PERFECT MATCHINGS IN CUBIC GRAPHS

SANKET BISWAS AND KALYANI GOHOKAR

ABSTRACT. In 1891, in connection to the Four Color Conjecture, Julius Petersen proved that every bridgeless cubic graph has a perfect matching. Tutte, in 1947, gave a complete characterization of graphs that have a perfect matching. Using Tutte's Theorem, one may easily deduce a stronger version of Petersen's Theorem: in a bridgeless cubic graph, each edge participates in a perfect matching. In this project, we intend to establish further strengthenings of Petersen's Theorem. One potential strengthening is the following conjecture — except for certain special graphs (such as the complete graph K_4, the triangular prism, etc.) and infinite families (such as the staircases that appear in the work of Norine and Thomas 2007) — in a bridgeless cubic graph, each edge participates in at least two distinct perfect matchings. The goal of the project is to prove this conjecture.

1. Background

Theorem 1.1 (Petersen's Theorem). *Every 2-connected cubic graph has a perfect matching.*

Lemma 1.2. *Let G be a 2-edge connected cubic graph, and let $S \subseteq V(G)$. Then $|\partial(S)| \geq 2$ whenever $S \neq \emptyset$, and $|\partial(S)| \geq 3$ whenever $|S|$ is odd.*

Theorem 1.3 (Strengthening of Petersen's Theorem). *Every 2-connected cubic graph is matching covered.*

Proof. Let G be any 2-connected cubic graph. Since G is 2-connected, G must be 2-edge connected as well. So, by Petersen's Theorem, G is matchable. Now let B be a barrier of G, and $K_1, K_2, \ldots, K_n$ be the odd components of $G - B$. Then, by Lemma 1.2, $|\partial(K_i)| \geq 3$ for all $1 \leq i \leq n$. This implies that

$$\sum_{v \in B} \deg(v) \geq \sum_{i=1}^{n} |\partial(K_i)| \geq 3n.$$

Now, we know that $|B| = n$. So, since G is cubic, $\sum_{v \in B} \deg(v) = 3n$. This implies that

$$\sum_{i=1}^{n} |\partial(K_i)| = 3n.$$

Note that this means that there are no edges between any pair of vertices in B. Hence, B is a stable set. Thus, using (2), G is matching covered. □

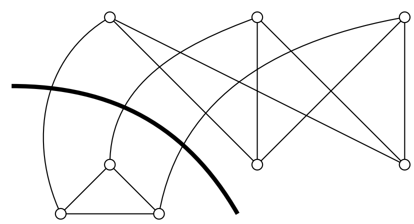
In the above proof, suppose that E is an even component of $G - B$, where B is a barrier of G. Then, note that, by the above argument there are no edges between E and B in G. This means that G has at least two components, which is a contradiction to the fact that G is connected. Hence, $G - B$ has no even components. So, we have the following proposition:

Proposition 1.4. *Let G be a 2-edge connected cubic graph. If B is a barrier of G then $G - B$ has no even components.*

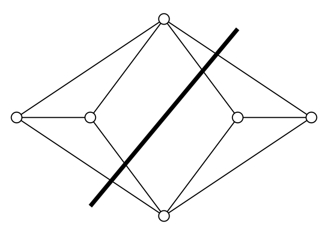
Theorem 1.5. *A cubic graph is matching covered if and only if it is 2-edge-connected.*

For a nonempty proper subset X of the vertices of a graph G, we denote by $\partial(X)$ the cut associated with X, that is, the set of all edges of G that have one end in X and the other end in $\overline{X} := V(G) - X$. We refer to X and $\overline{X}$ as the *shores* of $\partial(X)$. A cut is *trivial* if either of its shores is a singleton. We say that $\partial(X)$ is a *k-cut* if $|\partial(X)| = k$.

For a cut $\partial(X)$, we denote the graph obtained by contracting the shore X to a single vertex x by $G/(X \to x)$. The graph $G/(\overline{X} \to \overline{x})$ is defined analogously. In case the label of the contraction vertex x or $\overline{x}$ is irrelevant, we simply write G/X or $G/\overline{X}$, respectively. The two graphs G/X and $G/\overline{X}$ are called the *$\partial(X)$-contractions* of G. In Figure 1(a), the three edges crossing the bold line constitute a nontrivial cut, say $\partial(X)$, and the two $\partial(X)$-contractions are K_4 and $K_{3,3}$.



(a) A barrier cut



(b) A 2-separation cut

FIGURE 1. Nontrivial tight cuts

A cut $\partial(X)$ of a matching covered graph G is a *separating cut* if each $\partial(X)$-contraction of G is also matching covered. Clearly, each trivial cut is a separating cut. The triangular prism $\overline{C_6}$ has a (unique) nontrivial 3-cut $\partial(X)$ that is a separating cut. We have the following proposition:

Proposition 1.6. *In a cubic matching covered graph each* 3*-cut is a separating cut.*

However, a cubic matching covered graph may have a separating cut that is not a 3-cut. For instance, the Petersen graph has nontrivial separating cuts, each of which is a 5-cut; for any such cut $\partial(X)$, each of the $\partial(X)$-contractions is isomorphic to the odd wheel W_5.

Proposition 1.7. *Every tight cut of a matching covered graph is a separating cut.*

The converse is not necessarily true. For instance, as noted earlier, $\overline{C_6}$ has a nontrivial separating cut; however, $\overline{C_6}$ is free of nontrivial tight cuts.

Let G be a matching covered graph. If $\partial(X)$ is a nontrivial tight cut of G, then by Proposition 1.7, $\partial(X)$ is a nontrivial separating cut of G. This implies that each $\partial(X)$-contraction is a matching covered graph that has strictly fewer vertices than G. If either of the $\partial(X)$-contractions has a nontrivial tight cut, then that graph can be further decomposed into even smaller matching covered graphs. We can repeat this procedure until we obtain a list of matching covered graphs, each of which is free of nontrivial tight cuts. This procedure is known as a *tight cut decomposition* of G.

Let G be a matching covered graph free of nontrivial tight cuts. If G is bipartite then it is a *brace*; otherwise it is a *brick*. Thus, a tight cut decomposition of G results in a list of bricks and braces.

In general, a matching covered graph may admit several tight cut decompositions. However, Lovász (1987) proved the remarkable result that any two tight cut decompositions of a matching covered graph yield the same list of bricks and braces (except possibly for multiplicities of edges). In particular, any two tight cut decompositions of a matching covered graph G yield the same number of bricks; this number is denoted by $b(G)$.

Following Proposition 1.7 and the above discussion, we have the following proposition:

Proposition 1.8. *Let G be a bipartite matching covered graph. If $\partial(X)$ is a tight cut of G then both $\partial(X)$-contractions of G are bipartite and matching covered. Furthermore, any tight cut decomposition of G consists of only braces, that is, $b(G) = 0$.*

Theorem 1.9. *In a cubic matching covered graph each tight cut is a 3-cut.*

Let G be a cubic bipartite matching covered graph and $\partial(X)$ be a tight cut of G. Then, using Theorem 1.9 and Proposition 1.8, we can conclude that both G/X and $G/\overline{X}$ are cubic, bipartite, and matching covered. Following this discussion, we have the following corollary:

Corollary 1.10. *If G is a cubic bipartite matching covered graph, then any tight cut decomposition of G consists of only cubic braces.*

2. Main Results

Let G be a matching covered graph. An edge e of G is called *solitary* if it belongs to exactly one perfect matching of G.

Conjecture 2.1. *Except for a finite number of special graphs, and a finite number of special infinite families of graphs, each cubic 2-edge connected graph has the property that each edge belongs to at least two perfect matchings.*

Theorem 2.2. *If G is a 2-edge connected simple cubic bipartite graph, then G does not have a solitary edge.*

Proof. Note that for any edge xy in a bipartite graph H, the graph $H' := H - x - y$ is bipartite. Suppose that $H = H[X, Y]$ and $H' = H'[X', Y']$. xy is an edge in H, so wlog assume that $x \in X$ and $y \in Y$, and that $X' = X \setminus \{x\}$ and $Y' = Y \setminus \{y\}$. Now suppose that H is simple and cubic. Then deg $(x) =$ deg $(y) = 3$. So, suppose that the edges incident to x in H other than xy are xw and xz, where $w, z \in Y$ $(w \neq z)$. Then in H', all vertices in Y' have degree 3, except w and z, which have degree 2. By a similar reasoning on y we can show that all vertices in X' have degree 3, except for two vertices, which have degree 2.

Now for the sake of contradiction assume that uv is a solitary edge of G. Then the graph $F := G - u - v$ has exactly one perfect matching.

So, F has the following properties:

1. F is simple, since G is simple.
2. By the above reasoning F is bipartite, and if $F = F[A, B]$, then all vertices in F have degree 3, except for two vertices in X and two vertices in Y, which have degree 2.
3. F has exactly one perfect matching.

This implies that F is a matchable simple bipartite graph with deg $(s) \geq 2$ for all $s \in V(F)$. So, by Corollary 3.2, F has at least $2! = 2$ distinct perfect matchings, which is a contradiction to the fact that F has exactly one perfect matching. Hence, we are done. $\square$

Theorem 2.3. *Every brace with more than 6 vertices is a simple graph.*

Let G be a cubic bipartite matching covered graph and let $C := \partial(X)$ be a tight cut of G. Define $G_1 := G/X$ and $G_2 := G/\overline{X}$. Then we have the following lemmas:

Lemma 2.4. *Let $e \in C$. If e is solitary in both G_1 and G_2, then e is solitary in G.*

Lemma 2.5. *If e is solitary in G, then e is solitary in both G_1 and G_2, wherever it exits.*

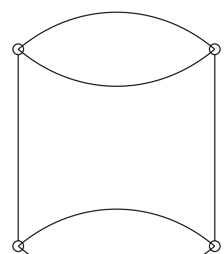
Lemma 2.6. *Let e be a solitary edge in G_1 and M be any perfect matching of G. Suppose that $e \in M$ and $M \cap C = e'$. If e' is solitary in G_2 then e is solitary in G.*

2.1. **Infinite Family.** Let $k \in \mathbb{Z}_{\geq 3}$ and $n \in \mathbb{Z}_{\geq 2}$ be arbitrary. Consider the even cycle C on $2n$ vertices; $C = v_1v_2v_3 \dots v_{2n}v_1$. Add $k-2$ edges between v_1 and v_2, v_3 and v_4, v_5 and $v_6, \dots,$ and v_{2n-1} and v_{2n}. Denote this resulting graph by $C_{k,n}$. Note that it is a k-regular matching covered graph on $2n$ vertices.

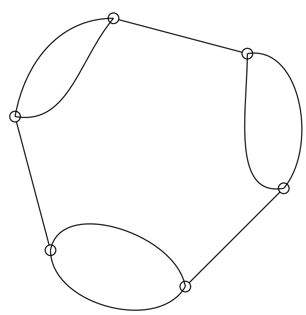
Now, for all $k \geq 3$, let

$$\mathcal{C}_k := \{C_{k,n} \mid n \in \mathbb{Z}_{\geq 2}\}.$$

For example, the following graphs belong to $\mathcal{C}_3$:



(a) $C_{3,2}$



(b) $C_{3,3}$

FIGURE 2.

Lemma 2.7. *Let $G \in$ infinite family. Then, if e is a single edge in G, then it is solitary in G. Otherwise, if e' is a one of the parallel edges in G, then e' is not solitary.*

2.2. **Operation of Splicing.** Let G_1 with a specified vertex u, and G_2 with a specified vertex v, be two disjoint graphs. Suppose that the degree of u in G_1 and the degree of v in G_2 are the same, and that π is a bijection between the set $\partial(u)$ of edges of G_1 incident with u, and the set $\partial(v)$ of edges of G_2 incident with v. We denote by $(G_1 \odot G_2)_{u,v,\pi}$ the graph obtained from the union of $G_1 - u$ and $G_2 - v$ by joining, for each edge e in $\partial(u)$, the end of e in $G_1 - u$ to the end of $\pi(e)$ in $G_2 - v$, and refer to it as the graph obtained by splicing G_1 at u with G_2 at v with respect to the bijection π.

Proposition 2.8. *The graph* $(G_1 \odot G_2)_{u,v,\pi}$ *obtained by splicing two matching covered graphs* G_1 *and* G_2 *is also matching covered.*

Let G_1 and G_2 be two arbitrary graphs in $\mathcal{C}_3$. Select any vertex u of G_1 and any vertex v of G_2. Note that $\deg(u) = 3 = \deg(v)$. Let the solitary edge incident with u and the solitary edge incident with v be denoted by e_3 and e'_3 respectively. Let the other two edges (parallel to each other) incident from u and v be denoted by e_1, e_2 and e'_1, e'_2 respectively. The number of distinct bijections π between $\partial(u)$ and $\partial(v)$ is $3! = 6$. Let π_1 be a bijection such that $\pi_1(e_i) = e'_i$ for all $i = 1, 2, 3$, and π_2 be a bijection such that $\pi_2(e_1) = e'_1$, $\pi_2(e_2) = e'_3$ and $\pi_2(e_3) = e'_2$. Suppose that the other four bijections are $\pi_3, \ldots, \pi_6$. Note that for all $i = 3, \ldots, 6$, the graph $(G_1 \odot G_2)_{u,v,\pi_i}$ is either isomorphic to $(G_1 \odot G_2)_{u,v,\pi_1}$ or to $(G_1 \odot G_2)_{u,v,\pi_2}$. So, from a structural point of view, to study all the possible splicings, it is enough for us to analyze the graphs $(G_1 \odot G_2)_{u,v,\pi_1}$ and $(G_1 \odot G_2)_{u,v,\pi_2}$. Moreover, by Proposition 2.8, $(G_1 \odot G_2)_{u,v,\pi_i}$ is matching covered for all $i = 1, \ldots, 6$. Figure 3(c) and 3(d) shows the graphs obtained by splicing G_1 at u and G_2 at v with respect to the bijections π_1 and π_2 respectively, where both G_1 and G_2 are isomorphic to $C_{3,2}$.

Theorem 2.9. *Let* G *be a cubic bipartite matching covered graph. Then,* G *has a solitary edge if and only if it belongs to* $\mathcal{C}_3$*.*

Proof. Suppose that G belongs to $\mathcal{C}_3$. Select any vertex u of G and consider $\partial(u)$, the set of edges of G incident with u. $\partial(u)$ consists of two edges parallel to each other and one more edge e. Note that e belongs to exactly one perfect matching of G, and hence is solitary in G. This completes the forward direction of the proof.

For the backward direction of the proof, we will proceed by induction on the number of vertices of G. Consider the base case of cubic bipartite Braces. By Theorem, 2.3, our claim has been proved in theorem 2.2. Hence, we look at cubic bipartite braces with 2 and 4 vertices. Note that, on 2 vertices the only possible cubic bipartite brace is the Theta graph which is same as $\mathcal{C}_{3,1}$. Also, the cubic bipartite braces on 4 vertices are isomorphic to $\mathcal{C}_{3,2}$. Hence, we have proved our claim for the base case.

Induction Hypothesis: If H has a solitary edge then it belongs $\mathcal{C}_3$ whenever $|V(H)| < |V(G)|$.

Suppose G is not a brace. Thus, G has a nontrivial tight cut, say $C = \partial(X)$. Let G_1, G_2 be the C-contractions of G. Using corollary 1.10, G_1, G_2 are cubic bipartite matching covered graphs.

Suppose G has a solitary edge. We will show that G belongs to the infinite family. Note that, $|V(G_i)| < |V(G)|$ for $i \in (0, 1)$. Hence, using induction hypothesis, $G_1, G_2 \in \mathcal{C}_\ni$.

Now, using 1.9, we have that $|C| = 3$. Note that there are two parallel edges say m_1, m_2 and one single edge say n in the cut C in G. Let the corresponding edges in G_1, G_2 be e_1, e_2, e_3 and e'_1, e'_2, e'_3

We have two types of splicings. $(G_1 \odot G_2)_{u,v,\pi_1}$ and $(G_1 \odot G_2)_{u,v,\pi_2}$.

Type 1 $(G_1 \odot G_2)_{u,v,\pi_1}$: Note that, there is exactly one solitary edge in C, that is the single edge n. Hence, using this splicing, parallel edges of G_1 are joined with respective parallel edges of G_2 and the single edge of G_1 with the single edge of G_2. Thus, observe that $(G_1 \odot G_2)_{u,v,\pi_1}$ gives $G \in \mathcal{C}_{3,x}$, where $x = |V(G_1)| + |V(G_2)| - 3$. Also, as e_3 is solitary in G_1 and e'_3 is solitary in G_2, the spliced edge n is also solitary in G using lemma 2.4. Hence, G has an solitary edge and $G \in \mathcal{C}_3$.

Type 2 $(G_1 \odot G_2)_{u,v,\pi_1}$:

Observe that we have three types of edges in G. Case 1: Suppose e is not solitary in one of G_1, G_2. If possible, let e be solitary in G. But, lemma 2.5 implies that e is solitary in both G_1, G_2, contradiction. Hence, e is not solitary in G.

Case 2: Let $e \in C$. Let e be the edge obtained by splicing $e_3, \pi(e_3)$. Note that, e_3 is solitary in G_1 then the corresponding edge $\pi(e_3) = e'_2$ is not solitary in G_2. Hence,by lemma 2.5 e is not solitary in G. Similarly, e_1, e_2 are not solitary in G_1. Thus, the corresponding spliced edges in G are not solitary. Hence, if $e \in C$, then e is not solitary in G.

Case 3: Suppose that e is solitary in G_1 such that $e \notin C$. Note that e is a single edge in G_1. Let $e \in M$, where M is a perfect matching in G_1. Let $M \cap C = e_1$, where $e_1 = u, v \in G_1$. Note that e_1 is a single edge in G_1. Otherwise e is not solitary in G_1. Thus, e is one of the parallel edges in G_2, due to the construction of the splicing. Thus, e is not solitary in G_2. Hence, using Lemma 2.4, e is not solitary in G.

Hence, by splicing of type 2, we get a graph G, such that none of the edges in G are solitary.

Thus, whenever G has a solitary edge, $G \in \mathcal{C}_3$. This proves our claim that, any cubic bipartite matching covered graph is solitary if an only if it belongs to $\mathcal{C}_3$.

$\square$

3. Appendix

Lemma 3.1 (Marshall Hall). *Let G be a simple bipartite graph with bipartition (A, B), and assume that each point in A has degree at least d. Then if G has at least one perfect matching, it has at least $d!$ perfect matchings.*

Corollary 3.2. *If G is a matchable simple bipartite graph with* $\deg(v) \geq d$ *for all $v \in V(G)$, then G has at least $d!$ distinct perfect matchings.*

References

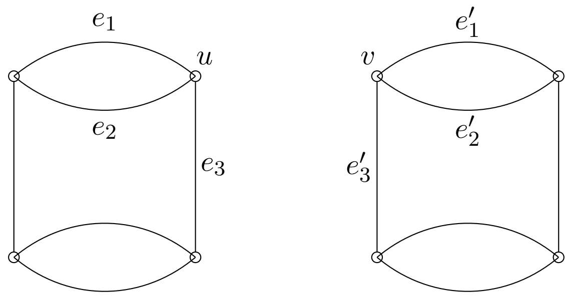
[1] M. Beck and S. Robins, *Computing the Continuous Discretely: Integer-Point Enumeration in Polyhedra*, Springer-Verlag, New York, 2007.

Department of Chemical Engineering, Sant Longowal Institute of Engineering and Technology, Sangrur, Punjab - 148106

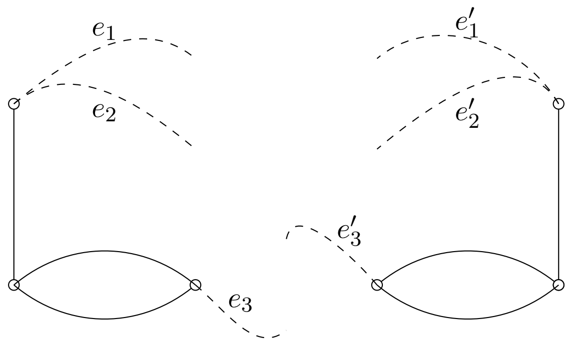
Email address: 1940108@sliet.ac.in, snktbiswas@gmail.com

Department of Mathematics, Fergusson College, Pune, Maharashtra - 411004

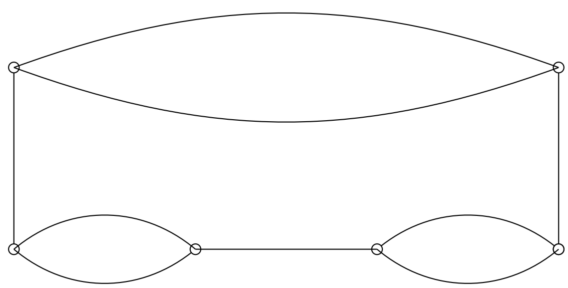
Email address: kalyani.gohokar2406@gmail.com



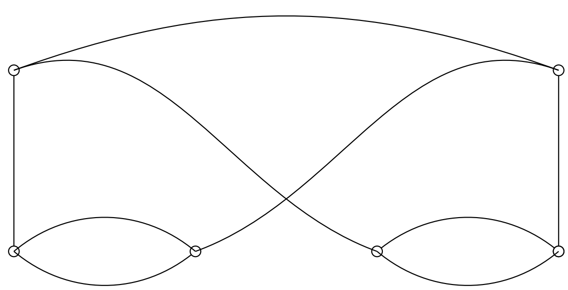
(a) G_1 and G_2, both isomorphic to $C_{3,2}$



(b) $G_1 - u$ and $G_2 - v$



(c) $(G_1 \odot G_2)_{u,v,\pi_1}$



(d) $(G_1 \odot G_2)_{u,v,\pi_2}$

FIGURE 3. Splicing of G_1 at u with G_2 at v