Unit 5 :- Calculus of Complex Function

Analytic function :

$F(z) = w = u + iv$ is an analytic function if it satisfied Cauchy Riemann's eq^n i.e

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

$$z = \phi(r, \theta)$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Harmonic Function

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Note :-

Every harmonic function is the analytic function and every analytic function need not be harmonic function.

$e^{i\theta} = \cos\theta + i\sin\theta$

DATE

* Examples -

1. Prove that $F(z) = \sinh z = \frac{e^z - e^{-z}}{2}$ for $z = x + iy$.

→ We have,

$$F(z) = \sinh z = \frac{e^z - e^{-z}}{z}$$

$$= \frac{e^{x+iy} - e^{-(x+iy)}}{2}$$

$$= \frac{1}{2}\left[e^x e^{iy} - e^{-x} e^{-iy}\right]$$

use $e^{i\theta} = \cos\theta + i\sin\theta$

$$= \frac{1}{2}\left[e^x[\cos y + i\sin y] - e^{-x}(\cos y - i\sin y)\right]$$

$$= \frac{1}{2}\left[e^x \cos y + ie^x \sin y - e^{-x}\cos y + ie^{-x}\sin y\right]$$

$$= \frac{1}{2}\cos y\,(e^x - e^{-x}) + \frac{1}{2} i\sin y\,(e^x + e^{-x})$$

$$= u + iv$$

$\therefore$ $u = \frac{1}{2}\cos y\,(e^x - e^{-x})$ & $v = \frac{1}{2}\sin y\,(e^x + e^{-x})$

diff w.r.t x and y

$$\frac{\partial u}{\partial x} = \frac{1}{2} \cos y \left(e^{x} - e^{-x}(-1)\right)$$

$$= \frac{1}{2} \cos y \left(e^{x} + e^{-x}\right) \quad \text{--- (I)}$$

$$\frac{\partial u}{\partial y} = -\frac{1}{2} \left(e^{x} - e^{-x}\right) \sin y \quad \text{--- (II)}$$

$$V = \frac{1}{2} \sin y \left(e^{x} + e^{-x}\right)$$

diff. w.r.t. x & y

$$\frac{\partial v}{\partial x} = \frac{1}{2} \sin y \left[e^{x} + e^{-x}(-1)\right]$$

$$= \frac{1}{2} \sin y \left[e^{x} - e^{-x}\right] \quad \text{--- (III)}$$

$$\frac{\partial v}{\partial y} = \frac{1}{2} \left(e^{x} + e^{-x}\right) \cos y \quad \text{--- (IV)}$$

From eq^n (I) & (IV)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{1}{2} \cos y \left(e^{x} + e^{-x}\right) \quad \text{--- (A)}$$

From eq^n (II) & (III)

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

$$\therefore \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{---} \; (B)$$

From eq^n (A) and (B)

Cauchy Riemann eq^n satisfied

$\therefore F(z) = \sinh z$ is analytic function.

2. Find the function $w = u + iv$ is analytic if $u = x^2 - y^2$

→

$w = u + iv$ is analytic

$\therefore$ C.R eq^n is satisfied

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{---} \; (A)$$

we have

$u = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x - 0$$

$$= 2x \quad \text{---} \; (I)$$

$$\frac{\partial u}{\partial y} = 0 - 2y \quad \text{---} \; (II)$$

by C.R eq^n

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial v}{\partial y} = 2x$$

$$\partial v = 2x \partial y$$

on integration

$$\int 1 \partial v = \int 2x \, \partial y$$

$$v = 2xy + F(x) \quad \text{— (B)}$$

But we know by C.R eq^n

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$-2y = -\frac{\partial v}{\partial x} \qquad \text{from (II)}$$

$$\boxed{\frac{\partial v}{\partial x} = 2y} \quad \text{— (C)}$$

Eq^n (B) diff. w.r.t. 'x'.

$$\frac{\partial v}{\partial x} = 2y \times 1 + F'(x)$$

From eq^n (C)

$$2y = 2y + F'(x)$$

$$F'(x) = 0$$

On integration

$$\int F'(x)\,dx = \int 0\,dx$$

$$F(x) = c$$

eq^n (B) becomes

$$\boxed{v = 2xy + c} \quad \text{— (D)}$$

$\therefore \quad w = u + iv$

$$\therefore \quad \boxed{w = x^2 - y^2 + i(2xy + c)}$$

3. Show that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is a harmonic function and hence determine the corresponding analytic function.

→ We have,

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

diff. w.r.t. x

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x + 0 + 0$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x \quad \text{— (1)}$$

diff. w.r.t. x

$$\frac{\partial^2 u}{\partial x^2} = 6x - 0 + 6$$

$$\frac{\partial^2 u}{\partial x^2} = 6x + 6 \quad \text{— (ii)}$$

$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

diff. w.r.t. y

$$\frac{\partial u}{\partial y} = 0 - 3x(2y) + 0 - 3(2y) + 0$$

$$\frac{\partial u}{\partial y} = -6xy - 6y \quad \text{— (III)}$$

Diff. w.r.t. y

$$\frac{\partial^2 u}{\partial y^2} = -6x - 6 \quad \text{— (IV)}$$

From eq^n (II) & (IV)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 6 + (-6x) - 6$$

$$= 0$$

$\therefore$ u is harmonic therefore u is analytic

$\therefore$ C-R eq^n is satisfied

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \longrightarrow (A)$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x = \frac{\partial v}{\partial y} \quad \text{... by eq}^n \text{ (A)}$$

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 6x$$

$$\partial v = (3x^2 - 3y^2 + 6x)\,\partial y$$

On integration

$$\int \partial v = \int (3x^2 - 3y^2 + 6x)\,\partial y$$

$$v = 3x^2 y - \frac{3y^3}{3} + 6xy + F(x) \longrightarrow \text{(B)}$$

From eq^n (II)

$$\frac{\partial u}{\partial y} = -6xy - 6y = -\frac{\partial v}{\partial x} \quad \text{by eq}^n \text{ (A)}$$

$$\frac{\partial v}{\partial x} = 6xy + 6y \longrightarrow \text{(C)}$$

diff eq^n (B) w.r.t. x

$$\frac{\partial v}{\partial x} = 6xy - 0 + 6y + F'(x)$$

$$6xy + 6y = 6xy + 6y + F'(x)$$

$$F'(x) = 0$$

$$\therefore \boxed{F(x) = c}$$

eq^n (B) becomes

$$\boxed{v = 3x^2 y - y^3 + 6xy + c}$$

$\therefore \quad w = u + iv$

$$\boxed{w = (x^3 - 3xy^2 + 3x^2 - 3y^2 + 1) + i(3x^2y - y^3 + 6xy + c)}$$

4. Find the analytic function whose imaginary part $e^x(x\sin y + y\cos y)$

$r^2\cos 2\theta - 2\cos\theta + 2$

→

~~We have given:~~

Let $F(z) = w = u + iv$ and

we have given.

V = Imaginary part = $e^x(x\sin y + y\cos y)$

$\therefore \quad V = e^x(x\sin y + y\cos y)$

diff. w.r.t. x

$$\frac{\partial v}{\partial x} = e^x[\sin y + 0] + (x\sin y + y\cos y)e^x$$

$$= e^x\sin y + xe^x\sin y + ye^x\cos y \quad \text{--- (I)}$$

Since $F(z) = w = u + iv$ is analytic

It satisfied C-R eq^n

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow (A)$$

From eq^n (I) & (A)

$$\frac{\partial v}{\partial x} = e^x\sin y + xe^x\sin y + ye^x\cos y = -\frac{\partial u}{\partial y}$$

$$-\frac{\partial u}{\partial y} = e^x \sin y + x e^x \sin y + y e^x \cos y$$

$$-\partial u = (e^x \sin y + x e^x \sin y + y e^x \cos y)\,\partial y$$

on integration

$$-\int 1\,\partial u = \int [e^x \sin y + x e^x \sin y + y e^x \cos y]\,\partial y$$

$$\cancel{= \int e^x [}$$

$$-u = -e^x \cos y + x e^x(-\cos y) + e^x\left[y \sin y - \int 1 \sin y\, dy\right] + F(x)$$

$$-u = -e^x \cos y - x e^x \cos y + e^x[y \sin y - (-\cos y)] + F(x)$$

$$-u = -\underline{e^x \cos y} - x e^x \cos y + y e^x \sin y + \underline{e^x \cos y} + F(x)$$

$$-u = -[x e^x \cos y - y e^x \sin y - F(x)]$$

$$u = x e^x \cos y - y e^x \sin y - F(x) \longrightarrow (B)$$

we know

$$v = e^x (x \sin y + y \cos y)$$

$$\frac{\partial v}{\partial y} = e^x [x \cos y + y(-\sin y) + \cos y \times 1]$$

$$\frac{\partial v}{\partial y} = e^x [x \cos y - y \sin y + \cos y] = \frac{\partial u}{\partial x}$$

--- from (A)

$$\frac{\partial u}{\partial x} = e^x x\cos y - ye^x\sin y + e^x\cos y \quad \text{--- (C)}$$

eq^n (B) diff. w.r.t. x

$$\frac{\partial u}{\partial x} = \cos y\,[xe^x + e^x \times 1] - y\sin y\, e^x - F'(x)$$

$$= xe^x\cos y + e^x\cos y - ye^x\sin y - F'(x)$$

From eq^n (C)

$$e^x x\cos y - ye^x\sin y + e^x\cos y = xe^x\cos y + e^x\cos y - ye^x\sin y - F'(x)$$

$$0 = -F'(x)$$

$$F'(x) = 0$$

on integration

$$\int F'(x) = \int 0 \, \partial x$$

$$\boxed{F(x) = c}$$

~~eq^n (B)(C) becomes~~

eq^n (B) becomes

$$u = xe^x\cos y - ye^x\sin y - c$$

$$F(z) = w = u + iv$$

$$\boxed{F(z) = (xe^x\cos y - ye^x\sin y - c) + i\,[e^x(x\sin y + y\cos y)]}$$

5) If $F(z)$ is analytic function with Constant modulus Show that $F(z)$ is Constant.

$\rightarrow$

Let $F(z) = u+iv$ is an analytic Function that implies if satisfy C.R eq^n.

i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ and also

modulus of $F(z) = \sqrt{u^2+v^2} = c^2$ that implies $u^2+v^2+c^2 = C_1$

$$u^2+v^2+c^2 = c_1$$

diff. w.r.t. x

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$$2\left[u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right] = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0$$

$$u\frac{\partial u}{\partial x} + v\left(-\frac{\partial u}{\partial y}\right) = 0 \quad \text{by C.R. eq}^n$$

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0 \quad \cdots \text{(A)}$$

again:

$$u^2+v^2=0$$

diff. w.r.t. y

$$2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}=0$$

$$2\left[u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}\right]=0$$

$$u\frac{\partial u}{\partial y}+\frac{v\,\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial y}+v\frac{\partial u}{\partial x}=0 \quad \text{.... (B)}$$

Squaring and adding eq^n A & B

$$\left(u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}\right)^2+\left(u\frac{\partial u}{\partial y}+v\frac{\partial u}{\partial x}\right)^2=0$$

$$u^2\left(\frac{\partial u}{\partial x}\right)^2-2uv\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}+v^2\left(\frac{\partial u}{\partial y}\right)^2+$$

$$u^2\left(\frac{\partial u}{\partial y}\right)^2+2uv\frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial x}+v^2\left(\frac{\partial u}{\partial x}\right)^2=0$$

$$\left(\frac{\partial u}{\partial x}\right)^2(u^2+v^2)+\left(\frac{\partial u}{\partial y}\right)^2(u^2+v^2)=0$$

$$(u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]=0$$

$$C_1\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] = 0$$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 0$$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(-\frac{\partial v}{\partial x}\right)^2 = 0$$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0$$

$$|F'(z)|^2 = 0$$

$$|F'(z)| = 0$$

On integration

$$\boxed{F(z) = c}$$

- **Cauchy Integral formula**

$$\oint \frac{F(z)}{(z-a)}\, dz = 2\pi i\, F(a)$$

where $z-a = Pe^{i\theta}$

$\therefore \partial z = iPe^{i\theta}\, d\theta$

and $a =$ Singular Point

Singular Point
- Isolated
  - Pole
  - Removable Pole
  - Essential Pole
- Non-Isolated

Formulae:

1. $$\oint_C \frac{F(z)}{z-a}\, \partial z = 2\pi i F(a)$$

e.g. $$\oint_C \frac{3z^2+2z}{(z-1)(z-3)(z-2)}\, \partial z$$ C is Circle

$z-1=0$

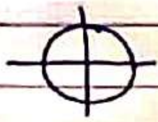
$z-3=0$

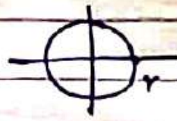
$z-2=0$

2. $$\oint_C \frac{F(z)}{(z-a)^{n+1}}\, \partial z = \frac{2\pi i^n F(a)}{n!}$$

$$\oint \frac{F(z)}{(z-a)^2}\, \partial z = \frac{2\pi i\, F'(a)}{2!}$$

$$\oint \frac{F(z)}{(z-a)^3}\, \partial z = \frac{2\pi i F''(a)}{2!}$$

1) $x^2 + y^2 = 1$  unique circle

2) $x^2 + y^2 = r^2$ 

3) $(x-h)^2 + (y-k)^2 = r^2$
where (h,k) is centre.

Examples.

Imp

1. $\oint_C \frac{e^{-z}}{z+1}\, dz$ C is circle

i) $|z| = 2$ and $|z| = 1/2$

→ The singular point

$z + 1 = 0$

$z = -1 = a$

C is circle for

i) $|z| = 2$

$|x + iy| = 2$

$|(x+0)+i(y+0)| = 2$

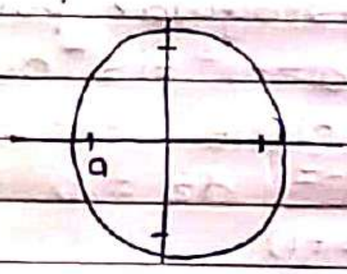
standard eq^n of circle

$(x-h)^2 + (y-k)^2 = r^2$

$(x+0)^2 + (y+0)^2 = 2^2$

Centre of circle

$C = (h,k) = (0,0)$ and radius $r = 2$



Since $z = a = -1$

inside the circle $|z| = 2$

$a = z = -1$

$= x+iy = -1$

$= -1+0i$

hence,

$$\oint \frac{F(z)}{(z-a)}\, dz = 2\pi i\, F(a) ; \quad z = -1, \Rightarrow (x,y) = (-1,0)$$

$$\oint \frac{e^{-z}}{(z+1)}\, dz = 2\pi i\, e^{-1(-1)}$$

$$= 2\pi i e$$

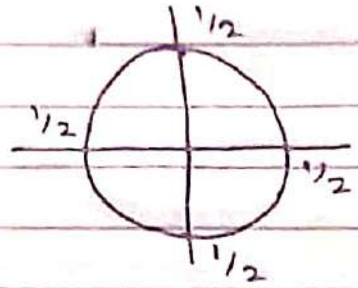
2) C is circle $|z| = 1/2$

$|x+iy| = 1/2$

$|(x+0)+i(y+0)| = 1/2$

$(x+0)^2 + (y+0)^2 = (1/2)^2$

centre $C = (0,0)$ $r = 1/2$



Since singular point $z = a = -1$ is outside the circle

$$\Rightarrow \oint_C \frac{e^{-z}}{(z+1)} dz = 0$$

2) $\oint \frac{\cos \pi z}{(z-1)(z-2)} dz$ C is circle $|z| = 3$

$\longrightarrow$ Singular points,

$z - 1 = 0 \Rightarrow z = 1 = a_1$

$z - 2 = 0 \Rightarrow z = 2 = a_2$

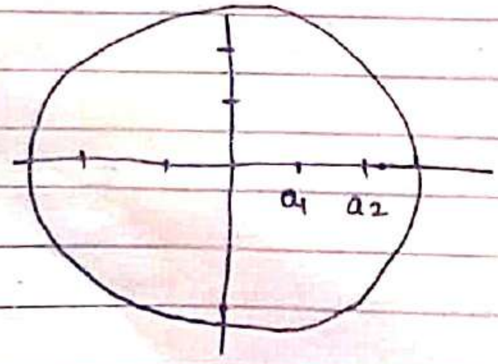
Since, C is circle $|z| = 3$

$|x + iy| = 3$

$|(x+0) + i(y+0)| = 3$

$(x+0)^2 + (y+0)^2 = 3^2$

centre $C = (0,0)$ & $r = 3$



Since both singular points

$z = a_1 = -1$ and $z = a_2 = 2$ are inside the circle

Consider,

$$\frac{1}{(z-1)(z-2)} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$\Rightarrow \oint_C \frac{\cos \pi z}{(z-1)(z-2)} dz = \oint_C \left[\frac{1}{(z-2)} - \frac{1}{(z-1)} \right] \cos \pi z \, dz$$

$$= \oint_C \frac{\cos \pi z}{(z-2)} dz - \oint \frac{\cos \pi z}{(z-1)} dz$$

use $\oint_C \frac{f(z)}{(z-a)} dz = 2\pi i f(a)$

$$= 2\pi i [\cos \pi (2) - \cos \pi (1)]$$

$$= 2\pi i [\cos 2\pi - \cos \pi]$$

$$= 2\pi i [1 - (-1)]$$

$$= 2\pi i [1+1]$$

$$= 4\pi i$$

3. Use Cauchy integral formula for $\oint_C \frac{e^{2z}}{(z+1)^4}$, C is circle $|z| = 2$

→ The singular point.

$$z+1=0$$

$$z=-1,-1,-1,-1$$

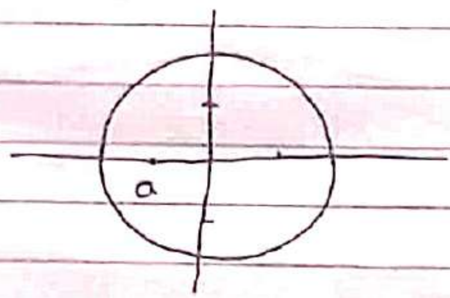
Since, C is circle $|z|=2$

$$|x+iy|=2$$

$$|(x+0)+i(y+0)|=2$$

$$(x+0)^2+(y+0)^2=2^2$$

Centre $C=(0,0)$, $r=2$



Since, all singular points are inside the circle $|z|=2$

$$\oint_C \frac{F(z)}{(z-a)^{n+1}}\,dz=\frac{2\pi i F^n(a)}{n!}$$

$$\oint_C \frac{e^{2z}}{(z+1)^4}\,dz=\frac{2\pi i F'''(a)}{3!} \longrightarrow A$$

Consider

$$F(z)=e^{2z}$$

$$F'(z)=2e^{2z}$$

$$F''(z)=4e^{2z}$$

$$f'''(z) = 8e^{2z}$$

$$f'''(a) = 8e^{2(-1)}$$
$$= 8e^{-2}$$

eq^n A becomes

$$\oint_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i}{3!}\, 8e^{-2}$$

$$= \frac{2\pi i}{3\times 2\times 1}\, 8e^{-2}$$

$$= \frac{8\pi i e^{-2}}{3}$$

Imp

4. $\oint \frac{\cos\pi z}{z^2-1}$ around rectangle whose vertices $2+i, -2\pm i$

→

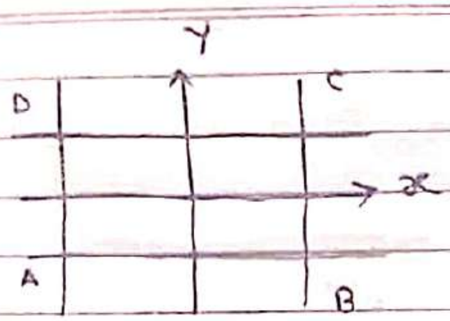
The Singular point

$$z^2-1=0$$
$$(z-1)(z+1)=0$$

$$z-1=0 \Rightarrow z=1$$
$$z+1=0 \Rightarrow z=-1$$

→ $a_1 = 1$, $a_2 = -1$

we have an rectangle whose vertices are $2\pm i$, $-2\pm i$



Since both the singular points are inside the circle i.e a_1 & a_2

& Consider,

$$\frac{1}{(z^2-1)} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left[\frac{1}{(z-1)} - \frac{1}{(z+1)}\right]$$

$$\oint \frac{f(z)}{(z-a)}\,dz = 2\pi i f(a)$$

$$\oint \frac{\cos\pi z}{(z^2-1)}\,dz = \oint_C \frac{\cos\pi z}{(z^2-1)(z+1)}\,dz$$

$$= \oint_C \frac{1}{2}\left[\frac{1}{z-1} - \frac{1}{z+1}\right]\cos\pi z\,dz$$

$$= \frac{1}{2}\oint \frac{\cos\pi z}{z-1}\,dz - \frac{1}{2}\oint \frac{\cos\pi z}{z+1}\,dz$$

$$= \frac{1}{2}\,2\pi i f(a_1) - \frac{1}{2}\,2\pi i f(a_2)$$

$$= \pi i \cos\pi(1) - \pi i \cos\pi(-1)$$

$$= \pi i\,[\cos\pi - \cos\pi]$$

$$= \pi i(0)$$

$$= 0$$

6) Solve $\int_0^{1+i} (x^2+iy)\,dz$ along the path

1) $y=x$, 2) $y=x^2$

→

Let $z=x+iy$

1) $z=x+iy$ ~~and~~ and at $y=x$

$z = x+ix$

$z = (1+i)x$

$dz = (1+i)dx$

as $z=0$, $x=0$

$z = x+ix$

$z = 0+0i = x+ix$

equating real and img.

$x=0$, $x=0$

as $z=1+i$ $x=1$

$1+i = x+ix$

$1+i = (1+i)x$

$x=1$

$$\int_0^{1+i} (x^2+iy)\,dz = \int_0^1 (x^2+ix)(1+i)\,dx$$

$$= (1+i)\left[\frac{x^3}{3} + i\frac{x^2}{2}\right]_0^1$$

$$= (1+i)\left[\frac{1}{3}+\frac{i}{2}\right] - 0$$

$$= (1+i)\frac{(2+3i)}{6}$$

$$= \frac{1}{6}\left[(1+i)(2+3i)\right]$$

$$= \frac{1}{6}\left[2+3i+2i+3i^2\right]$$

$$= \frac{1}{6}\left[2+5i-3\right]$$

$$= \frac{-1+5i}{6}$$

$$\int_0^{1+i} (x^2+iy)\,dz = \frac{-1+5i}{6}$$

2) $z = x+iy$ and at $y = x^2$

$z = x+ix^2$

$dz = dx + 2ix\,dx$

$dz = (1+2ix)\,dx$

as $z = 0$ $\quad x = 0$

$z = x+ix^2$

$z = 0+0i = x+ix^2$

equating real & img

$x = 0$, $x^2 = 0$

as $z = 1+i$ $\quad x = 1$

$1+i = x+ix^2$

$x = 1$, $x^2 = 1$

$$\int_0^{1+i} (x^2+iy)\,dz = \int_0^1 (x^2+ix^2)(1+2ix)\,dx$$

$$= \int_0^1 x^2(1+i)(1+2ix)\,dx$$

$$= (1+i)\int_0^1 x^2(1+2ix)\,dx$$

$$= (1+i)\int_0^1 (x^2+2ix^3)\,dx$$

$$= (1+i)\left[\frac{x^3}{3}+\frac{2ix^4}{4}\right]_0^1$$

$$= (1+i)\left[\frac{1}{3}+\frac{2i}{4}-0\right]$$

$$= (1+i)\left[\frac{1}{3}+\frac{i}{2}\right]$$

$$= (1+i)\frac{(2+3i)}{6}$$

$$= \frac{1}{6}\left[2+3i+2i+3i^2\right]$$

$$= \frac{1}{6}\left[2+5i-3\right]$$

$$\int_0^{1+i} (x^2+iy)\,dz = \frac{-1+5i}{6}$$

Residue Theorem

$$\oint_C F(z)\,dz = 2\pi i\left[\text{Sum of all residue at } z_1, z_2, z_3, z_4, \cdots\right]$$

Calculation methods of residue

A) If $F(z)$ is simple pole the

$$\operatorname{Res}_{z=z_0} F(z) = \lim_{z\to z_0} F(z)\,(z-z_0)$$

B) Let $F(z) = \dfrac{\phi(z)}{\psi(z)}$

then $$\operatorname{Res}_{z=z_0} F(z) = \lim_{z\to z_0}\left[(z-z_0)\frac{\phi(z)}{\psi(z)}\right]$$

$$= \lim_{z\to z_0}\left[\frac{(z-z_0)\phi(z)+\phi'(z)(z-z_0) + \cdots}{\psi(z)+(z-z_0)\psi'(z)}\right]$$

c) If $F(z)$ has a pole of order m then

$$\operatorname{Res}_{z=z_0} F(z) = \frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m F(z)\right]_{z=z_0}$$

* Examples -

1. Solve $\oint_C \frac{2z-1}{z(z+1)(z-3)} dz$ where c is circle where $|z| = 2$

→ We have,

$$F(z) = \frac{2z-1}{z(z+1)(z-3)}$$

Here it has three poles

$z = 0$, $z+1 = 0$, $z-3 = 0$

$z = 0$, $z = -1$, $z = 3$ and

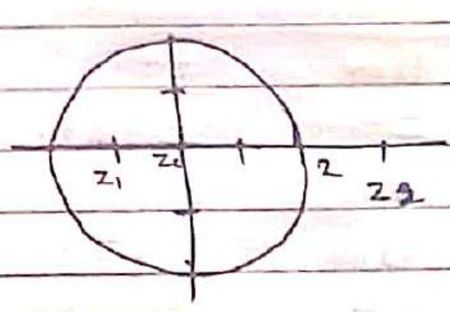
C is circle $|z| = 2$

$$|x+iy| = 0$$

$$|(x+0) + i(y+0)| = 2$$

$$(x+0)^2 + (y+0)^2 = 2^2$$

Centre $C = (0,0)$, $r = 2$



Since $z_0 = 0$, $z_1 = -1$ are inside the circle and $z_2 = 3$ outside the circle

∴ We can find residues for $z_0 = 0$ and $z_1 = -1$

$$\operatorname*{Res}_{z=z_0} F(z) = \lim_{z \to z_0} (z-z_0) F(z)$$

$$= \lim_{z \to 0} \frac{(z-0)(2z-1)}{z(z+1)(z-3)}$$

$$= \lim_{z \to 0} \frac{z(2z-1)}{z(z+1)(z-3)}$$

$$= \lim_{z \to 0} \frac{(2z-1)}{(z+1)(z-3)}$$

$$= \frac{2 \times 0 - 1}{(0+1)(0-3)} = \frac{-1}{-3}$$

$$= \frac{1}{3}$$

$$\operatorname*{Res}_{z=z_1} F(z) = \lim_{z \to -1} (z+1) F(z)$$

$$= \lim_{z \to -1} \frac{(z+1)(2z-1)}{z(z+1)(z-3)}$$

$$= \lim_{z \to -1} \frac{(2z-1)}{z(z-3)}$$

$$= \frac{2(-1)-1}{(-1)(-1-3)} = \frac{-2-1}{(-1)(-4)}$$

$$= -\frac{3}{4}$$

$$\oint_C F(z)\,dz = 2\pi i\,[\text{Sum of all Residues}]$$

$$= 2\pi i\left[\operatorname*{Res}_{z=z_0} F(z) + \operatorname*{Res}_{z=z_1} F(z)\right]$$

$$= 2\pi i\left[\frac{1}{3} + \left(\frac{-3}{4}\right)\right]$$

$$= 2\pi i\left[\frac{1}{3} - \frac{3}{4}\right]$$

$$= 2\pi i\left[\frac{4-9}{12}\right]$$

$$= 2\pi i\left[\frac{-5}{12}\right]$$

$$= \frac{-5\pi i}{6}$$

2. Solve $\oint_C \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z+1)^2 (z-2)}\,dz$ c is circle $|z|=3$

→

The poles are

$z+1=0 \qquad z-2=0$

$z=+1, +1 \qquad z = 2$

$\therefore \; z_0 = +1, +1 \;\&\; z = 2$

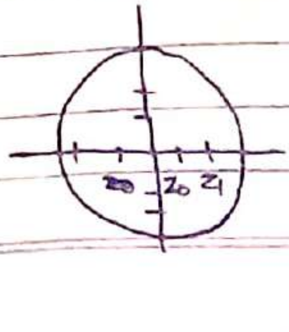
C is circle $|z| = 3$

$|x+iy| = 3$

$|(x+0) + i(y+0)| = 3$

$(x+0)^2 + (y+0)^2 = 3^2$

$C = (h,k) = (0,0) \;\&\; r = 3$



$$\operatorname*{Res}_{z=z_0} F(z) = \frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m F(z)\right]_{z=z_0}$$

$$= \frac{1}{1!}\left[\frac{d}{dz}\left[\frac{(z\mp 1)^2 \sin\pi z^2 + \cos\pi z^2}{(z\mp 1)^2 (z-2)}\right]\right]_{z=1}$$

$$= \frac{d}{dz}\left[\frac{\sin\pi z^2 + \cos\pi z^2}{(z-2)}\right]$$

$$= \frac{(z-2)\left(\cos\pi z^2 (2z\pi) + (-\sin\pi z^2 (2\pi z)\right) - (\sin\pi z^2 + \cos\pi z^2)(1-0)}{(z-2)^2}$$

$$= \frac{(+1-2)\left[\cos\pi(+1)^2 (2(+1)\pi) + (-\sin\pi(+1)^2 (2\pi(1))\right) - (\sin\pi(1)^2 + \cos\pi(1)^2)}{(1-2)^2}$$

$$= \frac{-1\,[2\pi(-1) + (-2\pi \times 0)] - (0 + (-1))}{(-1)^2}$$

$$= \frac{-1\,[-2\pi + \cancel{\pi}] + 1}{1}$$

$$= \cancel{4\pi+1}\ 2\pi + 1$$

$$\operatorname*{Res}_{z=z_1} F(z) = \lim_{z\to 2} (z-2) F(z)$$

$$= \lim_{z\to 2} \frac{\cancel{(z-2)}\,(\sin\pi z^2 + \cos\pi z^2)}{(z-1)^2 \cancel{(z-2)}}$$

$$\lim_{z\to 2} \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2}$$

$$= \frac{\sin\pi(2)^2 + \cos\pi(2)^2}{(2-1)^2}$$

$$= \frac{\sin 4\pi + \cos 4\pi}{1}$$

$$= 0 + 1$$

$$= 1$$

$$\oint_C F(z)\,dz = 2\pi i\,[\text{Sum of all residues}]$$

$$= 2\pi i\,[2\pi + 1 + 1]$$

$$= 2\pi i\,[2\pi + 2]$$

$$= 4\pi i\,[\pi + 1]$$

3. Solve $\oint_C \tan z\,dz$ C is circle $|z| = 2$

$\Rightarrow$ $F(z) = \tan z$

$$= \frac{\sin z}{\cos z}$$

The pole is $\cos z = 0$

$$z = \pm \frac{\pi}{2}$$

Both poles are inside the circle

$$z_0 = \frac{\pi}{2}, \quad z_1 = -\frac{\pi}{2}$$

$$\underset{z=z_0}{\text{Res}}\ F(z) = \lim_{z\to\pi/2} \frac{\sin z}{\frac{d}{dz}\cos z}$$

$$= \lim_{z \to \pi/2} \frac{\sin z}{-\sin z}$$

$$= \frac{\sin \pi/2}{-\sin \pi/2}$$

$$= \frac{1}{-1}$$

$$= -1$$

$$\operatorname*{Res}_{z=z_1} F(z) = \lim_{z \to -\pi/2} \frac{\sin z}{d/dz \cos z}$$

$$= \lim_{z \to -\pi/2} \frac{\sin z}{-\sin z}$$

$$= \frac{\sin(-\pi/2)}{-\sin(-\pi/2)}$$

$$= \frac{-\sin \pi/2}{-(-\sin \pi/2)}$$

$$= \frac{-1}{-1(-1)} = -1$$

$\oint F(z)\,dz = 2\pi i$ [Sum of all residues]

$$= 2\pi i\,[-1-1]$$

$$= -4\pi i$$