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9 Planar Graphs

9.1 PLANE AND PLANAR GRAPHS

A graph is said to be *embeddable in the plane*, or *planar*, if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing of a planar graph G is called a *planar embedding* of G. A planar embedding $\tilde{G}$ of G can itself be regarded as a graph isomorphic to G; the vertex set of $\tilde{G}$ is the set of points representing vertices of G, the edge set of $\tilde{G}$ is the set of lines representing edges of G, and a vertex of $\tilde{G}$ is incident with all the edges of $\tilde{G}$ that contain it. We therefore sometimes refer to a planar embedding of a planar graph as a *plane graph*. Figure 9.1*b* shows a planar embedding of the planar graph in figure 9.1*a*.

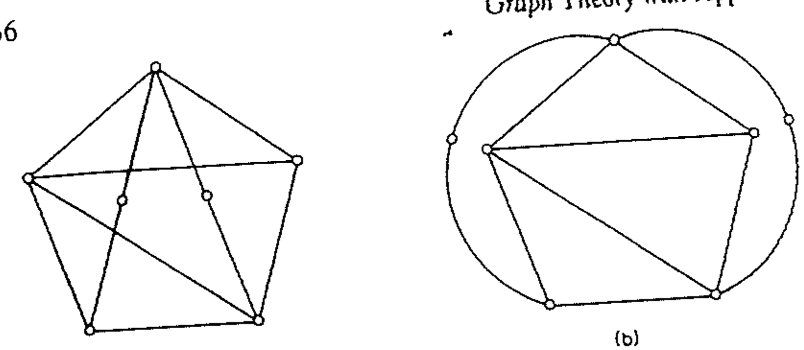
It is clear from the above definition that the study of planar graphs necessarily involves the topology of the plane. However, we shall not attempt here to be strictly rigorous in topological matters, and will be content to adopt a naive point of view toward them. This is done so as not to obscure the combinatorial aspect of the theory, which is our main interest.

The results of topology that are especially relevant in the study of planar graphs are those which deal with Jordan curves. (A *Jordan curve* is a continuous non-self-intersecting curve whose origin and terminus coincide.) The union of the edges in a cycle of a plane graph constitutes a Jordan curve; this is the reason why properties of Jordan curves come into play in planar graph theory. We shall recall a well-known theorem about Jordan curves and use it to demonstrate the nonplanarity of K_5.

Let J be a Jordan curve in the plane. Then the rest of the plane is partitioned into two disjoint open sets called the *interior* and *exterior* of J. We shall denote the interior and exterior of J, respectively, by $\operatorname{int} J$ and $\operatorname{ext} J$, and their closures by $\operatorname{Int} J$ and $\operatorname{Ext} J$. Clearly $\operatorname{Int} J \cap \operatorname{Ext} J = J$. The *Jordan curve theorem* states that any line joining a point in $\operatorname{int} J$ to a point in $\operatorname{ext} J$ must meet J in some point (see figure 9.2). Although this theorem is intuitively obvious, a formal proof of it is quite difficult.

Theorem 9.1 K_5 is nonplanar.

Proof By contradiction. If possible let G be a plane graph corresponding to K_5. Denote the vertices of G by v_1, v_2, v_3, v_4 and v_5. Since G is complete, any two of its vertices are joined by an edge. Now the cycle $C = v_1v_2v_3v_1$ is a Jordan curve in the plane, and the point v_4 must lie either in $\operatorname{int} C$ or $\operatorname{ext} C$.



(a) (b)

Figure 9.1. (*a*) A planar graph G; (*b*) a planar embedding of G

We shall suppose that $v_4 \in \text{int } C$. (The case where $v_4 \in \text{ext } C$ can be dealt with in a similar manner.) Then the edges v_4v_1, v_4v_2 and v_4v_3 divide int C into the three regions int C_1, int C_2 and int C_3, where $C_1 = v_1v_4v_2v_1$, $C_2 = v_2v_4v_3v_2$ and $C_3 = v_3v_4v_1v_3$ (see figure 9.3).

Now v_5 must lie in one of the four regions ext C, int C_1, int C_2 and int C_3. If $v_5 \in \text{ext } C$ then, since $v_4 \in \text{int } C$, it follows from the Jordan curve theorem that the edge v_4v_5 must meet C in some point. But this contradicts the assumption that G is a plane graph. The cases $v_5 \in \text{int } C_i$, $i = 1, 2, 3$, can be disposed of in like manner □

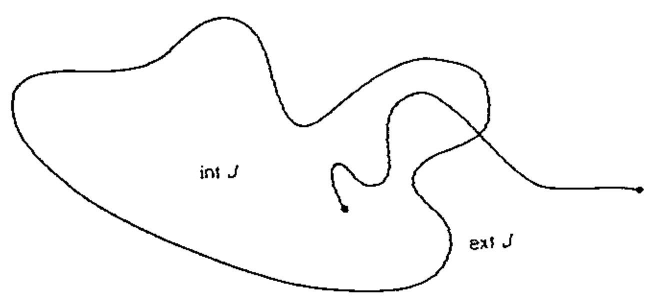


Figure 9.2

A similar argument can be used to establish that $K_{3,3}$, too, is nonplanar (exercise 9.1.1). We shall see in section 9.5 that, on the other hand, every nonplanar graph contains a subdivision of either K_5 or $K_{3,3}$.

The notion of a planar embedding extends to other surfaces.† A graph G is said to be *embeddable* on a surface S if it can be drawn in S so that its

† A *surface* is a 2-dimensional manifold. Closed surfaces are divided into two classes, orientable and non-orientable. The sphere and the torus are examples of orientable surfaces; the projective plane and the Möbius band are non-orientable. For a detailed account of embeddings of graphs on surfaces the reader is referred to Fréchet and Fan (1967).

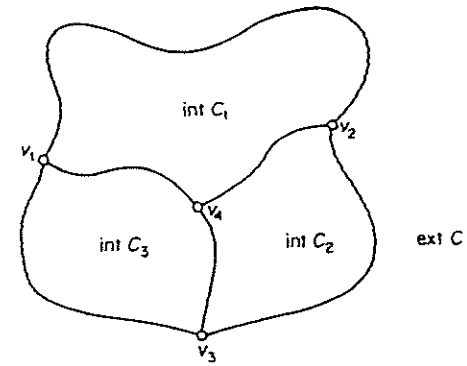
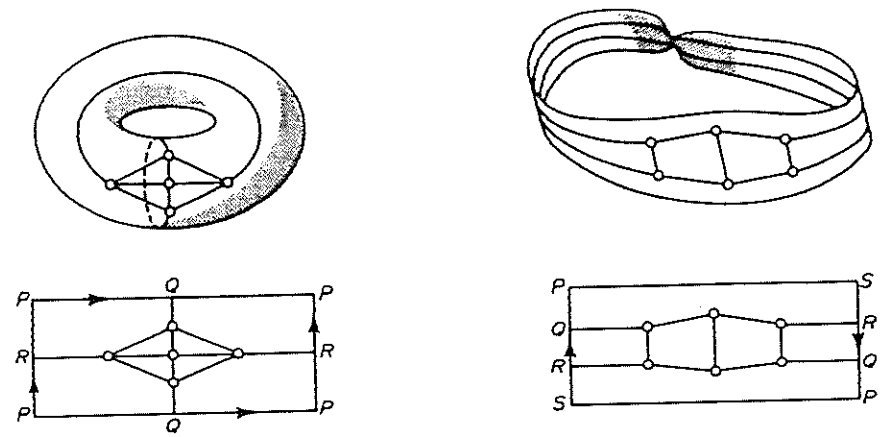


Figure 9.3

edges intersect only at their ends; such a drawing (if one exists) is called an *embedding* of G on S. Figure 9.4*a* shows an embedding of K_5 on the torus, and figure 9.4*b* an embedding of $K_{3,3}$ on the Möbius band. The torus is represented as a rectangle in which opposite sides are identified, and the Möbius band as a rectangle whose two ends are identified after one half-twist.

We have seen that not all graphs can be embedded in the plane; this is also true of other surfaces. It can be shown (see, for example, Fréchet and Fan, 1967) that, for every surface S, there exist graphs which are not embeddable on S. Every graph can, however, be 'embedded' in 3-dimensional space $\mathscr{R}^3$ (exercise 9.1.3).



(a) (b)

Figure 9.4. (*a*) An embedding of K_5 on the torus; (*b*) an embedding of $K_{3,3}$ on the Möbius band

Planar graphs and graphs embeddable on the sphere are one and the same. To show this we make use of a mapping known as stereographic projection. Consider a sphere S resting on a plane P, and denote by z the point of S that is diagonally opposite the point of contact of S and P. The mapping $\pi: S\backslash\{z\} \to P$, defined by $\pi(s) = p$ if and only if the points z, s and p are collinear, is called *stereographic projection* from z; it is illustrated in figure 9.5.

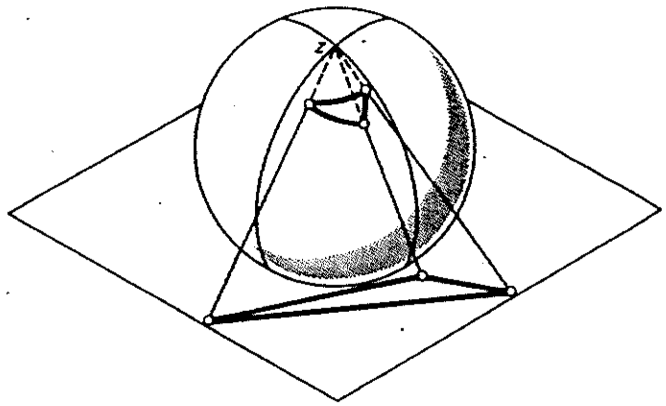


Figure 9.5. Stereographic projection

Theorem 9.2 A graph G is embeddable in the plane if and only if it is embeddable on the sphere.

Proof Suppose G has an embedding $\tilde{G}$ on the sphere. Choose a point z of the sphere not in $\tilde{G}$. Then the image of $\tilde{G}$ under stereographic projection from z is an embedding of G in the plane. The converse is proved similarly □

On many occasions it is advantageous to consider embeddings of planar graphs on the sphere; one instance is provided by the proof of theorem 9.3 in the next section.

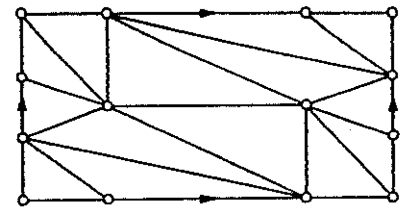
Exercises

9.1.1 Show that $K_{3,3}$ is nonplanar.

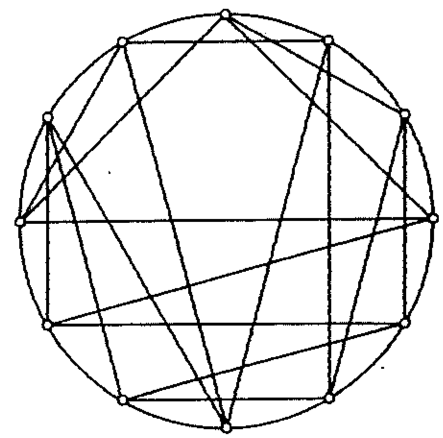
9.1.2 (*a*) Show that $K_5 - e$ is planar for any edge e of K_5.
(*b*) Show that $K_{3,3} - e$ is planar for any edge e of $K_{3,3}$.

9.1.3 Show that all graphs are 'embeddable' in $\mathscr{R}^3$.

9.1.4 Verify that the following is an embedding of K_7 on the torus:



9.1.5 Find a planar embedding of the following graph in which each edge is a straight line.
(Fáry, 1948 has proved that every simple planar graph has such an embedding.)



9.2 DUAL GRAPHS

A plane graph G partitions the rest of the plane into a number of connected regions; the closures of these regions are called the *faces* of G. Figure 9.6 shows a plane graph with six faces, f_1, f_2, f_3, f_4, f_5 and f_6. The notion of a face applies also to embeddings of graphs on other surfaces. We shall denote by $F(G)$ and $\phi(G)$, respectively, the set of faces and the number of faces of a plane graph G.

Each plane graph has exactly one unbounded face, called the *exterior face;* in the plane graph of figure 9.6, f_1 is the exterior face.

Theorem 9.3 Let v be a vertex of a planar graph G. Then G can be embedded in the plane in such a way that v is on the exterior face of the embedding.

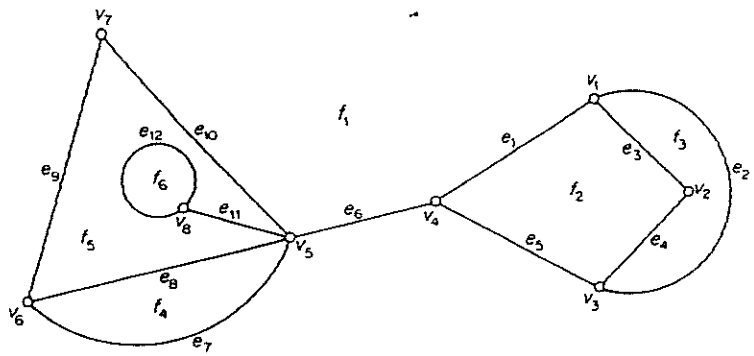


Figure 9.6. A plane graph with six faces

Proof Consider an embedding $\tilde{G}$ of G on the sphere; such an embedding exists by virtue of theorem 9.2. Let z be a point in the interior of some face containing v, and let $\pi(\tilde{G})$ be the image of $\tilde{G}$ under stereographic projection from z. Clearly $\pi(\tilde{G})$ is a planar embedding of G of the desired type □

We denote the boundary of a face f of a plane graph G by $b(f)$. If G is connected, then $b(f)$ can be regarded as a closed walk in which each cut edge of G in $b(f)$ is traversed twice; when $b(f)$ contains no cut edges, it is a cycle of G. For example, in the plane graph of figure 9.6,

$$b(f_2) = v_1e_3v_2e_4v_3e_5v_4e_1v_1$$

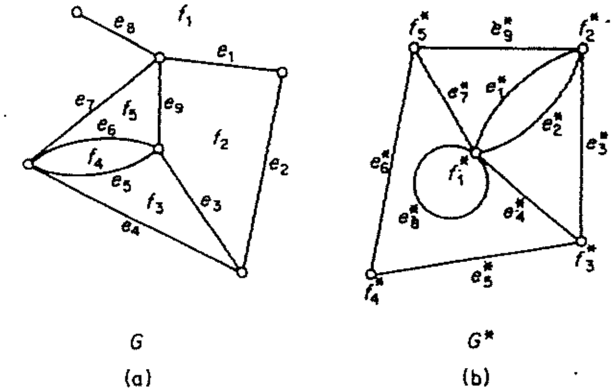
and

$$b(f_5) = v_7e_{10}v_5e_{11}v_8e_{12}v_8e_{11}v_5e_8v_6e_9v_7$$

A face f is said to be *incident* with the vertices and edges in its boundary. If e is a cut edge in a plane graph, just one face is incident with e; otherwise, there are two faces incident with e. We say that an edge *separates* the faces incident with it. The *degree*, $d_G(f)$, of a face f is the number of edges with which it is incident (that is, the number of edges in $b(f)$), cut edges being counted twice. In figure 9.6, f_1 is incident with the vertices v_1, v_3, v_4, v_5, v_6, v_7 and the edges e_1, e_2, e_5, e_6, e_7, e_9, e_{10}; e_1 separates f_1 from f_2 and e_{11} separates f_5 from f_5; $d(f_2) = 4$ and $d(f_5) = 6$.

Given a plane graph G, one can define another graph G^* as follows: corresponding to each face f of G there is a vertex f^* of G^*, and corresponding to each edge e of G there is an edge e^* of G^*; two vertices f^* and g^* are joined by the edge e^* in G^* if and only if their corresponding faces f and g are separated by the edge e in G. The graph G^* is called the *dual* of G. A plane graph and its dual are shown in figures 9.7*a* and 9.7*b*.

It is easy to see that the dual G^* of a plane graph G is planar; in fact,



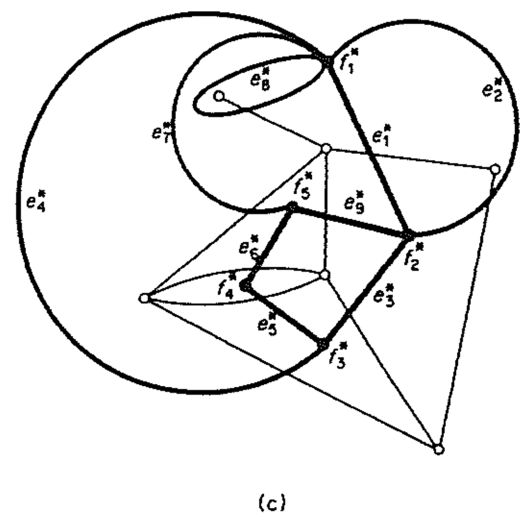
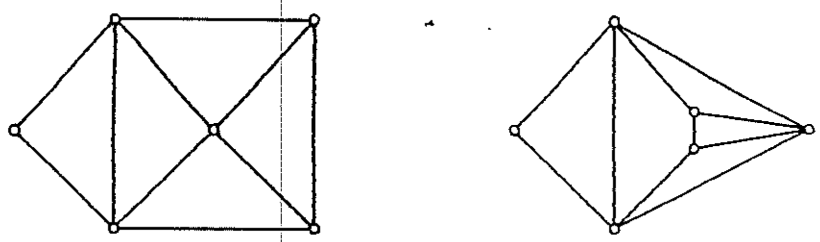


Figure 9.7. A plane graph and its dual

there is a natural way to embed G^* in the plane. We place each vertex f^* in the corresponding face f of G, and then draw each edge e^* in such a way that it crosses the corresponding edge e of G exactly once (and crosses no other edge of G). This procedure is illustrated in figure 9.7c, where the dual is indicated by heavy points and lines. It is intuitively clear that we can always draw the dual as a plane graph in this way, but we shall not prove this fact. Note that if e is a loop of G, then e^* is a cut edge of G^*, and vice versa.

Although defined abstractly, it is sometimes convenient to regard the dual



(a) (b)

Figure 9.8. Isomorphic plane graphs with nonisomorphic duals

G^* of a plane graph G as a plane graph (embedded as described above). One can then consider the dual G^{**} of G^*, and it is not difficult to prove that, when G is connected, $G^{**} \cong G$ (exercise 9.2.4); a glance at figure 9.7c will indicate why this is so.

It should be noted that isomorphic plane graphs may have nonisomorphic duals. For example, the plane graphs in figure 9.8 are isomorphic, but their duals are not—the plane graph of figure 9.8*a* has a face of degree five, whereas the plane graph of figure 9.8*b* has no such face. Thus the notion of a dual is meaningful only for plane graphs, and cannot be extended to planar graphs in general.

The following relations are direct consequences of the definition of G^*:

$$\begin{aligned} \nu(G^*) &= \phi(G) \\ \varepsilon(G^*) &= \varepsilon(G) \qquad (9.1) \\ d_{G^*}(f^*) &= d_G(f) \quad \text{for all} \quad f \in F(G) \end{aligned}$$

Theorem 9.4 If G is a plane graph, then

$$\sum_{f \in F} d(f) = 2\varepsilon$$

Proof Let G^* be the dual of G. Then

$$\begin{aligned} \sum_{f \in F(G)} d(f) &= \sum_{f^* \in V(G^*)} d(f^*) && \text{by (9.1)} \\ &= 2\varepsilon(G^*) && \text{by theorem 1.1} \\ &= 2\varepsilon(G) && \text{by (9.1)} \quad \square \end{aligned}$$

Exercises

9.2.1 (*a*) Show that a graph is planar if and only if each of its blocks is planar.

(*b*) Deduce that a minimal nonplanar graph is a simple block.

9.2.2 A plane graph is *self-dual* if it is isomorphic to its dual.

(*a*) Show that if G is self-dual, then $\varepsilon = 2\nu - 2$.

(*b*) For each $n \geq 4$, find a self-dual plane graph on n vertices.

9.2.3 (*a*) Show that B is a bond of a plane graph G if and only if $\{e^* \in E(G^*) \mid e \in B\}$ is a cycle of G^*.

(*b*) Deduce that the dual of an eulerian plane graph is bipartite.

9.2.4 Let G be a plane graph. Show that

(*a*) $G^{**} \cong G$ if and only if G is connected;

(*b*) $\chi(G^{**}) = \chi(G)$.

9.2.5 Let T be a spanning tree of a connected plane graph G, and let $E^* = \{e^* \in E(G^*) \mid e \notin E(T)\}$. Show that $T^* = G^*[E^*]$ is a spanning tree of G^*.

9.2.6 A *plane triangulation* is a plane graph in which each face has degree three. Show that every simple plane graph is a spanning subgraph of some simple plane triangulation ($\nu \geq 3$).

9.2.7 Let G be a simple plane triangulation with $\nu \geq 4$. Show that G^* is a simple 2-edge-connected 3-regular planar graph.

9.2.8* Show that any plane triangulation G contains a bipartite subgraph with $2\varepsilon(G)/3$ edges. (F. Harary, D. Matula)

9.3 EULER'S FORMULA

There is a simple formula relating the numbers of vertices, edges and faces in a connected plane graph. It is known as *Euler's formula* because Euler established it for those plane graphs defined by the vertices and edges of polyhedra.

Theorem 9.5 If G is a connected plane graph, then

$$\nu - \varepsilon + \phi = 2$$

Proof By induction on ϕ, the number of faces of G. If $\phi = 1$, then each edge of G is a cut edge and so G, being connected, is a tree. In this case $\varepsilon = \nu - 1$, by theorem 2.2, and the theorem clearly holds. Suppose that it is true for all connected plane graphs with fewer than n faces, and let G be a connected plane graph with $n \geq 2$ faces. Choose an edge e of G that is not a cut edge. Then $G - e$ is a connected plane graph and has $n - 1$ faces, since the two faces of G separated by e combine to form one face of $G - e$. By the induction hypothesis

$$\nu(G-e) - \varepsilon(G-e) + \phi(G-e) = 2$$

and, using the relations

$$\nu(G-e) = \nu(G) \qquad \varepsilon(G-e) = \varepsilon(G) - 1 \qquad \phi(G-e) = \phi(G) - 1$$

we obtain

$$\nu(G) - \varepsilon(G) + \phi(G) = 2$$

The theorem follows by the principle of induction $\square$

Corollary 9.5.1 All planar embeddings of a given connected planar graph have the same number of faces.

Proof Let G and H be two planar embeddings of a given connected planar graph. Since $G \cong H$, $\nu(G) = \nu(H)$ and $\varepsilon(G) = \varepsilon(H)$. Applying theorem 9.5, we have

$$\phi(G) = \varepsilon(G) - \nu(G) + 2 = \varepsilon(H) - \nu(H) + 2 = \phi(H) \quad \square$$

Corollary 9.5.2 If G is a simple planar graph with $\nu \geq 3$, then $\varepsilon \leq 3\nu - 6$.

Proof It clearly suffices to prove this for connected graphs. Let G be a simple connected graph with $\nu \geq 3$. Then $d(f) \geq 3$ for all $f \in F$, and

$$\sum_{f \in F} d(f) \geq 3\phi$$

By theorem 9.4

$$2\varepsilon \geq 3\phi$$

Thus, from theorem 9.5

$$\nu - \varepsilon + 2\varepsilon/3 \geq 2$$

or

$$\varepsilon \leq 3\nu - 6 \quad \square$$

Corollary 9.5.3 If G is a simple planar graph, then $\delta \leq 5$.

Proof This is trivial for $\nu = 1, 2$. If $\nu \geq 3$, then, by theorem 1.1 and corollary 9.5.2,

$$\delta\nu \leq \sum_{v \in V} d(v) = 2\varepsilon \leq 6\nu - 12$$

It follows that $\delta \leq 5$ $\square$

We have already seen that K_5 and $K_{3,3}$ are nonplanar (theorem 9.1 and exercise 9.1.1). Here, we shall derive these two results as corollaries of theorem 9.5.

Corollary 9.5.4 K_5 is nonplanar.

Proof If K_5 were planar then, by corollary 9.5.2, we would have

$$10 = \varepsilon(K_5) \leq 3\nu(K_5) - 6 = 9$$

Thus K_5 must be nonplanar $\square$

Corollary 9.5.5 $K_{3,3}$ is nonplanar.

Proof Suppose that $K_{3,3}$ is planar and let G be a planar embedding of $K_{3,3}$. Since $K_{3,3}$ has no cycles of length less than four, every face of G must

have degree at least four. Therefore, by theorem 9.4, we have

$$4\phi \leq \sum_{f \in F} d(f) = 2\varepsilon = 18$$

That is

$$\phi \leq 4$$

Theorem 9.5 now implies that

$$2 = \nu - \varepsilon + \phi \leq 6 - 9 + 4 = 1$$

which is absurd $\square$

Exercises

9.3.1 (*a*) Show that if G is a connected planar graph with girth $k \geq 3$, then $\varepsilon \leq k(\nu - 2)/(k - 2)$.
(*b*) Using (*a*), show that the Petersen graph is nonplanar.

9.3.2 Show that every planar graph is 6-vertex-colourable.

9.3.3 (*a*) Show that if G is a simple planar graph with $\nu \geq 11$, then G^c is nonplanar.
(*b*) Find a simple planar graph G with $\nu = 8$ such that G^c is also planar.

9.3.4 The *thickness* $\theta(G)$ of G is the minimum number of planar graphs whose union is G. (Thus $\theta(G) = 1$ if and only if G is planar.)
(*a*) Show that $\theta(G) \geq \{\varepsilon/(3\nu - 6)\}$.
(*b*) Deduce that $\theta(K_\nu) \geq \{\nu(\nu-1)/6(\nu-2)\}$ and show, using exercise 9.3.3*b*, that equality holds for all $\nu \leq 8$.

9.3.5 Use the result of exercise 9.2.5 to deduce Euler's formula.

9.3.6 Show that if G is a plane triangulation, then $\varepsilon = 3\nu - 6$.

9.3.7 Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of $n \geq 3$ points in the plane such that the distance between any two points is at least one. Show that there are at most $3n - 6$ pairs of points at distance exactly one.

9.4 BRIDGES

In the study of planar graphs, certain subgraphs, called bridges, play an important rôle. We shall discuss properties of these subgraphs in this section.

Let H be a given subgraph of a graph G. We define a relation $\sim$ on $E(G)\backslash E(H)$ by the condition that $e_1 \sim e_2$ if there exists a walk W such that

(i) the first and last edges of W are e_1 and e_2, respectively, and
(ii) W is internally-disjoint from H (that is, no internal vertex of W is a vertex of H).

It is easy to verify that $\sim$ is an equivalence relation on $E(G)\backslash E(H)$. A subgraph of $G - E(H)$ induced by an equivalence class under the relation $\sim$

is called a *bridge* of H in G. It follows immediately from the definition that if B is a bridge of H, then B is a connected graph and, moreover, that any two vertices of B are connected by a path that is internally-disjoint from H. It is also easy to see that two bridges of H have no vertices in common except, possibly, for vertices of H. For a bridge B of H, we write $V(B) \cap V(H) = V(B, H)$, and call the vertices in this set the *vertices of attachment* of B to H. Figure 9.9 shows a variety of bridges of a cycle in a graph; edges of different bridges are represented by different kinds of lines.

In this section we are concerned with the study of bridges of a cycle C. Thus, to avoid repetition, we shall abbreviate 'bridge of C' to 'bridge' in the coming discussion; all bridges will be understood to be bridges of a given cycle C.

In a connected graph every bridge has at least one vertex of attachment, and in a block every bridge has at least two vertices of attachment. A bridge with k vertices of attachment is called a *k-bridge*. Two k-bridges with the same vertices of attachment are *equivalent* k-bridges; for example, in figure 9.9, B_1 and B_2 are equivalent 3-bridges.

The vertices of attachment of a k-bridge B with $k \geq 2$ effect a partition of C into edge-disjoint paths, called the *segments* of B. Two bridges *avoid* one another if all the vertices of attachment of one bridge lie in a single segment of the other bridge; otherwise they *overlap*. In figure 9.9, B_2 and B_3 avoid one another, whereas B_1 and B_2 overlap. Two bridges B and B' are *skew* if there are four distinct vertices u, v, u' and v' of C such that u and v are vertices of attachment of B, u' and v' are vertices of attachment of B', and the four vertices appear in the cyclic order u, u', v, v' on C. In figure 9.9, B_3 and B_4 are skew, but B_1 and B_2 are not.

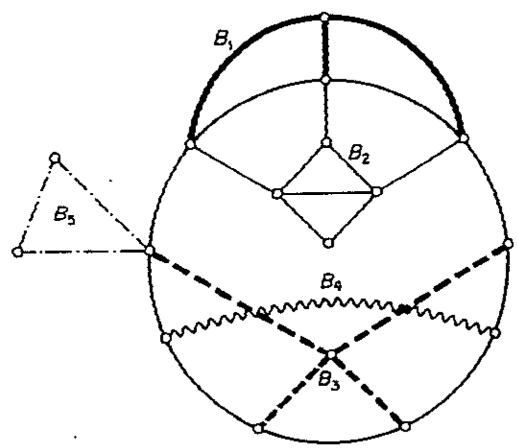


Figure 9.9. Bridges in a graph

Theorem 9.6 If two bridges overlap, then either they are skew or else they are equivalent 3-bridges.

Proof Suppose that the bridges B and B' overlap. Clearly, each must have at least two vertices of attachment. Now if either B or B' is a 2-bridge, it is easily verified that they must be skew. We may therefore assume that both B and B' have at least three vertices of attachment. There are two cases.

Case 1 B and B' are not equivalent bridges. Then B' has a vertex of attachment u' between two consecutive vertices of attachment u and v of B. Since B and B' overlap, some vertex of attachment v' of B' does not lie in the segment of B connecting u and v. It now follows that B and B' are skew.

Case 2 B and B' are equivalent k-bridges, $k \geq 3$. If $k \geq 4$, then B and B' are clearly skew; if $k = 3$, they are equivalent 3-bridges □

Theorem 9.7 If a bridge B has three vertices of attachment v_1, v_2 and v_3, then there exists a vertex v_0 in $V(B) \backslash V(C)$ and three paths P_1, P_2 and P_3 in B joining v_0 to v_1, v_2 and v_3, respectively, such that, for $i \neq j$, P_i and P_j have only the vertex v_0 in common (see figure 9.10).

Proof Let P be a (v_1, v_2)-path in B, internally-disjoint from C. P must have an internal vertex v, since otherwise the bridge B would be just P, and would not contain a third vertex v_3. Let Q be a (v_3, v)-path in B, internally-disjoint from C, and let v_0 be the first vertex of Q on P. Denote by P_1 the (v_0, v_1)-section of P^{-1}, by P_2 the (v_0, v_2)-section of P, and by P_3 the (v_0, v_3)-section of Q^{-1}. Clearly P_1, P_2 and P_3 satisfy the required conditions □

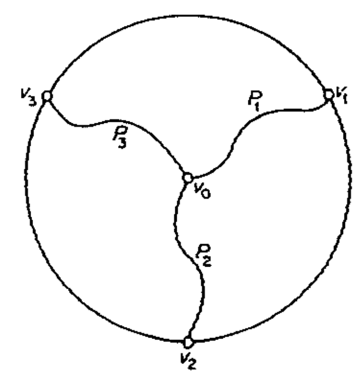


Figure 9.10

We shall now consider bridges in plane graphs. Suppose that G is a plane graph and that C is a cycle in G. Then C is a Jordan curve in the plane, and each edge of $E(G)\backslash E(C)$ is contained in one of the two regions Int C and Ext C. It follows that a bridge of C is contained entirely in Int C or Ext C. A bridge contained in Int C is called an *inner bridge*, and a bridge contained in Ext C, an *outer bridge*. In figure 9.11 B_1 and B_2 are inner bridges, and B_3 and B_4 are outer bridges.

Theorem 9.8 Inner (outer) bridges avoid one another.

Proof By contradiction. Let B and B' be two inner bridges that overlap. Then, by theorem 9.6, they must be either skew or equivalent 3-bridges.

Case 1 B and B' are skew. By definition, there exist distinct vertices u and v in B and u' and v' in B', appearing in the cyclic order u, u', v, v' on C. Let P be a (u, v)-path in B and P' a (u', v')-path in B', both internally-disjoint from C. The two paths P and P' cannot have an internal vertex in common because they belong to different bridges. At the same time, both P and P' must be contained in Int C because B and B' are inner bridges. By the Jordan curve theorem, G cannot be a plane graph, contrary to hypothesis (see figure 9.12).

Case 2 B and B' are equivalent 3-bridges. Let the common set of vertices of attachment be $\{v_1, v_2, v_3\}$. By theorem 9.7, there exist in B a vertex v_0 and three paths P_1, P_2 and P_3 joining v_0 to v_1, v_2 and v_3, respectively, such that, for $i \neq j$, P_i and P_j have only the vertex v_0 in common. Similarly, B' has a vertex v_0' and three paths P_1', P_2' and P_3' joining v_0' to v_1, v_2 and v_3, respectively, such that, for $i \neq j$, P_i' and P_j' have only the vertex v_0' in common (see figure 9.13).

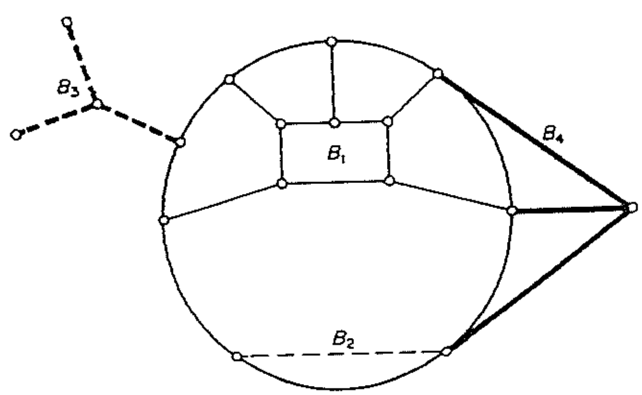


Figure 9.11. Bridges in a plane graph

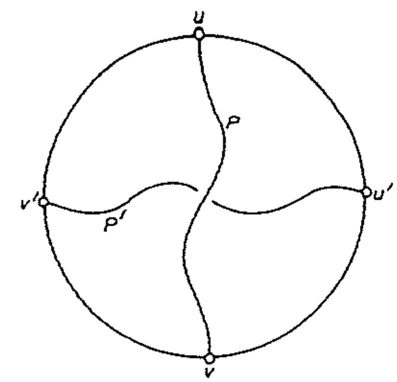


Figure 9.12

Now the paths P_1, P_2 and P_3 divide Int C into three regions, and v_0' must be in the interior of one of these regions. Since only two of the vertices v_1, v_2 and v_3 can lie on the boundary of the region containing v_0', we may assume, by symmetry, that v_3 is not on the boundary of this region. By the Jordan curve theorem, the path P_3' must cross either P_1, P_2 or C. But since B and B' are distinct inner bridges, this is clearly impossible.

We conclude that inner bridges avoid one another. Similarly, outer bridges avoid one another □

Let G be a plane graph. An inner bridge B of a cycle C in G is *transferable* if there exists a planar embedding $\tilde{G}$ of G which is identical to G itself, except that B is an outer bridge of C in $\tilde{G}$. The plane graph $\tilde{G}$ is said to be obtained from G by *transferring* B. Figure 9.14 illustrates the transfer of a bridge.

Theorem 9.9 An inner bridge that avoids every outer bridge is transferable.

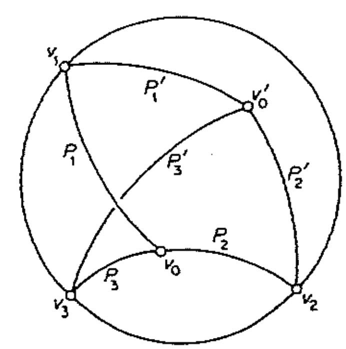


Figure 9.13

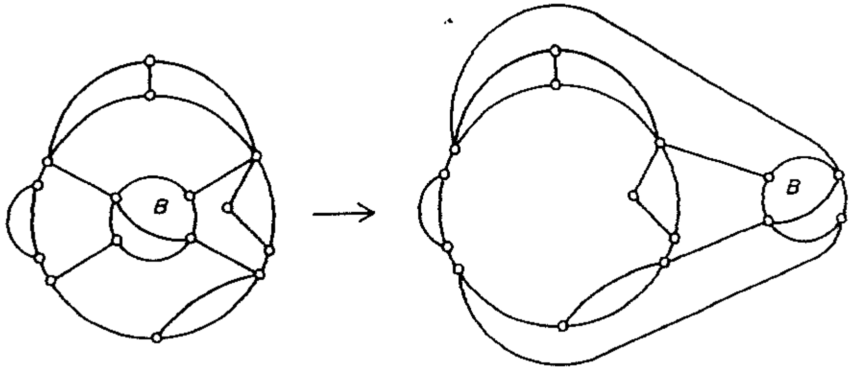


Figure 9.14. The transfer of a bridge

Proof Let B be an inner bridge that avoids every outer bridge. Then the vertices of attachment of B to C all lie on the boundary of some face of G contained in Ext C. B can now be drawn in this face, as shown in figure 9.15 □

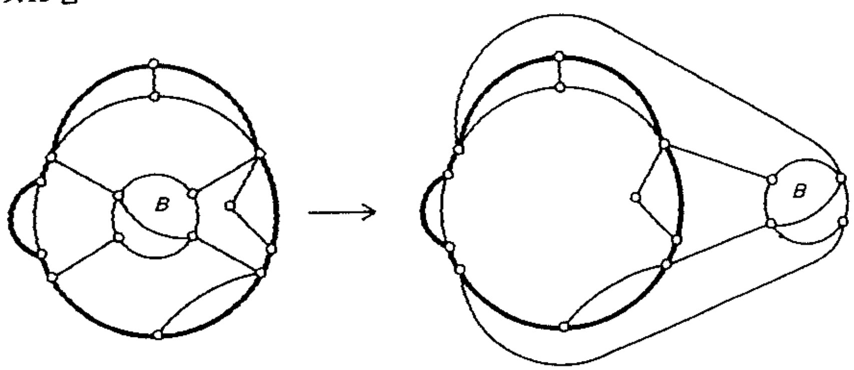


Figure 9.15

Theorem 9.9 is crucial to the proof of Kuratowski's theorem, which will be proved in the next section.

Exercises

9.4.1 Show that if B and B' are two distinct bridges, then $V(B)\cap V(B')\subseteq V(C)$.

9.4.2 Let u, x, v and y (in that cyclic order) be four distinct vertices of attachment of a bridge B to a cycle C in a plane graph. Show that there is a (u, v)-path P and an (x, y)-path Q in B such that (i) P and Q are internally-disjoint from C, and (ii) $|V(P)\cap V(Q)|\geq 1$.

9.4.3 (*a*) Let $C=v_1v_2\ldots v_nv_1$ be a longest cycle in a nonhamiltonian connected graph G. Show that

(i) there exists a bridge B such that $V(B)\backslash V(C)\neq\emptyset$;

(ii) if v_i and v_j are vertices of attachment of B, then $v_{i+1}v_{j+1}\notin E$.

(*b*) Deduce that if $\alpha\leq\kappa$, then G is hamiltonian.

(V. Chvátal and P. Erdös)

9.5 KURATOWSKI'S THEOREM

Since planarity is such a fundamental property, it is clearly of importance to know which graphs are planar and which are not. We have already noted that, in particular, K_5 and $K_{3,3}$ are nonplanar and that any proper subgraph of either of these graphs is planar (exercise 9.1.2). A remarkably simple characterisation of planar graphs was given by Kuratowski (1930). This section is devoted to a proof of Kuratowski's theorem.

The following lemmas are simple observations, and we leave their proofs as an exercise (9.5.1).

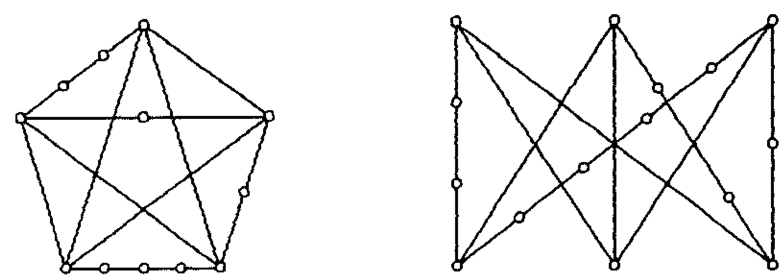
Lemma 9.10.1 If G is nonplanar, then every subdivision of G is nonplanar.

Lemma 9.10.2 If G is planar, then every subgraph of G is planar.

Since K_5 and $K_{3,3}$ are nonplanar, we see from these two lemmas that if G is planar, then G cannot contain a subdivision of K_5 or of $K_{3,3}$ (figure 9.16). Kuratowski showed that this necessary condition is also sufficient.

Before proving Kuratowski's theorem, we need to establish two more simple lemmas.

Let G be a graph with a 2-vertex cut $\{u, v\}$. Then there exist edge-disjoint subgraphs G_1 and G_2 such that $V(G_1)\cap V(G_2)=\{u, v\}$ and $G_1\cup G_2=G$. Consider such a separation of G into subgraphs. In both G_1 and G_2 join u



(a) (b)

Figure 9.16. (*a*) A subdivision of K_5; (*b*) a subdivision of $K_{3,3}$

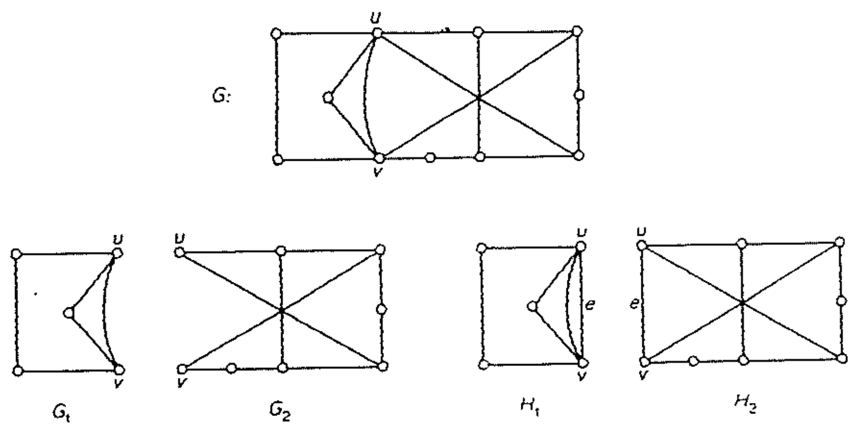


Figure 9.17

and v by a new edge e to obtain graphs H_1 and H_2, as in figure 9.17. Clearly $G = (H_1 \cup H_2) - e$. It is also easily seen that $\varepsilon(H_i) < \varepsilon(G)$ for $i = 1, 2$.

Lemma 9.10.3 If G is nonplanar, then at least one of H_1 and H_2 is also nonplanar.

Proof By contradiction. Suppose that both H_1 and H_2 are planar. Let $\tilde{H}_1$ be a planar embedding of H_1, and let f be a face of $\tilde{H}_1$ incident with e. If $\tilde{H}_2$ is an embedding of H_2 in f such that $\tilde{H}_1$ and $\tilde{H}_2$ have only the vertices u and v and the edge e in common, then $(\tilde{H}_1 \cup \tilde{H}_2) - e$ is a planar embedding of G. This contradicts the hypothesis that G is nonplanar □

Lemma 9.10.4 Let G be a nonplanar connected graph that contains no subdivision of K_5 or $K_{3,3}$ and has as few edges as possible. Then G is simple and 3-connected.

Proof By contradiction. Let G satisfy the hypotheses of the lemma. Then G is clearly a minimal nonplanar graph, and therefore (exercise 9.2.1*b*) must be a simple block. If G is not 3-connected, let $\{u, v\}$ be a 2-vertex cut of G and let H_1 and H_2 be the graphs obtained from this cut as described above. By lemma 9.10.3, at least one of H_1 and H_2, say H_1, is nonplanar. Since $\varepsilon(H_1) < \varepsilon(G)$, H_1 must contain a subgraph K which is a subdivision of K_5 or $K_{3,3}$; moreover $K \not\subseteq G$, and so the edge e is in K. Let P be a (u, v)-path in $H_2 - e$. Then G contains the subgraph $(K \cup P) - e$, which is a subdivision of K and hence a subdivision of K_5 or $K_{3,3}$. This contradiction establishes the lemma □

We shall find it convenient to adopt the following notation in the proof of Kuratowski's theorem. Suppose that C is a cycle in a plane graph. Then we can regard the two possible orientations of C as 'clockwise' and 'anticlockwise'. For any two vertices, u and v of C, we shall denote by $C[u, v]$ the (u, v)-path which follows the clockwise orientation of C; similarly we shall use the symbols $C(u, v]$, $C[u, v)$ and $C(u, v)$ to denote the paths $C[u, v] - u$, $C[u, v] - v$ and $C[u, v] - \{u, v\}$. We are now ready to prove Kuratowski's theorem. Our proof is based on that of Dirac and Schuster (1954).

Theorem 9.10 A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

Proof We have already noted that the necessity follows from lemmas 9.10.1 and 9.10.2. We shall prove the sufficiency by contradiction.

If possible, choose a nonplanar graph G that contains no subdivision of K_5 or $K_{3,3}$ and has as few edges as possible. From lemma 9.10.4 it follows that G is simple and 3-connected. Clearly G must also be a minimal nonplanar graph.

Let uv be an edge of G, and let H be a planar embedding of the planar graph $G - uv$. Since G is 3-connected, H is 2-connected and, by corollary 3.2.1, u and v are contained together in a cycle of H. Choose a cycle C of H that contains u and v and is such that the number of edges in $\text{Int}\, C$ is as large as possible.

Since H is simple and 2-connected, each bridge of C in H must have at least two vertices of attachment. Now all outer bridges of C must be 2-bridges that overlap uv because, if some outer bridge were a k-bridge for $k \geq 3$ or a 2-bridge that avoided uv, then there would be a cycle C' containing u and v with more edges in its interior than C, contradicting the choice of C. These two cases are illustrated in figure 9.18 (with C' indicated by heavy lines).

In fact, all outer bridges of C in H must be single edges. For if a 2-bridge with vertices of attachment x and y had a third vertex, the set $\{x, y\}$ would be a 2-vertex cut of G, contradicting the fact that G is 3-connected.

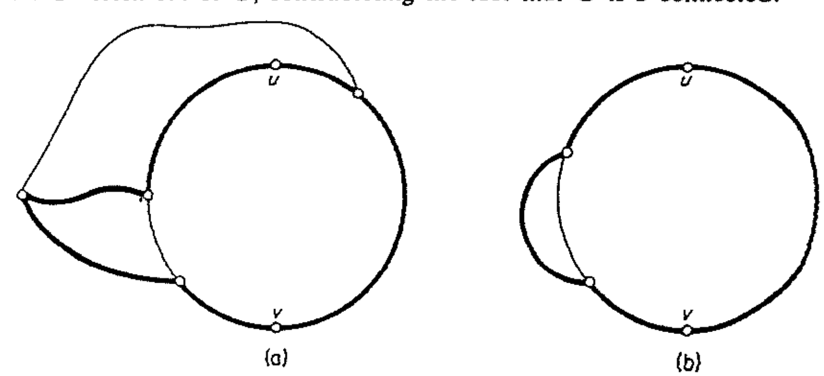


Figure 9.18

By theorem 9.8, no two inner bridges overlap. Therefore some inner bridge skew to uv must overlap some outer bridge. For otherwise, by theorem 9.9, all such bridges could be transferred (one by one), and then the edge uv could be drawn in Int C to obtain a planar embedding of G; since G is nonplanar, this is not possible. Therefore, there is an inner bridge B that is both skew to uv and skew to some outer bridge xy.

Two cases now arise, depending on whether B has a vertex of attachment different from u, v, x and y or not.

Case 1 B has a vertex of attachment different from u, v, x and y. We can choose the notation so that B has a vertex of attachment v_1 in $C(x, u)$ (see figure 9.19). We consider two subcases, depending on whether B has a vertex of attachment in $C(y, v)$ or not.

Case 1a B has a vertex of attachment v_2 in $C(y, v)$. In this case there is a (v_1, v_2)-path P in B that is internally-disjoint from C. But then $(C \cup P) + \{uv, xy\}$ is a subdivision of $K_{3,3}$ in G, a contradiction (see figure 9.19).

Case 1b B has no vertex of attachment in $C(y, v)$. Since B is skew to uv and to xy, B must have vertices of attachment v_2 in $C(u, y]$ and v_3 in $C[v, x)$. Thus B has three vertices of attachment v_1, v_2 and v_3. By theorem 9.7, there exists a vertex v_0 in $V(B)\backslash V(C)$ and three paths P_1, P_2 and P_3 in B joining v_0 to v_1, v_2 and v_3, respectively, such that, for $i \neq j$, P_i and P_j have only the vertex v_0 in common. But now $(C \cup P_1 \cup P_2 \cup P_3) + \{uv, xy\}$ contains a subdivision of $K_{3,3}$, a contradiction. This case is illustrated in figure 9.20. The subdivision of $K_{3,3}$ is indicated by heavy lines.

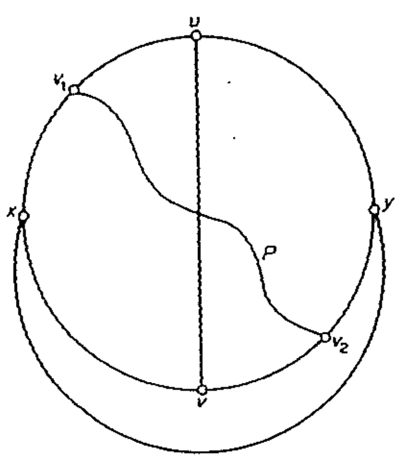


Figure 9.19

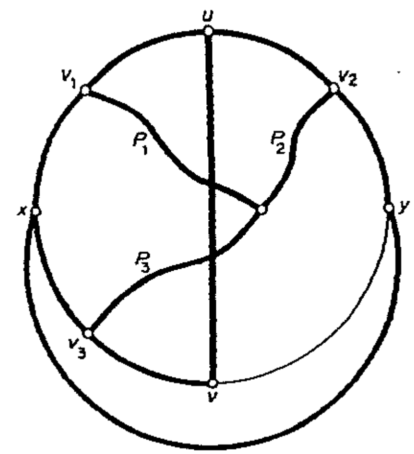


Figure 9.20

Case 2 B has no vertex of attachment other than u, v, x and y. Since B is skew to both uv and xy, it follows that u, v, x and y must all be vertices of attachment of B. Therefore (exercise 9.4.2) there exists a (u, v)-path P and an (x, y)-path Q in B such that (i) P and Q are internally-disjoint from C, and (ii) $|V(P) \cap V(Q)| \geq 1$. We consider two subcases, depending on whether P and Q have one or more vertices in common.

Case 2a $|V(P) \cap V(Q)| = 1$. In this case $(C \cup P \cup Q) + \{uv, xy\}$ is a subdivision of K_5 in G, again a contradiction (see figure 9.21).

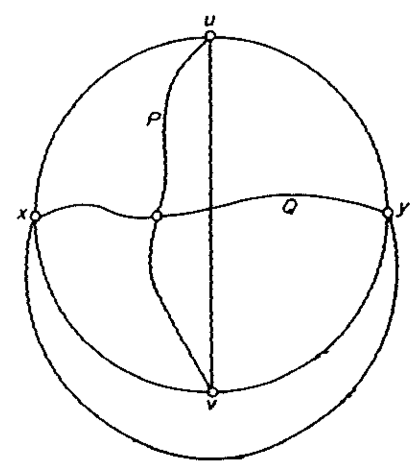


Figure 9.21

Case 2b $|V(P)\cap V(Q)|\geq 2$. Let u' and v' be the first and last vertices of P on Q, and let P_1 and P_2 denote the (u, u')- and (v', v)-sections of P. Then $(C\cup P_1\cup P_2\cup Q)+\{uv, xy\}$ contains a subdivision of $K_{3,3}$ in G, once more a contradiction (see figure 9.22).

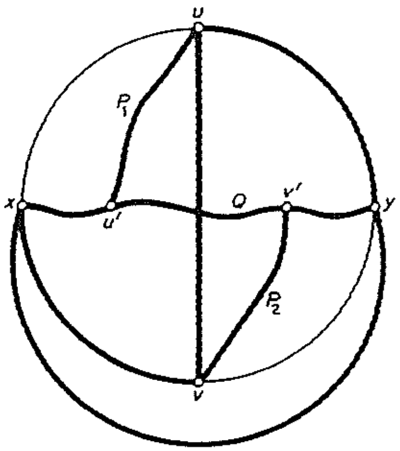


Figure 9.22

Thus all the possible cases lead to contradictions, and the proof is complete □

There are several other characterisations of planar graphs. For example, Wagner (1937) has shown that a graph is planar if and only if it contains no subgraph contractible to K_5 or $K_{3,3}$.

Exercises

9.5.1 Prove lemmas 9.10.1 and 9.10.2.

9.5.2 Show, using Kuratowski's theorem, that the Petersen graph is non-planar.

9.6 THE FIVE-COLOUR THEOREM AND THE FOUR-COLOUR CONJECTURE

As has already been noted (exercise 9.3.2), every planar graph is 6-vertex-colourable. Heawood (1890) improved upon this result by showing that one can always properly colour the vertices of a planar graph with at most five colours. This is known as the *five-colour theorem*.

Theorem 9.11 Every planar graph is 5-vertex-colourable.

Proof By contradiction. Suppose that the theorem is false. Then there exists a 6-critical plane graph G. Since a critical graph is simple, we see from

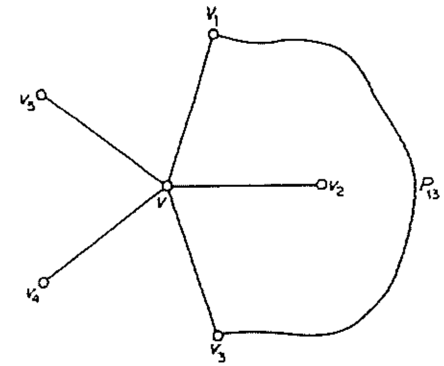


Figure 9.23

corollary 9.5.3 that $\delta\leq 5$. On the other hand we have, by theorem 8.1, that $\delta\geq 5$. Therefore $\delta = 5$. Let v be a vertex of degree five in G, and let $(V_1, V_2, V_3, V_4, V_5)$ be a proper 5-vertex colouring of $G-v$; such a colouring exists because G is 6-critical. Since G itself is not 5-vertex-colourable, v must be adjacent to a vertex of each of the five colours. Therefore we can assume that the neighbours of v in clockwise order about v are v_1, v_2, v_3, v_4 and v_5, where $v_i\in V_i$ for $1\leq i\leq 5$.

Denote by G_{ij} the subgraph $G[V_i\cup V_j]$ induced by $V_i\cup V_j$. Now v_i and v_j must belong to the same component of G_{ij}. For, otherwise, consider the component of G_{ij} that contains v_i. By interchanging the colours i and j in this component, we obtain a new proper 5-vertex colouring of $G-v$ in which only four colours (all but i) are assigned to the neighbours of v. We have already shown that this situation cannot arise. Therefore v_i and v_j must belong to the same component of G_{ij}. Let P_{ij} be a (v_i, v_j)-path in G_{ij}, and let C denote the cycle $vv_1P_{13}v_3v$ (see figure 9.23).

Since C separates v_2 and v_4 (in figure 9.23, $v_2\in \text{int } C$ and $v_4\in \text{ext } C$), it follows from the Jordan curve theorem that the path P_{24} must meet C in some point. Because G is a plane graph, this point must be a vertex. But this is impossible, since the vertices of P_{24} have colours 2 and 4, whereas no vertex of C has either of these colours □

The question now arises as to whether the five-colour theorem is best possible. It has been conjectured that every planar graph is 4-vertex-colourable; this is known as the *four-colour conjecture*. The four-colour conjecture has remained unsettled for more than a century, despite many attempts by major mathematicians to solve it. If it were true, then it would, of course, be best possible because there do exist planar graphs which are not 3-vertex-colourable (K_4 is the simplest such graph). For a history of the four-colour conjecture, see Ore (1967)†.

† The four-colour conjecture has now been settled in the affirmative by K. Appel and W. Haken; see page 253.

The problem of deciding whether the four-colour conjecture is true or false is called the *four-colour problem*.† There are several problems in graph theory that are equivalent to the four-colour problem; one of these is the case $n=5$ of Hadwiger's conjecture (see section 8.3). We now establish the equivalence of certain problems concerning edge and face colourings with the four-colour problem. A *k-face colouring* of a plane graph G is an assignment of k colours $1, 2, \ldots, k$ to the faces of G; the colouring is *proper* if no two faces that are separated by an edge have the same colour. G is *k-face-colourable* if it has a proper k-face colouring, and the minimum k for which G is k-face-colourable is the *face chromatic number* of G, denoted by $\chi^*(G)$. It follows immediately from these definitions that, for any plane graph G with dual G^*,

$$\chi^*(G)=\chi(G^*) \tag{9.2}$$

Theorem 9.12 The following three statements are equivalent:

(i) every planar graph is 4-vertex-colourable;
(ii) every plane graph is 4-face-colourable;
(iii) every simple 2-edge-connected 3-regular planar graph is 3-edge-colourable.

Proof We shall show that (i)⇒(ii)⇒(iii)⇒(i).

(*a*) (i)⇒(ii). This is a direct consequence of (9.2) and the fact that the dual of a plane graph is planar.

(*b*) (ii)⇒(iii). Suppose that (ii) holds, let G be a simple 2-edge-connected 3-regular planar graph, and let $\tilde{G}$ be a planar embedding of G. By (ii), $\tilde{G}$ has a proper 4-face-colouring. It is, of course, immaterial which symbols are used as the 'colours', and in this case we shall denote the four colours by the vectors $c_0=(0,0)$, $c_1=(1,0)$, $c_2=(0,1)$ and $c_3=(1,1)$, over the field of integers modulo 2. We now obtain a 3-edge-colouring of $\tilde{G}$ by assigning to each edge the sum of the colours of the faces it separates (see figure 9.24). If c_i, c_j and c_k are the three colours assigned to the three faces incident with a vertex v, then c_i+c_j, c_j+c_k and c_k+c_i are the colours assigned to the three edges incident with v. Since $\tilde{G}$ is 2-edge-connected, each edge separates two distinct faces, and it follows that no edge is assigned the colour c_0 under this scheme. It is also clear that the three edges incident with a given vertex are assigned different colours. Thus we have a proper 3-edge-colouring of $\tilde{G}$, and hence of G.

† The four-colour problem is often posed in the following terms: can the countries of any map be coloured in four colours so that no two countries which have a common boundary are assigned the same colour? The equivalence of this problem with the four-colour problem follows from theorem 9.12 on observing that a map can be regarded as a plane graph with its countries as the faces.

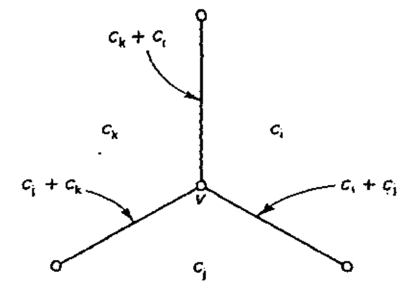


Figure 9.24

(c) (iii)⇒(i). Suppose that (iii) holds, but that (i) does not. Then there is a 5-critical planar graph G. Let $\tilde{G}$ be a planar embedding of G. Then (exercise 9.2.6) $\tilde{G}$ is a spanning subgraph of a simple plane triangulation H. The dual H^* of H is a simple 2-edge-connected 3-regular planar graph (exercise 9.2.7). By (iii), H^* has a proper 3-edge colouring (E_1, E_2, E_3). For $i\neq j$, let H^*_{ij} denote the subgraph of H^* induced by $E_i\cup E_j$. Since each vertex of H^* is incident with one edge of E_i and one edge of E_j, H^*_{ij} is a union of disjoint cycles and is therefore (exercise 9.6.1) 2-face-colourable. Now each face of H^* is the intersection of a face of H^*_{12} and a face of H^*_{23}. Given proper 2-face colourings of H^*_{12} and H^*_{23} we can obtain a 4-face colouring of H^* by assigning to each face f the pair of colours assigned to the faces whose intersection is f. Since $H^*=H^*_{12}\cup H^*_{23}$ it is easily verified that this 4-face colouring of H^* is proper. Since H is a supergraph of G we have

$$5=\chi(G)\leq\chi(H)=\chi^*(H^*)\leq 4$$

This contradiction shows that (i) does, in fact, hold □

That statement (iii) of theorem 9.12 is equivalent to the four-colour problem was first observed by Tait (1880). A proper 3-edge colouring of a 3-regular graph is often called a *Tait colouring*. In the next section we shall discuss Tait's ill-fated approach to the four-colour conjecture. Grötzsch (1958) has verified the four-colour conjecture for planar graphs without triangles. In fact, he has shown that every such graph is 3-vertex-colourable.

Exercises

9.6.1 Show that a plane graph G is 2-face-colourable if and only if G is eulerian.

9.6.2 Show that a plane triangulation G is 3-vertex colourable if and only if G is eulerian.

9.6.3 Show that every hamiltonian plane graph is 4-face-colourable.

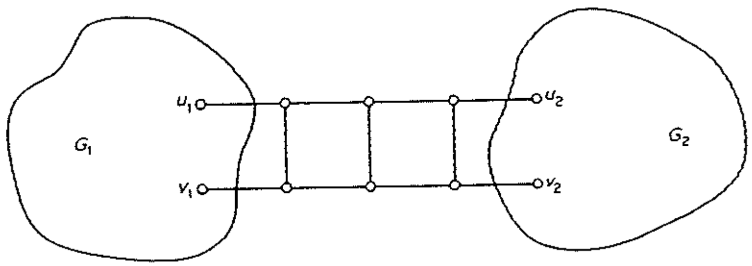
9.6.4 Show that every hamiltonian 3-regular graph has a Tait colouring.

† A planar graph G, having no cycle of length 3 can be coloured by 3 - colours

9.6.5 Prove theorem 9.12 by showing that (iii)$\Rightarrow$(ii)$\Rightarrow$(i)$\Rightarrow$(iii).

9.6.6 Let G be a 3-regular graph with $\kappa' = 2$.

(*a*) Show that there exist subgraphs G_1 and G_2 of G and nonadjacent pairs of vertices $u_1, v_1 \in V(G_1)$ and $u_2, v_2 \in V(G_2)$ such that G consists of the graphs G_1 and G_2 joined by a 'ladder' at the vertices u_1, v_1, u_2 and v_2.



(*b*) Show that if $G_1 + u_1v_1$ and $G_2 + u_2v_2$ both have Tait colourings, then so does G.

(*c*) Deduce, using theorem 9.12, that the four-colour conjecture is equivalent to *Tait's conjecture:* every simple 3-regular 3-connected planar graph has a Tait colouring.

9.6.7 Give an example of

(*a*) a 3-regular planar graph with no Tait colouring;

(*b*) a 3-regular 2-connected graph with no Tait colouring.

9.7 NONHAMILTONIAN PLANAR GRAPHS

In his attempt to prove the four-colour conjecture, Tait (1880) observed that it would be enough to show that every 3-regular 3-connected planar graph has a Tait colouring (exercise 9.6.6). By mistakenly assuming that every such graph is hamiltonian, he gave a 'proof' of the four-colour conjecture (see exercise 9.6.4). Over half a century later, Tutte (1946) showed Tait's proof to be invalid by constructing a nonhamiltonian 3-regular 3-connected planar graph; it is depicted in figure 9.25.

Tutte proved that his graph is nonhamiltonian by using ingenious *ad hoc* arguments (exercise 9.7.1), and for many years the Tutte graph was the only known example of a nonhamiltonian 3-regular 3-connected planar graph. However, Grinberg (1968) then discovered a necessary condition for a plane graph to be hamiltonian. His discovery has led to the construction of many nonhamiltonian planar graphs.

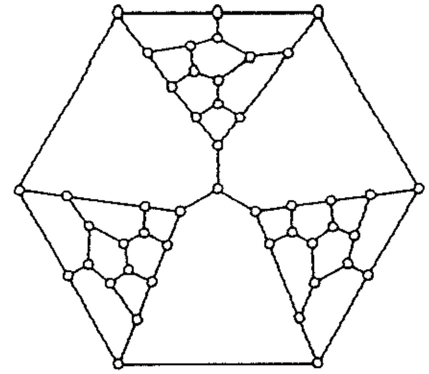


Figure 9.25. The Tutte graph

Theorem 9.13 Let G be a loopless plane graph with a Hamilton cycle C. Then

$$\sum_{i=1}^{\nu} (i-2)(\phi_i' - \phi_i'') = 0 \tag{9.3}$$

where ϕ_i' and ϕ_i'' are the numbers of faces of degree i contained in Int C and Ext C, respectively.

Proof Denote by E' the subset of $E(G)\backslash E(C)$ contained in Int C, and let $\varepsilon' = |E'|$. Then Int C contains exactly $\varepsilon' + 1$ faces (see figure 9.26), and so

$$\sum_{i=1}^{\nu} \phi_i' = \varepsilon' + 1 \tag{9.4}$$

Now each edge in E' is on the boundary of two faces in Int C, and each edge

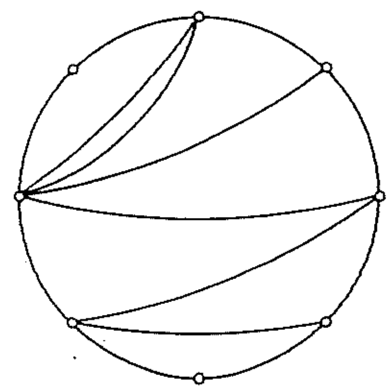


Figure 9.26

of C is on the boundary of exactly one face in Int C. Therefore

$$\sum_{i=1}^{\nu} i\phi_i' = 2\varepsilon' + \nu \qquad (9.5)$$

Using (9.4), we can eliminate ε' from (9.5) to obtain

$$\sum_{i=1}^{\nu} (i-2)\phi_i' = \nu - 2 \qquad (9.6)$$

Similarly

$$\sum_{i=1}^{\nu} (i-2)\phi_i'' = \nu - 2 \qquad (9.7)$$

Equations (9.6) and (9.7) now yield (9.3) □

With the aid of theorem 9.13, it is a simple matter to show, for example, that the Grinberg graph (figure 9.27) is nonhamiltonian.

Suppose that this graph is hamiltonian. Then, noting that it only has faces of degrees five, eight and nine, condition (9.3) yields

$$3(\phi_5' - \phi_5'') + 6(\phi_8' - \phi_8'') + 7(\phi_9' - \phi_9'') = 0$$

We deduce that

$$7(\phi_9' - \phi_9'') \equiv 0 \pmod{3}$$

But this is clearly impossible, since the value of the left-hand side is 7 or -7, depending on whether the face of degree nine is in Int C or Ext C. Therefore the graph cannot be hamiltonian.

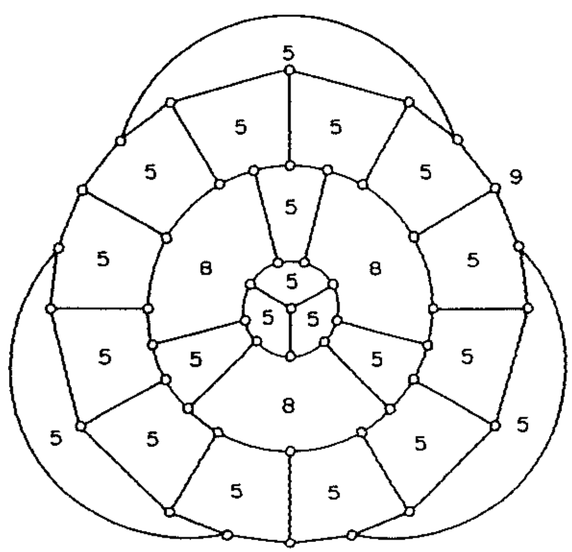


Figure 9.27. The Grinberg graph

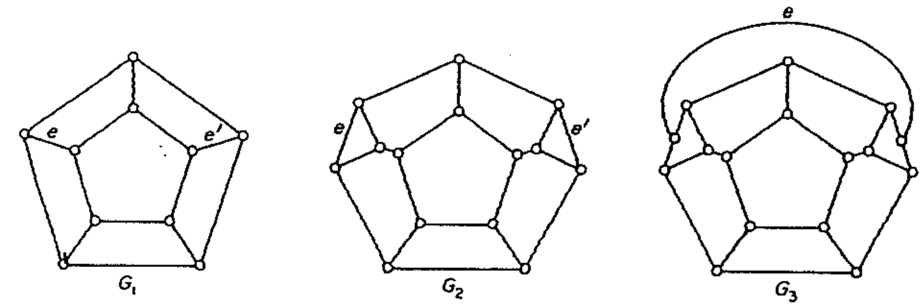
Although there exist nonhamiltonian 3-connected planar graphs, Tutte (1956) has shown that every 4-connected planar graph is hamiltonian.

Exercises

9.7.1 (*a*) Show that no Hamilton cycle in the graph G_1 below can contain both the edges e and e'.

(*b*) Using (*a*), show that no Hamilton cycle in the graph G_2 can contain both the edges e and e'.

(*c*) Using (*b*), show that every Hamilton cycle in the graph G_3 must contain the edge e.



(*d*) Deduce that the Tutte graph (figure 9.25) is nonhamiltonian.

9.7.2 Show, by applying theorem 9.13, that the Herschel graph (figure 4.2*b*) is nonhamiltonian. (It is, in fact, the smallest nonhamiltonian 3-connected planar graph.)

9.7.3 Give an example of a simple nonhamiltonian 3-regular planar graph with connectivity two.

APPLICATIONS

9.8 A PLANARITY ALGORITHM

There are many practical situations in which it is important to decide whether a given graph is planar, and, if so, to then find a planar embedding of the graph. For example, in the layout of printed circuits one is interested in knowing if a particular electrical network is planar. In this section, we shall present an algorithm for solving this problem, due to Demoucron, Malgrange and Pertuiset (1964).

Let H be a planar subgraph of a graph G and let $\tilde{H}$ be an embedding of H in the plane. We say that $\tilde{H}$ is *G-admissible* if G is planar and there is a planar embedding $\tilde{G}$ of G such that $\tilde{H} \subseteq \tilde{G}$. In figure 9.28, for example, two embeddings of a planar subgraph of G are shown; one is G-admissible and the other is not.

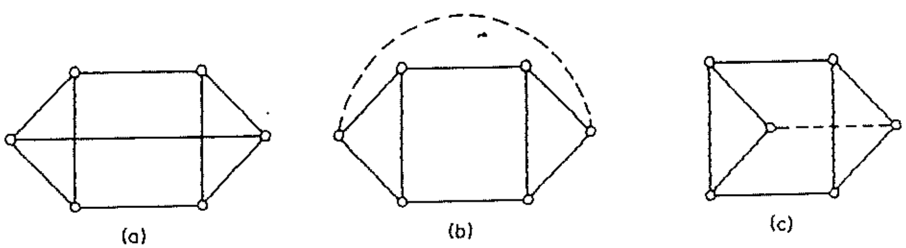


Figure 9.28. (a) G; (b) G-admissible; (c) G-inadmissible

If B is any bridge of H (in G), then B is said to be *drawable* in a face f of $\tilde{H}$ if the vertices of attachment of B to H are contained in the boundary of f. We write $F(B, \tilde{H})$ for the set of faces of $\tilde{H}$ in which B is drawable. The following theorem provides a necessary condition for G to be planar.

Theorem 9.14 If $\tilde{H}$ is G-admissible then, for every bridge B of H, $F(B, \tilde{H}) \neq \emptyset$.

Proof If $\tilde{H}$ is G-admissible then, by definition, there exists a planar embedding $\tilde{G}$ of G such that $\tilde{H} \subseteq \tilde{G}$. Clearly, the subgraph of $\tilde{G}$ which corresponds to a bridge B of H must be confined to one face of $\tilde{H}$. Hence $F(B, \tilde{H}) \neq \emptyset$ □

Since a graph is planar if and only if each block of its underlying simple graph is planar, it suffices to consider simple blocks. Given such a graph G, the algorithm determines an increasing sequence $G_1, G_2, \ldots$ of planar subgraphs of G, and corresponding planar embeddings $\tilde{G}_1, \tilde{G}_2, \ldots$. When G is planar, each $\tilde{G}_i$ is G-admissible and the sequence $\tilde{G}_1, \tilde{G}_2, \ldots$ terminates in a planar embedding of G. At each stage, the necessary condition in theorem 9.14 is used to test G for nonplanarity.

Planarity Algorithm

1. Let G_1 be a cycle in G. Find a planar embedding $\tilde{G}_1$ of G_1. Set $i = 1$.
2. If $E(G)\backslash E(G_i) = \emptyset$, stop. Otherwise, determine all bridges of G_i in G; for each such bridge B find the set $F(B, \tilde{G}_i)$.
3. If there exists a bridge B such that $F(B, \tilde{G}_i) = \emptyset$, stop; by theorem 9.14, G is nonplanar. If there exists a bridge B such that $|F(B, \tilde{G}_i)| = 1$, let $\{f\} = F(B, \tilde{G}_i)$. Otherwise, let B be any bridge and f any face such that $f \in F(B, \tilde{G}_i)$.
4. Choose a path $P_i \subseteq B$ connecting two vertices of attachment of B to G_i. Set $G_{i+1} = G_i \cup P_i$ and obtain a planar embedding $\tilde{G}_{i+1}$ of G_{i+1} by drawing P_i in the face f of $\tilde{G}_i$. Replace i by $i+1$ and go to step 2.

To illustrate this algorithm, we shall consider the graph G of figure 9.29. We start with the cycle $\tilde{G}_1 = 2345672$ and a list of its bridges (denoted, for

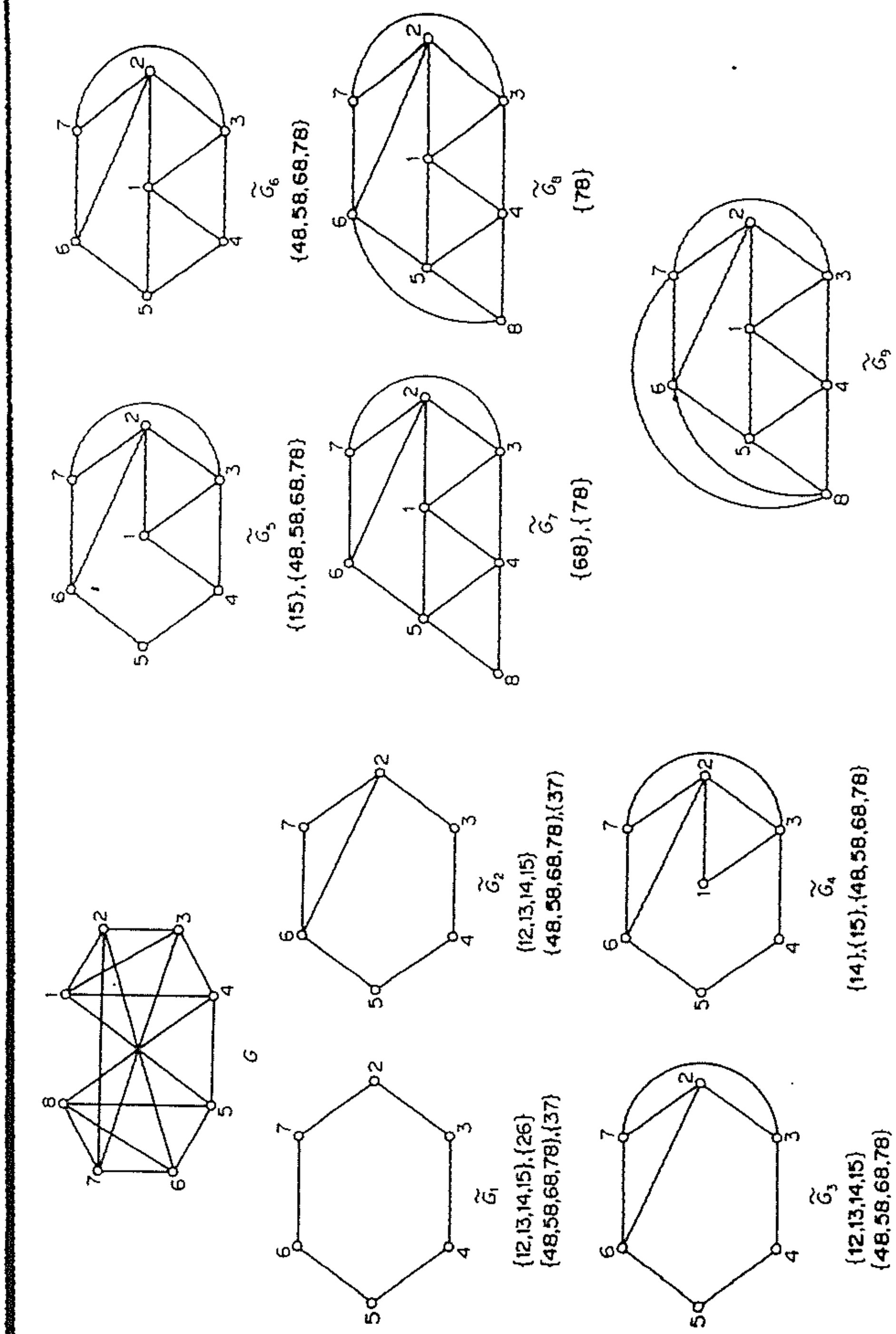


Figure 9.29

brevity, by their edge sets); at each stage, the bridges B for which $|F(B, \tilde{G}_i)| = 1$ are indicated in bold face. In this example, the algorithm terminates with a planar embedding $\tilde{G}_9$ of G. Thus G is planar.

Now let us apply the algorithm to the graph H obtained from G by deleting edge 45 and adding edge 36 (figure 9.30). Starting with the cycle 23672, we proceed as shown in figure 9.30. It can be seen that, having constructed $\tilde{H}_3$, we find a bridge $B = \{12, 13, 14, 15; 34, 48, 56, 58, 68, 78\}$

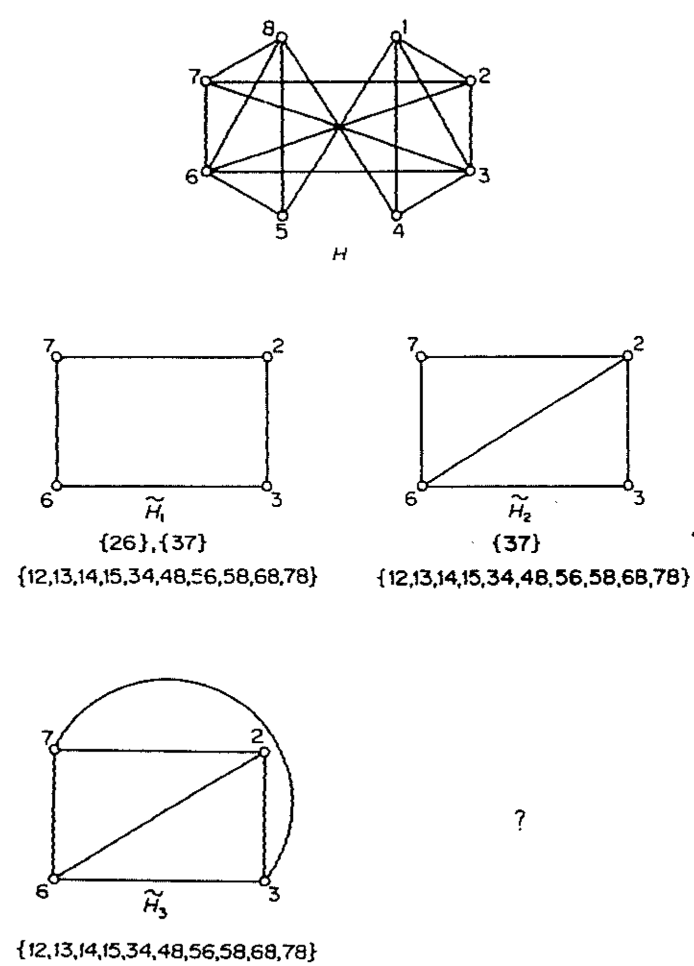


Figure 9.30

such that $F(B, \tilde{H}_3) = \emptyset$. At this point the algorithm stops (step 3), and we conclude that H is nonplanar.

In order to establish the validity of the algorithm, one needs to show that if G is planar, then each term of the sequence $\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_{\varepsilon-\nu+1}$ is G-admissible. Demoucron, Malgrange and Pertuiset prove this by induction. We shall give a general outline of their proof.

Suppose that G is planar. Clearly $\tilde{G}_1$ is G-admissible. Assume that $\tilde{G}_i$ is G-admissible for $1 \leq i \leq k < \varepsilon - \nu + 1$. By definition, there is a planar embedding $\tilde{G}$ of G such that $\tilde{G}_k \subset \tilde{G}$. We wish to show that $\tilde{G}_{k+1}$ is G-admissible. Let B and f be as defined in step 3 of the algorithm. If, in $\tilde{G}$, B is drawn in f, $\tilde{G}_{k+1}$ is clearly G-admissible. So assume that no bridge of G_k is drawable in only one face of $\tilde{G}_k$, and that, in $\tilde{G}$, B is drawn in some other face f'. Since no bridge is drawable in just one face, no bridge whose vertices of attachment are restricted to the common boundary of f and f' can be skew to a bridge not having this property. Hence we can interchange bridges across the common boundary of f and f' and thereby obtain a planar embedding of G in which B is drawn in f (see figure 9.31). Thus, again, $\tilde{G}_{k+1}$ is G-admissible.

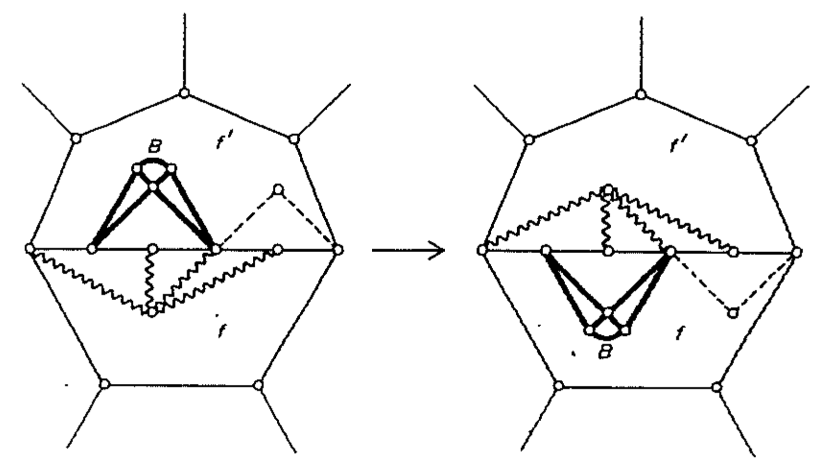


Figure 9.31

The algorithm that we have described is good. From the flow diagram (figure 9.32), one sees that the main operations involved are

(i) finding a cycle G_1 in the block G;
(ii) determining the bridges of G_i in G and their vertices of attachment to G_i;

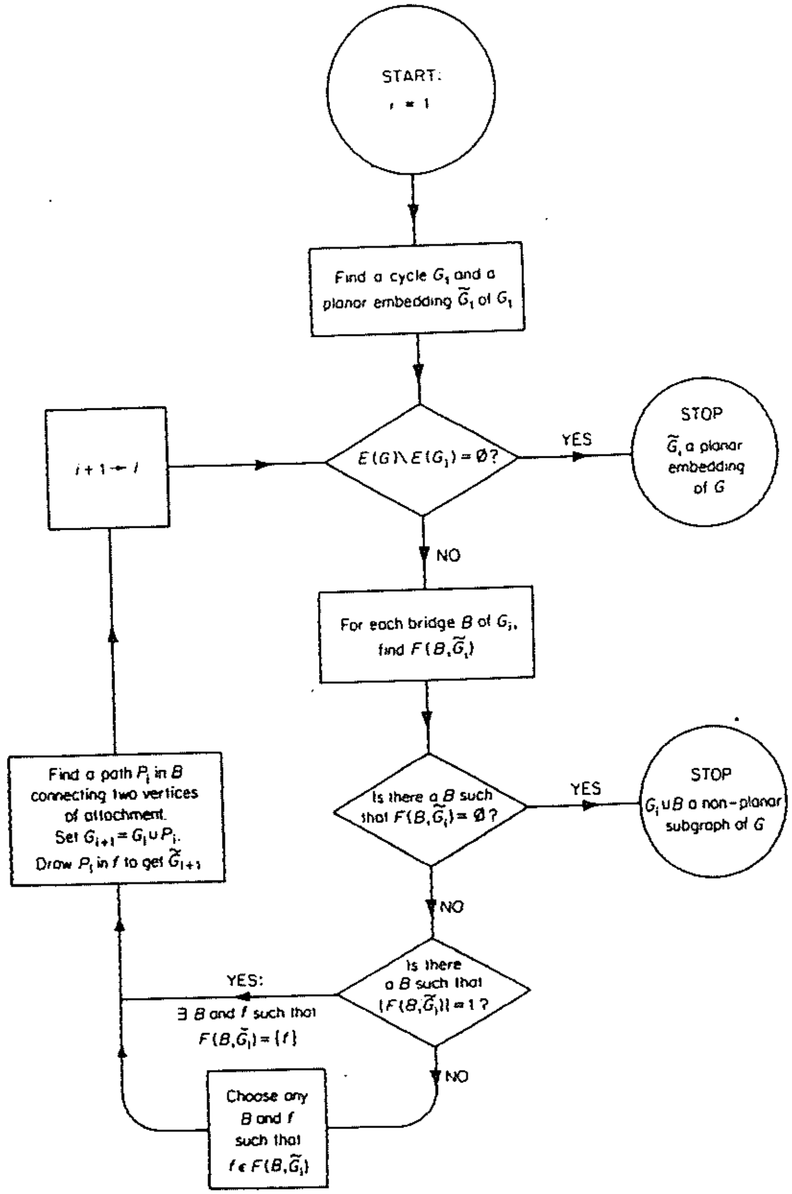


Figure 9.32. Planarity algorithm

(iii) determining $b(f)$ for each face f of $\tilde{G}_i$;
(iv) determining $F(B, \tilde{G}_i)$ for each bridge B of G_i;
(v) finding a path P_i in some bridge B of G_i between two vertices of $V(B, G_i)$.

There exists a good algorithm for each of these operations; we leave the details as an exercise.

More sophisticated algorithms for testing planarity than the above have since been obtained. See, for example, Hopcroft and Tarjan (1974).

Exercise

9.8.1 Show that the Petersen graph is nonplanar by applying the above algorithm.

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10 Directed Graphs

10.1 DIRECTED GRAPHS

Although many problems lend themselves naturally to a graph-theoretic formulation, the concept of a graph is sometimes not quite adequate. When dealing with problems of traffic flow, for example, it is necessary to know which roads in the network are one-way, and in which direction traffic is permitted. Clearly, a graph of the network is not of much use in such a situation. What we need is a graph in which each link has an assigned orientation—a directed graph. Formally, a *directed graph* D is an ordered triple $(V(D), A(D), \psi_D)$ consisting of a nonempty set $V(D)$ of *vertices*, a set $A(D)$, disjoint from $V(D)$, of *arcs*, and an *incidence function* ψ_D that associates with each arc of D an ordered pair of (not necessarily distinct) vertices of D. If a is an arc and u and v are vertices such that $\psi_D(a)=(u, v)$, then a is said to *join* u *to* v; u is the *tail* of a, and v is its *head*. For convenience, we shall abbreviate 'directed graph' to *digraph*. A digraph D' is a *subdigraph* of D if $V(D')\subseteq V(D)$, $A(D')\subseteq A(D)$ and $\psi_{D'}$ is the restriction of ψ_D to $A(D')$. The terminology and notation for subdigraphs is similar to that used for subgraphs.

With each digraph D we can associate a graph G on the same vertex set; corresponding to each arc of D there is an edge of G with the same ends. This graph is the *underlying graph* of D. Conversely, given any graph G, we can obtain a digraph from G by specifying, for each link, an order on its ends. Such a digraph is called an *orientation* of G.

Just as with graphs, digraphs have a simple pictorial representation. A digraph is represented by a diagram of its underlying graph together with arrows on its edges, each arrow pointing towards the head of the corresponding arc. A digraph and its underlying graph are shown in figure 10.1.

Every concept that is valid for graphs automatically applies to digraphs too. Thus the digraph of figure 10.1a is connected and has no cycle of length three because its underlying graph (figure 10.1b) has these properties. However, there are many concepts that involve the notion of orientation, and these apply only to digraphs.

A *directed walk* in D is a finite non-null sequence $W=(v_0, a_1, v_1, \ldots, a_k, v_k)$, whose terms are alternately vertices and arcs, such that, for $i=1, 2, \ldots, k$, the arc a_i has head v_i and tail v_{i-1}. As with walks in graphs, a directed walk $(v_0, a_1, v_1, \ldots, a_k, v_k)$ is often represented simply by