



Properties of Context-free Languages

Reading: Chapter 7



Topics

- 1) Simplifying CFGs, Normal forms
- 2) Pumping lemma for CFLs
- 3) Closure and decision properties of CFLs



How to “simplify” CFGs?



Three ways to simplify/clean a CFG

(clean)

1. Eliminate *useless symbols*

(simplify)

2. Eliminate ϵ -productions

$A \not\Rightarrow \epsilon$

3. Eliminate *unit productions*

$A \not\Rightarrow B$



Eliminating useless symbols

Grammar cleanup



Eliminating *useless symbols*

A symbol X is reachable if there exists:

- $S \rightarrow^* \alpha X \beta$

A symbol X is generating if there exists:

- $X \rightarrow^* w,$
 - for some $w \in T^*$

For a symbol X to be “useful”, it has to be both reachable *and* generating

- $S \rightarrow^* \alpha X \beta \rightarrow^* w',$ for some $w' \in T^*$

reachable generating



Algorithm to detect useless symbols

1. First, eliminate all symbols that are *not* generating
2. Next, eliminate all symbols that are *not* reachable

Is the order of these steps important,
or can we switch?



Example: Useless symbols

- $S \rightarrow AB \mid a$
- $A \rightarrow b$

1. A, S are generating
2. B is *not generating* (and therefore B is useless)
3. \Rightarrow Eliminating B ... (i.e., remove all productions that involve B)
 1. $S \rightarrow a$
 2. $A \rightarrow b$
4. Now, A is *not reachable* and therefore is useless

5. Simplified G:

1. $S \rightarrow a$

What would happen if you reverse the order:
i.e., test reachability before generating?

Will fail to remove:
 $A \rightarrow b$


$$X \rightarrow^* w$$

Algorithm to find all generating symbols

- Given: $G=(V,T,P,S)$
- Basis:
 - Every symbol in T is obviously generating.
- Induction:
 - Suppose for a production $A \rightarrow \alpha$, where α is generating
 - Then, A is also generating


$$S \rightarrow^* \alpha X \beta$$

Algorithm to find all reachable symbols

- Given: $G=(V,T,P,S)$
- Basis:
 - S is obviously reachable (from itself)
- Induction:
 - Suppose for a production $A \rightarrow \alpha_1 \alpha_2 \dots \alpha_k$, where A is reachable
 - Then, all symbols on the right hand side, $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ are also reachable.



Eliminating ε -productions

$A \Rightarrow \varepsilon$

X

What's the point of removing ε -productions?

$A \rightarrow \varepsilon$

Eliminating ε -productions

Caveat: It is *not* possible to eliminate ε -productions for languages which include ε in their word set

So we will target the grammar for the *rest of the language*

Theorem: If $G=(V,T,P,S)$ is a CFG for a language L , then $L \setminus \{\varepsilon\}$ has a CFG without ε -productions

Definition: A is “nullable” if $A \rightarrow^* \varepsilon$

- If A is nullable, then any production of the form “ $B \rightarrow CAD$ ” can be simulated by:
 - $B \rightarrow CD \mid CAD$
 - This can allow us to remove ε transitions for A



Algorithm to detect all nullable variables

- Basis:

- If $A \rightarrow \varepsilon$ is a production in G , then A is nullable
(note: A can still have other productions)

- Induction:

- If there is a production $B \rightarrow C_1 C_2 \dots C_k$, where *every* C_i is nullable, then B is also nullable



Eliminating ε -productions

Given: $G=(V,T,P,S)$

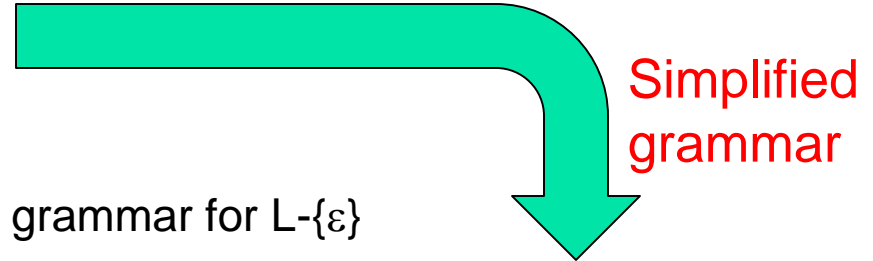
Algorithm:

1. Detect all nullable variables in G
2. Then construct $G_1=(V,T,P_1,S)$ as follows:
 - i. For each production of the form: $A \rightarrow X_1 X_2 \dots X_k$, where $k \geq 1$, suppose m out of the k X_i 's are nullable symbols
 - ii. Then G_1 will have 2^m versions for this production
 - i. i.e, all combinations where each X_i is either present or absent
 - iii. Alternatively, if a production is of the form: $A \rightarrow \varepsilon$, then remove it

Example: Eliminating ϵ -productions

- Let L be the language represented by the following CFG G :

- i. $S \rightarrow AB$
- ii. $A \rightarrow aAA \mid \epsilon$
- iii. $B \rightarrow bBB \mid \epsilon$



Goal: To construct G_1 , which is the grammar for $L - \{\epsilon\}$

- Nullable symbols: $\{A, B\}$
- G_1 can be constructed from G as follows:
 - $B \rightarrow b \mid bB \mid bB \mid bBB$
 - $\Rightarrow B \rightarrow b \mid bB \mid bBB$
 - Similarly, $A \rightarrow a \mid aA \mid aAA$
 - Similarly, $S \rightarrow A \mid B \mid AB$
- Note: $L(G) = L(G_1) \cup \{\epsilon\}$

G_1 :

- $S \rightarrow A \mid B \mid AB$
- $A \rightarrow a \mid aA \mid aAA$
- $B \rightarrow b \mid bB \mid bBB$

+

$S \rightarrow \epsilon$

Eliminating unit productions

$A \Rightarrow B$

X

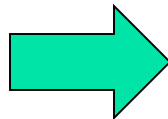
← B has to be a variable

What's the point of removing unit transitions ?

Will save #substitutions

E.g.,

$A \Rightarrow B \mid \dots$
$B \Rightarrow C \mid \dots$
$C \Rightarrow D \mid \dots$
$D \Rightarrow xxx \mid yyy \mid zzz$



$A \Rightarrow xxx \mid yyy \mid zzz \mid \dots$
$B \Rightarrow xxx \mid yyy \mid zzz \mid \dots$
$C \Rightarrow xxx \mid yyy \mid zzz \mid \dots$
$D \Rightarrow xxx \mid yyy \mid zzz$

before

after

$$A \rightarrow B$$

Eliminating unit productions

- Unit production is one which is of the form $A \rightarrow B$, where both A & B are variables

- E.g.,

1. $E \rightarrow T \mid E+T$
2. $T \rightarrow F \mid T^*F$
3. $F \rightarrow I \mid (E)$
4. $I \rightarrow a \mid b \mid Ia \mid Ib \mid I0 \mid I1$

- How to eliminate unit productions?

- Replace $E \rightarrow T$ with $E \rightarrow F \mid T^*F$

- Then, upon recursive application wherever there is a unit production:

- $E \rightarrow F \mid T^*F \mid E+T$ (substituting for T)
 - $E \rightarrow I \mid (E) \mid T^*F \mid E+T$ (substituting for F)
 - $E \rightarrow a \mid b \mid Ia \mid Ib \mid I0 \mid I1 \mid (E) \mid T^*F \mid E+T$ (substituting for I)
 - Now, E has no unit productions

- Similarly, eliminate for the remainder of the unit productions



The Unit Pair Algorithm: to remove unit productions

- Suppose $A \rightarrow B_1 \rightarrow B_2 \rightarrow \dots \rightarrow B_n \rightarrow \alpha$
- Action: Replace all intermediate productions to produce α directly
 - i.e., $A \rightarrow \alpha; B_1 \rightarrow \alpha; \dots B_n \rightarrow \alpha;$

Definition: (A, B) to be a “**unit pair**” if $A \rightarrow^* B$

- We can find all unit pairs inductively:
 - Basis: Every pair (A, A) is a unit pair (by definition). Similarly, if $A \rightarrow B$ is a production, then (A, B) is a unit pair.
 - Induction: If (A, B) and (B, C) are unit pairs, and $A \rightarrow C$ is also a unit pair.



The Unit Pair Algorithm: to remove unit productions

Input: $G=(V,T,P,S)$

Goal: to build $G_1=(V,T,P_1,S)$ devoid of unit productions

Algorithm:

1. Find all unit pairs in G
2. For each unit pair (A,B) in G :
 1. Add to P_1 a new production $A \rightarrow \alpha$, for every $B \rightarrow \alpha$ which is a *non-unit* production
 2. If a resulting production is already there in P , then there is no need to add it.

Example: eliminating unit productions

G:

1. $E \rightarrow T \mid E+T$
2. $T \rightarrow F \mid T^*F$
3. $F \rightarrow I \mid (E)$
4. $I \rightarrow a \mid b \mid Ia \mid Ib \mid I0 \mid I1$

Unit pairs

Only non-unit productions to be added to P_1

(E,E)	$E \rightarrow E+T$
(E,T)	$E \rightarrow T^*F$
(E,F)	$E \rightarrow (E)$
(E,I)	$E \rightarrow a \mid b \mid Ia \mid Ib \mid I0 \mid I1$
(T,T)	$T \rightarrow T^*F$
(T,F)	$T \rightarrow (E)$
(T,I)	$T \rightarrow a \mid b \mid Ia \mid Ib \mid I0 \mid I1$
(F,F)	$F \rightarrow (E)$
(F,I)	$F \rightarrow a \mid b \mid Ia \mid Ib \mid I0 \mid I1$
(I,I)	$I \rightarrow a \mid b \mid Ia \mid Ib \mid I0 \mid I1$

G₁:

1. $E \rightarrow E+T \mid T^*F \mid (E) \mid a \mid b \mid Ia \mid Ib \mid I0 \mid I1$
2. $T \rightarrow T^*F \mid (E) \mid a \mid b \mid Ia \mid Ib \mid I0 \mid I1$
3. $F \rightarrow (E) \mid a \mid b \mid Ia \mid Ib \mid I0 \mid I1$
4. $I \rightarrow a \mid b \mid Ia \mid Ib \mid I0 \mid I1$



Putting all this together...

- Theorem: If G is a CFG for a language that contains at least one string other than ε , then there is another CFG G_1 , such that $L(G_1) = L(G) - \varepsilon$, and G_1 has:
 - no ε -productions
 - no unit productions
 - no useless symbols
- Algorithm:
 - Step 1) eliminate ε -productions
 - Step 2) eliminate unit productions
 - Step 3) eliminate useless symbols

Again,
the order is
important!

Why?



Normal Forms



Why normal forms?

- If all productions of the grammar could be expressed in the same form(s), then:
 - a. It becomes easy to design algorithms that use the grammar
 - b. It becomes easy to show proofs and properties



Chomsky Normal Form (CNF)

Let G be a CFG for some $L - \{\epsilon\}$

Definition:

G is said to be in **Chomsky Normal Form** if all its productions are in one of the following two forms:

- i. $A \rightarrow BC$ where A, B, C are variables, or
- ii. $A \rightarrow a$ where a is a terminal

- G has no useless symbols
- G has no unit productions
- G has no ϵ -productions



CNF checklist

Is this grammar in CNF?

G_1 :

1. $E \rightarrow E+T \mid T^*F \mid (E) \mid Ia \mid Ib \mid I0 \mid I1$
2. $T \rightarrow T^*F \mid (E) \mid Ia \mid Ib \mid I0 \mid I1$
3. $F \rightarrow (E) \mid Ia \mid Ib \mid I0 \mid I1$
4. $I \rightarrow a \mid b \mid Ia \mid Ib \mid I0 \mid I1$

Checklist:

- G has no ε -productions ✓
- G has no unit productions ✓
- G has no useless symbols ✓
- But...
 - the normal form for productions is violated



So, the grammar is not in CNF

How to convert a G into CNF?

- Assumption: G has no ε -productions, unit productions or useless symbols
- 1) For every terminal **a** that appears in the body of a production:
 - i. create a unique variable, say X_a , with a production $X_a \rightarrow a$, and
 - ii. replace all other instances of **a** in G by X_a
- 2) Now, all productions will be in one of the following two forms:
 - $A \rightarrow B_1 B_2 \dots B_k$ ($k \geq 3$) or $A \rightarrow a$
- 3) Replace each production of the form $A \rightarrow B_1 B_2 B_3 \dots B_k$ by:

$\begin{array}{c} \xleftrightarrow{\hspace{1.5cm}} \\ \begin{array}{ccc} B_2 & C_2 & \\ \xleftrightarrow{\hspace{1.5cm}} \\ B_1 & C_1 & \end{array} \end{array}$

and so on...

 - $A \rightarrow B_1 C_1 \quad C_1 \rightarrow B_2 C_2 \quad \dots \quad C_{k-3} \rightarrow B_{k-2} C_{k-2} \quad C_{k-2} \rightarrow B_{k-1} B_k$

Example #1

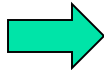
G:

$S \Rightarrow AS \mid BABC$

$A \Rightarrow A1 \mid 0A1 \mid 01$

$B \Rightarrow 0B \mid 0$

$C \Rightarrow 1C \mid 1$



G in CNF:

$X_0 \Rightarrow 0$

$X_1 \Rightarrow 1$

$S \Rightarrow AS \mid BY_1$

$Y_1 \Rightarrow AY_2$

$Y_2 \Rightarrow BC$

$A \Rightarrow AX_1 \mid X_0Y_3 \mid X_0X_1$

$Y_3 \Rightarrow AX_1$

$B \Rightarrow X_0B \mid 0$

$C \Rightarrow X_1C \mid 1$

All productions are of the form: $A \Rightarrow BC$ or $A \Rightarrow a$

Example #2

G:

1. $E \rightarrow E+T \mid T^*F \mid (E) \mid I_a \mid I_b \mid I_0 \mid I_1$
2. $T \rightarrow T^*F \mid (E) \mid I_a \mid I_b \mid I_0 \mid I_1$
3. $F \rightarrow (E) \mid I_a \mid I_b \mid I_0 \mid I_1$
4. $I \rightarrow a \mid b \mid I_a \mid I_b \mid I_0 \mid I_1$

Step (1)

1. $E \rightarrow EX_+T \mid TX_*F \mid X_((EX_)) \mid IX_a \mid IX_b \mid IX_0 \mid IX_1$
2. $T \rightarrow TX_*F \mid X_((EX_)) \mid IX_a \mid IX_b \mid IX_0 \mid IX_1$
3. $F \rightarrow X_((EX_)) \mid IX_a \mid IX_b \mid IX_0 \mid IX_1$
4. $I \rightarrow X_a \mid X_b \mid IX_a \mid IX_b \mid IX_0 \mid IX_1$
5. $X_+ \rightarrow +$
6. $X_* \rightarrow *$
7. $X_((\rightarrow ($
8. $X_((\rightarrow ($
9.

Step (2)

1. $E \rightarrow EC_1 \mid TC_2 \mid X_((C_3 \mid IX_a \mid IX_b \mid IX_0 \mid IX_1$
2. $C_1 \rightarrow X_+T$
3. $C_2 \rightarrow X_*F$
4. $C_3 \rightarrow X_((EX_))$
5. $T \rightarrow \dots\dots\dots$
6.

Languages with ε

- For languages that include ε ,
 - Write down the rest of grammar in CNF
 - Then add production “ $S \Rightarrow \varepsilon$ ” at the end

E.g., consider:

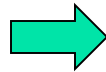
G:

$S \Rightarrow AS \mid BABC$

$A \Rightarrow A1 \mid 0A1 \mid 01 \mid \varepsilon$

$B \Rightarrow 0B \mid 0 \mid \varepsilon$

$C \Rightarrow 1C \mid 1 \mid \varepsilon$



G in CNF:

$X_0 \Rightarrow 0$

$X_1 \Rightarrow 1$

$S \Rightarrow AS \mid BY_1 \mid \varepsilon$

$Y_1 \Rightarrow AY_2$

$Y_2 \Rightarrow BC$

$A \Rightarrow AX_1 \mid X_0Y_3 \mid X_0X_1$

$Y_3 \Rightarrow AX_1$

$B \Rightarrow X_0B \mid 0$

$C \Rightarrow X_1C \mid 1$



Other Normal Forms

- Griebach Normal Form (GNF)
 - All productions of the form

$$A \Rightarrow a \alpha$$



Return of the Pumping Lemma !!

Think of languages that cannot be CFL

== think of languages for which a stack will not be enough

e.g., the language of strings of the form ww



Why pumping lemma?

- A result that will be useful in proving languages that *are not* CFLs
 - (just like we did for regular languages)
- But before we prove the pumping lemma for CFLs
 - Let us first prove an important property about parse trees

Observe that any parse tree generated by a CNF will be a binary tree, where all internal nodes have exactly two children (except those nodes connected to the leaves).

The “parse tree theorem”

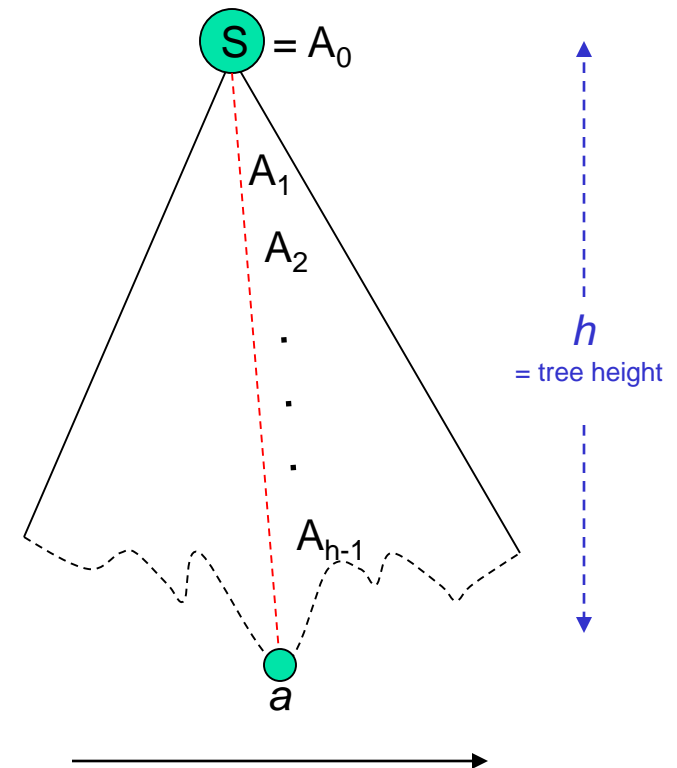
Given:

- Suppose we have a parse tree for a string w , according to a CNF grammar, $G=(V,T,P,S)$
- Let h be the height of the parse tree

Implies:

- $|w| \leq 2^{h-1}$

Parse tree for w



In other words, a CNF parse tree's string yield (w) can no longer be 2^{h-1}

To show: $|w| \leq 2^{h-1}$

Proof...The size of parse trees

Proof: (using induction on h)

Basis: $h = 1$

- Derivation will have to be " $S \rightarrow a$ "
- $|w| = 1 = 2^{1-1}$.

Ind. Hyp: $h = k-1$

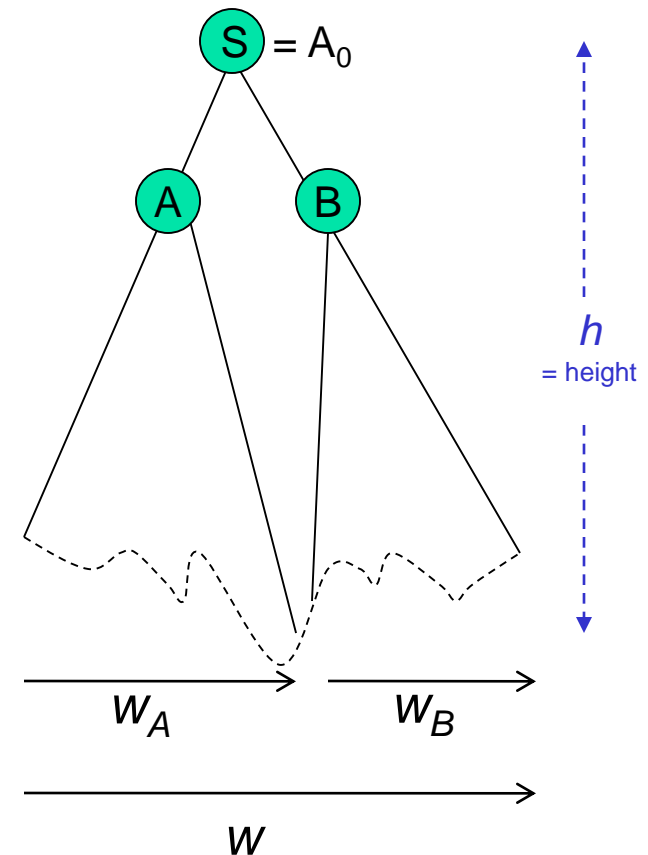
- $|w| \leq 2^{k-2}$

Ind. Step: $h = k$

S will have exactly two children:
 $S \rightarrow AB$

- Heights of A & B subtrees are at most $h-1$
- $w = w_A w_B$, where $|w_A| \leq 2^{k-2}$
and $|w_B| \leq 2^{k-2}$
- $|w| \leq 2^{k-1}$

Parse tree for w





Implication of the Parse Tree Theorem (assuming CNF)

Fact:

- If the height of a parse tree is h , then
 - $\implies |w| \leq 2^{h-1}$

Implication:

- If $|w| \geq 2^m$, then
 - Its parse tree's height is *at least* $m+1$



The Pumping Lemma for CFLs

Let L be a CFL.

Then there exists a constant N , s.t.,

- if $z \in L$ s.t. $|z| \geq N$, then we can write $z = uvwxy$, such that:

1. $|vwx| \leq N$
2. $vx \neq \varepsilon$
3. For all $k \geq 0$: $uv^kwx^ky \in L$

Note: we are pumping in two places (v & x)



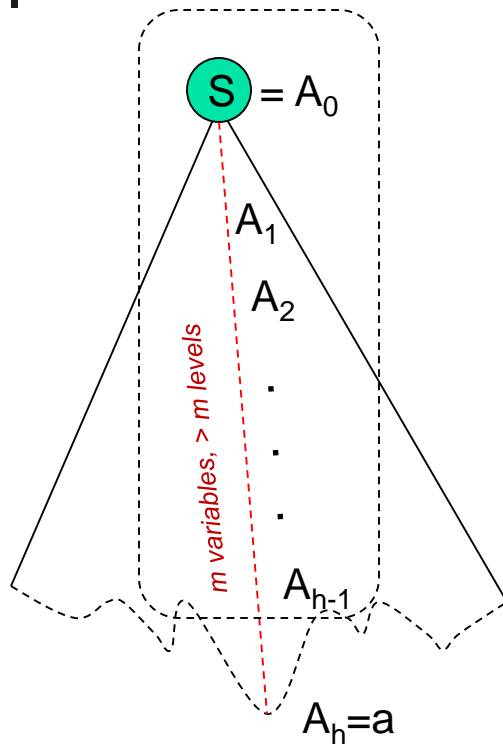
Proof: Pumping Lemma for CFL

- If $L = \Phi$ or contains only ε , then the lemma is trivially satisfied (as it cannot be violated)
- For any other L which is a CFL:
 - Let G be a CNF grammar for L
 - Let m = number of variables in G
 - Choose $N = 2^m$.
 - Pick any $z \in L$ s.t. $|z| \geq N$
 - ➔ the parse tree for z should have a height $\geq m+1$
(by the parse tree theorem)

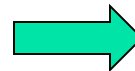
Parse tree for z

Meaning:
Repetition in the
last $m+1$ variables

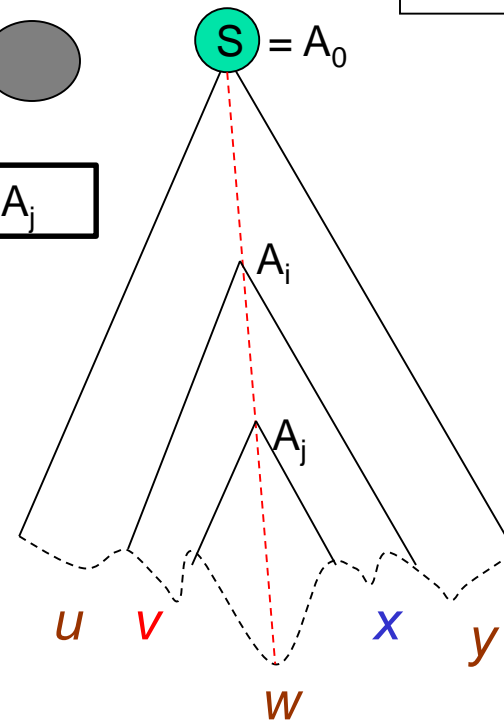
$$h-m \leq i < j \leq h$$



$$h \geq m+1$$



$$A_i = A_j$$

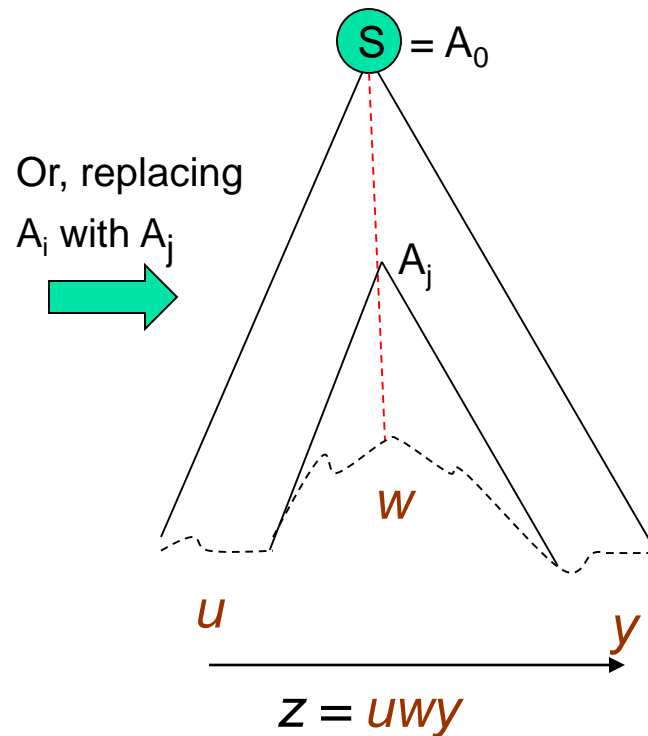
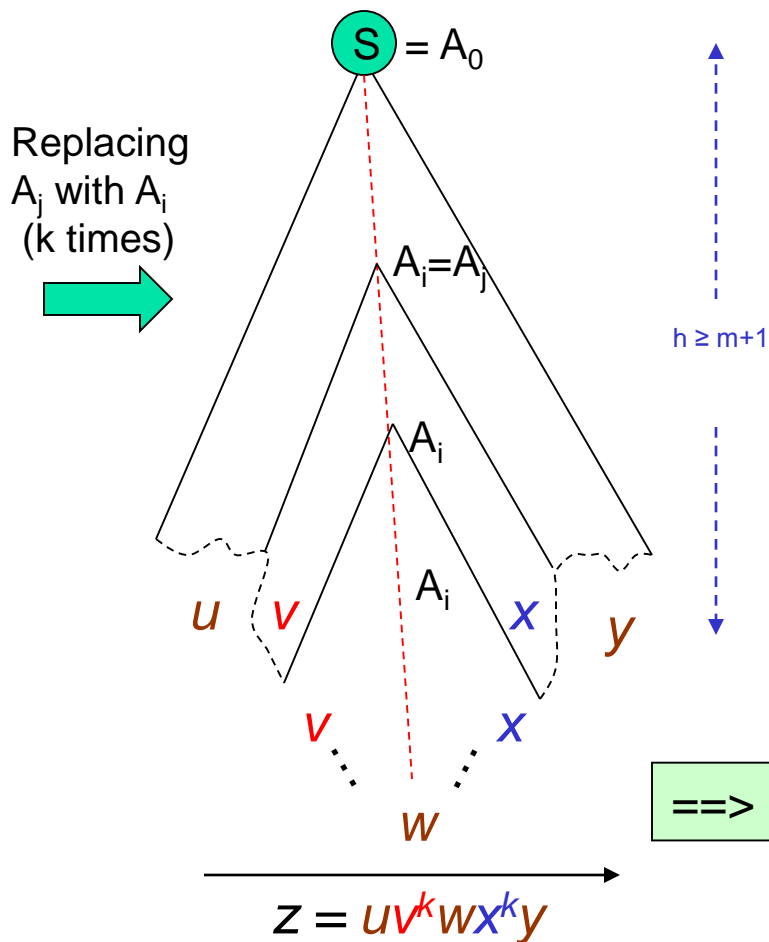


$$h \geq m+1$$

$$m+1$$

- Therefore, $vx \neq \varepsilon$

Extending the parse tree...



\Rightarrow For all $k \geq 0$: $uv^kwx^ky \in L$



Proof contd..

- Also, since A_i 's subtree no taller than $m+1$
 \implies the string generated under A_i 's subtree, which is $\textcolor{red}{v}\textcolor{brown}{w}\textcolor{blue}{x}$, cannot be longer than $2^m (=N)$

But, $2^m = N$

$$\implies |\textcolor{red}{v}\textcolor{brown}{w}\textcolor{blue}{x}| \leq N$$

This completes the proof for the pumping lemma.



Application of Pumping Lemma for CFLs

Example 1: $L = \{a^m b^m c^m \mid m > 0\}$

Claim: L is not a CFL

Proof:

- Let $N \leq P/L$ constant
- Pick $z = a^N b^N c^N$
- Apply pumping lemma to z and show that there exists at least one other string constructed from z (obtained by pumping up or down) that is $\notin L$



Proof contd...

- $z = uvwxy$
- As $z = a^N b^N c^N$ and $|vwx| \leq N$ and $vx \neq \varepsilon$
 - $\implies v, x$ cannot contain all three symbols (a, b, c)
 - \implies we can pump up or pump down to build another string which is $\notin L$



Example #2 for P/L application

- $L = \{ ww \mid w \text{ is in } \{0,1\}^* \}$
- Show that L is not a CFL
 - Try string $z = 0^N 0^N$
 - what happens?
 - Try string $z = 0^N 1^N 0^N 1^N$
 - what happens?



Example 3

- $L = \{ 0^{k^2} \mid k \text{ is any integer} \}$
- Prove L is not a CFL using Pumping Lemma



Example 4

- $L = \{a^i b^j c^k \mid i < j < k\}$
- Prove that L is not a CFL



CFL Closure Properties



Closure Property Results

- CFLs are closed under:
 - Union
 - Concatenation
 - Kleene closure operator
 - Substitution
 - Homomorphism, inverse homomorphism
 - reversal
-

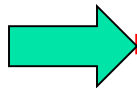
- CFLs are *not* closed under:
 - Intersection
 - Difference
 - Complementation

Note: Reg languages
are closed
under
these
operators

Strategy for Closure Property Proofs

- First prove “closure under **substitution**”
- Using the above result, prove other closure properties
- CFLs are closed under:
 - Union
 - Concatenation
 - Kleene closure operator
 - Substitution
 - Homomorphism, inverse homomorphism
 - Reversal

Prove
this first



Note: $s(L)$ can use
a different alphabet

The *Substitution* operation

For each $a \in \Sigma$, then let $s(a)$ be a language

If $w = a_1 a_2 \dots a_n \in L$, then:

$$\bullet s(w) = \{ x_1 x_2 \dots \} \in s(L), \text{ s.t., } x_i \in s(a_i)$$

Example:

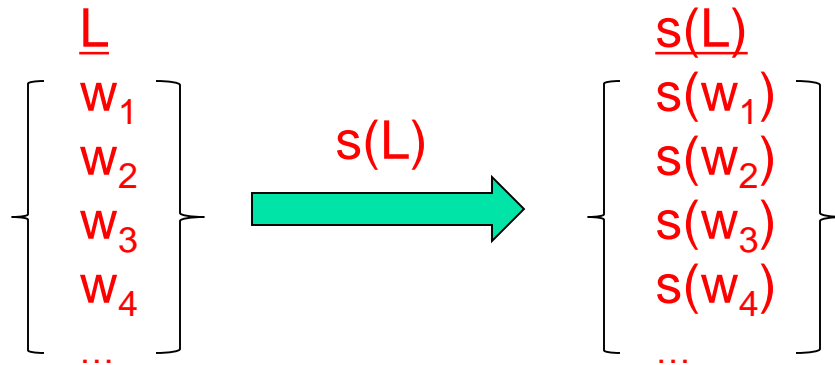
- Let $\Sigma = \{0, 1\}$
- Let: $s(0) = \{a^n b^n \mid n \geq 1\}$, $s(1) = \{aa, bb\}$
- If $w = 01$, $s(w) = s(0).s(1)$
 - E.g., $s(w)$ contains $a^1 b^1 aa$, $a^1 b^1 bb$,
 $a^2 b^2 aa$, $a^2 b^2 bb$,
... and so on.

CFLs are closed under Substitution

IF L is a CFL and a substitution defined on L , $s(L)$, is s.t., $s(a)$ is a CFL for every symbol a , THEN:

- $s(L)$ is also a CFL

What is $s(L)$?

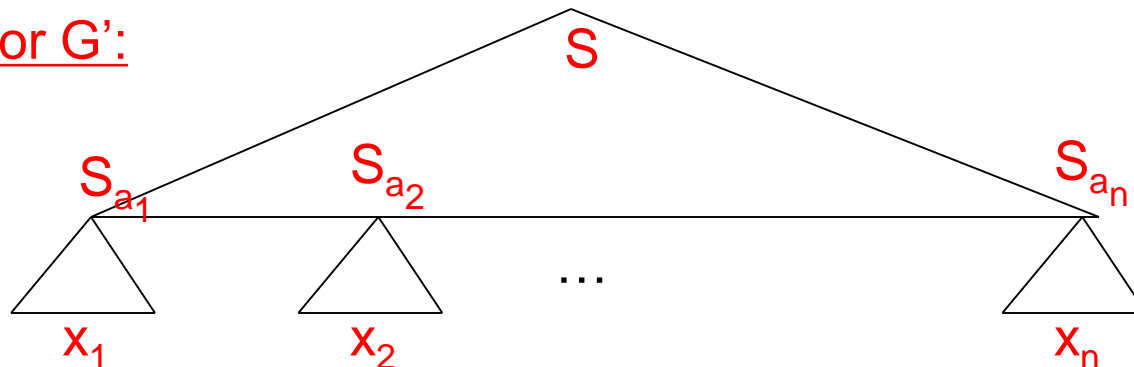


Note: each $s(w)$ is itself a set of strings

CFLs are closed under *Substitution*

- $G=(V,T,P,S)$: CFG for L
- Because every $s(a)$ is a CFL, there is a CFG for each $s(a)$
 - Let $G_a = (V_a, T_a, P_a, S_a)$
- Construct $G'=(V',T',P',S)$ for $s(L)$
- P' consists of:
 - The productions of P , but with every occurrence of terminal “ a ” in their bodies replaced by S_a .
 - All productions in any P_a , for any $a \in \Sigma$

Parse tree for G' :



Substitution of a CFL: example

- Let L = language of binary palindromes s.t., substitutions for 0 and 1 are defined as follows:
 - $s(0) = \{a^n b^n \mid n \geq 1\}$, $s(1) = \{xx, yy\}$
- Prove that $s(L)$ is also a CFL.

CFG for L :

$S \Rightarrow 0S0 \mid 1S1 \mid \varepsilon$

CFG for $s(0)$:

$S_0 \Rightarrow aS_0b \mid ab$

CFG for $s(1)$:

$S_1 \Rightarrow xx \mid yy$



Therefore, CFG for $s(L)$:

$S \Rightarrow S_0 S S_0 \mid S_1 S S_1 \mid \varepsilon$

$S_0 \Rightarrow aS_0b \mid ab$

$S_1 \Rightarrow xx \mid yy$



CFLs are closed under *union*

Let L_1 and L_2 be CFLs

To show: $L_1 \cup L_2$ is also a CFL

Let us show by using the result of *Substitution*

- Make a new language:

- $L_{\text{new}} = \{a, b\}$ s.t., $s(a) = L_1$ and $s(b) = L_2$

- $\implies s(L_{\text{new}}) == \text{same as } L_1 \cup L_2$



-
- A more direct, alternative proof

- Let S_1 and S_2 be the starting variables of the grammars for L_1 and L_2

- Then, $S_{\text{new}} \Rightarrow S_1 \mid S_2$



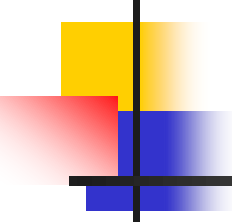
CFLs are closed under *concatenation*

- Let L_1 and L_2 be CFLs

Let us show by using the result of *Substitution*

- Make $L_{\text{new}} = \{ab\}$ s.t.,
 $s(a) = L_1$ and $s(b) = L_2$
 $\implies L_1 L_2 = s(L_{\text{new}})$

-
- A proof without using substitution?



CFLs are closed under *Kleene Closure*

- Let L be a CFL
- Let $L_{\text{new}} = \{a\}^*$ and $s(a) = L_1$
 - Then, $L^* = s(L_{\text{new}})$

CFLs are closed under *Reversal*

- Let L be a CFL, with grammar $G=(V,T,P,S)$
- For L^R , construct $G^R=(V,T,P^R,S)$ s.t.,
 - If $A \Rightarrow \alpha$ is in P , then:
 - $A \Rightarrow \alpha^R$ is in P^R
 - (that is, reverse every production)



CFLs are *not* closed under Intersection

- Existential proof:
 - $L_1 = \{0^n 1^n 2^i \mid n \geq 1, i \geq 1\}$
 - $L_2 = \{0^i 1^n 2^n \mid n \geq 1, i \geq 1\}$
- Both L_1 and L_2 are CFLs
 - Grammars?
- But $L_1 \cap L_2$ *cannot* be a CFL
 - Why?
- We have an example, where intersection is not closed.
- Therefore, CFLs are not closed under intersection

CFLs are not closed under complementation

- Follows from the fact that CFLs are not closed under intersection

- $$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$$

Logic: if CFLs were to be closed under complementation

- the whole right hand side becomes a CFL (because CFL is closed for union)
- the left hand side (intersection) is also a CFL
- but we just showed CFLs are NOT closed under intersection!
- CFLs cannot be closed under complementation.



CFLs are not closed under difference

- Follows from the fact that CFLs are not closed under complementation
- Because, if CFLs are closed under difference, then:
 - $\bar{L} = \Sigma^* - L$
 - So \bar{L} has to be a CFL too
 - Contradiction



Decision Properties

- Emptiness test
 - Generating test
 - Reachability test
- Membership test
 - PDA acceptance



“Undecidable” problems for CFL

- Is a given CFG G ambiguous?
- Is a given CFL inherently ambiguous?
- Is the intersection of two CFLs empty?
- Are two CFLs the same?
- Is a given $L(G)$ equal to Σ^* ?



Summary

- Normal Forms
 - Chomsky Normal Form
 - Griebach Normal Form
 - Useful in proving P/L
- Pumping Lemma for CFLs
 - Main difference: $z = uv^iwx^iy$
- Closure properties
 - Closed under: union, concatenation, reversal, Kleen closure, homomorphism, substitution
 - Not closed under: intersection, complementation, difference