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Steven A. Feller and Joseph E. Kasper

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How good are the common approximations used in physics?

Steven A. Feller and Joseph E. Kasper
Physics Department, Coe College, Cedar Rapids, Iowa 52402

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The qualities (relative errors) of several of the common approximations used in undergraduate physics are studied. Some novel features of the approximations are discussed. In some cases it is found that the approximations are substantially better than is usually thought to be the case.

I. INTRODUCTION

The use of approximate mathematical expressions is important in physics courses. It arises in estimating numerical values, as in "About how much is $e^{0.2}$?" Although the numerical evaluation of expressions is easy with a pocket calculator, the importance of making quick and dirty estimates remains with us.[1] Another kind of use of approximate expressions is illustrated by the well-known case in which $\sin\theta$ is replaced by θ in the treatment of the simple pendulum. In both cases the question of how good the approximation is will often occur to a thoughtful student.[2]

The purpose of this paper is to present in convenient form the relative error $|f_{\text{true}} - f_{\text{approx}}|/|f_{\text{true}}|$, made in replacing some function (f_{true}) by an approximate function (f_{approx}) for some common formulas, over ranges of the parameters that are involved. As will be seen, there are some curious features in the results, which seem not to be generally appreciated. We have found no similar discussion in the journals devoted to physics teaching.

II. SOME APPROXIMATIONS BASED ON TAYLOR SERIES EXPANSIONS

Table I shows results calculated numerically for the approximations $\cos x \cong 1 - \frac{1}{2}x^2$, $\sin x \cong x$, $\tan x \cong x$, and $e^x \cong 1 + x$. Table I shows that the cosine approximation is remarkably good over a remarkably large range of angles. A relative error of as much as 1% does not occur until the angle becomes even as large as $\frac{2}{3}$ rad, or about 38°. Even a 10% error does not occur until the angle gets as large as about 1.05 rad, or about 60°!

The Taylor Series expansions for cosine, sine, and tangent functions are well behaved. The cosine expansion is a rapidly convergent sum of even powers of x while the sine and tangent series are rapidly convergent sums of odd powers of x. The sine and tangent approximations are linear in x, with the first neglected term on the order of x^3, and so it is not surprising that the associated errors are greater than for the cosine approximation, which goes as x^2 with the first neglected term on the order of x^4. Nevertheless, these approximations are substantially better over a wider range of angles x than users of the approximations are probably aware of, by and large. For example, if the horizontal distance H from a flagpole to an observer on the ground at a point P is known, and the tangent of the angle subtended P is approximated by the angle itself (in rad), the consequent error in the deduced height of the flagpole is only about 5% at about 0.39 rad (22°), and only about 10% even when the angle is about 0.54 rad (31°).

These three trigonometric approximations result from truncation of Taylor Series expansions. Another much-used approximation with the same origin is $e^x \cong 1 + x$. It is interesting to note that this approximation is better for $x > 0$ than for $x < 0$.

Table I. Values of X for selected relative % errors: (a) $\cos X \cong 1 - \frac{1}{2}X^2$; (b) $\sin X \cong X$; (c) $\tan X \cong X$; (d) $e^X \cong 1 + X$.

(a) X (rad)	(b) X (rad)	(c) X (rad)	(d) X		Approximate relative % errors
0.00	0.00	0.00	0.00	0.00	0%
0.66	0.24	0.17	+0.15	−0.14	1%
0.93	0.54	0.39	+0.36	−0.29	5%
1.05	0.75	0.54	+0.53	−0.39	10%

III. BINOMIAL APPROXIMATION

Another useful approximation is $(1+x)^n \cong 1 + nx$. This can be used to handle the more general expression $(a+b)^n$ by writing the latter as $(a^n)(1+x)^n$, where $x = b/a$. Table II presents some calculated results for this approximation.

It appears that it is usual to use this approximation with confidence only when x is much smaller than 1 in magnitude. However, Table II shows that this restriction can be considerably relaxed. For example, the case of the square root of $(1+x)$, in which $n = 0.5$, clearly shows that x can be in the neighborhood of 1, or even greater than 1, with smallish errors which may well be tolerable.

The relative errors for the binomial approximation are somewhat curious. The approximation becomes exact for $n = 0$ or $n = 1$. Furthermore, one sees in Table II that the relative errors are not symmetric on reflection of x. The

Table II. Representative values of X and n for several relative % errors in the binomial approximation $(1+X)^n \cong 1 + nX$.

	n					Approximate relative % errors
	−1.0	−0.5	+0.5	+1.5	+2.0	
X	+0.10	+0.17	+0.32	+0.19	+0.11	1%
	−0.10	−0.16	−0.25	−0.14	−0.09	
	+0.23	+0.39	+0.88	+0.52	+0.29	5%
	−0.22	−0.34	−0.47	−0.28	−0.18	
	+0.32	+0.56	+1.43	+0.88	+0.46	10%
	−0.32	−0.47	−0.59	−0.36	−0.24	

nature of the asymmetry is such that the approximation is better for positive values of x than for negative values. Indeed, the approximation will, in general, not work for values of x less than -1. It is to be observed that as n gets large, the value of x needed to have a reasonable approximation becomes small.

There is included in these data some information about the function $(1 - v^2/c^2)^{-1/2}$, which is of substantial interest in special relativity theory. The binomial approximation for this expression is $(1 + \frac{1}{2}v^2/c^2)$. As an example, consider Table II in the case in which $n = -0.5$ and x is negative (as it must now be). This reveals that a 10% error occurs when v^2/c^2 is about 0.47. This implies that $v = 0.686c$, a speed that is within the region of interest in relativity. It is the case that students find the expression $(1 - v^2/c^2)^{-1/2}$ awkward, while the expression $(1 + v^2/2c^2)$ is much simpler.

IV. STIRLING APPROXIMATION

A remarkable approximation which is much used because of the intractable nature of factorials in general and the factorials of large numbers in particular, is the Stirling approximation $n! \cong \sqrt{(2\pi n)}\ n^n e^{-n}$. Even for n as small as unity, the approximation is only about 8% off, while for n greater than only about eight the error amounts to not more than 1%. For large values of n, the approximation is extremely good, as is well known. It is notoriously difficult to carry out numerical evaluations of the factorials of large values of n, but we can say that when $n = 1000$, the approximation to $n!$ is within 0.01% of the true value.[3]

Sometimes, and especially in statistical mechanics, the approximation $\ln(n!) \cong \ln(n) - n$ is used. For small values of n this approximation is not nearly as good as the one stated above, but yet even for $n = 10$ (for example) the approximation is only about 14% in error. For $n = 50$ the error is about 2%, while for $n = 1000$ the error is less than 0.1%.

[1] N. D. Mermin, Am. J. Phys. **46**, 101 (1978). This paper deals with logarithms, but contains interesting commentary about approximations and numerical work.

[2] It should be explained to the student that the question of the error made in using an approximate expression to evaluate another expression is not the same as the question of the error in the end result of a chain of reasoning in which an approximation is made earlier in the chain. Thus replacing the equation $\ddot{x} = -\sin(x)$ by the equation $\ddot{x} = -x$ involves an error *at this stage* which can be evaluated by the data given in this paper, but comparison of the solution $x = x(t)$ that follows from the approximate equation with that which follows from the exact equation is another matter.

[3] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1970), p. 276.

An investigation of CAI teaching methods in an electronics course

Kenneth W. Wood[a)]
Department of Physics, University of Utah, Salt Lake City, Utah 84112

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Computers are increasingly being used in the classroom. An investigation of several educational techniques in a computer-based version of an electronics course is reported. We found that, with a lesson for teaching virtual equality, students learned faster when using a general to specific approach. Students using a simulation of a Schmitt trigger before a qualitative analysis of the circuit performed the analysis faster and with less difficulty than the group performing the analysis first and then exploring the circuit with the simulation. Given a sizable amount of optional material in a computer lesson, most of the electronics students used all of the optional material.

I. INTRODUCTION

In a recent editorial,[1] Robert G. Fuller addresses the problem of improving physics teaching. Fuller points out that many physicists still treat teaching as an art rather than as a science. As David Hestenes pointed out in 1979,[2] an ample foundation for the science of teaching exists, but that few university professors have involved themselves in educational research. Fuller says: "The decreasing cost and increasing power of personal computers are already beginning to have some influence in the physics classroom." Because of the computer, educational research is both practical and essential.

Research is essential because, if computers are going to influence our teaching, then we must learn how best to use the computer before we are deluged with a proliferation of questionable lesson material. In turn, the computer itself makes such research practical for several reasons. First, the power of authoring systems for creating quality courseware also allows the educator to create more than one version of the material. Thus one can create several versions of a lesson and experimentally determine the educational implications of each version. Second, the computer can record data that would be difficult to obtain otherwise. These data includes question number, path number, time spent at the question, and student performance on the