JOURNAL OF FUNCTIONAL ANALYSIS 5, 218–235 (1970)

A Priori Estimates for the Wave Equation and Some Applications

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Communicated by Irving Segal

Received March 17, 1969

1. INTRODUCTION

We consider the wave operator

$$\Box u(x, t) = -\frac{\partial^2 u}{\partial t^2} + \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} = -u_{tt} + \Delta_x u \tag{1}$$

in n space variables $x \in E^n$, $n \geqslant 2$ and time t, $-\infty < t < +\infty$. In a previous work [15], we proved that the solution to the Cauchy problem

$$\Box u = 0 \tag{2}$$

$$u(x, 0) = f(x), \qquad u_t(x, 0) = g(x) \tag{3}$$

for sufficiently regular Cauchy data satisfied *a priori* estimates of the form

$$\| u(\cdot, t)\|_q \leqslant M(t) \left[\| g \|_p + \sum_{j=1}^{n} \left\| \frac{\partial f}{\partial x_j} \right\|_p\right] \qquad t \neq 0 \tag{4}$$

for certain values of p and q with $p \leqslant 2 \leqslant q$.

In this paper we use (4) to prove

$$\| u \|_q \leqslant M \| \Box u \|_p \tag{5}$$

(space-time integrals) for all sufficiently well behaved u with $u(x, 0) = u_t(x, 0) = 0$ for $p = 2(n + 1)/(n + 3)$, $q = 2(n + 1)/(n - 1)$.

We apply (5) to obtain existence and uniqueness theorems for solutions of

$$\square u = H(u, x, t) \tag{6}$$

with given Cauchy data for a wide class of functions H, including $V(x, t)\, u$ for $V \in L^{(n+1)/2}_{\text{loc}}(E^{n+1})$ and u^3 for $n = 3$.

In Section 2 we prove the basic estimate (5) and some variants. In Section 3 we define the spaces of distributions in which the Cauchy data is assumed to lie. We note that estimate (4) is inadequate for many purposes since, by a result of Littman [7], the assumptions $u_t(x, 0) \in L^p$, and $u_{x_j}(x, 0) \in L^p$ do not persist in time.

In Section 4 we apply the estimates to obtain existence and uniqueness theorems for the Cauchy problem. In Section 5 we obtain some results on the wave operators of $\square u = H(u)$ relative to $\square u = 0$; i.e., solutions of the first equation which are asymptotic to a given solution of the second as $t \to \infty$. The author is grateful to Professor Segal for having stimulated his interest in these problems. In Section 6 we conclude with some remarks on extending the techniques of this paper to other equations.

In the sequel we always assume $p = 2(n + 1)/(n + 3)$ and $q = 2(n + 1)/(n - 1)$. The results are equally valid whether we consider complex-valued functions or restrict ourselves to real-valued functions.

2. The Inhomogeneous Problem with Zero Cauchy Data

Let $\Omega = \{x \in E^n : |x - x_0| < T\}$ be a ball in space, and let $\Gamma = \{(x, t) : |x - x_0| < T - |t|, |t| < T\}$ be the union of the forward and backward cones with base Ω. We allow $T = \infty$, in which case Γ is all of space-time, and Ω is all of space. For a nonnegative integer k and p, $1 < p < \infty$ we define $L_k^p(\Gamma)$ to be the space of functions in $L^p(\Gamma)$ having distribution derivatives of order $\leqslant k$ in $L^p(\Gamma)$. Since Γ satisfies the hypotheses of Calderón's extension theorem [3] every function in $L_k^p(\Gamma)$ may be defined outside Γ in such a way that it is in $L_k^p(E^{n+1})$.

With small modifications we could consider more general regions Ω, and using the results of [1], [16], we could consider fractional Sobolev spaces $L_\alpha^p(\Gamma)$ for real $\alpha \geqslant 0$. We will not do so, however, in order not to complicate the arguments with unnecessary details.

In this section we consider solutions of

$$\square u = w \quad \text{in} \quad \Gamma \tag{7}$$

$$u(\cdot, 0) = u_t(\cdot, 0) = 0 \quad \text{in} \quad \Omega \tag{8}$$

in the distribution sense for $w \in L_k^p(\Gamma)$. The meaning of (7) is clear, but (8) must be interpreted as follows: if V is u set equal to zero for $t < 0$, then $\Box V$ is w set equal to zero for $t < 0$.

THEOREM 1. *Given $w \in L_k^p(\Gamma)$ there exists a unique solution $u \in L_k^q(\Gamma)$ of* (7) *and* (8), *where $p = 2(n+1)/(n+3)$, $q = 2(n+1)/(n-1)$, and*

$$\| u : L_k^q(\Gamma)\| \leqslant M \| w : L_k^p(\Gamma)\| \tag{9}$$

for some constant M.

Proof. First let $w \in C^\infty_{\text{com}}(E^{n+1})$. Then we can write down an explicit formula for the solution by means of the Fourier transform. Let $\hat{w}(\xi, t)$ and $\hat{u}(\xi, t)$ denote the Fourier transforms of w and u in the space variables for fixed time t. Then it is well-known [2] and easy to check that

$$\hat{u}(\xi, t) = \int_0^t \frac{\sin(t-s)|\xi|}{|\xi|} \hat{w}(\xi, s)\, ds \tag{10}$$

is a solution of (7) and (8), in the classical sense, in all of E^n. In [15] we proved the estimate

$$\left\| \mathscr{F}^{-1}\left[\frac{\sin|\xi|}{|\xi|}\hat{f}(\xi)\right]\right\|_q \leqslant M_1 \| f \|_p . \tag{11}$$

It follows from homogeneity considerations that we have

$$\left\| \mathscr{F}^{-1}\left[\frac{\sin t|\xi|}{|\xi|}\hat{f}(\xi)\right]\right\|_q \leqslant M_1 | t |^{1+n((1/q)-(1/p))} \| f \|_p , \tag{12}$$

for

$$\mathscr{F}^{-1}\left[\frac{\sin t|\xi|}{|\xi|}\hat{f}(\xi)\right] = t\,\delta(t^{-1})\left(\mathscr{F}^{-1}\left\{\frac{\sin|\xi|}{|\xi|}[\delta(t)f]\hat{\ }(\xi)\right\}\right)$$

where $\delta(t)$ is the dilation operator $\delta(t)f(x) = f(tx)$, and $\| \delta(t) f\|_p = | t |^{-n/p} \| f \|_p$.

Applying (12) to formula (10) we obtain

$$\| u(\cdot, t)\|_q \leqslant M_1 \int_0^t | t - s |^{1+n((1/q)-(1/p))} \| w(\cdot, s)\|_p\, ds. \tag{13}$$

Now we observe that for the particular values of p and q,

$$1 + n\left(\frac{1}{q} - \frac{1}{p}\right) = -\frac{n-1}{n+1} = -1 + \frac{1}{p} - \frac{1}{q}.$$

Thus the fractional integration theorem [5] says that convolution with $s^{1+n[(1/q)-(1/p)]}$ is a bounded operator from L^p to L^q, hence we obtain the global space-time estimate

$$\| u \|_q \leqslant M \| w \|_p . \tag{14}$$

Next we note that formula (10) still is meaningful for $w \in L^p(E^{n+1})$ and gives a weak solution of (7) and (8), which again satisfies (14). We obtain the local estimate (9) for $k = 0$ by extending w to be zero outside Γ and applying the global estimate (14). The uniqueness is well-known (see [4], Theorem 5.6.3) and depends on the hyperbolic property of $\Box$.

To establish (9) for $k > 0$ it suffices to dominate

$$\| \partial^k u/\partial x_j^k : L^q(\Gamma) \| \qquad \text{and} \qquad \| \partial^k u/\partial t^k : L^q(\Gamma) \| \quad \text{by} \quad \| w : L_k{}^p(\Gamma) \|.$$

The space derivatives are easy, since

$$\Box\, \partial^k u/\partial x_j^k = \partial^k w/\partial x_j^k \in L^p(\Gamma) \qquad \text{and} \qquad \partial^k u/\partial x_j^k$$

have zero Cauchy data, so we may apply (9) with $k = 0$. The time derivative no longer satisfies the initial conditions (8), however, so we need a special argument. Note that since $\partial^2 u/\partial t^2 = \Delta_x u - w$, we can express any even time derivative $\partial^k u/\partial t^k$ as a linear combination of space derivatives of u of order k and derivatives of w of order $\leqslant k - 2$. By Sobolev's inequality these derivatives of w are in $L^q(\Gamma)$ because $(1/p) - (1/q) = 2/(n + 1)$. This gives the desired estimate for $\partial^k u/\partial t^k$ for k even. If k is odd, the above argument shows $\partial^{k-1} u/\partial t^{k-1} = D_x u + v$ for $v \in L_1{}^p(\Gamma)$ and D_x is a differential operator of order $k - 1$ in the space variables. Now $\partial v/\partial t \in L^q(\Gamma)$ so it suffices to show $(\partial/\partial t)\, D_x u \in L^q(\Gamma)$. We have $\Box D_x u = D_x w \in L_1{}^q(\Gamma)$ and we extend $D_x w$ outside Γ to be in $L_1{}^q(E^{n+1})$. Then, since $D_x u$ has zero Cauchy data, (10) applies to give, for some extension u_0 of u to E^{n+1},

$$D_x u_0\hat{\ }(\xi, t) = \int_0^t \frac{\sin(t-s)|\,\xi\,|}{|\,\xi\,|}\, D_x w\hat{\ }(\xi, s)\, ds.$$

If we differentiate with respect to t and let

$$\hat{w}_0(\xi, t) = |\,\xi\,| D_x w\hat{\ }(\xi, t) \tag{15}$$

we get

$$\frac{\partial}{\partial t} D_x u_0\hat{\ }(\xi, t) = \int_0^t \frac{\cos(t-s)|\,\xi\,|}{|\,\xi\,|}\, \hat{w}_0(\xi, s)\, ds. \tag{16}$$

Now it follows from (15) that $w_0 \in L^q$, so we may repeat the argument going from (10) to (14) on (16) noting that the analog of (11) with cos replacing sin was also proved in [15]. We obtain

$$\|(\partial/\partial t)\, D_x u : L^q(\Gamma)\| \leqslant M\| w_0 : L^q(\Gamma)\| \tag{17}$$

which completes the proof.

Remarks. The constant M by which $\| w : L_k{}^p(\Gamma)\|$ dominates u and all space derivatives in $L^q(\Gamma)$ does not depend on Γ. The same is not true of time derivatives, however, as the example $u(x, t) = t^2$ shows.

The choice of p and q in the theorem deserves some comment. We proved in [15] that estimate (11) holds for a variety of choices of p and q. Here we have chosen those values of p and q which maximize the difference $(1/p) - (1/q)$ at $2/(n+1)$. It is a fortunate coincidence that this value is just right for the application of the fractional integration theorem to (13). We could, of course, use the other estimates of [15] to prove variants of the theorem if we were willing to introduce mixed L^p norms, i.e., taking first an L^p norm in space and then an L^q norm in time with $p \neq q$.

If $u \in L_2{}^p(\Gamma)$, then $\Box u \in L^p(\Gamma)$ and $u \in L^q(\Gamma)$ by Sobolev's inequality. Using the fact that Sobolev's inequality is sharp, we can easily construct examples to show that the value of q in the theorem cannot be increased.

Finally we obtain easily, by translation, the following

COROLLARY. *For all $u \in C^\infty_{\mathrm{com}}(E^{n+1})$ we have*

$$\| u : L_k{}^q(E^{n+1})\| \leqslant M\| \Box u : L_k{}^p(E^{n+1})\|. \tag{18}$$

Comparing this with the L^2 estimates in Hörmander [4], we have lost control of the first derivatives of u, but we have gained local integrability and we have a bound M independent of the support of u.

3. The Cauchy Data

We return to the study of nonzero Cauchy data. In contrast to the estimate (4), we seek conditions which will persist in time. We deal first with the global situation.

DEFINITION. We define $F_k(E^n)$, for a nonnegative integer k, to be the space of tempered distributions f such that the distributions u given by

$$\hat{u}(\xi, t) = \hat{f}(\xi) \cos t |\xi| \quad \text{is in} \quad L_k^q(E^{n+1}). \tag{19}$$

We define the norm $\| f : F_k(E^n) \|$ to be equal to $\| u : L_k^q(E^{n+1}) \|$.

We define $G_k(E^n)$ similarly, with

$$\hat{u}(\xi, t) = f(\xi) |\xi|^{-1} \sin t |\xi| \tag{20}$$

in place of (19).

Note that (19) and (20) make sense because $\cos t |\xi|$ and $|\xi|^{-1} \sin t |\xi|$ are bounded and C^∞.

LEMMA 1. *Let $u(x, t) \in L_k^q(E^{n+1})$ and satisfy $\square u = 0$. Then there exist $f \in F_k(E^n)$ and $g \in G_k(E^n)$ such that*

$$\hat{u}(\xi, t) = \hat{f}(\xi) \cos t |\xi| + \hat{g}(\xi) |\xi|^{-1} \sin t |\xi|. \tag{21}$$

Proof. Let

$$u_0(x, t) = \tfrac{1}{2}[u(x, t) + u(x, -t)] \qquad \text{and} \qquad u_1(x, t) = \tfrac{1}{2}[u(x, t) - u(x, -t)].$$

Then both u_0 and u_1 satisfy the same hypotheses that u does. Now if $k \geqslant 2$, then both $u_0(x, 0)$ and $(\partial u_1/\partial t)(x, 0)$ are well-defined tempered distributions; in fact they are in the Besov spaces $B^q_{k-(1/q)}$ and $B^q_{k-1-(1/q)}$, respectively, (see [13] for a proof and [17] for the other properties of the Besov spaces we will use). Here $B^q_{j+\alpha}$, j a nonnegative integer, and $0 < \alpha < 2$, is the space of functions in L_j^q satisfying

$$\iint |g(x + y) + g(x - y) - 2g(x)|^q \frac{dx\, dy}{|y|^{n+q\alpha}} < \infty \tag{22}$$

together with all derivatives of order $\leqslant j$.

In this case we set $f = u_0(x, 0)$ and $g = (\partial u_1/\partial t)(x, 0)$. If $k = 0$ or 1 we must first regularize in the space variables. Set

$$v_j(x,t) = \int u_j(x - y, t)\, G_2(y)\, dy \qquad \text{for} \quad j = 0, 1 \tag{23}$$

where G_2 is the Bessel potential of order 2

$$\hat{G}_2(\xi) = (1 + |\xi|^2)^{-1}. \tag{24}$$

We note that $\square v_j = 0$ and $(\partial^2/\partial x_i^2)\, v_j \in L_k^q$, hence $(\partial^2/\partial t^2)\, v_j \in L_k^q$, hence $v_j \in L^q_{k+2}$. Thus $v_0(x, 0) \in B^q_{k+2-(1/q)}$ and $(\partial v_1/\partial t)(x, 0) \in B^q_{k+1-(1/q)}$. We set

$$f = (1 - \Delta_x)\, v_0(x, 0) \in B^q_{k-(1/q)}$$

and

$$g = (1 - \Delta_x)(\partial v_1/\partial t)(x, 0) \in B^q_{k-1-(1/q)}$$

(note $(1 - \Delta_x)$ is the inverse of convolution with G_2).

Now let v be the difference of u and the right side of (21) with f and g as above.

If we consider

$$w(x, t) = \int v(x - y, t)\, G_{2N}(y)\, dy \tag{25}$$

for large enough N we can show, as above, that $w \in L^q_{k+2N} \subset C^2$, hence w is a classical solution of $\Box w = 0$. By the choice of f and g we see that w has zero Cauchy data, hence must be zero by Holmgren's uniqueness theorem. Since the transformation (25) is invertible by $(1 - \Delta_x)^N$ we have $v = 0$ which proves (21).

We consider next the local case of a finite ball Ω. We seek the analog of (19) and (20) for $f \in \mathscr{D}'(\Omega)$. Let Ω_j be an increasing sequence of balls whose closures are in Ω and whose union is Ω and denote by Γ_j the corresponding double cones in Γ. Let φ_j be a sequence of functions in $\mathscr{D}(\Omega)$ equal to one in a neighborhood of Ω_j. Then $f_j = \varphi_j f \in \mathscr{E}'(\Omega)$, the distributions of compact support in Ω, and hence f_j may be regarded naturally as a tempered distribution in E^n. We may thus define

$$\hat{u}_j(\xi, t) = \hat{f}_j(\xi) \cos t|\, \xi\, | \tag{26}$$

and

$$\hat{v}_j(\xi, t) = \hat{f}_j(\xi)\, |\, \xi\, |^{-1} \sin t|\, \xi\, | \tag{27}$$

The crucial remark is that, since the distributions whose Fourier transforms are $\cos t\, |\, \xi\, |$ and $|\, \xi\, |^{-1} \sin t\, |\, \xi\, |$ are supported in the ball of radius t, $u_j = u_k$ and $v_j = v_k$ in Γ_j for all $k \geqslant j$. The common values will be denoted u and v respectively.

Definition. We define $F_k(\Omega)$ [resp $G_k(\Omega)$] to be the space of distributions $f \in \mathscr{D}'(\Omega)$ for which u (resp v) given above is in $L_k{}^q(\Gamma)$. We define $\| f : F_k(\Omega)\| = \| u : L_k{}^q(\Gamma)\|$ and $\| f : G_k(\Omega)\| = \| v : L_k{}^q(\Gamma)\|$.

LEMMA 2. *Let* $u(x, t) \in L_k^q(\Gamma)$ *and satisfy* $\square u = 0$ *there. Then there exist* $f \in F_k(\Omega)$ *and* $g \in G_k(\Omega)$ *such that* $u(x, t) = u_j(x, t)$ *in* Γ_j *where*

$$\hat{u}_j(\xi, t) = \hat{f}_j(\xi) \cos t|\,\xi\,| + \hat{g}_j(\xi)\,|\,\xi\,|^{-1} \sin t|\,\xi\,| \quad \text{and} \quad f_j = \varphi_j f, \quad g_j = \varphi_j g. \tag{28}$$

Proof. As in the proof of Lemma 1 we form u_0 and u_1. If $k \geqslant 2$ we obtain f and g by restricting u_0 and $\partial u_1/\partial t$ to $t = 0$. If $k = 0$, or 1, the restrictions are not *a priori* well-defined. It follows from Theorem 4.3.1 of [4], however, that the restrictions can be defined as distributions because $t = 0$ is noncharacteristic for $\square$. If $\psi \in C_{\text{com}}^{\infty}(E^n)$ has sufficiently small support, then

$$R_\psi u = \int u(x - y, t)\, \psi(y)\, dy \tag{29}$$

will be well-defined and satisfy the wave equation in Γ_j. All space derivatives of $R_\psi u$ are in $L^q(\Gamma_j)$ and hence also all time derivatives. Thus $R_\psi u$ has well-defined restrictions to $t = 0$. The consequence of Theorem 4.3.1 of [4] which we use is the existence of a distribution of $f \in \mathscr{D}'(\Omega)$ such that

$$R_\psi u_0(x, 0) = \int f(x - y)\, \psi(y)\, dy \qquad \text{for} \quad x \in \Omega_j, \tag{30}$$

and every such ψ. A similar expression connects $(\partial/\partial t)\, R_\psi u_1(x, 0)$ with $g \in \mathscr{D}'(\Omega)$.

Having obtained f and g, we form u_j by (28) and consider $u - u_j$. If we form $R_\psi(u - u_j)$ we see that it satisfies the wave equation in the classical sense and has zero Cauchy data in Γ_j. Thus it vanishes in Γ_j. Letting ψ run through an approximate identity we prove that $u = u_j$ in Γ_j.

Remark. In the sequel when we say $u(x, t)$ in Γ has Cauchy data f and g we mean in the sense of (29) and (30).

LEMMA 3. *Let* $w \in L_k^p(\Gamma)$ *and* $f \in F_k(\Omega)$, $g \in G_k(\Omega)$. *Then the equation* $\square u = w$ *with Cauchy data* $u(x, 0) = f$, $u_t(x, 0) = g$ *has a unique solution in* $L_k^q(\Gamma)$, *and*

$$\|\, u : L_k^q(\Gamma)\| \leqslant M[\|\, w : L_k^p(\Gamma)\| + \|\, f : F_k(\Omega)\| + \|\, g : G_k(\Omega)\|]. \tag{31}$$

The bound for space derivatives is independent of Ω. *Conversely, let u be any function in* $L_k^q(\Gamma)$ *such that* $\square u = w \in L_k^p(\Gamma)$. *Then there exist*

$f \in F_k(\Omega)$ and $g \in G_k(\Omega)$ such that u has Cauchy data $u(x, 0) = f$, $u_t(x, 0) = g$ and

$$\|f : F_k(\Omega)\| + \|g : G_k(\Omega)\| \leqslant M[\|u : L_k^q(\Gamma)\| + \|w : L_k^p(\Gamma)\|]. \quad (32)$$

Proof. Let $f \in F_k(\Omega)$, $g \in G_k(\Omega)$. Then we can find $v \in L_k^q(\Gamma)$ with Cauchy data f and g,

$$\Box v = 0 \qquad \text{and} \qquad \|v : L_k^q(\Gamma)\| \leqslant \|f : F_k(\Omega) + \|g : G_k(\Omega)\|,$$

by the definition of $F_k(\Omega)$ and $G_k(\Omega)$. But by Theorem 1 we can find $v_0 \in L_k^q(\Gamma)$ satisfying $\Box v_0 = w$ with zero Cauchy data and

$$\|v_0 : L_k^q(\Gamma)\| \leqslant M\|w : L_k^p(\Gamma)\|.$$

The desired solution is thus $u = v + v_0$.

Conversely, if $u \in L_k^q(\Gamma)$ and $\Box u = w \in L_k^p(\Gamma)$ we construct v_0 as above and consider $u - v_0$. We have $\Box(u - v_0) = 0$ and $u - v_0 \in L_k^q(\Gamma)$, with $\|u - v_0 : L_k^q(\Gamma)\| \leqslant M' \|w : L_k^p(\Gamma)\| + \|u : L_k^q(\Gamma)\|$. Now Lemma 1 and Lemma 2 apply to $u - v_0$ and give the existence of Cauchy data $f \in F_k(\Omega)$ and $g \in G_k(\Omega)$ for $u - v_0$ satisfying (32). But v_0 has zero Cauchy data so u has f and g as Cauchy data.

LEMMA 4. *$B^p_{k+2-1/p}(E^n) \subseteq F_k(E^n)$ and $B^p_{k+1-1/p}(E^n) \subseteq G_k(E^n)$.*

Proof. Let

$$f \in B^p_{k+2-(1/p)}(E^n), \qquad g \in B^p_{k+1-(1/p)}(E^n).$$

Then there exists $u \in L^p_{k+2}(E^{n+1})$ such that $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$ (see [13] and [10]).

By Sobolev's inequality $u \in L_k^q(E^{n+1})$ since $(1/p) - (1/q) = 2/(n + 1)$, and obviously $\Box u \in L_k^p(E^{n+1})$. Lemma 3 applied to u gives the desired result.

LEMMA 5. *Let $f \in F_k(\Omega)$ and $g \in G_k(\Omega)$. Then $f|_{\Omega'} \in F_k(\Omega')$ and $g|_{\Omega'} \in G_k(\Omega')$ for any $\Omega' \subseteq \Omega$. Furthermore, given any $\epsilon > 0$ there exists a $\delta > 0$ (depending on f and g) such that if radius $(\Omega') \leqslant \delta$ and $\Omega' \subseteq \Omega$ then $\|f|_{\Omega'} : F_k(\Omega')\| + \|g|_{\Omega'} : G_k(\Omega')\| \leqslant \epsilon$.*

Proof. Let $f \in F_k(\Omega)$ and $g \in G_k(\Omega)$, and let u, $v \in L_k^q(\Gamma)$ be the solutions to $\Box u = \Box v = 0$ with Cauchy data f, 0 and 0, g, respectively. Then $u|_{\Gamma'}$ and $v|_{\Gamma'}$ are in $L_k^q(\Gamma')$, satisfy the wave equation, and have Cauchy data $f|_{\Omega'}$, 0 and 0, $g|_{\Omega'}$, respectively. This proves the first assertion. The second assertion follows from the fact that for fixed

$u \in L_k^q(\Gamma)$ and $\epsilon < 0$ there exists a $\delta > 0$ such that $\| u |_{\Gamma'} : L_k^q(\Gamma')\| \leqslant \epsilon/2$ provided diameter $(\Gamma') \leqslant \delta$.

Remarks. We may summarize the above as follows:

We have constructed spaces of distributions F_k, G_k which are well suited for the Cauchy data in solving $\Box u = w$ (Lemma 3). In fact, once we assume the initial data in these spaces we know that at all future times the data remains in these spaces (converse assertion in Lemma 3). We have identified large subspaces of F_k, G_k (Lemma 4) and, at least when $\Omega = E^n$, have shown that F_k, G_k are contained in other well-known classes of distributions (Lemma 1). Unfortunately, we have not been able to give a satisfactory characterization of these spaces. Thus Lemma 4 remains the best way of actually verifying that a particular distribution is in F_k or G_k. Lemma 5 establishes a trivial fact that will be useful in the sequel.

We note that it would not be possible to develop an adequate theory using the Besov spaces B_α^p alone, because they are not preserved by the Fourier multipliers $\sin |\xi|$ and $\cos |\xi|$.

Finally we note that if $k \geqslant (n+3)/2$, then $L_k^q \subseteq C^2$ and $L_k^p \subseteq C^0$ so the solutions exist in the classical sense.

4. $\Box u = H(u)$

It is now a fairly routine matter to prove existence and uniqueness for $\Box u = H(u)$, with Cauchy data in F_k, G_k and with H a Lipschitz continuous map of L_k^q into L_k^p, by the method of iteration. The one peculiarity of the present situation is that we must make the Lipschitz constant small by shrinking the domain. There are many variations on the same theme; we present a few that seem the most important.

THEOREM 2. *Let $H(\cdot, x, t)$ be a scalar function for each $x \in E^n$ and $t \in (T_0, T_1)$ where T_0, T_1 may be finite or infinite, and $T_0 \leqslant 0 \leqslant T_1$. Suppose that whenever $u(x, t) \in L_k^q[E^n \times (T_0, T_1)]$ we have $H[u(x,t), x, t] \in L_k^p[E^n \times (T_0, T_1)]$, and given any $\epsilon > 0$ there exists a decomposition $T_0 = t_0 < t_1 < \cdots < t_m = T_1$ such that, for $j = 1, \ldots, m-1$ we have*

$$\| H[u(x,t), x, t] - H[v(x,t), x, t] : L_k^p[E^n \times (t_j, t_{j+1})]\| \leqslant \epsilon \| u - v : L_k^q[E^n \times (t_j, t_{j+1})]\|. \tag{33}$$

Then the equation $\Box u(x,t) = H[u(x,t), x, t]$ with Cauchy data

$u(x, 0) = f \in F_k(E^n)$, $u_t(x, 0) = g \in G_k(E^n)$ has a unique solution in $L_k^q[E^n \times (T_0, T_1)]$.

Proof. We choose $\epsilon < M^{-1}$, where M is the constant that appears in Theorem 1. We construct the solution in each strip $t_j \leqslant t \leqslant t_{j+1}$, starting with the one containing $t = 0$. Once we have the solution in $t_j \leqslant t \leqslant t_{j+1}$ we can take new Cauchy data on $t = t_{j+1}$ which will be in F_k and G_k by Lemma 2, and continue the solution to $t_{j+1} \leqslant t \leqslant t_{j+2}$.

For simplicity assume $t_0 < 0 < t_1$ and that we wish to find the solution on $t_0 \leqslant t \leqslant t_1$. Let u_0 be the solution of $\Box u_0 = 0$ with Cauchy data f and g. Having found u_j let u_{j+1} be the solution of $\Box u_{j+1} = H(u_j, x, t)$ with Cauchy data f and g. Then $u_{j+1} - u_j$ has zero Cauchy data and satisfies $\Box(u_{j+1} - u_j) = H(u_j, x, t) - H(u_{j-1}, x, t)$ so

$$\begin{aligned} &\| u_{j+1} - u_j : L_k^q[E^n \times (t_0, t_1)]\| \\ &\quad \leqslant M\| H(u_j, x, t) - H(u_{j-1}, x, t) : L_k^p[E^n \times (t_0, t_1)]\| \\ &\quad \leqslant \epsilon M \| u_j - u_{j-1} : L_k^q[E^n \times (t_0, t_1)]\| \end{aligned} \tag{34}$$

by an immediate variant of Theorem 1 and (33). Iterating this inequality we obtain

$$\begin{aligned} &\| u_{j+1} - u_j : L_k^q[E^n \times (t_0, t_1)]\| \\ &\quad \leqslant (\epsilon M)^{j-1} \| u_1 - u_0 : L_k^q[E^n \times (t_0, t_1)]\| \end{aligned} \tag{35}$$

which shows that u_j is a Cauchy sequence since $\epsilon M < 1$. The limit is easily seen to be the desired solution.

To prove uniqueness assume u and v are two solutions. Then $u - v$ has zero Cauchy data and satisfies $\Box(u - v) = H(u, x, t) - H(v, x, t)$. Thus

$$\begin{aligned} &\| u - v : L_k^q[E^n \times (t_0, t_1)]\| \\ &\quad \leqslant M\| H(u, x, t) - H(v, x, t) : L_k^p[E^n \times (t_0, t_1)]\| \\ &\quad \leqslant \epsilon M \| u - v : L_k^q[E^n \times (t_0, t_1)]\| \end{aligned} \tag{36}$$

as before. Since $\epsilon M < 1$, this implies $u - v = 0$ in $t_0 \leqslant t \leqslant t_1$.

COROLLARY. *Let $v(x, t) \in L_k^r[E^n \times (t_0, t_1)]$ for every finite time interval (t_0, t_1), and let $w(x, t) \in L_k^p[E^n \times (t_0, t_1)]$, where*

$$r = \max[(n + 1)/(2 + k), p].$$

Then the equation $\Box u = vu + w$ with Cauchy data $u(x, 0) = f \in F_k(E^n)$, $u_t(x, 0) = g \in G_k(E^n)$ has a unique solution in E^{n+1} which is in $L_k^q[E^n \times (t_0, t_1)]$ for every finite time interval. If in addition $v \in L_k^r(E^{n+1})$ and $w \in L_k^p(E^{n+1})$, then $u \in L_k^q(E^{n+1})$.

Proof. We take $H(u, x, t) = vu + w$ and observe that

$$\| H(u, x, t) - H(u', x, t) : L_k^p[E^n \times (T_0, T_1)]\| \leqslant K\| v : L_k^r[E^n \times (T_0, T_1)]\| \, \| u - u' : L_k^q[E^n \times (T_0, T_1)]\|$$

by Liebnitz's formula and the Sobolev inequalities. Thus to satisfy the hypotheses of Theorem 2 we take the interval (T_0, T_1) sufficiently small so that $\| v : L_k^r[E^n \times (T_0, T_1)]\|$ is small. If $v \in L_k^r(E^{n+1})$, then only a finite number of intervals are needed to span $-\infty < t < +\infty$ so $u \in L_k^q(E^{n+1})$.

THEOREM 2′. *Let $H(\cdot, x, t)$ be a scalar function for each $(x, t) \in \Gamma$. Suppose that whenever $u(x, t) \in L^q_{k,\mathrm{loc}}(\Gamma)$ [by $u \in L^q_{k,\mathrm{loc}}(\Gamma)$ we mean $u \in L_k^q(\Gamma')$ for any bounded Γ' with $\bar{\Gamma}' \subset \Gamma$] that*

$$H[u(x, t), x, t] \in L^p_{k,\mathrm{loc}}(\Gamma),$$

and given any $\epsilon > 0$ and bounded Γ' with $\bar{\Gamma}' \subset \Gamma$ there exists a finite set of translates of double cones $\Gamma_1, \ldots, \Gamma_N$ covering Γ' such that

$$\| H[u(x, t), x, t] - H[v(x, t), x, t] : L_k^p(\Gamma_j')\| \leqslant \epsilon \| u - v : L_k^q(\Gamma_j')\|, \qquad \text{for any} \quad \Gamma_j' \subset \Gamma_j . \tag{37}$$

Then the equation $\square u = H(u, x, t)$ with Cauchy data $u(x, 0) = f \in F_k(\Omega)$ and $u_t(x, 0) = g \in G_k(\Omega)$ has a unique solution in $L^q_{k,\mathrm{loc}}(\Gamma)$.

Proof. The proof is a straightforward modification of the proof of Theorem 2 using Lemma 3 and Lemma 5.

COROLLARY. *Let $v \in L^r_{k,\mathrm{loc}}(\Gamma)$ for $r =$ $\max[(n + 1)/(2 + k), p]$, and let $w \in L^p_{k,\mathrm{loc}}(\Gamma)$. Then $\square u = vu + w$ with Cauchy data in $F_k(\Omega)$ and $G_k(\Omega)$ has a unique solution in $L^q_{k,\mathrm{loc}}(\Gamma)$.*

Proof. Follows just as the Corollary to Theorem 2.

DEFINITION. In order to apply the full strength of Lemma 3 we must deal with space derivatives only, since the bound M in (31) depends on Γ when time derivatives are included. Thus we shall denote by $\| u : L^q_{k,\mathrm{sp}}(\Gamma_j)\|$ the sum of the L^q norms of all space derivatives of u of order $\leqslant k$.

THEOREM 3. *Let $H(\cdot, x, t)$ be a scalar function for each $(x, t) \in E^{n+1}$.*

Suppose that whenever $u \in L_k^q(E^{n+1})$ we have $H[u(x, t), x, t] \in L_k^p(E^{n+1})$, and given any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\| H[u(x, t), x, t] - H[v(x, t), x, t] : L^q_{k,\mathrm{sp}}(\Gamma)\| \leqslant \epsilon \| u - v : L^q_{k,\mathrm{sp}}(\Gamma)\| \tag{38}$$

for every translate of a double cone Γ provided $\| u : L^q_{k,\mathrm{sp}}(\Gamma)\| \leqslant \delta$ and $\| v : L^q_{k,\mathrm{sp}}(\Gamma)\| \leqslant \delta$. Then given any $f \in F_k(E^n)$ and $g \in G_k(E^n)$ there exists a time interval $T_0 < t < T_1$ with $T_0 < 0 < T_1$ for which the equation $\Box u = H(u, x, t)$ with Cauchy data $u(x, 0) = f$, $u_t(x, 0) = g$ has a unique solution in $L_k^q[E^n \times (T_0, T_1)]$. Furthermore, the solution is unique as long as it exists, and if it does not exist globally, then the L_k^q norm of the solution tends to infinity as the maximal time interval of existence is approached.

Proof. Choose any $\epsilon < (1/3M)$ and consider the corresponding δ. Let $v(x, t) = H(0, x, t)$. Choose a finite covering of E^n by balls and half spaces $\Omega_1, \ldots, \Omega_N$ such that

$$\|f : F_k(\Omega_j)\| \leqslant \delta/6M, \quad \| g G_k(\Omega_j)\| \leqslant \delta/6M \quad \text{and} \quad \| v : L^p_{k,\mathrm{sp}}(\Gamma_j)\| \leqslant \delta/3M$$

(the modifications in dealing with half spaces rather than balls are trivial). We construct the solution in each Γ_j by iteration.

Let u_0 be the solution of $\Box u_0 = v$ with Cauchy data f, g, and let u_{j+1} satisfy $\Box u_{j+1} = H[u_j(x, t), x, t]$ with the same Cauchy data.

We claim that $\| u_j : L^q_{k,\mathrm{sp}}(\Gamma_i)\| \leqslant \delta$ so that we may apply (38). In fact

$$\| u_0 : L^q_{k,\mathrm{sp}}(\Gamma_i)\| \leqslant M[\| v : L^p_{k,\mathrm{sp}}(\Gamma_i)\| + \|f : F_k(\Omega_i)\| + \| g : G_k(\Omega_i)\|] \leqslant \tfrac{2}{3}\delta$$

by (31). Continuing by induction we have

$$\| u_{j+1} : L^q_{k,\mathrm{sp}}(\Gamma_i)\| \leqslant M\{\| H[u_j(x, t), x, t] : L^p_{k,\mathrm{sp}}(\Gamma_i)\| + \|f : F_k(\Omega_i)\| + \| g : G_k(\Omega_i)\|\}$$

from (31). But by (38) we have

$$\| H[u_j(x, t), x, t] : L^p_{k,\mathrm{sp}}(\Gamma_i)\| \leqslant \| v : L^p_{k,\mathrm{sp}}(\Gamma_i)\| + \epsilon \| u_j : L^q_{k,\mathrm{sp}}(\Gamma_i)\|$$

so $\| u_{j+1} : L^q_{k,\mathrm{sp}}(\Gamma_i)\| \leqslant M((\delta/3M) + \epsilon\delta + (\delta/6M) + (\delta/6M)) \leqslant \delta$.

Thus we may apply (38) to u_j and u_{j+1} to obtain

$$\| H(u_{j+1}) - H(u_j) : L^p_{k,\mathrm{sp}}(\Gamma_i)\| \leqslant \epsilon \| u_{j+1} - u_j : L^q_{k,\mathrm{sp}}(\Gamma_i)\|.$$

Combining this with (31) and the fact that $\Box(u_{j+1} - u_j) = H(u_j) - H(u_{j-1})$ and $u_{j+1} - u_j$ has zero Cauchy data we obtain $\| u_{j+1} - u_j : L^q_{k,\mathrm{sp}}(\Gamma_i)\| \leqslant \epsilon M \| u_j - u_{j-1} : L^q_{k,\mathrm{sp}}(\Gamma_i)\|$. Since $\epsilon M < 1$ we see that the iteration process converges to a solution in $L^q_{k,\mathrm{sp}}(\Gamma_i)$. To show that $u \in L_k{}^q(\Gamma_i)$ we repeat the argument at the end of Theorem 1.

To prove uniqueness assume we have two solutions u_0 and u_1. Choose $\epsilon < 1/M$ and let Γ be any double cone on which $\| u_j : L^q_{k,\mathrm{sp}}(\Gamma)\| \leqslant \delta$ for $j = 0, 1$. Then $\| u_0 - u_1 : L^q_{k,\mathrm{sp}}(\Gamma)\| \leqslant M \| H(u_0) - H(u_1) : L^p_{k,\mathrm{sp}}(\Gamma)\| \leqslant \epsilon M \| u_0 - u_1 : L^q_{k,\mathrm{sp}}(\Gamma)\|$, hence $u_0 = u_1$ on Γ.

Once we have proved existence and uniqueness in $T_0 < t < T_1$ we may take new Cauchy data at $t = T_0$ and $t = T_1$ which will again be in $F_k(E^n)$ and $G_k(E^n)$ by Theorem 1. Thus we may continue the unique solution until the $L_k{}^q$ norm blows up.

Example. When $n = 3$ the function $H(u) = u^3$ is easily seen to satisfy the hypothesis of Theorem 3.

5. The Wave Operators

We consider here the Cauchy problem at $-\infty$: Given a solution u_0 of $\Box u_0 = 0$ find a solution u of $\Box u = H(u)$ which is asymptotic to u_0 as $t \to -\infty$. If u exists globally we can also consider the problem of finding another solution u_1 of $\Box u_1 = 0$ which is asymptotic to u as $t \to +\infty$. The operators $u_0 \to u$ and $u \to u_1$ are called the wave operators of the equation $\Box u = H(u)$ with respect to the equation $\Box u = 0$, and the operator $u_0 \to u_1$ is called the scattering operator. These ideas have been considered before and we refer the reader to [6], [11], [14], for a discussion of their significance as well as results obtained by other methods.

Theorem 4. *Let $H(\cdot, x, t)$ be a scalar function for each $(x, t) \in E^{n+1}$. Suppose that whenever $u \in L_k{}^q[E^n \times (-\infty, T)]$ we have*

$$H[u(x, t), x, t] \in L_k{}^p[E^n \times (-\infty, T)]$$

for all real T, and given any $\epsilon > 0$ there exists a T_0 and $\delta > 0$ such that

$$\| H[u(x,t),x,t] - H[v(x,t),x,t] : L_k^p[E^n \times (-\infty, T)]\| \leqslant \epsilon \| u - v : L_k^q[E^n \times (-\infty, T)]\| \qquad (39)$$

provided $T \leqslant T_0$, $\| u : L_k^q[E^n \times (-\infty, T)]\| \leqslant \delta$ and

$$\| v : L_k^q[E^n \times (-\infty, T)]\| \leqslant \delta.$$

Then given any $u_0 \in L_k^q(E^{n+1})$ satisfying $\Box u_0 = 0$ there exists a unique solution of $\Box u = H(u)$ in $L_k^q[E^n \times (-\infty, T)]$ for T sufficiently small such that

$$\| u(\cdot, t) - u_0(\cdot, t) : F_k \| \to 0$$

and

$$\|(\partial u/\partial t)(\cdot, t) - (\partial u_0/\partial t)(\cdot, t) : G_k \| \to 0 \quad \text{as} \quad t \to -\infty.$$

If the solution exists globally and is in $L_k^q(E^{n+1})$, then there exists a unique $u_1 \in L_k^q(E^{n+1})$ satisfying $\Box u_1 = 0$ and $\| u(\cdot, t) - u_1(\cdot, t) : F_k \| \to 0$ and $\|(\partial u/\partial t)(\cdot, t) - (\partial u_1/\partial t)(\cdot, t) : G_k \| \to 0$ as $t \to +\infty$.

Proof. Let u be the desired solution. Consider the Cauchy problem $\Box v = 0$ with data $v(\cdot, r) = u_0(\cdot, r) - u(\cdot, r)$ and $(\partial v/\partial t)(\cdot, r) = (\partial u_0/\partial t)(\cdot, r) - (\partial u/\partial t)(\cdot, r)$. By the hypothesis and the definition of F_k and G_k we have $\| v : L_k^q(E^{n+1})\| \to 0$ as $r \to -\infty$. Now

$$\begin{aligned} \hat{v}(\xi, t) &= \hat{u}_0(\xi, t) - \hat{u}(\xi, r)\cos(t - r)\,|\,\xi\,| - (\partial\hat{u}/\partial t)(\xi, r)\,|\,\xi\,|^{-1}\sin(t - r)\,|\,\xi\,| \\ &= \hat{u}_0(\xi, t) - \hat{u}(\xi, t) + \int_r^t |\,\xi\,|^{-1}\sin(t - s)\,|\,\xi\,| H(u)\hat{\ }(\xi, s)\, ds \end{aligned}$$

by integrating by parts twice the last integral using $\Box u = H(u)$. Thus we see that u must satisfy

$$\hat{u}(\xi, t) = \hat{u}_0(\xi, t) + \int_{-\infty}^t |\,\xi\,|^{-1}\sin(t - s)\,|\,\xi\,| H(u)\hat{\ }(\xi, s)\, ds. \qquad (40)$$

Conversely, assume $u \in L_k^q[E^n \times (-\infty, T)]$ satisfies (40). Then an easy computation shows $\Box u = H(u)$. To establish the asymptotic estimate we must show that $\| v : L_k^q(E^{n+1})\| \to 0$ as $r \to -\infty$ where v is as defined above. Using (40) we see that

$$\hat{v}(\xi, r) = -\int_{-\infty}^r |\,\xi\,|^{-1}\sin(r - s)\,|\,\xi\,| H(u)(\xi, s)\, ds$$

and

$$\frac{\partial\hat{v}}{\partial t}(\xi, r) = -\int_{-\infty}^r \cos(r - s)\,|\,\xi\,| H(u)\hat{\ }(\xi, s)\, ds.$$

Applying (21) and simplifying we obtain

$$\hat{v}(\xi, t) = -\int_{-\infty}^{r} |\xi|^{-1} \sin(r + t - s) |\xi| H(u)^\wedge(\xi, s)\, ds. \tag{41}$$

Repeating the arguments in the proof of Theorem 1 we obtain

$$\| v : L_k{}^q(E^{n+1})\| \leqslant M \| H(u) : L_k{}^p[E^n \times (-\infty, r)]\|. \tag{42}$$

Since by assumption $H(u) \in L_k{}^p[E^n \times (-\infty, r)]$, the right side of (42) tends to zero as $r \to -\infty$. Thus we have reduced the problem to solving (40).

Now to solve (40) in $L_k{}^q[E^n \times (-\infty, T)]$ for sufficiently small T we again use an iteration argument and an estimate for the integral appearing in (40) analogous to Theorem 1. The same estimate also establish uniqueness for $t < T$ (cf. remarks below).

To establish the existence of u_1 if u exists globally and is in $L_k{}^q(E^{n+1})$ we replace $-\infty$ by $+\infty$ in the above argument and obtain

$$\hat{u}(\xi, t) = \hat{u}_1(\xi, t) - \int_{t}^{\infty} |\xi|^{-1} \sin(t - s) |\xi| H(u)^\wedge(\xi, s)\, ds \tag{43}$$

as an equivalent condition. But now u is known, so we have an explicit formula for u_1 from which it is clear that $u_1 \in L_k{}^q(E^{n+1})$.

Examples. If $H(u) = uv + w$ for $v \in L_k{}^r(E^{n+1})$ and $w \in L_k{}^p(E^{n+1})$ for $r = \max[(n+1)/(2+k), p]$, then both u and u_1 exist and the scattering operator is bounded on $L_k{}^q(E^{n+1})$.

If $H(u) = u^m$ for a positive integer $m \geqslant (n+3)/(n-1)$, then we have the existence of u for early times provided $u_0 \in L_k{}^q(E^{n+1})$ and we take $k \geqslant (n-1)/2 - 2/(m-1)$. For the Sobolev inequalities imply that u^m maps $L_k{}^q(E^{n+1})$ into $L_k{}^p(E^{n+1})$, and the Lipschitz condition follows from $u^m - v^m = (u - v)(u^{m-1} + u^{m-2}v + \cdots + v^{m-1})$.

Remarks. We can obtain uniqueness of u in Theorem 4 for as long as it exists if we assume, in addition to (39), an analogous condition with $(-\infty, T)$ replaced by any sufficiently small time interval. For if u and v both satisfy (40), then we have

$$\hat{u}(\xi, t) - \hat{v}(\xi, t) = \int_{-\infty}^{t} |\xi|^{-1} \sin(t - s) |\xi| [H(u)^\wedge(\xi, s) - H(v)^\wedge(\xi, s)]\, ds$$

from which we can conclude $u = v$, first on $(-\infty, T_0)$, and then piecewise on (T_0, T_1), (T_1, T_2), etc. This argument applies to the above examples.

The weakness of Theorem 4 for nonlinear scattering is that we have no sufficient conditions for the existence of global solutions, and even if they do exist we do not know if they are in $L_k^q(E^{n+1})$. In this respect the work of Strauss [14] using energy methods is more successful, despite his limitations on H and n.

6. Concluding Remarks

Let $P(D)$ be a homogeneous constant coefficient partial differential operator of degree m in $\partial/\partial t, \partial/\partial x_1, \dots, \partial/\partial x_n$ which is strictly hyperbolic with respect to time. It should be possible to extend the results for $\square$ to $P(D)$ for $p = 2(n+1)/(n+1+m)$ and $q = 2(n+1)/(n+1-m)$. Indeed, the kernel corresponding to $|\xi|^{-1} \sin t |\xi|$ for $\square$ has the form $\sum_{j=1}^{m} |\xi|^{1-m} \varphi_j(\xi)\, e^{it|\xi|\psi_j(\xi)}$ for φ_j, ψ_j, certain C^∞ functions homogeneous of degree zero (see [2], p. 75). Now φ_j is an L^p multiplier for $1 < p < \infty$ [5], so it sufficies to show $|\xi|^{1-m} e^{it|\xi|\psi_j(\xi)}$ is an $L^p - L^q$ multiplier. This can be shown by the techniques of [15] if $\psi_j(\xi) = 1$. This happens when $P(D)$ is a function of $\partial/\partial t$ and Δ_x alone, which is a very severe restriction. Thus the above multiplier problem is the only stumbling block to extending the theory to homogeneous strictly hyperbolic operators.

It is also of interest to consider the Klein–Gordon operator $\square u - m^2 u$, $m > 0$, on its own (rather than as a perturbation of $\square$ as can be done in Theorem 2′), especially regarding questions of scattering [11]. Here we remark that the analog of (4) holds, but the decay of the constant $M(t)$ as $t \to \pm\infty$ is not sufficiently rapid (or at least as far as we can show) to obtain the analog of Theorem 1.

In fact the solution to the Cauchy problem $\square u = m^2 u$, $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$ is given formally by

$$\hat{u}(\xi, t) = \hat{f}(\xi) \cos t \sqrt{m^2 + |\xi|^2} + \hat{g}(\xi)(m^2 + |\xi|^2)^{-1/2} \sin t \sqrt{m^2 + |\xi|^2}. \tag{44}$$

Thus we need an estimate

$$\| \mathscr{F}^{-1}(\hat{f}(\xi)(m^2 + |\xi|^2)^{-1/2} e^{it\sqrt{m^2+|\xi|^2}})\|_q \leqslant M(t) \|f\|_p . \tag{45}$$

Now $|\xi|(m^2 + |\xi|^2)^{-1/2}$ is a Fourier–Stieltjes transform [12] and $e^{it\sqrt{m^2+|\xi|^2}-|\xi|}$ satisfies the hypothesis of Hörmander's multiplier theorem [5] so

$$\begin{aligned}
&\| \mathscr{F}^{-1}[\hat{f}(\xi)(m^2 + |\xi|^2)^{-1/2} e^{it\sqrt{m^2+|\xi|^2}}]\|_q \\
&\quad = \| \mathscr{F}^{-1}[\hat{f}(\xi) |\xi|^{-1} e^{it|\xi|} |\xi|(m^2 + |\xi|^2)^{-1/2} e^{it\sqrt{m^2+|\xi|^2}-|\xi|}]\|_q \\
&\quad \leqslant K t^{(1-n)/(1+n)} \varphi(t) \|f\|_p
\end{aligned}$$

where $\varphi(t)$ is the norm of $e^{it\sqrt{m^2+|\xi|^2}-|\xi|}$ as an L^p multiplier. Unfortunately, the best estimate we can obtain for $\varphi(t)$ using Littman's refinement of Hörmander's Theorem [8] is $0(t^{n/n+1+\epsilon})$ as $t \to \infty$ whereas $0(1)$ is needed for the analog of Theorem 1. This may not be the last word on the question, however, because S. Nelson [9] has shown that for a related decay problem the Klein–Gordon equation is actually better behaved than the wave equation.

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