Main goals

The method
Introduction
The grid
The reconstruction process
The limiters

Tools
Mesh
Results

Application of the spectral volume method for fusion simulations

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Lightning talk, 05/29/14

Main Goals

- The dynamics of a plasma can be described by a set of fluid equations for density, flow and temperature(energy).
- They are similar to the Euler equations

$$\begin{cases} \dfrac{\partial}{\partial t} \underbrace{\begin{pmatrix} \rho \\ \rho u_x \\ \rho u_y \\ \rho \epsilon \end{pmatrix}}_{\vec{U}} + \dfrac{\partial}{\partial x} \underbrace{\begin{pmatrix} \rho u_x \\ \rho u_x^2 + p \\ \rho u_x u_y \\ u_x(\rho \epsilon + p) \end{pmatrix}}_{\vec{F_x}} + \dfrac{\partial}{\partial y} \underbrace{\begin{pmatrix} \rho u_y \\ \rho u_x u_y \\ \rho u_y^2 + p \\ u_y(\rho \epsilon + p) \end{pmatrix}}_{\vec{F_y}} = 0 \\ \rho \epsilon = \frac{p}{\gamma - 1} + \frac{1}{2}\rho(u_x^2 + u_y^2) \end{cases}$$

- Numerically solve hyperbolic systems with an *accurate*, *conservative* and *stable* scheme.

Why the SV method

Finite Volume method (Godunov method, MUSCL, (W)ENO):

- **High-order accurate finite volume schemes** can be obtained theoretically for an unstructured grid by using **high-order polynomial data reconstructions**.
- **Difficulty in finding valid (non-singular) stencils**, and the **enormous memory required** to store the coefficients used in the reconstruction process in higher than second-order schemes.

Discontinuous Galerkin method:

- The fluxes through the element boundaries, where the state variable is usually **not continuous**, are computed using an approximate **Riemann solver**: thanks to this, the DG method is **fully conservative**.
- **Very high-order surface and volume integrals are necessary**, which can be expensive to compute.

The **Spectral Volume method** is conservative, high-order accurate and efficient.

Quick overview of the Spectral Volume method

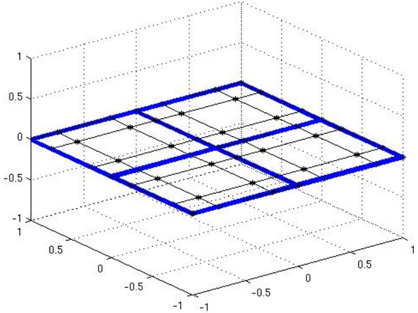
The grid

$$\Omega = \bigcup_{i=1}^{N} S_i$$

$$S_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$$

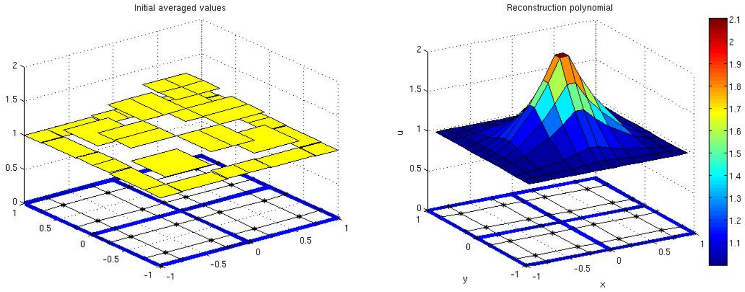
$$C_{ij} = (x_{i,j-\frac{1}{2}}, x_{i,j+\frac{1}{2}})$$

$$\forall j = 1, \ldots, k$$



- Solving for the **cell-averages of the unknowns** in each CV requires the **computation of fluxes** on the edges.
- The fluxes are calculated using the **reconstructed polynomial**

The reconstruction polynomial



$$\begin{pmatrix} \int_{C_1} \vec{e}_1(x,y)\,dxdy & \dots & \int_{C_1} \vec{e}_{k^2}(x,y)\,dxdy \\ \vdots & & \vdots \\ \int_{C_{k^2}} \vec{e}_1(x,y)\,dxdy & \dots & \int_{C_{k^2}} \vec{e}_{k^2}(x,y)\,dxdy \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{k^2} \end{pmatrix} = \begin{pmatrix} \overline{u}_1 \\ \vdots \\ \overline{u}_{k^2} \end{pmatrix}$$

$$p_k(x,y) = \alpha_1 \vec{e}_1(x,y) + \dots + \alpha_{k^2} \vec{e}_{k^2}(x,y)$$

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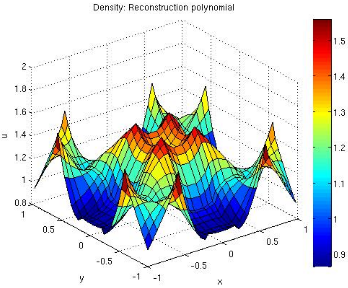
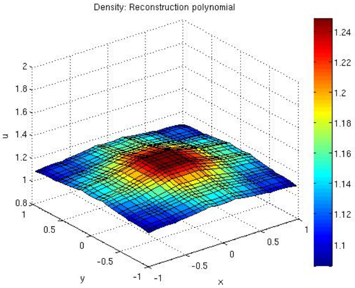
The limiters

High-order approximations of steep gradients or discontinuities may lead to spurious oscillations, known as *Gibbs phenomena*, thus losing the monotonicity in the solution.





$$\Delta u_{rq} = p_i(x_{rq}, y_{rq}) - \overline{u}_{ij} \qquad r = 1, \ldots, F \quad q = 1, \ldots, Q$$

IF $|\Delta u_{rq}| > MA_{ij}$, then it is assumed that the Control Volume is near a steep gradient and **limitation is required**

General procedure

The application of flux limiters consists of imposing a limitation on the reconstructed solution so that the monotonicity constraint

$$\overline{u}_{ij}^{min} \leq u_{ij}(x_{rq}, y_{rq}) \leq \overline{u}_{ij}^{max}$$

and the TVD condition

$$\sum_{j=1}^{N} ||\overline{u}_{ij}^{n} - \overline{u}_{i,j-1}^{n}|| \geq \sum_{j=1}^{N} ||\overline{u}_{ij}^{n+1} - \overline{u}_{i,j-1}^{n+1}||$$

are satisfied.

If one of the constraints is not satisfied, the solution in the cell C_{ij} is no longer approximated by a polynomial $p_i(x, y)$, but with:

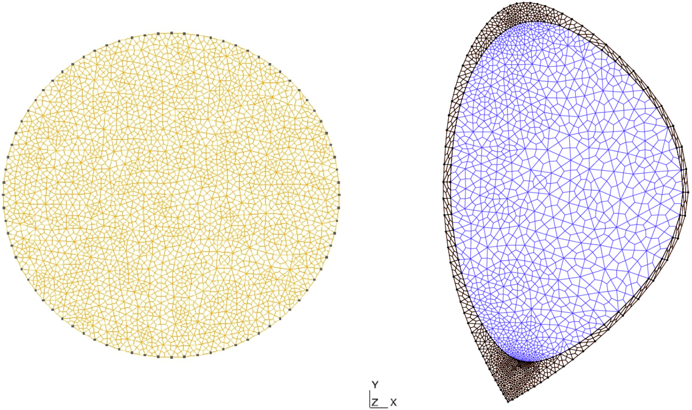
$$u_{ij}(x, y) = \overline{u}_{ij} + \varphi\vec{\nabla}u_{ij} \cdot (\vec{r} - \vec{r}_{ij}) \qquad \forall\vec{r} \in C_{ij}$$

Where the limiter function $\varphi \in [0, 1]$ allows the solution to be between the unlimited polynomial ($\varphi = 1$) and the mean value ($\varphi = 0$).

- *Minmod limiter*
- *Superbee limiter*

A C++ code using general mesh

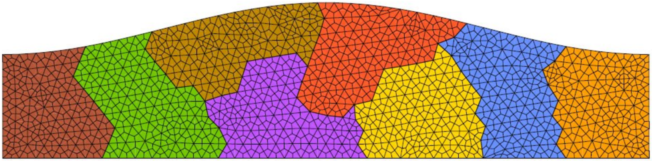
Meshes available



Unstructured mesh generated using `GMSH`

Parallel mesh management: main methods

The domain is too complex to perform the simulation with only one processor, therefore it has to be *parallelized*: all the information about the grid is distributed among N processes using `METIS` and `ParMETIS` softwares.



In order to make the computation less expensive, the mesh has to be optimized so that the communication between cores is reduced.

- **Pre-partitioning**
- **Partitioning**
- **Refinement**

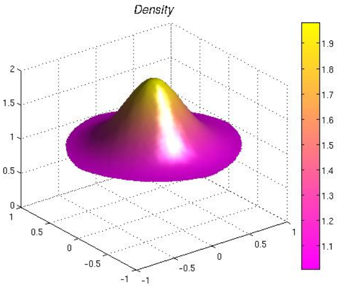
Results of the Euler problem

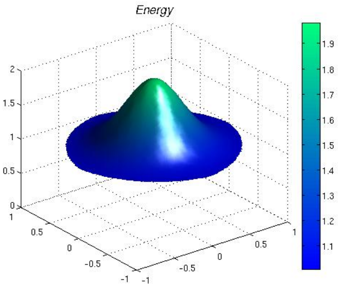
We recall that the problem we're trying to solve is:

$$\begin{cases} \frac{\partial}{\partial t} \underbrace{\begin{pmatrix} \rho \\ \rho u_x \\ \rho u_y \\ \rho \epsilon \end{pmatrix}}_{\vec{U}} + \frac{\partial}{\partial x} \underbrace{\begin{pmatrix} \rho u_x \\ \rho u_x^2 + p \\ \rho u_x u_y \\ u_x(\rho \epsilon + p) \end{pmatrix}}_{\vec{F_x}} + \frac{\partial}{\partial y} \underbrace{\begin{pmatrix} \rho u_y \\ \rho u_x u_y \\ \rho u_y^2 + p \\ u_y(\rho \epsilon + p) \end{pmatrix}}_{\vec{F_y}} = 0 \\ \rho \epsilon = \frac{p}{\gamma - 1} + \frac{1}{2}\rho(u_x^2 + u_y^2) \end{cases}$$

Initial condition:

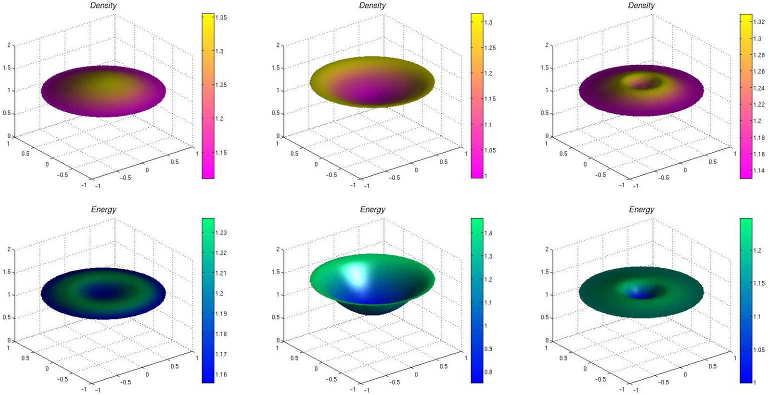
$$\vec{u}_0(x,y) = \begin{cases} \rho_0 = 1 + exp(-\frac{1}{2}(x^2+y^2)) \\ \rho u_0 = 0 \\ \rho v_0 = 0 \\ \rho\epsilon_0 = 1 + exp(-\frac{1}{2}(x^2+y^2)) \end{cases}$$





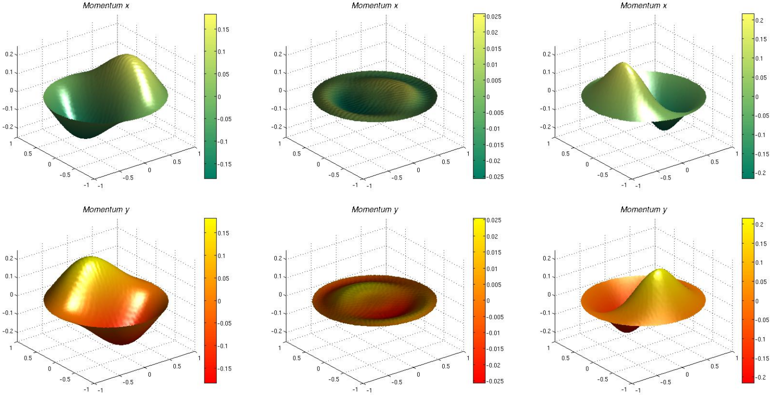
Boundary conditions: *reflective wall*, $n_x u_x + n_y u_y = 0$

Density and energy



t=110 t=220 t=330

Momentum over x and y



t=110 t=220 t=330

Now working on:

1. Improving the implementation of the limiters.
2. A viscosity term will be added to the Euler equations
3. The problem shall be converted to the plasma fluid equations in order to study the behaviors of ion and electron fluids