# Extended Essay

## *Mathematics*

### The Fourier Series and its applications in wave processing

*"How can a function be decomposed into simple sine and cosine waves and how can this knowledge be applied to process and distinguish between different waves?"*

**Word count:**

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## Introduction

During the summer break, I signed up for an online course on Android App Development with JAVA to explore my interest in computer languages and applications. After the completion of the course, I set out to make my first android application with the idea of creating a software that can filter out useful noise from background noise in real time which can be used for better quality of audio for online communications. To understand the concepts of sound analysis and noise filtering, I started researching and came across “The Fourier Transform” which was mentioned in many articles for its application in computer science for wave analysis. While exploring, I came across the Fourier Series and was fascinated by the fact that any periodic function can be broken up into the summation of simpler sine and cosine waves. I decided to learn more about it and its applications.

In my Extended Essay, I will be exploring the concept of Fourier Series and understand the different ways of representing the Fourier Series for a periodic function. Then, I will be explaining how it is used to convert a wave from time domain to frequency domain which helps greatly in wave analysis and to differentiate between different waves.

## Derivation of Fourier Series

Proposed by Joseph Fourier in the early 19$^{th}$ century, the Fourier Series[1] was based on the hypothesis that any periodic function can be defined as an infinite sum of sine and cosine functions having different coefficients (magnitude) and frequencies.

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1 Cheever, Erik. “Derivation of Fourier Series.” Swarthmore College. Accessed August 2, 2018. lpsa.swarthmore.edu/Fourier/Series/DerFS.html.

The math in Fourier series, on a very basic level, deals with calculating the values of the frequencies and their corresponding coefficients for the sine and cosine terms.

## *Periodic function:*

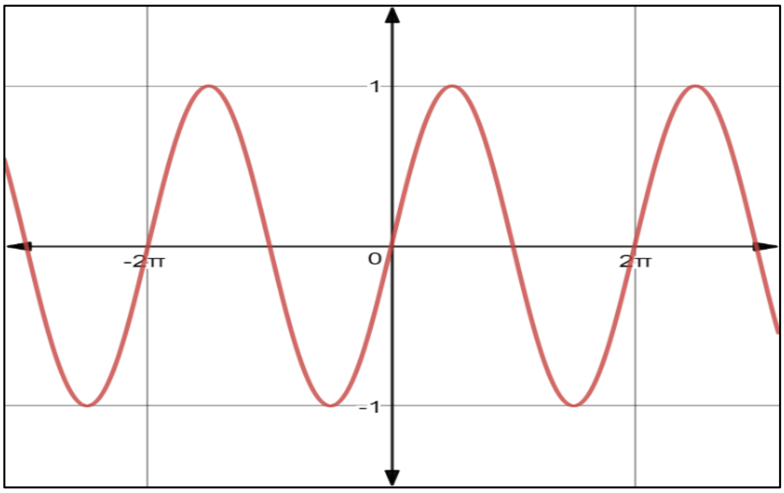
A function $f(t)$ is said to be periodic if it repeats its values, in the same order in which they appeared before, after set intervals of $t$ which is called the period of the function and can be represented as $T$. A periodic function must always hold the following true:

$$f(t) = f(t + kT)$$

Where $k$ is a constant and $k \in \mathbb{Z}$.

For example,

***Figure 1: graph of sin(x)***




In the $\sin x$ graph above, the values repeat itself after a set interval (or period) of $2\pi$ which makes the function periodic.

## *Odd and even functions:*

It is easier to comprehend the Fourier Series by first taking even and odd functions separately and then generalizing it to any function. So, what are odd and even functions and how can we find their Fourier series?

For $f(x)$ to be an even function, the expression $f(-x) = f(x)$ must hold true. For example, $\cos x$ is an even function as $\cos(-x) = \cos(x)$ (proven identity).

For $f(x)$ to be an odd function, the expression $f(-x) = -f(x)$ must hold true. For example, sin(x) is an odd function as $\sin(-x) = -\sin(x)$ (proven identity).

### *Even functions:*

An even function $f_e(t)$ with period $T$ can be written as a sum of cosine waves (as cosine waves are even functions) with different frequencies that are a multiple of the fundamental frequency of the function so that the zeros of the function coincide with the zeros of the cosine waves.

$$f_e(t) = a_0 \cos\left(\frac{2\pi}{T} \times 0 \times t\right) + a_1 \cos\left(\frac{2\pi}{T} \times 1 \times t\right) + a_2 \cos\left(\frac{2\pi}{T} \times 2 \times t\right) + \cdots$$

Where $\frac{2\pi}{T}$ is defined as the fundamental frequency[2] of the function $f_e(t)$.

This can be simplified to:

$$f_e(t) = \sum_{k=0}^{\infty} a_k \cos\left(\frac{2\pi}{T} \times k \times t\right)$$

When $k = 0$, $a_0\cos(0) = a_0$. Hence, the equation can be rewritten as:

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2 Wang, Ruye. "Fundamental Frequency of Continuous Signals." Fundamental Frequency, Harvey Mudd, 2nd Feb. 2009. Accessed August 2, 2018. fourier.eng.hmc.edu/e101/lectures/Fundamental_Frequency/node1.html.

$$f_e(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi}{T} \times k \times t\right)$$

This equation is also called the "synthesis" equation as it shows how the function is synthesized by the addition of the different cosines.

The next challenge is finding the magnitude of the coefficient $a_k$ in terms of the function $f_e(t)$. To start, both sides of the synthesis equation can be integrated and equated:

$$\int f_e(t)\,dt = \int \left(a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi}{T} \times k \times t\right)\right) dt$$

$$= a_0 \mathrm{t} + \sum_{k=1}^{\infty} a_k \left(\int \cos\left(\frac{2\pi}{T} \times k \times t\right) dt\right)$$

(Here, the integral of the cosine function is nested within the summation for $k$ going from $1 \to \infty$. This can be done as $a_k$ is a constant and can be removed outside the integration without affecting the result.)

Next, the integration of the cosine function can be taken separately and calculated:

$$\int \cos\left(\frac{2\pi}{T} \times k \times t\right) dt = \frac{\sin\left(\frac{2\pi}{T} \times k \times t\right)}{\frac{2\pi}{T} \times k} + c$$

(where $c$ is a constant)

If the integration is done over one period, let's say $0 \to T$,

$$\int_0^T \cos\left(\frac{2\pi}{T} \times k \times t\right) dt = \left[\frac{\sin\left(\frac{2\pi}{T} \times k \times t\right)}{\frac{2\pi}{T} \times k}\right]_0^T$$

$$= \frac{1}{\frac{2\pi}{T} \times k} \left(\sin(2\pi k) - \sin 0\right)$$

$$= 0$$

Since this period is taken for the integration of the cosine function, it must also be applicable to $a_0$ which was earlier integrated.

$$= [a_0 t]_0^T$$
$$= a_0 T - 0$$

Thus,

$$\int_0^T f_e(t)\, dt = a_0 T + 0$$

$$a_0 = \frac{1}{T} \int_0^T f_e(t)\, dt$$

$a_0$ can be determined to be the average value of the function as it will translate the average value of all the sine and cosine functions (which is 0) by $a_0$ along the Y-axis.

This is true for any intervals with the range being an integral multiple of the period of the function (e.g. $-T/_2 \to T/_2$ or $T \to 3T$).

Now to find $a_k$ for $k \geq 1$, both sides of the synthesis equation can be multiplied by a dummy cosine function: $\cos\left(\frac{2\pi}{T} \times j \times t\right)$, where $j \in \mathbb{Z}$, followed by integration of both sides. The limits of integration can be taken for one period $0 \to T$. Since k is the variable of the summation, the dummy cosine function can be considered a constant to the summation and can be moved in or out of the summation as a whole. This gives the following equation:

$$\int_0^T f_e(t)\, \cos\left(\frac{2\pi}{T} \times j \times t\right) dt = \int_0^T \left(\sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi}{T} \times k \times t\right) \cos\left(\frac{2\pi}{T} \times j \times t\right)\right) dt$$

$$= \sum_{k=1}^{\infty} a_k \left( \int_0^T \cos\left(\frac{2\pi}{T} \times k \times t\right) \cos\left(\frac{2\pi}{T} \times j \times t\right) dt \right)$$

Consider following trigonometrical identity:

$$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$$
$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$= 2\cos\alpha\cos\beta$$
$$\therefore \cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

This identity can be used on $\cos\left(\frac{2\pi}{T} \times k \times t\right)\cos\left(\frac{2\pi}{T} \times j \times t\right)$ in the equation to obtain:

$$\int_0^T f_e(t)\ \cos\left(\frac{2\pi}{T} \times j \times t\right) dt = \sum_{k=1}^{\infty} a_k \int_0^T \frac{1}{2}\left[\cos(j+k)\frac{2\pi}{T}t + \cos(j-k)\frac{2\pi}{T}t\right] dt$$
$$= \frac{1}{2}\sum_{k=1}^{\infty} a_k \int_0^T \left[\cos(j+k)\frac{2\pi}{T}t + \cos(j-k)\frac{2\pi}{T}t\right] dt$$

Since $j + k$ will always be an integer, $\int_0^T \cos\left((j+k)\frac{2\pi}{T}t\right) dt = 0$ as previously proven when finding $a_0$.

$$\int_0^T f_e(t)\ \cos\left(\frac{2\pi}{T} \times j \times t\right) dt = \frac{1}{2}\sum_{k=1}^{\infty} a_k \int_0^T \cos\left((j-k)\frac{2\pi}{T}t\right) dt$$

Due to cosine being an even function, $\cos\left((j-k)\frac{2\pi}{T}t\right) = \cos\left((k-j)\frac{2\pi}{T}t\right)$. $|j-k|$ will be an integer for every value $j \neq k$ except for when $j = k$ in which case $|j-k| = 0$. Integer results can be ignored as they result in the integral being equal to 0 and become redundant as they leave $a_0$ which has already been derived.

Considering the summation, every value of $k$ where $j \neq k$ is ignored and for value $k = j$, the equation can be written as follows:

$$\int_0^T f_e(t)\ \cos\left(\frac{2\pi}{T} \times k \times t\right) dt\ = \frac{1}{2} \times a_k \times \int_0^T \cos 0\ \ dt$$

$$= \frac{1}{2} \times a_k \times \int_0^T 1\ dt$$

$$= \frac{1}{2} \times a_k \times T$$

$$\therefore a_k = \frac{2}{T} \int_0^T f_e(t)\ \cos\left(\frac{2\pi}{T} \times k \times t\right) dt$$

### *Odd functions:*

Using the same idea used for even functions, an odd function $f_o(t)$ with period $T$ can be written as:

$$f_o(t) = \sum_{k=0}^{\infty} b_k \sin\left(\frac{2\pi}{T} \times k \times t\right)$$

A sine function is used as it is an odd function and adds up to form an odd function.

In this summation, there is no need for $b_0$ as $\sin 0 = 0$. Thus, the summation can be rewritten as:

$$f_o(t) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{2\pi}{T} \times k \times t\right)$$

Following the same steps as those for even functions but using sine instead of cosine, the following expression for $b_k$ can be obtained which is very similar to that for $a_k$:

$$b_k = \frac{2}{T}\int_0^T f_o(t)\ \sin\left(\frac{2\pi}{T} \times k \times t\right) dt$$

*Any periodic function:*

Any function can be expressed as the sum of an odd and an even function. To prove this, a function $f(t)$ can be used to formulate the following two expressions

$$f_e(t) = \frac{1}{2}\left(f(t) + f(-t)\right)$$
$$f_e(-t) = \frac{1}{2}\left(f(-t) + f(t)\right)$$
$$and$$
$$f_o(t) = \frac{1}{2}\left(f(t) - f(-t)\right)$$
$$f_o(-t) = \frac{1}{2}\left(f(-t) - f(t)\right)$$

From these equations, it can be clearly observed that $f_e(t) = f_e(-t)$ and $f_o(t) = f_o(-t)$ making $f_e(t)$ an even function and $f_o(t)$ an odd function. Upon adding $f_e(t) + f_o(t)$, the original function $f(t)$ is obtained hence proving that any equation can be written as a sum of an odd and an even function.

Now assume a periodic function $f(t)$ with period $T$. It can be written as the sum of an even and an odd function $f_e(t) + f_o(t)$ both of which are periodic with period $T$. Using the synthesis equation for an even and an odd function found before, the function $f(t)$ can be expressed as the following which is known as the synthesis equation:

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi}{T} \times k \times t\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2\pi}{T} \times k \times t\right)$$

$$\therefore \boldsymbol{f(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{2\pi}{T} \times k \times t\right) + b_k \sin\left(\frac{2\pi}{T} \times k \times t\right)\right]}$$

Where the coefficients can be derived using the analysis equations:

$$\boldsymbol{a_0 = \frac{1}{T} \int_0^T f(t)\, dt}$$

$$\boldsymbol{a_k = \frac{2}{T} \int_0^T f(t)\, \cos\left(\frac{2\pi}{T} \times k \times t\right) dt}$$

$$\boldsymbol{b_k = \frac{2}{T} \int_0^T f(t)\, \sin\left(\frac{2\pi}{T} \times k \times t\right) dt}$$

## *Exponential Representation[3]:*

The representation of Fourier series of a periodic function above is in the form of trigonometrical functions. However, this can be more concisely represented with the help of the exponential constant "*e*". At first, it is counter-intuitive to associate exponential equations with wave behavior but when complex numbers come in the picture, Euler's formula[4] can help make sense of it. Euler's formula can be used to convert the trigonometrical representation of Fourier Series to an exponential representation as it includes the exponential constant on one side of the equation and trigonometrical functions of sine and cosine on the other. Euler's formula states that:

$$e^{ix} = \cos x + i \sin x$$

3 Ghosh, Smarajit. "Chapter 2.5." Signals and Systems, Pearson India, 2005. Accessed August 29, 2018. https://www.safaribooksonline.com/library/view/signals-and-systems/9789332515147/xhtml/ch2_2-5.xhtml

4 Weisstein, Eric W. "Euler Formula." Wolfram MathWorld. Accessed August 30, 2018. mathworld.wolfram.com/EulerFormula.html.

Similarly, taking $x$ negative:

$$\begin{aligned} e^{-ix} &= \cos(-x) + i\sin(-x) \\ &= \cos x - i\sin x \end{aligned}$$

Now, it can be observed that:

$$e^{ix} + e^{-ix} = 2\cos x$$

$$\therefore \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\therefore \cos\left(\frac{2\pi}{T}kt\right) = \frac{e^{i\frac{2\pi}{T}kt} + e^{-i\frac{2\pi}{T}kt}}{2}$$

And,

$$e^{ix} - e^{-ix} = 2i\sin x$$

$$\therefore \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\therefore \sin\frac{2\pi}{T}kt = \frac{e^{i\frac{2\pi}{T}kt} - e^{-i\frac{2\pi}{T}kt}}{2i}$$

Using the expressions for sine and cosine found above in terms of the exponential constant, the synthesis equation for the Fourier series of a periodic function can be rewritten as:

$$\begin{aligned} f(t) &= a_0 + \sum_{k=1}^{\infty}\left[a_k\cos\left(\frac{2\pi}{T}\times k\times t\right) + b_k\sin\left(\frac{2\pi}{T}\times k\times t\right)\right] \\ &= a_0 + \sum_{k=1}^{\infty}\left[a_k\left(\frac{e^{i\frac{2\pi}{T}kt} + e^{-i\frac{2\pi}{T}kt}}{2}\right) + b_k\left(\frac{e^{i\frac{2\pi}{T}kt} - e^{-i\frac{2\pi}{T}kt}}{2i}\right)\right] \end{aligned}$$

To rationalize the denominator of the coefficient of $b_k$, it can be multiplied with $i$ in the numerator and denominator.

$$f(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k\left(\frac{e^{i\frac{2\pi}{T}kt} + e^{-i\frac{2\pi}{T}kt}}{2}\right) + b_k\left(\frac{e^{i\frac{2\pi}{T}kt} - e^{-i\frac{2\pi}{T}kt}}{2i}\right).\frac{i}{i}\right]$$

$$= a_0 + \sum_{k=1}^{\infty}\left[a_k\left(\frac{e^{i\frac{2\pi}{T}kt} + e^{-i\frac{2\pi}{T}kt}}{2}\right) + i.b_k\left(\frac{e^{i\frac{2\pi}{T}kt} - e^{-i\frac{2\pi}{T}kt}}{2i^2}\right)\right]$$

$$= a_0 + \sum_{k=1}^{\infty}\left[a_k\left(\frac{e^{i\frac{2\pi}{T}kt} + e^{-i\frac{2\pi}{T}kt}}{2}\right) + i.b_k\left(\frac{e^{i\frac{2\pi}{T}kt} - e^{-i\frac{2\pi}{T}kt}}{-2}\right)\right]$$

$$= a_0 + \sum_{k=1}^{\infty}\left[a_k\left(\frac{e^{i\frac{2\pi}{T}kt} + e^{-i\frac{2\pi}{T}kt}}{2}\right) + i.b_k\left(\frac{e^{-i\frac{2\pi}{T}kt} - e^{i\frac{2\pi}{T}kt}}{2}\right)\right]$$

$$= a_0 + \sum_{k=1}^{\infty}\left[\frac{e^{i\frac{2\pi}{T}kt}(a_k - i.b_k)}{2} + \frac{e^{-i\frac{2\pi}{T}kt}(a_k + i.b_k)}{2}\right]$$

Things become simpler if the two variables $a_k$ and $b_k$ are replaced by a single variable for the whole equation. The following can be assumed to do so:

$$F_k = \frac{1}{2}(a_k - i.b_k)$$

Taking $F_{-k}$, $b_k$ becomes negative because it is a coefficient for an odd function and $a_k$ remains positive because it is the coefficient of an even function,

$$F_{-k} = \frac{1}{2}(a_k + i.b_k)$$

Using these, the expression for the synthesis equation can be rewritten as:

$$f(t) = a_0 + \sum_{k=1}^{\infty}\left[F_k.e^{i\frac{2\pi}{T}kt} + F_{-k}.e^{-i\frac{2\pi}{T}kt}\right]$$

Where the analysis equation can be rewritten as,

$$F_k = \frac{1}{2}(a_k - i.b_k)$$

$$= \frac{1}{2}\left(\frac{2}{T}\int_{0}^{T} f(t)\ \cos\left(\frac{2\pi}{T}\times k\times t\right)dt - i.\frac{2}{T}\int_{0}^{T} f(t)\ \sin\left(\frac{2\pi}{T}\times k\times t\right)dt\right)$$

$$= \frac{1}{T}\int_{0}^{T} f(t)\ \left[\cos\left(\frac{2\pi}{T}\times k\times t\right) - i\sin\left(\frac{2\pi}{T}\times k\times t\right)\right]dt$$

$$\therefore \boldsymbol{F_k} = \frac{\boldsymbol{1}}{\boldsymbol{T}}\int_{\boldsymbol{0}}^{\boldsymbol{T}} \boldsymbol{f(t).e^{-i\frac{2\pi}{T}kt}dt},\ \ \boldsymbol{k\epsilon}\mathbb{Z}$$

Also,

$$F_0 = \frac{1}{T}\int_{0}^{T} f(t).e^{0}dt = a_0$$

Hence, the synthesis equation can be rewritten as:

$$f(t) = F_0 + \sum_{k=1}^{\infty} F_k.e^{i\frac{2\pi}{T}kt} + \sum_{k=1}^{\infty} F_{-k}.e^{-i\frac{2\pi}{T}kt}$$

$$= F_0 + \sum_{k=1}^{\infty} F_k.e^{i\frac{2\pi}{T}kt} + \sum_{k=-\infty}^{-1} F_k.e^{i\frac{2\pi}{T}kt}$$

$$\therefore \boldsymbol{f(t)} = \sum_{\boldsymbol{k=-\infty}}^{\boldsymbol{\infty}} \boldsymbol{F_k.e^{i\frac{2\pi}{T}kt}}$$

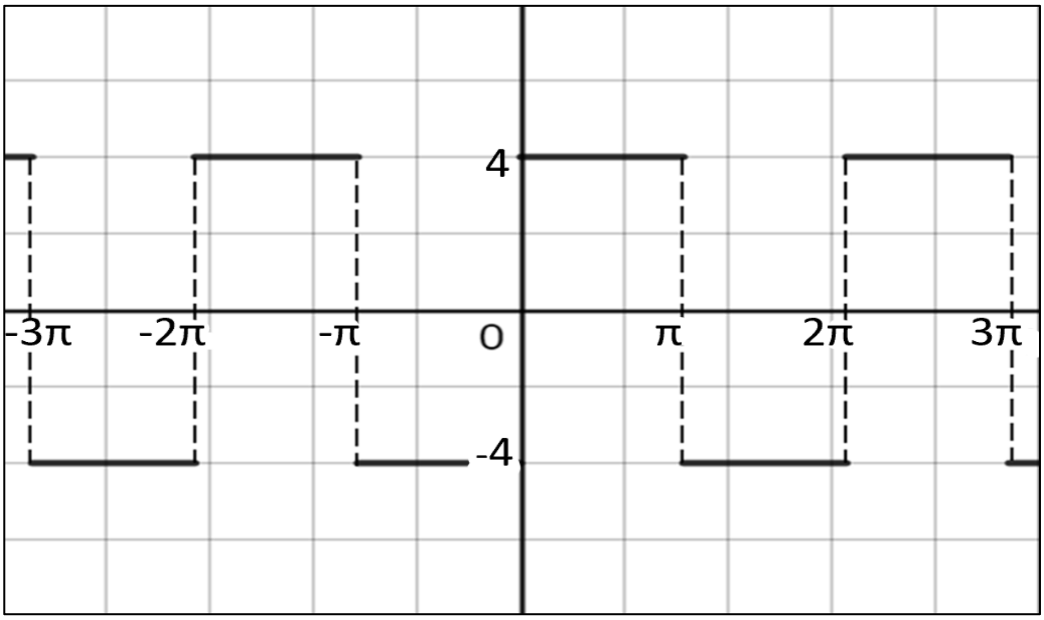
Comparatively, the exponential form of the Fourier Series is more compact and simpler to interpret as it deals with fewer variables. This helps computer scientists to form more compact algorithms for signal data processing while the trigonometric form can be helpful for gaining a more visual representation of the Fourier series using graphs. The reason why electrical engineers prefer exponential notation of Fourier Series in the field of communication and signals is because it helps in manipulation of waves in frequency domain (frequency domain is explained further in this essay) as it is easier to perform algebra with it.

## Application of Fourier Series

### *Fourier Series of a rectangular wave:*

The Fourier Synthesis and Analysis equations can be used to convert the graph of a wave from the time domain to the frequency domain. The two domains are basically the graphical representation of the wave when the domain is taken on the X-axis and amplitude is taken on the Y-axis. To see how this works, let the following wave be taken as an example for calculating its Fourier Series.

***Figure 2: Rectangular wave with an amplitude of 4***




The wave in the above graph is known as a rectangular wave.[5] It has an amplitude of 4 and the average value of 0 (it alternates between the values of 4 and -4 along set intervals). Let it be represented by a function $f(t)$ which will have a period of $T = 2\pi$. If this function is considered for a domain of one period, let's say $0 \to 2\pi$, it can be expressed as the following piecewise function:

5 "rectangular wave." McGraw-Hill Dictionary of Scientific & Technical Terms, 6E. 2003. The McGraw-Hill Companies, Inc. Accessed November 2, 2018. https://encyclopedia2.thefreedictionary.com/rectangular+wave

$$f(t) = \begin{cases} 4, & 0 \leq t < \pi \\ -4, & \pi \leq t < 2\pi \end{cases}$$

Now, the time period of $f(t)$ can be substituted in the synthesis equation of the Fourier series to get:

$$\begin{aligned} f(t) &= a_0 + \sum_{k=1}^{\infty} \left[ a_k \cos\left(\frac{2\pi}{T} \times k \times t\right) + b_k \sin\left(\frac{2\pi}{T} \times k \times t\right) \right] \\ &= a_0 + \sum_{k=1}^{\infty} \left[ a_k \cos\left(\frac{2\pi}{2\pi} \times k \times t\right) + b_k \sin\left(\frac{2\pi}{2\pi} \times k \times t\right) \right] \\ &= a_0 + \sum_{k=1}^{\infty} [a_k \cos(kt) + b_k \sin(kt)] \end{aligned}$$

Finding the coefficients can be tricky as the wave is represented by a piecewise function. Two separate functions exist within different domains over one period. To accommodate for this behavior of the wave, the integral (used while finding coefficients $a_k$ and $b_k$) must be split as an addition of two separate integrals of the two different functions that make up the piecewise function. The limits can be taken as the different domains of these functions and overall, the limits will cover one period of the wave. Keeping this in mind, the coefficient:

- $a_0$ is given by,

$$\begin{aligned} a_0 &= \frac{1}{T} \int_0^T f(t)\, dt \\ &= \frac{1}{2\pi} \left( \int_0^{\pi} 4\, dt + \int_{\pi}^{2\pi} -4\, dt \right) \\ &= \frac{1}{2\pi} \left( [4t]_0^{\pi} - [4t]_{\pi}^{2\pi} \right) \\ &= \frac{1}{2\pi} \left( 4\pi - (8\pi - 4\pi) \right) \\ &= 0 \end{aligned}$$

- $a_k$ is given by,

$$a_k = \frac{2}{T}\int_0^T f(t)\ \cos\left(\frac{2\pi}{T}\times k\times t\right)dt$$
$$= \frac{2}{2\pi}\left(\int_0^{\pi} 4\ \cos(kt)\,dt + \int_{\pi}^{2\pi} -4\ \cos(kt)\,dt\right)$$
$$= \frac{1}{\pi}\left(4\left[\frac{\sin(kt)}{k}\right]_0^{\pi} - 4\left[\frac{\sin(kt)}{k}\right]_{\pi}^{2\pi}\right)$$
$$= \frac{4}{\pi}\left(\frac{\sin(k\pi)}{k} - \frac{\sin(0)}{k} - \left(\frac{\sin(k.2\pi)}{k} - \frac{\sin(k\pi)}{k}\right)\right)$$

$\sin(k\pi) = 0$ for all integer values of $k$. Since $k$ only has discrete integer values according to the equation, all the terms inside the bracket in the equation above have a value of zero resulting in the value of $a_k$ being equal to zero for all values of $k$ inside the summation.

$$\therefore a_k = 0$$

Therefore, the Fourier Series of the wave represented as function $f(t)$ has no cosine terms. This also makes sense because the rectangular function is actually an odd function while the cosine is an even function.

- $b_k$ is given by,

$$b_k = \frac{2}{T}\int_0^T f(t)\ \sin\left(\frac{2\pi}{T}\times k\times t\right)dt$$
$$= \frac{2}{2\pi}\left(\int_0^{\pi} 4\ \sin(kt)\,dt + \int_{\pi}^{2\pi} -4\ \sin(kt)\,dt\right)$$
$$= \frac{1}{\pi}\left(4\left[\frac{-\cos(kt)}{k}\right]_0^{\pi} - 4\left[\frac{-\cos(kt)}{k}\right]_{\pi}^{2\pi}\right)$$

$$= \frac{4}{\pi}\left(\left(\frac{-\cos(k\pi)}{k} + \frac{\cos(0)}{k}\right) - \left(\frac{-\cos(k.2\pi)}{k} + \frac{\cos(k\pi)}{k}\right)\right)$$

$$= \frac{4}{\pi}\left(-\frac{\cos(k\pi)}{k} + \frac{\cos(0)}{k} + \frac{\cos(k.2\pi)}{k} - \frac{\cos(k\pi)}{k}\right)$$

$$= \frac{4}{\pi}\left(\frac{-2\cos(k\pi) + 1 + \cos(k.2\pi)}{k}\right)$$

The further simplify this expression, it can be split into two separate cases: for even values of $k$ and for odd values of $k$.

When $k$ is even, both the cosine terms in the expression, $\cos(k\pi)$ and $\cos(k.2\pi)$, will have an even multiple of $\pi$ as the angle and the cosine of an even multiple of $\pi$: $\cos(2n.\pi) = 1$.

When $k$ is odd, the first cosine term in the expression, $\cos(k\pi)$, will have an odd multiple of $\pi$ as the angle while the second cosine term, $\cos(k.2\pi)$, will still have an even multiple of $\pi$ as it is being multiplied by 2. When the angle in a cosine function is an odd multiple of $\pi$: $\cos((2n+1).\pi) = -1$.

This leads to the following set of expressions for when $k$ is odd and when $k$ is even:

$$b_k = \begin{cases} \frac{4}{\pi}\left(\frac{-2(1) + 1 + 1}{k}\right), & k \text{ is even} \\ \frac{4}{\pi}\left(\frac{-2(-1) + 1 + 1}{k}\right), & k \text{ is odd} \end{cases}$$

$$b_k = \begin{cases} 0, & k \text{ is even} \\ \frac{4 \times 4}{\pi.k}, & k \text{ is odd} \end{cases}$$

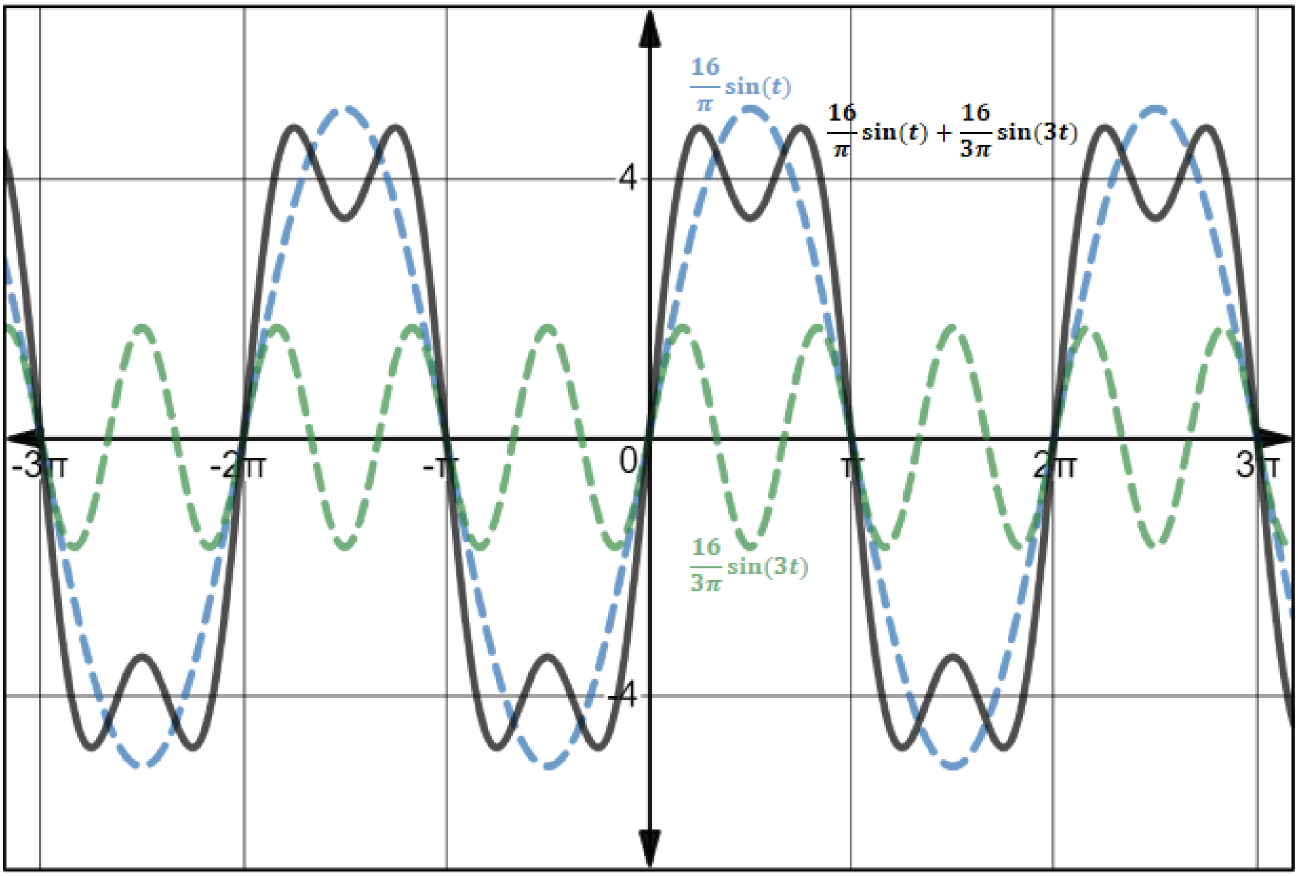
$$\therefore b_k = \frac{16}{\pi.k}, \qquad k \in \{1, 3, 5, 7 \ldots\}$$

Therefore, the Fourier Series of the wave consists of only sine functions for odd values of $k$. Substituting the values of $k$ in $\sin(kt)$ and $b_k$, the following expansion of the rectangular wave is obtained:

$$f(t) = \frac{16}{\pi}\sin(t) + \frac{16}{3\pi}\sin(3t) + \frac{16}{5\pi}\sin(5t) + \frac{16}{7\pi}\sin(7t) \ldots$$

## Graphical representation of Fourier Series synthesis:

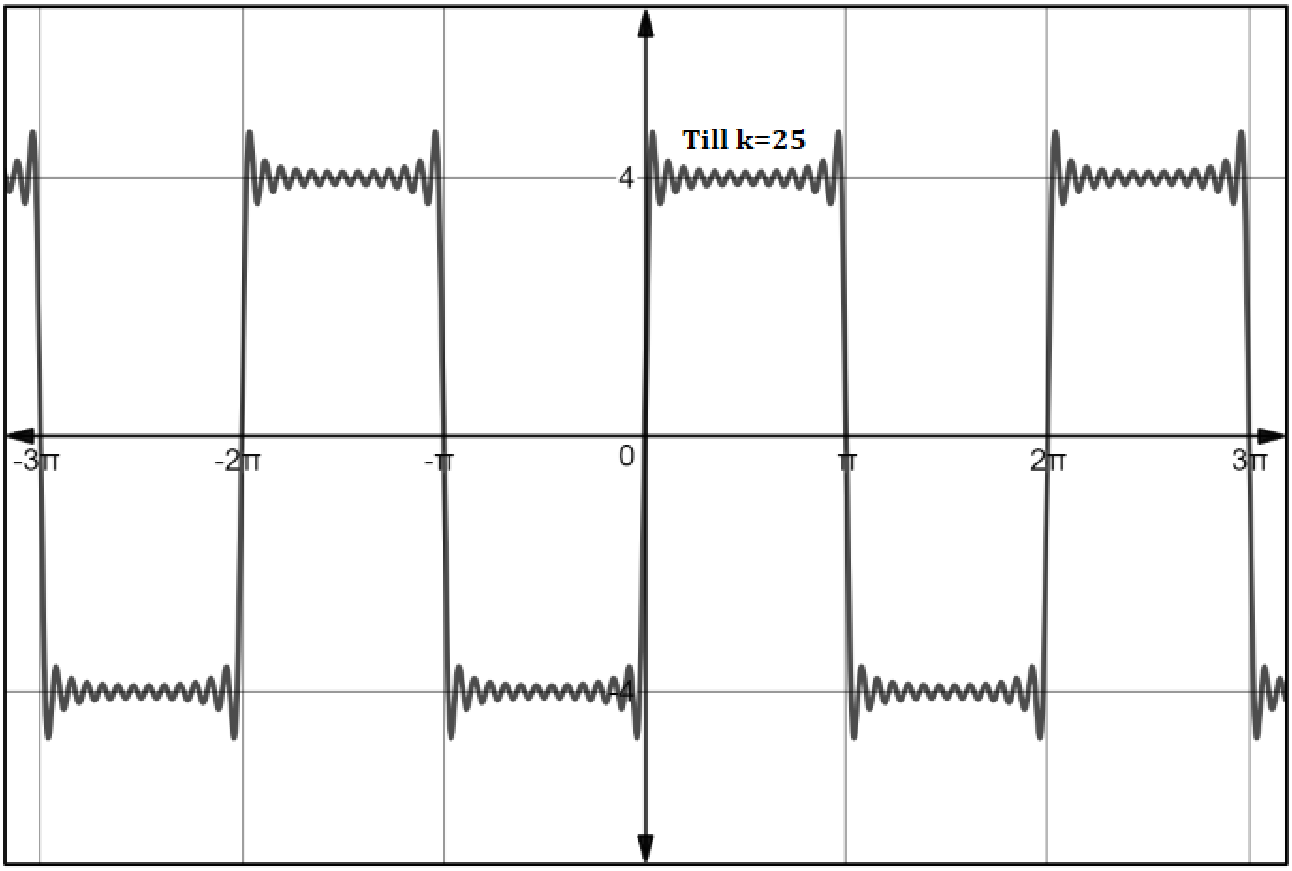
***Figure 3: The first two Fourier Terms of the rectangular wave with an amplitude of 4***




The graph above shows how the first two terms of the Fourier Series of the rectangular wave modeled by function $f(x)$ add up. As seen, with only the first and the third term (as derived from calculations), the wave already starts resembling the original rectangular wave.

Continuing, if the first few terms till $k = 25$ are taken, the following graph is obtained

*Figure 4: Addition of Fourier Series of the rectangular wave with amplitude of 4 till k=25*




It can be seen that with each term added, the Fourier series of the wave approaches the original wave with more accuracy. This shows convergent properties of the Fourier Series.

*Time domain – Frequency domain:*

Finding the Fourier Series of the rectangular wave can help plot a graph of the wave in the frequency domain.

The values of the frequencies (of the sine terms) can be taken on the X-axis and their corresponding coefficients (amplitudes) can be taken on the Y-axis and this forms a graph of the wave in the frequency domain.

Since the coefficient of $t$ in the sine and cosine functions of the Fourier Series is the frequency of the function in terms of fundamental frequency $\left(\frac{2\pi}{T}\right)$, the frequencies of the Fourier Series can be found by using the following information:

$$\frac{1}{T} = f$$

$$\frac{2\pi}{T} = 2\pi f$$

$$\therefore f = \frac{\left(\frac{2\pi}{T}\right)}{2\pi}$$

Therefore, the frequency when and $T = 2\pi$ comes out to be $f = \frac{1}{2\pi}$. As $k$ was multiplied by the fundamental frequencies, it should also be multiplied by the frequency to which it relates. This will lead to the following table of frequency values with their corresponding amplitudes (coefficients) for the Fourier Series of the rectangular wave:

$$f(t) = \frac{16}{\pi}\sin(t) + \frac{16}{3\pi}\sin(3t) + \frac{16}{5\pi}\sin(5t) + \frac{16}{7\pi}\sin(7t) \ldots$$

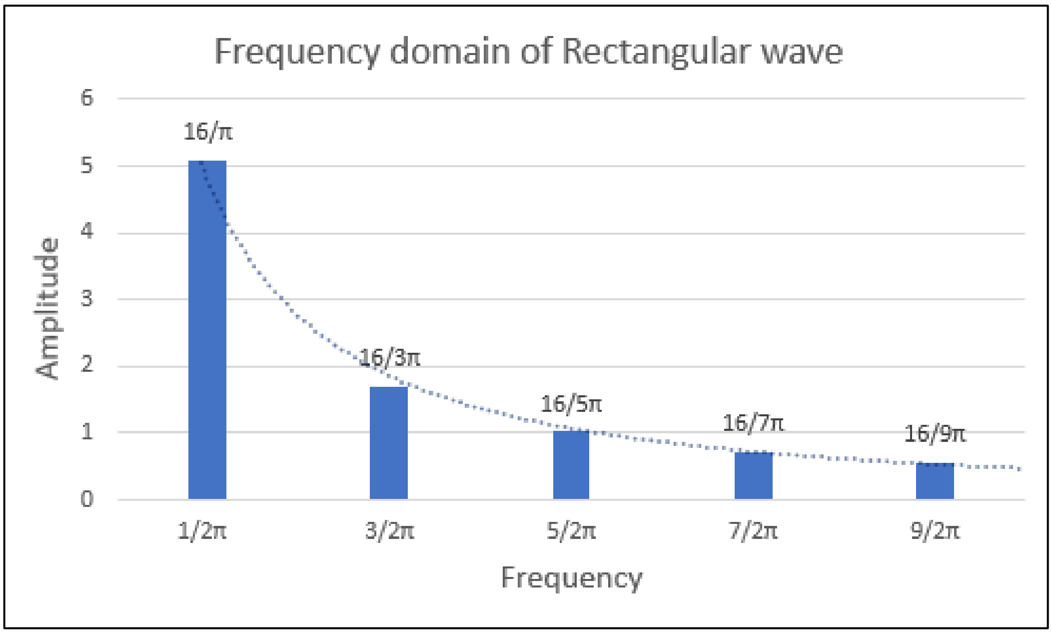
***Table 1: Amplitude and frequency of sine waves that constitute the rectangular wave with an amplitude of 4 for each value of k***

| **Amplitude** | **Frequency** | ***k*** |
|---|---|---|
| $\frac{16}{\pi}$ | $\frac{1}{2\pi}$ | 1 |
| $\frac{16}{3\pi}$ | $\frac{3}{2\pi}$ | 3 |
| $\frac{16}{5\pi}$ | $\frac{5}{2\pi}$ | 5 |
| $\frac{16}{7\pi}$ | $\frac{7}{2\pi}$ | 7 |
| $\cdots$ | $\cdots$ | |

| | | |
|---|---|---|
| $\frac{16}{(2n+1)\pi}$ | $\frac{(2n+1)}{2\pi}$ | 2n+1 |

This data helps make a graph of frequency vs amplitude which is a static representation of the rectangular wave in the frequency domain:

***Figure 5: Frequency Domain of the rectangular wave with an amplitude of 4***




Thus, Fourier series can help convert a wave from the time domain to its frequency domain. Compared to the time domain, the frequency domain does not need to be continuous and comparing only a few values of the frequency domain are enough to distinguish between waves.

Applications such as Shazam [6]make use of this idea wherein different songs have different frequency domains and can thus have their own "digital footprints". This

6 "How Does Shazam Work? Let's Understand Music Recognition Algorithms Together - Steemit." - Steemit, Phenom. Accessed November 20, 2018. steemit.com/technology/@phenom/how-does-shazam-work-let-s-understand-music-recognition-algorithms-together.

helps the application to identify incoming soundwaves by finding its Fourier Series to convert it from the time domain to frequency domain and then trying to match it with an existing database of frequency domains of different songs.

The frequency domain is also useful in many audio editing software where individual frequencies can be removed or added to remove background noise from audio or add different kinds of effects.

### *Convergence of Fourier Series:*

As seen from the frequency domain graph of the rectangular wave, when bigger frequencies are taken the amplitude keeps decreasing until it becomes insignificant. This is an illustration of the convergence of the Fourier Series and how only the first few terms have the most significant contributions to forming the original function (as they have greater amplitudes).

This property of Fourier Series proves to be extremely useful in audio file compression where the Fourier Series for a sound wave is found and only the frequencies with the highest amplitudes are taken while the others are removed. These frequencies with their amplitudes superimpose to closely resemble the original sound wave which becomes indistinguishable for the human ear from the original sound wave but now occupies lesser memory to be stored.

## The Fourier Transform

It is known that not all waves can be periodic, especially sound waves. Then how exactly do applications like Shazam make use of the Fourier Series?

It is through the concept of Fourier Transform. Fourier Transform is a part of the Fourier Series which deals with non-periodic waves. The Fourier Transform considers a wave to be periodic over a time interval with a time period, $T \to \infty$. This makes sense as a non-periodic wave will not repeat till "infinity". Taking this limit $T \to \infty$, the analysis equation and the synthesis equation of the Fourier Series can be changed which are then called as the Fourier Transform and the inverse Fourier Transform respectively.

Fourier Transform itself can be divided further into different types such as the Discrete Fourier Transform (DFT), the Fast Fourier Transform (FFT) and the Short-Term Fourier Transform (STFT) all of which have different applications in computer science when it comes to wave analysis based on their advantages such as faster processing or better accuracy.

## Conclusion

Thus, with the concept of Fourier Series and Fourier Transform, it is possible to decompose a periodic function into simple sine and cosine waves using the Fourier Synthesis equation:

$$= a_0 + \sum_{k=1}^{\infty} \left[ a_k \cos\left(\frac{2\pi}{T} \times k \times t\right) + b_k \sin\left(\frac{2\pi}{T} \times k \times t\right) \right]$$

And the analysis equations:

$$a_0 = \frac{1}{T} \int_0^T f(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T f(t)\, \cos\left(\frac{2\pi}{T} \times k \times t\right) dt$$

$$b_k = \frac{2}{T} \int_0^T f(t)\, \sin\left(\frac{2\pi}{T} \times k \times t\right) dt$$

These equations can help plot a graph of the wave in the frequency domain which is a static representation of the wave on a frequency vs amplitude graph. The frequency domain of different waves will be different which helps distinguish between different waves by comparing the frequencies and amplitudes.

This concept helps applications such as Shazam identify songs by comparing the frequency domain of an incoming sound wave with an existing database of frequency domains of different songs.

The Fourier series also converges very fast and can almost accurately approximate a wave. This property is advantageous in the field of file compression where only a few terms of the Fourier series can mimic the original function fairly accurately greatly reducing file sizes.

However, Fourier Series has the limitation that it is only applicable for periodic waves. To apply the concept of Fourier Series to any wave, the Fourier Transform can be derived putting parameters such as $T \to \infty$.

## Scope for further research

Further research is possible in the field of compression where images can be transformed into waves using mapping techniques such as the Hilbert's curve. The Fourier Transform can then be applied, and the convergence property of Fourier Series can be used to compress the file sizes of images. However, applying Fourier transform on every image can be slow for computers hence mathematicians came up with the Fast Fourier Transform. Using the foundation laid by this extended essay, further research can be done on Fast Fourier Transform and its advantages and disadvantages.

Another field where further research is possible is in the working of Shazam with the question: "How does Shazam use Fourier transform to identify songs efficiently and how does it differentiate between song audio and background noise?"

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