6 - Hidden Markov Models

CS 6320

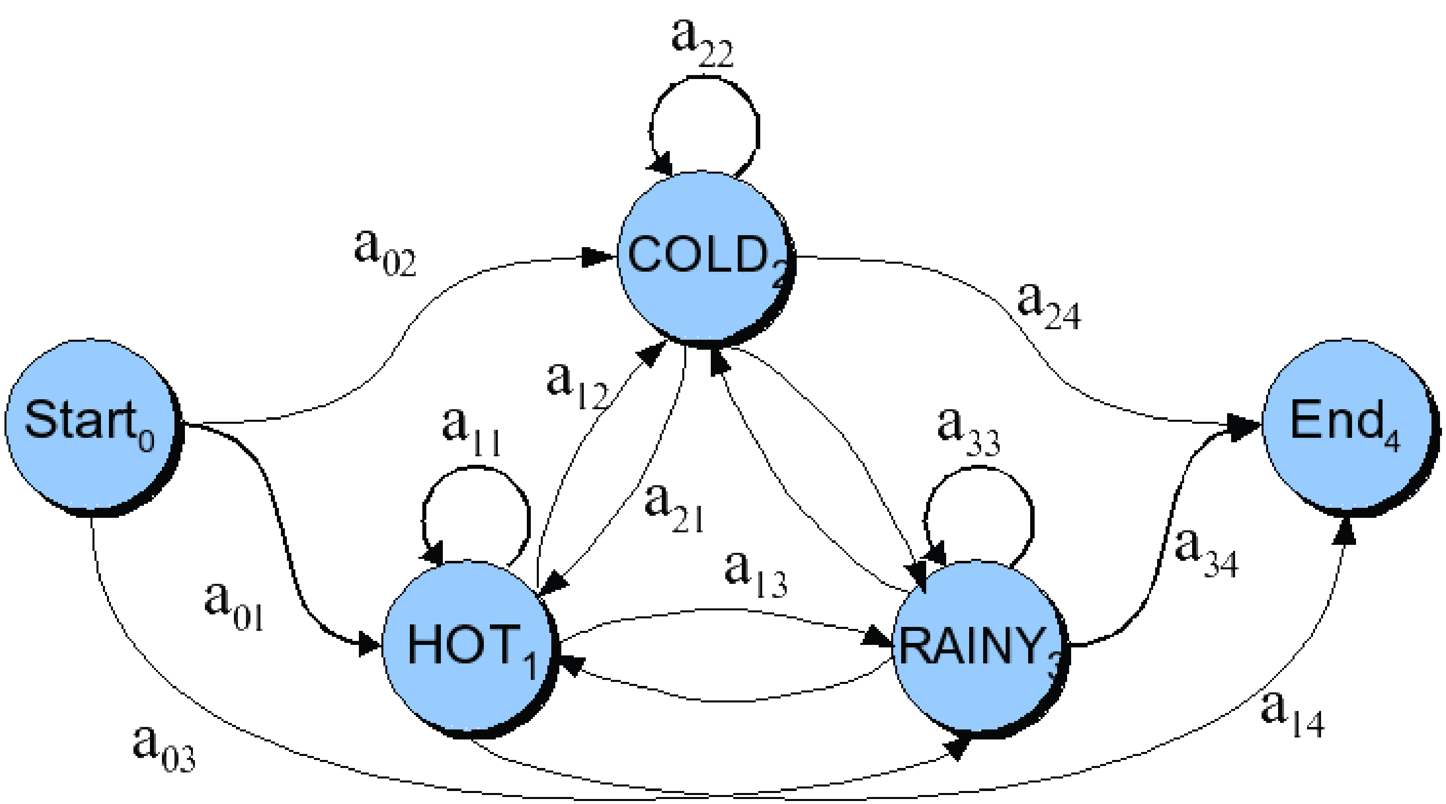
Outline

- Markov Chains
- Hidden Markov Model
- Likelihood: Forward Algorithm
- Decoding: Viterbi Algorithm

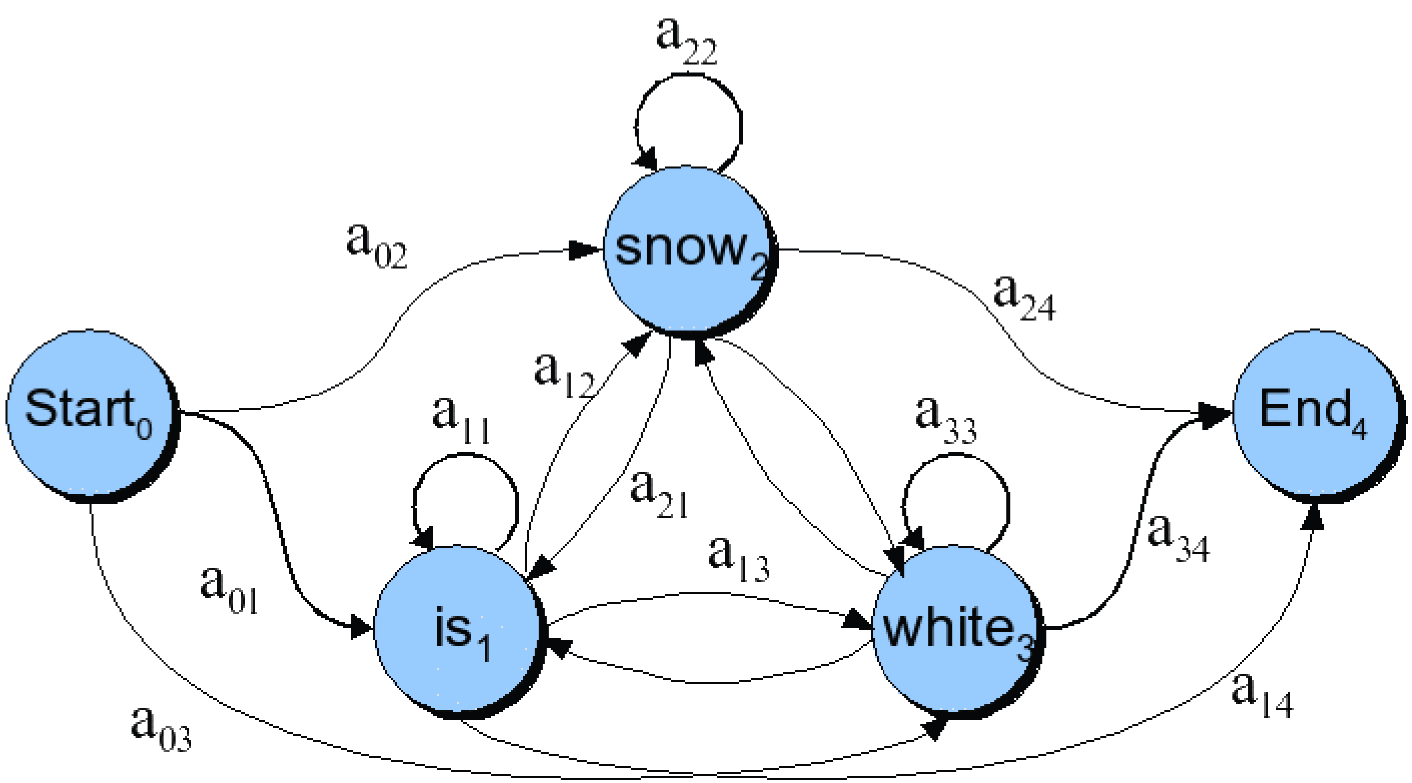
Definitions

- A weighted finite-state automaton adds probabilities to the arcs
  - The sum of the probabilities leaving any state must sum to one
- A Markov chain is a special case of a WFSA in which the input sequence uniquely determines which states the automaton will go through
- Markov chains can't represent inherently ambiguous problems
  - Useful for assigning probabilities to unambiguous sequences

Markov Chain for Weather



Markov Chain for Words



Markov Chain Model

- A set of states
  - $Q = q_1, q_2 \ldots q_{N;}$ the state at time t is q_t
- Transition probabilities:
  - a set of probabilities $A = a_{01} a_{02} \ldots a_{n1} \ldots a_{nn}$.
  - Each a_{ij} represents the probability of transitioning from state i to state j
  - The set of these is the transition probability matrix A

- Markov Assumption: Current state only depends on previous state

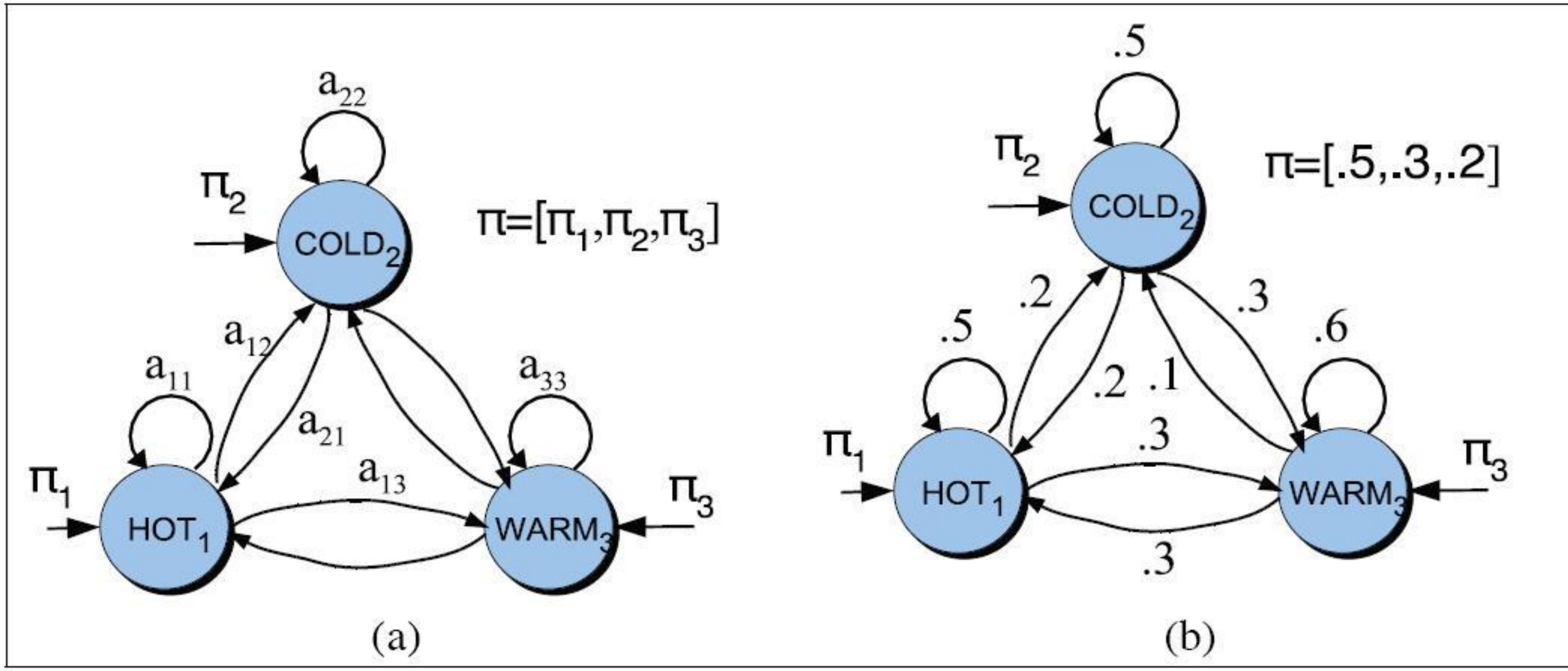
$$P(q_i \mid q_1 \ldots q_{i-1}) = P(q_i \mid q_{i-1})$$

Markov Chain Model

$$\sum_{j=1}^{n} a_{ij} = 1 \quad \forall i$$

$\pi = \pi_1, \pi_2, \ldots, \pi_N$	an **initial probability distribution** over states. π_i is the probability that the Markov chain will start in state i. Some states j may have $\pi_j = 0$, meaning that they cannot be initial states. Also, $\sum_{i=1}^{n} \pi_i = 1$
$QA = \{q_x, q_y \ldots\}$	a set $QA \subset Q$ of legal **accepting states**

Weather example



Markov chains are useful when we need to compute the probabilities for a sequence of events that are observable.

Markov Chain for Weather

- What is the probability of 4 consecutive warm days?
- Sequence is warm-warm-warm-warm
- I.e., state sequence is 3-3-3-3
- P(3,3,3,3) = $\pi_3 a_{33} a_{33} a_{33} a_{33}$ = 0.2 x $(0.6)^3$ = 0.0432

- But what about if states are not observable?

HMM for Ice Cream

- You are a climatologist in the year 2799
- Studying climate change
- You can't find any records of the weather in Baltimore, MA for summer of 2007
- But you find Jason Eisner's diary

- Which lists how many ice-creams Jason ate every date that summer
- Your job: figure out how hot it was

Hidden Markov Model

- For Markov chains, the output symbols are the same as the states.
  - See **hot** weather: we're in state **hot**
- But in part-of-speech tagging (and other things)
  - The output symbols are **words**
  - But the hidden states are **part-of-speech tags**
- So we need an extension!
- A Hidden Markov Model is an extension of a Markov chain in which the input symbols are not the same as the states.
- This means we don't know which state we are in.

Hidden Markov Models

- States $Q = q_1, q_2 \ldots q_{N;}$ the state at time t is q_t
- Observations $O= o_1, o_2 \ldots o_{T;}$
  - Each observation is a symbol from a vocabulary $V = \{v_1,v_2,\ldots v_V\}$
- Transition probabilities
  - Transition probability matrix $A = \{a_{ij}\}$

$$a_{ij} = P(q_t = j \mid q_{t-1} = i) \quad 1 \leq i,j \leq N$$

- Observation likelihoods
  - Output probability matrix $B=\{b_i(t)\}$

$$b_i(t) = P(X_t = o_t \mid q_t = i)$$

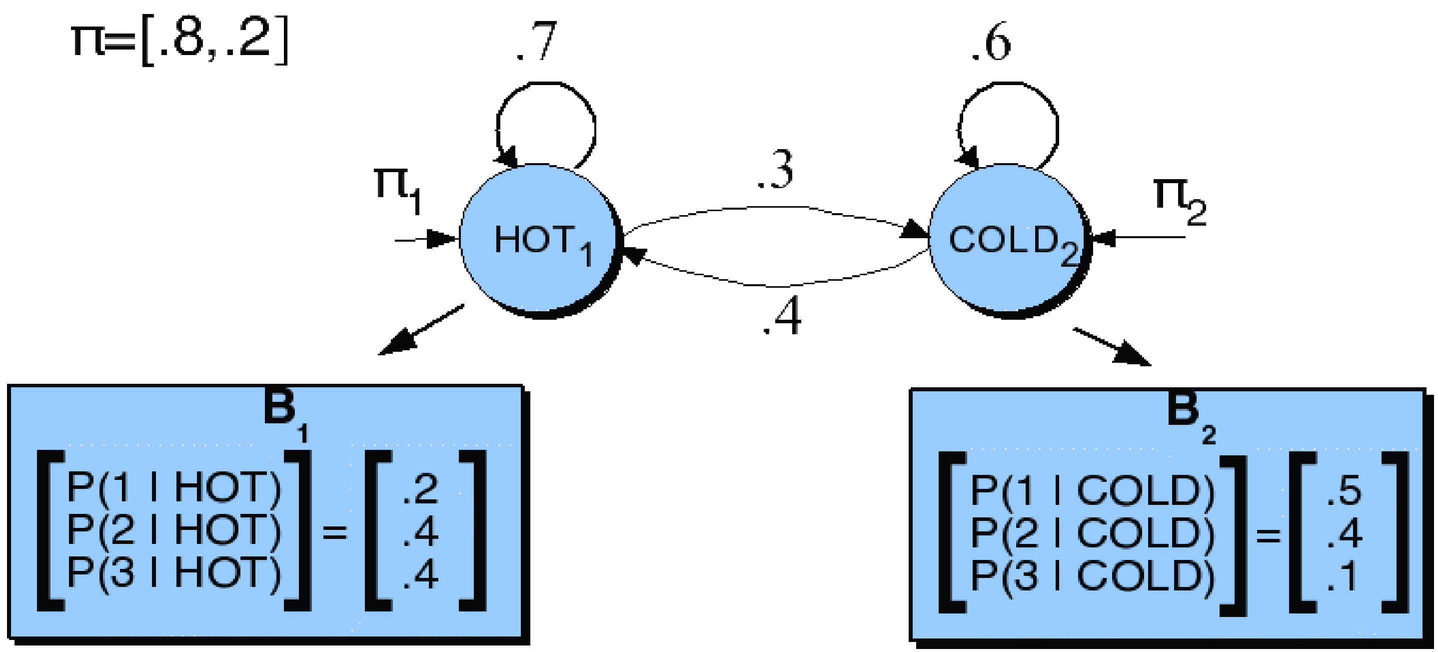
- Special initial probability vector π

$$\pi_i = P(q_{t=1} = i) \quad 1 \leq i \leq N$$

Eisner Task

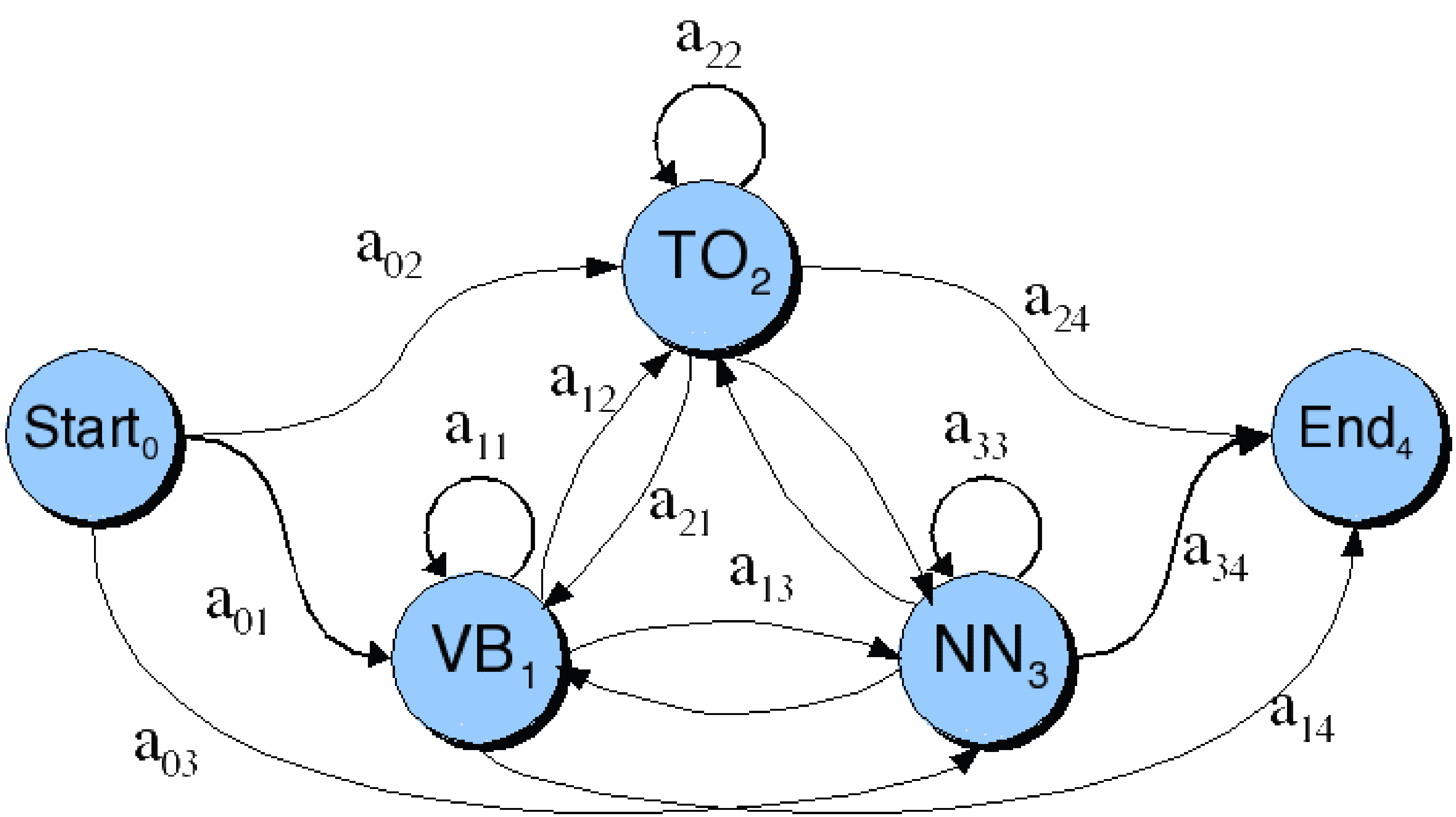
- Given
  - Ice Cream Observation Sequence. E.g.: 1,2,3,2,2,2,3…
- Produce:
  - Weather Sequence. E.g.: H,C,H,H,H,C…

HMM for Ice Cream

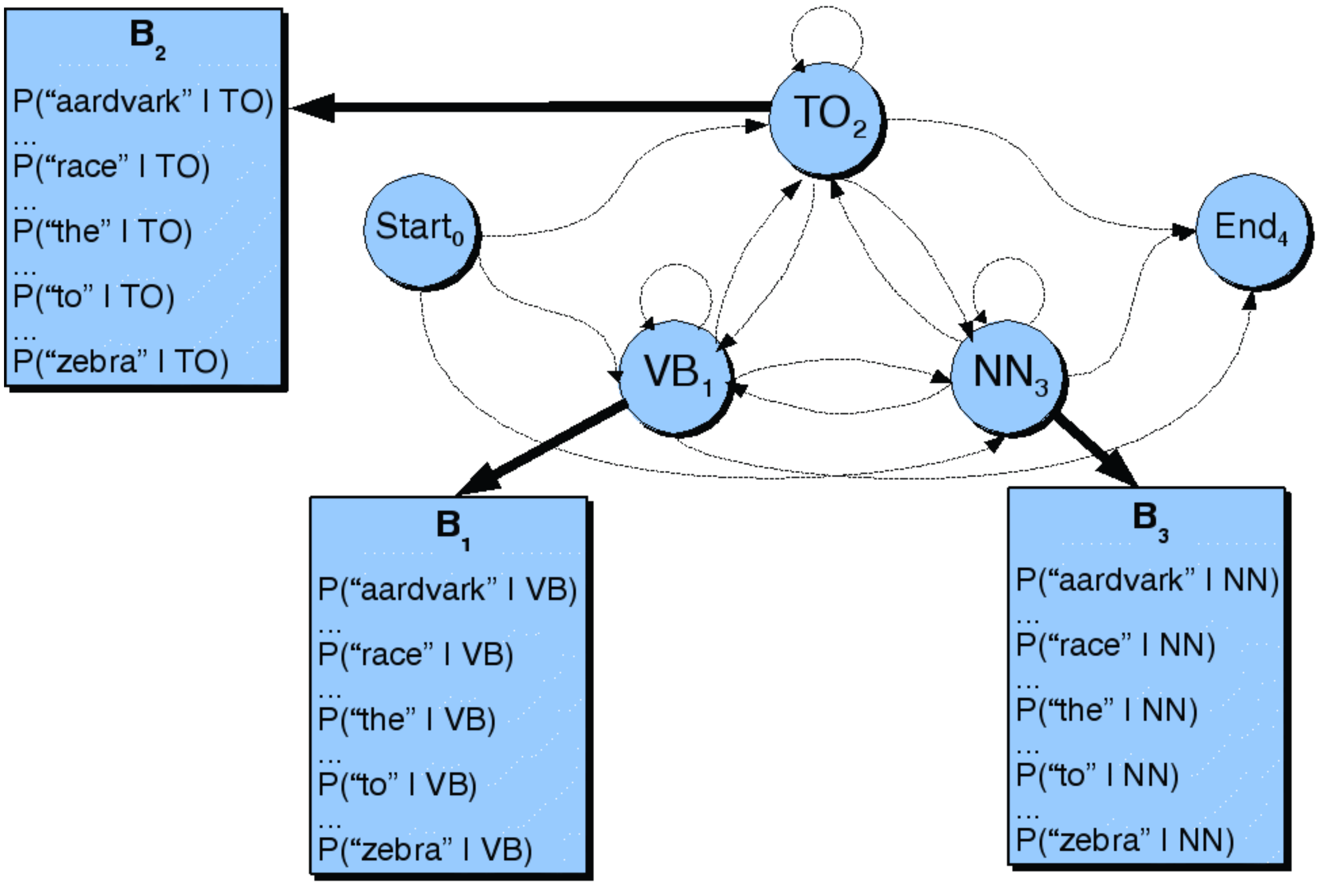


- There are two hidden states: Q = {hot,cold}
- Observations are the number of ice cream events: O = {1,2,3}

Transition Probabilities



Observation Likelihoods



HMM for Three Basic Problems

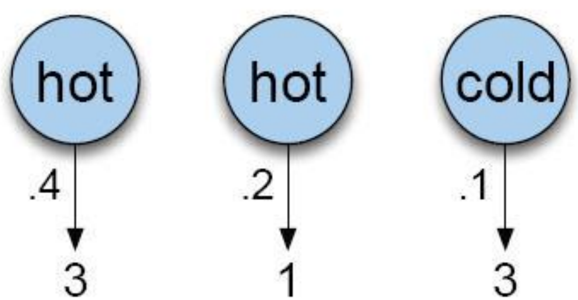
Problem 1 (Likelihood):	Given an HMM $\lambda = (A,B)$ and an observation sequence O, determine the likelihood $P(O\|\lambda)$.
Problem 2 (Decoding):	Given an observation sequence O and an HMM $\lambda = (A,B)$, discover the best hidden state sequence Q.
Problem 3 (Learning):	Given an observation sequence O and the set of states in the HMM, learn the HMM parameters A and B.

Likelihood Computation

Given an HMM $\lambda = (A, B)$ and an observation sequence O. Determine the likelihood $P(O|\lambda)$.

Problem 1: Compute the probability of eating 3 1 3 ice creams.

Problem 2: Compute the probability of eating 3 1 3 ice creams when the hidden sequence is hot hot cold.

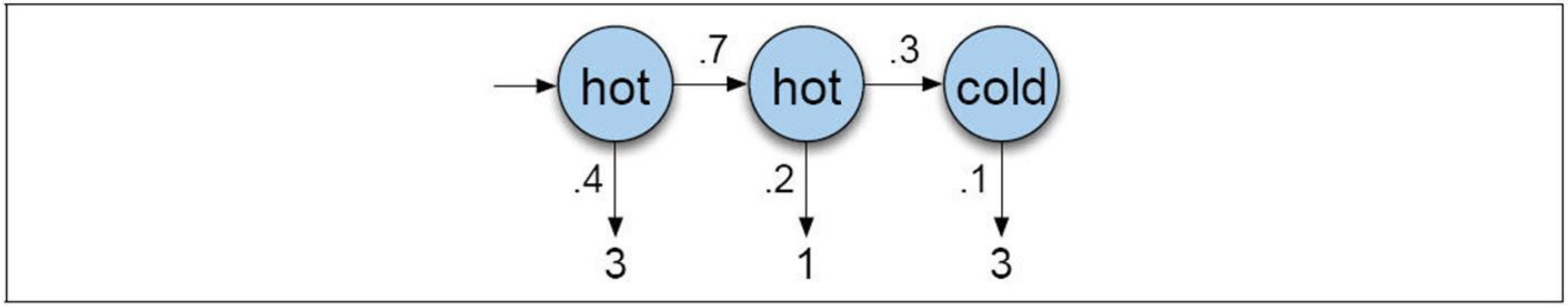


Likelihood Computation

- For a particular hidden state sequence Q
- And an observation sequence O
- The likelihood of the observation sequence is

$$P(O|Q) = \prod_{i=1}^{T} P(o_i|q_i)$$

$$P(3\ 1\ 3|\text{hot hot cold}) = P(3|\text{hot}) \text{ x } P(1|\text{hot}) \text{ x } P(3|\text{cold})$$



Likelihood Computation

Joint probability of being in a weather state sequence Q and a particular sequence of observations O of ice cream events is:

$$P(O,Q) = P(O|Q) \text{x}\, P(Q) = \prod_{i=1}^{n} P\left(o_{i|}q_i\right) \text{x} \prod_{i=1}^{n} P(q_i|q_{i-1})$$

$$P(3\ 1\ 3, \text{hot hot cold}) = P(\text{hot}|\text{start})\ \text{x}\ P(\text{hot}|\text{hot})\ \text{x}\ P(\text{cold}|\text{hot})$$

$$\text{x}\ P(3|\text{hot})\ \text{x}\ P(1|\text{hot})\ \text{x}\ \ P(3|\text{cold})$$

Likelihood Computation

We can compute now the probability of a sequence of observations O using the joint probabilities

$$P(O) = \sum_{Q} P\,(O, Q) = \sum_{Q} P(O|Q)P(Q)$$

P(3 1 3) = P(3 1 3 , cold cold cold) + P(3 1 3, cold cold hot) + . . .
. . . + P(3 1 3, hot hot hot)

Forward Algorithm

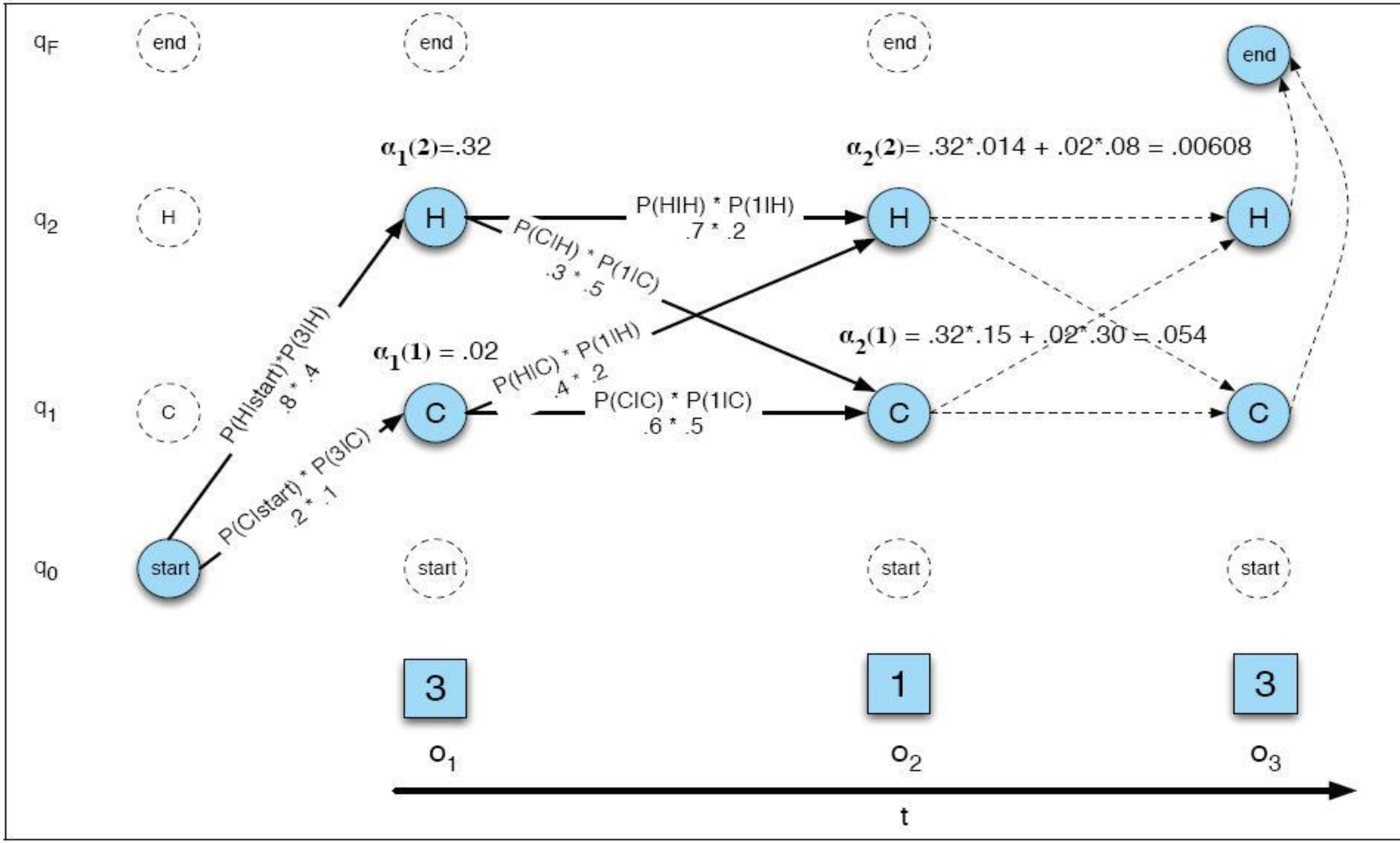
For N hidden states and a sequence of T observations Forward Algorithm uses $O(N^2T)$ operations instead of N^T

$a_t(j)$is the probability of being in state j after seing the first t observations

$$a_t(j) = P(o_1, o_2 \dots o_t, q_t = j|\lambda)$$

$$a_t(j) = \sum_{i=1}^{N} a_{t-1}(i)\mathrm{a}_{ij} b_j(o_t)$$

Forward trellis for ice cream example



Forward Algorithm

$\alpha_{t-1}(i)$	the **previous forward path probability** from the previous time step
a_{ij}	the **transition probability** from previous state q_i to current state q_j
$b_j(o_t)$	the **state observation likelihood** of the observation symbol o_t given the current state j

1. Initialization

$$a_1(j) = \mathrm{a}_{0j} b_j(o_1) \quad 1 \le j \le N$$

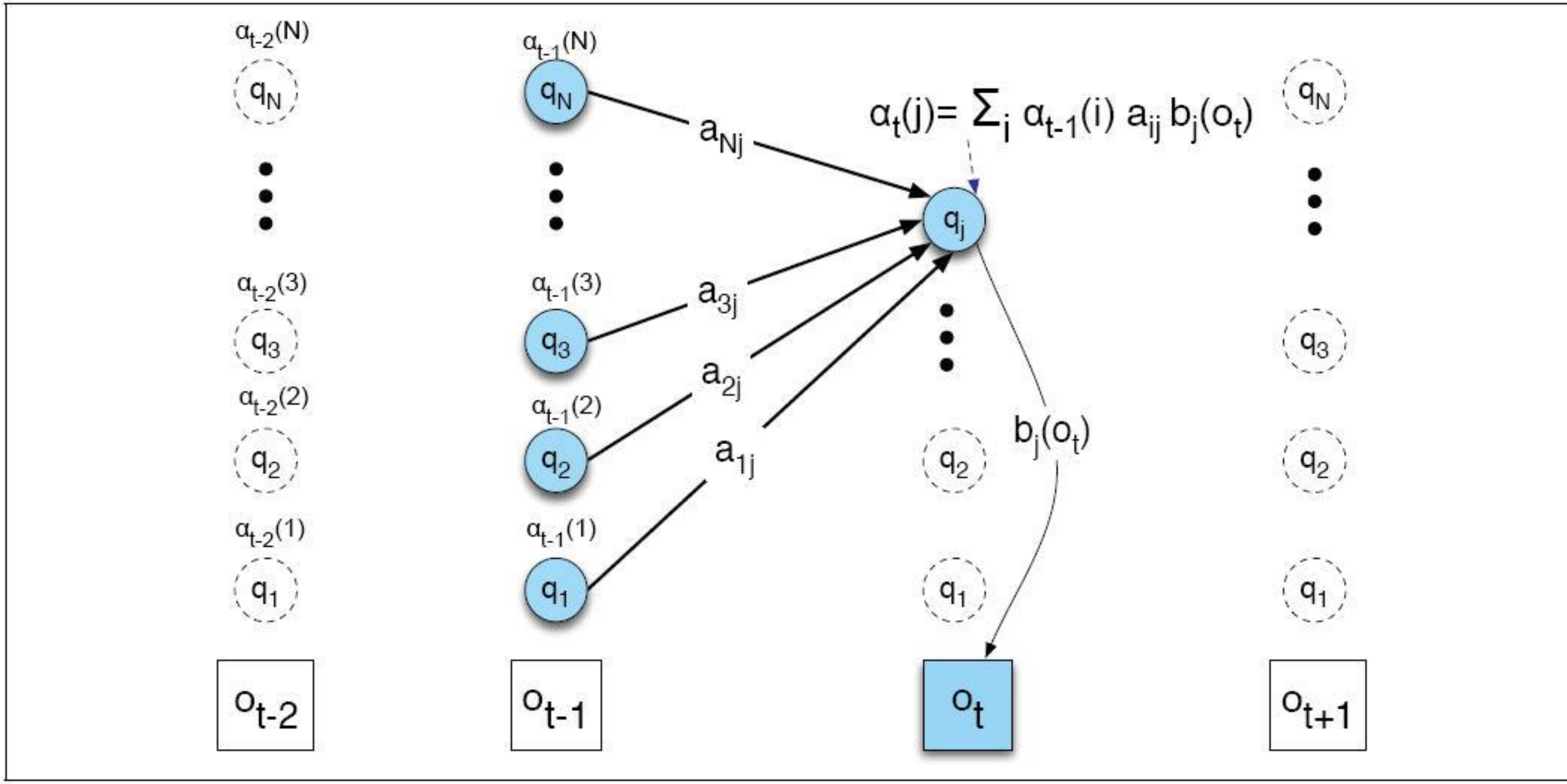
2. Recursion

$$a_t(j) = \sum_{i=1}^{N} a_{t-1}(i) \mathrm{a}_{ij} b_j(o_t); \quad 1 \le j \le N, 1 < t \le T$$

3. Termination

$$P(O|\lambda) = a_T(q_F) = \sum_{i=1}^{N} a_T(i) \mathrm{a}_{iF}$$

Forward Algorithm



Forward Algorithm

function FORWARD(*observations* of len T, *state-graph* of len N) **returns** *forward-prob*

create a probability matrix *forward[N+2,T]*
for each state s **from** 1 **to** N **do** ; initialization step

$$forward[s,1] \leftarrow a_{0,s} * b_s(o_1)$$

for each time step t **from** 2 **to** T **do** ; recursion step
for each state s **from** 1 **to** N **do**

$$forward[s,t] \leftarrow \sum_{s'=1}^{N} forward[s',t-1] * a_{s',s} * b_s(o_t)$$

$$forward[q_F,\mathrm{T}] \leftarrow \sum_{s=1}^{N} forward[s,T] * a_{s,q_F}$$; termination step

return $forward[q_F, T]$

Decoding

Decoding: given as input HMM $\lambda = (A, B)$ and a sequence of observations $O = o_1, o_2, \ldots, o_T$, find the most sequence of states $Q = q_1 q_2 q_3 \ldots q_T$

- POS tagging is such a problem, and so is the weather problem
- Recall that in the case of POS tagging we need to compute

$$\hat{t}_1^n = \underset{t_1^n}{\operatorname{argmax}} P(t_1^n \mid w_1^n)$$

- We could just enumerate all paths given the input and use the model to assign probabilities to each.
  - Not a good idea.
  - Luckily dynamic programming helps us here

Viterbi Algorithm

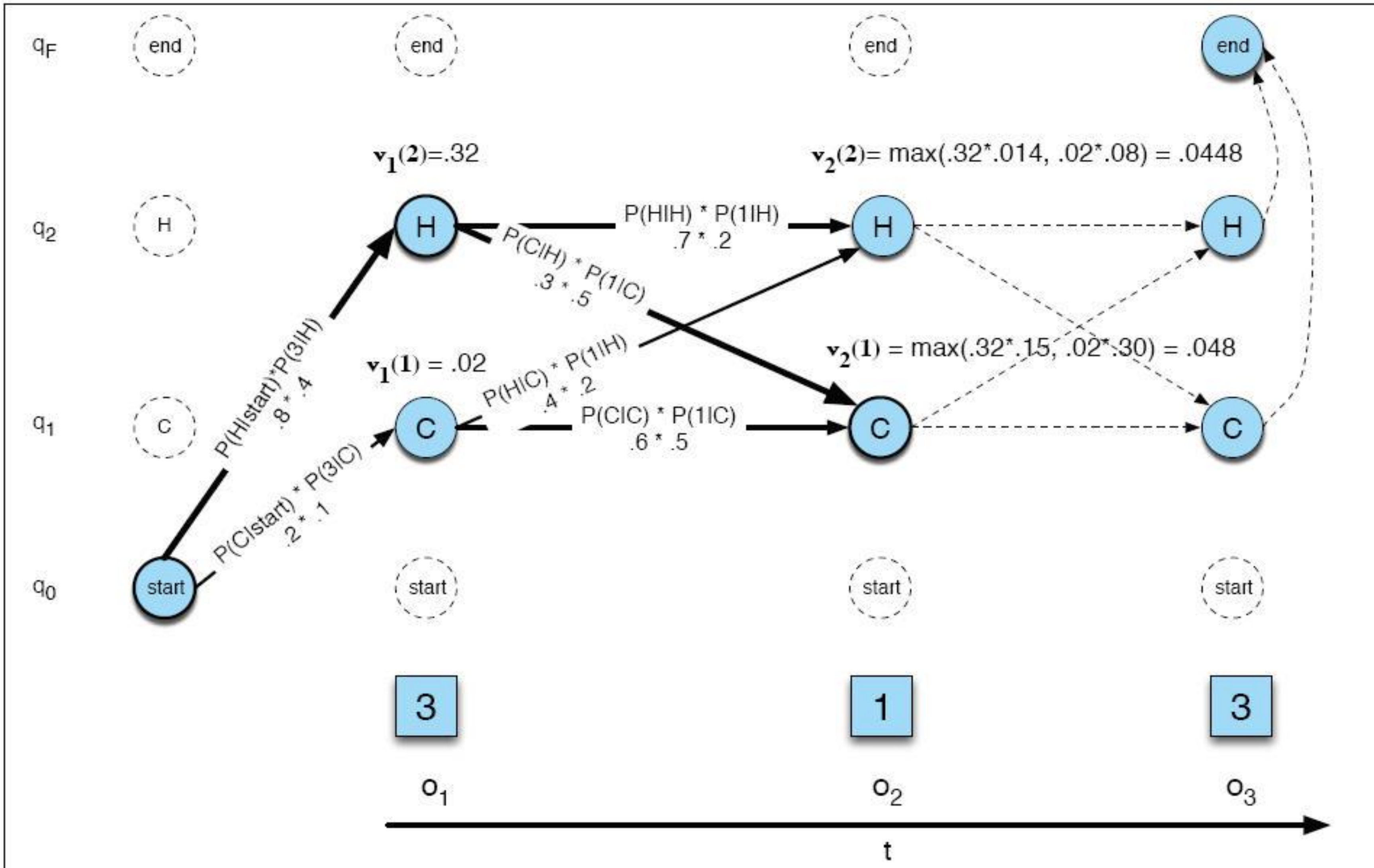
Viterbi algorithm computes a trellis using dynamic programming.
Observation is processed from left to right filling out a trellis of states
$v_t(j)$ is the probability that HMM is in state j after seeing the first t observations

$v_{t-1}(i)$	the **previous Viterbi path probability** from the previous time step
a_{ij}	the **transition probability** from previous state q_i to current state q_j
$b_j(o_t)$	the **state observation likelihood** of the observation symbol o_t given the current state j

$$v_t(j) = \max_{qo,qi\ldots qi-1} P(q_0, q_1 \ldots q_{t-1}, o_1, o_2 \ldots o_t q_t = j|\lambda)$$

$$v_t(j) = \max_{i=1}^{N} v_{t-1}(i)\, a_{ij} b_j(o_t)$$

Viterbi trellis for ice cream example



Viterbi Algorithm

1. Initialization

$$v_1(j) = a_{0j} b_j(o_1) 1 \le j \le N$$

$$bt_1(j) = 0$$

2. Recursion

$$v_t(j) = \max_{i=1}^{N} v_{t-1}(i)\, \mathrm{a}_{ij} b_j(o_t); \quad 1 \le j \le N, 1 < t \le T$$

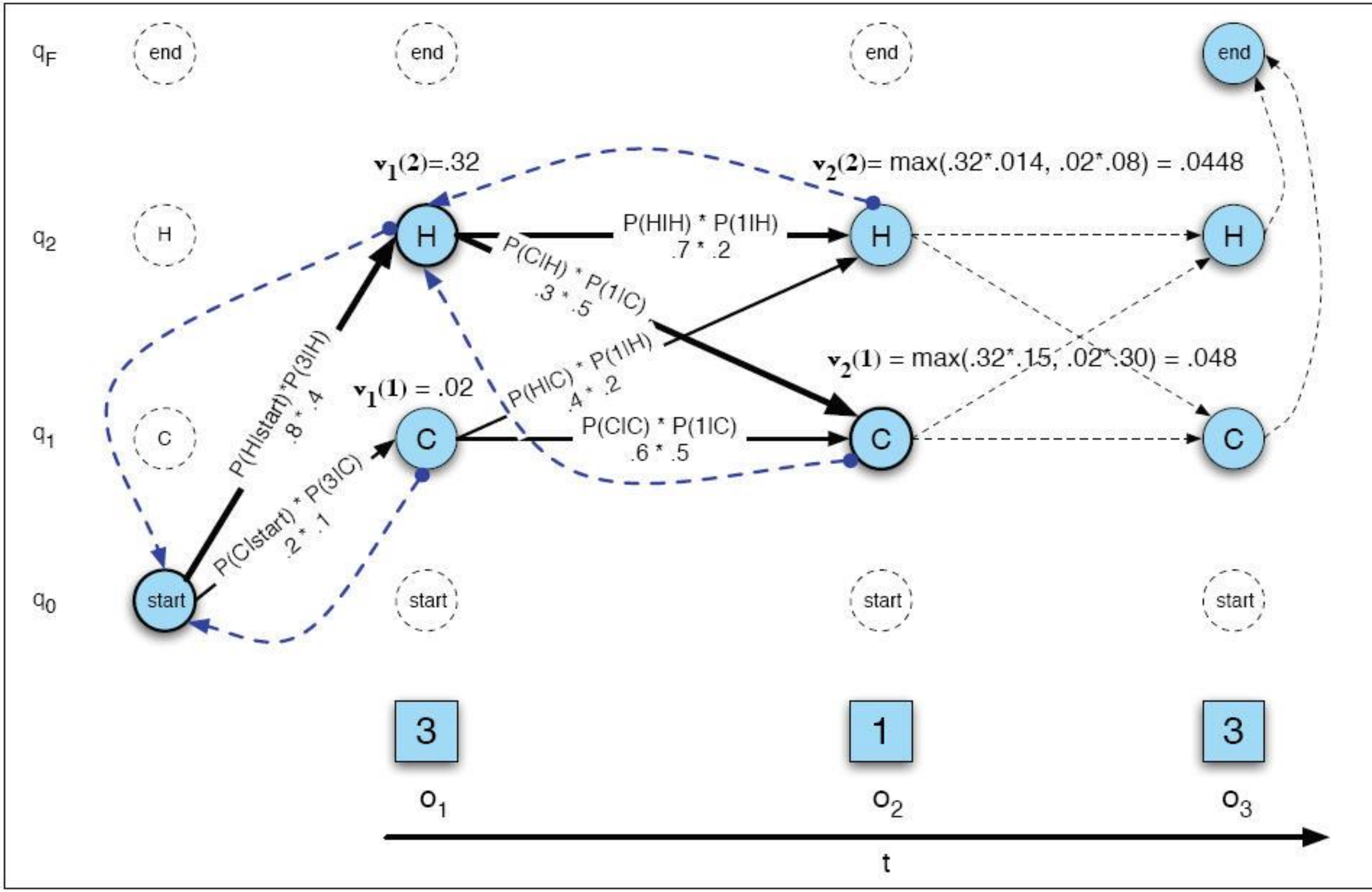
$$bt_t(j) = \operatorname{argmax}_{i=1}^{N} v_{t-1}(i)\, \mathrm{a}_{ij} b_j(o_t); \quad 1 \le j \le N, 1 < t \le T$$

3. Termination

The best score: $P* = v_t(q_F) = \max_{i=1}^{N} v_T(i) * \mathrm{a}_{i,F}$

The start of backtrace: $q_T* = b_{tT}(q_F) = \operatorname{argmax}_{i=1}^{N} v_T(i) * \mathrm{a}_{i,F}$

Viterbi Traceback



The Viterbi Algorithm

function VITERBI(*observations* of len T,*state-graph* of len N) **returns** *best-path*

create a path probability matrix *viterbi[N+2,T]*

for each state s **from** 1 **to** N **do** ; initialization step

$viterbi[s,1] \leftarrow a_{0,s} * b_s(o_1)$

$backpointer[s,1] \leftarrow 0$

for each time step t **from** 2 **to** T **do** ; recursion step

for each state s **from** 1 **to** N **do**

$$viterbi[s,t] \leftarrow \max_{s'=1}^{N} viterbi[s',t-1] * a_{s',s} * b_s(o_t)$$

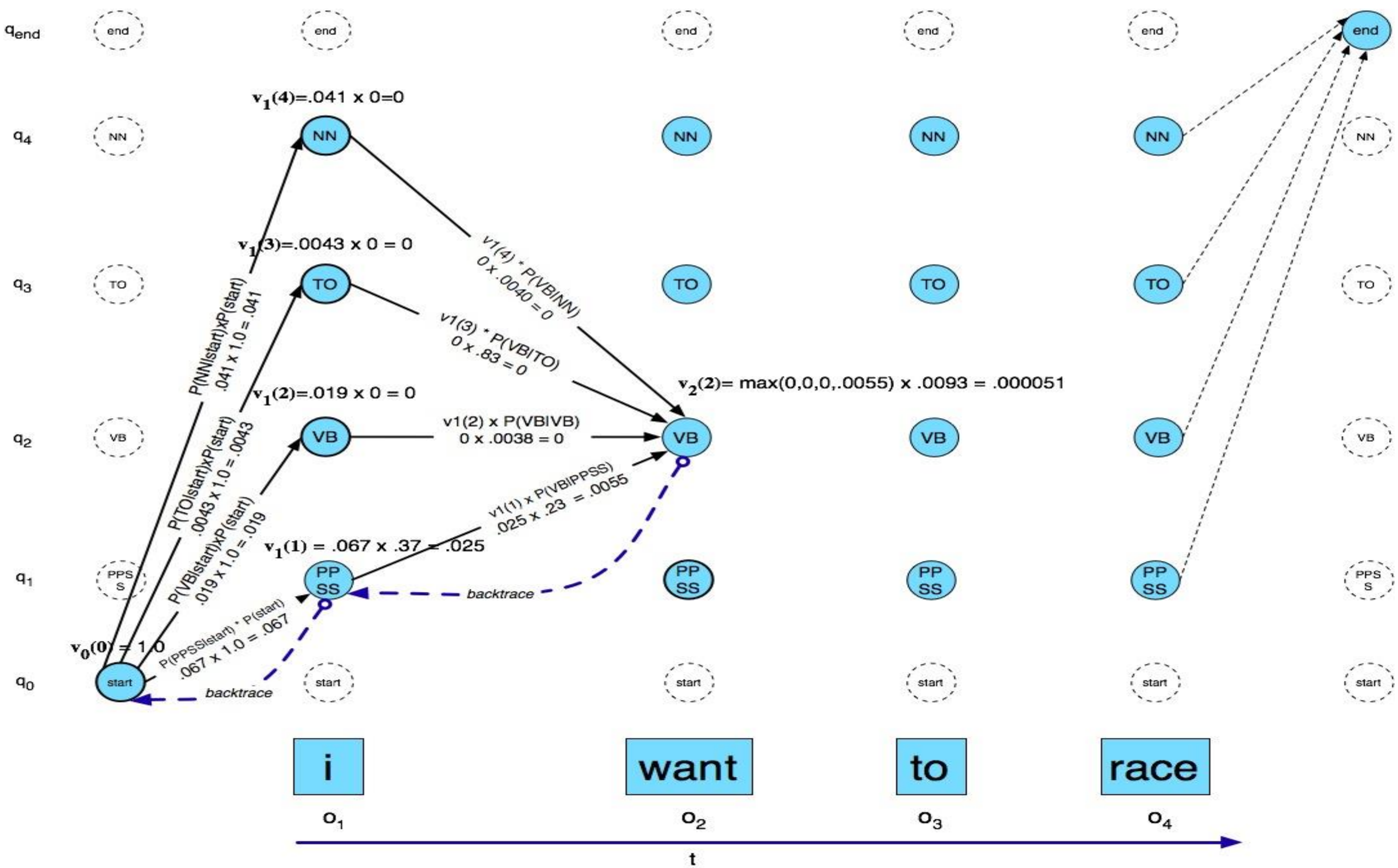
$$backpointer[s,t] \leftarrow \operatorname{argmax}_{s'=1}^{N} viterbi[s',t-1] * a_{s',s}$$

$$viterbi[q_F,T] \leftarrow \max_{s=1}^{N} viterbi[s,T] * a_{s,q_F}$$; termination step

$$backpointer[q_F,T] \leftarrow \operatorname{argmax}_{s=1}^{N} viterbi[s,T] * a_{s,q_F}$$; termination step

return the backtrace path by following backpointers to states back in time from $backpointer[q_F,T]$

Viterbi Example



Viterbi Summary

- Create an array
  - With columns corresponding to inputs
  - Rows corresponding to possible states
- Sweep through the array in one pass filling the columns left to right using our transition probs and observations probs
- Dynamic programming key is that we need only store the MAX prob path to each cell, (not all paths).

Evaluation

- So once you have your POS tagger running how do you evaluate it?
  - Overall error rate with respect to a gold-standard test set.
  - Error rates on particular tags
  - Error rates on particular words
  - Tag confusions...

Error Analysis

- Look at a confusion matrix

	IN	JJ	NN	NNP	RB	VBD	VBN
IN	—	.2			.7		
JJ	.2	—	**3.3**	2.1	1.7	.2	**2.7**
NN		**8.7**	—				.2
NNP	.2	**3.3**	**4.1**	—	.2		
RB	**2.2**	2.0	.5		—		
VBD		.3	.5			—	**4.4**
VBN		**2.8**				**2.6**	—

- See what errors are causing problems
  - Noun (NN) vs ProperNoun (NNP) vs Adj (JJ)
  - Preterite (VBD) vs Participle (VBN) vs Adjective (JJ)

Evaluation

- The result is compared with a manually coded "Gold Standard"
  - Typically accuracy reaches 96-97%
  - This may be compared with result for a baseline tagger (one that uses no context).
- Important: 100% is impossible even for human annotators.