Types, Type Inference and Unification

Mooly Sagiv

Slides by Kathleen Fisher and John Mitchell

Cornell CS 6110

Summary (Functional Programming)

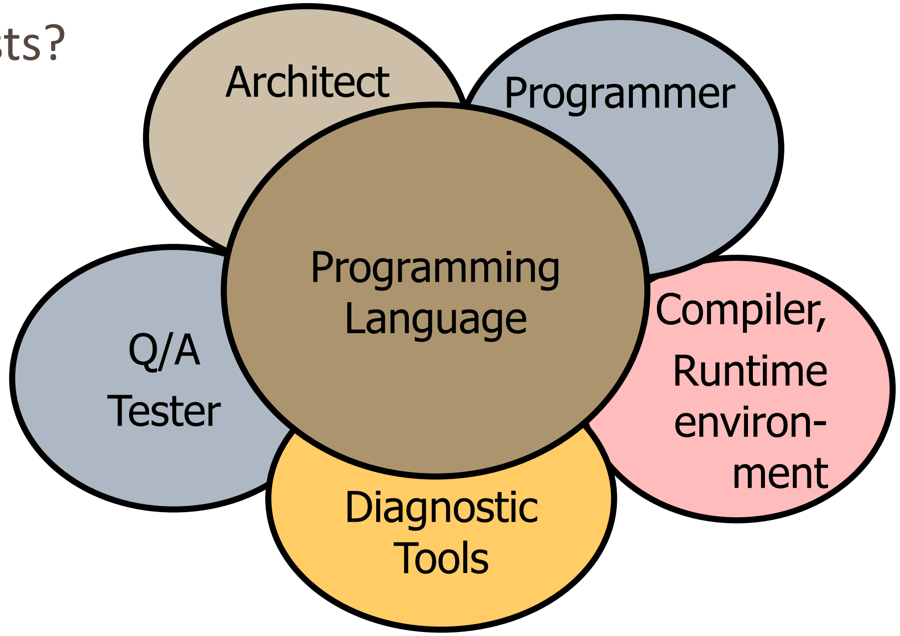
- Lambda Calculus
- Basic ML
- Advanced ML: Modules, References, Side-effects
- Closures and Scopes
- Type Inference and Type Checking

Outline

- General discussion of types
  - What is a type?
  - Compile-time versus run-time checking
  - Conservative program analysis
- Type inference
  - Discuss algorithm and examples
  - Illustrative example of static analysis algorithm
- Polymorphism
  - Uniform versus non-uniform implementations

Language Goals and Trade-offs

- Thoughts to keep in mind
  - What features are convenient for programmer?
  - What other features do they prevent?
  - What are design tradeoffs?
    - Easy to write but harder to read?
    - Easy to write but poorer error messages?
  - What are the implementation costs?



What is a type?

- A type is a collection of computable values that share some structural property.

Examples

```
int
string
int → bool
(int → int) → bool
[a] → a
[a]  * a → [a]
```

Non-examples

```
{3, True, \x->x}
Even integers
{f:int → int | x>3 =>
      f(x) > x *(x+1)}
```

Distinction between sets of values that are types and sets that are not types is *language dependent*

Advantages of Types

- Program organization and documentation
  - Separate types for separate concepts
    - Represent concepts from problem domain
  - Document intended use of declared identifiers
    - Types can be checked, unlike program comments
- Identify and prevent errors
  - Compile-time or run-time checking can prevent meaningless computations such as 3 + true – "Bill"
- Support optimization
  - Example: short integers require fewer bits
  - Access components of structures by known offset

What is a type error?

- Whatever the compiler/interpreter says it is?
- Something to do with bad bit sequences?
  - Floating point representation has specific form
  - An integer may not be a valid float
- Something about programmer intent and use?
  - A type error occurs when a value is used in a way that is inconsistent with its definition
    - Example: declare as character, use as integer

Type errors are language dependent

- Array out of bounds access
  - C/C++: run-time errors
  - OCaml/Java: dynamic type errors
- Null pointer dereference
  - C/C++: run-time errors
  - OCaml: pointers are hidden inside datatypes
    - Null pointer dereferences would be incorrect use of these datatypes, therefore static type errors

Compile-time vs Run-time Checking

- JavaScript and Lisp use run-time type checking
  - f(x)   Make sure f is a function before calling f

```
js> var f= 3;
js> f(2);
typein:3: TypeError: f is not a function
js>
```

- OCaml and Java use compile-time type checking
  - f(x)   Must have f: A $\rightarrow$ B and x : A
- Basic tradeoff
  - Both kinds of checking prevent type errors
  - Run-time checking slows down execution
  - Compile-time checking restricts program flexibility
    - JavaScript array: elements can have different types
    - OCaml list: all elements must have same type
  - Which gives better programmer diagnostics?

Expressiveness

- In JavaScript, we can write a function like

```
function f(x) { return x < 10 ? x : x(); }
```

Some uses will produce type error, some will not

- Static typing always conservative

```
if  (complicated-boolean-expression)
              then  f(5);
              else  f(15);
```

Type Safety

- Type safe programming languages protect its own abstractions
- Type safe programs cannot go wrong
- No run-time errors
- But exceptions are fine
- The small step semantics cannot get stuck
- Type safety is proven at language design time

Relative Type-Safety of Languages

- Not safe: BCPL family, including C and C++
  - Casts, unions, pointer arithmetic
- Almost safe: Algol family, Pascal, Ada
  - Dangling pointers
    - Allocate a pointer p to an integer, deallocate the memory referenced by p, then later use the value pointed to by p
    - Hard to make languages with explicit deallocation of memory fully type-safe
- Safe: Lisp, Smalltalk, ML, Haskell, Java, JavaScript
  - Dynamically typed: Lisp, Smalltalk, JavaScript
  - Statically typed: OCaml, Haskell, Java

If code accesses data, it is handled with the type associated with the creation and previous manipulation of that data

Type Checking vs Type Inference

- Standard type checking:

```
int f(int x) { return x+1; };
int g(int y) { return f(y+1)*2; };
```

  - Examine body of each function
  - Use declared types to check agreement
- Type inference:

```
int f(int x) { return x+1; };
int g(int y) { return f(y+1)*2; };
```

  - Examine code without type information
  - Infer the most general types that could have been declared

ML and Haskell are *designed* to make type inference feasible

The Type Inference Problem

- Input: A program without types (e.g., Lambda calculus)
- Output: A program with type for every expression (e.g., typed Lambda calculus)
  - Every expression is annotated with its most general type

Why study type inference?

- Types and type checking
  - Improved steadily since Algol 60
    - Eliminated sources of unsoundness
    - Become substantially more expressive
  - Important for modularity, reliability and compilation
- Type inference
  - Reduces syntactic overhead of expressive types
  - Guaranteed to produce most general type
  - Widely regarded as important language innovation
  - Illustrative example of a flow-insensitive static analysis algorithm

History

- Original type inference algorithm
  - Invented by Haskell Curry and Robert Feys for the simply typed lambda calculus in 1958
- In 1969, Hindley
  - extended the algorithm to a richer language and proved it always produced the most general type
- In 1978, Milner
  - independently developed equivalent algorithm, called algorithm W, during his work designing ML
- In 1982, Damas proved the algorithm was complete.
  - Currently used in many languages: ML, Ada, Haskell, C# 3.0, F#, Visual Basic .Net 9.0. Have been plans for Fortress, Perl 6, C++0x,…

Type Inference: Basic Idea

- Example

```
fun x -> 2 + x
  -: int -> int = <fun>
```

- What is the type of the expression?
  - `+` has type: int $\rightarrow$ int $\rightarrow$ int
  - 2 has type: int
  - Since we are applying + to x we need x : int
  - Therefore `fun x -> 2 + x` has type int $\rightarrow$ int

Imperative Example

```
x := b[z]
a [b[y]] := x
```

Type Inference: Basic Idea

- Example

```
fun f  =>  f 3
(int -> a) ->  a = <fun>
```

- What is the type of the expression?
  - 3 has type: int
  - Since we are applying f to 3 we need f : int → a and the result is of type a
  - Therefore `fun f -> f 3` has type (int → a) →a

Type Inference: Basic Idea

- Example

```
fun f  =>  f (f 3)
(int -> int) ->  int = <fun>
```

- What is the type of the expression?

Type Inference: Basic Idea

- Example

```
fun f  =>  f (f "hi")
(string -> string) -> string = <fun>
```

- What is the type of the expression?

Type Inference: Basic Idea

- Example

```
fun f  =>  f (f 3, f 4)
```

- What is the type of the expression?

Type Inference: Complex Example

let square = λz. z * z
    in
λf.λx. λy.
if (f x y)
  then (f (square x) y)
  else (f x (f x y))

$* : int \rightarrow (int \rightarrow int)$

$z : int$

$square : int \rightarrow int$

$f : a \rightarrow (b \rightarrow bool), x: a, y: b$

$a: int$

$b: bool$

$(int \rightarrow bool \rightarrow bool) \rightarrow int \rightarrow bool \rightarrow bool$

Unification

- Unifies two terms
- Used for pattern matching and type inference
- Simple examples
  - int * x and y * (bool * bool) are **unifiable** for y = int and x = (bool * bool)
  - int * int and int * bool are **not unifiable**

Substitution

Types:

<type> ::= int | float | bool |…
| <type> → <type>
| <type> * <type>
| variable

Terms:

<term> ::= constant
| variable
| f(<term>, …, <term>)

- The essential task of unification is to find a substitution that makes the two terms equal

$$f(x, h(x, y))\ \{x \mapsto g(y),\ y \mapsto z\} = f(g(y), h(g(y), z))$$

- The terms t_1 and t_2 are unifiable if there exists a substiitution S such that $t_1\ S = t_2\ S$
- Example: $t_1 = f(x, g(y))$, $t_2 = f(g(z), w)$

Most General Unifiers (mgu)

- It is possible that no unifier for given two terms exist
  - For example x and f(x) cannot be unified
- There may be several unifiers
  - Example: $t_1 = f(x, g(y))$, $t_2 = f(g(z), w)$
    - $S = \{x \mapsto g(z), y \mapsto w, w \mapsto g(w)\}$
    - $S' = \{x \mapsto g(f(a, b)), y \mapsto f(b, a), z \mapsto f(a, b), w \mapsto g(f(b, a)\}$
- When a unifier exists, there is always a **most general unifier** (mgu) that is unique up to renaming
- S is the most general unifier of t1 and t2 if
  - It is a unifier of t_1 and t_2
  - For every other unifier S' of t_1 and t_2 there exists a refinement of S to give S'
- mgu can be efficiently computed
  - $mgu(f(x, g(y)), f(g(z), w)) = \{x \mapsto g(z), y \mapsto w, w \mapsto g(w)\}$
  - $mgu(\{y \mapsto g(w)\}, f(x, g(y)), f(g(z), w)) = \{y \mapsto g(w), x \mapsto g(z), w \mapsto g(g(w))\}$

Type Inference with mgu

- Example

```
fun f  =>  f (f "hi")
```

(string -> string) -> string = <fun>

- What is the type of the expression?

$$\lambda \mathtt{f:T_1.apply(f:T_1,apply(f:T_1,\text{"hi"}:string):T_2):T_3}$$

$$\mathtt{mgu(T_1,string \to T_2)} = \{\mathtt{T_1 \mapsto string \to T_2}\} = \mathtt{S}$$

$$\mathtt{mgu(S,\ T_1,T_2 \to T_3)} =$$

$$\{\mathtt{T_1 \mapsto string \to T_2, T_2 \mapsto sring,\ T_3 \mapsto sring}\}$$

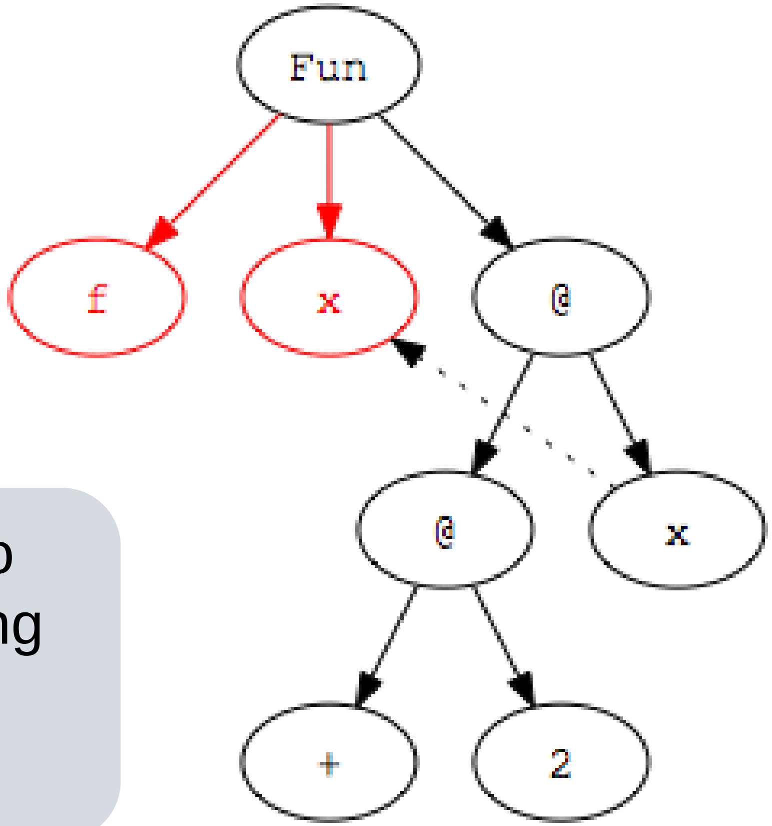
Type Inference Algorithm

- Parse program to build parse tree
- Assign type variables to nodes in tree
- Generate constraints:
  - From environment: literals (2), built-in operators (+), known functions (tail)
  - From form of parse tree: e.g., application and abstraction nodes
- Solve constraints using *unification*
- Determine types of top-level declarations

Step 1: Parse Program

- Parse program text to construct parse tree

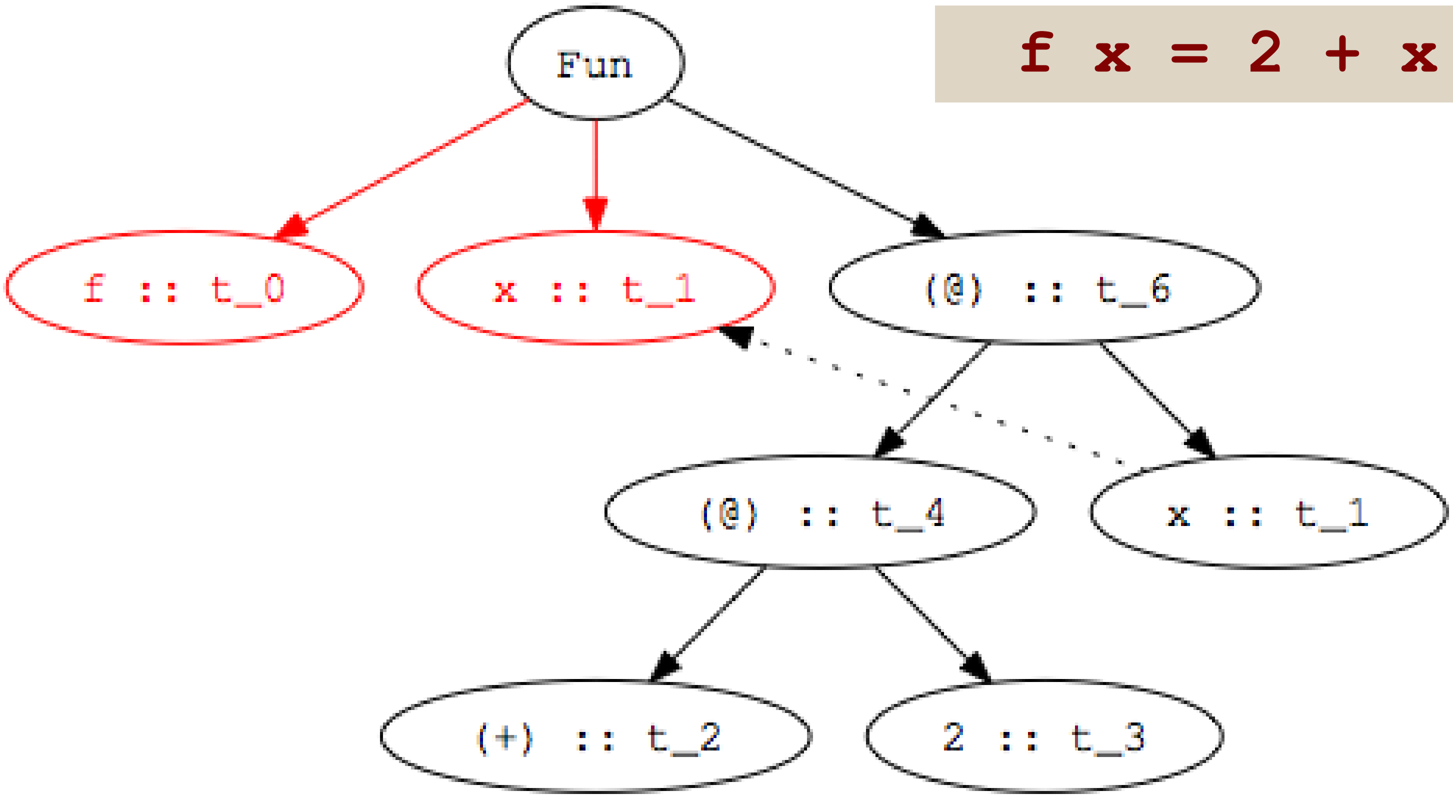
```
let f x = 2 + x
```



Infix operators are converted to Curied function application during parsing: (not necessary)

2 + x ➔ (+) 2 x

Step 2: Assign type variables to nodes

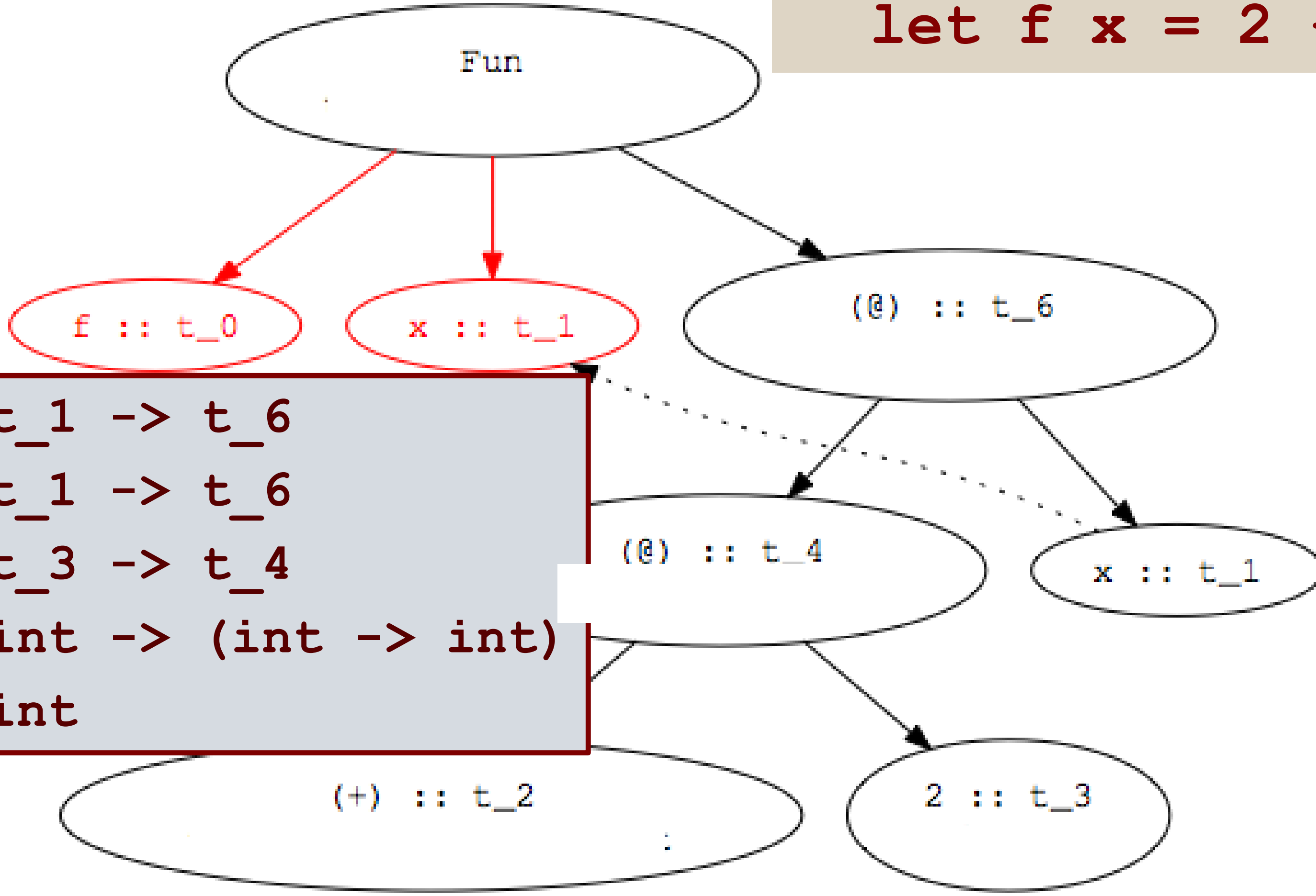



```
f x = 2 + x
```

Variables are given same type as binding occurrence

Step 3: Add Constraints

```
let f x = 2 + x
```




```
t_0 = t_1 -> t_6
t_4 = t_1 -> t_6
t_2 = t_3 -> t_4
t_2 = int -> (int -> int)
t_3 = int
```

Step 4: Solve Constraints

```
t_0 = t_1 -> t_6
t_4 = t_1 -> t_6
t_2 = t_3 -> t_4
t_2 = int -> (int -> int)
t_3 = int
```

```
t_3 -> t_4 = int -> (int -> int)
```

```
t_3 = int
t_4 = int -> int
```

```
t_0 = t_1 -> t_6
t_4 = t_1 -> t_6
t_4 = int -> int
t_2 = int -> (int -> int)
t_3 = int
```

```
t_1 -> t_6 = int -> int
```

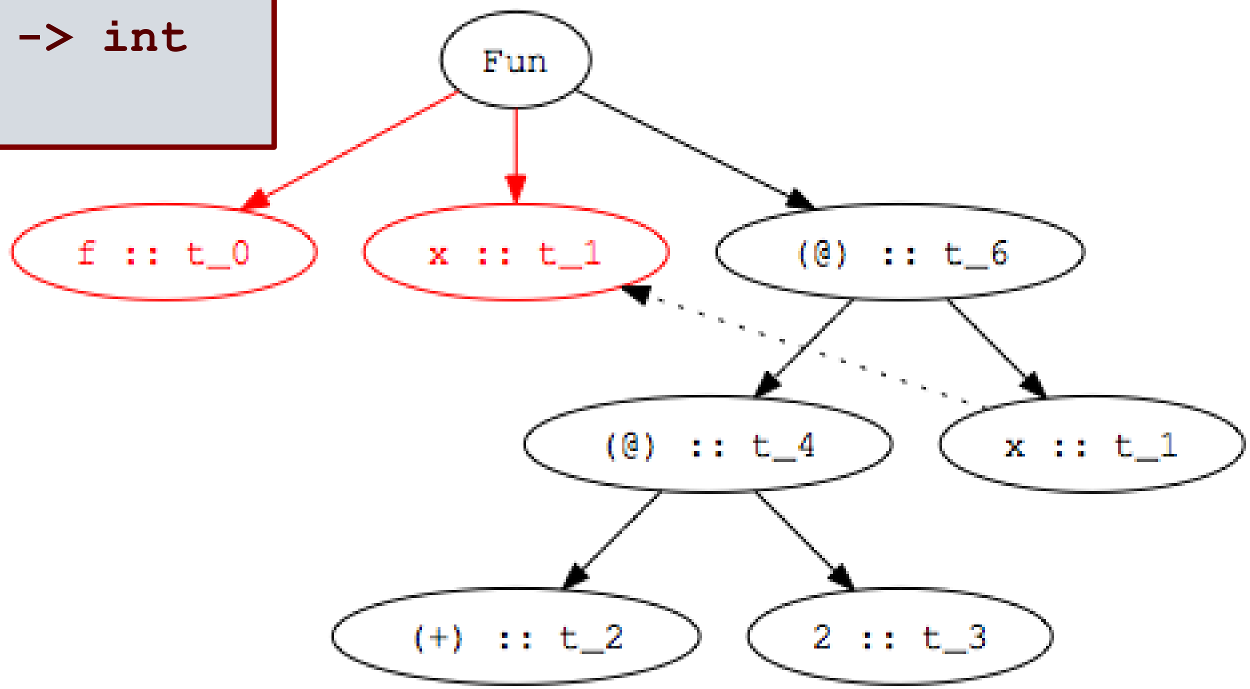
```
t_1 = int
t_6 = int
```

```
t_0 = int -> int
t_1 = int
t_6 = int
t_4 = int -> int
t_2 = int -> (int -> int)
t_3 = int
```

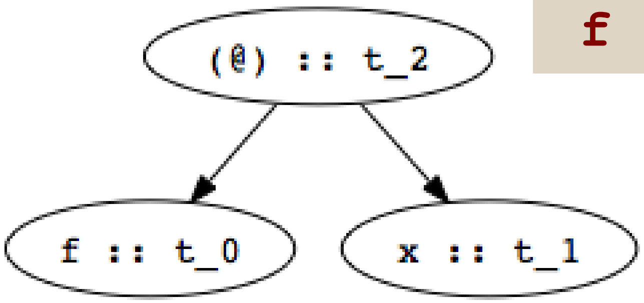
Step 5:
Determine type of declaration

```
t_0 = int -> int
t_1 = int
t_6 = int -> int
t_4 = int -> int
t_2 = int -> int -> int
t_3 = int
```

```
let f x = 2 + x
val f : int -> int =<fun>
```



Constraints from Application Nodes

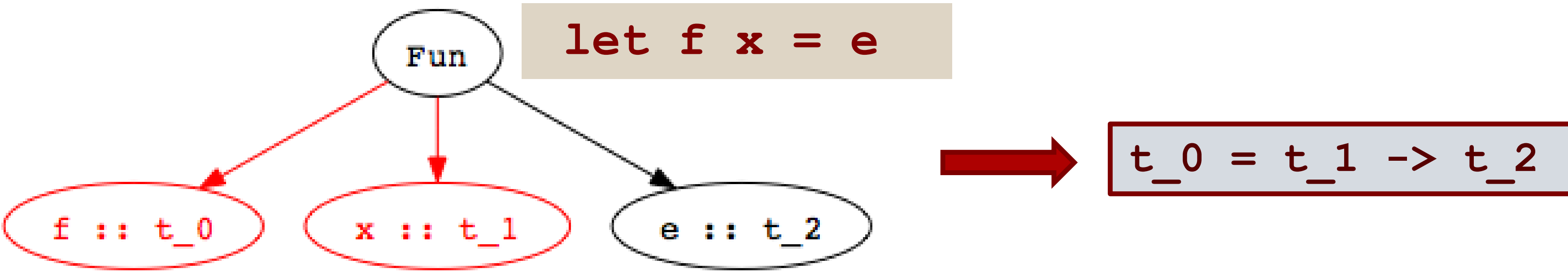


`f x`

`t_0 = t_1 -> t_2`

- Function application (apply f to x)
  - Type of f (t_0 in figure) must be domain $\rightarrow$ range
  - Domain of f must be type of argument x (t_1 in fig)
  - Range of f must be result of application (t_2 in fig)
  - Constraint: t_0 = t_1 -> t_2

Constraints from Abstractions



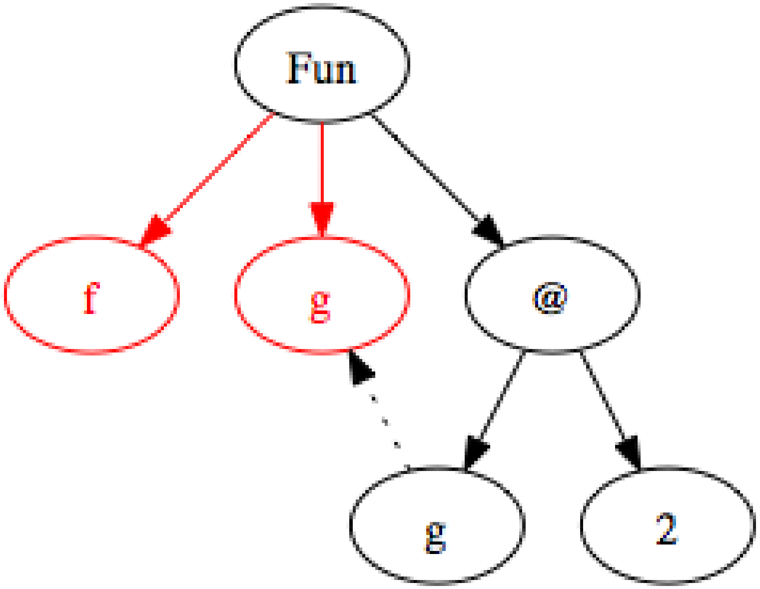
- Function declaration:
  - Type of f (t_0 in figure) must domain $\rightarrow$ range
  - Domain is type of abstracted variable x (t_1 in fig)
  - Range is type of function body e (t_2 in fig)
  - Constraint: t_0 = t_1 -> t_2

Inferring Polymorphic Types

- Example:

```
let f g = g 2
val f : (int -> t_4) -> t_4 = <fun>
```

- Step 1:
  Build Parse Tree

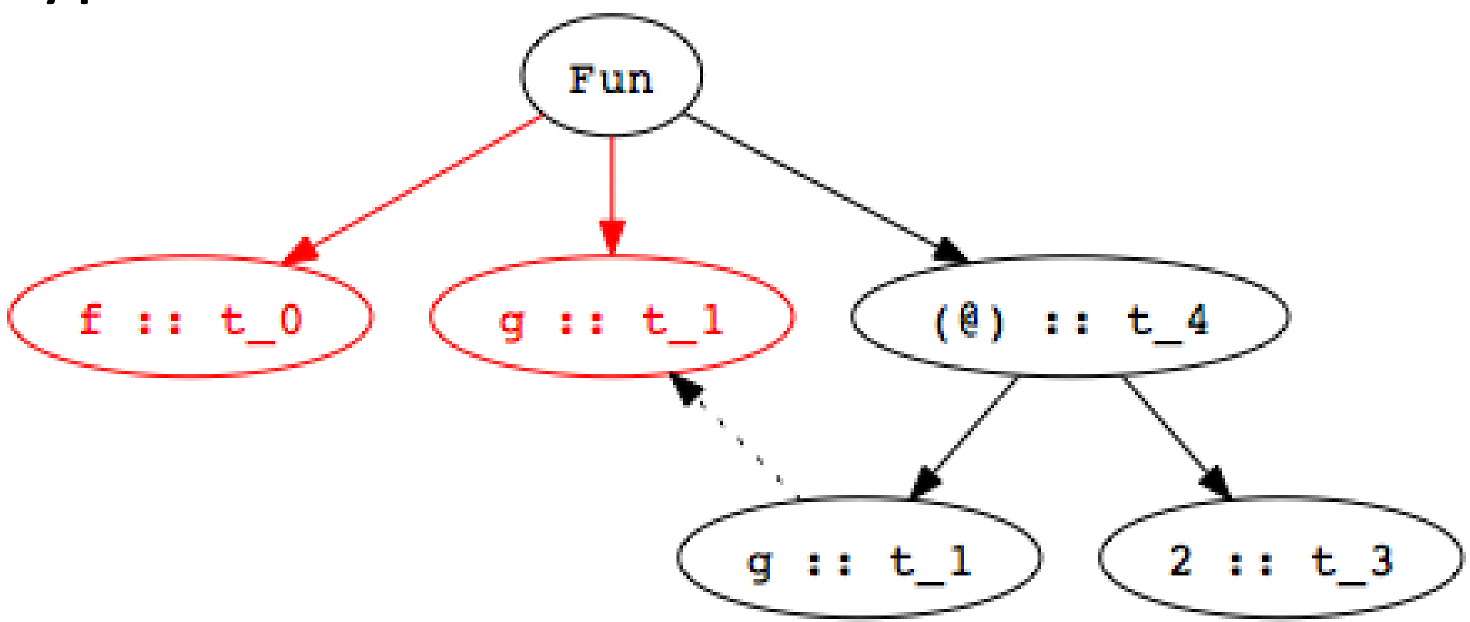


Inferring Polymorphic Types

- Example:

```
let f g = g 2
val f : (int -> t_4) -> t_4 = fun
```

- Step 2:
  Assign type variables



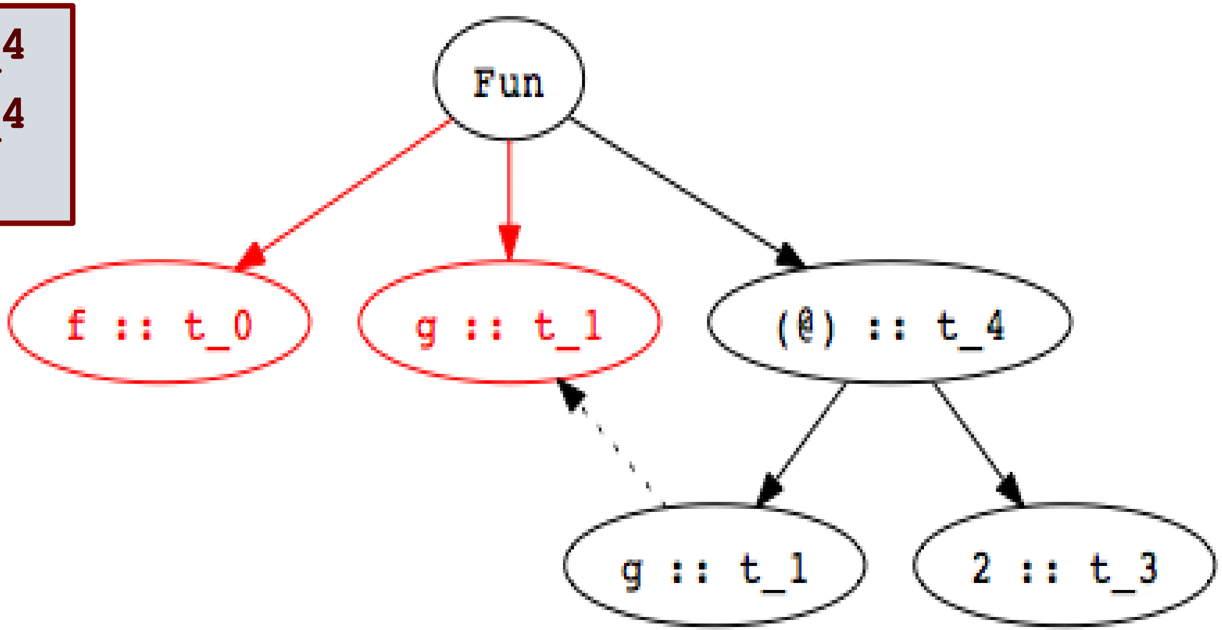
Inferring Polymorphic Types

- Example:

```
let f g = g 2
val f : (int -> t_4) -> t_4 = <fun>
```

- Step 3:
  Generate constraints

```
t_0 = t_1 -> t_4
t_1 = t_3 -> t_4
t_3 = int
```



Inferring Polymorphic Types

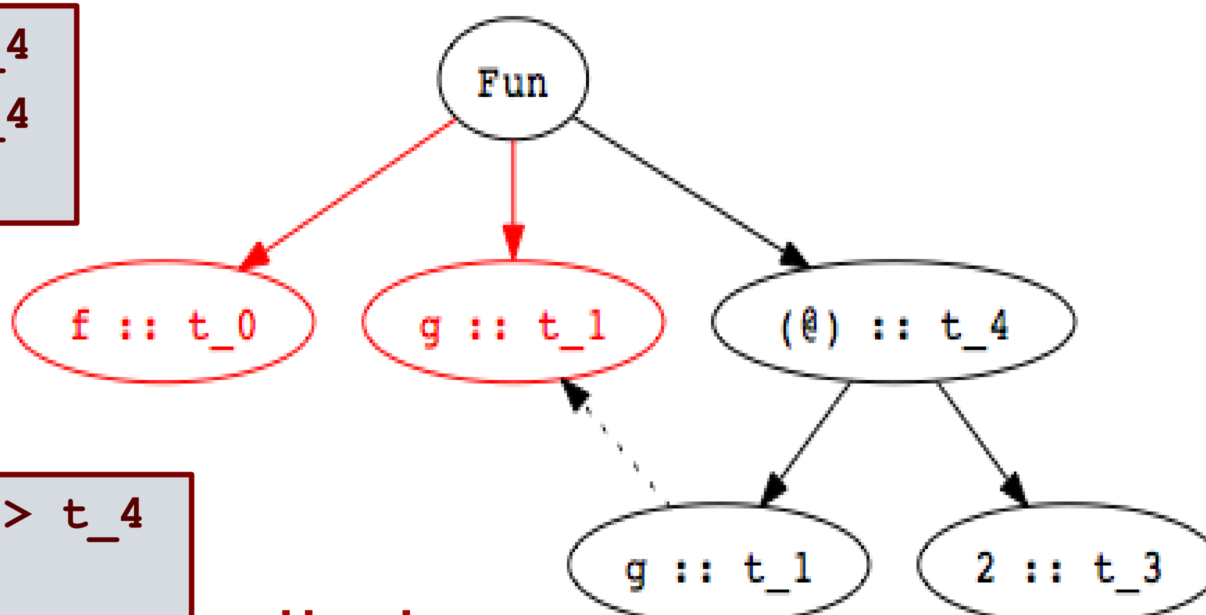
- Example:

```
let f g = g 2
val f : (int -> t_4) -> t_4 = <fun>
```

- Step 4:
  Solve constraints

```
t_0 = t_1 -> t_4
t_1 = t_3 -> t_4
t_3 = int
```

```
t_0 = (int -> t_4) -> t_4
t_1 =  int -> t_4
t_3 =  int
```



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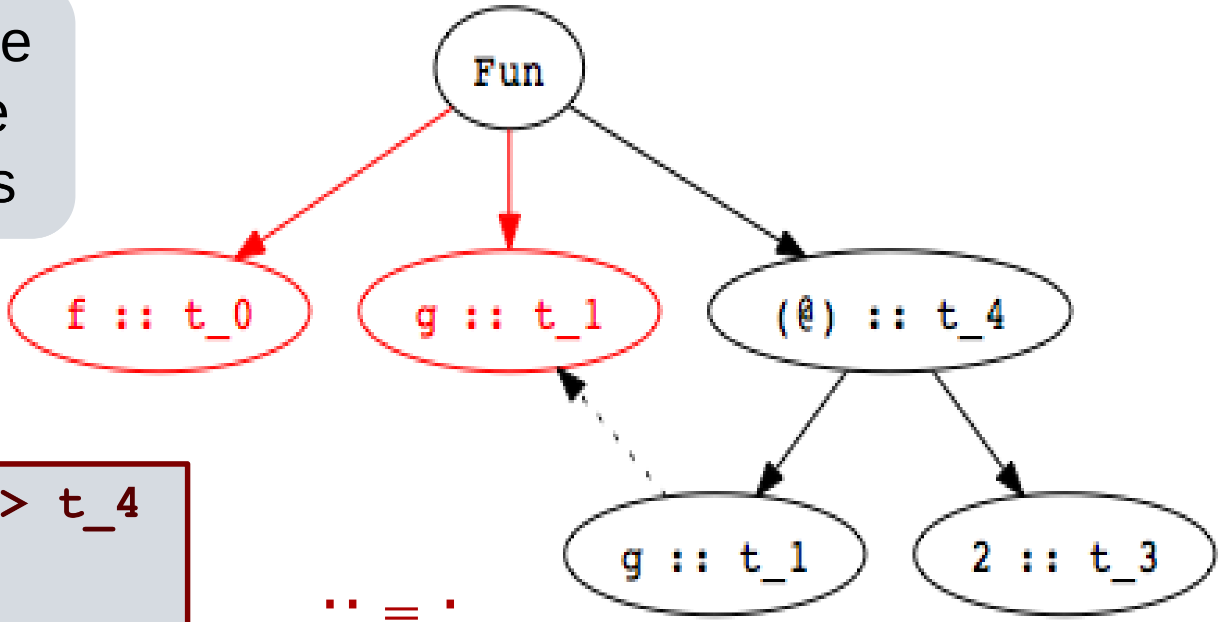
Inferring Polymorphic Types

- Example:

```
let f g = g 2
val f : (int -> t_4) -> t_4 = <fun>
```

- Step 5:
  Determine type of top-level declaration

Unconstrained type variables become polymorphic types




```
t_0 = (int -> t_4) -> t_4
t_1 =  int -> t_4
t_3 =  int
```

:: ≡ :

Using Polymorphic Functions

- Function:

```
let f g = g 2
val f : (int -> t_4) -> t_4 = <fun>
```

- Possible applications:

```
let add x = 2 + x
val add : int -> int = <fun>
f add
:- int = 4
```

```
let isEven x = mod (x, 2) == 0
val isEven: int -> bool = <fun>
f isEven
:- bool= true
```

Recognizing Type Errors

- Function:

```
let f g = g 2
val f : (int -> t_4) -> t_4 = <fun>
```

- Incorrect use

```
let not x = if x then true else false
val not : bool -> bool = <fun>
f not
> Error: operator and operand don't agree
  operator domain: int -> a
  operand:         bool-> bool
```

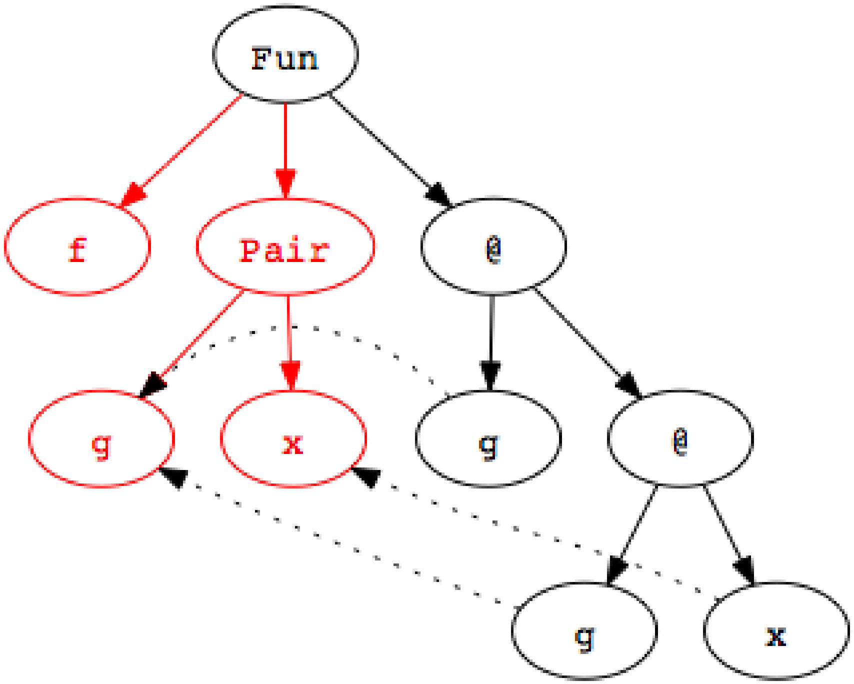
- Type error:
  cannot unify bool $\rightarrow$ bool and int $\rightarrow$ t

Another Example

- Example:

```
let f (g,x) = g (g x)
val f : ((t_8 -> t_8) * t_8) -> t_8
```

- Step 1:
  Build Parse Tree

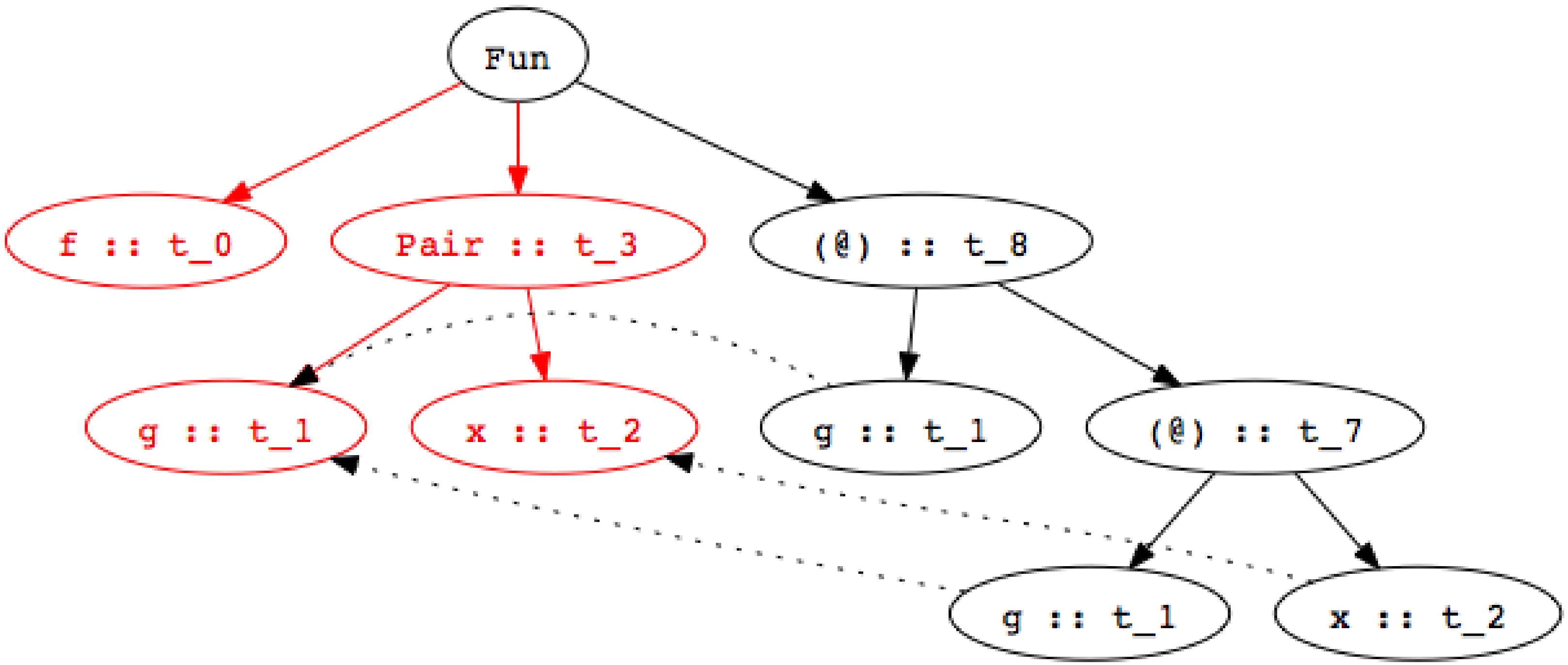


Another Example

- Example:

```
let f (g,x) = g (g x)
val f : ((t_8 -> t_8) * t_8) -> t_8
```

- Step 2:
  Assign type variables



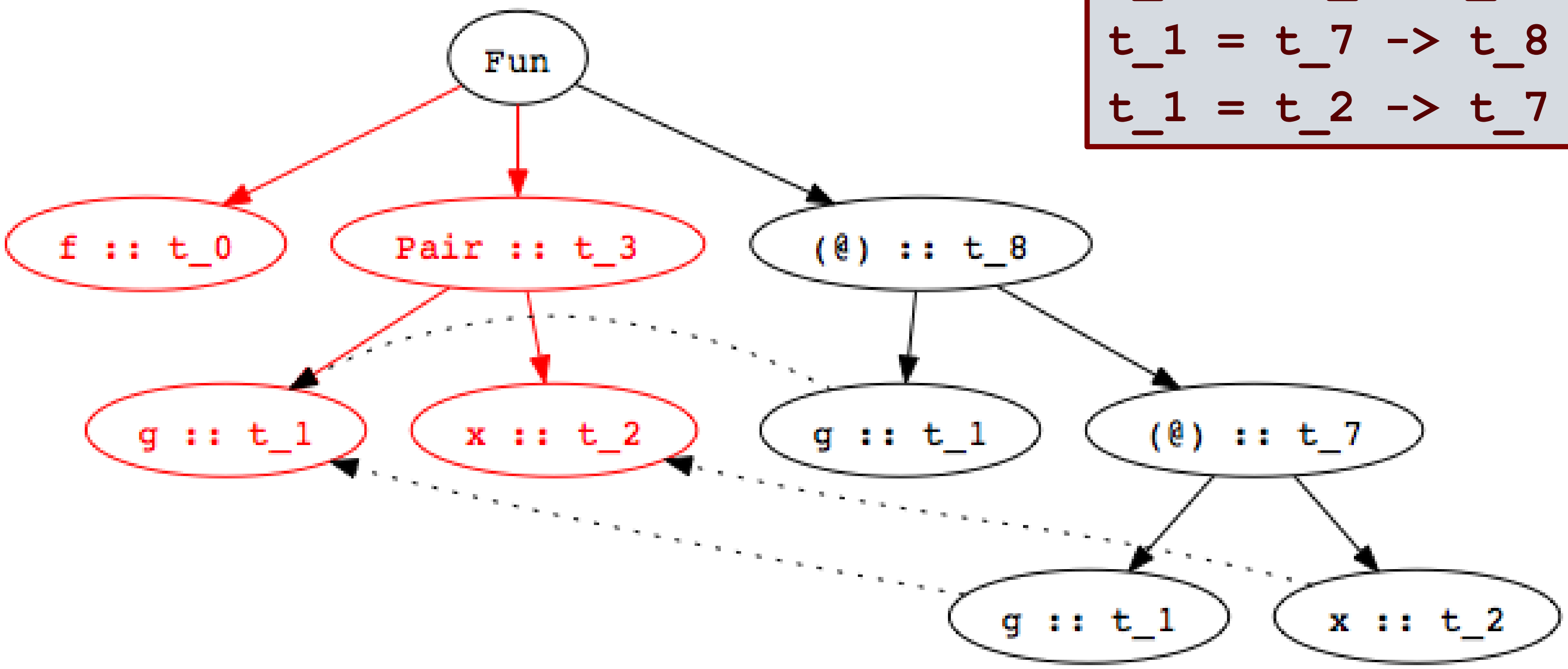
Another Example

- Example:

```
let f (g,x) = g (g x)
val f : ((t_8 -> t_8) * t_8) -> t_8
```

- Step 3:
  Generate constraints

```
t_0 = t_3 -> t_8
t_3 = (t_1, t_2)
t_1 = t_7 -> t_8
t_1 = t_2 -> t_7
```



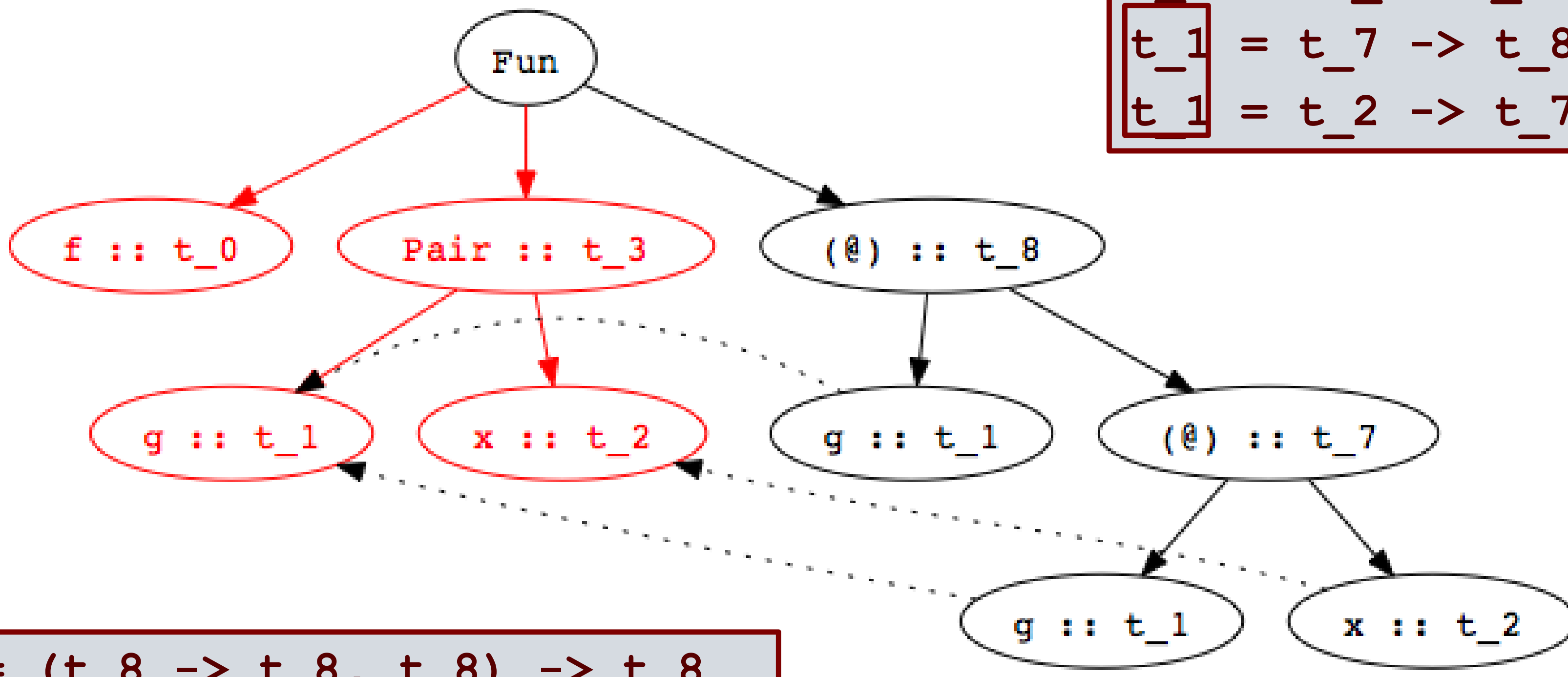
Another Example

- Example:

```
let f (g,x) = g (g x)
val f : ((t_8 -> t_8) * t_8) -> t_8
```

- Step 4:
  Solve constraints

```
t_0 = t_3 -> t_8
t_3 = (t_1, t_2)
t_1 = t_7 -> t_8
t_1 = t_2 -> t_7
```




```
t_0 = (t_8 -> t_8, t_8) -> t_8
```

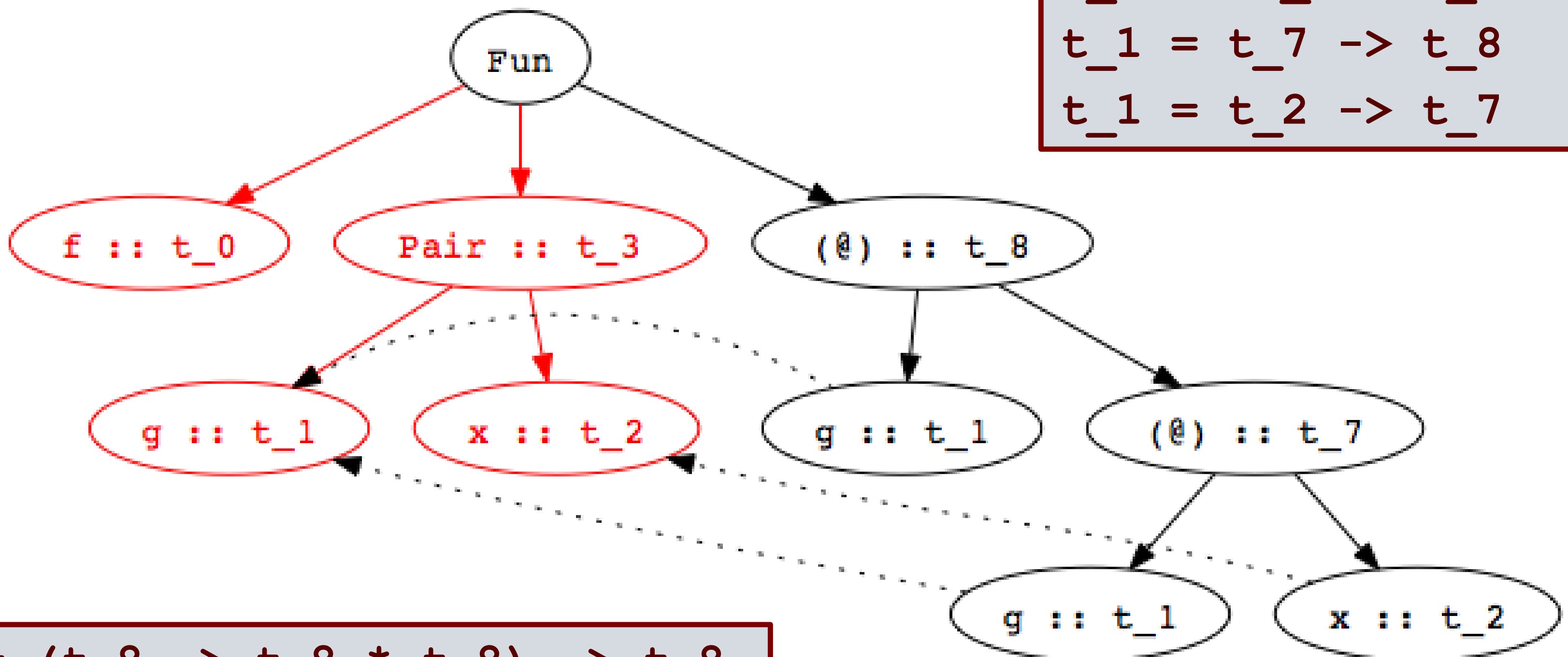
Another Example

- Example:

```
let f (g,x) = g (g x)
val f : ((t_8 -> t_8) * t_8) -> t_8
```

- Step 5:
  Determine type of f

```
t_0 = t_3 -> t_8
t_3 = (t_1 * t_2)
t_1 = t_7 -> t_8
t_1 = t_2 -> t_7
```




```
t_0 = (t_8 -> t_8 * t_8) -> t_8
```

Pattern Matching

- Matching with multiple cases

```
let isempty l = match l with
    |[] -> true
    | _ -> false
```

- Infer type of each case
  - First case:

```
[t_1] -> bool
```

  - Second case:

```
t_2 -> bool
```

- Combine by unification of the types of the cases

```
val isempty : [t_1] -> bool = <fun>
```

Bad Pattern Matching

- Matching with multiple cases

```
let isempty l = match l with
     |[] -> true
     | _ -> 0
```

- Infer type of each case
  - First case:

```
[t_1] -> bool
```

  - Second case:

```
t_2 -> int
```

- Combine by unification of the types of the cases

```
Type Error: cannot unify bool and int
```

Recursion

```
let rec concat a b = match a with
    | [] -> b
    | x::xs -> x :: concat xs b
```

- To handle recursion, introduce type variables for the function:

```
concat : t_1 -> t_2 -> t_3
```

- Use these types to conclude the type of the body:
  - Pattern matching first case:

```
[t_4] -> t_5 -> t_5
unify [t_4] with t_1,t_5 with t_2,
                     t_5 with t_3
t_1 =[t_4] and t_2 = t_3 = t_5
```

  - Pattern matching second case:

```
[t_6] -> t_7 -> [t_6]
unify [t_6] with t_1, t_7 with t_2,
                      [t_6] with t_3
```

```
unify [t_6] with t_1, t_7 with t_2,
 t_3  with [t_6]
```

Recursion

```
let rec concat a b = match a with
     | [] -> b
     | x::xs -> x :: concat xs b
```

- To handle recursion, introduce type variables for the function:

```
concat : t_1 -> t_2 -> t_3
```

- Conclude the type of the function:

```
val concat : [t_4] -> [t_4] -> [t_4] = <fun>
```

Most General Type

- Type inference produces the *most general type*

```
let rec map f arg = function
    [] -> []
   | hd :: tl -> f hd :: (map f tl)

val map : ('a -> 'b) -> 'a list -> 'b list = <fun>
```

- Functions may have many less general types

```
val map : (t_1  -> int, [t_1])  -> [int]
val map : (bool -> t_2, [bool]) -> [t_2]
val map : (char -> int, [cChar]) -> [int]
```

- Less general types are all instances of most general type, also called the *principal type*

Information from Type Inference

- Consider this function…

```
let reverse ls = match ls with
   [] -> []
   | x :: xs -> reverse xs
```

… and its most general type:

```
:- reverse :: list 't_1 -> list 't_2
```

- What does this type mean?

Reversing a list should not change its type, so there must be an error in the definition of reverse!

Complexity of Type Inference Algorithm

- When Hindley/Milner type inference algorithm was developed, its complexity was unknown
- In 1989, Kanellakis, Mairson, and Mitchell proved that the problem was exponential-time complete
- Usually linear in practice though…
  - Running time is exponential in the depth of polymorphic declarations

Type Inference: Key Points

- Type inference computes the types of expressions
  - Does not require type declarations for variables
  - Finds the most general type by solving constraints
  - Leads to polymorphism
- Sometimes better error detection than type checking
  - Type may indicate a programming error even if no type error
- Some costs
  - More difficult to identify program line that causes error
  - Natural implementation requires uniform representation sizes
  - Complications regarding assignment took years to work out
- Idea can be applied to other program properties
  - Discover properties of program using same kind of analysis

Parametric Polymorphism: OCaml vs C++

- OCaml polymorphic function
  – Declarations (generally) require no type information
  – Type inference uses type variables to type expressions
  – Type inference substitutes for type variables as needed to instantiate polymorphic code
- C++ function template
  – Programmer must declare the argument and result types of functions
  – Programmers must use explicit type parameters to express polymorphism
  – Function application: type checker does instantiation

Example: Swap Two Values

- OCaml

```
let swap (x, y) =
 let temp = !x in
   (x := !y; y := temp)
val swap : 'a ref * 'a ref -> unit = <fun>
```

- C++

```
template <typename T>
void swap(T& x, T& y){
      T tmp = x;  x=y;  y=tmp;
}
```

Declarations both swap two values polymorphically, but they are compiled very differently

Implementation

- OCaml
  - `swap` is compiled into one function
  - Typechecker determines how function can be used
- C++
  - `swap` is compiled differently for each instance (details beyond scope of this course …)
- Why the difference?
  - OCaml ref cell is passed by pointer. The local `x` is a pointer to value on heap, so its size is constant
  - C++ arguments passed by reference (pointer), but local `x` is on the stack, so its size depends on the type

Polymorphism vs Overloading

- Parametric polymorphism
  - Single algorithm may be given many types
  - Type variable may be replaced by any type
  - if `f:t→t` then `f:int→int`, `f:bool→bool`, ...
- Overloading
  - A single symbol may refer to more than one algorithm
  - Each algorithm may have different type
  - Choice of algorithm determined by type context
  - Types of symbol may be arbitrarily different
  - In ML, `+` has types `int*int→int, real*real→real`, no others
  - Haskel permits more general overloading and requires user assistance

Varieties of Polymorphism

- Parametric polymorphism A single piece of code is typed generically
  - Imperative or first-class polymorphism
  - ML-style or let-polymorphism
- Ad-hoc polymorphism The same expression exhibit different behaviors when viewed in different types
  - Overloading
  - Multi-method dispatch
  - intentional polymorphism
- Subtype polymorphism A single term may have many types using the rule of subsumption allowing to selectively forget information

Summary

- Types are important in modern languages
  - Program organization and documentation
  - Prevent program errors
  - Provide important information to compiler
- Type inference
  - Determine best type for an expression, based on known information about symbols in the expression
- Polymorphism
  - Single algorithm (function) can have many types