

Linear and Quadratic Programming (with CGAL)

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Based on slides by Bernd Gärtner

Today

- Quick introduction to Linear Programming (formulation and representation).
- Quick recap on techniques to solve it (no extensive knowledge needed).
- Linear Programming in CGAL.
- Examples
- Quadratic Convex Programming.

Linear Programming (LP)

- Linear programming is a central topic in optimization .
- It provides a powerful tool in modeling many applications .
- LP has attracted most of its attention in optimization during the last six decades for two main reasons :
 - Applicability : There are many real-world applications that can be modeled as linear programming ;
 - Solvability : There are theoretically and practically efficient techniques for solving large-scale problems.

Linear Programming (LP)

We face an optimization problem subject to some constraints.

- * **Variables:** describe our choices, the parameters that we are allowed to change;
- * **Objective function:** Describes the criterion that we wish to minimize (e.g. cost) or maximize (e.g. profit);
- * **Constraints:** Describe the limitations that we have for the choice of the values of the variables.

Linear Programming

Mathematical formulation

- ❖ **Problem:** Minimize a linear function in n variables subject to m linear (in)equality constraints!

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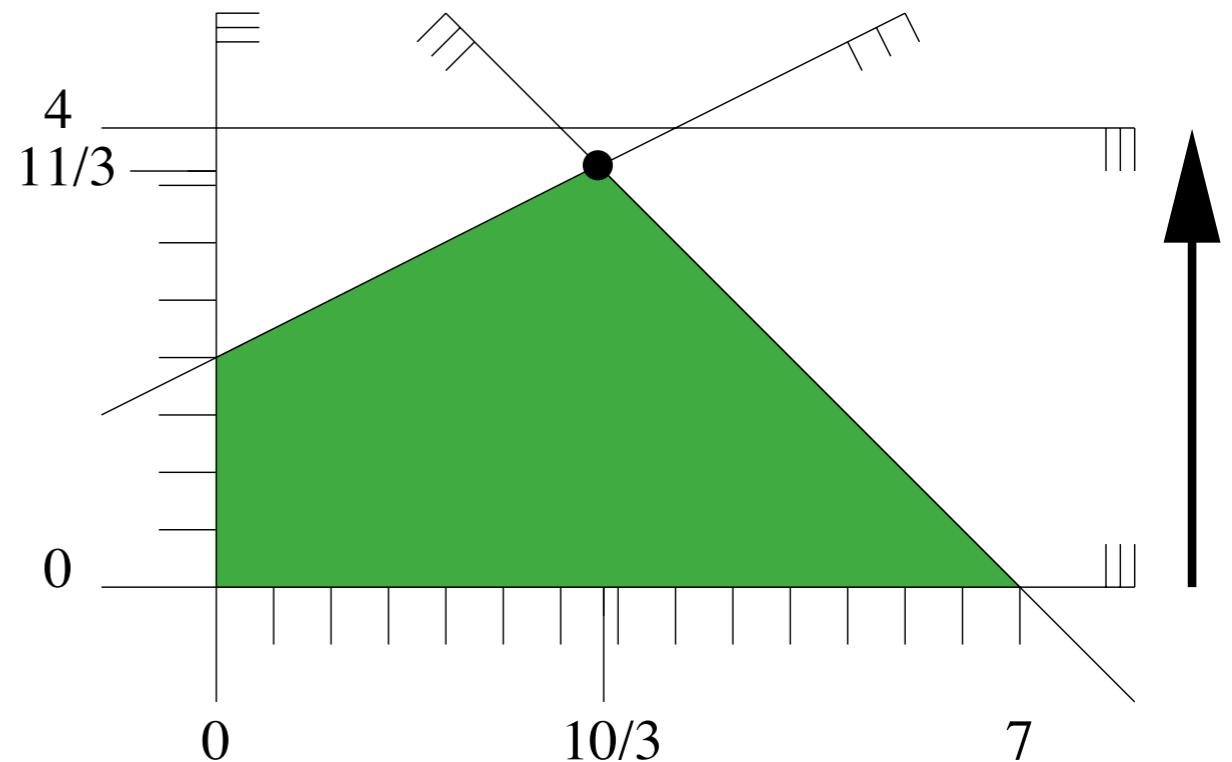
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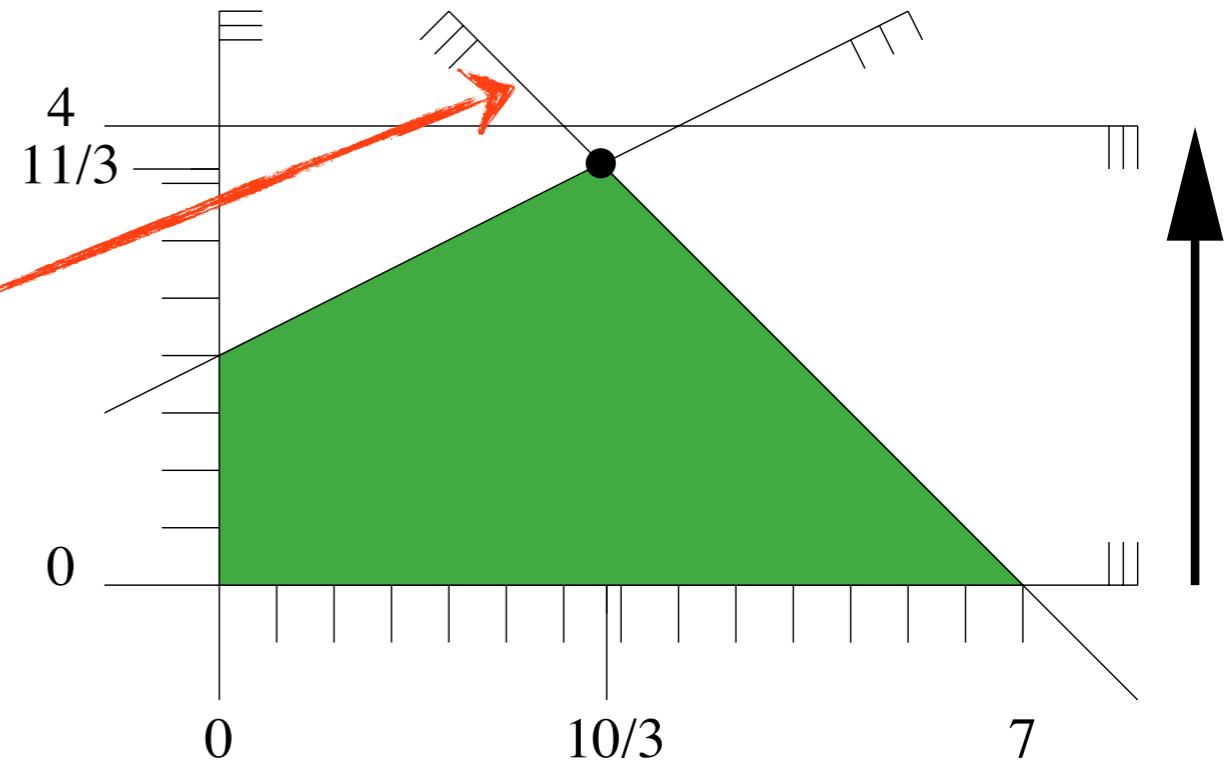


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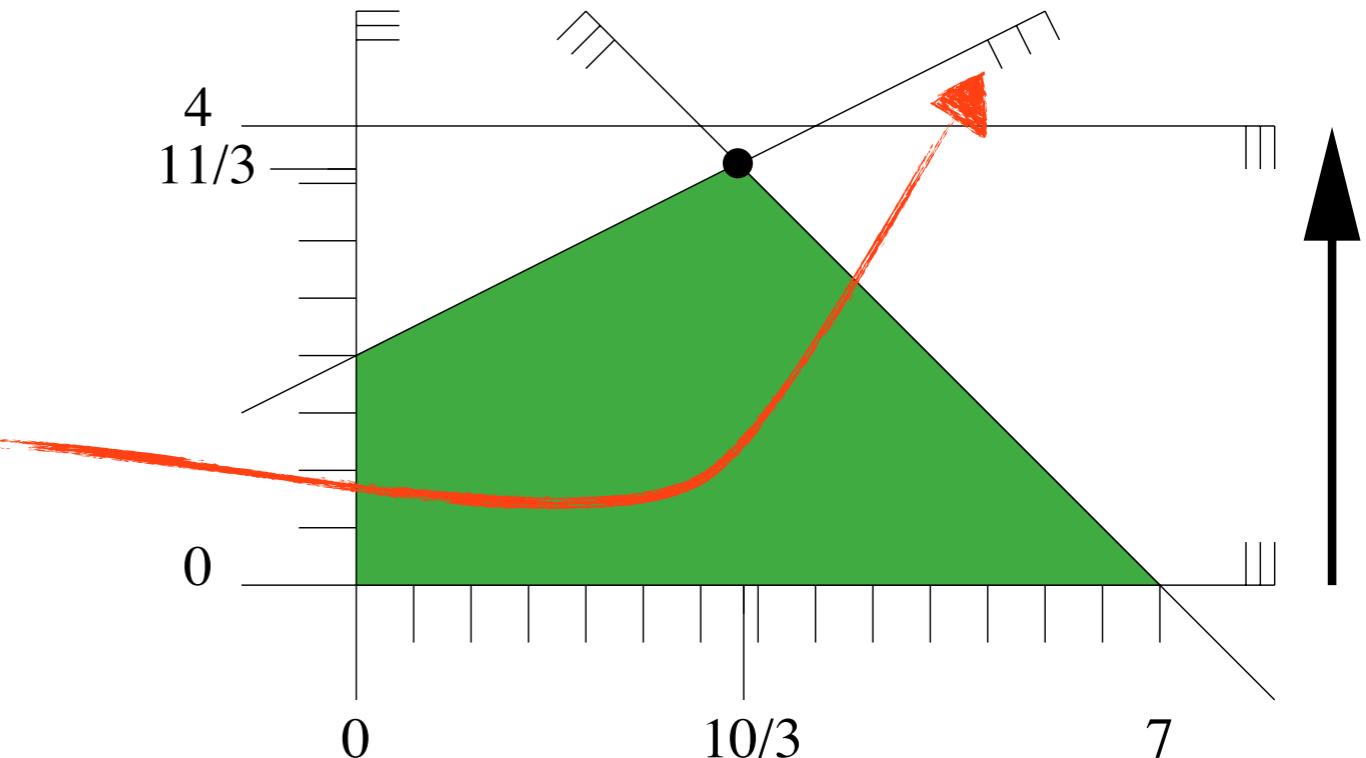
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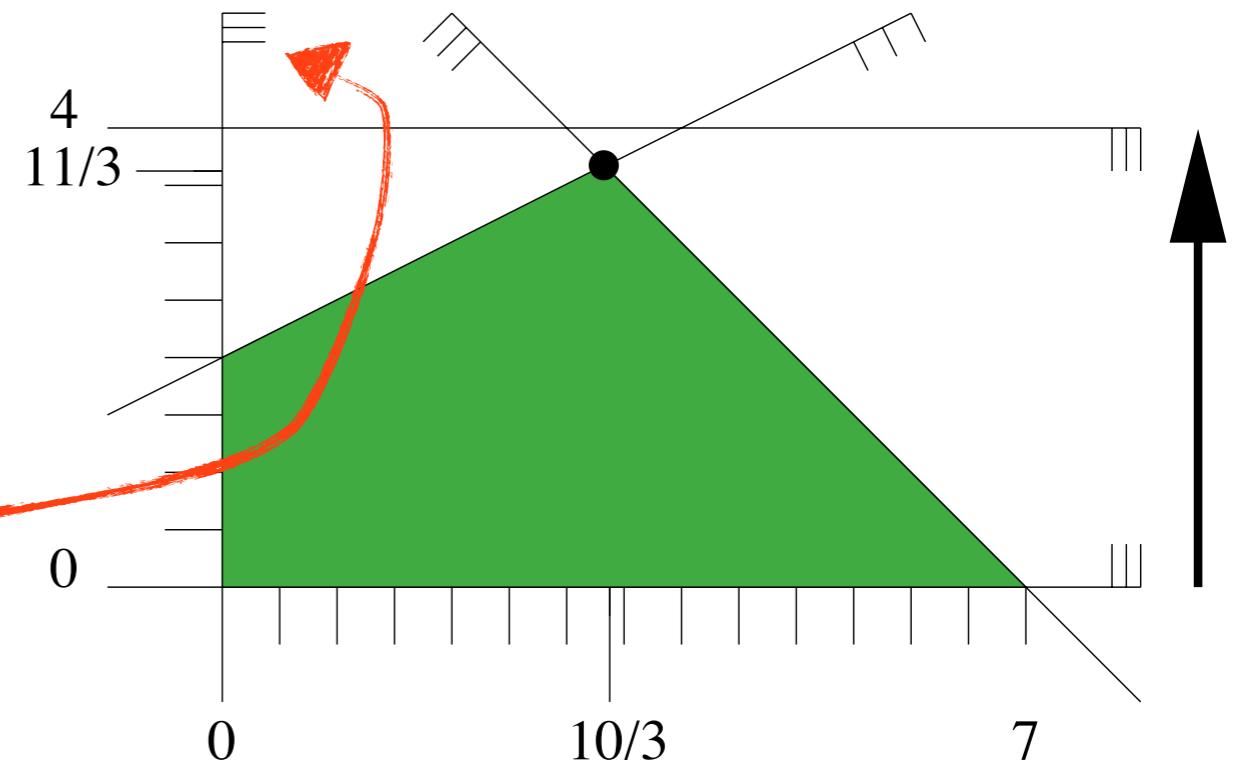
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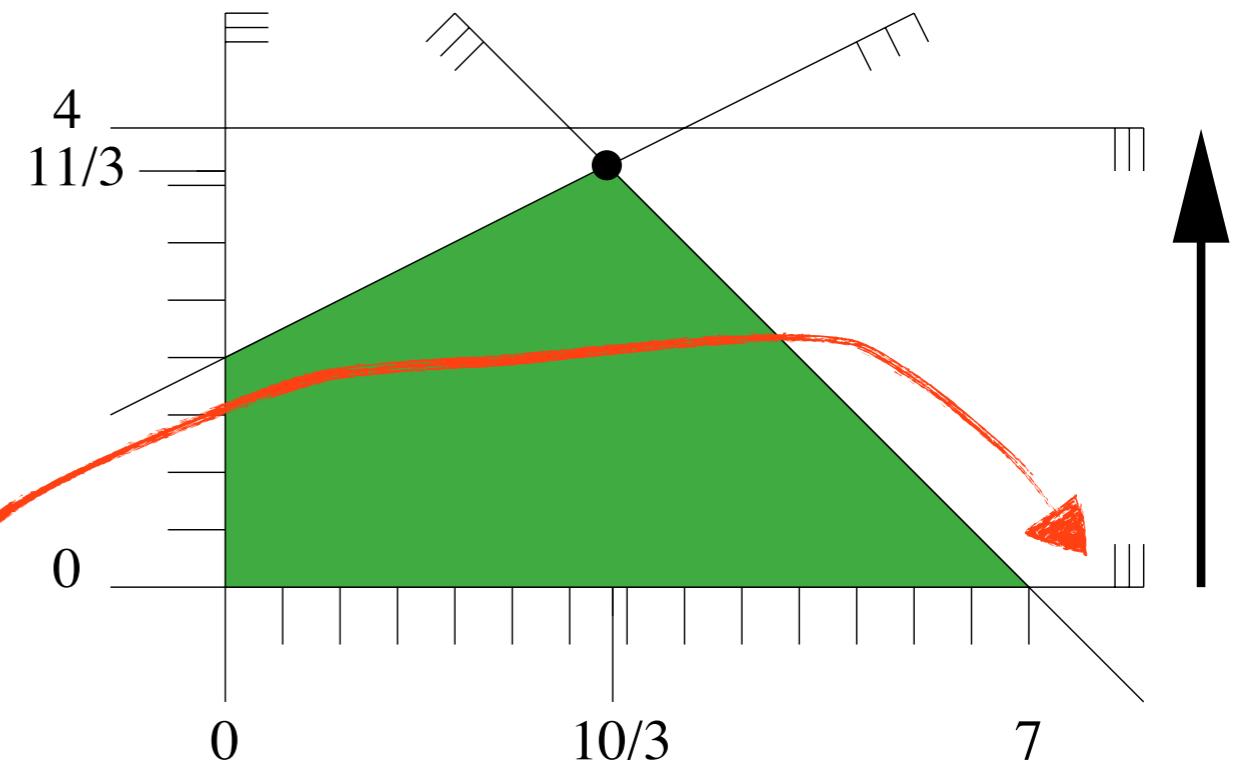
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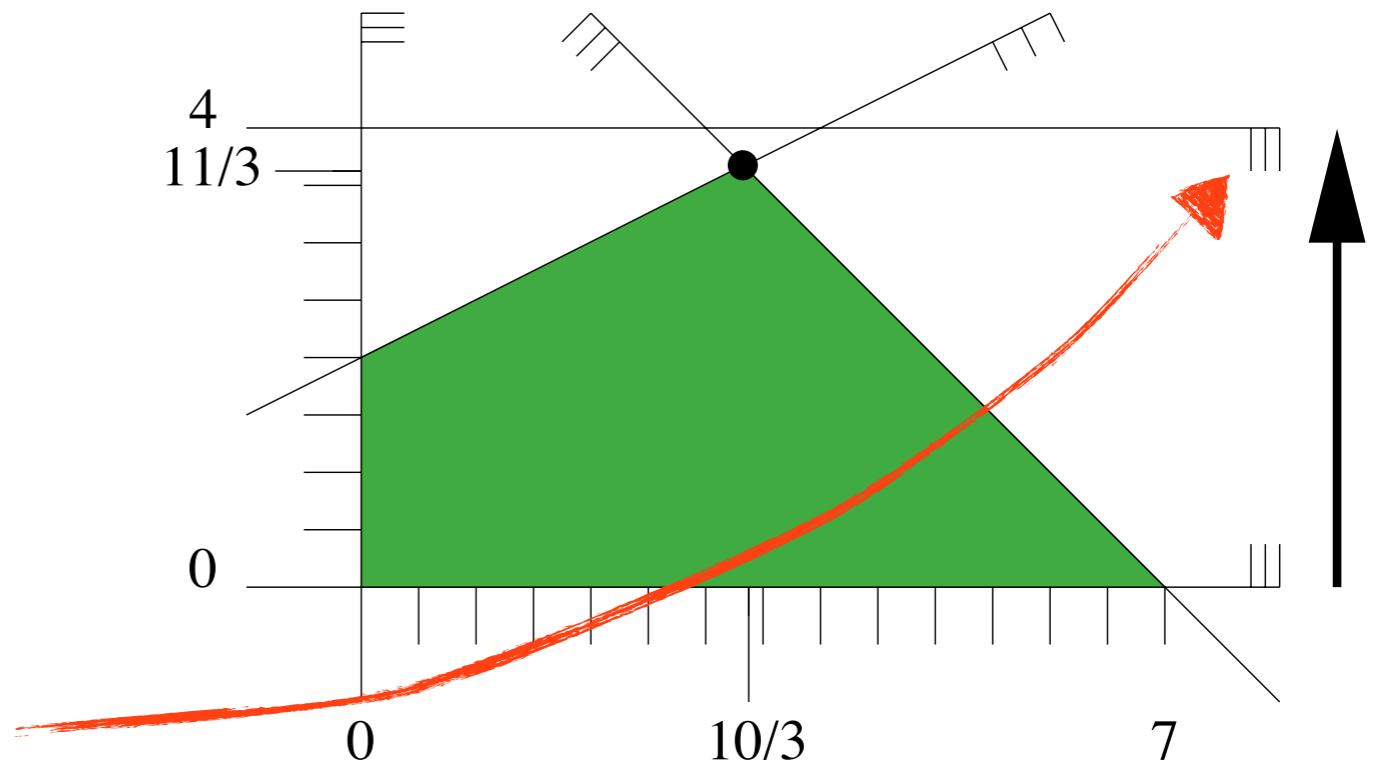


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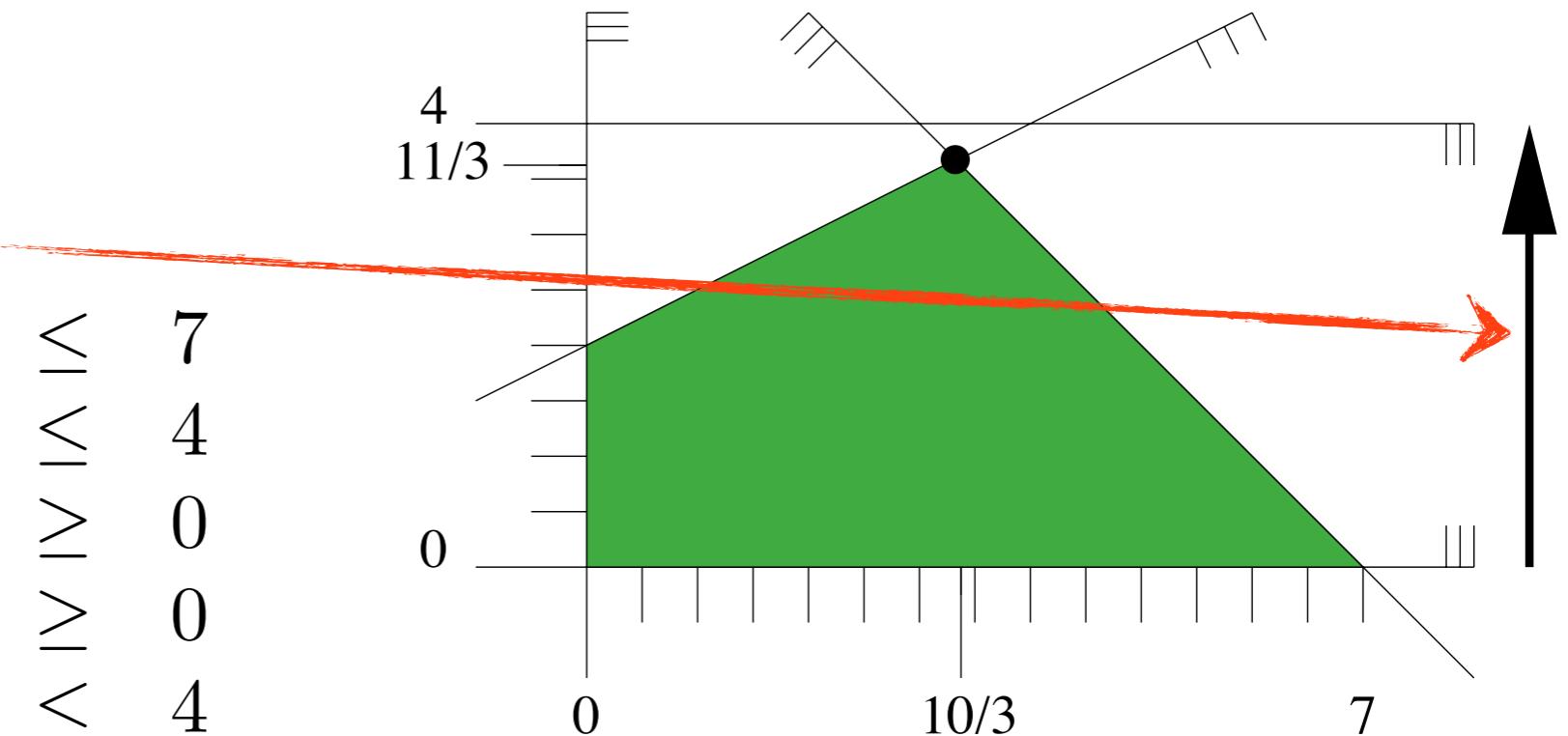


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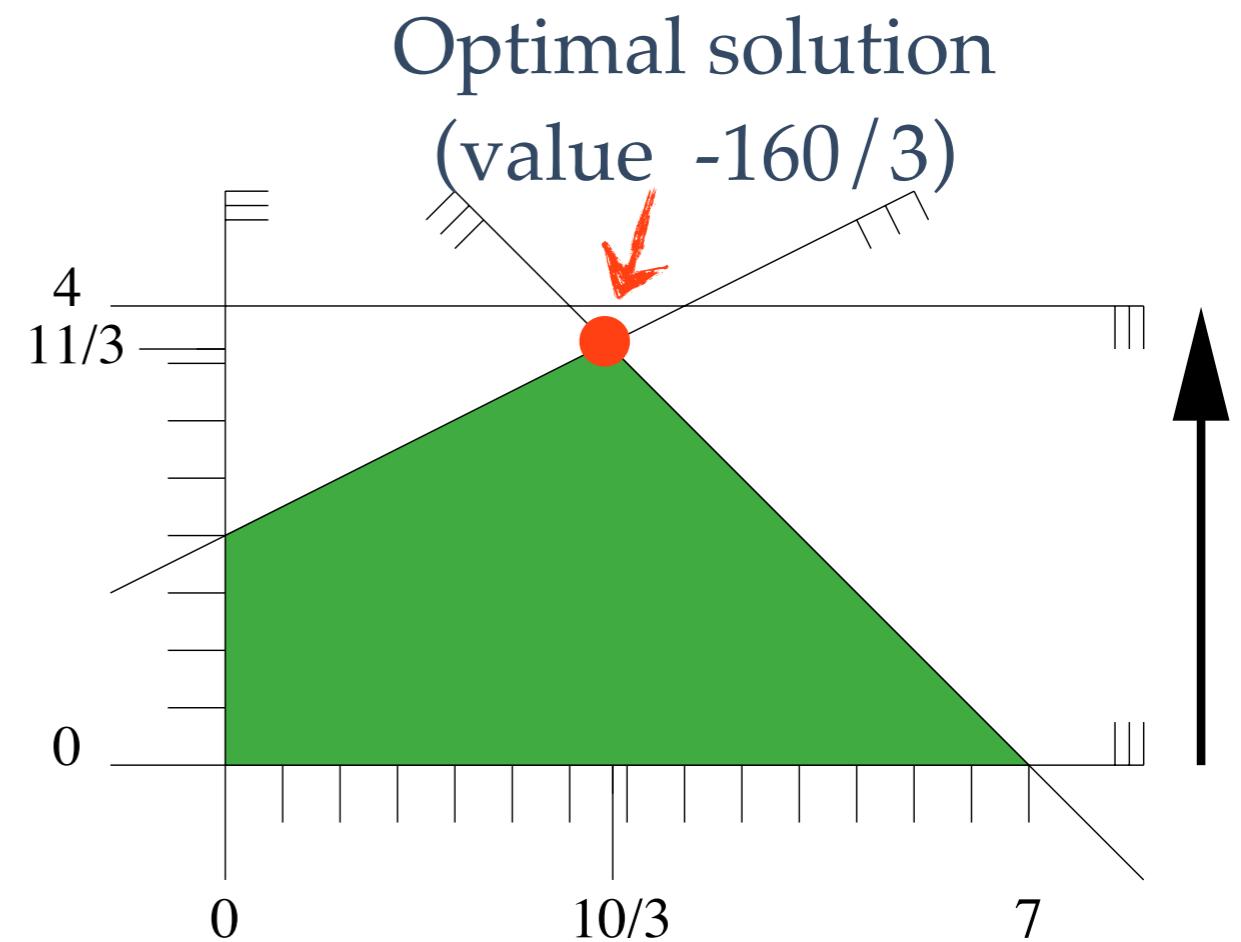


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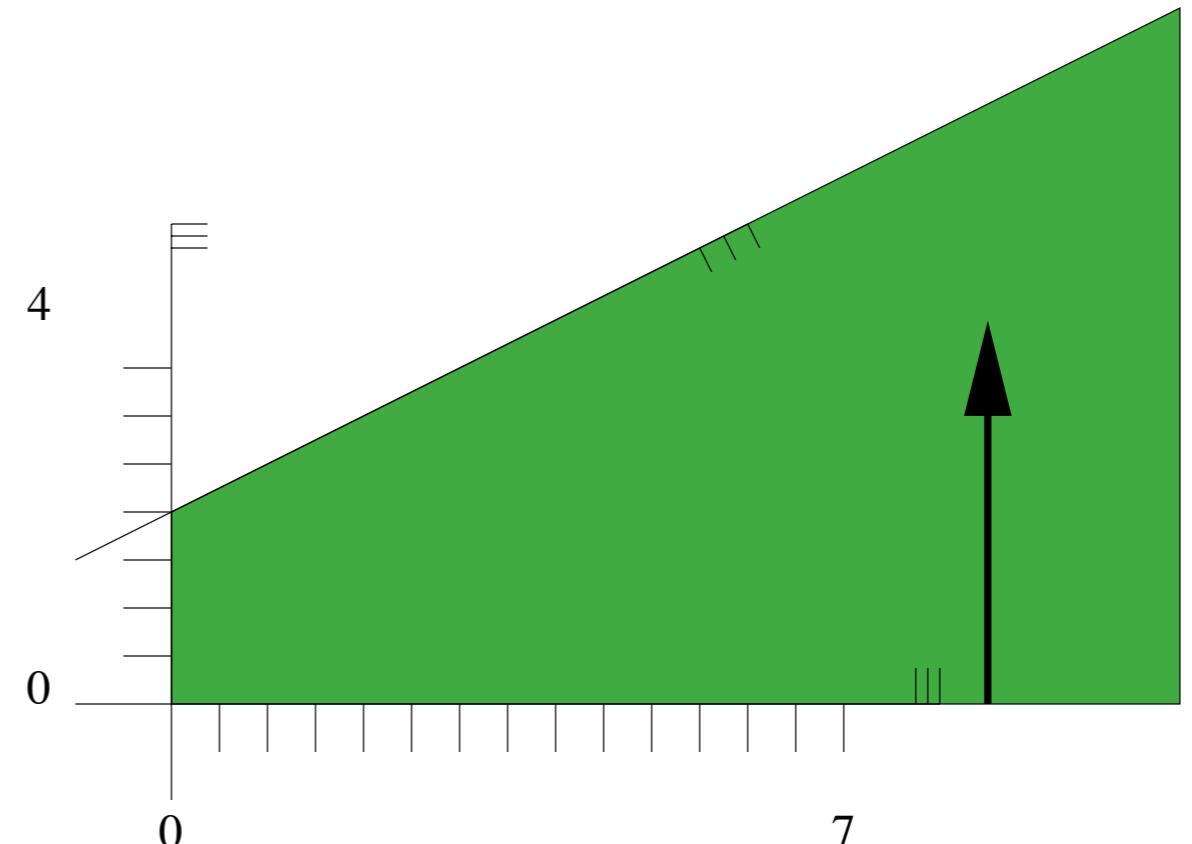
Linear Programming

Mathematical formulation

- ❖ **Problem:** Minimize a linear function in n variables subject to m linear (in)equation constraints!
- ❖ **Unbounded linear programs:**

minimize
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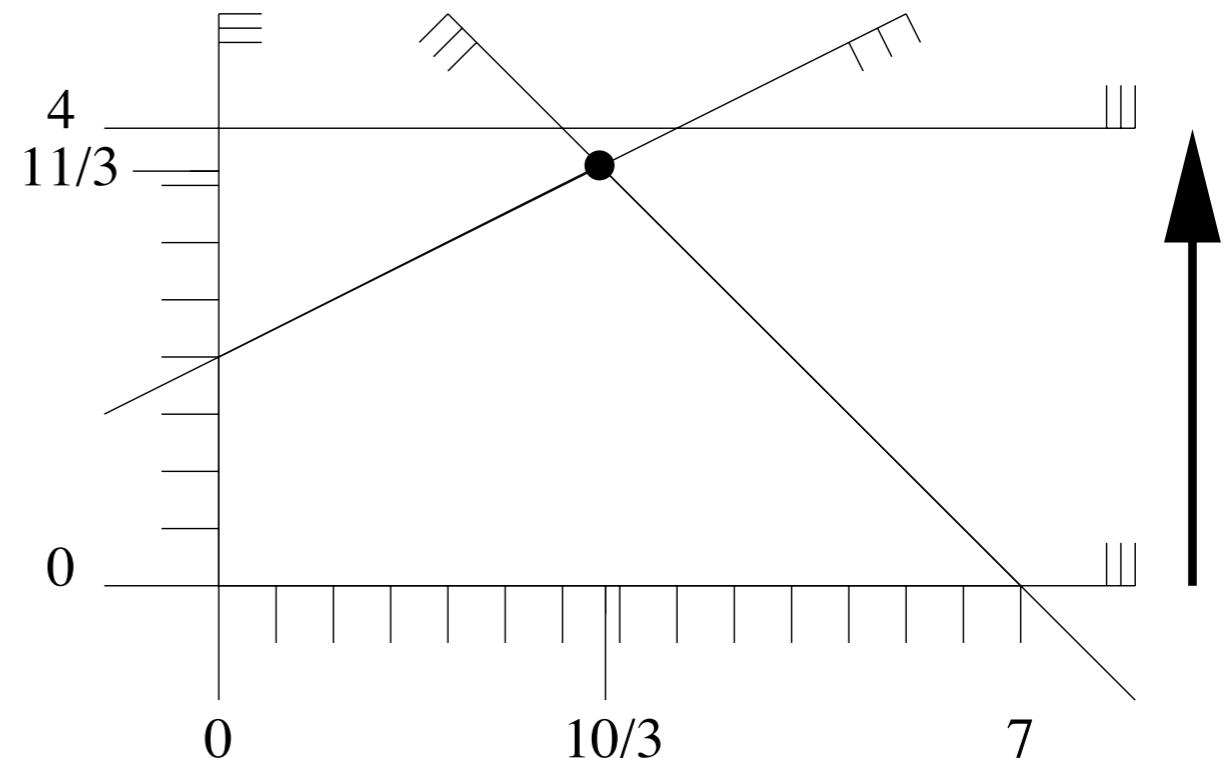


Linear Programming

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$$\begin{array}{ccccc} & 1 & 1 & & 7 \\ & -1 & 2 & x & 4 \\ & -1 & 0 & & 0 \\ & 0 & -1 & y & 0 \\ & 0 & 1 & & 4 \end{array} \leq \begin{array}{c} 7 \\ 4 \\ 0 \\ 0 \\ 4 \end{array}$$

A

b

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$$\begin{array}{c} \mathbf{c}^T \\ \hline \begin{matrix} 0 \\ -32 \end{matrix} \end{array} \begin{matrix} x & y \end{matrix} + \begin{matrix} \mathbf{c}_0 \\ 64 \end{matrix} \quad \begin{array}{c} \mathbf{A} \\ \hline \begin{matrix} 1 & 1 \\ -1 & 2 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{matrix} \end{array} \begin{matrix} x \\ y \end{matrix} \leq \begin{matrix} \mathbf{b} \\ \hline \begin{matrix} 7 \\ 4 \\ 0 \\ 0 \\ 4 \end{matrix} \end{array}$$

Linear Programming

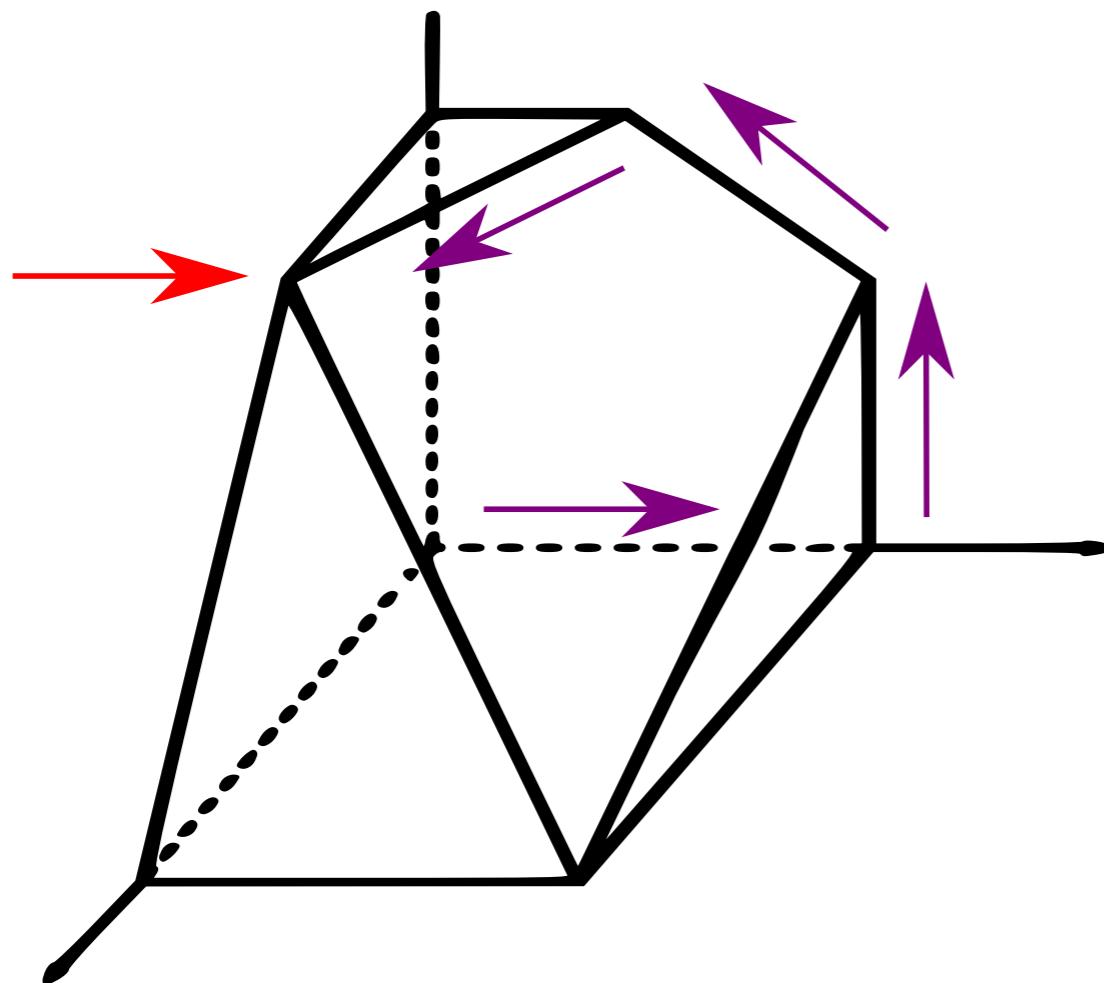
Mathematical formulation

- * **General form of LP:**

$$\begin{array}{ll}\text{minimize} & c^T x + c_0 \\ \text{subject to} & Ax \geq b \\ & l \leq x \leq u\end{array}$$

$$(x, c, l, u \in \mathbb{R}^n, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m, \quad c_0 \in \mathbb{R})$$

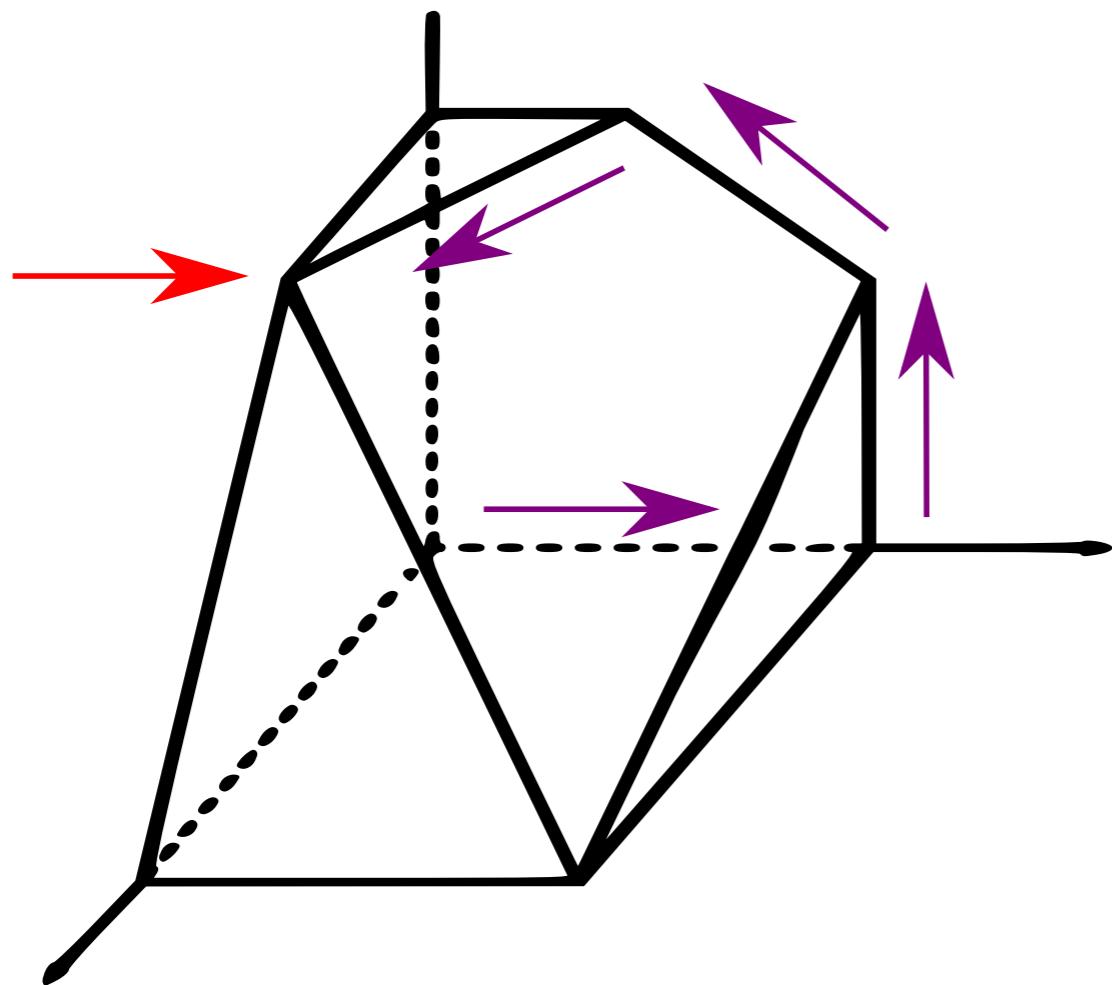
Linear Programming



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 $P = \{x | Ax \leq b\}$

Linear Programming



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- The constraints define a convex polyhedron
$$P = \{x | Ax \leq b\}$$
- Simplex-type algorithms walk along the edges of P to vertices with better objective value.
- To decide on the next edge we use a *pivot rule*.

Linear Programming

Complexity of Simplex-type algorithms
with n variables and m constraints.

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- $O(\max\{m, n\})$ if $\min\{m,n\}$ is $O(1)$
(hidden constant exponential in $\min\{m,n\}$)

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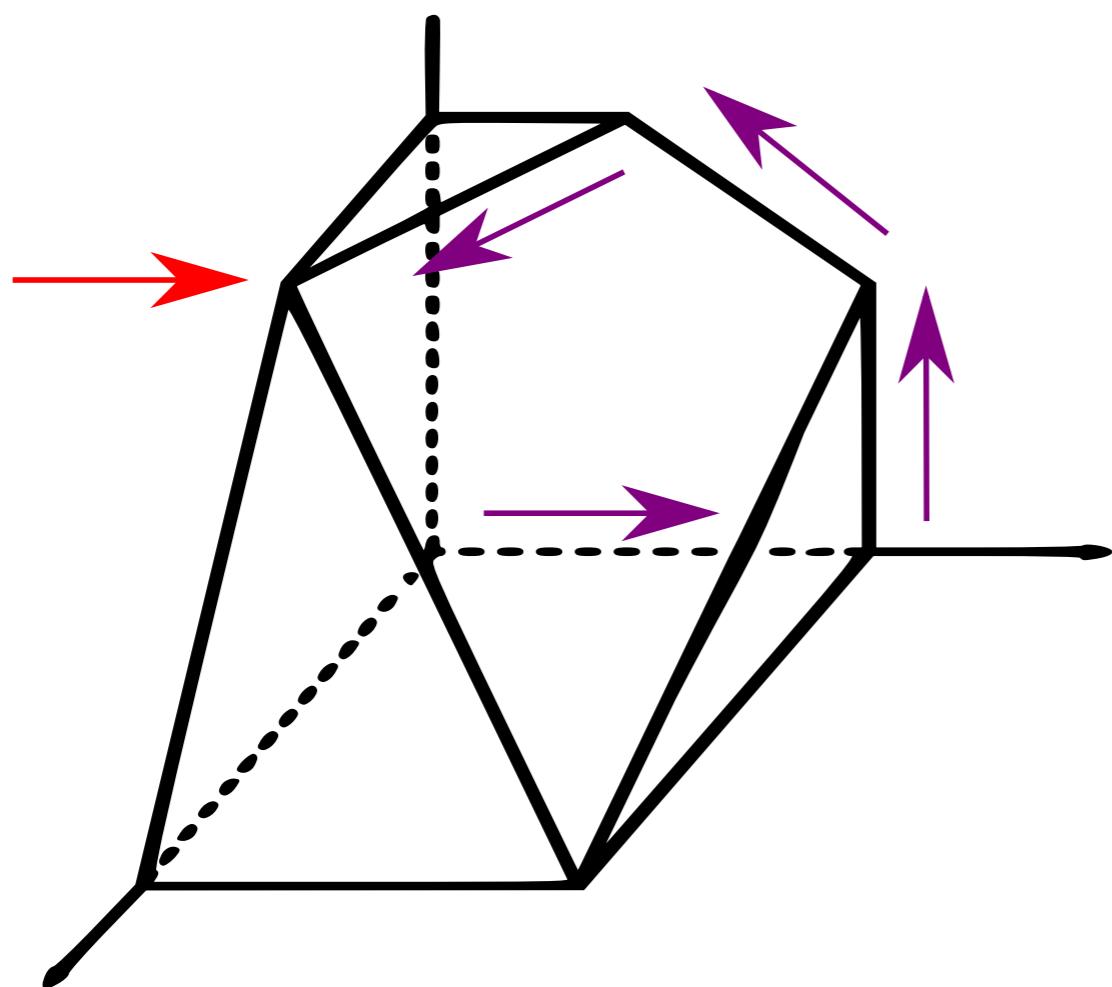
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- $O(\max\{m, n\})$ if $\min\{m,n\}$ is $O(1)$
(hidden constant exponential in $\min\{m,n\}$)
- CGAL: Gives exact solutions!
Works well as long as $\min\{m,n\} \leq 50$
(or up to 100)

$$\begin{array}{ll}\text{minimize} & c^T x + c_0 \\ \text{subject to} & Ax \geq b \\ & l \leq x \leq u\end{array}$$

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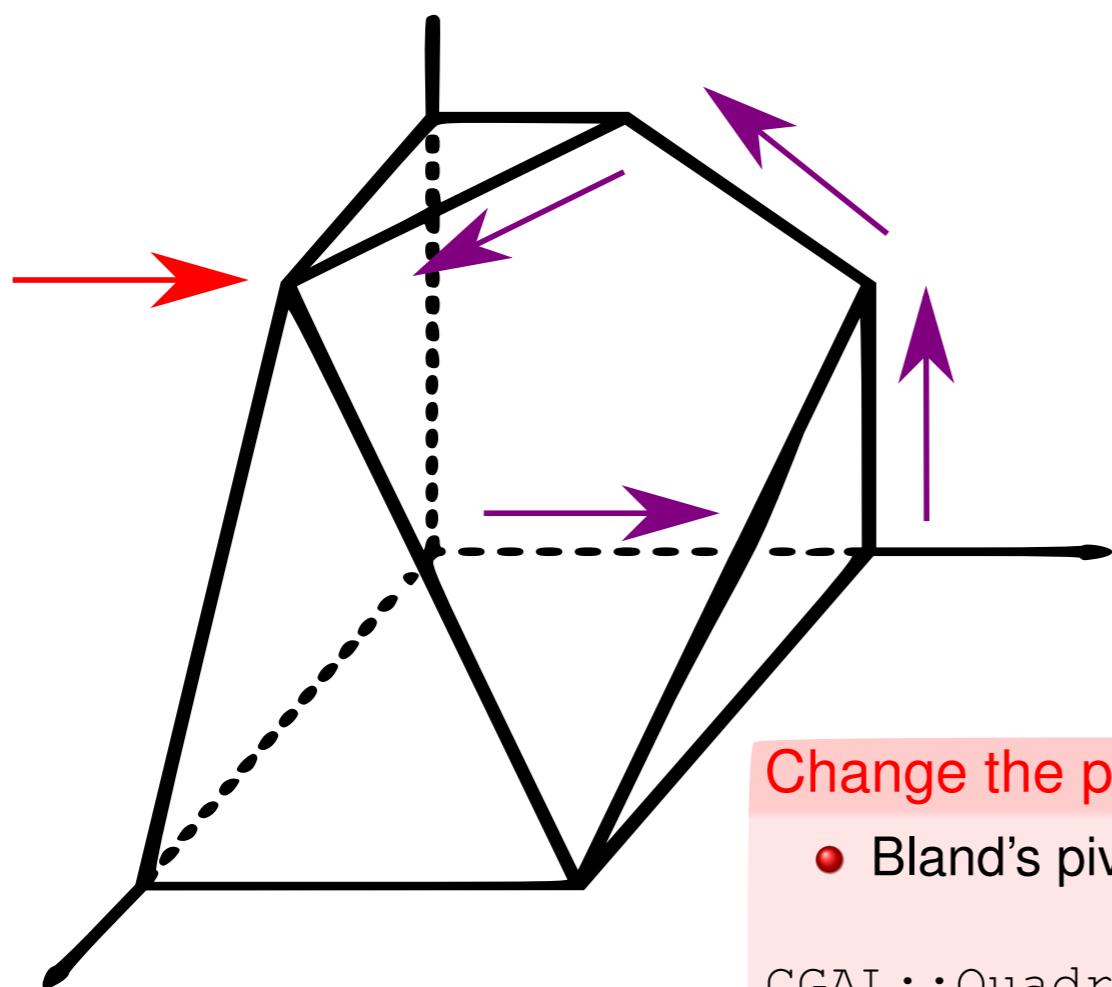
Linear Programming



The default pivot rule strategy might fail:

- it is deterministic, so it picks the same improving edge every time.
- If the algorithm reaches a vertex for the second time, then cycles forever.

Linear Programming



Change the pivot rule

- Bland's pivot rule avoids cycling (but it is slower...)

```
CGAL::Quadratic_program_options options;  
options.set_pricing_strategy(CGAL::QP_BLAND);  
Solution s = CGAL::SOLVER(program, ET(), options);
```

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Linear Programming

- * General form of LP in CGAL:

$$\begin{array}{ll} \text{minimize} & c^T x + c_0 \\ \text{subject to} & Ax \leq b \\ & l \leq x \leq u \end{array}$$

$\leq, =, \text{ or } \geq$ (individually
for each constraint)

variables → $(x, c, l, u \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c_0 \in \mathbb{R})$

objective function ↑ ↑ ↑ ↑

lower and upper bounds ↑ ↑

constraint matrix ↗

right-hand side ↗ ↗ ↗ ↗

shift ↗ ↗ ↗ ↗

Linear Programming ... in CGAL

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- ❖ **Preamble:** Choice of input type and exact internal number type

Gnu
Multi-
precision
Library
(GMP)

```
#include <iostream>
#include <cassert>
#include <CGAL/basic.h>
#include <CGAL/QP_models.h>
#include <CGAL/QP_functions.h>
#include <CGAL/Gmpz.h>

// choose exact integral type
typedef CGAL::Gmpz ET;

// program and solution types
typedef CGAL::Quadratic_program<int> Program;
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input type

exact internal type

for linear *and* quadratic programs

GMP used internally

Linear Programming ... in CGAL

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Inside, the LP solver uses quotients of
the data type specified here.

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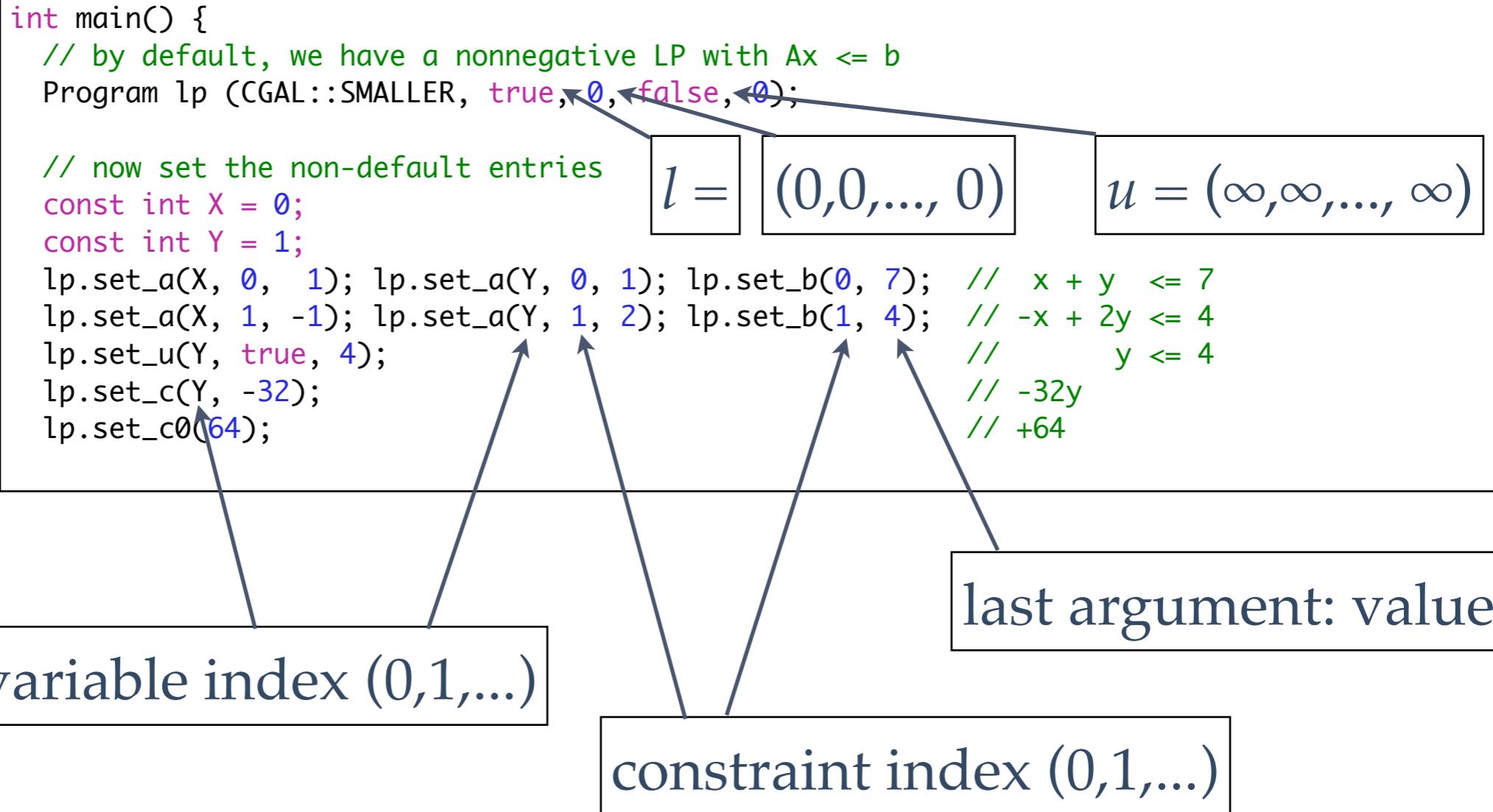
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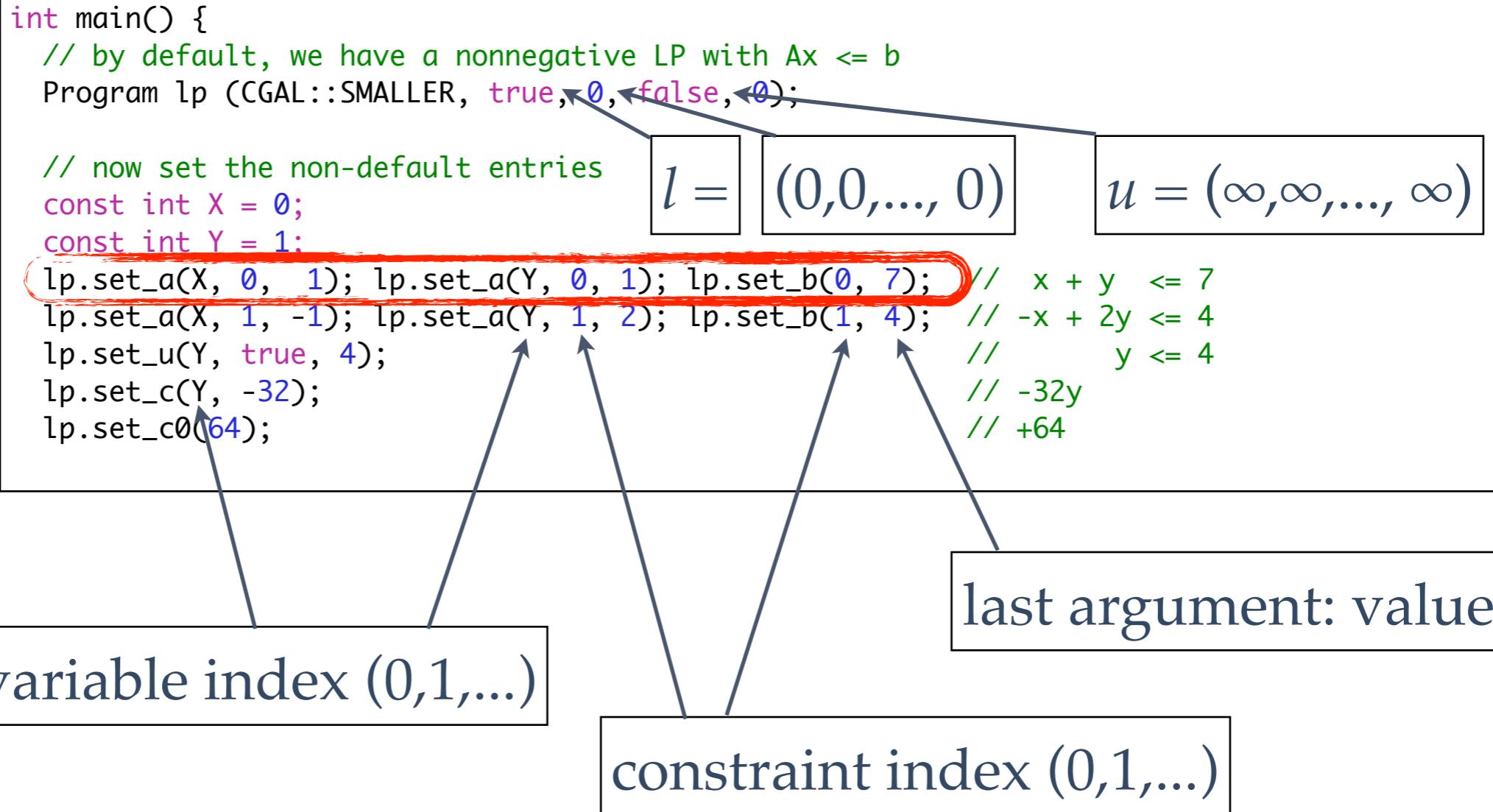
- ❖ **Setup:** Enter the program data



Linear Programming ... in CGAL

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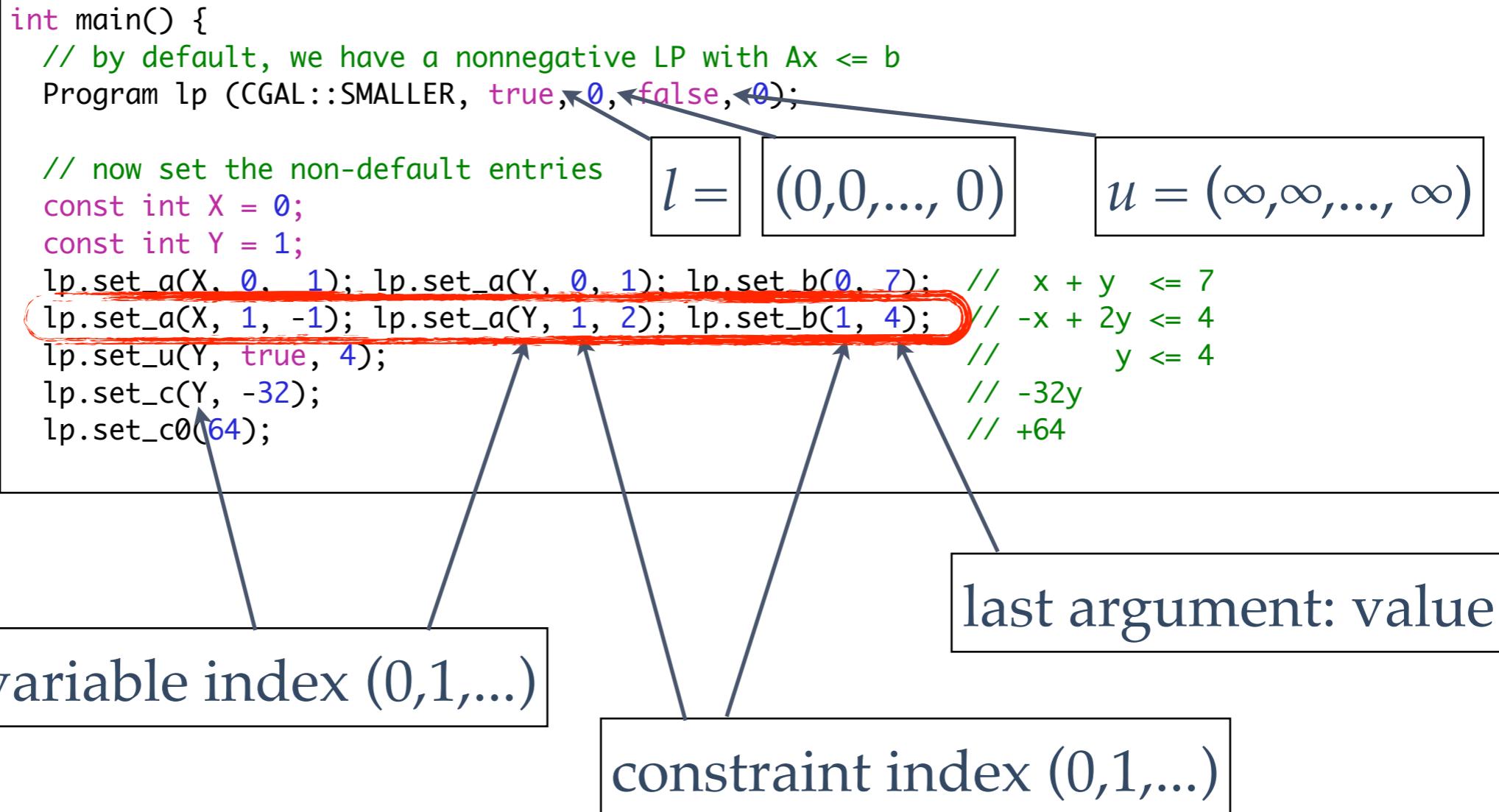
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- Setup: Enter the program data

```
int main() {
    // by default, we have a nonnegative LP with Ax <= b
    Program lp (CGAL::SMALLER, true, 0, false, 0);

    // now set the non-default entries
    const int X = 0;
    const int Y = 1;
    lp.set_a(X, 0, 1); lp.set_a(Y, 0, 1); lp.set_b(0, 7);    // x + y <= 7
    lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4);    // -x + 2y <= 4
    lp.set_u(Y, true, 4);                                     // y <= 4
    lp.set_c(Y, -32);                                         // -32y
    lp.set_c0(64);                                           // +64
```

$l =$

$(0, 0, \dots, 0)$

$u = (\infty, \infty, \dots, \infty)$

variable index (0,1,...)

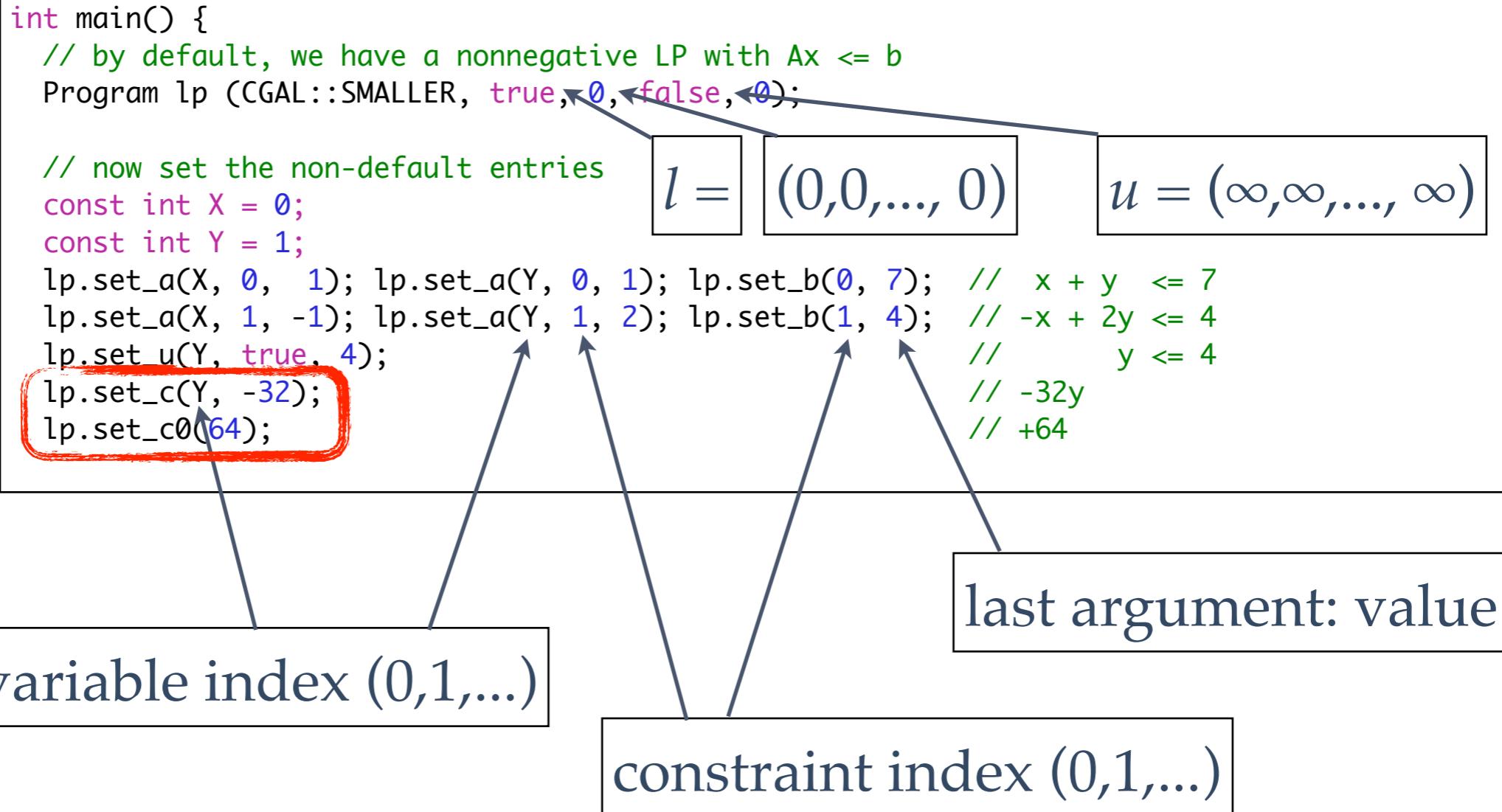
constraint index (0,1,...)

last argument: value

Linear Programming ... in CGAL

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- Setup: Enter the program data



Linear Programming ... in CGAL

- * **Solve:** Call the linear programming solver and output solution

```
// solve the program, using ET as the exact type
Solution s = CGAL::solve_linear_program(lp, ET());
assert (s.solves_linear_program(lp));

// output solution
std::cout << s;
return 0;
}
```

independent verification

Linear Programming ... in CGAL

- * **Solve:** Call the linear programming solver and output solution

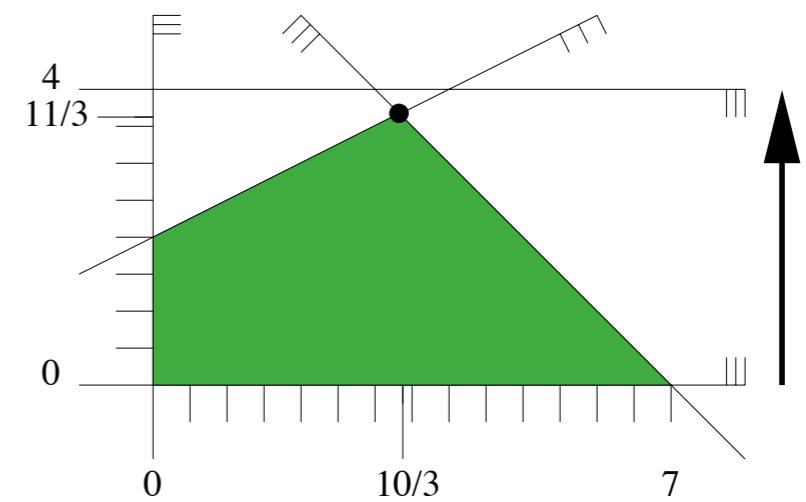
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- * **Output:**

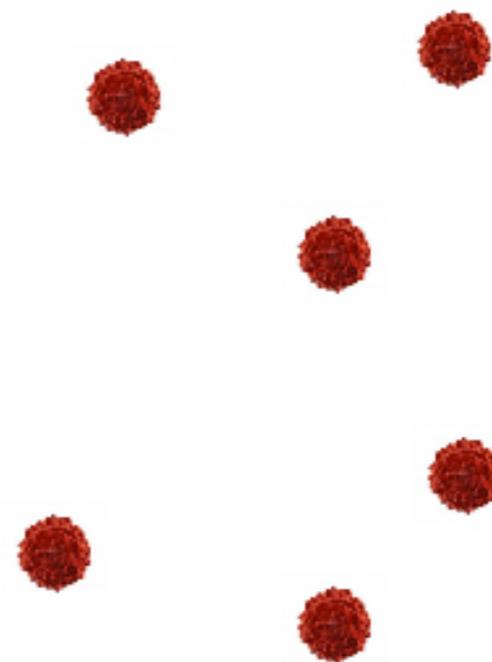
```
status: OPTIMAL
objective value: -160/3
variable values:
 0: 10/3
 1: 11/3
```



Linear Programming

Application I: Cancer Therapy

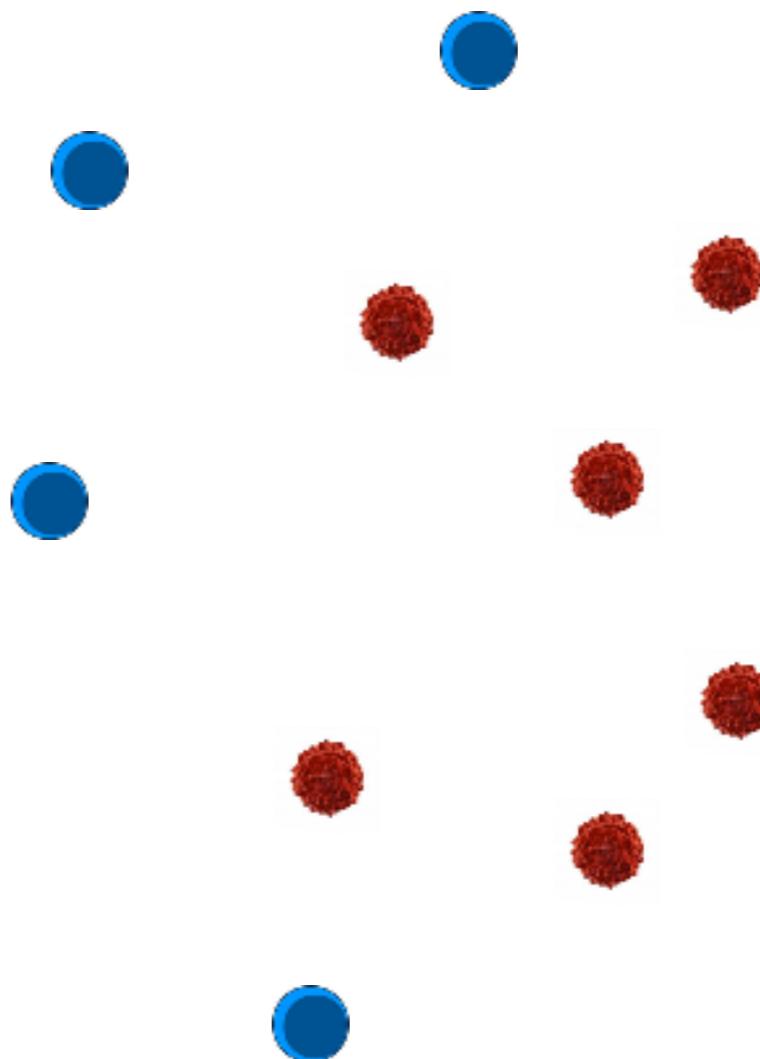
- * **Given:** locations of cancer cells (red)



Linear Programming

Application I: Cancer Therapy

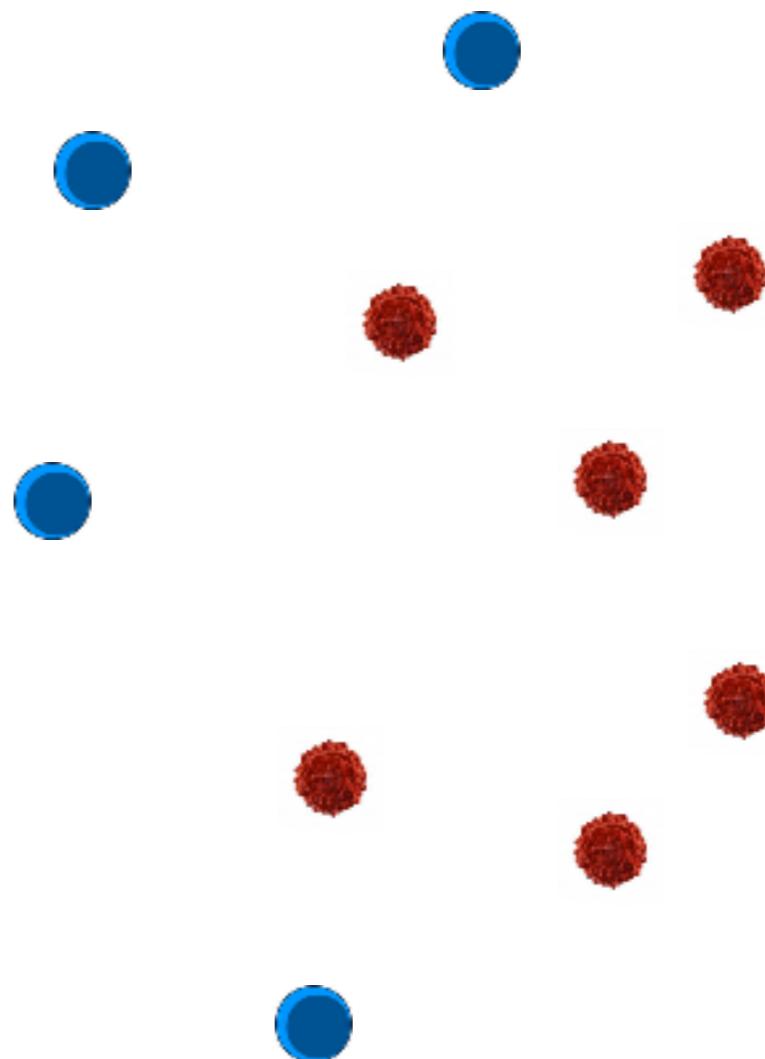
- * **Given:** locations of cancer cells (red) and healthy cells (blue)



Linear Programming

Application I: Cancer Therapy

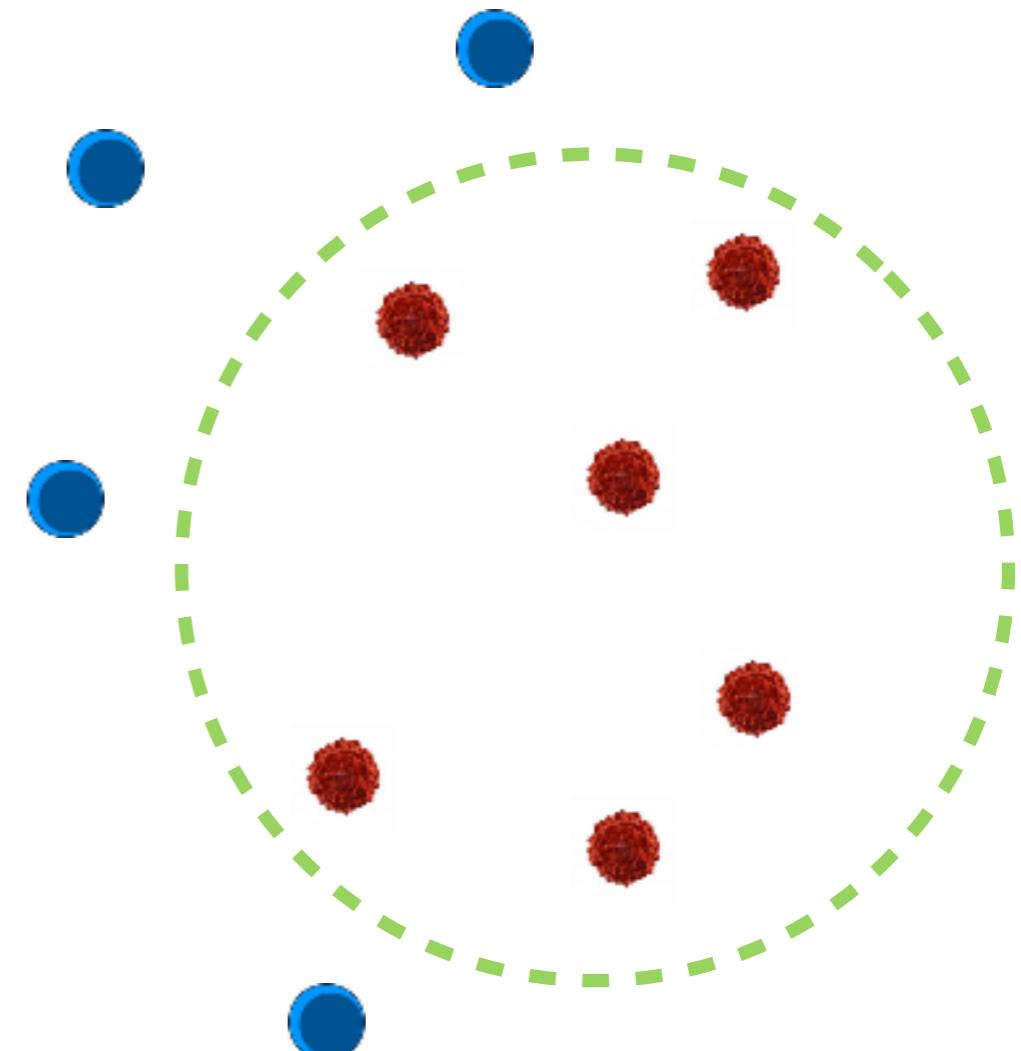
- ❖ **Given:** locations of cancer cells (red) and healthy cells (blue)
- ❖ **Wanted:** center and radius of exposure so that all cancer cells are killed and all healthy cells are unaffected.



Linear Programming

Application I: Cancer Therapy

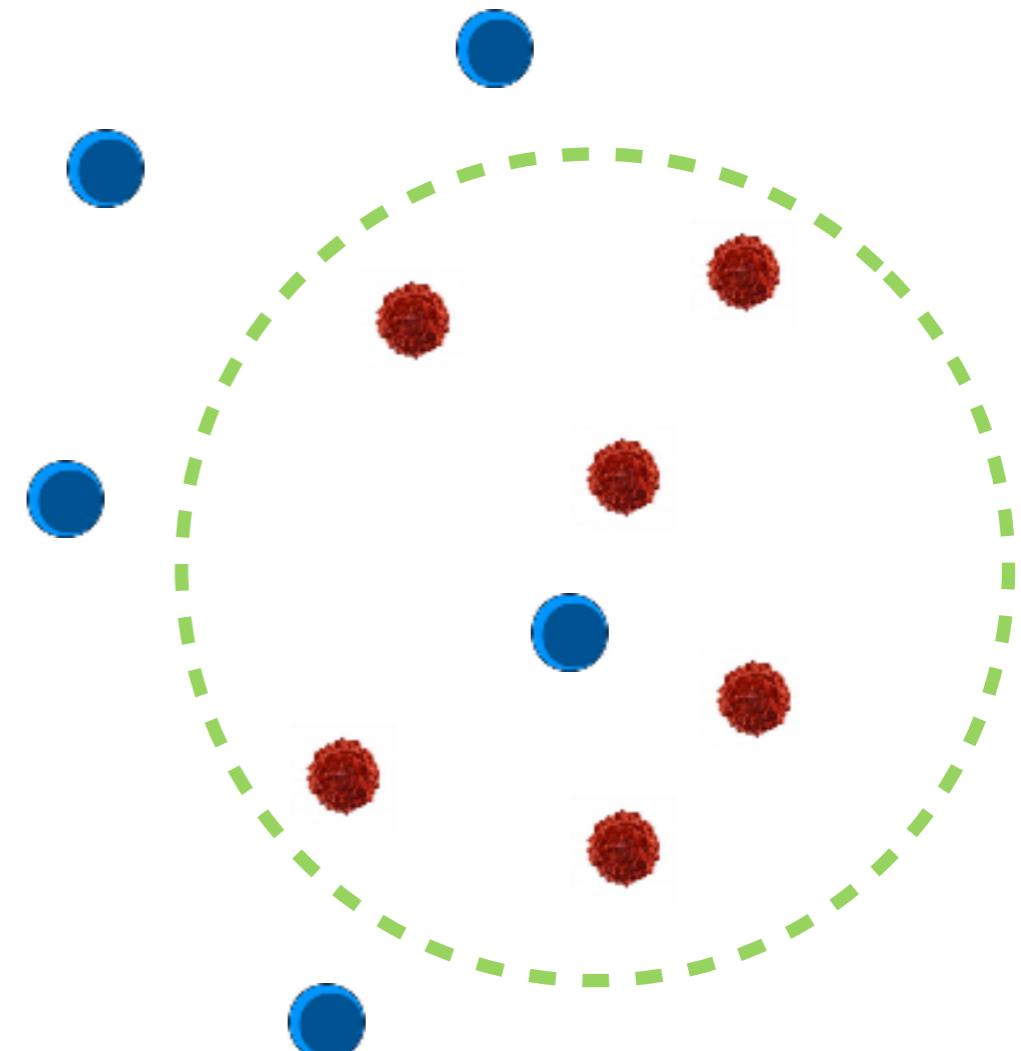
- ❖ **Given:** locations of cancer cells (red) and healthy cells (blue)
- ❖ **Wanted:** center and radius of exposure so that all cancer cells are killed and all healthy cells are unaffected.
- ❖ This may be possible...



Linear Programming

Application I: Cancer Therapy

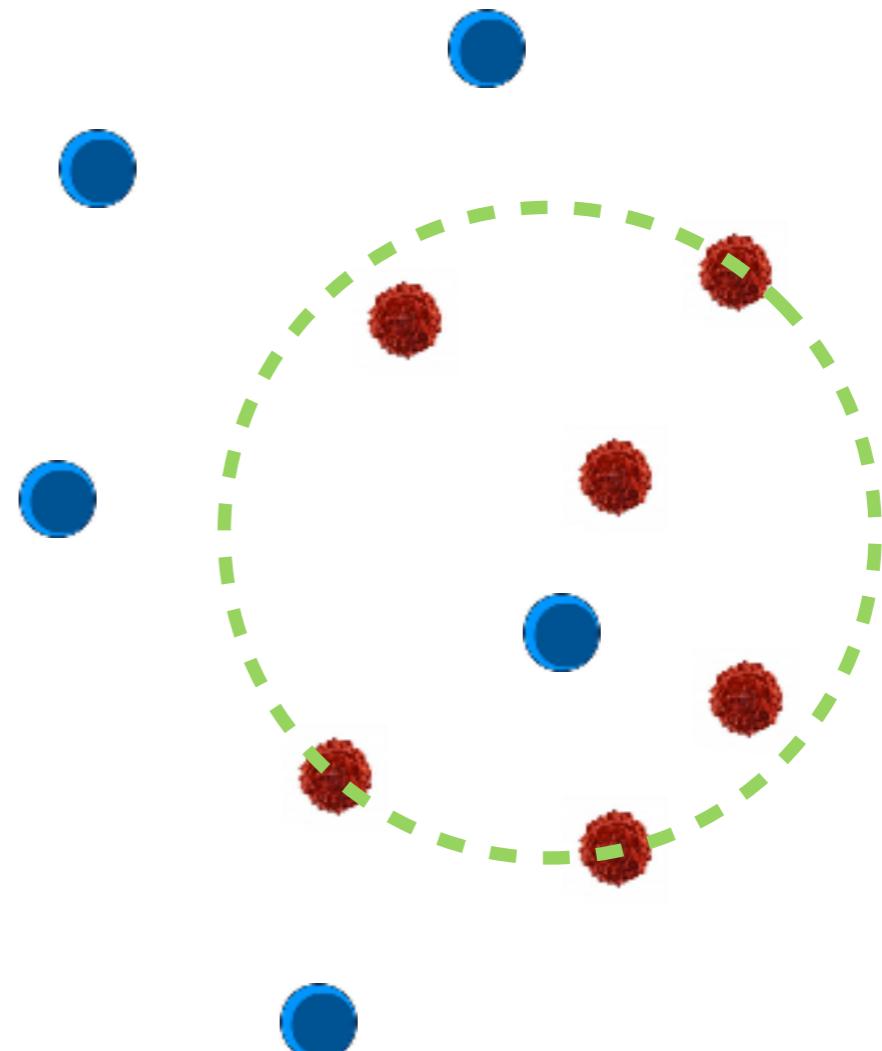
- ❖ **Given:** locations of cancer cells (red) and healthy cells (blue)
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- ❖ This may be possible... or not.



Linear Programming

Application I: Cancer Therapy

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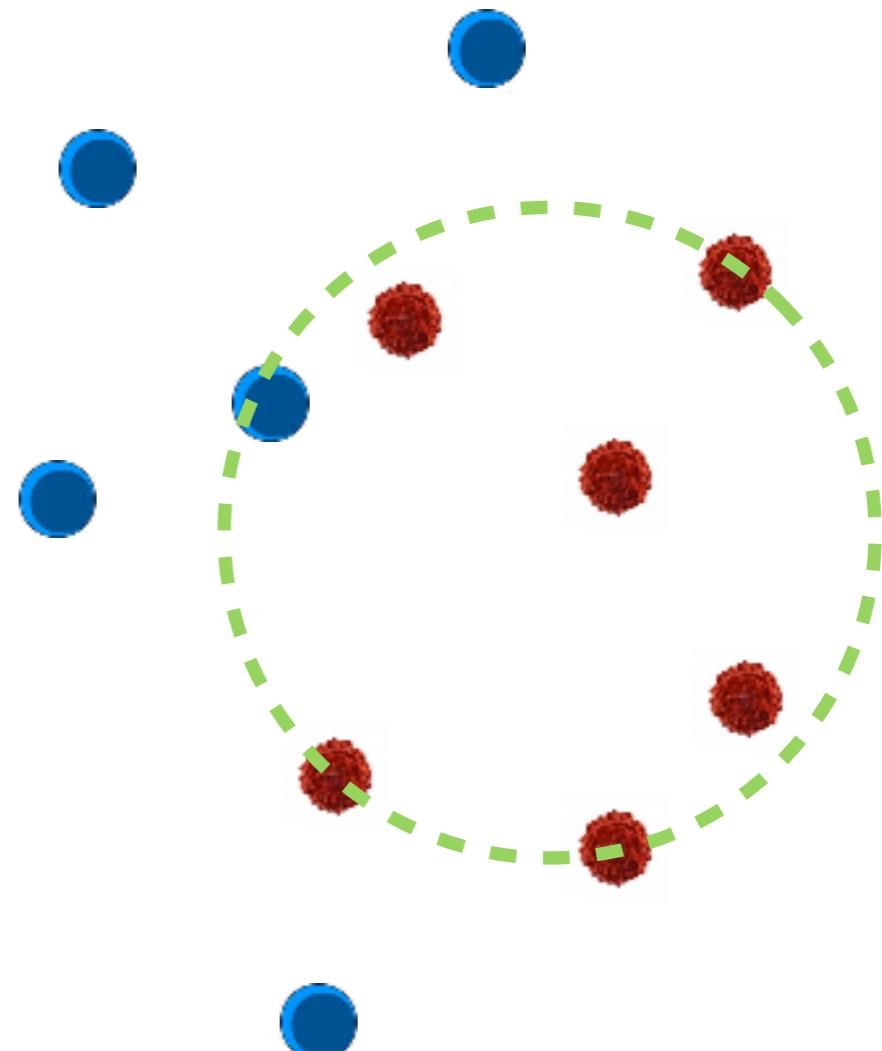
- This may be possible... or not.

- Minimum Enclosing circle is not enough to solve the problem.

Linear Programming

Application I: Cancer Therapy

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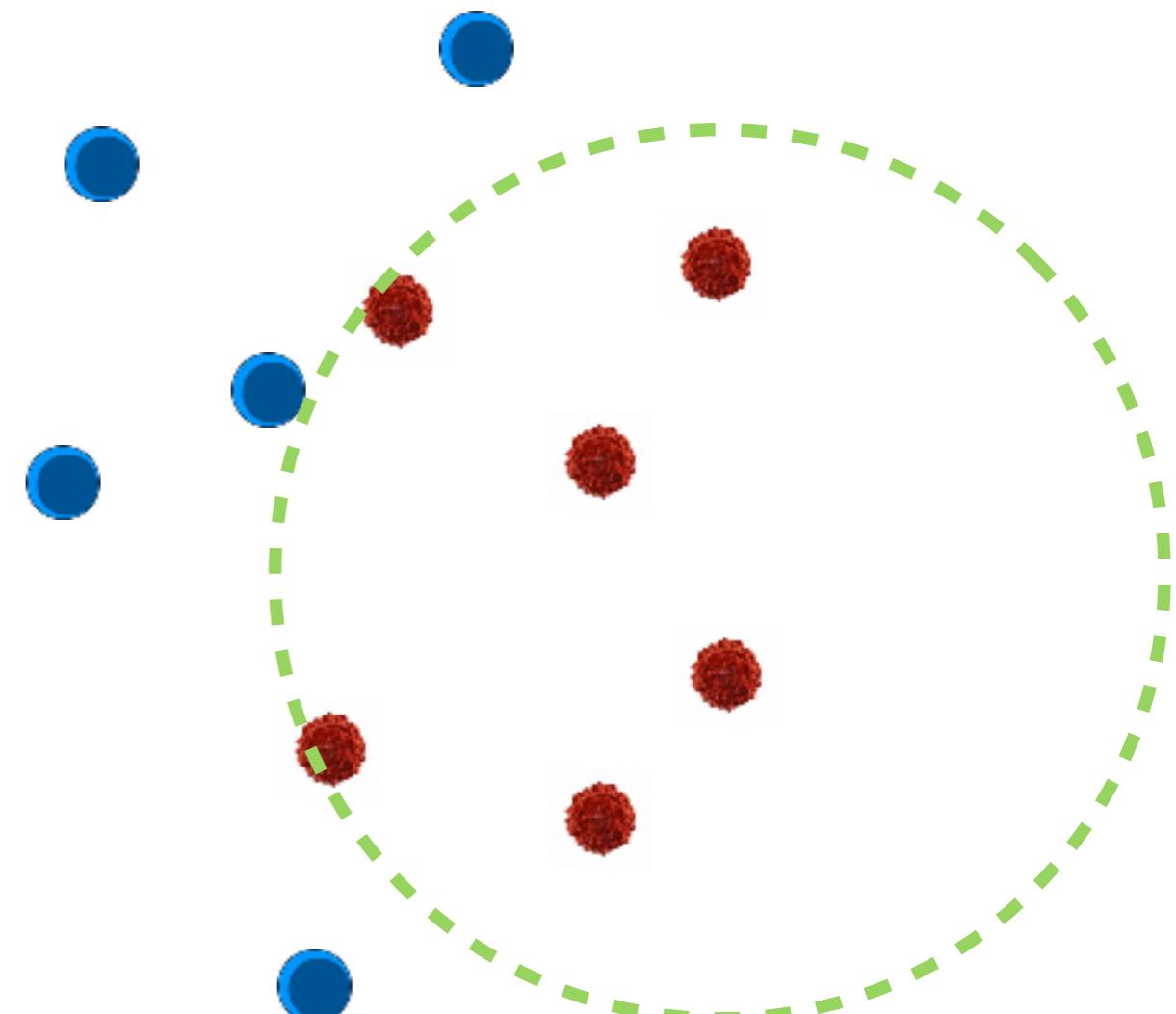
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Linear Programming

Application I: Cancer Therapy

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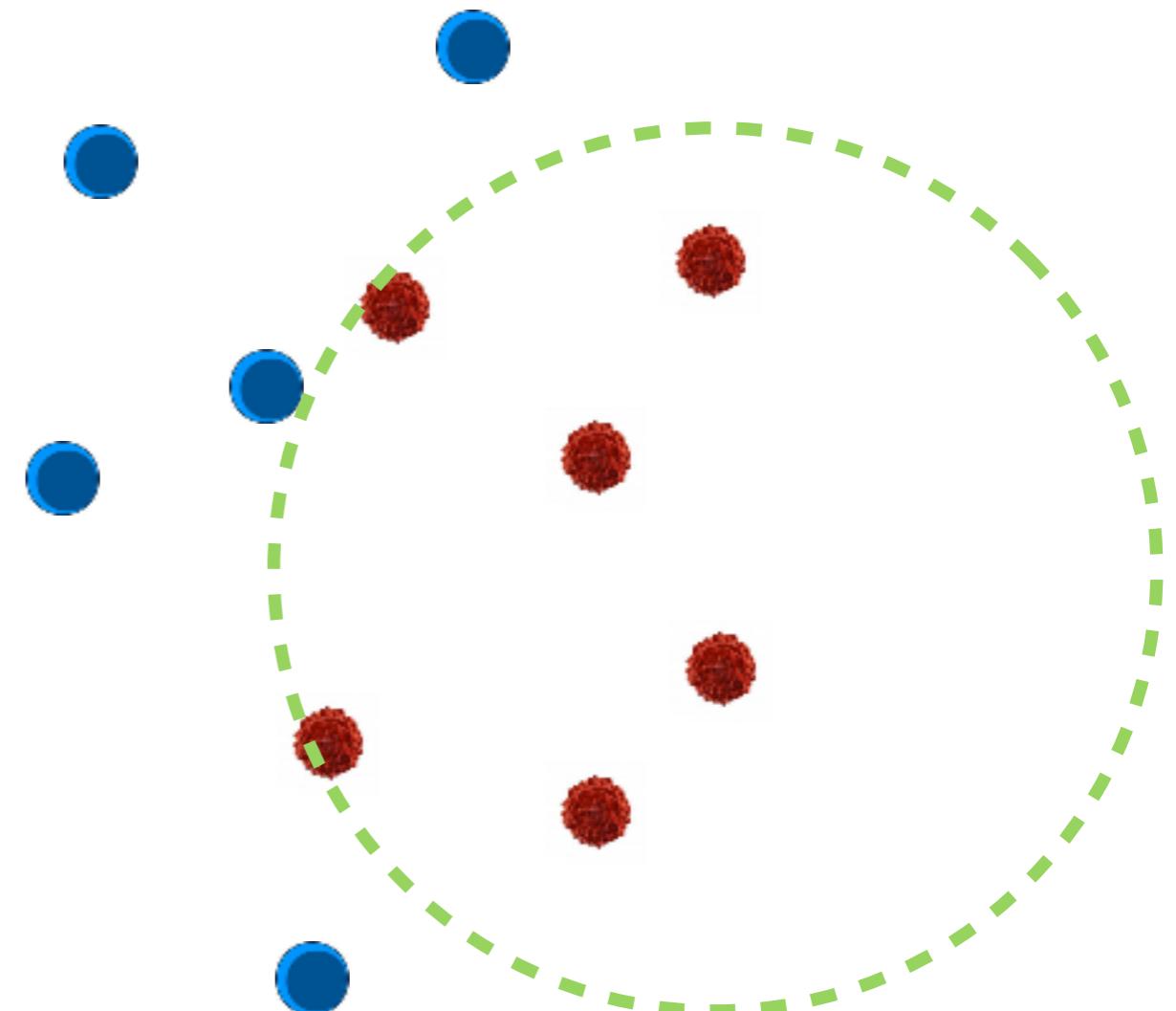


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Linear Programming

Application I: Cancer Therapy

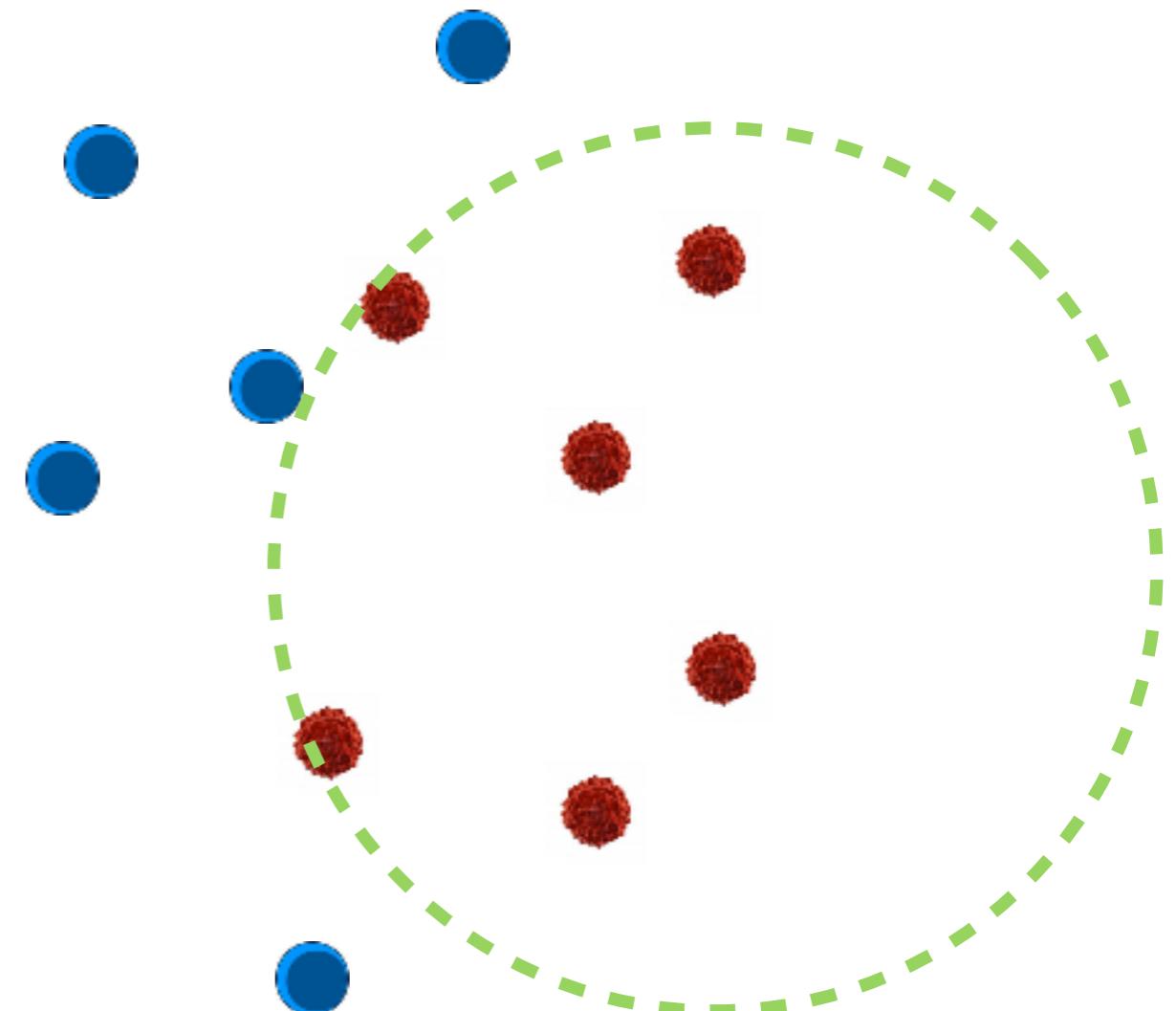
- ❖ **The geometric problem:** Given two finite sets R and B in the plane, does there exist a disk that contains R and is disjoint from B ?



Linear Programming

Application I: Cancer Therapy

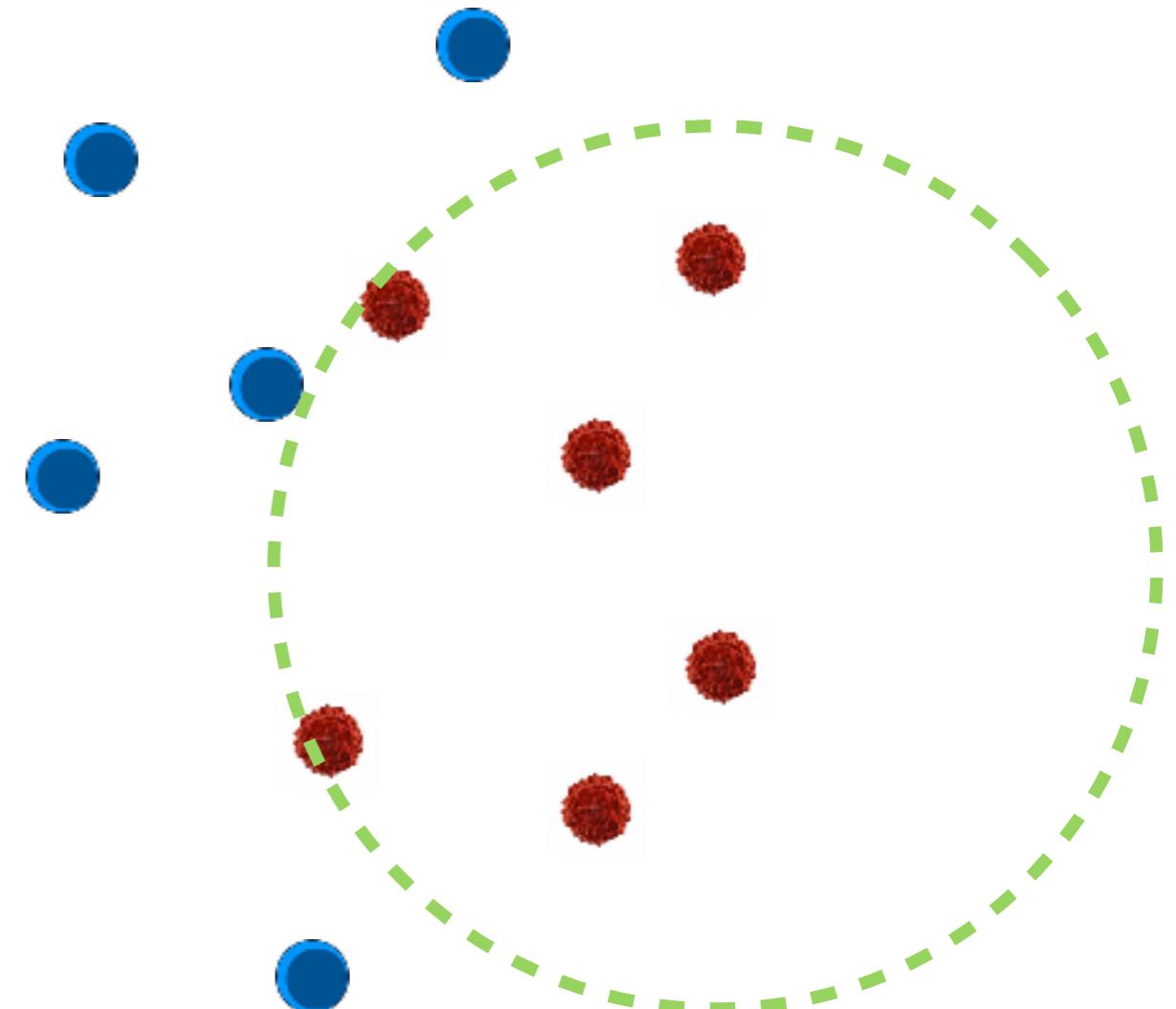
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Linear Programming

Application I: Cancer Therapy

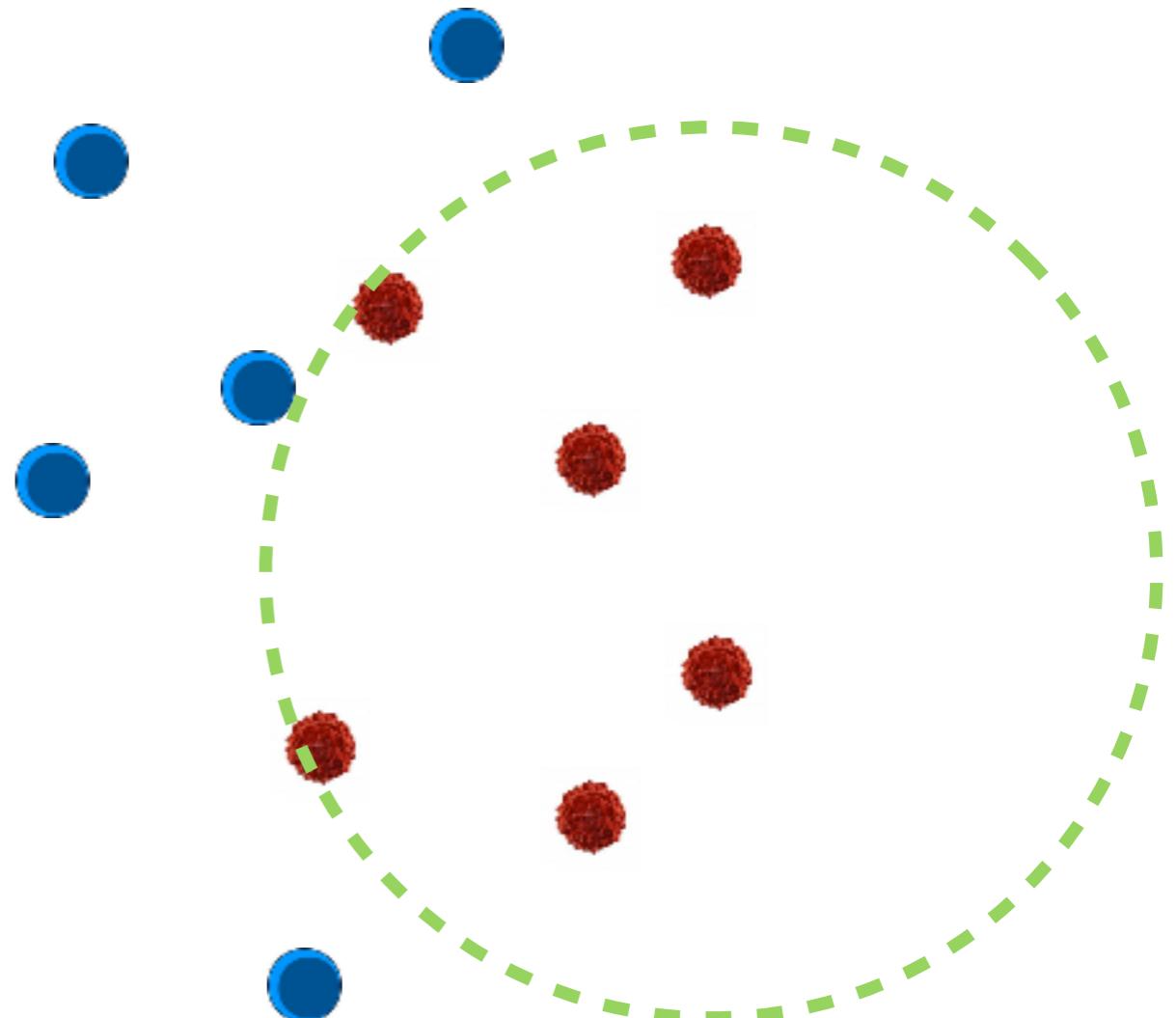
- ❖ **The geometric problem:** Given two finite sets R and B in the plane, does there exist a disk that contains R and is disjoint from B ?
- ❖ **Variables:** Parameters used to define a circle.
- ❖ **Constraints:** Blue points should be outside, red points inside.



Linear Programming

Application I: Cancer Therapy

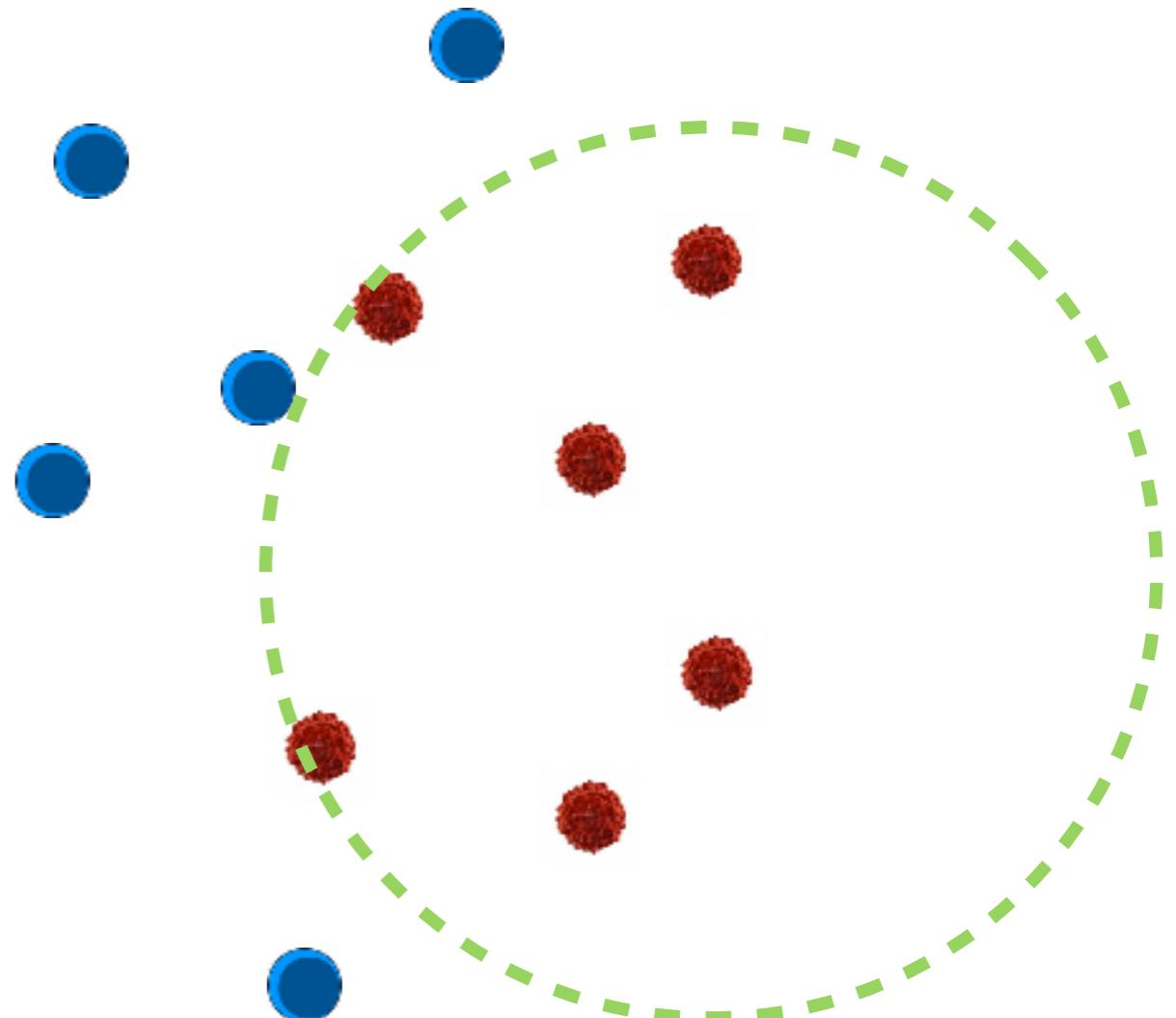
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- ❖ **Problem:** Constraints are not linear...



Linear Programming

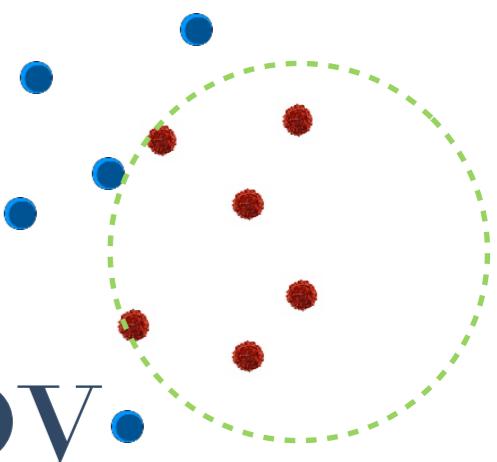
Application I: Cancer Therapy

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- ❖ **Problem:** Constraints are not linear... or are they?



Linear Programming

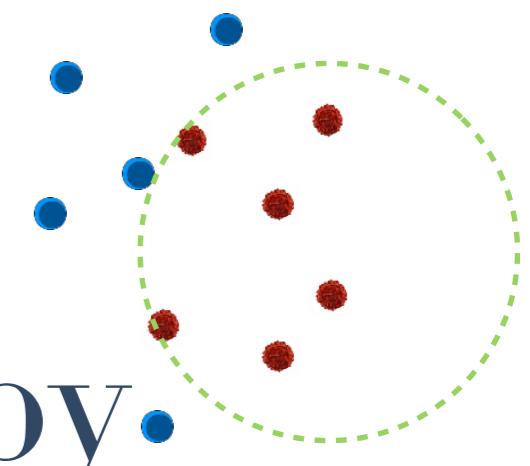
Application I: Cancer Therapy



- ❖ **Problem:** We want to represent the property of being inside of a circle as a linear constraint.

Linear Programming

Application I: Cancer Therapy



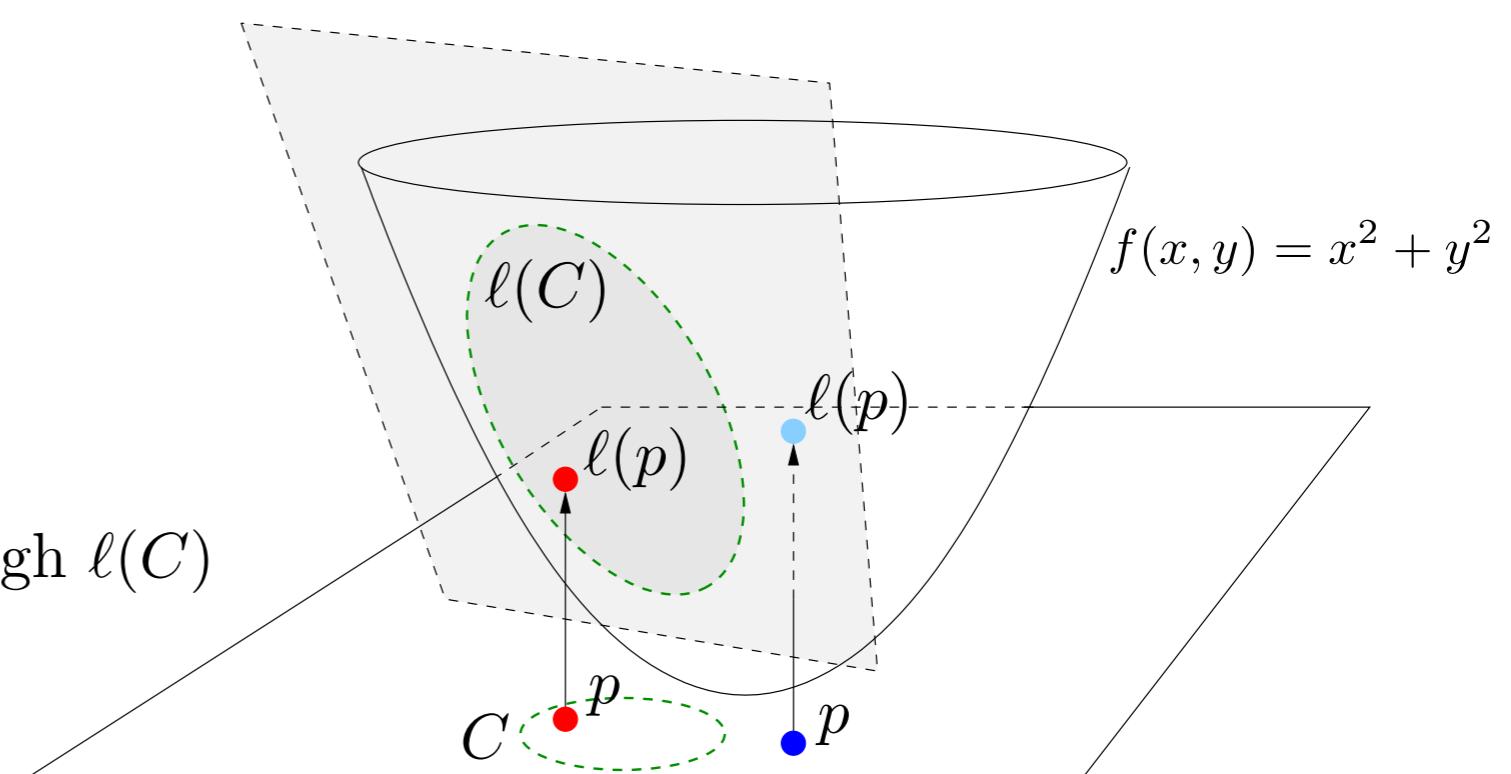
- **Problem:** We want to represent the property of being inside of a circle as a linear constraint.
 - Apply *lifting map* $\ell : (x, y) \mapsto (x, y, x^2 + y^2)$

$$p \left\{ \begin{array}{l} \text{inside} \\ \text{on} \\ \text{outside} \end{array} \right\} C$$

↔

$$\ell(p) \left\{ \begin{array}{l} \text{below} \\ \text{on} \\ \text{above} \end{array} \right\} \text{tl}$$

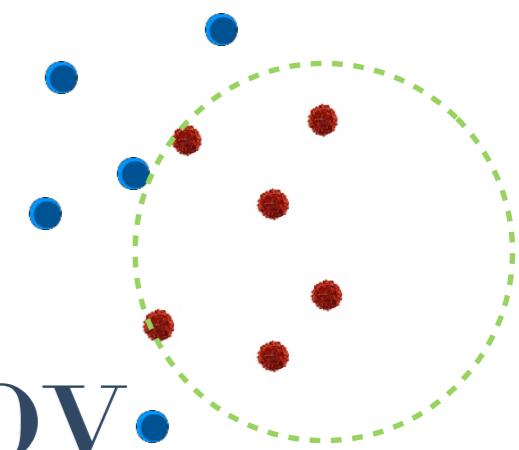
the plane through $\ell(C)$



[Ref: Section 2.3]

Linear Programming

Application I: Cancer Therapy

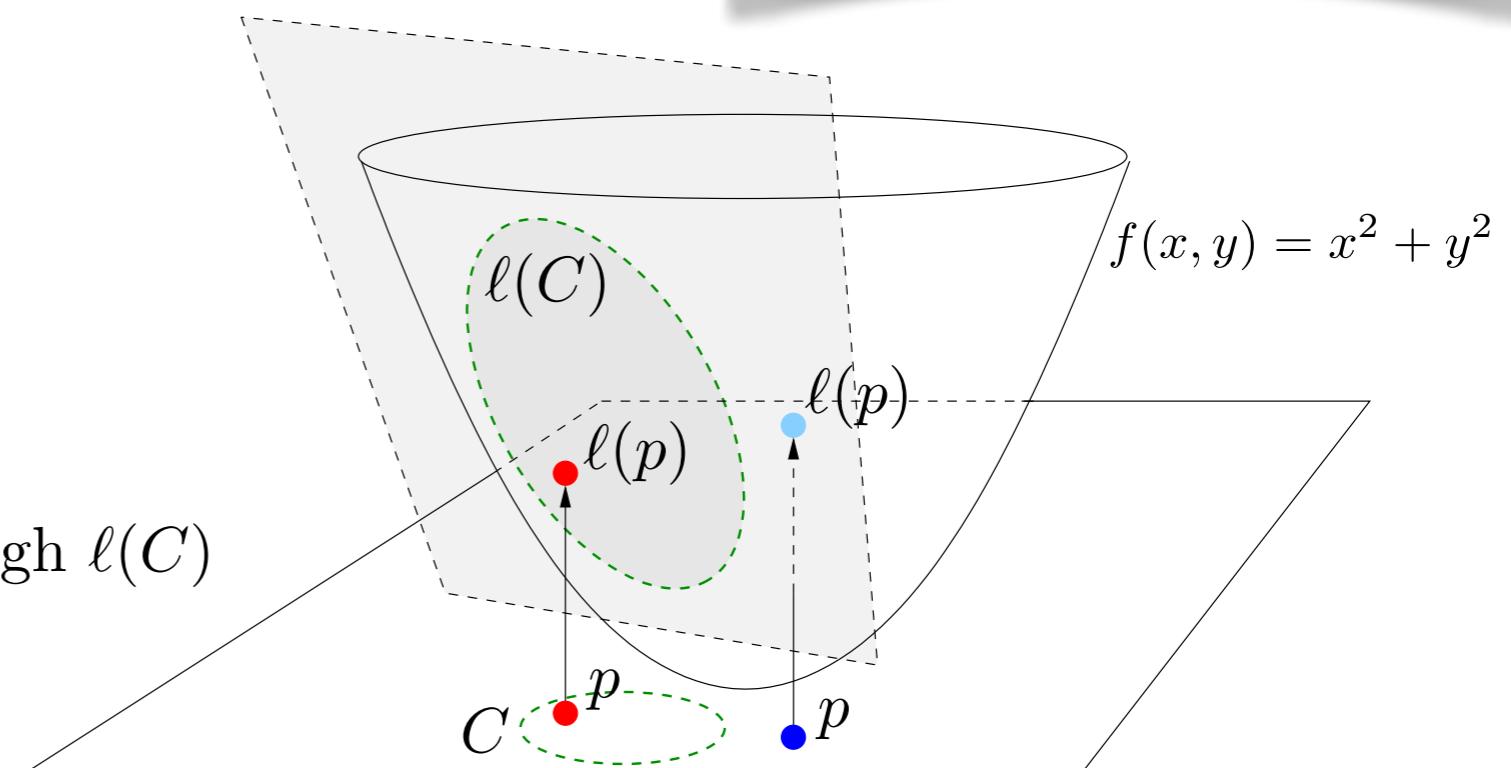


- ❖ **Problem:** We want to represent the property of being inside of a circle as a linear constraint.
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Every circle has a corresponding plane

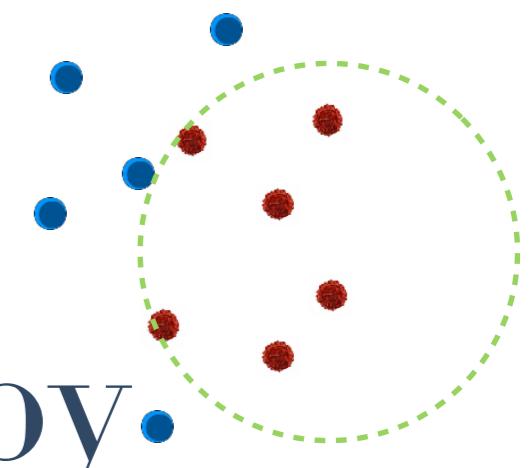
$$p \left\{ \begin{array}{l} \text{inside} \\ \text{on} \\ \text{outside} \end{array} \right\} C$$
$$\Updownarrow$$
$$\ell(p) \left\{ \begin{array}{l} \text{below} \\ \text{on} \\ \text{above} \end{array} \right\}$$

the plane through $\ell(C)$

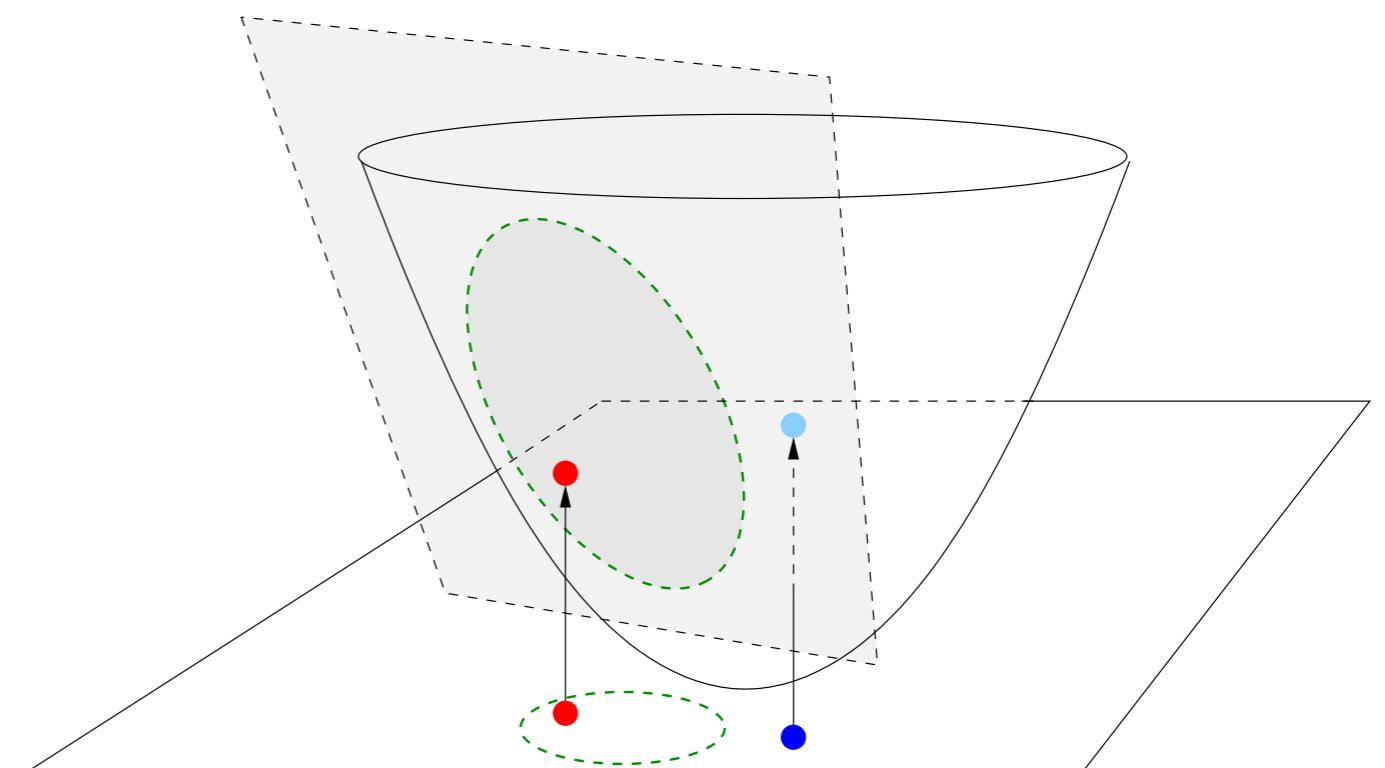


Linear Programming

Application I: Cancer Therapy



- ❖ **The geometric problem (lifted space):** Given the lifted sets R' and B' in space, is there a plane that has R' below/on it and B' above?



Linear Programming

Application I: Cancer Therapy

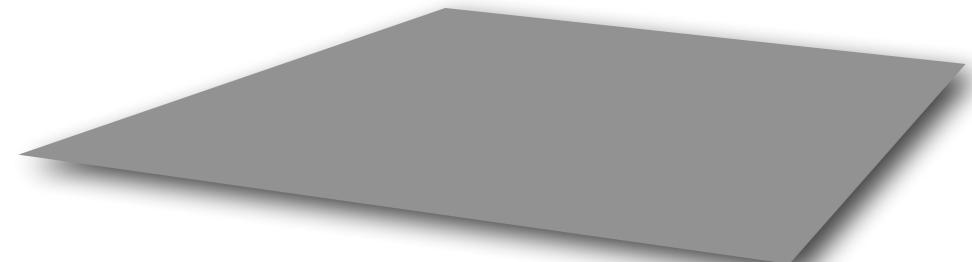
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Linear Programming

Application I: Cancer Therapy

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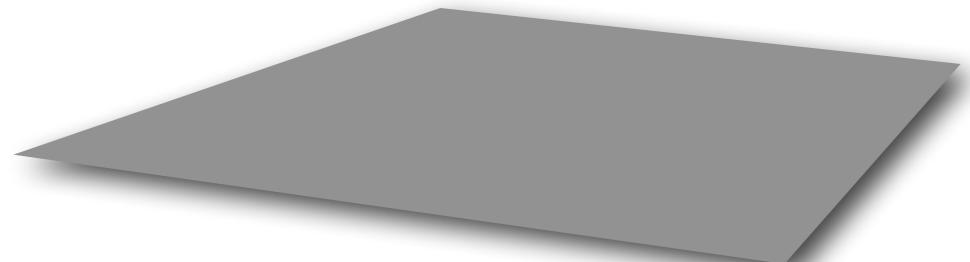
$$\text{plane: } z = \alpha x + \beta y + \gamma$$



Linear Programming

Application I: Cancer Therapy

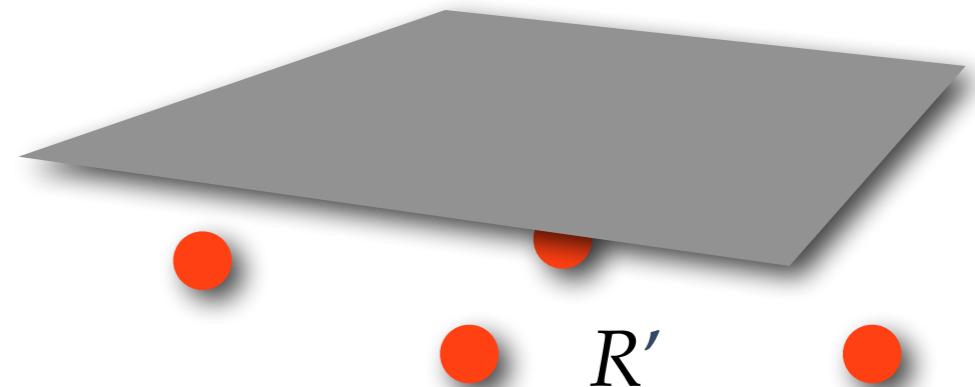
- ⊕ **The geometric problem (lifted space):** Given the lifted sets R' and B' in space, is there a plane that has R' below/on it and B' above?
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- ⊕ Find $\alpha, \beta, \gamma, \delta$ ($\delta > 0$) such that... plane: $z = \alpha x + \beta y + \gamma$



Linear Programming

Application I: Cancer Therapy

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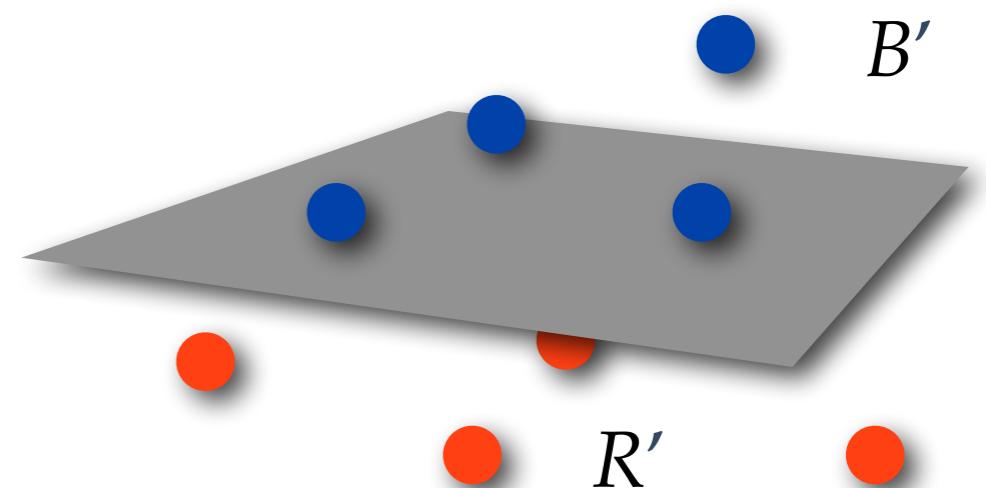
$$x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R$$

Linear Programming

Application I: Cancer Therapy

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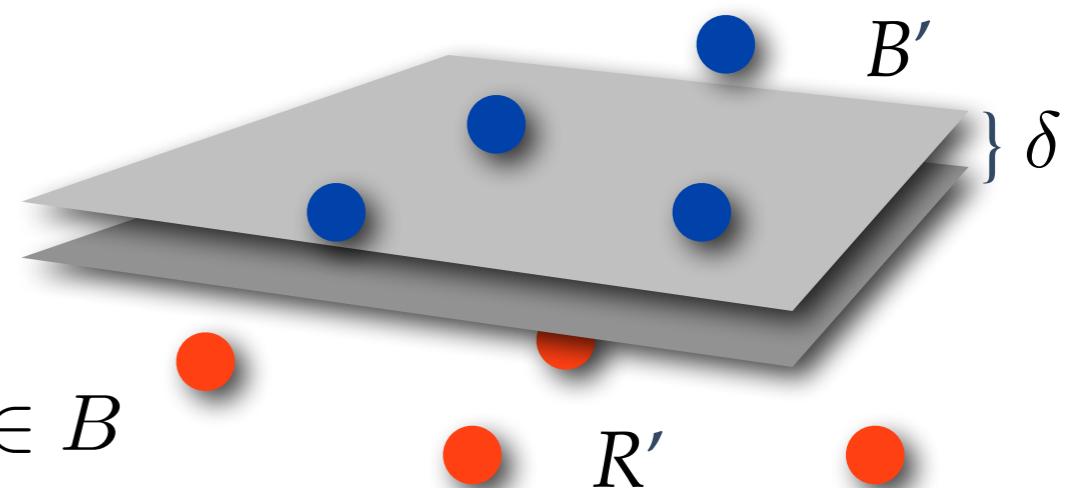
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Linear Programming

Application I: Cancer Therapy

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$$x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, \quad (\textcolor{red}{x}, \textcolor{red}{y}) \in B$$

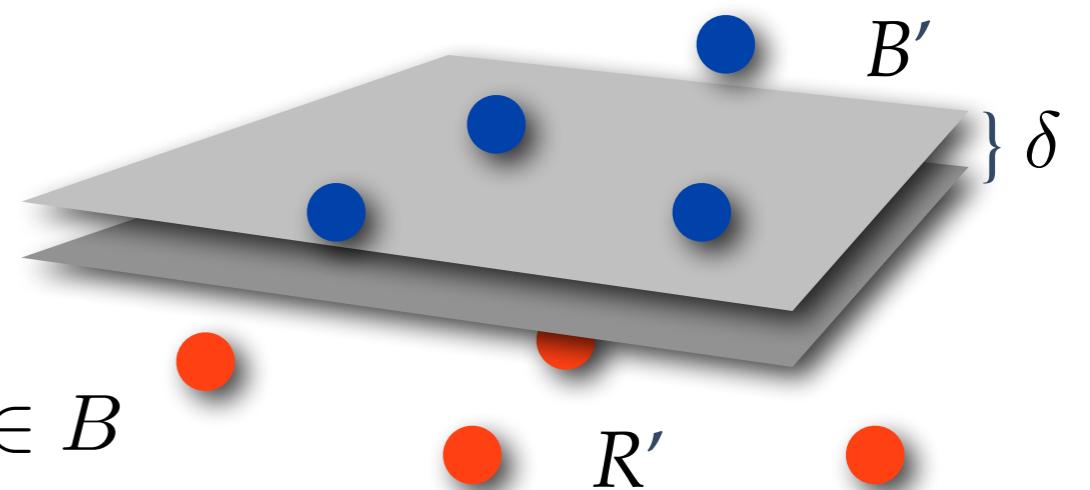
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Linear Programming

Application I: Cancer Therapy

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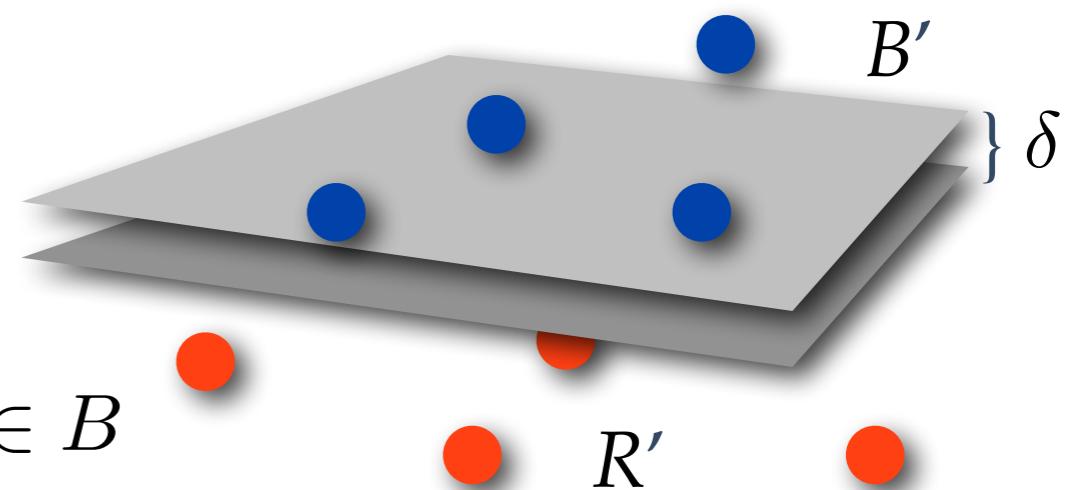
These are linear constraints!

Linear Programming

Application I: Cancer Therapy

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$$x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, \quad (\textcolor{red}{x}, \textcolor{red}{y}) \in B$$

$$x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (\textcolor{red}{x}, \textcolor{red}{y}) \in R$$

$$(\textcolor{red}{3}, \textcolor{red}{6}) \in R$$

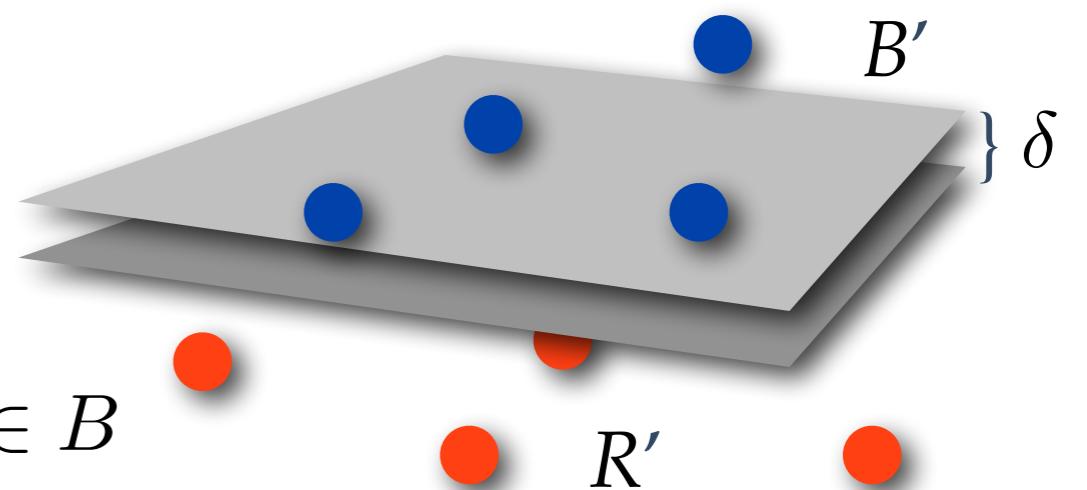
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Linear Programming

Application I: Cancer Therapy

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$$x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, \quad (x, y) \in B$$

$$x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R$$

$$(3, 6) \in R \longrightarrow 3^2 + 6^2 \leq 3\alpha + 6\beta + \gamma$$

These are linear constraints!

Linear Programming

Application I: Cancer Therapy

- **The geometric problem (lifted space):** Given the lifted sets R' and B' in space, is there a plane that has R' below/on it and B' above?

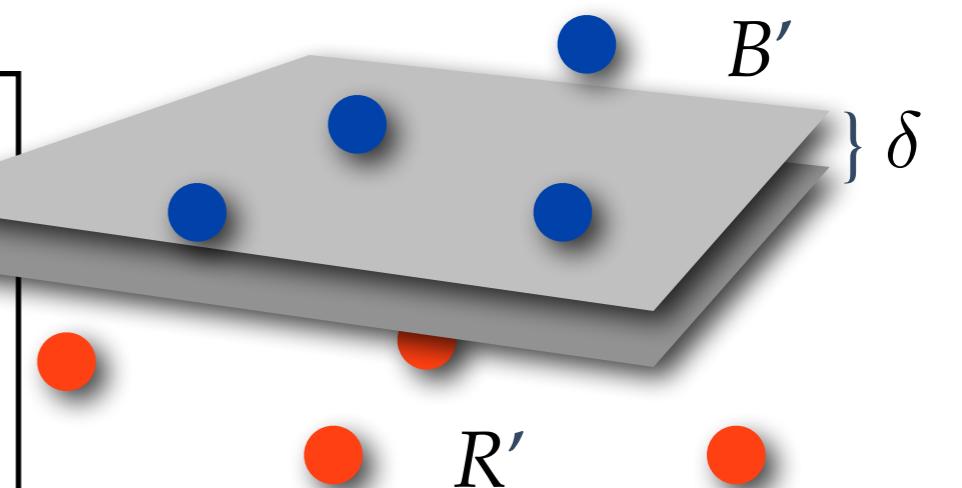
- This can be solved with linear programming!

- Find $\alpha, \beta, \gamma, \delta > 0$ such that...

$$\begin{aligned} & \text{maximize } \delta \\ & \text{subject to} \\ & \quad x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, \quad (x, y) \in B \\ & \quad x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R \end{aligned}$$

4 variables!!
 $|B| + |R|$ constraints

$$\text{plane: } z = \alpha x + \beta y + \gamma$$



Linear Programming

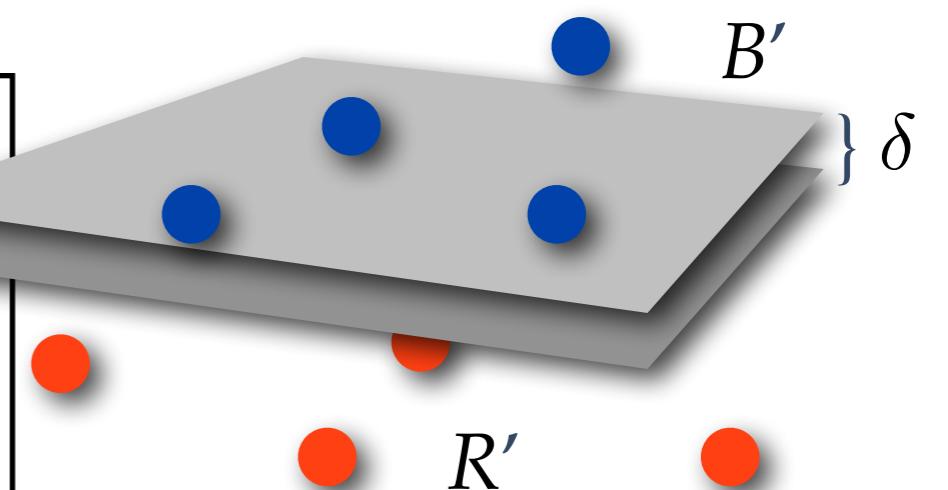
Application I: Cancer Therapy

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4 variables!!
 $|B'| + |R'|$ constraints

$$\text{plane: } z = \alpha x + \beta y + \gamma$$



Linear Programming

Application I: Cancer Therapy

- ❖ **Fact:** Exposure is possible if and only if the following linear program has positive value.

maximize δ

subject to

$$x^2 + y^2 \geq \alpha \textcolor{red}{x} + \beta \textcolor{red}{y} + \gamma + \delta, \quad (\textcolor{red}{x}, \textcolor{red}{y}) \in B$$

$$x^2 + y^2 \leq \alpha \textcolor{red}{x} + \beta \textcolor{red}{y} + \gamma, \quad (\textcolor{red}{x}, \textcolor{red}{y}) \in R$$

Linear Programming

Application I: Cancer Therapy

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- * **Reconstructing the exposure from an optimal solution $(\alpha, \beta, \gamma, \delta)$:**



$$= \{(x, y) : x^2 + y^2 = \alpha x + \beta y + \gamma\}$$

Linear Programming

Application I: Cancer Therapy

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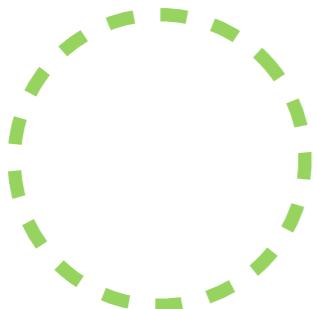
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- * **Reconstructing the exposure from an optimal solution $(\alpha, \beta, \gamma, \delta)$:**



$$= \{(x, y) : x^2 + y^2 = \alpha x + \beta y + \gamma\}$$

$$= \{(x, y) : (x - \frac{\alpha}{2})^2 + (y - \frac{\beta}{2})^2 = \gamma + \frac{\alpha^2}{4} + \frac{\beta^2}{4}\}$$

Linear Programming

Application I: Cancer Therapy

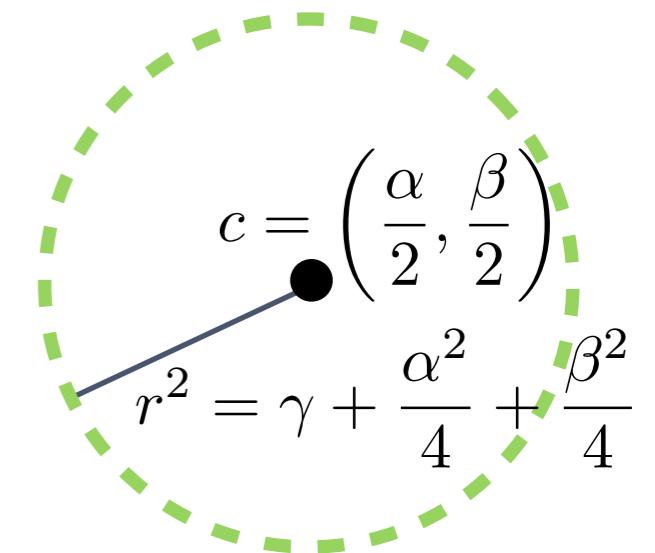
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Linear Programming

Application I: Cancer Therapy

- * **Implementation in CGAL:**

$$\begin{array}{lll} \text{minimize} & -\delta \\ \text{subject to} & x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, & (x, y) \in B \\ & x^2 + y^2 \leq \alpha x + \beta y + \gamma, & (x, y) \in R \\ & \delta \leq 1 \end{array}$$

Avoids unbounded program

maximize $c^T x \rightarrow$ minimize $-c^T x$ and negate resulting value

Linear Programming

Application I: Cancer Therapy

- Implementation in CGAL: Setup and Solve (Preamble as before)

```

int main() {
    // by default, we have an LP with Ax <= b and no bounds for
    // the four variables alpha, beta, gamma, delta
    Program lp (CGAL::SMALLER, false, 0, false, 0);
    const int alpha = 0;
    const int beta = 1;
    const int gamma = 2;
    const int delta = 3;

    // number of red and blue points
    int m; std::cin >> m;
    int n; std::cin >> n;

    // read the red points (cancer cells)
    for (int i=0; i<m; ++i) {
        int x; std::cin >> x;
        int y; std::cin >> y;
        // set up <= constraint for point inside/on circle:
        // -alpha x - beta y - gamma <= -x^2 - y^2
        lp.set_a (alpha, i, -x);
        lp.set_a (beta, i, -y);
        lp.set_a (gamma, i, -1);
        lp.set_b (i, -x*x - y*y);
    }
}

```

```

// read the blue points (healthy cells)
for (int j=0; j<n; ++j) {
    int x; std::cin >> x;
    int y; std::cin >> y;
    // set up <= constraint for point outside circle:
    // alpha x + beta y + gamma + delta <= x^2 + y^2
    lp.set_a (alpha, m+j, x);
    lp.set_a (beta, m+j, y);
    lp.set_a (gamma, m+j, 1);
    lp.set_a (delta, m+j, 1);
    lp.set_b (m+j, x*x + y*y);
}

// objective function: -delta (the solver minimizes)
lp.set_c(delta, -1);

// enforce a bounded problem:
lp.set_u (delta, true, 1);

// solve the program, using ET as the exact type
Solution s = CGAL::solve_linear_program(lp, ET());
assert (s.solves_linear_program(lp));

```

Linear Programming

Application I: Cancer Therapy

$$\begin{aligned}
 & \text{minimize} && -\delta \\
 & \text{subject to} && x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, \quad (x, y) \in B \\
 & && x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R \\
 & && \delta \leq 1
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    }
}

```

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    // alpha x + beta y + gamma + delta <= x^2 + y^2
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    lp.set_a (beta, m+j, y);
    lp.set_a (gamma, m+j, 1);
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Linear Programming

Application I: Cancer Therapy

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    const int gamma = 2;
    const int delta = 3;

    // number of red and blue points
    int m; std::cin >> m;
    int n; std::cin >> n;

    // read the red points (cancer cells)
    for (int i=0; i<m; ++i) {
        int x; std::cin >> x;
        int y; std::cin >> y;
        // set up <= constraint for point inside/on circle:
        // -alpha x - beta y - gamma <= -x^2 - y^2
        lp.set_a (alpha, i, -x);
        lp.set_a (beta, i, -y);
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```

$$\begin{aligned}
 & \text{minimize} && -\delta \\
 & \text{subject to} && x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, \quad (x, y) \in B \\
 & && x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R \\
 & && \delta \leq 1
 \end{aligned}$$

```

    // read the blue points (healthy cells)
    for (int j=0; j<n; ++j) {
        int x; std::cin >> x;
        int y; std::cin >> y;
        // set up <= constraint for point outside circle:
        // alpha x + beta y + gamma + delta <= x^2 + y^2
        lp.set_a (alpha, m+j, x);
        lp.set_a (beta, m+j, y);
        lp.set_a (gamma, m+j, 1);
        lp.set_a (delta, m+j, 1);
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    // objective function: -delta (the solver minimizes)
    lp.set_c(delta, -1);

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    lp.set_u (delta, true, 1);

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    Solution s = CGAL::solve_linear_program(lp, ET());
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Linear Programming

Application I: Cancer Therapy

- Implementation in CGAL: Setup and Solve (Preamble as before)

```

int main() {
    // by default, we have an LP with Ax <= b and no bounds for
    // the four variables alpha, beta, gamma, delta
    Program lp (CGAL::SMALLER, false, 0, false, 0);
    const int alpha = 0;
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Linear Programming

Application I: Cancer Therapy

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* Implementation in CGAL: Output

negate resulting value!

```
// output exposure center and radius, if they exist
if (s.is_optimal() && (s.objective_value() < 0)) {
    // *opt := alpha, *(opt+1) := beta, *(opt+2) := gamma
    CGAL::Quadratic_program_solution<ET>::Variable_value_iterator
        opt = s.variable_values_begin();
    CGAL::Quotient<ET> alpha_opt = *opt +alpha;// +0
    CGAL::Quotient<ET> beta_opt = *(opt+beta); // +1
    CGAL::Quotient<ET> gamma_opt = *(opt+gamma); // +2
    std::cout << "There is a valid exposure:\n";
    std::cout << " Center = (" // (alpha/2, beta/2)
        << alpha_opt/2 << ", " << beta_opt/2
        << ")\n";
    std::cout << " Squared Radius = " // gamma + alpha^2/4 + beta^2/4
        << gamma_opt + alpha_opt*alpha_opt/4 + beta_opt*beta_opt/4 << "\n";
} else
    std::cout << "There is no valid exposure.";
std::cout << "\n";
return 0;
}
```

Linear Programming

Application I: Cancer Therapy

$$\begin{array}{lll} \text{minimize} & -\delta \\ \text{subject to} & x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, & (x, y) \in B \\ & x^2 + y^2 \leq \alpha x + \beta y + \gamma, & (x, y) \in R \\ & \delta \leq 1 & \end{array}$$

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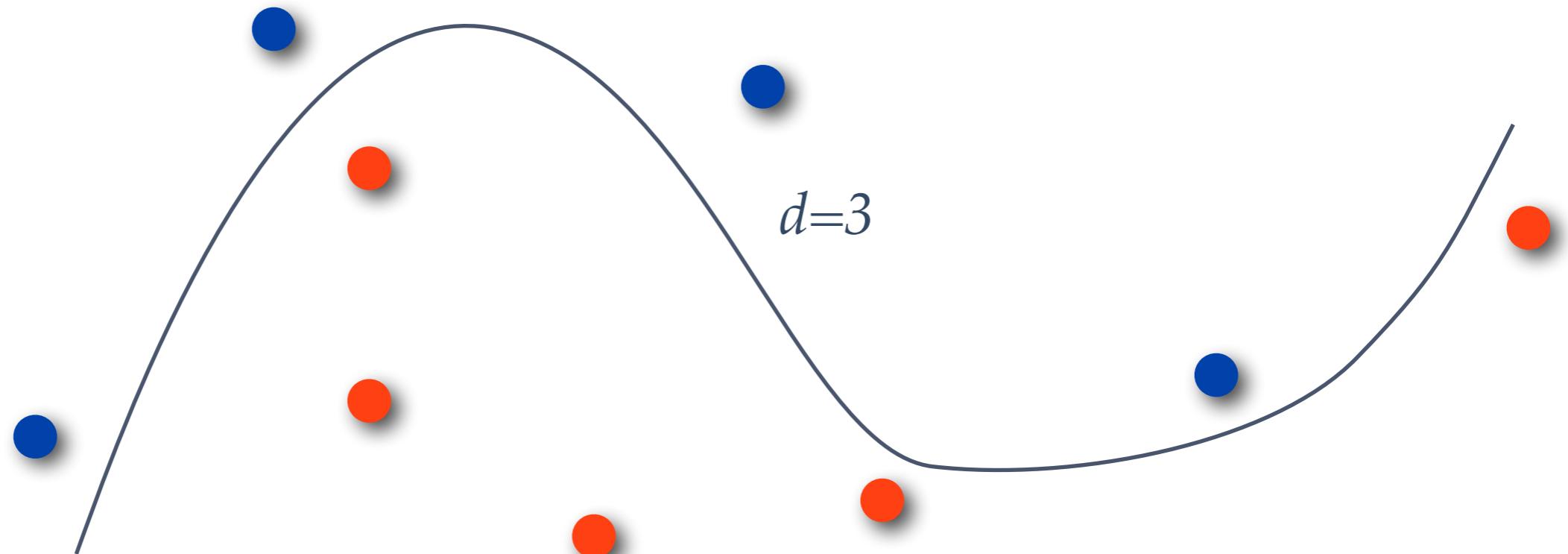
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    // *opt := alpha, *(opt+1) := beta, *(opt+2) := gamma
    CGAL::Quadratic_program_solution<ET>::Variable_value_iterator
        opt = s.variable_values_begin(); ← "Pointer" to first
    CGAL::Quotient<ET> alpha_opt = *opt +alpha;// +0 variable of optimal
    CGAL::Quotient<ET> beta_opt = *(opt+beta); // +1 solution
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} else
    std::cout << "There is no valid exposure.";
std::cout << "\n";
return 0;
}
```

"Pointer" to first
variable of optimal
solution

The quotient
*** (opt+i)** is
the value of the
variable x_i in the
optimal solution

Linear Programming Beyond Cancer Therapy

- * Given a set of R of red and a set B of blue points, can they be separated by the zero set of a polynomial of degree d ?



Linear Programming Beyond Cancer Therapy

- Given a set of R of red and a set B of blue points, can they be separated by the zero set of a polynomial of degree d ?
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$$ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j$$

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- Linear programming formulation: find a,b,c,d,e,f,g,h,i,j such that
$$ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j \leq 0, \quad (x, y) \in B$$
$$ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j \geq 0, \quad (x, y) \in R$$

Linear Programming Beyond Cancer Therapy

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- This is linear separability in 9-dimensional space, under the generalized lifting map $(x, y) \rightarrow (x^3, x^2y, xy^2, y^3, x^2, xy, y^2, x, y)$

Linear Programming Further Applications

Not necessary for the class, but good to know

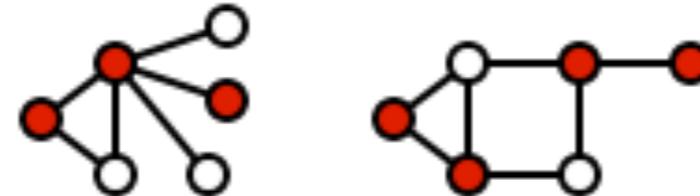
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Linear Programming

Further Applications

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- * Linear programming relaxations for hard combinatorial problems
- * **Vertex Cover:** Given a graph $G=(V,E)$, find a smallest subset of vertices (a vertex cover) such that every edge is incident to one vertex of the cover.

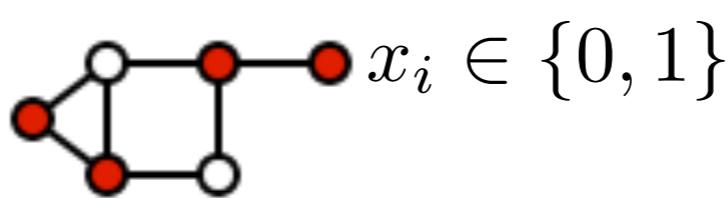


Linear Programming

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$$x_i \in \{0, 1\}$$

- * Formulation as “LP”: x_i indicates whether vertex i is in the cover (0: not in the cover, 1: in the cover):

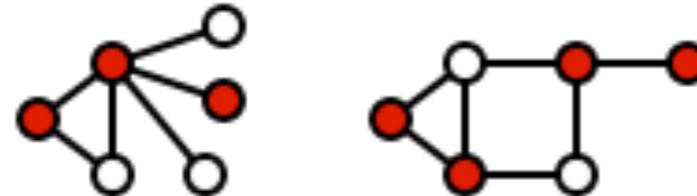
$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n x_i \\ \text{subject to} & x_i + x_j \geq 1 \quad \forall \{i, j\} \in E \\ & 0 \leq x_i \leq 1 \quad \forall i \in V \end{array}$$

Linear Programming

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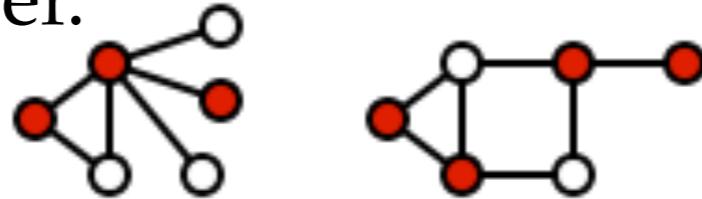
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Linear Programming

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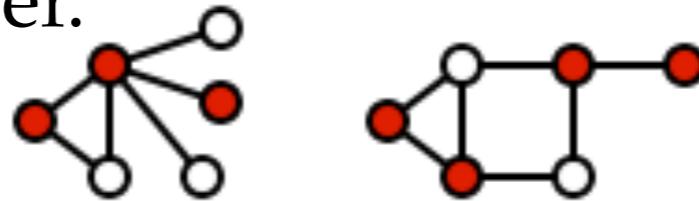
- ❖ Let $x_1^*, x_2^*, \dots, x_n^*$ be an optimal solution of the *LP relaxation*

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- **Theorem:** $C = \{i : x_i^* \geq 1/2\}$ is a vertex cover of size at most 2 opt.

Linear vs. Integer Programming

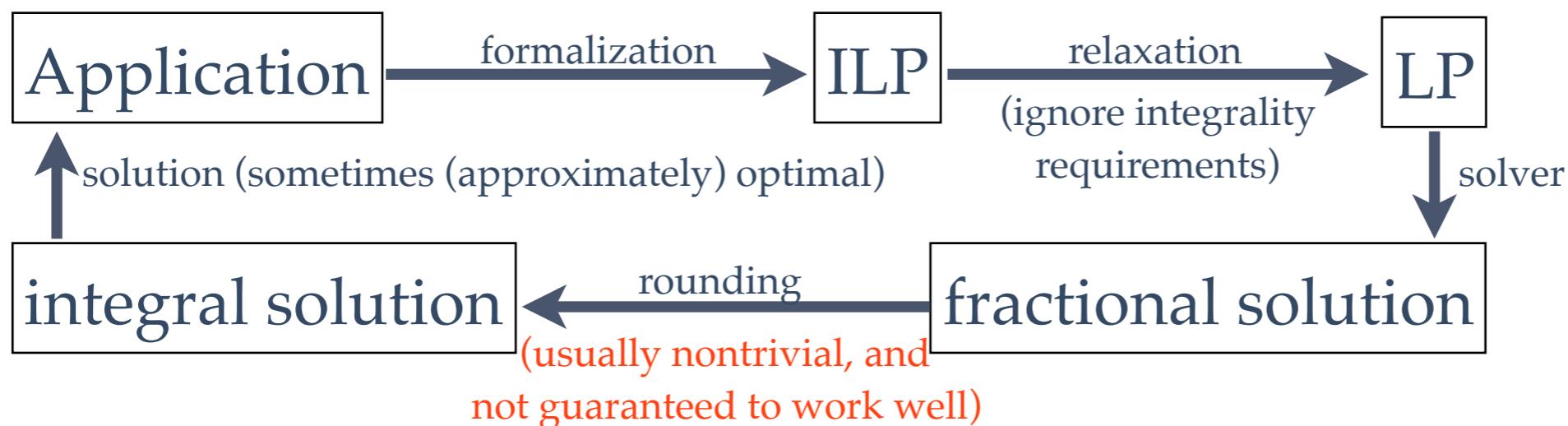
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Linear vs. Integer Programming

- Often, applications lead to linear programs with the additional requirement of *integral solutions* (e.g. vertex cover)
- Such programs are called *integer linear programs* (ILP) and are in general much harder to solve than linear programs (NP-hard)
- Typical approach (e.g. vertex cover):



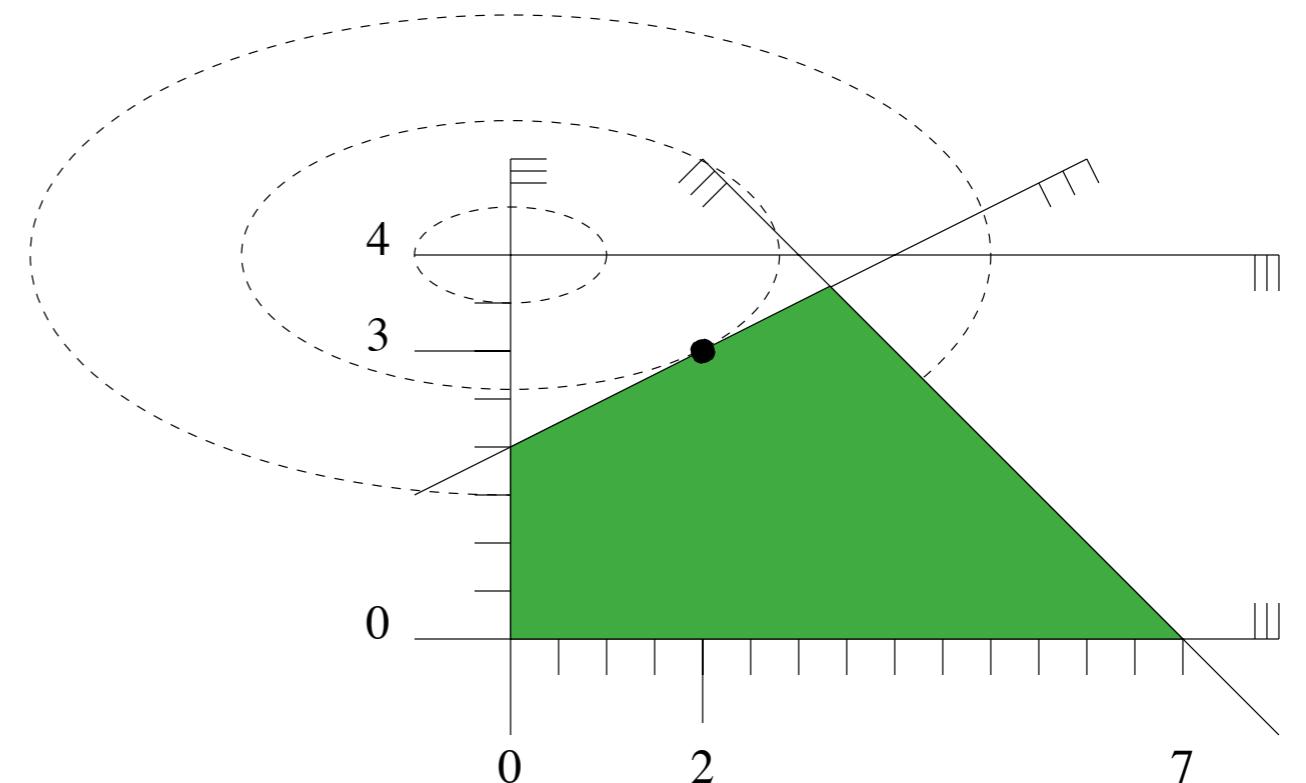
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Quadratic Programming

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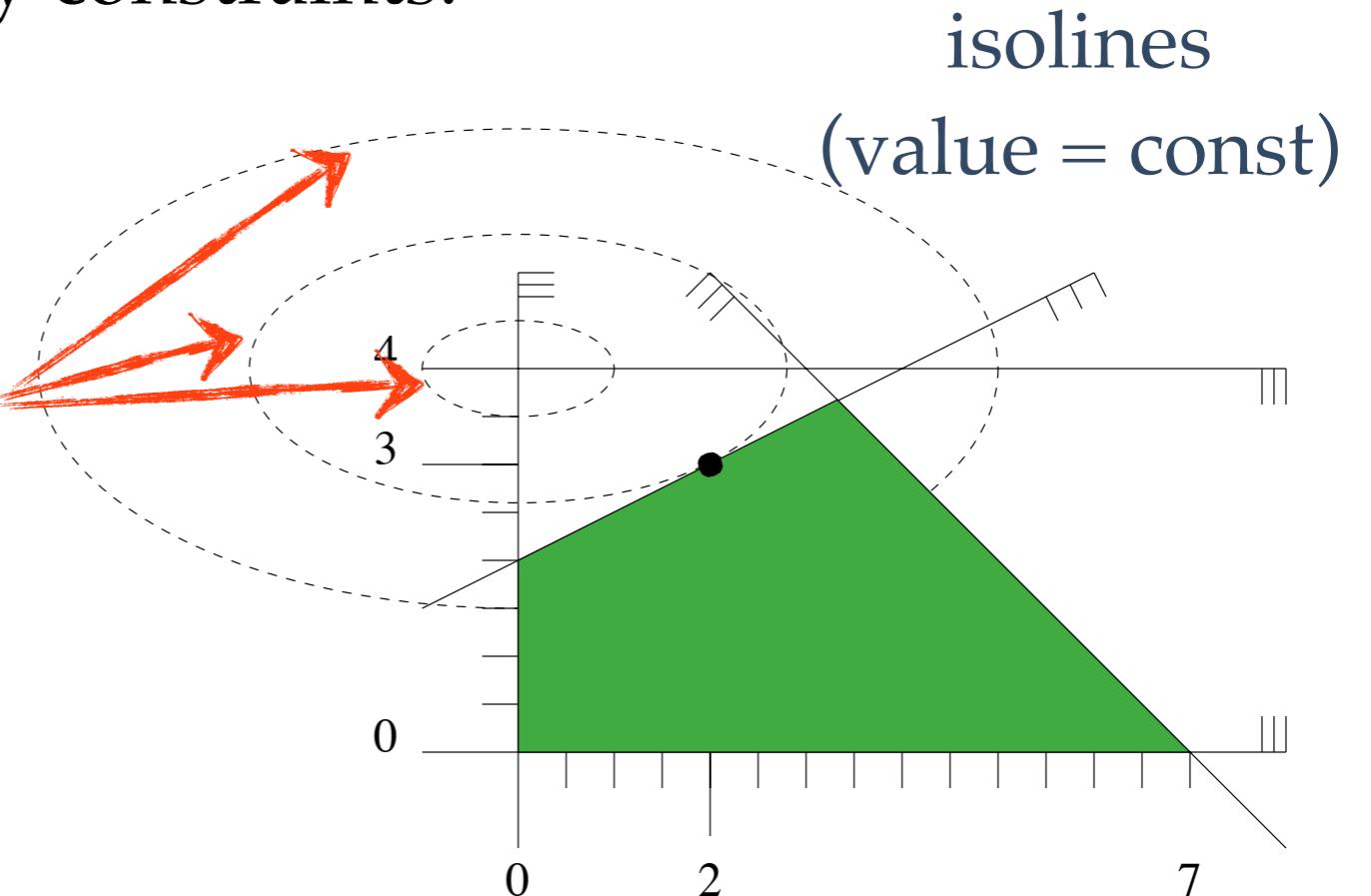
$$\begin{array}{ll} \text{minimize} & x^2 + 4y^2 - 32y + 64 \\ \text{subject to} & \begin{array}{lcl} x + y & \leq & 7 \\ -x + 2y & \leq & 4 \\ x & \geq & 0 \\ y & \geq & 0 \\ y & \leq & 4 \end{array} \end{array}$$



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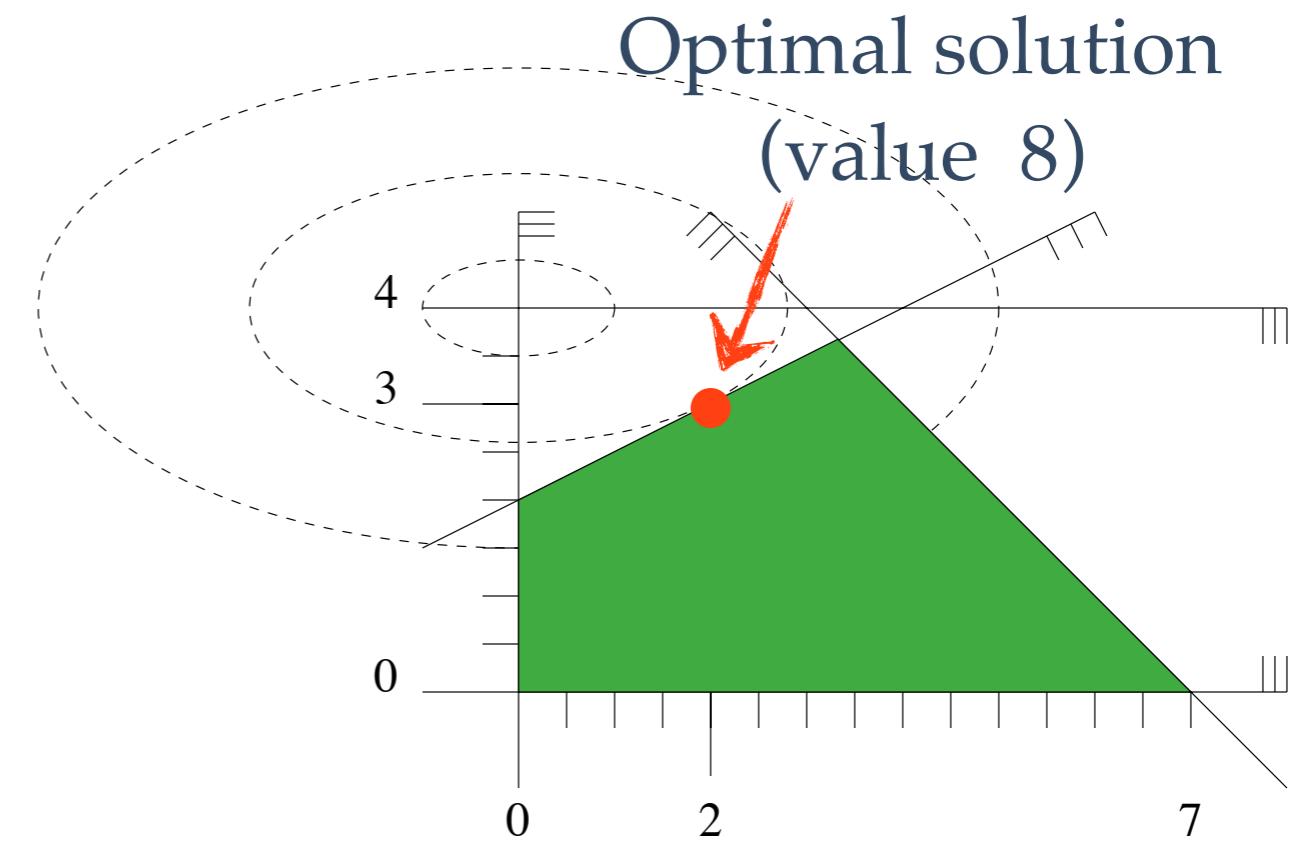


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Quadratic Programming ... in CGAL

- * **General form of QP in CGAL:**

$$\begin{array}{ll}\text{minimize} & x^T D x + c^T x + c_0 \\ \text{subject to} & Ax \gtrless b \\ & l \leq x \leq u\end{array}$$

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The CGAL solver might in this case return solutions that are not optimal, or it might crash.
- * **Relax:** In the applications, we know from theory that D is “good”

Quadratic Programming ... in CGAL

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- ❖ **Risk-tolerant strategy:** Minimize the risk under a given lower bound for the expected return!

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 - ❖ v_i : variance (“risk”) of R_i , $\text{Var} [R_i] := E [(R_i - E[R_i])^2]$

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 - ❖ $1, 2, \dots, n$ (e.g. 1 = Swatch shares, 2 = Credit Suisse shares,...)
- ❖ Investment Characteristics (not at all easy to know/estimate):
 - ❖ R_i : return rate of investment i (assumed to be a random variable)
 - ❖ r_i : expected return rate of investment i, $E [R_i]$
 - ❖ v_i : variance (“risk”) of R_i , $\text{Var} [R_i] := E [(R_i - E[R_i])^2]$
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Quadratic Programming Application: Low-Risk Investment

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$v_{ii} = v_i$
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Quadratic Programming Application: Low-Risk Investment

- ❖ **Example:** $n=2$

	r_i
Swatch shares	10% (0.1)
Credit Suisse shares	51% (0.51)

v_{ij}	Swatch shares	Credit Suisse shares
Swatch shares	0.09	-0.05
Credit Suisse shares	-0.05	0.25

Quadratic Programming Application: Low-Risk Investment

- Example: $n=2$

	r_i		
Swatch shares	10% (0.1)	Swatch shares	Credit Suisse shares
Credit Suisse shares	51% (0.51)	Credit Suisse shares	0.25

Negative correlation: if CS does worse than expected,
Swatch will probably do better, and vice versa

Quadratic Programming Application: Low-Risk Investment

- Example: $n=2$

	r_i		
Swatch shares	10% (0.1)	v_{ij}	Swatch shares
Credit Suisse shares	51% (0.51)	Credit Suisse shares	-0.05



Read as: standard deviation of return rate is $\sqrt{0.25} = 0.5$
(actual return rate could easily be off by 0.5)

Quadratic Programming Application: Low-Risk Investment

- ❖ **Investment strategy:**

$$(x_1, x_2, \dots, x_n), \quad \sum_{i=1}^n x_i = 1, \quad x_i \geq 0 \forall i$$

Meaning: An x_i fraction of your money goes into investment i

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- * **Example:** half the money in Swatch shares, half in Credit Suisse shares; expected return rate is

$$\frac{1}{2} \cdot 0.1 + \frac{1}{2} \cdot 0.51 = 0.305 = 30.5\%$$

Quadratic Programming Application: Low-Risk Investment

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- * **Risk of this strategy:**

$$\text{Var}\left[\sum_{i=1}^n x_i R_i\right] = \sum_{i=1}^n \sum_{j=1}^n v_{ij} x_i x_j = x^T D x$$

Straightforward calculations

$D = (v_{ij})_{1 \leq i, j \leq n}$ is the *covariance matrix*

Quadratic Programming Application: Low-Risk Investment

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- * **Example:** half-half Swatch/CS has risk $\frac{0.09 - 2 \cdot 0.05 + 0.25}{4} = 0.06$

Quadratic Programming Application: Low-Risk Investment

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Straightforward calculations

less than each individual risk!

- * **Example:** half-half Swatch/CS has risk $\frac{0.09 - 2 \cdot 0.05 + 0.25}{4} = 0.06$

Quadratic Programming Application: Low-Risk Investment

- * **The risk-tolerant case:** Find the investment strategy with lowest risk that guarantees expected return rate ρ at least!

$$\begin{aligned} & \text{minimize} && \sum_{i=1}^n \sum_{j=1}^n v_{ij} x_i x_j \leftarrow \boxed{\text{risk}} \\ & \text{subject to} && \sum_{i=1}^n r_i x_i \geq \rho \\ & && \sum_{i=1}^n x_i = 1 \quad \boxed{\text{expected return rate}} \\ & && x_i \geq 0, \quad i = 1, \dots, n \\ & \boxed{\text{strategy}} && \end{aligned}$$

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Fact: A covariance matrix is positive semidefinite, so this is indeed a convex QP.

Quadratic Programming Application: Low-Risk Investment

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Fact: A covariance matrix is positive semidefinite, so this is indeed a convex QP.

- * **Example:** $\rho = 0.4$: 26.8% Swatch, 73.2% Credit Suisse; risk = 0.121

Low-Risk Investment Example... in CGAL

$$\begin{aligned} & \text{minimize} && \sum_{i=1}^n \sum_{j=1}^n v_{ij} x_i x_j \\ & \text{subject to} && \sum_{i=1}^n r_i x_i \geq \rho \\ & && \sum_{i=1}^n x_i = 1 \\ & && x_i \geq 0, \quad i = 1, \dots, n \end{aligned}$$

- ❖ **Preamble:** This time, the input is rational...

Gnu
Multi-
precision
Library
(GMP)

```
#include <iostream>
#include <cassert>
#include <CGAL/basic.h>
#include <CGAL/QP_models.h>
#include <CGAL/QP_functions.h>
#include <CGAL/Gmpq.h>

// choose exact Rational type
typedef CGAL::Gmpq ET;

// program and solution types
typedef CGAL::Quadratic_program<ET> Program;
typedef CGAL::Quadratic_program_solution<ET> Solution;
```

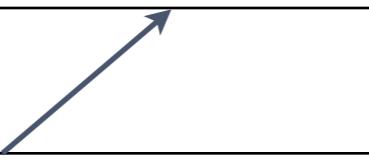
Low-Risk Investment Example... in CGAL

$$\begin{aligned} & \text{minimize} && \sum_{i=1}^n \sum_{j=1}^n v_{ij} x_i x_j \\ & \text{subject to} && \sum_{i=1}^n r_i x_i \geq \rho \\ & && \sum_{i=1}^n x_i = 1 \\ & && x_i \geq 0, \quad i = 1, \dots, n \end{aligned}$$

- ❖ **Input:** Desired expected return

```
int main() {
    // read minimum expected return rate
    std::cout << "What is your desired expected return rate? ";
    double rho; std::cin >> rho;
```

for example, $0.4 \approx 40\%$



Low-Risk Investment Example... in CGAL

$$\begin{aligned} & \text{minimize} && \sum_{i=1}^n \sum_{j=1}^n v_{ij} x_i x_j \\ & \text{subject to} && \sum_{i=1}^n r_i x_i \geq \rho \\ & && \sum_{i=1}^n x_i = 1 \\ & && x_i \geq 0, \quad i = 1, \dots, n \end{aligned}$$

- ❖ **Setup:** Make sure to enter matrix 2D (customary in QP solvers)!

```
// by default, we have a nonnegative QP with Ax >= b
Program qp(CGAL::LARGER, true, 0, false, 0);

// now set the non-default entries:
const int sw = 0;
const int cs = 1;

// constraint on expected return: 0.1 sw + 0.51 cs >= rho
qp.set_a(sw, 0, ET(1)/10);
qp.set_a(cs, 0, ET(51)/100);
qp.set_b( 0, rho);

// strategy constraint: sw + cs = 1
qp.set_a(sw, 1, 1);
qp.set_a(cs, 1, 1);
qp.set_b( 1, 1);
qp.set_r( 1, CGAL::EQUAL); // override default >=

// objective function: 0.09 sw^2 - 0.05 sw cs - 0.05 cs sw + 0.25 cs^2
// we need to specify the entries of the symmetric matrix 2D, on and below the diagonal
qp.set_d(sw, sw, ET(18)/100); // 0.09 sw^2
qp.set_d(cs, sw, -ET(1)/10); // -0.05 cs sw
qp.set_d(cs, cs, ET(1)/2); // 0.25 cs^2
```



j ≤ i in **set_d(i, j)**

Low-Risk Investment Example... in CGAL

$$\begin{aligned}
 & \text{minimize} && \sum_{i=1}^n \sum_{j=1}^n v_{ij} x_i x_j \\
 & \text{subject to} && \sum_{i=1}^n r_i x_i \geq \rho \\
 & && \sum_{i=1}^n x_i = 1 \\
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```

! j ≤ i in **set_d(i, j)**

Low-Risk Investment Example... in CGAL

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- ❖ **Solve:** ...as nonnegative quadratic program (a little faster)

```
// solve the program, using ET as the exact type
Solution s = CGAL::solve_nonnegative_quadratic_program(qp, ET());
assert (s.solves_quadratic_program(qp));
```

independent verification

The nonnegative solvers ignore global lower and upper bounds (as specified by the constructor) and replace them by nonnegativity constraints only. On the other hand, adding these bounds to the matrix explicitly blows up your LP.

In such a case, you are better off using global bounds and the "normal" solver (not nonnegative). Anyway, we recommend to use the "normal" solver as a default and switch to the nonnegative one only to check out whether it makes a difference (in many cases, it won't).

Low-Risk Investment Example... in CGAL

$$\begin{aligned} & \text{minimize} && \sum_{i=1}^n \sum_{j=1}^n v_{ij} x_i x_j \\ & \text{subject to} && \sum_{i=1}^n r_i x_i \geq \rho \\ & && \sum_{i=1}^n x_i = 1 \\ & && x_i \geq 0, \quad i = 1, \dots, n \end{aligned}$$

- ❖ **Output:** query solution status; if feasible, output strategy / risk

```
// output
if (s.status() == CGAL::QP_INFEASIBLE) {
    std::cout << "Expected return rate " << rho << " cannot be achieved.\n";
} else {
    assert (s.status() == CGAL::QP_OPTIMAL);
    Solution::Variable_value_iterator opt = s.variable_values_begin();
    double sw_percent = ceil_to_double(100 * *opt);
    //double cs_percent = ceil_to_double(100 * *(opt+1));
    double risk = ceil_to_double(1000 * s.objective_value());
    std::cout << "Minimum risk investment strategy:\n"
        << "~" << sw_percent << "%" << " Swatch\n"
        << "~" << 100-sw_percent << "%" << " Credit Suisse\n"
        << "Risk <= 0." << risk << "\n";
}
```

Known Bug :=(

- ❖ You can't reliably copy or assign instances of the class
CGAL::Quadratic_program_solution<ET>
- ❖ **Workaround 1:** If you want to pass or return such instances to / from a function, pass a pointer to the instance instead!
- ❖ **Workaround 2:** If you want to assign a new solution to an existing instance... don't do it!

Sources and Further Reading

- ❖ **LP/QP Solver:** Online manual at www.cgal.org: Online Manual
→ Combinatorial Algorithms → Linear and Quadratic Programming Solver
- ❖ **Cancer Therapy:** J. O'Rourke, S. Kosaraju, and N. Megiddo:
Computing Circular Separability, *Discrete & Computational Geometry* 1:105-113 (1986)
- ❖ **Low-Risk Investment:** H. Markowitz: Portfolio Selection, *Journal of Finance* 7(1): 77-91 (1952)