Russell's paradox

Xing Yan

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On an Island, there is only one barber, who is a male. Every men on this island keeps himself clean-shaven. On the window of the barber shop, there is a poster says I only shave people who don't shave themselves. Then does the barber shave himself?

If the barber shaves himself, then what he says in the poster is a lie. However, if he doesn't shave himself, then according to the poster, he should shave himself. This is a famous paradox called Barber Paradox. If we treat every man in that island as a set, and the element in that set is the person/people he shave. Then the barber set will contain all the person set that are not contained in themselves. Then the question becomes does the barber set contian itself?

Let's put it formally. Suppose there is a set U which contains all the sets in the world, so $U \in U$. Let R be the set which is $\{A \in U \mid A \notin A\}$. That means $\forall A \in U(A \in R \leftrightarrow A \notin A)$. Since U contains all the set in the universe, then $R \in U$, which make $R \in R \leftrightarrow R \notin R$. This is called Russell's Paradox discovered by Bertrand Russell in 1901.

Russell's Paradox showed that the naive set theory created by Georg Cantor leads to a contradiction. The main reason that the naive set theory leads to a contradiction is that the naive set theory assumes that a set can be formed by all classes of objects satisfying any particular defining condition. Russell's Paradox was a huge deal at that time cause it revealed the contradiction lead by the naive set theory. Everyone was trying to improve the set theory after Russell published the paradox. In 1908, Ernst Zermelo published a paper and constructed another set theory, Zermelo set theory, which is the ancestor of modern set theory. Zermelo set theory introduced the idea of using axioms to define sets. Zermelo used his set theory to show how to prevent Russell's Paradox from happening.

The third axioms in Zermelo set theory (Axiom of separation) states that given any set A, there is a set B such that, given any set x, x is a member of B if and only if x is a member of A and propositional function ϕ holds for x. We can use this axiom to prevent Russell's Paradox. The following is the steps to show how Russell's Paradox get fixed.

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\begin{aligned}
\forall A \exists B \forall x \ (x \in B \leftrightarrow (x \in A \wedge \phi(x))) \\
let\ \phi(x)\ be\ x \notin x, then\ \forall A \exists B \forall x \ (x \in B \leftrightarrow (x \in A \wedge (x \notin x))) \\
\exists B \forall x \ (x \in B \leftrightarrow (x \in R \wedge (x \notin x))) \\
let x = B\ then\ we\ have\ B \in B \leftrightarrow (B \in R \wedge (B \notin B))
\end{aligned}
$$

The truth of the last formula is seen by taking R itself as an appropiate B, for R is not a member of itself. We get:

$$R \in R \leftrightarrow (R \in R \wedge R \notin R)$$

Since both side is false, then the whole statement is true and no longer a contradiction. This is how Russell's Paradox get avioded in modern set thoery and because of Russell's Paradox our set theory became more mature.