# Mathematical Modeling & Analysis of 3D objects and their projections

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*Report 1*

**Abstract**

In this report, the main focus is on elucidation of mathematical model on computing projection from given $3D$ objects to $2D$ (Orthographic Projections) and vice versa. We figured out theoretically number of necessary views and How many of them are sufficient? It also includes the necessary interactions.

## 1 Introduction

Engineering Drawing (AutoCAD) Software is for visualizing the 3D objects. For getting a 3D model , here we introduce the mathematical approach for reconstructing 3D objects using some salient property of the matrix representation of solids. The proposed approach works as per following methodological steps : -

- The input data of orthographic views is taken. Then each view is separated and identified from input Engineering Drawing.
- Using matrix theory, analyzing the relationship between 3D solid edges and their projections onto the planes.

## 2 Matrix representation of solids (2D-3D)

In this section, we introduced the simple linear relation between solids and symmetric matrices. A general algebraic representation of conic(3D) in a plane can be given as

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0 \tag{1}$$

The matrix notation for this conic can be written in equation form as

$$f(u) = u^T A u = 0 \tag{2}$$

where $u = [x, y, 1]^T$, $u^T$ is the transpose of u and A is 3x3 symmetric matrix given by

$$A = \begin{bmatrix} a_{11} & b_{12} & c_{13} \\ a_{21} & b_{22} & c_{23} \\ a_{31} & b_{32} & c_{33} \end{bmatrix}$$

Matrix A should be non singular for f(u) due to which f(u) form a non degenerate solid . Any linear transformation, $u = Pu$ results in change of matrix form A to $A_p$ given by

$$A_p = P^T AP$$

Therefore, this represents the linear relationship between conic(3D) and its orthographic projection.

# 3 Minimum number of views

In this section we introduced minimum number of views possible to reconstruct 3D objects. For proving sufficient projections required, we have to build an argument on basis of non-degenerate parallel projection.

**Defination:-** If the plane containing the solid is not perpendicular to the projection plane, then the parallel projection is non-degenerate. All conics are equivalent and if at least one projection of a planar curve is conic, then the planar curve must also be conic. Therefore, we can determine the class of a conic from the class of its projection.

Consider a plane $p$ containing solid and an object coordinate system $c_p$ defined in a way such that it's $x_p$ and $y_p$ lies on the plane $p$. Let $c$ be a global coordinate system in space represented by

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

in $c^P$. Applying a transformation to $c$ of the form

$$x = Rx_p + t \tag{3}$$

where **R, t** are rotation and translation represented by matrix,

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_p \\ y_p \\ z_p \\ 1 \end{bmatrix} + \begin{bmatrix} t_0 \\ t_1 \\ t_2 \\ 0 \end{bmatrix}$$

For any point on the plane $p$, we have $z_p = 0$, thus we obtain

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_p \\ y_p \\ z_p \\ 1 \end{bmatrix} + \begin{bmatrix} t_0 \\ t_1 \\ t_2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} r_{00} & r_{01} & t_0 \\ r_{10} & r_{11} & t_1 \\ r_{20} & r_{21} & t_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_p \\ y_p \\ 1 \end{bmatrix} = Pu_p$$

$$x = Pu_p \tag{4}$$

where $u_p$= $[x_p, y_p, 1]^T$ and **P** is a 4x3 matrix given by

$$\mathbf{P} = \begin{bmatrix} r_{00} & r_{01} & t_0 \\ r_{10} & r_{11} & t_1 \\ r_{20} & r_{21} & t_2 \\ 0 & 0 & 1 \end{bmatrix}$$

Considering the relationship between a solid and its orthographic projections onto some projection planes. Let $c_i(i = 1, 2, ..., q)$ denote the 2D local coordinate system associated with $i$th porjection plane. If $C_i$ is a 3x4 matrix whose three columns form an orthogonal basis for this projection. Transformation from a point **x** in 3D to point $u_i$ is given by

$$u_i = C_i x \tag{5}$$

Matrix relating to $u_i$ and $u_p$ is obtained from eq(4)

$$u_i = C_i P u_p = G_i u_p \tag{6}$$

where $G_i = C_i P$

**Theorem** Three distinct orthographic projections are sufficient to uniquely recover a space conic.

**Proof.** Considering three non-degenerate parallel projections. Let **A** be a solid that lies on a plane $p$

$$u_p^T A u_p = 0 \tag{7}$$

and its projection curves $A_i$ are represented by

$$u_i^T A_i u_i = 0 \tag{8}$$

where $i = 1, 2, ..., q$ Substituting the linear transformation of the form $u_i = G_i u_p$ into eq(8), we obtain

$$u_p^T G_i^T A_i G_i u_p = 0 \tag{9}$$

From eq(7) and (9) it follows that

$$G_i^T A_i G_i = P^T C_i^T A_i C_i P = A \tag{10}$$

where $i = 1, 2, ..., q$, **A** and **P** are unknown matrices. It follows that three orthographic projections yield 18 equations in 15 unknowns. By Bernstein's seminal theorem [1], we can derive that the system of polynomial equations (10) is solvable. Hence, three distinct views are sufficient to uniquely recover a conic if none of its projections is degenerate. Further, consider the special case where at least one of the orthographic projections is degenerate. If an orthographic projection is degenerate, then the projection of a conic onto this projection plane is a straight line. By the defination of orthographic projections, we can determine the plane on which the conic lies, which is obtained by extruding the straight line along the degenerate projection direction. Since at least one of the projection of the conic is also conic, we locate the center point of the space conic by finding its corresponding points in the other two views. Without loss of generality, let the projection in the front view be conic. To reconstruct the space conic, we then need to solve the equation $P^T C_f^T A_f C_f P = A$ for **A**, where the subscript $f$ indicates the front view. Hence, if at least one of the orthographic projections is degenerate, three distinct orthogonal projections are also sufficiently to identify the space conic. Hence, this completes the proof.

# 4 Reconstruction of 3D solid using wired-frame

With the help of wired-frame oriented approach we recovered the 3D objects. In this approach, automatic interpretation of faces and solids is prepared and reconstructed solid is represented using B-rep model.

**Wired-frame oriented approach**

In this section, to create a 3D solid we construct all the possible 3D vertices and edges. A candidate vertex is created from 2D vertices of the front, top and side view. Let $N_f = (N_f(x), N_f(z)), N_t = (N_t(x), N_t(y)), N_s = (N_s(y), N_s(z))$ be 2D vertices in three views. If

$$|N_f(x) - N_t(x)| < \epsilon, |N_t(y) - N_s(y)| < \epsilon, |N_t(z) - N_s(z)| < \epsilon$$

Where epsilon is the tolerance factor for inexact match. We know that $N_f, N_t, N_s$ are the corresponding projections of a 3D vertex in each view. Now , for generating 3D edges we have to determine the symmetric matrix A using the equation given below :

$$G_i^T A_i G_i = P^T C_i^T A_i C_i P = A$$

For i=1,2,3 If the three orthographic views are the front view , top view and side view for the specified 3D solid , we can subsequently state that projection matrix for front view is matrix C2, for top view the projection matrix is C1 and for side view the projection matrix is C3.

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus the transformation matrix can be formulated as

$$G_i = C_i P$$

Henceforth , we obtain matrices of G for i=1,2,3 as follows :

$$G_1 = \begin{bmatrix} r_{00} & r_{01} & t_0 \\ r_{10} & r_{11} & t_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} r_{00} & r_{01} & t_0 \\ r_{20} & r_{21} & t_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G_3 = \begin{bmatrix} r_{00} & r_{01} & t_0 \\ r_{30} & r_{31} & t_3 \\ 0 & 0 & 1 \end{bmatrix}$$

From all the candidates faces found from the wired frame and pseudo faces that could be generated from back projection are detected and deleted. Finally, all true faces are assembled to form an oriented 3D object.

# 5 Projection of 3D solid

In this section, we are required to take projections of 3D object onto various planes (especially x-y, y-z and x-z planes). 3D object in a space constitutes some cases such as rotation along particular axes , translation and scaling . These cases change the projection of 3D objects on to the plane.
Let us know discuss the Mathematical modelling of these cases using Matrix-based approach.

**Mathematical model of Translation**

$$^2P =^1 P + [x_0, y_0, z_0]^t = T(x_0, y_0, z_0)^1P \tag{11}$$

$$^2P = \begin{bmatrix} ^2P_x \\ ^2P_y \\ ^2P_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ^1P_x \\ ^1P_y \\ ^1P_z \\ 1 \end{bmatrix}$$

**Mathematical model of Rotation along standard axes(x,y,z)**

Rotation of $\Theta$ about the X axis.

$$^2P = R(^1X, \Theta)^1P \tag{12}$$

$$\begin{bmatrix} ^2P_x \\ ^2P_y \\ ^2P_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & cos\Theta & -sin\Theta & 0 \\ 1 & sin\Theta & cos\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ^1P_x \\ ^1P_y \\ ^1P_z \\ 1 \end{bmatrix}$$

Rotation of $\Theta$ about the Y axis.

$$^2P = R(^1Y, \Theta)^1P \tag{13}$$

$$\begin{bmatrix} ^2P_x \\ ^2P_y \\ ^2P_z \\ 1 \end{bmatrix} = \begin{bmatrix} cos\Theta & 0 & sin\Theta & 0 \\ 0 & 1 & 0 & 0 \\ -sin\Theta & 0 & cos\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ^1P_x \\ ^1P_y \\ ^1P_z \\ 1 \end{bmatrix}$$

Rotation of $\Theta$ about the Z axis.

$$^2P = R(^1Z, \Theta)^1P \tag{14}$$

$$\begin{bmatrix} ^2P_x \\ ^2P_y \\ ^2P_z \\ 1 \end{bmatrix} = \begin{bmatrix} cos\Theta & -sin\Theta & 0 & 0 \\ sin\Theta & cos\Theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ^1P_x \\ ^1P_y \\ ^1P_z \\ 1 \end{bmatrix}$$

## Mathematical model of Scaling

$$^2P = S^1P = S(s_x, s_y, s_z)^1P \tag{15}$$

$$\begin{bmatrix} ^2P_x \\ ^2P_y \\ ^2P_z \\ 1 \end{bmatrix} = \begin{bmatrix} sx^2P_x \\ sx^2P_y \\ sx^2P_z \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ^1P_x \\ ^1P_y \\ ^1P_z \\ 1 \end{bmatrix}$$

## Mathematical model of Arbitrary Rotation

$$^2P = R(A, \Theta)^1P \tag{16}$$

$$\begin{bmatrix} ^2P_x \\ ^2P_y \\ ^2P_z \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_2 & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ^1P_x \\ ^1P_y \\ ^1P_z \\ 1 \end{bmatrix}$$

## Computing Projection of Point onto the plane

In this section,basis of a plane is taken. Since the plane is 2-dimensional, any two independent vectors in the plane will do, say, $(x_1, y_1, 0)$ and $(0, y_2, z_2)$. Set

$$A = \begin{bmatrix} x_1 & 0 \\ y_1 & y_2 \\ 0 & z_2 \end{bmatrix} and A^T A = \begin{bmatrix} x_1^2 + y_1^2 & y_1 y_2 \\ y_1 y_2 & y_2^2 + z_2^2 \end{bmatrix}$$

The projection matrix Q for the plane is

$$Q = A(A^T A)^{-1} A^T$$

We can now project any vector onto the plane by multiplying by Q:

$$Projection(v) = Qv$$

# References


[1] D.Klain, *Orthogonal Projections and Reflections*, (Version 2010.01.23).

[2] Stockman MSU/CSE.

[3] Shi-Xia Liu, Shi-Min Hu, Chiew-Lan Tai and Jia-Guang Sun *A matrix based approach to reconstruction of 3D objects from three orthographic projection*, (Tsinghua University, Beijing 100084).