Hydrodynamic Interactions of Microswimmers

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Far-Field Description of a Swimmer

- Mathematical framework necessary to answer our questions.
- This concept is useful for cases in which the flow field in close proximity of a swimmer is not of interest but the far-field behaviour is.
- It is equivalent to zooming-out and observing the swimmer over length scales much larger than its intrinsic length.
- The geometrical details of the swimmer are unimportant.

Concept of Stokeslet

We place a point force $f\mathbf{e}\delta(\mathbf{x})$ at the origin ($\mathbf{x} = \mathbf{0}$) in an otherwise quiescent infinite fluid, where

$$\delta(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} = \mathbf{0}, \\ 0, & \mathbf{x} \neq \mathbf{0}, \end{cases}$$

$\mathbf{e}$ a unit vector represents the direction of the point force, and f the magnitude of the force. The forced Stokes equations are given by

$$\nabla p = \mu \nabla^2 \mathbf{v} + f\mathbf{e}\delta,$$

$$\nabla \cdot \mathbf{v} = 0.$$

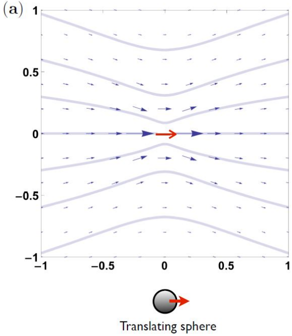
The velocity field is given by

$$\mathbf{v}(\mathbf{x}) = f\mathbf{G}(\mathbf{x}; \mathbf{e}) = \frac{f}{8\pi\mu}\left[\frac{\mathbf{e}}{r} + \frac{(\mathbf{e}\cdot\mathbf{x})\mathbf{x}}{r^3}\right],$$

where $r = |\mathbf{x}|$ is the distance from the origin and and $\mathbf{G}(\mathbf{x}; \mathbf{e})$ is the Green's function for the Stokes equations. This fundamental singular solution in viscous flows is called a **Stokeslet**.

Stokeslet - physical interpretation

- The flow field of a Stokeslet is the one given by a translating sphere in the far-field.



- As we increasingly zoom-out from a translating sphere, it becomes sufficiently small that it can be regarded as a point acted on by a force.
- A Stokeslet may thus be physically understood as the far-field approximation of a translating sphere.

Concept of Stokes dipole

The first derivative of an **e**-directed Stokeslet of strength f along the **d** direction is known as a **Stokes dipole**,

$$\mathbf{d} \cdot \nabla \mathbf{v}(\mathbf{x}) = \mathbf{d} \cdot \nabla f \mathbf{G}(\mathbf{x}; \mathbf{e}) = f \mathbf{G}_D(\mathbf{x}; \mathbf{d}, \mathbf{e})$$

$$= \frac{f}{8\pi\mu} \left\{ \frac{(\mathbf{d} \times \mathbf{e}) \times \mathbf{x}}{r^3} + \left[-\frac{(\mathbf{e} \cdot \mathbf{d})\mathbf{x}}{r^3} + 3\frac{(\mathbf{e} \cdot \mathbf{x})(\mathbf{d} \cdot \mathbf{x})\mathbf{x}}{r^5} \right] \right\}.$$

The flow due to the symmetric part of the Stokes dipole is termed a **stresslet**,

$$f \mathbf{S}(\mathbf{x}; \mathbf{d}, \mathbf{e}) = \frac{f}{8\pi\mu} \left[-\frac{(\mathbf{e} \cdot \mathbf{d})\mathbf{x}}{r^3} + 3\frac{(\mathbf{e} \cdot \mathbf{x})(\mathbf{d} \cdot \mathbf{x})\mathbf{x}}{r^5} \right],$$

whereas the flow due to the antisymmetric part is termed a **rotlet**,

$$f \mathbf{R}(\mathbf{x}; \mathbf{d}, \mathbf{e}) = \frac{f(\mathbf{d} \times \mathbf{e}) \times \mathbf{x}}{8\pi\mu r^3}.$$

So,

$$\mathbf{G}_D(\mathbf{x}; \mathbf{d}, \mathbf{e}) = \mathbf{S}(\mathbf{x}; \mathbf{d}, \mathbf{e}) + \mathbf{R}(\mathbf{x}; \mathbf{d}, \mathbf{e}).$$

Stokes dipole - physical interpretation

- A Stokes dipole can be understood as the limit when the distance between two Stokeslets of equal magnitudes but opposite directions becomes vanishingly small and the strength/magnitude is adjusted to give a finite flow field.
- Consider a Stokeslet, $\mathbf{v}^{+}(\mathbf{x})$, of strength $f\mathbf{e}$ acting at a small distance $d/2$ from the origin along the direction $\mathbf{d}$,

$$\mathbf{v}^{+}(\mathbf{x}) = f\mathbf{G}\left(\mathbf{x} - \frac{d}{2}\mathbf{d}; \mathbf{e}\right) = f\left[\mathbf{G}(\mathbf{x}; \mathbf{e}) - \frac{d}{2}\mathbf{d} \cdot \nabla\mathbf{G}(\mathbf{x}; \mathbf{e}) + \cdots\right].$$

- Consider another Stokeslet, $\mathbf{v}^{-}(\mathbf{x})$, of opposite strength $-f\mathbf{e}$ acting at an opposite position of $-\frac{d}{2}\mathbf{d}$ from the origin,

$$\mathbf{v}^{-}(\mathbf{x}) = f\mathbf{G}\left(\mathbf{x} + \frac{d}{2}\mathbf{d}; -\mathbf{e}\right) = f\left[\mathbf{G}(\mathbf{x}; -\mathbf{e}) + \frac{d}{2}\mathbf{d} \cdot \nabla\mathbf{G}(\mathbf{x}; -\mathbf{e}) + \cdots\right]$$

$$= f\left[-\mathbf{G}(\mathbf{x}; \mathbf{e}) - \frac{d}{2}\mathbf{d} \cdot \nabla\mathbf{G}(\mathbf{x}; \mathbf{e}) - \cdots\right].$$

Stokes dipole - physical interpretation contd...

- Superposing the two Stokeslets gives an overall flow field

$$\mathbf{v}(\mathbf{x}) = \mathbf{v}^{+}(\mathbf{x}) + \mathbf{v}^{-}(\mathbf{x}) = -df\,\mathbf{d}\cdot\nabla\mathbf{G}(\mathbf{x};\mathbf{e}) + \cdots ,$$

  where the leading-order contribution $-df\,\mathbf{d}\cdot\nabla\mathbf{G}(\mathbf{x};\mathbf{e})$ is a Stokes dipole.

- The distance d and strength f of the Stokeslets can be adjusted so that higher-order terms vanish upon taking the limit.

- A Stokes dipole may thus be physically understood as the leading-order contribution of the overall flow field due to two point forces of strength f acting at a fixed and sufficiently small separation distance d.

A special case of Stokes dipole

Consider a special case where the two point forces in a Stokes dipole are aligned in the same direction as the one along which derivatives are taken, i.e. $\mathbf{e} = \pm\mathbf{d}$. This implies that $\mathbf{d} \times \mathbf{e} = \mathbf{0} \implies \mathbf{R}(\mathbf{x}; \mathbf{d}, \mathbf{e}) = \mathbf{0}$. Setting $\alpha = d\mathbf{d} \cdot f\mathbf{e}$,

$$\mathbf{v}(\mathbf{x}) \approx -df\,\mathbf{G}_D(\mathbf{x}; \mathbf{d}, \mathbf{e}) = -df\,\mathbf{S}(\mathbf{x}; \mathbf{d}, \mathbf{e}) = \frac{\alpha}{8\pi\mu}\left[\frac{\mathbf{x}}{r^3} - 3\frac{(\mathbf{e}\cdot\mathbf{x})^2\mathbf{x}}{r^5}\right].$$

$\alpha = \pm df$ represents the two different opposite configurations:

1. $\alpha > 0$ represents the configuration where the two point forces act away from each other (**positive** Stokes dipole).
2. $\alpha < 0$ represents the configuration where the two point forces act towards each other (**negative** Stokes dipole).

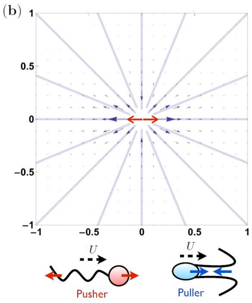
We call α the **strength** of the Stokes dipole.

Pusher and Puller

Self-propelled microorganisms can be broadly classified into two categories when observed from far-field:

- **Pusher:** One can identify two parts, namely the cell body and the flagellum. It exerts on the fluid a pair of forces acting away from each other, and thus generate a positive stokes dipole ($\alpha > 0$). Eg: Spermatozoon, *E. coli* cell.

- **Puller:** Obtain their thrust from the front part of the body and hence exerts a pair of force towards each other on the fluid, generating a negative Stokes dipole ($\alpha < 0$). Eg: *Chlamydomonas*.



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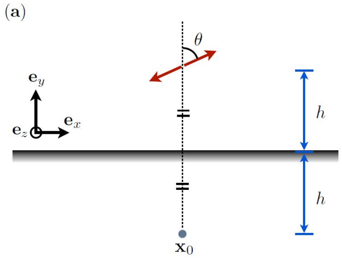
Swimming near a boundary

- Swimming microorganisms can be described as Stokes dipoles in an unbounded fluid under far-field approximations.
- However, the presence of rigid boundaries has an impact on the swimmer motion.

Consider a swimmer of strength α, oriented at an arbitrary angle θ and located at an arbitrary distance h from a wall (with zero curvature), and also assume that no other such walls are present nearby. Further, let us consider the swimmer to be of the shape of a prolate spheroid with major and minor axis lengths given by a and b respectively, and a body aspect ratio defined as $\gamma = a/b$.



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Expression for $\mathbf{U}$ and $\mathbf{\Omega}$

The (wall-induced) translational velocity, $\mathbf{U}$, and rotational velocity, $\mathbf{\Omega}$, of the swimmer due to an arbitrary (wall-induced) ambient flow is given by

$$\mathbf{U} = \frac{\alpha}{8\pi\mu}\left[\frac{3\sin(2\theta)}{8h^2}\mathbf{e}_x - \frac{3(1-3\cos^2\theta)}{8h^2}\mathbf{e}_y\right],$$

$$\mathbf{\Omega} = \frac{\alpha}{8\pi\mu}\left\{-\frac{3\sin(2\theta)}{16h^3}\left[1+\frac{\gamma^2-1}{2(\gamma^2+1)}(1+\cos^2\theta)\right]\right\}\mathbf{e}_z.$$

Hence, we have

$$U_x(\theta,h) = \frac{3\alpha\sin(2\theta)}{64\pi\mu h^2},\ \ U_y(\theta,h) = -\frac{3\alpha}{64\pi\mu h^2}(1-3\cos^2\theta),\ \ U_z(\theta,h)=0,$$

and

$$\Omega(\theta,h) = -\frac{3\alpha\sin(2\theta)}{128\pi\mu h^3}\left[1+\frac{\gamma^2-1}{2(\gamma^2+1)}(1+\cos^2\theta)\right].$$

Analysis of $U_x(\theta, h)$

Consider any arbitrary swimmer with strength α, body aspect ratio γ, swimming at a distance h from the wall, and oriented at an arbitrary angle θ.

- If $\alpha > 0$,

$$U_x(\theta, h) \begin{cases} = 0, & \theta = 0, \\ > 0, & 0 < \theta < \pi/2, \\ = 0, & \theta = \pi/2, \\ < 0, & \pi/2 < \theta < \pi. \end{cases}$$

- The opposite inequalities hold for a puller ($\alpha < 0$).
- In any case, there is no motion in the x direction when the swimmer is oriented parallel or perpendicular to the wall.

Analysis of $U_y(\theta, h)$

- For a pusher ($\alpha > 0$),

$$U_y(\theta, h) \begin{cases} > 0, & 0 \leq \theta < \sin^{-1}\sqrt{\frac{2}{3}}, \\ = 0, & \theta = \sin^{-1}\sqrt{\frac{2}{3}}, \\ < 0, & \sin^{-1}\sqrt{\frac{2}{3}} < \theta < \pi - \sin^{-1}\sqrt{\frac{2}{3}}, \\ = 0, & \theta = \pi - \sin^{-1}\sqrt{\frac{2}{3}}, \\ > 0, & \pi - \sin^{-1}\sqrt{\frac{2}{3}} < \theta < \pi. \end{cases}$$

  If the swimmer is swimming parallel to the wall ($\theta = \pi/2$), then it is attracted towards the wall.

- The opposite inequalities hold for a puller ($\alpha < 0$) and in that case swimming parallel to the wall leads to a repulsion.

Analysis of $\Omega(\theta, h)$

- In addition to inducing attraction or repulsion, the wall also hydrodynamically re-orients the swimmer.
- For any swimmer, $\gamma > 1$, implies

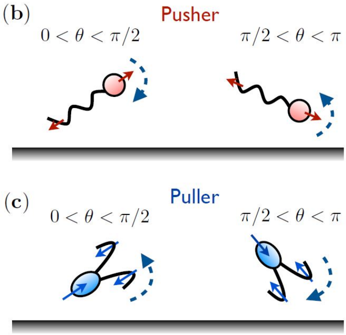
$$1 + \frac{\gamma^2 - 1}{2(\gamma^2 + 1)}(1 + \cos^2\theta) > 1 > 0.$$

  This means that the direction of $\mathbf{\Omega}$ is independent of γ, and therefore has the same sign as that of $-\alpha \sin(2\theta)$.
- So, if $\alpha > 0$,

$$\Omega(\theta, h) \begin{cases} = 0, & \theta = 0, \\ < 0, & 0 < \theta < \pi/2, \\ = 0, & \theta = \pi/2, \\ > 0, & \pi/2 < \theta < \pi. \end{cases}$$

  This means that the hydrodynamic interactions are expected to re-orient a pusher parallel to the wall.

Analysis of $\Omega(\theta, h)$ contd...



- They are expected to re-orient a puller in the direction perpendicular to the surface.

Framework for numerical simulation

Consider a swimmer of strength α and body aspect ratio γ, initially (at time $t = 0$) located at the point $\mathbf{x} = (x_0, y_0, 0)$ and oriented at an angle θ_0 $(0 \leq \theta_0 < \pi)$. We have the following system of ODEs:

$$U_x = \frac{dx}{dt} = \frac{3\alpha \sin(2\theta)}{64\pi\mu h^2} = \frac{\kappa \sin(2\theta)}{y^2},$$

$$U_y = \frac{dy}{dt} = -\frac{3\alpha}{64\pi\mu h^2}(1 - 3\cos^2\theta) = \frac{\kappa}{y^2}(3\cos^2\theta - 1), \;\; U_z = \frac{dz}{dt} = 0,$$

$$\Omega = -\frac{d\theta}{dt} = -\frac{3\alpha \sin(2\theta)}{128\pi\mu h^3}\left[1 + \frac{\gamma^2 - 1}{2(\gamma^2+1)}(1 + \cos^2\theta)\right]$$

$$\implies \frac{d\theta}{dt} = \frac{\kappa \sin(2\theta)}{2|y|^3}\left[1 + \beta(1 + \cos^2\theta)\right],$$

where $\kappa = \frac{3\alpha}{64\pi\mu}$ and $\beta = \frac{\gamma^2 - 1}{2(\gamma^2+1)}$.

So, $z(t) \equiv 0$ and the remaining system can be solved using various numerical methods of integration.

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Trajectories for different α

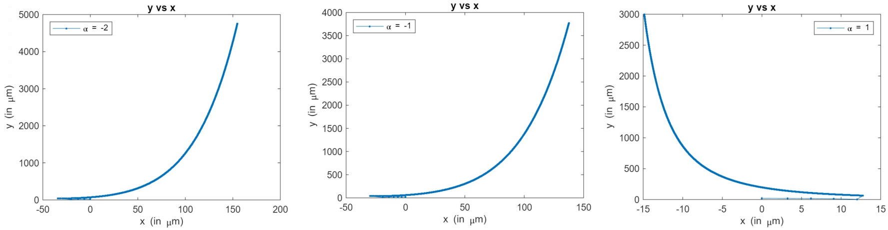

Figure: $\gamma = 3$, fluid of dynamic viscosity $\mu = 10^{-3}$ Pa.s, initially located at $\mathbf{x} = (0\mu\mathrm{m}, 20\mu\mathrm{m}, 0\mu\mathrm{m})$ and oriented at an angle $\theta = \pi/3$.

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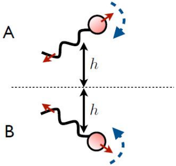
Interaction between swimmers

Consider swimmer A and its mirror-image swimmer B, separated by a distance $2h$.



The translational and rotational velocities induced on swimmer A by the flow created by swimmer B are given by

$$\mathbf{U} = -\frac{\alpha}{32\pi\mu h^2}(1 - 3\cos^2\theta)\mathbf{e}_y = -\frac{2\kappa}{3h^2}(1 - 3\cos^2\theta)\mathbf{e}_y,$$

$$\mathbf{\Omega} = -\frac{3\alpha\sin(2\theta)}{512\pi\mu h^3}\left(1 + \frac{\gamma^2 - 1}{\gamma^2 + 1}\cos^2\theta\right)\mathbf{e}_z = -\frac{\kappa\sin(2\theta)}{8h^3}(1 + 2\beta\cos^2\theta)\mathbf{e}_z.$$

Analysis of $U_y(\theta, h)$

- Suppose that A and B are pushers with strength $\alpha > 0$. Then

$$U_y(\theta, h) \begin{cases} > 0, & 0 \leq \theta < \sin^{-1}\sqrt{\frac{2}{3}}, \\ = 0, & \theta = \sin^{-1}\sqrt{\frac{2}{3}}, \\ < 0, & \sin^{-1}\sqrt{\frac{2}{3}} < \theta < \pi - \sin^{-1}\sqrt{\frac{2}{3}}, \\ = 0, & \theta = \pi - \sin^{-1}\sqrt{\frac{2}{3}}, \\ > 0, & \pi - \sin^{-1}\sqrt{\frac{2}{3}} < \theta < \pi. \end{cases}$$

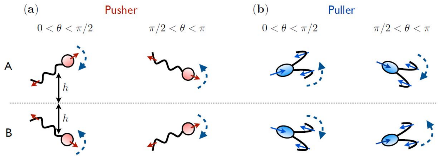
- When A and B swim side-by-side ($\theta = \pi/2$), the hydrodynamic interactions act to attract A and B. The opposite holds for the case where two cells swim head on ($\theta = 0$).
- The opposite inequalities hold when A and B are pullers ($\alpha < 0$) and hence opposite conclusions can be made.

Analysis of $\Omega(\theta, h)$

- The direction of the induced rotational velocity $\mathbf{\Omega}$ is independent of the shape of the swimmer.
- If A and B are pushers, we have

$$\Omega(\theta, h) \begin{cases} = 0, & \theta = 0, \\ < 0, & 0 < \theta < \pi/2, \\ = 0, & \theta = \pi/2, \\ > 0, & \pi/2 < \theta < \pi. \end{cases}$$

  This means that two pushers are reoriented to be perpendicular to their line of centers.
- Two pullers are reoriented so as to be parallel to their line of centers.



Framework for numerical simulation

We have the following governing system of ODEs:

$$\frac{dy}{dt} = \frac{2\kappa}{3y^2}(3\cos^2\theta - 1), \quad \frac{d\theta}{dt} = -\frac{\kappa\sin(2\theta)}{8|y|^3}(1 + 2\beta\cos^2\theta),$$

where $\kappa = \frac{3\alpha}{64\pi\mu}$ and $\beta = \frac{\gamma^2-1}{2(\gamma^2+1)}$, which can be numerically solved subject to given appropriate initial conditions.

Thank You!!!