Packings and Coverings

S. Biswas and C. Quincosis

Department of Pure Mathematics,
University of Waterloo, Waterloo, ON, Canada

sanket.biswas@uwaterloo.ca
cquincos@uwaterloo.ca

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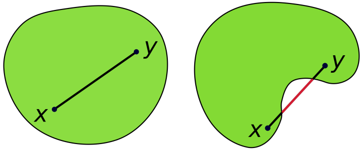

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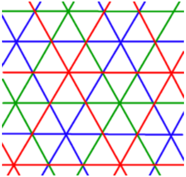

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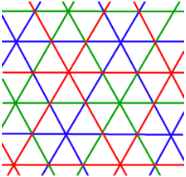

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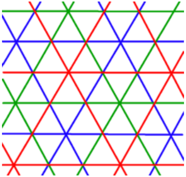

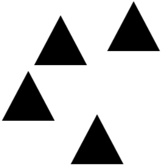

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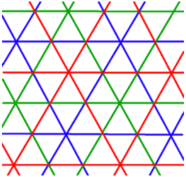

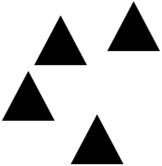

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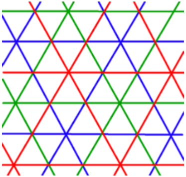

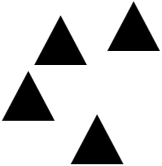

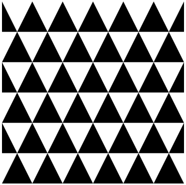

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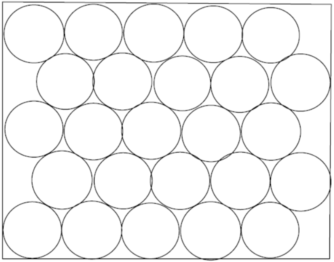

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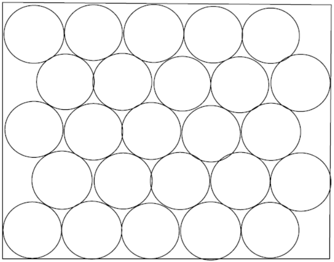

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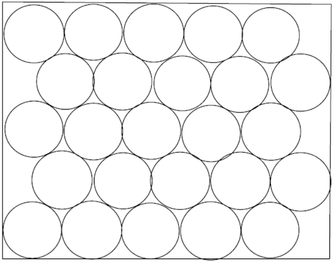

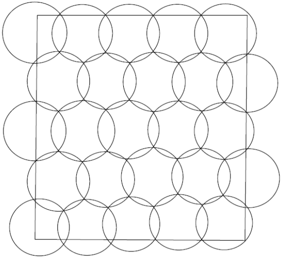

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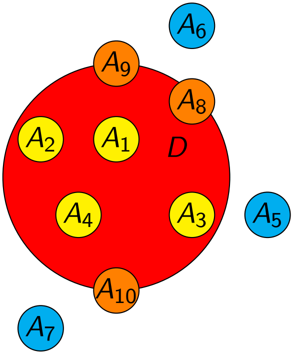

Example

$\mathcal{A} = \{A_1, \ldots, A_{10}\}$ and $D = B^2$,

$$d_{\mathrm{inn}}(\mathcal{A}|D) = \frac{V(A_1) + \cdots + V(A_4)}{V(D)}.$$

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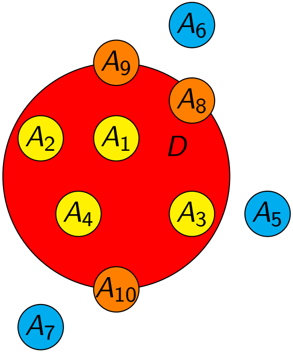

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$$d_{\text{out}}(\mathcal{A}|D) = d_{\text{inn}}(\mathcal{A}|D) + \frac{V(A_8) + V(A_9) + V(A_{10})}{V(D)},$$

and

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respectively, exists and are equal.
Moreover, $d(\mathcal{A}) = d_-(\mathcal{A}) = d_+(\mathcal{A})$.

Packing & Covering Density

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It is obvious that

$$\delta_L(K) \leq \delta_T(K) \leq \delta(K) \leq 1 \leq \theta(K) \leq \theta_T(K) \leq \theta_L(K).$$

Contents

Convex bodies $K \subset \mathbb{R}^2$ for which $\delta(K)$ is known

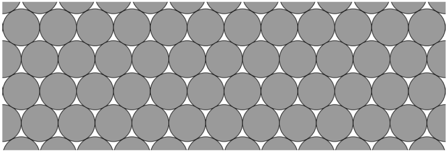

Figure: $\delta(B^2) = \pi/\sqrt{12}$ (A. Thue, 1910)

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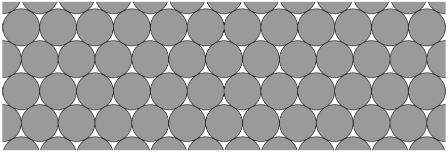

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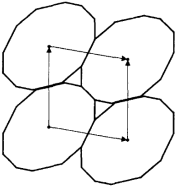

Figure: δ(cent. sym. n-gon) is solvable in $O(n)$ time (Mount & Silverman, 1990)

Convex bodies $K \subset \mathbb{R}^3$ for which $\delta(K)$ is known

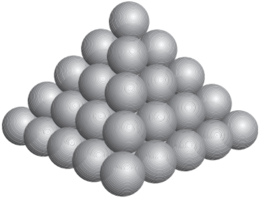

Figure: $\delta(B^3) = \pi/\sqrt{18}$ (T.C. Hales, 2005)

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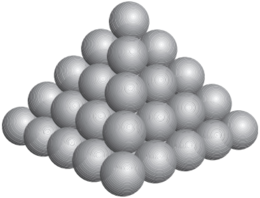

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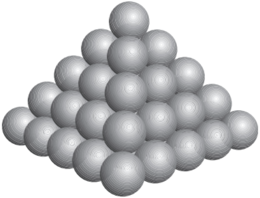

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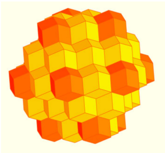

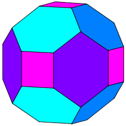

Figure: Densest packing with truncated rhombic dodecahedron (A. Bezdek, 1994)

Packing densities in higher dimensions

$$\delta(B^8) = \frac{\pi^4}{384}$$

(Cohn, Viazovska et al., 2017)

$$\delta(B^{24}) = \frac{\pi^{12}}{12!}$$

(M. Viazovska, 2017)

Convex bodies $K \subset \mathbb{R}^3$ for which $\delta_L(K)$ is known

CONVEX BODY K	$\delta_L(K)$	**AUTHOR(S)**
Tetrahedron	$18/49$	D.J. Hoylman (1970)
Octahedron	$18/19$	H. Minkowski (1911)
Dodecahedron	$(5+\sqrt{5})/8$	U. Betke and M. Henk (2000)
Icosahedron	$0.8363574\ldots$	U. Betke and M. Henk (2000)
Cuboctahedron	$45/49$	U. Betke and M. Henk (2000)
Icosidodecahedron	$(45+17\sqrt{5})/96$	U. Betke and M. Henk (2000)
Rhombic Cuboctahedron	$(16\sqrt{2}-20)/3$	U. Betke and M. Henk (2000)
Rhombic Icosidodecahedron	$0.8047084\ldots$	U. Betke and M. Henk (2000)
Truncated Cube	$9(5-3\sqrt{2})/7$	U. Betke and M. Henk (2000)
Truncated Dodecahedron	$(25+37\sqrt{5})/120$	U. Betke and M. Henk (2000)

Known values of $\delta_L(B^n)$ and $\theta_L(B^n)$

n	$\delta_L(B^n)$	$\theta_L(B^n)$
2	$\frac{\pi}{2\sqrt{3}}$	$\frac{2\pi}{3\sqrt{3}}$
3	$\frac{\pi}{\sqrt{18}}$	$\frac{5\sqrt{5}\pi}{24}$
4	$\frac{\pi^2}{16}$	$\frac{2\pi^2}{5\sqrt{5}}$
5	$\frac{\pi^2}{15\sqrt{2}}$	$\frac{245\sqrt{35}\pi^2}{3888\sqrt{3}}$
6	$\frac{\pi^3}{48\sqrt{3}}$	?
7	$\frac{\pi^3}{105}$	?
8	$\frac{\pi^4}{384}$	?
24	$\frac{\pi^{12}}{12!}$	?

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Thank You
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Juspaxar
Mamana
Kiitos
Maake
Terma Kasih
Grazie
Spasibo
Asante
Welalin
Chokrane
Mochchakkeram
Raibh Maith Agat
Ua Tsaug Rau Koj
Dankon
Matondo
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