5 Work, energy and power

Work and energy are concepts we use every day for any number of different things. For the purposes of formal study however, we need precise definitions that enable us to interpret the concepts of work and energy in a consistent way. From these definitions we deduce relations that are applicable to real systems.

5.1 The work done by a constant force

We consider a **constant** force of magnitude F acting on an object of mass m at an angle θ as shown in Figure 10. The force moves the object over a distance s.

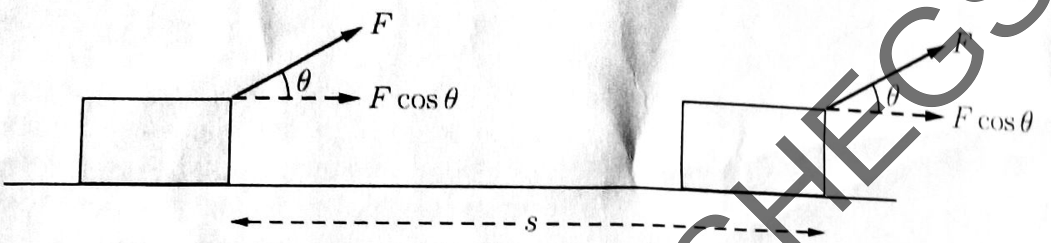


Figure 10: An object moved a distance s by a force F.

The amount of **work** done on an object by a force is equal to the product of the displacement and the **component** of the force in the **direction** of the displacement.

$$\boxed{W = Fs\cos\theta}, \tag{39}$$

where F and s are the magnitudes of $\mathbf{F}$ and $\mathbf{s}$. The angle θ is the included angle between $\mathbf{F}$ and $\mathbf{s}$. Note that:

1. If the applied force is in the direction of the displacement, then $\theta = 0°$ and $W = Fs$.
2. If the applied force is in the opposite direction of the displacement, then $\theta = 180°$ and $W = -Fs$.
3. If the applied force is perpendicular to the displacement, then $\theta = 90°$ and $W = 0$. (i.e. a force acting at right angles to a displacement does **no** work.)

Work is a scalar quantity and the unit of work is the newton-metre or joule.

One **joule** is the work done by a force of one newton when it moves its point of application through a distance of one metre in the direction of the force ($1\,\text{joule} \equiv 1\,\text{J} \equiv 1\,\text{N m}$).

Example 31: Work done on an object dragged over a distance

Find the work done when a trunk is dragged a distance of 10 m by a force of 50 N applied at an angle of 45° above the surface over which the trunk is moved.

Solution:

The work done may be obtained directly from Equation (39) with the values given. Hence

$$W = Fs\cos\theta = 50\,\text{N} \times 10\,\text{m} \times \cos 45° = 354\,\text{J}.$$

5.2 Energy

Different forms of energy are identified:

(a) Kinetic energy

(b) Potential energy

- gravitational
- elastic
- electrostatic

(c) Thermal and internal energy

(d) Radiant energy

(e) Chemical energy

(f) Nuclear energy

(g) Mass energy

On a microscopic scale, **all** forms of energy can be classified as either (a) or (b).

Changes occur between different forms of energy, and the amounts possessed by different bodies, but if we take all forms into account, we find there is no change in the total energy in the universe. This is the **law of conservation of energy**. Mathematically:

Total energy of a closed system before some event	=	Total energy of a closed system after the event

5.2.1 Kinetic energy and the work-energy theorem

Suppose a constant force F acts on an object of mass m. If the object moves a distance s in the direction of the force F, we can obtain the work done on the object by multiplying $F = ma$ on both sides by the displacement:

$$W = Fs = mas.$$

Since the force acting on the object is constant, the acceleration of the object is also constant and we may apply the kinematic equations of motion for constant acceleration. Substituting $v^2 = u^2 + 2as$ (with as the subject) in the equation above and using v_i and v_f for the initial and final velocities instead of u and v, we obtain

$$W = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2. \qquad (40)$$

If the **kinetic energy** of an object is defined as

$$\boxed{E_\mathrm{k} = \tfrac{1}{2}mv^2}, \qquad (41)$$

then the right hand side of Equation (40) represents the change in kinetic energy of the object when an amount of work W is done on it. Although we have derived Equation (40) for the work done by a constant force, it can be shown to hold in general for the work done by any type of force. Furthermore, if more than one force does work on an object, the total work done is equal to the work done by the **resultant** force:

$$\boxed{W(\text{due to the resultant force}) = \Delta E_\mathrm{k} = E_\mathrm{k}(\text{final}) - E_\mathrm{k}(\text{initial})}. \qquad (42)$$

Equation (42) is known as the **work-energy theorem** for an object. The work done on an object can be positive or negative depending on the size of the angle θ in Equation (39).

Note that the kinetic energy E_k is a *positive scalar quantity* that represents the energy associated with a body because of its motion. It is either

1. the work done *by* the resultant force in accelerating the body from rest to an instantaneous speed v, or
2. the work done *by* the body on some external agent which brings it to rest.

(1) and (2) are equivalent.

The work done on an object by the resultant force is the same as the sum of the work done by each force separately. Different types of energy are associate with the work done by different types of forces.

Example 32: The work done in accelerating a car

A 1000 kg car accelerates uniformly from rest to a speed of $30\,\mathrm{m\,s^{-1}}$ in a distance of 20 m. Determine

(a) the kinetic energy gained,
(b) the work done by the net force acting on the car, and
(c) the magnitude of the average net force.

Solution:

(a) The car starts from rest, so the initial kinetic energy is zero and the kinetic energy gained is the final kinetic energy. From Equation (11):

$$E_\mathrm{k} = \tfrac{1}{2}mv^2 = \tfrac{1}{2} \times 1000 \times 30^2 = 4.5 \times 10^5\,\mathrm{J}.$$

(b) By the work-energy theorem, the work done is equal to the kinetic energy gained, hence $W = 4.5 \times 10^5$ J.

(c) From Equation (39) with $\theta = 0$ (since we must assume the force accelerating the car acts in the direction of the displacement of the car),

$$F = \frac{W}{s} = \frac{4.5 \times 10^5\,\mathrm{J}}{20\,\mathrm{m}} = 2.25 \times 10^4\,\mathrm{N}.$$

5.2.2 Potential energy

Potential energy is the energy possessed by a system by virtue of the relative positions of its component parts

Gravitational potential energy

Suppose we exert forces on a body of mass m and on the earth, and thereby push the body m to a rest position a vertical distance h above its initial position. The work $W = Fs = mgh$ is done against the gravitational force. We say that the system has gained gravitational potential energy

$$\boxed{E_\mathrm{p} = mgh}. \tag{43}$$

There is no gain in kinetic energy. The pulls of the earth on the body and the body on the earth have done *negative work*.

When the system is released the two gravitational forces both do positive work on the body and on the earth. Both, in principle, acquire kinetic energy, but that gained by the earth is negligible. The potential energy is associated with the relative positions (i.e. separation) of the two masses making up the system.

Gravitational potential energy is a kind of energy that can be completely recovered and converted into kinetic energy.

Example 33: Work done in lifting an object

Find the work done in lifting a body whose mass is 5 kg through a vertical distance of 2 m.

Solution:

From Equation (43):

$$W = mgh = 5\,\mathrm{kg} \times 9.8\,\mathrm{m\,s^{-2}} \times 2\,\mathrm{m} = 98\,\mathrm{J}.$$

Elastic potential energy (stretching/compressing a spring)

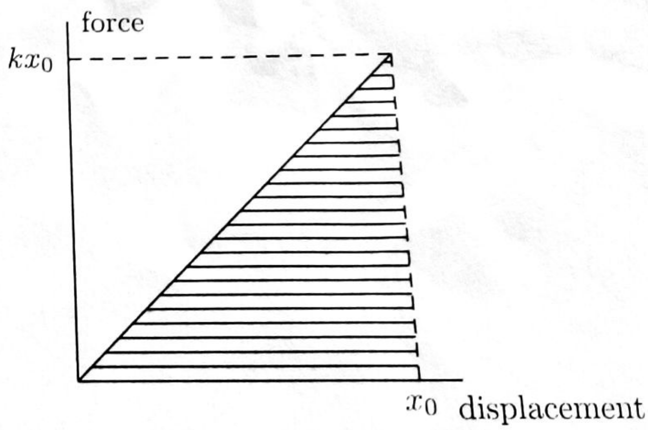
Consider a spring having natural length ℓ_0. Suppose the spring is stretched by an amount x to a new length ℓ (i.e. $x = \ell - \ell_0$). **Hooke's law** gives us the force F exerted by the spring. It is

$$\boxed{F = -kx}, \tag{44}$$

where the force constant k depends on the spring. The minus sign indicates that F points in the opposite direction to the displacement. This is a **restoring force**.

We cannot use $W = Fs$ to determine the work done in stretching this spring because F **is not constant**. It depends on the extension x.

To find the work done in stretching a spring by an amount x_0, we consider the graph alongside. This graph shows that the force varies linearly with x. It is **not** constant.



The work done is the area under a force–displacement graph. The shaded area is $W = \frac{1}{2} \times \text{base} \times \text{height}$.

$$W = \tfrac{1}{2} \times x_0 \times kx_0 = \tfrac{1}{2}kx_0^2.$$

This is the work done by an external agent in stretching the spring. This work is stored as elastic potential energy in the spring until we release the spring.

The potential energy for a spring is given by

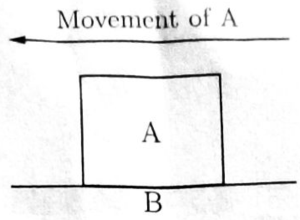
$$E_\mathrm{p} = \tfrac{1}{2}kx^2.$$

Note that x is the extension/compression from the spring's natural length.

Internal energy

A frictional force always opposes relative motion. When surfaces slide over one another, such a force always does negative work. This work represents energy being transferred to random molecular potential and kinetic energy (**internal energy**).

In the figure alongside, B exerts a frictional force on A to the right, which moves its point of application to the left, and so does negative work. Macroscopically we see that A experiences a force which reduces its speed. Microscopically work is being done on a molecular scale that results in an increase of the random kinetic and potential energies of individual molecules. We observe a temperature increase along the common surface.



5.3 Conservation of mechanical energy

The mechanical energy of an object is defined as the sum of its potential and kinetic energies:

$$E = E_\mathrm{p} + E_\mathrm{k}. \tag{45}$$

If there is no work done on an object by any applied forces, then the **mechanical energy** of the object is **conserved**. This means that the total mechanical energy of the object always remains the same. Hence $\Delta E = 0$ and

$$\Delta E_\mathrm{p} + \Delta E_\mathrm{k} = 0. \tag{46}$$

Equation (46) may be rewritten in terms of the final and initial kinetic and potential energies in the useful form

$$(E_\mathrm{p} + E_\mathrm{k})_\mathrm{final} = (E_\mathrm{p} + E_\mathrm{k})_\mathrm{initial}. \tag{47}$$

Example 34: Conservation of mechanical energy

A mass of 80 kg slides down a smooth inclined plane 16 m high and 80 m long. Neglecting friction,

(a) *calculate the potential energy of the mass at the top of the slope.*

(b) *How much kinetic energy does it have at the bottom of the slope?*

(c) *Determine the speed of the mass at the bottom of the slope.*

Solution:

(a) Relative to the bottom of the slope, the potential energy at the top is

$$\Delta E_\mathrm{p} = mgh = 80\,\mathrm{kg} \times 9.8\,\mathrm{m\,s^{-2}} \times 16\,\mathrm{m} = 1.25 \times 10^4\,\mathrm{J}.$$

(b) Mechanical energy is conserved, hence the potential energy lost equals the kinetic energy gained. The kinetic energy at the bottom of the slope is therefore $E_\mathrm{k} = 1.25 \times 10^4\,\mathrm{J}$.

(c) Rearranging Equation (41):

$$v = \sqrt{\frac{2 \times E_\mathrm{k}}{m}} = \sqrt{\frac{2 \times 1.25 \times 10^4\,\mathrm{J}}{80\,\mathrm{kg}}} = 17.7\,\mathrm{m\,s^{-1}}.$$

5.4 Power

Power is the **rate** at which work is done.

If work W is done in a time t, then the average power P for the time interval t is given by

$$\boxed{P = \frac{\text{work}}{\text{time}} = \frac{W}{t}}. \tag{48}$$

Power is not associated with any direction, and since work and time are scalar quantities, power is also a scalar quantity. The SI unit of power is the watt (W).

One **watt** is the power developed when one joule of work is done per second.

If the force doing work is in the same direction as the displacement, then Equation (39) becomes

$$W = Fs. \tag{49}$$

Substituting Equation (49) in Equation (48), we obtain the useful expression

$$P = \frac{W}{t} = \frac{Fs}{t} = F\bar{v}. \tag{50}$$

The velocity in Equation (50) is the **average velocity**, and the force is in the direction of the motion.

Example 35: The power generated by an accelerating car

A car whose mass is 1000 kg accelerates constantly from rest at $2.0\,\text{m}\,\text{s}^{-2}$ for 10 s. Determine the average power generated by the net force accelerating the car.

Solution:

We first find the force accelerating the car. Using Equation (22), we have

$$F = ma = 1000\,\text{kg} \times 2\,\text{m}\,\text{s}^{-2} = 2000\,\text{N}.$$

To find the power, we must either calculate the work done and use Equation (48), or the average velocity and use Equation (50). We will demonstrate both methods.

Method 1: To find the work done, we need to determine the distance travelled. Using Equation (15), the displacement

$$s = ut + \tfrac{1}{2}at^2 = 0 + \frac{2\,\text{m}\,\text{s}^{-2} \times (10\,\text{s})^2}{2} = 100\,\text{m}.$$

The work done is therefore

$$W = Fs = 2000\,\text{N} \times 100\,\text{m} = 2 \times 10^5\,\text{J},$$

and the power generated is

$$P = \frac{W}{t} = \frac{2 \times 10^5\,\text{J}}{10\,\text{s}} = 2 \times 10^4\,\text{W}.$$

Method 2: Since the acceleration is constant, the average velocity is half the initial plus final velocity. As the initial velocity is zero, we have

$$\bar{v} = \tfrac{1}{2}v.$$