Packing of $\mathbb{R}^n$ with n-dimensional symmetric convex bodies

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- It connects three areas central to our millennium: Mathematics, Statistics, and Computer Science.
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- Has access to institutes such as the Perimeter Institute, the Institute for Quantum Computing, and the Cybersecurity and Privacy Institute.
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Basic Definitions and Terminologies

- A *convex body* is a compact convex subset of the n-dimensional Euclidean space $\mathbb{R}^n$ with nonempty interior.
- An *arrangement* $\mathcal{A}$ in $\mathbb{R}^n$ is a collection of convex bodies in $\mathbb{R}^n$.
- A *packing* $\mathcal{P}$ is an arrangement with objects (elements) having mutually disjoint interiors.
- The *density* of a packing $\mathcal{P}$ in $\mathbb{R}^n$ is given by the limit

$$d(\mathcal{P}) := \lim_{r\to\infty} \frac{\mathrm{vol}(\mathcal{P} \cap B^n(r))}{\mathrm{vol}(B^n(r))},$$

where $B^n(r)$ is the n-dimensional open ball centred at the origin having radius r, and for a measurable subset $C \in \mathbb{R}^n$, $\mathrm{vol}(C)$ denotes the *volume* of C.

Basic Definitions and Terminologies Contd...

- The *bounded density* of a packing $\mathcal{P}$ in a *bounded domain* D of $\mathbb{R}^n$ is given by

$$d_D(\mathcal{P}) := \frac{\mathrm{vol}(\mathcal{P})}{\mathrm{vol}(D)}.$$

- For a convex body C in $\mathbb{R}^n$, the *packing density* of C is given by

$$\delta(C) := \max\{d(\mathcal{P}) : \mathcal{P} \text{ is a packing in } \mathbb{R}^n \text{ with congruent copies of C}\},$$

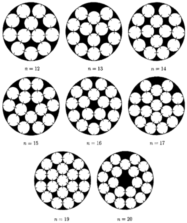
  and for a bounded domain D of $\mathbb{R}^n$, the *bounded packing density* of C is given by

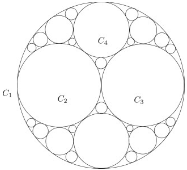
$$\delta_D(C) := \max\{d_D(\mathcal{P}) : \mathcal{P} \text{ is a packing in } D \text{ with congruent copies of C}\}.$$

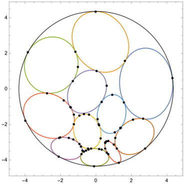
Objective: To find the optimal packing of $\mathbb{R}^n$ and its bounded domains using different convex bodies. Such problems are known as *object packing problems (OPPs)*.

ICP, GCP, and EPP

- An *identical circle packing (ICP)* is an optimized non-overlapping arrangement of n identical circles inside a container set such as a circle, square, or a general rectangle.
- A *general circle packing (GCP)* is an optimized non-overlapping arrangement of n arbitrary circles inside a container set.
- An *ellipse packing problem (EPP)* is an optimized non-overlapping arrangement of n ellipses with arbitrary size and orientation inside a container set.







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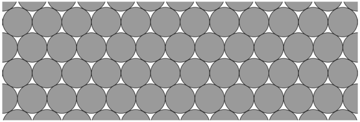
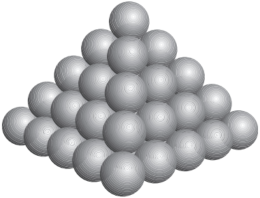
Background of Our Research

- The problem of finding the optimal packing of the n-dimensional Euclidean space $\mathbb{R}^n$ using congruent copies of an n-dimensional sphere has received considerable attention in the literature.
- The following table shows the values of n for which $\delta(B^n)$ is known, along with the values of $\delta(B^n)$.

n	2	3	4	5	6	7	8	24
$\delta(B^n)$	$\frac{\pi}{2\sqrt{3}}$	$\frac{\pi}{\sqrt{18}}$	$\frac{\pi^2}{16}$	$\frac{\pi^2}{15\sqrt{2}}$	$\frac{\pi^3}{48\sqrt{3}}$	$\frac{\pi^3}{105}$	$\frac{\pi^4}{384}$	$\frac{\pi^{12}}{12!}$





Results and Conjectures

- We are interested in finding the optimal packing of $\mathbb{R}^n$ using congruent copies of an n-dimensional ellipsoid—a convex body of the form

$$E^n[\mathbf{p},\mathbf{a}] := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{a} \cdot (\mathbf{x} - \mathbf{p})| \leq 1\},$$

  where $\mathbf{p} \in \mathbb{R}^n$ is the centre of the ellipsoid and $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ is the coefficient vector such that, $1/a_i$ is the length of the semi-axis along the i direction.
- **Result:** The packing density is independent of the centre of the ellipsoid, and hence we can denote it by just $\delta(E^n[\mathbf{a}])$.
- But, is this density also independent of the coefficient vector $\mathbf{a}$? We suspect that the answer is YES for any dimension n!!!
- **Conjecture:** For any dimension n and for any coefficient vector $\mathbf{a}$,

$$\delta(E^n[\mathbf{a}]) = \delta(E^n[\mathbf{1}]) = \delta(B^n),$$

  where $\mathbf{1} = (1, 1, \ldots, 1)$.

More on the Conjecture

If our conjecture holds true, then we would have the following:

n	2	3	4	5	6	7	8	24
$\delta(E^n)$	$\frac{\pi}{2\sqrt{3}}$	$\frac{\pi}{\sqrt{18}}$	$\frac{\pi^2}{16}$	$\frac{\pi^2}{15\sqrt{2}}$	$\frac{\pi^3}{48\sqrt{3}}$	$\frac{\pi^3}{105}$	$\frac{\pi^4}{384}$	$\frac{\pi^{12}}{12!}$

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Thank You