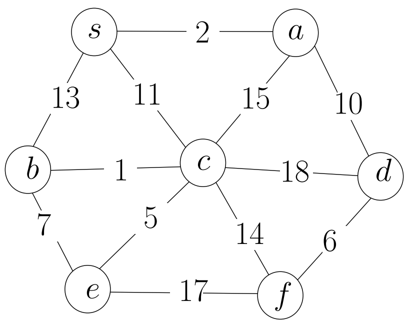
Homework 5 Solutions

Instructor: Shi Li **Deadline: 11/27/2022**

Your Name: ____________________ Your Student ID: ________________

Problems	1	2	3	Total
Max. Score	25	25	30	80
Your Score				

Problem 1. Consider the following graph G with non-negative edge weights. Use



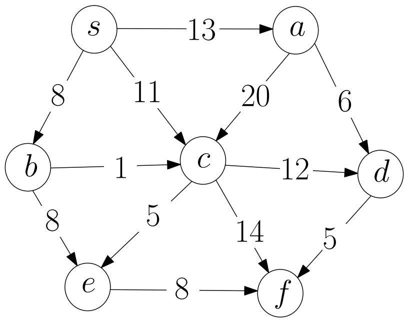
Prim's algorithm to compute the minimum spanning tree of G. You need to use the following table to describe the execution of the algorithm and give the minimum spanning tree and its weight.. If $d[v] = \infty$, then $\pi[v] =$"$\perp$". Also, when a vertex v has been added to S, you can leave its d and π values empty, to make the table clean (but it is not required to do so).

iteration	vertex added to S in iteration i	a		b		c		d		e		f	
		d	π	d	π	d	π	d	π	d	π	d	π
1	s	2	s	13	s	11	s	∞	$\perp$	∞	$\perp$	∞	$\perp$
2	a			13	s	11	s	10	a	∞	$\perp$	∞	$\perp$
3	d			13	s	11	s			∞	$\perp$	6	d
4	f			13	s	11	s			17	f		
5	c			1	c					5	c		
6	b									5	c		
7	e												

Table 1: Prim's Algorithm for Minimum Spanning Tree

The edges in the MST are $(s, a), (a, d), (d, f), (s, c), (c, b), (c, e)$.
Its weight is 35.

Problem 2. Consider the following directed graph G with non-negative edge weights. Use Dijkstra's algorithm to compute the shortest paths from s to all other vertices in G.



You need to fill the following table, and give the shortest path from s to f, and its length. When $d[v] = \infty$, we set $\pi[v] =$"$\perp$". Also, when a vertex v has been added to S, you can leave its d and π values empty, to make the table clean (but it is not required to do so).

iteration i	vertex added to S in iteration i	a		b		c		d		e		f	
		d	π	d	π	d	π	d	π	d	π	d	π
1	s	13	s	8	s	11	s	∞	$\perp$	∞	$\perp$	∞	$\perp$
2	b	13	s			9	b	∞	$\perp$	16	b	∞	$\perp$
3	c	13	s					21	c	14	c	23	c
4	a							19	a	14	c	23	c
5	e							19	a			22	e
6	d											22	e
7	f												

Table 2: Dijkstra's algorithm for Shortest Paths

The shortest path from s to f is $s \to b \to c \to e \to f$.
Its length is 22.

Problem 3. We are given an undirected graph $G = (V, E)$ with non-negative edge weights $(w_e)_{e \in E}$. Assume all the weights are different and G is connected.

(3a) True or False: The minimum spanning tree of G is unique. Justify your answer.

(3b) Let s and t be two distinct vertices in V. True or False: The shortest path from s to t in G is unique. Justify your answer.

By justifying your answer, we mean the following: If the answer is yes, you need to give a proof. If your answer is no, you need to give a counter-example.

(3a) Yes. Let e be the unique lightest edge in G. We prove that every minimum spanning tree must contain e, as follows. Suppose T is a MST that does not contain e. Then, the graph $T \cup \{e\}$ obtained by adding e to T will have a cycle C containing e. Let e' be the heaviest edge on the cycle; notice that $w(e') > w(e)$ as all edge weights are distinct. So, $T \cup \{e\} \setminus \{e'\}$ is a spanning tree of G with strictly smaller weight than T. Contradicting that T is the MST.

Therefore, we must include the unique lightest edge e in the MST. After including e in the spanning tree, the residual problem becomes finding the minimum spanning tree in the graph obtained by contracting e. Again in the graph, we must include the lightest edge in the MST. Therefore, in the end, the MST is unique.

(3b) No. Consider the graph $G = (V, E)$ with $V = \{s, t, a, b\}$, and edges $E = \{(s, a), (s, b), (a, t), (b, t)\}$ with weights $w(s, a) = 1, w(a, t) = 4, w(s, b) = 2$ and $w(b, t) = 3$. Then both s-a-t and s-b-t are shortest paths from s to t.