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WHICH ROOT DOES THE BISECTION ALGORITHM FIND?*

GEORGE CORLISS†

A student of elementary probability may be amused by an application of probability to numerical analysis.

Let f be a continuous function on the closed interval $[a, b]$ such that $f(a)f(b)<0$. Then f has at least one root α on (a, b). The well-known bisection algorithm [1, p. 28] generates a sequence $\{[a_k, b_k]\}$ of intervals on which a root is known to lie. Let $a_0=a$, $b_0=b$, and define $c_k=(a_k+b_k)/2$. If $f(c_k)=0$, $c_k=\alpha$, and the algorithm terminates. If $f(a_k)f(c_k)<0$, $\alpha\in(a_k, c_k)$, so let $[a_{k+1}, b_{k+1}]=[a_k, c_k]$. For the other possible case, if $f(a_k)f(c_k)>0$, $\alpha\in(c_k, b_k)$, so let $[a_{k+1}, b_{k+1}]=[c_k, b_k]$. $|b_k-a_k|=2^{-k}|b_0-a_0|$, so that the bisection algorithm is guaranteed to converge to some root of f on $[a, b]$.

If f has more than one root on $[a, b]$, a problem in [1, p. 35] asks which root the bisection algorithm usually locates. If f has n distinct, simple roots $x_1<x_2<\cdots<x_n$ on $[a, b]$ ($f(x_i)=0$ and $f'(x_i)\neq 0$), then it is well-known that the bisection algorithm finds the even numbered roots with probability zero. This paper shows that the probability of finding the odd numbered roots is uniform.

Let C_n denote the class of continuous functions which satisfy $f(a)f(b)<0$ with exactly n distinct, simple roots on (a, b). Let the roots of $f\in C_n$ be denoted by $x_1<x_2<\cdots<x_n$. We assume that the locations of the roots are independent and distributed according to a uniform random distribution on $[a, b]$. Let $x_0=a$ and $x_{n+1}=b$. Let $P_{i,n}$ denote the probability that the bisection algorithm converges to the ith root of f, given that $f\in C_n$. Let $Q_{i,n}$ denote the probability that

$$x_i<c_0=(a_0+b_0)/2<x_{i+1},$$

for $i=0, 1, \cdots, n$, given that $f\in C_n$.

We first note that n is odd and that $P_{i,n}=0$ for all even i since at each step, the bisection algorithm discards the subinterval of length $(b_k-a_k)/2$ which contains an even number of roots. Hence for even i, x_i will be found if and only if $c_k=(a_k+b_k)/2=x_i$ at some step of the algorithm.

THEOREM. *For n odd,*

$$P_{i,n}=\begin{cases}\dfrac{2}{n+1} & \text{for } i \text{ odd},\\[2ex] 0 & \text{for } i \text{ even}.\end{cases}$$

If i is even, $P_{i,n}=0$ was shown above. The proof for i odd is by induction on n. $P_{1,1}=1$, for if f has only one root on $[a, b]$, the bisection algorithm is guaranteed to find it.

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† Department of Mathematics and Statistics, University of Nebraska, Lincoln, Nebraska 68588.

Assume that

$$P_{i,m} = \begin{cases} \dfrac{2}{m+1} & \text{for } i \text{ odd,} \\ 0 & \text{for } i \text{ even,} \end{cases}$$

for all odd $m < n$.

$Q_{i,n} = \binom{n}{i}/2^n$ since it is equal to the probability that i of n trials (roots) land on the left half of $[a, b]$.

If $x_j < c_0 = (a_0 + b_0)/2 < x_{j+1}$ for some fixed even j, then the probability of finding x_i is given by

$$\begin{cases} P_{i-j,n-j} & \text{for } i > j, \\ 0 & \text{for } i \leqq j, \end{cases}$$

since the bisection algorithm will proceed on the interval $[a_1, b_1] = [c_0, b_0]$, which contains $n - j$ roots of f.

Similarly, if $x_j < c_0 < x_{j+1}$ for some fixed odd j, then the probability of finding x_i is given by

$$\begin{cases} 0 & \text{for } i > j, \\ P_{i,j} & \text{for } i \leqq j. \end{cases}$$

Then

$$\begin{aligned} P_{i,n} &= \sum_{j=0}^{n} P(\text{finding } x_i \text{ given } x_j < c_0 < x_{j+1}) \\ &= \sum_{\substack{j=0 \\ j \text{ even}}}^{i-1} Q_{j,n} P_{i-j,n-j} + \sum_{\substack{j=i \\ j \text{ odd}}}^{n} Q_{j,n} P_{i,j} \\ &= Q_{0,n} P_{i,n} + Q_{n,n} P_{i,n} + \sum_{\substack{j=2 \\ j \text{ even}}}^{i-1} Q_{j,n} P_{i-j,n-j} \\ &\quad + \sum_{\substack{j=i \\ j \text{ odd}}}^{n-2} Q_{j,n} P_{i,j}, \end{aligned}$$

where we assume that $\sum_2^0 = \sum_n^{n-2} = 0$. Hence,

$$\begin{aligned} P_{i,n} &= \frac{1}{1 - Q_{0,n} - Q_{n,n}} \left[\sum_{\substack{j=2 \\ j \text{ even}}}^{i-1} Q_{j,n} P_{i-j,n-j} + \sum_{\substack{j=i \\ j \text{ odd}}}^{n-2} Q_{j,n} P_{i,j} \right] \\ &= \frac{1}{2^n - 2} \left[\sum_{\substack{j=2 \\ j \text{ even}}}^{i-1} \binom{n}{j} \frac{2}{n-j+1} + \sum_{\substack{j=i \\ j \text{ odd}}}^{n-2} \binom{n}{j} \frac{2}{j+1} \right] \\ &= \frac{1}{2^n - 2} \left[\sum_{\substack{j=2 \\ j \text{ even}}}^{n-1} \binom{n+1}{j} \right] \frac{2}{n+1} \\ &= \frac{2}{n+1} \frac{1}{2^n - 2} \sum_{\substack{j=2 \\ j \text{ even}}}^{n-1} \left[\binom{n}{j-1} + \binom{n}{j} \right] = \frac{2}{n+1}. \end{aligned}$$

We have shown that if a function f has n distinct, simple roots, the bisection algorithm is equally likely to find each odd root.

REFERENCE

S. D. CONTE AND CARL DE BOOR, *Elementary Numerical Analysis: An Algorithmic Approach*, 2nd ed., McGraw-Hill, New York, 1965.