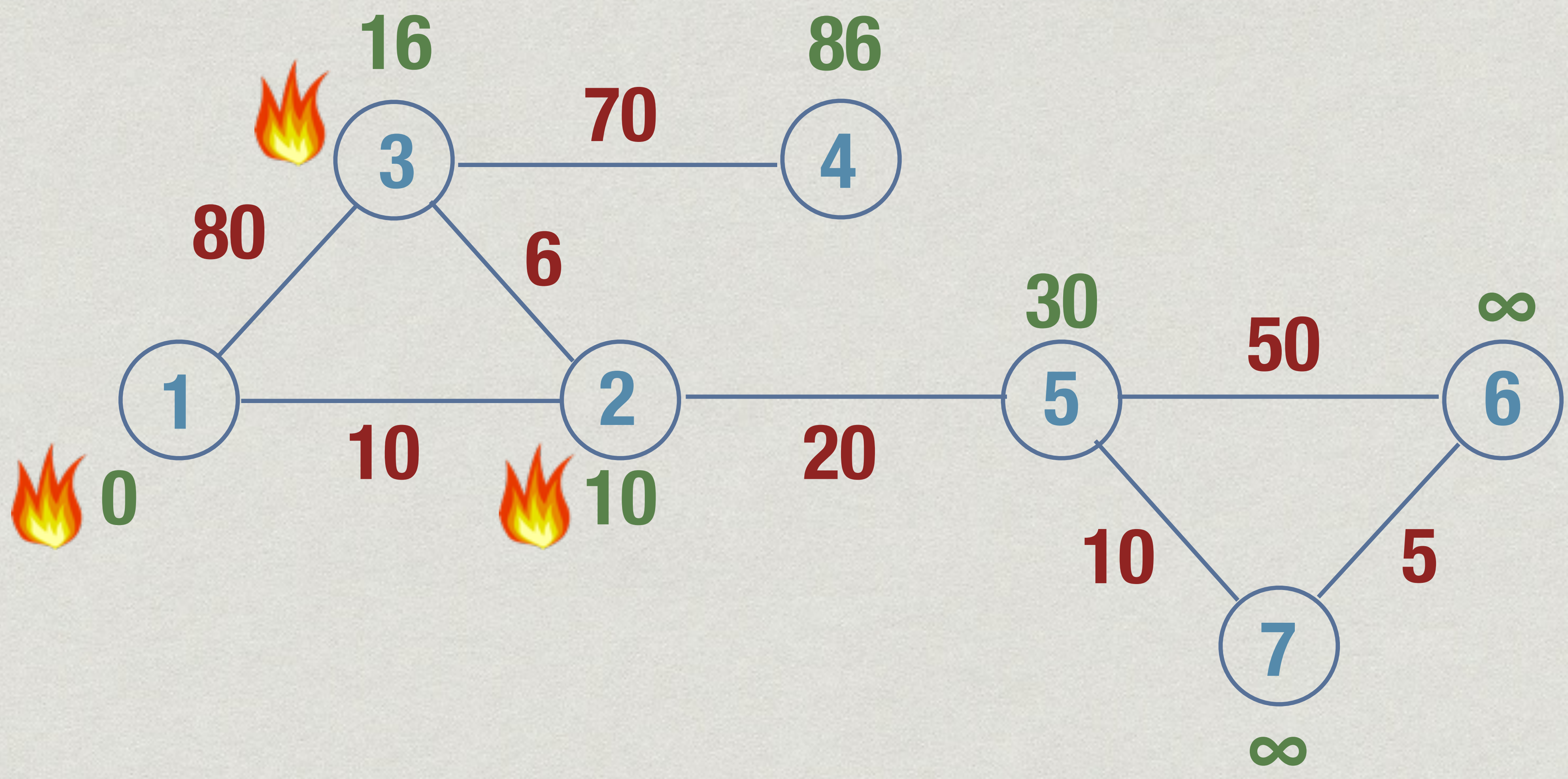

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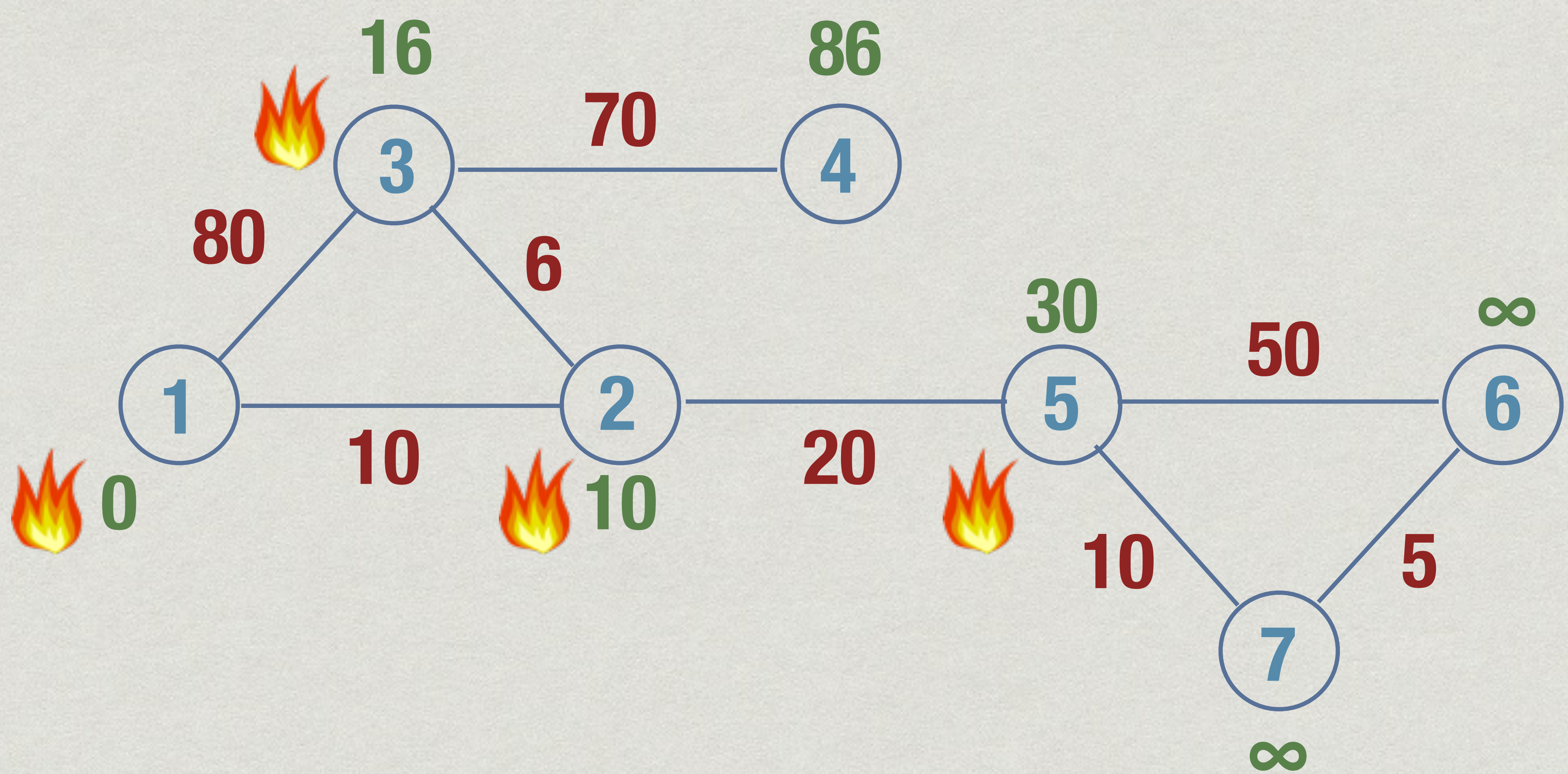

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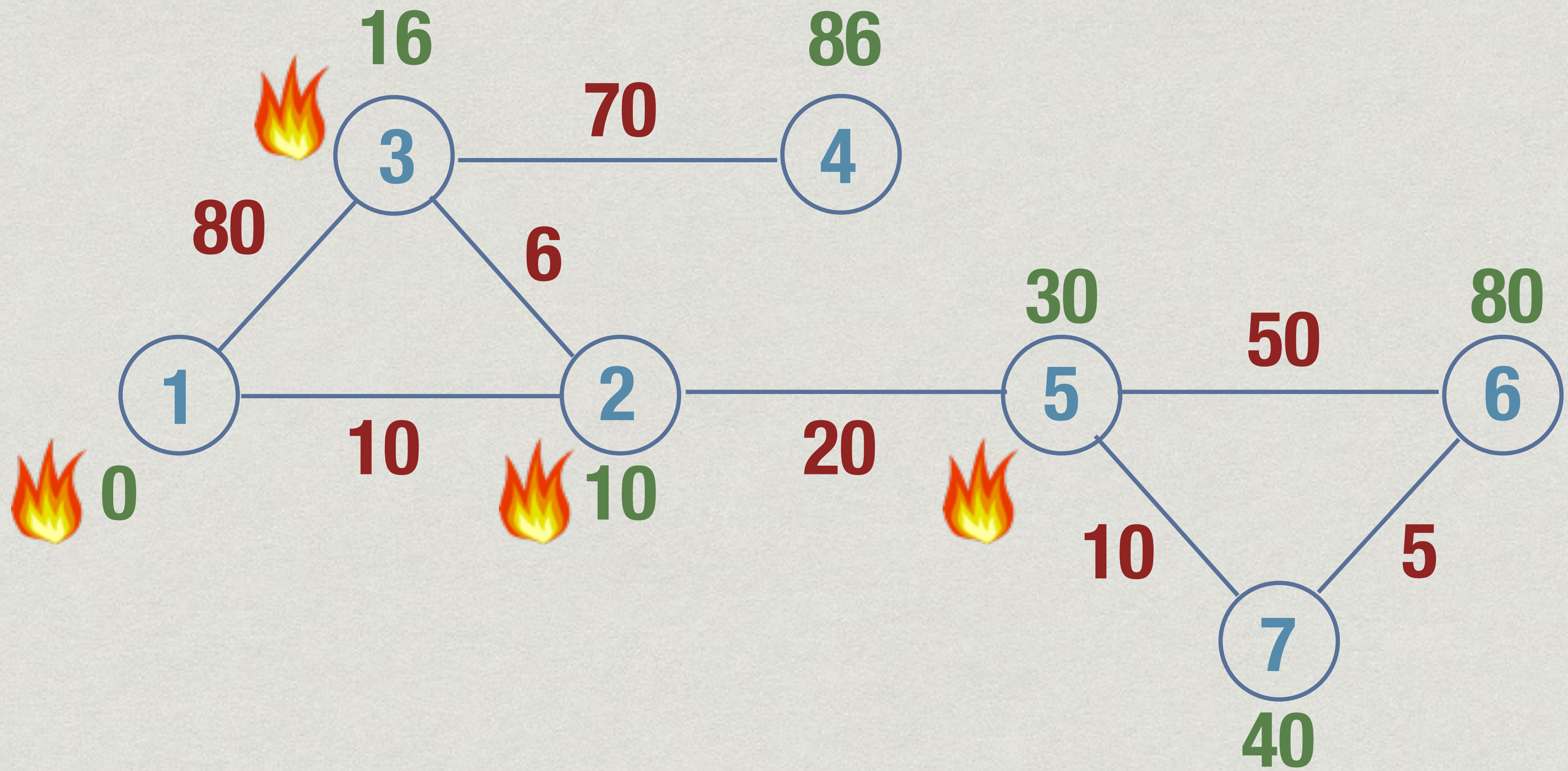

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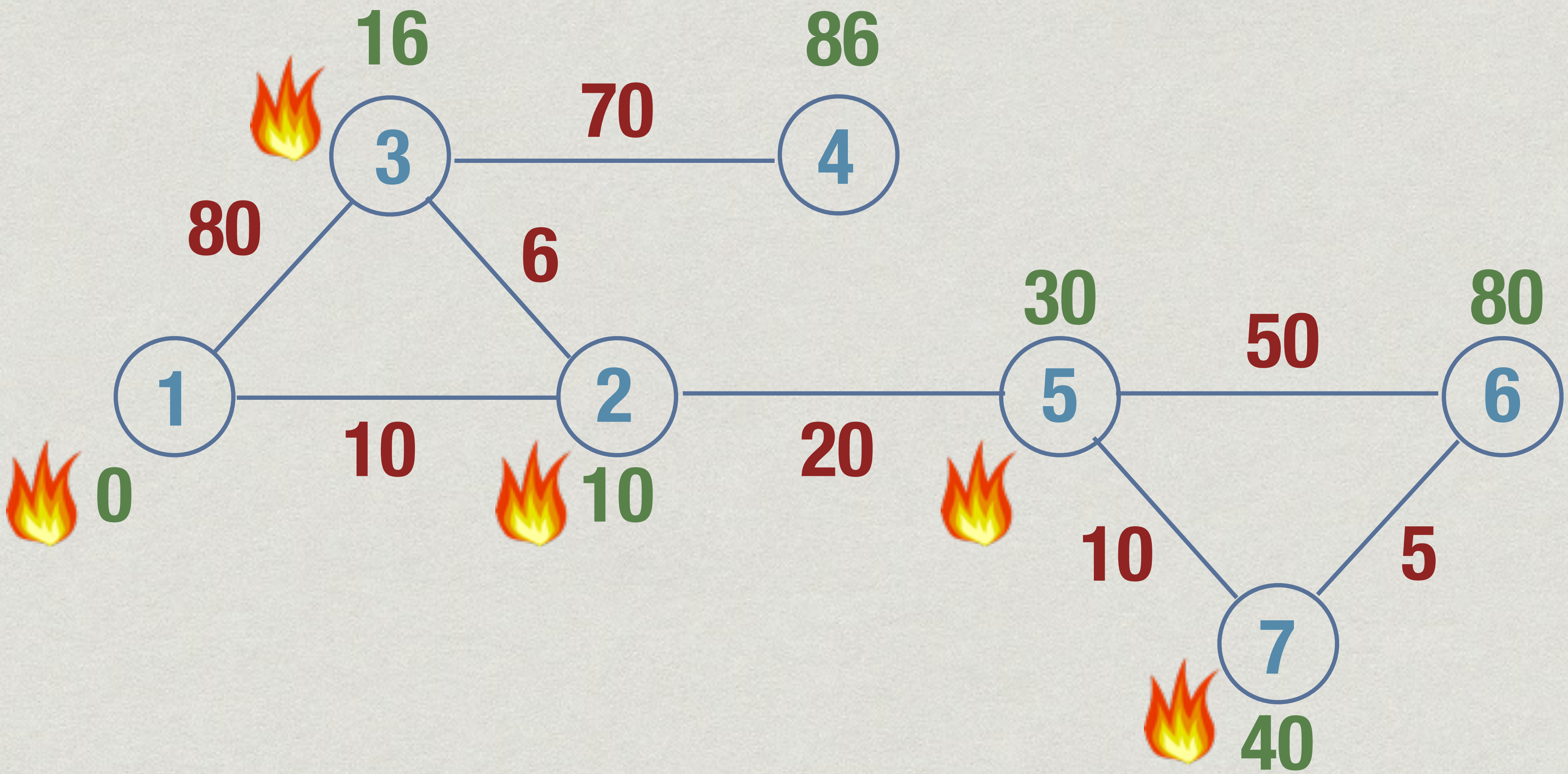

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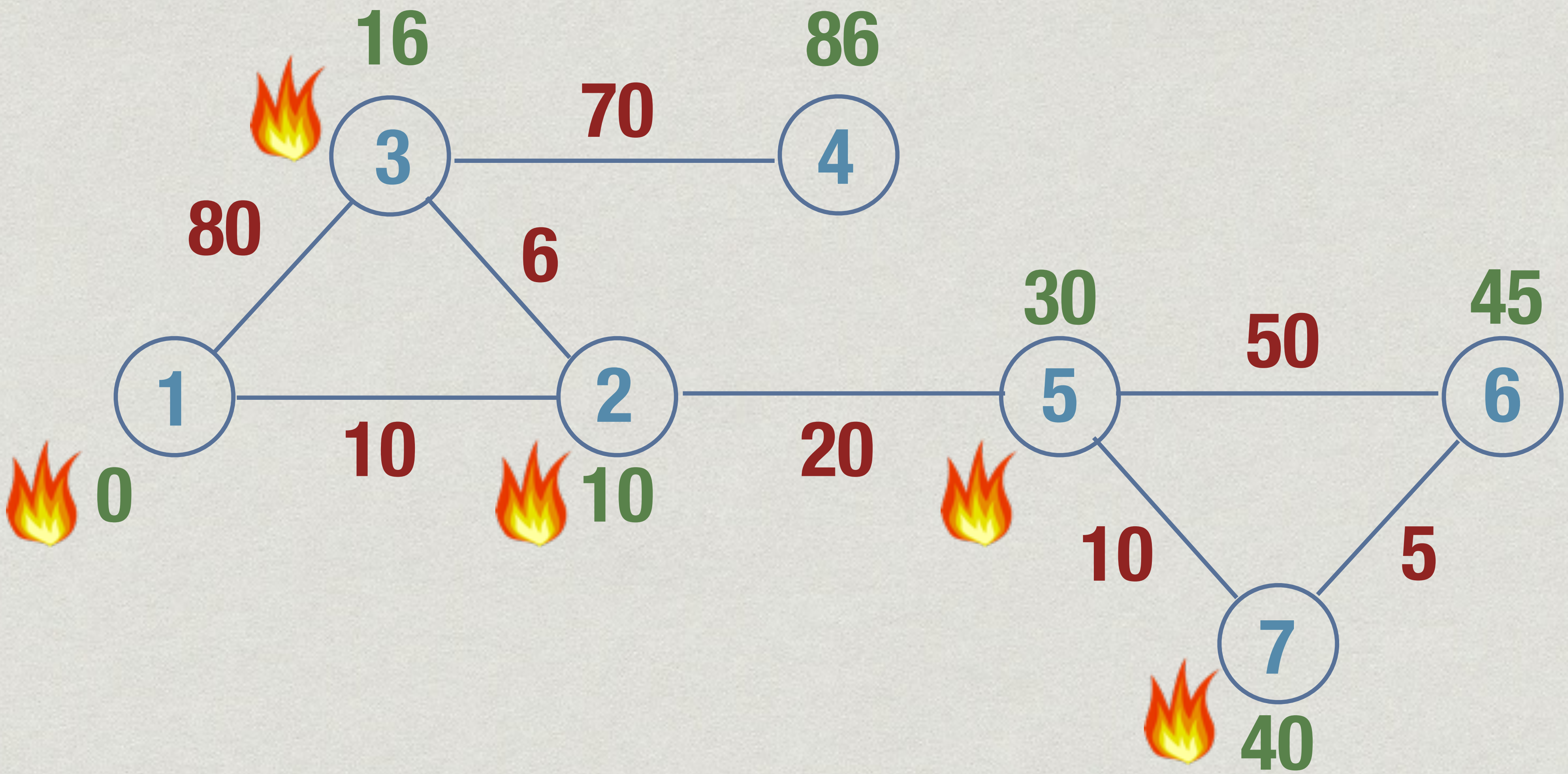

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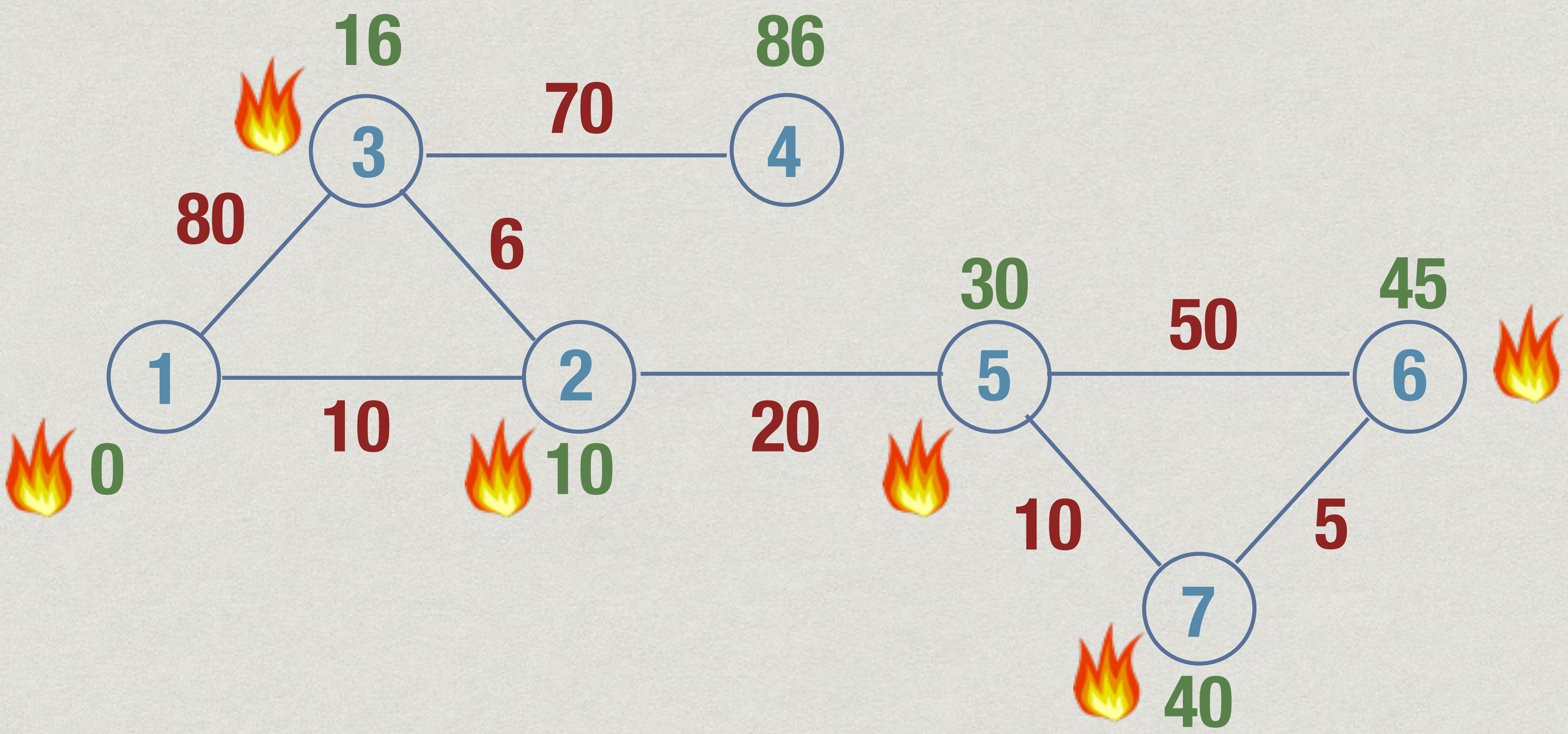

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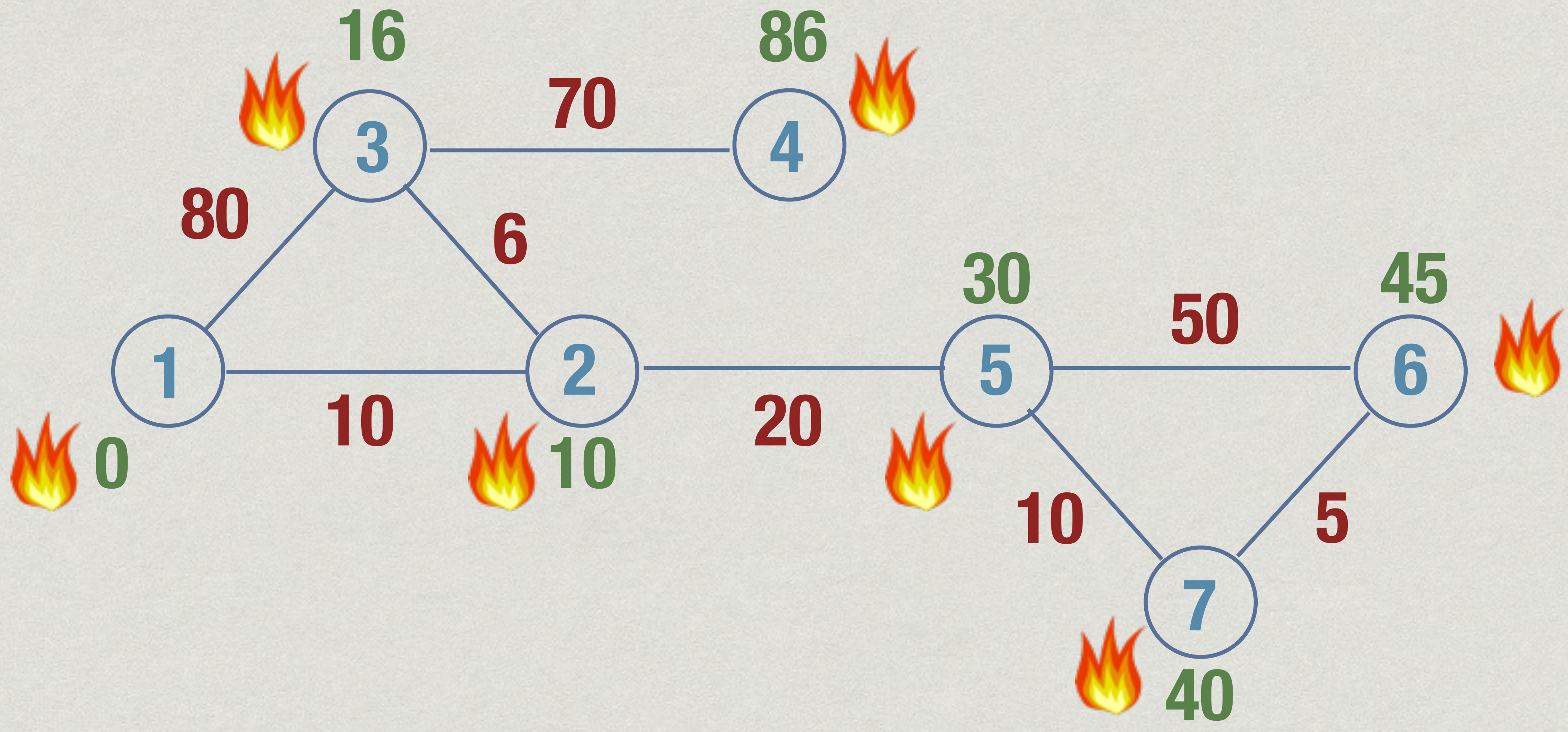

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- Repeat, until all vertices are burnt
  - Find `j` with minimum `ExpectedBurnTime`
  - Set `BurntVertices[j] = True`
  - Recompute `ExpectedBurnTime[k]` for each neighbour `k` of `j`

# Dijkstra’s algorithm

```
function ShortestPaths(s){ // assume source is s
  for i = 1 to n
    BV[i] = False; EBT[i] = infinity

  EBT[s] = 0

  for i = 1 to n
    Choose u such that BV[u] == False
                        and EBT[u] is minimum
    BV[u] = True
    for each edge (u,v) with BV[v] == False
      if EBT[v] > EBT[u] + weight(u,v)
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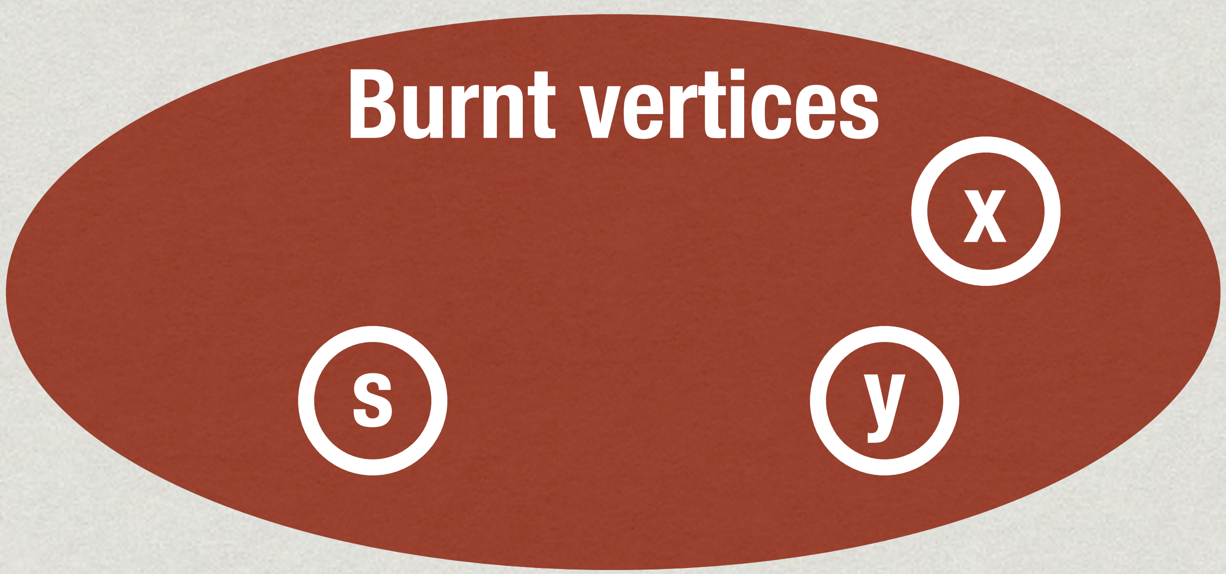

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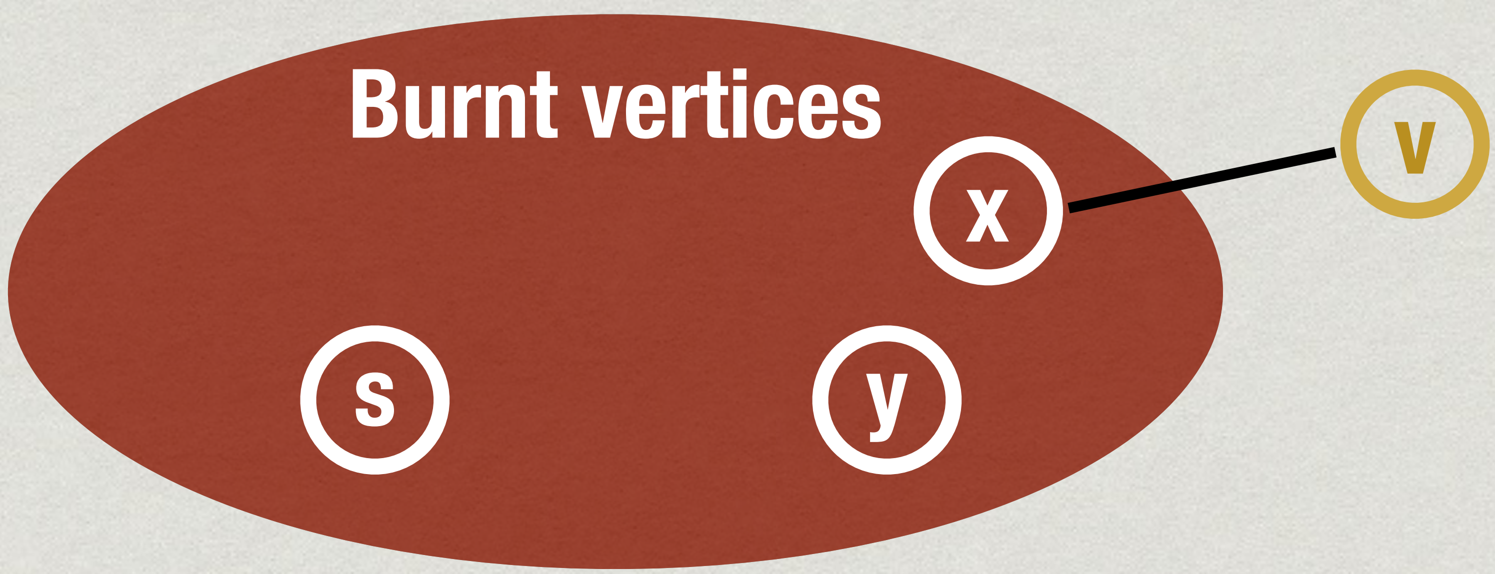


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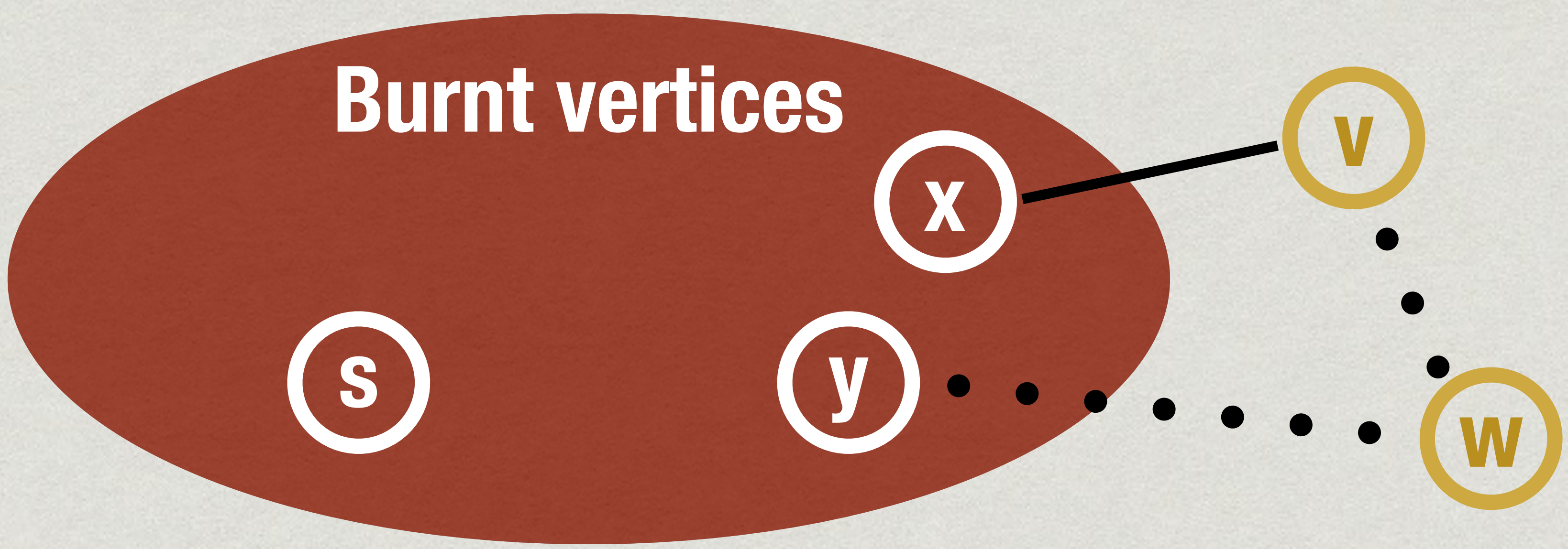


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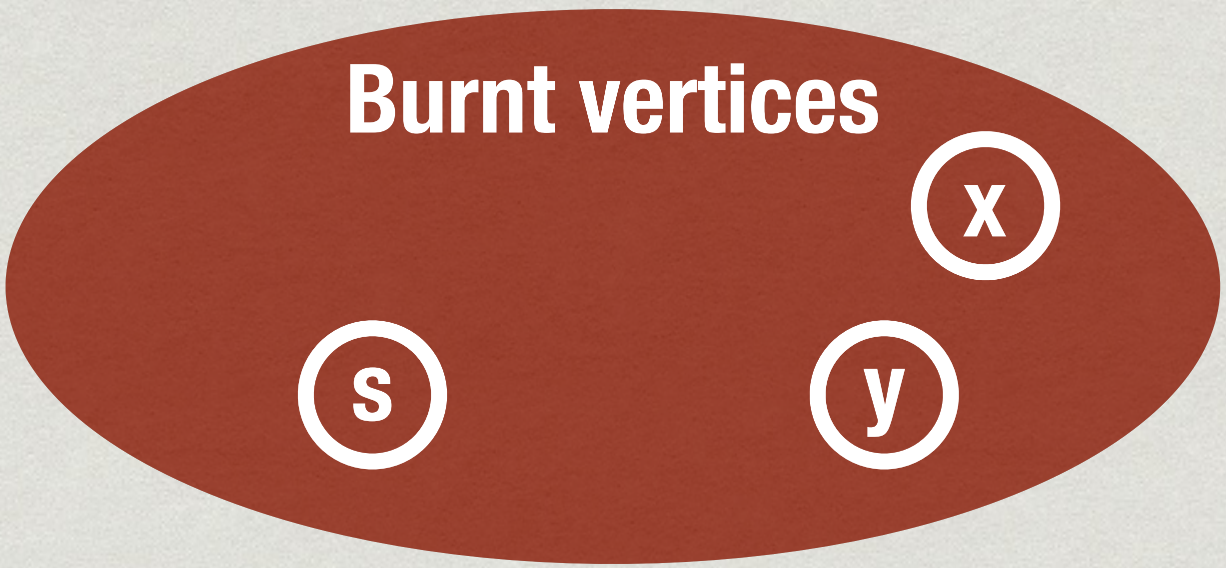

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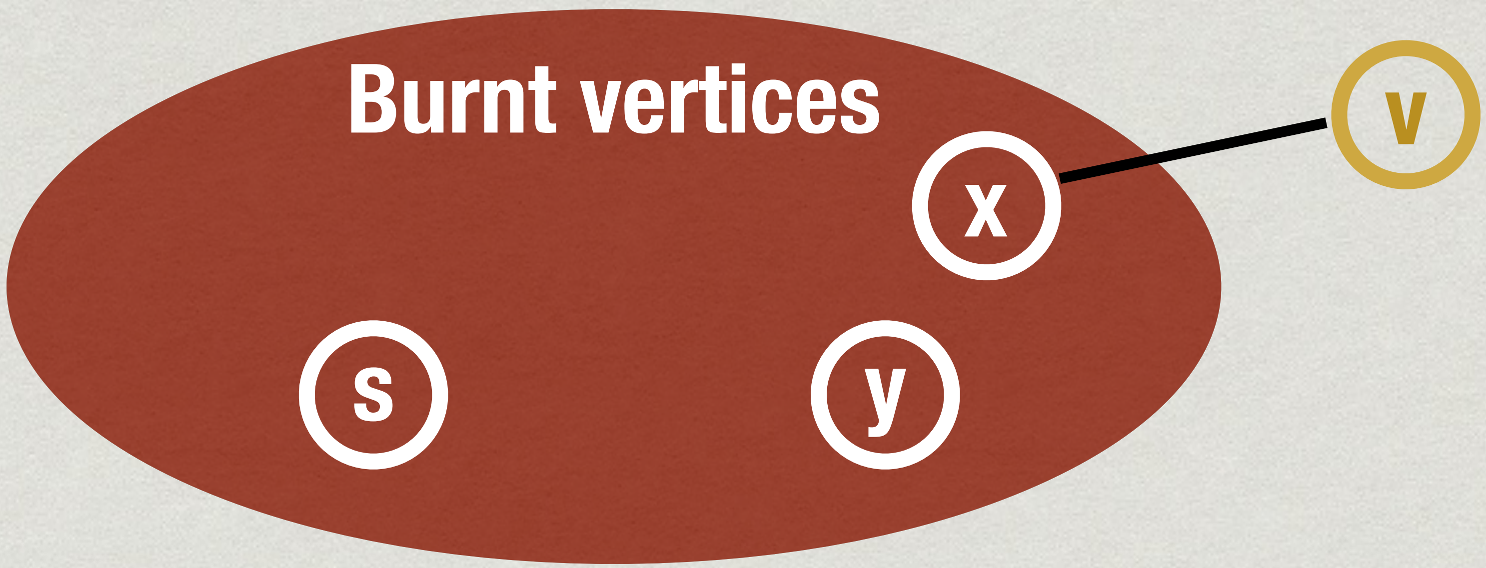


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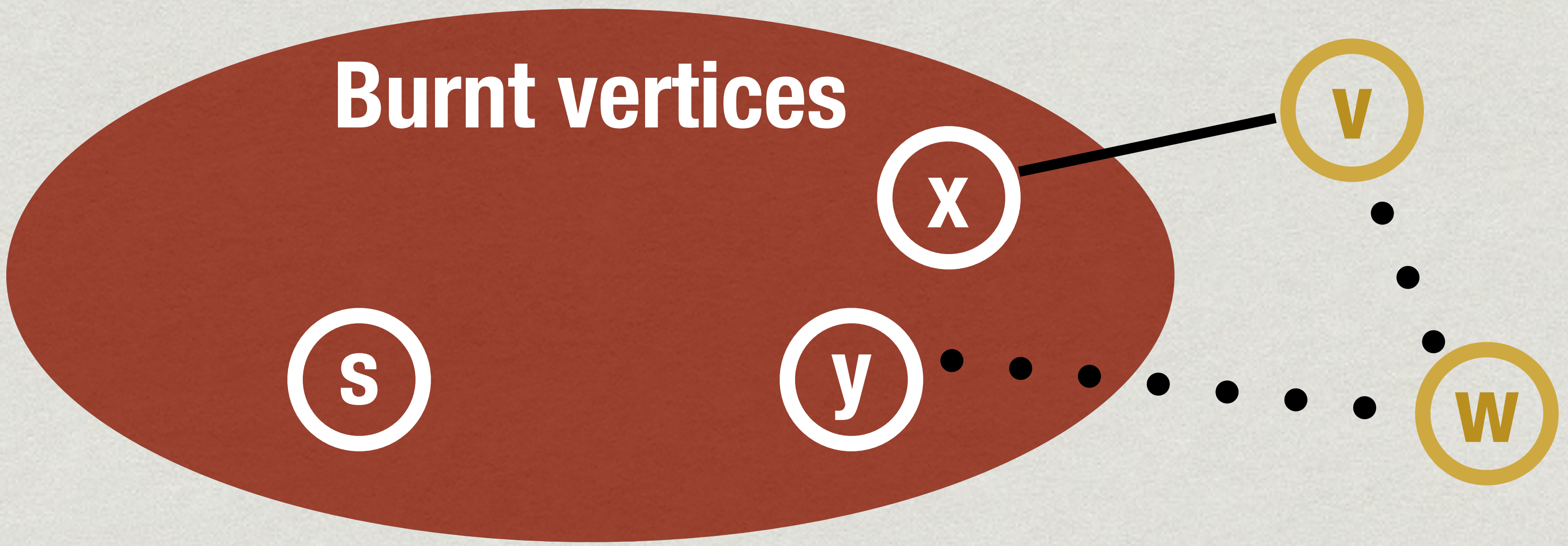


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  - Bellman-Ford
  - Floyd-Warshall all pairs shortest path (will see later)

# Summary

- Dijkstra's algorithm solves the single source shortest path problem, assuming no negative weights
  - Simple implementation is $O(n^2)$
  - Using clever trees, reduce to $O((n+m) \log n)$
- With negative edges, but without negative cycles, need to use other strategies

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- Most times, greedy approach fails
  - Current best choice may not be globally optimal