Introduction to Machine Learning

Maximum Margin Methods

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Outline

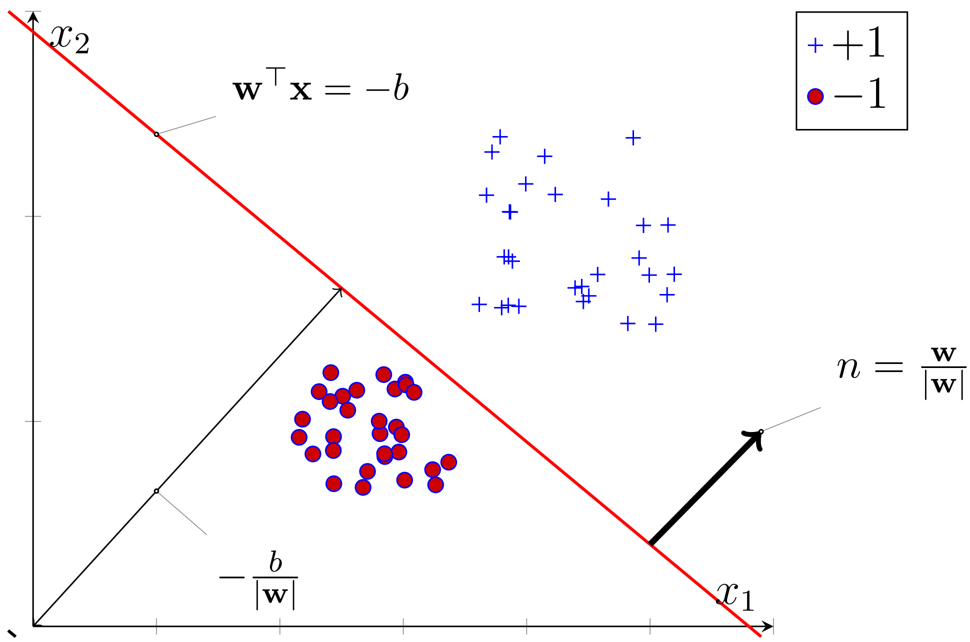
Contents

1 Maximum Margin Classifiers

$$y = \mathbf{w}^\top \mathbf{x} + b$$

- Remember the Perceptron!
- If data is linearly separable
  - Perceptron training guarantees learning the decision boundary

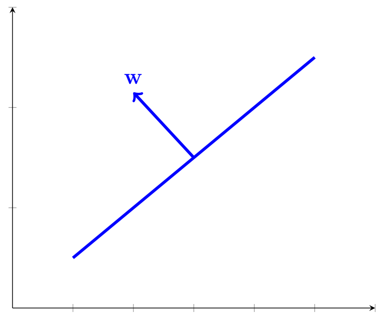
- There can be other boundaries
  - Depends on initial value for $\mathbf{w}$
- **But what is the best boundary?**



1.1 Linear Classification via Hyperplanes

- Separates a D-dimensional space into two half-spaces
- Defined by $\mathbf{w} \in \Re^D$
  - *Orthogonal* to the hyperplane
  - This $\mathbf{w}$ goes through the origin
  - How do you check if a point lies "above" or "below" $\mathbf{w}$?
  - What happens for points **on** $\mathbf{w}$?

For a hyperplane that passes through the origin, a point $\mathbf{x}$ will lie above the hyperplane if $\mathbf{w}^\top \mathbf{x} > 0$ and will lie below the plane if $\mathbf{w}^\top \mathbf{x} < 0$, otherwise.



This can be further understood by understanding that $bfw^\top \mathbf{x}$ is essentially equal to $|\mathbf{w}||\mathbf{x}|\cos\theta$, where θ is the angle between $\mathbf{w}$ and $\mathbf{x}$.

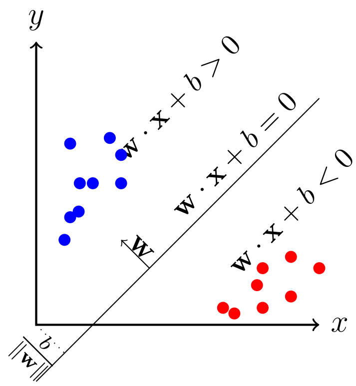
- Add a bias b
  - $b > 0$ - move along $\mathbf{w}$
  - $b < 0$ - move opposite to $\mathbf{w}$
- How to check if point lies above or below $\mathbf{w}$?
  - If $\mathbf{w}^\top \mathbf{x} + b > 0$ then $\mathbf{x}$ is *above*
  - Else, *below*
- Decision boundary represented by the hyperplane $\mathbf{w}$
- For binary classification, $\mathbf{w}$ points **towards** the positive class

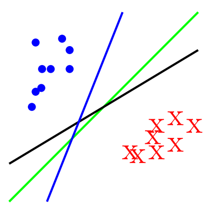
Decision Rule

$$y = sign(\mathbf{w}^\top \mathbf{x} + b)$$

- $\mathbf{w}^\top \mathbf{x} + b > 0 \Rightarrow y = +1$
- $\mathbf{w}^\top \mathbf{x} + b < 0 \Rightarrow y = -1$

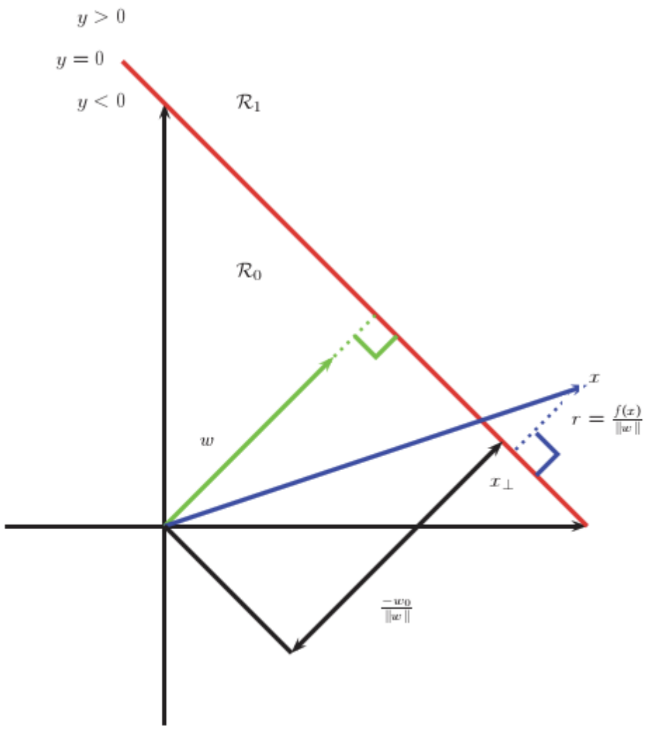
- **Perceptron** can find a hyperplane that separates the data
  - ... if the data is linearly separable
- But there can be many choices!





- Find the one with best separability (largest margin)
- Gives better generalization performance
  1. Intuitive reason
  2. Theoretical foundations

1.2 Concept of Margin



- **Margin** is the distance between an example and the decision line
- Denoted by γ
- For a positive point:

$$\gamma = \frac{\mathbf{w}^\top \mathbf{x} + b}{\|\mathbf{w}\|}$$

- For a negative point:

$$\gamma = -\frac{\mathbf{w}^\top \mathbf{x} + b}{\|\mathbf{w}\|}$$

To understand the margin from a geometric perspective, consider the projection of the vector connecting the origin to a point $\mathbf{x}$ on the decision line. Let the point be denoted as $\mathbf{x}'$. Obviously the vector $\mathbf{r}$ connecting $\mathbf{x}'$ and $\mathbf{x}$ is given by:

$$\mathbf{r} = \gamma \widehat{\mathbf{w}} = \gamma \frac{\mathbf{w}}{\|\mathbf{w}\|}$$

if $\mathbf{x}$ lies on the positive side of $\mathbf{w}$. But the same vector can be computed as:

$$\mathbf{r} = \mathbf{x} - \mathbf{x}'$$

Equating above two gives us $\mathbf{x}'$ as:

$$\mathbf{x}' = \mathbf{x} - \gamma \frac{\mathbf{w}}{\|\mathbf{w}\|}$$

Noting that, since $\mathbf{x}'$ lies on the hyperplane and hence:

$$\mathbf{w}^\top \mathbf{x}' + b = 0$$

Substituting $\mathbf{x}'$ from above:

$$\mathbf{w}^\top \mathbf{x} - \gamma \frac{\mathbf{w}^\top \mathbf{w}}{\|\mathbf{w}\|} + b = 0$$

Noting that $\frac{\mathbf{w}^\top \mathbf{w}}{\|\mathbf{w}\|} = \|\mathbf{w}\|$, we get γ as:

$$\gamma = \frac{\mathbf{w}^\top \mathbf{x} + b}{\|\mathbf{w}\|} \tag{1}$$

Similar analysis can be done for points on the negative side of $\mathbf{x}$. In general, one can write the expression for the margin as:

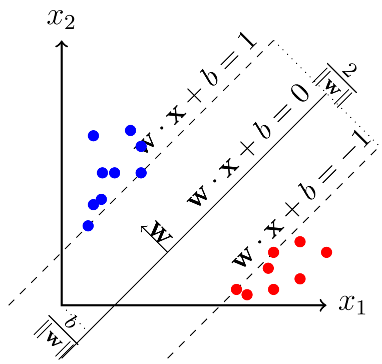
$$\gamma = y \frac{\mathbf{w}^\top \mathbf{x} + b}{\|\mathbf{w}\|} \tag{2}$$

where $y \in \{-1, +1\}$.

Functional Interpretation

- Margin **positive** if prediction is **correct**; **negative** if prediction is **incorrect**

From the figure one can note that the size of the margin is $\frac{2}{\|\mathbf{w}\|}$. We can show this as follows. Since the data is separable, we can get two parallel lines represented by $\mathbf{w}^\top \mathbf{x} + b = +1$ and $\mathbf{w}^\top \mathbf{x} + b = -1$. Using result from (1) and (2), the distance between the two lines is given by $2\gamma = \frac{2}{\|\mathbf{w}\|}$.



2 Support Vector Machines

- A hyperplane based classifier defined by $\mathbf{w}$ and b
- Like perceptron
- Find hyperplane with *maximum separation margin* on the training data
- Assume that data is linearly separable (will relax this later)
  - Zero training error (loss)

SVM Prediction Rule

$$y = sign(\mathbf{w}^\top \mathbf{x} + b)$$

SVM Learning

- **Input**: Training data $\{(\mathbf{x}_1, y_1), (\mathbf{x}_2, y_2), \ldots, (\mathbf{x}_N, y_N)\}$
- **Objective**: Learn $\mathbf{w}$ and b that maximizes the margin

2.1 SVM Learning

- SVM learning task as an optimization problem
- Find $\mathbf{w}$ and b that gives zero training error
- Maximizes the margin $(= \frac{2}{\|w\|})$

- Same as minimizing $\|\mathbf{w}\|$

Optimization Formulation

$$\begin{aligned} \underset{\mathbf{w},b}{\text{minimize}} \quad & \frac{\|\mathbf{w}\|^2}{2} \\ \text{subject to} \quad & y_n(\mathbf{w}^\top \mathbf{x}_n + b) \geq 1, \; n = 1, \ldots, N. \end{aligned}$$

- **Optimization** with N linear inequality constraint

A Different Interpretation of Margin

- What impact does the margin have on $\mathbf{w}$?
- Large margin $\Rightarrow$ Small $\|\mathbf{w}\|$
- Small $\|\mathbf{w}\|$ $\Rightarrow$ regularized/simple solutions
- Simple solutions $\Rightarrow$ Better generalizability (*Occam's Razor*)
- Computational Learning Theory provides a formal justification [1]

2.2 Solving SVM Optimization Problem

Optimization Formulation

$$\begin{aligned} \underset{\mathbf{w},b}{\text{minimize}} \quad & \frac{\|\mathbf{w}\|^2}{2} \\ \text{subject to} \quad & y_n(\mathbf{w}^\top \mathbf{x}_n + b) \geq 1, \; n = 1, \ldots, N. \end{aligned}$$

- There is an quadratic objective function to minimize with N inequality constraints
- "Off-the-shelf" packages - quadprog (MATLAB), CVXOPT
- Is that the best way?

3 Constrained Optimization and Lagrange Multipliers

$$\boxed{\underset{x,y}{\text{minimize}} \quad f(x,y) = \quad 2 - x^2 - 2y^2}$$

$$\boxed{\begin{aligned} \underset{x,y}{\text{minimize}} \quad f(x,y) &= \quad 2 - x^2 - 2y^2 \\ \text{subject to} \quad h(x,y) &= \quad x + y - 1 = 0. \end{aligned}}$$

- Tool for solving constrained optimization problems of differentiable functions

$$\boxed{\begin{aligned} \underset{x,y}{\text{minimize}} \quad f(x,y) &= \quad 2 - x^2 - 2y^2 \\ \text{subject to} \quad h(x,y) &: \quad x + y - 1 = 0. \end{aligned}}$$

- A Lagrangian multiplier (β) lets you combine the two equations into one

$$\boxed{\underset{x,y,\beta}{\text{minimize}} \quad L(x,y,\beta) = \quad f(x,y) + \beta h(x,y)}$$

Solution 1. *Writing the objective as Lagrangian.*

$$L(x,y,\beta) = 2 - x^2 - 2y^2 + \beta(x + y - 1)$$

Setting the gradient to 0 with respect to x, y and β will give us the optimal values.

$$\frac{\partial L}{\partial x} = -2x + \beta = 0$$

$$\frac{\partial L}{\partial y} = -4y + \beta = 0$$

$$\frac{\partial L}{\partial \beta} = x + y - 1 = 0$$

Multiple Constraints

$$\boxed{\begin{array}{lrr} \underset{x,y,z}{\text{minimize}} & f(x,y,z) = & x^2 + 4y^2 + 2z^2 + 6y + z \\ \text{subject to} & h_1(x,y,z): & x + z^2 - 1 = 0 \\ & h_2(x,y,z): & x^2 + y^2 - 1 = 0. \end{array}}$$

$$\boxed{L(x,y,z,\boldsymbol{\beta}) = f(x,y,z) + \sum_i \beta_i h_i(x,y,z)}$$

Handling Inequality Constraints

$$\boxed{\begin{array}{lrr} \underset{x,y}{\text{minimize}} & f(x,y) = & x^3 + y^2 \\ \text{subject to} & g(x): & x^2 - 1 \leq 0. \end{array}}$$

- Inequality constraints are **transferred** as constraints on the Lagrangian, α

The Lagrangian in the above example becomes:

$$\begin{aligned} L(x,y,\alpha) &= f(x,y) + \alpha g(x,y) \\ &= x^3 + y^2 + \alpha(x^2 - 1) \end{aligned}$$

Solving for the gradient of the Lagrangian gives us:

$$\begin{aligned} \frac{\partial}{\partial x} L(x,y,\alpha) = 3x^2 + 2\alpha x = 0 \\ \frac{\partial}{\partial y} L(x,y,\alpha) = 2y = 0 \\ \frac{\partial}{\partial \alpha_1} L(x,y,\alpha) = x^2 - 1 = 0 \end{aligned}$$

Furthermore we require that:

$$\alpha \geq 0$$

From above equations we get $y = 0$, $x = \pm 1$ and $\alpha = \pm\frac{3}{2}$. But since $\alpha \geq 0$, hence $\alpha = \frac{3}{2}$. This gives $x = 1$, $y = 0$, and $f = 1$.

Handling Both Types of Constraints

$$\begin{array}{|lll|}\hline \underset{\mathbf{w}}{\text{minimize}} & f(\mathbf{w}) & \\ \text{subject to} & g_i(\mathbf{w}) \leq 0 & i = 1, \ldots, k \\ \text{and} & h_i(\mathbf{w}) = 0 & i = 1, \ldots, l. \\ \hline \end{array}$$

Generalized Lagrangian

$$L(\mathbf{w}, \boldsymbol{\alpha}, \boldsymbol{\beta}) = f(\boldsymbol{w}) + \sum_{i=1}^{k} \alpha_i g_i(\mathbf{w}) + \sum_{j=1}^{l} \beta_j h_j(\mathbf{w})$$

subject to, $\alpha_i \geq 0, \forall i$

Primal and Dual Formulations

Primal Optimization

- Let θ_P be defined as:

$$\theta_P(\mathbf{w}) = \underset{\boldsymbol{\alpha},\boldsymbol{\beta}:\alpha_i \geq 0}{max} L(\mathbf{w}, \boldsymbol{\alpha}, \boldsymbol{\beta})$$

- One can prove that the optimal value for the original constrained problem is same as:

$$p^* = \underset{\mathbf{w}}{min} \theta_P(\mathbf{w}) = \underset{\mathbf{w}}{min} \underset{\boldsymbol{\alpha},\boldsymbol{\beta}:\alpha_i \geq 0}{max} L(\mathbf{w}, \boldsymbol{\alpha}, \boldsymbol{\beta})$$

  Consider

$$\begin{aligned} \theta_P(\mathbf{w}) &= \underset{\boldsymbol{\alpha},\boldsymbol{\beta}:\alpha_i \geq 0}{max} L(\mathbf{w}, \boldsymbol{\alpha}, \boldsymbol{\beta}) \\ &= \underset{\boldsymbol{\alpha},\boldsymbol{\beta}:\alpha_i \geq 0}{max} f(\mathbf{w}) + \sum_{i=1}^{k} \alpha_i g_i(\mathbf{w}) + \sum_{i-1}^{l} \beta_i h_i(\mathbf{w}) \end{aligned}$$

  It is easy to show that if any constraints are not satisfied, i.e., if either $g_i(\mathbf{w}) > 0$ or $h_i(\mathbf{w}) \neq 0$, then $\theta_P(\mathbf{w}) = \infty$. Which means that:

$$\theta_P(\mathbf{w}) = \begin{cases} f(\mathbf{w}) & \text{if primal constraints are satisfied} \\ \infty & \text{otherwise,} \end{cases}$$

Primal and Dual Formulations (II)

Dual Optimization

- Consider θ_D, defined as:

$$\theta_D(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \underset{\mathbf{w}}{min} L(\mathbf{w}, \boldsymbol{\alpha}, \boldsymbol{\beta})$$

- The **dual** optimization problem can be posed as:

$$d^* = \underset{\boldsymbol{\alpha},\boldsymbol{\beta}:\alpha_i \geq 0}{max} \theta_D(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \underset{\boldsymbol{\alpha},\boldsymbol{\beta}:\alpha_i \geq 0}{max} \underset{\mathbf{w}}{min} L(\mathbf{w}, \boldsymbol{\alpha}, \boldsymbol{\beta})$$

$d^* == p^*$**?**

- Note that $d^* \leq p^*$
- "Max min" of a function is always less than or equal to "Min max"
- When will they be equal?
  - $f(\mathbf{w})$ is convex
  - Constraints are affine

Relation between primal and dual

- In general $d^* \leq p^*$, for SVM optimization the equality holds
- Certain conditions should be true
- Known as the **Kahrun-Kuhn-Tucker** conditions
- For $d^* = p^* = L(\mathbf{w}^*, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$:

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{w}} L(\mathbf{w}^*, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*) & \quad = 0 \\
\frac{\partial}{\partial \beta_i} L(\mathbf{w}^*, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*) & = 0, \quad i = 1, \ldots, l \\
\alpha_i^* g_i(\mathbf{w}^*) & = 0, \quad i = 1, \ldots, k \\
g_i(\mathbf{w}^*) & \leq 0, \quad i = 1, \ldots, k \\
\alpha_i^* & \geq 0, \quad i = 1, \ldots, k
\end{aligned}$$

Optimization Formulation

$$\begin{aligned}
\underset{\mathbf{w},b}{\text{minimize}} \quad & \frac{\|\mathbf{w}\|^2}{2} \\
\text{subject to} \quad & y_n(\mathbf{w}^\top \mathbf{x}_n + b) \geq 1,\ n = 1, \ldots, N.
\end{aligned}$$

A Toy Example

- $\mathbf{x} \in \Re^2$
- Two training points:

$$\mathbf{x}_1, y_1 = (1,1), -1$$

$$\mathbf{x}_2, y_2 = (2,2), +1$$

- Find the best hyperplane $\mathbf{w} = (w_1, w_2)$

Optimization problem for the toy example

$$\boxed{\begin{aligned}
\underset{\mathbf{w}}{\text{minimize}} \quad f(\mathbf{w}) &= \frac{1}{2}\|\mathbf{w}\|^2 \\
\text{subject to} \quad g_1(\mathbf{w}, b) &= y_1(\mathbf{w}^\top \mathbf{x}_1 + b) - 1 \geq 0 \\
g_2(\mathbf{w}, b) &= y_2(\mathbf{w}^\top \mathbf{x}_2 + b) - 1 \geq 0.
\end{aligned}}$$

- Substituting actual values for $\mathbf{x}_1, y_1$ and $\mathbf{x}_2, y_2$.

$$\boxed{\begin{aligned}
\underset{\mathbf{w}}{\text{minimize}} \quad f(\mathbf{w}) &= \frac{1}{2}\|\mathbf{w}\|^2 \\
\text{subject to} \quad g_1(\mathbf{w}, b) &= -(\mathbf{w}^\top \mathbf{x}_1 + b) - 1 \geq 0 \\
g_2(\mathbf{w}, b) &= (\mathbf{w}^\top \mathbf{x}_2 + b) - 1 \geq 0.
\end{aligned}}$$

The above problem can be also written as:

$$\boxed{\begin{aligned}
\underset{w_1, w_2, b}{\text{minimize}} \quad f(w_1, w_2) &= \frac{1}{2}(w_1^2 + w_2^2) \\
\text{subject to} \quad g_1(w_1, w_2, b) &= -(w_1 + w_2 + b) - 1 \geq 0 \\
g_2(w_1, w_2, b) &= (2w_1 + 2w_2 + b) - 1 \geq 0.
\end{aligned}}$$

To solve the toy optimization problem, we rewrite it in the Lagrangian form:

$$L(w_1, w_2, b, \alpha) = \frac{1}{2}(w_1^2 + w_2^2) + \alpha_1(w_1 + w_2 + b + 1) - \alpha_2(2w_1 + 2w_2 + b - 1)$$

Setting $\nabla L = 0$, we get:

$$\begin{aligned}
\frac{\partial}{\partial w_1} L(w_1, w_2, b, \alpha) &= w_1 + \alpha_1 - 2\alpha_2 = 0 \\
\frac{\partial}{\partial w_2} L(w_1, w_2, b, \alpha) &= w_2 + \alpha_1 - 2\alpha_2 = 0 \\
\frac{\partial}{\partial b} L(w_1, w_2, b, \alpha) &= \alpha_1 - \alpha_2 = 0 \\
\frac{\partial}{\partial \alpha_1} L(w_1, w_2, b, \alpha) &= w_1 + w_2 + b + 1 = 0 \\
\frac{\partial}{\partial \alpha_2} L(w_1, w_2, b, \alpha) &= 2w_1 + 2w_2 + b - 1 = 0
\end{aligned}$$

Solving the above equations, we get, $w_1 = w_2 = 1$ and $b = -3$.

Back to SVM Optimization

Optimization Formulation

$$\begin{aligned}
\underset{\mathbf{w}, b}{\text{minimize}} \quad & \frac{\|\mathbf{w}\|^2}{2} \\
\text{subject to} \quad & y_n(\mathbf{w}^\top \mathbf{x}_n + b) \geq 1, \; n = 1, \ldots, N.
\end{aligned}$$

- Introducing Lagrange Multipliers,α_n, $n = 1, \ldots, N$

Rewriting as a (primal) Lagrangian

$$\begin{aligned}
\underset{\mathbf{w}, b, \alpha}{\text{minimize}} \quad & L_P(\mathbf{w}, b, \alpha) = \frac{\|\mathbf{w}\|^2}{2} + \sum_{n=1}^{N} \alpha_n \{1 - y_n(\mathbf{w}^\top \mathbf{x}_n + b)\} \\
\text{subject to} \quad & \alpha_n \geq 0 \; n = 1, \ldots, N.
\end{aligned}$$

Solving the Lagrangian

- Set gradient of L_P to 0

$$\frac{\partial L_P}{\partial \mathbf{w}} = 0 \Rightarrow \boxed{\mathbf{w} = \sum_{n=1}^{N} \alpha_n y_n \mathbf{x}_n}$$

$$\frac{\partial L_P}{\partial b} = 0 \Rightarrow \boxed{\sum_{n=1}^{N} \alpha_n y_n = 0}$$

- Substituting in L_P to get the dual L_D

Dual Lagrangian Formulation

$$\begin{aligned} \underset{\mathbf{w},b,\alpha}{\text{maximize}} \quad & L_D(\mathbf{w}, b, \alpha) = \sum_{n=1}^{N} \alpha_n - \frac{1}{2} \sum_{m,n=1}^{N} \alpha_m \alpha_n y_m y_n (\mathbf{x}_m^\top \mathbf{x}_n) \\ \text{subject to} \quad & \sum_{n=1}^{N} \alpha_n y_n = 0, \alpha_n \geq 0 \; n = 1, \ldots, N. \end{aligned}$$

- Dual Lagrangian is a *quadratic programming problem* in α_n's
  - Use "off-the-shelf" solvers
- Having found α_n's

$$\mathbf{w} = \sum_{n=1}^{N} \alpha_n y_n \mathbf{x}_n$$

- What will be the bias term b?

Investigating Kahrun Kuhn Tucker Conditions

- For the primal and dual formulations
- We can optimize the dual formulation (as shown earlier)
- Solution should satisfy the **Karush-Kuhn-Tucker** (KKT) Conditions

3.1 Kahrun-Kuhn-Tucker Conditions

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{w}} L_P(\mathbf{w}, b, \alpha) &= \mathbf{w} - \sum_{n=1}^{N} \alpha_n y_n \mathbf{x}_n = 0 & (3) \\
\frac{\partial}{\partial b} L_P(\mathbf{w}, b, \alpha) &= -\sum_{n=1}^{N} \alpha_n y_n = 0 & (4) \\
y_n\{\mathbf{w}^\top \mathbf{x}_n + b\} - 1 &\geq 0 & (5) \\
\alpha_n &\geq 0 & (6) \\
\alpha_n(y_n\{\mathbf{w}^\top \mathbf{x}_n + b\} - 1) &= 0 & (7)
\end{aligned}$$

- Use KKT condition #5
- For $\alpha_n > 0$

$$(y_n\{\mathbf{w}^\top \mathbf{x}_n + b\} - 1) = 0$$

- Which means that:

$$b = -\frac{\underset{n:y_n=-1}{max} \mathbf{w}^\top \mathbf{x}_n + \underset{n:y_n=1}{min} \mathbf{w}^\top \mathbf{x}_n}{2}$$

3.2 Support Vectors

Most α_n's are 0

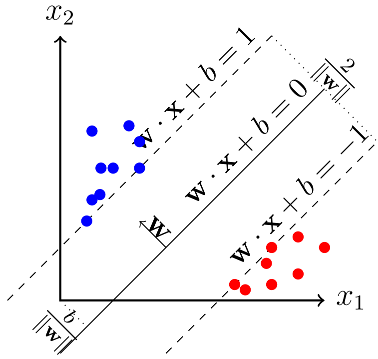
- KKT condition #5:

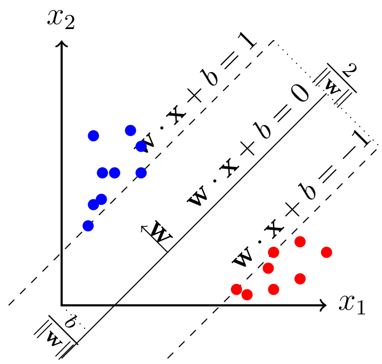
$$\alpha_n(y_n\{\mathbf{w}^\top \mathbf{x}_n + b\} - 1) = 0$$

- If $\mathbf{x}_n$ **not** on margin

$$\begin{aligned}
& y_n\{\mathbf{w}^\top \mathbf{x}_n + b\} > 1 \\
\Rightarrow \quad & \alpha_n = 0
\end{aligned}$$

- $\alpha_n \neq 0$ only for $\mathbf{x}_n$ on margin
- These are the **support vectors**
- Only need these for prediction


x_2
x_1
$\mathbf{w} \cdot \mathbf{x} + b = 1$
$\mathbf{w} \cdot \mathbf{x} + b = 0$
$\mathbf{w} \cdot \mathbf{x} + b = -1$
$\mathbf{w}$
$\frac{2}{\|\mathbf{w}\|}$
$\frac{b}{\|\mathbf{w}\|}$



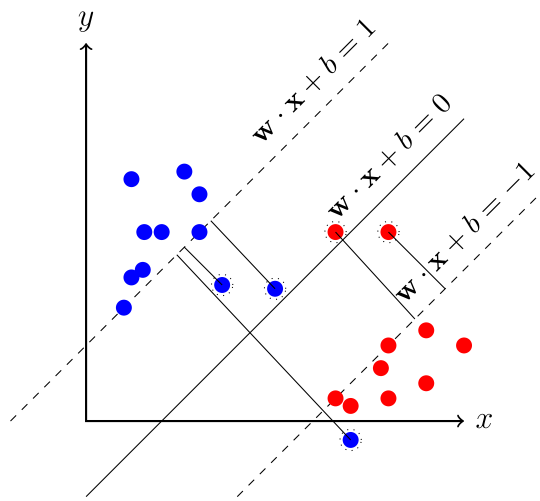
One can see from the prediction equation that:

$$y^* = sign(\sum_{n=1}^{N} \alpha_n y_n \, (\mathbf{x}_n^\top \mathbf{x}^*) \,)$$

In the summation, the entries for $\mathbf{x}_n$ that do not lie on the margin will have no contribution to the sum because α_n for those $\mathbf{x}_n$'s will be 0. Hence we only need to the non-zero input examples to get the prediction.

What have we seen so far?

- For linearly separable data, SVM learns a weight vector $\mathbf{w}$
- Maximizes the margin
- SVM training is a **constrained optimization problem**
  - Each training example should lie outside the margin
  - N constraints
- Cannot go for zero training error
- Still learn a maximum margin hyperplane
  1. Allow some examples to be misclassified
  2. Allow some examples to fall **inside** the margin
- How do you set up the optimization for SVM training



Introducing Slack Variables

- **Separable Case**: To ensure zero training loss, constraint was

$$y_n(\mathbf{w}^\top \mathbf{x}_n + b) \geq 1 \quad \forall n = 1 \ldots N$$

- **Non-separable Case**: Relax the constraint

$$y_n(\mathbf{w}^\top \mathbf{x}_n + b) \geq 1 - \xi_n \quad \forall n = 1 \ldots N$$

- ξ_n is called **slack variable** ($\xi_n \geq 0$)
- For misclassification, $\xi_n > 1$

3.3 Optimization Constraints

- It is OK to have some misclassified training examples
  - Some ξ_n's will be non-zero
- Minimize the number of such examples
  - Minimize $\sum_{n=1}^{N} \xi_n$

- Optimization Problem for Non-Separable Case

$$\boxed{\begin{array}{ll} \underset{\mathbf{w},b}{\text{minimize}} & f(\mathbf{w}, b) = \|\mathbf{w}\|^2 + C\sum_{n=1}^{N}\xi_n \\ \text{subject to} & y_n(\mathbf{w}^\top\mathbf{x}_n + b) \geq 1 - \xi_n, \xi_n \geq 0 \; n = 1, \ldots, N. \end{array}}$$

- C controls the impact of margin and the margin error.
- What is the role of C?
- Similar optimization procedure as for the separable case (QP for the dual)
- Weights have the same expression

$$\mathbf{w} = \sum_{n=1}^{N} \alpha_n y_n \mathbf{x}_n$$

- Support vectors are slightly different
  1. Points on the margin ($\xi_n = 0$)
  2. Inside the margin but on the correct side ($0 < \xi_n < 1$)
  3. On the wrong side of the hyperplane ($\xi_n \geq 1$)

C dictates if we focus more on maximizing the margin or reducing the training error.

- Training time for SVM training is $O(N^3)$
- Many *faster* but approximate approaches exist
  - Approximate QP solvers
  - Online training
- SVMs can be extended in different ways
  1. Non-linear boundaries (**kernel trick**)
  2. Multi-class classification
  3. Probabilistic output
  4. Regression (Support Vector Regression)

References

- Bishop Chapter 17.3

References

[1] V. Vapnik. *Statistical learning theory*. Wiley, 1998.