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Solution of the heat equation with variable properties by two-step Adomian decomposition method

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Abstract

Adomian Decomposition Method [G. Adomian, R. Rach, Equality of partial solutions in the decomposition method for linear or nonlinear partial differential equations, Appl. Math. Comput. 19 (1990) 9–12; G. Adomian, R. Rach, Analytical solution of non linear boundary value problem in several dimension by decomposition, J. Appl. Math. 174 (1993) 118–137; G. Adomian, R. Rach, Modified Adomian polynomials, Math. Comput. Modelling 24 (1996) 39–46] is useful to find the solution of linear and nonlinear equations. There is a renewed interest in the method [A.M. Wazwaz, A reliable modification of Adomian decomposition method, Appl. Math. Comput. 102 (1999) 77–86; A.M. Wazwaz, The modified decomposition method for analytic treatment of differential equations, Appl. Math. Comput. 173 (2006) 165–176; X.G. Luo, A two-step Adomian decomposition method, Appl. Math. Comput. 170 (2005) 570–583; B.Q. Zhang, X.G. Luo, Q.B. Wu, Experimentation with two-step Adomian decomposition method to solve evolution models, Appl. Math. Comput. 175 (2006) 1495–1502; B.Q. Zhang, X.G. Luo, Q.B. Wu, Revisit on partial solutions in the Adomian decomposition method: Solving heat and wave equations, J. Math. Anal. Appl. 321 (2006) 353–363; B.Q. Zhang, X.G. Luo, Q.B. Wu, The restrictions and improvement of the Adomian decomposition method: Solving heat and wave equations, Appl. Math. Comput. 177 (1999) 99–104; Necdet Bildik, Hatice Bayramoglu, The solution of two dimensional nonlinear differential equation by the Adomian decomposition method, Appl. Math. Comput. 163 (1999) 551–567] and a lot of research is being conducted using this method. We attempt to enlarge the scope of its application by presenting the solution of the diffusion equation with variable properties. In this, we present two problems dealing with the heat conduction with variable properties. The compression of the first problem with eigenfunction expansion is also made. The two analytical solution agree exactly with each other. Although, the two methods arrive at the same result, nevertheless, it is of much value to obtain the solution by ADM which provides a powerful method of finding the solution of both linear and nonlinear problems. To apply this method, we have shown that generalized Fourier series is required to build up the solution instead of trigonometric Fourier series.

Keywords: Heat equation; Adomian decomposition method; Eigenfunction expansion method

1. Introduction

Recently Zhang et al. [9] suggested a modification to ADM to solve a diffusion problem with zero initial condition and nonzero homogeneous boundary conditions. The modification was required, since the standard method [1–3]

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provides the trivial solution. In such situation essentially, the nonhomogeneous b.c. are transformed to homogeneous b.c. and the governing equation becomes nonhomogeneous. As a consequence the forcing function that appears in the equation is expressed in terms of trigonometric Fourier series and the nontrivial solution is obtained which agrees with the exact solution.

We propose to solve the diffusion equation with variable physical properties such as density and conductivity by ADM [4–9]. We observe that the method requires an extension to the idea, in the sense that the corresponding eigenfunctions of the related eigenvalue problem need to be used instead of trigonometric Fourier series. Here we present two problems using this concept of ADM [9,10] and content to show an agreement with the second problem with exact solution using eigenfunction expansion method. This extends the usefulness of ADM to a new class of problems concerning variable physical properties. The usefulness of the problem is extended to the problem which were hitherto not solvable by the usual ADM in which the trigonometric Fourier series is used.

2. Problem I

Consider the heat equation with variable properties

$$c\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right), \quad 1 < x < \mathrm{e}^{\pi},\ t > 0 \tag{1}$$

where the thermal coefficients ρ and k are taken as functions of x.

Specifically let us take $\rho(x) = \frac{1}{x}$ and $k(x) = x$. Then (1) takes the form

$$\frac{c}{x}\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right), \quad 1 < x < \mathrm{e}^{\pi},\ t > 0. \tag{2}$$

Consider the following boundary and initial conditions

$$\begin{aligned} &u(1,t) = 0, \quad t \geq 0,\\ &u\left(\mathrm{e}^{\pi}, t\right) = kt, \quad t \geq 0,\\ &u(x,0) = 0, \quad 1 \leq x \leq \mathrm{e}^{\pi}. \end{aligned}$$

We propose the following solution

$$u(x,t) = V(x,t) + W(x,t) \tag{3}$$

where $W(x,t)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}W}{\mathrm{d}x}\right) = 0 \tag{4}$$

with nonhomogeneous boundary conditions given by

$$\begin{aligned} &W(1,t) = 0, \quad t \geq 0,\\ &W\left(\mathrm{e}^{\pi}, t\right) = kt, \quad t \geq 0. \end{aligned} \tag{5}$$

The solution of Eq. (4) satisfying the conditions (5) is given by

$$W(x,t) = \frac{kt}{\pi}\ln x. \tag{6}$$

Using (3) in (2), we get

$$\frac{c}{x}\left[\frac{\partial V}{\partial t} + \frac{\partial W}{\partial t}\right] = \frac{\partial}{\partial x}\left(x\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial x}\left(x\frac{\partial W}{\partial x}\right). \tag{7}$$

Eqs. (6) and (7) together give

$$c\frac{\partial V}{\partial t} = x\frac{\partial}{\partial x}\left(x\frac{\partial V}{\partial x}\right) - \frac{ck}{\pi}\ln x \tag{8}$$

with the corresponding boundary and initial conditions

$$V(1,t)=0, \quad t\geq 0,$$
$$V\left(e^{\pi},t\right)=0, \quad t\geq 0,$$
$$V(x,0)=0, \quad 1\leq x\leq e^{\pi}.$$

We use the modified ADM [6–9] to obtain the solution of Eq. (5).

In an operator form, see [1–10] (8) can be written as

$$L_t V = L_x V + cH(x)$$

where

$$L_t = c\frac{\partial}{\partial t}, \qquad L_x = x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right) \qquad \text{and} \qquad H(x)=-\frac{k}{\pi}\ln x, \qquad L_t^{-1}=\frac{1}{c}\int_0^t ()\,\mathrm{d}t.$$

We express $H(x)$ as generalized Fourier series in terms of the eigenfunctions of the S-L boundary value problem. In this case, we cannot express the source function $H(x)$ in terms of the trigonometric Fourier series and thus we suggest generalized Fourier series of the eigenfunctions of the following S-L boundary value problem;

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)+\frac{\lambda}{x}\phi=0$$
$$\phi(1)=0$$
$$\phi\left(e^{\pi}\right)=0.$$

The corresponding eigenvalues and eigenfunctions are

$$\lambda_n = n^2$$
$$\phi_n(x)=\sin(n\ln x).$$

Thus, expressing $H(x)$ in terms of $\phi_n(x)$, we have

$$H(x)=\sum_{n=1}^{\infty}h_n\sin(n\ln x), \qquad h_n=\frac{\int_1^{e^{\pi}}H(x)\frac{\sin(n\ln x)}{x}\mathrm{d}x}{\int_1^{e^{\pi}}\frac{\sin^2(n\ln x)}{x}\mathrm{d}x}.$$

The standard ADM [1–3] requires that

$$V=\sum_{m=0}^{\infty}V_m. \tag{9}$$

The zeroth component is obtained as

$$L_t^{-1}\left[L_t V_0\right]=cL_t^{-1}H(x).$$

This gives

$$V_0 = V(x,0)+cL_t^{-1}H(x)$$
$$V_0 = t\sum_{n=1}^{\infty}h_n\sin(n\ln x).$$

The mth component is given by

$$V_m=\left[L_t^{-1}L_x\right]^m V_0, \quad m\geq 0.$$

Since V_0 is known

$$V_m = L_t^{-1}V_{m-1}=\frac{t^{m+1}}{(m+1)!}\sum_{n=1}^{\infty}\left(\frac{-n^2}{c}\right)^m h_n\sin(n\ln x).$$

The solution V is now expressed through Eq. (9) as

$$V = \sum_{n=1}^{\infty} h_n \sin(n \ln x) \left[\frac{1 - \mathrm{e}^{\frac{-n^2}{c} t}}{\frac{n^2}{c}} \right].$$

The solution of the partial differential equation (1) is finally given by

$$u(x,t) = \frac{kt}{\pi} \ln x + \sum_{n=1}^{\infty} h_n \sin(n \ln x) \left[\frac{1 - \mathrm{e}^{\frac{-n^2}{c} t}}{\frac{n^2}{c}} \right].$$

3. Problem II

In this we consider the heat equation with variable properties,

$$c\rho(x) \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(k(x) \frac{\partial u}{\partial x} \right), \quad -1 < x < 1, \ t > 0 \tag{10}$$

where $\rho(x) = \frac{1}{\sqrt{1-x^2}}$ and $k(x) = \sqrt{1-x^2}$. Thus (10) takes the form

$$\frac{c}{\sqrt{1-x^2}} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(\sqrt{1-x^2} \frac{\partial u}{\partial x} \right), \quad -1 < x < 1, \ t > 0 \tag{11}$$

with the boundary and initial conditions given by

$$\begin{aligned} &u(-1,t) = 0, \quad t \geq 0, \\ &u(1,t) = kt, \quad t \geq 0, \\ &u(x,0) = 0, \quad -1 \leq x \leq 1. \end{aligned}$$

First, we let

$$u(x,t) = V(x,t) + W(x,t) \tag{12}$$

where $W(x,t)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\sqrt{1-x^2} \frac{\mathrm{d}W}{\mathrm{d}x} \right) = 0 \tag{13}$$

with nonhomogeneous boundary conditions

$$\begin{aligned} &W(-1,t) = 0, \quad t \geq 0, \\ &W(1,t) = kt, \quad t \geq 0. \end{aligned} \tag{14}$$

The Eq. (13) can be readily solved to give

$$W(x,t) = -\frac{kt}{\pi} \sin^{-1} x + \frac{3kt}{2}. \tag{15}$$

Using (12) in (11) gives

$$\frac{c}{\sqrt{1-x^2}} \left[\frac{\partial V}{\partial t} + \frac{\partial W}{\partial t} \right] = \frac{\partial}{\partial x} \left(\sqrt{1-x^2} \frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial x} \left(\sqrt{1-x^2} \frac{\partial W}{\partial x} \right). \tag{16}$$

Eqs. (15) and (16) together give

$$c \frac{\partial V}{\partial t} = \sqrt{1-x^2} \frac{\partial}{\partial x} \left(\sqrt{1-x^2} \frac{\partial V}{\partial x} \right) + c \left(\frac{k}{\pi} \sin^{-1} x - \frac{3k}{2} \right) \tag{17}$$

with the corresponding boundary and initial conditions

$$V(-1,t) = 0, \quad t \ge 0,$$
$$V(1,t) = 0, \quad t \ge 0,$$
$$V(x,0) = 0, \quad -1 \le x \le 1.$$

In an operator form, (17) can be written as

$$L_t V = L_x V + cH(x)$$

where

$$L_t = c\frac{\partial}{\partial t}, \qquad L_x = \sqrt{1-x^2}\frac{\partial}{\partial x}\left(\sqrt{1-x^2}\frac{\partial}{\partial x}\right) \quad \text{and} \quad H(x) = \frac{k}{\pi}\sin^{-1}x - \frac{3k}{2}, \qquad L_t^{-1} = \frac{1}{c}\int_0^t ()\,\mathrm{d}t.$$

We express $H(x)$ as generalized Fourier series in terms of the eigenfunctions of the S-L boundary value problem.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{1-x^2}\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) + \frac{\lambda}{\sqrt{1-x^2}}\phi = 0$$
$$\phi(-1) = 0$$
$$\phi(1) = 0.$$

The eigenvalues and eigenfunctions are given by

$$\lambda_n = n^2$$
$$\phi_n(x) = \sin(n\cos^{-1}x).$$

Thus, expressing $H(x)$ in terms of $\phi_n(x)$, we have

$$H(x) = \sum_{n=1}^{\infty} h_n \sin(n\cos^{-1}x), \qquad h_n = \frac{\int_{-1}^{1} H(x)\frac{\sin(n\cos^{-1}x)}{\sqrt{1-x^2}}\mathrm{d}x}{\int_{-1}^{1}\frac{\sin^2(n\cos^{-1}x)}{\sqrt{1-x^2}}\mathrm{d}x}.$$

The standard ADM [1–3] requires that

$$V = \sum_{m=0}^{\infty} V_m. \tag{18}$$

The *zeroth* component is obtained as

$$L_t^{-1}[L_t V_0] = cL_t^{-1}H(x).$$

This gives

$$V_0 = V(x,0) + cL_t^{-1}H(x)$$
$$V_0 = t\sum_{n=1}^{\infty} h_n \sin(n\cos^{-1}x).$$

The mth component is given by

$$V_m = \left[L_t^{-1}L_x\right]^m V_0, \quad m \ge 0.$$

Since V_0 is known, we can calculate the mth component

$$V_m = L_t^{-1}V_{m-1} = \frac{t^{m+1}}{(m+1)!}\sum_{n=1}^{\infty}\left(\frac{-n^2}{c}\right)^m h_n \sin(n\cos^{-1}x).$$

The solution V is now expressed through Eq. (9) as

$$V = \sum_{n=1}^{\infty} h_n \sin(n \cos^{-1} x) \left[\frac{1 - \mathrm{e}^{\frac{-n^2}{c} t}}{\frac{n^2}{c}} \right]$$

and the solution of the partial differential equation (10) is finally given by

$$u(x,t) = -\frac{kt}{\pi} \sin^{-1} x + \frac{3kt}{2} + \sum_{n=1}^{\infty} h_n \sin(n \cos^{-1} x) \left[\frac{1 - \mathrm{e}^{\frac{-n^2}{c} t}}{\frac{n^2}{c}} \right].$$

For the sake of completeness, we will prefer to obtain a solution of one problem, say problem I, by eigenfunction expansion method and show the exact agreement with the two solutions.

4. Eigenfunction expansion method

In this section we are solving problem I by Eigenfunction expansion method.

Consider

$$c\rho(x) \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(k(x) \frac{\partial u}{\partial x} \right), \quad 1 < x < \mathrm{e}^{\pi},\ t > 0 \tag{19}$$

with

$$\begin{aligned} &u(1,t) = 0, \quad t \geq 0, \\ &u\left(\mathrm{e}^{\pi}, t\right) = kt, \quad t \geq 0, \\ &u(x,0) = 0, \quad 1 \leq x \leq \mathrm{e}^{\pi}. \end{aligned}$$

First we let

$$u(x,t) = V(x,t) + W(x,t), \tag{20}$$

where $W(x,t)$ can be determined from

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(x \frac{\mathrm{d}W}{\mathrm{d}x} \right) = 0 \tag{21}$$

subject to the nonhomogeneous boundary conditions

$$\begin{aligned} &W(1,t) = 0, \quad t \geq 0, \\ &W\left(\mathrm{e}^{\pi}, t\right) = kt, \quad t \geq 0. \end{aligned} \tag{22}$$

The solution of (21) satisfying the conditions (22) is given by

$$W(x,t) = \frac{kt}{\pi} \ln x. \tag{23}$$

Eqs. (20) and (23) give

$$\frac{c}{x} \frac{\partial V}{\partial t} = \frac{\partial}{\partial x} \left(x \frac{\partial V}{\partial x} \right) + \frac{c}{x} H(x), \quad H(x) = -\frac{k}{\pi} \ln x \tag{24}$$

with the corresponding boundary and initial conditions

$$\begin{aligned} &V(1,t) = 0, \quad t \geq 0, \\ &V\left(\mathrm{e}^{\pi}, t\right) = 0, \quad t \geq 0, \\ &V(x,0) = 0, \quad 1 \leq x \leq \mathrm{e}^{\pi}. \end{aligned}$$

The eigenfunctions of the related homogeneous problem are given by

$$\frac{d}{dx}\left(x\frac{d\phi}{dx}\right)+\frac{\lambda}{x}\phi=0 \qquad (25)$$
$$\phi(1)=0$$
$$\phi(e^{\pi})=0.$$

The Eq. (25) give

$$\phi_n(x)=\sin(n\ln x)$$
$$\lambda_n=n^2.$$

Let us write

$$V(x,t)=\sum_{n=1}^{\infty}a_n(t)\sin(n\ln x). \qquad (26)$$

Using Eqs. (25) and (26) in (24) gives

$$\frac{c}{x}\sum_{n=1}^{\infty}\frac{d}{dt}a_n(t)\,\phi_n(x)=\sum_{n=1}^{\infty}a_n(t)\frac{d}{dx}\left(x\frac{d\phi_n(x)}{dx}\right)+\frac{c}{x}H(x)$$
$$\sum_{n=1}^{\infty}\left(\frac{d}{dt}a_n(t)+\frac{n^2}{c}a_n(t)\right)\frac{\sin(n\ln x)}{x}=\frac{1}{x}H(x)$$
$$\frac{d}{dt}a_n(t)+\frac{n^2}{c}a_n(t)=h_n,\quad \text{where } h_n=\frac{\int_1^{e^{\pi}}H(x)\frac{\sin(n\ln x)}{x}dx}{\int_1^{e^{\pi}}\frac{\sin^2(n\ln x)}{x}dx}. \qquad (27)$$

The solution of (27) is

$$a_n(t)=a_n(0)e^{\frac{-n^2}{c}t}+h_n\left(\frac{1-e^{\frac{-n^2}{c}t}}{\frac{n^2}{c}}\right)$$

where $a_n(0)$ is expressed as

$$a_n(0)=\frac{\int_1^{e^{\pi}}V(x,0)\frac{\sin(n\ln x)}{x}dx}{\int_1^{e^{\pi}}\frac{\sin^2(n\ln x)}{x}dx}=0.$$

Thus

$$a_n(t)=h_n\left(\frac{1-e^{\frac{-n^2}{c}t}}{\frac{n^2}{c}}\right).$$

This gives

$$V(x,t)=\sum_{n=1}^{\infty}h_n\sin(n\ln x)\left(\frac{1-e^{\frac{-n^2}{c}t}}{\frac{n^2}{c}}\right)$$

and

$$u(x,t)=\frac{kt}{\pi}\ln x+\sum_{n=1}^{\infty}h_n\sin(n\ln x)\left[\frac{1-e^{\frac{-n^2}{c}t}}{\frac{n^2}{c}}\right].$$

5. Conclusion

In this paper, we solve two problems of heat conduction with variable properties by using TSADM [6–9]. The problem I is solved by eigenfunction expansion method as well to show the complete agreement with the exact solution.

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