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Physicist's version of traveling salesman problem: statistical analysis

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We report a statistical calculation of the average distance covered by the traveling salesman, per customer visited, in its dependence upon the width of the "swaths" of landscape covered. We find a "practical person's minimum of (0.921) (density of points)$^{-1/2}$ for a swath width of (1.732) (density of points)$^{-1/2}$. The point of greatest physical interest is the foliation induced in the random pattern of cities to be visited by the "regulating principle" of minimum distance.

As one illustration of the theme of "order out of disorder via the action of a regulating principle" dealt with in an accompanying paper by one of us,[1] we report here a statistical analysis of the "physicist's version" of the old problem of the traveling salesman.

How far does the salesman have to travel, on the average, per customer visited when the cites are scattered with "uniform randomness" over a rectangular landscape deprived of boundaries by identification of opposite edges ("two-torus topology")? We do not ask for the absolute minimum. We ask for a "practical person's" approach to this minimum. For this purpose we foliate the cities into winrows, swaths, or ribbons of identical width. The "separate" ribbons constitute a single ribbon wound around and around the torus. The salesman visits each city within the swath in the rigorously enforced order of progression along the swath coordinate x. We report here a statistical analysis of "average distance per customer" in its dependence on the swath width. For a swath width that is too wide the salesman must execute mostly zig-zag motion back and forth between the swath boundaries. When the swath width is too narrow, he travels in an almost straight line long distances before chancing upon a city. What is the average distance per customer for the general value of the swath width L?

In terms of the surface density ρ of cities to be visited, the chance that there shall be one more city to be visited in a forward progression dx along the direction of the swath is $\rho L\, dx$. The chance that no other city shall have been encountered in the distance x from the last city to this next one is $\exp(-\rho Lx)$. The chance that the last city shall be in the interval of y coordinate dy_1 is dy_1/L; and the chance that the next city shall be in the interval dy_2 is dy_2/L; and the chance that the next city shall be in the interval dy_2 is dy_2/L. Consequently, in conventional or distance units the average distance per customer visited, D_{conv}, is

$$\int_{x=0}^{x=\infty}\int_{y_2=0}^{y_2=L}\int_{y_1=0}^{y_1=L}[x^2+(y_2-y_1)^2]^{1/2}(dy_1/L)(dy_2/L)\times\exp(-\rho Lx)\rho L\,dx. \tag{1}$$

We introduce the dimensionless measures of distance, $D=\rho^{1/2}D_{\text{conv}}$, and swath width, $L=\rho^{1/2}L_{\text{conv}}$, and measures of lateral and forward motion, $v=|y_2-y_1|/L$ and $u=\rho L_{\text{conv}}x$. We integrate first over y, for a given v, and find

$$D=\frac{1}{L}\int_{v=0}^{v=1}\int_{u=0}^{u=\infty}(u^2+L^4v^2)^{1/2}\exp(-u)du(1-v)2\,dv. \tag{2}$$

We evaluate the integral over u, via the integral tables of Gradshteyn and Ryzhik,[2]

$$D=\frac{1}{L}\int_0^1\frac{\pi}{2}L^2v[H_1(L^2v)-N_1(L^2v)](1-v)2\,dv. \tag{3}$$

Here the first-order Struve function H_1 and the first-order Neumann function N_1 are defined and tabulated in Jahnke–Emde.[3] The final integration we did numerically, obtaining the results listed in Table I of this paper and plotted in Fig. 10 of Ref. 1. We interpolated between the calculated points by Lagrangian six-point interpolation and found the practical person's minimum $D=0.921$, at $\sim L=1.73$.

The cities being random in their arrangement, we can see that cases will come up where it will pay to depart from the rigid rules "visit each city in a given swath in the order of increasing x" and "never trespass across swath boundaries." But we do not. That is part of what we mean by the term "practical person's minimum" as contrasted with an absolute minimum.

L. Few[4] has proved the theorem: "given n ($\geqslant 2$) points in a unit square, there is a path through the n points of length not exceeding $2^{1/2}n^{1/2}+1.75$." This theorem implies, in our notation, a value of the dimensionless measure D of distance per customer for large n not exceeding $D=1.414\cdots$. L. Fejes Tóth finds ($D=2^{1/2})/(3^{1/4})=1.07\cdots$ when the n points (large n!) "consist of all points of a regular hexagonal lattice [close-packed pennies!] lying in the unit square" (wording of L. Few). Comparing these results with our figure, $D=0.921$ for a "random" arrangement of cities, we conclude that "foliation looks good." However, it

Table I. Distance per customer visited in its dependence on swath width [both in units of (density)$^{-1/2}$]. The final column is an attempt at an approximate analytic fit to the results obtained by numerical integration (middle column).

L	D	$[(1/L)^{1.484}+(L/3)^{1.484}]^{1/1.484}$
Small L	$1/L$	$1/L$
0.775	1.365	1.369
1.414	0.949	0.949
1.732	0.921	0.921
2.000	0.935	0.935
2.098	0.946	0.946
3.162	1.166	1.170
4.000	1.402	1.407
Large L	$L/3$	$L/3$

seems reasonable to believe that a still closer analysis along the present statistical lines, allowing for some slight departure from perfect foliation, will produce an even lower figure. We can still say that the disorder of the "random" arrangement of cities, plus the "regulating principle" of minimum length, suggest in a natural way the "ordering principle of foliation"—example of the theme of the accompanying paper, "law without law."

ACKNOWLEDGMENTS

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[1] J. A. Wheeler, Am. J. Phys. **51**, 396 (1983).

[2] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products* (Academic, New York, 1980), pp. 322 and 938.

[3] *Jahncke–Emde–Losch Tables of Higher Functions*, 6th ed., revised by F. Losch (McGraw-Hill, New York and Teubner, Stuttgart, West Germany, 1960), pp. 186–191 and 254–261.

[4] L. Few, Mathematika **2**, 141–144 (1955); see also F. R. K. Chung and R. L. Graham, Geometriae Dedicata **II**, 353–361 (1981).

PROBLEM

Suppose that a visible-light photon starting from the center of the sun random walks to the surface by repeated scatterings from the electrons.

(1) Write a formula for the photon's escape time t_{es} to reach the visible surface, at radius R_s. Assume that:

(a) The sun is pure hydrogen, completely ionized.

(b) Its density is $\rho(r) = \rho_0 \exp(-r^2/R_0^2)$ where R_0 is a "core radius."

(c) The scatterings are instantaneous and backward or forward scattering is equally likely.

(d) The distance traveled between scatterings is $l(r) = [n(r)\sigma_T]^{-1}$ where

$$\sigma_T = (8\pi/3)(e^2/m_e c^2) = 6.65 \times 10^{-25}\ \text{cm}^2$$

is the Thomson scattering cross section and $n(r) = \rho(r)/m_H$ is the number density of protons or electrons.

(e) The distance of a typical photon from the center after time t is the rms radius of all photons released at time 0.

(2) Calculate t_{es} for[1] $\rho_0 = 160\ \text{g cm}^{-3}$, $R_S = 7.0 \times 10^5$ km, and $R_0 = 1.5 \times 10^5$ km.

(Solution is on page 455).

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[1] M. Zeilik, *Astronomy: the Evolving Universe* (Harper and Row, New York, 1982), Fig. 13.29.