

CSE 431/531: Algorithm Analysis and Design (Fall 2022)

Divide-and-Conquer

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Outline

- 1 Divide-and-Conquer
- 2 Counting Inversions
- 3 Quicksort and Selection
 - Quicksort
 - Lower Bound for Comparison-Based Sorting Algorithms
 - Selection Problem
- 4 Polynomial Multiplication
- 5 Other Classic Algorithms using Divide-and-Conquer
- 6 Solving Recurrences
- 7 Computing n -th Fibonacci Number

Greedy Algorithm

- mainly for combinatorial optimization problems
- trivial algorithm runs in exponential time
- greedy algorithm gives an efficient algorithm
- main focus of analysis: correctness of algorithm

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Divide-and-Conquer

- not necessarily for combinatorial optimization problems
- trivial algorithm already runs in polynomial time
- divide-and-conquer gives a more efficient algorithm
- main focus of analysis: running time

Divide-and-Conquer

- **Divide:** Divide instance into many smaller instances
- **Conquer:** Solve each of smaller instances recursively and separately
- **Combine:** Combine solutions to small instances to obtain a solution for the original big instance

merge-sort(A, n)

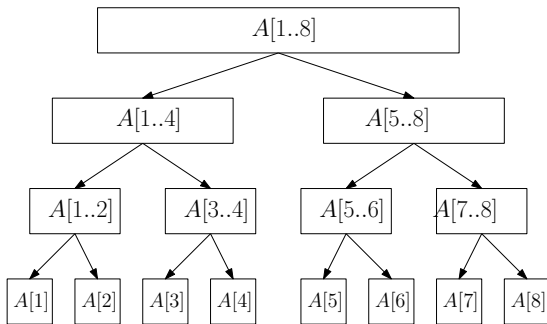
```
1: if  $n = 1$  then  
2:   return  $A$   
3: else  
4:    $B \leftarrow \text{merge-sort}(A[1..\lfloor n/2 \rfloor], \lfloor n/2 \rfloor)$   
5:    $C \leftarrow \text{merge-sort}(A[\lfloor n/2 \rfloor + 1..n], \lceil n/2 \rceil)$   
6:   return  $\text{merge}(B, C, \lfloor n/2 \rfloor, \lceil n/2 \rceil)$ 
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- Divide: trivial
- Conquer: 4, 5
- Combine: 6

Running Time for Merge-Sort



- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels
- Running time = $O(n \lg n)$
- Better than insertion sort

Running Time for Merge-Sort Using Recurrence

- $T(n)$ = running time for sorting n numbers, then

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + O(n) & \text{if } n \geq 2 \end{cases}$$

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- Solving this recurrence, we have $T(n) = O(n \lg n)$ (we shall show how later)

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Example:

10

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15

9

12

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Example:

10	8	15	9	12
8	9	10	12	15

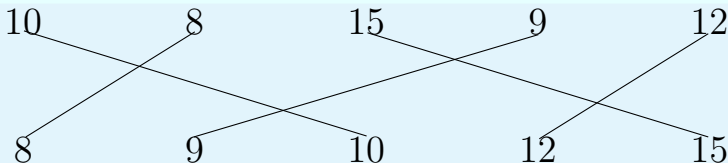
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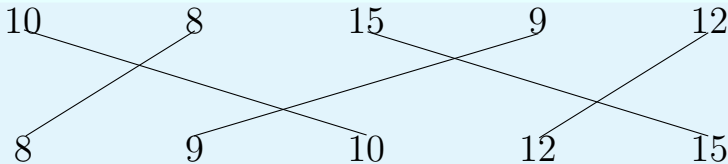
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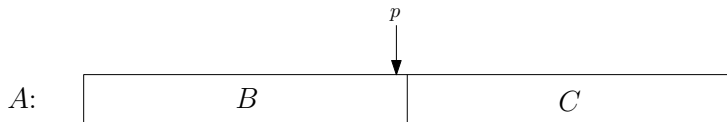
- 4 inversions (for convenience, using numbers, not indices):
(10, 8), (10, 9), (15, 9), (15, 12)

Naive Algorithm for Counting Inversions

count-inversions(A, n)

```
1:  $c \leftarrow 0$ 
2: for every  $i \leftarrow 1$  to  $n - 1$  do
3:   for every  $j \leftarrow i + 1$  to  $n$  do
4:     if  $A[i] > A[j]$  then  $c \leftarrow c + 1$ 
5: return  $c$ 
```

Divide-and-Conquer



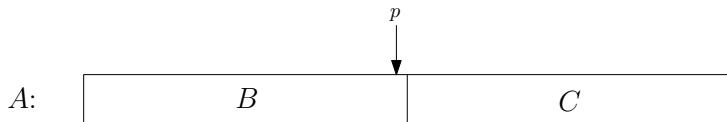
- $p = \lfloor n/2 \rfloor, B = A[1..p], C = A[p + 1..n]$
- $$\#invs(A) = \#invs(B) + \#invs(C) + m$$
$$m = |\{(i, j) : B[i] > C[j]\}|$$

Q: How fast can we compute m , via trivial algorithm?

A: $O(n^2)$

- Can not improve the $O(n^2)$ time for counting inversions.

Divide-and-Conquer



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Lemma If both B and C are sorted, then we can compute m in $O(n)$ time!

Counting Inversions between B and C

Count pairs i, j such that $B[i] > C[j]$:

B :

3	8	12	20	32	48
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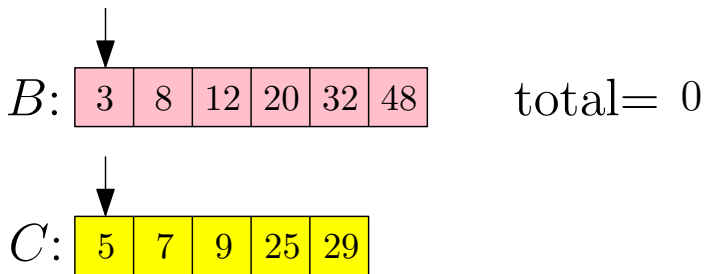
 total = 0

C :

5	7	9	25	29
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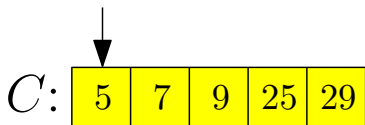
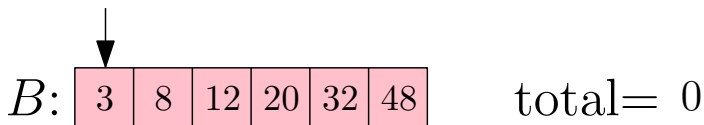
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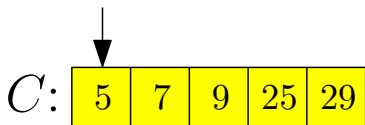
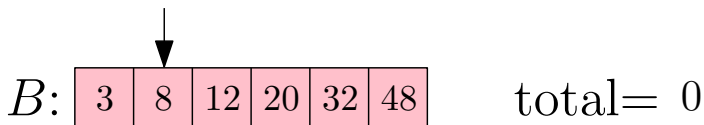


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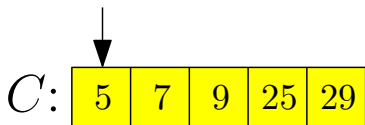
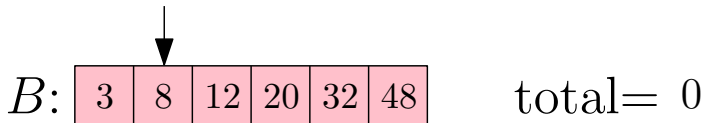


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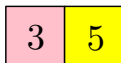


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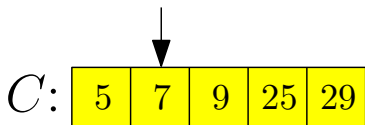
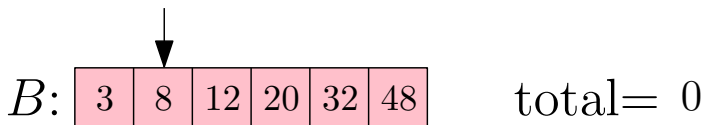


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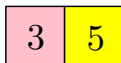


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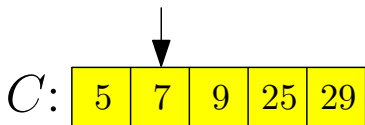
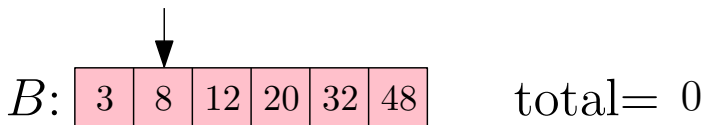


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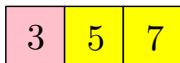


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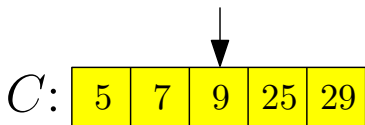
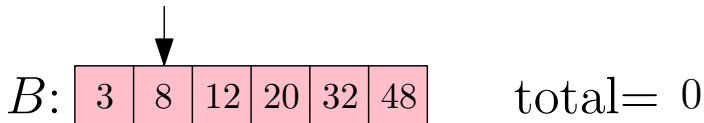


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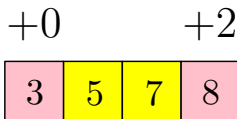
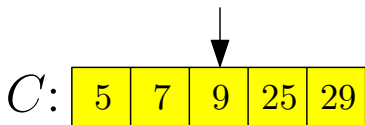
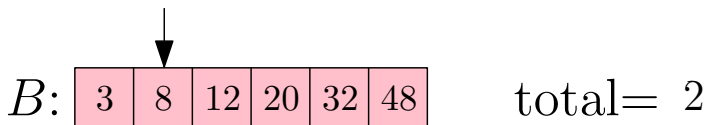


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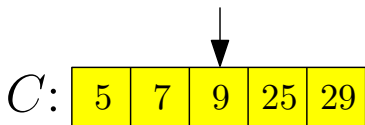
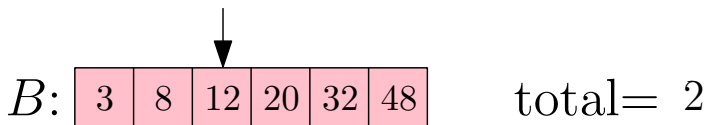
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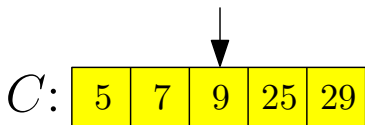
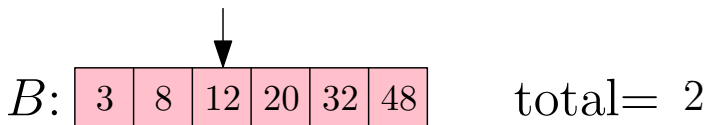


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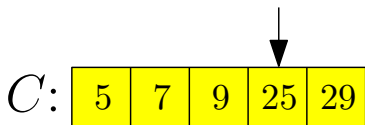
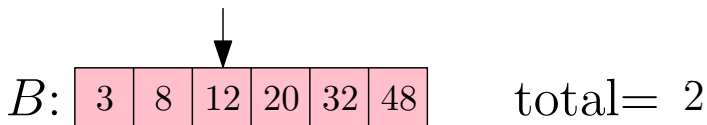
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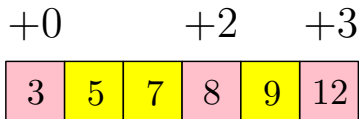
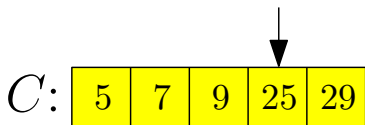
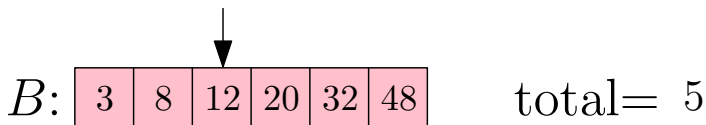
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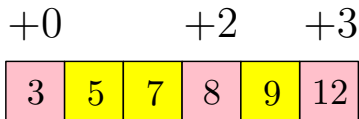
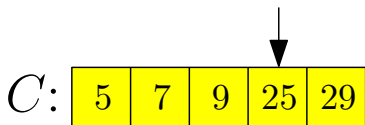
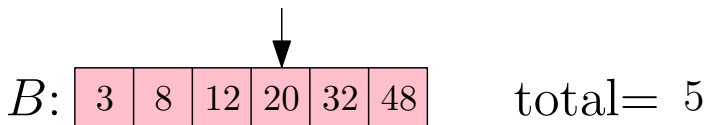
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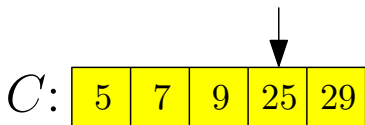
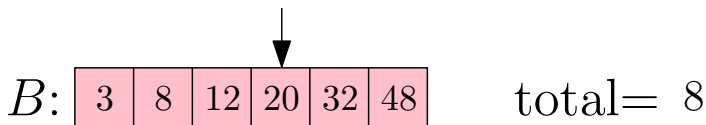
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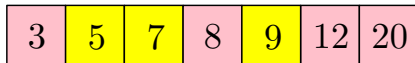


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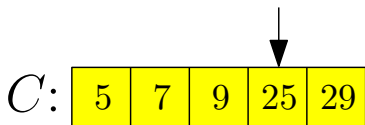
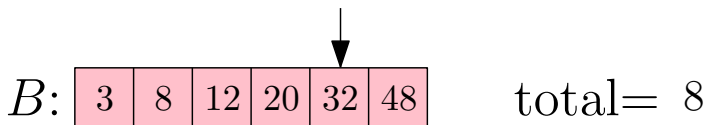


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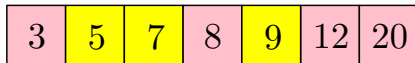


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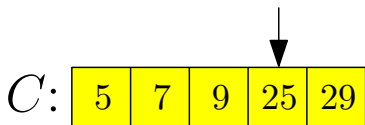
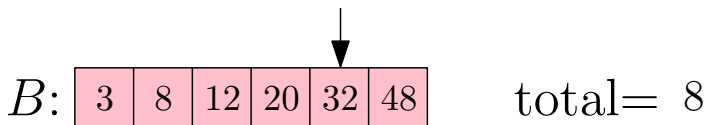


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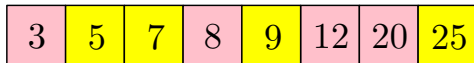


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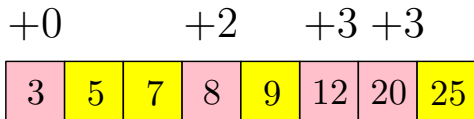
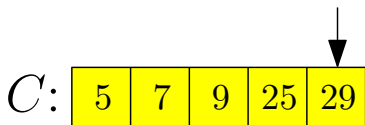
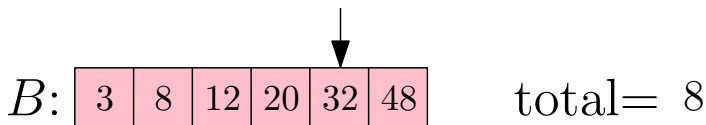


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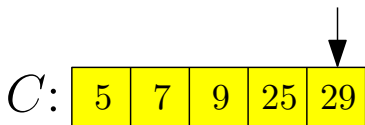
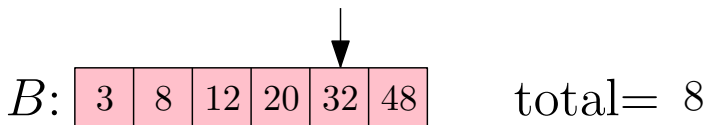
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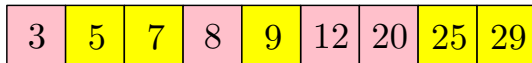


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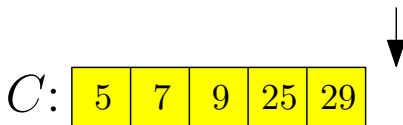
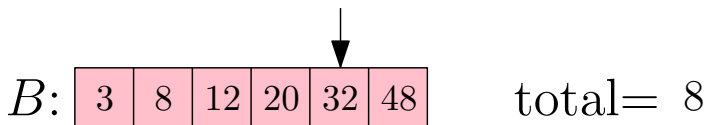


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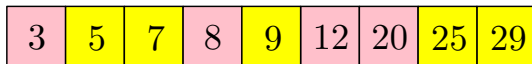


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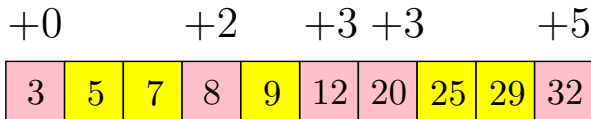
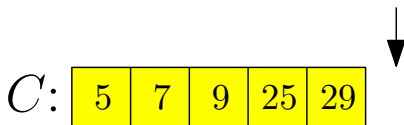
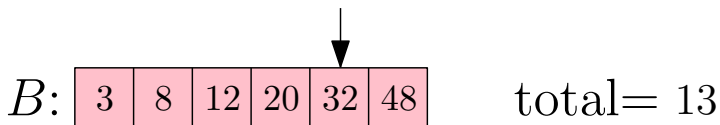


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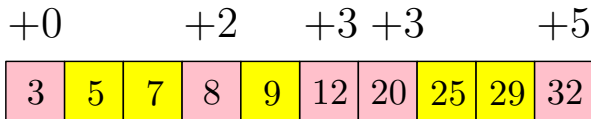
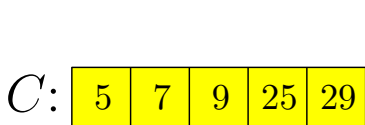
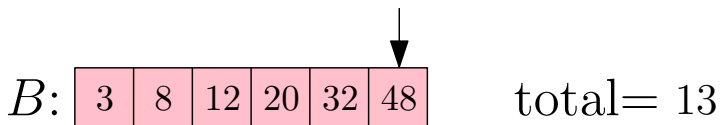
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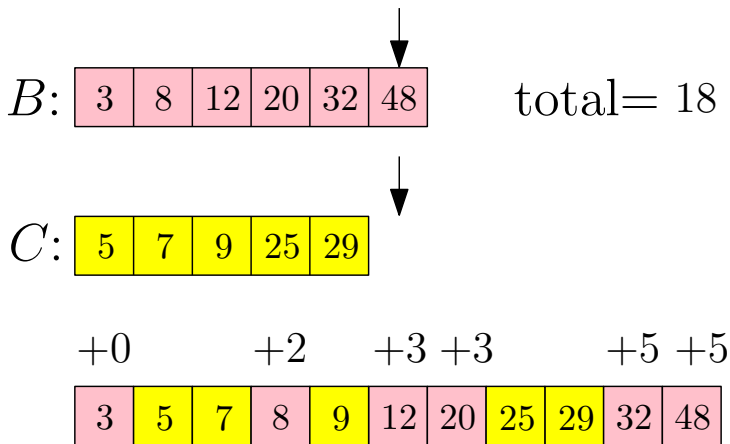
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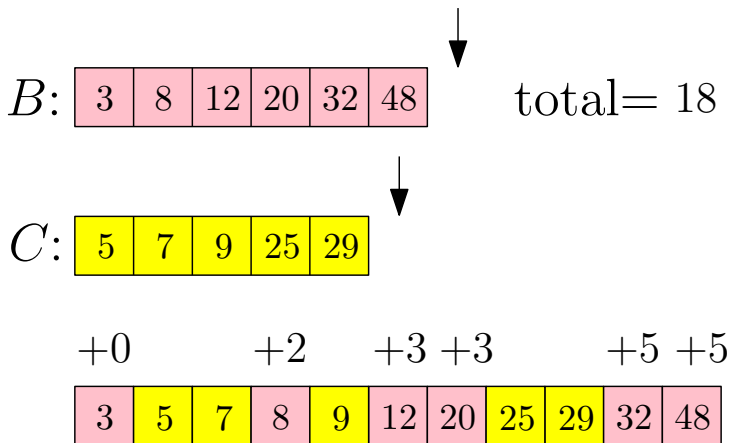
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Count Inversions between B and C

- Procedure that merges B and C and counts inversions between B and C at the same time

merge-and-count(B, C, n_1, n_2)

```
1:  $count \leftarrow 0$ ;  
2:  $A \leftarrow$  array of size  $n_1 + n_2$ ;  $i \leftarrow 1$ ;  $j \leftarrow 1$   
3: while  $i \leq n_1$  or  $j \leq n_2$  do  
4:   if  $j > n_2$  or ( $i \leq n_1$  and  $B[i] \leq C[j]$ ) then  
5:      $A[i + j - 1] \leftarrow B[i]$ ;  $i \leftarrow i + 1$   
6:      $count \leftarrow count + (j - 1)$   
7:   else  
8:      $A[i + j - 1] \leftarrow C[j]$ ;  $j \leftarrow j + 1$   
9: return ( $A, count$ )
```

Sort and Count Inversions in A

- A procedure that returns the sorted array of A and counts the number of inversions in A :

sort-and-count(A, n)

```
1: if  $n = 1$  then  
2:   return ( $A, 0$ )  
3: else  
4:    $(B, m_1) \leftarrow \text{sort-and-count}(A[1..\lfloor n/2 \rfloor], \lfloor n/2 \rfloor)$   
5:    $(C, m_2) \leftarrow \text{sort-and-count}(A[\lfloor n/2 \rfloor + 1..n], \lceil n/2 \rceil)$   
6:    $(A, m_3) \leftarrow \text{merge-and-count}(B, C, \lfloor n/2 \rfloor, \lceil n/2 \rceil)$   
7:   return ( $A, m_1 + m_2 + m_3$ )
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7: **return** ($A, m_1 + m_2 + m_3$)

- Divide: trivial

- Conquer: 4, 5

- Combine: 6, 7

sort-and-count(A, n)

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Outline

- 1 Divide-and-Conquer
- 2 Counting Inversions
- 3 Quicksort and Selection
 - Quicksort
 - Lower Bound for Comparison-Based Sorting Algorithms
 - Selection Problem
- 4 Polynomial Multiplication
- 5 Other Classic Algorithms using Divide-and-Conquer
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Quicksort vs Merge-Sort

	Merge Sort	Quicksort
Divide	Trivial	Separate small and big numbers
Conquer	Recurse	Recurse
Combine	Merge 2 sorted arrays	Trivial

Quicksort Example

Assumption We can choose median of an array of size n in $O(n)$ time.

29	82	75	64	38	45	94	69	25	76	15	92	37	17	85
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quicksort(A, n)

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Q: How to remove this assumption?

A:

- 1 There is an algorithm to find median in $O(n)$ time, using divide-and-conquer (we shall not talk about it; it is complicated and not practical)
- 2 Choose a **pivot randomly** and pretend it is the median (it is practical)

Quicksort Using A Random Pivot

quicksort(A, n)

- 1: **if** $n \leq 1$ **then return** A
- 2: $x \leftarrow$ a random element of A (x is called a **pivot**)
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- In practice: use **pseudo-random-generator**, a deterministic algorithm returning numbers that “look like” random
- In theory: assume they can.

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Lemma The **expected** running time of the algorithm is $O(n \lg n)$.

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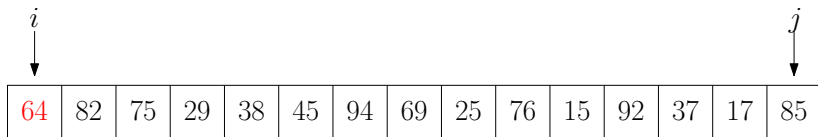
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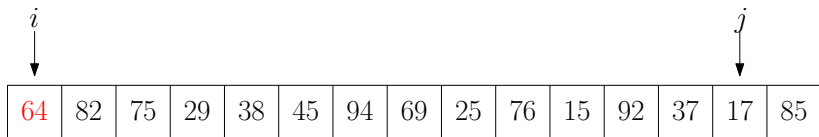
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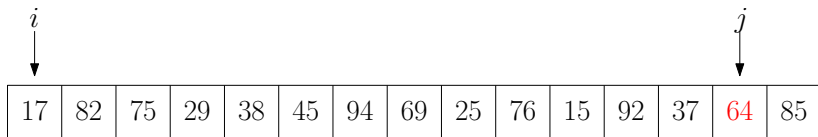
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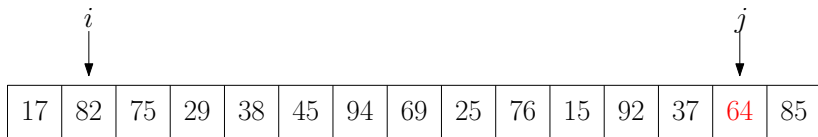
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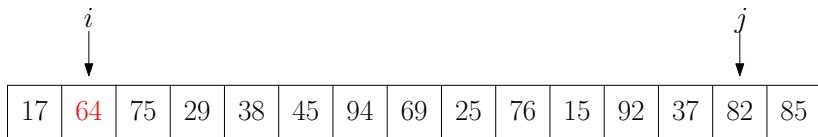
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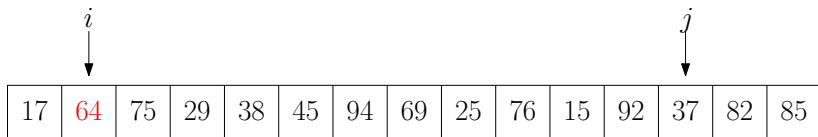
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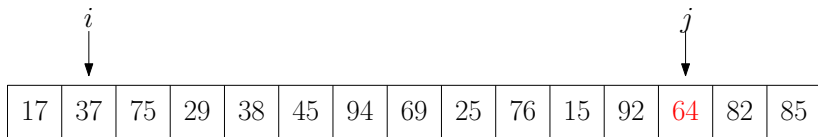
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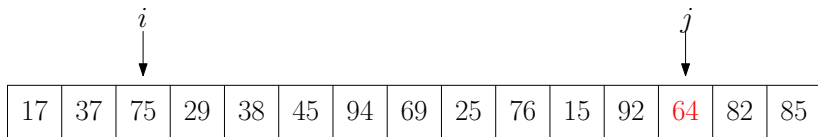
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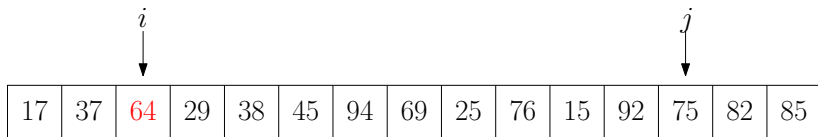
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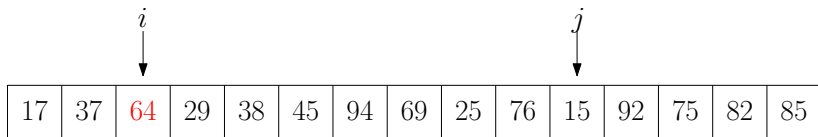
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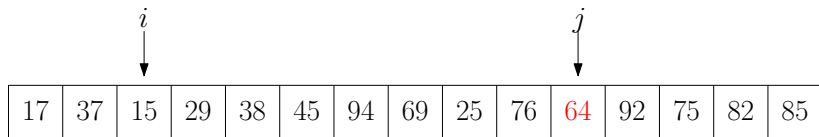
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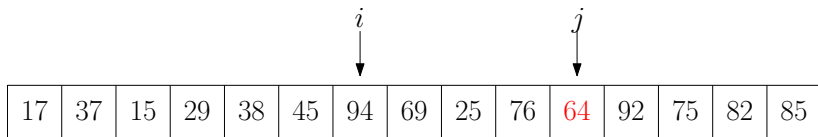
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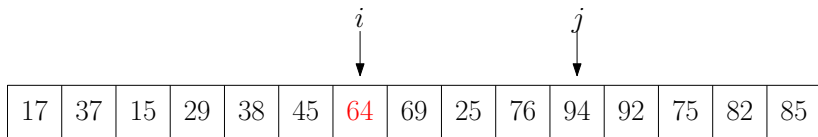
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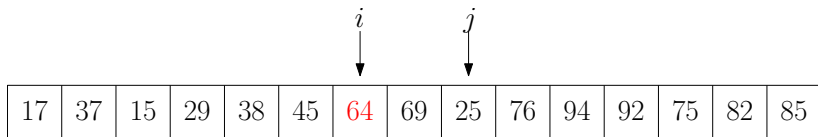
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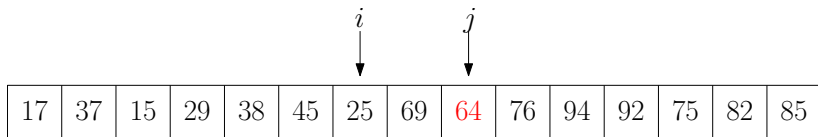
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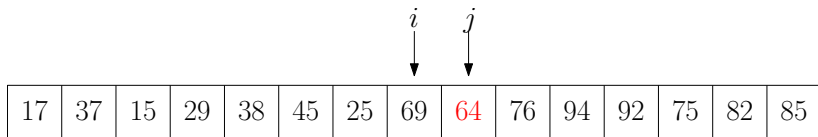
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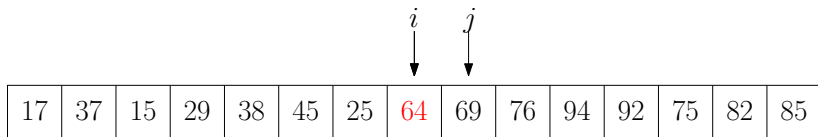
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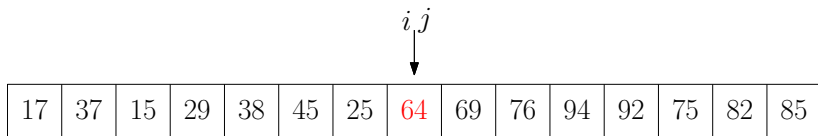
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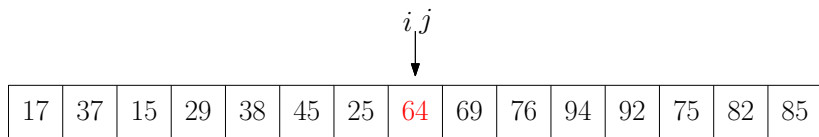
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- To partition the array into two parts, we only need $O(1)$ extra space.

partition(A, ℓ, r)

```
1:  $p \leftarrow$  random integer between  $\ell$  and  $r$ , swap  $A[p]$  and  $A[\ell]$ 
2:  $i \leftarrow \ell, j \leftarrow r$ 
3: while true do
4:   while  $i < j$  and  $A[i] < A[j]$  do  $j \leftarrow j - 1$ 
5:   if  $i = j$  then break
6:   swap  $A[i]$  and  $A[j]; i \leftarrow i + 1$ 
7:   while  $i < j$  and  $A[i] < A[j]$  do  $i \leftarrow i + 1$ 
8:   if  $i = j$  then break
9:   swap  $A[i]$  and  $A[j]; j \leftarrow j - 1$ 
10: return  $i$ 
```

In-Place Implementation of Quick-Sort

quicksort(A, ℓ, r)

- 1: **if** $\ell \geq r$ **then return**
- 2: $m \leftarrow \text{partition}(A, \ell, r)$
- 3: **quicksort**($A, \ell, m - 1$)
- 4: **quicksort**($A, m + 1, r$)

- To sort an array A of size n , call **quicksort**($A, 1, n$).

Note: We pass the array A by reference, instead of by copying.

Merge-Sort is Not In-Place

- To merge two arrays, we need a third array with size equaling the total size of two arrays

Merge-Sort is Not In-Place

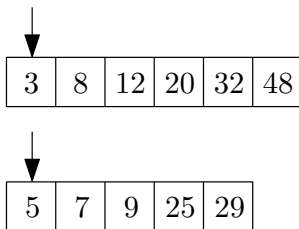
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3	8	12	20	32	48
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5	7	9	25	29
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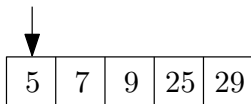
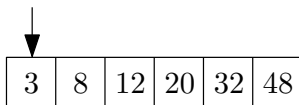
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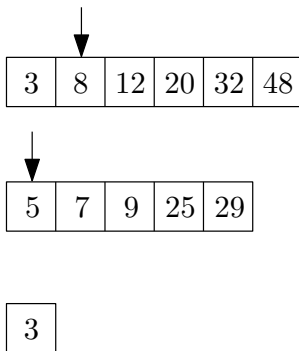
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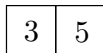
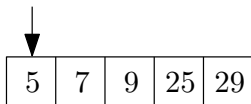
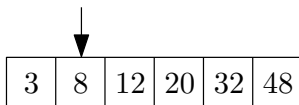
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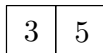
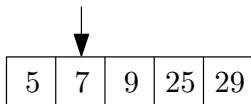
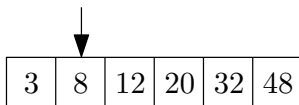
Merge-Sort is Not In-Place

- To merge two arrays, we need a third array with size equaling the total size of two arrays



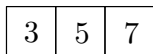
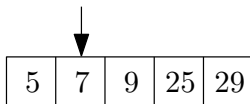
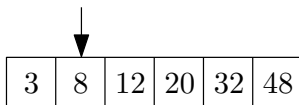
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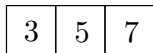
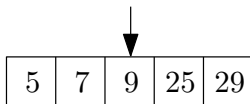
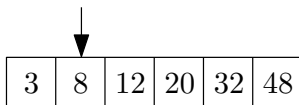
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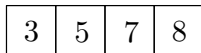
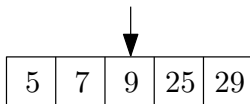
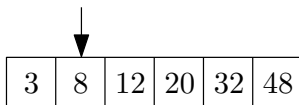
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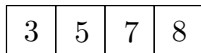
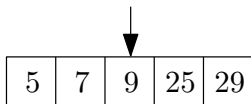
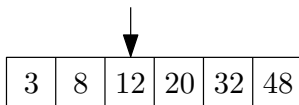
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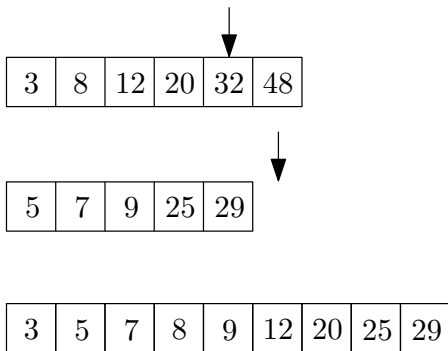
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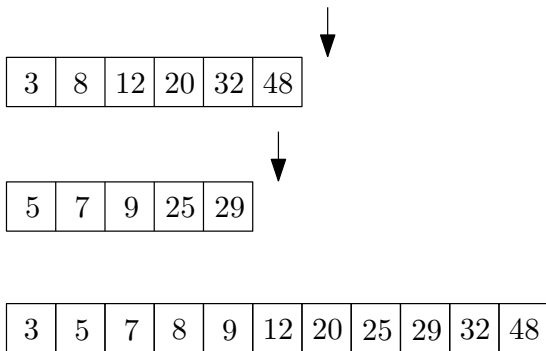
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Outline

- 1 Divide-and-Conquer
- 2 Counting Inversions
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Comparison-Based Sorting Algorithms

Q: Can we do better than $O(n \log n)$ for sorting?

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Comparison-Based Sorting Algorithms

- To sort, we are only allowed to **compare** two elements
- We can not use “internal structures” of the elements

Lemma The (worst-case) running time of any comparison-based sorting algorithm is $\Omega(n \lg n)$.

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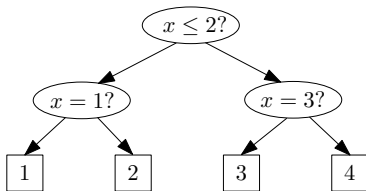
A: $\lceil \log_2 N \rceil$.

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Comparison-Based Sorting Algorithms

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A: No, for comparison-based sorting algorithms.

- Bob has a permutation π over $\{1, 2, 3, \dots, n\}$ in his hand.
- You can ask Bob questions of the form “does i appear before j in π ?”

Q: How many questions do you need to ask in order to get the permutation π ?

A: At least $\log_2 n! = \Theta(n \lg n)$

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Selection Problem

Input: a set A of n numbers, and $1 \leq i \leq n$

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- Our goal: $O(n)$ running time

Recall: Quicksort with Median Finder

quicksort(A, n)

- 1: **if** $n \leq 1$ **then return** A
- 2: $x \leftarrow$ lower median of A
- 3: $A_L \leftarrow$ elements in A that are less than x ▷ Divide
- 4: $A_R \leftarrow$ elements in A that are greater than x ▷ Divide
- 5: $B_L \leftarrow$ quicksort($A_L, A_L.size$) ▷ Conquer
- 6: $B_R \leftarrow$ quicksort($A_R, A_R.size$) ▷ Conquer
- 7: $t \leftarrow$ number of times x appear A
- 8: **return** the array obtained by concatenating B_L , the array containing t copies of x , and B_R

Selection Algorithm with Median Finder

selection(A, n, i)

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- 3: $A_L \leftarrow$ elements in A that are less than x ▷ Divide
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- 5: **if** $i \leq A_L.\text{size}$ **then**
- 6: **return** selection($A_L, A_L.\text{size}, i$) ▷ Conquer
- 7: **else if** $i > n - A_R.\text{size}$ **then**
- 8: **return** selection($A_R, A_R.\text{size}, i - (n - A_R.\text{size})$) ▷ Conquer
- 9: **else**
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- Recurrence for selection: $T(n) = T(n/2) + O(n)$
- Solving recurrence: $T(n) = O(n)$

Randomized Selection Algorithm

selection(A, n, i)

- 1: **if** $n = 1$ **then return** A
- 2: $x \leftarrow$ **random element** of A (called **pivot**)
- 3: $A_L \leftarrow$ elements in A that are less than x ▷ Divide
- 4: $A_R \leftarrow$ elements in A that are greater than x ▷ Divide
- 5: **if** $i \leq A_L.\text{size}$ **then**
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```

- **expected** running time = $O(n)$

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Input: two polynomials of degree $n - 1$

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- **Input:** $(4, -5, 2, 3), (-5, 6, -3, 2)$
- **Output:** $(-20, 49, -52, 20, 2, -5, 6)$

Naïve Algorithm

polynomial-multiplication(A, B, n)

```
1: let  $C[k] \leftarrow 0$  for every  $k = 0, 1, 2, \dots, 2n - 2$   
2: for  $i \leftarrow 0$  to  $n - 1$  do  
3:   for  $j \leftarrow 0$  to  $n - 1$  do  
4:      $C[i + j] \leftarrow C[i + j] + A[i] \times B[j]$   
5: return  $C$ 
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Running time: $O(n^2)$

Divide-and-Conquer for Polynomial Multiplication

$$p(x) = 3x^3 + 2x^2 - 5x + 4 = (3x + 2)x^2 + (-5x + 4)$$

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- $p(x)$: degree of $n - 1$ (assume n is even)
- $p(x) = p_H(x)x^{n/2} + p_L(x)$,
- $p_H(x), p_L(x)$: polynomials of degree $n/2 - 1$.

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$$\begin{aligned}pq &= (p_Hx^{n/2} + p_L)(q_Hx^{n/2} + q_L) \\ &= p_Hq_Hx^n + (p_Hq_L + p_Lq_H)x^{n/2} + p_Lq_L\end{aligned}$$

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Reduce Number from 4 to 3

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- $p_H q_L + p_L q_H = (p_H + p_L)(q_H + q_L) - p_H q_H - p_L q_L$

Divide-and-Conquer for Polynomial Multiplication

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$$+ r_L$$

Divide-and-Conquer for Polynomial Multiplication

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Divide-and-Conquer for Polynomial Multiplication

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- Solving Recurrence: $T(n) = 3T(n/2) + O(n)$
- $T(n) = O(n^{\lg_2 3}) = O(n^{1.585})$

Assumption n is a power of 2. Arrays are 0-indexed.

multiply(A, B, n)

- 1: if $n = 1$ then return ($A[0]B[0]$)
- 2: $A_L \leftarrow A[0 .. n/2 - 1], A_H \leftarrow A[n/2 .. n - 1]$
- 3: $B_L \leftarrow B[0 .. n/2 - 1], B_H \leftarrow B[n/2 .. n - 1]$
- 4: $C_L \leftarrow \text{multiply}(A_L, B_L, n/2)$
- 5: $C_H \leftarrow \text{multiply}(A_H, B_H, n/2)$
- 6: $C_M \leftarrow \text{multiply}(A_L + A_H, B_L + B_H, n/2)$
- 7: $C \leftarrow$ array of $(2n - 1)$ 0's
- 8: **for** $i \leftarrow 0$ to $n - 2$ **do**
- 9: $C[i] \leftarrow C[i] + C_L[i]$
- 10: $C[i + n] \leftarrow C[i + n] + C_H[i]$
- 11: $C[i + n/2] \leftarrow C[i + n/2] + C_M[i] - C_L[i] - C_H[i]$
- 12: **return** C

Outline

- 1 Divide-and-Conquer
- 2 Counting Inversions
- 3 Quicksort and Selection
 - Quicksort
 - Lower Bound for Comparison-Based Sorting Algorithms
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- 5 Other Classic Algorithms using Divide-and-Conquer
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- 7 Computing n -th Fibonacci Number

- Closest pair
- Convex hull
- Matrix multiplication
- FFT(Fast Fourier Transform): polynomial multiplication in $O(n \lg n)$ time

Closest Pair

Input: n points in plane: $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$

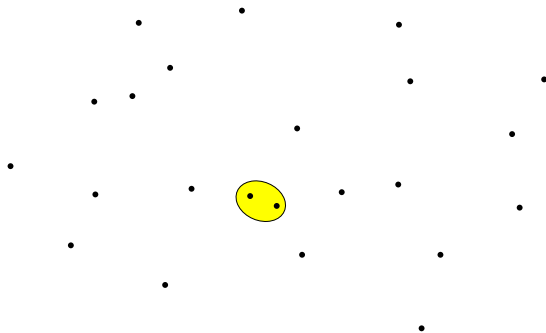
Output: the pair of points that are closest



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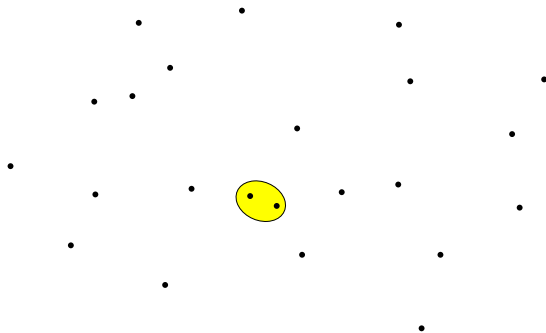
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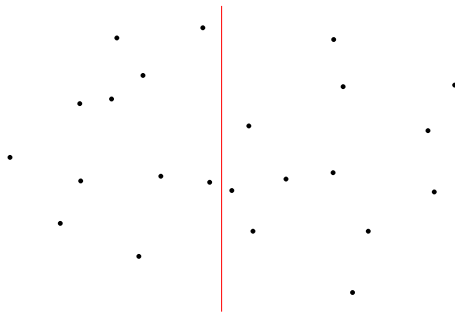
Output: the pair of points that are closest



- Trivial algorithm: $O(n^2)$ running time

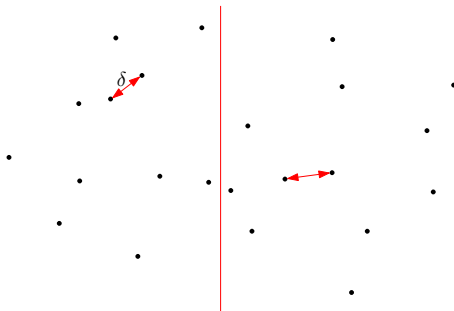
Divide-and-Conquer Algorithm for Closest Pair

- **Divide:** Divide the points into two halves via a vertical line



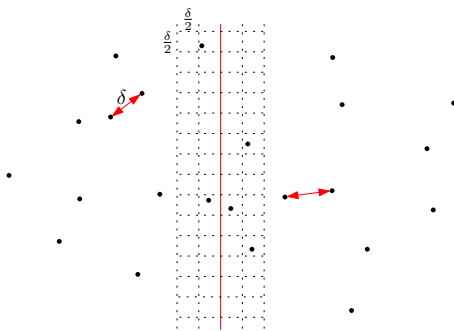
Divide-and-Conquer Algorithm for Closest Pair

- **Divide:** Divide the points into two halves via a vertical line
- **Conquer:** Solve two sub-instances recursively

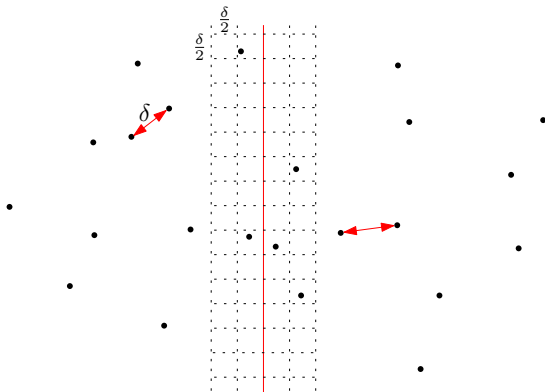


Divide-and-Conquer Algorithm for Closest Pair

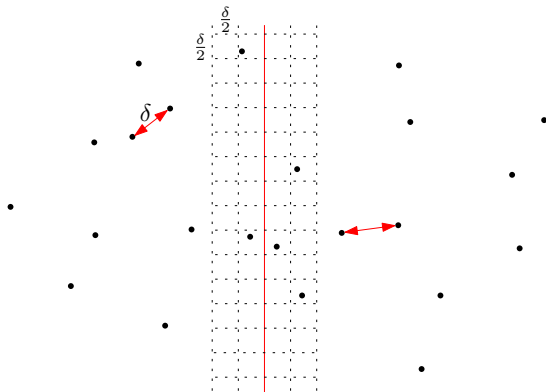
- **Divide:** Divide the points into two halves via a vertical line
- **Conquer:** Solve two sub-instances recursively
- **Combine:** Check if there is a closer pair between left-half and right-half



Divide-and-Conquer Algorithm for Closest Pair

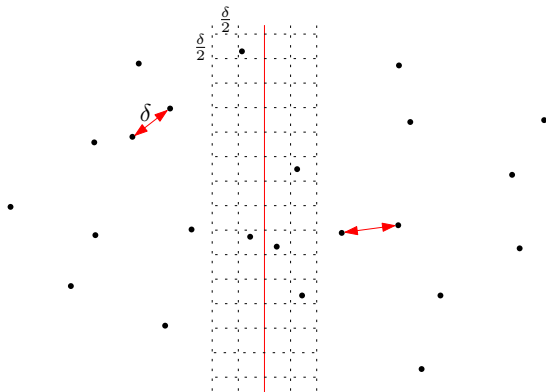


Divide-and-Conquer Algorithm for Closest Pair



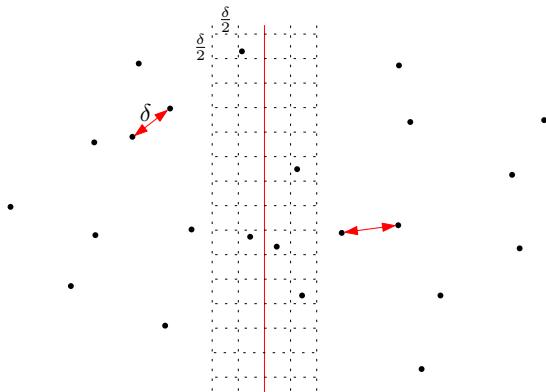
- Each box contains at most one pair

Divide-and-Conquer Algorithm for Closest Pair



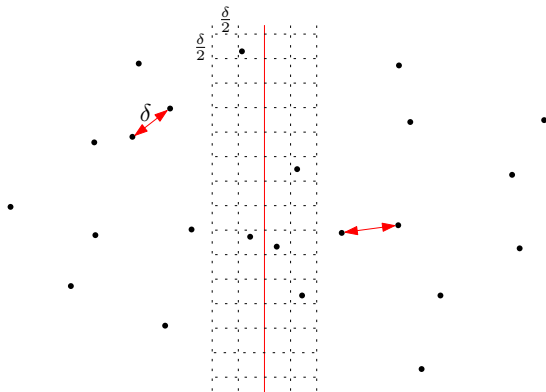
- Each box contains at most one pair
- For each point, only need to consider $O(1)$ boxes nearby

Divide-and-Conquer Algorithm for Closest Pair



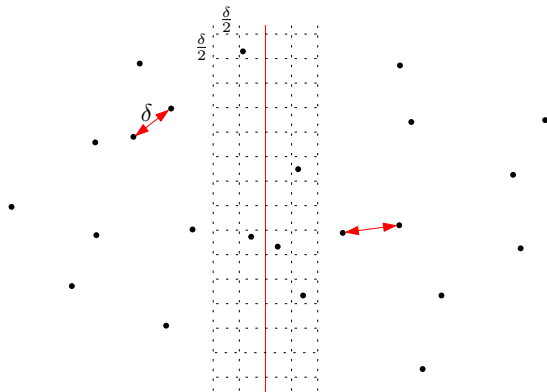
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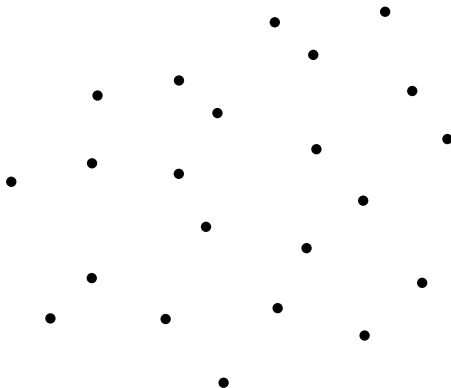
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Divide-and-Conquer Algorithm for Closest Pair

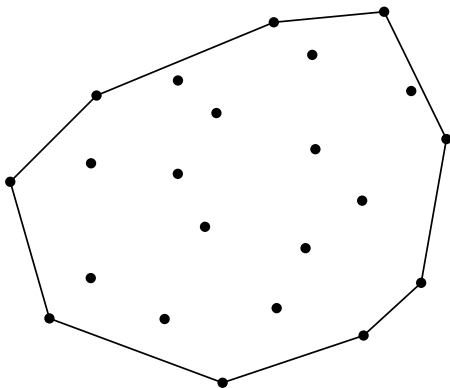


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- Running time: $O(n \lg n)$

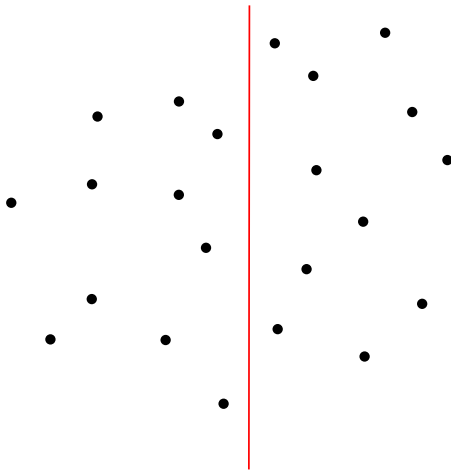
$O(n \lg n)$ -Time Algorithm for Convex Hull



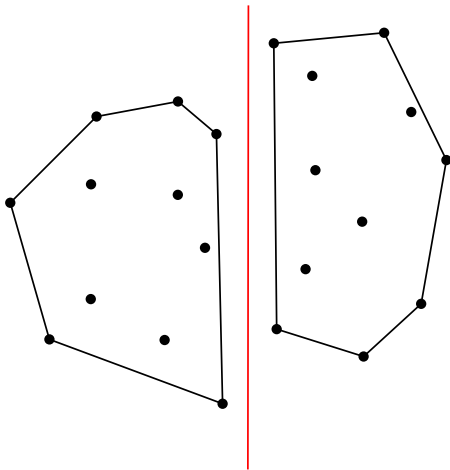
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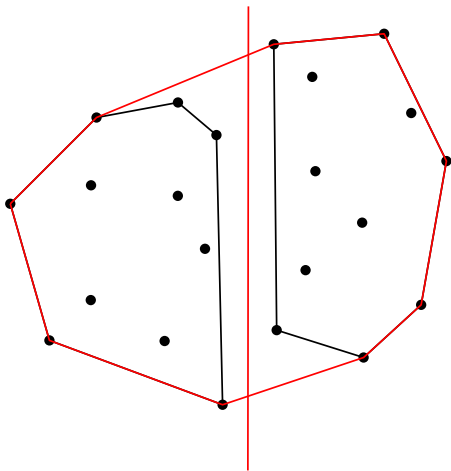
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Strassen's Algorithm for Matrix Multiplication

Matrix Multiplication

Input: two $n \times n$ matrices A and B

Output: $C = AB$

Strassen's Algorithm for Matrix Multiplication

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Input: two $n \times n$ matrices A and B

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Naive Algorithm: matrix-multiplication(A, B, n)

```
1: for  $i \leftarrow 1$  to  $n$  do
2:   for  $j \leftarrow 1$  to  $n$  do
3:      $C[i, j] \leftarrow 0$ 
4:     for  $k \leftarrow 1$  to  $n$  do
5:        $C[i, j] \leftarrow C[i, j] + A[i, k] \times B[k, j]$ 
6: return  $C$ 
```

Strassen's Algorithm for Matrix Multiplication

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6: return  $C$ 
```

- running time = $O(n^3)$

Try to Use Divide-and-Conquer

$$A = \begin{array}{|c|c|} \hline A_{11} & A_{12} \\ \hline A_{21} & A_{22} \\ \hline \end{array} \quad \begin{array}{c} \overbrace{\hspace{1cm}}^{n/2} \\ \end{array} \quad \begin{array}{c} \left. \vphantom{\begin{array}{|c|c|} \hline A_{11} & A_{12} \\ \hline A_{21} & A_{22} \\ \hline \end{array}} \right\} n/2 \end{array} \quad B = \begin{array}{|c|c|} \hline B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline \end{array} \quad \begin{array}{c} \overbrace{\hspace{1cm}}^{n/2} \\ \end{array} \quad \begin{array}{c} \left. \vphantom{\begin{array}{|c|c|} \hline B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline \end{array}} \right\} n/2 \end{array}$$

- $C = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$
- `matrix_multiplication(A, B)` recursively calls
`matrix_multiplication(A11, B11)`, `matrix_multiplication(A12, B21)`,
...

Try to Use Divide-and-Conquer

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- `matrix_multiplication(A, B)` recursively calls
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...
- Recurrence for running time: $T(n) = 8T(n/2) + O(n^2)$
- $T(n) = O(n^3)$

Strassen's Algorithm

- $T(n) = 8T(n/2) + O(n^2)$
- Strassen's Algorithm: improve the number of multiplications from 8 to 7!
- New recurrence: $T(n) = 7T(n/2) + O(n^2)$

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- Solving Recurrence $T(n) = O(n^{\log_2 7}) = O(n^{2.808})$

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Methods for Solving Recurrences

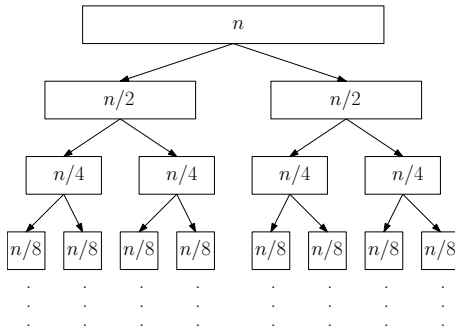
- The recursion-tree method
- The master theorem

Recursion-Tree Method

- $T(n) = 2T(n/2) + O(n)$

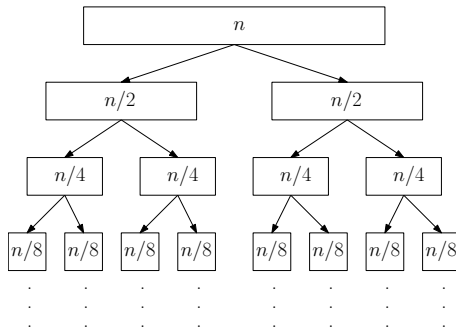
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Recursion-Tree Method

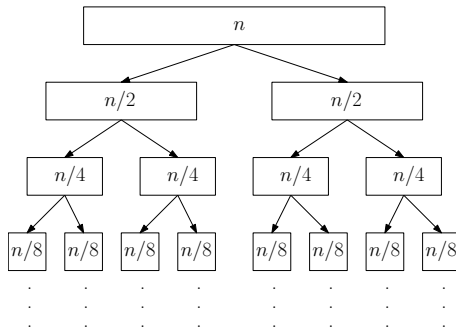
- $T(n) = 2T(n/2) + O(n)$



- Each level takes running time $O(n)$

Recursion-Tree Method

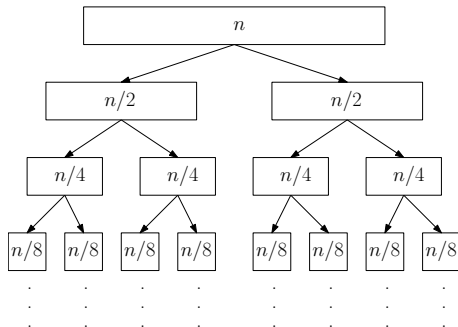
- $T(n) = 2T(n/2) + O(n)$



- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels

Recursion-Tree Method

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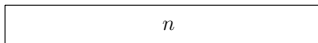
- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels
- Running time = $O(n \lg n)$

Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n)$

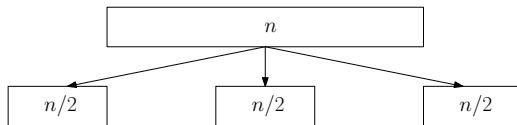
Recursion-Tree Method

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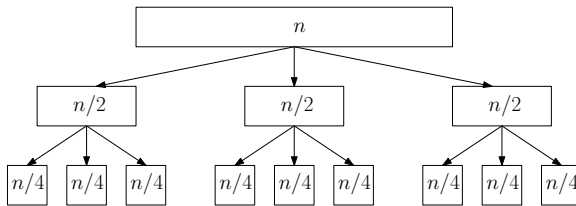
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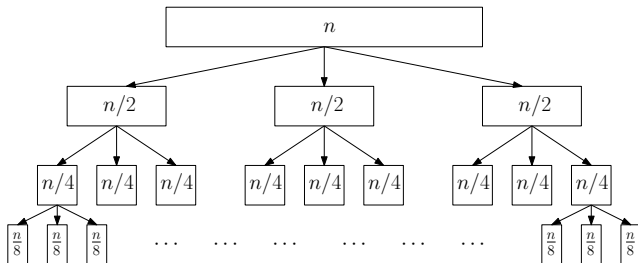
Recursion-Tree Method

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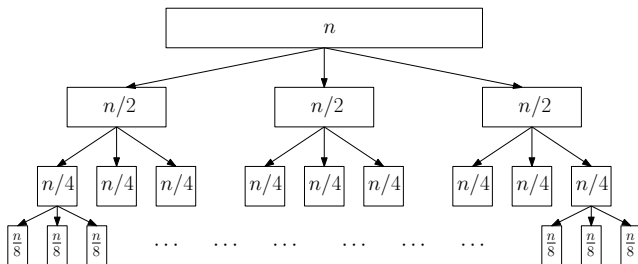
Recursion-Tree Method

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Recursion-Tree Method

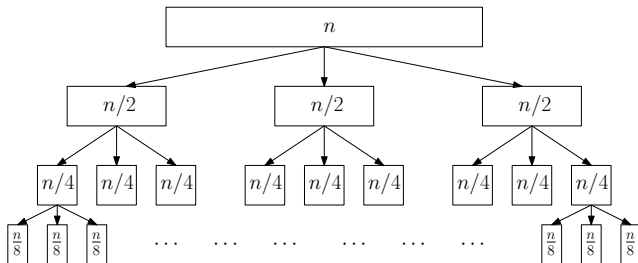
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- Total running time at level i ?

Recursion-Tree Method

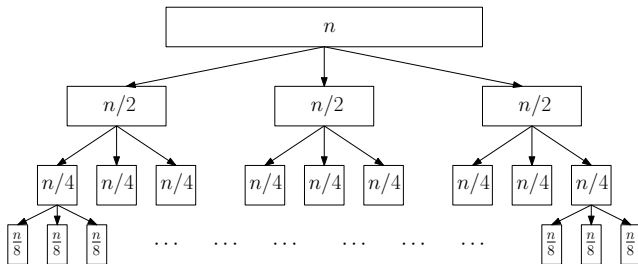
- $T(n) = 3T(n/2) + O(n)$



- Total running time at level i ? $\frac{n}{2^i} \times 3^i = \left(\frac{3}{2}\right)^i n$

Recursion-Tree Method

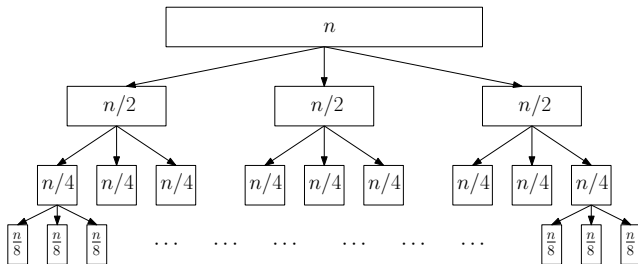
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Recursion-Tree Method

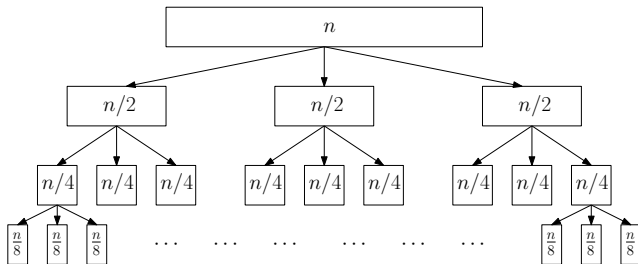
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Recursion-Tree Method

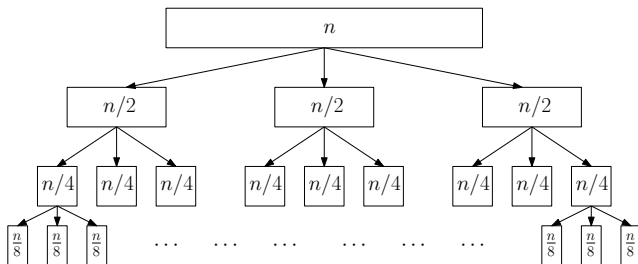
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$$\sum_{i=0}^{\lg_2 n} \left(\frac{3}{2}\right)^i n = O\left(n \left(\frac{3}{2}\right)^{\lg_2 n}\right) = O(3^{\lg_2 n}) = O(n^{\lg_2 3}).$$

Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n^2)$

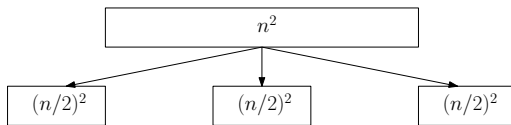
Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n^2)$

n^2

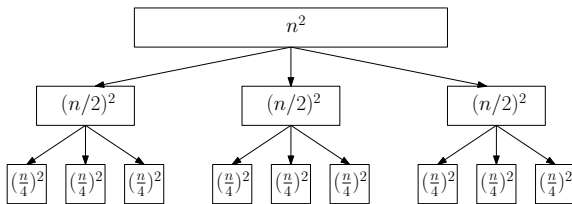
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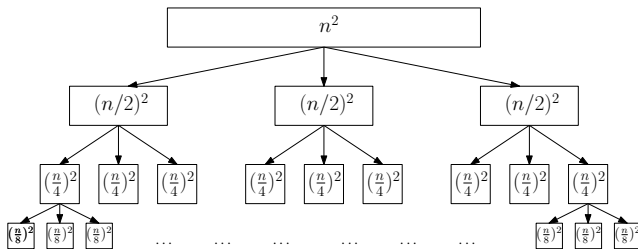
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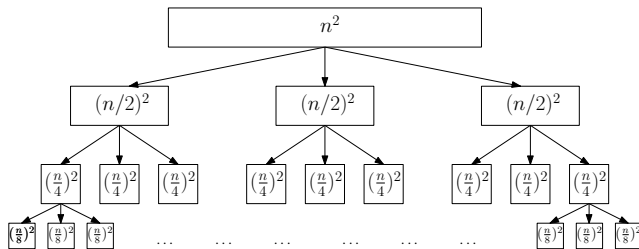
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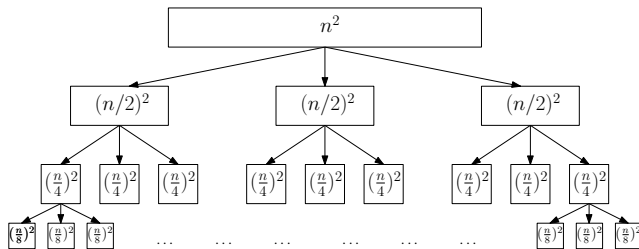
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- Total running time at level i ?

Recursion-Tree Method

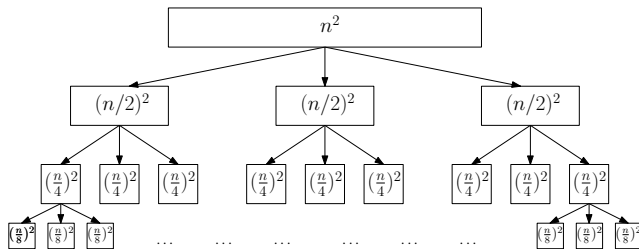
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- Total running time at level i ? $\left(\frac{n}{2^i}\right)^2 \times 3^i = \left(\frac{3}{4}\right)^i n^2$

Recursion-Tree Method

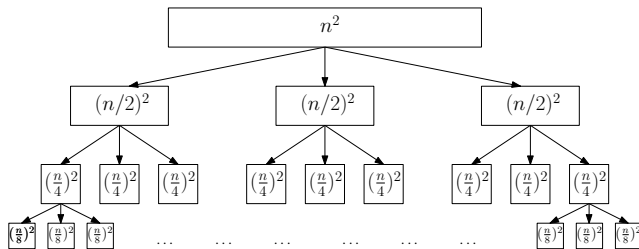
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Recursion-Tree Method

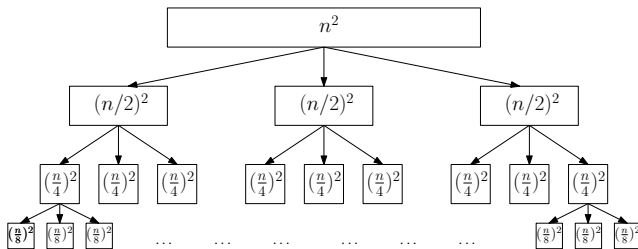
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Recursion-Tree Method

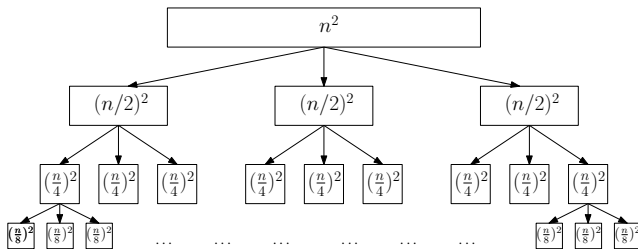
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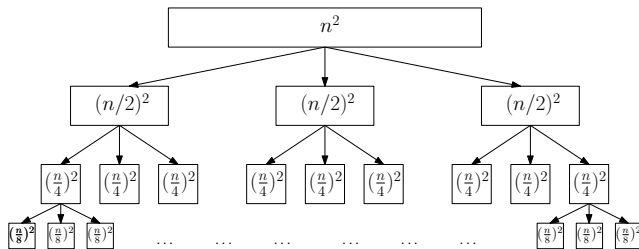


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$$\sum_{i=0}^{\lg_2 n} \left(\frac{3}{4}\right)^i n^2 =$$

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$$\sum_{i=0}^{\lg_2 n} \left(\frac{3}{4}\right)^i n^2 = O(n^2).$$

Master Theorem

Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$				$O(n \lg n)$
$T(n) = 3T(n/2) + O(n)$				$O(n^{\lg_2 3})$
$T(n) = 3T(n/2) + O(n^2)$				$O(n^2)$

Theorem $T(n) = aT(n/b) + O(n^c)$, where $a \geq 1, b > 1, c \geq 0$ are constants. Then,

Master Theorem

Recurrences	a	b	c	time
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$T(n) = 2T(n/2) + O(n)$	2	2	1	$O(n \lg n)$
$T(n) = 3T(n/2) + O(n)$	3	2	1	$O(n^{\lg_2 3})$
$T(n) = 3T(n/2) + O(n^2)$				$O(n^2)$

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- Ex: $T(n) = 2T(n/2) + O(n^2)$. Case 3.

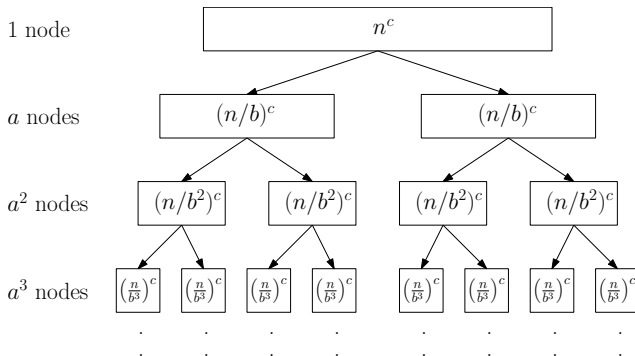
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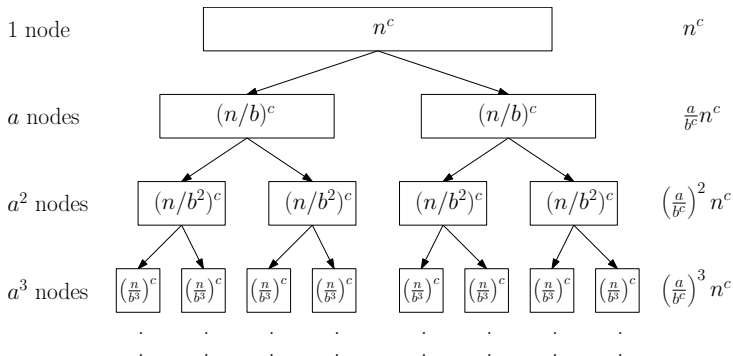
Proof of Master Theorem Using Recursion Tree

$$T(n) = aT(n/b) + O(n^c)$$



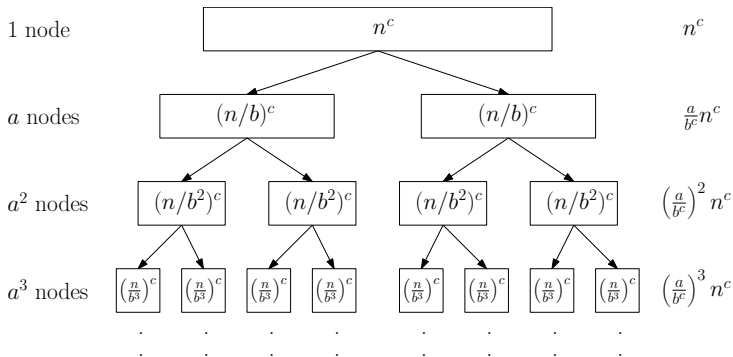
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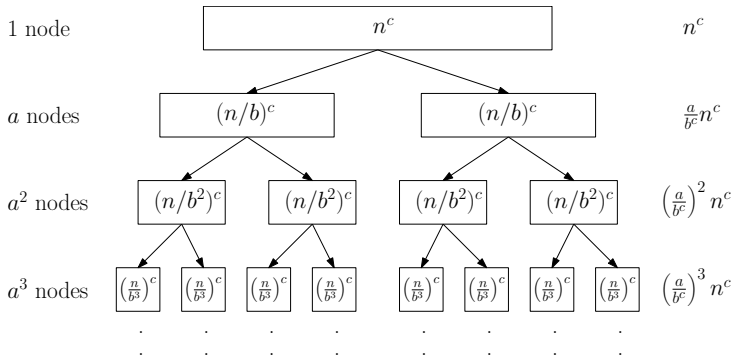
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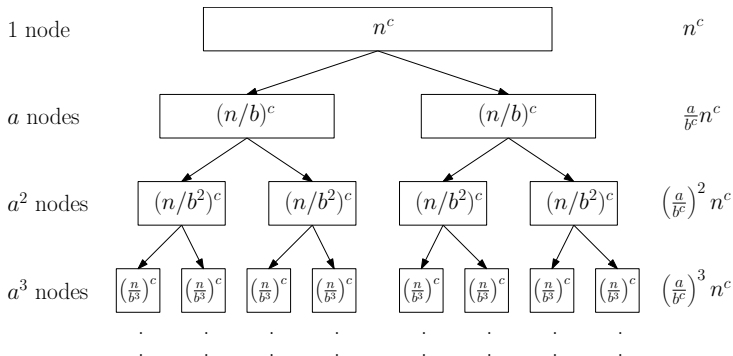
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- $c > \lg_b a$: top-level dominates: $O(n^c)$

Outline

- 1 Divide-and-Conquer
- 2 Counting Inversions
- 3 Quicksort and Selection
 - Quicksort
 - Lower Bound for Comparison-Based Sorting Algorithms
 - Selection Problem
- 4 Polynomial Multiplication
- 5 Other Classic Algorithms using Divide-and-Conquer
- 6 Solving Recurrences
- 7 Computing n -th Fibonacci Number

Fibonacci Numbers

- $F_0 = 0, F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}, \forall n \geq 2$
- Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots

n -th Fibonacci Number

Input: integer $n > 0$

Output: F_n

Computing F_n : Stupid Divide-and-Conquer Algorithm

Fib(n)

- 1: if $n = 0$ return 0
- 2: if $n = 1$ return 1
- 3: return $\text{Fib}(n - 1) + \text{Fib}(n - 2)$

Q: Is the running time of the algorithm polynomial or exponential in n ?

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- Running time is at least $\Omega(F_n)$

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- Running time is at least $\Omega(F_n)$
- F_n is exponential in n

Computing F_n : Reasonable Algorithm

Fib(n)

```
1:  $F[0] \leftarrow 0$   
2:  $F[1] \leftarrow 1$   
3: for  $i \leftarrow 2$  to  $n$  do  
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5: return  $F[n]$ 
```

- Dynamic Programming

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- Dynamic Programming
- Running time = $O(n)$

Computing F_n : Even Better Algorithm

$$\begin{aligned}\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix} \\ \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix} \\ &\dots \\ \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}\end{aligned}$$

power(n)

- 1: if $n = 0$ then return $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- 2: $R \leftarrow \text{power}(\lfloor n/2 \rfloor)$
- 3: $R \leftarrow R \times R$
- 4: if n is odd then $R \leftarrow R \times \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
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- We can not add (or multiply) two integers of $\Theta(n)$ bits in $O(1)$ time
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Fixing the Problem

To compute F_n , we need $O(\lg n)$ basic arithmetic operations on integers

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- **Divide:** Divide instance into many smaller instances
- **Conquer:** Solve each of smaller instances recursively and separately
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- **Divide:** Divide instance into many smaller instances
- **Conquer:** Solve each of smaller instances recursively and separately
- **Combine:** Combine solutions to small instances to obtain a solution for the original big instance
- Write down recurrence for running time
- Solve recurrence using master theorem

Summary: Divide-and-Conquer

- Merge sort, quicksort, count-inversions, closest pair, \dots :
 $T(n) = 2T(n/2) + O(n) \Rightarrow T(n) = O(n \lg n)$

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- Integer Multiplication:

$$T(n) = 3T(n/2) + O(n) \Rightarrow T(n) = O(n^{\lg_2 3})$$

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- Integer Multiplication:
 $T(n) = 3T(n/2) + O(n) \Rightarrow T(n) = O(n^{\lg_2 3})$
- Matrix Multiplication:
 $T(n) = 7T(n/2) + O(n^2) \Rightarrow T(n) = O(n^{\lg_2 7})$

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- Usually, designing better algorithm for “combine” step is key to improve running time