NP-Completeness

Reference: Computers and Intractability: A Guide to the Theory of NP-Completeness
by Garey and Johnson,
W.H. Freeman and Company, 1979.

General Problems, Input Size and Time Complexity

- Time complexity of algorithms :
  polynomial time algorithm ("efficient algorithm") v.s.
  exponential time algorithm ("inefficient algorithm")

f(n) \ n	**10**	**30**	**50**
n	0.00001 sec	0.00003 sec	0.00005 sec
n^5	0.1 sec	24.3 sec	5.2 mins
2^n	0.001 sec	17.9 mins	35.7 yrs

"Hard" and "easy' Problems

- Sometimes the dividing line between "easy" and "hard" problems is a fine one. For example
  - Find the shortest path in a graph from X to Y. (easy)
  - Find the longest path in a graph from X to Y. (with no cycles) (hard)
- View another way – as "yes/no" problems
  - Is there a simple path from X to Y with weight <= M? (easy)
  - Is there a simple path from X to Y with weight >= M? (hard)
  - First problem can be solved in polynomial time.
  - All known algorithms for the second problem (could) take exponential time .

- Decision problem: The solution to the problem is "yes" or "no". Most optimization problems can be phrased as decision problems (still have the same time complexity).

  Example :
  Assume we have a decision algorithm X for 0/1 Knapsack problem with capacity M,
  i.e. Algorithm X returns "Yes" or "No" to the question
  "Is there a solution with profit $\geq P$ subject to knapsack capacity $\leq M$?"

We can repeatedly run algorithm X for various profits(P values) to find an optimal solution.
Example : Use binary search to get the optimal profit,
maximum of lg $\sum p_i$ runs.
(where M is the capacity of the knapsack optimization problem)

Min Bound Optimal Profit Max Bound

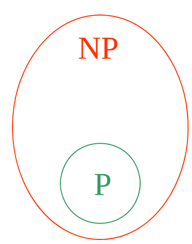
0 Search for the optimal solution $\sum p_i$

The Classes of P and NP

- The class P and Deterministic Turing Machine
  - Given a decision problem X, if there is a polynomial time Deterministic Turing Machine program that solves X, then X is belong to P
  - ***Informally, there is a polynomial time algorithm to solve the problem***

- The class NP and Non-deterministic Turing Machine
  - Given a decision problem X.
    If there is a polynomial time Non-deterministic Turing machine program that solves X, then X belongs to NP
  - ***Given a decision problem X.***
    For every instance I of X,
    (a) guess solution S for I, and
    (b) check "is S a solution to I?"
    If (a) and (b) can be done in polynomial time, then X belongs to NP.

- Obvious : P $\subseteq$ NP, i.e. A (decision) problem in P does not need "guess solution". The correct solution can be computed in polynomial time.



- Some problems which are in NP, but may not in P :
  - 0/1 Knapsack Problem
  - PARTITION Problem : Given a finite set of positive integers Z.
    Question : Is there a subset Z' of Z such that
    Sum of all numbers in Z' = Sum of all numbers in Z-Z' ?
    i.e. $\sum Z' = \sum (Z-Z')$

- One of the most important open problem in theoretical compute science :

Is P=NP ?

Most likely “No”.

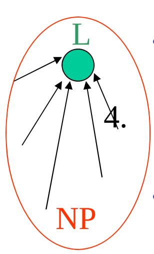
Currently, there are many known (decision) problems in NP, and there is no solution to show anyone of them in P.

NP-Complete Problems

- Stephen Cook introduced the notion of NP-Complete Problems.
  - This makes the problem “P = NP ?” much more interesting to study.
- The following are several important things presented by Cook :

1. Polynomial Transformation (" $\propto$ ")
   - **L1 $\propto$ L2** :
     There is a polynomial time transformation that transforms arbitrary instance of L1 to some instance of L2.
   - **If L1 $\propto$ L2 then L2 is in P implies L1 is in P**
     (or L1 is not in P implies L2 is not in P)
   - **If L1 $\propto$ L2 and L2 $\propto$ L3 then L1 $\propto$ L3**

2. Focus on the class of NP – **decision** problems only. Many intractable problems, when phrased as decision problems, belong to this class.
3. L is **NP-Complete**
   if (#1) L $\in$NP & (#2) for all other L' $\in$ NP, L' $\propto$ L
   - If an NP-complete problem can be solved in polynomial time then all problems in NP can be solved in polynomial time.
   - If a problem in NP cannot be solved in polynomial time then all problems in NP-complete cannot be solved in polynomial time.
   - Note that an NP-complete problem is one of those hardest problems in NP.



4. L is **NP-Hard** if
   (#2 of NP-Complete) for all other L' $\in$ NP, L' $\propto$ L
   - Note that an NP-Hard problem is a problem which is as hard as an NP-Complete problem and it's not necessary a decision problem.
5.

- So, if an NP-complete problem is in P then P=NP
- if P != NP then all NP-complete problems are in NP-P

4. Question : How can we obtain the first NP-complete problem L?

Cook Theorem : SATISFIABILITY is NP-Complete. (The first NP-Complete problem)

Instance : Given a set of variables, U, and a collection of clauses, C, over U.

Question : Is there a truth assignment for U that satisfies all clauses in C?

Example :

$\mathbf{U = \{x_1, x_2\}}$

$\mathbf{C_1 = \{(x_1, \neg x_2), (\neg x_1, x_2)\}}$

$\mathbf{= (x_1 \text{ OR } \neg x_2) \text{ AND } (\neg x_1 \text{ OR } x_2)}$

$\mathbf{\text{if } x_1 = x_2 = \text{True} \rightarrow C_1 = \text{True}}$

$\mathbf{C_2 = (x_1, x_2)\ (x_1, \neg x_2)\ (\neg x_1) \rightarrow}$ **not satisfiable**

“$\neg x_i$” = “not x_i”   “OR” = “logical or”   “AND” = “logical and”

This problem is also called “CNF-Satisfiability” since the expression is in CNF – Conjunctive Normal Form (the product of sums).

- With the Cook Theorem, we have the following property :

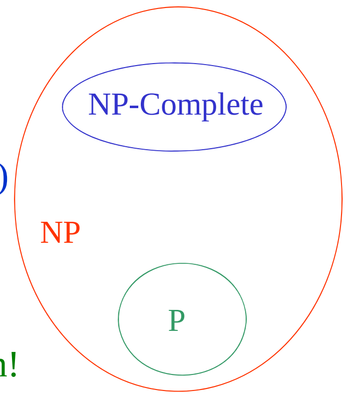
  Lemma :
  If L1 and L2 belong to NP,
  L1 is NP-complete, and L1 $\propto$ L2
  then L2 is NP-complete.

    i.e. L1, L2 $\in$NP and for all other L' $\in$ NP, L' $\propto$ L1 and L1 $\propto$ L2 $\rightarrow$ L' $\propto$ L2

- So now, to prove
  a (decision) problem L to be NP-complete, we need to
  - show L is in NP
  - select a known NP-complete problem L'
  - construct a polynomial time transformation f from L' to L
  - prove the correctness of f (i.e. L' has a solution if and only if L has a solution) and that f is a polynomial transformation

- P: (Decision) problems solvable by deterministic algorithms in polynomial time
- NP: (Decision) problems solved by non-deterministic algorithms in polynomial time
- A group of (decision) problems, including all of the ones we have discussed (Satisfiability, 0/1 Knapsack, Longest Path, Partition) have an additional important property:

  If any of them can be solved in polynomial time, then they all can!



- These problems are called NP-complete problems.

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