The Medvedev and Muchnik degrees

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Logique à Paris 2023
Paris, France
11 May 2023

Turing reducibility

Let $f, g\colon \mathbb{N} \to \mathbb{N}$.

Say that f **Turing reduces to** g ($f \leq_{\mathrm{T}} g$) if there is a program computing f that uses g as an **oracle / black box**.

To make sense of this:

- Add an instruction called `query` to the programming language.
- Equip program Φ with oracle g: Φ^g.
- When Φ^g executes $\mathtt{query}(n)$, it evaluates to $g(n)$.

Example: Let Φ be the following oracle machine.

Input: n

$y \coloneqq \mathtt{query}(n)$;
$y \coloneqq n + 1$;
return y;

Then $\Phi^g(n) = g(n) + 1$ for every oracle g.

The Turing degrees

Let $f, g \colon \mathbb{N} \to \mathbb{N}$.

If $f \leq_{\mathrm{T}} g$, we say that:

- f is **recursive in / computable from** g
- g **computes / knows** f.

The relation $f \leq_{\mathrm{T}} g$ is a quasi-order:

- $f \leq_{\mathrm{T}} f$
- $(f \leq_{\mathrm{T}} g \ \& \ g \leq_{\mathrm{T}} h) \ \Rightarrow \ f \leq_{\mathrm{T}} h$.

Functions f and g are **Turing equivalent** ($\boldsymbol{f \equiv_{\mathrm{T}} g}$) if $f \leq_{\mathrm{T}} g \ \& \ g \leq_{\mathrm{T}} f$.

The **Turing degree** of f is $\deg_{\mathrm{T}}(f) = \{g : g \equiv_{\mathrm{T}} f\}$.

The **Turing degrees** are $\mathcal{D}_{\mathrm{T}} = \{\deg_{\mathrm{T}}(f) : f \in \mathbb{N}^{\mathbb{N}}\}$.

The Turing degrees as an upper semi-lattice

Turing reducibility $\leq_{\mathrm{T}}$ induces a partial order on $\mathcal{D}_{\mathrm{T}}$:

$$\deg_{\mathrm{T}}(f) \leq_{\mathrm{T}} \deg_{\mathrm{T}}(g) \quad \Leftrightarrow \quad f \leq_{\mathrm{T}} g.$$

For $f, g \colon \mathbb{N} \to \mathbb{N}$, define the **join** $f \oplus g$ by:

$$\begin{aligned} (f \oplus g)(2n) &= f(n) \\ (f \oplus g)(2n+1) &= g(n). \end{aligned}$$

Then:

- $(f_0 \equiv_{\mathrm{T}} f_1 \;\&\; g_0 \equiv_{\mathrm{T}} g_1) \Rightarrow f_0 \oplus g_0 \equiv_{\mathrm{T}} f_1 \oplus g_1$
- $f \leq_{\mathrm{T}} f \oplus g \quad \& \quad g \leq_{\mathrm{T}} f \oplus g$
- $(f \leq_{\mathrm{T}} h \;\&\; g \leq_{\mathrm{T}} h) \Rightarrow f \oplus g \leq_{\mathrm{T}} h$.

The Turing degrees as an upper semi-lattice

Recall:

$$(f \oplus g)(2n) \;=\; f(n) \qquad\qquad (f \oplus g)(2n+1) \;=\; g(n).$$

Let

$$\deg_{\mathrm{T}}(f) \vee \deg_{\mathrm{T}}(g) \;=\; \deg_{\mathrm{T}}(f \oplus g).$$

Then:

- $\deg_{\mathrm{T}}(f) \vee \deg_{\mathrm{T}}(g)$ is **well-defined**
- $\deg_{\mathrm{T}}(f) \vee \deg_{\mathrm{T}}(g)$ is the **$\leq_{\mathrm{T}}$-least upper bound** of $\deg_{\mathrm{T}}(f)$ and $\deg_{\mathrm{T}}(g)$.

Thus $(\mathcal{D}_{\mathrm{T}}; \leq_{\mathrm{T}})$ is an **upper semi-lattice**.
I.e., a partial order where every pair of elements has a least upper bound.

Also, $\mathcal{D}_{\mathrm{T}}$ has least element $\mathbf{0} = \deg_{\mathrm{T}}(0) = \{f : f \text{ is recursive}\}$.

The Turing jump

The **Turing jump** of $f\colon \mathbb{N} \to \mathbb{N}$ is the **halting problem** relative to f.

Let $\Phi_0, \Phi_1, \Phi_2, \ldots$ be a computable list of all oracle programs.

Let

$$f' = \{e : \Phi_e^f(e) \text{ halts}\}.$$

Then:

- $f <_{\mathrm{T}} f'$
- $f \leq_{\mathrm{T}} g \ \Rightarrow\ f' \leq_{\mathrm{T}} g'$

Therefore the Turing jump is well-defined on $\mathcal{D}_{\mathrm{T}}$:

$$\deg_{\mathrm{T}}(f)' = \deg_{\mathrm{T}}(f').$$

The Turing degrees are not a lattice

Exact pair theorem:

Let $\boldsymbol{a}_0 \leq_{\mathrm{T}} \boldsymbol{a}_1 \leq_{\mathrm{T}} \boldsymbol{a}_2 \leq_{\mathrm{T}} \cdots$ be a countable increasing sequence from $\mathcal{D}_{\mathrm{T}}$.
Then there are $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{D}_{\mathrm{T}}$ such that

$$\forall \boldsymbol{d} \ \ (\exists n \ \boldsymbol{d} \leq_{\mathrm{T}} \boldsymbol{a}_n \ \ \Leftrightarrow \ \ \boldsymbol{d} \leq_{\mathrm{T}} \boldsymbol{x} \ \& \ \boldsymbol{d} \leq_{\mathrm{T}} \boldsymbol{y}).$$

The $\boldsymbol{x}$ and $\boldsymbol{y}$ are called an **exact pair** for $\boldsymbol{a}_0 \leq_{\mathrm{T}} \boldsymbol{a}_1 \leq_{\mathrm{T}} \boldsymbol{a}_2 \leq_{\mathrm{T}} \cdots$.

It follows that $\mathcal{D}_{\mathrm{T}}$ is **not** a lattice.

- Consider the sequence $\mathbf{0} <_{\mathrm{T}} \mathbf{0}' <_{\mathrm{T}} \mathbf{0}'' <_{\mathrm{T}} \cdots$.
- Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be an exact pair for this sequence.
- Then $\boldsymbol{x}$ and $\boldsymbol{y}$ **do not** have a $\leq_{\mathrm{T}}$-greatest lower bound:
    - If $\boldsymbol{z} \leq_{\mathrm{T}} \boldsymbol{x}, \boldsymbol{y}$, then $\boldsymbol{z} \leq_{\mathrm{T}} \mathbf{0}^{(n)}$ for some n.
    - But then $\boldsymbol{z} \leq_{\mathrm{T}} \mathbf{0}^{(n)} <_{\mathrm{T}} \mathbf{0}^{(n+1)} \leq_{\mathrm{T}} \boldsymbol{x}, \boldsymbol{y}$.
    - So $\boldsymbol{z}$ is not the greatest lower bound.

Embedding partial orders into the Turing degrees

$\mathcal{D}_{\mathrm{T}}$ has a rich structure.

$\mathcal{D}_{\mathrm{T}}$ has size $\mathfrak{c} = 2^{\aleph_0}$ and has antichains of size $\mathfrak{c}$.

$\mathcal{D}_{\mathrm{T}}$ has **countable predecessors**:
For every $\boldsymbol{d} \in \mathcal{D}_{\mathrm{T}}$, the initial interval $[\mathbf{0}, \boldsymbol{d}]$ is countable (or finite).

If partial order P embeds into $\mathcal{D}_{\mathrm{T}}$ ($P \hookrightarrow \mathcal{D}_{\mathrm{T}}$), then P has countable predecessors.

Theorem (Sacks)
For a P of size $|P| \leq \aleph_1$:

$$P \hookrightarrow \mathcal{D}_{\mathrm{T}} \quad \Leftrightarrow \quad P \text{ has countable predecessors.}$$

Thus under CH: $\quad P \hookrightarrow \mathcal{D}_{\mathrm{T}} \quad \Leftrightarrow \quad |P| \leq \mathfrak{c}$ and P has countable predecessors.

Theorem (Groszek & Slaman)
The following is consistent:
There is P of size $\mathfrak{c}$ that has countable predecessors but does **not** embed into $\mathcal{D}_{\mathrm{T}}$.

Ideals in the Turing degrees

An **ideal** in an upper semi-lattice $(U, \leq, \vee)$ is a set $I \subseteq U$ that is:

- Downward closed under $\leq$: $a \in I \ \& \ b \leq a \ \Rightarrow \ b \in I$
- Closed under joins: $a, b \in I \ \Rightarrow \ a \vee b \in I$.

Theorem (Lerman)

- *Every finite lattice embeds into $\mathcal{D}_{\mathrm{T}}$ as an initial segment.*
- *Thus the finite ideals of $\mathcal{D}_{\mathrm{T}}$ are exactly the finite lattices.*

Theorem (Lachlan & Lebeuf)

- *Every countable upper semi-lattice with a least element embeds into $\mathcal{D}_{\mathrm{T}}$ as an initial segment.*
- *Thus the countable ideals of $\mathcal{D}_{\mathrm{T}}$ are exactly the countable upper semi-lattices with least elements.*

The first-order theory of the Turing degrees

$\mathcal{D}_{\mathrm{T}}$ is as complicated as possible, in the following sense. Let:

- $\mathbf{Th}(\mathcal{D}_{\mathrm{T}})$ denote the **first-order theory** of $\mathcal{D}_{\mathrm{T}}$.
- $\mathbf{Th}_2(\mathbb{N})$ denote the **second-order theory** of $\mathbb{N}$.

$$\mathrm{Th}(\mathcal{D}_{\mathrm{T}}) = \{1^{\text{st}}\text{-order sentences } \varphi \text{ in the language of p.o.'s } : \mathcal{D}_{\mathrm{T}} \models \varphi\}$$
$$\mathrm{Th}_2(\mathbb{N}) = \{2^{\text{nd}}\text{-order sentences } \varphi \text{ in the language of arithmetic } : \mathbb{N} \models \varphi\}.$$

Theorem (Simpson)

$$\mathrm{Th}(\mathcal{D}_{\mathrm{T}}) \equiv_1 \mathrm{Th}_2(\mathbb{N}).$$

This means that there is a recursive bijection between $\mathrm{Th}(\mathcal{D}_{\mathrm{T}})$ and $\mathrm{Th}_2(\mathbb{N})$.

Determining whether a 1^{st}-order sentence is true of $\mathcal{D}_{\mathrm{T}}$ is exactly as hard as determining whether a 2^{nd}-order sentence is true of $\mathbb{N}$.

Sets of functions as mass problems

The **Turing degrees** are about **computing one function from another**.

The Medvedev and Muchnik degrees are about **computing one set of functions from another**.

In this context, a set $\mathcal{A} \subseteq \mathbb{N}^{\mathbb{N}}$ is called a **mass problem**.

Idea:

- An $\mathcal{A} \subseteq \mathbb{N}^{\mathbb{N}}$ represents the set of solutions to an abstract mathematical problem.
- **Solve $\mathcal{A}$** means **find a member of $\mathcal{A}$**.

Intuition:

If $\mathcal{B} \subseteq \mathcal{A}$, then problem $\mathcal{A}$ is easier than problem $\mathcal{B}$ because $\mathcal{A}$ has more solutions.

Some example mass problems

Note that we can compute on domains other than $\mathbb{N}$, like $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{N}^{<\mathbb{N}}$, etc.

Problem	**Mass problem**
Enumerate $A \subseteq \mathbb{N}$	$\{f \in \mathbb{N}^{\mathbb{N}} : \operatorname{ran} f = A\}$
Find a path through tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$	$\{f \in \mathbb{N}^{\mathbb{N}} : f$ is a path through $T\}$
Find an infinite homogeneous set for $f \colon \mathbb{N}^2 \to 2$	$\{\chi_H \in 2^{\mathbb{N}} : H$ is infinite homogeneous$\}$
Find a fixed point of continuous $F \colon [0,1]^2 \to [0,1]^2$	$\{(q_n) \in (\mathbb{Q}^2)^{\mathbb{N}} : (q_n)$ is a Cauchy sequence of pairs of rationals converging to a fixed point of $F\}$
Find a prime ideal in countable commutative ring R encoded over $\mathbb{N}$	$\{\chi_I \in 2^{\mathbb{N}} : I$ is a prime ideal in $R\}$
Find a representation of countable linear order $(L, \prec)$	$\{\chi_R \in 2^{(\mathbb{N}^2)} : (\mathbb{N}, R) \cong (L, \prec)\}$

Mass problems vs. Π^1_2 sentences

In **reverse mathematics** and the **Weihrauch degrees** we look at a Π^1_2 sentence

$$\forall X \; \exists Y \; \varphi(X, Y)$$

such as

"For every countable commutative ring R, there is a prime ideal $I \subseteq R$"

as a single object and study the complexity of producing a Y from a given X.

With reverse mathematics / the Weihrauch degrees

- $\{(R, I) : I$ is a prime ideal in countable commutative ring $R\}$ counts as a single problem.

With the mass problems

- For each countable commutative ring R, $\{I : I$ is a prime ideal in $R\}$ counts as its own problem.
- If R and S are two countable commutative rings, it might be harder to find a prime ideal in R than in S.

Reducibilities between mass problems

Recall: $\mathcal{A} \subseteq \mathbb{N}^{\mathbb{N}}$ represents (the solutions to) a mathematical problem.

Basic idea: $\mathcal{A}$ is easier than $\mathcal{B}$ if $\mathcal{A}$ has more solutions: $\mathcal{B} \subseteq \mathcal{A}$.

Refined idea: $\mathcal{A}$ is easier than $\mathcal{B}$ if every solution to $\mathcal{B}$ computes a solution to $\mathcal{A}$.

But how uniformly?

Medvedev (strong) reductions:

$$\mathcal{A} \leq_{\mathrm{s}} \mathcal{B} \quad \text{if} \quad \text{there is an oracle program } \Phi \text{ such that } \Phi(\mathcal{B}) \subseteq \mathcal{A}.$$

Here '$\Phi(\mathcal{B}) \subseteq \mathcal{A}$' means $\Phi(f)$ is total and in $\mathcal{A}$ for all $f \in \mathcal{B}$.
(We now write $\Phi(f)$ in place of Φ^f.)

Muchnik (weak) reductions:

$$\mathcal{A} \leq_{\mathrm{w}} \mathcal{B} \quad \text{if} \quad \forall f \in \mathcal{B} \ \exists g \in \mathcal{A} \ \ g \leq_{\mathrm{T}} f.$$

The Medvedev and Muchnik degrees

$\mathcal{A} \leq_{\mathrm{s}} \mathcal{B}$	if	there is a program Φ such that $\Phi(\mathcal{B}) \subseteq \mathcal{A}$
$\mathcal{A} \leq_{\mathrm{w}} \mathcal{B}$	if	$\forall f \in \mathcal{B} \;\; \exists g \in \mathcal{A} \;\; g \leq_{\mathrm{T}} f$

The relations $\mathcal{A} \leq_{\mathrm{s}} \mathcal{B}$ and $\mathcal{A} \leq_{\mathrm{w}} \mathcal{B}$ are quasi-orders. For $\leq_{\mathrm{s}}$:

- $\mathcal{A} \leq_{\mathrm{s}} \mathcal{A}$ via the identity $\Phi(f) = f$.
- Say $\mathcal{A} \leq_{\mathrm{s}} \mathcal{B} \leq_{\mathrm{s}} \mathcal{C}$. Let $\Psi(\mathcal{C}) \subseteq \mathcal{B}$ and $\Phi(\mathcal{B}) \subseteq \mathcal{A}$. Let $\Theta = \Phi \circ \Psi$. Then $\Theta(C) = \Phi(\Psi(\mathcal{C})) \subseteq \mathcal{A}$, so $\mathcal{A} \leq_{\mathrm{s}} \mathcal{C}$.

The Medvedev and Muchnik degrees

- Mass problems $\mathcal{A}$ and $\mathcal{B}$ are **Medvedev/Muchnik equivalent** ($\mathcal{A} \equiv_{\bullet} \mathcal{B}$) if $\mathcal{A} \leq_{\bullet} \mathcal{B} \;\&\; \mathcal{B} \leq_{\bullet} \mathcal{A}$.
- The **Medvedev/Muchnik degree** of $\mathcal{A}$ is $\deg_{\bullet}(\mathcal{A}) = \{\mathcal{B} \subseteq \mathbb{N}^{\mathbb{N}} : \mathcal{B} \equiv_{\bullet} \mathcal{A}\}$.
- The **Medvedev/Muchnik degrees** are $\mathcal{M}_{\bullet} = \{\deg_{\bullet}(\mathcal{A}) : \mathcal{A} \subseteq \mathbb{N}^{\mathbb{N}}\}$.

A calculus of problems

Kolmogorov wanted an interpretation of propositional logic as a **logic of problem-solving** or a **calculus of problems**.

Medvedev introduced his degrees to provide semantics for propositional logic.

Muchnik introduced his degrees as a non-uniform alternative.

Here **truth** corresponds to **solvability by a Turing machine** and **falsehood** corresponds to **impossibility**.

The hope was that $\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{w}$ would give semantics for intuitionistic logic.

It turns out that $\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{w}$ give semantics for the logic of **weak excluded middle**:

$$\neg p \text{ or } \neg\neg p.$$

$\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{w}$ as bounded distributive lattices

$\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{w}$ are **bounded distributive lattices**.

Moreover, the **lattice operations** correspond to **logical operations**.

$\mathbf{0} = \deg_\bullet(\mathbb{N}^\mathbb{N})$		**true**
$\mathbf{1} = \deg_\bullet(\emptyset)$		**false**
$\mathcal{A} \vee \mathcal{B} = \{f \oplus g : f \in \mathcal{A} \ \&\ g \in \mathcal{B}\}$		**and**
$\mathcal{A} \wedge \mathcal{B} = 0^\frown\mathcal{A} \cup 1^\frown\mathcal{B}$		**or**

For the meet operation:

- $n^\frown f$ means think of f as an infinite string and prepend n to f.
- Then $n^\frown\mathcal{A} = \{n^\frown f : f \in \mathcal{A}\}$.
- In the Muchnik degrees: $0^\frown\mathcal{A} \cup 1^\frown\mathcal{B} \equiv_\mathrm{w} \mathcal{A} \cup \mathcal{B}$.

Meets in the Medvedev degrees

The operation $\mathcal{A} \wedge \mathcal{B} \;=\; 0^\frown\mathcal{A} \,\cup\, 1^\frown\mathcal{B}$ gives greatest lower bounds in $\mathcal{M}_\mathrm{s}$.

Lower bound

- $0^\frown\mathcal{A} \,\cup\, 1^\frown\mathcal{B} \;\leq_\mathrm{s}\; \mathcal{A}$ via $\Phi(f) = 0^\frown f$.
- Similarly, $0^\frown\mathcal{A} \,\cup\, 1^\frown\mathcal{B} \;\leq_\mathrm{s}\; \mathcal{B}$.

Greatest lower bound

- Suppose $\mathcal{C} \leq_\mathrm{s} \mathcal{A}$ and $\mathcal{C} \leq_\mathrm{s} \mathcal{B}$.
- There are Φ, Ψ such that $\Phi(\mathcal{A}) \subseteq C$ and $\Psi(\mathcal{B}) \subseteq \mathcal{C}$.
- Let f^- denote the result of shifting f to the left: $f^-(n) = f(n+1)$. Let

$$\Theta(f) = \begin{cases} \Phi(f^-) & \text{if } f(0) = 0 \\ \Psi(f^-) & \text{if } f(0) = 1. \end{cases}$$

- Then $\Theta(0^\frown\mathcal{A} \,\cup\, 1^\frown\mathcal{B}) \subseteq \mathcal{C}$. So $\mathcal{C} \;\leq_\mathrm{s}\; 0^\frown\mathcal{A} \,\cup\, 1^\frown\mathcal{B}$.

A difference between $\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{w}$

Given $\mathcal{A} \subseteq \mathbb{N}^\mathbb{N}$, let **$C(\mathcal{A})$** denote the **Turing upward closure** of $\mathcal{A}$:

$$C(\mathcal{A}) = \{g : \exists f \in \mathcal{A} \;\; f \leq_\mathrm{T} g\}.$$

Then $\mathcal{A} \equiv_\mathrm{w} C(\mathcal{A})$ for every $\mathcal{A} \subseteq \mathbb{N}^\mathbb{N}$.

$\mathcal{M}_\mathrm{w}$ is a **complete lattice**. The join and meet of $(\mathcal{A}_\alpha : \alpha < \kappa)$ are computed by:

$$\bigvee_{\alpha<\kappa} \mathcal{A}_\alpha \;=\; \bigcap_{\alpha<\kappa} C(\mathcal{A}_\alpha) \qquad\qquad \bigwedge_{\alpha<\kappa} \mathcal{A}_\alpha \;=\; \bigcup_{\alpha<\kappa} C(\mathcal{A}_\alpha).$$

In a sense, the **Muchnik degrees** are a **completion** of the **Turing degrees**.

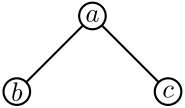
$\mathcal{M}_\mathrm{s}$ is **not** a complete lattice (**Dyment**).

- There are countable collections with no least upper bound.
- There are countable collections with no greatest lower bound.

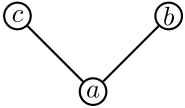
Join- and meet- reducibility

Let L be a lattice.

- $a \in L$ is **join-reducible** if $\exists\, b, c < a \;\; (a = b \vee c)$.
- $a \in L$ is **meet-reducible** if $\exists\, b, c > a \;\; (a = b \wedge c)$.



This a is join-reducible.



This a is meet-reducible.

In both $\mathcal{M}_{\mathrm{s}}$ and $\mathcal{M}_{\mathrm{w}}$:

- $\mathbf{0} = \deg_{\bullet}(\mathbb{N}^{\mathbb{N}})$ is meet-**ir**reducible. If $\mathcal{A} \wedge \mathcal{B} \;=\; 0^\frown\mathcal{A} \cup 1^\frown\mathcal{B}$ has a recursive element, then either $\mathcal{A}$ or $\mathcal{B}$ has a recursive element.
- $\mathbf{1} = \deg_{\bullet}(\emptyset)$ is join-**ir**reducible. If $\mathcal{A}$ and $\mathcal{B}$ are non-empty, then $\mathcal{A} \vee \mathcal{B} \;=\; \{f \oplus g : f \in \mathcal{A} \;\&\; g \in \mathcal{B}\}$ is non-empty.

An elementary difference between $\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{w}$

$\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{w}$ also have a **second-least element** called $\mathbf{0}'$:

$$\mathbf{0}' \;=\; \deg_\bullet(\mathrm{NON}) \qquad\qquad \text{where } \mathrm{NON} = \{f : f \text{ is } \textbf{not} \text{ recursive}\}.$$

$\mathbf{0}'$ is second-least: if $\mathcal{A} >_\bullet \mathbb{N}^\mathbb{N}$, then $\mathcal{A} \subseteq \mathrm{NON}$, so $\mathcal{A} \geq_\bullet \mathrm{NON}$.

In $\mathcal{M}_\mathrm{s}$, the element $\mathbf{0}'$ is **meet-irreducible**.

In $\mathcal{M}_\mathrm{w}$, the element $\mathbf{0}'$ is **meet-reducible**.

Thus $\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{w}$ are **not** elementarily equivalent because

the second-least element is meet-reducible

is expressible by a first-order sentence in the language of partial orders.

$\mathbf{0}'$ is meet-irreducible in $\mathcal{M}_{\mathrm{s}}$

$\mathbf{0}' \ = \ \deg_{\mathrm{s}}(\mathrm{NON})$ where $\mathrm{NON} = \{f : f$ is **not** recursive$\}$.

Suppose that $\mathrm{NON} \geq_{\mathrm{s}} \mathcal{A} \wedge \mathcal{B}$. Show that $\mathrm{NON} \geq_{\mathrm{s}} \mathcal{A}$ or $\mathrm{NON} \geq_{\mathrm{s}} \mathcal{B}$.

Let Φ be such that $\Phi(\mathrm{NON}) \ \subseteq \ 0^\frown\mathcal{A} \cup 1^\frown\mathcal{B}$.

Let $f \in \mathrm{NON}$. **Suppose that $\Phi(f) \in 0^\frown\mathcal{A}$**.

- Then $\Phi(f)(0) = 0$.
- Let $\sigma \sqsubseteq f$ be an initial segment of f such that $\Phi(\sigma)(0) = 0$.

Let Ψ be the functional $\Psi(g) = \Phi(\sigma^\frown g)^-$. Let $g \in \mathrm{NON}$.

- Then $\sigma^\frown g \in \mathrm{NON}$, so $\Phi(\sigma^\frown g) \ \in \ 0^\frown\mathcal{A} \cup 1^\frown\mathcal{B}$.
- Also, $\Phi(\sigma^\frown g)(0) = 0$, so $\Phi(\sigma^\frown g) \in 0^\frown\mathcal{A}$.
- Thus $\Psi(g) = \Phi(\sigma^\frown g)^- \in \mathcal{A}$.

Thus $\Psi(\mathrm{NON}) \subseteq \mathcal{A}$, so $\mathrm{NON} \geq_{\mathrm{s}} \mathcal{A}$.

$\mathbf{0}'$ is meet-reducible in $\mathcal{M}_{\mathrm{w}}$

$\mathbf{0}' \; = \; \deg_{\mathrm{w}}(\mathrm{NON})$ where $\mathrm{NON} = \{f : f$ is **not** recursive$\}$.

Let $f \in \mathrm{NON}$ have **minimal Turing degree**:
If $h \leq_{\mathrm{T}} f$, then either $h \equiv_{\mathrm{T}} f$ or h is recursive.

Let:

$$\mathcal{A} = \{f\} \qquad\qquad \mathcal{B} = \{g : g \nleq_{\mathrm{T}} f\}$$

Then:

- $\mathcal{A} >_{\mathrm{w}} \mathrm{NON}$ because $\exists g \in \mathrm{NON} \;\; f \nleq_{\mathrm{T}} g$.
- $\mathcal{B} >_{\mathrm{w}} \mathrm{NON}$ because $f \in \mathrm{NON}$ and $\forall g \in \mathcal{B} \;\; g \nleq_{\mathrm{T}} f$.

However, $\mathrm{NON} \; \geq_{\mathrm{w}} \; \mathcal{A} \cup \mathcal{B} \; \equiv_{\mathrm{w}} \; \mathcal{A} \wedge \mathcal{B}$. So $\mathrm{NON} \; \equiv_{\mathrm{w}} \; \mathcal{A} \wedge \mathcal{B}$.

Let $g \in \mathrm{NON}$.

- If $g \leq_{\mathrm{T}} f$, then $g \equiv_{\mathrm{T}} f$ because f has minimal Turing degree, and $f \in \mathcal{A}$.
- If $g \nleq_{\mathrm{T}} f$, then $g \in \mathcal{B}$.

More on reducible / irreducible Medvedev degrees

Theorem (Dyment)

Degree $\boldsymbol{a} \in \mathcal{M}_{\mathrm{s}}$ *is meet-reducible* $\Leftrightarrow$ $\boldsymbol{a} = \deg_{\mathrm{s}}(\mathcal{A})$ *for an* $\mathcal{A}$ *for which there are r.e. sets* $U, V \subseteq \mathbb{N}^{<\mathbb{N}}$ *such that:*

- i $\forall f \in \mathcal{A} \;\; \exists \sigma \in U \cup V \;\; \sigma \sqsubseteq f$
- ii $\{f \in \mathcal{A} : \exists \sigma \in U \;\; \sigma \sqsubseteq f\} \;|_{\mathrm{s}}\; \{f \in \mathcal{A} : \exists \sigma \in V \;\; \sigma \sqsubseteq f\}$.

Here, $|_{\mathrm{s}}$ is Medvedev incomparability: $\mathcal{X} \;|_{\mathrm{s}}\; \mathcal{Y} \;\Leftrightarrow\; \mathcal{X} \nleq_{\mathrm{s}} \mathcal{Y} \;\&\; \mathcal{Y} \nleq_{\mathrm{s}} \mathcal{X}$.

Theorem (S)

Degree $\boldsymbol{a} \in \mathcal{M}_{\mathrm{s}}$ *is join-irreducible* $\Leftrightarrow$ $\boldsymbol{a} = \deg_{\mathrm{s}}(\mathbb{N}^{\mathbb{N}} \setminus \mathcal{I})$ *for a Turing ideal* $\mathcal{I}$.

Here, $\mathcal{I} \subseteq \mathbb{N}^{\mathbb{N}}$ is a **Turing ideal** if it is:

- Downward closed under $\leq_{\mathrm{T}}$: $f \in \mathcal{I} \;\&\; g \leq_{\mathrm{T}} f \;\Rightarrow\; g \in \mathcal{I}$
- Closed under Turing joins: $f, g \in \mathcal{I} \;\Rightarrow\; f \oplus g \in \mathcal{I}$.

$\mathcal{M}_{\mathrm{s}}$ and $\mathcal{M}_{\mathrm{w}}$ as Brouwer algebras

We have interpretations of **true**, **false**, **and**, and **or**:

$$\begin{array}{rcll} \mathbf{0} & = & \deg_\bullet(\mathbb{N}^\mathbb{N}) & \qquad\qquad \textbf{true} \\ \mathbf{1} & = & \deg_\bullet(\emptyset) & \qquad\qquad \textbf{false} \\ \mathcal{A} \vee \mathcal{B} & = & \{f \oplus g : f \in \mathcal{A} \ \& \ g \in \mathcal{B}\} & \qquad\qquad \textbf{and} \\ \mathcal{A} \wedge \mathcal{B} & = & 0^\frown\mathcal{A} \ \cup \ 1^\frown\mathcal{B} & \qquad\qquad \textbf{or} \end{array}$$

To interpret propositional logic, we also need an interpretation of **implies**.

A **Brouwer algebra** is a bounded distributive lattice such that:

$$\forall a, b \ \exists \text{ least } c \ \ (a \vee c \ \geq \ b).$$

The witnessing c is written $\boldsymbol{a \to b}$.

Brouwer algebras are the duals of the **Heyting algebras**. They provide semantics for propositional logics between **intuitionistic logic** and **classical logic**.

$\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{w}$ as Brouwer algebras

A **Brouwer algebra** is a bounded distributive lattice such that:

$$\forall a, b \;\; \exists \text{ least } \underbrace{c}_{a \to b} \; (a \vee c \;\geq\; b).$$

The following operations make $\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{w}$ into Brouwer algebras.

$$\text{In } \mathcal{M}_\mathrm{s}: \qquad \mathcal{A} \to_\mathrm{s} \mathcal{B} \;=\; \{e^\frown g : \forall f \in \mathcal{A} \;\; \Phi_e(f \oplus g) \in \mathcal{B}\}$$

$$\text{In } \mathcal{M}_\mathrm{w}: \qquad \mathcal{A} \to_\mathrm{w} \mathcal{B} \;=\; \{g : \forall f \in \mathcal{A} \;\; \exists h \in \mathcal{B} \;\; h \leq_\mathrm{T} f \oplus g\}$$

Intuition:

- $\mathcal{A} \to \mathcal{B}$ is the least information one must add to $\mathcal{A}$ in order to know $\mathcal{B}$.
- $\mathcal{A} \to \mathcal{B}$ represents the problem of converting solutions to $\mathcal{A}$ into solutions to $\mathcal{B}$.

Implication in the Medvedev degrees

In $\mathcal{M}_{\mathrm{s}}$, implication is $\mathcal{A} \to \mathcal{B} \;=\; \{e^\frown g : \forall f \in \mathcal{A} \;\; \Phi_e(f \oplus g) \in \mathcal{B}\}$.

$\boldsymbol{\mathcal{A} \vee (\mathcal{A} \to \mathcal{B}) \;\geq_{\mathrm{s}}\; \mathcal{B}}$

- Let Ψ be

$$\Psi(f \oplus g) = \Phi_{g(0)}(f \oplus g^-).$$

- If $f \in \mathcal{A}$ and $e^\frown g \in \mathcal{A} \to \mathcal{B}$, then

$$\Psi(f \oplus e^\frown g) = \Phi_e(f \oplus g) \in \mathcal{B}.$$

- Thus $\Psi(\mathcal{A} \vee (\mathcal{A} \to \mathcal{B})) \subseteq \mathcal{B}$.
- So $\mathcal{A} \vee (\mathcal{A} \to \mathcal{B}) \;\geq_{\mathrm{s}}\; \mathcal{B}$.

Implication in the Medvedev degrees

In $\mathcal{M}_{\mathrm{s}}$, implication is $\mathcal{A} \to \mathcal{B} \ = \ \{e^\frown g : \forall f \in \mathcal{A} \ \ \Phi_e(f \oplus g) \in \mathcal{B}\}$.

$\mathcal{A} \vee (\mathcal{A} \to \mathcal{B})$ is least

- Suppose $\mathcal{A} \vee \mathcal{X} \geq_{\mathrm{s}} \mathcal{B}$.
- Some Φ_e witness the reduction: $\Phi_e(\mathcal{A} \vee \mathcal{X}) \subseteq \mathcal{B}$.
- This means that:

$$\forall f \in \mathcal{A} \ \ \forall g \in \mathcal{X} \ \ \Phi_e(f \oplus g) \in \mathcal{B}.$$

- Let Ψ be $\Psi(g) = e^\frown g$.
- If $g \in \mathcal{X}$, then $\Psi(g) = e^\frown g \ \in \ \mathcal{A} \to \mathcal{B}$. So $\Psi(\mathcal{X}) \ \subseteq \ \mathcal{A} \to \mathcal{B}$. So $\mathcal{A} \to \mathcal{B} \ \leq_{\mathrm{s}} \ \mathcal{X}$.

Could also phrase the argument as:
$\mathcal{X} \equiv_{\mathrm{s}} e^\frown \mathcal{X}$ and $e^\frown X \ \subseteq \ \mathcal{A} \to \mathcal{B}$, so $\mathcal{A} \to \mathcal{B} \ \leq_{\mathrm{s}} \ \mathcal{X}$.

Interpreting propositional formulas in Brouwer algebras

Let $\mathfrak{B}$ be a Brouwer algebra. A **valuation** is a function

$$\nu\colon \text{propositional variables} \to \mathfrak{B}.$$

Valuations extend to all propositional formulas by:

$$\begin{aligned} \nu(\varphi \ \& \ \psi) &= \nu(\varphi) \vee \nu(\psi) \\ \nu(\varphi \text{ or } \psi) &= \nu(\varphi) \wedge \nu(\psi) \\ \nu(\varphi \to \psi) &= \nu(\varphi) \to \nu(\psi) \\ \nu(\neg\varphi) &= \nu(\varphi) \to 1. \end{aligned}$$

Propositional formula φ is **valid** in $\mathfrak{B}$ if $\nu(\varphi) = 0$ for every valuation ν.

$\textbf{Prop-Th}(\mathfrak{B})$ denotes the **propositional theory** given by $\mathfrak{B}$.

$$\text{Prop-Th}(\mathfrak{B}) = \{\varphi : \varphi \text{ is valid in } \mathfrak{B}\}$$

$\text{Prop-Th}(\mathfrak{B})$ is always some logic between intuitionistic and classical logic.

Join-irreducibility and weak excluded middle

Weak excluded middle ($\mathbf{WEM}$) is the law $\neg p \text{ or } \neg\neg p$.

Fact

If $\mathfrak{B}$ is a Brouwer algebra where 1 is join-irreducible, then $\mathfrak{B}$ validates WEM.

Let $b \in \mathfrak{B}$.

- If $b = 1$, then $(b \to 1) \;=\; (1 \to 1) \;=\; 0$.
- If $b < 1$, then $b \to 1 \;=\; 1$ because 1 is join-irreducible.
  Thus $(b \to 1) \to 1 \;=\; (1 \to 1) \;=\; 0$.
- Therefore $(b \to 1) \wedge ((b \to 1) \to 1) = 0$.

Thus if φ is any formula and ν is any valuation for $\mathfrak{B}$:

$$\nu(\neg\varphi \text{ or } \neg\neg\varphi) \;=\; (\nu(\varphi) \to 1) \wedge ((\nu(\varphi) \to 1) \to 1) \;=\; 0.$$

So $\neg\varphi \text{ or } \neg\neg\varphi$ is valid in $\mathfrak{B}$.

Weak excluded middle in the logic of problem-solving

In $\mathcal{M}_{\mathrm{s}}$ and $\mathcal{M}_{\mathrm{w}}$:

- $\mathbf{0} = \deg_\bullet(\mathbb{N}^\mathbb{N})$ is the problem solvable by a computer.
- $\mathbf{1} = \deg_\bullet(\emptyset)$ is the impossible problem.
- All other problems are possible, but not solvable by computers.
- p means that p is solvable by a computer.
- $\neg p$ means that p is impossible.
- $p \to q$ means that solutions to p can compute solutions to q.
- $p \text{ or } \neg p$ means that p is either solvable by a computer or impossible.
- $\neg p \text{ or } \neg\neg p$ means that p is either possible or impossible.

$\mathbf{1}$ is join-irreducible in $\mathcal{M}_{\mathrm{s}}$ and $\mathcal{M}_{\mathrm{w}}$, so they both validate WEM.

$\mathcal{M}_\mathrm{s}$, $\mathcal{M}_\mathrm{w}$, and weak excluded middle

Here

- **IPC** denotes intuitionistic logic
- **WEM** denotes IPC plus the scheme $\neg p \text{ or } \neg\neg p$.

Theorem

- $\text{Prop-Th}(\mathcal{M}_\mathrm{s}) = \text{WEM}$. **(Medvedev / Sorbi)**
- $\text{Prop-Th}(\mathcal{M}_\mathrm{w}) = \text{WEM}$. **(Sorbi)**

We know that $\mathbf{1}$ is join-irreducible in $\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{w}$.

Thus $\text{WEM} \subseteq \text{Prop-Th}(\mathcal{M}_\mathrm{s})$ and $\text{WEM} \subseteq \text{Prop-Th}(\mathcal{M}_\mathrm{w})$.

How do we show the reverse inclusions?

Semantics for weak excluded middle

Semantics for IPC:

$$\mathrm{IPC} \quad = \quad \bigcap \Big\{ \mathrm{Prop\text{-}Th}(\mathfrak{B}) \;:\; \mathfrak{B} \text{ is a finite Brouwer algebra} \Big\}$$

Semantics for WEM (Jankov):

$$\mathrm{WEM} \quad = \quad \bigcap \Big\{ \mathrm{Prop\text{-}Th}(\mathfrak{B}) \;:\; \mathfrak{B} \text{ is a finite Brouwer algebra with } 0 \text{ meet-irreducible and } 1 \text{ join-irreducible} \Big\}$$

Fact:
For Brouwer algebras $\mathfrak{A}$ and $\mathfrak{B}$:

$$\mathfrak{A} \hookrightarrow \mathfrak{B} \quad \Rightarrow \quad \mathrm{Prop\text{-}Th}(\mathfrak{B}) \subseteq \mathrm{Prop\text{-}Th}(\mathfrak{A})$$

So we want to embed certain finite Brouwer algebras into $\mathcal{M}_{\mathrm{s}}$.

Embedding finite algebras with irreducible 0 and 1 into $\mathcal{M}_s$

Theorem (Sorbi)

A finite Brouwer algebra embeds into $\mathcal{M}_s \quad \Leftrightarrow \quad 0$ *is meet-irreducible and* 1 *is join-irreducible.*

It follows that $\text{Prop-Th}(\mathcal{M}_s) \subseteq \text{WEM}$. *Thus* $\text{Prop-Th}(\mathcal{M}_s) = \text{WEM}$.

To prove this:

- Every finite Brouwer algebra with meet-irred. 0 and join-irred. 1 embeds into a Brouwer algebra of the form $0 \oplus \mathbb{F}(P) \oplus 1$ for a **finite partial order P**.

  Here $0 \oplus \mathbb{F}(P) \oplus 1$ is the **free distributive lattice** generated by P with new bottom and top elements.

- Every finite partial order embeds into $\mathcal{D}_T$.
- Thus for every finite partial order P,

$$0 \oplus \mathbb{F}(P) \oplus 1 \quad \hookrightarrow \quad 0 \oplus \mathbb{F}(\mathcal{D}_T) \oplus 1$$

- So we want that $0 \oplus \mathbb{F}(\mathcal{D}_T) \oplus 1 \hookrightarrow \mathcal{M}_s$.

The free distributive lattice generated by a partial order

Let $(P, \leq)$ be a partial order. The elements of $\mathbb{F}(P)$ are expressions

$$\bigvee_{j \in J} \bigwedge_{i \in I_j} p_i^j$$

where J and the I_j are finite sets of indices and each p_i^j is in P.

Define

$$\bigvee_{v \in V} \bigwedge_{u \in U_v} q_u^v \quad \leq \quad \bigvee_{j \in J} \bigwedge_{i \in I_j} p_i^j$$

if and only if

$$\forall v \in V \;\; \exists j \in J \;\; \forall i \in I_j \;\; \exists u \in U_v \;\; (q_u^v \leq p_i^j)$$

(Then take the quotient of the equivalence relation induced by $\leq$.)

$0 \oplus \mathbb{F}(P) \oplus 1$ is a Brouwer algebra

Let

$$a \;=\; \bigvee_{v \in V} \; \bigwedge_{u \in U_v} \; q_u^v \qquad\qquad b \;=\; \bigvee_{j \in J} \; \bigwedge_{i \in I_j} \; p_i^j.$$

If $a \not\geq b$, then $a \to b$ is the join of meets of b missing from a:

$$a \to b \quad = \quad \bigvee \left\{ \bigwedge_{i \in I_j} p_i^j \;:\; \forall v \in V \;\; \left(\bigwedge_{i \in I_j} p_i^j \;\not\leq\; \bigwedge_{u \in U_v} q_u^v \right) \right\}$$

If $a \geq b$, then $a \to b$ should be 0.

Thus $0 \oplus \mathbb{F}(P) \oplus 1$ is a Brouwer algebra.

Embedding $0 \oplus \mathbb{F}(\mathcal{D}_\mathrm{T}) \oplus 1$ into $\mathcal{M}_\mathrm{s}$

For $f\colon \mathbb{N} \to \mathbb{N}$, let $\mathcal{B}_f$ be NON relativized to f:

$$\mathcal{B}_f \;=\; \{h : h \not\leq_\mathrm{T} f\} \qquad\qquad \boldsymbol{b}_f \;=\; \deg_\mathrm{s}(\mathcal{B}_f).$$

Then

$$f \leq_\mathrm{T} g \quad \Leftrightarrow \quad \mathcal{B}_g \subseteq \mathcal{B}_f \quad \Leftrightarrow \quad \mathcal{B}_f \leq_\mathrm{s} \mathcal{B}_g$$

Thus the map

$$\deg_\mathrm{T}(f) \mapsto \boldsymbol{b}_f$$

embeds $\mathcal{D}_\mathrm{T}$ into $\mathcal{M}_\mathrm{s}$ **as a partial order**.

Embedding $0 \oplus \mathbb{F}(\mathcal{D}_{\mathrm{T}}) \oplus 1$ into $\mathcal{M}_{\mathrm{s}}$

Recall: $\mathcal{B}_f = \{h : h \nleq_{\mathrm{T}} f\} \quad \boldsymbol{b}_f = \deg_{\mathrm{s}}(\mathcal{B}_f)$.

The degrees $\boldsymbol{b}_f$ are join- and meet-irreducible.

Moreover:

$$\bigvee_{v \in V} \bigwedge_{u \in U_v} \boldsymbol{b}_{g_u^v} \quad \leq_{\mathrm{s}} \quad \bigvee_{j \in J} \bigwedge_{i \in I_j} \boldsymbol{b}_{f_i^j}$$

if and only if

$$\forall v \in V \;\; \exists j \in J \;\; \forall i \in I_j \;\; \exists u \in U_v \;\; (\boldsymbol{b}_{g_u^v} \leq_{\mathrm{s}} \boldsymbol{b}_{f_i^j}).$$

Also:

$$\bigvee_{v \in V} \bigwedge_{u \in U_v} \boldsymbol{b}_{g_u^v} \quad \to \quad \bigvee_{j \in J} \bigwedge_{i \in I_j} \boldsymbol{b}_{f_i^j}$$

$$= \quad \bigvee \left\{ \bigwedge_{i \in I_j} \boldsymbol{b}_{f_i^j} \;:\; \forall v \in V \;\left(\bigwedge_{i \in I_j} \boldsymbol{b}_{f_i^j} \nleq_{\mathrm{s}} \bigwedge_{u \in U_v} \boldsymbol{b}_{g_u^v} \right) \right\}$$

Embedding $0 \oplus \mathbb{F}(\mathcal{D}_{\mathrm{T}}) \oplus 1$ into $\mathcal{M}_{\mathrm{s}}$

Thus $0 \oplus \mathbb{F}(\mathcal{D}_{\mathrm{T}}) \oplus 1$ embeds into $\mathcal{M}_{\mathrm{s}}$ as a Brouwer algebra:

$$\begin{aligned} 0 &\mapsto \mathbf{0} \\ \bigvee_{j \in J} \bigwedge_{i \in I_j} \deg_{\mathrm{T}}(f_i^j) &\mapsto \bigvee_{j \in J} \bigwedge_{i \in I_j} \boldsymbol{b}_{f_i^j} \\ 1 &\mapsto \mathbf{1} \end{aligned}$$

This shows that $\text{Prop-Th}(\mathcal{M}_{\mathrm{s}}) \subseteq \mathrm{WEM}$.

Thus $\text{Prop-Th}(\mathcal{M}_{\mathrm{s}}) = \mathrm{WEM}$.

$\text{Prop-Th}(\mathcal{M}_{\mathrm{w}}) = \mathrm{WEM}$ is also true. **(Sorbi)**

Initial intervals as semantics for propositional logic

Let $\mathfrak{B}$ be a Brouwer algebra, and let $a < b$ be elements of $\mathfrak{B}$.

Then the interval $[a, b] = \{x \in \mathfrak{B} : a \leq x \leq b\}$ is also a Brouwer algebra.

Thus for every $b > 0$, the initial interval $[0, b]$ is a Brouwer algebra.

It is possible to realize IPC as the **logic of an initial interval** of $\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{w}$.

Theorem

- $\exists \boldsymbol{b} \in \mathcal{M}_\mathrm{s}$ *such that* $\mathrm{Prop\text{-}Th}(\mathcal{M}_\mathrm{s}[\mathbf{0}, \boldsymbol{b}]) = \mathrm{IPC}$. **(Skvortsova = Dyment)**
- $\exists \boldsymbol{b} \in \mathcal{M}_\mathrm{w}$ *such that* $\mathrm{Prop\text{-}Th}(\mathcal{M}_\mathrm{w}[\mathbf{0}, \boldsymbol{b}]) = \mathrm{IPC}$. **(Sorbi & Terwijn)**

Alternate proofs of these theorems are given by **Kuyper**.

Initial intervals as semantics for propositional logic

In $\mathcal{M}_\mathrm{s}$, every non-trivial initial interval yields a logic **between $\mathbf{IPC}$ and $\mathbf{WEM}$**.

Theorem (Kuyper)

Let $\boldsymbol{b} \in \mathcal{M}_\mathrm{s}$ be a degree with $\boldsymbol{b} >_\mathrm{s} \mathbf{0}'$. Then

$$\mathrm{IPC} \quad \subseteq \quad \text{Prop-Th}(\mathcal{M}_\mathrm{s}[\mathbf{0}, \boldsymbol{b}]) \quad \subseteq \quad \mathrm{WEM}.$$

(The above theorem is **false** for $\mathcal{M}_\mathrm{w}$.)

Infinitely many different logics are obtained from initial segments of $\mathcal{M}_\mathrm{s}$.

Theorem (Sorbi & Terwijn)

There is an ascending sequence $\boldsymbol{b}_0 <_\mathrm{s} \boldsymbol{b}_1 <_\mathrm{s} \boldsymbol{b}_2 <_\mathrm{s} \cdots$ in $\mathcal{M}_\mathrm{s}$ such that

$$\text{Prop-Th}(\mathcal{M}_\mathrm{s}[\mathbf{0}, \boldsymbol{b}_0]) \supsetneq \text{Prop-Th}(\mathcal{M}_\mathrm{s}[\mathbf{0}, \boldsymbol{b}_1]) \supsetneq \text{Prop-Th}(\mathcal{M}_\mathrm{s}[\mathbf{0}, \boldsymbol{b}_2]) \supsetneq \cdots$$

Embedding large objects into $\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{w}$

$\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{w}$ have antichains of size $2^\mathfrak{c}$. **(Platek)**

That $\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{w}$ have chains of size $2^\mathfrak{c}$ is consistent with ZFC. **(Terwijn)**

That $\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{w}$ **do not** have chains of size $2^\mathfrak{c}$ is also consistent with ZFC. **(S)**

In fact, $\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{w}$ have chains of size κ if and only if $(\mathcal{P}(\mathfrak{c}), \subseteq)$ does. **(S)**

$(\mathcal{P}(\mathfrak{c}), \supseteq)$ embeds into $\mathcal{M}_\mathrm{s}$ as an **upper semi-lattice**. **(Terwijn)**

But only **countable** Boolean algebras embed into $\mathcal{M}_\mathrm{s}$ as **lattices**. **(Terwijn)**

$(\mathcal{P}(\mathfrak{c}), \supseteq)$ embeds into $\mathcal{M}_\mathrm{w}$ as a **lattice**. **(Terwijn)**

Embedding $\mathcal{D}_\mathrm{T}$ into $\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{w}$

$\mathcal{D}_\mathrm{T}$ embeds into $\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{w}$ as an upper semi-lattice with 0:

$$\deg_\mathrm{T}(f) \mapsto \deg_\mathrm{s}(\{f\}) \qquad\qquad \deg_\mathrm{T}(f) \mapsto \deg_\mathrm{w}(\{f\})$$

Theorem (Medvedev / Muchnik / Dyment)

For both $\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{w}$, the range of the embedding of $\mathcal{D}_\mathrm{T} \hookrightarrow \mathcal{M}_\bullet$ is defined by the following formula $\varphi(\boldsymbol{x})$ saying that $\boldsymbol{x}$ has an immediate successor:

$$\exists \boldsymbol{a}\ \ (\boldsymbol{x} <_\bullet \boldsymbol{a}\ \ \&\ \ \forall \boldsymbol{b}\, (\boldsymbol{x} <_\bullet \boldsymbol{b} \ \rightarrow\ \boldsymbol{a} \leq_\bullet \boldsymbol{b})).$$

For $\deg_\bullet(\{f\})$, the witnessing $\boldsymbol{a}$'s are:

$$\deg_\mathrm{s}\big(\{e^\frown g : g >_\mathrm{T} f \ \&\ \Phi_e(g) = f\}\big)$$

$$\deg_\mathrm{w}\big(\{g : g >_\mathrm{T} f\}\big).$$

Embedding $\mathcal{M}_\mathrm{w}$ into $\mathcal{M}_\mathrm{s}$

Recall that $C(\mathcal{A}) = \{g : \exists f \in \mathcal{A}\ \ f \leq_\mathrm{T} g\}$ is the upward closure of $\mathcal{A} \subseteq \mathbb{N}^\mathbb{N}$.

Theorem (Muchnik)
$\mathcal{M}_\mathrm{w}$ embeds into $\mathcal{M}_\mathrm{s}$ as a lattice with 0 and 1 via the following map.

$$\deg_\mathrm{w}(\mathcal{A}) \mapsto \deg_\mathrm{s}(C(\mathcal{A}))$$

Theorem (Dyment)
The range of the embedding $\mathcal{M}_\mathrm{w} \hookrightarrow \mathcal{M}_\mathrm{s}$ is definable in $\mathcal{M}_\mathrm{s}$.

The formula $\psi(\boldsymbol{x})$ defining $\mathcal{M}_\mathrm{w}$ in $\mathcal{M}_\mathrm{s}$ says:

For every degree $\boldsymbol{a}$, if $\boldsymbol{s} \geq_\mathrm{s} \boldsymbol{x}$ whenever $\boldsymbol{s} \geq_\mathrm{s} \boldsymbol{a}$ is a singleton degree, then $\boldsymbol{x} \leq_\mathrm{s} \boldsymbol{a}$.

The first-order theories of $\mathcal{M}_{\mathrm{s}}$ and $\mathcal{M}_{\mathrm{w}}$

$\mathcal{M}_{\mathrm{s}}$ and $\mathcal{M}_{\mathrm{w}}$ are as complicated as possible. Let:

- $\mathbf{Th}(\mathcal{M}_\bullet)$ denote the **first-order theory** of $\mathcal{M}_\bullet$.
- $\mathbf{Th_3}(\mathbb{N})$ denote the **third-order theory** of $\mathbb{N}$.

$$\mathrm{Th}(\mathcal{M}_\bullet) = \{1^{\text{st}}\text{-order sentences } \varphi \text{ in the language of p.o.'s} : \mathcal{M}_\bullet \models \varphi\}$$
$$\mathrm{Th}_3(\mathbb{N}) = \{3^{\text{rd}}\text{-order sentences } \varphi \text{ in the language of arithmetic} : \mathbb{N} \models \varphi\}.$$

Theorem (S; independently Lewis-Pye, Nies, Sorbi)

$$\mathrm{Th}(\mathcal{M}_{\mathrm{s}}) \quad \equiv_1 \quad \mathrm{Th}(\mathcal{M}_{\mathrm{w}}) \quad \equiv_1 \quad \mathrm{Th}_3(\mathbb{N}).$$

Determining whether a 1^{st}-order sentence is true of $\mathcal{M}_{\mathrm{s}}$ or $\mathcal{M}_{\mathrm{w}}$ is exactly as hard as determining whether a 3^{rd}-order sentence is true of $\mathbb{N}$.

Compact mass problems

Here we focus on mass problems that are closed subsets of $\mathbf{2}^{\mathbb{N}}$.

Mass problem $\mathcal{A} \subseteq 2^{\mathbb{N}}$ is **closed** if there is a tree $T \subseteq 2^{<\mathbb{N}}$ such that

$$\mathcal{A} \quad = \quad [T] \quad = \quad \text{the set of infinite paths through } T.$$

Mass problem $\mathcal{A} \subseteq 2^{\mathbb{N}}$ is **effectively closed** if there is a **recursive** tree $T \subseteq 2^{<\mathbb{N}}$ such that $\mathcal{A} = [T]$.

Closed / effectively closed mass problems yield natural sub-lattices of $\mathcal{M}_{\mathrm{s}}$ and $\mathcal{M}_{\mathrm{w}}$.

$$\begin{aligned}
\mathcal{M}^{01}_{\mathrm{s,cl}} &= \{\deg_{\mathrm{s}}(\mathcal{A}) : \mathcal{A} \subseteq 2^{\mathbb{N}} \text{ is closed}\} \\
\mathcal{M}^{01}_{\mathrm{w,cl}} &= \{\deg_{\mathrm{w}}(\mathcal{A}) : \mathcal{A} \subseteq 2^{\mathbb{N}} \text{ is closed}\} \\
\mathcal{E}^{01}_{\mathrm{s}} &= \{\deg_{\mathrm{s}}(\mathcal{A}) : \mathcal{A} \subseteq 2^{\mathbb{N}} \text{ is effectively closed}\} \\
\mathcal{E}^{01}_{\mathrm{w}} &= \{\deg_{\mathrm{w}}(\mathcal{A}) : \mathcal{A} \subseteq 2^{\mathbb{N}} \text{ is effectively closed}\}
\end{aligned}$$

Theorem (Lewis-Pye, Shore, Sorbi / Higuchi / Simpson)
These sub-lattices are **not** Brouwer algebras.

The first-order theories of the closed degrees

The closed and effectively closed degrees are as complicated as possible.

Theorem (S)

$$\mathrm{Th}(\mathcal{M}^{01}_{\mathrm{s,cl}}) \quad \equiv_1 \quad \mathrm{Th}(\mathcal{M}^{01}_{\mathrm{w,cl}}) \quad \equiv_1 \quad \mathrm{Th}_2(\mathbb{N})$$
$$\mathrm{Th}(\mathcal{E}^{01}_{\mathrm{s}}) \quad \equiv_1 \quad \mathrm{Th}(\mathbb{N})$$

Furthermore, $\mathrm{Th}(\mathcal{E}^{01}_{\mathrm{w}})$ *is undecidable.*

For $\mathcal{M}_{\mathrm{s}}$, $\mathcal{M}_{\mathrm{w}}$, and their closed and effectively closed substructures:

- the 3-quantifier theory in the language of lattices is undecidable
- the 4-quantifier theory in the language of partial orders is undecidable.

Theorem (Cole & Kihara)
The 2-quantifier theory of $\boldsymbol{\mathcal{E}^{01}_{\mathrm{s}}}$ in the language of partial orders is **decidable**.

Merci!

Thank you for attending my talk!
Do you have a question about it?

Further reading:

[1] Peter G. Hinman, *A survey of Mučnik and Medvedev degrees*, Bulletin of Symbolic Logic **18** (2012), no. 2, 161–229.

[2] Stephen G. Simpson, *Mass problems associated with effectively closed sets*, Tohoku Mathematical Journal, Second Series **63** (2011), no. 4, 489–517.

[3] Andrea Sorbi, *The Medvedev lattice of degrees of difficulty*, Computability, Enumerability, Unsolvability, 1996, pp. 289–312. MR1395886