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Some Notes on Regular Graphs as Applied to Distributed Index Design

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Abstract Some interesting properties of regular graphs that are relevant to distributed index design, are summarized in this paper. Although regular graphs have a long history, some of its properties relevant to design issues like lookup complexity, extensibility and resilience don't seem to have received enough attention. The purpose of this paper is to provide an easy introduction to regular graphs as well as present some interesting theoretical results pertinent to distributed index design. We also provide some conjectures of our own regarding regular graphs.

Keywords Regular Graphs · Distributed Hash Tables · Data-centric Networking · Distributed Index · Symmetry Property

1 Introduction

A distributed index is the core element of any distributed hash table (DHT), which in turn is the basic data structure of a data-centric network. In the recent past, data-centric networking has received enormous research interest due to its promise of separating network related concerns with application concerns. A data-centric network makes it much easier to build distributed applications, as the application designer no longer has to worry about network-related concerns like connection management, host addresses, etc. The network appears as a database to the application designer allowing applications to use database connections to access data that lie anywhere in the network. A good survey of DHTs as applied to both fixed and wireless networks is provided by Araújo and Rodrigues [1].

Design of a distributed index is a challenging task as it has to address several conflicting objectives. These include handling "churn" (or the frequent arrival and departures of nodes), frequent network topology changes, unreliable connections, bandwith bottlenecks, etc. In addition, there may be restrictions on storage and computing abilities of the participating nodes. Finally, the topology of the distributed index has to be such that the lookup complexity is optimized, or at least bounded to a reasonable maximum number of lookups per key.

The most common data structure for a distributed index is a graph whose vertices are distributed across all hosts in the distributed system. Index lookups can originate anywhere in the network and seek data elements that are located anywhere else in the network. The lookup complexity hence is bounded by the *diameter* of the index graph. The primary design objective of a distributed index is hence *diameter reduction* under constraints.

In this paper we concentrate on one specific challenge in distributed index design. This is the *symmetry* constraint, where each node is expected to take the same amount of bookkeeping cost in managing the distributed index.

Bookkeeping cost arises from keeping track of other nodes and their addresses. Symmetry is addressed by

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modeling the index in the form of a *regular graph.* A graph $G = (V, E)$ is called *r-regular* if

$\forall v \in V(G), degree(v) = r$

The term r is called the "regularity" of the graph G. If the graph is a directed graph, then r-regularity means that each node has r incoming and r outgoing edges.

The symmetry constraint in conjunction with other requirements like diameter reduction, resilience and churn poses several challenges. Hence it is worthwile to address regular graphs separately as applied to distributed index design.

That is the focus of this paper. The emphasis is on regular graphs and their topological properties, rather than other relevant issues like key distribution, etc. Although the paper may appear more like a graph theory paper, we believe it is relevant to an audience of researchers in databases and distributed systems. These graph theoretic results are motivated directly by the problem of distributed index design, and may provide significant theoretical underpinnings for distributed index design under the symmetry constraint.

Since we are not addressing any other characteristic of DHTs, we shall be using the terms "distributed index" and "DHT" interchangably in this paper. We shall use the term "node" to represent a vertex in the index graph and the term "host" to represent a machine participating in the distributed system.

2 The Regularity Problem in DHTs

Regular graphs have been used extensively in designing DHT topologies. Examples include Chord [26], CAN [22], Pastry [24] and Hypercup [25].

In Chord [26], an identifier space is formed as a logical circle to which both hosts and keys are mapped using a hash function. The Chord topology is a skip list, wherein each node connects to $\log n$ other nodes on the circle to achieve a diameter of $\log n$ and a symmetric degree distribution. Thus, the Chord topology is $\log n$-regular in its outdegree. If the number of nodes in the network is a power of 2, Chord is also indegree regular.

Koorde [12], which is based on Chord, describes an implementation of a network based on *de Bruijn* [28] graphs. Here, a $\log n$ diameter is achieved with a fixed degree of 2. The paper also describes an extension to a degree-m de Bruijn graph, using which a diameter of $\log_m n$ can be achieved. D2B [7] is another implementation of the de Bruijn graph that is similar to Koorde.

A de Bruijn Graph is a directed regular graph, where each node is mapped onto an identifier in the identifier space formed by all m-length strings of an alphabet of length b. Every node has exactly m outgoing edges. The m edges are drawn by right shifting each node identifier by 1 position, and adding each of the b symbols in the alphabet at the end. A de Bruijn graph guarantees a diameter of $\log_b(b^m)$. It is a regular directed graph where every node has the same indegree and outdegree.

De Bruijn graphs are claimed to be diameter optimal when retaining regularity [17]. However, Datta et. al [5] argue that while the topology of de Bruijn graphs are attractive, they are not well suited for DHT applications where key distributions are typically uneven.

The "Moore Bound" [3] defines an upper bound on the number of nodes that can be packed in a graph of fixed degree m and diameter d. In other words, the Moore Bound gives us the size of the largest regular graph of regularity m and diameter d. To estimate this, imagine a m-ary tree of depth d. So, the maximum number of nodes, N_{max} can be estimated as: $N_{max} = 1 + m + m^2 + m^3 + ... + m^d = \frac{(m^{d+1}-1)}{m-1}$.

As a consequence, the lower bound on the diameter of a graph with n nodes and a fixed degree m is estimated as: $D_{min} = \lceil \log_m(n(m-1)+1) \rceil - 1$.

A Moore Graph is an m-regular graph that has a diameter d, in which the number of nodes is equal to the Moore Bound. Though Moore graphs are optimal structures, it is only possible to construct trivial Moore graphs.

Loguinov [17] provides a detailed graph-theoretic analysis of peer-to-peer networks, with respect to routing distances and resilience to faults. The paper argues that de Bruijn graphs offer the optimal diameter topology among the class of practically useful graphs. They offer a low diameter and they come close to satisfying the Moore Bound. Being directed graphs, de Bruijn graphs suit very well for applications like DHTs. However, de Bruijn graphs are defined only for specific networks where the number of nodes is a power of 2 (or a power of the arity of the connection network). It can also be shown that an undirected de Bruijn graph is not the best possible topology in terms of diameter optimality. Therefore, the symmetry problem in DHT design cannot be completely addressed with de Bruijn graphs.

A hypercube graph is a regular graph of 2^m nodes, represented by all m-length binary strings. Each node connects to all other nodes that are at a *Hamming Distance* of 1, forming a m-regular graph.

In Hypercup [25], a hypercube graph is constructed in a distributed manner by assuming that each node in an evolving hypercube takes more than one position in the hypercube. That is, the topology of the next dimensional hypercube is implicitly present in the present hypercube, with some of the hosts hosting "virtual" nodes to complete the hypercube graph. Similarly, when hosts depart, some of the existing hosts take their position along with their own.

Viceroy [18] is an implementation of an approximate butterfly network (cf. [10]). Nodes are arranged in $\log n$ levels, with the nodes at each level forming a ring topology with each node having an outgoing link to a successor and a predecessor. Apart from the "neighbors" on the ring, each node has long range outgoing links to five other nodes across the $\log n$ levels. These levels and nodes are chosen by a randomized process. Viceroy claims to achieve a $\log n$ diameter with a fixed degree of 7. Ulysses [14] is another implementation of the butterfly, though with its $\log n$ neighbours, it is not a fixed degree graph.

A distributed trie based approach is proposed in [19]. It is based on Plaxton et al.'s [21] prefix based routing, wherein a k-ary prefix tree is maintained in a distributed manner. The maximum degree of a node is $k+1$ and the diameter is $2\lceil \log_k n \rceil$. Content Addressable Network (CAN) by Ratnasamy et. al [22] forms an identifier space over n nodes with an approximation of a m-dimensional torus. CAN has a fixed degree of $2m$ and provides a diameter of $\frac{mn^{\frac{1}{m}}}{2}$.

Law et al. [16] take up the problem of distributed construction of random regular graphs that have m Hamiltonian circuits. The graph is $2m$-regular with each node having a degree of $2m$, and m Hamiltonian circuits passing though each node. With a high probability these graphs will achieve a diameter of $\log_m n$. Though the approach yields a multi-graph, this graph can be viewed as a multiple layered r-regular graph with the different layers acting as different regular graphs for appropriate implementation levels.

Another interesting class of regular graphs suitable for distributed indexing are von Neumann grids (cf. [8]). Von Neumann grids are grids in which every node is part of two circles, one "horizontal" and another "vertical." Thus it is a grid in which every row and every column is a circle. As a result, every node has a degree of 4 and the grid is 4-regular. The grid is also 4-connected. If the grid is $m \times m$ big, the diameter of the grid is the maximum manhattan distance, which is m ($\frac{m}{2}$ in the horizontal direction + $\frac{m}{2}$ in the vertical direction).

Regular graphs were also optimal topologies in the face of high resilience requirements, resulting from genetic algorithms based optimization in our larger project called topology breeding [20].

3 Regular graphs and DHTs

Regular graphs have interesting properties that are important for DHT design. We first begin by examining properties of *undirected* regular graphs even though several DHT index structures are directed graphs. An undirected edge can represent a pair of directed edges in either directions. In applications like DHTs, the extra cost of hosting a back link is negligible, compared to the analytical simplicity of regular undirected graphs. We will also be briefly addressing regular directed graphs later on in the paper.

We see that a 0-regular graph is a set of disconnected nodes; a 1-regular graph is a set of disconnected edges and a 2-regular graph is one big cycle encompassing all the nodes, or a set of one or more disconnected cycles.

An undirected graph G is said to be *connected* if there exists a path from any node to any other node in G. When dealing with DHT design, we always require regular *connected* graphs. A regular graph with disconnected components may indicate a problem – like network partitioning for instance.

In the following sections, whenever we refer to regular graphs, we shall be refering to regular connected graphs, unless specified otherwise.

We begin with the following observations about regular connected graphs:

Proposition 1 *For any given set of nodes $n > 2$, the smallest possible regularity of a connected graph over all the n nodes is 2.*

In other words, for any connected graph that is bigger than a single edge, the smallest regularity possible is 2. We shall hence not be considering 0 and 1-regular graphs further on.

Lemma 1 *It is always possible to build a 2-regular connected graph for any given set of nodes $n > 2$.*

Proof *Such a 2-regular graph is in the form of a single hamiltonian circuit encompassing all the n nodes in one big cycle.*

During the course of our work, we encountered some conjectures like the following:

1. Every r-regular graph is also $(r-1)$-regular. That is, it is possible to obtain an $(r-1)$-regular graph from an r-regular graph by removing one or more edges.
2. Since a connected 2-regular graph is a hamiltonian circuit, every connected r-regular graph with $r > 2$ has a hamiltonian circuit.

If the first conjecture were to be true, it has applications in handling failures in distributed systems. If one or more connections fail, and it is not possible to construct an r-regular graph with the set of nodes, we should always be able to build an $r-1$ regular graph and retain the symmetry property.

If the second conjecture were to be true, we can guarantee that an r-regular graph with n nodes with $r > 2$ has a diameter less than or equal to $\frac{n}{2}$. Also, a hamiltonian undirected graph is at least 2-connected. That is, there are at least two edge-independent paths between any pair of nodes in the graph.

However, we shall refute both these conjectures and also relate it with some old results concerning hamiltonian circuits and cubic graphs (or 3-regular graphs). We start with the following observations:

Lemma 2 *The smallest number of nodes required to build an r-regular connected graph is* $(r+1)$.

Proof *The fully connected* $(r+1)$ *clique where each node connects the other* r *nodes directly is the smallest r-regular connected graph possible.*

Lemma 3 *It is not possible to build an r-regular graph with n nodes, where both r and n are odd numbers. This is true even if we allow the graph to have disconnected components.*

Proof *This follows from one of the basic results in graph theory that the total degree in a graph is even. When both r and n are odd, the total degree becomes odd, which makes such a graph impossible.*

If the graph has k disconnected components, there should be at least one component (or an odd number of components) that is made up of an odd number of nodes, since the total number of nodes should add up to the odd number n. Such components cannot exist if the regularity r is an odd number.

Using Lemma 3 we can refute the first assertion that every r-regular graph is also $(r-1)$-regular. Suppose r were to be an even number and we are considering a graph having an odd number of nodes, clearly no $(r-1)$-regular graph can be built by removing some edges from the original graph.

However, this poses the question whether in the first place, can we could have indeed built that r-regular graphs with n nodes, where r is even and n is odd. Lemma 2 shows that it is possible, where an example is an $(r+1)$-clique. Later on, in Theorem 1 we show that it is possible to build graphs with even regularity r for any n nodes, as long as $n \geq r+1$.

Figure 1 refutes the second assertion. It shows a 3-regular graph that does not have a hamiltonian circuit.

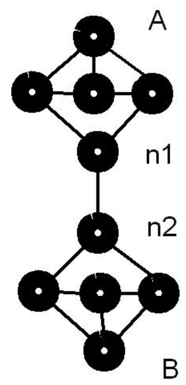

Fig. 1 A 3-regular graph that has no hamiltonian circuit nor having 3 edge independent paths between nodes

We can see that the edge (n_1, n_2) is a bridge that needs to be traversed twice in order to build a circuit encompassing the regions A and B.

The debate about the existence of hamiltonian circuits has an interesting history [27]. In 1886, P.G. Tait made a conjecture about polyhedra, that was eventually extended to cubic graphs or 3-regular graphs. The conjecture was that every cubic graph without a bridge has a hamiltonian circuit. This conjecture, if true, would have implied the four-colour theorem for planar graphs[1]. However, this conjecture was disproved in 1946 by W.T. Tutte with a counter-example graph having 46 vertices.

We have also encountered an alternate definition of regular graphs in the literature (an example is Lamport et al. [15] in their seminal paper on byzantine consensus in distributed systems). Here, an r-regular graph is defined as a graph that is r-connected – that is, having r edge independent paths between any pair of nodes.

This poses the question whether r-regularity implies r-connectedness. However, Figure 1 is again a counter example. The graph is 3-regular, but it is not possible to construct three edge independent paths from any node in component A to any node in component B. Every set of paths between these components has the bridge (n_1, n_2) as the common edge.

We will have more to say about the relation between regularity, connectivity and Hamiltonicity in Section 5.

4 Extending Regular Graphs

In this section, we address the question whether we can build a regular graph given a set of nodes and a regularity number. We also address the question of how

[1] From the Wikipedia page: http://en.wikipedia.org/wiki/Tait% 27s_conjecture

to extend a given regular graph to accommodate host arrivals, without affecting the regularity.

Theorem 1 *If r is even, it is always possible to build an r-regular graph over n nodes, where $n \geq r+1$.*

Proof *We shall prove this using induction. From Lemma 2 we see that the minimum number of nodes required for building an r-regular graph is $r+1$. This regular graph is in the form of an $r+1$ clique.*

Now consider that we have an r-regular graph over k nodes where $k \geq r+1$. We now show a technique for building an r-regular graph over $k+1$ nodes.

Let the graph be denoted as $G_k = (V, E)$ where $|V| = k$. In order to build G_{k+1}, bring a new node v' into V. We can think of this new node as having r "holes" that need to be filled by connecting r edges.

Take an arbitrary edge from G_k and disconnect it from one of its end nodes and connect it to v'. Now G_k has 1 hole and v' has $r-1$ holes.

Continue this process for a total of $\frac{r}{2}$ times so that there are $\frac{r}{2}$ holes each in G_k and v'. Now add $\frac{r}{2}$ edges from v' to each of the holes in G_k. We have hence obtained an r-regular graph with $k+1$ nodes.

Note that the $\frac{r}{2}$ holes in G_k need to be on distinct nodes. Also, they need to be on nodes from which no edge is joining v' already. Otherwise, there will be multiple edges between v' and one or more nodes in G_k, making it into a multi-graph.

To ensure that such a situation can be avoided, we need to have separate sets of $\frac{r}{2}$ nodes in G_k from which there are edges to v' and which have holes. This means that if G_k has at least r nodes, then the graph can be extended by one node using the above procedure.

This condition is trivially satisfied because the smallest r-regular graph has $r+1$ nodes. Hence the proof.

Hence, when r is even, there are no constraints on the number of nodes, except for the lower bound of $r+1$. This has implications in DHT design where it is quite common to find questions of being able to extend DHT in the face of node arrivals.

Figure 2 illustrates Theorem 1 being applied on a 4-regular graph with 5 nodes. It is extended by one node by rearranging two edges and adding two more edges.

For odd regularities, the technique used in Theorem 1 does not work. There will always be a difference of at least 1 between the number of holes in v' and the graph G_k.

For this, we can extend Theorem 1 for odd regularities as shown below in Theorem 2.

Theorem 2 *Given an r-regular graph with $n \geq r+1$ nodes where r is odd, there exists an r-regular graph with $n+2$ nodes.*

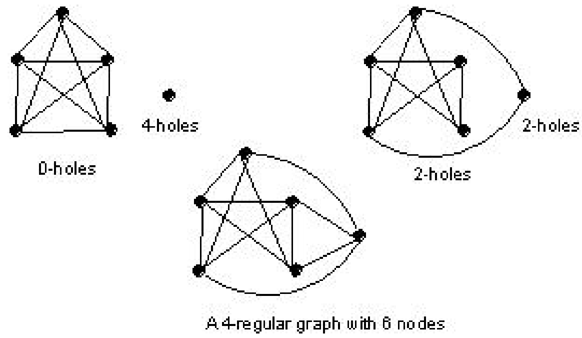

Fig. 2 Extending a 5-node 4-regular graph by one node

Proof *Given an r-regular graph $G_k = (V, E)$, where r is odd and $|V| = k$, we show how to extend it by adding two nodes v' and v'' respectively to V.*

Consider both v' and v'' to have r holes respectively. Take r arbitrary edges from G_k and disconnect them from one of their nodes and connect them to v'. There are now r holes each in G_k and v''. Introduce r new edges to connect v'' to the holes in G_k.

If $V_i \subseteq V$ is the set of all nodes in G_k that have incoming edges to v' and $V_h \subseteq V$ is the set of all nodes in G_k that have holes, we need to ensure that $|V_i| = |V_h| = r$ in order to prevent the formation of multi-graphs. Note that $V_i \cap V_h$ need not be null. Unlike in the case of Theorem 1 new edges are added from a different node v'', rather than v' where the edges from G_k were attached.

Hence, the minimum number of nodes required in G_k is still r, which is trivially satisfied.

Note that Theorem 2 represents the *minimal extension* we can make to a graph with odd regularity. If an r-regular graph with n nodes has to exist where r is odd, then n should be even and can only be extended in steps of 2.

While extending a regular graph in the minimal possible fashion is fairly straightforward, contracting a regular graph on the departure of a node (for even regular graphs) and the departure of a pair of nodes (for odd regular graphs) is amazingly very complex.

Given an r-regular graph over n where r is even and $n > r+1$, it is clear that there exists an r-regular graph topology even after losing one node. Similarly, for an r-regular graph where r is odd and the number of nodes $n > r+1$ holds, it is clear that an odd regular topology exists with two lesser nodes.

However, obtaining the smaller graph with the minimum amount of changes to the network seems to be an amazingly complex problem. As of now, while we can prove the existence or non-existence of a regular graph in the face of node failures, we don't have efficient algo-

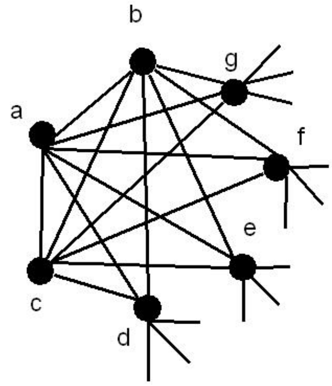

Fig. 3 Deleting node 'a' creates holes on 'b' and 'c', which cannot be paired with any other node containing holes

rithms to compute the same. Below, we briefly explain the challenges in obtaining this algorithm.

Suppose we consider the simpler of the two problems – namely, even regular graphs that can afford to lose a node. That is, an r regular graph over n nodes where r is even and $n > r + 1$. When a node is removed from such a graph, it creates r holes in the graph. Since r is even, the holes can be paired by adding edges.

However, the complication comes from the following: given a pair of nodes (u, v) that have holes in them, an edge can be added across u and v iff there didn't already exist an edge across them. This requires us to search different sequences of pairings to find a solution. Given r holes, there are a total of $\frac{r!}{2}$ permutations to search. However, even this does not guarantee a solution.

In contrast to the problem of extending regular graphs, we do not have the freedom to choose the location of the holes when a node is removed. This can lead to a situation where holes may appear on nodes which cannot be paired with any other nodes.

Figure 3 shows one such situation in a 6-regular graph fragment. Deletion of node a creates holes in nodes b, c, d, e, f and g. However, no edges can be added from either node b or c to any of the other nodes since they are already connected to the rest of the nodes.

Hence, enumerating the $\frac{r!}{2}$ permutations need not guarantee a solution. For this reason, we have not addressed node deletions in this paper. In a similar vein, we will also not be addressing node cutsets when we address connectivity of regular graphs in Section 5.

Extending regular graphs by embeddings: While the minimal extension theorems show theoretical underpinnings for extending regular graphs, in practice it is often desirable to build regular graphs much faster than extending them by the minimal extension possible.

If there are a large number of nodes over which a DHT has to be constructed, it is often desirable to start the DHT construction process parallely involving several subsets of nodes. These subsets can then merge into one another to finally result in a single DHT over the entire network. An example of this is GHT [23], a geographic hash table implemented over wireless sensor networks, where locality plays a major role in the DHT performance. It is easier for local nodes to form networks amongst themselves and then merge into a global network.

Similarly, merging is also necessary when a network has to repair itself from a partitioning.

Here we address merging of regular graphs without losing the regularity property. It is addressed in the form of graph "embeddings."

Theorem 3 *Given any r-regular graph with n nodes and $r \geq 2$, there exists an r-regular graph of $n + r - 1$ nodes.*

Proof *We prove this by embedding an $(r-2)$-regular clique into an r regular graph. Note that an $(r-2)$-regular clique has $r-1$ nodes and is fully connected. So the combined graph would have $n + r - 1$ nodes.*

Let G^r be the given graph and C^{r-2} be the $(r-2)$-regular clique to be embedded into G^r. In order to perform the embedding, take each node v_i of C^{r-2} in sequence. For each such v_i pick an arbitrary edge from G^r and embed v_i in the middle of the edge. We now note that v_i, which already had degree $r-2$, now has degree r after inserting it in the middle of an existing edge. Once all nodes from C^{r-2} are embedded, we get one combined graph with regularity r.

However, we need to address a caveat here. Suppose there is an edge $(u, v) \in G^r$. This edge can take embeddings from nodes $v_i, v_j \in C^{r-2}$ iff (v_i, v_j) is not an edge in C^{r-2}. Failing which, we would have multiple edges between v_i and v_j in the combined graph.

In order to ensure that the formation of multi-graphs can always be prevented, G^r needs to have at least as many edges as the number of nodes in C^{r-2}. Since C^{r-2} is an $(r-2)$-regular clique, it has $r-1$ nodes. If the number of nodes in G^r is n, then it has $\frac{nr}{2}$ edges.

Since n has a lower bound of $(r+1)$ (Lemma 2), the inequality $\frac{nr}{2} > r - 1$ is satisfied trivially.

We can extend Theorem 3 by allowing the embedding of any $(r-2)$-regular graph (not necessarily a clique) into an r-regular graph as long as the number of nodes in the graph being embedded is no more than the number of edges in the graph into which the embedding is done.

It is possible to embed graphs with a larger number of nodes than that specified above, but the maximum node constraint is a safe upper bound below which, embedding is always possible.

5 Connectivity, Hamiltonicity and Graph Powers

The *edge-connectivity* of a graph is the smallest number of edges (or the edge cutset) whose removal would partition the graph. We can note that an edge cutset partitions the graph into exactly 2 components. This is in contrast to a node cutset whose removal can partition the graph into any number of components.

Edge connectivity has direct correspondence with the resilience of a DHT in the face of connection failures. In the following sections, when we refer to the connectivity of a graph, we are refering to edge-connectivity only. We shall not be addressing the counter part of node-connectivity.

Menger's theorem [29] states that the connectivity of a graph is equal to the maximum number of pairwise independent paths between any two nodes in the graph. Also as we had noted earlier, the presence of a hamiltonian circuit guarantees connectivity of at least 2. In this section we discuss some observations relating regularity, connectivity and hamiltonicity of graphs.

5.1 Regularity and Connectivity

We observed earlier that a regularity of r need not mean r-connectivity. We can find graphs that are r-regular but k-connected where $k < r$. Based on this, we have the following observations to make.

Proposition 2 *The maximum connectivity of an r-regular graph is r.*

This follows from the following two facts: (a). the maximum connectivity of a graph is equal to the minimum degree in the graph [9], and (b). in a regular graph all nodes have the same degree equal to the regularity; hence the minimum degree is the regularity.

Proposition 3 *A connected 2-regular graph is exactly 2-connected.*

This follows from the fact that all connected 2-regular graphs are hamiltonian circles and circles are 2-connected.

Theorem 4 *Given an r-regular, k-connected graph ($k \leq r$) over n nodes with the following constraints: r is even and $n \geq 2r$; it is always possible to extend this graph by any number of nodes without altering the regularity r or edge connectivity k.*

Proof *A k-connected, r-regular graph $G^{k,r}$ is one that is r-regular and there exists at least one set of k edges, whose removal partitions the graph into two disconnected components.*

Let the graph be represented as two components C_1 and C_2 connected with k bridges across them. Since r is even, from the extension theorem (Theorem 1) we can extend either C_1 or C_2 arbitrarily without adding any edges across C_1 and C_2. This keeps the graph r-regular and k-connected after extension.

However, for the extension theorem to work for any component C_i, we see that C_i should have at least r nodes. This brings us to the constraint that the number of nodes $n \geq 2r$. Given this, at least one of C_1 or C_2 is guaranteed to have at least r nodes and can be arbitrarily extended.

Since the smallest number of nodes n to create an r regular graph is only $r + 1$, it is interesting to explore whether we can extend k-connected, r-regular graphs having $n < 2r$ nodes. As of now, we do not have any proof or refutation for this conjecture.

The significance of the above theorem for DHTs is as follows. Often, it is cheaper to build r-regular graphs with a connectivity less than r. This is especially true when the distributed system is modeled as a metric space and there is a distance function or a cost associated with each edge that is proportional to the (physical or logical) distance between the incident nodes. In such situations, it is easier to maintain r-regularity using short-range connections, rather than connecting nodes in far away corners of the network. The connectivity k, which even though is less than r, may be the minimal connectivity guarantees that the DHT design offers for such a network.

The extension theorem for k-connected regular graphs is significant in this respect, where connectivity can be maintained (or at least is not compromised) when extending the graph.

The extension theorem for k-connected graphs can be extended to odd regularity in an analogous fashion as follows.

Theorem 5 *Given an r-regular, k-connected graph, where r is odd and $n \geq 2r$, there exists an r-regular, k-connected graph with $n + 2$ nodes.*

Proof *The proof is similar to the one above, except that in this case we consider the extension theorem for odd regularity (Theorem 2).*

Both the above extension theorems require the number of nodes n to be at least twice the regularity r. This value is much higher than the minimum required value of $r + 1$ for forming an r-regular graph.

We see that the smallest connected r-regular graph, which is a clique of $r + 1$ nodes is exactly r-connected. If the connectivity has to decrease, it means that there has to be more nodes added to this graph increasing the value of n.

This brings us to the conjecture that given a value of r there is a maximum value of n below which a regular graph of r-regularity has to be at least k-connected. It would be interesting to analytically find this threshold. As of now we do not have any proof or refutation for this conjecture. Imase et al. [11] have developed some interesting results in this connection by studying regular *directed graphs*. They derive relations linking the number of nodes n, regularity r, diameter d, and edge connectivity k. They show, for example, that the edge connectivity k is equal to r in an r-regular graph if $n \geq r^{d-1} + d^2 - 2$. They also develop similar results for connectivity less than r given a diameter d.

A side effect of Theorems 4 and 5 is also a guarantee that we can always build and extend r-regular graphs that are r-connected, whenever r is even. This is explained by the following theorem.

Theorem 6 *Starting from the r-regular clique, which is r-connected, it is always possible to build r-regular, r-connected graphs over any n, when r is even.*

Proof *Note that, this theorem does not trivially entail from Theorem 4 for k-connected graphs due to the constraint on the number of nodes $n > 2r$. An r-regular clique has only $r + 1$ nodes.*

We prove the theorem using the extension theorem for even regular graphs as follows. Consider the way extension is done. When a new node needs to be added to the graph, first, $\frac{r}{2}$ edges in the existing graph are "broken". To guarantee extensibility, these edges are all on distinct pairs of nodes.

Therefore, each of the $\frac{r}{2}$ pairs of nodes now have one path less between them. As a result, the connectivity of the graph reduces to $r - 1$. Next, the r dangling ends of the broken edges are connected to the new node to obtain the extended graph of regularity r. Now, each of the $\frac{r}{2}$ pairs of nodes have a new path between them through the newly added node, thus restoring their pairwise r edge independence. Also, the newly added node is connected to r distinct nodes in the graph, each of which has r edge independent paths to all nodes. Thus, the connectivity of the graph remains r. Hence the result follows.

Note that in this case of r-connectvity, there is no constraint on the number of nodes n to be at least $2r$.

We can extend this result to odd regular graphs analogously.

Theorem 7 *Starting from the r-regular, $r + 1$ node clique, which is r-connected (where r is odd), it is always possible to build r-regular, r-connected graphs over any $n + 2$ using the extension theorem.*

Proof *The proof is similar to the above proof, except that we consider the extension lemma with odd regularity.*

Hence, when connectivity is r, extension without compromising on the connectivity, is always possible. However, for regular graphs where the connectivity is less than r, there is a constraint on the minimum number of nodes in order to guarantee extensibility.

Connectivity of regular graphs also has important applications in consensus problems in distributed systems, where nodes are allowed to fail in a byzantine fashion (failure due to arbitrariness in behavior) [15].

5.2 Regularity and Hamiltonicity

We have similar observations with respect to regularity and hamiltonicity and the absence of it.

In an undirected graph of n nodes, the presence of a hamiltonian circuit bounds the diameter of the graph to $\frac{n}{2}$. It also guarantees 2-connectivity.

For directed graphs, a hamiltonian circuit is the smallest regular graph (with one directed edge per node) that keeps the graph strongly connected. Hamiltonicity hence plays an important role in DHT design.

An interesting family of regular graphs that are hamiltonian are *Kneser graphs* [2]. The nodes of a Kneser graph $K(n, k)$ represent the set of all k-element subsets of a set of n elements. Two nodes are connected only if they represent disjoint subsets. We can see that $K(n, k)$ has $\binom{n}{k}$ nodes and is a regular graph, where the regularity is $\binom{n-k}{k}$. Kneser graphs are shown to be hamiltonian when $n \geq 3k$ [4].

Kneser graphs seem to have interesting underpinnings for key distributions in distributed hash tables. However, we have not found any approaches using Kneser graphs for DHTs.

In any case, hamiltonicity in itself has interesting properties for DHT design. Given a set of n nodes and a regularity number r, it is quite natural to expect the question whether is it possible to construct a hamiltonian r-regular graph over these n nodes.

To answer this, we start with the following observation.

Proposition 4 *The smallest r-regular graph, i.e. a clique of $n = r + 1$ nodes is hamiltonian.*

Further, we can state the following for even regular graphs.

Theorem 8 *It is always possible to build a hamiltonian r-regular graph over n nodes, where r is even and $n \geq r + 1$.*

Proof *We prove this by induction. We start with the smallest r regular graph – a clique of $r+1$ nodes which is trivially hamiltonian.*

We now consider a graph of n nodes that is hamiltonian and show how to extend it by one node and retain the hamiltonicity property. Let our graph be called G, and let $h = v_1, v_2, \dots v_n, v_1$ represent a hamiltonian cycle encompassing all the nodes once and ending up at the starting node without repeating an edge.

Now extend G using the extension theorem for even graphs (Theorem 1) as follows. When a new node v' is to be added into G, break at least one edge (v_i, v_j) that was on the hamiltonian cycle h and connect v_i to v'. Now v_j is guaranteed to have a hole resulting from breaking the edge and a new edge is guaranteed between v' and v_j.

We have hence successfully embedded an extra node into the hamiltonian cycle and retained the hamiltonicity property the new graph.

In contrast to other results before, the technique shown in Theorem 8 **cannot** be extended analogously to odd regular graphs. In Theorem 2, an odd regular graph was extended by two nodes v' and v''. One of the nodes received connections from the graph and the other node added extra connections to the holes in the graph. So there is no guarantee of retaining the hamiltonicity property using the extension technique of Theorem 2.

We can however, prove a slightly weaker result for odd regular graphs using graph embeddings.

Theorem 9 *Given a hamiltonian r-regular graph over n nodes where r is odd, there exists a hamiltonian r-regular graph over $n + r - 1$ nodes.*

Proof *Consider a hamiltonian circuit in the given graph G denoted by $h = v_1, v_2, \dots, v_n, v_1$. Using the clique embedding theorem (Theorem 3), we now extend this by embedding an $r-1$ clique, which is the smallest $(r-2)$-regular graph.*

In order to retain the hamiltonicity property of this embedding, we need to ensure that every node in the clique is embedded into a unique edge in h. The existence of such an edge is guaranteed as the smallest length of h is $r+1$ edges and the clique to be embedded has only $r-1$ nodes.

Hence for any r-regular graph where r is odd, we can start with the $r+1$ clique which is hamiltonian and keep extending the graph in steps of $r-1$ to retain the hamiltonicity property.

There is also a theorem by Nash-Williams stating that every r-regular graph up to $n = 2r+1$ nodes are hamiltonian[2]. However, we have not been able to locate this theorem ourselves.

While hamiltonicity is a desirable property in DHT design, there are cases where *not* having a hamiltonian circuit is desirable. An example is of hybrid networks that have mix of fixed and mobile components. For such networks it is desirable to not have a single overarching data structure over the entire network. It may be desirable to model the DHT for hybrid networks as a set of distinct components connected by bridges, with an appropriate key distribution mechanism.

Changes to a DHT structure in the face of churn is usually expensive. In hybrid networks, it is desirable to isolate the mobile part of the network from the fixed part. All these may require to be performed in a way that does not affect the symmetry property of the DHT. In such cases the absence of hamiltonicity becomes an important property.

For addressing such issues, we begin with the following observation.

Proposition 5 *It is not possible to have a connected 2-regular graph that is non-hamiltonian.*

In a 2-regular graph, all nodes have a degree 2. Unless there are an infinite number of nodes, it is not possible to have anything other than a logical cycle encompassing all the nodes, to build a connected 2-regular graph.

Hence if a network has hybrid components, and it is required to build a symmetric DHT where changes in one part of the network do not affect other parts, the DHT needs to have a regularity more than 2.

Corollary 1 *Given an r-regular, non-hamiltonian graph with n nodes, where r is even and $n \geq 2r$, it is always possible to extend this graph over any number of nodes and retain the regularity and non-hamiltonian properties.*

Proof *The proof for this can be seen as a corollary to Theorem 4.*

In a non-hamiltonian graph, there exists at least one edge e' that needs to be traversed twice before completing a circuit that encompasses all the nodes in the graph.

Even though the removal of this edge does not paritition the graph, we can consider this edge to divide the graph into two parts C_1 and C_2, where C_1 is the part that was traversed before the first traversal of e' combined with the part traversed after the second traversal; and C_2 is the part traversed in between the first traversal and the second.

[2] From the Wikipedia page: http://en.wikipedia.org/wiki/Regular_graph. Last accessed on 16 July 2008

We can readily see that C_2 is a connected component and we can also show that C_1 is a connected component since it has at least one node of the bridge in common between the first and third traversals.

Once we show that e$'$ logically divides the graph into two, with at least one part containing at least r nodes, we can use Theorem 4 to extend the graph.

We can analogously extend this to graphs with odd regularity.

Corollary 2 *Given an r-regular, non-hamiltonian graph over n nodes, where r is odd and $n \geq 2r$, there exists an r-regular, non-hamiltonian graph with $n+2$ nodes.*

5.3 Regularity and Graph Powers

The p^{th} power of a graph G is the graph that results by adding edges between nodes that are separated by a maximum path length p in G while maintaining the same number of nodes.

Computing power graphs of a given graph have important relevance to DHT design. A power graph reduces the diameter of the original graph by adding long distance edges across nodes. The extra edges that are added also increases the resilience of the DHT against failures. The extraneous edges however add to the infrastructure and bookkeeping costs and may affect the symmetry property of the DHT.

We start addressing graph powers with the following observation.

Lemma 4 *Given a non-complete graph G (that is, a graph that is not a clique) of diameter d, the p^{th} power G^p of G has diameter $\lceil \frac{d}{p} \rceil$.*

Proof *This follows from the fact that every pair of nodes on the diameter of G which were separated by path length p are separated by pathlength 1 in G^p. Thus the diameter of G^p is $\lceil \frac{d}{p} \rceil$.*

Specifically, raising a graph G to its d^{th} power results in a complete graph. In addition, there are also interesting results regarding the hamiltonian property of graph powers. The following are significant [6].

– The square of a 2-connected graph is hamiltonian.
– The cube of a connected graph is hamiltonian.

For regular graphs, the power graph alters the original regularity due to the extra edges that are added. In fact, the power graph of a regular graph need not be regular at all. To see this, consider the graph shown in Figure 4. This is a 3-regular graph in which some of the nodes are labeled. We can see that between nodes

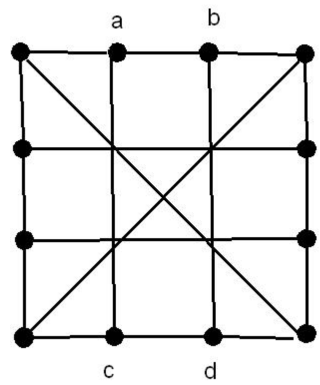

Fig. 4 Raising this graph to the power 2 introduces two edges between nodes 'a' and 'd'

'a' and 'd' for example, there are two paths of length 2, passing via nodes 'b' and 'c' respectively. Hence, raising this graph to the power of 2 makes it into a multigraph. And when we remove extraneous edges in the multi graph, the graph is no longer regular.

For regular graphs, we shall be considering a specific kind of power operation, which we shall call the "Hamiltonian power" operation.

The hamiltonian power operation is defined only on graphs that are hamiltonian. It is the power operation applied to any one of the hamiltonian circuits in the graph. Alternatively, the hamiltonian power is the graph power operation over connected 2-regular graphs.

Theorem 10 *The p^{th} power of a connected 2-regular graph of diameter d is $2p$-regular, when $p < d$.*

Proof *A connected 2-regular graph is a hamiltonian circle. In a circle, every node has exactly 2 nodes sepatared by a path length one, 2 nodes separated by a pathlength 2, and so on upto a pathlength of $d-1$. (Every node has only 1 other node separated by a pathlength of d.)*

Specifically, the number of edges that need to be added to get the square of a circle is (since we are dealing with undirected graphs) $\frac{2n}{2}$ or n. The n edges increase the total degree of the graph by $2n$ which gets equally distributed among the n nodes, which increases the degree of each node by 2, thus making it a 4-regular graph. We can extend this to show that the pth power of a graph is $2p$-regular.

The d^{th} power of a circle is of course a complete graph.

An alternative representation of a hamiltonian circuit is in the form of an auto-bijective function or a *permutation.* If V is the set of all nodes in a graph G, we can define a hamiltonian circuit h as a function $h : V \rightarrow V$, which is one-one and onto.

Each node $v_i \in V$ maps onto the next node as indicated in the hamiltonian circuit h.

Given this, graph powers can be regarded as compositions of permutations. The set of new edges that are added when G is raised to the power of 2, is represented by the permutation $h \circ h$.

Permutations on a finite set are known to be closed under the composition operation (cf. [13]). Hence, raising a hamiltonian circuit to power 2 results in another permutation, which can be one of the following: (a). another hamiltonian circuit, or (b). a pair of disconnected cycles.

If the number of nodes $n > 2$ in the hamiltonian circuit, there will be no trivial cycles (self loops from a node to itself) when the circuit is raised to the power of 2. Specifically, we see that if $n > 2$ and is odd, we obtain another hamiltonian circuit and if n is even, we obtain a pair of cycles.

We can relate the latter result with a technique called *polygon embedding* that we had proposed earlier in order to obtain optimal or near-optimal topologies (in terms of diameter reduction) given a set of nodes and edges [20].

Specifically, when n is a power of 2, each higher power of the graph will result in cycles that are half as big as the previous cycles. Even when we retain only one of the cycles in each iteration and discard the rest, we can show that the resultant graph (called *polygon halving embedding* in [20]), has a diameter $O(\log n)$.

6 Regular Directed Graphs

Until now, we have only addressed regular graphs that are undirected. An undirected edge may be seen as a pair of directed edges in opposite directions.

In DHT design, undirected edges pose extra bookkeeping cost in the form of back links that need to be maintained. However, this extra cost is negligible compared to the analytical simplicity that undirected graphs provide. Directed graphs involve several issues like differences between indegree and outdegree, strong connectivity and weak connectivity, etc. that make them harder to analyze than undirected graphs.

However, since a number of DHT topologies have been based on directed graphs, we briefly explore the symmetry property for directed graphs in this section.

As noted earlier, symmetry or regularity for directed graphs have two facets: indegree regularity and outdegree regularity.

A directed graph G has indegree regularity r if all nodes have r incoming edges to them, and it is said to have outdegree regularity r if all nodes have r outgoing edges from them. Hence, when we mention r-regularity of a directed graph, we mean a graph that has regularity r in both indegree and outdegree. We shall use the terms r-inregular and r-outregular to refer to graphs having regularity only in their indegrees and outdegrees respectively.

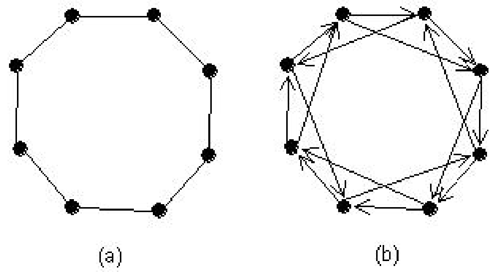

Fig. 5 Diameter can be reduced in directed regular graphs, without affecting regularity

The Chord DHT [26] is outdegree regular as each node has the same number of outgoing links specified by the finger table. It is indegree regular only when the number of nodes is a power of 2.

Also, we shall not be considering multiple edges of the same direction among nodes in the graph, as well as self loops, (i.e. a directed edge from a node to itself) when reasoning about directed regular graphs.

For directed graphs, it helps to think about outgoing edges as being "owned" by the nodes from which they originate and incoming edges as connections that are made by other nodes.

In the context of undirected graphs, given a node v, if all other nodes of the graph can be reached, the graph is said to be connected. Connectedness implies the existence of a path from any node to any other node in the graph. However, for directed graphs, we will need to explicitly ensure *strong connectedness* – that is, reachability of every node from every other node of the graph.

The smallest number of edges required for building a connected undirected graph of n nodes is $n-1$, where the graph is in the form of a logical line. For directed graphs, the smallest number of edges required to build a strongly connected graph of n nodes is n, where the graph is in the form of a hamiltonian circle encompassing all nodes.

Given n nodes, the existence of a hamiltonian cycle bounds the diameter of an undirected graph at $\lceil \frac{n}{2} \rceil$, while for a directed graph, the bound is only $n-1$.

An undirected graph can be rewritten as a directed graph by representing each undirected edge as a pair of directed edges in opposite directions. Given this, we face the question of whether designing a diameter optimal r-regular directed graph necessarily entails representing directed edges in pairs, making the graph into an undirected graph.

Figure 5 refutes such an assertion. Given 8 nodes and 8 undirected edges (or 16 pairs of directed edges), the best we can do for regularity-preserving diameter reduction, is to form a 2-regular hamiltonian cycle having a diameter 4. This is shown in Figure 5(a). However, when the pairwise coupling of the directed edges are broken, we can obtain a directed 2-regular graph with diameter 3, as shown in Figure 5(b).

Further, we can make the following observations over directed r-regular graphs.

Proposition 6 *The smallest number of nodes required for a strongly connected, directed r-regular graph is $r+1$.*

Such a graph is in the form of a clique, where every node has r outgoing edges connecting every other node in the graph, directly. Given this, we have the following important extension theorem for directed graphs.

Theorem 11 *An r-regular directed graph over n nodes, where $n \geq r+1$, can be extended by a single node, regardless of whether the regularity is odd or even.*

Proof *The extension theorem for directed graphs is similar to Theorem 2 concerning odd regular undirected graphs.*

Consider an r-regular directed graph G and an incoming node v. We can think of v as having r holes for incoming connections and r dangling outgoing edges looking for connections.

We now break r edges from G and connect them to v, such that they create holes on r distinct nodes in G. The r outgoing edges from v are then connected to these holes.

Note that any node in G can host both an outgoing connection, as well as an incoming connection from v. This requires a maximum of only r nodes in G, which is satisfied trivially.

In directed regular graphs, we do not face incompatibility between regularity and number of nodes. If the graph is r-regular in both indegree and outdegree, the total degree of every node is always even. This makes it possible to have r-regular directed graphs over any number of nodes, as long as the number of nodes are at least $r+1$.

The connectivity of a directed graph is the size of the smallest edge cutset, whose removal will no longer make the graph strongly connected. Given this, we can prove results analogous to Theorem 4 for directed graphs as well. Similarly, hamiltonian circuits in directed graphs can be extended in the same spirit as of Theorem 8.

7 Conclusions

The symmetry requirement is very common in many DHT design problems. The underlying basis of a symmetric topology is the regular graph. The objective of this paper has been to explore theoretical underpinnings of regular graphs that are pertinent to DHT design. This work is part of a larger effort to obtain a set of meta-heuristics for designing diameter-optimal DHTs in the face of arbitrary constraints.

The primary tool that we use for exploration is genetic algorithms. When the symmetry constraint is relaxed, we get a family of topologies for various configurations of infrastructure cost (described by number of edges) and resilience (described by edge and node cutsets). The family of regular graph topologies forms an interesting subset having its own unique properties that deserves to be studied separately.

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