Chapter One

1. $AC\{D,B\} = ACDB + ACBD$, $A\{C,B\}D = ACBD + ABCD$, $C\{D,A\}B = CDAB + CADB$, and $\{C,A\}DB = CADB + ACDB$. Therefore $-AC\{D,B\} + A\{C,B\}D - C\{D,A\}B + \{C,A\}DB = -ACDB + ABCD - CDAB + ACDB = ABCD - CDAB = [AB, CD]$

In preparing this solution manual, I have realized that problems 2 and 3 in are misplaced in this chapter. They belong in Chapter Three. The Pauli matrices are not even defined in Chapter One, nor is the math used in previous solution manual. – *Jim Napolitano*

2. (a) $\mathsf{Tr}(X) = a_0\mathsf{Tr}(1) + \sum_\ell \mathsf{Tr}(\sigma_\ell)a_\ell = 2a_0$ since $\mathsf{Tr}(\sigma_\ell) = 0$. Also $\mathsf{Tr}(\sigma_k X) = a_0\mathsf{Tr}(\sigma_k) + \sum_\ell \mathsf{Tr}(\sigma_k\sigma_\ell)a_\ell = \frac{1}{2}\sum_\ell \mathsf{Tr}(\sigma_k\sigma_\ell + \sigma_\ell\sigma_k)a_\ell = \sum_\ell \delta_{k\ell}\mathsf{Tr}(1)a_\ell = 2a_k$. So, $a_0 = \frac{1}{2}\mathsf{Tr}(X)$ and $a_k = \frac{1}{2}\mathsf{Tr}(\sigma_k X)$. (b) Just do the algebra to find $a_0 = (X_{11} + X_{22})/2$, $a_1 = (X_{12} + X_{21})/2$, $a_2 = i(-X_{21} + X_{12})/2$, and $a_3 = (X_{11} - X_{22})/2$.

3. Since $\det(\boldsymbol{\sigma}\cdot\mathbf{a}) = -a_z^2 - (a_x^2 + a_y^2) = -|\mathbf{a}|^2$, the cognoscenti realize that this problem really has to do with rotation operators. From this result, and (3.2.44), we write

$$\det\left[\exp\left(\pm\frac{i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\phi}{2}\right)\right] = \cos\left(\frac{\phi}{2}\right) \pm i\sin\left(\frac{\phi}{2}\right)$$

and multiplying out determinants makes it clear that $\det(\boldsymbol{\sigma}\cdot\mathbf{a}') = \det(\boldsymbol{\sigma}\cdot\mathbf{a})$. Similarly, use (3.2.44) to explicitly write out the matrix $\boldsymbol{\sigma}\cdot\mathbf{a}'$ and equate the elements to those of $\boldsymbol{\sigma}\cdot\mathbf{a}$. With $\hat{\mathbf{n}}$ in the z-direction, it is clear that we have just performed a rotation (of the spin vector) through the angle ϕ.

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4. (a) $\mathsf{Tr}(XY) \equiv \sum_a \langle a|XY|a\rangle = \sum_a\sum_b \langle a|X|b\rangle\langle b|Y|a\rangle$ by inserting the identity operator. Then commute and reverse, so $\mathsf{Tr}(XY) = \sum_b\sum_a \langle b|Y|a\rangle\langle a|X|b\rangle = \sum_b \langle b|YX|b\rangle = \mathsf{Tr}(YX)$.
(b) $XY|\alpha\rangle = X[Y|\alpha\rangle]$ is dual to $\langle\alpha|(XY)^\dagger$, but $Y|\alpha\rangle \equiv |\beta\rangle$ is dual to $\langle\alpha|Y^\dagger \equiv \langle\beta|$ and $X|\beta\rangle$ is dual to $\langle\beta|X^\dagger$ so that $X[Y|\alpha\rangle]$ is dual to $\langle\alpha|Y^\dagger X^\dagger$. Therefore $(XY)^\dagger = Y^\dagger X^\dagger$.
(c) $\exp[if(A)] = \sum_a \exp[if(A)]|a\rangle\langle a| = \sum_a \exp[if(a)]|a\rangle\langle a|$
(d) $\sum_a \psi_a^*(\mathbf{x}')\psi_a(\mathbf{x}'') = \sum_a \langle\mathbf{x}'|a\rangle^*\langle\mathbf{x}''|a\rangle = \sum_a \langle\mathbf{x}''|a\rangle\langle a|\mathbf{x}'\rangle = \langle\mathbf{x}''|\mathbf{x}'\rangle = \delta(\mathbf{x}'' - \mathbf{x}')$

5. For basis kets $|a_i\rangle$, matrix elements of $X \equiv |\alpha\rangle\langle\beta|$ are $X_{ij} = \langle a_i|\alpha\rangle\langle\beta|a_j\rangle = \langle a_i|\alpha\rangle\langle a_j|\beta\rangle^*$. For spin-1/2 in the $|\pm z\rangle$ basis, $\langle +|S_z = \hbar/2\rangle = 1$, $\langle -|S_z = \hbar/2\rangle = 0$, and, using (1.4.17a), $\langle\pm|S_x = \hbar/2\rangle = 1/\sqrt{2}$. Therefore

$$|S_z = \hbar/2\rangle\langle S_x = \hbar/2| \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

6. $A[|i\rangle + |j\rangle] = a_i|i\rangle + a_j|j\rangle \neq [|i\rangle + |j\rangle]$ so in general it is not an eigenvector, unless $a_i = a_j$. That is, $|i\rangle + |j\rangle$ is not an eigenvector of A unless the eigenvalues are degenerate.

7. Since the product is over a complete set, the operator $\prod_{a'}(A-a')$ will always encounter a state $|a_i\rangle$ such that $a'=a_i$ in which case the result is zero. Hence for any state $|\alpha\rangle$

$$\prod_{a'}(A-a')|\alpha\rangle = \prod_{a'}(A-a')\sum_i |a_i\rangle\langle a_i|\alpha\rangle = \sum_i\prod_{a'}(a_i-a')|a_i\rangle\langle a_i|\alpha\rangle = \sum_i 0 = 0$$

If the product instead is over all $a'\neq a_j$ then the only surviving term in the sum is

$$\prod_{a'}(a_j-a')|a_i\rangle\langle a_i|\alpha\rangle$$

and dividing by the factors (a_j-a') just gives the projection of $|\alpha\rangle$ on the direction $|a'\rangle$. For the operator $A\equiv S_z$ and $\{|a'\rangle\}\equiv\{|+\rangle,|-\rangle\}$, we have

$$\begin{aligned}
& \prod_{a'}(A-a') &=& \left(S_z-\frac{\hbar}{2}\right)\left(S_z+\frac{\hbar}{2}\right) & \\
\text{and} \quad & \prod_{a'\neq a''}\frac{A-a'}{a''-a'} &=& \frac{S_z+\hbar/2}{\hbar} & \text{for } a''=+\frac{\hbar}{2} \\
& \text{or} &=& \frac{S_z-\hbar/2}{-\hbar} & \text{for } a''=-\frac{\hbar}{2}
\end{aligned}$$

It is trivial to see that the first operator is the null operator. For the second and third, you can work these out explicitly using (1.3.35) and (1.3.36), for example

$$\frac{S_z+\hbar/2}{\hbar} = \frac{1}{\hbar}\left[S_z+\frac{\hbar}{2}1\right] = \frac{1}{2}\left[(|+\rangle\langle+|)-(|-\rangle\langle-|)+(|+\rangle\langle+|)+(|-\rangle\langle-|)\right] = |+\rangle\langle+|$$

which is just the projection operator for the state $|+\rangle$.

8. I don't see any way to do this problem other than by brute force, and neither did the previous solutions manual. So, make use of $\langle+|+\rangle=1=\langle-|-\rangle$ and $\langle+|-\rangle=0=\langle-|+\rangle$ and carry through six independent calculations of $[S_i,S_j]$ (along with $[S_i,S_j]=-[S_j,S_i]$) and the six for $\{S_i,S_j\}$ (along with $\{S_i,S_j\}=+\{S_j,S_i\}$).

9. From the figure $\hat{\mathbf{n}}=\hat{\mathbf{i}}\cos\alpha\sin\beta+\hat{\mathbf{j}}\sin\alpha\sin\beta+\hat{\mathbf{k}}\cos\beta$ so we need to find the matrix representation of the operator $\mathbf{S}\cdot\hat{\mathbf{n}}=S_x\cos\alpha\sin\beta+S_y\sin\alpha\sin\beta+S_z\cos\beta$. This means we need the matrix representations of S_x, S_y, and S_z. Get these from the prescription (1.3.19) and the operators represented as outer products in (1.4.18) and (1.3.36), along with the association (1.3.39a) to define which element is which. Thus

$$S_x \doteq \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} \qquad S_y \doteq \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix} \qquad S_z \doteq \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

We therefore need to find the (normalized) eigenvector for the matrix

$$\begin{pmatrix}\cos\beta & \cos\alpha\sin\beta-i\sin\alpha\sin\beta\\ \cos\alpha\sin\beta+i\sin\alpha\sin\beta & -\cos\beta\end{pmatrix} = \begin{pmatrix}\cos\beta & e^{-i\alpha}\sin\beta\\ e^{i\alpha}\sin\beta & -\cos\beta\end{pmatrix}$$

with eigenvalue $+1$. If the upper and lower elements of the eigenvector are a and b, respectively, then we have the equations $|a|^2 + |b|^2 = 1$ and

$$\begin{aligned} a\cos\beta + be^{-i\alpha}\sin\beta &= a \\ ae^{i\alpha}\sin\beta - b\cos\beta &= b \end{aligned}$$

Choose the phase so that a is real and positive. Work with the first equation. (The two equations should be equivalent, since we picked a valid eigenvalue. You should check.) Then

$$\begin{aligned} a^2(1-\cos\beta)^2 &= |b|^2\sin^2\beta = (1-a^2)\sin^2\beta \\ 4a^2\sin^4(\beta/2) &= (1-a^2)4\sin^2(\beta/2)\cos^2(\beta/2) \\ a^2[\sin^2(\beta/2)+\cos^2(\beta/2)] &= \cos^2(\beta/2) \\ a &= \cos(\beta/2) \\ \text{and so}\qquad b &= ae^{i\alpha}\frac{1-\cos\beta}{\sin\beta} = \cos(\beta/2)e^{i\alpha}\frac{2\sin^2(\beta/2)}{2\sin(\beta/2)\cos(\beta/2)} \\ &= e^{i\alpha}\sin(\beta/2) \end{aligned}$$

which agrees with the answer given in the problem.

10. Use simple matrix techniques for this problem. The matrix representation for H is

$$H \doteq \begin{bmatrix} a & a \\ a & -a \end{bmatrix}$$

Eigenvalues E satisfy $(a-E)(-a-E) - a^2 = -2a^2 + E^2 = 0$ or $E = \pm a\sqrt{2}$. Let x_1 and x_2 be the two elements of the eigenvector. For $E = +a\sqrt{2} \equiv E^{(1)}$, $(1-\sqrt{2})x_1^{(1)} + x_2^{(1)} = 0$, and for $E = -a\sqrt{2} \equiv E^{(2)}$, $(1+\sqrt{2})x_1^{(2)} + x_2^{(2)} = 0$. So the eigenstates are represented by

$$|E^{(1)}\rangle \doteq N^{(1)}\begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix} \qquad \text{and} \qquad |E^{(2)}\rangle \doteq N^{(2)}\begin{bmatrix} -1 \\ \sqrt{2}+1 \end{bmatrix}$$

where ${N^{(1)}}^2 = 1/(4-2\sqrt{2})$ and ${N^{(2)}}^2 = 1/(4+2\sqrt{2})$.

11. It is of course possible to solve this using simple matrix techniques. For example, the characteristic equation and eigenvalues are

$$\begin{aligned} 0 &= (H_{11}-\lambda)(H_{22}-\lambda) - H_{12}^2 \\ \lambda &= \frac{H_{11}+H_{22}}{2} \pm \left[\left(\frac{H_{11}-H_{22}}{2}\right)^2 + H_{12}^2\right]^{1/2} \equiv \lambda_\pm \end{aligned}$$

You can go ahead and solve for the eigenvectors, but it is tedious and messy. However, there is a strong hint given that you can make use of spin algebra to solve this problem, another two-state system. The Hamiltonian can be rewritten as

$$H \doteq A\mathbf{1} + B\sigma_z + C\sigma_x$$

where $A \equiv (H_{11}+H_{22})/2$, $B \equiv (H_{11}-H_{22})/2$, and $C \equiv H_{12}$. The eigenvalues of the first term are both A, and the eigenvalues for the sum of the second and third terms are those of $\pm(2/\hbar)$ times a spin vector multiplied by $\sqrt{B^2+C^2}$. In other words, the eigenvalues of the full Hamiltonian are just $A \pm \sqrt{B^2+C^2}$ in full agreement with what we got with usual matrix techniques, above. From the hint (or Problem 9) the eigenvectors must be

$$|\lambda_+\rangle = \cos\frac{\beta}{2}|1\rangle + \sin\frac{\beta}{2}|2\rangle \qquad \text{and} \qquad |\lambda_-\rangle = -\sin\frac{\beta}{2}|1\rangle + \cos\frac{\beta}{2}|2\rangle$$

where $\alpha = 0$, $\tan\beta = C/B = 2H_{12}/(H_{11}-H_{22})$, and we do $\beta \to \pi - \beta$ to "flip the spin."

12. Using the result of Problem 9, the probability of measuring $+\hbar/2$ is

$$\left|\left[\frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}}\langle -|\right]\left[\cos\frac{\gamma}{2}|+\rangle + \sin\frac{\gamma}{2}|-\rangle\right]\right|^2 = \frac{1}{2}\left[\sqrt{\frac{1+\cos\gamma}{2}} + \sqrt{\frac{1-\cos\gamma}{2}}\right]^2 = \frac{1+\sin\gamma}{2}$$

The results for $\gamma = 0$ (i.e. $|+\rangle$), $\gamma = \pi/2$ (i.e. $|S_x+\rangle$), and $\gamma = \pi$ (i.e. $|-\rangle$) are 1/2, 1, and 1/2, as expected. Now $\langle (S_x - \langle S_x\rangle)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2$, but $S_x^2 = \hbar^2/4$ from Problem 8 and

$$\begin{aligned}
\langle S_x\rangle &= \left[\cos\frac{\gamma}{2}\langle +| + \sin\frac{\gamma}{2}\langle -|\right]\frac{\hbar}{2}\left[|+\rangle\langle -| + |-\rangle\langle +|\right]\left[\cos\frac{\gamma}{2}|+\rangle + \sin\frac{\gamma}{2}|-\rangle\right] \\
&= \frac{\hbar}{2}\left[\cos\frac{\gamma}{2}\langle -| + \sin\frac{\gamma}{2}\langle +|\right]\left[\cos\frac{\gamma}{2}|+\rangle + \sin\frac{\gamma}{2}|-\rangle\right] = \hbar\cos\frac{\gamma}{2}\sin\frac{\gamma}{2} = \frac{\hbar}{2}\sin\gamma
\end{aligned}$$

so $\langle (S_x - \langle S_x\rangle)^2\rangle = \hbar^2(1-\sin^2\gamma)/4 = \hbar^2\cos^2\gamma/4 = \hbar^2/4, 0, \hbar^2 4$ for $\gamma = 0, \pi/2, \pi$.

13. All atoms are in the state $|+\rangle$ after emerging from the first apparatus. The second apparatus projects out the state $|S_n+\rangle$. That is, it acts as the projection operator

$$|S_n+\rangle\langle S_n+| = \left[\cos\frac{\beta}{2}|+\rangle + \sin\frac{\beta}{2}|-\rangle\right]\left[\cos\frac{\beta}{2}\langle +| + \sin\frac{\beta}{2}\langle -|\right]$$

and the third apparatus projects out $|-\rangle$. Therefore, the probability of measuring $-\hbar/2$ after the third apparatus is

$$P(\beta) = |\langle +|S_n+\rangle\langle S_n+|-\rangle|^2 = \cos^2\frac{\beta}{2}\sin^2\frac{\beta}{2} = \frac{1}{4}\sin^2\beta$$

The maximum transmission is for $\beta = 90°$, when 25% of the atoms make it through.

14. The characteristic equation is $-\lambda^3 - 2(-\lambda)(1/\sqrt{2})^2 = \lambda(1-\lambda^2) = 0$ so the eigenvalues are $\lambda = 0, \pm 1$ and there is no degeneracy. The eigenvectors corresponding to these are

$$\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} \qquad \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

The matrix algebra is not hard, but I did this with MATLAB using

```
M=[[0 1 0];[1 0 1];[0 1 0]]/sqrt(2)
[V,D]=eig(M)
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These are the eigenvectors corresponding to the a spin-one system, for a measurement in the x-direction in terms of a basis defined in the z-direction. I'm not sure if there is enough information in Chapter One, though, in order to deduce this.

15. The answer is *yes*. The identity operator is $1 = \sum_{a',b'} |a',b'\rangle\langle a',b'|$ so

$$AB = AB1 = AB\sum_{a',b'} |a',b'\rangle\langle a',b'| = A\sum_{a',b'} b'|a',b'\rangle\langle a',b'| = \sum_{a',b'} b'a'|a',b'\rangle\langle a',b'| = BA$$

Completeness is powerful. It is important to note that the sum must be over both a' and b' in order to span the complete set of sets.

16. Since $AB = -BA$ and $AB|a,b\rangle = ab|a,b\rangle = BA|a,b\rangle$, we must have $ab = -ba$ where both a and b are real numbers. This can only be satisfied if $a = 0$ or $b = 0$ or both.

17. Assume there is no degeneracy and look for an inconsistency with our assumptions. If $|n\rangle$ is a nondegenerate energy eigenstate with eigenvalue E_n, then it is the *only* state with this energy. Since $[H.A_1] = 0$, we must have $HA_1|n\rangle = A_1H|n\rangle = E_nA_1|n\rangle$. That is, $A_1|n\rangle$ is an eigenstate of energy with eigenvalue E_n. Since H and A_1 commute, though, they may have simultaneous eigenstates. Therefore, $A_1|n\rangle = a_1|n\rangle$ since there is only one energy eigenstate.

Similarly, $A_2|n\rangle$ is also an eigenstate of energy with eigenvalue E_n, and $A_2|n\rangle = a_2|n\rangle$. But $A_1A_2|n\rangle = a_2A_1|n\rangle = a_2a_1|n\rangle$ and $A_2A_1|n\rangle = a_1a_2|n\rangle$, where a_1 and a_2 are real numbers. This cannot be true, in general, if $A_1A_2 \neq A_2A_1$ so our assumption of "no degeneracy" must be wrong. There is an out, though, if $a_1 = 0$ or $a_2 = 0$, since one operator acts on zero.

The example given is from a "central forces" Hamiltonian. (See Chapter Three.) The Hamiltonian commutes with the orbital angular momentum operators L_x and L_y, but $[L_x, L_y] \neq 0$. Therefore, in general, there is a degeneracy in these problems. The degeneracy is avoided, though for S-states, where the quantum numbers of L_x and L_y are both necessarily zero.

18. The positivity postulate says that $\langle\gamma|\gamma\rangle \geq 0$, and we apply this to $|\gamma\rangle \equiv |\alpha\rangle + \lambda|\beta\rangle$. The text shows how to apply this to prove the Schwarz Innequality $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$, from which one derives the generalized uncertainty relation (1.4.53), namely

$$\langle(\Delta A)^2(\Delta B)^2\rangle \geq \frac{1}{4}\left|\langle[A,B]\rangle\right|^2$$

Note that $[\Delta A, \Delta B] = [A - \langle A\rangle, B - \langle B\rangle] = [A, B]$. Taking $\Delta A|\alpha\rangle = \lambda\Delta B|\alpha\rangle$ with $\lambda^* = -\lambda$, as suggested, so $\langle\alpha|\Delta A = -\lambda\langle\alpha|\Delta B$, for a particular state $|\alpha\rangle$. Then

$$\langle\alpha|[A,B]|\alpha\rangle = \langle\alpha|\Delta A\Delta B - \Delta B\Delta A|\alpha\rangle = -2\lambda\langle\alpha|(\Delta B)^2|\alpha\rangle$$

and the equality is clearly satisfied in (1.4.53). We are now asked to verify this relationship for a state $|\alpha\rangle$ that is a gaussian wave packet when expressed as a wave function $\langle x'|\alpha\rangle$. Use

$$\begin{aligned}
\langle x'|\Delta x|\alpha\rangle &= \langle x'|x|\alpha\rangle - \langle x\rangle\langle x'|\alpha\rangle = (x'-\langle x\rangle)\langle x'|\alpha\rangle \\
\text{and}\qquad \langle x'|\Delta p|\alpha\rangle &= \langle x'|p|\alpha\rangle - \langle p\rangle\langle x'|\alpha\rangle = \frac{\hbar}{i}\frac{d}{dx'}\langle x'|\alpha\rangle - \langle p\rangle\langle x'|\alpha\rangle \\
\text{with}\qquad \langle x'|\alpha\rangle &= (2\pi d^2)^{-1/4}\exp\left[\frac{i\langle p\rangle x'}{\hbar} - \frac{(x'-\langle x\rangle)^2}{4d^2}\right] \\
\text{to get}\qquad \frac{\hbar}{i}\frac{d}{dx'}\langle x'|\alpha\rangle &= \left[\langle p\rangle - \frac{\hbar}{i}\frac{1}{2d^2}(x'-\langle x\rangle)\right]\langle x'|\alpha\rangle \\
\text{and so}\qquad \langle x'|\Delta p|\alpha\rangle &= i\frac{\hbar}{2d^2}(x'-\langle x\rangle)\langle x'|\alpha\rangle = \lambda\langle x'|\Delta x|\alpha\rangle
\end{aligned}$$

where λ is a purely imaginary number. The conjecture is satisfied.

It is very simple to show that this condition is satisfied for the ground state of the harmonic oscillator. Refer to (2.3.24) and (2.3.25). Clearly $\langle x\rangle = 0 = \langle p\rangle$ for any eigenstate $|n\rangle$, and $x|0\rangle$ is proportional to $p|0\rangle$, with a proportionality constant that is purely imaginary.

19. Note the obvious typographical error, i.e. S_2^x should be S_x^2. Have $S_x^2 = \hbar^2/4 = S_y^2 = S_z^2$, also $[S_x, S_y] = i\hbar S_z$, all from Problem 8. Now $\langle S_x\rangle = \langle S_y\rangle = 0$ for the $|+\rangle$ state. Then $\langle(\Delta S_x)^2\rangle = \hbar^2/4 = \langle(\Delta S_y)^2\rangle$, and $\langle(\Delta S_x)^2\langle(\Delta S_y)^2 = \hbar^4/16$. Also $|\langle[S_x, S_y]\rangle|^2/4 = \hbar^2|\langle S_z\rangle|^2/4 = \hbar^4/16$ and the generalized uncertainty principle is satisfied by the equality. On the other hand, for the $|S_x+\rangle$ state, $\langle(\Delta S_x)^2\rangle = 0$ and $\langle S_z\rangle = 0$, and again the generalized uncertainty principle is satisfied with an equality.

20. Refer to Problems 8 and 9. Parameterize the state as $|\rangle = \cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle$, so

$$\begin{aligned}
\langle S_x\rangle &= \frac{\hbar}{2}\left[\cos\frac{\beta}{2}\langle +| + e^{-i\alpha}\sin\frac{\beta}{2}\langle -|\right][|+\rangle\langle -| + |-\rangle\langle +|]\left[\cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle\right] \\
&= \frac{\hbar}{2}\sin\frac{\beta}{2}\cos\frac{\beta}{2}(e^{i\alpha}+e^{-i\alpha}) = \frac{\hbar}{2}\sin\beta\cos\alpha \\
\langle(\Delta S_x)^2\rangle &= \langle S_x^2\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4}(1-\sin^2\beta\cos^2\alpha) \qquad \text{(see prob 12)} \\
\langle S_y\rangle &= i\frac{\hbar}{2}\left[\cos\frac{\beta}{2}\langle +| + e^{-i\alpha}\sin\frac{\beta}{2}\langle -|\right][-|+\rangle\langle -| + |-\rangle\langle +|]\left[\cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle\right] \\
&= i\frac{\hbar}{2}\sin\frac{\beta}{2}\cos\frac{\beta}{2}(e^{i\alpha}-e^{-i\alpha}) = -\frac{\hbar}{2}\sin\beta\sin\alpha \\
\langle(\Delta S_y)^2\rangle &= \langle S_y^2\rangle - \langle S_y\rangle^2 = \frac{\hbar^2}{4}(1-\sin^2\beta\sin^2\alpha)
\end{aligned}$$

Therefore, the left side of the uncertainty relation is

$$\begin{aligned}
\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle &= \frac{\hbar^4}{16}(1-\sin^2\beta\cos^2\alpha)(1-\sin^2\beta\sin^2\alpha) \\
&= \frac{\hbar^4}{16}\left(1-\sin^2\beta+\frac{1}{4}\sin^4\beta\sin^2 2\alpha\right) \\
&= \frac{\hbar^4}{16}\left(\cos^2\beta+\frac{1}{4}\sin^4\beta\sin^2 2\alpha\right) \equiv P(\alpha,\beta)
\end{aligned}$$

which is clearly maximized when $\sin 2\alpha = \pm 1$ for any value of β. In other words, the uncertainty product is a maximum when the state is pointing in a direction that is 45° with respect to the x or y axes in any quadrant, for any tilt angle β relative to the z-axis. This makes sense. The maximum tilt angle is derived from

$$\frac{\partial P}{\partial \beta} \propto -2\cos\beta\sin\beta+\sin^3\beta\cos\beta(1) = \cos\beta\sin\beta(-2+\sin^2\beta) = 0$$

or $\sin\beta = \pm 1/\sqrt{2}$, that is, 45° with respect to the z-axis. It all hangs together. The maximum uncertainty product is

$$\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle = \frac{\hbar^4}{16}\left(\frac{1}{2}+\frac{1}{4}\frac{1}{4}\right) = \frac{9}{256}\hbar^4$$

The right side of the uncertainty relation is $|\langle[S_x,S_y]\rangle|^2/4 = \hbar^2|\langle S_z\rangle|^2/4$, so we also need

$$\langle S_z\rangle = \frac{\hbar}{2}\left[\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right] = \frac{\hbar}{2}\cos\beta$$

so the value of the right hand side at maximum is

$$\frac{\hbar^2}{4}|\langle S_z\rangle|^2 = \frac{\hbar^2}{4}\frac{\hbar^2}{4}\frac{1}{2} = \frac{8}{256}\hbar^4$$

and the uncertainty principle is indeed satisfied.

21. The wave function is $\langle x|n\rangle = \sqrt{2/a}\sin(n\pi x/a)$ for $n = 1,2,3,\ldots$, so we calculate

$$\begin{aligned}
\langle x|x|n\rangle &= \int_0^a \langle n|x\rangle x\langle x|n\rangle dx = \frac{a}{2} \\
\langle x|x^2|n\rangle &= \int_0^a \langle n|x\rangle x^2\langle x|n\rangle dx = \frac{a^2}{6}\left(-\frac{3}{n^2\pi^2}+2\right) \\
(\Delta x)^2 &= \frac{a^2}{6}\left(-\frac{3}{n^2\pi^2}+2-\frac{6}{4}\right) = \frac{a^2}{6}\left(-\frac{3}{n^2\pi^2}+\frac{1}{2}\right) \\
\langle x|p|n\rangle &= \int_0^a \langle n|x\rangle \frac{\hbar}{i}\frac{d}{dx}\langle x|n\rangle dx = 0 \\
\langle x|p^2|n\rangle &= -\hbar^2\int_0^a \langle n|x\rangle \frac{d^2}{dx^2}\langle x|n\rangle dx = \frac{n^2\pi^2\hbar^2}{a^2} = (\Delta p)^2
\end{aligned}$$

(I did these with MAPLE.) Since $[x,p]=i\hbar$, we compare $(\Delta x)^2(\Delta p)^2$ to $\hbar^2/4$ with

$$(\Delta x)^2(\Delta p)^2 = \frac{\hbar^2}{6}\left(-3+\frac{n^2\pi^2}{2}\right) = \frac{\hbar^2}{4}\left(\frac{n^2\pi^2}{3}-2\right)$$

which shows that the uncertainty principle is satisfied, since $n\pi^2/3 > n\pi > 3$ for all n.

22. We're looking for a "rough order of magnitude" estimate, so go crazy with the approximations. Model the ice pick as a mass m and length L, standing vertically on the point, i.e. and inverted pendulum. The angular acceleration is $\ddot{\theta}$, the moment of inertia is mL^2 and the torque is $mgL\sin\theta$ where θ is the angle from the vertical. So $mL^2\ddot{\theta} = mgL\sin\theta$ or $\ddot{\theta} = \sqrt{g/L}\sin\theta$. Since $\theta \ll 0$ as the pick starts to fall, take $\sin\theta = \theta$ so

$$\begin{aligned}
\theta(t) &= A\exp\left(\sqrt{\frac{g}{L}}t\right) + B\exp\left(-\sqrt{\frac{g}{L}}t\right) \\
x_0 \equiv \theta(0)L &= (A+B)L \\
p_0 \equiv m\dot{\theta}(0)L &= m\sqrt{\frac{g}{L}}(A-B)L = \sqrt{m^2gL}(A-B)
\end{aligned}$$

Let the uncertainty principle relate x_0 and p_0, i.e. $x_0p_0 = \sqrt{m^2gL^3}(A^2-B^2) = \hbar$. Now ignore B; the exponential decay will become irrelevant quickly. You can notice that the pick is falling when it is tilting by something like $1^\circ = \pi/180$, so solve for a time T where $\theta(T) = \pi/180$. Then

$$T = \sqrt{\frac{L}{g}}\ln\frac{\pi/180}{A} = \sqrt{\frac{L}{g}}\left(\frac{1}{4}\ln\frac{m^2gL^3}{\hbar^2} - \ln\frac{180}{\pi}\right)$$

Take $L = 10$ cm, so $\sqrt{L/g} \approx 0.1$ sec, but the action is in the logarithms. (It is worth your time to confirm that the argument of the logarithm in the first term is indeed dimensionless.) Now $\ln(180/\pi) \approx 4$ but the first term appears to be much larger. This is good, since it means that quantum mechanics is driving the result. For $m = 0.1$ kg, find $m^2gL^3/\hbar^2 = 10^{64}$, and so $T = 0.1\text{ sec} \times (147/4 - 4) \sim 3$ sec. I'd say that's a surprising and interesting result.

23. The eigenvalues of A are obviously $\pm a$, with $-a$ twice. The characteristic equation for B is $(b-\lambda)(-\lambda)^2 - (b-\lambda)(ib)(-ib) = (b-\lambda)(\lambda^2-b^2) = 0$, so its eigenvalues are $\pm b$ with b twice. (Yes, B has degenerate eigenvalues.) It is easy enough to show that

$$AB = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix} = BA$$

so A and B commute, and therefore must have simultaneous eigenvectors. To find these, write the eigenvector components as u_i, $i = 1,2,3$. Clearly, the basis states $|1\rangle$, $|2\rangle$, and $|3\rangle$ are eigenvectors of A with eigenvalues a, $-a$, and $-a$ respectively. So, do the math to find the eigenvectors for B in this basis. Presumably, some freedom will appear that allows us to linear combinations that are also eigenvectors of A. One of these is obviously $|1\rangle \equiv |a,b\rangle$, so just work with the reduced 2×2 basis of states $|2\rangle$ and $|3\rangle$. Indeed, both of these states have eigenvalues a for A, so one linear combinations should have eigenvalue $+b$ for B, and orthogonal combination with eigenvalue $-b$.

Let the eigenvector components be u_2 and u_3. Then, for eigenvalue $+b$,

$$-ibu_3 = +bu_2 \qquad \text{and} \qquad ibu_2 = +bu_3$$

both of which imply $u_3 = iu_2$. For eigenvalue $-b$,

$$-ibu_3 = -bu_2 \qquad \text{and} \qquad ibu_2 = -bu_3$$

both of which imply $u_3 = -iu_2$. Choosing u_2 to be real, then ("No, the eigenvalue alone does not completely characterize the eigenket.") we have the set of simultaneous eigenstates

Eigenvalue of A	B	Eigenstate
a	b	$\|1\rangle$
$-a$	b	$\frac{1}{\sqrt{2}}(\|2\rangle + i\|3\rangle)$
$-a$	$-b$	$\frac{1}{\sqrt{2}}(\|2\rangle - i\|3\rangle)$

24. *This problem also appears to belong in Chapter Three. The Pauli matrices are not defined in Chapter One, but perhaps one could simply define these matrices, here and in Problems 2 and 3.*

Operating on the spinor representation of $|+\rangle$ with $(1\sqrt{2})(1+i\sigma_x)$ gives

$$\frac{1}{\sqrt{2}}\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}+i\begin{pmatrix}0&1\\1&0\end{pmatrix}\right]\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1&i\\i&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$$

So, for an operator U such that $U \doteq (1\sqrt{2})(1+i\sigma_x)$, we observe that $U|+\rangle = |S_y;+\rangle$, defined in (1.4.17b). Similarly operating on the spinor representation of $|-\rangle$ gives

$$\frac{1}{\sqrt{2}}\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}+i\begin{pmatrix}0&1\\1&0\end{pmatrix}\right]\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1&i\\i&1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}i\\1\end{pmatrix} = \frac{i}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

that is, $U|-\rangle = i|S_y;-\rangle$. This is what we would mean by a "rotation" about the x-axis by 90°. The sense of the rotation is about the $+x$ direction vector, so this would actually be a rotation of $-\pi/2$. (See the diagram following Problem Nine.) The phase factor $i = e^{i\pi/2}$ does not affect this conclusions, and in fact leads to observable quantum mechanical effects. (This is all discussed in Chapter Three.) The matrix elements of S_z in the S_y basis are then

$$\begin{aligned}
\langle S_y;+|S_z|S_y;+\rangle &= \langle +|U^\dagger S_z U|+\rangle \\
\langle S_y;+|S_z|S_y;-\rangle &= -i\langle +|U^\dagger S_z U|-\rangle \\
\langle S_y;-|S_z|S_y;+\rangle &= i\langle -|U^\dagger S_z U|+\rangle \\
\langle S_y;-|S_z|S_y;-\rangle &= \langle -|U^\dagger S_z U|-\rangle
\end{aligned}$$

Note that $\sigma_x^\dagger = \sigma_x$ and $\sigma_x^2 = 1$, so $U^\dagger U \doteq (1\sqrt{2})(1 - i\sigma_x)(1\sqrt{2})(1 + i\sigma_x) = (1/2)(1 + \sigma_x^2) = 1$ and U is therefore unitary. (This is no accident, as will be discussed when rotation operators are presented in Chapter Three.) Furthermore $\sigma_z\sigma_x = -\sigma_x\sigma_z$, so

$$\begin{aligned}
U^\dagger S_z U &\doteq \frac{1}{\sqrt{2}}(1 - i\sigma_x)\frac{\hbar}{2}\sigma_z\frac{1}{\sqrt{2}}(1 + i\sigma_x) = \frac{\hbar}{2}\frac{1}{2}(1 - i\sigma_x)^2\sigma_z = -i\frac{\hbar}{2}\sigma_x\sigma_z \\
&= -i\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\
\text{so} \qquad S_z &\doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_x
\end{aligned}$$

in the $|S_y; \pm\rangle$ *basis.* This can be easily checked directly with (1.4.17b), that is

$$S_z|S_y; \pm\rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}[|+\rangle \mp i|-\rangle = \frac{\hbar}{2}|S_y; \mp\rangle$$

There seems to be a mistake in the old solution manual, finding $S_z = (\hbar/2)\sigma_y$ instead of σ_x.

25. Transforming to another representation, say the basis $|c\rangle$, we carry out the calculation

$$\langle c'|A|c''\rangle = \sum_{b'}\sum_{b''}\langle c'|b'\rangle\langle b'|A|b''\rangle\langle b''|c''\rangle$$

There is no principle which says that the $\langle c'|b'\rangle$ need to be real, so $\langle c'|A|c''\rangle$ is not necessarily real if $\langle b'|A|b''\rangle$ is real. The problem alludes to Problem 24 as an example, but not that specific question (assuming my solution is correct.) Still, it is obvious, for example, that the operator S_y is "real" in the $|S_y; \pm\rangle$ basis, but is not in the $|\pm\rangle$ basis.

For another example, also suggested in the text, if you calculate

$$\langle p'|x|p''\rangle = \int \langle p'|x|x'\rangle\langle x'|p''\rangle dx' = \int x'\langle p'|x'\rangle\langle x'|p''\rangle dx' = \frac{1}{2\pi\hbar}\int x' e^{i(p''-p')x'/\hbar}dx'$$

and then define $q \equiv p'' - p'$ and $y \equiv x'/\hbar$, then

$$\langle p'|x|p''\rangle = \frac{\hbar}{2\pi i}\frac{d}{dq}\int e^{iqy}dy = \frac{\hbar}{i}\frac{d}{dq}\delta(q)$$

so you can also see that although x is real in the $|x'\rangle$ basis, it is not so in the $|p'\rangle$ basis.

26. From (1.4.17a), $|S_x; \pm\rangle = (|+\rangle \pm |-\rangle)/\sqrt{2}$, so clearly

$$\begin{aligned}
U \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \\
&= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}[1\ 0] + \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}[0\ 1] \\
\Longrightarrow \quad &= |S_x : +\rangle\langle +| + |S_x : -\rangle\langle -| \doteq \sum_r |b^{(r)}\rangle\langle a^{(r)}|
\end{aligned}$$

27. The idea here is simple. Just insert a complete set of states. Firstly,

$$\langle b''|f(A)|b'\rangle = \sum_{a'} \langle b''|f(A)|a'\rangle\langle a'|b'\rangle = \sum_{a'} f(a')\langle b''|a'\rangle\langle a'|b'\rangle$$

The numbers $\langle a'|b'\rangle$ (and $\langle b''|a'\rangle$) constitute the "transformation matrix" between the two sets of basis states. Similarly for the continuum case,

$$\begin{aligned}\langle \mathbf{p}''|F(r)|\mathbf{p}'\rangle &= \int \langle \mathbf{p}''|F(r)|\mathbf{x}'\rangle\langle \mathbf{x}'|\mathbf{p}'\rangle d^3x' = \int F(r')\langle \mathbf{p}''|\mathbf{x}'\rangle\langle \mathbf{x}'|\mathbf{p}'\rangle d^3x' \\ &= \frac{1}{(2\pi\hbar)^3}\int F(r')e^{i(\mathbf{p}'-\mathbf{p}'')\cdot\mathbf{x}'/\hbar}d^3x'\end{aligned}$$

The angular parts of the integral can be done explicitly. Let $\mathbf{q} \equiv \mathbf{p}' - \mathbf{p}''$ define the "z"-direction. Then

$$\begin{aligned}\langle \mathbf{p}''|F(r)|\mathbf{p}'\rangle &= \frac{2\pi}{(2\pi\hbar)^3}\int dr' F(r')\int_0^\pi \sin\theta d\theta e^{iqr'\cos\theta/\hbar} = \frac{1}{4\pi^2\hbar^3}\int dr' F(r')\int_{-1}^{1} d\mu\ e^{iqr'\mu/\hbar} \\ &= \frac{1}{4\pi^2\hbar^3}\int dr' F(r')\frac{\hbar}{iqr'}2i\sin(qr'/\hbar) = \frac{1}{2\pi^2\hbar^2}\int dr' F(r')\frac{\sin(qr'/\hbar)}{qr'}\end{aligned}$$

28. For functions $f(q,p)$ and $g(q,p)$, where q and p are conjugate position and momentum, respectively, the Poisson bracket from classical physics is

$$[f,g]_{\text{classical}} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} \qquad \text{so} \qquad [x, F(p_x)]_{\text{classical}} = \frac{\partial F}{\partial p_x}$$

Using (1.6.47), then, we have

$$\left[x, \exp\left(\frac{ip_x a}{\hbar}\right)\right] = i\hbar\left[x, \exp\left(\frac{ip_x a}{\hbar}\right)\right]_{\text{classical}} = i\hbar\frac{\partial}{\partial p_x}\exp\left(\frac{ip_x a}{\hbar}\right) = -a\exp\left(\frac{ip_x a}{\hbar}\right)$$

To show that $\exp(ip_x a/\hbar)|x'\rangle$ is an eigenstate of position, act on it with x. So

$$x\exp\left(\frac{ip_x a}{\hbar}\right)|x'\rangle = \left[\exp\left(\frac{ip_x a}{\hbar}\right)x - a\exp\left(\frac{ip_x a}{\hbar}\right)\right]|x'\rangle = (x'-a)\exp\left(\frac{ip_x a}{\hbar}\right)|x'\rangle$$

In other words, $\exp(ip_x a/\hbar)|x'\rangle$ is an eigenstate of x with eigenvalue $x'-a$. That is $\exp(ip_x a/\hbar)|x'\rangle$ is the translation operator with $\Delta x' = -a$, but we knew that. See (1.6.36).

29. I wouldn't say this is "easily derived", but it is straightforward. Expressing $G(\mathbf{p})$ as a power series means $G(\mathbf{p}) = \sum_{nm\ell} a_{nm\ell}p_i^n p_j^m p_k^\ell$. Now

$$\begin{aligned}[x_i, p_i^n] = x_i p_i p_i^{n-1} - p_i^n x_i &= i\hbar p_i^{n-1} + p_i x_i p_i^{n-1} - p_i^n x_i \\ &= 2i\hbar p_i^{n-1} + p_i^2 x_i p_i^{n-2} - p_i^n x_i \\ &\cdots \\ &= ni\hbar p_i^{n-1} \\ \text{so} \qquad [x_i, G(\mathbf{p})] &= i\hbar\frac{\partial G}{\partial p_i}\end{aligned}$$

The procedure is essentially identical to prove that $[p_i, F(\mathbf{x})] = -i\hbar\partial F/\partial x_i$. As for

$$[x^2, p^2] = x^2p^2 - p^2x^2 = x^2p^2 - xp^2x + xp^2x - p^2x^2 = x[x, p^2] + [x, p^2]x$$

make use of $[x, p^2] = i\hbar\partial(p^2)/\partial p = 2i\hbar p$ so that $[x^2, p^2] = 2i\hbar(xp+px)$. The classical Poisson bracket is $[x^2, p^2]_{\text{classical}} = (2x)(2p) - 0 = 4xp$ and so $[x^2, p^2] = i\hbar[x^2, p^2]_{\text{classical}}$ when we let the (classical quantities) x and p commute.

30. This is very similar to problem 28. Using problem 29,

$$[x_i, \mathcal{J}(\mathbf{l})] = \left[x_i, \exp\left(\frac{-i\mathbf{p}\cdot\mathbf{l}}{\hbar}\right)\right] = i\hbar\frac{\partial}{\partial p_i}\exp\left(\frac{-i\mathbf{p}\cdot\mathbf{l}}{\hbar}\right) = l_i \exp\left(\frac{-i\mathbf{p}\cdot\mathbf{l}}{\hbar}\right) = l_i\mathcal{J}(\mathbf{l})$$

We can use this result to calculate the expectation value of x_i. First note that

$$\begin{aligned}
\mathcal{J}^\dagger(\mathbf{l})\,[x_i, \mathcal{J}(\mathbf{l})] &= \mathcal{J}^\dagger(\mathbf{l})x_i\mathcal{J}(\mathbf{l}) - \mathcal{J}^\dagger(\mathbf{l})\mathcal{J}(\mathbf{l})x_i = \mathcal{J}^\dagger(\mathbf{l})x_i\mathcal{J}(\mathbf{l}) - x_i \\
&= \mathcal{J}^\dagger(\mathbf{l})l_i\mathcal{J}(\mathbf{l}) = l_i
\end{aligned}$$

Therefore, under translation,

$$\langle x_i\rangle = \langle\alpha|x_i|\alpha\rangle \to \langle\alpha|\mathcal{J}^\dagger(\mathbf{l})x_i\mathcal{J}(\mathbf{l})|\alpha\rangle = \langle\alpha|\mathcal{J}^\dagger(\mathbf{l})x_i\mathcal{J}(\mathbf{l})|\alpha\rangle = \langle\alpha|(x_i + l_i)|\alpha\rangle = \langle x_i\rangle + l_i$$

which is exactly what you expect from a translation operator.

31. This is a continued rehash of the last few problems. Since $[\mathbf{x}, \mathcal{J}(\mathbf{dx}')] = \mathbf{dx}'$ by (1.6.25), and since $\mathcal{J}^\dagger[\mathbf{x}, \mathcal{J}] = \mathcal{J}^\dagger\mathbf{x}\mathcal{J} - \mathbf{x}$, we have $\mathcal{J}^\dagger(\mathbf{dx}')\mathbf{x}\mathcal{J}(\mathbf{dx}') = \mathbf{x} + \mathcal{J}^\dagger(\mathbf{dx}')\mathbf{dx}' = \mathbf{x} + \mathbf{dx}'$ since we only keep the lowest order in $\mathbf{dx}'$. Therefore $\langle\mathbf{x}\rangle \to \langle\mathbf{x}\rangle + \mathbf{dx}'$. Similarly, from (1.6.45), $[\mathbf{p}, \mathcal{J}(\mathbf{dx}')] = 0$, so $\mathcal{J}^\dagger[\mathbf{p}, \mathcal{J}] = \mathcal{J}^\dagger\mathbf{p}\mathcal{J} - \mathbf{p} = 0$. That is $\mathcal{J}^\dagger\mathbf{p}\mathcal{J} = \mathbf{p}$ and $\langle\mathbf{p}\rangle \to \langle\mathbf{p}\rangle$.

32. These are all straightforward. In the following, all integrals are taken with limits from $-\infty$ to ∞. One thing to keep in mind is that odd integrands give zero for the integral, so the right change of variables can be very useful. Also recall that $\int \exp(-ax^2)dx = \sqrt{\pi/a}$, and $\int x^2\exp(-ax^2)dx = -(d/da)\int\exp(-ax^2)dx = \sqrt{\pi}/2a^{3/2}$. So, for the x-space case,

$$\begin{aligned}
\langle p\rangle &= \int\langle\alpha|x'\rangle\langle x'|p|\alpha\rangle dx' = \int\langle\alpha|x'\rangle\frac{\hbar}{i}\frac{d}{dx'}\langle x'|\alpha\rangle dx' = \frac{1}{d\sqrt{\pi}}\int\hbar k\exp\left(-\frac{x'^2}{d^2}\right)dx' = \hbar k \\
\langle p^2\rangle &= -\hbar^2\int\langle\alpha|x'\rangle\frac{d^2}{dx'^2}\langle x'|\alpha\rangle dx' \\
&= -\frac{\hbar^2}{d\sqrt{\pi}}\int\exp\left(-ikx' - \frac{x'^2}{2d^2}\right)\frac{d}{dx'}\left[\left(ik - \frac{x'}{d^2}\right)\exp\left(ikx' - \frac{x'^2}{2d^2}\right)\right]dx' \\
&= -\frac{\hbar^2}{d\sqrt{\pi}}\int\left[-\frac{1}{d^2} + \left(ik - \frac{x'}{d^2}\right)^2\right]\exp\left(-\frac{x'^2}{d^2}\right)dx' \\
&= \hbar^2\left[\frac{1}{d^2} + k^2\right] - \frac{\hbar^2}{d^5\sqrt{\pi}}\int x'^2\exp\left(-\frac{x'^2}{d^2}\right)dx' = \hbar^2\left[\frac{1}{d^2} + k^2\right] - \frac{\hbar^2}{2d^2} = \frac{\hbar^2}{2d^2} + \hbar^2k^2
\end{aligned}$$

Using instead the momentum space wave function (1.7.42), we have

$$
\begin{aligned}
\langle p\rangle &= \int \langle\alpha|p|p'\rangle\langle p'|\alpha\rangle dp' = \int p'|\langle p'|\alpha\rangle|^2 dp' \\
&= \frac{d}{\hbar\sqrt{\pi}}\int p' \exp\left[-\frac{(p'-\hbar k)^2 d^2}{\hbar^2}\right] dp' = \frac{d}{\hbar\sqrt{\pi}}\int (q+\hbar k)\exp\left[-\frac{q^2 d^2}{\hbar^2}\right] dq = \hbar k \\
\langle p^2\rangle &= \frac{d}{\hbar\sqrt{\pi}}\int (q+\hbar k)^2 \exp\left[-\frac{q^2d^2}{\hbar^2}\right] dq = \frac{d}{\hbar\sqrt{\pi}}\frac{\sqrt{\pi}}{2}\frac{\hbar^3}{d^3} + (\hbar k)^2 = \frac{\hbar^2}{2d^2} + \hbar^2 k^2
\end{aligned}
$$

33. I can't help but think this problem can be done by creating a "momentum translation" operator, but instead I will follow the original solution manual. This approach uses the position space representation and Fourier transform to arrive the answer. Start with

$$
\begin{aligned}
\langle p'|x|p''\rangle &= \int \langle p'|x|x'\rangle\langle x'|p''\rangle dx' = \int x'\langle p'|x'\rangle\langle x'|p''\rangle dx' \\
&= \frac{1}{2\pi\hbar}\int x' \exp\left[-i\frac{(p'-p'')\cdot x'}{\hbar}\right] dx' = i\frac{\partial}{\partial p'}\frac{1}{2\pi}\int \exp\left[-i\frac{(p'-p'')\cdot x'}{\hbar}\right] dx' \\
&= i\hbar\frac{\partial}{\partial p'}\delta(p'-p'')
\end{aligned}
$$

Now find $\langle p'|x|\alpha\rangle$ by inserting a complete set of states $|p''\rangle$, that is

$$
\langle p'|x|\alpha\rangle = \int \langle p'|x|p''\rangle\langle p''|\alpha\rangle dp'' = i\hbar\frac{\partial}{\partial p'}\int \delta(p'-p'')\langle p''|\alpha\rangle dp'' = i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle
$$

Given this, the next expression is simple to prove, namely

$$
\langle\beta|x|\alpha\rangle = \int dp'\langle\beta|p'\rangle\langle p'|x|\alpha\rangle = \int dp'\phi_\beta^*(p')i\hbar\frac{\partial}{\partial p'}\phi_\alpha(p')
$$

using the standard definition $\phi_\gamma(p') \equiv \langle p'|\gamma\rangle$.

Certainly the operator $\mathcal{T}(\Xi) \equiv \exp(ix\Xi/\hbar)$ looks like a momentum translation operator. So, we should try to work out $p\mathcal{T}(\Xi)|p'\rangle = p\exp(ix\Xi/\hbar)|p'\rangle$ and see if we get $|p'+\Xi\rangle$. Take a lesson from problem 28, and make use of the result from problem 29, and we have

$$
p\mathcal{T}(\Xi)|p'\rangle = \{\mathcal{T}(\Xi)p + [p,\mathcal{T}(\Xi)]\}|p'\rangle = \left\{p'\mathcal{T}(\Xi) - i\hbar\frac{\partial}{\partial x}\mathcal{T}(\Xi)\right\}|p'\rangle = (p'+\Xi)\mathcal{T}(\Xi)|p'\rangle
$$

and, indeed, $\mathcal{T}(\Xi)|p'\rangle$ is an eigenstate of p with eigenvalue $p'+\Xi$. In fact, this could have been done first, and then write down the translation operator for infinitesimal momenta, and derive the expression for $\langle p'|x|\alpha\rangle$ the same way as done in the text for infinitesimal spacial translations. (I like this way of wording the problem, and maybe it will be changed in the next edition.)

Chapter Two

1. The equation of motion for an operator in the Heisenberg picture is given by (2.2.19), so

$$\dot{S}_x = \frac{1}{i\hbar}[S_x, H] = -\frac{1}{i\hbar}\frac{eB}{mc}[S_x, S_z] = \frac{eB}{mc}S_y \qquad \dot{S}_y = -\frac{eB}{mc}S_x \quad \dot{S}_z = 0$$

and $\ddot{S}_{x,y} = -\omega^2 S_{x,y}$ for $\omega \equiv eB/mc$. Thus S_x and S_y are sinusoidal with frequency ω and S_z is a constant.

2. The Hamiltonian is not Hermitian, so the time evolution operator will not be unitary, and probability will not be conserved as a state evolves in time. As suggested, set $H_{11} = H_{22} = 0$. Then $H = a|1\rangle\langle 2|$ in which case $H^2 = a^2|1\rangle\langle 2|1\rangle\langle 1| = 0$. Since H is time-independent,

$$\mathcal{U}(t) = \exp\left(-\frac{i}{\hbar}Ht\right) = 1 - \frac{i}{\hbar}Ht = 1 - \frac{i}{\hbar}at|1\rangle\langle 2|$$

even for finite times t. Thus a state $|\alpha, t\rangle \equiv \mathcal{U}(t)|2\rangle = |2\rangle - (iat/\hbar)|1\rangle$ has a time-dependent norm. Indeed $\langle\alpha|\alpha\rangle = 1 + a^2t^2/\hbar^2$ which is nonsense. In words, it says that if you start out in the state $|2\rangle$, then the probability of finding the system in this state is unity at $t = 0$ and then grows with time. You can be more formal, and talk about an initial state $c_1|1\rangle + c_2|2\rangle$, but the bottom line is the same; probability is no longer conserved in time.

3. We have $\hat{\mathbf{n}} = \sin\beta\hat{\mathbf{x}} + \cos\beta\hat{\mathbf{z}}$ and $\mathbf{S} \doteq (\hbar/2)\boldsymbol{\sigma}$, so $\mathbf{S}\cdot\mathbf{n} \doteq (\hbar/2)(\sin\beta\sigma_x + \cos\beta\sigma_z)$ and we want to solve the matrix equation $\mathbf{S}\cdot\mathbf{n}\psi = (\hbar/2)\psi$ in order to find the initial state column vector ψ. This is, once again, a problem whose solution best makes use of the Pauli matrices, which are not introduced until Section 3.2. On the other hand, we can also make use of Problem 1.9 to write down the initial state. Either way, we find

$$\begin{aligned}
|\alpha, t=0\rangle &= \cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)|-\rangle \qquad \text{so,} \\
|\alpha, t\rangle &= \exp\left[-\frac{i}{\hbar}\frac{eB}{mc}tS_z\right]|\alpha, t=0\rangle = e^{-i\omega t/2}\cos\left(\frac{\beta}{2}\right)|+\rangle + e^{i\omega t/2}\sin\left(\frac{\beta}{2}\right)|-\rangle
\end{aligned}$$

for $\omega \equiv eB/mc$. From (1.4.17a), the state $|S_x; +\rangle = (1/\sqrt{2})|+\rangle + (1/\sqrt{2})|-\rangle$, so

$$\begin{aligned}
|\langle S_x; +|\alpha, t\rangle|^2 &= \left|\frac{1}{\sqrt{2}}e^{-i\omega t/2}\cos\left(\frac{\beta}{2}\right) + \frac{1}{\sqrt{2}}e^{i\omega t/2}\sin\left(\frac{\beta}{2}\right)\right|^2 \\
&= \frac{1}{2}\cos^2\left(\frac{\beta}{2}\right) + \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\beta}{2}\right) + \frac{1}{2}\sin^2\left(\frac{\beta}{2}\right) \\
&= \frac{1}{2} + \frac{1}{2}\cos(\omega t)\sin\beta = \frac{1}{2}(1 + \sin\beta\cos\omega t)
\end{aligned}$$

which makes sense. For $\beta = 0$, the initial state is a z-eigenket, and there is no precession, so you just get 1/2 for the probability of measuring S_x in the positive direction. The same works out for $\beta = \pi$. For $\beta = \pi/2$, the initial state is $|S_x; +\rangle$ so the probabilty is $+1$ at $t = 0$ and 0 at $t = \pi/\omega = T/2$. Now from (1.4.18a), $S_x = (\hbar/2)[|+\rangle\langle-| + |-\rangle\langle+|]$, so

$$\begin{aligned}
\langle\alpha, t|S_x|\alpha, t\rangle &= \left[e^{i\omega t/2}\cos\left(\frac{\beta}{2}\right)\langle+| + e^{-i\omega t/2}\sin\left(\frac{\beta}{2}\right)\langle-|\right] \\
&\quad \frac{\hbar}{2}\left[e^{i\omega t/2}\sin\left(\frac{\beta}{2}\right)|+\rangle + e^{-i\omega t/2}\cos\left(\frac{\beta}{2}\right)|-\rangle\right] \\
&= \frac{\hbar}{2}\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)\left[e^{i\omega t} + e^{-\omega t}\right] = \frac{\hbar}{2}\sin\beta\cos\omega t
\end{aligned}$$

Again, this makes perfect sense. The expectation value is zero for $\beta = 0$ and $\beta = \pi$, but for $\beta = \pi/2$, you get the classical precession of a vector that lies in the xy-plane.

4. First, restating equations from the textbook,

$$\begin{aligned}
|\nu_e\rangle &= \cos\theta|\nu_1\rangle - \sin\theta|\nu_2\rangle \\
|\nu_\mu\rangle &= \sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle \\
\text{and} \qquad E &= pc\left(1 + \frac{m^2c^2}{2p^2}\right)
\end{aligned}$$

Now, let the initial state $|\nu_e\rangle$ evolve in time to become a state $|\alpha, t\rangle$ in the usual fashion

$$\begin{aligned}
|\alpha, t\rangle &= e^{-iHt/\hbar}|\nu_e\rangle \\
&= \cos\theta e^{-iE_1t/\hbar}|\nu_1\rangle - \sin\theta e^{-iE_2t/\hbar}|\nu_2\rangle \\
&= e^{-ipct/\hbar}\left[e^{-im_1^2c^3t/2p\hbar}\cos\theta|\nu_1\rangle - e^{-im_2^2c^3t/2p\hbar}\sin\theta|\nu_2\rangle\right]
\end{aligned}$$

The probability that this state is observed to be a $|\nu_e\rangle$ is

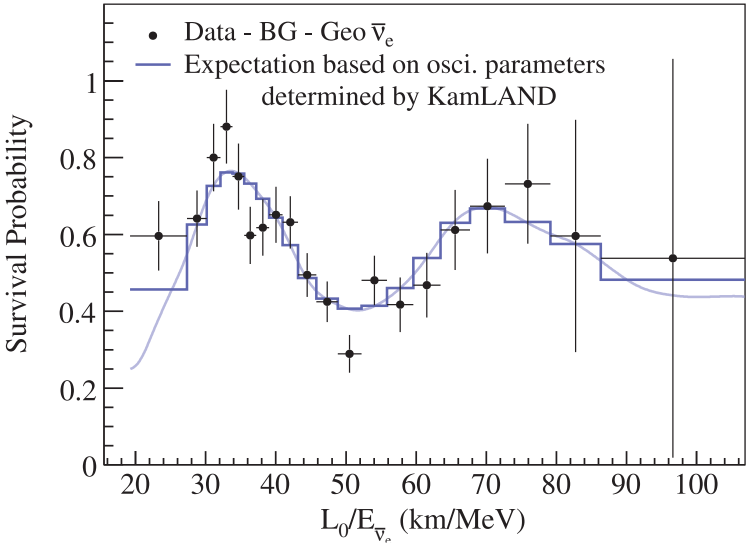
$$\begin{aligned}
P(\nu_e \to \nu_e) = |\langle\nu_e|\alpha, t\rangle|^2 &= \left|e^{-im_1^2c^3t/2p\hbar}\cos^2\theta + e^{-im_2^2c^3t/2p\hbar}\sin^2\theta\right|^2 \\
&= \left|\cos^2\theta + e^{i\Delta m^2c^3t/2p\hbar}\sin^2\theta\right|^2 \\
&= \cos^4\theta + \sin^4\theta + 2\cos^2\theta\sin^2\theta\cos\left[\frac{\Delta m^2c^3t}{2p\hbar}\right] \\
&= 1 - \sin^2 2\theta\sin^2\left[\frac{\Delta m^2c^3t}{4p\hbar}\right]
\end{aligned}$$

Writing the nominal neutrino energy as $E = pc$ and the flight distance $L = ct$ we have

$$P(\nu_e \to \nu_e) = 1 - \sin^2 2\theta\sin^2\left[\Delta m^2c^4\frac{L}{4E\hbar c}\right]$$

It is quite customary to ignore the factor of c^4 and agree to measure mass in units of energy, typically eV.

The neutrino oscillation probability from KamLAND is plotted here:



The minimum in the oscillation probability directly gives us $\sin^2 2\theta$, that is

$$1 - \sin^2 2\theta \approx 0.4 \qquad \text{so} \qquad \theta \approx 25^\circ$$

The wavelength gives the mass difference parameter. We have

$$40 \frac{\text{km}}{\text{MeV}} = 2\pi \frac{4\hbar c}{\Delta m^2} = \frac{8\pi \times 200 \text{ MeV fm}}{\Delta m^2}$$

where we explicitly agree to measure Δm^2 in eV^2. Therefore

$$\Delta m^2 = 40\pi \times 10^{12}\text{eV}^2 \times 10^{-15}/10^3 = 1.2 \times 10^{-4} \text{ eV}^2$$

The results from a detailed analysis by the collaboration, in Phys.Rev.Lett.100(2008)221803, are $\tan^2\theta = 0.56$ ($\theta = 37^\circ$) and $\Delta m^2 = 7.6 \times 10^{-5}$ eV^2. The full analysis not only includes the fact that the source reactors are at varying distances (although clustered at a nominal distance), but also that neutrino oscillations are over three generations.

5. *Note: This problem is worked through rather thoroughly in the text. See page 85.* First, $\dot{x} = (1/i\hbar)[x, H] = (1/i\hbar)[x, p^2/2m] = p/m$ (using Problem 1.29). However $\dot{p} = (1/i\hbar)[p, p^2/2m] = 0$ so $p(t) = p(0)$, a constant. Therefore $x(t) = x(0) + p(0)t/m$, and $[x(t), x(0)] = [x(0) + p(0)t/m, x(0)] = [p(0), x(0)]t/m = -i\hbar t/m$. By the generalized uncertainty principle(1.4.53), this means that the uncertainty in position grows with time. This conclusion is also a consequence of a study of "wave packets."

6. This is the proof of the so-called "dipole sum rule." Using Problem 1.29,

$$[H,x]=\left[\frac{p^2}{2m}+V(x),x\right]=-i\hbar\frac{p}{m}\qquad\text{so}\qquad[[H,x],x]=-\frac{\hbar^2}{m}$$

Now $[[H,x],x]=[H,x]x-x[H,x]=Hx^2-xHx-xHx+x^2H=Hx^2+x^2H-2xHx$, and so $\langle a''|[[H,x],x]|a''\rangle=2E''\langle a''|x^2|a''\rangle-2\langle a''|xHx|a''\rangle=-\hbar^2/m$ from above. Inserting a complete set of states $|a'\rangle$ into each of the two terms on the left, we come up with

$$\begin{aligned}\frac{\hbar^2}{2m}&=\langle a''|xHx|a''\rangle-E''\langle a''|x^2|a''\rangle\\&=\sum_{a'}\left[\langle a''|xH|a'\rangle\langle a'|x|a''\rangle-E''\langle a''|x|a'\rangle\langle a'|x|a''\rangle\right]=\sum_{a'}(E'-E'')|\langle a''|x|a'\rangle|^2\end{aligned}$$

7. We solve this in the Heisenberg picture, letting the operators be time dependent. Then

$$\begin{aligned}\frac{d}{dt}\mathbf{x}\cdot\mathbf{p}&=\frac{1}{i\hbar}[\mathbf{x}\cdot\mathbf{p},H]=\frac{1}{i\hbar}\left[xp_x+yp_y+zp_z,\frac{1}{2m}(p_x^2+p_y^2+p_z^2)+V(\mathbf{x})\right]\\&=\frac{1}{2i\hbar m}\left\{[x,p_x^2]p_x+[y,p_y^2]p_y+[z,p_z^2]p_z\right\}+\frac{1}{i\hbar}\mathbf{x}\cdot[\mathbf{p},V(\mathbf{x})]\\&=\frac{1}{m}(p_x^2+p_y^2+p_z^2)-x\frac{\partial V}{\partial x}-y\frac{\partial V}{\partial y}-z\frac{\partial V}{\partial z}=\frac{\mathbf{p}^2}{m}-\mathbf{x}\cdot\boldsymbol{\nabla}V\end{aligned}$$

using (2.2.23). What does this mean if $d\mathbf{x}\cdot\mathbf{p}/dt=0$? The original solution manual is elusive, so I'm not sure what Sakurai was getting at. In Chapter Three, we show that for the orbital angular momentum operator $\mathbf{L}$, one has $\mathbf{L}^2=\mathbf{x}^2\mathbf{p}^2-(\mathbf{x}\cdot\mathbf{p})^2+i\hbar\mathbf{x}\cdot\mathbf{p}$, so it appears that there is a link between this quantity and conservation of angular momentum. So,...?

8. Firstly, $\langle(\Delta x)^2\rangle=\langle x^2\rangle-\langle x\rangle^2$ and (from Problem 5 above) $x(t)=x(0)+(p(0)/m)t$, so $\langle x(t)\rangle=\langle x(0)\rangle+(\langle p(0)\rangle/m)t=0$ and $\langle(\Delta x)^2\rangle=\langle x^2\rangle$ at all times. Therefore we want

$$\langle(\Delta x)^2\rangle=\langle x^2(t)\rangle=\langle x^2(0)\rangle+\frac{t}{m}\langle x(0)p(0)+p(0)x(0)\rangle+\frac{t^2}{m^2}\langle p^2(0)\rangle$$

where the expectation value can be calculated for the state at $t=0$. For this (minimum uncertainty) state, we have $\Delta x=x(0)-\langle x(0)\rangle=x(0)$ and $\Delta p=p(0)-\langle p(0)\rangle=p(0)$, so from Problem 1.18(b) we have $\Delta p(0)|\rangle=ia\Delta x(0)|\rangle$ where a is real. Therefore

$$\langle(\Delta x)^2\rangle=\langle x^2(0)\rangle+\frac{t}{m}\left[ia\langle x^2(0)\rangle-ia\langle x^2(0)\rangle\right]+\frac{t^2}{m^2}(-ia)(ia)\langle x^2(0)\rangle=\langle x^2(0)\rangle\left[1+\frac{a^2t^2}{m^2}\right]$$

where $\hbar^2/4=\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle=a^2\langle x^2(0)\rangle$ sets $a^2=\hbar^2/4\langle(\Delta x)^2\rangle|_{t=0}$. San Fu Tuan's original solution manual states that this agrees with the expansion of wave packets calculated using wave mechanics. This point should probably be investigated further.

9. The matrix representation of H in the $|a'\rangle$, $|a''\rangle$ basis is $H = \begin{bmatrix} 0 & \delta \\ \delta & 0 \end{bmatrix}$, so the characteristic equation for the eigenvalues is $(-E)^2 - \delta^2 = 0$ and $E = \pm\delta \equiv E_\pm$ with eigenstates $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \pm 1 \end{bmatrix}$. This gives $|a'\rangle = (|E_+\rangle + |E_-\rangle)/\sqrt{2}$ and $|a''\rangle = (|E_+\rangle - |E_-\rangle)/\sqrt{2}$. Since the Hamiltonian is time-independent, the time evolved state is $\exp(-iHt/\hbar)|a'\rangle = (e^{-i\delta t/\hbar}|E_+\rangle + e^{i\delta t/\hbar}|E_-\rangle)/\sqrt{2}$. The probability to find this state at time t in the state $|a''\rangle$ is $|\langle a''|\exp(-iHt/\hbar)|a'\rangle|^2$, or

$$\frac{1}{4}\left|\left(\langle E_+| - \langle E_-|\right)\left(e^{-i\delta t/\hbar}|E_+\rangle + e^{i\delta t/\hbar}|E_-\rangle\right)\right|^2 = \frac{1}{4}\left|e^{-i\delta t/\hbar} - e^{i\delta t/\hbar}\right|^2 = \sin^2\left(\frac{\delta t}{\hbar}\right)$$

This is the classic two-state problem. Spin-1/2 is one example. Another is ammonia.

10. *This problem is nearly identical to Problem 9, only instead speciying two ways to determine the time-evolved state, plus Problem 2 tossed in at the end. Perhaps it should be removed from the next edition.*

(a) The energy eigenvalues are $E_\pm \equiv \pm\Delta$ with normalized eigenstates $|E_\pm\rangle = (|R\rangle \pm |L\rangle)/\sqrt{2}$.

(b) We have $|R\rangle = (|E_+\rangle + |E_-\rangle)/\sqrt{2}$ and $|L\rangle = (|E_+\rangle - |E_-\rangle)/\sqrt{2}$, so, with $\omega \equiv \Delta/\hbar$,

$$\begin{aligned} |\alpha, t\rangle &= e^{-iHt/\hbar}|\alpha, t=0\rangle = e^{-iHt/\hbar}|R\rangle\langle R|\alpha\rangle + e^{-iHt/\hbar}|L\rangle\langle L|\alpha\rangle \\ &= \frac{1}{\sqrt{2}}\left[e^{-i\omega t}|E_+\rangle + e^{i\omega t}|E_-\rangle\right]\langle R|\alpha\rangle + \frac{1}{\sqrt{2}}\left[e^{-i\omega t}|E_+\rangle - e^{i\omega t}|E_-\rangle\right]\langle L|\alpha\rangle \end{aligned}$$

(c) The initial condition means that $\langle R|\alpha\rangle = 1$ and $\langle L|\alpha\rangle = 0$, so we calculate

$$|\langle L|\alpha, t\rangle|^2 = \frac{1}{4}\left|\left(\langle E_+| - \langle E_-|\right)\left(e^{-i\omega t}|E_+\rangle + e^{i\omega t}|E_-\rangle\right)\right|^2 = \frac{1}{4}\left|e^{-i\omega t} - e^{i\omega t}\right|^2 = \sin^2\omega t$$

(d) *This is the only part of the problem that is "new." Indeed, Problem 9 could have been done this way, instead of using the time propagation operator.* Using (2.1.27) we write

$$i\hbar\frac{\partial}{\partial t}\langle R|\alpha, t\rangle = \langle R|H|\alpha, t\rangle \qquad \text{and} \qquad i\hbar\frac{\partial}{\partial t}\langle L|\alpha, t\rangle = \langle L|H|\alpha, t\rangle$$

Let $\psi_R(t) \equiv \langle R|\alpha, t\rangle$ and $\psi_L(t) \equiv \langle L|\alpha, t\rangle$. These coupled equations become

$$i\hbar\dot{\psi}_R = \frac{1}{\sqrt{2}}\left(\Delta\langle E_+| - \Delta\langle E_-|\right)|\alpha, t\rangle = \Delta\psi_L \qquad \text{and} \qquad i\hbar\dot{\psi}_L = \Delta\psi_R$$

or $\dot{\psi}_R = -i\omega\psi_L$ and $\dot{\psi}_L = -i\omega\psi_R$, so $\psi_R(t) = Ae^{i\omega t} + B^{-i\omega t}$ and $\psi_L(t) = Ce^{i\omega t} + D^{-i\omega t}$. These are just (b) where $A = \langle R|E_+\rangle$, $B = \langle R|E_-\rangle$, $C = \langle L|E_+\rangle$, and $D = \langle L|E_-\rangle$.

(e) See Problem 2. It can be embellished by in fact solving the most general time-evolution problem, but in the end, the point will still be that probability is not conserved.

11. Restating this problem: *Using the one-dimensional simple harmonic oscillator as an example, illustrate the difference between the Heisenberg picture and the Schrödinger picture. Discuss in particular how (a) the dynamic variables x and p and (b) the most general state vector evolve with time in each of the two pictures.*

This problem, namely 2.10 in the previous edition, is rather open ended, atypical for most of the problems in the book. Perhaps it should be revised. Most of the problem is in fact covered on pages 94 to 96. Anyway, we start from the Hamiltonian

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 = \left(N + \frac{1}{2}\right)\hbar\omega$$

(a) In the Schrödinger picture, x and p do not evolve in time. In the Heisenberg picture

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{i\hbar}[x, H] = \frac{1}{2im\hbar}[x, p^2] = \frac{1}{2im\hbar}i\hbar(2p) = \frac{p}{m} \\ \frac{dp}{dt} &= \frac{1}{i\hbar}[p, H] = \frac{m\omega^2}{2i\hbar}[p, x^2] = \frac{m\omega^2}{2i\hbar}(-i\hbar)(2x) = -m\omega^2 x\end{aligned}$$

using Problem 1.29. These are just the classical Hamilton's equations, with a force $-\omega^2 x$. Solving these coupled equations are simple, yielding sinusoidal motion at frequency ω for x and p. One can also recognize that the two pictures coincide at $t = 0$, and then get Heisenberg from Schrödinger using $x_H(t) = \exp(iHt/\hbar)x(0)\exp(-iHt/\hbar)$ and expanding the exponentials. Similarly for momentum.

(b) In the Heisenberg picture, state vectors are stationary. For the Schrödinger picture, it is easiest to expand in terms of eigenstates of N, that is $|\alpha, t\rangle = \sum c_n(t)|n\rangle$, so (2.1.27) gives

$$i\hbar \sum_n \dot{c}_n(t)|n\rangle = H|\alpha, t\rangle = \sum_n \left(n + \frac{1}{2}\right)\hbar\omega c_n(t)|n\rangle$$

in which case $c_n(t) = \exp[-i(n + 1/2)\omega t]$, using orthonormality of the $|n\rangle$.

12. *Not enough information is given in the problem statement. The state $|0\rangle$ is one for which $\langle x\rangle = 0 = \langle p\rangle$.* As described in the solution to Problem 11, in the Heisenberg picture, the position operator is $x(t) = x(0)\cos(\omega t) + (p(0)/m)\sin(\omega t)$, and $\langle x\rangle = \langle t = 0|x(t)|t = 0\rangle$. Since $e^{ip/\hbar}xe^{-ipa/\hbar} = e^{ip/\hbar}\{[x, e^{-ipa/\hbar}] + e^{-ipa/\hbar}x\} = e^{ip/\hbar}i\hbar(-ia/\hbar)e^{-ipa/\hbar} + x = x + a$, using Problem 1.29, the expectation value of position is

$$\begin{aligned}\langle x\rangle &= \langle 0|e^{ip/\hbar}x(0)e^{-ipa/\hbar}|0\rangle\cos(\omega t) + \langle 0|e^{ip/\hbar}p(0)e^{-ipa/\hbar}|0\rangle\sin(\omega t) \\ &= \langle 0|[x(0) + a]|0\rangle\cos(\omega t) + \langle 0|p(0)|0\rangle\sin(\omega t) = a\cos(\omega t)\end{aligned}$$

Since the state $e^{-ipa/\hbar}|0\rangle$ represents a position displaced by a distance a (See Problem 1.28), we have the classical motion of a harmonic oscillator starting from rest with amplitude a.

13. Making use of (1.6.36), we recognize $\mathcal{T}(a) = \exp(-ipa/\hbar)$ as the operator that translates in x by a distance a. Therefore $\langle x'|\mathcal{T}(a) = \langle x' - a|$ and

$$\langle x'|e^{-ipa/\hbar}|0\rangle = \langle x' - a|0\rangle = \frac{1}{\pi^{1/4}}\frac{1}{x_0^{1/2}}\exp\left[-\frac{1}{2}\left(\frac{x'-a}{x_0}\right)^2\right]$$

The probability to find the state $e^{-ipa/\hbar}|0\rangle$ in the ground state $|0\rangle$ is the square of

$$\langle 0|e^{-ipa/\hbar}|0\rangle = \int dx'\langle 0|x'\rangle\langle x'|e^{-ipa/\hbar}|0\rangle = \frac{1}{\pi^{1/2}}\frac{1}{x_0}\int_{-\infty}^{\infty} dx' e^{-[(x'-a)^2+x'^2]/2x_0^2}$$

The integral is simple to do by completing the square. Write

$$(x'-a)^2 + x'^2 = 2\left[x'^2 - ax' + \frac{a^2}{2}\right] = 2\left[\left(x' - \frac{a}{2}\right)^2\right] + \frac{a^2}{2}$$

and shift the integration variable by $a/2$. You end up with

$$\langle 0|e^{-ipa/\hbar}|0\rangle = \frac{1}{\pi^{1/2}}\frac{1}{x_0}e^{-a^2/4x_0^2}\int_{-\infty}^{\infty} dy e^{-y^2/x_0^2} = e^{-a^2/4x_0^2}$$

so the probability is just $e^{-a^2/2x_0^2}$. This is indeed time-independent.

14. Rearranging, we have $x = \sqrt{\hbar/2m\omega}(a + a^\dagger)$ and $p = i\sqrt{\hbar m\omega/2}(a^\dagger - a)$, therefore

$$\begin{aligned}
x|n\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}|n-1\rangle + \sqrt{n+1}|n+1\rangle\right] \\
p|n\rangle &= i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n+1}|n+1\rangle - \sqrt{n}|n-1\rangle\right] \\
\langle m|x|n\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\langle m|(a+a^\dagger)|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m.n+1}\right] \\
\langle m|p|n\rangle &= i\sqrt{\frac{\hbar m\omega}{2}}\langle m|(a^\dagger - a)|n\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n+1}\delta_{m.n+1} - \sqrt{n}\delta_{m,n-1}\right] \\
\langle m|\{x,p\}|n\rangle &= \langle m|(xp+px)|n\rangle = \langle m|xp|n\rangle + \langle n|xp|m\rangle^* \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n+1}\langle m|x|n+1\rangle - \sqrt{n}\langle m|x|n-1\rangle\right. \\
&\quad \left. -\sqrt{m+1}\langle n|x|m+1\rangle + \sqrt{m}\langle n|x|m-1\rangle\right] \\
&= i\frac{\hbar}{2}\left[(n+1)\delta_{nm} + \sqrt{(n+1)(n+2)}\delta_{n+2,m} - \sqrt{n(n-1)}\delta_{n-2,m} - n\delta_{nm}\right. \\
&\quad \left. -(m+1)\delta_{nm} - \sqrt{(m+1)(m+2)}\delta_{n,m+2} + \sqrt{m(m-1)}\delta_{n,m-2} + m\delta_{nm}\right] \\
&= i\hbar\left[\sqrt{(n+1)(n+2)}\delta_{n+2,m} - \sqrt{n(n-1)}\delta_{n-2,m}\right]
\end{aligned}$$

$$\begin{aligned}
\langle m|x^2|n\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\langle m|x|n-1\rangle + \sqrt{n+1}\langle m|x|n+1\rangle\right] \\
&= \frac{\hbar}{2m\omega}\left[\sqrt{n(n-1)}\delta_{n-2,m} + (2n+1)\delta_{nm} + \sqrt{(n+1)(n+2)}\delta_{n+2,m}\right] \\
\langle m|p^2|n\rangle &= i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n+1}\langle m|p|n+1\rangle - \sqrt{n}\langle m|p|n-1\rangle\right] \\
&= -\frac{\hbar m\omega}{2}\left[\sqrt{(n+1)(n+2)}\delta_{n+2,m} - (2n+1)\delta_{nm} + \sqrt{n(n-1)}\delta_{n-2,m}\right]
\end{aligned}$$

Now, the virial theorem in three dimensions is quoted as

$$\left\langle \frac{\mathbf{p}^2}{m} \right\rangle = \langle \mathbf{x}\cdot\boldsymbol{\nabla} V\rangle \qquad \text{or} \qquad \left\langle \frac{p^2}{m} \right\rangle = \left\langle x\frac{dV}{dx} \right\rangle$$

in one dimension. For the harmonic oscillator, $xdV/dx = m\omega^2 x^2$. So, evaluating the expectation value in the state $|n\rangle$ using the calculations above, we have

$$\left\langle \frac{p^2}{m} \right\rangle = \frac{\hbar\omega}{2}(2n+1) = \hbar\omega\left(n+\frac{1}{2}\right) \qquad \text{and} \qquad \left\langle x\frac{dV}{dx} \right\rangle = \frac{\hbar\omega}{2}(2n+1) = \hbar\omega\left(n+\frac{1}{2}\right)$$

and the virial theorem is indeed satisfied.

15. Turning around what is given, $\langle p'|x'\rangle = (2\pi\hbar)^{-1/2}e^{-ip'x'/\hbar}$. Then

$$\begin{aligned}
\langle p'|x|\alpha\rangle &= \int dx' \langle p'|x'\rangle\langle x'|x|\alpha\rangle = \int dx' x' \langle p'|x'\rangle\langle x'|\alpha\rangle \\
&= i\hbar\int dx' \frac{\partial}{\partial p'}\langle p'|x'\rangle\langle x'|\alpha\rangle = i\hbar\frac{\partial}{\partial p'}\int dx' \langle p'|x'\rangle\langle x'|\alpha\rangle = i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle
\end{aligned}$$

For the Hamiltonian $H = p^2/2m + m\omega^2x^2/2$ with eigenvalues E, the wave equation in momentum space is $\langle p'|H|\alpha\rangle = E\langle p'|\alpha\rangle \equiv Eu_\alpha(p')$, and the second term in $\langle p'|H|\alpha\rangle$ is

$$\frac{m\omega^2}{2}\langle p'|x^2|\alpha\rangle = \frac{m\omega^2}{2}i\hbar\frac{\partial}{\partial p'}\langle p'|x|\alpha\rangle = -\frac{m\hbar^2\omega^2}{2}\frac{\partial^2}{\partial p'^2}\langle p'|\alpha\rangle = -\frac{m\hbar^2\omega^2}{2}\frac{d^2u_\alpha}{dp'^2}$$

With a little rearranging, the wave equation becomes

$$-\frac{m\hbar^2\omega^2}{2}\frac{d^2u_\alpha}{dp'^2} + \frac{1}{2m}p'^2 u_\alpha(p') = Eu_\alpha(p')$$

which is the same as (2.5.13) but with $m\omega^2$ replaced with $1/m$. Inserting this same substitution into (2.5.28) therefore gives the wave functions in momentum space.

16. From (2.3.45a), $x(t) = x(0)\cos\omega t + [p(0)/m\omega]\sin\omega t$, so

$$C(t) \equiv \langle 0|x(t)x(0)|0\rangle = \langle 0|x(0)x(0)|0\rangle \cos\omega t + (1/m\omega)\langle 0|p(0)x(0)|0\rangle \sin\omega t$$

The matrix elements can be calculated by the techniques in Problem 14. You find that $\langle 0|x(0)x(0)|0\rangle = \hbar/2m\omega$ and $\langle 0|p(0)x(0)|0\rangle = 0$. Therefore $C(t) = (\hbar/2m\omega)\cos\omega t$.

17. Write $|\alpha\rangle = a|0\rangle + b|1\rangle$, with a, b real and $a^2 + b^2 = 1$. Using Problem 14,

$$\langle\alpha|x|\alpha\rangle = a^2\langle 0|x|0\rangle + ab\langle 0|x|1\rangle + ab\langle 1|x|0\rangle + b^2\langle 1|x|1\rangle = 2ab\sqrt{\frac{\hbar}{2m\omega}}$$

The maximum is obtained when $a = b = 1/\sqrt{2}$ so $\langle x\rangle = \sqrt{\hbar/2m\omega}$.

The state vector in the Schrödinger picture is $|\alpha, t\rangle = e^{-iHt/\hbar}|\alpha\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\omega t/2}|0\rangle + e^{-3\omega t/2}|1\rangle\right]$ and the expectation value $\langle\alpha, t|x|\alpha, t\rangle$, again using Problem 14, is

$$\langle x\rangle = \frac{1}{2}e^{-i\omega t}\langle 0|x|1\rangle + \frac{1}{2}e^{i\omega t}\langle 1|x|0\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}(e^{-i\omega t} + e^{i\omega t}) = \sqrt{\frac{\hbar}{2m\omega}}\cos\omega t$$

In the Heisenberg picture, use $x(t)$ from (2.3.45a), and again Problem 14. In this case, we note that $\langle 0|p|1\rangle = \langle 1|p|0\rangle = 0$, so we read off $\langle x\rangle = \sqrt{\hbar/2m\omega}\cos\omega t$.

To evaluate $\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$, we just need to calculate $\langle x^2\rangle$. Use the state vector in the Schrödinger picture, and read off matrix elements of x^2 from Problem 14, to get

$$\langle x^2\rangle = \frac{1}{2}\langle 0|x^2|0\rangle + \frac{1}{2}e^{-i\omega t}\langle 0|x^2|1\rangle + \frac{1}{2}e^{i\omega t}\langle 1|x^2|0\rangle + \frac{1}{2}\langle 1|x^2|1\rangle = \frac{1}{2}\frac{\hbar}{2m\omega}[1+3] = \frac{\hbar}{m\omega}$$

so $\langle(\Delta x)^2\rangle = (\hbar/m\omega)(1 - \frac{1}{2}\cos^2\omega t)$.

18. Somehow, it seems this problem should be worked by considering $\langle 0|x^{2n}|0\rangle$, but I don't see it. So, instead, work the left and right sides separately. For the right side, from Problem 14, $\exp[-k^2\langle 0|x^2|0\rangle/2] = \exp[-k^2\hbar/4m\omega]$. For the left side, use position space to write

$$\langle 0|e^{ikx}|0\rangle = \int dx'\langle 0|e^{ikx}|x'\rangle\langle x'|0\rangle = \int dx' e^{ikx'}|\langle x|0\rangle|^2 = \sqrt{\frac{m\omega}{\pi\hbar}}\int dx' e^{ikx'}e^{-m\omega x'^2/\hbar}$$

Put $x' = u\sqrt{\hbar/m\omega}$ and write $-u^2 + iku\sqrt{\hbar/m\omega} = -(u - ik\sqrt{\hbar/m\omega}/2)^2 - \hbar k^2/4m\omega$. Then, putting $w = u - ik\sqrt{\hbar/m\omega}/2$, we have

$$\langle 0|e^{ikx}|0\rangle = \sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{\hbar}{m\omega}}e^{-\hbar k^2/4m\omega}\int dw e^{-w^2} = \frac{1}{\sqrt{\pi}}e^{-\hbar k^2/4m\omega}\sqrt{\pi} = e^{-\hbar k^2/4m\omega}$$

and the two sides are indeed equal.

19. It will be useful to note that, from (2.3.21), $(a^\dagger)^n|0\rangle = \sqrt{n!}|n\rangle$. So

$$\begin{aligned} a\left[e^{\lambda a^\dagger}|0\rangle\right] &= a\left[\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}(a^\dagger)^n|0\rangle\right] = a\left[\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle\right] = \sum_{n=1}^{\infty}\frac{\lambda^n}{\sqrt{n!}}a|n\rangle \\ &= \sum_{n=1}^{\infty}\frac{\lambda^n}{\sqrt{(n-1)!}}|n-1\rangle = \lambda\sum_{m=0}^{\infty}\frac{\lambda^m}{\sqrt{m!}}|m\rangle = \lambda\left[e^{\lambda a^\dagger}|0\rangle\right] \end{aligned}$$

so $e^{\lambda a^\dagger}|0\rangle$ is an eigenvector of a with eigenvalue λ. For the normalization, we need the inner produce of $e^{\lambda a^\dagger}|0\rangle$ with itself. However, $\langle 0|e^{\lambda^* a}e^{\lambda a^\dagger}|0\rangle = \langle 0|e^{\lambda^*\lambda}|0\rangle = e^{|\lambda|^2}$ since $e^{\lambda a^\dagger}|0\rangle$ is an eigenvector of a with eigenvalue λ. Thus $|\lambda\rangle = e^{-|\lambda|^2/2}e^{\lambda a^\dagger}|0\rangle$ is the normalized eigenvector.

Now we have $a|\lambda\rangle = \lambda|\lambda\rangle$ and $\langle\lambda|a^\dagger = \langle\lambda|\lambda^*$, so $\langle\lambda|(a^\dagger \pm a)|\lambda\rangle = \lambda^* \pm \lambda$; $\langle\lambda|(a)^2|\lambda\rangle = \lambda^2$; $\langle\lambda|(a^\dagger)^2|\lambda\rangle = (\lambda^*)^2$; $\langle\lambda|a^\dagger a|\lambda\rangle = \lambda^*\lambda$; and $\langle\lambda|aa^\dagger|\lambda\rangle = \langle\lambda|(1+a^\dagger a)|\lambda\rangle = 1+\lambda^*\lambda$. Therefore

$$\begin{aligned} \langle x\rangle &= \langle\lambda|x|\lambda\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\lambda^*+\lambda) \\ \langle x^2\rangle &= \frac{\hbar}{2m\omega}\left[\lambda^2+(\lambda^*)^2+\lambda^*\lambda+(1+\lambda^*\lambda)\right] \\ (\Delta x)^2 &= \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{2m\omega} \\ \langle p\rangle &= \langle\lambda|p|\lambda\rangle = i\sqrt{\frac{m\hbar\omega}{2}}(\lambda^*-\lambda) \\ \langle p^2\rangle &= -\frac{m\hbar\omega}{2}\left[\lambda^2+(\lambda^*)^2-\lambda^*\lambda-(1+\lambda^*\lambda)\right] \\ (\Delta p)^2 &= \langle p^2\rangle - \langle p\rangle^2 = \frac{m\hbar\omega}{2} \end{aligned}$$

so $\Delta x\Delta p = \hbar/2$ and the minimum uncertainty relation is indeed satisfied. Now, from above,

$$|\lambda\rangle = e^{-|\lambda|^2/2}\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle = \sum_{n=0}^{\infty}f(n)|n\rangle \qquad \text{so} \qquad |f(n)|^2 = e^{-|\lambda|^2}\frac{|\lambda|^{2n}}{n!}$$

which is a Poisson distribution $P_n(\mu) = e^{-\mu}\mu^n/n!$ with mean $\mu \equiv |\lambda|^2$. Note that the mean value of n is not the same as the most probable value, which is an integer, although they approach the same value for large μ, when the Poisson distribution approaches a Gaussian. However, $P_n(\mu)/P_{n-1}(\mu) = \mu/n > 1$ only if $n < \mu$, so the most probable value of n is the largest integer n_m less than $|\lambda^2|$, and the energy is $(n_m+1)\hbar\omega$. To evaluate $e^{-ip\ell/\hbar}|0\rangle = e^{\ell\sqrt{m\omega/2\hbar}(a^\dagger - a)}|0\rangle$, use $e^{A+B} = e^A e^B e^{-[A,B]/2}$ where A and B each commute with $[A,B]$. (See Gottfried, 1966, page 262; Gottfried, 2003, problem 2.13; or R. J. Glauber, Phys. Rev. 84(1951)399, equation 39.) With $\lambda \equiv \ell\sqrt{m\omega/2\hbar}$, we then easily prove the last part, as

$$e^{-ip\ell/\hbar}|0\rangle = e^{\ell\sqrt{m\omega/2\hbar}a^\dagger}e^{-\ell\sqrt{m\omega/2\hbar}a}e^{-\ell^2 m\omega/4\hbar}|0\rangle = e^{-m\ell^2\omega/4\hbar}e^{\ell\sqrt{m\omega/2\hbar}a^\dagger}|0\rangle = e^{-\lambda^2/2}e^{\lambda a^\dagger}|0\rangle$$

20. *Note the entry in the errata;* $\mathbf{J}^2$ *is not yet defined at this point in the text.* The solution is straightforward. We have $[a_\pm, a_\pm^\dagger] = 1$ and $[a_\pm, a_\mp^\dagger] = 0 = [a_\pm^\dagger, a_\mp^\dagger] = [a_\pm, a_\mp]$. Then

$$\begin{aligned}
[J_z, J_+] &= \frac{\hbar^2}{2}\left(a_+^\dagger a_+ a_+^\dagger a_- - a_+^\dagger a_- a_+^\dagger a_+ - a_-^\dagger a_- a_+^\dagger a_- + a_+^\dagger a_- a_-^\dagger a_-\right) \\
&= \frac{\hbar^2}{2}\left(a_+^\dagger a_+ a_+^\dagger a_- - a_+^\dagger a_-(a_+ a_+^\dagger - 1) - a_-^\dagger a_- a_+^\dagger a_- + a_+^\dagger a_- a_-^\dagger a_-\right) \\
&= \frac{\hbar^2}{2}\left(a_+^\dagger a_- - a_-^\dagger a_- a_+^\dagger a_- + a_+^\dagger a_- a_-^\dagger a_-\right) = \frac{\hbar^2}{2} a_+^\dagger\left(a_- - a_-^\dagger a_- a_- + a_- a_-^\dagger a_-\right) \\
&= \frac{\hbar^2}{2} a_+^\dagger\left(a_- - a_-^\dagger a_- a_- + (1 + a_-^\dagger a_-) a_-\right) = \hbar^2 a_+^\dagger a_- = +\hbar J_+
\end{aligned}$$

and similarly for $[J_z, J_-]$. Put $N_\pm = a_\pm^\dagger a_\pm$ so $J_z = (\hbar/2)(N_+ - N_-)$ with $[N_+, N_-] = 0$. From (3.5.24), $\mathbf{J}^2 = J_+J_- + J_z^2 - \hbar J_z$, so $J_+J_- = \hbar^2 a_+^\dagger a_- a_-^\dagger a_+ = \hbar^2 N_+(1 + a_-^\dagger a_-) = \hbar^2 N_+(1 + N_-)$, so $\mathbf{J}^2 = \frac{\hbar^2}{4}(N_+^2 + 2N_+N_- + N_-^2 + 2N_+ + 2N_-) = \frac{\hbar^2}{4}(N^2 + 2N) = \frac{\hbar^2}{2}N\left(\frac{N}{2} + 1\right)$. Finally, noting that we can write both $\mathbf{J}^2$ and J_z in terms of $N_\pm$, which commute, we clearly have $[\mathbf{J}^2, J_z] = 0$.

21. Starting with (2.5.17a), namely $g(x,t) = \exp(-t^2 + 2tx)$, carry out the suggested integral

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x,t)g(x,s)e^{-x^2}dx &= = \int_{-\infty}^{\infty} e^{2st-(t+s)^2+2x(t+s)-x^2}dx \\
&= e^{2st}\int_{-\infty}^{\infty} e^{-[x-(t+s)]^2}dx = \pi^{1/2}e^{2st}
\end{aligned}$$

$$\text{i.e.}\quad \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx\right]\frac{1}{(n!)^2}t^n s^m = \pi^{1/2}\sum_{n=0}^{\infty}\frac{2^n}{n!}t^n s^n$$

The sum on the right only includes terms where t and s have the same power, so the normalization integral on the left must be zero if $n \neq m$. When $n = m$ this gives

$$\begin{aligned}
\left[\int_{-\infty}^{\infty} H_n(x)H_n(x)e^{-x^2}dx\right]\frac{1}{(n!)^2} &= \pi^{1/2}\frac{2^n}{n!} \\
\text{or}\qquad \int_{-\infty}^{\infty} H_n^2(x)e^{-x^2}dx &= \pi^{1/2}2^n n!
\end{aligned}$$

which is (2.5.29). In order to normalize the wave function (2.5.28), we compute

$$\int_{-\infty}^{\infty} u_n^*(x)u_n(x)dx = |c_n|^2\int_{-\infty}^{\infty} H_n^2\left(x\sqrt{\frac{m\omega}{\hbar}}\right)e^{-m\omega x^2/\hbar}dx = |c_n|^2\sqrt{\frac{\hbar}{m\omega}}\pi^{1/2}2^n n! = 1$$

so that $c_n = (m\omega/\pi\hbar)^{1/4}(2^n n!)^{-1/2}$, taking c_n to be real. Compare to (B.4.3).

22. This is a harmonic oscillator with $\omega = \sqrt{k/m}$ for $x > 0$, with $\langle x|n\rangle = 0$ at $x = 0$, that is, solutions with odd n. So, the ground state has energy $3\hbar\omega/2$. The wave function is given by (B.4.3), times $\sqrt{2}$ for normalization, that is $u(x) = 2(m\omega/\pi\hbar)^{1/4}e^{-m\omega x^2/2\hbar}x\sqrt{m\omega/\hbar}$, for $x > 0$, and $u(x) = 0$ for $x < 0$. We than calculate the expectation value

$$\langle x^2\rangle = \frac{4m\omega}{\hbar}\sqrt{\frac{m\omega}{\pi\hbar}}\int_0^\infty x^4 e^{-m\omega x^2/\hbar}dx = \frac{4m\omega}{\hbar}\sqrt{\frac{m\omega}{\pi\hbar}}\frac{3}{8}\left(\frac{\hbar}{m\omega}\right)^2\sqrt{\frac{\pi\hbar}{m\omega}} = \frac{3}{2}\frac{\hbar}{m\omega}$$

23. From (B.2.4), $u_n(x) = \langle x|n\rangle = \sqrt{2/L}\sin(n\pi x/L)$ and $E_n = n^2\pi^2\hbar^2/2mL^2$, so

$$\psi(x,t) = \langle x|\alpha,t\rangle = \langle x|e^{-iHt/\hbar}|\alpha,0\rangle = \sum_n \langle x|e^{-iHt/\hbar}|n\rangle\langle n|\alpha,0\rangle = \sum_n c_n e^{-iE_nt/\hbar}u_n(x)$$

where $c_n \equiv \langle n|\alpha,0\rangle$. Now, I take a hint from the previous solutions manual, that "known to be exactly at $x = L/2$ with certainty" and "You need not worry about normalizations" mean that $\langle x|\alpha,0\rangle \equiv \psi(x,0) = \delta(x - L/2)$, so $c_n = \int_0^L \psi(x,0)u_n(x)dx = \sqrt{2/L}\sin(n\pi/2)$. I don't like this; it seems that $\psi(x,0) = \sqrt{\delta(x - L/2)}$ is a better choice, but how well defined is "known with certainty"? Anyway, $c_n = 0$ if n is even, and $c_n = \sqrt{2/L}(-1)^{(n-1)/2}$ if n is odd, and $|c_n|^2 = 0$ or $|c_n|^2 = 2/L$, i.e. independent of n, for n odd. Then, insert in above.

24. Write the energy eigenvalue as $-E < 0$ for a bound state, so the Schrödinger Equation is $(-\hbar^2/2m)d^2u/dx^2 - \nu_0\delta(x)u(x) = -Eu(x)$. Thus $u(x) = A\exp(-x\sqrt{2mE}/\hbar)$ for $x > 0$, and $u(x) = A\exp(+x\sqrt{2mE}/\hbar)$ for $x < 0$, and $du/dx = \mp(\sqrt{2mE}/\hbar)u(x)$. Now integrate the Schrödiner Equation from $-\varepsilon$ to $+\varepsilon$, and then take $\varepsilon \to 0$. You end up with

$$\lim_{\varepsilon\to 0}\left\{-\frac{\hbar^2}{2m}\frac{\sqrt{2mE}}{\hbar}[-u(\varepsilon) - u(-\varepsilon)]\right\} - \nu_0 u(0) = \frac{\hbar^2}{m}\frac{\sqrt{2mE}}{\hbar}u(0) - \nu_0 u(0) = 0$$

which gives $E = m\nu_0^2/2\hbar^2$. This is unique, so there is only the ground state.

25. For this problem, I just reproduce the solution from the manual for the revised edition. (Note that "problem 22" means "problem 24" here.) See the errata for some comments.

Using the result of problem 22, where $2mE/\hbar^2 = \lambda^2m^2/\hbar^4$ in our notation, we have $\psi(x,t=0) = A\exp[-m\lambda|x|/\hbar^2]$. The normalization is then $2A^2\int_0^\infty \exp[-2m\lambda x/\hbar^2]dx = 1$ or $2A^2[\hbar^2/2m\lambda] = 1$ and hence $A = (m\lambda/\hbar^2)^{1/2}$. From (2.5.7) and (2.5.16), we have

$$\psi(x,t>0) = \int_{-\infty}^{\infty} dx'\psi(x',0)K(x,x';t)$$

$$= (m\lambda/\hbar^2)^{1/2}(m/2\pi i\hbar t)^{1/2}\int_{-\infty}^{\infty}\exp[-m\lambda|x'|/\hbar^2]\exp[i(x-x')^2m/2\hbar t]dx'$$

where we have used $\psi(x',0) = (m\lambda/\hbar^2)^{1/2}\exp[-m\lambda|x'|/\hbar^2]$.

26. With $V(x) = \lambda x$, $\lambda > 0$ and $-\infty < x < \infty$, the eigenvalues E are continuous. The wave function is oscillatory for $x < a$ and decaying for $x > a$, where $a \equiv E/\lambda$ is the classical turning point. Indeed, the wave function is proportional to the Airy function $Ai(z)$ where $z \propto (x-a)$. See Figure 2.3. On the other hand, for $V(x) = \lambda|x|$, there are now quantized bound states. This parity-symmetric potential has even and odd wave functions. The even wave functions have $Ai'(z) = 0$ at $x = 0$, and the odd wave functions have $Ai(z) = 0$ at $x = 0$. These conditions lead to quantized energies through (2.5.34) and (2.5.35). As shown in Figure 2.4, the odd energy levels have been confirmed by "bouncing neutrons."

27. *Note: This was Problem 36 in Chapter Five in the Revised Edition. It was moved to this chapter because "density of states" is explicitly worked out now in this chapter. It seems, though, that I should have reworded the problem a bit. See the errata.*

Refer back to the discussion in Section 2.5. The wave function is

$$u_E(\mathbf{x}) = \frac{1}{L}e^{i\mathbf{k}\cdot\mathbf{x}} \qquad \text{where} \qquad k_x = \frac{2\pi}{L}n_x \qquad \text{and} \qquad k_y = \frac{2\pi}{L}n_y$$

and n_x and n_y are integers, with $\mathbf{p} = \hbar\mathbf{k}$. The energy is

$$\begin{aligned} E &= \frac{\mathbf{p}^2}{2m} = \frac{\hbar^2}{2m}(k_x^2 + k_y^2) = \frac{2\pi^2\hbar^2}{mL^2}(n_x^2 + n_y^2) = \frac{2\pi^2\hbar^2}{mL^2}\mathbf{n}^2 \\ \text{so} \qquad dE &= \frac{4\pi^2\hbar^2}{mL^2}ndn \end{aligned}$$

The number of states with $|\mathbf{n}|$ between n and $n + dn$, and ϕ and $\phi + d\phi$, is

$$dN = ndnd\phi = m\left(\frac{L}{2\pi\hbar}\right)^2 dEd\phi$$

so the density of states is just $m(L/2\pi\hbar)^2$. Remarkably, this result is independent of energy.

28. We want to solve (2.5.1) in cylindrical coordinates, that is find $u(\rho, \phi, z)$ where

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{2m_e E}{\hbar^2}u \equiv -k^2 u$$

subject to $u(\rho_a, \phi, z) = u(\rho_b, \phi, z) = u(\rho, \phi, 0) = u(\rho, \phi, L) = 0$. For $u(\rho, \phi, z) = w(\rho, z)\Phi(\phi)$,

$$\frac{1}{w}\left[\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial w}{\partial\rho}\right) + \rho^2\frac{\partial^2 w}{\partial z^2}\right] + \rho^2 k^2 + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0$$

The first two terms are independent of ϕ, and the third term is independent of ρ and z, so they both must equal some constant but with opposite sign. Write $(1/\Phi)\partial^2\Phi/\partial\phi^2 = -m^2$, giving $\Phi(\phi) = e^{\pm im\phi}$ with m an integer so that $\Phi(\phi + 2\pi) = \Phi(\phi)$. Now with $w(\rho, z) = R(\rho)Z(z)$,

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{\rho^2}{Z}\frac{\partial^2 Z}{\partial z^2} + \rho^2 k^2 = m^2 \qquad \text{so} \qquad \frac{1}{\rho R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) - \frac{m^2}{\rho^2} + k^2 + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

and similarly put $(1/Z)\partial^2 Z/\partial z^2 = -\alpha^2$ so that $Z(\alpha) = e^{\pm i\alpha z}$. Enforcing $Z(0) = 0 = Z(L)$ leads to $Z(z) = \sin\alpha_\ell z$ where $\alpha_\ell = \ell\pi/L$ and $\ell = 1, 2, 3\ldots$. The ρ equation is therefore

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \left(k^2 - \alpha_\ell^2 - \frac{m^2}{\rho^2}\right) R = 0$$

Now define $\kappa^2 \equiv k^2 - \alpha_\ell^2$ and $x \equiv \kappa\rho$. Multiply through by x^2 and this becomes

$$x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - m^2)R = 0$$

i.e., Bessel's equation, with solution $R(\rho) = A_m J_m(\kappa\rho) + B_m N_m(\kappa\rho)$, where $J_m(x)$ and $N_m(x)$ are Bessel functions of the first and second kind, respectively. The cylinder wall boundary conditions tell us that for each m we must have $A_m J_m(\kappa\rho_a) + B_m N_m(\kappa\rho_a) = 0$ and $A_m J_m(\kappa\rho_b) + B_m N_m(\kappa\rho_b) = 0$. Set the determinant to zero, and so we would solve

$$J_m(\kappa\rho_a)N_m(\kappa\rho_b) - J_m(\kappa\rho_b)N_m(\kappa\rho_a)$$

for κ. Denote with k_{mn} the nth solution for κ for a given m. Then

$$E = \frac{\hbar^2}{2m_e}k^2 = \frac{\hbar^2}{2m_e}[\kappa^2 + \alpha_\ell^2] \qquad \text{or} \qquad E_{\ell mn} = \frac{\hbar^2}{2m_e}\left[k_{mn}^2 + \left(\frac{\ell\pi}{L}\right)^2\right]$$

In the presence of a magnetic field, the Hamiltonian becomes (2.7.20), with $\phi = 0$. We recover the problem already solved, essentially, using the gauge transformation (2.7.36), but we need to multiply the wave function by the phase factor $\exp[ie\Lambda(\mathbf{x})/\hbar c]$ as in (2.7.55). In this case, $\mathbf{A} = \boldsymbol{\nabla}\Lambda = \hat{\boldsymbol{\phi}}(1/\rho)\partial\Lambda/\partial\phi$ is given by (2.7.62), so $\Lambda(\mathbf{x}) = B\rho_a^2\phi/2 \equiv \hbar c g\phi/e$, and

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} \longrightarrow e^{-ig\phi}\frac{1}{\Phi}\frac{d^2}{d\phi^2}\left(e^{ig\phi}\Phi\right) = \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + \frac{2ig}{\Phi}\frac{d\Phi}{d\phi} - g^2 = -m^2 \mp 2gm - g^2 = -(m \pm g)^2$$

for $\Phi(\phi) = e^{\pm im\phi}$. Consequently, the solution is the same, but with (integer) m replaced by $\gamma \equiv m \pm g$. (The solutions to Bessel's equation are perfectly valid for non-integral indices.) The ground state is $\ell = 1$ and $n = 1$, so $E_0 = (\hbar^2/2m_e)(k_{01} + \pi^2/L^2)$ for $B = 0$, and $E_0 = (\hbar^2/2m_e)(k_{\gamma 1} + \pi^2/L^2)$ for $B \neq 0$. For these to be equal, $m \pm g = 0$ for integer m, so

$$g \equiv \frac{e}{\hbar c}\frac{B\rho_a^2}{2} = \pm m \qquad \text{or} \qquad B \times \pi\rho_a^2 = \pm 2\pi\frac{\hbar c}{e}m = \pm\frac{hc}{e}m$$

which is the "flux quantization" condition.

The history of flux quantization is quite fascinating. The original discovery can be found in B. S. Deaver and W. M. Fairbank, "Experimental Evidence for Quantized Flux in Superconducting Cylinders", Phys. Rev. Lett. 7(1961)43. The flux quantum worked out to be $hc/2e$, but it was later appreciated that the charge carriers were Cooper pairs of electrons. See also articles by Deaver and others in "Near Zero: new frontiers of physics", by Fairbank, J. D.; Deaver, B. S., Jr.; Everitt, C. W. F.; Michelson, P. F.. Freeman, 1988.

29. The hardest part of this problem is to identify the Hamilton-Jacobi Equation. See Chapter 10 in Goldstein, Poole, and Safko. With one spacial dimension, this equation is $H(x, \partial S/\partial x, t) + \partial S/\partial t = 0$ to be solved for $S(x,t)$, called Hamilton's Principle Function. So, $H\psi = -(\hbar^2/2m)\partial^2\psi/\partial t^2\psi + V(x)\psi = i\hbar\partial\psi/\partial t$ with $\psi(x,t) = \exp[iS(x,t)/\hbar]$ becomes

$$-\frac{\hbar^2}{2m}\left[\frac{i}{\hbar}\frac{\partial^2 S}{\partial x^2} + \left(\frac{i}{\hbar}\frac{\partial S}{\partial x}\right)^2\right]\psi + V(x)\psi = -\frac{\partial S}{\partial t}\psi$$

If $\hbar$ is "small" then the second term in square brackets dominates. Dividing out ψ then leaves us with the Hamilton-Jacobi Equation. Putting $V(x) = 0$ and trying $S(x,t) = X(x) + T(t)$, find $(X'')^2/2m = -T' = \alpha$ (a constant). Thus $T(t) = a - \alpha t$ and $X(x) = \pm\sqrt{2m\alpha x + b}$, where a and b are constants that can be discarded when forming $\psi(x,t) = \exp[i(X+T)/\hbar]$. Hence $\psi(x,t) = \exp[i(\pm\sqrt{2m\alpha}x - \alpha t)/\hbar]$, a plane wave. This exact solution comes about because S is linear in x, so $\partial^2 S/\partial x^2 = 0$ and the first term in the Schrödinger Equation, above, is manifestly zero.

30. You could argue this should be in Chapter 3, but what you need to know about the hydrogen atom is so basic, it would surely be covered in an undergraduate quantum physics class. (See, for example, Appendix B.5.) The wave function for the atom looks like $\psi(r,\theta,\phi) = R_{nl}(r)Y_l^m(\theta,\phi) = C_{lm}R_{nl}(r)P_l^m(\cos\theta)e^{im\phi}$ where C_{lm}, $R_{nl}(r)$, and $P_l^m(\cos\theta)$ are all real. Since $\boldsymbol{\nabla} = \hat{\mathbf{r}}\partial/\partial r + \hat{\boldsymbol{\theta}}(1/r)\partial/\partial\theta + \hat{\boldsymbol{\phi}}(1/r\sin\theta)\partial/\partial\phi$, we have from (2.4.16)

$$\mathbf{j} = \frac{\hbar}{m_e}\mathrm{Im}\left[\psi^\star\boldsymbol{\nabla}\psi\right] = \hat{\boldsymbol{\phi}}\frac{m\hbar}{m_e r\sin\theta}|\psi|^2$$

so $\mathbf{j} = 0$ if $m = 0$, and is in the positive (negative) ϕ direction if m is positive (negative).

31. Write $ibp' - iap'^2 = -ia(p'^2 - bp'/a + b^2/4a^2) + ib^2/4a = -ia(p' - b/2a)^2 + ib^2/4a$, translate p' in the integral, and use $\int_{-\infty}^{\infty} e^{-cx^2}dx = \sqrt{\pi/c}$. Then

$$\begin{aligned} K(x'', t; x', t_0) &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp' \exp\left[\frac{ip(x''-x')}{\hbar} - \frac{ip'^2(t-t_0)}{2m\hbar}\right] \\ &= \frac{1}{2\pi\hbar}\sqrt{\frac{2\pi\hbar m}{i(t-t_0)}}\exp\left[i\frac{m(x''-x')^2}{2\hbar(t-t_0)}\right] = \sqrt{\frac{m}{2\pi\hbar i(t-t_0)}}\exp\left[i\frac{m(x''-x')^2}{2\hbar(t-t_0)}\right] \end{aligned}$$

To generalize to three dimensions, just realize that the length along the x-axis is invariant under rotations. Therefore, we have

$$K(\mathbf{x}'', t; \mathbf{x}', t_0) = \sqrt{\frac{m}{2\pi\hbar i(t-t_0)}}\exp\left[i\frac{m(\mathbf{x}''-\mathbf{x}')^2}{2\hbar(t-t_0)}\right]$$

32. From (2.6.22), $Z = \sum_{a'} \exp[-\beta E_{a'}]$, so, defining E_0 to be the ground state energy,

$$\lim_{\beta\to\infty}\left\{-\frac{1}{Z}\frac{\partial Z}{\partial\beta}\right\} = \lim_{\beta\to\infty}\left\{\frac{\sum_{a'} E_{a'}\exp[-\beta E_{a'}]}{\sum_{a'}\exp[-\beta E_{a'}]}\right\} = \lim_{\beta\to\infty}\left\{\frac{\sum_{a'} E_{a'}\exp[-\beta(E_{a'}-E_0)]}{\sum_{a'}\exp[-\beta(E_{a'}-E_0)]}\right\} = E_0$$

where we multiply top and bottom by $\exp(\beta E_0)$ in the penultimate step. The limit is easy to take because for all terms in which $E_{a'} \neq E_0$, the exponent is negative as $\beta \to \infty$ and the term vanishes. For the term $E_{a'} = E_0$, the numerator is E_0 and the denominator is unity.

To "illustrate this for a particle in a one-dimensional box" is trivial. Just replace $E_{a'}$ with $E_n = \hbar^2\pi^2 n^2/2mL^2$ for $n = 1, 2, 3\ldots$ (B.2.4) and the work above carries through. The old solution manual has a peculiar approach, though, replacing the sum by an integral, presumably valid as $\beta \to \infty$, but I don't really get the point.

33. Recall that, in the treatment (2.6.26) for the propagator, position (or momentum) bras and kets are taken to be in the Heisenberg picture. So, one should recall the discussion on pages 86–88, regarding the time dependence of base kets. In particular, $|a', t\rangle_H = \mathcal{U}^\dagger(t)|a'\rangle$, that is, base kets are time dependent and evolve "backwards" relative to state kets in the Schrödinger picture. So, for a free particle with $H = \mathbf{p}^2/2m$, we have

$$\langle \mathbf{p}'', t|\mathbf{p}', t_0\rangle = \langle \mathbf{p}''|e^{-iHt/\hbar}e^{iHt_0/\hbar}|\mathbf{p}'\rangle = \exp\left[-\frac{i}{\hbar}\frac{\mathbf{p}'^2}{2m}(t-t_0)\right]\delta^{(3)}(\mathbf{p}''-\mathbf{p}')$$

The solution in the old manual confuses me.

34. The classical action is $S(t_a, t_b) = \int_{t_a}^{t_b} dt\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2\right)$. Approximating this for the time interval $\Delta t \equiv t_b - t_a$, defining $\Delta x \equiv x_b - x_a$, and writing $x_a + x_b = 2x_b - \Delta x$, we have

$$S(t_a, t_b) \approx \Delta t\left[\frac{1}{2}m\left(\frac{\Delta x}{\Delta t}\right)^2 - \frac{1}{2}m\omega^2\left(x_b - \frac{\Delta x}{2}\right)^2\right] \approx \frac{1}{2}m\left(\frac{\Delta x}{\Delta t}\right)\Delta x - \frac{1}{2}m\omega^2 x_b^2\Delta t$$

keeping only lowest order terms. Combine this with (2.6.46) (and sum over all paths) to get the Feynman propagator. Now the problem says to show this is the same as (2.6.26), but (2.6.18) is the solution for the harmonic oscillator. Taking this limit for $\Delta t \to 0$, one gets

$$\begin{aligned} K(x_b, t_b; x_a, t_a) &= \sqrt{\frac{m}{2\pi i\hbar\Delta t}}\exp\left[\left\{\frac{im}{2\hbar\Delta t}\right\}\left\{(x_b^2 + x_a^2)\left(1 - \frac{\omega^2\Delta t^2}{2}\right) - 2x_a x_b\right\}\right] \\ &= \sqrt{\frac{m}{2\pi i\hbar\Delta t}}\exp\left[\frac{i}{\hbar}\left\{\frac{1}{2}m\frac{(\Delta x)^2}{\Delta t} - \frac{1}{2}m\omega^2(x_a^2 + x_b^2)\Delta t\right\}\right] \end{aligned}$$

Taking the limit $\Delta x \to 0$ clearly gives the same expression as inserting our classical action, above, into (2.6.46).

I'm not sure I understand the point of this problem.

35. The "Schwinger action principle" does not seem to be treated in modern references, and also not in (this version of) this textbook. So, I just reprint here San Fu Tuan's old solution.

The Schwinger action principle states that the following condition determines the transformation function $\langle x_2 t_2 | x_1 t_1 \rangle$ in terms of a given quantum mechanical Lagrangian L

$$\delta\langle x_2 t_2 | x_1 t_1 \rangle = (i/\hbar)\langle x_2 t_2 | \delta\int_{t_1}^{t_2} L dt | x_1 t_1 \rangle.$$

To obtain $\langle x_2 t_2 | x_1 t_1 \rangle$, let $\delta\langle x_2 t_2 | x_1 t_1 \rangle = (i/\hbar)\langle x_2 t_2 | \delta W_{21} | x_1 t_1 \rangle$ where W_{21} is action in going from initial state $x_1 t_1$ to final state $x_2 t_2$. Also, let $\delta W_{21} = \delta\omega_{21}$ where $\delta\omega_{21}$ is the well-ordered form (c.f. Finkelstein (1973), p.164) of δW_{21}. Then $\delta\langle x_2 t_2 | x_1 t_1 \rangle = \frac{i}{\hbar}\langle x_2 t_2 | \ \delta\omega_{21} | x_1 t_1 \rangle = \frac{i}{\hbar}\ \delta\omega_{21}' \langle x_2 t_2 | x_1 t_1 \rangle$ and thus $\delta \ln\langle x_2 t_2 | x_1 t_1 \rangle = \frac{i}{\hbar}\delta\omega_{21}'$ or

$$\langle x_2 t_2 | x_1 t_1 \rangle = \exp[\tfrac{i}{\hbar}\omega_{21}'].\qquad (1)$$

The corresponding Feynman expression for $\langle x_2 t_2 | x_1 t_1 \rangle$ [c.f. Finkelstein (1973), p.144] is

$$\langle x_2 t_2 | x_1 t_1 \rangle = \frac{1}{N}\sum_{\text{paths}} \exp[(i/\hbar)S_{21}].\qquad (2)$$

The classical limit of (2) is such that as $\hbar/S \to$ small, the probability amplitude $\langle x_2 t_2 | x_1 t_1 \rangle$ will be important only for those varied paths which lie in a narrow tube between $x_1 t_1$ and $x_2 t_2$ enclosing the classical path. On the other hand, to describe the classical limit for (1) (which has a well-ordered exponent instead of a sum over paths), is to consider first the operator Hamilton-Jacobi equation (c.f. Finkelstein (1973), p.166)

$$H(\tfrac{\partial\omega}{\partial x}, \ldots, x, \ldots) + \partial\omega/\partial t = 0.\qquad (3)$$

Since ω_{21}' satisfies (3), which arises from a variation of the final state (and is similar to the Schrödinger picture), it is seen that the correspondence limit of ω_{21}' is S, i.e. the probability amplitude (1) approaches the consideration of all possible paths as in the Feynman path integral case (2). Thus in the classical limit, (1) and (2) become equal provided they both are modulated by the factor 1/N (N = total number of individual steps in going from $x_1 t_1 \to x_2 t_2$).

36. Wave mechanically, the phase difference comes about because, approximating the neutron by a plane wave, the factor $\exp[-i(\omega t - px/\hbar)]$ (where x is the direction AC or BD in Figure 2.9) is different because p (and $v = p/m_n$) will depend on the height. That is, $p_{BD}^2/2m_n = p_{AC}^2/2m_n - m_n gz$ where $z = l_2 \sin\delta$. The accumulated phase difference is

$$\phi_{BD} - \phi_{AC} = \left[\frac{p_{BD} - p_{AC}}{\hbar} - \omega\left(\frac{1}{v_{BD}} - \frac{1}{v_{AC}}\right)\right] l_1 = \frac{p_{BD} - p_{AC}}{\hbar}\left[1 + \frac{\hbar\omega}{m_n v_{BD} v_{AC}}\right] l_1$$

The experiment in Figure 2.10 was performed with $\lambda = 1.445$Å neutrons. (The book has $\lambda = 1.42$Å?) So $p = h/\lambda = 2\pi\hbar c/c\lambda = 2\pi(200 \times 10^6 \times 10^{-5}\ \text{eV} - \text{Å})/c\lambda = 8.7$ keV$/c$ and $E = \hbar\omega = p^2/2m_n = 4.05 \times 10^{-2}$ eV, whereas $m_n gh = (m_n c^2)gh/c^2 \approx 10^{-9}$ eV for $h = 10$ cm. Thus the change in momentum is very small and $\hbar\omega/m_n v_{BD} v_{AC} = m_n E/p^2 = 1/2$. Therefore

$$\phi_{BD} - \phi_{AC} = \frac{p_{BD} - p_{AC}}{\hbar}\frac{3}{2}l_1 \approx \frac{p_{BD}^2 - p_{AC}^2}{2\hbar p}\frac{3}{2}l_1 = -\frac{2m_n^2 gz}{2\hbar p}\frac{3}{2}l_1 = -\frac{3}{2}\frac{m_n^2 g(\lambda/2\pi)l_1 z}{\hbar^2}$$

This differs from (2.7.17) by the factor 3/2, which comes from the ωt contribution to the phase. San Fu Tuan's solution starts with the same expression as I do, but ignores the ωt term when calculating the phase. My thought is that this is in fact a more complicated problem than meets the eye, and I need to think about it more.

37. Since $\mathbf{A} = \mathbf{A}(\mathbf{x})$, write $p_i = (\hbar/i)\partial/\partial x_i$ and work in position space. Then

$$\begin{aligned}
[\Pi_i, \Pi_j]\psi(\mathbf{x}) &= \left[\frac{\hbar}{i}\frac{\partial}{\partial x_i} - \frac{eA_i}{c}, \frac{\hbar}{i}\frac{\partial}{\partial x_j} - \frac{eA_j}{c}\right]\psi(\mathbf{x}) = -\frac{\hbar}{i}\frac{e}{c}\left\{\left[\frac{\partial}{\partial x_i}, A_j\right] - \left[A_i, \frac{\partial}{\partial x_j}\right]\right\}\psi(\mathbf{x}) \\
&= -\frac{\hbar}{i}\frac{e}{c}\left\{\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right\}\psi(\mathbf{x}) = \frac{i\hbar e}{c}\varepsilon_{ijk}\left(\boldsymbol{\nabla} \times \mathbf{A}\right)_k \psi(\mathbf{x}) = \frac{i\hbar e}{c}\varepsilon_{ijk}B_k\psi(\mathbf{x})
\end{aligned}$$

$$m\frac{d^2x_i}{dt^2} = \frac{d\Pi_i}{dt} = \frac{1}{i\hbar}[\Pi_i, H] = \frac{1}{i\hbar}\left[\Pi_i, \frac{1}{2m}\boldsymbol{\Pi}^2 + e\phi\right] = \frac{1}{2im\hbar}\sum_j\left[\Pi_i, \Pi_j^2\right] + \frac{1}{i\hbar}[p_i, e\phi]$$

Now from Problem 1.29(a), $(1/i\hbar)[p_i, e\phi] = -e\partial\phi/\partial x_i = eE_i$. Also $[\Pi_i, \Pi_j^2] = [\Pi_i, \Pi_j]\Pi_j + \Pi_j[\Pi_i, \Pi_j]$ so $(1/2im\hbar)[\Pi_i, \Pi_j^2] = (e/2mc)(\varepsilon_{ijk}B_k p_j + p_j\varepsilon_{ijk}B_k)$. This amounts to

$$m\frac{d^2\mathbf{x}}{dt^2} = e\mathbf{E} + \frac{e}{2mc}[-\mathbf{B} \times \mathbf{p} + \mathbf{p} \times \mathbf{B}] = e\left[\mathbf{E} + \frac{1}{2c}\left(\frac{d\mathbf{x}}{dt} \times \mathbf{B} - \mathbf{B} \times \frac{\mathbf{x}}{dt}\right)\right]$$

As for showing that (2.7.30) follows from (2.7.29) with $\mathbf{j}$ defined as in (2.7.31), just follow the same steps used to prove (2.4.15) with the definition (2.4.16). That is, multiply the Schrödinger equation by ψ^*, and then multiply its complex conjugate by ψ, and subtract the two equations. You just need to use some extra care when writing out (2.7.29) to make sure the $\mathbf{A}(\mathbf{x}')$ is appropriately differentiated. Indeed, the Schrödinger equation becomes

$$-\frac{\hbar^2}{2m}\boldsymbol{\nabla}'^2\psi + \frac{i\hbar e}{mc}\mathbf{A} \cdot \boldsymbol{\nabla}'\psi + \frac{i\hbar e}{2mc}\left(\boldsymbol{\nabla}' \cdot \mathbf{A}\right)\psi + \frac{e^2}{2mc^2}\mathbf{A}^2\psi + e\phi\psi = i\hbar\frac{\partial\psi}{\partial t}$$

The remainder of the proof is simple from here.

38. The vector potential $\mathbf{A} = -\frac{1}{2}By\hat{\mathbf{x}} + \frac{1}{2}Bx\hat{\mathbf{y}}$ gives $\mathbf{B} = B\hat{\mathbf{z}}$ in a gauge where $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$. Reading the Hamiltonian from the previous problem solution, we are led to an interaction

$$\frac{i\hbar e}{mc}\mathbf{A}\cdot\boldsymbol{\nabla}+\frac{e^2}{2mc^2}\mathbf{A}^2 = -\frac{e}{mc}\left(-\frac{1}{2}By\hat{\mathbf{x}}+\frac{1}{2}Bx\hat{\mathbf{y}}\right)\cdot\mathbf{p}+\frac{e^2B^2}{8mc^2}(x^2+y^2) = \frac{eB}{2mc}L_z+\frac{e^2B^2}{8mc^2}(x^2+y^2)$$

where $\mathbf{L} \equiv \mathbf{r}\times\mathbf{p}$. The first term is just $\boldsymbol{\mu}\cdot\mathbf{B}$ for $\boldsymbol{\mu} \equiv (e/2mc)\mathbf{L}$, the magnetic moment of an orbiting electron. The second term gives rise to the quadratic Zeeman effect. See pages 328–330 and Problems 5.18 and 5.19 in the textbook.

39. See the solution to Prob.37. We find $[\Pi_x, \Pi_y] = (i\hbar e/c)B_z = i\hbar eB/c$ or $[Y, \Pi_y] = i\hbar$ for $Y \equiv c\Pi_x/eB$. As in the solution to Prob.38, $A_z = 0$. So, as in Prob.37, the Hamiltonian is

$$H = \frac{\Pi_x^2}{2m}+\frac{\Pi_y^2}{2m}+\frac{p_z^2}{2m} = \frac{p_z^2}{2m}+\frac{\Pi_y^2}{2m}+\frac{1}{2}m\frac{e^2B^2}{m^2c^2}Y^2$$

The second two terms constitute the one dimensional harmonic oscillator Hamiltonian, by virtue of the commutation relation $[Y, \Pi_y] = i\hbar$, with ω replaced by eB/mc.

40. One requires that the phase change $\mu BT/\hbar$ be 2π after traversing a field B of length $l = vT$. The speed $v = p/m = h/\lambda m$. Since $\mu = g_n(e\hbar/2mc)$, we have

$$\frac{\mu BT}{\hbar} = g_n\frac{e\hbar}{2mc}\frac{B}{\hbar}\frac{lm\lambda}{h} = 2\pi \qquad \text{or} \qquad B = \frac{4\pi hc}{eg_nl\lambda}$$

See also (3.2.25). San Fu Tuan's solution is much more complicated. I may be misunderstanding something.

Chapter Three

1. *Note: The original solution manual does not answer this problem correctly.* The eigenvalues λ satisfy $\lambda^2 - i(-i) = \lambda^2 - 1 = 0$, i.e. $\lambda = \pm 1$, as they must be, since $S_y \doteq (\hbar/2)\sigma_y$ has eigenvalues $\pm\hbar/2$. The eigenvectors are well known by now, namely

$$|S_y;+\rangle \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} \qquad \text{and} \qquad |S_y;-\rangle \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad \text{so, for} \qquad |\psi\rangle \doteq \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

where $|\alpha|^2+|\beta|^2 = 1$, the probability of finding $S_y = +\hbar/2$ is $|\langle S_y;+|\psi\rangle|^2 = |(\alpha - i\beta)/\sqrt{2}|^2 = (1 + \mathrm{Im}(\alpha^*\beta))/2$. Clearly this gives the right answer for $|\psi\rangle = |S_y;\pm\rangle$. It might have been more interesting, though, to ask for the expectation value of S_y, namely

$$\langle\psi|S_y|\psi\rangle = \frac{\hbar}{2}\begin{bmatrix} \alpha^* & \beta^* \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = i\frac{\hbar}{2}(\beta^*\alpha - \alpha^*\beta) = \frac{\hbar}{2}\mathrm{Im}(\alpha^*\beta)$$

2. Since $\mathbf{S} \doteq (\hbar/2)\boldsymbol{\sigma}$, the matrix representation of the Hamiltonian is

$$H \doteq \mu\begin{pmatrix} -B_z & -B_x + iB_y \\ -B_x - iB_y & B_z \end{pmatrix}$$

Therefore, the characteristic equation for the eigenvalues λ is

$$(-\mu B_z - \lambda)(\mu B_z - \lambda) - \mu^2(-B_x - iB_y)(-B_x + iB_y) = -\mu^2(B_z^2 + B_x^2 + B_y^2) + \lambda^2 = 0$$

so the eigenvalues are $\lambda = \pm\mu B$ where $B^2 = B_x^2 + B_y^2 + B_z^2$. Of course.

3. We have $U = A(A^\dagger)^{-1}$ where $A \equiv a_0 + i\boldsymbol{\sigma}\cdot\mathbf{a}$ and $AA^\dagger = a_0^2 + (\boldsymbol{\sigma}\cdot\mathbf{a})^2 = a_0^2 + \mathbf{a}^2 \equiv \alpha^2$, using (3.2.41). So $UU^\dagger = A(A^\dagger)^{-1}A^{-1}A^\dagger = A(AA^\dagger)^{-1}A^\dagger = A(1/\alpha^2)A^\dagger = \alpha^2/\alpha^2 = 1$ and U is unitary. Now $\det U = \det A/\det A^\dagger$, so writing these out as

$$A = \begin{pmatrix} a_0 + ia_3 & ia_1 + a_2 \\ ia_1 - a_2 & a_0 - ia_3 \end{pmatrix} \qquad \text{and} \qquad A^\dagger = \begin{pmatrix} a_0 - ia_3 & -ia_1 - a_2 \\ -ia_1 + a_2 & a_0 + ia_3 \end{pmatrix}$$

we see that $\det A = \alpha^2 = \det A^\dagger$, so $\det U = \alpha^2/\alpha^2 = 1$ and U is unimodular.

See (3.3.7) and (3.3.10). We want to find expressions for the complex numbers a and b in terms of our real parameters a_0, a_1, a_2, and a_3. To do this, write

$$U = AAA^{-1}(A^\dagger)^{-1} = A^2(A^\dagger A)^{-1} = \frac{1}{\alpha^2}A^2 = \frac{1}{\alpha^2}\begin{pmatrix} a_0 - \mathbf{a}^2 + 2ia_0a_3 & 2a_0a_2 + 2ia_0a_1 \\ -2a_0a_2 + 2ia_0a_1 & a_0 - \mathbf{a}^2 - 2ia_0a_3 \end{pmatrix}$$

so $\cos(\phi/2) = \mathrm{Re}(a) = (a_0 - \mathbf{a}^2)/\alpha^2$ which gives $\sin(\phi/2) = \sqrt{1-\cos^2(\phi/2)} = 2a_0|\mathbf{a}|/\alpha^2$, and $n_x = -\mathrm{Im}(b)/\sin(\phi/2) = -a_1/|\mathbf{a}|$; $n_y = -\mathrm{Re}(b)/\sin(\phi/2) = -a_2/|\mathbf{a}|$; and $n_z = -\mathrm{Im}(a)/\sin(\phi/2) = -a_3/|\mathbf{a}|$.

4. In principle, this could be solved by diagonalizing the Hamiltonian for $A \neq 0$ and $eB/mc \neq 0$, and then taking limits. However, the problem is not posed that way, so work it the way we are led. Write the "spin function" as $|+-\rangle$. Then for $A = 0$, $eB/mc \neq 0$,

$$H|+-\rangle = \frac{eB}{mc}\left[S_z^{(e^-)} - S_z^{(e^+)}\right]|+-\rangle = \frac{eB}{mc}\left[\frac{\hbar}{2} - \left(-\frac{\hbar}{2}\right)\right]|+-\rangle = \frac{eB\hbar}{mc}|+-\rangle$$

and this is an eigenstate, with eigenvalue $eB\hbar/mc$. For $A \neq 0$, $B = 0$, see (3.8.19) to write

$$\begin{aligned} H &= A\mathbf{S}^{(e^-)} \cdot \mathbf{S}^{(e^+)}|+-\rangle = A\left[S_z^{(e^-)}S_z^{(e^+)} + \frac{1}{2}S_+^{(e^-)}S_-^{(e^+)} + \frac{1}{2}S_-^{(e^-)}S_+^{(e^+)}\right]|+-\rangle \\ &= A\left(\frac{\hbar}{2}\right)\left(-\frac{\hbar}{2}\right)|+-\rangle + 0 + A\frac{1}{2}\hbar\hbar|-+\rangle = A\frac{\hbar^2}{4}\left[-|+-\rangle + 2|-+\rangle\right] \end{aligned}$$

using (3.5.39) and (3.5.40). So, $|+-\rangle$ is not an eigenvector. In this case, the expectation value is $\langle +-|H|+-\rangle = -A\hbar^2/4$.

5. The answer is zero in both cases. For $S_z(S_z + \hbar I)(S_z - \hbar I) = S_z(S_z^2 - \hbar^2 I)$, use the basis states $|\pm 1\rangle, |0\rangle$ with quantization in the z-direction. Then $(S_z^2 - \hbar^2 I)|\pm 1\rangle = 0$ and $S_z|0\rangle = 0$ and any expectation value works out to be zero. The case is the same for S_x.

6. Start with $d\mathbf{K}/dt = (i/\hbar)[H, \mathbf{K}] = (i/2\hbar)[K_1^2/I_1 + K_2^2/I_2 + K_3^2/I_3, K_1\hat{\mathbf{x}} + K_2\hat{\mathbf{y}} + K_3\hat{\mathbf{z}}]$ and first consider $dK_1/dt = (i/2\hbar)[K_2^2/I_2 + K_3^2/I_3, K_1]$. Before we make use of the relation $[A^2, B] = A^2B - BA^2 = A^2B - ABA + ABA - BA^2 = A[A, B] + [A, B]A$, realize that for a system in which the axes rotate, $[K_1, K_2] = -i\hbar K_3$. (Other than the brief mention of "active" versus "passive" rotations on page 158, I don't think this is discussed in the book. I see this point in San Fu Tuan's original solution manual.) Therefore

$$\begin{aligned} \frac{dK_1}{dt} &= \frac{i}{2\hbar}\left\{\frac{1}{I_2}\left(i\hbar K_2K_3 + i\hbar K_3K_2\right) - \frac{1}{I_3}\left(i\hbar K_3K_2 + i\hbar K_2K_3\right)\right\} \\ &= \frac{1}{2}\left\{K_2K_3\left(\frac{1}{I_3} - \frac{1}{I_2}\right) + K_3K_2\left(\frac{1}{I_3} - \frac{1}{I_2}\right)\right\} = \frac{I_2 - I_3}{2I_2I_3}\{K_2, K_3\} \end{aligned}$$

and similarly for the other components. In the "correspondence limit", the operators are just observed variables, so $K_iK_j = K_jK_i$ and, for example,

$$\frac{dK_1}{dt} = \frac{I_2 - I_3}{I_2I_3}K_2K_3 = (I_2 - I_3)\omega_2\omega_3$$

which is, in fact, the Euler equation (for K_1) for rotational motion.

7. For this problem, I just reprint San Fu Tuan's solution from the manual for the previous edition. I don't understand it, however. It refers, for example, to G_i, $i = 1, 2, 3$, but the problem only refers to G_2 and G_3. Indeed, comparison with (3.3.19) and (3.1.15) & (3.1.16) argues that $G_2 = -J_y/\hbar$ and $G_3 = -J_z/\hbar$. Either it is trivial, or I misunderstand the point.

If U represents the rotation with Euler angles α, β, γ, then U must satisfy for infinitesimal rotation angle ϵ (c.f. (3.1.7)) $U_x(\epsilon)U_y(\epsilon) - U_y(\epsilon)U_x(\epsilon) = U_z(\epsilon^2) - 1$. Obviously $U_x(\epsilon) = e^{iG_1\epsilon}$, $U_y(\epsilon) = e^{iG_2\epsilon}$, and $U_z(\epsilon) = e^{iG_3\epsilon}$, and represent infinitesimal rotations around x,y,z axes respectively. In terms of Euler angle rotation $U_x(\epsilon) = e^{-iG_3\pi/2}e^{iG_2\epsilon}e^{iG_3\pi/2}$, etc. where we have used (3.3.19). Expand $e^{iG_1\epsilon}$, $e^{iG_2\epsilon}$, and $e^{iG_3\epsilon^2}$ in terms of Taylor series in $U_x(\epsilon)U_y(\epsilon)-U_y(\epsilon)U_x(\epsilon) = U_z(\epsilon^2) - 1$, and compare coefficients of ϵ^2 on both sides. We have $[G_1, G_2] = iG_3$, and similarly $[G_2, G_3] = iG_1$ and $[G_3, G_1] = iG_2$, i.e. $[G_i, G_j] = i\epsilon_{ijk}G_k$. Compare with commutation relations for $\vec{J}$, viz:- $[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$, we find $G_i = J_i/\hbar$.

8. A_ℓ are unrotated operators while $U^{-1}A_kU$ are operators under rotation. So, $U^{-1}A_kU = \sum_\ell R_{k\ell}A_\ell$ is the connecting equation between unrotated operators and operators obtained after rotation. The operators after rotation are just combinations of unrotated operators. From $U^{-1}A_kU = A'_k = \sum_\ell R_{k\ell}A_\ell$ we obtain for matrix elements $\langle m|A'_k|n\rangle = \sum_\ell R_{k\ell}\langle m|A_\ell|n\rangle$. But this is the same as the vector transformation $V'_k = \sum_\ell R_{k\ell}V_\ell$, hence $\langle m|A_\ell|n\rangle$ transforms like a vector. (I just copied this from the old solutions manual.)

9. This problem amounts to equating (3.3.21) and (3.2.45), using θ for ϕ. So,

$$\begin{bmatrix} \cos\left(\frac{\theta}{2}\right) - in_z\sin\left(\frac{\theta}{2}\right) & (-in_x - n_y)\sin\left(\frac{\theta}{2}\right) \\ (-in_x + n_y)\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) + in_z\sin\left(\frac{\theta}{2}\right) \end{bmatrix} = \begin{bmatrix} e^{-i(\alpha+\gamma)/2}\cos\left(\frac{\beta}{2}\right) & -e^{-i(\alpha-\gamma)/2}\sin\left(\frac{\beta}{2}\right) \\ e^{i(\alpha-\gamma)/2}\sin\left(\frac{\beta}{2}\right) & e^{i(\alpha+\gamma)/2}\cos\left(\frac{\beta}{2}\right) \end{bmatrix}$$

This could be used to determine $\hat{\mathbf{n}}$ as well as θ, but the problem only asks for the angle. So, equate the traces of these two matrices.

$$\begin{aligned} 2\cos\left(\frac{\theta}{2}\right) &= \left[e^{-i(\alpha+\gamma)/2} + e^{i(\alpha+\gamma)/2}\right]\cos\left(\frac{\beta}{2}\right) = 2\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right) \\ \theta &= 2\cos^{-1}\left[\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right)\right] \end{aligned}$$

10. A pure ensemble consists of spin-1/2 systems, all in the same state. We have seen the general expression for a spin-1/2 state many times. For example, see Problem 1.11 or (3.2.52). In the latter case, we have, making use of (1.3.38) and (1.4.19),

$$\begin{aligned}
|\alpha\rangle &= \cos\left(\frac{\beta}{2}\right)|+\rangle + e^{i\alpha}\sin\left(\frac{\beta}{2}\right)|-\rangle \qquad \text{so,} \\
\langle S_z\rangle &= \frac{\hbar}{2}\left[\cos^2\left(\frac{\beta}{2}\right) - \sin^2\left(\frac{\beta}{2}\right)\right] = \frac{\hbar}{2}\cos\beta \\
\langle S_x\rangle &= \frac{\hbar}{2}(e^{i\alpha}+e^{-i\alpha})\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\beta}{2}\right) = \frac{\hbar}{2}\cos\alpha\sin\beta \\
\langle S_y\rangle &= \frac{\hbar}{2i}(-e^{i\alpha}+e^{-i\alpha})\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\beta}{2}\right) = -\frac{\hbar}{2}\sin\alpha\sin\beta
\end{aligned}$$

Note that $0 \le \beta \le \pi$ and $0 \le \alpha \le 2\pi$, so β is determined directly from $\langle S_z\rangle$, and $(\hbar/2)\cos\alpha = \pm\sqrt{\langle S_x\rangle^2/[(\hbar/2)^2 - \langle S_z\rangle^2]}$. The sign ambiguity is between α and $\pi-\alpha$, both of which have the same value of $\sin\alpha$, and hence is resolved by measuring $\langle S_y\rangle$.

For a mixed ensemble, use (3.4.10), i.e $[A] = \mathsf{Tr}(\rho A)$ with $A = S_x$, S_y, S_z, and 1. Let $\rho = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and use $\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$ to find

$$\begin{aligned}
\mathsf{Tr}(\rho S_x) = \frac{\hbar}{2}\mathsf{Tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \frac{\hbar}{2}(b+c) &= [S_x] \\
\mathsf{Tr}(\rho S_y) = \frac{\hbar}{2}\mathsf{Tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\right) = \frac{i\hbar}{2}(b-c) &= [S_y] \\
\mathsf{Tr}(\rho S_z) = \frac{\hbar}{2}\mathsf{Tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \frac{\hbar}{2}(a-d) &= [S_z] \\
\mathsf{Tr}(\rho) = \frac{\hbar}{2}\mathsf{Tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+d) &= 1
\end{aligned}$$

These are easily solved for the elements of the density matrix. One finds

$$\begin{aligned}
a &= \frac{1}{2}\left(1 + \frac{2}{\hbar}[S_z]\right) \\
b &= \frac{1}{\hbar}\left([S_x] - i[S_y]\right) \\
b &= \frac{1}{\hbar}\left([S_x] + i[S_y]\right) \\
d &= \frac{1}{2}\left(1 - \frac{2}{\hbar}[S_z]\right)
\end{aligned}$$

11. Start with the definition (3.4.27) of the density operator, and use (2.1.5). Then

$$\begin{aligned}
\rho(t) &= \sum_i w_i|\alpha^{(i)},t\rangle\langle\alpha^{(i)},t| = \sum_i w_i\,\mathcal{U}(t,t_0)|\alpha^{(i)},t_0\rangle\langle\alpha^{(i)},t_0|\mathcal{U}^\dagger(t,t_0) \\
&= \mathcal{U}(t,t_0)\left[\sum_i w_i|\alpha^{(i)},t_0\rangle\langle\alpha^{(i)},t_0|\right]\mathcal{U}^\dagger(t,t_0) = \mathcal{U}(t,t_0)\rho_0\mathcal{U}^\dagger(t,t_0)
\end{aligned}$$

A pure ensemble is one for which $\rho^2 = 1$. Therefore

$$\rho^2(t) = \mathcal{U}(t,t_0)\rho_0\mathcal{U}^\dagger(t,t_0)\mathcal{U}(t,t_0)\rho_0\mathcal{U}^\dagger(t,t_0) = \mathcal{U}(t,t_0)\rho_0^2\mathcal{U}^\dagger(t,t_0) = \mathcal{U}(t,t_0)\mathcal{U}^\dagger(t,t_0) = 1$$

This of course makes perfect sense. If all the particles are in the same state, then whatever I do to one of them, I do identically to them all, and result is still a pure ensemble.

12. Since there are three basis states, the density matrix is 3×3. See (3.4.9). It is also Hermitian, and has zero trace. The most general form is therefore

$$\begin{bmatrix} a & x & y \\ x^* & b & z \\ y^* & z^* & 1-a-b \end{bmatrix}$$

where a and b are real. That is, two real numbers and three complex numbers, in other words *eight* real numbers. In addition to the $[S_i]$, we would also use various combinations $[S_iS_j]$, for example $[S_x^2]$, $[S_y^2]$, $[S_xS_y]$, $[S_xS_z]$, $[S_yS_z]$. (Recall that $[\mathbf{S}^2] = 2\hbar^2$ for an ensemble of spin-1 particles.) We can interpret $[\mathbf{S}]$ as the average spin vector of the ensemble, and the other five as elements of the quadrupole tensor.

13. The rotated state is $\mathcal{D}_y(\varepsilon)|jj\rangle = \exp(-iJ_y\varepsilon/\hbar)|jj\rangle$, so with $J_y = (J_+ - J_-)/2i$, get

$$\begin{aligned}
\langle jj|\mathcal{D}_y(\varepsilon)|jj\rangle &= \langle jj|\left[1 - \frac{iJ_y\varepsilon}{\hbar} - \frac{J_y^2\varepsilon^2}{2\hbar^2} + \cdots\right]|jj\rangle = 1 - \frac{\varepsilon^2}{8\hbar^2}\langle jj|J_+J_-|jj\rangle \\
&= 1 - \frac{\varepsilon^2}{8\hbar^2}\left(\sqrt{2j}\hbar\right)\left(\sqrt{2j}\hbar\right) = 1 - \frac{\varepsilon^2 j}{4}
\end{aligned}$$

so the probability is $|\langle jj|\mathcal{D}_y(\varepsilon)|jj\rangle|^2 = 1 - \varepsilon^2 j/2$. You can instead consider all of the states, to order ε, which can mix into the rotates states. This is clearly just the state $|j,j-1\rangle$. So, from above, this amplitude is $\langle j,j-1|(-iJ_y\varepsilon/\hbar)|jj\rangle = (\varepsilon/2\hbar)\langle j,j-1|J_-|jj\rangle = \varepsilon\sqrt{2j}/2$, and the probability to be in all states *other than* $|jj\rangle$ is $\varepsilon^2 j/2$.

14. It is easy enough to prove the commutation relations using the algebras of δ_{ij} and ε_{ijk}:

$$\begin{aligned}
[G_i, G_j]_{ln} &= (G_iG_j - G_jG_i)_{ln} = (G_i)_{lm}(G_j)_{mn} - (G_j)_{lm}(G_i)_{mn} \\
&= -\hbar^2(\varepsilon_{ilm}\varepsilon_{jmn} - \varepsilon_{jlm}\varepsilon_{imn}) = \hbar^2(\varepsilon_{ilm}\varepsilon_{jnm} - \varepsilon_{jlm}\varepsilon_{inm}) \\
&= \hbar^2(\delta_{ij}\delta_{ln} - \delta_{in}\delta_{lj} - \delta_{ij}\delta_{ln} + \delta_{jn}\delta_{il}) = \hbar^2(\delta_{jn}\delta_{il} - \delta_{in}\delta_{lj}) \\
&= \hbar^2\varepsilon_{ijk}\varepsilon_{lnk} = \hbar^2\varepsilon_{ijk}\varepsilon_{kln} = i\hbar\varepsilon_{ijk}(-i\hbar\varepsilon_{kln}) = i\hbar\varepsilon_{ijk}(G_k)_{ln}
\end{aligned}$$

As for the rest of the problem, the intent is not clear to me, and I do not find the old solutions manual enlightening. However, I think the point is more or less the following. Since we can write a vector cross product, as $(\mathbf{A} \times \mathbf{B})_i = \varepsilon_{ijk}A_jB_k$, write the infinitesimal rotation $\mathbf{V}_i \to \mathbf{V}_i + (\hat{\mathbf{n}}\delta\phi \times \mathbf{V})_i$ as $V_i + \delta\phi\varepsilon_{ikj}n_kV_j = (\delta_{ij} - \delta\phi\varepsilon_{ijk}n_k)V_j$ which leads to a spin-one representation of the rotation operator as $1 - \delta\phi\ \varepsilon_{ijk}n_k = 1 - i\delta\phi(G_i)_{jk}n_k/\hbar$. However, the G_i are not representations of the J_i in a basis where any of them are diagonal. (All of the G_i are antisymmetric.) The eigenvectors which diagonalize, for example, G_3 are $\mathbf{r}_0 = \hat{\mathbf{z}}$ and $\mathbf{r}_\pm = \hat{\mathbf{x}} \pm i\hat{\mathbf{y}}$. Thus, it appears that there are generators for circular polarization states.

I would welcome feedback on this problem and the solution.

15. First, $J_+J_- = (J_x + iJ_y)(J_x - iJ_y) = J_x^2 + J_y^2 - i[J_x, J_y] = J_x^2 + J_y^2 + \hbar J_z = \mathbf{J}^2 - J_z^2 + \hbar J_z$. Why the second part is written as a wave function, I don't know, but famous or otherwise,

$$\begin{aligned}
|c_-|^2 &= (\langle j, m|J_-^\dagger)(J_-|j.m\rangle) = \langle j, m|J_+J_-|j.m\rangle = \langle j, m|(\mathbf{J}^2 - J_z^2 + \hbar J_z)|j.m\rangle \\
&= j(j+1)\hbar^2 - m^2\hbar^2 + m\hbar^2 = [j^2 - m^2 + j + m]\hbar^2 = [(j+m)(j-m+1)]\hbar^2
\end{aligned}$$

and, by convention, we choose $c_- = \hbar\sqrt{(j+m)(j-m+1)}$.

16. These proofs are straightforward. Just work with separate components. For example

$$\begin{aligned}
[L_z, \mathbf{p}^2] = [xp_y - yp_x, p_x^2 + p_y^2 + p_z^2] &= [xp_y, p_x^2] - [yp_x, p_y^2] \\
&= \left(i\hbar\frac{\partial}{\partial p_x}p_x^2\right)p_y - \left(i\hbar\frac{\partial}{\partial p_y}p_y^2\right)p_x \\
&= 2i\hbar[p_x, p_y] = 0
\end{aligned}$$

where I have made use of problem 1.29, although it is easy enough just to write it out. It works similarly for L_x and L_y. The commutator with $\mathbf{x}^2$ is done the same way, that is

$$\begin{aligned}
[L_z, \mathbf{x}^2] = [xp_y - yp_x, x^2 + y^2 + z^2] &= [xp_y, y^2] - [yp_x, x^2] \\
&= x\left(-i\hbar\frac{\partial}{\partial y}y^2\right) - y\left(-i\hbar\frac{\partial}{\partial x}x^2\right) \\
&= -2i\hbar[x, y] = 0
\end{aligned}$$

17. The key here is to write $\psi(\mathbf{x})$ in terms of spherical harmonics. Using (B.5.7), we have

$$\begin{aligned}\psi(\mathbf{x}) &= r[\cos\phi\sin\theta+\sin\phi\sin\theta+3\cos\theta]f(r)\\ &= \sqrt{\frac{8\pi}{3}}\left[\frac{Y_1^1(\theta,\phi)+Y_1^{-1}(\theta,\phi)}{2}+\frac{Y_1^1(\theta,\phi)-Y_1^{-1}(\theta,\phi)}{2i}+\frac{3}{\sqrt{2}}Y_1^0(\theta,\phi)\right]rf(r)\end{aligned}$$

So, yes, this is an eigenstate of $\mathbf{L}^2$ with $l=1$. It is a mixture of $m=\pm1,0$ states, with probabilities $(1/2)\div(1+9/2)=1/11$, $(9/2)\div(1+9/2)=9/11$ respectively. Given $\psi(\mathbf{x})$ and the energy eigenvalue E, you can find the potential $V(r)$ from the Schrödinger equation easily enough. Write $\psi(\mathbf{x})=F_1(\theta,\phi)rf(r)$ where $F_1(\theta,\phi)$ is a linear combination of $Y_1^m(\theta,\phi)$. Then, substitute this into (3.7.7) with $l=1$ and $R_{El}(r)=rf(r)$ and solve for $V(r)$.

18. We are looking for expectation values of $L_x=(L_++L_-)/2$ and $L_y=(L_+-L_-)/2i$ in eigenstates $|lm\rangle$. Since $L_\pm|lm\rangle\propto|l,m\pm1\rangle$, it is obvious that $\langle L_x\rangle=0=\langle L_y\rangle$. Now

$$L_x^2=\frac{1}{4}(L_+^2+L_+L_-+L_-L_++L_-^2)\qquad\text{and}\qquad L_y^2=-\frac{1}{4}(L_+^2-L_+L_--L_-L_++L_-^2)$$

so $\langle L_x^2\rangle=\langle L_+L_-+L_-L_+\rangle/4=\langle\mathbf{L}^2-L_z^2\rangle/2-[l(l+1)-m^2]\hbar^2/2=\langle L_y^2\rangle$ using (3.6.14). This is fine, semiclassically. The x- and y-components of $\mathbf{L}$ are as much positive as negative, and $\langle\mathbf{L}^2\rangle=\langle L_x^2\rangle+\langle L_y^2\rangle+\langle L_z^2\rangle=\langle L_x^2\rangle+\langle L_y^2\rangle+m^2\hbar^2$ with $(\Delta L_x)^2=\langle L_x^2\rangle-\langle L_x\rangle^2=(\Delta L_y)^2$.

19. From (3.6.13), we take $L_\pm=-i\hbar e^{\pm i\phi}(\pm i\partial/\partial\theta-\cot\theta\partial/\partial\phi)$ as an operator in coordinate space. The prescription (3.6.34) gives $Y_{1/2}^{1/2}(\theta,\phi)=ce^{i\phi/2}\sqrt{\sin\theta}$ which works. That is

$$L_+Y_{1/2}^{1/2}(\theta,\phi)=-i\hbar e^{i\phi}ice^{i\phi/2}\frac{1}{2}\frac{\cos\theta}{\sqrt{\sin\theta}}+i\hbar e^{i\phi}\frac{\cos\theta}{\sin\theta}c\frac{i}{2}ce^{i\phi/2}\sqrt{\sin\theta}=0$$

From (3.5.40) we would expect $L_-Y_{1/2}^{1/2}(\theta\phi)=\sqrt{(1)(1)}Y_{1/2}^{-1/2}(\theta,\phi)=Y_{1/2}^{-1/2}(\theta,\phi)$, so try it:

$$Y_{1/2}^{-1/2}(\theta,\phi)=-i\hbar e^{-i\phi}(-i)ce^{i\phi/2}\frac{1}{2}\frac{\cos\theta}{\sqrt{\sin\theta}}+i\hbar e^{-i\phi}\frac{\cos\theta}{\sin\theta}c\frac{i}{2}ce^{i\phi/2}\sqrt{\sin\theta}=-c\hbar e^{-i\phi/2}\frac{\cos\theta}{\sqrt{\sin\theta}}$$

So far, so good, but now check that we have $L_-Y_{1/2}^{-1/2}(\theta,\phi)=0$:

$$\begin{aligned}L_-Y_{1/2}^{-1/2}(\theta,\phi) &= ic\hbar^2e^{-i\phi}\left[-i\left(-\frac{\sin\theta}{\sqrt{\sin\theta}}-\frac{1}{2}\frac{\cos^2\theta}{\sqrt{\sin^3\theta}}\right)e^{-i\phi/2}-\frac{\cos\theta}{\sin\theta}\left(-\frac{i}{2}\right)e^{-i\phi/2}\sqrt{\sin\theta}\right]\\ &= c\hbar^2e^{-3i\phi/2}\frac{1}{2\sqrt{\sin^3\theta}}\left[\cos\theta\sin\theta-2\sin^2\theta-\cos^2\theta\right]\neq0\end{aligned}$$

for all θ. (Just examine $\theta=\pi/2$, for example.) So, there is an internal inconsistency for half-integer l spherical harmonics.

20. The rotated state is $\mathcal{D}(R)|l=2,m=0\rangle = \mathcal{D}(\alpha=0,\beta,\gamma=0)|20\rangle$ and the probability is $|\langle 2m|\mathcal{D}(R)|20\rangle|^2$, for $m=0,\pm1,\pm2$. We use (3.6.52) and (B.5.7). Then

$$\begin{aligned}
\mathcal{D}(R)|l=2,m=0\rangle &= \sum_{m'}|2,m'\rangle\langle 2,m'|\mathcal{D}(R)|l=2,m=0\rangle \\
&= \sum_{m'}|2,m'\rangle\mathcal{D}^{(2)}_{m',0}(\alpha=0,\beta,\gamma=0) = \sum_{m'}|2,m'\rangle\sqrt{\frac{4\pi}{5}}Y_2^{m'*}(\beta,0) \\
|\langle 2m|\mathcal{D}(R)|20\rangle|^2 &= \frac{4\pi}{5}\left|Y_2^{m*}(\beta,0)\right|^2
\end{aligned}$$

and, therefore, $|\langle 20|\mathcal{D}(R)|20\rangle|^2 = \frac{1}{4}(3\cos^2\beta-1)^2$, $|\langle 2,\pm1|\mathcal{D}(R)|20\rangle|^2 = \frac{3}{2}\sin^2\beta\cos^2\beta$, and $|\langle 2,\pm2|\mathcal{D}(R)|20\rangle|^2 = \frac{3}{8}\sin^4\beta$. We can do a reality check with

$$\sum_m |\langle 2m|\mathcal{D}(R)|20\rangle|^2 = \frac{1}{4}(3\cos^2\beta-1)^2 + 3\sin^2\beta\cos^2\beta + \frac{3}{4}\sin^4\beta$$

just by using $\sin^2\beta = 1-\cos^2\beta$ and expanding. The algebra is simple.

21. We need to determine the quantities $\langle\mathbf{n}|qlm\rangle$. We only care about states degenerate in energy, which is simple to see by inserting H and operating both left and right. If energies are different, the inner product has to be zero. (Same old proof of orthogonality of states for hermitian operators.) For degenerate energies, we get the equation

$$n_x+n_y+n_z = 2q+l \equiv N$$

Below, we won't distinguish between the operator N and the value N. Also, to avoid confusion, we will always write inner products with the spherical state on the right and the cartesian state on the left.

First, work out the form of the angular momentum operator in terms of creation and annihilation operators.

$$\begin{aligned}
L_i &= \epsilon_{ijk}x_jp_k \\
&= i\epsilon_{ijk}\frac{\hbar}{2}(a_j+a_j^\dagger)(-a_k+a_k^\dagger) \\
&= i\hbar\epsilon_{ijk}a_ja_k^\dagger
\end{aligned} \tag{1}$$

Summation of repeated indices is implied. Write this out explicitly for L_z, so

$$\begin{aligned}
L_z &= xp_y - yp_x \\
&= i\frac{\hbar}{2}\left[(a_x+a_x^\dagger)(-a_y+a_y^\dagger) - (a_y+a_y^\dagger)(-a_x+a_x^\dagger)\right] \\
&= i\hbar\left[a_xa_y^\dagger - a_x^\dagger a_y\right]
\end{aligned}$$

Sandwich the left and right sides of this equation with the spherical and cartesian states, that is

$$\begin{aligned} \langle n_x n_y n_z | L_z | qlm \rangle &= m\hbar \langle n_x n_y n_z | qlm \rangle \\ \text{and} \qquad \langle n_x n_y n_z | L_z | qlm \rangle &= i\hbar \langle n_x n_y n_z | \left[a_x a_y^\dagger - a_x^\dagger a_y \right] | qlm \rangle \end{aligned}$$

which leads us to the equation

$$\begin{aligned} m \langle n_x n_y n_z | qlm \rangle &= i\sqrt{(n_x+1)n_y} \langle n_x+1, n_y-1, n_z | qlm \rangle \\ &- i\sqrt{n_x(n_y+1)} \langle n_x-1, n_y+1, n_z | qlm \rangle \end{aligned} \tag{2}$$

This is enough to decompose the first excited state, with $N = 1$, that has threefold degeneracy. We have

$$\begin{aligned} m\langle 100|01m \rangle &= -i \langle 010|01m \rangle \\ m\langle 010|01m \rangle &= +i \langle 100|01m \rangle \\ m\langle 001|01m \rangle &= 0 \end{aligned}$$

Therefore, since

$$|qlm\rangle = \sum_{n_x n_y n_z} |n_x n_y n_z\rangle \langle n_x n_y n_z | qlm \rangle$$

we can write

$$\begin{aligned} |011\rangle &= \frac{1}{\sqrt{2}}|100\rangle + \frac{i}{\sqrt{2}}|010\rangle \\ |010\rangle &= |001\rangle \\ |01,-1\rangle &= \frac{1}{\sqrt{2}}|100\rangle - \frac{i}{\sqrt{2}}|010\rangle \end{aligned}$$

We can check that these are correct by considering the angular dependence in coordinate space, and remembering that for Hermite polynomials $H_1(w) = 2w$. Thus, these three equations say, in turn,

$$\begin{aligned} Y_1^1(\theta,\phi) &\propto x + iy = re^{i\phi}\sin\theta \\ Y_1^0(\theta,\phi) &\propto z = r\cos\theta \\ Y_1^{-1}(\theta,\phi) &\propto x - iy = re^{-i\phi}\sin\theta \end{aligned}$$

which are indeed correct.

For the second excited state, we need to find coefficients to find the five states $|02m\rangle$ plus the one state $|200\rangle$ in terms of the six states $|n_x n_y n_z\rangle = |200\rangle$, $|020\rangle$, $|002\rangle$, $|110\rangle$, $|101\rangle$, and

$|011\rangle$. Consider first the state $|qlm\rangle = |200\rangle$. Equation (2) give us

$$
\begin{aligned}
0 &= \langle 110|200\rangle && \text{(3a)}\\
0 &= \langle 011|200\rangle && \text{(3b)}\\
0 &= \langle 101|200\rangle && \text{(3c)}\\
0 &= \langle 200|200\rangle - \langle 020|200\rangle && \text{(3d)}
\end{aligned}
$$

but no information on the inner product $\langle 002|200\rangle$.

For more information we have to look for an equation using $\mathbf{L}^2$. We will need to flip operator order, so use

$$\left[a_i, a_j^\dagger\right] = a_i a_j^\dagger - a_j^\dagger a_i = \delta_{ij}$$

We will also need the "double epsilon" formula; see the derivation step in the second line of (3.6.17). Put all this together with (1) to find

$$
\begin{aligned}
\mathbf{L}^2 = L_i^2 &= (-\hbar^2)\epsilon_{ijk} a_j a_k^\dagger \epsilon_{ilm} a_l a_m^\dagger \\
&= (-\hbar^2)\left[(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\, a_j a_k^\dagger a_l a_m^\dagger\right] \\
&= (-\hbar^2)\left[a_j a_k^\dagger a_j a_k^\dagger - a_j a_k^\dagger a_k a_j^\dagger\right] \\
&= (-\hbar^2)\left[\left(a_k^\dagger a_j + \delta_{jk}\right)\left(a_k^\dagger a_j + \delta_{jk}\right) - a_j a_k^\dagger\left(a_j^\dagger a_k + \delta_{jk}\right)\right] \\
&= (-\hbar^2)\left[a_k^\dagger a_j a_k^\dagger a_j + 2N + 3 - a_j a_k^\dagger a_j^\dagger a_k - a_j a_j^\dagger\right] \\
&= (-\hbar^2)\left[a_k^\dagger\left(a_k^\dagger a_j + \delta_{jk}\right) a_j + 2N + 3 - a_j a_j^\dagger a_k^\dagger a_k - a_j a_j^\dagger\right] \\
&= (-\hbar^2)\left[a_k^\dagger a_k^\dagger a_j a_j + 3N + 3 - a_j a_j^\dagger (N+1)\right] \\
&= (-\hbar^2)\left[a_k^\dagger a_k^\dagger a_j a_j + 3N + 3 - (N+3)(N+1)\right] \\
\text{or} \qquad \mathbf{L}^2 &= (\hbar^2)\left[N(N+1) - a_k^\dagger a_k^\dagger a_j a_j\right] && \text{(4)}
\end{aligned}
$$

Don't forget the implied summation of repeated indices.

Now (4) gives us information on the inner product $\langle 002|200\rangle$. We know that

$$
\begin{aligned}
\langle 002| a_k^\dagger a_k^\dagger a_j a_j &= \langle 002|\left(a_x^\dagger a_x^\dagger + a_y^\dagger a_y^\dagger + a_z^\dagger a_z^\dagger\right)(a_x a_x + a_y a_y + a_z a_z) \\
&= \sqrt{2}\langle 000|\,(a_x a_x + a_y a_y + a_z a_z) \\
&= 2\left(\langle 200| + \langle 020| + \langle 002|\right) && \text{(5)}
\end{aligned}
$$

and so (4) gives us

$$0 = 6\langle 002|200\rangle - 2\left(\langle 200|200\rangle + \langle 020|200\rangle + \langle 002|200\rangle\right)$$

which combined with (3) tells us that

$$|200\rangle = \frac{1}{\sqrt{3}}|200\rangle + \frac{1}{\sqrt{3}}|020\rangle + \frac{1}{\sqrt{3}}|002\rangle$$

Since $H_2(w) = 4w^2 - 2$, converting this to coordinate space combines all the angular dependence into $x^2 + y^2 + z^2 = r^2$, that is, no dependence on θ or ϕ. Since the spherical harmonic here is $Y_0^0(\theta, \phi)$, this is once again correct.

For second excited states $|qlm\rangle = |02m\rangle$ we have

$$\mathbf{L}^2 = 6\hbar^2 = N(N+1)\hbar^2$$

so that application of (4) tells us that

$$0 = \langle n_x n_y n_z| \left(a_x^\dagger a_x^\dagger + a_y^\dagger a_y^\dagger + a_z^\dagger a_z^\dagger\right) \left(a_x a_x + a_y a_y + a_z a_z\right) |02m\rangle$$

This equation yields no information for the states $\langle n_x n_y n_z| = \langle 110|$, $\langle 101|$, and $\langle 011|$. However for states $\langle n_x n_y n_z| = \langle 200|$, $\langle 020|$, and $\langle 002|$, (5) gives the same result each time. So

$$0 = \langle 200|02m\rangle + \langle 020|02m\rangle + \langle 002|02m\rangle \tag{6}$$

We need to go back to (2) for enough information to solve for these inner products. We have

$$\begin{aligned}
m\langle 110|02m\rangle &= i\sqrt{2}\left(\langle 200|02m\rangle - \langle 020|02m\rangle\right) && \text{(7a)}\\
m\langle 101|02m\rangle &= -i\langle 011|02m\rangle && \text{(7b)}\\
m\langle 011|02m\rangle &= +i\langle 101|02m\rangle && \text{(7c)}\\
m\langle 200|02m\rangle &= -i\sqrt{2}\langle 110|02m\rangle && \text{(7d)}\\
m\langle 020|02m\rangle &= +i\sqrt{2}\langle 110|02m\rangle && \text{(7e)}\\
m\langle 002|02m\rangle &= 0 && \text{(7f)}
\end{aligned}$$

Now we can consider the state $|qlm\rangle = |020\rangle$. We have

$$\begin{aligned}
0 &= \langle 101|020\rangle = \langle 011|020\rangle = \langle 110|020\rangle\\
0 &= \langle 200|020\rangle - \langle 020|020\rangle\\
\text{and}\quad 0 &= \langle 200|020\rangle + \langle 020|020\rangle + \langle 002|020\rangle
\end{aligned}$$

which means that

$$|020\rangle = \frac{1}{\sqrt{6}}|200\rangle + \frac{1}{\sqrt{6}}|020\rangle - \frac{2}{\sqrt{6}}|002\rangle$$

and the behavior in coordinate space has angular dependence

$$(4x^2 - 2) + (4y^2 - 2) - 2(4z^2 - 2) = 4(\sin^2\theta - 2\cos^2\theta) = 4(1 - 3\cos^2\theta)$$

which is proportional to $Y_2^0(\theta, \phi)$, once again, as it should be.

Next, consider the state $|qlm\rangle = |022\rangle$. From (7) we have $\langle 020|022\rangle = -\langle 200|022\rangle$ and $\langle 110|022\rangle = i\sqrt{2}\langle 200|022\rangle$. We also find $\langle 002|022\rangle = \langle 101|022\rangle = \langle 011|022\rangle = 0$. Therefore

$$|022\rangle = \frac{1}{\sqrt{10}}|200\rangle - \frac{1}{\sqrt{10}}|020\rangle + i\sqrt{\frac{2}{10}}|110\rangle$$

which gives a (relatively normalized) angular dependence in coordinate space

$$\begin{aligned}
& \frac{1}{2\sqrt{2}}(4x^2-2) - \frac{1}{2\sqrt{2}}(4y^2-2) + i\sqrt{2}\frac{1}{\sqrt{2}}2x\frac{1}{\sqrt{2}}2y \\
= \; & \sqrt{2}\left[(x^2-y^2)+2ixy\right] \\
= \; & \sqrt{2}r^2\left[(\cos^2\phi - \sin^2\phi)\sin^2\theta + 2i\cos\phi\sin\phi\sin^2\theta\right] \\
= \; & \sqrt{2}r^2\left[\cos 2\phi + i\sin 2\phi\right]\sin^2\theta = \sqrt{2}r^2e^{2i\phi}\sin^2\theta \propto Y_2^2(\theta,\phi)
\end{aligned}$$

Finally, consider the state $|qlm\rangle = |021\rangle$. From (6) and (7) we find that all inner products are zero except for $\langle 011|021\rangle = i\langle 101|021\rangle$. Therefore

$$|021\rangle = \frac{1}{2\sqrt{2}}|101\rangle + \frac{i}{2\sqrt{2}}|011\rangle$$

which has an angular dependence in coordinate space

$$xz + iyz = r^2(\cos\phi + i\sin\phi)\sin\theta\cos\theta = r^2e^{i\phi}\sin\theta \propto Y_2^1(\theta,\phi)$$

that is, once again, the correct answer.

22. *Note: See the correction discussed at the end of this solution.* For convenience, here is reproduced the generating function of the Laguerre polynomials:

$$g(x,t) \equiv \frac{e^{-xt/(1-t)}}{1-t} \equiv \sum_{n=0}^{\infty} L_n(x)\frac{t^n}{n!}$$

Therefore

$$g(0,t) = \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} L_n(0)\frac{t^n}{n!}$$

which shows that $L_n(0) = n!$. Also

$$g(x,t) = \left[1 - \frac{xt}{1-t} + \cdots\right] \times \left[1 + t + t^2 + \cdots\right]$$

which shows that the coefficient of t^0 is just unity for all x, so $L_0(x) = 1$. Now differentiate with respect to x and proceed

$$\begin{aligned}
\frac{\partial g}{\partial x} = -\frac{t}{1-t}g(x,t) &= \sum_{n=0}^{\infty} L_n'(x)\frac{t^n}{n!} \\
-\sum_{n=0}^{\infty} L_n(x)\frac{t^{n+1}}{n!} &= \sum_{n=0}^{\infty} L_n'(x)\frac{t^n}{n!} - \sum_{n=0}^{\infty} L_n'(x)\frac{t^{n+1}}{n!} \\
-\sum_{m=1}^{\infty} L_{m-1}(x)m\frac{t^m}{m!} &= \sum_{n=0}^{\infty} L_n'(x)\frac{t^n}{n!} - \sum_{m=1}^{\infty} L_{m-1}'(x)m\frac{t^m}{m!}
\end{aligned}$$

where we set $m = n-1$ in two of the summations. Now, the $n = 0$ term in the first summation on the right is just $L_0'(x) = 0$ since $L_0(x) = 1$ for all x. Therefore all summations can be taken from $n, m = 1$. So, combining this expression and equating term by term gives

$$L_n'(x) = nL_{n-1}'(x) - nL_{n-1}(x)$$

We now have what we need in order to calculate the $L_n(x)$ for $n \geq 1$. Proceeding

$$\begin{aligned}
L_1'(x) = L_0'(x) - L_0(x) &= -1 &\quad \text{so} \quad && L_1(x) &= 1 - x \\
L_2'(x) = 2L_1'(x) - 2L_1(x) &= -4 + 2x &\quad \text{so} \quad && L_2(x) &= 2 - 4x + x^2 \\
L_3'(x) = 3L_2'(x) - 3L_2(x) &= -18 + 18x - 3x^2 &\quad \text{so} \quad && L_3(x) &= 6 - 18x + 9x^2 - x^3
\end{aligned}$$

Note that mathematicians frequently define the $L_n(x)$ such that they are smaller by a factor of $n!$. Now differentiate with respect to t to get $\partial g/\partial t$ both ways, i.e.

$$\begin{aligned}
\left[\frac{1}{1-t} - \frac{xt}{(1-t)^2} - \frac{x}{1-t}\right] g(x,t) &= \sum_{n=0}^{\infty} L_n(x)\frac{nt^{n-1}}{n!} \\
\frac{1-x-t}{(1-t)^2}\sum_{n=0}^{\infty} L_n(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} L_n(x)n\frac{t^{n-1}}{n!} = \sum_{n=0}^{\infty} L_{n+1}(x)\frac{t^n}{n!} \\
\sum_{n=0}^{\infty} L_n(x)\frac{1}{n!}\left[(1-x)t^n - t^{n+1}\right] &= \sum_{n=0}^{\infty} L_{n+1}(x)\frac{1}{n!}\left[t^n - 2t^{n+1} + t^{n+2}\right] \\
\sum_{n=0}^{\infty}\left[(1-x)L_n(x) - nL_{n-1}(x)\right]\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\left[L_{n+1}(x) - 2nL_n(x) + n(n-1)L_{n-1}(x)\right]\frac{t^n}{n!}
\end{aligned}$$

where we note that the first two terms in the summation for the second and third terms on the right are explicitly zero. So, equating term by term, the recursion relation becomes

$$L_{n+1}(x) = (2n+1-x)L_n(x) - n^2L_{n-1}(x)$$

Now combine the recursion relations to get a differential equation for $L_n(x)$. First, use the two recursion relations to get different expressions for $L_{n+1}'(x)$, namely

$$\begin{aligned}
L_{n+1}' &= (n+1)(L_n' - L_n) \\
\text{and} \qquad L_{n+1}' &= (2n+1-x)L_n' - L_n - n^2L_{n-1}'
\end{aligned}$$

Equate these two expressions and cancel common terms to find

$$-nL_n = (n-x)L_n' - n^2 L_{n-1}' = n(L_n' - nL_{n-1}') - xL_n' = -n^2 L_{n-1} - xL_n'$$

where we used the first recursion relation once again. Now we have a recursion relation that just needs the function one order below, not two. Simplify, differentiate, and subtract to get

$$\begin{aligned}
xL_n' &= nL_n - n^2 L_{n-1} \\
xL_n'' + L_n' &= nL_n' - n^2 L_{n-1}' \\
xL_n'' + (1-x)L_n' &= nL_n' - nL_n - n^2(L_{n-1}' - L_{n-1}) = nL_n' - nL_n - nL_n' = -nL_n
\end{aligned}$$

where the first recursion relation is once again used, in the second-to-last step. This is what we are going for, namely a differential equation for $L_n(x)$:

$$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0$$

Contrary to what is said in the problem statement, this is *not* Kummer's equation. Instead, one needs to work with the "associated" Laguerre polynomials, which can be defined as

$$L_n^k(x) = (-1)^k \frac{d^k}{dx^k}\left[L_{n+k}(x)\right]$$

and which satisfy instead the equation

$$xL_n^{k\prime\prime}(x) + (k+1-x)L_n^{k\prime}(x) + nL_n^k(x) = 0$$

This is easy enough to see, starting from the differential equation for $L_{n+k}(x)$, namely

$$xL_{n+k}'' + (1-x)L_{n+k}' + (n+k)L_{n+k} = 0$$

and taking the derivative of the left side k times. The first term will give xd^kL_{n+k}''/dx^k plus k occurrences of d^kL_{n+k}'/dx^k, which adds a k inside the parentheses in the second term. The $-x$ inside theses parentheses will remain for half of the k derivatives, and also contribute k terms d^kL_{n+k}/dx^k with alternating signs. These alternating signs exactly cancel the factor of k in the third term, when the $(-1)^k$ is included in the definition of the associated Laguerre polynomial, so the associated Laguerre equation is satisfied. (Work it out for a couple of cases like $k=1$ and $k=2$ and watch this in action.) This is Kummer's Equation (3.7.46) with $c = 2(l+1) = k+1$, i.e. $k = 2l+1$, and $a = l+1-\rho_0/2 = -n$, or $n = \rho_0/2 - l - 1 = N$ as defined in (3.7.50).

23. From (3.9.5) $K_+|n_+,n_-\rangle = a_+^\dagger a_-^\dagger|n_+,n_-\rangle = \sqrt{(n_++1)(n_-+1)}|n_++1,n_-+1\rangle$ and $K_-|n_+,n_-\rangle = a_+a_-|n_+,n_-\rangle = \sqrt{n_+n_-}|n_+-1,n_--1\rangle$. Therefore, from (3.9.17), $j = (n_++n_-)/2$ increases (decreases) by one, and $m = (n_+-n_-)/2$ is unchanged for K_+ (K_-). Writing $n_+ = j+m$ and $n_- = j-m$, we can write the action of $K_\pm$ on the state $|jm\rangle$ as

$$K_+|jm\rangle = \sqrt{(j+m+1)(j-m+1)}|j+1,m\rangle \qquad K_-|jm\rangle = \sqrt{(j+m)(j-m)}|j-1,m\rangle$$

In other words, $K_\pm$ act as raising or lowering operators for the total angular momentum quantum number. Since $2j$ is interpreted as the number of spin-1/2 "particles", they raise or lower the number of these quanta by two. The nonvanishing matrix elements are just

$$\begin{aligned} \langle j'm'|K_+|jm\rangle &= \sqrt{(j+m+1)(j-m+1)}\delta_{j',j+1}\delta_{m',m} \\ \text{and} \qquad \langle j'm'|K_-|jm\rangle &= \sqrt{(j+m)(j-m)}\delta_{j',j-1}\delta_{m',m} \end{aligned}$$

24. Our job is to express $|j,m\rangle$, $j=0,1,2$, in terms of $|j_1,j_2;m_1,m_2\rangle = |1,1;m_1,m_2\rangle$, using the notation "$\pm,0$" for $m_{1,2}=\pm1,0$ with $j_1=j_2=1$. Since $m=m_1+m_2$, we must have $|j,m\rangle = |2,2\rangle = |++\rangle$ and $|2,-2\rangle = |--\rangle$. From this, we can build the other $j=2$ states. Recall that $J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}|j,m-1\rangle$ and $J_- = J_{1-}+J_{2-}$. Therefore, both $J_-|2,2\rangle = \sqrt{4}|2,1\rangle = 2|2,1\rangle$ and $J_-|2,2\rangle = (J_{1-}+J_{2-})|++\rangle = \sqrt{2}(|0+\rangle + |+0\rangle)$, so

$$\begin{aligned} |2,1\rangle &= \frac{1}{\sqrt{2}}\left(|0+\rangle + |+0\rangle\right) \\ J_-|2,1\rangle &= \sqrt{6}|2,0\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{2}|-+\rangle + \sqrt{2}|00\rangle + \sqrt{2}|00\rangle + \sqrt{2}|+-\rangle\right) \\ |2,0\rangle &= \frac{1}{\sqrt{6}}\left(|-+\rangle + 2|00\rangle + |+-\rangle\right) \\ J_-|2,0\rangle &= \sqrt{6}|2,-1\rangle = \frac{1}{\sqrt{6}}\left(\sqrt{2}|-0\rangle + 2\sqrt{2}|-0\rangle + 2\sqrt{2}|0-\rangle + \sqrt{2}|0-\rangle\right) \\ |2,-1\rangle &= \frac{1}{\sqrt{2}}\left(|-0\rangle + |0-\rangle\right) \\ \text{Also} \qquad J_-|2,-1\rangle &= \sqrt{4}|2,-2\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{2}|--\rangle + \sqrt{2}|--\rangle\right) \end{aligned}$$

which just confirms that $|2,-2\rangle = |--\rangle$. For $j=1$, write $|1,\pm1\rangle = a|0\pm\rangle + b|\pm0\rangle$ with a and b real, and $a^2+b^2=1$. Since $\langle 2,\pm1|1,\pm1\rangle = 0$, $a+b=0$ so $|1,\pm1\rangle = \frac{1}{\sqrt{2}}\left(|\pm0\rangle - |0\pm\rangle\right)$. Then, take $J_-|1,1\rangle = \sqrt{2}|1,0\rangle$ and $J_-|1,1\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{2}|+-\rangle + \sqrt{2}|00\rangle - \sqrt{2}|00\rangle - \sqrt{2}|-+\rangle\right)$ so that $|1,0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$. Finally, put $|0,0\rangle = \alpha|+-\rangle + \beta|00\rangle + \gamma|-+\rangle$, normalize, and use $\langle 2,0|0,0\rangle = 0 = \langle 1,0|0,0\rangle$. Then $\alpha+2\beta+\gamma=0$, $\alpha-\gamma=0$, and $\alpha^2+\beta^2+\gamma^2=1$. Thus $\beta = -(\alpha+\gamma)/2 = -\alpha = -1/\sqrt{3}$ and so $|0,0\rangle = \frac{1}{\sqrt{3}}\left(|+-\rangle - |00\rangle + |-+\rangle\right)$.

These results can be easily checked using a table of Clebsch-Gordan coefficients, for example
http://en.wikipedia.org/wiki/Table_of_ClebschGordan_coefficients
http://pdg.lbl.gov/2011/reviews/rpp2011-rev-clebsch-gordan-coefs.pdf
Perhaps the next edition of the book should include a version of this table.

25. Do (a) straightforwardly with angular momentum algebra, making use of (3.2.7).

$$\sum_{m=-j}^{j} |d^{(j)}_{mm'}(\beta)|^2 m = \sum_{m=-j}^{j} \langle jm'|e^{iJ_y\beta/\hbar}|jm\rangle m \langle jm|e^{-iJ_y\beta/\hbar}|jm'\rangle$$

$$= \frac{1}{\hbar}\sum_{m=-j}^{j} \langle jm'|e^{iJ_y\beta/\hbar}J_z|jm\rangle\langle jm|e^{-iJ_y\beta/\hbar}|jm'\rangle = \frac{1}{\hbar}\langle jm'|e^{iJ_y\beta/\hbar}J_z e^{-iJ_y\beta/\hbar}|jm'\rangle$$

$$= \frac{1}{\hbar}\langle jm'|(J_z\cos\beta + J_x\sin\beta)|jm'\rangle = m'\cos\beta$$

From (3.5.52), $d^{(\frac{1}{2})}_{\frac{1}{2},\frac{1}{2}}(\beta) = \cos\left(\frac{\beta}{2}\right) = d^{(\frac{1}{2})}_{-\frac{1}{2},-\frac{1}{2}}(\beta)$ and $d^{(\frac{1}{2})}_{\frac{1}{2},-\frac{1}{2}}(\beta) = -\sin\left(\frac{\beta}{2}\right) = -d^{(\frac{1}{2})}_{-\frac{1}{2},\frac{1}{2}}(\beta)$. So,

$$\begin{aligned}\sum_{m=-j}^{j} |d^{(j)}_{mm'}(\beta)|^2 m &= \cos^2\left(\frac{\beta}{2}\right)\left[\frac{1}{2}\right] + \sin^2\left(\frac{\beta}{2}\right)\left[-\frac{1}{2}\right] = +\frac{1}{2}\cos\beta \qquad \text{for } m' = +\frac{1}{2}\\ &= \sin^2\left(\frac{\beta}{2}\right)\left[\frac{1}{2}\right] + \cos^2\left(\frac{\beta}{2}\right)\left[-\frac{1}{2}\right] = -\frac{1}{2}\cos\beta \qquad \text{for } m' = -\frac{1}{2}\end{aligned}$$

For part (b), we start out the same way

$$\sum_{m=-j}^{j} |d^{(j)}_{m'm}(\beta)|^2 m^2 = \sum_{m=-j}^{j} \langle jm'|e^{-iJ_y\beta/\hbar}|jm\rangle m^2 \langle jm|e^{iJ_y\beta/\hbar}|jm'\rangle = \langle jm'|e^{-iJ_y\beta/\hbar}\frac{J_z^2}{\hbar^2}e^{iJ_y\beta/\hbar}|jm'\rangle$$

Rather than finding an analog for (3.2.7), we make use of the properties of tensor operators. [The original solution manual solves Part (a) this way.] First, $e^{-iJ_y\beta/\hbar}J_z^2 e^{iJ_y\beta/\hbar} = \mathcal{D}^\dagger(R)J_z^2\mathcal{D}(R)$ rotates J_z^2. (R is a rotation through $-\beta$ about the y-axis.) As in (3.11.12), J_z^2 is the "zz" component of the Cartesian tensor J_iJ_j. Following (3.11.13), we write

$$J_z^2 = \frac{1}{3}\mathbf{J}^2 + \left(J_z^2 - \frac{1}{3}\mathbf{J}^2\right) \equiv \frac{1}{3}\mathbf{J}^2 + T_0^{(2)}$$

decomposing J_z^2 into a scalar and the $q = 0$ component of a rank 2 spherical tensor $T_q^{(2)}$. The scalar is unchanged by rotation, and from (3.11.22), the spherical tensor rotates as $\mathcal{D}^\dagger(R)T_q^{(2)}\mathcal{D}(R) = \sum_{q'=-2}^{2}\mathcal{D}^{(2)}_{q'q}T^{(2)}_{q'}$. We need the expectation value in the state $|jm'\rangle$. From (3.11.26), only $T_0^{(2)}$ gives a nonzero result. Therefore, using (3.5.50), (3.5.51), and (3.6.53),

$$\begin{aligned}\sum_{m=-j}^{j} |d^{(j)}_{m'm}(\beta)|^2 m^2 &= \frac{1}{3}j(j+1) + \frac{1}{\hbar^2}\mathcal{D}^{(2)}_{00}(\beta)\langle jm'|\left[J_z^2 - \frac{1}{3}\mathbf{J}^2\right]|jm'\rangle\\ &= \frac{1}{3}j(j+1) + P_2(\cos\beta)\left[m'^2 - \frac{1}{3}j(j+1)\right]\\ &= \frac{1}{3}j(j+1)\left[1 + \frac{1}{2} - 3\cos^2\beta\right] + \frac{1}{2}(3\cos^2\beta - 1)m'^2\\ &= \frac{1}{2}j(j+1)\sin^2\beta + m'^2\frac{1}{2}(3\cos^2\beta - 1)\end{aligned}$$

26. Since $J_y = (J_+ - J_-)/2i$, just use (3.5.41) to find the matrix elements

$$\langle 1m'|J_y|1m\rangle = -\frac{\hbar}{2}\left[\sqrt{(1-m)(2+m)}\delta_{m',m+1} - \sqrt{(1+m)(2-m)}\delta_{m',m-1}\right] i$$

Expressing this as a matrix gives (3.5.54). By multiplying this matrix by itself several times, you can easily show that $(J_y/\hbar)^3 = J_y/\hbar$. The rotation operator is therefore, for $j = 1$,

$$\begin{aligned}
\exp\left(-i\frac{J_y}{\hbar}\beta\right) &= 1 - i\frac{J_y}{\hbar}\beta - \left(\frac{J_y}{\hbar}\right)^2\frac{\beta^2}{2!} + i\left(\frac{J_y}{\hbar}\right)^3\frac{\beta^3}{3!} + \left(\frac{J_y}{\hbar}\right)^4\frac{\beta^4}{4!} - i\left(\frac{J_y}{\hbar}\right)^5\frac{\beta^5}{5!} + \cdots \\
&= 1 - i\frac{J_y}{\hbar}\beta - \left(\frac{J_y}{\hbar}\right)^2\frac{\beta^2}{2!} + i\frac{J_y}{\hbar}\frac{\beta^3}{3!} + \left(\frac{J_y}{\hbar}\right)^2\frac{\beta^4}{4!} - i\frac{J_y}{\hbar}\frac{\beta^5}{5!} + \cdots \\
&= 1 - i\frac{J_y}{\hbar}\left(\beta - \frac{\beta^3}{3!} + \frac{\beta^5}{5!} + \cdots\right) - \left(\frac{J_y}{\hbar}\right)^2\left(1 - 1 + \frac{\beta^2}{2!} - \frac{\beta^4}{4!} + \cdots\right) \\
&= 1 - i\left(\frac{J_y}{\hbar}\right)\sin\beta - \left(\frac{J_y}{\hbar}\right)^2(1-\cos\beta)
\end{aligned}$$

Now $d^{(1)}_{m'm}(\beta) = \langle 1m'|\exp(-iJ_y\beta/\hbar)|1m\rangle$, so use the matrices $(J_y/\hbar)$ and $(J_y/\hbar)^2$ to write

$$\begin{aligned}
d^{(1)}_{m'm}(\beta) &= 1 - i\begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}\sin\beta - \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}(1-\cos\beta) \\
&= \begin{pmatrix} \cos^2\left(\frac{\beta}{2}\right) & -\frac{\sin(\beta)}{\sqrt{2}} & \sin^2\left(\frac{\beta}{2}\right) \\ \frac{\sin(\beta)}{\sqrt{2}} & \cos(\beta) & -\frac{\sin(\beta)}{\sqrt{2}} \\ \sin^2\left(\frac{\beta}{2}\right) & \frac{\sin(\beta)}{\sqrt{2}} & \cos^2\left(\frac{\beta}{2}\right) \end{pmatrix}
\end{aligned}$$

same as the problem, and (3.5.57), since $1 - \cos\beta = 2\sin^2(\beta/2)$ and $1 + \cos\beta = 2\cos^2(\beta/2)$.

27. Just insert a complete set of states on both left and right. Then

$$\begin{aligned}
\langle\alpha_2\beta_2\gamma_2|J_3^2|\alpha_1\beta_1\gamma_1\rangle &= \sum_{j,j'}\langle\alpha_2\beta_2\gamma_2|jmn\rangle\langle jmn|J_3^2|j'm'n'\rangle\langle j'm'n'|\alpha_1\beta_1\gamma_1\rangle \\
&= \sum_{jmn} n^2\mathcal{D}^j_{mn}(\alpha_2\beta_2\gamma_2)\mathcal{D}^{j*}_{mn}(\alpha_1\beta_1\gamma_1)
\end{aligned}$$

I just copied this from the old manual, and the solution is attributed to Prof. Thomas Fulton who passed away in April, 2011. It isn't clear to me what is meant by the state $|jmn\rangle$, so I can't be sure of the matrix element of J_3^2.

28. If B makes no measurement, then his existence is irrelevant for this problem. The probability that A measures $s_z = +\hbar/2$ (i.e. "spin up" in the z-direction) is 1/2. Her probability to measure $s_x = +\hbar/2$ is also 1/2; one cannot distinguish x- and z-directions in this case. Now, according to (3.8.15d), the state with total spin zero is given by

$$|00\rangle = \frac{1}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right]$$

So, if B measures $s_z = +\hbar/2$, then A measures $s_z = +\hbar/2$ ($s_z = -\hbar/2$) with zero (100%) probability. That is, $|\langle ++|-+\rangle|^2 = 0$ and $|\langle -+|-+\rangle|^2 = 1$. In, instead, she measures s_x, then the probability of measuring either $s_x = \pm\hbar/2$ is 1/2. That is $|\langle S_x; \pm, +| -+\rangle|^2 = 0$. The explicit state construction of $|S_x; \pm\rangle$ is given in (1.14.17a).

29. For a rotation $\mathbf{V} \to \mathbf{V}'$ through β about the y-axis, we have $V_x' = V_x \cos\beta + V_z \sin\beta$, $V_y' = V_y$, and $V_z' = -V_x \sin\beta + V_z \cos\beta$. Therefore, $V_{\pm 1}^{(1)'} = \mp(V_x \cos\beta + V_z \sin\beta \pm iV_y)/\sqrt{2}$, and $V_0^{(1)'} = -V_x \sin\beta + V_z \cos\beta$. Now doing $\sum_{q'} d_{qq'}^{(1)}(\beta) V_{q'}^{(1)}$ component by component,

$$\begin{aligned}
\sum_{q'} d_{+1,q'}^{(1)}(\beta) V_{q'}^{(1)} &= -\frac{1}{2}(1+\cos\beta)\frac{V_x + iV_y}{\sqrt{2}} - \frac{1}{\sqrt{2}}\sin\beta V_z + \frac{1}{2}(1-\cos\beta)\frac{V_x + iV_y}{\sqrt{2}} \\
&= -\frac{V_x\cos\beta + V_z \sin\beta - iV_y}{\sqrt{2}} \\
\sum_{q'} d_{0,q'}^{(1)}(\beta) V_{q'}^{(1)} &= -\frac{1}{\sqrt{2}}\sin\beta\frac{V_x + iV_y}{\sqrt{2}} + \cos\beta V_z - \frac{1}{\sqrt{2}}\sin\beta\frac{V_x - iV_y}{\sqrt{2}} \\
&= -V_x \sin\beta + V_z\cos\beta \\
\sum_{q'} d_{-1,q'}^{(1)}(\beta) V_{q'}^{(1)} &= -\frac{1}{2}(1-\cos\beta)\frac{V_x + iV_y}{\sqrt{2}} + \frac{1}{\sqrt{2}}\sin\beta V_z + \frac{1}{2}(1+\cos\beta)\frac{V_x - iV_y}{\sqrt{2}} \\
&= +\frac{V_x\cos\beta + V_z \sin\beta - iV_y}{\sqrt{2}}
\end{aligned}$$

and it all checks out.

30. Following (3.11.27) and forming $U_q^{(1)}$ and $V_q^{(1)}$ as in (3.11.16), write

$$T_q^{(k)} = \sum_{q_1=-1}^{1}\sum_{q_2=-1}^{1} \langle 11; q_1 q_2 | 11; kq\rangle U_{q_1}^{(1)} V_{q_2}^{(1)}$$

where $U_0^{(1)} = U_z$, $U_{\pm 1}^{(1)} = \mp(U_x \pm iU_y)/\sqrt{2}$, and similarly for $V_q^{(1)}$. (We needn't worry about the overall normalization.) The Clebsch-Gordan coefficients $\langle 11; q_1 q_2 | 11; kq\rangle$ (which are, of course, zero unless $q_1 + q_2 = q$) can be looked up, but also note that they were worked out in Problem 24. At this point, both parts (a) and (b) of this problem are reduced to algebra. For completeness, we work out this algebra below. It is a bit tedious, but there is some merit in comparing the resulting forms with, for example, (3.11.13) and (3.11.26).

$$
\begin{aligned}
T^{(1)}_{+1} &= \langle 11;01|11;11\rangle U^{(1)}_0 V^{(1)}_{+1} + \langle 11;10|11;11\rangle U^{(1)}_{+1} V^{(1)}_0 \\
&= \frac{1}{\sqrt{2}}\left[-U^{(1)}_0 V^{(1)}_{+1} + U^{(1)}_{+1} V^{(1)}_0\right] \\
&= \frac{1}{2}\left[U_z(V_x+iV_y) - (U_x+iU_y)V_z\right] = \frac{1}{2}\left[U_zV_x - U_xV_z + i(U_zV_y - U_yV_z)\right] \\
T^{(1)}_{0} &= \langle 11;-1,1|11;10\rangle U^{(1)}_{-1} V^{(1)}_{+1} + \langle 11;00|11;10\rangle U^{(1)}_0 V^{(1)}_0 \langle 11;1,-1|11;10\rangle U^{(1)}_{+1} V^{(1)}_{-1} \\
&= \frac{1}{\sqrt{2}}\left[-U^{(1)}_{-1} V^{(1)}_{+1} + U^{(1)}_{+1} V^{(1)}_{-1}\right] \\
&= \frac{1}{2\sqrt{2}}\left[(U_x-iU_y)(V_x+iV_y) - (U_x+iU_y)(V_x-iV_y)\right] = \frac{i}{\sqrt{2}}\left[U_xV_y - U_yV_x\right] \\
T^{(1)}_{-1} &= \langle 11;-1,0|11;1,-1\rangle U^{(1)}_{-1} V^{(1)}_0 + \langle 11;0,-1|11;1,-1\rangle U^{(1)}_0 V^{(1)}_{-1} \\
&= \frac{1}{\sqrt{2}}\left[-U^{(1)}_{-1} V^{(1)}_0 + U^{(1)}_0 V^{(1)}_{-1}\right] \\
&= \frac{1}{2}\left[-(U_x-iU_y)V_z + U_z(V_x-iV_y)\right] = \frac{1}{2}\left[U_zV_x - U_xV_z + i(U_yV_z - U_zV_y)\right] \\
T^{(2)}_{+2} &= U^{(1)}_{+1} V^{(1)}_{+1} = \frac{1}{2}\left[(U_x+iU_y)(V_x+iV_y)\right] = \frac{1}{2}\left[U_xV_x - U_yV_y + i(U_yV_x + U_xV_y)\right] \\
T^{(2)}_{+1} &= \langle 11;01|11;21\rangle U^{(1)}_0 V^{(1)}_{+1} + \langle 11;10|11;21\rangle U^{(1)}_{+1} V^{(1)}_0 \\
&= \frac{1}{\sqrt{2}}\left[U^{(1)}_0 V^{(1)}_{+1} + U^{(1)}_{+1} V^{(1)}_0\right] \\
&= -\frac{1}{2}\left[U_z(V_x+iV_y) + (U_x+iU_y)V_z\right] \\
T^{(2)}_{0} &= \langle 11;-1,1|11;20\rangle U^{(1)}_{-1} V^{(1)}_{+1} + \langle 11;00|11;20\rangle U^{(1)}_0 V^{(1)}_0 \langle 11;1,-1|11;20\rangle U^{(1)}_{+1} V^{(1)}_{-1} \\
&= \frac{1}{\sqrt{6}}\left[U^{(1)}_{-1} V^{(1)}_{+1} + 2U^{(1)}_0 V^{(1)}_0 + U^{(1)}_{+1} V^{(1)}_{-1}\right] \\
&= \frac{1}{2\sqrt{6}}\left[-(U_x-iU_y)(V_x+iV_y) + 4U_zV_z - (U_x+iU_y)(V_x-iV_y)\right] \\
&= -\frac{1}{\sqrt{6}}\left[U_xV_x + U_yV_y + 2U_zV_z\right] \\
T^{(2)}_{-1} &= \langle 11;-1,0|11;2,-1\rangle U^{(1)}_{-1} V^{(1)}_0 + \langle 11;0,-1|11;2,-1\rangle U^{(1)}_0 V^{(1)}_{-1} \\
&= \frac{1}{\sqrt{2}}\left[U^{(1)}_{-1} V^{(1)}_0 + U^{(1)}_0 V^{(1)}_{-1}\right] \\
&= \frac{1}{2}\left[(U_x-iU_y)V_z + U_z(V_x-iV_y)\right] = \frac{1}{2}\left[U_zV_x + U_xV_z - i(U_yV_z + U_zV_y)\right] \\
T^{(2)}_{-2} &= U^{(1)}_{-1} V^{(1)}_{-1} = \frac{1}{2}\left[(U_x-iU_y)(V_x-iV_y)\right] = \frac{1}{2}\left[U_xV_x - U_yV_y - i(U_yV_x + U_xV_y)\right]
\end{aligned}
$$

I'd be lucky to have gotten through all of this without any mistakes! Please email me if you find errors.

31. In the language of spherical tensors, $\mp(x \pm iy)/\sqrt{2} = T^{(1)}_{\pm 1}$ and $z = T^{(1)}_0$. According to the Wigner-Eckart Theorem, these three operators are related by

$$\langle n'l'm'|T^{(1)}_q|nlm\rangle = \langle l1; mq|l1; l'm'\rangle \frac{\langle n'l'||T^{(1)}||nl\rangle}{\sqrt{2l+1}}$$

where all the dependence on m, m', and q is contained in the Clebsch-Gordan Coefficient, so

$$\frac{\langle n'l'm'|T^{(1)}_{\pm 1}|nlm\rangle}{\langle n'l'm'|T^{(1)}_0|nlm\rangle} = \frac{\langle l1; m, \pm 1|l1; l'm'\rangle}{\langle l1; m0|l1; l'm'\rangle}$$

All three matrix elements are, of course, zero unless $m' = m + q$ and $|l-1| \le l' \le l+1$. I've looked around for symmetry relations among CG coefficients (not covered in the textbook) that could simplify this expression, but nothing is apparent to me. (The old solution manual offers some reduction based on parity, but this is not covered until Chapter Four.)

In wave mechanics, insert $1 = \int d^3x|\mathbf{x}\rangle\langle\mathbf{x}|$ and let the operators $T^{(1)}_q$ give spherical harmonics according to (3.11.16). Defining $r^3_{n'n;l'l} \equiv \int_0^\infty r^3 dr R_{n'l'}(r)R_{nl}(r)$ then gives, using (3.8.73),

$$\begin{aligned}\langle n'l'm'|T^{(1)}_q|nlm\rangle &= r^3_{n'n;l'l} \times \sqrt{\frac{4\pi}{3}} \int d\Omega Y^{m'*}_{l'}(\theta,\phi) Y^m_l(\theta,\phi) Y^q_1(\theta,\phi) \\ &= r^3_{n'n;l'l} \times \sqrt{\frac{2l+1}{2l'+1}} \langle l1; 00|l1; l'0\rangle\langle l1; mq|l1; l'm'\rangle\end{aligned}$$

The second CG coefficient is the same as in the Wigner-Eckart Theorem. The rest is absorbed into the reduced matrix element.

32. Use $Y^m_2(\mathbf{x})$ from (B.5.7) to construct the tensor. So $Y^0_2(\mathbf{x}) = \sqrt{5/16\pi}(3z^2 - r^2)/r^2$, $Y^{\pm 1}_2(\mathbf{x}) = \mp\sqrt{15/8\pi}(x \pm iy)z/r^2$, and $Y^{\pm 2}_2(\mathbf{x}) = \sqrt{15/32\pi}(x^2 - y^2 \pm 2ixy)/r^2$. So we rearrange these to find $xy = i\sqrt{2\pi/15}\left[Y^{-2}_2(\mathbf{x}) - Y^2_2(\mathbf{x})\right] r^2$, $xz = \sqrt{2\pi/15}\left[Y^{-1}_2(\mathbf{x}) - Y^1_2(\mathbf{x})\right] r^2$, and $x^2 - y^2 = \sqrt{8\pi/15}\left[Y^{-2}_2(\mathbf{x}) + Y^2_2(\mathbf{x})\right] r^2$. Now, using the Wigner-Eckart Theorem,

$$\begin{aligned}e\langle\alpha jm'|(x^2-y^2)|\alpha jj\rangle &= e\sqrt{\frac{8\pi}{15}}\langle\alpha jm'|\left[Y^{-2}_2(\mathbf{x}) + Y^2_2(\mathbf{x})\right] r^2|\alpha jj\rangle \\ &= e\sqrt{\frac{8\pi}{15}}\frac{\langle\alpha j||Y^{(2)}||\alpha j\rangle}{\sqrt{2j+1}}\left[\langle j2; j, -2|j2; jm'\rangle + \langle j2; j, 2|j2; jm'\rangle\right] \\ \text{But} \quad \langle j2; j, 2|j2; jm'\rangle &= 0 \quad \text{since} \quad m' = j+2 \text{ is not possible} \\ \text{and} \quad Q &= e\sqrt{\frac{16\pi}{5}}\langle\alpha jj|Y^0_2(\mathbf{x})r^2|\alpha jj\rangle \\ &= e\sqrt{\frac{16\pi}{5}}\frac{\langle\alpha j||Y^{(2)}||\alpha j\rangle}{\sqrt{2j+1}}\langle j2; j, 0|j2; jj\rangle \quad \text{so,} \\ e\langle\alpha jm'|(x^2-y^2)|\alpha jj\rangle &= \frac{Q}{\sqrt{2}}\frac{\langle j2; j, -2|j2; jm'\rangle}{\langle j2; j, 0|j2; jj\rangle}\end{aligned}$$

33. The trick is to group terms in the Hamiltonian so that we can put $\nabla^2\phi = 0$ while replacing S_x^2 and S_y^2 with $S_\pm^2$. First use $S_\pm^2 = (S_x \pm iS_y)^2 = S_x^2 - S_y^2 \pm i(S_xS_y + S_yS_x)$ so that $S_x^2 - S_y^2 = (S_+^2 + S_-^2)/2$. Also $S_x^2 + S_y^2 = \mathbf{S}^2 - S_z^2$. So, solving for S_x^2 and S_y^2, with $s = 3/2$,

$$
\begin{aligned}
H_{\text{int}} &= \frac{eQ}{2s(s-1)\hbar^2} \times \\
&\left\{ \left(\frac{\partial^2\phi}{\partial x^2}\right)_0 \frac{S_+^2 + S_-^2 + 2(\mathbf{S}^2 - S_z^2)}{4} - \left(\frac{\partial^2\phi}{\partial y^2}\right)_0 \frac{S_+^2 + S_-^2 - 2(\mathbf{S}^2 - S_z^2)}{4} + \left(\frac{\partial^2\phi}{\partial z^2}\right)_0 S_z^2 \right\} \\
&= \frac{2eQ}{3\hbar^2} \times \\
&\left\{ \frac{1}{4}\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 + \left(\frac{\partial^2\phi}{\partial y^2}\right)_0\right](2\mathbf{S}^2 - 2S_z^2 - 4S_z^2) + \frac{1}{4}\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 - \left(\frac{\partial^2\phi}{\partial y^2}\right)_0\right](S_+^2 + S_-^2) \right\} \\
&= \frac{A}{\hbar^2}(3S_z^2 - \mathbf{S}^2) + \frac{B}{\hbar^2}(S_+^2 + S_-^2)
\end{aligned}
$$

where $A = -\frac{eQ}{3}\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 + \left(\frac{\partial^2\phi}{\partial y^2}\right)_0\right]$ and $B = \frac{eQ}{6}\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 - \left(\frac{\partial^2\phi}{\partial y^2}\right)_0\right]$. (This differs from the old manual, but I don't see my error and I have trouble following their algebra.) Now,

$$
\begin{aligned}
(3S_z^2 - \mathbf{S}^2)|m\rangle &= \hbar^2(3m^2 - 15/4) \\
\text{and} \qquad S_\pm^2|m\rangle &= \hbar^2\sqrt{(s \mp m - 1)(s \pm m + 2)(s \mp m)(s \pm m + 1)}|m \pm 2\rangle
\end{aligned}
$$

so, labeling the basis states $m = \{+3/2, -1/2, -3/2, +1/2\}$, the Hamiltonian matrix is

$$
H_{\text{int}} \doteq \begin{pmatrix} 3A & 2B\sqrt{3} & 0 & 0 \\ 2B\sqrt{3} & -3A & 0 & 0 \\ 0 & 0 & 3A & 2B\sqrt{3} \\ 0 & 0 & 2B\sqrt{3} & -3A \end{pmatrix}
$$

which is block-diagonal 2×2 with the identity matrix, so there is twofold degeneracy in all eigenvalues. It is simple to diagonalize the 2×2 matrix. One finds that the eigenvalues are $\lambda_\pm = \pm\sqrt{9A^2 + 12B^2}$ and the (unnormalized) eigenvectors are

$$
|\lambda_\pm\rangle = 2\sqrt{3}B \left|\pm\frac{3}{2}, \pm\frac{3}{2}\right\rangle + (\lambda_\pm \pm 3A)\left|\pm\frac{3}{2}, \mp\frac{1}{2}\right\rangle
$$

To be sure, I don't quite understand the point of this problem, other than exercising some of the basic matters in angular momentum theory. Perhaps I'm missing something.

Chapter Four

1. **(a)** The Schrödinger equation, in coordinate space, is simple and has a separable solution:

$$H\psi(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = \left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}_1^2 - \frac{\hbar^2}{2m}\boldsymbol{\nabla}_2^2 - \frac{\hbar^2}{2m}\boldsymbol{\nabla}_2^2\right]\psi(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = E\psi(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$$

Putting $\psi(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = u^{(1)}(\mathbf{x}_1)u^{(2)}(\mathbf{x}_2)u^{(3)}(\mathbf{x}_3)$ and $E = E^{(1)} + E^{(2)} + E^{(3)}$ allows us to solve separately $-(\hbar^2/2m)\boldsymbol{\nabla}_i^2 u^{(i)}(\mathbf{x}_i) = E^{(i)}u^{(i)}(\mathbf{x}_i)$, $i = 1,2,3$. The eigenvalues are well known, namely $E^{(i)} = (\hbar^2\pi^2/2mL^2)\sum_{j=1}^3 \left(n_j^{(i)}\right)^2$. Therefore $E = (\hbar^2\pi^2/2mL^2)\sum_{i=1}^3\sum_{j=1}^3 \left(n_j^{(i)}\right)^2$. For the lowest energy level, all nine of the $n_j^{(i)} = 1$, so $E_1 = 9(\hbar^2\pi^2/2mL^2) \equiv 9E_0$. For the next level, one n is 2, so $E_2 = 12E_0$, and for the third level, two n's are 2, so $E_3 = 15E_0$.

Each level has an eight-fold degeneracy, that is the spin states $|\pm,\pm,\pm\rangle$. Level 1 has only one spatial wave function, so its degeneracy is 9. Level 2 has nine spatial wave function possibilities, since any of the nine $n_j^{(i)}$ can be 2, so level 2 has degeneracy $9 \times 8 = 72$. For level 3, any two of the $n_j^{(i)}$ can be 2, and the number of ways to take nine things two at a time is $9!/7!2! = 9 \cdot 8/2 = 36$, so the total degeracy is $36 \times 8 = 288$.

(b) With four electrons there is a 16-fold spin degeneracy. Schrödinger's Equation has four terms, and the wave function is $\psi(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) = u^{(1)}(\mathbf{x}_1)u^{(2)}(\mathbf{x}_2)u^{(3)}(\mathbf{x}_3)u^{(4)}(\mathbf{x}_4)$ and $E = E^{(1)} + E^{(2)} + E^{(3)} + E^{(4)}$. So, $E_1 = 12E_0$, $E_2 = 15E_0$, and $E_3 = 18E_0$. There is a $2^4 = 16$ fold spin degeneracy. With twelve factors in the spatial wave function, the spatial degeneracy is $12!/k!(12-k)!$ for the states with $E = E_{k+1}$. So, the degeneracy is $1 \times 16 = 16$ for level 1, $12 \times 16 = 192$ for level 2, and $66 \times 16 = 1056$ for level 3.

2. **(a)** See Sec.1.6. $\mathcal{T}_{\mathbf{d}}\mathcal{T}_{\mathbf{d}'} = \exp(-i\mathbf{p}\cdot\mathbf{d})\exp(-i\mathbf{p}\cdot\mathbf{d}') = \exp(-i\mathbf{p}\cdot\mathbf{d}')\exp(-i\mathbf{p}\cdot\mathbf{d}) = \mathcal{T}_{\mathbf{d}'}\mathcal{T}_{\mathbf{d}}$ since all components of $\mathbf{p}$ commute. Therefore, $\mathcal{T}_{\mathbf{d}}$ and $\mathcal{T}_{\mathbf{d}'}$ commute. **(b)** See Sec.3.1. Rotations do not commute with each other. This is what led us to commutation relations for $\mathbf{J}$. **(c)** Work in coordinate space. Using (4.2.5), we have $\mathcal{T}_{\mathbf{d}}\pi|\mathbf{x}'\rangle = \mathcal{T}_{\mathbf{d}}|-\mathbf{x}'\rangle = |-\mathbf{x}'+\mathbf{d}\rangle$ but we also have $\pi\mathcal{T}_{\mathbf{d}}|\mathbf{x}'\rangle = \pi|\mathbf{x}'+\mathbf{d}\rangle = |-\mathbf{x}'-\mathbf{d}\rangle$. In other words $\mathcal{T}_{\mathbf{d}}$ and π do not commute. **(d)** A rotation operator $\mathcal{D}(R) = \mathcal{D}(\hat{\mathbf{n}},\phi)$ that takes $\mathbf{x}' \to \mathbf{x}''$ also takes $-\mathbf{x}' \to -\mathbf{x}''$. Therefore $\mathcal{D}(R)\pi|\mathbf{x}'\rangle = |-\mathbf{x}''\rangle = \pi\mathcal{D}(R)|\mathbf{x}'\rangle$, and the operators commute.

3. Write $|\Psi\rangle = |a,b\rangle$ where a and b are eigenvalues of the operators A and B, respectively. Then $(AB+BA)|a,b\rangle = (ab+ba)|a,b\rangle = 0$ implies that $ab = -ba = -ab$. That is, either $a = 0$ or $b = 0$. For $A = \pi$ and $B = \mathbf{p}$, which indeed anticommute, as is easily shown using (4.2.10) and (4.2.7), we can conclude that only a $\mathbf{p} = 0$ state can be a parity eigenstate. (The parity operator can only have eigenvalues ± 1.)

4. *This problem is a special case of the discussion on pages 508–509.* Starting with (3.8.64),

$$\begin{aligned}
\mathcal{Y}_0^{1/2,1/2}(\theta,\phi) &= \begin{bmatrix} Y_0^0(\theta,\phi) \\ 0 \end{bmatrix} = \frac{1}{\sqrt{4\pi}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
(\boldsymbol{\sigma}\cdot\mathbf{x})\mathcal{Y}_0^{1/2,1/2} &= \frac{1}{\sqrt{4\pi}}\begin{bmatrix} z & x-iy \\ x+iy & -z \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{r}{\sqrt{4\pi}}\begin{bmatrix} \cos\theta \\ e^{i\phi}\sin\theta \end{bmatrix} = \frac{r}{\sqrt{3}}\begin{bmatrix} Y_1^0 \\ -\sqrt{2}Y_1^1 \end{bmatrix}
\end{aligned}$$

using (B.5.7). Referring to (3.8.64), then, $(\boldsymbol{\sigma}\cdot\mathbf{x})\mathcal{Y}_0^{1/2,1/2} = -r\mathcal{Y}_1^{1/2,1/2}$. That is, j and m are unchanged, but $l = 0 \to 1$. In terms of the operator $T_0^{(0)} \equiv \mathbf{S}\cdot\mathbf{x}$, this is not surprising. Since $T_0^{(0)}$ is a scalar under rotations, from (3.11.31) (the Wigner-Eckart Theorem), it can only connect states with $j' = j$ and $m' = m$. However, from (4.2.17), the operator connects states of opposite parity. This is accomplished by changing l by one unit.

5. From first order perturbation theory (which is not reached until Chapter 5, but which we will take as given here) the $C_{n'l'j'm'} = \langle n'l'j'm'|V|nljm\rangle/(E_{nlj} - E_{n'l'j'})$. The symmetry of V is determined by $T_0^{(0)} \equiv \mathbf{S}\cdot\mathbf{p}$, which is a pseudoscalar, as discussed above in Problem 4. Therefore, the matrix element needed for the $C_{n'l'j'm'}$ is zero unless $j' = j$, $m' = m$, and $l' = l \pm 1$. From (3.7.14), the radial wave functions $R_{nl}(r)$ go like r^l, so, since $V(\mathbf{x}) \propto \delta^{(3)}(\mathbf{x})$, the matrix element will only be nonzero for states with $l = 0$. Consequently, this interaction only connects $S_{1/2}$ and $P_{1/2}$ states.

6. First imagine that the barrier is infinitely high. Then, the lowest two energy levels are the degenerate case of sinusoidal wave functions with $\lambda = 2b$ in either the $a \le x \le a+b$ or $-a-b \le x \le -a$ regions. Now invoke parity. With a finite barrier, we take the odd and even linear combinations, tied together with exponential wave functions inside the barrier. So, consider the solution only for $x \ge 0$. Put $u(x) = A\sin[k(x-a-b)]$ for $a \le x \le a+b$, with $\hbar^2k^2/2m = E$. This form satisfies $u(a+b) = 0$ and is valid for either the symmetric ("s") or antisymmetric ("a") solution. Put $u_s(x) = B\cosh(\kappa x)$ and $u_a(x) = B\sinh(\kappa x)$ for $0 \le x \le a$, with $\hbar^2\kappa^2/2m = V_0 - E$. Matching $u(x)$ and $u'(x)$ at $x = a$ gives

$$\begin{aligned}
A_s\sin k_sb + B_s\cosh\kappa_sa &= 0 \\
k_sA_s\cos k_sb - \kappa_sB\sinh\kappa_sa &= 0
\end{aligned}
\qquad\text{and}\qquad
\begin{aligned}
A_a\sin k_ab + B_a\sinh\kappa_aa &= 0 \\
k_aA_a\cos k_ab - \kappa_aB_a\cosh\kappa_aa &= 0
\end{aligned}$$

Next set the determinants to zero. Since $E \ll V_0$, write $\kappa_s = \kappa_a = \sqrt{2mV_0}/\hbar \equiv \kappa$. Therefore, $(1/k_s)\tan k_sb = -(1/\kappa)\coth\kappa a$ and $(1/k_a)\tan k_ab = -(1/\kappa)\tanh\kappa a$. Now, since we expect λ to be only slightly larger than $2b$, put $\lambda = (1+\epsilon)2b$ or $kb = 2\pi b/\lambda = \pi/(1+\epsilon) \approx \pi(1-\epsilon)$. Thus, $\tan kb = \sin kb/\cos kb = kb - \pi$. The quantization conditions then become

$$\frac{k_sb-\pi}{k_s} = -\frac{1}{\kappa}\coth\kappa a \qquad\text{and}\qquad \frac{k_ab-\pi}{k_a} = -\frac{1}{\kappa}\tanh\kappa a$$

which are easily solved for the energy levels $E = \hbar^2k^2/2m$. The splitting is

$$\Delta E \equiv E_a - E_s = \frac{\hbar^2\pi^2\kappa^2}{2m}\left[\frac{1}{(\kappa b + \tanh\kappa a)^2} - \frac{1}{(\kappa b + \coth\kappa a)^2}\right]$$

Further simplification is possible if the barrier is "narrow", i.e. $\kappa a \ll 1$.

7. **(a)** $\psi_{\mathbf{p}}(\mathbf{x},t) = e^{i(\mathbf{p}\cdot\mathbf{x}-Et)/\hbar}$ so $\psi^*_{\mathbf{p}}(\mathbf{x},-t) = e^{-i(\mathbf{p}\cdot\mathbf{x}+Et)/\hbar} = e^{i(-\mathbf{p}\cdot\mathbf{x}-Et)/\hbar} = \psi_{-\mathbf{p}}(\mathbf{x},t)$

(b) Equation (3.2.52) constructs $\chi_+(\hat{\mathbf{n}}) = \begin{bmatrix} e^{-i\gamma/2}\cos(\beta/2) \\ e^{i\gamma/2}\sin(\beta/2) \end{bmatrix}$. Following that example,

$$\chi_-(\hat{\mathbf{n}}) = \begin{bmatrix} \cos\frac{\gamma}{2} - i\sin\frac{\gamma}{2} & 0 \\ 0 & \cos\frac{\gamma}{2} + i\sin\frac{\gamma}{2} \end{bmatrix} \begin{bmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -e^{-i\gamma/2}\sin\frac{\beta}{2} \\ e^{i\gamma/2}\cos\frac{\beta}{2} \end{bmatrix}$$

$$\text{So,} \qquad -i\sigma_2\chi^*_+(\hat{\mathbf{n}}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{i\gamma/2}\cos\frac{\beta}{2} \\ e^{-i\gamma/2}\sin\frac{\beta}{2} \end{bmatrix} = \begin{bmatrix} -e^{-i\gamma/2}\sin\frac{\beta}{2} \\ e^{i\gamma/2}\cos\frac{\beta}{2} \end{bmatrix} = \chi_-(\hat{\mathbf{n}})$$

See the discussion surrounding (4.4.66).

8. The statement in **(a)** is just Theorem 4.2, proven in the text. As it says just following this proof, on page 295, the theorem would appear to be violated by the plane wave $e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}$ except that this state is degenerate with $e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar}$, so violates the assumptions of the theorem.

9. This problem is also, essentially, worked in the text. See (4.4.61). We have

$$\begin{aligned} \langle \mathbf{p}'|\Theta|\alpha\rangle &= \langle \mathbf{p}'|\Theta \int d^3p''|\mathbf{p}''\rangle\langle\mathbf{p}''|\alpha\rangle = \langle \mathbf{p}'| \int d^3p''|-\mathbf{p}''\rangle\langle\mathbf{p}''|\alpha\rangle^* \\ &= \langle \mathbf{p}'| \int d^3p''|\mathbf{p}''\rangle\langle-\mathbf{p}''|\alpha\rangle^* = \int d^3p''\langle\mathbf{p}'|\mathbf{p}''\rangle\langle-\mathbf{p}''|\alpha\rangle^* = \langle-\mathbf{p}'|\alpha\rangle^* = \phi^*(-\mathbf{p}') \end{aligned}$$

We used the antiunitary property (4.4.13b) of Θ.

10. See the rewritten version of this problem in the Errata. (By the way, the idiosyncrasies connected with Problems 8 and 9 are that they are in fact solved in the text.)

(a) Equation (4.4.53) just says that Θ anticommutes with all components of $\mathbf{J}$. Therefore $J_z\Theta|jm\rangle = J_z[\Theta|jm\rangle] = -\Theta J_z|jm\rangle = (-m)[\Theta|jm\rangle]$, so $\Theta|jm\rangle \propto |j,-m\rangle$. Similarly, $J_\pm[\Theta|jm\rangle] = -c_\pm(j,m)[\Theta|j,m\pm1\rangle]$, and in a convention such as (3.5.39) and (3.5.40) where the $c_\pm(j,m)$ are real and non-negative for all j and m, $\Theta|jm\rangle \propto (-1)^m|j,-m\rangle$ in order for the sign to change when m is changed by one. Thus $\Theta|jm\rangle = e^{i\delta}(-1)^m|j,-m\rangle$ where δ is real, in order to maintain norm.

(b) Using the infinitesimal form (3.1.15) for $\mathcal{D}(\hat{\mathbf{n}},d\phi) = 1 - i(\mathbf{J}\cdot\hat{\mathbf{n}}/\hbar)d\phi$, we have

$$\Theta\mathcal{D}(R)\Theta^{-1} = \Theta\left[1 - i\frac{\mathbf{J}\cdot\hat{\mathbf{n}}}{\hbar}d\phi\right]\Theta^{-1} = 1 + i\frac{\Theta\mathbf{J}\Theta^{-1}\cdot\hat{\mathbf{n}}}{\hbar}d\phi = 1 - i\frac{\mathbf{J}\cdot\hat{\mathbf{n}}}{\hbar}d\phi = \mathcal{D}(R)$$

So, $\Theta\mathcal{D}(R)|jm\rangle = \Theta\mathcal{D}(R)\Theta^{-1}\Theta|jm\rangle = \mathcal{D}(R)\Theta|jm\rangle$.

(c) The trick here is to write $\langle j,-m'|\mathcal{D}\Theta|j,m\rangle$ two different ways, and then equate them.

$$\begin{aligned}
\langle j,-m'|\mathcal{D}\Theta|j,m\rangle &= e^{i\delta}(-1)^m\langle j,-m'|\mathcal{D}|j,-m\rangle = e^{i\delta}(-1)^m\mathcal{D}^{(j)}_{-m',-m} \\
\text{and} \qquad &= \langle j,-m'|\Theta\mathcal{D}|j,m\rangle = \langle j,-m'|\Theta\sum_{m''}|j,m''\rangle\langle j,m''\mathcal{D}|j,m\rangle \\
&= \sum_{m''}\mathcal{D}^{(j)*}_{m'',m}\langle j,-m'|\Theta|j,m''\rangle \\
&= \sum_{m''}\mathcal{D}^{(j)*}_{m'',m}e^{i\delta}(-1)^{m''}\delta_{-m',-m''} = \mathcal{D}^{(j)*}_{m',m}e^{i\delta}(-1)^{m'} \\
\text{so} \qquad \mathcal{D}^{(j)*}_{m',m} &= (-1)^{m-m'}\mathcal{D}^{(j)}_{-m',-m}
\end{aligned}$$

for any value of δ. So **(d)**, the properties of rotations are satisfied and, so long as we stick with the "real and non-negative" phase convention for $J_\pm$, we can set $\delta=0$ and take $\Theta|jm\rangle = (-1)^m|j,-m\rangle = i^{2m}|j,-m\rangle$.

11. Time reversal invariance means $\Theta H\Theta^{-1}=H$, or $\Theta H = H\Theta$. So for a state $|E\rangle$ with $H|E\rangle = E|E\rangle$, we have $H[\Theta|E\rangle] = E[\Theta|E\rangle]$ which says that if $|E\rangle$ is an energy eigenstate, then so is $|\tilde{E}\rangle \equiv \Theta|E\rangle$. Since there is no degeneracy, $|\tilde{E}\rangle = e^{i\delta}|E\rangle$ for a real phase δ. Now

$$\langle E|\mathbf{L}|E\rangle = \langle\tilde{E}|\Theta\mathbf{L}\Theta^{-1}|\tilde{E}\rangle = -\langle\tilde{E}|\mathbf{L}|\tilde{E}\rangle = -e^{-i\delta}\langle E|\mathbf{L}|E\rangle e^{i\delta} = -\langle E|\mathbf{L}|E\rangle$$

and so $\langle E|\mathbf{L}|E\rangle = 0$, where the first step makes use of (4.4.36). Then, writing the wave function for an eigenstate as $\psi_E(\mathbf{x}) \equiv \langle\mathbf{x}|E\rangle = \sum_{lm}F_{lm}(r)Y_l^m(\theta,\phi)$, the time reversed wave function is $\tilde{\psi}(\mathbf{x}) = \langle\mathbf{x}|\tilde{E}\rangle = e^{i\delta}\langle\mathbf{x}|E\rangle = e^{i\delta}\psi_E(\mathbf{x})$, but also, from (4.4.56), $\tilde{\psi}_E(\mathbf{x}) = \psi_E^*(\mathbf{x}) = \sum_{lm}F^*_{lm}(r)\left(Y_l^m(\theta,\phi)\right)^* = \sum_{lm}F^*_{lm}(r)(-1)^{-m}Y_l^{-m}(\theta,\phi) = \sum_{lm}F^*_{l,-m}(r)(-1)^mY_l^m(\theta,\phi)$ so, finally, $F_{lm}(r) = (-1)^m e^{-i\delta}F^*_{l,-m}(r)$.

12. It is easy to build the matrix representation of H using S_z and $S_\pm = S_x \pm iS_y$. Find

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix} \quad S_y = \frac{\hbar}{\sqrt{2}}\begin{bmatrix}0&-i&0\\i&0&-i\\0&i&0\end{bmatrix} \quad S_z = \hbar\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix} \quad H = \hbar^2\begin{bmatrix}A&0&B\\0&0&0\\B&0&A\end{bmatrix}$$

so energy eigenvalues $E=\hbar^2(A\pm B)$ and 0 with eigenvectors $|E_\pm\rangle \equiv [|1,1\rangle \pm |1,-1\rangle]/\sqrt{2}$ and $|E_0\rangle \equiv |1,0\rangle$. Now for a spin component S_i, have $\Theta S_i^2\Theta^{-1} = \Theta S_i\Theta^{-1}\Theta S_i\Theta^{-1} = (-S_i)(-S_i) = S_i^2$, and A and B are real, so H is time reversal invariant. From (4.4.78),

$$\Theta|E_\pm\rangle = \frac{1}{\sqrt{2}}\left[(-1)^1|1,1\rangle \pm (-1)^{-1}|1,-1\rangle\right] = -|E_\pm\rangle \qquad \text{and} \qquad \Theta|E_0\rangle = (-1)^0|E_0\rangle = |E_0\rangle$$

Chapter Five

1. The first order correction is proportional to $\langle 0|x|\rangle = 0$. The second order correction is $\Delta E_0 = \sum_{k=1}^{\infty} |V_{0k}|^2/(E_0^{(0)} - E_k^{(0)}) = -b^2 \sum_{k=1}^{\infty} |\langle k|x|0\rangle|^2/(k\hbar\omega)$. From (2.3.25a), or using the equation given, albeit written in a strange notation, you find $\langle k|x|0\rangle = \sqrt{\hbar/2m\omega}\delta_{k1}$. Hence $\Delta E_0 = (-b^2)(\hbar/2m\omega)/(\hbar\omega) = -b^2/2m\omega^2$, and $E_0 = \hbar\omega/2 - b^2/2m\omega^2$ to second order. To solve exactly, write $V(x) = m\omega^2 x^2/2 + bx = m\omega^2/2(x + b/m\omega^2)^2 - b^2/2m\omega^2$. If you define $x' \equiv x + b/m\omega^2$, then you see that it is still the simple harmonic oscillator, but with the equilibrium point shifted and with an overall energy shift of $-b^2/2m\omega^2$. That is, second order perturbation theory gives the exact answer in this case.

2. The problem should ask for terms up to λ^2, not g^2, for the notation to be consistent. Now, in the notation leading to (5.1.44), we want the quantity $|\langle n^{(0)}|n\rangle|^2$, where the $|n\rangle$ are properly normalized, i.e. $|\langle n^{(0)}|n\rangle|^2/|\langle n|n\rangle|^2$. Using (5.1.44), $\langle n^{(0)}|n\rangle = 1$ and

$$\begin{aligned}
\langle n|n\rangle &= \left[\langle n^{(0)}| + \lambda \sum_{k\neq n} \frac{V_{kn}^* \langle k^{(0)}|}{E_n^{(0)} - E_k^{(0)}} + \mathcal{O}(\lambda^2)\right]\left[|n^{(0)}\rangle + \lambda \sum_{\ell\neq n} \frac{|\ell^{(0)}\rangle V_{\ell n}}{E_n^{(0)} - E_\ell^{(0)}} + \mathcal{O}(\lambda^2)\right] \\
&= 1 + \lambda^2 \sum_{k\neq n}\sum_{\ell\neq n} \frac{V_{kn}^* V_{\ell n}}{(E_n^{(0)} - E_k^{(0)})(E_n^{(0)} - E_\ell^{(0)})}\delta_{k\ell} + \mathcal{O}(\lambda^3) = 1 + \lambda^2 \sum_{k\neq n} \frac{|V_{kn}|^2}{(E_n^{(0)} - E_k^{(0)})^2}
\end{aligned}$$

since the $|n^{(0)}\rangle$ are orthonormal. Note also that this precludes any of the $\mathcal{O}(\lambda^2)$ terms in the first multiplication from yielding any other $\mathcal{O}(\lambda^2)$ terms in the second line. Hence

$$\frac{|\langle n^{(0)}|n\rangle|^2}{|\langle n|n\rangle|^2} = 1 - \lambda^2 \sum_{k\neq n} \frac{|V_{kn}|^2}{(E_n^{(0)} - E_k^{(0)})^2} + \mathcal{O}(\lambda^3)$$

3. The unperturbed ground state wave function is $\psi_0^{(0)}(x,y) = (2/L)\sin(\pi x/L)\sin(\pi y/L)$, so $\Delta_0^{(1)} = 4\lambda/L^2 \int_0^L \int_0^L xy \sin^2(\pi x/L)\sin^2(\pi y/L)dxdy = \lambda L^2/4$, and the zeroth order eigenfunction is just $\psi_0^{(0)}$. For the first excited state, $\psi_{1a}^{(0)}(x,y) = (2/L)\sin(\pi x/L)\sin(2\pi y/L)$ and $\psi_{1b}^{(0)}(x,y) = (2/L)\sin(2\pi x/L)\sin(\pi y/L)$, that is twice degenerate, so we construct a 2×2 matrix and diagonalize. The diagonal elements are $V_{aa} = \int \psi_{1a}^{(0)} V \psi_{1a}^{(0)} dxdy = 4\lambda/L^2 \int_0^L \int_0^L xy \sin^2(\pi x/L)\sin^2(2\pi y/L)dxdy = \lambda L^2/4 = V_{bb}$, and the off diagonal elements are $V_{ab} = \int \psi_{1a}^{(0)} V \psi_{1b}^{(0)} dxdy = 128\lambda L^2/81\pi^4 = V_{ba}$. Therefore, following (5.2.9), the first order energy shifts in the first excited are the eigenvalues $\Delta_1^{(1)}$ of

$$\mathsf{V} = \frac{\lambda L^2}{4\pi^4}\begin{bmatrix} \pi^4 & 1024/81 \\ 1024/81 & \pi^4 \end{bmatrix} \qquad \text{so} \qquad \left(\frac{\lambda L^2}{4} - \Delta_1^{(1)}\right)^2 - \left(\frac{128\lambda L^2}{81\pi^4}\right)^2 = 0$$

and $\Delta_1^{(1)} = \lambda L^2(1/4 \pm 128/81\pi^4) = \lambda L^2\{0.266, 0.233\}$. The eigenvectors are easy to find for a simple matrix of this form. They are $(\psi_{1a}^{(0)} \pm \psi_{1b}^{(0)})/\sqrt{2}$.

4. The problem separates into independent x- and y-harmonic oscillators, i.e $H_0 = H_x + H_y$, with states $|n_x, n_y\rangle$ and energy eigenvalues $E = (n_x + n_y + 1)\hbar\omega$. So the three lowest energy eigenstates are $|0,0\rangle$, with $E = \hbar\omega$, and $|1,0\rangle$ and $|0,1\rangle$, each with $E = 2\hbar\omega$. That is, the first excited state is doubly degenerate. The first order energy shift for the ground state is zero, since the operators x and y only connect states that differ by one quantum, i.e. $\langle 0,0|xy|0,0\rangle = 0$. The first excited state requires us to diagonalize the perturbation for the first order energy shift, but $\langle 1,0|xy|1,0\rangle = 0 = \langle 0,1|xy|0,1\rangle$. The off diagonal elements are

$$\delta m\omega^2\langle 1,0|xy|0,1\rangle = \delta m\omega^2\langle 1,0|\left[\sqrt{\frac{\hbar}{2m\omega}}\sqrt{1}\sqrt{\frac{\hbar}{2m\omega}}\sqrt{1}\right]|1,0\rangle = \delta\frac{\hbar\omega}{2} = \langle 0,1|xy|1,0\rangle$$

So, following (5.2.9), the first order energy shifts in the first excited are the eigenvalues $\Delta_1^{(1)}$ where $\left(\Delta_1^{(1)}\right)^2 - (\delta\hbar\omega/2)^2 = 0$ or $\Delta_1^{(1)} = \pm\delta\hbar\omega/2$ and $E = (2 \pm \delta/2)\hbar\omega$ for the degenerate first excited state. The corresponding eigenstates are $(|1,0\rangle \pm |0,1\rangle)/\sqrt{2}$.

To solve the problem exactly, observe that we can rewrite the potential energy as

$$\frac{1}{2}m\omega^2(x^2 + y^2 + 2\delta xy) = \frac{1}{2}m\omega^2\left[(1+\delta)\frac{(x+y)^2}{2} + (1-\delta)\frac{(x-y)^2}{2}\right]$$

and then rotate the x, y axes by 45°. This anharmonic oscillator has normal coordinate $x' \equiv (x+y)/\sqrt{2}$ with frequency $\omega(1+\delta)^{1/2}$, and $y' \equiv (x-y)/\sqrt{2}$ with $\omega(1-\delta)^{1/2}$. Therefore

$\lvert n_{x'}, n_{y'}\rangle$	Energy
$\lvert 0,0\rangle$	$\frac{1}{2}\hbar\omega(1+\delta)^{1/2} + \frac{1}{2}\hbar\omega(1-\delta)^{1/2} \approx \frac{1}{2}\hbar\omega(1+\delta/2+1-\delta/2) = \hbar\omega$
$\lvert 1,0\rangle$	$\frac{3}{2}\hbar\omega(1+\delta)^{1/2} + \frac{1}{2}\hbar\omega(1-\delta)^{1/2} \approx \frac{1}{2}\hbar\omega(3+3\delta/2+1-\delta/2) = (2+\delta/2)\hbar\omega$
$\lvert 1,0\rangle$	$\frac{1}{2}\hbar\omega(1+\delta)^{1/2} + \frac{3}{2}\hbar\omega(1-\delta)^{1/2} \approx \frac{1}{2}\hbar\omega(1+1\delta/2+3-3\delta/2) = (2-\delta/2)\hbar\omega$

in perfect agreement with our lowest order result from perturbation theory.

5. We need the matrix elements $V_{k0} = \varepsilon m\omega^2\langle k|x^2|0\rangle/2$. The algebra for matrix elements for harmonic oscillator states was worked out in Problem 2.14. Using that result, we have

$$V_{k0} = \frac{1}{2}\varepsilon m\omega^2\langle k|x^2|0\rangle = \frac{1}{2}\varepsilon m\omega^2\frac{\hbar}{2m\omega}\left[\delta_{0k} + \sqrt{2}\delta_{2,k}\right] = \frac{\varepsilon}{4}\hbar\omega\left[\delta_{0k} + \sqrt{2}\delta_{2,k}\right]$$

and so $V_{00} = \varepsilon\hbar\omega/4$ and $V_{20} = \varepsilon\hbar\omega/2\sqrt{2}$, in agreement with (5.1.54).

6. Put $\omega_x = \omega_y \equiv \omega$ and $\omega_z = \omega(1+\epsilon)$ with $\epsilon \ll 1$. Then, following (2.7.20),

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 + \frac{m\omega^2}{2}(x^2+y^2) + \frac{m\omega^2}{2}(1+\epsilon)^2z^2 = \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 + \frac{m\omega^2}{2}r^2 + \epsilon m\omega^2 z^2$$

For fixed $\mathbf{B}$, $\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r}$, from (5.3.32) or any textbook on electromagnetism. So for $\mathbf{B} = B\hat{\mathbf{x}}$,

$$\begin{aligned}
\frac{1}{2m}\left(\mathbf{p}-\frac{q}{c}\mathbf{A}\right)^2 &= \frac{\mathbf{p}^2}{2m} - \frac{q}{4mc}\left[\mathbf{p}\cdot(\mathbf{B}\times\mathbf{r}) + (\mathbf{B}\times\mathbf{r})\cdot\mathbf{p}\right] + \frac{q^2}{8mc^2}(\mathbf{B}\times\mathbf{r})^2 \\
&= \frac{\mathbf{p}^2}{2m} - \frac{qB}{4mc}\left[p_z y - p_y z + y p_z - z p_y\right] + \frac{q^2B^2}{8mc^2}(y\hat{\mathbf{z}} - z\hat{\mathbf{y}})^2 \\
&= \frac{\mathbf{p}^2}{2m} - \frac{qB}{2mc}L_x + \frac{q^2B^2}{8mc^2}(y^2+z^2)
\end{aligned}$$

Now, "Zeeman splitting is comparable to the splitting produced by the anisotropy" means that the second term above $\sim qB\hbar/mc$ is about the same size as $\epsilon\hbar\omega$, that is $\epsilon \sim qB/mc\omega$. The third term above is $\sim (q^2B^2/mc^2)(\hbar/m\omega) = \epsilon^2(\hbar\omega)$, and we can disregard it with respect to the first term. Finally, to make the angular momentum algebra easier, rotate $x \to z$ and $z \to -x$. Therefore the Hamiltonian becomes

$$H = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2 r^2 - \frac{qB}{2mc}L_z + \epsilon m\omega^2 x^2 \equiv H_0 + V$$

where $H_0 \equiv \mathbf{p}^2/2m + m\omega^2 r^2/2$ and $V \equiv -qBL_z/2mc + \epsilon m\omega^2 x^2$ is an order ϵ perturbation.

The eigenstates of H_0 are derived and discussed in Section 3.7. See also Problem 3.21. The first excited state has $E = 5\hbar\omega/2$ and is threefold degenerate. All three states have angular momentum eigenvalue $l = 1$, so label them in order $m = +1$, $m = 0$, and $m = -1$. In this basis, the first term in V is diagonal, with values $-qB\hbar/2mc$, 0, and $+qB\hbar/2mc$, respectively. For the second term, write these basis states in terms of the basis $|n_x n_y n_z\rangle$. See the solution to Problem 3.21. The result is $|+\rangle = (|100\rangle + i|010\rangle)/\sqrt{2}$, $|0\rangle = |001\rangle$, and $|-\rangle = (|100\rangle - i|010\rangle)/\sqrt{2}$. See Problem 2.14 for matrix elements of x^2 in the $|n_x n_y n_z\rangle$ basis; the essential result is $\langle m|x^2|n\rangle = (2n+1)(\hbar/2m\omega)\delta_{nm}$ for the states considered here. So for example $\langle +|x^2|+\rangle = [\langle 100|x^2|100\rangle + \langle 010|x^2|010\rangle]/2 = \hbar/m\omega$. The perturbation becomes

$$V \doteq \begin{bmatrix} -qB\hbar/2mc + \epsilon\hbar\omega & 0 & \epsilon\hbar\omega/2 \\ 0 & \epsilon\hbar\omega/2 & 0 \\ \epsilon\hbar\omega/2 & 0 & +qB\hbar/2mc + \epsilon\hbar\omega \end{bmatrix} = \hbar \begin{bmatrix} -\alpha + \epsilon\omega & 0 & \epsilon\omega/2 \\ 0 & \epsilon\omega/2 & 0 \\ \epsilon\omega/2 & 0 & \alpha + \epsilon\omega \end{bmatrix}$$

with $\alpha \equiv qB/2mc$. The energy shifts $\hbar\delta$ come from diagonalizing this matrix. One eigenvalue is $\delta = \epsilon\omega/2$. The others solve $(-\alpha+\epsilon\omega-\delta)(\alpha+\epsilon\omega-\delta)-(\epsilon\omega/2)^2 = -\alpha^2+(\epsilon\omega-\delta)^2-(\epsilon\omega/2)^2 = 0$ and the energy shifts are given by

$$\frac{1}{2}\epsilon\hbar\omega \qquad \text{and} \qquad \epsilon\hbar\omega \pm \frac{1}{2}\hbar\left[\left(\frac{qB}{mc}\right)^2 + \epsilon^2\omega^2\right]^{1/2}$$

In the limit $\epsilon = 0$, we simply have the Zeeman splitting of the levels according to m eigenvalues for orbital angular momentum. On the other hand, for $B = 0$, the degeneracy is only partial lifted, i.e., one shift is $(3/2)\epsilon\hbar\omega$, but the other two are both $(1/2)\epsilon\hbar\omega$. This makes sense, since the anisotropy is in only one direction, leaving symmetry between the other two.

7. Use (5.1.44) to write the first order approximation to the ground state as

$$|100^{(1)}\rangle = |100^{(0)}\rangle - e|\mathbf{E}|\sum_{nlm}|nlm^{(0)}\rangle\frac{\langle nlm^{(0)}|z|100^{(0)}\rangle}{E_{100}-E_{nlm}}$$

Now find the expectation value of the electric dipole moment (to lowest nonzero order) making use of the fact that $\langle 100^{(0)}|z|100^{(0)}\rangle = 0$. We have

$$\begin{aligned}
&\langle 100^{(1)}|ez|100^{(1)}\rangle \\
&= -e^2|\mathbf{E}|\left[\sum_{nlm}\langle 100^{(0)}|z|nlm^{(0)}\rangle\frac{\langle nlm^{(0)}|z|100^{(0)}\rangle}{E_{100}-E_{nlm}} + \sum_{nlm}\langle nlm^{(0)}|z|100^{(0)}\rangle\frac{\langle nlm^{(0)}|z|100^{(0)}\rangle^*}{E_{100}-E_{nlm}}\right] \\
&= -2e^2|\mathbf{E}|\sum_{nlm}\frac{|\langle nlm^{(0)}|z|100^{(0)}\rangle|^2}{E_{100}-E_{nlm}} \equiv \alpha|\mathbf{E}| \qquad \text{so} \qquad \alpha = -2e^2\sum_{nlm}\frac{|\langle nlm^{(0)}|z|100^{(0)}\rangle|^2}{E_{100}-E_{nlm}}
\end{aligned}$$

defining the polarizability α, the same as (5.1.68) from the second order energy shift.

8. I'm not exactly sure what these have to do with "approximation methods", but OK.

(a) From (B.5.7) $x = r\sin\theta\cos\phi = -(2\pi/3)^{1/2}(Y_1^1 - Y_1^{-1})$. So, the matrix element vanishes. In position space, it is proportional to integrals of $Y_0^1 Y_1^{\pm 1}$ which are zero by orthogonality. Alternately, by (3.11.28), since x combines spherical tensors $T_q^{(k)}$ with $q = 1$, the matrix element must vanish since $0 \neq 1 + 0$. More colloquially, the matrix element is between two states with new orientation perpendicular to the z-axis, so its value must vanish.

(b) From (2.2.25) and (2.2.26), $p_z = m\dot{z} = (m/i\hbar)[z, H]$, so the matrix element is proportional to $(E_{200} - E_{210})\langle 210|z|200\rangle = 0$ since $E_{200} = E_{210}$. More physically, this matrix element is between states that have no up/down asymmetry, so $\langle p_z\rangle = 0$.

(c) First express the state $|lsjm\rangle = \left|4\frac{1}{2}\frac{9}{2}\frac{7}{2}\right\rangle$ in terms of states $|ls; m_l m_s\rangle = \left|4\frac{1}{2}; m_l, \frac{7}{2} - m_l\right\rangle$. This transformation matrix is given by (3.8.62) and using $j = l + 1/2$. The result is

$$\left|4\frac{1}{2}\frac{9}{2}\frac{7}{2}\right\rangle = \frac{1}{3}\sqrt{8}\left|4\frac{1}{2}; 3, +\frac{1}{2}\right\rangle + \frac{1}{3}\left|4\frac{1}{2}; 4, -\frac{1}{2}\right\rangle$$

So $\left\langle 4\frac{1}{2}\frac{9}{2}\frac{7}{2}\right| L_z \left|4\frac{1}{2}\frac{9}{2}\frac{7}{2}\right\rangle = (8/9)3\hbar + (1/9)4\hbar = (28/9)\hbar$.

(d) See the solution to Problem 3.4. The singlet (triplet) state is $\frac{1}{\sqrt{2}}\left[|+-\rangle \mp |-+\rangle\right]$, so the matrix element is $\frac{1}{2}\left[\langle +-| - \langle -+|\right]\left[+\hbar|+-\rangle - \hbar|-+\rangle\right] = \hbar$.

(e) One is apparently supposed to be aware that the ground state of the hydrogen molecule puts the spin part in a singlet state, so-called "homopolar binding" with a symmetric spatial wave function. Therefore $\langle \mathbf{S}_1\cdot\mathbf{S}_2\rangle = \frac{1}{2}\left[\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2\right] = \frac{1}{2}\left[0 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2\right] = -\frac{3}{4}\hbar^2$.

9. Write $x^2 - y^2 = r^2 \sin^2\theta(\cos^2\phi - \sin^2\phi) = r^2\sin^2\theta\cos 2\phi = r^2\sin^2\theta(e^{2i\phi} + e^{-2i\phi})/2$, so from (B.5.7) V is a combination of $Y_2^{\pm 2}$, i.e. the sum of tensor operators $T_2^{\pm 2}$. Therefore by (3.11.28), the perturbation only connects states with m differing by 2. The integrals $I = \frac{\lambda}{2}\int \sin^2\theta e^{\pm\phi} e^{\mp 2i\phi} \sin^2\theta e^{\pm\phi} d\Omega \int r^2 R_{n1}^2 r^2 dr$ are the same for the cases connecting $m = \pm 1$ to $m = \mp 1$ respectively. So, labeling the states $m = 1, 0, -1$, the perturbation is

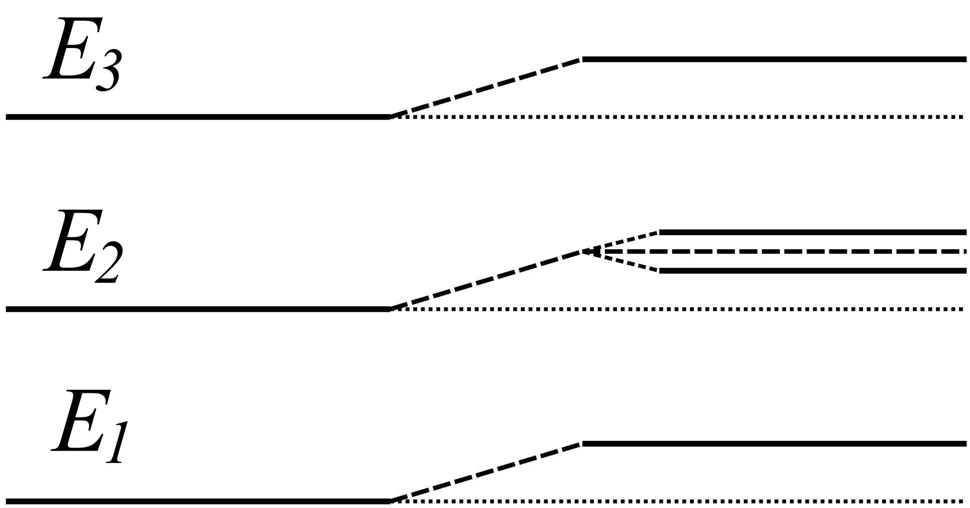
$$V \doteq \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad |n10\rangle, \frac{1}{\sqrt{2}}[|n11\rangle \pm |n1,-1\rangle]$$

are the "correct" zeroth order eigenstates. From (4.4.58), under time reversal, a state $|l,m\rangle$ goes to a state $|l,-m\rangle$ and picks up a phase $(-1)^m$. Thus, the eigenstates go into each other.

10. (a) The lowest order solution is well known. For $n \equiv \{n_x, n_y\}$, the energy eigenvalues are $E_n = (\hbar^2\pi^2/2ma^2)(n_x^2 + n_y^2)$ and $\psi_n(x,y) = (2/a)\sin(n_x\pi x/a)\sin(n_y\pi y/a)$ are the wave functions. The (nondegenerate) ground state is $n = \{1,1\}$, the (doubly denerage) first excited state is $n = \{1,2\}$ and $n = \{1,2\}$, and the (nondegenerate) third state is $n = \{2,2\}$.
(b) Let $E_1 = 2(\hbar^2\pi^2/2ma^2)$, $E_2 = 5(\hbar^2\pi^2/2ma^2)$, and $E_3 = 8(\hbar^2\pi^2/2ma^2)$ be the three unperturbed energies. For the first and third energy levels, the first order shifts are

$$\begin{aligned} \Delta_1^{(1)} &= \lambda\left[\frac{2}{a}\int_0^a u\sin^2\left(\frac{\pi u}{a}\right)du\right]^2 = \frac{1}{4}\lambda a^2 = \lambda a^2(0.250) \\ \Delta_3^{(1)} &= \lambda\left[\frac{2}{a}\int_0^a u\sin^2\left(\frac{2\pi u}{a}\right)du\right]^2 = \frac{1}{4}\lambda a^2 = \lambda a^2(0.250) \end{aligned}$$

For the degenerate first excited state, see the solution to Problem 3. The two energy shifts are $\Delta_{2a}^{(1)} = \lambda a^2(0.233)$ and $\Delta_{2b}^{(1)} = \lambda a^2(0.266, 0.233)$. The energy level diagram is



11. See Problem 1.11. Find $E_{1,2} = (E_1^0 + E_2^0)/2 \pm \sqrt{(E_1^0 - E_2^0)^2/4 + \lambda^2\Delta^2}$, and

$$\psi_1 = \begin{bmatrix} \cos(\beta/2) \\ \sin(\beta/2) \end{bmatrix} \quad \text{and} \quad \psi_2 = \begin{bmatrix} -\sin(\beta/2) \\ \cos(\beta/2) \end{bmatrix} \quad \text{where} \quad \tan\beta = \frac{2\lambda\Delta}{E_1^0 - E_2^0}$$

are the exact eigenstates. Now in terms of perturbation theory with $\lambda\Delta \ll (E_1^0 - E_2^0)$,

$$H_0 = \begin{bmatrix} E_1^0 & 0 \\ 0 & E_2^0 \end{bmatrix} \quad \text{with} \quad \phi_1^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; , \quad \phi_2^{(0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 0 & \lambda\Delta \\ \lambda\Delta & 0 \end{bmatrix}$$

The first order energy shifts are $\tilde{\phi}_1^{(0)} V \phi_1^{(0)} = 0 = \tilde{\phi}_2^{(0)} V \phi_2^{(0)}$. Use (5.1.42) for the second order shifts to find $\Delta_1^{(1)} = |\tilde{\phi}_2^{(0)} V \phi_1^{(0)}|^2/(E_1^0 - E_2^0) = \lambda^2\Delta^2/(E_1^0 - E_2^0)$ and $\Delta_2^{(1)} = \lambda^2\Delta^2/(E_2^0 - E_1^0)$ which agree with (5.1.14). From (5.1.44), the first order eigenstates are

$$\phi_1^{(1)} = \phi_1^{(0)} + \phi_2^{(0)} \frac{\tilde{\phi}_2^{(0)} V \phi_1^{(0)}}{E_1^0 - E_2^0} = \begin{bmatrix} 1 \\ \frac{\lambda\Delta}{E_1^0 - E_2^0} \end{bmatrix} \quad \text{and} \quad \phi_2^{(1)} = \phi_2^{(0)} + \phi_1^{(0)} \frac{\tilde{\phi}_1^{(0)} V \phi_2^{(0)}}{E_2^0 - E_1^0} = \begin{bmatrix} \frac{\lambda\Delta}{E_2^0 - E_1^0} \\ 1 \end{bmatrix}$$

In the limit $\lambda\Delta \ll (E_1^0 - E_2^0)$, these are clearly the same as the exact eigenstates because $\cos(\beta/2) \to 1$ and $\sin(\beta/2) \to \beta/2$. As for the energy eigenvalues in this limit,

$$E_{1,2} = \frac{E_1^0 + E_2^0}{2} \pm \frac{E_1^0 - E_2^0}{2}\sqrt{1 + \frac{4\lambda^2\Delta^2}{(E_1^0 - E_2^0)^2}} \to \frac{E_1^0 + E_2^0}{2} \pm \frac{E_1^0 - E_2^0}{2}\left[1 + \frac{2\lambda^2\Delta^2}{(E_1^0 - E_2^0)^2}\right]$$

and the energy shifts are $\pm\lambda^2\Delta^2/(E_1^0 - E_2^0)$, the same as the values $\Delta_{1,2}^{(1)}$ obtained from second order perturbation theory.

Now in the opposite limit, i.e. $\lambda\Delta \gg (E_1^0 - E_2^0)$, $\beta \to 90°$ and $\psi_{1,2} \to \begin{bmatrix} \pm 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, and

$$E_{1,2} = \frac{E_1^0 + E_2^0}{2} \pm \lambda\Delta\sqrt{1 + \frac{(E_1^0 - E_2^0)^2}{4\lambda^2\Delta^2}} \to \frac{E_1^0 + E_2^0}{2} \pm \lambda\Delta\left[1 + \frac{(E_1^0 - E_2^0)^2}{8\lambda^2\Delta^2}\right]$$

which means *first order* splits $\pm\lambda\Delta$ for the degenerate state with $E_1^0 = E_2^0$. Both of these are exactly what you get if you diagonalize the perturbation matrix V, that is, in agreement with the treatment of degenerate perturbation theory.

12. I will treat a and b as real numbers. It is useful to know where we're going, so let's find the exact eigenvalues λ first, which satisfy $(E_1 - \lambda)^2(E_2 - \lambda) - a^2(E_1 - \lambda) - b^2(E_1 - \lambda) = 0$. Therefor $\lambda = E_1 \equiv \lambda_0$ or $\lambda^2 - (E_1 + E_2)\lambda + E_1E_2 - (a^2 + b^2) = 0$, that is

$$\lambda = \frac{E_1 + E_2 \pm \sqrt{(E_2 - E_1)^2 + 4(a^2 + b^2)}}{2} \approx \frac{E_1 + E_2}{2} \pm \frac{E_2 - E_1}{2}\left[1 + 2\frac{a^2 + b^2}{(E_2 - E_1)^2}\right]$$

so the others are $\lambda = E_2 + (a^2 + b^2)/(E_2 - E_1) \equiv \lambda_+$ and $\lambda = E_1 - (a^2 + b^2)/(E_2 - E_1) \equiv \lambda_-$.

Now with $\phi_1^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\phi_2^{(0)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\phi_3^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, with $V = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a & b & 0 \end{bmatrix}$, the only nonzero matrix elements $V_{nk} = \tilde{\phi}_n^{(0)} V \phi_k^{(0)}$ are $V_{31} = a^2 = V_{13}$ and $V_{32} = b^2 = V_{23}$, so $V_{21} = 0 = V_{12}$ and the E_1 degeneracy is not removed in first order. If we use (5.1.42) to calculate the shift using non-generate second order perturbation theory, we get $\Delta_1 = a^2/(E_1 - E_2)$, $\Delta_2 = b^2/(E_1 - E_2)$, and $\Delta_3 = (a^2 + b^2)/(E_2 - E_1)$. The energies $E_n + \Delta_n$ do not agree with the values λ derived above, except for $E_2 + \Delta_3 = \lambda_-$. Non-degenerate perturbation theory is not applicable to degenerate states, even taking it to second order.

Use the formula in Problem 1, Page 397 of Gottfried (1966). The energy shifts Δ satisfy

$$\left(\Delta - \frac{a^2}{E_1 - E_2}\right)\left(\Delta - \frac{b^2}{E_1 - E_2}\right) = \frac{a^2 b^2}{(E_1 - E_2)^2}$$

That is, $\Delta = 0$ and $\Delta = (a^2 + b^2)/(E_1 - E_2)$. These energy shifts give eigenvalues that agree with λ_0 and λ_+ above, from the exact solution.

13. The perturbation is $V = -e\varepsilon z$, so $\langle 2S_{1/2}|V|2S_{1/2}\rangle = 0 = \langle 2P_{1/2}|V|2P_{1/2}\rangle$, which should be obvious from parity considerations. Also $\langle 2S_{1/2}|V|2P_{1/2}\rangle = 3ea_0\varepsilon$ from (5.2.19). So

$$H \doteq \begin{bmatrix} E_2 + \delta & 3ea_0\varepsilon \\ 3ea_0\varepsilon & E_2 \end{bmatrix} \qquad \text{and} \qquad (E_2 + \delta - E)(E_2 - E) - 9e^2 a_0^2 \varepsilon^2 = 0$$

gives the eigenvalues E, where δ is the Lamb shift. Solving for the eigenvalues gives

$$E = \frac{2E_2 + \delta \pm \sqrt{(2E_2 + \delta)^2 + 36e^2 a_0^2 \varepsilon^2 - 4E_2(E_2 + \delta)}}{2} = E_2 + \frac{\delta}{2} \pm \sqrt{\left(\frac{\delta}{2}\right)^2 + 9e^2 a_0^2 \varepsilon^2}$$

So, for $ea_0\varepsilon \ll \delta$, have $E \approx E_2 + (\delta/2)(1 \pm 18e^2 a_0^2 \varepsilon^2/\delta^2) = E_2 + (\delta/2) \pm 9e^2 a_0^2 \varepsilon^2/\delta$ and the energy shifts are quadratic in ε. For $ea_0\varepsilon \gg \delta$ find $\pm 3ea_0\varepsilon(1 + \delta^2/36e^2 a_0^2 \varepsilon^2) \approx \pm 3ea_0\varepsilon$ for the energy shifts, which are linear in ε.

Regarding time reversal, I will just quote from the original solutions manual. "Whereas parity restricts $\langle 2S_{1/2}|V|2S_{1/2}\rangle = 0 = \langle 2P_{1/2}|V|2P_{1/2}\rangle$, time reversal invariance of our Hamiltonian places no similar restriction. Nevertheless, for example from (4.4.84), it imposes the restriction that expectation value $\langle \mathbf{x} \rangle$ (hence $\langle z \rangle$ as a special case) vanishes when taken with respect to eigenstates of j, m. For example, $|j, m\rangle$ of our problem need not be parity eigenkets, and could be $c_S|2S_{1/2}\rangle + c_P|2P_{1/2}\rangle$. yet it remains true that $\langle j, m|\mathbf{x}|j, m\rangle = 0$ under time reversal invariance, i.e. no electric dipole moment."

14. This is the linear Stark effect (see pages 319 through 321) but with a larger (nine dimensional) degenerate subspace. Let us write $V = -ez\varepsilon = -e\varepsilon r\cos\theta$ and first determine the nonzero matrix elements $\langle 3l'm'|V|3lm\rangle$. Since $V \propto Y_1^0 \sim T_0^{(1)}$, from (3.11.28) the matrix elements are zero unless $m = m'$. Also, from the Wigner-Eckart Theorem (3.11.31), the matrix elements are proportional to $\langle l1; m0|l1; l'm\rangle$, so, as in (3.11.32), we need $|l-1| \leq l' \leq l+1$. Finally, since V is odd parity, the matrix elements are only nonzero between states where l and l' differ by an odd number, and we have $l, l' = 0, 1, 2$ in this problem. The integrals can be done using (B.6.3) with (B.5.7). I'll do the math with MATHEMATICA, which defines internally the spherical harmonics and associated Laguerre polynomials. Proceeding,

$$\begin{aligned}
\langle 321|V|311\rangle &= \frac{9}{2}e\varepsilon a_0 = \langle 311|V|321\rangle = \langle 32,-1|V|31,-1\rangle = \langle 31,-1|V|32,-1\rangle \\
\langle 320|V|310\rangle &= 3\sqrt{3}e\varepsilon a_0 = \langle 310|V|320\rangle \\
\langle 310|V|300\rangle &= 3\sqrt{6}e\varepsilon a_0 = \langle 300|V|310\rangle
\end{aligned}$$

which, for some reason, are all three times smaller than the values in the original solutions manual. The code which finds the wave functions and calculates (one of) the matrix elements follows. Note the convention MATHEMATICA uses for the associated Laguerre polynomials.

```
R1eAtom[n_, l_] := (2/n^2) Sqrt[1/a^3] Sqrt[(n - l - 1)!/(n + l)!]*
  Exp[-Z r/(n a)] (2 Z r/(n a))^l *
  LaguerreL[n - l - 1, 2 l + 1, 2 Z r/(n a)]
\[Psi][n_, l_, m_] :=
 R1eAtom[n, l] SphericalHarmonicY[l, m, \[Theta], \[Phi]]

Z = 1;
\[Psi]300 = \[Psi][3, 0, 0];
\[Psi]31p1 = \[Psi][3, 1, 1];
\[Psi]310 = \[Psi][3, 1, 0];
\[Psi]31m1 = \[Psi][3, 1, -1];
\[Psi]32p2 = \[Psi][3, 2, 2];
\[Psi]32p1 = \[Psi][3, 2, 1];
\[Psi]320 = \[Psi][3, 2, 0];
\[Psi]32m1 = \[Psi][3, 2, -1];
\[Psi]32m2 = \[Psi][3, 2, -2];

p = Simplify[Conjugate[\[Psi]32p1] r Cos[\[Theta]] \[Psi]31p1, a > 0 && r > 0];
V2p11p1 = Integrate[
  p r^2 Sin[\[Theta]], {r, 0, \[Infinity]}, {\[Theta], 0, Pi}, {\[Phi], 0,
   2 Pi}]
```

Writing the eigenvalues as $-3\lambda e\varepsilon a_0$, with $a \equiv 3/2$, $b \equiv \sqrt{3}$, and $c \equiv \sqrt{6}$, and labeling $\{l=0\{m=0\}, l=1\{m=-1,0,1\}, 2\{m=-2,-1,0,1,2\}\}$, the eigenvalue equation is

$$3e\varepsilon a_0 \begin{bmatrix} \lambda & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & a & 0 & 0 & 0 \\ c & 0 & \lambda & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \\ A_9 \end{bmatrix} = 0$$

The characteristic equation is $\lambda^3 \left(a^2-\lambda^2\right)^2 \left(b^2+c^2-\lambda^2\right) = 0$, so the energy shifts are

$$\begin{aligned} \Delta_{1,2,3} &= 0 \\ \Delta_{4,5} &= -3e\varepsilon a_0(-3/2) = 9e\varepsilon a_0/2 \\ \Delta_{6,7} &= -3e\varepsilon a_0(+3/2) = -9e\varepsilon a_0/2 \\ \Delta_8 &= -3e\varepsilon a_0(-3) = 9e\varepsilon a_0 \\ \Delta_9 &= -3e\varepsilon a_0(+3) = -9e\varepsilon a_0 \end{aligned}$$

and the corresponding eigenstates are

$$\begin{aligned} |1,2,3\rangle &= |32,\pm 2\rangle \qquad \text{and} \qquad \sqrt{\frac{2}{3}}|320\rangle - \sqrt{\frac{1}{3}}|300\rangle \\ |4,5\rangle &= \frac{1}{\sqrt{2}}\left[|321\rangle + |311\rangle\right] \qquad \text{and} \qquad \frac{1}{\sqrt{2}}\left[|32,-1\rangle + |31,-1\rangle\right] \\ |6,7\rangle &= \frac{1}{\sqrt{2}}\left[|321\rangle - |311\rangle\right] \qquad \text{and} \qquad \frac{1}{\sqrt{2}}\left[|32,-1\rangle - |31,-1\rangle\right] \\ |8\rangle &= \frac{1}{\sqrt{3}}|300\rangle + \frac{1}{\sqrt{2}}|310\rangle + \frac{1}{\sqrt{6}}|300\rangle \\ |9\rangle &= \frac{1}{\sqrt{3}}|300\rangle - \frac{1}{\sqrt{2}}|310\rangle + \frac{1}{\sqrt{6}}|300\rangle \end{aligned}$$

Following is the relevant MATHEMATICA code, with `mV` defined as the matrix above with zeros on the diagonal. Also, I'm including the code for only the first two normalized eigenvectors, just to save space.

```
Simplify[CharacteristicPolynomial[mV, \[Lambda]]] // TeXForm
Eigenvalues[mV] /. {a -> 3/2, b -> Sqrt[3], c -> Sqrt[6]}
eiv = Eigenvectors[mV] /. {a -> 3/2, b -> Sqrt[3], c -> Sqrt[6]};
Normalize[eiv[[1]]]
Normalize[eiv[[2]]]
```

15. *This problem is a bit open-ended, so I am copying the solution here pretty much directly from the original solutions manual.* For an electric dipole $\boldsymbol{\mu}_e = \mu_e \boldsymbol{\sigma}$, have $V = -\boldsymbol{\mu}_e \cdot \mathbf{E}$ where the $\mathbf{E} = -(1/e)\hat{\mathbf{r}}dV_c/dr$ and $V_c(r)$ is the Coulomb potential energy of the nucleus. Writing

$$\boldsymbol{\sigma} \cdot \hat{\mathbf{r}} = \frac{1}{r}\left[\sigma_+(x-iy) + \sigma_-(x+iy) + \sigma_z z\right] = \sqrt{\frac{4\pi}{3}}\left[\sqrt{2}(\sigma_+ Y_1^{-1} + \sigma_- Y_1^1) + \sigma_z Y_a^0\right]$$

we see that the Wigner-Eckart theorem tells us which matrix elements are nonzero. For $\Delta m = 0$, the matrix elements of Y_1^0 are needed, and these vanish unless $\Delta\ell = \pm 1$. For $\Delta m = \pm 1$, we still need $\Delta \ell = \pm 1$. This is expected since $\hat{\mathbf{r}}$ is a vector operator and connects states of different parity. The radial contribution is proportional to $\int_0^\infty R_{n\ell}\frac{dV_C}{dr}R_{n\ell'}r^2 dr = -\int_0^\infty R_{n\ell}R_{n\ell'}dr$. One may verify that for $\ell - \ell' = \pm 1$, this integral vanishes for $n = n'$.

The ground state of Na has $n = 3$ (degeneracy $n^2 = 9$), but from the above, we know that $\Delta n = 0$. Therefore the effects of this perturbation V on the energy levels are seen in second order. Mixing occurs between $3s$ and $4p$ states, between $4s$ and $3p$, $3d$, and $4p$, and so on. Using eigenstates of $\mathbf{L}^2$, L_z, $\mathbf{S}^2$, S_z, the following expression for $\langle 3s|V|4p\rangle$ is true for $\Delta\ell_z = 0$:

$$\begin{aligned}
\langle 3s|V|4p\rangle_{\Delta\ell_z=0} &= \frac{Z}{(-e)}\int_0^\infty dr R_{30}(r)R_{41}(r)\sqrt{\frac{4\pi}{3}}\left\langle 00\frac{1}{2}\frac{1}{2}|Y_1^0\sigma_z|10\frac{1}{2}\frac{1}{2}\right\rangle \\
&= -\frac{Z}{e}\int_0^\infty dr R_{30}R_{41}\sqrt{\frac{4\pi}{3}}\int_{-1}^{1} d\phi \int_0^{2\pi} d(\cos\theta)\sqrt{\frac{1}{4\pi}}\sqrt{\frac{3}{4\pi}}\cos\theta \\
&= -\frac{Z}{e}\sqrt{\frac{1}{3}}\int_0^\infty dr R_{30}R_{41} \equiv -\frac{Z}{e}\sqrt{\frac{1}{3}}I_R
\end{aligned}$$

Therefore, using (5.2.15), we determine the second (lowest) order shift in the $3s$ states of Na to be $\Delta_{3s} = \left(-\frac{\mu_e Z I_R}{e\sqrt{3}}\right)^2/(E_{n=3} - E_{n=4})$ where $E_n = -Z^2 m_e e^4/2\hbar^2 n^2$.

16. For any $l = 0$ state, we can write (real valued) $\psi(\mathbf{r}) = u(r)/r$ where u satisfies (3.7.9) and $u(r) \to 0$ as $r \to 0$ or $r \to \infty$. Multiply (3.7.9) by $du/dr \equiv u'(r)$ and rewrite it as

$$-\frac{\hbar^2}{2m}\frac{1}{2}\frac{d}{dr}(u')^2 + \frac{1}{2}\frac{d}{dr}(Vu^2) - \frac{1}{2}\frac{dV}{dr}u^2 = E\frac{1}{2}\frac{d}{dr}u^2$$

Now integrate over r from zero to ∞, leaving $-\frac{\hbar^2}{4m}(u')^2|_0^\infty - \frac{1}{2}\int_0^\infty \frac{dV}{dr}u^2 dr = 0$. Since $u = r\psi$, $u'(r) = \psi(r) + r\psi'(r)$, where both $\psi(r) \to 0$ and $\psi'(r) \to 0$ as $r \to \infty$. Therefore

$$\frac{\hbar^2}{4m}\psi(0)^2 = \frac{1}{2}\int_0^\infty \frac{dV}{dr}\psi^2(r)r^2 dr = \frac{1}{2}\frac{1}{4\pi}\left\langle\frac{dV}{dr}\right\rangle \qquad \text{so} \qquad \psi(0)^2 = \frac{m}{2\pi\hbar^2}\left\langle\frac{dV}{dr}\right\rangle$$

The wave function of the **hydrogen ground state** is, from (B.6.3), (B.6.7), and (B.5.7), $\psi(\mathbf{r}) = \exp(-r/a)/\sqrt{\pi a^3}$. So, $\psi(0)^2 = 1/\pi a^3$ and

$$\left\langle\frac{dV}{dr}\right\rangle = 4\pi\frac{e^2}{\pi a^3}\int_0^\infty \frac{1}{r^2}e^{-2r/a}r^2 dr = 4\pi\frac{\hbar^2/ma}{\pi a^3}\frac{a}{2} = \frac{2\pi\hbar^2}{m}\frac{1}{\pi a^3} = \frac{2\pi\hbar^2}{m}\psi(0)^2$$

For the **3D isotropic harmonic oscillator ground state** we have, from (3.7.8), (3.7.40), (3.7.31), and (3.7.33), $\psi(\mathbf{r}) = N\exp(-m\omega r^2/2\hbar)$ where the normalization constant was never determined. Nevertheless, $\psi(0)^2 = N^2$ and (using MATHEMATICA for the integral)

$$\left\langle \frac{dV}{dr} \right\rangle = 4\pi N^2 m\omega^2 \int_0^\infty r e^{-m\omega r^2/\hbar} r^2 dr = 4\pi N^2 m\omega^2 \frac{\hbar^2}{2m^2\omega^2} = \frac{2\pi\hbar^2}{m} N^2$$

17a. Using the notation of Section 5.1, we write $H_0 = A\mathbf{L}^2 + BL_z$, so $|n^{(0)}\rangle = |lm\rangle$, and $V = CL_y = (C/2i)(L_+ - L_-)$. So $E_{lm}^{(0)} = A\hbar^2 l(l+1) + B\hbar m$, $\Delta_{lm}^{(1)} = \langle n^{(0)}|V|n^{(0)}\rangle = 0$, and

$$\Delta_{lm}^{(2)} = \sum_{l'm'} \frac{|\langle lm|V|l'm'\rangle|^2}{E_{lm}^{(0)} - E_{l'm'}^{(0)}} = \frac{C^2\hbar^2}{4}\left[\frac{(l-m+1)(l+m)}{E_{lm}^{(0)} - E_{l,m-1}^{(0)}} + \frac{(l+m+1)(l-m)}{E_{lm}^{(0)} - E_{l,m+1}^{(0)}}\right] = \frac{C^2\hbar m}{2B}$$

Note that the problem can be solved exactly by rotating about the x-axis through an angle θ such that $\tan\theta = L_y/L_z = C/B$. That is, $H = A\mathbf{L}^2 + (B^2+C^2)^{1/2}L_z'$, with eigenvalues

$$E = A\hbar^2 l(l+1) + (B^2+C^2)^{1/2}\hbar m = A\hbar^2 l(l+1) + B\hbar m + \frac{C^2}{2B}\hbar m + \cdots$$

which agrees with the result from second order perturbation theory.

17b. First of all, both operators are spin independent, so $\Delta m_s = 0$. For the rest, see Problem (3.32). Since $3z^2 - r^2 = r^2(\cos^2\theta - 1) \propto Y_2^0$, this operator is a spherical tensor $T_0^{(2)}$, and $\langle l'm'|(3z^2 - r^2)|lm\rangle$ is proportional to $\langle l2; m0|l2; l'm'\rangle$ by the Wigner-Eckart Theorem (3.11.31). Therefore $m = m'$ and $|l-2| \le l' \le l+2$. (Also, $\Delta l = l - l'$ must be even since Y_2^0 has even parity.) Now $xy = r\sin^2\theta\cos\phi\sin\phi \propto \sin^2\theta\sin 2\phi \propto \left(Y_2^2 - Y_2^{-2}\right)$, i.e $T_2^{(2)} - T_{-2}^{(2)}$. So, similarly, $\langle l'm'|xy|lm\rangle$ is proportional to $\langle l2; m2|l2; l'm'\rangle$ if $m' - m = 2$, or proportional to $\langle l2; m, -2|l2; l'm'\rangle$ if $m' - m = -2$, and zero otherwise. The same rules hold for l and l'.

18. The perturbation is $V = e^2\mathbf{A}^2/2m_ec^2 = e^2B^2(x^2+y^2)/8m_ec^2$ where $\mathbf{A}$ is given by (5.3.33). For the spherically symmetric ground state $\langle x^2\rangle = \langle y^2\rangle = \langle z^2\rangle = \langle r^2\rangle/3$. Therefore

$$\Delta = \frac{e^2B^2}{8m_ec^2}\frac{2}{3}\frac{1}{\pi a_0^3}4\pi\int_0^\infty r^2 e^{-2r/a_0} r^2 dr = \frac{e^2B^2}{m_ec^2}\frac{1}{3}\frac{1}{a_0^3}\frac{4!}{(2/a_0)^5} = \frac{e^2B^2a_0^2}{4m_ec^2} = -\frac{1}{2}\chi B^2$$

so the diagmagnetic susceptibility is $\chi = -e^2a_0^2/2m_ec^2$.

19. This is a numerical comparison based on Problem 18; the two problems should be combined. With $a_0 \to a_0/Z_{\text{eff}}$, find $\chi = 2\times(-e^2a_0^2/2Z_{\text{eff}}^2 m_ec^2) = -e^2a_0^2/Z_{\text{eff}}^2 m_ec^2$ where we recognize that this is a two-electron atom, each of which behaves independently in this approximation. Put $e^2 = \hbar c/137 = (200/137)\text{MeV}\times 10^{-5}\text{Å}$, $a_0 = 0.53\text{Å}$, $m_ec^2 = 0.511$ MeV, and $Z_{\text{eff}} = 2 - 5/16 = 1.69$, to find $\chi = -0.281\times 10^{-5}\text{Å}^3/\text{atom}{=}1.69\times 10^{-6}\text{cm}^3/\text{mole}$, in good agreement with the measured value of $1.88\times 10^{-6}\text{cm}^3/\text{mole}$.

20. Follow (5.4.1) and calculate the approximate energy $\bar{H}(\beta)$ as follows:

$$\bar{H} = \frac{(-\hbar^2/2m)\int_{-\infty}^{\infty}\exp(-\beta|x|)\frac{d^2}{dx^2}\exp(-\beta|x|)dx + \int_{-\infty}^{\infty}\exp(-2\beta|x|)(m\omega^2x^2/2)dx}{\int_{-\infty}^{\infty}\exp(-2\beta|x|)dx}$$

The denominator is $\int_{-\infty}^{\infty}\exp(-2\beta|x|)dx = 2\int_0^{\infty}\exp(-2\beta x)dx = 1/\beta$, and the second integral in the numerator is $\int_{-\infty}^{\infty}\exp(-2\beta|x|)(m\omega^2x^2/2)dx = m\omega^2\int_0^{\infty}\exp(-2\beta x)x^2dx = m\omega^2(2/8\beta^3)$. The first integral in the numerator is tricky since the first derivative is discontinuous. Write

$$\int_{-\infty}^{\infty} e^{-\beta|x|}\frac{d^2}{dx^2}e^{-\beta|x|}dx = 2\int_{\epsilon}^{\infty} e^{-\beta x}\frac{d^2}{dx^2}e^{-\beta x}dx + \int_{-\epsilon}^{\epsilon} e^{-\beta|x|}\frac{d^2}{dx^2}e^{-\beta|x|}dx$$

and let $\epsilon \to 0$. The first term is just β. For the second, the integrand factor $e^{-\beta|x|} \to 1$ leaving the integral $\int_{-\epsilon}^{\epsilon}\frac{d^2}{dx^2}e^{-\beta|x|}dx = \frac{d}{dx}e^{-\beta|x|}\Big|_{-\epsilon}^{\epsilon} = -\beta e^{-\beta\epsilon} - (+\beta)e^{-\beta\epsilon} \to -2\beta$. Therefore

$$\bar{H} = \frac{(-\hbar^2/2m)\beta + (-\hbar^2/2m)(-2\beta) + m\omega^2(2/8\beta^3)}{1/\beta} = \frac{\hbar^2\beta^2}{2m} + \frac{m\omega^2}{4\beta^2}$$

Putting $d\bar{H}/d\beta = \hbar^2\beta/m - m\omega^2/2\beta^3 = 0$ find $\beta^2 = m\omega/\hbar\sqrt{2}$, so the minimum value of $\bar{H}$ is $\bar{H}_{\min} = \hbar\omega/2\sqrt{2} + \hbar\omega\sqrt{2}/4 = \hbar\omega\sqrt{2}/2$, compared to the correct ground state energy $\hbar\omega/2$.

21. Solve this using (5.4.1) as for a "Hamiltonian" $H = -d^2/dx^2 + |x|$ with eigenvalue λ and trial wave function $\psi(x) = c(\alpha - |x|)$ for $|x| \leq \alpha$ and zero otherwise. The denominator is $2\int_0^{\alpha}c^2(\alpha - x)^2dx = 2\alpha^3c^2/3$. For the numerator, we need to deal with the discontinuity in $d\psi/dx$ at $x = 0$. Skipping this for the moment, we take $d^2\psi/dx^2 = 0$ for $|x| > 0$ and are left with $2\int_0^{\alpha}c^2(\alpha - x)^2xdx = \alpha^4c^2/6$. For the discontinuity, see Problem 20. We need

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}\psi\frac{d^2\psi}{dx^2}dx = \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}c\alpha\frac{d^2\psi}{dx^2}dx = \lim_{\epsilon\to 0}c\alpha\frac{d\psi}{dx}\Big|_{-\epsilon}^{\epsilon} = c\alpha[-c-(+c)] = -2c^2\alpha$$

Therefore $\bar{H} = (2\alpha + \alpha^4/6)/(2\alpha^3/3) = 3/\alpha^2 + \alpha/4$. Minimizing, $d\bar{H}/d\alpha = -6/\alpha^3 + 1/4 = 0$, so $\alpha = 24^{1/3} = 2\sqrt[3]{3}$ and $\bar{H}_{\min} = 3/4\sqrt[3]{9} + \sqrt[3]{3}/2 = 3\sqrt[3]{3}/4 = 1.0817$, indeed larger than 1.019.

22. From (5.7.17) we have $c_0^{(0)} = 1$ and, for lowest nonvanishing order,

$$\begin{aligned} c_n^{(1)} &= -\frac{i}{\hbar}\int_0^t e^{i(E_n-E_0)t'/\hbar}\langle n|F_0x\cos\omega t'|0\rangle dt' = -\frac{i}{\hbar}F_0\sqrt{\frac{\hbar}{2m\omega_0}}\delta_{n1}\int_0^t e^{i\omega_0t'}\cos\omega t'dt' \\ \text{so}\quad c_1^{(1)} &= -\frac{i}{2\hbar}F_0\sqrt{\frac{\hbar}{2m\omega_0}}\int_0^t e^{i\omega_0t'}\left(e^{i\omega t'} + e^{-i\omega t'}\right)dt' \\ &= -\frac{1}{2\hbar}F_0\sqrt{\frac{\hbar}{2m\omega_0}}\left[\frac{e^{i(\omega_0+\omega)t}-1}{\omega_0+\omega} + \frac{e^{i(\omega_0-\omega)t}-1}{\omega_0-\omega}\right] \equiv c_1(t) \end{aligned}$$

and $c_n^{(1)} = 0$ for all $n \neq 1$. Therefore, from (5.5.13), $|\alpha, t\rangle_I = |0\rangle + c_1(t)|1\rangle$, and so, from (5.5.5), $|\alpha, t\rangle_S = e^{-iH_0 t/\hbar}|\alpha, t\rangle_I = e^{-i\omega_0 t/2}|0\rangle + c_1(t)e^{-3i\omega_0 t/2}|1\rangle$. We can now calculate

$$
\begin{aligned}
\langle x\rangle &= \langle \alpha, t|x|\alpha, t\rangle_S \\
&= \left[e^{i\omega_0 t/2}\langle 0| + c_1^* e^{3i\omega_0 t/2}\langle 1|\right]\sqrt{\frac{\hbar}{2m\omega_0}}\left[a + a^\dagger\right]\left[e^{-i\omega_0 t/2}|0\rangle + c_1 e^{-3i\omega_0 t/2}|1\rangle\right] \\
&= \left[e^{i\omega_0 t/2}\langle 0| + c_1^* e^{3i\omega_0 t/2}\langle 1|\right]\sqrt{\frac{\hbar}{2m\omega_0}}\left[c_1 e^{-3i\omega_0 t/2}|0\rangle + e^{-i\omega_0 t/2}|1\rangle + c_1 e^{-3i\omega_0 t/2}\sqrt{2}|2\rangle\right] \\
&= \sqrt{\frac{\hbar}{2m\omega_0}}\left[c_1 e^{-i\omega_0 t} + c_1^* e^{i\omega_0 t}\right] \\
&= -\frac{F_0}{2\hbar}\frac{\hbar}{2m\omega_0}\left[\frac{e^{i\omega t} - e^{-i\omega_0 t}}{\omega_0 + \omega} + \frac{e^{-i\omega t} - e^{-i\omega_0 t}}{\omega_0 - \omega} + \frac{e^{-i\omega t} - e^{i\omega_0 t}}{\omega_0 + \omega} + \frac{e^{i\omega t} - e^{i\omega_0 t}}{\omega_0 - \omega}\right] \\
&= -\frac{F_0}{2m\omega_0}\left[\frac{\cos\omega t - \cos\omega_0 t}{\omega_0 + \omega} + \frac{\cos\omega t - \cos\omega_0 t}{\omega_0 - \omega}\right] = -\frac{F_0}{m}\frac{\cos\omega t - \cos\omega_0 t}{\omega_0^2 - \omega^2}
\end{aligned}
$$

Classically, this is a harmonic oscillator with a "forcing function" $F(t) = -F_0\cos\omega t$. The classical equation of motion is $m\ddot{x} = -m\omega_0^2 x + F(t)$ or $\ddot{x} + \omega_0^2 x = -(F_0/m)\cos\omega t$. The homogeneous solution is $x_h(t) = A\cos\omega_0 t + B\sin\omega_0 t$. A particular solution $x_p(t) = C\cos\omega t$ implies $C(-\omega^2 + \omega_0^2) = -F_0/m$, or $C = -(F_0/m)/(\omega_0^2 - \omega^2)$. Therefore

$$
x(t) = x_h(t) + x_p(t) = A\cos\omega_0 t + B\sin\omega_0 t - \frac{F_0}{m}\frac{\cos\omega t}{\omega_0^2 - \omega^2}
$$

is the general classical solution. For $t \le 0$ the oscillator has $\langle x\rangle = 0 = \langle p\rangle$, so for initial conditions take $x(0) = 0 = \dot{x}(0)$. Hence $A = (F_0/m)/(\omega_0^2 - \omega^2)$ and $B = 0$, and we see that the classical solution is the same as the quantum mechanical result for $\langle x\rangle$ as a function of time. Of course, the result is invalid at "resonance", i.e. $\omega = \omega_0$, since the response tends to infinity for any finite F_0 and perturbation theory breaks down.

23. The probability for the transition $|0\rangle \to |n\rangle$ from (5.7.19) is $|c_n^{(1)}|^2$, to first order, where $c_n^{(1)}$ is given by (5.7.17). In this case, $V(x,t) = -F_0 x e^{-t/\tau}$ for $t \ge 0$, but zero for $t < 0$. So,

$$
\begin{aligned}
c_n^{(1)} &= \frac{i}{\hbar}\int_0^t e^{i(E_n - E_0)t'/\hbar}\langle n|F_0 x e^{-t'/\tau}|0\rangle dt' = \frac{i}{\hbar}F_0\sqrt{\frac{\hbar}{2m\omega_0}}\delta_{n1}\int_0^t e^{i\omega_0 t'}e^{-t'/\tau}dt' \\
\text{so}\quad c_1^{(1)} &= \frac{i}{\hbar}F_0\sqrt{\frac{\hbar}{2m\omega_0}}\frac{e^{(i\omega_0 - 1/\tau)t} - 1}{i\omega_0 - 1/\tau} \qquad \text{and} \qquad c_n^{(1)} = 0 \quad \text{for} \quad n \ge 2 \\
\text{Hence}\quad \left|c_1^{(1)}\right|^2 &= \frac{F_0^2}{2m\omega_0}\frac{e^{-2t/\tau} - 2e^{-t/\tau}\cos\omega_0 t + 1}{\omega_0^2 + 1/\tau^2} \longrightarrow \frac{1}{2m\omega_0}\frac{F_0^2\tau^2}{1 + \omega_0^2\tau^2} \quad \text{for } t \to \infty
\end{aligned}
$$

which is independent of t, not unexpected since the force turns off. More interestingly, the result only depends on the (finite) impulse $F_0\tau$ as $\tau \to 0$. Higher excited states are not possible in first order, but from the expression for $c_n^{(2)}$ in (5.7.17), we see that $|2\rangle$ can be reached in second order. Apparently, $|0\rangle \to |n\rangle$ transitions can occur in nth order.

24. Follow the solution to Problems 22 and (especially) 23. Also use Problem 2.14 which gives $\langle m|x^2|n\rangle = (\hbar/2m\omega_0)[\sqrt{n(n-1)}\delta_{n-2,m} + (2n+1)\delta_{nm} + \sqrt{(n+1)(n+2)}\delta_{n+2,m}]$. So,

$$\begin{aligned} c_n^{(1)} &= -\frac{i}{\hbar}A\frac{\hbar}{2m\omega_0}\sqrt{2}\delta_{n2}\int_0^t e^{i\omega_0 t'}e^{-t'/\tau}dt' = -\frac{iA}{m\omega_0\sqrt{2}}\delta_{n2}\frac{e^{(i\omega_0-1/\tau)t}-1}{i\omega_0-1/\tau} \\ \text{and} \quad \left|c_2^{(1)}\right|^2 &= \frac{A^2}{2m^2\omega_0^2}\frac{e^{-2t/\tau}-2e^{-t/\tau}\cos\omega_0 t+1}{\omega_0^2+1/\tau^2} \longrightarrow \frac{1}{2m^2\omega_0^2}\frac{A^2\tau^2}{1+\omega_0^2\tau^2} \quad \text{for } t\to\infty \end{aligned}$$

and transitions (to first order) to states other than $|n\rangle = |2\rangle$ do not occur. Apparently, however, from (5.7.17), $|0\rangle \to |2n\rangle$ transitions can occur in nth order, but transitions to states with odd n are forbidden.

25. I first address whether or not this problem can be solved exactly. (The original solutions manual says it can be.) Indeed, it is essentially the "magnetic resonance" problem, for an oscillating field $B_x\cos\omega t$ in a constant, stronger holding field B_z. Many books solve this problem, but in the approximation $\omega \approx (E_1^0 - E_2^0)/\hbar$, that is, near resonance. If, instead, the oscillating field rotates, that is $B_x\cos\omega t + B_y\sin\omega t$, then you can find an exact solution for all ω. Just expand $|\alpha,t\rangle = a(t)|\uparrow\rangle + b(t)|\downarrow\rangle$, substitute into (2.1.27), and express as a matrix to arrive at coupled, first order differential equations for $a(t)$ and $b(t)$. With the rotating field, the ansatz $a(t) = a_0\exp(+i\omega t/2)$ and $b(t) = b_0\exp(-i\omega t/2)$ leads to finding a_0 and b_0 by enforcing nontrivial solutions to the homogenous linear equations. This does not work, though, for the linearly oscillating field, and I cannot find any other solutions.

So until an exact solution is found (or I delete this problem in favor of 5.30) I will not be able to compare the perturbation theory result to the exact solution. Too bad, that would be instructive. For now, then, just do the first order perturbation solution. We then have

$$|\alpha,t\rangle = \exp(-iE_1^0 t/\hbar)|\uparrow\rangle + c_2^{(1)}(t)\exp(-iE_2^0 t/\hbar)|\downarrow\rangle$$

from (5.5.4) and (5.5.17). With $\omega_0 \equiv (E_1^0 - E_2^0)/\hbar$ and $c_2^{(1)}(t) = (-i/\hbar)\int_0^t e^{i\omega_0 t'}\lambda\cos\omega t'$, find

$$\left|c_2^{(1)}(t)\right|^2 = \frac{\lambda^2}{\hbar^2}\left[\frac{\sin^2(\omega_0+\omega)t/2}{(\omega_0+\omega)^2} + \frac{\sin^2(\omega_0-\omega)t/2}{(\omega_0-\omega)^2} + \frac{\cos\omega t(\cos\omega t - \cos\omega_0 t)}{(\omega_0-\omega)^2}\right]$$

If $E_1^0 - E_2^0$ is close to $\pm\hbar\omega$ then one or more denominators in the above expression goes to zero and the coefficient is large, so the perturbation expansion breaks down. Of course, this is just the resonance condition.

26. The perturbation is $V = -F(t)x$ and $\omega_{10} = [(3/2)\hbar\omega - (1/2)\hbar\omega]/\hbar = \omega$, so from (5.7.17)

$$c_1^{(1)}(\infty) = -\frac{i}{\hbar}\left(-\frac{F_0\tau}{\omega}\right)\langle 1|x|0\rangle \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\tau^2+t^2}dt = \frac{i}{\hbar}\frac{F_0\tau}{\omega}\sqrt{\frac{\hbar}{2m\omega}}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{\tau^2+t^2}dt$$

The integral can be done using complex analysis; see, for example, Section 6.2 of this textbook. Since $\omega > 0$, replace the integral with a semicircular contour integral closed in the upper plane. The contribution along the curved portion tends to zero as $t \to i\infty$, leaving the integral we're after. Only the pole at $t = +i\tau$ contributes, so by the residue theorem, the integral equals $2\pi i e^{i\omega(i\tau)}/(i\tau + i\tau) = (\pi/\tau)e^{-\omega\tau}$, and $|c_1^{(1)}(\infty)|^2 = (\pi^2 F_0^2/2m\hbar\omega^3)e^{-2\omega\tau}$.

Does it make sense that the probability is zero for $\tau \gg 1/\omega$, even though the impulse is independent of τ? Yes, it does. In this limit, the perturbation is turned on and then off very slowly, and the oscillator is, essentially, always in the ground state.

27. Once again, use (5.7.17). Insert $\int dx|x\rangle\langle x|$ and change variables in the t' integral to get

$$c_f^{(1)} = -\frac{i}{\hbar}\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}dx\; e^{i\omega_{fi}t}A\delta(x-ct')u_f^*(x)u_i(x) = -\frac{iA}{\hbar c}\int_{-\infty}^{\infty}dx\; e^{i\omega_{fi}x/c}u_f^*(x)u_i(x)$$

and the probability to end up in state $|f\rangle$ is $|c_f^{(1)}|^2$, where $\omega_{fi} \equiv (E_f - E_i)/\hbar$. *The following explanation copied from the original solutions manual.* The pulse can be regarded as a superposition of the form $e^{i\omega x/c}e^{-\omega t}$ with $\omega > 0$ (absorption) and $\omega < 0$ (emission). Our result shows that the traveling pulse can give up energy $\hbar\omega_{fi}$ so that the particle gets excited to $|f\rangle$, and that only that part of the harmonic perturbation with the "right" frequency is relevant, as expected from energy conservation. Note that the space integral $\int u_f^* u_i dx$ is identical to the case where only one frequency component is present.

28. The potential energy ($e < 0$) is $V = -eE_0 z e^{-t/\tau}$. The matrix element $\langle 200|z|100\rangle = 0$ by parity symmetry, and $\langle 21,\pm 1|z|100\rangle = 0$ by the Wigner-Eckart Theorem. I calculate with MATHEMATICA $\langle 210|z|100\rangle = (128\sqrt{2}/243)a_0$. Returning once again to (5.7.17), we have

$$\begin{aligned} c_{2\mathrm{p}}^{(1)}(t) &= -\frac{i}{\hbar}\int_0^t dt'(-eE_0)\frac{128\sqrt{2}}{243}a_0 e^{(i\omega-1/\tau)t'} = \frac{2^{15/2}}{3^5}\frac{ieE_0a_0}{\hbar}\frac{e^{(i\omega-1/\tau)t}-1}{i\omega-1/\tau} \\ \left|c_{2\mathrm{p}}^{(1)}(t)\right|^2 &= \frac{2^{15}}{3^{10}}\frac{e^2E_0^2a_0^2}{\hbar^2}\frac{e^{-2t/\tau}-2e^{-t/\tau}\cos\omega t+1}{\omega^2+1/\tau^2} \to \frac{2^{15}}{3^{10}}\frac{e^2E_0^2a_0^2}{\hbar^2}\frac{1}{\omega^2+1/\tau^2} \end{aligned}$$

with $\omega = (E_2 - E_1)/\hbar = -(e^2/2a_0\hbar)(-3/4) = 3e^2/8a_0\hbar$ and $a_0 = \hbar^2/m_e e^2$, as $t \to \infty$. In the limit $\tau \to \infty$, that is, a step function perturbation, the probability of transition is

$$\left|c_{2\mathrm{p}}^{(1)}(\infty)\right|^2 = \frac{2^{15}}{3^{10}}\frac{e^2E_0^2a_0^2}{\hbar^2}\frac{1}{\omega^2} = \frac{2^{21}}{3^{12}}\frac{E_0^2a_0^4}{e^2}$$

29. Note that $\mathbf{S}_1\cdot\mathbf{S}_2 = (\mathbf{S}^2-\mathbf{S}_1^2-\mathbf{S}_2^2)/2 = \hbar^2/4$ for triplet state $|1,0\rangle = [|+-\rangle+|-+\rangle]/\sqrt{2}$, and $= -3\hbar^2/4$ for a singlet state $|0,0\rangle = [|+-\rangle-|-+\rangle]/\sqrt{2}$. Therefore $H|1,0\rangle = +\Delta|1,0\rangle$ and $H|0,0\rangle = -3\Delta|0,0\rangle$. Also, for $t\le 0$, the system is in the state $|+-\rangle = [|1,0\rangle+|0,0\rangle]/\sqrt{2}$.

To solve exactly, go back to basics and use (2.1.5), using (2.1.28) and initial conditions at $t_0 = 0$. That is, put $|\alpha,0\rangle = |+-\rangle = [|1,0\rangle+|0,0\rangle]/\sqrt{2}$, so that at time t the state vector is

$$
\begin{aligned}
|\alpha,t\rangle &= e^{-iHt/\hbar}|\alpha,0\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\Delta t/\hbar}|1,0\rangle + e^{3i\Delta t/\hbar}|0,0\rangle\right] \qquad \text{Therefore,}\\
|\langle +-|\alpha,t\rangle|^2 &= \frac{1}{4}\left|e^{-i\Delta t/\hbar}+e^{3i\Delta t/\hbar}\right|^2 = \frac{1+\cos(4\Delta t/\hbar)}{2} \quad ,\\
|\langle -+|\alpha,t\rangle|^2 &= \frac{1}{4}\left|e^{-i\Delta t/\hbar}-e^{3i\Delta t/\hbar}\right|^2 = \frac{1-\cos(4\Delta t/\hbar)}{2} \quad ,
\end{aligned}
$$

and $|\langle ++|\alpha,t\rangle|^2 = 0 = |\langle --|\alpha,t\rangle|^2$.

For perturbation theory, back to (5.7.17). For the states $|n\rangle = \{|++\rangle,|+-\rangle,|-+\rangle,|--\rangle\}$, we need the matrix elements $\langle n|H|+-\rangle = [|1,0\rangle+|0,0\rangle]/\sqrt{2}$. Also note that $|++\rangle = |1,1\rangle$ and $|--\rangle = |1,-1\rangle$. Therefore $\langle ++|H|+-\rangle = 0 = \langle --|H|+-\rangle$ and there is no (first order) transition to these states, in agreement with the exact solution. We are left with

$$
c^{(1)}_{-+}(t) = -\frac{i}{\hbar}\int_0^t e^{i(0)t'}\langle -+|H|+-\rangle dt' = -\frac{i}{\hbar}\frac{t}{2}\left[\langle 1,0|-\langle 0,0|\right]\left[\Delta|1,0\rangle - 3\Delta|0,0\rangle\right] = -\frac{2i\Delta t}{\hbar}
$$

so the transition probability is $|c^{(1)}_{-+}(t)|^2 = 4\Delta^2t^2/\hbar^2$. Expanding the exact solution for small times gives $|\langle -+|\alpha,t\rangle|^2 = (1/2)(4\Delta t/\hbar)^2/2 = 4\Delta^2t^2/\hbar^2$, in agreement with first order perturbation theory. In the sense that the probabilities sum to one, this also agrees with the probability $|\langle +-|\alpha,t\rangle|^2$, but applying the first order formula in (5.7.17) to the state $|n\rangle = |+-\rangle = |i\rangle$ does not give agreement.

30. The exact solution follows from (5.5.15); note the missing "=" sign. We have

$$
\begin{aligned}
i\hbar\dot{c}_1 &= V_{12}(t)e^{i(E_1-E_2)t/\hbar}c_2 = \gamma e^{i(\omega-\omega_0)t}c_2 \qquad \text{with} \qquad c_1(0)=1\\
\text{and} \qquad i\hbar\dot{c}_2 &= V_{21}(t)e^{i(E_2-E_1)t/\hbar}c_1 = \gamma e^{-i(\omega-\omega_0)t}c_1 \qquad \text{with} \qquad c_2(0)=0
\end{aligned}
$$

where $\omega_0 \equiv (E_2-E_1)/\hbar = \omega_{21}$ and $|c_1(t)|^2+|c_2(t)|^2 = 1$. Recast these equations using the changes $c_1(t) = a_1(t)e^{i(\omega-\omega_0)t/2}$ and $c_2(t) = a_2(t)e^{-i(\omega-\omega_0)t/2}$ with $|a_1(t)|^2+|a_2(t)|^2 = 1$ to find

$$
\begin{aligned}
i\hbar\dot{a}_1 - \hbar[(\omega-\omega_0)/2]a_1 &= \gamma a_2 \qquad \text{with} \qquad a_1(0)=1\\
\text{and} \qquad i\hbar\dot{a}_2 + \hbar[(\omega-\omega_0)/2]a_2 &= \gamma a_1 \qquad \text{with} \qquad a_2(0)=0
\end{aligned}
$$

To solve, write $a_1 = a_1^0e^{i\Omega t}$ and $a_2 = a_2^0e^{i\Omega t}$ for constants a_1^0 and a_2^0. Then

$$
\begin{aligned}
\hbar[\Omega+(\omega-\omega_0)/2]a_1^0 + \gamma a_2^0 &= 0\\
\gamma a_1^0 + \hbar[\Omega-(\omega-\omega_0)/2]a_2^0 &= 0
\end{aligned}
$$

Nontrivial solutions are only possible for $\Omega = \pm[\gamma^2/\hbar^2 + (\omega-\omega_0)^2/4]^{1/2}$. Taking $\Omega > 0$ we write the solutions as $a_1(t) = \alpha e^{i\Omega t} + \beta e^{-i\Omega t}$ and $a_2(t) = r_\alpha \alpha e^{i\Omega t} + r_\beta \beta e^{-i\Omega t}$ where

$$r_\alpha = \frac{\Omega + (\omega-\omega_0)/2}{\gamma/\hbar} = -\frac{\gamma/\hbar}{\Omega - (\omega-\omega_0)/2} \text{ and } r_\beta = -\frac{\Omega - (\omega-\omega_0)/2}{\gamma/\hbar} = \frac{\gamma/\hbar}{\Omega + (\omega-\omega_0)/2}$$

Now $a_1(0) = \alpha + \beta = 1$ and $a_2(0) = r_\alpha\alpha + r_\beta\beta = 0$, so $\alpha - r_\alpha\alpha/r_\beta = \alpha(1 - r_\alpha/r_\beta) = 1$. Therefore $a_2(t) = 2ir_\alpha\alpha \sin\Omega t$ where

$$2ir_\alpha\alpha = 2i\frac{r_\alpha}{1 - r_\alpha/r_\beta} = 2i\frac{r_\alpha r_\beta}{r_\beta - r_\alpha} = \frac{-2i}{2\Omega/(\gamma/\hbar)} = \frac{\gamma}{i\hbar\Omega}$$

We therefore have $i\hbar\dot{c}_2(0) = i\hbar a_2^0\Omega = \gamma c_1(0) = \gamma$ and

$$c_2(t) = \frac{\gamma}{i\hbar\Omega}e^{-i(\omega-\omega_0)t/2}\sin\Omega t \quad \text{so} \quad |c_2(t)|^2 = \frac{\gamma^2}{\hbar^2\Omega^2}\sin^2\Omega t \quad \text{and} \quad |c_1(t)|^2 = 1 - |c_2(t)|^2$$

which agree with the solution given in the problem and also in (5.5.21). We can also find $c_1(t)$ directly, as $\alpha = r_\beta/(r_\beta - r_\alpha)$ and $\beta = -r_\alpha/(r_\beta - r_\alpha)$, so

$$c_1(t) = (\alpha e^{i\Omega t} + \beta e^{-i\Omega t})e^{i(\omega-\omega_0)t/2} = \frac{\gamma}{2\hbar\Omega}(r_\beta e^{i\Omega t} + r_\alpha e^{-i\Omega t})e^{i(\omega-\omega_0)t/2}$$

It is worthwhile to check that the normalization condition is maintained. We have

$$\begin{aligned}
|c_1(t)|^2 &= \frac{\gamma^2}{4\hbar^2\Omega^2}\left[r_\beta^2 + r_\alpha^2 + r_\alpha r_\beta(e^{2i\Omega t} + e^{-2i\Omega t})\right] = \frac{\gamma^2}{4\hbar^2\Omega^2}\left[r_\beta^2 + r_\alpha^2 - 2\cos 2\Omega t\right] \\
&= \frac{\gamma^2}{4\hbar^2\Omega^2}\left[\frac{\Omega + (\omega-\omega_0)/2}{\Omega - (\omega-\omega_0)/2} + \frac{\Omega - (\omega-\omega_0)/2}{\Omega + (\omega-\omega_0)/2} - 2\cos^2\Omega t + 2\sin^2\Omega t\right] \\
&= \frac{\gamma^2}{2\hbar^2\Omega^2}\frac{\hbar^2}{\gamma^2}\left[\Omega^2 + \frac{(\omega-\omega_0)^2}{4} + \frac{\gamma^2}{\hbar^2}\left(1 - 2\sin^2\Omega t\right)\right] \\
&= \frac{1}{2\Omega^2}\left[2\Omega^2 - 2\frac{\gamma^2}{\hbar^2}\sin^2\Omega t\right] = 1 - \frac{\gamma^2}{\hbar^2\Omega^2}\sin^2\Omega t = 1 - |c_2(t)|^2
\end{aligned}$$

So much for the exact solution. For perturbation theory, go back to (5.7.17), that is

$$\begin{aligned}
c_2^{(1)} &= -\frac{i}{\hbar}\int_0^t e^{i\omega_0 t'}\gamma e^{-i\omega t'}dt' = \gamma\frac{e^{-i(\omega-\omega_0)} - 1}{\hbar(\omega-\omega_0)} \quad \text{and therefore,} \\
\left|c_2^{(1)}\right|^2 &= \frac{\gamma^2}{\hbar^2(\omega-\omega_0)^2}[2 - 2\cos(\omega-\omega_0)t] = \frac{\gamma^2}{\hbar^2\frac{(\omega-\omega_0)^2}{4}}\sin^2\left(\frac{\omega-\omega_0}{2}t\right)
\end{aligned}$$

This agrees with the exact solution only for $\gamma \ll |\omega-\omega_0|/2$, in which case $\Omega \approx |\omega-\omega_0|/2$. This makes sense. For $\omega \approx \omega_0$, that is near resonance, the effect will be large and we don't expect perturbation theory to hold.

31. Take $V(t) \to V e^{\eta t}$ and consider the second order expression in (5.7.17). Then

$$\begin{aligned}
c_n^{(2)}(t) &= \left(\frac{-i}{\hbar}\right)^2 \sum_m \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' V_{nm} e^{(i\omega_{nm}+\eta)t'} V_{mi} e^{(i\omega_{mi}+\eta)t''} \\
&= \left(\frac{-i}{\hbar}\right)^2 \sum_m V_{nm} V_{mi} \frac{1}{i\omega_{mi}+\eta} \int_{-\infty}^{t} dt' e^{(i\omega_{ni}+2\eta)t'} \\
&= \left(\frac{i}{\hbar}\right)^2 \frac{e^{(i\omega_{ni}+2\eta)t}}{i\omega_{ni}+2\eta} \sum_m \frac{V_{nm}V_{mi}}{i\omega_{mi}+\eta} = \frac{e^{i\omega_{ni}t}e^{2\eta t}}{E_n - E_i - 2i\hbar\eta} \sum_m \frac{V_{nm}V_{mi}}{E_m - E_i - i\hbar\eta} \\
&= \sum_m \frac{V_{nm}V_{mi}}{(E_m - E_i - i\hbar\eta)(E_n - E_i - 2i\hbar\eta)} + \frac{e^{i\omega_{ni}t}e^{2\eta t} - 1}{E_n - E_i - 2i\hbar\eta} \sum_m \frac{V_{nm}V_{mi}}{E_m - E_i - i\hbar\eta}
\end{aligned}$$

Now we can let $\eta \to 0$. The first term above becomes the first term in (5.7.36). In the text, the second term in (5.7.36) is argued to give no contribution to a transition probability that grows with t. However, this is not the case for the second term above. Letting $\eta \to 0$ gives the factor $[e^{i\omega_{ni}t} - 1]/(E_n - E_i) \to t/\hbar$ as $\omega_{ni} \to 0$.

32. The eigenvalues and eigenstates of $H_0 = A\mathbf{S}_1 \cdot \mathbf{S}_2 = (A/2)(\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S} - 2^2)$ are well known. They are $-3A\hbar^2/4$ for $|0,0\rangle = [|+-\rangle - |-+\rangle]/\sqrt{2} \equiv |1^{(0)}\rangle$, and $A\hbar^2/4$ for $|1,0\rangle = [|+-\rangle + |-+\rangle]/\sqrt{2} \equiv |2^{(0)}\rangle$, $|1,-1\rangle = |--\rangle \equiv |3^{(0)}\rangle$, and $|1,+1\rangle = |++\rangle \equiv |4^{(0)}\rangle$. The perturbation $V = (eB/m_e c)(S_{1_z} - S_{2_z})$ gives no first order energy shifts, since all matrix elements $\langle n^{(0)}|V|n^{(0)}\rangle = 0$. This is easy to see, since $(S_{1_z} - S_{2_z})|1^{(0)}\rangle = \hbar|2^{(0)}\rangle$, $(S_{1_z} - S_{2_z})|2^{(0)}\rangle = \hbar|1^{(0)}\rangle$, and $(S_{1_z} - S_{2_z})|3^{(0)}\rangle = 0 = (S_{1_z} - S_{2_z})|4^{(0)}\rangle$. This also shows that all matrix elements $\langle m^{(0)}|V|n^{(0)}\rangle = 0$ for the degenerate subspace $m, n = 2, 3, 4$. Therefore there are no zero energy denominators in (5.1.42) and we can use nondegenerate second order perturbation theory to find the energy eigenvalues. We find energy shifts $\Delta_3 = 0 = \Delta_4$ and

$$\begin{aligned}
\Delta_1 &= \frac{|V_{12}|^2}{E_1^{(0)} - E_2^{(0)}} = \frac{e^2B^2}{m_e^2c^2} \frac{|\langle 1^{(0)}|(S_{1_z} - S_{2_z})|2^{(0)}\rangle|^2}{-3A\hbar^2/4 - A\hbar^2/4} = -\frac{e^2B^2}{m_e^2c^2A} \\
\Delta_2 &= \frac{|V_{21}|^2}{E_2^{(0)} - E_1^{(0)}} = -\Delta_1 = +\frac{e^2B^2}{m_e^2c^2A}
\end{aligned}$$

These agree exactly with the exact eigenvalues for the triplet $m = \pm 1$ states $|3^{(0)}\rangle$ and $|4^{(0)}\rangle$. Expanding the exact solutions for the two $m = 0$ states for $eB\hbar/m_e c \ll A\hbar^2$,

$$E = -\frac{\hbar^2 A}{4}\left\{1 \pm 2\left[1 + \frac{1}{2}4\left(\frac{eB}{m_e c\hbar A}\right)^2\right]\right\} = -\frac{\hbar^2 A}{4}(1 \pm 2) \mp \frac{e^2B^2}{m_e^2c^2A}$$

which agrees with the zeroth order eigenvalues plus the second order energy shifts. The first order wave functions from (5.1.44) are $|3\rangle = |3^{(0)}\rangle$, $|4\rangle = |4^{(0)}\rangle$,

$$|1\rangle = |1^{(0)}\rangle + |2^{(0)}\rangle \frac{V_{21}}{E_1^{(0)} - E_2^{(0)}} = |1^{(0)}\rangle - |2^{(0)}\rangle \frac{eB}{m_e cA\hbar} \quad \text{and} \quad |2\rangle = |2^{(0)}\rangle + |1^{(0)}\rangle \frac{eB}{m_e cA\hbar}$$

Now introduce an oscillating magnetic field $\tilde{\mathbf{B}} = \tilde{B}e^{i\omega t}\hat{\mathbf{n}}$ where $\hbar\omega = E_1 - E_2$, and the problem asks for which direction $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ or $\hat{\mathbf{n}} = \hat{\mathbf{x}}$ is needed to induce transitions between the states we've labeled $|1\rangle$ and $|2\rangle$. The interaction Hamiltonian is $\boldsymbol{\mu} \cdot \tilde{\mathbf{B}} = (e\tilde{B}/m_e c)e^{i\omega t}(\mathbf{S}_1 - \mathbf{S}_2) \cdot \hat{\mathbf{n}}$, so we need to examine the matrix elements $\langle 1|(S_{1_z} - S_{2_z})|2\rangle$ and $\langle 1|(S_{1_x} - S_{2_x})|2\rangle$. Note from above that $|1\rangle$ and $|2\rangle$ each contain components in the direction of $|1^{(0)}\rangle$ and $|2^{(0)}\rangle$.

First consider $\langle 1|(S_{1_z} - S_{2_z})|2\rangle$ since most of that work is done above. We have already seen that $\langle 1^{(0)}|(S_{1_z} - S_{2_z})|1^{(0)}\rangle = 0 = \langle 2^{(0)}|(S_{1_z} - S_{2_z})|2^{(0)}\rangle$. However

$$\begin{aligned} \langle 1^{(0)}|(S_{1_z} - S_{2_z})|1^{(0)}\rangle &= \frac{1}{2}\left[\langle +-| - \langle -+|\right](S_{1_z} - S_{2_z})\left[|+-\rangle + |-+\rangle\right] \\ &= \frac{\hbar}{4}\left[\langle +-| - \langle -+|\right]\left[|+-\rangle - |-+\rangle + |+-\rangle - |-+\rangle\right] = \frac{\hbar}{2} \neq 0 \end{aligned}$$

So, an oscillating magnetic field in the z-direction *will* cause transitions between $|1\rangle$ and $|2\rangle$.

For $\langle 1|(S_{1_x} - S_{2_x})|2\rangle$ calculate $(S_{1_x} - S_{2_x})|1^{(0)}\rangle$. Use $S_x = (\hbar/2)[|+\rangle\langle -| + |-\rangle\langle +|]$ from (3.2.1). Then calculate $S_{1_x}|1^{(0)}\rangle = (\hbar/2\sqrt{2})[-|++\rangle + |--\rangle] = (\hbar/2\sqrt{2})[|3^{(0)}\rangle - |4^{(0)}\rangle]$ and $S_{2_x}|1^{(0)}\rangle = -(\hbar/2\sqrt{2})[|3^{(0)}\rangle - |4^{(0)}\rangle]$, so $(S_{1_x} - S_{2_x})|1^{(0)}\rangle = (\hbar/\sqrt{2})[|3^{(0)}\rangle - |4^{(0)}\rangle]$. However, all of the $|n^{(0)}\rangle$ are orthogonal. Therefore $\langle 1|(S_{1_x} - S_{2_x})|2\rangle = 0$ and there will be no transitions for an oscillating magnetic field in the x-direction. Similarly for the y-direction.

33. *The essential physics in this problem has to do with the proton magnetic moment being much smaller than that of the electron. Therefore, the interaction of the external magnetic field with the proton is neglected. Mathematically, the degeneracy ends up getting treated differently, but pretty much everything else is the same. I am just reproducing here what was published in the old solutions manual, which does not deal with a comparison to the exact eigenvalues, but I want to look into reformulating the problem at some point in the future.*

We need to digress here on time independent degenerate perturbation theory. Let $(H_o+V)\phi_n = E_n\phi_n \equiv (E_n^{(o)} + E_n^{(1)} + E_n^{(2)} + \ldots\ldots)\phi_n$. Write $\phi_n = \phi_n^{(o)} + \phi_n^{(1)} + \phi_n^{(2)}\ldots$ We have 1. $H_o\phi_n^{(1)} + V\phi_n^{(o)} = E_n^{(o)}\phi_n^{(1)} + E_n^{(1)}\phi_n^{(o)}$; 2. $H_o\phi_n^{(2)} + V\phi_n^{(1)} = E_n^{(o)}\phi_n^{(2)} + E_n^{(1)}\phi_n^{(1)} + E_n^{(2)}\phi_n^{(o)}$; 3. $H_o\phi_n^{(3)} + V\phi_n^{(2)} = E_n^{(o)}\phi_n^{(3)} + E_n^{(1)}\phi_n^{(2)} + E_n^{(2)}\phi_n^{(1)} + E_n^{(3)}\phi_n^{(o)}$ for the first three orders of perturbation. Let $\phi_n^{(o)}, \phi_{n'}^{(o)}\ldots$ be degenerate eigenfunctions of H_o with eigenvalue $E_n^{(o)}$. We saw earlier that we need to choose these to be eigen-functions of V so these would not be mixed by the perturbation. We want to solve $\phi_n^{(1)} = \sum_k C_k\phi_k^{(o)}$. Take scalar product of first order relation 1. above with $\phi_n^{(o)}$ we have $E_n^{(1)} = V_{nn}$. Take scalar product of 1. with $\phi_p^{(o)}$ where $E_p^{(o)} \neq E_n^{(o)}$, we have $C_p = V_{pn}/[E_n^{(o)}-E_p^{(o)}]$ which is correct to 1st order. Note if we had taken scalar product with $\phi_{n'}^{(o)}$ in 1. we would have got $C_{n'}E_n^{(o)} = E_n^{(o)}C_{n'}$, because $V_{n'n}=0$ and hence $C_{n'}$ is <u>not</u> fixed by the 1st order equation. Multiply second order eqn. 2. by $\phi_{n'}^{(o)}$ we have $C_{n'}(V_{nn} - V_{n'n'}) = \sum_{k\neq n,n'\ldots} C_k V_{n'k}$. Although this is formally a 2nd equation, note however that if the perturbation <u>removes</u> the degeneracy, i.e. $V_{nn}\neq V_{n'n'}$ then $C_{n'} = \sum_{k\neq n,n'\ldots} C_k V_{n'k}/[V_{nn}-V_{n'n'}] = \sum_{k\neq n,n'\ldots} \frac{V_{kn}V_{n'k}}{(E_n^{(o)}-E_k^{(o)})(V_{nn}-V_{n'n'})}$ which is formally 1st order! The reason is that 1. above does <u>not</u> determine $C_{n'}$, hence we need to go to second order equation 2. and obtain the curious $V_{nn}-V_{n'n'}$ denominator for $C_{n'}$. But for our case, n & n' were nondegenerate only in 1st order, i.e. $V_{nn}-V_{n'n'}$ is 1st order so that we must need a 2nd order numerator (and hence use 2nd order eqn.) to get first order answer for the wavefunction/eigenvector. This was the reason why an arbitrarily <u>small</u> perturbation makes a big effect <u>unless</u> we work in the correct basis.

(a) Write $\vec{S}_1\cdot\vec{S}_2 = \frac{1}{2}[\vec{S}^2-\vec{S}_1^2-\vec{S}_2^2]$. Again eigenstates are triplets and singlets. Treat $V \equiv eBS_{1z}/m_ec \equiv \alpha S_{1z}$ as a perturbation and denote state as $|e,p\rangle$. Then $S_{1z}[|+-\rangle - |-+\rangle]/2^{\frac{1}{2}} = \frac{1}{2}\hbar[|+-\rangle + |-+\rangle]/2^{\frac{1}{2}}$ and thus $S_{1z}|\text{singlet}\rangle = \frac{1}{2}\hbar|\text{triplet}\rangle_{m=0}$ $S_{1z}|\text{triplet},m=0\rangle = \frac{1}{2}\hbar|\text{singlet}\rangle$, $S_{1z}|++\rangle = \frac{1}{2}\hbar|++\rangle$, $S_{1z}|--\rangle = -\frac{1}{2}\hbar|--\rangle$. The singlet state clearly obtains no shift to first order in perturbation theory. For the triplet state, we must again use degenerate perturbation theory and choose a bas where V is diagonal with respect to the triplets. Fortunately, our basis alread has this property. The m = ±1 components of the triplet suffer a first order energy shift $E^{(1)}_{m=\pm1} = \langle\pm|V|\pm\rangle = \frac{1}{2}\hbar\alpha$.

<u>Second order perturbation theory</u>. For the singlet state, usual perturbation the formula is okay. We have

$$\Delta E^{(2)}_{\text{singlet}} = \sum_k V_{nk}V_{kn}/(E_n^{(o)}-E_k^{(o)}),\ |n\rangle = \text{singlet}$$

This has only a non-vanishing matrix element with the m = 0 triplet, i.e. $|k\rangle$ m be triplet m = 0 state. We have

$\langle m=0,\text{triplet}|V|\text{singlet}\rangle = \tfrac{1}{2}\hbar\alpha$

and since $E^{(o)}_{\text{singlet}} = -3A\hbar^2/4$, $E^{(o)}_{\text{triplet}} = A\hbar^2/4$, we have

$$\Delta E^{(2)}_{\text{singlet}} = (\tfrac{1}{2}\hbar\alpha)^2/-A\hbar^2$$

Shift of triplet wavefunction in 1st order. The $|m=\pm,\text{triplet}\rangle$ states are eigenfunctions of V and so they don't mix. The $|m=0,\text{triplet}\rangle$ does mix with the sing and $C_{\text{singlet}} = \tfrac{1}{2}\hbar\alpha/[A\hbar^2/4 - (-3A\hbar^2/4)] = \tfrac{1}{2}\hbar\ /A\hbar^2$. There is no mixing with the other triplet components as can be seen unambiguously from our expression for C equation above. Writing "$|m=0,\text{triplet}\rangle$" for the first order shift, we have

$$"|m=0,\text{triplet}\rangle" = |m=0,\text{triplet}\rangle^{(o)} + \frac{\tfrac{1}{2}\hbar}{A\hbar^2}|\text{singlet}\rangle$$

Finally, multiplying Eq. 2. by $\phi^{(o)}_n$, we have

$$(\phi^{(o)}_n, V\phi^{(1)}_n) = E^{(2)}_n \quad \text{(where we have used } E^{(1)}_n = 0)$$

and thus $E^{(2)}_{\text{triplet}}(m=0) = C_{\text{singlet}}\langle m=0,\text{triplet}|V|\text{singlet}\rangle = \tfrac{1}{2}\hbar/A\hbar^2 \times \tfrac{1}{2}\hbar\alpha =$ $(\hbar\alpha)^2/4A\hbar^2$.

(b) The new time dependent perturbative term is for this problem

$$V'(t) = (eB'/m_e c)e^{i\omega t}\vec{S}_1\cdot\hat{B}'$$

Using the expressions for χ_1 and χ_o in terms of ϕ^o_1 and ϕ^o_o and Eq.(9) and Eq.(1) of solution to Problem 32(b), we find readily that $\langle\chi_1|S_{1x}|\chi_o\rangle = 0$, and similar $\langle\chi_1|S_{1y}|\chi_o\rangle = 0$. However again like Problem 32(b) $\langle\chi_1|S_{1z}|\chi_o\rangle$ does not vanish. Therefore the $\hat{B}'$ field should again be in the z-direction to cause m=0 transitions.

(c) The first order eigenvector "$|m=0,\text{triplet}\rangle$" was already given in part (a), for "$|m=0,\text{singlet}\rangle$" $= |\phi^o_o\rangle + [\langle\phi^o_1|V|\phi^o_o\rangle]/(E^{(o)}_o - E^{(o)}_1)]|\phi^o_1\rangle = |\phi^o_o\rangle + (\tfrac{1}{2}\hbar\alpha/-A\hbar^2)|\phi^o_1\rangle$.
Note for this problem n = triplet, m=0; n' = triplet, m=±1; k = singlet; hence $V_{n'k} = 0$ and thus $C_{n'} = 0$ and does not actually contribute here in first order.

34. Following (5.8.1), the interaction piece of the Hamiltonian is $V = -(e/m_e c)\mathbf{A}\cdot\mathbf{p}$, for the gauge condition $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$, and ignoring $\mathbf{A}^2$ terms. As seen in (5.8.5) and the attendant discussion, we take for emission $\mathbf{A}(\mathbf{x},t) = A_0\hat{\boldsymbol{\varepsilon}}\exp(-i\omega\hat{\mathbf{n}}\cdot\mathbf{x}/c)\exp(+i\omega t)$ for polarization vector $\hat{\boldsymbol{\varepsilon}}$, although the photon field is in fact real, given by (5.8.3). Since the process "is known to be an $E1$ transition", we invoke the long wavelength approximation and take the leading term in $\exp(-i\omega\hat{\mathbf{n}}\cdot\mathbf{x}/c) \to 1$. The transition rate for *emission*, following (5.8.8), is

$$w_{i\to n} = \frac{2\pi}{\hbar}\frac{e^2}{m_e^2c^2}|A_0|^2\,|\langle n|(\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{p})|i\rangle|^2\,\delta(E_n - E_i + \hbar\omega),$$

Next, as in (5.8.21), make use of $[\mathbf{x}, H_0] = [\mathbf{x}, \mathbf{p}^2]/2m_e = i\hbar\mathbf{p}/m_e$, and therefore

$$\langle n|\mathbf{p}|i\rangle = \frac{m_e}{i\hbar}\langle n|[\mathbf{x}, H_0]|i\rangle = \frac{m_e}{i\hbar}(E_i - E_n)\langle n|\mathbf{x}|i\rangle = im_e\omega_{ni}\langle n|\mathbf{x}|i\rangle \equiv im_e\omega_{ni}\mathbf{d}_{ni}$$

and the quantity $|\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{d}_{ni}|^2$ should tell us the angular distribution. We'll take $\mathbf{d}_{ni}$ as the angular momentum quantization ("z-") axis. (Since "the magnetic quantum number of the atom decreases by one", we must be able to distinguish m states. Thus, some small perturbation breaks the degeneracy, and defines the z-axis.) Let the photon be emitted in the xz-plane. Then $\hat{\mathbf{n}} = \hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta$ for polar angle θ. The polarization unit vector $\hat{\boldsymbol{\varepsilon}}$ is normal to $\hat{\mathbf{n}}$ so we can write it as $\hat{\boldsymbol{\varepsilon}} = -\sin\alpha\cos\theta\hat{\mathbf{x}} + \cos\alpha\hat{\mathbf{y}} + \sin\alpha\sin\theta\hat{\mathbf{z}}$.

Now write $|i\rangle = |l, m\rangle$ and $|n\rangle = |l', m-1\rangle$ where we know that l and l' must differ by an odd integer. Also write $\mathbf{x} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} = r(\hat{\mathbf{x}}\cos\phi\sin\theta + \hat{\mathbf{y}}\sin\phi\sin\theta + \hat{\mathbf{z}}\cos\theta)$ as

$$\mathbf{x} = r\sqrt{2\pi/3}Y_1^{-1}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) - r\sqrt{2\pi/3}Y_1^{1}(\hat{\mathbf{x}} - i\hat{\mathbf{y}}) + r\sqrt{4\pi/3}Y_1^{0}\hat{\mathbf{z}}$$

These terms are proportional to a spherical tensor $T_q^{(1)}$ for $q = -1, +1, 0$, respectively, so by (3.11.28), $\langle n|T_q^{(1)}|i\rangle = \langle l', m-1|T_q^{(1)}|l, m\rangle$ is nonzero only for $m - 1 = q + m$ or $q = -1$. So

$$\hat{\boldsymbol{\varepsilon}}\cdot\langle n|\mathbf{x}|i\rangle \propto \boldsymbol{\varepsilon}\cdot(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) = \varepsilon_x + i\varepsilon_y = -\sin\alpha\cos\theta + i\cos\alpha = \frac{i}{2}e^{i\alpha}(1+\cos\theta) + \frac{i}{2}e^{-i\alpha}(1-\cos\theta)$$

and a photon emitted in the $+\hat{\mathbf{z}}$ ($-\hat{\mathbf{z}}$) direction, seems to have right (left) handed circular polarization. This makes sense, since the L_z for the atom decreased by one unit, and the photon carries off the difference. For emission angles $0 < \theta < 180°$, the photon also carries off some "orbital" angular momentum. Since $\langle(\varepsilon_x^2 + \varepsilon_y^2)\rangle = \langle\sin^2\alpha\sin^2\theta\rangle \propto \sin^2\theta$, the (polarization averaged) angular distribution is $\sin^2\theta$, the same as for classical dipole radiation.

35. Have $\langle\mathbf{x}|i\rangle = (1/a_0)^{3/2}\exp(-r/a_0)/\sqrt{\pi}$ and $\langle\mathbf{x}|f\rangle = (2/a_0)^{3/2}\exp(-2r/a_0)/\sqrt{\pi}$. So, the probability amplitude is $\langle i|f\rangle = \int d^3x\langle i|\mathbf{x}\rangle\langle\mathbf{x}|f\rangle = 4\pi(2^{3/2}/\pi a_0^3)\int r^2dr\exp(-3r/a_0)$. Using MATHEMATICA I find $\int r^2dr\exp(-3r/a_0) = 2a_0^3/27$ so $\langle i|f\rangle = 8\sqrt{2}\times(2/27)$ and $|\langle i|f\rangle|^2 = 2^9/3^6 = 0.702$ is the probability to find the atom in the $^3He^+$ ground state. The decay electron leaves the neighborhood in a time $T = 1$Å$/v$ where $m_ev^2/2 \approx 10$ keV. Therefore $v/c \approx (20/511)^{1/2}$ and $T \approx 10^{-18}$ sec, whereas (see page 346) $2\pi/\omega_{ab} \sim h/10$ eV$\sim 10^{-15}$ sec. Thus $T \ll 2\pi/\omega_{ab}$ and the condition for the sudden approximation is satisfied.

36. This problem shows that Berry's Phase is a real number, and it is not hard, but the notation is a little tricky. Remember that a differential operator acts to the right, and that you can differentiate a ket (or a bra) with respect to the parameters on which it depends, and get a different ket (or bra). Being very explicit to make this clear, we have

$$0 = \boldsymbol{\nabla}_{\mathbf{R}}\left[1\right] = \boldsymbol{\nabla}_{\mathbf{R}}\left[\langle n;t|n;t\rangle\right] = \boldsymbol{\nabla}_{\mathbf{R}}\left[\left(\langle n;t|\right)\left(|n;t\rangle\right)\right] = \left(\boldsymbol{\nabla}_{\mathbf{R}}\langle n;t|\right)|n;t\rangle + \langle n;t|\left(\boldsymbol{\nabla}_{\mathbf{R}}|n;t\rangle\right)$$

However $\left(\boldsymbol{\nabla}_{\mathbf{R}}\langle n;t|\right)|n;t\rangle = \left(\langle n;t|\left[\boldsymbol{\nabla}_{\mathbf{R}}|n;t\rangle\right]\right)^*$. So $\langle n;t|\left[\boldsymbol{\nabla}_{\mathbf{R}}|n;t\rangle\right] = -\left(\langle n;t|\left[\boldsymbol{\nabla}_{\mathbf{R}}|n;t\rangle\right]\right)^*$, in which case it is a purely imaginary quantity. Therefor $\mathbf{A}_n(\mathbf{R})$ in (5.6.23) must be real.

37. The state vector is well known, in Problem 1.11 and (3.2.52). In spherical coordinates,

$$|n;t\rangle = \cos\left(\frac{\theta}{2}\right)|+\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|-\rangle$$

To be sure, we want the state $|n;t\rangle$ which depends on the vector-of-parameters $\mathbf{R}(t)$ that is the magnetic field. However, the state does not depend on the magnitude of the field, only on the field's coordinates, in the usual three spatial coordinates. Consequently, a gradient with respect to B_θ or B_ϕ is the same as the usual three dimensional spatial gradient. Therefore

$$\begin{aligned}
\boldsymbol{\nabla}_{\mathbf{R}}|n;t\rangle &= \left[\hat{\boldsymbol{\theta}}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]|n;t\rangle \\
&= -\frac{1}{2}\sin\left(\frac{\theta}{2}\right)\hat{\boldsymbol{\theta}}|+\rangle + e^{i\phi}\frac{1}{2}\cos\left(\frac{\theta}{2}\right)\hat{\boldsymbol{\theta}}|-\rangle + \frac{i}{\sin\theta}e^{i\phi}\sin\left(\frac{\theta}{2}\right)\hat{\boldsymbol{\phi}}|-\rangle \\
\langle n;t|\left[\boldsymbol{\nabla}_{\mathbf{R}}|n;t\rangle\right] &= \frac{i}{\sin\theta}\sin^2\left(\frac{\theta}{2}\right)\hat{\boldsymbol{\phi}} \\
\mathbf{A}_n(\mathbf{R}) &= i\langle n;t|\left[\boldsymbol{\nabla}_{\mathbf{R}}|n;t\rangle\right] = -\frac{1}{\sin\theta}\sin^2\left(\frac{\theta}{2}\right)\hat{\boldsymbol{\phi}} \equiv A_\phi(\theta)\hat{\boldsymbol{\phi}} \\
\boldsymbol{\nabla}_{\mathbf{R}}\times\mathbf{A}_n(\mathbf{R}) &= \hat{\mathbf{r}}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) = -\hat{\mathbf{r}}\frac{1}{\sin\theta}2\frac{1}{2}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = -\hat{\mathbf{r}}\frac{1}{2}
\end{aligned}$$

At this point, you will notice that we have essentially derived the first part of (5.6.42) for this particular state vector. The rest follows from the definition of the solid angle, but for the sake of completeness, we can carry it through. So

$$\gamma_n(C) = \int\left[\boldsymbol{\nabla}_{\mathbf{R}}\times\mathbf{A}_n(\mathbf{R})\right]\cdot d\mathbf{a} = -\frac{1}{2}\int\hat{\mathbf{r}}\cdot d\mathbf{a} = -\frac{1}{2}\Omega$$

If you really want, you could carry out the integral, at which point you will simply derive the expression for the solid angle subtended by a cone of half-angle θ, namely $2\pi(1-\cos\theta)$.

38. Write $V(\mathbf{x},t) = [\exp(kz-\omega t)+\exp(-kz+\omega t)]/2$. Following (5.8.6), only keep the first term since $E_i < E_f$ and only absorption of $\hbar\omega$ is important. The absorption rate (5.7.35) is

$$w_{i\to f} = \frac{2\pi}{\hbar}\left(\frac{V_0}{2}\right)^2 \left|\langle\mathbf{p}|e^{ikz}|0\rangle\right|^2 \delta(E_{\mathbf{p}}-E_0-\hbar\omega)$$

where E_0 is the energy of the ground state $|0\rangle$, and $E_{\mathbf{p}}$ is the energy of the state $|\mathbf{p}\rangle$, taken to be a plane wave with momentum $\mathbf{p}$. Inserting $1=\int d^3x'|\mathbf{x}'\rangle\langle\mathbf{x}'|$, the matrix element is

$$\langle\mathbf{p}|e^{ikz}|0\rangle = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0L}\right)^{3/2}\int d^3x'\; e^{-i\mathbf{p}\cdot\mathbf{x}'/\hbar}e^{ikz'}e^{-r'/a_0} = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0L}\right)^{3/2}\int d^3x'\; e^{-i\mathbf{q}\cdot\mathbf{x}'}e^{-r'/a_0}$$

where $\mathbf{q}\equiv k\hat{\mathbf{z}}-\mathbf{p}/\hbar$. To evaluate this integral, let $\mathbf{q}$ define the z' direction. Then,

$$\int d^3x'\; e^{-i\mathbf{q}\cdot\mathbf{x}'}e^{-r'/a_0} = 2\pi\int_0^\infty r'^2\,dr'e^{-r'/a_0}\int_{-1}^{1}d(\cos\theta')e^{-iqr'\cos\theta'} = \frac{4\pi a_0^3}{(1+a_0^2q^2)^2}$$

(For more detail, see the solution to Problem 41.)

The problem asks for a comparison to the photoelectric effect, so let's first retrace the steps that lead to the differential cross section (5.8.36). We start with the absorption cross section (5.8.14) and integrate with the correct density of states $\rho(E)$. We are concerned with the number of states for ejected electron energies between E and $E+dE$ which move into a solid angle $d\Omega$. As in Section 2.5, we quantize in a "big box" of side length L, the energy (5.8.30) is $E=\mathbf{p}^2/2m_e=\hbar^2k_f^2/2m_e=(2\pi\hbar)^2n^2/2m_eL^2$ where $n^2=n_x^2+n_y^2+n_z^2$ and n_x, n_y, and n_z are (positive, negative, or zero) integers, and following the textbook we put $\mathbf{p}^2=\hbar^2k_f^2$ with $k_f\equiv(2\pi/L)n$. Note that a value of n uniquely specifies the value of the energy E.

Now consider the number of states in n space, for large values of n. There is one state for each point (n_x,n_y,n_z). Also, the electron momentum vector $\mathbf{p}$ points in the direction (n_x,n_y,n_z), so the solid angle $d\Omega$ is in n space. Therefore, the number of states between E and $E+dE$ ejected into solid angle $d\Omega$ is obtained by counting the number of states in a thin spherical shell of thickness dn for this solid angle. That is

$$\rho(E) = n^2dnd\Omega = n^2\frac{dn}{dE}dEd\Omega = n^2\left(\frac{L}{2\pi\hbar}\right)^2\frac{m_e}{n}dEd\Omega = \left(\frac{L}{2\pi}\right)^3\frac{m_e}{\hbar^2}k_fdEd\Omega$$

which is (5.8.31). Then, multiplying (5.8.14) by $\rho(E)$ and integrating over the energy E

$$\frac{d\sigma}{d\Omega} = \frac{4\pi^2\hbar}{m_e^2\omega}\left(\frac{e^2}{\hbar c}\right)\left|\langle n|e^{i(\omega/c)(\hat{\mathbf{n}}\cdot\mathbf{x})}\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{p}|i\rangle\right|^2\left(\frac{L}{2\pi}\right)^3\frac{m_e}{\hbar^2}k_f$$

which in fact is the same as (5.8.32), with $\alpha\equiv e^2/\hbar c$. At this point, the book's description is straightforward. Integrating by parts, one turns $\langle n|e^{i(\omega/c)(\hat{\mathbf{n}}\cdot\mathbf{x})}\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{p}|i\rangle$ into $\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{k}_f$ times our integral, above. The result is (5.8.36).

So, now return to the problem at hand. Integrating over $E_{\mathbf{p}}$ after inserting the same density of states, the decay rate $w_{i\to f}$ becomes

$$\frac{dw}{d\Omega}=\frac{2\pi}{\hbar}\left(\frac{V_0}{2}\right)^2\frac{1}{\pi}\left(\frac{1}{a_0L}\right)^3\left[\frac{4\pi a_0^3}{(1+a_0^2q^2)^2}\right]^2\left(\frac{L}{2\pi}\right)^3\frac{m_e}{\hbar^2}k_f=\frac{m_eV_0^2a_0^3}{\pi\hbar^3}\frac{k_f}{(1+a_0^2q^2)^4}$$

Checking dimensionality, $(ME^2L^3/E^3T^3)(1/L)=ML^2/ET^3=ML^2/[(ML^2/T^2)T^3=1/T$. The angular distribution here is contained in the dependence on q^2. It is similar to the angular distribution (5.8.36), except for the factor $(\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{k}_f)^2=k_f^2\sin^2\theta\cos^2\phi$.

This is a peculiar perturbation, a wave traveling in the z-direction but with no polarization. Indeed, the polarization of a true electromagnetic wave leads to the additional angular dependence. It isn't clear to me how one might create such a perturbation, independent of the a magnetic field that would be generated Maxwell's Equations.

39. From (B.2.4) the energy is $E=\hbar^2n^2\pi^2/2mL^2$ for $n=1,2,3,\ldots$ and we consider $n\gg 1$. There are dn states between n and $n+dn$, and $dE=\hbar^2ndn\pi^2/mL^2$, so we have

$$\frac{dn}{dE}=\frac{mL^2}{\hbar^2\pi^2}\frac{1}{n}=\frac{mL^2}{\hbar^2\pi^2}\frac{\hbar\pi}{L}\left(\frac{1}{2mE}\right)^{1/2}=\frac{L}{\hbar\pi}\left(\frac{m}{2E}\right)^{1/2}$$

Note that $L/\hbar$ has dimensions of 1/momentum and m/E has dimensions 1/velocity^2 so that dn/dE has dimensions 1/momentum$\times$velocity= 1/energy. *Note: This does not agree with the old solutions manual, but they count states as ndn, which is the two-dimensional case.*

40. We start from (5.8.32). As in (5.8.33) we need to evaluate

$$\langle\mathbf{k}_f|e^{i(\omega/c)(\hat{\mathbf{n}}\cdot\mathbf{x})}\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{p}|i\rangle=\hat{\boldsymbol{\varepsilon}}\cdot\int d^3x\langle\mathbf{k}_f|\mathbf{x}\rangle e^{i(\omega/c)(\hat{\mathbf{n}}\cdot\mathbf{x})}\langle\mathbf{x}|\mathbf{p}|i\rangle=-i\hbar\hat{\boldsymbol{\varepsilon}}\cdot\int d^3x\frac{e^{-i\mathbf{k}_f\cdot\mathbf{x}}}{L^{3/2}}e^{i(\omega/c)(\hat{\mathbf{n}}\cdot\mathbf{x})}\boldsymbol{\nabla}\psi(\mathbf{x})$$

where $\psi(\mathbf{x})\equiv\langle\mathbf{x}|i\rangle$. Following the text and integrating by parts, we are led to evaluate

$$\hat{\boldsymbol{\varepsilon}}\cdot\boldsymbol{\nabla}[e^{-i\mathbf{k}_f\cdot\mathbf{x}}e^{i(\omega/c)(\hat{\mathbf{n}}\cdot\mathbf{x})}]=\hat{\boldsymbol{\varepsilon}}\cdot[-i\mathbf{k}_f+i(\omega/c)\hat{\mathbf{n}}]e^{-i\mathbf{k}_f\cdot\mathbf{x}}e^{i(\omega/c)(\hat{\mathbf{n}}\cdot\mathbf{x})}=-i(\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{k}_f)e^{-i\mathbf{k}_f\cdot\mathbf{x}}e^{i(\omega/c)(\hat{\mathbf{n}}\cdot\mathbf{x})}$$

since $\hat{\boldsymbol{\varepsilon}}\cdot\hat{\mathbf{n}}$ for the electromagnetic wave. Now $\psi(\mathbf{x})$ is the ground state wave function of the 3D harmonic oscillator with Hamiltonian is $H=\mathbf{p}^2/2m+m\omega_0^2\mathbf{x}^2/2=H_x+H_y+H_z$ where the H_i are the corresponding 1D Hamiltonians. Therefore $\psi(\mathbf{x})=(m\omega_0/\pi\hbar)^{3/4}\exp(-m\omega_0r^2/2\hbar)$ using (B.4.3). Defining $\mathbf{q}\equiv\mathbf{k}_f-(\omega/c)\hat{\mathbf{n}}$ we have

$$\langle\mathbf{k}_f|e^{i(\omega/c)(\hat{\mathbf{n}}\cdot\mathbf{x})}\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{p}|i\rangle=i\hbar(\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{k}_f)\frac{1}{L^{3/2}}\left(\frac{m\omega_0}{\pi\hbar}\right)^{3/4}\int d^3x\;e^{-i\mathbf{q}\cdot\mathbf{x}}e^{-m\omega_0r^2/2\hbar}$$

The integral can be written as $I_xI_yI_z$ where $I_x\equiv\int dx\exp(-iq_xx)\exp(-m\omega_0x^2/2\hbar)$, etc. With $iq_xx+m\omega_0x^2/2\hbar=(m\omega_0/2\hbar)(x^2+2i\hbar q_xx/m\omega_0-\hbar^2q_x^2/m^2\omega_0^2)+\hbar q_x^2/2m\omega_0$, we have

$$I_x=\exp(-\hbar q_x^2/2m\omega_0)\int_{-\infty}^{\infty}dx\exp[-(m\omega_0/2\hbar)(x+i\hbar q_x/m\omega_0)^2]=\left(\frac{2\pi\hbar}{m\omega_0}\right)^{1/2}e^{-\hbar q_x^2/2m\omega_0}$$

So now return to (5.8.32). The differential cross section becomes

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{4\pi^2\alpha\hbar^3}{m^2\omega}(\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{k}_f)^2\frac{1}{L^3}\left(\frac{m\omega_0}{\pi\hbar}\right)^{3/2}\left(\frac{2\pi\hbar}{m\omega_0}\right)^3\exp\left[-\frac{\hbar\mathbf{q}^2}{m\omega_0}\right]\frac{mk_fL^3}{\hbar^2(2\pi)^3}\\ &= \frac{4\alpha\hbar^2}{m^2\omega\omega_0}k_f(\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{k}_f)^2\left(\frac{\pi\hbar}{m\omega_0}\right)^{1/2}\exp\left[-\frac{\hbar\mathbf{q}^2}{m\omega_0}\right]\end{aligned}$$

Using the coordinate system in Fig. 5.12, we find (5.8.37), that is $(\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{k}_f)^2k_f^2\sin^2\theta\cos^2\phi$ and $\mathbf{q}^2 = k_f^2 - 2k_f(\omega/c)\cos\theta + (\omega/c)^2$. Putting this all together,

$$\frac{d\sigma}{d\Omega} = \frac{4\alpha\hbar^2k_f^3}{m^2\omega\omega_0}\sin^2\theta\cos^2\phi\left(\frac{\pi\hbar}{m\omega_0}\right)^{1/2}\exp\left\{-\frac{\hbar}{m\omega_0}\left[k_f^2+\left(\frac{\omega}{c}\right)^2\right]\right\}\exp\left[2\hbar k_f\frac{\omega}{mc\omega_0}\cos\theta\right]$$

41. The real point is to calculate the Fourier transform $\phi(\mathbf{p})$ of the $1S$ state of the hydrogen atom. It is relevant for deriving (5.8.36). We have (with $\mathbf{p}$ defining the "z" axis)

$$\begin{aligned}\phi(\mathbf{p}) &= \frac{1}{(2\pi\hbar)^{3/2}}\int d^3x\; e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar}\psi(\mathbf{x}) = \frac{1}{(8\pi^4a^3\hbar^3)^{1/2}}\int d^3x\; e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar}e^{-r/a}\\ &= \frac{1}{(2\pi^2a^3\hbar^3)^{1/2}}\int_0^\infty r^2dr\; e^{-r/a}\int_{-1}^1 d(\cos\theta)\; e^{-ipr\cos\theta/\hbar}\\ &= \frac{1}{(2\pi^2a^3\hbar^3)^{1/2}}\frac{\hbar}{p}\int_0^\infty r\sin\left(\frac{p}{\hbar}r\right)e^{-r/a}dr = \frac{1}{(2\pi^2a^3\hbar^3)^{1/2}}\frac{2\hbar^4a^3}{(\hbar^2+a^2p^2)^2}\\ \text{so}\qquad |\phi(\mathbf{p})|^2 &= \frac{2\hbar^5}{\pi^2}\frac{a^3}{(\hbar^2+a^2p^2)^4}\end{aligned}$$

where the integral was evaluated with Mathematica.

42. *Note. The previous solution manual refers to Sakurai "Advanced Quantum Mechanics", pages 41–44, for the solution to this problem. That uses a quantized electromagnetic field, however. Here I give a solution within the context of the present textbook.*

The lifetime $\tau(2p \to 1s)$ is the inverse of the transition rate (5.8.8), evaluating the matrix element and integrating over final states. An important ingredient, though, is the electromagnetic field normalization A_0, which is not discussed in the book. We can determine this, however, by integrating the energy density over our "big box" of slide length L, and setting it equal to $\hbar\omega$. That is, the electromagnetic energy must equal that of the emitted photon.

Start with (5.8.3) with $\mathbf{k} \equiv \omega\hat{\mathbf{n}}/c = (2\pi/L)(n_x\hat{\mathbf{x}} + n_y\hat{\mathbf{y}} + n_z\hat{\mathbf{z}})$. That is, imposed periodic boundary conditions in our big box, with $L \to \infty$ at the end of the calculation. One finds

$$\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} = 2A_0k\hat{\boldsymbol{\varepsilon}}\sin(\mathbf{k}\cdot\mathbf{x}-\omega t) \qquad\text{and}\qquad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} = 2A_0(\hat{\boldsymbol{\varepsilon}}\times\mathbf{k})\sin(\mathbf{k}\cdot\mathbf{x}-\omega t)$$

The energy density is $u = (\mathbf{E}^2 + \mathbf{B}^2)/8\pi$. Note that $(\hat{\boldsymbol{\varepsilon}} \times \mathbf{k})^2 = \hat{\boldsymbol{\varepsilon}}^2\mathbf{k}^2 - (\hat{\boldsymbol{\varepsilon}} \cdot \mathbf{k})2 = k^2$ since $\hat{\boldsymbol{\varepsilon}}$ is a unit vector perpendicular to the propagation direction $\mathbf{k}$. Therefore

$$u = \frac{k^2}{\pi^2}A_0^2 \sin^2(\mathbf{k}\cdot\mathbf{x} - \omega t) = \frac{k^2}{2\pi}A_0^2\left[1 - \cos 2(\mathbf{k}\cdot\mathbf{x} - \omega t)\right]$$

The cosine term integrates to zero with our periodic boundary conditions. Therefore

$$\int_{L^3} dx\ u = \frac{k^2}{2\pi}A_0^2 L^3 = \frac{\omega^2}{2\pi c^2}A_0^2 L^3 = \hbar\omega \qquad \text{so} \qquad A_0 = \sqrt{\frac{2\pi\hbar c^2}{\omega L^3}}$$

which agrees with (7.6.21). To get the density of states for the photon, write the energy $\mathcal{E} = \hbar\omega = \hbar kc = 2\pi\hbar cn/L$. As in Problem 38, but this time integrating over solid angle,

$$\rho(\mathcal{E}) = 4\pi n^2 dn = 4\pi n^2 \frac{dn}{d\mathcal{E}}d\mathcal{E} = 4\pi\frac{\hbar^2\omega^2 L^2}{4\pi^2\hbar^2 c^2}\frac{L}{2\pi\hbar c}d\mathcal{E} = \frac{\omega^2}{2\pi^2\hbar c^3}L^3 d\mathcal{E}$$

Now the matrix element. The photon wavelength is hundreds of nm, whereas the atom has a size ~ 0.1 nm, so use the first term in (5.8.15); indeed, the $2p \to 1s$ transition is $E1$. So,

$$\langle 1s|e^{i(\omega/c)(\hat{\mathbf{n}}\cdot\mathbf{x})}\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{p}|2p\rangle = \langle 1s|\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{p}|2p\rangle = \frac{m_e}{i\hbar}\langle 1s|\hat{\boldsymbol{\varepsilon}}\cdot[\mathbf{x}, H_0]|2p\rangle = -im_e\omega\langle 1s|\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{x}|2p\rangle$$

But the state $|2p\rangle$ could be any one of three states with $m = 0, \pm 1$. We separate these using

$$\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{x} = \frac{\varepsilon_x - i\varepsilon_y}{\sqrt{2}}\frac{x+iy}{\sqrt{2}} + \frac{\varepsilon_x + i\varepsilon_y}{\sqrt{2}}\frac{x-iy}{\sqrt{2}} + \varepsilon_z z = \sqrt{\frac{4\pi}{3}}\left(-\varepsilon_- rY_1^{+1} + \varepsilon_+ rY_1^{-1} + \varepsilon_0 rY_1^0\right)$$

where $\varepsilon_\pm \equiv (\varepsilon_x \pm i\varepsilon_y)/\sqrt{2}$ and $\varepsilon_0 \equiv \varepsilon_z$. Since the Y_l^m are spherical tensor operators, by (3.11.28) the matrix elements $\langle 1s|Y_1^q|2p, m\rangle$ are nonzero only if $m = -q$. Furthermore, by (3.11.31) $\langle 1s|Y_1^q|2p, m\rangle = \langle 11; -q, q|11; 00\rangle\langle 1s||Y_1||2p\rangle/\sqrt{3}$, so only calculate for $q = 0$, i.e.

$$\begin{aligned}
\langle 1s|z|2p, m=0\rangle &= 2\pi\frac{\sqrt{3}}{4\pi}\frac{2}{2^{3/2}a_0^3}\frac{1}{a_0\sqrt{3}}\int_0^\infty r^2 dr\int_{-1}^1 d(\cos\theta)\ e^{-r/a_0}\ r\cos\theta\ re^{-r/2a_0} \\
&= \sqrt{\frac{1}{8}}\frac{1}{a_0^4}\frac{2}{3}\int_0^\infty r^4 e^{-3r/2a_0}dr = \frac{1}{3\sqrt{2}}\frac{1}{a_0^4}\frac{2^8}{3^4}a_0^5 = \frac{2^{15/2}}{3^5}a_0 \\
\text{so} \qquad \langle 1s||rY_1||2p\rangle &= \frac{\sqrt{3}\langle 1s|rY_1^0|2p, 0\rangle}{\langle 11; 00|11; 00\rangle} = \frac{\sqrt{3}}{-1/\sqrt{3}}\sqrt{\frac{3}{4\pi}}\langle 1s|z|2p, 0\rangle = -\sqrt{\frac{3}{4\pi}}\frac{2^{15/2}}{3^4}a_0 \\
\text{and} \qquad \langle 1s|\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{x}|2p, m\rangle &= \frac{1}{\sqrt{3}}\left(-\varepsilon_-\frac{1}{\sqrt{3}}\delta_{m,-1} + \varepsilon_+\frac{1}{\sqrt{3}}\delta_{m,+1} - \varepsilon_0\frac{1}{\sqrt{3}}\delta_{m,0}\right)\left(-\frac{2^{15/2}}{3^4}a_0\right)
\end{aligned}$$

Note that all three matrix elements have the same value. For the transition rate, average over these, with $|\varepsilon_-|^2 + |\varepsilon_+|^2 + |\varepsilon_0|^2 = 1$. Finally, integrate (5.8.8) over final states to get

$$\begin{aligned}
w &= 2\times\frac{2\pi}{\hbar}\frac{e^2}{m_e^2 c^2}\frac{2\pi\hbar c^2}{\omega L^3}m_e^2\omega^2\frac{1}{3}\left(a_0^2\frac{2^{15}}{3^{10}}\right)\frac{\omega^2}{2\pi^2\hbar c^3}L^3 = \frac{2^{17}}{3^{11}}\frac{e^2}{\hbar c}\frac{a_0^2}{c^2}\omega^3 \\
&= \frac{2^{17}}{3^{11}}\frac{e^2}{\hbar c}\frac{a_0^2}{c^2}\left[\frac{1}{\hbar}\frac{e^2}{2a_0}\left(1 - \frac{1}{2^2}\right)\right]^3 = \frac{2^8}{3^8}\left(\frac{e^2}{\hbar c}\right)^4\frac{c}{a_0} = \frac{2^8}{3^8}\left(\frac{e^2}{\hbar c}\right)^5\frac{m_e c^2}{\hbar} = \frac{1}{\tau}
\end{aligned}$$

This agrees with, for example, Townsend (2012) Eq. (14.168). Using $m_e c^2 = 0.511$ MeV and $\hbar = 6.58 \times 10^{-16}$ eV·s, we find $\tau = (6.58/0.511) \times 10^{-22} \cdot 137^5 \cdot (3/2)^8 = 1.59 \times 10^{-9}$ sec.

Chapter Six

1. The problem is to solve the Lippman-Schwinger equation (6.2.2) for a one-dimensional potential. Ignore the comment about "Is the prescription $E \to E + i\varepsilon$?" and just consider scattering forward in time. That is, we want the solutions $\psi^{(+)}(x) \equiv \langle x|\psi^{+)}\rangle$ to the equation

$$\begin{aligned}\psi^{(+)}(x) &= \phi(x) + \int dx' \left\langle x \left| \frac{1}{E - H_0 + i\varepsilon} \right| x' \right\rangle \langle x'|V|\psi^{(\pm)}\rangle \\ &= \phi(x) + \frac{2m}{\hbar^2} \int_{-a}^{a} dx' G^{(+)}(x,x')V(x')\psi^{(\pm)}(x')\end{aligned}$$

where $\phi(x) \equiv \langle x|i\rangle = e^{ikx}/\sqrt{2\pi}$ and $G^{(+)}(x,x') \equiv \frac{\hbar^2}{2m}\left\langle x \left| \frac{1}{E-H_0+i\varepsilon} \right| x' \right\rangle$. Now drop the superscript "(+)". To find $G(x,x')$, insert complete sets of (continuous) momentum states. We have $E = \hbar^2 k^2/2m$, and make the definition $p \equiv \hbar q$. Therefore

$$\begin{aligned}G(x,x') &= \frac{\hbar^2}{2m} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dp' \, \langle x|p\rangle \left\langle p \left| \frac{1}{E - H_0 + i\varepsilon} \right| p' \right\rangle \langle p'|x'\rangle \\ &= \frac{\hbar^2}{2m} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dp' \, \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}} \frac{\langle p|p'\rangle}{E - p'^2/2m + i\varepsilon} \frac{e^{-ip'x'/\hbar}}{\sqrt{2\pi\hbar}} \\ &= \frac{1}{2\pi\hbar}\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} dp \frac{e^{ip(x-x')/\hbar}}{E - p^2/2m + i\varepsilon} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} dq \frac{e^{iq(x-x')}}{(q-q_0)(q+q_0)}\end{aligned}$$

with $q_0 \equiv k(1 + i\varepsilon)$, redefining ε but keeping the same sign. Do the integral with a half-infinite-complex-plane contour. For $x > x'$, close the contour (counter clockwise) in the upper plane and the exponential factor goes to zero along the semicircle; this picks up a pole at $q = +q_0$. For $x < x'$, close (clockwise) in the lower plane, with pole at $q = -q_0$. Therefore

$$\begin{aligned} G(x,x') &= -\frac{1}{2\pi}(+2\pi i)\frac{e^{ik(x-x')}}{k+k} = \frac{1}{2ik}e^{ik(x-x')} \qquad \text{for} \qquad x > x' \\ \text{and} \qquad G(x,x') &= -\frac{1}{2\pi}(-2\pi i)\frac{e^{-ik(x-x')}}{-k-k} = \frac{1}{2ik}e^{-ik(x-x')} \qquad \text{for} \qquad x < x'\end{aligned}$$

which agrees with (239) in Sec. 4.4 of Gottfried and Yan (2003). For $V(x) = -\gamma(\hbar^2/2m)\delta(x)$, find $\psi(x) = \phi(x) - \gamma G(x,0)\psi(0)$. Since $G(0,0) = 1/2ik$, have $\psi(0) = \phi(0)/(1 + \gamma/2ik)$. So,

$$\begin{aligned}\psi(x) &= \frac{1}{\sqrt{2\pi}}\left[e^{ikx} - \frac{\gamma}{2ik+\gamma}e^{ikx}\right] = \frac{1}{\sqrt{2\pi}}\frac{2ik}{2ik+\gamma}e^{ikx} \qquad \text{for} \qquad x > 0 \\ \text{and} \qquad \psi(x) &= \frac{1}{\sqrt{2\pi}}\left[e^{ikx} - \frac{\gamma}{2ik+\gamma}e^{-ikx}\right] \qquad \text{for} \qquad x < 0\end{aligned}$$

so $T(k) = 2ik/(2ik + \gamma)$ and $R(k) = -\gamma/(2ik + \gamma)$, in agreement with (4.4.267) of Gottfried and Yan (2003). Note that $|T(k)|^2 + |R(k)|^2 = 1$. The attractive δ-function potential was solved in Problem 2.24, with $V(x) = -\nu_0\delta(x)$. The bound state energy was found to be $E = -m\nu_0^2/2\hbar^2 = -\hbar^2\gamma^2/8m$, i.e. $k = i\gamma/2$, i.e. the poles of $T(k)$ and $R(k)$.

2. The first order Born approximation is $\langle\mathbf{k}'|V|\psi^{(+)}\rangle = \langle\mathbf{k}'|T|\mathbf{k}\rangle \approx \langle\mathbf{k}'|V|\mathbf{k}\rangle$ using (6.3.1) and (6.3.2). Using (6.2.6), (6.2.7), (6.2.22), and (6.2.23) the total cross section is

$$\begin{aligned}
\sigma_{\text{tot}} &= \left(\frac{mL^3}{2\pi\hbar^2}\right)^2 \int d\Omega_{\mathbf{k}'} \langle\mathbf{k}'|V|\mathbf{k}\rangle\langle\mathbf{k}|V|\mathbf{k}'\rangle \\
&= \left(\frac{mL^3}{2\pi\hbar^2}\right)^2 \int d\Omega_{\mathbf{k}'}\, d^3x\, d^3x' \, \langle\mathbf{k}'|V|\mathbf{x}\rangle\langle\mathbf{x}|\mathbf{k}\rangle\langle\mathbf{k}|V|\mathbf{x}'\rangle\langle\mathbf{x}'|\mathbf{k}'\rangle \\
&= \left(\frac{mL^3}{2\pi\hbar^2}\right)^2 \int d\Omega_{\mathbf{k}'}\, d^3x\, d^3x' \, V(\mathbf{x})V(\mathbf{x}')\langle\mathbf{k}'|\mathbf{x}\rangle\langle\mathbf{x}|\mathbf{k}\rangle\langle\mathbf{k}|\mathbf{x}'\rangle\langle\mathbf{x}'|\mathbf{k}'\rangle \\
&= \left(\frac{m}{2\pi\hbar^2}\right)^2 \int d\Omega_{\mathbf{k}'}\, d^3x\, d^3x' \, V(\mathbf{x})V(\mathbf{x}')\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}e^{-i\mathbf{k}'\cdot(\mathbf{x}-\mathbf{x}')}
\end{aligned}$$

The integral over $\Omega_{\mathbf{k}'}$ is easy. Put $\hat{\mathbf{k}}'_z$ in the direction of $\mathbf{x}-\mathbf{x}'$, so $\mathbf{k}'\cdot(\mathbf{x}-\mathbf{x}') = k|\mathbf{x}-\mathbf{x}'|\cos\theta_{\mathbf{k}'}$. Then $\int d\Omega_{\mathbf{k}'} e^{-i\mathbf{k}'\cdot(\mathbf{x}-\mathbf{x}')} = 2\pi\int_{-1}^{1} d(\cos\theta_{\mathbf{k}'})e^{-ik|\mathbf{x}-\mathbf{x}'|\cos\theta_{\mathbf{k}'}} = 4\pi\sin[k|\mathbf{x}-\mathbf{x}'|]/k|\mathbf{x}-\mathbf{x}'|$. (Recall that $|\mathbf{k}| \equiv k = |\mathbf{k}'|$.) We can reduce the total cross section further if the potential is spherically symmetric. In this case, every spatial direction $\mathbf{x}-\mathbf{x}'$ contributes equally to the double position integral. So, we can average over all directions $\mathbf{k}$, picking up a factor $(1/4\pi)\int d\Omega_{\mathbf{k}}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = \sin[k|\mathbf{x}-\mathbf{x}'|]/k|\mathbf{x}-\mathbf{x}'|$. The result is

$$\sigma_{\text{tot}} = \frac{m^2}{\pi\hbar^4}\int d^3x\, d^3x'\, V(r)V(r')\frac{\sin^2[k|\mathbf{x}-\mathbf{x}'|]}{k^2|\mathbf{x}-\mathbf{x}'|^2}$$

The Optical Theorem (6.2.24), or (6.2.33), using (6.2.22) and (6.3.1), reads

$$\sigma_{\text{tot}} = \frac{4\pi}{k}\text{Im}f(\mathbf{k},\mathbf{k}) = -\frac{2mL^3}{\hbar^2 k}\text{Im}\langle\mathbf{k}|T|\mathbf{k}\rangle$$

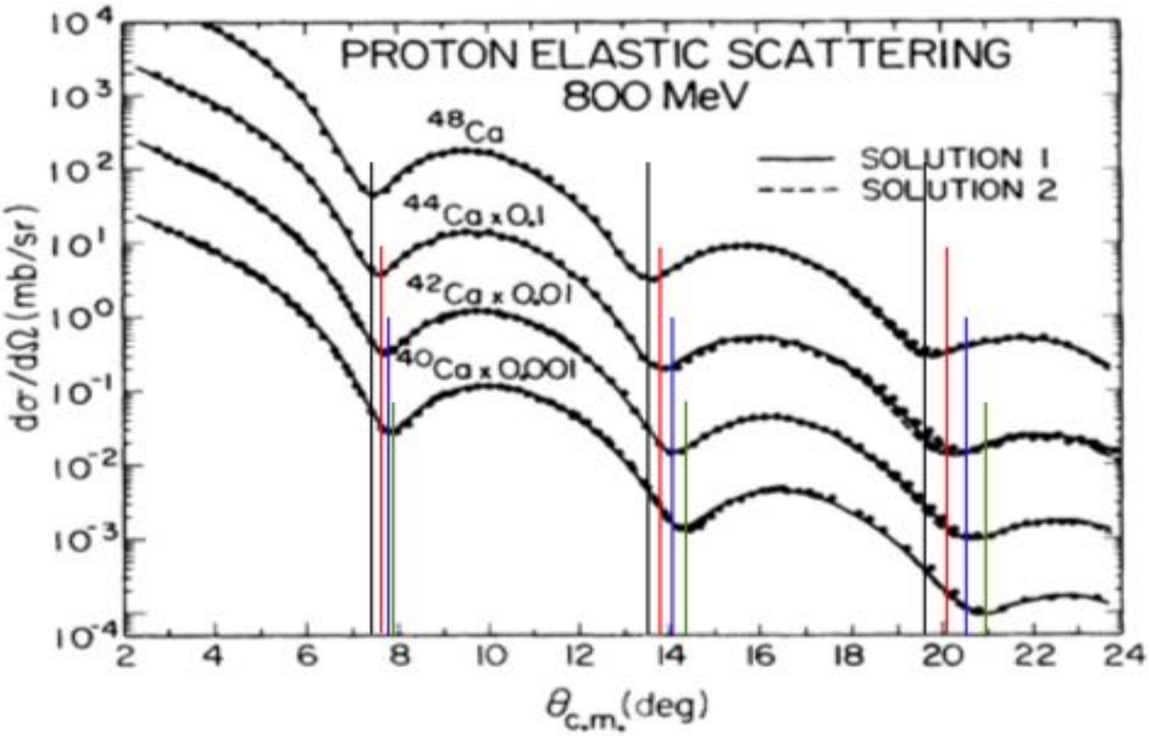
The first order Born approximation from (6.3.2) gives zero, since $T = V$ is real. Therefore

$$\begin{aligned}
\sigma_{\text{tot}} &= -\frac{2mL^3}{\hbar^2 k}\text{Im}\left\langle \mathbf{k}\left| V\frac{1}{E-H_0+i\varepsilon}V\right|\mathbf{k}\right\rangle \\
&= -\frac{2mL^3}{\hbar^2 k}\text{Im}\int d^3x\, d^3x'\, \langle\mathbf{k}|V|\mathbf{x}\rangle\left\langle\mathbf{x}\left|\frac{1}{E-H_0+i\varepsilon}\right|\mathbf{x}'\right\rangle\langle\mathbf{x}'|V|\mathbf{k}\rangle \\
&= -\frac{2m}{\hbar^2 k}\text{Im}\int d^3x\, d^3x'\, V(r)V(r')e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\frac{2m}{\hbar^2}G_+(\mathbf{x},\mathbf{x}') \\
&= \frac{4m^2}{\hbar^4 k}\text{Im}\int d^3x\, d^3x'\, V(r)V(r')\frac{\sin[k|\mathbf{x}-\mathbf{x}'|]}{k|\mathbf{x}-\mathbf{x}'|}\,\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} \\
&= \frac{m^2}{\pi\hbar^4}\int d^3x\, d^3x'\, V(r)V(r')\frac{\sin^2[k|\mathbf{x}-\mathbf{x}'|]}{k^2|\mathbf{x}-\mathbf{x}'|^2}
\end{aligned}$$

making use of (6.2.3) and (6.2.11), and the same "averaging over $\mathbf{k}$" argument we used above. That is, the second order Born approximation gives the same answer for the total cross section when applied to the Optical Theorem.

3. The figure is reprinted here from the original paper, along with vertical lines to help read off the positions of the minima. We determine momentum transfer from $q = 2k\sin(\theta/2)$ where $\hbar^2k^2/2m = (\hbar c)^2k^2/mc^2 = 800$ MeV, so $k = \sqrt{800 \cdot 934}/200 = 4.32/\text{fm}$. For a square well of radius a, the first three minima are at $qa = 4.49$, 7.73, and 10.9. (See page 401.) So,

		Minimum #1			Minimum #2			Minimum #3		
Isotope	$1.4A^{1/3}$	θ	q	a	θ	q	a	θ	q	a
^{40}Ca	4.79	7.95°	1.20	3.76	14.2°	2.12	3.65	20.9°	3.08	3.54
^{42}Ca	4.87	7.85°	1.18	3.80	14.0°	2.09	3.70	20.5°	3.03	3.60
^{44}Ca	4.94	7.6°	1.14	3.93	13.8°	2.06	3.75	20.1°	2.97	3.67
^{48}Ca	5.09	7.3°	1.10	4.09	13.4°	2.00	3.86	19.5°	2.88	3.78



Several remarks are in order. Firstly, as mentioned in the text, the minimum shifts to lower angles as the number of neutrons is increased. In other words, the radius increases with neutron number, just as expected. The quantitative agreement with the liquid drop formula $a = 1.4A^{1/3}$ is marginal, but you can only expect so much when comparing one crude approximation (liquid drop) with another (square well). The position of the minima, though, are reasonably consistent with each other, each leading to a radius that is within $\sim 5\%$ of the others for a given isotope.

4. This problem, low energy scattering from a weak potential, makes a lot of use of the properties of the spherical Bessel functions. Chapter 10 of the NIST Digital Library of Mathematical Functions at http://dlmf.nist.gov/ is a good online reference.

We need to solve the radial Schrödinger Equation (6.4.55) for $u_l(r) = rA_l(r)$ in the region $r \leq R$ where $V = V_0$. This is easy, since $E - V_0 \equiv \hbar^2\kappa^2/2m$, that is $A_l(r) = j_l(\kappa r)$ and the logarithmic derivative (6.4.53) "just inside" $r = R$ is $\beta_l = \kappa R j_l'(\kappa R)/j_l(\kappa R)$. Note that $|V_0| \ll E$, so $k \sim \kappa$ and $kR \ll 1$ implies that $\kappa R \ll 1$. This means we expand both to lowest order, but this is tricky because (6.4.54) involves the Bessel functions and their derivatives, which mix different orders. It turns out to be very useful to use the identity

$$f_l'(x) = \frac{l}{x} f_l(x) - f_{l+1}(x)$$

where $f_l(x)$ is any spherical Bessel function. Therefore, the logarithmic derivative is

$$\beta_l = \frac{\kappa R}{j_l(\kappa R)}\, j_l'(\kappa R) = \frac{\kappa R}{j_l(\kappa R)}\left[\frac{l}{\kappa R} j_l(\kappa R) - j_{l+1}(\kappa R)\right] = l - \kappa R\,\frac{j_{l+1}(\kappa R)}{j_l(\kappa R)}$$

Also, for both the numerator and denominator in (6.4.54) we have

$$\begin{aligned} kR\, f_l'(kR) - \beta_l\, f_l(kR) &= kR\left[\frac{l}{kR} f_l(kR) - f_{l+1}(kR)\right] - \left[l - \kappa R\,\frac{j_{l+1}(\kappa R)}{j_l(\kappa R)}\right] f_l(kR) \\ &= \kappa R\,\frac{j_{l+1}(\kappa R)}{j_l(\kappa R)} f_l(kR) - kR\, f_{l+1}(kR) \end{aligned}$$

Now $j_l(x) \approx x^l/(2l+1)!!$ for $x \ll 1$, so $j_{l+1}(\kappa R)/j_l(\kappa R) = \kappa R/(2l+3)$ to leading order. We also know that $n_l(x) = -(2l-1)!!/x^{l+1}$ for $x \ll 1$. Therefore (6.4.54) becomes

$$\begin{aligned} \tan\delta_l &= \frac{(\kappa R)^2 j_l(kR)/(2l+3) - kR\, j_{l+1}(kR)}{(\kappa R)^2 n_l(kR)/(2l+3) - kR\, n_{l+1}(kR)} \\ &= \frac{(\kappa R)^2 (kR)^l/(2l+3)!! - (kR)^{l+2}/(2l+3)!!}{-(2l-1)!!(\kappa R)^2/[(2l+3)(kR)^{l+1}] + (2l+1)!!/(kR)^{l+1}} \\ &\approx (kR)^{2l+1}\frac{(\kappa R)^2 - (kR)^2}{(2l+3)!!(2l+1)!!} = \frac{(kR)^{2l+3}}{(2l+3)!!(2l+1)!!}\left[\frac{\kappa^2}{k^2} - 1\right] \end{aligned}$$

where we ignore the first term in the denominator for $kR \ll 1$. Clearly $l = 0$ dominates, and

$$\tan\delta_0 = \frac{1}{3}(kR)^3\left[\frac{E - V_0}{E} - 1\right] = -\frac{1}{3}(kR)^3\frac{V_0}{E} = -\frac{1}{3}k\frac{2mV_0R^3}{\hbar^2} \approx \delta_0 \approx \sin\delta_0$$

That is, (6.4.40) gives an isotropic angular distribution. The total cross section (6.4.41) is

$$\sigma_{\text{tot}} = \frac{4\pi}{k^2}\sin^2\delta_0 = \frac{16\pi}{9}\frac{m^2V_0^2R^6}{\hbar^4}$$

The next most important term is the p-wave. The phase shift is

$$\tan\delta_1 = -\frac{1}{45}(kR)^5\frac{V_0}{E} \approx \delta_1 \approx \sin\delta_1 \ll \sin\delta_0$$

With just s- and p-waves, the differential cross section from (6.2.23) and (6.4.40) is

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2}\left|e^{i\delta_0}\sin\delta_0 + 3e^{i\delta_1}\sin\delta_1\cos\theta\right|^2 \approx \frac{1}{k^2}\left[\sin^2\delta_0 + 6\cos(\delta_0-\delta_1)\sin\delta_0\sin\delta_1\cos\theta\right]$$

which is of the form $d\sigma/d\Omega = A + B\cos\theta$. Since $\delta_1 \ll \delta_0 \ll 1$, we have $\cos(\delta_0-\delta_1)\approx 1$ and

$$\frac{B}{A} = \frac{6\sin\delta_1}{\sin\delta_0} = \frac{6\cdot 3}{45}(kR)^2 = \frac{2}{5}(kR)^2$$

5. Use orthogonality of the Legendre polynomials, i.e. $\int_{-1}^{1} P_m(x)P_n(x)dx = 2m\delta_{mn}/(2n+1)$, to find an expression for the expansion coefficients in (6.4.40). That is

$$\frac{2}{k}e^{i\delta_l}\sin\delta_l = -\frac{2m}{\hbar^2}\frac{V_0}{\mu}\int_{-1}^{1}\frac{P_l(x)dx}{2k^2(1-x)+\mu^2} = -\frac{2m}{\hbar^2k^2}\frac{V_0}{\mu}\frac{1}{2}\int_{-1}^{1}\frac{P_l(x)dx}{1+\mu^2/2k^2-x} = -\frac{V_0}{\mu E}Q_l(\alpha)$$

where $\alpha \equiv 1+\mu^2/2k^2 > 1$ and $E = \hbar^2k^2/2m$. For $|\delta_l| \ll 1$, $e^{i\delta_l}\sin\delta_l \approx \delta_l$, so

$$\delta_l = -\frac{V_0}{E}\frac{k}{2\mu}\frac{l!}{(2l+1)!!}\left[\frac{1}{\alpha^{l+1}} + \frac{(l+1)(l+2)}{2(2l+3)}\frac{1}{\alpha^{l+3}} + \frac{(l+1)(l+2)(l+3)(l+4)}{2\cdot 4\cdot(2l+3)(2l+5)}\frac{1}{\alpha^{l+5}} + \cdots\right]$$

All terms on the right are positive, so if $V_0 > 0$ (i.e. repulsive), then $\delta_l < 0$, but if $V_0 < 0$, then $\delta_l > 0$. If $\lambda = 2\pi/k \gg 1/\mu$, then $\mu/k \gg 1$, so $\alpha \approx \mu^2/2k^2 \gg 1$, and

$$\delta_l \approx -\frac{V_0}{E}\frac{k}{2\mu}\frac{l!}{(2l+1)!!}\frac{1}{\alpha^{l+1}} = -\frac{mV_0}{\hbar^2k\mu}\frac{l!}{(2l+1)!!}\frac{2^{l+1}k^{2l+2}}{\mu^{2l+2}} = -\frac{2^{l+1}l!}{(2l+1)!!}\frac{mV_0}{\hbar^2\mu^{2l+3}}k^{2l+1}$$

6. The ground state wave function is $\psi(\mathbf{x}) = A\sin(kr)/kr$, with $k = \pi/a$. (See Section 3.7, especially Page 210.) Use $\int d^3x\,\psi^2(\mathbf{x}) = A^2(4\pi/k^2)\int_0^a \sin^2 kr\,dr = A^2(2\pi a/k^2) = 1$, so $A^2 = k^2/2\pi a = \pi/2a^3$. We need $(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2$ and $(\Delta p_x)^2 = \langle p_x^2\rangle - \langle p_x\rangle^2$, but by spherical symmetry, $\langle x\rangle = 0 = \langle p_x\rangle$. Putting $x = r\sin\theta\cos\phi$, we have

$$\begin{aligned}\langle x^2\rangle &= A^2\int_0^a r^2dr\; r^2\frac{\sin^2 kr}{k^2r^2}\int_0^\pi \sin\theta d\theta\;\sin^2\theta\int_0^{2\pi}d\phi\;\cos^2\phi \\ &= \frac{\pi}{2a^3}\frac{a^2}{\pi^2}\frac{a^3(2\pi^2-3)}{12\pi^2}\frac{4}{3}\pi = \frac{a^2}{9}\left(1-\frac{3}{2\pi^2}\right)\end{aligned}$$

where I used MATHEMATICA to do the integrals. For $\langle p_x^2\rangle = -\hbar^2\int d^3x\,\psi(\mathbf{x})\partial^2\psi(\mathbf{x})/\partial x^2$, we need to take derivatives with respect to x. Since $r = (x^2+y^2+z^2)^{1/2}$, we have $\partial r/\partial x = x/r$.

Proceeding straightforwardly, we first find the second derivative of the wave function:

$$\begin{aligned}
\frac{\partial\psi}{\partial x} &= A\left[\frac{k\cos kr}{kr} - \frac{\sin kr}{kr^2}\right]\frac{x}{r} = \frac{A}{k}\frac{x}{r^3}\left[kr\cos kr - \sin kr\right] \\
\frac{\partial^2\psi}{\partial x^2} &= \frac{A}{k}\left\{\left[\frac{1}{r^3} - 3\frac{x}{r^4}\frac{x}{r}\right]\left[kr\cos kr - \sin kr\right] + \frac{x}{r^3}\left[k\cos kr - k^2 r\sin kr - k\cos kr\right]\frac{x}{r}\right\} \\
&= \frac{A}{k}\frac{1}{r^3}\left\{\left[1 - 3\sin^2\theta\cos^2\phi\right]\left[kr\cos kr - \sin kr\right] - k^2r^2\sin^2\theta\cos^2\phi\sin kr\right\}
\end{aligned}$$

Now $\int_0^\pi \sin\theta\, d\theta\ \sin^2\theta = \int_{-1}^1 d\mu(1-\mu^2) = 4/3$ and $\int_0^{2\pi} d\phi\ \cos^2\phi = \pi$, so the solid angle integrals $\int d\Omega[1 - 3\sin^2\theta\cos^2\phi] = 4\pi - 3(4/3)\pi = 0$ and $\int d\Omega\sin^2\theta\cos^2\phi = 4\pi/3$. Thus

$$\langle p_x^2\rangle = \hbar^2\frac{4\pi}{3}\frac{A^2}{k}\int_0^a r^2 dr\ \frac{\sin kr}{kr}\frac{k^2}{r}\sin kr = \hbar^2\frac{4\pi}{3}\frac{\pi}{2a^3}\int_0^a \sin^2\left(\frac{\pi r}{a}\right)\, dr = \frac{\hbar^2}{a^2}\frac{\pi^2}{3}$$

Therefore $(\Delta x)^2(\Delta p_x)^2 = \hbar^2\pi^2(1 - 3/2\pi^2)/27 = (\hbar/1.796)^2 > (\hbar/2)^2$.

7. Oddly, this problem is worked out thoroughly in the text, but with a replaced by R, on pages 416–417. The theorems you "may assume without proof" are in fact derived in the book. See (6.2.23) and (6.4.40). The s-wave phase shift from (6.4.63) is $\delta_0 = -ka$. The very low energy total cross section is $\sigma = 4\pi a^2$ from (6.4.48). There is some discussion about the difference between this and the geometric cross section in the textbook.

8. We need to evaluate $\Delta(b = l/k)$ from (6.5.14). For $V(r) = V_0\exp(-r^2/a^2)$,

$$\Delta(b) = -\frac{m}{2k\hbar^2}\int_{-\infty}^{\infty} V_0 e^{-(b^2+z^2)/a^2}dz = -\frac{mV_0}{2k\hbar^2}e^{-b^2/a^2}\int_{-\infty}^{\infty} e^{-z^2/a^2}dz = -\frac{mV_0 a\sqrt{\pi}}{2k\hbar^2}e^{-b^2/a^2}$$

so $\delta_l = -(mV_0 a\sqrt{\pi}/2k\hbar^2)\exp(-l^2/k^2a^2)$. Clearly, $\delta_l \to 0$ "very rapidly," i.e. exponentially in the square of l, as $l \gg ka$.

For $V(r) = V_0\exp(-\mu r)/\mu r$, rewrite with $r^2 = b^2 + z^2$, so $rdr = zdz$ and

$$\Delta(b) = -\frac{m}{2k\hbar^2}\frac{V_0}{\mu}\int_{-\infty}^{\infty} e^{-\mu r}\frac{dz}{r} = -\frac{m}{k\hbar^2}\frac{V_0}{\mu}\int_b^\infty e^{-\mu r}\frac{dr}{(r^2-b^2)^{1/2}} = -\frac{m}{k\hbar^2}\frac{V_0}{\mu}\int_1^\infty \frac{e^{-\mu b s}ds}{(s^2-1)^{1/2}}$$

Mathematica says that the integral is $K_0(\mu b)$, the modified Bessel function of the second kind, zero order, but I can't find this explicit representation anywhere. I suppose one could derive it from a contour integration of some of the other forms, but I'm not going to worry about that. So, $\delta_l = -(mV_0/\mu k\hbar^2)K_0(\mu l/k) \to -(mV_0/\mu k\hbar^2)\sqrt{\pi k/\mu l}\exp(-\mu l/k)$, and once again, the phase shift goes to zero rapidly for $l \gg k/\mu$.

We don't need to know about $K_0(z)$ to find the behavior for $l \gg k/\mu$, i.e. $\mu b = \mu l/k \gg 1$. In this case, the integrand only contributes for the minimum value of s, i.e. $s = 1$, and the $\exp(-\mu b)$ behavior of $\Delta(b)$ is evident.

9. The left side of the equation in part **(a)** is just $G(\mathbf{x},\mathbf{x}')=-\exp(ik|\mathbf{x}-\mathbf{x}'|)/|\mathbf{x}-\mathbf{x}'|$, i.e. the Green's function (6.2.3) and (6.2.11). Equation (6.2.12) mentions that $G(\mathbf{x},\mathbf{x}')$ satisfies the Helmholtz Equation for a δ-function source, but it is worth proving that here. Write $G(r)=-e^{ikr}/4\pi r$ with $r\equiv|\mathbf{x}-\mathbf{x}'|$ so that $\partial r/\partial x=(x-x')/r$ and then, so long as $r\neq 0$,

$$\frac{\partial G}{\partial x}=\frac{\partial G}{\partial r}\frac{\partial r}{\partial x}=-\frac{1}{4\pi}\left[\frac{ik}{r}-\frac{1}{r^2}\right]e^{ikr}=\left[ik-\frac{1}{r}\right]G(r)$$

and similarly for $\partial G/\partial y$ and $\partial G/\partial z$. This form reduces the tedium to find $\partial^2 G/\partial x^2$ and then to determine $\boldsymbol{\nabla}^2 G=-k^2G$. Thus $(\boldsymbol{\nabla}^2+k^2)G=0$. For $r\to 0$, $G(r)\to -1/4\pi r$ and $(\boldsymbol{\nabla}^2+k^2)G\to\boldsymbol{\nabla}^2 G$. The rest just follows from an introductory course in electromagnetic theory. That is, $\int\boldsymbol{\nabla}^2 G d^3x=\oint\boldsymbol{\nabla}G\cdot d\mathbf{A}=(1/4\pi r^2)(4\pi r^2)=1$. Thus $(\boldsymbol{\nabla}^2+k^2)G=\delta(\mathbf{x}-\mathbf{x}')$.

Use (6.2.12) to find the coefficients (as a function of r) of an expansion of the Green's function in spherical harmonics. (Eigenfunction expansions of Green's functions is discussed in many books on mathematical physics, but we can get there with our own formalism.) Consider the transition from (6.2.3) to (6.2.4), where complete sets $|\mathbf{k}'\rangle$ and $|\mathbf{k}''\rangle$ are inserted. Instead insert states $|\alpha lm\rangle$ and $|\alpha' l' m'\rangle$. A similar collapse happens because H_0 (6.1.2) is spherically symmetric, leaving functions $\langle\mathbf{x}|\alpha lm\rangle=R_{\alpha l}(r)Y_l^m(\theta,\phi)$ and sums (or integrals) over α and α' of the Lippman-Schwinger operator. Absorb all this into a function $g_l(r,r')$, that is

$$G(\mathbf{x},\mathbf{x}')=\sum_l\sum_m g_l(r,r')Y_l^m(\theta,\phi)Y_l^{m^*}(\theta',\phi')$$

Another result from mathematical physics is the so-called Closure Relation for spherical harmonics, namely $\sum_{lm}Y_l^m(\Omega)Y_l^{m^*}(\Omega')=\delta(\Omega-\Omega')$. This allows us to write

$$\delta(\mathbf{x}-\mathbf{x}')=\frac{1}{r^2}\delta(r-r')\sum_l\sum_m Y_l^m(\theta,\phi)Y_l^{m^*}(\theta',\phi')$$

With everything in spherical coordinates, we can now apply (6.2.12). Following (3.6.21), and equating term by term in l and m, we have

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+k^2-\frac{l(l+1)}{r^2}\right]g_l(r,r')=\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+k^2-\frac{l(l+1)}{r^2}\right]g_l=\frac{1}{r^2}\delta(r-r')$$

For $r\neq r'$, this is just the spherical Bessel equation in the variable kr. The solution must be finite at $r\to 0$ and represent an outgoing spherical wave as $r\to\infty$, so we need $g_l(r,r')=C_l j_l(r_<)h_l^{(1)}(r_>)$ where $r_<$ ($r_>$) is the lesser (greater) of r and r'. Integrating both sides with r^2dr from $r=r'-\varepsilon$ to $r=r'+\varepsilon$, followed by $\varepsilon\to 0$, gives

$$\begin{aligned}&C_lr^2\left[j_l(kr)kh_l^{(1)'}(kr)-kj_l'(kr)h_l^{(1)}(kr)\right]\\ =\;&iC_lr^2k\left[j_l(kr)n_l'(kr)-j_l'(kr)n_l(kr)\right]=iC_lkr^2\left[(kr)^{-2}\right]=1\qquad\text{so}\qquad C_l=-ik\end{aligned}$$

Note that we used the Wronskian $\mathcal{W}[j_l(z),n_l(z)]=z^{-2}$. See http://dlmf.nist.gov/10.50.

From (6.2.2) and (6.2.3) with (6.2.14), we have, as in the text,

$$\langle \mathbf{x}|\psi\rangle = \langle \mathbf{x}|i\rangle + \frac{2m}{\hbar^2}\int d^3x'\; G(\mathbf{x},\mathbf{x}')V(\mathbf{x}')\langle \mathbf{x}|\psi\rangle$$

but now $|\psi\rangle = |Elm(+)\rangle$ and $|i\rangle = |Elm\rangle$ are angular momentum eigenstates, and $G(\mathbf{x},\mathbf{x}')$ is taken from above. Using (6.4.21b) for $\langle \mathbf{x}|i\rangle$ and the same normalization for $\langle \mathbf{x}|\psi\rangle$,

$$\begin{aligned} A_l(k;r)Y_l^m(\Omega) &= j_l(kr)Y_l^m(\Omega) \\ &- ik\frac{2m}{\hbar^2}\int r'^2\,dr'\;d\Omega'\sum_{l',m'} j_{l'}(kr')h_{l'}^{(1)}(kr)Y_{l'}^{m'}(\Omega)Y_{l'}^{m'*}(\Omega')V(r')A_l(k;r')Y_l^m(\Omega') \end{aligned}$$

where $r_< = r'$, that is within the potential volume, and r is outside. Now

$$\delta_{ll'}\delta_{mm'} = \langle l'm'|lm\rangle = \int d\Omega'\langle l'm'|\Omega'\rangle\langle\Omega'|lm\rangle = \int d\Omega' Y_{l'}^{m'*}(\Omega')Y_l^m(\Omega')$$

so the summation collapses and the integral over Ω' is gone, leaving a factor $Y_l^m(\Omega)$ in the radial integral. Dividing out this factor over the whole equation leaves

$$A_l(k;r) = j_l(kr) - \frac{2mik}{\hbar^2}h_l^{(1)}(kr)\int_0^\infty j_l(kr')V(r')A_l(k;r')r'^2\,dr'$$

Now take $r\to\infty$. From (B.5.15) and (B.5.19), we see that $h_l^{(1)}(kr)\to e^{i[kr-(l+1)\pi/2]}/kr$ and $j_l(kr)\to\{e^{i[kr-(l+1)\pi/2]}+e^{-i[kr-(l+1)\pi/2]}\}/2kr = \{e^{i[kr-l\pi/2]}-e^{-i[kr-l\pi/2]}\}/2ikr$. Therefore

$$A_l(k;r) = \frac{e^{-il\pi/2}}{2ik}\left\{\left[1-\frac{4mik}{\hbar^2}\int_0^\infty j_l(kr')V(r')A_l(k;r')r'^2\,dr'\right]\frac{e^{ikr}}{r} - \frac{e^{-i(kr-l\pi)}}{r}\right\}$$

Compare to (6.4.31). The factor in square brackets above, multiplying the outgoing spherical wave, is written in terms of partial waves as $[1+2ikf_l(k)]$. That is,

$$f_l(k) = \frac{1}{2ik}\left[-\frac{4mik}{\hbar^2}\int_0^\infty j_l(kr)V(r)A_l(k;r)r^2dr\right] = -\frac{2m}{\hbar^2}\int_0^\infty j_l(kr)V(r)A_l(k;r)r^2dr$$

Thus the assertion is proved. The relation $f_l(k) = e^{i\delta_l}\sin\delta_l/k$ is just (6.4.39).

10. We could take a first principles approach, solving the Schrödinger equation and matching solutions at $r=R$, but this potential makes it easy to use the results of Problem 9, namely

$$f_0(k) = \frac{e^{i\delta_0}\sin\delta_0}{k} = -\frac{2m}{\hbar^2}\int_0^\infty j_0(kr)V(r)A_0(k;r)r^2dr = -\gamma j_0(kR)A_0(k;R)R^2$$

We determine $A_0(k;R)$ from the integral equation also derived in Problem 9:

$$\begin{aligned} A_0(k;R) &= j_0(kR) - \frac{2mik}{\hbar^2}h_0^{(1)}(kR)\int_0^\infty j_0(kr')V(r')A_0(k;r')r'^2\,dr' \\ &= j_0(kR) - ik\gamma h_0^{(1)}(kR)j_0(kR)A_0(k;R)R^2 \end{aligned}$$

which we can solve for $A_0(k;R)$. Note, however, that $\tan\delta_0 = \Im(f_0)/\Re(f_0)$ so it is handy to find the real and imaginary parts of $A_0(k;R)$ just to within common factors. So make use of $-ih_0^{(1)}(kR) = -i[j_0(kR)+in_0(kR)] = n_0(kR)-ij_0(kR) = -e^{ikR}/kR$ to write

$$
\begin{aligned}
A_0(k;R) &= \frac{j_0(kR)}{1+\gamma R j_0(kR)e^{ikR}} \cdot \frac{1+\gamma R j_0(kR)e^{-ikR}}{1+\gamma R j_0(kR)e^{-ikR}} \\
&= \text{Real Expression} \times \left[1+(\gamma/k)\sin(kR)\cos(kR) - i(\gamma/k)\sin^2(kR)\right] \\
\tan\delta_0 &= -\frac{(\gamma/k)\sin^2(kR)}{1+(\gamma/k)\sin(kR)\cos(kR)} = \frac{1}{\cot\delta_0}
\end{aligned}
$$

For $\gamma \gg k$, or in the case if $kR \ll 1$ then $(\gamma/k)kR = \gamma R \gg 1$, the second term in the denominator dominates unity and $\tan\delta_0 = -\sin^2(kR)/\sin(kR)\cos(kR) = -\tan(kR)$. Thus $\delta_0 = -kR$, in agreement with the result (6.4.63) for hard sphere scattering. The resonance condition $\cot\delta_0 = 0$ is satisfied when $\sin(kR)\cos(kR) = -k/\gamma$. That is, when kR is near $n\pi$ or $(n+1/2)\pi$ where n is an integer. Resonance also requires that $\cot\delta_0 = 0$ pass through zero "from the positive side" so first consider its behavior away from a zero. Since $\gamma \gg k$, $\cot\delta_0 \approx -\cos(kR)/\sin(kR)$. We can make the following table:

		Below			Above		
kR	Angle	$\cos(kR)$	$\sin(kR)$	$\cot\delta_0$	$\cos(kR)$	$\sin(kR)$	$\cot\delta_0$
$(\text{even}+1/2)\pi$	90°	+	+	−	−	+	+
$(\text{odd}+1/2)\pi$	270°	−	−	−	+	−	+
$(\text{even})\pi$	0°	+	−	+	+	+	−
$(\text{odd})\pi$	180°	−	+	+	−	−	−

So we have resonance only for the zeros at $kR = n\pi$, for both even and odd n. To find the energies, put $kR = n\pi - k_r R$ where $|k_r R| \ll 1$. Therefore

$$
-\frac{k}{\gamma} = \sin(kR)\cos(kR) = \frac{1}{2}\sin(2kR) = -\frac{1}{2}\sin(2k_r R) \approx -k_r R = kR - n\pi
$$

That is, $kR(1+1/\gamma R) = n\pi$ or $k \approx (n\pi/R)(1-1/\gamma R)$ and the energies are

$$
E_n = \frac{\hbar^2 k^2}{2m} = \frac{n^2\pi^2\hbar^2}{2mR^2}\left(1-\frac{2}{\gamma R}\right)
$$

which are the same as for the infinite spherical well, i.e. (3.7.25), up to the factor $(1 - 2/\gamma R)$. Finally, $(d\cos\delta_0/dE) = (d\cos\delta_0/dk)(dk/dE) = (m/\hbar^2 k)(d\cos\delta_0/dk)$. Noting that $\sin(kR) = \sin(n\pi - n\pi/\gamma R) \approx (-1)^{n+1} n\pi/\gamma R$, we have, using MATHEMATICA,

$$
\frac{d\cos\delta_0}{dk} = \frac{R}{\sin^2(kR)}\left[1+2\frac{\cos(kR)}{\sin(kR)}\right] \approx \frac{2R\cos(n\pi)}{(-1)^{3n+3}(n\pi/\gamma R)^3} = \frac{(-1)^n}{-(-1)^{3n}}\frac{2\gamma^3 R^4}{(n\pi)^3} = -\frac{2\gamma^3 R^4}{(n\pi)^3}
$$

Hence $\Gamma = -2/[(mR/n\pi\hbar^2)(-2\gamma^3R^4/(n\pi)^3)] = (n\pi)^4\hbar^2/m\gamma^3R^5 \to 0$ (rapidly) as $\gamma \to \infty$.

11. The perturbation is $V(\mathbf{x})\cos\omega t = V(\mathbf{x})[e^{i\omega t}+e^{-i\omega t}]/2$ so the first order transition amplitude from an initial state $|i\rangle$ to a final state $|f\rangle$ is, from (5.7.17) with $\omega_{fi}\equiv(E_f-E_i)/\hbar$,

$$\begin{aligned}
c_f^{(1)}(t) &= -\frac{i}{\hbar}\int_0^t e^{i\omega_{fi}t}\langle f|V(\mathbf{x})|i\rangle\frac{1}{2}\left[e^{i\omega t}+e^{-i\omega t}\right]dt \\
&= \frac{1}{2\hbar}\langle f|V(\mathbf{x})|i\rangle\left[\frac{1-\exp[i(\omega_{fi}+\omega)t]}{\omega_{fi}+\omega}+\frac{1-\exp[i(\omega_{fi}-\omega)t]}{\omega_{fi}-\omega}\right]
\end{aligned}$$

where we assume the perturbation turns on at $t=0$. As discussed in Chapter 5, the only appreciable contributions after long times come from $\omega_{fi}\mp\omega=0$ or $E_f=E_i\pm\hbar\omega$. Following Section 5.7, in particular (5.7.43), and writing $V_{fi}\equiv\langle f|V(\mathbf{x})|i\rangle$, we have

$$w_{i\to f}=\frac{2\pi}{\hbar}|V_{fi}|^2\left[\rho(E_f)|_{E_f=E_i+\hbar\omega}+\rho(E_f)|_{E_f=E_i-\hbar\omega}\right]$$

From (6.1.19) $\rho(E)=(L/2\pi)^3(mk_f/\hbar^2)d\Omega$. The incident flux $j_i=\hbar k_i/mL^3$ from (6.1.21). The cross section $d\sigma$ is the transition rate per incident flux, so

$$\frac{d\sigma}{d\Omega}=\frac{1}{d\Omega}\frac{w_{i\to f}}{j_i}=\frac{2\pi}{\hbar}|V_{fi}|^2\left(\frac{L}{2\pi}\right)^3\frac{m\sum k_f/\hbar^2}{\hbar k_i/mL^3}=\frac{m^2}{4\pi^2\hbar^4}\frac{\sum k_f}{k_i}|L^3V_{fi}|^2$$

where $\sum k_f\equiv k_f|_{E_f=E_i+\hbar\omega}+k_f|_{E_f=E_i-\hbar\omega}$. Assuming initial and final state plane waves,

$$\begin{aligned}
L^3V_{fi} &= L^3\langle k_f|V(\mathbf{x})|k_i\rangle = L^3\int d^3x''\int d^3x'\langle\mathbf{k}_f|\mathbf{x}''\rangle\langle\mathbf{x}''|V(\mathbf{x})|\mathbf{x}'\rangle\langle\mathbf{x}'|\mathbf{k}_i\rangle \\
&= \int d^3x'\ e^{i(\mathbf{k}_f-\mathbf{k}_i)}V(\mathbf{x}')\equiv\mathcal{V}(\mathbf{q})
\end{aligned}$$

using (6.2.6) and (6.2.7), with $\mathbf{q}\equiv\mathbf{k}_f-\mathbf{k}_i$. Since $\hbar^2k_f^2/2m=\hbar^2k_i^2/2m\pm\hbar\omega$, we have

$$\frac{d\sigma}{d\Omega}=\frac{m^2}{4\pi^2\hbar^4}\frac{1}{k_i}\left[\left(k_i^2+\frac{2m\omega}{\hbar}\right)^{1/2}+\left(k_i^2-\frac{2m\omega}{\hbar}\right)^{1/2}\right]|\mathcal{V}(\mathbf{q})|^2$$

For higher order terms, return to (5.7.17). The second order amplitude $c_f^{(2)}(t)$ will have denominators as in $c_f^{(1)}(t)$ above, replaced with $\omega_{fm}\pm\omega+\omega_{mi}\pm\omega=\omega_{fi}\pm 2\omega$. In other words, the final state will get contributions from the second harmonics, that is $\omega_{fi}=\pm 2\omega$. The third order amplitude would get contributions from the third harmonics, and so on.

12. We need to calculate the elastic scattering matrix element $\langle \mathbf{k}', 0|V(\mathbf{x}, \mathbf{x}')|\mathbf{k}, 0\rangle$ where $|0\rangle$ is the ground state of the hydrogen atom, and $V(\mathbf{x}, \mathbf{x}') = -e^2/|\mathbf{x}| + e^2/|\mathbf{x} - \mathbf{x}'|$. (In this notation, $\mathbf{x}$ locates the "fast" electron, while $\mathbf{x}'$ locates the atomic electron.) From (6.2.6) and (B.6.3) we know that $\langle \mathbf{x}, \mathbf{x}'|\mathbf{k}, 0\rangle = [\exp(i\mathbf{k}\cdot\mathbf{x})/L^{3/2}][2\exp(-r'/a)/a^{3/2}\sqrt{\pi}]$. So, inserting complete sets of states and defining $\mathbf{q} \equiv \mathbf{k} - \mathbf{k}'$, we have

$$
\begin{aligned}
\langle \mathbf{k}', 0|\frac{1}{|\mathbf{x}|}|\mathbf{k}, 0\rangle &= \int d^3x \int d^3x' \, \langle \mathbf{k}', 0|\frac{1}{|\mathbf{x}|}|\mathbf{x}, \mathbf{x}'\rangle\langle \mathbf{x}, \mathbf{x}'|\mathbf{k}, 0\rangle \\
&= \frac{1}{\pi a^3 L^3}\int d^3x \; e^{i\mathbf{q}\cdot\mathbf{x}}\frac{1}{|\mathbf{x}|}\int d^3x' \; e^{-2r'/a} \\
&= \frac{1}{\pi a^3 L^3}\left\{\frac{4\pi}{q^2} \text{ [See (6.9.10)]}\right\}\left\{4\pi\int_0^\infty r'^2\,dr' e^{-2r'/a}\right\} = \frac{4\pi}{q^2}\frac{1}{L^3} \\
\langle \mathbf{k}', 0|\frac{1}{|\mathbf{x}-\mathbf{x}'|}|\mathbf{k}, 0\rangle &= \int d^3x \int d^3x' \, \langle \mathbf{k}', 0|\frac{1}{|\mathbf{x}-\mathbf{x}'|}|\mathbf{x}, \mathbf{x}'\rangle\langle \mathbf{x}, \mathbf{x}'|\mathbf{k}, 0\rangle \\
&= \frac{1}{\pi a^3 L^3}\int d^3x \int d^3x' \; e^{i\mathbf{q}\cdot\mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}e^{-2r'/a} \\
&= \frac{1}{\pi a^3 L^3}\int d^3\xi \; e^{i\mathbf{q}\cdot\boldsymbol{\xi}}\frac{1}{\xi}\int d^3x' \; e^{i\mathbf{q}\cdot\mathbf{x}'}e^{-2r'/a} \\
&= \frac{1}{\pi a^3 L^3}\frac{4\pi}{q^2}\left\{2\pi\int_0^\infty r'^2 \; dr' \int_{-1}^1 d\mu \; e^{iqr'\mu}e^{-2r'/a}\right\} \\
&= \frac{4}{a^3 L^3}\frac{4\pi}{q^3}\int_0^\infty r' e^{-2r'/a}\sin(qr') \; dr' = \frac{64\pi}{L^3}\frac{1}{q^2}\frac{1}{(4+q^2a^2)^2} \\
\langle \mathbf{k}', 0|V(\mathbf{x}, \mathbf{x}')|\mathbf{k}, 0\rangle &= -\frac{4\pi e^2}{L^3}\frac{1}{q^2}\left[1 - \frac{16}{(4+q^2a^2)^2}\right]
\end{aligned}
$$

The differential cross section, following (6.9.6), is therefore

$$
\frac{d\sigma}{d\Omega} = L^6\left(\frac{1}{4\pi}\frac{2m}{\hbar^2}\right)^2 |\langle \mathbf{k}', 0|V(\mathbf{x}, \mathbf{x}')|\mathbf{k}, 0\rangle|^2 = \frac{4m^2e^4}{\hbar^4}\frac{1}{q^4}\left[1 - \frac{16}{(4+q^2a^2)^2}\right]^2
$$

13. See the comments in the *Errata*. I don't really know what this problem is doing here, and it will likely be eliminated in future editions. The original solutions manual refers to Finkelstein, "Non Relativistic Mechanics" (1973) for background, but I am not familiar with that book. As I mention in the *Errata*, it seems to me that this is about using angle-action variables to expose the $SO(4)$ symmetry in the Coulomb problem, discussed in Section 4.1 of this textbook. In any case, I don't find the original solution enlightening, and am not reproducing it here.

Chapter Seven

1. With $E \approx 3kT/2$, have $\lambda = h/p = hc/\sqrt{2(mc^2)E} = 2\pi(\hbar c)/\sqrt{3(mc^2)kT}$ $= 2\pi(2\times 10^{-7})/\sqrt{3\cdot 4\times 10^9\cdot 8.6\times 10^{-5}\cdot 2.17}$ m $= 8.4$Å, for helium. As the size of a helium atom is around 1Å, at this temperature, the DeBroglie wavelength spans many atoms, and that is the key. For heavier elements, the temperature needs to be proportionally smaller to get the same wavelength, and the atoms are larger, suggesting that even longer wavelengths are required. However, higher Z noble gases have interactions that prevent their remaining liquid at very low temperatures. Neon, for example, freezes at 27K.

2. For non-interacting particles, just add up the energies with two (i.e. $2s+1$) particles per energy level. If $N = 2m$ is even, then

$$E_{\text{Total}}^{\text{Even }N} = 2\times\left[\frac{1}{2}\hbar\omega + \frac{3}{2}\hbar\omega + \cdots + \frac{2m-1}{2}\hbar\omega\right] = \left[\sum_{i=1}^{N/2}(2i-1)\right]\hbar\omega = \frac{N^2}{4}\hbar\omega$$

For N odd, take the sum above to $(N-1)/2$ and add one unit at the top level, so

$$E_{\text{Total}}^{\text{Odd }N} = \frac{(N-1)^2}{4}\hbar\omega + \left(2\frac{N-1}{2}+1\right)\frac{\hbar\omega}{2} = \frac{N^2+1}{4}\hbar\omega$$

The *Fermi Energy* (see page 464) is the highest occupied level. This means $E_F = (N-1)\hbar\omega/2$ if N is even, and $E_F = N\hbar\omega/2$ if N is odd. For large N, $E_{\text{Total}} = N^2\hbar\omega/4$ and $E_F = N\hbar\omega/2$.

3. First write down the nine states $|jm\rangle = \sum_{m1,m2}|m_1,m_2\rangle\langle m_1,m_2|jm\rangle$ and then inspect their symmetry under $1\leftrightarrow 2$. Using Clebsch-Gordan Coefficients from pdg.lbl.gov, we find

m	$j=2$	$j=1$
2	$\|1,1\rangle$	
1	$\frac{1}{\sqrt{2}}\|1,0\rangle + \frac{1}{\sqrt{2}}\|0,1\rangle$	$\frac{1}{\sqrt{2}}\|1,0\rangle - \frac{1}{\sqrt{2}}\|0,1\rangle$
0	$\frac{1}{\sqrt{6}}\|1,-1\rangle + \sqrt{\frac{2}{3}}\|0,0\rangle + \frac{1}{\sqrt{6}}\|-1,1\rangle$	$\frac{1}{\sqrt{2}}\|1,-1\rangle - \frac{1}{\sqrt{2}}\|-1,1\rangle$
-1	$\frac{1}{\sqrt{2}}\|0,-1\rangle + \frac{1}{\sqrt{2}}\|-1,0\rangle$	$\frac{1}{\sqrt{2}}\|0,-1\rangle - \frac{1}{\sqrt{2}}\|-1,0\rangle$
-2	$\|-1,-1\rangle$	

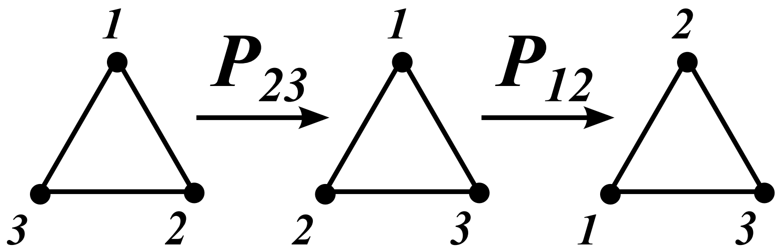
along with $|j=0\rangle = \frac{1}{\sqrt{3}}|1,-1\rangle - \frac{1}{\sqrt{3}}|0,0\rangle + \frac{1}{\sqrt{3}}|-1,1\rangle$. Two spin-one particles must obey Bose statistics, and since there is no orbital angular momentum, the state must be symmetric under the exchange $m_1 \leftrightarrow m_2$. Clearly this is true for $j=2$ and $j=0$, but the states are antisymmetric for $j=0$. Thus, two identical spin-one particles can only form s-states with $j=0$ or $j=2$.

4. If the electron were a spinless boson, then the total wave function (now with no spin part) must be symmetric, namely

$$\begin{aligned}\psi(x_1,x_2) &= \frac{1}{\sqrt{2}}\left[\psi_\alpha(x_1)\psi_\beta(x_2)+\psi_\beta(x_1)\psi_\alpha(x_2)\right] \qquad \text{if} \qquad \alpha\neq\beta \\ &= \psi_\alpha(x_1)\psi_\alpha(x_2) \qquad \text{if} \qquad \alpha=\beta\end{aligned}$$

We have only "singlet" parahelium. If we assume that the interaction due to spin is small, then there is no "triplet" orthohelium and the levels of parahelium remains the same.

5. The rotation operator about the z-axis from (3.1.16) is $\mathcal{D}(\phi)=\exp(-iJ_z\phi/\hbar)$. Since the particles at the triangle vertices are identical, $\mathcal{D}(2\pi/3)$ returns an indistinguishable state. Therefore we must have $\mathcal{D}(2\pi/3)|\alpha\rangle = \text{constant}|\alpha\rangle$. Consider, however, the following:



That is, the double permutation operation $P_{23}P_{12}$ is equivalent to a rotation through $2\pi/3$. Since the particles are spin-zero, $P_{ij}|\alpha\rangle=+|\alpha\rangle$, and the "constant" must be unity, so

$$e^{-iJ_z(2\pi/3)/\hbar}|jm\rangle = e^{-i2\pi m/3}|jm\rangle = +|jm\rangle$$

Thus $m/3$ must be an integer, i.e. $m=0,\pm3,\pm6,\ldots$.

6. The particles have spin-one, so the spin states must be symmetric (antisymmetric) if the spatial wave function is symmetric (antisymmetric). **(a)** For the symmetric case first,

(i) The state is simply $|+\rangle|+\rangle|+\rangle$. This is obviously $|jm\rangle=|3,3\rangle$, but let's prove it.

$$\begin{aligned}S_z|+\rangle|+\rangle|+\rangle &= (S_{1_z}+S_{2_z}+S_{3_z})|+\rangle|+\rangle|+\rangle = 3\hbar|+\rangle|+\rangle|+\rangle \\ \mathbf{S}^2|+\rangle|+\rangle|+\rangle &= [S_1^2+2S_{1_z}S_{2_z}+S_{1_+}S_{2_-}+S_{1_-}S_{2_+} \\ &\quad +S_2^2+2S_{1_z}S_{3_z}+S_{1_+}S_{3_-}+S_{1_-}S_{3_+} \\ &\quad +S_3^2+2S_{2_z}S_{3_z}+S_{2_+}S_{3_-}+S_{2_-}S_{3_+}]|+\rangle|+\rangle|+\rangle \\ &= [2+2+0+0+2+2+0+0+2+2+0+0]\hbar^2|+\rangle|+\rangle|+\rangle \\ &= 12\hbar^2|+\rangle|+\rangle|+\rangle = 3(3+1)\hbar^2|+\rangle|+\rangle|+\rangle\end{aligned}$$

(ii) The state is just $\frac{1}{\sqrt{3}}[|0\rangle|+\rangle|+\rangle+|+\rangle|0\rangle|+\rangle+|+\rangle|+\rangle|0\rangle]\equiv|\alpha\rangle$ for which $S_z|\alpha\rangle=2\hbar|\alpha\rangle$ is easy to prove. Also, you can reach this state from $S_-|+\rangle|+\rangle|+\rangle=[S_{1_-}+S_{2_-}+S_{3_-}]|+\rangle|+\rangle|+\rangle$, again showing that $m=2$. Also, since S_- and $\mathbf{S}^2$ commute, $\mathbf{S}^2|\alpha\rangle\propto\mathbf{S}^2S_-|+\rangle|+\rangle|+\rangle = S_-\mathbf{S}^2|+\rangle|+\rangle|+\rangle = 12\hbar^2S_-|+\rangle|+\rangle|+\rangle$, and so $|\alpha\rangle=|j=3,m=2\rangle$.

(iii) The state is all six combinations of the three S_z possibilities, that is

$$|\beta\rangle = \frac{1}{\sqrt{6}}\left[|+\rangle|0\rangle|-\rangle + |-\rangle|+\rangle|0\rangle + |0\rangle|-\rangle|+\rangle + |0\rangle|+\rangle|-\rangle + |-\rangle|0\rangle|+\rangle + |+\rangle|-\rangle|0\rangle\right]$$

Clearly $S_z|\beta\rangle = 0$, but we should be suspicious before suggesting this is a $j = 0$ state, as this state should include the combination $|0\rangle|0\rangle|0\rangle$. Indeed, if we carried out $\mathbf{S}^2|\beta\rangle$ as we did in **(a)**, then we would find components where all three particles were *not* in different states. That is, $|\beta\rangle$ is not an eigenstate of $\mathbf{S}^2$.

(b) Now consider the antisymmetric case. Cases **(i)** and **(ii)** are clearly not possible, since any component with two spins the same will remain the same under particle interchange, and we need the overall sign to change. The only possible state is

$$|\gamma\rangle = \frac{1}{\sqrt{6}}\left[|+\rangle|0\rangle|-\rangle + |-\rangle|+\rangle|0\rangle + |0\rangle|-\rangle|+\rangle - |0\rangle|+\rangle|-\rangle - |-\rangle|0\rangle|+\rangle - |+\rangle|-\rangle|0\rangle\right]$$

Clearly $S_z|\gamma\rangle = 0$, and in this case, because of the minus signs, it is not obvious that $\mathbf{S}^2|\gamma\rangle$ will involve components with two spins the same. So, let's work it out in detail and see.

$$\begin{aligned}
S_1^2|\gamma\rangle &= 2\hbar^2|\gamma\rangle = S_2^2|\gamma\rangle = S_3^2|\gamma\rangle \\
S_{1_z}S_{2_z}|\gamma\rangle &= \frac{1}{\sqrt{6}}\hbar^2\left[-|-\rangle|+\rangle|0\rangle + |+\rangle|-\rangle|0\rangle\right] \\
\left[S_{1_z}S_{2_z} + S_{1_z}S_{3_z} + S_{2_z}S_{3_z}\right]|\gamma\rangle &= -\hbar^2|\gamma\rangle \\
S_{1_+}S_{2_-}|\gamma\rangle &= 2[0 + |0\rangle|0\rangle|0\rangle + 0 - |+\rangle|0\rangle|-\rangle - |0\rangle|-\rangle|+\rangle - 0]\hbar^2 \\
S_{1_-}S_{2_+}|\gamma\rangle &= 2[|0\rangle|+\rangle|-\rangle + 0 + |0\rangle|-\rangle|+\rangle - 0 - 0 - |0\rangle|0\rangle|0\rangle]\hbar^2 \\
S_{1_+}S_{3_-}|\gamma\rangle &= 2[0 + |0\rangle|+\rangle|-\rangle + |+\rangle|-\rangle|0\rangle - 0 - |0\rangle|0\rangle|0\rangle - 0]\hbar^2 \\
S_{1_-}S_{3_+}|\gamma\rangle &= 2[|0\rangle|0\rangle|0\rangle + 0 + 0 - |-\rangle|+\rangle|0\rangle - 0 - |0\rangle|-\rangle|+\rangle]\hbar^2 \\
S_{2_+}S_{3_-}|\gamma\rangle &= 2[0 + 0 + |0\rangle|0\rangle|0\rangle - 0 - |-\rangle|+\rangle|0\rangle - |+\rangle|0\rangle|-\rangle]\hbar^2 \\
S_{2_-}S_{3_+}|\gamma\rangle &= 2[|+\rangle|-\rangle|0\rangle + |-\rangle|0\rangle|+\rangle + 0 - |0\rangle|0\rangle|0\rangle - 0 - 0]\hbar^2 \\
\text{``}\sum S_{i_\pm}S_{j_\mp}\text{''} &= -4\hbar^2|\gamma\rangle \\
\mathbf{S}^2|\gamma\rangle &= [2 + 2 + 2 - 2 - 4]\hbar^2|\gamma\rangle = 0
\end{aligned}$$

and, indeed, $S = 0$.

7. Obviously N is Hermitian, so $N|\eta\rangle = \eta|\eta\rangle$ where η is a real number. We know that the eigenvalues η of N cannot be negative since $\eta = \langle\eta|N|\eta\rangle = \left[\langle\eta|a^\dagger\right]\left[a|\eta\rangle\right] = \langle\alpha|\alpha\rangle \geq 0$ based on the "positivity postulate" of quantum mechanics. This was our starting point for the algebra of the simple harmonic oscillator, establishing a minimum value for η.

Now, consider $a^\dagger|\eta\rangle$. We have $N\left[a^\dagger|\eta\rangle\right] = a^\dagger a a^\dagger|\eta\rangle = a^\dagger\left[1 - N\right]|\eta\rangle = (1-\eta)\left[a^\dagger|\eta\rangle\right]$. Therefore $a^\dagger|\eta\rangle$ is also an eigenstate of N but with eigenvalue $(1-\eta)$ which must also be positive. Therefore, there is a *maximum* value of η as well.

8. The number of possible spin states is $2s+1=4$ for spin-3/2, so the configuration would be $(1s)^(2s)^4(2p)^{12}$; the $2p$ state can accommodate four electrons in each of $2l+1=3$ orbital p states. With $Z=10$, only two electrons are in the $2p$ state, so the degeneracy is $\begin{pmatrix} 12 \\ 2 \end{pmatrix} = 12!/2!10! = 66$. We would indeed call this "highly degenerate." The ground state should have spin states as symmetric as possible, and spacial states as antisymmetric as possible. The only antisymmetric spacial states are p-wave, that is $l_1 = l_2 = 1$ with total orbital angular momentum $L=1$. The total spin is $S = 3/2+3/2 = 3$, that is a spin 7-plet. For the total angular momentum $\mathbf{L}$ and $\mathbf{S}$ should be as "antiparallel" as possible, and this implies that $J=2$. Therefore the ground state would be $^7P_2$.

9. The single particle wave function is $\psi_n(x) = \sqrt{2/L}\sin(n\pi x/L)$, and the single particle energy is $E_n = n^2\pi^2\hbar^2/2mL^2$. The triplet spin state is symmetric, so in this case, the spatial state must be antisymmetric. Therefore (with $H = p_1^2/2m + p_2^2/2m$),

$$\begin{aligned}
\psi_{\text{gs}}(x_1,x_2)|_{\text{triplet}} &= \frac{1}{\sqrt{2}}\left[\psi_1(x_1)\psi_2(x_2) - \psi(x_2)\psi(x_1)\right] \\
H\ \psi_{\text{gs}}(x_1,x_2)|_{\text{triplet}} &= \left[\frac{\pi^2\hbar^2}{2mL^2} + \frac{4\pi^2\hbar^2}{2mL^2}\right]\psi_{\text{gs}}(x_1,x_2)|_{\text{triplet}} = \frac{5\pi^2\hbar^2}{2mL^2}\ \psi_{\text{gs}}(x_1,x_2)|_{\text{triplet}}
\end{aligned}$$

For the singlet spin state, the spatial state is symmetric, so

$$\begin{aligned}
\psi_{\text{gs}}(x_1,x_2)|_{\text{singlet}} &= \psi_1(x_1)\psi_1(x_2) \\
H\ \psi_{\text{gs}}(x_1,x_2)|_{\text{singlet}} &= \left[\frac{\pi^2\hbar^2}{2mL^2} + \frac{\pi^2\hbar^2}{2mL^2}\right]\psi_{\text{gs}}(x_1,x_2)|_{\text{singlet}} = \frac{\pi^2\hbar^2}{mL^2}\ \psi_{\text{gs}}(x_1,x_2)|_{\text{singlet}}
\end{aligned}$$

In first order perturbation theory, the energy shift is

$$\Delta E = \langle \text{gs}|V|\text{gs}\rangle = \int dx_1 \int dx_2\ \psi_{\text{gs}}(x_1,x_2)V(x_1,x_2)\psi_{\text{gs}}(x_1,x_2) = -\lambda \int dx\ \psi_{\text{gs}}^2(x,x)$$

In the triplet case, obviously $\Delta E = 0$. The antisymmetric spatial wave function never allows the two particles to be at the same place, so they never feel the δ-function potential. On the other hand, in the singlet case,

$$\Delta E_{\text{singlet}} = -\lambda\left(\frac{2}{L}\right)^2 \int_0^L \sin^4\left(\frac{\pi x}{L}\right)dx = -\lambda\left(\frac{2}{L}\right)^2\frac{3L}{8} = -\frac{3\lambda}{2L}$$

10. To prove the orthogonality relations (7.6.11), start with the definitions

$$
\begin{aligned}
\hat{\mathbf{e}}_{\mathbf{k}\pm} &= \mp\frac{1}{\sqrt{2}}\left(\hat{\mathbf{e}}_{\mathbf{k}}^{(1)} \pm i\hat{\mathbf{e}}_{\mathbf{k}}^{(2)}\right) \qquad \text{so we have} \\
\hat{\mathbf{e}}_{\mathbf{k}\lambda}^{*} \cdot \hat{\mathbf{e}}_{\pm\mathbf{k}\lambda'} &= \left[-\lambda\frac{1}{\sqrt{2}}\left(\hat{\mathbf{e}}_{\mathbf{k}}^{(1)} - \lambda i\hat{\mathbf{e}}_{\mathbf{k}}^{(2)}\right)\right] \cdot \left[-\lambda'\frac{1}{\sqrt{2}}\left(\hat{\mathbf{e}}_{\pm\mathbf{k}}^{(1)} + \lambda' i\hat{\mathbf{e}}_{\pm\mathbf{k}}^{(2)}\right)\right] \\
&= \frac{\lambda\lambda'}{2}\left[\hat{\mathbf{e}}_{\mathbf{k}}^{(1)} \cdot \hat{\mathbf{e}}_{\pm\mathbf{k}}^{(1)} + i\lambda'\hat{\mathbf{e}}_{\mathbf{k}}^{(1)} \cdot \hat{\mathbf{e}}_{\pm\mathbf{k}}^{(2)} - i\lambda\hat{\mathbf{e}}_{\mathbf{k}}^{(2)} \cdot \hat{\mathbf{e}}_{\pm\mathbf{k}}^{(1)} + \lambda\lambda'\hat{\mathbf{e}}_{\mathbf{k}}^{(2)} \cdot \hat{\mathbf{e}}_{\pm\mathbf{k}}^{(2)}\right] \\
&= \frac{\lambda\lambda'}{2}\left[\pm 1 + 0 - 0 \pm \lambda\lambda'\right] \\
&= \pm 1 \qquad \text{if} \lambda = \lambda' \\
&= 0 \qquad \text{if} \lambda \neq \lambda' \\
\hat{\mathbf{e}}_{\mathbf{k}\lambda}^{*} \times \hat{\mathbf{e}}_{\pm\mathbf{k}\lambda'} &= \left[-\lambda\frac{1}{\sqrt{2}}\left(\hat{\mathbf{e}}_{\mathbf{k}}^{(1)} - \lambda i\hat{\mathbf{e}}_{\mathbf{k}}^{(2)}\right)\right] \times \left[-\lambda'\frac{1}{\sqrt{2}}\left(\hat{\mathbf{e}}_{\pm\mathbf{k}}^{(1)} + \lambda' i\hat{\mathbf{e}}_{\pm\mathbf{k}}^{(2)}\right)\right] \\
&= \frac{\lambda\lambda'}{2}\left[\hat{\mathbf{e}}_{\mathbf{k}}^{(1)} \times \hat{\mathbf{e}}_{\pm\mathbf{k}}^{(1)} + i\lambda'\hat{\mathbf{e}}_{\mathbf{k}}^{(1)} \times \hat{\mathbf{e}}_{\pm\mathbf{k}}^{(2)} - i\lambda\hat{\mathbf{e}}_{\mathbf{k}}^{(2)} \times \hat{\mathbf{e}}_{\pm\mathbf{k}}^{(1)} + \lambda\lambda'\hat{\mathbf{e}}_{\mathbf{k}}^{(2)} \times \hat{\mathbf{e}}_{\pm\mathbf{k}}^{(2)}\right] \\
&= \frac{\lambda\lambda'}{2}\left[0 \pm i\lambda'\hat{\mathbf{k}} \pm i\lambda\hat{\mathbf{k}} + 0\right] \\
&= \pm i\hat{\mathbf{k}} \qquad \text{if} \lambda = \lambda' \\
&= 0 \qquad \text{if} \lambda \neq \lambda'
\end{aligned}
$$

The first result (7.6.11a) serves to collapse the two sums over λ and λ' into one, when calculating $|\mathbf{E}|^2 = \mathbf{E}^* \cdot \mathbf{E}$ from (7.6.14), and the integral (7.6.15) collapses the two sums over $\mathbf{k}$ and $\mathbf{k}'$ into one, leading to (7.6.16). The expression for the magnetic field is

$$
\mathbf{B}(\mathbf{x}, t) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{x}, t) = \frac{i}{c}\sum_{\mathbf{k},\lambda} \omega_k \left[\mathbf{A}_{\mathbf{k},\lambda} e^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} - \mathbf{A}_{\mathbf{k},\lambda}^{*} e^{i(\omega_k t - \mathbf{k}\cdot\mathbf{x})}\right] \hat{\mathbf{k}} \times \hat{\mathbf{e}}_{\mathbf{k}}^{(\lambda)}
$$

which is very similar to (7.6.14), differing by the presence of $\hat{\mathbf{k}} \times \hat{\mathbf{e}}_{\mathbf{k},\lambda}$ instead of $\hat{\mathbf{e}}_{\mathbf{k},\lambda}$. But

$$
\hat{\mathbf{k}} \times \hat{\mathbf{e}}_{\mathbf{k},\pm} = -\mp\frac{1}{\sqrt{2}}\left[\hat{\mathbf{k}} \times \hat{\mathbf{e}}_{\mathbf{k}}^{(1)} \pm i\hat{\mathbf{k}} \times \hat{\mathbf{e}}_{\mathbf{k}}^{(2)}\right] = -\mp\frac{1}{\sqrt{2}}\left[\hat{\mathbf{e}}_{\mathbf{k}}^{(2)} \mp i\hat{\mathbf{e}}_{\mathbf{k}}^{(1)}\right] = i\hat{\mathbf{e}}_{\mathbf{k},\pm}
$$

so that the calculation of $|\mathbf{B}|^2 = \mathbf{B}^* \cdot \mathbf{B}$ carries through directly as for the electric field. The cross terms, however, have opposite sign, and therefore cancel when adding the contributions to the energy from electric and magnetic fields, leading to (7.6.17).

Chapter Eight

1. (a) (Note the typo in the exponent.) $m_pc^2 = (1.67 \times 10^{-27}$ kg$)(3.00 \times 10^8$ m/s$)^2 = 1.5 \times 10^{-10}$ joule (1 eV/1.60×10^{-19} joule $= 9.39 \times 10^8$ eV $= 0.939$ GeV.
(b) $E = pc \sim (\hbar/1 \text{ fm})c = 200 \text{ MeV} \cdot \text{fm}/1 \text{ fm} = 200$ MeV, which is about the same as the pion mass. The point is that, at these distances, mass and energy are not easily distinguished. That is, in this regime (called "particle physics") the formalism has to be relativistic.
(c) Using "$[a]$" to mean "dimensions of a", i.e. M, L, or T for mass, length or time, we note that $[G] = L^3M^{-1}T^{-2}$, $[\hbar] = ML^2T^{-1}$, and $[c] = LT^{-1}$. Writing $M_P = G^x\hbar^y c^z$ we must have $1 = -x + y$, $0 = 3x + 2y + z$, and $0 = -2x - y - z$, so $x = -1/2$, $y = 1/2$, and $z = 1/2$. So $M_Pc^2 = \sqrt{\hbar c^5/G} = \sqrt{(1.05 \times 10^{-34})(3 \times 10^8)^5/(6.67 \times 10^{-11})} = 1.96 \times 10^9$ j$=1.2 \times 10^{19}$ GeV.

2. This problem is trivial, but the implications are important. Since

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

the metric tensor is its own inverse, i.e. $\eta^{\mu\lambda}\eta_{\lambda\nu} = \delta^\mu_\nu$. It is therefore simple to show that the contravariant form of the metric follows appropriately from the covariant form, that is $\eta^{\mu\lambda}\eta^{\nu\sigma}\eta_{\lambda\sigma} = \delta^\mu_\sigma\eta^{\nu\sigma} = \eta^{\nu\mu} = \eta^{\mu\nu}$ since it is also symmetric. Also $a^\mu b_\mu = a_\nu\eta^{\mu\nu}b^\lambda\eta_{\lambda\mu} = a_\nu b^\lambda\delta^\nu_\lambda = a_\nu b^\nu = a_\mu b^\mu$.

3. For (8.1.11) to be a conserved current, we must show that $\partial_\mu j^\mu = 0$:

$$\begin{aligned} \partial_\mu j^\mu &= \frac{i}{2m}\left[\partial_\mu(\Psi^*\partial^\mu\Psi) - \partial_\mu\left((\partial^\mu\Psi)^*\Psi\right)\right] \\ &= \frac{i}{2m}\left[(\partial_\mu\Psi^*)(\partial^\mu\Psi) + \Psi^*(\partial^2\Psi) - (\partial^2\Psi^*)\Psi - (\partial^\mu\Psi)^*(\partial_\mu\Psi)\right] \\ &= \frac{i}{2m}\left[\Psi^*(-m^2\Psi) - (-m^2\Psi^*)\Psi\right] = 0 \end{aligned}$$

4. This is a silly, trivial problem. The Klein-Gordon equation basically comes from writing $E^2 = p^2 + m^2$ with E replaced by $i\partial^0$ and p replaced by $-i\boldsymbol{\nabla}$. In other words

$$\left[E^2 - p^2\right]\Psi = \left[-(\partial^0)^2 + \boldsymbol{\nabla}^2\right]\Psi = -\partial^\mu\partial_\mu\Psi = m^2\Psi \qquad \text{or} \qquad (\partial^\mu\partial_\mu + m^2)\Psi = 0$$

which is (8.1.8). This can be read as replacing $p^\mu p_\mu = E^2 - p^2$ with the operator $-\partial^\mu\partial_\mu$, i.e. p^μ with $-i\partial^\mu$. So, the minimal electromagnetic substitution $p^\mu \to p^\mu - eA^\mu$ becomes $-i\partial^\mu \to -i\partial^\mu - eA^\mu = -i(\partial^\mu - ieA^\mu) = -iD^\mu$ where $D^\mu \equiv \partial^\mu - ieA^\mu$. (Did I make a sign error in the definition of D^μ in the textbook? I guess so.)

5. Rewrite (8.1.14) using $D_\mu D^\mu = D_t^2 - \mathbf{D}^2$ as $D_t^2\Psi = \mathbf{D}^2\Psi - m^2\Psi$. Then, using (8.1.15),

$$\begin{aligned}
iD_t\phi &= \frac{i}{2}D_t\Psi - \frac{1}{2m}D_t^2\Psi = \frac{i}{2}D_t\Psi - \frac{1}{2m}\mathbf{D}^2\Psi + \frac{m}{2}\Psi \\
&= -\frac{1}{2m}\mathbf{D}^2\Psi + \frac{m}{2}\left[\Psi + \frac{i}{m}D_t\Psi\right] = -\frac{1}{2m}\mathbf{D}^2\Psi + m\phi \\
\text{and} \qquad iD_t\chi &= \frac{i}{2}D_t\Psi + \frac{1}{2m}D_t^2\Psi = \frac{i}{2}D_t\Psi + \frac{1}{2m}\mathbf{D}^2\Psi - \frac{m}{2}\Psi \\
&= +\frac{1}{2m}\mathbf{D}^2\Psi - \frac{m}{2}\left[\Psi - \frac{i}{m}D_t\Psi\right] = +\frac{1}{2m}\mathbf{D}^2\Psi - m\chi
\end{aligned}$$

which are (8.1.16) since $\Psi = \phi + \chi$. These two equations obviously become (8.1.18) since

$$\tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \text{and} \qquad \tau_3 + i\tau 2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

6. Write the solutions as

$$\Upsilon(\boldsymbol{x}, t) = \begin{pmatrix} a \\ b \end{pmatrix} e^{-iEt + i\boldsymbol{p}\cdot\boldsymbol{x}}$$

which results in the matrix equation

$$E\begin{pmatrix} a \\ b \end{pmatrix} = \left[\frac{p^2}{2m}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + m\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]\begin{pmatrix} a \\ b \end{pmatrix}$$

or

$$\begin{pmatrix} \frac{p^2}{2m} + m - E & \frac{p^2}{2m} \\ -\frac{p^2}{2m} & -\frac{p^2}{2m} - m - E \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Take the determinant to get the characteristic equation

$$-\left(\frac{p^2}{2m} + m\right)^2 + E^2 + \left(\frac{p^2}{2m}\right)^2 = 0$$

so that

$$E^2 = 2\frac{p^2}{2m}m + m^2 = p^2 + m^2$$

in which case the energy eigenvalues are

$$E = \pm E_p \qquad \text{where} \qquad E_p = \sqrt{p^2 + m^2}$$

In order to find the eigenfunctions, first rewrite the characteristic equation by multiplying through by $2m$ and also writing $p^2 = E_p^2 - m^2$, so

$$\begin{pmatrix} E_p^2 + m^2 - 2mE & E_p^2 - m^2 \\ m^2 - E_p^2 & -E_p^2 - m^2 - 2mE \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

In the case of *positive energy eigenvalues* $E = +E_p$, we have

$$\begin{bmatrix} (E_p - m)^2 & (E_p + m)(E_p - m) \\ (m - E_p)(m + E_p) & -(E_p + m)^2 \end{bmatrix} \begin{bmatrix} a^+ \\ b^+ \end{bmatrix} = 0$$

which implies that $(E_p - m)a^+ + (E_p + m)b^+ = 0$. Normalizing using the relation $\Upsilon^\dagger \tau_3 \Upsilon = +1$ we have

$$\left(a^+\right)^2 - \left(b^+\right)^2 = \left(a^+\right)^2 \left[1 - \frac{(E_p - m)^2}{(E_p + m)^2}\right] = \left(a^+\right)^2 \left[\frac{4mE_p}{(E_p + m)^2}\right] = +1$$

in which case

$$a^+ = \frac{E_p + m}{2\sqrt{mE_p}} \qquad \text{and} \qquad b^+ = \frac{m - E_p}{2\sqrt{mE_p}}$$

Similarly, for *negative energy eigenvalues* $E = -E_p$, we have

$$\begin{bmatrix} (E_p + m)^2 & (E_p + m)(E_p - m) \\ (m - E_p)(m + E_p) & -(E_p - m)^2 \end{bmatrix} \begin{bmatrix} a^- \\ b^- \end{bmatrix} = 0$$

which implies that $(E_p + m)a^- + (E_p - m)b^- = 0$. This time we normalize using the relation $\Upsilon^\dagger \tau_3 \Upsilon = -1$ and therefore

$$\left(a^-\right)^2 - \left(b^-\right)^2 = \left(a^-\right)^2 \left[1 - \frac{(E_p + m)^2}{(E_p - m)^2}\right] = \left(a^-\right)^2 \left[\frac{-4mE_p}{(E_p - m)^2}\right] = -1$$

in which case

$$a^- = \frac{m - E_p}{2\sqrt{mE_p}} \qquad \text{and} \qquad b^- = \frac{E_p + m}{2\sqrt{mE_p}}$$

7. First, a mea culpa. I wrote this problem (from Landau's book) years before the manuscript was completed and I came to work out this solution. There are a few mistakes, I realize, in the problem and a little in the text. One has to do with the sign of e, which in this book (unlike any others I know) is negative. Therefore $D_\mu \equiv \partial_\mu - ieA_\mu$ (see problem 4) and $A_0 = \Phi = -Z|e|/r = +Ze/r$. In the problem statement, I misused k in the argument of $u(r)$ and also in the definition of ρ. Also, I wrote "Work the upper component", but I don't recall why. It may be the default when using the positive energy solution.

Anyway, we start with the Klein-Gordon Equation (8.1.14), namely

$$\left[D_\mu D^\mu + m^2\right] \Psi(\mathbf{x}, t) = 0$$

where $D_\mu \equiv \partial_\mu - ieA_\mu$. With $\mathbf{A} = 0$ and $eA_0 = Ze^2/r = Z\alpha/r$, this becomes

$$\left[\left(\partial_t - i\frac{Z\alpha}{r}\right)^2 - \boldsymbol{\nabla}^2 + m^2\right] \Psi(\mathbf{x}, t) = 0$$

Next, as suggested, put $\Psi(\mathbf{x},t) = Ne^{-iEt}[u_l(r)/r]Y_{lm}(\theta,\phi)$. Then

$$\left[\left(E + \frac{Z\alpha}{r}\right)^2 + \boldsymbol{\nabla}^2 - m^2\right]\frac{u_l(r)}{r}Y_{lm}(\theta,\phi) = 0$$

From (3.6.21) the Laplacian $\boldsymbol{\nabla}^2$ can be written as

$$\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\mathbf{L}^2$$

where, in this context, $\mathbf{L}^2$ is a differential operator in θ and ϕ. Therefore

$$\left[\left(E + \frac{Z\alpha}{r}\right)^2 - m^2 - \frac{l(l+1)}{r^2}\right]\frac{u}{r} + \frac{1}{r}\frac{d^2u}{dr^2} = 0$$

Finally, with $\gamma^2 \equiv 4(m^2 - E^2)$ and $\rho \equiv \gamma r$, this becomes

$$\frac{d^2u}{d\rho^2} + \left[\frac{2EZ\alpha}{\gamma\rho} - \frac{1}{4} - \frac{l(l+1) - (Z\alpha)^2}{\rho^2}\right]u = 0$$

For $\rho \to \infty$, this becomes $d^2u/d\rho^2 = u/4$ or $u(\rho) = \exp(\pm\rho/2)$. Only the negative sign gives a normalizable solution, so we write $u(\rho) = w(\rho)\exp(-\rho/2)$ in which case

$$\frac{d^2w}{d\rho^2} - \frac{dw}{d\rho} + \left[\frac{2EZ\alpha}{\gamma\rho} - \frac{l(l+1) - (Z\alpha)^2}{\rho^2}\right]w = 0$$

Now substitute $w(\rho) = \sum_{q=0}^{\infty} C_n\rho^{k+q}$ and collect terms into the same power of ρ by redefining the index of summation. This gives

$$\begin{aligned}
&[k(k-1) - l(l+1) + (Z\alpha)^2]C_0\rho^{k-2} \\
+\textstyle\sum_{q=0}^{\infty} &\{[(k+q+1)(k+q) - l(l+1) + (Z\alpha)^2]C_{q+1} \\
&- \left[(k+q) - \frac{2EZ\alpha}{\gamma}\right]C_q\Big\}\,\rho^{k+q-1} = 0
\end{aligned}$$

This series is set to zero term by term. Solving $k(k-1) - l(l+1) + (Z\alpha)^2 = 0$ for k gives

$$k = \frac{1}{2} \pm \frac{1}{2}\left[1 + 4l(l+1) - 4(Z\alpha)^2\right]^{1/2} = \frac{1}{2} \pm \left[\left(l + \frac{1}{2}\right)^2 - (Z\alpha)^2\right]^{1/2}$$

Near the origin, the wave function goes like $u(r)/r \propto r^{k-1}$, so the expectation value of kinetic energy goes like $r^{k-1}r^{k-3}r^2 = r^{2k-2}$. Consider the negative sign solution for k. For $l = 0$, k is close to zero, and negative for nonzero l, so the kinetic energy diverges too rapidly. For the positive sign solution, with $Z\alpha \ll 1$, $k \approx l+1$ and the wave function goes like r^l, which is the nonrelativistic result from the Schrödinger equation. All this points to taking the positive sign in the solution for k.

Now set the higher power terms of ρ to zero. We have

$$C_{q+1} = \frac{k+q-2EZ\alpha/\gamma}{(k+q+1)(k+q)-l(l+1)+(Z\alpha)^2}C_q \sim \frac{1}{q}C_n \qquad \text{as } q\to\infty$$

so that the series approaches e^ρ. In other words, the wave function is proportional to $e^{+\rho/2}$ for large ρ which is unacceptably divergent. Let the series terminate at $q=N$, then

$$k+N-2EZ\alpha/\gamma = k+N-EZ\alpha/\sqrt{m^2-E^2}=0$$

Note that $N=0$ is possible. As in (3.7.51), then, define the principle quantum number $n=N=l+1$, and so $(m^2-E^2)(k+n-l-1)^2 = E^2(Z\alpha)^2$. Solving for E we find

$$E = \frac{m}{\left(1+(Z\alpha)^2\left[n-l-\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-(Z\alpha)^2}\right]^{-2}\right)^{1/2}}$$

A Taylor expansion of this expression is straightforward, but I used MAPLE instead:

$$\begin{aligned} E &= m-\frac{m(Z\alpha)^2}{2n^2} \\ &\quad +\frac{m(Z\alpha)^2}{2n^2}\left[\frac{3}{4n^2}-\frac{1}{n(l+1/2)}\right](Z\alpha)^2 \\ &\quad +\frac{m(Z\alpha)^2}{2n^2}\left[\frac{3}{2n^3(l+1/2)}-\frac{n+3(l+1/2)}{4n^2(l+1/2)^3}-\frac{5}{8n^4}\right](Z\alpha)^4+\cdots \end{aligned}$$

The first term is just the rest energy, the second is the Balmer formula, and the third is the relativistic correction to the kinetic energy; the spin-orbit term is, of course, missing. (See the solution to Problem 16.)

Jenkins and Kunselman give a large number of transitions, both experimental values and Klein-Gordon solutions, for π^- atoms. For example, the $3D\to 2P$ transition in ^{59}Co is 384.6 ± 1.0 keV, while the "Klein-Gordon energy" is listed as 378.6 keV. With $m=m_\pi=$ 139.57 MeV and $\alpha=1/137.036$ (from PDG 2010) and $Z=27$, the Balmer transition energy is $-2709.1\times(1/9-1/4)=376.3$ keV, and the relativistic correction to the kinetic energy adds to this an amount

$$376.3\text{ keV}\times\left[\frac{3}{36}-\frac{3}{16}-\frac{1}{15/2}+\frac{1}{3}\right]\times(27\alpha)^2 = 1.4\text{ keV}$$

for a total (to first order) transition energy 377.7 keV. The next order correction will be smaller by $\sim (Z\alpha)^2 = 4/100$ and will not account for the difference between this and the number in the paper; theirs is likely due to older values for the pion mass.

8. All of these follow from equations (8.2.4), and $\mathsf{Tr}(AB) = \mathsf{Tr}(BA)$. For example

$$\mathsf{Tr}(\beta) = \mathsf{Tr}(\gamma^0) = -\mathsf{Tr}(\gamma^i\gamma^i\gamma^0) = -\mathsf{Tr}(\gamma^i\gamma^0\gamma^i) = +\mathsf{Tr}(\gamma^0\gamma^i\gamma^i) = -\mathsf{Tr}(\gamma^0)$$

and so $\mathsf{Tr}(\gamma^0) = 0$. In other words, insert an appropriate γ^μ twice to get a factor ± 1, then split the pair using the commutativity property above, then reverse the order using (8.2.4c) to pick up a minus sign, then contract the two γ^μ that you inserted in the first place. You always get that the trace equals its own negative, so must be zero.

9. We are to construct the γ matrices from

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{bmatrix} \qquad \text{and} \qquad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

where

$$\alpha_i \equiv \gamma^0\gamma^i \qquad \text{and} \qquad \beta \equiv \gamma^0$$

so that

$$\gamma^0 = \beta \qquad \text{and} \qquad \gamma^i = \gamma^0\alpha^i$$

and we can write the γ matrices explicitly as

$$\begin{aligned}
\gamma^0 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\
\gamma^1 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{bmatrix} \\
\gamma^2 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix} \\
\gamma^3 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{bmatrix}
\end{aligned}$$

It is simple enough to multiply out the 4×4 matrices, or even to use the compact 2×2 form we derive here, and show that the Clifford Algebra is satisfied.

10. The Schrödinger equation with the Dirac Hamiltonian is

$$\begin{aligned}
i\partial_t\Psi &= H\Psi = (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)\Psi \\
&= -i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\Psi + \beta m\Psi \\
\text{so that} \qquad -i\partial_t\Psi^\dagger &= +i(\boldsymbol{\nabla}\Psi^\dagger)\cdot\boldsymbol{\alpha} + \beta m\Psi^\dagger
\end{aligned}$$

Note that $\boldsymbol{\alpha}$ and β are Hermitian. Now take the time derivative of the probability density

$$\frac{\partial \rho}{\partial t} = \partial_t(\Psi^\dagger \Psi) = (\partial_t \Psi^\dagger)\Psi + \Psi^\dagger \partial_t \Psi = \left[-(\boldsymbol{\nabla}\Psi^\dagger)\cdot\boldsymbol{\alpha}\Psi - \Psi^\dagger \boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\Psi\right] = -\boldsymbol{\nabla}\cdot\left(\Psi^\dagger\boldsymbol{\alpha}\Psi\right)$$

Therefore $\rho \equiv \Psi^\dagger\Psi$ satisfies the continuity equation for the current $\mathbf{j} \equiv \Psi^\dagger\boldsymbol{\alpha}\Psi$.

11. As indicated in the text, this decouples into two eigenvalue problems, one for u_1 and u_3, and the other for u_2 and u_4. That is, we have

$$\begin{bmatrix} m & p \\ p & -m \end{bmatrix}\begin{bmatrix} u_1 \\ u_3 \end{bmatrix} = E\begin{bmatrix} u_1 \\ u_3 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} m & -p \\ -p & -m \end{bmatrix}\begin{bmatrix} u_2 \\ u_4 \end{bmatrix} = E\begin{bmatrix} u_2 \\ u_4 \end{bmatrix}$$

The first equation implies that $(m-E)(-m-E)-p^2 = -(m^2-E^2)-p^2 = 0$ which implies that $E = \pm E_p$ where $E_p \equiv \sqrt{p^2+m^2}$. The second gives the same characteristic equation, so the eigenvalues are once again $E = \pm E_p$.

12. Since $j^\mu \equiv \bar{\Psi}\gamma^\mu\Psi$ with $\bar{\Psi} \equiv \Psi^\dagger\gamma^0$, $j^0 = \Psi^\dagger\gamma^0\gamma^0\Psi = \Psi^\dagger\Psi$ and $\mathbf{j} = \Psi^\dagger\gamma^0\boldsymbol{\gamma}\Psi = \Psi^\dagger\boldsymbol{\alpha}\Psi$. So, for if Ψ has four (real) components a, b, c, and d, then

$$\begin{aligned}
j^0 &= a^2+b^2+c^2+d^2 \\
j^1 &= \begin{bmatrix} a & b & c & d \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 2(ad+bc) \\
j^2 &= \begin{bmatrix} a & b & c & d \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0 \\
j^3 &= \begin{bmatrix} a & b & c & d \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 2(ac-bd)
\end{aligned}$$

For each of the spinors (8.2.22), the (normalized) probability density is

$$j^0 = \frac{E_p+m}{2E_p}\left[1+\frac{p^2}{(E_p+m)^2}\right] = 1$$

that is, a constant, independent of momentum. (Note that the exponential plane wave factors multiply to one in any combination of $\Psi^\dagger\Psi$.) Also for each of the spinors, $j^1 = 0$. For both positive energy solutions $u_R^{(+)}(p)$ and $u_L^{(+)}(p)$, we find

$$j^3 = \frac{E_p+m}{2E_p}\,\frac{2p}{E_p+m} = \frac{p}{E_p}$$

that is, the velocity of the particle. For both negative energy solutions $u_R^{(-)}(p)$ and $u_L^{(-)}(p)$, we find

$$j^3 = \frac{E_p + m}{2E_p}\frac{-2p}{E_p + m} = -\frac{p}{E_p}$$

that is, the velocity of the particle moving in the direction opposite from the momentum.

13. Work out $U_T = \gamma^1\gamma^3$ as follows:

$$U_T = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{pmatrix} = i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} = i\sigma^2 \otimes I$$

where (3.2.34) and (3.2.35) imply that $\sigma^1\sigma^3 = i\varepsilon_{13k}\sigma^k = -i\sigma^2$.

14. Refer to **(9)** for the γ matrices in explicit form. The positive helicity, positive energy electron free particle Dirac wave function is

$$\Psi(\mathbf{x}, t) = u_R^{(+)}(p)e^{-ip_\mu x^\mu} = \begin{bmatrix} 1 \\ 0 \\ p/(E_p + m) \\ 0 \end{bmatrix} e^{-i(E_p t - \mathbf{p}\cdot\mathbf{x})}$$

We can therefore construct the following

$$
\begin{aligned}
\mathcal{P}\Psi(\mathbf{x},t) = \gamma^0\Psi(-\mathbf{x},t) &= \begin{bmatrix} 1 \\ 0 \\ -p/(E_p+m) \\ 0 \end{bmatrix} e^{-i(E_p t+\mathbf{p}\cdot\mathbf{x})} \\
\mathcal{C}\Psi(\mathbf{x},t) = i\gamma^2\Psi^*(\mathbf{x},t) &= \begin{bmatrix} 0 \\ -p/(E_p+m) \\ 0 \\ 1 \end{bmatrix} e^{+i(E_p t-\mathbf{p}\cdot\mathbf{x})} \\
\mathcal{C}\mathcal{P}\Psi(\mathbf{x},t) = i\gamma^2\gamma^0\Psi^*(-\mathbf{x},t) &= \begin{bmatrix} 0 \\ p/(E_p+m) \\ 0 \\ 1 \end{bmatrix} e^{+i(E_p t+\mathbf{p}\cdot\mathbf{x})} \\
\mathcal{T}\Psi(\mathbf{x},t) = \gamma^1\gamma^3\Psi^*(\mathbf{x},t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\Psi^*(\mathbf{x},t) &= \begin{bmatrix} 0 \\ -1 \\ 0 \\ -p/(E_p+m) \end{bmatrix} e^{+i(E_p t-\mathbf{p}\cdot\mathbf{x})} \\
\mathcal{P}\mathcal{T}\Psi(\mathbf{x},t) &= \begin{bmatrix} 0 \\ -1 \\ 0 \\ p/(E_p+m) \end{bmatrix} e^{+i(E_p t+\mathbf{p}\cdot\mathbf{x})} \\
\mathcal{C}\mathcal{P}\mathcal{T}\Psi(\mathbf{x},t) &= \begin{bmatrix} p/(E_p+m) \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{-i(E_p t+\mathbf{p}\cdot\mathbf{x})}
\end{aligned}
$$

We see that $\mathcal{C}\mathcal{P}\mathcal{T}\Psi(\mathbf{x},t)$ is the wave function for a negative energy, right-handed electron with momentum $-\mathbf{p}$. This is the "hole" that we call a positron.

15. This is also a silly problem, with the solution pretty much outlined in the text. For large i, (8.4.39) shows that b_i is proportional to a_i. Furthermore, for large i, equations (8.4.38) show that a_i is proportional to $+1/i$. Therefore, each of the series (8.4.32) or (8.4.33) look like $x^i/i!$, that is e^x for large x. As they are multiplied by factors $\exp[-(1-\varepsilon^2)^{1/2}x]$ but $\varepsilon < 1$, the functions $u(x)$ and $v(x)$ will grow without bound for large x unless the series terminates.

16. First, expand the second term in the denominator of (8.4.43) to second order in $(Z\alpha)^2$:

$$
\begin{aligned}
\frac{(Z\alpha)^2}{\left[\sqrt{(j+1/2)^2-(Z\alpha)^2}+n'\right]^2} &= (Z\alpha)^2\left[\left(j+\frac{1}{2}\right)\left(1-\frac{1}{2}\frac{(Z\alpha)^2}{(j+1/2)^2}\right)+n'\right]^{-2} \\
&= (Z\alpha)^2\left[n-\frac{1}{2}\frac{(Z\alpha)^2}{j+1/2}\right]^{-2} = \frac{(Z\alpha)^2}{n^2}\left[1+\frac{(Z\alpha)^2}{n(j+1/2)}\right]
\end{aligned}
$$

where $n \equiv j+1/2+n'$ as defined in the text. Now on to (8.4.43). Recall that

$$
\begin{aligned}
(1+x)^{-1/2} &= 1-\frac{1}{2}x+\frac{3}{8}x^2+\cdots \qquad \text{and, so} \\
E &= mc^2\left\{1-\frac{1}{2}\frac{(Z\alpha)^2}{n^2}\left[1+\frac{(Z\alpha)^2}{n(j+1/2)}\right]+\frac{3}{8}\frac{(Z\alpha)^4}{n^4}+\cdots\right\} \\
&= mc^2-\frac{mc^2(Z\alpha)^2}{2n^2}-\frac{mc^2(Z\alpha)^4}{2n^2}\left[\frac{1}{n(j+1/2)}-\frac{3}{4n^2}\right]
\end{aligned}
$$

In other words, to this order, the energy is shifted by an amount

$$
\Delta = E_0(Z\alpha)^2\left[\frac{1}{n(j+1/2)}-\frac{3}{4n^2}\right]
$$

where $E_0 = -mc^2(Z\alpha)^2/2n^2$ is the energy level to lowest order. Perturbatively, the energy shift is given by the sum of the relativistic correction to the kinetic energy (5.3.10) and the spin-orbit energy (5.3.31). That is, we expect $\Delta = \Delta_{\text{rel}}+\Delta_{\text{so}}$ where

$$
\begin{aligned}
\Delta_{\text{rel}} &= E_0(Z\alpha)^2\left[-\frac{3}{4n^2}+\frac{1}{n(l+1/2)}\right] \\
\text{and} \qquad \Delta_{\text{so}} &= -E_0(Z\alpha)^2\frac{1}{2nl(l+1)(l+1/2)}\begin{cases} l & \text{for } j=l+1/2 \\ -(l+1) & \text{for } j=l-1/2 \end{cases}
\end{aligned}
$$

Adding l-dependent terms for $j=l+1/2$ gives

$$
\frac{1}{n(l+1/2)}-\frac{1}{2n(l+1)(l+1/2)} = \frac{1}{2nj}\left[2-\frac{1}{j+1/2}\right] = \frac{1}{n(j+1/2)}
$$

Adding l-dependent terms for $j=l-1/2$ gives

$$
\frac{1}{n(l+1/2)}+\frac{1}{2nl(l+1/2)} = \frac{1}{2n(j+1)}\left[2+\frac{1}{j+1/2}\right] = \frac{1}{n(j+1/2)}
$$

Therefore, for both $j=l\pm 1/2$ we find

$$
\Delta_{\text{rel}}+\Delta_{\text{so}} = E_0(Z\alpha)^2\left[-\frac{3}{4n^2}+\frac{1}{n(j+1/2)}\right]
$$

in agreement with our second order expansion of the Dirac energy level.

17. For (a) and (b) we can make the comparison using the second order approximation derived in Problem 16. However, we need to take into account that the energies of the two states with the same j but different l are not the same. See the discussion of Lamb Shift, part (d) of this problem. If we just average those two levels in each case, we find (in eV),

n	j_1	j_2	Experiment	Theory
2	1/2	3/2	4.318×10^{-5}	4.528×10^{-5}
4	5/2	7/2	9.4338985×10^{-7}	9.4338739×10^{-7}

Clearly, Dirac's theory does a much better job for the higher lying energy levels. As for the $1S \to 2S$ transition energy, the question is leading. If we tabulate the answer for the Balmer formula and for the Dirac formula using the next order approximation from Problem 16, and then also the exact Dirac formula, we find (in cm^{-1})

Balmer	82302.98684444
Dirac (approx)	82303.99122362
Dirac (exact)	82303.99125026
Experiment	82258.95439928

The Dirac formula makes a small correction to the Balmer formula, but even the exact form for Dirac is (relatively) far from the precise value. Now the Lamb Shift is a direct violation of the Dirac formula, resulting from quantum field effects. It shows up six times in the table, for any two states with the same n and j but different l values. We have (in (in cm^{-1})

n	Splitting	Value
2	SP	0.0353
3	SP	0.0105
3	PD	1.78×10^{-4}
4	SP	0.00444
4	PD	7.631×10^{-5}
4	DF	2.700×10^{-5}

The Lamb Shift gets smaller with increasing n, and apparently very much smaller for higher angular momenta. The moral of the story is that quantum field theory is important for understanding the energy levels of the hydrogen atom, especially for the lower lying ones.

Following is the MATLAB code used to calculate the numbers in this solution:

```
clear all
%Fundamental constants from 2010 PDG
hc=2*pi*197.3269631*1.0E6*1.0E-13; %From h-bar c
alpha=1/137.035999679;
mc2=0.510998910E6;
%
E0n1=-mc2*alpha^2/2;
%
%Analyzes precision hydrogen atomic energy differences
%
load EnljHAtom.dat
n=EnljHAtom(:,1);
l=EnljHAtom(:,2);
j2=2*EnljHAtom(:,3);
EDel=hc*EnljHAtom(:,4);
clear EnljHAtom
%
% Fine structure split in n=2, j=1/2 and 3/2, using 1st order expression
nset=2
DEexpt=EDel(find(n==nset & j2==3))-mean(EDel(find(n==nset & j2==1)))
DEcalc=(E0n1/nset^2)*alpha^2*(1/2-1)/nset
%
% Fine structure split in n=4, j=5/2 and 7/2, using 1st order expression
nset=4
DEexpt=EDel(find(n==nset & j2==7))-mean(EDel(find(n==nset & j2==5)))
DEcalc=(E0n1/nset^2)*alpha^2*(1/4-1/3)/nset
%
% 1S-2S energy difference using balmer, approximate, and exact formulas
format('long')
DE12balmr=E0n1*(1/4-1)/hc
DE12apprx=(E0n1/4)*(1+alpha^2*(1/2-3/16))-E0n1*(1+alpha^2*(1-3/4));
DE12apprx=DE12apprx/hc
DE12exact=mc2*(1/sqrt(1+alpha^2/(sqrt(1-alpha^2)+1)^2)-1/sqrt(1+alpha^2/(1-alpha^2)));
DE12exact=DE12exact/hc
%
% Lamb Shift data
LS2SP=EDel(find(n==2 & j2==1 & l==0))-EDel(find(n==2 & j2==1 & l==1));
LS2SP=LS2SP/hc
%
LS3SP=EDel(find(n==3 & j2==1 & l==0))-EDel(find(n==3 & j2==1 & l==1));
LS3SP=LS3SP/hc
LS3PD=EDel(find(n==3 & j2==3 & l==1))-EDel(find(n==3 & j2==3 & l==2));
LS3PD=LS3PD/hc
%
LS4SP=EDel(find(n==4 & j2==1 & l==0))-EDel(find(n==4 & j2==1 & l==1));
LS4SP=LS4SP/hc
LS4PD=EDel(find(n==4 & j2==3 & l==1))-EDel(find(n==4 & j2==3 & l==2));
LS4PD=LS4PD/hc
LS4DF=EDel(find(n==4 & j2==5 & l==2))-EDel(find(n==4 & j2==5 & l==3));
LS4DF=LS4DF/hc
```