CSE 431/531: Algorithm Analysis and Design — Spring 2022

Homework 6

Instructor: Shi Li — **Deadline: No**

Your Name: ____________________ Your Student ID: ______________

Problem 1 For each of the following 5 problems, state (i) whether the problem is known to be in NP, and (ii) whether the problem is known to be in Co-NP. For problems (1c), (1d) and (1e), if your answer for (i) (or (ii)) is yes, you need to give the certificate and the certifier that establishes that the problem is in NP (or Co-NP).

(1a) Given a graph $G = (V, E)$ and $s \leq |V|$, the problem asks whether G contains an independent set of size s.

NP: yes Co-NP: no

(1b) Given two circuits C_1 and C_2, each with m input variables $z_1, z_2, \cdots, z_m$, decide if the two circuits compute the same function. That is, whether C_1 and C_2 give the same output for every boolean assignment of z-variables.

NP: no Co-NP: yes

(1c) Given a graph $G = (V, E)$, decide if G is 3-colorable.

NP: yes Co-NP: no

Certificate for NP: a valid 3-coloring of V, that is, a coloring of V using 3 colors such that no two adjacent vertices have the same color.

Certifier for NP: Given the graph G and the 3-coloring, the algorithm checks if it is a valid 3-coloring for G.

(1d) Given a graph $G = (V, E)$, decide if G is 2-colorable.

NP: yes Co-NP: yes

Deciding if G is 2-colorable is the same as deciding if G is bipartite. So the problem can be solved in polynomial time and thus is in both NP and Co-NP. There is no need to define the certificates. The certifier takes the graph G, and checks in polynomial time if it is (not) 2-colorable.

(1e) An undirected graph $G = (V, E)$ is called a 1-expander if for every $U \subseteq V$, the number of edges between U and $V \setminus U$ in G is at least $\min\{|U|, |V \setminus U|\}$. Given a graph G, decide if G is a 1-expander.

NP: no Co-NP: yes

Certificate for Co-NP: a subset $U \subseteq V$, such that the number of edges between U and $V \setminus U$ is less than $|\min\{|U|, |V \setminus U|\}|$.

Certifier for Co-NP: Given the graph G and the set U, the certifier checks if the number of edges between U and $V \setminus U$ is less than $|\min\{|U|, |V \setminus U|\}|$.

Problem 2 Let NPC be the set of NP-Complete problems. Prove the following statements:

- If $\mathrm{P} \neq \mathrm{NP}$, then $\mathrm{P} \cap \mathrm{NPC} = \emptyset$.

  Proof. We prove the contra-positive of the statement: If $\mathrm{P} \cap \mathrm{NPC} \neq \emptyset$ then $\mathrm{P} = \mathrm{NP}$. If $\mathrm{P} \cap \mathrm{NPC} \neq \emptyset$, then there is a problem X in NPC and in P. Then by our definition of NPC, every problem Y in NP is polynomial time reducible to X and thus can be solved in polynomial time. So $\mathrm{P} = \mathrm{NP}$. □

- If $\mathrm{P} = \mathrm{NP}$, then $\mathrm{P} = \mathrm{NPC}$.

  Proof. If $\mathrm{P} = \mathrm{NP}$, then every problem in NP has a polynomial time algorithm. This implies that every problem Y in NP is polynomial time reducible to every problem X in NP. (In particular, there is no need to call the black-box algorithm for X in the algorithm for Y.) Therefore every problem X in $\mathrm{P} = \mathrm{NP}$ is NP-complete. □

- If $\mathrm{P} = \mathrm{NP}$ then $\mathrm{P} = \text{Co-NP}$.

  Proof. We know that $\mathrm{P} \subseteq \text{Co-NP}$ and it suffices to prove that $\text{Co-NP} \subseteq \mathrm{P}$.

  - Let X be any problem in Co-NP.
  - Then by the relation between NP and Co-NP, we have $\bar{X} \in \mathrm{NP}$.
  - Then, $\bar{X} \in \mathrm{P}$ because of the condition $\mathrm{P} = \mathrm{NP}$.
  - So, there is a polynomial time algorithm A that solves the problem $\bar{X}$.
  - Running the algorithm A and taking the negation of the output, we can solve the problem X in polynomial time.
  - Therefore, $X \in \mathrm{P}$. So, $\text{Co-NP} \subseteq \mathrm{P}$, which implies $\mathrm{P} = \text{Co-NP}$.

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