Random Graphs and Complex Networks

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Contents

Preface

These lecture notes are intended to be used for master courses, where the students have a limited prior knowledge of special topics in probability. Therefore, we have included many of the preliminaries, such as convergence of random variables, probabilistic bounds, coupling, martingales and branching processes. These notes are aimed to be self-contained, and to give the readers an insight in the history of the field of random graphs.

The field of random graphs was started in 1959-1960 by Erdős and Rényi, see [87, 88, 89, 90]. At first, the study of random graphs was used to prove deterministic properties of graphs. For example, if we can show that a random graph has with a positive probability a certain property, then a graph must exist with this property. The method of proving deterministic statements using probabilistic arguments is called the *probabilistic method*, and goes back a long way. See among others the preface of a standard work in random graphs by Bollobás [40], or the work devoted to it *The Probabilistic Method* [14]. Erdős was one of the first to use this method, see e.g., [85], where it was shown that the Ramsey number $R(k)$ is at least $2^{k/2}$. The Ramsey number $R(k)$ is the value n for which any graph of size at least n is such that either itself or its complement contains a complete graph of size at least k. Erdős in [85] shows that for $n \leq 2^{k/2}$, the fraction of graphs for which the graph or its complement contains a complete graph of size k is bounded by $1/2$, so that there must be graphs of size $n \leq 2^{k/2}$ for which the graph nor its complement contains a complete graph of size k.

The initial work by Erdős and Rényi on random graphs has incited a great amount of work on the field. See the standard references on the subject [40] and [124] for the state of the art. Particularly [88] is a highly impressive paper. In it, a rather complete picture is given of the various phase transitions that occur on the Erdős-Rényi random graph. An interesting quote appearing in [88, Page 2-3] is the following:

> "It seems to us worthwhile to consider besides graphs also more complex structures from the same point of view, i.e. to investigate the laws governing their evolution in a similar spirit. This may be interesting not only from a purely mathematical point of view. In fact, the evolution of graphs can be seen as a rather simplified model of the evolution of certain communication nets..."

This was an excellent prediction indeed! Later, interest in random graphs of a different nature arose. Due to the increase of computer power, it has become possible to study so-called *real networks*. Many of these networks turned out to share similar properties, such as the fact that they are *small worlds*, and are *scale-free*, which means that they have degrees obeying *power laws*. The Erdős-Rényi random graph does not obey these properties, and, therefore, new graph models were invented. In fact, already in [88], Erdős and Rényi remark that

> "Of course, if one aims at describing such a real situation, one should replace the hypothesis of equiprobability of all connection by some more realistic hypothesis."

See [167] and [8] for two reviews of real networks and their properties to see what 'more realistic' could mean. These other models are also partly covered in the classical works [40] and [124], but up to today, there is no text treating random graphs and random graph models for complex networks in a relatively elementary way. See [79] for the most recent book on random graph, and, particularly, dynamical processes living on them. Durrett covers part of the material in this book, and much more, but the intended audience is rather different. The goal of these notes is to provide a source for a 'Random graphs' course at master level.

In these notes, we treat both results for the Erdős-Rényi random graph, as well as for the random graph models for complex networks. The aim is to give the simplest possible proofs for classical results, such as the phase transition for the largest connected component in the Erdős-Rényi random graph. Some proofs are more technical and difficult, and the sections containing these proofs will be indicated with a star *. These sections can be omitted without losing the logic in the results. We also try to give many exercises that help the reader to obtain a deeper understanding of the material by working at their solutions.

These notes contain two main parts. In the first part consisting of Chapters 1–8, we give complete and self-contained proofs for some of the main results on random graphs. In the second part consisting of Chapters 9–11, we state further results giving many, but not all, proofs. We sometimes add heuristic arguments for the validity of the results and the logic behind the proof, without giving all details. In the first part, we start by investigating the Erdős-Rényi random graph, and describe some of its key aspects, such as the phase transition for the largest connected component, the connectivity transition, as well as degrees. On the basis of these results, we shall see that the Erdős-Rényi random graph is inappropriate as a model for many real networks, primarily due to the fact that the degrees in the Erdős-Rényi random graph have too thin tails. Therefore, we investigate several other models for real networks, focussing on their degree sequences. In the second part, we investigate related properties of the random graph models, focussing on their connectivity and largest connected components, as well as on the distances in such graphs. The investigation of degree sequences allows us to verify the scale-free nature of the random graph models for complex networks, while the distance results allow us to quantify the small-world behavior in such graphs. These two properties together give insight into what a realistic model for a real network should behave like.

These notes would not have been possible without the help of many people. I thank Gerard Hooghiemstra for the encouragement to write these notes, and for using them at Delft University of Technology almost simultaneously while I used these notes at Eindhoven University of Technology in the spring of 2006. Together with Piet Van Mieghem, we entered the world of random graphs in 2001, and I have tremendously enjoyed exploring this field together with you, and Henri van den Esker, Dmitri Znamenski and Mia Deijfen. I particularly wish to thank Gerard for many useful comments on these notes, solutions to exercises and suggestions for improvements of the presentation of particularly Chapters 2–5.

I thank Christian Borgs, Jennifer Chayes, Gordon Slade and Joel Spencer for joint work on random graphs which are like the Erdős-Rényi random graph, but do have geometry. This work has deepened my understanding of the basic properties of random graphs, and many of the proofs presented here have been inspired by our work in [47, 48, 49]. Special thanks go to Gordon Slade, who has introduced me to the world of percolation, which is a close neighbor of random graphs (see also [101]). It is peculiar to see that two communities work on two so related topics with quite different methods and even terminology, while it has taken such a long time to build bridges between the subjects.

Further I wish to thank Finbar Bogerd, Mia Deijfen, Michel Dekking, Henri van den Esker, Markus Heydenreich, Martin van Jole, Willemien Kets, Xiaotin Yu for remarks, corrections of typos and ideas that have improved the content and presentation of these notes substantially. Wouter Kager has entirely read the February 2007 version of the notes, giving many ideas for improvements of the arguments and

of the methodology. I especially wish to thank Dennis Timmers, who, as a student-assistant, has been a great help in the development of these notes. He has been responsible for the references, many figures and proofs, as well as first versions of the solutions to many of the exercises.

I have tried to give as many references to the literature as possible. However, the number of papers on random graphs is currently exploding. In MathSciNet (see `http://www.ams.org/mathscinet/`), there are, on December 21, 2006, a total of 1,428 papers that contain the phrase 'random graphs' in the review text. These are the papers on the topic in the math community only. What is special about random graph theory is that it is extremely multidisciplinary, and many papers are currently written in economics, biology, theoretical physics and computer science, using random graph models. For example, in Scopus (see `http://www.scopus.com/scopus/home.url`), again on December 21, 2006, there are 5,403 papers that contain the phrase 'random graph' in the title, abstract or keywords. It can be expected that these numbers will increase even faster in the coming period, rendering it impossible to review most of the literature.

Course outline

The course 'Random Graphs and Complex Networks' is organized as follows:

Week	Topic	Sections
Week 1	Introduction and Preliminaries	Chapters 1 and 2
Week 2	Branching Processes	Chapter 3
Week 3	Subcritical Erdős-Rényi Graphs	Sections 4.1–4.3
Week 4	Supercritical Erdős-Rényi Graphs	Section 4.4–4.5
Week 5	Critical Erdős-Rényi Graphs and Connectivity Transition	Chapter 5
Week 6	Degree Sequences in Generalized Random Graph	Chapter 6
Week 7	Degree Sequences in Configuration Model	Chapter 7
Week 8	Power Law Degree Sequences in Preferential Attachment Models	Chapter 8
Week 9	Average Distances in GRG and CM	Chapters 9 and further

Chapter 1

Introduction

In this first chapter, we give an introduction to random graphs and complex networks. The advent of the computer age has incited an increasing interest in the fundamental properties of real networks. Due to the fast computers, large data sets can now be stored and investigated. Many networks share fascinating features. Many are *small worlds*, in the sense that most vertices are separated by a relatively short chain of edges. From an efficiency point of view, this general property could perhaps be expected. More surprisingly, many networks are *scale free*, which means that their degrees are size independent, in the sense that the empirical degree distribution is almost independent of the size of the graph. In particular, many real networks appear to have *power-law degree sequences*, which means that the number of vertices with degree k is proportional to an inverse power of k. These realisations have had a fundamental implication for scientific research on networks. This research is aimed to both understand *why* many networks share these fascinating features, and also what the *properties* of these networks are.

The study of complex networks plays an increasingly important role in science. Examples of such networks are electrical power grids and telephony networks, social relations, the World-Wide Web and Internet, co-authorship and citation networks of scientists, etc. The structure of these networks affects their performance. For instance, the topology of social networks affects the spread of information and disease (see e.g., [196]). The rapid evolution in, and the success of, the Internet have incited fundamental research on the topology of networks. See [20] and [201] for expository accounts of the discovery of network properties by Barabási, Watts and co-authors. In [170], you can find some of the original papers on network modeling, as well as on the empirical findings on them.

One main feature of complex networks is that they are *large*. As a result, their complete description is utterly impossible, and researchers, both in the applications and in mathematics, have turned to their *local description*: how many vertices do they have, and by which local rules are vertices connected to one another? These local rules are *probabilistic*, which leads us to consider *random graphs*. The simplest imaginable random graph is the Erdős-Rényi random graph, which arises by taking n vertices, and placing an edge between any pair of distinct vertices with some fixed probability p. We give an introduction to the classical Erdős-Rényi random graph and informally describe the scaling behavior when the size of the graph is large in Section 1.4. As it turns out, the Erdős-Rényi random graph is *not* a good model for a complex network, and in these notes, we shall also study extensions of it that take the above two key features of real networks into account. These will be introduced and discussed informally in Section 1.5.

1.1   Complex networks

Complex networks have received a tremendous amount of attention in the past decade. In this section, we use the Internet as an example of a real network, and illustrate the properties of real networks using
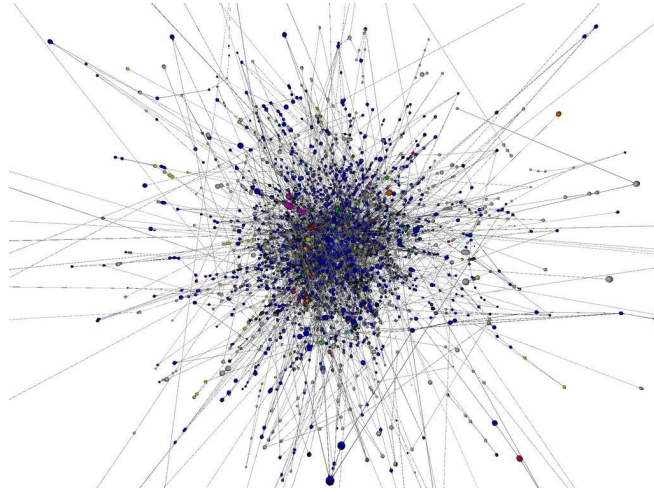
Figure 1.1: The Internet topology in 2001 taken from `http://www.fractalus.com/steve/stuff/ipmap/`.

the Internet as a key example. For an artist's impression of the Internet, see Figure 1.1.

Measurements have shown that many real networks share two fundamental properties. The first fundamental network property is the fact that typical distances between vertices are small. This is called the 'small-world' phenomenon (see [200]). For example, in Internet, IP-packets cannot use more than a threshold of physical links, and if distances in the Internet would be larger than this threshold, e-mail service would simply break down. Thus, the graph of the Internet has evolved in such a way that typical distances are relatively small, even though the Internet is rather large. For example, as seen in Figure 1.2, the AS count, which is the number of Autonomous Systems (AS) which are traversed by an e-mail data set, is most often bounded by 7. In Figure 1.3, the hopcount, which is the number of routers traversed by an e-mail message between two uniformly chosen routers, is depicted.

The second, maybe more surprising, fundamental property of many real networks is that the number of vertices with degree k falls off as an inverse power of k. This is called a 'power-law degree sequence', and resulting graphs often go under the name 'scale-free graphs', which refers to the fact that the asymptotics of the degree sequence is independent of its size. We refer to [8, 77, 167] and the references therein for an introduction to complex networks and many examples where the above two properties hold. The second fundamental property is visualized in Figure 1.4, where the degree distribution is plotted on log-log scale. Thus, we see a plot of $\log k \mapsto \log N_k$, where N_k is the number of vertices with degree k. When N_k is proportional to an inverse power of k, i.e., when, for some normalizing constant c_n and some exponent τ,

$$N_k \sim c_n k^{-\tau}, \tag{1.1.1}$$

then

$$\log N_k \sim \log c_n - \tau \log k, \tag{1.1.2}$$

so that the plot of $\log k \mapsto \log N_k$ is close to a straight line. Here, and in the remainder of this section, we write $\sim$ to denote an uncontrolled approximation. Also, the power exponent τ can be estimated by the slope of the line, and, for the AS-data, this given as estimate of $\tau \approx 2.15 - 2.20$. Naturally, we must have that

$$\sum_k N_k = n, \tag{1.1.3}$$

so that it is reasonable to assume that $\tau > 1$.
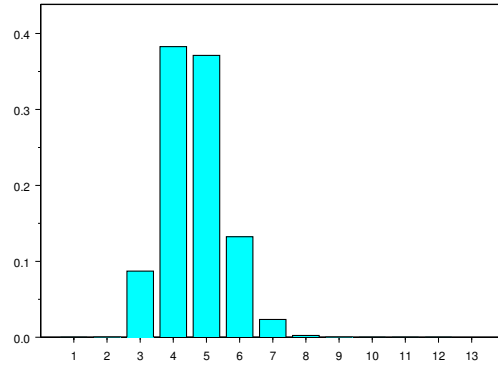
Figure 1.2: Number of AS traversed in hopcount data. Data courtesy of Hongsuda Tangmunarunkit.
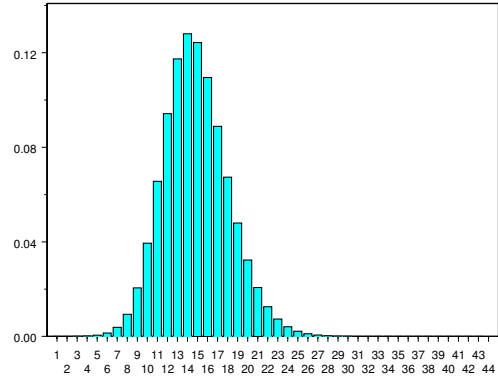
Figure 1.3: Internet hopcount data. Data courtesy of Hongsuda Tangmunarunkit.
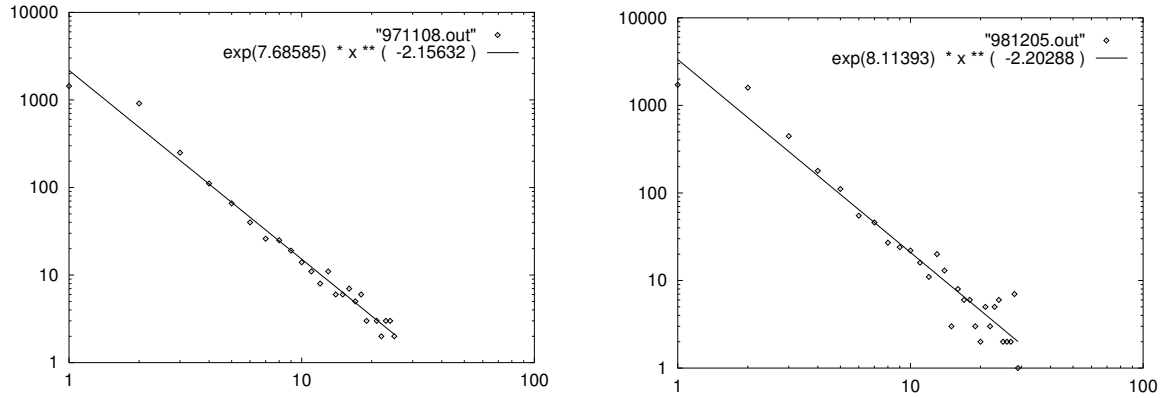
Figure 1.4: Degree sequences AS domains on 11-97 and 12-98 on log-log scale [95]: Power-law degrees with exponent $\approx 2.15 - 2.20$.

	E[h$_{A_s}$]	Var[h$_{A_s}$]	alfa	# points
RIPE	2.81	1.04	2.70	1163687
AMSIX	3.13	1.06	2.95	366075
LINX	2.91	0.98	2.97	168398
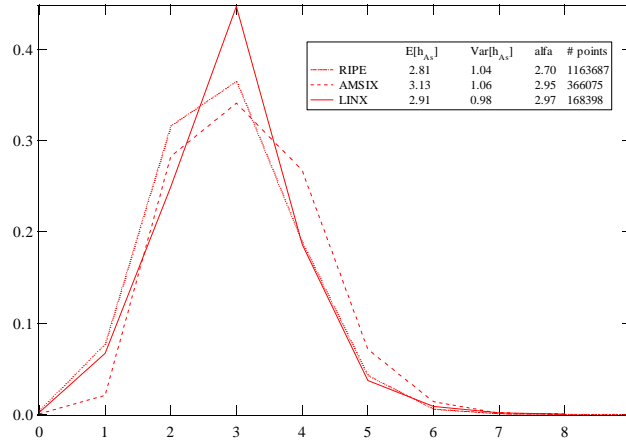
Figure 1.5: Number of AS traversed in various data sets. Data courtesy of Piet Van Mieghem.

Interestingly, in the AS-count, various different data sets (which focus on different parts of the Internet) show roughly the same picture for the AS-count. This shows that the AS-count is somewhat robust, and it hints at the fact that the AS graph is somewhat homogenous. See also Figure 1.5. For example, the AS-count between AS's in North-America on the one hand, and between AS's in Europe, are quite close to the one of the entire AS. This implies that the dependence on geometry of the AS-count is rather weak, even though one would expect geometry to play a role. As a result, most of the models for the Internet, as well as for the AS graph, ignore geometry altogether.

The observation that many real networks have the above properties have incited a burst of activity in network modeling. Most of the models use random graphs as a way to model the uncertainty and the lack of regularity in real networks. In these notes, we survey some of the proposals for network models. These models can be divided into two distinct types: 'static' models, where we model a graph of a given size as a time snap of a real network, and 'dynamical' models, where we model the growth of the network. Static models aim to describe real networks and their topology at a given time instant, and to share properties with the networks under consideration. Dynamical models aim to *explain* how the networks came to be as they are. Such explanations often focus on the growth of the network as a way to explain the power law degree sequences by means of 'preferential attachment' growth rules, where added vertices and links are more likely to be attached to vertices that already have large degrees.

When we would like to model a power-law relationship between the number of vertices with degree k and k, the question is how to model this appropriately. In Chapters 6, 7 and 8, we discuss a number of models which have been proposed for graphs with roughly a given degree sequence. For this, we let F_X be the distribution function of an integer random variable X, and we denote its probability mass function by $\{f_k\}_{k=1}^{\infty}$, so that

$$F_X(x) = \mathbb{P}(X \leq x) = \sum_{k \leq x} f_k. \tag{1.1.4}$$

We wish to obtain a random graph model where N_k, the number of vertices with degree k, is roughly equal to nf_k, where we recall that n is the size of the network. For a power-law relationship as in (1.1.1), we should have that

$$N_k \sim nf_k, \tag{1.1.5}$$

so that

$$f_k \propto k^{-\tau}, \tag{1.1.6}$$

where, to make $f = \{f_k\}_{k=1}^{\infty}$ a probability measure, we take $\tau > 1$, and $\propto$ in (1.1.6) denotes that the left-hand side is proportional to the right-hand side. Now, often (1.1.6) is too restrictive, and we wish to formulate a power-law relationship in a weaker sense. A different formulation could be to require that

$$1 - F_X(x) = \sum_{k>x} f_k \propto x^{1-\tau}, \qquad\qquad \text{(1.1.7)}$$

for some power-law exponent $\tau > 1$. Indeed, (1.1.7) is strictly weaker than (1.1.6), as indicated in the following exercise:

Exercise 1.1. *Show that when* (1.1.6) *holds with equality, then* (1.1.7) *holds. Find an example where* (1.1.7) *holds in the form that there exists a constant C such that*

$$1 - F_X(x) = Cx^{1-\tau}(1 + o(1)), \qquad\qquad \text{(1.1.8)}$$

but that (1.1.6) *fails.*

An even weaker form of a power-law relation is to require that

$$1 - F_X(x) = L_X(x)x^{1-\tau}, \qquad\qquad \text{(1.1.9)}$$

where the function $x \mapsto L_X(x)$ is a so-called *slowly varying function*. Here, a function $x \mapsto \ell(x)$ is called *slowly varying* when, for all constants $c > 0$,

$$\lim_{x \to \infty} \frac{\ell(cx)}{\ell(x)} = 1. \qquad\qquad \text{(1.1.10)}$$

Exercise 1.2. *Show that $x \mapsto \log x$ and, for $\gamma \in (0, 1)$, $x \mapsto e^{(\log x)^{\gamma}}$ are slowly varying, but that when $\gamma = 1$, $x \mapsto e^{(\log x)^{\gamma}}$ is not slowly varying.*

The above discussion on real networks has been illustrated by using the Internet as a prime example. We close the discussion by giving references to the literature on the empirical properties of the Internet:

1. Siganos, Faloutsos, Faloutsos and Faloutsos [190] take up where [95] have left, and further study power-laws arising in Internet.

2. In [126], Jin and Bestavros summarize various Internet measurements and study how the small-world properties of the AS graph can be obtained from the degree properties and a suitable way of connecting the vertices.

3. In [208], Yook, Jeong and Barabási find that the Internet topology depends on geometry, and find that the fractal dimension is equal to $D_f = 1.5$. They continue to propose a model for the Internet growth that predicts this behavior using *preferential attachment* including geometry. We shall discuss this in more detail in Chapter 8.

4. A critical look at the proposed models for the Internet, and particularly the suggestion of preferential attachment in Internet was given by Willinger, Govindan, Paxson and Shenker in [205]. Preferential attachment models shall be described informally in Section 1.1, and are investigated in more detail in Chapters 8 and 11. The authors conclude that the Barabási-Albert model does not model the growth of the AS graph appropriately, particularly since the degrees of the receiving vertices in the AS graph is even larger than for the Barabási-Albert model. This might also explain why the power-law exponent, which is around 2.2 for the AS-graph, is smaller than the power-law exponent in the Barabási-Albert model, which is 3 (see Chapter 8 for this result).

5. An interesting topic of research receiving substantial attention is how the Internet behaves under malicious attacks or random breakdown [66, 67]. The conclusion is that the topology is critical for the vulnerability under intentional attacks. When vertices with high degrees are taken out, then the connectivity properties of random graph models for the Internet cease to have the necessary connectivity properties.

In the remainder of this section, we shall describe a number of other examples of real networks where the small-world phenomenon and the power-law degree sequence phenomenon are observed:

1. 'Six Degrees of Separation' and social networks.

2. Kevin Bacon Game and the movie actor network.

3. Erdős numbers and collaboration networks.

4. The World-Wide Web.

In this section, we shall discuss some of the empirical findings in the above applications, and discuss the key publications on their empirical properties. Needless to say, one could easily write a whole book on each of these examples separately, so we cannot dive into the details too much.

1.1.1   Six degrees of separation and social networks

In 1967, Stanley Milgram performed an interesting experiment. See

<div align="center">

`http://www.stanleymilgram.com/milgram.php`

</div>

for more background on the psychologist Milgram, whose main topic of study was the obedience of people, for which he used a very controversial 'shock machine'.

In his experiment, Milgram sent 60 letters to various recruits in Wichita, Kansas, U.S.A., who were asked to forward the letter to the wife of a divinity student living at a specified location in Cambridge, Massachusetts. The participants could only pass the letters (by hand) to personal acquaintances who they thought might be able to reach the target, either directly, or via a "friend of a friend". While fifty people responded to the challenge, only three letters (or roughly 5%) eventually reached their destination. In later experiments, Milgram managed to increase the success rate to 35% and even 95%, by pretending that the value of the package was high, and by adding more clues about the recipient, such as his/her occupation. See [155, 199] for more details.

The main conclusion from the work of Milgram was that most people in the world are connected by a chain of at most 6 "friends of friends", and this phrase was dubbed "Six Degrees of Separation". The idea was first proposed in 1929 by the Hungarian writer Frigyes Karinthy in a short story called 'Chains' [128], see also [170] where a translation of the story is reproduced.   Playwright John Guare popularized the phrase when he chose it as the title for his 1990 play. In it, Ousa, one of the main characters says:

> "*Everybody on this planet is separated only by six other people. Six degrees of separation. Between us and everybody else on this planet. The president of the United states. A gondolier in Venice... It's not just the big names. It's anyone. A native in the rain forest. (...) An Eskimo. I am bound to everyone on this planet by a trail of six people. It is a profound thought.*".

The fact that any number of people can be reached by a chain of at most 6 intermediaries is rather striking. It would imply that two people in as remote areas as Greenland and the Amazone could be linked by a sequence of at most 6 "friends of friends". This makes the phrase "It's a small world!" very appropriate indeed! Another key reference in the small-world work in social sciences is the paper by Pool and Kochen [182]$^{\text{PooKoc78}}$, which was written in 1958, and has been circulating around the social sciences ever since, before it was finally published in 1978.

The idea of Milgram was taken up afresh in 2001, with the added possibilities of the computer era. In 2001, Duncan Watts, a professor at Columbia University, recreated Milgram's experiment using an e-mail message as the "package" that needed to be delivered. Surprisingly, after reviewing the data collected by 48,000 senders and 19 targets in 157 different countries, Watts found that again the average number of intermediaries was six. Watts' research, and the advent of the computer age, has opened up new areas of inquiry related to six degrees of separation in diverse areas of network theory such as power grid analysis, disease transmission, graph theory, corporate communication, and computer circuitry. See the web site

http://smallworld.columbia.edu/project.html

for more information on the Small-World Project conducted by Watts. See [200]$^{\text{Watt99}}$ for a popular account of the small-world phenomenon. Related examples of the small-world phenomenon can be found in [8]$^{\text{AlbBar02}}$ and [167]$^{\text{Newm03a}}$.

To put the idea of a small-world into a network language, we define the vertices of the social graph to be the inhabitants of the world (so that $n \approx 6$ billion), and we draw an edge between two people when they know each other. Needless to say, we should make it more precise what it means to "know each other". Possibilities here are various. We could mean that the two people involved have shaken hands at some point, or that they know each other on a first name basis.

One of the main difficulties of social networks is that they are notoriously hard to measure. Indeed, questionaires can not be trusted easily, since people have a different idea what a certain social relation is. Also, questionaires are quite physical, and they take time to collect. As a result, researchers are quite interested in examples of social networks that *can* be measured, for example due to the fact that they are *electronic*. Examples are e-mail networks or social networks such as Hyves. Below, I shall give a number of references to the literature for studies of social networks.

1. Amaral, Scala, Bartélémy and Stanley [15]$^{\text{AmaScaBarSta00}}$ calculated degree distributions of several networks, among others a friendship network of 417 junior high school students and a social network of friendships between Mormons in Provo, Utah. For these examples, the degree distributions turn out to be closer to a normal distribution than to a power law.

2. In [83]$^{\text{EbeMieBor02}}$, Ebel, Mielsch and Bornholdt investigate the topology of an e-mail network of an e-mail server at the University of Kiel over a period of 112 days. The authors conclude that the degree sequence obeys a power law, with an exponential cut-off for degrees larger than 200. The estimated degree exponent is 1.81. The authors note that since this data set is gathered at a server, the observed degree of the *external* vertices is an underestimation of their true degree. When only the *internal* vertices are taken into account, the estimate for the power-law exponent decreases to 1.32. When taking into account that the network is in fact *directed*, the power-law exponent of the in-degree is estimated at 1.49, while the out-degrees have an exponent of 2.03. The reported errors in the estimation of the exponents are between 0.10 and 0.18.

3. There are many references to the social science literature on social networks in the book by Watts [201]$^{\text{Watt03}}$, who now has a position in social sciences. In [169]$^{\text{NewStrWat02}}$, Newman, Watts and Strogatz survey

Kevin Bacon Number	# of actors
0	1
1	1902
2	160463
3	457231
4	111310
5	8168
6	810
7	81
8	14

Table 1.1: Kevin Bacon Numbers.

various models for social networks that have appeared in their papers. Many of the original references can also be found in the collection in [170], along with an introduction explaining their relevance.

4. Liljeros, Edling, Amaral and Stanley [146] investigated sexual networks in Sweden, where two people are connected when they have had a sexual relation in the previous year, finding that the degree distributions of males and females obey power laws, with estimated exponents of $\tau_{\text{fem}} \approx 2.5$ and $\tau_{\text{mal}} \approx 2.3$. When extending to the entire lifetime of the Swedish population, the estimated exponents decrease to $\tau_{\text{fem}} \approx 2.1$ and $\tau_{\text{mal}} \approx 1.6$. The latter only holds in the range between 20 and 400 contacts, after which it is truncated. Clearly, this has important implications for the transmittal of sexual diseases.

1.1.2   Kevin Bacon Game and movie actor network

A second example of a large network in the movie actor network. In this example, the vertices are movie actors, and two actors share an edge when they have played in the same movie. This network has attracted some attention in connection to Kevin Bacon, who appears to be reasonably central in this network. The Computer Science Department at Virginia University has an interesting web site on this example, see The Oracle of Bacon at Virginia web site on

$$\texttt{http://www.cs.virginia.edu/oracle/}.$$

See Table 1.1 for a table of the Kevin Bacon Numbers of all the actors in this network. Thus, there is one actor at distance 0 from Kevin Bacon (namely, Kevin Bacon himself), 1902 actors have played in a movie starring Kevin Bacon, and 160463 actors have played in a movie in which another movie star played who had played in a movie starring Kevin Bacon. In total, the number of linkable actors is equal to 739980, and the Average Kevin Bacon number is 2.954. In search for "Six Degrees of Separation", one could say that most pairs of actors are related by a chain of co-actors of length at most 6.

It turns out that Kevin Bacon is not the most central vertex in the graph. A more central actor is Sean Connery. See See Table 1.2 for a table of the Sean Connery Numbers. By computing the average of these numbers we see that the average Connery Number is about 2.731, so that Connery a better center than Bacon. Mr. Bacon himself is the 1049th best center out of nearly 800,000 movie actors, which makes Bacon a better center than 99% of the people who have ever appeared in a feature film.

On the web site `http://www.cs.virginia.edu/oracle/`, one can also try out one's own favorite actors to see what Bacon number they have, or what the distance is between them.

We now list further studies of the movie actor network.

Sean Connery Number	# of actors
0	1
1	2272
2	218560
3	380721
4	40263
5	3537
6	535
7	66
8	2

Table 1.2: Sean Connery Numbers

1. Watts and Strogatz [202] investigate the small-world nature of the movie-actor network, finding that it has more clustering and shorter distances than a random graph with equal edge density. Amaral et al. looked closer at the degree distribution to conclude that the power-law in fact has an exponential cut-off.

2. Albert and Barabási [21] use the movie actor network as a prime example of a network showing power-law degrees. The estimated power-law exponent is 2.3.

1.1.3 Erdős numbers and collaboration networks

A further example of a complex network that has drawn substantial attention is the collaboration graph in mathematics. This is popularized under the name "Erdős number project". In this network, the vertices are mathematicians, and there is an edge between two mathematicians when they have co-authored a paper. See

http://www.ams.org/msnmain/cgd/index.html

for more information. The Erdős number of a mathematician is how many papers that mathematician is away from the legendary mathematician Paul Erdős, who was extremely prolific with around 1500 papers and 509 collaborators. Of those that are connected by a trail of collaborators to Erdős, the maximal Erdős number is claimed to be 15.

On the above web site, one can see how far one's own professors are from Erdős. Also, it is possible to see the distance between any two mathematicians.

The Erdős numbers has also attracted attention in the literature. In [73, 74], the authors investigate the Erdős numbers of Nobel prize laureates, as well as Fields medal winners, to come to the conclusion that Nobel prize laureates have Erdős numbers of at most 8 and averaging 4-5, while Fields medal winners have Erdős numbers of at most 5 and averaging 3-4. See also

http://www.oakland.edu/enp

for more information on the web, where we also found the following summary of the collaboration graph. This summary dates back to July, 2004. An update is expected somewhere in 2006.

In July, 2004, the collaboration graph consisted of about 1.9 million authored papers in the Math Reviews database, by a total of about 401,000 different authors. Approximately 62.4% of these items

Erdős Number	# of Mathematicians
0	1
1	504
2	6593
3	33605
4	83642
5	87760
6	40014
7	11591
8	3146
9	819
10	244
11	68
12	23
13	5

Table 1.3: Erdős Numbers

are by a single author, 27.4% by two authors, 8.0% by three authors, 1.7% by four authors, 0.4% by five authors, and 0.1% by six or more authors. The largest number of authors shown for a single item is in the 20s. Sometimes the author list includes "et al." so that in fact, the number of co-authors is not always precisely known.

The fraction of items authored by just one person has steadily decreased over time, starting out above 90% in the 1940s and currently standing at under 50%. The entire graph has about 676,000 edges, so that the average number of collaborators per person is 3.36. In the collaboration graph, there is one large component consisting of about 268,000 vertices. Of the remaining 133,000 authors, 84,000 of them have written no joint papers, and these authors correspond to isolated vertices. The average number of collaborators for people who have collaborated is 4.25. The average number of collaborators for people in the large component is 4.73. Finally, the average number of collaborators for people who have collaborated but are not in the large component is 1.65. There are only 5 mathematicians with degree at least 200, the largest degree is for Erdős, who has 509 co-authors. The diameter of the largest connected component is 23.

The clustering coefficient of a graph is equal to the fraction of ordered triples of vertices a, b, c in which edges ab and bc are present that have edge ac present. In other words, the clustering coefficient describes how often are two neighbors of a vertex adjacent to each other. The clustering coefficient of the collaboration graph of the first kind is $1308045/9125801 = 0.14$. The high value of this figure, together with the fact that average path lengths are small, indicates that this graph is a *small world* graph.

For the Erdős numbers, we refer to Table 1.3. The median Erdős number is 5, the mean is 4.65, and the standard deviation is 1.21. We note that the Erdős number is finite if and only if the corresponding mathematician is in the largest connected component of the collaboration graph.

See Figure 1.6 for an artistic impression of the collaboration graph in mathematics taken from

<div align="center"><code>http://www.orgnet.com/Erdos.html</code></div>

and Figure 1.7 for the degree sequence in the collaboration graph.

We close this section by listing interesting papers on collaboration graphs.
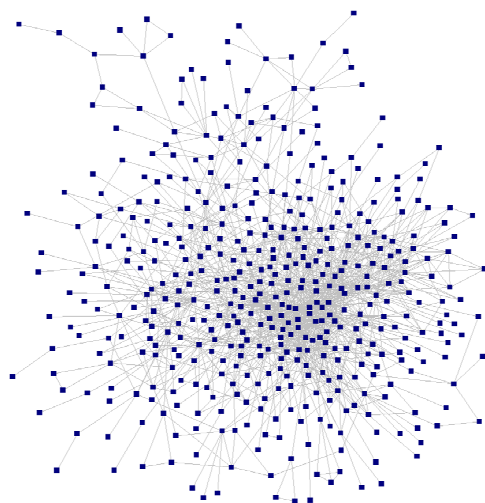
Figure 1.6: An artist impression of the collaboration graph in mathematics.
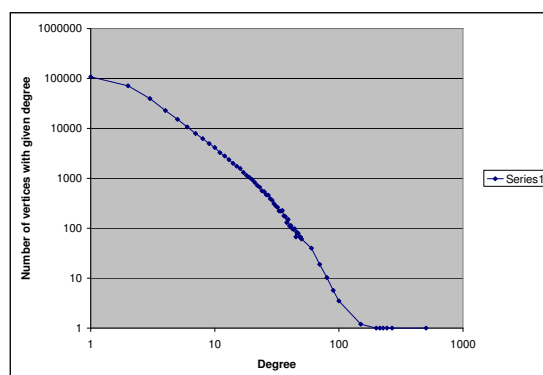
Figure 1.7: The degree sequence in the collaboration graph.

1. In [26], Batagelj and Mrvar use techniques for the analysis of large networks, such as techniques to identify interesting subgroups and hierarchical clustering techniques, to visualize further aspects of the Erdős collaboration graph.

2. Newman has studied several collaboration graphs in a sequence of papers that we shall discuss now. In [166], he finds that several of these data bases are such that the degrees have power-laws with exponential cut-offs. The data bases are various arXiv data bases in mathematics and theoretical physics, the MEDLINE data base in medicine, and the ones in high-energy physics and theoretical computer science. Also, the average distance between scientists is shown to be rather small, which is a sign of the small-world nature of these networks. Finally, the average distance is compared to $\log n/\log z$, where n is the size of the collaboration graph and z is the average degree. The fit shows that these are quite close. Further results are given in [165].

3. In Barabási et al. [23], the evolution of scientific collaboration graphs is investigated. The main conclusion is that scientists are more likely to write papers with other scientists who have written many papers, i.e., there is a tendency to write papers with others who have already written many. This preferential attachment is shown to be a possible explanation for the existence of power laws in collaboration networks (see Chapter 8).

1.1.4 The World-Wide Web

A final example of a complex network that has attracted enormous attention is the World-Wide Web (WWW). The elements of the WWW are web pages, and there is a directed connection between two web pages when the first links to the second. Thus, while the WWW is virtual, the Internet is physical. With the world becoming ever more virtual, and the WWW growing at tremendous speed, the study of properties of the WWW has grown as well. It is of great practical importance to know what the structure of the WWW is, for example, in order for search engines to be able to explore it. A notorious, but rather interesting, problem is the Page-Rank problem, which is the problem to rank web pages on related topics such that the most important pages come first. Page-Rank is claimed to be the main reason of the success of Google, and the inventors of Page-Rank were also the founders of Google (see [52] for the original reference).

In [9], the authors Albert, Jeong and Barabási study the degree distribution to find that the in-degrees obey a power law with exponent $\tau_{\mathrm{in}} \approx 2.1$ and the out-degrees obey a power law with exponent $\tau_{\mathrm{out}} \approx 2.45$, on the basis of several Web domains, such as `nd.edu`, `mit.edu` and `whitehouse.gov`, respectively the Web domain of the home university of Barabási at Notre Dame, the Web domain of MIT and of the White House. Further, they investigated the distances between vertices in these domains, to find that distances within domains grown linearly with the log of the size of the domains, with an estimated dependence of $d = 0.35 + 2.06 \log n$, where d is the average distance between elements in the part of the WWW under consideration, and n is the size of the subset of the WWW. Extrapolating this relation to the estimated size of the WWW at the time, $n = 8 \cdot 10^8$, Albert, Jeong and Barabási [9] concluded that the diameter of the WWW was 19 at the time, which prompted the authors to the following quote:

> "*Fortunately, the surprisingly small diameter of the web means that all information is just a few clicks away.*"

In [143], it was first observed that the WWW also has power-law degree sequences. In fact, the WWW is a *directed* graph, and in [143] it was shown that the in-degree follows a power-law with power-law exponent quite close to 2. See also Figure 1.8.

Indegree Distribution
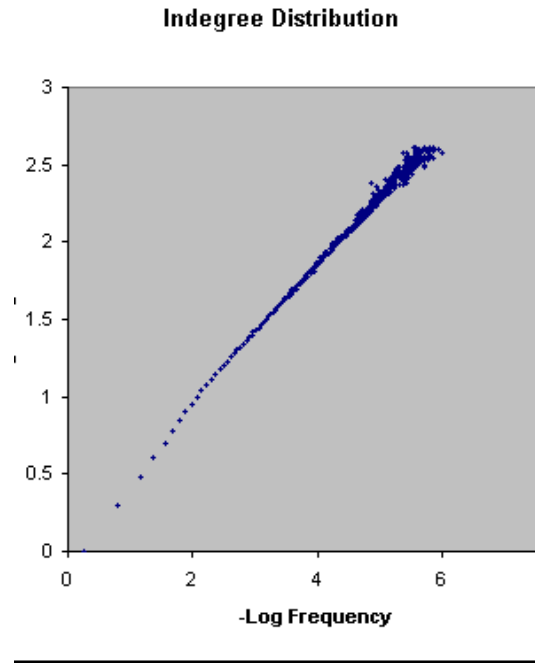
Figure 1.8: The in-degree sequence in the WWW taken from [143].[KumRagRajTom99]

The most substantial analysis of the WWW was performed by Broder et al. [54],[Broetal00] following up on earlier work in [143, 142][KumRagRajTom99, KumRagRajSivTomUpf00] in which the authors divide the WWW into several distinct parts. See Figure 1.9 for the details. The division is roughly into four parts:

(a) The central core or Strongly Connected Component (SCC), consisting of those web pages that can reach each other along the directed links (28% of the pages);

(b) The IN part, consisting of pages that can reach the SCC, but cannot be reached from it (21% of the pages);

(c) The OUT part, consisting of pages that can be reached from the SCC, but do not link back into it (21% of the pages);

(d) The TENDRILS and other components, consisting of pages that can neither reach the SCC, nor be reached from it (30% of the pages).

Broder et al. [54][Broetal00] also investigate the diameter of the WWW, finding that the SCC has diameter at least 28, but the WWW as a whole has diameter at least 500. This is partly due to the fact that the graph is *directed*. When the WWW is considered to be an *undirected* graph, the average distance between vertices decreases to around 7. Further, it was shown that both the in- and out-degrees in the WWW follow a power-law, with power-law exponents estimated as $\tau_{\text{in}} \approx 2.1, \tau_{\text{out}} \approx 2.5$.

In [3],[Adam99] distances in the WWW are discussed even further. When considering the WWW as a directed graph, it is seen that the distances between most pairs of vertices within the SCC is quite small. See Figure 1.10 for a histogram of pairwise distances in the sample. Distances between pairs of vertices in the SCC tend to be at most 7: Six Degrees of Separation.

We close this section by discussing further literature on the WWW:
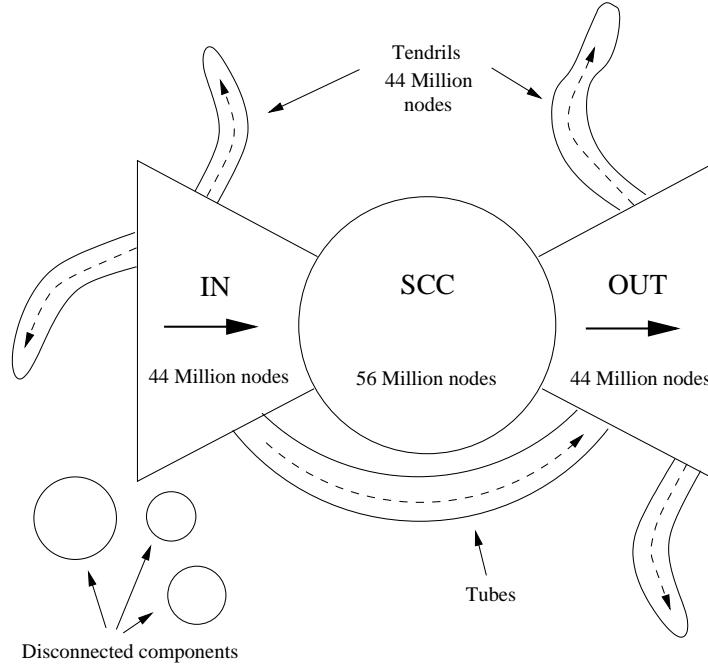
Figure 1.9: The WWW according to Broder et al [54].[Broetal00]

1. In [21],[BarAlb99] it is argued that new web pages are more likely to attach to web pages that already have a high degree, giving a bias towards popular web pages. This is proposed as an explanation for the occurrences of power laws. We shall expand this explanation in Section 1.5, and make the discussion rigorous in Chapter 8.

2. In [142],[KumRagRajSivTomUpf00] models for the WWW are introduced, by adding vertices which copy the links of older vertices in the graph. This is called an *evolving copying model*. In some cases, depending on the precise copying rules, the model is shown to lead to power-law degree sequences. The paper [138][Kleietal99] is a nice survey of measurements, models and methods for obtaining information on the WWW, by analyzing how Web crawling works.

3. Barábasi, Albert and Jeong [22][BarAlbJeo00] investigate the scale-free nature of the WWW, and propose a preferential attachment model for it. In the proposed model for the WWW in [21, 22],[BarAlb99, BarAlbJeo00] older vertices tend to have the highest degrees. On the WWW this is not necessarily the case, as Adamic and Huberman [4][AdaHub00] demonstrate. For example, Google is a late arrival on the WWW, but has yet managed to become one of the most popular web sites. A possible fix for this problem is given in [34][BiaBar01b] through a notion of *fitness* of vertices, which enhance or decrease their preferential power.

4. The works by Kleinberg [135, 136, 137][Klei99, Klei00, Klei00b] investigate the WWW and other networks from a computer science point of view. In [135, 136],[Klei99, Klei00] the problem is addressed how hard it is to find short paths in small-world networks on the d-dimensional lattice, finding that navigation sensitively depends upon how likely it is for large edges to be present. Indeed, the delivery time of any local algorithm can be bounded below by a positive power of the width of the box, except for one special value of the parameters, in which case it is of the order of the square of the log of the width of the box. Naturally, this has important implications for the WWW, even though the WWW may depend less sensitively on geometry. In Milgram's work discussed in Section 1.1.1, on the one hand, it is striking
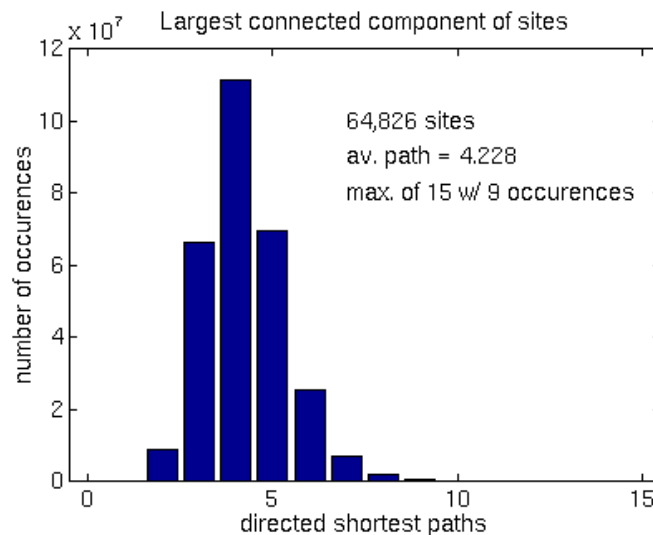
Figure 1.10: Average distances in the Strongly Connected Component of the WWW taken from [3].

that there exist short paths between most pairs of individuals, but, on the other hand, it may be even more striking that people actually succeed in *finding* them. In [135], the problem is addressed how "authoritative sources" for the search on the WWW can be quantified. These authoritative sources can be found in an algorithmic way by relating them to the hubs in the network. Clearly, this problem is intimately connected to the Page-Rank problem.

1.2 Tales of tails

In this section, we discuss the occurrence of power laws. In Section 1.2.1, we discuss the literature on this topic, which dates back to the twenties of the last century. In Section 1.2.2, we describe the new view points on power laws in real networks.

1.2.1 Old tales of tails

Mathematicians are always drawn to simple relations, believing that they explain the rules that gave rise to them. Thus, finding such relations uncovers the hidden structure behind the often chaotic appearance. A power-law relationship is such a simple relation. We say that there is a power-law relationship between two variables when one is proportional to a power of the other. Or, in more mathematical language, the variable k and the characteristic $f(k)$ are in a power-law relation when $f(k)$ is proportional to a power of k, that is, for some number τ,

$$f(k) = Ck^{-\tau}. \tag{1.2.1}$$

Power laws are intimately connected to so-called 80/20 rules. For example, when studying the wealth in populations, already Pareto observed a huge variability [176]. Most individuals do not earn so much, but there are these rare individuals that earn a substantial part of the total income. Pareto's principle was best known under the name '80/20 rule', indicating, for example, that 20 percent of the people earn 80 percent of the total income. This law appears to be true much more generally. For example, 20 percent of the people own 80 percent of the land, 20 percent of the employees earn 80 percent of the profit of large companies, and maybe even 20 percent of the scientists write 80 percent of the papers. In each of

these cases, no typical size exists due to the high variability present, which explains why these properties are called 'scale-free'.

Intuitively, when a 80/20 rule holds, a power law must be hidden in the background! Power laws play a crucial role in mathematics, as well as in many applications. Power laws have a long history. Zipf [210] was one of the first to find one in the study of the frequencies of occurrence of words in large pieces of text. He found that the relative frequency of words is roughly inversely proportional to its rank in the frequency table of all words. Thus, the most frequent word is about twice as frequent as the second most frequent word, and about three times as frequent as the third most frequent word, etc. In short, with k the rank of the word and $f(k)$ the relative frequency of kth most frequent word, $f(k) \propto k^{-\tau}$ where τ is close to 1. This is called *Zipf's law*.

Already in the twenties, several other examples of power laws were found. Lotka [148] investigated papers that were referred to in the Chemical Abstracts in the period from 1901-1916, and found that the number of scientists appearing with 2 entries is close to $1/2^2 = 1/4$ of the number of scientists with just one entry. The number of scientists appearing with 3 entries is close to $1/3^2 = 1/9$ times the number of scientists appearing with 1 entry, etc. Again, with $f(k)$ denoting the number of scientists appearing in k entries, $f(k) \propto k^{-\tau}$, where τ now is close to 2. This is dubbed *Lotka's Law*. Recently, effort has been put into explaining power-laws using 'self-organization'. Per Bak, one of the central figures in this area, called his book on the topic "How nature works" [19].

Power-law relations are one-step extensions of linear relations. Conveniently, even when one does not understand the mathematical definition of a power law too well, one can still *observe* them in a simple way: in a log-log plot, power laws are turned into straight lines! Indeed, taking the log of the power-law relationship (1.2.1) yields

$$\log f(k) = \log C - \tau \log k, \tag{1.2.2}$$

so that $\log f(k)$ is in a linear relationship with $\log k$, with coefficient equal to $-\tau$. Thus, not only does this allow us to visually inspect whether $f(k)$ is in a power-law relationship to k, it also allows us to estimate the exponent τ! Naturally, this is precisely what has been done in order to obtain the power-law exponents in the examples in Section 1.1. An interesting account of the history of power-laws can be found in [156], where possible explanations why power laws arise so frequently are also discussed.

1.2.2   New tales of tails

In this section, we discuss the occurrence of power-law degree sequences in real networks. We start by giving a heuristic explanation for the occurrence of power law degree sequences, in the setting of exponentially growing graphs. This heuristic is based on some assumptions that we formulate now.

We assume that

(1) the number of vertices $V(t)$ is growing *exponentially* at some rate $\rho > 0$, i.e., $V(t) \approx e^{\rho t}$;

(2) the number $N(t)$ of links *into* a vertex at some time t after its creation is $N(t) \approx e^{\beta t}$. (Note that we then must have that $\beta \leq \rho$, since the number of links into a vertex must be bounded above by the number of vertices.) Thus, also the number of links into a vertex grows exponentially with time.

We note that assumption (1) is equivalent to the assumption that

(1') the lifetime T since its creation of a random vertex has law

$$\mathbb{P}(T > t) = e^{-\rho t}, \tag{1.2.3}$$

so that the density of the lifetime of a random vertex is equal to

$$f_T(t) = \rho e^{-\rho t}. \tag{1.2.4}$$

Then, using the above assumptions, the number of links *into* a random vertex X equals

$$\mathbb{P}(X > i) \approx i^{-\rho/\beta}, \tag{1.2.5}$$

since it is equal to

$$
\begin{aligned}
\mathbb{P}(X > i) &= \int_0^\infty f_T(t)\mathbb{P}(X > i | T = t)dt \\
&= \int_0^\infty \rho e^{-t\rho}\mathbb{P}(X > i | T = t)dt \\
&= \rho \int_0^\infty e^{-t\rho}\mathbb{1}_{\{e^{\beta t} > i\}}dt \\
&\sim \rho \int_{(\log i)/\beta}^\infty e^{-t\rho}dt \sim e^{-(\log i)\rho/\beta} \sim i^{-\rho/\beta},
\end{aligned}
$$

where $\mathbb{1}_{\mathcal{E}}$ denotes the indicator of the event $\mathcal{E}$. Stretching the above heuristic a bit further yields

$$\mathbb{P}(X = i) = \mathbb{P}(X > i - 1) - \mathbb{P}(X > i) \sim i^{-(\rho/\beta+1)}. \tag{1.2.6}$$

This heuristic suggests a power law for the in-degrees of the graph, with power-law exponent $\tau = \rho/\beta + 1 \geq 2$. Peculiarly, this heuristic does not only explain the occurrence of power laws, but even of power laws with exponents that are at least 2.

The above heuristic only explains why the *in-degree* of a vertex has a power law. An alternative reason why power laws occur so generally will be given in Chapter 8. Interestingly, so far, also in this explanation only power laws that are at least 2 are obtained.

While power-law degree sequences are claimed to occur quite generally in real networks, there are also some critical observations, particularly about he measurements that produce power laws in Internet. In [144], it is argued that `traceroute`-measurements, by which the Internet-topology is uncovered, could be partially responsible for the fact that power-law degree sequences are observed in Internet. Indeed, it was shown that applying similar methods as `traceroute`-measurements to certain subgraphs of the Erdős-Rényi random graph exhibit power-law degree sequences. Clearly, the Erdős-Rényi random graph does *not* have power-law degree sequences, so that this observation is an *artefact* of the way the measurements are performed. The point is that in Internet measurements, subgraphs are typically obtained by exploring the paths between sets of pairs of vertices. Indeed, we obtain a subgraph of the Internet by only taking that part of the network that appear along a path between the various starting points and destinations, and this is the way how `traceroute` is used in Internet. Assuming that paths are all *shortest-paths*, i.e., there is shortest-path routing, vertices with a high degree are far more likely to appear in one of the shortest paths between our initial set of pairs of vertices. Therefore, such data sets tend to *overestimate* the degrees in the complete network. This bias in `traceroute` data was further studied in [2, 65], in which both for Erdős-Rényi random graphs and for random regular graphs, it was shown that subgraphs appear to obey a power-law.

While the above criticism may be serious for the Internet, and possibly for the World-Wide Web, where degree distributions are investigated using web-crawling, there are many networks which are completely available that also show power-law degree sequences. When the network is completely described, the observed power-laws can not be so easily dismissed. However, one needs to be careful in using and analyzing data confirming power-law degree sequences. Particularly, it could be that many estimates of the power-law degree exponent τ are *biased*, and that the true values of τ are substantially larger. Possibly, this criticism may give an argument why so often power laws are observed with exponents in the interval $(2, 3)$.

1.3   Notation

In these notes, we frequently make use of certain notation, and we strive to be as consistent as possible. We shall denote events by calligraphic letters, such as $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{E}$. We shall use $\mathbb{1}_{\mathcal{E}}$ to denote the indicator function of the event $\mathcal{E}$. We shall use capital letters, such as X, Y, Z, U, V, W, to denote random variables. There are some exceptions, for example, F_X and M_X denote the distribution function and moment generating function of a random variable X, and we emphasize this by writing the subscript X explicitly. We say that a sequence of events $\{\mathcal{E}_n\}_{n=0}^{\infty}$ occurs *with high probability* when $\lim_{n\to\infty} \mathbb{P}(\mathcal{E}_n) = 1$.

We shall use special notion for certain random variables, and write $X \sim \mathrm{BE}(p)$ when X has a Bernoulli distribution with success probability p, i.e., $\mathbb{P}(X = 1) = 1 - \mathbb{P}(X = 0) = p$. We write $X \sim \mathrm{BIN}(n, p)$ when the random variable X has a binomial distribution with parameters n and p, and we write $X \sim \mathrm{Poi}(\lambda)$ when X has a Poisson distribution with parameter λ.

Furthermore, we write $f(n) = o(g(n))$ as $n \to \infty$ when $g(n) > 0$ and $\lim_{n\to\infty} |f(n)|/g(n) = 0$. We write $f(n) = O(g(n))$ as $n \to \infty$ when $g(n) > 0$ and $\limsup_{n\to\infty} |f(n)|/g(n) < \infty$. Finally, we write $f(n) = \Theta(g(n))$ as $n \to \infty$ if $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

1.4   The Erdős-Rényi Random Graph: Introduction of the model

In the previous sections, we have described properties of real networks. These networks are quite large, and in most cases, it is utterly impossible to describe them explicitly. To circumvent this problem, random graph models have been considered as network models. These random graphs describe by which *local* and *probabilistic* rules vertices are connected to one another. The use of probabilistic rules is to be able to describe the *complexity* of the networks. In deterministic models, often too much structure is present, making the arising networks unsuitable to describe real networks. This approach introduces *randomness* in network theory, and leads us to consider random graphs as network models. However, it does not tell us *what* these random graph models should look like.

The field of random graphs was established in the late fifties and early sixties of the last century. While there were a few papers appearing around (and even before) that time, one paper is generally considered to have founded the field [88]. The authors Erdős and Rényi studied the simplest imaginable random graph, which is now named after them. Their graph has n elements, and each pair of elements is independently connected with a fixed probability. When we think of this graph as describing a social network, then the elements denote the individuals, while two individuals are connected when they know one another. The probability for elements to be connected is sometimes called the *edge probability*. Let $\mathrm{ER}(n, p)$ denote the resulting random graph. This random graph is named after its inventors Erdős and Rényi who introduced a version of it in [88] in 1960. Note that the precise model above is introduced by Gilbert in [99], and in [88] a model was formulated with a fixed number of edges (rather than a binomial number of edges). It is not hard to see that the two models are intimately related (see e.g., Section 4.6, where the history is explained in a bit more detail). The Erdős-Rényi random graph was named after Erdős and Rényi due to the deep and striking results proved in [88], which opened up an entirely new field. Earlier papers investigating random graphs are [85], using the probabilistic method to prove graph properties, and [193], where the model is introduced as a model for neurons.

Despite the fact that $\mathrm{ER}(n, p)$ is the simplest imaginable model of a random network, it has a fascinating *phase transition* when p varies. Phase transitions are well known in physics. The paradigm example is the solid-fluid transition of water, which occurs when we move the temperature from below $0°$ to above $0°$. Similar phase transitions occur in various real phenomena, such as magnetism or the conductance properties of porous materials. Many models have been invented that describe and explain such phase transitions, and we shall see some examples in these notes. As we will see, the Erdős-Rényi random graph

exhibits a phase transition in the size of the maximal component, as well as in the connectivity of the arising random graph.

Indeed, if $p = \lambda/n$ with $\lambda < 1$, then $\mathrm{ER}(n, p)$ consists of many small components having at most size $\Theta(\log n)$. If, otherwise, $\lambda > 1$ the graph consists of one giant component of $\Theta(n)$ and some small components which have size $\Theta(\log n)$. (Recall the notation in Section 1.3.) These properties shall be explained and proved in full detail in Chapter 4. In Chapter 5, we shall also investigate the size for the largest connected component when $\lambda = 1$, and for which λ the Erdős-Rényi random graph is connected.

A rough outline of the ideas behind the proof in Chapters 4–5 is given below. The necessary probabilistic ingredients are described in Chapter 2, for example, stochastic orderings, convergence of random variables, and couplings. In Chapter 3, we describe *branching processes*, which prove to be extremely useful in the analysis of the Erdős-Rényi random graph and many other related random graph models.

To describe these preliminaries, let's investigate the clusters of a vertex in an Erdős-Rényi random graph. We say that $u, v \in \{1, \ldots, n\}$ are *connected* when there exists a path of occupied bonds connecting the two vertices u and v, and we write this as $u \longleftrightarrow v$. We let the cluster of v, i.e., the connected component containing v, be equal to

$$C(v) = \{y : v \longleftrightarrow y\}, \qquad (1.4.1)$$

where, by convention, v is connected to v, so that $v \in C(v)$. Let $|C(v)|$ denote the number of vertices in $C(v)$. Then, the size of the largest connected component of $\mathrm{ER}(n, p)$ is equal to

$$|C_{\max}| = \max_{v \in \{1,\ldots,n\}} |C(v)|. \qquad (1.4.2)$$

Naturally, the law of $C(x)$, and, therefore also of $|C_{\max}|$, depends sensitively on the value of p.

To describe the largest connected component, we explore the different clusters one by one. We start with vertex 1, and explore all the edges that are incident to 1. The endpoints of these edges are clearly elements of the cluster $C(1)$. Therefore, the exploration of the edges starting from 1 gives rise to a subset of vertices that are in $C(1)$, namely, precisely the vertices that are at distance 1 in the random graph $\mathrm{ER}(n, p)$ from the vertex 1, i.e., the direct neighbors. Denote the number of different neighbors by X_1. Note that the distribution of the number of direct neighbors X_1 is equal to a binomial random variable with parameters $n - 1$ and p, i.e., $X_1 \sim \mathrm{BIN}(n - 1, p)$.

When $X_1 = 0$, then $C(1) = \{1\}$, and we have explored the entire cluster of vertex 1. However, when $X_1 \geq 1$, then there is at least one direct neighbor of 1, and we next explore its direct neighbors. We denote $i_1, \ldots, i_{X_1}$ the vertices that are direct neighbors of 1, where we order these such that $i_1 < i_2 < \ldots < i_{X_1}$.

We now explore the neighbors of i_1. Naturally, when we wish to explore the elements of $C(1)$, we are only interested in those neighbors of 1 for which we do not yet know that they are part of $C(1)$. When we fix the number of direct neighbors X_1, then this number of neighbors of i_1 again has a binomial distribution, now with parameters $n - 1 - X_1$ and probability of success p. Denote the number of vertices by X_2. We emphasize here that the *conditional distribution* of X_2 given X_1 is $\mathrm{BIN}(n - 1 - X_1, p)$, but the marginal distribution of X_2 is *not* binomial.

When $X_1 \geq 2$, we can also explore the direct neighbors of i_2 that are not yet part of $C(1)$, and this number, which we denote by X_3, has, conditionally on X_1 and X_2, distribution $\mathrm{BIN}(n - 1 - X_1 - X_2, p)$. This is called *breadth-first search*. In general, when we explore the $(i + 1)^{\mathrm{st}}$ vertex of the cluster of vertex 1, we obtain a random number of newly added vertices, denoted by X_{i+1}, which are part of $C(1)$, and of which the law is $\mathrm{BIN}(N_i, p)$, where

$$N_i = n - 1 - X_1 - \cdots - X_i, \qquad i = 1, 2, \ldots. \qquad (1.4.3)$$

Before exploring the ith vertex, the number of vertices whose neighbors we have not yet investigated is equal to

$$1 + X_1 + \ldots + X_i - i, \qquad (1.4.4)$$

that is, the number of vertices of which we have decided that they are part of the cluster of vertex 1 minus the number of vertices which have been fully explored. This process continues as long as there are unexplored or active vertices, i.e., it continues as long as

$$1 + X_1 + \ldots + X_i - i \geq 1. \tag{1.4.5}$$

Since finally we explore all vertices in the cluster, we obtain that

$$|C(1)| = \min\{i : X_1 + \ldots + X_i = i - 1\}. \tag{1.4.6}$$

Similarly, we can explore the clusters of the other vertices that are not elements of $C(1)$. Say that $j \in \{1, \ldots, n\}$ is the smallest element that does not belong to $C(1)$. Then, in a similar way as above, we can explore $C(j)$, the cluster of j. Note, however, that, since $j \not\in C(1)$, the vertices in $C(1)$ should now be removed from the procedure. Therefore, the number of available vertices decreases. This phenomenon is sometimes called the *depletion of points* effect.

It is well known that when n is large, then the binomial distribution with parameters n and $p = \lambda/n$ is close to the Poisson distribution with parameter λ. More precisely, we have that

$$\mathbb{P}\Big(\mathrm{BIN}(n, \lambda/n) = k\Big) = e^{-\lambda}\frac{\lambda^k}{k!} + o(1), \qquad k = 0, 1, \ldots. \tag{1.4.7}$$

The probability mass function $f_k = e^{-\lambda}\frac{\lambda^k}{k!}$ is the probability mass function of the Poisson distribution with parameter λ. In particular, for every fixed i, if we were to know that $X_1, \ldots, X_i$ are not too large (which is true if X_j were Poisson random variables with parameter λ), then

$$N_i = n - 1 - X_1 - \cdots - X_i \approx n. \tag{1.4.8}$$

Thus, we have that a binomial random variable with parameters N_i and success probability $p = \lambda/n$ is approximately Poisson distributed with parameter λ. With this approximation, the random variables $\{X_j\}_{j=1}^{\infty}$ are *independent and identically distributed,* which is often abbreviated by i.i.d. in these notes. In this approximation, we see that the number of unexplored vertices satisfies a recurrence relation given by

$$S_i^* \sim 1 + X_1^* + \ldots + X_i^* - i, \tag{1.4.9}$$

up to the point where $S_i^* = 0$, and where $\{X_i^*\}_{i=1}^{\infty}$ are i.i.d. Poisson random variables with parameter λ. We write

$$T^* = \min\{i : S_i^* = 0\} = \min\{i : X_1^* + \ldots + X_i^* = i - 1\} \tag{1.4.10}$$

for the first time at which $S_i = 0$. In the above simplified model, the random variable T^* could be *infinite,* while in (1.4.6) this is clearly impossible. In (1.4.10), we explore vertices in a tree, and the ith explored individual gives rise to X_i^* children, where $\{X_j^*\}_{j=1}^{\infty}$ are i.i.d. Poisson random variables with parameter λ. The above process is called a *branching process* with a Poisson offspring distribution with parameter or mean λ.

Branching processes are simple models for the evolution of a population, and have received considerable attention in the mathematical literature. See [17, 106, 117] for introductions to the subject. Branching processes have a *phase transition* when the expected offspring varies. When the expected offspring exceeds 1, then there is a positive probability of survival forever, while if the expected offspring is at most 1, then the population dies out with probability one. This phase transition for branching processes is intimately connected to the phase transition on the random graph.

We describe the phase transition on the random graph in Chapter 4. In that chapter, the exploration description of connected components described above will be crucial. In order to make the above steps

rigorous, we need some preliminaries. In Chapter 2, we describe the probabilistic preliminaries, such as stochastic ordering, convergence of random variables, coupling theory and martingales. For example, stochastic domination allows us to make the intuition that $X_{i+1} \sim \text{BIN}(N_i, p)$ when $N_i \leq n$ is smaller than a binomial random variable with parameters n and p precise. Convergence of random variables is the right notion to show that a binomial distribution with parameters n and $p = \lambda/n$ is close to the Poisson distribution with parameter λ. A coupling of these two random variables allows us to give a bound on their difference. In Chapter 3, we describe branching processes. We prove the phase transition, and relate super critical branching processes conditioned to die out with subcritical branching processes. We pay particular attention to branching processes with a Poisson offspring distribution.

While the Erdős-Rényi random graph is a beautiful model displaying fascinating scaling behavior for large graphs and varying edge probabilities, its degrees are *not* scale-free, rendering it unrealistic as a network model. Indeed, its typical degree size is the average degree, and there is little variability in its degrees. In particular, no hubs exist. More precisely, the degree of any vertex in an Erdős-Rényi random graph with edge probability $p = \lambda/n$ is precisely equal to a binomial random variable with parameters $n - 1$ and success probability $p = \lambda/n$. As a result, the limiting degree of any vertex is equal to a Poisson random variable with mean λ. It is well known that Poisson random variables have thinner tails than power laws. In fact, Poisson random variables have *exponential tails*.

Therefore, to model networks more appropriately, we are on the hunt for scale-free random graph models! Remarkably, the fact that the Erdős-Rényi random graph is not a suitable network model was already foreseen by the masters themselves [88]:

> "*Of course, if one aims at describing such a real situation, one should replace the hypothesis of equiprobability of all connections by some more realistic hypothesis.*".

How do power laws arise then in networks, and what can we learn from that? In the next section, we shall describe three models for scale-free networks.

1.5 Random graph models for complex networks

As explained in Section 1.4, Erdős-Rényi random graphs are *not* scale free, whereas, as explained in Section 1.1, many real networks are scale free. In Chapters 6, 7 and 8, we describe three scale-free random graph models. In Chapter 6, we describe the *generalized random graph*. The philosophy of this model is simple: we adapt the random graph in such a way that it becomes scale free. For this, we note that the degrees of the Erdős-Rényi random graph with edge probability $p = \lambda/n$ are close to to a Poisson random variable with mean λ. As mentioned before, these are *not* scale free. However, we can make these degrees scale free by taking the parameter λ to be a random variable with a power law. Thus, to each vertex i, we associate a random variable W_i, and, conditionally on W_i, the edges emanating from i will be occupied with a probability depending on i. There are many ways in which this can be done. For example, in the generalized random graph [53], the probability that edge between vertices s and t, which we denote by st, is occupied, conditionally on the weights $\{W_i\}_{i=1}^n$, is equal to

$$p_{st} = \frac{W_s W_t}{W_s W_t + L_n}, \tag{1.5.1}$$

where $L_n = \sum_{i=1}^n W_i$ is the total weight of the graph, and different edges are conditionally independent given $\{W_i\}_{i=1}^n$. In Chapter 6, we shall prove that this further randomization of the Erdős-Rényi random graph does, in the case when the W_i are i.i.d. and satisfy a power law, lead to scale-free graphs. There are various other possibilities to generalize the Erdős-Rényi random graph, some of which will also be

discussed. See [61, 171] for two specific examples, and [43] for the most general set-up of generalized random graphs.

In the second scale-free random graph model, the idea is that we should take the degrees as a start for the model. Thus, to each vertex i, we associate a degree D_i, and in some way connect up the different edges. Clearly, we need that the sum of the degrees $L_n = \sum_{i=1}^{n} D_i$ is *even*, and we shall assume this from now on. Then we think of placing D_i half-edges or stubs incident to vertex i, and connecting all the stubs in a certain way to yield a graph. One way to do this is to attach all the stub *uniformly*, and this leads to the *configuration model*. Naturally, it is possible that the above procedure does not lead to a simple graph, since self-loops and multiple edges can occur. As it turns out, when the degrees are not too large, more precisely, when they have *finite variance*, then the graph is with positive probability simple. By conditioning on the graph being simple, we end up with a uniform graph with the specified degrees. Sometimes this is also referred to as the *repeated configuration model*, since we can think of conditioning as repeatedly forming the graph until it is simple, which happens with strictly positive probability. A second approach to dealing with self-loops and multiple edges is simply to remove them, leading to the so-called *erased configuration model*. In Chapter 7, we investigate these two models, and show that the degrees are given by the degree distribution, when the graph size tends to infinity. Thus, the erasing and the conditioning do not alter the degrees too much.

The generalized random graph and configuration models *describe* networks, in some sense, quite satisfactorily. Indeed, they give rise to models with degrees that can be matched to degree distributions found in real networks. However, they do not *explain* how the networks came to be as they are. A possible explanation for the occurrence of scale-free behavior was given by Albert and Barabási [21], by a feature called *preferential attachment*. Most real networks grow. For example, the WWW has increased from a few web pages in 1990 to an estimated size of a few billion now. *Growth* is an aspect that is not taken into account in Erdős-Rényi random graphs, but it would not be hard to reformulate them as a growth process where elements are successively added, and connections are added and removed. Thus, growth by itself is not enough to explain the occurrence of power laws. However, viewing real networks as evolving in time does give us the possibility to investigate just *how* they grow.

So, how do real networks grow? Think of a social network describing a certain population in which a newcomer arrives, increasing it by one element. He/She will start to socialize with people in the population, and this process is responsible for the connections to the newly arrived person. In an Erdős-Rényi random graph, the connections to the newcomer will be spread uniformly over the population. Is this realistic? Is the newcomer not more likely to get to know people who are socially active, and, therefore, already have a larger degree? Probably so! We do not live in a perfectly egalitarian world. Rather, we live in a self-reinforcing world, where people who are successful are more likely to become even more successful! Therefore, rather than equal probabilities for our newcomer to acquaint him-/herself to other individuals in the population, there is a bias towards individuals who already know many people. When we think of the degree of elements as describing the *wealth* of the individuals in the population, we live in a world where the rich get richer!

Phrased in a more mathematical way, preferential attachment models are such that new elements are more likely to attach to elements with high degree compared to elements with small degree. For example, suppose that new elements are born with a fixed amount of edges to the older elements. Each edge is connected to a specific older element with a probability which is proportional to the degree of that older element. This phenomenon is now mostly called *preferential attachment*, and was first described informally by Albert and Barabási [21]. See also the book [20] for a highly readable and enthusiastic personal account by Barabási. Albert and Barabási have been two of the major players in the investigation of the similarities of real networks, and their papers have proved to be very influential. See [8, 9, 10, 21]. The notion of preferential attachment in networks has lead the theoretical physics and the mathematics communities

to study the structure of preferential attachment models in numerous papers. For some of the references, see Chapter 8.

While the above explanation is for social networks, also in other examples some form of preferential attachment is likely to be present. For example, in the WWW, when a new web page is created, it is more likely to link to an already popular site, such as Google, than to my personal web page. For the Internet, it may be profitable for new routers to be connected to highly connected routers, since these give rise to short distances. Even in biological networks, a more subtle form of preferential attachment exists.

In Chapter 8, we shall introduce and study preferential attachment models, and show that preferential attachment leads to scale-free random graphs. The power-law exponent of the degrees depends sensitively on the precise parameters of the model, such as the number of added edges and how dominant the preferential attachment effect is, in a similar way as the suggested power law exponent in the heuristic derivation in (1.2.6) depends on the parameters of that model.

In Chapters 6, 7 and 8, we investigate the *degrees* of the proposed random graph models. This explains the scale-free nature of the models. In Chapters 9, 10 and 11, we investigate further properties of these models, focussing on the *connected components* and the *distances* in the graphs. As observed in Section 1.1, most real networks are *small worlds*. As a result, one would hope that random graph models for real networks are such that distances between their elements are small. In Chapters 9, 10 and 11, we shall quantify this, and relate graph distances to the properties of the degrees. A further property we shall investigate is the *phase transition* of the largest connected component, as described in detail for the Erdős-Rényi random graph in Chapter 4.

Chapter 2

Probabilistic methods

In this chapter, we describe basic results in probability theory that we shall rely on in these notes. We describe convergence of random variables in Section 2.1, coupling in Section 2.2 and stochastic domination in Section 2.3. In Section 2.4 we describe bounds on random variables, namely the Markov inequality, the Chebychev inequality and the Chernoff bound. Particular attention will be given to binomial random variables, as they play a crucial role throughout these notes. In Section 2.5, we describe a few results on martingales. Finally, in Section 2.6, we describe some extreme value theory of random variables. In this chapter, not all proofs are given.

2.1 Convergence of random variables

In the random graph with $p = \lambda/n$, for some $\lambda > 0$, we note that the degree of a vertex is distributed as a $\mathrm{BIN}(n-1, p)$ random variable. When n is large, and $np = \lambda$ is fixed, then it is well known that a $\mathrm{BIN}(n-1, p)$ is close to a Poisson random variable with mean λ. In Chapter 4, we make heavy use of this convergence result, and a version of it is stated in Theorem 2.9 below.

In order to formalize that

$$\mathrm{BIN}(n, p) \approx \mathrm{Poi}(np), \tag{2.1.1}$$

we need to introduce the notions of convergence of random variables. For this, we note that random variables are defined to be functions on a sample space. It is well known that there are several possible notions for convergence of functions on function spaces. In a similar fashion, there are several notions of convergence of random variables, three of which we state in the following definition. For more background on the convergence of random variables, we refer the reader to [35].

Definition 2.1 (Convergence of random variables)**.**

(a) *A sequence X_n of random variables converges in distribution to a limiting random variable X when*

$$\lim_{n\to\infty} \mathbb{P}(X_n \leq x) = \mathbb{P}(X \leq x), \tag{2.1.2}$$

for every x for which $F(x) = \mathbb{P}(X \leq x)$ is continuous. We write this as $X_n \xrightarrow{d} X$.

(b) *A sequence X_n of random variables converges in probability to a limiting random variable X when, for every $\varepsilon > 0$*

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0. \tag{2.1.3}$$

We write this as $X_n \xrightarrow{\mathbb{P}} X$.

(c) *A sequence X_n of random variables converges almost surely to a limiting random variable X when*

$$\mathbb{P}(\lim_{n \to \infty} X_n = X) = 1. \tag{2.1.4}$$

We write this as $X_n \xrightarrow{a.s.} X$.

It is not hard to see that convergence in probability implies convergence in distribution. The notion of convergence almost surely is clearly the most difficult to grasp. It turns out that convergence almost surely implies convergence in probability, making it the strongest version of convergence to be discussed in these notes. We shall mainly work with convergence in distribution and convergence in probability.

There are also further forms of convergence that we do not discuss, such as convergence in L^1 or L^2. We again refer to [35], or to introductory books in probability, such as [36, 96, 97, 102].

There are examples where convergence in distribution holds, but convergence in probability fails:

Exercise 2.1. *Find an example of a sequence of random variables where convergence in distribution occurs, but convergence in probability does not.*

Exercise 2.2. *Show that the sequence of random variables $\{X_n\}_{n=1}^{\infty}$, where X_n takes the value n with probability $\frac{1}{n}$ and 0 with probability $1 - \frac{1}{n}$ converges both in distribution and in probability to 0.*

We next state some theorems that give convenient criterions by which we can conclude that random variables converge in distribution. In their statement, we make use of a number of functions of random variables that we introduce now.

Definition 2.2 (Generating functions of random variables). *Let X be a random variable. Then*

(a) *The characteristic function of X is the function*

$$\phi_X(t) = \mathbb{E}[e^{itX}], \qquad t \in \mathbb{R}. \tag{2.1.5}$$

(b) *The probability generating function of X is the function*

$$G_X(t) = \mathbb{E}[t^X], \qquad t \in \mathbb{R}. \tag{2.1.6}$$

(c) *The moment generating function of X is the function*

$$M_X(t) = \mathbb{E}[e^{tX}], \qquad t \in \mathbb{R}. \tag{2.1.7}$$

We note that the characteristic function exists for every random variable X, since $|e^{itX}| = 1$ for every t. The moment generating function, however, does not always exist.

Exercise 2.3. *Find a random variable for which the moment generating function is equal to $+\infty$ for every $t \neq 0$.*

Theorem 2.3 (Criteria for convergence in distribution). *The sequence of random variables $\{X_n\}_{n=1}^{\infty}$ converges in distribution to a random variable X*

(a) *if and only if the characteristic functions $\phi_n(t)$ of X_n converge to the characteristic function $\phi_X(t)$ of X for all $t \in \mathbb{R}$.*

(b) when, for some $\varepsilon > 0$, the moment generating functions $M_n(t)$ of X_n converge to the moment generating function $M_X(t)$ of X for all $|t| < \varepsilon$.

(c) when, for some $\varepsilon > 0$, the probability generating functions $G_n(t)$ of X_n converge to the probability generating function $G_X(t)$ of X for all $|t| < 1 + \varepsilon$ for some $\varepsilon > 0$.

(d) when the X_n are non-negative and integer-valued, and the moments $\mathbb{E}[X_n^r]$ converge to the moments $\mathbb{E}[X^r]$ of X for each $r = 1, 2, \ldots$, provided the moments of X satisfy

$$\lim_{r \to \infty} \frac{\mathbb{E}[X^r] r^m}{r!} = 0 \qquad \forall m = 0, 1, \ldots \tag{2.1.8}$$

(e) when the moments $\mathbb{E}[X_n^r]$ converge to the moments $\mathbb{E}[X^r]$ of X for each $r = 1, 2, \ldots$, and $M_X(t)$, the moment generating function of X, is finite for t in some neighborhood of the origin.

Exercise 2.4. *Show that a Poisson random variable satisfies the moment condition in* (2.1.8).

Exercise 2.5. *Compute the factorial moments of a binomial random variable with parameters n and $p = \lambda/n$ and the ones of a Poisson random variable with mean λ, and use this to conclude that a binomial random variable with parameters n and $p = \lambda/n$ converges in distribution to a Poisson random variable with mean λ.*

Exercise 2.6. *Show that the moments of a Poisson random variable X with mean λ satisfy the recursion*

$$\mathbb{E}[X^m] = \lambda \mathbb{E}[(X+1)^{m-1}]. \tag{2.1.9}$$

We finally discuss a special case of convergence in distribution, namely, when we deal with a sum of indicators, and the limit is a Poisson random variable. We write $(X)_r = X(X-1)\cdots(X-r+1)$, so that $\mathbb{E}[(X)_r]$ is the rth *factorial moment* of X.

For a random variable X taking values in $\{0, 1, \ldots, n\}$, the factorial moments of X uniquely determine the probability mass function, since

$$\mathbb{P}(X = k) = \sum_{r=k}^{n} (-1)^{k+r} \frac{\mathbb{E}[(X)_r]}{(r-k)! k!}, \tag{2.1.10}$$

see e.g. [40, Corollary 1.11]. To see (2.1.10), we write

$$\mathbb{1}_{\{X=k\}} = \binom{X}{k} \left(1 - 1\right)^{X-k}, \tag{2.1.11}$$

using the convention that $0^0 = 1$. Then, by Newton's binomial, we obtain

$$\mathbb{1}_{\{X=k\}} = \binom{X}{k} \sum_{i=0}^{X-k} (-1)^i \binom{X-k}{i} = \sum_{i=0}^{\infty} (-1)^i \binom{X}{k} \binom{X-k}{i}, \tag{2.1.12}$$

where, by convention, we take that $\binom{n}{k} = 0$ when $k < 0$ or $k > n$. Rearranging the binomials, we arrive at

$$\mathbb{1}_{\{X=k\}} = \sum_{r=k}^{\infty} (-1)^{k+r} \frac{(X)_r}{(r-k)! k!}, \tag{2.1.13}$$

where $r = k + i$, and taking expectations yields

$$\mathbb{P}(X = k) = \sum_{r=k}^{\infty} (-1)^{k+r} \frac{\mathbb{E}[(X)_r]}{(r-k)!k!}, \tag{2.1.14}$$

which is (2.1.10). Similar results also hold for unbounded random variables, since the sum

$$\sum_{r=k}^{n} (-1)^{k+r} \frac{\mathbb{E}[(X)_r]}{(r-k)!k!} \tag{2.1.15}$$

is alternatingly smaller than $\mathbb{P}(X = k)$ (for n even) and larger than $\mathbb{P}(X = k)$ (for n odd). This implies the following result:

Theorem 2.4 (Convergence to a Poisson random variable). *A sequence of integer-valued random variables* $\{X_n\}_{n=1}^{\infty}$ *converges in distribution to a Poisson random variable with parameter λ when, for all $r = 1, 2, \ldots$,*

$$\lim_{n \to \infty} \mathbb{E}[(X_n)_r] = \lambda^r. \tag{2.1.16}$$

Exercise 2.7. *Show that if*

$$\lim_{n \to \infty} \sum_{r \geq n} \frac{\mathbb{E}[(X)_r]}{(r-k)!} = 0, \tag{2.1.17}$$

then also

$$\mathbb{P}(X = k) = \sum_{r=k}^{\infty} (-1)^{k+r} \frac{\mathbb{E}[(X)_r]}{(r-k)!k!}, \tag{2.1.18}$$

and use this to conclude that when $\lim_{n \to \infty} \mathbb{E}[(X_n)_r] = \mathbb{E}[(X)_r]$ for all $r \geq 1$, where X_n and X are all integer-valued non-negative random variables, then also $X_n \xrightarrow{d} X$.

Theorem 2.4 is particularly convenient when dealing with sums of indicators, i.e., when

$$X_n = \sum_{i \in \mathcal{I}_n} I_{i,n}, \tag{2.1.19}$$

where $I_{i,n}$ takes the values 0 and 1 only, as the following result shows:

Theorem 2.5 (Factorial moments of sums of indicators). *When $X = \sum_{i \in \mathcal{I}} I_i$ is a sum of indicators, then*

$$\mathbb{E}[(X)_r] = \sum_{i_1, \ldots, i_r \in \mathcal{I}}^{*} \mathbb{E}[\prod_{l=1}^{r} I_{i_l}] = \sum_{i_1, \ldots, i_r \in \mathcal{I}}^{*} \mathbb{P}(I_{i_1} = \cdots = I_{i_r} = 1), \tag{2.1.20}$$

where $\sum_{i_1, \ldots, i_r \in \mathcal{I}}^{*}$ denotes a sum over distinct indices.

Exercise 2.8. *Prove (2.1.20) for $r = 2$.*

Exercise 2.9. *Prove that when X is a Poisson random variable with mean λ, then*

$$\mathbb{E}[(X)_r] = \lambda^r. \tag{2.1.21}$$

Proof of Theorem 2.5. We prove (2.1.20) by induction on $r \geq 1$ and for all probability measures $\mathbb{P}$ and corresponding expectations $\mathbb{E}$. For $r = 1$, we have that $(X)_1 = X$, and (2.1.20) follows from the fact that the expectation of a sum of random variables is the sum of expectations. This initializes the induction hypothesis.

In order to advance the induction hypothesis, we first note that it suffices to prove the statement for indicators I_i for which $\mathbb{P}(I_i = 1) > 0$. Then, for $r \geq 2$, we write out

$$\mathbb{E}[(X)_r] = \sum_{i_1 \in \mathcal{I}} \mathbb{E}\big[I_{i_1}(X-1)\cdots(X-r+1)\big]. \tag{2.1.22}$$

Denote by $\mathbb{P}_{i_1}$ the conditional distribution given that $I_{i_1} = 1$, i.e., for any event E, we have

$$\mathbb{P}_{i_1}(E) = \frac{\mathbb{P}(E \cap \{I_{i_1} = 1\})}{\mathbb{P}(I_{i_1} = 1)}. \tag{2.1.23}$$

Then we can rewrite

$$\mathbb{E}\big[I_{i_1}(X-1)\cdots(X-r+1)\big] = \mathbb{P}(I_{i_1} = 1)\mathbb{E}_{i_1}\big[(X-1)\cdots(X-r+1)\big]. \tag{2.1.24}$$

We define

$$Y = X - I_{i_1} = \sum_{j \in \mathcal{I}\setminus\{i_1\}} I_j, \tag{2.1.25}$$

and note that, conditionally on $I_{i_1} = 1$, we have that $X = Y + 1$. As a result, we obtain that

$$\mathbb{E}_{i_1}\big[(X-1)\cdots(X-r+1)\big] = \mathbb{E}_{i_1}\big[Y\cdots(Y-r+2)\big] = \mathbb{E}_{i_1}\big[(Y)_{r-1}\big]. \tag{2.1.26}$$

We now apply the induction hypothesis to $\mathbb{E}_{i_1}\big[(Y)_{r-1}\big]$, to obtain

$$\mathbb{E}_{i_1}\big[(Y)_{r-1}\big] = \sum_{i_2,\ldots,i_r \in \mathcal{I}\setminus\{i_1\}}^{*} \mathbb{P}_{i_1}\big(I_{i_2} = \cdots = I_{i_r} = 1\big). \tag{2.1.27}$$

As a result, we arrive at

$$\mathbb{E}[(X)_r] = \sum_{i_1 \in \mathcal{I}} \mathbb{P}(I_{i_1} = 1) \sum_{i_2,\ldots,i_r \in \mathcal{I}\setminus\{i_1\}}^{*} \mathbb{P}_{i_1}\big(I_{i_2} = \cdots = I_{i_r} = 1\big). \tag{2.1.28}$$

We complete the proof by noting that

$$\mathbb{P}(I_{i_1} = 1)\mathbb{P}_{i_1}\big(I_{i_2} = \cdots = I_{i_r} = 1\big) = \mathbb{P}\big(I_{i_1} = I_{i_2} = \cdots = I_{i_r} = 1\big), \tag{2.1.29}$$

and that

$$\sum_{i_1 \in \mathcal{I}} \sum_{i_2,\ldots,i_r \in \mathcal{I}\setminus\{i_1\}}^{*} = \sum_{i_1,\ldots,i_r \in \mathcal{I}}^{*}. \tag{2.1.30}$$

$\square$

There also exist multidimensional versions of Theorems 2.4 and 2.5:

Theorem 2.6 (Convergence to independent Poisson random variables). *A vector of integer-valued random variables $\{(X_{1,n},\ldots,X_{d,n})\}_{n=1}^{\infty}$ converges in distribution to a vector of independent Poisson random variable with parameters $\lambda_1,\ldots,\lambda_d$ when, for all $r_1,\ldots,r_d \in \mathbb{N}$,*

e:convPoifactmul

$$\lim_{n\to\infty} \mathbb{E}[(X_{1,n})_{r_1}\cdots(X_{d,n})_{r_d}] = \lambda_1^{r_1}\cdots\lambda_d^{r_d}. \tag{2.1.31}$$

Theorem 2.7 (Factorial moments of sums of indicators). *When $X_l = \sum_{i \in \mathcal{I}_l} I_{i,l}$ for all $l = 1, \ldots, d$ are sums of indicators, then*

$$\mathbb{E}[(X_{1,n})_{r_1} \cdots (X_{d,n})_{r_d}] = \sum_{i_1^{(1)}, \ldots, i_{r_1}^{(1)} \in \mathcal{I}_1}^{*} \cdots \sum_{i_1^{(d)}, \ldots, i_{r_d}^{(d)} \in \mathcal{I}_d}^{*} \mathbb{P}(I_{i_s}^{(l)} = 1 \forall l = 1, \ldots, d \,\&\, s = 1, \ldots, r_l).$$

<div align="right">e:factmomsumindmul
(2.1.32)</div>

Exercise 2.10. *Prove Theorem 2.7 using Theorem 2.5.*

The fact that the convergence of moments as in Theorems 2.3, 2.4 and 2.6 yields convergence in distribution is sometimes called the *method of moments*, and is a good way of proving convergence results.

2.2   Coupling

For any λ fixed, it is well known that, when $n \to \infty$,

$$\mathrm{BIN}(n, \lambda/n) \xrightarrow{\mathbb{P}} \mathrm{Poi}(\lambda).$$

<div align="right">BINPoi
(2.2.1)</div>

In general, convergence in probability implies convergence in distribution, so that also convergence in distribution follows. To prove this convergence, we will use a *coupling* proof. Couplings will be quite useful in what follows, so we will discuss couplings, as well as the related topic of *stochastic orderings*, in detail. An excellent treatment of coupling theory is given in [198], to which we refer for more details.

In general, two random variables X and Y are *coupled* when they are defined on the same probability space. This means that there is one probability law $\mathbb{P}$ such that $\mathbb{P}(X \in E, Y \in F)$ are defined for all events E and F. This is formalized in the following definition, where it is also generalized to more than one random variable:

Definition 2.8 (Coupling of random variables). *The random variables $(\hat{X}_1, \ldots, \hat{X}_n)$ are a coupling of the random variables $X_1, \ldots, X_n$ when $(\hat{X}_1, \ldots, \hat{X}_n)$ are defined on the same probability space, and are such that the marginal distribution of $\hat{X}_i$ is the same as the distribution of X_i for all $i = 1, \ldots, n$, i.e., for all measurable subsets E of $\mathbb{R}$, we have*

$$\mathbb{P}(\hat{X}_i \in E) = \mathbb{P}(X_i \in E). \tag{2.2.2}$$

The key point of Definition 2.8 is that while the random variables $X_1, \ldots, X_n$ may be defined on *different* probability spaces, the coupled random variables $(\hat{X}_1, \ldots, \hat{X}_n)$ are defined on the *same* probability space. The coupled random variables $(\hat{X}_1, \ldots, \hat{X}_n)$ are related to the original random variables $X_1, \ldots, X_n$ by the fact that the marginal distributions of $(\hat{X}_1, \ldots, \hat{X}_n)$ are equal to the random variables $X_1, \ldots, X_n$. Note that one coupling arises by taking $(\hat{X}_1, \ldots, \hat{X}_n)$ to be independent, with $\hat{X}_i$ having the same distribution as X_i. However, in our proofs, we shall often make use of more elaborate couplings, which give rise to stronger results.

Couplings are very useful to prove that random variables are somehow related. We now describe a general coupling between two random variables which makes two random variables be with high probability *equal*. We let X and Y be two random variables with

$$\mathbb{P}(X = x) = p_x, \qquad \mathbb{P}(Y = y) = q_y, \qquad x \in \mathcal{X}, y \in \mathcal{Y} \tag{2.2.3}$$

where $\{p_x\}_{x\in\mathcal{X}}$ and $\{q_y\}_{y\in\mathcal{Y}}$ are any two probability mass functions on two subsets $\mathcal{X}$ and $\mathcal{Y}$ of the same space. Then, we define the random vector $(\hat{X}, \hat{Y})$ by

$$\mathbb{P}(\hat{X} = \hat{Y} = x) = \min\{p_x, q_x\}, \tag{2.2.4}$$

$$\mathbb{P}(\hat{X} = x, \hat{Y} = y) = \frac{(p_x - \min\{p_x, q_x\})(q_y - \min\{p_y, q_y\})}{\frac{1}{2}\sum_z |p_z - q_z|}, \qquad x \neq y. \tag{2.2.5}$$

First of all, this is a probability distribution, since

$$\sum_x (p_x - \min\{p_x, q_x\}) = \sum_x (q_x - \min\{p_x, q_x\}) = \frac{1}{2}\sum_x |p_x - q_x|. \tag{2.2.6}$$

Exercise 2.11. *Prove* (2.2.6).

The distance between discrete probability distributions $\{p_x\}_{x\in\mathcal{X}}$ and $\{q_x\}_{x\in\mathcal{X}}$ in (2.2.6) is called the *total variation distance* between the discrete probability mass functions $\{p_x\}_{x\in\mathcal{X}}$ and $\{q_x\}_{x\in\mathcal{X}}$. In general, for two probability measures μ and ν, the total variation distance is given by

$$d_{\mathrm{TV}}(\mu, \nu) = \max_A |\mu(A) - \nu(A)|, \tag{2.2.7}$$

where $\mu(A)$ and $\nu(A)$ are the probabilities of the event A under the measures μ and ν.

When μ and ν are the distribution functions corresponding to two discrete probability mass functions $p = \{p_x\}_{x\in\mathcal{X}}$ and $q = \{q_x\}_{x\in\mathcal{X}}$, so that, for every measurable A with $A \subset \mathcal{X}$, we have

$$\mu(A) = \sum_{x\in A} p_x, \qquad \nu(A) = \sum_{x\in A} q_x, \tag{2.2.8}$$

then it is not hard to see that

$$d_{\mathrm{TV}}(p, q) = \frac{1}{2}\sum_x |p_x - q_x|. \tag{2.2.9}$$

When F and G are the distribution functions corresponding to two continuous densities $f = \{f(x)\}_{x\in\mathbb{R}}$ and $g = \{g(x)\}_{x\in\mathbb{R}}$, so that for every measurable $A \subseteq \mathbb{R}$,

$$\mu(A) = \int_A f(x)dx, \qquad \nu(A) = \int_A g(x)dx, \tag{2.2.10}$$

then we obtain

$$d_{\mathrm{TV}}(f, g) = \frac{1}{2}\int_{-\infty}^{\infty} |f(x) - g(x)|dx. \tag{2.2.11}$$

Exercise 2.12. *Prove* (2.2.9) *and* (2.2.11).

We now continue investigating the coupling in (2.2.4) for two discrete random variables. By construction,

$$\mathbb{P}(\hat{X} = x) = p_x, \qquad \mathbb{P}(\hat{Y} = y) = q_y, \tag{2.2.12}$$

so that $\hat{X}$ and $\hat{Y}$ have the right marginal distributions as required in Definition 2.8. Moreover, we have that, by (2.2.6),

$$\mathbb{P}(\hat{X} \neq \hat{Y}) = \sum_{x,y} \frac{(p_x - \min\{p_x, q_x\})(q_y - \min\{p_y, q_y\})}{\frac{1}{2}\sum_z |p_z - q_z|}$$

$$= \frac{1}{2}\sum_x |p_x - q_x| = d_{\mathrm{TV}}(p, q). \tag{2.2.13}$$

It turns out that this is an optimal or maximal coupling. See [198] for details. Indeed, we have that for all x,

$$\mathbb{P}(\hat{X} = \hat{Y} = x) \le \mathbb{P}(\hat{X} = x) = \mathbb{P}(X = x) = p_x, \tag{2.2.14}$$

and also

$$\mathbb{P}(\hat{X} = \hat{Y} = x) \le \mathbb{P}(\hat{Y} = x) = \mathbb{P}(Y = x) = q_x, \tag{2.2.15}$$

so that for any coupling we must have that

$$\mathbb{P}(\hat{X} = \hat{Y} = x) \le \min\{p_x, q_x\}. \tag{2.2.16}$$

Therefore, any coupling must be such that

$$\mathbb{P}(\hat{X} = \hat{Y}) = \sum_x \mathbb{P}(\hat{X} = \hat{Y} = x) \le \sum_x \min\{p_x, q_x\}. \tag{2.2.17}$$

As a result, we have that, for *any* coupling,

$$\mathbb{P}(\hat{X} \ne \hat{Y}) \ge 1 - \sum_x \min\{p_x, q_x\} = \frac{1}{2} \sum_x |p_x - q_x|. \tag{2.2.18}$$

The coupling in (2.2.4) attains this equality, which makes it the best coupling possible, in the sense that it maximizes $\mathbb{P}(\hat{X} = \hat{Y})$.

In these notes, we will often work with binomial random variables which we wish to compare to Poisson random variables. We will make use of the following theorem, which will be proved using a coupling argument:

Theorem 2.9 (Poisson limit for binomial random variables). *let $\{I_i\}_{i=1}^n$ be independent with $I_i \sim \mathrm{BE}(p_i)$, and let $\lambda = \sum_{i=1}^n p_i$. Let $X = \sum_{i=1}^n I_i$ and let Y be a Poisson random variable with parameter λ. Then, there exists a coupling $(\hat{X}, \hat{Y})$ of (X, Y) such that*

$$\mathbb{P}(\hat{X} \ne \hat{Y}) \le \sum_{i=1}^n p_i^2. \tag{2.2.19}$$

Consequently, for every $\lambda \ge 0$ and $n \in \mathbb{N}$, there exists a coupling $(\hat{X}, \hat{Y})$, where $\hat{X} \sim \mathrm{BIN}(n, \lambda/n)$ and $\hat{Y} \sim \mathrm{Poi}(\lambda)$ such that

$$\mathbb{P}(\hat{X} \ne \hat{Y}) \le \frac{\lambda^2}{n}. \tag{2.2.20}$$

Exercise 2.13. *Let $X \sim \mathrm{BIN}(n, \lambda/n)$ and $Y \sim \mathrm{Poi}(\lambda)$. Write $f_i = \mathbb{P}(X = i)$ and $g_i = \mathbb{P}(Y = i)$. Prove that Theorem 2.9 implies that $d_{\mathrm{TV}}(f, g) \le \lambda^2/n$. Conclude also that, for every $i \in \mathbb{N}$,*

$$\left| \mathbb{P}(X = i) - \mathbb{P}(Y = i) \right| \le \lambda^2/n. \tag{2.2.21}$$

Proof of Theorem 2.9. Throughout the proof, we let $I_i \sim \mathrm{BE}(p_i)$ and assume that $\{I_i\}_{i=1}^n$ are independent, and we let $J_i \sim \mathrm{Poi}(p_i)$ and assume that $\{J_i\}_{i=1}^n$ are independent. In the proof, we write

$$p_{i,x} = \mathbb{P}(I_i = x) = p_i \mathbb{1}_{\{x=1\}} + (1 - p_i) \mathbb{1}_{\{x=0\}}, \qquad q_{i,x} = \mathbb{P}(J_i = x) = e^{-p_i} \frac{p_i^x}{x!} \tag{2.2.22}$$

for the Bernoulli and Poisson probability mass functions.

For each pair I_i, J_i, the maximal coupling $(\hat{I}_i, \hat{J}_i)$ described above satisfies

$$\mathbb{P}(\hat{I}_i = \hat{J}_i = x) = \min\{p_{1,x}, q_{1,x}\} = \begin{cases} 1 - p_i & \text{for } x = 0, \\ p_i e^{-p_i} & \text{for } x = 1, \\ 0 & \text{for } x \geq 2, \end{cases} \qquad (2.2.23)$$

where we have used the inequality $1 - p_i \leq e^{-p_i}$ for $x = 0$. Thus, now using that $1 - e^{-p_i} \leq p_i$,

$$\mathbb{P}(\hat{I}_i \neq \hat{J}_i) = 1 - \mathbb{P}(\hat{I}_i = \hat{J}_i) = 1 - (1 - p_i) - p_i e^{-p_i} = p_i(1 - e^{-p_i}) \leq p_i^2. \qquad \overset{\text{e:couplingn=1}}{(2.2.24)}$$

Next, let $\hat{X} = \sum_{i=1}^n \hat{I}_i$ and $\hat{Y} = \sum_{i=1}^n \hat{J}_i$. Then, $\hat{X}$ has the same distribution as $X = \sum_{i=1}^n I_i$, and $\hat{Y}$ has the same distribution as $Y = \sum_{i=1}^n J_i \sim \text{Poi}(p_1 + \cdots + p_n)$. Finally, by Boole's inequality and (2.2.24),

$$\mathbb{P}(\hat{X} \neq \hat{Y}) \leq \mathbb{P}\Big(\bigcup_{i=1}^n \{\hat{I}_i \neq \hat{J}_i\} \Big) \leq \sum_{i=1}^n \mathbb{P}(\hat{I}_i \neq \hat{J}_i) \leq \sum_{i=1}^n p_i^2. \qquad (2.2.25)$$

This completes the proof of Theorem 2.9. $\qquad \square$

For $p = \{p_x\}$ and $q = \{q_x\}$, the total variation distance $d_{\text{TV}}(p, q)$ is obviously larger than $\frac{1}{2}|p_x - g_x|$, so that convergence in total variation distance of $p(n) = \{p_x(n)\}$ to a probability mass function $p = \{p_x\}$ implies pointwise convergence of the probability mass functions $\lim_{n\to\infty} p_x(n) = p_x$ for every x. Interestingly, it turns out that these notions are equivalent:

Exercise 2.14. *Show that if* $\lim_{n\to\infty} p_x(n) = p_x$ *for every* x, *and* $p = \{p_x\}$ *is a probability mass function, then also* $\lim_{n\to\infty} d_{\text{TV}}(p(n), p) = 0$.

2.3 Stochastic ordering

To compare random variables, we will rely on the notion of *stochastic ordering*, which is defined as follows:

Definition 2.10 (Stochastic domination). *Let* X *and* Y *be two random variables, not necessarily living on the same probability space. The random variable* X *is stochastically smaller than the random variable* Y *when, for every* $x \in \mathbb{R}$, *the inequality*

$$\mathbb{P}(X \leq x) \geq \mathbb{P}(Y \leq x) \qquad \overset{\text{stochdom}}{(2.3.1)}$$

holds. We denote this by $X \preceq Y$.

A nice coupling reformulation of stochastic ordering is presented in the following lemma:

Lemma 2.11 (Coupling definition of stochastic domination). *The random variable* X *is stochastically smaller than the random variable* Y *if and only if there exists a coupling* $(\hat{X}, \hat{Y})$ *of* X, Y *such that*

$$\mathbb{P}(\hat{X} \leq \hat{Y}) = 1. \qquad (2.3.2)$$

Proof. When $\mathbb{P}(\hat{X} \leq \hat{Y}) = 1$, then

$$\mathbb{P}(Y \leq x) = \mathbb{P}(\hat{Y} \leq x) = \mathbb{P}(\hat{X} \leq \hat{Y} \leq x)$$
$$\leq \mathbb{P}(\hat{X} \leq x) = \mathbb{P}(X \leq x), \qquad (2.3.3)$$

so that X is stochastically smaller than Y.

For the other direction, suppose that X is stochastically smaller than Y. We define the generalized inverse of a distribution function F by

$$F^{-1}(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\}, \qquad\qquad (2.3.4)$$

where $u \in [0, 1]$. If U is a uniform random variable on $[0, 1]$, then it is well-known that the random variable $F^{-1}(U)$ has distribution function F. This follows since the function F^{-1} is such that

$$F^{-1}(u) > x \qquad \text{precisely when} \qquad u > F(x). \qquad\qquad (2.3.5)$$

Denote by F_X and F_Y the marginal distribution functions of X and Y. Then (2.3.1) is equivalent to

$$F_X(x) \geq F_Y(x) \qquad\qquad (2.3.6)$$

for all x. It follows that, for all $u \in [0, 1]$,

$$F_X^{-1}(u) \leq F_Y^{-1}(u). \qquad\qquad (2.3.7)$$

Therefore, since $\hat{X} = F_X^{-1}(U)$ and $\hat{Y} = F_Y^{-1}(U)$ have the same marginal distributions as X and Y, respectively, it follows that

$$\mathbb{P}(\hat{X} \leq \hat{Y}) = \mathbb{P}(F_X^{-1}(U) \leq F_Y^{-1}(U)) = 1. \qquad\qquad (2.3.8)$$

$$\square$$

There are many examples of pairs of random variables which are stochastically ordered, and we will now describe a few.

Binomial random variables. A simple example of random variables which are stochastically ordered is as follows. Let $m, n \in \mathbb{N}$ be integers such that $m \leq n$. Let $X \sim \text{BIN}(m, p)$ and $Y \sim \text{BIN}(n, p)$. Then, we claim that $X \preceq Y$. To see this, let $\hat{X} = \sum_{i=1}^{m} I_i$ and $\hat{Y} = \sum_{i=1}^{n} I_i$, where $\{I_i\}_{i=1}^{\infty}$ is an i.i.d. sequence of Bernoulli random variables, i.e.,

$$\mathbb{P}(I_i = 1) = 1 - \mathbb{P}(I_i = 0) = p, \qquad\qquad i = 1, \dots, n, \qquad\qquad (2.3.9)$$

and $I_1, I_2, \dots, I_n$ are mutually independent. Then, since $I_i \geq 0$ for each i, we have that

$$\mathbb{P}(\hat{X} \leq \hat{Y}) = 1. \qquad\qquad (2.3.10)$$

Therefore, $X \preceq Y$.

The stochastic domination above also holds when $X = \text{BIN}(n - Z, p)$ and $Y = \text{BIN}(n, p)$, when Z is any random variable that takes non-negative integer values. This domination result will prove to be useful in the investigation of the Erdős-Rényi random graph.

Poisson random variables. Another example of random variables which are stochastically ordered is as follows. Let $\lambda, \mu \in \mathbb{R}$ be such that $\lambda \leq \mu$. Let $X \sim \text{Poi}(\lambda)$ and $Y \sim \text{Poi}(\mu)$. Then, $X \preceq Y$. To see this, let $\hat{X} \sim \text{Poi}(\lambda), \hat{Z} \sim \text{Poi}(\mu - \lambda)$, where $\hat{X}$ and $\hat{Z}$ are independent, and let $\hat{Y} = \hat{X} + \hat{Z}$. Then, $\hat{Y} \sim \text{Poi}(\mu)$. Moreover, since $\hat{Z} \geq 0$ for each i, we have that

$$\mathbb{P}(\hat{X} \leq \hat{Y}) = 1. \qquad\qquad (2.3.11)$$

Therefore, $X \preceq Y$.

Exercise 2.15. *Let X and Y be normal distributions with equal variances σ^2 and means $\mu_X \leq \mu_Y$. Is $X \preceq Y$?*

Exercise 2.16. *Let X and Y be normal distributions with variances $\sigma_X^2 < \sigma_Y^2$ and equal means μ. Is $X \preceq Y$?*

2.3.1 Consequences of stochastic domination

In this section, we discuss a number of consequences of stochastic domination, such as the fact that the means of a stochastically ordered pair of random variables is ordered as well.

Theorem 2.12 (Ordering of means for stochastically ordered random variables)**.** *Suppose $X \preceq Y$. Then*

$$\mathbb{E}[X] \leq \mathbb{E}[Y].\tag{2.3.12}$$

Proof. We apply Lemma 2.11. Let $\hat{X}$ and $\hat{Y}$ have the same law as X and Y, and be such that $\hat{X} \leq \hat{Y}$ with probability 1. Then

$$\mathbb{E}[X] = \mathbb{E}[\hat{X}] \leq \mathbb{E}[\hat{Y}] = \mathbb{E}[Y].\tag{2.3.13}$$

$\square$

Theorem 2.13 (Preservation of ordering under monotone functions)**.** *Suppose $X \preceq Y$, and $g\colon \mathbb{R} \to \mathbb{R}$ is non-decreasing. Then $g(X) \preceq g(Y)$.*

Proof. Let $\hat{X}$ and $\hat{Y}$ have the same laws as X and Y and be such that $\hat{X} \leq \hat{Y}$ (see Lemma 2.11). Then, $g(\hat{X})$ and $g(\hat{Y})$ have the same distributions as $g(X)$ and $g(Y)$, and $g(\hat{X}) \leq g(\hat{Y})$ with probability one, by the fact that g is non-decreasing. Therefore, by Lemma 2.11, the claim follows. $\square$

2.4 Probabilistic bounds

We will often make use of a number of probabilistic bounds, which we will summarise and prove in this section.

Theorem 2.14 (Markov inequality)**.** *Let X be a non-negative random variable with $\mathbb{E}[X] < \infty$. Then,*

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}.\tag{2.4.1}$$

In particular, when X is integer valued with $\mathbb{E}[X] \leq m$, then

$$\mathbb{P}(X = 0) \geq 1 - m.\tag{2.4.2}$$

By (2.4.2), if the integer random variable has a small mean, then it must be equal to 0 with high probability. This is called the *first moment method*, and is a powerful tool to prove results.

Proof. Equation (2.4.1) follows by

$$a\mathbb{P}(X \geq a) \leq \mathbb{E}[X\mathbb{1}_{\{X \geq a\}}] \leq \mathbb{E}[X].\tag{2.4.3}$$

$\square$

Theorem 2.15 (Chebychev inequality)**.** *Assume that X is integer valued with $\mathrm{Var}(X) = \sigma^2$. Then,*

$$\mathbb{P}\Big(|X - \mathbb{E}[X]| \geq a\Big) \leq \frac{\sigma^2}{a^2}.\tag{2.4.4}$$

In particular, when X is integer valued with $\mathbb{E}[X] \geq m$ and $\mathrm{Var}(X) = \sigma^2$, then

$$\mathbb{P}(X = 0) \leq \frac{\sigma^2}{m^2}.\tag{2.4.5}$$

By (2.4.5), if the integer random variable has a large mean, and a variance which is small compared to the square of the mean, then it must be positive with high probability. This is called the *second moment method.*

Proof. For (2.4.4), we note that

$$\mathbb{P}\Big(|X - \mathbb{E}[X]| \geq a\Big) = \mathbb{P}\Big((X - \mathbb{E}[X])^2 \geq a^2\Big), \tag{2.4.6}$$

and apply the Markov inequality. For (2.4.5), we note that

$$\mathbb{P}(X = 0) \leq \mathbb{P}(|X - \mathbb{E}[X]| \geq \mathbb{E}[X]) \leq \frac{\text{Var}(X)}{\mathbb{E}[X]^2} \leq \frac{\sigma^2}{m^2}. \tag{2.4.7}$$

$\square$

We will often rely on bounds on the probability that a sum of independent random variables is larger than its expectation. For such probabilities, *large deviation theory* gives good bounds. We will describe these bounds here. For more background on large deviations, we refer the reader to [76, 116, 174].

Theorem 2.16 (Cramér's upper bound, Chernoff bound)**.** *Let* $\{X_i\}_{i=1}^\infty$ *be a sequence of i.i.d. random variables. Then, for all* $a \geq \mathbb{E}[X_1]$,

$$\mathbb{P}\Big(\sum_{i=1}^n X_i \geq na\Big) \leq e^{-nI(a)}, \tag{2.4.8}$$

while, for all $a \leq \mathbb{E}[X_1]$,

$$\mathbb{P}\Big(\sum_{i=1}^n X_i \leq na\Big) \leq e^{-nI(a)}, \tag{2.4.9}$$

where, for $a \geq \mathbb{E}[X_1]$,

$$I(a) = \sup_{t \geq 0} \Big(ta - \log \mathbb{E}[e^{tX_1}]\Big), \tag{2.4.10}$$

while, for $a \leq \mathbb{E}[X_1]$,

$$I(a) = \sup_{t \leq 0} \Big(ta - \log \mathbb{E}[e^{tX_1}]\Big). \tag{2.4.11}$$

Note that the function $t \mapsto ta - \log \mathbb{E}[e^{tX_1}]$ is concave, and the derivative in 0 is $a - \mathbb{E}[X_1] \geq 0$ for $a \geq \mathbb{E}[X_1]$. Therefore, for $a \geq \mathbb{E}[X_1]$, the supremum of $t \mapsto (ta - \log \mathbb{E}[e^{tX_1}])$ will be attained for a $t \geq 0$ when $\mathbb{E}[e^{tX_1}]$ exists in a neighborhood of $t = 0$. As a result, (2.4.10)–(2.4.11) can be combined as

$$I(a) = \sup_t \Big(ta - \log \mathbb{E}[e^{tX_1}]\Big). \tag{2.4.12}$$

Proof. We only prove (2.4.8), the proof of (2.4.9) is identical when we replace X_i by $-X_i$. We rewrite, for every $t \geq 0$,

$$\mathbb{P}\Big(\sum_{i=1}^n X_i \geq na\Big) = \mathbb{P}\Big(e^{t\sum_{i=1}^n X_i} \geq e^{tna}\Big) \leq e^{-nta}\mathbb{E}\Big[e^{t\sum_{i=1}^n X_i}\Big], \tag{2.4.13}$$

where we have used Markov's inequality in Theorem 2.14. Since $\{X_i\}_{i=1}^\infty$ is a sequence of i.i.d. random variables, we have

$$\mathbb{E}\Big[e^{t\sum_{i=1}^n X_i}\Big] = \mathbb{E}[e^{tX_1}]^n, \tag{2.4.14}$$

so that, for every $t \geq 0$,

$$\mathbb{P}\Big(\sum_{i=1}^{n} X_i \geq na\Big) \leq \Big(e^{-ta}\mathbb{E}[e^{tX_1}]\Big)^n. \tag{2.4.15}$$

Minimizing the right-hand side over $t \geq 0$ gives that

$$\mathbb{P}\Big(\sum_{i=1}^{n} X_i \geq na\Big) \leq e^{-n\sup_{t\geq 0}\big(ta-\log\mathbb{E}[e^{tX_1}]\big)}. \tag{2.4.16}$$

This proves (2.4.8). $\qquad\square$

Exercise 2.17. *Compute $I(a)$ for $\{X_i\}_{i=1}^{\infty}$ being independent Poisson random variables with mean λ. Show that, for $a > \lambda$,*

$$\mathbb{P}\Big(\sum_{i=1}^{n} X_i \geq na\Big) \leq e^{-nI_\lambda(a)}, \tag{2.4.17}$$

where $I_\lambda(a) = a(\log a/\lambda - 1) + \lambda$. Show also that, for $a < \lambda$

$$\mathbb{P}\Big(\sum_{i=1}^{n} X_i \leq na\Big) \leq e^{-nI_\lambda(a)}. \tag{2.4.18}$$

Prove that $I_\lambda(a) > 0$ for all $a \neq \lambda$.

2.4.1 Bounds on binomial random variables

In this section, we investigate the tails of the binomial distribution. We start by a corollary of Theorem 2.16:

Corollary 2.17 (Large deviations for binomial distribution)**.** *Let X_n be a binomial random variable with parameters p and n. Then, for all $a \in (p, 1]$,*

$$\mathbb{P}\Big(X_n \geq na\Big) \leq e^{-nI(a)}, \tag{2.4.19}$$

where

$$I(a) = a\log\Big(\frac{a}{p}\Big) + (1-a)\log\Big(\frac{1-a}{1-p}\Big). \tag{2.4.20}$$

Moreover,

$$I(a) \geq I_p(a) \tag{2.4.21}$$

where

$$I_\lambda(a) = \lambda - a - a\log\left(\lambda/a\right) \tag{2.4.22}$$

is the rate function of a Poisson random variable with mean λ.

We can recognize (2.4.22) as the rate function of a Poisson random variable with mean λ (recall Exercise 2.17). Thus, Corollary 2.17 suggests that the upper tail of a binomial random variable is thinner than the one of a Poisson random variable.

Proof. We start by proving (2.4.19), using (2.4.8). We note that, by (2.4.10), we obtain a bound with $I(a)$ instead of I_p, where, with $X_1 \sim \mathrm{BE}(p)$,

$$I(a) = \sup_{t \geq 0} \left(ta - \log \mathbb{E}[e^{tX_1}]\right) = \sup_t \left(ta - \log\left(pe^t + (1-p)\right)\right) = a\log\left(\frac{a}{p}\right) + (1-a)\log\left(\frac{1-a}{1-p}\right). \quad \text{(2.4.23)}$$

We note that, for $t \geq 0$,

$$pe^t + (1-p) = 1 + p(e^t - 1) \leq e^{p(e^t-1)}, \quad \text{(2.4.24)}$$

so that

$$I(a) \geq \sup_t \left(ta - p(e^t - 1)\right) = p - a - a\log(p/a) = I_p(a). \quad \text{(2.4.25)}$$

$\square$

We continue to study tails of the binomial distribution, following [124]. The main bound is the following:

Theorem 2.18. *Let $X_i \sim \mathrm{BE}(p_i)$, $i = 1, 2, \ldots, n$, be independent Bernoulli distributed random variables, and write $X = \sum_{i=1}^n X_i$ and $\lambda = \mathbb{E}[X] = \sum_{i=1}^n p_i$. Then*

$$\mathbb{P}(X \geq \mathbb{E}X + t) \leq \exp\left(-\frac{t^2}{2(\lambda + t/3)}\right), \qquad\qquad t \geq 0; \quad \text{(2.4.26)}$$

$$\mathbb{P}(X \leq \mathbb{E}X - t) \leq \exp\left(-\frac{t^2}{2\lambda}\right), \qquad\qquad t \geq 0. \quad \text{(2.4.27)}$$

Further similar bounds under the same conditions and, even more generally, for independent random variables X_i such that $0 \leq X_i \leq 1$, are given, for example, in [28, 107] and [14, Appendix A].

Exercise 2.18. *Prove that Theorem 2.18 also holds for the Poisson distribution by a suitable limiting argument.*

Proof. Let $Y \sim \mathrm{BIN}(n, \lambda/n)$ where we recall that $\lambda = \sum_{i=1}^n p_i$. Since $x \mapsto \log x$ is concave, we have that for every $x_1, \ldots, x_n \in \mathbb{R}$,

$$\sum_{i=1}^n \frac{1}{n}\log(x_i) \leq \log\left(\frac{1}{n}\sum_{i=1}^n x_i\right). \quad \text{(2.4.28)}$$

As a result, for every real u, upon taking the logarithm,

$$\mathbb{E}[e^{uX}] = \prod_{i=1}^n (1 + (e^u - 1)p_i) = e^{n\sum_{i=1}^n \frac{1}{n}\log(1+(e^u-1)p_i)} \quad \text{(2.4.29)}$$

$$\leq e^{n\log(1+(e^u-1)\lambda/n)} = \left(1 + (e^u - 1)\lambda/n\right)^n = \mathbb{E}[e^{uY}].$$

Then we compute that, for all $u \geq 0$, by the Markov inequality,

$$\mathbb{P}(X \geq \mathbb{E}[X] + t) \leq e^{-u(\mathbb{E}[X]+t)}\mathbb{E}[e^{uX}] \leq e^{-u(\mathbb{E}[X]+t)}\mathbb{E}[e^{uY}] = e^{-u(\lambda+t)}(1 - p + pe^u)^n, \quad \text{(2.4.30)}$$

where $p = \lambda/n$ and using that $\mathbb{E}[X] = \lambda$.

When $t > n - \lambda$, the left-hand side of (2.4.30) equals 0, and there is nothing to prove. For $\lambda + t < n$, the right-hand side of (2.4.30) attains its minimum for the u satisfying $e^u = (\lambda + t)(1 - p)/(n - \lambda - t)p$. This yields, for $0 \leq t \leq n - \lambda$,

$$\mathbb{P}(X \geq \lambda + t) \leq \left(\frac{\lambda}{\lambda + t}\right)^{\lambda+t}\left(\frac{n - \lambda}{n - \lambda - t}\right)^{n-\lambda-t}. \quad \text{(2.4.31)}$$

The bound is implicit in [58]$^{\text{Cher52}}$ and is often called the *Chernoff bound*, appearing for the first time explicitly in [172]$^{\text{Okam58}}$.

For $0 \leq t \leq n - \lambda$,, we can rewrite (2.4.31) as

$$\mathbb{P}(X \geq \lambda + t) \leq \exp\left(-\lambda\varphi\left(\frac{t}{\lambda}\right) - (n - \lambda)\varphi\left(\frac{-t}{n - \lambda}\right)\right), \tag{2.4.32}$$

where $\varphi(x) = (1 + x)\log(1 + x) - x$ for $x \geq -1$ (and $\varphi(x) = \infty$ for $x < -1$). Replacing X by $n - X$, we also obtain, again for $0 \leq t \leq n - \lambda$,

$$\mathbb{P}(X \leq \lambda - t) \leq \exp\left(-\lambda\varphi\left(\frac{t}{\lambda}\right) - (n - \lambda)\varphi\left(\frac{t}{n - \lambda}\right)\right). \tag{2.4.33}$$

Since $\varphi(x) \geq 0$ for every x we can ignore the second term in the exponent. Furthermore, $\varphi(0) = 0$ and $\varphi'(x) = \log(1 + x) \leq x$, so that $\varphi(x) \geq x^2/2$ for $x \in [-1, 0]$, which proves (2.4.27). Similarly, $\varphi(0) = \varphi'(0) = 0$ and, for $x \in [0, 1]$,

$$\varphi''(x) = \frac{1}{1 + x} \geq \frac{1}{(1 + x/3)^3} = \left(\frac{x^2}{2(1 + x/3)}\right)'', \tag{2.4.34}$$

so that $\varphi(x) \geq x^2/(2(1 + x/3))$, which proves (2.4.26). $\qquad\square$

2.5 Martingales

In this section, we state and prove some results concerning martingales. These results will be used in the remainder of the text. For more details on martingales, we refer the reader to [102, 204]$^{\text{GriSti01, Will91}}$.

We assume some familiarity with conditional expectations. For the readers who are unfamiliar with filtrations and conditional expectations given a σ-algebra, we start by giving the simplest case of a martingale:

Definition 2.19 (Martingale). *A stochastic process $\{M_n\}_{n=0}^{\infty}$ is a martingale process if*

$$\mathbb{E}[|M_n|] < \infty \qquad \text{for all } n \geq 0, \tag{2.5.1}$$

and

$$\mathbb{E}[M_{n+1}|M_0, M_1, \ldots, M_n] = M_n \qquad \text{for all } n \geq 0. \tag{2.5.2}$$

As can be seen from (2.5.2), a martingale can be interpreted as a fair game. Indeed, when M_n denotes the profit after the nth game has been played, then (2.5.2) tells us that the expected profit at time $n + 1$ given the profits up to time n is equal to the profit at time n.

Exercise 2.19. *Show that when $\{M_n\}_{n=0}^{\infty}$ is a martingale process, then $\mu = \mathbb{E}[M_n]$ is independent of n.*

We now give a second definition, which we will need in Chapter 8, where a martingale is defined with respect to a more general filtration.

Definition 2.20 (Martingale definition general). *A stochastic process $\{M_n\}_{n=0}^{\infty}$ is a martingale process with respect to $\{X_n\}_{n=0}^{\infty}$ if*

$$\mathbb{E}[|M_n|] < \infty \qquad \text{for all } n \geq 0, \tag{2.5.3}$$

M_n is measurable with respect to the σ-algebra generated by $(X_0, \ldots, X_n)$, and

$$\mathbb{E}[M_{n+1}|X_0, \ldots, X_n] = M_n \qquad \text{for all } n \geq 0. \tag{2.5.4}$$

For $X_n = M_n$, the definitions in (2.5.2) and (2.5.4) coincide.

Exercise 2.20. *Let $\{X_i\}_{0=1}^{\infty}$ be an independent sequence of random variables with $\mathbb{E}[X_i] = 1$. Show that, for $n \geq 0$,*

$$M_n = \prod_{i=0}^{n} X_i \tag{2.5.5}$$

is a martingale.

Exercise 2.21. *Let $\{X_i\}_{i=0}^{\infty}$ be an independent sequence of random variables with $\mathbb{E}[X_i] = 0$. Show that, for $n \geq 0$,*

$$M_n = \sum_{i=0}^{n} X_i \tag{2.5.6}$$

is a martingale.

Exercise 2.22. *Let $M_n = \mathbb{E}[Y|X_0, \ldots, X_n]$ for some random variables $\{X_i\}_{i=0}^{\infty}$ and Y with $\mathbb{E}[|Y|] < \infty$ and $\{X_n\}_{n=0}^{\infty}$. Show that $\{M_n\}_{n=0}^{\infty}$ is a martingale process with respect to $\{X_n\}_{n=0}^{\infty}$. $\{M_n\}_{n=0}^{\infty}$ is called a Doob martingale.*

In the following two sections, we state and prove two key results for martingales, the *martingale convergence theorem* and the *Azuma-Hoeffding inequality*. These results are a sign of the power of martingales. Martingale techniques play a central role in modern probability theory, partly due to these results.

2.5.1   Martingale convergence theorem

We start with the martingale convergence theorem:

Theorem 2.21 (Martingale convergence theorem)**.** *Let $\{M_n\}_{n=0}^{\infty}$ be a martingale process with respect to $\{X_n\}_{n=0}^{\infty}$ satisfying*

$$\mathbb{E}[|M_n|] \leq B \qquad \text{for all } n \geq 0. \tag{2.5.7}$$

<div align="right">_{e:Mn-condMCT}</div>

Then, $M_n \xrightarrow{a.s.} M_\infty$, for some limiting random variable M_∞ which is finite with probability 1.

The martingale convergence theorem comes in various forms. There also is an L^2-version, for which it is assumed that $\mathbb{E}[M_n^2] \leq M$ uniformly for all $n \geq 1$. In this case, one also obtains the convergence $\lim_{n\to\infty} \mathbb{E}[M_n^2] = \mathbb{E}[M_\infty^2]$. Theorem 2.21 is an adaptation of the L^1-martingale convergence theorem, for which one only needs that $\{M_n\}_{n=0}^{\infty}$ is a *submartingale*, i.e., when we assume (2.5.7), but (2.5.4) is replaced with

$$\mathbb{E}[M_{n+1}|X_0, \ldots, X_n] \geq M_n \qquad \text{for all } n \geq 0. \tag{2.5.8}$$

<div align="right">_{e:SubMartCE}</div>

See e.g., [102, Section 12.3].

Exercise 2.23. *Prove that when the martingale $\{M_n\}_{n=0}^{\infty}$ is non-negative, i.e., when $M_n \geq 0$ with probability 1 for all $n \geq 1$, then $M_n \xrightarrow{a.s.} M_\infty$ to some limiting random variable M_∞ which is finite with probability 1.*

Exercise 2.24. *Let $\{X_i\}_{i=0}^{\infty}$ be an independent sequence of random variables with $\mathbb{E}[X_i] = 1$ and for which $X_i \geq 0$ with probability 1. Show that the martingale*

$$M_n = \prod_{i=0}^{n} X_i \tag{2.5.9}$$
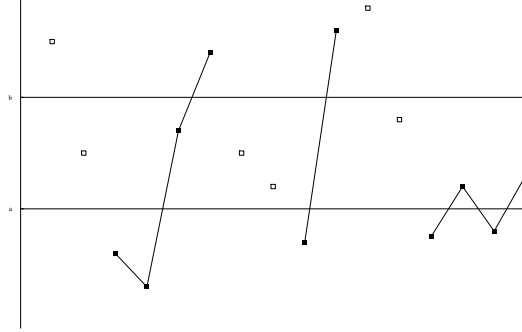
Figure 2.1: Up-crossings

converges in probability to a random variable which is finite with probability 1. Hint: Prove that $\mathbb{E}[|M_n|] = \mathbb{E}[M_n] = 1$, and apply Exercise 2.23.

Exercise 2.25. *For $i = 1, \ldots, m$, let $\{M_n^{(i)}\}_{n=0}^{\infty}$ be a sequence of martingales with respect to $\{X_n\}_{n=0}^{\infty}$. Show that*

$$M_n = \max_{i=0}^{m} M_n^{(i)} \tag{2.5.10}$$

is a submartingale with respect to $\{X_n\}_{n=0}^{\infty}$.

Proof of Theorem 2.21. We shall prove Theorem 2.21 in the case where M_n is a *submartingale*.

We follow the proof of the martingale convergence theorem in [102,$^{\text{GriSti01}}$ Section 12.3]. The key step in this classical probabilistic proof is 'Snell's up-crossings inequality'. Suppose that $\{m_n : n \geq 0\}$ is a real sequence, and $[a, b]$ is a real interval. An *up-crossing* of $[a, b]$ is defined to be a crossing by m of $[a, b]$ in the upwards direction. More precisely, let $T_1 = \min\{n : m_n \leq a\}$, the first time m hits the interval $(-\infty, a]$, and $T_2 = \min\{n > T_1 : m_n \geq b\}$, the first subsequent time when m hits $[b, \infty)$; we call the interval $[T_1, T_2]$ an up-crossing of $[a, b]$. In addition, for $k > 1$, define the stopping times T_n by

$$T_{2k-1} = \min\{n > T_{2k-2} : m_n \leq a\}, \qquad T_{2k} = \min\{n > T_{2k-1} : m_n \geq b\}, \tag{2.5.11}$$

so that the number of up-crossings of $[a, b]$ is equal to the number of intervals $[T_{2k-1}, T_{2k}]$ for $k \geq 1$. Let $U_n(a, b; m)$ be the number of up-crossings of $[a, b]$ by the subsequence $m_0, m_1, \ldots, m_n$, and let $U(a, b; m) = \lim_{n \to \infty} U_n(a, b; m)$ be the total number of up-crossings of m.

Let $\{M_n\}_{n=0}^{\infty}$ be a submartingale, and let $U_n(a, b; M)$ be the number of up-crossings of $[a, b]$ by M up to time n. Then the up-crossing inequality gives a bound on the expected number of up-crossings of an interval $[a, b]$:

Proposition 2.22 (Up-crossing inequality)**.** *If $a < b$ then*

$$\mathbb{E}[U_n(a, b; M)] \leq \frac{\mathbb{E}[(M_n - a)^+]}{b - a},$$

where $(M_n - a)^+ = \max\{0, M_n - a\}$.

Proof. Setting $Z_n = (M_n - a)^+$, we have that Z_n is a non-negative submartingale because $\mathbb{E}[|M_n|] \leq \mathbb{E}[|M_n|] + |a| < \infty$. Furthermore, for every random variable X and $a \in \mathbb{R}$,

$$\mathbb{E}[(X - a)_+] \geq \mathbb{E}[X - a]_+, \qquad \qquad (2.5.12)$$

so that

$$Z_n \leq \left(\mathbb{E}[M_{n+1}|X_0, \ldots, X_n] - a\right)_+ \leq \mathbb{E}[(M_{n+1} - a)_+|X_0, \ldots, X_n] = \mathbb{E}[Z_{n+1}|X_0, \ldots, X_n], \qquad (2.5.13)$$

where we first used the submartingale property $\mathbb{E}[M_{n+1}|X_0, \ldots, X_n] \geq M_n$, followed by (2.5.12). Up-crossings of $[a, b]$ by M correspond to up-crossings of $[0, b - a]$ by Z, so that $U_n(a, b; M) = U_n(0, b - a; Z)$.

Let $[T_{2k-1}, T_{2k}]$, for $k \geq 1$, be the up-crossings of Z of $[0, b - a]$, and define the indicator functions

$$I_i = \begin{cases} 1 & \text{if } i \in (T_{2k-1}, T_{2k}] \text{ for some } k, \\ 0 & \text{otherwise} \end{cases} \qquad \qquad (2.5.14)$$

Note that the event $\{I_i = 1\}$ depends on $M_0, M_1, \ldots, M_{i-1}$ only. Since $M_0, M_1, \ldots, M_{i-1}$ are measurable with respect to the the σ-algebra generated by $(X_0, \ldots, X_{i-1})$, also I_i is measurable with respect to the σ-algebra generated by $(X_0, \ldots, X_{i-1})$. Now

$$(b - a)U_n(0, b - a; Z) \leq \sum_{i=1}^{n}(Z_i - Z_{i-1})I_i \qquad \qquad (2.5.15)$$

since each up-crossing of $[0, b - a]$ by Z contributes an amount of at least $b - a$ to the summation. The expectation of the summands on the right-hand side of (2.5.15) is equal to

$$\mathbb{E}[(Z_i - Z_{i-1})I_i] = \mathbb{E}\Big[\mathbb{E}[(Z_i - Z_{i-1})I_i|X_0, \ldots, X_{i-1}]\Big] = \mathbb{E}[I_i(\mathbb{E}[Z_i|X_0, \ldots, X_{i-1}] - Z_{i-1})]$$

$$\leq \mathbb{E}[\mathbb{E}[Z_i|X_0, \ldots, X_{i-1}] - Z_{i-1}] = \mathbb{E}[Z_i] - \mathbb{E}[Z_{i-1}],$$

where we use that I_i is measurable with respect to the σ-algebra generated by $(X_0, \ldots, X_{i-1})$ for the second equality, and we use that Z is a submartingale and $0 \leq I_i \leq 1$ to obtain the inequality. Summing over i and take expectations on both sides of (2.5.15), we obtain

$$(b - a)\mathbb{E}[U_n(0, b - a; Z)] \leq \mathbb{E}[Z_n] - \mathbb{E}[Z_0] \leq \mathbb{E}[Z_n], \qquad \qquad (2.5.16)$$

which completes the proof of Proposition 2.22. $\qquad \qquad \square$

Now we have the tools to give the proof of Theorem 2.21.

Proof of Theorem 2.21. Suppose $\{M_n\}_{n=0}^{\infty}$ is a submartingale and $\mathbb{E}[|M_n|] \leq B$ for all n. Let Λ be defined as follows

$$\Lambda = \{\omega : M_n(\omega) \text{ does not converge to a limit in } [-\infty, \infty]\}.$$

The claim that M_n converges is proved if we show that $\mathbb{P}(\Lambda) = 0$. The set Λ has an equivalent definition

$$
\begin{aligned}
\Lambda &= \{\omega : \liminf M_n(\omega) < \limsup M_n(\omega)\} \\
&= \bigcup_{a,b \in \mathbb{Q}: a<b} \{\omega : \liminf M_n(\omega) < a < b < \limsup M_n(\omega)\} \\
&= \bigcup_{a,b \in \mathbb{Q}: a<b} \Lambda_{a,b}.
\end{aligned}
$$

However,

$$
\Lambda_{a,b} \subseteq \{\omega : U(a,b;M) = \infty\},
$$

so that, by Proposition 2.22, $\mathbb{P}(\Lambda_{a,b}) = 0$. Since Λ is a countable union of sets $\Lambda_{a,b}$, it follows that $\mathbb{P}(\Lambda) = 0$. This concludes the first part of the proof that M_n converges almost surely to a limit M_∞.

To show that the limit is bounded, we use Fatou's lemma (see Theorem A.1.4 in the appendix) to conclude

$$
\mathbb{E}[|M_\infty|] = \mathbb{E}[\liminf_{n\to\infty} |M_n|] \leq \liminf_{n\to\infty} \mathbb{E}[|M_n|] \leq \sup_{n \geq 0} \mathbb{E}[|M_n|] < \infty,
$$

so that, by Markov's inequality (recall Theorem 2.14),

$$
\mathbb{P}(M_\infty < \infty) = 1.
$$

This completes the proof of Theorem 2.21. $\qquad \square$

2.5.2 Azuma-Hoeffding inequality

We continue with the Azuma-Hoeffding inequality, which provides exponential bounds for the tails of a special class of martingales:

Theorem 2.23 (Azuma-Hoeffding inequality). *Let $\{M_n\}_{n=0}^\infty$ be a martingale process with the property that, with probability 1, there exists $K_n \geq 0$ such that*

$$
|M_n - M_{n-1}| \leq K_n \qquad \text{for all } n \geq 0, \tag{2.5.17}
$$

where, by convention, we define $M_{-1} = \mu = \mathbb{E}[M_n]$ (recall also Exercise 2.19). Then, for every $a \geq 0$,

$$
\mathbb{P}(|M_n - \mu| \geq a) \leq 2\exp\left\{-\frac{a^2}{2\sum_{i=0}^n K_i^2}\right\}. \tag{2.5.18}
$$

Theorem 2.23 is very powerful, as it provides tails on the distribution of M_n. In many cases, the bounds are close to optimal. The particular strength of Theorem 2.23 is that the bound is valid *for all $n \geq 1$*.

Proof. For $\psi > 0$, the function $g(d) = e^{\psi d}$ is convex, so that, for all d with $|d| \leq 1$,

$$
e^{\psi d} \leq \frac{1}{2}(1-d)e^{-\psi} + \frac{1}{2}(1+d)e^{\psi}. \tag{2.5.19}
$$

Applying this with $d = D$ to a random variable D having mean 0 and satisfying $\mathbb{P}(|D| \leq 1) = 1$, we obtain

$$
\mathbb{E}[e^{\psi D}] \leq \mathbb{E}[\frac{1}{2}(1-D)e^{-\psi} + \frac{1}{2}(1+D)e^{\psi}] = \frac{1}{2}(e^{-\psi} + e^{\psi}). \tag{2.5.20}
$$

We can use that $(2n)! \geq 2^n n!$ for all $n \geq 0$ to obtain that

$$\frac{1}{2} \left(e^{-\psi} + e^{\psi} \right) = \sum_{n \geq 0} \frac{\psi^{2n}}{(2n)!} \leq \sum_{n \geq 0} \frac{\psi^{2n}}{2^n n!} = e^{\psi^2/2}. \tag{2.5.21}$$

By Markov's inequality in Theorem 2.14, for any $\theta > 0$,

$$\mathbb{P}(M_n - \mu \geq x) = \mathbb{P}(e^{\theta(M_n - \mu)} \geq e^{\theta x}) \leq e^{-\theta x} \mathbb{E}[e^{\theta(M_n - \mu)}]. \tag{2.5.22}$$

Writing $D_n = M_n - M_{n-1}$, we obtain

$$\mathbb{E}[e^{\theta(M_n - \mu)}] = \mathbb{E}[e^{\theta(M_{n-1} - \mu)} e^{\theta D_n}].$$

Conditioning on $X_0, \ldots, X_{n-1}$ yields

$$\mathbb{E}[e^{\theta(M_n - \mu)} \mid X_0, \ldots, X_{n-1}] = e^{\theta(M_{n-1} - \mu)} \mathbb{E}[e^{\theta D_n} \mid X_0, \ldots, X_{n-1}] \leq e^{\theta(M_{n-1} - \mu)} \exp(\frac{1}{2} \theta^2 K_n^2), \tag{2.5.23}$$

where (2.5.20) and (2.5.21) are applied to the random variable D_n/K_n which satisfies

$$\mathbb{E}[D_n | X_0, \ldots, X_{n-1}] = \mathbb{E}[M_n | X_0, \ldots, X_{n-1}] - \mathbb{E}[M_{n-1} | X_0, \ldots, X_{n-1}] = M_{n-1} - M_{n-1} = 0. \tag{2.5.24}$$

Taking expectations on both sides of (2.5.23) and iterate to find

$$\mathbb{E}[e^{\theta(M_n - \mu)}] \leq \mathbb{E}[e^{\theta(M_{n-1} - \mu)}] \exp(\frac{1}{2} \theta^2 K_n^2) \leq \exp\left(\frac{1}{2} \theta^2 \sum_{i=0}^{n} K_n^2 \right). \tag{2.5.25}$$

Therefore, by (2.5.22), for all $\theta > 0$,

$$\mathbb{P}(M_n - \mu \geq x) \leq \exp\left(-\theta x + \frac{1}{2} \theta^2 \sum_{i=0}^{n} K_n^2 \right). \tag{2.5.26}$$

The exponential is minimized, with respect to θ, by setting $\theta = x/\sum_{i=0}^{n} K_i^2$. Hence,

$$\mathbb{P}(M_n - \mu \geq x) \leq \exp\left(-\frac{x^2}{\sum_{i=0}^{n} K_i^2} \right). \tag{2.5.27}$$

Using that also $-M_n$ is a martingale, we obtain by symmetry that

$$\mathbb{P}(M_n - \mu \leq -x) \leq \exp\left(-\frac{x^2}{\sum_{i=0}^{n} K_i^2} \right). \tag{2.5.28}$$

Adding the two bounds completes the proof. $\square$

Exercise 2.26. *Show that Theorem 2.23 implies that for $X \sim \mathrm{BIN}(n, p)$,*

$$\mathbb{P}(|X - np| \geq a) \leq 2 \exp\left\{ -\frac{a^2}{2n(1-p)^2} \right\}. \tag{2.5.29}$$

Exercise 2.27. *Let $\{X_i\}_{i=0}^{\infty}$ be an independent identically distributed sequence of random variables with $\mathbb{E}[X_i] = 0$ and $|X_i| \leq 1$, and define the martingale $\{M_n\}_{n=0}^{\infty}$ by*

$$M_n = \sum_{i=0}^{n} X_i. \tag{2.5.30}$$

Show that

$$\mathbb{P}(|M_n| \geq a) \leq 2 \exp\left(-\frac{a^2}{2n} \right). \tag{2.5.31}$$

Take $a = x\sqrt{n}$, and prove by using the central limit theorem that $\mathbb{P}(|M_n| \geq a)$ converges. Compare the limit to the bound in (2.5.31).

2.6 Order statistics and extreme value theory

In this section, we study the largest values of a sequence of i.i.d. random variables. For more background on order statistics, we refer the reader to [84]. We will be particularly interested in the case where the random variables in question have heavy tails. We let $\{X_i\}_{i=1}^n$ be an i.i.d. sequence, and introduce the *order statistics* of $\{X_i\}_{i=1}^n$ by

$$X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}, \tag{2.6.1}$$

so that $X_{(1)} = \min\{X_1, \ldots, X_n\}$, $X_{(2)}$ is the second smallest of $\{X_i\}_{i=1}^n$, etc. In the notation in (2.6.1), we ignore the fact that the distribution of $X_{(i)}$ depends on n. Sometimes the notation $X_{(1:n)} \le X_{(2:n)} \le \cdots \le X_{(n:n)}$ is used instead to make the dependence on n explicit. In this section, we shall mainly investigate $X_{(n)}$, i.e., the maximum of $X_1, \ldots, X_n$. We note that the results immediately translate to $X_{(1)}$, by changing to $-X_i$.

We denote the distribution function of the random variables $\{X_i\}_{i=1}^n$ by

$$F_X(x) = \mathbb{P}(X_1 \le x). \tag{2.6.2}$$

Before stating the results, we introduce a number of special distributions. We say that the random variable Y has a *Fréchet distribution* if there exists an $\alpha > 0$ such that

$$\mathbb{P}(Y \le y) = \begin{cases} 0, & y \le 0, \\ \exp\{-y^{-\alpha}\} & y > 0. \end{cases} \tag{2.6.3}$$

We say that the random variable Y has a *Weibull distribution* if there exists an $\alpha > 0$ such that

$$\mathbb{P}(Y \le y) = \begin{cases} \exp\{-(-y)^\alpha\}, & y \le 0, \\ 1 & y > 0. \end{cases} \tag{2.6.4}$$

We say that the random variable Y has a *Gumbel distribution* if

$$\mathbb{P}(Y \le y) = \exp\{-\exp\{-y\}\}, \qquad y \in \mathbb{R}. \tag{2.6.5}$$

One of the fundamental results in extreme value theory is the following characterization of possible limit distributions of $X_{(n)}$:

Theorem 2.24 (Fisher-Tippett theorem, limit laws for maxima)**.** *Let $\{X_n\}_{n=0}^\infty$ be a sequence of i.i.d. random variables. If there exists norming constants $c_n > 0$ and $d_n \in \mathbb{R}$ and some non-degenerate distribution function H such that*

$$\frac{X_{(n)} - c_n}{d_n} \xrightarrow{d} Y, \tag{2.6.6}$$

where Y has distribution function H, then H is the distribution function of a Fréchet, Weibull or Gumbel distribution.

A fundamental role in extreme value statistics is played by approximate solutions u_n of $[1 - F_X(u_n)] = 1/n$. More precisely, we define u_n by

$$u_n = \inf\{u : 1 - F_X(u) \ge 1/n\}. \tag{2.6.7}$$

We shall often deal with random variables which have a power-law distribution. For such random variables, the following theorem identifies the Fréchet distribution as the only possible extreme value limit:

Theorem 2.25 (Maxima of heavy-tailed random variables). *Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of i.i.d. unbounded random variables satisfying*

$$1 - F_X(x) = x^{1-\tau} L_X(x), \tag{2.6.8}$$

where $x \mapsto L_X(x)$ is a slowly varying function, and where $\tau > 1$. Then

$$\frac{X_{(n)}}{u_n} \overset{d}{\longrightarrow} Y, \tag{2.6.9}$$

where Y has a Fréchet distribution with parameter $\alpha = \tau - 1$ and u_n is defined in (2.6.7).

Exercise 2.28. *Show that when (2.6.8) holds, then u_n is regularly varying with exponent $\frac{1}{\tau-1}$.*

For completeness, we also state two theorems identifying when the Weibull distribution or Gumbel distribution occur as the limiting distribution in extreme value theory:

Theorem 2.26 (Maxima of bounded random variables). *Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of i.i.d. random variables satisfying that $F_X(x_X) = 1$ for some $x_X \in \mathbb{R}$ and*

$$1 - F_X(x_X - x^{-1}) = x^{-\alpha} L_X(x), \tag{2.6.10}$$

where $x \mapsto L_X(x)$ is a slowly varying function, and where $\alpha > 1$. Then

$$\frac{X_{(n)} - x_X}{d_n} \overset{d}{\longrightarrow} Y, \tag{2.6.11}$$

where Y has a Weibull distribution with parameter α, and $d_n = x_X - u_n$ where u_n is defined in (2.6.7).

Theorem 2.27 (Maxima of random variables with thin tails). *Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of i.i.d. bounded random variables satisfying that $F(x_F) = 1$ for some $x_F \in [0, \infty]$, and*

$$\lim_{x \uparrow x_F} \frac{1 - F(x + ta(x))}{1 - F(x)} = e^{-t}, \qquad t \in \mathbb{R}, \tag{2.6.12}$$

where $x \mapsto a(x)$ is given by

$$a(x) = \int_x^{x_F} \frac{1 - F(t)}{1 - F(x)} dt. \tag{2.6.13}$$

Then

$$\frac{X_{(n)} - u_n}{d_n} \overset{d}{\longrightarrow} Y, \tag{2.6.14}$$

where Y has a Gumbel distribution, and $d_n = a(u_n)$ where u_n is defined in (2.6.7).

We next assume that the random variables $\{X_i\}_{i=1}^{n}$ have *infinite mean*. It is well known that the order statistics of the random variables, as well as their sum, are governed by u_n in the case that $\tau \in (1, 2)$. The following theorem shows this in detail. In the theorem below, $E_1, E_2, \ldots$ is an i.i.d. sequence of exponential random variables with unit mean and $\Gamma_j = E_1 + E_2 + \ldots + E_j$, so that Γ_j has a Gamma distribution with parameters j and 1.

It is well known that when the distribution function F of $\{X_i\}_{i=1}^{n}$ satisfies (2.6.8), then $\sum_{i=1}^{n} X_i$ has size approximately $n^{1/(\tau-1)}$, just as holds for the maximum, and the rescaled sum $n^{-1/(\tau-1)} \sum_{i=1}^{n} X_i$ converges to a stable distribution. The next result generalizes this statement to convergence of the sum together with the first order statistics:

Theorem 2.28 (Convergence in distribution of order statistics and sum). *$\{X_n\}_{n=0}^{\infty}$ be a sequence of i.i.d. random variables satisfying* (2.6.8) *for some $\tau \in (1,2)$. Then, for any $k \in \mathbb{N}$,*

$$\left(\frac{L_n}{u_n}, \left\{\frac{X_{(n+1-i)}}{u_n}\right\}_{i=1}^{n}\right) \overset{d}{\longrightarrow} (\eta, \{\xi_i\}_{i=1}^{\infty}), \;\; as \; n \to \infty, \tag{2.6.15}$$

where $(\eta, \{\xi_i\}_{i=1}^{\infty})$ is a random vector which can be represented by

$$\eta = \sum_{j=1}^{\infty} \Gamma_j^{-1/(\tau-1)}, \qquad \xi_i = \Gamma_i^{-1/(\tau-1)}, \tag{2.6.16}$$

and where u_n is slowly varying with exponent $1/(\tau-1)$ (recall Exercise 2.28). Moreover,

$$\xi_k k^{1/(\tau-1)} \overset{\mathbb{P}}{\longrightarrow} 1 \; as \; k \to \infty. \tag{2.6.17}$$

Proof. Because $\tau - 1 \in (0,1)$, the proof is a direct consequence of [145, Theorem 1'], and the continuous mapping theorem [35, Theorem 5.1], which together yield that on $\mathbb{R} \times \mathbb{R}^{\infty}$, equipped with the product topology, we have

$$(S_n^{\#}, Z^{(n)}) \overset{d}{\longrightarrow} (S^{\#}, Z), \tag{2.6.18}$$

where $S_n^{\#} = u_n^{-1} L_n$, $Z^{(n)} = u_n^{-1}(D_{(n:n)}, \ldots, D_{(1:n)}, 0, 0, \ldots)$, and $Z_j = \Gamma_j^{-1/(\tau-1)}$, $j \geq 1$. Finally, (2.6.17) follows because by the weak law of large numbers,

$$\frac{\Gamma_k}{k} \overset{\mathbb{P}}{\longrightarrow} 1, \tag{2.6.19}$$

and $\xi_k = \Gamma_k^{-1/(\tau-1)}$. $\square$

2.7   Discussion and notes

Notes on Section 2.1.   For a through discussion on convergence issues of integer random variables including Theorems 2.4–2.6 and much more, see [40, Section 1.4].

Notes on Section 2.4.   Theorem 2.16 has a long history. See e.g., [76, Theorem 2.2.3] for a more precise version of Cramér's Theorem, which states that (2.4.8)–(2.4.9) are sharp, in the sense that $-\frac{1}{n} \log \mathbb{P}(\frac{1}{n} \sum_{i=1}^{n} X_i \leq a)$ converges to $I(a)$. See [174, Theorem 1.1] for a version of Cramér's Theorem that includes also the Chernoff bound.

Notes on Section 2.5.   This discussion is adapted after [102]. For interesting examples of martingale argument, as well as adaptations of the Azuma-Hoeffding inequality in Theorem 2.23, see [63].

Notes on Section 2.6.   Theorem 2.24 is [84, Theorem 3.2.3]. Theorem 2.25 is [84, Theorem 3.3.7]. Theorem 2.26 is [84, Theorem 3.3.12]. Theorem 2.27 is [84, Theorem 3.3.27]. For a thorough discussion of extreme value results, as well as many examples, we refer to the standard work on the topic [84].

Chapter 3

Branching processes

Branching processes will be used in an essential way throughout these notes to describe the connected components of various random graphs. To prepare for this, we describe branching processes in quite some detail here. Special attention will be given to branching processes with a Poisson offspring distribution, as well as to branching processes with a binomial offspring distribution and their relation (see Sections 3.5 and 3.6 below). We start by describing the survival versus extinction transition in Section 3.1, and provide a useful random walk perspective on branching processes in Section 3.3. For more information about branching processes, we refer to the books [17, 106, 117].

3.1 Survival versus extinction

A branching process is the simplest possible model for a population evolving in time. Suppose each organism independently gives birth to a number of children with the same distribution. We denote the offspring distribution by $\{p_i\}_{i=0}^{\infty}$, where

$$p_i = \mathbb{P}(\text{individual has } i \text{ children}). \tag{3.1.1}$$

We denote by Z_n the number of individuals in the nth generation, where, by convention, we let $Z_0 = 1$. Then Z_n satisfies the recursion relation

$$Z_n = \sum_{i=1}^{Z_{n-1}} X_{n,i}, \tag{3.1.2}$$

where $\{X_{n,i}\}_{n,i\geq 1}$ is a doubly infinite array of i.i.d. random variables. We will often write X for the offspring distribution, so that $\{X_{n,i}\}_{n,i\geq 1}$ is a doubly infinite array of i.i.d. random variables with $X_{n,i} \sim X$ for all n, i.

One of the major results of branching processes is that when $\mathbb{E}[X] \leq 1$, the population dies out with probability one (unless $X_{1,1} = 1$ with probability one), while if $\mathbb{E}[X] > 1$, there is a non-zero probability that the population will not become extinct. In order to state the result, we denote the extinction probability by

$$\eta = \mathbb{P}(\exists n : Z_n = 0). \tag{3.1.3}$$

Theorem 3.1 (Survival v.s. extinction for branching processes). *For a branching process with i.i.d. offspring X, $\eta = 1$ when $\mathbb{E}[X] < 1$, while $\eta < 1$ when $\mathbb{E}[X] > 1$. When $\mathbb{E}[X] = 1$, and $\mathbb{P}(X = 1) < 1$, then $\eta = 1$. Moreover, with G_X the probability generating function of the offspring distribution X, i.e.,*

$$G_X(s) = \mathbb{E}[s^X], \tag{3.1.4}$$

61
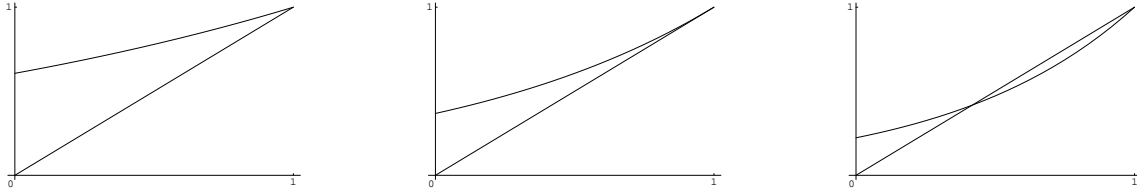
Figure 3.1:   The solution of $s = G_X(s)$ when $\mathbb{E}[X] < 1, \mathbb{E}[X] = 1, \mathbb{E}[X] > 1$ respectively. Note that $\mathbb{E}[X] = G'_X(1)$, and $G'_X(1) > 1$ precisely when there is a solution $\eta < 1$ to $\eta = G_X(\eta)$.

the extinction probability η is the smallest solution in $[0, 1]$ of

$$\eta = G_X(\eta). \tag{3.1.5}$$

<div align="right">e:extinc</div>

Proof. We write

$$\eta_n = \mathbb{P}(Z_n = 0). \tag{3.1.6}$$

Because $\{Z_n = 0\} \subseteq \{Z_{n+1} = 0\}$, we have that $\eta_n \uparrow \eta$. Let

$$G_n(s) = \mathbb{E}[s^{Z_n}] \tag{3.1.7}$$

denote the generating function of the nth generation. Then, since for an integer-valued random variable X, $\mathbb{P}(X = 0) = G_X(0)$, we have that

$$\eta_n = G_n(0). \tag{3.1.8}$$

By conditioning on the first generation, we obtain that

$$G_n(s) = \mathbb{E}[s^{Z_n}] = \sum_{i=0}^{\infty} p_i \mathbb{E}[s^{Z_n} | Z_1 = i] = \sum_{i=0}^{\infty} p_i G_{n-1}(s)^i. \tag{3.1.9}$$

Therefore, writing $G_X = G_1$ for the generating function of $X_{1,1}$, we have that

$$G_n(s) = G_X(G_{n-1}(s)). \tag{3.1.10}$$

When we substitute $s = 0$, we obtain that η_n satisfies the recurrence relation

$$\eta_n = G_X(\eta_{n-1}). \tag{3.1.11}$$

<div align="right">e:eta-rec</div>

See Figure 3.2 for the evolution of $n \mapsto \eta_n$.

When $n \to \infty$, we have that $\eta_n \uparrow \eta$, so that, by continuity of $s \mapsto G_X(s)$, we have

$$\eta = G_X(\eta). \tag{3.1.12}$$

When $\mathbb{P}(X = 1) = 1$, then $Z_n = 1$ a.s., and there is nothing to prove. Therefore, for the remainder of this proof, we shall assume that $\mathbb{P}(X = 1) < 1$.

Suppose that $\psi \in [0, 1]$ satisfies $\psi = G_X(\psi)$. We claim that $\eta \leq \psi$. We use induction to prove that $\eta_n \leq \psi$ for all n. Indeed, $\eta_0 = 0 \leq \psi$, which initializes the induction hypothesis. To advance the induction, we use (3.1.11), the induction hypothesis, as well as the fact that $s \mapsto G_X(s)$ is increasing on $[0, 1]$, to see that

$$\eta_n = G_X(\eta_{n-1}) \leq G_X(\psi) = \psi, \tag{3.1.13}$$
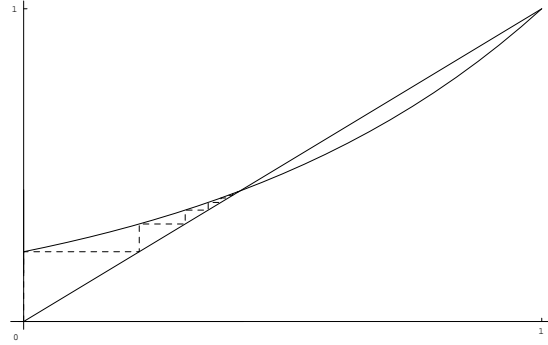
Figure 3.2: The iteration for $n \mapsto \eta_n$ in (3.1.11).

where the final conclusion comes from the fact that ψ is a solution of $\psi = G_X(\psi)$. Therefore, $\eta_n \leq \psi$, which advances the induction. Since $\eta_n \uparrow \eta$, we conclude that $\eta \leq \psi$ for all solutions ψ of $\psi = G_X(\psi)$. Therefore, η is the smallest such solution.

We note that $s \mapsto G_X(s)$ is increasing and convex for $s \geq 0$, since

$$G_X''(s) = \mathbb{E}[X(X-1)s^{X-2}] \geq 0. \tag{3.1.14}$$

When $\mathbb{P}(X = 1) < 1$, then $\mathbb{E}[X(X-1)s^{X-2}] > 0$, so that $s \mapsto G_X(s)$ is strictly increasing and strictly convex for $s > 0$. Therefore, there can be at most two solutions of $s = G_X(s)$ in $[0,1]$. Note that $s = 1$ is always a solution of $s = G_X(s)$, since G is a probability generating function. Since $G_X(0) > 0$, there is precisely one solution when $G_X'(1) < 1$, while there are two solutions when $G_X'(1) > 1$. The former implies that $\eta = 1$ when $G_X'(1) > 1$, while the latter implies that $\eta < 1$ when $G_X'(1) < 1$. When $G_X'(1) = 1$, again there is precisely one solution, except when $G_X(s) = s$, which is equivalent to $\mathbb{P}(X = 1) = 1$. Since $G_X'(1) = \mathbb{E}[X]$, this proves the claim. $\qquad \square$

In many cases, we shall be interested in the *survival probability*, denoted by $\zeta = 1 - \eta$, which is the probability that the branching process survives forever, i.e., $\zeta = \mathbb{P}(Z_n > 0 \ \forall n \geq 0)$. See Figure 3.3 for the survival probability of a Poisson branching process with parameter λ, as a function of λ.

Exercise 3.1. *Show that $\eta = 0$ precisely when $p_0 = 0$.*

Exercise 3.2. *When the offspring distribution is given by*

$$p_x = (1 - p)\mathbb{1}_{\{x=0\}} + p\mathbb{1}_{\{x=2\}}, \tag{3.1.15}$$
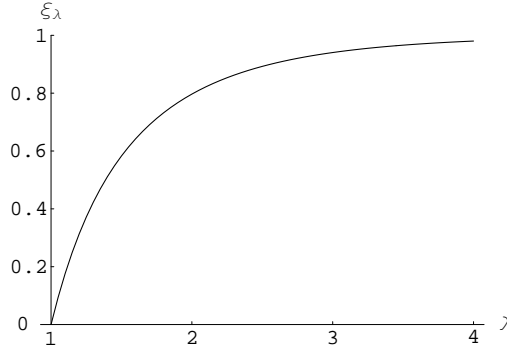
Figure 3.3:   The survival probability $\zeta = \zeta_\lambda$ for a Poisson branching process with mean offspring equal to λ. The survival probability equals $\zeta = 1 - \eta$, where η is the extinction probability.

we speak of binary branching. Prove that $\eta = 1$ when $p \leq 1/2$ and, for $p > 1/2$,

$$\eta = \frac{1 - p}{p}. \tag{3.1.16}$$

Exercise 3.3 ([17], Pages 6-7.)**.** *Let the probability distribution $\{p_k\}_{k=0}^{\infty}$ be given by*

$$\begin{cases} p_k = b(1 - p)^{k-1} & \text{for } k = 1, 2, \ldots; \\ p_0 = 1 - b/p & \text{for } k = 0, \end{cases} \tag{3.1.17}$$

so that, for $b = p$, the offspring distribution has a geometric distribution with success probability p. Show that the extinction probability η is given by $\eta = 1$ if $\mu = \mathbb{E}[X] = b/p^2 \leq 1$, while, with the abbreviation $q = 1 - p$,

$$\eta = \frac{1 - \mu p}{q}. \tag{3.1.18}$$

Exercise 3.4 (Exercise 3.3 cont.)**.** *Let the probability distribution $\{p_k\}_{k=0}^{\infty}$ be given by (3.1.17). Show that $G_n(s)$, the generating function of Z_n is given by*

$$G_n(s) = \begin{cases} 1 - \mu^n \frac{1-\eta}{\mu^n - \eta} + \dfrac{\mu^n \left(\frac{1-\eta}{\mu^n - \eta} \right)^2 s}{1 - \left(\frac{\mu^n - 1}{\mu^n - \eta} \right)} & \text{when } b \neq p^2; \\[2em] \frac{nq - (nq - p)s}{p + nq - nps} & \text{when } b = p^2. \end{cases} \tag{3.1.19}$$

Exercise 3.5 (Exercise 3.4 cont.)**.** *Conclude from Exercise 3.4 that, for $\{p_k\}_{k=0}^{\infty}$ in (3.1.17),*

$$\mathbb{P}(Z_n > 0, \ \exists m > n \text{ such that } Z_m = 0) = \begin{cases} \mu^n \frac{1-\eta}{\mu^n - \eta} & \text{when } b < p^2; \\ \frac{p}{p + nq} & \text{when } b = p^2; \\ \frac{(1-\eta)\eta}{\mu^n - \eta} & \text{when } b > p^2. \end{cases} \tag{3.1.20}$$

We continue by studying the total progeny T of the branching process, which is defined as

$$T = \sum_{n=0}^{\infty} Z_n. \tag{3.1.21}$$

We denote by $G_T(s)$ the probability generating function of T, i.e.,

$$G_T(s) = \mathbb{E}[s^T].\tag{3.1.22}$$

The main result is the following:

Theorem 3.2 (Total progeny probability generating function)**.** *For a branching process with i.i.d. offspring X having probability generating function $G(s) = \mathbb{E}[s^X]$, the probability generating function of the total progeny T satisfies the relation*

$$G_T(s) = sG(G_T(s)).\tag{3.1.23}$$

e:GFtotprog

Proof. We again condition on the size of the first generation, and use that when $Z_1 = i$, for $j = 1, \ldots, i$, the total progeny T_j of the jth child of the initial individual satisfies that $\{T_j\}_{j=1}^i$ is an i.i.d. sequence of random variables with law equal to the one of T. Therefore, using also that

$$T = 1 + \sum_{j=1}^{i} T_j,\tag{3.1.24}$$

where, by convention, the empty sum, arising when $i = 0$, is equal to zero, we obtain

$$G_T(s) = \sum_{i=0}^{\infty} p_i \mathbb{E}[s^T | Z_1 = i] = s \sum_{i=0}^{\infty} p_i \mathbb{E}[s^{T_1 + \cdots + T_i}] = s \sum_{i=0}^{\infty} p_i G_T(s)^i = sG(G_T(s)).\tag{3.1.25}$$

This completes the proof. $\qquad\square$

Exercise 3.6 (Exercise 3.2 cont.)**.** *In the case of binary branching, i.e., when p is given by (3.1.15), show that*

$$G_T(s) = \frac{1 - \sqrt{1 - 4s^2pq}}{2sp}.\tag{3.1.26}$$

Exercise 3.7 (Exercise 3.5 cont.)**.** *Show, using Theorem 3.2, that, for $\{p_k\}_{k=0}^{\infty}$ in (3.1.17),*

$$G_T(s) = \frac{\sqrt{(p + s(b - pq))^2 - 4pqs(p - b)} - (p + sbq)}{2pq}\tag{3.1.27}$$

e:TGF-comp

3.2   Family moments

In this section, we compute the mean generation size of a branching process, and use this to compute the mean family size or the mean total progeny. The main result is the following theorem:

Theorem 3.3 (Moments of generation sizes)**.** *For all $n \geq 0$, and with $\mu = \mathbb{E}[Z_1] = \mathbb{E}[X]$ the expected offspring of a given individual,*

$$\mathbb{E}[Z_n] = \mu^n.\tag{3.2.1}$$

e:momBPgeneration

Proof. Recall that

$$Z_n = \sum_{i=1}^{Z_{n-1}} X_{n,i},\tag{3.2.2}$$

e:Zn-recrep

where $\{X_{n,i}\}_{n,i \geq 1}$ is a doubly infinite array of i.i.d. random variables. In particular, $\{X_{n,i}\}_{i \geq 1}$ is independent of Z_{n-1}.

Exercise 3.8. *Complete the proof of Theorem 3.3 by conditioning on Z_{n-1} and showing that*

$$\mathbb{E}\big[\sum_{i=1}^{Z_{n-1}} X_{n,i}|Z_{n-1} = m\big] = m\mu, \qquad \text{(3.2.3)}$$

so that

$$\mathbb{E}[Z_n] = \mu\mathbb{E}[Z_{n-1}]. \qquad \text{(3.2.4)}$$

Exercise 3.9. *Prove that $\{\mu^{-n}Z_n\}_{n\geq 1}$ is a martingale.*

Exercise 3.10. *When the branching process is critical, note that $Z_n \xrightarrow{\mathbb{P}} 0$. On the other hand, conclude that $\mathbb{E}[Z_n] = 1$ for all $n \geq 1$.*

Theorem 3.4. *Fix $n \geq 0$. Let $\mu = \mathbb{E}[Z_1] = \mathbb{E}[X]$ be the expected offspring of a given individual, and assume that $\mu < 1$. Then*

$$\mathbb{P}(Z_n > 0) \leq \mu^n. \qquad \text{(3.2.5)}$$

Exercise 3.11. *Prove Theorem 3.4 by using Theorem 3.3, together with the Markov inequality (2.4.1).*

Theorem 3.4 implies that in the subcritical regime, i.e., when the expected offspring $\mu < 1$, the probability that the population survives up to time n is exponentially small in n.

Theorem 3.5 (Expected total progeny)**.** *For a branching process with i.i.d. offspring X having mean offspring $\mu < 1$,*

$$\mathbb{E}[T] = \frac{1}{1 - \mu}. \qquad \text{(3.2.6)}$$

Exercise 3.12. *Prove (3.2.6).*

3.3   Random-walk perspective to branching processes

In branching processes, it is common to study the number of descendants of each generation. For random graph purposes, it is often convenient to use a different construction of a branching process by sequentially investigating the number of children of each member of the population. This picture leads to a *random walk formulation of branching processes.* For more background on random walks, we refer the reader to [195] or [102, Section 5.3].

We now give the random walk representation of a branching process. Let $X_1, X_2, \ldots$ be independent random variables with the same distribution as $X_{1,1}$ in (3.1.2). Define $S_0, S_1, \ldots$ by the recursion

$$\begin{aligned} S_0 &= 1, \\ S_i &= S_{i-1} + X_i - 1 = X_1 + \ldots + X_i - (i - 1). \end{aligned} \qquad \text{(3.3.1)}$$

Let T be the smallest t for which $S_t = 0$, i.e., (recall (1.4.10))

$$T = \min\{t : S_t = 0\} = \min\{t : X_1 + \ldots + X_t = t - 1\}. \qquad \text{(3.3.2)}$$

If such a t does not exist, then we define $T = +\infty$.

The above description is *equivalent* to the normal definition of a branching process, but records the branching process tree in a different manner. For example, in the random walk picture, it is slightly more difficult to extract the distribution of the generation sizes. To see that the two pictures agree, we shall show that the distribution of the random variable T is equal to the *total progeny* of the branching process as defined in (3.1.21), and it is equal to the total number of individuals in the family tree of the initial individual.

To see this, we note that we can explore the branching process family tree as follows. We let X_1 denote the children of the original individual, and set S_1 as in (3.3.1). Then, there are $S_1 = X_1 - 1$ unexplored individuals, i.e., individuals of whom we have not yet explored how many children they have. We claim that after exploring i individuals, and on the event that there are at least i individuals in the family tree, S_i denotes the number of individuals of whom the children have not yet been explored. Indeed, suppose this is true for S_{i-1}. We are done when $S_{i-1} = 0$, since then all individuals have been explored, and the total number of explored individuals is clearly equal to the size of the family tree, which is T by definition. Thus, assume that $S_{i-1} > 0$. Then we pick an arbitrary unexplored individual and denote the number of its children by X_i. By the independence property of the offspring of different individuals in a branching process, we have that the distribution of X_i is equal to the distribution of Z_1, say. Also, after exploring the children of the ith individual, we have added X_i individuals that still need to be explored, and have explored a single individual, so that now the total number of unexplored individuals is equal to $S_{i-1} + X_i - 1$, which, by (3.3.1) is equal to S_i. This completes the proof using induction. Also, this gives a nice interpretation of the random process $\{S_i\}_{i=0}^{\infty}$ in (3.3.1). Finally, since the branching process total progeny is explored *precisely* at the moment that all of its individuals have been explored, it follows that T in (3.3.2) has the same distribution as the total progeny of the branching process.

Exercise 3.13. *Compute $\mathbb{P}(T = k)$ for T in (3.3.2) and $\mathbb{P}(T = k)$ for T in (3.1.21) explicitly, for $k = 1, 2$ and 3.*

The branching process belonging to the recursion in (3.3.1) is the following. The population starts with one active individual. At each time unit, we select one of the active individuals in the population, and give it X_i children. The children (if any) are set to active, and the individual becomes inactive.

This process is continued as long as there are active individuals in the population. Then, the process S_i describes the number of active individuals after the first i individuals have been explored. The process stops when $S_t = 0$, but the recursion can be defined for all t since this leaves the value of T unaffected. Note that, for a branching process, (3.3.1) only makes sense as long as $i \leq T$, since only then $S_i \geq 0$ for all $i \leq T$. However, (3.3.1) in itself can de defined for *all $i \geq 0$*, also when $S_i < 0$. This fact will be useful in the sequel.

Exercise 3.14 (Exercise 3.2 cont.)**.** *In the case of binary branching, i.e., when the offspring distribution is given by (3.1.15), show that*

$$\mathbb{P}(T = k) = \frac{1}{p}\mathbb{P}\big(S_0 = S_{k+1} = 0, S_i > 0 \ \forall 1 \leq i \leq k\big), \tag{3.3.3}$$

where $\{S_i\}_{i=1}^{\infty}$ is a simple random walk, i.e.,

$$S_i = Y_1 + \cdots + Y_i, \tag{3.3.4}$$

where $\{Y_i\}_{i=1}^{\infty}$ are i.i.d. random variables with distribution

$$\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = p. \tag{3.3.5}$$

This gives a one-to-one relation between random walks excursions and the total progeny of a binary branching process.

Denote by $H = (X_1, \ldots, X_T)$ the history of the process up to time T. We include the case where $T = \infty$, in which case the vector H has infinite length. A sequence $(x_1, \ldots, x_t)$ is a possible history if and only if the sequence x_i satisfies (3.3.1), i.e., when $s_i > 0$ for all $i < t$, while $s_t = 0$, where $s_i = x_1 + \cdots + x_i - (i - 1)$. Then, for any $t < \infty$,

$$\mathbb{P}(H = (x_1, \ldots, x_t)) = \prod_{i=1}^{t} p_{x_i}.$$

<div align="right">e:history
(3.3.6)</div>

Note that (3.3.6) determines the law of the branching process.

We will use the random walk perspective in order to describe the distribution of a branching process *conditioned on extinction*. Call the distributions p and p' a conjugate pair if

$$p_x' = \eta^{x-1} p_x,$$

<div align="right">e:conjugate
(3.3.7)</div>

where η is the extinction probability belonging to the offspring distribution $\{p_x\}_{x=0}^{\infty}$, so that $\eta = G(\eta)$.

Exercise 3.15. *Prove that $p' = \{p_x'\}_{x=0}^{\infty}$ defined in (3.3.7) is a probability distribution.*

The relation between a supercritical branching process conditioned on extinction and its conjugate branching process is as follows:

Theorem 3.6 (Duality principle for branching processes). *Let p and p' be conjugate offspring distributions. The branching process with distribution p, conditional on extinction, has the same distribution as the branching process with offspring distribution p'.*

Proof. It suffices to show that for every history $H = (x_1, \ldots, x_t)$, the probability (3.3.6) is the same for the branching process with offspring distribution p, when conditioned on extinction, and the branching process with offspring distribution p'. Fix a $t < \infty$. First observe that

$$\mathbb{P}(H = (x_1, \ldots, x_t) | \text{extinction}) = \frac{\mathbb{P}(\{H = (x_1, \ldots, x_t)\} \cap \text{extinction})}{\mathbb{P}(\text{extinction})}$$
$$= \eta^{-1} \mathbb{P}(H = (x_1, \ldots, x_t)),$$

<div align="right">e:historysub
(3.3.8)</div>

since a finite history implies that the population becomes extinct. Then, we use (3.3.6), together with the fact that

$$\prod_{i=1}^{t} p_{x_i} = \prod_{i=1}^{t} p_{x_i}' \eta^{-(x_i - 1)} = \eta^{t - \sum_{i=1}^{t} x_i} \prod_{i=1}^{t} p_{x_i}'$$
$$= \eta \prod_{i=1}^{t} p_{x_i}', \tag{3.3.9}$$

since $x_1 + \ldots + x_t = t - 1$. Substitution into (3.3.8) yields that

$$\mathbb{P}(H = (x_1, \ldots, x_t) | \text{extinction}) = \mathbb{P}'(H = (x_1, \ldots, x_t)), \tag{3.3.10}$$

where $\mathbb{P}'$ is the distribution of the branching process with offspring distribution p'. $\square$

Exercise 3.16. *Let $G_d(s) = \mathbb{E}'[s^{X_1}]$ be the probability generating function of the offspring of the dual branching process. Show that*

$$G_d(s) = \frac{1}{\eta} G(\eta s). \tag{3.3.11}$$

Exercise 3.17. *Let X' have probability mass function $p' = \{p'_x\}_{x=0}^{\infty}$ defined in (3.3.7). Show that when $\eta < 1$, then*

$$\mathbb{E}[X'] < 1. \qquad (3.3.12)$$

Thus, the branching process with offspring distribution p' is subcritical.

Another convenient feature of the random walk perspective for branching processes is that it allows one to study what the probability is of extinction when the family tree has at least a given size. The main result in this respect is given below:

Theorem 3.7 (Extinction probability with large total progeny). *For a branching process with i.i.d. offspring X having mean $\mu = \mathbb{E}[X] > 1$,*

$$\mathbb{P}(k \leq T < \infty) \leq \frac{e^{-Ik}}{1 - e^{-I}}, \qquad \overset{\text{e:LDTP}}{(3.3.13)}$$

where the exponential rate I is given by

$$I = \sup_{t \leq 0} \left(t - \log \mathbb{E}[e^{tX}]\right) > 0. \qquad (3.3.14)$$

Theorem 3.7 can be reformulated by saying that when the total progeny is large, then the branching process will survive with high probability.

Note that when $\mu = \mathbb{E}[X] > 1$ and when $\mathbb{E}[e^{tX}] < \infty$ for all $t \in \mathbb{R}$, then we can also write

$$I = \sup_{t} \left(t - \log \mathbb{E}[e^{tX}]\right), \qquad (3.3.15)$$

(see also (2.4.12)). However, in Theorem 3.7, it is *not* assumed that $\mathbb{E}[e^{tX}] < \infty$ for all $t \in \mathbb{R}$! Since $X \geq 0$, we clearly *do* have that $\mathbb{E}[e^{tX}] < \infty$ for all $t \leq 0$. Therefore, since also the derivative of $t \mapsto t - \log \mathbb{E}[e^{tX}]$ in $t = 0$ is equal to $1 - \mathbb{E}[X] < 0$, the supremum is attained at a $t < 0$, and, therefore, we obtain that $I > 0$ under no assumptions on the existence of the moment generating function of the offspring distribution. We now give the full proof:

Proof. We use the fact that $T = s$ implies that $S_s = 0$, which in turn implies that $X_1 + \ldots + X_s = s - 1 \leq s$. Therefore,

$$\mathbb{P}(k \leq T < \infty) \leq \sum_{s=k}^{\infty} \mathbb{P}(S_s = 0) \leq \sum_{s=k}^{\infty} \mathbb{P}(X_1 + \ldots + X_s \leq s). \qquad (3.3.16)$$

For the latter probability, we use (2.4.9) and (2.4.11) in Theorem 2.16 with $a = 1 < \mathbb{E}[X]$. Then, we arrive at

$$\mathbb{P}(k \leq T < \infty) \leq \sum_{s=k}^{\infty} e^{-sI} = \frac{e^{-Ik}}{1 - e^{-I}}. \qquad (3.3.17)$$

$\square$

3.4 Supercritical branching processes

In this section, we prove a convergence result for the population in the nth generation. Clearly, in the (sub)critical case, the limit of Z_n is equal to 0, and there is nothing to prove. In the supercritical case, when the expected offspring is equal to $\mu > 1$, it is also known that (see e.g., [17, Theorem 2, p. 8]) $\lim_{n \to \infty} \mathbb{P}(Z_n = k) = 0$ unless $k = 0$, and $\mathbb{P}(\lim_{n \to \infty} Z_n = 0) = 1 - \mathbb{P}(\lim_{n \to \infty} Z_n = \infty) = \eta$, where η

is the extinction probability of the branching process. In particular, the branching process population cannot stabilize. It remains to investigate what happens when $\eta < 1$, in which case $\lim_{n\to\infty} Z_n = \infty$ occurs with positive probability. We prove the following convergence result:

Theorem 3.8 (Convergence for supercritical branching processes). *For a branching process with i.i.d. offspring X having mean $\mu = \mathbb{E}[X] > 1$, $\mu^{-n} Z_n \xrightarrow{a.s.} W_\infty$ for some random variable W_∞ which is finite with probability 1.*

Proof. We use the martingale convergence theorem, Theorem 2.21, and, in particular, its consequence formulated in Exercise 2.23. Denote $M_n = \mu^{-n} Z_n$, and recall that by Exercise 3.9, $\{M_n\}_{n=1}^\infty$ is a martingale. By Theorem 3.3, we have that $\mathbb{E}[|M_n|] = \mathbb{E}[M_n] = 1$, so that Theorem 2.21 gives the result. $\qquad\square$

Unfortunately, not much is known about the limiting distribution W_∞. Its probability generating function $G_W(s) = \mathbb{E}[s^{W_\infty}]$ satisfies the implicit relation, for $s \in [0, 1]$,

$$G_W(s) = G(G_W(s^{1/\mu})). \tag{3.4.1}$$

<div align="right">_{e:Wrel}</div>

Exercise 3.18. *Prove* (3.4.1).

We next investigate when $\mathbb{P}(W_\infty > 0) = 1 - \eta = \zeta$:

Theorem 3.9 (Kesten-Stigum Theorem). *For a branching process with i.i.d. offspring X having mean $\mu = \mathbb{E}[X] > 1$, $\mathbb{P}(W_\infty = 0) = \eta$ precisely when $\mathbb{E}[X \log X] < \infty$. When $\mathbb{E}[X \log X] < \infty$, also $\mathbb{E}[W_\infty] = 1$, while, when $\mathbb{E}[X \log X] = \infty$, $\mathbb{P}(W_\infty = 0) = 1$.*

Theorem 3.9 implies that $\mathbb{P}(W_\infty > 0) = 1 - \eta$, where η is the extinction probability of the branching process, so that conditionally on survival, the probability that $W_\infty > 0$ is equal to one. Theorem 3.9 was first proved by Kesten and Stigum in [131, 132, 133]. It is remarkable that the precise condition when $W_\infty = 0$ a.s. can be so easily expressed in terms of a moment condition on the offspring distribution. An alternative proof of Theorem 3.9 is given in [17, Pages 24-26]. We shall present a proof from [152] below.
   Theorem 3.9 leaves us with the question what happens when $\mathbb{E}[X \log X] = \infty$. In this case, Seneta [187] has shown that there always exists a proper renormalization, i.e., there exists a sequence $\{c_n\}_{n=1}^\infty$ with $\lim_{n\to\infty} c_n^{1/n} = \mu$ such that Z_n/c_n converges to a non-degenerate limit. However, $c_n = o(\mu^n)$, so that $\mathbb{P}(W_\infty = 0) = 1$.

Exercise 3.19. *Prove that $\mathbb{P}(W_\infty > 0) = 1 - \eta$ implies that $\mathbb{P}(W_\infty > 0| \text{ survival}) = 1$.*

Exercise 3.20. *Prove, using Fatou's Lemma, that $\mathbb{E}[W_\infty] \leq 1$ always holds.*

Proof of Theorem 3.9.
Add proof from [152]! $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We continue by studying the number of particles with an infinite line of descent, i.e., the particles of whom the family tree survives forever. Interestingly, these particles form a branching process again, as we describe now. In order to state the result, we start with some definitions. We let $Z_n^{(1)}$ denote those particles from the nth generation of $\{Z_k\}_{k=0}^\infty$ that survive forever. Then, the main result is as follows:

Theorem 3.10 (Individuals with an infinite line of descent)**.** *Conditionally on survival, the process* $\{Z_n^{(\infty)}\}_{n=0}^{\infty}$ *is again a branching process with offspring distribution* $p^{(\infty)} = \{p_k^{(\infty)}\}_{k=0}^{\infty}$ *given by* $p_0^{(\infty)} = 0$ *and, for* $k \geq 1$,

$$p_k^{(\infty)} = \frac{1}{\zeta} \sum_{j=k}^{\infty} \binom{j}{k} \eta^{j-k} (1-\eta)^j p_j. \qquad \text{(3.4.2)}$$

<div align="right"><code>e:p1-def</code></div>

Moreover, since

$$\mu^{(\infty)} = \mathbb{E}[Z_1^{(\infty)}] = \mu = \mathbb{E}[Z_1], \qquad \text{(3.4.3)}$$

<div align="right"><code>e:mu-infline</code></div>

this branching process is supercritical with the same expected offspring distribution as $\{Z_n\}_{n=0}^{\infty}$ *itself.*

Comparing Theorem 3.10 to Theorem 3.6, we see that in the supercritical regime, the branching process conditioned on extinction is a branching process with the dual (subcritical) offspring distribution, while, conditional on survival, the individuals with an infinite line of descent for a (supercritical) branching process.

Exercise 3.21. *Prove that* $p^{(\infty)}$ *is a probability distribution.*

Proof of Theorem 3.10. We let $\mathcal{A}_{\infty}$ be the event that $Z_n \to \infty$, and $B = \mathcal{A}_{\infty}^c$. We shall prove, by induction on $n \geq 0$, that the distribution of $\{Z_k^{(\infty)}\}_{k=0}^n$ conditionally on $\mathcal{A}_{\infty}$ is equal to that of a $\{\hat{Z}_k\}_{k=0}^n$, where $\{\hat{Z}_k\}_{k=0}^{\infty}$ is a branching process with offspring distribution $p^{(\infty)}$ given in (3.4.2). We start by initializing the induction hypothesis. For this, we note that, on $\mathcal{A}_{\infty}$, we have that $Z_0^{(\infty)} = 1$, whereas, by convention, $\hat{Z}_0 = 1$. This initializes the induction hypothesis.

To advance the induction hypothesis, we argue as follows. Suppose that the distribution of $\{Z_k^{(\infty)}\}_{k=0}^n$, conditionally on $\mathcal{A}_{\infty}$, is equal to that of $\{\hat{Z}_k\}_{k=0}^n$. Then, we shall show that also the distribution of $\{Z_k^{(\infty)}\}_{k=0}^{n+1}$, conditionally on $\mathcal{A}_{\infty}$, is equal to that of $\{\hat{Z}_k\}_{k=0}^{n+1}$. By the induction hypothesis, this immediately follows if the *conditional* distributions of $Z_{n+1}^{(\infty)}$ given $\{Z_k^{(\infty)}\}_{k=0}^n$ is equal to the conditional distribution of $\hat{Z}_{n+1}$ given $\{\hat{Z}_k\}_{k=0}^n$.

The law of $\hat{Z}_{n+1}$ given $\{\hat{Z}_k\}_{k=0}^n$ is that of an independent sum of $\hat{Z}_n$ i.i.d. random variables with law $p^{(\infty)}$. Now, the law of $Z_{n+1}^{(\infty)}$ given $\{Z_k^{(\infty)}\}_{k=0}^n$ is equal to the law of $Z_{n+1}^{(\infty)}$ given $Z_n^{(\infty)}$, and each individual with infinite line of descent in the nth generation gives rise to a random and i.i.d. number of individuals with infinite line of descent in the $(n+1)^{\text{st}}$ generation with the same law as $Z_1^{(\infty)}$ conditionally on $\mathcal{A}_{\infty}$. As a result, to complete the proof of (3.4.2), we must show that

$$\mathbb{P}(Z_1^{(\infty)} = k \,|\, \mathcal{A}_{\infty}) = p_k^{(\infty)}. \qquad \text{(3.4.4)}$$

For $k = 0$, this is trivial, since, conditionally on $\mathcal{A}_{\infty}$, we have that $Z_1^{(\infty)} \geq 1$, so that both sides are equal to 0 for $k = 0$. For $k \geq 1$, on the other hand, the proof follows by conditioning on Z_1. We have that, for $k \geq 1$, $Z_1^{(\infty)} = k$ implies that $Z_1 \geq k$ and that $\mathcal{A}_{\infty}$ occurs, so that

$$\mathbb{P}(Z_1^{(\infty)} = k \,|\, \mathcal{A}_{\infty}) = \zeta^{-1} \mathbb{P}(Z_1^{(\infty)} = k) = \zeta^{-1} \sum_{j \geq k} \mathbb{P}(Z_1^{(\infty)} = k \,|\, Z_1 = j) \mathbb{P}(Z_1 = j) = \zeta^{-1} \sum_{j \geq k} \binom{j}{k} \eta^{j-k} (1-\eta)^j p_j,$$

$$\text{(3.4.5)}$$

since each of the j particles has infinite line of descent with probability $\zeta = 1 - \eta$, so that $\mathbb{P}(Z_1^{(\infty)} = k \,|\, Z_1 = j) = \mathbb{P}(\text{BIN}(j, 1 - \eta) = k)$.

We complete the proof of Theorem 3.10 by proving (3.4.3). We start by proving (3.4.2) when $\mu < \infty$. For this, we write, using that for $k = 0$, we may substitute the right-hand side of (3.4.2) instead of

$p_0^{(\infty)} = 0$, to obtain

$$\mu^{(\infty)} = \sum_{k=0}^{\infty} k p_k^{(\infty)} = \sum_{k=0}^{\infty} k \frac{1}{\zeta} \sum_{j=k}^{\infty} \binom{j}{k} \eta^{j-k} (1-\eta)^j p_j$$

$$= \frac{1}{\zeta} \sum_{j=0}^{\infty} p_j \sum_{k=0}^{j} k \binom{j}{k} \eta^{j-k} (1-\eta)^j = \frac{1}{\zeta} \sum_{j=0}^{\infty} p_j (\zeta j) = \sum_{j=0}^{\infty} j p_j = \mu. \qquad (3.4.6)$$

This proves (3.4.2) when $\mu < \infty$. When $\mu = \infty$, on the other hand, we only need to show that $\mu^{(\infty)} = \infty$ as well. This can easily be seen by an appropriate truncation argument, and is left to the reader. $\qquad \square$

Exercise 3.22. *Prove* (3.4.2) *when* $\mu = \infty$.

With Theorems 3.10 and 3.8 at hand, we see an interesting picture emerging. Indeed, by Theorem 3.8, we have that $Z_n \mu^{-n} \xrightarrow{a.s.} W_\infty$, where, if the $X \log X$-condition in Theorem 3.9 is satisfied, $\mathbb{P}(W_\infty > 0) = \zeta$, the branching process survival probability. On the other hand, by Theorem 3.10 and conditionally on $\mathcal{A}_\infty$, $\{Z_n^{(\infty)}\}_{n=0}^{\infty}$ is also a branching process with expected offspring μ, which survives with probability 1. As a result, $Z_n^{(\infty)} \mu^{-n} \xrightarrow{a.s.} W_\infty^{(\infty)}$, where, conditionally on $\mathcal{A}_\infty$, $\mathbb{P}(W_\infty^{(\infty)} > 0) = 1$, while, yet, $Z_n^{(\infty)} \leq Z_n$ for all $n \geq 0$, by definition. This raises the question what the relative size is of $Z_n^{(\infty)}$ and Z_n, conditionally on $\mathcal{A}_\infty$. This question is answered in the following theorem:

Theorem 3.11 (Proportion of particles with infinite line of descent)**.** *Conditionally on survival,*

$$\frac{Z_n^{(\infty)}}{Z_n} \xrightarrow{a.s.} \zeta. \qquad (3.4.7)$$

Theorem 3.11 will prove to be quite useful, since it allows us sometimes to transfer results on branching processes which survive with probability 1, such as $\{Z_n^{(\infty)}\}_{n=0}^{\infty}$ conditionally on survival, to branching processes which have a non-zero extinction probability, such as $\{Z_n\}_{n=0}^{\infty}$.

Proof of Theorem 3.11.

Add, maybe from [17]? $\qquad \square$

3.5 Properties of Poisson branching processes

In this section, we specialize the discussion of branching processes to branching processes with Poisson offspring distributions. We will denote the distribution of a Poisson branching process by $\mathbb{P}_\lambda^*$. We also write T^* for the total progeny of the Poisson branching process, and X^* for a Poisson random variable.

For a Poisson random variable X^* with mean λ, we have that the probability generating function of the offspring distribution is equal to

$$G_\lambda^*(s) = \mathbb{E}_\lambda^*[s^{X^*}] = \sum_{i=0}^{\infty} s^i e^{-\lambda} \frac{\lambda^i}{i!} = e^{\lambda(s-1)}. \qquad (3.5.1)$$

Therefore, the relation for the extinction probability η in (3.1.5) becomes

$$\eta_\lambda = e^{\lambda(\eta_\lambda - 1)}, \qquad (3.5.2)$$

where we add the subscript λ to make the dependence on λ explicit.

For $\lambda \leq 1$, the equation (3.5.2) has the unique solution $\eta_\lambda = 1$, which corresponds to certain extinction. For $\lambda > 1$ there are two solutions, of which the smallest satisfies $\eta_\lambda \in (0, 1)$. As $\mathbb{P}_\lambda^*(T^* < \infty) < 1$, we know

$$\mathbb{P}_\lambda^*(T^* < \infty) = \eta_\lambda. \tag{3.5.3}$$

We recall that $H = (X_1^*, \ldots, X_T^*)$ is the history of the branching process, where again we have added superscripts $*$ to indicate that we mean a Poisson branching process. Then, conditionally on extinction, a Poisson branching process has law p' given by

$$p_i' = \eta_\lambda^{i-1} p_i = e^{-\lambda\eta_\lambda} \frac{(\lambda\eta_\lambda)^i}{i!}, \qquad\qquad \text{\texttt{e:conjugatePois}} \tag{3.5.4}$$

where we have used (3.5.2). Note that this offspring distribution is again Poisson with mean

$$\mu_\lambda = \lambda\eta_\lambda, \qquad\qquad \text{\texttt{e:conjmu}} \tag{3.5.5}$$

and, again by (3.5.2),

$$\mu_\lambda e^{-\mu_\lambda} = \lambda\eta_\lambda e^{-\lambda\eta_\lambda} = \lambda e^{-\lambda}. \tag{3.5.6}$$

Therefore, we call $\mu < 1 < \lambda$ a *conjugate pair* if

$$\mu e^{-\mu} = \lambda e^{-\lambda}. \qquad\qquad \text{\texttt{e:conjugatepairPois}} \tag{3.5.7}$$

Since $x \mapsto xe^{-x}$ is first increasing and then decreasing, with a maximum of e^{-1} at $x = 1$, the equation $\mu e^{-\mu} = \lambda e^{-\lambda}$ has precisely two solutions, a solution $\mu < 1$ and a solution $\lambda > 1$. Therefore, for Poisson offspring distributions, the duality principle in Theorem 3.6 can be reformulated as follows:

Theorem 3.12 (Poisson duality principle). *Let $\mu < 1 < \lambda$ be conjugates. The Poisson branching process with mean λ, conditional on extinction, has the same distribution as a Poisson branching process with mean μ.*

We further describe the law of the total progeny of a Poisson branching process:

Theorem 3.13 (Total progeny for Poisson BP). *For a branching process with i.i.d. offspring X, where X has a Poisson distribution with mean λ,*

$$\mathbb{P}_\lambda^*(T^* = n) = \frac{(\lambda n)^{n-1}}{n!} e^{-\lambda n}, \qquad (n \geq 1). \qquad\qquad \text{\texttt{e:totprogPois}} \tag{3.5.8}$$

In the proof below, we make heavy use of combinatorial results, more in particular, of Cayley's Theorem. In Section 3.7 below, we give a more general version of Theorem 3.13 valid for *any* offspring distribution, using the random-walk *Hitting-time theorem*.

Exercise 3.23. *Use Theorem 3.13 to show that, for any λ, and for k sufficiently large,*

$$\mathbb{P}_\lambda^*(k \leq T^* < \infty) \leq e^{-I_\lambda k}, \tag{3.5.9}$$

where $I_\lambda = \lambda - 1 - \log \lambda$.

Proof of Theorem 3.13. The proof of Theorem 3.13 relies on Cayley's Theorem on the number of labeled trees [56]. In its statement, we define a *labeled tree* on $\{1, \ldots, n\}$ to be a tree of size n where all vertices have a label in $\{1, \ldots, n\}$ and each label occurs precisely once. We now make this definition precise. An *edge* of a labeled tree is a pair $\{v_1, v_2\}$, where v_1 and v_2 are the labels of two connected vertices in the tree. The *edge set* of a tree of size n is the collection of its $n-1$ edges. Two labeled trees are equal if and only if they consist of the same edge sets. A labeled tree of n vertices is equivalent to a spanning tree of the complete graph K_n on the vertices $\{1, \ldots, n\}$. Cayley's Theorem reads as follows:

Theorem 3.14 (Cayley's Theorem). *The number of labeled trees of size n is equal to n^{n-2}. Equivalently, the number of spanning trees of the complete graph of size n equals n^{n-2}.*

Proof. We need to prove that the number of spanning trees in a complete graph on n points is equal to n^{n-2}. We first show that any spanning tree has at least one vertex of degree 1. Suppose, on the contrary, that all vertices in the spanning tree T have degree at least 2. Fix an initial vertex, and perform a walk by traversing one of its edges. Then again choose an edge that we have not yet chosen from the current position, and repeat. Since the graph is finite, there must be a moment that, from the current position, we cannot choose an edge we have not chosen before. Since each vertex has at least *two* edges, this means that the final vertex has been visited at least twice, which, in turn, means that there exists a cycle contradicting the statement that T is a tree.

We now complete the proof. Let $r_1, r_2, \ldots, r_k$ be non-negative integers with sum n. Then the multinomial coefficient $\binom{n}{r_1, r_2, \ldots, r_k}$ is defined by the relation

$$(x_1 + x_2 + \ldots + x_k)^n = \sum \binom{n}{r_1, r_2, \ldots, r_k} x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k}, \tag{3.5.10}$$

where the sum is over all k-tuples $(r_1, r_2, \ldots, r_k)$ which sum to n.

Since $(x_1 + x_2 + \ldots + x_k)^n = (x_1 + x_2 + \ldots + x_k)^{n-1}(x_1 + x_2 + \ldots + x_k)$, it follows that

$$\binom{n}{r_1, r_2, \ldots, r_k} = \sum_{i=1}^{k} \binom{n-1}{r_1, \ldots, r_i - 1, \ldots, r_k}. \tag{3.5.11}$$

Denote the number of spanning trees on the complete graph of size n for which the degrees are $d_1, d_2, \ldots, d_n$, i.e., the degree of vertex i equals d_i, by $t(n; d_1, d_2, \ldots, d_n)$. Then, the total number of spanning trees equals $\sum_{d_1, \ldots, d_n} t(n; d_1, d_2, \ldots, d_n)$.

Clearly, $t(n; d_1, d_2, \ldots, d_n)$ is 0 if one of the d_i is zero. The value of $t(n; d_1, d_2, \ldots, d_n)$ depends only on the multiset of numbers d_i and not on their order. Therefore, we may assume without loss of generality that $d_1 \geq d_2 \geq \ldots \geq d_n$, so $d_n = 1$ since there is at least one vertex with degree equal to one. Take the vertex v_n corresponding to d_n. This vertex is joined to some vertex v_i of degree $d_i \geq 2$, and any of the remaining vertices is a candidate. Therefore,

$$t(n; d_1, d_2, \ldots, d_n) = \sum_{i=1}^{n-1} t(n-1; d_1, \ldots, d_i - 1, \ldots, d_{n-1}) \tag{3.5.12}$$

It is trivial to check by hand that

$$t(n; d_1, d_2, \ldots, d_n) = \binom{n-2}{d_1 - 1, \ldots, d_n - 1} \tag{3.5.13}$$

for $n = 3$. Since the numbers of the left-hand side, respectively on the right-hand side, of (3.5.13) satisfy the recurrence relation (3.5.12), respectively (3.5.11) it follows by induction that (3.5.13) is true for all n, that is,

$$
\begin{aligned}
t(n; d_1, d_2, \ldots, d_n) &= \sum_{i=1}^{n-1} t(n-1; d_1, \ldots, d_i - 1, \ldots, d_{n-1}) \\
&= \sum_{i=1}^{n-1} \binom{n-3}{d_1 - 1, \ldots, d_i - 2, \ldots, d_{n-1} - 1} \\
&= \binom{n-2}{d_1 - 1, \ldots, d_i - 1, \ldots, d_{n-1} - 1, d_n - 1}.
\end{aligned}
\tag{3.5.14}
$$

We have added an extra argument $d_n - 1$ to the multinomial coefficient, which does not alter its value since $d_n - 1 = 0$. In (3.5.10), replace n by $n - 2$, take $r_i = d_i - 1$ and $x_i = 1$, to find

$$
n^{n-2} = \sum_{d_1, \ldots, d_n} t(n; d_1, d_2, \ldots, d_n).
\tag{3.5.15}
$$

This completes the proof. $\square$

Exercise 3.24. *Use the above proof to show that the number of labeled trees with degrees $d_1, \ldots, d_n$, where d_i is the degree of vertex i, equals*

$$
t(n; d_1, d_2, \ldots, d_n) = \binom{n-2}{d_1 - 1, \ldots, d_i - 1, \ldots, d_{n-1} - 1, d_n - 1}.
$$

We show that Cayley's Theorem is equivalent to the following equality:

Lemma 3.15. *For every $n \geq 2$, the following equality holds:*

$$
\sum_{i=1}^{n-1} \frac{1}{i!} \sum_{n_1 + \ldots + n_i = n-1} \prod_{j=1}^{i} \frac{n_j^{n_j - 1}}{n_j!} = \frac{n^{n-1}}{n!}.
\tag{3.5.16}
$$

Proof. We use the fact that a tree of size n is in one-to-one correspondence with the degree of vertex 1, which we denote by i, and with the labeled subtrees that hang off the i direct neighbors of vertex 1, where we distinguish the vertices which are direct neighbors of 1. See Figure 3.4.

Denote the size of the subtrees by $n_1, \ldots, n_i$, so that $n_1 + \ldots + n_i = n - 1$. There are

$$
\frac{(n-1)!}{n_1! \cdots n_i!}
\tag{3.5.17}
$$

ways of dividing the labels $2, \ldots, n$ into i groups. There are $n_j^{n_j - 2}$ trees of size n_j, so that there are $n_j^{n_j - 1}$ possible trees of size n_j with a distinguished vertex. Since the tree of size n does not change by permuting the i trees located at the neighbors of 1 and since there are $i!$ of such permutations, in total, we have

$$
\frac{1}{i!} \frac{(n-1)!}{n_1! \cdots n_i!} \prod_{j=1}^{i} n_j^{n_j - 1}
\tag{3.5.18}
$$

ways of choosing the i trees located at the direct neighbors of 1 together with their distinguished neighbors of 1.
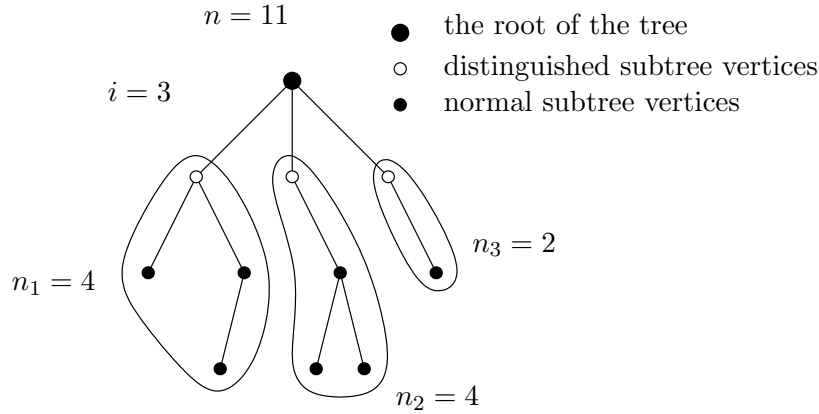
Figure 3.4: One-to-one correspondence of a labeled tree and $(i, T_1, \ldots, T_i)$.

By summing over i, we obtain the total number of trees of size n is equal to

$$\sum_{i=1}^{n-1} \frac{1}{i!} \sum_{n_1+\ldots+n_i=n-1} (n-1)! \prod_{j=1}^{i} \frac{n_j^{n_j-1}}{n_j!}. \tag{3.5.19}$$

By Cayley's Theorem, Theorem 3.14, we therefore obtain that

$$n^{n-2} = \sum_{i=1}^{n-1} \frac{1}{i!} \sum_{n_1+\ldots+n_i=n-1} (n-1)! \prod_{j=1}^{i} \frac{n_j^{n_j-1}}{n_j!}. \tag{3.5.20}$$

Dividing by $(n-1)!$ and using that $\frac{n^{n-2}}{(n-1)!} = \frac{n^{n-1}}{n!}$, we arrive at the claim. $\qquad\square$

We now complete the proof of Theorem 3.13. We use induction. For $n = 1$, the first individual of the branching process must die immediately, which has probability $e^{-\lambda}$. Since (3.5.8) is also equal to $e^{-\lambda}$ for $n = 1$, this initializes the induction hypotheses (3.5.8).

To advance the induction, we condition on the number i of children of the initial individual at time 0. We denote the size of the total progenies of the i children by $n_1, \ldots, n_i$, respectively, so that $T^* = n$ is equivalent to $n_1 + \ldots + n_i = n - 1$. Therefore,

$$\mathbb{P}_\lambda^*(T^* = n) = \sum_{i=1}^{n-1} e^{-\lambda} \frac{\lambda^i}{i!} \sum_{n_1+\ldots+n_i=n-1} \prod_{j=1}^{i} \mathbb{P}_\lambda^*(T^* = n_j). \tag{3.5.21}$$

By the induction hypothesis, and since $n_j \leq n - 1$, we obtain that

$$\mathbb{P}_\lambda^*(T^* = n_j) = \frac{(\lambda n_j)^{n_j-1}}{n_j!} e^{-\lambda n_j}. \tag{3.5.22}$$

Therefore, using that $\sum_{j=1}^{i}(n_j - 1) = n - i - 1$,

$$\mathbb{P}_\lambda^*(T^* = n) = \sum_{i=1}^{n-1} e^{-\lambda} \frac{\lambda^i}{i!} \sum_{n_1+\ldots+n_i=n-1} \prod_{j=1}^{i} \frac{(\lambda n_j)^{n_j-1}}{n_j!} e^{-\lambda n_j}$$

$$= e^{-\lambda n} \lambda^{n-1} \sum_{i=1}^{n-1} \frac{1}{i!} \sum_{n_1+\ldots+n_i=n-1} \prod_{j=1}^{i} \frac{n_j^{n_j-1}}{n_j!}. \tag{3.5.23}$$

Lemma 3.15 now completes the first proof of Theorem 3.13. □

Exercise 3.25. *Verify* (3.5.8) *for $n = 1, 2$ and $n = 3$.*

When the Poisson branching process is *critical*, i.e., when $\lambda = 1$, then we can use Stirling's Formula to see that

$$\mathbb{P}^*_\lambda(T^* = n) = (2\pi)^{-1/2} n^{-3/2} [1 + O(n^{-1})]. \tag{3.5.24}$$

This is an example of a *power-law* relationship that often holds at criticality. The above $n^{-3/2}$ behavior is associated more generally with the distribution of the total progeny whose offspring distribution has finite variance (see e.g., [12, Proposition 24]).

In Chapter 4, we will investigate the behavior of the Erdős-Rényi random graph by making use of couplings to branching processes. There, we also need the fact that, for $\lambda > 1$, the extinction probability is sufficiently smooth (see Section 4.4):

Corollary 3.16 (Differentiability of the extinction probability). *Let η_λ denote the extinction probability of a branching process with a mean λ Poisson offspring distribution. Then, for all $\lambda > 1$,*

$$|\frac{d}{d\lambda}\eta_\lambda| = \frac{\eta_\lambda(\lambda - \mu_\lambda)}{\lambda(1 - \mu_\lambda)} < \infty, \tag{3.5.25}$$

where μ_λ is the dual of λ.

Proof. The function η_λ, which we denote in this proof by $\eta(\lambda)$, is decreasing and satisfies

$$\eta(\lambda) = \mathbb{P}^*_\lambda(T^* < \infty) = \sum_{n=1}^{\infty} e^{-\lambda n} \frac{(\lambda n)^{n-1}}{n!}, \tag{3.5.26}$$

and thus

$$0 \le -\frac{d}{d\lambda}\eta(\lambda) = \sum_{n=1}^{\infty} e^{-n\lambda} \left[\frac{(\lambda n)^{n-1}}{(n-1)!}\right] - \sum_{n=2}^{\infty} e^{-n\lambda} \left[\frac{(\lambda n)^{n-2}}{(n-2)!}\right]. \tag{3.5.27}$$

On the other hand,

$$\mathbb{E}^*_\lambda[T^*|T^* < \infty] = \frac{1}{\mathbb{P}^*_\lambda(T^* < \infty)} \sum_{n=1}^{\infty} n \cdot e^{-\lambda n} \frac{(\lambda n)^{n-1}}{n!} = \frac{1}{\eta(\lambda)} \sum_{n=1}^{\infty} e^{-\lambda n} \frac{(\lambda n)^{n-1}}{(n-1)!}, \tag{3.5.28}$$

so that

$$-\frac{d}{d\lambda}\eta(\lambda) = \eta(\lambda)\mathbb{E}^*_\lambda[T^*|T^* < \infty] - \frac{\eta(\lambda)}{\lambda}\mathbb{E}^*_\lambda[T^*|T^* < \infty] + \frac{\eta(\lambda)}{\lambda}, \tag{3.5.29}$$

where we have made use of the fact that

$$\sum_{n=2}^{\infty} e^{-\lambda n} \frac{(\lambda n)^{n-2}}{(n-2)!} = \sum_{n=1}^{\infty} e^{-\lambda n}(n-1)\frac{(\lambda n)^{n-2}}{(n-1)!} = \sum_{n=1}^{\infty} e^{-\lambda n} \frac{1}{\lambda}\frac{(\lambda n)^{n-1}}{(n-1)!} - \sum_{n=1}^{\infty} e^{-\lambda n} \frac{(\lambda n)^{n-2}}{(n-1)!}$$

$$= \frac{\eta(\lambda)}{\lambda}\mathbb{E}^*_\lambda[T^*|T^* < \infty] - \sum_{n=1}^{\infty} e^{-\lambda n} \frac{1}{\lambda}\frac{(\lambda n)^{n-1}}{n!}$$

$$= \frac{\eta(\lambda)}{\lambda}\mathbb{E}^*_\lambda[T^*|T^* < \infty] - \frac{1}{\lambda}\mathbb{P}^*_\lambda(T^* < \infty). \tag{3.5.30}$$

By the duality principle and Theorem 3.5,

$$\mathbb{E}[T^*|T^* < \infty] = \frac{1}{1 - \mu_\lambda}$$

where $\mu_\lambda = \lambda\eta(\lambda)$, by (3.5.5). Hence,

$$\begin{aligned}
0 \leq -\frac{d}{d\lambda}\eta(\lambda) &= \frac{\eta(\lambda)}{1 - \mu_\lambda}\left(1 - \frac{1}{\lambda}\right) + \frac{\eta(\lambda)}{\lambda} \\
&= \frac{\eta(\lambda)(\lambda - \mu(\lambda))}{\lambda(1 - \mu_\lambda)}.
\end{aligned} \tag{3.5.31}$$

$\square$

3.6   Binomial and Poisson branching processes

When dealing with random graphs where the probability of keeping an edge is λ/n, the total number of vertices incident to a given vertex has a binomial distribution with parameters n and success probability λ/n. By Theorem 2.9, this distribution is close to a Poisson distribution with parameter λ. This suggests that also the corresponding branching processes, the one with a binomial offspring distribution with parameters n and λ/n, and the one with Poisson offspring distribution with mean λ, are close. In the following theorem, we make this statement more precise. In its statement, we write $\mathbb{P}_{n,p}$ for the law of a Binomial branching process with parameters n and success probability p.

Theorem 3.17 (Poisson and binomial branching processes). *For a branching process with binomial offspring distribution with parameters n and p, and the branching process with Poisson offspring distribution with parameter $\lambda = np$, for each $k \geq 1$,*

$$\mathbb{P}_{n,p}(T \geq k) = \mathbb{P}_\lambda^*(T^* \geq k) + e_k(n), \tag{3.6.1}$$

where T and T^* are the total progenies of the binomial and Poisson branching processes, respectively, and where

$$|e_k(n)| \leq \frac{2\lambda^2}{n}\sum_{s=1}^{k-1}\mathbb{P}_\lambda^*(T^* \geq s). \tag{3.6.2}$$

In particular, $|e_k(n)| \leq \frac{2k\lambda^2}{n}$.

Proof. We use a coupling proof. The branching processes are described by their offspring distributions, which are binomial and Poisson random variables respectively. We use the coupling in Theorem 2.9 for each of the random variables X_i and X_i^* determining the branching processes, where $X_i \sim \mathrm{BIN}(n, \lambda/n)$, $X_i^* \sim \mathrm{Poi}(\lambda)$, and where

$$\mathbb{P}(X_i \neq X_i^*) \leq \frac{\lambda^2}{n}. \tag{3.6.3}$$

We use $\mathbb{P}$ to denote the joint probability distributions of the binomial and Poisson branching processes, where the offspring is coupled in the above way.

We start by noting that

$$\mathbb{P}_{n,p}(T \geq k) = \mathbb{P}(T \geq k, T^* \geq k) + \mathbb{P}(T \geq k, T^* < k), \tag{3.6.4}$$

and

$$\mathbb{P}_\lambda^*(T^* \geq k) = \mathbb{P}(T \geq k, T^* \geq k) + \mathbb{P}(T^* \geq k, T < k). \tag{3.6.5}$$

Subtracting the two probabilities yields

$$|\mathbb{P}_{n,p}(T \geq k) - \mathbb{P}_\lambda^*(T^* \geq k)| \leq \mathbb{P}(T \geq k, T^* < k) + \mathbb{P}(T^* \geq k, T < k). \tag{3.6.6}$$

We then use Theorem 2.9, as well as the fact that the event $\{T \geq k\}$ is determined by the values of $X_1, \ldots, X_{k-1}$ only. Indeed, by (3.3.1), by investigating $X_1, \ldots, X_{k-1}$, we can verify whether there exists a $t < k$ such that $X_1 + \cdots + X_t = t-1$, implying that $T < k$. When there is no such t, then $T \geq k$. Similarly, by investigating $X_1^*, \ldots, X_{k-1}^*$, we can verify whether there exists a $t < k$ such that $X_1^* + \cdots + X_t^* = t-1$, implying that $T < k$.

When $T \geq k$ and $T^* < k$, or when $T^* \geq k$ and $T < k$, there must be a value of $s < k$ for which $X_s \neq X_s^*$. Therefore, we can bound, by splitting depending on the first value $s < k$ where $X_s \neq X_s^*$,

$$\mathbb{P}(T \geq k, T^* < k) \leq \sum_{s=1}^{k-1} \mathbb{P}(X_i = X_i^* \; \forall i \leq s-1, X_s \neq X_s^*, T \geq k), \tag{3.6.7}$$

where $\{X_i^*\}_{i=1}^\infty$ are i.i.d. Poisson random variables with mean λ and $\{X_i\}_{i=1}^\infty$ are i.i.d. binomial random variables with parameters n and p. Now we note that when $X_i = X_i^*$ for all $i \leq s-1$ and $T \geq k$, this implies in particular that $X_1^* + \ldots + X_i^* \geq i$ for all $i \leq s-1$, which in turn implies that $T^* \geq s$. Moreover, the event $\{T^* \geq s\}$ depends only on $X_1^*, \ldots, X_{s-1}^*$, and, therefore, is independent of the event that $X_s \neq X_s^*$. Thus, we arrive at the fact that

$$\mathbb{P}(T \geq k, T^* < k) \leq \sum_{s=1}^{k-1} \mathbb{P}(T^* \geq s, X_s \neq X_s^*)$$
$$= \sum_{s=1}^{k-1} \mathbb{P}(T^* \geq s) \mathbb{P}(X_s \neq X_s^*). \tag{3.6.8}$$

By Theorem 2.9,

$$\mathbb{P}(X_s \neq X_s^*) \leq \frac{\lambda^2}{n}, \tag{3.6.9}$$

so that

$$\mathbb{P}(T \geq k, T^* < k) \leq \frac{\lambda^2}{n} \sum_{s=1}^{k-1} \mathbb{P}(T^* \geq s). \tag{3.6.10}$$

An identical argument yields that

$$\mathbb{P}(T^* \geq k, T < k) \leq \sum_{s=1}^{k-1} \mathbb{P}(T^* \geq s) \mathbb{P}(X_s \neq X_s^*)$$
$$\leq \frac{\lambda^2}{n} \sum_{s=1}^{k-1} \mathbb{P}(T^* \geq s). \tag{3.6.11}$$

We conclude from (3.6.6) that

$$|\mathbb{P}_{n,p}(T \geq k) - \mathbb{P}_\lambda^*(T^* \geq k)| \leq \frac{2\lambda^2}{n} \sum_{s=1}^{k-1} \mathbb{P}_\lambda^*(T^* \geq s). \tag{3.6.12}$$

This completes the proof of Theorem 3.17. $\qquad\qquad\square$

3.7   Hitting-time theorem and the total progeny

In this section, we derive a general result for the law of the total progeny for branching processes, by making use of the Hitting-time theorem for random walks. The main result is the following:

Theorem 3.18 (Law of total progeny). *For a branching process with i.i.d. offspring distribution $Z_1 = X$,*

$$\mathbb{P}(T = n) = \frac{1}{n}\mathbb{P}(X_1 + \cdots + X_n = n - 1), \qquad\qquad \text{(3.7.1)}$$

e:totproggen

where $\{X_i\}_{i=1}^n$ are i.i.d. copies of X.

Exercise 3.26. *Prove Theorem 3.13 using Theorem 3.18.*

Exercise 3.27. *Compute the probability mass function of a branching process with a binomial offspring distribution using Theorem 3.18.*

Exercise 3.28. *Compute the probability mass function of a branching process with a geometric offspring distribution using Theorem 3.18. Hint: note that when $\{X_i\}_{i=1}^n$ are i.i.d. geometric, then $X_1 + \cdots + X_n$ has a negative binomial distribution.*

We shall prove Theorem 3.18 below. In fact, we shall prove a more general version of Theorem 3.18, which states that

$$\mathbb{P}(T_1 + \cdots + T_k = n) = \frac{k}{n}\mathbb{P}(X_1 + \cdots + X_n = n - k), \qquad\qquad \text{(3.7.2)}$$

e:gen-TP

where $T_1, \ldots, T_k$ are k independent random variables with the same distribution as T. Alternatively, we can think of $T_1 + \cdots + T_k$ as being the total progeny of a branching process starting with k individuals, i.e., when $Z_0 = k$.

The proof is based on the random walk representation of a branching process, together with the random-walk hitting time theorem. In its statement, we write $\mathbb{P}_k$ for the law of a random walk starting in k, we let $\{Y_i\}_{i=1}^\infty$ be the i.i.d. steps of the random walk, and we let $S_n = k + Y_1 + \cdots + Y_n$ be the position of the walk, starting in k, after n steps. We finally let

$$T_0 = \inf\{n : S_n = 0\} \qquad\qquad \text{(3.7.3)}$$

e:HitTime

denote the first hitting time of the origin of the walk. Then, the hitting-time theorem is the following result:

Theorem 3.19 (Hitting-time theorem). *For a random walk with i.i.d. steps $\{Y_i\}_{i=1}^\infty$ satisfying that*

$$\mathbb{P}(Y_i \geq -1) = 1, \qquad\qquad \text{(3.7.4)}$$

e:Yi-ass

the distribution of T_0 is given by

$$\mathbb{P}_k(T_0 = n) = \frac{k}{n}\mathbb{P}_k(S_n = 0). \qquad\qquad \text{(3.7.5)}$$

e:Tk-law

Theorem 3.19 is a remarkable result, since it states that, conditionally on the event $\{S_n = 0\}$, and regardless of the precise distribution of the steps of the walk $\{Y_i\}_{i=1}^\infty$ satisfying (3.7.4), the probability of the walk to be at 0 *for the first time* at time n is equal to $\frac{k}{n}$. Equation (3.7.2) follows from Theorem 3.19 since the law of $T_1 + \cdots + T_k$ is that of a the hitting time of a random walk starting in k with step distribution $Y_i = X_i - 1$, where $\{X_i\}_{i=1}^\infty$ are the offsprings of the vertices. Since $X_i \geq 0$, we have that $Y_i \geq -1$, which completes the proof of (3.7.2) and hence of Theorem 3.18. The details are left as an exercise:

Exercise 3.29. *Prove that Theorem 3.19 implies* (3.7.2).

Exercise 3.30. *Is Theorem 3.19 still true when the restriction that $\mathbb{P}(Y_i \geq -1) = 1$ is dropped?*

Proof of Theorem 3.19. We prove (3.7.5) for all $k \geq 0$ by induction on $n \geq 1$. When $n = 1$, then both sides are equal to 0 when $k > 1$ and $k = 0$, and are equal to $\mathbb{P}(Y_1 = -1)$ when $k = 1$. This initializes the induction.

To advance the induction, we take $n \geq 2$, and note that both sides are equal to 0 when $k = 0$. Thus, we may assume that $k \geq 1$. We condition on the first step to obtain

$$\mathbb{P}_k(T_0 = n) = \sum_{s=-1}^{\infty} \mathbb{P}_k(T_0 = n | Y_1 = s)\mathbb{P}(Y_1 = s). \tag{3.7.6}$$

By the random-walk Markov property,

$$\mathbb{P}_k(T_0 = n | Y_1 = s) = \mathbb{P}_{k+s}(T_0 = n - 1) = \frac{k + s}{n - 1}\mathbb{P}_{k+s}(S_{n-1} = 0), \tag{3.7.7}$$

where in the last equality we used the induction hypothesis, which is allowed since $k \geq 1$ and $s \geq -1$, so that $k + s \geq 0$. This leads to

$$\mathbb{P}_k(T_0 = n) = \sum_{s=-1}^{\infty} \frac{k + s}{n - 1}\mathbb{P}_{k+s}(S_{n-1} = 0)\mathbb{P}(Y_1 = s). \tag{3.7.8}$$

We undo the law of total probability, using that $\mathbb{P}_{k+s}(S_{n-1} = 0) = \mathbb{P}_k(S_n = 0 | Y_1 = s)$, to arrive at

$$\mathbb{P}_k(T_0 = n) = \sum_{s=-1}^{\infty} (k + s)\mathbb{P}_k(S_n = 0 | Y_1 = s)\mathbb{P}(Y_1 = s) = \mathbb{P}_k(S_n = 0)\Big(k + \mathbb{E}_k[Y_1 | S_n = 0]\Big), \tag{3.7.9}$$

where $\mathbb{E}_k[Y_1 | S_n = 0]$ is the conditional expectation of Y_1 given that $S_n = 0$ occurs. We next note that the conditional expectation of $\mathbb{E}_k[Y_i | S_n = 0]$ is independent of i, so that

$$\mathbb{E}_k[Y_1 | S_n = 0] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}_k[Y_i | S_n = 0] = \frac{1}{n}\mathbb{E}_k\Big[\sum_{i=1}^{n} Y_i \Big| S_n = 0\Big] = -\frac{k}{n}, \tag{3.7.10}$$

since $\sum_{i=1}^{n} Y_i = S_n - k = -k$ when $S_n = 0$. Therefore, we arrive at

$$\mathbb{P}_k(T_0 = n) = \frac{1}{n - 1}\Big[k - \frac{k}{n}\Big]\mathbb{P}_k(S_n = 0) = \frac{k}{n}\mathbb{P}_k(S_n = 0). \tag{3.7.11}$$

This advances the induction, and completes the proof of Theorem 3.19. $\qquad\square$

Exercise 3.31. *Extend the hitting-time theorem, Theorem 3.19, to the case where $\{Y_i\}_{i=1}^{n}$ is an exchangeable sequence rather than an i.i.d. sequence, where a sequence $\{Y_i\}_{i=1}^{n}$ is called exchangeable when its distribution is the same as the distribution of any permutation of the sequence. Hint: if $\{Y_i\}_{i=1}^{n}$ is exchangeable, then so is $\{Y_i\}_{i=1}^{n}$ conditioned on $\sum_{i=1}^{n} Y_i = -k$.*

3.8   Branching processes with infinite mean

In the analysis of the configuration model, we shall also encounter branching processes with infinite mean. Such branching processes are always supercritical, as the following exercise shows:

Exercise 3.32. *Prove that a branching process with infinite mean is always supercritical, e.g., the probability that it survives forever is strictly positive.*

In this section, we shall give a scaling result for the generation sizes for such branching processes. The main result we shall make use of is the following theorem:

Theorem 3.20 (Branching processes with infinite mean)**.** *Let $\{Z_n\}_{n=0}^{\infty}$ be a branching process with offspring distribution $Z_1 = X$ having distribution function F_X. Assume that there exist a $\tau \in (1, 2)$ and a non-negative, non-increasing function $x \mapsto \gamma(x)$, such that*

$$x^{-(\tau-1)-\gamma(x)} \leq 1 - F_X(x) \leq x^{-(\tau-1)+\gamma(x)}, \qquad \text{for large } x, \tag{3.8.1}$$

<div align="right">e:FSV-def</div>

where $x \mapsto \gamma(x)$ satisfies

   (i) $x \mapsto x^{\gamma(x)}$ *is non-decreasing,*

   (ii) $\int_0^{\infty} \gamma(e^{e^x})\, dx < \infty$, *or, equivalently, $\int_e^{\infty} \frac{\gamma(y)}{y \log y}\, dy < \infty$.*

Then $(\tau - 1)^n \log(Z_n \vee 1) \xrightarrow{a.s.} Y$, with $\mathbb{P}(Y = 0)$ equal to the extinction probability of $\{Z_n\}$, whereas Y admits a density on $(0, \infty)$.

Theorem 3.20 covers the case where the branching process has an offspring which has very thick tails. Indeed, it is not hard to show that Theorem 3.20 implies that $\mathbb{E}[X^s] = \infty$ for every $s > \alpha \in (0, 1)$ (see Exercise 3.34 below).

We shall not prove Theorem 3.20 in full generality. Rather, we shall prove it in a simpler, yet still quite general case, in which $\gamma(x) = (\log x)^{\gamma-1}$ for some $\gamma \in [0, 1)$.

Exercise 3.33. *Prove that $\gamma(x) = (\log x)^{\gamma-1}$ for some $\gamma \in [0, 1)$ satisfies the assumptions in Theorem 3.20.*

Proof of Theorem 3.20 for $\gamma(x) = (\log x)^{\gamma-1}$. The proof shall be divided into four main steps. Let $\alpha = \tau - 1$. Define

$$M_n = \alpha^n \log(Z_n \vee 1). \tag{3.8.2}$$

We shall first assume that $\mathbb{P}(Z_1 \geq 1) = 1$, so that $\eta = 1$. We start by splitting M_n in a suitable way.

The split.   For $i \geq 1$, we define

$$Y_i = \alpha^i \log \left(\frac{Z_i \vee 1}{(Z_{i-1} \vee 1)^{1/\alpha}} \right). \tag{3.8.3}$$

<div align="right">e:Yi-def</div>

We can write

$$M_n = Y_1 + Y_2 + \cdots + Y_n. \tag{3.8.4}$$

<div align="right">e:splitBPIM</div>

From this split, it is clear that almost sure convergence of M_n follows when the sum $\sum_{i=0}^{\infty} Y_i$ converges, which, in turn, is the case when

$$\sum_{i=1}^{\infty} \mathbb{E}[|Y_i|] < \infty. \tag{3.8.5}$$

This is what we shall prove in the following steps.

Inserting normalization sequences. We shall next investigate $\mathbb{E}[|Y_i|]$. We shall prove by induction on i that there exist constants $\kappa < 1$ and $C > 0$ such that

$$\mathbb{E}[|Y_i|] \leq K\kappa^i. \tag{3.8.6}$$

For $i = 0$, this follows from the fact that, when (3.8.1) holds, the random variable $Y_1 = \alpha \log(Z_1 \vee 1)$ has a bounded absolute expectation. This initializes the induction hypothesis. We next turn to the advancement of the induction hypothesis. For this, we recall the definition of u_n in (2.6.7) and define

$$U_i = \alpha^i \log\Big(\frac{Z_i \vee 1}{u_{Z_{i-1} \vee 1}}\Big), \qquad V_i = \alpha^i \log\Big(\frac{u_{Z_{i-1} \vee 1}}{(Z_{i-1} \vee 1)^{1/\alpha}}\Big). \tag{3.8.7}$$

Then we have that $Y_i = U_i + V_i$, so that

$$\mathbb{E}[|Y_i|] \leq \mathbb{E}[|U_i|] + \mathbb{E}[|V_i|]. \tag{3.8.8}$$

We shall bound each of these terms separately.

Bounding the normalizing constants. In this step, we analyse the normalizing constants $n \mapsto u_n$, assuming (3.8.1), and use this, as well as the induction hypothesis, to bound $\mathbb{E}[|U_i|]$.

When (3.8.1) holds, then it follows from [37] that there exists a slowly varying function $n \mapsto \ell(n)$ such that

$$u_n = n^{1/\alpha}\ell(n). \tag{3.8.9}$$

Since $n \mapsto \ell(n)$ is slowly varying, by Potter's Theorem, we have that, for every $\varepsilon > 0$, there exists a constant $C_\varepsilon \geq 1$ such that, for all $n \geq 1$,

$$C_\varepsilon^{-1} n^{-\varepsilon} \leq \ell(n) \leq C_\varepsilon n^\varepsilon, \tag{3.8.10}$$

so that, in turn, $u_n \leq C_\varepsilon n^{\frac{1}{\alpha}+\varepsilon}$. This gives a first bound on $n \mapsto u_n$. We next substitute this bound into (3.8.1) and use that $x \mapsto x^{\gamma(x)}$ is non-decreasing together with $\gamma(x) = (\log x)^{\gamma-1}$, to obtain that

$$1 + o(1) = n[1 - F_X(u_n)] \geq n\big[u_n^{-(\tau-1)-\gamma(u_n)}\big] \geq n\Big[u_n^{-(\tau-1)} e^{\log\big(C_\varepsilon n^{\frac{1}{\alpha}+\varepsilon}\big)^\gamma}\Big], \tag{3.8.11}$$

which, in turn, implies that there exists a constant $c > 0$ such that

$$u_n \leq n^{1/\alpha} e^{c(\log n)^\gamma}. \tag{3.8.12}$$

In a similar way, we can show the matching lower bound $u_n \geq n^{1/\alpha} e^{-c(\log n)^\gamma}$. As a result, we obtain that

$$\mathbb{E}[|U_i|] \leq c\alpha^i \mathbb{E}[(\log(Z_{i-1} \vee 1))^\gamma]. \tag{3.8.13}$$

Using the concavity of $x \mapsto x^\gamma$ for $\gamma \in [0, 1)$, as well as Jensen's Inequality, we arrive at

$$\mathbb{E}[|U_i|] \leq c\alpha^i \Big(\mathbb{E}[(\log(Z_{i-1} \vee 1))]\Big)^\gamma = \alpha^{i(1-\gamma)} \mathbb{E}[M_{i-1}]^\gamma. \tag{3.8.14}$$

By (3.8.4) and (3.8.6), which implies that $\mathbb{E}[M_{i-1}] \leq K\kappa/(1-\kappa)$, we arrive at

$$\mathbb{E}[|U_i|] \leq \alpha^{i(1-\gamma)} c\Big(\frac{K\kappa}{1-\kappa}\Big)^\gamma. \tag{3.8.15}$$

Bounding the logarithmic moment of an asymptotic stable random variable.   In this step, we bound $\mathbb{E}[|V_i|]$. We note that by Theorem 2.28 and for Z_i quite large, the random variable $(Z_i \vee 1)/(u_{Z_{i-1}} \vee 1)$ should be close to a stable random variable. We make use of this fact by bounding

$$\mathbb{E}[|V_i|] \leq \alpha^i \sup_{m \geq 1} \mathbb{E}[|\log(S_m/u_m)|], \qquad \text{(3.8.16)}$$

where $S_m = X_1 + \cdots + X_m$, and $\{X_i\}_{i=1}^m$ are i.i.d. copies of the offspring distribution X. We shall prove that there exists a constant $C > 0$ such that, for all $m \geq 1$,

$$\mathbb{E}[|\log(S_m/u_m)|] \leq C. \qquad \text{(3.8.17)}$$

In order to prove (3.8.17), we note that it suffices to bound

$$\mathbb{E}[(\log(S_m/u_m))_+] \leq C_+, \qquad \mathbb{E}[(\log(S_m/u_m))_-] \leq C_-, \qquad \text{(3.8.18)}$$

where, for $x \in \mathbb{R}$, $x_+ = \max\{x,0\}$ and $x_- = \max\{-x,0\}$. Since $|x| = x_+ + x_-$, we then obtain (3.8.17) with $C = C_+ + C_-$. In order to prove (3.8.17), we start by investigating $\mathbb{E}[(\log(S_m/u_m))_-]$. We note that $(\log x)_- = \log(x^{-1} \vee 1)$, so that

$$\mathbb{E}[(\log(S_m/u_m))_-] = \mathbb{E}[\log(u_m/(S_m \wedge u_m))], \qquad \text{(3.8.19)}$$

where $x \wedge y = \min\{x,y\}$. The function $x \mapsto \log((u_m/(x \wedge u_m))$ is non-increasing, and, since $S_m \geq X_{(m)}$, where $X_{(m)} = \max_{1 \leq i \leq m} X_i$, we arrive at

$$\mathbb{E}[\log(u_m/(S_m \wedge u_m))] \leq \mathbb{E}[\log(u_m/(X_{(m)} \wedge u_m))]. \qquad \text{(3.8.20)}$$

We next use that, for $x \geq 1$, $x \mapsto \log(x)$ is concave, so that, for every s,

$$\mathbb{E}[\log(u_m/(X_{(m)} \wedge u_m))] = \frac{1}{s}\mathbb{E}[\log((u_m/(X_{(m)} \wedge u_m))^s)] \leq \frac{1}{s}\log\left(\mathbb{E}[(u_m/(X_{(m)} \wedge u_m))^s]\right)$$
$$\leq \frac{1}{s} + \frac{1}{s}\log\left(u_m^p \mathbb{E}[X_{(m)}^{-s}]\right), \qquad \text{(3.8.21)}$$

where, in the last step, we made us of the fact that $u_m/(x \wedge u_m) \leq 1 + u_m/x$. Now rewrite $X_{(m)}^{-s} = (-Y_{(m)})^s$, where $Y_j = -X_j^{-1}$ and $Y_{(m)} = \max_{1 \leq j \leq m} Y_j$. Clearly, since $X_i \geq 1$, we have that $Y_j \in [-1,0]$, so that $\mathbb{E}[(-Y_1)^s] < \infty$. Also, $u_m Y_{(m)} = -u_m/X_{(m)}$ converges in distribution to $-E^{-1/(\tau-1)}$, where E is exponential with mean 1, so it follows from [178, Theorem 2.1] that, as $m \to \infty$,

$$\mathbb{E}[(u_m Y_{(m)})^p] \to \mathbb{E}[E^{-1/(\tau-1)}] < \infty. \qquad \text{(3.8.22)}$$

We proceed with $\mathbb{E}[(\log(S_m/u_m))_+]$, for which the proof is a slight adaptation of the above argument. Now we make use of the fact that $(\log x)_+ = \log(x \vee 1) \leq 1 + x$ for $x \geq 0$, so that we must bound

$$\mathbb{E}[\log(S_m \vee u_m/u_m)] = \frac{1}{s}\mathbb{E}[\log((S_m \vee u_m/u_m))^s)] \leq \frac{1}{s} + \log\left(\mathbb{E}[(\frac{S_m}{u_m})^s]\right). \qquad \text{(3.8.23)}$$

The discussion on page 565 and Corollary 1 of [104] yields, for $s < \tau - 1$, $\mathbb{E}[S_m^s] = \mathbb{E}[|S_m|^s] \leq 2^{s/2}\lambda_s(t)$, for some function $\lambda_s(m)$ depending on s, m and F_X. Using the discussion on [104, Page 564], we have that $\lambda_s(m) \leq C_s m^{s/(\tau-1)} l(m^{1/(\tau-1)})^s$, where $l(\cdot)$ is a slowly varying function. With some more effort, it can be shown that we can replace $l(m^{1/(\tau-1)})$ by $\ell(m)$, which gives

$$\mathbb{E}[\log(S_m \vee u_m/u_m)] \leq \frac{1}{s} + \log\left(\mathbb{E}[(\frac{S_m}{u_m})^s]\right) \leq \frac{1}{s} + \frac{C_s}{s}m^{s/(\tau-1)}\ell(m)^s u_m^{-s} = \frac{1}{s} + 2^{s/2}\frac{C_s}{s}, \qquad \text{(3.8.24)}$$

which proves the first bound in (3.8.18) with $C_+ = \frac{1}{s} + 2^{s/2}\frac{C_s}{s}$.

Completion of the proof of Theorem 3.20 when $X \geq 1$. Combining (3.8.8) with (3.8.15) and (3.8.16)–(3.8.17), we arrive at

$$\mathbb{E}[|Y_i|] \leq \alpha^{i(1-\gamma)} c \Big(\frac{K\kappa}{1-\kappa} \Big)^\gamma + C\alpha^i \leq K\kappa^i, \tag{3.8.25}$$

when we take $\kappa = \alpha^{1-\gamma}$ and we take K to be sufficiently large, for example $K \geq 2C$ and $K \geq 2c\Big(\frac{K\kappa}{1-\kappa}\Big)^\gamma$. We shall not prove that Y admits a density. This completes the proof when the offspring distribution X satisfies $X \geq 1$.

Completion of the proof of Theorem 3.20. We finally extend the result to the setting where $X = 0$ with positive probability. Since $\mathbb{E}[X] = \infty$, we have that the survival probability $\zeta > 0$ (recall Exercise 3.32). Conditionally on extinction, we have that $Z_n \xrightarrow{a.s.} 0$, so that, on the survival event, $(\tau-1)^n \log(Z_n \vee 1) \xrightarrow{a.s.} Y$, where, conditionally on extinction, $Y = 0$.

It remains to prove that $(\tau-1)^n \log(Z_n \vee 1) \xrightarrow{a.s.} Y$ on the survival event. By Theorem 3.11, we have that, conditionally on survival,

$$\frac{Z_n^{(\infty)}}{Z_n} \xrightarrow{a.s.} \zeta > 0, \tag{3.8.26}$$

where we recall that $Z_n^{(\infty)}$ are the individuals in the nth generation which have an infinite line of descent. By Theorem 3.10 and conditionally on survival, $\{Z_n^{(\infty)}\}_{n=0}^\infty$ is again a branching process, now with offspring distribution $p^{(\infty)}$ given in (3.4.2). Note that, in particular, $\mathbb{P}(Z_1^{(\infty)} \geq 1) = 1$, and we wish to apply Theorem 3.20 to $Z_n^{(\infty)}$ instead of Z_n. We shall assume that also $p^{(\infty)}$ in (3.4.2) satisfies the conditions in Theorem 3.20 with the function $x \mapsto \gamma^*(x)$, given by $\gamma^*(x) = \gamma(x) + c$ for some constant $c \geq 0$. Thus, we have that, conditionally on survival,

$$(\tau-1)^n \log(Z_n^{(\infty)} \vee 1) \xrightarrow{a.s.} Y^{(\infty)}, \tag{3.8.27}$$

and combining (3.8.26) and (3.8.27), it immediately follows that, conditionally on survival, we also obtain

$$(\tau-1)^n \log(Z_n \vee 1) \xrightarrow{a.s.} Y^{(\infty)}. \tag{3.8.28}$$

We conclude that Theorem 3.11 holds, where $Y = 0$ with probability $\eta = 1 - \zeta$ and $Y = Y^{(\infty)}$ with probability ζ. □

Exercise 3.34. *Prove that $\mathbb{E}[X] = \infty$ when the conditions in Theorem 3.20 are satisfied. Extend this to show that $\mathbb{E}[X^s] = \infty$ for every $s > \alpha \in (0,1)$.*

Exercise 3.35. *Show that, under the conditions of Theorem 3.20, also $\alpha^n \log(Z_n + 1)$ converges to Y almost surely.*

When $1 - F_X(x)$ is slowly varying with exponent 1, Theorem 3.20 does not apply. In this case, the generation sizes grow rather differently:

Theorem 3.21 (Branching processes with exponent $\tau = 2$). *Suppose that there exists a constant $c > 0$ such that*

$$1 - F_X(x) = cx^{-1}(1 + o(1)). \tag{3.8.29}$$

Then

$$\frac{\log Z_n \vee 1}{n \log n} \xrightarrow{\mathbb{P}} 1. \tag{3.8.30}$$

It would be of interest to extend Theorem 3.21 to other cases where $\tau = 2$. Unlike in Theorem 3.20, where the precise limiting behavior of Z_n is independent of the slowly varying function of $x^{\tau-1}[1 - F_X(x)]$, one would expect the limiting behavior to depend on the precise asymptotics of $x^{\tau-1}[1 - F_X(x)]$.

While Theorem 3.21 resolves what the leading order growth of the branching process is when $\alpha = 1$, it does not show what the fluctuations behave like (compare, for example to, Theorems 3.8 and 3.20, where the a.s. limits are *random*):

Open problem 3.22 (Branching processes with exponent $\tau = 2$). *Suppose that there exists a constant $c > 0$ such that*

$$1 - F_X(x) = cx^{-1}(1 + o(1)). \tag{3.8.31}$$

Determine the scale κ_n for which there exists a non-trivial random variable Y such that

$$\kappa_n^{-1}\Big(\log Z_n \vee 1 - n \log n \Big) \xrightarrow{a.s.} Y. \tag{3.8.32}$$

We shall not prove Theorem 3.21 entirely here, but rather only the lower bound, which is the simplest of the two:

Proof of the lower bound in Theorem 3.21.
   Add proof from notes.

$\square$

3.9   Discussion and notes

Notes on Section 3.5.   The proof of Theorem 3.14 is taken from [147]. Theorem 3.13, together with (3.5.2), can also be proved making use of Lambert's W function. Indeed, we use that the generating function of the total progeny in (3.1.23), for Poisson branching process, reduces to

$$G_T(s) = se^{\lambda(G_T(s)-1)}. \tag{3.9.1}$$

Equation (3.9.1) actually defines a function analytic in $\mathbb{C}\backslash[1, \infty)$, and we are taking the principal branch. Equation (3.9.1) can be written in terms of the Lambert W function, which is defined by $W(x)e^{W(x)} = x$, as $G_T(s) = -W(-s\lambda e^{-\lambda})/\lambda$. The branches of W are described in [70], where also the fact that

$$W(x) = -\sum_{n=1}^{\infty} \frac{n^{n-1}}{n!}(-x)^n. \tag{3.9.2}$$

is derived. Theorem 3.14 follows immediately from this equation upon substituting $x = \lambda e^{-\lambda}$ and using that the coefficient of s^n in $G_T(s)$ equals $\mathbb{P}(T = n)$. Also, since $\eta_\lambda = \lim_{s\uparrow 1} G_T(s) = -W(-\lambda e^{-\lambda})/\lambda$. This also allows for a more direct proof of Corollary 3.16, since

$$\frac{d}{d\lambda}\eta_\lambda = -\frac{d}{d\lambda}\Big[\frac{W(-\lambda e^{-\lambda})}{\lambda}\Big], \tag{3.9.3}$$

and where, since $W(x)e^{W(x)} = x$,

$$W'(x) = \frac{1}{x}\frac{W(x)}{1 + W(x)}. \tag{3.9.4}$$

We omit the details of this proof, taking a more combinatorial approach instead.

Notes on Section 3.7. The current proof is taken from [113]$^{\text{HofKea07}}$, where also an extension is proved by conditioning on the numbers of steps of various sizes. The first proof of the special case of Theorem 3.19 for $k = 1$ can be found in [175]$^{\text{Otte49}}$. The extension to $k \geq 2$ is in [130]$^{\text{Kemp61}}$, or in [82]$^{\text{Dwas69}}$ using a result in [81]$^{\text{Dwas68}}$. Most of these proofs make unnecessary use of generating functions, in particular, the Lagrange inversion formula, which the simple proof given here does not employ. See also [102$^{\text{GriSti01}}$, Page 165-167] for a more recent version of the generating function proof. In [203]$^{\text{Wend75}}$, various proofs of the hitting-time theorem are given, including a combinatorial proof making use of a relation in [80]$^{\text{Dwas62}}$. A proof for random walks making only steps of size ± 1 using the reflection principle can for example be found in [102$^{\text{GriSti01}}$, Page 79].

The hitting-time theorem is closely related to the ballot theorem, which has a long history dating back to Bertrand in 1887 (see [139]$^{\text{Kons95}}$ for an excellent overview of the history and literature). The version of the ballot theorem in [139]$^{\text{Kons95}}$ states that, for a random walk $\{S_n\}_{n=0}^{\infty}$ starting at 0, with exchangeable, nonnegative steps, the probability that $S_m < m$ for all $m = 1, \dots, n$, conditionally on $S_n = k$, equals k/n. This proof borrows upon queueing theory methodology, and is related to, yet slightly different from, our proof.

The ballot theorem for random walks with independent steps is the following result:

Theorem 3.23 (Ballot theorem). *Consider a random walk with i.i.d. steps $\{X_i\}_{i=1}^{\infty}$ taking non-negative integer values. Then, with $S_m = X_1 + \cdots + X_m$ the position of the walk after m steps,*

$$\mathbb{P}_0(S_m < m \text{ for all } 1 \leq m \leq n | S_n = n - k) = \frac{k}{n}. \tag{3.9.5}$$
$$^{\text{e:ballot}}$$

Exercise 3.36. *Prove the ballot theorem using the hitting-time theorem. Hint: Let $S'_m = k + (S_n - n) - (S_{n-m} - n + m)$, and note that $S_m < m$ for all $1 \leq m \leq n$ precisely when $S'_m > 0$ for all $0 \leq m < n$, and $\{S'_m\}_{m=0}^{n}$ is a random walk taking steps $Y_m = S'_m - S'_{m-1} = X_{n-m} - 1$.*

Notes on Section 3.8. Theorem 3.20 is proved in [72]$^{\text{Davi78}}$. A related result, under stronger conditions, appeared in [71]$^{\text{Darl70}}$. Branching processes with infinite mean have attracted considerable attention, see e.g., [186, 188]$^{\text{SchBar77, Sene73}}$ and the references therein. There is a balance between the generality of the results and the conditions on the offspring distribution, and, in our opinion, Theorem 3.20 strikes a nice balance in that the result is relatively simple and the conditions fairly general.

Theorem 3.21 proves [79$^{\text{Durr06}}$, Open problem].

Chapter 4

The phase transition for the Erdős-Rényi random graph

In this chapter, we study the connected components of the Erdős-Rényi random graph. In the introduction in Section 4.1, we will show that these connected components can be described in a similar way as for branching processes. As we have seen in Chapter 3, branching processes have a phase transition: when the mean offspring is below 1, the branching process dies out almost surely, while when the expected offspring exceeds 1, then it will survive with positive probability. The Erdős-Rényi random graph has a related phase transition. Indeed, when the expected degree is smaller than 1, the components are small, the largest one being of order $\log n$. On the other hand, when the expected degree exceeds 1, the there is a giant connected component which contains a positive proportion of all vertices. This phase transition can already be observed for relatively small graphs. For example, Figure 4.1 shows two realizations of Erdős-Rényi random graphs with 100 elements and expected degree close to $1/2$, respectively, $3/2$. The left picture is in the subcritical regime, and the connected components are tiny, while the right picture is in the supercritical regime, and the largest connected component is already substantial. The aim of this chapter is to quantify these facts.

We will also show that the Erdős-Rényi random graph is monotone increasing in the edge probability p. The link to branching processes is described in more detail in Section 4.2, where we prove upper and lower bounds for the tails of the cluster size (or connected component size) distribution. The *connected component containing v* is denoted by $C(v)$, and consists of all vertices that can be reached from v using occupied edges. We sometimes also call $C(v)$ the *cluster of v*. The connection between branching processes and clusters is used extensively in the later sections, Section 4.3–4.5. In Section 4.3, we study the subcritical regime of the Erdős-Rényi random graph. In Sections 4.4 and 4.5 we study the supercritical regime of the Erdős-Rényi random graph, by proving a law of large numbers for the largest connected component in Section 4.4 and a central limit theorem in Section 4.5.

In Chapter 5, we shall investigate several more properties of the Erdős-Rényi random graph. In particular, in Section 5.1, we study the component sizes of the critical Erdős-Rényi random graph, as well as the connectivity threshold.

4.1 Introduction

In this section, we introduce some notation for the Erdős-Rényi random graph, and prove some elementary properties. We recall from Section 1.4 that the Erdős-Rényi random graph has vertex set $\{1, \ldots, n\}$, and, denoting the edge between vertices $s, t \in \{1, \ldots, n\}$ by st, st is occupied or present with probability p, and absent or vacant otherwise, independently of all the other edges. The parameter p is called the *edge*
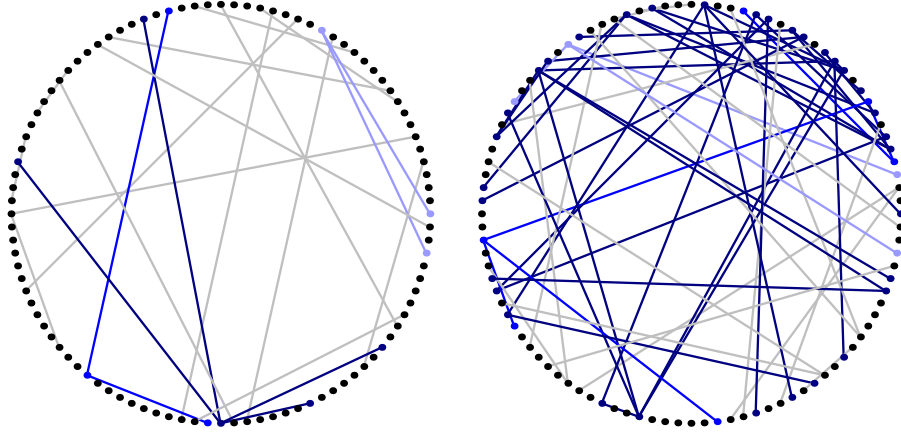
Figure 4.1: Two realizations of Erdős-Rényi random graphs with 100 elements and edge probabilities 1/200, respectively, 3/200. The three largest connected components are ordered by the darkness of their edge colors, the remaining connected components have edges with the lightest shade.

probability. The above random graph is denoted by $\mathrm{ER}(n, p)$.

We now introduce some notation. For two vertices $s, t \in \{1, \ldots, n\}$, we write $s \longleftrightarrow t$ when there exists a path of occupied edges connecting s and t. By convention, we always assume that $v \longleftrightarrow v$. For $v \in \{1, \ldots, n\}$, we denote the *connected component containing v* or *cluster of v* by

$$C(v) = \{x \in \{1, \ldots, n\} \colon v \longleftrightarrow x\}. \tag{4.1.1}$$

We denote the size of $C(v)$ by $|C(v)|$. The *largest connected component* is equal to any cluster $C(v)$ for which $|C(v)|$ is maximal, so that

$$|C_{\max}| = \max\{|C(v)| \colon v = 1, \ldots, n\}. \tag{4.1.2}$$

Note that the above definition does identify $|C_{\max}|$ uniquely, but it may not identify $C_{\max}$ uniquely. We can make this definition unique, by requiring that $C_{\max}$ is the cluster of maximal size containing the vertex with the smallest label. As we will see, the typical size of $C_{\max}$ will depend sensitively on the value λ.

We first define a procedure to find the connected component $C(v)$ containing a given vertex v in a given graph G. This procedure is closely related to the random walk perspective for branching processes described in Section 3.3, and works as follows. In the course of the exploration, vertices can have three different statuses: vertices are active, neutral or inactive. The status of vertices is changed in the course of the exploration of the connected component of v, as follows. At time $t = 0$, only v is active and all other vertices are neutral, and we set $S_0 = 1$. At each time t, we choose an active vertex w in an arbitrary way (for example, by taking the smallest active vertex) and explore all the edges ww, where w' runs over all the neutral vertices. If there is an edge in G connecting the active vertex w and some neutral vertex w', then we set w' active, otherwise it remains neutral. After searching the entire set of neutral vertices, we set w inactive and we let S_t equal the new number of active vertices at time t. When there are no more active vertices, i.e., when $S_t = 0$, the process terminates and $C(v)$ is the set of all inactive vertices, i.e., $|C(v)| = t$. Note that at any stage of the process, the size of $C(v)$ is bounded from below by the sum of the number of active and inactive vertices.

Let w_t be the tth active vertex of which all edges to neutral vertices are explored. Let X_t denote the number of neutral vertices w' with $w_t w' \in G$. Let S_t be the total number of active vertices at time

t. Similarly as for the branching process in (3.3.1), we can represent this procedure with the recursive relation

$$S_0 = 1,$$
$$S_t = S_{t-1} + X_t - 1. \tag{4.1.3}$$

The variable X_t is the number of vertices that become active due to the exploration of the tth vertex, and after its exploration, the tth explored vertex becomes inactive. Thus, if S_{t-1} denotes the number of active vertices after the exploration of $(t-1)$ vertices, $S_t = S_{t-1} + X_t - 1$ denotes the number of active vertices after the exploration of t vertices. This explains (4.1.3).

The above description is true for *any* graph G. We now specialize to the random graph $\text{ER}(n, p)$, where each edge can be independently occupied or vacant. As a result, the distribution of X_t depends on the number of active vertices at time $t - 1$, i.e., on S_{t-1}, and not in any other way on which vertices are active, inactive or neutral. More precisely, each neutral w' in the random graph has probability p to become active. The edges ww' are examined precisely once, so that the conditional probability for $ww' \in \text{ER}(n, p)$ is always equal to p. After $t - 1$ explorations of active vertices, we have $t - 1$ inactive vertices and S_{t-1} active vertices. This leaves $n - (t-1) - S_{t-1}$ neutral vertices. Therefore, conditionally on S_{t-1},

$$X_t \sim \text{BIN}\big(n - (t-1) - S_{t-1}, p\big). \tag{4.1.4}$$

We note that the recursion in (4.1.3) is identical to the recursive relation (3.3.1). The only difference is the distribution of the process $\{X_i\}_{i=1}^n$, as described in (4.1.4). For branching processes, $\{X_i\}_{i=1}^n$ is an i.i.d. sequence, but for the exploration of connected components, we see that this is not quite true. However, by (4.1.4), it is 'almost' true as long as the number of active vertices is not too large. We see in (4.1.4) that the parameter of the binomial distribution decreases. This is due to the fact that after more explorations, fewer neutral vertices remain, and is sometimes called the *depletion of points* effect.

Let T be the least t for which $S_t = 0$, then $|C(v)| = T$, see also (1.4.10). This describes the exploration of a single connected component. While of course the recursion in (4.1.3) and (4.1.4) only makes sense when $S_{t-1} \geq 1$, that is, when $t \leq T$, there is no harm in continuing it formally for $t > T$. This will be prove to be extremely useful later on.

Exercise 4.1. *Verify that $T = |C(v)|$ by computing the probabilities of the events that $\{|C(v)| = 1\}$, $\{|C(v)| = 2\}$ and $\{|C(v)| = 3\}$ directly, and by using (4.1.4), (1.4.9) and (1.4.10).*

We end this section by introducing some notation. For the Erdős-Rényi random graph, the status of all edges $\{st : 1 \leq s < t \leq n\}$ are i.i.d. random variables taking the value 1 with probability p and the value 0 with probability $1 - p$, 1 denoting that the edge is occupied and 0 that it is vacant. We will sometimes call the edge probability p, and sometimes λ/n. We will always use the convention that

$$p = \frac{\lambda}{n}. \tag{4.1.5}$$

We shall write $\mathbb{P}_\lambda$ for the distribution of $\text{ER}(n, p) = \text{ER}(n, \lambda/n)$.

Exercise 4.2. *We say that the distinct vertices (i, j, k) form an occupied triangle when the edges ij, jk and ki are all occupied. Note that (i, j, k) is the same triangle as (i, k, j) and as any other permutation. Compute the expected number of occupied triangles.*

Exercise 4.3. *We say that the distinct vertices (i, j, k, l) form an occupied square when the edges ij, jk, kl and li are all occupied. Note that the squares (i, j, k, l) and (i, k, j, l) are different. Compute the expected number of occupied squares.*

Exercise 4.4. *Show that the number of occupied triangles in an Erdős-Rényi random graph with edge probability $p = \lambda/n$ has an asymptotic Poisson distribution. Do the same for the number of occupied squares. Hint: use the method of moments in Theorem 2.4.*

Exercise 4.5. *Define the clustering coefficient of a random graph G to be*

$$CC_G = \frac{\mathbb{E}[\Delta_G]}{\mathbb{E}[W_G]}, \tag{4.1.6}$$

where

$$\Delta_G = \sum_{i,j,k \in G} I[ij, ik, jk \text{ occupied}], \qquad W_G = \sum_{i,j,k \in G} I[ij, ik \text{ occupied}]. \tag{4.1.7}$$

Thus, Δ_G is six times the number of triangles in G, and W_G is two times the number of open wedges in G, and CC_G is the ratio of the number of expected closed triangles to the expected number of open wedges. Compute CC_G for Erdős-Rényi random graph with edge probability $p = \lambda/n$.

Exercise 4.6. *Show that $W_G/n \xrightarrow{\mathbb{P}} \lambda^2$ by using the second moment method. Use Exercise 4.4 to conclude that*

$$\frac{n\Delta_G}{W_G} \xrightarrow{d} \frac{3}{\lambda^3} Y, \tag{4.1.8}$$

where $Y \sim \text{Poi}(\lambda^3/6)$.

4.1.1   Monotonicity of Erdős-Rényi random graphs in the edge probability

In this section, we investigate Erdős-Rényi random graphs with different values of p, and show that the Erdős-Rényi random graph is *monotonically increasing in p*, using a coupling argument. The material in this section makes it clear that components of the Erdős-Rényi random graph are growing with the edge probability p, as one would intuitively expect. This material shall also play a crucial role in determining the critical behavior of the Erdős-Rényi random graph in Section 5.1 below.

We use a coupling of all random graphs $\text{ER}(n, p)$ for all $p \in [0, 1]$. For this, we draw independent uniform random variables for each edge st, and, for fixed p, we declare an edge to be p-occupied if and only if $U_{st} \leq p$. The above coupling shows that the number of occupied bonds increases when p increases. Therefore, the Erdős-Rényi random graph $\text{ER}(n, p)$ is monotonically increasing in p. Because of the monotone nature of $\text{ER}(n, p)$ one expects that certain events and random variables grow larger when p increases. This is formalized in the following definition:

Definition 4.1. *We say that an event is increasing when, if the event occurs for a given set of occupied edges, it remains to hold when we make some more edges occupied.*
We say that a random variable X is increasing when the events $\{X \geq x\}$ are increasing for each $x \in \mathbb{R}$.

An example of an increasing event is $\{s \longleftrightarrow t\}$. An example of a monotone random variable is $|C(v)|$ and the maximal cluster $|C_{\max}|$, where

$$|C_{\max}| = \max_{v=1}^{n} |C(v)|. \tag{4.1.9}$$

Exercise 4.7. *Show that $|C_{\max}|$ is an increasing random variable.*

Exercise 4.8. *Is the event $\{v \in C_{\max}\}$ an increasing event?*

4.1.2 Informal link to Poisson branching processes

We now describe the link to Poisson branching processes in an informal manner. The results in this section will not be used in the remainder of the chapter, even though the philosophy forms the core of the argument. Fix $\lambda > 0$. Let $S_0^*, S_1^*, \ldots, X_1^*, X_2^*, \ldots, H^*$ refer to the history of a branching process with Poisson offspring distribution with mean λ and $S_0, S_1, \ldots, X_1, X_2, \ldots, H$ refer to the history of the random graph, where $S_0, S_1, \ldots$ are defined in (4.1.3) above. The event $\{H^* = (x_1, \ldots, x_t)\}$ is the event that the total progeny T^* of the Poisson branching process is equal to t, and the values of $X_1^*, \ldots, X_t^*$ are given by $x_1, \ldots, x_t$. Naturally, by (3.3.2), we have that

$$t = \min\{i : s_i = 0\} = \min\{i : x_1 + \ldots + x_i = i - 1\}, \tag{4.1.10}$$

where

$$s_0 = 1, \qquad s_i = s_{i-1} + x_i - 1. \tag{4.1.11}$$

For any possible history $(x_1, \ldots, x_t)$, we have that (recall (3.3.6))

$$\mathbb{P}_\lambda^*(H^* = (x_1, \ldots, x_t)) = \prod_{i=1}^t \mathbb{P}_\lambda^*(X_i^* = x_i), \tag{4.1.12}$$

where $\{X_i^*\}_{i=1}^\infty$ are i.i.d. Poisson random variables with mean λ, while

$$\mathbb{P}_\lambda(H = (x_1, \ldots, x_t)) = \prod_{i=1}^t \mathbb{P}_\lambda(X_i = x_i | X_1 = x_1, \ldots, X_{i-1} = x_{i-1}),$$

where, conditionally on $X_1 = x_1, \ldots, X_{i-1} = x_{i-1}$, the random variable X_i is binomially distributed $\mathrm{BIN}(n - (i-1) - s_{i-1}, \lambda/n)$, recall (4.1.4).

As shown in Theorem 2.9, the Poisson distribution is the limiting distribution of binomials when n is large and $p = \lambda/n$. When $m(n) = n(1 + o(1))$ and λ, i are fixed, then we can extend this to

$$\lim_{n\to\infty} \mathbb{P}\big(\mathrm{BIN}(m(n), \lambda/n) = i\big) = e^{-\lambda} \frac{\lambda^i}{i!}. \tag{4.1.13}$$

Therefore, for every $t < \infty$,

$$\lim_{n\to\infty} \mathbb{P}_\lambda(H = (x_1, \ldots, x_t)) = \mathbb{P}_\lambda^*(H^* = (x_1, \ldots, x_t)). \tag{4.1.14}$$

Thus, the distribution of finite connected components in the random graph $\mathrm{ER}(n, \lambda/n)$ is closely related to a Poisson branching process with mean λ. This relation shall be explored further in the remainder of this chapter.

4.2 Comparisons to branching processes

In this section, we investigate the relation between connected components and binomial branching processes. We start by proving two stochastic domination results for connected components in the Erdős-Rényi random graph. In Theorem 4.2, we give a stochastic upper bound on $|C(v)|$, and in Theorem 4.3 a lower bound on the cluster tails. These bounds will be used in the following sections to prove results concerning $|C_{\max}|$.

4.2.1   Stochastic domination of connected components

We prove the following upper bound, which shows that each connected component is bounded from above by the total progeny of a branching process with binomial offspring distribution:

Theorem 4.2 (Stochastic domination of the cluster size). *For each $k \geq 1$,*

$$\mathbb{P}_{np}(|C(1)| \geq k) \leq \mathbb{P}_{n,p}(T \geq k), \qquad i.e., \qquad |C(1)| \preceq T, \qquad \text{(4.2.1)}$$

<div align="right">e:CBP1</div>

where T is the total progeny of a binomial branching process with parameters n and p.

Proof. We note that the only distinction between the recursions (4.1.3) and (3.3.1), where X_i has a binomial distribution with parameters n and p, is that the parameter of the binomial distribution decreases in (4.1.3), see in particular (4.1.4), while it remains fixed in (3.3.1). The conditional distribution of X_i given $X_1, \ldots, X_{i-1}$ is stochastically dominated by $\hat{X}_i \sim \text{BIN}(n, p)$, which is independent of $X_1, \ldots, X_{i-1}$. Furthermore, the event $\{T \geq k\}$ is increasing in the variables $(X_1, \ldots, X_k)$, i.e., when $\{T \geq k\}$ occurs, and we make any of the random variables X_i larger, then $\{T \geq k\}$ continues to occur. As a result, $\{T \geq k\} \subseteq \{\hat{T} \geq k\}$, where $\hat{T} = \min\{i : \hat{X}_1 + \ldots + \hat{X}_i = i - 1\}$. Finally, by (3.3.1), $\hat{T}$ is the total progeny of a branching process with binomial offspring distribution with parameters n and p. $\square$

Exercise 4.9. *Show that, for $\lambda < 1$, $\mathbb{E}_\lambda[|C(v)|] \leq 1/(1 - \lambda)$.*

4.2.2   Lower bound on the cluster tail

We prove the following lower bound, which shows that the probability that a connected component has size at least k is bounded from below by the probability that the total progeny of a branching process with binomial offspring distribution exceeds k, where now the parameters of the binomial distribution are $n - k$ and p:

Theorem 4.3 (Lower bound on cluster tail). *For every $k \in \{1, \ldots, n\}$, with T the total progeny of a branching process with binomial distribution with parameters $n - k$ and success probability $p = \lambda/n$,*

$$\mathbb{P}_\lambda(|C(1)| \geq k) \geq \mathbb{P}_{n-k,p}(T \geq k). \qquad \text{(4.2.2)}$$

<div align="right">e:CBP2</div>

Note that, since the parameter $n-k$ on the right-hand side of (4.2.2) explicitly depends on k, Theorem 4.3 does *not* imply a stochastic lower bound on $|C(1)|$.

Proof. We again use a coupling approach. We explore the component of 1, and initially classify the vertices $n - k + 2, \ldots, n$ as *forbidden*, which means that we do *not* explore any edges that are incident to them.

During the exploration process, we will adjust this pool of forbidden vertices in such a way that the total number of forbidden, active and inactive vertices is fixed to k. Note that, initially, the vertex 1 is the only active vertex, there are no inactive vertices and the initial pool of forbidden vertices $\{n-k+2, \ldots, n\}$ has size $k - 1$. Thus, initially, the total number of forbidden, active and inactive vertices is fixed to k. We can only keep the total number of forbidden, active and inactive vertices fixed to k as long as the total number of active and inactive vertices is at most k. This poses no problems to us, because, in order to determine whether the event $\{|C(1)| \geq k\}$ occurs, we may stop the exploration at the first moment that the number of active and inactive vertices together is at least k, since $|C(v)|$ is at least as large as the number of active and inactive vertices at any moment in the exploration process.

We only explore edges to vertices that are not forbidden, active or inactive. We call these vertices the *allowed vertices*, so that the allowed vertices consist of the neutral vertices with the forbidden vertices

removed. When an edge to an allowed vertex is explored and found to be occupied, then the vertex becomes active, and we make the largest forbidden vertex neutral. As a result, for each vertex that turns active, we move one vertex from the forbidden vertices to the neutral vertices, thus keeping the number of allowed vertices fixed at $n - k$.

As long as the number of vertices that are active or inactive is at most k, we have that the total number of forbidden, active and inactive vertices is precisely equal to k. We arrive at a binomial branching process with the specified parameters $n - k$ and success probability p. Since the connected component $C(1)$ contains all the vertices that are found to be active or inactive in this process, we arrive at the claim. $\square$

The general strategy for the investigation of the largest connected component $|C_{\max}|$ is as follows. We make use of the stochastic bounds in Theorems 4.2–4.3 in order to compare the cluster sizes to binomial branching processes. Then, using Theorem 3.17, we can make the comparison to a Poisson branching process with a parameter that is close to the parameter λ in $\mathrm{ER}(n, \lambda/n)$. Using the results on branching processes in Chapter 3 then allows us to complete the proofs.

By Theorems 4.2–4.3, the connected components of the Erdős-Rényi random graph are closely related to binomial branching processes with a binomial offspring with parameters n and $p = \lambda/n$. By Theorem 3.1, the behavior of branching processes is rather different when the expected offspring is larger than 1 or smaller than or equal to 1. In Theorems 4.2–4.3, when $k = o(n)$, the expected offspring is close to $np \approx \lambda$. Therefore, for the Erdős-Rényi random graph, we expect different behavior in the subcritical regime $\lambda < 1$, in the supercritical regime $\lambda > 1$ and in the critical regime $\lambda = 1$.

The proof of the behavior of the largest connected component $|C_{\max}|$ is substantially different in the subcritical regime where $\lambda < 1$, which is treated in Section 4.3, compared to the supercritical regime $\lambda > 1$, which is treated in Section 4.4. In Section 4.5, we prove a central limit theorem for the giant supercritical component. The critical regime $\lambda = 1$ requires some new ideas, and is treated in Section 5.1.

4.3 The subcritical regime

In this section, we derive bounds for the size of the largest connected component for the Erdős-Rényi random graph in the subcritical regime, i.e., when $\lambda = np < 1$. Let I_λ denote the large deviation rate function for Poisson random variables with mean λ, given by

$$I_\lambda = \lambda - 1 - \log(\lambda). \tag{4.3.1}$$

Recall Exercise 2.17 to see an upper bound on Poisson random variables involving I_λ, as well as the fact that $I_\lambda > 0$ for all $\lambda \neq 1$.

The main results when $\lambda < 1$ are Theorem 4.4, which proves that $|C_{\max}| \leq a \log n$ with high probability, for any $a > I_\lambda^{-1}$, and Theorem 4.5, where a matching lower bound on $|C_{\max}|$ is provided by proving that $|C_{\max}| \geq a \log n$ with high probability, for any $a < I_\lambda^{-1}$. These results are stated now:

Theorem 4.4 (Upper bound on largest subcritical component). *Fix $\lambda < 1$. Then, for every $a > I_\lambda^{-1}$, there exists a $\delta = \delta(a, \lambda) > 0$ such that*

$$\mathbb{P}_\lambda(|C_{\max}| \geq a \log n) = O(n^{-\delta}). \tag{4.3.2}$$

Theorem 4.5 (Lower bound on largest subcritical component). *Fix $\lambda < 1$. Then, for every $a < I_\lambda^{-1}$, there exists a $\delta = \delta(a, \lambda) > 0$ such that*

$$\mathbb{P}_\lambda(|C_{\max}| \leq a \log n) = O(n^{-\delta}). \tag{4.3.3}$$

Theorems 4.4 and 4.5 will be proved in Sections 4.3.2 and 4.3.3 below. Together, they prove that $|C_{\max}|/\log n \xrightarrow{\mathbb{P}} I_{\lambda}^{-1}$:

Exercise 4.10. *Prove that Theorems 4.4 and 4.5 imply $|C_{\max}|/\log n \xrightarrow{\mathbb{P}} I_{\lambda}^{-1}$.*

We start in Section 4.3.1 by explaining the strategy of the proof. We close this section by stating an open problem:

Open problem 4.6 (Convergence of the largest connected component, [114]). *Show that there exists a sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n \leq 1$ and $\liminf_{n\to\infty} a_n > 0$ and a constant $\xi_{\lambda} > 0$ such that*

$$\mathbb{P}_{\lambda}(|C_{\max}| > \lceil I_{\lambda}^{-1}\log n \rceil + k) = e^{-a_n e^{-\xi_{\lambda} k}} + o(1). \tag{4.3.4}$$

The probability distribution in (4.3.4) is very alike a Gumbel distribution, recall (2.6.5). However, the Gumbel distribution is continuous, while $|C_{\max}|$ is an integer-valued random variable. This explain the need for the $\{a_n\}_{n=1}^{\infty}$ sequence in Open problem 4.6. A similar problem occurs for the maximum of n i.i.d. geometric random variables:

Exercise 4.11. *Show that when $\{X_i\}_{i=1}^{\infty}$ is an i.i.d. sequence of geometric random variables with parameter p, i.e., $\mathbb{P}(X_1 > k) = (1-p)^k$ for all $k = 0, 1, \ldots$, then there exists a sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n \leq 1$ and $\liminf_{n\to\infty} a_n = 1 - p$ and a constant $\xi_{\lambda} > 0$, such that, with $X_{(n)} = \max_{1 \leq i \leq n} X_i$ the maximum of $\{X_i\}_{i=1}^{\infty}$,*

$$\mathbb{P}\Big(X_{(n)} > \lceil \frac{\log n}{\log(1-p)} \rceil + k\Big) = e^{-a_n e^{-\xi_p k}} + o(1). \tag{4.3.5}$$

4.3.1   Largest subcritical cluster: strategy of proof of Theorems 4.4 and 4.5

We start by describing the strategy of proof. We denote by

$$Z_{\geq k} = \sum_{v=1}^{n} \mathbb{1}_{\{|C(v)| \geq k\}} \tag{4.3.6}$$

the number of vertices that are contained in connected components of size at least k. The random variable $Z_{\geq k}$ will be used to prove the asymptotics of $|C_{\max}|$. This can be understood by noting that $\{|C_{\max}| \geq k\} = \{Z_{\geq k} \geq k\}$:

Exercise 4.12. *Prove that $\{|C_{\max}| \geq k\} = \{Z_{\geq k} \geq k\}$.*

We conclude that we can identify $|C_{\max}|$ as

$$|C_{\max}| = \max\{k : Z_{\geq k} \geq k\}, \tag{4.3.7}$$

This allows us to prove bounds on $|C_{\max}|$ by investigating $Z_{\geq k}$ for an appropriately chosen k.

To prove Theorem 4.4, we use the first moment method or Markov inequality (Theorem 2.14). we compute that

$$\mathbb{E}_{\lambda}[Z_{\geq k}] = n\mathbb{P}_{\lambda}(|C(1)| \geq k), \tag{4.3.8}$$

and we use Theorem 4.2 to bound $\mathbb{P}_{\lambda}(|C(1)| \geq k_n)$ for $k_n = a \log n$ for any $a > I_{\lambda}^{-1}$. Therefore, with high probability, $Z_{\geq k_n} = 0$, so that, again with high probability, $|C_{\max}| \leq k_n$. This proves Theorem 4.4. For the details we refer to the formal argument in Section 4.3.2.

To prove Theorem 4.5, we use the second moment method or Chebychev inequality (Theorem 2.15). In order to be able to apply this result, we first prove an upper bound on the variance of $Z_{\geq k}$, see Proposition

4.8 below. We further use Theorem 4.3 to prove a lower bound on $\mathbb{E}_\lambda[Z_{\geq k_n}]$, now for $k_n = a \log n$ for any $a < I_\lambda^{-1}$. Then, (2.4.5) in Theorem 2.14 proves that with high probability, $Z_{\geq k_n} > 0$, so that, again with high probability, $|C_{\max}| \geq k_n$.

We now present the details of the proofs.

4.3.2 Upper bound on the largest subcritical cluster: proof of Theorem 4.4

By Theorem 4.2,

$$\mathbb{P}_\lambda(|C(v)| > t) \leq \mathbb{P}_{n,p}(T > t), \tag{4.3.9}$$

where T is the total progeny of a branching process with a binomial offspring distribution with parameters n and $p = \lambda/n$. To study $\mathbb{P}_{n,p}(T > t)$, we let $\{\hat{X}_i\}_{i=1}^\infty$ be an i.i.d. sequence of binomial random variables with parameters n and success probability p, and let

$$\hat{S}_t = \hat{X}_1 + \ldots + \hat{X}_t - (t-1). \tag{4.3.10}$$

Then, by (3.3.2) and (3.3.1), we have that

$$\mathbb{P}_{n,p}(T > t) \leq \mathbb{P}(\hat{S}_t > 0) = \mathbb{P}_{n,p}(\hat{X}_1 + \ldots + \hat{X}_t \geq t) \leq e^{-tI_\lambda}, \tag{4.3.11}$$

by Corollary 2.17 and using the fact that $\hat{X}_1 + \ldots + \hat{X}_t \sim \mathrm{BIN}(nt, \lambda/n)$. We conclude that

$$\mathbb{P}_\lambda(|C(v)| > t) \leq e^{-tI_\lambda}. \tag{4.3.12}$$

Therefore, by Boole's inequality,

$$\mathbb{P}_\lambda(|C_{\max}| \geq a \log n) = \mathbb{P}_\lambda\Big(\bigcup_{v=1}^n \{|C(v)| \geq a \log n\}\Big)$$

$$\leq \sum_{v=1}^n \mathbb{P}_\lambda(|C(v)| \geq a \log n)$$

$$\leq n\mathbb{P}_\lambda(|C(1)| \geq a \log n) \leq n^{1-aI_\lambda}e^{I_\lambda} = O(n^{-\delta}), \tag{4.3.13}$$

whenever $a > 1/I_\lambda$ and with $\delta = aI_\lambda - 1$. This proves that with high probability the largest connected component is bounded by $a \log n$ for every $a > I_\lambda^{-1}$.

We note that (4.3.13) is equivalent to the bound

$$\mathbb{P}_\lambda(|C_{\max}| \geq a \log n) \leq \mathbb{P}_\lambda(Z_{\geq k_n} \geq 1)$$

$$\leq \mathbb{E}[Z_{\geq k_n}]$$

$$= n\mathbb{P}_\lambda(|C(1)| \geq a \log n), \tag{4.3.14}$$

so that the proof in (4.3.13) is equivalent to the strategy of proof sketched in Section 4.3.1. $\square$

We now give a second proof of (4.3.12), which is based on a distributional equality of S_t, and which turns out to be useful in the analysis of the Erdős-Rényi random graph with $\lambda > 1$ as well. The result states that S_t is also binomially distributed, but with a different success probability. This identification will also be used to find the asymptotics of $|C_{\max}|$ when $\lambda > 1$. In the statement of Proposition 4.7 below, we make essential use of the formal continuation of the recursions in (4.1.3) and (4.1.4) for the breadth-first search, defined right below (4.1.4). Note that, in particular, S_t need not be non-negative.

Proposition 4.7 (The law of S_t). *For all $t \in \{1, \ldots, n\}$,*

$$S_t + (t - 1) \sim \mathrm{BIN}(n - 1, 1 - (1 - p)^t). \tag{4.3.15}$$

We shall only make use of Proposition 4.7 when $T \geq t$, in which case $S_t \geq 0$ *does* hold.

Proof. Let N_t represent the number of unexplored vertices, i.e.,

$$N_t = n - t - S_t. \tag{4.3.16}$$

It is more convenient to show the equivalent statement that for all t

$$N_t \sim \mathrm{BIN}(n - 1, (1 - p)^t). \tag{4.3.17}$$

Heuristically, this can be understood by noting that each of the vertices $\{2, \ldots, n\}$ has, independently of all other vertices, probability $(1 - p)^t$ to stay neutral in the first t explorations. More formally, conditionally on S_t, we have that $X_t \sim \mathrm{BIN}(n - (t - 1) - S_{t-1}, p)$ by (4.1.4). Thus, noting that $N_0 = n - 1$ and

$$\begin{aligned}
N_t = n - t - S_t &= n - t - S_{t-1} - X_t + 1 \\
&= n - (t - 1) - S_{t-1} - \mathrm{BIN}(n - (t - 1) - S_{t-1}, p) \\
&= N_{t-1} - \mathrm{BIN}(N_{t-1}, p) = \mathrm{BIN}(N_{t-1}, 1 - p),
\end{aligned} \tag{4.3.18}$$

the conclusion follows by recursion on t. $\qquad\square$

Exercise 4.13. *Show that if $N \sim \mathrm{BIN}(n, p)$ and, conditionally on N, $M \sim \mathrm{BIN}(N, q)$, then $M \sim \mathrm{BIN}(n, pq)$. Use this to complete the proof that $N_t \sim \mathrm{BIN}(n - 1, (1 - p)^t)$.*

To give a second proof of (4.3.12), we use Proposition 4.7 to see that

$$\mathbb{P}_\lambda(|C(v)| > t) \leq \mathbb{P}(S_t > 0) \leq \mathbb{P}(\mathrm{BIN}(n - 1, 1 - (1 - p)^t) \geq t). \tag{4.3.19}$$

Using Bernoulli's inequality $1 - (1 - p)^t \leq tp$, we therefore arrive at

$$\begin{aligned}
\mathbb{P}_\lambda(|C(v)| > t) &\leq \mathbb{P}(\mathrm{BIN}(n, \frac{t\lambda}{n}) \geq t) \\
&\leq \min_{s \geq 0} e^{-st} \mathbb{E}[e^{s\mathrm{BIN}(n, \frac{t\lambda}{n})}] \\
&= \min_{s \geq 0} e^{-st} [1 + \frac{t\lambda}{n}(e^s - 1)]^n \\
&\leq \min_{s \geq 0} e^{-st} e^{t\lambda(e^s - 1)},
\end{aligned} \tag{4.3.20}$$

where we have used the Markov inequality (Theorem 2.14) in the second inequality, and $1 + x \leq e^x$ in the last. We arrive at the bound

$$\mathbb{P}_\lambda(|C(v)| > t) \leq e^{-I_\lambda t}, \tag{4.3.21}$$

which reproves (4.3.12). $\qquad\square$

4.3.3 Lower bound on the largest subcritical cluster: proof of Theorem 4.5

The proof of Theorem 4.5 makes use of a variance estimate on $Z_{\geq k}$. We we use the notation

$$\chi_{\geq k}(\lambda) = \mathbb{E}_\lambda[|C(v)|\mathbb{1}_{\{|C(v)|\geq k\}}].$$

e:chi>-def
(4.3.22)

Note that, by exchangeability of the vertices, $\chi_{\geq k}(\lambda)$ does not depend on v.

Proposition 4.8 (A variance estimate for $Z_{\geq k}$). *For every n and $k \in \{1, \ldots, n\}$,*

$$\mathrm{Var}_\lambda(Z_{\geq k}) \leq n\chi_{\geq k}(\lambda).$$

(4.3.23)

Proof. We use that

$$\mathrm{Var}_\lambda(Z_{\geq k}) = \sum_{i,j=1}^{n} \Big[\mathbb{P}_\lambda(|C(i)| \geq k, |C(j)| \geq k) - \mathbb{P}_\lambda(|C(i)| \geq k)\mathbb{P}_\lambda(|C(j)| \geq k)\Big].$$

(4.3.24)

We split the probability $\mathbb{P}_\lambda(|C(i)| \geq k, |C(j)| \geq k)$, depending on whether $i \longleftrightarrow j$ or not. Clearly,

$$\mathbb{P}_\lambda(|C(i)| \geq k, |C(j)| \geq k, i \longleftrightarrow\!\!\!\!\!/\ \, j) = \sum_{l=k}^{n} \mathbb{P}_\lambda(|C(i)| = l, i \longleftrightarrow\!\!\!\!\!/\ \, j)\mathbb{P}_\lambda(|C(j)| \geq k \mid |C(i)| = l, i \longleftrightarrow\!\!\!\!\!/\ \, j).$$

(4.3.25)

When $|C(i)| = l$ and $i \longleftrightarrow\!\!\!\!\!/\ \, j$, then all vertices in the components different from the one of i, which includes the components of j, form a random graph where the size n is replaced by $n - l$. Since the probability that $|C(j)| \geq k$ in $\mathrm{ER}(n, p)$ is increasing in n, we have that

$$\mathbb{P}_\lambda(|C(j)| \geq k \mid |C(i)| = l, i \longleftrightarrow\!\!\!\!\!/\ \, j) \leq \mathbb{P}_\lambda(|C(j)| \geq k).$$

(4.3.26)

We conclude that

$$\mathbb{P}_\lambda(|C(i)| = l, |C(j)| \geq k, i \longleftrightarrow\!\!\!\!\!/\ \, j) - \mathbb{P}_\lambda(|C(i)| = l)\mathbb{P}_\lambda(|C(j)| \geq k) \leq 0,$$

(4.3.27)

which in turn implies that

$$\mathrm{Var}_\lambda(Z_{\geq k}) \leq \sum_{i,j=1}^{n} \mathbb{P}_\lambda(|C(i)| \geq k, |C(j)| \geq k, i \longleftrightarrow j).$$

(4.3.28)

Therefore, we arrive at the fact that, by the exchangeability of the vertices,

$$\begin{aligned}
\mathrm{Var}_\lambda(Z_{\geq k}) &\leq \sum_{i,j} \mathbb{P}_\lambda(|C(i)| \geq k, |C(j)| \geq k, i \longleftrightarrow j) \\
&= \sum_i \sum_j \mathbb{E}_\lambda\big[I[|C(i)| \geq k]\mathbb{1}_{\{j \in C(i)\}}\big] \\
&= \sum_i \mathbb{E}_\lambda\Big[I[|C(i)| \geq k]\sum_j \mathbb{1}_{\{j \in C(i)\}}\Big].
\end{aligned}$$

(4.3.29)

Since $\sum_j \mathbb{1}_{\{j \in C(i)\}} = |C(i)|$, we thus arrive at

$$\begin{aligned}
\mathrm{Var}_\lambda(Z_{\geq k}) &\leq \sum_i \mathbb{E}_\lambda[|C(i)|\mathbb{1}_{\{|C(i)|\geq k\}}] \\
&= n\mathbb{E}_\lambda[|C(1)|\mathbb{1}_{\{|C(1)|\geq k\}}] \\
&= n\chi_{\geq k}(\lambda).
\end{aligned}$$

(4.3.30)

$$\square$$

Proof of Theorem 4.5. To prove Theorem 4.5, it suffices to prove that $\mathbb{P}_\lambda(Z_{\geq k_n} = 0) = O(n^{-\delta})$, where $k_n = a \log n$ with $a < I_\lambda^{-1}$. For this, we use the Chebychev inequality (Theorem 2.15). In order to apply Theorem 2.15, we need to derive a lower bound on $\mathbb{E}_\lambda[Z_{\geq k}]$ and an upper bound on $\mathrm{Var}_\lambda(Z_{\geq k})$.

We start by giving a lower bound on $\mathbb{E}_\lambda[Z_{\geq k}]$. We use that

$$\mathbb{E}_\lambda[Z_{\geq k}] = nP_{\geq k}(\lambda), \qquad \text{where} \qquad P_{\geq k}(\lambda) = \mathbb{P}_\lambda(|C(v)| \geq k). \tag{4.3.31}$$

We take $k = k_n = a \log n$. We use Theorem 4.3 to see that, with T a binomial branching process with parameters $n - k_n$ and $p = \lambda/n$,

$$P_{\geq k}(\lambda) \geq \mathbb{P}_{n-k_n,p}(T \geq a \log n). \tag{4.3.32}$$

By Theorem 3.17, with T^* the total progeny of a Poisson branching process with mean $\lambda_n = \lambda \frac{n-k_n}{n}$,

$$\mathbb{P}_{n-k_n,p}(T \geq a \log n) = \mathbb{P}_{\lambda_n}^*(T^* \geq a \log n) + O\left(\frac{a\lambda^2 \log n}{n}\right). \tag{4.3.33}$$

Also, by Theorem 3.13, we have that

$$\begin{aligned}
\mathbb{P}_{\lambda_n}^*(T^* \geq a \log n) &= \sum_{k=a \log n}^{\infty} \mathbb{P}_{\lambda_n}^*(T^* = k) \\
&= \sum_{k=a \log n}^{\infty} \frac{(\lambda_n k)^{k-1}}{k!} e^{-\lambda_n k}.
\end{aligned} \tag{4.3.34}$$

By Stirling's formula,

$$k! = \left(\frac{k}{e}\right)^k \sqrt{2\pi k}(1 + o(1)), \tag{4.3.35}$$

so that, recalling (4.3.1), and using that $I_{\lambda_n} = I_\lambda + o(1)$,

$$\begin{aligned}
\mathbb{P}(T^* \geq a \log n) &= \lambda^{-1} \sum_{k=a \log n}^{\infty} \frac{1}{\sqrt{2\pi k^3}} e^{-I_{\lambda_n} k}(1 + o(1)) \\
&= e^{-I_\lambda a \log n(1+o(1))}.
\end{aligned} \tag{4.3.36}$$

As a result, it follows that, with $k_n = a \log n$ and any $0 < \alpha < 1 - I_\lambda a$,

$$\mathbb{E}_\lambda[Z_{\geq k_n}] = nP_{\geq k_n}(\lambda) \geq n^{(1-I_\lambda a)(1+o(1))} \geq n^\alpha. \tag{4.3.37}$$

We next bound the variance of $Z_{\geq k_n}$ using Proposition 4.8. By (4.3.12),

$$\begin{aligned}
\chi_{\geq k_n}(\lambda) &= \sum_{t=k_n}^{n} P_{\geq t}(\lambda) \\
&\leq \sum_{t=k_n}^{n} e^{-I_\lambda(t-1)} \\
&\leq \frac{e^{-(k_n-1)I_\lambda}}{1 - e^{-I_\lambda}} \\
&= O(n^{-aI_\lambda}).
\end{aligned} \tag{4.3.38}$$

We conclude that, by Proposition 4.8,

$$\text{Var}_\lambda(Z_{\geq k_n}) \leq n\chi_{\geq k_n}(\lambda) \leq O(n^{1-aI_\lambda}), \tag{4.3.39}$$

while

$$\mathbb{E}_\lambda[Z_{\geq k_n}] \geq n^\alpha. \tag{4.3.40}$$

Therefore, by the Chebychev inequality (Theorem 2.14),

$$\begin{aligned}
\mathbb{P}_\lambda(Z_{\geq k_n} = 0) &\leq \frac{\text{Var}_\lambda(Z_{\geq k_n})}{\mathbb{E}_\lambda[Z_{\geq k_n}]^2} \\
&\leq O(n^{1-aI-2\alpha}) = O(n^{-\delta}), \tag{4.3.41}
\end{aligned}$$

when we pick $\delta = 2\alpha - (1 - I_\lambda a)$, and $0 < \alpha < 1 - I_\lambda a$ such that $\delta = 2\alpha - (1 - I_\lambda a) > 0$. Finally, we use that

$$\mathbb{P}_\lambda(|C_{\max}| < k_n) = \mathbb{P}_\lambda(Z_{\geq k_n} = 0), \tag{4.3.42}$$

to complete the proof of Theorem 4.5. $\qquad\square$

4.4 The supercritical regime

In this section, we fix $\lambda > 1$. The main result proved in this section is the following theorem. In its statement, we write $\zeta_\lambda = 1 - \eta_\lambda$ for the survival probability of a Poisson branching process with mean offspring λ.

Theorem 4.9 (Law of large numbers for giant component)**.** *Fix $\lambda > 1$. Then, for every $\nu \in (\frac{1}{2}, 1)$, there exists a $\delta = \delta(\nu, \lambda) > 0$ such that*

$$\mathbb{P}_\lambda\Big(\big||C_{\max}| - \zeta_\lambda n\big| \geq n^\nu\Big) = O(n^{-\delta}). \tag{4.4.1}$$

Theorem 4.9 can be interpreted as follows. A vertex has a large connected component with probability ζ_λ. Therefore, there are of the order $\zeta_\lambda n$ vertices with large connected components. Theorem 4.9 implies that all these vertices in large components are in fact in the *same* connected component, which is called the *giant component*.

We first give an overview of the proof of Theorem 4.9.

4.4.1 Strategy of proof of law of large numbers for the giant component

In this section, we give an overview of the proof of Theorem 4.9. We again crucially rely on an analysis of the number of vertices in connected components of size at least k,

$$Z_{\geq k} = \sum_{v=1}^n \mathbb{1}_{\{|C(v)| \geq k\}}. \tag{4.4.2}$$

We first pick $k = k_n = K \log n$ for some $K > 0$ sufficiently large. Note that

$$\mathbb{E}[Z_{\geq k_n}] = n\mathbb{P}_\lambda(|C(v)| \geq k_n). \tag{4.4.3}$$

We evaluate $\mathbb{P}_\lambda(|C(v)| \geq k_n)$ using the bound in Theorem 4.3. Indeed,, we prove an estimate on the cluster size distribution in Proposition 4.10 below, which states that for $k_n = K \log n$ and K sufficiently large

$$\mathbb{P}_\lambda(|C(v)| \geq k_n) = \zeta_\lambda(1 + o(1)). \tag{4.4.4}$$

Then we show that, for $k = k_n = K \log n$, for some $K > 0$ sufficiently large, there is with high probability no connected component with size in between k_n and αn for any $\alpha < \zeta_\lambda$. This is done by a first moment argument: the expected number of vertices in such connected components is equal to $\mathbb{E}_\lambda[Z_{\geq k_n} - Z_{\geq \alpha n}]$, and we use the bound in Proposition 4.10 described above, as well as Proposition 4.11, which states that, for any $\alpha < \zeta_\lambda$, there exists $J > 0$ such that

$$\mathbb{P}_\lambda\big(k_n \leq |C(v)| < \alpha n\big) \leq e^{-k_n J}. \tag{4.4.5}$$

Therefore, for $K > 0$ sufficiently large, there is, with high probability, no cluster with size in between k_n and αn.

We next use a variance estimate on $Z_{\geq k}$ in Proposition 4.13, which implies that with high probability, and for all $\nu \in (\frac{1}{2}, 1)$,

$$|Z_{\geq k_n} - \mathbb{E}_\lambda[Z_{\geq k_n}]| \leq n^\nu. \tag{4.4.6}$$

We finally use that for $2\alpha > \zeta_\lambda$, and on the event that there are no clusters with size in between k_n and αn, and on the event in (4.4.6), we have

$$Z_{\geq k_n} = |C_{\max}|. \tag{4.4.7}$$

The proof of Theorem 4.9 follows by combining (4.4.3), (4.4.6) and (4.4.7). The details of the proof of Theorem 4.9 are given in Section 4.4.4 below. We start by describing the cluster size distribution in Section 4.4.2, and the variance estimate on $Z_{\geq k}$ in Section 4.4.3.

4.4.2   The supercritical cluster size distribution

In this section, we prove two propositions that investigate the tails of the cluster size distribution. In Proposition 4.10, we show that the probability that $|C(v)| \geq k$ is, for $k_n \geq a \log n$, close to the survival probability of a Poisson branching process with mean λ. Proposition 4.10 implies (4.4.4).

Proposition 4.10 (Cluster tail is branching process survival probability). *Fix $\lambda > 1$ and let $n \to \infty$. Then, for $k_n \geq a \log n$ where $a > I_\lambda^{-1}$ and I_λ is defined in (4.3.1),*

$$\mathbb{P}_\lambda(|C(v)| \geq k_n) = \zeta_\lambda + O\Big(\frac{k_n}{n}\Big). \tag{4.4.8}$$

Proof. For the upper bound on $\mathbb{P}_\lambda(|C(v)| \geq k)$, we first use Theorem 4.2, followed by Theorem 3.17, to deduce

$$\mathbb{P}_\lambda(|C(v)| \geq k_n) \leq \mathbb{P}_{n,\lambda/n}(T \geq k_n) \leq \mathbb{P}_\lambda^*(T^* \geq k_n) + O(\frac{k_n}{n}), \tag{4.4.9}$$

where T and T^*, respectively, are the the total progeny of a binomial branching process with parameters n and λ/n and a Poisson mean λ branching process, respectively. To complete the upper bound, we use Theorem 3.7 to see that

$$\mathbb{P}_\lambda^*(T^* \geq k_n) = \mathbb{P}_\lambda^*(T^* = \infty) + \mathbb{P}_\lambda^*(k_n \leq T^* < \infty)$$

$$= \zeta_\lambda + O(e^{-k_n I_\lambda}) = \zeta_\lambda + O\Big(\frac{k_n}{n}\Big), \tag{4.4.10}$$

as required.

For the lower bound, we use Theorem 4.3 again followed by Theorem 3.17, so that, with $\lambda_n = \lambda(1 - \frac{k_n}{n})$,

$$\mathbb{P}_\lambda(|C(v)| \geq k_n) \geq \mathbb{P}_{n,\lambda_n/n}(T \geq k_n) \geq \mathbb{P}_{\lambda_n}^*(T^* \geq k_n) + O\Big(\frac{k_n}{n}\Big), \tag{4.4.11}$$

where now T and T^*, respectively, are the the total progeny of a binomial branching process with parameters $n - k_n$ and λ/n and a Poisson mean λ_n branching process, respectively.

By Exercise 3.23 for $k_n \geq a \log n$ with $a > I_\lambda^{-1}$,

$$\mathbb{P}_{\lambda_n}^*(T^* \geq k_n) = \zeta_{\lambda_n} + O(e^{-k_n I_{\lambda_n}}) = \zeta_{\lambda_n} + O\left(\frac{k_n}{n}\right). \tag{4.4.12}$$

Now, furthermore, by Corollary 3.16, when $\lambda > 1$, and because $\lambda_n - \lambda = \frac{k_n}{n}$,

$$\eta_{\lambda_n} = \eta_\lambda + O\left(\frac{k_n}{n}\right). \tag{4.4.13}$$

Therefore, also $\zeta_{\lambda_n} = \zeta_\lambda + O(\frac{k_n}{n})$. Putting these estimates together proves the lower bound. Together, the upper and lower bound complete the proof of Proposition 4.10. $\qquad\square$

We next show that the probability that $k_n \leq |C(v)| \leq \alpha n$ is exponentially small in k_n:

Proposition 4.11 (Exponential tails for supercritical clusters smaller than $\zeta_\lambda n$). *Fix $\lambda > 1$ and let $k_n \to \infty$. Then, for any $\alpha < \zeta_\lambda$, there exists a $J = J(\alpha, \lambda) > 0$ such that*

$$\mathbb{P}_\lambda(k_n \leq |C(v)| \leq \alpha n) \leq C e^{-k_n J}. \tag{4.4.14}$$

Proof. We start by bounding

$$\mathbb{P}_\lambda(k_n \leq |C(v)| \leq \alpha n) = \sum_{t=k_n}^{\alpha n} \mathbb{P}_\lambda(|C(v)| = t) \leq \sum_{t=k_n}^{\alpha n} \mathbb{P}_\lambda(S_t = 0), \tag{4.4.15}$$

where we recall (4.1.3). By Proposition 4.7, we have that $S_t \sim \text{BIN}(n-1, 1-(1-p)^t) + 1 - t$. Therefore, with $p = \lambda/n$,

$$\mathbb{P}_\lambda(S_t = 0) = \mathbb{P}_\lambda\Big(\text{BIN}(n-1, 1-(1-p)^t) = t - 1\Big). \tag{4.4.16}$$

To explain the exponential decay, we note that, for $p = \lambda/n$ and $t = \alpha n$,

$$1 - (1-p)^t = 1 - \left(1 - \frac{\lambda}{n}\right)^{\alpha n} = (1 - e^{-\lambda\alpha})(1 + o(1)) \qquad \text{for large } n. \tag{4.4.17}$$

The unique solution to the equation $1 - e^{-\lambda\alpha} = \alpha$ is $\alpha = \zeta_\lambda$:

Exercise 4.14. *Prove that the unique solution to the equation $1 - e^{-\lambda\alpha} = \alpha$ is $\alpha = \zeta_\lambda$, where ζ_λ is the survival probability of a Poisson branching process with parameter λ.*

If $\alpha < \zeta_\lambda$, then $\alpha < 1 - e^{-\lambda\alpha}$, and thus the probability in (4.4.16) drops exponentially. We now fill in the details.

First, by (4.4.16) and using that $1 - p \leq e^{-p}$, so that $1 - (1-p)^t \geq 1 - e^{-pt}$,

$$\mathbb{P}_\lambda(S_t = 0) = \mathbb{P}_\lambda\Big(\text{BIN}(n-1, 1-(1-p)^t) = t - 1\Big) \leq \mathbb{P}_\lambda\Big(\text{BIN}(n, 1 - e^{-pt}) \leq t\Big). \tag{4.4.18}$$

By Exercise 4.14, the solution $\alpha = nt$ to $1 - e^{-\lambda\alpha} = \alpha$ is given by $\alpha = \zeta_\lambda$. It is easy to verify that if $\alpha < \zeta_\lambda$, then there exists $\delta = \delta(\alpha, \lambda) > 0$ such that, for all $\beta \leq \alpha$,

$$1 - \lambda\beta \leq e^{-\lambda\beta} \leq 1 - (1 + \delta)\beta. \tag{4.4.19}$$

Write $X \sim \mathrm{BIN}(n, 1 - e^{-pt})$ and $t = \beta n$, where $k_n/n \leq \beta \leq \alpha$. Then, by (4.4.19),

$$\beta(1 + \delta)n \leq \mathbb{E}[X] \leq \lambda \beta n. \tag{4.4.20}$$

Hence,

$$\mathbb{P}_\lambda(S_t \leq 0) \leq \mathbb{P}_\lambda\Big(\mathrm{BIN}(n, 1 - e^{-pt}) \leq t\Big) \leq \mathbb{P}_\lambda\big(X \leq \mathbb{E}[X] - \beta \delta n\big), \tag{4.4.21}$$

and Theorem 2.18 gives that, for every $t \leq \alpha n$,

$$\mathbb{P}_\lambda(S_t \leq 0) \leq e^{-\beta \delta^2 n/2\lambda} = e^{-t\delta^2/2\lambda}. \tag{4.4.22}$$

We conclude that, with $J = J(\alpha, \lambda) = \delta^2/2\lambda$,

$$\mathbb{P}_\lambda(k_n \leq |C(v)| \leq \alpha n) \leq \sum_{t=k_n}^{\alpha n} \mathbb{P}_\lambda(S_t = 0) \leq \sum_{t=k_n}^{\alpha n} e^{-Jt} \leq [1 - e^{-J}]^{-1} e^{-Jk_n}. \tag{4.4.23}$$

This completes the proof of Proposition 4.11.   □

We finally state a consequence of Proposition 4.11 that shows that there is, with high probability, no cluster with size in between k_n and αn. Corollary 4.12 implies (4.4.5).

Corollary 4.12. *Fix $k_n = K \log n$ and $\alpha < \zeta_\lambda$. Then, for K sufficiently large, and with probability at least $1 - n^{-\delta}$, there is no connected component with size in between k_n and αn.*

Proof. We use that the expected number of clusters with sizes in between k_n and αn, for any $\alpha < \zeta_\lambda$, is equal to

$$\mathbb{E}_\lambda[Z_{\geq k_n} - Z_{\geq \alpha n+1}] = n\mathbb{P}_\lambda(k_n \leq |C(v)| \leq \alpha n) \leq Cne^{-k_n J}, \tag{4.4.24}$$

where we have used Proposition 4.11 for the last estimate. When $k_n = K \log n$, and K is sufficiently large, the right-hand side is $O(n^{-\delta})$. By the Markov inequality (Theorem 2.14),

$$\mathbb{P}_\lambda(\exists v : k_n \leq |C(v)| \leq \alpha n) = \mathbb{P}_\lambda(Z_{\geq k_n} - Z_{\geq \alpha n+1} \geq 1) \leq \mathbb{E}_\lambda[Z_{\geq k_n} - Z_{\geq \alpha n+1}] = O(n^{-\delta}). \tag{4.4.25}$$

This completes the proof of Corollary 4.12.   □

Exercise 4.15. *Prove that the expected cluster size of a given vertex*

$$\chi(\lambda) = \mathbb{E}_\lambda[|C(1)|], \tag{4.4.26}$$

satisfies

$$\chi(\lambda) = 1 + (n - 1)\mathbb{P}_\lambda(1 \longleftrightarrow 2). \tag{4.4.27}$$

Exercise 4.16. *Prove that (4.4.1) and Corollary 4.12 imply that, for $\lambda > 1$,*

$$\mathbb{P}_\lambda(1 \longleftrightarrow 2) = \zeta_\lambda^2[1 + o(1)]. \tag{4.4.28}$$

Exercise 4.17. *Prove that (4.4.1) implies that the expected cluster size satisfies, for $\lambda > 1$,*

$$\chi(\lambda) = \zeta_\lambda^2 n(1 + o(1)). \tag{4.4.29}$$

4.4.3 A second variance estimate on the number of vertices in large components

The proof of Theorem 4.9 makes use of a variance estimate on $Z_{\geq k}$ for an appropriate choice of k. In its statement, we use the notation

$$\chi_{<k}(\lambda) = \mathbb{E}_\lambda[|C(v)|\mathbb{1}_{\{|C(v)|<k\}}]. \qquad \text{e:chi<-def} \qquad (4.4.30)$$

Proposition 4.13 (A second variance estimate on $Z_{\geq k}$). *For every n and $k \in \{1,\ldots,n\}$,*

$$\mathrm{Var}_\lambda(Z_{\geq k}) \leq (\lambda k + 1)n\chi_{<k}(\lambda). \qquad (4.4.31)$$

Note that the variance estimate in Proposition 4.13 is, in the supercritical regime, much better than the variance estimate in Proposition 4.8. Indeed, the bound in Proposition 4.8 reads

$$\mathrm{Var}_\lambda(Z_{\geq k}) \leq n\chi_{\geq k}(\lambda). \qquad (4.4.32)$$

However, when $\lambda > 1$, according to Theorem 4.9 (which is currently not yet proved), $|C(1)| = \Theta(n)$ with positive probability. Therefore,

$$n\chi_{\geq k}(\lambda) = \Theta(n^2), \qquad (4.4.33)$$

which is a trivial bound. The bound in Proposition 4.13 is at most $\Theta(k^2 n)$, which is much smaller when k is not too large. We will pick $k = k_n = \Theta(\log n)$, for which the estimate in Proposition 4.13 is much better. In Section 4.4.4, we shall see that Proposition 4.13 implies (4.4.6).

Proof. Define

$$Z_{<k} = \sum_{v=1}^n \mathbb{1}_{\{|C(v)|<k\}}. \qquad \text{e:Z<k-def} \qquad (4.4.34)$$

Then, since $Z_{<k} = n - Z_{\geq k}$, we have

$$\mathrm{Var}_\lambda(Z_{\geq k}) = \mathrm{Var}_\lambda(Z_{<k}). \qquad (4.4.35)$$

Therefore, it suffices to prove that $\mathrm{Var}(Z_{<k}) \leq (\lambda k + 1)n\chi_{<k}(\lambda)$. For this, we compute

$$\mathrm{Var}_\lambda(Z_{<k}) = \sum_{i,j} \big[\mathbb{P}_\lambda(|C(i)| < k, |C(j)| < k) - \mathbb{P}_\lambda(|C(i)| < k)\mathbb{P}_\lambda(|C(j)| < k)\big]. \qquad (4.4.36)$$

We again split, depending on whether $i \longleftrightarrow j$ or not:

$$\mathrm{Var}_\lambda(Z_{<k}) = \sum_{i,j} \big[\mathbb{P}_\lambda(|C(i)| < k, |C(j)| < k, i \longleftrightarrow\!\!\!\!\!/\ j) - \mathbb{P}_\lambda(|C(i)| < k)\mathbb{P}_\lambda(|C(j)| < k)\big]$$
$$+ \sum_{i,j} \mathbb{P}_\lambda(|C(i)| < k, |C(j)| < k, i \longleftrightarrow j). \qquad \text{e:splitVar} \qquad (4.4.37)$$

We again compute explicitly

$$\sum_{i,j} \mathbb{P}_\lambda(|C(i)| < k, |C(j)| < k, i \longleftrightarrow j) = \sum_{i,j} \mathbb{E}_\lambda\big[\mathbb{1}_{\{|C(i)|<k\}}\mathbb{1}_{\{i\longleftrightarrow j\}}\big] = \sum_{i=1}^n \mathbb{E}_\lambda\big[\mathbb{1}_{\{|C(i)|<k\}}\sum_{j=1}^n \mathbb{1}_{\{i\longleftrightarrow j\}}\big]$$
$$= \sum_{i=1}^n \mathbb{E}_\lambda[|C(i)|\mathbb{1}_{\{|C(i)|<k\}}] = n\chi_{<k}(\lambda). \qquad \text{e:sumsup1} \qquad (4.4.38)$$

To compute the first sum on the right hand-side of (4.4.37) we write that, for $l < k$,

$$\mathbb{P}_\lambda(|C(i)| = l, |C(j)| < k, i \longleftrightarrow\!\!\!\!/\ j) = \mathbb{P}_\lambda(|C(i)| = l)\mathbb{P}_\lambda(i \longleftrightarrow\!\!\!\!/\ j||C(i)| = l)\mathbb{P}_\lambda(|C(j)| < k||C(i)| = l, i \longleftrightarrow\!\!\!\!/\ j).$$
(4.4.39)

See Exercise 4.18 for an explicit formula for $\mathbb{P}_\lambda(i \longleftrightarrow\!\!\!\!/\ j||C(i)| = l)$. We bound $\mathbb{P}_\lambda(i \longleftrightarrow\!\!\!\!/\ j||C(i)| = l) \leq 1$, to obtain

$$\mathbb{P}_\lambda(|C(i)| = l, |C(j)| < k, i \longleftrightarrow\!\!\!\!/\ j) \leq \mathbb{P}_\lambda(|C(i)| = l)\mathbb{P}_\lambda(|C(j)| < k||C(i)| = l, i \longleftrightarrow\!\!\!\!/\ j).$$
(4.4.40)

Now we use that, when $|C(i)| = l$ and when $i \longleftrightarrow\!\!\!\!/\ j$, the law of $|C(j)|$ is identical to the law of $|C(1)|$ in a random graph with $n - l$ vertices and edge probability $p = \lambda/n$, i.e.,

$$\mathbb{P}_{n,\lambda}(|C(j)| < k||C(i)| = l, i \longleftrightarrow\!\!\!\!/\ j) = \mathbb{P}_{n-l,\lambda}(|C(1)| < k),$$
(4.4.41)

where we write $\mathbb{P}_{m,\lambda}$ for the distribution of $\mathrm{ER}(m, \lambda/n)$. Therefore,

$$\mathbb{P}_\lambda(|C(j)| < k||C(i)| = l, i \longleftrightarrow\!\!\!\!/\ j) = \mathbb{P}_{n-l,\lambda}(|C(1)| < k)$$
(4.4.42)
$$= \mathbb{P}_{n,\lambda}(|C(1)| < k) + \mathbb{P}_{n-l,\lambda}(|C(1)| < k) - \mathbb{P}_{n,\lambda}(|C(1)| < k).$$

We can couple $\mathrm{ER}(n - l, p)$ and $\mathrm{ER}(n, p)$ by adding the vertices $\{n + l + 1, \ldots, n\}$, and by letting st, for $s \in \{n + l + 1, \ldots, n\}$ and $t \in \{1, \ldots, n\}$ be independently occupied with probability p. In this coupling, we note that $\mathbb{P}_{n-l,\lambda}(|C(1)| < k) - \mathbb{P}_{n,\lambda}(|C(1)| < k)$ is equal to the probability of the event that $|C(1)| < k$ in $\mathrm{ER}(n - l, p)$, but $|C(1)| \geq k$ in $\mathrm{ER}(n, p)$. If $|C(1)| < k$ in $\mathrm{ER}(n - l, p)$, but $|C(1)| \geq k$ in $\mathrm{ER}(n, p)$, it follows that al least one of the vertices $\{n - l + 1, \ldots, n\}$ must be connected to one of the at most k vertices in the connected component of vertex 1 in $\mathrm{ER}(n - l, p)$. This has probability at most lkp, so that, by Boole's inequality,

$$\mathbb{P}_\lambda(|C(j)| < k, i \longleftrightarrow\!\!\!\!/\ j||C(i)| = l) - \mathbb{P}_\lambda(|C(j)| < k) \leq \frac{lk\lambda}{n}.$$
(4.4.43)

Therefore,

$$\sum_{i,j} \left[\mathbb{P}_\lambda(|C(i)| < k, |C(j)| < k, i \longleftrightarrow\!\!\!\!/\ j) - \mathbb{P}_\lambda(|C(i)| < k)\mathbb{P}_\lambda(|C(j)| < k)\right]$$
$$\leq \sum_{l=1}^{k-1} \sum_{i,j} \frac{\lambda k l}{n} \mathbb{P}_\lambda(|C(i)| = l)$$
$$= \frac{\lambda k}{n} \sum_{i,j} \mathbb{E}_\lambda[|C(i)| \mathbb{1}_{\{|C(i)| < k\}}]$$
$$= nk\lambda \chi_{<k}(\lambda),$$
(4.4.44)

which, together with (4.4.38) and (4.4.37), completes the proof. $\qquad\square$

Exercise 4.18. *Show that, for $i \neq j$,*

$$\mathbb{P}_\lambda(i \longleftrightarrow\!\!\!\!/\ j||C(i)| = l) = 1 - \frac{l - 1}{n - 1}.$$
(4.4.45)

4.4.4 Proof of law of large numbers of the giant component in Theorem 4.9

We fix $\nu \in (\frac{1}{2}, 1)$ and take $k_n = K \log n$ with K sufficiently large.

We use Proposition 4.10 to note that

$$\mathbb{E}_\lambda[Z_{\geq k_n}] = n\mathbb{P}_\lambda(|C(v)| \geq k_n) = n\zeta_\lambda + O(k_n), \tag{4.4.46}$$

and therefore, for n sufficiently large and since $k_n = o(n^\nu)$,

$$\{|Z_{\geq k_n} - \mathbb{E}_\lambda[Z_{\geq k_n}]| \leq n^\nu/2\} \subseteq \{|Z_{\geq k_n} - n\zeta_\lambda| \leq n^\nu\}. \tag{4.4.47}$$

By the Chebychev inequality (Theorem 2.15), and using Proposition 4.13 as well as $\chi_{<k_n}(\lambda) \leq k_n$, we then obtain that

$$\begin{aligned}
\mathbb{P}_\lambda(|Z_{\geq k_n} - n\zeta_\lambda| \leq n^\nu) &\geq \mathbb{P}_\lambda(|Z_{\geq k_n} - \mathbb{E}_\lambda[Z_{\geq k_n}]| \leq n^\nu/2) \\
&\geq 1 - 4n^{-2\nu}\mathrm{Var}(Z_{\geq k_n}) \\
&\geq 1 - 4n^{1-2\nu}(\lambda k_n^2 + k_n) \geq 1 - n^{-\delta},
\end{aligned} \tag{4.4.48}$$

for any $\delta < 2\nu - 1$ and n sufficiently large, since $k_n = K \log n$.

By Corollary 4.12, the event

$$\mathcal{E}_n = \{\nexists v : k_n \leq |C(v)| \leq \alpha n\} \tag{4.4.49}$$

occurs with high probability:

$$\mathbb{P}_\lambda(\mathcal{E}_n^c) \leq n^{-\delta}. \tag{4.4.50}$$

Take $2\alpha > \zeta_\lambda$. Then, we claim that on the event $\mathcal{E}_n \cap \{|Z_{\geq k_n} - \zeta_\lambda n| \leq n^\nu\}$,

$$|C_{\max}| = Z_{\geq k_n}. \tag{4.4.51}$$

Equation (4.4.51) is one of the most crucial steps in the proof. To see (4.4.51), we start by noting that $\{|Z_{\geq k_n} - \zeta_\lambda n| \leq n^\nu\} \subseteq \{Z_{\geq k_n} \geq 1\}$. Next, we note that when $|C_{\max}| \neq Z_{\geq k_n}$, then, since $|C_{\max}| \leq Z_{\geq k_n}$ whenever $Z_{\geq k_n} \geq 1$, we must have that $|C_{\max}| < Z_{\geq k_n}$. In turn, $|C_{\max}| < Z_{\geq k_n}$ implies that there are two connected components with size at least k_n. Furthermore, since $\mathcal{E}_n$ occurs, there are no connected components with sizes in between k_n and αn. Therefore, there must be two connected components with size at least αn, which in turn implies that $Z_{\geq k_n} \geq 2\alpha n$. When $2\alpha > \zeta_\lambda$ and n is sufficiently large, this is in contradiction with $Z_{\geq k_n} \leq \zeta_\lambda n + n^\nu$. We conclude that (4.4.51) holds.

By (4.4.51), we have

$$\begin{aligned}
\mathbb{P}_\lambda\Big(||C_{\max}| - \zeta_\lambda n| \leq n^\nu\Big) &\geq \mathbb{P}_\lambda\Big(\{||C_{\max}| - \zeta_\lambda n| \leq n^\nu\} \cap E_n \cap \{|Z_{\geq k_n} - \zeta_\lambda n| \leq n^\nu\}\Big) \\
&= \mathbb{P}_\lambda(\{|Z_{\geq k_n} - \zeta_\lambda n| \leq n^\nu\} \cap \mathcal{E}_n).
\end{aligned} \tag{4.4.52}$$

We conclude by using that

$$\begin{aligned}
\mathbb{P}_\lambda(||C_{\max}| - \zeta_\lambda n| \leq n^\nu) &\geq \mathbb{P}_\lambda(\{|Z_{\geq k_n} - \zeta_\lambda n| \leq n^\nu\} \cap \mathcal{E}_n) \\
&\geq \mathbb{P}_\lambda(|Z_{\geq k_n} - \zeta_\lambda n| \leq n^\nu) - \mathbb{P}_\lambda(\mathcal{E}_n^c) \\
&\geq 1 - 2n^{-\delta},
\end{aligned} \tag{4.4.53}$$

by (4.4.48) and (4.4.50). This completes the proof of the law of large number of the giant component in Theorem 4.9. $\qquad\square$

4.4.5   The discrete duality principle

Using the results we can construct a duality principle for Erdős-Rényi random graphs similar to the duality principle for branching processes:

Theorem 4.14 (Discrete duality principle). *Let $\mu_\lambda < 1 < \lambda$ be conjugates as in (3.5.7). Conditionally, the graph $\mathrm{ER}(n, \lambda/n)$ with the giant component removed is close in law to the random graph $\mathrm{ER}(m, \frac{\mu_\lambda}{m})$, where the variable $m = \lceil n\eta_\lambda \rceil$ is the asymptotic number of vertices outside the giant component.*

We will see that the proof follows from Theorem 4.9, since this implies that the giant component has size $n - m = \zeta_\lambda n(1 + o(1))$. In the statement of Theorem 4.14 we make use of the informal notion 'close in law'. This notion can be made precise as follows. Let $\mathrm{ER}(n, \lambda/n)'$ be $\mathrm{ER}(n, \lambda/n)$ with the giant component removed. We write $\mathbb{P}'_\lambda$ for the law of $\mathrm{ER}(n, \lambda/n)'$, and we recall that $\mathbb{P}_{m,\mu}$ denotes the law of $\mathrm{ER}(m, \mu)$. Let $\mathcal{E}$ be an event which is determined by the edges variables. Then, if $\lim_{m \to \infty} \mathbb{P}_{m,\mu_\lambda}(\mathcal{E})$ exists, then

$$\lim_{n \to \infty} \mathbb{P}'_{n,\lambda}(\mathcal{E}) = \lim_{m \to \infty} \mathbb{P}_{m,\mu_\lambda}(\mathcal{E}). \tag{4.4.54}$$

We shall sketch a proof of Theorem 4.14. First of all, all the edges in the complement of the giant component in $\mathrm{ER}(n, p)$ are independent. Furthermore, the conditional probability that an edge st is occupied in $\mathrm{ER}(n, p)$ with the giant component removed is, conditionally on $|C_{\max}| = n - m$, equal to

$$\frac{\lambda}{n} = \frac{\lambda}{m}\frac{m}{n}. \tag{4.4.55}$$

Now, $m \approx \eta_\lambda n$, so that the conditional probability that an edge st is occupied in $\mathrm{ER}(n, p)$ with the giant component removed, conditionally on $|C_{\max}| \approx \zeta_\lambda n$, is equal to

$$\frac{\lambda}{n} \approx \frac{\lambda\eta_\lambda}{m} = \frac{\mu_\lambda}{m}, \tag{4.4.56}$$

where we have used (3.5.2) and (3.5.5), which implies that $\lambda\eta_\lambda = \mu_\lambda$. Therefore, the conditional probability that an edge st is occupied in $\mathrm{ER}(n, p)$ with the giant component removed, conditionally on $|C_{\max}| \approx \zeta_\lambda n$, is equal to $\frac{\mu_\lambda}{m}$.

Exercise 4.19. *Use the duality principle to show that the second largest component of a supercritical Erdős-Rényi random graph $\mathcal{C}_{(2)}$ satisfies*

$$\frac{|\mathcal{C}_{(2)}|}{\log n} \xrightarrow{\mathbb{P}} I_{\mu_\lambda}. \tag{4.4.57}$$

4.5   The CLT for the giant component

In this section, we prove a central limit theorem for the giant component in the supercritical regime, extending the law of large numbers for the giant component in Theorem 4.9. The main result is as follows:

Theorem 4.15 (Central Limit Theorem for the Giant Component). *Fix $\lambda > 1$. Then,*

$$\frac{|C_{\max}| - \zeta_\lambda n}{\sqrt{n}} \xrightarrow{d} Z, \tag{4.5.1}$$

where Z is a normal random variable with mean 0 and variance $\sigma_\lambda^2 = \frac{\zeta_\lambda(1-\zeta_\lambda)}{(1-\lambda+\lambda\zeta_\lambda)^2}$.

We shall make use of the exploration of connected components to prove Theorem 4.15. In the proof, we shall make essential use of Theorem 4.9.

In order to present the proof, we start with some introductions. Fix $k = k_n$, which will be chosen later on. We shall explore the union of the connected components of the vertices $\{1, \ldots, k\}$. When $k \to \infty$ and using Theorem 4.9, this union contains the largest connected component $C_{\max}$, and it cannot be larger than $|C_{\max}| + k b_n$, where $b_n \leq K \log n$ is an upper bound on the second largest component. As a result, when k is $o(n^\nu)$ with $\nu < \frac{1}{2}$, this union of components is equal to $|C_{\max}| + o(\sqrt{n})$. As a result, a central limit theorem for the union of components implies one for $|C_{\max}|$. We now describe the size of the union of the components of $\{1, \ldots, k\}$.

Let S_1 be the total number of vertices in $\{k+1, \ldots, n\}$ that are connected to the vertices $\{1, \ldots, k\}$. Then, it is not hard to see that

$$S_1 \sim \text{BIN}(n - k, 1 - (1-p)^k). \tag{4.5.2}$$

Then, for $m \geq 1$, let

$$S_m = S_{m-1} + X_m - 1, \tag{4.5.3}$$

where

$$X_m \sim \text{BIN}\Big(n - S_{m-1} - (m + k - 2), p\Big). \tag{4.5.4}$$

Equations (4.5.3) and (4.5.4) are similar to the ones in (4.1.3) and (4.1.4), but they are adapted to the case where we explore the connected component of more than one vertex. We next derive the distribution of S_t in a similar way as in Proposition 4.7.

Proposition 4.16 (The law of S_t revisited). *For all $t \in \{1, \ldots, n\}$,*

$$S_t + (t - 1) \sim \text{BIN}(n - k, 1 - (1-p)^{t+k-1}). \tag{4.5.5}$$

Moreover, for all $l, m \in \{1, \ldots, n\}$ satisfying $l \geq m$, and conditionally on S_m,

$$S_l + (l - m) - S_m \sim \text{BIN}\Big(n - (m + k - 1) - S_m, 1 - (1-p)^{l-m}\Big). \tag{4.5.6}$$

For $k = 1$, the equality in distribution (4.5.5) in Proposition 4.16 reduces to Proposition 4.7.

Proof. For $t = 1$, the claim in (4.5.5) follows from (4.5.2). For $t \geq 1$, let N_t represent the number of unexplored vertices, i.e.,

$$N_t = n - (t + k - 1) - S_t. \tag{4.5.7}$$

It is more convenient to show the equivalent statement that for all $t \geq 1$

$$N_t \sim \text{BIN}(n - k, (1-p)^{t+k-1}). \tag{4.5.8}$$

To see this, we note that each of the vertices $\{k + 1, \ldots, n\}$ has, independently of all other vertices, probability $(1-p)^{t+k-1}$ to stay neutral in the first t explorations. More formally, conditionally on S_{t-1}, and by (4.5.4), we have that $X_t \sim \text{BIN}(n - S_{t-1} - (t + k - 2), p) = \text{BIN}(N_{t-1}, p)$ by (4.5.4). Thus, noting that $N_1 \sim \text{BIN}(n - k, (1-p)^k)$ and

$$
\begin{aligned}
N_t &= n - (t + k - 1) - S_t = n - (t + k - 1) - S_{t-1} - X_t + 1 \\
&= n - (t + k - 2) - S_{t-1} - \text{BIN}(N_{t-1}, p) \\
&= N_{t-1} - \text{BIN}(N_{t-1}, p) = \text{BIN}(N_{t-1}, 1 - p),
\end{aligned} \tag{4.5.9}
$$

the conclusion follows by recursion on $m \geq 2$ and Exercise 4.13. We note that (4.5.9) also implies that for any $l \geq m$,

$$N_l \sim \mathrm{BIN}(N_m, (1-p)^{l-m}).\tag{4.5.10}$$

Substituting $N_m = n - (m+k-1) - S_m$, this implies that

$$n - (l+k-1) - S_l \sim \mathrm{BIN}\Big(n - (m+k-1) - S_m, (1-p)^{l-m}\Big)\tag{4.5.11}$$

$$= n - (m+k-1) - S_m - \mathrm{BIN}\Big(n - (m+k-1) - S_m, 1 - (1-p)^{l-m}\Big),$$

which, in turn, is equivalent to the statement that, for all $l \geq m$ and, conditionally on S_m,

$$S_l + (l-m) - S_m \sim \mathrm{BIN}\Big(n - (m+k-1) - S_m, 1 - (1-p)^{l-m}\Big),\tag{4.5.12}$$

$\square$

We now state a corollary of Proposition 4.16 which states that $S_{\lfloor nt \rfloor}$ satisfies a central limit theorem. By convention, we let $S_0 = k$. In its statement, we make use of the asymptotic mean

$$\mu_t = 1 - t - e^{-\lambda t}\tag{4.5.13}$$

<div align="right" style="font-size:smaller">e:MeantdefCLT</div>

and asymptotic variance

$$v_t = e^{-\lambda t}(1 - e^{-\lambda t}).\tag{4.5.14}$$

<div align="right" style="font-size:smaller">e:VartdefCLT</div>

Th central limit theorem for S_m reads as follows:

Corollary 4.17 (CLT for S_m). *Fix $k = k_n = o(\sqrt{n})$. Then, for every $t \in [0,1]$, the random variable $\frac{S_{\lfloor nt \rfloor} - n\mu_t}{\sqrt{nv_t}}$ converges in distribution to a standard normal random variable.*

Proof. The statement follows immediately from the central limit theorem for the binomial distribution when we can show that

$$\mathbb{E}[S_{\lfloor nt \rfloor}] = n\mu_t + o(\sqrt{n}), \qquad \mathrm{Var}(S_{\lfloor nt \rfloor}) = nv_t + o(n).\tag{4.5.15}$$

<div align="right" style="font-size:smaller">e:meanvarCLT</div>

Indeed, by the central limit theorem for the binomial distribution we have that

$$\frac{S_{\lfloor nt \rfloor} - \mathbb{E}[S_{\lfloor nt \rfloor}]}{\sqrt{\mathrm{Var}(S_{\lfloor nt \rfloor})}} \xrightarrow{d} Z,\tag{4.5.16}$$

<div align="right" style="font-size:smaller">e:CLTbin</div>

where Z is a standard normal random variable.

Exercise 4.20. *Prove that if $X_n = \mathrm{BIN}(a_n, p_n)$, where $\mathrm{Var}(X) = a_n p_n (1 - p_n) \to \infty$, then*

$$\frac{X_n - a_n p_n}{\sqrt{a_n p_n (1 - p_n)}} \xrightarrow{d} Z,\tag{4.5.17}$$

where Z is a standard normal random variable. Use this to conclude that (4.5.15) implies (4.5.16).

Now we can further write

$$\frac{S_{\lfloor nt \rfloor} - n\mu_t}{\sqrt{nv_t}} = \sqrt{\frac{\mathrm{Var}(S_{\lfloor nt \rfloor})}{nv_t}} \frac{S_{\lfloor nt \rfloor} - \mathbb{E}[S_{\lfloor nt \rfloor}]}{\sqrt{\mathrm{Var}(S_{\lfloor nt \rfloor})}} + \frac{\mathbb{E}[S_{\lfloor nt \rfloor}] - n\mu_t}{\sqrt{\mathrm{Var}(S_{\lfloor nt \rfloor})}}.\tag{4.5.18}$$

By (4.5.15), we have that the last term converges to zero, and the factor $\sqrt{\frac{\text{Var}(S_{\lfloor nt \rfloor})}{nv_t}}$ converges to one. Therefore, (4.5.15) implies the central limit theorem.

To see the asymptotics of the mean in (4.5.15), we note that

$$\mathbb{E}[S_{\lfloor nt \rfloor}] = (n-k)\Big(1 - (1 - \frac{\lambda}{n})^{\lfloor nt \rfloor + k - 1}\Big) - \big(\lfloor nt \rfloor - 1\big) = n\mu_t + o(\sqrt{n}), \qquad (4.5.19)$$

as long as $k = o(\sqrt{n})$. For the asymptotics of the variance in (4.5.15), we note that

$$\text{Var}(S_{\lfloor nt \rfloor}) = (n-k)(1 - \frac{\lambda}{n})^{\lfloor nt \rfloor + k - 1}\big(1 - (1 - \frac{\lambda}{n})^{\lfloor nt \rfloor + k - 1}\big) = nv_t + o(n), \qquad (4.5.20)$$

as long as $k = o(n)$. □

Proof of Theorem 4.15. Let $|\mathcal{C}_{\leq k}|$ be the size of the union of the components of the vertices $1, \ldots, k$. Then we have that

$$|\mathcal{C}_{\leq k}| \sim \min\{m : S_m = 0\}. \qquad \overset{\text{e:Ck-distr}}{(4.5.21)}$$

Let $k = k_n = \log n$. Then, by Theorem 4.9, the probability that none of the first k_n vertices is in the largest connected component is bounded above by

$$\mathbb{E}_\lambda\Big[\Big(\frac{n - |C_{\max}|}{n}\Big)^{k_n}\Big] = o(1). \qquad (4.5.22)$$

Therefore, with high probability, $|\mathcal{C}_{\leq k}| \geq |C_{\max}|$. On the other hand, by Corollary 4.12 for $2\alpha > \zeta_\lambda$ and Theorem 4.9, with high probability, the second largest cluster has size at most $K \log n$. Hence, with high probability,

$$|\mathcal{C}_{\leq k}| \leq |C_{\max}| + (k-1)K \log n. \qquad (4.5.23)$$

We conclude that a central limit theorem for $|C_{\max}|$ follows from one for $|\mathcal{C}_{\leq k}|$ with $k = \log n$.

The central limit theorem for $|\mathcal{C}_{\leq k}|$ is proved by upper and upper bounds on the probabilities

$$\mathbb{P}_\lambda\Big(\frac{|\mathcal{C}_{\leq k}| - \zeta_\lambda n}{\sqrt{n}} > x\Big).$$

For the upper bound, we use that (4.5.21) implies that, for every ℓ,

$$\mathbb{P}_\lambda(|\mathcal{C}_{\leq k}| > \ell) = \mathbb{P}_\lambda(\forall m \leq \ell : S_m > 0). \qquad \overset{\text{e:Ck-tails}}{(4.5.24)}$$

Applying (4.5.24) to $\ell = m_x = \lfloor n\zeta_\lambda + x\sqrt{n} \rfloor$, we obtain

$$\mathbb{P}_\lambda\Big(\frac{|\mathcal{C}_{\leq k}| - \zeta_\lambda n}{\sqrt{n}} > x\Big) = \mathbb{P}_\lambda(\forall m \leq m_x : S_m > 0) \leq \mathbb{P}_\lambda(S_{m_x} > 0). \qquad \overset{\text{e:CLTupperbd1}}{(4.5.25)}$$

Now we use (4.5.13), (4.5.15) and $\mu_{\zeta_\lambda} = 0$, and writing μ'_t for the derivative of $t \mapsto \mu_t$, to see that

$$\mathbb{E}[S_{m_x}] = n\mu_{\zeta_\lambda} + \sqrt{n}x\mu'_{\zeta_\lambda} + o(\sqrt{n}) = \sqrt{n}x(\lambda e^{-\lambda\zeta_\lambda} - 1) + o(\sqrt{n}) = \sqrt{n}x(\lambda e^{-\lambda\zeta_\lambda} - 1) + o(\sqrt{n}), \quad (4.5.26)$$

where we note that $\lambda e^{-\lambda\zeta_\lambda} - 1 < 0$ for $\lambda > 1$.

Exercise 4.21. *Prove that, for $\lambda > 1$, we have $\mu_{\zeta_\lambda} = 0$ and $\mu'_{\zeta_\lambda} = \lambda e^{-\lambda\zeta_\lambda} - 1 < 0$.*

The variance of S_{m_x} is, by (4.5.14) and (4.5.15),

$$\text{Var}(S_{m_x}) = nv_{\zeta_\lambda} + o(n). \tag{4.5.27}$$

As a result, we have that

$$\mathbb{P}_\lambda(S_{m_x} > 0) = \mathbb{P}_\lambda\Big(\frac{S_{m_x} - \mathbb{E}[S_{m_x}]}{\sqrt{\text{Var}(S_{m_x})}} > \frac{x(1 - \lambda e^{-\lambda \zeta_\lambda})}{\sqrt{v_{\zeta_\lambda}}}\Big) + o(1). \tag{4.5.28}$$

By Corollary 4.17, the right-hand side converges to

$$\mathbb{P}\Big(Z > \frac{x(1 - \lambda e^{-\lambda \zeta_\lambda})}{\sqrt{v_{\zeta_\lambda}}}\Big) = \mathbb{P}(Z' > x), \tag{4.5.29}$$

where Z' has a normal distribution with mean 0 and variance $v_{\zeta_\lambda}(1 - \lambda e^{-\lambda \zeta_\lambda})^{-2}$. We finally note that, by (3.5.2) and $\zeta_\lambda = 1 - \eta_\lambda$, we have that $1 - \zeta_\lambda = e^{-\lambda \zeta_\lambda}$, so that

$$v_{\zeta_\lambda} = e^{-\lambda \zeta_\lambda}(1 - e^{-\lambda \zeta_\lambda}) = \zeta_\lambda(1 - \zeta_\lambda). \tag{4.5.30}$$

By (4.5.30), the variance of the normal distribution appearing in the lower bound can be rewritten as

$$\frac{v_{\zeta_\lambda}}{(1 - \lambda e^{-\lambda \zeta_\lambda})^2} = \frac{\zeta_\lambda(1 - \zeta_\lambda)}{(1 - \lambda + \lambda \zeta_\lambda)^2}. \tag{4.5.31}$$

By (4.5.25), this completes the upper bound.

For the lower bound, we again use the fact that

$$\mathbb{P}_\lambda\Big(|\mathcal{C}_{\leq k}| - \zeta_\lambda n > x\Big) = \mathbb{P}_\lambda(\forall m \leq m_x : S_m > 0), \tag{4.5.32}$$

where we recall that $m_x = \lfloor n\zeta_\lambda + x\sqrt{n}\rfloor$. Then, for any $\varepsilon > 0$, we bound from below

$$\mathbb{P}_\lambda(\forall m \leq m_x : S_m > 0) \geq \mathbb{P}_\lambda(\forall m < m_x : S_m > 0, S_{m_x} > \varepsilon\sqrt{n})$$
$$= \mathbb{P}_\lambda(S_{m_x} > \varepsilon\sqrt{n}) - \mathbb{P}_\lambda(S_{m_x} > \varepsilon\sqrt{n}, \exists m < m_x : S_m = 0). \tag{4.5.33}$$

The first term can be handled in a similar way as for the upper bound. Indeed, repeating the steps in the upper bound, we obtain that, for every $\varepsilon > 0$,

$$\mathbb{P}_\lambda(S_{m_x} > \varepsilon\sqrt{n}) = \mathbb{P}\Big(Z > \frac{x(1 - \lambda e^{-\lambda \zeta_\lambda}) + \varepsilon}{\sqrt{v_{\zeta_\lambda}}}\Big) + o(1). \tag{4.5.34}$$

The quantity in (4.5.34) converges to $\mathbb{P}(Z' > x)$, where Z' has a normal distribution with mean 0 and variance σ_λ^2, as $\varepsilon \downarrow 0$.

We conclude that it suffices to prove that

$$\mathbb{P}_\lambda(S_{m_x} > \varepsilon\sqrt{n}, \exists m < m_x : S_m = 0) = o(1). \tag{4.5.35}$$

To bound the probability in (4.5.35), we first use Boole's inequality to get

$$\mathbb{P}_\lambda(S_{m_x} > \varepsilon\sqrt{n}, \exists m < m_x : S_m = 0) \leq \sum_{m=1}^{m_x - 1} \mathbb{P}_\lambda(S_m = 0, S_{m_x} > \varepsilon\sqrt{n}). \tag{4.5.36}$$

For $m \leq \alpha n$ with $\alpha < \zeta_\lambda$, we can show that, when $k = K \log n$ and K sufficiently large, and uniformly in $m \leq \alpha n$,

$$\mathbb{P}_\lambda(S_m = 0) = o(n^{-1}). \tag{4.5.37}$$

Exercise 4.22. *Prove that, uniformly in $m \leq \alpha n$ with $\alpha < \zeta_\lambda$, and when $k = K \log n$ with K sufficiently large. Hint: make use of (4.5.5) in Proposition 4.16.*

We continue by proving a similar bound for $m > \alpha n$, where $\alpha < \zeta_\lambda$ can be chosen arbitrarily close to ζ_λ. Here we shall make use of the fact that, for m close to $\zeta_\lambda n$, $\mathbb{E}_\lambda[X_m] < 1$, so that $m \mapsto S_m$, for $m \geq \alpha n$ is close to a random walk with negative drift. As a result, the probability that $S_m = 0$, yet $S_{m_x} > \varepsilon\sqrt{n}$ is exponentially small.

We now present the details of this argument. We bound

$$\mathbb{P}_\lambda(S_m = 0, S_{m_x} > \varepsilon\sqrt{n}) \leq \mathbb{P}_\lambda(S_{m_x} > \varepsilon\sqrt{n} \mid S_m = 0) \qquad (4.5.38)$$
$$= \mathbb{P}_\lambda\Big(\mathrm{BIN}(n - (m + k - 1), 1 - (1 - p)^{m_x - m}) > (m_x - m) + \varepsilon\sqrt{n}\Big),$$

since, by (4.5.6) in Proposition 4.16 and conditionally on $S_m = 0$,

$$S_l + (l - m) \sim \mathrm{BIN}\Big(n - (m + k - 1), 1 - (1 - p)^{l - m}\Big).$$

We pick $\kappa = \zeta_\lambda - \varepsilon$, for some $\varepsilon > 0$ which is very small. Then, using that $1 - (1 - a)^b \leq ab$ for every a, b with $0 < a < 1, b \geq 1$, we arrive at

$$1 - (1 - p)^{m_x - m} = 1 - \left(1 - \frac{\lambda}{n}\right)^{m_x - m} \leq \frac{\lambda(m_x - m)}{n}. \qquad (4.5.39)$$

As a result, with $X = \mathrm{BIN}\Big(n - (m + k - 1), 1 - (1 - p)^{m_x - m}\Big)$, and using that $n - (m + k - 1) \leq n - m \leq n(1 - \zeta_\lambda + \varepsilon)$ and $p = \lambda/n$,

$$\mathbb{E}_\lambda[X] = [n - (m + k - 1)][1 - (1 - p)^{m_x - m}] \leq (m_x - m)\lambda(1 - \zeta_\lambda + \varepsilon). \qquad \overset{\text{e:meanX}}{(4.5.40)}$$

Since $\lambda > 1$, we can use that $\lambda(1 - \zeta_\lambda) = \lambda e^{-\lambda\zeta_\lambda} < 1$ by Exercise 4.21, so that, taking $\varepsilon > 0$ so small that $\lambda(1 - \zeta_\lambda + \varepsilon) < 1 - \varepsilon$, we have

$$\mathbb{E}[X] \leq (1 - \varepsilon)(m_x - m). \qquad \overset{\text{e:meanXrep}}{(4.5.41)}$$

Therefore,

$$\mathbb{P}_\lambda(S_m = 0, S_{m_x} > \varepsilon\sqrt{n}) \leq \mathbb{P}_\lambda\Big(X - \mathbb{E}[X] > \varepsilon((m_x - m) + \sqrt{n})\Big). \qquad (4.5.42)$$

By Theorem 2.18, with $t = \varepsilon((m_x - m) + \sqrt{n})$ and using (4.5.41), we obtain

$$\mathbb{P}_\lambda(S_m = 0, S_{m_x} > \varepsilon\sqrt{n}) \leq \exp\left(-\frac{t^2}{2((1 - \varepsilon)(m_x - m) + t/3)}\right) \leq \exp\left(-\frac{t^2}{2((m_x - m) + 2\varepsilon\sqrt{n}/3)}\right). \qquad (4.5.43)$$

Thus, for $m_x - m \geq \varepsilon\sqrt{n}$, since $t \geq \varepsilon(m_x - m)$, we have

$$\mathbb{P}_\lambda(S_m = 0, S_{m_x} > \varepsilon\sqrt{n}) \leq \exp\left(-3\varepsilon^2(m_x - m)/8\right) = o(n^{-1}), \qquad \overset{\text{e:bdm2}}{(4.5.44)}$$

while, for $m_x - m \leq \varepsilon\sqrt{n}$, since $t \geq \varepsilon\sqrt{n}$, we have

$$\mathbb{P}_\lambda(S_m = 0, S_{m_x} > \varepsilon\sqrt{n}) \leq \exp\left(-3\varepsilon\sqrt{n}/8\right) = \exp\left(-\varepsilon\sqrt{n}/2\right) = o(n^{-1}). \qquad \overset{\text{e:bdm3}}{(4.5.45)}$$

The bounds (4.5.37), (4.5.44) and (4.5.45) complete the proof of Theorem 4.15. $\qquad \square$

4.6   Notes and discussion

Notes on Section 4.1.   There are several possible definitions of the Erdős-Rényi random graph. Many of the classical results are proved for $\mathrm{ER}(n, M)$, which is the random graph on the vertices $\{1, \ldots, n\}$ obtained by adding M edges uniformly at random. Since the number of edges in the Erdős-Rényi random graph has a binomial distribution with parameters $n(n-1)/2$ and p, we should think of M corresponding roughly to $pn(n-1)/2$. Also, writing $\mathbb{P}_M$ for the distribution of $ER(n, M)$, we have that $\mathbb{P}_\lambda$ and $\mathbb{P}_M$ are related as

$$\mathbb{P}_\lambda(E) = \sum_{M=1}^{n(n-1)/2} \mathbb{P}_M(E)\mathbb{P}\big(\mathrm{BIN}(n(n-1)/2, p) = M\big), \qquad \text{(4.6.1)}$$

where E is any event. This allows one to deduce results for $\mathrm{ER}(n, M)$ from the ones for $\mathrm{ER}(n, p)$ and vice versa. The model $\mathrm{ER}(n, M)$ was first studied in [87], the model $\mathrm{ER}(n, p)$ was introduced in [99], and a model with possibly multiple edges between vertices in [18].

The random graph $\mathrm{ER}(n, M)$ has the advantage that we can think of the graph as evolving as a process, by adding the edges one at a time, which also allows us to investigate *dynamical properties*, such as when the first cycle appears. This is also possible for $\mathrm{ER}(n, p)$ using the coupling in Section 4.1.1, but is slightly less appealing.

We refer to the books [14, 40, 124] for more detailed references of the early literature on random graphs.

Notes on Section 4.2.

Notes on Section 4.3.   The strategy in the proof of Theorems 4.4 and 4.5 is close in spirit to the proof in [14], with ingredients taken from [47], which, in turn, was inspired by [50, 51]. In particular, the use of the random variable $Z_{\geq k}$ has appeared in these references. The random variable $Z_{\geq k}$ also plays a crucial role in the analysis of $|C_{\max}|$ both when $\lambda > 1$ and when $\lambda = 1$.

Exercise 4.23.   *Use (4.6.1) and Theorems 4.4–4.5 to show that* $|C_{\max}|/\log n \overset{\mathbb{P}}{\longrightarrow} I_\lambda^{-1}$ *for* $\mathrm{ER}(n, M)$ *when* $M = n\lambda/2$.

Notes on Section 4.4.

Exercise 4.24.   *Use (4.6.1) and Theorem 4.9 to show that* $|C_{\max}|/n \overset{\mathbb{P}}{\longrightarrow} \zeta_\lambda$ *for* $\mathrm{ER}(n, M)$ *when* $M = n\lambda/2$.

Exercises 4.23 and 4.24 show that $\mathrm{ER}(n, M)$ has a phase transition when $M = n\lambda/2$ at $\lambda = 1$.

Notes on Section 4.5.   The central limit theorem for the largest supercritical cluster was proved in [153],[179] and [24]. In [179], the result follows as a corollary of the main result, involving central limit theorems for various random graph quantities, such as the number tree components of various size. Martin-Löf [153] studies the giant component in the context of epidemics. His proof makes clever use of a connection to asymptotic stochastic differential equations, and is reproduced in [79]. Since we do not assume familiarity with stochastic differential equations, we have produced an independent proof which only relies on elementary techniques.

Chapter 5

The Erdős-Rényi random graph revisited

In the previous chapter, we have proved that the largest connected component of the Erdős-Rényi random graph exhibits a phase transition. In this chapter, we investigate several more properties of the Erdős-Rényi random graph. We start by investigating the critical behavior of the size of largest connected component in the Erdős-Rényi random graph by studying $p = 1/n$ in Section 5.1. After this, in Section 5.2, we investigate the phase transition for the connectivity of $\mathrm{ER}(n,p)$, and for p inside the *critical window*, compute the asymptotic probability that the Erdős-Rényi random graph is connected. Finally, in Section 5.3, we study the degree sequence of an Erdős-Rényi random graph.

5.1 The critical behavior

In this section, we study the behavior of the largest connected component for the critical value $p = \frac{1}{n}$. In this case, it turns out that there is interesting behavior, where the size of the largest connected component is large, yet much smaller than the size of the volume.

Theorem 5.1 (Largest critical cluster). *Fix $\lambda = 1$. There exists a constant $b > 0$ such that for all $\omega \geq 1$ and for n sufficiently large,*

$$\mathbb{P}_1\Big(\omega^{-1}n^{2/3} \leq |C_{\max}| \leq \omega n^{2/3}\Big) \geq 1 - \frac{b}{\omega}. \qquad \begin{smallmatrix}\texttt{e:LCBd1-win}\end{smallmatrix} \tag{5.1.1}$$

Theorem 5.1 shows that the largest critical cluster obeys a non-trivial scaling result. While $|C_{\max}|$ is logarithmically small in the subcritical regime $\lambda < 1$ by Theorem 4.4, and $|C_{\max}| = \Theta(n)$ in the supercritical regime $\lambda > 1$ by Theorem 4.9, at the critical value $\lambda = 1$, we see that the largest cluster is $\Theta(n^{2/3})$. The result in Theorem 5.1 shows that the random variable $|C_{\max}|n^{-2/3}$ is *tight*, in the sense that with high probability, we have $|C_{\max}|n^{-2/3} \leq \omega$ for ω sufficiently large. Also, with high probability, $|C_{\max}|n^{-2/3} \geq \omega^{-1}$, so that with substantial probability, $|C_{\max}| = \Theta(n^{2/3})$.

5.1.1 Strategy of the proof

In the proof, we make essential use of bounds on the expected cluster size, as well as on the tail of the cluster size distribution. We will formulate these results now. We define the tail of the cluster size distribution by

$$P_{\geq k}(\lambda) = \mathbb{P}_\lambda(|C(v)| \geq k). \qquad \begin{smallmatrix}\texttt{e:Pgeqkdef}\end{smallmatrix} \tag{5.1.2}$$

We study the tail of the distribution of $|C(v)|$ for the critical case $\lambda = 1$ in the following theorem:

Proposition 5.2 (Critical cluster tails). *Fix $\lambda = 1$. For $k \leq rn^{2/3}$, there exist constants $0 < c_1 < c_2 < \infty$ with $c_1 = c_1(r)$ such that $\min_{r \leq \kappa} c_1(r) > 0$ for some $\kappa > 0$, and c_2 independent of r, such that for n sufficiently large*

$$\frac{c_1}{\sqrt{k}} \leq P_{\geq k}(1) \leq \frac{c_2}{\sqrt{k}}. \tag{5.1.3}$$

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Proposition 5.2 implies that the tails of the critical cluster size distribution obey similar asymptotics as the tails of the total progeny of a critical branching process. See (3.5.24), from which it follows that

$$\mathbb{P}^*_\lambda(T^* \geq k) = \left(\frac{2}{\pi}\right)^{1/2} k^{-1/2}[1 + O(k^{-1})]. \tag{5.1.4}$$

<div align="right">_{e:TPPoisrep}</div>

However, the tail in (5.1.3) is only valid for values of k that are not too large. Indeed, when $k > n$, then $P_{\geq k}(1) = 0$. Therefore, there must be a cut-off above which the asymptotics fails to hold. As it turns out, this cut-off is given by $rn^{2/3}$. The upper bound in (5.1.3) holds for a wider range of k, in fact, the proof yields that (5.1.3) is valid for *all k*.

Exercise 5.1. *Prove (5.1.4) using (3.5.24).*

Proposition 5.3 (Bound on critical expected cluster size). *There exists a constant $K > 0$ such that for all $\lambda \leq 1$ and $n \geq 1$,*

$$\mathbb{E}_\lambda[|C(1)|] = \chi(\lambda) \leq Kn^{1/3}. \tag{5.1.5}$$

<div align="right">_{e:chikbd}</div>

Proposition 5.3 is intuitively consistent with Theorem 5.1. Indeed, in the critical regime, the expected cluster size receives a substantial amount from the largest cluster. Therefore, intuitively, for any $v \in \{1, \ldots, n\}$,

$$\chi(1) \sim \mathbb{E}_1[|C(v)|\mathbb{1}_{\{v \in C_{\max}\}}] = \mathbb{E}_1[|C_{\max}|\mathbb{1}_{\{v \in C_{\max}\}}], \tag{5.1.6}$$

where $\sim$ denotes an equality with an uncontrolled error.

When $|C_{\max}| = \Theta(n^{2/3})$, then

$$\mathbb{E}_1[|C_{\max}|\mathbb{1}_{\{v \in C_{\max}\}}] \sim n^{2/3}\mathbb{P}_1(v \in C_{\max}). \tag{5.1.7}$$

Furthermore, when $|C_{\max}| = \Theta(n^{2/3})$, then

$$\mathbb{P}_1(v \in C_{\max}) \sim \frac{n^{2/3}}{n} = n^{-1/3}. \tag{5.1.8}$$

Therefore, one is intuitively lead to the conclusion

$$\chi(1) \sim n^{1/3}. \tag{5.1.9}$$

Exercise 5.2. *Prove that Proposition 5.2 also implies that $\chi(1) \geq cn^{1/3}$ for some $c > 0$. Therefore, for $\lambda = 1$, the bound in Proposition 5.3 is asymptotically sharp.*

Propositions 5.2 and 5.3 are proved in Section 5.1.2 below. We will first prove Theorem 5.1 subject to them.

Proof of Theorem 5.1 subject to Propositions 5.2 and 5.3. We start with the upper bound on $|C_{\max}|$. We again make use of the fundamental equality

$$\{|C_{\max}| \geq k\} = \{Z_{\geq k} \geq k\}, \tag{5.1.10}$$

where we recall that

$$Z_{\geq k} = \sum_{v=1}^{n} \mathbb{1}_{\{|C(v)| \geq k\}}. \tag{5.1.11}$$

Therefore, by the Markov inequality (Theorem 2.14), we obtain

$$\mathbb{P}_1\big(|C_{\max}| \geq \omega n^{2/3}\big) = \mathbb{P}_1\big(Z_{\geq \omega n^{2/3}} \geq \omega n^{2/3}\big) \leq \omega^{-1} n^{-2/3} \mathbb{E}_1[Z_{\geq \omega n^{2/3}}]. \tag{5.1.12}$$

By Proposition 5.2,

$$\mathbb{E}_1[Z_{\geq \omega n^{2/3}}] = n P_{\geq \omega n^{2/3}}(1) \leq n^{2/3} \frac{c_2}{\sqrt{\omega}}, \tag{5.1.13}$$

so that

$$\mathbb{P}_1\big(|C_{\max}| > \omega n^{2/3}\big) \leq \frac{c_2}{\omega^{3/2}}. \tag{5.1.14}$$

Equation (5.1.14) proves a stronger bound than the one in Theorem 5.1, particularly for $\omega \geq 1$ large.

For the lower bound on $|C_{\max}|$, we first note that for $\omega < b$, there is nothing to prove. The constant $b > 0$ will be taken large, so that web shall assume that $\omega > \kappa^{-1}$, where $\kappa > 0$ is the constant appearing in Proposition 5.2.

We use the Chebychev inequality (Theorem 2.15), as well as

$$\{|C_{\max}| < k\} = \{Z_{\geq k} = 0\}, \tag{5.1.15}$$

to obtain that

$$\mathbb{P}_1\big(|C_{\max}| < \omega^{-1} n^{2/3}\big) = \mathbb{P}_1\big(Z_{\geq \omega^{-1} n^{2/3}} = 0\big) \leq \frac{\text{Var}_1(Z_{\geq \omega^{-1} n^{2/3}})}{\mathbb{E}_1[Z_{\geq \omega^{-1} n^{2/3}}]^2}. \tag{5.1.16}$$

By (5.1.3), we have that

$$\mathbb{E}_1[Z_{\geq \omega^{-1} n^{2/3}}] = n P_{\geq \omega^{-1} n^{2/3}}(1) \geq c_1 \sqrt{\omega} n^{2/3}, \tag{5.1.17}$$

where we used that $\omega \geq \kappa^{-1}$, and $c_1 = \min_{r \leq \kappa} c_1(r) > 0$. Also, by Proposition 4.8, with $k_n = \omega^{-1} n^{2/3}$,

$$\text{Var}_1(Z_{\geq \omega^{-1} n^{2/3}}) \leq n \chi_{\geq \omega^{-1} n^{2/3}}(1) = n \mathbb{E}_1[|C(1)| \mathbb{1}_{\{|C(1)| \geq \omega^{-1} n^{2/3}\}}]. \tag{5.1.18}$$

By Proposition 5.3, we can further bound

$$\text{Var}_1(Z_{\geq \omega^{-1} n^{2/3}}) \leq n \chi_{\geq \omega^{-1} n^{2/3}}(1) \leq n \chi(1) \leq K n^{4/3}. \tag{5.1.19}$$

Substituting (5.1.16)–(5.1.19), we obtain, for n sufficiently large,

$$\mathbb{P}_1\big(|C_{\max}| < \omega^{-1} n^{2/3}\big) \leq \frac{K n^{4/3}}{c_1^2 \omega n^{4/3}} = \frac{K}{c_1^2 \omega}. \tag{5.1.20}$$

This completes the proof of Theorem 5.1 subject to Propositions 5.2 and 5.3. □

5.1.2 Proofs of Propositions 5.2 and 5.3

We start by proving Proposition 5.2.

Proof of Proposition 5.2.   We fix $\lambda \leq 1$. Theorem 4.2 gives

$$P_{\geq k}(\lambda) \leq \mathbb{P}_{n,p}(T \geq k), \tag{5.1.21}$$

where we recall that $\mathbb{P}_{n,p}$ is the law of a binomial branching process with parameters n and $p = \lambda/n$, and T its total progeny. By Theorem 3.17, for $\lambda = 1$,

$$P_{\geq k}(\lambda) \leq \mathbb{P}^*_\lambda(T^* \geq k) + e_k(n), \tag{5.1.22}$$

where, by (3.6.2),

$$|e_k(n)| \leq \frac{2}{n} \sum_{s=1}^{k} \mathbb{P}^*_\lambda(T^* \geq s), \tag{5.1.23}$$

and where we recall that $\mathbb{P}^*_\lambda$ is the law of a (critical) Poisson branching process, i.e., a branching process with Poisson offspring distribution with parameter λ, and T^* is its total progeny.

By (3.5.24), it follows that there exists a $C > 0$ such that for all $\lambda \leq 1$ and $s \geq 1$,

$$\mathbb{P}^*_\lambda(T^* \geq s) \leq \mathbb{P}^*_1(T^* \geq s) \leq \frac{C}{\sqrt{s}}. \tag{5.1.24}$$

Therefore, we can also bound $|e_k(n)|$ for all $\lambda \leq 1$ and $k \leq n$ by

$$|e_k(n)| \leq \frac{4}{n} \sum_{s=1}^{k} \frac{C}{\sqrt{s}} \leq 4C \frac{\sqrt{k}}{n} \leq \frac{4C}{\sqrt{k}}, \tag{5.1.25}$$

since $k \leq n$. We conclude that, for all $\lambda \leq 1$ and $k \leq n$,

$$P_{\geq k}(\lambda) \leq \frac{5C}{\sqrt{k}}. \tag{5.1.26}$$

In particular, taking $\lambda = 1$ prove the upper bound in (5.1.3).

We proceed with the lower bound in (5.1.3), for which we make use of Theorem 4.3 with $k \leq rn^{2/3}$. This gives that

$$\mathbb{P}_1(|C(1)| \geq k) \geq \mathbb{P}_{n-k,p}(T \geq k). \tag{5.1.27}$$

where T is the total progeny of a binomial branching process with parameters $n - k \leq n - rn^{2/3}$ and $p = 1/n$. We again use Theorem 3.17 for $\lambda_n = 1 - rn^{-1/3} \geq 1 - k/n$, as in (5.1.22) and (5.1.23). We apply the one-but-last bound in (5.1.25), so that

$$\mathbb{P}_1(|C(1)| \geq k) \geq \mathbb{P}^*_{\lambda_n}(T^* \geq k) - \frac{4C\sqrt{k}}{n} \geq \mathbb{P}^*_{\lambda_n}(T^* \geq k) - \frac{4C\sqrt{r}}{n^{2/3}}. \tag{5.1.28}$$

We then use Theorem 3.13 to obtain, since $\lambda_n \leq 1$,

$$\mathbb{P}_\lambda(|C(1)| \geq k) \geq \sum_{t=k}^{\infty} \mathbb{P}^*_{\lambda_n}(T^* = t) - \frac{4C\sqrt{r}}{n^{2/3}}$$

$$= \sum_{t=k}^{\infty} \frac{(\lambda_n t)^{t-1}}{t!} e^{-\lambda_n t} - \frac{4C\sqrt{r}}{n^{2/3}}$$

$$\geq \sum_{t=k}^{\infty} \mathbb{P}^*_1(T^* = t) e^{-I_{\lambda_n} t} - \frac{4C\sqrt{r}}{n^{2/3}}, \tag{5.1.29}$$

where, for $\lambda_n = 1 - rn^{-1/3}$ and by (4.3.1),

$$I_{\lambda_n} = \lambda_n - 1 - \log \lambda_n = \frac{1}{2}(\lambda_n - 1)^2 + O(|\lambda_n - 1|^3). \tag{5.1.30}$$

Exercise 5.3. *Prove that*

$$\frac{(\lambda t)^{t-1}}{t!}e^{-\lambda t} = \frac{1}{\lambda}e^{-I_\lambda t}\mathbb{P}_1^*(T^* = t).$$ (5.1.31)

Therefore, for n sufficiently large,

$$\mathbb{P}_\lambda(|C(1)| \geq k) \geq \sum_{t=k}^{2k}\mathbb{P}_1^*(T^* = t)e^{-\frac{1}{2}(\lambda_n-1)^2 t(1+o(1))} - \frac{4C\sqrt{r}}{n^{2/3}}$$

$$\geq \sum_{t=k}^{2k}\frac{C}{\sqrt{t^3}}e^{-\frac{1}{2}(\lambda_n-1)^2 t(1+o(1))} - \frac{4C\sqrt{r}}{n^{2/3}}$$

$$\geq \frac{2^{-3/2}C}{\sqrt{k}}e^{-k(\lambda_n-1)^2(1+o(1))} - \frac{4C\sqrt{r}}{n^{2/3}}$$

$$\geq \frac{c_1(r)}{\sqrt{k}},$$ (5.1.32)

since $\lambda_n - 1 = -rn^{-1/3}$, and where $c_1(r) = C(2^{-3/2}e^{-r^3} - 4\sqrt{r}) > 0$ for $r \leq \kappa$, for some $\kappa > 0$ sufficiently small. This completes the proof of Proposition 5.2. $\qquad\square$

Exercise 5.4. *Adapt the proof of Proposition 5.2 to the case where $p = (1 + \theta n^{-1/3})/n$, where $\theta \in \mathbb{R}$.*

Proof of Proposition 5.3.[*]   Define

$$\tau_n(\lambda) = \mathbb{P}_\lambda(1 \longleftrightarrow 2),$$ (5.1.33)

where we have added a subscript n to make the dependence on the graph size explicit. We add a subscript n to $\chi_n(\lambda)$ as well to make its dependence on the size of the graph explicit, and we recall

$$\chi_n(\lambda) = \mathbb{E}_\lambda[|C(v)|].$$ (5.1.34)

Note that, by exchangeability of the vertices, (see also Exercise 4.15)

$$\chi_n(\lambda) = \mathbb{E}_\lambda\Big[\sum_{v=1}^{n}\mathbb{1}_{\{v\longleftrightarrow 1\}}\Big]$$

$$= \sum_{v=1}^{n}\mathbb{P}_\lambda(v \longleftrightarrow 1)$$

$$= (n-1)\tau_n(\lambda) + 1.$$ (5.1.35)

Therefore, Proposition 5.3 follows from the bound

$$\tau_n(1) \leq Kn^{-2/3}.$$ (5.1.36)

For this, we will use a bound on the derivative of $\lambda \mapsto \tau_n(\lambda)$ with respect to λ. This derivative exists, see also Exercise 5.5 below.

Exercise 5.5. *Show that $\lambda \mapsto \tau_n(\lambda)$ is differentiable. Hint: $\tau_n(\lambda)$ is a polynomial of bounded degree in λ.*

Fix $\lambda = 1 - n^{-1/3}$, and note that

$$\tau_n(1) = \tau_n(\lambda) + \int_\lambda^1 \tau'_n(\alpha)d\alpha, \tag{5.1.37}$$

where $\tau'_n(\lambda)$ denotes the derivative of $\tau_n(\lambda)$ with respect to λ.

Since $|C(v)|$ is stochastically smaller than the total progeny T of a binomial branching process with parameters n and $p = \lambda/n$, we have that by Theorem 2.12, $\chi_n(\lambda) \leq \mathbb{E}_{n,p}[T]$. By Theorem 3.5 (see also Exercise 4.9),

$$\chi_n(\lambda) \leq \mathbb{E}_{n,p}[T] = \frac{1}{1-\lambda} = n^{1/3}. \tag{5.1.38}$$

Using (5.1.35), we conclude that

$$\tau_n(\lambda) = \frac{1}{n-1}(\chi_n(\lambda) - 1) \leq \frac{n^{1/3}-1}{n-1} \leq n^{-2/3}. \tag{5.1.39}$$

This bounds the first term in (5.1.37). For the second term we will make use of a bound on the derivative of $\lambda \mapsto \tau_n(\lambda)$, which is formulated in the following lemma:

Lemma 5.4. *There exists a constant $C_\tau > 0$ independent of a, λ and n such that, or all $\lambda \leq 1$, $n \geq 1$ and $a < 1$,*

$$\tau'_n(\lambda) \leq an^{-2/3}\chi_n(1) + \frac{C_\tau}{a}n^{-1/3}. \tag{5.1.40}$$

Before proving Lemma 5.4, we complete the proof of Proposition 5.3 subject to Lemma 5.4:

Proof of Proposition 5.3 subject to Lemma 5.4. Substituting the bound in Lemma 5.4 into (5.1.37) and using (5.1.39), we obtain

$$\tau_n(1) \leq n^{-2/3} + \frac{a}{n}\chi_n(1) + \frac{C_\tau}{a}n^{-2/3}. \tag{5.1.41}$$

For n sufficiently large and for $a = \frac{1}{2}$, and by (5.1.35), we have $\frac{a}{n}\chi_n(1) \leq 3\tau_n(1)/4$. Thus we obtain, for n sufficiently large,

$$\tau_n(1) \leq \frac{3}{4}\tau_n(1) + (2C_\tau + 1)n^{-2/3}, \tag{5.1.42}$$

so that, again for n sufficiently large,

$$\tau_n(1) \leq (8C_\tau + 4)n^{-2/3}. \tag{5.1.43}$$

This completes the proof of (5.1.36), with $K = 8C_\tau + 4$, and (5.1.36) in turn implies Proposition 5.3.   □

Proof of Lemma 5.4. We need to bound

$$\tau'_n(\lambda) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}[\tau_n(\lambda + \varepsilon) - \tau_n(\lambda)]. \tag{5.1.44}$$

We use the coupling of all random graphs $\mathrm{ER}(n, p)$ for all $p \in [0, 1]$ in Section 4.1.1, which we briefly recall here. For this coupling, we take $n(n-1)/2$ independent uniform random variables U_{st} for each edge st. For fixed λ, we declare an edge st to be λ-occupied when $U_{st} \leq \lambda/n$.

The above coupling shows that the number of occupied bonds increases when λ increases. We recall that an event is *increasing* when, if the event occurs for a given set of occupied edges, it remains to hold when we make some more edges occupied. For example, the event $\{1 \longleftrightarrow 2\}$ that there exists an

occupied path from 1 to 2, is an increasing event. As a consequence of the above coupling, we obtain that $\lambda \mapsto \tau_n(\lambda)$ is increasing, and that

$$\tau_n(\lambda + \varepsilon) - \tau_n(\lambda) = \mathbb{P}(1 \xleftrightarrow{\lambda+\varepsilon} 2, 1 \overset{\lambda}{\nleftrightarrow} 2), \tag{5.1.45}$$

<div style="text-align:right">e:dertau1</div>

where we write $1 \xleftrightarrow{\lambda} 2$ for the event that 1 is connected to 2 in the λ-occupied edges.

Equation (5.1.45) implies that there must be at least one edge that is $(\lambda + \varepsilon)$-occupied, but not λ-occupied. When $\varepsilon \downarrow 0$, this edge becomes *unique* with high probability, since the probability that there are at least *two* of such edges is bounded by ε^2.

The probability that a given edge is $(\lambda + \varepsilon)$-occupied, but not λ-occupied is equal to ε/n, of which the factor ε is canceled by the factor $\frac{1}{\varepsilon}$ in (5.1.44). Moreover, this edge, which we denote by st, must be such that $1 \overset{\lambda}{\nleftrightarrow} 2$, but if we turn st occupied, then $1 \xleftrightarrow{\lambda} 2$ does occur. Thus, we must have that $1 \xleftrightarrow{\lambda} s$ and $2 \xleftrightarrow{\lambda} t$, but $1 \overset{\lambda}{\nleftrightarrow} 2$. Therefore, we obtain the remarkable identity

$$\tau_n'(\lambda) = \frac{1}{n} \sum_{st} \mathbb{P}_\lambda(1 \longleftrightarrow s, 2 \longleftrightarrow t, 1 \nleftrightarrow 2). \tag{5.1.46}$$

<div style="text-align:right">e:russo</div>

We can perform the sums over s and t to obtain

$$\tau_n'(\lambda) = \frac{1}{n} \mathbb{E}_\lambda \Big[\sum_{st} \mathbb{1}_{\{1 \longleftrightarrow s\}} \mathbb{1}_{\{2 \longleftrightarrow t\}} \mathbb{1}_{\{1 \nleftrightarrow 2\}} \Big]$$

$$= \frac{1}{n} \mathbb{E}_\lambda \big[|C(1)||C(2)| \mathbb{1}_{\{1 \nleftrightarrow 2\}} \big]. \tag{5.1.47}$$

<div style="text-align:right">e:russores</div>

We condition on $|C(1)| = l$ and $1 \overset{2}{\nleftrightarrow}$, and note that

$$\mathbb{E}_\lambda \big[|C(2)| \mid |C(1)| = l, 1 \nleftrightarrow 2 \big] = \chi_{n-l}(\lambda_{n,l}), \tag{5.1.48}$$

where we write $\lambda_{n,l} = \lambda \frac{n-l}{n}$. Therefore, we arrive at

$$\tau_n'(\lambda) \leq \frac{1}{n} \sum_{l=1}^{n} l \mathbb{P}_\lambda(|C(1)| = l) \chi_{n-l}(\lambda_{n,l}). \tag{5.1.49}$$

<div style="text-align:right">e:russobd</div>

We split the sum over l between $l \leq a^2 n^{2/3}$ and $l > a^2 n^{2/3}$. For $l \leq a^2 n^{2/3}$, we use that

$$\chi_{n-l}(\lambda_{n,l}) \leq \chi_n(\lambda) \leq \chi_n(1), \tag{5.1.50}$$

since $l \mapsto \chi_{n-l}(\lambda_{n,l})$ is decreasing.

Exercise 5.6. *Prove that $l \mapsto \chi_{n-l}(\lambda_{n,l})$ is non-increasing and decreasing for $\lambda > 0$. Hint: $\chi_{n-l}(\lambda \frac{n-l}{n})$ is equal to the expected cluster size in the random graph $\mathrm{ER}(n - l, \lambda/n)$.*

We conclude that

$$\tau_n'(\lambda) \leq \frac{1}{n} \chi_n(1) \sum_{l \leq a^2 n^{2/3}} l \mathbb{P}_\lambda(|C(1)| = l) + \frac{1}{n} \sum_{l > a^2 n^{2/3}}^{n} l \mathbb{P}_\lambda(|C(1)| = l) \chi_{n-l}(\lambda_{n,l}). \tag{5.1.51}$$

<div style="text-align:right">e:inttauder1</div>

For $l > a^2 n^{2/3}$, we use that $\chi_n(\lambda) \leq (1 - \lambda)^{-1}$ (compare to the argument in (5.1.38)), so that,

$$\chi_{n-l}(\lambda_{n,l}) \leq \frac{1}{1 - \lambda_{n,l}} = \frac{n}{l} \frac{1}{\frac{n}{l}(1 - \lambda) + \lambda} \leq \frac{n}{l}, \tag{5.1.52}$$

since $n/l \geq 1$. Therefore,

$$\tau'_n(\lambda) \leq \frac{1}{n} \chi_n(1) \sum_{l \leq a^2 n^{2/3}} l \mathbb{P}_\lambda(|C(1)| = l) + \sum_{l > a^2 n^{2/3}}^{n} \mathbb{P}_\lambda(|C(1)| = l). \tag{5.1.53}$$

<div style="text-align: right;"></div>

We bound, using Proposition 5.2,

$$\begin{aligned}
\sum_{l \leq a^2 n^{2/3}} l \mathbb{P}_\lambda(|C(1)| = l) &= \sum_{l \leq a^2 n^{2/3}} \sum_{i=1}^{l} \mathbb{P}_\lambda(|C(1)| = l) \\
&\leq \sum_{i \leq a^2 n^{2/3}} \sum_{l=i}^{\infty} \mathbb{P}_\lambda(|C(1)| = l) \\
&= \sum_{i \leq a^2 n^{2/3}} \mathbb{P}_\lambda(|C(1)| \geq i) \\
&\leq \sum_{i \leq a^2 n^{2/3}} \mathbb{P}_1(|C(1)| \geq i) \\
&\leq \sum_{i \leq a^2 n^{2/3}} \frac{c_2}{\sqrt{i}} \leq Can^{1/3},
\end{aligned} \tag{5.1.54}$$

where we use that c_2 in Proposition 5.2 is independent of r. Furthermore, we again use Proposition 5.2 to bound

$$\sum_{l > a^2 n^{2/3}}^{n} \mathbb{P}_\lambda(|C(1)| = l) = P_{\geq a^2 n^{2/3}}(\lambda) \leq P_{\geq a^2 n^{2/3}}(1) \leq \frac{C}{a} n^{-1/3}. \tag{5.1.55}$$

Substitution of (5.1.54)–(5.1.55) into (5.1.53) proves

$$\tau'_n(\lambda) \leq Can^{-2/3} \chi_n(1) + \frac{C}{a} n^{-1/3}. \tag{5.1.56}$$

Replacing Ca by a, this is equivalent to

$$\tau'_n(\lambda) \leq an^{-2/3} \chi_n(1) + \frac{C^2}{a} n^{-1/3}, \tag{5.1.57}$$

which, in turn, is equivalent to (5.1.40). $\qquad \square$

Exercise 5.7. * *Prove that*

$$\frac{\partial}{\partial \lambda} \chi(\lambda) \leq \chi(\lambda)^2, \tag{5.1.58}$$

and use this inequality to deduce that, for all $\lambda \leq 1$,

$$\chi(\lambda) \geq \frac{1}{\chi(1)^{-1} + (\lambda - 1)}. \tag{5.1.59}$$

5.2 Connectivity threshold

In this section, we investigate the connectivity threshold for the Erdős-Rényi random graph. As we can see in Theorem 4.9, for every $1 < \lambda < \infty$, the largest cluster for the Erdős-Rényi Random Graph when $p = \lambda/n$ is $\zeta_\lambda n(1 + o(1))$, where $\zeta_\lambda > 0$ is the survival probability of a Poisson branching process with parameter λ. Since extinction is certain when the root has no offspring, we have

$$\zeta_\lambda \leq 1 - \mathbb{P}^*(Z_1^* = 0) = 1 - e^{-\lambda} < 1. \tag{5.2.1}$$

Therefore, the Erdős-Rényi random graph with edge probability $p = \lambda/n$ is with high probability *disconnected* for each fixed $\lambda < \infty$. Here, we use the terminology "with high probability" to denote an event of which the probability tends to 1. We now investigate the threshold for connectivity for an appropriate choice $\lambda = \lambda_n \to \infty$. Theorem 5.5 and its extension, Theorem 5.8, were first proved in [87].

Theorem 5.5 (Connectivity Threshold). *For $\lambda - \log n \to \infty$, the Erdős-Rényi random graph is with high probability connected, while for $\lambda - \log n \to -\infty$, the Erdős-Rényi random graph is with high probability disconnected.*

In the proof, we investigate the number of isolated vertices. Define

$$Y = \sum_{i=1}^{n} I_i, \qquad \text{where} \qquad I_i = \mathbb{1}_{\{|C(i)|=1\}} \tag{5.2.2}$$

for the number of isolated vertices. Clearly, when $Y \geq 1$, then there exists at least one isolated vertex, so that the graph is disconnected. Remarkably, it turns out that when there is no isolated vertex, i.e., when $Y = 0$, then the random graph is also with high probability connected. See Proposition 5.7 below for the precise formulation of this result. By Proposition 5.7, we need to investigate the probability that $Y \geq 1$. In the case where $|\lambda - \log n| \to \infty$, we make use of the Markov and Chebychev inequality (Theorems 2.14 and 2.15) combined with a first and second moment argument using a variance estimate in Proposition 5.6. We will extend the result to the case that $\lambda = \log n + t$, in which case we need a more precise result in Theorem 5.8 below. The main ingredient to the proof of Theorem 5.8 is to show that, for $\lambda = \log n + t$, Y converges to a Poisson random variable with parameter e^{-t} when $n \to \infty$.

To prove that $Y \geq 1$ with high probability when $\lambda - \log n \to -\infty$, and $Y = 0$ with high probability when $\lambda - \log n \to \infty$ we use the Markov inequality (Theorem 2.14). We make use of an estimate on the mean and variance of Y:

Proposition 5.6. *For every $\lambda \leq n/2$,*

$$\mathbb{E}_\lambda[Y] = ne^{-\lambda}(1 + O(e^{\lambda^2}n)), \tag{5.2.3}$$

and, for every $\lambda \leq n$,

$$\mathrm{Var}_\lambda(Y) \leq \mathbb{E}_\lambda[Y] + \frac{\lambda}{n - \lambda}\mathbb{E}_\lambda[Y]^2. \tag{5.2.4}$$

Proof. Since $|C(i)| = 1$ precisely when all edges emanating from i are vacant, we have, using $1 - x \leq e^{-x}$,

$$\mathbb{E}_\lambda[Y] = n\mathbb{P}_\lambda(|C(1)| = 1) = n(1 - \frac{\lambda}{n})^{n-1} \leq ne^{-\lambda}e^{\frac{\lambda}{n}}. \tag{5.2.5}$$

Also, using that $1 - x \geq e^{-x-x^2}$ for $0 \leq x \leq \frac{1}{2}$, we obtain

$$\mathbb{E}_\lambda[Y] = n\mathbb{P}_\lambda(|C(1)| = 1) \geq ne^{-(n-1)\frac{\lambda}{n}(1+\frac{\lambda}{n})}$$
$$\geq ne^{-\lambda(1+\frac{\lambda}{n})} = ne^{-\lambda}e^{-\frac{\lambda^2}{n}}. \tag{5.2.6}$$

This proves (5.2.3).

To prove (5.2.4), we use the exchangeability of the vertices to compute

$$\mathbb{E}_\lambda[Y^2] = n\mathbb{P}_\lambda(|C(1)| = 1) + n(n-1)\mathbb{P}_\lambda(|C(1)| = 1, |C(2)| = 1). \tag{5.2.7}$$

Therefore, we obtain

$$\text{Var}_\lambda(Y) = n\big[\mathbb{P}_\lambda(|C(1)| = 1) - \mathbb{P}_\lambda(|C(1)| = 1, |C(2)| = 1)\big]$$
$$+ n^2\big[\mathbb{P}_\lambda(|C(1)| = 1, |C(2)| = 1) - \mathbb{P}_\lambda(|C(1)| = 1)^2\big]. \tag{5.2.8}$$

The first term is bounded above by $\mathbb{E}_\lambda[Y]$. The second term can be computed by using (5.2.5), together with

$$\mathbb{P}_\lambda(|C(1)| = 1, |C(2)| = 1) = (1 - \frac{\lambda}{n})^{2n-3}. \tag{5.2.9}$$

Therefore, by (5.2.5) and (5.2.8), we obtain

$$\mathbb{P}_\lambda(|C(1)| = 1, |C(2)| = 1) - \mathbb{P}_\lambda(|C(1)| = 1)^2 = \mathbb{P}_\lambda(|C(1)| = 1)^2\big[(1 - \frac{\lambda}{n})^{-1} - 1\big]$$
$$= \frac{\lambda}{n(1 - \frac{\lambda}{n})}\mathbb{P}_\lambda(|C(1)| = 1)^2. \tag{5.2.10}$$

We conclude that

$$\text{Var}_\lambda(Y) \leq \mathbb{E}_\lambda[Y] + \frac{\lambda}{n - \lambda}\mathbb{E}_\lambda[Y]^2. \tag{5.2.11}$$

$\square$

Proposition 5.7. *For all $0 \leq \lambda \leq n$,*

$$\mathbb{P}_\lambda\big(\text{ER}(n, \lambda/n) \text{ connected}\big) \leq \mathbb{P}_\lambda(Y = 0). \tag{5.2.12}$$

Moreover, if there exists an $a > 1/2$ such that $\lambda \geq a\log n$, then, for $n \to \infty$,

$$\mathbb{P}_\lambda\big(\text{ER}(n, \lambda/n) \text{ connected}\big) = \mathbb{P}_\lambda(Y = 0) + o(1). \tag{5.2.13}$$

Proof. We use that

$$\mathbb{P}_\lambda\big(\text{ER}(n, \lambda/n) \text{ disconnected}\big) = \mathbb{P}_\lambda(Y > 0) + \mathbb{P}_\lambda\big(\text{ER}(n, \lambda/n) \text{ disconnected}, Y = 0\big). \tag{5.2.14}$$

This immediately proves (5.2.12).

To prove (5.2.13), we make use of a computation involving *trees*. For $k = 2, \ldots, n$, we denote by X_k the number of occupied trees of size equal to k on the vertices $1, \ldots, n$ that cannot be extended to a tree of larger size. Thus, each tree which is counted in X_k has size precisely equal to k, and when we denote it's vertices by $v_1, \ldots, v_k$, then all the edges between v_i and $v \notin \{v_1, \ldots, v_k\}$ are vacant. Moreover, there are precisely $k - 1$ occupied edges between the v_i that are such that these occupied edges form a tree. Note that a connected component of size k can contain more than one tree of size k, since the connected component may contain cycles. Note furthermore that, when $\text{ER}(n, \lambda/n)$ is disconnected, but $Y = 0$, there must be a $k \in \{2, \ldots, n/2\}$ for which $X_k \geq 1$.

We conclude from Boole's inequality and the Markov inequality (Theorem 2.14) that

$$\mathbb{P}_\lambda\big(\text{ER}(n, \lambda/n) \text{ disconnected}, Y = 0\big) \leq \sum_{k=2}^{n/2} \mathbb{P}_\lambda(X_k \geq 1) \leq \sum_{k=2}^{n/2} \mathbb{E}_\lambda[X_k]. \tag{5.2.15}$$

Therefore, we need to bound $\mathbb{E}_\lambda[X_k]$. For this, we note that there are $\binom{n}{k}$ ways of choosing k vertices, and, by Cayley's Theorem 3.14, there are k^{k-2} labeled trees containing k vertices. Therefore,

$$\mathbb{E}_\lambda[X_k] = \binom{n}{k} k^{k-2} q_k, \tag{5.2.16}$$

where q_k is the probability that any tree of size k is occupied and all the edges from the tree to other vertices are vacant, which is equal to

$$q_k = \left(\frac{\lambda}{n}\right)^{k-1}\left(1 - \frac{\lambda}{n}\right)^{k(n-k)} \le \left(\frac{\lambda}{n}\right)^{k-1} e^{-\lambda k(n-k)/n}. \tag{5.2.17}$$

We conclude that

$$\mathbb{E}_\lambda[X_k] \le n\lambda^{k-1}\frac{k^{k-2}}{k!}e^{-\frac{\lambda}{n}k(n-k)}. \tag{5.2.18}$$

If we further use that $k! \ge k^k e^{-k}$, and also use that $\lambda \ge 1$, then we arrive at

$$\mathbb{E}_\lambda[X_k] \le n(e\lambda)^k\frac{1}{k^2}e^{-\frac{\lambda}{n}k(n-k)}. \tag{5.2.19}$$

Since $\lambda \mapsto e^{-\frac{\lambda}{n}k(n-k)}$ is decreasing in λ, it suffices to investigate $\lambda = a\log n$ for some $a > 1/2$. For $k \in \{2,3,4\}$, for $\lambda = a\log n$ for some $a > 1/2$,

$$\mathbb{E}_\lambda[X_k] \le n(e\lambda)^4 e^{-\lambda k}e^{o(1)} = o(1). \tag{5.2.20}$$

For all $k \le n/2$, we bound $k(n-k) \ge kn/2$, so that

$$\mathbb{E}_\lambda[X_k] \le n(e\lambda e^{-\lambda/2})^k. \tag{5.2.21}$$

As a result, for $\lambda = a\log n$ with $a > 1/2$, and all $k \ge 5$,

$$\mathbb{E}_\lambda[X_k] \le n^{1-k/4} = o(1). \tag{5.2.22}$$

We conclude that

$$\mathbb{P}_\lambda\Big(\mathrm{ER}(n,\lambda/n) \text{ disconnected}, Y = 0\Big) \le \sum_{k=2}^{n/2}\mathbb{E}_\lambda[X_k] = o(1). \tag{5.2.23}$$

$\square$

Proof of Theorem 5.5. The proof makes essential use of Proposition 5.7.

We start by proving that for $\lambda - \log n \to -\infty$, the Erdős-Rényi random graph is with high probability disconnected. We use (5.2.3) to note that

$$\mathbb{E}_\lambda[Y] = ne^{-\lambda}(1 + o(1)) = e^{-\lambda + \log n}(1 + o(1)) \to \infty. \tag{5.2.24}$$

By the Chebychev inequality (Theorem 2.15),

$$\mathbb{P}_\lambda(Y = 0) \le \frac{\mathbb{E}_\lambda[Y] + \frac{2\lambda}{n}\mathbb{E}_\lambda[Y]^2}{\mathbb{E}_\lambda[Y]^2} = \mathbb{E}_\lambda[Y]^{-1} + \frac{2\lambda}{n} \to 0. \tag{5.2.25}$$

Proposition 5.7 completes the proof that for $\lambda - \log n \to -\infty$, the Erdős-Rényi random graph is with high probability disconnected.

When $\lambda - \log n \to \infty$, then, by the Markov inequality (Theorem 2.14) and (5.2.5),

$$\mathbb{P}_\lambda(Y = 0) = 1 - \mathbb{P}_\lambda(Y \ge 1) \ge 1 - \mathbb{E}_\lambda[Y] \ge 1 - ne^{-\lambda}O(1) \to 1. \tag{5.2.26}$$

Therefore, the claim again follows from Proposition 5.7. $\square$

5.2.1   Critical window for connectivity*

In this section, we investigate the critical window for connectivity, by considering connectivity of $\mathrm{ER}(n, \lambda/n)$ when $\lambda = \log n + t$ for *fixed* $t \in \mathbb{R}$. The main result in this section is as follows:

Theorem 5.8 (Critical Window for Connectivity). *For $\lambda = \log n + t \to \infty$, the Erdős-Rényi random graph is connected with probability $e^{-e^{-t}}(1 + o(1))$.*

Proof. In the proof, we again rely on Proposition 5.7. We fix $\lambda = \log n + t$.

We prove a Poisson approximation for Y that reads that $Y \xrightarrow{d} Z$, where Z is a Poisson random variable with parameter

$$\lim_{n\to\infty} \mathbb{E}_\lambda[Y] = e^{-t}, \tag{5.2.27}$$

where we recall (5.2.3). Therefore, convergence in distribution implies that

$$\mathbb{P}_\lambda(Y = 0) = e^{-\lim_{n\to\infty}\mathbb{E}_\lambda[Y]} + o(1) = e^{-e^{-t}} + o(1), \tag{5.2.28}$$

and the result follows by Proposition 5.7.

In order to show that $Y \xrightarrow{d} Z$, we use Theorem 2.4 and Theorem 2.5, so that it suffices to prove that, recalling that $I_i = \mathbb{1}_{\{|C(i)|=1\}}$, we have that, for all $r \geq 1$,

$$\lim_{n\to\infty} \mathbb{E}[(Y)_r] = \lim_{n\to\infty} \sum_{i_1,\dots,i_r}^* \mathbb{P}_\lambda(I_{i_1} = \cdots = I_{i_r} = 1) = e^{-tr}, \tag{5.2.29}$$

where the sum ranges over all $i_1, \dots, i_r \in \{1, \dots, n\}$ which are *distinct*. By exchangeability of the vertices, $\mathbb{P}_\lambda(I_{i_1} = \cdots = I_{i_r} = 1)$ is independent of the precise choice of the indices $i_1, \dots, i_r$, so that

$$\mathbb{P}_\lambda(I_{i_1} = \cdots = I_{i_r} = 1) = \mathbb{P}_\lambda(I_1 = \cdots = I_r = 1). \tag{5.2.30}$$

Using that there are $n(n-1)\cdots(n-r+1)$ distinct choices of $i_1, \dots, i_r \in \{1, \dots, n\}$, we arrive at

$$\mathbb{E}[(Y)_r] = \frac{n!}{(n-r)!}\mathbb{P}_\lambda(I_1 = \cdots = I_r = 1). \tag{5.2.31}$$

The event $\{I_1 = \cdots = I_r = 1\}$ occurs precisely when all edges st with $s \in \{1, \dots, r\}$ and $t \in \{1, \dots, n\}$ are vacant. There are $r(r-1)/2 + r(n-r) = r(2n-r-1)/2$ of such edges since these edges are all independent, we arrive at

$$\mathbb{P}_\lambda(I_1 = \cdots = I_r = 1) = (1 - \frac{\lambda}{n})^{r(2n-r-1)/2} = (1 - \frac{\lambda}{n})^r(1 - \frac{\lambda}{n})^{-r(r+1)/2} = n^{-r}\mathbb{E}_\lambda[Y]^r(1 + o(1)). \tag{5.2.32}$$

We conclude that

$$\lim_{n\to\infty} \mathbb{E}[(Y)_r] = \lim_{n\to\infty} \frac{n!}{(n-r)!}\mathbb{E}_\lambda[Y]^r = e^{-tr}, \tag{5.2.33}$$

where we use (5.2.27). This completes the proof of Theorem 5.8.   □

Exercise 5.8. *Prove directly that the second moment of Y converges to the second moment of Z, by using* (5.2.10).

5.3 Degree sequence of the Erdős-Rényi random graph

As described in Chapter 1, the degree sequences of various real networks obey power laws. Therefore, in this section, we investigate the degree sequence of the Erdős-Rényi random graph for fixed $\lambda > 0$. In order to be able to state the result, we first introduce some notation. We write

$$p_k = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k \geq 0, \tag{5.3.1}$$

for the Poisson distribution with parameter λ. Let D_i denote the degree of vertex i and write

$$P_k^{(n)} = \frac{1}{n}\sum_{i=1}^n \mathbb{1}_{\{D_i = k\}} \tag{5.3.2}$$

for the empirical degree distribution of the degrees. The main result is as follows:

Theorem 5.9 (Degree sequence of the Erdős-Rényi random graph). *Fix $\lambda > 0$. Then, for every ε_n such that $\sqrt{n}\varepsilon_n \to \infty$,*

$$\mathbb{P}_\lambda\big(\max_k |p_k^{(n)} - p_k| \geq \varepsilon_n\big) \to 0. \tag{5.3.3}$$

We now prove Theorem 5.9:

Proof of Theorem 5.9. We note that

$$\mathbb{E}_\lambda[P_k^{(n)}] = \mathbb{P}_\lambda(D_1 = k) = \binom{n-1}{k}\Big(\frac{\lambda}{n}\Big)^k\Big(1 - \frac{\lambda}{n}\Big)^{n-k-1}. \tag{5.3.4}$$

Furthermore,

$$\sum_{k=0}^\infty \Big| p_k - \binom{n-1}{k}(\frac{\lambda}{n})^k(1 - \frac{\lambda}{n})^{n-k-1}\Big| = \sum_{k=0}^\infty |\mathbb{P}(X^* = k) - \mathbb{P}(X_n = k)|, \tag{5.3.5}$$

where X^* is a Poisson random variable with mean λ, and X_n is a binomial random variable with parameters $n - 1$ and $p = \lambda/n$. We will use a coupling argument to bound this difference. Indeed, we let X denote a binomial random variable with parameters n and $p = \lambda/n$. Since we can couple X and X_n such that the probability that they are different is precisely equal to $p = \lambda/n$, we obtain

$$\sum_{k=0}^\infty |\mathbb{P}(X_n = k) - \mathbb{P}(X = k)| \leq \frac{\lambda}{n}. \tag{5.3.6}$$

Therefore, for all $k \geq 0$,

$$\sum_{k=0}^\infty |\mathbb{P}(X^* = k) - \mathbb{P}(X_n = k)| \leq \frac{\lambda}{n} + \mathbb{P}(X^* \neq X) \leq \frac{\lambda + \lambda^2}{n}, \tag{5.3.7}$$

where we have also used Theorem 2.9. Since $\frac{\lambda + \lambda^2}{n} \leq \frac{\varepsilon_n}{2}$, we have just shown that $\sum_{k=0}^\infty |p_k - \mathbb{E}_\lambda[P_k^{(n)}]| \leq \varepsilon_n/2$ for n sufficiently large. Thus, it suffices to prove that

$$\mathbb{P}_\lambda\Big(\sum_k |P_k^{(n)} - \mathbb{E}_\lambda[P_k^{(n)}]| \geq \frac{\varepsilon_n}{2}\Big) = o(1). \tag{5.3.8}$$

For this, we use Boole's inequality to bound

$$\mathbb{P}_\lambda\Big(\max_k |P_k^{(n)} - \mathbb{E}_\lambda[P_k^{(n)}]| \geq \frac{\varepsilon_n}{2}\Big) \leq \sum_{k=1}^{\infty} \mathbb{P}_\lambda(|P_k^{(n)} - \mathbb{E}_\lambda[P_k^{(n)}]| \geq \frac{\varepsilon_n}{2}). \tag{5.3.9}$$

By the Chebychev inequality (Theorem 2.15),

$$\mathbb{P}_\lambda(|P_k^{(n)} - \mathbb{E}_\lambda[P_k^{(n)}]| \geq \frac{\varepsilon_n}{2}) \leq 4\varepsilon_n^{-2}\mathrm{Var}_\lambda(P_k^{(n)}). \tag{5.3.10}$$

We then note that

$$\mathrm{Var}_\lambda(P_k^{(n)}) = \frac{1}{n}\Big[\mathbb{P}_\lambda(D_1 = k) - \mathbb{P}_\lambda(D_1 = k)^2\Big]$$
$$+ \frac{n-1}{n}\Big[\mathbb{P}_\lambda(D_1 = D_2 = k) - \mathbb{P}_\lambda(D_1 = k)^2\Big]. \tag{5.3.11}$$

We next use a coupling argument. We let X_1, X_2 be two independent $\mathrm{BIN}(n-2, \lambda/n)$ random variables, and I_1, I_2 two independent Bernoulli random variables with success probability λ/n. Then, the law of (D_1, D_2) is the same as the one of $(X_1 + I_1, X_2 + I_1)$ while $(X_1 + I_1, X_2 + I_2)$ are two independent copies of the D_1. Then,

$$\mathbb{P}_\lambda(D_1 = D_2 = k) = \mathbb{P}_\lambda\Big((X_1 + I_1, X_2 + I_1) = (k, k)\Big), \tag{5.3.12}$$

$$\mathbb{P}_\lambda(D_1 = k)^2 = \mathbb{P}_\lambda\Big((X_1 + I_1, X_2 + I_2) = (k, k)\Big), \tag{5.3.13}$$

so that

$$\mathbb{P}_\lambda(D_1 = D_2 = k) - \mathbb{P}_\lambda(D_1 = k)^2 \leq \mathbb{P}_\lambda\Big((X_1 + I_1, X_2 + I_1) = (k, k), (X_1 + I_1, X_2 + I_2) \neq (k, k)\Big). \tag{5.3.14}$$

When $(X_1 + I_1, X_2 + I_1) = (k, k)$, but $(X_1 + I_1, X_2 + I_2) \neq (k, k)$, we must have that $I_1 \neq I_2$. If $I_1 = 1$, then $I_2 = 0$ and $X_2 = k$, while, if $I_1 = 0$, then $I_2 = 1$ and $X_1 = k$. Therefore, since X_1 and X_2 have the same distribution,

$$\mathbb{P}_\lambda(D_1 = D_2 = k) - \mathbb{P}_\lambda(D_1 = k)^2 \leq \frac{2\lambda}{n}\mathbb{P}_\lambda(X_1 = k). \tag{5.3.15}$$

We conclude from (5.3.11) that

$$\mathrm{Var}_\lambda(P_k^{(n)}) \leq \frac{(2\lambda + 1)}{n}\mathbb{P}_\lambda(X_1 = k), \tag{5.3.16}$$

so that, by (5.3.9)–(5.3.10),

$$\mathbb{P}_\lambda\big(\max_k |P_k^{(n)} - \mathbb{E}_\lambda[P_k^{(n)}]| \geq \varepsilon_n/2\big) \leq \frac{4(2\lambda + 1)}{\varepsilon_n^2 n}\sum_{k=0}^{\infty}\mathbb{P}_\lambda(X_1 = k)$$
$$= \frac{4(2\lambda + 1)}{\varepsilon_n^2 n} \to 0. \tag{5.3.17}$$

This completes the proof of Theorem 5.9.   □

In Chapter 6 below, we give an alternative proof of Theorem 5.9, allowing for weaker bounds on ε_n. In that proof, we use that the Erdős-Rényi random graph is a special case of the generalized random graph with equal weights. See Theorem 6.7 below.

5.4 Notes and discussion

Notes on Section 5.1. There are many more results known about the scaling of large connected components in the random graph $ER(n,p)$ when $p = \frac{1}{n}$. The most complete picture is given by Aldous in [13]. Aldous proves that the connected components ordered in size and multiplied by $n^{-2/3}$ converge as a sequence of random variables. In particular, his result implies that the distribution of $|C_{\max}|n^{-2/3}$ at $p = \frac{1}{n}$ converges in distribution to some non-trivial distribution function. Aldous also considers the number of cycles in the large connected components, and proves that these converge as well.

There have been many papers proving preliminary versions of the result by Aldous. See in particular [122, 151], as well as the books [40, 124] for more detailed references.

We finally list some more recent results. In [125], a point process description is given of the sizes and number of components of size $\varepsilon n^{2/3}$. In [181], an explicit, yet involved, description is given for the distribution of the limit of $|C_{\max}|n^{-2/3}$. The proof makes use of generating functions, and the relation between the largest connected component and the number of labeled graphs with a given complexity l. Here, the complexity of a graph is its number of edges minus its number of vertices. Relations between Erdős-Rényi random graphs and the problem of counting the number of labeled graphs has received considerable attention, see e.g. [39, 115, 150, 194, 206, 207] and the references therein. Consequences of the result by Pittel [181] are for example that the probability that $|C_{\max}|n^{-2/3}$ exceeds a for large a decays as $e^{-a^3/8}$ (in fact, the asymptotics are much stronger than this!), and for very small $a > 0$, the probability that $|C_{\max}|n^{-2/3}$ is smaller than a decays as $e^{-ca^{-3/2}}$ for some explicit constant $c > 0$. The bound on the upper tails of $|C_{\max}|n^{-2/3}$ is also proved in [162], and is valid for all n and a, with the help of relatively simple martingale arguments. In [162], the bound (5.1.14) is also explicitly proved.

The equality in (5.1.46) is a special example of *Russo's formula*, see [101]. Russo's Formula has played a crucial role in the study of percolation on general graphs, and states that for any increasing event E on $ER(n,p)$, we have that

$$\frac{\partial \mathbb{P}(E)}{\partial p} = \sum_{st} \mathbb{P}(st \text{ is pivotal for } E), \tag{5.4.1}$$

where we say that an edge st is *pivotal* for an increasing event E when the event E occurs in the (possibly modified) configuration of edges where st is turned occupied, and the event E does not occur in the (possibly modified) configuration of edges where st is turned vacant. See [6, 7, 25] for examples where pivotal edges play a crucial role.

Notes on Section 5.2. Connectivity of the Erdős-Rényi random graph was investigated in the early papers on the subject. In [87], versions of Theorems 5.5–5.8 were proved for $ER(n,M)$. Bollobás gives two separate proofs in [40, Pages 164-165].

Notes on Section 5.3. The degrees of Erdős-Rényi random graphs have attracted considerable attention. In particular, when ordering the degrees by size as $d_1 \geq d_2 \geq \cdots \geq d_n$, various properties have been shown, such as the fact that there is, with high probability, a *unique* vertex with degree d_1 [91]. See [38] or [40] for more details. The result on the degree sequence proved here is a weak consequence of the result in [120, Theorem 4.1], where even asymptotic normality was shown for the number of vertices with degree k, for all k simultaneously.

Intermezzo: Back to real networks I...

Theorem 5.9 shows that the degree sequence of the Erdős-Rényi random graph is close to a Poisson distribution with parameter λ. A Poisson distribution has thin tails, for example, its moment generating function is always finite. As a result, the Erdős-Rényi random graph cannot be used to model real networks where power law degree sequences are observed. Therefore, several related models have been proposed. In this intermezzo, we shall discuss three of them.

The first model is the so-called *generalized random graph* (GRG), and was first introduced in [53].[BriDeiMar-Lof05] In this model, each vertex $i \in \{1, \ldots, n\}$ receives a *weight W_i*. Given the weights, edges are present independently, but the occupation probabilities for different edges are not identical, but moderated by the weights of the vertices. Naturally, this can be done in several different ways. The most general version is presented in [43],[BolJanRio06] which we explain in detail in Chapter 9. In the generalized random graph, the edge probability of the edge between vertex i and j (conditionally on the weights $\{W_i\}_{i=1}^n$) is equal to

$$p_{ij} = \frac{W_i W_j}{L_n + W_i W_j}, \tag{I.1}$$

where the random variables $\{W_i\}_{i=1}^n$ are the *weights of the vertices,* and L_n is the total weight of all vertices given by

$$L_n = \sum_{i=1}^n W_i. \tag{I.2}$$

We shall assume that the weights $\{W_i\}_{i=1}^n$ are independent and identically distributed.

The second model is the *configuration model*, in which the degrees of the vertices are fixed. Indeed, we write D_i for the degree of vertex i, and let, similarly to (I.2), $L_n = \sum_{i=1}^n D_i$ denote the total degree. We assume that L_n is even. We will make a graph where vertex i has degree D_i. For this, we think of each vertex having D_i stubs attached to it. Two stubs can be connected to each other to form an edge. The configuration model is the model where all stubs are connected in a uniform fashion, i.e., where the stubs are uniformly matched.

The third model is the so-called *preferential attachment model*, in which the *growth of the random graph* is modeled by adding edges to the already existing graph in such a way that vertices with large degree are more likely to be connected to the newly added edges. See Chapter 8 for details.

All these models have in common that the degree sequence converges to some limiting distribution which can have various shapes, particularly including power laws. For the generalized random graph and the configuration model, this is proved in Chapter 6 and Chapter 7 respectively. For the preferential attachment models, we will defer this proof to Chapter 8. In Chapters 6–8, we shall focus on properties of the degree sequence of the random graphs involved. We shall study further properties, namely, the connected components and distances in these models, in Chapters 9–11, respectively.

In Chapters 6–8 we shall be interested in the properties of the *degree sequence of a graph*. A natural question is which sequences of numbers can occur as the degree sequence of a simple graph. A sequence $\{d_1, d_2, \ldots, d_n\}$ with $d_1 \leq d_2 \leq \cdots \leq d_n$ is called *graphic* if it is the degree sequence of a simple graph.

Thus, the question is which degree sequences are graphic? Erdős and Gallai [86] proved that a degree sequence $\{d_1, d_2, \ldots, d_n\}$ is graphic if and only if $\sum_{i=1}^{n} d_i$ is even and

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(k, d_i), \tag{I.3}$$

for each integer $k \leq n-1$. The fact that the total degree of a graph needs to be even is fairly obvious:

Exercise 5.9. *Show that for every graph, and d_j the degree of vertex j we have that $\sum_{j=1}^{n} d_j$ is even. This fact is sometimes called the handshake lemma.*

The necessity of (I.3) is relatively easy to see. The left side of (I.3) is the degree of the first k vertices. The first term on the right-hand side of (I.3) is the twice the maximal number of edges between the vertices in $\{1, \ldots, k\}$. The second term is a bound on the total degree of the vertices $\{1, \ldots, k\}$ coming from edges that connect to vertices in $\{k+1, \ldots, n\}$. The sufficiency is harder to see, see [59] for a simple proof of this fact, and [189] for seven different proofs. Arratia and Liggett [16] investigate the asymptotic probability that an i.i.d. sequence of n integer random variables is graphical, the result being in many cases equal to 0 or 1/2, at least when $\mathbb{P}(D \text{ even}) \neq 1$. The limit is equal to 0 when $\lim_{n \to \infty} n\mathbb{P}(D_i \geq n) = \infty$ and 1/2 when $\lim_{n \to \infty} n\mathbb{P}(D_i \geq n) = 0$. Interestingly, when $\lim_{n \to \infty} n\mathbb{P}(D_i \geq n) = c$ for some constant $c > 0$, then the set of limit points of the probability that $\{D_1, \ldots, D_n\}$ is graphical is a subset of $(0, 1/2)$. The proof is by verifying that (I.3) holds.

Chapter 6

Generalized random graphs

The *generalized random graph* is a model in which the equal edge probabilities of the Erdős-Rényi random graph are replaced by edge occupation statuses that are *conditionally* independent, given some vertex weights. These weights can be taken to be *random* or *fixed*, and both options have been considered in the literature. We shall see that this model gives rise to random graphs where the degrees can have power laws. More precisely, this happens precisely when the weights have an appropriate power law. As such, this is the simplest adaption of the Erdős-Rényi random graph having a power-law degree sequence.

This chapter is organised as follows. In Section 6.1, we introduce the model. In Section 6.2, we investigate the degree sequence of the generalized random graph with fixed weights, as well as a close brother of the generalized random graph, the so-called Chung-Lu model or expected degree model. In Section 6.3, we investigate the degree sequence of the generalized random graph. In Section 6.4, we study the generalized random graph with i.i.d. weights. In Section 6.5 we show that the generalized random graph, conditioned on its degrees, is a uniform random graph with these degrees. Finally, in Section 6.6, we introduce several more models of inhomogeneous random graphs similar to the generalized random graph that have been studied in the literature. In Chapter 9, we shall generalize such models for inhomogeneous random graphs, following [43] and discuss the results proved in this impressive paper. We close this chapter with notes and discussion in Section 6.7.

6.1   Introduction of the model

In the generalized random graph, each vertex has a weight associated to it. Edges are present independently given these weights, but the occupation probabilities for edges are *not* identical, but are rather moderated by the weights of the vertices. These weights can be *fixed* or *deterministic*. When the weights are themselves random variables, they introduce a *double randomness*: firstly there is the randomness introduced by the weights, and secondly there is the randomness introduced by the edge occupations, which are conditionally independent given the weights.

In the generalized random graph model with random weights, the edge probability of the edge between vertices i and j is equal to

$$p_{ij} = p_{ij}^{(\mathrm{GRG})} = \frac{w_i w_j}{l_n + w_i w_j}, \qquad (6.1.1)$$

where $\{w_i\}_{i=1}^n$ are the *weights of the vertices,* and L_n is the total weight of all vertices given by

$$l_n = \sum_{i=1}^n w_i. \qquad (6.1.2)$$

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We denote the resulting graph by $\mathrm{GRG}(n, \{w_i\}_{i=1}^n)$. To avoid possible difficulties with weights which are equal to 0, we shall assume that $w_i > 0$. Note that when, for a particular $i \in \{1, \ldots, n\}$, $w_i = 0$, then vertex i will be isolated with probability 1, and, therefore, we can omit i from the graph. Thus, the assumption that $w_i > 0$ can be made without loss of generality. Often, we shall assume that $\{w_i\}_{i=1}^n = \{W_i\}_{i=1}^n$ are independent and identically distributed, and in this case, the resulting graph is denoted by $\mathrm{GRG}(n, \{W_i\}_{i=1}^n)$. The great advantage of independent and identically distributed weights is that in the vertices in the resulting graph are, in distribution, the same. Therefore, the vertices are completely exchangeable.

In [53], L_n in (6.1.1) is replaced by n, which only leads to a minor change. Indeed, when the weights have finite mean denoted by μ, then $L_n = \mu n(1 + o(1))$, by the law of large numbers. If we would replace L_n by μn in (6.1.1), then

$$\frac{W_i W_j}{\mu n + W_i W_j} = \frac{\mu^{-1} W_i W_j}{n + \mu^{-1} W_i W_j}, \tag{6.1.3}$$

so that this change amounts to replacing W_i by $W_i/\sqrt{\mu}$. Therefore, at least on a heuristic level, there is hardly any difference between the definition of p_{ij} in (6.1.1), and the choice $p_{ij} = \frac{W_i W_j}{n + W_i W_j}$ in [53].

A special case of the generalized random graph is when we take $W_i \equiv \frac{n\lambda}{n-\lambda}$, in which case $p_{ij} = \lambda/n$ for all $i, j \in \{1, \ldots, n\}$. Therefore, the generalized random graph with constant weights is closely related to the Erdős-Rényi random graph, which makes the name generalized random graph appropriate.

Exercise 6.1. *Prove that $p_{ij} = \lambda/n$ when $W_i \equiv n\lambda/(n - \lambda)$.*

The generalized random graph is an adaptation of a model studied by Chung and Lu [60, 61, 62, 64, 149], in which the edge probabilities are given by $p_{ij} = \frac{w_i w_j}{l_n}$, where the (deterministic) weights are chosen appropriately. Naturally, here we need to impose that $\frac{w_i w_j}{l_n} \leq 1$ for all i, j, which is implied by the condition that $\max_i w_i^2 \leq l_n$.

We start by discussing the relation between the generalized random graph and the Chung-Lu model. In the generalized random graph, the empirical distribution F_n of the weights is given by

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{W_i \leq x\}}. \tag{6.1.4}$$

When the weights are independently and identically distributed with distribution function F, then it is well-known that this empirical distribution function is close to F (this is the Glivenko-Cantelli Theorem).

Naturally, the graph structure only depends on the empirical properties of the weights, rather then the weights of each of the vertices. Thus, the topology of the graph with weights $\{w_i\}_{i=1}^n$ is equal to the topology of the graph with weights $\{w_{(i)}\}_{i=1}^n$, where $w_{(i)}$ is the ith largest weight. We now fix

$$w_{(i)} = F^{-1}(1 - \frac{i}{n}), \tag{6.1.5}$$

where F^{-1} is the generalized inverse function defined in (2.3.4), where, by convention, we set $F^{-1}(0) = 0$.

Then, clearly, the number of weights bounded above by x is equal to

$$
\sum_{i=1}^{n} \mathbb{1}_{\{w_i \leq x\}} = \sum_{i=1}^{n} \mathbb{1}_{\{F^{-1}(1-\frac{i}{n}) \leq x\}}
$$

$$
= \sum_{i=1}^{n} \mathbb{1}_{\{1-\frac{i}{n} \leq F(x)\}}
$$

$$
= \sum_{i=1}^{n} \mathbb{1}_{\{\frac{i}{n} \geq 1-F(x)\}}
$$

$$
= \min\{n, \lfloor nF(x) \rfloor + 1\}. \tag{6.1.6}
$$

Exercise 6.2. *Prove that $\sum_{i=1}^{n} \mathbb{1}_{\{\frac{i}{n} \geq 1-F(x)\}} = \min\{n, \lfloor nF(x) \rfloor + 1\}$.*

We conclude that the empirical distribution $F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{w_i \leq x\}}$ of the weights in (6.1.5) is close to $F(x)$ for every $x \geq 0$. This suggests that the graph structure in both cases to be similar. For example, when, for $x \geq 1$,

$$
F(x) = 1 - x^{1-\tau} \tag{6.1.7}
$$

for some $\tau > 1$, then

$$
F^{-1}(1-x) = x^{-1/(\tau-1)}, \tag{6.1.8}
$$

so that

$$
w_{(i)} = \left(\frac{i}{n}\right)^{-1/(\tau-1)}. \tag{6.1.9}
$$

Clearly, the GRG can be studied both with fixed weight as well as with independent and identically distributed (i.i.d.) weights. The GRG with fixed weights is denoted by $\mathrm{GRG}(n, \{w_i\}_{i=1}^n)$, the GRG with i.i.d. weights by $\mathrm{GRG}(n, \{W_i\}_{i=1}^n)$. Both models have their own merits (see Section 6.7 for more details). In this chapter, we shall investigate both. In Section 6.2, we shall investigate fixed weights, and in Section 6.4, we study i.i.d. weights.

6.2 Generalized random graph with fixed weights

In this section, we study the generalized random graph with deterministic degrees. We start by investigating the degrees of vertices in $\mathrm{GRG}(n, \{w_i\}_{i=1}^n)$ in Section 6.2.1. The proof of the main result is in Section 6.5. Along the way, we prove an interesting property of $\mathrm{GRG}(n, \{w_i\}_{i=1}^n)$, namely, that $\mathrm{GRG}(n, \{w_i\}_{i=1}^n)$ conditioned on the degrees is a uniform simple graph with these degrees. We continue by studying the degree sequence in Section 6.3.1.

6.2.1 Degrees of the generalized random graph

In this section, we study the degrees of vertices in $\mathrm{GRG}(n, \{w_i\}_{i=1}^n)$. In order to state the main results, we start with some definitions. Given weights $\{w_i\}_{i=1}^n$, we let the probability that the edge ij is occupied be equal to p_{ij} in (6.1.1), and where we recall that $l_n = \sum_{i=1}^n w_i$. Sometimes, we write $w_i^{(n)} = w_i$ to indicate that the deterministic weights $\{w_i\}_{i=1}^n$ are allowed to depend on n. The occupation statuses of different edges are independent. We write $D_k = D_k^{(n)}$ for the degree of vertex k in the GRG. Thus, D_k is given by

$$
D_k = \sum_{j=1}^{n} X_{kj}, \tag{6.2.1}
$$

<div align="right"><code>e:Dk-GRG</code></div>

where X_{kj} is the indicator that the edge kj is occupied. By convention, we set $X_{ij} = X_{ji}$. The main result concerning the degrees is as follows:

Theorem 6.1 (Degree of GRG with fixed weights).

(a) *For deterministic weights $\{w_i\}_{i=1}^n = \{w_i^{(n)}\}_{i=1}^n$, there exists a Poisson random variable Z_k with parameter $w_k^{(n)}$, such that the degree D_k of vertex k satisfies*

$$\mathbb{P}(D_k \neq Z_k) \leq \frac{\left(w_k^{(n)}\right)^2}{l_n}\Big(1 + 2\sum_{j=1}^n \frac{\left(w_j^{(n)}\right)^2}{l_n}\Big). \qquad \text{(6.2.2)}$$

In particular, when $w_k = \lim_{n \to \infty} w_k^{(n)}$ exists, then D_k converges in distribution to a Poisson random variable with parameter w_k, whenever

$$\sum_{i=1}^n \left(w_i^{(n)}\right)^2 = o(l_n^2). \qquad \text{(6.2.3)}$$

(b) *When p_{ij} given by (6.1.1) are all such that $\lim_{n \to \infty} p_{ij} = 0$, the degrees $D_1, \ldots, D_m$ of vertices $1, \ldots, m$ are asymptotically independent.*

Before proving Theorem 6.1 , we state a consequence for the degree sequence when the weights are given by (6.1.5). To be able to state this consequence, we need the following definition:

Definition 6.2. *A random variable X has a mixed Poisson distribution with mixing distribution F when, for every $k \in \mathbb{N}$,*

$$\mathbb{P}(X = k) = \mathbb{E}[e^{-W}\frac{W^k}{k!}], \qquad \text{(6.2.4)}$$

where W is a random variable with distribution function F.

Corollary 6.3 (Degree of GRG with weights given by (6.1.5)). *When $\{w_i\}_{i=1}^n$ are weights given by (6.1.5), i.e.,*

$$w_i = F_W^{-1}(1 - \frac{i}{n}), \qquad \text{(6.2.5)}$$

for some distribution function F_W with a finite mean, then

(a) *the degree of a uniformly chosen vertex converges in distribution to a mixed Poisson random variable with distribution F_W;*

(b) *the degrees of m uniformly chosen vertices in $\{1, \ldots, n\}$ are asymptotically independent.*

We now prove Theorem 6.1 and Corollary 6.3:

Proof of Theorem 6.1. We make essential use of Theorem 2.9, in particular, the coupling of a sum of Bernoulli random variables with a Poisson random variable in (2.2.19). Throughout this proof, we shall omit the dependence on n of the weights, and abbreviate $w_i = w_i^{(n)}$. We recall that

$$D_k = \sum_{i=1}^n X_{kj}, \qquad \text{(6.2.6)}$$

where X_{kj} are independent Bernoulli random variables with success probabilities $p_{kj} = \frac{w_k w_j}{l_n + w_k w_j}$. By Theorem 2.9, there exists a Poisson random variable $\hat{Y}_k$ with parameter

$$\lambda_k = \sum_{j \neq k} \frac{w_k w_j}{l_n + w_k w_j}, \tag{6.2.7}$$

and a random variable $\hat{D}_k$ where $\hat{D}_k$ has the same distribution as D_k, such that

$$\mathbb{P}(\hat{D}_k \neq \hat{Y}_k) \leq \sum_{j \neq k} p_{kj}^2 = \sum_{j \neq k} \frac{w_k^2 w_j^2}{(l_n + w_k w_j)^2} \leq w_k^2 \sum_{j=1}^n \frac{w_j^2}{l_n^2}. \tag{6.2.8}$$

<div style="text-align:right">e:DkYk-diff</div>

Thus, in order to prove the claim, it suffices to prove that we can, in turn, couple $\hat{Y}_k$ to a Poisson random variable $\hat{Z}_k$ with parameter w_k, such that

$$\mathbb{P}(\hat{Y}_k \neq \hat{Z}_k) \leq w_k^2 \sum_{j=1}^n \frac{w_j^2}{l_n^2}. \tag{6.2.9}$$

For this, we note that $\lambda_k \leq w_k$, and we let $\varepsilon_k = w_k - \lambda_k$. Then, we let $\hat{V}_k \sim \text{Poi}(\varepsilon_k)$ be independent of $\hat{Y}_k$, and write $\hat{Z}_k = \hat{Z}_k + \hat{V}_k$. Then,

$$\mathbb{P}(\hat{Y}_k \neq \hat{Z}_k) = \mathbb{P}(\hat{V}_k \neq 0) = \mathbb{P}(\hat{V}_k \geq 1) \leq \mathbb{E}[\hat{V}_k] = \varepsilon_k. \tag{6.2.10}$$

<div style="text-align:right">e:YkVk-diff</div>

To bound ε_k, we note that

$$\varepsilon_k = w_k - \sum_{j \neq k} \frac{w_k w_j}{l_n + w_k w_j} = \frac{w_k^2}{l_n + w_k^2} + \sum_{j=1}^n w_k w_j \left(\frac{1}{l_n} - \frac{1}{l_n + w_k w_j} \right) = \frac{w_k^2}{l_n + w_k^2} + \sum_{j=1}^n \frac{w_j^2 w_k^2}{l_n(l_n + w_k w_j)}$$

$$\leq \frac{w_k^2}{l_n} + \sum_{j=1}^n \frac{w_j^2 w_k^2}{l_n^2}. \tag{6.2.11}$$

We conclude that

$$\mathbb{P}(\hat{D}_k \neq \hat{Z}_k) \leq \mathbb{P}(\hat{D}_k \neq \hat{Y}_k) + \mathbb{P}(\hat{Y}_k \neq \hat{Z}_k) \leq w_k^2 \sum_{j=1}^n \frac{w_j^2}{l_n^2} + \frac{w_k^2}{l_n} + \sum_{j=1}^n \frac{w_j^2 w_k^2}{l_n^2}, \tag{6.2.12}$$

as required. This proves Theorem 6.1(a).

To prove Theorem 6.1(b), it suffices to prove that we can couple $(D_1, \ldots, D_m)$ to an *independent* vector $(\hat{D}_1, \ldots, \hat{D}_m)$ such that

$$\mathbb{P}\Big((D_1, \ldots, D_m) \neq (\hat{D}_1, \ldots, \hat{D}_m) \Big) = o(1). \tag{6.2.13}$$

<div style="text-align:right">e:couplingdiffbd</div>

To this end, we recall that X_{ij} denotes the indicator that the edge ij is occupied. The random variables $\{X_{ij}\}_{1 \leq i < j \leq n}$ are independent Bernoulli random variables with parameters $\{p_{ij}\}_{1 \leq i < j \leq n}$ given in (6.1.1). We let $\{X'_{ij}\}_{1 \leq i < j \leq n}$ denote an *independent* copy of $\{X_{ij}\}_{1 \leq i < j \leq n}$, and let, for $i = 1, \ldots, n$,

$$D'_i = \sum_{j<i} X'_{ij} + \sum_{j=i+1}^n X_{ij}. \tag{6.2.14}$$

Then, we observe the following. (1) Since $\{X'_{ij}\}_{1 \leq i < j \leq n}$ is an independent copy of $\{X_{ij}\}_{1 \leq i < j \leq n}$, the distribution of D'_i is equal to the one of D_i, for every $i = 1, \ldots, n$. (2) Set $i < j$. While D_i and D_j are

dependent since they both contain $X_{ij} = X_{ji}$, D_i' contains X_{ij}', while D_j' contains $X_{ji} = X_{ij}$, which is an independent copy of X_{ij}. We conclude that $(D_1', \ldots, D_m')'$ are sums of *independent* Bernoulli random variables, and, therefore, are *independent*. (3) Finally, $(D_1, \ldots, D_m) \neq (\hat{D}_1, \ldots, \hat{D}_m)$ precisely when there exists an ij with $i, j \in \{1, \ldots, m\}$ such that $X_{ij} \neq X_{ij}'$. Since X_{ij} and X_{ij}' are Bernoulli random variables, $X_{ij} \neq X_{ij}'$ implies that either $X_{ij} = 0, X_{ij}' = 1$ or $X_{ij} = 1, X_{ij}' = 0$. Thus, by Boole's inequality, we obtain that

$$\mathbb{P}\Big((D_1, \ldots, D_m) \neq (\hat{D}_1, \ldots, \hat{D}_m)\Big) \leq 2 \sum_{i,j=1}^m \mathbb{P}(X_{ij} = 1) = 2 \sum_{i,j=1}^m p_{ij}. \qquad (6.2.15)$$

By assumption, $\lim_{n \to \infty} p_{ij} = 0$, so that (6.2.13) holds for every $m \geq 2$ fixed. This proves Theorem 6.1(b). $\qquad \square$

Exercise 6.3. *Suppose that there exists a $\varepsilon > 0$ such that $\varepsilon \leq w_i \leq \varepsilon^{-1}$ for every i, so that the weights are uniformly bounded from above and below. Then, prove that we can couple $(D_1, \ldots, D_m)$ to an independent vector $(\hat{D}_1, \ldots, \hat{D}_m)$ such that (6.2.13) holds whenever $m = o(\sqrt{n})$. As a result, even the degrees of a growing number of vertices can be coupled to independent degrees.*

Proof of Corollary 6.3. We show that (6.2.3) is satisfied, and that the law of the weight of a uniformly chosen vertex has approximate distribution function F.

For the first step, we use that, since $x \mapsto F_W^{-1}(x)$ is non-decreasing,

$$\begin{aligned}
\frac{1}{n} \sum_{i=1}^n w_i^2 &= \frac{1}{n} \sum_{i=1}^n F_W^{-1}(1 - \frac{i}{n})^2 \\
&= \mathbb{E}[F_W^{-1}(U_n)^2] \\
&\leq F_W^{-1}(1 - \frac{1}{n}) \mathbb{E}[F_W^{-1}(U_n)], \qquad \text{(6.2.16)}
\end{aligned}$$

where U_n is a discrete uniform random variable taking values $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\}$. Now we claim that

$$F_W^{-1}(1 - \frac{1}{n}) = o(n), \qquad \text{(6.2.17)}$$

since (6.2.17) is equivalent to $F^{-1}(1 - \frac{1}{n}) \leq \varepsilon n$ for every $\varepsilon > 0$ and n sufficiently large. In turn, $F^{-1}(1 - \frac{1}{n}) \leq \varepsilon n$ is, by (2.3.5), equivalent to

$$1 - \frac{1}{n} \leq F_W(\varepsilon n), \qquad (6.2.18)$$

or

$$1 - F_W(\varepsilon n) = \mathbb{P}(W_1 > \varepsilon n) \leq \frac{1}{n}. \qquad (6.2.19)$$

However, we have that $\mathbb{P}(W_1 > \varepsilon n) = o(n^{-1})$ since the mean of W_1 is finite:

$$\mathbb{P}(W_1 > \varepsilon n) \leq (\varepsilon n)^{-1} \mathbb{E}[W_1 \mathbb{1}_{\{W_1 > \varepsilon n\}}] = o(n^{-1}). \qquad \text{(6.2.20)}$$

Furthermore, since $U_n \preceq U$, where U is a uniform random variable on $[0, 1]$, by Theorem 2.13 and the fact that $x \mapsto F_W^{-1}(x)$ is non-decreasing, we also have that $F_W^{-1}(U_n) \preceq F_W^{-1}(U)$. By Theorem 2.12, we then obtain that

$$\mathbb{E}[F_W^{-1}(U_n)] \leq \mathbb{E}[F_W^{-1}(U)] = \mathbb{E}[W], \qquad \text{(6.2.21)}$$

where W has distribution function F, since $F^{-1}(U)$ has distribution function F.

Since $\mathbb{E}[W] < \infty$, we obtain that, using (6.2.16) and (6.2.21)

$$\frac{1}{n}\sum_{i=1}^{n} w_i^2 \le F_W^{-1}(1 - \frac{1}{n})\mathbb{E}[W] = o(n), \tag{6.2.22}$$

since $l_n = \Theta(n)$. Thus, by (6.2.2), we have that the degree of any vertex for which $w_k^{(n)}$ is bounded is close to a Poisson random variable with parameter $w_k^{(n)}$.

Thus, we note that the degree of a uniform vertex is close in distribution to a Poisson random variable with parameter $w_{B^{(n)}}^{(n)}$, where $B^{(n)}$ is a uniform random variable in $\{1, \ldots, n\}$. This is a mixed Poisson distribution with mixing distribution equal to $w_{B^{(n)}}^{(n)}$. We start by noting that, with high probability, $w_{B^{(n)}}^{(n)}$ is bounded. Thus, we are allowed to make use of Theorem 6.1(a).

Since the a mixed Poisson random variable converges to a limiting mixed Poisson random variable follows when the mixing distribution converges in distribution, it suffices to show that the weight of a uniform vertex has a limiting distribution given by F.

Let D be the degree of a uniformly chosen vertex. Then, we note that, by Theorem 6.1, with $B^{(n)}$ a uniform choice in $\{1, \ldots, n\}$,

$$\mathbb{P}(D = x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{P}(D_i = x | B^{(n)} = i)$$

$$= \frac{1}{n}\sum_{i=1}^{n} \mathbb{P}(\mathrm{Poi}(w_i^{(n)}) = x) + o(1). \tag{6.2.23}$$

When the weights $\{w_i^{(n)}\}_{i=1}^n$ are given in (6.2.5),

$$\mathbb{P}(D = x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{P}(\mathrm{Poi}(w_i^{(n)}) = x) + o(1)$$

$$= \frac{1}{n}\sum_{i=1}^{n} \mathbb{P}\Big(\mathrm{Poi}(F^{-1}(1 - \frac{i}{n})) = x\Big) + o(1)$$

$$= \mathbb{P}(\mathrm{Poi}(F^{-1}(U_n)) = x) + o(1). \tag{6.2.24}$$

where again U_n is a discrete uniform random variable taking values $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\}$. Therefore,

$$\mathbb{P}(D = x) = \mathbb{E}\big[e^{-F^{-1}(U_n)}\frac{F^{-1}(U_n)^x}{x!}\big] + o(1). \tag{6.2.25}$$

Since $U_n \xrightarrow{d} U$, and since, for every $x \ge 0$, $y \mapsto e^{-y}\frac{y^x}{x!}$ is a continuous and bounded function, we obtain that

$$e^{-F^{-1}(U_n)}\frac{F^{-1}(U_n)^x}{x!} \xrightarrow{d} e^{-F^{-1}(U)}\frac{F^{-1}(U)^x}{x!}. \tag{6.2.26}$$

Moreover, we also have that $e^{-F^{-1}(U_n)}\frac{F^{-1}(U_n)^x}{x!} \le 1$, so that by the Dominated convergence theorem, Theorem A.1.2, we also have that

$$\mathbb{P}(D = x) = \mathbb{E}\big[e^{-F^{-1}(U)}\frac{F^{-1}(U)^x}{x!}\big] + o(1)$$

$$= \mathbb{E}\big[e^{-W}\frac{W^x}{x!}\big] + o(1), \tag{6.2.27}$$

where W has distribution function F. This proves that D has an asymptotic mixed Poisson distribution, with mixing distribution F.

We shall not spell out all details of the proof of part (b), which is a minor adaptation of the proof of Theorem 6.1(b). We shall only discuss the asymptotic independence. Let $B_1^{(n)}, \ldots, B_m^{(n)}$ be independent uniform random variables. Then, the dependence between the degrees of the vertices $B_1^{(n)}, \ldots, B_m^{(n)}$ arises only through the edges between the vertices $B_1^{(n)}, \ldots, B_m^{(n)}$. Now, the expected number of occupied edges between the vertices $B_1^{(n)}, \ldots, B_m^{(n)}$, conditionally on $B_1^{(n)}, \ldots, B_m^{(n)}$, is bounded by to

$$\sum_{i,j=1}^{m} \frac{w_{B_i^{(n)}} w_{B_j^{(n)}}}{l_n + w_{B_i^{(n)}} w_{B_j^{(n)}}} \leq \sum_{i,j=1}^{m} \frac{w_{B_i^{(n)}} w_{B_j^{(n)}}}{l_n} = \frac{1}{l_n} \Big(\sum_{i=1}^{m} w_{B_i^{(n)}} \Big)^2. \tag{6.2.28}$$

As a result, the expected number of occupied edges between m uniformly chosen vertices is bounded by

$$\frac{\mathbb{E}[w_{B_i^{(n)}}]^2}{l_n} = \frac{1}{n^2 l_n} \Big(\sum_{i=1}^{n} w_i \Big)^2 = o(1), \tag{6.2.29}$$

so that we can couple the degrees of m uniform vertices to m *independent* mixed Poisson random variables with mixing distribution $w_{B^{(n)}}^{(n)}$. Since these random variables converge in distribution to independent mixed Poisson random variables with mixing distribution F, this completes the argument. □

Naturally, not every random variable can be obtained as a mixed Poisson distribution (recall Definition 6.2). In the following exercises, aspects of mixed Poisson distributions are further investigated.

Exercise 6.4. *Give an example of a random variable that cannot be represented as a mixed Poisson distribution.*

Exercise 6.5. *Let X have a mixed Poisson distribution with mixing distribution F and moment generating function M_W, i.e., for $t \in \mathbb{C}$,*

$$M_W(t) = \mathbb{E}[e^{tW}], \tag{6.2.30}$$

where W has distribution function F. Show that the characteristic function of X is given by

$$\phi_X(t) = \mathbb{E}[e^{itX}] = M_W(e^{it} - 1). \tag{6.2.31}$$

Exercise 6.6. *Let X have a mixed Poisson distribution with mixing distribution F. Express the mean and variance of X into the moments of W, where W has distribution function F.*

Exercise 6.7. *Suppose that there exist constants $0 < c_1 < c_2 < \infty$ such that*

$$c_1 x^{1-\tau} \leq 1 - F(x) \leq c_2 x^{1-\tau}. \tag{6.2.32}$$

Show that there exist $0 < c_1' < c_2' < \infty$ such that the distribution function G of a mixed Poisson distribution with mixing distribution F satisfies

$$c_1' x^{1-\tau} \leq 1 - G(x) \leq c_2' x^{1-\tau}. \tag{6.2.33}$$

6.2.2 Degrees of the Chung-Lu model

In this section, we prove a coupling result for the degrees of the Chung-Lu random graph, where the edge probabilities are given by

$$p_{ij}^{(\mathrm{CL})} = \frac{w_i w_j}{l_n} \wedge 1, \tag{6.2.34}$$

where again

$$l_n = \sum_{i=1}^{n} w_i. \tag{6.2.35}$$

When $\max_{i=1}^n w_i^2 \le l_n$, we may forget about the maximum with 1 in (6.2.34). We shall assume $\max_{i=1}^n w_i^2 \le l_n$ throughout this section, and denote the resulting graph by $\mathrm{CL}(n, \{w_i\}_{i=1}^n)$.

Naturally, when $\frac{w_i}{\sqrt{l_n}}$ is quite small, then there is hardly any difference between edge weights $p_{ij} = \frac{w_i w_j}{l_n + w_i w_j}$ and $p_{ij} = \frac{w_i w_j}{l_n}$. Therefore, one would expect that the same degree sequence appears. This is the content of the following theorem:

Theorem 6.4 (Degree sequence of Chung-Lu RG with deterministic weights)**.**

(a) *For deterministic weights* $\{w_i^{(n)}\}_{i=1}^n$, *the degree of vertex* k *in the Chung-Lu random graph, denoted by* D_k', *can be coupled to a Poisson random variable* Z_k *with parameter* $w_k^{(n)}$, *such that*

$$\mathbb{P}(D_k \neq Z_k) \le \frac{(w_k^{(n)})^2}{l_n}\Big(1 + 3\sum_{j=1}^n \frac{(w_j^{(n)})^2}{l_n}\Big). \tag{6.2.36}$$

(b) *When* $p_{ij}^{(\mathrm{CL})}$ *given by* (6.2.34) *are all such that for each fixed edge* ij, $\lim_{n\to\infty} p_{ij}^{(\mathrm{CL})} = 0$, *the degrees* $D_1', \ldots, D_m'$ *of vertices* $1, \ldots, m$ *are asymptotically independent.*

Proof. We use a coupling argument, reminiscent of the coupling of different Erdős-Rényi random graphs presented in Section 4.1.1. To each edge ij, we associate an independent uniform random variable U_{ij} on [0,1]. We let the edge weights of the Chung-Lu random graph $p_{ij}^{(\mathrm{CL})}$ be given by (6.2.34), and the ones for the generalized random graph by (6.1.1). In $\mathrm{GRG}(n, \{w_i\}_{i=1}^n)$ with p_{ij} given by (6.1.1), we say that an edge ij is *occupied* when $U_{ij} \le p_{ij}$, and the edge is vacant otherwise. In $\mathrm{CL}(n, \{w_i\}_{i=1}^n)$, on the other hand, we say that an edge ij is *occupied* when $U_{ij} \le p_{ij}^{(\mathrm{CL})}$, and the edge is vacant otherwise. By (2.2.13), the above coupling is in fact the maximal coupling of the edge variables. Let D_k denote the degree of vertex k in $\mathrm{GRG}(n, \{w_i\}_{i=1}^n)$ and D_k' the degree of vertex k in $\mathrm{CL}(n, \{w_i\}_{i=1}^n)$. We shall make use of the fact that $p_{ij}^{(\mathrm{CL})} \ge p_{ij}$ for each edge ij.

Using the above coupling, we note that the probability that the edge ij has a different occupation status in the Chung-Lu random graph and in the generalized random graph is equal to

$$\mathbb{P}(\text{status of edge } ij \text{ different in CL and GRG}) = \mathbb{P}(U_{ij} \in [p_{ij}, p_{ij}^{(\mathrm{CL})}]) = p_{ij}^{(\mathrm{CL})} - p_{ij}. \tag{6.2.37}$$

We compute that, for fixed ij,

$$p_{ij}^{(\mathrm{CL})} - p_{ij} = \frac{w_i w_j}{l_n} - \frac{w_i w_j}{l_n + w_i w_j} = \frac{w_i w_j}{l_n}\Big[1 - \frac{1}{1 + \frac{w_i w_j}{l_n}}\Big] \le \frac{w_i^2 w_j^2}{l_n^2}. \tag{6.2.38}$$

For fixed k, we have that the probability that the degree D_k' of the Chung-Lu random graph and the degree D_k of $\mathrm{GRG}(n, \{w_i\}_{i=1}^n)$ are different with probability at most

$$\mathbb{P}\Big(\exists j : X_{kj} \neq X_{kj}'\Big) \le \sum_{j=1}^n \mathbb{P}(X_{kj} \neq X_{kj}'), \tag{6.2.39}$$

where $\{X_{ij}\}_{1\leq j<j\leq n}$ are the edge occupation statuses for $\mathrm{GRG}(n,\{w_i\}_{i=1}^n)$ and $\{X'_{ij}\}_{1\leq j<j\leq n}$ the ones for $\mathrm{CL}(n,\{w_i\}_{i=1}^n)$.

By (6.2.37) and (6.2.38), we arrive at

$$
\begin{aligned}
\mathbb{P}(D_k \neq D'_k) &\leq \sum_{j=1}^n |p'_{kj} - p_{kj}| \\
&\leq \sum_{j=1}^n \frac{w_k^2 w_j^2}{l_n^2} \\
&= w_k^2 \sum_{j=1}^n \frac{w_j^2}{l_n^2}.
\end{aligned}
\tag{6.2.40}
$$

Thus, Theorem 6.4(a) follows from (6.2.40) together with Theorem 6.1(a). We omit the proof of Theorem 6.4(b). $\qquad\square$

6.3   Degree sequences

In this section, we investigate the degree sequences in inhomogeneous random graph models.

6.3.1   Degree sequence in the generalized random graph

Theorem 6.1 investigates the degree of a single vertex in the generalized random graph. In this section, we extend the result to the convergence of the empirical degree sequence. For $k \geq 0$, we let

$$
P_k^{(n)} = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{D_i=k\}}
\tag{6.3.1}
$$

denote the degree sequence of $\mathrm{GRG}(n,\{w_i\}_{i=1}^n)$. Due to Theorem 6.1, one would expect that this degree sequence is close to a mixed Poisson distribution. We denote the probability mass function of such a mixed Poisson distribution by p_k, i.e., for $k \geq 0$,

$$
p_k = \mathbb{E}\big[e^{-W} \frac{W^k}{k!}\big].
\tag{6.3.2}
$$

Theorem 6.5 shows that indeed the degree sequence $\{P_k^{(n)}\}_{k\geq 0}$ is close to the mixed Poisson distribution $\{p_k\}_{k\geq 0}$:

Theorem 6.5 (Degree sequence of $\mathrm{GRG}(n,\{w_i\}_{i=1}^n)$). *When $\{w_i\}_{i=1}^n$ are weights given by (6.1.5), i.e., when*

$$
w_i = F_W^{-1}(1 - \frac{i}{n}),
\tag{6.3.3}
$$

for some distribution function F with a finite mean, then, for every $\varepsilon > 0$,

$$
\mathbb{P}\Big(\sum_k |P_k^{(n)} - p_k| \geq \varepsilon\Big) \to 0,
\tag{6.3.4}
$$

where $\{p_k\}_{k=0}^\infty$ is given by (6.3.2).

Proof of Theorems 6.5. By Exercise 2.14 and the fact that $\{p_k\}_{k=0}^{\infty}$ is a probability mass function, we have that $\sum_k |P_k^{(n)} - p_k| = 2d_{\mathrm{TV}}(P^{(n)}, p) \to 0$ if and only if $\max_k |P_k^{(n)} - p_k| \to 0$. Thus, we need to show that, for every $\varepsilon > 0$, $\mathbb{P}(\sum_k |P_k^{(n)} - p_k| \geq \varepsilon)$ converges to 0. We use that

$$\mathbb{P}\Big(\sum_k |P_k^{(n)} - p_k| \geq \varepsilon\Big) \leq \sum_{k=0}^{\infty} \mathbb{P}(|P_k^{(n)} - p_k| \geq \varepsilon). \tag{6.3.5}$$

Note that

$$\mathbb{E}[P_k^{(n)}] = \mathbb{P}(D_1 = k), \tag{6.3.6}$$

and, by Corollary 6.6(a), we have that

$$\lim_{n\to\infty} \mathbb{P}(D_1 = k) = p_k. \tag{6.3.7}$$

Also, it is not hard to see that the convergence is *uniform in k*, that is, for every $\varepsilon > 0$, and for n sufficiently large, we have

$$\max_k |\mathbb{E}[p_k^{(n)}] - p_k| \leq \frac{\varepsilon}{2}. \tag{6.3.8}$$

Exercise 6.8. *Prove* (6.3.8).

By (6.3.5) and (6.3.8), it follows that, for n sufficiently large,

$$\mathbb{P}\Big(\max_k |P_k^{(n)} - p_k| \geq \varepsilon\Big) \leq \sum_{k=0}^{\infty} \mathbb{P}\Big(|P_k^{(n)} - \mathbb{E}[P_k^{(n)}]| \geq \varepsilon/2\Big). \tag{6.3.9}$$

Note that, by Chebychev inequality (Theorem 2.15),

$$\mathbb{P}\big(|P_k^{(n)} - \mathbb{E}[P_k^{(n)}]| \geq \varepsilon/2\big) \leq \frac{4}{\varepsilon^2}\mathrm{Var}(P_k^{(n)}). \tag{6.3.10}$$

Then we use the definition in (6.3.1) to see that

$$\mathbb{E}[(P_k^{(n)})^2] = \frac{1}{n^2} \sum_{1 \leq i,j \leq n} \mathbb{P}(D_i = D_j = k)$$

$$= \frac{1}{n}\mathbb{P}(D_1 = k) + \frac{n-1}{n}\mathbb{P}(D_1 = D_2 = k). \tag{6.3.11}$$

Therefore,

$$\mathrm{Var}(P_k^{(n)}) \leq \frac{1}{n}\mathbb{P}(D_1 = k) + [\mathbb{P}(D_1 = D_2 = k) - \mathbb{P}(D_1 = k)^2]. \tag{6.3.12}$$

By Corollary 6.6(b), we have that D_1 and D_2 are asymptotically independent, so that, when we note that

$$\mathbb{P}(D_1 = D_2 = k) - \mathbb{P}(D_1 = k)^2 = \mathbb{P}(D_1 = k)\big[\mathbb{P}(D_2 = k|D_1 = k) - \mathbb{P}(D_1 = k)\big], \tag{6.3.13}$$

the independence gives

$$\mathbb{P}(D_2 = k|D_1 = k) - \mathbb{P}(D_1 = k) = o(1). \tag{6.3.14}$$

Therefore,

$$\mathbb{P}\big(\max_k |P_k^{(n)} - \mathbb{E}[P_k^{(n)}]| \geq \varepsilon/2\big) \leq \frac{4}{\varepsilon^2} \sum_k \mathrm{Var}(P_k^{(n)})$$

$$\leq \frac{4}{\varepsilon^2} \sum_k \mathbb{P}(D_1 = k)\Big[\frac{1}{n} + \mathbb{P}(D_2 = k|D_1 = k) - \mathbb{P}(D_1 = k)\Big]. \tag{6.3.15}$$

By dominated convergence, the latter sum converges to 0.

Exercise 6.9. *Use the Dominated convergence theorem (Theorem A.1.2), to show that*

$$\sum_k \mathbb{P}(D_1 = k)[\mathbb{P}(D_2 = k|D_1 = k) - \mathbb{P}(D_1 = k)] \to 0. \tag{6.3.16}$$

$\square$

6.4   Generalized random graph with i.i.d. weights

We next state a consequence of Theorem 6.1, where we treat the special case where $\{W_i\}_{i=1}^n$ are independent and identically distributed. In this case, we have that, conditionally on the weights $\{W_i\}_{i=1}^n$, the edge ij is occupied is equal to

$$p_{ij} = \frac{W_i W_j}{L_n + W_i W_j}, \tag{6.4.1}$$

$^{\text{e:edgeprobiidweights}}$

where

$$L_n = \sum_{i=1}^n W_i \tag{6.4.2}$$

denotes the total weight. Note that there now is *double randomness*. Indeed, there is randomness due to the fact that the weights $\{W_i\}_{i=1}^n$ are random, and then there is the randomness in the occupation status of the edges conditionally on the weights $\{W_i\}_{i=1}^n$. We denote the resulting graph by $\mathrm{GRG}(n, \{W_i\}_{i=1}^n)$. In $\mathrm{GRG}(n, \{W_i\}_{i=1}^n)$, the edge statuses are *not* independent:

Exercise 6.10. *Prove that in* $\mathrm{GRG}(n, \{W_i\}_{i=1}^n)$ *the status of two edges is not necessarily independent.*

We now investigate the degrees and degree sequence of $\mathrm{GRG}(n, \{W_i\}_{i=1}^n)$:

Corollary 6.6 (Degrees of $\mathrm{GRG}(n, \{W_i\}_{i=1}^n)$). *When* $\{W_i\}_{i=1}^\infty$ *are i.i.d. random variables with distribution function* F_W *with a finite mean, then*

(a) *the degree* D_k *of vertex* k *converges in distribution to a mixed Poisson random variable with mixing distribution* F_W;

(b) *the degrees* $D_1, \dots, D_m$ *of vertices* $1, \dots, m$ *are asymptotically independent.*

To see that Corollary 6.6 follows from Theorem 6.1, we note that when $\{w_i\}_{i=1}^n = \{W_i\}_{i=1}^n$, where W_i are i.i.d. with distribution function F_W, the condition (6.2.3) follows when W_1 has a finite mean:

Exercise 6.11. *Show that when* $\{W_i\}_{i=1}^n$ *are i.i.d. random variables with distribution function* F_W *with a finite mean, then, almost surely,*

$$\frac{1}{n^2} \sum_{i=1}^n W_i^2 \to 0. \tag{6.4.3}$$

Hint: Show that $\max_{i=1}^n W_i = o(n)$ *with high probability, by using that*

$$\mathbb{P}(\max_{i=1}^n W_i \geq \varepsilon n) \leq \sum_{i=1}^n \mathbb{P}(W_i \geq \varepsilon n)$$

$$= n\mathbb{P}(W_1 \geq \varepsilon n). \tag{6.4.4}$$

Then use a variant of the Markov inequality (2.4.1) to show that $\mathbb{P}(W_1 \geq \varepsilon n) = o(\frac{1}{n})$.

Therefore, Theorem 6.1 is an extension of [53, Theorem 3.1], in which Corollary 6.6 was proved under the extra assumption that W_i have a finite $(1 + \varepsilon)$−moment.

Theorem 6.7 (Degree sequence of $\mathrm{GRG}(n, \{W_i\}_{i=1}^n)$). *When $\{W_i\}_{i=1}^n$ are i.i.d. random variables with distribution function F with a finite mean, then, for every $\varepsilon > 0$,*

$$\mathbb{P}\Big(\max_k |P_k^{(n)} - p_k| \geq \varepsilon \Big) \to 0, \tag{6.4.5}$$

where

$$p_k = \mathbb{P}(\mathrm{Poi}(W_1) = k) = \mathbb{E}\Big[e^{-W_1} \frac{W_1^k}{k!}\Big] \tag{6.4.6}$$

is the probability mass function of a mixed Poisson distribution with mixing distribution F_W.

We leave the proof of Theorem 6.7, which is quite similar to the proof of Theorem 6.5, to the reader:

Exercise 6.12. *Complete the proof of Theorem 6.7, now using Corollary 6.3, as well as the equality*

$$\mathbb{E}[(P_k^{(n)})^2] = \frac{1}{n^2} \sum_{1 \leq i,j \leq n} \mathbb{P}(D_i = D_j = k)$$

$$= \frac{1}{n}\mathbb{P}(D_1 = k) + \frac{2}{n^2} \sum_{1 \leq i < j \leq n} \mathbb{P}(D_i = D_j = k). \tag{6.4.7}$$

We next turn attention to the case where the weights $\{W_i\}_{i=1}^n$ are i.i.d. with infinite mean. We denote the distribution of W_i by F.

Exercise 6.13. *Show that the mean of W_1 is infinite precisely when the distribution function F_W of W_1 satisfies*

$$\int_0^\infty [1 - F_W(x)]dx = \infty. \tag{6.4.8}$$

Our next goal is to obtain a random graph which has a power-law degree sequence with a power-law exponent $\tau \in (1, 2)$. We shall see that this is a non-trivial issue.

Theorem 6.8 (Degrees of $\mathrm{GRG}(n, \{W_i\}_{i=1}^n)$ with i.i.d. conditioned weights). *When $\{W_i\}_{i=1}^n$ are i.i.d. random variables with distribution function F_W, and let $\{W_i^{(n)}\}_{i=1}^n$ be i.i.d. copies of the random variable W_1 conditioned on $W_1 \leq a_n$. Then, for every $a_n \to \infty$ such that $a_n = o(n)$,*

(a) *the degree $D_k^{(n)}$ of vertex k in the GRG with weights $\{W_i^{(n)}\}_{i=1}^n$, converges in distribution to a mixed Poisson random variable with mixing distribution F_W;*

(b) *the degrees $D_1^{(n)}, \ldots, D_m^{(n)}$ of vertices $1, \ldots, m$ are asymptotically independent.*

Proof. Theorem 6.8 follows by a simple adaptation of the proof of Theorem 6.1 and will be left as an exercise:

Exercise 6.14. *Prove Theorem 6.8.*

We finally show that the conditioning in Theorem 6.8 is necessary by proving that if we do not condition the weights to be at most a_n, then the degree distribution changes:

Theorem 6.9 (Degrees of $\mathrm{GRG}(n, \{W_i\}_{i=1}^n)$ with i.i.d. infinite mean weights). *When $\{W_i\}_{i=1}^n$ are i.i.d. random variables with distribution function F_W satisfying that for some $\tau \in (1, 2)$,*

$$1 - F_W(x) = cx^{1-\tau}(1 + o(1)), \tag{6.4.9}$$

and $x^{\tau-1}[1 - F_W(x)]$ is uniformly bounded. Let the edge probabilities $\{p_{ij}\}_{1 \leq i < j \leq n}$ conditionally on the weights $\{W_i\}_{i=1}^n$ be given by

$$p_{ij} = \frac{W_i W_j}{n^{\frac{1}{\tau-1}} + W_i W_j}. \tag{6.4.10}$$

Then

(a) the degree D_k of vertex k converges in distribution to a mixed Poisson random variable with parameter $\gamma W_k^{\tau-1}$, where

$$\gamma = c \int_0^\infty (1+x)^{-2} x^{-(\tau-1)} dx. \tag{6.4.11}$$

(b) the degrees $D_1, \ldots, D_m$ of vertices $1, \ldots, m$ are asymptotically independent.

We note that a mixed Poisson distribution with mixing distribution γW_k^α does not obey a power law with exponent τ:

Exercise 6.15. *Let the distribution function F_W satisfy that, for some $\tau \in (1, 2)$ and some $c_F > 0$,*

$$1 - F_W(x) = c_F x^{1-\tau}(1 + o(1)), \tag{6.4.12}$$

and let Y be a mixed Poisson random variable with parameter $W^{\tau-1}$, where W has distribution function F. Show that Y is such that there exists a constant $c > 0$ such that

$$\mathbb{P}(Y \geq y) = cy^{-1}(1 + o(1)). \tag{6.4.13}$$

As a result of Exercise 6.15, we see that if we do not condition on the weights to be at most a_n, and if the distribution function F of the weights satisfies (6.4.9), then the degree distribution always obeys a power law with exponent $\tau = 2$.

We note that the choice of the edge probabilities in (6.4.10) is different from the choice in (6.4.1). Indeed, the term L_n in the denominator in (6.4.1) is replaced by $n^{\frac{1}{\tau-1}}$ in (6.4.10). Since, when (6.4.9) is satisfied,

$$L_n n^{-\frac{1}{\tau-1}} \xrightarrow{d} S, \tag{6.4.14}$$

where S is a stable random variable with parameter $\tau - 1 \in (0, 1)$, we expect that the behavior for the choice (6.4.1) is similar (recall Theorem 2.28). However, we are not able to prove this.

Open problem 6.10. *Extend Theorem 6.9 to the case in* (6.1.1).

The proof of Theorem 6.9 is given in Section 6.5 below.

6.5 The generalized random graph conditioned on its degree sequence

In this section, we investigate the distribution of $\mathrm{GRG}(n, \{w_i\}_{i=1}^n)$ in more detail. The main result in this section is that the generalized random graph conditioned on its degree sequence is a uniform random graph with that degree sequence (see Theorem

We start by introducing some notation. We let $X = \{X_{ij}\}_{1 \leq i < j \leq n}$, where X_{ij} are independent random variables with

$$\mathbb{P}(X_{ij} = 1) = 1 - \mathbb{P}(X_{ij} = 0) = p_{ij}, \tag{6.5.1}$$

where p_{ij} is given in (6.1.1). Then, with $q_{ij} = 1 - p_{ij}$, we have that, for $x = \{x_{ij}\}_{1 \leq i < j \leq n}$,

$$\mathbb{P}(X = x) = \prod_{1 \leq i < j \leq n} p_{ij}^{x_{ij}} q_{ij}^{1-x_{ij}}. \tag{6.5.2}$$

We define the *odd-ratios* $\{r_{ij}\}_{1 \leq i < j \leq n}$ by

$$r_{ij} = \frac{p_{ij}}{q_{ij}}. \tag{6.5.3}$$

Then

$$p_{ij} = \frac{r_{ij}}{1 + r_{ij}}, \qquad q_{ij} = \frac{1}{1 + r_{ij}}, \tag{6.5.4}$$

so that

$$\mathbb{P}(X = x) = \prod_{1 \leq i < j \leq n} \frac{1}{1 + r_{ij}} \prod_{1 \leq i < j \leq n} r_{ij}^{x_{ij}}. \tag{6.5.5}$$

We now specialize to the setting of the generalized random graph, and choose

$$r_{ij} = u_i u_j, \tag{6.5.6}$$

for some weights $\{u_i\}_{i=1}^n$. Later, we shall choose

$$u_i = \frac{w_i}{\sqrt{l_n}}, \tag{6.5.7}$$

in which case we return to (6.1.1) since

$$p_{ij} = \frac{r_{ij}}{1 + r_{ij}} = \frac{u_i u_j}{1 + u_i u_j} = \frac{w_i w_j}{l_n + w_i w_j}. \tag{6.5.8}$$

Then, with

$$G(u) = \prod_{1 \leq i < j \leq n} (1 + u_i u_j), \tag{6.5.9}$$

we obtain

$$\mathbb{P}(X = x) = G(u)^{-1} \prod_{1 \leq i < j \leq n} (u_i u_j)^{x_{ij}}$$

$$= G(u)^{-1} \prod_{i=1}^n u_i^{d_i(x)}, \tag{6.5.10}$$

where $\{d_i(x)\}_{i=1}^n$ is given by

$$d_i(x) = \sum_{j=1}^n x_{ij}, \tag{6.5.11}$$

i.e., $d_i(x)$ is the degree of vertex i in the generalized random graph configuration $x = \{x_{ij}\}_{1 \leq i < j \leq n}$. By convention, we assume that $x_{ii} = 0$, and we recall that $x_{ij} = x_{ji}$.

Exercise 6.16. *Prove the last equality in* (6.5.10).

From (6.5.10), and using that $\sum_x \mathbb{P}(X = x) = 1$, it follows that

$$\prod_{1 \leq i < j \leq n} (1 + u_i u_j) = G(u) = \sum_x \prod_{i=1}^n u_i^{d_i(x)}. \tag{6.5.12}$$

Furthermore, it also follows from (6.5.10) that the distribution of X conditionally on $\{d_i(X) = d_i \forall 1 \leq i \leq n\}$ is *uniform*. That is, all graphs with the same degree sequence have the same probability. This wonderful result is formulated in the following theorem:

Theorem 6.11 (GRG conditioned on degrees has uniform law). *The GRG with edge probabilities* $\{p_{ij}\}_{1 \leq i < j \leq n}$ *given by*

$$p_{ij} = \frac{u_i u_j}{1 + u_i u_j}, \tag{6.5.13}$$

conditioned on $\{d_i(X) = d_i \forall i = 1, \ldots, n\}$, *is uniform over all graphs with degree sequence* $\{d_i\}_{i=1}^n$.

Proof. For x satisfying $d_i(x) = d_i$ for all $i = 1, \ldots, n$, we can write out

$$\begin{aligned}
\mathbb{P}(X = x | d_i(X) = d_i \forall i = 1, \ldots, n) &= \frac{\mathbb{P}(X = x)}{\mathbb{P}(d_i(X) = d_i \forall i = 1, \ldots, n)} \\
&= \frac{\mathbb{P}(X = x)}{\sum_{y : d_i(y) = d_i \forall i} \mathbb{P}(X = y)}. \tag{6.5.14}
\end{aligned}$$

By (6.5.10), we have that (6.5.14) simplifies to

$$\begin{aligned}
\mathbb{P}(X = x | d_i(X) = d_i \forall i = 1, \ldots, n) &= \frac{\prod_{i=1}^n u_i^{d_i(x)}}{\sum_{y : d_i(y) = d_i \forall i} \prod_{i=1}^n u_i^{d_i(y)}} \\
&= \frac{\prod_{i=1}^n u_i^{d_i}}{\sum_{y : d_i(y) = d_i \forall i} \prod_{i=1}^n u_i^{d_i}} \\
&= \frac{1}{\#\{y : d_i(y) = d_i \forall i = 1, \ldots, n\}}, \tag{6.5.15}
\end{aligned}$$

that is, the distribution is uniform over all graphs with the prescribed degree sequence. $\square$

We next compute the generating function of all degrees, that is, for $t_1, \ldots, t_n \in \mathbb{R}$, we compute, with $D_i = d_i(X)$,

$$\mathbb{E}\Big[\prod_{i=1}^n t_i^{D_i}\Big] = \sum_x \mathbb{P}(X = x) \prod_{i=1}^n t_i^{d_i(x)}. \tag{6.5.16}$$

By (6.5.10) and (6.5.12),

$$\mathbb{E}\Big[\prod_{i=1}^n t_i^{D_i}\Big] = G(u)^{-1} \sum_x \prod_{i=1}^n (u_i t_i)^{d_i(x)} = \frac{G(tu)}{G(u)}, \tag{6.5.17}$$

where $(tu)_i = t_i u_i$. By (6.5.9), we obtain

$$\mathbb{E}\Big[\prod_{i=1}^n t_i^{D_i}\Big] = \prod_{1 \leq i < j \leq n} \frac{1 + u_i t_i u_j t_j}{1 + u_i u_j}. \tag{6.5.18}$$

Therefore, we have proved the following nice property:

Proposition 6.12 (Generating function of degrees of $\mathrm{GRG}(n, \{w_i\}_{i=1}^n)$). *For the edge probabilities given by (6.1.1) and (6.5.7),*

$$\mathbb{E}\Big[\prod_{i=1}^n t_i^{D_i}\Big] = \prod_{1 \le i < j \le n} \frac{l_n + w_i t_i w_j t_j}{l_n + w_i w_j}. \tag{6.5.19}$$

Exercise 6.17. *Use Proposition 6.12 to give an alternative proof of Theorem 6.1.*

Exercise 6.18. *Show that for the Erdős-Rényi random graph with $p = \lambda/n$, the degree of vertex 1 is close to a Poisson random variable with mean λ by using (B.0.104). Hint: Use that the Erdős-Rényi random graph is obtained by taking $W_i \equiv \frac{\lambda}{1-\frac{\lambda}{n}}$.*

Exercise 6.19. *Show that for the Erdős-Rényi random graph with $p = \lambda/n$, the degrees of vertices $1,\dots,m$ are asymptotically independent by using (B.0.105).*

We finally make use of Proposition 6.12 to prove Theorem 6.9:

Proof of Theorem 6.9. We study the generating function of the degree D_k. We note that

$$\mathbb{E}[t^{D_k}] = \mathbb{E}\Big[\prod_{i \ne k} \frac{1 + t W_i W_k n^{-\frac{1}{\tau-1}}}{1 + W_i W_k n^{-\frac{1}{\tau-1}}}\Big]. \tag{6.5.20}$$

Denote $\phi_w : \mathbb{R} \mapsto \mathbb{R}$ by

$$\phi_w(x) = \frac{1 + twx}{1 + wx}. \tag{6.5.21}$$

Then, by the independence of the weights $\{W_i\}_{i=1}^n$, we have that

$$\mathbb{E}[t^{D_k} | W_k = w] = \mathbb{E}\Big[\prod_{i \ne k} \phi_w\big(W_i n^{-\frac{1}{\tau-1}}\big)\Big] = \psi_n(w)^{n-1}, \tag{6.5.22}$$

where

$$\psi_n(w) = \mathbb{E}\Big[\phi_w\big(\frac{W_1}{n^{-\frac{1}{\tau-1}}}\big)\Big]. \tag{6.5.23}$$

We claim that

$$\psi_n(w) = 1 + \frac{1}{n}(t-1)\gamma w^{\tau-1} + o(n^{-1}). \tag{6.5.24}$$

This completes the proof since it implies that

$$\mathbb{E}[t^{D_k} | W_k = w] = \psi_n(w)^{n-1} = e^{(t-1)\gamma w^{\tau-1}}(1 + o(1)), \tag{6.5.25}$$

which in turn implies that

$$\lim_{n \to \infty} \mathbb{E}[t^{D_k}] = \mathbb{E}[e^{(t-1)\gamma W_k^{\tau-1}}]. \tag{6.5.26}$$

Since $\mathbb{E}[e^{(t-1)\gamma W_k^{\tau-1}}]$ is the probability generating function of a mixed Poisson random variable with mixing distribution $\gamma W_k^{\tau-1}$ (see Exercise 6.20), (6.5.24) indeed completes the proof.

Exercise 6.20. *Prove that $\mathbb{E}[e^{(t-1)\gamma W^{\tau-1}}]$ is the probability generating function of a mixed Poisson random variable with mixing distribution $\gamma W^{\tau-1}$*

We complete the proof of Theorem 6.9 by showing that (6.5.24) holds. For this, we first note

$$\psi_n(w) = \mathbb{E}\left[\phi_w\left(W_1 n^{-\frac{1}{\tau-1}}\right)\right]$$

$$= 1 + \mathbb{E}\left[\phi_w\left(W_1 n^{-\frac{1}{\tau-1}}\right) - 1\right]. \tag{6.5.27}$$

We make use of the fact, for every function $f\colon [0,\infty) \to \mathbb{R}$, with $f(0) = 0$ and every random variable $X \geq 0$ with distribution function F_X, the partial integration formula

$$\mathbb{E}[f(X)] = \int_0^\infty f'(x)[1 - F_X(x)]dx \tag{6.5.28}$$

holds.

Exercise 6.21. *Prove* (6.5.28).

Applying (6.5.28) to $f(x) = \phi_w\left(xn^{-\frac{1}{\tau-1}}\right) - 1$ and $X = W_1$ yields

$$\psi_n(w) = 1 + n^{-\frac{1}{\tau-1}} \int_0^\infty \phi_w'\left(xn^{-\frac{1}{\tau-1}}\right)[1 - F_W(x)]dx$$

$$= 1 + \int_0^\infty \phi_w'(x)[1 - F_W(xn^{\frac{1}{\tau-1}})]dx. \tag{6.5.29}$$

Thus,

$$n(\psi_n(w) - 1) = \int_0^\infty \frac{\phi_w'(x)}{x^{\tau-1}}(n^{\frac{1}{\tau-1}}x)^{\tau-1}[1 - F_W(xn^{\frac{1}{\tau-1}})]dx. \tag{6.5.30}$$

By assumption, $x^{\tau-1}[1 - F(x)]$ is a bounded function that converges to c. As a result, by the Dominated convergence theorem (Theorem A.1.1),

$$\lim_{n\to\infty} \int_0^\infty \frac{\phi_w'(x)}{x^{\tau-1}}(n^{\frac{1}{\tau-1}}x)^{\tau-1}[1 - F(xn^{\frac{1}{\tau-1}})]dx = c\int_0^\infty \frac{\phi_w'(x)}{x^{\tau-1}}dx. \tag{6.5.31}$$

Exercise 6.22. *Verify the conditions for dominated convergence.*

Finally, we note that

$$\phi_w'(x) = \frac{tw}{1 + wx} - \frac{w(1 + twx)}{(1 + wx)^2} = \frac{w(t - 1)}{(1 + wx)^2}, \tag{6.5.32}$$

so that

$$c\int_0^\infty \frac{\phi_w'(x)}{x^{\tau-1}}dx = c\int_0^\infty \frac{w(t - 1)}{(1 + wx)^2 x^{\tau-1}}dx = \gamma(t - 1)w^{\tau-1}. \tag{6.5.33}$$

This completes the proof of (6.5.24). $\square$

6.6   Related inhomogeneous random graph models

There are numerous papers that introduce models along the lines of the generalized random graph, in that they have (conditionally) independent edge statuses.

The most general model has appeared in [43]. In this paper, the properties of such random graphs (such as diameter, phase transition and average distances) have been studied using comparisons to multitype branching processes. We shall return to [43] in Chapter 9.

Norros-Reittu model. In [171], the authors introduce a random multigraph with an Poissonean number of edges in between any two vertices i and j, with parameter equal to $W_i W_j / L_n$. The graph is defined as a graph process, where at each time t, a new vertex is born with an associated weight W_t. The number of edges between i and t is $\text{Poi}(W_i W_t / L_t)$ distributed. Furthermore, at each time each of the older edges is erased with probability equal to W_t / L_t. As a result, the number of edges between i, j at time t is a Poisson random variable with mean

$$\frac{W_i W_j}{L_{t-1}}\Big(1 - \frac{W_t}{L_t}\Big) = \frac{W_i W_j}{L_t}, \tag{6.6.1}$$

as it should be.

Exercise 6.23. *Let X be a Poisson random variable with mean λ, and let $\{I_i\}_{i=1}^{\infty}$ be an independent and identically distributed sequence of $\text{BE}(p)$ random variables. Prove that*

$$Y = \sum_{i=1}^{X} I_i \tag{6.6.2}$$

has a Poisson distribution with mean λp. Use this to prove (6.6.1).

The Norros-Reittu graph process produces a multigraph. However, when the weights are sufficiently bounded, it can be seen that the resulting graph is with positive probability simple:

Exercise 6.24. *Compute the probability that the Norros-Reittu graph is simple at time t.*

Exercise 6.25. *Prove that the degree of vertex k in the Norros-Reittu graph at time t has an asymptotic mixed Poisson distribution with mixing distribution F, the distribution function of W_1.*

We now discuss the Norros-Reittu model at time n, ignoring the dynamic formulation given above. We shall denote this graph by $\text{NR}(n, \{W_i\}_{i=1}^{n})$. The Norros-Reittu is a *multigraph*, for which the probability that there is at least one edge between vertices i and j exists is, conditionally on the weights $\{W_i\}_{i=1}^{n}$, given by

$$p_{ij}^{(\text{NR})} = 1 - e^{-\frac{W_i W_j}{L_n}}, \tag{6.6.3}$$

and the occupation status of different edges is, conditionally on the weights $\{W_i\}_{i=1}^{n}$, independent. Naturally, also the setting where the weights of the vertices in the Norros-Reittu model are non-random, can be studied. We will refrain from giving these details.

We next return to the relation between the various random graph models discussed in this section. We shall fix the weights to be equal to $\{w_i\}_{i=1}^{n}$, and compare the generalized random graph, Chung-Lu model and Norros-Reittu model with these weights. The latter is denoted by $\text{NR}(n, \{w_i\}_{i=1}^{n})$. We say that a random graph G_n is stochastically dominated by the random graph G_n' when, with $\{X_{ij}\}_{1 \le i < j \le n}$ and $\{X_{ij}'\}_{1 \le i < j \le n}$ denoting the occupation statuses of the edges in G_n and G_n' respectively, there exists a *coupling* $\left(\{\hat{X}_{ij}\}_{1 \le i < j \le n}, \{\hat{X}_{ij}'\}_{1 \le i < j \le n}\right)$ of $\{X_{ij}\}_{1 \le i < j \le n}$ and $\{X_{ij}'\}_{1 \le i < j \le n}$ such that

$$\mathbb{P}(\hat{X}_{ij} \le \hat{X}_{ij}') = 1. \tag{6.6.4}$$

When the statuses of the edges are independent, then (6.6.4) is equivalent to

$$p_{ij} = \mathbb{P}(X_{ij} = 1) \le p_{ij}' = \mathbb{P}(X_{ij}' = 1). \tag{6.6.5}$$

We write $G_n \preceq G'_n$ when G_n is stochastically dominated by the random graph G'_n. We note that, by (6.6.5) and the fact that, for every $x \geq 0$,

$$\frac{x}{1+x} \leq 1 - e^{-x} \leq \max x, 1, \tag{6.6.6}$$

we have that

$$\mathrm{GRG}(n, \{w_i\}_{i=1}^n) \preceq \mathrm{NR}(n, \{w_i\}_{i=1}^n) \preceq \mathrm{CL}(n, \{w_i\}_{i=1}^n). \tag{6.6.7}$$

This provides a good way of comparing the various inhomogeneous random graph models discussed in this chapter.

Exercise 6.26. *Prove that, when the weights obey that, for some $\tau > 3$,*

$$1 - F_W(x) \leq cx^{1-\tau}, \tag{6.6.8}$$

then the Norros-Reittu graph at time n can be coupled with $\mathrm{GRG}(n, \{W_i\}_{i=1}^n)$ in such a way that the probability that the degree of a fixed vertex k is different in the two graphs is negligible.

6.7   Notes and discussion

Notes on Section 6.1.   The generalized random graph can clearly be investigated with *fixed* weights and with independent and identically distributed weights. Clearly, these choices are intimately related. In [60, 61, 62, 64, 149], the Chung-Lu model is studied with fixed weights. In [53, 93], on the other hand, the situation is studied where the weights are i.i.d. The advantage of fixed weights is that there is no double randomness, which makes the model easier to analyse. The results are also more general, since often the results for random weights are a simple consequence of the ones for fixed weights. On the other hand, the advantage of working with i.i.d. weights is that the vertices are exchangeable, and, in contrast to the fixed weights case, not many assumptions need to be made. For fixed weights, one often has to make detailed assumptions concerning the precise structure of the weights.

Notes on Section 6.4.   Theorem 6.9 is [53, Proof of Theorem 3.2], whose proof we follow.

Notes on Section 6.5.   The proof in Section 6.5 follows the argument in [53, Section 3].

Chapter 7

Configuration model

In this chapter, we investigate graphs with fixed degrees. Ideally, we would like to investigate uniform graphs having a prescribed degree sequence. As it turns out, it is not a trivial task to generate graphs having prescribed degrees, in particular, because they may not exist (recall (I.3) on page 132). We shall therefore introduce a model that produces a uniform *multigraph* with the prescribed degrees. This is called the *configuration model*. We shall discuss the connections between the configuration model and a uniform random graph having the same degree sequence, and give an asymptotic formula for the number of graphs with a given degree sequence.

This chapter is organized as follows. In Section 7.1, we shall introduce the model. In Sections 7.2 and 7.3, we shall investigate two ways of turning the configuration model into a simple graph, namely, by erasing the self-loops and multiple edges, or by conditioning on obtaining a simple graph. For the latter, we compute the asymptotic probability of the configuration model to be simple. This also allows us to compute the asymptotic number of graphs with a given degree sequence in the case where the degrees are not too large. In Section 7.2, we again focus on deterministic or fixed degrees, and in Section 7.3 on i.i.d. degrees. We close this chapter in Section 7.4 with notes and discussion.

7.1   Introduction to the model

Fix an integer n. Consider a sequence $d_1, d_2, \ldots, d_n$. The aim is to construct an undirected graph with n vertices where vertex j has degree d_j. Without loss of generality, we shall assume that $d_j \geq 1$ for all $j \in \{1, \ldots, n\}$. The random graph model is then to take the uniform measure over such undirected graphs. However, the set of undirected graphs with n vertices where vertex j has degree d_j may be empty. For example, in order for such a graph to exist, we must assume that the total degree

$$l_n = \sum_{j=1}^{n} d_j \tag{7.1.1}$$

is even. We wish to construct a *simple* graph such that $\{d_i\}_{i=1}^{n}$ are the degrees of the n vertices. Here, we call a graph simple when it has no self-loops and no multiple edges between any pair of vertices. However, even when $l_n = \sum_{j=1}^{n} d_j$ *is* even, this is not always possible, as explained in more detail in (I.3) on page 132.

Exercise 7.1. *Find a simple example of a $\{d_i\}_{i=1}^{n}$ satisfying that $l_n = \sum_{j=1}^{n} d_j$ is even, for which there is no simple graph where vertex i has degree d_i.*

Since it is not always possible to construct a simple graph with a given degree sequence, instead, we can construct a *multigraph*, that is, a graph *possibly* having self-loops and multiple edges between pairs

of vertices. One way of obtaining a uniform multigraph with the given degree sequence is to pair the half-edges attached to the different vertices in a uniform way. Two half-edges together form an edge, thus creating the edges in the graph. We shall refer to the half-edges as *stubs*.

To construct the uniform multigraph where vertex j has degree d_j for all $j \in \{1, \ldots, n\}$, we have n separate vertices and incident to vertex j, we have d_j stubs. Every stub needs to be connected to another stub to build the graph. The stubs are numbered in an arbitrary order from 1 to l_n. We start by connecting at random the first stub with one of the $l_n - 1$ remaining stubs. Once paired, two stubs form a single edge of the multigraph. Hence, a stub can be seen as the left or the right half of an edge. We continue the procedure of randomly choosing and pairing the stubs until all stubs are connected Unfortunately, vertices having self-loops, as well as multiple edges may occur. However, we shall see that self-loops and multiple edges are scarce when $n \to \infty$. The above model is called the *configuration model with degree sequence* $\{d_i\}_{i=1}^n$. Clearly, when the total degree $l_n = \sum_{j=1}^n d_j$ is even, then the above procedure produces a multigraph with the right degree sequence.

To explain the term configuration model, we now present an equivalent way of defining the configuration model. For this, we construct a *second graph*, with vertices $1, \ldots, l_n$. These vertices in the new graph will correspond to the edges of the random multigraph in the configuration model. We pair the vertices in a uniform way to produce a *uniform matching*. For this, we pair vertex 1 with a uniform other vertex. After this, we pair the first not yet paired vertex to a uniform vertex which is not yet paired. The procedure stops when all vertices are paired to another (unique) vertex. We denote the resulting graph by F, and the graph in the configuration model by $CM(n, \{d_i\}_{i=1}^n)$. The pairing of the vertices $1, \ldots, l_n$ is called a *configuration*.

Exercise 7.2. *Prove that there are* $(2m-1)!! = (2m-1)(2m-3)\cdots 3 \cdot 1$ *different ways of pairing vertices* $1, \ldots, 2m$.

To construct the graph of the configuration model from the above configuration, we identify vertices $1, \ldots, d_1$ in F to form vertex 1 in $CM(n, \{d_i\}_{i=1}^n)$, and vertices $d_1 + 1, \ldots, d_1 + d_2$ in F to form vertex 2 in $CM(n, \{d_i\}_{i=1}^n)$, etc. Therefore, precisely d_j vertices in F are identified with vertex j in G. In the above identification, the number of edges in $CM(n, \{d_i\}_{i=1}^n)$ between vertices $i, j \in \{1, \ldots, n\}$ is the number of vertices in F that are identified with i and are paired to a vertex in F that is identified with j. As a consequence, the degree of vertex j in $CM(n, \{d_i\}_{i=1}^n)$ is *precisely* equal to d_j. The resulting graph is a *multigraph*, since both self-loops and multiple edges between vertices are possible.

It is not hard to see that each such multigraph is equally likely. Since the uniform matching of the l_n vertices in F is sometimes referred to as the *configuration*, the resulting graph $CM(n, \{d_i\}_{i=1}^n)$ is called *the configuration model*.

Similarly to the generalized random graph, there are again two versions of the model. In the first, the degree sequence is fixed, and obeys certain asymptotic relations. In the second, the degrees $\{D_i\}_{i=1}^n$ are taken such that $\{D_i\}_{i=1}^\infty$ is an i.i.d. sequence of random variables. The configuration model with fixed degrees has a long history, see e.g. [40, Section 2.4]. One specific example is to take the degrees all equal, in which case we speak of a *random regular graph*.

When we extend the construction of the configuration model to degrees $\{D_i\}_{i=1}^n$ where $\{D_i\}_{i=1}^\infty$ is an i.i.d. sequence of random variables, then we should bear in mind that the total degree

$$L_n = \sum_{i=1}^n D_i \tag{7.1.2}$$

is odd with probability close to 1/2, as the following exercise shows:

Exercise 7.3. *Assume that $\{D_i\}_{i=1}^{\infty}$ is an i.i.d. sequence of random variables. Prove that $L_n = \sum_{i=1}^{n} D_i$ is odd with probability close to $1/2$. For this, note that*

$$\mathbb{P}(L_n \text{ is odd}) = \frac{1}{2}\Big[1 - \mathbb{E}[(-1)^{L_n}]\Big]. \tag{7.1.3}$$

Then compute

$$\mathbb{E}[(-1)^{L_n}] = \phi_{D_1}(\pi)^n, \tag{7.1.4}$$

where

$$\phi_{D_1}(t) = \mathbb{E}[e^{itD_1}] \tag{7.1.5}$$

is the characteristic function of the degree D_1. Prove that, when $\mathbb{P}(D \text{ even}) \neq 1$, $|\phi_{D_1}(\pi)| < 1$, so that $\mathbb{P}(L_n \text{ is odd})$ is exponentially close to $\frac{1}{2}$.

In order to overcome the problem of an odd total degree L_n, there are different possible solutions, each producing a graph with similar characteristics. We make use of the following solution: If L_n is odd, then we add a stub to the nth vertex, so that D_n is increased by 1. This single stub will make hardly any difference in what follows, and we will ignore this effect. We will later specify the distribution of D_1.

In this chapter, we study the configuration model both with fixed degrees, as well as with i.i.d. degrees. We focus on two main results. The first main result shows that when we erase all self-loops and combine the multiple edges into one, then we obtain a graph with asymptotically the same degree sequence. This model is also referred to as the *erased configuration model*, see also [53, Section 2.1].[BriDeiMar-Lof05] In the second main result, we investigate the probability that the configuration model actually produces a *simple graph*. We note that the configuration model produces a uniform multigraph, which implies that, conditionally on this graph being simple, implies that the result is a uniform simple graph with the prescribed degree sequence. Remarkably, even though there *could* be many self-loops and multiple edges, in the case when the degrees are not too large, there is an asymptotically *positive* probability that the configuration model produces a simple graph. Therefore, we may obtain a uniform *simple* random by repeating the procedure until we obtain a simple graph. As a result, this model is sometimes called the *repeated* configuration model. The fact that the configuration model yields a simple graph with asymptotically positive probability has many interesting consequences that we shall explain in some detail. For example, it allows us to compute the asymptotics of the number of simple graph with a given degree sequence.

7.2   Configuration model with fixed degrees

In this section, we investigate the configuration model with fixed or deterministic degrees. We start by investigating the erased configuration model, in which loops and multiple edges are erased, in Section 7.2.1. In Section 7.2.2, we compute the asymptotic probability that the configuration model is simple. In Section 7.2.3, we study the repeated configuration model, which is the configuration model conditioned on being simple.

7.2.1   Degree sequence of the erased configuration model

We first define the erased configuration model. We fix the degrees $\{d_i\}_{i=1}^{n}$. We start with the multigraph produced in the configuration model with degrees $\{d_i\}_{i=1}^{n}$ and erase all self-loops, if any exist. After this, we merge all multiple edges into a single edge. Therefore, the erased configuration model yields a simple random graph, where two vertices are connected by an edge if and only if there is (at least one) edge connecting them in the original multigraph definition of the configuration model.

We next introduce some notation. We denote the degrees in the erased configuration model by $\{\tilde{D}_i\}_{i=1}^n$, so that

$$\tilde{D}_i = d_i - 2s_i - m_i, \tag{7.2.1}$$

where $\{d_i\}_{i=1}^n$ are the degrees in the configuration model, s_i is the number of self-loops of vertex i in the configuration model, and m_i is the number of multiple edges removed from i. Similarly to the weights in the generalized random graph, we allow for the possibility that $\{d_i\}_{i=1}^n = \{d_i^{(n)}\}_{i=1}^n$, so that the degree of vertex i depends on n. In the sequel, however, we shall omit this dependence on n and simply write $\{d_i\}_{i=1}^n$ for the degrees in the configuration model.

Denote the empirical degree sequence $\{p_k^{(n)}\}_{k=1}^\infty$ in the configuration model by

$$p_k^{(n)} = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{d_i = k\}}, \tag{7.2.2}$$

and denote the related degree sequence in the erased configuration model $\{\tilde{P}_k^{(n)}\}_{k=1}^\infty$ by

$$\tilde{P}_k^{(n)} = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{\tilde{D}_i = k\}}. \tag{7.2.3}$$

From the notation it is clear that $\{p_k^{(n)}\}_{k=1}^\infty$ is a *deterministic* sequence since $\{d_i\}_{i=1}^n$ is deterministic, while $\{\tilde{P}_k^{(n)}\}_{k=1}^\infty$ is a *random* sequence, since the erased degrees $\{\tilde{D}_i\}_{i=1}^n$ is a random vector.

Before we can state the main theorem on the degree sequence of the erased configuration model, we need some assumptions. We assume that, for every k, the degree sequence $\{d_i\}_{i=1}^n$ satisfies that

$$\lim_{n\to\infty} p_k^{(n)} = p_k, \tag{7.2.4}$$

for some probability distribution $\{p_k\}_{k=1}^\infty$. Equation (7.2.4) is equivalent to the fact that the proportion of vertices with degree k in the configuration model converges to some limit.

Further, we assume that

$$\mu = \sum_{k=1}^\infty k p_k < \infty, \tag{7.2.5}$$

so that the average degree in the configuration model remains constant.

Exercise 7.4. *Prove that*

$$\sum_{k=1}^\infty k p_k^{(n)} = \frac{1}{n} \sum_{i=1}^n d_i = \frac{l_n}{n}. \tag{7.2.6}$$

Exercise 7.5. *Prove that for i.i.d. degrees, the limit in (7.2.4) exists in an almost sure sense. Prove that (7.2.5) is satisfied when the degree distribution has finite mean.*

Now we are ready to state the main result concerning the degree sequence of the erased configuration model:

Theorem 7.1 (Degree sequence of erased configuration model with fixed degrees)**.** *For fixed degrees $\{d_i\}_{i=1}^n$ satisfying (7.2.4) and (7.2.5), and*

$$\max_{i=1}^n d_i = o(\sqrt{n}), \tag{7.2.7}$$

the degree sequence of the erased configuration model $\{\tilde{P}_k^{(n)}\}_{k=1}^\infty$ converges to $\{p_k\}_{k=1}^\infty$. More precisely, for every $\varepsilon > 0$,

$$\mathbb{P}\Big(\sum_{k=1}^\infty |\tilde{P}_k^{(n)} - p_k| \geq \varepsilon \Big) \to 0. \tag{7.2.8}$$

Proof. By (7.2.4) and the fact that pointwise convergence of a probability mass function is equivalent to convergence in total variation distance (recall Exercise 2.14), we obtain that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |p_k^{(n)} - p_k| = 0. \tag{7.2.9}$$

Therefore, we can take n so large that

$$\sum_{k=1}^{\infty} |p_k^{(n)} - p_k| \le \varepsilon/2. \tag{7.2.10}$$

We continue to bound $\mathbb{P}(\sum_{k=1}^{\infty} |\tilde{P}_k^{(n)} - p_k^{(n)}| \ge \varepsilon/2)$. For this, we use (7.2.1), which implies that $\tilde{D}_i \ne d_i$ if and only if $2s_i + m_i \ge 1$. We use

$$\sum_{k=1}^{\infty} |\tilde{P}_k^{(n)} - p_k^{(n)}|$$
$$\le \frac{1}{n} \sum_{k=1}^{\infty} \sum_i |\mathbb{1}_{\{\tilde{D}_i=k\}} - \mathbb{1}_{\{d_i=k\}}|, \tag{7.2.11}$$

and write out that

$$\mathbb{1}_{\{\tilde{D}_i=k\}} - \mathbb{1}_{\{d_i=k\}} = \mathbb{1}_{\{\tilde{D}_i=k,d_i>k\}} - \mathbb{1}_{\{\tilde{D}_i<k,d_i=k\}}$$
$$= \mathbb{1}_{\{s_i+m_i>0\}}\left(\mathbb{1}_{\{\tilde{D}_i=k\}} - \mathbb{1}_{\{d_i=k\}}\right). \tag{7.2.12}$$

Therefore,

$$|\mathbb{1}_{\{\tilde{D}_i=k\}} - \mathbb{1}_{\{d_i=k\}}| \le \mathbb{1}_{\{s_i+m_i>0\}}\left(\mathbb{1}_{\{\tilde{D}_i=k\}} + \mathbb{1}_{\{d_i=k\}}\right), \tag{7.2.13}$$

so that

$$\sum_{k=1}^{\infty} |\tilde{P}_k^{(n)} - p_k^{(n)}| \le \frac{1}{n} \sum_{k=1}^{\infty} \sum_{i=1}^{n} |\mathbb{1}_{\{\tilde{D}_i=k\}} - \mathbb{1}_{\{d_i=k\}}|$$
$$\le \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{s_i+m_i>0\}} \sum_{k=1}^{\infty} \left(\mathbb{1}_{\{\tilde{D}_i=k\}} + \mathbb{1}_{\{d_i=k\}}\right)$$
$$= \frac{2}{n} \sum_{i=1}^{n} \mathbb{1}_{\{s_i+m_i>0\}}$$
$$\le \frac{2}{n} \sum_{i=1}^{n} (s_i + m_i). \tag{7.2.14}$$

We denote the number of self-loops by S_n and the number of multiple edges by M_n, that is

$$S_n = \sum_{i=1}^{n} s_i, \qquad M_n = \frac{1}{2} \sum_{i=1}^{n} m_i. \tag{7.2.15}$$

Then, by (7.2.14),

$$\mathbb{P}\left(\sum_{k=1}^{\infty} |\tilde{P}_k^{(n)} - p_k^{(n)}| \ge \varepsilon/2\right) \le \mathbb{P}(2S_n + 4M_n \ge \varepsilon n/2), \tag{7.2.16}$$

so that Theorem 7.1 follows if

$$\mathbb{P}(2S_n + 4M_n \ge \varepsilon n/2) \to 0. \tag{7.2.17}$$

By the Markov inequality (Theorem 2.14), we obtain

$$\mathbb{P}(2S_n + 4M_n \geq \varepsilon n/2) \leq \frac{4}{\varepsilon n}\big(\mathbb{E}[S_n] + 2\mathbb{E}[M_n]\big), \tag{7.2.18}$$

so that it suffices to prove that

$$\mathbb{E}[S_n] + \mathbb{E}[M_n] = o(n). \tag{7.2.19}$$

Bounds on $\mathbb{E}[S_n]$ and $\mathbb{E}[M_n]$ are provided in the following proposition:

Proposition 7.2. *For fixed degrees $\{d_i\}_{i=1}^n$, the expected number of self-loops S_n in the configuration model satisfies*

$$\mathbb{E}[S_n] \leq \sum_{i=1}^n \frac{d_i^2}{l_n}, \tag{7.2.20}$$

while the expected number of multiple edges M_n satisfies

$$\mathbb{E}[M_n] \leq 2\Big(\sum_{i=1}^n \frac{d_i^2}{l_n}\Big)^2. \tag{7.2.21}$$

Proof. For a vertex i, and for $1 \leq s < t \leq d_i$, we define $I_{st,i}$ to be the indicator of the event that the half-edge s is paired to the half-edge t. Here we number the half-edges, or stubs, of the vertices in an arbitrary way. Then

$$S_n = \sum_{i=1}^n \sum_{1 \leq s < t \leq d_i} I_{st,i}. \tag{7.2.22}$$

Therefore,

$$\begin{aligned}
\mathbb{E}[S_n] &= \sum_{i=1}^n \sum_{1 \leq s < t \leq d_i} \mathbb{E}[I_{st,i}] \\
&= \sum_{i=1}^n \frac{1}{2} d_i(d_i - 1)\mathbb{E}[I_{12,i}],
\end{aligned} \tag{7.2.23}$$

since the probability of producing a self-loop by pairing the stubs s and t does not depend on s and t. Now, $\mathbb{E}[I_{12,i}]$ is equal to the probability that stubs 1 and 2 are paired to each other, which is equal to $(l_n - 1)^{-1}$. Therefore,

$$\mathbb{E}[S_n]\frac{1}{2}\sum_{i=1}^n \frac{d_i(d_i - 1)}{l_n - 1} \leq \sum_{i=1}^n \frac{d_i^2}{l_n}. \tag{7.2.24}$$

Similarly, for vertices i and j, and for $1 \leq s_1 < s_2 \leq d_i$ and $1 \leq t_1 \neq t_2 \leq d_j$, we define $I_{s_1 t_1, s_2 t_2, ij}$ to be the indicator of the event that the stub s_1 is paired to the stub t_1 and stub s_2 is paired to the stub t_2. If $I_{s_1 t_1, s_2 t_2, ij} = 1$ for some $s_1 t_1$ and $s_2 t_2$, then there are multiple edges between vertices i and j. It follows that

$$M_n \leq \frac{1}{2} \sum_{1 \leq i \neq j \leq n} \sum_{1 \leq s_1 < s_2 \leq d_i} \sum_{1 \leq t_1 \neq t_2 \leq d_j} I_{s_1 t_1, s_2 t_2, ij}, \tag{7.2.25}$$

so that

$$\begin{aligned}
\mathbb{E}[M_n] &\leq \frac{1}{2} \sum_{1 \leq i \neq j \leq n} \sum_{1 \leq s_1 < s_2 \leq d_i} \sum_{1 \leq t_1 \neq t_2 \leq d_j} \mathbb{E}[I_{s_1 t_1, s_2 t_2, ij}] \\
&= \frac{1}{4} \sum_{1 \leq i \neq j \leq n} d_i(d_i - 1)d_j(d_j - 1)\mathbb{E}[I_{11,22,ij}].
\end{aligned} \tag{7.2.26}$$

Now, since $I_{11,22,ij}$ is an indicator, $\mathbb{E}[I_{11,22,ij}]$ is the probability that $I_{11,22,ij} = 1$, which is equal to the probability that stub 1 of vertex i and stub 1 of vertex j, as well as stub 2 of vertex i and stub 2 of vertex j are paired, which is equal to

$$\mathbb{E}[I_{11,22,ij}] = \frac{1}{(l_n - 1)(l_n - 3)}. \tag{7.2.27}$$

Therefore,

$$
\begin{aligned}
\mathbb{E}[M_n] &\leq \sum_{i,j=1}^{n} \frac{d_i(d_i - 1)d_j(d_j - 1)}{4(l_n - 1)(l_n - 3)} \\
&= \frac{\left(\sum_{i=1}^{n} d_i(d_i - 1) \right)^2}{4(l_n - 1)(l_n - 3)} \\
&\leq \frac{2 \left(\sum_{i=1}^{n} d_i(d_i - 1) \right)^2}{4l_n^2},
\end{aligned}
\tag{7.2.28}
$$

where we use that $8(l_n - 1)(l_n - 3) \geq l_n^2$ since $l_n \geq 4$. Since $M_n = 0$ with probability one when $l_n \leq 3$, the claim follows. $\qquad\square$

To complete the proof of Theorem 7.1, we note that, by (7.2.7), we have that $\max_{i=1}^{n} d_i = o(\sqrt{n})$. Therefore, applying Proposition 7.2, we obtain that

$$\mathbb{E}[S_n] \leq \sum_{i=1}^{n} \frac{d_i^2}{l_n} \leq \max_{i=1}^{n} d_i = o(\sqrt{n}), \tag{7.2.29}$$

which proves the bound on $\mathbb{E}[S_n]$ in (7.2.19). The bound on $\mathbb{E}[M_n]$ is similar. $\qquad\square$

7.2.2 Probability that the configuration model yields a simple graph

In this section, we investigate the probability that the configuration model yields a simple graph, i.e., the probability that the graph produced in the configuration model has no self-loops nor multiple edges. We assume that the limit

$$\nu = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{d_i(d_i - 1)}{l_n} = \frac{1}{\mu} \sum_{j=1}^{\infty} j(j-1)p_j, \tag{7.2.30}$$

is finite, where μ is defined in (7.2.5). Then the asymptotics of the probability that the configuration model is simple is derived in the following theorem:

Theorem 7.3 (Probability of simplicity of $\mathrm{CM}(n, \{d_i\}_{i=1}^{n})$). *Assume that $\{d_i\}_{i=1}^{n}$ satisfies (7.2.4) and (7.2.30). Then, the probability that $\mathrm{CM}(n, \{d_i\}_{i=1}^{n})$ is a simple graph is asymptotically equal to $e^{-\frac{\nu}{2} - \frac{\nu^2}{4}}$.*

In Theorem 7.3, we have not assumed that (7.2.5) holds. In fact, it turns out that (7.2.5) follows from (7.2.4) and (7.2.30):

Exercise 7.6. *Prove that (7.2.5) follows from (7.2.4) and (7.2.30).*

Theorem 7.3 is a consequence of the following result:

Proposition 7.4. *Assume that the fixed degree sequence $\{d_i\}_{i=1}^{n}$ satisfies (7.2.4) and (7.2.30). Then (S_n, M_n) converges in distribution to (S, M), where S and M are two independent Poisson random variables with means $\frac{\nu}{2}$ and $\frac{\nu^2}{4}$.*

Indeed, Theorem 7.3 is a simple consequence of Proposition 7.4, since $CM(n, \{d_i\}_{i=1}^n)$ is simple precisely when $S_n = M_n = 0$. By the weak convergence result stated in Proposition 7.4 and the independence of S and M, the probability that $S_n = M_n = 0$ converges to $e^{-\mu_S - \mu_M}$, where μ_S and μ_M are the means of S and M. Using the identification of the means of S and M in Proposition 7.4, this completes the proof of Theorem 7.3. We are left to prove Proposition 7.4.

Proof of Proposition 7.4. Throughout the proof, we shall assume that S and M are two independent Poisson random variables with means $\frac{\nu}{2}$ and $\frac{\nu^2}{4}$.

We make us of Theorem 2.6 which imply that it suffices to prove that the factorial moments converge. Also, S_n is a sum of indicators, so that we can use Theorem 2.7 to identify its factorial moments. For M_n, this is not so clear. However, we define

$$\tilde{M}_n = \sum_{1 \le i < j \le n} \sum_{1 \le s_1 < s_2 \le d_i} \sum_{1 \le t_1 \ne t_2 \le d_j} I_{s_1 t_1, s_2 t_2, ij}, \qquad (7.2.31)$$

so that, by (7.2.25), $M_n \le \tilde{M}_n$. We shall first show that with high probability $M_n = M_n'$. Note that $M_n < \tilde{M}_n$ precisely when there exist vertices $i \ne j$ such that there are at least *three* edges between i and j. The probability that there are at least three edges between i and j is bounded above by

$$\frac{d_i(d_i - 1)(d_i - 2)d_j(d_j - 1)(d_j - 2)}{(l_n - 1)(l_n - 3)(l_n - 5)}. \qquad (7.2.32)$$

Thus, by Boole's inequality, the probability that there exist vertices $i \ne j$ such that there are at least *three* edges between i and j is bounded above by

$$\sum_{i,j=1}^n \frac{d_i(d_i - 1)(d_i - 2)d_j(d_j - 1)(d_j - 2)}{(l_n - 1)(l_n - 3)(l_n - 5)} = o(1), \qquad (7.2.33)$$

since $d_i = o(\sqrt{n})$ as well as $l_n \ge n$. We conclude that the probability that there are i, j such that there are at least three edges between i and j is $o(1)$ as $n \to \infty$. As a result, (S_n, M_n) converges in distribution to (S, M) precisely when $(S_n, \tilde{M}_n)$ converges in distribution to (S, M).

To prove that $(S_n, \tilde{M}_n)$ converges in distribution to (S, M), we use Theorem 2.6 to see that we are left to prove that, for every $s, r = 0, 1, \ldots$,

$$\lim_{n \to \infty} \mathbb{E}[(S_n)_s (\tilde{M}_n)_r] = \left(\frac{\nu}{2}\right)^s \left(\frac{\nu^2}{4}\right)^r. \qquad (7.2.34)$$

By Theorem 2.7,

$$\mathbb{E}[(S_n)_s (\tilde{M}_n)_r] = \sideset{}{^*}\sum_{m_1^{(1)}, \ldots, m_s^{(1)} \in \mathcal{I}_1} \sideset{}{^*}\sum_{m_1^{(2)}, \ldots, m_r^{(2)} \in \mathcal{I}_2} \mathbb{P}\big(I_{m_1^{(1)}}^{(1)} = \ldots = I_{m_s^{(1)}}^{(1)} = I_{m_1^{(2)}}^{(2)} = \ldots = I_{m_r^{(2)}}^{(2)} = 1\big), \qquad (7.2.35)$$

where

$$\mathcal{I}_1 = \{(st, i) : i \in \{1, \ldots, n\}, 1 \le s < t \le d_i\}, \qquad (7.2.36)$$
$$\mathcal{I}_2 = \{(s_1 t_1, s_2 t_2, i, j) : 1 \le i < j \le n, 1 \le s_1 < s_2 \le d_i, 1 \le t_1 \ne t_2 \le d_j\}, \qquad (7.2.37)$$

and where, for $m^{(1)} = (st, i) \in \mathcal{I}_1$ and $m^{(2)} = (s_1 t_1, s_2 t_2, i, j) \in \mathcal{I}_2$,

$$I_{m^{(1)}}^{(1)} = I_{st, i}, \qquad (7.2.38)$$
$$I_{m^{(2)}}^{(2)} = I_{s_1 t_1, s_2 t_2, ij}. \qquad (7.2.39)$$

Now, by the fact that all stubs are uniformly paired, we have that

$$\mathbb{P}\big(I^{(1)}_{m^{(1)}_1} = \ldots = I^{(1)}_{m^{(1)}_s} = I^{(2)}_{m^{(2)}_1} = \ldots = I^{(2)}_{m^{(2)}_r} = 1\big) = \frac{1}{(l_n - 1)(l_n - 3)\cdots(l_n - 1 - 2s - 4r)}, \qquad (7.2.40)$$

unless there is a conflict in the attachment rules, in which case

$$\mathbb{P}\big(I^{(1)}_{m^{(1)}_1} = \ldots = I^{(1)}_{m^{(1)}_s} = I^{(2)}_{m^{(2)}_1} = \ldots = I^{(2)}_{m^{(2)}_r} = 1\big) = 0. \qquad (7.2.41)$$

Such a conflict arises precisely when a stub is required to be paired to *two* different other half-stubs. Since the upper bound in (7.2.40) always holds, we arrive at

$$\mathbb{E}[(S_n)_s(\tilde{M}_n)_r] \leq \sum^*_{m^{(1)}_1,\ldots,m^{(1)}_s\in\mathcal{I}_1}\sum^*_{m^{(2)}_1,\ldots,m^{(2)}_r\in\mathcal{I}_2} \frac{1}{(l_n - 1)(l_n - 3)\cdots(l_n - 1 - 2s - 4r)}$$

$$= \frac{|\mathcal{I}_1|(|\mathcal{I}_1| - 1)\cdots(|\mathcal{I}_1| - s + 1)|\mathcal{I}_2|(|\mathcal{I}_2| - 1)\cdots(|\mathcal{I}_2| - r + 1)}{(l_n - 1)(l_n - 3)\cdots(l_n - 1 - 2s - 4r)}. \qquad (7.2.42)$$

Since $|\mathcal{I}_1|, |\mathcal{I}_2|, l_n$ all tend to infinity, and s, r remain fixed, we have that

$$\limsup_{n\to\infty} \mathbb{E}[(S_n)_s(\tilde{M}_n)_r] = \Big(\lim_{n\to\infty}\frac{|\mathcal{I}_1|}{l_n}\Big)^s\Big(\lim_{n\to\infty}\frac{|\mathcal{I}_2|}{l_n^2}\Big)^r. \qquad (7.2.43)$$

Now,

$$\lim_{n\to\infty}\frac{|\mathcal{I}_1|}{l_n} = \lim_{n\to\infty}\frac{1}{l_n}\sum_{i=1}^{n}\frac{d_i(d_i - 1)}{2} = \frac{\nu}{2}, \qquad (7.2.44)$$

by (7.2.30). Further, again by (7.2.30) and also using that $d_i = o(\sqrt{n})$ while $l_n \geq n$,

$$\lim_{n\to\infty}\frac{|\mathcal{I}_2|}{l_n^2} = \lim_{n\to\infty}\frac{1}{l_n^2}\sum_{1\leq i<j\leq n}\frac{d_i(d_i - 1)}{2}d_j(d_j - 1)$$

$$= \Big(\lim_{n\to\infty}\frac{1}{l_n}\sum_{i=1}^{n}\frac{d_i(d_i - 1)}{2}\Big)^2 - \lim_{n\to\infty}\sum_{i=1}^{n}\frac{d_i^2(d_i - 1)^2}{2l_n^2} = \Big(\frac{\nu}{2}\Big)^2. \qquad (7.2.45)$$

This provides the required upper bound.

To prove the matching lower bound, we note that, by (7.2.41),

$$\sum^*_{m^{(1)}_1,\ldots,m^{(1)}_s\in\mathcal{I}_1}\sum^*_{m^{(2)}_1,\ldots,m^{(2)}_r\in\mathcal{I}_2}\frac{1}{(l_n - 1)(l_n - 3)\cdots(l_n - 1 - 2s - 4r)} - \mathbb{E}[(S_n)_s(\tilde{M}_n)_r]$$

$$= \sum^*_{m^{(1)}_1,\ldots,m^{(1)}_s\in\mathcal{I}_1}\sum^*_{m^{(2)}_1,\ldots,m^{(2)}_r\in\mathcal{I}_2}\frac{I_{m^{(1)}_1,\ldots,m^{(1)}_s,m^{(2)}_1,\ldots,m^{(2)}_r}}{(l_n - 1)(l_n - 3)\cdots(l_n - 1 - 2s - 4r)}, \qquad (7.2.46)$$

where $I_{m^{(1)}_1,\ldots,m^{(1)}_s\in\mathcal{I}_1,m^{(2)}_1,\ldots,m^{(2)}_r} = 1$ precisely when there is a conflict in $m^{(1)}_1,\ldots,m^{(1)}_s,m^{(2)}_1,\ldots,m^{(2)}_r$. There is a conflict precisely when there exist a vertex i and one of its stubs s must be paired to two *different* stubs. For this, there has to be a pair of indices in $m^{(1)}_1,\ldots,m^{(1)}_s,m^{(2)}_1,\ldots,m^{(2)}_r$ which create the conflict. There are three such possibilities: (a) the conflict is created by $m^{(1)}_a, m^{(1)}_b$ for some a, b; (b) the conflict is created by $m^{(1)}_a, m^{(2)}_b$ for some a, b; and (c) the conflict is created by $m^{(2)}_a, m^{(2)}_b$ for some a, b. We shall bound each of these possibilities separately.

In case (a), the number of $m_c^{(1)}, c \in \{1, \ldots, s\} \backslash \{a, b\}$ and $m_d^{(2)}, d \in \{1, \ldots, r\}$ is bounded by $|\mathcal{I}_1|^{s-2}|\mathcal{I}_2|^r$. Thus, comparing with (7.2.42), we see that it suffices to prove that the number of conflicting $m_a^{(1)}, m_b^{(1)}$ is $o(|\mathcal{I}_1|^2)$. Now, the number of conflicting $m_a^{(1)}, m_b^{(1)}$ is bounded by

$$\sum_{i=1}^n d_i^3 = o\Big(\sum_{i=1}^n d_i(d_i - 1) \Big)^2, \tag{7.2.47}$$

where we use that $d_i = o(\sqrt{n})$, as required.

In case (b), the number of $m_c^{(1)}, c \in \{1, \ldots, s\} \backslash \{a\}$ and $m_d^{(2)}, d \in \{1, \ldots, r\} \backslash \{b\}$ is bounded by $|\mathcal{I}_1|^{s-1}|\mathcal{I}_2|^{r-1}$, while the number of conflicting $m_a^{(1)}, m_b^{(2)}$ is bounded by

$$\sum_{i=1}^n d_i^3 \sum_{j=1}^n d_j^2 = o\Big(\sum_{i=1}^n d_i(d_i - 1) \Big)^3, \tag{7.2.48}$$

where we again use that $d_i = o(\sqrt{n})$, as required.

In case (c), the number of $m_c^{(1)}, c \in \{1, \ldots, s\}$ and $m_d^{(2)}, d \in \{1, \ldots, r\} \backslash \{a, b\}$ is bounded by $|\mathcal{I}_1|^s|\mathcal{I}_2|^{r-2}$, while the number of conflicting $m_a^{(2)}, m_b^{(2)}$ is bounded by

$$\sum_{i=1}^n d_i^3 \sum_{j=1}^n d_j^2 \sum_{k=1}^n d_k^2 = o\Big(\sum_{i=1}^n d_i(d_i - 1) \Big)^4, \tag{7.2.49}$$

where we again use that $d_i = o(\sqrt{n})$, as required. This completes the proof. $\qquad\square$

Exercise 7.7. *Show that the moments of (X, Y), where (X, Y) are independent Poisson random variables with parameters μ_X and μ_Y are identified by the relations, for $r \geq 1$,*

$$\mathbb{E}[X^r] = \mu_X \mathbb{E}[(X + 1)^{r-1}], \tag{7.2.50}$$

and, for $r, s \geq 1$,

$$\mathbb{E}[X^r Y^s] = \mu_Y \mathbb{E}[X^r (Y + 1)^{s-1}]. \tag{7.2.51}$$

Exercise 7.8. *Give an alternative proof of Proposition 7.4 by using Theorem 2.3(e) together with Exercise 7.7 and the fact that all joint moments of (S_n, M_n) converge to those of (S, M), where S and M are two independent Poisson random variables with means $\frac{\nu}{2}$ and $\frac{\nu^2}{4}$.*

Exercise 7.9. *Compute the average number of occupied triangles in $\mathrm{CM}(n, \{d_i\}_{i=1}^n)$.*

Exercise 7.10. *Show that the number of occupied triangles in $\mathrm{CM}(n, \{d_i\}_{i=1}^n)$ converges to a Poisson random variable when (7.2.30) holds.*

7.2.3   The configuration model conditioned on simplicity

In this section, we describe some results for the configuration model when we condition on being simple.

Proposition 7.5 (Uniform graphs with given degree sequence). *For any degree sequence $\{d_i\}_{i=1}^n$, and conditionally on the event $\{\mathrm{CM}(n, \{d_i\}_{i=1}^n) \text{ is a simple graph}\}$, $\mathrm{CM}(n, \{d_i\}_{i=1}^n)$ is a uniform simple random graph with the prescribed degree sequence.*

Proof. We recall that the graph in the configuration model is produced by a uniform matching of the corresponding configuration of stubs. By Exercise 7.11 below, we note that, conditionally on the matching producing a simple graph, the conditional distribution of the configuration is uniform over all configurations which are such that the corresponding graph is simple:

Exercise 7.11. *Let $\mathbb{P}$ be a uniform distribution on some finite state space X, and let U be a uniform random variable on X. Let $Y \subseteq X$ be a non-empty subset of X. Show that the conditional probability $\mathbb{P}(\cdot | U \in Y)$ given that U is in Y is the uniform distribution on Y.*

We conclude that Proposition 7.5 is equivalent to the statement that every simple graph has an *equal* number of configurations contributing to it. To show this, we note that we obtain the *same* resulting graph when we permute the stubs. When the graph is *simple* to start with, the *only* configurations that produce the same simple graph are obtained by permuting the stubs of any configuration that produces this graph. There are precisely

$$\prod_{i=1}^{n} d_i! \tag{7.2.52}$$

ways of permuting the different stubs. Since this number does not depend on the resulting graph apart from its degrees, we obtain that to any simple graph with the right degree sequence, there are an equal number of configurations. This completes the proof of Proposition 7.5. $\qquad \square$

Exercise 7.12. *Assume that the fixed degree sequence $\{d_i\}_{i=1}^n$ satisfies (7.2.4) and (7.2.5). Furthermore, we assume that (7.2.30) is satisfied. Let T_n denote the number of triangles in $\mathrm{CM}(n, \{d_i\}_{i=1}^n)$, i.e., the number of i, j, k such that $i < j < j$ and such that there are edges between i and j, between j and k and between k and i. Show that (S_n, M_n, T_n) converges to three independent Poisson random variables and compute their asymptotic parameters.*

An important corollary of Theorem 7.3 is about the number of graphs with a given degree sequence:

Corollary 7.6 (The number of graphs with given degree sequence). *Assume that the degree sequence $\{d_i^{(n)}\}_{i=1}^n = \{d_i\}_{i=1}^n$ satisfies (7.2.4) and (7.2.30), and that $l_n = \sum_{i=1}^n d_i$ is even. Then, the number of simple graphs with degree sequence $\{d_i\}_{i=1}^n$ is equal to*

$$e^{-\frac{\nu}{2} - \frac{\nu^2}{4}} \frac{(l_n - 1)!!}{\prod_{i=1}^n d_i!} (1 + o(1)). \tag{7.2.53}$$

e:numb-SG

In Corollary 7.6, we *explicitly* write the dependence on n in $\{d_i^{(n)}\}_{i=1}^n$ to avoid confusion. Indeed, when $\{d_i\}_{i=1}^n$ were *independent* on n, and when $l_n = \sum_{i=1}^n d_i$ would be even for all n, then we must have that d_i is even for all $i = 1, 2, \dots$, which is not what we have in mind.

A special case of the configuration model is when all degrees are equal to some r. In this case, when we condition on the fact that the resulting graph in the configuration model to be simple, we obtain a *uniform regular random graph*. Uniform regular random graphs can be seen as a finite approximation of a regular tree. In particular, Corollary 7.6 implies that, when nr is even, the number of regular r-ary graphs is equal to

$$e^{-\frac{(r-1)}{2} - \frac{(r-1)^2}{4}} \frac{(rn - 1)!!}{(r!)^n} (1 + o(1)). \tag{7.2.54}$$

e:numb-rarySG

Exercise 7.13. *Prove (7.2.54).*

Exercise 7.14. *Assume that the fixed degree sequence $\{d_i\}_{i=1}^n$ satisfies (7.2.4) and (7.2.30). Compute the number of simple graphs with degree sequence $\{d_i\}_{i=1}^n$ not containing any triangle. Hint: use Exercise 7.12.*

7.2.4   Configuration model, uniform simple random graphs and the GRG

In this section, we shall investigate the relations between the configuration model, uniform simple random graphs with given degrees, and the generalized random graph with given weights. These results are 'folklore' in the random graph community, and allow to use the configuration model to prove results for several other models.

A further consequence of Theorem 7.3 is that it allows to prove a property for uniform graphs with a given degree sequence by proving it for the configuration model with that degree sequence:

Corollary 7.7 (Uniform graphs with given degree sequence and $\mathrm{CM}(n, \{d_i\}_{i=1}^n)$). *Assume that the fixed degree sequence $\{d_i^{(n)}\}_{i=1}^n = \{d_i\}_{i=1}^n$ satisfies (7.2.4) and (7.2.30), and that $l_n = \sum_{i=1}^n d_i$ is even. Then, an event $\mathcal{E}_n$ occurs with high probability for a uniform simple random graph with degrees $\{d_i\}_{i=1}^n$ when it occurs with high probability for $\mathrm{CM}(n, \{d_i\}_{i=1}^n)$.*

Corollary 7.7 allows a simple strategy to study proporties of uniform simple random graphs with a prescribed degree sequence. Indeed, $\mathrm{CM}(n, \{d_i\}_{i=1}^n)$ can be constructed in a rather simple manner, which makes it easier to prove properties for $\mathrm{CM}(n, \{d_i\}_{i=1}^n)$ than it is for a uniform random graph with degrees $\{d_i\}_{i=1}^n$. For completeness, we now prove the above statement. Also, when (7.2.4) and (7.2.30) hold, and that $l_n = \sum_{i=1}^n d_i$ is even, then

Proof. Let $\mathbb{P}'$ denote the law of a uniform simple random graph with degrees $\{d_i\}_{i=1}^n$, and let $\mathbb{P}$ denote the law of $\mathrm{CM}(n, \{d_i\}_{i=1}^n)$. We note that we need to prove that if $\lim_{n\to\infty} \mathbb{P}(\mathcal{E}_n^c) = 0$, then also $\lim_{n\to\infty} \mathbb{P}'(\mathcal{E}_n^c) = 0$. Let $\mathcal{S}_n$ denote the event that $\mathrm{CM}(n, \{d_i\}_{i=1}^n)$ is simple. We note that, by Proposition 7.5,

$$\mathbb{P}'(\mathcal{E}_n^c) = \mathbb{P}(\mathcal{E}_n^c | \mathcal{S}_n) = \frac{\mathbb{P}(\mathcal{E}_n^c \cap \mathcal{S}_n)}{\mathbb{P}(\mathcal{S}_n)} \leq \frac{\mathbb{P}(\mathcal{E}_n^c)}{\mathbb{P}(\mathcal{S}_n)}. \tag{7.2.55}$$

By Theorem 7.3, for which the assumptions are satisfied by the hypotheses in Corollary 7.7, $\limsup_{n\to\infty} \mathbb{P}(\mathcal{S}_n) > 0$. Moreover, $\lim_{n\to\infty} \mathbb{P}(\mathcal{E}_n^c) = 0$, so that

$$\lim_{n\to\infty} \frac{\mathbb{P}(\mathcal{E}_n^c)}{\mathbb{P}(\mathcal{S}_n)} = 0.$$

This completes the proof of Corollary 7.7. $\square$

As a consequence of Proposition 7.5 and Theorem 6.11, we see that the GRG conditionally on its degrees, and the Configuration Model with the same degrees and conditionally on producing a simple graph, have identically the same distribution. This also partially explains the popularity of the configuration model: Some results for the Erdős-Rényi random graph are more easily proved by conditioning on the degree sequence, proving the result for the configuration model, and using that the degree distribution of the Erdős-Rényi random graph is very close to a sequence of independent Poisson random variables. See Chapters 9 and 10. We shall formalize this 'folklore' result in the following theorem.

Theorem 7.8 (Relation between $\mathrm{GRG}(n, \{w_i\}_{i=1}^n)$ and $\mathrm{CM}(n, \{d_i\}_{i=1}^n)$). *Let D_i be the degree of vertex i in $\mathrm{GRG}(n, \{w_i\}_{i=1}^n)$ defined in (6.2.1). Assume that $\{D_i\}_{i=1}^n$ satisfies that the limits in (7.2.4) and (7.2.30) hold in probability. Assume that the event $\mathcal{E}_n$ occurs with high probability for $\mathrm{CM}(n, \{D_i\}_{i=1}^n)$. Then $\mathcal{E}_n$ also occurs with high probability for $\mathrm{GRG}(n, \{w_i\}_{i=1}^n)$.*

We note that, by Theorem 6.5, in many cases, the assumption (7.2.4) is satisfied. Also, we note that (7.2.30) is equivalent to the fact that the second moment of the degrees converges to the second moment of the asymptotic degree sequence. These properties are often easier to verify than the event $\mathcal{E}_n$ itself. We also remark that related versions of Theorem 7.8 can be stated with stronger hypotheses on the degrees.

Then, the statement becomes that, when an event $\mathcal{E}_n$ occurs with high probability for $\mathrm{CM}(n, \{d_i\}_{i=1}^n)$ under the assumptions on the degrees, $\mathcal{E}_n$ also occurs with high probability for $\mathrm{GRG}(n, \{w_i\}_{i=1}^n)$.

Proof. We again let $\mathcal{S}_n$ denote the event that $\mathrm{CM}(n, \{D_i\}_{i=1}^n)$ is simple, let $\mathbb{P}'$ denote the law of a uniform simple random graph with degrees $\{D_i\}_{i=1}^n$, and let $\mathbb{P}$ denote the law of $\mathrm{CM}(n, \{D_i\}_{i=1}^n)$. Finally, we let $\mathbb{Q}_{\vec{d}}$ denote the law of $\mathrm{GRG}(n, \{w_i\}_{i=1}^n)$ conditionally on the degrees being equal to $\{d_i\}_{i=1}^n$, and $\mathbb{Q}$ denotes the unconditional law of $\mathrm{GRG}(n, \{w_i\}_{i=1}^n)$. By Theorem 6.11, we have that, conditionally on $\{D_i\}_{i=1}^n$,

$$\mathbb{Q}_{\vec{D}}(\mathcal{E}_n^c) = \mathbb{P}'(\mathcal{E}_n^c). \tag{7.2.56}$$

<div align="right" style="font-family:monospace">e:relGRGURG</div>

Let $\mathcal{P}_n$ denote the event that the degrees of $\mathrm{GRG}(n, \{w_i\}_{i=1}^n)$ satisfy (7.2.4) and (7.2.30). More precisely, we fix an $\varepsilon_n \to 0$ and let

$$\mathcal{P}_n = \Big\{ \big| \nu - \sum_{i=1}^n \frac{d_i(d_i-1)}{l_n} \big| \le \varepsilon_n \Big\} \cap \Big\{ \sum_{k=0}^{\infty} |p_k^{(n)} - p_k| \le \varepsilon_n \Big\}. \tag{7.2.57}$$

By (7.2.4) and Exercise 2.14, we have that there exists an $\varepsilon_n \to 0$ such that $\Big\{ \sum_{k=0}^{\infty} |p_k^{(n)} - p_k| \le \varepsilon_n \Big\}$ has $\mathbb{Q}$-probability converging to 1. Also, by (7.2.30), there exists an $\varepsilon_n \to 0$ such that $\Big\{ \big| \nu - \sum_{i=1}^n \frac{d_i(d_i-1)}{l_n} \big| \le \varepsilon_n \Big\}$ has $\mathbb{Q}$-probability converging to 1. Therefore, for such a ε_n, we have that $\lim_{n\to\infty} \mathbb{Q}(\mathcal{P}_n^c) = 0$.

Then, we split

$$\mathbb{Q}(\mathcal{E}_n^c) \le \mathbb{E}_{\mathbb{Q}}\Big[\mathbb{1}_{\mathcal{P}_n} \mathbb{Q}_{\vec{D}}(\mathcal{E}_n^c) \Big] + \mathbb{Q}(\mathcal{P}_n^c). \tag{7.2.58}$$

By (7.2.56) and Corollary 7.7, and conditionally on $\{D_i\}_{i=1}^n$ such that $\mathcal{P}_n$ holds, $\mathbb{Q}_{\vec{D}}(\mathcal{E}_n^c) \xrightarrow{\mathbb{P}} 0$. Therefore, by dominated convergence (Theorem A.1.2), we obtain that

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}}\Big[\mathbb{1}_{\mathcal{P}_n} \mathbb{Q}_{\vec{D}}(\mathcal{E}_n^c) \Big] = 0,$$

so that we conclude that $\lim_{n\to\infty} \mathbb{Q}(\mathcal{E}_n^c) = 0$, as required. $\square$

We shall make use of Theorem 7.8 in Chapter 10 below.

7.3 Configuration model with i.i.d. degrees

In this section, we apply the results of the previous section to the configuration model with i.i.d. degrees. Indeed, we take the degrees $\{D_i\}_{i=1}^{\infty}$ to be an i.i.d. sequence. Since the total degree $\sum_{i=1}^n D_i$ is with probability close to $1/2$ odd (recall Exercise 7.3), we need to make sure that the total degree is even. Therefore, by convention, we set

$$D_n^{(n)} = D_n + \mathbb{1}_{\{\sum_{i=1}^n D_i \text{ odd}\}}, \tag{7.3.1}$$

<div align="right" style="font-family:monospace">e:Dn-def</div>

and set

$$L_n = \sum_{i=1}^n D_i^{(n)} = \sum_{i=1}^n D_i + \mathbb{1}_{\{\sum_{i=1}^n D_i \text{ odd}\}}. \tag{7.3.2}$$

<div align="right" style="font-family:monospace">e:Ln-def</div>

Often, we shall ignore the effect of the added indicator in the definition of $D_n^{(n)}$, since it shall hardly make any difference.

We note that, similarly to the generalized random graph with i.i.d. weights, the introduction of randomness in the degrees introduces a *double randomness* in the model: firstly the randomness of the

weights, and secondly, the randomness of the pairing of the edges *given* the degrees. Due to this double randomness, we need to investigate the degree sequence $\{P_k^{(n)}\}_{k=1}^{\infty}$ defined by

$$P_k^{(n)} = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{D_i^{(n)}=k\}}. \tag{7.3.3}$$

If we ignore the dependence on n of $D_n^{(n)}$, then we see that $\{P_k^{(n)}\}_{k=1}^{\infty}$ is precisely equal to the empirical distribution of the degrees, which is an i.i.d. sequence. As a result, by the Strong Law of Large Numbers, we have that

$$P_k^{(n)} \xrightarrow{a.s.} p_k \equiv \mathbb{P}(D_1 = k), \tag{7.3.4}$$

so that the empirical distribution of i.i.d. degrees is almost surely close to the probability distribution of each of the degrees. By Exercise 2.14, the above convergence also implies that $d_{\mathrm{TV}}(P^{(n)}, p) \xrightarrow{a.s.} 0$, where $p = \{p_k\}_{k=1}^{\infty}$ and $P^{(n)} = \{P_k^{(n)}\}_{k=1}^{\infty}$.

The main result is the following:

Theorem 7.9 (Degree sequence of erased configuration model with i.i.d. degrees). *Let $\{D_i\}_{i=1}^{n}$ be an i.i.d. sequence of degrees, where D_i has a finite mean. The degree sequence of the erased configuration model $\{\tilde{P}_k^{(n)}\}_{k=1}^{\infty}$ with degrees $\{D_i^{(n)}\}_{i=1}^{n}$ converges to $\{p_k\}_{k=1}^{\infty}$. More precisely,*

$$\mathbb{P}(\sum_{k=1}^{\infty} |\tilde{P}_k^{(n)} - p_k| \geq \varepsilon) \to 0. \tag{7.3.5}$$

Proof. By Exercise 2.14, we see that

$$\sum_{k=1}^{\infty} |\tilde{P}_k^{(n)} - p_k| \xrightarrow{\mathbb{P}} 0 \tag{7.3.6}$$

precisely when

$$\max_{k=1}^{\infty} |\tilde{P}_k^{(n)} - p_k| \xrightarrow{\mathbb{P}} 0. \tag{7.3.7}$$

As a result, we need to prove that

$$\mathbb{P}(\max_{k=1}^{\infty} |\tilde{P}_k^{(n)} - p_k| \geq \varepsilon) \to 0. \tag{7.3.8}$$

For this, we note that

$$\mathbb{P}(\max_{k=1}^{\infty} |\tilde{P}_k^{(n)} - p_k| \geq \varepsilon) \leq \mathbb{P}(\max_{k=1}^{\infty} |\tilde{P}_k^{(n)} - P_k^{(n)}| \geq \varepsilon/2) + \mathbb{P}(\max_{k=1}^{\infty} |P_k^{(n)} - p_k| \geq \varepsilon/2). \tag{7.3.9}$$

Since $d_{\mathrm{TV}}(P^{(n)}, p) \xrightarrow{a.s.} 0$, also $d_{\mathrm{TV}}(P^{(n)}, p) \xrightarrow{\mathbb{P}} 0$, so that

$$\mathbb{P}(\max_{k=1}^{\infty} |P_k^{(n)} - p_k| \geq \varepsilon/2) \to 0. \tag{7.3.10}$$

We continue to study $\mathbb{P}(\max_{k=1}^{\infty} |\tilde{P}_k^{(n)} - P_k^{(n)}| \geq \varepsilon/2)$, We first give the proof when we assume that the degrees are bounded. In this case, we note that, by (7.2.14) and (7.2.18)

$$\mathbb{P}(\max_{k=1}^{\infty} |\tilde{P}_k^{(n)} - P_k^{(n)}| \geq \varepsilon/2) \leq \mathbb{P}(2(S_n + 2M_n) \geq \varepsilon n/2) \leq \frac{4}{\varepsilon n} (\mathbb{E}[S_n] + 2\mathbb{E}[M_n]). \tag{7.3.11}$$

By Proposition 7.2, $\mathbb{E}[S_n]$ and $\mathbb{E}[M_n]$ are bounded.

When the degrees D_1 are unbounded, we need to refine the above argument. Note that, for any $a_n \to \infty$,

$$\sum_{k>a_n} |\tilde{P}_n^{(n)} - P_n^{(n)}| \leq \frac{1}{n} \sum_{k>a_n} \sum_{i=1}^n \left(\mathbb{1}_{\{\tilde{D}_i=k\}} + \mathbb{1}_{\{D_i=k\}} \right)$$

$$\leq \frac{2}{n} \sum_{i=1}^n \mathbb{1}_{\{D_i \geq a_n\}}, \tag{7.3.12}$$

so that, by the Markov inequality (Theorem 2.14), we have that, for any $a_n \to \infty$,

$$\mathbb{P}(\max_{k>a_n} |\tilde{P}_n^{(n)} - P_n^{(n)}| \geq \varepsilon/2) \leq \mathbb{P}\left(\frac{2}{n} \sum_{i=1}^n \mathbb{1}_{\{D_i \geq a_n\}} \geq \varepsilon n/2 \right) \tag{7.3.13}$$

$$\leq \frac{4}{\varepsilon} \mathbb{P}(D_1 \geq a_n) = o(1).$$

We shall choose a_n appropriately below, and first conclude that we need to show that, for some $a_n \to \infty$,

$$\mathbb{P}(\max_{k \leq a_n} |\tilde{P}_k^{(n)} - P_k^{(n)}| \geq \varepsilon/2) = o(1). \tag{7.3.14}$$

For this, we note that, by (7.2.29), the expected number of self-loops is bounded by $\max_{i=1}^n D_i = o(n)$. However, by (7.2.28), the expected number of multiple edges is not necessarily $o(n)$. We denote by M_{n,a_n} the number of multiple edges between vertices with degree at most a_n and any other vertex. Then,

$$M_{n,a_n} \leq \frac{1}{2} \sum_{1 \leq i \neq j \leq n} \mathbb{1}_{\{D_i \leq a_n\}} \sum_{1 \leq s_1 < s_2 \leq D_i} \sum_{1 \leq t_1 \neq t_2 \leq D_j} I_{s_1 t_1, s_2 t_2, ij}, \tag{7.3.15}$$

where we recall that $I_{s_1 t_1, s_2 t_2, ij}$ is defined above (7.2.25). Similarly as in (7.2.28), we obtain

$$\mathbb{E}[M_{n,a_n}] = \sum_{1 \leq i \neq j \leq n} \mathbb{E}\left[\mathbb{1}_{\{D_i \leq a_n\}} \frac{D_i(D_i-1)D_j(D_j-1)}{4(L_n-1)(L_n-3)} \right]$$

$$\leq a_n \mathbb{E}\left[\frac{D_j(D_j-1)}{4(L_n-3)} \right]$$

$$\leq n a_n \mathbb{E}\left[\max_{j=1}^n \frac{D_j}{n} \right]. \tag{7.3.16}$$

We have that $\max_{j=1}^n \frac{D_j}{n}$ converges to zero in probability. As a consequence, by the dominated convergence theorem (Theorem A.1.2), we have that

$$\mathbb{E}\left[\max_{j=1}^n \frac{D_j}{n} \right] = o(1). \tag{7.3.17}$$

We pick

$$a_n = \mathbb{E}\left[\max_{j=1}^n \frac{D_j}{n} \right]^{-1/2} \to \infty, \tag{7.3.18}$$

then we obtain that the expected number of multiple edges originating from vertices with degree at most a_n is bounded by (see (7.3.16))

$$\mathbb{E}[M_{n,a_n}] \leq n \mathbb{E}\left[\max_{j=1}^n \frac{D_j}{n} \right]^{1/2} = o(n). \tag{7.3.19}$$

Thus, we arrive at the conclusion that the erased configuration model satisfies that

$$\lim_{n\to\infty} \mathbb{P}(\max_{k\le a_n} |\tilde{P}_k^{(n)} - P_k^{(n)}| \ge \varepsilon/2) = 0. \tag{7.3.20}$$

$\square$

We next investigate the probability of obtaining a simple graph in $\mathrm{CM}(n, \{D_i\}_{i=1}^n)$:

Theorem 7.10 (Probability of simplicity in $\mathrm{CM}(n, \{D_i\}_{i=1}^n)$). *Let $\{D_i\}_{i=1}^\infty$ be an i.i.d. sequence of degrees, so that D_i are independent copies of a random variable D with $\mathrm{Var}(D) < \infty$ and $\mathbb{P}(D \ge 1) = 1$. Then, the probability that the configuration model yields a simple graph is asymptotically equal to $e^{-\frac{\nu}{2} - \frac{\nu^2}{4}}$, where*

$$\nu = \frac{\mathbb{E}[D(D-1)]}{\mathbb{E}[D]}. \tag{7.3.21}$$

e:nu-defrep

Proof. We note that

$$\frac{1}{n} L_n = \frac{1}{n} \sum_{i=1}^n D_i \xrightarrow{a.s.} \mathbb{E}[D] \ge 1, \tag{7.3.22}$$

while

$$\frac{1}{n} \sum_{i=1}^n D_i(D_i - 1) \xrightarrow{a.s.} \mathbb{E}[D(D-1)]. \tag{7.3.23}$$

Therefore, with probability 1, (7.2.30) is satisfied, and Theorem 7.10 follows from Theorem 7.3. $\square$

We finally investigate the case where the mean is infinite. We start by investigating the case where we condition the degrees to be bounded above by some $a_n = o(n)$.

Theorem 7.11 (Degree sequence of erased configuration model with i.i.d. conditioned infinite mean degrees). *Let $\{D_i^{(n)}\}_{i=1}^n$ be i.i.d. copies of the random variable D_1 conditioned on $D_1 \le a_n$. Then, for every $a_n = o(n)$, the empirical degree distribution of the erased configuration model $\{\tilde{p}_k^{(n)}\}_{k=1}^\infty$ with degrees $\{D_i^{(n)}\}_{i=1}^n$ converges to $\{p_k\}_{k=1}^\infty$. More precisely, with $p_k = \mathbb{P}(D_1 = k)$,*

$$\mathbb{P}(\max_{k=1}^\infty |\tilde{P}_k^{(n)} - p_k| \ge \varepsilon) \to 0. \tag{7.3.24}$$

Theorem 7.11 is similar in spirit to Theorem 6.8 for the generalized random graph, and is left as an exercise:

Exercise 7.15. *Adapt the proof of Theorem 7.9 to prove Theorem 7.11.*

$\square$

We continue by studying the erased configuration model with infinite mean degrees in the unconditioned case. We assume that (2.6.8) holds, i.e., that there exists a slowly varying function $x \mapsto L_D(x)$ for which

$$1 - F_D(x) = x^{1-\tau} L_D(x), \tag{7.3.25}$$

e:[1-FD]assinfmeanrep

where $\tau \in (1, 2)$. We now investigate the degree sequence in the configuration model with infinite mean degrees, where we do not condition the degrees to be at most a_n. We shall make substantial use of Theorem 2.28. In order to describe the result, we need a few definitions. We define the (random)

probability distribution $P = \{P_i\}_{i=1}^{\infty}$ as follows. Let, as in Theorem 2.28, $\{E_i\}_{i=1}^{\infty}$ be i.i.d. exponential random variables with parameter 1, and define $\Gamma_i = \sum_{j=1}^{i} E_j$. Let $\{D_i\}_{i=1}^{\infty}$ be an i.i.d. sequence of random variables with distribution function F_D in (7.3.25), and let $D_{(n:n)} \geq D_{(n-1:n)} \geq \cdots \geq D_{(1:n)}$ be the order statistics of $\{D_i\}_{i=1}^{n}$. We recall from Theorem 2.28 that there exists a sequence u_n, with $u_n n^{-1/(\tau-1)}$ slowly varying, such that

$$u_n^{-1}\left(L_n, \{D_{(i)}\}_{i=1}^{\infty}\right) \xrightarrow{d} \left(\sum_{j=1}^{\infty} \Gamma_j^{-1/(\tau-1)}, \{\Gamma_i^{-1/(\tau-1)}\}_{i=1}^{\infty}\right). \tag{7.3.26}$$

<div align="right">e:EVTexpsrep</div>

We abbreviate $\eta = \sum_{j=1}^{\infty} \Gamma_j^{-1/(\tau-1)}$ and $\xi_i = \Gamma_i^{-1/(\tau-1)}$, and let

$$P_i = \frac{\xi_i}{\eta}, \tag{7.3.27}$$

<div align="right">e:PRPDdef</div>

so that, by (7.3.26),

$$\sum_{i=1}^{\infty} P_i = 1. \tag{7.3.28}$$

However, the P_i are all random variables, so that $P = \{P_i\}_{i=1}^{\infty}$ is a random probability distribution. We further write $M_{P,k}$ for a multinomial distribution with parameters k and probabilities $P = \{P_i\}_{i=1}^{\infty}$, and U_{P,D_k} is the number of distinct outcomes of the random variable M_{P,D_k}, where D_k is independent of $P = \{P_i\}_{i=1}^{\infty}$ and the multinomial trials.

Theorem 7.12 (Degree sequence of erased configuration model with i.i.d. infinite mean degrees)**.** *Let $\{D_i\}_{i=1}^{\infty}$ be i.i.d. copies of a random variable D_1 having distribution function F_D satisfying (7.3.25). Fix $k \in \mathbb{N}$. The degree of vertex k in the erased configuration model with degrees $\{D_i\}_{i=1}^{n}$ converges in distribution to the random variable U_{P,D_k}, where $P = \{P_i\}_{i=1}^{\infty}$ is given by (7.3.27), and the random variables D_k and $P = \{P_i\}_{i=1}^{\infty}$ are independent.*

Theorem 7.12 is similar in spirit to Theorem 6.9 for the generalized random graph.

Proof. We fix vertex k, and note that its degree is given by D_k. With high probability, we have that $D_k \leq \log n$, so that D_k is not one of the largest order statistics. Therefore, D_k is independent of $(\eta, \xi_1, \xi_2, \dots)$. The vertex k now has D_k stubs, which need to be connected to other stubs. The probability that any stub is connected to the vertex with the jth largest degree is asymptotic to

$$P_j^{(n)} = \frac{D_{(n-j+1:n)}}{L_n}, \tag{7.3.29}$$

where, by Theorem 2.28 (see also (7.3.26)),

$$\{P_j^{(n)}\}_{j=1}^{\infty} \xrightarrow{d} \{\frac{\xi_j}{\eta}\}_{j=1}^{\infty}. \tag{7.3.30}$$

Moreover, the vertices to which the D_k stubs are connected are close to being independent, when $D_k \leq \log n$. As a result, the D_k stubs of vertex k are paired to D_k vertices, and the number of edges of vertex k that are paired to the vertex with the ith largest degree are asymptotically given by the ith coordinate of $M_{P^{(n)},D_k}$. The random variable $M_{P^{(n)},D_k}$ converges in distribution to M_{P,D_k}. We note that in the erased configuration model, the degree of the vertex k is equal to the number of *distinct* vertices to which k is connected, which is therefore equal to the number of distinct outcomes of the random variable M_{P,D_k}, which, by definition, is equal to U_{P,D_k}. $\qquad \square$

We finally investigate the properties of the degree distribution, to obtain an equivalent result as in Exercise 6.15.

Theorem 7.13 (Power law with exponent 2 for erased configuration model with infinite mean degrees)**.** *Let the distribution function F_D of D_k satisfy (7.3.25) with $L_D(x) = 1$. Then, the asymptotic degree of vertex k, which is given by U_{P,D_k} satisfies that*

$$\mathbb{P}(U_{P,D_k} \geq x) \leq x^{-1}. \tag{7.3.31}$$

The result in Theorem 7.13 is similar in spirit to Exercise 6.15. It would be of interest to prove a precise identity here as well.

Proof. We give a sketch of proof only. We condition on $D_k = \lceil x^b \rceil$, for some $b \geq 1$. Then, in order that $U_{P,D_k} \geq x$, at least $x/2$ values larger than $x/2$ need to be chosen. By (2.6.17), we have that the probability that value k is chosen, for some large value k, is close to $\frac{k^{-1/(\tau-1)}}{\eta}$. Therefore, the probability that a value at least k is chosen is close to

$$\frac{k^{-1/(\tau-1)+1}}{\eta} = \frac{k^{\frac{\tau-2}{\tau-1}}}{\eta}. \tag{7.3.32}$$

Moreover, conditionally on $D_k = \lceil x^b \rceil$, the number of values larger than $x/2$ that are chosen is equal to a Binomial random variable with $\lceil x^b \rceil$ trials and success probability

$$q_x = x^{\frac{\tau-2}{\tau-1}}/\eta. \tag{7.3.33}$$

Therefore, conditionally on $D_k = \lceil x^b \rceil$, and using Theorem 2.18, the probability that at least $x/2$ values larger than $x/2$ are chosen is negligible when, for some sufficiently large $C > 0$,

$$|x^b q_x - \frac{x}{2}| \geq C \log x \sqrt{x^b q_x}. \tag{7.3.34}$$

Equations (7.3.33) and (7.3.34) above imply that $b = 1 - \frac{\tau-2}{\tau-1} = \frac{1}{\tau-1}$. As a result, we obtain that

$$\mathbb{P}(U_{P,D_k} \geq x) \leq \mathbb{P}(D_k \geq \lceil x^b \rceil) \leq x^{-b(\tau-1)} = x^{-1}. \tag{7.3.35}$$

$\square$

7.4   Notes and discussion

Notes on Section 7.1.   The configuration model has a long history. It was introduced in [27] to study uniform random graphs with a given degree sequence (see also [40, Section 2.4]). This also explains the emphasis on the study of the probability for the graph to be simple. It was further studied in [157, 158], where it was investigated when the resulting graph has a giant component. The setting in [157, 158] is slightly different, since in [157, 158], it is assumed that the degrees are *fixed*, while we assume that the degrees are i.i.d.

Notes on Section 7.2.   The result in Theorem 7.1 can be found in [121]. The term erased configuration model is first used in [53, Section 2.1].

Notes on Section 7.3.   A version of Theorem 7.9 can be found in [53].

Chapter 8

Preferential attachment models

The generalized random graph model and the configuration model described in Chapters 6 and 7 are *static* models, i.e., the size of the graph is *fixed*, and we have not modeled the *growth* of the graph. There is a large body of work investigating *dynamical* models for complex networks, often in the context of the World-Wide Web. In various forms, such models have been shown to lead to power-law degree sequences, and, thus, they *explain* the occurrence of power law degree sequences in random graphs. The existence of power-law degree sequences in various real networks is quite striking, and models offering a convincing explanation can teach us about the mechanisms which give rise to their scale-free nature.

A possible and convincing explanation for the occurrence of power-law degree sequences is offered by the *preferential attachment paradigm*. In the preferential attachment model, vertices are added sequentially with a number of edges connected to them. These edges are attached to a receiving vertex with a probability proportional to the degree of the receiving vertex at that time, thus favoring vertices with large degrees. For this model, it is shown that the number of vertices with degree k decays proportionally to k^{-3} [46], and this result is a special case of the more general result that we shall describe in this chapter.

The idea behind preferential attachment is simple. In an evolving graph, i.e., a graph that evolves in time, the newly added vertices are connected to the already existing vertices. In an Erdős-Rényi random graph, which can also be formulated as an evolving graph, where edges are added and removed, these edges would be connected to each individual with equal probability.

Exercise 8.1. *Give a dynamical model for the Erdős-Rényi random graph, where at each time n we add a single individual, and where at time n the graph is equal to* $\mathrm{ER}(n, p)$. *See also the dynamic description of the Norros-Reittu model on Page 151.*

Now think of the newly added vertex as a new individual in a social population, which we model as a graph by letting the individuals be the vertices and the edges be the acquaintance relations. Is it then realistic that the edges connect to each already present individual with equal probability, or is the newcomer more likely to get to know socially active individuals, who already know many people? If the latter is true, then we should forget about equal probabilities for receiving ends of the edges of the newcomer, and introduce a bias in his/her connections towards more social individuals. Phrased in a mathematical way, it should be more likely that the edges be connected to vertices that already have a high degree. A possible model for such a growing graph was proposed by Barabási and Albert [21], and has incited an enormous research effort since.

Strictly speaking, Barabási and Albert in [21] were not the first to propose such a model, and we shall start by referring to the old literature on the subject. Yule [209] was the first to propose a growing model

where preferential attachment is present, in the context of the evolution of species. He derives the power law distribution that we shall also find in this chapter. Simon [191] provides a more modern version of the preferential attachment model, as he puts it

> *"Because Yule's paper predates the modern theory of stochastic processes, his derivation was necessarily more involved than the one we shall employ here."*

The stochastic model of Simon is formulated in the context of the occurrence of words in large pieces of text (as in [210]), and is based on two assumptions, namely (i) that the probability that the $(k+1)^{\text{st}}$ word is a word that has already appeared exactly i times is proportional to the number of occurrences of words that have occurred exactly i times, and (ii) that there is a constant probability that the $(k+1)^{\text{st}}$ word is a new word. Together, these two assumptions give rise to frequency distributions of words that obey a power law, with a power-law exponent that is a simple function of the probability of adding a new vertex. We shall see a similar effect occurring in this chapter. A second place where the model studied by Simon and Yule can be found is in work by Champernowne [57], in the context of income distributions in populations.

In their paper, Barabási and Albert [21] describe the preferential attachment graph informally as follows:

> *"To incorporate the growing character of the network, starting with a small number (m_0) of vertices, at every time step we add a new vertex with $m(\leq m_0)$ edges that link the new vertex to m different vertices already present in the system. To incorporate preferential attachment, we assume that the probability Π that a new vertex will be connected to a vertex i depends on the connectivity k_i of that vertex, so that $\Pi(k_i) = k_i / \sum_j k_j$. After t time steps, the model leads to a random network with $t + m_0$ vertices and mt edges."*

This description of the model is informal, but it must have been given precise meaning in [21] (since, in particular, Barabási and Albert present simulations of the model predicting a power-law degree sequence with exponent close to $\tau = 3$). The model description does not explain how the first edge is connected (note that at time $t = 1$, there are no edges, so the first edge can not be attached according to the degrees of the existing vertices), and does not give the dependencies between the m edges added at time t. We are left wondering whether these edges are independent, whether we allow for self-loops, whether we should update the degrees after each attachment of a single edge, etc. In fact, each of these choices has, by now, been considered in the literature, and the results, in particular the occurrence of power laws and the power-law exponent, do not depend sensitively on the respective choices. See Section 8.7 for an extensive overview of the literature on preferential attachment models.

The first to investigate the model rigorously, were Bollobás, Riordan, Spencer and Tusnady [46]. They complain heavily about the lack of a formal definition in [21], arguing that

> *"The description of the random graph process quoted above (i.e, in [21], edt.) is rather imprecise. First, as the degrees are initially zero, it is not clear how the process is started. More seriously, the expected number of edges linking a new vertex v to earlier vertices is $\sum_i \Pi(k_i) = 1$, rather than m. Also, when choosing in one go a set S of m earlier vertices as the neighbors of v, the distribution of S is not specified by giving the marginal probability that each vertex lies in S."*

One could say that these differences in formulations form the heart of much confusion between mathematicians and theoretical physicists. To resolve these problems, choices had to be made, and these choices were, according to [46], made first in [44], by specifying the initial graph to consist of a vertex with m self-loops, and that the degrees will be updated in the process of attaching the m edges. This model will be described in full detail in Section 8.1 below.

This chapter is organized as follows. In Section 8.1, we introduce the model. In Section 8.2, we investigate how the degrees of fixed vertices evolve as the graph grows. In Section 8.3, we investigate the degree sequences in preferential attachment models. The main result is Theorem 8.2, which states that the preferential attachment model has a power-law degree sequence. The proof of Theorem 8.2 consists of two key steps, which are formulated and proved in Sections 8.4 and 8.5, respectively. In Section 8.6, we investigate the maximal degree in a preferential attachment model. In Section 8.7, we also discuss many related preferential attachment models. We close this chapter with notes and discussion in Section 8.8.

8.1 Introduction to the model

In this chapter, we prove that the preferential attachment modes has a power-law degree sequence. We start by introducing the model. The model we investigate produces a *graph sequence* which we denote by $\{\mathrm{PA}_{m,\delta}(t)\}_{t=1}^{\infty}$, which for every t yields a graph of t vertices and mt edges for some $m = 1, 2, \ldots$ We start by defining the model for $m = 1$. In this case, $\mathrm{PA}_{1,\delta}(1)$ consists of a single vertex with a single self-loop. We denote the vertices of the graph by $v_1, v_2, v_3, \ldots$, so that the vertices of $\mathrm{PA}_{1,\delta}(t)$ are equal to $\{v_1, \ldots, v_t\}$. We denote the degree of vertex v_i in $\mathrm{PA}_{1,\delta}(t)$ by $D_i(t)$, where a self-loop increases the degree by 2.

Then, conditionally on $\mathrm{PA}_{1,\delta}(t)$, the growth rule to obtain $\mathrm{PA}_{1,\delta}(t + 1)$ is as follows. We add a single vertex v_{t+1} having a single edge. This edge is connected to a second end point, which is equal to v_{t+1} with probability $(1 + \delta)/(t(2 + \delta) + (1 + \delta))$, and to a vertex $v_i \in \mathrm{PA}_{1,\delta}(t)$ with probability $(D_i(t) + \delta)/(t(2 + \delta) + (1 + \delta))$, where $\delta \geq -1$ is a parameter of the model. Thus,

$$\mathbb{P}\big(v_{t+1} \to v_i | \mathrm{PA}_{1,\delta}(t)\big) = \begin{cases} \frac{1+\delta}{t(2+\delta)+(1+\delta)} & \text{for } i = t + 1, \\ \frac{D_i(t)+\delta}{t(2+\delta)+(1+\delta)} & \text{for } i = 1, \ldots, t. \end{cases} \qquad \text{(8.1.1)}$$

<div align="right" style="font-size:small">e:growthrulePA</div>

Exercise 8.2. *Verify that $D_i(t) \geq 1$ for all i and t, so that $D_i(t) + \delta \geq 0$ for all $\delta \geq -1$.*

Exercise 8.3. *Verify that the probabilities in* (8.1.1) *sum up to one.*

The model with $m > 1$, is defined in terms of the model for $m = 1$ as follows. We start with $\mathrm{PA}_{1,\delta'}(mt)$ with δ' given by $\delta' = \delta/m$, and denote the vertices in $\mathrm{PA}_{1,\delta'}(mt)$ by $v_1', \ldots, v_{mt}'$. Then we identify $v_1', \ldots, v_m'$ in $\mathrm{PA}_{1,\delta'}(mt)$ to be v_1 in $\mathrm{PA}_{m,\delta}(t)$, and $v_{m+1}', \ldots, v_{2m}'$ in $\mathrm{PA}_{1,\delta'}(mt)$ to be v_2 in $\mathrm{PA}_{m,\delta}(t)$, and, more generally, $v_{(j-1)m+1}', \ldots, v_{jm}'$ in $\mathrm{PA}_{1,\delta'}(mt)$ to be v_j in $\mathrm{PA}_{m,\delta}(t)$. This defines the model for general $m \geq 1$. We note that $\mathrm{PA}_{m,\delta}(t)$ is a multigraph with precisely t vertices and mt edges, so that the total degree is equal to $2mt$.

In order to explain the description of $\mathrm{PA}_{m,\delta}(t)$ in terms of $\mathrm{PA}_{1,\delta'}(mt)$, we note that an edge in $\mathrm{PA}_{1,\delta'}(mt)$ is attached to vertex v_k' with probability proportional to the weight of vertex v_k', where the weight is equal to the degree of vertex v_k' plus δ'. Now, vertices $v_{(j-1)m+1}', \ldots, v_{jm}'$ in $\mathrm{PA}_{1,\delta'}(mt)$ are identified or collapsed to vertex v_j in $\mathrm{PA}_{m,\delta}(t)$. Thus, an edge in $\mathrm{PA}_{m,\delta}(t)$ is attached to vertex v_j with probability proportional to the total weight of the vertices $v_{(j-1)m+1}', \ldots, v_{jm}'$. Since the sum of the degrees of the vertices $v_{(j-1)m+1}', \ldots, v_{jm}'$ is equal to the degree of vertex v_j, this probability is proportional to the degree of vertex v_j in $\mathrm{PA}_{m,\delta}(t)$ plus δ precisely when $\delta' = \delta/m$. We note that in the above construction and for $m \geq 2$, the degrees are updated after each edge is attached. This is what we refer to as *intermediate updating of the degrees.*

The important feature of the model is that edges are more likely to be connected to vertices with large degrees, thus making the degrees even larger. This effect is called *preferential attachment.* Preferential attachment may explain *why* there are quite large degrees. Therefore, the preferential attachment model
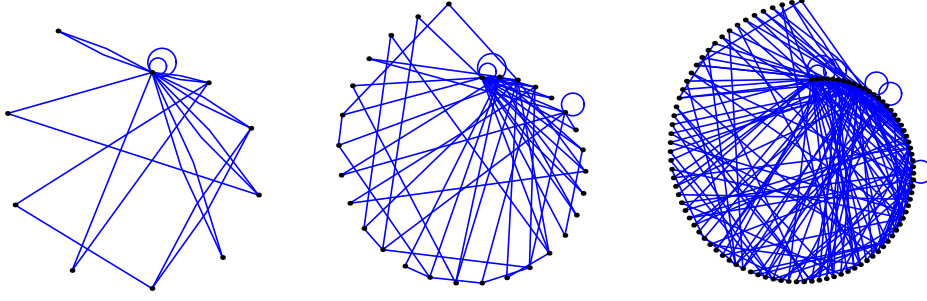
Figure 8.1: Preferential attachment random graph with $m = 2$ and $\delta = 0$ of sizes 10, 30 and 100.

is sometimes called the *Rich-get-Richer model*. It is quite natural to believe in preferential attachment in many real networks. For example, one is more likely to get to know a person who already knows many people, making preferential attachment not unlikely in social networks. However, the precise form of preferential attachment in (8.1.1) is only one possible example.

The above model is a slight variation of models that have appeared in the literature. The model with $\delta = 0$ is the *Barabási-Albert model*, which has received substantial attention in the literature and which was first formally defined in [44]. We have added the extra parameter δ to make the model more general. We denote the probability measure of the random graph $\{\mathrm{PA}_{m,\delta}(t)\}_{t=1}^{\infty}$, where $\delta > -m$ is fixed, by $\mathbb{P}$, and the corresponding expectation by $\mathbb{E}$.

The definition of $\{\mathrm{PA}_{m,\delta}(t)\}_{t=1}^{\infty}$ in terms of $\{\mathrm{PA}_{1,\delta'}(t)\}_{t=1}^{\infty}$ is quite convenient. However, we can also equivalently define the model for $m \geq 2$ directly. We start with $\mathrm{PA}_{m,\delta}(1)$ consisting of a single vertex with m self-loops. To construct $\mathrm{PA}_{m,\delta}(t+1)$ from $\mathrm{PA}_{m,\delta}(t)$, we add a single vertex with m edges attached to it. These edges are attached *sequentially* with intermediate updating of the degrees as follows. The eth edge is connected to vertex v_i, for $i = 1, \ldots, t$, with probability proportional to $(D_i(e-1, t) + \delta)$, where, for $e = 1, \ldots, m$, $D_i(e, t)$ is the degree of vertex i after the eth edge is attached, and to vertex v_{t+1} with probability proportional to $(D_{t+1}(e-1, t) + e\delta/m)$, with the convention that $D_{t+1}(0, t) = 1$. This alternative definition makes it perfectly clear how the choices missing in [21] are made. Indeed, the degrees are updated during the process of attaching the edges, and the initial graph at time 1 consists of a single vertex with m self-loops. Naturally, the edges could also be attached sequentially by a different rule, for example by attaching the edges independently according to the distribution for the first edge. Also, one has the choice to allow for self-loops or not. See Figure 8.1 for a realization of $\{\mathrm{PA}_{m,\delta}(t)\}_{t=1}^{\infty}$ for $m = 2$ and $\delta = 0$, and Figure 8.2 for a realization of $\{\mathrm{PA}_{m,\delta}(t)\}_{t=1}^{\infty}$ for $m = 2$ and $\delta = -1$.

Exercise 8.4. *Prove that the alternative definition of $\{\mathrm{PA}_{m,\delta}(t)\}_{t=1}^{\infty}$ is indeed equal to the one obtained by collapsing m consecutive vertices in $\{\mathrm{PA}_{1,\delta'}(t)\}_{t=1}^{\infty}$ when $\delta' = \delta/m$.*

Exercise 8.5. *Show that when $\delta = -1$, the graph $\mathrm{PA}_{1,\delta}(t)$ consists of a self-loop at vertex v_1, and each other vertex is connected to v_1 with precisely one edge. What is the implication of this result for $m > 1$?*

In some cases, it will be convenient to consider a slight variation on the above model where, for $m = 1$, self-loops do not occur. We shall denote this variation by $\{\mathrm{PA}_{m,\delta}^{(b)}(t)\}_{t \geq 2}$ and sometimes refer to this model by *model (b)*. To define $\mathrm{PA}_{m,\delta}^{(b)}(t)$, we let $\mathrm{PA}_{m,\delta}^{(b)}(2)$ consist of two vertices with two edges between them, and we replace the growth rule in (8.1.1) by the rule that, for all $i = 1, \ldots, t$,

$$\mathbb{P}(v_{t+1} \to v_i | \mathrm{PA}_{1,\delta}^{(b)}(t)) = \frac{D_i(t) + \delta}{t(2 + \delta)}. \tag{8.1.2}$$
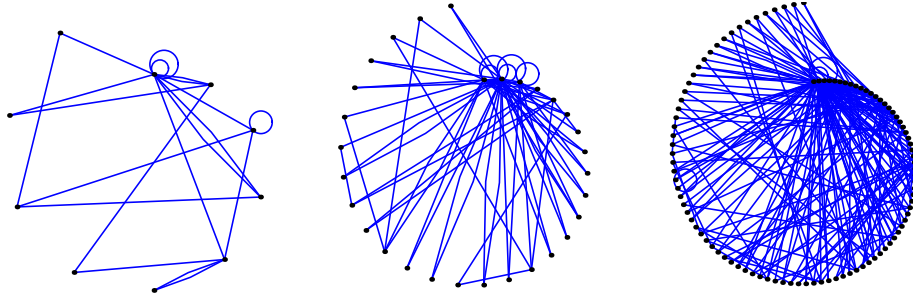
Figure 8.2: Preferential attachment random graph with $m = 2$ and $\delta = -1$ of sizes 10, 30 and 100.

The advantage of this model is that it leads to a *connected graph*. We again define the model with $m \geq 2$ and $\delta > -m$ in terms of the model with $m = 1$ and $\delta' = \delta/m > -1$ as below Exercise 8.3. We also note that the differences between $\{\mathrm{PA}_{m,\delta}(t)\}_{t \geq 1}$ and $\{\mathrm{PA}_{m,\delta}^{(b)}(t)\}_{t \geq 2}$ are minor, since the probability of a self-loop in $\mathrm{PA}_{m,\delta}(t)$ is quite small when t is large. Thus, most of the results we shall prove in this chapter for $\{\mathrm{PA}_{m,\delta}(t)\}_{t \geq 1}$ shall also apply to $\{\mathrm{PA}_{m,\delta}^{(b)}(t)\}_{t \geq 2}$, but we shall not state these extensions explicitly.

Interestingly, the above model with $\delta \geq 0$ can be viewed as a mix between the model with $\delta = 0$ and $\delta = \infty$. We again let the graph at time 2 consist of two vertices with two edges between them. We fix $\alpha \in [0, 1]$. Then, we first draw an random variable X_{t+1} taking values 0 with probability α and $X_{t+1} = 1$ with probability $1 - \alpha$. The random variables $\{X_t\}_{t=1}^{\infty}$ are independent. When $X_{t+1} = 0$, then we attach the $(t+1)^{\mathrm{st}}$ edge to a *uniform* vertex in $\{1, \ldots, t\}$. When $X_{t+1} = 1$, then we attach the $(t+1)^{\mathrm{st}}$ edge to vertex $i \in \{1, \ldots, t\}$ with probability $(D_i(t) + \delta)/(t(2 + \delta))$. We denote this model by $\{\mathrm{PA}_{1,\alpha}^{(b')}(t)\}_{t=1}^{\infty}$.

When $\alpha \geq 0$ is chosen appropriately, then this is precisely the above preferential attachment model:

Exercise 8.6. *For $\alpha = \frac{\delta}{2+\delta}$, the law of $\{\mathrm{PA}_{1,\alpha}^{(b')}(t)\}_{t=2}^{\infty}$ is equal to the one of $\{\mathrm{PA}_{1,\delta}(t)\}_{t=1}^{\infty}$.*

Exercise 8.7. *Prove that $D_i(t) \xrightarrow{a.s.} \infty$. Hint: use that, with $\{I_t\}_{t=i}^{\infty}$ a sequence of independent Bernoulli random variables with $\mathbb{P}(I_t = 1) = (1 + \delta)/(t(2 + \delta) + 1 + \delta)$, we have that $\sum_{s=i}^{t} I_s \preceq D_i(t)$.*

8.2 Degrees of fixed vertices

We start by investigating the degrees of given vertices. The main result in this section is the following:

Theorem 8.1 (Degrees of fixed vertices). *Fix $m = 1$ and $\delta > -1$. Then, $D_i(t)/t^{\frac{1}{2+\delta}}$ converges almost surely to a random variable ξ_i as $t \to \infty$, and*

$$\mathbb{E}[D_i(t) + \delta] = (1 + \delta)\frac{\Gamma(t+1)\Gamma(i + \frac{1+\delta}{2+\delta})}{\Gamma(t + \frac{1+\delta}{2+\delta})\Gamma(i+1)}. \tag{8.2.1}$$

In Section 8.6, we shall considerably extend the result in Theorem 8.1. For example, we shall also prove the almost sure convergence of *maximal* degree.

Proof. Fix $m = 1$. We compute that

$$\mathbb{E}[D_i(t+1) + \delta | D_i(t)] = D_i(t) + \delta + \mathbb{E}[D_i(t+1) - D_i(t) | D_i(t)]$$

$$= D_i(t) + \delta + \frac{D_i(t) + \delta}{(2+\delta)t + 1 + \delta}$$

$$= (D_i(t) + \delta)\frac{(2+\delta)t + 2 + \delta}{(2+\delta)t + 1 + \delta}$$

$$= (D_i(t) + \delta)\frac{(2+\delta)(t+1)}{(2+\delta)t + 1 + \delta}. \tag{8.2.2}$$

Using also that

$$\mathbb{E}[D_i(i) + \delta] = 1 + \delta + \frac{1+\delta}{(2+\delta)(i-1) + 1 + \delta} = (1+\delta)\frac{(2+\delta)(i-1) + 2 + \delta}{(2+\delta)(i-1) + 1 + \delta} = (1+\delta)\frac{(2+\delta)i}{(2+\delta)(i-1) + 1 + \delta},$$

we obtain that

$$M_i(t) = \frac{D_i(t) + \delta}{1 + \delta} \prod_{s=i-1}^{t-1} \frac{(2+\delta)s + 1 + \delta}{(2+\delta)(s+1)} \tag{8.2.3}$$

is a non-negative martingale with mean 1. As a consequence of the martingale convergence theorem (Theorem 2.21), as $t \to \infty$, $M_i(t)$ converges almost surely to a limiting random variable ξ_i.

We can compute that

$$\prod_{s=i}^{t-1} \frac{(2+\delta)s + 1 + \delta}{(2+\delta)s + 2 + \delta} = \prod_{s=i}^{t-1} \frac{s + \frac{1+\delta}{2+\delta}}{s + 1}$$

$$= \frac{\Gamma(t + \frac{1+\delta}{2+\delta})\Gamma(i+1)}{\Gamma(t+1)\Gamma(i + \frac{1+\delta}{2+\delta})}, \tag{8.2.4}$$

where we define the *Gamma-function* $t \mapsto \Gamma(t)$ for $t > 0$ by

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx. \tag{8.2.5}$$

We also make use of the recursion formula

$$\Gamma(t+1) = t\Gamma(t). \tag{8.2.6}$$

Exercise 8.8. *Prove (8.2.6) using partial integration, and also prove that $\Gamma(n) = (n-1)!$ for $n = 1, 2, \ldots$.*

It is not hard to see that, using Stirling's formula, (see e.g. [1, 6.1.47])

$$\frac{\Gamma(t+a)}{\Gamma(t)} = t^a (1 + O(1/t)). \tag{8.2.7}$$

Therefore, we have that $D_i(t)/t^{\frac{1}{2+\delta}}$ converges in distribution to a random variable ξ_i having expected value $(1 + \delta)\frac{\Gamma(i + \frac{1+\delta}{2+\delta})}{\Gamma(i+1)}$. In particular, the degrees of the first i vertices at time t is at most of order $t^{\frac{1}{2+\delta}}$. Note, however, that we do not yet know whether $\mathbb{P}(\xi_i = 0) = 0$ or not!

Exercise 8.9. *Prove* (8.2.7), *using that* [100, 8.327]$^{\text{GraRyz65}}$

$$e^{-t}t^{t+\frac{1}{2}}\sqrt{2\pi} \leq \Gamma(t+1) \leq e^{-t}t^{t+\frac{1}{2}}\sqrt{2\pi}e^{\frac{1}{12t}}.$$

<div align="right">(e:Stirlingineq)
(8.2.8)</div>

□

Note that we can extend the above result to the case when $m \geq 1$, by using the relation between $\mathrm{PA}_{m,\delta}(t)$ and $\mathrm{PA}_{1,\delta/m}(mt)$. This implies in particular that

$$\mathbb{E}_m^\delta[D_i(t)] = \sum_{s=1}^{m} \mathbb{E}_1^{\delta/m}[D_i(m(t-1)+s)],$$

<div align="right">(e:meandim)
(8.2.9)</div>

where we have added a subscript m and a superscript δ to denote the values of m and δ involved.

Exercise 8.10. *Prove* (8.2.9) *and use it to compute* $\mathbb{E}_m^\delta[D_i(t)]$.

Exercise 8.11. *Prove that, for* $m \geq 2$ *and any* $i \geq 1$, $D_i(t)(mt)^{-1/(2+\delta/m)} \xrightarrow{a.s.} \xi_i'$, *where*

$$\xi_i' = \sum_{j=(i-1)m+1}^{mi} \xi_j,$$

<div align="right">(8.2.10)</div>

and ξ_j *is the almost sure limit of* $D_j(t)$ *in* $\{\mathrm{PA}_{1,\delta/m}(t)\}_{t=1}^\infty$.

Exercise 8.12. *Prove that for* $\mathrm{PA}_{1,\delta}^{(b)}(t)$, (8.2.1) *is adapted to*

$$\mathbb{E}[D_i(t)+\delta] = (1+\delta)\frac{\Gamma(t+\frac{1}{2+\delta})\Gamma(i)}{\Gamma(t)\Gamma(i+\frac{1}{2+\delta})}.$$

<div align="right">(e:meandib)
(8.2.11)</div>

We close this section by giving a heuristic explanation for the occurrence of a power-law degree sequence in preferential attachment models. Theorem 8.1 in conjunction with Exercise 8.11 implies that there exists an a_m such that, for i, t large, and any $m \geq 1$,

$$\mathbb{E}[D_i(t)] \sim a_m \left(\frac{t}{i}\right)^{1/(2+\delta/m)}.$$

<div align="right">(8.2.12)</div>

When the graph indeed has a power-law degree sequence, then the number of vertices with degrees at least k be close to $ctk^{-(\tau-1)}$ for some $\tau > 1$ and some $c > 0$. The number of vertices with degree at least k at time t is equal to $N_{\geq k}(t) = \sum_{i=1}^{t} \mathbb{1}_{\{D_i(t) \geq k\}}$. Now, assume that in the above formula, we are allowed to replace $\mathbb{1}_{\{D_i(t) \geq k\}}$ by $\mathbb{1}_{\{\mathbb{E}[D_i(t)] \geq k\}}$ (there is a big leap of faith here). Then we would obtain that

$$N_{\geq k}(t) \sim \sum_{i=1}^{t} \mathbb{1}_{\{\mathbb{E}[D_i(t)] \geq k\}} \sim \sum_{i=1}^{t} \mathbb{1}_{\{a_m(\frac{t}{i})^{1/(2+\delta/m)} \geq k\}}$$

$$= \sum_{i=1}^{t} \mathbb{1}_{\{i \leq ta_m^{2+\delta/m}k^{-(2+\delta/m)}\}} = ta_m^{2+\delta/m}k^{-(2+\delta/m)},$$

<div align="right">(8.2.13)</div>

so that we obtain a power-law with exponent $\tau - 1 = 2 + \delta/m$, so that $\tau = 3 + \delta/m$. The above heuristic shall be made precise in the following section, but the proof will be quite a bit more subtle than the above heuristic!

8.3   Degree sequences of preferential attachment models

The main result in this chapter is that degree sequence obeys a power law, which establishes the scale-free nature of preferential attachment graphs. In order to state the result, we need some notation. We write

$$P_k(t) = \frac{1}{t} \sum_{i=1}^{t} \mathbb{1}_{\{d_i(t)=k\}} \tag{8.3.1}$$

for the (random) proportion of vertices with degree k at time t. For $m \geq 1$ and $\delta > -m$, we define $\{p_k\}_{k=0}^{\infty}$ to be the probability distribution given by $p_k = 0$ for $k = 0, \dots, m-1$ and, for $k \geq m$,

$$p_k = (2 + \frac{\delta}{m}) \frac{\Gamma(k+\delta)\Gamma(m+2+\delta+\frac{\delta}{m})}{\Gamma(m+\delta)\Gamma(k+3+\delta+\frac{\delta}{m})} \tag{8.3.2}$$

e:pkm2def

For $m = 1$, (8.3.2) reduces to

$$p_k = (2+\delta) \frac{\Gamma(k+\delta)\Gamma(3+2\delta)}{\Gamma(k+3+2\delta)\Gamma(1+\delta)}. \tag{8.3.3}$$

e:pkdefPAm1

Also, when $\delta = 0$ and $k \geq m$, (8.3.2) simplifies to

$$p_k = \frac{2\Gamma(k)\Gamma(m+2)}{\Gamma(k+3)\Gamma(m)} = \frac{2m(m+1)}{k(k+1)(k+2)}. \tag{8.3.4}$$

e:pkdefPAm1AB

We start by proving that $\{p_k\}_{k=1}^{\infty}$ is a *probability distribution*. For this, we note that, by (8.2.6),

$$\frac{\Gamma(k+a)}{\Gamma(k+b)} = \frac{1}{b-a-1}\Big(\frac{\Gamma(k+a)}{\Gamma(k-1+b)} - \frac{\Gamma(k+1+a)}{\Gamma(k+b)}\Big). \tag{8.3.5}$$

e:Gamma-rec-cons

Applying (8.3.5) to $a = \delta$, $b = 3 + \delta + \frac{\delta}{m}$, we obtain that, for $k \geq m$,

$$p_k = \frac{\Gamma(m+2+\delta+\frac{\delta}{m})}{\Gamma(m+\delta)}\Big(\frac{\Gamma(k+\delta)}{\Gamma(k+2+\delta+\frac{\delta}{m})} - \frac{\Gamma(k+1+\delta)}{\Gamma(k+3+\delta+\frac{\delta}{m})}\Big). \tag{8.3.6}$$

e:pk-alt

Using that $p_k = 0$ for $k < m$, and by a telescoping sum identity,

$$\sum_{k \geq 1} p_k = \sum_{k \geq m} p_k = \frac{\Gamma(m+2+\delta+\frac{\delta}{m})}{\Gamma(m+\delta)} \frac{\Gamma(m+\delta)}{\Gamma(m+2+\delta+\frac{\delta}{m})} = 1. \tag{8.3.7}$$

Thus, since also $p_k \geq 0$, we obtain that $\{p_k\}_{k=1}^{\infty}$ indeed is a probability distribution. We shall see that $\{p_k\}_{k=1}^{\infty}$ arises as the limiting degree distribution for $\mathrm{PA}_{m,\delta}(t)$:

Theorem 8.2 (Degree sequence in preferential attachment model)**.** *Fix $\delta > -m$ and $m \geq 1$. Then, there exists a constant $C > 0$ such that, as $t \to \infty$,*

$$\mathbb{P}\Big(\max_k |P_k(t) - p_k| \geq C \sqrt{\frac{\log t}{t}} \Big) = o(1). \tag{8.3.8}$$

Theorem 8.2 identifies the asymptotic degree sequence of $\mathrm{PA}_{m,\delta}(t)$ as $\{p_k\}_{k=1}^{\infty}$. We next show that, for k large, p_k is close to a power-law distribution. For this, we first note that from (8.3.2) and (8.2.7), as $k \to \infty$,

$$p_k = c_m k^{-\tau}(1 + O(\frac{1}{k})), \tag{8.3.9}$$
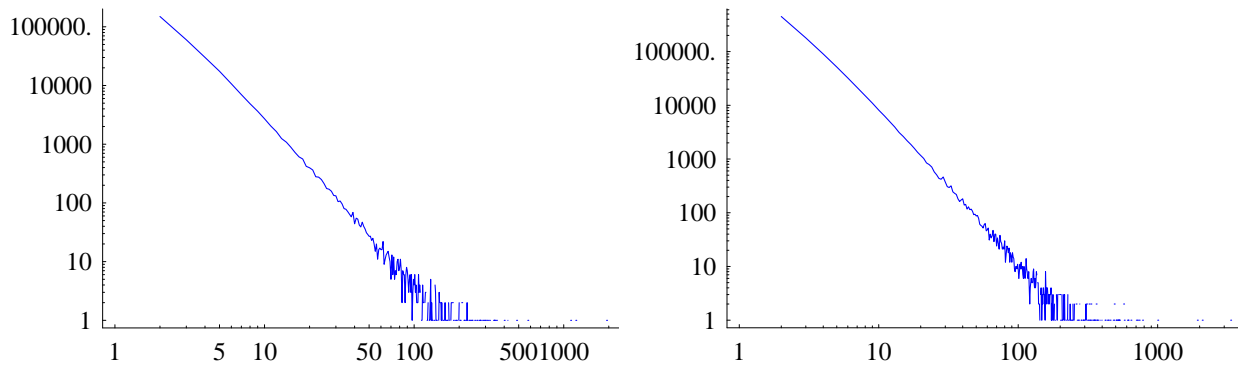
e:powerlawPA

Figure 8.3: The degree sequences of a preferential attachment random graph with $m = 2, \delta = 0$ of sizes 300,000 and 1,000,000 in log-log scale.

where

$$\tau = 3 + \frac{\delta}{m} > 2, \tag{8.3.10}$$

and

$$c_m = \frac{(2 + \frac{\delta}{m})\Gamma(m + 2 + \delta + \frac{\delta}{m})}{\Gamma(m + \delta)}. \tag{8.3.11}$$

Therefore, by Theorem 8.2 and (8.3.9), the asymptotic degree sequence of $\mathrm{PA}_{m,\delta}(t)$ is close to a power law with exponent $\tau = 3 + \delta/m$. We note that any exponent $\tau > 2$ is possible by choosing $\delta > -m$ and $m \geq 1$ appropriately. The power-law degree sequence can clearly be observed in a simulation, see Figure 8.3, where a realization of the degree sequence of $\mathrm{PA}_{m,\delta}(t)$ is shown for $m = 2$, $\delta = 0$ and $t = 300,000$ and $t = 1,000,000$.

The important feature of the preferential attachment model is that, unlike the configuration model and the generalized random graph, the power law in $\mathrm{PA}_{m,\delta}(t)$ is *explained* by giving a model for the growth of the graph that produces power-law degrees. Therefore, preferential attachment offers a convincing explanation as to *why* power-law degree sequences occur. As Barabási puts it [20]

> "...the scale-free topology is evidence of organizing principles acting at each stage of the network formation. (...) No matter how large and complex a network becomes, as long as preferential attachment and growth are present it will maintain its hub-dominated scale-free topology.".

Many more possible explanations have been given for *why* power laws occur in real networks, and many adaptations of the above simple preferential attachment model have been studied in the literature, all giving rise to power-law degrees. See Section 8.7 for an overview of the literature.

The remainder of this chapter shall be primarily devoted to the proof of Theorem 8.2, which is divided into two main parts. In Section 8.4, we prove that the degree sequence is concentrated around its mean, and in Section 8.5, we identify the mean of the degree sequence. In the course of the proof, we also prove results related to Theorem 8.2.

Exercise 8.13. *Prove that Theorem 8.2 implies that the degree at time t of a uniform vertex in $\{1, \ldots, t\}$ has asymptotic distribution $\{p_k\}_{k=1}^{\infty}$.*

Exercise 8.14. *In a uniform recursive tree we attach each vertex to a uniformly chosen old vertex. This can be seen as the case where $m = 1$ and $\delta = \infty$ of $\{\mathrm{PA}_{m,\delta}^{(b)}(t)\}_{t \geq 2}$. Show that Theorem 8.2 remains true, but now with $p_k = 2^{-(k+1)}$ (see e.g. [119]).*

8.4   Concentration of the degree sequence

In this section, we prove that the (random) degree sequence is sufficiently concentrated around its expected degree sequence. We use a martingale argument which first appeared in [46],^{BolRioSpeTus01} and has been used in basically all subsequent works proving power law degree sequences for preferential attachment models. The argument is very pretty and general, and we spend some time explaining the details of it.

We start by stating the main result in this section. In its statement, we use the notation

$$N_k(t) = \sum_{i=1}^{t} \mathbb{1}_{\{d_i(t)=k\}} = tP_k(t) \qquad\qquad \text{(8.4.1)}$$

for the total number of vertices with degree k at time t.

Proposition 8.3 (Concentration of the degrees). *Fix $\delta \geq -m$ and $m \geq 1$. Then, for any $C > \sqrt{8m}$, as $t \to \infty$,*

$$\mathbb{P}\Big(\max_k |N_k(t) - \mathbb{E}[N_k(t)]| \geq C\sqrt{t \log t} \Big) = o(1). \qquad\qquad \text{(8.4.2)}$$

We note that Theorem 8.2 predicts that $N_k(t) \approx tp_k$. Thus, at least for k for which p_k is not too small, i.e., $tp_k \gg \sqrt{t \log t}$, Proposition 8.3 suggests that the number of vertices with degree equal to k is very close to its expected value. Needless to say, in order to prove Theorem 8.2, we still need to investigate $\mathbb{E}[N_k(t)]$, and prove that it is quite close to tp_k. This is the second main ingredient in the proof of Theorem 8.2. We first prove Proposition 8.3.

Proof. We start by reducing the proof. First of all, $N_k(t) = 0$ when $k > mt$. Therefore,

$$\mathbb{P}\Big(\max_k |N_k(t) - \mathbb{E}[N_k(t)]| \geq C\sqrt{t \log t} \Big) = \mathbb{P}\Big(\max_{k \leq mt} |N_k(t) - \mathbb{E}[N_k(t)]| \geq C\sqrt{t \log t} \Big)$$

$$\leq \sum_{k=1}^{mt} \mathbb{P}\Big(|N_k(t) - \mathbb{E}[N_k(t)]| \geq C\sqrt{t \log t} \Big). \qquad \text{(8.4.3)}$$

We shall prove that for any $C > \sqrt{8m}$, uniformly in $k \leq mt$,

$$\mathbb{P}\Big(|N_k(t) - \mathbb{E}[N_k(t)]| \geq C\sqrt{t \log t} \Big) = o(t^{-1}), \qquad\qquad \text{(8.4.4)}$$

which would complete the proof of Proposition 8.3.

For $n = 0, \ldots, t$, we denote by

$$M_n = \mathbb{E}\big[N_k(t) | \mathrm{PA}_{m,\delta}(n) \big] \qquad\qquad \text{(8.4.5)}$$

the conditional expected number of vertices with degree k at time t, conditionally on the graph $\mathrm{PA}_{m,\delta}(n)$ at time $n \in \{0, \ldots, t\}$. By the tower property of conditional expectations, and the fact that $\mathrm{PA}_{m,\delta}(n)$ can be deduced from $\mathrm{PA}_{m,\delta}(n+1)$, we have that, for all $n \leq t-1$,

$$\mathbb{E}[M_{n+1}|\mathrm{PA}_{m,\delta}(n)] = \mathbb{E}\Big[\mathbb{E}\big[N_k(t)|\mathrm{PA}_{m,\delta}(n+1) \big] \Big| \mathrm{PA}_{m,\delta}(n) \Big]$$

$$= \mathbb{E}\big[N_k(t)|\mathrm{PA}_{m,\delta}(n) \big]$$

$$= M_n, \qquad\qquad \text{(8.4.6)}$$

so that $\{M_n\}_{n=0}^{t}$ satisfies the conditional expectation requirement for a martingale. In fact, $\{M_n\}_{n=0}^{t}$ is a so-called *Doob martingale* (see also Exercise 2.22).

Furthermore, since $N_k(t)$ is bounded by the total number of vertices at time t, we have $N_k(t) \leq t$, so that

$$\mathbb{E}[|M_n|] = \mathbb{E}[M_n] = \mathbb{E}[N_k(t)] \leq t < \infty. \tag{8.4.7}$$

Therefore, $\{M_n\}_{n=0}^t$ also satisfies the moment condition for martingales. We conclude that $\{M_n\}_{n=0}^t$ is a martingale process with respect to $\{\mathrm{PA}_{m,\delta}(n)\}_{n=0}^t$. This is the first main ingredient of the martingale proof of (8.4.4).

For the second ingredient, we note that M_0 is identified as

$$M_0 = \mathbb{E}\big[N_k(t)|\mathrm{PA}_{m,\delta}(0)\big] = \mathbb{E}[N_k(t)], \tag{8.4.8}$$

Since $\mathrm{PA}_{m,\delta}(0)$ is the empty graph. Furthermore, M_t is trivially identified as

$$M_t = \mathbb{E}\big[N_k(t)|\mathrm{PA}_{m,\delta}(t)\big] = N_k(t), \tag{8.4.9}$$

since one can determine $N_k(t)$ from $\mathrm{PA}_{m,\delta}(t)$. Therefore, we have that

$$N_k(t) - \mathbb{E}[N_k(t)] = M_t - M_0. \tag{8.4.10}$$

This completes the second key ingredient in the martingale proof of (8.4.4).

The third key ingredient is the Azuma-Hoeffding inequality, Theorem 2.23. For this, we need to investigate the support of $|M_n - M_{n-1}|$. We claim that

$$|M_n - M_{n-1}| \leq 2m. \tag{8.4.11}$$

In order to prove this, we note that

$$M_n - M_{n-1} = \mathbb{E}[N_k(t)|\mathrm{PA}_{m,\delta}(n)] - \mathbb{E}[N_k(t)|\mathrm{PA}_{m,\delta}(n-1)]. \tag{8.4.12}$$

Thus, we need to investigate the influence of the extra information contained in $\mathrm{PA}_{m,\delta}(n)$ compared to the information contained in $\mathrm{PA}_{m,\delta}(n-1)$. For any $s = 1, \ldots, t$, conditioning on $\mathrm{PA}_{m,\delta}(s)$ is the *same* as conditioning to which vertices the first sm edges are attached. Thus, in $\mathrm{PA}_{m,\delta}(n-1)$, we know where the edges of the vertices $v_1, \ldots, v_{n-1}$ are attached to. In $\mathrm{PA}_{m,\delta}(n)$, we have the additional information of where the m edges originating from the vertex v_n are attached to. These m edges effect the degrees of at most m other vertices, namely, the receiving ends of these m edges.

For the conditional expectations given $\mathrm{PA}_{m,\delta}(s)$, we need to take the expectation with respect to all possible ways of attaching the remaining edges originating from the vertices $v_{s+1}, \ldots, v_t$. As explained above, only the distribution of the degrees of the vertices in $\mathrm{PA}_{m,\delta}(t)$ to which the m edges originating from v_n are connected are effected by the knowledge of $\mathrm{PA}_{m,\delta}(n)$ compared to $\mathrm{PA}_{m,\delta}(n-1)$. This number of vertices is at most m, so that the distribution of the degrees of at most $2m$ vertices is different in the law of $\mathrm{PA}_{m,\delta}(t)$ conditionally on $\mathrm{PA}_{m,\delta}(n-1)$ compared to the law of $\mathrm{PA}_{m,\delta}(t)$ conditionally on $\mathrm{PA}_{m,\delta}(n)$. This implies (8.4.11).

The Azuma-Hoeffding's Inequality in Theorem 2.23 then yields that, for any $a > 0$,

$$\mathbb{P}\Big(|N_k(t) - \mathbb{E}[N_k(t)]| \geq a\Big) \leq 2e^{-\frac{a^2}{8m^2 t}}. \tag{8.4.13}$$

Taking $a = C\sqrt{t \log t}$ for any C with $C^2 > 8m$ then proves that

$$\mathbb{P}\Big(|N_k(t) - \mathbb{E}[N_k(t)]| \geq C\sqrt{t \log t}\Big) \leq 2e^{-(\log t)\frac{C^2}{8m}} = o(t^{-1}). \tag{8.4.14}$$

This completes the proof of (8.4.4), and thus of Proposition 8.3. $\qquad\square$

The above proof is rather general, and can also be used to prove concentration around the mean of other graph properties that are related to the degrees. An example is the following. Denote by

$$N_{\geq k}(t) = \sum_{l=k}^{\infty} N_l(t) \qquad (8.4.15)$$

the total number of vertices with degrees at least k. Then we can also prove that $N_{\geq k}(t)$ concentrates. Indeed, for $C > \sqrt{8m}$,

$$\mathbb{P}\Big(|N_{\geq k}(t) - \mathbb{E}[N_{\geq k}(t)]| \geq C\sqrt{t\log t}\Big) = o(t^{-1}). \qquad \text{e:aimconcPgeqPA} \atop (8.4.16)$$

The proof uses the same ingredients as given above for $N_{\geq k}(t)$, where now we can make use of the martingale

$$M_n' = \mathbb{E}[N_{\geq k}(t)|\mathrm{PA}_{m,\delta}(n)]. \qquad (8.4.17)$$

Exercise 8.15. *Prove* (8.4.16) *by adapting the proof of Proposition 8.3.*

8.5   Expected degree sequence

The main result in this section investigates the expected number of vertices with degree equal to k. We denote the expected number of vertices of degree k in $\mathrm{PA}_{m,\delta}(t)$ by

$$\bar{N}_k(t) = \mathbb{E}[P_k(t)]. \qquad (8.5.1)$$

The main aim is to prove that $\bar{N}_k(t)$ is close to $p_k t$, where p_k is defined in (8.3.3). This is the content of the following proposition:

Proposition 8.4 (Expected degree sequence). *Fix $\delta > -m$ and $m \geq 1$. Then, there exists a constant $C = C(\delta, m)$ such that for all $t \geq 1$ and all $k \in \mathbb{N}$,*

$$|\bar{N}_k(t) - p_k t| \leq C. \qquad (8.5.2)$$

The proof of Proposition 8.4 is split into two separate cases. We first prove the claim for $m = 1$ in Section 8.5.1, and extend the proof to $m > 1$ in Section 8.5.2.

Exercise 8.16. *Use Propositions 8.4 and 8.3 to prove that for $l = l(t) \to \infty$ as $t \to \infty$ such that $tl^{2-\tau} \geq K\sqrt{t\log t}$ for some $K > 0$ sufficiently large, there exists a constant $B > 0$ such that with probability exceeding $1 - o(t^{-1})$, for all such l,*

$$\sum_{i:D_i(t)\geq l} D_i(t) \geq Btl^{2-\tau}. \qquad \text{e:lowerbddegrees} \atop (8.5.3)$$

Show further that, with probability exceeding $1 - o(t^{-1})$, for all such l,

$$N_{\geq l}(t) \gg \sqrt{t}. \qquad \text{e:Nt-lb-l} \atop (8.5.4)$$

8.5.1 Expected degree sequence for $m = 1$

In this section, we study the expected degree sequence when $m = 1$. We adapt the argument in [41].[BolBorChaRio03] We start by writing

$$\mathbb{E}[N_k(t+1)|\mathrm{PA}_{1,\delta}(t)] = N_k(t) + \mathbb{E}[N_k(t+1) - N_k(t)|\mathrm{PA}_{1,\delta}(t)]. \tag{8.5.5}$$

Conditionally on $\mathrm{PA}_{1,\delta}(t)$, there are four ways how $N_k(t+1) - N_k(t)$ can be unequal to zero:

(a) The end vertex of the (unique) edge incident to vertex v_{t+1} had degree $k-1$, so that its degree is increased to k, which happens with probability $\frac{k-1+\delta}{t(2+\delta)+(1+\delta)}$. Note that there are $N_{k-1}(t)$ end vertices with degree $k-1$ at time t;

(b) The end vertex of the (unique) edge incident to vertex v_{t+1} had degree k, so that its degree is increased to $k+1$, which happens with probability $\frac{k+\delta}{t(2+\delta)+(1+\delta)}$. Note that there are $N_k(t)$ end vertices with degree k at time t;

(c) The degree of vertex v_{t+1} is one, so that $N_1(t)$ is increased by one, when the end vertex of the (unique) edge incident to vertex v_{t+1} is not v_{t+1}, which happens with probability $1 - \frac{1+\delta}{t(2+\delta)+(1+\delta)}$;

(d) The degree of vertex v_{t+1} is equal to two, so that $N_2(t)$ is increased by one, when the end vertex of the (unique) edge incident to vertex v_{t+1} is equal to v_{t+1}, which happens with probability $\frac{1+\delta}{t(2+\delta)+(1+\delta)}$.

The changes in the degree sequence in cases (a) and (b) arise due to the attachment of the edge (thus, the degree of one of the vertices $v_1, \dots, v_t$ is changed), whereas in cases (c) and (d) we determine the degree of the added vertex v_{t+1}.

Taking all these cases into account, we arrive at the key identity

$$\begin{aligned}
\mathbb{E}[N_k(t+1) - N_k(t)|\mathrm{PA}_{1,\delta}(t)] &= \frac{k-1+\delta}{t(2+\delta)+(1+\delta)} N_{k-1}(t) \\
&\quad - \frac{k+\delta}{t(2+\delta)+(1+\delta)} N_k(t) \\
&\quad + \mathbb{1}_{\{k=1\}} \Big(1 - \frac{1+\delta}{t(2+\delta)+(1+\delta)}\Big) \\
&\quad + \mathbb{1}_{\{k=2\}} \frac{1+\delta}{t(2+\delta)+(1+\delta)}.
\end{aligned} \tag{8.5.6}$$

Here, $k \geq 1$, and for $k = 0$, by convention, we define

$$N_0(t) = 0. \tag{8.5.7}$$

By taking the expectation on both sides of (8.5.6), we arrive at the explicit recurrence relation that, for $k \geq 1$,

$$\begin{aligned}
\bar{N}_k(t+1) &= \bar{N}_k(t) + \frac{k-1+\delta}{t(2+\delta)+(1+\delta)} \bar{N}_{k-1}(t) \\
&\quad - \frac{k+\delta}{t(2+\delta)+(1+\delta)} \bar{N}_k(t) \\
&\quad + \mathbb{1}_{\{k=1\}} \Big(1 - \frac{1+\delta}{t(2+\delta)+(1+\delta)}\Big) \\
&\quad + \mathbb{1}_{\{k=2\}} \frac{1+\delta}{t(2+\delta)+(1+\delta)}.
\end{aligned} \tag{8.5.8}$$

Equation (8.5.8) will the the key to the proof of Proposition 8.4 for $m = 1$. We start by explaining its relation to (8.3.3). Indeed, when $\bar{N}_k(t) \approx tp_k$, then one might expect that $\bar{N}_k(t+1) - \bar{N}_k(t) \approx p_k$. Substituting these approximations into (8.5.8), and approximating $t/(t(2+\delta) + (1+\delta)) \approx 1/(2+\delta)$ and $\frac{1+\delta}{t(2+\delta)+(1+\delta)} \approx 0$, we arrive at the fact that p_k must satisfy the recurrence relation, for $k \geq 1$,

$$p_k = \frac{k-1+\delta}{2+\delta}p_{k-1} - \frac{k+\delta}{2+\delta}p_k + \mathbb{1}_{\{k=1\}}, \tag{8.5.9}$$

where we define $p_0 = 0$. We now show that the unique solution to (8.5.9) is (8.3.3). We can rewrite

$$p_k = \frac{k-1+\delta}{k+2+2\delta}p_{k-1} + \frac{2+\delta}{k+2+2\delta}\mathbb{1}_{\{k=1\}}. \tag{8.5.10}$$

When $k = 1$, using that $p_0 = 0$, we obtain

$$p_1 = \frac{2+\delta}{3+2\delta}. \tag{8.5.11}$$

On the other hand, when $k > 1$, we arrive at

$$p_k = \frac{k-1+\delta}{k+2+2\delta}p_{k-1}. \tag{8.5.12}$$

Therefore, using (8.2.6) repeatedly,

$$
\begin{aligned}
p_k &= \frac{\Gamma(k+\delta)\Gamma(4+2\delta)}{\Gamma(k+3+2\delta)\Gamma(1+\delta)}p_1 \\
&= \frac{(2+\delta)\Gamma(k+\delta)\Gamma(4+2\delta)}{(3+2\delta)\Gamma(k+3+2\delta)\Gamma(1+\delta)} \\
&= \frac{(2+\delta)\Gamma(k+\delta)\Gamma(3+2\delta)}{\Gamma(k+3+2\delta)\Gamma(1+\delta)},
\end{aligned}
\tag{8.5.13}
$$

and we see that the unique solution of (8.5.9) is p_k in (8.3.3).

The next step is to use (8.5.8) and (8.5.9) to prove Proposition 8.4 for $m = 1$. To this end, we define

$$\varepsilon_k(t) = \bar{N}_k(t) - tp_k. \tag{8.5.14}$$

Then, in order to prove Proposition 8.4 for $m = 1$, we are left to prove that there exists a constant $C = C(\delta)$ such that

$$\max_k |\varepsilon_k(t)| \leq C. \tag{8.5.15}$$

The value of C will be determined in the course of the proof.

Now we deviate from the proof in [41]. In [41], induction in k was performed. Instead, we use induction in t. First of all, we note that we can rewrite (8.5.9) as

$$
\begin{aligned}
(t+1)p_k &= tp_k + p_k \\
&= tp_k + \frac{k-1+\delta}{2+\delta}p_{k-1} - \frac{k+\delta}{2+\delta}p_k + \mathbb{1}_{\{k=1\}} \\
&= tp_k + \frac{k-1+\delta}{t(2+\delta)+(1+\delta)}tp_{k-1} - \frac{k+\delta}{t(2+\delta)+(1+\delta)}tp_k + \mathbb{1}_{\{k=1\}} \\
&\quad + \left(\frac{1}{2+\delta} - \frac{t}{t(2+\delta)+(1+\delta)}\right)(k-1+\delta)p_{k-1} \\
&\quad - \left(\frac{1}{2+\delta} - \frac{t}{t(2+\delta)+(1+\delta)}\right)(k+\delta)p_k.
\end{aligned}
\tag{8.5.16}
$$

We abbreviate

$$\kappa_k(t) = -\Big(\frac{1}{2+\delta} - \frac{t}{t(2+\delta)+(1+\delta)}\Big)(k-1+\delta)p_{k-1} + \Big(\frac{1}{2+\delta} - \frac{t}{t(2+\delta)+(1+\delta)}\Big)(k+\delta)p_k,$$

<div align="right">e:kappadefPA
(8.5.17)</div>

$$\gamma_k(t) = -\mathbb{1}_{\{k=1\}}\frac{1+\delta}{t(2+\delta)+(1+\delta)} + \mathbb{1}_{\{k=2\}}\frac{1+\delta}{t(2+\delta)+(1+\delta)}.$$

<div align="right">e:gammadefPA
(8.5.18)</div>

Then, (8.5.8) and (8.5.16) can be combined to yield that

$$\varepsilon_k(t+1) = \Big(1 - \frac{k+\delta}{t(2+\delta)+(1+\delta)}\Big)\varepsilon_k(t) + \frac{k-1+\delta}{t(2+\delta)+(1+\delta)}\varepsilon_{k-1}(t) + \kappa_k(t) + \gamma_k(t).$$

<div align="right">e:linrecvepPA
(8.5.19)</div>

We prove the bounds on $\varepsilon_k(t)$ in (8.5.15) by induction on $t \geq 1$. We start by initializing the induction hypothesis. When $t = 1$, we have that $\mathrm{PA}_{1,\delta}(1)$ consists of a vertex with a single self-loop. Thus,

$$\bar{N}_k(1) = \mathbb{1}_{\{k=2\}}.$$

<div align="right">(8.5.20)</div>

Therefore, since also $p_k \leq 1$, we arrive at the estimate that, uniformly in $k \geq 1$,

$$|\varepsilon_k(1)| = |\bar{N}_k(1) - p_k| \leq \max\{\bar{N}_k(1), p_k\} \leq 1.$$

<div align="right">(8.5.21)</div>

We have initialized the induction hypothesis for $t = 1$ in (8.5.15) for any $C \geq 1$.

We next advance the induction hypothesis. We start with $k = 1$. In this case, we have that $\varepsilon_0(t) = N_0(t) - p_0 = 0$ by convention, so that (8.5.19) reduces to

$$\varepsilon_1(t+1) = \Big(1 - \frac{1+\delta}{t(2+\delta)+(1+\delta)}\Big)\varepsilon_1(t) + \kappa_1(t) + \gamma_1(t).$$

<div align="right">(8.5.22)</div>

We note that

$$1 - \frac{1+\delta}{t(2+\delta)+(1+\delta)} \geq 0,$$

<div align="right">e:probbdPA
(8.5.23)</div>

so that

$$|\varepsilon_1(t+1)| \leq \Big(1 - \frac{1+\delta}{t(2+\delta)+(1+\delta)}\Big)|\varepsilon_1(t)| + |\kappa_1(t)| + |\gamma_1(t)|.$$

<div align="right">(8.5.24)</div>

Using the explicit forms in (8.5.17) and (8.5.18), it is not hard to see that there are universal constants $C_\kappa = C_\kappa(\delta)$ and $C_\gamma = C_\gamma(\delta)$ such that, uniformly in $k \geq 1$,

$$|\kappa_k(t)| \leq C_\kappa(t+1)^{-1}, \qquad |\gamma_k(t)| \leq C_\gamma(t+1)^{-1}.$$

<div align="right">e:kappa-gammabdPAm1
(8.5.25)</div>

Exercise 8.17. *Show that $C_\gamma = 1$ does the job, and $C_\kappa = \sup_{k \geq 1}(k+\delta)p_k = (1+\delta)p_1 = \frac{(1+\delta)(2+\delta)}{3+2\delta}$.*

Using the induction hypothesis (8.5.15), as well as (8.5.25), we arrive at

$$|\varepsilon_1(t+1)| \leq C\Big(1 - \frac{1+\delta}{t(2+\delta)+(1+\delta)}\Big) + (C_\kappa + C_\gamma)(t+1)^{-1}.$$

<div align="right">(8.5.26)</div>

Next, we use that $t(2+\delta) + (1+\delta) \leq (t+1)(2+\delta)$, so that

$$|\varepsilon_1(t+1)| \leq C - (t+1)^{-1}\Big(C\frac{1+\delta}{2+\delta} - (C_\kappa + C_\gamma)\Big) \leq C,$$

<div align="right">(8.5.27)</div>

whenever

$$C \geq \frac{2+\delta}{1+\delta}(C_\kappa + C_\gamma). \tag{8.5.28}$$

This advances the induction hypothesis for $k = 1$.

We now extend the argument to $k \geq 2$. We again use (8.5.19). We note that

$$1 - \frac{k+\delta}{t(2+\delta)+(1+\delta)} \geq 0 \tag{8.5.29}$$

as long as

$$k \leq t(2+\delta) + 1. \tag{8.5.30}$$

We will assume (8.5.30) for the time being, and deal with $k \geq t(2+\delta) + 2$ later.

By (8.5.19) and (8.5.30), we obtain that, for $k \geq 2$ and $\delta > -1$, so that $k - 1 + \delta \geq 0$,

$$|\varepsilon_k(t+1)| \leq \Big(1 - \frac{k+\delta}{t(2+\delta)+(1+\delta)}\Big)|\varepsilon_k(t)| + \frac{k-1+\delta}{t(2+\delta)+(1+\delta)}|\varepsilon_{k-1}(t)| + |\kappa_k(t)| + |\gamma_k(t)|. \tag{8.5.31}$$

Again using the induction hypothesis (8.5.15), as well as (8.5.25), we arrive at

$$|\varepsilon_k(t+1)| \leq C\Big(1 - \frac{k+\delta}{t(2+\delta)+(1+\delta)}\Big) + C\frac{k-1+\delta}{t(2+\delta)+(1+\delta)} + (C_\kappa + C_\gamma)(t+1)^{-1}$$

$$= C\Big(1 - \frac{1}{t(2+\delta)+(1+\delta)}\Big) + (C_\kappa + C_\gamma)(t+1)^{-1}. \tag{8.5.32}$$

Next, we use that,

$$t(2+\delta) + (1+\delta) \leq (t+1)(2+\delta), \tag{8.5.33}$$

so that

$$|\varepsilon_k(t+1)| \leq C - (t+1)^{-1}\Big(\frac{C}{2+\delta} - (C_\kappa + C_\gamma)\Big) \leq C, \tag{8.5.34}$$

whenever

$$C \geq (2+\delta)(C_\kappa + C_\gamma). \tag{8.5.35}$$

Finally, we deal with the case that $k \geq t(2+\delta) + 2$. Note that $k \geq t(2+\delta) + 2 > t + 2$ when $\delta > -1$. Since the maximal degree of $\mathrm{PA}_{1,\delta}(t)$ is $t + 2$ (which happens precisely when all edges are connected to the initial vertex), we have that $\bar{N}_k(t+1) = 0$ for $k \geq t(2+\delta) + 2$. Therefore, for $k \geq t(2+\delta) + 2$,

$$|\varepsilon_k(t+1)| = (t+1)p_k. \tag{8.5.36}$$

By (8.3.9) and (8.3.10), uniformly for $k \geq t(2+\delta) + 2 \geq t + 2$ for $\delta \geq -1$, there exists a $C_p = C_p(\delta)$ such that

$$p_k \leq C_p(t+1)^{-(3+\delta)}. \tag{8.5.37}$$

For $\delta \geq -1$, and again uniformly for $k \geq t + 2$,

$$(t+1)p_k \leq C_p(t+1)^{-(2+\delta)} \leq C_p. \tag{8.5.38}$$

Therefore, if $C \geq C_p$, then also the claim follows for $k \geq t(2+\delta) + 2$. Comparing to (8.5.28) and (8.5.28), we choose

$$C = \max\Big\{(2+\delta)(C_\kappa + C_\gamma), \frac{(2+\delta)(C_\kappa + C_\gamma)}{1+\delta}, C_p\Big\}. \tag{8.5.39}$$

This advances the induction hypothesis for $k \geq 2$, and completes the proof of Proposition 8.4 when $m = 1$ and $\delta > -1$.   □

8.5.2 Expected degree sequence for $m > 1^*$

In this section, we prove Proposition 8.4 for $m > 1$. We adapt the argument in Section 8.5.1 above. In Section 8.5.1, we have been rather explicit in the derivation of the recursion relation in (8.5.8), which in turn gives the explicit recursion relation on the errors $\varepsilon_k(t)$ in (8.5.19). In this section, we make the derivation more abstract, since the explicit derivations become too involved when $m > 1$. The current argument is rather flexible, and can, e.g., be extended to different preferential attachment models.

We make use of the fact that to go from $\mathrm{PA}_{m,\delta}(t)$ to $\mathrm{PA}_{m,\delta}(t+1)$, we add precisely m edges in a preferential way. This process can be described in terms of certain operators. For a sequence of numbers $Q = \{Q_k\}_{k=1}^{\infty}$, we define the operator $T_{t+1}\colon \mathbb{R}^{\infty} \mapsto \mathbb{R}^{\infty}$ by

$$(T_{t+1}Q)_k = \Big(1 - \frac{k+\delta}{t(2+\delta') + (1+\delta')}\Big)Q_k + \frac{k-1+\delta}{t(2+\delta') + (1+\delta')}Q_{k-1}, \qquad \text{(8.5.40)}$$

where we recall that $\delta' = \delta/m$. Then, writing $\bar{N}(t) = \{\bar{N}_k(t)\}_{k=1}^{\infty}$, we can rewrite (8.5.8) when $m = 1$ so that $\delta' = \delta$,

$$\bar{N}_k(t+1) = (T_{t+1}\bar{N}(t))_k + \mathbb{1}_{\{k=1\}}\Big(1 - \frac{1+\delta}{t(2+\delta) + (1+\delta)}\Big) + \mathbb{1}_{\{k=2\}}\frac{1+\delta}{t(2+\delta) + (1+\delta)}. \qquad \text{(8.5.41)}$$

Thus, as remarked above (8.5.6), the operator T_{t+1} describes the effect to the degree sequence of a single addition of the $(t+1)^{\mathrm{st}}$ edge, apart from the degree of the newly added vertex. The latter degree is equal to 1 with probability $1 - \frac{1+\delta}{t(2+\delta)+(1+\delta)}$, and equal to 2 with probability $\frac{1+\delta}{t(2+\delta)+(1+\delta)}$. This explains the origin of each of the terms appearing in (8.5.8).

In the case when $m > 1$, every vertex has m edges that are each connected in a preferential way. Therefore, we need to investigate the effect of attaching m edges in sequel. Due to the fact that we update the degrees after attaching an edge, the effect of attaching the $(j+1)^{\mathrm{st}}$ edge is described by applying the operator T_j to $\bar{N}(j)$. When we add the edges incident to the tth vertex, this corresponds to attaching the edges $m(t-1)+1, \ldots, mt$ in sequel with intermediate updating. The effect on the degrees of vertices $v_1, \ldots, v_t$ is described precisely by applying first T_{mt+1} to describe the effect of the addition of the first edge, followed by T_{mt+2} to describe the effect of the addition of the second edge, etc. Therefore, the recurrence relation of the expected number of vertices with degree k is changed to

$$\bar{N}_k(t+1) = (T_{t+1}^{(m)}\bar{N}(t))_k + \alpha_k(t), \qquad \text{(8.5.42)}$$

where

$$T_{t+1}^{(m)} = T_{m(t+1)} \circ \cdots \circ T_{mt+1}, \qquad \text{(8.5.43)}$$

and where, for $k = m, \ldots, 2m$, we have that $\alpha_k(t)$ is equal to the probability that the degree of the $(t+1)^{\mathrm{st}}$ added vertex is precisely equal to n. Indeed, when t changes to $t+1$, then the graph grows by one vertex. Its degree is equal to n with probability $q_n(t)$, so that the contribution of this vertex is equal to $\alpha_k(t)$. On the other hand, the edges that are connected from the $(t+1)^{\mathrm{st}}$ vertex also change the degrees of the other vertices. The expected number of vertices with degree k among vertices $v_1, \ldots, v_t$ is precisely given by $(T_{t+1}^{(m)}\bar{N}(t))_k$. Thus, the operator $T_{t+1}^{(m)}$ describes the effect to the degrees of vertices $v_1, \ldots, v_t$ of the attachment of the edges emanating from vertex v_{t+1}. This explains (8.5.42).

When t grows large, then the probability distribution $k \mapsto \alpha_k(t)$ is such that $\alpha_m(t)$ is very close to 1, while $\alpha_k(t)$ is close to zero when $k > m$. Indeed, for $k > m$, at least one of the m edges should be connected to its brother half-edge, so that

$$\sum_{k=m+1}^{2m} \alpha_k(t) \leq \frac{m^2(1+\delta)}{mt(2+\delta') + (1+\delta')}. \qquad \text{(8.5.44)}$$

We define

$$\gamma_k(t) = \alpha_k(t) - \mathbb{1}_{\{k=m\}}, \tag{8.5.45}$$

then we obtain from (8.5.44) that there exists a constant $C_\gamma = C_\gamma(\delta, m)$ such that

$$|\gamma_k(t)| \leq C_\gamma (t+1)^{-1}. \tag{8.5.46}$$

The bound in (8.5.46) replaces the bound on $|\gamma_k(t)|$ for $m = 1$ in (8.5.25).

Denote the operator $S^{(m)}$ on sequences of numbers $Q = \{Q_k\}_{k=1}^\infty$ by

$$(S^{(m)}Q)_k = m\frac{k-1+\delta}{2m+\delta}Q_{k-1} - m\frac{k+\delta}{2m+\delta}Q_k. \tag{8.5.47}$$

Then, for $m = 1$, we have that (8.5.9) is equivalent to

$$p_k = (S^{(1)}p)_k + \mathbb{1}_{\{k=1\}}. \tag{8.5.48}$$

For $m > 1$, we replace the above recursion on p by $p_k = 0$ for $k < m$ and, for $k \geq m$,

$$p_k = (S^{(m)}p)_k + \mathbb{1}_{\{k=m\}}. \tag{8.5.49}$$

Again, we can explicitly solve for $p = \{p_k\}_{k=1}^\infty$. The solution is given in the following lemma:

Lemma 8.5. *Fix $\delta > -1$ and $m \geq 1$. Then, the solution to (8.5.49) is given by (8.3.2).*

Proof. We start by noting that $p_k = 0$ for $k < m$, and identify p_m as

$$p_m = -m\frac{m+\delta}{2m+\delta}p_m + 1, \tag{8.5.50}$$

so that

$$p_m = \frac{2m+\delta}{m(m+\delta) + 2m+\delta} = \frac{2 + \frac{\delta}{m}}{(m+\delta) + 2 + \frac{\delta}{m}}. \tag{8.5.51}$$

For $k > m$, the recursion relation in (8.5.49) becomes

$$p_k = \frac{m(k-1+\delta)}{m(k+\delta) + 2m+\delta}p_{k-1} = \frac{k-1+\delta}{k+\delta+2+\frac{\delta}{m}}p_{k-1}. \tag{8.5.52}$$

As a result, we obtain that, again repeatedly using (8.2.6),

$$\begin{aligned}
p_k &= \frac{\Gamma(k+\delta)\Gamma(m+3+\delta+\frac{\delta}{m})}{\Gamma(m+\delta)\Gamma(k+3+\delta+\frac{\delta}{m})}p_m \\
&= \frac{\Gamma(k+\delta)\Gamma(m+3+\delta+\frac{\delta}{m})}{\Gamma(m+\delta)\Gamma(k+3+\delta+\frac{\delta}{m})} \frac{(2+\frac{\delta}{m})}{(m+\delta+2+\frac{\delta}{m})} \\
&= \frac{(2+\frac{\delta}{m})\Gamma(k+\delta)\Gamma(m+2+\delta+\frac{\delta}{m})}{\Gamma(m+\delta)\Gamma(k+3+\delta+\frac{\delta}{m})}.
\end{aligned} \tag{8.5.53}$$

$\square$

Similarly to (8.5.16), we can rewrite (8.5.49) as

$$\begin{aligned}
(t+1)p_k &= tp_k + p_k \\
&= tp_k + (S^{(m)}p)_k + \mathbb{1}_{\{k=m\}} \\
&= (T_{t+1}^{(m)}tp)_k + \mathbb{1}_{\{k=m\}} - \kappa_k(t),
\end{aligned} \tag{8.5.54}$$

where, writing I for the identity operator,

$$\kappa_k(t) = -\big([S^{(m)} + t(I - T_{t+1}^{(m)})]p\big)_k. \tag{8.5.55}$$

<div align="right">e:kappadefm2PA</div>

While (8.5.55) is not very explicit, a similar argument as the ones leading to (8.5.25) can be used to deduce an identical bound. That is the content of the following lemma:

Lemma 8.6. *Fix $\delta \geq -1$ and $m \geq 1$. Then there exists a constant $C_\kappa = C_\kappa(\delta, m)$ such that*

$$|\kappa_k(t)| \leq C_\kappa(t+1)^{-1}. \tag{8.5.56}$$

<div align="right">e:kappabdm2PA</div>

We defer the proof of Lemma 8.6 to the end of this section, and continue with the proof of Proposition 8.4 for $m > 1$.

We define, for $k \geq m$,

$$\varepsilon_k(t) = \bar{N}_k(t) - tp_k. \tag{8.5.57}$$

<div align="right">e:vepkdefm2</div>

Subtracting (8.5.54) from (8.5.42) and writing $\varepsilon(t) = \{\varepsilon_k(t)\}_{k=1}^\infty$ leads to

$$\varepsilon_k(t+1) = (T_{t+1}^{(m)}\varepsilon(t))_k + \kappa_k(t) + \gamma_k(t). \tag{8.5.58}$$

<div align="right">e:vepkrecm2</div>

In order to study the recurrence relation (8.5.58) in more detail, we investigate the properties of the operator $T_t^{(m)}$. To state the result, we introduce some notation. We let $Q = \{Q_k\}_{k=1}^\infty$ be a sequence of real numbers, and we let $\mathcal{Q} = \mathbb{R}^\infty$ denote the set of all such sequences. For $Q \in \mathcal{Q}$, we define the supremum-norm to be

$$\|Q\|_\infty = \sup_{k=1}^\infty |Q_k|. \tag{8.5.59}$$

<div align="right">e:supnorm</div>

Thus, in functional analytic terms, we consider the ℓ^∞ norm on $\mathcal{Q} = \mathbb{R}^\infty$.

Furthermore, we let $\mathcal{Q}_m(t) \subseteq \mathcal{Q}$ be the subset of sequences for which $Q_k = 0$ for $k > m(t+1)$, i.e.,

$$\mathcal{Q}_m(t) = \{Q \in \mathcal{Q} : Q_k = 0 \ \ \forall k > m(t+1)\}. \tag{8.5.60}$$

Clearly, $\bar{N}(t) \in \mathcal{Q}_m(t)$.

We regard $T_{t+1}^{(m)}$ in (8.5.43) as an operator on $\mathcal{Q}$. We now derive its functional analytic properties:

Lemma 8.7. *Fix $\delta \geq -1$ and $m \geq 1$. Then $T_{t+1}^{(m)}$ maps $\mathcal{Q}_m(t)$ into $\mathcal{Q}_m(t+1)$ and, for every $Q \in \mathcal{Q}_m(t)$,*

$$\|T_{t+1}^{(m)}Q\|_\infty \leq \Big(1 - \frac{1}{t(2m+\delta) + (m+\delta)}\Big)\|Q\|_\infty. \tag{8.5.61}$$

<div align="right">e:operatorbdm2PA</div>

Lemma 8.7 implies that $T_{t+1}^{(m)}$ acts as a *contraction* on elements of $\mathcal{Q}_m(t)$. Using Lemmas 8.6 and 8.7, as well as (8.5.46) allows us to complete the proof of Proposition 8.4:

Proof of Proposition 8.4. We use (8.5.58). We define the sequence $\varepsilon'(t) = \{\varepsilon'_k(t)\}_{k=1}^\infty$ by

$$\varepsilon'_k(t) = \varepsilon_k(t)\mathbb{1}_{\{k \leq m(t+1)\}}. \tag{8.5.62}$$

Then, by construction, $\varepsilon'(t) \in \mathcal{Q}_m(t)$. Therefore, by Lemma 8.7,

$$\|\varepsilon(t+1)\|_\infty \leq \|T_{t+1}^{(m)}\varepsilon'(t)\|_\infty + \|\varepsilon'(t+1) - \varepsilon(t+1)\|_\infty + \|\kappa(t)\|_\infty + \|\gamma(t)\|_\infty$$

$$\leq \Big(1 - \frac{1}{(2m+\delta) + (m+\delta)}\Big)\|\varepsilon'(t)\|_\infty + \|\varepsilon'(t+1) - \varepsilon(t+1)\|_\infty + \|\kappa(t)\|_\infty + \|\gamma(t)\|_\infty.$$

$$\tag{8.5.63}$$

Equation (8.5.46) is equivalent to the statement that

$$\|\gamma(t)\|_\infty \le \frac{C_\gamma}{t+1}. \tag{8.5.64}$$

Lemma 8.6 implies that

$$\|\kappa(t)\|_\infty \le \frac{C_\kappa}{t+1}. \tag{8.5.65}$$

It is not hard to see that

$$\|\varepsilon'(t+1) - \varepsilon(t+1)\|_\infty \le C_{\varepsilon'}(t+1)^{-(\tau-1)}, \tag{8.5.66}$$

where $\tau > 2$ is defined in (8.3.10). See (8.5.37)–(8.5.38) for the analogous proof for $m = 1$.

Therefore,

$$\|\varepsilon(t+1)\|_\infty \le \Big(1 - \frac{1}{t(2m+\delta)+(m+\delta)}\Big)\|\varepsilon(t)\|_\infty + \frac{(C_\gamma + C_\kappa + C_{\varepsilon'})}{t+2}. \tag{8.5.67}$$

Using further that, for $m \ge 1$ and $\delta > -m$,

$$t(2m+\delta) + (m+\delta) \le (2m+\delta)(t+1) \tag{8.5.68}$$

we arrive at

$$\|\varepsilon(t+1)\|_\infty \le \Big(1 - \frac{1}{(t+1)(2m+\delta)}\Big)\|\varepsilon(t)\|_\infty + \frac{(C_\gamma + C_\kappa + C_{\varepsilon'})}{t+1}. \tag*{e:vepsupm2 \\ (8.5.69)}$$

Now we can advance the induction hypothesis

$$\|\varepsilon(t)\|_\infty \le C. \tag{8.5.70}$$

For some $C > 0$ sufficiently large, this statement trivially holds for $t = 1$. To advance it, we use (8.5.69), to see that

$$\|\varepsilon(t+1)\|_\infty \le \Big(1 - \frac{1}{(2m+\delta)(t+1)}\Big)C + \frac{(C_\gamma + C_\kappa + C_{\varepsilon'})}{t+1} \le C, \tag*{e:vepsupm2rep \\ (8.5.71)}$$

whenever

$$C \ge (2m+\delta)(C_\gamma + C_\kappa + C_{\varepsilon'}). \tag{8.5.72}$$

This advances the induction hypothesis, and completes the proof that $|\bar{N}_k(t) - p_k t| \le C$ for $m \ge 2$. $\square$

Proof of Lemmas 8.6 and 8.7.   We first prove Lemma 8.7, and then Lemma 8.6.

Proof of Lemma 8.7. We recall that

$$T_{t+1}^{(m)} = T_{m(t+1)} \circ \cdots \circ T_{mt+1}, \tag*{e:Tt-defrep \\ (8.5.73)}$$

Thus, the fact that $T_{t+1}^{(m)}$ maps $\mathcal{Q}_m(t)$ into $\mathcal{Q}_m(t+1)$ follows from the fact that T_{t+1} maps $\mathcal{Q}_1(t)$ into $\mathcal{Q}_1(t+1)$. This proves the first claim in Lemma 8.7.

To prove that the contraction property of $T_{t+1}^{(m)}$ in (8.5.61), we shall first prove that, for all $Q \in \mathcal{Q}_1(mt + a - 1)$, $a = 1, \ldots, m$, $\delta > -m$ and $\delta' = \delta/m > -1$, we have

$$\|(T_{mt+a}Q)\|_\infty \le \Big(1 - \frac{1}{t(2+\delta)+(1+\delta)}\Big)\|Q\|_\infty. \tag*{e:Ttbdfinal \\ (8.5.74)}$$

For this, we recall from (8.5.40) that

$$(T_{mt+a}Q)_k = \left(1 - \frac{k+\delta}{(mt+a)(2+\delta') + (1+\delta')}\right)Q_k + \frac{k-1+\delta}{(mt+a)(2+\delta') + (1+\delta')}Q_{k-1}. \qquad (8.5.75)$$

When $Q \in \mathcal{Q}_1(mt+a)$, then, for all k for which $Q_k \neq 0$,

$$1 - \frac{k+\delta}{(mt+a-1)(2+\delta') + (1+\delta')} \in [0,1], \qquad (8.5.76)$$

and, for $k \geq 2$, also

$$\frac{k-1+\delta}{(mt+a-1)(2+\delta') + (1+\delta')} \in [0,1]. \qquad (8.5.77)$$

As a consequence, we have that

$$\|T_{mt+a}Q\|_\infty \leq \sup_k \left[\left(1 - \frac{k+\delta}{(mt+a-1)(2+\delta') + (1+\delta')}\right)\|Q\|_\infty + \frac{k-1+\delta}{(mt+a-1)(2+\delta') + (1+\delta')}\|Q\|_\infty\right]$$

$$= \left(1 - \frac{1}{(mt+a-1)(2+\delta') + (1+\delta')}\right)\|Q\|_\infty. \qquad \substack{\text{e:Tt-m1contr} \\ (8.5.78)}$$

Now, by (8.5.78), the application of T_{mt+a} to an element Q of $\mathcal{Q}_1(mt+a-1)$ reduces its norm. By (8.5.73), we therefore conclude that, for every $Q \in \mathcal{Q}_m(t)$,

$$\|T_{t+1}^{(m)}Q\|_\infty \leq \|T_{mt+1}Q\|_\infty \leq \left(1 - \frac{1}{mt(2+\delta') + (1+\delta')}\right)\|Q\|_\infty = \left(1 - \frac{1}{t(2m+\delta) + (m+\delta)}\right)\|Q\|_\infty, \substack{\text{e:operatorbdm2PArep} \\ (8.5.79)}$$

since $\delta' = \delta/m$. This completes the proof of Lemma 8.7. $\qquad\qquad\square$

Proof of Lemma 8.6. We recall

$$\kappa_k(t) = \left(\left[S^{(m)} + t(I - T_{t+1}^{(m)})\right]p\right)_k. \qquad (8.5.80)$$

We start with

$$T_{t+1}^{(m)} = T_{m(t+1)} \circ \cdots \circ T_{mt+1} = \left(I + (T_{m(t+1)} - I)\right) \circ \cdots \circ \left(I + (T_{mt+1} - I)\right). \qquad \substack{\text{e:Ttrew} \\ (8.5.81)}$$

Clearly,

$$((T_{t+1} - I)Q)_k = -\frac{k+\delta}{t(2+\delta') + (1+\delta')}Q_k + \frac{k-1+\delta}{t(2+\delta') + (1+\delta')}Q_{k-1}. \qquad (8.5.82)$$

When $\sup_k k|Q_k| \leq K$, then there exists a constant C such that

$$\sup_k \left|((T_{t+1} - I)Q)_k\right| \leq \frac{C}{t+1}. \qquad (8.5.83)$$

Moreover, when $\sup_k k^2|Q_k| \leq K$, then there exists a constant $C = C_K$ such that, when $u, v \geq t$,

$$\sup_k \left|((T_{u+1} - I) \circ (T_{v+1} - I)Q)_k\right| \leq \frac{C}{(t+1)^2}. \qquad \substack{\text{e:squareterms} \\ (8.5.84)}$$

We expand out the brackets in (8.5.81), and note that, by (8.5.84) and the fact that the operators T_u are contractions that the terms where we have at least two factors $T_u - I$ lead to error terms. More precisely, we conclude that, when $\sup_k k^2|Q_k| \leq K$,

$$(T_{t+1}^{(m)}Q)_k = Q_k + \sum_{a=1}^m \left((T_{mt+a} - I)Q\right)_k + E_k(t, Q), \qquad (8.5.85)$$

where, uniformly in k and Q for which $\sup_k k^2|Q_k| \leq K$,

$$|E_k(t,Q)| \leq \frac{C_K}{(t+1)^2}. \tag{8.5.86}$$

As a result, we obtain that

$$((I - T_{t+1}^{(m)})Q)_k = -\sum_{a=1}^{m} ((T_{mt+a} - I)Q)_k - E_k(t,Q). \tag{8.5.87}$$

Furthermore, for every $a = 1, \ldots, m$,

$$((T_{mt+a} - I)Q)_k = \frac{1}{mt}(S^{(m)}Q)_k + F_{k,a}(t,Q), \tag{8.5.88}$$

where, uniformly in k, Q for which $\sup_k k|Q_k| \leq K$ and $a = 1, \ldots, m$,

$$|F_{k,a}(t,Q)| \leq \frac{C'_K}{(t+1)^2}. \tag{8.5.89}$$

Therefore, we also obtain that

$$\sum_{a=1}^{m} ((T_{mt+a} - I)Q)_k = \frac{1}{t}(S^{(m)}Q)_k + F_k(t,Q), \tag{8.5.90}$$

where

$$F_k(t,Q) = \sum_{a=1}^{m} F_{k,a}(t,Q). \tag{8.5.91}$$

We summarize from (8.5.87) and (8.5.90) that

$$([S^{(m)} + t(I - T_{t+1}^{(m)})]Q)_k = -tF_k(t,Q) - tE_k(t,Q), \tag{8.5.92}$$

so that

$$\kappa_k(t) = ([S^{(m)} + t(I - T_{t+1}^{(m)})]p)_k = -tF_k(t,p) - tE_k(t,p). \tag{8.5.93}$$

We note that by (8.3.9) and (8.3.10), p satisfies that

$$\sup_k k^2 p_k \leq C_p, \tag{8.5.94}$$

so that we conclude that

$$\begin{aligned}
\|\kappa(t)\|_\infty &= \sup_k \left| ([S^{(m)} + t(I - T_{t+1}^{(m)})p])_k \right| \\
&\leq \sup_k t(|E_k(t,p)| + |F_k(t,p)|) \\
&\leq \frac{t(C_K + C'_K)}{(t+1)^2} \\
&\leq \frac{C_K + C'_K}{t+1}.
\end{aligned} \tag{8.5.95}$$

$\square$

8.5.3 Degree sequence in preferential attachment model: Proof of Theorem 8.2

We only prove the result for $m = 1$, the proof for $m > 1$ being identical. By Proposition 8.4, we obtain that

$$\max_k |\mathbb{E}[N_k(t)] - p_k t| \leq C. \tag{8.5.96}$$

Therefore, by Proposition 8.3 we obtain

$$\mathbb{P}\Big(\max_k |N_k(t) - p_k t| \geq C(1 + \sqrt{t \log t})\Big) = o(1), \tag{8.5.97}$$

which, since $P_k(t) = N_k(t)/t$, implies that

$$\mathbb{P}\Big(\max_k |P_k(t) - p_k| \geq \frac{C}{t}(1 + \sqrt{t \log t})\Big) = o(1). \tag{8.5.98}$$

Equation (8.5.98) in turn implies Theorem 8.2. □

8.6 Maximal degree in preferential attachment models

In this section, we shall investigate the maximal degree and the clustering of the graph $\mathrm{PA}_{m,\delta}(t)$. In order to state the results on the maximal degree, we denote

$$M_t = \max_{i=1}^{t} D_i(t). \tag{8.6.1}$$

The main result on the maximal degree is the following theorem:

Theorem 8.8 (The maximal degree of $\mathrm{PA}_{m,\delta}(t)$). *Fix $m \geq 1$ and $\delta < -m$. Then,*

$$M_t t^{-\frac{1}{\tau-1}} \xrightarrow{a.s.} \mu, \tag{8.6.2}$$

with $\mathbb{P}(\mu = 0) = 0$.

Below, we shall be able to compute all moments of the limiting random variables ξ_i of $D_i(t)t^{-1/(2+\delta)}$. We do not recognize these moments as the moments of a continuous random variable.

Exercise 8.18 ([108]). *Fix $m = 1$ and $\delta > -1$. Then, prove that for all $t \geq i$*

$$\mathbb{P}(D_i(t) = j) \leq C_j \frac{\Gamma(t)\Gamma(i + \frac{1+\delta}{2+\delta})}{\Gamma(t + \frac{1+\delta}{2+\delta})\Gamma(i)}, \tag{8.6.3}$$

where $C_1 = 1$ and

$$C_j = \frac{j - 1 + \delta}{j - 1} C_{j-1}. \tag{8.6.4}$$

Mori [159] studied various martingales related to degrees, and used them to prove that the maximal degree of $\{\mathrm{PA}_{m,\delta}(t)\}_{t=1}^{\infty}$ converges a.s. We shall reproduce his argument here, applied to a slightly different model. See also [79, Section 4.3]. We fix $m = 1$ for the time being, and extend the results to $m \geq 2$ at the end of this section.

In [159], the graph at time 1 consists of two vertices, 0 and 1, connected by a single edge. In the attachment scheme, no self-loops are created, so that the resulting graph is a tree. The proof generalizes easily to other initial configurations and attachment rules, and we shall adapt the argument here to the usual preferential attachment model in which self-loops *do* occur and $\mathrm{PA}_{1,\delta}(1)$ consists of one vertex with

a single self-loop. At the tth step, a new vertex is added and connected to an existing vertex. A vertex of degree k is chosen with probability $(k+\delta)/n(t)$ where $\delta > -1$ and $n(t) = t(2+\delta) + 1 + \delta$ is the sum of the weights for the random graph with t edges and t vertices.

Let $X_j(t) = D_j(t) + \delta$ be the weight of vertex j at time t, let $\Delta_j(t+1) = X_j(t+1) - X_j(t)$. If $j \le t$, then

$$\mathbb{P}\left(\Delta_j(t+1) = 1 | \mathrm{PA}_{1,\delta}(t)\right) = X_j(t)/n(t) \tag{8.6.5}$$

From this, we get

$$\mathbb{E}\left(X_j(t+1) | \mathrm{PA}_{1,\delta}(t)\right) = X_j(t)\left(1 + \frac{1}{n(t)}\right) \tag{8.6.6}$$

so $c_t X_j(t)$ will be a martingale if and only if $c_t/c_{t+1} = n(t)/(1 + n(t))$. Anticipating the definition of a larger collection of martingales we let

$$c_k(t) = \frac{\Gamma(t + \frac{1+\delta}{2+\delta})}{\Gamma(t + \frac{k+1+\delta}{2+\delta})} \quad t \ge 1, k \ge 0, \tag{8.6.7}$$

For fixed $k \ge 0$, by (8.2.7),

$$c_k(t) = t^{-k/(2+\delta)}(1 + o(1)) \quad \text{as } t \to \infty \tag{8.6.8}$$

Using the recursion $\Gamma(r) = (r-1)\Gamma(r-1)$ we have

$$\frac{c_k(t+1)}{c_k(t)} = \frac{t + \frac{1+\delta}{2+\delta}}{t + \frac{k+1+\delta}{2+\delta}} = \frac{n(t)}{n(t)+k}. \tag{8.6.9}$$

In particular, it follows that $c_1(t)X_j(t)$ is a martingale for $t \ge j$. Being a positive martingale it will converge a.s. to a random variable ξ_j, as discussed in full detail in Theorem 8.1. To study the joint distribution of the $X_j(t)$ we make use of a whole class of martingales. We first introduce some notation. For $a, b > -1$ with $a - b > -1$, where a, b are not necessarily integers, we write

$$\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}. \tag{8.6.10}$$

The restriction on a, b is such that the arguments of the Gamma-function are all strictly positive. Then the following proposition identifies a whole class of useful martingales related to the degrees of the vertices:

Proposition 8.9. *Let $r \ge 0$ be a non-negative integer, $k_1, k_2, \ldots, k_r > \min\{-1, -1-\delta\}$, and $0 \le j_1 < \ldots < j_r$ be non-negative integers. Then, with $k = \sum_i k_i$,*

$$Z_{\vec{j},\vec{k}}(t) = c_k(t) \prod_{i=1}^{r} \binom{X_{j_i}(t) + k_i - 1}{k_i} \tag{8.6.11}$$

is a martingale for $t \ge \max\{j_r, 1\}$.

The restriction $k_i > \min\{-1, -1-\delta\}$ is to satisfy the restrictions $a, b, a - b > -1$ in (8.6.10), since $X_j(t) \ge 1 + \delta$. Since $\delta > -1$, this means that Proposition 8.9 also holds for certain $k_i < 0$.

Exercise 8.19. *Use Proposition 8.9 to show that, for all $t \ge \max\{j_r, 1\}$,*

$$\mathbb{E}[Z_{\vec{j},\vec{k}}(t)] = \prod_{i=1}^{r} \frac{c_{K_i}(j_i)}{c_{K_{i-1}}(j_i)} \binom{k_i + \delta}{k_i}, \tag{8.6.12}$$

where $K_i = \sum_{a=1}^{i} k_a$.

Proof. By considering the two cases $\Delta_j(t) = 0$ or $\Delta_j(t) = 1$, and using (8.6.10) and $\Gamma(r) = (r-1)\Gamma(r-1)$, it is easy to check that, for all k,

$$\binom{X_j(t+1)+k-1}{k} = \binom{X_j(t)+k-1}{k} \frac{\Gamma(X_j(t+1)+k)}{\Gamma(X_j(t)+k)} = \binom{X_j(t)+k-1}{k}\left(1 + \frac{k\Delta_j(t)}{X_j(t)}\right). \quad \text{(8.6.13)}$$

e:Durrett_c43_6

At most one $X_j(t)$ can change, so that

$$\prod_{i=1}^{r}\left(1 + \frac{k_i\Delta_{j_i}(t)}{X_{j_i}(t)}\right) = \left(1 + \sum_{i=1}^{r}\frac{k_i\Delta_{j_i}(t)}{X_{j_i}(t)}\right). \quad \text{(8.6.14)}$$

e:prodsumPAdegrees

Together, (8.6.13) and (8.6.14) imply that

$$\prod_{i=1}^{r}\binom{X_{j_i}(t+1)+k_i-1}{k_i} = \left(1 + \sum_{i=1}^{r}\frac{k_i\Delta_{j_i}(t)}{X_{j_i}(t)}\right)\prod_{i=1}^{r}\binom{X_{j_i}(t)+k_i-1}{k_i}. \quad \text{(8.6.15)}$$

e:Durrett_c43_7

Since $\mathbb{P}(\Delta_j(t+1) = 1|\mathrm{PA}_{1,\delta}(t)) = X_j(t)/n(t)$, using the definition of $Z_{\vec{j},\vec{k}}(t)$ and taking expected value,

$$\mathbb{E}\left(Z_{\vec{j},\vec{k}}(t+1)|\mathrm{PA}_{1,\delta}(t)\right) = Z_{\vec{j},\vec{k}}(t) \cdot \frac{c_k(t+1)}{c_k(t)}\left(1 + \frac{\sum_i k_i}{n(t)}\right) = Z_{\vec{j},\vec{k}}(t), \quad \text{(8.6.16)}$$

e:Durrett_c43_8

where $k = \sum_i k_i$ and the last equality follows from (8.6.9). $\qquad\square$

Being a non-negative martingale $Z_{\vec{j},\vec{k}}(t)$ converges. From the form of the martingale, the convergence result for the factors, and the asymptotics for the normalizing constants in (8.6.8), the limit must be $\prod_{i=1}^{r}\xi_i^{k_i}/\Gamma(k_i+1)$, where we recall that ξ_i is the almost sure limit of $D_i(t)t^{-1/(2+\delta)}$. Here we make use of (8.2.7), which implies that

$$\binom{X_j(t)+k-1}{k} = X_j(t)^k(1 + O(1/X_j(t))), \quad \text{(8.6.17)}$$

together with the fact that $D_i(t) \xrightarrow{a.s.} \infty$ (see Exercise 8.7).

Our next step is to check that the martingale converges in L^1. To do this we begin by observing that (8.6.8) implies $c_m(t)^2/c_{2m}(t) \to 1$ and we have

$$\binom{x+k-1}{k}^2 = \left(\frac{\Gamma(x+k)}{\Gamma(x)\Gamma(k+1)}\right)^2 = \frac{\Gamma(x+k)}{\Gamma(x)}\frac{\Gamma(x+k)}{\Gamma(x)\Gamma(k+1)^2}. \quad \text{(8.6.18)}$$

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Now we use that $x \mapsto \Gamma(x+k)/\Gamma(x)$ is increasing for $k \geq 0$, so that

$$\binom{x+k-1}{k}^2 \leq \frac{\Gamma(x+2k)}{\Gamma(x+k)}\frac{\Gamma(x+k)}{\Gamma(x)\Gamma(k+1)^2} = \binom{x+2k-1}{2k}\cdot\binom{2k}{k}. \quad \text{(8.6.19)}$$

From this it follows that

$$Z_{\vec{j},\vec{k}}(t)^2 \leq C_{\vec{k}}Z_{\vec{j},2\vec{k}}(t), \quad \text{(8.6.20)}$$

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where

$$C_{\vec{k}} = \prod_{i=1}^{r}\binom{2k_i}{k_i}. \quad \text{(8.6.21)}$$

Therefore, $Z_{\vec{j},\vec{k}}(t)$ is an L^2−bounded martingale, and hence converge in L^1.

Taking $r = 1$ we have, for all $j \geq 1$ integer and $k \in \mathbb{R}$ with $k \geq 0$,

$$\mathbb{E}[\xi_j^k / \Gamma(k+1)] = \lim_{t \to \infty} \mathbb{E}[Z_{j,k}(t)] = \mathbb{E}[Z_{j,k}(j)] = c_k(j)\binom{k+\delta}{k}. \qquad \text{(8.6.22)}$$

Recalling that $c_k(j) = \frac{\Gamma(j+\frac{1+\delta}{2+\delta})}{\Gamma(j+\frac{k+1+\delta}{2+\delta})}$, we thus arrive at the fact that, for all j non-negative integers, and all k non-negative,

$$\mathbb{E}[\xi_j^k] = \frac{\Gamma(j+\frac{1+\delta}{2+\delta})}{\Gamma(j+\frac{k+1+\delta}{2+\delta})}\frac{\Gamma(k+1+\delta)}{\Gamma(1+\delta)}. \qquad \text{(8.6.23)}$$

It is, as far as we know, unknown which random variable has these moments, but we *can* see that the above moments identify the distribution:

Exercise 8.20. *Prove that the moments in* (8.6.23) *identify the distribution of ξ_j uniquely. Prove also that $\mathbb{P}(\xi_j > x) > 0$ for every $x > 0$, so that ξ_j has unbounded support.*

Exercise 8.21. *Show that ξ_j has the same distribution as*

$$\xi_1 \prod_{k=1}^{j} B_k, \qquad \text{(8.6.24)}$$

where B_k has a Beta$(1, (2+\delta)k - 1)$-distribution.

Exercise 8.22. *([159], [79, Section 4.3]) Prove that when the graph at time 0 is given by two vertices with a single edge between them, and we do not allow for self-loops, then* (8.6.22) *remains valid when we instead define*

$$c_k(t) = \frac{\Gamma(t+\frac{\delta}{2+\delta})}{\Gamma(t+\frac{k+\delta}{2+\delta})} \quad t \geq 1, k \geq 0. \qquad \text{(8.6.25)}$$

In the special case where $\delta = 0$, we are lead to

$$\mathbb{E}[\xi_j^k] = k!\frac{\Gamma(j+\frac{1}{2})}{\Gamma(j+\frac{k+1}{2})}. \qquad \text{(8.6.26)}$$

Thus, in particular,

$$\mathbb{E}[\xi_1^k] = k!\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{k+3}{2})} = 2^k\frac{\Gamma(\frac{k+2}{2})}{k+1}. \qquad \text{(8.6.27)}$$

If we denote $\mathbb{E}[\xi_j^k] = m_k$, then we have that

$$m_k = \frac{b_{k+1}}{b_1(k+1)}, \qquad \text{(8.6.28)}$$

where

$$b_k = 2^k\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{1}{2})}. \qquad \text{(8.6.29)}$$

Now, let $Y = 2|Z|$, where Z be a standard normal distribution. Then

$$\mathbb{E}[Y^k] = 2^k\sqrt{\frac{2}{\pi}}\int_0^\infty y^k e^{-y^2/2}dy = \sqrt{\frac{1}{\pi}}\int_0^\infty z^{\frac{k}{2}-\frac{1}{2}}e^{-z}dz = 2^k\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{1}{2})} = b_k, \qquad \text{(8.6.30)}$$

so that $\{b_k\}_{k=0}^\infty$ are the moments of the non-negative random variable Y with density $f_Y(y) = \sqrt{2/\pi}e^{-2y^2}$. The relation $m_k = b_{k+1}/[b_1(k+1)]$ in turn implies that $\{m_k\}_{k=0}^\infty$ are the moments of a random variable X with density f_X given by

$$f_X(x) = \frac{1}{\mathbb{E}[Y]}\int_x^\infty f_Y(y)dy, \tag{8.6.31}$$

so that the density of ξ_1 is given by

$$f_{\xi_1}(x) = \frac{1}{\sqrt{2}}\int_x^\infty e^{-2y^2}dy. \tag{8.6.32}$$

In particular, for $\delta = 0$, $\mathbb{P}(\xi_1 = 0) = 0$. We expect that the latter is also true when $\delta \neq 0$, but we do not have a proof.

Exercise 8.23. *Prove that the moments $\{m_k\}_{k=0}^\infty$ of X with density f_X defined in (8.6.31) are given by $m_k = b_{k+1}/[b_1(k+1)]$, where $\{b_k\}_{k=0}^\infty$ are the moments of Y.*

We complete this discussion by showing that $\mathbb{P}(\xi_j = 0) = 0$ for all $j \geq 1$. For this, we use (8.2.7), which implies that, for $k > \min\{-1, -1 - \delta\}$,

$$\limsup_{t\to\infty}\mathbb{E}\Big[\Big(\frac{X_j(t)}{t^{1/(2+\delta)}}\Big)^k\Big] \leq A_k\limsup_{t\to}\mathbb{E}[Z_{j,k}(t)] < \infty. \tag{8.6.33}$$

Since $\delta > -1$, we have $-1 - \delta < 0$, so that the a negative moment of $X_j(t)/t^{1/(2+\delta)}$ remains uniformly bounded. This implies that $\mathbb{P}(\xi_j = 0) = 0$. Indeed, we use that $X_j(t)/t^{1/(2+\delta)} \xrightarrow{a.s.} \xi_j$, which implies that $X_j(t)/t^{1/(2+\delta)} \xrightarrow{d} \xi_j$, so that, using the Markov property (Theorem 2.14), for every $\varepsilon > 0$ and $k \in (\min\{-1, -1 - \delta\}, 0)$,

$$\mathbb{P}(\xi_j \leq \varepsilon) = \limsup_{t\to\infty}\mathbb{P}\Big(X_j(t)/t^{1/(2+\delta)} \leq \varepsilon\Big) \leq \limsup_{t\to\infty}\varepsilon^{-k}\mathbb{E}\Big[\Big(\frac{X_j(t)}{t^{1/(2+\delta)}}\Big)^k\Big] = O(\varepsilon^{-k}). \tag{8.6.34}$$

Letting $\varepsilon \downarrow 0$, we obtain that $\mathbb{P}(\xi_j = 0) = 0$.

We next move on to study the maximal degree M_t. Let M_t denote the maximal degree in our random graph after t steps, and, for $t \geq j$, let

$$M_j(t) = \max_{0 \leq i \leq j} Z_{i,1}(t). \tag{8.6.35}$$

Note that $M_t(t) = c_1(t)(M_t + \delta)$. We shall now prove that $M_t(t) \xrightarrow{a.s.} \sup_{j=1}^\infty \xi_j$:

Proof of Theorem 8.8 for $m = 1$. We start by proving Theorem 8.8 for $m = 1$. Being a maximum of martingales, $\{M_t(t)\}_{t=1}^\infty$ is a non-negative submartingale. Therefore, $M_t(t) \xrightarrow{a.s.} \mu$ for some limiting random variable μ, and we are left to prove that $\mu = \sup_{j\geq 0} \xi_j$.

Since $Z_{j,1}(t)^k$ is a submartingale for every $k \geq 1$, and $Z_{j,1}(t)^k$ converges in L^1 to ξ_j^k, we further have that

$$\mathbb{E}[Z_{j,1}(t)^k] \leq \mathbb{E}[\xi_j^k]. \tag{8.6.36}$$

Then, using the trivial inequality

$$M_t(t)^k = \max_{0 \leq i \leq t} Z_{i,1}(t)^k \leq \sum_{j=0}^t Z_{j,1}(t)^k, \tag{8.6.37}$$

and (8.6.36), we obtain

$$\mathbb{E}[M_t(t)^k] \leq \sum_{j=0}^{t} \mathbb{E}[Z_{j,1}(t)^k] \leq \sum_{j=0}^{\infty} \mathbb{E}[\xi_j^k] = \Gamma(k+1)\binom{k+\delta}{k}\sum_{j=0}^{\infty} c_k(j), \qquad \text{(8.6.38)}$$

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which is finite by (8.6.8) if $k > 2 + \delta$. Thus $M_t(t)$ is bounded in L^k for every integer $k > 2 + \delta$, and hence bounded and convergent in L^p for any $p \geq 1$. Therefore, to prove that $\mu = \sup_{j \geq 0} \xi_j$, we are left to prove that $M_t(t)$ converges to $\sup_{j \geq 0} \xi_j$ in L^k for some k.

Let $k > 2 + \delta$ be fixed. Then, by a similar inequality as in (8.6.37),

$$\mathbb{E}\big[(M_t(t) - M_j(t))^k\big] \leq \sum_{i=j+1}^{t} \mathbb{E}[Z_{i,1}(t)^k] \qquad \text{(8.6.39)}$$

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Since $M_j(t)$ is a *finite* maximum of martingales, it is again a non-negative submartingale which each converge almost surely and in L^k for any $k > 2 + \delta$, its almost sure limit is equal to $\max_{0 \leq i \leq j} \xi_i = \mu_j$, Therefore, the limit of the left-hand side of (8.6.39) is

$$\mathbb{E}\Big[\Big(\lim_{t \to \infty} t^{-1/(2+\delta)} M_t - \mu_j\Big)^k\Big] \qquad \text{(8.6.40)}$$

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while the right-hand side of (8.6.39) increases to (compare to (8.6.36))

$$\sum_{i=j+1}^{\infty} \mathbb{E}[\xi_i^k] = k!\binom{k+\delta}{k}\sum_{i=j+1}^{\infty} c_k(i), \qquad \text{(8.6.41)}$$

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which is small if j is large by (8.6.8). Recall that $t^{-1/(2+\delta)} M_t \xrightarrow{a.s.} \mu$. Therefore, we obtain that

$$\lim_{j \to \infty} \mathbb{E}\Big[(\mu - \mu_j)^k\Big] = 0. \qquad \text{(8.6.42)}$$

Hence $\lim_{t \to \infty} t^{-1/(2+\delta)} M_t = \mu$ as claimed.

When $m \geq 2$, then the above can be used as well. Indeed, in this case, we have that by Exercise 8.11, $D_i(t)(mt)^{-1/(2+\delta/m)} \xrightarrow{a.s.} \xi_i'$, where

$$\xi_i' = \sum_{j=(i-1)m+1}^{mi} \xi_j, \qquad \text{(8.6.43)}$$

and ξ_j is the almost sure limit of $D_j(t)$ in $\{\text{PA}_{1,\delta/m}(t)\}_{t=1}^{\infty}$. This implies that $M_t \xrightarrow{a.s.} \mu = \sup_{j=1}^{\infty} \xi_j$. We omit the details. $\qquad \square$

Since $\mathbb{P}(\xi_1 = 0) = 0$, we have that $\mathbb{P}(\mu = 0) = \mathbb{P}(\sup_{j=1}^{\infty} \xi_j = 0) \leq \mathbb{P}(\xi_1 = 0) = 0$. Thus, we see that M_t really is of order $t^{1/(2+\delta)}$, and is not smaller.

8.7   Related models

There are numerous related preferential attachment models in the literature. Here we discuss a few of them:

A directed preferential attachment model. In [41],[BolBorChaRio03] a directed preferential attachment model is investigated, and it is proved that the degrees obey a power law similar to the one in Theorem 8.2. We first describe the model. Let G_0 be any fixed initial directed graph with t_0 edges. Fix some non-negative parameters $\alpha, \beta, \gamma, \delta_{\text{in}}$ and δ_{out}, where $\alpha + \beta + \gamma = 1$.

We next define $G(t)$. In order to do so, we say that we choose a vertex *according to* $f_i(t)$ when we choose vertex i with probability

$$\frac{f_i(t)}{\sum_j f_j(t)}. \tag{8.7.1}$$

Thus, the probability that we choose a vertex i is proportional to the value of the function $f_i(t)$. Also, we denote the in-degree of vertex i in $G(t)$ by $D_{\text{in},i}(t)$, and the out-degree of vertex i in $G(t)$ by $D_{\text{out},i}(t)$.

We let $G(t_0) = G_0$, where t_0 is chosen appropriately, as we will indicate below. For $t \geq t_0$, we form $G(t+1)$ from $G(t)$ according to the following growth rules:

(A) With probability α, we add a new vertex v together with an edge from v to an existing vertex which is chosen according to $D_{\text{in},i}(t) + \delta_{\text{in}}$.

(B) With probability β, we add an edge between the existing vertices v and w, where v and w are chosen independently, v according to $D_{\text{in},i}(t) + \delta_{\text{in}}$ and w according to $D_{\text{out},i}(t) + \delta_{\text{out}}$.

(C) With probability γ, we add a vertex w and an edge from an existing vertex v to w according to $D_{\text{out},i}(t) + \delta_{\text{out}}$.

The above growth rule produces a graph process $\{G(t)\}_{t \geq t_0}$ where $G(t)$ has precisely t edges. The number of vertices in $G(t)$ is denoted by $T(t)$, where $T(t) \sim \text{BIN}(t, \alpha + \gamma)$.

It is not hard to see that if $\alpha \delta_{\text{in}} + \gamma = 0$, then all vertices outside of G_0 will have in-degree zero, while if $\gamma = 1$ all vertices outside of G_0 will have in-degree one. Similar trivial graph processes arise when $\gamma \delta_{\text{out}} + \alpha = 0$ or $\alpha = 1$.

Exercise 8.24. *Prove that if $\alpha \delta_{\text{in}} + \gamma = 0$, then all vertices outside of G_0 will have in-degree zero, while if $\gamma = 1$ all vertices outside of G_0 will have in-degree one.*

We exclude the above cases. Then, [41][BolBorChaRio03] show that both the in-degree and the out degree of the graph converge, in the sense that we will explain now. Denote by $X_i(t)$ the in-degree sequence of $G(t)$, so that

$$X_k(t) = \sum_{v \in G(t)} \mathbb{1}_{\{D_{\text{in},v}(t) = k\}}, \tag{8.7.2}$$

and, similarly, let $Y_i(t)$ be the out-degree sequence of $G(t)$, so that

$$Y_k(t) = \sum_{v \in G(t)} \mathbb{1}_{\{D_{\text{out},v}(t) = k\}}. \tag{8.7.3}$$

Denote

$$\tau_{\text{in}} = 1 + \frac{1 + \delta_{\text{in}}(\alpha + \beta)}{\alpha + \beta}, \qquad \tau_{\text{out}} = 1 + \frac{1 + \delta_{\text{out}}(\gamma + \beta)}{\gamma + \beta}. \tag{8.7.4}$$

Then [41, Theorem 3.1][BolBorChaRio03] shows that there exist probability distributions $p = \{p_k\}_{k=0}^{\infty}$ and $q = \{q_k\}_{k=0}^{\infty}$ such that with high probability

$$X_k(t) - p_k t = o(t), \qquad Y_k(t) - q_k t = o(t), \tag{8.7.5}$$

while, for $k \to \infty$,

$$p_k = C_{\text{in}}k^{-\tau_{\text{in}}}(1 + o(1)), \qquad q_k = C_{\text{out}}k^{-\tau_{\text{out}}}(1 + o(1)). \tag{8.7.6}$$

In fact, the probability distributions p and q are determined explicitly, as in (8.3.2) above, and p and q have a similar shape as p in (8.3.2). Also, since $\delta_{\text{in}}, \delta_{\text{out}} \geq 0$, and $\alpha + \beta, \gamma + \beta \leq 1$, we again have that $\tau_{\text{in}}, \tau_{\text{out}} \in (2, \infty)$. In [41], there is also a result on the joint distribution of the in- and out-degrees of $G(t)$, which we shall not state here.

The proof in [41] is similar to the one chosen here. Again the proof is split into a concentration result as in Proposition 8.3, and a determination of the expected empirical degree sequence in Proposition 8.4. In fact, the proof Proposition 8.4 is adapted after the proof in [41], which also writes down the recurrence relation in (8.5.19), but analyses it in a different way, by performing induction on k, rather than on t as we do in Sections 8.5.1 and 8.5.2. As a result, the result proved in Proposition 8.4 is slightly stronger. A related result on a directed preferential attachment model can be found in [55]. In this model, the preferential attachment probabilities only depend on the in-degrees, rather than on the total degree, and power-law in-degrees are proved.

A general preferential attachment model.   A quite general version of preferential attachment models is presented in [69]. In this paper, an undirected graph process is defined. At time 0, there is a single initial vertex v_0. Then, to go from $G(t)$ to $G(t + 1)$, either a new vertex can be added or a number of edges between existing vertices. The first case is called NEW, the second OLD. With probability α, we choose to apply the procedure OLD, and with probability $1 - \alpha$ we apply the procedure NEW.

In the procedure NEW, we add a single vertex, and let $f = \{f_i\}_{i=1}^{\infty}$ be such that f_i is the probability that the new vertex generates i edges. With probability β, the end vertices of these edges are chosen *uniformly* among the vertices, and, with probability $1 - \beta$, the end vertices of the added edges are chosen proportionally to the degree.

In the procedure OLD, we choose a single old vertex. With probability δ, this vertex is chosen uniformly, and with probability $1 - \delta$, it is chosen with probability proportionally to the degree. We let $g = \{g_i\}_{i=1}^{\infty}$ be such that g_i is the probability that the old vertex generates i edges. With probability γ, the end vertices of these edges are chosen *uniformly* among the vertices, and, with probability $1 - \gamma$, the end vertices of the added edges are chosen proportionally to the degree.

The main result in [69] states that the empirical degree distribution converges to a probability distribution which obeys a power law with a certain exponent τ which depends on the parameters of the model. More precisely, a result such as in Theorem 8.2 is proved, at least for $k \leq t^{1/21}$. Also, a version of Proposition 8.4 is proved, where the error term $\mathbb{E}[P_k(t)] - tp_k$ is proved to be at most $Mt^{1/2}\log t$. For this result, some technical conditions need to be made on the first moment of f, as well as on the distribution g. The result is nice, because it is quite general. The precise bounds are a bit weaker than the ones presented here.

Interestingly, also the maximal degree is investigated, and it is shown that the maximal degree is of order $\Theta(t^{1/(\tau-1)})$ as one would expect. This result is proved as long as $\tau < 3$. [1] Finally, the results close to those that we present here are given in [5]. In fact, the error bound in Proposition 8.4 is proved there for $m = 1$ for several models. The result for $m > 1$ is, however, not contained there.

Non-linear preferential attachment.   There is also work on preferential attachment models where the probability of connecting to a vertex with degree k depends in a non-linear way on k. In [141],

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[1] On [69, Page 318], it is mentioned that when the power law holds with power law exponent τ, that this suggests that the maximal degree should grow like $t^{1/\tau}$. However, when the degrees are independent and identically distributed with a power law exponent equal to τ, then the maximal degree should grow like $\Theta(t^{1/(\tau-1)})$, which is precisely what is proved in [69, Theorems 2 and 5].

the attachment probabilities have been chosen proportional to k^γ for some γ. The linear case was non-rigorously investigated in [140], and the cases where $\gamma \neq 1$ in [141]. As one can expect, the results depend dramatically in the choice of γ. When $\gamma < 1$, the degree sequence is predicted to have a power law with a certain stretched exponential cut-off. Indeed, the number of vertices with degree k at time t is predicted to be roughly equal to $t\alpha_k$, where

$$\alpha_k = \frac{\mu}{k^\gamma} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{j^\gamma}}, \tag{8.7.7}$$

and where μ satisfies the implicit equation that $\sum_k \alpha_k = 1$. When $\gamma > 1$, then [140] predicts that there is a single vertex that is connected to nearly all the other vertices. In more detail, when $\gamma \in (1 + \frac{1}{m+1}, 1 + \frac{1}{m})$, it is predicted that there are only finitely many vertices that receive more than $m + 1$ links, while there are, asymptotically, infinitely many vertices that receive at least m links. This was proved rigorously in [173].

In [185], random trees with possibly non-linear preferential attachment are studied by relating them to continuous-time branching processes and using properties of such branching processes. Their analysis can be seen as a way to make the heuristic in Section 1.2.2 precise. To explain their results, let w_i be the *weight* of a vertex of degree i. The random tree evolves, conditionally on the tree at time t, by attaching the $(t + 1)^{\text{st}}$ vertex to vertex i with probability proportional to $w_{D_i(t)-1}$. Let λ^* be the solution, if it exists, of the equation

$$1 = \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{w_i}{w_i + \lambda}. \tag{8.7.8}$$

Then, it is proved in [185] that the degree distribution converges to $p_w = \{p_w(k)\}_{k=1}^{\infty}$, where[2]

$$p_w(k) = \frac{\lambda^*}{w_k + \lambda^*} \prod_{i=0}^{k} \frac{w_i}{w_i + \lambda^*}. \tag{8.7.9}$$

For linear preferential attachment models where $w_i = i + 1 + \delta$, we have that $\lambda^* = \delta$, so that (8.7.9) reduces to (8.3.3):

Exercise 8.25. *Prove that, when $\lambda^* = \delta$ and $w_i = i + 1 + \delta$, (8.7.9) reduces to (8.3.3).*

Interestingly, in [185] not only the degree of a uniformly chosen vertex is studied, but also its neighborhood. We refrain from describing these results here.

Preferential attachment and power-law exponents in $(1, 2)$**.** In all models, and similarly to Theorem 8.2, the power law exponents τ are limited to the range $(2, \infty)$. It would be of interest to find simple examples where the power law exponent can lie in the interval $(1, 2)$. A possible solution to this is presented in [75], where a preferential attachment model is presented in which a *random* number of edges can be added which is, unlike [69], not bounded. In this case, when the number of edges obeys a power law, then there is a cross-over between a preferential attachment power law and the power law from the edges, the one with the smallest exponent winning. Unfortunately, the case where the weights have degrees with power-law exponent in $(1, 2)$ is not entirely analyzed.

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[2]The notion of degree used in [185] is slightly different since [185] makes use of the in-degree only. For trees, we have that the degree is the in-degree plus 1, which explains the apparent difference in the formulas.

8.8   Notes and discussion

Notes on Section 8.1.   There are various ways of modeling the Rich-get-Richer or preferential attachment phenomenon, and in these notes, we shall describe some related models. The most general model is studied in [69], the main result being that the degrees obey a power law. A model where the added edges are conditionally independent given the degrees is given in [129]. A directed preferential attachment model is presented in [29].

Notes on Section 8.2.   The degrees of fixed vertices plays a crucial role in the analysis of preferential attachment models, see e.g. [46]. In [197], several moments of the degrees are computed for the Albert-Barabási model, including the result in Theorem 8.1 and several extensions.

Notes on Section 8.3.   Most papers on specific preferential attachment models prove that the degree sequences obey a power law. We shall refer in more detail to the various papers on the topic when we discuss the various different ways of proving Proposition 8.4. General results in this direction can be found for example in [32].

Notes on Section 8.4.   The proof of Theorem 8.2 relies on two key propositions, namely, Propositions 8.3 and 8.4. Proposition 8.3 is a key ingredient in the investigation of the degrees in preferential attachment models, and is used in many related results for other models. The first version, as far as we know, of this proof is in [46].

Notes on Section 8.5.   The proof of the expected empirical degree sequence in Proposition 8.4 is new, and proves a stronger result than the one for $\delta = 0$ appearing in [46]. The proof of Proposition 8.4 is also quite flexible. For example, instead of the growth rule in (8.1.1), we could attach the m edges of the newly added vertex v_{t+1} each independently and with equal probability to a vertex $i = 1, \ldots, t$ with probability proportional to $D_i(t) + \delta$. More precisely, this means that, for $t \geq 3$,

$$\mathbb{P}\big(v_{t+1} \to v_i \big| \mathrm{PA}_{1,\delta}(t)\big) = \frac{D_i(t) + \delta}{t(2m + \delta)} \qquad \text{for } i = 1, \ldots, t. \tag{8.8.1}$$

In this new model, we do not create any self-loops. We can define $\mathrm{PA}_{m,\delta}(2)$ to consist of 2 vertices connected by m edges.

   It is not hard to see that the proof of Proposition 8.3 applies verbatim:

Exercise 8.26. *Adapt the proof of Proposition 8.3 showing the concentration of the degrees to the preferential attachment model defined in* (8.8.1).

It is not hard to see that also the proof of Proposition 8.4 applies by making the obvious changes. In fact, the limiting degree sequence remains unaltered. A second slightly different model, in which edges are added independently without intermediate updating, is studied by Jordan in [127].

   The original proof in [46] of the asymptotics of the expected empirical degree sequence for $\delta = 0$ makes use of an interesting relation between this model and so-called *n-pairings*. An n-pairing is a partition of the set $\{1, \ldots, 2n\}$ into pairs. We can think about the pairs as being points on the x-axis, and the pairs as chords joining them. This allows us to speak of the left- and right-endpoints of the pairs.

   The link between an n-pairing and the preferential attachment model with $\delta = 0$ and $m = 1$ is obtained as follows. We start from the left, and merge all left-endpoints up to and including the first right endpoint into the single vertex v_1. Then, we merge all further left-endpoints up to the next right endpoint into

vertex v_2, etc. For the edges, we replace each pair by a directed edge from the vertex corresponding to its right endpoint to the vertex corresponding to its left endpoint. Then, as noted in [44]_{BolRio04b}, the resulting graph has the same distribution as $G_1(t)$. The proof in [46]_{BolRioSpeTus01} then uses explicit computations to prove that for $k \leq t^{1/15}$,

$$\mathbb{E}[N_k(t)] = tp_k(1 + o(1)). \qquad \text{(8.8.2)}$$

The advantage of the current proof is that the restriction on k in $k \leq t^{1/15}$ is absent, that the error term in (8.8.2) is bounded uniformly by a constant, and that the proof applies to $\delta = 0$ and $\delta \neq 0$.

The approach of Hagberg and Wiuf in [103]_{HagWiu06} is closest to ours. In it, the authors assume that the model is a preferential attachment model, where the expected number of vertices of degree k in the graph at time $t + 1$, conditionally on the graph at time t solves

$$\mathbb{E}[N_k(t+1)|N(t)] = (1 - \frac{a_k}{t})N_k(t) - \frac{a_{k-1}}{t}N_{k-1}(t) + c_k, \qquad \text{(8.8.3)}$$

where $N_k(t)$ is the number of vertices of degree k at time t, $N(t) = \{N_k(t)\}_{k=0}^{\infty}$ and it is assumed that $a_{-1} = 0$, and where $c_k \geq 0$ and $a_k \geq a_{k-1}$. Also, it is assumed that $|N_k(t) - N_k(t-1)|$ is uniformly bounded. This is almost true for the model considered in this chapter. Finally, $\{N(t)\}_{t=0}^{\infty}$ is assumed to be a Markov process, starting at some time t_0 in a configuration $N(t_0)$. Then, with

$$\alpha_k = \sum_{j=0}^{k} \frac{c_j}{1 + a_j} \prod_{i=j+1}^{\infty} \frac{a_{i-1}}{1 + a_i}, \qquad \text{(8.8.4)}$$

it is shown that $N_t(k)/t$ converges to α_k.

Exercise 8.27. *Show that*

$$\max_{j=1}^{k} |\mathbb{E}[N_t(j)] - \alpha_j t| \qquad \text{(8.8.5)}$$

is non-increasing.

Notes on Section 8.6. The beautiful martingale description in Proposition 8.9 is due to Mori [159]_{Mori02} (see also [160]_{Mori05}). We largely follow the presentation in [79, Section 4.3]_{Durr06}, adapting it to the setting of preferential attachment models in Section 8.1. The fact that Proposition 8.9 also holds for *non-integer* k_i is, as far as we know, new. This is relevant, since it identifies *all* moments of the limiting random variables ξ_j, which might prove useful in order to identify their distribution. This, however, has not been done yet. In particular, it would be of interest to prove that $\mathbb{P}(\xi_j = 0) = 0$. The latter is, by Exercise 8.21, equivalent to proving the statement for $j = 1$. In turn, $\mathbb{P}(\xi_j = 0) = 0$ would imply that $\mathbb{P}(\mu = 0) = 0$, which proves that the maximal degree at time t is indeed of order $t^{1/(2+\delta)}$, and not smaller.

Intermezzo: Back to real networks II...

In the previous sections, we have discussed various models having flexible degree sequences. The generalized random graph and the configuration model give us static flexible models for random graphs with various degree sequences. Preferential attachment models give us a convincing explanation of the abundance of power-law degree sequences in various applications. In Chapters 6–8, we have focussed on the properties of the *degrees* of such graphs. However, we have noted in Chapter 1 that many real networks not only have degree sequences that are rather different from the ones of the Erdős-Rényi random graph, also many examples are *small worlds* and have a *giant connected component*.

In the Chapters 9–11, we shall return to the models discussed in Chapters 6–8, and focus on their critical behavior as well as on the distances in these random graph models. Interestingly, a large chunk of the non-rigorous physics literature suggests that the behavior in various *different* random graph models can be described by only a *few* essential parameters. The key parameter of each of these models in the power-law degree exponent, and the physics literature predicts that the behavior in random graph models with similar degree sequences is similar. This is an example of the notion of *universality*, a notion which is central in statistical physics. Despite its importance, there are only few example of universality that can be rigorously proved. In Chapters 9–11, we shall investigate the level of universality present in random graph models.

Chapter 9

Generalized random graphs revisited

In this chapter, we discuss the phase transition and distances in inhomogeneous random graphs. This chapter is organized as follows. In Section 9.1, we discuss the general setting of inhomogeneous random graphs from [43]$^{\text{BolJanRio06}}$ with its detailed and deep results. For random graphs with power-law degrees where the exponent τ satisfies $\tau \in (1,3)$, it turns out that [43]$^{\text{BolJanRio06}}$ does not prove detailed distance results. Such results have been proved for specific examples, most notably for the Chung-Lu model, the generalized random graph and the Norros-Reittu model. Therefore, in Section 9.2, we shall investigate distances in these models.

9.1 Inhomogeneous random graphs

9.2 Distances in inhomogeneous random graphs for $\tau \in (2,3)$

In this section, we study the distances of special inhomogeneous random graph models in the case where the degrees obey a power-law with degree exponent τ satisfying that $\tau \in (2,3)$. In Section 9.1, we have seen that often distances in inhomogeneous random graphs are of order $\log n$, where n denotes the size of the graph. In certain cases, particularly when the degrees of the graph are quite large, distances can be quite a bit smaller. Distances have not been considered in the generality of inhomogeneous random graphs discussed in Section 9.1. In this section, we shall therefore specialize to specific cases which have been investigated in the literature, namely, the *Chung-Lu model* [60, 61, 62, 64, 149]$^{\text{ChuLu02a,ChuLu02b,ChuLu03,ChuLu06, Lu02}}$, the *Generalized random graph* [93]$^{\text{EskHofHoo06}}$ and the *Norros-Reittu model* [171]$^{\text{NorRei06}}$. We shall discuss the setting where the weights $\{W_i\}_{i=1}^n$ are independent and identically distributed random variables having a heavy-tailed distribution function F_W with

$$1 - F_W(x) = x^{\tau-1} L_W(x), \qquad \text{(9.2.1)}$$

<div align="right">e:FW-asstau(2,3)NR</div>

where the exponent satisfies $\tau \in (2,3)$, and where $x \mapsto L_W(x)$ is slowly varying. We shall also discuss the case where the weights $\{w_i\}_{i=1}^n$ are non-random and chosen according to (6.1.5) where F_W satisfies (9.2.1).

We let H_n be the graph distance between two uniformly chosen *connected* vertices. The main result to be proved in this section is as follows:

Theorem 9.1 (Average distances in $\mathrm{NR}(n, \{W_i\}_{i=1}^n)$ for $\tau \in (2,3)$)**.** *In the Norros-Reittu model* $\mathrm{NR}(n, \{W_i\}_{i=1}^n)$, *where the weights* $\{W_i\}_{i=1}^n$ *are independent and identically distributed with distribution function* F_W *satisfying the conditions in Theorem 3.20, i.e., (9.2.1),*

$$\frac{H_n}{\log\log n} \xrightarrow{\mathbb{P}} \frac{2}{|\log(\tau - 2)|}. \qquad \text{(9.2.2)}$$

<div align="right">e:Hn-conv-NR</div>

To prove Theorem 9.1, for $\varepsilon \in (0,1)$, we take

$$k_\varepsilon^+ = (1+\varepsilon)\frac{2\log\log n}{|\log(\tau-2)|}, \qquad k_\varepsilon^- = (1-\varepsilon)\frac{2\log\log n}{|\log(\tau-2)|}. \tag{9.2.3}$$

Then it suffices to prove that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(k_\varepsilon^- \leq H_n \leq k_\varepsilon^+ \mid H_n < \infty) = 1. \tag{9.2.4}$$

Since

$$\mathbb{P}(k_\varepsilon^- \leq H_n \leq k_\varepsilon^+ \mid H_n < \infty) = \frac{\mathbb{P}(k_\varepsilon^- \leq H_n \leq k_\varepsilon^+)}{\mathbb{P}(H_n < \infty)}, \tag{9.2.5}$$

this follows from the bounds

$$\liminf_{n\to\infty} \mathbb{P}(H_n < \infty) \leq \zeta^2, \tag{9.2.6}$$

$$\limsup_{n\to\infty} \mathbb{P}(H_n \leq k_\varepsilon^-) = 0, \tag{9.2.7}$$

$$\limsup_{n\to\infty} \mathbb{P}(H_n \leq k_\varepsilon^+) \geq \zeta^2, \tag{9.2.8}$$

where ζ is arbitrary. Since $\mathrm{NR}(n, \{W_i\}_{i=1}^n)$ is a special case of the inhomogeneous random graph model studie in Section 9.1, we see that (9.2.6) follows from the results obtained there, with $\zeta > 0$ the survival probability of the underlying branching process approximation of the neighborhood shells of $\mathrm{NR}(n, \{W_i\}_{i=1}^n)$. Thus, we shall not prove (9.2.6), and rather focus on (9.2.7) and (9.2.8).

This section is organized as follows. The main tool to study distances in $\mathrm{NR}(n, \{w_i\}_{i=1}^n)$ shall be a comparison to branching processes, which is particularly pretty for $\mathrm{NR}(n, \{w_i\}_{i=1}^n)$, and shall be described in Section 9.2.1. In Section 9.2.2, we make use of these results to prove (9.2.7). Here we shall crucially rely on Theorem 3.20. In Section 9.2.3, we give an upper bound on the diameter of the *core* of $\mathrm{NR}(n, \{w_i\}_{i=1}^n)$ consisting of those vertices that have a weight that is larger than a positive power of $\log n$. These are the vertices with the high degrees, and it turns out that the *diameter* of the core is bounded by $\frac{2\log\log n}{|\log(\tau-2)|}(1+o(1))$ with high probability. This result also sets the stage for a proof of (9.2.8), which follows if we show that the probability that two uniformly chosen vertices are connected to the core is bounded below by $\zeta^2(1+o(1))$.

Complete proof of Theorem 9.1.

9.2.1 Comparison to branching processes

In this section, we describe a beautiful comparison of the neighborhoods of a uniformly chosen vertex in inhomogeneous random graphs, such as the generalized random graph, the Chung-Lu model and the Norros-Reittu model, and a marked branching process. This comparison is particularly pretty when considering the Norros-Reittu model, where there is an explicit stochastic domination result of these neighborhoods are bounded by a so-called *two-stage* branching process with a mixed Poissonian offspring distribution. [1]

Two-stage branching processes with mixed Poissonian offspring. We start by introducing two-stage branching processes with a mixed Poissonian offspring. We define the branching process $\{Z_l\}_{l\geq 0}$ as starting from $Z_0 = 1$, where in the first generation the offspring distribution is equal to

$$f_k = \mathbb{E}\big[e^{-W}\frac{W^k}{k!}\big], \, k \geq 0, \tag{9.2.9}$$

---

[1]In [93], the two-stage branching process is called a delayed branching process.

whereas in the second and further generations the offspring is chosen in accordance to

$$g_k = \frac{(k+1)f_{k+1}}{\mu} = \frac{1}{\mu}\mathbb{E}\big[e^{-W}\frac{W^{k+1}}{k!}\big], \ k \geq 0, \qquad \text{(9.2.10)}$$

where W is the mixing distribution, and μ is the expected number of offspring in the first generation, which is

$$\mu = \sum_{k=1}^{\infty} k f_k = \mathbb{E}\big[\sum_{k=1}^{\infty} k e^{-W}\frac{W^k}{k!}\big] = \mathbb{E}[W]. \qquad \text{(9.2.11)}$$

Similarly, the expected number of offspring in the second and further generations is given by

$$\nu = \sum_{k=1}^{\infty} k g_k = \frac{1}{\mu}\mathbb{E}\big[\sum_{k=1}^{\infty} k(k+1)e^{-W}\frac{W^{k+1}}{(k+1)!}\big] = \frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}. \qquad \text{(9.2.12)}$$

In particular, a branching process with mixed Poisson offspring distribution is supercritical when $\mathbb{E}[W^2] > \mathbb{E}[W]$. In this section, we shall assume that (9.2.1) holds with an exponent τ satisfying $\tau \in (2, 3)$. In particular, in this case, $\mathbb{E}[W^2] = \infty$, so this branching process is supercritical. Later, we shall make stronger assumptions on $1 - F_W$ than the one in (9.2.1), since we shall rely on results on branching processes with infinite mean as described in Theorem 3.20.

Stochastic domination of neighborhoods by a branching process. In this section, we shall dominate the neighborhoods of a vertex in the Norros-Reittu model by a two-stage branching processes with mixed Poissonian offspring.

In order to describe this relation, we consider the neighborhood shells of a uniformly chosen vertex $V \in \{1, 2, \ldots, n\}$, i.e., all vertices on a fixed graph distance of vertex V. More precisely,

$$\partial\mathcal{N}_0 = \{V\} \qquad \text{and} \qquad \partial\mathcal{N}_l = \{1 \leq j \leq n : d(V, j) = l\}, \qquad \text{(9.2.13)}$$

where $d(i, j)$ denotes the graph distance between vertices i and j in $\mathrm{NR}(n, \{w_i\}_{i=1}^n)$, i.e., the minimum number of edges in a path between the vertices i and j. Define the set of vertices reachable in at most j steps from vertex V by

$$\mathcal{N}_l = \{1 \leq j \leq n : d(V, j) \leq l\} = \bigcup_{k=0}^{l} \partial\mathcal{N}_k. \qquad \text{(9.2.14)}$$

The main idea is that we can explicitly view the neighborhood sizes $\{|\mathcal{N}_l|\}_{l=0}^{\infty}$ as a marked Poisson branching process where repeated vertices are thinned. For this, let us first describe the marked Poisson branching process, which will be denoted by

The NR-process is a marked two-stage branching process denoted by $\{Z_l, \mathbf{M}\}_{l \geq 0}$, where Z_l denotes the number of individuals of generation l, and where the vector

$$\mathbf{M} = (M_{l,1}, M_{l,2}, \ldots, M_{l,Z_l}) \in \{1, 2, \ldots, n\}^{Z_l}, \qquad \text{(9.2.15)}$$

denotes the marks of the individuals in generation l. These marks shall label to which vertex in $\mathrm{NR}(n, \{w_i\}_{i=1}^n)$ an individual in the branching process corresponds. We now give a more precise definition of the NR-process and describe its connection with $\mathrm{NR}(n, \{w_i\}_{i=1}^n)$.

We define $Z_0 = 1$ and take $M_{0,1}$ uniformly from the set $\{1, 2, \ldots, n\}$, corresponding to the choice of A_1, which is uniformly over all the vertices. The offspring of an individual with mark $m \in \{1, 2, \ldots, n\}$ is as follows: the total number of children has a Poisson distribution with parameter w_m, of which, for each $i \in \{1, 2, \ldots, n\}$, a Poisson distributed number with parameter

$$\frac{w_i w_m}{l_n}, \qquad \text{(9.2.16)}$$

bears mark i, independently of the other individuals. Since

$$\sum_{i=1}^{n} \frac{w_i w_m}{l_n} = \frac{w_m}{l_n} \sum_{i=1}^{n} w_i = w_m, \tag{9.2.17}$$

and sums of independent Poisson random variables are again Poissonian, we may take the number of children with different marks mutually independent. As a result of this definition, the marks of the children of an individual in $\{Z_l, \mathbf{M}\}_{l \geq 0}$ can be seen as independent realizations of a random variable M, with distribution

$$\mathbb{P}(M = m) = \frac{w_m}{l_n}, \quad 1 \leq m \leq n, \tag{9.2.18}$$

and, consequently,

$$\mathbb{E}[w_M] = \sum_{m=1}^{n} w_m \mathbb{P}(M = m) = \frac{1}{l_n} \sum_{m=1}^{n} w_m^2. \tag{9.2.19}$$

For the definition of the $\underline{\text{NR}}$-process, we start with a copy of the NR-process $\{Z_l, \mathbf{M}\}_{l \geq 0}$, and reduce this process generation by generation, i.e., in the order

$$M_{0,1}, M_{1,1}, M_{1,2}, \ldots, M_{1,Z_1}, M_{2,1}, \ldots \tag{9.2.20}$$

by discarding each individual and all its descendants whose mark has been chosen before, i.e., if $M_{i_1, j_1} = M_{i_2, j_2}$ for some i_1, j_1 which has appeared before i_2, j_2, then the individual corresponding to i_2, j_2, as well as its entire offspring, is erased. The process obtained in this way is called the $\underline{\text{NR}}$-process and is denoted by the sequence $\{\underline{Z}_l, \underline{\mathbf{M}}_l\}_{l \geq 0}$. One of the main results of [171, Proposition 3.1] is the fact that the distribution of $\{\partial \mathcal{N}_l\}_{l \geq 0}$ is equal to that of :

Proposition 9.2 (Neighborhoods are a thinned marked branching process). *Let $\{\underline{Z}_l, \underline{\mathbf{M}}_l\}_{l \geq 0}$ be the $\underline{\text{NR}}$-process and let $\underline{\mathbf{M}}_l$ be the set of marks in the lth generation, then the sequence of sets $\{\underline{\mathbf{M}}_l\}_{l \geq 0}$ has the same distribution as the sequence $\{\partial \mathcal{N}_l\}_{l \geq 0}$ given by (9.2.13).*

As a consequence of Proposition 9.2, we can couple the $\underline{\text{NR}}$-process to the neighborhood shells of a uniformly chosen vertex $V \in \{1, 2, \ldots, n\}$, i.e., all vertices on a fixed graph distance of V, see (9.2.13) and note that $V \sim M_{0,1}$. Thus, using Proposition 9.2, we can couple the expansion of the neighborhood shells and the $\underline{\text{NR}}$-process in such a way that

$$\underline{\mathbf{M}}_l = \partial \mathcal{N}_l \qquad \text{and} \qquad \underline{Z}_l = |\partial \mathcal{N}_l|, \quad l \geq 0. \tag{9.2.21}$$

Furthermore, we see that an individual with mark m in the $\underline{\text{NR}}$-process is identified with vertex m in the graph $\mathrm{NR}(n, \{w_i\}_{i=1}^{n})$ whose capacity is w_m.

For given weights $\{w_i\}_{i=1}^{n}$, we now describe the distribution of the marked Poisson process. For this, we note that since the marks are mutually *independent*, the marked Poisson process is a branching process. The offspring distribution $f^{(n)}$ of Z_1, i.e., the first generation of $\{Z_l\}_{l \geq 0}$, is given by

$$f_k^{(n)} = \mathbb{P}\big(\mathrm{Poi}(w_V) = k\big) = \sum_{m=1}^{n} \mathbb{P}(\mathrm{Poi}(w_m) = k | V = m)\mathbb{P}(M = m) = \frac{1}{k} \sum_{m=1}^{n} e^{-w_m} \frac{w_m^k}{k!}, \tag{9.2.22}$$

for $n \geq 0$. Recall that individuals in the second and further generations have a random mark distributed as an independent copy of M given by (9.2.18). Hence, if we denote the offspring distribution of the second and further generations by $g^{(n)}$, then we obtain

$$g_k^{(n)} = \mathbb{P}\big(\mathrm{Poi}(w_M) = k\big) = \sum_{m=1}^{n} \mathbb{P}(\mathrm{Poi}(w_m) = k | M = m)\mathbb{P}(M = m) = \frac{1}{l_n} \sum_{m=1}^{n} e^{-w_m} \frac{w_m^{k+1}}{k!}, \tag{9.2.23}$$

for $k \geq 0$. We recognize the offspring distribution in (9.2.23) as a mixed Poisson distribution, where the mixing random variable has distribution w_M. Together, (9.2.22) and (9.2.23) identify the distribution $\{Z_l\}_{l \geq 0}$ as a two-stage branching process, where the first generation has a mixed Poisson distribution with mixing random variable w_V, where $V \in \{1, \ldots, n\}$ is a uniform random variable, and where the offspring distribution of the second and further generations has a a mixed Poisson distribution with mixing random variable w_M, where $M \in \{1, \ldots, n\}$ has distribution given by (9.2.18). This describes a stochastic upper bound on the neighborhood shells of a uniform vertex $V \in \{1, \ldots, n\}$ in the Norros-Reittu model $\mathrm{NR}(n, \{w_i\}_{i=1}^n)$.

Convergence of the branching processes. Both $f^{(n)}$ and $g^{(n)}$ are probability laws of certain mixed Poisson random variables, where the dependence on n appears only in the distribution of the mixing random variable. This allows us to bound the total variation distances between $f^{(n)}$ and f, and the one between $g^{(n)}$ and g in terms of the total variation distance between the mixing distributions:

Lemma 9.3 (Total variation distance for mixed Poisson variables)**.** *Let X and Y be two mixed Poisson random variables with mixing random variables V and W respectively. Then,*

$$d_{\mathrm{TV}}(F_X, F_Y) \leq d_{\mathrm{TV}}(F_V, F_W). \tag{9.2.24}$$

Proof. We make use of coupling theory. We recall that for the maximal coupling $(\hat{V}, \hat{W})$ of V and W, we have

$$\mathbb{P}(\hat{V} \neq \hat{W}) = d_{\mathrm{TV}}(F_V, F_W). \tag{9.2.25}$$

Then, we let $\hat{Z} \sim \mathrm{Poi}(\min\{\hat{V}, \hat{W}\})$, and $\hat{Z}_X \sim \mathrm{Poi}(\hat{V} - \min\{\hat{V}, \hat{W}\})$, $\hat{Z}_Y \sim \mathrm{Poi}(\hat{W} - \min\{\hat{V}, \hat{W}\})$, where the different Poisson random variables are independent. Then, we set $\hat{X} = \hat{Z} + \hat{Z}_X, \hat{Y} = \hat{Z} + \hat{Z}_Y$. By the independence of the Poisson variables, $\hat{X}$ and $\hat{Y}$ have a mixed Poisson distribution with mixing random variables $\hat{V}$ and $\hat{W}$. Since $(\hat{V}, \hat{W})$ are a coupling of V and W, it follows that $\hat{X}$ and $\hat{Y}$ have the same marginal distributions as X and Y, respectively. Furthermore, we have that

$$\mathbb{P}(\hat{X} \neq \hat{Y}) \leq \mathbb{P}(\hat{V} \neq \hat{W}) = d_{\mathrm{TV}}(F_V, F_W). \tag{9.2.26}$$

Since

$$\mathbb{P}(\hat{X} \neq \hat{Y}) \geq d_{\mathrm{TV}}(F_X, F_Y) \tag{9.2.27}$$

for every coupling $(\hat{X}, \hat{Y})$ of X and Y, this proves the claim. $\square$

9.2.2 A $\log\log$ lower bound on average distances in $\mathrm{NR}\big(n, \{w_i\}_{i=1}^n\big)$ for $\tau \in (2,3)$

In this section, we prove a $\log\log$-lower bound on the average distances of $\mathrm{NR}(n, \{w_i\}_{i=1}^n)$ for $\tau \in (2,3)$. The main result we shall prove here is the following theorem:

Theorem 9.4 (Lower bound on average distances in $\mathrm{NR}(n, \{w_i\}_{i=1}^n)$ for $\tau \in (2,3)$)**.** *With high probability, $H_n \geq (1 - \varepsilon)\frac{2\log\log n}{|\log(\tau-2)|}$.*

In order to study the average distances in $\mathrm{NR}(n, \{w_i\}_{i=1}^n)$, we shall grow *two* independent $\underline{\mathrm{NR}}$-processes $\{\underline{Z}_l^{(i)}, \mathbf{M}_l^{(i)}\}_{l \geq 0}$, for $i = 1, 2$. The root of $\{\underline{Z}_l^{(i)}, \mathbf{M}_l^{(i)}\}_{l \geq 0}$ starts from a uniformly chosen vertex $V_i \in \{1, 2, \ldots, n\}$. These two vertices are with high probability different, because

$$\mathbb{P}(V_1 = V_2) = \frac{1}{n}. \tag{9.2.28}$$

By Proposition 9.2 the two $\underline{\mathrm{NR}}$-processes can be coupled to the neighborhood expansion shells $\{\mathcal{N}_l^{(1)}\}_{l \geq 0}$ and $\{\mathcal{N}_l^{(2)}\}_{l \geq 0}$ of the vertices V_1 and V_2.

Neighborhood exploration shells and capacities.   Introduce the *capacity* of the lth generation of the NR-process $\{Z_l, \mathbf{M}_l\}_{l \geq 0}$ and the $\underline{\text{NR}}$-process $\{\underline{Z}_l, \underline{\mathbf{M}}_l\}_{l \geq 0}$ as, respectively,

$$C_{l+1} = \sum_{i=1}^{Z_l} w_{M_{l,i}}, \qquad \text{and} \qquad \underline{C}_{l+1} = \sum_{i=1}^{\underline{Z}_l} w_{\underline{M}_{l,i}}, \ l \geq 0. \qquad \text{(9.2.29)}$$

<div align="right">e:def-CandCt</div>

By the coupling in Proposition 9.2, we can rewrite the capacity $\underline{C}_{l+1}$ as

$$\underline{C}_{l+1} = \sum_{i \in \partial \mathcal{N}_l} w_i. \qquad \text{(9.2.30)}$$

<div align="right">e:def-CinN</div>

In the following lemma we compute the distribution of the number of edges between two different shells, i.e., between $\mathcal{N}_k^{(1)}$ and $\mathcal{N}_l^{(2)}$.

Lemma 9.5. *Consider the neighborhood expansion shells $\{\mathcal{N}_l^{(1)}\}_{l \geq 0}$ and $\{\mathcal{N}_l^{(2)}\}_{l \geq 0}$. Then, conditionally on $\mathcal{N}_k^{(1)}$ and $\mathcal{N}_l^{(2)}$ and given that $\mathcal{N}_k^{(1)} \cap \mathcal{N}_l^{(2)} = \varnothing$, the number of edges between the vertices in $\mathcal{N}_k^{(1)}$ and $\mathcal{N}_l^{(2)}$, for fixed positive integers k and l, has a mixed Poisson random variable with mixing distribution equal to*

$$\frac{\underline{C}_{k+1}^{(1)} \underline{C}_{l+1}^{(2)}}{l_n}. \qquad \text{(9.2.31)}$$

<div align="right">e:mcimcj</div>

Proof. Conditioned on $\mathcal{N}_k^{(1)}$, $\mathcal{N}_l^{(2)}$ and the fact that $\mathcal{N}_k^{(1)} \cap \mathcal{N}_l^{(2)} = \varnothing$, the number of edges between the vertices in $\mathcal{N}_k^{(1)}$ and $\mathcal{N}_l^{(2)}$ is given by

$$\sum_{i \in \partial \mathcal{N}_k^{(1)}} \sum_{j \in \partial \mathcal{N}_l^{(2)}} E_{ij}, \qquad \text{(9.2.32)}$$

<div align="right">e:poissonSum</div>

where E_{ij} are independent Poisson random variables with mean $w_i w_j / l_n$. Sums of independent Poisson random variables are again Poissonian, thus (9.2.32) is a mixed Poisson random variable with mixing random variable equal to

$$\sum_{i \in \partial \mathcal{N}_k^{(1)}} \sum_{j \in \partial \mathcal{N}_l^{(2)}} \frac{w_i w_j}{l_n} = \frac{1}{l_n} \Big(\sum_{i \in \partial \mathcal{N}_k^{(1)}} w_i \Big) \Big(\sum_{j \in \partial \mathcal{N}_l^{(2)}} w_j \Big) = \frac{\underline{C}_{k+1}^{(1)} \underline{C}_{l+1}^{(2)}}{l_n}, \qquad \text{(9.2.33)}$$

where we have used (9.2.30) in the last step. $\square$

Expressing the distribution of the average distances in terms of capacities.   We have $H_n > 1$ if and only if there are no edges between the nodes V_1 and V_2, which is equivalent to the fact that $X_{V_1 V_2} = 0$. Given the capacities $\underline{C}_1^{(1)}$ and $\underline{C}_1^{(2)}$, the number of edges between vertices V_1 and V_2 has, according to Lemma 9.5, a mixed Poisson distribution with mixing random variable

$$\frac{\underline{C}_1^{(1)} \underline{C}_1^{(2)}}{l_n}. \qquad \text{(9.2.34)}$$

<div align="right">e:poissoncc</div>

We note that $H_n > l$ precisely when there are no edges between the neighborhood shells $\mathcal{N}_{\lceil k/2 \rceil}^{(1)}$ and $\mathcal{N}_{\lfloor k/2 \rfloor}^{(2)}$ for every $2 \leq k \leq l+1$.

Lemma 9.6 (The tail probability of H_n and capacities). *For every $l \geq 1$,*

$$\mathbb{P}(H_n > l) = \mathbb{E}\Big[\exp \Big\{ -\frac{\sum_{k=2}^{l+1} \underline{C}_{\lceil k/2 \rceil}^{(1)} \underline{C}_{\lfloor k/2 \rfloor}^{(2)}}{l_n} \Big\} \Big]. \qquad \text{(9.2.35)}$$

<div align="right">e:compeven</div>

By the coupling in Proposition 9.2, we immediately obtain that

$$\mathbb{P}(H_n > l) \geq \mathbb{E}\Big[\exp\Big\{ - \frac{\sum_{k=2}^{l+1} C^{(1)}_{\lceil k/2 \rceil} C^{(2)}_{\lfloor k/2 \rfloor}}{l_n} \Big\}\Big]. \tag{9.2.36}$$

<div align="right">e:compevenlbd</div>

9.2.3 A $\log\log$ upper bound on the diameter of the core for $\tau \in (2,3)$

In Theorem 9.7, we give a bound on the diameter of the *core* of the Norros-Reittu model for $\tau \in (2,3)$. The core consists of all vertices with weight at least a certain power of $\log n$. Throughout this section, we shall assume that the distribution function of the weights F_W satisfies that there exists a $c_W > 0$ such that

$$1 - F_W(x) \geq c_W x^{\tau-1}. \tag{9.2.37}$$

<div align="right">e:infvarassNR</div>

While (9.2.37) appears to be rather different as (9.2.1), they are quite related. Indeed, assume that (9.2.1) is true for some $\tau \in (2,3)$. Then, since $x \mapsto L_W(x)$ is slowly varying, by Potter's Theorem, there exists a constant $c(\varepsilon)$ such that for every $x \geq 1$,

$$1 - F_W(x) \geq c(\varepsilon)x^{\tau-1+\varepsilon}. \tag{9.2.38}$$

As a result, (9.2.37) holds for any $\tau' > \tau$. Naturally, when $L_W(x) \geq c_W > 0$ for all $x \geq 1$, then (9.2.37) immediately follows from (9.2.1).

We take $\sigma > \frac{1}{3-\tau}$ and define the *core* Core_n of the Norros-Reittu model to be

$$\mathrm{Core}_n = \{i : W_i \geq (\log n)^\sigma\}, \tag{9.2.39}$$

<div align="right">e:core-NR</div>

i.e., the set of vertices with weight at least $(\log n)^\sigma$. In this section, we present a result on the diameter of the core Core_n of the Norros-Reittu model. We shall spend some time on discussing the argument, as variations of it shall be used in Chapter 10 to study distances in the configuration model when $\tau \in (2,3)$ and in Chapter 11 to study distances in the preferential attachment model when $\tau = 3 + \delta/m \in (2,3)$ or $\delta < 0$. Thus, this argument is quite flexible. The main result of this section is the following theorem:

Theorem 9.7 (Diameter of the NR-core). *For every* $\sigma > \frac{1}{3-\tau}$, *the diameter of* Core_n *is bounded above by*

$$\frac{2\log\log n}{|\log(\tau-2)|}(1 + o(1)). \tag{9.2.40}$$

Before proving Theorem 9.7, we introduce some preliminaries. We note that (9.2.37) implies that **whp** the largest weight $W_{(n)}$ satisfies

$$W_{(n)} \geq u_1, \quad \text{where} \quad u_1 = n^{\frac{1}{\tau-1}}(\log n)^{-1}, \tag{9.2.41}$$

because for $n \to \infty$,

$$\mathbb{P}(W_{(n)} > u_1) = 1 - \mathbb{P}(W_{(n)} \leq u_1) = 1 - [F_W(u_1)]^n \geq 1 - (1 - cu_1^{1-\tau})^n$$

$$= 1 - \Big(1 - c\frac{(\log n)^{\tau-1}}{n}\Big)^n \sim 1 - \exp(-c(\log n)^{\tau-1}) \to 1. \tag{9.2.42}$$

<div align="right">e:verdmaxNR</div>

Define

$$\Gamma_1 = \{i : W_i \geq u_1\}, \tag{9.2.43}$$

so that by (9.2.42), with high probability, $\Gamma_1 \neq \varnothing$. For some constant $C > 0$, which will be specified later, and $k \geq 2$, we recursively define

$$u_k = C \log n (u_{k-1})^{\tau-2}, \qquad\qquad (9.2.44)$$

and

$$\Gamma_k = \{i : W_i \geq u_k\}. \qquad\qquad (9.2.45)$$

We start by identifying u_k:

Lemma 9.8 (Identification of u_k). *For each $k \in \mathbb{N}$,*

$$u_k = C^{a_k} (\log n)^{b_k} n^{c_k}, \qquad\qquad (9.2.46)$$

with

$$c_k = \frac{(\tau-2)^{k-2}}{\tau-1}, \qquad b_k = \frac{1}{3-\tau} - \frac{4-\tau}{3-\tau}(\tau-2)^{k-1}, \qquad a_k = \frac{1-(\tau-2)^{k-1}}{3-\tau}. \qquad (9.2.47)$$

Proof. We will identify a_k, b_k and c_k recursively. We note that $c_1 = \frac{1}{\tau-1}, b_1 = -1, a_1 = 0$. By (9.2.44), we can, for $k \geq 2$, relate a_k, b_k, c_k to $a_{k-1}, b_{k-1}, c_{k-1}$ as follows:

$$c_k = (\tau-2)c_{k-1}, \qquad b_k = 1 + (\tau-2)b_{k-1}, \qquad a_k = 1 + (\tau-2)a_{k-1}. \qquad (9.2.48)$$

As a result, we obtain

$$c_k = (\tau-2)^{k-1} c_1 = \frac{(\tau-2)^{k-1}}{\tau-1}, \qquad\qquad (9.2.49)$$

$$b_k = b_1(\tau-2)^{k-1} + \sum_{i=0}^{k-2}(\tau-2)^i = \frac{1-(\tau-2)^{k-1}}{3-\tau} - (\tau-2)^{k-1}, \qquad\qquad (9.2.50)$$

$$a_k = \frac{1-(\tau-2)^{k-1}}{3-\tau}. \qquad\qquad (9.2.51)$$

$\square$

The key step in the proof of Theorem 9.7 is the following lemma:

Lemma 9.9 (Connectivity between Γ_{k-1} and Γ_k). *Fix $k \geq 2$, and $C > 4\mu/c$ (see (??), and (9.2.37) respectively). Then, the probability that there exists an $i \in \Gamma_k$ that is not directly connected to Γ_{k-1} is $o(n^{-\delta})$, for some $\delta > 0$ independent of k.*

Proof. We note that, by definition,

$$\sum_{i \in \Gamma_{k-1}} W_i \geq u_{k-1}|\Gamma_{k-1}|. \qquad\qquad (9.2.52)$$

Also,

$$|\Gamma_{k-1}| \sim \mathrm{Bin}(n, 1 - F(u_{k-1})), \qquad\qquad (9.2.53)$$

and we have that, by (9.2.37),

$$n[1 - F(u_{k-1})] \geq cn(u_{k-1})^{1-\tau}, \qquad\qquad (9.2.54)$$

which, by Lemma 9.8, grows as a positive power of n, since $c_k \leq c_2 = \frac{\tau-2}{\tau-1} < \frac{1}{\tau-1}$. Using Theorem 2.18, we obtain that the probability that $|\Gamma_{k-1}|$ is bounded below by $n[1 - F(u_{k-1})]/2$ is exponentially small in n. As a result, we obtain that, for every k and with high probability,

$$\sum_{i \in \Gamma_k} W_i \geq \frac{c}{2}n(u_k)^{2-\tau}. \qquad\qquad (9.2.55)$$

We note that for any two sets of vertices A, B, we have that

$$\mathbb{P}_n(A \text{ not directly connected to } B) \leq e^{-\frac{W_A W_B}{L_n}},$$

<div align="right">(e:noconnAB-NR)</div>
<div align="right">(9.2.56)</div>

where, for any $A \subseteq \{1, \ldots, n\}$, we write

$$W_A = \sum_{i \in A} W_i.$$

<div align="right">(9.2.57)</div>

On the event where $|\Gamma_{k-1}| \geq n[1 - F(u_{k-1})]/2$ and where $L_n \leq 2\mu n$, we then obtain by (9.2.56), and Boole's inequality that the $\mathbb{P}_n$-probability that there exists an $i \in \Gamma_k$ such that i is not directly connected to Γ_{k-1} is bounded by

$$ne^{-\frac{u_k n u_{k-1}[1-F(u_{k-1})]}{2L_n}} \leq ne^{-\frac{cu_k(u_{k-1})^{2-\tau}}{4\mu}} = n^{1-\frac{cC}{4\mu}},$$

<div align="right">(9.2.58)</div>

where we have used (9.2.44). Taking $C > 4\mu/c$ proves the claim. $\qquad\square$

Now we are ready to complete the proof of Theorem 9.7:

Proof of Theorem 9.7. Fix

$$k^* = \frac{2 \log \log n}{|\log (\tau - 2)|}.$$

<div align="right">(e:k*-defNR)</div>
<div align="right">(9.2.59)</div>

As a result of Lemma 9.9, we have that, with high probability, the diameter of Γ_{k^*} is at most $2k^*$, because the distance between any vertex in Γ_{k^*} and the vertex with maximal weight $W_{(n)}$ is at most k^*. Therefore, we are done when we can show that

$$\text{Core}_n \subseteq \Gamma_{k^*}.$$

<div align="right">(e:coresubNR)</div>
<div align="right">(9.2.60)</div>

To prove (9.2.60), we note that

$$\Gamma_{k^*} = \{i : W_i \geq u_{k^*}\},$$

<div align="right">(9.2.61)</div>

so that it suffices to prove that $u_{k^*} \geq (\log n)^\sigma$, for any $\sigma > \frac{1}{3-\tau}$. According to Lemma 9.8,

$$u_{k^*} = C^{a_{k^*}} (\log n)^{b_{k^*}} n^{c_{k^*}}.$$

<div align="right">(e:uk*-asyNR)</div>
<div align="right">(9.2.62)</div>

Because for $x \to \infty$, and $2 < \tau < 3$,

$$x(\tau - 2)^{\frac{2 \log x}{|\log (\tau-2)|}} = x \cdot x^{-2} = o(\log x),$$

<div align="right">(e:toegerard1)</div>
<div align="right">(9.2.63)</div>

we find, with $x = \log n$, that

$$\log n \cdot (\tau - 2)^{\frac{2 \log \log n}{|\log (\tau-2)|}} = o(\log \log n).$$

<div align="right">(e:toegerard2)</div>
<div align="right">(9.2.64)</div>

In turn, (9.2.64) implies that

$$n^{c_{k^*}} = (\log n)^{o(1)}, \quad (\log n)^{b_{k^*}} = (\log n)^{\frac{1}{3-\tau}+o(1)}, \quad \text{and} \quad C^{a_{k^*}} = (\log n)^{o(1)},$$

<div align="right">(e:uk*-asyNRrep)</div>
<div align="right">(9.2.65)</div>

so that

$$u_{k^*} = (\log n)^{\frac{1}{3-\tau}+o(1)},$$

<div align="right">(e:Wkcore-NR)</div>
<div align="right">(9.2.66)</div>

so that, by picking n sufficiently large, we can make $\frac{1}{3-\tau} + o(1) \leq \sigma$. This completes the proof of Theorem 9.7. $\qquad\square$

Exercise 9.1. *Use* (6.6.7) *to prove Theorem 9.7 for the Chung-Lu model under the same conditions as in Theorem 9.7.*

Chapter 10

Configuration model revisited

In this chapter, we state some further results on the configuration model. We will not give the entire proofs. This chapter is organized as follows. In Section 10.1, we study when a giant component exists for the configuration model. In Sections 10.2–10.5, we study the average graph distance in the configuration model.

Firstly, we describe the phase transition in the configuration model. As we will see, also for the configuration model, similar results as the ones presented in Chapter 4 for the Erdős-Rényi random graph, also hold for the configuration model. When the degrees satisfy some supercriticality condition, then the largest connected component is of order $\Theta(n)$, while, in the subcritical regime, the largest connected component is much smaller. As a second result, we state a discrete duality principle alike the one in Section 4.4.5. These results are due to Molloy and Reed, see [157, 158].

Secondly, we investigate distances in the supercritical configuration model. We focus on the *average graph distance*, which is the graph distance between two uniformly chosen vertices in the configuration model. We restrict this discussion to the setting where the degrees are i.i.d. random variable with some distribution function F. As it turns out, this average graph distance depends sensitively on the tail behavior of the degrees. Here the tail behavior is the asymptotics of $x \mapsto 1 - F(x)$, as $x \to \infty$. When the degrees have thick tails, meaning that $1 - F(x)$ is relatively large, then the graph distance is much smaller than when the degrees have thin tails. These results are proved in a sequence of papers, see [109, 111, 110]. See Chapter 10 for details.

10.1 Phase transition in the configuration model

In this section, we investigate the connected components in the configuration model. We shall show that, alike for the Erdős-Rényi random graph, there is a condition for the configuration model such that with high probability a giant component exists. Again, this condition has the interpretation that the survival probability of an underlying branching process describing the exploration of a cluster, has a strictly positive survival probability.

We recall the definition of the configuration model in Chapter 7. Recall that

$$\nu = \frac{\sum_{k=1}^{\infty} k(k-1)p_k}{\sum_{k=1}^{\infty} kp_k}, \tag{10.1.1}$$

where $\{p_k\}_{k=1}^{\infty}$ is the limiting probability distribution of the degrees defined in (7.2.4). Recall that the empirical degree sequence of the configuration model is given by

$$p_k^{(n)} = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{d_i = k\}}, \tag{10.1.2}$$

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so that $\lim_{n\to\infty} p_k^{(n)} = p_k$. We further denote the distribution function corresponding to $\{p_k\}_{k=1}^{\infty}$ by

$$F(x) = \sum_{k=1}^{\lfloor x \rfloor} p_k, \qquad\qquad (10.1.3)$$

where $\lfloor x \rfloor$ is the largest integer smaller than or equal to x. We write $\mathcal{D}$ for the empirical degree sequence of the graph, so that

$$\mathcal{D} = \{p_k^{(n)}\}_{k=1}^{\infty}. \qquad\qquad (10.1.4)$$

There are several cases of interest. We can pick $\{p_k^{(n)}\}_{k=1}^{\infty}$ *deterministic* in such a way that

$$p_k^{(n)} \to p_k. \qquad\qquad (10.1.5)$$

On the other hand, we can also pick the $\{D_i\}_{i=1}^n$ to be i.i.d. with distribution $\{p_k\}_{k=1}^{\infty}$. Clearly for i.i.d. degrees and every k, $np_k^{(n)} \sim \text{BIN}(n, p_k)$, where $p_k = \mathbb{P}(D_i = k)$. Some results, however, require that $p_k^{(n)} = 0$ for all $k \geq n^a$ and some $a < 1$. In this case, we could also pick t he degrees $\{D_i^{(n)}\}_{i=1}^n$, where $D_i^{(n)}$ is the degree of vertex i, to be i.i.d. with distribution

$$p_k(n) = \mathbb{P}(D = k | D \leq n^a), \qquad\qquad (10.1.6)$$

i.e., we can condition on the degrees to be at most n^a. For the results of [157, 158] to be valid, we will often work with $\{p_k(n)\}_{k=1}^{\infty}$ rather than with $\{p_k\}_{k=1}^{\infty}$. Note that

$$\lim_{n\to\infty} p_k(n) = p_k = \mathbb{P}(D = k), \qquad\qquad (10.1.7)$$

whenever $a > 0$. In the following exercises, we relate the different choices of the degrees:

Exercise 10.1. *Denote the total variation distance between $p = \{p_k\}_{k=1}^{\infty}$ and $p(n) = \{p_k(n)\}_{k=1}^{\infty}$ by*

$$d_{\text{TV}}(p, p(n)) = \frac{1}{2} \sum_{k=1}^{\infty} |p_k - p_k(n)|. \qquad\qquad (10.1.8)$$

Show that

$$d_{\text{TV}}(p, p(n)) = 1 - F(n^a) = \sum_{k>n^a} p_k. \qquad\qquad (10.1.9)$$

Exercise 10.2. *Let $\{D_i\}_{i=1}^n$ be i.i.d. with distribution $\{p_k\}_{k=1}^{\infty}$ and $\{D_i^{(n)}\}_{i=1}^n$ i.i.d. with distribution $\{p_k(n)\}_{k=1}^{\infty}$. When there exists a $c > 0$ such that*

$$1 - F(x) = \sum_{k>x} p_k \leq cx^{1-\tau} \qquad\qquad (10.1.10)$$

for some $\tau > 1$, show that with high probability, the degrees $\{D_i^{(n)}\}_{i=1}^n$ and $\{D_i\}_{i=1}^n$ can be coupled such that the probability that $\{D_i^{(n)}\}_{i=1}^n \neq \{D_i\}_{i=1}^n$ is bounded by $n^{1-a(\tau-1)}$. In particular, for $a > \frac{1}{\tau-1}$, with high probability, $\{D_i^{(n)}\}_{i=1}^n = \{D_i\}_{i=1}^n$.

Before we state the results, we start by introducing some notation. Let $G_p(t)$ be the probability generating function of $\{p_k\}_{k=1}^{\infty}$, i.e.,

$$G_p(t) = \sum_{k=1}^{\infty} p_k t^k. \qquad\qquad (10.1.11)$$

Furthermore, for a probability distribution $\{p_k\}_{k=1}^{\infty}$, we define the *size-biased law of* $\{p_k\}_{k=1}^{\infty}$ by $\{q_k\}_{k=0}^{\infty}$, where

$$q_k = \frac{1}{\mu}(k+1)p_{k+1}, \qquad \text{with} \quad \mu = \sum_{k=1}^{\infty} kp_k, \tag{10.1.12}$$

and let $G_q(t)$ be the probability generating function of $\{q_k\}_{k=0}^{\infty}$, i.e.,

$$G_q(t) = \sum_{k=0}^{\infty} q_k t^k. \tag{10.1.13}$$

Exercise 10.3. *Show that*

$$G_q(t) = \frac{G_p'(t)}{\mu}. \tag{10.1.14}$$

Let $\beta_{\mathcal{D}}$ denote the extinction probability of the branching process with distribution $\{q_k\}_{k=0}^{\infty}$, so that $\beta_{\mathcal{D}}$ is the smallest solution of

$$\beta_{\mathcal{D}} = G_q(\beta_{\mathcal{D}}). \tag{10.1.15}$$

Set

$$\eta_{\mathcal{D}} = \sum_{k=1}^{\infty} p_k \beta_{\mathcal{D}}^k. \tag{10.1.16}$$

Then $\eta_{\mathcal{D}}$ is the extinction probability of a branching process where the probability mass function of Z_1 is equal to $\{p_k\}_{k=1}^{\infty}$, while the distribution of all further offspring is equal to $\{q_k\}_{k=0}^{\infty}$. We call such a branching process a *delayed branching process*.

Exercise 10.4. *Prove that $\eta_{\mathcal{D}}$ is indeed the extinction probability of the delayed branching process.*

Exercise 10.5. *Show that ν in* (10.1.1) *is equal to the expected value of a random variable with distribution* $\{q_k\}_{k=0}^{\infty}$, *so that the delayed branching process is supercritical precisely when $\nu > 1$.*

In this section, we shall work with *deterministic* degree sequences, of which we shall pose a number of assumptions which we formulate now.

Assumption 10.1. (i) *A degree sequence is* **feasible** *when there exists at least one graph with that degree sequence.*

(ii) *A degree sequence is* **smooth** *when, for all $k \geq 1$,*

$$\lim_{n\to\infty} p_k^{(n)} = p_k. \tag{10.1.17}$$

(iii) *A degree sequence is* **sparse** *when*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} kp_k^{(n)} = \mu = \sum_{k=1}^{\infty} kp_k. \tag{10.1.18}$$

(iv) *A degree sequence is* **well-behaved** *when it is feasible and smooth, and further,*

$$\lim_{n\to\infty} \sup_{k=1}^{\infty} \left| k(k-2)p_k^{(n)} - k(k-2)p_k \right| = 0, \tag{10.1.19}$$

and, when $\nu < \infty$,

$$\lim_{i\to\infty} \lim_{n\to\infty} \sum_{k=1}^{i} k(k-2)p_k^{(n)} - \nu = 0, \tag{10.1.20}$$

while, if $\nu = \infty$,

$$\lim_{i \to \infty} \lim_{n \to \infty} \sum_{k=1}^{i} k(k-2) p_k^{(n)} = \infty. \tag{10.1.21}$$

Exercise 10.6. *Show that when the degrees $\{D_i\}_{i=1}^n$ are i.i.d. and there exist $c > 0$ and $\tau > 2$ such that*

$$1 - F(x) \le cx^{1-\tau}, \tag{10.1.22}$$

then the degree sequence is smooth, well-behaved and sparse.

The main result about the size of the giant component is the following:

Theorem 10.2 (Giant component in $\mathrm{CM}(n, \{d_i\}_{i=1}^n)$). *Let $\mathcal{D}$ be a well-behaved and sparse degree sequence for which there exists an $\varepsilon > 0$ such that $p_k^{(n)} = 0$ for all $k > n^{1/4-\varepsilon}$. Then,*

(i) *if $\nu > 1$, then with high probability the giant component of the configuration model has size $\zeta_{\mathcal{D}} n + o(n)$, where $\zeta_{\mathcal{D}} = 1 - \eta_{\mathcal{D}}$ is the survival probability of the delayed branching process. Furthermore, if $\nu < \infty$, then, with high probability, the configuration model has exactly one component of size greater than $\gamma \log n$ for some constant γ dependent on $\mathcal{D}$.*

(ii) *If $\nu < 1$ and $p_k^{(n)} = 0$ for all $k \ge \omega(n)$, where $0 \le \omega(n) \le n^{\frac{1}{8}-\epsilon}$, then for some constant R dependent on ν, the configuration model with high probability has no component with at least $R\omega(n)^2 \log n$ vertices.*

Exercise 10.7. ([109])[HofHooVan05a] *Show that when the degrees $\{D_i\}_{i=1}^n$ are i.i.d. and there exist $c > 0$ and $\tau > 5$ such that*

$$1 - F(x) \le cx^{1-\tau}, \tag{10.1.23}$$

then, with high probability, there exists $\epsilon > 0$ such that $p_k^{(n)} = 0$ for all $k > n^{1/4-\epsilon}$.

We next state a duality principle like the one in Theorem 4.14. Let

$$p_k' = \frac{p_k}{\eta_{\mathcal{D}}} \beta_{\mathcal{D}}^k \tag{10.1.24}$$

[e:dualDS]

be the dual probability distribution (recall (3.3.7)).

Exercise 10.8. *Show that $p' = \{p_k'\}_{k=1}^\infty$ is a probability distribution, and is such that*

$$\nu' = \frac{\sum_{k=1}^\infty k(k-1) p_k'}{\sum_{k=1}^\infty k p_k} < 1. \tag{10.1.25}$$

[e:nu'-def]

Theorem 10.3 (Duality principle for configuration model). *The structure of the configuration model with degree sequence $\mathcal{D}$, formed by deleting the largest component $C_{\max}$, is the same as that of the configuration model with $m = n - |C_{\max}|$ vertices and with degree sequence $\mathcal{D}'$, for $\mathcal{D}' = p_1'^{(n)}, p_2'^{(n)}, \ldots$, which are such that $\lim_{n \to \infty} p_k'^{(n)} = p_k'$ and p_k' is defined in (10.1.24).*

The definition of $p' = \{p_k'\}_{k=1}^\infty$ can again be explained with help of a branching process. The p_i sequence belongs to a branching process which will not become extinct, which implies that the random graph with that degree sequence has a giant component. Thus, the removal of the giant component alters the accompanying branching process. In Theorem 3.6, the dual branching process is described. As a

result, with D_1 the degree of vertex 1 conditioned on 1 not belonging to the giant component, we would expect that

$$p'_k = \mathbb{P}(D_1 = k | 1 \text{ not in giant component}) \approx \mathbb{P}(Z_1 = k | \text{extinction}) = \frac{\mathbb{P}(\{Z_1 = k\} \cap \{\text{extinction}\})}{\mathbb{P}(\text{extinction})},$$
(10.1.26)

where Z_1 is the law of the first generation of the delayed branching process. The children in the first generation cause Z_1 independent branching processes, which all have to become extinct, and this happens with probability $\beta_{\mathcal{D}}^{Z_1}$. The probability for extinction in the delayed branching process equals $\eta_{\mathcal{D}}$, so that

$$p'_k = \frac{p_k \beta_{\mathcal{D}}^k}{\eta_{\mathcal{D}}}.$$
(10.1.27)

This heuristically explains the form of the dual asymptotic degree distribution in (10.1.24). We shall describe the connection of $\mathrm{CM}(n, \{d_i\}_{i=1}^n)$ and $\mathrm{CM}(n, \{D_i\}_{i=1}^n)$ to branching processes in much more detail in Sections 10.3 and 10.4 below.

We continue to investigate when the configuration model yields a connected graph:

Theorem 10.4 (Connectivity of $\mathrm{CM}(n, \{d_i\}_{i=1}^n)$)**.** *Assume that the degree sequence $\mathcal{D}$ is feasible, smooth and sparse, and that $p_1^{(n)} = p_2^{(n)} = 0$. Furthermore, assume the conditions of Theorem 10.2, in particular, that there exists an $\varepsilon > 0$ such that $p_k^{(n)} = 0$ for all $k > n^{1/4-\varepsilon}$. Finally, we assume that $\nu < \infty$. Then, with high probability, the configuration model with degree sequence $\mathcal{D}$ is connected.*

We remark that when $p_1 = p_2 = 0$, then $\nu \geq 2 > 1$ is immediate, so we are always in the supercritical regime. Also, when $p_1 = p_2 = 0$, we have that $\eta_{\mathcal{D}} = 1$, since survival of the delayed branching process occurs with probability 1. Therefore, Theorem 10.2 implies that the largest connected component has size $n(1 + o(1))$. Theorem 10.4 extends this to the statement that $\mathrm{CM}(n, \{d_i\}_{i=1}^n)$ is with high probability *connected*. We note that this is an important difference between the generalized random graph and the configuration model. Indeed, for the generalized random graph to be connected with high probability, the degrees must tend to infinity. This can be observed already for $\mathrm{ER}(n, p)$ in Theorem 5.5. For the configuration model, it is possible that the graph is connected while the average degree is bounded.

Many different versions of Theorem 10.4 exist in the literature. We have chosen a version that combines well with Theorem 10.2, so that the proof of Theorem 10.4 is fairly straightforward.

Proof. We already know, from Theorem 10.2, that the largest connected component has size $n(1 + o(1))$, and that any connected component which is not the giant component is bounded by $O(\log n)$. We now proceed as follows. Fix a vertex $i \in \{1, \ldots, n\}$. We shall show that the probability that the connected component of i is less than $K \log n$ for some $K > 0$ sufficiently large is bounded above by $o(n^{-1})$. To formalize this, we introduce some notation. We let the *cluster* or *connected component* of i be given by

$$C(i) = \{j \in \{1, \ldots, n\} : i \longleftrightarrow j\}. \tag{10.1.28}$$

We shall show that

$$\mathbb{P}\Big(|C(i)| \leq K \log n\Big) = o(n^{-1}). \tag{10.1.29}$$

We first show that this is sufficient. Indeed, by (10.1.29) and Boole's inequality,

$$\mathbb{P}\Big(\exists i : |C(i)| \leq K \log n\Big) \leq \sum_{i=1}^n \mathbb{P}\Big(|C(i)| \leq K \log n\Big) = o(1), \tag{10.1.30}$$

so that all connected component contain with high probability more than $K \log n$ vertices. By Theorem 10.2, with high probability, the second largest component is bounded above by $K \log n$, so that all connected components are part of the giant component, which completes the proof subject to (10.1.29).

To prove (10.1.29), we explore the neighborhood of i as follows. Let $m = \min\{k : p_k^{(n)} > 0\} - 1$, so that $m \geq 2$ since $p_1^{(n)} = p_2^{(n)} = 0$. From i, we pair the first $m + 1$ stubs (ignoring the other ones). Then, successively, we pair the first m stubs from the vertices connected to i in the $m + 1$ vertices explored. We continue by exploring the first m edges of the above vertices, and repeat the above process iteratively. We call the arising process when we have explored up to distance k from the initial vertex i the k-*exploration tree*. We stress that we do *not* explore all the neighbors of the vertices in the k-exploration tree, since we only pair the first m stubs of each vertex in the k-exploration tree.

When we never connect a stub to a stub already in the k-exploration tree, then the number of vertices we can reach in k steps is *precisely* equal to $(m + 1)m^{k-1}$. We call an event where a stub on the k-exploration tree connects to a stub incident to a vertex in the k-exploration tree a *collision*. The number of collisions in the k-exploration tree is the number of cycles or self-loops in it. When k increases, the probability of a collision increases since more and more vertices become part of the k-exploration tree. However, for k not too large, the probability that at least *two* collisions occur in the k-exploration tree is small, as we shall prove now:

Lemma 10.5 (At most one collision). *Take $k = 1, 2, \ldots$. Then, under the hypotheses of Theorem 10.4, the probability that there exists a vertex of which the k-exploration tree has at least two collisions is bounded by $(m + 1)^2 m^{2k} n^{-2\varepsilon - 3/2}$.*

Proof. For any stub in the k-exploration tree, the probability that it will create a collision is bounded above by $(m + 1)m^{k-1}n^{1/4-\varepsilon}L_n^{-1}$, where we recall that

$$L_n = n \sum_{k=1}^{\infty} k p_k^{(n)} \tag{10.1.31}$$

denotes the total degree in the model. The probability that two stubs will both create a collision is, by similar arguments, bounded above by $\left[(m + 1)m^{k-1}n^{1/4-\varepsilon}L_n^{-1}\right]^2$. The total number of possible pairs of stubs in the k-exploration tree is bounded by

$$[(m + 1)(1 + m + \ldots + m^{k-1})]^2 \leq [(m + 1)m^k]^2,$$

so that by Boole's inequality, the probability that the k-exploration tree has at least two collisions is bounded above by

$$\left[(m + 1)m^k\right]^4 n^{1/2-2\varepsilon} L_n^{-2} \leq \left[(m + 1)m^k\right]^4 n^{-2\varepsilon - 3/2}. \tag{10.1.32}$$

$\square$

We now complete the proof of Theorem 10.4. Fix $k = \lceil \varepsilon \log_m n \rceil$, so that the probability that there exists a vertex of which the k-exploration tree has at least two collisions is bounded by $(m + 1)^4 n^{2\varepsilon - 3/2}$. When the k-exploration tree has at most one collision, then $|C(i)| \geq m^k \geq n^\varepsilon$. As a result, we obtain that

$$\mathbb{P}\Big(|C(i)| \leq n^\varepsilon\Big) \leq \mathbb{P}\Big(k\text{-exploration tree has at least two collisions}\Big)$$
$$\leq (m + 1)^4 n^{2\varepsilon - 3/2} = o(n^{-1}). \tag{10.1.33}$$

This implies (10.1.29), and thus completes the proof of Theorem 10.4. $\square$

We close this section by giving two open problems on the giant component in the configuration model:

Open problem 10.6 (Largest subcritical cluster). *Let $\{D_i\}_{i=1}^n$ be i.i.d. with distribution $\{p_k\}_{k=1}^\infty$, and let the distribution function F in (10.1.3) satisfy*

$$1 - F(x) = \sum_{k>x} p_k = cx^{1-\tau}(1 + o(1)), \tag{10.1.34}$$

for some $\tau > 3$. Suppose $\nu < 1$, so that there is no giant component. Show that the largest connected component has size $\Theta(n^{1/(\tau-1)})$.

We note that the maximal degree in the case of (10.1.34) is also equal to $\Theta(n^{1/(\tau-1)})$ (see Theorem 2.25 and Exercise 2.28). Thus, the lower bound on the largest connected component is relatively simple.

Exercise 10.9. *Show that when* (10.1.34) *holds, then, with high probability, there exists a connected component of size $\Theta(n^{1/(\tau-1)})$.*

As a result, an answer to Open problem 10.6 should include a proof that the cluster of the vertex with maximal degree is also $\Theta(n^{1/(\tau-1)})$.

Open problem 10.7 (Largest critical cluster). *Let $\{D_i\}_{i=1}^n$ be i.i.d. with distribution $\{p_k\}_{k=1}^\infty$. Suppose $\nu = 1$, so that the configuration model is* critical. *What is the size of the largest connected component?*

Naturally, the answer to the question in Open problem 10.7 should depend sensitively on properties of the degree distribution. It is tempting to conjecture that, when the degrees have exponential tails (such as for the Erdős-Rényi random graph), the maximal component has size $\Theta(n^{2/3})$, as in Theorem 5.1.

10.2 Distances in configuration model with i.i.d. degrees: overview

In this section, we study distances in the configuration model. This is relevant from a practical point of view. Indeed, in Chapter 1, we have seen that real networks often satisfy two main properties: they are *scale-free* and *small worlds*.

By the results in Chapter 7, when the degree distribution is chosen appropriately, the degree sequence in the configuration model can be arbitrary, including a power law. As a result, by choosing the degree distribution to obey a power law, we obtain a scale-free random graph. Thus, the first main feature of real networks is satisfied. For the small-world phenomenon, we need to investigate the distances in the configuration model. The goal in the remainder of this chapter is to prove results for these distances not only to *verify* the small-world behavior in the configuration model, but also to *quantify* it.

We start with some notation and definitions. The probability mass function and the distribution function of the vertex degree distribution are denoted by

$$\mathbb{P}(D_1 = j) = p_j, \quad j = 1, 2, \ldots, \quad \text{and} \quad F(x) = \sum_{j=1}^{\lfloor x \rfloor} p_j, \qquad \text{(10.2.1)}$$

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where $\lfloor x \rfloor$ is the largest integer smaller than or equal to x. We often, though not exclusively, consider distributions of the form

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$$1 - F(x) = x^{-\tau+1} L(x), \qquad \text{(10.2.2)}$$

where $\tau > 1$ and L is slowly varying at infinity. This means that the random variables D_i obey a power law, and the factor L is meant to generalize the model. In particular, by Theorem 7.1, the configuration model with i.i.d. degrees with distribution given by (10.2.2) is a scale-free random graph. We shall later assume more specific conditions.

We define the graph distance H_n between the vertices 1 and 2 as the minimum number of edges that form a path from 1 to 2. By convention, the distance equals ∞ if 1 and 2 are not connected. Observe that the distance between two *randomly* chosen vertices is equal in distribution to H_n, because the vertices are exchangeable. We shall also be interested in the *diameter* of the configuration model, which is defined to be the maximal graph distance between any pair of vertices. Naturally, the diameter of the configuration model is infinite when $\mathrm{CM}(n, \{d_i\}_{i=1}^n)$ or $\mathrm{CM}(n, \{D_i\}_{i=1}^n)$ is not connected, which is one of the reasons why we have investigated the probability that the configuration model is connected in the previous section. To

avoid trivial cases where the diameter is infinite, we shall instead define the diameter to be the maximal graph distance between any pair of connected vertices.

In three sections below, we shall describe the results on the distances in the configuration model, splitting between the case of finite variance degrees, finite mean and infinite variance degrees and infinite mean degrees. It turns out that distances in $\mathrm{CM}(n, \{D_i\}_{i=1}^n)$ in these three cases are rather different. Indeed, in the case of finite variance degrees, the distances turn out to be of the order $\log_\nu n$ when $\nu > 1$. Thus distances grow logarithmically in the size of the graph. Since $n \mapsto \log n$ grows rather slowly with n, distances for realistic graphs are not 'too' large. However, when $n = 10^8$ (like for the network of social relations on this planet), the average distances can still be large. On the other hand, empirical studies have shown that for many real networks, the exponent τ of the power-law degree sequence is in between 2 and 3, i.e., $\tau \in (2, 3)$. Therefore, in such cases, the degrees have *infinite variance*, and, more precisely, $\mathbb{E}[D_i^\alpha] = \infty$ for any $\alpha > \tau - 1 \in (1, 2)$. In such cases, the average distance in the graph turns out to be of order $\log \log n$, which is much smaller than $\log n$. For most realistic networks, even the network of social relations on the planet, $\log \log n$ is quite moderate still. This may be an explanation of the 'Six Degrees of Separation' paradigm discussed in detail in Chapter 1. One of the first papers on the subject calls such distances quite appropriately *ultra-small* (see [68]). Finally, when the power-law exponent is in between 1 and 2, in which case degrees have infinite mean, distances remain uniformly bounded. However, in the configuration model with infinite mean degrees, one should remark that the graph is a multigraph, and is even such that the erased configuration model has rather *different* degrees (see e.g., Theorem 6.9).

In the following three sections, we shall go over these three cases, and describe the above results in full detail. We shall also give overviews of the proofs, and present proofs of partial results to give the flavor. We shall both describe the *diameter* of the configuration model as well as the *average distances*. In Section 10.3, we discuss the finite variance case, in Section 10.4 the finite mean and infinite variance case, and, finally, in Section 10.5, the infinite mean case. In each of these cases, we shall make more precise assumptions on the degree distributions involved.

10.3   Finite variance degrees

In this section, we give an overview of the results on the average graph distance in the configuration model $\mathrm{CM}(n, \{D_i\}_{i=1}^n)$ or $\mathrm{CM}(n, \{d_i\}_{i=1}^n)$ when the (asymptotic) degree sequence has a finite variance. We start by analyzing the average graph distance. For $\tau > 3$, we assume that there exists a constant $c > 0$ such that

$$1 - F(x) \leq cx^{-\tau+1}, \qquad \text{for all } x \geq 1, \tag{10.3.1}$$

and that $\nu > 1$, where we recall that ν is given by

$$\nu = \frac{\mathbb{E}[D(D-1)]}{\mathbb{E}[D]}. \tag{10.3.2}$$

Distributions satisfying (10.3.1) include distributions which have a lighter tail than a power law, and (10.3.1) is only slightly stronger than assuming finite variance. The assumption that $\nu > 1$ is reasonable in the light of Theorem 10.2. Indeed, when $\nu < 1$, then there is no giant component, so it does not make much sense to speak of 'average distances'.

We start by formulating a convergence in probability statement for the average distance:

Theorem 10.8 (Typical average distance in $\mathrm{CM}(n, \{D_i\}_{i=1}^n)$). *Assume that* (10.3.1) *is satisfied with* $\tau > 3$, *and let* $\nu > 1$. *Then,*

$$\frac{H_n}{\log n} \xrightarrow{\mathbb{P}} \frac{1}{\log \nu}. \tag{10.3.3}$$

Theorem 10.8 shows that the average distances in $\mathrm{CM}(n, \{D_i\}_{i=1}^n)$ are of order $\log_\nu n$. We shall heuristically explain this result in Section 10.3.1 below.

We continue the discussion of distances in the configuration model by investigating the diameter in the model. Before stating the main result, we introduce some notation. Recall that by (10.1.13), $G_q(s) = \sum_{k=0}^\infty q_k s^k = \sum_{k=0}^\infty \frac{(k+1)}{\mu} p_{k+1} s^k$ is the probability generating function of the probability distribution $q = \{q_k\}_{k=0}^\infty$ defined in (10.1.12). We recall that $\beta_\mathcal{D}$ is the extinction probability of the branching process with offspring distribution q defined in (10.1.15) and further define

$$\nu' = G_q'(\beta_\mathcal{D}) = \sum_{k=1}^\infty k \beta_\mathcal{D}^{k-1} q_k. \tag{10.3.4}$$

We shall again assume that for every $k = 1, 2, \ldots$,

$$\lim_{n\to\infty} p_k^{(n)} = p_k, \tag{10.3.5}$$

and that the limit

$$\nu = \lim_{n\to\infty} \sum_{i=1}^n \frac{d_i(d_i - 1)}{l_n} = \frac{1}{\mu} \sum_{j=1}^\infty j(j-1)p_j, \tag{10.3.6}$$

is finite, where

$$\mu = \sum_{j=1}^\infty j p_j \tag{10.3.7}$$

is the mean of the probability distribution $p = \{p_k\}_{k=1}^\infty$. Then, the main result is as follows:

Theorem 10.9 (Diameter of the configuration model). *Assume that* (10.3.5) *holds, and that* (10.3.6) *holds when $\nu < \infty$, or that*

$$\frac{l_n}{n} = \sum_{k=1}^\infty k p_k^{(n)} \to \mu = \sum_{k=1}^\infty k p_k \tag{10.3.8}$$

when $\nu = \infty$. Finally, assume that $p_1^{(n)} = 0$ when $p_1 = 0$, and that $p_2^{(n)} = 0$ when $p_2 = 0$. Then,

$$\frac{\mathrm{diam}(\mathrm{CM}(n, \{d_i\}_{i=1}^n))}{\log n} \xrightarrow{\mathbb{P}} \frac{1}{\log \nu} + \frac{(2 - \mathbb{1}_{\{p_1=0\}} - \mathbb{1}_{\{p_2=0\}})}{|\log \nu'|}. \tag{10.3.9}$$

We note that, by Theorem 10.8 and Theorem 10.9, the diameter of the configuration model is strictly larger than the average distances, except when $p_1 = p_2 = 0$. In the latter case, the degrees are at least three, so that thin lines are not possible, and the configuration model is with high probability *connected* (recall Theorem 10.4). We also remark that Theorem 10.9 applies not only to the finite variance case, but also to the finite mean and infinite variance case. In the latter case, the diameter is of order $\log n$ unless when $p_1 = p_2 = 0$, in which case Theorem 10.9 implies that the diameter is $o(\log n)$. What the scaling of the diameter is in this case will be discussed in detail in Section 10.4 below. By Corollary 7.7, Theorem 10.9 also applies to uniform random graphs with a given degree sequence. This shall be used in the examples below.

We start by giving a few examples of Theorem 10.9:

Random regular graphs. Let r be the degree of the random regular graph, where $r \geq 2$. By Corollary 7.7, the diameter of a random regular r-graph has with high probability the same asymptotics as the diameter of $\mathrm{CM}(n, \{d_i\}_{i=1}^n)$, where $d_i = r$ with probability 1. Thus, $p_r = 1$ and $p_i = 0$ for any $i \neq r$. We

assume that nr is even, so that the degree sequence is feasible. It is not hard to see that all assumptions of Theorem 10.9 are satisfied. Moreover, we have that $\nu = r - 1$. When $r \geq 3$, we thus obtain that

$$\frac{\operatorname{diam}(\mathrm{CM}(n, \{d_i\}_{i=1}^n))}{\log n} \xrightarrow{\ \mathbb{P}\ } \frac{1}{\log(r-1)}. \tag{10.3.10}$$

When $r = 2$, on the other hand, the graph is *critical*, so that there is no giant component. It would be of interest to investigate the size of the largest connected component in this case. Since $\nu = 1$, we have that $\nu' = \nu = 1$, so that $\operatorname{diam}(\mathrm{CM}(n, \{d_i\}_{i=1}^n)) \gg \log n$. This is quite reasonable, since the graph will consist of a collection of *cycles*. The diameter of such a graph is equal to half the longest cycle.

Exercise 10.10. *Prove that in a graph consisting solely of cycles, the diameter is equal to the longest cycle divided by 2.*

Exercise 10.11. *What is the size of the longest cycle of the 2-regular graph?*

Erdős-Rényi random graph.   We next study the diameter of $\mathrm{ER}(n, \lambda/n)$. We let $\lambda > 1$. In this case, we have that (10.3.5) holds with

$$p_k = e^{-\lambda} \frac{\lambda^k}{k!}. \tag{10.3.11}$$

Also, (10.3.6) holds with $\nu = \lambda$ and $\nu' = \mu_\lambda$, the dual parameter in (3.5.7).

Exercise 10.12. *Verify that* (7.2.4) *and* (7.2.30) *hold, and prove that $\nu = \lambda$ and $\nu' = \mu_\lambda$.*

We make essential use of Theorem 7.8, which related the configuration model and the generalized random graph. We note that $\mathrm{ER}(n, \lambda/n)$ is the same as $\mathrm{GRG}(n, \{w_i\}_{i=1}^n)$, where (recall Exercise 6.1)

$$w_i = \frac{n\lambda}{n - \lambda}. \tag{10.3.12}$$

From the above identifications and using Theorem 7.8, we find that

$$\frac{\operatorname{diam}(\mathrm{ER}(n, \lambda/n))}{\log n} \xrightarrow{\ \mathbb{P}\ } \frac{1}{\log \lambda} + \frac{2}{|\log \mu_\lambda|}. \tag{10.3.13}$$

This identifies the diameter of the Erdős-Rényi random graph.

We continue to study the *fluctuations* of the distances in the configuration model. We need a limit result from branching process theory before we can identify the limiting random variables $(R_a)_{a \in (-1, 0]}$. We introduce a *delayed* branching process $\{\mathcal{Z}_k\}$, where in the first generation, the offspring distribution is chosen according to (**??**) and in the second and further generations, the offspring is chosen in accordance to q given by

$$q_j = \frac{(j+1)p_{j+1}}{\mu}, \quad j = 0, 1, \ldots. \tag{10.3.14}$$

The process $\{\mathcal{Z}_k / \mu \nu^{k-1}\}$ is a martingale with uniformly bounded expectation and consequently converges almost surely to a limit (see e.g., Theorem 2.21 and Exercise 2.23):

$$\lim_{n \to \infty} \frac{\mathcal{Z}_n}{\mu \nu^{n-1}} = \mathcal{W} \qquad a.s. \tag{10.3.15}$$

In the theorem below we need two independent copies $\mathcal{W}^{(1)}$ and $\mathcal{W}^{(2)}$ of $\mathcal{W}$.

Theorem 10.10 (Limit law for the average distance). *Assume that (10.3.1) is satisfied, and let $\nu > 1$. For $k \geq 1$, let $a_k = \lfloor \log_\nu k \rfloor - \log_\nu k \in (-1, 0]$. Then, there exist random variables $(R_a)_{a \in (-1,0]}$ such that as $n \to \infty$,*

$$\mathbb{P}(H_n - \lfloor \log_\nu n \rfloor = k \mid H_n < \infty) = \mathbb{P}(R_{a_n} = k) + o(1), \qquad k \in \mathbb{Z}, \tag{10.3.16}$$

where, for $a \in (-1, 0]$,

$$\mathbb{P}(R_a > k) = \mathbb{E}\big[\exp\{-\kappa \nu^{a+k} \mathcal{W}^{(1)} \mathcal{W}^{(2)}\}\big| \mathcal{W}^{(1)} \mathcal{W}^{(2)} > 0\big], \tag{10.3.17}$$

where $\mathcal{W}^{(1)}$ and $\mathcal{W}^{(2)}$ are independent limit copies of $\mathcal{W}$ in (10.3.15) and where $\kappa = \mu(\nu - 1)^{-1}$.

In words, Theorem 10.10 states that for $\tau > 3$, the graph distance H_n between two randomly chosen connected vertices grows like the $\log_\nu n$, where n is the size of the graph, and that the fluctuations around this mean remain uniformly bounded in n. Theorem 10.10 proves the conjecture in Newman, Strogatz and Watts [168, Section II.F, (54)], where a heuristic is given that the number of edges between arbitrary vertices grows like $\log_\nu n$. In addition, Theorem 10.10 improves upon that conjecture by specifying the fluctuations around the value $\log_\nu n$.

The law of R_a is involved, and can in most cases not be computed exactly. The reason for this is the fact that the random variables $\mathcal{W}$ that appear in its statement are hard to compute explicitly. For example, for the configuration model with $\tau > 3$, we do not know the law of $\mathcal{W}$. See also Chapter 3. There are two examples where the law of $\mathcal{W}$ is known. The first is when all degrees in the graph are equal to some $r > 2$, and we obtain the r-regular graph (see also [42], where the diameter of this graph is studied). In this case, we have that $\mu = r, \nu = r - 1$, and $\mathcal{W} = 1$ a.s. In particular, $\mathbb{P}(H_n < \infty) = 1 + o(1)$. Therefore, we obtain that

$$\mathbb{P}(R_a > k) = \exp\{-\frac{r}{r-2}(r-1)^{a+k}\}, \tag{10.3.18}$$

and H_n is asymptotically equal to $\log_{r-1} n$. The second example is when g is the probability mass function of a geometric random variable, in which case the branching process with offspring q conditioned to be positive converges to an exponential random variable with parameter 1. This example corresponds to

$$q_j = p(1-p)^{j-1}, \quad \text{so that} \quad p_j = \frac{1}{jc_p} p(1-p)^{j-2}, \quad \forall j \geq 1, \tag{10.3.19}$$

and c_p is a normalization constant. For $p > \frac{1}{2}$, the law of $\mathcal{W}$ has the same law as the sum of D_1 copies of a random variable $\mathcal{Y}$, where $\mathcal{Y} = 0$ with probability $\frac{1-p}{p}$ and equal to an exponential random variable with parameter 1 with probability $\frac{2p-1}{p}$. Even in this simple case, the computation of the exact law of R_a is non-trivial.

We finally give two exercises discussing consequences of Theorem 10.10:

Exercise 10.13. ([109]) *Fix an integer n_1. Prove that, under the assumptions in Theorem 10.10, and conditionally on $H_n < \infty$, along the subsequence $n_k = \lfloor n_1 \nu^{k-1} \rfloor$, the sequence of random variables $H_{n_k} - \lfloor \log_\nu n_k \rfloor$ converges in distribution to $R_{a_{n_1}}$ as $k \to \infty$.*

Exercise 10.14. ([109]) *Prove that, under the assumptions in Theorem 10.10,*

(i) *with probability $1 - o(1)$ and conditionally on $H_n < \infty$, the random variable H_n is in between $(1 \pm \varepsilon) \log_\nu n$ for any $\varepsilon > 0$;*

(ii) *conditionally on $H_n < \infty$, the random variables $H_n - \log_\nu n$ form a tight sequence, i.e.,*

$$\lim_{K \to \infty} \limsup_{n \to \infty} \mathbb{P}(|H_n - \log_\nu n| \leq K | H_n < \infty) = 1. \tag{10.3.20}$$

As a consequence, prove that the same result applies to a uniform random graph with degrees $\{D_i\}_{i=1}^n$. Hint: Make use of Theorem 7.10.

10.3.1   An overview of the proof for the average distances

In this section, we present a heuristic explanation of Theorem 10.10. We also indicate how the limiting random variable in Theorem 10.10 comes about.

There are two basic ingredients underlying these results. The first is that for two disjoint sets of stubs of sizes n and m out of a total of L, the probability that none of the stubs in the first set is attached to a stub in the second set, is approximately equal to

$$\prod_{i=0}^{n-1} \left(1 - \frac{m}{L - n - 2i - 1}\right). \tag{10.3.21}$$

In fact, the product in (10.3.21) is precisely equal to the probability that none of the n stubs in the first set of stubs is attached to a stub in the second set, given that no two stubs in the first set are attached to one another. When $n = o(L)$, $L \to \infty$, however, these two probabilities are asymptotically equal. We approximate (10.3.21) further as

$$\prod_{i=0}^{n-1} \left(1 - \frac{m}{L - n - 2i - 1}\right) = \exp\left\{\sum_{i=0}^{n-1} \log\left(1 - \frac{m}{L}\left(1 + \frac{n + 2i + 1}{L}\right)\right)\right\} \approx e^{-\frac{mn}{L}}, \tag{10.3.22}$$

where the approximation is valid as long as $nm(n + m) = o(L^2)$ when $L \to \infty$.

The *shortest path tree* (SPG) from vertex 1 is the random graph as observed from vertex 1, and consists of the shortest paths between vertex 1 and all other vertices $\{2, \ldots, n\}$. We define the SPG from vertex 2 in a similar fashion. We apply the above heuristic asymptotics to the growth of the SPG. Let $Z_j^{(1)}$ denote the number of stubs that are attached to vertices precisely $j - 1$ steps away from vertex 1, and similarly for $Z_j^{(2)}$. We then apply (10.3.22) to $n = Z_j^{(1)}, m = Z_j^{(2)}$ and $L = L_n$. Let $\mathbb{Q}_Z^{(k,l)}$ be the conditional probabilities given $\{Z_s^{(1)}\}_{s=1}^k$ and $\{Z_s^{(2)}\}_{s=1}^l$. For $l = 0$, we only condition on $\{Z_s^{(1)}\}_{s=1}^k$. For $j \geq 1$, we have the multiplication rule (see [109, Lemma 4.1])

$$\mathbb{P}(H_n > j) = \mathbb{E}\left[\prod_{i=2}^{j+1} \mathbb{Q}_Z^{(\lceil i/2 \rceil, \lfloor i/2 \rfloor)}(H_n > i - 1 | H_n > i - 2)\right]. \tag{10.3.23}$$

where $\lceil x \rceil$ is the smallest integer greater than or equal to x and $\lfloor x \rfloor$ the largest integer smaller than or equal to x. Now we approximate, using (10.3.22),

$$\mathbb{Q}_Z^{(\lceil i/2 \rceil, \lfloor i/2 \rfloor)}(H_n > i - 1 | H_n > i - 2) \approx \exp\left\{-\frac{Z_{\lceil i/2 \rceil}^{(1)} Z_{\lfloor i/2 \rfloor}^{(2)}}{L_n}\right\}. \tag{10.3.24}$$

This asymptotic identity follows because the event $\{H_n > i - 1 | H_n > 1 - 2\}$ occurs precisely when none of the stubs $Z_{\lceil i/2 \rceil}^{(1)}$ attaches to one of those of $Z_{\lfloor i/2 \rfloor}^{(2)}$. A typical value of the average distance between two connected vertices H_n is the value j for which

$$\frac{1}{L_n} \sum_{i=2}^{j+1} Z_{\lceil i/2 \rceil}^{(1)} Z_{\lfloor i/2 \rfloor}^{(2)} \approx 1.$$

This is the first ingredient of the heuristic.

The second ingredient is the connection to branching processes. Given any vertex i and a stub attached to this vertex, we attach the stub to a second stub to create an edge of the graph. This chosen stub is attached to a certain vertex, and we wish to investigate how many further stubs this vertex has. The

conditional probability that this number of stubs, excluding the stub that is attached to the stub at vertex i, equals n given $D_1, \ldots, D_n$, is approximately equal to the probability that a random stub from all $L_n = D_1 + \ldots + D_n$ stubs is attached to a vertex with in total $n + 1$ stubs. Since there are precisely $\sum_{j=1}^{n} (n+1) \mathbb{1}_{\{D_j = n+1\}}$ stubs that belong to a vertex with degree $n + 1$, we find for the latter probability

$$q_n^{(n)} = \frac{1}{L_n} \sum_{j=1}^{n} (n+1) \mathbb{1}_{\{D_j = n+1\}} = \frac{n+1}{L_n} \sum_{j=1}^{n} \mathbb{1}_{\{D_j = n+1\}}. \tag{10.3.25}$$

The above formula comes from sampling the stubs *with* replacement, whereas in the configuration model, and hence in SPG, the sampling is performed without replacement. Now, as we grow the SPGs from vertices 1 and 2, of course the number of stubs that can still be chosen decreases. However, when the sizes of both SPGs are much smaller than n, for instance at most $\sqrt{n}$, or slightly bigger, this dependence can be neglected, and it is as if we choose each time *independently* and with replacement. Thus, the growth of the SPGs is closely related to a *branching process* with offspring distribution $\{g_n^{(n)}\}$.

When $\tau > 2$, using the strong law of large numbers for $n \to \infty$,

$$\frac{L_n}{n} \xrightarrow{a.s.} \mu = \mathbb{E}[D], \quad \text{and} \quad \frac{1}{n} \sum_{j=1}^{n} \mathbb{1}_{\{D_j = n+1\}} \to p_{n+1} \xrightarrow{a.s.} \mathbb{P}(D = n+1),$$

so that, almost surely,

$$q_n^{(n)} \to \frac{(n+1)\mathbb{P}(D = n+1)}{\mu} = \frac{(n+1)p_{n+1}}{\mu} \equiv q_n, \quad n \to \infty. \tag{10.3.26}$$

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Therefore, the growth of the shortest path tree should be well described by a branching process with offspring distribution $\{g_n\}$, and we come to the question what is a typical value of j for which

$$\sum_{i=2}^{j+1} \mathcal{Z}_{\lceil i/2 \rceil}^{(1)} \mathcal{Z}_{\lfloor i/2 \rfloor}^{(2)} \approx \mu n, \tag{10.3.27}$$

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where $\{\mathcal{Z}_j^{(1)}\}$ and $\{\mathcal{Z}_j^{(2)}\}$ denote two independent copies of a two-stage branching process with offspring distribution $\{p_n\}$, $p_n = \mathbb{P}(D = n)$, $n = 1, 2, \ldots$, in the first generation and offspring distribution $\{q_n\}$ in all further generations.

When $\tau > 3$, then $\nu = \sum_n n q_n < \infty$. Assume also that $\nu > 1$, so that the branching process is supercritical. In this case, the delayed branching process $\mathcal{Z}_j / \mu\nu^{j-1}$ converges almost surely to a random variable $\mathcal{W}$. Hence for the two *independent* branching processes $\{\mathcal{Z}_j^{(i)}\}$, $i = 1, 2$, that locally describe the number of stubs attached to vertices on distance $j - 1$, we find that for $j \to \infty$,

$$\mathcal{Z}_j^{(i)} \sim \mu\nu^{j-1} \mathcal{W}^{(i)}. \tag{10.3.28}$$

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This explains why the average value of $Z_j^{(i)}$ grows like $\mathbb{E}[\mathcal{Z}_j^{(i)}] = \mu\nu^{j-1} = \mu \exp((j-1)\log\nu)$, that is exponential in j for $\nu > 1$, so that a typical value of j for which (10.3.27) holds, satisfies

$$\mu \cdot \nu^{j-1} = n, \quad \text{or} \quad j = \log_\nu(n/\mu) + 1, \tag{10.3.29}$$

where $c = 1 + \log\mu$. We can extend this argument to describe the fluctuation around the asymptotic mean. Since (10.3.28) describes the fluctuations of $Z_j^{(i)}$ around the mean value $\mu\nu^{j-1}$, we are able to describe the random fluctuations of H_n around $\log_\nu n$. In Section 10.4, we shall also apply the same heuristic argument to the case where the degrees have infinite variance.

When $\tau \in (1, 2)$, the sequence q_n is not defined since $\mathbb{P}(L_n/n \to \infty) = 1$, as $n \to \infty$. Thus the limiting branching process is not defined. In this case, L_n is the i.i.d. sum of n random variables $D_1, D_2, \ldots, D_n$, with infinite mean. From extreme value theory, it is well known that then the bulk of the contribution to L_n comes from a *finite* number of vertices which have giant degrees (the so-called *giant vertices*). Since these giant vertices have degree roughly $n^{1/(\tau-1)}$, which is much larger than $\sqrt{n}$, they are all connected to each other, thus forming a complete graph of giant vertices. Each stub is with probability close to 1 attached to a giant vertex, and therefore, the distance between any two vertices is, with high probability, at most 3. In fact, this distance equals 2 precisely when the two vertices are connected to the *same* giant vertex, and is 3 otherwise.

10.4   Finite mean, infinite variance degrees

In this section, we study the average distance and diameter of the configuration model with degrees having finite mean and infinite variance. We start by assuming that there exists $\tau \in (2, 3)$ such that

$$1 - F(x) = L(x)x^{-\tau+1}. \tag{10.4.1}$$

Then, the average distance of $\mathrm{CM}(n, \{D_i\}_{i=1}^n)$ is identified in the following theorem:

Theorem 10.11 (Average graph distance of $\mathrm{CM}(n, \{D_i\}_{i=1}^n)$ for infinite variance degrees). *Assume that* (10.4.1) *holds. Fix* $\tau \in (2, 3)$. *Then,*

$$\frac{H_n}{\log \log n} \overset{\mathbb{P}}{\longrightarrow} \frac{2}{|\log(\tau - 2)|}. \tag{10.4.2}$$

We explain the asymptotics above using the heuristics in Section 10.3.1. When $\tau \in (2, 3)$, the branching processes $\{\mathcal{Z}_j^{(1)}\}$ and $\{\mathcal{Z}_j^{(2)}\}$ are well-defined, but has infinite mean in generations 2, 3, etc. Under certain conditions on the underlying offspring distribution, Davies [72] proves for this case that $(\tau - 2)^j \log(1 + \mathcal{Z}_j)$ converges almost surely, as $j \to \infty$, to some random variable Y. Moreover, $\mathbb{P}(Y = 0) = 1 - \eta$, the extinction probability of $\{\mathcal{Z}_j\}_{j=0}^\infty$. Therefore, also $(\tau - 2)^j \log(\mathcal{Z}_j \vee 1)$ converges almost surely to Y.

Since $\tau > 2$, we still have that $L_n \approx \mu n$, so that the typical value of j for which (10.3.27) holds satisfies

$$\mathcal{Z}_{\lceil (j+1)/2 \rceil}^{(1)} \mathcal{Z}_{\lfloor (j+1)/2 \rfloor}^{(2)} \approx \mu n, \qquad \text{or} \qquad \log(\mathcal{Z}_{\lceil (j+1)/2 \rceil}^{(1)} \vee 1) + \log(\mathcal{Z}_{\lfloor (j+1)/2 \rfloor}^{(2)} \vee 1) \approx \log n.$$

This indicates that the typical value of j is of order

$$m_{\tau,n} = 2\frac{\log \log n}{|\log(\tau - 2)|}, \tag{10.4.3}$$

as formulated in Theorem 10.11.

We continue by studying the diameter of $\mathrm{CM}(n, \{D_i\}_{i=1}^n)$ in the case where $\tau \in (2, 3)$. Note that when $p_1 + p_2 > 0$, then the diameter is equal to a positive constant times $\log n$ by Theorem 10.9. Therefore, we turn to the case where $p_1 = p_2 = 0$. For this result, we assume that there exists a $\tau \in (2, 3)$ such that, for some $c > 0$ and all $x \geq 1$,

$$1 - F(x) \geq cx^{1-\tau}. \tag{10.4.4}$$

Observe that (10.4.4) is strictly weaker than (10.4.5). The main result is as follows:

Theorem 10.12 (Upper bound on diameter of $\mathrm{CM}(n, \{D_i\}_{i=1}^n)$). *Assume that* $p_1 = p_2 = 0$ *and that* (10.4.4) *holds. Then, there exists a positive constant* C_F *such that with high probability the diameter of* $\mathrm{CM}(n, \{D_i\}_{i=1}^n)$ *is bounded above by* $C_F \log \log n$, *as* $n \to \infty$.
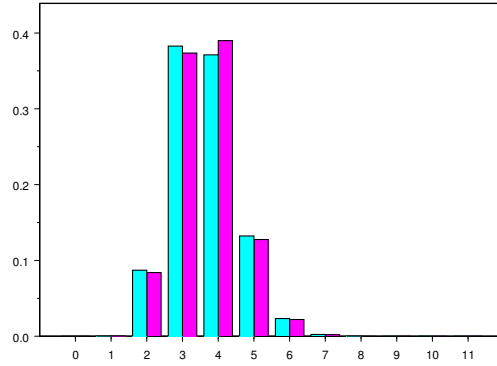
Figure 10.1: Number of AS traversed in hopcount data (blue) compared to the model (purple) with $\tau = 2.25, n = 10,940$.

We next study the fluctuations of the average distance in $\mathrm{CM}(n, \{D_i\}_{i=1}^n)$, for which we need to make a stronger assumption than in (10.4.1). For $\tau \in (2,3)$, we assume that there exists $\gamma \in [0,1)$ and $C < \infty$ such that

$$x^{-\tau+1-C(\log x)^{\gamma-1}} \leq 1 - F(x) \leq x^{-\tau+1+C(\log x)^{\gamma-1}}, \qquad \text{for large } x. \tag{10.4.5}$$

The condition in (10.4.5) is slightly stronger than (10.4.1), and is such that the results in Theorem 3.20 apply. Then, we can identify the fluctuations of the average graph distance in $\mathrm{CM}(n, \{D_i\}_{i=1}^n)$ as follows:

Theorem 10.13 (Fluctuations of the graph distance of $\mathrm{CM}(n, \{D_i\}_{i=1}^n)$ for infinite variance degrees). *Fix $\tau \in (2,3)$ and assume that (10.4.5) holds. Then there exist random variables $(R_a)_{a \in (-1,0]}$ such that, as $n \to \infty$,*

$$\mathbb{P}\Big(H_n = 2\Big\lfloor \frac{\log\log n}{|\log(\tau-2)|} \Big\rfloor + l \,\Big|\, H_n < \infty\Big) = \mathbb{P}(R_{a_n} = l) + o(1), \qquad l \in \mathbb{Z}, \tag{10.4.6}$$

where $a_n = \lfloor \frac{\log\log n}{|\log(\tau-2)|} \rfloor - \frac{\log\log n}{|\log(\tau-2)|} \in (-1,0]$, and where, for $a \in (-1,0]$,

$$\mathbb{P}(R_a > l) = \mathbb{P}\Big(\min_{s \in \mathbb{Z}} \big[(\tau-2)^{-s}Y^{(1)} + (\tau-2)^{s-c_l}Y^{(2)} \big] \leq (\tau-2)^{\lceil l/2 \rceil + a} \,\Big|\, Y^{(1)}Y^{(2)} > 0 \Big),$$

where $c_l = 1$ if l is even, and zero otherwise, and $Y^{(1)}, Y^{(2)}$ are two independent copies of the limit random variable in Theorem 3.20.

In words, Theorem 10.11 states that for $\tau \in (2,3)$, the graph distance H_n between two randomly chosen connected vertices grows proportional to $\log\log$ of the size of the graph, and that the fluctuations around this mean remain uniformly bounded in n.

Simulations indicating the properties of the average graph distance for $\tau \in (2,3)$ can bee seen in Figure 10.1. In it, the distances of the AS-graph in Figure 1.2, and these distances are compared to the ones in $\mathrm{CM}(n, \{D_i\}_{i=1}^n)$ where n is equal to the number of AS and the best approximation to the exponent of the power-law for the degree sequence of the AS-graph, which is $\tau = 2.25$.

10.4.1   A $\log\log$ upper bound on the diameter of the core for $\tau \in (2, 3)$

In this section, we define the *core* of the configuration model for $\tau \in (2, 3)$, and investigate the diameter of the core. This argument is reminiscent of the proof of Theorem 9.7, in which the core of the Norros-Reittu model was investigated.

The proof is divided into two key steps. In the first, in Proposition 11.17, we give a bound on the diameter of the *core* of the configuration model consisting of all vertices with degree at least a certain power of $\log n$. This argument is very close in spirit to the one in [183],[ReiNor04] the only difference being that we have simplified the argument slightly. After this, in Proposition 11.21, we derive a bound on the distance between vertices with small degree and the core. We note that Proposition 11.17 only relies on the assumption in (10.4.4), while Proposition 11.21 only relies on the fact that $\mathbb{P}(D \geq 3) = 1$. We start by investigating the core of the configuration model.

We take $\sigma > \frac{1}{3-\tau}$ and define the *core* Core_n of the configuration model to be

$$\mathrm{Core}_n = \{i : D_i \geq (\log n)^\sigma\}, \tag{10.4.7}$$

i.e., the set of vertices with degree at least $(\log n)^\sigma$. Then, the diameter of the core is bounded in the following theorem:

Theorem 10.14 (Diameter of the core). *Fix* $\tau \in (2, 3)$ *and assume that* (10.4.5) *holds. For every* $\sigma > \frac{1}{3-\tau}$, *the diameter of* Core_n *is with high probability bounded above by*

$$\frac{2 \log\log n}{|\log(\tau - 2)|}(1 + o(1)). \tag{10.4.8}$$

Proof. We note that (10.4.4) implies that with high probability the largest degree $D_{(n)}$ satisfies

$$D_{(n)} \geq u_1, \quad \text{where} \quad u_1 = n^{\frac{1}{\tau-1}}(\log n)^{-1}, \tag{10.4.9}$$

because for $n \to \infty$ and since $1 - x \leq e^{-x}$,

$$\mathbb{P}(D_{(n)} > u_1) = 1 - \mathbb{P}(D_{(n)} \leq u_1) = 1 - [F(u_1)]^n \geq 1 - (1 - cu_1^{1-\tau})^n$$

$$= 1 - \left(1 - c\frac{(\log n)^{\tau-1}}{n}\right)^n \geq 1 - \exp(-c(\log n)^{\tau-1}) \to 1. \tag{10.4.10}$$

Define

$$\Gamma_1 = \{i : D_i \geq u_1\}, \tag{10.4.11}$$

so that, with high probability, $\Gamma_1 \neq \varnothing$. For some constant $C > 0$, which will be specified later, and $k \geq 2$ we define recursively

$$u_k = C \log n (u_{k-1})^{\tau-2}. \tag{10.4.12}$$

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Then, we define

$$\Gamma_k = \{i : D_i \geq u_k\}. \tag{10.4.13}$$

Lemma 9.8 identifies u_k as $u_k = C^{a_k}(\log n)^{b_k} n^{c_k}$, where

$$c_k = \frac{(\tau - 2)^{k-2}}{\tau - 1}, \qquad b_k = \frac{1}{3-\tau} - \frac{4-\tau}{3-\tau}(\tau - 2)^{k-1}, \qquad a_k = \frac{1 - (\tau - 2)^{k-1}}{3-\tau}. \tag{10.4.14}$$

The key step in the proof of Theorem 10.14 is the following lemma:

Lemma 10.15 (Connectivity between Γ_{k-1} and Γ_k). *Fix $\tau \in (2,3)$ and assume that (10.4.5) holds. Fix $k \geq 2$, and take $C > 4\mu/c$. Then, the probability that there exists an $i \in \Gamma_k$ that is not directly connected to Γ_{k-1} is $o(n^{-\delta})$, for some $\delta > 0$ independent of k.*

Proof. We note that, by definition,

$$\sum_{i \in \Gamma_{k-1}} D_i \geq u_{k-1} |\Gamma_{k-1}|. \tag{10.4.15}$$

Also,

$$|\Gamma_{k-1}| \sim \text{Bin}(n, 1 - F(u_{k-1})), \tag{10.4.16}$$

and we have that, by (10.4.4),

$$n[1 - F(u_{k-1})] \geq cn(u_{k-1})^{1-\tau}, \tag{10.4.17}$$

which, by Lemma 9.8, grows as a positive power of n, since $c_k \leq c_2 = \frac{\tau-2}{\tau-1} < \frac{1}{\tau-1}$. Using Theorem 2.18, we obtain that the probability that $|\Gamma_{k-1}|$ is bounded above by $n[1 - F(u_{k-1})]/2$ is exponentially small in n. As a result, we obtain that for every k, and with high probability

$$\sum_{i \in \Gamma_k} D_i \geq \frac{c}{2} n(u_k)^{2-\tau}. \tag{10.4.18}$$

We note (see e.g., [111, (4.34)]) that for any two sets of vertices A, B, we have that

$$\mathbb{P}_n(A \text{ not directly connected to } B) \leq e^{-\frac{D_A D_B}{L_n}}, \tag{10.4.19}$$

where, for any $A \subseteq \{1, \ldots, n\}$, we write

$$D_A = \sum_{i \in A} D_i. \tag{10.4.20}$$

On the event where $|\Gamma_{k-1}| \geq n[1 - F(u_{k-1})]/2$ and where $L_n \leq 2\mu n$, we then obtain by (10.4.19), and Boole's inequality that the $\mathbb{P}_n$-probability that there exists an $i \in \Gamma_k$ such that i is not directly connected to Γ_{k-1} is bounded by

$$ne^{-\frac{u_k n u_{k-1}[1-F(u_{k-1})]}{2L_n}} \leq ne^{-\frac{cu_k(u_{k-1})^{2-\tau}}{4\mu}} = n^{1-\frac{cC}{4\mu}}, \tag{10.4.21}$$

where we have used (10.4.12). Taking $C > 4\mu/c$ proves the claim. $\qquad \square$

We now complete the proof of Theorem 10.14. Fix

$$k^* = \frac{2 \log \log n}{|\log(\tau-2)|}. \tag{10.4.22}$$

As a result of Lemma 10.15, we have with high probability that the diameter of Γ_{k^*} is at most $2k^* + 1$, because the distance between any vertex in Γ_{k^*} and Γ_1 is at most k^*. Since $u_1 = n^{\frac{1}{\tau-1}}(\log n)^{-1} \gg \sqrt{n} \log n$, we have that, again with high probability, each pair of vertices in Γ_1 is directly connected. Therefore, we are done when we can show that

$$\text{Core}_n \subseteq \Gamma_{k^*}. \tag{10.4.23}$$

For this, we note that

$$\Gamma_{k^*} = \{i : D_i \geq u_{k^*}\}, \tag{10.4.24}$$

so that it suffices to prove that $u_{k^*} \geq (\log n)^\sigma$, for any $\sigma > \frac{1}{3-\tau}$. According to Lemma 9.8,

$$u_{k^*} = C^{a_{k^*}}(\log n)^{b_{k^*}} n^{c_{k^*}}. \tag{10.4.25}$$

Because for $x \to \infty$, and $2 < \tau < 3$,

$$x(\tau - 2)^{\frac{2 \log x}{|\log (\tau - 2)|}} = x \cdot x^{-2} = o(\log x), \tag{10.4.26}$$

we find with $x = \log n$ that

$$\log n \cdot (\tau - 2)^{\frac{2 \log \log n}{|\log (\tau - 2)|}} = o(\log \log n), \tag{10.4.27}$$

which implies that $n^{c_{k^*}} = (\log n)^{o(1)}$, $(\log n)^{b_{k^*}} = (\log n)^{\frac{1}{3 - \tau} + o(1)}$, and $C^{a_{k^*}} = (\log n)^{o(1)}$. Thus,

$$u_{k^*} = (\log n)^{\frac{1}{3 - \tau} + o(1)}, \tag{10.4.28}$$

so that, by picking n sufficiently large, we can make $\frac{1}{3 - \tau} + o(1) \leq \sigma$. This completes the proof of Theorem 10.14. $\qquad\square$

10.4.2 Diameter of $\mathrm{CM}\big(n, \{D_i\}_{i=1}^n\big)$ for $\tau \in (2, 3)$: Proof of Theorem 10.12

In this section, we use Theorem 10.14 to complete the proof of Theorem 10.12. More precisely, we investigate the configuration model when $p_1 + p_2 = 0$, or equivalently $\mathbb{P}(D \geq 3) = 1$. We assume (10.4.4) for some $\tau \in (2, 3)$. We will show that under these assumptions $C_F \log \log n$ is an upper bound on the diameter of G for some sufficiently large constant C_F (see Theorem 10.12). Note that Theorem 10.14 implies that, for $\tau \in (2, 3)$, the diameter of the core is bounded above by $\frac{2 \log \log n}{|\log (\tau - 2)|}(1 + o(1))$. Therefore, all we are left to do in this section is to prove that a given vertex, when $\mathbb{P}(D \geq 3) = 1$, is within distance $O(\log \log n)$ of the core. This will be the strategy of the proof in this section.

Define

$$C(m, \varepsilon) = \left(\frac{\tau - 2}{3 - \tau} + 1 + \varepsilon\right) / \log m, \tag{10.4.29}$$

where $\varepsilon > 0$ and $m \geq 2$ is an integer.

Theorem 10.16 (Maximal distance between periphery and core). *Fix $\tau \in (2, 3)$ and assume that (10.4.5) holds. Assume that $\mathbb{P}(D \geq m + 1) = 1$ for some $m \geq 2$, and take $\varepsilon > 0$. Then, with high probability, the maximal distance between any vertex and the core is bounded from above by $C(m, \varepsilon) \log \log n$.*

Proof. We start from a vertex i and will show that the probability that the distance between i and Core_n is at least $C(m, \varepsilon) \log \log n$ is $o(n^{-1})$. This proves the claim. For this, we explore the neighborhood of i as follows. From i, we connect the first $m + 1$ stubs (ignoring the other ones). Then, successively, we connect the first m stubs from the closest vertex to i that we have connected to and have not yet been explored. We call the arising process when we have explored up to distance k from the initial vertex i the k-*exploration tree*.

When we never connect two stubs between vertices we have connected to, then the number of vertices we can reach in k steps is *precisely* equal to $(m + 1)m^{k-1}$. We call an event where a stub on the k-exploration tree connects to a stub incident to a vertex in the k-exploration tree a *collision*. The number of collisions in the k-exploration tree is the number of cycles or self-loops in it. When k increases, the probability of a collision increases. However, for k of order $\log \log n$, the probability that more than *two* collisions occur in the k-exploration tree is small, as we will prove now:

Lemma 10.17 (At most one collision. II.). *Fix $\tau \in (2, 3)$ and assume that (10.4.5) holds. Assume that $\mathbb{P}(D \geq m + 1) = 1$ for some $m \geq 2$. Take $k = \lceil C(m, \varepsilon) \log \log n \rceil$. Then, the probability that there exists a vertex of which the k-exploration tree has at least two collisions, before hitting the core Core_n, is bounded by $(m + 1)^4 (\log n)^d L_n^{-2}$, for $d = 4C(m, \varepsilon) \log m + 2\sigma$.*

Proof. For any stub in the k-exploration tree, the probability that it will create a collision before hitting the core is bounded above by $(m + 1)m^{k-1}(\log n)^\sigma L_n^{-1}$. The probability that two stubs will both create a collision is, by similar arguments, bounded above by $[(m + 1)m^{k-1}(\log n)^\sigma L_n^{-1}]^2$. The total number of possible pairs of stubs in the k-exploration tree is bounded by

$$[(m + 1)(1 + m + \ldots + m^{k-1})]^2 \leq [(m + 1)m^k]^2,$$

so that by Boole's inequality, the probability that the k-exploration tree has at least two collisions is bounded by

$$[(m + 1)m^k]^4 (\log n)^{2\sigma} L_n^{-2}. \tag{10.4.30}$$

When $k = C(m, \varepsilon) \log \log n$, we have that $[(m + 1)m^k]^4 (\log n)^{2\sigma} \leq (m + 1)^4 (\log n)^d$, where $d = 4C(m, \varepsilon) \log (m + 1) + 2\sigma$. $\square$

Finally, we show that for $k = C(m, \varepsilon) \log \log n$, the k-exploration tree will, with high probability, connect to Core_n:

Lemma 10.18 (Connecting the exploration tree to the core). *Fix $\tau \in (2, 3)$ and assume that (10.4.5) holds. Assume that $\mathbb{P}(D \geq m + 1) = 1$ for some $m \geq 2$. Take $k = C(m, \varepsilon) \log \log n$. Then, the probability that there exists an i such that the distance of i to the core is at least k is $o(n^{-1})$.*

Proof. Since $\mu < \infty$ we have that $L_n/n \sim \mu$. Then, by Lemma 10.17, the probability that there exists a vertex for which the k-exploration tree has at least 2 collisions before hitting the core is $o(n^{-1})$. When the k-exploration tree from a vertex i does *not* have two collisions, then there are at least $(m - 1)m^{k-1}$ stubs in the kth layer that have not yet been connected. When $k = C(m, \varepsilon) \log \log n$ this number is at least equal to $(\log n)^{C(m,\varepsilon) \log m + o(1)}$. Furthermore, the number of stubs incident to the core Core_n is stochastically bounded from below by $(\log n)^\sigma$ times a binomial distribution with parameters n and success probability $\mathbb{P}(D_1 \geq (\log n)^\sigma)$. The expected number of stubs incident to Core_n is therefore at least $n(\log n)^\sigma \mathbb{P}(D_1 \geq (\log n)^\sigma)$ so that with high probability, the number of stubs incident to Core_n is at least (by Theorem 2.18)

$$\frac{1}{2}n(\log n)^\sigma \mathbb{P}(D_1 \geq (\log n)^\sigma) \geq \frac{c}{2}n(\log n)^{\sigma(2-\tau)} \geq \frac{c}{2}n(\log n)^{\frac{2-\tau}{3-\tau}}. \tag{10.4.31}$$

By (10.4.19), the probability that we connect none of the stubs of the k-exploration tree to one of the stubs incident to Core_n is bounded by

$$\exp\left\{-\frac{cn(\log n)^{\frac{2-\tau}{3-\tau} + C(m,\varepsilon) \log m}}{2L_n}\right\} \leq \exp\left\{-\frac{c}{4\mu}(\log n)^{\frac{2-\tau}{3-\tau} + C(m,\varepsilon) \log m}\right\} = o(n^{-1}), \tag{10.4.32}$$

because $L_n/n \leq 2\mu$ with high probability, and since $\frac{2-\tau}{3-\tau} + C(m, \varepsilon) \log m = 1 + \varepsilon$. $\square$

Theorems 10.14 and 10.16 prove that **whp** the diameter of the configuration model is bounded above by $C_F \log \log n$, where

$$C_F = \frac{2}{|\log (\tau - 2)|} + \frac{2(\frac{\tau-2}{3-\tau} + 1 + \varepsilon)}{\log m}. \tag{10.4.33}$$

This completes the proof of Theorem 10.12. $\square$
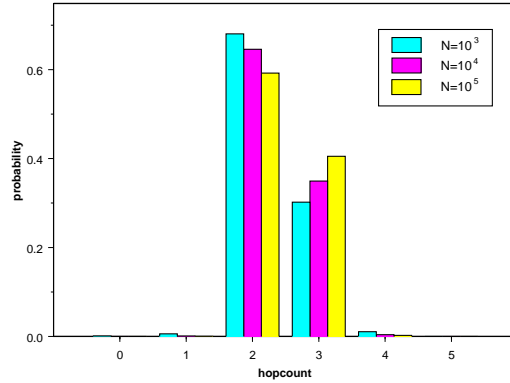
Figure 10.2: Empirical probability mass function of the hopcount for $\tau = 1.8$ and $N = 10^3, 10^4, 10^5$.

10.5   Infinite mean degrees

In this section, we investigate the distances in $\text{CM}(n, \{D_i\}_{i=1}^n)$ when the degrees have infinite mean. More specifically, we shall assume that (??) holds with $\tau \in (1, 2)$. In the statement, we shall make heavy use of the notation used in Theorem 7.12, which we first recall. Recall that the random probability distribution $P = \{P_i\}_{i=1}^\infty$ is given by

$$P_i = \frac{\xi_i}{\eta}, \qquad\qquad (10.5.1)$$

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where $\xi_i = \Gamma_i^{-1/(\tau-1)}$ and $\Gamma_i = \sum_{j=1}^i E_i$ with $\{E_i\}_{i=1}^\infty$ an i.i.d. sequence of exponential random variables with parameter 1. Recall further that $M_{P,k}$ is a multinomial distribution with parameters k and probabilities $P = \{P_i\}_{i=1}^\infty$. Thus, $M_{P,k} = (B_1, B_2, \ldots)$, where, conditionally on $P = \{P_i\}_{i=1}^\infty$, B_i is the number of outcomes i in k independent trials such that each outcome is equal to i with probability P_i. In Theorem 7.12, the random variable M_{P,D_1} appears, where D_1 is independent of $P = \{P_i\}_{i=1}^\infty$. We let $M_{P,D_1}^{(1)}$ and $M_{P,D_2}^{(2)}$ be two random variables which are conditionally independent given $P = \{P_i\}_{i=1}^\infty$.

In terms of this notation, the main result on distances in $\text{CM}(n, \{D_i\}_{i=1}^n)$ when the degrees have infinite mean is the following:

Theorem 10.19. *Fix $\tau \in (1, 2)$ in (??) and let $D_1, D_2, \ldots, D_n$ be a sequence of i.i.d. copies of D. Then,*

$$\lim_{n\to\infty} \mathbb{P}(H_n = 2) = 1 - \lim_{n\to\infty} \mathbb{P}(H_n = 3) = p_F \in (0, 1). \qquad (10.5.2)$$

<div align="right">limit lawrep</div>

The probability p_F can be identifies as the probability that $M_{P,D_1}^{(1)}$ and $M_{P,D_2}^{(2)}$ have an identical outcome, i.e., there is an outcome that occurs both in $M_{P,D_1}^{(1)}$ and $M_{P,D_2}^{(2)}$.

Proof. We sketch the proof of Theorem 10.19. First, with high probability, both $D_1 \leq \log n$ and $D_2 \leq \log n$, and . The event that $H_n = 1$ occurs precisely when one of the D_1 stubs of vertex 1 is attached to one of the D_2 stubs of vertex 2. Also, with high probability, $L_n \geq n^{1/(\tau-1)-\varepsilon}$. Therefore, on the event that $L_n \geq n^{1/(\tau-1)-\varepsilon}$ and $D_1 \leq \log n$ and $D_2 \leq \log n$, the probability that $H_n = 1$ is bounded above by

$$\frac{(\log n)^2}{n^{1/(\tau-1)-\varepsilon}} = o(1). \qquad\qquad (10.5.3)$$

Exercise 10.15. *Complete the argument that* $\mathbb{P}(H_n = 1) = o(1)$.

We note that the proof of Theorem 7.12 implies that $M_{P,D_1}^{(1)}$ denotes the number of edges between vertex 1 and the largest order statistics. Indeed, $M_{P,D_1}^{(1)} = (B_1^{(1)}, B_2^{(1)}, \ldots)$, where $B_i^{(1)}$ is the number of edges between vertex i and the vertex with degree $D_{(n+1-i)}$. The same applies to vertex 2. As a result, when $M_{P,D_1}^{(1)}$ and $M_{P,D_2}^{(2)}$ have an identical outcome, then the average graph distance equals 2. We are left to prove that the average graph distance is bounded by 3 with high probability. By (2.6.17), we have that $\xi_k k^{1/(\tau-1)} \overset{\mathbb{P}}{\longrightarrow} 1$ as $k \to \infty$. Thus, when K is large, the probability that vertex 1 is not connected to any of the vertices corresponding to $\{D_{(n+1-i)}\}_{i=1}^K$ converges to 0 when K tends to infinity.

Recall that $\mathbb{P}_n$ denotes the conditional probability given the degrees $\{D_i\}_{i=1}^n$. For $i = 1, 2, \ldots,$, we let v_i be the vertex corresponding to the ith order statistic $D_{(n+1-i)}$. By (11.4.11), we have that

$$\mathbb{P}_n(v_i \text{ not directly connected to } v_j) \leq e^{-\frac{D_{(n+1-i)}D_{(n+1-j)}}{L_n}}. \tag{10.5.4}$$

Moreover, $D_{(n+1-i)}, D_{(n+1-j)} \geq n^{1/(\tau-1)-\varepsilon}$ with high probability for n sufficiently large and any $\varepsilon > 0$, while $L_n \leq n^{1/(\tau-1)+\varepsilon}$ with high probability. As a result, with high probability,

$$\mathbb{P}_n(v_i \text{ not directly connected to } v_j) \leq e^{-n^{1/(\tau-1)-3\varepsilon}}, \tag{10.5.5}$$

which converges to 0 exponentially fast. Therefore, for every $i, j \in \{1, \ldots, K\}$, the vertices v_i and v_j are connected with high probability. This implies that the vertices corresponding to the high order statistics form a *complete graph*. We have already concluded that 1 is connected to v_i for some $i \leq K$. In the same way, we conclude that vertex 2 is connected to v_j for some $j \leq K$. Since v_i is with high probability connected to v_j, we conclude that

$$\mathbb{P}_n(H_n \leq 3) = 1 - o(1). \tag{10.5.6}$$

This completes the proof. $\qquad\square$

10.6 Notes and discussion

Notes on Section 10.1 This section is based on [157, 158].[MolRee95,MolRee98] More precisely, Theorem 10.2 is a combination of [157, Theorem 1][MolRee95] and [158, Theorem 1][MolRee98]. We note that in [157, 158],[MolRee95,MolRee98] the results are applied to a *uniform* random graph with degree sequence $\mathcal{D}$. However, due to Theorem 7.3, the results for the configuration model carry over to the one for a uniform random graph under the hypotheses of Theorem 10.2:

Exercise 10.16. *Show that Theorem 10.2 remains to be valid for a uniform random graph with degree sequence* $\mathcal{D}$.

Theorem 10.3 is [158, Theorem 2].[MolRee98] The results in [157, 158][MolRee95,MolRee98] are not phrased in terms of branching processes, which makes their results a bit more difficult to grasp. I have chosen to reformulate them using branching process terminology.

Notes on Section 10.2 Distances in the configuration model were first obtained in [168, 169].[NewStrWat00, NewStrWat02]

Notes on Section 10.3 Theorem 10.9 is [98, Theorem 4.1].[FerRam04] A $\log n$ lower bound on the diameter is also proved in [112].[HofHooZna04d] Theorem 10.10 is proved in [109].[HofHooVan05a]

Notes on Section 10.4 Theorem 10.13 is proved in [111].[HofHooZna04a] Theorem 11.16 is proved in [112].[HofHooZna04d]

Notes on Section 10.5   Theorem 10.19 is proved in [94]$^{\text{EskHofHooZna04}}$. The identification of $\mathbb{P}(H_n = 2)$ is new.

One might argue that including degrees larger than $n - 1$ is artificial in a network with n vertices. In fact, in many real networks, the degree is bounded by a physical constant. Therefore, in [94]$^{\text{EskHofHooZna04}}$, also the case where the degrees are conditioned to be smaller than n^α is considered, where α is an arbitrary positive number. Of course, we cannot condition on the degrees to be at most M, where M is fixed and independent on n, since in this case, the degrees are uniformly bounded, and this case is treated in [109]$^{\text{HofHooVan05a}}$. Therefore, [94]$^{\text{EskHofHooZna04}}$ considers cases where the degrees are conditioned to be at most a given power of n. In this setting, it turns out that the average distance is equal to $k + 3$ with high probability, whenever $\alpha \in (1/(\tau + k), 1/(\tau + k - 1))$.

Chapter 11

Preferential attachment revisited

In this chapter, we state some further results on preferential attachment models. We start by discussing an important tool in this chapter: exchangeable random variables and their distribution described in De Finetti's Theorem. These results shall be applied to Polya urn schemes, which can be used to describe the distribution of the degrees in preferential attachment models. In Section 11.2, we investigate the connectivity of $\mathrm{PA}_{m,\delta}(t)$. In Section 11.2.1, we investigate graph distances in $\mathrm{PA}_{m,\delta}(t)$.

Throughout this chapter, we shall work with the preferential attachment model defined in Section 8.1 and denoted by $\{\mathrm{PA}_{m,\delta}(t)\}_{t=1}^{\infty}$, unless stated otherwise. We recall that this model starts with a single vertex with m self-loops at time $t = 1$, and at each time, a vertex is added with m edges which are attached to the vertices in the graph with probabilities given in (8.1.1) for $m = 1$, and as described on page 174 for $m \geq 2$. This model can also be obtained by identifying blocks of m vertices in $\{\mathrm{PA}_{1,\delta}(t)\}_{t=1}^{\infty}$. We shall sometimes also discuss other variants of the model, such as $\{\mathrm{PA}_{m,\delta}(t)\}_{t=1}^{\infty}$, in which the $m = 1$ model does not have any self-loops.

11.1 Exchangeable random variables and Polya urn schemes

In this section, we discuss the distribution of infinite sequences of exchangeable random variables and their applications to Polya urn schemes. We start by discussing De Finetti's Theorem.

De Finetti's Theorem for infinite sequences of exchangeable random variables. A sequence of random variables $\{X_i\}_{i=1}^{\infty}$ is called *exchangeable* when, for every $n \geq 1$, the distribution of $\{X_i\}_{i=1}^{n}$ is the same as the one of $\{X_{\sigma(i)}\}_{i=1}^{n}$ for any permutation $\sigma \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ of $\{1, \ldots, n\}$. Clearly, when a sequence of random variables is independent and identically distributed, then it is also exchangeable. Remarkably, however, the distribution of an infinite sequence of random variables is always a *mixture* of independent and identically distributed random variables. This is the content of De Finetti's Theorem, which we shall state and prove here in the case where $\{X_i\}_{i=1}^{\infty}$ are indicator variables:

Theorem 11.1 (De Finetti's Theorem). *Let $\{X_i\}_{i=1}^{\infty}$ be an infinite sequence of exchangeable random variables, and assume that $X_i \in \{0, 1\}$. Then there exists a random variable U with $\mathbb{P}(U \in [0, 1]) = 1$ such that, for all $1 \leq k \leq n$,*

$$\mathbb{P}(X_1 = \cdots = X_k = 1, X_{k+1} = \cdots = X_n = 0) = \mathbb{E}[U^k(1 - U)^{n-k}]. \qquad \text{(11.1.1)}$$

De Finetti's Theorem (Theorem 11.2) states that an infinite exchangeable sequence of indicators has the same distribution as an independent Bernoulli sequence with a *random* success probability U. Thus, the

different elements of the sequence are *not* independent, but their dependence enters only through the random success probability.

From the proof, we shall see that Theorem 11.1 in fact holds more generally, for example, when X_i takes on at most a *finite* number of values. Since we shall use Theorem 11.1 only for indicator variables, we refrain from stating this version.

When writing

$$S_n = \sum_{k=1}^{n} X_k, \tag{11.1.2}$$

Theorem 11.1 implies that

$$\mathbb{P}(S_n = k) = \mathbb{E}\Big[\mathbb{P}(\mathrm{BIN}(n, U) = k)\Big]. \tag{11.1.3}$$

Equation (11.1.3) also allows us to compute the distribution of U. Indeed, when we would have that

$$\lim_{n \to \infty} n\mathbb{P}(S_n = \lceil un \rceil) = f(u), \tag{11.1.4}$$

where f is a density, then (11.1.3) implies that f is in fact the density of the random variable U. This shall prove to be useful in applications of De Finetti's Theorem (Theorem 11.1). Furthermore, by the strong law of large numbers applied to the conditional law given U, we have that $S_n/n \xrightarrow{a.s.} U$.

Exercise 11.1. *Prove* (11.1.3).

Proof of Theorem 11.1. The proof of this deep result is in fact quite simple. It makes use of Helly's Theorem, which states that any sequence of *bounded* random variables has a weakly converging subsequence.

We fix $m \geq n$ and condition on S_m to write

$$\mathbb{P}(X_1 = \cdots = X_k = 1, X_{k+1} = \cdots = X_n = 0)$$

$$= \sum_{j=k}^{m} \mathbb{P}(X_1 = \cdots = X_k = 1, X_{k+1} = \cdots = X_n = 0 | S_m = j)\mathbb{P}(S_m = j). \tag{11.1.5}$$

Now, when conditioning on $S_m = j$ and by exchangeability, each sequence $\{X_i\}_{i=1}^{m}$ containing precisely j ones is equally likely. Since there are precisely $\binom{m}{j}$ such sequences, and precisely $\binom{n}{k}\binom{j-k}{m-n}$ of them start with k ones and $n - k$ zeros, we obtain

$$\mathbb{P}(X_1 = \cdots = X_k = 1, X_{k+1} = \cdots = X_n = 0 | S_m = j) = \frac{\binom{n}{k}\binom{j-k}{m-n}}{\binom{m}{j}}. \tag{11.1.6}$$

We therefore arrive at

$$\mathbb{P}(X_1 = \cdots = X_k = 1, X_{k+1} = \cdots = X_n = 0)$$

$$= \sum_{j=k}^{m} \frac{j \cdots (j - k + 1) \cdot (m - j) \cdots (m - j - (n - k) + 1)}{m \cdots (m - n + 1)} \mathbb{P}(S_m = j). \tag{11.1.7}$$

When $m \to \infty$, we see that, for $k \leq n$ fixed,

$$\frac{j \cdots (j - k + 1) \cdot (m - j) \cdots (m - j - (n - k) + 1)}{m \cdots (m - n + 1)} = \Big(\frac{j}{m}\Big)^k\Big(1 - \frac{j}{m}\Big)^{n-k} + o(1). \tag{11.1.8}$$

Recall that $S_m = j$, so that

$$\mathbb{P}(X_1 = \cdots = X_k = 1, X_{k+1} = \cdots = X_n = 0) = \lim_{m \to \infty} \mathbb{E}[Y_m^k(1 - Y_m)^{n-k}], \tag{11.1.9}$$

where we write $Y_m = S_m/m$. Note that it is here that we make use of the fact that $\{X_i\}_{i=1}^{\infty}$ is an *infinite* exchangeable sequence of random variables.

Since $0 \leq S_m \leq m$, we have that $0 \leq Y_m \leq 1$, so that the sequence of random variables $\{Y_m\}_{m=1}^{\infty}$ is a bounded sequence. By Helly's Theorem, it contains a weakly converging subsequence, i.e., there exists a $\{Y_{m_l}\}_{l=1}^{\infty}$ with $\lim_{l\to\infty} m_l = \infty$ and a random variable U such that $Y_{m_l} \xrightarrow{d} U$. Since, for each k, n, the random variable $Y_m^k(1 - Y_m)^{n-k}$ is uniformly bounded, we thus obtain by bounded convergence that

$$\lim_{m\to\infty} \mathbb{E}\big[Y_m^k(1 - Y_m)^{n-k}\big] = \lim_{l\to\infty} \mathbb{E}\big[Y_{m_l}^k(1 - Y_{m_l})^{n-k}\big] = \mathbb{E}\big[U^k(1 - U)^{n-k}\big] \qquad (11.1.10)$$

This completes the proof. $\qquad\qquad\square$

We close this paragraph by discussing some consequences of De Finetti's Theorem. De Finetti's Theorem implies that when X_k and X_n are coordinates of an infinite exchangeable sequence of indicators, then they are *positively correlated*:

Exercise 11.2. *Let $\{X_i\}_{i=1}^{\infty}$ be an infinite sequence of exchangeable random variables. Prove that*

$$\mathbb{P}(X_k = X_n = 1) \geq \mathbb{P}(X_k = 1)\mathbb{P}(X_n = 1). \qquad (11.1.11)$$

Prove that equality holds if and only if there exists a p such that $\mathbb{P}(U = p) = 1$.

By Exercise 11.2, an infinite sequence of exchangeable random variables is positively correlated. This also allows us to discuss examples of *finite* exchangeable sequences of random variables and their relation to De Finetti's Theorem (Theorem 11.1). Indeed, in the proof of De Finetti's Theorem, it is imperative that the sequence $\{X_i\}_{i=1}^{\infty}$ is infinite. This is not mere a technicality of the proof. Rather, there are finite exchangeable sequences of random variables for which the equality (11.1.1) does *not* hold. Indeed, take an urn filled with b blue and r red balls and draw balls successively without replacement. Let X_i denote the indicator that the ith ball drawn is blue. Then, clearly, the sequence $\{X_i\}_{i=1}^{r+b}$ is exchangeable. However,

$$\mathbb{P}(X_1 = X_2 = 1) = \frac{b(b - 1)}{(b + r)(b + r - 1)} < \Big(\frac{b}{b + r}\Big)^2 = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1), \qquad (11.1.12)$$

so that X_1 and X_2 are *negatively* correlated. As a result, (11.1.1) fails.

Polya urn schemes. An important application of De Finetti's Theorem (Theorem 11.1) arises in Polya urn schemes. An urn consists of a number of balls, and we successively draw balls and replace them in the urn. We start with $B_0 = b_0$ blue balls and $R_0 = r_0$ red balls at time $n = 0$. Let $W_b, W_r \colon \mathbb{N} \to (0, \infty)$ be two weight functions. Then, at time $n + 1$, the probability of drawing a blue ball, conditionally on the number B_n of blue balls at time n, is proportional to the weight of the blue balls at time n, i.e., the probability of drawing a blue ball is equal to

$$\frac{W_b(B_n)}{W_b(B_n) + W_r(R_n)}. \qquad\qquad \text{e:PUS-probs} \atop (11.1.13)$$

After drawing a ball, it is replaced together with a *second* ball of the same color. We shall denote this Polya urn scheme by $\{(B_n, R_n)\}_{n=1}^{\infty}$. Naturally, we have that $B_n + R_n = b_0 + r_0 + n$.

In this section, we shall restrict to the case where there are $a_r, a_b \geq 0$ such that

$$W_b(k) = a_b + k, \qquad W_r(k) = a_r + k, \qquad\qquad \text{e:PUSlinear} \atop (11.1.14)$$

i.e., both weight functions are linear with the same slope, but possibly a different intercept. The main result concerning Polya urn schemes is the following theorem:

Theorem 11.2 (Limit theorem for linear Polya urn schemes). *Let $\{(B_n, R_n)\}_{n=1}^{\infty}$ be a Polya urn scheme with linear weight functions W_b and W_r as in (11.1.14). Then, as $n \to \infty$,*

$$\frac{B_n}{B_n + R_n} \xrightarrow{a.s.} U,$$

(11.1.15)

where U has a Beta-distribution with parameters $a = b_0 + a_b$ and $b = r_0 + a_r$, and

$$\mathbb{P}(B_n = B_0 + k) = \mathbb{E}\Big[\mathbb{P}(\mathrm{BIN}(n, U) = k)\Big].$$

(11.1.16)

Proof of Theorem 11.2. Let X_n denote the indicator that the nth ball drawn is blue. We first show that $\{X_n\}_{n=1}^{\infty}$ is an infinite exchangeable sequence. For this, we note that

$$B_n = b_0 + \sum_{j=1}^{n} X_j, \qquad R_n = r_0 + \sum_{j=1}^{n}(1 - X_j) = r_0 - b_0 + n - B_n.$$

(11.1.17)

Now, for any sequence $\{x_t\}_{t=1}^{n}$, we have that

$$\mathbb{P}(\{X_t\}_{t=1}^{n} = \{x_t\}_{t=1}^{n}) = \prod_{t=1}^{n} \frac{W_b(b_t)^{x_t} W_r(r_t)^{1-x_t}}{W_b(b_t) + W_r(r_t)},$$

(11.1.18)

where $b_t = b_0 + \sum_{j=1}^{t} x_j$ and $r_t = R_0 - B_0 + t - b_t$. Denote $k = \sum_{t=1}^{n} x_t$. Then, by (11.1.14) and (11.1.17), we have that

$$\prod_{t=1}^{n}(W_b(b_t) + W_r(r_t)) = \prod_{t=0}^{n-1}(b_0 + r_0 + a_b + a_r + t),$$

(11.1.19)

while

$$\prod_{t=1}^{n} W_b(b_t)^{x_t} = \prod_{m=0}^{k-1}(b_0 + a_b + m), \qquad \prod_{t=1}^{n} W_r(r_t)^{1-x_t} = \prod_{j=0}^{n-k-1}(r_0 + a_r + j).$$

(11.1.20)

Thus, we arrive at

$$\mathbb{P}(\{X_t\}_{t=1}^{n} = \{x_t\}_{t=1}^{n}) = \frac{\prod_{m=0}^{k-1}(b + m) \prod_{j=0}^{n-k-1}(r + j)}{\prod_{t=0}^{n-1}(b + r + t)},$$

(11.1.21)

where $b = b_0 + a_b$ and $r = r_0 + a_r$. In particular, (11.1.21) does not depend on the *order* in which the elements of $\{x_t\}_{t=1}^{n}$ appear, so that the sequence $\{X_n\}_{n=1}^{\infty}$ is an infinite exchangeable sequence. Thus, by De Finetti's Theorem (Theorem 11.1), the sequence $\{X_n\}_{n=1}^{\infty}$ is a mixture of Bernoulli random variables with a random success probability U, and we are left to compute the distribution of U. We also observe that the distribution of depends only on b_0, r_0, a_b, a_r through $b = b_0 + a_b$ and $r = r_0 + a_r$.

We next verify (11.1.4). For fixed $0 \le k \le n$, there are $\binom{n}{k}$ sequences of k ones and $n - k$ zeros. Each sequence has the same probability given by (11.1.21). Thus,

$$\begin{aligned}
\mathbb{P}(S_n = k) &= \binom{n}{k} \frac{\prod_{m=0}^{k-1}(b + m) \prod_{j=0}^{n-k-1}(r + j)}{\prod_{t=0}^{n-1}(b + r + t)} \\
&= \frac{\Gamma(n + 1)}{\Gamma(k + 1)\Gamma(n - k + 1)} \times \frac{\Gamma(k + b)}{\Gamma(b)} \times \frac{\Gamma(n - k + r)}{\Gamma(r)} \times \frac{\Gamma(b + r)}{\Gamma(n + b + r)} \\
&= \frac{\Gamma(b + r)}{\Gamma(r)\Gamma(b)} \times \frac{\Gamma(k + b)}{\Gamma(k + 1)} \times \frac{\Gamma(n - k + r)}{\Gamma(n - k + 1)} \times \frac{\Gamma(n + 1)}{\Gamma(n + b + r)}.
\end{aligned}$$

(11.1.22)

For k and $n - k$ large, by (8.2.7),

$$\mathbb{P}(S_n = k) = \frac{\Gamma(b+r)}{\Gamma(r)\Gamma(b)} \frac{k^{b-1}(n-k)^{r-1}}{n^{b+r-1}}(1+o(1)). \tag{11.1.23}$$

Taking $k = \lceil un \rceil$ (recall (11.1.4))

$$\lim_{n\to\infty} n\mathbb{P}(S_n = \lceil un \rceil) = \frac{\Gamma(b+r)}{\Gamma(r)\Gamma(b)} u^{b-1}(1-u)^{r-1}, \tag{11.1.24}$$

which is the density of a Beta-distribution with parameters b and r. $\qquad\square$

Applications to scale-free trees. We close this section by discussing applications of Polya urn schemes to scale-free trees. We start at time $t = 2$ with two vertices of which vertex 1 has degree d_1 and vertex 2 has degree d_2. After this, we successively attach vertices to older vertices with probability proportional to the degree plus $\delta > -1$. We do not allow for self-loops, so that indeed we obtain a tree. This is a generalization of $\{\mathrm{PA}_{1,\delta}(t)\}_{t=2}^{\infty}$, in which we are are more flexible in choosing the initial graph.

We now decompose the growing tree in two trees. For $i = 1, 2$, we let $T_i(t)$ be the tree of vertices which are closer to i than to $2 - i$. Thus, the tree $T_2(t)$ consists of those vertices for which the path in the tree from the vertex to the root passes through vertex 2, and $T_1(t)$ consists of the remainder of the scale-free tree. Let $S_i(t) = |T_i(t)|$ denote the number of vertices in $T_i(t)$. Clearly, $S_1(t) + S_2(t) = t$, which is the total number of vertices of the tree at time t.

Theorem 11.3 (Tree decomposition for scale-free trees). *As $t \to \infty$,*

$$\frac{S_1(t)}{S_1(t) + S_2(t)} \xrightarrow{a.s.} U, \tag{11.1.25}$$

where U has a Beta-distribution with parameters $a = \frac{d_1+\delta}{2+\delta}$ and $b = \frac{d_2+\delta}{2+\delta}$, and

$$\mathbb{P}(S_1(t) = k) = \mathbb{E}\Big[\mathbb{P}(\mathrm{BIN}(t-1, U) = k-1)\Big]. \tag{11.1.26}$$

Theorem 11.3 implies that we can decompose a scale-free tree into two disjoint scale-free trees, each of which contains a positive proportion of the vertices. This proportion converges almost surely to a Beta-distribution with parameters $a = \frac{d_1+\delta}{2+\delta}$ and $b = \frac{d_2+\delta}{2+\delta}$, which is strictly bounded away from 0 and 1.

Proof of Theorem 11.3. The evolution of $\{S_1(t)\}_{t=2}^{\infty}$ can be viewed as a Polya urn scheme. Indeed, when $S_1(t) = s_1(t)$, then the probability of attaching the $(t+1)^{\mathrm{st}}$ vertex to $T_1(t)$ is equal to

$$\frac{(2s_1(t) + d_1 - 2) + \delta s_1(t)}{(2s_1(t) + d_1 - 2) + \delta s_1(t) + 2(s_2(t) + d_2) + \delta s_2(t)}, \tag{11.1.27}$$

since the number of vertices in $T_i(t)$ equals $S_i(t)$, while the total degree of $T_i(t)$ equals $(2S_i(t) + d_i - 2)$. We can rewrite this as

$$\frac{s_1(t) + \frac{d_1-2}{2+\delta}}{s_1(t) + s_2(t) + \frac{d_1+d_2-4}{2+\delta}}, \tag{11.1.28}$$

which is equal to (11.1.13) in the case (11.1.14) when $r_0 = b_0 = 1$ and $a_b = \frac{d_1-2}{2+\delta}, a_r = \frac{d_2-2}{2+\delta}$. Therefore, the proof of Theorem 11.3 follows directly from Theorem 11.2. $\qquad\square$

Exercise 11.3. *A uniform recursive tree is obtained by starting with a single vertex, and successively attaching the $(n+1)^{\text{st}}$ vertex to a uniformly chosen vertex in $\{1, \ldots, n\}$. Prove that for uniform recursive trees, the tree decomposition described above is such that*

$$\frac{S_1(n)}{S_1(n) + S_2(n)} \xrightarrow{a.s.} U, \tag{11.1.29}$$

where U is uniform on $[0,1]$. Use this to prove that, for each $k \in \{1, \ldots, n\}$,

$$\mathbb{P}(S_1(n) = k) = \frac{1}{n}. \tag{11.1.30}$$

We continue by discussing an application of Polya urn schemes to the relative sizes of the initial degrees. For this, we fix an integer $k \geq 2$, and only regard times $t \geq k$ at which an edge is attached to one of the k initial vertices. We work with $\{\mathrm{PA}_{m,\delta}(t)\}_{t \geq 1}$, so that we start at time $t = 1$ with one vertices with one self-loop, after which we successively attach vertices to older vertices with probability proportional to the degree plus $\delta > -1$, allowing for self-loops. The main result is as follows:

Theorem 11.4 (Relative degrees in scale-free trees). *As $t \to \infty$,*

$$\frac{D_k(t)}{D_1(t) + \cdots + D_k(t)} \xrightarrow{a.s.} B_k, \tag{11.1.31}$$

<div align="right"><code>e:PUSasconvRelDegr</code></div>

where B_k has a Beta-distribution with parameters $a = 1 + \delta$ and $b = (k-1)(2+\delta)$.

By Theorem 8.8, $B_k = \xi_k/(\xi_1 + \cdots + \xi_k)$ (where we also use that these random variables are positive almost surely). Theorem 11.4 allows to identify properties of the law of the limiting degrees.

Proof of Theorem 11.4. Denote the sequence of stopping times $\{\tau_k(n)\}_{n=2k-1}^{\infty}$, by $\tau_k(2k-1) = k-1$, and

$$\tau_k(n) = \inf\{t : D_1(t) + \cdots + D_k(t) = n\}, \tag{11.1.32}$$

i.e., $\tau_k(n)$ is the time where the total degree of vertices $1, \ldots, k$ equals n. Since $\mathbb{P}(\xi_j = 0) = 0$ for every j, $\tau_k(n) < \infty$ for every n. Moreover, since $\lim_{n \to \infty} \tau_k(n) = \infty$, we have that

$$\lim_{t \to \infty} \frac{D_k(t)}{D_1(t) + \cdots + D_k(t)} = \lim_{n \to \infty} \frac{D_k(\tau_k(n))}{D_1(\tau_k(n)) + \cdots + D_k(\tau_k(n))} = \lim_{n \to \infty} \frac{D_k(\tau_k(n))}{n}. \tag{11.1.33}$$

<div align="right"><code>e:RelDegrtau</code></div>

Now, $\{(D_k(\tau_k(n)), D_1(\tau_k(n)) + \cdots + D_{k-1}(\tau_k(n)))\}_{n=2k-1}^{\infty}$ is a Polya urn scheme, with $D_k(\tau_k(2k-1)) = 1$, and $D_1(\tau_k(2k-1)) + \cdots + D_{k-1}(\tau_k(2k-1)) = 2k-2$. The edge at time $\tau_k(n)$ is attached to vertex k with probability

$$\frac{D_k(t) + \delta}{n + k\delta}, \tag{11.1.34}$$

which are the probabilities of a Polya urn scheme in the linear weight case in (11.1.14) when $a_b = \delta, a_r = (k-1)\delta$, $b_0 = 1$, $r_0 = 2(k-1)$. Thus, the statement follows from Theorem 11.2. $\qquad \square$

11.2   Connectivity of preferential attachment models

In this section, we investigate the connectivity of $\{\mathrm{PA}_{m,\delta}(t)\}_{t \geq 1}$. We start by describing the connectivity when $m = 1$, which is special. For $m = 1$, we note that N_t, the number of connected components of $\mathrm{PA}_{1,\delta}(t)$, has distribution given by

$$N_t = I_1 + I_2 + \cdots + I_t, \tag{11.2.1}$$

where I_i is the indicator that the ith edge connects to itself, so that $\{I_i\}_{i=1}^t$ are independent indicator variables with

$$\mathbb{P}(I_i = 1) = \frac{1+\delta}{(2+\delta)(i-1)+1+\delta}. \tag{11.2.2}$$

As a result, it is not hard to see that $N_t/\log t$ converges in probability to $(1+\delta)/(2+\delta) < 1$, so that **whp** there exists a largest connected component of size at least $t/\log t$. As a result, we see that $\mathrm{PA}_{1,\delta}(t)$ is not connected, but has few connected components which are almost all quite large. We shall not elaborate more on the connectivity properties when $m = 1$, but leave the asymptotics of the number of connected components as an exercise:

Exercise 11.4. *Show that the number of connected components N_t in $\mathrm{PA}_{1,\delta}(t)$ satisfies a central limit theorem with equal asymptotic mean and variance given by*

$$\mathbb{E}[N_t] = \frac{1+\delta}{2+\delta}\log t(1+o(1)), \qquad \mathrm{Var}(N_t) = \frac{1+\delta}{2+\delta}\log t(1+o(1)). \tag{11.2.3}$$

When $m \geq 2$, then the situation is entirely different. Indeed, in this case, $\mathrm{PA}_{m,\delta}(t)$ turns out to be *connected* with high probability:

Theorem 11.5 (Connectivity of $\mathrm{PA}_{m,\delta}(t)$ for $m \geq 2$). *Fix $m \geq 2$. Then, with high probability, $\mathrm{PA}_{m,\delta}(t)$ is connected.*

Proof of Theorem 11.5. Again we let N_t denote the number of connected components of $\mathrm{PA}_{m,\delta}(t)$. We note that, $I_t = N_t - N_{t-1} = 1$ precisely when all m edges of vertex t are attached to vertex t. Thus,

$$\mathbb{P}(I_t = 1) = \prod_{e=1}^m \frac{2e-1+\delta}{(2m+\delta)t+(2e-1+\delta)}. \tag{11.2.4}$$

When $m \geq 2$, then

$$\sum_{t=2}^\infty \mathbb{P}(I_t = 1) < \infty, \tag{11.2.5}$$

so that, almost surely, $I_t = 1$ only occurs *finitely* often. As a result, since also $N_t \leq 1 + \sum_{t=2}^\infty I_t$, we have that $N_t < \infty$ almost surely. This implies that, for $m \geq 2$, $\mathrm{PA}_{m,\delta}(t)$ almost surely contains only finitely many connected components. However, $\mathrm{PA}_{m,\delta}(t)$ has a *positive* probability of being disconnected at a certain time $t \geq 2$ (see Exercise 11.5 below). We next extend the proof to show that, with probability converging to 1, $N_t = 1$. For this, we first fix $K \geq 1$ large. Then, with probability converging to 1 as $K \to \infty$, $\sum_{t=K}^\infty I_t = 0$. We condition on $\sum_{t=K}^\infty I_t = 0$, so that no new connected components are formed after time K, and we are left to prove that the vertices $1, \ldots, K$ are all connected in $\mathrm{PA}_{m,\delta}(t)$ for any $m \geq 2$. This proof proceeds in two steps. In the first step, we prove that with high probability, the number of vertices that are connected to $i \in \{1, \ldots, K\}$ is large when $t \to \infty$. In the next step, we show that this implies that in fact *all* vertices in $\{1, \ldots, K\}$ are connected to i in $\mathrm{PA}_{m,\delta}(t)$.

To prove a lower bound on the number of vertices attached to vertex $i \in \{1, \ldots, K\}$, we lower bound this number by a Polya urn scheme. We denote $b_0 = 1$ and $r_0 = i$. We recursively define the Polya urn scheme $\{(B_n, R_n)\}_{n=1}^\infty$ as follows. We draw a ball, where the color of the ball is blue with probability proportional to the number of the blue balls. If we draw a blue ball, we replace it with an extra blue ball. If we draw a red ball, then we replace it with an extra red ball. Then, we claim that $C_i(t)$, the number of vertices connected to vertex i at time t, is bounded below by B_{t-i}. We shall prove this by induction on $t \geq i$. At time i, the number of vertices connected to vertex i is at least i (namely, vertex i itself), and this proves the claim since $B_0 = b_0 = 1$. Suppose that $C_i(t) \geq B_{t-i}$. Then, the total weight

of the vertices connected to vertex i at time t is *at least* $B_{t-i}(2m + \delta)$, while the total weight is equal to $t(2m+\delta)+(1+\delta) \leq (t+1)(2m+\delta) = (2m+\delta)(B_{t-i} + R_{t-i})$, since $B_{t-i} + R_{t-i} = (t-i)+b_0+r_0 = t+1$. Now, we say that at time $t + 1$, we draw a red ball when the first edge of vertex $t + 1$ is attached to a vertex which is connected to vertex i at time t. If this happens, then indeed vertex $t+1$ *will* be connected to vertex i, and $C_i(t)$ is increased by (at least) one. Thus, this advances the induction hypothesis, and proves the claim.

Now, by Theorem 11.2, we have that $C_i(t) \geq B_{t-i}$, and $B_t/t \xrightarrow{a.s.} U$, where U has a Beta-distribution with parameters $a = 1$ and $b = i$. Since $\mathbb{P}(U = 0) = 0$, we have that, with high probability, $C_i(t) \geq \varepsilon(t-i)$ for all t sufficiently large when $\varepsilon > 0$ is small. We conclude that, as $t \to \infty$,

$$\mathbb{P}(\liminf_{t\to\infty} C_i(t)/t \geq \varepsilon \ \ \forall i = 1,\dots,K) = 1 - o(1) \tag{11.2.6}$$

as $\varepsilon \downarrow 0$. We claim that when $C_i(t)/t \geq \varepsilon$ and $C_j(t)/t \geq \varepsilon$ for $i \neq j$, then, with high probability, i is connected to j in $\mathrm{PA}_{m,\delta}(2t)$. Indeed, let $l \in \{t+1,\dots,2t\}$. Then, when the first edge of vertex l is attached to a vertex connected to vertex i, and the second to a vertex connected to vertex j, then i and j will be connected in $\mathrm{PA}_{m,\delta}(2t)$. In this case, we say that l is a *t-connector* for vertices i and j. Independently of the attachment of the edges of vertices $t+1,\dots,l-1$, and conditionally on $\mathrm{PA}_{m,\delta}(2t)$, the probability that l is *t-connector* for vertices i and j is at least

$$\frac{C_i(t)(2m+\delta)}{l(2m+\delta)+1+\delta}\frac{C_j(t)(2m+\delta)}{l(2m+\delta)+2+\delta} \geq \frac{C_i(t)C_j(t)}{(2t+1)^2}. \tag{11.2.7}$$

When $C_i(t)/t \geq \varepsilon$ and $C_j(t)/t \geq \varepsilon$, this is at least $\varepsilon^2/16$. Thus, the probability that there is no *t-connector* for vertices i and j is at most

$$\left(1 - \frac{\varepsilon^2}{16}\right)^t \leq e^{-\frac{\varepsilon^2}{16}t}. \tag{11.2.8}$$

This tends to 0 exponentially when $t \to \infty$, so that we obtain that the probability that the probability that there exists $i, j \in \{1,\dots,K\}$ that are not connected in $\mathrm{PA}_{m,\delta}(2t)$ tends to 0 as $t \to \infty$. This proves that $\mathrm{PA}_{m,\delta}(2t)$ is connected with high probability for t large, which implies Theorem 11.5. $\square$

Exercise 11.5. *Compute the probability that $\{\mathrm{PA}_{m,\delta}(t)\}_{t=1}^{\infty}$ is connected for all times $t \geq 1$, and show that this probability is in $(0,1)$.*

11.2.1   Small-world effect in preferential attachment models

In the next sections, we shall investigate distances in preferential attachment models. While the results are far from being as complete as the results for inhomogeneous random, graphs or the configuration model discussed in Chapter 9 and 10, respectively, the available results are all consistent with the fact that distances in preferential attachment models have the same asymptotics as distances in the configuration model with the same degree sequence. This suggest a strong form of *universality*, which is interesting in its own right. For distances in preferential attachment models, we shall focus on the *diameter*, since results on the distances between a uniform pair of vertices have not yet been obtained. By Theorem 11.5, $\mathrm{PA}_{m,\delta}(t)$ is connected with high probability for $m \geq 2$. When $m = 1$, this is not true for $\mathrm{PA}_{m,\delta}(t)$, but $\mathrm{PA}_{1,\delta}^{(b)}(t)$ is, by definition, connected. Thus, we shall define the diameter of $\mathrm{PA}_{m,\delta}(t)$ to be the maximum graph distance between any pair of vertices that lie in the same connected component.

The results on distances are organized as follows. In Section 11.3, we state and prove logarithmic bounds on the diameter of $\mathrm{PA}_{m,\delta}(t)$ when $\delta \geq 0$ and when $m = 1$. In Section 11.4, we prove doubly logarithmic bounds on the diameter of $\mathrm{PA}_{m,\delta}(t)$ when $\delta < 0$ and $m \geq 2$.

11.3 Small-world effect in PA models: logarithmic bounds

In this section, we prove logarithmic bounds on distances in $PA_{m,\delta}(t)$. In Section 11.3.1, we investigate the case where $m = 1$, in which case $PA_{1,\delta}(t)$ consists of a collection of trees. The main results in this case are proved in [180]. In Section 11.3, we prove logarithmic bound on the diameter in $PA_{m,\delta}^{(b)}(t)$ when $m \geq 2$.

11.3.1 Logarithmic distances in preferential attachment trees

When $m = 1$, one of the main results in [180] is as follows:

Theorem 11.6 (The diameter of scale-free trees). *Fix $m = 1$ and $\delta > -1$. Let γ be the non-negative solution of*

$$\gamma + (1 + \delta)(1 + \log \gamma) = 0. \tag{11.3.1}$$

Then, for every $\varepsilon > 0$ and, with high probability, the diameter of $PA_{1,\delta}(t)$ is in between $(\frac{2(1+\delta)}{(2+\delta)\gamma} - \varepsilon) \log t$ and $(\frac{2(1+\delta)}{(2+\delta)\gamma} + \varepsilon) \log t$.

In order to prove Theorem 11.6, it will be convenient to first consider a variation of the $PA_{1,\delta}(t)$ model where there are no self-loops. Recall the growth rule in (8.1.2) and the discussion below it. The main result for $\{PA_{1,\delta}^{(b)}(t)\}_{t=1}^{\infty}$ is as follows:

Theorem 11.7 (The diameter of scale-free trees $PA_{1,\delta}^{(b)}(t)$). *Fix $m = 1$ and $\delta > -1$, and let γ be the solution of (11.3.1). Then, for every $\varepsilon > 0$ and with high probability, the diameter of $PA_{1,\delta}^{(b)}(t)$ is in between $(\frac{2(1+\delta)}{(2+\delta)\gamma} - \varepsilon) \log t$ and $(\frac{2(1+\delta)}{(2+\delta)\gamma} + \varepsilon) \log t$.*

Theorem 11.6 implies that for $m = 1$, the diameter grows proportionally to the logarithm of the graph size. Scale-free trees have received substantial attention in the literature, we refer to [45, 180] and the references therein.

The proof of Theorem 11.6 relies on the fact that the case when $m = 1$ is special. In this case, $PA_{1,\delta}(t)$ consists of a collection of trees each with precisely one self-loop. There is a close analogy between these trees and so-called *uniform recursive trees*. In uniform recursive trees, we grow a tree such that at time 1, we have a unique vertex called the root, with label 1. At time t, we add a vertex and connect it to a uniformly chosen vertex in the tree. See [192] for a survey of recursive trees.

A variant of a uniform recursive tree is the case where the probability that a newly added vertex is attached to a vertex is proportional to the degree of the vertices (and, for the root, the degree of the root plus one). This process is called a *random plane-oriented recursive tree*. For a uniform recursive tree of size t, it is proved by Pittel in [180] that the maximal distance between the root and any other vertex is with high probability equal to $\frac{1}{2\gamma} \log t(1 + o(1))$, where γ satisfies (11.3.1) with $\delta = 0$. It is not hard to see that this implies that the maximal graph distance between any two vertices in the uniform recursive tree is equal to $\frac{1}{\gamma} \log t(1 + o(1))$. We shall make use of similar arguments below to prove Theorem 11.6.

In order to prove Theorem 11.7, we make use of a result on the *height* of scale-free trees, which is proved by Pittel in [180]. The height of a tree is the maximal distance between any of the vertices of the tree to the root. For a tree T, we denote the height of T by $H(T)$. Then the asymptotics of the height of scale-free trees is as follows:

Theorem 11.8 (The height of scale-free trees). *Fix $m = 1$ and $\delta > -1$, and let γ be the solution of (11.3.1). Then,*

$$\frac{H\big(\mathrm{PA}^{(b)}_{1,\delta}(t)\big)}{\log t} \xrightarrow{a.s.} \frac{(1+\delta)}{(2+\delta)\gamma}. \tag{11.3.2}$$

We shall only prove the upper bound in Theorem 11.8. The lower bound, which is somewhat more delicate, is proved in [180] using a result by Kingman in [134]. We remark that the almost sure limit of the height in Theorem 11.8 does not depend on the precise starting configuration of the graph $\mathrm{PA}^{(b)}_{1,\delta}(2)$. We shall crucially rely on this fact.

In the proof of the upper bound, we shall make use of the following result which computes the probability mass function of the distance between vertex v_s and the root v_1. Before stating the result, we need some more notation. We write $t \longrightarrow s$ when in $\{\mathrm{PA}^{(b)}_{1,\delta}(t)\}_{t \geq 1}$ one of the edges of vertex v_t is connected to vertex v_s. Note that for this to happen, we need that $t > s$. For $s_1 = t > s_2 > \cdots > s_k = 1$, and denoting $\vec{s}_k = (s_1, s_2, \ldots, s_k)$, we write

$$E_{\vec{s}_k} = \bigcap_{i=1}^{k-1} \{s_i \longrightarrow s_{i+1}\}. \tag{11.3.3}$$

For a configuration of $\mathrm{PA}^{(b)}_{1,\delta}(t)$, we let $\mathrm{dist}(v_t, v_1)$ denote the unique value of k such that $t = s_1 \longrightarrow s_2 \longrightarrow \cdots \longrightarrow s_{k-1} \longrightarrow s_k = 1$. Then the probability mass function of $\mathrm{dist}(v_t, v_1)$ can be identified as follows:

Proposition 11.9 (The distribution of $\mathrm{dist}(v_t, v_1)$ in $\mathrm{PA}^{(b)}_{1,\delta}(t)$). *Fix $m = 1$ and $\delta > -1$. Then,*

$$\mathbb{P}(dist(v_t, v_1) = k) = \Big(\frac{1+\delta}{2+\delta}\Big)^{k-1} \frac{\Gamma(s + \frac{1}{2+\delta})}{\Gamma(1 + \frac{1}{2+\delta})\Gamma(s+1)} \sum_{\vec{s}_k} \prod_{i=2}^{k-1} \frac{1}{s_i}, \tag{11.3.4}$$

<div align="right">e:probdst1</div>

where the sum is over ordered vectors $\vec{s}_k$ of length k, for which $s_1 = t$ and $s_k = 1$.

Proof of Proposition 11.7. We note that since the path between vertex v_t and v_1 is *unique*, we have

$$\mathbb{P}(\mathrm{dist}(v_t, v_1) = k) = \sum_{\vec{s}_k} \mathbb{P}\Big(\bigcap_{i=1}^{k-1} \{s_i \longrightarrow s_{i+1}\}\Big), \tag{11.3.5}$$

<div align="right">e:probdst1a</div>

where the sum is over all *ordered* vectors $\vec{s}_k$ of length k, for which $s_1 = t$ and $s_k = 1$.

We claim that the events $\{s_i \longrightarrow s_{i+1}\}$ are *independent*, i.e.,

$$\mathbb{P}\Big(\bigcap_{i=1}^{k-1} \{s_i \longrightarrow s_{i+1}\}\Big) = \prod_{i=1}^{k-1} \mathbb{P}(s_i \longrightarrow s_{i+1}). \tag{11.3.6}$$

<div align="right">e:indepattach</div>

We prove this independence by induction. First note that

$$\mathbb{P}\Big(\bigcap_{i=1}^{k-1} \{s_i \longrightarrow s_{i+1}\}\Big) = \mathbb{E}\Big[\mathbb{P}\Big(\bigcap_{i=1}^{k-1} \{s_i \longrightarrow s_{i+1}\}\Big| \mathrm{PA}^{(b)}_{1,\delta}(s_1 - 1)\Big)\Big]$$

$$= \mathbb{E}\Big[\mathbb{1}_{\{\bigcap_{i=2}^{k-1}\{s_i \longrightarrow s_{i+1}\}\}} \mathbb{P}\big(s_1 \longrightarrow s_2 | \mathrm{PA}^{(b)}_{1,\delta}(s_1 - 1)\big)\Big], \tag{11.3.7}$$

since the event $\bigcap_{i=2}^{l-1}\{s_i \longrightarrow s_{i+1}\}$ is measurable with respect to $\mathrm{PA}^{(b)}_{1,\delta}(s_1 - 1)$. Furthermore, from (8.1.2),

$$\mathbb{P}\big(s_1 \longrightarrow s_2 | \mathrm{PA}^{(b)}_{1,\delta}(s_1 - 1)\big) = \frac{D_{s_2}(s_1 - 1) + \delta}{(2+\delta)(s_1 - 1)}. \tag{11.3.8}$$

In particular, we have that

$$\mathbb{P}(s_1 \longrightarrow s_2) = \mathbb{E}\Big[\frac{D_{s_2}(s_1 - 1) + \delta}{(2 + \delta)(s_1 - 1)}\Big]. \tag{11.3.9}$$

Therefore,

$$\mathbb{P}\Big(\bigcap_{i=1}^{k-1}\{s_i \longrightarrow s_{i+1}\}\Big) = \mathbb{E}\Big[\mathbb{1}_{\{\bigcap_{i=2}^{k-1}\{s_i \longrightarrow s_{i+1}\}\}}\frac{D_{s_2}(s_1 - 1) + \delta}{(2 + \delta)(s_1 - 1)}\Big]$$

$$= \mathbb{P}\Big(\bigcap_{i=2}^{k-1}\{s_i \longrightarrow s_{i+1}\}\Big)\mathbb{E}\Big[\frac{D_{s_2}(s_1 - 1) + \delta}{(2 + \delta)(s_1 - 1)}\Big], \tag{11.3.10}$$

since the random variable $D_{s_2}(s_1 - 1)$ only depends on how many edges are connected to s_2 after time s_2, which is independent of the event $\bigcap_{i=2}^{l-1}\{s_i \longrightarrow s_{i+1}\}$, which depends on the attachment of the edges *up to and including* time s_2 only. We conclude that

$$\mathbb{P}\Big(\bigcap_{i=1}^{k-1}\{s_i \longrightarrow s_{i+1}\}\Big) = \mathbb{P}(\{s_1 \longrightarrow s_2\})\mathbb{P}\Big(\bigcap_{i=2}^{k-1}\{s_i \longrightarrow s_{i+1}\}\Big). \tag{11.3.11}$$

Iteration leads to the independence claim in (11.3.6).

Combining (11.3.5) with (11.3.6) and (11.3.9), we obtain that

$$\mathbb{P}(\mathrm{dist}(v_t, v_1) = k) = \sum_{\vec{s}_k}\prod_{i=1}^{k-1}\mathbb{E}\Big[\frac{D_{s_{i+1}}(s_i - 1) + \delta}{(2 + \delta)(s_i - 1)}\Big]. \tag{11.3.12}$$

By (8.2.11),

$$\mathbb{E}\Big[\frac{D_i(t) + \delta}{(2 + \delta)t}\Big] = (1 + \delta)\frac{\Gamma(t + \frac{1}{2+\delta})\Gamma(i)}{(2 + \delta)t\Gamma(t)\Gamma(i + \frac{1}{2+\delta})} = \frac{1 + \delta}{2 + \delta}\frac{\Gamma(t + \frac{1}{2+\delta})\Gamma(i)}{\Gamma(t + 1)\Gamma(i + \frac{1}{2+\delta})}, \tag{11.3.13}$$

so that

$$\mathbb{P}(\mathrm{dist}(v_t, v_1) = k) = \sum_{\vec{s}_k}\Big(\frac{1 + \delta}{2 + \delta}\Big)^{k-1}\prod_{i=1}^{k-1}\frac{\Gamma(s_i + \frac{1}{2+\delta})\Gamma(s_{i+1})}{\Gamma(s_i + 1)\Gamma(s_{i+1} + \frac{1}{2+\delta})}$$

$$= \Big(\frac{1 + \delta}{2 + \delta}\Big)^{k-1}\sum_{\vec{s}_k}\frac{\Gamma(s_1 + \frac{1}{2+\delta})\Gamma(s_k)}{\Gamma(s_k + \frac{1}{2+\delta})\Gamma(s_1)}\prod_{i=1}^{k-1}\frac{1}{s_i}$$

$$= \Big(\frac{1 + \delta}{2 + \delta}\Big)^{k-1}\frac{\Gamma(t + \frac{1}{2+\delta})}{\Gamma(1 + \frac{1}{2+\delta})\Gamma(t + 1)}\sum_{\vec{s}_k}\prod_{i=2}^{k-1}\frac{1}{s_i} \tag{11.3.14}$$

where we recall that the sum is over ordered vectors $\vec{s}_k$ of length k, for which $s_1 = t$ and $s_k = 1$. This completes the proof of Proposition 11.7.

Proof of the upper bound in Theorem 11.8. We first use Proposition 11.7 to prove that, almost surely,

$$\limsup_{t \to \infty}\frac{\mathrm{dist}(v_t, v_1)}{\log t} \leq \frac{(1 + \delta)}{(2 + \delta)\gamma}. \tag{11.3.15}$$

To see (11.3.15), by (11.3.4) and by symmetry,

$$\mathbb{P}(\mathrm{dist}(v_t, v_1) = k) = \Big(\frac{1 + \delta}{2 + \delta}\Big)^{k-1}\frac{\Gamma(t + \frac{1}{2+\delta})}{\Gamma(1 + \frac{1}{2+\delta})\Gamma(t + 1)}\sum_{\vec{t}_{k-2}}^{*}\frac{1}{(k - 2)!}\prod_{i=1}^{k-2}\frac{1}{t_i}, \tag{11.3.16}$$

where the sum now is over all vectors $\vec{t}_{k-2} = (t_1, \ldots, t_{k-2})$ with $1 < t_i < t$ with *distinct* coordinates. We can upper bound this sum by leaving out the restriction that the coordinates of $\vec{t}_{k-2}$ are distinct, so that

$$\mathbb{P}(\text{dist}(v_t, v_1) = k) \leq \left(\frac{1+\delta}{2+\delta}\right)^{k-1} \frac{\Gamma(t + \frac{1}{2+\delta})}{\Gamma(1 + \frac{1}{2+\delta})\Gamma(t+1)} \frac{1}{(k-2)!} \left(\sum_{s=2}^{t-1} \frac{1}{s}\right)^{k-2}. \tag{11.3.17}$$

Since $x \mapsto 1/x$ is monotonically decreasing, we have that

$$\sum_{s=2}^{t-1} \frac{1}{s} \leq \int_1^t \frac{1}{x} dx = \log t. \tag{11.3.18}$$

Also, we use (8.2.7) to bound, for some constant $C_\delta > 0$,

$$\mathbb{P}(\text{dist}(v_t, v_1) = k) \leq C_\delta t^{-\frac{1+\delta}{2+\delta}} \frac{\left(\frac{1+\delta}{2+\delta}\log t\right)^{k-2}}{(k-2)!}. \tag{11.3.19}$$

The solution γ to (11.3.1) is strictly in between $(0, e^{-1})$, so that, for any $k \geq \frac{(1+\delta)}{(2+\delta)\gamma} \log t + 2$, we have that

$$\frac{\frac{1+\delta}{2+\delta}\log t}{k} \leq \gamma, \tag{11.3.20}$$

and we conclude that, for $k \geq \frac{(1+\delta)}{(2+\delta)\gamma} \log t + 1$,

$$\mathbb{P}(\text{dist}(v_t, v_1) > k) \leq C_\delta t^{-\frac{1+\delta}{2+\delta}} \frac{\left(\frac{1+\delta}{2+\delta}\log t\right)^{k-1}}{(k-1)!} \sum_{l=0}^{\infty} \frac{\left(\frac{1+\delta}{2+\delta}\log t\right)^l (k-1)!}{(k+l-1)!}$$

$$\leq C_\delta t^{-\frac{1+\delta}{2+\delta}} \frac{\left(\frac{1+\delta}{2+\delta}\log t\right)^{k-1}}{(k-1)!} \sum_{l=0}^{\infty} \gamma^l = \frac{C_\delta}{1-\gamma} t^{-\frac{1+\delta}{2+\delta}} \frac{\left(\frac{1+\delta}{2+\delta}\log t\right)^{k-1}}{(k-1)!}. \tag{11.3.21}$$

Now we fix $\varepsilon > 0$ and we take $k_t = \frac{(1+\delta)}{(2+\delta)}(\frac{1}{\gamma} + \varepsilon)\log t + 1$. We use

$$k! \geq k^k e^{-k}, \tag{11.3.22}$$

to arrive at

$$\mathbb{P}(\text{dist}(v_t, v_1) > k_t) \leq \frac{C_\delta}{1-\gamma} t^{-\frac{1+\delta}{2+\delta}} \left(\frac{e(1+\delta)\log t}{(k_t-1)}\right)^{k_t-1} = \frac{C_\delta}{1-\gamma} t^{-\frac{1+\delta}{2+\delta}} \left(\frac{e\gamma}{1+\gamma\varepsilon}\right)^{\frac{(1+\delta)}{(2+\delta)}(\frac{1}{\gamma}+\varepsilon)\log t}$$

$$\leq \frac{C_\delta}{1-\gamma} t^{-\frac{1+\delta}{2+\delta}} \left(e\gamma\right)^{\frac{(1+\delta)}{(2+\delta)}(\frac{1}{\gamma}+\varepsilon)\log t}$$

$$= \frac{C_\delta}{1-\gamma} t^{-\frac{1+\delta}{2+\delta}\left(1-(\frac{1}{\gamma}+\varepsilon)[1+\log\gamma]\right)} = \frac{C_\delta}{1-\gamma} t^{-p}, \tag{11.3.23}$$

where

$$p = \frac{1+\delta}{2+\delta}\left(1 - (\frac{1}{\gamma} + \varepsilon)[1 + \log\gamma]\right). \tag{11.3.24}$$

Since $\gamma < e^{-1}$, $1 + \log\gamma < 0$, so that

$$p > \frac{1+\delta}{2+\delta}\left(1 - \frac{1}{\gamma}[1 + \log\gamma]\right) = \frac{1+\delta}{2+\delta}\left(1 + \frac{1}{1+\delta}\right) = 1, \tag{11.3.25}$$

where the equality follows from (11.3.1). Thus, we conclude that, for $k_t = \frac{(1+\delta)}{(2+\delta)}(\frac{1}{\gamma} + \varepsilon)\log t + 1$ with $\varepsilon >$, we have

$$\mathbb{P}(\mathrm{dist}(v_t, v_1) > k_t) = O(t^{-p}), \tag{11.3.26}$$

for some $p > 1$. As a result, by the Borel-Cantelli Lemma, the event $\{\mathrm{dist}(v_t, v_1) > k_t\}$ occurs only finitely often, and we conclude that (11.3.15) holds. We finally complete the proof of Theorem 11.8. For this, we note that

$$H(\mathrm{PA}_{1,\delta}^{(b)}(t)) = \max_{s=1}^{t} \mathrm{dist}(v_s, v_1), \tag{11.3.27}$$

so that when $\{\mathrm{dist}(v_t, v_1) > k_t\}$ occurs only finitely often, we also have that $\{H(\mathrm{PA}_{1,\delta}^{(b)}(t)) > k_t\}$ occurs only finitely often. Since $\varepsilon > 0$ was arbitrary, we obtain Theorem 11.8. $\qquad\square$

We next prove Theorems 11.6–11.7:

Proof of Theorems 11.6 and 11.7. We first prove the upper bound on the diameter of $\mathrm{PA}_{1,\delta}^{(b)}(t)$ in Theorem 11.7, for which we use that

$$\mathrm{diam}(\mathrm{PA}_{1,\delta}^{(b)}(t)) \leq 2H(\mathrm{PA}_{1,\delta}^{(b)}(t)). \tag{11.3.28}$$

Equation (11.3.28) together with the upper bound in Theorem 11.8 imply that

$$\limsup_{t\to\infty} \frac{\mathrm{diam}(\mathrm{PA}_{1,\delta}^{(b)}(t))}{\log t} \leq \frac{2(1+\delta)}{\gamma(2+\delta)}. \tag{11.3.29}$$

For the lower bound, we make use of the lower bound on the diameter of $\mathrm{PA}_{1,\delta}^{(b)}(t)$ in Theorem 11.7, we make use of the decomposition of scale-free trees in Theorem 11.3, which states that the scale-free tree $\mathrm{PA}_{1,\delta}^{(b)}(t)$ can be decomposed into two scale-free trees, having a similar distribution as copies $\mathrm{PA}_{1,\delta}^{(b1)}(S_1(t))$ and $\mathrm{PA}_{1,\delta}^{(b2)}(t - S_1(t))$, where $\{\mathrm{PA}_{1,\delta}^{(b1)}(t)\}_{t=1}^{\infty}$ and $\{\mathrm{PA}_{1,\delta}^{(b2)}(t)\}_{t=1}^{\infty}$ are independent scale-free trees. By this tree decomposition, we have

$$\mathrm{diam}(\mathrm{PA}_{1,\delta}^{(b)}(t)) \geq H(\mathrm{PA}_{1,\delta}^{(b1)}(S_1(t))) + H(\mathrm{PA}_{1,\delta}^{(b2)}(t - S_1(t))). \tag{11.3.30}$$

The two trees $\{\mathrm{PA}_{1,\delta}^{(b1)}(t)\}_{t=1}^{\infty}$ and $\{\mathrm{PA}_{1,\delta}^{(b2)}(t)\}_{t=1}^{\infty}$ are not *exactly* equal in distribution to $\mathrm{PA}_{1,\delta}^{(b)}(t)$, because the initial degree of the starting vertices at time $t = 2$ is different. However, the precise almost sure scaling in Theorem 11.3 does not depend sensitively on d_1 and d_2, and also the height of the scale-free tree in Theorem 11.8 does not depend on the starting graphs $\mathrm{PA}_{1,\delta}^{(b1)}(2)$ and $\mathrm{PA}_{1,\delta}^{(b1)}(2)$ (see the remark below Theorem 11.8). Since $S_1(t)/t \xrightarrow{a.s.} U$, with U having a Beta-distribution, $H(\mathrm{PA}_{1,\delta}^{(b1)}(S_1(t)))/\log t \xrightarrow{a.s.} \frac{(1+\delta)}{(2+\delta)\gamma}$ and $H(\mathrm{PA}_{1,\delta}^{(b2)}(t - S_1(t)))/\log t \xrightarrow{a.s.} \frac{(1+\delta)}{(2+\delta)\gamma}$, we conclude that

$$\limsup_{t\to\infty} \frac{\mathrm{diam}(\mathrm{PA}_{1,\delta}^{(b)}(t))}{\log t} \geq \frac{2(1+\delta)}{(2+\delta)\gamma}. \tag{11.3.31}$$

Combining (11.3.29) and (11.3.31) proves Theorem 11.7.

To prove Theorem 11.6, we note that the connected components of $\mathrm{PA}_{1,\delta}(t)$ are similar in distribution to single scale-free tree $\mathrm{PA}_{1,\delta}^{(b1)}(t_1), \ldots, \mathrm{PA}_{1,\delta}^{(bN_t)}(t_{N_t})$, apart from the initial degree of the root. Since $N_t/\log t \xrightarrow{d} (1 + \delta)/(2 + \delta)$ (recall Exercise 11.4), we have that, with high probability, the largest connected component has size at least $\varepsilon t/\log t$. Since

$$\log\left(\varepsilon t/\log t\right) = \log t(1 + o(1)), \tag{11.3.32}$$

the result follows along the same lines as in the proof of Theorem 11.7. $\qquad\square$

We complete this section by proving that the logarithmic bounds on the diameter for $\{\text{PA}_{1,\delta}^{(b)}(t)\}_{t=1}^{\infty}$ imply similar upper bounds for $\{\text{PA}_{m,\delta}^{(b)}(t)\}_{t=1}^{\infty}$:

Theorem 11.10 (A $\log t$ upper bound for the diameter in $\{\text{PA}_{m,\delta}^{(b)}(t)\}_{t=1}^{\infty}$). *Fix $m \geq 1$ and $\delta > -m$. Then, with γ the solution of* (11.3.1),

$$\mathbb{P}\Big(\text{diam}(\text{PA}_{m,\delta}^{(b)}(t)) > \big(\frac{2(m+\delta)}{(2m+\delta)\gamma} + \varepsilon\big)\log t\Big) = o(1), \quad t \to \infty. \tag{11.3.33}$$

Proof of Theorem 11.10. Since $\{\text{PA}_{m,\delta}^{(b)}(t)\}_{t=1}^{\infty}$ is obtained from $\{\text{PA}_{1,\delta/m}^{(b)}(mt)\}_{t=1}^{\infty}$ by collapsing m successive vertices, we have that $\text{diam}(\text{PA}_{m,\delta}^{(b)}(t)) \preceq \text{diam}(\text{PA}_{1,\delta/m}^{(b)}(mt))$, and the result follows from Theorem 11.7. $\qquad\square$

The result in Theorem 11.10 for $\{\text{PA}_{m,\delta}^{(b)}(t)\}_{t=1}^{\infty}$ does not follow in the same way for $\{\text{PA}_{m,\delta}(t)\}_{t=1}^{\infty}$. The reason for this is that $\{\text{PA}_{1,\delta}(t)\}_{t=1}^{\infty}$ is disconnected, while by Theorem 11.5, $\{\text{PA}_{m,\delta}(t)\}_{t=1}^{\infty}$ is connected. This immediately implies that $\text{diam}(\text{PA}_{m,\delta}(t)) \leq N_t \text{diam}(\text{PA}_{m,\delta/m}(mt))$, which is of order $(\log t)^2$ instead of $\log t$. We strongly believe that $\text{diam}(\text{PA}_{m,\delta}(t))/\log t \xrightarrow{a.s.} \frac{2(m+\delta)}{(2m+\delta)\gamma}$, but we lack a rigorous proof for this fact.

11.3.2   Logarithmic bounds on distances when $m \geq 2$

In this section, we investigate logarithmic bounds on the distances in $\text{PA}_{m,\delta}(t)$ when $m \geq 2$.

Theorem 11.11 (Diameter of $\text{PA}_{m,\delta}(t)$). *Fix $m \geq 2$, $\delta \geq 0$ and let $\varepsilon > 0$ be fixed. Then, with high probability, the largest distance between two vertices in the same component of $\text{PA}_{m,\delta}(t)$ is bounded below by $(1-\varepsilon)\frac{\log t}{\log \log t}$, i.e.,*

$$\text{diam}(\text{PA}_{m,\delta}(t)) \geq (1-\varepsilon)\frac{\log t}{\log \log t}. \tag{11.3.34}$$

The proof of Theorem 11.11 proceeds in two steps. The first step is to use the collapsing of vertices to obtain

$$\text{PA}_{m,\delta}(t)$$

from $\text{PA}_{1,\delta/m}(mt)$, so that we fix $m = 1$ for the time being. For the lower bound, it is proved that the probability that vertex j sends it (unique) edge to vertex i is $O\big((ij)^{-1/2}\big)$. This suggests that the model is related to the Chung-Lu model with $w_i = ci^{-1/2}n^{-1/4}$ and an appropriately chosen constant c. However, the edge statuses are *not* independent. A second result says that the probability that j and k both send their edges to i is $O\big((ij)^{-1/2}(ik)^{-1/2}\big)$, so that a weak form of independence is valid.

We finally state the complementary result from [44],[BolRio04b] which states that, when $\delta = 0$, the diameter of $\text{PA}_{m,\delta}(t)$ is asymptotically equal to $\frac{\log t}{\log \log t}(1 + o(1))$:

Theorem 11.12 (Diameter of $\mathrm{PA}_{m,0}(t)$). *Fix $m \geq 2$, $\delta = 0$ and let $\varepsilon > 0$ be fixed. Then, with high probability, the largest distance between two vertices in the same component in $\mathrm{PA}_{m,\delta}(t)$ is bounded below by*

$$\mathrm{diam}(\mathrm{PA}_{m,0}(t)) \leq (1 + \varepsilon)\frac{\log t}{\log \log t}. \tag{11.3.35}$$

The proof of Theorem 11.12 is given in [44]$^{\text{BolRio04b}}$ and is considerably more involved than the one of Theorem 11.11, and will be omitted here. We now give the proof of Theorem 11.11.

Proof of Theorem 11.11. Denote by

$$\{g(t,j) = s\}, \quad 1 \leq j \leq m, \tag{11.3.36}$$

the event that at time t the jth edge of vertex t is attached to the earlier vertex s. For the models (a) and (b), this event means that in $\{\mathrm{PA}_{1,\delta}(mt)\}$ the edge from vertex $m(t-1) + j$ is attached to one of the vertices $m(s-1)+1, \ldots, ms$. It is a direct consequence of the definition of PA-models that the event (11.3.36) increases the preference for vertex s, and hence decreases (in a relative way) the preference for the vertices u, $1 \leq u \leq t$, $u \neq s$. It should be intuitively clear that another way of expressing this effect is to say that, for different $s_1 \neq s_2$, the events $\{g(t_1, j_1) = s_1\}$ and $\{g(t_2, j_2) = s_2\}$ are negatively correlated. In order to state such a result, we introduce some notation. For integer $n_s \geq 1$, we denote by

$$E_s = \bigcap_{i=1}^{n_s} \{g(t_i^{(s)}, j_i^{(s)}) = s\}, \tag{11.3.37}$$

the event that at time t_i the j_i^{th} edge of vertex t_i is attached to the earlier vertex s. We will start by proving that for each $k \geq 1$ and all possible choices of $t_i^{(s)}, j_i^{(s)}$, the events E_s, for different s, are negatively correlated:

Lemma 11.13. *For distinct $s_1, s_2, \ldots, s_k$,*

$$\mathbb{P}\Big(\bigcap_{i=1}^{k} E_{s_i} \Big) \leq \prod_{i=1}^{k} \mathbb{P}(E_{s_i}). \tag{11.3.38}$$

Proof. We will use induction on the largest edge number present in the events E_s. Here, for an event $\{g(t,j) = s\}$, we let the edge number be $m(t-1) + j$, which is the order of the edge when we consider the edges as being attached in sequence. The induction hypothesis is that (11.3.38) holds for all k and all choices of $t_i^{(s)}, j_i^{(s)}$ such that $\max_{i,s} m(t_i^{(s)} - 1) + j_i^{(s)} \leq e$, where induction is performed with respect to e. We now complete the induction argument. To initialize the induction, we note that for $e = 1$, the induction hypothesis holds trivially, since $\bigcap_{i=1}^{k} E_{s_i}$ can be empty or consist of exactly one event, and in the latter case there is nothing to prove. This initializes the induction.

To advance the induction, we assume that (11.3.38) holds for all k and all choices of $t_i^{(s)}, j_i^{(s)}$ such that $\max_{i,s} m(t_i^{(s)} - 1) + j_i^{(s)} \leq e - 1$, and we extend it to all k and all choices of $t_i^{(s)}, j_i^{(s)}$ such that $\max_{i,s} m(t_i^{(s)} - 1) + j_i^{(s)} \leq e$. Clearly, for k and $t_i^{(s)}, j_i^{(s)}$ such that $\max_{i,s} m(t_i^{(s)} - 1) + j_i^{(s)} \leq e - 1$, the bound follows from the induction hypothesis, so we may restrict attention to the case that $\max_{i,s} m(t_i^{(s)} - 1) + j_i^{(s)} = e$. We note that there is a unique choice of t, j such that $m(t-1) + j = e$. In this case, there are again two possibilities. Either there is exactly one choice of s and $t_i^{(s)}, j_i^{(s)}$ such that $t_i^{(s)} = t, j_i^{(s)} = j$, or there are at least *two* of such choices. In the latter case, we immediately have that $\bigcap_{s=1}^{k} E_s = \varnothing$, since the eth

edge can only be connected to a *unique* vertex. Hence, there is nothing to prove. Thus, we are left to investigate the case where there exists unique s and $t_i^{(s)}, j_i^{(s)}$ such that $t_i^{(s)} = t, j_i^{(s)} = j$. Denote by

$$E_s' = \bigcap_{i=1:(t_i^{(s)}, j_i^{(s)}) \neq (t,j)}^{n_s} \{g(t_i^{(s)}, j_i^{(s)}) = s\}, \tag{11.3.39}$$

the restriction of E_s to the *other* edges. Then we can write

$$\bigcap_{i=1}^{k} E_{s_i} = \{g(t,j) = s\} \cap E_s' \cap \bigcap_{i=1:s_i \neq s}^{k} E_{s_i}. \tag{11.3.40}$$

By construction, all the edge numbers of the events in $E_s' \cap \bigcap_{i=1:s_i \neq s}^{k} E_{s_i}$ are at most $e - 1$. Thus, we obtain

$$\mathbb{P}\Big(\bigcap_{i=1}^{k} E_{s_i}\Big) \leq \mathbb{E}\Big[\mathbb{1}_{E_s' \cap \bigcap_{i=1:s_i \neq s}^{k} E_{s_i}} \mathbb{P}_{e-1}(g(t,j) = s)\Big], \tag{11.3.41}$$

where $\mathbb{P}_{e-1}$ denotes the conditional probability given the edge attachments up to the $(e-1)^{\text{st}}$ edge connection.

We can compute

$$\mathbb{P}_{e-1}(g(t,j) = s) = \frac{D_s(t-1, e-1) + \delta}{(2m+\delta)(t-1) + (e-1)(2+\delta/m) + 1 + \delta}. \tag{11.3.42}$$

We wish to use the induction hypothesis. For this, we note that

$$D_s(t-1, e-1) = m + \sum_{(t',j'):t' \leq t-1} \mathbb{1}_{\{g(t',j')=s\}}. \tag{11.3.43}$$

We note that each of the terms in (11.3.43) has edge number strictly smaller than e and occurs with a non-negative multiplicative constant. As a result, we may use the induction hypothesis for each of these terms. Thus, we obtain, using also $m + \delta \geq 0$, that,

$$(2m+\delta)(t-1)\mathbb{P}\Big(\bigcap_{i=1}^{k} E_{s_i}\Big) \leq (m+\delta)\mathbb{P}(E_s') \prod_{i=1:s_i \neq s}^{k} \mathbb{P}(E_{s_i})$$

$$+ \sum_{(t',j'):t' \leq t-1} \mathbb{P}(E_s' \cap \{g(t',j') = s\}) \prod_{i=1:s_i \neq s}^{k} \mathbb{P}(E_{s_i}). \tag{11.3.44}$$

We can recombine to obtain

$$\mathbb{P}\Big(\bigcap_{i=1}^{k} E_{s_i}\Big) \leq \mathbb{E}\Big[\mathbb{1}_{E_s'} \frac{D_s(t-1, e-1) + \delta}{(2m+\delta)(t-1) + (e-1)(2+\delta/m) + 1 + \delta}\Big] \prod_{i=1:s_i \neq s}^{k} \mathbb{P}(E_{s_i}), \tag{11.3.45}$$

and the advancement is completed when we note that

$$\mathbb{E}\Big[\mathbb{1}_{E_s'} \frac{D_s(t-1, e-1) + \delta}{(2m+\delta)(t-1) + (e-1)(2+\delta/m) + 1 + \delta}\Big] = \mathbb{P}(E_s). \tag{11.3.46}$$

$\square$

We next study the probabilities of E_s when $n_s \leq 2$:

Lemma 11.14. *There exist absolute constants M_1, M_2, such that*
(i) for each $1 \leq j \leq m$, and $t > s$,

$$\mathbb{P}(g(t,j) = s) \leq \frac{M_1}{t^{1-a}s^a}, \tag{11.3.47}$$

and
(ii) for $t_2 > t_1 > s$, and any $1 \leq j_1, j_2 \leq m$,

$$\mathbb{P}\Big(g(t_1,j_1) = s, g(t_2,j_2) = s\Big) \leq \frac{M_2}{(t_1 t_2)^{1-a}s^{2a}}, \tag{11.3.48}$$

where we abbreviate $a = \frac{m}{2m+\delta}$.

Proof. We prove (11.3.47) and (11.3.48) only for $m = 1$, for which we abbreviate $g(t) = g(t,1)$. By the definition of $\{\mathrm{PA}_{m,\delta}(t)\}_{t=1}^\infty$ in terms of $\{\mathrm{PA}_{1,\delta/m}(t)\}_{t=1}^\infty$, this implies the result for general $m \geq 1$. Note, in particular, that $g(t,j) = s$ for $m \geq 1$ in $\{\mathrm{PA}_{m,\delta}(t)\}_{t=1}^\infty$ is equivalent to $g(m(t-1)+j) \in \{m(s-1)+1, \ldots, ms\}$ in $\{\mathrm{PA}_{1,\delta/m}(t)\}_{t=1}^\infty$.

For (11.3.47), we use Theorem 8.1 to compute

$$\mathbb{P}(g(t) = s) = \mathbb{E}\big[\mathbb{E}[\mathbb{1}_{\{g(t)=s\}} \,|\mathrm{PA}_{1,\delta}(t-1)]\big] = \mathbb{E}\big[\frac{D_s(t-1)+\delta}{(2+\delta)(t-1)+1+\delta}\big]$$

$$= (1+\delta)\frac{\Gamma(t)\Gamma(i+\frac{1+\delta}{2+\delta})}{\Gamma(t+\frac{1+\delta}{2+\delta})\Gamma(i+1)}, \tag{11.3.49}$$

so that the result follows from (8.2.7).

We proceed with the proof of (11.3.48) for $m = 1$ by computing

$$\mathbb{P}\Big(g(t_1) = s, g(t_2) = s\Big) = \mathbb{E}\Big[\mathbb{P}(g(t_1) = s, g(t_2) = s|\mathrm{PA}_{m,\delta}(t_2-1))\Big]$$

$$= \mathbb{E}\left[\mathbb{1}_{\{g(t_1)=s\}}\left(\frac{D_s(t_2-1)+\delta}{(t_2-1)(2+\delta)+1+\delta}\right)\right]$$

$$= \frac{1}{(t_2-1)(2+\delta)+1+\delta}\frac{\Gamma(t_2)\Gamma(t_1+\frac{1+\delta}{2+\delta})}{\Gamma(t_2-1+\frac{1+\delta}{2+\delta})\Gamma(t_1+1)}\mathbb{E}\left[\mathbb{1}_{\{g(t_1)=s\}}(D_s(t_1)+\delta)\right]$$

$$= \frac{\Gamma(t_2)\Gamma(t_1+\frac{1+\delta}{2+\delta})}{\Gamma(t_2+\frac{1+\delta}{2+\delta})\Gamma(t_1+1)}\mathbb{E}\left[\mathbb{1}_{\{g(t_1)=s\}}(D_s(t_1)+\delta)\right]$$

$$\leq \frac{M_3}{t_1^a t_2^{1-a}}\mathbb{E}\left[\mathbb{1}_{\{g(t_1)=s\}}(D_s(t_1)+\delta)\right], \tag{11.3.50}$$

where we use, for $t_1 < u \leq t_2 - 1$, the iteration

$$\mathbb{E}\left[\mathbb{1}_{\{g(t_1)=s\}}(D_s(u)+\delta)\right] = \left(1+\frac{1}{(2+\delta)(u-1)+1+\delta}\right)\mathbb{E}\left[\mathbb{1}_{\{g(t_1)=s\}}(D_s(u-1)+\delta)\right]. \tag{11.3.51}$$

We are lead to compute $\mathbb{E}\left[\mathbb{1}_{\{g(t_1)=s\}}(d_s(t_1)+\delta)\right]$. We do so by recursion as follows

$$\mathbb{E}\left[\mathbb{1}_{\{g(t_1)=s\}}(D_s(t_1)+\delta)\,|\mathrm{PA}_{m,\delta}(t_1-1)\right] = \mathbb{E}\left[\mathbb{1}_{\{g(t_1)=s\}}(D_s(t_1)-D_s(t_1-1))\,|\mathrm{PA}_{m,\delta}(t_1-1)\right] + \mathbb{E}\left[\mathbb{1}_{\{g(t_1)=s\}}(D_s(t_1-1)\right.$$

$$= \frac{(D_s(t_1-1)+\delta)(D_s(t_1-1)+1+\delta)}{(t_1-1)(2+\delta)+1+\delta}. \tag{11.3.52}$$

By Proposition 8.9, there exists a uniform constant M_4 such that

$$\mathbb{E}[(D_s(t) + \delta)(D_s(t) + 1 + \delta)] = \frac{2}{c_2(t)}\mathbb{E}[Z_{s,2}(t)] \le M_4(t/s)^{2a}. \tag{11.3.53}$$

Combining (11.3.50), (11.3.52) and (11.3.53), we find

$$\mathbb{P}(g(t_1) = s, g(t_2) = s) \le \left(\frac{M_3}{t_1^a t_2^{1-a}}\right)\left[\frac{M_4}{(2+\delta)(t_1 - 1) + 1 + \delta}\left(\frac{t_1 - 1}{s}\right)^{2a}\right] \le \frac{M_2}{(t_1 t_2)^{1-a}s^{2a}}, \tag{11.3.54}$$

for some M_2, as required.   □

We combine the results of Lemmas 11.13 and 11.14 into the following corollary, yielding an upper bound for the probability of the existence of a path. In its statement, we call a path $\Gamma = (s_0, s_1, \ldots, s_l)$ self-avoiding when $s_i \ne s_j$ for all $1 \le i < j \le l$.

Corollary 11.15. *Let $\Gamma = (s_0, s_1, \ldots, s_l)$ be a self-avoiding path of length l consisting of the $l+1$ unordered vertices $s_0, s_1, \ldots, s_l$, then*

$$\mathbb{P}(\Gamma \in \mathrm{PA}_{m,\delta}(t)) \le (m^2 C)^l \prod_{i=0}^{l-1} \frac{1}{(s_i \wedge s_{i+1})^a(s_i \vee s_{i+1})^{1-a}}. \tag{11.3.55}$$

Proof. Since Γ is self-avoiding, we can write $\{\Gamma \in G\} = \cap_{i=1}^k E_{s_i}$, where either

$$E_s = \{g(t, j) = s\} \tag{11.3.56}$$

for some $t > s$ and some $1 \le j \le m$, or

$$E_s = \{g(t_1, j_1) = g(t_2, j_2) = s\}, \tag{11.3.57}$$

for some $t_1, t_2 > s$ and some $1 \le j_1, j_2 \le m$. In the first case we have according to (11.3.47),

$$\mathbb{P}(E_s) = \mathbb{P}(g(t, j) = s) \le \frac{M_1}{t^{1-a}s^a}, \tag{11.3.58}$$

whereas in the second case, according to (11.3.48),

$$\mathbb{P}(E_s) = \mathbb{P}(g(t_1, j_1) = s, g(t_2, j_2) = s) \le \frac{M_2}{(t_1 t_2)^{1-a}s^{2a}} = \frac{M_2}{t_1^{1-a}s^a t_2^{1-a}s^a}. \tag{11.3.59}$$

In both cases M_i, $i = 1, 2$, is an absolute constant. Lemma 11.13 then yields (11.3.55), where the factor m^{2l} originates from the choices of $j \in \{1, 2, \ldots, m\}$.   □

Observe that for integers $j > i$ we have

$$\left(\frac{j}{i}\right)^{a - \frac{1}{2}} \le 1, \tag{11.3.60}$$

if and only if $a \le \frac{1}{2}$, which happens if and only if $\delta \ge 0$, because $a = \frac{m}{2m+\delta}$. Hence, $j^{a-1}i^{-a} \le (ij)^{-1/2}$ precisely when $\delta \ge 0$. It now follows from (11.3.55) and the above inequality that for $\delta \ge 0$,

$$\mathbb{P}(\Gamma \in \mathrm{PA}_{m,\delta}(t)) \le (m^2 C)^l \prod_{i=0}^{l-1} \frac{1}{\sqrt{s_i s_{i+1}}}. \tag{11.3.61}$$

We finally prove that (11.3.61) implies that for $\delta \geq 0$,

$$L = \frac{\log(t-1)}{\log(3Cm^2 \log t)} \qquad \text{(11.3.62)}$$

is a lower bound for the diameter of $\mathrm{PA}_{m,\delta}(t)$. Consider a path of length l consisting of the vertices $v_0, v_1 \ldots, v_l$, then (11.3.61) implies that

$$\mathbb{P}(\Gamma \in \mathrm{PA}_{m,\delta}(t)) \leq (Cm^2)^l \prod_{k=0}^{l-1} \frac{1}{\sqrt{v_k v_{k+1}}} = \frac{(Cm^2)^l}{\sqrt{v_0 v_l}} \prod_{k=1}^{l-1} \frac{1}{v_k}. \qquad \text{(11.3.63)}$$

Thus, the expected number of paths of length l between $v_0 = t$ and $v_l = t-1$ is bounded by

$$\frac{(Cm^2)^l}{\sqrt{t(t-1)}} \sum_{1 \leq v_1, \ldots, v_{l-1} \leq t-2} \prod_{k=1}^{l-1} \frac{1}{v_k} = \frac{(Cm^2)^l}{\sqrt{t(t-1)}} \left(\sum_1^{t-2} \frac{1}{v_k} \right)^{l-1}$$

$$\leq \frac{(Cm^2)^l}{t-1} (\log t)^{l-1} \leq (1/2)^l (\log t)^{-1} \to 0$$

precisely when $(2Cm^2 \log t)^l \leq t-1$, or, equivalently,

$$l \leq \frac{\log(t-1)}{\log((2Cm^2) \log t)}. \qquad \text{(11.3.64)}$$

Equality in (11.3.64) holds for L in (11.3.62). This implies that the diameter is at least L in (11.3.62), and completes the proof of Theorem 11.11. □

11.4 Small-world effect in PA models: doubly logarithmic bounds

In this section, we investigate $\mathrm{PA}_{m,\delta}(t)$ with $m \geq 2$ and $\delta \in (-m, 0)$. Since the degree power-law exponent equals $\tau = 3 + \delta/m$, such graphs are such that $\tau \in (2, 3)$. We know, for $\mathrm{CM}(n, \{D_i\}_{i=1}^n)$, that when $\tau \in (2, 3)$, the graph distances grow doubly logarithmically in the size of the graph. We now state and prove a result from [108] that proves a similar result for $\mathrm{PA}_{m,\delta}(t)$ when $m \geq 2$:

Theorem 11.16 (A $\log \log t$ upper bound on the diameter for $\delta < 0$). *Fix $m \geq 2$ and assume that $\delta \in (-m, 0)$. Then, for every $\sigma > \frac{1}{3-\tau}$ and with*

$$C_G = \frac{4}{|\log(\tau-2)|} + \frac{4\sigma}{\log m}. \qquad \text{(11.4.1)}$$

*the diameter of $\mathrm{PA}_{m,\delta}(t)$ is, **whp**, bounded above by $C_G \log \log t$, as $t \to \infty$.*

The proof of Theorem 11.16 is divided into two key steps. In the first, in Theorem 11.17, we give a bound on the diameter of the *core* which consists of the vertices with degree at least a certain power of $\log t$. This argument is close in spirit to the argument in [183] used to prove bounds on the average distance for the configuration model, but substantial adaptations are necessary to deal with preferential attachment. After this, in Theorem 11.21, we derive a bound on the distance between vertices with a small degree and the core. We start by defining and investigating the core of the preferential attachment model. In the sequel, it will be convenient to prove Theorem 11.16 for $2t$ rather than for t. Clearly, this does not make any difference for the results.

11.4.1 The diameter of the core

We adapt the proof of Theorem 9.7 to $\text{PA}_{m,\delta}(t)$. We recall that

$$\tau = 3 + \frac{\delta}{m}, \tag{11.4.2}$$

so that $-m < \delta < 0$ corresponds to $\tau \in (2,3)$. Throughout this section, we fix $m \geq 2$.

We take $\sigma > \frac{1}{3-\tau} = -\frac{m}{\delta} > 1$ and define the *core* Core_t of the PA-model $\text{PA}_{m,\delta}(2t)$ to be

$$\text{Core}_t = \{i \in \{1,2,\ldots,t\} : D_i(t) \geq (\log t)^\sigma\}, \tag{11.4.3}$$

i.e., all the vertices which at time t have degree at least $(\log t)^\sigma$.

For a graph G with vertex set $\{1,2,\ldots,t\}$ and a given edge set, we write $d_G(i,j)$ for the shortest-path distance between i and j in the graph G. Also, for $A \subseteq \{1,2,\ldots,t\}$, we write

$$\text{diam}_t(A) = \max_{i,j \in A} d_{\text{PA}_{m,\delta}(t)}(i,j). \tag{11.4.4}$$

Then, the diameter of the core in the graph $\text{PA}_{m,\delta}(2t)$, which we denote by $\text{diam}_{2t}(\text{Core}_t)$, is bounded in the following theorem:

Theorem 11.17 (The diameter of the core). *Fix $m \geq 2$. For every $\sigma > \frac{1}{3-\tau}$,* **whp**,

$$\text{diam}_{2t}(\text{Core}_t) \leq (1 + o(1))\frac{4 \log \log t}{|\log(\tau - 2)|}. \tag{11.4.5}$$

The proof of Theorem 11.17 is divided into several smaller steps. We start by proving that the diameter of the *inner core* Inner_t, which is defined by

$$\text{Inner}_t = \{i \in \{1,2,\ldots,t\} : D_i(t) \geq t^{\frac{1}{2(\tau-1)}}(\log t)^{-\frac{1}{2}}\}, \tag{11.4.6}$$

is, **whp** bounded by $C_\delta < \infty$. After this, we will show that the distance from the *outer core*, which is defined to be equal to $\text{Outer}_t = \text{Core}_n \backslash \text{Inner}_t$, to the inner core can be bounded by a fixed constant times $\log \log t$. This also shows that the diameter of the outer core is bounded by a different constant times $\log \log t$. We now give the details.

Proposition 11.18 (The diameter of the inner core). *Fix $m \geq 2$ and $\delta \in (-m, 0)$. Then,* **whp**,

$$\text{diam}_{2t}(\text{Inner}_t) < C_\delta. \tag{11.4.7}$$

Proof. We first introduce the important notion of a t-connector between a vertex $i \in \{1,2,\ldots,t\}$ and a set of vertices $A \subseteq \{1,2,\ldots,t\}$, which plays a crucial role throughout the proof. Fix a set of vertices A and a vertex i. We say that the vertex $j \in \{t+1, t+2, \ldots, 2t\}$ is a *t-connector* between i and A if one of the edges incident to j connects to i and another edge incident to j connects to a vertex in A. Thus, when there exists a t-connector between i and A, the distance between i and A in $\text{PA}_{m,\delta}(2t)$ is at most 2.

We note that for a set of vertices A and a vertex i with degree at time t equal to $D_i(t)$, we have that, conditionally on $\text{PA}_{m,\delta}(t)$, the probability that $j \in \{t+1, t+2, \ldots, 2t\}$ is a t-connector for i and A is at least

$$\frac{(D_A(t) + \delta|A|)(D_i(t) + \delta)}{[2t(2m + \delta)]^2}, \tag{11.4.8}$$

independently of the fact whether the other vertices are t-connectors or not, and where, for any $A \subseteq \{1,2,\ldots,t\}$, we write

$$D_A(t) = \sum_{i \in A} d_i(t). \tag{11.4.9}$$

Since $d_i(t) + \delta \geq m + \delta > 0$ for every $i \leq t$, and $\delta < 0$, we have that

$$D_i(t) + \delta = D_i(t)\Big(1 + \frac{\delta}{D_i(t)}\Big) \geq D_i(t)(1 + \frac{\delta}{m}) = D_i(t)\frac{m+\delta}{m}, \tag{11.4.10}$$

and, thus, also $D_A(t)+\delta|A| \geq D_A(t)\frac{m+\delta}{m}$. As a result, for $\eta = (m+\delta)^2/(2m(2m+\delta))^2 > 0$, the probability that $j \in \{t+1, t+2, \ldots, 2t\}$ is a t-connector for i and A is at least $\frac{\eta D_A(t)D_i(t)}{t^2}$, independently of the fact whether the other vertices are t-connectors or not. Therefore, the probability that there is no t-connector for i and A is, conditionally on $\mathrm{PA}_{m,\delta}(t)$, bounded above by

$$\Big(1 - \frac{\eta D_A(t)D_i(t)}{t^2}\Big)^t \leq \exp\Big\{-\frac{\eta D_A(t)D_i(t)}{t}\Big\}. \tag{11.4.11}$$

<div style="text-align:right"><code>e:noconnAB-PA</code></div>

We shall make use of (11.4.11) in several places throughout the proof.

From Theorem 8.2 **whp**, Inner_t contains at least $\sqrt{t}$ vertices and denote the first $\sqrt{t}$ vertices of Inner_t by I. Observe that for $\tau > 2$ we have $t^{(\tau-1)^{-1}-1} \downarrow 0$ so that, for any $i, j \in I$, the probability that there exists a t-connector for i and j is bounded below by

$$1 - \exp\{-\eta t^{\frac{1}{\tau-1}-1}(\log t)^{-1}\} \geq p_t \equiv t^{\frac{1}{\tau-1}-1}(\log t)^{-2}, \tag{11.4.12}$$

<div style="text-align:right"><code>e:gerard9</code></div>

for t sufficiently large.

We wish to couple Inner_t with an Erdős-Rényi random graph with $n_t = \sqrt{t}$ vertices and edge probability p_t, which we denote by $\mathrm{ER}(n_t, p_t)$. For this, for $i, j \in \{1, 2, \ldots, n_t\}$, we say that an edge between i and j is present when there exists a t-connector connecting the ith and j^{th} vertex in I. We now prove that this graph is bounded below by $\mathrm{ER}(n_t, p_t)$. Note that (11.4.12) does not guarantee this coupling, instead we should prove that the lower bound holds uniformly, when i and j belong to I.

For this, we order the $n_t(n_t - 1)/2$ edges in an arbitrary way, and bound the conditional probability that the lth edge is present conditionally on the previous edges from below by p_t, for every l. This would prove the claimed stochastic domination by $\mathrm{ER}(n_t, p_t)$.

Indeed, the lth edge is present precisely when there exists a t-connector connecting the corresponding vertices which we call i and j in I. Moreover, we shall not make use of the first vertices which were used to t-connect the previous edges. This removes at most $n_t(n_t - 1)/2 \leq t/2$ possible t-connectors, after which at least another $t/2$ remain. The probability that one of them is a t-connector for the ith and j^{th} vertex in I is bounded below by, for t sufficiently large,

$$1 - \exp\{-\eta t^{\frac{1}{\tau-1}-2}(\log t)^{-1}t/2\} = 1 - \exp\{-\eta t^{\frac{1}{\tau-1}-1}(\log t)^{-1}/2\} \geq p_t \equiv t^{\frac{1}{\tau-1}-1}(\log t)^{-2}, \tag{11.4.13}$$

using $1 - e^{-x} \geq x/2$ for $x \in [0, 1]$ and $\eta/2 \geq \log t^{-1}$ for t sufficiently large.

This proves the claimed stochastic domination of the random graph on the vertices I and $\mathrm{ER}(n_t, p_t)$. Next, we show that $\mathrm{diam}(\mathrm{ER}(n_t, p_t))$ is, **whp**, bounded by a uniform constant.

For this we use the result in [40, Corollary 10.12], which gives sharp bounds on the diameter of an Erdős-Rényi random graph. Indeed, this result implies that if $p^d n^{d-1} - 2\log n \to \infty$, while $p^{d-1}n^{d-2} - 2\log n \to -\infty$, then $\mathrm{diam}(\mathrm{ER}(n, p)) = d$, **whp**. In our case, $n = n_t = t^{1/2}$ and $p = p_t = t^{\frac{1}{\tau-1}-1}(\log t)^{-2}$, which implies that, **whp**, $\frac{\tau-1}{3-\tau} < d \leq \frac{\tau-1}{3-\tau} + 1$. Thus, we obtain that the diameter of I in $\mathrm{PA}_{m,\delta}(2t)$ is **whp** bounded by $2(\frac{\tau-1}{3-\tau} + 1)$ in this case.

We finally show that for any $i \in \mathrm{Inner}_t \setminus I$, the probability that there does not exist a t-connector connecting i and I is small. Indeed, this probability is, since $D_I(t) \geq \sqrt{t}t^{\frac{1}{\tau-1}}(\log t)^{-1/2}$, and $D_i(t) \geq t^{\frac{1}{2(\tau-1)}}(\log t)^{-1/2}$, the probability of there not existing a t-connector is bounded above by $e^{-\eta t^{1/(\tau-1)-1/2}(\log t)^{-1}}$, which is tiny since $\tau < 3$. This proves that **whp** the distance between any vertex $i \in \mathrm{Inner}_t \setminus I$ and I is bounded by 2, and, together with the fact that $\mathrm{diam}_{2t}(I) \leq 2(\frac{\tau-1}{3-\tau} + 1)$ thus implies that $\mathrm{diam}_{2t}(\mathrm{Inner}_t) \leq 2(\frac{\tau-1}{3-\tau} + 2)$. $\qquad\square$

Proposition 11.19 (Distance between outer and inner core). *Fix $m \geq 2$. With high probability, the inner core Inner_t can be reached from any vertex in the outer core Outer_t using no more than $\frac{2 \log \log t}{|\log (\tau - 2)|}$ edges in $\mathrm{PA}_{m,\delta}(2t)$. More precisely, with high probability,*

$$\max_{i \in \mathrm{Outer}_t} \min_{j \in \mathrm{Inner}_t} d_{\mathrm{PA}_{m,\delta}(2t)}(i, j) \leq \frac{2 \log \log t}{|\log (\tau - 2)|}. \tag{11.4.14}$$

Proof. Recall that

$$\mathrm{Outer}_t = \mathrm{Core}_t \setminus \mathrm{Inner}_t. \tag{11.4.15}$$

and define

$$\mathcal{N}_1 = \mathrm{Inner}_t = \{i : D_i(t) \geq \mathcal{W}_1\}, \tag{11.4.16}$$

where

$$\mathcal{W}_1 = l_t = t^{\frac{1}{2(\tau-1)}} (\log t)^{-\frac{1}{2}}. \tag{11.4.17}$$

We now recursively define a sequence u_k, for $k \geq 2$, so that for any vertex $i \in \{1, 2, \ldots, t\}$ with degree at least u_k, the probability that there is no t-connector for the vertex i and the set

$$\mathcal{N}_{k-1} = \{j : D_j(t) \geq \mathcal{W}_{k-1}\}, \tag{11.4.18}$$

conditionally on $\mathrm{PA}_{m,\delta}(t)$ is tiny. According to (11.4.11) and Exercise 8.16, this probability is at most

$$\exp\left\{ -\frac{\eta B t [u_{k-1}]^{2-\tau} u_k}{t} \right\} = o(t^{-1}), \tag{11.4.19}$$

where we define

$$\mathcal{W}_k = D \log t (\mathcal{W}_{k-1})^{\tau-2}, \tag{11.4.20}$$

with D exceeding $(\eta B)^{-1}$. By Lemma 9.8 we have

$$\mathcal{W}_k = D^{a_k} (\log t)^{b_k} t^{c_k}, \tag{11.4.21}$$

where

$$c_k = \frac{(\tau - 2)^{k-1}}{2(\tau - 1)}, \qquad b_k = \frac{1 - (\tau - 2)^{k-1}}{3 - \tau} - \frac{1}{2}(\tau - 2)^{k-1}, \qquad a_k = \frac{1 - (\tau - 2)^{k-1}}{3 - \tau}. \tag{11.4.22}$$

Then, the key step in the proof of Proposition 11.19 is the following lemma:

Lemma 11.20 (Connectivity between $\mathcal{N}_{k-1}$ and $\mathcal{N}_k$). *Fix $m, k \geq 2$. Then the probability that there exists an $i \in \mathcal{N}_k$ that is not at distance two from $\mathcal{N}_{k-1}$ in $\mathrm{PA}_{m,\delta}(2t)$ is $o(t^{-1})$.*

Proof. We note that, by Exercise 8.16, with probability exceeding $1 - o(t^{-1})$, for all k,

$$\sum_{i \in \mathcal{N}_{k-1}} D_i(t) \geq B t [\mathcal{W}_{k-1}]^{2-\tau}. \tag{11.4.23}$$

On the event that the bounds in (11.4.23) hold, we obtain by (11.4.11) that the conditional probability, given $\mathrm{PA}m, \delta(t)$, that there exists an $i \in \mathcal{N}_k$ such that there is no t-connector between i and $\mathcal{N}_{k-1}$ is bounded, using Boole's inequality, by

$$t e^{-\eta B [\mathcal{W}_{k-1}]^{2-\tau} \mathcal{W}_k} = t e^{-\eta B D \log t} = o(t^{-1}), \tag{11.4.24}$$

where we have used (11.4.20) and we have taken $D > 2(\eta B)^{-1}$. $\qquad \square$

We now complete the proof of Proposition 11.19. Fix

$$k^* = \left\lfloor \frac{\log \log t}{|\log (\tau - 2)|} \right\rfloor. \tag{11.4.25}$$

As a result of Lemma 11.20, we have that the distance between $\mathcal{N}_{k^*}$ and Inner_t is at most $2k^*$. Therefore, we are done when we can show that

$$\mathrm{Outer}_t \subseteq \{i : D_i(t) \geq (\log t)^\sigma\} \subseteq \mathcal{N}_{k^*} = \{i : D_i(t) \geq \mathcal{W}_{k^*}\}, \tag{11.4.26}$$

so that it suffices to prove that $(\log t)^\sigma \geq \mathcal{W}_{k^*}$, for any $\sigma > \frac{1}{3-\tau}$. For this, we note that, by Lemma 9.8, we have that

$$\mathcal{W}_{k^*} = D^{a_{k^*}} (\log t)^{b_{k^*}} t^{c_{k^*}}. \tag{11.4.27}$$

We have that $t^{c_{k^*}} = O(1) = (\log t)^{o(1)}$, $(\log t)^{b_{k^*}} = (\log t)^{\frac{1}{3-\tau}+o(1)}$, and $D^{a_{k^*}} = (\log t)^{o(1)}$. Thus,

$$\mathcal{W}_{k^*} = (\log t)^{\frac{1}{3-\tau}+o(1)}, \tag{11.4.28}$$

so that, by picking t sufficiently large, we can make $\sigma \geq \frac{1}{3-\tau}+o(1)$. This completes the proof of Proposition 11.19. $\qquad \square$

Proof of Theorem 11.17. We note that **whp** $\mathrm{diam}_{2t}(\mathrm{Core}_t) \leq C_\delta + 2k^*$, where k^* is the upper bound on $\max_{i \in \mathrm{Outer}_t} \min_{j \in \mathrm{Inner}_t} d_{\mathrm{PA}_{m,\delta}(2t)}(i,j)$ in Proposition 11.19, and we have made use of Proposition 11.18. This proves Theorem 11.17. $\qquad \square$

11.4.2 Connecting the periphery to the core

In this section, we extend the results of the previous section and, in particular, study the distance between the vertices not in the core Core_n and the core. The main result in this section is the following theorem:

Theorem 11.21 (Connecting the periphery to the core). *Fix $m \geq 2$. For every $\sigma > \frac{1}{3-\tau}$, **whp**, the maximal distance between any vertex and Core_t in $G_m(2t)$ is bounded from above by $2\sigma \log \log t/\log m$.*

Together with Theorem 11.17, Theorem 11.21 proves the main result in Theorem 11.16.

The proof of Theorem 11.21 again consists of two key steps. The first in Proposition 11.22 states that, for any two vertices $i, j \in \{1, 2, \ldots, t\}$, the distance $d_{\mathrm{PA}_{m,\delta}(2t)}(i,j)$ is bounded by a constant times $\log \log t$, i.e., $\mathrm{diam}_{2t}(\mathrm{PA}_{m,\delta}(t))$ is bounded by some constant times $\log \log t$. The second in Proposition 11.25 shows that the distance between any vertex in $\{t+1, t+2, \ldots, 2t\}$ and $\{1, 2, \ldots, t\}$ is bounded by another constant times $\log \log t$.

Proposition 11.22 (Connecting half of the periphery to the core). *Fix $m \geq 2$. For every $\sigma > \frac{1}{3-\tau}$, **whp**, the maximal distance between any vertex in $\{1, 2, \ldots, t\}$ and the core Core_t in $\mathrm{PA}_{m,\delta}(2t)$ is bounded from above by $\sigma \log \log t/\log m$.*

Proof. We start from a vertex $i \in \{1, 2, \ldots, t\}$ and will show that the probability that the distance between i and Core_t is at least $C \log \log t$ is $o(t^{-1})$ where $C = \sigma/\log m$. This proves the claim. For this, we explore the neighborhood of i as follows. From i, we connect its $m \geq 2$ edges. Then, successively, we connect the m edges from each of the at most m vertices that i has connected to and have not yet been explored. We continue in the same fashion. We call the arising process when we have explored up to distance k from the initial vertex i the *k-exploration tree*.

When we never connect two edges to the same vertex, then the number of vertices we can reach within k steps is *precisely* equal to m^k. We call an event where an edge connects to a vertex which already was in the exploration tree a *collision*. When k increases, the probability of a collision increases. However, the probability that there exists a vertex for which *many* collisions occur in the k-exploration tree before it hits the core is small, as we prove now:

Lemma 11.23 (A bound on the probability of multiple collisions). *Fix $m \geq 2$ and $\delta \in (-m, 0)$. Fix $C = \sigma/\log m$, $l \geq 1$, $b \in (0, 1]$ and take $k \leq C \log\log t$. Then, for every vertex $i \in \{1, 2, \ldots, t\}$, the probability that its k-exploration tree has at least l collisions before it hits $\mathrm{Core}_t \cup \{j : j \leq t^b\}$ is bounded above by*

$$\left((\log t)^d t^{-b}\right)^l = (\log t)^{dl} t^{-bl},$$

for some $d > 0$.

Proof. Take $i \in \{\lceil t^b \rceil + 1, \lceil t^b \rceil + 2, \ldots, t\}$ and consider its k-exploration tree $\mathcal{T}_i^{(k)}$. Since we add edges after time t^b the denominator in (8.1.1) is at least t^b. Moreover, before hitting the core, any vertex in the k-exploration tree has degree at most $(\log t)^\sigma$. Hence, for $l = 1$, the probability mentioned in the statement of the lemma is at most

$$\sum_{v \in \mathcal{T}_i^{(k)}} \frac{D_v(t) + \delta}{t^b} \leq \sum_{v \in \mathcal{T}_i^{(k)}} \frac{(\log t)^\sigma}{t^b} \leq \frac{m^k (\log t)^\sigma}{t^b} \qquad\qquad \text{e:gerard11} \tag{11.4.29}$$

where the bound follows from $\delta < 0$ and $|\mathcal{T}_i^{(k)}| \leq m^k$. For general l this upper bound becomes

$$\left(\frac{m^k (\log t)^\sigma}{t^b}\right)^l$$

When $k = C \log\log t$ with $C = \sigma/\log m$, we have that $m^{lk} = (\log t)^{l\sigma}$. Therefore, the claim in Lemma 11.23 holds with $d = 2\sigma$. $\qquad\square$

Lemma 11.23 shall prove to be extremely useful, as it will imply that the shortest path graph from any vertex is, with high probability, close to a tree up to the moment when $\mathrm{Core}_t \cup \{j : j \leq t^b\}$ is hit. We recall that Core_t consists of the vertices with high degree, and it is reasonable to suspect that the early vertices are precisely the ones which have high degree. This suggests that $\{j : j \leq t^b\} \subseteq \mathrm{Core}_t$. We next prove that indeed, **whp**, $\{j : j \leq t^b\}$ is a subset of the core Core_t:

Lemma 11.24 (Early vertices have large degrees **whp**). *Fix $m \geq 1$. There exists a $b > 0$ such that, **whp**, $\min_{j \leq t^b} d_j(t) \geq (\log t)^\sigma$, for some $\sigma > \frac{1}{3-\tau}$. As a result, **whp**, $\{j : j \leq t^b\} \subseteq \mathrm{Core}_t$.*

Proof. Note that, by Exercise 8.18, for all $m \geq 1$ and for $\delta < 0$,

$$\mathbb{P}(D_i(t) \leq j) \leq j \frac{\Gamma(mt)\Gamma(mi + \frac{m+\delta}{2m+\delta})}{\Gamma(mt + \frac{m+\delta}{2m+\delta})\Gamma(mi)}. \qquad\qquad \text{e:summdjbd} \tag{11.4.30}$$

Indeed, note that by (8.6.4), and when $\delta \leq 0$, we have that $C_j \leq 1$. Thus, for $m = 1$, the bound in (11.4.30) follows in this case. Furthermore, to conclude (11.4.30) from Exercise 8.18 for $m \geq 2$, we note that, by the relation between $\mathrm{PA}_{m,\delta}(t)$ and $\mathrm{PA}_{1,\delta/m}(mt)$, the degree of vertex i in $\mathrm{PA}_{m,\delta}(t)$ is bounded from below by the degree of vertex im in $\mathrm{PA}_{1,\delta}(mt)$. As a result,

$$\mathbb{P}(D_i^{(m)}(t) \leq j) \leq \mathbb{P}(D_{im}^{(1)}(mt) \leq j), \tag{11.4.31}$$

after which the result follows from (11.4.30) for $m = 1$.

Below, we shall rely on the obvious consequence of (11.4.30) that

$$\mathbb{P}(D_i(t) \leq j) \leq j \frac{\Gamma(mt)\Gamma(mi + \frac{m+\delta}{2m+\delta})}{\Gamma(mt + \frac{m+\delta}{2m+\delta})\Gamma(mi)}. \qquad \text{(11.4.32)}$$

Obviously, for t and i large, we have that

$$\mathbb{P}(D_i(t) \leq j) \leq j t^{-\frac{m+\delta}{2m+\delta}} i^{\frac{m+\delta}{2m+\delta}} (1 + o(1)). \qquad \text{(11.4.33)}$$

We finally use (11.4.32) to complete the proof of Lemma 11.24. Take $0 < b < \frac{\frac{m+\delta}{2m+\delta}}{\frac{m+\delta}{2m+\delta}+1} = \frac{m+\delta}{3m+2\delta}$. Then, by Boole's inequality,

$$\mathbb{P}(\exists i \leq t^b : D_i(t) \leq (\log t)^\sigma) \leq \sum_{i=1}^{t^b} \mathbb{P}(d_i(t) \leq (\log t)^\sigma) \leq (\log t)^\sigma \frac{\Gamma(mt)}{\Gamma(mt + \frac{m+\delta}{2m+\delta})} \sum_{i=1}^{t^b} \frac{\Gamma(mi + \frac{m+\delta}{2m+\delta})}{\Gamma(mi)}$$

$$\leq (\log t)^\sigma (\frac{m+\delta}{2m+\delta} + 1)^{-1} \frac{\Gamma(mt)}{\Gamma(mt + \frac{m+\delta}{2m+\delta})} \frac{\Gamma(mt^b + \frac{m+\delta}{2m+\delta} + 1)}{\Gamma(mt^b)} = o(1),$$

$$\text{(11.4.34)}$$

by a similar equality as in (11.4.33). This completes the proof of Lemma 11.24. $\qquad \square$

Now we are ready to complete the proof of Proposition 11.22:

Proof of Proposition 11.22. By combining Lemmas 11.23 and 11.24, the probability that there exists an $i \in \{1, 2, \ldots, t\}$ for which the exploration tree $\mathcal{T}_i^{(k)}$ has at least l collisions before hitting the core is $o(1)$, whenever $l > 1/b$, since, by Boole's inequality, it is bounded by

$$\sum_{i=1}^{t} (\log t)^{dl} t^{-bl} = (\log t)^{2\sigma l} t^{-bl+1} = o(1), \qquad \text{(11.4.35)}$$

precisely when $l > \frac{1}{b}$. When the k-exploration tree hits the core, then we are done by Theorem 11.17. When the k-exploration tree from a vertex i does not hit the core, but has less than l collisions, then there are at least m^{k-l} vertices in k-exploration tree. Indeed, when there are at most l collisions, the minimal size of the tree is obtained by identifying at most l vertices and their complete offspring, and the size of the pruned tree has size at least m^{k-l}.

When $k = C \log \log t$ with $C = \sigma / \log m$, this number is at least equal to $(\log t)^{\sigma+o(1)}$. The total weight of the core is, by Exercise 8.16, at least

$$\sum_{i \in \text{Core}_t} (D_i(t) + \delta) \geq Bt(\log t)^{-(\tau-2)\sigma}. \qquad \text{(11.4.36)}$$

The probability that there does not exist a t-connector between the k-exploration tree and the core is, by (11.4.11) bounded above by

$$\exp\left\{-\frac{\eta Bt(\log t)^{-(\tau-2)\sigma}(\log t)^{\sigma+o(1)}}{t}\right\} = o(t^{-1}), \qquad \text{(11.4.37)}$$

by picking B sufficiently large, since $\sigma > 1/(3 - \tau)$. This completes the proof. $\qquad \square$

Proposition 11.25. *Fix $m \geq 2$ and $\delta \in (-m, 0)$. For every $\sigma > \frac{1}{3-\tau}$, **whp**, the maximal distance between any vertex and $\mathrm{Core}_n \cup \{1, 2, \ldots, t\}$ in $\mathrm{PA}_{m,\delta}(2t)$ is bounded from above by $\frac{\sigma \log \log t}{\log m}$.*

Proof. Denote $k = \lfloor \frac{\sigma \log \log t}{\log m} \rfloor - 1$. We again grow the k-exploration trees from the vertices $i \in \{t+1, t+2, \ldots, 2t\}$.

By Lemma 11.23 for $b = 1$, the probability that there exists a vertex whose k-exploration tree contains at least two collisions before hitting the vertex set $\mathrm{Core}_t \cup \{1, 2, \ldots, t\}$ is bounded above by $t^{-2}(\log t)^{d_1}$ for some d_1 sufficiently large. When the k-exploration tree contains a vertex in $\mathrm{Core}_t \cup \{1, 2, \ldots, t\}$, then we are done by Proposition 11.22 and Theorem 11.17. If not, and there are at most 2 collisions, then there are at least $m_k = (m-1)m^{k-1}$ vertices in $\{t+1, t+2, \ldots, 2t\}$ at distance precisely equal to k from the original vertex. Denote these vertices by $i_1, \ldots, i_{m_k}$, and denote the k-exploration tree of vertex $i \in \{t+1, t+2, \ldots, 2t\}$ by $\mathcal{T}_i^{(k)}$. We write

$$\mathbb{P}(\nexists j \in \{1, 2, \ldots, m_k\} \text{ such that } i_j \longrightarrow \{1, 2, \ldots, t\} | \mathcal{T}_i^{(k)}) \tag{11.4.38}$$

$$= \prod_{j=1}^{m_k} \mathbb{P}(i_j \longleftrightarrow\!\!\!\!/\ \{1, 2, \ldots, t\} | i_s \longleftrightarrow\!\!\!\!/\ \{1, 2, \ldots, t\} \forall s < j, \mathcal{T}_i^{(k)}).$$

Now we note that, uniformly in the way all edges in $\mathrm{PA}_{m,\delta}(2t)$ are formed, we have that for every $s \in \{t+1, t+2, \ldots, 2t\}$,

$$\frac{\sum_{i=1}^{t}(D_i(s) + \delta)}{(2m + \delta)s} \geq \frac{1}{2}. \tag{11.4.39}$$

Thus, for any vertex i_j in the boundary of $\mathcal{T}_i^{(k)}$, the probability that it will be directly connected to $\{1, 2, \ldots, t\}$ is at least $1/2$. As a result, we have that, uniformly in t, i and j,

$$\mathbb{P}(i_j \longleftrightarrow\!\!\!\!/\ \{1, 2, \ldots, t\} | i_s \longleftrightarrow\!\!\!\!/\ \{1, 2, \ldots, t\} \forall s < j, \mathcal{T}_i^{(k)}) \leq \frac{(2m + \delta)t}{(2m + \delta)(2t)} = \frac{1}{2}. \tag{11.4.40}$$

Therefore, we obtain that

$$\mathbb{P}(\nexists j = 1, \ldots, m_k \text{ such that } i_j \longleftrightarrow \{1, 2, \ldots, t\} | \mathcal{T}_i^{(k)}) \leq 2^{-m_k}. \tag{11.4.41}$$

Since $m_k = \frac{m-1}{m}(\log t)^\sigma$, with $\sigma = \frac{1}{3-\tau} > 1$, we have that $2^{-m_k} = o(t^{-1})$. Therefore, any vertex $i \in \{t+1, t+2, \ldots, 2t\}$ is, **whp**, within distance $k+1$ from $\{1, 2, \ldots, t\}$. $\qquad\square$

Proof of Theorem 11.21. Proposition 11.25 states that **whp** every vertex in $\mathrm{PA}_{m,\delta}(2t)$ is within distance $k+1 = \lfloor \frac{\sigma \log \log t}{\log m} \rfloor$ of $\mathrm{Core}_t \cup \{1, 2, \ldots, t\}$. Proposition 11.22 states that **whp** every vertex in $\{1, 2, \ldots, t\}$ is at most distance $k+1$ from the core Core_t. This shows that every vertex in $\mathrm{PA}_{m,\delta}(2t)$ is **whp** within distance $2(k+1)$ from the core. $\qquad\square$

Proof of Theorem 11.16. Theorem 11.21 states that every vertex in $\mathrm{PA}_{m,\delta}(2t)$ is within distance $\frac{2\sigma \log \log t}{\log m}$ of the core Core_t. Theorem 11.17 states that the diameter of the core is at most $\frac{4 \log \log t}{|\log(\tau-2)|}(1 + o(1))$, so that the diameter of $\mathrm{PA}_{m,\delta}(2t)$ is at most $C_G \log \log t$, where C_G is given in (11.4.1). This completes the proof of Theorem 11.16. $\qquad\square$

11.5   Notes and discussion

Notes on Section 11.1.   The proof of Theorem 11.1 is adapted from [184]^{Ross96}. More recent discussions on exchangeable random variables and their properties can be found in [11]^{Aldo85} and [177]^{Pema07}, the latter focussing

on random walks with self-interaction, where exchangeability is a crucial tool. There is a lot of work on urn schemes, also in cases where the weight functions are not linear with equal slope, in which case the limits can be seen to obey rather different characteristics. See e.g., [17, Chapter 9].

Notes on Section 11.3. There is a beautiful result on the height of trees using branching processes due to Kingman [134], which Pittel [180] makes crucial use of. This approach is based on exponential martingales, and allows for a relatively short proof of the lower bound on the height of the tree.

Chapter 12

Related models

To be added in these notes:

Chapter 1.

  1. Extra section on sums of i.i.d. random variables?

Chapter 2.

  1. Add exercise that maximum of martingales is submartingale.

Chapter 3.

  1. Add the martingale limit and the $Z \log Z$ criterium. Check proofs in [152].[LyoPemPer95]

  2. Introduction and results of multitype branching processes.

Chapter 4.

  1.

Chapter 5.

  1. A discussion of the critical Erdős-Rényi random graph in Section 5.1 to be taken from [13],[Aldo97] [122].[JanKnuLucPit93]

  2. Add results and proof of [161][NacPer05] on critical random graph.

Chapter 6.

  1.

Chapter 7.

  1.

Chapter 8.

  1. Add discussion on other explanations for power-law degrees (e.g., competition induced in [30, 31]).[BerBorChaSouKle04, BerBorChaSouKle]

  2. Add discussion fitness models [33, 34, 92].[BiaBar01a, BiaBar01b,ErgRod02]

Structure Chapters 9–11.

  1. The phase transition: Connectivity structure and, if applicable, the phase transition.

  2. The Small-World Effect: distances, first for $\tau \in (1, 2)$ if applicable, then $\tau \in (2, 3)$ and finally $\tau > 3$. Only prove convergence in probability of average graph distance, possibly state and explain, but not prove, results on diameters.

  3. Other properties, such as attack vulnerability. No proofs.

Chapter 9.

1. Add results in [43]: Phase transition, degrees, and distances results. Proofs?

2. Add result [93] in Chapter 9, but without full proof.

3. Add results in [164] on diameter of critical random graph in notes.

Chapter 10.

1. Add proof of Theorem 10.2 from [157, Theorem 1]. Check also paper [123] for methodology...

2. Add proof of Theorem 10.11 in Section 10.4. Also proof of $\log n$ asymptotics in Theorem 10.8.

3. Add remark on results in [163], which are random graph asymptotics for critical percolation on random regular graphs.

Chapter 11.

1. Add connection to urn schemes in [29].

2. Investigate connection to multitype BP's.

Chapter 12.

1. Small-world models and results.

2. CHKNS model and the infinite order of the phase transition.

3. Percolation, in particular on finite tori.

4. Uniform recursive trees and shortest path routing on the complete graph with exponential weights:
   (a) SPT from given vertex is URT;
   (a1) Degree vertex (proof);
   (a2) Distance between two vertices from [154] (proof);
   (a3) Size subtree (no proof);
   (a4) Distance results in [118];
   (b) Open problems for other weights.

Chapter A

Appendix

A.1 Some measure and integration results

In this section, we give some classical results from the theory of measure and integration, which will be used in the course of the proofs. For details and proofs of these results, we refer to the books [36, 97, 78, 105].[Bill95, Fell71, Dudl02, Halm50] For the statements of the results below, we refer to [97, Pages 110-111].[Fell71]

Theorem A.1.1 (Lebesque's dominated convergence theorem). *Let X_n and Y satisfy $\mathbb{E}[Y] < \infty$, $X_n \xrightarrow{a.s.} X$, and $|X_n| \leq Y$ almost surely. Then*

$$\mathbb{E}[X_n] \to \mathbb{E}[X], \tag{A.1.1}$$

and $\mathbb{E}[|X|] < \infty$.

We shall also make use of a slight extension, where almost sure convergence is replaced with convergence in distribution:

Theorem A.1.2 (Lebesque's dominated convergence theorem). *Let X_n and Y satisfy $\mathbb{E}[|X_n|] < \infty$, $\mathbb{E}[Y] < \infty$, $X_n \xrightarrow{d} X$, and $|X_n| \leq Y$. Then*

$$\mathbb{E}[X_n] \to \mathbb{E}[X], \tag{A.1.2}$$

and $\mathbb{E}[|X|] < \infty$.

Theorem A.1.3 (Monotone convergence theorem). *Let X_n be a monotonically increasing sequence, i.e., $X_n \leq X_{n+1}$ such that $\mathbb{E}[|X_n|] < \infty$. Then $X_n(\omega) \uparrow X(\omega)$ for all ω and some limiting random variable X, and*

$$\mathbb{E}[X_n] \uparrow \mathbb{E}[X]. \tag{A.1.3}$$

In particular, when $\mathbb{E}[X] = \infty$, then $\mathbb{E}[X_n] \uparrow \infty$.

Theorem A.1.4 (Fatou's lemma). *If $X_n \geq 0$ and $\mathbb{E}[|X_n|] < \infty$, then*

$$\mathbb{E}[\liminf_{n \to \infty} X_n] \leq \liminf_{n \to \infty} \mathbb{E}[X_n]. \tag{A.1.4}$$

In particular, if $X_n(\omega) \to X(\omega)$ for every ω, then

$$\mathbb{E}[X] \leq \liminf_{n \to \infty} \mathbb{E}[X_n]. \tag{A.1.5}$$

Chapter B

Solutions to selected exercises

Solutions to the exercises of Chapter 1.

Exercise 1.1. Show that when (1.1.6) holds with equality, then (1.1.7) holds. Find an example where (1.1.7) holds in the form that there exists a constant C such that

$$1 - F(x) = Cx^{1-\tau}(1 + o(1)), \tag{B.0.1}$$

but that (1.1.6) fails.

Solution to Exercise 1.1. When (1.1.6) holds with equality, then

$$1 - F_X(x) = \sum_{k=x+1}^{\infty} f_k = \sum_{k=x+1}^{\infty} k^{-\tau}.$$

Therefore, by monotonicity of $x \mapsto x^{-\tau}$,

$$1 - F_X(x) \leq \int_x^{\infty} y^{-\tau} dy = \frac{x^{1-\tau}}{\tau - 1},$$

while

$$1 - F_X(x) \geq \int_{x+1}^{\infty} y^{-\tau} dy = \frac{(x+1)^{1-\tau}}{\tau - 1}.$$

As a result, we obtain that

$$1 - F_X(x) = \frac{x^{1-\tau}}{\tau - 1}(1 + O(\frac{1}{x})).$$

For an example where (B.0.1) holds, but (1.1.6) fails, we can take $f_{2k+1} = 0$ for $k \geq 0$ and, for $k \geq 1$,

$$f_{2k} = \frac{1}{k^{\tau-1}} - \frac{1}{(k+1)^{\tau-1}}.$$

Then (1.1.6) fails, while

$$1 - F_X(x) = \sum_{k>x} f_k \sim \frac{1}{\lfloor x/2 \rfloor^{\tau-1}} \sim \frac{1}{x^{\tau-1}}.$$

$\square$

Exercise 1.2. Show that $x \mapsto \log x$ and, for $\gamma \in (0,1)$, $x \mapsto e^{(\log x)^\gamma}$ are slowly varying, but that when $\gamma = 1$, $x \mapsto e^{(\log x)^\gamma}$ is *not* slowly varying.

Solution to Exercise 1.2.   Recall that a function $x \mapsto L(x)$ is slowly varying when, for every $c > 0$,

$$\lim_{x \to \infty} \frac{L(cx)}{L(x)} = 1.$$

For $L(x) = \log x$, we can compute

$$\lim_{x \to \infty} \frac{L(cx)}{L(x)} = \lim_{x \to \infty} \frac{\log(cx)}{\log x} = \lim_{x \to \infty} \frac{\log x + \log c}{\log x} = 1.$$

For $L(x) = e^{(\log x)^\gamma}$, we compute similarly

$$
\begin{aligned}
\lim_{x \to \infty} \frac{L(cx)}{L(x)} &= \lim_{x \to \infty} e^{(\log(cx))^\gamma - (\log x)^\gamma} \\
&= \lim_{x \to \infty} e^{\log(x)^\gamma \left((1 + \frac{\log c}{\log x})^\gamma - 1 \right)} \\
&= \lim_{x \to \infty} e^{\log(x)^{\gamma - 1} \gamma \log c} = 1.
\end{aligned}
$$

When $\gamma = 1$, however, we have that $L(x) = e^{\log x} = x$, which is regularly varying with exponent 1.   □

Solutions to the exercises of Chapter 2.

Exercise 2.1.   Find an example of a sequence of random variables where convergence in distribution occurs, but convergence in probability does not hold.

Solution to Exercise 2.1.   Take
$$
X_n = \begin{cases} Y_1 & \text{for } n \text{ even,} \\ Y_2 & \text{for } n \text{ odd,} \end{cases}
$$
where Y_1 and Y_2 are two independent copies of a random variable which is such that $\mathbb{P}(Y_i = \mathbb{E}[Y_i]) < 1$. Then, since Y_1 and Y_2 are identical in distribution, the sequence $\{X_n\}_{n=1}^\infty$ converges in distribution. In fact, $\{X_n\}_{n=1}^\infty$ is constant in distribution.

Moreover, $X_{2n} \equiv Y_1$ and $X_{2n+1} \equiv Y_2$. Since subsequences of converging sequences are again converging, if $\{X_n\}_{n=1}^\infty$ converges in probability, the limit of $\{X_n\}_{n=1}^\infty$ should be equal to Y_1 and to Y_2. Since $\mathbb{P}(Y_1 \neq Y_2) > 0$, we obtain a contradiction.   □

Exercise 2.2.   Show that the sequence of random variables $\{X_n\}_{n=1}^\infty$, where X_n takes the value n with probability $\frac{1}{n}$ and 0 with probability $1 - \frac{1}{n}$ converges both in distribution and in probability to 0.

Solution to Exercise 2.2.   Note that for any $\varepsilon > 0$, we have

$$\mathbb{P}(|X_n| > \varepsilon) = \mathbb{P}(X_n = n) = \frac{1}{n} \to 0. \tag{B.0.2}$$

Therefore, $X_n \xrightarrow{\mathbb{P}} 0$, which in turn implies that $X_n \xrightarrow{d} 0$.   □

Exercise 2.3.   Find a random variable for which the moment generating function is equal to $+\infty$ for every $t \neq 0$.

Solution to Exercise 2.3. The random variable X with density

$$f_X(x) = \frac{1}{\pi(1 + x^2)},$$

which is a *Cauchy random variable*, does the job. □

Exercise 2.3. Find a random variable for which the moment generating function is equal to $+\infty$ for every $t \neq 0$.

Solution to Exercise 2.3. The density of the so-called *Pareto distribution* is equal to

$$f_\alpha(y) = \begin{cases} \frac{\alpha}{y^{\alpha+1}} & \text{for } y > 1, \\ 0 & \text{otherwise.} \end{cases}$$

The moment generating function of Y is

$$\mathbb{E}[e^{tX}] = \int_1^\infty \frac{\alpha e^{ty}}{y^{\alpha+1}} \, dy,$$

which equals $+\infty$ for any $t > 0$. Then,

$$g_\alpha(y) = \begin{cases} \frac{1}{2} \frac{\alpha}{y^{\alpha+1}} & \text{for } y > 1, \\ 0 & \text{for } -1 \leq y \leq 1, \\ \frac{1}{2} \frac{\alpha}{|y|^{\alpha+1}} & \text{for } y < -1, \end{cases}$$

gives an example of a density so that $M(t) = +\infty$ for all $t \neq 0$. □

Exercise 2.4. Show that a Poisson random variable satisfies the moment condition in (2.1.8).

Solution to Exercise 2.4. Note that, by a Taylor expansion of the moment generating function, if $M_X(t) < \infty$ for all t, we have that

$$M_X(t) = \sum_{r=0}^\infty \mathbb{E}[X^r] \frac{t^r}{r!}.$$

As a result, when $M_X(t) < \infty$ for all t, we must have that

$$\lim_{r \to \infty} \mathbb{E}[X^r] \frac{t^r}{r!} = 0.$$

Thus, when $t > 1$, (2.1.8) follows. Thus, it is sufficient to show that the moment generating function $M_X(t)$ of the Poisson distribution is finite for all t. For this, we compute

$$M_X(t) = \mathbb{E}[e^{tX}] = \sum_{k=0}^\infty e^{tk} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^\infty \frac{(\lambda e^t)^k}{k!} = \exp\{-\lambda(1 - e^t)\} < \infty,$$

for all t. □

Exercise 2.5. Compute the factorial moments of a binomial random variable with parameters n and $p = \lambda/n$ and the ones of a Poisson random variable with mean λ, and use this to conclude that a binomial random variable with parameters n and $p = \lambda/n$ converges in distribution to a Poisson random variable with mean λ.

Solution to Exercise 2.5.   For the Poisson distribution factorial moments are given by

$$\mathbb{E}[(X)_k] = \mathbb{E}[X(X-1)\cdots(X-k+1)] = \lambda^k$$

(recall Exercise We make use of Theorem 2.4. Now if X_n is binomial with parameters n and $p_n = \lambda/n$, then

$$\mathbb{E}[(X_n)_k] = \mathbb{E}[X_n(X_n-1)\cdots(X_n-k+1)] = n(n-1)\ldots(n-k+1)p^k \to \lambda^k,$$

when $p = \lambda/n$ and $n \to \infty$.   □

Exercise 2.6.   Show that the moments of a Poisson random variable X with mean λ satisfy the recursion

$$\mathbb{E}[X^m] = \lambda\mathbb{E}[(X+1)^{m-1}]. \tag{B.0.3}$$

<div align="right">_{e:momPois-ex}</div>

Solution to Exercise 2.6.   Compute that

$$\mathbb{E}[X^m] = e^{-\lambda}\sum_{k=1}^{\infty} k^m\frac{\lambda^k}{k!} = \lambda e^{-\lambda}\sum_{k=1}^{\infty} k^{m-1}\frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda}\sum_{l=0}^{\infty}(l+1)^{m-1}\frac{\lambda^l}{(l)!} = \lambda\mathbb{E}[(X+1)^{m-1}].$$

<div align="right">□</div>

Exercise 2.7.   Show that if

$$\lim_{n\to\infty}\sum_{r\geq n}\frac{\mathbb{E}[(X)_r]}{(r-k)!} = 0, \tag{B.0.4}$$

<div align="right">_{e:alternatingass}</div>

then also

$$\mathbb{P}(X = k) = \sum_{r=k}^{\infty}(-1)^{k+r}\frac{\mathbb{E}[(X)_r]}{(r-k)!k!}, \tag{B.0.5}$$

and use this to conclude that when $\lim_{n\to\infty}\mathbb{E}[(X_n)_r] = \mathbb{E}[(X)_r]$ for all $r \geq 1$, where X_n and X are all integer-valued non-negative random variables, then also $X_n \xrightarrow{d} X$.

Solution to Exercise 2.7.   By the discussion around (2.1.15), we have that the sum $\sum_{r=k}^{n}(-1)^{k+r}\frac{\mathbb{E}[(X)_r]}{(r-k)!k!}$ is alternatingly larger and smaller than $\mathbb{P}(X = k)$. Thus, it suffices to prove that, when (B.0.4) holds, then

$$\lim_{n\to\infty}\sum_{r=k}^{n}(-1)^{k+r}\frac{\mathbb{E}[(X)_r]}{(r-k)!k!} = \sum_{r=k}^{\infty}(-1)^{k+r}\frac{\mathbb{E}[(X)_r]}{(r-k)!k!}. \tag{B.0.6}$$

This is equivalent to the statement that

$$\lim_{n\to\infty}\sum_{r=n}^{\infty}(-1)^{k+r}\frac{\mathbb{E}[(X)_r]}{(r-k)!k!} = 0. \tag{B.0.7}$$

<div align="right">_{e:alternatingequiv}</div>

To prove (B.0.7), we bound

$$\Big|\sum_{r=n}^{\infty}(-1)^{k+r}\frac{\mathbb{E}[(X)_r]}{(r-k)!k!}\Big| \leq \sum_{r=n}^{\infty}\frac{\mathbb{E}[(X)_r]}{(r-k)!k!} \to 0, \tag{B.0.8}$$

by (B.0.4).   □

Exercise 2.8. Prove (2.1.20) for $r = 2$.

Solution to Exercise 2.8. For $r = 2$, we note that

$$\mathbb{E}[(X)_r] = \mathbb{E}[X^2] - \mathbb{E}[X], \tag{B.0.9}$$

and, for $X = \sum_{i \in \mathcal{I}} I_i$ a sum of indicators,

$$\mathbb{E}[X^2] = \sum_{i,j} \mathbb{E}[I_i I_j] = \sum_{i \neq j} \mathbb{P}(I_i = I_j = 1) + \sum_i \mathbb{P}(I_i = 1). \tag{B.0.10}$$

Using that $\mathbb{E}[X] = \sum_i \mathbb{P}(I_i = 1)$, we thus arrive at

$$\mathbb{E}[(X)_r] = \sum_{i \neq j} \mathbb{P}(I_i = I_j = 1), \tag{B.0.11}$$

which is (2.1.20) for $r = 2$. $\qquad\square$

Exercise 2.9. Prove that when X is a Poisson random variable with mean λ, then

$$\mathbb{E}[(X)_r] = \lambda^r. \tag{B.0.12}$$

Solution to Exercise 2.9. We write out

$$\mathbb{E}[(X)_r] = \mathbb{E}[X(X-1)\cdots(X-r+1)] = \sum_{x=0}^{\infty} x(x-1)\cdots(x-r+1)\mathbb{P}(X=x)$$

$$= \sum_{x=r}^{\infty} x(x-1)\cdots(x-r+1)e^{-\lambda}\frac{\lambda^x}{x!}$$

$$= \lambda^r \sum_{x=r}^{\infty} e^{-\lambda}\frac{\lambda^{x-r}}{(x-r)!} = \lambda^r. \tag{B.0.13}$$

$\qquad\square$

Exercise 2.10. Prove Theorem 2.7 using Theorem 2.5.

Solution to Exercise 2.10. We prove Theorem 2.7 by induction on $d \geq 1$. The induction hypothesis is that (2.1.20) holds for all measures $\mathbb{P}$ with corresponding expectations $\mathbb{E}$ and all $r_1, \ldots, r_d$.

Theorem 2.7 for $d = 1$ is Theorem 2.5, which initializes the induction hypothesis. We next advance the induction hypothesis by proving (2.1.20) for $d + 1$. For this, we first note that we may assume that $\mathbb{E}[(X_{d+1,n})_{r_{d+1}}] > 0$, since $(X_{d+1,n})_{r_{d+1}} \geq 0$ and when $\mathbb{E}[(X_{d+1,n})_{r_{d+1}}] = 0$, then $(X_{d+1,n})_{r_{d+1}} \equiv 0$, so that (2.1.20) follows. Then, we define the measure $\mathbb{P}_{X,d}$ by

$$\mathbb{P}_{X,d}(E) = \frac{\mathbb{E}\big[(X_{d+1,n})_{r_{d+1}} \mathbb{1}_E\big]}{\mathbb{E}[(X_{d+1,n})_{r_{d+1}}]}, \tag{B.0.14}$$

for all possible measurable events E. Then,

$$\mathbb{E}[(X_{1,n})_{r_1} \cdots (X_{d,n})_{r_d} (X_{d+1,n})_{r_{d+1}}] = \mathbb{E}[(X_{d+1,n})_{r_{d+1}}] \mathbb{E}_{X,d}\big[(X_{1,n})_{r_1} \cdots (X_{d,n})_{r_d}\big]. \tag{B.0.15}$$

By the induction hypothesis applied to the measure $\mathbb{P}_{X,d}$, we have that

$$\mathbb{E}_{X,d}\Big[(X_{1,n})_{r_1}\cdots(X_{d,n})_{r_d}\Big] = \sideset{}{^*}\sum_{i_1^{(1)},\ldots,i_{r_1}^{(1)}\in\mathcal{I}_1}\cdots\sideset{}{^*}\sum_{i_1^{(d)},\ldots,i_{r_d}^{(d)}\in\mathcal{I}_d}\mathbb{P}_{X,d}(I_{i_s}^{(l)}=1\forall l=1,\ldots,d\&s=1,\ldots,r_l). \tag{B.0.16}$$

Next, we define the measure $\mathbb{P}_{X,d}$ by

$$\mathbb{P}_{\vec{i}_d}(E) = \frac{\mathbb{E}\Big[\prod_{l=1}^d I_{i_s}^{(l)}\mathbb{1}_E\Big]}{\mathbb{P}(I_{i_s}^{(l)}=1\ \forall l=1,\ldots,d,\ s=1,\ldots,r_l)}, \tag{B.0.17}$$

so that

$$\mathbb{E}[(X_{d+1,n})_{r_{d+1}}]\mathbb{P}_{X,d}\big(I_{i_s}^{(l)}=1\ \forall l=1,\ldots,d,\ s=1,\ldots,r_l\big)$$
$$= \mathbb{E}_{\vec{i}_d}[(X_{d+1,n})_{r_{d+1}}]\mathbb{P}\big(I_{i_s}^{(l)}=1\ \forall l=1,\ldots,d,\ s=1,\ldots,r_l\big). \tag{B.0.18}$$

Again by Theorem 2.5,

$$\mathbb{E}_{\vec{i}_d}[(X_{d+1,n})_{r_{d+1}}] = \sideset{}{^*}\sum_{i_1^{(d+1)},\ldots,i_{r_1}^{(d+1)}\in\mathcal{I}_{d+1}}\mathbb{P}_{\vec{i}_d}\big(I_{i_1}^{(d+1)}=\cdots=I_{i_{r_{d+1}}}^{(d+1)}=1\big). \tag{B.0.19}$$

Then, the claim for $d+1$ follows by noting that

$$\mathbb{P}\big(I_{i_s}^{(l)}=1\ \forall l=1,\ldots,d,\ s=1,\ldots,r_l\big)\mathbb{P}_{\vec{i}_d}\big(I_{i_1}^{(d+1)}=\cdots=I_{i_{r_{d+1}}}^{(d+1)}=1\big) \tag{B.0.20}$$
$$= \mathbb{P}\big(I_{i_s}^{(l)}=1\ \forall l=1,\ldots,d+1,\ s=1,\ldots,r_l\big).$$

$\square$

Exercise 2.11. Prove (2.2.6).

Solution to Exercise 2.11. Observe that

$$\sum_x|p_x-q_x| = \sum_x(p_x-q_x)\mathbb{1}_{\{p_x>q_x\}} + \sum_x(q_x-p_x)\mathbb{1}_{\{q_x>p_x\}} \tag{B.0.21}$$

$$0 = 1-1 = \sum_x(p_x-q_x) = \sum_x(p_x-q_x)\mathbb{1}_{\{p_x>q_x\}} + \sum_x(p_x-q_x)\mathbb{1}_{\{q_x>p_x\}}. \tag{B.0.22}$$

We add the two equalities to obtain

$$\sum_x|p_x-q_x| = 2\sum_x(p_x-q_x)\mathbb{1}_{\{p_x>q_x\}}.$$

Complete the solution by observing that

$$\sum_x(p_x-\min(p_x,q_x)) = \sum_x(p_x-q_x)\mathbb{1}_{\{p_x>q_x\}}.$$

$\square$

Exercise 2.12. Prove (2.2.9) and (2.2.11).

Solution to Exercise 2.12. The proof of (2.2.11) is the continuous equivalent of the proof of (2.2.9). Therefore, we will only prove (2.2.9).

Let Ω be the set of possible outcomes of the probability mass functions $\{p_x\}$ and $\{q_x\}$. The set Ω can be partitioned into two subsets

$$\Omega_1 = \{x \in \Omega : p_x \geq q_x\} \qquad \text{and} \qquad \Omega_2 = \{x \in \Omega : p_x < q_x\}.$$

Since $\{p_x\}$ and $\{q_x\}$ are probability distribution functions, the sum $\sum_{x \in \Omega}(p_x - q_x)$ equals zero. Therefore,

$$\sum_{x \in \Omega} |p_x - q_x| = \sum_{x \in \Omega_1} (p_x - q_x) - \sum_{x \in \Omega_2} (p_x - q_x)$$
$$0 = \sum_{x \in \Omega} (p_x - q_x) = \sum_{x \in \Omega_1} (p_x - q_x) + \sum_{x \in \Omega_2} (p_x - q_x)$$

Adding and subtracting the above equations yields

$$\sum_{x \in \Omega} |p_x - q_x| = 2 \sum_{x \in \Omega_1} (p_x - q_x) = -2 \sum_{x \in \Omega_2} (p_x - q_x).$$

Hence, there exists a set $A \subseteq \Omega$ such that $|F(A) - G(A)| \geq \frac{1}{2} \sum_{x \in \Omega} |p_x - q_x|$. It remains to show that $|F(A) - G(A)| \leq \frac{1}{2} \sum_{x \in \Omega} |p_x - q_x|$ for all $A \subseteq \Omega$.

Let A be any subset of Ω. Just as the set Ω, the set A can be partitioned into two subsets

$$A_1 = A \cap \Omega_1 \qquad \text{and} \qquad A_2 = A \cap \Omega_2,$$

so that

$$|F(A) - G(A)| = |\sum_{x \in A_1} (p_x - q_x) + \sum_{x \in A_2} (p_x - q_x)| = |\alpha_A + \beta_A|.$$

Since α_A is non-negative and β_A non-positive, it holds that

$$|\alpha_A + \beta_A| \leq \max_A (\alpha_A, -\beta_A).$$

The quantity α_A satisfies

$$\alpha_A \leq \sum_{x \in \Omega_1} (p_x - q_x) = \frac{1}{2} \sum_{x \in \Omega} |p_x - q_x|,$$

while β_A satisfies

$$\beta_A \geq \sum_{x \in \Omega_2} (p_x - q_x) = -\frac{1}{2} \sum_{x \in \Omega} |p_x - q_x|.$$

Therefore,

$$|F(A) - G(A)| \leq \frac{1}{2} \sum_{x \in \Omega} |p_x - q_x| \qquad \forall A \subseteq \Omega,$$

which completes the proof. $\qquad\qquad\square$

Exercise 2.13. Let $X \sim \text{BIN}(n, \lambda/n)$ and $Y \sim \text{Poi}(\lambda)$. Write $f_i = \mathbb{P}(X = i)$ and $g_i = \mathbb{P}(Y = i)$. Prove that Theorem 2.9 implies that $d_{\text{TV}}(f, g) \leq \lambda^2/n$. Conclude also that, for every $i \in \mathbb{N}$,

$$|\mathbb{P}(X = i) - \mathbb{P}(Y = i)| \leq 2\lambda^2/n. \tag{B.0.23}$$

Solution to Exercise 2.13. By (2.2.13) and (2.2.18)

$$d_{\mathrm{TV}}(f, g) \leq \mathbb{P}(\hat{X} \neq \hat{Y}). \tag{B.0.24}$$

Therefore, the first claim follows directly from Theorem 2.9. The second claim follows by (2.2.9).

Exercise 2.15. Let X and Y be normal distributions with equal variances σ^2 and means $\mu_X \leq \mu_Y$. Is $X \preceq Y$?

Solution to Exercise 2.15. Without any loss of generality we can take $\sigma^2 = 1$. Then for each t, and with Z a standard normal variate

$$\mathbb{P}(X \geq t) = \mathbb{P}(Z \geq t - \mu_X) \leq \mathbb{P}(Z \geq t - \mu_Y) = \mathbb{P}(Y \geq t),$$

whence $X \preceq Y$. $\qquad\square$

Exercise 2.16. Let X and Y be normal distributions with variances $\sigma_X^2 < \sigma_Y^2$ and equal means μ. Is $X \preceq Y$?

Solution to Exercise 2.16. The answer is negative. Take X standard normal and $Y \sim N(0, 2)$, then $X \preceq Y$ implies

$$\mathbb{P}(Y \geq t) \geq \mathbb{P}(X \geq t) = \mathbb{P}(Y \geq t\sqrt{2}),$$

for each t. However, this is false for $t < 0$. $\qquad\square$

Exercise 2.17. Compute $I(a)$ for $\{X_i\}_{i=1}^{\infty}$ being independent Poisson random variables with mean λ. Show that, for $a > \lambda$,

$$\mathbb{P}\Big(\sum_{i=1}^{n} X_i \geq na \Big) \leq e^{-nI_\lambda(a)}, \tag{B.0.25}$$

where $I_\lambda(a) = a(\log a/\lambda - 1) + \lambda$. Show also that, for $a < \lambda$

$$\mathbb{P}\Big(\sum_{i=1}^{n} X_i \leq na \Big) \leq e^{-nI_\lambda(a)}. \tag{B.0.26}$$

Prove that $I_\lambda(a) > 0$ for all $a \neq \lambda$.

Solution to Exercise 2.17. Let X be Poisson distributed with parameter λ, then

$$\mathbb{E}[e^{tX}] = \sum_{n=0}^{\infty} e^{tn} e^{-\lambda} \frac{\lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} = e^{\lambda(e^t - 1)}.$$

Put

$$g(t) = at - \log \mathbb{E}[e^{tX}] = at + \lambda - \lambda e^t$$

then $g'(t) = a - \lambda e^t = 0 \Leftrightarrow t = \log(a/\lambda)$. Hence, $I(a)$ in (2.4.12) is equal to $I(a) = I_\lambda(a) = a(\log(a/\lambda) - 1) + \lambda$ and with $a > \lambda$ we obtain from (2.4.9),

$$\mathbb{P}(\sum_{i=1}^{n} X_i \geq an) \leq e^{-nI_\lambda(a)}.$$

This proves (B.0.25). For $a < \lambda$, we get $g'(t) = a - \lambda e^t = 0$ for $t = \log(a/\lambda) < 0$ and we get again

$$I_\lambda(a) = a(\log a/\lambda - 1) + \lambda.$$

By (2.4.9), with $a < \lambda$, we obtain (B.0.26).

$I_\lambda(\lambda) = 0$ and $\frac{d}{da}I_\lambda(a) = \log a - \log \lambda$, so that for $a < \lambda$ the function $a \mapsto I_\lambda(a)$ decreases, whereas for $a > \lambda$ the function $a \mapsto I_\lambda(a)$ increases. Because $I_\lambda(\lambda) = 0$, this shows that for all $a \neq \lambda$, we have $I_\lambda(a) > 0$. $\qquad\square$

Exercise 2.19. Show that when $\{M_n\}_{n=0}^\infty$ is a martingale process, then $\mu = \mathbb{E}[M_n]$ is independent of n.

Solution to Exercise 2.19. By taking expectations on both sides of (2.5.2),

$$\mathbb{E}[M_n] = \mathbb{E}[\mathbb{E}[M_{n+1}|M_1, M_2, \ldots, M_n]] = \mathbb{E}[M_{n+1}],$$

since according to the theorem of total probability:

$$\mathbb{E}[\mathbb{E}[X|Y_1, \ldots, Y_n]] = \mathbb{E}[X].$$

$\square$

Exercise 2.20. Let $\{X_i\}_{i=0}^\infty$ be an independent sequence of random variables with $\mathbb{E}[X_i] = 1$. Show that

$$M_n = \prod_{i=0}^n X_i$$

is a martingale.

Solution to Exercise 2.20. First we show that $\mathbb{E}[|M_n|] < \infty$. Indeed, since $\mathbb{E}[|X_i|] < \infty$, $\forall i$, and since the fact that X_i is an independent sequence implies that the sequence $|X_i|$ is independent we get

$$\mathbb{E}[|M_n|] = \prod_{i=0}^n \mathbb{E}[|X_i|] < \infty.$$

To verify the martingale condition, we write

$$
\begin{aligned}
\mathbb{E}[M_{n+1}|X_1, X_2, \ldots, X_n] &= \mathbb{E}\Big[\prod_{i=1}^{n+1} X_i \Big| X_1, X_2, \ldots, X_n\Big] \\
&= \prod_{i=1}^n X_i \cdot \mathbb{E}[X_{n+1}|X_1, X_2, \ldots, X_n] = M_n \mathbb{E}[X_{n+1}] = M_n \qquad \text{a.s.}
\end{aligned}
$$

$\square$

Exercise 2.21. Let $\{X_i\}_{i=0}^\infty$ be an independent sequence of random variables with $\mathbb{E}[X_i] = 0$. Show that

$$M_n = \sum_{i=0}^n X_i$$

is a martingale.

Solution to Exercise 2.21.   First we show that $\mathbb{E}[|M_n|] < \infty$. Indeed, since $\mathbb{E}[|X_i|] < \infty$, $\forall i$,

$$\mathbb{E}[|M_n|] = \mathbb{E}\Big|\sum_{i=1}^{n} X_i\Big| \leq \sum_{i=1}^{n} \mathbb{E}|X_i| < \infty.$$

To verify the martingale condition, we write

$$
\begin{aligned}
\mathbb{E}[M_{n+1}|M_1, M_2, \ldots, M_n] &= \mathbb{E}[\sum_{i=1}^{n+1} X_i | X_0, X_1, \ldots, X_n] \\
&= \sum_{i=1}^{n} X_i + \mathbb{E}[X_{n+1}|X_0, X_1, \ldots, X_n] = M_n + \mathbb{E}[X_{n+1}] = M_n \qquad \text{a.s.}
\end{aligned}
$$

$\square$

Exercise 2.22.   Let $M_n = \mathbb{E}[Y|X_0, \ldots, X_n]$ for some random variables $\{X_n\}_{n=1}^{\infty}$ and Y with $\mathbb{E}[|Y|] < \infty$. Show that $\{M_n\}_{n=0}^{\infty}$ is a martingale process with respect to $\{X_n\}_{n=0}^{\infty}$.

Solution to Exercise 2.22.   Again we first that $\mathbb{E}[|M_n|] < \infty$. Indeed, since $\mathbb{E}[|X_i|] < \infty$, $\forall i$,

$$\mathbb{E}[|M_n|] = \mathbb{E}\Big|\mathbb{E}[Y|X_0, \ldots, X_n]\Big| \leq \mathbb{E}\Big[\mathbb{E}[|Y||X_0, \ldots, X_n]\Big] = \mathbb{E}[|Y|] < \infty.$$

To verify the martingale condition, we write

$$
\begin{aligned}
\mathbb{E}[M_{n+1}|X_0, \ldots, X_n] &= \mathbb{E}\Big[\mathbb{E}[Y|X_0, \ldots, X_{n+1}]\Big|X_0, \ldots, X_n\Big] \\
&= \mathbb{E}[Y|X_0, \ldots, X_n] = M_n + \mathbb{E}[X_{n+1}] = M_n \qquad \text{a.s.}
\end{aligned}
$$

$\square$

Exercise 2.23.   Prove that when the martingale $\{M_n\}_{n=0}^{\infty}$ is *non-negative*, i.e., when $M_n \geq 0$ with probability 1 for all $n \geq 1$, then $M_n \xrightarrow{a.s.} M_\infty$ for some limiting random variable M_∞ which is finite with probability 1.

Solution to Exercise 2.23.   Since M_n is non-negative we have $\mathbb{E}[|M_n|] = \mathbb{E}[M_n] = \mu \leq M$, by Exercise 2.19. Hence, according to Theorem 2.21 we have convergence to some limiting random variable M_∞. $\square$

Exercise 2.24.   Let $\{X_i\}_{i=0}^{\infty}$ be an independent sequence of random variables with $\mathbb{E}[X_i] = 1$ and for which $X_i \geq 0$ with probability 1. Show that the martingale

$$M_n = \prod_{i=0}^{n} X_i$$

converges in probability to a random variable which is finite with probability 1. Hint: Prove that $\mathbb{E}[|M_n|] = \mathbb{E}[M_n] = 1$, and apply Exercise 2.23.

Solution to Exercise 2.24.   Since $X_i \geq 0$, we have $M_n = \prod_{i=0}^{n} X_i \geq 0$, hence the claim is immediate from Exercise 2.23. $\square$

Exercise 2.25. For $i = 1, \ldots, m$, let $\{M_n^{(i)}\}_{n=0}^{\infty}$ be a sequence of martingales with respect to $\{X_n\}_{n=0}^{\infty}$. Show that

$$M_n = \max_{i=0}^{m} M_n^{(i)} \tag{B.0.27}$$

is a submartingale with respect to $\{X_n\}_{n=0}^{\infty}$.

Solution to Exercise 2.25. First,

$$\mathbb{E}[|M_n|] \leq \sum_{i=1}^{m} \mathbb{E}[|M_n^{(i)}|] < \infty. \tag{B.0.28}$$

Secondly, since $\mathbb{E}[\max\{X, Y\}] \geq \max\{\mathbb{E}[X], \mathbb{E}[Y]\}$, we obtain

$$\mathbb{E}[M_{n+1}|X_0, \ldots, X_n] = \mathbb{E}\Big[\max_{i=0}^{m} M_{n+1}^{(i)}|X_0, \ldots, X_n\Big] \geq \max_{i=0}^{m} \mathbb{E}[M_{n+1}^{(i)}|X_0, \ldots, X_n] \tag{B.0.29}$$

$$= \max_{i=0}^{m} M_n^{(i)} = M_n, \tag{B.0.30}$$

where we use that $\{M_n^{(i)}\}_{n=0}^{\infty}$ is a sequence of martingales with respect to $\{X_n\}_{n=0}^{\infty}$.

Exercise 2.27. Let $\{X_i\}_{i=0}^{\infty}$ be an independent identically distributed sequence of random variables with $\mathbb{E}[X_i] = 0$ and $|X_i| \leq 1$, and define the martingale $\{M_n\}_{n=0}^{\infty}$ by

$$M_n = \sum_{i=0}^{n} X_i. \tag{B.0.31}$$

Show that

$$\mathbb{P}(|M_n| \geq a) \leq 2\exp\Big(-\frac{a^2}{2n}\Big). \tag{B.0.32}$$

e:AH-speccase-ex

Take $a = x\sqrt{n}$, and prove by using the central limit theorem that $\mathbb{P}(|M_n| \geq a)$ converges. Compare the limit to the bound in (B.0.32).

Solution to Exercise 2.27. Since $\mathbb{E}[X_i] = 0$, we have, by Exercise 2.21, that $M_n = \sum_1^n X_i$ is a martingale, with by hypothesis,

$$-1 \leq M_n - M_{n-1} = X_n \leq 1,$$

so that the condition of Theorem 2.23 is satisfied with $\alpha_i = \beta_i = 1$. Since $\mathbb{E}[M_n] = 0$, we have $\mu = 0$ and $\sum_{i=0}^{n}(\alpha_i + \beta_i)^2 = 4(n+1)$, hence from (2.5.18) we get (B.0.32).

We now compare the Azuma-Hoeffding bound (B.0.32) with the central limit approximation. With $a = x\sqrt{n+1}$, and $\sigma^2 = \text{Var}(X_i)$,

$$\mathbb{P}(|M_n| \geq a) = \mathbb{P}(|M_n| \geq x\sqrt{n+1}) = \mathbb{P}(|M_n|/\sigma\sqrt{n+1} \geq x/\sigma) \to 2(1 - \Phi(x/\sigma)),$$

where $\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} \, du$. A well-known approximation tells us that

$$2(1 - \Phi(t)) \sim 2\phi(t)/t = \frac{\sqrt{2}}{t\sqrt{\pi}} e^{-t^2/2},$$

so that by the central limit theorem and this approximation

$$\mathbb{P}(|M_n| \geq a) \sim \frac{\sigma\sqrt{2}}{x\sqrt{\sigma\pi}} e^{-x^2/2\sigma^2} = \frac{\sigma\sqrt{2(n+1)}}{a\sqrt{\pi}} e^{-a^2/2(n+1)\sigma^2}$$

Finally $\sigma^2 \leq 1$, so that the leading order term and with $a = x\sqrt{n+1}$, the inequality of Azuma-Hoefding is quite sharp! $\qquad\square$

Solutions to the exercises of Chapter 3.

Exercise 3.1.   Show that $\eta = 0$ precisely when $p_0 = 0$.

Solution to Exercise 3.1.   When $\eta = 0$, then, since η is a solution of $\eta = G_X(\eta)$, we must have that

$$p_0 = G_X(0) = 0. \tag{B.0.33}$$

$\square$

Exercise 3.2.   When the offspring distribution is given by

$$p_x = (1-p)\mathbb{1}_{\{x=0\}} + p\mathbb{1}_{\{x=2\}}, \tag{B.0.34}$$

<div align="right">_{e:p-binbranch-ex}</div>

we speak of *binary branching*. Prove that $\eta = 1$ when $p \leq 1/2$ and, for $p > 1/2$,

$$\eta = \frac{1-p}{p}. \tag{B.0.35}$$

<div align="right">_{e:eta-binbranch-ex}</div>

Solution to Exercise 3.2.   We note that for $p = \{p_x\}_{x=0}^{\infty}$ given in (B.0.34), and writing $q = 1 - p$, we have that $\mathbb{E}[X] = 2p$, so that $\eta = 1$ when $p \leq 1/2$, and

$$G_X(s) = q + ps^2. \tag{B.0.36}$$

<div align="right">_{e:G-binbranch}</div>

Since η satisfies $\eta = G(\eta)$, we obtain that

$$\eta = q + p\eta^2, \tag{B.0.37}$$

of which the solutions are

$$\eta = \frac{1 \pm \sqrt{1 - 4pq}}{2p}. \tag{B.0.38}$$

Noting further that $1 - 4pq = 1 - 4p(1-p) = 4p^2 - 4p + 1 = (2p-1)^2$, and $p > 1/2$, we arrive at

$$\eta = \frac{1 \pm (2p-1)}{2p}. \tag{B.0.39}$$

Since $\eta \in [0, 1)$ for $p > 1/2$, we must have that

$$\eta = \frac{1 - (2p-1)}{2p} = \frac{1-p}{p}. \tag{B.0.40}$$

$\square$

Exercise 3.3.   Let the probability distribution $\{p_k\}_{k=0}^{\infty}$ be given by

$$\begin{cases} p_k = b(1-p)^{k-1} & \text{for } k = 1, 2, \ldots; \\ p_0 = 1 - b/p & \text{for } k = 0, \end{cases} \tag{B.0.41}$$

<div align="right">_{e:pk-geom-ex}</div>

so that, for $b = p$, the offspring distribution has a geometric distribution with success probability p. Show that the extinction probability η is given by $\eta = 1$ if $\mu = \mathbb{E}[X] = b/p^2 \leq 1$, while, with the abbreviation $q = 1 - p$,

$$\eta = \frac{1 - \mu p}{q}. \tag{B.0.42}$$

<div align="right">_{e:eta-geom-ex}</div>

Solution to Exercise 3.3. We compute that

$$G_X(s) = 1 - b/p + \sum_{k=1}^{\infty} b(1-p)^{k-1} s^k = 1 - \frac{b}{p} + \frac{bs}{1-qs}, \tag{B.0.43}$$

so that

$$\mu = G'_X(1) = \frac{b}{p^2}. \tag{B.0.44}$$

As a result, $\eta = 1$ if $\mu = b/p^2 \leq 1$ follows from Theorem 3.1. Now, when $\mu = b/p^2 > 1$, then $\eta < 1$ is the solution of $G_X(\eta) = \eta$, which becomes

$$1 - \frac{b}{p} + \frac{b\eta}{1-q\eta} = \eta, \tag{B.0.45}$$

which has the solution given by (B.0.42). □

Exercise 3.4. Let the probability distribution $\{p_k\}_{k=0}^{\infty}$ be given by (3.1.17). Show that $G_n(s)$, the generating function of Z_n is given by

$$G_n(s) = \begin{cases} 1 - \mu^n \frac{1-\eta}{\mu^n - \eta} + \frac{\mu^n \left(\frac{1-\eta}{\mu^n - \eta}\right)^2 s}{1 - \left(\frac{\mu^n - 1}{\mu^n - \eta}\right)} & \text{when } b \neq p^2; \\ \frac{nq - (nq-p)s}{p + nq - nqs} & \text{when } b = p^2. \end{cases} \tag{B.0.46}$$

Solution to Exercise 3.4. We note that $s \mapsto G_X(s)$ in (B.0.43) has the property that for any points s, u, v

$$\frac{G_X(s) - G_X(u)}{G_X(s) - G_X(v)} = \frac{s-u}{s-v} \frac{1-qv}{1-qu}. \tag{B.0.47}$$

Taking $u = \eta, v = 1$ and using that $G_X(\eta) = \eta$ by Theorem 3.1, we obtain that, if $\eta < 1$,

$$\frac{G_X(s) - \eta}{G_X(s) - 1} = \frac{s-\eta}{s-1} \frac{p}{1-q\eta}. \tag{B.0.48}$$

By (B.0.42), we further obtain that

$$\frac{p}{1-q\eta} = \mu^{-1} = p^2/b, \tag{B.0.49}$$

so that we arrive at

$$\frac{G_X(s) - \eta}{G_X(s) - 1} = \frac{1}{\mu} \frac{s-\eta}{s-1}. \tag{B.0.50}$$

Since $G_n(s)$ is the n-fold iteration of $s \mapsto G_X(s)$, we thus arrive at

$$\frac{G_n(s) - \eta}{G_n(s) - 1} = \frac{1}{\mu^n} \frac{s-\eta}{s-1}, \tag{B.0.51}$$

of which the solution is given by the first line of (B.0.46).

When $\mu = 1$, then we have that $b = p^2$, so that

$$G_X(s) = \frac{q - (q-p)s}{1-qs}. \tag{B.0.52}$$

We now prove by induction that $G_n(s)$ is equal to the second line of (B.0.46). For $n = 1$, we have that $G_1(s) = G_X(s)$, so that the induction is initialized by (B.0.52).

To advance the induction, we assume it for n and advance it to $n + 1$. For this, we note that, since $G_n(s)$ is the n-fold iteration of $s \mapsto G_X(s)$, we have

$$G_{n+1}(s) = G_n(G_X(s)). \tag{B.0.53}$$

By the induction hypothesis, we have that $G_n(s)$ is equal to the second line of (B.0.46), so that

$$G_{n+1}(s) = \frac{nq - (nq - p)G(s)}{p + nq - nqG_X(s)} = \frac{nq(1 - qs) - (nq - p)(q - (q - p)s)}{(p + nq)(1 - qs) - nq(q - (q - p)s)}. \tag{B.0.54}$$

Note that, using $p = 1 - q$,

$$nq(1 - qs) - (nq - p)(q - (q - p)s) = [nq - (nq - p)q] + s[(q - p)(nq - p) - nq^2] \tag{B.0.55}$$
$$= (n + 1)qp - s[qp(n + 1) - p^2],$$

while

$$(p + nq)(1 - qs) - nq(q - (q - p)s) = [(p + nq) - nq^2] + s[(q - p)nq - (p + nq)q] \tag{B.0.56}$$
$$= [p + nqp] - s(n + 1)pq = p[p + (n + 1)q] - s(n + 1)pq,$$

and dividing (B.0.55) by (B.0.56) advances the induction hypothesis. □

Exercise 3.5. Conclude from Exercise 3.4 that, for $\{p_k\}_{k=0}^{\infty}$ in (3.1.17),

$$\mathbb{P}(Z_n > 0, \ \exists m > n \text{ such that } Z_m = 0) = \begin{cases} \mu^n \frac{1 - \eta}{\mu^n - \eta} & \text{when } b < p^2; \\ \frac{p}{p + nq} & \text{when } b = p^2; \\ \frac{(1 - \eta)\eta}{\mu^n - \eta} & \text{when } b > p^2. \end{cases} \tag{B.0.57}$$

Solution to Exercise 3.5. We first note that

$$\mathbb{P}(Z_n > 0, \ \exists m > n \text{ such that } Z_m = 0) = \mathbb{P}(\exists m > n \text{ such that } Z_m = 0) - \mathbb{P}(Z_n = 0) = \eta - \mathbb{P}(Z_n = 0). \tag{B.0.58}$$

We next compute, using (3.1.19),

$$\mathbb{P}(Z_n = 0) = G_n(0) = \begin{cases} 1 - \mu^n \frac{1 - \eta}{\mu^n - \eta} & \text{when } b \neq p^2; \\ \frac{nq}{p + nq} & \text{when } b = p^2. \end{cases} \tag{B.0.59}$$

Using that $\eta = 1$ when $b \leq p^2$ gives the first two lines of (B.0.57). When $\eta < 1$, so that $\mu > 1$, we thus obtain

$$\mathbb{P}(Z_n > 0, \ \exists m > n \text{ such that } Z_m = 0) = (1 - \eta)\left[\frac{\mu^n}{\mu^n - \eta} - 1\right] = \frac{(1 - \eta)\eta}{\mu^n - \eta}. \tag{B.0.60}$$

This proves the third line of (B.0.57). □

Exercise 3.6. In the case of binary branching, i.e., when p is given by (3.1.15), show that

$$G_T(s) = \frac{1 - \sqrt{1 - 4s^2 pq}}{2sp}. \tag{B.0.61}$$

Solution to Exercise 3.6. By (B.0.36), we have that $G(s) = q + ps^2$. Thus, by (3.1.23), we obtain

$$G_T(s) = s(q + pG_T(s)^2), \tag{B.0.62}$$

of which the solutions are given by

$$G_T(s) = \frac{1 \pm \sqrt{1 - 4s^2pq}}{2sp}. \tag{B.0.63}$$

Since $G_T(0) = 0$, we must that that

$$G_T(s) = \frac{1 - \sqrt{1 - 4s^2pq}}{2sp}. \tag{B.0.64}$$

$\square$

Exercise 3.7. Show, using Theorem 3.2, that, for $\{p_k\}_{k=0}^{\infty}$ in (3.1.17),

$$G_T(s) = \frac{\sqrt{(p + s(b - pq))^2 - 4pqs(p - b)} - (p + sbq)}{2pq} \tag{B.0.65}$$

e:TGF-comp-ex

Solution to Exercise 3.7. By (B.0.43), we have $G_X(s) = 1 - \frac{b}{p} + \frac{bs}{1 - qs}$. Thus, by (3.1.23), we obtain

$$G_T(s) = s\left[1 - \frac{b}{p} + \frac{bG_T(s)}{1 - qG_T(s)}\right]. \tag{B.0.66}$$

Multiplying by $p(1 - qG_T(s))$, and using that $p + q = 1$, leads to

$$pG_T(s)(1 - qG_T(s)) = s[(p - b)(1 - qG_T(s)) + bpG_T(s)] = s[(p - b) + (b - pq)G_T(s)]. \tag{B.0.67}$$

We can simplify the above to

$$pqG_T(s)^2 + (p + s(b - pq))G_T(s) + s(p - b) = 0, \tag{B.0.68}$$

of which the two solutions are given by

$$G_T(s) = \frac{-(p + sbq) \pm \sqrt{(p + s(b - pq))^2 - 4pqs(p - b)}}{2pq}. \tag{B.0.69}$$

Since $G_T(s) \geq 0$ for all $s \geq 0$, we thus arrive at

$$G_T(s) = \frac{\sqrt{(p + s(b - pq))^2 - 4pqs(p - b)} - (p + sbq)}{2pq}. \tag{B.0.70}$$

$\square$

Exercise 3.8. Complete the proof of Theorem 3.3 by conditioning on Z_{n-1} and showing that

$$\mathbb{E}\left[\sum_{i=1}^{Z_{n-1}} X_{n,i} \mid Z_{n-1} = m\right] = m\mu, \tag{B.0.71}$$

e:Znmart-ex

so that

$$\mathbb{E}[Z_n] = \mu\mathbb{E}[Z_{n-1}].$$

Solution to Exercise 3.8.   Compute

$$\mathbb{E}[Z_n|Z_{n-1} = m] \quad = \mathbb{E}[\sum_{i=1}^{Z_{n-1}} X_{n,i}|Z_{n-1} = m] = \mathbb{E}[\sum_{i=1}^{m} X_{n,i}|Z_{n-1} = m]$$
$$= \sum_{i=1}^{m} \mathbb{E}[X_{n,i}] = m\mu,$$

so that, by taking double expectations,

$$\mathbb{E}[Z_n] = \mathbb{E}[\mathbb{E}[Z_n|Z_{n-1}]] = \mathbb{E}[\mu Z_{n-1}] = \mu\mathbb{E}[Z_{n-1}].$$

$\square$

Exercise 3.9.   Prove that $\{\mu^{-n}Z_n\}_{n\geq 1}$ is a martingale.

Solution to Exercise 3.9.   Using induction we conclude from the previous exercise that

$$\mathbb{E}[Z_n] = \mu\mathbb{E}[Z_{n-1}] = \mu^2\mathbb{E}[Z_{n-2}] = \ldots = \mu^n\mathbb{E}[Z_0] = \mu^n.$$

Hence,

$$\mathbb{E}[\mu^{-n}Z_n] = \mu^{-n}\mathbb{E}[Z_n] = 1.$$

Therefore, we have that, for all $n \geq 0$, $\mathbb{E}[|\mu^{-n}Z_n|] = \mathbb{E}[\mu^{-n}Z_n] < \infty$
    By the Markov property and the calculations in the previous exercise

$$\mathbb{E}[Z_n|Z_1,\ldots,Z_{n-1}] = \mathbb{E}[Z_n|Z_{n-1}] = \mu Z_{n-1},$$

so that, with $M_n = Z_n/\mu^n$,

$$\mathbb{E}[M_n|Z_1,\ldots,Z_{n-1}] = \mathbb{E}[M_n|Z_{n-1}] = \frac{1}{\mu^n}\mu Z_{n-1} = M_{n-1},$$

almost surely. Therefore, $M_n = \mu^{-n}Z_n$ is a martingale with respect to $\{Z_n\}_{n=1}^{\infty}$. $\square$

Exercise 3.10.   When the branching process is *critical*, note that $Z_n \to 0$ in probability. On the other hand, conclude that $\mathbb{E}[Z_n] = 1$ for all $n \geq 1$.

Solution to Exercise 3.10.   For a critical BP we have $\mu = 1$, and so Z_n is a martingale. Therefore, for all n,

$$\mathbb{E}[Z_n] = \mathbb{E}[Z_0] = 1.$$

On the other hand, if $\mathbb{P}(X = 1) < 1$, then, $\eta = 1$ by Theorem 3.1, and by monotonicity,

$$\lim_{n\to\infty}\mathbb{P}(Z_n = 0) = \mathbb{P}(\lim_{n\to\infty} Z_n = 0) = \eta = 1.$$

$\square$

Exercise 3.11.   Prove Theorem 3.4 by using Theorem 3.3, together with the Markov inequality (Theorem 2.14).

Solution to Exercise 3.11.

$$\mathbb{P}(Z_n > 0) = \mathbb{P}(Z_n \geq 1) \leq \mathbb{E}[Z_n] = \mu^n,$$

by Theorem 3.3. $\square$

Exercise 3.12. Prove (3.2.6).

Solution to Exercise 3.12. Since $T = 1 + \sum_{n=1}^{\infty} Z_n$, we obtain by (3.2.1) that

$$\mathbb{E}[T] = 1 + \sum_{n=1}^{\infty} \mathbb{E}[Z_n] = 1 + \sum_{n=1}^{\infty} \mu^n = 1/(1 - \mu). \tag{B.0.72}$$

$\square$

Exercise 3.15. Prove that $p' = \{p'_x\}_{x=0}^{\infty}$ defined in (3.3.7) is a probability distribution.

Solution to Exercise 3.15. We note that $p'_x \geq 0$ for all $x \in \mathbb{N}$. Furthermore,

$$\sum_{x=0}^{\infty} p'_x = \sum_{x=0}^{\infty} \eta^{x-1} p_x = \eta^{-1} \sum_{x=0}^{\infty} \eta^x p_x = \eta^{-1} G(\eta). \tag{B.0.73}$$

Since η satisfies $\eta = G(\eta)$, it follows also that $p' = \{p'_x\}_{x=0}^{\infty}$ sums up to 1, so that p' is a probability distribution. $\square$

Exercise 3.17. Let X' have probability mass function $p' = \{p'_x\}_{x=0}^{\infty}$ defined in (3.3.7). Show that when $\eta > 0$, then

$$\mathbb{E}[X'] < 1. \tag{B.0.74}$$

Thus, the branching process with offspring distribution p' is *subcritical*.

Solution to Exercise 3.17. We note that

$$\mathbb{E}[X'] = \sum_{x=0}^{\infty} x p'_x = \sum_{x=0}^{\infty} x \eta^{x-1} p_x = G'_X(\eta). \tag{B.0.75}$$

Now, η is the smallest solution of $\eta = G_X(\eta)$, and, when $\eta > 0$, $G_X(0) = p_0 > 0$ by Exercise 3.1. Therefore, since $s \mapsto G'_X(s)$ is increasing, we must have that $G'_X(\eta) < 1$. $\square$

Exercise 3.16. Let $G_d(s) = \mathbb{E}'[s^{X_1}]$ be the probability generating function of the offspring of the dual branching process. Show that

$$G_d(s) = \frac{1}{\eta} G_X(\eta s). \tag{B.0.76}$$

Solution to Exercise 3.16. We compute

$$G_d(s) = \sum_{x=0}^{\infty} s^x p'_x = \sum_{x=0}^{\infty} s^x \eta^{x-1} p_x = \eta^{-1} \sum_{x=0}^{\infty} (\eta s)^x p_x = \frac{1}{\eta} G_X(\eta s). \tag{B.0.77}$$

$\square$

Exercise 3.18. Prove (3.4.1).

Solution to Exercise 3.18.   Since $M_n = \mu^{-n} Z_n \xrightarrow{a.s.} W_\infty$ by Theorem 3.8, by Lebesques dominated convergence theorem and the fact that, for $y \geq 0$ and $s \in [0, 1]$, we have that $s^y \leq 1$, it follows that

$$\mathbb{E}[s^{M_n}] \to \mathbb{E}[s^{W_\infty}]. \qquad\qquad \text{(B.0.78)}$$

e:conGFMnW

However,

$$\mathbb{E}[s^{M_n}] = \mathbb{E}[s^{Z_n/\mu_n}] = G_n(s^{\mu^{-n}}). \qquad\qquad \text{(B.0.79)}$$

Since $G_n(s) = G_X(G_{n-1}(s))$, we thus obtain

$$\mathbb{E}[s^{M_n}] = G_X\big(G_{n-1}(s^{\mu^{-n}})\big) = G_X\Big(G_{n-1}\big((s^{\mu^{-1}})^{\mu^{-n-1}}\big)\Big) \to G_X\big(G_W(s^{1/\mu})\big), \qquad \text{(B.0.80)}$$

again by (B.0.78).

Exercise 3.20.   Prove, using Fubini's theorem, that $\mathbb{E}[W_\infty] \leq 1$ always holds.

Solution to Exercise 3.20.   By Theorem 3.8, we have that $M_n = \mu^{-n} Z_n \xrightarrow{a.s.} W_\infty$. By Fubini's theorem, we thus obtain that

$$\mathbb{E}[W_\infty] \leq \lim_{n\to\infty} \mathbb{E}[M_n] = 1, \qquad\qquad \text{(B.0.81)}$$

where the equality follows from Theorem 3.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Exercise 3.13.   Compute $\mathbb{P}(T = k)$ for T in (3.3.2) and $\mathbb{P}(T = k)$ for T in (3.1.21) explicitly, for $k = 1, 2$ and 3.

Solution to Exercise 3.13.   For $k = 1$, we note that, in (3.3.2), $\{T = 1\} = \{X_1 = 0\}$, so that

$$\mathbb{P}(T = 1) = p_0. \qquad\qquad \text{(B.0.82)}$$

On the other hand, in (3.1.21), $T = 1$ precisely when $Z_1 = X_{1,1} = 0$, which occurs with probability p_0 as well.

For $k = 2$, since $X_i \geq 0$, we have that $\{T = 2\} = \{X_1 = 1, X_2 = 0\}$, so that

$$\mathbb{P}(T = 2) = p_0 p_1. \qquad\qquad \text{(B.0.83)}$$

On the other hand, in (3.1.21), $T = 2$ precisely when $Z_1 = X_{1,1} = 1$ and $Z_2 = X_{2,1} = 0$, which occurs with probability $p_0 p_1$ as well, as required.

For $k = 3$, since $X_i \geq 0$, we have that $\{T = 3\} = \{X_1 = 2, X_2 = X_3 = 0\} \cup \{X_1 = X_2 = 1, X_3 = 0\}$, so that

$$\mathbb{P}(T = 3) = p_0^2 p_2 + p_0 p_1^2. \qquad\qquad \text{(B.0.84)}$$

On the other hand, in (3.1.21),

$$\{T = 3\} = \{Z_1 = Z_2 = 1, Z_3 = 0\} \cup \{Z_1 = 2, Z_2 = 0\}, \qquad\qquad \text{(B.0.85)}$$

so that $\{T = 3\} = \{X_{1,1} = X_{2,1} = 1, X_{3,1} = 0\} \cup \{X_{1,1} = 2, X_{2,1} = X_{2,2} = 0\}$, which occurs with probability $p_0^2 p_2 + p_0 p_1^2$ as well, as required. This proves the equality of $\mathbb{P}(T = k)$ for T in (3.3.2) and (3.1.21) and $k = 1, 2$ and 3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Exercise 3.14. In the case of binary branching, i.e., when the offspring distribution is given by (3.1.15), show that

$$\mathbb{P}(T = k) = \frac{1}{p}\mathbb{P}(S_0 = S_{k+1} = 0, S_i > 0 \ \forall 1 \leq i \leq k), \qquad (B.0.86)$$

where $\{S_i\}_{i=1}^{\infty}$ is a simple random walk, i.e.,

$$S_i = Y_1 + \cdots + Y_i, \qquad (B.0.87)$$

where $\{Y_i\}_{i=1}^{\infty}$ are i.i.d. random variables with distribution

$$\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = p. \qquad (B.0.88)$$

This gives a one-to-one relation between random walks excursions and the total progeny of a binary branching process.

Solution to Exercise 3.14. We note that

$$\mathbb{P}(S_0 = S_{k+1} = 0, S_i > 0 \ \forall 1 \leq i \leq k) = p\mathbb{P}(S_1 = 1, S_i > 0 \ \forall 1 \leq i \leq k, S_{k+1} = 0), \qquad (B.0.89)$$

since the first step must be upwards. By (3.3.2),

$$\mathbb{P}(S_1 = 1, S_i > 0 \ \forall 1 \leq i \leq k, S_{k+1} = 0) = \mathbb{P}(T = k), \qquad (B.0.90)$$

which completes the proof. $\qquad\square$

Exercise 3.19. Prove that $\mathbb{P}(W_\infty > 0) = 1 - \eta$ implies that $\mathbb{P}(W_\infty > 0| \text{ survival}) = 1$.

Solution to Exercise 3.19. If $M_n = 0$, then $M_m = 0$ for all $m \geq n$, so that

$$\{M_\infty = 0\} = \lim_{n \to \infty}\{M_n = 0\} = \cap_{n=0}^{\infty}\{M_n = 0\}.$$

On the other hand, $\{\text{extinction}\} = \{\exists n : M_n = 0\}$ or $\{\text{survival}\} = \{\forall n, M_n > 0\}$. We hence conclude that $\{\text{survival}\} \subset \{M_\infty > 0\} = \cup_{n=0}^{\infty}\{M_n > 0\}$, and so

$$\mathbb{P}(M_\infty > 0|\text{survival}) = \frac{\mathbb{P}(M_\infty > 0 \cap \{\text{survival}\})}{\mathbb{P}(\text{survival})} = \frac{\mathbb{P}(M_\infty > 0)}{1 - \eta} = 1,$$

because it is given that $\mathbb{P}(W_\infty > 0) = 1 - \eta$. $\qquad\square$

Exercise 3.25. Verify (3.5.8) for $n = 1, 2$ and $n = 3$.

Solution to Exercise 3.25. The total offspring equals $T = 1 + \sum_{n=1}^{\infty} Z_n$, see (3.1.21). Since we search for $T \leq 3$, we must have $\sum_{n=1}^{\infty} Z_n \leq 2$ or $\sum_{n=1}^{2} Z_n \leq 2$, because $Z_k > 0$ for some $k \geq 3$ implies $Z_3 \geq 1, Z_2 \geq 1, Z_1 \geq 1$, so that $\sum_{n=1}^{\infty} Z_n \geq \sum_{n=1}^{3} Z_n \geq 3$. Then, we can write out

$$
\begin{aligned}
\mathbb{P}(T = 1) &= \mathbb{P}(\sum_{n=1}^{2} Z_n = 0) = \mathbb{P}(Z_1 = 0) = e^{-\lambda}, \\
\mathbb{P}(T = 2) &= \mathbb{P}(\sum_{n=1}^{2} Z_n = 1) = \mathbb{P}(Z_1 = 1, Z_2 = 0) = \mathbb{P}(X_{1,1} = 1)\mathbb{P}(X_{2,1} = 0) = \lambda e^{-2\lambda} \\
\mathbb{P}(T = 3) &= \mathbb{P}(\sum_{n=1}^{2} Z_n = 2) = \mathbb{P}(Z_1 = 1, Z_2 = 1, Z_3 = 0) + \mathbb{P}(Z_1 = 2, Z_2 = 0) \\
&= \mathbb{P}(X_{1,1} = 1, X_{2,1} = 1, X_{3,1} = 0) + \mathbb{P}(X_{1,1} = 2, X_{2,1} = 0, X_{2,2} = 0) \\
&= (\lambda e^{-\lambda})^2 \cdot e^{-\lambda} + e^{-\lambda}(\lambda^2/2) \cdot e^{-\lambda} \cdot e^{-\lambda} = e^{-3\lambda}\frac{3\lambda^2}{2}.
\end{aligned}
$$

These answers do coincide with $\mathbb{P}(T = n) = e^{-n\lambda} \frac{(n\lambda)^{n-1}}{n!}$, for $n \le 3$.

$\square$

Solutions to the exercises of Chapter 4

Exercise 4.1.   Verify that $T = |\mathcal{C}(v)|$ by computing the probabilities of the events that $\{|\mathcal{C}(v)| = 1\}$, $\{|\mathcal{C}(v)| = 2\}$ and $\{|\mathcal{C}(v)| = 3\}$ directly, and by using (4.1.4) and (1.4.9) and (1.4.10).

Solution to Exercise 4.1.   We start by computing $\mathbb{P}(T = m)$ for $m = 1, 2, 3$. For $m = 1$, we get

$$\mathbb{P}(T = 1) = \mathbb{P}(S_1 = 0) = \mathbb{P}(X_1 = 0) = \mathbb{P}(\mathrm{BIN}(n-1,p) = 0) = (1-p)^{n-1}.$$

For $m = 2$, we get

$$\begin{aligned}
\mathbb{P}(T = 2) &= \mathbb{P}(S_1 > 0, S_2 = 0) = \mathbb{P}(X_1 > 0, X_1 + X_2 = 1) = \mathbb{P}(X_1 = 1, X_2 = 0) \\
&= \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 0 | X_1 = 1) = \mathbb{P}(\mathrm{BIN}(n-1,p) = 1)\mathbb{P}(\mathrm{BIN}(n-2,p) = 0) \\
&= (n-1)p(1-p)^{n-2} \cdot (1-p)^{n-2} = (n-1)p(1-p)^{2n-4}.
\end{aligned}$$

For $m = 3$, we get

$$\begin{aligned}
\mathbb{P}(T = 3) &= \mathbb{P}(S_1 > 0, S_2 > 0, S_3 = 0) = \mathbb{P}(X_1 > 0, X_1 + X_2 > 1, X_1 + X_2 + X_3 = 2) \\
&= \mathbb{P}(X_1 = 1, X_2 = 1, X_3 = 0) + \mathbb{P}(X_1 = 2, X_2 = 0, X_3 = 0) \\
&= \mathbb{P}(X_3 = 0 | X_2 = 1, X_1 = 1)\mathbb{P}(X_2 = 1 | X_1 = 1)\mathbb{P}(X_1 = 1) \\
&\qquad + \mathbb{P}(X_3 = 0 | X_2 = 0, X_1 = 2)\mathbb{P}(X_2 = 0 | X_1 = 2)\mathbb{P}(X_1 = 2) \\
&= \mathbb{P}(X_3 = 0 | S_2 = 1)\mathbb{P}(X_2 = 1 | S_1 = 1)\mathbb{P}(X_1 = 1) \\
&\qquad + \mathbb{P}(X_3 = 0 | S_2 = 1)\mathbb{P}(X_2 = 0 | S_1 = 2)\mathbb{P}(X_1 = 2) \\
&= \mathbb{P}(\mathrm{BIN}(n-3,p) = 0)\mathbb{P}(\mathrm{BIN}(n-2,p) = 1)\mathbb{P}(\mathrm{BIN}(n-1,p) = 1) \\
&\qquad + \mathbb{P}(\mathrm{BIN}(n-3,p) = 0)\mathbb{P}(\mathrm{BIN}(n-3,p) = 0)\mathbb{P}(\mathrm{BIN}(n-1,p) = 2) \\
&= (1-p)^{n-3}(n-2)p(1-p)^{n-3}(n-1)p(1-p)^{n-2} \\
&\qquad + (1-p)^{n-3}(1-p)^{n-3}(n-1)(n-2)p^2(1-p)^{n-3}/2 \\
&= (n-1)(n-2)p^2(1-p)^{3n-8} + (n-1)(n-2)p^2(1-p)^{3n-9}/2 \\
&= (n-1)(n-2)p^2(1-p)^{3n-9}\left(\frac{3}{2} - p\right).
\end{aligned}$$

We now give the combinatoric proof. For $m = 1$,

$$\mathbb{P}(|\mathcal{C}(v)| = 1) = (1-p)^{n-1},$$

because all connections from vertex 1 have to be closed. For $m = 2$,

$$\mathbb{P}(|\mathcal{C}(v)| = 2) = (n-1)p(1-p)^{2n-4}$$

because you must connect one of $n - 1$ vertices to vertex v and then isolate these two vertices which means that $2n - 4$ connections should not be present.

For $m = 3$, the first possibility is to attach one vertex a to 1 and then a second vertex b to a, with the edge vb being closed. This gives

$$(n-1)p(1-p)^{n-2}(n-2)p(1-p)^{n-3}(1-p)^{n-3} = (n-1)(n-2)p^2(1-p)^{3n-8}.$$

The second possibility is to attach one vertex a to v and then a second vertex b to a, with the edge vb being occupied. This gives

$$\binom{n-1}{2}p(1-p)^{n-3}p(1-p)^{n-3}(1-p)^{n-3}p = \binom{n-1}{2}p^3(1-p)^{3n-9}.$$

The final possibility is that you pick two vertices attached to vertex v, and then leave both vertices without any further attachments to the other $n-3$ and being unconnected (the connected case is part of the second possibility)

$$\binom{n-1}{2}p^2(1-p)^{n-3} \cdot (1-p)^{2n-5} = \binom{n-1}{2}p^2(1-p)^{3n-8}.$$

In total, this gives

$$(n-1)(n-2)p^2(1-p)^{3n-8} + \binom{n-1}{2}p^3(1-p)^{3n-9} + \binom{n-1}{2}p^2(1-p)^{3n-9} \qquad \text{(B.0.91)}$$

$$= (n-1)(n-2)p^2(1-p)^{3n-9}(1-p+\frac{p}{2}+\frac{(1-p)}{2}) = (n-1)(n-2)p^2(1-p)^{3n-9}(\frac{3}{2}-p).$$

$\square$

Exercise 4.2. We say that i, j, k form an occupied triangle when ij, jk and ki are all occupied. Compute the expected number of occupied triangles.

Solution to Exercise 4.2. We first pick 3 different elements i, j, k from $\{1, 2, \ldots, n\}$ without order. This can be done in

$$\binom{n}{3}$$

different ways. Then all three edges ij, ik, jk have to be present, which has probability p^3. The number of triangles is the sum of indicators running over all unordered triples. These indicators are dependent, but that is of no importance for the expectation, because the expectation of a sum of dependent random variables equals the sum of the expected values. Hence the expected number of occupied triangles equals:

$$\binom{n}{3}p^3.$$

$\square$

Exercise 4.7. Show that $|C_{\max}|$ is an increasing random variable.

Solution to Exercise 4.7. We have to show that for each x,

$$\{|\mathcal{C}(v)| \geq x\}$$

remains true if the the number of edges increases.

Obviously by increasing the number of edges the number $|\mathcal{C}(v)|$ increases or stays the same depending on whether or not some of the added edges connect new vertices to the cluster. In both cases $\{|\mathcal{C}(v)| \geq x\}$ remains true.

$\square$

Exercise 4.8.   Is the event $\{v \in C_{\max}\}$ an increasing event?

Solution to Exercise 4.8.   This is not true. Take two disjoint clusters which differ by one in size, and suppose that the larger component equals $C_{\max}$, before adding the edges. Take any $v \in C_{\max}$. Now add edges between the second largest component and isolated vertices. If you add two of such edges, then the new $C_{\max}$ equals the union of the second largest component and the two isolated vertices. Since originally v did not belong to the second largest component and v was not isolated, because it was a member of the previous largest component, we now have $v \notin C_{\max}$.

<div align="right">□</div>

Exercise 4.12.   Prove that $|C_{\max}| \geq k$ precisely when $Z_{\geq k} \geq k$.

Solution to Exercise 4.12.   We recall that $Z_{\geq k} = \sum_{i=1}^{n} \mathbb{1}_{\{|\mathcal{C}(i)| \geq k\}}$.

$$|C_{\max}| < k \Rightarrow |\mathcal{C}(i)| < k \, \forall i, \text{ which implies that } Z_{\geq k} = 0$$
$$|C_{\max}| \geq k \Rightarrow |\mathcal{C}(i)| \geq k \text{ for at least } k \text{ vertices } \Rightarrow Z_{\geq k} \geq k.$$

<div align="right">□</div>

Exercise 4.14.   Prove that the unique solution β to the equation $1 - e^{-\lambda\beta} = \beta$ is $\beta = \zeta_\lambda$, where ζ_λ is the survival probability of a Poisson branching process with parameter λ.

Solution to Exercise 4.14.   The extinction probability η satisfies

$$\eta_\lambda = G_X(\eta_\lambda) = \mathbb{E}[\eta_\lambda^X] = e^{-\lambda + \lambda\eta_\lambda}$$

Hence,

$$\zeta_\lambda = 1 - \eta_\lambda = 1 - e^{-\lambda + \lambda\eta} = 1 - e^{-\lambda\zeta_\lambda}.$$

This equation has only two solutions, one of which is $\zeta_\lambda = 0$, the other must be the survival probability.   □

Exercise 4.15.   Prove that the expected cluster size of a given vertex

$$\chi(\lambda) = \mathbb{E}_\lambda[|\mathcal{C}(1)|],$$

satisfies

$$\chi(\lambda) = 1 + (n-1)\mathbb{P}_\lambda(1 \longleftrightarrow 2).$$

Solution to Exercise 4.15.   We compute that

$$
\begin{aligned}
\chi(\lambda) &= \mathbb{E}_\lambda[|\mathcal{C}(1)|] = \mathbb{E}_\lambda\left[\sum_{j=1}^{n} \mathbb{1}_{\{j \in \mathcal{C}(1)\}}\right] \\
&= 1 + \sum_{j=2}^{n} \mathbb{E}_\lambda[\mathbb{1}_{\{j \in \mathcal{C}(1)\}}] = 1 + \sum_{j=2}^{n} \mathbb{E}_\lambda[\mathbb{1}_{\{1 \leftrightarrow j\}}] \\
&= 1 + \sum_{j=2}^{n} \mathbb{P}_\lambda(1 \leftrightarrow j) = 1 + (n-1)\mathbb{P}_\lambda(1 \leftrightarrow 2).
\end{aligned}
$$

<div align="right">□</div>

Exercise 4.16. Prove that (4.4.1) and Corollary 4.12 imply that, for $\lambda > 1$,

$$\mathbb{P}_\lambda(1 \longleftrightarrow 2) = \zeta_\lambda^2[1 + o(1)].$$

Solution to Exercise 4.16. In this exercise we denote by $|\mathcal{C}_{(1)}| \geq |\mathcal{C}_{(2)}| \geq \ldots$, the components ordered by their size. Relation (4.4.1) reads that for $\nu \in (\frac{1}{2}, 1)$:

$$\mathbb{P}\Big(\big||C_{\max}| - n\zeta_\lambda\big| \geq n^\nu\Big) = O(n^{-\delta}).$$

Observe that

$$
\begin{aligned}
\mathbb{P}_\lambda(1 \leftrightarrow 2) &= \mathbb{P}_\lambda(\exists \mathcal{C}(k) : 1 \in \mathcal{C}(k), 2 \in \mathcal{C}(k)) \\
&= \sum_{l \geq 1} \mathbb{P}_\lambda(1, 2 \in \mathcal{C}_{(l)}) = \mathbb{P}_\lambda(1, 2 \in \mathcal{C}_{(1)}) + \sum_{l \geq 2} \mathbb{P}_\lambda(1, 2 \in \mathcal{C}_{(l)}) \\
&= \frac{(n\zeta_\lambda \pm n^\nu)^2}{n^2} + O(n^{-\delta}) + \sum_{l \geq 2} \mathbb{P}_\lambda(1, 2 \in \mathcal{C}_{(l)}).
\end{aligned}
$$

For $l \geq 2$, we have $|\mathcal{C}_{(l)}| \leq K \log n$ with high probability, hence

$$\mathbb{P}_\lambda(1, 2 \in \mathcal{C}_{(l)}) \leq \frac{K^2 \log^2 n}{n^2} + O(n^{-2}),$$

so that

$$\sum_{l \geq 2} \mathbb{P}_\lambda(1, 2 \in \mathcal{C}_{(l)}) \leq \frac{K^2 \log^2 n}{n} + O(n^{-1}) \to 0.$$

Together, this shows that

$$\mathbb{P}_\lambda(1 \leftrightarrow 2) = \zeta_\lambda^2 + O(n^{-\delta}),$$

for some $\delta > 0$. $\qquad\square$

Exercise 4.17. Prove that (4.4.1) implies that for $\lambda > 1$ the expected cluster size satisfies

$$\chi(\lambda) = \zeta_\lambda^2 n(1 + o(1)).$$

Solution to Exercise 4.17. Combining Exercise 4.15 and Exercise 4.16, yields

$$\chi(\lambda) = 1 + (n-1)\zeta_\lambda^2(1 + o(1)) = n\zeta_\lambda^2(1 + o(1)).$$

$\qquad\square$

Solutions to the exercises of Chapter 5

Exercise 5.1. Prove (5.1.4) using (3.5.24).

Solution to Exercise 5.1.   Using (3.5.24) we see that

$$\mathbb{P}_\lambda^*(T^* \geq k) = (2\pi)^{-1/2} \sum_{n=k}^{\infty} n^{-3/2}[1 + O(n^{-1})]. \tag{B.0.92}$$

The sum can be bounded from above and below by an integral as follows

$$\int_k^{\infty} x^{-3/2} dx \leq \sum_{n=k}^{\infty} n^{-3/2} \leq \int_{k-1}^{\infty} x^{-3/2} dx$$

Computing these integrals gives

$$2k^{-1/2} \leq \sum_{n=k}^{\infty} n^{-3/2} \leq 2(k-1)^{-1/2}$$

Similar bounds can be derived such that

$$\sum_{n=k}^{\infty} n^{-3/2} O(n^{-1}) = O(k^{-3/2}).$$

Combining both bounds, it follows that

$$\mathbb{P}_\lambda^*(T^* \geq k) = \left(\frac{2}{\pi}\right)^{1/2} k^{-1/2}[1 + O(k^{-1})].$$

$\square$

Exercise 5.2.   Prove that Proposition 5.2 also implies that $\chi(1) \geq cn^{1/3}$ for some $c > 0$. Therefore, the bound in Proposition 5.3 is asymptotically sharp for $\lambda = 1$.

Solution to Exercise 5.2.   Fix some $r > 0$, then

$$\chi(1) \geq \sum_{k=1}^{rn^{2/3}} \mathbb{P}(|\mathcal{C}(1)| \geq k) = \sum_{k=1}^{rn^{2/3}} P_{\geq k}(1). \tag{B.0.93}$$

By Proposition 5.2, we have the bounds

$$P_{\geq k}(1) \geq \frac{c_1}{\sqrt{k}}.$$

Substituting this bounds into (B.0.93) yields

$$\chi(1) \geq \sum_{k=1}^{rn^{2/3}} \frac{c_1}{\sqrt{k}} \geq c_1' rn^{1/3},$$

where $c_1' > 0$ and $r > 0$.

$\square$

Exercise 5.3.   Prove that

$$\frac{(\lambda t)^{t-1}}{t!} e^{-\lambda t} = \frac{1}{\lambda} e^{-I_\lambda t} \mathbb{P}_1^*(T^* = t). \tag{B.0.94}$$

Solution to Exercise 5.3. By Theorem 3.13, we have that

$$\frac{1}{\lambda}\, e^{-I_\lambda t}\mathbb{P}_1^*(T^* = t) = \frac{1}{\lambda}\, e^{-(\lambda - 1 - \log \lambda)t}\frac{t^{t-1}}{t!}e^{-t}.$$

Rearranging the terms in this equation we get

$$\frac{1}{\lambda}\, e^{-I_\lambda t}\mathbb{P}_1^*(T^* = t) = \frac{1}{\lambda}\, \left(e^{\log \lambda}\right)^t \frac{t^{t-1}}{t!}\, e^{-\lambda t} = \frac{(\lambda t)^{t-1}}{t!}\, e^{-\lambda t}.$$

$\square$

Exercise 5.5. Show that $\lambda \mapsto \tau_n(\lambda)$ is differentiable. Hint: $\tau_n(\lambda)$ is a polynomial of bounded degree in λ.

Solution to Exercise 5.5. Let $\mathcal{G}(n)$ be the collection of all possible simple graphs on n points. The set $\mathcal{G}(n,m)$ is the subset of $\mathcal{G}$ which contains all possible simple graphs on n points which have m edges. Then,

$$\mathbb{P}(1 \longleftrightarrow 2) = |\mathcal{G}(n)|^{-1} \sum_{m=1}^{\binom{n}{2}} \sum_{G \in \mathcal{G}(n,m)} \mathbb{P}(G)\mathbb{1}_{\{1 \longleftrightarrow 2 \text{ in } G\}}$$

$$= 2^{-\binom{n}{2}} \sum_{m=1}^{n} \sum_{G \in \mathcal{G}(n,m)} \left(\frac{\lambda}{n}\right)^m \left(1 - \frac{\lambda}{n}\right)^{\binom{n}{2}-m} \mathbb{1}_{\{1 \longleftrightarrow 2 \text{ in } G\}},$$

which is polynomial in λ. Furthermore, the maximal degree of the polynoom is $\binom{n}{2}$.

Exercise 5.6. Prove that $l \mapsto \chi_{n-l}(\lambda\frac{n-l}{n})$ is decreasing. Hint: $\chi_{n-l}(\lambda\frac{n-l}{n})$ is equal to the expected cluster size in the random graph $\mathrm{ER}(n - l, \lambda/n)$.

Solution to Exercise 5.6. Take some $l \in \mathbb{N}$ such that $l < n$, then $\chi_{n-l}(\lambda\frac{n-l}{n})$ is the expected component size in the graph $\mathrm{ER}(n - l, p)$. We have to prove that the expected component size in the graph $\mathrm{ER}(n - l, p)$ is smaller than the expected component size in the graph $\mathrm{ER}(n - l + 1, p)$ for all $0 < p \le 1$. Consider the graph $\mathrm{ER}(n - l + 1, p)$. This graph can be created from $\mathrm{ER}(n - l, p)$ by adding the vertex $n - l + 1$ and independently connecting this vertex to each of the vertices $1, 2, \ldots, n - l$.

Let $\mathcal{C}'(1)$ denote the component of $\mathrm{ER}(n - l, p)$ which contains vertex 1 and $\mathcal{C}(1)$ represents the component of $\mathrm{ER}(n - l + 1, p)$ which contains vertex 1. By the construction of $\mathrm{ER}(n - l + 1, p)$, it follows that

$$\mathbb{P}(|\mathcal{C}(1)| = k) = \begin{cases} (1-p)^{n-l+1} & \text{if } k = 1, \\ \mathbb{P}(|\mathcal{C}'(1)| = k)(1-p)^k + \mathbb{P}(|\mathcal{C}'(1)| = k-1)(1-(1-p)^{k-1}) & \text{if } 2 \le k \le n, \\ \mathbb{P}(|\mathcal{C}'(1)| = n)(1-(1-p)^n) & \text{if } k = n+1. \end{cases}$$

Hence, the expected size of $\mathcal{C}(1)$ is

$$\mathbb{E}[|\mathcal{C}(1)|] = \sum_{k=1}^{n+1} \mathbb{P}(|\mathcal{C}(1)| = k)k$$

$$= (1-p)^{n-l+1} + \sum_{k=2}^{n} \left[\mathbb{P}(|\mathcal{C}'(1)| = k)(1-p)^k + \mathbb{P}(|\mathcal{C}'(1)| = k-1)(1-(1-p)^{k-1})\right]k$$

$$+ \mathbb{P}(|\mathcal{C}'(1)| = n)(1-(1-p)^n)(n+1).$$

Rewriting this expression for the expected size of $\mathcal{C}(1)$ yields

$$\mathbb{E}[|\mathcal{C}(1)|] = (1-p)^{n-l+1} + \mathbb{P}(|\mathcal{C}'(1)| = 1)2p + \sum_{k=2}^{n-1} \mathbb{P}(|\mathcal{C}'(1)| = k)k + \sum_{k=2}^{n-1} \mathbb{P}(\mathcal{C}'(1) = k)(1 - (1-p)^{k-1})$$

$$+ \mathbb{P}(|\mathcal{C}'(1)| = n)(n + (1 - (1-p)^n))$$

$$\geq (1+p)\mathbb{P}(|\mathcal{C}'(1)| = 1) + \sum_{k=2}^{n-1} k\mathbb{P}(\mathcal{C}'(1) = k) \geq \mathbb{E}[|\mathcal{C}(1)'|].$$

□

Exercise 5.7.   Prove that

$$\frac{\partial}{\partial \lambda} \chi_n(\lambda) \leq \chi_n(\lambda)^2, \tag{B.0.95}$$

and use this inequality to deduce that for all $\lambda \leq 1$

$$\chi_n(\lambda) \geq \frac{1}{\chi_n(1)^{-1} + (\lambda - 1)}. \tag{B.0.96}$$

Solution to Exercise 5.7.   By (5.1.35), we have that

$$\frac{\partial}{\partial \lambda} \chi_n(\lambda) = (n - 1)\frac{\partial}{\partial \lambda} \tau_n(\lambda).$$

For the derivative of $\tau_n(\lambda)$ we use (5.1.49) to obtain

$$\frac{\partial}{\partial \lambda} \chi_n(\lambda) \leq \sum_{l=1}^{n} l\mathbb{P}_\lambda(|\mathcal{C}(1)| = l)\chi_{n-l}(\lambda \frac{n-l}{n}).$$

The function $l \mapsto \chi_{n-l}(\lambda \frac{n-l}{n})$ is decreasing (see Exercise 5.6), hence

$$\frac{\partial}{\partial \lambda} \chi_n(\lambda) \leq \chi_n(\lambda) \sum_{l=1}^{n} l\mathbb{P}_\lambda(|\mathcal{C}(1)| = l) = \chi_n(\lambda)^2,$$

or

$$\frac{\frac{\partial}{\partial \lambda} \chi_n(\lambda)}{\chi_n(\lambda)^2} \leq 1. \tag{B.0.97}$$

The second part of the exercise relies on integration. Integrate both the left-hand and the right-hand side of (B.0.97) between λ and 1.

$$\frac{1}{\chi_n(\lambda)} - \frac{1}{\chi_n(1)} \leq 1 - \lambda$$

Bring a term to the other side to obtain

$$\frac{1}{\chi_n(\lambda)} \leq \frac{1}{\chi_n(1)} + 1 - \lambda,$$

which is equivalent to

$$\chi_n(\lambda) \geq \frac{1}{\chi_n(1)^{-1} + (1 - \lambda)}.$$

□

Exercise 5.8. Prove directly that the second moment of Y converges to the second moment of Z, by using (5.2.10).

Solution to Exercise 5.8. Using (5.2.8) and (5.2.10) we see that

$$\mathbb{E}_\lambda[Y^2] = n\mathbb{P}_\lambda(|\mathcal{C}(1)| = 1) + n(n-1)\left(\frac{\lambda}{n(1-\frac{\lambda}{n})} + 1\right)\mathbb{P}_\lambda(|\mathcal{C}(1)| = 1)^2$$

$$= n\left(1 - \frac{\lambda}{n}\right)^{n-1} + n(n-1)\left(1 - \frac{\lambda}{n}\right)^{2n-3}$$

$$= n\left(1 - \frac{\lambda}{n}\right)^{n-1}\left(1 + (n-1)\left(1 - \frac{\lambda}{n}\right)^{n-2}\right).$$

Consider the first power, taking the logarithm yields

$$\log n + (n-1)\log(1 - \frac{\lambda}{n}) = \log n + (n-1)\log(1 - \frac{\log n + t}{n}).$$

Taylor expanding the logarithm gives

$$\log n + (n-1)\log(1 - \frac{\log n + t}{n}) = \log n - (n-1)\left[\frac{\log n + t}{n} + O\left((\frac{\log n + t}{n})^2\right)\right].$$

The latter expression can be simplified to

$$\log n - (n-1)\left[\frac{\log n + t}{n} + O\left((\frac{\log n + t}{n})^2\right)\right] = \log n - \frac{n-1}{n}\log n - \frac{n-1}{n}t + O\left(\frac{(\log n + t)^2}{n}\right)$$

$$= -t + \frac{\log n}{n} + \frac{t}{n} + O\left(\frac{(\log n + t)^2}{n}\right),$$

and, as n tends to infinity,

$$-t + \frac{\log n}{n} + \frac{t}{n} + O\left(\frac{(\log n + t)^2}{n}\right) \to -t.$$

Hence,

$$\lim_{n\to\infty} n\left(1 - \frac{\lambda}{n}\right)^{n-1} = e^{-t}.$$

A similar argument gives that as $n \to \infty$

$$\lim_{n\to\infty}\left(1 - \frac{\lambda}{n}\right)^{n-2} = e^{-t}.$$

Therefore, we conclude

$$\lim_{n\to\infty}\mathbb{E}_\lambda[Y^2] = e^{-t}(1 - e^{-t}),$$

which is the second moment of a Poisson random variable with mean e^{-t}.

$\square$

Solutions to the exercises of Chapter 6.

Exercise 6.1. Prove that $p_{ij} = \frac{\lambda}{n}$ when $W_i \equiv \frac{n\lambda}{n-\lambda}$.

Solution to Exercise 6.1.   By the definition of p_{ij} (6.1.1), the numerator of p_{ij} is $(n\lambda)^2(n-\lambda)^{-2}$. The denominator of p_{ij} is

$$\sum_{i=1}^{n}\frac{n\lambda}{n-\lambda}+\left(\frac{n\lambda}{n-\lambda}\right)^2 = \frac{n^2\lambda}{n-\lambda}+\left(\frac{n\lambda}{n-\lambda}\right)^2 = \frac{n^2\lambda(n-\lambda)+(n\lambda)^2}{(n-\lambda)^2} = \frac{n^3\lambda}{(n-\lambda)^2}.$$

Dividing the numerator of p_{ij} by its denominator gives

$$p_{ij} = \frac{(n\lambda)^2}{n^3\lambda} = \frac{\lambda}{n}.$$

$\square$

Exercise 6.2.   Prove that $\sum_{i=1}^{n}\mathbb{1}_{\{\frac{i}{n}\geq 1-F_W(x)\}} = \min\{n, \lfloor nF_W(x)\rfloor+1\}$.

Solution to Exercise 6.2.   We rewrite, with $j = n-i$,

$$\sum_{i=1}^{n}\mathbb{1}_{\{\frac{i}{n}\geq 1-F_W(x)\}} = \sum_{j=0}^{n-1}\mathbb{1}_{\{j\leq nF_W(x)\}} = \sum_{j=0}^{n-1}\mathbb{1}_{\{j\leq\lfloor nF_W(x)\rfloor\}} = \min\{n, \lfloor nF_W(x)\rfloor+1\}.$$

$\square$

Exercise 6.5.   Let X have a mixed Poisson distribution with mixing distribution F and moment generating function M_W, i.e., for $t \in \mathbb{C}$,
$$M_W(t) = \mathbb{E}[e^{tW}], \tag{B.0.98}$$
where W has distribution function F. Show that the characteristic function of X is given by
$$\phi_X(t) = \mathbb{E}[e^{itX}] = M_W(e^{it}-1). \tag{B.0.99}$$

Solution to Exercise 6.5.   By definition, the characteristic function of X is

$$\mathbb{E}[e^{itX}] = \sum_{n=0}^{\infty}e^{itn}\mathbb{P}(X=n) = \sum_{n=0}^{\infty}e^{itn}\left(\int_0^{\infty}f_W(w)\frac{e^{-w}w^n}{n!}dw\right),$$

where $f_W(w)$ is the density function of W evaluated in w. Since all terms are non-negative we can interchange summation and integration. Rearranging the terms gives

$$\mathbb{E}[e^{itX}] = \int_0^{\infty}f_W(w)e^{-w}\left(\sum_{n=0}^{\infty}\frac{(e^{it}w)^n}{n!}\right)dw = \int_0^{\infty}f_W(w)e^{-w}\exp(e^{it}w)dw$$
$$= \int_0^{\infty}f_W(w)\exp((e^{it}-1)w)dw.$$

The latter expression is the moment generating function of W evaluated in $e^{it}-1$. $\square$

Exercise 6.17.   Use Proposition 6.12 to give an alternative proof of Theorem 6.1.

Solution to Exercise 6.17. We pick $t_k = t$ and $t_i = 1$ for all $i \neq k$. Then,

$$
\mathbb{E}[t^{D_k}] = \prod_{1 \leq i \leq n : i \neq k} \frac{l_n + w_i w_k t}{l_n + w_i w_k}
$$
$$
= e^{w_k(t-1) \sum_{1 \leq i \leq n : i \neq k} \frac{w_i}{l_n} + R_n}, \tag{B.0.100}
$$

where

$$
R_n = \sum_{1 \leq i \leq n : i \neq k} \log\left(1 + \frac{w_i w_k t}{l_n}\right) - \log\left(1 + \frac{w_i w_k}{l_n}\right) - w_k(t-1) \sum_{1 \leq i \leq n : i \neq k} \frac{w_i}{l_n}
$$
$$
= \sum_{1 \leq i \leq n : i \neq k} \log(l_n + w_i w_k t) - \log(l_n + w_i w_k) - w_k(t-1) \sum_{1 \leq i \leq n : i \neq k} \frac{w_i}{l_n}. \tag{B.0.101}
$$

A Taylor expansion of $x \mapsto \log(a+x)$ yields that

$$
\log(a+x) = \log(a) + \frac{x}{a} + O\left(\frac{x^2}{a^2}\right). \tag{B.0.102}
$$

Therefore, applying the above with $a = l_n$ and $x = w_i w_k$, yields that, for t bounded,

$$
R_n = O\left(w_k^2 \sum_{i=1}^{n} \frac{w_i^2}{l_n^2}\right) = o(1), \tag{B.0.103}
$$

by (6.2.3), so that

$$
\mathbb{E}[t^{D_k}] = e^{w_k(t-1) \sum_{1 \leq i \leq n : i \neq k} \frac{w_i}{l_n}}(1 + o(1))
$$
$$
= e^{w_k(t-1)}(1 + o(1)), \tag{B.0.104}
$$

since w_k is fixed. Since the generating function of the degree converges, the degree of vertex k converges in distribution to a random variable with generating function $e^{w_k(t-1)}$ (recall Theorem 2.3(c)). The probability generating function of a Poisson random variable with mean λ is given by $e^{\lambda(t-1)}$, which completes the proof of Theorem 6.1(a).

For Theorem 6.1(b), we use similar ideas, now taking $t_i = t_i$ for $i \leq m$ and $t_i = 0$ for $i > m$. Then,

$$
\mathbb{E}[\prod_{i=1}^{m} t_i^{D_i}] = \prod_{1 \leq i \leq m, i < j \leq n} \frac{l_n + w_i w_j t_i}{l_n + w_i w_j} = \prod_{i=1}^{m} e^{w_i(t_i - 1)}(1 + o(1)), \tag{B.0.105}
$$

so that the claim follows. $\qquad\square$

Solutions to the exercises of Chapter 7.

Exercise 7.8. Give an alternative proof of Proposition 7.4 by using Theorem 2.3(e) together with Exercise 7.7 and the fact that all joint moments of (S_n, M_n) converge to those of (S, M), where S and M are two independent Poisson random variables with means $\frac{\nu}{2}$ and $\frac{\nu^2}{4}$.

Solution to Exercise 7.8. We use a two-dimensional extension of Theorem 2.3(e), stating that when the mixed moments $\mathbb{E}[X_n^r Y_n^s]$ converge to the moments $\mathbb{E}[X^r Y^s]$ for each $r, s = 0, 1, 2, \ldots$, and the moments of X and Y satisfy (2.1.8), then (X_n, Y_n) converges in distribution to (X, Y). See also Theorem 2.6 for the equivalent statement for the factorial moments instead of the normal moments, from which the above claim actually follows. Therefore, we are left to prove the asymptotics of the mixed moments of (S_n, M_n).

To prove that $\mathbb{E}[S_n^r M_n^s]$ converge to the moments $\mathbb{E}[S^r M^s]$, we again make use of induction, now in both r and s.

Proposition 7.4 follows when we prove that

$$\lim_{n\to\infty} \mathbb{E}[S_n^r] = \mathbb{E}[S^r] = \mu_S \mathbb{E}[(S+1)^{r-1}], \tag{B.0.106}$$

and

$$\lim_{n\to\infty} \mathbb{E}[S_n^r M_n^s] = \mathbb{E}[S^r M^s] = \mu_M \mathbb{E}[S^r (M+1)^{s-1}], \tag{B.0.107}$$

where the second equalities in (B.0.106) and (B.0.107) follow from (7.2.50) and (7.2.51).

To prove (B.0.106), we use the shape of S_n in (7.2.22), which we restate here as

$$S_n = \sum_{i=1}^{n} \sum_{1 \le a < b \le d_i} I_{ab,i}. \tag{B.0.108}$$

Then, we prove by induction on r that

$$\lim_{n\to\infty} \mathbb{E}[S_n^r] = \mathbb{E}[S^r]. \tag{B.0.109}$$

The induction hypothesis is that (B.0.109) is true for all $r' \le r - 1$, for $\mathrm{CM}(n, \{d_i\}_{i=1}^n)$ when $n \to \infty$ and for all $\{d_i\}_{i=1}^n$ satisfying (7.2.30). We prove (B.0.109) by induction on r. For $r = 0$, the statement is trivial, which initializes the induction hypothesis.

To advance the induction hypothesis, we write out

$$\mathbb{E}[S_n^r] = \sum_{i=1}^{n} \sum_{1 \le a < b \le d_i} \mathbb{E}[I_{ab,i} S_n^{r-1}]$$
$$= \sum_{i=1}^{n} \sum_{1 \le a < b \le d_i} \mathbb{P}(I_{ab,i} = 1) \mathbb{E}[S_n^{r-1} | I_{ab,i} = 1]. \tag{B.0.110}$$

When $I_{ab,i} = 1$, then the *remaining* stubs need to be paired in a uniform manner. The number of self-loops in the total graph in this pairing has the same distribution as

$$1 + S_n', \tag{B.0.111}$$

where S_n' is the number of self-loops in the configuration model where with degrees $\{d_i'\}_{i=1}^n$, where $d_i' = d_i - 2$, and $d_j' = d_j$ for all $j \ne i$. The added 1 in (B.0.111) originates from $I_{ab,i}$. By construction, the degrees $\{d_i'\}_{i=1}^n$ still satisfy (7.2.30). By the induction hypothesis, for all $k \le r - 1$

$$\lim_{n\to\infty} \mathbb{E}[(S_n')^k] = \mathbb{E}[S^k]. \tag{B.0.112}$$

As a result,

$$\lim_{n\to\infty} \mathbb{E}[(1 + S_n')^{r-1}] = \mathbb{E}[(1 + S)^{r-1}]. \tag{B.0.113}$$

Since the limit does not depend on i, we obtain that

$$\lim_{n\to\infty} \mathbb{E}[S_n^r] = \mathbb{E}[(1+S)^{r-1}] \lim_{n\to\infty} \sum_{i=1}^n \sum_{1\leq a<b\leq d_i} \mathbb{P}(I_{ab,i}=1)$$

$$\mathbb{E}[(1+S)^{r-1}] \lim_{n\to\infty} \sum_{i=1}^n \frac{d_i(d_i-1)}{2}$$

$$= \frac{\nu}{2}\mathbb{E}[(1+S)^{r-1}] = \mathbb{E}[S^r]. \tag{B.0.114}$$

This advances the induction hypothesis, and completes the proof of (B.0.106).

To prove (B.0.107), we perform a similar induction scheme. Now we prove that, for all $r \geq 0$, $\mathbb{E}[S_n^r M_n^s]$ converges to $\mathbb{E}[S^r M^s]$ by induction on s. The claim for $s=0$ follows from (B.0.106), which initializes the induction hypothesis, so we are left to advance the induction hypothesis. We follow the argument for S_n above. It is not hard to see that it suffices to prove that, for every ij,

$$\lim_{n\to\infty} \mathbb{E}[S_n^r M_n^{s-1}|I_{s_1t_1,s_2t_2,ij}=1] = \mathbb{E}[S^r(1+M)^{s-1}]. \tag{B.0.115}$$

Note that when $I_{s_1t_1,s_2t_2,ij}=1$, then we know that two edges are paired together to form a multiple edge. Removing these two edges leaves us with a graph which is very close to the configuration model with degrees $\{d_i'\}_{i=1}^n$, where $d_i' = d_i - 2$, and $d_j' = d_j - 2$ and $d_t' = d_t$ for all $t \neq i, j$. The only difference is that when a stub connected to i is attached to a stub connected to j, then this creates an additional number of multiple edges. Ignoring this effect creates the lower bound

$$\mathbb{E}[S_n^r M_n^{s-1}|I_{s_1t_1,s_2t_2,ij}=1] \geq \mathbb{E}[S_n^r(M_n+1)^{s-1}], \tag{B.0.116}$$

which, by the induction hypothesis, converges to $\mathbb{E}[S^r(1+M)^{s-1},]$ as required.

Let $I_{s_1t_1,s_2t_2,ij}'$ denote the indicator that stub s_1 is connected to t_1, s_2 to t_2 and *no other stub* of vertex i is connected to a stub of vertex j. Then,

$$\frac{1}{2} \sum_{1\leq i\neq j\leq n} \sum_{1\leq s_1<s_2\leq d_i} \sum_{1\leq t_1\neq t_2\leq d_j} I_{s_1t_1,s_2t_2,ij}' \leq M_n \leq \frac{1}{2} \sum_{1\leq i\neq j\leq n} \sum_{1\leq s_1<s_2\leq d_i} \sum_{1\leq t_1\neq t_2\leq d_j} I_{s_1t_1,s_2t_2,ij}. \tag{B.0.117}$$

Hence,

$$\mathbb{E}[S_n^r M_n^s] \leq \frac{1}{2} \sum_{1\leq i\neq j\leq n} \sum_{1\leq s_1<s_2\leq d_i} \sum_{1\leq t_1\neq t_2\leq d_j} \mathbb{P}(I_{s_1t_1,s_2t_2,ij}=1)\mathbb{E}[S_n^r M_n^{s-1}|I_{s_1t_1,s_2t_2,ij}=1], \tag{B.0.118}$$

and

$$\mathbb{E}[S_n^r M_n^s] \leq \frac{1}{2} \sum_{1\leq i\neq j\leq n} \sum_{1\leq s_1<s_2\leq d_i} \sum_{1\leq t_1\neq t_2\leq d_j} \mathbb{P}(I_{s_1t_1,s_2t_2,ij}'=1)\mathbb{E}[S_n^r M_n^{s-1}|I_{s_1t_1,s_2t_2,ij}'=1]. \tag{B.0.119}$$

Now, by the above, $\mathbb{E}[S_n^r M_n^{s-1}|I_{s_1t_1,s_2t_2,ij}=1]$ and $\mathbb{E}[S_n^r M_n^{s-1}|I_{s_1t_1,s_2t_2,ij}'=1]$ converge to $\mathbb{E}[S^r(M_++1)^{s-1}]$, independently of s_1t_1, s_2t_2, ij. Further,

$$\frac{1}{2} \sum_{1\leq i\neq j\leq n} \sum_{1\leq s_1<s_2\leq d_i} \sum_{1\leq t_1\neq t_2\leq d_j} \mathbb{P}(I_{s_1t_1,s_2t_2,ij}'=1) \to \nu^2/2, \tag{B.0.120}$$

and also

$$\frac{1}{2} \sum_{1\leq i\neq j\leq n} \sum_{1\leq s_1<s_2\leq d_i} \sum_{1\leq t_1\neq t_2\leq d_j} \mathbb{P}(I_{s_1t_1,s_2t_2,ij}=1) \to \nu^2/2. \tag{B.0.121}$$

This implies that

$$\mathbb{E}[S_n^r M_n^{s-1}|I_{s_1t_1,s_2t_2,ij}=1] = \mathbb{E}[S_{n-1}^r M_{n-1}^{s-1}] + o(1). \tag{B.0.122}$$

The remainder of the proof is identical to the one leading to (B.0.114). $\qquad\square$

Solutions to the exercises of Chapter 8.

Exercise 8.6. For $\alpha = \frac{\delta}{2+\delta}$, the law of $\{\mathrm{PA}_{1,\alpha}^{(b')}(t)\}_{t=2}^{\infty}$ is equal to the one of $\{\mathrm{PA}_{1,\delta}(t)\}_{t=1}^{\infty}$.

Solution to Exercise 8.6. The proof is by induction on $t \geq 1$. For $t = 1$, the statement is correct, since, at time 2, both graphs consist of two vertices with two edges between them. This initializes the induction hypothesis.

To advance the induction hypothesis, we assume that the law of $\{\mathrm{PA}_{1,\alpha}^{(b')}(t)\}_{s=1}^{t}$ is equal to the one of $\{\mathrm{PA}_{1,\delta}^{(b)}(s)\}_{s=1}^{t}$, and, from this, prove that the law of $\{\mathrm{PA}_{1,\alpha}^{(b')}(s)\}_{s=1}^{t}$ is equal to the one of $\{\mathrm{PA}_{1,\delta}^{(b)}(s)\}_{s=1}^{t}$. The only difference between $\mathrm{PA}_{1,\delta}^{(b)}(t+1)$ and $\mathrm{PA}_{1,\delta}^{(b)}(t)$ and between $\mathrm{PA}_{1,\alpha}^{(b')}(t+1)$ and $\mathrm{PA}_{1,\alpha}^{(b')}(t)$ is to what vertex the $(t+1)^{\mathrm{st}}$ edge is attached. For $\{\mathrm{PA}_{1,\delta}^{(b)}(t)\}_{t=1}^{\infty}$ and conditionally on $\mathrm{PA}_{1,\delta}^{(b)}(t)$, this edge is attached to vertex i with probability

$$\frac{D_i(t) + \delta}{t(2 + \delta)}, \tag{B.0.123}$$

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while, for $\{\mathrm{PA}'_{1,\alpha}(t)\}_{t=1}^{\infty}$ and conditionally on $\mathrm{PA}'_{1,\alpha}(t)$, this edge is attached to vertex i with probability

$$\alpha \frac{1}{t} + (1 - \alpha)\frac{D_i(t)}{2t}. \tag{B.0.124}$$

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Bringing the terms in (B.0.124) onto a single denominator yields

$$\frac{D_i(t) + 2\frac{\alpha}{1-\alpha}}{\frac{2}{1-\alpha}t}, \tag{B.0.125}$$

<div align="right" style="font-size:small">e:attachementprobalpharew</div>

which agrees with (B.0.123) precisely when $2\frac{\alpha}{1-\alpha} = \delta$, so that

$$\alpha = \frac{\delta}{2 + \delta}. \tag{B.0.126}$$

$\square$

Exercise 8.16. Use Propositions 8.4 and 8.3 to prove that for $l = l(t) \to \infty$ as $t \to \infty$ such that $tl^{2-\tau} \geq K\sqrt{t \log t}$ for some $K > 0$ sufficiently large, there exists a constant $B > 0$ such that with probability exceeding $1 - o(t^{-1})$, for all such l,

$$\sum_{i:D_i(t)\geq l} D_i(t) \geq Btl^{2-\tau}. \tag{B.0.127}$$

<div align="right" style="font-size:small">e:lowerbddegreesrep</div>

Show further that, with probability exceeding $1 - o(t^{-1})$, for all such l,

$$N_{\geq l}(t) \gg \sqrt{t}. \tag{B.0.128}$$

<div align="right" style="font-size:small">e:Nt-lb-lrep</div>

Solution to Exercise 8.16. We note that

$$\sum_{i:D_i(t)\geq l} D_i(t) \geq lN_{\geq l}(t), \tag{B.0.129}$$

where we recall that $N_{\geq l}(t) = \#\{i \leq t : D_i(t) \geq l\}$ is the number of vertices with degree at least l.

By the proof of Proposition 8.3 (see also Exercise 8.15), there exists C_1 such that uniformly for all l,

$$\mathbb{P}\Big(|N_{\geq l}(t) - \mathbb{E}[N_{\geq l}(t)]| \geq C_1 \sqrt{t \log t}\Big) = o(t^{-1}). \tag{B.0.130}$$

By Proposition 8.4, there exists a constant C_2 such that

$$\sup_{l \geq 1} |\mathbb{E}[P_l(t)] - tp_l| \leq C_2. \tag{B.0.131}$$

Therefore, we obtain that, with probability exceeding $1 - o(t^{-1})$,

$$N_{\geq l}(t) \geq \mathbb{E}[N_{\geq l}(t)] - C_1 \sqrt{t \log t} \geq \mathbb{E}[N_{\geq l}(t)] - \mathbb{E}[N_{\geq 2l}(t)] - C_1 \sqrt{t \log t}$$

$$\geq \sum_{l=l}^{2l-1} [tp_l - C_2] - C_1 \sqrt{t \log t} \geq C_3 t l^{1-\tau} - C_2 l - C_1 \sqrt{t \log t} \geq B t l^{2-\tau}, \tag{B.0.132}$$

whenever l is such that

$$t l^{1-\tau} \gg l, \qquad \text{and} \qquad t l_t^{1-\tau} \gg \sqrt{t \log t}. \tag{B.0.133}$$

The first condition is equivalent to $l \ll t^{\frac{1}{\tau}}$, and the second to $l \ll t^{\frac{1}{2(\tau-1)}} (\log t)^{-\frac{1}{2(\tau-1)}}$. Note that $\frac{1}{\tau} \geq \frac{1}{2(\tau-1)}$ for all $\tau > 2$, so the second condition is the strongest, and follows when $t l^{2-\tau} \geq K \sqrt{t \log t}$ for some K sufficiently large.

Then, for l satisfying $t l^{2-\tau} \geq K \sqrt{t \log t}$, we have with probability exceeding $1 - o(t^{-1})$,

$$\sum_{i:D_i(t) \geq l} D_i(t) \geq B t l^{2-\tau}. \tag{B.0.134}$$

Also, with probability exceeding $1 - o(t^{-1})$, for all such l, $N_{\geq l}(t) \gg \sqrt{t}$. $\qquad\square$

Exercise 8.18. Fix $m = 1$ and $\delta > -m$. Then, prove that for all $t \geq i$

$$\mathbb{P}(D_i(t) = j) \leq C_j \frac{\Gamma(t)\Gamma(i + \frac{1+\delta}{2+\delta})}{\Gamma(t + \frac{1+\delta}{2+\delta})\Gamma(i)}, \tag{B.0.135}$$

where $C_1 = 1$ and

$$C_j = \frac{j - 1 + \delta}{j - 1} C_{j-1}. \tag{B.0.136}$$

Solution to Exercise 8.18. We prove (B.0.135) by induction on $j \geq 1$. Clearly, for every $t \geq i$,

$$\mathbb{P}(D_i(t) = 1) = \prod_{s=i+1}^{t} \Big(1 - \frac{1+\delta}{(2+\delta)(s-1) + (1+\delta)}\Big) = \prod_{s=i+1}^{t} \Big(\frac{s-1}{s - 1 + \frac{1+\delta}{2+\delta}}\Big) = \frac{\Gamma(t)\Gamma(i + \frac{1+\delta}{2+\delta})}{\Gamma(t + \frac{1+\delta}{2+\delta})\Gamma(i)}, \tag{B.0.137}$$

which initializes the induction hypothesis, since $C_1 = 1$.

To advance the induction, we let $s \leq t$ be the last time at which a vertex is added to i. Then we have that

$$\mathbb{P}(D_i(t) = j) = \sum_{s=i+j-1}^{t} \mathbb{P}(D_i(s-1) = j - 1) \frac{j - 1 + \delta}{(2+\delta)(s-1) + 1 + \delta} \mathbb{P}(D_i(t) = j | D_i(s) = j). \tag{B.0.138}$$

By the induction hypothesis, we have that

$$\mathbb{P}(D_i(s-1) = j-1) \le C_{j-1} \frac{\Gamma(s-1)\Gamma(i+\frac{1+\delta}{2+\delta})}{\Gamma(s-1+\frac{1+\delta}{2+\delta})\Gamma(i)}. \tag{B.0.139}$$

<div style="text-align:right">e:bdInda</div>

Moreover, analogously to (B.0.137), we have that

$$\mathbb{P}(D_i(t) = j | D_i(s) = j) = \prod_{q=s+1}^{t} \left(1 - \frac{j+\delta}{(2+\delta)(q-1)+(1+\delta)}\right) \tag{B.0.140}$$

<div style="text-align:right">e:bdIndb</div>

$$= \prod_{q=s+1}^{t} \left(\frac{q-1-\frac{j-1}{2+\delta}}{q-1+\frac{1+\delta}{2+\delta}}\right) = \frac{\Gamma(t-\frac{j-1}{2+\delta})\Gamma(s+\frac{1+\delta}{2+\delta})}{\Gamma(t+\frac{1+\delta}{2+\delta})\Gamma(s-\frac{j-1}{2+\delta})}.$$

Combining (B.0.139) and (B.0.140), we arrive at

$$\mathbb{P}(D_i(t) = j) \le \sum_{s=i+j-1}^{t} \left(C_{j-1} \frac{\Gamma(s-1)\Gamma(i+\frac{1+\delta}{2+\delta})}{\Gamma(s-1+\frac{1+\delta}{2+\delta})\Gamma(i)}\right)\left(\frac{j-1+\delta}{(2+\delta)(s-1)+(1+\delta)}\right)\left(\frac{\Gamma(t-\frac{j-1}{2+\delta})\Gamma(s+\frac{1+\delta}{2+\delta})}{\Gamma(t+\frac{1+\delta}{2+\delta})\Gamma(s-\frac{j-1}{2+\delta})}\right). \tag{B.0.141}$$

We next use that

$$\Gamma(s-1+\frac{1+\delta}{2+\delta})((2+\delta)(s-1)+(1+\delta)) = (2+\delta)\Gamma(s+\frac{1+\delta}{2+\delta}), \tag{B.0.142}$$

to arrive at

$$\mathbb{P}(D_i(t) = j) \le C_{j-1} \frac{j-1+\delta}{2+\delta} \frac{\Gamma(i+\frac{1+\delta}{2+\delta})}{\Gamma(i)} \frac{\Gamma(t-\frac{j-1}{2+\delta})}{\Gamma(t+\frac{1+\delta}{2+\delta})} \sum_{s=i+j-1}^{t} \frac{\Gamma(s-1)}{\Gamma(s-\frac{j-1}{2+\delta})}. \tag{B.0.143}$$

We note that, whenever $l+b, l+1+a > 0$ and $a-b+1 > 0$,

$$\sum_{s=l}^{t} \frac{\Gamma(s+a)}{\Gamma(s+b)} = \frac{1}{a-b+1}\left[\frac{\Gamma(t+1+a)}{\Gamma(t+b)} - \frac{\Gamma(l+1+a)}{\Gamma(l+b)}\right] \le \frac{1}{a-b+1} \frac{\Gamma(t+1+a)}{\Gamma(t+b)}. \tag{B.0.144}$$

<div style="text-align:right">e:Gammasum</div>

Application of (B.0.144) for $a = -1, b = -\frac{j-1}{2+\delta}, l = i+j-1$, so that $a-b+1 = \frac{j-1}{2+\delta} > 0$ when $j > 1$, leads to

$$\mathbb{P}(D_i(t) = j) \le C_{j-1} \frac{j-1+\delta}{2+\delta} \frac{\Gamma(i+\frac{1+\delta}{2+\delta})}{\Gamma(i)} \frac{\Gamma(t-\frac{j-1}{2+\delta})}{\Gamma(t+\frac{1+\delta}{2+\delta})} \frac{1}{\frac{j-1}{2+\delta}} \frac{\Gamma(t)}{\Gamma(t-\frac{j-1}{2+\delta})} \tag{B.0.145}$$

<div style="text-align:right">e:IndAdv</div>

$$= C_{j-1} \frac{j-1+\delta}{j-1} \frac{\Gamma(i+\frac{1+\delta}{2+\delta})}{\Gamma(i)} \frac{\Gamma(t)}{\Gamma(t+\frac{1+\delta}{2+\delta})}.$$

Equation (B.0.145) advances the induction by (B.0.136).   □

Solutions to the exercises of Chapter 9.

Solutions to the exercises of Chapter 10.

Exercise 10.1. Denote the total variation distance between $\{p_k(n)\}_{k=1}^{\infty}$ and $\{p_k\}_{k=1}^{\infty}$ by

$$d_{\text{TV}}(p, p(n)) = \frac{1}{2} \sum_{k=1}^{\infty} |p_k - p_k(n)|.$$

Show that

$$d_{\mathrm{TV}}(p, p(n)) = 1 - F(n^a).$$

Solution to Exercise 10.1. We note that $p_k \leq p_k(n)$ when $k \leq n^a$, while $p_k(n) = 0$ for $k > n^a$. Therefore, we obtain that

$$d_{\mathrm{TV}}(p, p(n)) = \frac{1}{2} \sum_{k=1}^{n^a} p_k \left(\frac{1}{F(n^a)} - 1 \right) + \frac{1}{2} \sum_{k > n^a} p_k = \frac{1}{2} F(n^a) \left(\frac{1}{F(n^a)} - 1 \right) + \frac{1}{2} [1 - F(n^a)] = 1 - F(n^a).$$

Exercise 10.2. When

$$1 - F(x) = \sum_{k > x} p_k \leq x^{1-\tau} \tag{B.0.146}$$

for some $\tau > 1$, show that with high probability, the degree sequences $\{D_i^{(n)}\}_{i=1}^n$ and $\{D_i\}_{i=1}^n$ can be coupled to $\{\hat{D}_i^{(n)}\}_{i=1}^n$ and $\{\hat{D}_i\}_{i=1}^n$ such that the probability that $\{\hat{D}_i^{(n)}\}_{i=1}^n \neq \{\hat{D}_i\}_{i=1}^n$ is bounded by $n^{1-a(\tau-1)}$. In particular, for $a > \frac{1}{\tau-1}$, with high probability, $\{\hat{D}_i^{(n)}\}_{i=1}^n = \{\hat{D}_i\}_{i=1}^n$.

Solution to Exercise 10.2. We use Exercise 10.1 and a coupling argument. We take $\{D_i\}_{i=1}^n$ and $\{D_i^{(n)}\}_{i=1}^n$ to be two i.i.d. sequences of random variables, where D_i has probability mass function $\{p_k\}_{k=1}^\infty$, while $D_i^{(n)}$ has probability mass function $\{p_k(n)\}_{k=1}^\infty$.

By the coupling in Section 2.2, we can couple each of the $D_i, D_i^{(n)}$ with $(\hat{D}_i, \hat{D}_i^{(n)})$ such that

$$\mathbb{P}(\hat{D}_i \neq \hat{D}_i^{(n)}) = d_{\mathrm{TV}}(p, p(n)).$$

By Exercise 10.1, and (B.0.146), we obtain that

$$\mathbb{P}(\hat{D}_i \neq \hat{D}_i^{(n)}) \leq n^{a(1-\tau)}.$$

Therefore, by Boole's inequality,

$$\begin{aligned}
\mathbb{P}(\{\hat{D}_i^{(n)}\}_{i=1}^n \neq \{\hat{D}_i\}_{i=1}^n) &\leq \sum_{i=1}^n \mathbb{P}(\hat{D}_i \neq \hat{D}_i^{(n)}) \\
&= n\mathbb{P}(\hat{D}_1 \neq \hat{D}_1^{(n)}) \leq n^{1+a(1-\tau)}.
\end{aligned}$$

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