

Ehrhart-Equivalence and $GL_4(\mathbb{Z})$ -Equidecomposability in Dimension 4

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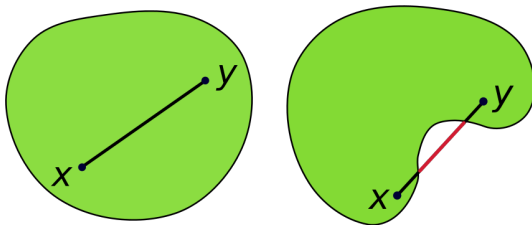
PROMYS

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Convex Set

Definition (Convex Subset)

A subset M of \mathbb{R}^n is called convex if for any pair of points $x, y \in M$, the points $z = \lambda x + (1 - \lambda)y \in M$, for all $0 \leq \lambda \leq 1$; or equivalently, if $x, y \in M$, then the segment $[x, y] \subseteq M$.



Convex Hull

Definition (Convex Hull)

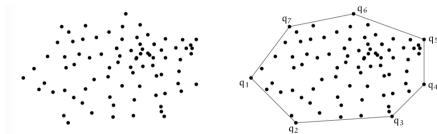
Let M be a nonempty subset of \mathbb{R}^n . Then among all convex sets containing M there exists the smallest one, namely, the intersection of all convex sets containing M . This set is called the convex hull of M and is denoted by $\text{conv}(M)$.

Alternatively, if $M := \{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_m\} \subset \mathbb{R}^n$, then

$$\text{conv}(M) := \left\{ \mathbf{y} = \sum_{i=1}^m \lambda_i \mathbf{y}_i \mid \lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1 \right\}.$$

Example

The following shows an example of the convex hull of a set of points in \mathbb{R}^2 .



Polytope

Definition (Polytope)

A convex polytope $P \subset \mathbb{R}^n$ is the convex hull of finitely affinely-independent many points in \mathbb{R}^n . We call a convex polytope rational if it is the convex hull of finitely many points in \mathbb{Q}^n (or integral if in \mathbb{Z}^n , respectively).



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Definition (Dimension of a Polytope)

The dimension of a polytope P is the dimension of the affine space

$$\text{span}P := \{\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} : \mathbf{x}, \mathbf{y} \in P, \lambda \in \mathbb{R}\}$$

spanned by P . If P has dimension d , we use the notation $\dim P = d$ and call P a d -polytope. Note that $P \subset \mathbb{R}^d$ does not necessarily have dimension d .

Definition (d -Simplex)

A d -simplex is a convex d -polytope that is the convex hull of exactly $d + 1$ affinely independent vertices.

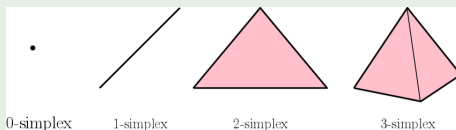
Simplex

Definition (d -Simplex)

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Example

The following image shows d -simplices up to dimension 3.



Definition (Dilation)

Suppose $P \subset \mathbb{R}^n$ is a polytope. Then the t -th dilation of P is the polytope denoted by tP and defined by

$$tP := \{(tx_1, tx_2, \dots, tx_n) : (x_1, x_2, \dots, x_n) \in P\}.$$

Discrete Volume & Lattice-Pt. Enumerator Function

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Definition (Discrete Volume)

The discrete volume of any polytope $P \subset \mathbb{R}^n$ is defined to be the number of integer (lattice) points contained in P , that is, it is equal to $\#(P \cap \mathbb{Z}^n)$.

Discrete Volume & Lattice-Pt. Enumerator Function

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Definition (Lattice-Point Enumerator Function)

For any polytope $P \subset \mathbb{R}^n$, the lattice-point enumerator function of P is denoted by L_P and is defined by $L_P : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$, such that

$$L_P(t) = \#(tP \cap \mathbb{Z}^n), \forall t \geq 0.$$

Ehrhart's Theorem and Ehrhart Polynomial

Theorem (Ehrhart's Theorem)

If $P \subset \mathbb{R}^n$ is a d -dimensional **integral** convex polytope, then

$$|tP \cap \mathbb{Z}^n| = L_P(t) = p_d t^d + \cdots + p_1 t + 1, \quad t \in \mathbb{Z}_{>0}.$$

We call L_P the Ehrhart polynomial of P .

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Example

The following figure depicts lattice-point diagram of the 1^{st} , 2^{nd} , 3^{rd} and 4^{th} dilations of the 2-polytope $P := \text{conv}\{0, e_1, e_2\}$ respectively with the Ehrhart polynomial $L_P(t) = \frac{1}{2}t^2 + \frac{3}{2}t + 1$. Note that $L_P(t)$ is a polynomial in t of degree 2, which is as expected by the Ehrhart's theorem, since $\dim P = 2$.



Ehrhart-MacDonald Reciprocity Theorem

Theorem (Ehrhart-Macdonald Reciprocity)

Suppose $P \subset \mathbb{R}^n$ is a convex integral polytope. Then the evaluation of the polynomial L_P at negative integers yields

$$L_P(-t) = (-1)^{\dim P} L_{P^\circ}(t),$$

where P° denotes the strict interior of the polytope P and $L_{P^\circ}(t) = |tP^\circ \cap \mathbb{Z}^n|$.

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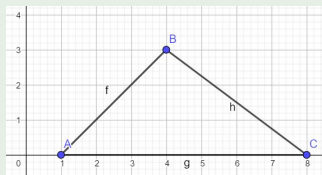
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where P° denotes the strict interior of the polytope P and $L_{P^\circ}(t) = |tP^\circ \cap \mathbb{Z}^n|$.

Example

The following 2-polytope P has $L_P(t) = \frac{21}{2}t^2 + \frac{11}{2}t + 1$, and hence

$$L_P(-1) = 6 = (-1)^2 |P^\circ \cap \mathbb{Z}^2| = (-1)^2 L_{P^\circ}(1).$$



Ehrhart Equivalence

Definition (Ehrhart Equivalence)

Two integral polytopes $Q_1, Q_2 \subset \mathbb{R}^d$ are called Ehrhart-equivalent if they have the same Ehrhart polynomial. That is, if

$$|kQ_1 \cap \mathbb{Z}^d| = |kQ_2 \cap \mathbb{Z}^d|$$

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Example

The lattice 2-polytopes $P_1, P_2 \subset \mathbb{R}^2$ defined by

$$P_1 := \text{conv} \left\{ (-4, 0), (-1, 0), \left(-3, \frac{2}{3} \right) \right\} \quad \text{and} \quad P_2 := \text{conv} \{ (1, 0), (3, 0), (1, 1) \}$$

are Ehrhart-equivalent since they have the same Ehrhart polynomial

$$L_{P_1}(t) = L_{P_2}(t) = L_P(t) = 9t^2 + 6t + 1.$$

Algorithm to Generate Ehrhart-Equivalent Polytopes

- Select a random finite vertex set $V \subset \mathbb{R}^4$, with $|V| \geq 5$.
- Obtain the polytope $S := \text{conv}(V)$.
- Evaluate the Ehrhart polynomial of S . Call it $L_S(t)$ and store it.
- Apply the "method of hash collisions" on V to obtain a vertex set $U \neq V$, and thereby a polytope $P := \text{conv}(U) (\neq S)$ with the Ehrhart polynomial $L_P(t) = L_S(t)$.

Example

Using this algorithm we found that the lattice 4-polytopes $P_1, P_2 \subset \mathbb{R}^4$ defined by

$$P_1 := \text{conv}\{(0, 0, 0, 0), (1, 1, 1, 1), (1, 1, 2, 2), (3, 2, 1, 2), (2, 3, 1, 3)\},$$

$$P_2 := \text{conv}\{(0, 0, 0, 0), (2, 1, 1, 1), (1, 1, 2, 3), (2, 2, 1, 2), (2, 2, 1, 3)\},$$

have the same Ehrhart polynomial

$$L_{P_1}(t) = L_{P_2}(t) = L_P(t) = \frac{1}{8}t^4 + \frac{3}{4}t^3 + \frac{15}{8}t^2 + \frac{9}{4}t + 1.$$

The General Linear Group Over Integers- $GL_n(\mathbb{Z})$

Definition ($GL_n(\mathbb{Z})$)

$GL_n(\mathbb{Z})$ is defined as the group of all invertible $n \times n$ over the ring of integers, under matrix multiplication.

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Remark

Since the determinant of a matrix is multiplicative and the only invertible integers are ± 1 , $GL_n(\mathbb{Z})$ can be equivalently defined as the set of all matrices $M \in M_n(\mathbb{Z})$ having $\det(M) = \pm 1$.

Example

$$M := \begin{pmatrix} 2 & -16 & 3 & -1 \\ 1 & -2 & 0 & 0 \\ 4 & 5 & -3 & 1 \\ 0 & 35 & -8 & 3 \end{pmatrix} \in GL_4(\mathbb{Z}).$$

Unimodular & Affine-Unimodular Transformation

Definition (Unimodular Transformation)

A unimodular transformation is a linear transformation $U : S_1 \rightarrow S_2$, $S_1, S_2 \subseteq \mathbb{R}^n$, such that

$$U(\mathbf{v}) = \mathbf{A}\mathbf{v},$$

$\forall \mathbf{v} \in S_1$ and some $\mathbf{A} \in GL_n(\mathbb{Z})$.

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Definition (Affine-Unimodular Transformation)

An affine-unimodular transformation $U : S_1 \rightarrow S_2$, $S_1, S_2 \subseteq \mathbb{R}^n$ is an unimodular transformation composed with a translation by a lattice vector, i.e., if $\mathbf{A} \in GL_n(\mathbb{Z})$ and $\mathbf{b} \in \mathbb{Z}^n$, then the following is such a transformation:

$$U(\mathbf{v}) = \mathbf{A}\mathbf{v} + \mathbf{b},$$

$\forall \mathbf{v} \in S_1$.

Unimodular Equivalence

Definition (Unimodular Equivalence)

Two polytopes $P, Q \subset \mathbb{R}^n$ are said to be unimodularly equivalent if there exists some affine-unimodular transformation $U : P \rightarrow Q$, that is there exists some $\mathbf{A} \in GL_n(\mathbb{Z})$ and $\mathbf{b} \in \mathbb{Z}^n$, such that for each $\mathbf{x} \in P$, there exists a unique $\mathbf{y} \in Q$ satisfying

$$\mathbf{y} = U(\mathbf{x}) = \mathbf{Ax} + \mathbf{b}.$$

In this case we write $U(P) = Q$.

Example

The lattice 4-polytopes $P, Q \subset \mathbb{R}^4$ defined by

$$P := \text{conv}\{(4, 3, 2, 1), (6, 7, 5, 8), (10, 9, 12, 11), (13, 14, 15, 16), (18, 19, 17, 20)\},$$

$$Q := \text{conv}\{(5, 4, 3, 2), (7, 8, 6, 9), (11, 10, 13, 12), (14, 15, 16, 17), (19, 20, 18, 21)\},$$

are unimodularly equivalent by the affine-unimodular transformation $U : P \rightarrow Q$ such that

$$U(\mathbf{x}) = \begin{pmatrix} -8 & 7 & 1 & -1 \\ -9 & 8 & 1 & -1 \\ -12 & 12 & 3 & -4 \\ -12 & 12 & 2 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 21 \\ 21 \\ 23 \\ 23 \end{pmatrix}, \forall \mathbf{x} \in P.$$

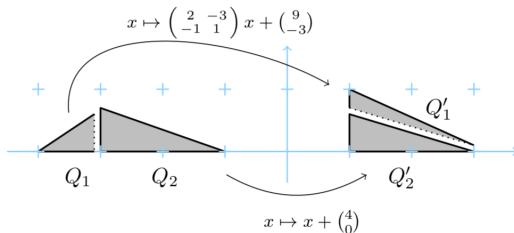
$GL_n(\mathbb{Z})$ -Equidecomposability

Definition ($GL_n(\mathbb{Z})$ -Equidecomposability)

Two polytopes $P, Q \subset \mathbb{R}^n$ are $GL_n(\mathbb{Z})$ -equidecomposable if there are relatively open simplices T_1, \dots, T_r and affine-unimodular transformations U_1, \dots, U_r such that

$$P = \bigsqcup_{i=1}^r T_i \quad \text{and} \quad Q = \bigsqcup_{i=1}^r U_i(T_i).$$

(Here, \bigsqcup indicates disjoint union.)



$GL_n(\mathbb{Z})$ -Equidecomposability and Ehrhart Equivalence

Theorem

If two integral polytopes $P, Q \subset \mathbb{R}^n$ are $GL_n(\mathbb{Z})$ -equidecomposable, then they are Ehrhart-equivalent.

$GL_n(\mathbb{Z})$ -Equidecomposability and Ehrhart Equivalence

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If two integral polytopes $P, Q \subset \mathbb{R}^n$ are $GL_n(\mathbb{Z})$ -equidecomposable, then they are Ehrhart-equivalent.

Proof.

Decomposition of P and Q into unimodularly equivalent simplices P_1, \dots, P_k and Q_1, \dots, Q_k yields a bijection between the lattice points in P and Q and hence there exists a bijection between the lattice points of kP and kQ resulting in

$$|kP \cap \mathbb{Z}^n| = |kQ \cap \mathbb{Z}^n|, \forall k \in \mathbb{Z}_{\geq 1}.$$



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We might hope that the converse also holds true. This is indeed true for two dimensional rational polytopes, but is not necessarily true for rational polytopes having dimension greater than or equal to 3.

Weakly $GL_n(\mathbb{Z})$ -equidecomposability

Definition

Two rational polytopes $P, Q \subset \mathbb{R}^n$ are weakly $GL_n(\mathbb{Z})$ -equidecomposable if they can be decomposed into rational polytopes P_1, P_2, \dots, P_k and Q_1, Q_2, \dots, Q_k so that P_i and Q_i is equivalent via affine-unimodular transformations having rational translation vector.

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The above definition is equivalent to stating that there is a dilation factor $k \in \mathbb{Z}_{>0}$ such that kP and kQ are (ordinarily) $GL_n(\mathbb{Z})$ -equidecomposable.

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Theorem (Haase-McAllister)

Two integral (lattice) polytopes $P, Q \subset \mathbb{R}^n$ are weakly $GL_n(\mathbb{Z})$ -equidecomposable if and only if they are Ehrhart-equivalent.

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Two integral (lattice) polytopes $P, Q \subset \mathbb{R}^n$ are weakly $GL_n(\mathbb{Z})$ -equidecomposable if and only if they are Ehrhart-equivalent.

Note that this implies that given two arbitrary Ehrhart-equivalent lattice polytopes $P, Q \subset \mathbb{R}^n$, there is always some dilation factor $k \in \mathbb{Z}_{\geq 1}$ such that kP and kQ are $GL_n(\mathbb{Z})$ -equidecomposable.

More on $GL_n(\mathbb{Z})$ –equidecomposability

Remark

If two lattice simplices $S_1, S_2 \subset \mathbb{R}^n$ are unimodularly equivalent, then by definition they are $GL_n(\mathbb{Z})$ –equidecomposable, and hence Ehrhart-equivalent.

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Remark

If two lattice simplices $S_1, S_2 \subset \mathbb{R}^n$ are unimodularly equivalent, then by definition they are $GL_n(\mathbb{Z})$ –equidecomposable, and hence Ehrhart-equivalent.

Proposition

If two lattice polytopes $P, Q \subset \mathbb{R}^n$ are Ehrhart equivalent, then there exists infinitely many $k \in \mathbb{Z}_{\geq 1}$ such that kP and kQ are $GL_n(\mathbb{Z})$ –equidecomposable.

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Theorem (Erbe, Haase, Santos)

If two lattice 3-polytopes $P, Q \subset \mathbb{R}^3$ are Ehrhart-equivalent, then $2P$ and $2Q$ are $GL_3(\mathbb{Z})$ -equidecomposable.

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If two lattice simplices $S_1, S_2 \subset \mathbb{R}^n$ are unimodularly equivalent, then by definition they are $GL_n(\mathbb{Z})$ –equidecomposable, and hence Ehrhart-equivalent.

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Theorem (Erbe, Haase, Santos)

If two lattice 3-polytopes $P, Q \subset \mathbb{R}^3$ are Ehrhart-equivalent, then $2P$ and $2Q$ are $GL_3(\mathbb{Z})$ -equidecomposable.

Following this theorem we came up with the following corollary:

Corollary

If two lattice 3-polytopes $P, Q \subset \mathbb{R}^3$ are Ehrhart equivalent, then kP and kQ are $GL_3(\mathbb{Z})$ -equidecomposable, where $k = 2, 4, 8, 12, \dots$.

Kedlaya's Conjecture

The results of the previous slide motivated us and Professor Kiran Kedlaya to come up with the following conjecture:

Conjecture (due to Kedlaya)

For all $n \in \mathbb{N}$, suppose P and Q are two Ehrhart-equivalent lattice n -polytopes, then $(n-1)!P$ and $(n-1)!Q$ are $GL_n(\mathbb{Z})$ -equidecomposable.

We hope to prove this conjecture for $n = 4$. We began by considering unimodular equivalence of Ehrhart-equivalent 4-simplices contained in \mathbb{R}^4 .

Some Motivation

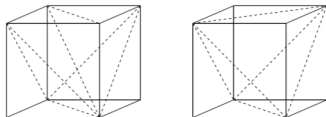
Definition (Triangulation)

A triangulation of a convex d -polytope P is a finite collection T of d -simplices with the following properties:

- 1 $P = \bigcup_{\Delta \in T} \Delta$.
- 2 For every $\Delta_1, \Delta_2 \in T$, $\Delta_1 \cap \Delta_2$ is a face common to both Δ_1 and Δ_2 .

Theorem (Existence of Triangulations)

Every convex polytope can be triangulated using no new vertices.



To work with 4-dimensional lattice polytopes, it makes sense for us to begin by working with 4-simplices.

Equivalent Form of Affine-Unimodular Transformation

Let $S_1, S_2 \subset \mathbb{R}^n$ and $U : S_1 \rightarrow S_2$ be an affine-unimodular transformation such that

$$U(\mathbf{x}) := \mathbf{B}\mathbf{x} + \mathbf{c}, \forall \mathbf{x} \in S_1$$

where $\mathbf{B} \in GL_n(\mathbb{Z})$ and $\mathbf{c} \in \mathbb{Z}^n$. Note that U can be equivalently expressed as

$$U\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \mathbf{A}\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}, \forall \mathbf{x} \in S_1$$

where

$$\mathbf{A} := \begin{pmatrix} \mathbf{B} & \mathbf{c} \\ \mathbf{0} & 1 \end{pmatrix},$$

since

$$\mathbf{A}\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{B} & \mathbf{c} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{B}\mathbf{x} + \mathbf{c} \\ 1 \end{pmatrix}.$$

We will use this exact result in \mathbb{R}^4 for testing unimodular equivalence (defined later) of two lattice polytopes $P, Q \subset \mathbb{R}^4$.

Equivalent Form of Affine-Unimodular Transformation

Example

Consider the affine-unimodular transformation $U : S_1 \rightarrow S_2$, $S_1, S_2 \subseteq \mathbb{R}^2$, such that

$$U \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 9 \\ -3 \end{pmatrix}.$$

Suppose that $\begin{pmatrix} -9 \\ 2 \end{pmatrix} \in S_1$, and thus

$$U \begin{pmatrix} -9 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -9 \\ 2 \end{pmatrix} + \begin{pmatrix} 9 \\ -3 \end{pmatrix} = \begin{pmatrix} -15 \\ 8 \end{pmatrix}.$$

Now note that

$$\left(\begin{array}{cc|c} 2 & -3 & 9 \\ -1 & 1 & -3 \\ \hline 0 & 0 & 1 \end{array} \right) \begin{pmatrix} -9 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -15 \\ 8 \\ 1 \end{pmatrix}.$$

Unimodular Equivalence of 4-Simplices

Two lattice 4-simplices $S_1, T_1 \subset \mathbb{R}^4$ defined by

$$S_1 := \text{conv}\{(s_{11}, s_{21}, s_{31}, s_{41}), (s_{12}, s_{22}, s_{32}, s_{42}), \dots, (s_{15}, s_{25}, s_{35}, s_{45})\},$$

and

$$T_1 := \text{conv}\{(t_{11}, t_{21}, t_{31}, t_{41}), (t_{12}, t_{22}, t_{32}, t_{42}), \dots, (t_{15}, t_{25}, t_{35}, t_{45})\}$$

are unimodularly equivalent if and only if there exists $\mathbf{A} \in \mathbb{M}_5(\mathbb{Z})$, with

$$\mathbf{A} = \begin{pmatrix} \mathbf{B} & \mathbf{c} \\ \mathbf{0} & 1 \end{pmatrix},$$

where $\mathbf{B} \in GL_4(\mathbb{Z})$ and $\mathbf{c} \in \mathbb{Z}^4$, such that

$$\mathbf{AS} = \mathbf{T},$$

where

$$\mathbf{S} = \begin{pmatrix} s_{11} & s_{12} & \dots & s_{15} \\ \vdots & \vdots & \ddots & \vdots \\ s_{41} & s_{42} & \dots & s_{45} \\ 1 & 1 & \dots & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{T} = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{15} \\ \vdots & \vdots & \ddots & \vdots \\ t_{41} & t_{42} & \dots & t_{45} \\ 1 & 1 & \dots & 1 \end{pmatrix},$$

and the columns of \mathbf{S} and \mathbf{T} can be freely permuted, since any of the five vertices of S_1 can be mapped to any vertex of T_1 .

Unimodular Equivalence of Ehrhart-Equivalent 4–Simplices

Keeping all the notations the same as in the previous slide, we have the following proposition:

Proposition

If S_1 and T_1 are Ehrhart-equivalent, then S_1 and T_1 are unimodularly equivalent if and only if there exists $\mathbf{A} \in \mathbb{M}_5(\mathbb{Z})$, such that

$$\mathbf{AS} = \mathbf{T},$$

under the free column permutations of \mathbf{S} and \mathbf{T} .

Note that this proposition holds true for any $n \in \mathbb{N}$ and for any two Ehrhart-equivalent n –simplices $S_1, T_1 \subset \mathbb{R}^n$.

Keeping this proposition and the result stated in the previous slide as a reference and the basis to our algorithm we wrote up the following SageMath code to check the unimodular equivalence of two arbitrary lattice 4-simplices $S_1, T_1 \subset \mathbb{R}^4$.

SageMath Code to Check Unimodular Equivalence

Suppose that we have two lattice 4-simplices $S_1, T_1 \subset \mathbb{R}^4$ defined by

$$S_1 := \text{conv}\{(s_{11}, s_{21}, s_{31}, s_{41}), (s_{12}, s_{22}, s_{32}, s_{42}), \dots, (s_{15}, s_{25}, s_{35}, s_{45})\},$$

and

$$T_1 := \text{conv}\{(t_{11}, t_{21}, t_{31}, t_{41}), (t_{12}, t_{22}, t_{32}, t_{42}), \dots, (t_{15}, t_{25}, t_{35}, t_{45})\}$$

and we wish to check the unimodular equivalence of S_1 and T_1 . We wrote the following code to check the same.

```
sage: M_1=matrix([(s_11,s_21,s_31,s_41,1),(s_12,s_22,s_32,
s_42,1),(s_13,s_23,s_33,s_43,1),(s_14,s_24,s_34,
s_44,1),(s_15,s_25,s_35,s_45,1)])
sage: S=M_1.transpose()
sage: N_1=matrix([(t_11,t_21,t_31,t_41,1),(t_12,t_22,t_32,
t_42,1),(t_13,t_23,t_33,t_43,1),(t_14,t_24,t_34,
t_44,1),(t_15,t_25,t_35,t_45,1)])
sage: T=N_1.transpose()
sage: for s in Permutations([1,2,3,4,5]):
    M = s.to_matrix()
    print(S*M)
    print(T*(S*M).inverse())
    print('next')
```

The Algorithm

Let \mathbf{S} and \mathbf{T} be the matrices representing the simplices as in previous slide and \mathbf{A} represent the potential unimodular transformation. Then

$$\mathbf{A}\mathbf{S}\mathbf{M} = \mathbf{T},$$

for some permutation matrix \mathbf{M} .

Algorithm:

- Iterates through all 5-dimensional permutation matrices \mathbf{M} and calculates

$$\mathbf{A} = \mathbf{T}(\mathbf{S}\mathbf{M})^{-1}.$$

- Tests whether \mathbf{A} is of the form

$$\mathbf{A} = \begin{pmatrix} \mathbf{B} & \mathbf{c} \\ \mathbf{0} & 1 \end{pmatrix}$$

where $\mathbf{B} \in GL_4(\mathbb{Z})$ and $\mathbf{c} \in \mathbb{Z}^4$, i.e., if \mathbf{A} has all integer entries.

Sample Testing and a Conjecture

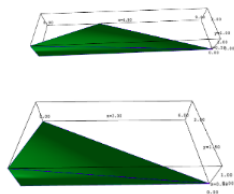
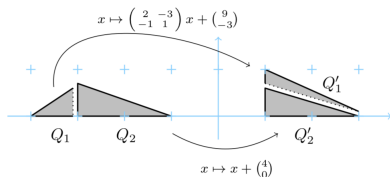
We tested over fifty simplices with 5 different Ehrhart polynomials and found that all of the Ehrhart equivalent ones were unimodularly equivalent, leading us to the following conjecture:

Conjecture (False)

Two lattice 4-simplices $S_1, T_1 \subset \mathbb{R}^4$ are Ehrhart-equivalent if and only if they are unimodularly equivalent.

Interestingly, motivation for the counterexample came from 2D case...

Counterexample in 2D



Shifted them to origin:

$$P_1 := \text{conv}\{(0,0), (9,0), (3,2)\},$$

$$P_2 := \text{conv}\{(0,0), (6,0), (0,3)\},$$

Turned into 4D pyramids by adding basis vectors $\mathbf{e}_3 = (0,0,1,0)$ and $\mathbf{e}_4 = (0,0,0,1)$:

$$R_1 := \text{conv}\{(0,0,0,0), (9,0,0,0), (3,2,0,0), (0,0,1,0), (0,0,0,1)\},$$

$$R_2 := \text{conv}\{(0,0,0,0), (6,0,0,0), (0,3,0,0), (0,0,1,0), (0,0,0,1)\}.$$

Turns out that these are:

- $GL_4(\mathbb{Z})$ -equidecomposable \implies Ehrhart equivalent (follows from 2D case).
- **Not** unimodularly equivalent (doesn't follow immediately from 2D case – tested this with our program).

More on Unimodular Equivalence

Proposition

Let P_1, P_2 be two 2-simplices and R_1, R_2 be their n -dimensional pyramids. Then R_1 and R_2 are unimodularly equivalent if and only if P_1 and P_2 are unimodularly equivalent.

- some vector of original triangles maps to \mathbf{e}_k
- but k th component of all other vertices in image is 0
- implies that P_1 and P_2 are unimodular simplices dilated in either x or y direction
 $\implies P_1$ and P_2 are unimodularly equivalent

Corollary

For any $n \in \mathbb{N}$ there exists two Ehrhart-equivalent n -simplices $S_1, S_2 \in \mathbb{R}^n$ that are not unimodularly equivalent.

Problems for Further Investigation

- Is there an algorithm to find enough Ehrhart-equivalent lattice 4-simplices $S_1, S_2 \subset \mathbb{R}^4$, which are not unimodularly equivalent?
- Find example of two Ehrhart equivalent polytopes that are not unimodularly equidecomposable.
- Given two lattice 4-polytopes $P, Q \subset \mathbb{R}^4$, is there an algorithm to check their $GL_4(\mathbb{Z})$ -equidecomposability?
- Prove or disprove Kedlaya's conjecture for $n = 4$ (factor for weak equidecomposability is $k = 6$) and generalize to higher dimensions

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