Axiomatisabilité et décidabilité de corps valués henséliens

Sylvy ANSCOMBE

Habilitation à Diriger des Recherches (HDR)

Institut de Mathématiques de Jussieu–
Paris Rive Gauche

Université Paris Cité

Le jury est composé de :

Antoine CHAMBERT-LOIR	Rapporteur interne
Raf CLUCKERS	Rapporteur externe
Françoise DELON	Examinatrice
Charlotte HARDOUIN	Examinatrice
Martin HILS	Rapporteur externe
Frank WAGNER	Président et examinateur

INSTITUT DE MATHÉMATIQUES JUSSIEU - PARIS RIVE GAUCHE

Université Paris Cité

Winter sceal geweorpan
weder eft cuman
sumor swegle hat

Maxims I , Exeter Book

Little darling, it’s been a long, cold, lonely winter
Little darling, it feels like years since it’s been here
Here comes the sun
Here comes the sun
And I say, it’s alright

George Harrison

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Résumé

Dans ce mémoire, nous explorons des questions de théorie des modèles autour de l'axiomatabilité et de la décidabilité de théories et de fragments de théories de corps valués henséliens, en particulier ceux de caractéristique résiduelle positive. Le problème ouvert le plus important dans ce domaine est peut-être la décidabilité des corps $\mathbb{F}_q(\!(t)\!)$ de séries formelles sur des corps finis. Si les théories complètes de ces corps restent mystérieuses, leurs théories existentielles sont de mieux en mieux comprises : en 2003, Denef et Schoutens ont fourni un algorithme, basé sur l'hypothèse de la résolution des singularités en caractéristique positive, pour déterminer si un énoncé existentielle dans la langage des anneaux avec un symbole pour t est vraie dans $\mathbb{F}_q(\!(t)\!)$. Avec Fehm, en 2016, nous avons fourni inconditionnellement une procédure de décision pour décider de la théorie existentielle dans un langage sans le symbole supplémentaire pour t. Plus tard en 2023, toujours avec Dittmann, nous avons donné une procédure de décision pour la théorie existentielle avec t, conditionnel à une hypothèse appelée **(R4)**, dont nous savons qu'elle découle de la résolution des singularités, mais qui semble en principe nettement plus faible. Nous présentons ici une présentation unifiée de ces deux résultats.

En caractéristique mixte, avec Dittmann et Jahnke en 2024, nous avons fourni une axiomatisation des théories des corps valués henséliens à ramification finie de caractéristique mixte, en termes de groupe de valeurs et de structure induite sur le corps résiduel. Nous présentons brièvement ce résultat et le représentons succinctement pour donner un énoncé resplendissant, essentiellement uniforme en caractéristique résiduelle.

Deux autres sujets explorés dans ce texte sont les corps « existentiellement henséliens » — ces corps existentiellement équivalents à un corps admettant une valuation hensélienne non triviale — et les théories des corps valués « separably tame », que nous étendons pour tenir compte d'un degré infini d'imperfection.

Abstract

In this mémoire we explore model theoretic questions around the axiomatizability and decidability of theories and fragments of theories of henselian valued fields, particularly those of positive residue characteristic. Perhaps the most important open problem in this area is the question of the decidability of fields $\mathbb{F}_q(\!(t)\!)$ of formal power series over finite fields. While the full theories of such fields remain mysterious, their existential theories are increasingly well understood: in 2003, Denef and Schoutens gave an algorithm, based on the assumption of Resolution of Singularities in positive characteristic, to determine whether or not an existential sentence in the language of rings together with a symbol for t holds in $\mathbb{F}_q(\!(t)\!)$. With Fehm, in 2016, we gave unconditionally a decision procedure to decide the existential theory in a language without the additional symbol for t. Later in 2023, also with Dittmann, we gave a decision procedure to existential theory with t conditionally on an assumption called **(R4)**, which we know follows from Resolution of Singularities, but seems in principle significantly weaker. We present here a unified exposition of these two results.

In mixed characteristic, with Dittmann and Jahnke in 2024, we gave an axiomatization of the theories of finitely ramified henselian valued fields of mixed characteristic, in terms of the value group and of the structure induced on the residue field. We give a brief exposition of this result, and represent it slightly to give a resplendent statement that is essentially uniform in the residue characteristic.

Two other topics explored in this text are "existentially henselian" fields — those fields existentially equivalent to a field admitting a nontrivial henselian valuation —, and theories of separably tame valued fields, which we extend to allow infinite imperfection degree.

Acknowledgements

I'll walk where my own nature would be leading:
It vexes me to choose another guide:
Where the grey flocks in ferny glens are feeding;
Where the wild wind blows on the mountain side.

Emily Brontë

But you gotta make your own kind of music
Sing your own special song
Make your own kind of music
Even if nobody else sings along

Cass Elliot

Where to begin? *Ich danke allen, die ich kenne, und denen, die ich vergessen habe.* It feels dangerous to start listing people, but here we go.

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This is for you three.

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Chapter 1

Introduction

Far out in the uncharted backwaters of the unfashionable end of the western spiral arm of the Galaxy lies a small unregarded yellow sun.

The Hitchhiker's Guide to the Galaxy
Douglas Adams

This mémoire presents research around the topic of the model theory of henselian valued fields. It is a survey, of sorts, of some of my work since my doctoral thesis. Inevitably technical, I want to begin with the following short introduction, which I hope is somewhat accessible to a general mathematical audience. More specialist readers may wish to jump ahead to section 1.2 or 1.3.

1.1 *Dramatis personae*

"**Henselian valued fields**" — these three words need some explaining. From the point-of-view of first-order logic, and model theory, fields are structures in the language of rings, modelling the usual axioms for the theory of fields. The first-order language of rings, denoted $\mathfrak{L}_{\mathrm{ring}}$, is the first-order language built from the signature $\sigma_{\mathrm{ring}} = \{+, \cdot, -, 0, 1\}$, where $+$ and $\cdot$ are binary function symbols, $-$ is a unary function symbol, and 0 and 1 are constant symbols. A **field** is then a tuple[1] $(F, +_F, \cdot_F, -_F, 0_F, 1_F)$ in which F is a set, $+_F$ ("addition") and $\cdot_F$ ("multiplication") are both binary operations $F \times F \to F$, $-_F$ ("minus") is a unary operation $F \to F$, and 0_F ("zero") and 1_F ("one") are elements of F. But that is not all: this tuple is required to satisfy the usual axioms for fields, namely that the addition and multiplication be associative and commutative, that minus be the inverse of addition, that 0_F be the neutral element for addition, that 1_F be the neutral element for multiplication, that 0_F is not equal to 1_F, that every nonzero element admits a multiplicative inverse, and that multiplication distributes over addition.

Prominent examples of fields include the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$. Slightly less familiarly, there are also the fields $\mathbb{F}_p$ of integers modulo p, for each prime number p, and their finite extensions $\mathbb{F}_{p^k}$, where $k \in \mathbb{N}_{>0}$. The ring of integers $\mathbb{Z}$ is not a field, as not every nonzero element has a multiplicative inverse.

Each field F admits a ring morphism $\mathbb{Z} \to F$, i.e. a function respecting the operations $+, \cdot, -$ as well as the constants $0, 1$, the kernel of which is a prime ideal $p\mathbb{Z}$, for a prime number p, or $p = 0$. This p

[1] *Fear not* — I won't maintain the notational distinction between symbols in a signature and their interpretations in structures (e.g. $+$ and $+_F$), unless strictly necessary. Though in the standard setup of first-order logic we take care that the (function/relation/constant) symbols are generally not the same as their interpretations, there is usually no need for this to be visible at the level of notation. For example, where the language of rings is concerned, we will not use different notation for the symbol for addition and its interpretations, beyond the introductory paragraph, above. But just occassionally this issue rears its head: for example when one discusses several different valuations on the same field.

In a similar vein, we will usually identify a structure with its domain — the underlying set —, except in cases where this practice is problematic, such as for valued fields, where it is often helpful to explicitly name the valuation. Thus, for example, a field may be written F, K, L, etc., but a valued field will usually be written $(F, u), (K, v), (L, w)$, etc.

is then called the **characteristic** of F, and is denoted $\mathrm{char}(F)$. We then obtain a unique embedding (an injective ring morphism) $\mathbb{F}_p \to F$, and this $\mathbb{F}_p$ is the **prime field** of F, where $\mathbb{F}_p$ denotes the finite field of p elements if p is a prime number, or $\mathbb{F}_0 = \mathbb{Q}$.

At the other end of the scale, in some sense, a field F is **algebraically closed** if every irreducible polynomial (in one variable) over F is linear. A related, weaker notion is that of **separably closed** fields: those F for which every separable irreducible polynomial over F is linear. For example, $\mathbb{C}$ is algebraically closed, and every field has an **algebraic closure**: an algebraically closed field that is an algebraic extension of F. In fact the algebraic closure of F is unique up to isomorphism over F, though the isomorphism is not unique in general.

Fields of formal series. We will now introduce three families of fields whose elements are formal series of different kinds: formal power series, Newton–Puiseux series, and Hahn series. Of these, the formal power series are the most fundamental, and are a particular case of Hahn series. Given any field F, the **field of formal power series** over F (sometimes called the **field of formal Laurent series**), is denoted $F(\!(t)\!)$: Its elements are **formal power series** $\sum_{i=n}^{\infty} a_i t^i$, where each a_i is an element of F, and i ranges through integers greater than or equal to some $n \in \mathbb{Z}$. The word "formal" here means that we disregard issues of convergence. In fact formal power series are not *a priori* considered to be functions at all: from another point-of-view each is just a sequence of coefficients $(a_i)_{i\in\mathbb{Z}}$, with the proviso that the **support** $\mathrm{supp}(a) = \{i \in \mathbb{Z} \mid a_i \neq 0\}$ is bounded below[2]. For the sake of readability, and to match our intuitions, we package such sequences into the expressions $\sum_{i=n}^{\infty} a_i t^i$. Addition and multiplication are defined on $F(\!(t)\!)$ as follows:

$$\left(\sum_i a_i t^i\right) + \left(\sum_j b_j t^j\right) := \sum_k (a_k + b_k) t^k$$
$$\left(\sum_i a_i t^i\right) \cdot \left(\sum_j b_j t^j\right) := \sum_k \left(\sum_{i+j=k} a_i b_j\right) t^k.$$

Some checking is required to be sure that these series on the right-hand side have support bounded below. With respect to this addition and multiplication, $F(\!(t)\!)$ becomes a field. For example, the multiplicative inverse of $1 - t$ is $\sum_{i=0}^{\infty} t^i$, as one expects by analogy with power series expansions of real and complex functions. Fields of formal power series are never algebraically closed: for example the element t has no n-th root in $F(\!(t)\!)$, for all $n > 1$. Adjoining such an n-th root, we obtain the field extension $F(\!(t^{1/n})\!)/F(\!(t)\!)$ of degree n. Taking the union of these extensions by n-th roots of t, we arrive at the field **Newton–Puiseux series** (or simply the **Puiseux series**):

$$F(\!(t)\!)^{\mathrm{Px}} := \bigcup_{n>0} F(\!(t^{1/n})\!).$$

Puiseux's Theorem, expressed in more modern language, is the assertion that if F is an algebraically closed field of characteristic zero, then $F(\!(t)\!)^{\mathrm{Px}}$ is also algebraically closed. On the other hand, if F is an algebraically closed field of characteristic $p > 0$, then $F(\!(t)\!)^{\mathrm{Px}}$ is not even separably closed: for example the Artin–Schreier polynomial $X^p - X - t^{-1}$ has no root in $F(\!(t)\!)^{\mathrm{Px}}$.

Thirdly, for each field F and for each ordered abelian group Γ, the **field of Hahn series**, denoted $F(\!(t^\Gamma)\!)$, or sometimes just $F(\!(\Gamma)\!)$, consists of those formal series $\sum_{\gamma\in\Gamma} a_\gamma t^\gamma$, such that $\mathrm{supp}_\Gamma(a) = \{\gamma \in \Gamma \mid a_\gamma \neq 0\}$ is well ordered. The equations $\left(\sum_\gamma a_\gamma t^\gamma\right) + \left(\sum_\delta b_\delta t^\delta\right) := \sum_\epsilon (a_\epsilon + b_\epsilon) t^\epsilon$ and $\left(\sum_\gamma a_\gamma t^\gamma\right) \cdot \left(\sum_\delta b_\delta t^\delta\right) := \sum_\epsilon \left(\sum_{\gamma+\delta=\epsilon} a_\gamma b_\delta\right) t^\epsilon$ again define addition and multiplication in $F(\!(t^\Gamma)\!)$. Note that $F(\!(t)\!)^{\mathrm{Px}}$ is a proper subfield $F(\!(t^{\mathbb{Q}})\!)$: for example the series $\sum_{n=1}^{\infty} t^{p^{-n}}$ is a root of $X^p - X - t^{-1}$. A field of Hahn series $F(\!(t^\Gamma)\!)$ is algebraically closed if and only if F is algebraically closed and Γ is divisible.

[2] The empty set is bounded below.

Valuations. A **valuation** on a field F is a surjective function $v : F \to \Gamma \cup \{\infty\}$, where Γ is a totally ordered abelian group, and ∞ is an extra symbol (satisfying $\gamma < \infty$, $\gamma + \infty = \infty$, and $\infty + \infty = \infty$, for all $\gamma \in \Gamma$), such that

(i) $v(x) = \infty \Leftrightarrow x = 0$

(ii) $v(xy) = v(x) + v(y)$, and

(iii) $v(x + y) \geq \min\{v(x), v(y)\}$,

for all $x, y \in F$. A field F with a valuation v is called a **valued field**, the subset $\mathcal{O}_v = \{x \in F \mid v(x) \geq 0\}$ is called the **valuation ring** of v, it is indeed a subring of F. Moreover it is a valuation ring in an abstract sense: it is an integral domain that contains either x or x^{-1} for every $x \neq 0$ in its field of fractions. In particular, it is a local ring, that is it has a unique maximal ideal, which in this case is the **valuation ideal** $\mathfrak{m}_v = \{x \in F \mid v(x) > 0\}$. The quotient $k_v := \mathcal{O}_v/\mathfrak{m}_v$ is the **residue field**, and Γ is called the **value group**. Among others, there are two short exact sequences worth bearing in mind:

$$1 \to 1 + \mathfrak{m}_v \to \mathcal{O}_v^\times \to k_v^\times \to 1$$

and

$$1 \to \mathcal{O}_v^\times \to F^\times \to \Gamma_v \to 0.$$

We will also use the notation $vF = \Gamma_v$ and $Fv = k_v$. A valuation v is **discrete**[3] if its value group Γ_v has a smallest positive element, so for example $\mathbb{Z}$-valuations are discrete.

For fields of power series, the **t-adic valuation** $v_t : F(\!(t)\!) \to \mathbb{Z} \cup \{\infty\}$ is defined by

$$v_t(a) := \begin{cases} \min \operatorname{supp}(a) & \text{if } \operatorname{supp}(a) \neq \emptyset, \\ \infty & \text{if } \operatorname{supp}(a) = \emptyset. \end{cases}$$

The axioms of valuations hold for v_t by the same sort of calculations we performed to check that $F(\!(t)\!)$ really is a field. The value group of v_t is $\mathbb{Z}$, so in particular v_t is discrete. The valuation ring of v_t is denoted $F[\![t]\!]$, and it consists of the formal power series a for which the support is a subset of $\mathbb{N}_{\geq 0}$, so these are the series of the form $\sum_{i=0}^{\infty} a_i t^i$. The unique maximal ideal of $F[\![t]\!]$ is $tF[\![t]\!]$, and so the residue field is $F[\![t]\!]/tF[\![t]\!]$, which is canonically isomorphic to F. Moreover, v_t may be naturally extended to a valuation v_Γ on the field $F(\!(t^\Gamma)\!)$ of Hahn series: for an element $a = \sum_\gamma a_\gamma t^\gamma$, we define $v_\Gamma(a)$ to be $\min \operatorname{supp}_\Gamma(a)$, or to be ∞ when $a = 0$. The value group of v_Γ is Γ, and the residue field is F. The restriction of this valuation from $F(\!(t^\mathbb{Q})\!)$ to $F(\!(t)\!)^{\mathrm{Px}}$ is also a valuation, and it has value group $\mathbb{Q}$ and residue field F.

A valued field (L, w) extends another (K, v) if firstly L/K is a field extension and also $\mathcal{O}_w \cap K = \mathcal{O}_v$. The latter condition is equivalent to requiring that (under a suitable normalization), w is literally an extension of v. Such an extension induces inclusions $\Gamma_v \subseteq \Gamma_w$ of value groups and $k_v \subseteq k_w$ of residue fields. The index $e(w/v) := (\Gamma_w : \Gamma_v)$ and degree $f(w/v) := [k_w : k_v]$ are important quantities attached to the extension, but they do not completely describe it. There are in fact proper **immediate** extensions of valued fields: those for which $e(w/v) = f(w/v) = 1$. In broad terms, some proper immediate extensions are harmless: for example, the completion of (K, v) gives an immediate extension. On the other hand, especially in positive residue characteristic, there are proper immediate extensions not coming from the completion, and these can be problematic. For example the Artin–Schreier root of t^{-1} in $\mathbb{F}_q(\!(t^{p^{-\infty}\mathbb{Z}})\!)$, discussed above, generates a proper immediate extension of degree p of $\mathbb{F}_q(\!(t)\!)^{\mathrm{Px}}$.

Model theory of fields and valued fields. Let $\mathfrak{L}$ be any first-order language, for example the languages of groups, or rings, or partial orders. The basic task of model theoretic algebra is, given an

[3]In fact, some prefer "discrete" to mean $\mathbb{Z}$-valued, i.e. $\Gamma_v \cong \mathbb{Z}$, but I do not. I will, however, write **discrete valuation ring** (DVR) [respectively, **complete discrete valuation ring** (CDVR)] to mean the valuation ring of a $\mathbb{Z}$-valuation [resp., complete $\mathbb{Z}$-valuation], especially in chapter 3, since the notions of DVR and CDVR are so well-established in the literature.

$\mathfrak{L}$-structure M, to find a characterization of the **theory** $\mathrm{Th}(M)$ of $\mathfrak{L}$-sentences that are true in M. This characterization might come in the form of an **axiomatization**: another set of $\mathfrak{L}$-sentences T which is logically equivalent to $\mathrm{Th}(M)$, meaning that the latter is exactly the set of formal consequences of T. Such an axiomatization is only useful if it has an *a priori* simpler description than $\mathrm{Th}(M)$. Perhaps, for example, we can find a finite axiomatization T of $\mathrm{Th}(M)$: sometimes this is possible, for example the total order $(\mathbb{Q}, <)$; and sometimes it is not, for example the theory of the field $\mathbb{C}$ admits an axiomatization by countably many axioms, but not finitely many. A weaker aim might be to find a computable axiomatization: an explicit list of axioms that can be recognized by a Turing machine. From such a computable axiomatization we gain a decision procedure to determine the truth or falsehood of any given $\mathfrak{L}$-sentence in M — such a theory is then **decidable**. This exists for the field $\mathbb{C}$ for example, but does not for the ring $\mathbb{Z}$. From the perspective of a posssible axiomatization T (a theory perhaps not known to be complete), the task is to identify its completions.

For this task, and of deep interest in its own right, we seek to gain some control over the kinds of sets defined in M by formulas $\varphi(x_1, \dots, x_n)$: the subset of A^n, where A is the domain of M, defined by $\varphi(x_1, \dots, x_n)$ is

$$\varphi(M) = \{(a_1, \dots, a_n) \in M^n \mid M \models \varphi(a_1, \dots, a_n)\}.$$

A theory T in the language $\mathfrak{L}$ has **quantifier elimination** if for every $\mathfrak{L}$-formula $\varphi(x)$, with free variables among those in the tuple $x = (x_1, \dots, x_n)$, there is a quantifier-free $\mathfrak{L}$-formula $\psi(x)$, also with free variables among x, such that $T \models \forall x\ (\varphi(x) \leftrightarrow \psi(x))$. Two $\mathfrak{L}$-structures M, N are **elementarily equivalent**, written $M \equiv N$, if $M \models \varphi \Leftrightarrow N \models \varphi$ for every $\mathfrak{L}$-sentence φ.

Tarski's foundational work on the model theory of the complex numbers and real numbers, both viewed in the language of rings, includes computable axiomatizations for those theories, and thus proves their decidability. The $\mathfrak{L}_{\mathrm{ring}}$-theory **ACF** consists of the axioms for fields, together with an axiom demanding that every polynomial of degree n in one variable has a root, for each $n \geq 1$. The $\mathfrak{L}_{\mathrm{ring}}$-theory **RCF** of real-closed fields consists of the axioms for fields, together with an axiom for each odd $n \geq 1$, requiring that every polynomial over the field, in one variable and of degree n, has a root in the field; and moreover an axiom that requires for every element a in the field that a or $-a$ be a square, and that -1 is not a square.

Theorem 1.1.1 (Tarski, Tarski–Seidenberg, see for example [Tar49]).

(i) *The theory* ACF *admits quantifier-elimination in* $\mathfrak{L}_{\mathrm{ring}}$. *For each* $\ell \in \mathbb{P}$, *let* χ_ℓ *be the* $\mathfrak{L}_{\mathrm{ring}}$*-sentence*

$$\underbrace{1 + \ldots + 1}_{\ell \text{ times}} = 0.$$

Then the completions of ACF *are the theories* $\mathrm{ACF}_p = \mathrm{ACF} \cup \{\chi_p\}$, *for* $p \in \mathbb{P}$, *and* $\mathrm{ACF}_0 = \mathrm{ACF} \cup \{\neg\chi_\ell \mid \ell \in \mathbb{P}\}$.

(ii) *The theory* RCF *is complete, and its expansion to and* $\mathfrak{L}_{\mathrm{oring}}$*-theory by the extra axiom* $\forall x \forall y\ (x \leq y \leftrightarrow \exists z\ y = x + z^2)$ *admits quantifier-elimination in* $\mathfrak{L}_{\mathrm{oring}}$, *where* $\mathfrak{L}_{\mathrm{oring}}$ *is the language of rings augmented by an extra binary relation symbol* $\leq$ *for the ordering.*

If $\mathfrak{L}$ is a countable language, equipped with a standard injection of its symbols into $\mathbb{N}$, then we say that an $\mathfrak{L}$-theory T is **decidable** if the set of Gödel codes (under some standard coding) of sentences in T is a computable (equivalently, recursive) subset of $\mathbb{N}$. Throughout we denote by $T^{\vdash}$ the deductive closure of T, i.e. the set of $\mathfrak{L}$-sentences that are entailed by T. The following corollary states the decidability of the deductive closures of the theories discussed in the above theorem.

Corollary 1.1.2. *The theories* $\mathrm{ACF}^{\vdash}$, $\mathrm{ACF}_p^{\vdash}$, *and* $\mathrm{RCF}^{\vdash}$ *are decidable, for every* $p \in \mathbb{P} \cup \{0\}$.

A. Robinson was the first to extend this kind of model theory to valued fields:

Theorem 1.1.3 (Robinson, [Rob56])**.** *The theory* **ACVF** *of algebraically closed fields with a nontrivial valuation has quantifier-elimination in the language of valued fields consisting of the language of rings augmented by a binary relation symbol $V(x, y)$ interpreted so that $V(a, b)$ is true in a valued field (F, v) if and only if $v(a) \leq v(b)$, for $a, b \in F$. The completions of* **ACVF** *are given by adjoining axioms specifying the characteristics of the field and of the residue field, in the form of a pair $(0, 0)$, $(0, p)$, or (p, p), for a prime number p.*

As before, we have the following corollary regarding decidability:

Corollary 1.1.4. *The theories* $\mathbf{ACVF}^{\vdash}_{(0,0)}$, $\mathbf{ACVF}^{\vdash}_{(0,p)}$, *and* $\mathbf{ACVF}^{\vdash}_{(p,p)}$ *are decidable, for every* $p \in \mathbb{P}$.

Henselianity. A valued field (F, v) is **henselian** if the valuation has a unique extension to the algebraic closure of F. Equivalently, (F, v) is henselian if simple zeros of single-variable polynomials in the residue field lift uniquely to zeros in F; equivalently, if all polynomials of the form $X^{n+1} + X^n + a_{n-1}X^{n-1} + \ldots + a_0$, where $a_i \in \mathfrak{m}_v$, have a zero in F. In one form or another, henselianity is the modern form of an idea going back at least as far as the work of Newton. Such a claim needs significant justification! Newton, like others of his era, was extending the methods to solve algebraic equations by taking series expansions. This involved inverting algebraic and analytic functions, often defined implicitly. All of this machinery, we would now put in terms of Taylor expansions, power series, convergence, etc. But Newton's approach yielded results, regardless of the formalism. The "Newton–Raphson Method" of solving equations by successive approximation bears a great similarity to the property of henselianity, though they are not directly equivalent. A tighter relationship is given by the Implicit Function Theorem (IFT): Prestel and Ziegler showed in [PZ78] that when restricted to the domain of polynomials, and expressed suitably for topological fields, the IFT turns out to be equivalent to the henselianity of the topology: a field topology on F is **henselian** if for every $n \geq 1$ there is a neighbourhood U of 0 such that every polynomial of the form $X^{n+1} + X^n + a_{n-1}X^{n-1} + \ldots + a_0$, with $a_i \in U$, has a zero in F. Roughly, a henselian topology is one that is induced by a nontrivial henselian valuation, albeit perhaps on an "ω-completion".

1.2 Ax, Kochen, Ershov, *et al.*

In the mid 1960s, Ax, Kochen, and (independently) Ershov, showed that complete theories of henselian valued fields of equal characteristic zero are parameterized by the theories of their residue fields and value groups. This is the original "*Ax–Kochen/Ershov principle*":

Theorem 1.2.1 (Ax–Kochen/Ershov, equal characteristic 0)**.** *Let (K, v) and (L, w) be henselian valued fields of equal characteristic zero. Then (K, v) and (L, w) are elementarily equivalently in the language of valued fields if and only if their residue fields k_v and k_w are elementarily equivalent in the language of rings and their value groups Γ_v and Γ_w are elementarily equivalent in the language of ordered abelian groups.*

Combining this understanding of the case of equal characteristic zero with the classical structure theorem for complete discrete valuation rings of mixed characteristic, one deduces the following:

Theorem 1.2.2 (Ax–Kochen/Ershov, finitely ramified, perfect residue field)**.** *Let (K, v) and (L, w) be unramified henselian valued fields of mixed characteristic $(0, p)$, with $p > 0$, and perfect residue fields. Then (K, v) and (L, w) are elementarily equivalently in the language of valued fields if and only if their residue fields k_v and k_w are elementarily equivalent in the language of rings and their value groups Γ_v and Γ_w are elementarily equivalent in the language of ordered abelian groups.*

Here, **unramified** means that $v(p)$ is the smallest positive element of the value group. In the next section we will see results that involve **finite ramification**, which means that there are finitely many elements of the value group between 0 and the value of p. The perfection of the residue field is important, because in that case there is a unique multiplicative choice of representatives in K of elements of the residue field Kv. It is this fact that renders the structure theory particularly straightforward, using the rings of Witt vectors.

Ostrowski and Kaplansky's work on pseudo-Cauchy[4] sequences fills in the other major part of the puzzle (at least, according to modern accounts), giving an analysis of simple immediate extensions, those finite extensions for which the induced extensions of value groups and residue fields are in fact equalities. We say a valued field of residue characteristic exponent[5] p is **Kaplansky** (or that it satisfies Kaplansky's **"Hypothesis A"**) if it has p-divisible value group and residue field that admits no finite extension of degree divisible by p. Kaplansky proved the following:

Theorem 1.2.3 (Kaplansky, [Kap42]). *Every Kaplansky valued field has a maximal immediate extension that is unique up to isomorphism.*

Such valued fields are then a setting for Ax-Kochen/Ershov Principles:

Theorem 1.2.4 (Delon, [Del82]). *The theory* **SAMK** *of separably algebraically maximal Kaplansky valued fields is complete relative to the characteristic and elementary imperfection degree*[6]*, and to the theories of residue field and value group.*

Generalizing the setting of algebraically maximal Kaplansky fields, **tame valued fields** are those valued fields that admit no purely wild extensions. Equivalently, a valued field (K, v) of residue characteristic exponent p is tame if it is henselian, has no proper algebraic immediate extensions, has a p-divisible value group, and a perfect residue field.

Theorem 1.2.5 (Kuhlmann, [Kuh16]; Kuhlmann–Pal, [KP16]). *Let (K, v) and (L, w) be equicharacteristic separably tame valued fields of the same finite imperfection degree. Then (K, v) and (L, w) are elementarily equivalent in the language of valued fields if and only if their residue fields are elementarily equivalent in the language of rings and their value groups are elementarily equivalent in the language of ordered abelian groups.*

Perhaps the biggest open problem in this area is to axiomatize the theory of $\mathbb{F}_q((t))$ for q a prime power. There is an "obvious" theory one might consider, given by a system of axioms for a henselian defectless valued field of equicharacteristic p, of imperfection degree 1, with residue field $\mathbb{F}_q$, and value group elementarily equivalent to $\mathbb{Z}$. However it is a theorem of Kuhlmann, from [Kuh01], that this theory is incomplete. Nevertheless, there is the following result for the existential fragment of the theory, i.e. the set of existential sentences in the theory:

Theorem 1.2.6 (Denef–Schoutens, [DS03]). *Suppose that Resolution of Singularities holds in positive characteristic $p > 0$, and let $q = p^k$. Then the theory $\mathrm{Th}_\exists(\mathbb{F}_q((t)), t)$ is decidable.*

Note that this result is for the theory in the language of rings expanded by a constant symbol, interpreted in $\mathbb{F}_q((t))$ as t.

1.3 The results presented in this mémoire

The work in this mémoire has one broad, underlying goal: to extend the reach of the Ax–Kochen/Ershov theorems. In principle this might be achieved by proving that the same kind of statements apply to ever more general classes of valued fields, or perhaps by specializing the theories under consideration to particular fragments of the language, for example the existential fragment. We take both of these paths. However, we are not only interested in axiomatizability and decidability, there is always the accompanying study of definable sets, their geometric and combinatorial properties, and the place of valued fields within the model theoretic universe.

[4]Pseudo-Cauchy sequences are also called pseduo-convergent sequences, or Ostrowski nets.

[5]The residue characteristic exponent is simply the residue characteristic, in case this is positive, or is 1 if the residue characteristic is 0.

[6]I.e. the imperfection degree if finite, or ∞ otherwise.

1.3.1 Existential theories of equicharacteristic henselian valued fields

In [AF16], together with Arno Fehm, we studied the $\mathfrak{L}_{\mathrm{val}}$-theory $\mathbf{H}^{\mathrm{e}\prime}$ of equicharacteristic henselian nontrivially valued fields (K, v). We showed the following:

Theorem 1.3.1 ([AF16, Theorem 1.1]). *The existential and universal $\mathfrak{L}_{\mathrm{val}}$-theory of a model $(K, v) \models \mathbf{H}^{\mathrm{e}\prime}$ is axiomatized by the axioms for the existential and universal theories of its residue field k_v, applied to the residue sort, together of course with $\mathbf{H}^{\mathrm{e}\prime}$ itself.*

Here, axioms for a residue field are applied to the residue sort by a standard interpretation $\iota_{\mathbf{k}}$ (see Definition 4.4.1): this is simply a function from $\mathfrak{L}_{\mathrm{ring}}$-formulas to $\mathfrak{L}_{\mathrm{val}}$-formulas with the property that $(K, v) \models \iota_{\mathbf{k}}\varphi$ if and only if $k_v \models \varphi$, for every $\mathfrak{L}_{\mathrm{ring}}$-sentence φ. Since we work in a three-sorted language in which one sort is for the residue field, a map $\iota_{\mathbf{k}}$ can be constructed very easily by simply relativizing each quantifier to the sort $\mathbf{k}$. For example $\exists x\ \varphi(x, y)$ becomes $\exists \alpha \in \mathbf{k}\ \varphi(\alpha, \beta)$, where we have ensured that α, β are new variable symbols (or tuples of such) belonging to the residue sort $\mathbf{k}$. The axiomatization statement is equivalent to showing that $\mathbf{H}^{\mathrm{e}\prime}$ is "existentially complete" relative to the residue sort, meaning that $\mathbf{H}^{\mathrm{e}\prime} \cup \iota_{\mathbf{k}}\mathrm{Th}_{\exists}(k_v)$ already entails $\mathrm{Th}_{\exists}(K, v)$. This yields the following corollary:

Corollary 1.3.2 ([AF16, Corollary 1.3]). *For each model $(K, v) \models \mathbf{H}^{\mathrm{e}\prime}$, $\mathrm{Th}_{\exists}(K, v)$ is decidable if and only if $\mathrm{Th}_{\exists}(k_v)$ is decidable.*

As a particular example:

Corollary 1.3.3 ([AF16, Corollary 7.7]). *The existential $\mathfrak{L}_{\mathrm{val}}$-theory $\mathrm{Th}_{\exists}(\mathbb{F}_q((t)), v_t)$ is decidable, for each prime power q.*

Nowadays, I would rather emphasize the following "monotonicity" statement, not explicitly stated in [AF16], but essentially proved there:

Theorem 1.3.4 ([AF26+a]). *For $(K, v), (L, w) \models \mathbf{H}^{\mathrm{e}\prime}$, we have*

$$\mathrm{Th}_{\exists}(K, v) \subseteq \mathrm{Th}_{\exists}(L, w) \Leftrightarrow \mathrm{Th}_{\exists}(k_v) \subseteq \mathrm{Th}_{\exists}(k_w).$$

By combining this with a standard compactnesss argument, in the guise of the Separation Lemma, we deduced in [AF26+a] the existence of a computable "elimination" function $\epsilon_{\mathbf{k}} : \mathrm{Sent}_{\exists}(\mathfrak{L}_{\mathrm{val}}) \to \mathrm{Sent}_{\exists}(\mathfrak{L}_{\mathrm{ring}})$ such that $\mathbf{H}^{\mathrm{e}\prime} \models \varphi \leftrightarrow \iota_{\mathbf{k}}\epsilon_{\mathbf{k}}\varphi$, for every $\varphi \in \mathrm{Sent}_{\exists}(\mathfrak{L}_{\mathrm{val}})$. In turn this yields the following theorem.

Main Theorem 1 ([AF26+a, Theorem 1.1(a)], see chapter 4). *The $\mathfrak{L}_{\mathrm{val}}$-theory $\mathbf{H}^{\mathrm{e}\prime}$ is existentially complete relative to the residue field sort, and $\mathbf{H}^{\mathrm{e}\prime}(R)_{\exists}$ is many-one equivalent to $R_{\exists}$ for every theory R of fields.*

Here, for theories T_1 and T_2, there is a **many-one reduction** $T_1 \leq_{\mathrm{m}} T_2$ if there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $\alpha_1 T_1$ is the preimage under f of $\alpha_2 T_2$ for suitable enumerations α_i. There is a **many-one equivalence** $T_1 \simeq_{\mathrm{m}} T_2$ if there are many-one reductions in both directions. See section 2.3.3 for more detail. Also, we have written $R_{\exists}$ to mean the subset of $R^{\vdash}$ consisting of existential sentences, and we denote $\mathbf{H}^{\mathrm{e}\prime}(R) = \mathbf{H}^{\mathrm{e}\prime} \cup \iota_{\mathbf{k}}R$. It is worth highlighting that this theorem doesn't require R to be a complete theory of fields: the many-one reduction is uniform over possibly incomplete theories of residue fields.

In chapter 4 we have made use of the present opportunity to partially unify the proofs of Theorem 1.3.1 (via Main Theorem 1) and of Theorem 1.3.5 (via Main Theorem 2). The proof we give is very close to the proof of Theorem 1.3.5 given in [ADF23]. The unified result we prove is Theorem 4.2.2.

With Arno Fehm and Philip Dittmann, in [ADF23], we studied the $\mathfrak{L}_{\mathrm{val}}(\varpi)$-theory $\mathbf{H}^{\mathrm{e},\varpi}$ of equicharacteristic henselian discretely valued fields (K, v, t) with a distinguished uniformizer t. The language $\mathfrak{L}_{\mathrm{val}}(\varpi)$ expands $\mathfrak{L}_{\mathrm{val}}$ by a new constant symbol ϖ, which we then interpret in (K, v, t) by t. Important models of this theory are the power series fields $(F((t)), v_t, t)$, described above. We sought to strengthen the result of Denef and Schoutens (Theorem 1.2.6) that $\mathrm{Th}_\exists(F((t)), t)$ is decidable if and only if $\mathrm{Th}_\exists(F)$ is decidable, on the hypothesis of Resolution of Singularities in characteristic $p > 0$. Replacing their explicit decision procedure with an axiomatization, like in [AF16], we proved the same conclusion, but with an *a priori* weaker[7] hypothesis, which we call **(R4)**, or sometimes "**Non-local Uniformization**": *that every large field F is existentially closed in an extension K/F whenever K admits an F-rational F-place.* Here, a field F is **large** if it is existentially closed in $F((t))$ in the language of rings, see [Pop96]. This is equivalent to the statement, called **(R3)**: *for every field K and every nontrivial finitely generated extension F/K such that there exists a valuation v on F/K with residue field $Fv = K$, there also exists a valuation with value group $\mathbb{Z}$ which has that property.* The hypotheses **(R4)** and **(R3)** are implied by **Local Uniformization**: *Let F/K be a finitely generated field extension and let v be a valuation on F trivial on K. There exists a local uniformization of v, which is a K-variety Y and an isomorphism $F \to K(Y)$ over K, such that v is centered on a regular point y of Y, under the identification with F.* Moreover, **(R4)** is equivalent to an assertion about the existential completeness of certain theories of henselian valued fields, see [ADF23, Remark 4.17].

Theorem 1.3.5 ([ADF23, Theorem 1.5]). *Suppose that* **(R4)** *holds. Let (K, v) be an equicharacteristic henselian valued field with distinguished uniformizer π. Then the union of the universal and existential $\mathfrak{L}_{\mathrm{val}}(\varpi)$-theories of (K, v, π) is entailed by*

- **(i)** *$\mathfrak{L}_{\mathrm{val}}$-axioms for equicharacteristic henselian valued fields,*
- **(ii)** *the $\mathfrak{L}_{\mathrm{val}}(\varpi)$-axiom expressing that π has smallest positive value, and*
- **(iii)** *$\mathfrak{L}_{\mathrm{val}}$-axioms expressing that the residue field models the union of the universal and existential $\mathfrak{L}_{\mathrm{ring}}$-theories of Kv.*

It follows that the existential $\mathfrak{L}_{\mathrm{val}}(\varpi)$-theory $\mathrm{Th}_\exists(K, v, \pi)$ of (K, v, π) is decidable relative to the existential $\mathfrak{L}_{\mathrm{ring}}$-theory $\mathrm{Th}_\exists(Kv)$ of Kv.

This also straightforwardly yields a monotonicity statement, and the following corollary:

Corollary 1.3.6 ([ADF23, Corollary 1.6]). *Suppose that* **(R4)** *holds. Then the existential $\mathfrak{L}_{\mathrm{val}}(\varpi)$-theory $\mathrm{Th}_\exists(\mathbb{F}_q((t)), v_t, t)$ is decidable, for each prime power q.*

Combining the results of [ADF23] with a uniform approach, in [AF26+a] we deduced the following theorem.

Main Theorem 2 ([AF26+a, Theorem 1.1(b)], see chapter 4). *Suppose that* **(R4)** *holds. The $\mathfrak{L}_{\mathrm{val}}(\varpi)$-theory $\mathbf{H}^{\mathrm{e},\varpi}$ is existentially complete relative to the residue field sort, and $\mathbf{H}^{\mathrm{e},\varpi}(R)_\exists$ is many-one equivalent to $R_\exists$ for every theory R of fields.*

This "uniform approach" of [AF26+a] is a formalism of so-called contexts, bridges, and arches: these are gadgets to facilitate the study of interpretations between theories in certain fragments of their languages, and they are introduced properly in 2.4.1. Limits of these gadgets are studied in Chapter 5, and this allows us for example to work with existential theories of henselian valued fields stratified by their characteristic. As a concrete example we obtain a list of various theories many-one equivalent to the existential theory of $\mathbb{Q}$, whose decidability is a famous open question also known under the name Hilbert's tenth problem for $\mathbb{Q}$, see for example [Koe14, Poo03]:

Main Theorem 3 ([AF26+a, Theorem 1.2], see chapter 5, p. 59). *The following theories are many-one equivalent:*

- **(a)** *The existential theory of $\mathbb{Q}$ in the language of rings.*
- **(b)** *The existential theory of $\mathbb{Q}((t))$ in the language of rings.*
- **(c)** *The existential theory of $\mathbb{Q}((t))$ in the language of valued fields.*

[7] By "*a priori* weaker" we mean that Resolution of Singularities implies **(R4)**, via Local Uniformization, and the converse implication is open.

- **(d)** *The existential theory of $\mathbb{Q}(\!(t)\!)$ in the language of valued fields with constant t.*
- **(e)** *The existential theory of large fields of characteristic zero in the language of rings.*
- **(f)** *The existential theory of large fields in the language of rings.*
- **(g)** *The existential theory of fields in the language of rings.*

Another application of this formalism can be found in [DF24].

1.3.2 The model theory of finitely ramified henselian valued fields

Turning to the world of henselian valued fields of mixed characteristic $(0, p)$, there are two obvious regimes of interest: those with **finite ramification** (here we mean that the initial ramification is finite, i.e. there are only finitely many elements between 0 and $v(p)$), and those with **infinite ramification**. In full generality, the infinite ramification world seems a bit beyond current methods, though there are valuable special cases that are increasingly well understood, such as algebraically maximal Kaplansky fields, tame valued fields, and perfectoid (or 'tameable') valued fields, See [Del82, Kuh16, JK23], among many others. In the **unramified** case (i.e. $v(p)$ is minimum positive in Γ_v), there are Ax–Kochen/Ershov principles that were developed by Ax–Kochen, Ershov, and many others, but they relied on the hypothesis that the residue field was perfect, and used rings of Witt vectors. Together with Franziska Jahnke, in [AJ22] we extended this picture to allow arbitrary residue fields, using the classical structure theory due to Mac Lane and Cohen. We showed, for example that the theory $\mathbf{H}^{\mathrm{ur}}$ of such a henselian unramified valued field (K, v) is decidable if and only if the theories of its residue field and value group are decidable, and we gave a new embedding lemma. With Philip Dittmann and Franziska Jahnke, [ADJ24] we extended this analysis to the realm of arbitrary finite ramification, and we identified the structure induced on the residue field: the so-called **†-residue field** structure $k_v^\dagger$ that expands the residue field k_v. We denote by $\mathbf{H}_e^{\mathrm{fr}}$ the $\mathfrak{L}_{\mathrm{val}}$-theory of henselian valued fields of characteristic zero, and which are either of equal characteristic, or are of mixed characteristic and are finitely ramified of initial ramification bounded by $e \in \mathbb{N}_{>0}$. Indeed, these valued fields form an $\mathfrak{L}_{\mathrm{val}}$-elementary class. Synthesising these results with the uniform approach of [AF26+a], we have the following.

Main Theorem 4 (Chapter 6, see p. 67)**.** *For each $e \in \mathbb{N}_{>0}$, the $\mathfrak{L}_{\mathrm{val}}$-theory $\mathbf{H}_e^{\mathrm{fr}}$ is complete relative to the †-residue field and to the value group. Moreover, the deductive closure $\mathbf{H}_e^{\mathrm{fr}}(R, G)^\vdash$ is Turing equivalent to the Turing sum $R^\vdash \oplus_T G^\vdash$, and also $\mathbf{H}_e^{\mathrm{fr}}(R, G)_\exists$ is many-one equivalent to $R_{\exists+}$, for every theory R of †-fields and every consistent theory G of discrete ordered abelian groups.*

Here, for theories T_1 and T_2, there is a **Turing reduction** $T_1 \leq_{\mathrm{T}} T_2$ if there is a Turing machine that decides the set $\alpha_1 T_1$, given an oracle for $\alpha_2 T_2$, where α_i is a suitable enumeration of T_i. There is a **Turing equivalence** $T_1 \simeq_{\mathrm{T}} T_2$ if there are Turing reductions in both directions. The **Turing sum** $T_1 \oplus T_2$ of two theories is a standard coding of the disjoint union. See section 2.3.3 for more detail. We have also written $\mathbf{H}_e^{\mathrm{fr}}(R, G) = \mathbf{H}_e^{\mathrm{fr}} \cup \iota_{\mathrm{k}} R \cup \iota_\Gamma G$, where ι_Γ is the standard intepretation of the value group sort in $\mathfrak{L}_{\mathrm{val}}$, and we have denoted by $\exists+$ the **positive-existential** fragment (see p. 62).

1.3.3 The canonical equicharacteristic henselian valuation and $\mathbb{Z}$-largeness

For a field K, let $\mathcal{H}^e(K)$ denote the set of equicharacteristic henselian valuations on K, taken up to equivalence, and let $\mathcal{H}_1^e(K)$ denote the subset of those that are comparable to all elements of $\mathcal{H}^e(K)$. Then $\mathcal{H}_1^e(K)$ is an up-set (i.e. closed under coarsening), and is nonempty. By analogy with the construction of the usual canonical henselian valuation, there is a finest element v_K^e of $\mathcal{H}_1^e(K)$, which we call the **canonical equicharacteristic henselian valuation** of K. The following is just little result, but I want to mention it here because it shows the simplicity of the overall picture.

Main Theorem 5 (Chapter 7, see p. 72)**.** *Let K be any field. For all non-trivial $v \in \mathcal{H}^e(K) \setminus \{v_K^e\}$, the valued fields (K, v) share the same existential theory. Namely:*

$$\mathrm{Th}_\exists(K, v) = \mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists = \mathbf{H}^{e\prime}(\underbrace{\mathbf{H}^{e\prime}(\mathrm{Th}(k))^\vdash \cap \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})}_{\textit{the deductive closure in } \mathfrak{L}_{\mathrm{ring}} \textit{ of } \mathbf{H}^{e\prime}(\mathrm{Th}(k))})_\exists,$$

where k denotes the residue field of v_K^e. In particular, each field admits at most three expansions by equicharacteristic henselian nontrivial valuations, up to the equivalence of having the same existential $\mathfrak{L}_{\mathrm{val}}$-theory.

1.3.4 The model theory of separably tame valued fields

Finally, in the short note [Ans26b], I adapted the embedding arguments of Kuhlmann ([Kuh16]) and Kuhlmann–Pal ([KP16]) to give a treatment of equicharacteristic separably tame valued fields that is essentially uniform in the imperfection degree. In particular, we allow arbitrary imperfection degree, not necessarily finite. We denote by $\mathbf{STVF}^{\mathrm{eq}}$ the $\mathfrak{L}_{\mathrm{val}}$-theory of equicharacteristic separably tame valued fields, and we prove the following theorem, which I stress is only novel insofar as it is uniform in the imperfection degree, and in particular it includes the case of infinite imperfection degree:

Main Theorem 6 (Chapter 8, see p. 83)**.** *The $\mathfrak{L}_{\mathrm{val}}$-theory $\mathbf{STVF}^{\mathrm{eq}}$ of separably tame valued fields of equal characteristic is complete relative to the residue field and value group sorts, and the elementary imperfection degree. Moreover, the deductive closure $\mathbf{STVF}^{\mathrm{eq}}_I(R, G)^\vdash$ is Turing equivalent to the Turing sum $R^\vdash \oplus_T G^\vdash$, and also $\mathbf{STVF}^{\mathrm{eq}}_I(R, G)_\exists$ is many-one equivalent to $R_\exists$, for every theory R of fields, every consistent theory G of nontrivial ordered abelian groups, and every finite or cofinite set I of elementary imperfection degrees.*

Here $\mathbf{STVF}^{\mathrm{eq}}_I(R, G)$ denotes $\mathbf{STVF}^{\mathrm{eq}} \cup \iota_{\mathbf{k}} R \cup \iota_\Gamma G$ together with axioms to specify that the elementary imperfection degree is in I.

This result on separably tame valued fields extends those of [KP16] in two ways: firstly, we allow infinite imperfection degree, and secondly, our results are resplendent, though resplendency in finite imperfection degree can be read from the arguments presented in [KP16].

1.4 A remark on attribution and authorship

Most of the work presented here is either drawn from the literature or is from my work with collaborators. Even that portion that is due to me, I am sure has benefitted from hundreds of informal discussions with colleagues, too many to name. Every effort has been made to ensure that credit is correctly attributed, nevertheless I am sure mistakes remain. For these and other errors, please accept my sincere apologies.

There is no new thing under the sun.

Chapter 2

Preliminaries

Our Sun is a second- or third-generation star. All of the rocky and metallic material we stand on, the iron in our blood, the calcium in our teeth, the carbon in our genes were produced billions of years ago in the interior of a red giant star. We are made of star-stuff.

Carl Sagan

The majority of the definitions in this chapter are standard in algebra and model theory, and they can be found for example in [Bou07, FJ08, Lan87, Ser79], in [CK90, Hod93, Hod97, Mar02, PD11, TZ12], or in many other places.

2.1 Inseparability

The reader unfamiliar with the general theory of separable and inseparable field extensions, and p-independence, is encouraged to consult [Mac39], [Tei36], and [Lan87, Chapter VIII, Proposition 4.1].

2.1.1 p-independence

This section is drawn from [Ans26b].

We fix a prime number p and let $\mathfrak{i}$ be any cardinal. The set of **finitely supported multi-indices** (indexed by $\mathfrak{i}$, with each index $< p$) is

$$p^{[\mathfrak{i}]} = \{I = (i_\alpha)_{\alpha<\mathfrak{i}} \in \{0, \dots, p-1\}^{\mathfrak{i}} \mid \mathrm{supp}(I) \text{ is finite}\},$$

where $\mathrm{supp}(I) := \{\alpha < \mathfrak{i} \mid i_\alpha \neq 0\}$. Given a subset $b = (b_\alpha)_{\alpha<\mathfrak{i}}$ of a ring, indexed by $\mathfrak{i}$, a **p-monomial** in b is the product $b^I := \prod_{\alpha<\mathfrak{i}} b_\alpha^{i_\alpha}$, for some $I = (i_\alpha)_{\alpha<\mathfrak{i}} \in p^{[\mathfrak{i}]}$. Similarly, let $\mathbb{Z}^{[\mathfrak{i}]}$ be the set of finitely supported $\mathfrak{i}$-tuples of integers, elements of which are also called **multi-indices** of length $\mathfrak{i}$.

Let F/C be a field extension of characteristic exponent p. A subset A of F is **p-independent over** C if $a \notin F^{(p)}C(A \setminus \{a\})$, for all $a \in A$. It is **p-spanning over** C if $F = F^{(p)}C(A)$, and it is a **p-basis over** C if it is both p-independent and p-spanning over C. Mostly we will be interested in the absolute versions of these notions, i.e. when C is a prime field $\mathbb{F}$, in which case we simply say **p-independent**, **p-spanning**, and **p-basis**. We see immediately that p-independence is of finite character: $A \subseteq F$ is p-independent in F over C if and only if every finite subset of A is p-independent in F over C, since $F^{(p)}C(A)$ is the union of subfields $F^{(p)}C(a)$ for finite subsets $a \subseteq A$. We write $(F/C)_{[p]}$ (respectively $(F/C)_{[\![p]\!]}$) for the set of subsets of F that are p-independent (resp. p-bases) in F over C, and we write $F_{[p]}$ (resp. $F_{[\![p]\!]}$) in the absolute case when $C = \mathbb{F}$.

Remark 2.1.1. In [Ans26b] we defined $(F/C)_{[p]}$ (resp. $(F/C)_{[\![p]\!]}$) to be the set of well-ordered tuples that are p-independent (resp. p-bases). However, in the present work the role of the ordering is diminished, so we have adapted this definition accordingly,

The relation of p-independence in F over C satisfies the exchange property: that is $a \in F^{(p)}C(A,b) \setminus F^{(p)}C(A)$ implies $b \in F^{(p)}C(A,a)$. It defines a pre-geometry on subsets of F, and any two p-bases (in F over C) have the same cardinality: thus we may define the **(relative) imperfection degree** of F over C, denoted $\mathfrak{imp}(F/C)$, to be the cardinality of a p-basis of F over C. The **imperfection degree** of F, denoted $\mathfrak{imp}(F) := \mathfrak{imp}(F/\mathbb{F})$, is the cardinality of a p-basis of F in the absolute case. If $\mathfrak{imp}(F/C)$ is finite then $[F : F^{(p)}C] = p^{\mathfrak{imp}(F/C)}$, and $[F : F^{(p)}C] = \mathfrak{imp}(F/C)$ otherwise.

2.1.2 Lambda functions and lambda closure

Let F be a field of characteristic exponent p.

Definition 2.1.2 (Lambda functions)**.** For $b \in F_{[p]}$ that is indexed by $\mathfrak{i}$, and for $a \in F^{(p)}(b)$, there is a unique family $(\lambda_I^b(a))_{I \in p^{[|b|]}}$ of elements of F such that

$$a = \sum_{I \in p^{[\mathfrak{i}]}} b^I \lambda_I^b(a)^p, \tag{Λ}$$

Thus for each $I \in p^{[\mathfrak{i}]}$ there is a function

$$\begin{aligned} \lambda_I^b : F^{(p)}(b) &\to F \\ a &\mapsto \lambda_I^b(a). \end{aligned}$$

We write λ^b for the function $a \mapsto (\lambda_I^b(a))_{I \in p^{[|b|]}}$ from $F^{(p)}(b)$ to the set of subsets of F indexed by $p^{[|b|]}$. On the other hand, the **parameterized lambda functions** are the partial functions $\lambda_I : F \times F_{[p]} \to F$, for $I \in p^{[\mathfrak{i}]}$, that are defined by $\lambda_I(a,b) := \lambda_I^b(a)$ when $|b| = \mathfrak{i}$ and $a \in F^{(p)}(b)$, and are undefined otherwise. Finally, for any set $A \subseteq F$, we will write $\lambda^b(A)$ to mean the union $\bigcup_{I \in p^{[\mathfrak{i}]}} \lambda_I^b(A \cap F^{(p)}(b))$, where each $\lambda_I^b(A \cap F^{(p)}(b))$ is simply the set $\{\lambda_I^b(a) \mid a \in A \cap F^{(p)}(b)\}$.

Remark 2.1.3. Note that the sum in Equation (Λ) is finite, i.e. $\lambda_I^b(a)$ is zero for all but finitely many $I \in p^{[\mathfrak{i}]}$. Also, for each $I \in p^{[\mathfrak{i}]}$, the restriction of the map λ_I^b to $F^{(p)}(b)$ is $\mathbb{F}_p$-linear.

Let F/C be a field extension. The **lambda closure** of C in F, which we denote by $\Lambda_F C$, is the minimal subfield of F that contains C such that $F/\Lambda_F C$ is separable. The existence of $\Lambda_F C$ is given by [DM77, Theorem 1.1]. Generators for $\Lambda_F C$ are described in [Ans26b, Theorem 1.1.]. For any set $A \subseteq F$, we write $\Lambda_F A := \Lambda_F \mathbb{F}(A)$, where $\mathbb{F}(A)$ is the subfield of F generated by A. The following statement collects a few basic properties:

Fact 2.1.4 ([Ans19, Lemmas 7, 8, and 9])**.**

- **(i)** *Let $F \preceq F^*$ be an elementary extension. Then $\Lambda_F(C) = \Lambda_{F^*}(C)$.*
- **(ii)** *Let $A \subseteq F$ be a finite set. Then $|\Lambda_F A| \leq \aleph_0$.*
- **(iii)** *Suppose that F/C is separable and let $a \in F^{(p^\infty)}$. Then $\Lambda_F C(a) = C(a^{p^{-n}} \mid n < \omega)$.*

2.2 Valuation theory

The reader unfamiliar with valuation theory is encouraged to consult [EP05, Ers01, FJ08]. We briefly recall several important facts and definitions.

Fact 2.2.1 (Fundamental equality)**.** *Let L/K be a finite extension of fields, and let v be a valuation on K. We have*

$$[L : K] = \sum_{w \in \mathcal{V}_v(L)} e(w/v) \cdot f(w/v) \cdot p^{d(w/v)},$$

where the sum ranges over the finitely many valuations on L that extend v, and p denotes the characteristic exponent of the residue field of v.

Definition 2.2.2 (Henselianity)**.** A valuation v on a field K is **henselian** if for every monic $f \in \mathcal{O}_v[X]$ and every $a \in \mathcal{O}_v$, if $f(a) \in \mathfrak{m}_v$ and $f'(a) \in \mathcal{O}_v^\times$, there exists a unique $b \in a + \mathfrak{m}_v$ such that $f(b) = 0$.

See [EP05, Chapter 4] for more general algebraic information on henselian valuations.

Definition 2.2.3. We say that an extension $(L, v_L)/(K, v_K)$ of valued fields is

(i) **immediate** if $e(v_L/v_K) = f(v_L/v_K) = 1$,

(ii) **defectless** if $d(v_L/v_K) = 1$, otherwise it is a **defect** extension.

Definition 2.2.4. We say that a valued field (K, v) is

(i) **(separably) algebraically maximal** if it admits no nontrivial immediate (separable) algebraic extensions,

(ii) **(separably) defectless** if every finite (separable) extension is defectless,

(iii) **maximal** if every extension is defectless.

For a valued field we have the following implications:

$$\begin{aligned} \text{maximal} &\Rightarrow \text{henselian and defectless} \\ &\Rightarrow \text{algebraically maximal} \\ &\Rightarrow \text{separably algebraically maximal} \\ &\Rightarrow \text{henselian} \end{aligned}$$

In his landmark paper [Kap42], Kaplansky analysed simple immediate extensions of valued fields, and he gave certain hypotheses under which a given valued field admits a unique maximal immediate extension, up to isomorphism.

Definition 2.2.5. A valued field (K, v) of residue characteristic exponent p is **Kaplansky**[1] if Γ_v is p-divisible and k_v admits no proper finite extension of degree divisible by p.

Theorem 2.2.6 (Kaplansky)**.** *If (K, v) is Kaplansky then it admits a unique maximal immediate extension (K^m, v^m), up to isomorphism over K.*

2.3 Languages and fragments

This section is based on [AF26+a, AF25].

A **language** $\mathfrak{L}$ is simply a first-order language which we allow to be multi-sorted, built from a **signature** that consists of relation symbols, function symbols, and constant symbols, with given arities. Though it is an abuse of notation, we frequently identify a signature with the first-order language it defines. In case that $\mathfrak{L}$ is multi-sorted, each symbol determines its corresponding finite sequence of sorts (this generalizes the notion of arity). The sets of **$\mathfrak{L}$-terms**, **$\mathfrak{L}$-formulas**, and **$\mathfrak{L}$-sentences** are built recursively, according to their usual definitions which we omit. We follow the convention that $\top, \bot$ are sentences of every language. We allow the propositional connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$, an unlimited (or sufficiently large) family of variables in each sort, and both existential $\exists$ and universal $\forall$ quantification for every variable in every sort. We denote by $\mathrm{Form}(\mathfrak{L})$ the set of $\mathfrak{L}$-formulas, and by $\mathrm{Sent}(\mathfrak{L})$ the set of $\mathfrak{L}$-sentences. A subset $T \subseteq \mathrm{Sent}(\mathfrak{L})$ is called an **$\mathfrak{L}$-theory**. In particular we do not require theories to be deductively closed, i.e. closed under entailment. Indeed, we let $T^\vdash$ denote the deductive closure of an $\mathfrak{L}$-theory T. We omit the standard definition of an $\mathfrak{L}$-structure. We denote by $\mathrm{Th}(M)$ the $\mathfrak{L}$-theory of an $\mathfrak{L}$-structure M. The class of models of a theory T, always understood in a fixed language, is denoted $\mathbf{Mod}(T)$.

[1]Or, is **kaplanskian**, or satisfies **Kaplansky's hypothesis**.

Example 2.3.1. There are several important languages that will be used throughout this mémoire.

- Let $\mathfrak{L}_{\text{ring}} = \{+, \cdot, -, 0, 1\}$ be the first-order language of rings[2].
- Let $\mathfrak{L}_{\text{oag}} = \{+, 0, \infty, <\}$ be the first-order language of ordered abelian groups augmented by a symbol for an infinite element.
- Let $\mathfrak{L}_{\text{val}}$ denote the three-sorted language of valued fields, with a sort $\mathbf{K}$ for the field itself, a sort $\mathbf{k}$ for the residue field, and a sort $\mathbf{\Gamma}$ for the value group, equipped respectively with the languages $\mathfrak{L}_{\text{ring}}$, $\mathfrak{L}_{\text{ring}}$, and $\mathfrak{L}_{\text{oag}}$, and with the additional function symbols $\mathbf{val} : \mathbf{K} \to \mathbf{\Gamma}$: and $\mathbf{res} : \mathbf{K} \to \mathbf{k}$:

$$\mathfrak{L}_{\text{val}} = \{+^{\mathbf{K}}, -^{\mathbf{K}}, \cdot^{\mathbf{K}}, 0^{\mathbf{K}}, 1^{\mathbf{K}}, +^{\mathbf{k}}, -^{\mathbf{k}}, \cdot^{\mathbf{k}}, 0^{\mathbf{k}}, 1^{\mathbf{k}}, +^{\mathbf{\Gamma}}, -^{\mathbf{\Gamma}}, <^{\mathbf{\Gamma}}, 0^{\mathbf{\Gamma}}, \infty^{\mathbf{\Gamma}}, \mathbf{val}, \mathbf{res}\}.$$

- Let $\mathfrak{L}_{\text{val}}(\varpi)$ be the expansion of $\mathfrak{L}_{\text{val}}$ by a new constant symbol ϖ in the sort $\mathbf{K}$.
- For a ring C, let $\mathfrak{L}_{\text{val}}(C)$ denote the expansion of $\mathfrak{L}_{\text{val}}$ by new constant symbols, one for each element of C, in the sort $\mathbf{K}$.

Common mathematical structures are viewed as $\mathfrak{L}_{\text{ring}}$, $\mathfrak{L}_{\text{val}}$, etc., -structures in the obvious way: a ring is naturally seen as an $\mathfrak{L}_{\text{ring}}$-structure, a valued field (K, v) is seen as an $\mathfrak{L}_{\text{val}}$-structure

$$(K, Kv, vK \cup \{\infty\}, v, \text{res}),$$

where Kv is the residue field, vK is the value group, and res is the residue map (for convenience set to $0 \in Kv$ outside the valuation ring). For notational simplicity, we will usually continue to write (K, v) to refer to the $\mathfrak{L}_{\text{val}}$-structure it induces.

2.3.1 $\mathfrak{L}$-fragments and fragments

This section is based on [AF25, §2].

Disclaimer 2.3.2. I like fragments, contexts, bridges, and arches, more than I should, and I devote the next two sections to introducing them with their accompanying notation and preservation criteria. There are several variations on the theme of fragments already in the literature, and the notions of contexts, bridges, and arches, are somewhat technical and *ad hoc*. Nevertheless I find they help greatly with the framing of precise questions, especially in the context of structures whose full theories are at present beyond our comprehension. I therefore hope the reader can forgive me for labouring the introduction of these gadgets over the next several pages.

For $F \subseteq \text{Form}(\mathfrak{L})$, we write $\text{Form}_F(\mathfrak{L}) = F$ and $\text{Sent}_F(\mathfrak{L}) = \text{Sent}(\mathfrak{L}) \cap F$. The F-**theory** of an $\mathfrak{L}$-structure M is $\text{Th}_F(M) := \text{Th}(M) \cap F$, and the F-**type** of a tuple a in M is $\text{tp}_F^M(a) := \text{tp}^M(a) \cap F$. For an $\mathfrak{L}$-theory T, we write $T_F = T^{\vdash} \cap F$ for the F-**consequences** of T.

Definition 2.3.3. A set $F \subseteq \text{Form}(\mathfrak{L})$ is an $\mathfrak{L}$-**fragment** if it contains $\top$ and $\bot$, it is closed under finite conjunctions and disjunctions, and it is

(f1) closed under change of variables: $\varphi(\mathbf{x}) \in F \Rightarrow \varphi(\mathbf{y}) \in F$.

Remark 2.3.4. For an $\mathfrak{L}$-fragment F, and for a consistent F-theory T we have that $T \models \text{Th}_F(M)$ for some M if and only if T is "F-complete", meaning that $T \models \varphi \vee \psi$ implies $T \models \varphi$ or $T \models \psi$, for all $\varphi, \psi \in F$. We might say that T is "F-deductively closed" if $T = T_F$.

Remark 2.3.5. The definition of $\mathfrak{L}$-fragments suggests other properties of sets $F \subseteq \text{Form}(\mathfrak{L})$:

(f2) closed under substitution of variables by existing constants: $\varphi(\mathbf{x}) \in F \Rightarrow \varphi(\mathbf{c}) \in F$,

(f3) closed under substitution of constants by variables: $\varphi(\mathbf{c}) \in F \Rightarrow \varphi(\mathbf{x}) \in F$,

[2]Sometimes we use the symbol $\times$ instead of $\cdot$ for multiplication in rings.

(f4) closed under commutation of disjunctions and conjunctions: $(\varphi \blacksquare \psi) \in F \Leftrightarrow (\psi \blacksquare \varphi) \in F$,

(f5) closed under association of disjunctions and conjunctions: $(\varphi \blacksquare (\psi \blacksquare \chi)) \in F \Leftrightarrow ((\varphi \blacksquare \psi) \blacksquare \chi) \in F$,

(f6) closed under negation: $\varphi \in F \Rightarrow \neg\varphi \in F$,

(f7) closed under logical equivalence: if $\models (\varphi \leftrightarrow \psi)$, then $\varphi F \Leftrightarrow \psi \in F$,

(f8) closed under subformulas: φ is a subformula of $\psi \in F \Rightarrow \varphi \in F$,

where in **(f5)** and **(f6)**, the symbol $\blacksquare$ denotes both conjunction and disjunction.

We may view the class function $\mathfrak{L} \mapsto \mathrm{Form}(\mathfrak{L})$ as a covariant functor $\mathsf{Form} : \mathbf{Lang} \to \mathbf{Sets}$, from the category of languages (with inclusion) to the category of sets (with inclusion). A **functor of formulas** is a subfunctor $\mathsf{F} : \mathbf{L} \to \mathbf{Sets}$ of Form that is defined on a full subcategory $\mathbf{L}$ of $\mathbf{Lang}$.

Convention 2.3.6. For a functor of formulas F, we write $\mathrm{Form}_{\mathsf{F}}(\mathfrak{L}) = \mathsf{F}(\mathfrak{L})$. and $\mathrm{Sent}_{\mathsf{F}}(\mathfrak{L}) = \mathrm{Sent}(\mathfrak{L}) \cap \mathsf{F}(\mathfrak{L})$. The F-**theory** of an $\mathfrak{L}$-structure M is $\mathrm{Th}_{\mathsf{F}}(M) := \mathrm{Th}_{\mathsf{F}(\mathfrak{L})}(M)$, and the F-**type** of a tuple a in M is $\mathrm{tp}^M_{\mathsf{F}}(a) := \mathrm{tp}^M_{\mathsf{F}(\mathfrak{L})}(a)$.

Definition 2.3.7. The **functor of quantifier-free formulas** is F_0, defined so that $\mathsf{F}_0(\mathfrak{L})$ is the set of quantifier-free $\mathfrak{L}$-formulas. For $Q \in \{\forall, \exists\}$ and a functor of formulas F we inductively define functors of formulas $[Q_n; \mathsf{F}]$, for $n \in \mathbb{N}_{>0}$, by

- $[Q_1; \mathsf{F}](\mathfrak{L}) = \{Qx\ \psi : \psi \in \mathsf{F}(\mathfrak{L}), x \text{ a variable of } \mathfrak{L}\} \cup \mathsf{F}(\mathfrak{L})$,
- $[Q_{n+1}; \mathsf{F}] = [Q_1; [Q_n; \mathsf{F}]]$.

We also define $[Q; \mathsf{F}]$ by

- $[Q; \mathsf{F}] = \bigcup_n [Q_n; \mathsf{F}]$.

Note that we have not defined $[Q_0, \mathsf{F}]$. For finite (possibly empty) sequences $(Q_{(0)}, \dots, Q_{(m)})$, with $Q_{(i)} \in \{\exists_n, \exists, \forall_n, \forall \mid n \in \mathbb{N}_{>0}\}$, we inductively define the associated functors of formulas, called the **prenex-form functors**, by

- $[\emptyset; \mathsf{F}] = \mathsf{F}$,
- $[Q_{(0)}, \dots, Q_{(m)}; \mathsf{F}] = [Q_{(0)}; [Q_{(1)}, \dots, Q_{(m)}; \mathsf{F}]]$.

We abbreviate $[Q_{(0)}, \dots, Q_{(m)}] := [Q_{(0)}, \dots, Q_{(m)}; \mathsf{F}_0]$.

These finite sequences $[Q_{(0)}, \dots, Q_{(m)}]$ are **words** in the alphabet $\{\exists_n, \exists, \forall_n, \forall \mid n \in \mathbb{N}_{>0}\}$, and such words may be composed by concatenation: the concatenation of $\tilde{Q} = [Q_{(0)}, \dots, Q_{(m)}]$ and $\tilde{R} = [R_{(0)}, \dots, R_{(n)}]$ yields $\tilde{Q}^\frown\tilde{R} = [Q_{(0)}, \dots, Q_{(m)}, R_{(0)}, \dots, R_{(n)}]$, and it easily follows that $[\tilde{Q}^\frown\tilde{R}; \mathsf{F}] = [\tilde{Q}; [\tilde{R}; \mathsf{F}]]$. Moreover, there are some obvious redundancies in this notational scheme, for example $[\exists_1, \exists_1; \mathsf{F}] = [\exists_2; \mathsf{F}]$.

Definition 2.3.8. A **fragment** is a functor of formulas such that $\mathsf{F}(\mathfrak{L})$ is an $\mathfrak{L}$-fragment, whenever F is defined on $\mathfrak{L}$. We denote by $\times\mathsf{F}$ the fragment generated by a functor of formulas F, i.e. $\times\mathsf{F}(\mathfrak{L})$ is the closure of $\mathsf{F}(\mathfrak{L})$ under finite conjunctions and disjunctions, whenever F is defined on $\mathfrak{L}$.

Remark 2.3.9. We can define various special properties of a functor of formulas F: for $\mathbf{i} \in \{1, \dots, 8\}$, we say that F satisfies **(Fi)** to mean that, for every $\mathfrak{L}$ in the domain of F, **(fi)** holds for $\mathsf{F}(\mathfrak{L})$. We may even define further properties of F:

(F2') closed under substitution of variables by new constants: $\varphi(\mathbf{x}) \in \mathsf{F}(\mathfrak{L}) \Rightarrow \varphi(\mathbf{c}) \in \mathsf{F}(\mathfrak{L}(\mathbf{c}))$.

(F3') closed under substitution of constants by variables with reduction: $\varphi(\mathbf{c}) \in \mathsf{F}(\mathfrak{L}(\mathbf{c})) \Rightarrow \varphi(\mathbf{x}) \in \mathsf{F}(\mathfrak{L})$

(F9) compatible with restriction to sublanguages: $\mathfrak{L} \subseteq \mathfrak{L}' \Rightarrow \mathsf{F}(\mathfrak{L}') = \mathsf{F}(\mathfrak{L}) \cap \mathrm{Form}(\mathfrak{L})$.

Lemma 2.3.10. *Let* F *be a functor of formulas,* $\mathfrak{L}$ *a language on which* F *is defined, and* M *an* $\mathfrak{L}$*-structure. Then* $\mathrm{Th}_{\mathsf{F}}(M) \models \mathrm{Th}_{\times\mathsf{F}}(M)$.

Definition 2.3.11. For $Q \in \{\forall, \exists\}$ and a functor of formulas F we inductively define functors of formulas $Q_n\mathsf{F}$, $Q^n\mathsf{F}$, $Q_{<\omega}\mathsf{F}$, and $Q\mathsf{F}$ by

- $Q_n\mathsf{F} = \rtimes[Q_n; \mathsf{F}]$ for $n > 0$,
- $Q_{<\omega}\mathsf{F} = \rtimes[Q; \mathsf{F}]$,
- $Q^1\mathsf{F} = Q_1\mathsf{F}$,
- $Q^{n+1}\mathsf{F} = Q_1 Q^n\mathsf{F}$,
- $Q\mathsf{F} = \bigcup_n Q^n\mathsf{F}$.

Note that we have defined neither $Q_0\mathsf{F}$ nor $Q^0\mathsf{F}$. Abusing notation, we abbreviate the functors of formulas $\exists_n\mathsf{F}_0$, $\forall_n\mathsf{F}_0$, $\exists^n\mathsf{F}_0$, $\forall^n\mathsf{F}_0$, $\exists\mathsf{F}_0$, $\forall\mathsf{F}_0$ by $\exists_n$, $\forall_n$, $\exists^n$, $\forall^n$, $\exists$, $\forall$.

Remark 2.3.12. Definitions 2.3.7 and 2.3.11 are slightly modified from their analogues in [AF25], though their substance remains the same.

To see the difference between Q_n and Q^n, we refer to [AF25, Example 3.5]. Notice that $Q\mathsf{F} = \rtimes[Q; \mathsf{F}]$, for $Q \in \{\exists_n, \exists_{<\omega}, \forall_n, \forall_{<\omega} \mid n \in \mathbb{N}_{>0}\}$. Indeed, for a fragment F, all of $Q_n\mathsf{F}$, $Q^n\mathsf{F}$, $Q_{<\omega}\mathsf{F}$, and $Q\mathsf{F}$ are fragments, for $Q \in \{\forall, \exists\}$. Similarly to Definition 2.3.7, Definition 2.3.11 defines a fragment $\bar{Q}\mathsf{F}$ for all (possibly empty) **words** $\bar{Q}$ in the alphabet $\{\exists_n, \exists_{<\omega}, \exists^n, \exists, \forall_n, \forall_{<\omega}, \forall^n, \forall \mid n \in \mathbb{N}_{>0}\}$. However, the reader should be warned that this is a very different identification of sequences with words: $\exists_1$ is a fragment, and is entirely different to $[\exists_1]$, which is not.

Definition 2.3.13. A **classical fragment** is one of the form $\bar{Q}$, for a (possibly empty) word $\bar{Q}$ in the alphabet $\{\exists, \forall\}$. A **baroque fragment** is one of the form $\bar{Q}$, for a (possibly empty) word $\bar{Q}$ in the alphabet $\{\exists_n, \exists_{<\omega}, \exists^n, \exists, \forall_n, \forall_{<\omega}, \forall^n, \forall \mid n \in \mathbb{N}_{>0}\}$.

Figure 2.3.1 illustrates inclusions between some baroque (though not too complex) fragments.

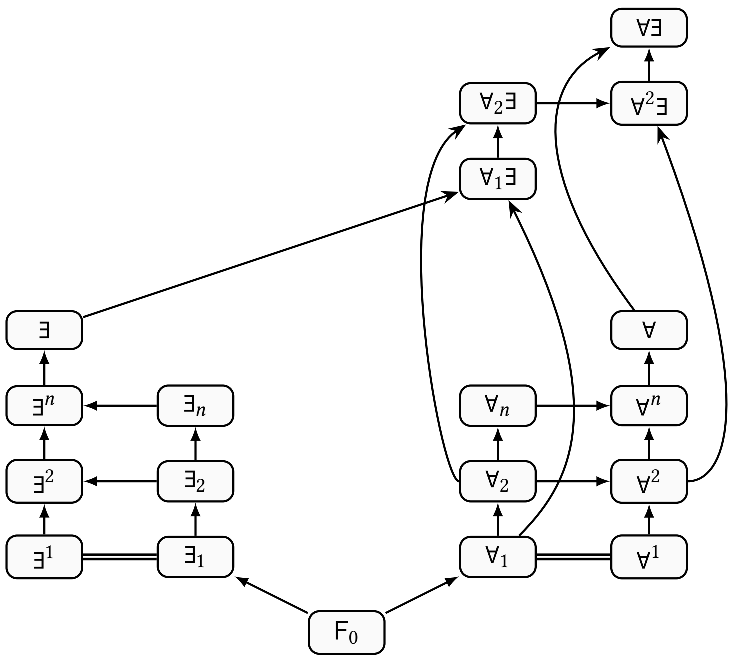

Figure 2.1: Some baroque fragments

First we recall a standard "Preservation Theorem":

Lemma 2.3.14. *Let M, N be $\mathfrak{L}$-structures. Then $\mathrm{Th}_{\exists}(M) \subseteq \mathrm{Th}_{\exists}(N)$ if and only if there is $N^* \succeq N$ and an $\mathfrak{L}$-embedding $M \to N^*$.*

Lemma 2.3.15 (Prepending $\exists_n$)**.** *Let F be a fragment, let $\mathfrak{L}$ be a language, let $n \in \mathbb{N}_{>0}$, and let M, N be $\mathfrak{L}$-structures. The following are equivalent.*

- **(i)** $\mathrm{Th}_{\exists_n \mathsf{F}}(M) \subseteq \mathrm{Th}_{\exists_n \mathsf{F}}(N)$.
- **(ii)** *For all $a \in M^n$ there exists $N^* \succcurlyeq N$ and $b \in (N^*)^n$ such that $\mathrm{tp}_{\mathsf{F}}^M(a) \subseteq \mathrm{tp}_{\mathsf{F}}^{N^*}(b)$.*
- **(iii)** *There exists $N^* \succcurlyeq N$ such that for all $a \in M^n$ there exists $b \in (N^*)^n$ such that $\mathrm{tp}_{\mathsf{F}}^M(a) \subseteq \mathrm{tp}_{\mathsf{F}}^{N^*}(b)$.*

If moreover F satisfies **(F2')** *and* **(F3')***, then also* **(i)***,* **(ii)***, and* **(iii)** *are equivalent to the following.*

- **(ii')** *For all $a \in M^n$ there exists $N^* \succcurlyeq N$ and $b \in (N^*)^n$ such that $\mathrm{Th}_{\mathsf{F}}(M, a) \subseteq \mathrm{Th}_{\mathsf{F}}(N^*, b)$.*
- **(iii')** *There exists $N^* \succcurlyeq N$ such that for all $a \in M^n$ there exists $b \in (N^*)^n$ such that $\mathrm{Th}_{\mathsf{F}}(M, a) \subseteq \mathrm{Th}_{\mathsf{F}}(N^*, b)$.*

Dually, the following are equivalent.

- **(iv)** $\mathrm{Th}_{\forall_n \mathsf{F}}(M) \subseteq \mathrm{Th}_{\forall_n \mathsf{F}}(N)$
- **(v)** *For all $b \in N^n$ there exists $M^* \succcurlyeq M$ and $a \in (M^*)^n$ such that $\mathrm{tp}_{\mathsf{F}}^{M^*}(a) \subseteq \mathrm{tp}_{\mathsf{F}}^N(b)$.*
- **(vi)** *There exists $M^* \succcurlyeq M$ such that for all $b \in N^n$ there exists $a \in (M^*)^n$ such that $\mathrm{tp}_{\mathsf{F}}^{M^*}(a) \subseteq \mathrm{tp}_{\mathsf{F}}^N(b)$.*

If moreover F satisfies **(F2')** *and* **(F3')***, then also* **(iv)***,* **(v)***, and* **(vi)** *are equivalent to the following.*

- **(v')** *For all $b \in N^n$ there exists $M^* \succcurlyeq M$ and $a \in (M^*)^n$ such that $\mathrm{Th}_{\mathsf{F}}(M^*, a) \subseteq \mathrm{Th}_{\mathsf{F}}(N, b)$.*
- **(vi')** *There exists $M^* \succcurlyeq M$ such that for all $b \in N^n$ there exists $a \in (M^*)^n$ such that $\mathrm{Th}_{\mathsf{F}}(M^*, a) \subseteq \mathrm{Th}_{\mathsf{F}}(N, b)$.*

Proof. **(i)⇒(ii)**: Let $a \in M^n$ and let $\varphi(x) \in \mathrm{tp}_{\mathsf{F}}^M(a)$, so that $\varphi(x) \in \mathsf{F}(\mathfrak{L})$, with x an n-tuple of variables. Thus $M \models \varphi(a)$ and so $M \models \exists x\ \varphi(x)$. Since $\exists x\ \varphi(x) \in \exists_n[\mathsf{F}](\mathfrak{L}) \subseteq \exists_n \mathsf{F}(\mathfrak{L})$, we have $\exists x\ \varphi(x) \in \mathrm{Th}_{\exists_n \mathsf{F}}(M)$. Therefore $\exists x\ \varphi \in \mathrm{Th}_{\exists_n \mathsf{F}}(N)$ by **(i)**. This shows that each formula in the type $\mathrm{tp}_{\mathsf{F}}^M(a)$ is consistent with the elementary diagram of N. Since F is a fragment, the type $\mathrm{tp}_{\mathsf{F}}^M(a)$ is finitely consistent, and thus consistent, with the elementary diagram of N. Thus there exists $N^* \succeq N$ and $b \in (N^*)^n$ such that b realises $\mathrm{tp}_{\mathsf{F}}^M(a)$ in N, i.e. $\mathrm{tp}_{\mathsf{F}}^M(a) \subseteq \mathrm{tp}_{\mathsf{F}}^{N^*}(b)$.

(ii)⇒(i): Let $\varphi \in \mathrm{Th}_{[\exists_n;\mathsf{F}]}(M)$. Then φ is of the form $\exists x\ \psi(x)$, where x is an n-tuple and $\psi(x) \in \mathsf{F}(\mathfrak{L})$. There exists an x-tuple a in M such that $M \models \psi(a)$. Thus $\psi(x) \in \mathrm{tp}_{\mathsf{F}}^M(a)$, so by **(ii)** there exists $N^* \succcurlyeq N$ and an x-tuple b in N^* such that $\mathrm{tp}_{\mathsf{F}}^M(a) \subseteq \mathrm{tp}_{\mathsf{F}}^{N^*}(b)$. Therefore $N^* \models \psi(b)$, and in particular $N^* \models \exists x\ \psi(x)$. Since $\varphi = \exists x\ \psi(x) \in [\exists_n; \mathsf{F}](\mathfrak{L})$, we have $\varphi \in \mathrm{Th}_{[\exists_n;\mathsf{F}]}(N)$, which shows that $N \models \mathrm{Th}_{[\exists_n;\mathsf{F}]}(M)$. We have proved $\mathrm{Th}_{[\exists_n;\mathsf{F}]}(M) \subseteq \mathrm{Th}_{[\exists_n;\mathsf{F}]}(N)$, which by Lemma 2.3.10 implies **(i)**.

(ii)⇒(iii): Compactness.

(iii)⇒(ii): Clear.

Claim 2.3.15.1. *Under hypotheses* **(F2')** *and* **(F3')***, we have $\mathrm{tp}_{\mathsf{F}}^M(a) \subseteq \mathrm{tp}_{\mathsf{F}}^N(b) \Leftrightarrow \mathrm{Th}_{\mathsf{F}}(M, a) \subseteq \mathrm{Th}_{\mathsf{F}}(N, b)$.*

Proof of claim. (⇒): Let $\varphi(c) \in \mathrm{Th}_{\mathsf{F}}(M, a)$, i.e. both $(M, a) \models \varphi(c)$ and $\varphi(c) \in \mathsf{F}(\mathfrak{L}(c))$. Thus $\varphi(x) \in \mathrm{tp}^M(a)$ and by **(F3')** we have $\varphi(x) \in \mathsf{F}(\mathfrak{L})$. Therefore $\varphi(x) \subseteq \mathrm{tp}_{\mathsf{F}}^M(a) \subseteq \mathrm{tp}_{\mathsf{F}}^N(b)$. Thus $(N, b) \models \varphi(c)$, and since we already know $\varphi(c) \in \mathsf{F}(\mathfrak{L}(c))$, we conclude $\varphi(c) \in \mathrm{Th}_{\mathsf{F}}(N, b)$.

(⇐): Let $\varphi(x) \in \mathrm{tp}_{\mathsf{F}}^M(a)$, i.e. both $M \models \varphi(a)$ and $\varphi(x) \in \mathsf{F}(\mathfrak{L})$. Thus $\varphi(c) \in \mathrm{Th}(M, a)$ and by **(F2')** we have $\varphi(c) \in \mathsf{F}(\mathfrak{L}(c))$. Therefore $\varphi(c) \subseteq \mathrm{Th}_{\mathsf{F}}(M, a) \subseteq \mathrm{Th}_{\mathsf{F}}(N, b)$. Thus $N \models \varphi(b)$, and since we already know $\varphi(x) \in \mathsf{F}(\mathfrak{L})$, we conclude $\varphi(x) \in \mathrm{tp}_{\mathsf{F}}^N(b)$. $\blacksquare_{\text{claim}}$

Both **(ii)⇔(ii')** and **(iii)⇔(iii')** follow from the claim. The proofs of the "dual" statements are very similar. □

Proposition 2.3.16 (Recursive preservation)**.** *Let F be an $\mathfrak{L}$-fragment and let M, N be $\mathfrak{L}$-structures. We have $\mathrm{Th}_{\exists F}(M) \subseteq \mathrm{Th}_{\exists F}(N)$ if and only if there exists N^* and a map $M \to N^*$ that preserves F-formulas.*

Proposition 2.3.17 ($\exists_n$-preservation, [AF26+a, Lemma 3.22]). *Let $\mathfrak{L}$ be a language and let M, N be $\mathfrak{L}$-structures. If $\mathfrak{L}$ contains no constant symbols then assume that $n \geq 1$.*

(a) $\mathrm{Th}_\exists(M) \subseteq \mathrm{Th}_\exists(N)$ *if and only if* $\mathrm{Th}_\exists(M') \subseteq \mathrm{Th}_\exists(N)$ *for every finitely generated substructure $M' \subseteq M$.*

(b) $\mathrm{Th}_{\exists_n}(M) \subseteq \mathrm{Th}_{\exists_n}(N)$ *if and only if* $\mathrm{Th}_\exists(M') \subseteq \mathrm{Th}_\exists(N)$ *for every substructure $M' \subseteq M$ generated by at most n elements.*

2.3.2 Existential formulas

Remark 2.3.18. For a language $\mathfrak{L}$, an $\mathfrak{L}$-formula $\varphi(x_1, \dots, x_m)$ is **prenex existential** (resp. **prenex universal**) if it is of the form

$$\begin{aligned} &\exists y_1 \dots \exists y_n \ \psi(x_1, \dots, x_m, y_1, \dots, y_n) \\ \text{(resp.} \quad &\forall y_1 \dots \forall y_n \ \psi(x_1, \dots, x_m, y_1, \dots, y_n)), \end{aligned}$$

for some $n \geq 0$ and an $\mathfrak{L}$-formula $\psi(x_1, \dots, x_m, y_1, \dots, y_n)$ without quantifiers. Note that the quantifiers appearing in φ each range over one of the sorts of $\mathfrak{L}$. Thus, a prenex existential (resp. universal) $\mathfrak{L}$-formula is an element of $[\exists](\mathfrak{L})$ (resp. $[\forall](\mathfrak{L})$), and the notion of a prenex existential (resp. universal) formula is different from the notion of an existential (resp. universal) formula, as I would now prefer to think of it, i.e. an element of $\exists(\mathfrak{L}) = \mathrm{Form}_\exists(\mathfrak{L})$ (resp. $\forall(\mathfrak{L}) = \mathrm{Form}_\forall(\mathfrak{L})$), for some language $\mathfrak{L}$. These types of distinctions, though varying widely in the literature, generally do us very little harm. See [AF25, Remark 2.6] for a discussion on the computability of prenexing relative to particular functors of formulas. For example this shows that $T_\exists$ and $T_{[\exists]}$ are many-one equivalent, for any theory T.

Remark 2.3.19. For each $\mathfrak{L}$, the $\mathfrak{L}$-fragments $\exists_n(\mathfrak{L})$ form a chain, ordered by inclusion, as do the $\mathfrak{L}$-fragments $\exists^n(\mathfrak{L})$. The union $\bigcup_{n<\omega} \exists_n(\mathfrak{L})$ is the $\mathfrak{L}$-fragment $\exists_{<\omega}(\mathfrak{L})$, which is a proper subset of the union $\bigcup_{n<\omega} \exists^n(\mathfrak{L})$ which is the $\mathfrak{L}$-fragment $\exists(\mathfrak{L})$. Similarly for the universal quantifier in place of the existential quantifier. We remark that our use of $\exists_n$ and $\forall_n$ is very different from the one in [Hod97], where for example an "$\exists_1$-formula" is an element of what we call $\exists(\mathfrak{L})$.

2.3.3 Computability

Definition 2.3.20 (See [AF26+a]). For a countable language $\mathfrak{L}$, when discussing the computability of an $\mathfrak{L}$-theory T, we will always assume that $\mathfrak{L}$ is **presented** in the sense that it comes with a fixed injection mapping the symbols of $\mathfrak{L}$ to $\mathbb{N}$ (the precise form of which is irrelevant and will not be given explicitly when $\mathfrak{L}$ is finite). By a standard Gödel coding, this induces an injection $\alpha : \mathrm{Form}(\mathfrak{L}) \to \mathbb{N}$, and we always assume that $\alpha(\mathrm{Form}(\mathfrak{L}))$ is computable. A set of formulas $T \subseteq \mathrm{Form}(\mathfrak{L})$ is then **computable** (or **decidable**) respectively **computably enumerable** if $\alpha(T) \subseteq \mathbb{N}$ is.

If $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are two countable languages with corresponding injections $\alpha_i : \mathrm{Form}(\mathfrak{L}_i) \to \mathbb{N}$, and $T_i \subseteq \mathrm{Form}(\mathfrak{L}_i)$, we say that a map $f : T_1 \to T_2$ is **computable** if the induced function $\alpha_1(T_1) \to \alpha_2(T_2)$ is. Similarly, we say that T_1 is **many-one reducible** to T_2, and write $T_1 \leq_{\mathrm{m}} T_2$, if $\alpha_1(T_1)$ is many-one reducible to $\alpha_2(T_2)$, i.e. there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $n \in \alpha_1(T_1)$ if and only if $f(n) \in \alpha_2(T_2)$, see [Soa16, Definition 1.6.8]. We say that T_1 and T_2 are **many-one equivalent**, and write $T_1 \simeq_{\mathrm{m}} T_2$, if $T_1 \leq_{\mathrm{m}} T_2$ and $T_2 \leq_{\mathrm{m}} T_1$. We say that T_1 is **Turing reducible** to T_2, and write $T_1 \leq_{\mathrm{T}} T_2$, if $\alpha_1(T_1)$ is Turing reducible to $\alpha_2(T_2)$, i.e. there exists a Turing machine that decides $\alpha_1(T_1)$ with an oracle for $\alpha_2(T_2)$, see [Tur39] and [Pos44, §11]. We say that T_1 and T_2 are **Turing equivalent**, and write $T_1 \simeq_{\mathrm{T}} T_2$, if $T_1 \leq_{\mathrm{T}} T_2$ and $T_2 \leq_{\mathrm{T}} T_1$. When we write the **Turing sum** $T_1 \oplus T_2$ on one of the sides of $\leq_{\mathrm{T}}$ or $\leq_{\mathrm{m}}$ we mean some standard coding of the disjoint union, e.g. the set $2\alpha_1(T_1) \cup (2\alpha_2(T_2) + 1) \subseteq \mathbb{N}$.

Fact 2.3.21. *If F is a computable fragment, then so are $[Q_n; \mathsf{F}]$, $Q_n\mathsf{F}$, $Q^n\mathsf{F}$, $Q_{<\omega}\mathsf{F}$, and $Q\mathsf{F}$.*

Remark 2.3.22. Just as in Remark 2.3.4, when deciding the existential fragment of a *complete* $\mathfrak{L}$-theory, i.e. $\mathrm{Th}_\exists(M)$ for some $\mathfrak{L}$-structure M, by induction on the structure of formulas it suffices to decide which existential $\mathfrak{L}$-sentences hold in M: $\varphi \wedge \psi \in \mathrm{Th}_\exists(M)$ if and only if $\varphi \in \mathrm{Th}_\exists(M)$ and $\psi \in \mathrm{Th}_\exists(M)$, and $\varphi \vee \psi \in \mathrm{Th}_\exists(M)$ if and only if $\varphi \in \mathrm{Th}_\exists(M)$ or $\psi \in \mathrm{Th}_\exists(M)$. (This does not hold in general for possibly incomplete theories.) The same goes for checking inclusions like $\mathrm{Th}_\exists(M) \subseteq \mathrm{Th}_\exists(M')$ for $\mathfrak{L}$-structures M, M'.

2.4 An introduction to interpretations of theories

This section is closely based on [AF26+a].

The notion of an **interpretation** of one structure N in a language $\mathfrak{L}_1$ in another structure M in a language $\mathfrak{L}_2$ is well established in contemporary model theory. Usually this refers to an isomorphism between N and a definable quotient inside M, cf. [Hod93, Hod97, Poi00, Mar02]. For example, every field interprets every linear algebraic group over it (even definably so, i.e. without the need to pass to a quotient), and every valued field interprets its value group and its residue field. The main importance of this notion of interpretation derives from the fact that many properties (like stability, or decidability) are transferred from the interpreting structure M to the interpreted structure N.

There seems to be less agreement on what an interpretation of an $\mathfrak{L}_1$-theory T_1 in an $\mathfrak{L}_2$-theory T_2 is. Older definitions seem to be of little use nowadays, as Hodges remarks in his discussion in [Hod93, §5.4]. One possible notion relates to that of a **uniform interpretation** Γ of the models of T_1 in the models of T_2, in the sense that there is one $\mathfrak{L}_2$-formula δ_Γ and for every unnested atomic $\mathfrak{L}_1$-formula φ a corresponding $\mathfrak{L}_2$-formula φ_Γ such that for every $M \models T_2$ there exists $N \models T_1$ and a surjective map $f_\Gamma : \delta_\Gamma(M) \to N$ such that

$$N \models \varphi(f_\Gamma(a_1), \dots, f_\Gamma(a_m)) \iff M \models \varphi_\Gamma(a_1, \dots, a_m), \quad a_1, \dots, a_m \in \delta_\Gamma(M),$$

so that f_Γ induces an $\mathfrak{L}_1$-isomorphism $\delta_\Gamma(M)/E_\Gamma \cong N$ (where E_Γ is the equivalence relation defined by the $\mathfrak{L}_2$-formula assigned to the $\mathfrak{L}_1$-formula $x_1 = x_2$), and thus an interpretation of N in M. Many variants of this are discussed in the above mentioned [Hod93, §5.4].

Such a uniform interpretation yields in particular the following two things: A class function $\sigma : \mathbf{Mod}(T_2) \to \mathbf{Mod}(T_1)$ assigning to every model M of T_2 the model $\sigma M = \delta_\Gamma(M)/E_\Gamma$ of T_1 that the given formulas produce, and a translation map $\iota : \mathrm{Sent}(\mathfrak{L}_1) \to \mathrm{Sent}(\mathfrak{L}_2)$ from the set of $\mathfrak{L}_1$-sentences to the set of $\mathfrak{L}_2$-sentences which are related via

$$\sigma M \models \varphi \iff M \models \iota\varphi \tag{2.4.1}$$

for every $M \models T_2$ and every $\mathfrak{L}_1$-sentence φ. If the structure map σ is surjective (say up to isomorphism, or just up to elementary equivalence), this then gives in particular an interpretation of the theory T_1 in the theory T_2 in the sense that

$$T_1 \models \varphi \iff T_2 \models \iota\varphi. \tag{2.4.2}$$

In particular, one obtains (assuming moreover that the translation map ι is computable) a Turing reduction, and more precisely even a many-one reduction from T_1 to T_2, and in fact a whole family of reductions of $\mathfrak{L}_1$-theories extending T_1 to corresponding $\mathfrak{L}_2$-theories extending T_2. We remark that there are important examples where a situation as in (2.4.1) arises without the existence of an actual interpretation of structures, like when σ assigns to a field the inverse system of its absolute Galois group.

Building on these remarks, we wish to study interpretations of theories, but more precisely focused on fragments of those theories. Generally speaking, our goal is to obtain translation maps as in (2.4.2) but now with φ in a given $\mathfrak{L}_1$-fragment and $\iota\varphi$ in a given $\mathfrak{L}_2$-fragment. For example, when the formulas δ_Γ, φ_Γ in a uniform interpretation Γ are all say quantifier-free, then the translation map ι will map existential ($\mathfrak{L}_1$-)sentences to existential ($\mathfrak{L}_2$-)sentences, thereby giving (many-one) reductions between existential theories, see for example [Pas22] for foundational work on interpretations in the context of existential theories. Moreover, one obtains a monotonicity statement for in this case the existential fragment:

$$M, M' \models T_2, \mathrm{Th}_\exists(M) \subseteq \mathrm{Th}_\exists(M') \implies \mathrm{Th}_\exists(\sigma M) \subseteq \mathrm{Th}_\exists(\sigma M'). \tag{2.4.3}$$

In the rest of this section we recall a general formalism, presented in [AF26+a], to deal with such interpretations of fragments, which for applicability still works with a structure map σ but accommodates for the fact that, when requiring (2.4.1) only for certain sentences (like existential sentences), asking for an actual interpretation (in the usual sense) of the whole structure σM in M is certainly far more than necessary, and in fact there are many more settings where a translation ι between fragments of theories arises without being induced by an interpretation of structures (again, in the usual sense).

Our most important setting (both for motivation and for applications) is that of valued fields, where as mentioned above every valued field interprets its residue field (in the sense of a uniform intepretation with T_1 the theory of fields and T_2 the theory of valued fields), and its value group (again in the sense of a uniform interpretation, this time with T_1 the theory of ordered abelian groups). However, in general neither the residue field nor the value group interprets the valued field. Nevertheless, strikingly, at least in the context of equicharacteristic henselian valued fields, the existential theory of the residue field determines the existential theory of the valued field (this is the main result of [AF16]), and in fact monotonically so in the sense of (2.4.3) with the arrow reversed, which in turn gives an interpretation of the existential theory of the valued field in the existential theory of the residue field in the sense of (2.4.1) and (2.4.2), with a translation map ι going in the other direction (which is what we will call an elimination map). So together one obtains something like a mutual interpretation of the existential theory of the valued field with the existential theory of the residue field, without the presence of a mutual interpretation of structures, and our formalism is designed to deal with such situations.

2.4.1 Interpretations, eliminations, contexts, bridges, and arches

We now set up our formalism in a general setting. Concrete examples and applications are provided in 4.4. For the rest of this section let $\mathfrak{L}, \mathfrak{L}_1, \mathfrak{L}_2$ be languages.

Definition 2.4.1. A pair (L, T) consisting of an $\mathfrak{L}$-fragment L and an $\mathfrak{L}$-theory T will be called an $\mathfrak{L}$-**context**. We call an $\mathfrak{L}$-context (L_1, T_1) a **subcontext** of the $\mathfrak{L}$-context (L_2, T_2) if $L_1 \subseteq L_2$ and $T_1 \subseteq T_2$. If $C_1 = (L_1, T_1)$ is an $\mathfrak{L}_1$-context and $C_2 = (L_2, T_2)$ an $\mathfrak{L}_2$-context, a **translation** from C_1 to C_2 is a map $\tau : L_1 \to L_2$ with

$$T_1 \models \varphi \iff T_2 \models \tau\varphi \quad \text{for all } \varphi \in L_1.$$

A **bitranslation** from C_1 to C_2 is a pair (τ_1, τ_2) where τ_1 is a translation from C_1 to C_2 and τ_2 is a translation from C_2 to C_1 such that

$$T_1 \models \varphi \leftrightarrow \tau_2\tau_1\varphi \quad \text{for all } \varphi \in L_1, \tag{2.4.4}$$

$$T_2 \models \psi \leftrightarrow \tau_1\tau_2\psi \quad \text{for all } \psi \in L_2. \tag{2.4.5}$$

The following lemma is immediate from the definitions:

Lemma 2.4.2. *Let τ be a translation from the context (L_1, T_1) to the context (L_2, T_2). Then $(T_1)_{L_1} = \tau^{-1}((T_2)_{L_2})$. If moreover L_1, L_2 and τ are computable, then $(T_1)_{L_1} \leq_{\mathrm{m}} (T_2)_{L_2}$.*

Remark 2.4.3. For a context $C = (L, T)$, let $[\varphi]_C$ denote the equivalence class of $\varphi \in L$ under the equivalence relation $\sim_C$ defined on L by $\varphi \sim_C \varphi' :\Leftrightarrow T \models \varphi \leftrightarrow \varphi'$. Denote by $\mathrm{LT}(C) = \{[\varphi]_C \mid \varphi \in L\}$ the quotient of L by $\sim_C$, which we view as a lattice isomorphic to the sublattice of the Lindenbaum–Tarski algebra of T generated by the fragment L, cf. [Sik69, Ch. 1, §.1, Example D, p. 5].

Given contexts C_1, C_2, a function $\tau : L_1 \to L_2$ is a translation from C_1 to C_2 if and only if $\tau^{-1}([\top]_{C_2}) = [\top]_{C_1}$. A pair (τ_1, τ_2) of functions $\tau_1 : L_1 \to L_2$, $\tau_2 : L_2 \to L_1$ satisfies (2.4.4) if and only if $\tau_2 \circ \tau_1$ fixes each $\sim_{C_1}$-class setwise, i.e. $\varphi \sim_{C_1} \tau_2\tau_1\varphi$ for all $\varphi \in L_1$. In particular, there exist pairs (τ_1, τ_2) which are not bitranslations but both τ_i are translations. Also there exist pairs (τ_1, τ_2) satisfying both (2.4.4) and (2.4.5) but which are not bitranslations.

Definition 2.4.4. A **bridge** is a triple $B = (C_1, C_2, \sigma)$ where each $C_i = (L_i, T_i)$ is an $\mathfrak{L}_i$-context and $\sigma : \mathbf{Mod}(T_2) \to \mathbf{Mod}(T_1)$ is a class function. We write $C_1.C_2$ for B when σ is clear from the context. An **interpretation** for the bridge B is a map $\iota : L_1 \to L_2$ such that

$$\sigma M \models \varphi \quad \Longleftrightarrow \quad M \models \iota\varphi \quad \text{for all } \varphi \in L_1, M \models T_2, \tag{2.4.6}$$

and an **elimination** for B is a map $\epsilon : L_2 \to L_1$ such that

$$\sigma M \models \epsilon\psi \quad \Longleftrightarrow \quad M \models \psi \quad \text{for all } \psi \in L_2, M \models T_2. \tag{2.4.7}$$

A bridge $B' = (C_1', C_2', \sigma')$, with $C_i' = (L_i', T_i')$, **extends** B if C_i is a subcontext of C_i', for $i = 1, 2$, and $\sigma|_{\mathbf{Mod}(T_2')} = \sigma'$. We denote this by $B \sqsubseteq B'$. Such an extension is a **fragment extension** if $T_i' = T_i$, for $i = 1, 2$; or it is a **theory extension** if $L_i' = L_i$, for $i = 1, 2$.

Example 2.4.5. Our main example is that of valued fields: Here $\mathfrak{L}_1$ is the language of rings, $\mathfrak{L}_2$ is some language of valued fields, T_2 some theory of valued fields whose residue fields are models of the theory T_1, and σ maps a valued field to its residue field. As there is a uniform interpretation of the residue field in the valued field, if L_1 and L_2 are compatible fragments (like all $\mathfrak{L}_i$-sentences, or all existential $\mathfrak{L}_i$-sentences), the bridge $((L_1, T_1), (L_2, T_2), \sigma)$ has an interpretation. If one uses the usual three-sorted language of valued fields, this interpretation is just relativizing to the residue field sort.

For of the rest of this section let $C_i = (L_i, T_i)$, for $i = 1, 2$, be an $\mathfrak{L}_i$-context, and let $B = (C_1, C_2, \sigma)$ be a bridge.

Remark 2.4.6. A function $\iota : L_1 \to L_2$ is an interpretation for B if and only if it is a translation from $(L_1, \mathrm{Th}_{L_1}(\sigma M))$ to $(L_2, \mathrm{Th}_{L_2}(M))$, for each $M \models T_2$. A function $\varepsilon : L_2 \to L_1$ is an elimination for B if and only if it is a translation from $(L_2, \mathrm{Th}_{L_2}(M))$ to $(L_1, \mathrm{Th}_{L_1}(\sigma M))$, for each $M \models T_2$.

Remark 2.4.7. Let τ be a translation from C_1 to C_2 such that $T_1 \models \varphi \leftrightarrow \varphi' \implies T_2 \models \tau\varphi \leftrightarrow \tau\varphi'$, so that τ induces a map $\mathrm{LT}(C_1) \to \mathrm{LT}(C_2)$. One can show that if this is a lattice homomorphism, then there exists a class function $\tilde{\sigma} : \mathbf{Mod}(T_2) \to \mathbf{Mod}(T_1)$ such that τ is an interpretation for the bridge $(C_1, C_2, \tilde{\sigma})$.

The following two lemmas are immediate from the definitions:

Lemma 2.4.8. *For $i = 1, 2$ suppose that C_i is a subcontext of an $\mathfrak{L}_i$-context $\hat{C}_i = (\hat{L}_i, T_i)$. Then also $\hat{B} = (\hat{C}_1, \hat{C}_2, \sigma)$ is a bridge, it is a fragment extension of B, and the following holds:*

(*a*) *If ι is an interpretation for $\hat{B}$ and $\iota(L_1) \subseteq L_2$, then $\iota|_{L_1}$ is an interpretation for B.*

(*b*) *If ϵ is an elimination for $\hat{B}$ and $\epsilon(L_2) \subseteq L_1$, then $\epsilon|_{L_2}$ is an elimination for B.*

Lemma 2.4.9. *For $i = 1, 2$ suppose that C_i is a subcontext of an $\mathfrak{L}_i$-context $\check{C}_i = (L_i, \check{T}_i)$ such that $\sigma(\mathbf{Mod}(\check{T}_2)) \subseteq \mathbf{Mod}(\check{T}_1)$. Then also $\check{B} = (\check{C}_1, \check{C}_2, \check{\sigma})$ is a bridge, where $\check{\sigma} = \sigma|_{\mathbf{Mod}(\check{T}_2)}$, and it is a theory extension of B. Moreover, the following holds:*

(*a*) *If ι is an interpretation for B, then ι is an interpretation for $\check{B}$.*

(*b*) *If ϵ is an elimination for B, then ϵ is an elimination for $\check{B}$.*

Definition 2.4.10. We consider the following property of the bridge B:

(sur) For all $N \models T_1$ there exists $M \models T_2$ such that $\mathrm{Th}_{L_1}(N) = \mathrm{Th}_{L_1}(\sigma M)$.

Lemma 2.4.11. (*a*) *For every interpretation ι for B we have*

$$T_1 \models \varphi \quad \Longrightarrow \quad T_2 \models \iota\varphi, \quad \textit{for all } \varphi \in L_1, \tag{2.4.8}$$

and if B satisfies **(sur)**, *then also*

$$T_1 \models \varphi \quad \Longleftarrow \quad T_2 \models \iota\varphi, \quad \textit{for all } \varphi \in L_1, \tag{2.4.9}$$

and so then ι is a translation from C_1 to C_2.

(*b*) *For every elimination ϵ for B we have*

$$T_1 \models \epsilon\psi \quad \Longrightarrow \quad T_2 \models \psi, \quad \textit{for all } \psi \in L_2, \tag{2.4.10}$$

and if B satisfies **(sur)**, *then also*

$$T_1 \models \epsilon\psi \quad \Longleftarrow \quad T_2 \models \psi, \quad \textit{for all } \psi \in L_2, \tag{2.4.11}$$

and so then ϵ is a translation from C_2 to C_1.

Proof. In both (a) and (b) the implication $\Longrightarrow$ is trivial from the definition. For $\Longleftarrow$ in (a) let $\varphi \in L_1$ with $T_2 \models \iota\varphi$. For every $N \models T_1$, there exists $M \models T_2$ with $\mathrm{Th}_{L_1}(N) = \mathrm{Th}_{L_1}(\sigma M)$ by **(sur)**, thus $M \models \iota\varphi$, hence $\sigma M \models \varphi$, and therefore $N \models \varphi$. The implication $\Longleftarrow$ for (b) is proven similarly. □

Proposition 2.4.12. *If the bridge B has both an interpretation ι and an elimination ϵ, then (ι, ϵ) satisfies part (2.4.5) of the definition a bitranslation (but ι and ϵ need not be translations between C_1 and C_2). If in addition B satisfies* **(sur)**, *then (ι, ϵ) is a bitranslation between C_1 and C_2.*

Proof. For $\psi \in L_2$, by (2.4.6) and (2.4.7) we have for every $M \models T_2$ that $M \models \psi$ if and only if $M \models \iota\epsilon\psi$, which shows (2.4.5). If B satisfies **(sur)**, then Lemma 2.4.11(a) and (b) give that ι, respectively ϵ, is a translation, and for $\varphi \in L_1$ and $N \models T_1$ we have

$$N \models \varphi \quad \Longleftrightarrow \quad \sigma M \models \varphi \quad \Longleftrightarrow \quad M \models \iota\varphi \quad \Longleftrightarrow \quad \sigma M \models \epsilon\iota\varphi \quad \Longleftrightarrow \quad N \models \epsilon\iota\varphi,$$

where M is some model of T_2 with $\mathrm{Th}_{L_1}(N) = \mathrm{Th}_{L_1}(\sigma M)$, whose existence is guaranteed by **(sur)**. □

Definition 2.4.13. We consider the following monotonicity property of the bridge B:

(mon) For all $M, M' \models T_2$ with $\mathrm{Th}_{L_1}(\sigma M) \subseteq \mathrm{Th}_{L_1}(\sigma M')$ we have $\mathrm{Th}_{L_2}(M) \subseteq \mathrm{Th}_{L_2}(M')$.

Lemma 2.4.14. *Suppose ι is an interpretation for B. Then for every $\varphi, \varphi' \in L_1$, $T_2 \models \iota(\varphi \wedge \varphi') \leftrightarrow (\iota\varphi \wedge \iota\varphi')$ and $T_2 \models \iota(\varphi \vee \varphi') \leftrightarrow (\iota\varphi \vee \iota\varphi')$.*

Proof. This is very easy check. □

Proposition 2.4.15. *If B admits an elimination, then it satisfies* **(mon)**. *Conversely, if B admits an interpretation ι and satisfies* **(mon)**, *then it admits an elimination. In that case, if in addition L_1 and L_2 are computable, T_2 is computably enumerable and ι is computable, then B admits a computable elimination.*

Proof. If ϵ is an elimination for B, then for $M, M' \models T_2$ with $\mathrm{Th}_{L_1}(\sigma M) \subseteq \mathrm{Th}_{L_1}(\sigma M')$ we have

$$M \models \psi \quad \Longleftrightarrow \quad \sigma M \models \epsilon\psi \quad \Longrightarrow \quad \sigma M' \models \epsilon\psi \quad \Longleftrightarrow \quad M' \models \psi$$

for every $\psi \in L_2$, hence $\mathrm{Th}_{L_2}(M) \subseteq \mathrm{Th}_{L_2}(M')$.

Now suppose that B satisfies **(mon)** and admits an interpretation ι. We claim that for every $\psi \in L_2$ there exists $\varphi \in L_1$ with $T_2 \models \psi \leftrightarrow \iota\varphi$. Indeed, for every $M, M' \models T_2$ with $M \models \psi$ and $M' \not\models \psi$, by **(mon)** there exists $\alpha \in L_1$ with $\sigma M \models \alpha$ and $\sigma M' \not\models \alpha$. Thus $M \models \iota\alpha$ and $M' \not\models \iota\alpha$. Therefore, the separation lemma [PD11, Lemma 3.1.3] gives that there exist $\alpha_{ij} \in L_1$ such that $T_2 \models \psi \leftrightarrow \bigwedge_i \bigvee_j \iota\alpha_{ij}$. By Lemma 2.4.14, $T_2 \models \bigwedge_i \bigvee_j \iota\alpha_{ij} \leftrightarrow \iota(\bigwedge_i \bigvee_j \alpha_{ij})$. Thus the choice $\varphi := \bigwedge_i \bigvee_j \alpha_{ij}$ satisfies the claim. By defining $\epsilon\psi$ to be φ for any such φ, we have $M \models \psi \Leftrightarrow M \models \iota\varphi \Leftrightarrow \sigma M \models \epsilon\psi$ for any $M \models T_2$.

Assume now in addition that L_1 and L_2 are computable, T_2 is computably enumerable and ι is computable. Then we can obtain a computable ϵ as follows: Fix a computable enumeration $\varphi_1, \varphi_2, \dots$ of L_1 and a computable enumeration $P_1, P_2, \dots$ of the proofs from T_2 (here we use $\alpha : \mathrm{Form}(\mathfrak{L}_2) \to \mathbb{N}$ to obtain an injection from the set of finite sequences of $\mathfrak{L}_2$-formulas into $\mathbb{N}$ in the usual way). The ordering $\prec$ on $\mathbb{N} \times \mathbb{N}$ defined by

$$(n, m) \prec (n', m') \iff n + m < n' + m' \vee (n + m = n' + m' \wedge m < m') \tag{2.4.12}$$

is computable and has order type ω. Given $\psi \in L_2$, by the previous paragraph there exist n and m such that P_n is a proof of $T_2 \vdash \psi \leftrightarrow \iota\varphi_m$. We take the $\prec$-minimal such pair (n, m) and define $\epsilon\psi := \varphi_m$. □

2.4.2 Extensions of bridges

Definition 2.4.16. An **arch** is a triple $A = (B, \hat{B}, \iota)$ such that $B \sqsubseteq \hat{B}$ is a fragment extension, ι is an interpretation for $\hat{B}$, and $\iota|_{L_1}$ is an interpretation for B. We write $B|\hat{B}$ for A when ι is clear from the context.

Example 2.4.17. In our main example of valued fields (see Example 2.4.5), we use arches to capture the fact that we have an interpretation ι defined on arbitrary sentences although the eliminations are usually defined only on existential or universal sentences (see, for example, Corollary 4.4.4), but restricting to existential (or universal) sentences throughout would be too limiting, as we would like to work for example in the class of valued fields with pseudofinite residue field, which is axiomatized neither by existential nor by universal sentences. Most of our applications of arches will involve Corollary 2.4.20 below.

Lemma 2.4.18. *Suppose that B admits an interpretation ι. Let $R \subseteq L_1$ and write $\check{T}_1 = T_1 \cup R$, $\check{T}_2 = T_2 \cup \iota R$, $\check{\sigma} = \sigma|_{\mathbf{Mod}(\check{T}_2)}$, and $\check{C}_i = (L_i, \check{T}_i)$. Then $\check{B} = (\check{C}_1, \check{C}_2, \check{\sigma})$ is a bridge, it is a theory extension of B, and in particular $\check{T}_2^{\vdash} = (T_2 \cup \iota(R_{L_1}))^{\vdash} = (T_2 \cup \iota(\check{T}_{1,L_1}))^{\vdash}$. Moreover, if B satisfies* **(sur)**, *then so does $\check{B}$.*

Proof. If $M \models \check{T}_2 = T_2 \cup \iota R$, then in particular $\sigma M \models R$, but also $\sigma M \models T_1$ as B is a bridge, hence $\sigma M \models \check{T}_1$, and so $\check{B}$ is a bridge. In particular, $\sigma M \models \check{T}_{1,L_1}$, hence $\check{T}_2^{\vdash} \supseteq (T_2 \cup \iota(\check{T}_{1,L_1}))^{\vdash}$, and the inclusions $\check{T}_2^{\vdash} \subseteq (T_2 \cup \iota(R_{L_1}))^{\vdash} \subseteq (T_2 \cup \iota(\check{T}_{1,L_1}))^{\vdash}$ are trivial as $R \subseteq L_1$. Now suppose that B satisfies **(sur)**, and let $N \models \check{T}_1 = T_1 \cup R$. By **(sur)** there exists $M \models T_2$ with $\mathrm{Th}_{L_1}(N) = \mathrm{Th}_{L_1}(\sigma M)$. Since $R \subseteq L_1$, this implies that $\sigma M \models R$, and so $M \models \iota R$. Therefore $\check{B}$ satisfies **(sur)**. □

For the rest of this section $\hat{B}$ will denote a fragment extension of $B = (C_1, C_2, \sigma)$, and we write $\hat{B} = (\hat{C}_1, \hat{C}_2, \sigma)$, with $\hat{C}_i = (\hat{L}_i, T_i)$.

Proposition 2.4.19. *Let $A = (B, \hat{B}, \iota)$ be an arch where B admits an elimination ϵ. We suppose that $\hat{B}$ satisfies* **(sur)**. *Then for each $R \subseteq \hat{L}_1$, $(\iota|_{L_1}, \epsilon)$ is a bitranslation between $(L_1, T_1 \cup R)$ and $(L_2, T_2 \cup \iota R)$.*

Proof. Define $\check{T}_1 = T_1 \cup R$, $\check{T}_2 = T_2 \cup \iota R$. Since $\sigma(\mathbf{Mod}(\check{T}_2)) \subseteq \mathbf{Mod}(\check{T}_1)$, both $B(R) := ((L_1, \check{T}_1), (L_2, \check{T}_2), \check{\sigma})$ and $\hat{B}(R) := ((\hat{L}_1, \check{T}_1), (\hat{L}_2, \check{T}_2), \check{\sigma})$ are bridges, where $\check{\sigma} = \sigma|_{\mathbf{Mod}(\check{T}_2)}$ (Lemma 2.4.9). Moreover, $\iota|_{L_1}$ is an interpretation for $B(R)$ (Lemmas 2.4.8(a), 2.4.9(a)) and also ϵ is an elimination for $B(R)$ (Lemma 2.4.9(b)). Furthermore, since $\hat{B}$ satisfies **(sur)**, so does $\hat{B}(R)$, by Lemma 2.4.18, which trivially implies that also $B(R)$ satisfies **(sur)**. The claim follows from Proposition 2.4.12. □

Corollary 2.4.20. *Let $A = (B, \hat{B}, \iota)$ be an arch. We suppose*

(i) *$L_1, L_2, \hat{L}_1, \hat{L}_2$ are computable, ι is computable, and T_2 is computably enumerable*[3]*,*

(ii) *$\hat{B}$ satisfies* **(sur)**, *and*

(iii) *B satisfies* **(mon)**.

Then

(I) *B admits a computable elimination.*

(II) *For each $R \subseteq \hat{L}_1$, $(T_1 \cup R)_{L_1} \simeq_{\mathrm{m}} (T_2 \cup \iota R)_{L_2}$.*

(III) *$\mathrm{Th}_{L_2}(M) = (T_2 \cup \iota\mathrm{Th}_{\hat{L}_1}(\sigma M))_{L_2} = (T_2 \cup \iota\mathrm{Th}_{L_1}(\sigma M))_{L_2}$, for each $M \models T_2$.*

[3]The computability of L_1 and $\hat{L}_1$ is to be understood with respect to the same injection $\alpha_1 : \mathrm{Form}(\mathfrak{L}_1) \to \mathbb{N}$, as in Definition 2.3.20; and similarly for L_2, $\hat{L}_2$ and T_2.

Proof. Since B satisfies **(mon)** and admits an interpretation, it admits an elimination ϵ (Proposition 2.4.15), which, together with the computability assumptions, may be taken to be computable.

For $R \subseteq \hat{L}_1$, $(\iota|_{L_1}, \epsilon)$ is a bitranslation between $(L_1, T_1 \cup R)$ and $(L_2, T_2 \cup \iota R)$ by Proposition 2.4.19. Since both $\iota|_{L_1}$ and ϵ are computable, this gives a many-one equivalence by Lemma 2.4.2.

For $M \models T_2$ the inclusions $\mathrm{Th}_{L_2}(M) \supseteq (T_2 \cup \iota\mathrm{Th}_{\hat{L}_1}(\sigma M))_{L_2} \supseteq (T_2 \cup \iota\mathrm{Th}_{L_1}(\sigma M))_{L_2}$ are trivial. Let $M' \models T_2 \cup \iota\mathrm{Th}_{L_1}(\sigma M)$, so that $\sigma M' \models \mathrm{Th}_{L_1}(\sigma M)$. Then $M' \models \mathrm{Th}_{L_2}(M)$, since B satisfies **(mon)**, which proves (III). $\square$

Lemma 2.4.21. *Let $A = (B, \hat{B}, \iota)$ be an arch. Suppose that $\hat{B}$ satisfies* **(sur)**, *and that B admits an elimination ϵ. Let $I \neq \emptyset$ and $R_i \subseteq \hat{L}_1$, for $i \in I$. Then*

$$(T_2 \cup \iota(\bigcap_{i\in I}(T_1 \cup R_i)_{L_1}))_{L_2} = (T_2 \cup \iota(\bigcap_{i\in I}(T_1 \cup R_i)_{\hat{L}_1}))_{L_2} = \bigcap_{i\in I}(T_2 \cup \iota R_i)_{L_2}.$$

In particular, $(T_2 \cup \iota((T_1 \cup R)_{L_1}))_{L_2} = (T_2 \cup \iota((T_1 \cup R)_{\hat{L}_1}))_{L_2} = (T_2 \cup \iota R)_{L_2}$ for every $R \subseteq \hat{L}_1$.

Proof. For $R \subseteq \hat{L}_1$, we write $\bar{R} = (T_1 \cup R)_{\hat{L}_1}$. First of all note that $R \subseteq \bar{R} = \bar{\bar{R}}$, as $(T_1 \cup R)_{\hat{L}_1} \subseteq (T_1 \cup \bar{R})_{\hat{L}_1} \subseteq (T_1^{\vdash} \cup (T_1 \cup R)^{\vdash})_{\hat{L}_1} \subseteq ((T_1 \cup T_1 \cup R)^{\vdash})_{\hat{L}_1} = (T_1 \cup R)_{\hat{L}_1}$. In particular $(T_1 \cup \bar{R})_{L_1} = (T_1 \cup R)_{L_1}$. Moreover, $(T_2 \cup \iota R)^{\vdash} = (T_2 \cup \iota\bar{R})^{\vdash}$, and in particular, $(T_2 \cup \iota R)_{L_2} = (T_2 \cup \iota\bar{R})_{L_2}$, by Lemma 2.4.18 applied to $\hat{B}$. Note that by Lemma 2.4.18, for any $R \subseteq \hat{L}_1$, $((\hat{L}_1, T_1 \cup R), (\hat{L}_2, T_2 \cup \iota R), \sigma|_{\mathbf{Mod}(T_2 \cup \iota R)})$ is a bridge satisfying **(sur)**. In particular, $((L_1, T_1 \cup R), (L_2, T_2 \cup \iota R), \sigma|_{\mathbf{Mod}(T_2 \cup \iota R)})$ is a bridge satisfying **(sur)**. By Lemma 2.4.9(b), ϵ is an elimination also for this bridge.

By the first paragraph, we may assume that $R_i = \bar{R}_i$, for each $i \in I$. Hence we want to prove that

$$(T_2 \cup \iota(\bigcap_{i\in I}(R_i)_{L_1}))_{L_2} = (T_2 \cup \iota(\bigcap_{i\in I} R_i))_{L_2} = \bigcap_{i\in I}(T_2 \cup \iota R_i)_{L_2}.$$

Both inclusions $\subseteq$ are trivial. It remains to show that $\bigcap_{i\in I}(T_2 \cup \iota R_i)_{L_2} \subseteq (T_2 \cup \iota(\bigcap_{i\in I}(R_i)_{L_1}))_{L_2}$, so let $\varphi \in \bigcap_{i\in I}(T_2 \cup \iota R_i)_{L_2}$. For each $i \in I$, $T_2 \cup \iota R_i \models \varphi$. Since $((L_1, T_1 \cup R_i), (L_2, T_2 \cup \iota R_i), \sigma|_{\mathbf{Mod}(T_2 \cup \iota R_i)})$ satisfies **(sur)** and has elimination ϵ for each $i \in I$, we have $T_1 \cup R_i \models \epsilon\varphi$ by (2.4.11). Thus $\epsilon\varphi \in \bigcap_{i\in I}(T_1 \cup R_i)_{L_1} = \bigcap_{i\in I}(R_i)_{L_1} \subseteq T_1 \cup \bigcap_{i\in I}(R_i)_{L_1}$. Then $T_2 \cup \iota\bigcap_{i\in I}(R_i)_{L_1} \models \varphi$ by (2.4.10). $\square$

For an $\mathfrak{L}$-fragment L write $\neg L = \{\neg\varphi : \varphi \in L\}$ and let $\bar{L}$ be the smallest fragment that contains $L \cup \neg L$. For a context $C = (L, T)$ let $\bar{C} = (\bar{L}, T)$, and for a bridge $B = (C_1, C_2, \sigma)$ let $\bar{B} = (\bar{C}_1, \bar{C}_2, \sigma)$.

Lemma 2.4.22. *If B satisfies* **(mon)** *then so does $\bar{B}$. In particular, if $A = (B, \hat{B}, \iota)$ is an arch that satisfies the assumptions of Corollary 2.4.20, with $\neg L_i \subseteq \hat{L}_i$ for $i = 1, 2$ and $\iota(\neg L_1) \subseteq \neg L_2$, then also $(\bar{B}, \hat{B}, \iota)$ is an arch that satisfies these assumptions.*

Proof. For $M, M' \models T_2$, $\mathrm{Th}_{\bar{L}_1}(\sigma M) \subseteq \mathrm{Th}_{\bar{L}_1}(\sigma M')$ implies $\mathrm{Th}_{L_1}(\sigma M) \subseteq \mathrm{Th}_{L_1}(\sigma M')$ and $\mathrm{Th}_{\neg L_1}(\sigma M) \subseteq \mathrm{Th}_{\neg L_1}(\sigma M')$, equivalently $\mathrm{Th}_{L_1}(\sigma M') \subseteq \mathrm{Th}_{L_1}(\sigma M)$, hence by **(mon)** $\mathrm{Th}_{L_2}(M) \subseteq \mathrm{Th}_{L_2}(M')$ and $\mathrm{Th}_{L_2}(M') \subseteq \mathrm{Th}_{L_2}(M)$, and therefore $\mathrm{Th}_{\bar{L}_2}(M) = \mathrm{Th}_{\bar{L}_2}(M')$, in particular $\mathrm{Th}_{\bar{L}_2}(M) \subseteq \mathrm{Th}_{\bar{L}_2}(M')$. $\square$

Remark 2.4.23. Regarding the consequences of Corollary 2.4.20 for $(\bar{B}, \hat{B}, \iota)$, of course one can easily construct an elimination for $\bar{B}$ from an elimination for B, but it is not clear how to deduce $(T_1 \cup R)_{\bar{L}_1} \simeq_{\mathrm{m}} (T_2 \cup \iota R)_{\bar{L}_2}$ directly from $(T_1 \cup R)_{L_1} \simeq_{\mathrm{m}} (T_2 \cup \iota R)_{L_2}$.

2.4.3 Weak monotonicity

We conclude this section with a weak variant of **(mon)** in the context of a fragment extension:

Definition 2.4.24. We consider the following property of the fragment extension $B \sqsubseteq \hat{B}$:

(wm) For all $M \models T_2$ and all $N \models T_1$, if $\mathrm{Th}_{L_1}(N) \subseteq \mathrm{Th}_{L_1}(\sigma M)$ then there exists $M' \models T_2$ with $\mathrm{Th}_{\hat{L}_1}(\sigma M') = \mathrm{Th}_{\hat{L}_1}(N)$ and $\mathrm{Th}_{L_2}(M') \subseteq \mathrm{Th}_{L_2}(M)$.

Lemma 2.4.25. *Suppose $\hat{B}$ satisfies* **(sur)** *and B satisfies* **(mon)**. *Then $B \sqsubseteq \hat{B}$ satisfies* **(wmon)**.

Proof. Let $M \vDash T_2$ and $N \vDash T_1$ with $\mathrm{Th}_{L_1}(N) \subseteq \mathrm{Th}_{L_1}(\sigma M)$. By **(sur)**, there exists $M' \vDash T_2$ with $\mathrm{Th}_{\hat{L}_1}(N) = \mathrm{Th}_{\hat{L}_1}(\sigma M')$. In particular $\mathrm{Th}_{L_1}(\sigma M') \subseteq \mathrm{Th}_{L_1}(\sigma M)$. By **(mon)**, we have $\mathrm{Th}_{L_2}(M') \subseteq \mathrm{Th}_{L_2}(M)$. □

Lemma 2.4.26. *Let T be an $\mathfrak{L}$-theory and let L be an $\mathfrak{L}$-fragment. Suppose that $M \vDash T_L$. Then there exists $N \vDash T$ such that $M \vDash \mathrm{Th}_L(N)$.*

Proof. It suffices to show that $T \cup \mathrm{Th}_{(\neg)L}(M)$ is consistent, because any model $N \vDash T \cup \mathrm{Th}_{(\neg)L}(M)$ satisfies $M \vDash \mathrm{Th}_L(N)$. By the compactness theorem, it suffices to show that $T \cup \mathrm{Th}_{(\neg)L}(M)$ is finitely consistent. Let $\neg\varphi_1, \dots, \neg\varphi_n \in \mathrm{Th}_{(\neg)L}(M)$, so then $M \vDash \neg \bigvee_{i\leq n} \varphi_i$. Since L is a fragment, we have $\bigvee_{i\leq n} \varphi_i \in L$ and so $\neg \bigvee_{i\leq n} \varphi_i \in \mathrm{Th}_{(\neg)L}(M)$. Therefore $\bigvee_{i\leq n} \varphi_i \notin \mathrm{Th}_L(M)$, and in particular $T \nvDash \bigvee_{i\leq n} \varphi_i$, which proves that $T \cup \mathrm{Th}_{(\neg)L}(M)$ is finitely consisten □

Lemma 2.4.27. *Let $A = (B, \hat{B}, \iota)$ be an arch such that $B \sqsubseteq \hat{B}$ satisfies* **(wmon)**. *Then for all $R, R' \subseteq \hat{L}_1$, if $(T_1 \cup R)_{L_1} \subseteq (T_1 \cup R')_{L_1}$ then $(T_2 \cup \iota R)_{L_2} \subseteq (T_2 \cup \iota R')_{L_2}$.*

Proof. Let $M \vDash T_2 \cup \iota R'$. Then $\sigma M \vDash T_1 \cup R'$, and in particular $\sigma M \vDash (T_1 \cup R')_{L_1}$, and so $(T_1 \cup R)_{L_1} \subseteq (T_1 \cup R')_{L_1} \subseteq \mathrm{Th}_{L_1}(\sigma M)$. By Lemma 2.4.26, there exists $N \vDash T_1 \cup R$ such that $\mathrm{Th}_{L_1}(N) \subseteq \mathrm{Th}_{L_1}(\sigma M)$. By **(wmon)** there exists $M' \vDash T_2$ with $\mathrm{Th}_{\hat{L}_1}(\sigma M') = \mathrm{Th}_{\hat{L}_1}(N)$ and $\mathrm{Th}_{L_2}(M') \subseteq \mathrm{Th}_{L_2}(M)$. Thus $\sigma M' \vDash R$, whence $M' \vDash T_2 \cup \iota R$. Therefore $M \vDash (T_2 \cup \iota R)_{L_2}$, from which it follows that $T_2 \cup \iota R' \vDash (T_2 \cup \iota R)_{L_2}$. □

Remark 2.4.28. To clarify the relation between **(mon)** and **(wmon)**, we remark that with B and $\hat{B}$ as above and $\hat{B}$ satisfying **(sur)**, one can show that B satisfies **(mon)** if and only if the pair $\hat{B}, B$ satisfies **(wmon)** and in addition B satisfies the following symmetric weakening of **(mon)**: For all $M, M' \vDash T_2$ with $\mathrm{Th}_{L_1}(\sigma M) = \mathrm{Th}_{L_1}(\sigma M')$ we have $\mathrm{Th}_{L_2}(M) = \mathrm{Th}_{L_2}(M')$.

Remark 2.4.29. Under the assumptions of Corollary 2.4.20, we have that

(II') For each $\mathfrak{L}_1$-theory R', $(T_1 \cup R')_{L_1} \simeq_{\mathrm{m}} (T_2 \cup \iota((T_1 \cup R')_{\hat{L}_1}))_{L_2}$,

which follows from applying Corollary 2.4.20(II) to $R = (T_1 \cup R')_{\hat{L}_1}$. Similar adaptations can be made to the conclusions of Proposition 2.4.19 and Lemmas 2.4.21 and 2.4.27 to allow $\mathfrak{L}_1$-theories.

Chapter 3

Representatives of residue fields

Oh, let the Sun beat down upon my face
With stars to fill my dreams
I am a traveler of both time and space
To be where I have been

Kashmir, Led Zeppelin

This is a short chapter of tools, to be applied principally in chapters 4 and 6. In this chapter we are concerned with functions $\zeta : k \dashrightarrow \mathcal{O}$ that are partial right (set-theoretic) inverses of the residue map $\mathrm{res} : \mathcal{O} \to k$ corresponding to a valuation ring $\mathcal{O}$ with residue field k. We call such a function a choice of **representatives**. Usually but not always the domain will be either a subring of k or it will be a p-basis of a subfield of k. A choice of representatives that is defined on a subring of k is a **partial section** when it is a ring homomorphism, and a **section** if moreover its domain is k. Note that in the case of mixed characteristic there is no chance of a choice of representatives being a section.

In the case that the residue is perfect, the existence an uniqueness of multiplicative representatives i.e. that are multiplicative homomorphisms, is well-established.

Lemma 3.0.1 (Teichmüller)**.** *Let $\mathcal{O}$ be a complete $\mathbb{Z}$-valuation ring with perfect residue field k. There is a unique multiplicative choice of representatives $\zeta : k \to \mathcal{O}$.*

Thus, the main issues we deal with in this chapter are those pertaining to imperfection and inseparability. Even without the hypothesis that l is perfect, there are classical results:

Lemma 3.0.2 (cf [Ser79])**.** *Let $\mathcal{O}$ be a complete $\mathbb{Z}$-valuation ring of equal characteristic and residue field k. Let $E_0 \subseteq k$ and let $\zeta_0 : E_0 \to \mathcal{O}$ be a partial section of the residue map. If k/E_0 is separable then ζ_0 may be extended to a section of the residue map $\zeta : k \to \mathcal{O}$.*

For this chapter we have two main settings. In the first we consider an equicharacteristic henselian nontrivially valued field (K, v), with valuation ring $\mathcal{O}_v$ and residue field k. In the second setting we consider a discrete valuation ring B (i.e. corresponding to a $\mathbb{Z}$-valuation) of residue characteristic p, with a uniformizer π, maximal ideal $\mathfrak{m}$, residue field l, and field of fractions L. Both B and L are equipped with the $\mathfrak{m}$**-adic** (or π**-adic**) topology, metric $|\cdot|_\pi$, and valuation v_π. We suppose that the corresponding valuation v is either of equal characteristic p, or is finitely ramified of initial ramification $\leq e$, so in that case $v_\pi(\pi) \leq v_\pi(p) \leq v_\pi(\pi^e)$. We do not yet assume B to be complete, but later it will be important. Using the conventions and notation of section 2.1, we let $k \subseteq l$ be a subfield of imperfection degree[1] $\mathfrak{i} = \mathfrak{imp}(k)$. Let $B^k_{\{\!\{p\}\!\}}$ be the set of choices of representatives $s : b \to B$ for p-bases $b \in k_{[\![p]\!]}$ of k. We allow the domain of each s to be extended multiplicatively to include all the p-monomials b^I in b, for $I \in p^{[\mathfrak{i}]}$, by writing $s(b^I) = \prod_{\alpha<\mathfrak{i}} s(b_\alpha)^{i_\alpha}$. Note that we do not in general suppose l/k to be separable.

[1] Recall that the **imperfection degree** of k is the cardinality of a maximal p-independent subset of k. This is equal to $\log_p[k : k^{(p)}]$ when this is finite, or simply $[k : k^{(p)}]$ when this is infinite.

3.1 Sections of residue maps in equal characteristic HVFs

We work in the first setting: (K,v) is an equal characteristic henselian nontrivially valued field. The first lemma is well-known, for example it appears in [Kuh11], but for the sake of completeness we gave a proof in [AF16]. It allows us to extend partial sections of the residue map through separably generated field extensions.

Lemma 3.1.1 ([AF16, Lemma 2.3]). *Let (K,v) be an equicharacteristic henselian valued field. Let $E_0 \subseteq E \subseteq Kv$ and let $\zeta_0 : E_0 \to K$ be a partial section of the residue map. If E/E_0 is separably generated then ζ_0 may be extended to a partial section of the residue map $\zeta : E \to K$.*

Corollary 3.1.2. *Every henselian valued field of equal characteristic zero admits a section of the residue map.*

Partial sections of the residue map can be extended through arbitrary separable extensions at the cost of passing to an elementary extension, as shown in the next proposition.

Lemma 3.1.3 ([ADF23, Lemma 4.4]). *Let (K,v) be an equicharacteristic henselian valued field. Let $E_0 \subseteq E \subseteq Kv$ and let $\zeta_0 : E_0 \to K$ be a partial section of the residue map. If E/E_0 is separable then there exists an elementary extension $(K^*,v^*) \succeq (K,v)$ and an extension of ζ_0 to a partial section of the residue map $\zeta : E \to K^*$.*

Iterating this construction, we may obtain a full section of the residue map, under suitable separability assumptions, again at the cost of passing to a sufficiently saturated elementary extension.

Lemma 3.1.4 ([ADF23, Proposition 4.5]). *Let (K,v) be an equicharacteristic henselian valued field. Let $E_0 \subseteq Kv$ and let $\zeta_0 : E_0 \to K$ be a partial section of the residue map. If Kv/E_0 is separable then there exists an elementary extension $(K^*,v^*) \succeq (K,v)$ and a section of the residue map $\zeta : K^*v^* \to K^*$ that extends ζ_0.*

So far, so good: the preceding lemmas will be important in the arguments of chapter 4. For a valuation v, we denote by v^+ the finest proper coarsening of v, when such a valuation exists. In particular v^+ exists when v is discrete. If v is equal characteristic then so is v^+, and if v is henselian then so is v^+. The following two lemmas were part of a discarded approach to proving the main theorem in [ADF23], but retain an interest of their own and will also be used in a very slightly new presentation that we present in chapter 4.

Lemma 3.1.5. *Let (K,v) be an equicharacteristic henselian valued field with uniformizer π_K, and let $k_0 \subseteq Kv^+$ be a subfield of the residue field of v^+. Suppose that Kv^+/k_0 is separable and that $\xi_0 : k_0 \longrightarrow K$ is a partial section of the residue map of v^+. There exists an elementary extension $(K^*,v^*) \succeq (K,v)$ that admits a partial section $\xi : Kv^+ \longrightarrow K^*$ of the residue map of v^{*+} that extends ξ_0.*

Proof. We expand (K,v) by adding the valuation v^+ to obtain the structure (K,v^+,v). By Lemma 3.1.3 (perhaps using an ultraproduct instead) there is an elementary extension $(K^*,w^*,v^*) \succeq (K,v^+,v)$ that admits a partial section $\xi : Kv^+ \longrightarrow K^*$ of the residue map of w^* that extends ξ_0. Note that π_K is also a uniformizer of v^*, and in particular v^{*+} exists. Also Kv^+ is a subfield both of K^*w^* and of K^*v^{*+}: more precisely, the valuation v^{*+} induces $\bar{v}^{*+}$ on K^*w^*, and the restriction of the residue map of $\bar{v}^{*+}$ to Kv^+ is an isomorphism. Therefore ξ is simultaneously a partial section of the residue maps of w^* and of v^{*+}, and of course it still extends ξ_0. This is illustrated in Figure 3.1. □

Lemma 3.1.6. *Let (K,v) be an equicharacteristic henselian valued field with uniformizer π_K, and let $k_0 \subseteq Kv^+$ be a subfield of the residue field of v^+. Suppose that Kv^+/k_0 is separable and that $\xi_0 : k_0 \longrightarrow K$ is a partial section of the residue map of v^+. There is an elementary extension $(K^*,v^*) \succeq (K,v)$ that admits a section $\xi_* : K^*v^{*+} \longrightarrow K^*$ of the residue map of v^{*+} that extends ξ_0.*

$$
\begin{tikzcd}
K^* \arrow[r] \arrow[d, dash] & K^* w^* \arrow[r] \arrow[dr, dash] & K^* v^{*+} \arrow[r] \arrow[d, dash] & K^* v^* \arrow[d, dash] \\
K \arrow[rr] & & Kv^+ \arrow[r] \arrow[d, dash] \arrow[ull, dotted, "\xi"'] & Kv \\
& & k_0 \arrow[ull, dotted, "\xi_0"] &
\end{tikzcd}
$$

Figure 3.1: Illustration of Lemma 3.1.5.

Proof. We construct a chain $(K_\alpha, v_\alpha, k_\alpha, \xi_\alpha)_{\alpha \le \omega}$ of quadruples, where each (K_α, v_α) is an equicharacteristic henselian valued field, k_α is a subfield of the residue field $K_\alpha v_\alpha^+$ such that $K_\alpha v_\alpha^+/k_\alpha$ is separable, and $\xi_\alpha : k_\alpha \longrightarrow K_\alpha$ is a partial section of the residue map of v_α^+.

We begin the chain with the quadruple $(K_0, v_0, k_0, \xi_0) := (K, v, k_0, \xi_0)$. Note that Kv_0^+/k_0 is separable by assumption. Each $(K_{\alpha+1}, v_{\alpha+1}, k_{\alpha+1}, \xi_{\alpha+1})$ is constructed by an application of Lemma 3.1.5 to (K_α, v_α) and the partial section $\xi_\alpha : k_\alpha \longrightarrow K_\alpha$ of the residue map of v_α^+, by which we obtain an extension $\xi_{\alpha+1}$ of ξ_α to the residue field $k_{\alpha+1} := K_\alpha v_\alpha^+$. Note that $(K_{\alpha+1}, v_{\alpha+1})$ is an elementary extension of (K_α, v_α). It follows that $K_{\alpha+1}v_{\alpha+1}/K_\alpha v_\alpha$ is separable. By the structure theorem for complete discretely valued fields in equicharacteristic [Ser79, II, §4, Theorem 2], also $K_{\alpha+1}v_{\alpha+1}^+/K_\alpha v_\alpha^+$ is separable. At limit ordinals we take a union, i.e. a direct limit. At stage ω_1, the direct limit $(K_{\omega_1}, v_{\omega_1}, k_{\omega_1}, \xi_{\omega_1})$ is such that $(K_{\omega_1}, v_{\omega_1})$ is an elementary extension of $(K_0, v_0) = (K, v)$ that is equipped with a section $\xi_{\omega_1} : k_{\omega_1} \longrightarrow K_{\omega_1}$ of the residue map of $v_{\omega_1}^+$, and also that $k_{\omega_1} = \bigcup_{\alpha<\omega_1} k_\alpha$ is the residue field $K_{\omega_1}v_{\omega_1}^+$ of $v_{\omega_1}^+$. This is illustrated in Figure 3.1. Since the cofinality of ω_1 is ω_1, v_{ω_1} induces a complete $\mathbb{Z}$-valuation on $K_{\omega_1}v_{\omega_1}^+$. $\square$

$$
\begin{tikzcd}
(K_{\omega_1}, v_{\omega_1}) \arrow[r, dashed] \arrow[d, dash, dashed] & K_{\omega_1} v_{\omega_1}^+ \arrow[r, dashed] \arrow[d, dash, dashed] \arrow[l, bend right, dotted, "\xi_{\omega_1}"'] & K_{\omega_1} v_{\omega_1} \arrow[d, dash, dashed] \\
(K_{\alpha+1}, v_{\alpha+1}) \arrow[r, dashed] \arrow[d, dash] & K_{\alpha+1} v_{\alpha+1}^+ \arrow[r, dashed] \arrow[d, dash] & K_{\alpha+1} v_{\alpha+1} \arrow[d, dash] \\
(K_\alpha, v_\alpha) \arrow[r, dashed] \arrow[d, dash, dashed] & K_\alpha v_\alpha^+ \arrow[r, dashed] \arrow[d, dash, dashed] \arrow[ul, dotted, "\xi_{\alpha+1}"'] & K_\alpha v_\alpha \arrow[d, dash, dashed] \\
(K_1, v_1) \arrow[r, dashed] \arrow[d, dash] & K_1 v_1^+ \arrow[r, dashed] \arrow[d, dash] & K_1 v_1 \arrow[d, dash] \\
(K_0, v_0) \arrow[r, dashed] & K_0 v_0^+ \arrow[r, dashed] \arrow[d, dash] \arrow[ul, dotted, "\xi_1"'] & K_0 v_0 \\
& k_0 \arrow[ul, dotted, "\xi_0"'] &
\end{tikzcd}
$$

Figure 3.2: Illustration of Lemma 3.1.6.

3.2 Representatives of residue fields in CDVRs

The material in this section might appear in [AF26+b, AJ26+], which at the present time are work-in-progress.

We adopt the second setting: B is a $\mathbb{Z}$-valuation ring of residue characteristic p, with field of fractions L, corresponding to a valuation v with residue field l and maximal ideal $\mathfrak{m}$. Thus either B is also of characteristic p or B is of characteristic 0, and so v is finitely ramified with initial ramification bounded by some $e \in \mathbb{N}$, say. This is the setting of DVRs, i.e. discrete valuation rings. A CDVR is a complete discrete valuation ring.

Throughout, we write $\lambda_{k,b,I}$ for the function $k^{(p)}(b) \to k$ otherwise denoted λ_I^b, as for example in section 2.1.2. The reason is simple: there are two fields of characteristic p in play, namely k and l, and it is important to specify which one we mean.

Our main aim in this section is to establish the following theorem.

Theorem 3.2.1 (S-maps)**.** *Let B be a CDVR. Let $k \subseteq l$ be a subfield with a p-basis $b \in k_{[\![p]\!]}$, and let $s : b \to B$ be a choice of representatives. There exists a unique map of representatives $k \to B$ that extends s and which satisfies*

$$S(a) = \sum_{I \in p^{[\mathrm{i}]}} S(b)^I S(\lambda_{k,b,I}(a))^p, \tag{S}$$

for all $a \in k$. Moreover, if B is of characteristic p, then S is a ring morphism, and thus a section of the residue map.

The proof is assembled after Proposition 3.2.10.

The following is a version of the usual Teichmüller Lemma, really coming from Fermat's Little Theorem.

Lemma 3.2.2. *For $a, b \in B$ and $n > 0$, if $a \equiv b \pmod{\mathfrak{m}^n}$ then $a^p \equiv b^p \pmod{\mathfrak{m}^{1+n}}$. If $\mathrm{char}(B) = p$ then in fact $a \equiv b \pmod{\mathfrak{m}^n}$ implies $a^p \equiv b^p \pmod{\mathfrak{m}^{pn}}$.*

Remark 3.2.3. By Lemma 3.2.2, when B is complete, there is a unique of choice of p^∞-**representatives** $s_{(p^\infty)} : l^{(p^\infty)} \to B^{(p^\infty)}$, which is itself a partial section of the residue map.

In the "Level $n+1$" definition of $\mathrm{U}_{n+1,s}(a)$, below, we need a notion of an infinite sum of subsets of B.

Definition 3.2.4. Let $(A_j)_{j\in J}$ be an indexed family of subsets of a group, written additively. We define $\sum_{j\in J} A_j$ to be the set of sums $\sum_{j\in J} a_j$, where $a_j \in A_j$, such that a_j is nonzero for only finitely many $j \in J$.

Remark 3.2.5. $\sum_{j\in J} A_j$ is nonempty if and only if A_j contains 0 for all but finitely many $j \in J$.

In the following definition and arguments we have adapted quite closely [Ser79, Proposition 8] to this setting.

Definition 3.2.6 (U-sets)**.** We define a function

$$\begin{aligned} \mathrm{U}^{k,B} : (\omega \cup \{\infty\}) \times B^k_{\{\!\{p\}\!\}} \times k &\to \mathcal{P}(B) \\ (n, s, a) &\mapsto \mathrm{U}^{k,B}_{n,s}(a) := \mathrm{U}^{k,B}(n, s, a). \end{aligned}$$

Usually k and B are understood, and so omitted from notation to minimize clutter. This definition goes by recursion on n, uniformly in s and a.

- (**"Level** 0") We define $\mathrm{U}_{0,s}(a) := \mathrm{res}^{-1}(a)$, where res is the residue map $B \to l$.

- (**"Level** $n+1$**"**) We define

$$\mathrm{U}_{n+1,s}(a) := \sum_{I \in p^{[\mathrm{i}]}} s(b^I)\mathrm{U}_{n,s}(\lambda_{k,b,I}(a))^{(p)}.$$

- (**"Level** ∞**"**) We define

$$\mathrm{U}_{\infty,s}(a) := \bigcap_{n \in \mathbb{N}} \mathrm{U}_{n,s}(a).$$

A $\lambda(s)$-**representative** of a is any element of $\mathrm{U}_{\infty,s}(a)$.

Lemma 3.2.7 (Level $n < \infty$)**.** *For all $s \in B^k_{\{\!\{p\}\!\}}$ and $a \in k$, we have*

(a) $\mathrm{U}_{n,s}(a)$ *is nonempty,*

(b) $\mathrm{U}_{n+1,s}(a) \subseteq \mathrm{U}_{n,s}(a)$*, and*

(c) $\mathrm{U}_{n,s}(a) - \mathrm{U}_{n,s}(a) \subseteq \mathfrak{m}^{1+n}$.

Proof. We proceed by induction on n, working uniformly in s and a. To that end, we write $(a)_n$, $(b)_n$, $(c)_n$, for the statements (a), (b), (c), at level n. For $n = 0$, we have $\mathrm{U}_{0,s}(a) = \mathrm{res}^{-1}(a)$, and so $(a)_0$ and $(c)_0$ are trivial. For $(b)_0$ we let $c \in \mathrm{U}_{1,s}(a)$. So there exist elements $l_I \in \mathrm{U}_{0,s}(\lambda_{k,b,I}(a)) = \mathrm{res}^{-1}(\lambda_{k,b,I}(a))$, for $I \in p^{[\mathrm{i}]}$, such that only finitely many are nonzero, and

$$c = \sum_{I \in p^{[\mathrm{i}]}} s(b^I) l_I^p.$$

Taking the residue, we have

$$\mathrm{res}(c) = \sum_{I \in p^{[\mathrm{i}]}} b^I \lambda_{k,b,I}(a)^p,$$

which equals a by Equation (Λ). Therefore indeed $c \in \mathrm{U}_{0,s}(a)$.

For the inductive step we let $n \in \mathbb{N}$ be arbitrary and suppose that $(a)_n$, $(b)_n$, and $(c)_n$ are true.

For $(a)_{n+1}$, by the inductive hypothesis $(a)_n$, the set $\mathrm{U}_{n,s}(\lambda_{k,b,I}(a))$ is nonempty, for each $I \in p^{[\mathrm{i}]}$. Moreover, by Remark 2.1.3, $\lambda_{k,b,I}(a)$ is zero for all but finitely many $I \in p^{[\mathrm{i}]}$. Since $\mathrm{U}_{n,s}(0)$ contains 0, for all $n \in \mathbb{N}$, and so by Remark 3.2.5, $\mathrm{U}_{n+1,s}(a)$ is nonempty.

For $(b)_{n+1}$, note that $\lambda_{k,b,I}(a) \in k$, for each $I \in p^{[\mathrm{i}]}$. By inductive hypothesis $(b)_n$, we have

$$\mathrm{U}_{n+1,s}(\lambda_{k,b,I}(a)) \subseteq \mathrm{U}_{n,s}(\lambda_{k,b,I}(a)),$$

for all $I \in p^{[\mathrm{i}]}$. Therefore

$$\begin{aligned}
\mathrm{U}_{n+2,s}(a) &= \sum_{I \in p^{[\mathrm{i}]}} s(b^I)\mathrm{U}_{n+1,s}(\lambda_{k,b,I}(a))^{(p)} \\
&\subseteq \sum_{I \in p^{[\mathrm{i}]}} s(b^I)\mathrm{U}_{n,s}(\lambda_{k,b,I}(a))^{(p)} \\
&= \mathrm{U}_{n+1,s}(a),
\end{aligned}$$

since each $\lambda_{k,b,I}(a)$ is an element of k.

For $(c)_{n+1}$, again note that $\lambda_{k,b,I}(a) \in k$, for each $I \in p^{[\mathrm{i}]}$. By inductive hypothesis $(c)_n$, we have

$$\mathrm{U}_{n,s}(\lambda_{k,b,I}(a)) - \mathrm{U}_{n,s}(\lambda_{k,b,I}(a)) \subseteq \mathfrak{m}^{1+n},$$

and so by Lemma 3.2.2 we have

$$\mathrm{U}_{n,s}(\lambda_{k,b,I}(a))^{(p)} - \mathrm{U}_{n,s}(\lambda_{k,b,I}(a))^{(p)} \subseteq \mathfrak{m}^{2+n},$$

for $I \in p^{[\mathrm{i}]}$. We have

$$\begin{aligned}
\mathrm{U}_{n+1,s}(a) - \mathrm{U}_{n+1,s}(a) &= \sum_{I \in p^{[\mathrm{i}]}} s(b^I)\Big(\mathrm{U}_{n,s}(\lambda_{k,b,I}(a))^{(p)} - \mathrm{U}_{n,s}(\lambda_{k,b,I}(a))^{(p)}\Big) \\
&\subseteq \sum_{I \in p^{[\mathrm{i}]}} s(b^I)\mathfrak{m}^{2+n} \\
&\subseteq \mathfrak{m}^{2+n},
\end{aligned}$$

as required to prove $(c)_{n+1}$. □

Lemma 3.2.8 (Level ∞). *For all $s \in B^k_{\{\!\{p\}\!\}}$ and $a \in k$, we have*

- **(a)** $\mathrm{U}_{\infty,s}(a) \subseteq \mathrm{U}_{n,s}(a)$,
- **(b)** $\mathrm{U}_{\infty,s}(a)$ *is either empty or is a singleton, and*
- **(c)** *if B is complete then* $\mathrm{U}_{\infty,s}(a)$ *is nonempty.*

Proof. For (a), this is definitional. For (b) and (c), we observe first that the valuation induces the $\mathfrak{m}$-adic topology on B, which is a metric topology. Then $\mathrm{U}_{\infty,s}(a)$ is the intersection of a descending chain of nonempty sets. By Lemma 3.2.7, any choice of a sequence (a_n), with $a_n \in \mathrm{U}_{n,s}(a)$ for $n \in \mathbb{N}$, is a Cauchy sequence, which can have at most one limit in B. Indeed any element of $\mathrm{U}_{\infty,s}(A)$ is even the common limit of all such sequences. □

By this lemma, there is at most one choice of $\lambda(s)$-representative for each $a \in k$. For the rest of this section, we make the following assumption: **We suppose B to be complete.**

Definition 3.2.9. For $s \in B^k_{\{\!\{p\}\!\}}$, we define the map of $\lambda(s)$-**representatives** $\mathrm{S}^{k,B}_s : k \to B$ by declaring $\mathrm{S}^{k,B}_s(a)$ to be the unique element of $\mathrm{U}_{\infty,s}(a)$.

For each $s \in B^k_{\{\!\{p\}\!\}}$, the map $\mathrm{S}^{k,B}_s$ is defined and has domain k, by Lemma 3.2.8. Usually we omit k, B. Right away we note that S_s really is a partial right inverse of the residue map, since in particular $\mathrm{S}_s(a) \in \mathrm{U}_{0,s}(a) = \mathrm{res}^{-1}(a)$, for each $a \in k$.

Proposition 3.2.10. *For all $s \in B^k_{\{\!\{p\}\!\}}$ and $a \in k$, we have*

$$\mathrm{S}_s(a) = \sum_{I \in p^{[\mathrm{i}]}} s(b^I)\mathrm{S}_s(\lambda_{k,b,I}(a))^p. \tag{S1}$$

Proof. For each $n \in \mathbb{N}$, we have

$$\sum_{I \in p^{[\mathrm{i}]}} s(b^I)\mathrm{S}_s(\lambda_{k,b,I}(a))^p \in \sum_{I \in p^{[\mathrm{i}]}} s(b^I)\mathrm{U}_{n,s}(\lambda_{k,b,I}(a))^{(p)} = \mathrm{U}_{n+1,s}(a).$$

Therefore

$$\sum_{I \in p^{[\mathrm{i}]}} s(b^I)\mathrm{S}_s(\lambda_{k,b,I}(a))^p \in \bigcap_{n \in \mathbb{N}} \mathrm{U}_{n,s}(a) = \mathrm{U}_{\infty,s}(a) = \{\mathrm{S}_s(a)\},$$

which gives the required equality. □

Remark 3.2.11. An easy argument now proves that for all $s \in B^k_{\{\!\{p\}\!\}}$, $a \in k$, and $n \in \mathbb{N}$, we have

$$\mathrm{S}_s(a) = \sum_{I \in p^{n[\mathrm{i}]}} s(b^I)\mathrm{S}_s(\lambda_{k,b,I}(a))^{p^n}, \tag{S2}$$

where for $I \in p^{n[\mathrm{i}]}$, the coefficients $\lambda_{k,b,I}(a)$ are defined so that $a = \sum_{I \in p^{n[\mathrm{i}]}} b^I \lambda_{k,b,I}(a)^{p^n}$.

The following two corollaries are straight from Proposition 3.2.10.

Corollary 3.2.12 (Frobenius). *For all $s \in B^k_{\{\!\{p\}\!\}}$ and $a \in k$, we have $\mathrm{S}_s(a^p) = \mathrm{S}_s(a)^p$.*

Corollary 3.2.13. *The map S_s of $\lambda(s)$-representatives extends s, i.e. $s(b^I) = \mathrm{S}_s(b^I)$, for all $I \in p^{[\mathrm{i}]}$.*

Proof of Theorem 3.2.1. Let $s \in B^k_{\{\!\{p\}\!\}}$, and define S_s as in Definition 3.2.9. The uniqueness of S_s comes straight from the definition, which in turns uses Lemma 3.2.8**(b,c)**. The equation (S1), combined with Corollary 3.2.13 gives (S). If B has characteristic p, both additivity and multiplicatively follows by induction from (S). ☐

There are two questions are potential applications.

Question 3.2.14 (Problem 9.1)**.** Use the S-maps to give a clear account of the structure of the automorphism groups of $C(k)$ and of $k(\!(t)\!)$.

Question 3.2.15 (Problem 9.2)**.** How many 1-types are there in $\mathbb{F}_p(\!(t)\!)$ of elements of value n?

Chapter 4

Existential theories of equal characteristic HVFs

Widespread and rapid changes in the atmosphere, ocean, cryosphere and biosphere have occurred. Human-caused climate change is already affecting many weather and climate extremes in every region across the globe. This has led to widespread adverse impacts and related losses and damages to nature and people. Vulnerable communities who have historically contributed the least to current climate change are disproportionately affected.

Climate Change 2023, Synthesis Report, Summary for Policymakers, IPCC

Our aims in this section are firstly to redevelop the main results from [AF16] and [ADF23], and then to take advantage of the present opportunity to extend them just a little.

We recall from 2.3 several important first-order languages: $\mathfrak{L}_{\mathrm{ring}} = \{+, \cdot, -, 0, 1\}$ is the language of rings, $\mathfrak{L}_{\mathrm{oag}} = \{+, -, 0, \infty, <\}$ is the language of (totally) ordered abelian groups expanded by an infinite element, and $\mathfrak{L}_{\mathrm{val}}$ is the three-sorted language of valued fields, which has three sorts $\mathbf{K}$, $\mathbf{k}$, and $\mathbf{\Gamma}$, which are equipped with $\mathfrak{L}_{\mathrm{ring}}$, $\mathfrak{L}_{\mathrm{ring}}$, and $\mathfrak{L}_{\mathrm{oag}}$, respectively, with two connecting maps $\mathbf{val} : \mathbf{K} \to \mathbf{\Gamma}$ and $\mathbf{res} : \mathbf{K} \to \mathbf{k}$. Finally, let $\mathfrak{L}_{\mathrm{val}}(\varpi) = \mathfrak{L}_{\mathrm{val}} \cup \{\varpi\}$ be the expansion of $\mathfrak{L}_{\mathrm{val}}$ by a constant symbol ϖ in the sort $\mathbf{K}$, which is intended to be interpreted by a uniformizer. Let $\mathbf{H}^{\mathrm{e}\prime}$ be the $\mathfrak{L}_{\mathrm{val}}$-theory of equicharacteristic henselian non-trivially valued fields, and let $\mathbf{H}^{\mathrm{e},\varpi}$ be $\mathfrak{L}_{\mathrm{val}}(\varpi)$-theory expanding $\mathbf{H}^{\mathrm{e}\prime}$ by asserting that the interpretation of ϖ is a uniformizer.

The results we wish to recover are the following.

Theorem 4.0.1 ([AF16])**.** *Let (K, v) be an equicharacteristic henselian nontrivially valued field. Then the $\mathfrak{L}_{\mathrm{val}}$-theory $\mathrm{Th}_{\exists}(K, v)$ is axiomatized by the theory $\mathbf{H}^{\mathrm{e}\prime}$ of equicharacteristic henselian nontrivially valued fields, together with the $\mathfrak{L}_{\mathrm{ring}}$-theory $\mathrm{Th}_{\exists}(Kv)$ imposed on the residue field. In particular $\mathrm{Th}_{\exists}(K, v)$ is decidable if and only if $\mathrm{Th}_{\exists}(Kv)$ is decidable.*

Theorem 4.0.2 ([ADF23])**.** *Suppose* **(R4)**[1]*. Let (K, v, π) be an equicharacteristic henselian valued field with uniformizer π. Then the $\mathfrak{L}_{\mathrm{val}}(\varpi)$-theory $\mathrm{Th}_{\exists}(K, v, \pi)$ is axiomatized by the theory $\mathbf{H}^{\mathrm{e},\varpi}$ of equicharacteristic henselian valued fields with uniformizer, together with the $\mathfrak{L}_{\mathrm{ring}}$-theory $\mathrm{Th}_{\exists}(Kv)$ imposed on the residue field. In particular $\mathrm{Th}_{\exists}(K, v, \pi)$ is decidable if and only if $\mathrm{Th}_{\exists}(Kv)$ is decidable.*

In his thesis [Kar22], Kartas also proved related results conditional on variants of Local Uniformization. See also [Kar21, Kar23].

[1] See section 4.1

Remark 4.0.3 (cf [AF26+a]). We might instead work in a one-sorted language valued fields $\mathfrak{L}_O = \mathfrak{L}_{\text{ring}} \cup \{O\}$, where O is a unary predicate and a valued field (K, v) gives rise to an $\mathfrak{L}_O$-structure by interpreting O by the valuation ring $\mathcal{O}_v$. As there is a uniform biinterpretation Γ between the $\mathfrak{L}_{\text{val}}$-structure (K, v) and the $\mathfrak{L}_O$-structure $(K, \mathcal{O}_v)$ in which each defining formula φ_Γ (see section 2.4) is equivalent modulo the theory of valued fields to both an existential and a universal formula, all our results for the $\mathfrak{L}_{\text{val}}$-fragment $\text{Sent}_\exists(\mathfrak{L}_{\text{val}})$ carry over to the $\mathfrak{L}_O$-fragment $\text{Sent}_\exists(\mathfrak{L}_O)$.

Remark 4.0.4 (cf [AF26+a]). Since we are going to discuss results from [AF16] we should point out that the definition of $\exists$-*sentence* used there, namely "logically equivalent to a formula in prenex normal form with only existential quantifiers" is not suitable for the arguments about decidability given there, nor for our use here. In fact, in general it is not possible to decide whether a given sentence is an $\exists$-sentence in that sense: For example, if $\mathfrak{L} = \{R\}$ with a binary predicate symbol R, and $T = \emptyset$, then $T^\vdash$ is undecidable since it encodes graph theory (which is undecidable, see e.g. [FJ08, Corollary 28.5.3]), but $T_\exists$ is trivially decidable, which implies that it is not possible to decide whether a given sentence φ is equivalent to some $\psi \in \text{Sent}_\exists(\mathfrak{L})$, as such a ψ could be found effectively. However, this poses no problems for either the results in [AF16] or our application here, as one can simply replace the definition of $\exists$-sentence by being an element of what we here call the existential fragment $\text{Sent}_\exists$, and then everything goes through.

Sketch of original proof for $\mathbf{H}^{e\prime}$. We sketch the original argument from [AF16] for Theorem 4.0.1 that goes via the model theory of tame valued fields, from [Kuh16].

Sketch proof of Theorem 4.0.1. Let $(K, v) \models \mathbf{H}^{e\prime}$ have residue field k. The aim is to show that (K, v) and $k(\!(t)\!)$ have the same existential theories. Firstly, there is an elementary extension $(K^*, v^*) \succeq (K, v)$ that admits a partial section $\zeta_0 : k \to K^*$. Choosing any nonzero π in the maximal ideal of v^*, we obtain an $\mathfrak{L}_{\text{val}}(\varpi)$-embedding $\zeta_1 : (k(t)^h, v_t) \to (K^*, v^*, \pi)$. Therefore $\text{Th}_\exists(k(t)^h, v_t) \subseteq \text{Th}_\exists(K, v)$.

In the other direction, by [KPR86], there exists a "tamification": a tame valued field $(K_1, v_1) \supseteq (K, v)$ with $K_1 v_1 = (Kv)^{\text{perf}}$ and $v_1 K_1 = p^{-\infty} vK$. This tamification in turn admits an algebraic extension $(K_2, v_2) \supseteq (K_1, v_1)$ that is also tame and is such that $K_2 v_2 = K_1 v_1$ and $v_2 K_2$ is the divisible hull of $v_1 K_1$. By the Ax–Kochen/Ershov principle for tame valued fields, $\text{Th}(K_2, v_2) = \text{Th}(k^{\text{perf}}(\!(t^{\mathbb{Q}})\!), v_t)$. So far this shows that $\text{Th}_\exists(K, v) = \text{Th}_\exists(k^{\text{perf}}(\!(t^{\mathbb{Q}})\!), v_t)$. Now, the relative algebraic closure $(k^{\text{perf}}(\!(t)\!)^{\mathbb{Q}}, v_t)$ of $k^{\text{perf}}(\!(t)\!)$ in $k^{\text{perf}}(\!(t^{\mathbb{Q}})\!)$ is in fact an elementary subextension of $(k^{\text{perf}}(\!(t^{\mathbb{Q}})\!), v_t)$, by the "going down" principle for tame valued fields and the elementary extension form of the Ax–Kochen/Ershov principle for tame valued fields. Therefore $\text{Th}(k^{\text{perf}}(\!(t)\!)^{\mathbb{Q}}, v_t) = \text{Th}(k^{\text{perf}}(\!(t^{\mathbb{Q}})\!), v_t)$. Next, we observe that $k^{\text{perf}}(\!(t)\!)^{\mathbb{Q}}$ is the direct limit of a family of totally ramified extensions of $k^{\text{perf}}(\!(t)\!)$, in particular they all have residue field k^{perf}. By the general structure theory of complete $\mathbb{Z}$-valued fields, each of these finite extensions is isomorphic to $k^{\text{perf}}(\!(t)\!)$, thus $\text{Th}_\exists(k^{\text{perf}}(\!(t)\!), v_t) = \text{Th}_\exists(k^{\text{perf}}(\!(t)\!)^{\mathbb{Q}}, v_t)$. By a theorem of Ershov, $(k^{\text{perf}}(t)^h, v_t)$ is existentially closed in $(k^{\text{perf}}(\!(t)\!), v_t)$, and therefore $\text{Th}_\exists(k^{\text{perf}}(t)^h, v_t) = \text{Th}_\exists(k^{\text{perf}}(\!(t)\!), v_t)$. Finally, $(k^{\text{perf}}(t)^h, v_t)$ is a direct limit of valued fields isomorphic to $(k(t)^h, v_t)$. Therefore $\text{Th}_\exists(k(t)^h, v_t) = \text{Th}_\exists(k^{\text{perf}}(t)^h, v_t)$. Putting all of this together gives the required inclusion $\text{Th}_\exists(K, v) \subseteq \text{Th}_\exists(k(t)^h, v_t)$. This shows that the $\exists$-theory of (K, v) depends only on its residue field. That is, all $(K, v) \models \mathbf{H}^{e\prime}$ with residue field k have the same existential theory. Next, if k and k' are elementarily equivalent, then there are isomorphic ultrapowers $k^{\mathcal{U}} \cong (k')^{\mathcal{U}}$, which yields an isomorphism $k(\!(t)\!)^{\mathcal{U}} \cong k'(\!(t)\!)^{\mathcal{U}}$. This shows that the $\exists$-theory of (K, v) depends only on the theory of its residue field. It remains to show that if l models the $\exists$-theory of k, then $l(\!(t)\!)$ models the $\exists$-theory of $k(\!(t)\!)$. By the method of diagrams and compactness, there is an elementary extension $l^* \succeq l$ and an embedding $k(\!(t)\!) \to l^*(\!(t)\!)$, from which we deduce that $l^*(\!(t)\!)$ models the $\exists$-theory of $k(\!(t)\!)$. This shows that the $\exists$-theory of (K, v) depends only on the $\exists$-theory of its residue field. $\square$

Remark 4.0.5. If we are concerned only with the existential theories of power series fields $k(\!(t)\!)$ for perfect k, then a small piece of the previous argument is enough: $k(\!(t)\!)$ has the same existential theory as $k(\!(t)\!)^{\mathbb{Q}}$, since the latter is a direct limit of isomorphic copies of the former. By the "going down" principle, and the Ax–Kochen/Ershov principle for tame valued fields, we have $k(\!(t)\!)^{\mathbb{Q}} \preceq k(\!(t^{\mathbb{Q}})\!)$. Therefore the existential theory of $k(\!(t)\!)$ is $k(\!(t^{\mathbb{Q}})\!)$, which is well understood. In particular, the full theory of $k(\!(t^{\mathbb{Q}})\!)$ is decidable if and only if the theory of k is decidable.

4.1 The hypothesis (R4) — "Non-Local Uniformization"

A field K is **large** if it is existentially closed in $K(\!(t)\!)$, see [Pop96, Pop10, BF13, Pop14] for background on large fields. To apply prove our main theorem of section 4.2, we will work under the following hypothesis, introduced in [ADF23, §2], and also called **non-local uniformization**:

(R4) Every large field K is existentially closed in every extension F/K for which there exists a valuation v on F/K with residue field $Fv = K$.

Here the existential closure is meant in the language of rings, and we note that a valuation v on F/K with residue field $Fv = K$ is one that corresponds to a K-rational K-place of F. As proved in [Kuh04, Theorem 13] and [Feh11, Lemma 9], **(R4)** is a consequence of Local Uniformization, which is in turn implied by Resolution of Singularities. See [ADF23, Proposition 2.3] for more detail on these implications, and for a proof that **(R4)** is equivalent to

(R3) For every field K and every nontrivial finitely generated extension F/K such that there exists a valuation v on F/K with residue field $Fv = K$, there also exists a valuation with value group $\mathbb{Z}$ which has that property.

Also see the discussion in [ADF23, Remark 2.4]. It is known that **(R4)** holds for perfect large fields K:

Proposition 4.1.1 ([Kuh04, Theorem 17])**.** *For any large perfect field K, and any extension F/K that admits a K-rational K-place, we have $K \preceq_\exists F$.*

For a ring C we also consider the hypothesis $\textbf{(R4)}_C$, which we define as **(R4)** restricted to large fields K that admit a ring homomorphism $C \to K$. Since **(R4)** holds when restricted to perfect fields K, in particular $(\mathrm{R4})_{\mathbb{Q}}$ holds.

$\textbf{(R4)}_C$ Every large field $K \supseteq C$ has the same existential $\mathfrak{L}_{\mathrm{ring}}(C)$-theory as every extension F/K for which there exists a valuation v on F/K with residue field $Fv = K$.

Proposition 4.1.2. *Let $K/k(\!(t)\!)$ be a field extension and suppose that $K/k(\!(t)\!)$ admits a $k(\!(t)\!)$-rational $k(\!(t)\!)$-place. Suppose that K is endowed with an $\mathfrak{L}$-structure. We then endow $k(\!(t)\!)$ is the induced $\mathfrak{L}$-structure. If $k(\!(t)\!) \preceq_\exists K$ in $\mathfrak{L}_{\mathrm{ring}}$ then $k(\!(t)\!) \preceq_\exists K$ in $\mathfrak{L}$.*

4.2 Strengthening

We hope to accomplish two new things:

A1 **Uniformity**: treat uniformly the hitherto separate cases of $\mathbf{H}^{e\prime}$ and $\mathbf{H}^{e,\varpi}$, the latter with a suitable version of the hypothesis **(R4)**, and in both cases allowing constants from an excellent $\mathbb{Z}$-valued (or trivially valued) subfield (C, u), again subject to suitable hypotheses.

A2 **Resplendence**: allow expansions $\mathfrak{L}_{\mathrm{k}}$ of the language of rings on the residue field, with corresponding $\mathfrak{L}$-theory R, subject to suitable hypotheses.

Let C be an $\mathfrak{L}_{\mathrm{val}}$-structure expanding an integral domain. In particular, C is equipped with a valuation u which will be equicharacteristic throughout. If u is nontrivial then it will be discrete, and we distinguish a uniformizer π_C. We consider the setting of C-fields, C-valued fields, etc., in which essentially every map is supposed to commute with the structure maps from C. More precisely, a C-**ring** is simply a ring R and a morphism $\xi_R : C \to R$ called the **structure morphism**. A C-**field** is a C-ring that is a field, and a morphism of C-rings is just a ring morphism that commutes with the structure morphisms. Analogously, when C is already equipped with a valuation u, a C-**valuation** is a valuation

v on a C-field F that restricts to the image of u on the image of C, i.e. $u = v \circ \xi_F$. A C-field F equipped with a C-valuation is called a **valued C-field**. All rings, fields, and valued fields, etc., in this section will be C-rings, C-fields, C-valued fields, etc., sometimes without explicit mention.

Setting 4.2.1. Let $\mathbf{H}^{\mathrm{e},C}$ be the $\mathfrak{L}_{\mathrm{val}}(C)$-theory of equicharacteristic henselian valued C-fields for which π_C is a uniformizer. Let $\mathfrak{L}_{\mathbf{k}}$ be an expansion of $\mathfrak{L}_{\mathrm{ring}}$ (still one-sorted), and let $\mathfrak{L}$ be the expansion of $\mathfrak{L}_{\mathrm{val}}(C)$ by $\mathfrak{L}_{\mathbf{k}}$ on the residue sort $\mathbf{k}$, i.e. $\mathfrak{L} = \mathfrak{L}_{\mathrm{val}}(C) + \iota_{\mathbf{k}}\mathfrak{L}_{\mathbf{k}}$. For each $\mathfrak{L}_{\mathbf{k}}$-theory R, denote by $\mathbf{H}^{\mathrm{e},C}(R)$ the $\mathfrak{L}$-theory $\mathbf{H}^{\mathrm{e},C} \cup \iota_{\mathbf{k}}R$. For each $\mathfrak{L}$-structure K, let $k_K = \mathbf{k}(K)$ be the reduct of K to the $\mathfrak{L}_{\mathbf{k}}$-structure on the sort $\mathbf{k}$.

4.2.1 Hypothesis (Υ)

We do not expect to get these strengthened results for free. We will require supplementary hypotheses, principally on the $\mathfrak{L}_{\mathbf{k}}$-theory R that we impose on the residue field. The key hypothesis is the following:

(Υ)

- (Υ**i**) **<u>either</u>** C is trivially valued and
  - **a.** either k_K is perfect,
  - **b.** or some N-th power of the Frobenius map $k_K \to k_K^{(p^N)}$ given by $x \mapsto x^{p^N}$, is an $\mathfrak{L}_{\mathbf{k}}$-isomorphism, where $k_K^{(p^N)}$ is the $\mathfrak{L}_{\mathbf{k}}$-substructure of k_K induced on the subset $\{x^{p^N} \mid x \in k_K\}$;
- (Υ**ii**) **<u>or</u>** (C, u) is a $\mathbb{Z}$-valued field with uniformizer π, with $\mathcal{O}_u$ excellent, and $(K, v)/(C, u)$ having no initial ramification (i.e. π is also a uniformizer of (K, v)).

4.2.2 Resplendent monotonicity for $\mathrm{H}^{\mathrm{e},C}$

With aims **A1** and **A2** in mind, in this chapter we prove the following theorem.

Theorem 4.2.2 (Resplendent monotonicity). *Suppose* (Υ) *and in case* (Υ**ii**) *suppose also* **(R4)**. *Let* $K, L \models \mathbf{H}^{\mathrm{e},C}$. *Then*

$$\underbrace{\mathrm{Th}_\exists(k_K) \subseteq \mathrm{Th}_\exists(k_L)}_{\textit{in } \mathfrak{L}_{\mathbf{k}}} \implies \underbrace{\mathrm{Th}_\exists(K) \subseteq \mathrm{Th}_\exists(L)}_{\textit{in } \mathfrak{L}}.$$

Remark 4.2.3. Theorem 4.0.1 now follows by taking $C = \mathbb{F}_p$ and $\mathfrak{L}_{\mathbf{k}} = \mathfrak{L}_{\mathrm{ring}}$. Also Theorem 4.0.2 follows by taking $C = \mathbb{F}_p(\pi)$, with the π-adic valuation $u = v_\pi$ and distinguished uniformizer π, and we take $\mathfrak{L}_{\mathbf{k}} = \mathfrak{L}_{\mathrm{ring}}$. Note that $\mathcal{O}_u$ is excellent.

Indeed, the other principal results of [AF16] and [ADF23] also follow from Theorem 4.2.2.

Recovering results for $\mathrm{H}^{\mathrm{e}\prime}$ and $\mathrm{H}^{\mathrm{e},\varpi}$.

Remark 4.2.4. The hypothesis on C and k in [AF16] was that either k is perfect or C is algebraic. This hypothesis is subsumed into (Υ).

The following statements are not resplendent, in that throughout $\mathfrak{L}_{\mathbf{k}} = \mathfrak{L}_{\mathrm{ring}}$. Thus $\mathfrak{L} = \mathfrak{L}_{\mathrm{val}}$ when discussing $\mathbf{H}^{\mathrm{e}\prime}$, and $\mathfrak{L} = \mathfrak{L}_{\mathrm{val}}(\varpi)$ when discussing $\mathbf{H}^{\mathrm{e},\varpi}$. As discussed in section 1.3, the two parts of the following theorem are not exactly to be found in either [AF16] or [ADF23], but are close restatements of theorems in those papers.

Theorem 4.2.5 ($\exists$-monotonicity, see [AF16, ADF23]).

- **(i)** *Let* $K, L \models \mathbf{H}^{\mathrm{e}\prime}$ *be* $\mathfrak{L}_{\mathrm{val}}$*-structures. Suppose that either* C *is algebraic or both* k_K *and* k_L *are perfect. We have* $\mathrm{Th}_\exists(k_K) \subseteq \mathrm{Th}_\exists(k_L) \implies \mathrm{Th}_\exists(K) \subseteq \mathrm{Th}_\exists(L)$.
- **(ii)** *Assume* **(R4)**. *Suppose that* C *is an equicharacteristic* $\mathbb{Z}$*-valued field with an excellent valuation ring. Let* $K, L \models \mathbf{H}^{\mathrm{e},\varpi}$ *be* $\mathfrak{L}_{\mathrm{val}}(\varpi)$*-structures. We have* $\mathrm{Th}_\exists(k_K) \subseteq \mathrm{Th}_\exists(k_L) \implies \mathrm{Th}_\exists(K) \subseteq \mathrm{Th}_\exists(L)$.

Proof. This is a special case of Theorem 4.2.2. As already mentioned, we take $\mathfrak{L}_{\mathrm{k}} = \mathfrak{L}_{\mathrm{ring}}$. For **(i)**, we aim to verify (Υ**i**), and indeed if k is perfect then (Υ**ia**) holds, otherwise if C is algebraic then in particular any finitely generated subfield of it is invariant under a sufficiently high power of Frobenius, and so (Υ**ib**) holds. For **(ii)**, we have exactly the hypotheses of (Υ**ii**). □

The results around "∃-completeness" and transfer of decidability now follow, in both $\mathbf{H}^{\mathrm{e}\prime}$ and $\mathbf{H}^{\mathrm{e},\varpi}$ contexts, with hypothesis **(R4)** in the latter, of course.

Theorem 4.2.6 (∃-completeness relative to **k**)**.** *Let k be a field.*

- **(i)** $\mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}_\exists(k))$ *is ∃-complete, i.e.* $\mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}_\exists(k))$ *already entails the existential theory of one of its models. Consequently, for existential $\mathfrak{L}_{\mathrm{val}}$-sentences φ, ψ, if* $\mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}_\exists(k)) \models \varphi \vee \psi$ *then either* $\mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}_\exists(k)) \models \varphi$ *or* $\mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}_\exists(k)) \models \psi$.
- **(ii)** *Assume* **(R4)**. *Then* $\mathbf{H}^{\mathrm{e},\varpi}(\mathrm{Th}_\exists(k))$ *is ∃-complete.*

Theorem 4.2.7 ($\mathbf{H}^{\mathrm{e}\prime}$ ∃-decidability transfer)**.** *Let k be a field.*

- **(i)** $\mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}_\exists(k))_\exists \simeq_{\mathrm{m}} \mathrm{Th}_\exists(k)$.
- **(ii)** *Assume* **(R4)**. *Then* $\mathbf{H}^{\mathrm{e},\varpi}(\mathrm{Th}_\exists(k))_\exists \simeq_{\mathrm{m}} \mathrm{Th}_\exists(k)$.

Remark 4.2.8. When k is perfect, the "finitely ramified" case in which we suppose that $(K, v)/(C, u)$ has finite initial ramification (instead of none), is easy.

Synthesis: a proof of monotonicity. We begin by recalling two classical lemmas.

Lemma 4.2.9. *Let (K, v) be an equicharacteristic henselian valued field, let $\pi \in \mathfrak{m}_v \setminus \{0\}$, and let $\zeta : k \to K$ be a partial section of the residue map defined on a subfield k of the residue field k_v. There is a unique extension of ζ to an $\mathfrak{L}_{\mathrm{val}}(C, \varpi)$-embedding $\varphi : (k(t)^h, v_t, t) \to (K, v, \pi)$. Moreover, φ induces ζ on the residue field.*

Lemma 4.2.10 (Ershov)**.** *For every field k, we have $(k(t)^h, v_t) \preceq_\exists (k(\!(t)\!), v_t)$ as an inclusion of $\mathfrak{L}_{\mathrm{val}}(C)$-structures for $C = k$.*

Lemma 4.2.11 (Main Lemma 1 for $\mathbf{H}^{\mathrm{e},C}$)**.** *Let k, l be two $\mathfrak{L}_{\mathrm{k}}$-structures expanding fields. Then* $\mathrm{Th}_\exists^{\mathfrak{L}_{\mathrm{k}}}(k) \subseteq \mathrm{Th}_\exists^{\mathfrak{L}_{\mathrm{k}}}(l) \implies \mathrm{Th}_\exists^{\mathfrak{L}}(k(\!(t)\!)) \subseteq \mathrm{Th}_\exists^{\mathfrak{L}}(l(\!(t)\!))$.

Proof. We suppose that $\mathrm{Th}_\exists^{\mathfrak{L}_{\mathrm{k}}}(k) \subseteq \mathrm{Th}_\exists^{\mathfrak{L}_{\mathrm{k}}}(l)$. There are three simple steps to this argument. Firstly, it follows from the hypothesis that there is an ultrafilter $\mathcal{U}$ and an $\mathfrak{L}_{\mathrm{k}}$-embedding $\varphi_{\mathrm{k}} : k \to l^{\mathcal{U}}$. This lifts to an $\mathfrak{L}$-embedding $\varphi : k(\!(t)\!) \to l^{\mathcal{U}}(\!(t)\!)$valuation, and so $\mathrm{Th}_\exists^{\mathfrak{L}}(k(\!(t)\!)) \subseteq \mathrm{Th}_\exists^{\mathfrak{L}}(l^{\mathcal{U}}(\!(t)\!))$. Secondly $l^{\mathcal{U}}(t)^h \preceq_\exists l^{\mathcal{U}}(\!(t)\!)$ is existentially closed as an extension of $\mathfrak{L}$-structures, by Ershov's Theorem (Lemma 4.2.10). Thirdly, there is an obvious $\mathfrak{L}$-embedding $\psi : l^{\mathcal{U}}(t)^h \to (l(\!(t)\!))^{\mathcal{U}}$ extending the identity on $l^{\mathcal{U}}$ and mapping t to the image of t. Therefore

$$\mathrm{Th}_\exists^{\mathfrak{L}}(k(\!(t)\!)) \subseteq \mathrm{Th}_\exists^{\mathfrak{L}}(l^{\mathcal{U}}(\!(t)\!)) = \mathrm{Th}_\exists^{\mathfrak{L}}(l^{\mathcal{U}}(t)^h) \subseteq \mathrm{Th}_\exists^{\mathfrak{L}}(l(\!(t)\!)^{\mathcal{U}}) \subseteq \mathrm{Th}_\exists^{\mathfrak{L}}(l(\!(t)\!)),$$

as required. □

Constraining our attention to $\mathbb{Z}$-valued K, things are easy:

Fact 4.2.12. *If $K \models \mathbf{H}^{\mathrm{e},C}$ is $\mathbb{Z}$-valued with residue field k_K, then* $\mathrm{Th}_\exists^{\mathfrak{L}}(k_K(\!(t)\!)) = \mathrm{Th}_\exists^{\mathfrak{L}}(K)$.

We will not need this fact, so we omit the proof, but it is interesting to note that we do not need to suppose excellence.

Lemma 4.2.13 (Main Lemma 2 for $\mathbf{H}^{\mathrm{e},C}$)**.** *Let K be an $\mathfrak{L}$-structure that models* $\mathbf{H}^{\mathrm{e},C}$, *and let k_K be the $\mathfrak{L}_{\mathrm{k}}$-stucture of the residue field of K.*

- **(i)** *Suppose that C is trivially valued. Then* $\mathrm{Th}_\exists^{\mathfrak{L}}(k_K(\!(t)\!)) \subseteq \mathrm{Th}_\exists^{\mathfrak{L}}(K)$.

(ii) *Suppose that C is trivially valued. There is a discretely valued extension K^{Disc}/K of $\mathfrak{L}$-structures for which the residue field extension is trivial, i.e. $k_{K^{\mathrm{Disc}}} = k_K$.*

(iii) *Suppose that K is discretely valued with uniformizer π, and that either C is trivially valued or is $\mathbb{Z}$-valued, also with uniformizer π, and $\mathcal{O}_u$ is excellent. There is an $\mathfrak{L}$-elementary extension $K \preceq K'$ that has an $\mathfrak{L}$-substructure D such that*

a. *the $\mathfrak{L}_{\mathrm{val}}$-reduct of D is a field complete with respect to a $\mathbb{Z}$-valuation v_D, which also has uniformizer π,*

b. *the natural extension $k_D \subseteq k_{K'}$ of the $\mathfrak{L}_{\mathbf{k}}$-structures on the residue fields is an equality,*

c. *D and $k_{K'}((t))$ are isomorphic $\mathfrak{L}$-structures, and*

d. $\mathrm{Th}^{\mathfrak{L}}_{\exists}(D) \subseteq \mathrm{Th}^{\mathfrak{L}}_{\exists}(K)$.

[Note also that the finest proper coarsening $(v')^+$ of v' corresponds to a D-rational D-place.]

(iv) *Suppose that C is trivially valued or that* $(\Upsilon$**ii**$)$ *holds. Then $\mathrm{Th}^{\mathfrak{L}}_{\exists}(k_K((t))) \subseteq \mathrm{Th}^{\mathfrak{L}}_{\exists}(K)$.*

(v) *Suppose that* $(\Upsilon$**i**$)$ *holds. Then $\mathrm{Th}^{\mathfrak{L}}_{\exists}(k_K((t))) = \mathrm{Th}^{\mathfrak{L}}_{\exists}(k_K((t))^{\mathrm{perf}})$.*

Perhaps for **(v)** we should first observe that in fact $k_K((t))^{\mathrm{perf}}$ we *mean* the structure described in the proof.

Proof. For **(i)**: Applying Lemma 3.1.4 to K (with respect to the given valuation v_K) and the partial section $\zeta_0 = \xi_K \circ \xi_{k_K}^{-1} : \xi_{k_K}(C) \to K$, we find $K^* \succeq K$ that admits a section $\zeta : k_{K^*} \to K^*$ extending ζ_0, Choosing any nonzero π in the maximal ideal of v^* (i.e. the valuation on K^*), we obtain by Lemma 4.2.9 an $\mathfrak{L}$-embedding $k_{K^*}(t)^h \to K^*$ that maps $t \mapsto \pi$ and that both extends and induces ζ. Combining this with Ershov's Theorem (Lemma 4.2.10), and with Lemma 4.2.11, we have

$$\mathrm{Th}^{\mathfrak{L}}_{\exists}(k_K((t))) = \mathrm{Th}^{\mathfrak{L}}_{\exists}(k_{K^*}((t))) = \mathrm{Th}^{\mathfrak{L}}_{\exists}(k_{K^*}(t)^h) \subseteq \mathrm{Th}^{\mathfrak{L}}_{\exists}(K^*) = \mathrm{Th}^{\mathfrak{L}}_{\exists}(K),$$

which proves **(i)**.

For **(ii)**: We let K^{Disc} be the henselization $K(\pi)^h$ of a purely transcendental extension $K(\pi)$ of K, with respect to the π-adic valuation v_π. We let v^{Disc} be the composition $v_K \circ v_\pi$ of v_K with v_π, and we endow K^{Disc} with v^{Disc} and the original $\mathfrak{L}_{\mathbf{k}}$-structure on the residue field k_K (so that $k_{K^{\mathrm{Disc}}} = k_K$). Then K^{Disc} is as required.

For **(iii)**: Let $w := (v_K)^+$ be the finest proper coarsening of v_K. Our hypothesis on K guarantees that w exists, and our hypothesis on C guarantees that w is trivial on C. Then v_K induces on the residue field Kw a $\mathbb{Z}$-valuation $\bar{v}$ of which the image of π is a uniformizer. Note that k_K/Cu is separable by hypothesis, and so it follows from excellence that Kw/Cw is also separable. By Lemma 3.1.6, there is an elementary extension $K \preceq K'$, which we may assume to be $\aleph_1$-saturated, with valuation $v' = v_{K'}$, that admits a section $\zeta : K'w' \to K'$ with respect to the finest proper coarsening $w' := (v')^+$ of v'. The image of ζ in K' is the required subfield D: For the valuation v_D we of course take the image under ζ of $\bar{v}'$, the valuation induced on $K'w'$ by v', of which the image in $K'w'$ of π is still a uniformizer. With respect to $\bar{v}'$, $K'w'$ is complete by the $\aleph_1$-saturation of K'. This proves **(a)**. The residue field of v' is an elementary extension of k_K, and since w' is trivial on D, the residue field of D is $(K'w')\bar{v}' = K'v' = k_{K'}$. This proves **(b)**. Point **(c)** follows by the general structure theory for complete discretely valued fields. Point **(d)** follows since existential sentences "go up".

For **(iv)**: this now easy. If C is trivially valued we apply **(i)**. Otherwise $(\Upsilon$**ii**$)$ implies the hypotheses of **(iii)**, so we get that $\mathrm{Th}^{\mathfrak{L}}_{\exists}(k_{K'}((t))) \subseteq \mathrm{Th}^{\mathfrak{L}}_{\exists}(K)$ and by Lemma 4.2.11 we have $\mathrm{Th}^{\mathfrak{L}}_{\exists}(k_K((t))) = \mathrm{Th}^{\mathfrak{L}}_{\exists}(k_{K'}((t)))$.

For **(v)**: Consider the inclusion $k_K((t)) \subseteq k_K((t))^{\mathrm{perf}}$. If $(\Upsilon$**ia**$)$, holds, i.e. C is trivially valued and k_K is perfect, then $k_K((t))^{\mathrm{perf}}$ is the direct limit of a chain of isomorphic $\mathfrak{L}$-structures:

$$k_K((t))^{\mathrm{perf}} = \bigcup_{n<\omega} k_K((t))^{(p^{1/n})}.$$

$$\begin{tikzcd}
K' \arrow[r, dashed, "w'"] \arrow[d, dash] & K'w' \arrow[r, dashed, "\bar{v}'"] \arrow[d, dash] \arrow[l, dotted, bend left, "\zeta"] & k_{K'} \arrow[d, dash] \\
K \arrow[r, dashed, "w"] \arrow[d, dash] & Kw \arrow[r, dashed, "\bar{v}"] \arrow[d, dash, "\mathrm{sep}"] & k_K \arrow[d, dash, "\mathrm{sep}"] \\
C \arrow[r, "\sim"] & Cw \arrow[r, dashed, "\bar{u}"] & Cu
\end{tikzcd}$$

Figure 4.1: Illustration of the proof Lemma 4.2.13**(iii)**

Otherwise if (Υ**ib**), holds, i.e. C is trivially valued and some N-th power of Frobenius gives an $\mathfrak{L}_{\mathbf{k}}$-embedding $k_K \to k_K$ (equivalently, $k_K \subseteq k_K^{(p^{-N})}$ is an inclusion of $\mathfrak{L}_{\mathbf{k}}$-structures), then $k_K(\!(t)\!)^{\mathrm{perf}}$ is again the direct limit of a chain of isomorphic $\mathfrak{L}$-structures:

$$k_K(\!(t)\!)^{\mathrm{perf}} = \bigcup_{n<\omega} k_K(\!(t)\!)^{(p^{1/Nn})}.$$

Therefore in either case, we have $\mathrm{Th}_{\exists}^{\mathfrak{L}}(k_K(\!(t)\!)) = \mathrm{Th}_{\exists}^{\mathfrak{L}}(k_K(\!(t)\!)^{\mathrm{perf}})$. □

Proof of Resplendent Monotonicity, Theorem 4.2.2. By hypothesis, k_L models the existential $\mathfrak{L}_{\mathbf{k}}$-theory of k_K. Therefore there is an elementary extension $k_L \succeq k_L^*$ and an $\mathfrak{L}_{\mathbf{k}}$-embedding $\varphi_{\mathbf{k}} : k_K \to k_L^*$. This extends to an $\mathfrak{L}$-embedding $\varphi : k_K(\!(t)\!) \to k_L^*(\!(t)\!)$, which shows that $\mathrm{Th}_{\exists}^{\mathfrak{L}}(k_K(\!(t)\!)) \subseteq \mathrm{Th}_{\exists}^{\mathfrak{L}}(k_L^*(\!(t)\!))$. Since k_L and k_L^* are elementarily equivalent, $\mathrm{Th}_{\exists}^{\mathfrak{L}}(k_L(\!(t)\!)) = \mathrm{Th}_{\exists}^{\mathfrak{L}}(k_L^*(\!(t)\!))$. Therefore $\mathrm{Th}_{\exists}^{\mathfrak{L}}(k_k(\!(t)\!)) \subseteq \mathrm{Th}_{\exists}^{\mathfrak{L}}(k_L(\!(t)\!))$. By Lemma 4.2.13**(iv)**, $\mathrm{Th}_{\exists}^{\mathfrak{L}}(k_K(\!(t)\!)) \subseteq \mathrm{Th}_{\exists}^{\mathfrak{L}}(K)$ and $\mathrm{Th}_{\exists}^{\mathfrak{L}}(k_L(\!(t)\!)) \subseteq \mathrm{Th}_{\exists}^{\mathfrak{L}}(L)$. It remains to argue that $\mathrm{Th}_{\exists}^{\mathfrak{L}}(k_K(\!(t)\!)) \supseteq \mathrm{Th}_{\exists}^{\mathfrak{L}}(K)$.

We may take an extension K'/K of $\mathfrak{L}$-structures such that K' is discretely valued with $k_{K'} = k_K$ and, in case that C is not trivially valued, K'/C is unramified: this is automatic in Otherwise choose K' to be $K^{\mathbf{Disc}}$ from Lemma 4.2.13**(ii)**.

Thus K' satisfies the hypotheses of Lemma 4.2.13**(iii)** to obtain a substructure $D \subseteq K'$ with $k_D = k_{K'}$ and that D is isomorphic to $k_K(\!(t)\!)$. Note that D is large. If **(R4)** holds, then $D \preceq K'$ is existentially closed. Otherwise (Υ**i**) holds, and we note that $D^{\mathrm{perf}} \subseteq (K')^{\mathrm{perf}}$ and $(K')^{\mathrm{perf}}$ admits a D^{perf}-rational D^{perf}-place. Since D^{perf} is large and perfect, and since **(R4)** holds for such fields, D^{perf} is existentially closed in $(K')^{\mathrm{perf}}$ as $\mathfrak{L}$-structures. By Lemma 4.2.13**(v)**, $\mathrm{Th}_{\exists}^{\mathfrak{L}}(D) \supseteq \mathrm{Th}_{\exists}^{\mathfrak{L}}(D^{\mathrm{perf}})$. Therefore $\mathrm{Th}_{\exists}^{\mathfrak{L}}(k_K(\!(t)\!)) \supseteq \mathrm{Th}_{\exists}^{\mathfrak{L}}(K)$, as required. □

$$\begin{tikzcd}
K' \arrow[r, dash] \arrow[d, dash, "(\star)"'] \arrow[d, dotted, bend left] & (K')^{\mathrm{perf}} \arrow[d, dash, "(\star\star)"'] \arrow[d, dotted, bend left] \\
D \arrow[r, dash, "(\dagger)"'] \arrow[d, dash] & D^{\mathrm{perf}} \\
C &
\end{tikzcd}$$

Figure 4.2: Illustration of the proof of Resplendent Monotonicity, Theorem 4.2.2

Question 4.2.14 (Problem 9.3)**.** Can the hypothesis (Υ) on C and k_K be replaced in Theorem 4.2.2 by an *a priori* weaker hypotheses?

Remark 4.2.15. The above results are a little bit resplendent, but only a little. In reality, hypothesis (Υ) imposes serious constraints on the $\mathfrak{L}_{\mathrm{k}}$-structure of k_K. For example, we may expand by constants, but only as long as *either* k is perfect, *or* these constants are invariant under Frobenius. This is exactly the hypothesis that appears in [AF16]. The situation is not hopeless, as there are plenty of predicates that are invariant under Frobenius, and we may expand by these.

4.3 Monotonicity of $\exists_n$ and $\exists^n$ in $\mathbf{H}^{e\prime}$ and $\mathbf{H}^{0,\varpi}$, respectively

Based on work from [AF26+a], we investigate monotonicity of $\mathbf{H}^{e\prime}$ with respect to certain fragments between $\exists_1$ and $\exists$.

Proposition 4.3.1 ($\exists_n$-monotonicity for $\mathbf{H}^{e\prime}$, cf [AF26+a, §3.4])**.** *Let k_1, k_2 be fields with common subfield k_0 such that k_1/k_0 is separable. Then*

$$\mathrm{Th}_{\exists_n}(k_1, k_0) \subseteq \mathrm{Th}_{\exists_n}(k_2, k_0) \iff \mathrm{Th}_{\exists_n}(k_1(t)^h, v_t, t, k_0) \subseteq \mathrm{Th}_{\exists_n}(k_2(t)^h, v_t, t, k_0).$$

Proof sketch. Let $A \subseteq k_1(t)^h$ be generated by n elements. We consider the substructure $A_1 := k_0A(t)$ generated by A. Observe that the residue field of A_1 is generated by at most n elements over k_0: Either the transcendence degree is n, in which case the extension is generated by n algebraically independent elements, or the transcendence degree is $< n$, and the remaining separable algebraic extension is generated by a single element by the Primitive Element Theorem. $\square$

We switch to working with $\mathbf{H}^{0,\varpi}$: the equal characteristic zero case with distinguished uniformizer.

Theorem 4.3.2 ($\exists^n$-monotonicity for $\mathbf{H}^{0,\varpi}$)**.** *Let k_1, k_2 be fields with common $\mathfrak{L}_{\mathrm{ring}}$-substructure k_0. Then*

$$\mathrm{Th}^{\mathfrak{L}_{\mathrm{ring}}}_{\exists^n}(k_1, k_0) \subseteq \mathrm{Th}^{\mathfrak{L}_{\mathrm{ring}}}_{\exists^n}(k_2, k_0) \iff \mathrm{Th}^{\mathfrak{L}_{\mathrm{val}}}_{\exists^n}(k_1(t)^h, v_t, t, k_0) \subseteq \mathrm{Th}^{\mathfrak{L}_{\mathrm{val}}}_{\exists^n}(k_2(t)^h, v_t, t, k_0).$$

Proof sketch. This is similar to the previous proof, except that we proceed by induction, making several uses of Lemma 2.3.15. The base case is trivial. For $n > 0$, we may also suppose that k_0 is relatively algebraically closed in k_1, by enlarging k_0 if necessary. Suppose that the claim holds for some $n \geq 0$ and that $\mathrm{Th}_{\exists^{n+1}}(k_1, k_0) \subseteq \mathrm{Th}_{\exists^{n+1}}(k_2, k_0)$. Note that $\exists^{n+1} = \exists_1\exists^n$. Let $a \in k_1(t)^h$. Let A_1 be the substructure of $k_1(t)^h$ generated by a, t, and k_0. If $A_1 \subseteq k_0(t)^h$ then we are done, otherwise $A_1 \not\subseteq k_0(t)^h$ and the residue extension is transcendental. Therefore, choosing $b \in k_2$ by the hypothesis on the residue fields, we have $\mathrm{Th}_{\exists^n}(k_1, k_0(a)) \subseteq \mathrm{Th}_{\exists^n}(k_2, k_0(b))$. Now by the inductive hypothesis, we conclude that $\mathrm{Th}_{\exists^n}(k_1(t)^h, v_t, t, k_0(a)) \subseteq \mathrm{Th}_{\exists^n}(k_2(t)^h, v_t, t, k_0(b))$. $\square$

Question 4.3.3. Can these results for the $\exists_n$ and $\exists^n$ fragments be improved to a "fully-fragmented" monotonicity, for fragments inside $\exists$? The first step is to work out what this should mean, i.e. for which fragments we could reasonably hope to prove monotonicity. Perhaps an even more preliminary task is to explore the fragments between F_0 and $\exists_1$. There might be an inductive argument that for a positive fragment F, monotonicity with respect to F might imply monotonicity with respect to $\exists_1\mathsf{F}$. However, we caution the reader that we have not defined "positive fragment". And what about $\exists^n$ in $\mathbf{H}^{e\prime}$?

4.4 Consequences of monotonicity

This section is drawn from [AF26+a].

4.4.1 Equicharacteristic henselian valued fields: $\exists$-theories

We denote by $\mathbf{H}^{\mathrm{e}\prime}$ the $\mathfrak{L}_{\mathrm{val}}$-theory of equicharacterstic henselian nontrivially valued fields, and by $\mathbf{H}^{\mathrm{e},\varpi}$ (resp., $\mathbf{H}_0^{\mathrm{e},\varpi}$) the $\mathfrak{L}_{\mathrm{val}}(\varpi)$-theory of equicharacteristic (resp., equicharacteristic zero) henselian valued fields in which the interpretation of ϖ is a uniformizer (i.e. an element of minimal positive value). We define the contexts $\mathrm{H}^{\mathrm{e}\prime} := (\mathrm{Sent}(\mathfrak{L}_{\mathrm{val}}), \mathbf{H}^{\mathrm{e}\prime})$ and $\mathrm{H}^{\mathrm{e}\prime}_{\exists} := (\mathrm{Sent}_{\exists}(\mathfrak{L}_{\mathrm{val}}), \mathbf{H}^{\mathrm{e}\prime})$, and analogously $\mathrm{H}^{\mathrm{e},\varpi}$ and $\mathrm{H}_0^{\mathrm{e},\varpi}$ with fragment $\mathrm{Sent}(\mathfrak{L}_{\mathrm{val}}(\varpi))$, and $\mathrm{H}^{\mathrm{e},\varpi}_{\exists}$ and $\mathrm{H}^{\mathrm{e},\varpi}_{0,\exists}$ with fragment $\mathrm{Sent}_{\exists}(\mathfrak{L}_{\mathrm{val}}(\varpi))$.

We denote by $\mathbf{F}_0$ the theory of fields of characteristic zero and define the contexts $\mathrm{F}_0 = (\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}}), \mathbf{F}_0)$ and analogously $\mathrm{F}_{0,\exists}$. We denote by $\sigma_{\mathbf{k}}$ the map that sends a valued field (K, v) to its residue field Kv. Thus by equipping each with an appropriate restriction of $\sigma_{\mathbf{k}}$ we have the bridges $\mathrm{F}.\mathrm{H}^{\mathrm{e}\prime}$, $\mathrm{F}_{\exists}.\mathrm{H}^{\mathrm{e}\prime}_{\exists}$, $\mathrm{F}_0.\mathrm{H}_0^{\mathrm{e},\varpi}$, $\mathrm{F}_{0,\exists}.\mathrm{H}^{\mathrm{e},\varpi}_{0,\exists}$, $\mathrm{F}.\mathrm{H}^{\mathrm{e},\varpi}$, and $\mathrm{F}_{\exists}.\mathrm{H}^{\mathrm{e},\varpi}_{\exists}$.

Definition 4.4.1. For an $\mathfrak{L}_{\mathrm{ring}}$-term t we define an $\mathfrak{L}_{\mathrm{val}}$-term $t_{\mathbf{k}}$ by recursion on the length of t as follows: for a variable x we let $x_{\mathbf{k}}$ be a variable of the residue field sort; we let $0_{\mathbf{k}} := 0^{\mathbf{k}}$ and $1_{\mathbf{k}} := 1^{\mathbf{k}}$; for $t = t_1 + t_2$ we let $t_{\mathbf{k}} = (t_1)_{\mathbf{k}} +^{\mathbf{k}} (t_2)_{\mathbf{k}}$, similarly for $t = t_1 - t_2$ and $t = t_1 \cdot t_2$. For an $\mathfrak{L}_{\mathrm{ring}}$-formula φ we define an $\mathfrak{L}_{\mathrm{val}}$-formula $\varphi_{\mathbf{k}}$ by recursion on the length of φ as follows: if $\varphi = t_1 \doteq t_2$, we let $\varphi_{\mathbf{k}} = (t_1)_{\mathbf{k}} \doteq (t_2)_{\mathbf{k}}$; if $\varphi = \neg\psi$, we define $\varphi_{\mathbf{k}} := \neg\psi_{\mathbf{k}}$; if $\varphi = (\psi \wedge \rho)$, we define $\varphi_{\mathbf{k}} := (\psi_{\mathbf{k}} \wedge \rho_{\mathbf{k}})$; and if $\varphi = \exists x\ \psi$, we define $\varphi_{\mathbf{k}} := \exists x \in \mathbf{k}\ \psi_{\mathbf{k}}$. Finally, define $\iota_{\mathbf{k}} : \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}}) \to \mathrm{Sent}(\mathfrak{L}_{\mathrm{val}}) \subseteq \mathrm{Sent}(\mathfrak{L}_{\mathrm{val}}(\varpi))$, given by $\varphi \longmapsto \varphi_{\mathbf{k}}$.

Remark 4.4.2. The map $\iota_{\mathbf{k}}$ is an interpretation for the bridge $(\mathrm{F}, \mathrm{VF}, \sigma_{\mathbf{k}})$, where $\mathbf{VF}$ is the $\mathfrak{L}_{\mathrm{val}}$-theory of valued fields, and VF is the context $(\mathrm{Sent}(\mathfrak{L}_{\mathrm{val}}), \mathbf{VF})$. Equipped with the appropriate restrictions of $\iota_{\mathbf{k}}$ we get arches $\mathrm{F}_{\exists}.\mathrm{H}^{\mathrm{e}\prime}_{\exists}|\mathrm{F}.\mathrm{H}^{\mathrm{e}\prime}$, $\mathrm{F}_{0,\exists}.\mathrm{H}^{\mathrm{e},\varpi}_{0,\exists}|\mathrm{F}_0.\mathrm{H}_0^{\mathrm{e},\varpi}$ and $\mathrm{F}_{\exists}.\mathrm{H}^{\mathrm{e},\varpi}_{\exists}|\mathrm{F}.\mathrm{H}^{\mathrm{e},\varpi}$, where we use that $\iota_{\mathbf{k}}(\mathrm{Sent}_{\exists}(\mathfrak{L}_{\mathrm{ring}})) \subseteq \mathrm{Sent}_{\exists}(\mathfrak{L}_{\mathrm{val}}) \subseteq \mathrm{Sent}_{\exists}(\mathfrak{L}_{\mathrm{val}}(\varpi))$.

Lemma 4.4.3.

(a) *Hypotheses (i)–(iii) of Corollary 2.4.20 hold for* $\mathrm{F}_{\exists}.\mathrm{H}^{\mathrm{e}\prime}_{\exists}|\mathrm{F}.\mathrm{H}^{\mathrm{e}\prime}$.

(b) *Hypotheses (i)–(iii) of Corollary 2.4.20 hold for* $\mathrm{F}_{0,\exists}.\mathrm{H}^{\mathrm{e},\varpi}_{0,\exists}|\mathrm{F}_0.\mathrm{H}_0^{\mathrm{e},\varpi}$.

(c) *Hypotheses (i)–(ii) of Corollary 2.4.20 hold for* $\mathrm{F}_{\exists}.\mathrm{H}^{\mathrm{e},\varpi}_{\exists}|\mathrm{F}.\mathrm{H}^{\mathrm{e},\varpi}$*, and if in addition* **(R4)** *holds, then so does hypothesis (iii).*

Proof. For (i): The fact that the fragments and $\iota_{\mathbf{k}}$ are computable is clear in both cases. The theories $\mathbf{H}^{\mathrm{e}\prime}$ and $\mathbf{H}^{\mathrm{e},\varpi}$ are computably enumerable since they have computable axiomatizations, see e.g. [ADF23, Remark 4.8], taking Remark 4.0.3 into account.

For (ii): let k be a field. Then $(k((t)), v_t)$ is an equicharacteristic henselian nontrivially valued field with residue field k, of which t is a uniformizer. In particular, $\sigma_{\mathbf{k}} : \mathbf{Mod}(\mathbf{H}^{\mathrm{e},\varpi}) \to \mathbf{Mod}(\mathbf{F})$ is surjective, and so (ii) follows in both cases.

For (iii) in (a): let $(K, v), (L, w) \models \mathbf{H}^{\mathrm{e}\prime}$ and suppose that $Kv \models \mathrm{Th}_{\exists}(Lw)$. That is $(K, v) \models \mathbf{H}^{\mathrm{e}\prime} \cup \iota_{\mathbf{k}}\mathrm{Th}_{\exists}(Lw)$. By [AF16, Lemma 6.3][2], $\mathbf{H}^{\mathrm{e}\prime} \cup \iota_{\mathbf{k}}\mathrm{Th}_{\exists}(Lw) \models (\mathbf{H}^{\mathrm{e}\prime} \cup \iota_{\mathbf{k}}\mathrm{Th}(Lw))_{\exists}$. By [AF16, Lemma 6.1], $\mathbf{H}^{\mathrm{e}\prime} \cup \iota_{\mathbf{k}}\mathrm{Th}(Lw) \models \mathrm{Th}_{\exists}(L, w)$. Then $(K, v) \models \mathrm{Th}_{\exists}(L, w)$, and therefore $\mathrm{F}_{\exists}.\mathrm{H}^{\mathrm{e}\prime}_{\exists}$ satisfies **(mon)**.

For (iii) in (c): **(R4)** implies that $\mathrm{F}_{\exists}.\mathrm{H}^{\mathrm{e},\varpi}_{\exists}$ satisfies **(mon)** by [ADF23, Proposition 4.11] (again, note Remark 4.0.3), taking C to be the subfield $\mathbb{F}(\pi)$ generated by the uniformizer π and the prime subfield $\mathbb{F}$, together with the π-adic valuation v_π. Note that the valuation ring of v_π is excellent.

For (iii) in (b): as for (c) except applying $(\mathrm{R4})_{\mathbb{Q}}$ instead of **(R4)**: by [ADF23, Remark 4.18] we may apply [ADF23, Proposition 4.11] to $(C, u) = (\mathbb{Q}(\pi), v_\pi)$, noting that the valuation ring of v_π is excellent, the separability hypotheses are satisfied in characteristic zero. □

Corollary 4.4.4.

(a) (I) $\mathrm{F}_{\exists}.\mathrm{H}^{\mathrm{e}\prime}_{\exists}$ *admits a computable elimination* $\mathrm{Sent}_{\exists}(\mathfrak{L}_{\mathrm{val}}) \to \mathrm{Sent}_{\exists}(\mathfrak{L}_{\mathrm{ring}})$.

(II) *For every* $R \subseteq \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})$, $(\mathbf{F} \cup R)_{\exists} \simeq_{\mathrm{m}} (\mathbf{H}^{\mathrm{e}\prime} \cup \iota_{\mathbf{k}}R)_{\exists}$.

(III) *For every* $(K, v) \models \mathbf{H}^{\mathrm{e}\prime}$, $\mathrm{Th}_{\exists}(K, v) = (\mathbf{H}^{\mathrm{e}\prime} \cup \iota_{\mathbf{k}}\mathrm{Th}_{\exists}(Kv))_{\exists}$.

[2] Alternatively, one may apply Lemma 2.4.21 taking $R = \mathrm{Th}(Lw)$.

(*b*) (I) $\mathrm{F}_{0,\exists}.\mathrm{H}^{\mathrm{e},\varpi}_{0,\exists}$ *admits a computable elimination* $\mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{val}}(\varpi)) \to \mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{ring}})$.

(II) *For every* $R \subseteq \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})$*,* $(\mathbf{F}_0 \cup R)_\exists \simeq_{\mathrm{m}} (\mathbf{H}^{\mathrm{e},\varpi}_0 \cup \iota_{\mathrm{k}} R)_\exists$.

(III) *For every* $(K, v, \pi_K) \models \mathbf{H}^{\mathrm{e},\varpi}_0$, $\mathrm{Th}_\exists(K, v, \pi_K) = (\mathbf{H}^{\mathrm{e},\varpi}_0 \cup \iota_{\mathrm{k}}\mathrm{Th}_\exists(Kv))_\exists$.

(*c*) *Suppose* **(R4)***.*

(I) $\mathrm{F}_\exists.\mathrm{H}^{\mathrm{e},\varpi}_\exists$ *admits a computable elimination* $\mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{val}}(\varpi)) \to \mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{ring}})$.

(II) *For every* $R \subseteq \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})$*,* $(\mathbf{F} \cup R)_\exists \simeq_{\mathrm{m}} (\mathbf{H}^{\mathrm{e},\varpi} \cup \iota_{\mathrm{k}} R)_\exists$.

(III) *For every* $(K, v, \pi_K) \models \mathbf{H}^{\mathrm{e},\varpi}$, $\mathrm{Th}_\exists(K, v, \pi_K) = (\mathbf{H}^{\mathrm{e},\varpi} \cup \iota_{\mathrm{k}}\mathrm{Th}_\exists(Kv))_\exists$.

Remark 4.4.5. By Lemma 2.4.22, Lemma 4.4.3 and therefore also each of the parts of Corollary 4.4.4 remains true if we replace each $\exists$ by $\forall/\exists$, denoting, in each of the respective languages, the set of finite conjunctions and disjunctions of existential or universal sentences. For example, for every $R \subseteq \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})$, $(\mathbf{F} \cup R)_{\forall/\exists} \simeq_{\mathrm{m}} (\mathbf{H}^{\mathrm{e}\prime} \cup \iota_{\mathrm{k}} R)_{\forall/\exists}$. Similar extensions hold for several of the results in the rest of this paper, but we will not spell them out each time.

Although we have only proved **(mon)** for the bridge $\mathrm{F}_\exists.\mathrm{H}^{\mathrm{e},\varpi}_\exists$ under the hypothesis **(R4)**, we are able to prove **(wmon)** unconditionally:

Lemma 4.4.6. *The fragment extension* $\mathrm{F}_\exists.\mathrm{H}^{\mathrm{e},\varpi}_\exists \sqsubseteq \mathrm{F}.\mathrm{H}^{\mathrm{e},\varpi}$ *satisfies* **(wmon)***.*

Proof. Let $(K, v, \pi) \models \mathbf{H}^{\mathrm{e},\varpi}$ and $k \models \mathbf{F}$ with $\mathrm{Th}_\exists(k) \subseteq \mathrm{Th}_\exists(Kv)$. Without loss of generality, (K, v, π) is $|k|^+$-saturated, and then so is Kv. This implies that there exists an embedding $\varphi : k \to Kv$, see [CK90, Lemma 5.2.1]. By [ADF23, Lemma 4.4] there exists an elementary extension $(K, v) \prec (K^*, v^*)$ with a partial section $\zeta : Kv \to K^*$ of res_{v^*}. The composition $\zeta \circ \varphi : k \to K^*$ is an embedding of fields, and moreover is an embedding of valued fields when k is equipped with the trivial valuation. Mapping $t \mapsto \pi$ and using the universal property of henselisations, we may extend $\zeta \circ \varphi$ to an embedding of valued fields $(k(t)^h, v_t, t) \to (K^*, v^*, \pi)$. Thus $\mathrm{Th}_\exists(k(t)^h, v_t, t) \subseteq \mathrm{Th}_\exists(K^*, v^*, \pi_K) = \mathrm{Th}_\exists(K, v, \pi_K)$. Note that the residue field of $(k(t)^h, v_t, t)$ is k, in particular it is elementarily equivalent to k. □

4.4.2 Equicharacteristic henselian valued fields: $\exists_n$-theories

For $n \geq 0$, we define the contexts $\mathrm{H}^{\mathrm{e}\prime}_{\exists_n} := (\mathrm{Sent}_{\exists_n}(\mathfrak{L}_{\mathrm{val}}), \mathbf{H}^{\mathrm{e}\prime})$ and $\mathrm{F}_{\exists_n} := (\mathrm{Sent}_{\exists_n}(\mathfrak{L}_{\mathrm{ring}}), \mathbf{F})$, as well as the bridge $\mathrm{F}_{\exists_n}.\mathrm{H}^{\mathrm{e}\prime}_{\exists_n}$. As $\iota_{\mathrm{k}}(\mathrm{Sent}_{\exists_n}(\mathfrak{L}_{\mathrm{ring}})) \subseteq \mathrm{Sent}_{\exists_n}(\mathfrak{L}_{\mathrm{val}})$, we obtain an arch $\mathrm{F}_{\exists_n}.\mathrm{H}^{\mathrm{e}\prime}_{\exists_n}|\mathrm{F}.\mathrm{H}^{\mathrm{e}\prime}$.

Lemma 4.4.7. *Let* $\mathfrak{L}$ *be a language and let* M, N *be* $\mathfrak{L}$*-structures. If* $\mathfrak{L}$ *contain no constant symbols assume that* $n \geq 1$*.*

(*a*) $\mathrm{Th}_\exists(M) \subseteq \mathrm{Th}_\exists(N)$ *if and only if* $\mathrm{Th}_\exists(M') \subseteq \mathrm{Th}_\exists(N)$ *for every finitely generated substructure* $M' \subseteq M$*.*

(*b*) $\mathrm{Th}_{\exists_n}(M) \subseteq \mathrm{Th}_{\exists_n}(N)$ *if and only if* $\mathrm{Th}_\exists(M') \subseteq \mathrm{Th}_\exists(N)$ *for every substructure* $M' \subseteq M$ *generated by at most* n *elements.*

Proof. Part (a) follows immediately from (b), and the latter is what we prove: For $\Leftarrow$ let $\varphi \in \mathrm{Th}_{\exists_n}(M)$. Without loss of generality, φ is of the form $\exists y_1, \dots, y_n \psi(\underline{y})$ with ψ quantifier-free. Then $M \models \psi(\underline{b})$ for some $\underline{b} \in M^n$, and $\underline{b}$ generates a substructure M' of M with $M' \models \varphi$, hence $\varphi \in \mathrm{Th}_{\exists_n}(M') \subseteq \mathrm{Th}_{\exists_n}(N)$.

For $\Rightarrow$, let $M' \subseteq M$ be generated by $a_1, \dots, a_n$, and let $\varphi \in \mathrm{Th}_\exists(M')$, without loss of generality of the form $\exists y_1, \dots, y_m \psi(\underline{y})$ for some m. Then $M' \models \psi(\underline{b})$ for some $b_1, \dots, b_m \in M'$, and $b_i = t_i(a_1, \dots, a_n)$ for an $\mathfrak{L}$-term t_i, for each i. Thus $\exists x_1, \dots, x_n \psi(t_1(\underline{x}), \dots, t_m(\underline{x})) \in \mathrm{Th}_{\exists_n}(M) \subseteq \mathrm{Th}_{\exists_n}(N)$, and so there are $c_1, \dots, c_n \in$ with $N \models \psi(t_1(\underline{c}), \dots, t_m(\underline{c}))$, in particular $\varphi \in \mathrm{Th}_\exists(N)$. □

Lemma 4.4.8. *Hypotheses (i)–(iii) of Corollary 2.4.20 hold for* $\mathrm{F}_{\exists_n}.\mathrm{H}^{\mathrm{e}\prime}_{\exists_n}|\mathrm{F}.\mathrm{H}^{\mathrm{e}\prime}$*.*

Proof. For *(i)*: this is again clear.

For *(ii)*: this is again clear, see Lemma 4.4.3(a).

For *(iii)*: Let $(K, v), (L, w) \models \mathbf{H}^{\mathrm{e}\prime}$ and suppose $\mathrm{Th}_{\exists_n}(Kv) \subseteq \mathrm{Th}_{\exists_n}(Lw)$. Without loss of generality, (L, w) is $\aleph_0$-saturated. We will show that $\mathrm{Th}_\exists(M') \subseteq \mathrm{Th}_\exists(L, w)$ for every $\mathfrak{L}_{\mathrm{val}}$-substructure M' of (K, v)

generated by at most n elements, which by Lemma 4.4.7(b) will imply that $\mathrm{Th}_{\exists_n}(K,v) \subseteq \mathrm{Th}_{\exists_n}(L,w)$. Without loss of generality assume that M' is generated by at most n elements in the sort $\mathbf{K}$, in particular $R_0 = M'^{\mathbf{K}}$ is a ring generated by at most n elements. Let $E = \mathrm{Quot}(R_0)$, let $u = v|_E$ and note that $\mathrm{Th}_\exists(M') \subseteq \mathrm{Th}_\exists(E,u)$.

First we consider the case that u is trivial. Then (E,u) is isomorphic to the trivially valued field (Eu, v_{triv}), and this isomorphism carries R_0 to a subring of Eu, which we denote R_0u. Indeed, R_0u is a subring of Kv generated by at most n elements, so $\mathrm{Th}_\exists(R_0u) \subseteq \mathrm{Th}_\exists(Lw)$ by Lemma 4.4.7(b). As Lw is $\aleph_0$-saturated and R_0 is countable, there exists an $\mathfrak{L}_{\mathrm{ring}}$-embedding $R_0u \to Lw$ ([CK90, Lemma 5.2.1]). This extends to an $\mathfrak{L}_{\mathrm{ring}}$-embedding $\eta : Eu = \mathrm{Quot}(R_0u) \to Lw$, which already shows that $\mathrm{Th}_\exists(Eu) \subseteq \mathrm{Th}_\exists(Lw)$. Since $\eta(Eu)$ is a finitely generated field extension of its prime field, it is separably generated, so there is an embedding $\zeta : \eta(Eu) \to L$ such that $\mathrm{res}_w \circ \zeta = \mathrm{id}_{\eta(Eu)}$, see [AF16, Lemma 2.3]. It follows that w is trivial on the image of $\eta(Eu)$, thus we have an $\mathfrak{L}_{\mathrm{val}}$-embedding $(\eta(Eu), v_{\mathrm{triv}}) \to (L,w)$. This shows that $\mathrm{Th}_\exists(E,u) = \mathrm{Th}_\exists(Eu, v_{\mathrm{triv}}) = \mathrm{Th}_\exists(\eta(Eu), v_{\mathrm{triv}}) \subseteq \mathrm{Th}_\exists(L,w)$.

Second we consider the case that u is nontrivial. Let $\mathbb{F}$ be the prime field of Eu, and let $R \subseteq Eu$ be a finitely generated subring of Eu. The field of fractions $F = \mathrm{Quot}(R)$ is a finitely generated subfield of Eu. Since u is nontrivial, $\mathrm{trdeg}(F/\mathbb{F}) < \mathrm{trdeg}(E/\mathbb{F}) \leq n$ by the Abhyankar inequality [EP05, Theorem 3.4.3]. Since $F/\mathbb{F}$ is separable and finitely generated, there exists a separating transcendence base $T \subseteq F$. Thus the field extension $F/\mathbb{F}(T)$ is finite and separable, and therefore generated by a single element a, by the primitive element theorem. Let $S \subseteq F$ be the subring generated by $T \cup \{a\}$. Then S is a subring of Kv that is generated by at most $|T| + 1 \leq n$ elements, and $\mathrm{Quot}(S) = F$. Thus, by Lemma 4.4.7(b), $\mathrm{Th}_\exists(S) \subseteq \mathrm{Th}_\exists(Lw)$, which as argued for R_0u and Eu in the previous paragraph, implies that $\mathrm{Th}_\exists(F) \subseteq \mathrm{Th}_\exists(Lw)$. We have thus shown that $\mathrm{Th}_\exists(R) \subseteq \mathrm{Th}_\exists(Lw)$ for every finitely generated $\mathfrak{L}_{\mathrm{ring}}$-substructure R of Eu, which by Lemma 4.4.7(a) gives that $\mathrm{Th}_\exists(Eu) \subseteq \mathrm{Th}_\exists(Lw)$. Since by Lemma 4.4.3(a) $\mathrm{F}_\exists.\mathrm{H}^{\mathrm{e}\prime}_\exists$ satisfies **(mon)**, we have $\mathrm{Th}_\exists(E^h, u^h) \subseteq \mathrm{Th}_\exists(L,w)$, where $(E^h, u^h) \supseteq (E,u)$ denotes the henselization, which is also of equal characteristic and has residue field Eu. Therefore $\mathrm{Th}_\exists(E,u) \subseteq \mathrm{Th}_\exists(L,w)$. □

Corollary 4.4.9.

(I) $\mathrm{F}_{\exists_n}.\mathrm{H}^{\mathrm{e}\prime}_{\exists_n}$ *admits a computable elimination* $\mathrm{Sent}_{\exists_n}(\mathfrak{L}_{\mathrm{val}}) \to \mathrm{Sent}_{\exists_n}(\mathfrak{L}_{\mathrm{ring}})$.

(II) *For every* $R \subseteq \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})$, $(\mathbf{F} \cup R)_{\exists_n} \simeq_{\mathrm{m}} (\mathbf{H}^{\mathrm{e}\prime} \cup \iota_{\mathrm{k}} R)_{\exists_n}$.

(III) *For every* $(K,v) \models \mathbf{H}^{\mathrm{e}\prime}$, $\mathrm{Th}_{\exists_n}(K,v) = (\mathbf{H}^{\mathrm{e}\prime} \cup \iota_{\mathrm{k}}\mathrm{Th}_{\exists_n}(Kv))_{\exists_n}$.

Proposition 4.4.10. $\mathrm{F}_\exists.\mathrm{H}^{\mathrm{e}\prime}_\exists$ *admits a computable elimination* $\epsilon : \mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{val}}) \to \mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{ring}})$ *that for each* $n \in \mathbb{N}$ *restricts to an elimination* $\mathrm{Sent}_{\exists_n}(\mathfrak{L}_{\mathrm{val}}) \to \mathrm{Sent}_{\exists_n}(\mathfrak{L}_{\mathrm{ring}})$ *of* $\mathrm{F}_{\exists_n}.\mathrm{H}^{\mathrm{e}\prime}_{\exists_n}$.

Proof. This is an adaptation of the proof of Proposition 2.4.15. We use two uniform aspects of the computability of the fragments of existential sentences. Firstly, the function $e : \mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{val}}) \to \mathbb{N} \cup \{0\}$, $\psi \mapsto \min\{n \in \mathbb{N} \mid \psi \in \mathrm{Sent}_{\exists_n}(\mathfrak{L}_{\mathrm{val}})\}$ is computable. Secondly, we may fix a computable function $\mathbb{N} \times \mathbb{N} \to \mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{ring}})$, $(n,m) \mapsto \varphi_{n,m}$, such that $\varphi_{n,1}, \varphi_{n,2}, \ldots$ is an enumeration of $\mathrm{Sent}_{\exists_n}(\mathfrak{L}_{\mathrm{ring}})$, for each n. We can also fix a computable enumeration $P_1, P_2, \ldots$ of the proofs from $\mathbf{H}^{\mathrm{e}\prime}$, since the latter is computably enumerable. By Corollary 4.4.9(I), given $\psi \in \mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{val}})$ there exist l and m such that P_l is a proof of $\mathbf{H}^{\mathrm{e}\prime} \vdash \psi \leftrightarrow \iota_{\mathrm{k}}\varphi_{e(\varphi),m}$. We take such pair (l,m) minimial with respect to the ordering (2.4.12) and define $\epsilon\psi := \varphi_{e(\psi),m}$. □

Chapter 5

Existential theories of HVFs in uniform settings

The sun in the west was a drop of burning gold that slid nearer and nearer the sill of the world.

Lord of the Flies, William Golding

This chapter is drawn from [AF26+a].

5.1 Limit theories

By analogy with the notion of characteristic in the theory of fields, we study extensions of theories defined by a sequence of sentences, including limit cases.

5.1.1 All, almost all, and uniform

Fix a language $\mathfrak{L}$ and a sequence of $\mathfrak{L}$-sentences $(\rho_n)_{n\in\mathbb{N}}$.

Definition 5.1.1. For any $\mathfrak{L}$-theory T, we define

$$\begin{aligned}
T_0 &:= (T \cup \{\rho_n \mid n \in \mathbb{N}\})^{\vdash} \\
T_n &:= (T \cup \{\rho_1, \dots, \rho_{n-1}, \neg\rho_n\})^{\vdash}, \quad n \in \mathbb{N} \\
T_{>m} &:= \bigcap_{n>m} T_n, \quad m \in \mathbb{N} \cup \{0\} \\
T_{\gg 0} &:= \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} T_n \\
T_N &:= \{(\varphi, n) \mid n \in N, \varphi \in T_n\}, \quad N \subseteq \mathbb{N} \cup \{0\}.
\end{aligned}$$

For an $\mathfrak{L}$-fragment L, by analogy with above we write $T_{N,L} = T_N \cap (L \times (\mathbb{N} \cup \{0\}))$.

Example 5.1.2. We will apply this definition in Section 5.2 mainly in the case where T is some theory of (possibly valued) fields and ρ_n is a quantifier-free sentence expressing that the characteristic is not n, so that T_0, T_n, $T_{>m}$ and $T_{\gg 0}$ are, respectively, the sets of sentences true in all models of T of characteristic zero, all models of T of characteristic n, all models of T of characteristic greater than m, and all models of T of sufficiently large positive characteristic.

Note that every model of T is a model of T_n for exactly one $n \in \mathbb{N} \cup \{0\}$. The models of $T_{>0}$ and $T_{\gg 0}$ can be described as follows:

Proposition 5.1.3. *Let T be an $\mathfrak{L}$-theory, and let M be an $\mathfrak{L}$-structure.*

(a) $M \models T_{>0}$ if and only if $M \equiv \prod_{i \in I} M_i/\mathcal{U}$ for some ultrafilter $\mathcal{U}$ on a set I, and $M_i \models T_{n_i}$ for some $n_i \in \mathbb{N}$.

(b) $M \models T_{\gg 0}$ if and only if $M \models T_{>0} \cup T_0$

Proof. (a): $T_{>0}$ is the common theory of the models of T_n, for $n \in \mathbb{N}$, so the implication $\Rightarrow$ follows from [CK90, Exercise 4.1.18], and $\Leftarrow$ follows from Łoś's theorem since $T_{n_i} \supseteq T_{>0}$.

(b): $\Rightarrow$ is trivial from the definitions. Suppose $M \models T_{>0} \cup T_0$. Then $M \equiv \prod_{i \in I} M_i/\mathcal{U}$, as in (a). For $l > 0$, $M \models \rho_l$, therefore by Łoś's theorem $\{i \in I \mid n_i = l\} \notin \mathcal{U}$. Thus for every $m \geq 0$, $\{i \in I \mid n_i > m\} \in \mathcal{U}$, and so $M \models T_{>m}$, again by Łoś's theorem. $\square$

Proposition 5.1.4. *Let T, S be $\mathfrak{L}$-theories such that $\neg\rho \in L$ for every $\rho \in S$. Then*

(a) $(T_L \cup S)_L = (T \cup S)_L$.

Consequently, if $\neg\rho_n \in L$ for every $n \in \mathbb{N}$, then

(b) $(T_L)_{0,L} = T_{0,L}$,

(c) $(T_L)_{n,L} = T_{n,L}$ *for every* $n \in \mathbb{N}$,

(d) $(T_L)_{>m,L} = T_{>m,L}$ *for every* $m \in \mathbb{Z}_{\geq 0}$,

(e) $(T_L)_{\gg 0,L} = T_{\gg 0,L}$, *and*

(f) $(T_L)_{N,L} = T_{N,L}$ *for every* $N \subseteq \mathbb{N} \cup \{0\}$.

Proof. For (a), the inclusion $\subseteq$ is clear since $(T \cup S)_L = (T^{\vdash} \cup S)_L$ and $T_L \subseteq T^{\vdash}$. Now let $\varphi \in (T \cup S)_L$. Then there exist $\rho_1, \dots, \rho_m \in S$ such that $T \cup \{\rho_1, \dots, \rho_m\} \vdash \varphi$, and equivalently $T \vdash (\bigvee_{j=1}^m \neg\rho_j) \vee \varphi$. Since $(\bigvee_{j=1}^m \neg\rho_j) \vee \varphi \in L$, we have $(\bigvee_{j=1}^m \neg\rho_j) \vee \varphi \in T_L$, in particular $T_L \cup \{\rho_1, \dots, \rho_m\} \vdash \varphi$. Thus $\varphi \in (T_L \cup S)_L$, which proves (a). Both (b) and (c) are special cases of (a), and (d) and (e) follow easily. Finally, (f) follows directly from (b) and (c). $\square$

5.1.2 Turing reductions

We keep notations and definitions from the previous subsection.

Proposition 5.1.5. *Let T be an $\mathfrak{L}$-theory, let L be a computable $\mathfrak{L}$-fragment with $\rho_n, \neg\rho_n \in L$ for every n, and assume that the function $n \mapsto \rho_n$ is computable. Then*

(a) $T_{0,L} \leq_{\mathrm{T}} T_L \oplus T_{>0,L} \oplus T_{\gg 0,L}$, *and*

(b) *if $T_{>0,L}$ and $T_{\gg 0,L}$ are decidable, then $T_L \simeq_{\mathrm{m}} T_{0,L}$.*

Let in addition T' be an $\mathfrak{L}$-theory with $T'_L \subseteq T_L$ and $T_{\gg 0,L} = T'_{\gg 0,L}$. Then

(c) $T_{>0,L} \leq_{\mathrm{T}} T_{\gg 0,L} \oplus T'_{>0,L} \oplus T_{\mathbb{N},L}$, *and*

(d) *if $T'_{\gg 0,L}$ and $T'_{>0,L}$ are decidable, then $T_{>0,L} \simeq_{\mathrm{T}} T_{\mathbb{N},L}$ and $T_L \simeq_{\mathrm{T}} T_{0,L} \oplus T_{>0,L} \simeq_{\mathrm{T}} T_{\mathbb{N} \cup \{0\},L}$.*

The rest of this subsection is devoted to the proof of Proposition 5.1.5, and for this we will denote[1] $\Sigma := T_L$, $\Sigma_0 := (T_L)_0 \cap L$, $\Sigma_n := (T_L)_n \cap L$, $\Sigma_{>m} := (T_L)_{>m} \cap L$, $\Sigma_{\gg 0} := (T_L)_{\gg 0} \cap L$, $\Sigma_N := \{(\varphi, n) : n \in N, \varphi \in \Sigma_n\}$. For $\varphi \in L$ and $n \in \mathbb{N} \cup \{0\}$ we will (only in this section) denote

$$\varphi_n \ := \ \varphi \vee \bigvee_{i=1}^{n} \neg\rho_i \ \in L$$

where it is understood that $\varphi_0 = \varphi$.

Lemma 5.1.6. *We have $\Sigma = \Sigma_0 \cap \Sigma_{>0}$. In particular, if $\Sigma_{>0}$ is decidable, then $\Sigma \leq_{\mathrm{m}} \Sigma_0$.*

[1] Note that e.g. by Σ_0 we do not mean the first part of Definition 5.1.1 applied to $T = \Sigma$, instead Σ_0 is $(T_L)_{0,L}$.

Proof. The inclusion from left to right is trivial. For the converse inclusion, let $\varphi \in \Sigma_0 \cap \Sigma_{>0}$. From $\varphi \in \Sigma_0$ we see that $\Sigma \vdash \varphi_n$ for some minimal n. So if $n > 0$, then $\Sigma \vdash \varphi_{n-1} \vee \neg\rho_n$, and $\varphi \in \Sigma_{>0} \subseteq \Sigma_n$ implies that $\Sigma \vdash \varphi_{n-1} \vee \rho_n$, hence $\Sigma \vdash \varphi_{n-1}$, contradicting the minimality of n. Thus $n = 0$ and $\varphi = \varphi_0 \in \Sigma$. □

Lemma 5.1.7. *We have* $\Sigma_0 \leq_{\mathrm{T}} \Sigma \oplus \Sigma_{>0} \oplus \Sigma_{\gg 0}$. *Moreover, if* $\Sigma_{>0}$ *and* $\Sigma_{\gg 0}$ *are decidable, then* $\Sigma_0 \leq_{\mathrm{m}} \Sigma$.

Proof. Suppose that there are algorithms $A_\Sigma, A_{\Sigma_{>0}}, A_{\Sigma_{\gg 0}}$ to decide the theories $\Sigma, \Sigma_{>0}, \Sigma_{\gg 0}$. We outline an algorithm to decide Σ_0.

- **Input.** $\varphi \in L$
- **Step 1.** Apply $A_{\Sigma_{\gg 0}}$ to decide whether or not $\varphi \in \Sigma_{\gg 0}$. If 'NO' then in particular we have $\varphi \notin \Sigma_0$, so we **output** 'NO' and finish. If else 'YES' then we continue.
- **Step 2.** Since $\varphi \in \Sigma_{\gg 0}$ there exists $m \in \mathbb{N}$ such that $\varphi \in \bigcap_{n \geq m} \Sigma_n$. So since $\Sigma_n \models \neg\rho_n$ we obtain $\varphi_m \in \bigcap_{n>0} \Sigma_n = \Sigma_{>0}$. Apply $A_{\Sigma_{>0}}$ to $\varphi_0, \varphi_1, \dots$ until we find the minimal m_0 with $\varphi_{m_0} \in \Sigma_{>0}$.
- **Step 3.** Apply A_Σ to decide whether or not $\varphi_{m_0} \in \Sigma$.
  - If 'YES' then in particular we have $\varphi_{m_0} \in \Sigma_0$. Since also $\Sigma_0 \vdash \bigwedge_{i=1}^{m_0} \rho_i$, it follows that $\Sigma_0 \vdash \varphi$, i.e. $\varphi \in \Sigma_0$. We **output** 'YES' and finish.
  - If 'NO', on the other hand, then $\varphi_{m_0} \notin \Sigma$. Since we already know $\varphi_{m_0} \in \Sigma_{>0}$, it follows from Lemma 5.1.6 that $\varphi_{m_0} \notin \Sigma_0$. Therefore $\varphi \notin \Sigma_0$. We **output** 'NO' and finish. □

This proves the Turing reduction. If both $\Sigma_{>0}$ and $\Sigma_{\gg 0}$ are decidable, A_Σ is the only oracle occurring in the above algorithm, and it gives the many-one reduction $\Sigma_0 \leq_{\mathrm{m}} \Sigma$.

Lemma 5.1.8. $\Sigma_{\mathbb{N}} \leq_{\mathrm{m}} \Sigma$ *and* $\Sigma_{\mathbb{N}} \leq_{\mathrm{m}} \Sigma_{>0}$.

Proof. For $\varphi \in L$ and $n \in \mathbb{N}$, let $\psi = \varphi_{n-1} \vee \rho_n$. Then $(\varphi, n) \in \Sigma_{\mathbb{N}}$ if and only if $\psi \in \Sigma$. Moreover, $\psi \in \Sigma$ if and only if $\psi \in \Sigma_{>0}$. Indeed, $\Sigma \subseteq \Sigma_{>0} \subseteq \Sigma_n$, and $\Sigma \vdash \psi$ if and only if $\Sigma_n \vdash \psi$. □

Lemma 5.1.9. *Let* $\Sigma' \subseteq \Sigma$ *with* $\Sigma'_{\gg 0} = \Sigma_{\gg 0}$. *The*$\Sigma_{>0} \leq_{\mathrm{T}} \Sigma_{\gg 0} \oplus \Sigma'_{>0} \oplus \Sigma_{\mathbb{N}}$.

Proof. Suppose there are algorithms $A_{\Sigma_{\mathbb{N}}}$, $A_{\Sigma'_{>0}}$ and $A_{\Sigma_{\gg 0}}$ to decide $\Sigma_{\mathbb{N}}$, $\Sigma'_{>0}$ and $\Sigma_{\gg 0}$. We outline an algorithm to decide $\Sigma_{>0}$.

- **Input.** $\varphi \in L$.
- **Step 1.** Apply $A_{\Sigma_{\gg 0}}$ to decide whether or not $\varphi \in \Sigma_{\gg 0}$. If 'NO' then in particular $\varphi \notin \Sigma_{>0}$, so we **output** 'NO' and finish. If else 'YES' then we continue.
- **Step 2.** Since $\Sigma'_{\gg 0} = \Sigma_{\gg 0} \models \varphi$ there exists $m \in \mathbb{N}$ such that $\Sigma'_n \models \varphi$ for every $n \geq m$. In particular, $\Sigma'_n \models \varphi_m$ for every $n > 0$, i.e. $\Sigma'_{>0} \models \varphi_m$. Apply $A_{\Sigma'_{>0}}$ to $\varphi_0, \varphi_1, \dots$ until we find the minimal m_0 with $\varphi_{m_0} \in \Sigma'_{>0}$.
- **Step 3.** Since $\Sigma' \subseteq \Sigma$ and $\varphi_{m_0} \in \Sigma'_{>0}$, also $\varphi_{m_0} \in \Sigma_{>0}$, hence $\varphi \in \bigcap_{n>m_0} \Sigma_n$. We now use $A_{\Sigma_{\mathbb{N}}}$ to check if $(\varphi, 1), \dots, (\varphi, m_0) \in \Sigma_{\mathbb{N}}$. If yes then $\varphi \in \bigcap_{n>0} \Sigma_n = \Sigma_{>0}$ and we **output** 'YES', otherwise we **output** 'NO'. □

Proof of Proposition 5.1.5. By Proposition 5.1.4 it suffices to prove statements with $T_{0,L}$ replaced by $(T_L)_{0,L} = \Sigma_0$, $T_{>0,L}$ replaced by $(T_L)_{>0,L} = \Sigma_{>0}$, etc., which we do now:

(a): This is Lemma 5.1.7.

(b): If $\Sigma_{>0}$ and $\Sigma_{\gg 0}$ are decidable, $\Sigma \leq_{\mathrm{m}} \Sigma_0$ follows from Lemma 5.1.6, and $\Sigma_0 \leq_{\mathrm{m}} \Sigma$ from Lemma 5.1.7.

(c): This is Lemma 5.1.9.

(d): $\Sigma_{>0} \leq_{\mathrm{T}} \Sigma_{\mathbb{N}}$ is (c) and $\Sigma_{>0} \geq_T \Sigma_{\mathbb{N}}$ is Lemma 5.1.8. The reduction $\Sigma \leq_{\mathrm{T}} \Sigma_0 \oplus \Sigma_{>0}$ is Lemma 5.1.6, and $\Sigma \geq_{\mathrm{T}} \Sigma_{>0}$ follows from Lemma 5.1.8 and (c), and $\Sigma \oplus \Sigma_{>0} \geq_{\mathrm{T}} \Sigma_0$ from (a). Thus $\Sigma \simeq_{\mathrm{T}} \Sigma_0 \oplus \Sigma_{>0}$. Trivially $\Sigma_0 \oplus \Sigma_{>0} \simeq_{\mathrm{T}} \Sigma_{\mathbb{N} \cup \{0\}}$. □

5.1.3 Limit arches

Let $A = (B, \hat{B}, \iota)$ be an arch. We write $B = (C_1, C_2, \sigma)$ and $\hat{B} = (\hat{C}_1, \hat{C}_2, \sigma)$, where $C_i = (L_i, T_i)$ is a subcontext of $\hat{C}_i = (\hat{L}_i, T_i)$, for $i = 1, 2$. Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of $\mathfrak{L}_1$-sentences, with $\rho_n, \neg\rho_n \in L_1$ for every n. Let $(\iota\rho_n)_{n\in\mathbb{N}}$ be the corresponding sequence of $\mathfrak{L}_2$-sentences.

For an $\mathfrak{L}_1$-theory R, we write $T_2(R) = T_2 \cup \iota(R_{\hat{L}_1})$. We will use the notation introduced in Definition 5.1.1, first with $T = R$ and the sentences $(\rho_n)_{n\in\mathbb{N}}$, leading to theories R_0, R_n, $R_{>m}$, $R_{\gg 0}$, and second with $T = T_2(R)$ and the sentences $(\iota\rho_n)_{n\in\mathbb{N}}$, leading to theories $T_2(R)_0$, $T_2(R)_n$, $T_2(R)_{>m}$, $T_2(R)_{\gg 0}$. In particular we have theories $T_{1,0}$, $T_{1,n}$, $T_{1,>m}$, $T_{1,\gg 0}$, $T_{2,0}$, $T_{2,n}$, $T_{2,>m}$, $T_{2,\gg 0}$. We moreover extend one of these notations to contexts and bridges: For $i = 1, 2$, let $C_{i,0} = (L_i, T_{i,0})$ and note that $B_0 := (C_{1,0}, C_{2,0}, \sigma|_{\mathrm{Mod}(T_{2,0})})$ is a bridge.

Remark 5.1.10. Note that the deductive closure of $T_2(R)$ does not depend on the choice of interpretation ι: we have $M \models T_2(R)$ if and only if $M \models T_2$ and $\sigma M \models R_{\hat{L}_1}$. Moreover, if $R \subseteq \hat{L}_1$ then $(T_2 \cup \iota R)^{\vdash} = T_2(R)^{\vdash} = (T_2 \cup \iota((T_1 \cup R)_{\hat{L}_1}))^{\vdash}$ by Lemma 2.4.18; and if instead $R \supseteq T_1$ then again $T_2(R)^{\vdash} = (T_2 \cup \iota((T_1 \cup R)_{\hat{L}_1})^{\vdash}$.

Proposition 5.1.11.

(*a*) $(T_2)^{\vdash} = T_2(T_1)^{\vdash}$.

(*b*) $T_{2,n} = T_2(T_{1,n})^{\vdash}$, *for all* $n \in \mathbb{N} \cup \{0\}$.

(*c*) *If* B *satisfies* **(sur)** *and* B_0 *satisfies* **(mon)**, *then* $(T_{2,>m})_{L_2} = T_2(T_{1,>m})_{L_2}$, *for all* $m \geq 0$.

(*d*) *If* B *satisfies* **(sur)** *and* B_0 *satisfies* **(mon)**, *then* $(T_{2,\gg 0})_{L_2} = T_2(T_{1,\gg 0})_{L_2}$.

Proof. (*a*) We apply Lemma 2.4.18 to the bridge $\hat{B}$ and $R = \emptyset$.

(*b*) Let $n \in \mathbb{N}$. We have assumed that $\rho_n, \neg\rho_n \in L_1 \subseteq \hat{L}_1$, thus $\iota\rho_n, \iota\neg\rho_n \in \hat{L}_2$, since ι is an interpretation for $\hat{B}$. For $M \models T_2$, we have $M \models \neg\iota\rho_n \Leftrightarrow \sigma M \models \neg\rho_n \Leftrightarrow M \models \iota\neg\rho_n$. Thus (*): $T_2 \models (\neg\iota\rho_n \leftrightarrow \iota\neg\rho_n)$. Together with $T_2(R)^{\vdash} = (T_2 \cup \iota((T_1 \cup R)_{\hat{L}_1}))^{\vdash}$ for $R = \{\rho_1, \ldots, \rho_{n-1}, \neg\rho_n\}$ by Lemma 2.4.18, we obtain $T_{2,n} = T_2(T_{1,n})^{\vdash}$, and similarly $T_{2,0} = T_2(T_{1,0})^{\vdash}$.

(*c*) By (b), for each $n > m$ we have $T_{1,n} \supseteq T_{1,>m}$, and therefore $T_{2,n} = T_2(T_{1,n})^{\vdash} \supseteq T_2(T_{1,>m})^{\vdash}$. Thus $(T_{2,>m})_{L_2} \supseteq T_2(T_{1,>m})_{L_2}$.

Conversely, let $M \models T_2(T_{1,>m})$ so that $\sigma M \models (T_{1,>m})_{\hat{L}_1}$. Note that $T_{1,>m} \models \rho_1 \wedge \ldots \wedge \rho_m$ and $\rho_1 \wedge \ldots \wedge \rho_m \in L_1 \subseteq \hat{L}_1$. Thus $\sigma M \models \rho_1 \wedge \ldots \wedge \rho_m$, and therefore also $M \models \iota\rho_1 \wedge \ldots \wedge \iota\rho_m$ (Lemma 2.4.14). There are two cases:

First, if $M \models T_{2,n}$ for some $n > 0$, then necessarily $n > m$, and so $M \models T_{2,>m}$.

Otherwise, in the second case, we have $M \models T_{2,0}$, whence $\sigma M \models (T_{1,0})_{\hat{L}_1}$ by (b). In particular $\sigma M \models \rho_n$, for all $n > 0$. Since σM is a model of $(T_{1,>m})_{\hat{L}_1}$, by Lemma 2.4.26 there exists $N \models T_{1,>m}$ such that $\sigma M \models \mathrm{Th}_{\hat{L}_1}(N)$. Since $\neg\rho_n \in L_1 \subseteq \hat{L}_1$ and $\sigma M \not\models \neg\rho_n$, also $N \not\models \neg\rho_n$, i.e. $N \models \rho_n$, for all $n > 0$. It follows that $N \models T_{1,0}$. Let $S = T_1 \cup \{\rho_1, \ldots, \rho_m\}$ and write $\sigma_n = \rho_{m+n}$, for $n \in \mathbb{N}$. We apply the notation from Definition 5.1.1 to the $\mathfrak{L}_1$-theory S and the sequence of $\mathfrak{L}_1$-sentences $(\sigma_n)_{n\in\mathbb{N}}$. Then $S_n = T_{1,m+n}$ for each $n > 0$, and so $S_{>0} = T_{1,>m}$ and $S_0 = T_{1,0}$. It follows that $N \models S_{>0} \cup S_0$. By Proposition 5.1.3(a), $N \equiv \prod_{i\in I} H_i/\mathcal{U}$ for a set I, an ultrafilter $\mathcal{U}$ on I and $H_i \models S_{n_i}$ for some $n_i \in \mathbb{N}$. By **(sur)** for B, for each $i \in I$ there exists $K_i \models T_2$ with $\mathrm{Th}_{L_1}(H_i) = \mathrm{Th}_{L_1}(\sigma K_i)$. It followthat $\sigma K_i \models S_{n_i}$, hence $K_i \models T_{2,m+n_i}$, for each $i \in I$, and so $K_i \models T_{2,>m}$. Denote $K := \prod_{i\in I} K_i/\mathcal{U}$. By Łoś's theorem, $\mathrm{Th}_{L_1}(\sigma K) = \mathrm{Th}_{L_1}(\prod_{i\in I} \sigma K_i/\mathcal{U}) = \mathrm{Th}_{L_1}(N)$. So since $N \models S_0$ it follows that $K \models T_{2,0} \cup T_{2,>m}$. Thus, $\sigma M \models \mathrm{Th}_{L_1}(\sigma K)$, in particular $\mathrm{Th}_{L_1}(\sigma K) \subseteq \mathrm{Th}_{L_1}(\sigma M)$. Since B_0 satisfies **(mon)**, we conclude that $\mathrm{Th}_{L_2}(K) \subseteq \mathrm{Th}_{L_2}(M)$.

Therefore, in both cases, $M \models (T_{2,>m})_{L_2}$. Thus $T_2(T_{1,>m}) \models (T_{2,>m})_{L_2}$, and hence $(T_{2,>m})_{L_2} \subseteq T_2(T_{1,>m})_{L_2}$.

(*d*) Both $T_{2,\gg 0}$ and $T_2(T_{1,\gg 0})$ are unions of chains. Thus

$$(T_{2,\gg 0})_{L_2} = \bigcup_{m\geq 0} (T_{2,>m})_{L_2} \overset{\text{(c)}}{=} \bigcup_{m\geq 0} T_2(T_{1,>m})_{L_2} = \Big(\bigcup_{m\geq 0} T_2(T_{1,>m})\Big)_{L_2} = T_2(T_{1,\gg 0})_{L_2},$$

as required. $\square$

5.2 Existential theories of fields with varying characteristic

Throughout this section we consider the language $\mathfrak{L}_1 = \mathfrak{L}_{\mathrm{ring}}$ and we fix the sequence $(\rho_n)_{n\in\mathbb{N}}$ defined by writing ρ_n for the quantifier-free $\mathfrak{L}_{\mathrm{ring}}$-sentence

$$\neg\underbrace{1+\ldots+1}_{n}\doteq 0,$$

for each $n \in \mathbb{N}$. Note that $n \mapsto \rho_n$ is computable.

For an arch $A = (B, \hat{B}, \iota)$ we write $B = (C_1, C_2, \sigma)$ and $\hat{B} = (\hat{C}_1, \hat{C}_2, \sigma)$, where $C_i = (L_i, T_i)$ is a subcontext of $\hat{C}_i = (\hat{L}_i, T_i)$, for $i = 1, 2$. For all such arches A considered in this section, C_1 and $\hat{C}_1$ are $\mathfrak{L}_1$-contexts, L_1 contains the quantifier-free $\mathfrak{L}_{\mathrm{ring}}$-sentences (thus $\rho_n, \neg\rho_n \in L_1$ for every n), $\iota(L_1) \subseteq L_2$, and B satisfies **(sur)**. Thus we may use the notation from Section 5.1.3, building on Definition 5.1.1: first with an $\mathfrak{L}_1$-theory $T = R$ and the sentences $(\rho_n)_{n\in\mathbb{N}}$, and second with $T = T_2(R)$ and the sentences $(\iota\rho_n)_{n\in\mathbb{N}}$.

Remark 5.2.1. For an $\mathfrak{L}_1$-theory R of fields, let S denote either R or $T_2(R)$. Then S_n is consistent only if n is prime or 0; and $S_{>0} = \bigcap_{p\in\mathbb{P}} S_p$ and $S_{\gg 0} = \bigcup_{m\in\mathbb{N}}\bigcap_{p\geq m, p\in\mathbb{P}} S_p$. We also follow Convention 2.3.6 by writing $S_\exists$, $S_{n,\exists}$, etc., for S_L, $(S_n)_L$, etc., when $L = \mathrm{Sent}_\exists(\mathfrak{L}_i)$. Moreover by Proposition 5.1.4(b,d) and Lemma 5.1.6 we have $S_\exists = \bigcap_{p\in\mathbb{P}\cup\{0\}} S_{p,\exists}$.

Remark 5.2.2. Consider the arch $A = B|\hat{B} = \mathrm{F}_\exists.\mathrm{H}^{\mathrm{e},\varpi}_\exists|\mathrm{F}.\mathrm{H}^{\mathrm{e},\varpi}$ together with the sentences $(\rho_n)_{n\in\mathbb{N}}$ defined above. Applying the extended notations of Section 5.1.3 we have the bridge $B_0 = (C_{1,0}, C_{2,0}, \sigma|_{\mathbf{Mod}((\mathbf{H}^{\mathrm{e},\varpi})_0)})$ where $C_{2,0} = (\mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{val}}), (\mathbf{H}^{\mathrm{e},\varpi})_0)$ and $(\mathbf{H}^{\mathrm{e},\varpi})_0 = (\mathbf{H}^{\mathrm{e},\varpi} \cup \{\iota_{\mathbf{k}}\rho_n \mid n \in \mathbb{N}\})^{\vdash}$.

In Section 4.4.1 we had already defined the theory $\mathbf{H}^{\mathrm{e},\varpi}_0$ to be the theory of equicharacteristic zero models of $\mathbf{H}^{\mathrm{e},\varpi}$, which is clearly axiomatized by $\mathbf{H}^{\mathrm{e},\varpi} \cup \{\iota_{\mathbf{k}}\rho_n \mid n \in \mathbb{N}\}$. Thus $\mathbf{H}^{\mathrm{e},\varpi}_0 = (\mathbf{H}^{\mathrm{e},\varpi})_0$. Similarly we had defined the contexts $\mathrm{H}^{\mathrm{e},\varpi}_0 = (\mathrm{Sent}(\mathfrak{L}_{\mathrm{val}}), \mathbf{H}^{\mathrm{e},\varpi}_0)$ and $\mathrm{H}^{\mathrm{e},\varpi}_{0,\exists} = (\mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{val}}), \mathbf{H}^{\mathrm{e},\varpi}_0)$. We observe that there is no ambiguity between the notation introduced in sections 4.4.1 and 5.1.3. In particular, we have $B_0 = \mathrm{F}_{0,\exists}.\mathrm{H}^{\mathrm{e},\varpi}_{0,\exists}$.

Remark 5.2.3. We make use of the following basic facts: If C_1 and C_2 are two classes structures in some language $\mathfrak{L}$, such that for every $M \in C_1$ there exists $N \in C_2$ with either $N \subseteq M$ or $M \preceq_\exists N$, then $\mathrm{Th}_\exists(C_1) \supseteq \mathrm{Th}_\exists(C_2)$. If in addition $C_1 \supseteq C_2$, then $\mathrm{Th}_\exists(C_1) = \mathrm{Th}_\exists(C_2)$. In particular, if R_1, R_2 are two $\mathfrak{L}$-theories such that for every $M \models R_1$ there exists $N \models R_2$ with either $N \subseteq M$ or $M \preceq_\exists N$, then $R_{1,\exists} \supseteq R_{2,\exists}$, and if in addition $R_1 \subseteq R_2$, then $R_{1,\exists} = R_{2,\exists}$.

Remark 5.2.4. For any theory T of fields we have $T_\exists \subseteq T_{0,\exists}, T_{>0,\exists} \subseteq T_{\gg 0,\exists}$ (cf. Proposition 5.1.3). If T has models of every characteristic, then the sentence $1 + 1 \neq 0$ is in $T_{0,\exists}$ but not in $T_{>0,\exists}$, hence $T_\exists \subsetneqq T_{0,\exists}$ and $T_{>0,\exists} \subsetneqq T_{\gg 0,\exists}$. If T has a model of characteristic zero in which $\mathbb{Q}$ is algebraically closed, then $\exists x(x^2 = 2 \vee x^2 = 3 \vee x^2 = 6)$ is in $T_{>0,\exists}$ but not in $T_{0,\exists}$, hence $T_\exists \subsetneqq T_{>0,\exists}$ and $T_{0,\exists} \subsetneqq T_{\gg 0,\exists}$.

5.2.1 Prime fields

Recall that $\mathbf{F}$ denotes the theory of fields. Denote $\mathbb{F}_0 = \mathbb{Q}$ and let $\mathbf{P} = \bigcap_{p\in\mathbb{P}\cup\{0\}} \mathrm{Th}(\mathbb{F}_p)$ be the theory of prime fields.

Remark 5.2.5. Ax proved in [Ax68] that $\mathbf{P}_{>0}$ and $\mathbf{P}_{\gg 0}$ are decidable [FJ08, Corollary 20.9.6, Theorem 20.9.7]. Moreover, with **Fin** the theory of finite fields and **Psf** the theory of pseudofinite fields (i.e. the infinite models of the theory of finite fields), Ax's work also shows that $\mathbf{Psf}_0 = \mathbf{Psf}_{\gg 0} = \mathbf{P}_{\gg 0} = \mathbf{Fin}_{\gg 0}$: Indeed, for every $K \models \mathbf{Psf}_0$ there is an ultrafilter $\mathcal{U}$ on $\mathbb{P}$ such that $K \equiv \prod_{p\in\mathbb{P}} \mathbb{F}_p/\mathcal{U} \models \mathbf{P}_{\gg 0}$ [Ax68, Thm. 8"]. Moreover, every $K \models \mathbf{Fin}_{\gg 0}$ is elementarily equivalent to an ultraproduct $\prod_{p\in\mathbb{P}} K_p/\mathcal{U}$ with each K_p a finite field of characteristic p (Proposition 5.1.3), and if F_p is any pseudofinite field in which K_p is algebraically closed (which exists by [Ax68, Thm. 7,8]), then $\prod_{p\in\mathbb{P}} K_p/\mathcal{U} \equiv \prod_{p\in\mathbb{P}} F_p/\mathcal{U} \models \mathbf{Psf}_{\gg 0}$

[Ax68, Thm. 4]. Thus, $\mathbf{Psf}_0 \supseteq \mathbf{P}_{\gg 0} \supseteq \mathbf{Fin}_{\gg 0} \supseteq \mathbf{Psf}_{\gg 0} \supseteq \mathbf{Psf}_0$ (cf. Proposition 5.1.3). Finally, by Weil's Riemann hypothesis for curves, $\mathbf{C} \subseteq \mathbf{Psf}$ [FJ08, Cor. 20.10.5].

Proposition 5.2.6.

- **(a)** $\mathbf{F}_{p,\exists} = \mathbf{P}_{p,\exists} = \mathrm{Th}_{\exists}(\mathbb{F}_p)$ *for all* $p \in \mathbb{P} \cup \{0\}$,
- **(b)** $\mathbf{F}_{\exists} = \mathbf{P}_{\exists}$,
- **(c)** $\mathbf{F}_{>0,\exists} = \mathbf{P}_{>0,\exists}$,
- **(d)** $\mathbf{F}_{\gg 0,\exists} = \mathbf{P}_{\gg 0,\exists}$.

Proof. For $p \in \mathbb{P} \cup \{0\}$, since $\mathbf{F}_p \subseteq \mathbf{P}_p \subseteq \mathrm{Th}(\mathbb{F}_p)$ and every $K \models \mathbf{F}_p$ is a field of characteristic p, hence has prime field isomorphic to $\mathbb{F}_p$, (a) follows from Remark 5.2.3. This implies (b) by Remark 5.2.1, and (c) and (d) follow immediately from (b) and Proposition 5.1.4. □

Corollary 5.2.7.

- **(a)** $\mathbf{F}_{0,\exists} = \mathrm{Th}_{\exists}(\mathbb{Q})$*, and* $\mathbf{F}_{p,\exists}$ *is decidable for every* $p \in \mathbb{P}$.
- **(b)** $\mathbf{F}_{\exists} \simeq_{\mathrm{m}} \mathbf{F}_{0,\exists}$*, and* $\mathbf{F}_{\exists_1}$ *is decidable.*
- **(c)** $\mathbf{F}_{>0,\exists}$ *is decidable.*
- **(d)** $\mathbf{F}_{\gg 0,\exists}$ *is decidable.*

Proof. Part (a) follows from Proposition 5.2.6(a) since $\mathrm{Th}(\mathbb{F}_p)$ is decidable. Parts (c) and (d) follow from the decidability of $\mathbf{P}_{>0}$ and $\mathbf{P}_{\gg 0}$ via Proposition 5.2.6(c,d), and $\mathbf{F}_{\exists} \simeq_{\mathrm{m}} \mathbf{F}_{0,\exists}$ is then a consequence of Proposition 5.1.5(b), applied to $T = \mathbf{F}$ and $L = \mathrm{Sent}_{\exists}(\mathfrak{L}_{\mathrm{ring}})$. Similarly, applying Proposition 5.1.5(b) to $T = \mathbf{F}$ and $L = \mathrm{Sent}_{\exists_1}(\mathfrak{L}_{\mathrm{ring}})$ shows $\mathbf{F}_{\exists_1} \simeq_{\mathrm{m}} \mathbf{F}_{0,\exists_1}$, and $\mathbf{F}_{0,\exists_1} = \mathrm{Th}_{\exists_1}(\mathbb{Q})$ by Proposition 5.2.6(a), which is decidable, see e.g. [FJ08, Lemma 19.1.3]. □

5.2.2 Henselian fields

We use the above notational conventions, applied to the arches $\mathrm{F}_{\exists}.\mathrm{H}^{\mathrm{e}\prime}_{\exists}|\mathrm{F}.\mathrm{H}^{\mathrm{e}\prime}$ and $\mathrm{F}_{\exists}.\mathrm{H}^{\mathrm{e},\varpi}_{\exists}|\mathrm{F}.\mathrm{H}^{\mathrm{e},\varpi}$, together with $\mathfrak{L}_{\mathrm{ring}}$-theories $R = \mathbf{F}, \mathbf{P}, \mathrm{Th}(\mathbb{F}_p)$, etc.

Proposition 5.2.8.

(1) **(a)** $\mathbf{H}^{\mathrm{e}\prime}_{p,\exists} = \mathbf{H}^{\mathrm{e}\prime}(\mathbf{F}_p)_{\exists} = \mathbf{H}^{\mathrm{e}\prime}(\mathbf{P}_p)_{\exists} = \mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}(\mathbb{F}_p))_{\exists} = \mathrm{Th}_{\exists}(\mathbb{F}_p(\!(t)\!), v_t)$ *for all* $p \in \mathbb{P} \cup \{0\}$,

(b) $\mathbf{H}^{\mathrm{e}\prime}_{\exists} = \mathbf{H}^{\mathrm{e}\prime}(\mathbf{F})_{\exists} = \mathbf{H}^{\mathrm{e}\prime}(\mathbf{P})_{\exists}$,

(c) $\mathbf{H}^{\mathrm{e}\prime}_{>0,\exists} = \mathbf{H}^{\mathrm{e}\prime}(\mathbf{F}_{>0})_{\exists} = \mathbf{H}^{\mathrm{e}\prime}(\mathbf{P}_{>0})_{\exists}$,

(d) $\mathbf{H}^{\mathrm{e}\prime}_{\gg 0,\exists} = \mathbf{H}^{\mathrm{e}\prime}(\mathbf{F}_{\gg 0})_{\exists} = \mathbf{H}^{\mathrm{e}\prime}(\mathbf{P}_{\gg 0})_{\exists}$

(2) **(a)** $\mathbf{H}^{\mathrm{e},\varpi}_{p,\exists} = \mathbf{H}^{\mathrm{e},\varpi}(\mathbf{F}_p)_{\exists} = \mathbf{H}^{\mathrm{e},\varpi}(\mathbf{P}_p)_{\exists} = \mathbf{H}^{\mathrm{e},\varpi}(\mathrm{Th}(\mathbb{F}_p))_{\exists} = \mathrm{Th}_{\exists}(\mathbb{F}_p(\!(t)\!), v_t, t)$ *for all* $p \in \mathbb{P} \cup \{0\}$*, where for the last equality in case* $p \in \mathbb{P}$ *we require* **(R4)***,*

(b) $\mathbf{H}^{\mathrm{e},\varpi}_{\exists} = \mathbf{H}^{\mathrm{e},\varpi}(\mathbf{F})_{\exists} = \mathbf{H}^{\mathrm{e},\varpi}(\mathbf{P})_{\exists}$,

(c) $\mathbf{H}^{\mathrm{e},\varpi}_{>0,\exists} = \mathbf{H}^{\mathrm{e},\varpi}(\mathbf{F}_{>0})_{\exists} = \mathbf{H}^{\mathrm{e},\varpi}(\mathbf{P}_{>0})_{\exists}$,

(d) $\mathbf{H}^{\mathrm{e},\varpi}_{\gg 0,\exists} = \mathbf{H}^{\mathrm{e},\varpi}(\mathbf{F}_{\gg 0})_{\exists} = \mathbf{H}^{\mathrm{e},\varpi}(\mathbf{P}_{\gg 0})_{\exists}$

Proof. For (1) we consider the arch $B|\hat{B} = \mathrm{F}_{\exists}.\mathrm{H}^{\mathrm{e}\prime}_{\exists}|\mathrm{F}.\mathrm{H}^{\mathrm{e}\prime}$, and for (2) we consider the arch $B|\hat{B} = \mathrm{F}_{\exists}.\mathrm{H}^{\mathrm{e},\varpi}_{\exists}|\mathrm{F}.\mathrm{H}^{\mathrm{e},\varpi}$, just as in Lemma 4.4.3(a,c). Henceforth we can argue simultaneously in both cases. The bridge B_0 satisfies **(mon)** by Lemma 4.4.3(a) in case (1) and Lemma 4.4.3(b) in case (2) (note for the latter that $B_0 = \mathrm{F}_{0,\exists}.\mathrm{H}^{\mathrm{e},\varpi}_{0,\exists}$, as discussed in Remark 5.2.2). Thus we may apply Proposition 5.1.11 to get the first equality in each subcase. Since $B \sqsubseteq \hat{B}$ satisfies **(wmon)**, by Lemma 2.4.25 and Lemma 4.4.3(a) in case (1), and Lemma 4.4.6 in case (2), we may apply Lemma 2.4.27 and Proposition 5.2.6 to deduce the remaining equalities, except for the final equality in (1a) (respectively, (2a)) for which we apply Corollary 4.4.4(aIII) (respectively, Corollary 4.4.4(bIII,cIII)). □

Remark 5.2.9. Alternatively, in case (1) of Proposition 5.2.8 we may argue via Lemma 2.4.21: for the arch $\mathrm{F}_\exists.\mathrm{H}^{\mathrm{e}\prime}_\exists|\mathrm{F}.\mathrm{H}^{\mathrm{e}\prime}$, the hypotheses of the lemma hold by Lemma 4.4.3(a). Then by Proposition 5.2.6(a), for every $p \in \mathbb{P}\cup\{0\}$, we have $\mathbf{H}^{\mathrm{e}\prime}(\mathbf{F}_{p,\exists}) = \mathbf{H}^{\mathrm{e}\prime}(\mathbf{P}_{p,\exists}) = \mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}_\exists(\mathbb{F}_p))$. By three applications of Lemma 2.4.21, we have $\mathbf{H}^{\mathrm{e}\prime}(\mathbf{F}_p)_\exists = \mathbf{H}^{\mathrm{e}\prime}(\mathbf{P}_p)_\exists = \mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}(\mathbb{F}_p))_\exists$. Moreover $(\mathbf{H}^{\mathrm{e}\prime}_p)^\vdash = \mathbf{H}^{\mathrm{e}\prime}(\mathbf{F}_p)^\vdash$ by Proposition 5.1.11(b), and so in particular we have $\mathbf{H}^{\mathrm{e}\prime}_{p,\exists} = \mathbf{H}^{\mathrm{e}\prime}(\mathbf{F}_p)_\exists$, which proves (1a). From this (1b) follows, as

$$\mathbf{H}^{\mathrm{e}\prime}_\exists = \bigcap_{p\in\mathbb{P}\cup\{0\}} \mathbf{H}^{\mathrm{e}\prime}_{p,\exists} = \bigcap_{p\in\mathbb{P}\cup\{0\}} \mathbf{H}^{\mathrm{e}\prime}(\mathbf{P}_p)_\exists = \mathbf{H}^{\mathrm{e}\prime}(\bigcap_{p\in\mathbb{P}\cup\{0\}} \mathbf{P}_p)_\exists = \mathbf{H}^{\mathrm{e}\prime}(\mathbf{P})_\exists,$$

where the first and fourth equalities hold by Remark 5.2.1 and the third equality holds by by Lemma 2.4.21. Similarly, Lemma 2.4.21 gives that

$$\mathbf{H}^{\mathrm{e}\prime}_{>0,\exists} = \bigcap_{p>0} \mathbf{H}^{\mathrm{e}\prime}_{p,\exists} = \bigcap_{p>0} \mathbf{H}^{\mathrm{e}\prime}(\mathbf{P}_p)_\exists = \mathbf{H}^{\mathrm{e}\prime}(\bigcap_{p>0} \mathbf{P}_p)_\exists = \mathbf{H}^{\mathrm{e}\prime}(\mathbf{P}_{>0})_\exists$$

and

$$\mathbf{H}^{\mathrm{e}\prime}_{\gg 0,\exists} = \bigcup_{\ell>0}\bigcap_{p>\ell} \mathbf{H}^{\mathrm{e}\prime}_{p,\exists} = \bigcup_{\ell>0} \mathbf{H}^{\mathrm{e}\prime}(\bigcap_{p>\ell} \mathbf{P}_p)_\exists = \mathbf{H}^{\mathrm{e}\prime}(\bigcup_{\ell>0}\bigcap_{p>\ell} \mathbf{P}_p)_\exists = \mathbf{H}^{\mathrm{e}\prime}(\mathbf{P}_{\gg 0})_\exists,$$

which proves (1c,d).

For case (2), we consider the arch $B|\hat{B} = \mathrm{F}_\exists.\mathrm{H}^{\mathrm{e},\varpi}_\exists|\mathrm{F}.\mathrm{H}^{\mathrm{e},\varpi}$. Then a similar argument is possible provided that we suppose **(R4)**. Under this hypothesis, B satisfies **(mon)**. Thus Lemma 2.4.21 applies and the above proof goes through *mutatis mutandis*. The advantage of the proof given above is that it does not rely on **(R4)**.

Corollary 5.2.10.

(1) **(a)** $\mathbf{H}^{\mathrm{e}\prime}_{0,\exists} \simeq_{\mathrm{m}} \mathrm{Th}_\exists(\mathbb{Q})$, *and* $\mathbf{H}^{\mathrm{e}\prime}_{p,\exists}$ *is decidable for every* $p \in \mathbb{P}$.

(b) $\mathbf{H}^{\mathrm{e}\prime}_\exists \simeq_{\mathrm{m}} \mathbf{H}^{\mathrm{e}\prime}_{0,\exists}$.

(c) $\mathbf{H}^{\mathrm{e}\prime}_{>0,\exists}$ *is decidable.*

(d) $\mathbf{H}^{\mathrm{e}\prime}_{\gg 0,\exists}$ *is decidable.*

(2) **(a)** $\mathbf{H}^{\mathrm{e},\varpi}_{0,\exists} \simeq_{\mathrm{m}} \mathrm{Th}_\exists(\mathbb{Q})$, *and if* **(R4)** *holds then* $\mathbf{H}^{\mathrm{e},\varpi}_{p,\exists}$ *is decidable for every* $p \in \mathbb{P}$.

(b) *If* **(R4)** *holds, then* $\mathbf{H}^{\mathrm{e},\varpi}_\exists \simeq_{\mathrm{m}} \mathbf{H}^{\mathrm{e},\varpi}_{0,\exists}$.

(c) *If* **(R4)** *holds, then* $\mathbf{H}^{\mathrm{e},\varpi}_{>0,\exists}$ *is decidable.*

(d) $\mathbf{H}^{\mathrm{e},\varpi}_{\gg 0,\exists}$ *is decidable.*

Proof. Part (1a) (respectively (2a)) for $p = 0$ follows from Proposition 5.2.6(a) and Corollary 4.4.4(aII) (respectively Corollary 4.4.4(bII)) applied to $R = \emptyset$, and for $p \in \mathbb{P}$ follows from the decidability of $\mathbf{P}_p$ via Proposition 5.2.8(1a) and Corollary 4.4.4(aII) (resp. Proposition 5.2.8(2a) and Corollary 4.4.4(cII)). Similarly, (1c,d) (resp. (2c,d) assuming **(R4)**) follow from Corollary 5.2.7(c,d) via Proposition 5.2.8(1c,d) and Corollary 4.4.4(aII) (resp. Proposition 5.2.8(2c,d) and Corollary 4.4.4(cII)). To obtain (2d) as written (i.e. without assuming **(R4)**) we argue as follows: by Proposition 5.2.8(2d), we have $\mathbf{H}^{\mathrm{e},\varpi}_{\gg 0,\exists} = \mathbf{H}^{\mathrm{e},\varpi}(\mathbf{F}_{\gg 0})_\exists$, and by Corollary 4.4.4(bII) $\mathbf{H}^{\mathrm{e},\varpi}_0(\mathbf{F}_{\gg 0})_\exists \simeq_{\mathrm{m}} \mathbf{F}_{\gg 0,\exists}$, which is decidable by Corollary 5.2.7(d). This proves (2d) since $\mathbf{H}^{\mathrm{e},\varpi}(\mathbf{F}_{\gg 0}) = \mathbf{H}^{\mathrm{e},\varpi}_0(\mathbf{F}_{\gg 0})$. Furthermore (1c) (resp. (2c)) is a consequence of Proposition 5.1.5(b), applied to $T = \mathbf{H}^{\mathrm{e}\prime}$ (resp. $T = \mathbf{H}^{\mathrm{e},\varpi}$) and $L = \mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{val}})$ (resp. $L = \mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{val}}(\varpi))$), or can alternatively be deduced from Corollary 5.2.7(c). □

5.2.3 Large fields

Let $\mathbf{L}$ denote the $\mathfrak{L}_{\mathrm{ring}}$-theory of large fields (cf. section 4.1). As every PAC field is large, and every nontrivially valued henselian field is large [Jar11, Example 5.6.2], $\mathbf{L} \subseteq \mathbf{PAC}$ and $\mathbf{L} \subseteq \mathbf{H}^{\mathrm{e}\prime} \cap \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})$.

Proposition 5.2.11.

(a) $\mathbf{L}_{p,\exists} = \mathbf{H}^{\mathrm{e}\prime}_{p,\exists} \cap \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}}) = \mathrm{Th}_\exists(\mathbb{F}_p(\!(t)\!))$ *for all* $p \in \mathbb{P} \cup \{0\}$.

(b) $\mathbf{L}_{\exists} = \mathbf{H}^{e\prime}_{\exists} \cap \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})$.

(c) $\mathbf{L}_{>0,\exists} = \mathbf{H}^{e\prime}_{>0,\exists} \cap \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})$.

(d) $\mathbf{L}_{\gg 0,\exists} = \mathbf{H}^{e\prime}_{\gg 0,\exists} \cap \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})$.

Proof. By Proposition 5.2.8(1a), $\mathbf{H}^{e\prime}_{p,\exists} \cap \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}}) = \mathrm{Th}_{\exists}(\mathbb{F}_p((t)))$. As every nontrivially valued henselian field is large, we have $\mathbf{L} \subseteq \mathbf{H}^{e\prime}$, in particular $\mathbf{L}_{p,\exists} \subseteq \mathbf{H}^{e\prime}_{p,\exists} \cap \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})$. Conversely, if $K \models \mathbf{L}_p$, then $K \prec_{\exists} K((t)) \models \mathbf{H}^{e\prime}_{p,\exists} \cap \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})$, hence $K \models \mathbf{H}^{e\prime}_{p,\exists} \cap \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})$, which proves (a). Then (b) follows via Remark 5.2.1, and (c) and (d) follow immediately from (a) and Proposition 5.1.4. □

Corollary 5.2.12.

(a) $\mathbf{L}_{0,\exists} \simeq_{\mathrm{m}} \mathrm{Th}_{\exists}(\mathbb{Q})$*, and* $\mathbf{L}_{p,\exists}$ *is decidable for every* $p \in \mathbb{P}$.

(b) $\mathbf{L}_{\exists} \simeq_{\mathrm{m}} \mathbf{L}_{0,\exists}$

(c) $\mathbf{L}_{>0,\exists}$ *is decidable.*

(d) $\mathbf{L}_{\gg 0,\exists}$ *is decidable.*

Proof. The decidability of $\mathbf{L}_{>0,\exists}$ and $\mathbf{L}_{\gg 0,\exists}$ follow from Corollary 5.2.10(1c,d) and Proposition 5.2.11(c,d). For $p \in \mathbb{P} \cup \{0\}$, $\mathbf{L}_{p,\exists} = \mathbf{H}^{e\prime}_{p,\exists} \cap \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})$ by Proposition 5.2.11(a), and by Corollary 5.2.10(1a) this is decidable when $p \in \mathbb{P}$. For the case $p = 0$, $\mathbf{L}_{\exists} \simeq_{\mathrm{m}} \mathbf{L}_{0,\exists}$ is a consequence of Proposition 5.1.5(b), applied to $T = \mathbf{L}$ and $L = \mathrm{Sent}_{\exists}(\mathfrak{L}_{\mathrm{ring}})$, and $\mathbf{L}_{0,\exists} = \mathbf{H}^{e\prime}(\mathrm{Th}(\mathbb{Q}))_{\exists} \cap \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})$ by Proposition 5.2.11(a). The latter is many-one equivalent to $\mathbf{H}^{e\prime}(\mathrm{Th}(\mathbb{Q}))_{\exists}$ since by [AF17, Corollary 6.2 and Corollary 6.18] there are two $\mathfrak{L}_{\mathrm{ring}}$-formulas — one existential and one universal — which each define the valuation ring in every model of $\mathbf{H}^{e\prime}(\mathrm{Th}(\mathbb{Q}))$. Finally, $\mathbf{H}^{e\prime}(\mathrm{Th}(\mathbb{Q}))_{\exists} \simeq_{\mathrm{m}} \mathrm{Th}_{\exists}(\mathbb{Q})$ by Corollary 5.2.8(1a) and Corollary 5.2.10(1a). □

Remark 5.2.13. We proved both $\mathbf{H}^{e\prime}_{0,\exists} \simeq_{\mathrm{T}} \mathbf{H}^{e\prime}_{\exists}$ and $\mathbf{H}^{e\prime}_{0,\exists} \cap \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}}) \simeq_{\mathrm{T}} \mathbf{H}^{e\prime}_{\exists} \cap \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})$ but neither statement seems to imply the other immediately, as there is no uniform existential (or universal) definition of the valuation ring in models of $\mathbf{H}^{e\prime}$ (or even $\mathbf{H}^{e\prime} \cup \iota_{\mathbf{k}}\mathbf{P}$). In particular, we obtain the somewhat surprising Turing equivalence $\mathbf{H}^{e\prime}_{\exists} \simeq_{\mathrm{T}} \mathbf{H}^{e\prime}_{\exists} \cap \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})$.

Remark 5.2.14. The statement $\mathbf{L}_{0,\exists} = \mathrm{Th}_{\exists}(\mathbb{Q}((t)))$, which is part of Proposition 5.2.11(a) was also proven by Sander in [San96, Proposition 2.25]. Sander continues to remark (with references but without detailed proof) that $\mathrm{Th}_{\exists}(\mathbb{Q}((t))) \leq_{\mathrm{T}} \mathrm{Th}_{\exists}(\mathbb{Q})$ follows from work of Weispfenning, and he asks in [San96, Problem 2.26] whether $\mathbf{L}_{0,\exists}$ is decidable. The statement $\mathbf{L}_{0,\exists} \simeq_{\mathrm{m}} \mathrm{Th}_{\exists}(\mathbb{Q})$ (Corollary 5.2.12(a)) shows that this question is in fact equivalent to a famous and still open problem.

Proof of Main Theorem 3. By Corollary 5.2.8(1a,2a), we have $\mathrm{Th}_{\exists}(\mathbb{Q}((t)), v_t) = \mathbf{H}^{e\prime}(\mathrm{Th}(\mathbb{Q}))_{\exists}$ and $\mathrm{Th}_{\exists}(\mathbb{Q}((t)), v_t, t) = \mathbf{H}^{e,\varpi}(\mathrm{Th}(\mathbb{Q}))_{\exists}$. By Corollary 5.2.10(1a,2a) these theories are many-one equivalent to $\mathrm{Th}_{\exists}(\mathbb{Q})$. Thus (a), (c), and (d) are many-one equivalent. By Proposition 5.2.11(a), (b) and (e) are many-one equivalent. By Corollary 5.2.7(a,b), (a) and (g) are many-one equivalent. Finally by Corollary 5.2.12(a,b), (a), (e), and (f) are many-one equivalent. □

5.2.4 Global fields

Let $\mathbf{F}^{\infty}$ be the theory of infinite fields and $\mathbf{G}$ the theory of global fields (i.e. finitely generated fields of characteristic zero and transcendence degree zero, or positive characteristic and transcendence degree one).

Proposition 5.2.15. *We have* $\mathbf{G}_{\exists} = (\mathbf{F}^{\infty})_{\exists}$. *Consequently,* $\mathbf{G}_{\gg 0,\exists} = (\mathbf{F}^{\infty})_{\gg 0,\exists}$, $\mathbf{G}_{p,\exists} = (\mathbf{F}^{\infty})_{p,\exists} = \mathrm{Th}_{\exists}(\mathbb{F}_p(t))$ *for every* $p \in \mathbb{P}$*, and* $\mathbf{G}_{0,\exists} = (\mathbf{F}^{\infty})_{0,\exists} = \mathbf{F}_{0,\exists} = \mathrm{Th}_{\exists}(\mathbb{Q})$.

Proof. Every global field is infinite. Conversely, let K be an infinite field of characteristic $p \geq 0$. If $p = 0$, then the global field $\mathbb{Q}$ can be embedded into K, and if $p > 0$ then the global field $\mathbb{F}_p(t)$ can be embedded into any uncountable elementary extension $K \prec K^*$. It follows that $\mathbf{G}_\exists = (\mathbf{F}^\infty)_\exists$ (Remark 5.2.3). Using Proposition 5.1.4, we conclude that $\mathbf{G}_{p,\exists} = (\mathbf{G}_\exists)_{p,\exists} = ((\mathbf{F}^\infty)_\exists)_{p,\exists} = (\mathbf{F}^\infty)_{p,\exists}$ for every $p \in \mathbb{P} \cup \{0\}$, similarly $\mathbf{G}_{\gg 0,\exists} = (\mathbf{F}^\infty)_{\gg 0,\exists}$. As $(\mathbf{F}^\infty)_0 = \mathbf{F}_0$, we conclude that $(\mathbf{F}^\infty)_{0,\exists} = \mathbf{F}_{0,\exists}$, and $\mathbf{F}_{0,\exists} = \mathrm{Th}_\exists(\mathbb{Q})$ follows from Proposition 5.2.6(a). Similarly, $\mathbb{F}_p(t)$ is infinite, and if K is any infinite field of characteristic $p > 0$, then $\mathbb{F}_p(t)$ can be embedded into any proper elementary extension $K \prec K^*$. $\square$

Lemma 5.2.16. $\mathbf{F}_{\gg 0} = (\mathbf{F}^\infty)_{\gg 0}$

Proof. Since $\mathbf{F} = \mathbf{F}^\infty \cap \mathbf{Fin}$, also $\mathbf{F}_{\gg 0} = (\mathbf{F}^\infty)_{\gg 0} \cap \mathbf{Fin}_{\gg 0}$. So as $\mathbf{Fin}_{\gg 0} = \mathbf{Psf}_{\gg 0}$ (Remark 5.2.5) and $\mathbf{Psf} \supseteq \mathbf{F}^\infty$, in particular $\mathbf{Psf}_{\gg 0} \supseteq (\mathbf{F}^\infty)_{\gg 0}$, we get that $\mathbf{F}_{\gg 0} = (\mathbf{F}^\infty)_{\gg 0} \cap \mathbf{Fin}_{\gg 0} = (\mathbf{F}^\infty)_{\gg 0}$. $\square$

Remark 5.2.17. In particular, using Proposition 5.2.6(d), Proposition 5.2.15, and Lemma 5.2.16 we obtain $\mathbf{P}_{\gg 0,\exists} = \mathbf{F}_{\gg 0,\exists} = (\mathbf{F}^\infty)_{\gg 0,\exists} = \mathbf{G}_{\gg 0,\exists}$. We can also show that all of these theories are equal to $\mathbf{L}_{\gg 0,\exists}$ and to $\mathbf{PAC}_{\gg 0,\exists}$: As $\mathbb{F}_p \subseteq \mathbb{F}_p(t) \subseteq \mathbb{F}((t))$, from Propositions 5.2.6(a), 5.2.15 and 5.2.11(a) we get $\mathbf{F}_{p,\exists} \subseteq \mathbf{G}_{p,\exists} \subseteq \mathbf{H}^{e\prime}_{p,\exists} \cap \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})$, and therefore

$$\mathbf{P}_{\gg 0,\exists} = \mathbf{F}_{\gg 0,\exists} \subseteq \mathbf{G}_{\gg 0,\exists} \subseteq \mathbf{H}^{e\prime}_{\gg 0,\exists} \cap \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}}) = \mathbf{L}_{\gg 0,\exists} \subseteq \mathbf{PAC}_{\gg 0,\exists}$$

using Proposition 5.2.11(d). By Remark 5.2.5 we have $\mathbf{PAC}_{\gg 0} \subseteq \mathbf{Psf}_{\gg 0} = \mathbf{Psf}_0 = \mathbf{P}_{\gg 0}$, and thus $\mathbf{PAC}_{\gg 0,\exists} = \mathbf{P}_{\gg 0,\exists}$ (although $\mathbf{PAC}_{\gg 0} \neq \mathbf{P}_{\gg 0}$). Note that $\mathbf{F}_{\gg 0,\exists} \supsetneqq \mathbf{F}_{0,\exists}$, $\mathbf{PAC}_{\gg 0,\exists} \supsetneqq \mathbf{PAC}_{0,\exists}$ and $\mathbf{H}^{e\prime}_{\gg 0,\exists} \supsetneqq \mathbf{H}^{e\prime}_{0,\exists}$, as each of $\mathbf{F}$, $\mathbf{PAC}$ and $\mathbf{H}^{e\prime}$ has models of characteristic zero in which $\mathbb{Q}$ is algebraically closed (cf. Remark 5.2.4).

Corollary 5.2.18. $\mathbf{G}_{\mathbb{N},\exists} \simeq_{\mathrm{T}} \mathbf{G}_{>0,\exists}$ *and* $\mathbf{G}_{\mathbb{N}\cup\{0\},\exists} \simeq_{\mathrm{T}} \mathbf{G}_\exists$.

Proof. It follows from Lemma 5.2.16 that $\mathbf{F}_{\gg 0,\exists} = (\mathbf{F}^\infty)_{\gg 0,\exists}$. Therefore, using Corollary 5.2.7(c,d), Proposition 5.1.5(d) for $T = \mathbf{F}^\infty$, $T' = \mathbf{F}$ and $L = \mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{ring}})$ gives that $(\mathbf{F}^\infty)_{>0,\exists} \simeq_{\mathrm{T}} (\mathbf{F}^\infty)_{\mathbb{N},\exists}$ and $(\mathbf{F}^\infty)_\exists \simeq_{\mathrm{T}} (\mathbf{F}^\infty)_{\mathbb{N}\cup\{0\},\exists}$. As $(\mathbf{F}^\infty)_\exists = \mathbf{G}_\exists$ (Proposition 5.2.15), the claim follows via Proposition 5.1.4. $\square$

Chapter 6

Theories and existential theories of mixed characteristic HVFs

Rows and floes of angel hair
And ice cream castles in the air
And feather canyons everywhere
I looked at clouds that way

But now they only block the sun
They rain and they snow on everyone
So many things I would have done
But clouds got in my way

I've looked at clouds from both sides now
From up and down and still somehow
It's cloud illusions I recall
I really don't know clouds at all

Both Sides Now, Joni Mitchell

This chapter is a short exposition of the main results of [AJ22, ADJ24], touching also on material from [AJ26+].

A valued field (K, v) of mixed characteristic is **finitely ramified** if, for some $e \in \mathbb{N}_{>0}$, there are exactly e elements $\gamma \in \Gamma_v$ with $0 < \gamma \leq v(p)$. This number e is the **initial ramification** of (K, v). We say (K, v) is **unramified** when $e = 1$, i.e. when $v(p)$ is minimum positive in the value group.

Already in [AK65b], Ax and Kochen proved that the theory of $\mathbb{Q}_p$ is axiomatized by the theory of henselian unramified valued fields of mixed characteristic, with value group elementarily equivalent to $\mathbb{Z}$ and residue field isomorphic to $\mathbb{F}_p$. See [PR84] for a detailed exposition of the general p-adic case: those finitely ramified henselian valued fields with value group a $\mathbb{Z}$-group, and finite residue fields. See also Macintyre's survey [Mac86]. The general unramified case has been worked out in [Ers65], in the case of perfect residue field, and in [AJ22] in general. The finitely ramified case is more subtle, but both Ershov ([Ers01]) and Ziegler ([Zie72]) showed that an embedding of finitely ramified henselian valued fields is elementary if and only if the induced embeddings of value group and residue field are elementary: in terminology we will introduce, the class of finitely ramified henselian valued fields is an AKE$^{\preceq}$-class. However, without further structure on the residue field, reducing elementary equivalence and existential equivalence to corresponding properties of residue field and value group fails: this is illustrated in Examples 6.1.1 and 6.1.2. It was shown in [Dit25, Example 5.12] that the transfer of existential decidability from residue field to valued field is false.

Let $\mathbf{H}^{\mathrm{ur}}$ be the $\mathfrak{L}_{\mathrm{val}}$-theory of unramified henselian valued fields, and for each $e \in \mathbb{N}_{>0}$ let $\mathbf{H}^{\mathrm{fr}}_e$ be the $\mathfrak{L}_{\mathrm{val}}$-theory of finitely ramified henselian valued fields of initial ramification e. Of course, $\mathbf{H}^{\mathrm{ur}}$ is the same as $\mathbf{H}^{\mathrm{fr}}_e$ with $e = 1$.

In [AJ22] with Jahnke, and in [ADJ24] with Dittmann and Jahnke, we proved several AKE-type theorems on the model theory of unramified and finitely ramified henselian valued fields.

Theorem 6.0.1 ([AJ22, Theorems 1.2 and 1.3]). *Let* $(K, v), (L, w) \models \mathbf{H}^{\mathrm{ur}}$. *Then*

$$\underbrace{(K, v) \equiv (L, w)}_{\text{in } \mathfrak{L}_{\mathrm{val}}} \Leftrightarrow \underbrace{k_v \equiv k_w}_{\text{in } \mathfrak{L}_{\mathrm{ring}}} \text{ and } \underbrace{\Gamma_v \equiv \Gamma_w}_{\text{in } \mathfrak{L}_{\mathrm{oag}}},$$

and if $(K, v) \subseteq (L, w)$ *then also*

$$\underbrace{(K, v) \preceq (L, w)}_{\text{in } \mathfrak{L}_{\mathrm{val}}} \Leftrightarrow \underbrace{k_v \preceq k_w}_{\text{in } \mathfrak{L}_{\mathrm{ring}}} \text{ and } \underbrace{\Gamma_v \preceq \Gamma_w}_{\text{in } \mathfrak{L}_{\mathrm{oag}}}.$$

We denote by $\exists+$ the **positive-existential** fragment, i.e. $\exists+ = \exists \mathsf{F}_{0,+}$, where $\mathsf{F}_{0,+}(\mathfrak{L})$ is the $\mathfrak{L}$-fragment generated by atomic formulas, for each language $\mathfrak{L}$. In section 6.2, we introduce a language $\mathfrak{L}_{p,e}$ expanding $\mathfrak{L}$, and a corresponding expansion $k_v^\dagger$ of k_v that is definable in (K, v), in order to account for the extra structure on the residue field and so obtain the following theorem.

Theorem 6.0.2 (Cf [ADJ24, Main theorem]). *Let* $(K, v), (L, w) \models \mathbf{H}^{\mathrm{fr}}_{e,p}$ *be finitely ramified valued fields of mixed characteristic* $(0, p)$ *and of initial ramification e. Then, we have*

$$\underbrace{(K, v) \equiv (L, w)}_{\text{in } \mathfrak{L}_{\mathrm{val}}} \Leftrightarrow \underbrace{k_v^\dagger \equiv k_w^\dagger}_{\text{in } \mathfrak{L}_{p,e}} \text{ and } \underbrace{\Gamma_v \equiv \Gamma_w}_{\text{in } \mathfrak{L}_{\mathrm{oag}}},$$

and

$$\underbrace{(K, v) \equiv_{\exists} (L, w)}_{\text{in } \mathfrak{L}_{\mathrm{val}}} \Leftrightarrow \underbrace{k_v^\dagger \equiv_{\exists+} k_w^\dagger}_{\text{in } \mathfrak{L}_{p,e}}.$$

If $(K, v) \subseteq (L, w)$ *then also*

$$\underbrace{(K, v) \preceq_{\exists} (L, w)}_{\text{in } \mathfrak{L}_{\mathrm{val}}} \Leftrightarrow \underbrace{k_v^\dagger \preceq_{\exists} k_w^\dagger}_{\text{in } \mathfrak{L}_{\mathrm{ring}}} \text{ and } \underbrace{\Gamma_w \preceq_{\exists} \Gamma_w}_{\text{in } \mathfrak{L}_{\mathrm{oag}}}.$$

Note that $k_v^\dagger$ depends on (K, v). Also, since in the unramified case $k_v^\dagger$ amounts to just the $\mathfrak{L}_{\mathrm{ring}}$-structure k_v, Theorem 6.0.2 subsumes Theorem 6.0.1. We can strengthen the above results, just a little, giving a view that is suitably resplendent and uniform in the residue characteristic. Recall (or, rather, look ahead to) the notion of an $\mathrm{sAKE}^{\blacklozenge}$-class, for $\blacklozenge \in \{\equiv, \equiv_\exists, \preceq, \preceq_\exists\}$, from Definition 8.2.2. Since our valued fields in this chapter are of characteristic zero, we may safely ignore the issues around separability (of valued field extensions), so we write $\mathrm{AKE}^{\blacklozenge}$ for the property $\mathrm{sAKE}^{\blacklozenge}$ but removing the restrictions on imperfection degree (which anyway trivialize in characteristic zero).

Theorem 6.0.3 (Resplendent Ax–Kochen/Ershov for $\mathbf{H}^{\mathrm{fr}}_e$). *Let* $\mathfrak{L} = \mathfrak{L}_{\mathrm{val},\lambda}(\mathfrak{L}_{\mathbf{k}}, \mathfrak{L}_\Gamma)$ *be a* $(\mathbf{k}, \Gamma)$*-expansion of* $\mathfrak{L}_{\mathrm{val},\lambda}$*, expanding* $\mathfrak{L}_e$ *on* $\mathbf{k}$*. Let* $\blacklozenge \in \{\equiv, \preceq, \preceq_\exists\}$*. Then the class* $\mathbf{Mod}_{\mathfrak{L}}(\mathbf{H}^{\mathrm{fr}}_e)$ *is an* $\mathrm{sAKE}^{\blacklozenge}$*-class for* $(\mathfrak{L}, \mathfrak{L}_{\mathbf{k}}, \mathfrak{L}_\Gamma)$*. Furthermore, for* $K, L \models \mathbf{H}^{\mathrm{fr}}_e$*, with valuations v and w respectively, we have*

$$\underbrace{K \equiv_{\exists} L}_{\text{in } \mathfrak{L}} \Leftrightarrow \underbrace{k_v^\dagger \equiv_{\exists+} k_w^\dagger}_{\text{in } \mathfrak{L}_{p,e}}.$$

Turning to questions around stable embeddedness, we proved several results:

Observation 6.0.4 ([AJ22, Example 11.5], [ADJ24, Remark 6.3]). k_v is not canonically embedded in $\mathbf{H}^{\mathrm{fr}}_e$.

Theorem 6.0.5 ([AJ22, Theorem 1.4], [ADJ24, Theorem 6.2, Remark 6.3]). *For* $e \in \mathbb{N}_{>0}$*, we let* $(K, v) \models \mathbf{H}^{\mathrm{fr}}_e$*. The sorts* $\mathbf{k}$ *of the residue field and* Γ *of the value group are both stably embedded and are orthogonal, as a pure field and pure ordered abelian group, respectively.*

6.1 Two important examples

We quote directly from [ADJ24] two examples that show one cannot hope for AKE principles in the finitely ramified case without taking account of extra structure. The first example shows that simply axiomatizing the residue field and value group is not enough to determine the algebraic part of a model. The second example shows that even axiomatizing the algebraic part of the field is not enough to determine even the existential fragment of the theory of a model.

Example 6.1.1 ([ADJ24, Example 2.3])**.** Suppose $p \neq 2$ and let $c \in \mathbb{Z}$ be a quadratic non-residue modulo p. Then the quadratic extensions $\mathbb{Q}_p(\sqrt{p})$ and $\mathbb{Q}_p(\sqrt{cp})$ of $\mathbb{Q}_p$ are distinct, since the elements p and cp of $\mathbb{Q}_p$ lie in distinct square classes by construction. Now $\mathbb{Q}_p(\sqrt{p})$ and $\mathbb{Q}_p(\sqrt{cp})$ with the unique extensions of the p-adic valuation on $\mathbb{Q}_p$ are henselian valued fields with value group isomorphic to $\mathbb{Z}$, residue field $\mathbb{F}_p$ and initial ramification 2. As they have different algebraic parts – indeed, only one of them contains a square root of p –, they do not have the same existential $\mathfrak{L}_{\mathrm{ring}}$-theory, and in particular are not $\mathfrak{L}_{\mathrm{ring}}$-elementarily equivalent. (Compare also [AF16, Remark 7.4] for related observations.)

Example 6.1.2 ([ADJ24, Example 2.4])**.** Again we suppose $p \neq 2$. Consider $F = \mathbb{F}_p(t)^{\mathrm{perf}}$ (the perfect hull of $\mathbb{F}_p(t)$), and let (K_0, v) be the fraction field of the ring of Witt vectors over F with its natural valuation with value group $\mathbb{Z}$ and residue field F. Let $\tau : F \longrightarrow K_0$ be the unique multiplicative map choosing Teichmüller representatives as in [AJ22, Theorem 3.3]. Let $\alpha_1 = \sqrt{p\tau(t)}$, and $\alpha_2 = \sqrt{p\tau(t^3+1)}$. Consider $K_1 := K_0(\alpha_1)$ and $K_2 := K_0(\alpha_2)$, each endowed with the unique extension v_i of v to K_i. Then (K_1, v_1) and (K_2, v_2) are both complete (and in particular henselian), and they both have the same value group (isomorphic to $\mathbb{Z}$), the same initial ramification (namely 2), and the same residue field F. Moreover, we argue that they also have the same algebraic part, namely $\mathbb{Q}_{p,\mathrm{alg}} = \mathbb{Q}_p \cap \mathbb{Q}^{\mathrm{alg}}$. A priori it is clear that the algebraic parts are either $\mathbb{Q}_{p,\mathrm{alg}}$ or ramified quadratic extensions thereof, since K_0 has algebraic part $\mathbb{Q}_{p,\mathrm{alg}}$. Consider first K_1. If the algebraic part $K_{1,\mathrm{alg}}$ of K_1 is a quadratic extension of $\mathbb{Q}_{p,\mathrm{alg}}$ with uniformizer π, we have $p = \tau(a)u\pi^2$ for some $a \in \mathbb{F}_p^\times$ and some $u \in K_{1,\mathrm{alg}}$ with $v_1(u-1) > 0$. Then u has a square root in $K_{1,\mathrm{alg}}$ by Hensel's Lemma, so $\sqrt{p\tau(a)} \in K_{1,\mathrm{alg}}$ and hence $\sqrt{\tau(at)} \in K_1$. It follows that in the residue field of K_1 (i.e. in F), at has a square root — this is a contradiction. Similarly, for K_2, if the algebraic part $K_{2,\mathrm{alg}}$ were a proper extension of $\mathbb{Q}_{p,\mathrm{alg}}$, then for some $a \in \mathbb{F}_p^\times$, $a(t^3+1)$ would have a square root in F, which is again a contradiction.

Nevertheless, K_1 and K_2 do not have the same existential $\mathfrak{L}_{\mathrm{val}}$-theory: by construction the curve $C : Y^2 = X^3 + 1$ has a rational point with X-coordinate t in the residue field $F(\sqrt{t^3+1})$ of $K_2(\sqrt{p})$; however, in the residue field $F(\sqrt{t})$ of $K_1(\sqrt{p})$, all rational points on the curve C of genus 1 have coordinates in $\mathbb{F}_p$ since $F(\sqrt{t}) = \bigcup_{n\geq 0} \mathbb{F}_p(t^{1/(2p^n)})$ is an increasing union of function fields over $\mathbb{F}_p$ of genus 0 (see [Koe02, Lemma 3.2], as well as the analogous argument in [DJKK23, Example 3.7]). In fact K_1 and K_2 do not even have the same existential $\mathfrak{L}_{\mathrm{ring}}$-theory.

6.2 The language $\mathfrak{L}_e$

We describe an expansion $\mathfrak{L}_e$ of $\mathfrak{L}_{\mathrm{ring}}$ by one new predicate that captures the induced structure on the residue field of a finitely ramified henselian valued field. In [ADJ24, Definition 3.7], two fast-growing primitive recursive functions $M_0(p,-,-,-), M_1(p,-,-,-) : \mathbb{N}^3 \to \mathbb{N}$ are defined in order to bound the arity of the new predicate. We also write $d(p,e) = e(1+v_p(e))$.

Definition 6.2.1 (Expanded language $\mathfrak{L}_e$)**.** Let $d = d(p,e)$ and $m = M_1(p,e,de-1,d)$. We define $\mathfrak{L}_{p,e}$ to be the expansion of $\mathfrak{L}_{\mathrm{ring}}$ by an $(e(d-1)p^{(de-1)m} + m)$-ary predicate symbol[1] Ω. Let $\mathfrak{L}_e$ denote the union of the languages $\mathfrak{L}_{p,e}$, for all $p \in \mathbb{P}$.

[1]For the basic case $e = 1$, this means we have a 0-ary predicate symbol. Such symbols are traditionally not considered in model theory (although see [ADH17, Section B.2]), however this causes us no problem.

For an m-tuple $b = (b_j)_{1\le j\le m}$ in some ring and a multi-index $I = (i_j)_{1\le j\le m}$ we let $b^I := \prod_{j=1}^m b_j^{i_j}$ be the I-th monomial in the b_j. We denote by $p^{n[m]}$ the set of such multi-indices consisting of indices $i_j < p^n$.

Definition 6.2.2 ([ADJ24, Definition 3.6])**.** Let (K, v) be a finitely ramified valued field of initial ramification e and residue field k. For $d > 0$, we write $\mathrm{res}_d : \mathcal{O}_v \to \mathcal{O}_v/(p^d)$ for the d-th higher residue map. For $d > 0$ and $m \ge 0$, we define:

$$\Omega_{d,e,m}(K,v) = \left\{ (\gamma_{i,j,I}, \beta_l)_{\substack{0\le i<e,\\ 0<j<d,\\ I\in p^{(de-1)[m]},\\ 0\le l<m}} \in k^{e(d-1)p^{(de-1)m}+m} \;\middle|\; \begin{array}{l} \exists b \in \mathcal{O}_v^m \text{ such that } \mathrm{res}(b) = \beta \text{ and} \\ \text{for } c_{ij} := \sum\limits_{I\in p^{(de-1)[m]}} \mathrm{res}_d(b^I)\tau(\gamma_{i,j,I}^{p^{de-1}}) \text{ the} \\ \text{polynomial } X^e + \sum\limits_{i=0}^{e-1}(\sum\limits_{j=1}^{d-1} c_{ij}p^j)X^i \\ \text{has a root in } \mathcal{O}_v/p^d. \end{array} \right\}$$

Given a valued field (K, v) of initial ramification e, we consider the expansion $k_v^\dagger$ of the residue field k_v to an $\mathfrak{L}_e$-structure by interpreting Ω as the set $\Omega_{d,e,m}(K, v)$. We call this new structure the **$\dagger$-residue field** of (K, v). Although it is not obvious from the definition, each of the sets $\Omega_{d,e,m}(K, v)$, and therefore the entire $\mathfrak{L}_e$-structure, is $\mathfrak{L}_{\mathrm{ring}}$-definable in Kv using parameters.

Remark 6.2.3 ([ADJ24, Remark 3.9]). It is clear from the definition that the sets $\Omega_{d,e,m}(K, v)$ only depend on the ring $\mathcal{O}_v/p^d$. This has the following consequence: If v_0 is a proper coarsening of v and $\bar{v}$ the valuation induced on the residue field Kv_0, then $\Omega_{d,e,m}(K, v) = \Omega_{d,e,m}(Kv_0, \bar{v})$ since $\mathcal{O}_{\bar{v}}/p^d = \mathcal{O}_v/p^d$. Therefore the $\mathfrak{L}_e$-structure induced on the residue field $Kv = (Kv_0)\bar{v}$ is the same for (K, v) and $(Kv_0, \bar{v})$. Furthermore, since the $\mathfrak{L}_e$-structure Kv is completely determined by $\mathcal{O}_v/p^{d(e)}$, our Ax–Kochen/Ershov style results formally imply results in terms of residue rings $\mathcal{O}_v/p^n$ (see [ADJ24, Corollaries 5.5 and 5.13]) in the style of Basarab [Bas78, Theorem 3.1] and Lee–Lee [LL21, Theorem 5.2].

Lemma 6.2.4 ([ADJ24, Lemma 3.10])**.** *Let (K, v) be a finitely ramified valued field with residue field k and initial ramification e. Each set $\Omega_{d,e,m}(K, v)$ is existentially definable without parameters in the $\mathfrak{L}_{\mathrm{val}}$-structure (K, v) by a formula depending only on e and p. In particular, the $\mathfrak{L}_e$-structure is definable on the residue sort of (K, v).*

This lemma, together with the uniformity of the definitions of $\Omega_{d,e,m}$ in models of $\mathbf{H}_e^{\mathrm{fr}}$, means that the "standard interpretation" of the residue field $\iota_{\mathbf{k}} : \mathrm{Form}(\mathfrak{L}_{\mathrm{ring}}) \to \mathrm{Form}(\mathfrak{L}_{\mathrm{val}})$ can be extended to an interpretation $\iota_{\mathbf{k}} : \mathrm{Form}(\mathfrak{L}_e) \to \mathrm{Form}(\mathfrak{L}_{\mathrm{val}})$ of the $\dagger$-residue field, that moreover restricts to an interpretation $\mathrm{Form}_{\exists^+}(\mathfrak{L}_e) \to \mathrm{Form}_{\exists}(\mathfrak{L}_{\mathrm{val}})$.

6.3 The embedding lemmas and monotonicity

In order to prove Ax–Kochen/Ershov principles for $\mathbf{H}_e^{\mathrm{fr}}$, we first need to prove that the extra structure given by the predicate $\Omega_{d,e,m}(K, v)$ really does ensure the existence of embeddings between models. Just as in the unramified setting, the existentence of such embeddings boils down to their existence at the level of complete $\mathbb{Z}$-valued fields, which appear as residue fields of the finest proper coarsening of v, together with the valuation induced by v. The extra difficulty in our case is that we need to find an embedding under the hypothesis that the domain is unramified, but which is compatible with given lifts of a p-basis of the residue field of the domain.

There are many results in the literature around embeddings between CDVRs, for example by Witt, Teichmüller, Mac Lane, and Cohen. Our setting is typically that we have two $\mathbb{Z}$-valued fields (K, v) and (L, w) of mixed characteristic with residue fields $k = k_v$ and $l = k_w$, and a given ring homomorphism $\varphi_{\mathbf{k}} : k \to l$. We are given to suppose that (K, v) is unramified. We let $b \subseteq k$ be a p-basis, and we are given choices of representatives $s_K : b \to \mathcal{O}_v$ and $s_L : \varphi_{\mathbf{k}}(b) \to \mathcal{O}_w$. We seek an embedding of valued fields $\varphi : (K, v) \to (L, w)$ that induces $\varphi_{\mathbf{k}}$ and is compatible with s_K and s_L insofar as $s_L \circ \varphi_{\mathbf{k}} = \varphi \circ s_K$.

What was clear from the literature was how to deal with the case that $l/\varphi_{\mathrm{k}}k$ is separable. Thus we proved the following Lemma and Proposition. Nevertheless, after the fact, Dittmann pointed out a more abstract expression of the same principle, from [SP]. We fix $e \in \mathbb{N}_{>0}$.

Lemma 6.3.1 ([ADJ24, Lemma 4.2]). *Let (K, v) and (L, w) be $\mathbb{Z}$-valued fields with residue fields k, l. Assume that K is unramified and L is complete. Let $\varphi : k \to l$ be a field embedding. Let b be a (possibly infinite) tuple of p-independent elements of k, $s_K : b \to \mathcal{O}_v$ a choice of representatives in K, and $s_L : \varphi_{\mathrm{k}}(b) \to \mathcal{O}_w$ a choice of representatives in L. Then there exists an embedding $\varphi : (K, v) \to (L, w)$ of valued fields compatible with s_K and s_L, i.e. one sending $s_K(b)$ to $s_L(\varphi_{\mathrm{k}}(b))$.*

Proof idea. By now we have multiple proof strategies. The proof we give in [ADJ24] is to first deal with the special case that b is a separating trascendence basis of k. In this case, the subring of $\mathcal{O}_v$ generated by $s_K(b)$ may be mapped across to $\mathcal{O}_w$ by an obvious ring homomorphism, which is compatible with s_K and s_L, and which can be seen to preserve the valuation. The remaining extension of k over the separating transcendence basis is separably algebraic, and now henselianity ensures that φ extends to $\mathcal{O}_v \to \mathcal{O}_w$. The general case is now an amalgamation problem, for which we use an ultraproduct: every field is a direct limit of fields that admit separating transcendence bases. This yields a suitable embedding of $\mathcal{O}_v \to \mathcal{O}_w^*$. We recover the sought-after embedding $\mathcal{O}_v \to \mathcal{O}_w$ by composing with the place corresponding to the finest proper coarsening of w. $\square$

Proposition 6.3.2 ([ADJ24, Lemma 4.6]). *Let (K, v) and (L, w) be complete $\mathbb{Z}$-valued fields with $\dagger$-residue fields k and l, respectively, each of initial ramification e. Then, every $\mathfrak{L}_e$-homomorphism $k \to l$ is induced by an $\mathfrak{L}_{\mathrm{val}}$-embedding $(K, v) \to (L, w)$. Moreover every $\mathfrak{L}_e$-isomorphism $k \to l$ is induced by an $\mathfrak{L}_{\mathrm{val}}$-isomorphism $(K, v) \to (L, w)$.*

The above embedding lemma is suitable for the $\mathbb{Z}$-valued case. Moving to arbitrary rank, we have the following general embedding result, which we give in a relative version, over a suitable subfield.

Proposition 6.3.3 (Embedding Lemma for $\mathbf{H}_e^{\mathrm{fr}}$, [ADJ24, Lemma 4.12]). *Let (K, v) and (L, w) be two extensions of a valued field (K_0, v_0) and suppose that all three fields are henselian and finitely ramified of initial ramification e. Assume that v_0K_0 is pure in vK and that Kv/K_0v_0 is separable. Moreover, assume that (K_0, v_0) and (K, v) are $\aleph_1$-saturated, and that (L, w) is $|K|^+$-saturated. Then every pair of an $\mathfrak{L}_{\mathrm{oag}}$-embedding $\varphi_\Gamma : vK \to wL$ over v_0K_0 and an $\mathfrak{L}_{\mathrm{ring}}$-embedding $\varphi_{\mathrm{k}} : Kv \to Lw$ over K_0v_0 is induced by an embedding $\varphi : K \to L$ over K_0.*

This is proved as a combination of Proposition 6.3.1 and the analogous principles in equal characteristic zero. Note that we are only given an $\mathfrak{L}_{\mathrm{ring}}$-embedding between residue fields — this is a reflection of the stable embeddedness of the residue field.

6.4 Consequences of the embedding lemma

Copying the strategy one uses in equicharacteristic, we use the embedding lemma to get a variety of Ax–Kochen/Ershov principles, i.e. Theorem 6.0.3, as follows:

Proof sketch of Theorem 6.0.3. Now that we have the embedding lemma, this is all fairly standard. We sketch the "furthermore" claim. Let $(K, v), (L, w) \models \mathbf{H}_e^{\mathrm{fr}}$. We may assume that both are $\aleph_1$-saturated, let v^+ and w^+ denote the finest proper coarsenings of v and w, respectively. On the residue fields Kv^+ and Lw^+ are induced complete $\mathbb{Z}$-valuations $\bar{v}$ and $\bar{w}$, respectively. If $\mathrm{Th}_{\exists^+}(k_v^\dagger) \subseteq \mathrm{Th}_{\exists^+}(k_w^\dagger)$ (which also holds for the identical $\dagger$-residue fields of $(Kv^+, \bar{v})$ and $(Lw^+, \bar{w})$), then passing to an elementary extension of (L, w) if necessary, there is an $\mathfrak{L}_e$ homomorphism $\varphi_{\mathrm{k}} : k_v^\dagger \to k_w^\dagger$. By Proposition 6.3.2, this extends to an $\mathfrak{L}_{\mathrm{val}}$-embedding $(Kv^+, \bar{v}) \to (Lw^+, \bar{w})$, which shows that $\mathrm{Th}_\exists(Kv^+, \bar{v}) \subseteq \mathrm{Th}_\exists(Lw^+, \bar{w})$. By the AKE principle in equal characteristic zero, we deduce $\mathrm{Th}_\exists(K, v^+) \subseteq \mathrm{Th}_\exists(L, w^+)$. In particular $\mathrm{Th}_\exists(K) \subseteq \mathrm{Th}_\exists(L)$. By the existential and universal $\mathfrak{L}_{\mathrm{ring}}$-definabiltiy (without parameters) of the valuations v and w in K and L, respectively, we conclude $\mathrm{Th}_\exists(K, v) \subseteq \mathrm{Th}_\exists(L, w)$ as required. $\square$

Corollary 6.4.1 (Monotonicity). *Let* $(K,v),(L,w)\models\mathbf{H}^{\mathrm{fr}}_{e,p}$. *Then*

$$\underbrace{\mathrm{Th}_{\exists+}(k_v^\dagger)\subseteq\mathrm{Th}_{\exists+}(k_w^\dagger)}_{in\ \mathfrak{L}_e}\Rightarrow\underbrace{\mathrm{Th}_\exists(K,v)\subseteq\mathrm{Th}_\exists(L,w)}_{in\ \mathfrak{L}_{\mathrm{val}}}$$

and

$$\underbrace{\mathrm{Th}(k_v^\dagger)=\mathrm{Th}(k_w^\dagger)}_{in\ \mathfrak{L}_e}\ and\ \underbrace{\mathrm{Th}(\Gamma_v)=\mathrm{Th}(\Gamma_w)}_{in\ \mathfrak{L}_{\mathrm{oag}}}\Rightarrow\underbrace{\mathrm{Th}(K,v)=\mathrm{Th}(L,w).}_{in\ \mathfrak{L}_{\mathrm{val}}}$$

Proof idea. For the sake of exposition, we sketch an argument directly from the embedding lemma, rather than applying the Ax–Kochen/Ershov principles, which would need to be slightly reformulated in any case. The first statement is rather easy: if $\mathrm{Th}_{\exists+}(k_v^\dagger)\subseteq\mathrm{Th}_{\exists+}(k_w^\dagger)$, then there is a $\mathfrak{L}_e$-embedding $\varphi_{\mathbf{k}}$ of $k_v^\dagger$ into a sufficiently saturated elementary extension of $k_w^\dagger$. Since all discrete ordered abelian groups share the same existential theory, there is also a $\mathfrak{L}_{\mathrm{oag}}$-embedding φ_Γ of Γ_v into a sufficiently saturated elementary extension of Γ_w, preserving the smallest positive element. Using now Proposition 6.3.3, and replacing (L,w) be a suitable elementary extension, we get an $\mathfrak{L}_{\mathrm{val}}$-embedding $(K,v)\to(L,w)$ inducing $\varphi_{\mathbf{k}}$ and φ_Γ. This shows in particular that $\mathrm{Th}_\exists(K,v)\subseteq\mathrm{Th}_\exists(L,w)$ in $\mathfrak{L}_{\mathrm{val}}$. The second statement is apparently harder, but we can take the Keisler–Shelah shortcut to find isomorphisms $k_v^\dagger\to k_w^\dagger$ and $\Gamma_v\to\Gamma_w$, perhaps by passing again to sufficiently saturated elementary extensions. Now applying the structure theory of CDVRs and the Ax–Kochen/Ershov theory in equal characteristic zero, we that that K and L are elementarily equivalent in $\mathfrak{L}_{\mathrm{ring}}$. The elementary equivalence in $\mathfrak{L}_{\mathrm{val}}$ follows from the $\emptyset$-definability of the valuations. □

6.5 Consequences of monotonicity

We denote by $\mathbf{VF}^{\mathrm{fr}}_e$ the $\mathfrak{L}_{\mathrm{val}}$-theory of valued fields of mixed characteristic, finitely ramified with initial ramification e, and let $\mathrm{VF}^{\mathrm{fr}}_e$ be the context $(\mathrm{Sent}(\mathfrak{L}_{\mathrm{val}}),\mathbf{VF}^{\mathrm{fr}}_e)$. As above we denote by $\mathbf{H}^{\mathrm{fr}}_e$ the $\mathfrak{L}_{\mathrm{val}}$-theory of finitely ramified henselian valued fields of initial ramification e. We define the contexts $\mathrm{H}^{\mathrm{fr}}_e:=(\mathrm{Sent}(\mathfrak{L}_{\mathrm{val}}),\mathbf{H}^{\mathrm{fr}}_e)$ and analogously $\mathrm{H}^{\mathrm{fr}}_{e,\exists}:=(\mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{val}}),\mathbf{H}^{\mathrm{fr}}_e)$. We denote by $\mathbf{F}^{\dagger,e}$ the $\mathfrak{L}_e$-theory of $\dagger$-fields suitable for initial ramification e, and the contexts $\mathrm{F}^{\dagger,e}=(\mathrm{Sent}(\mathfrak{L}_e),\mathbf{F}^{\dagger,e})$ and analogously $\mathrm{F}^{\dagger,e}_{\exists+}=(\mathrm{Sent}_{\exists+}(\mathfrak{L}_e),\mathbf{F}^{\dagger,e})$. We denote by **DiscOAG** the $\mathfrak{L}_{\mathrm{oag}}$-theory of discrete ordered abelian groups and the context $\mathrm{F}^{\dagger,e}\mathrm{DiscOAG}=(\mathrm{Sent}(\mathfrak{L}_e\sqcup\mathfrak{L}_{\mathrm{oag}}),\mathbf{F}^{\dagger,e}\cup\mathbf{DiscOAG})$ where $\mathfrak{L}_e\sqcup\mathfrak{L}_{\mathrm{oag}}$ denotes the two-sorted language with sorts $\mathbf{k}$ and $\mathbf{\Gamma}$, the former equipped with $\mathfrak{L}_e$ and the latter with $\mathfrak{L}_{\mathrm{oag}}$. Like above, we denote by $\sigma_{\mathbf{k}}^\dagger$ the map that sends a valued field (K,v) of finite ramification to its $\dagger$-residue field Kv, and we denote by $\sigma_{\mathbf{k},\Gamma}^\dagger$ the map that sends (K,v) to the pair of its $\dagger$-residue field and its value group. Thus by equipping each with an appropriate restriction of $\sigma_{\mathbf{k}}^\dagger$ or $\sigma_{\mathbf{k},\Gamma}^\dagger$, we have the bridges $\mathrm{F}^{\dagger,e}.\mathrm{H}^{\mathrm{fr}}_e$ and $\mathrm{F}^{\dagger,e}_{\exists+}.\mathrm{H}^{\mathrm{fr}}_{e,\exists}$, as well as $\mathrm{F}^{\dagger,e}\mathrm{DiscOAG}.\mathrm{H}^{\mathrm{fr}}_e$.

Again following the lead of above, we define $\iota_{\mathbf{k}}^\dagger$ to be the standard interpretation of $\mathfrak{L}_e$-formulas in $\mathfrak{L}_{\mathrm{val}}$-formulas, and define ι_Γ to be the standard interpretation of $\mathfrak{L}_{\mathrm{oag}}$-formulas in $\mathfrak{L}_{\mathrm{val}}$, both suitable for the case of describing $\dagger$-residue fields and value groups in valued fields of mixed characteristic and initial ramification e. Of course, these may be combined into $\iota_{\mathbf{k},\Gamma}^\dagger$ which is an interpretation of $\mathfrak{L}_e\cup\mathfrak{L}_{\mathrm{oag}}$-formulas in $\mathfrak{L}_{\mathrm{val}}$-formulas. More precisely: the map $\iota_{\mathbf{k}}^\dagger$ is an interpretation for the bridge $(\mathrm{F}^{\dagger,e},\mathrm{VF}^{\mathrm{fr}}_e,\sigma_{\mathbf{k}})$, that restricts to an interpretation for the bridge $(\mathrm{F}^{\dagger,e}_{\exists+},\mathrm{VF}^{\mathrm{fr}}_{e,\exists},\sigma_{\mathbf{k}})$, and $\iota_{\mathbf{k},\Gamma}^\dagger$ is an interpretation for the bridge $(\mathrm{F}^{\dagger,e}\mathrm{DiscOAG},\mathrm{VF}^{\mathrm{fr}}_e,\sigma_{\mathbf{k},\Gamma}^\dagger)$.

Equipped with the appropriate restrictions of $\iota_{\mathbf{k}}^\dagger$ and $\iota_{\mathbf{k},\Gamma}^\dagger$, we get arches

(i) $\mathrm{F}^{\dagger,e}_{\exists+}.\mathrm{H}^{\mathrm{fr}}_{e,\exists}|\mathrm{F}^{\dagger,e}.\mathrm{H}^{\mathrm{fr}}_e$ and

(ii) $\mathrm{F}^{\dagger,e}\mathrm{DiscOAG}.\mathrm{H}^{\mathrm{fr}}_e|\mathrm{F}^{\dagger,e}\mathrm{DiscOAG}.\mathrm{H}^{\mathrm{fr}}_e$, where we use that $\iota_{\mathbf{k}}^{\dagger}(\mathrm{Sent}_{\exists+}(\mathfrak{L}_e)) \subseteq \mathrm{Sent}_{\exists}(\mathfrak{L}_{\mathrm{val}})$.

We observe that the notion of an arch becomes redundant in situations like **(ii)**, where one has two copies of the same bridge, in this case $\mathrm{F}^{\dagger,e}\mathrm{DiscOAG.H}_e^{\mathrm{fr}}$. For any arch of the form $B.B$, we might as well express the hypotheses and conclusions of Corollary 2.4.20 (and similar results) as properties of simply the bridge B, as we will in part **(b)** of the following lemma.

Lemma 6.5.1.

(a) *Hypotheses (i)–(iii) of Corollary 2.4.20 hold for* $\mathrm{F}_{\exists+}^{\dagger,e}.\mathrm{H}_{e,\exists}^{\mathrm{fr}}|\mathrm{F}^{\dagger,e}.\mathrm{H}_e^{\mathrm{fr}}$.

(b) *Hypotheses (i)–(iii) of Corollary 2.4.20 hold for* $\mathrm{F}^{\dagger,e}\mathrm{DiscOAG.H}_e^{\mathrm{fr}}$.

Corollary 6.5.2.

(a) (I) $\mathrm{F}_{\exists+}^{\dagger,e}.\mathrm{H}_{e,\exists}^{\mathrm{fr}}$ *admits a computable elimination* $\mathrm{Sent}_{\exists}(\mathfrak{L}_{\mathrm{val}}) \to \mathrm{Sent}_{\exists+}(\mathfrak{L}_e)$.

(II) *For every* $R \subseteq \mathrm{Sent}(\mathfrak{L}_e)$, $(\mathbf{F}^{\dagger,e} \cup R)_{\exists+} \simeq_{\mathrm{m}} (\mathbf{H}_e^{\mathrm{fr}} \cup \iota_{\mathbf{k}}^{\dagger} R)_{\exists}$.

(III) *For every* $(K,v) \models \mathbf{H}_e^{\mathrm{fr}}$, $\mathrm{Th}_{\exists}(K,v) = (\mathbf{H}_e^{\mathrm{fr}} \cup \iota_{\mathbf{k}}^{\dagger}\mathrm{Th}_{\exists+}(Kv))_{\exists}$.

(b) (I) $\mathrm{F}^{\dagger,e}\mathrm{DiscOAG.H}_e^{\mathrm{fr}}$ *admits a computable elimination* $\mathrm{Sent}(\mathfrak{L}_{\mathrm{val}}) \longrightarrow \mathrm{Sent}(\mathfrak{L}_e \cup \mathfrak{L}_{\mathrm{oag}})$.

(II) *For every* $R \subseteq \mathrm{Sent}(\mathfrak{L}_e)$ *and* $G \subseteq \mathrm{Sent}(\mathfrak{L}_{\mathrm{oag}})$, $(\mathbf{F}^{\dagger,e} \cup R) \oplus_T (\mathbf{DiscOAG} \cup G) \simeq_{\mathrm{T}} (\mathbf{H}_0^{e,\varpi} \cup \iota_{\mathbf{k}} R \cup \iota_{\Gamma} G)^{\vdash}$.

(III) *For every* $(K,v) \models \mathbf{H}_e^{\mathrm{fr}}$, $\mathrm{Th}(K,v) = (\mathbf{H}_e^{\mathrm{fr}} \cup \iota_{\mathbf{k}}\mathrm{Th}(k_v^{\dagger}) \cup \iota_{\Gamma}\mathrm{Th}(\Gamma_v))^{\vdash}$.

One obvious flaw of this formalism is that the computable elimination in **(b)**(I) of the corollary takes its image in the fragment generated by $\iota_{\mathbf{k}}\mathrm{Sent}(\mathfrak{L}_e) \cup \iota_{\Gamma}\mathrm{Sent}(\mathfrak{L}_{\mathrm{oag}})$, and we can dispense with the much larger given codomain $\mathrm{Sent}(\mathfrak{L}_e \cup \mathfrak{L}_{\mathrm{oag}})$.

Main Theorem 4 follows from Corollary 6.5.2. Completions of $\mathbf{H}_e^{\mathrm{fr}}$ are now described by complete theories of discrete ordered abelian groups, together with complete theories of †-fields (for initial ramification e). We note that †-fields for a fixed initial ramification do form an elementary class, as can be seen by arguing with ultraproducts. This is actually important for forming the bridges used above. Likewise, existential completions are given by existential-positive completions of the theory of †-fields.

The original motivation for exploring unramified and finitely ramified henselian valued fields was to prove — as part of work on NIP valued fields with Jahnke — the following statement:

Theorem 6.5.3 ([AJ24])**.** *Let* (K,v) *be a finitely ramified henselian valued field and suppose that* Kv *is NIP. Then* (K,v) *is NIP.*

This result does indeed follow from the work discussed in this chapter: see [AJ24]. Here it is important that the residue field and value group are stably embedded.

Chapter 7

Diophantine equicharacteristic henselian valuation rings and ideals

Hey little baby, don't you cry
We got that sunny morning waiting on us now
There's a light at the end of the tunnel
We can be worry-free, just take it from me
Honey child, let me tell you now, child

Morning Sun, Melody Gardot

Beginning with [CDLM13], a short sequence of papers around Diophantine (here meaning existential without parameters) $\mathfrak{L}_{\mathrm{ring}}$-definability of henselian valuation rings in their fields of fractions continued with [AKo14, Feh15, Pre15], culminating in [AF17]. This chapter is a short exposition of the principal result in [AF17], though it also relates to the earlier papers, and includes a small new section related to—but disjoint from—[AF26+a].

7.1 Diophantine henselian valuation rings

In this section we consider fields K and equicharacteristic henselian nontrivial valuations v on K, with corresponding valuation rings $\mathcal{O}$. See [FJ17] for a survey of the recent progress on the subject of $\mathfrak{L}_{\mathrm{ring}}$-definable henselian valuation rings. The main theorem in this direction is the following.

Theorem 7.1.1 ([AF17, Theorem 1.1])**.** *Let F be a field. Then the following are equivalent.*

- **(i)** *There is an $\exists$-$\mathfrak{L}_{\mathrm{ring}}$-formula that defines $\mathcal{O}_v$ [respectively, $\mathfrak{m}_v$] in K for some equicharacteristic henselian nontrivially valued field (K, v) with residue field F.*
- **(ii)** *There is an $\exists$-$\mathfrak{L}_{\mathrm{ring}}$-formula that defines $\mathcal{O}_v$ [respectively, $\mathfrak{m}_v$] in K for every henselian valued field (K, v) with residue field elementarily equivalent to F.*
- **(iii)** *There is no elementary extension $F \preceq F^*$ with a nontrivial valuation v on F^* for which the residue field F^*v embeds into F^* [respectively, with a nontrivial henselian valuation v on a subfield E of F^* with $Ev \cong F^*$].*

We begin with several negative examples of $\mathcal{O}$ that are not Diophantine in K.

Example 7.1.2 (Negative examples)**.**

- **(i)** $\mathbb{C}[\![t]\!]$ is not Diophantine in $\mathbb{C}(\!(t)\!)$: this folkloric result is explained in [AKo14, Appendix A].
- **(ii)** $\mathbb{Q}_p[\![t]\!]$ is not Diophantine in $\mathbb{Q}_p(\!(t)\!)$: this is also explained in [AKo14, Appendix A].
- **(iii)** $\mathbb{R}[\![t]\!]$ is not Diophantine in $\mathbb{R}(\!(t)\!)$: this is a similar direct limit argument. Any existential formula defining $\mathbb{R}[\![t]\!]$ in $\mathbb{R}(\!(t)\!)$ must also define the (nontrivial) valuation ring $\mathbb{R}[\![t]\!]^{\mathrm{Px}}$ in the Puiseux series $\mathbb{R}(\!(t)\!)^{\mathrm{Px}}$, but this is a real closed field.

This example yields the following observation.

Observation 7.1.3. $\mathbb{F}[\![t]\!]$ is not Diophantine in $\mathbb{F}(\!(t)\!)$, for any algebraically closed field $\mathbb{F}$.

Proof. Again we give a direct limit argument: in the algebraic closure of $\mathbb{F}(\!(t)\!)$ the unique prolongation of $\mathbb{F}[\![t]\!]$ is again nontrivial and must be defined by any existential formula defining $\mathbb{F}[\![t]\!]$ in $\mathbb{F}(\!(t)\!)$ since the algebraic closure is a direct limit of isomorphic copies of $\mathbb{F}(\!(t)\!)$. $\square$

The arguments here can be easily extended to show that $F[\![t]\!]$ is not Diophantine in $F(\!(t)\!)$ whenever F is a characteristic zero t-henselian field, i.e. elementarily equivalent to one admitting a nontrivial henselian valuation. In fact, as the main theorem will show, the characteristic assumption may be removed.

Turning to positive examples, we have the following.

Example 7.1.4 (Positive examples)**.**

(i) $\mathbb{F}_q[\![t]\!]$ is Diophantine in $\mathbb{F}_q(\!(t)\!)$ for all prime powers q ([AKo14]).

(ii) $F[\![t]\!]$ is Diophantine in $F(\!(t)\!)$, for F a PAC field not containing the algebraic closure of its prime subfield ([Feh15]).

(iii) $\mathbb{Q}[\![t]\!]$ is Diophantine in $\mathbb{Q}(\!(t)\!)$ ([AF17]).

Each of these can be seen in a rather concrete fashion, with explicit formulas.

Recall the notion of a C-field: a field equipped with a distinguished morphism from C, for a given integral domain C.

Definition 7.1.5 ([AF17, Definition 3.5])**.** Let $\mathcal{F}$ be a class of C-fields.

(i) We say that $\mathcal{F}$ has [**equicharacteristic**] **embedded residue** if there exist $F_1, F_2 \in \mathcal{F}$ and a nontrivial [equicharacteristic] valuation w on F_1 with an embedding of F_1w into F_2. For a single C-field F, we say F has **embedded residue** if the class of C-fields elementarily equivalent to F has embedded residue.

(ii) We say that $\mathcal{F}$ is [**equicharacteristic**] **large** if there exist $F_1, F_2 \in \mathcal{F}$, a C-subfield $E \subseteq F_2$, and a nontrivial [equicharacteristic] henselian valuation w on E such that Ew is isomorphic to F_1. For a single C-field F, we say F is **large**[1] if the class of C-fields elementarily equivalent to F is large.

There is an underlying duality between the pictures for existential and universal definability of Diophantine henselian valuation rings, but somewhat twisted (note the appearance of the henselian valuation in the definition of largeness).

Theorem 7.1.6 ([AF17, Theorem 5.1])**.** *Let $\mathcal{K}$ be a class of equicharacteristic henselian nontrivially valued C-fields, let $\mathcal{F}$ be the smallest elementary class of C-fields that contains $\mathcal{K}v$, and suppose that*

$$(*)\begin{cases} (a) & C \text{ is integral over its prime ring, or} \\ (b) & C \text{ is a perfect field and every } F \in \mathcal{F} \text{ is perfect.} \end{cases}$$

Then for $Q \in \{\exists, \forall\}$ the following hold:

(i) *The following properties are equivalent.*

(0^e_Q) *The valuation ring is uniformly Q-C-definable in $\mathcal{K}$.*

($1^{e\prime}_Q$) *The valuation ring is uniformly Q-C-definable in $\mathcal{H}^{e\prime}(\mathcal{F})$.*

(1^e_Q) *The valuation ring is uniformly Q-C-definable in $\mathcal{H}^{e}(\mathcal{F})$.*

(2^e_Q) *$\mathcal{F}$ does not have equicharacteristic embedded residue, if $Q = \exists$ (respectively is not equicharacteristic large, if $Q = \forall$).*

[1] This notion of a large C-field is related to the notion of a large field in the sense of Pop [Pop96]. We discuss this connection in Section 6.2.

(ii) *The following properties are equivalent.*

($\mathbf{1}'_Q$) *The valuation ring is uniformly Q-C-definable in $\mathcal{H}'(\mathcal{F})$.*

($\mathbf{1}_Q$) *The valuation ring is uniformly Q-C-definable in $\mathcal{H}(\mathcal{F})$.*

($\mathbf{2}_Q$) *$\mathcal{F}$ does not have embedded residue, if $Q = \exists$ (respectively is not equicharacteristic large, if $Q = \forall$).*

(iii) *Moreover, if $\mathcal{F}$ does not both contain fields of characteristic zero and fields of positive characteristic, then all seven conditions $(\mathbf{0}^e_Q)$, $(\mathbf{1}^{e\prime}_Q)$, $(\mathbf{1}^e_Q)$, $(\mathbf{1}'_Q)$, $(\mathbf{1}_Q)$, $(\mathbf{2}^e_Q)$, and $(\mathbf{2}_Q)$ are equivalent.*

Theorem 7.1.1 follows as the special case when $\mathcal{F}$ is the class of fields elementarily equivalent to a given F.

Proof idea for Theorem 7.1.6. We discuss only the existential case, i.e. when $Q = \exists$, which corresponds to the property of embedded residue, and only the equicharacteristic statements. Moreover we consider only the case $C = \mathbb{Z}$. Thus, we give a brief justification of $(\mathbf{0}^e_Q) \Leftrightarrow (\mathbf{1}^{e\prime}_Q) \Leftrightarrow (\mathbf{1}^e_Q) \Leftrightarrow (\mathbf{2}^e_Q)$. Firstly, the implications $(\mathbf{0}^e_Q) \Leftarrow (\mathbf{1}^{e\prime}_Q) \Leftarrow (\mathbf{1}^e_Q)$ are easy. If $\mathcal{F}$ has equicharacteristic embedded residue, i.e. the negation of $(\mathbf{2}^e_Q)$ holds, then there are $F_1, F_2 \in \mathcal{F}$ with a nontrivial valuation w on F_1 and a field embedding $F_1 w \subseteq F_2$. Pick any $K_1 \in \mathcal{H}^{e\prime}$ with residue field F_1, e.g. $F_1(\!(t)\!)$, and then construct $K_2 \in \mathcal{H}^{e\prime}$ extending K_1 with residue field F_2, matching the diagram of places, Figure 7.1. It is clear from the construction

$$\begin{tikzcd}
K_2 \arrow[rr, "v_2"] \arrow[d, dash] & & F_2 \arrow[d, dash] \\
F_1(\!(t)\!) \arrow[r, "v_1"] \arrow[dr, dash] & F_1 \arrow[r, "w"] \arrow[d, dash] & E \arrow[dl, dash] \\
& C &
\end{tikzcd}$$

Figure 7.1: Diagram of places [AF17, Lemma 3.9]

that the valuation ring $\mathcal{O}_{v_1}$ is not contained in $\mathcal{O}_{v_2}$. Applying a combination of Beth Definability and the Preservation Theorem, as described by Prestel [Pre15], this shows that the valuation ring is not uniformly $\exists$-C-definable in $\mathcal{H}^{e\prime}$, i.e. the negation of $(\mathbf{1}^{e\prime}_Q)$. This proves $(\mathbf{1}^{e\prime}_Q) \Rightarrow (\mathbf{2}^e_Q)$. To prove $(\mathbf{0}^e_Q) \Rightarrow (\mathbf{1}^{e\prime}_Q)$ is a question of existential and universal *transfer*: if the valuation ring is $\exists$-C-definable in some $K \in \mathcal{H}^{e\prime}$ with $Kv = F$, we must show that there is an $\exists$-C-formula (perhaps a different one) that defines the valuation ring uniformly in all $L \in \mathcal{H}^{e\prime}$ with residue field a model of $\mathrm{Th}(F)$. The key ingredient here are the results of Chapter 4, specifically those around the existential completeness of $\mathbf{H}^{e\prime}$ relative to the residue field, see also Main Theorem 1.

It remains to prove $(\mathbf{1}^e_Q) \Leftarrow (\mathbf{2}^e_Q)$. We suppose that $\mathcal{F}$ does not have equicharacteristic embedded residue, and we want to test the hypothesis of Prestel's Criterion ([Pre15]). Thus we take two $K_1, K_2 \in \mathcal{H}^e$ with K_1 and $\mathfrak{L}_{\mathrm{ring}}(C)$-substructure of K_2. The aim is to show that v_1 refines the restriction of v_2 to K_1, where v_i is the valuation given on K_i. Let u denote this restriction of v_2 to K_1. Let F_i be the residue field $K_i v_i$ and let E be the residue field $K_1 u$. Since K_2 is henselian, the henselization L of K_1 with respect to u may be identified with a subfield of K_2 containing K_1. The restriction of v_2 to L we denote by w_2, and the unique prologation of v_1 to L we denote by w_1. Figure 7.1 illustrates the setup. Let w_3 be the finest common coarsening of w_1 and w_2. If v_1 and u are incomparable then so are w_1 and w_2. Thus Lw_3, E, and Lw_1 are separably closed. Moreover w_2 induces a nontrivial valuation $\bar{w}_2$ on Lw_3, so the residue field $(Lw_3)\bar{w}_2 = Lw_2$ is algebraically closed. By [AF17, Lemma 3.8], F_2 has equicharacteristic embedded residue, contradicting our assumption. Thus v_1 and u are comparable. If v_1 is a proper coarsening of u then u induces a nontrivial equicharacteristic valuation $\bar{u}$ on $K_1 v_1$ with residue field a subfield of $K_2 v_2$. This again contradicts the assumption. Therefore v_1 is a refinement of u, as required. □

$$\begin{tikzcd}
K_2 \arrow[r, "v_2"] \arrow[d, dash] & F_2 \arrow[d, dash] & \\
L \arrow[r, "w_2"] \arrow[rr, bend left, "w_1"] \arrow[d, dash] & E \arrow[d, dash] & Lw_1 \arrow[d, dash] \\
K_1 \arrow[r, "u"] \arrow[rr, bend left, "v_1"] \arrow[dr, dash] & E \arrow[d, dash] & F_1 \arrow[dl, dash] \\
& C &
\end{tikzcd}$$

Figure 7.2: Diagram of places, cf [AF17, Lemma 3.10]

We show as a corollary that there are at most two Diophantine equicharacteristic henselian valuation rings on any field (including the trivial one!). The story in mixed characteristic is a little different, most obviously in the case of a finitely ramified henselian valuation ring, which is always Diophantine by the formula of J. Robinson.

Question 7.1.7 (Problem 9.4)**.** Are there at most 3 existentially $\emptyset$-definable henselian valuation rings on any field?

7.2 Existentially t-henselian fields

Recall from above (also [AF17, Definition 3.5]) that a class $\mathcal{F}$ of fields is **$\mathbb{Z}$-large** if there exist $F_1, F_2 \in \mathcal{F}$ a subfield $F_0 \subseteq F_2$, and a non-trivial henselian valuation w on F_0 such that $F_0 w$ is isomorphic to F_1. An individual field F is **$\mathbb{Z}$-large** if the class $\{F' \mid F \equiv F'\}$ is $\mathbb{Z}$-large. For a field K, we denote by $\mathcal{H}^e(K)$ the set of (equivalence classes of) equicharacteristic henselian valuations on K, partially ordered by the coarsening relation, with largest element v^{triv}, the trivial valuation on K. Let E denote the fragment $\mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{ring}})$. We say that F satisfies **(ð)** (or is **existentially t-henselian**[2]) if it satisfies the equivalent conditions in the following lemma. Compare the following with [AF17, Section 6.3].

Recall that for all $\mathfrak{L}_{\mathrm{ring}}$-theories T we have $\mathbf{H}^{e\prime}(T)_\exists = \mathbf{H}^{e\prime}(T_\exists)_\exists$, which follows from Lemma 2.4.21.

Lemma 7.2.1. *For a field F, the following are equivalent.*

(a) $\mathrm{Th}_\exists(F) = \mathrm{Th}_\exists(F(\!(t)\!))$.

(b) $\mathrm{Th}_\exists(F) = \mathbf{H}^{e\prime}(\mathrm{Th}(F))_E$.

(c) $\mathrm{Th}_\exists(F) = \mathbf{H}^{e\prime}(\mathrm{Th}_\exists(F))_E$.

(d) $\mathrm{Th}_\exists(F)$ *is a fixed point of the map* $T \longmapsto \mathbf{H}^{e\prime}(T)_E$ *from the power set of* E *to itself.*

(e) *There exists a henselian field* F' *such that* $\mathrm{Th}_\exists(F) = \mathrm{Th}_\exists(F')$.

(f) F *is* $\mathbb{Z}$*-large.*

(g) $\{F' \mid \mathrm{Th}_\exists(F) = \mathrm{Th}_\exists(F')\}$ *is* $\mathbb{Z}$*-large.*

Proof. We already know that $\mathrm{Th}_\exists(F(\!(t)\!)) = \mathbf{H}^{e\prime}(\mathrm{Th}(F))_\exists$ by Corollary 4.4.4(a)(iii), thus (a, b, c, d) are equivalent. If there exists $v \in \mathcal{H}^e(F)$ non-trivial, then $\mathrm{Th}_\exists(F, v) = \mathrm{Th}_\exists(F(\!(t)\!), v \circ v_t)$, and so $\mathrm{Th}_\exists(F) = \mathbf{H}^{e\prime}(\mathrm{Th}(F))_E$. Next suppose that there exists henselian F' such that $\mathrm{Th}_\exists(F) = \mathrm{Th}_\exists(F')$. Replacing F' with an elementary extension if necessary, we may assume there exists $v \in \mathcal{H}^e(F')$ non-trivial. Then $\mathrm{Th}_\exists(F) = \mathrm{Th}_\exists(F') = \mathbf{H}^{e\prime}(\mathrm{Th}(F'))_E = \mathbf{H}^{e\prime}(\mathrm{Th}(F))_E$, which shows that $(e) \Rightarrow (b)$. Conversely, if $\mathrm{Th}_\exists(F) = \mathbf{H}^{e\prime}(\mathrm{Th}(F))_E$ then $\mathrm{Th}_\exists(F) = \mathrm{Th}_\exists(F(\!(t)\!))$. Thus $(e) \Rightarrow (a)$.

[2]The letter ð is pronounced "eth".

By [AF17, Proposition 6.10 (1 $\Leftrightarrow$ 4)], $\mathrm{Th}_\exists(F) = \mathrm{Th}_\exists(F(\!(t)\!))$ is equivalent to $\mathbb{Z}$-largeness, which proves $(a) \Leftrightarrow (f)$. Clearly $(f) \Rightarrow (g)$.

Suppose that for some $F_1, F_2 \in \{F' \mid \mathrm{Th}_\exists(F) = \mathrm{Th}_\exists(F')\}$ there exists $F_0 \subseteq F_2$ and a non-trivial henselian valuation w on F_0 with $F_0 w$ isomorphic to F_1. Then w is equicharacteristic and henselian, so we may embed $F_1(t)^h$ into F_2. This shows that $\mathrm{Th}_\exists(F_1) \subseteq \mathrm{Th}_\exists(F_1(t)^h) \subseteq \mathrm{Th}_\exists(F_2)$. By [AF17, Proposition 6.10 (4 $\Rightarrow$ 1)], we conclude that F is $\mathbb{Z}$-large, i.e. $(g) \Rightarrow (f)$. □

Lemma 7.2.2. *Let K be a field and $v \in \mathcal{H}^e(K)$ non-trivial. Then Kv satisfies* **(ð)** *if and only if* $\mathrm{Th}_\exists(K, v) = \mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists$.

Proof. For the implication $\Rightarrow$ we have

$$\mathrm{Th}_\exists(K,v) = \mathbf{H}^{e\prime}(\mathrm{Th}_\exists(Kv))_\exists = \mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(\mathrm{Th}_\exists(Kv))_E)_\exists = \mathbf{H}^{e\prime}(\mathrm{Th}_\exists(K))_\exists = \mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists,$$

by Lemma 7.2.1. For the converse, from the hypothesis $\mathrm{Th}_\exists(K,v) = \mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists$ it follows that $\mathrm{Th}_\exists(Kv) = \mathrm{Th}_\exists(K)$, and K admits a non-trivial henselian valuation, thus Kv satisfies **(ð)**. □

Note that any field with an algebraically closed subfield satisfies **(ð)**. We are now in a position to describe the theories $\mathrm{Th}_\exists(K,v)$ for all $v \in \mathcal{H}^e(K)$.

Theorem 7.2.3 (Dichotomy)**.** *Let K be a field of characteristic $p \in \mathbb{P} \cup \{0\}$. We have the following dichotomy:*

(a) Either $\mathcal{H}^e(K)$ is not linearly ordered, in which case $\mathbb{F}_p^{\mathrm{alg}} \subseteq K$, and for all non-trivial $v \in \mathcal{H}^e(K)$ we have that Kv satisfies **(ð)** *and*

$$\mathrm{Th}_\exists(K,v) = \mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists = \mathbf{H}^{e\prime}(\mathbf{ACF}_p)_\exists.$$

(b) Or $\mathcal{H}^e(K)$ is linearly ordered, in which case it admits a smallest element v_K^e, and for all non-trivial $v \in \mathcal{H}^e(K) \setminus \{v_K^e\}$ we have Kv satisfies **(ð)** *and*

$$\mathrm{Th}_\exists(K,v) = \mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists = \mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(\mathrm{Th}(Kv_K^e))_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})_\exists.$$

Moreover if v_K^e is non-trivial then $\mathrm{Th}_\exists(K, v_K^e) = \mathbf{H}^{e\prime}(\mathrm{Th}(Kv_K^e))_\exists$, and this equals $\mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists$ if and only if Kv_K^e satisfies **(ð)**.

In particular, Kv does not satisfy **(ð)** *for at most one $v \in \mathcal{H}^e(K)$. If such a v exists, it is the finest element v_K^e of $\mathcal{H}^e(K)$, which is totally ordered.*

Main Theorem 5 follows.

Proof. If $\mathcal{H}^e(K)$ is not linearly ordered then there exist equicharacteristic henselian non-trivial valuations v on K with Kv separably closed. By Hensel's Lemma, $\mathbb{F}_p^{\mathrm{alg}} \subseteq K$, and thus for all $v \in \mathcal{H}^e(K)$ the residue field Kv contains $\mathbb{F}_p^{\mathrm{alg}}$, thus $\mathrm{Th}_\exists(K,v) = \mathbf{H}^{e\prime}(\mathbf{ACF}_p)_\exists$. Moreover Kv satisfies **(ð)**, so $\mathrm{Th}_\exists(K,v) = \mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists$ by Lemma 7.2.2.

Suppose that $\mathcal{H}^e(K)$ is linearly ordered. Let v_K^e denote the finest element of $\mathcal{H}^e(K)$: the valuation ring corresponding to v_K^e is the intersection of all the valuations rings of elements of $\mathcal{H}^e(K)$, and such intersections of chains of equicharacteristic henselian valuation rings are again equicharacteristic henselian valuation rings. For every non-trivial $v \in \mathcal{H}^e(K) \setminus \{v_K^e\}$, v_K^e induces a non-trivial equicharacteristic henselian valuation $\bar{v}_K^e$ on Kv. Straight away we have $\mathrm{Th}_\exists(K,v) = \mathbf{H}^{e\prime}(\mathrm{Th}(Kv))_\exists$ and $\mathrm{Th}_\exists(Kv, \bar{v}_K^e) = \mathbf{H}^{e\prime}(\mathrm{Th}(Kv_K^e))_\exists$. Combining these equalities we have $\mathrm{Th}_\exists(K,v) = \mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(\mathrm{Th}(Kv_K^e))_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})_\exists$. Moreover Kv satisfies **(ð)**, so $\mathrm{Th}_\exists(K,v) = \mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists$ by Lemma 7.2.2.

Finally we suppose that v_K^e is non-trivial. Then straight away we have $\mathrm{Th}_\exists(K, v_K^e) = \mathbf{H}^{e\prime}(\mathrm{Th}(Kv_K^e))_\exists$. It only remains to argue that Kv_K^e satisfies **(ð)** if and only if $\mathrm{Th}_\exists(K, v_K^e) = \mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists$, but this is simply Lemma 7.2.2 applied to $v = v_K^e$. □

Corollary 7.2.4. *In case (a) of the theorem, all non-trivial $v \in \mathcal{H}^e(K)$ share the same existential theory, namely $\mathbf{H}^{e\prime}(\mathbf{ACF}_p)_\exists$. In case (b) of the theorem, all non-trivial $v \in \mathcal{H}^e(K)$ not equal to v_K^e share the same existential theory, namely $\mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(\mathrm{Th}(Kv_K^e))_E)_\exists$.*

Corollary 7.2.5. *For any field K, there are at most three distinct existential theories of the valued fields (K, v), for $v \in \mathcal{H}^e(K)$:*

(I) the trivial case $v = v_{\mathrm{triv}}$,

(II) v is non-trivial and Kv satisfies **(ð)**,

(III) v is non-trivial and Kv does not satisfy **(ð)**.

Cases (I, II, III) are always distinct. Case (I) exists for every K, but both cases (II) and (III) may be void, independently, according to the following examples. However, since at most one element of $\mathcal{H}^e(K)$ satisfies (III), (II) is void if and only if either $\mathcal{H}^e(K) = \{v^{\mathrm{triv}}\}$ or $\mathcal{H}^e(K) = \{v^{\mathrm{triv}}, v_K^e\}$ with Kv_K^e not satisfying **(ð)**.

Example 7.2.6.

(a) Let $K = \mathbb{Q}$. There are no non-trivial elements of $\mathcal{H}^e(\mathbb{Q})$. In this case $\mathbb{Q}$ does not satisfy **(ð)**. Both (II) and (III) are void.

(b) Let $K = \mathbb{Q}^{\mathrm{alg}}$. There are no non-trivial elements of $\mathcal{H}^e(\mathbb{Q}^{\mathrm{alg}})$. In this case $\mathbb{Q}^{\mathrm{alg}}$ satisfies **(ð)**. Nevertheless, both (II) and (III) are void.

(c) Let $K = \mathbb{Q}((t))$. The only non-trivial element of $\mathcal{H}^e(\mathbb{Q}((t)))$ is v_t, which is in case (III) since $\mathbb{Q}$ does not satisfy **(ð)**. Thus (II) is void.

(d) Let $K = \mathbb{Q}^{\mathrm{alg}}((t))$. The only non-trivial element of $\mathcal{H}^e(\mathbb{Q}^{\mathrm{alg}}((t)))$ is v_t, which is in case (II) since $\mathbb{Q}^{\mathrm{alg}}$ satisfies **(ð)**. Thus (III) is void.

(e) Let $K = \mathbb{Q}((s))((t))$. There are two non-trivial elements of $\mathcal{H}^e(\mathbb{Q}((s))((t)))$, namely v_t, which is in case (II), and $v_s \circ v_t$, which is in case (III). Note that $\mathbb{Q}$ and $\mathbb{Q}((s))$ have different existential theories, thus so do $(\mathbb{Q}((s))((t)), v_t)$ and $(\mathbb{Q}((s))((t)), v_s \circ v_t)$.

Let $A = (B, \hat{B}, \iota)$ be an arch. Suppose $(\star)$ that $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$, $\hat{L}_1 = \mathrm{Sent}(\mathfrak{L}_1) \cap \hat{L}_2$, and $L_1 = \mathrm{Sent}(\mathfrak{L}_1) \cap L_2$. We do not suppose ι to be the identity map. The set of L_1-deductively closed L_1-theories forms a complete lattice. Consider the map from $\mathfrak{L}_1$-theories to L_1-theories given by $R \mapsto T_2(R)_{L_1}$, and let Φ_A denote the restriction of this map to L_1-theories.

Definition 7.2.7. We say that Φ_A is **increasing** if $R \subseteq \Phi_A(R)$, for all L_1-theories R; and that Φ_A is **idempotent** if $\Phi_A(R) = \Phi_A \circ \Phi_A(R)$, for all R.

It is easy to see that $R \subseteq S$ implies $\Phi_A(R) \subseteq \Phi_A(S)$. Now we consider the arch $A = \mathrm{F}_\exists/\mathrm{H}_\exists^{e\prime} \parallel \mathrm{F}/\mathrm{H}^{e\prime}$. Note that the hypotheses $(\star)$ are satisfied for A. We continue to denote $E = L_1 = \mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{ring}})$ and note that in this case $\iota = \iota_{\mathbf{k}}$ is not the identity map. By Lemma 7.2.1, a theory $\mathrm{Th}_\exists(k)$ is a fixed point of Φ_A if and only if k satisfies **(ð)**.

It is easy to see that Φ_A is increasing, at least when $R = \mathrm{Th}_\exists(k)$ is the existential theory of a field: since $k \subseteq k((t))$ and $\mathrm{Th}_\exists(k((t))) = \mathbf{H}^{e\prime}(\mathrm{Th}_\exists(k))_E$, we have $\mathrm{Th}_\exists(k) \subseteq \mathbf{H}^{e\prime}(\mathrm{Th}_\exists(k))_E$.

Lemma 7.2.8. *Φ_A is increasing.*

Proof. Let R be an E-theory, so that $R = R_\exists$. We want $R \subseteq \mathbf{H}^{e\prime}(R)_E$, i.e. $\mathbf{H}^{e\prime}(R) \models R$. Let $(K, v) \models \mathbf{H}^{e\prime}(R)$. Then $Kv \models R$. By [ADF23] there exists an elementary extension $(K, v) \preceq (K^*, v^*)$ and a partial section $Kv \to K^*$ of the residue map of v^*. Thus $\mathrm{Th}_\exists(Kv) \subseteq \mathrm{Th}_\exists(K^*, v^*)$. Thus $(K^*, v^*) \models R$. □

Lemma 7.2.9. $\mathbf{H}^{e\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})} = \mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})}$ *for each $\mathfrak{L}_{\mathrm{ring}}$-theory R.*

Proof. Since Φ_A is increasing we have $R \subseteq \mathbf{H}^{e\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})}$. Thus $\mathbf{H}^{e\prime}(R) \subseteq \mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})$. Let $(K, v) \models \mathbf{H}^{e\prime}(R)$, so that $Kv \models R$. There exists an elementary extension $(K, v) \preceq (K^*, v^*)$ such that v^* admits a nontrivial proper coarsening w. Then $K^*v^* \models R$, so $(K^*w, \bar{v}) \models \mathbf{H}^{e\prime}(R)$ and $K^*w \models \mathbf{H}^{e\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})}$. Therefore $(K^*, w) \models \mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})$, and so $K^* \models \mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})}$. This proves that $\mathbf{H}^{e\prime}(R) \models \mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})}$. □

Lemma 7.2.10. Φ_A *is idempotent.*

Proof. Let R be an E-theory. Taking the existential consequences of the previous lemma we have $\mathbf{H}^{e\prime}(R)_E = \mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})_E$. The latter is equal to $\mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(R)_E)_E$. $\square$

Question 7.2.11 (Problem 9.5)**.** What can we say about $\mathfrak{L}_{\mathrm{ring}}$-theories R that are fixed points of Φ_A?

Observation 7.2.12. Let R be an E-theory of fields. The following are equivalent.

(*a*) R is a fixed point of Φ_A.

(*b*) R is in the image of Φ_A.

(*c*) Every model of R is $\mathbb{Z}$-large and $R = R_\exists$.

Proof. The equivalence between (a) and (b) is trivial since Φ_A is idempotent. For $(c) \Rightarrow (a)$: by monotonicity we have $R \subseteq \mathbf{H}^{e\prime}(R)_E$. If $F \models R$ then F is $\mathbb{Z}$-large, so $\mathrm{Th}_\exists(F) = \mathbf{H}^{e\prime}(\mathrm{Th}_\exists(F))_E$ by Lemma 7.2.1. Therefore $\mathrm{Th}_\exists(F) = \mathbf{H}^{e\prime}(\mathrm{Th}_\exists(F))_E \supseteq \mathbf{H}^{e\prime}(R)_E$, and so $R \models \mathbf{H}^{e\prime}(R)_E$. Since also $R = R_E$, we have $R_E = \mathbf{H}^{e\prime}(R)_E$, which proves (a). Next, suppose (a) and let $F \models R = \mathbf{H}^{e\prime}(R)_E$. Then F is a model of the existential $\mathfrak{L}_{\mathrm{ring}}$-theory of a henselian field, thus it is $\mathbb{Z}$-large. $\square$

Chapter 8

Separably tame valued fields

Sounds of laughter, shades of life
are ringing through my open ears
Inciting and inviting me
Limitless undying love
which shines around me like a million suns
It calls me on and on across the universe

Across the universe, The Beatles

This chapter is an exposition of part of [Ans26b].

The main aim of this section is to extend the account of separably tame valued fields, as developed by Kuhlmann and Pal ([KP16]) to allow infinite imperfection degree. For this we will make (rather mild) use of Lambda closure Λ_F from section 2.1.2. We continue (following Example 2.3.1) to let $\mathfrak{L}_{\text{ring}} = \{+, \times, -, 0, 1\}$ be the language of rings, to let $\mathfrak{L}_{\text{oag}}$ the language of ordered abelian groups, and to let $\mathfrak{L}_{\text{val}}$ be the three-sorted language of valued fields with sorts $\mathbf{K}$, $\mathbf{k}$, and $\mathbf{\Gamma}$. The first two are endowed with $\mathfrak{L}_{\text{ring}}$, and the last with $\mathfrak{L}_{\text{oag}}$, moreover there is a symbol for the valuation map from $\mathbf{K}$ to $\mathbf{\Gamma}$, and for the residue map from $\mathbf{K}$ to $\mathbf{k}$.

Remark 8.0.1. In this section we write $K, K_i, \ldots$, etc., for expansions of valued fields. The valuation will be usually be denoted by v, with subscripts or other decorations used to indicate to which valued field the valuation belongs, e.g. v_i is the valuation from K_i. Likewise $\Gamma_i = v_i K_i$ and $k_i = K_i v_i$ are the value group and residue field, respectively, of K_i.

We will be concerned with valued fields K of equal characteristic, and p will always represent the characteristic (of both K and of its residue field k), and $\hat{p}$ the corresponding characteristic exponent:

Convention 8.0.2. For $p \in \mathbb{P} \cup \{0\}$, we let $\hat{p} = p$ if $p \in \mathbb{P}$, and $\hat{p} = 1$ if $p = 0$.

We observe that in equal characteristic zero, i.e. the case $p = 0$, everything in this chapter reduces to the classical Ax–Kochen/Ershov theory of henselian valued fields of equal characteristic zero, see [AK65a, Ers65, AK66]. Thus our interest is focused on the case $p > 0$.

Remark 8.0.3. When formalizing valued fields in model theory, we have the usual choice of alternative languages. Instead of $\mathfrak{L}_{\text{val}}$, we might use a one-sorted language $\mathfrak{L}^1_{\text{val}} := \mathfrak{L}_{\text{ring}} \cup \{O\}$, where O is a unary predicate symbol, intended to be interpreted by the valuation ring; or we might use a two-sorted language $\mathfrak{L}^2_{\text{val}}$ with sorts $\mathbf{K}$ and $\mathbf{\Gamma}$, and with a function symbol from $\mathbf{K}$ to $\mathbf{\Gamma}$. For the results of this paper, the precise choice of language of valued fields does not matter. For example, both theorems 8.2.1 and 8.2.3 remain true when replacing $\mathfrak{L}_{\text{val}}$ by another language $\mathfrak{L}$, provided that the $\mathfrak{L}_{\text{val}}$- and $\mathfrak{L}$-structures are biinterpretable, and that the interpretations of both the value group and residue field are by both existential and universal formulas. This latter condition ensures that, for example, existential $\mathfrak{L}_{\text{ring}}$-sentences in the theory of the residue field are interpreted by existential sentences in the $\mathfrak{L}$-theory of the valued field. In particular, these conditions hold for the languages $\mathfrak{L}^1_{\text{val}}$ and $\mathfrak{L}^2_{\text{val}}$.

We denote by $\mathfrak{L}_{\mathrm{val},\lambda} = \mathfrak{L}_{\mathrm{val}} \cup \mathfrak{L}_\lambda$ the expansion of $\mathfrak{L}_{\mathrm{val}}$ by symbols for the parameterized lambda functions, uniform across all characteristics, as introduced in [Ans26b, Section 2.3]. For an expansion $\mathfrak{L} \supseteq \mathfrak{L}_{\mathrm{val}}$, any $\mathfrak{L}$-theory T of valued fields is in particular an expansion of an $\mathfrak{L}_{\mathrm{ring}}$-theory of fields, thus T_λ denotes its natural $(\mathfrak{L} \cup \mathfrak{L}_\lambda)$-expansion, as described in [Ans26b, Definition 2.38].

Fact 8.0.4. *Each valued field F admits a natural expansion $\tilde{F} \models \mathrm{Th}(F)_\lambda$ to an $\mathfrak{L}_{\mathrm{val},\lambda}$-structure, and an $\mathfrak{L}_{\mathrm{val}}$-embedding between F_1 and F_2 is in fact an $\mathfrak{L}_{\mathrm{val},\lambda}$-embedding between $\tilde{F}_1$ and $\tilde{F}_2$ if and only if it is separable, as an embedding of fields.*

Definition 8.0.5. A valued field K is **separably tame** if it is separably defectless, has perfect residue field, and $\hat{p}$-divisible value group. Let **STVF** be the $\mathfrak{L}_{\mathrm{val}}$-theory of separably tame valued fields. For $p \in \mathbb{P} \cup \{0\}$, we let $\mathbf{STVF}_p := \mathbf{STVF} \cup \mathbf{X}_p$ be the theory of separably tame valued fields of equal characteristic p. For $(p, \mathfrak{I}) \in \mathbb{P}\times(\mathbb{N}\cup\{\infty\})$, we let $\mathbf{STVF}_{p,\mathfrak{I}} := \mathbf{STVF}_p \cup \mathbf{X}_{p,\mathfrak{I}}$ be the theory $\mathbf{STVF}_p$ extended by axioms for the elementary imperfection degree to be $\mathfrak{I}$. To any of these theories the superscript "$^{\mathrm{eq}}$" will indicate the addition of axioms to ensure that the valued field is of equal characteristic, though of course in the case of positive characteristic, which is our main fare, equal characteristic is automatic.

For example, $\mathbf{STVF}_0^{\mathrm{eq}}$ is the $\mathfrak{L}_{\mathrm{val}}$-theory of separably tame valued fields of equal characteristic zero (which are in fact automatically tame). We recall the following theorem.

Theorem 8.0.6 ([KP16, Theorem 1.2])**.** *The class* $\mathbf{Mod}(\mathbf{STVF}_{p,\mathfrak{i}})$ *of all separably tame valued fields of fixed characteristic $p > 0$ and fixed finite imperfection degree $\mathfrak{i} \in \mathbb{N}$ is an* $\mathrm{AKE}^{\exists}$*-class in $\mathfrak{L}_Q$, an* $\mathrm{AKE}^{\prec}$*-class in $\mathfrak{L}_Q$, and an* $\mathrm{AKE}^{\equiv}$*-class in $\mathfrak{L}_{\mathrm{val}}$.*

First, a small detail: we prefer to write $\mathrm{AKE}^{\preceq_\exists}$ where Kuhlmann and Kuhlmann–Pal write $\mathrm{AKE}^{\exists}$ because we wish to include principles like $\mathrm{AKE}^{\equiv_\exists}$, and the earlier notation risks ambiguity.

We extend this theorem by strengthening the underlying embedding lemma from [KP16], which is closely based on the one from [Kuh16]. The AKE principles may then be stated uniformly for the class of separably tame valued fields of equal characteristic. In particular, this extends the known Ax–Kochen/Ershov phenomena to the case of infinite imperfection degree.

Remark 8.0.7. Let **SCVF** be the theory of separably closed valued fields, in the language $\mathfrak{L}_{\mathrm{val}}$ of valued fields. It is known since work of Delon (e.g. [Del82]) that the completions of **SCVF** are $\mathbf{SCVF}_0$ and $\mathbf{SCVF}_{p,\mathfrak{I}}$, for $(p, \mathfrak{I}) \in \mathbb{P} \times (\mathbb{N} \cup \{\infty\})$. Indeed, Hong showed in [Hon16] that **SCVF** has QE in $\mathfrak{L}_{\mathrm{val},\lambda}$, for $p > 0$.

The following two theorems, due to Kuhlmann and Knaf–Kuhlmann, are the most powerful ingredients of the Embedding Lemma, in all of its forms: those from [Kuh16, KP16] and Theorem 8.1.10.

Theorem 8.0.8 (Strong Inertial Generation, [KK05, Theorem 3.4], [Kuh16, Theorem 1.9])**.** *Let L/K be a function field without transcendence defect, where K is defectless. Suppose also that Lv/Kv is separable and vL/vK is torsion free. Then L/K is strongly inertially generated.*

Theorem 8.0.9 (Henselian Rationality, [Kuh16, Theorem 1.10])**.** *Let L/K be an immediate function field (a finitely generated and regular extension) of dimension 1, where K is separably tame. Then $L \subseteq K(b)^h$ for some $b \in L^h$.*

8.1 The Lambda relative embedding property of separably tame valued fields

Let $\mathfrak{L}$ be an expansion of $\mathfrak{L}_{\mathrm{val}}$, and let C be a class of $\mathfrak{L}$-structures expanding valued fields.

Definition 8.1.1 (Lambda relative embedding property)**.** We say that C has the **Lambda relative embedding property** (ΛREP) if

– for all $K_1, K_2 \in \mathrm{C}$ that extend a separably tame K for which

(i) K_1/K and $\underline{K_2/K}$ are separable,

(ii) $\underline{\mathfrak{imp}(K_1/K) \leq \mathfrak{imp}(K_2/K)}$,

(iii) K_1 is $\aleph_0$-saturated, K_2 is $|K_1|^+$-saturated,

(iv) vK_1/vK is torsion free and K_1v/Kv is separable, and

(v) $\rho : vK_1 \underset{vK}{\to} vK_2$ and $\sigma : K_1v \underset{Kv}{\to} K_2v$;

– there exists a $\underline{\text{separable}}$ embedding $\iota : K_1 \to K_2$ inducing ρ and σ.

Remark 8.1.2. We compare the ΛREP with the SREP, as expressed in [KP16, §4]. The points at which ΛREP differs from SREP are underlined, above, with the key strengthened conclusion of ΛREP also underlined. Strictly speaking, the two properties are incomparable: the hypotheses are stronger, i.e. the extension K_2/K is separable and we suppose an inequality between imperfection degrees, but the conclusion of the ΛREP is also stronger, i.e. the embedding ι is separable.

Remark 8.1.3. The hypothesis **(ii)** on imperfection degrees is a natural one, given that our aim is to separably embed K_1 into K_2 over K. Regarding **(iv)**, note that both ΛREP and SREP suppose K_1v/Kv to be separable, but this is redundant in the case that v is nontrivial on K, because then Kv is perfect, since K is separably tame. Similarly, note that σ is not assumed to be separable in **(v)**, however this is automatic when v is nontrivial on K, for then again Kv is perfect.

Remark 8.1.4. The REP, as expressed in [Kuh16], appears to have weaker hypotheses than SREP and ΛREP (aside from the obvious issues around separability), but this is not a material distinction. The conjunction of the hypothesis that K is defectless with the shared hypothesis **(iv)** is essentially equivalent to our hypothesis that K is separably tame: whenever REP is to be verified in a class of (separably) tame valued fields, any common valued subfield K satisfying the hypotheses is necessarily (separably) tame.

Lemma 8.1.5 (Separable going down, [KP16, Lemma 2.17]/[Kuh16, Lemma 3.15])**.** *Let L be a separably tame valued field, and let $K \subseteq L$ be a relatively algebraically closed subfield, equipped with the restriction of the valuation on L. If the residue field extension $Lv|Kv$ is algebraic, then (K, v) is also a separably tame valued field, and moreover, vL/vK is torsion free and $Lv = Kv$.*

The following lemma is preparation for the new step in the proof of Theorem 8.1.10. This method, informally termed "*wiggling*", is applied in the thesis of van der Schaaf [vdS25], as well as in in forthcoming papers by Jahnke and van der Schaaf on separable taming [JS25], and by Soto Moreno [SM25] on relative quantifier elimination in separably algebraically maximal Kaplansky valued fields,

Lemma 8.1.6. *Let U be a nonempty open set in a topological field L and let $K \subset L$ be a proper subfield. Then $U \setminus K$ is not empty.*

Proof. The subfield generated by any nontrivial open set B in a field topology is the entire field, since $L = (U - U) \cdot ((U - U) \setminus \{0\})^{-1}$. □

We say that K_2/K_1 is **separated** if there exists a p-basis of K_1 that is also a p-basis of K_2, equivalently if K_2/K_1 is separable and $K_2 = K_2^{(p)} K_1$.

Lemma 8.1.7. *Let K_1, K_2 be two separable field extensions of K, and suppose that K_1/K is separated. Then every embedding $\iota : K_1 \to K_2$ over K is separable.*

Proof. Let c be a p-basis of K. Since K_1/K is separated, by [Ans26b, Lemma 2.3], c is a p-basis of K_1. Since K_2/K is separable, by [Ans26b, Lemma 2.2], c is p-independent in K_2. Since ι is the identity on K, $\iota(c) = c$, which shows that c is already a p-basis of the image of ι. □

The following two lemmas provide cross-sections and sections (respectively) in sufficiently saturated henselian valued fields. Such maps give a additional structure to a valued field, and cross-sections especially have been part of standard approaches to Ax–Kochen/Ershov phenomena since the first papers.

Lemma 8.1.8 ([vdD14, Proposition 5.4]). *Let $s_0 : \Delta \to K^\times$ be a partial cross-section of v such that Δ is pure in Γ_v. Then there is an elementary extension $K \preceq K^*$ of valued fields, with a cross-section $s : vK^* \to K^{*\times}$ of the valuation on K^* that extends s_0.*

Lemma 8.1.9 ([ADF23, Proposition 4.5]). *Let K be a henselian valued field. For every partial section $\zeta_0 : k_0 \to K$ of res_v with Kv/k_0 separable there exists an elementary extension $K \preceq K^*$ of valued fields, with a section $\zeta : K^*v^* \to K^*$ of the residue map res_{v^*} on K^* that extends ζ_0.*

In the following, when we speak of embeddings we mean embeddings of valued fields.

Theorem 8.1.10. $\mathbf{Mod}(\mathbf{STVF}^{\mathrm{eq}})$ *has the ΛREP.*

We follow very closely the embedding arguments in [Kuh16, KP16], only giving full details and making changes where necessary. Thus the reader might want to read this proof alongside those others.

Proof. We work only in equal positive characteristic, since in equal characteristic zero (and even in mixed characteristic) every separably tame valued field is tame, and the ΛREP becomes equivalent to the REP. Suppose that we have $K, K_1, K_2 \in \mathbf{Mod}(\mathbf{STVF}^{\mathrm{eq}})$, where K is a common valued field of K_1 and K_2, satisfying the hypotheses **(i)–(v)** of the ΛREP.

By saturating the triple (K, K_1, K_2), if necessary, we may assume by Lemma 8.1.8 that there is a cross-section $\chi : vK \to K^\times$, and by Lemma 8.1.9 that there is a section $\zeta : Kv \to K$. By hypothesis **(iv)**, v_1K_1/vK is torsion-free and K_1v_1/Kv is separable. Since ρ and σ are embeddings, also $\rho(v_1K_1)/vK$ is torsion-free and $\sigma(K_1v_1)/Kv$ is separable. Thus, by further saturating K_1 and K_2, if necessary, again by Lemmas 8.1.8 and 8.1.9, there are extensions $\chi_1 : vK_1 \to K_1^\times$ and $\chi_2 : \rho(vK_1) \to K_2^\times$ of χ, and extensions $\zeta_1 : K_1v \to K_1$ and $\zeta_2 : \sigma(K_1v) \to K_2$ of ζ. Note that these saturation steps preserve the hypotheses of the ΛREP, so these reductions are without loss of generality.

Let $K_0 := K(\zeta_1(K_1v), \chi_1(vK_1))^{\mathrm{rac}}$ be the relative algebraic closure in K_1 of the field generated over K by the images of the section and cross-section. As argued in [KP16], K_0/K is without transcendence defect, and so every finitely generated subextension F/K of K_0/K is strongly inertially generated, by Theorem 8.0.8. By compactness, as in [Kuh16], there is an $\mathfrak{L}_{\mathrm{val},\lambda}$-embedding $\iota_0 : K_0 \to K_2$ such that $\iota_0 \circ \chi_1 = \chi_2 \circ \rho$ and $\iota_0 \circ \zeta_1 = \zeta_2 \circ \sigma$. Thus ι_0 induces ρ and σ.

Note that K_0/K is separated, since K_1v is perfect and vK_1 is p-divisible. Moreover K_2/K is separable by hypothesis **(i)**, so ι_0 is automatically separable, by Lemma 8.1.7, i.e. $K_2/\iota_0(K_0)$ is separable. By Lemma 8.1.5, K_0 is also separably tame, and K_1/K_0 is immediate.

Let $b = (b_\mu)_{\mu<\nu} \subseteq K_1$ be a p-basis of K_1 over K_0. For $\mu \leq \nu$, let $K_{0,\mu} := K_0(b_\kappa)_{\kappa<\mu}^{\mathrm{rac}}$ be the relative algebraic closure of $K_0(b_\kappa)_{\kappa<\mu}$ in K_1. Note that each $K_{0,\mu}$ is separably tame, by Lemma 8.1.5. Since $K_{0,\nu}$ is the relative algebraic closure in K_1 of $K_0(b)$, and b is a p-basis of $K_{0,\nu}$ over K_0, thus $K_1/K_{0,\nu}$ is separated, using hypothesis **(i)**. We will prove the following claim.

Claim 8.1.10.1. *There is a separable $\mathfrak{L}_{\mathrm{val},\lambda}$-embedding $\iota_{0,\nu} : K_{0,\nu} \to K_2$ extending ι_0.*

Proof of claim. We will build a chain of separable $\mathfrak{L}_{\mathrm{val},\lambda}$-embeddings $\iota_{0,\mu} : K_{0,\mu} \to K_2$ for $\mu \leq \nu$. We proceed inductively, noting that the base case is trivial since $K_{0,0} = K_0$. The limit stage is also easy: a union of a chain of $\mathfrak{L}_{\mathrm{val},\lambda}$-embeddings is an $\mathfrak{L}_{\mathrm{val},\lambda}$-embedding. We assume as an inductive hypothesis that ι_0 is already extended to a separable $\mathfrak{L}_{\mathrm{val},\lambda}$-embedding $\iota_{0,\mu} : K_{0,\mu} \to K_2$. Let $c \in K_{0,\mu+1}$. Then c is separably algebraic over $K_{0,\mu}(b_\mu)$. By Theorem 8.0.9 (Henselian Rationality), there exists $d \in K_{0,\mu}(b_\mu, c)$ such that $K_{0,\mu}(b_\mu, c)^h = K_{0,\mu}(d)^h$. It is clear that d is inter-p-dependent with b_μ in K_1 over $K_{0,\mu}$. In particular, d is p-independent in K_1 over $K_{0,\mu}$.

Let $(d_\delta)_{\delta<\alpha}$ be a pseudo-Cauchy sequence in $K_{0,\mu}$, without pseudo-limit there, of which d is a pseudo-limit. By Kuhlmann–Pal (specifically by [KP16, Lemma 3.11] — see Remark 8.1.12), and since $K_{0,\mu}$ is separably tame (so in particular separably algebraically maximal), $(d_\delta)_{\delta<\alpha}$ is of trancendental type. By Kaplansky's second theorem, [Kap42, Theorem 2], its quantifier-free $\mathfrak{L}_{\mathrm{val}}$-type $q(x)$ over $K_{0,\mu}$ is implied by formulas of the form $v(x - d_\delta) \geq \gamma_\delta$, where $\gamma_\delta = v(d_{\delta+1} - d_\delta)$. Any finitely many such formulas are already realised in $K_{0,\mu}$. Let $q_\iota(x)$ be the image of $q(x)$ by translating the parameters in each formula by $\iota_{0,\mu}$. Then $q_\iota(x)$ is implied by formulas of the form $v(x-\iota_{0,\mu}(d_\delta)) \geq \rho(\gamma_\delta)$. Any finitely many such formulas are realised in $\iota_{0,\mu}(K_{0,\mu})$, and in particular $q_\iota(x)$ is consistent. By saturation of K_2, there is even a nontrivial ball B in K_2 which is the set of realisations of $q_\iota(x)$. Since $\mathfrak{imp}(K_1/K) \leq \mathfrak{imp}(K_2/K)$, and by saturation, i.e. by hypotheses **(ii,iii)**, we have that $K_2^{(p)}\iota_{0,\mu}(K_{0,\mu})$ is a proper subfield of K_2. Now comes the *wiggling*: there exists $d' \in B \setminus K_2^{(p)}\iota_{0,\mu}(K_{0,\mu})$, by Lemma 8.1.6. Then d' is p-independent in K_2 over $\iota_{0,\mu}(K_{0,\mu})$ and also realises $q_\iota(x)$. Via the assignment $d \mapsto d'$ we extend $\iota_{0,\mu}$ to a separable $\mathfrak{L}_{\mathrm{val},\lambda}$-embedding $K_{0,\mu}(d)^h \to K_2$.

By the Primitive Element Theorem, this already shows how to extend $\iota_{0,\mu}$ to a separable $\mathfrak{L}_{\mathrm{val},\lambda}$-embedding into K_2 of any finite separably algebraic extension of $K_{0,\mu}(b_\mu)$ inside $K_{0,\mu+1}$. By the Compactness Theorem, we extend $\iota_{0,\mu}$ to a separable embedding $\iota_{0,\mu+1} : K_{0,\mu+1} \to K_2$, as required for the inductive step. By induction, there is a separable $\mathfrak{L}_{\mathrm{val},\lambda}$-embedding $\iota_{0,\nu} : K_{0,\nu} \to K_2$ extending ι_0. $\blacksquare_{\text{claim}}$

The remaining extension of $\iota_{0,\nu}$ to $\iota_1 : K_1 \to K_2$ is almost the same. We construct an $\mathfrak{L}_{\mathrm{val}}$-embedding $\iota_1 : K_1 \to K_2$, extending $\iota_{0,\nu}$, by following the analogous arguments in [Kuh16, KP16], that is by Henselian Rationality and Kaplansky's theory, but without the "*wiggling*" argument. Finally, we see that ι_1 is automatically separable since $K_1/K_{0,\nu}$ is separated, by Lemma 8.1.7, so ι_1 is automatically an $\mathfrak{L}_{\mathrm{val},\lambda}$-embedding. $\square$

Remark 8.1.11. A similar Embedding Lemma is applied in the case of separably algebraically maximal Kaplansky fields by Soto Moreno ([SM25]) to yield a relative quantifier elimination.

Remark 8.1.12. During my discussions with Soto Moreno regarding his QE argument for separably algebraically maximal Kaplansky ("SAMK") valued fields, we heard from F. V. Kuhlmann about a small error in the statement of [KP16, Lemma 3.11]. There is a hypothesis missing: the pseudo-Cauchy sequence should be supposed to not have a pseudo-limit in M. This causes no significant problem for the above argument, which in any case is only a sketch. Many thanks to Franz-Viktor Kuhlmann for this communication.

8.2 The resplendent model theory of separably tame valued fields

The Embedding Lemma yields model theoretic results, specifically Ax–Kochen/Ershov principles and a transfer of decidability. Moreover these results are resplendent over the sorts $\mathbf{k}$ for the residue field, and $\mathbf{\Gamma}$ for the value group, as we explain in this final section.

For any expansion $\mathfrak{L}_0$ of $\mathfrak{L}_{\mathrm{val}}$, an **$(\mathbf{k}, \mathbf{\Gamma})$-expansion** of $\mathfrak{L}_0$ is any expansion in which

- the residue field sort $\mathbf{k}$ is expanded to a language $\mathfrak{L}_{\mathbf{k}} \supseteq \mathfrak{L}_{\mathrm{ring}}$, and
- the value group sort $\mathbf{\Gamma}$ is expanded to a language, $\mathfrak{L}_{\mathbf{\Gamma}} \supseteq \mathfrak{L}_{\mathrm{oag}}$.

We emphasise that such a language is simply $\mathfrak{L}_0$ expanded *by and only by* $\mathfrak{L}_{\mathbf{k}}$ on the residue field sort and $\mathfrak{L}_{\mathbf{\Gamma}}$ on the value group sort. Such an expansion will be denoted $\mathfrak{L}_0(\mathfrak{L}_{\mathbf{k}}, \mathfrak{L}_{\mathbf{\Gamma}})$. Usually $\mathfrak{L}_0$ is either $\mathfrak{L}_{\mathrm{val}}$ or $\mathfrak{L}_{\mathrm{val},\lambda}$.

Recall from section 2.3 (and [AF25, §2]) the notion of an **$\mathfrak{L}$-fragment**[1], for a language $\mathfrak{L}$: a set F of $\mathfrak{L}$-formulas that contains $\top$ and $\bot$, that is closed under (finite) conjunctions and disjunctions, and is closed under the substitution of one free variable for another. For an $\mathfrak{L}$-theory T and an $\mathfrak{L}$-fragment F, we let T_{F} denote the intersection of the deductive closure $T^{\vdash}$ with F. Recall also that a **fragment** is a functor F from a subcategory $\mathbb{L}$ of the category of languages with inclusion to the category of sets with functions, such that $\mathsf{F}(\mathfrak{L}) \subseteq \mathrm{Form}(\mathfrak{L})$, for each $\mathfrak{L} \in \mathbb{L}$. If T is an $\mathfrak{L}$-theory where $\mathfrak{L} \in \mathbb{L}$, we write $T_{\mathsf{F}} = T_{\mathsf{F}(\mathfrak{L})}$, similarly if M is an $\mathfrak{L}$-structure we write $\mathrm{Th}_{\mathsf{F}}(M) = \mathrm{Th}_{\mathsf{F}(\mathfrak{L})}(M) = \mathrm{Th}(M) \cap \mathsf{F}(\mathfrak{L})$. For any

[1]What are here called $\mathfrak{L}$-fragments are also discussed in [AF26+a], though in that paper they are constrained to sentences and are just called "fragments".

language $\mathfrak{L}$, for any fragment F, and for any $\mathfrak{L}$-structures M_1, M_2 with common substructure M, we write

$$M_1 \Rrightarrow_M M_2 \text{ in } \mathsf{F}(\mathfrak{L})$$

to mean that F is defined on both $\mathfrak{L}$ and $\mathfrak{L}(M)$, and moreover that $\mathrm{Th}_{\mathsf{F}}(M_{1,M}) \subseteq \mathrm{Th}_{\mathsf{F}}(M_{2,M})$, where $M_{i,M}$ denotes the $\mathfrak{L}(M)$ expansion of M_i in which we interpret each new constant symbol by its corresponding element from M.

Theorem 8.2.1 (Main Theorem for Separably Tame Valued Fields)**.** *Let $\mathfrak{L} = \mathfrak{L}_{\mathrm{val},\lambda}(\mathfrak{L}_{\mathbf{k}}, \mathfrak{L}_{\Gamma})$ be a $(\mathbf{k}, \mathbf{\Gamma})$-expansion of $\mathfrak{L}_{\mathrm{val},\lambda}$. Let $K_1, K_2 \in \mathbf{Mod}_{\mathfrak{L}}(\mathbf{STVF}^{\mathrm{eq}})$ have common $\mathfrak{L}$-substructure K_0 which as a valued field is defectless, and v_1K_1/v_0K_0 is torsion-free and K_1v_1/K_0v_0 is separable.*

- **(I)** *$K_1 \Rrightarrow_{K_0} K_2$ in $\mathrm{Sent}_{\exists}(\mathfrak{L})$ if and only if*
  - **(i)** *$k_1 \Rrightarrow_{k_0} k_2$ in $\mathrm{Sent}_{\exists}(\mathfrak{L}_{\mathbf{k}})$,*
  - **(ii)** *$\Gamma_1 \Rrightarrow_{\Gamma_0} \Gamma_2$ in $\mathrm{Sent}_{\exists}(\mathfrak{L}_{\Gamma})$, and*
  - **(iii)** *$\mathfrak{Imp}(K_1/K_0) \leq \mathfrak{Imp}(K_2/K_0)$.*
- **(II)** *$K_1 \Rrightarrow_{K_0} K_2$ in $\mathrm{Sent}(\mathfrak{L})$ if and only if*
  - **(i)** *$k_1 \Rrightarrow_{k_0} k_2$ in $\mathrm{Sent}(\mathfrak{L}_{\mathbf{k}})$,*
  - **(ii)** *$\Gamma_1 \Rrightarrow_{\Gamma_0} \Gamma_2$ in $\mathrm{Sent}(\mathfrak{L}_{\Gamma})$, and*
  - **(iii)** *$\mathfrak{Imp}(K_1/K_0) = \mathfrak{Imp}(K_2/K_0)$.*

Proof. For **(I)**, the direction $\Rightarrow$ is almost trivial: the interpretations of both $\mathbf{k}$ and $\mathbf{\Gamma}$ map existential formulas to existential formulas. Moreover, if $\mathfrak{Imp}(K_0) = \infty$, then certainly $\mathfrak{Imp}(K_1/K_0) = \mathfrak{Imp}(K_2/K_0) = \infty$. Otherwise, suppose that $\mathfrak{Imp}(K_0) = m$ and let $c \in (K_0)_{[\![p]\!]}$ be a p-basis of K_0. If $\mathfrak{Imp}(K_1/K_0) \geq n$ then $K_1 \models \exists b = (b_0, \dots, b_{n-1})\ \lambda_{\mathbf{0}}^{cb}(1) = 1$, where $\mathbf{0}$ is the multi-index that is constantly zero. By hypothesis, K_2 also models this sentence. Therefore $\mathfrak{Imp}(K_2/K_0) \geq n$. For the converse direction we suppose that **(I:i,ii,iii)** hold. Let $K_1^* \succeq K_1$ be an $\aleph_0$-saturated elementary extension, and let $K_2^* \succeq K_2$ be an $|K_1^*|^+$-saturated elementary extension. By saturation hypotheses, there is an $\mathfrak{L}_{\mathbf{k}}$-embedding $k_1^* \to k_2^*$ over k_0 and an $\mathfrak{L}_{\Gamma}$-embedding $\Gamma_1^* \to \Gamma_2^*$ over Γ_0. Then the three valued fields K_1^*, K_2^*, with common valued subfield K_0, satisfy the hypotheses of ΛREP. The proof of **(II)** is a standard back-and-forth argument, making use of Theorem 8.1.10, for example following the proof of [Kuh16, Lemma 6.1] or [KP16, Lemma 4.1]. $\square$

In Theorem 8.2.3 we will deduce that the class $\mathbf{Mod}(\mathbf{STVF}^{\mathrm{eq}})$ satisfies the separable AKE principles $\mathrm{sAKE}^{\blacklozenge}$, for $\blacklozenge \in \{\equiv, \equiv_{\exists}, \preceq, \preceq_{\exists}\}$, resplendently. These principles are defined as follows.

Definition 8.2.2. Let $\mathfrak{L}$ be an expansion of a $(\mathbf{k}, \mathbf{\Gamma})$-expansion $\mathfrak{L}_{\mathrm{val},\lambda}(\mathfrak{L}_{\mathbf{k}}, \mathfrak{L}_{\Gamma})$ of $\mathfrak{L}_{\mathrm{val},\lambda}$, and let $\mathfrak{L}_0 \subseteq \mathfrak{L}$. Let $\mathbf{C}$ be a class of $\mathfrak{L}$-structures and let $\blacklozenge \in \{\equiv, \equiv_{\exists}, \preceq, \preceq_{\exists}\}$. We say that $\mathbf{C}$ is an $\mathrm{sAKE}^{\blacklozenge}$**-class** for the triple of languages $(\mathfrak{L}_0, \mathfrak{L}_{\mathbf{k}}, \mathfrak{L}_{\Gamma})$, if for all $K_1, K_2 \in \mathbf{C}$ (where we additionally suppose $K_1 \subseteq K_2$ in case $\blacklozenge$ is either $\preceq$ or $\preceq_{\exists}$) we have

- $K_1 \blacklozenge K_2$ in $\mathfrak{L}_0$

if and only if

- $k_1 \blacklozenge k_2$ in $\mathfrak{L}_{\mathbf{k}}$,
- $\Gamma_1 \blacklozenge \Gamma_2$ in $\mathfrak{L}_{\Gamma}$, and
- $\mathfrak{Imp}(K_1) = \mathfrak{Imp}(K_2)$.

In this case we say that $\mathbf{C}$ satisfies the **separable Ax–Kochen/Ershov principle** $\mathrm{sAKE}^{\blacklozenge}$ for the languages $\mathfrak{L}_0$, $\mathfrak{L}_{\mathbf{k}}$, and $\mathfrak{L}_{\Gamma}$.

Theorem 8.2.3 (Resplendent Ax–Kochen/Ershov for **STVF**, [Ans26b,]). *Let $\mathfrak{L} = \mathfrak{L}_{\mathrm{val},\lambda}(\mathfrak{L}_{\mathbf{k}}, \mathfrak{L}_{\Gamma})$ be a $(\mathbf{k}, \Gamma)$-expansion of $\mathfrak{L}_{\mathrm{val},\lambda}$. Let $\blacklozenge \in \{\equiv, \equiv_{\exists}, \preceq, \preceq_{\exists}\}$. The class $\mathbf{Mod}_{\mathfrak{L}}(\mathbf{STVF}^{\mathrm{eq}})$ of all $\mathfrak{L}$-structures which expand separably tame valued fields of equal characteristic is an* sAKE$^{\blacklozenge}$*-class for $(\mathfrak{L}, \mathfrak{L}_{\mathbf{k}}, \mathfrak{L}_{\Gamma})$.*

Proof. Firstly, if $\blacklozenge$ is $\equiv$, the result follows from Theorem 8.2.1 **(II)** applied with $K_0 = \mathbb{F}_p$ trivially valued. Secondly, if $\blacklozenge$ is $\equiv_{\exists}$, the result follows from Theorem 8.2.1 **(I)** applied twice, with $K_0 = \mathbb{F}_p$ trivially valued. Thirdly, if $\blacklozenge$ is $\preceq$, the result follows from Theorem 8.2.1 **(II)** applied to $K_0 = K_1$. Finally, if $\blacklozenge$ is $\preceq_{\exists}$, the result follows from Theorem 8.2.1 **(I)** applied to $K_0 = K_1$. □

Corollary 8.2.4. *For $p \in \mathbb{P}$ and $\mathfrak{I} \in \mathbb{N} \cup \{\infty\}$.*

(i) $\mathbf{STVF}^{\mathrm{eq}}_{p,\mathfrak{I}}$ *is resplendently complete relative to the value group and residue field.*

(ii) $\mathbf{STVF}^{\mathrm{eq}}_{\lambda,p,\mathfrak{I}}$ *is resplendently model complete relative to the value group and residue field.*

Proof. **(i)** is just a particular case of the sAKE$^{\equiv}$ principle, while **(ii)** is a particular case of the sAKE$^{\preceq}$ principle. □

We define several $\mathfrak{L}_{\mathrm{ring}}$-sentences and theories:

Definition 8.2.5. For each $p \in \mathbb{P}$, let χ_p be the $\mathfrak{L}_{\mathrm{ring}}$-sentence

$$\underbrace{1 + \ldots + 1}_{p \text{ times}} = 0,$$

and, for each $\mathrm{i} \in \mathbb{N}$, let $\iota_{p,\leq \mathrm{i}}$ be the $\mathfrak{L}_{\mathrm{ring}}$-sentence

$$\exists b = (b_0, \ldots, b_{\mathrm{i}-1})\ \forall a\ \exists y = (y_I)_{I \in p^{[\mathrm{i}]}}\ :\ x = \sum_{I \in p^{[\mathrm{i}]}} b^I y_I^p,$$

and let $\iota_{p,\mathrm{i}}$ be the $\mathfrak{L}_{\mathrm{ring}}$-sentence

$$\begin{cases} \iota_{p,\leq 0} & \text{if } \mathrm{i} = 0, \\ (\iota_{p,\leq \mathrm{i}} \wedge \neg \iota_{p,\leq \mathrm{i}-1}) & \text{if } \mathrm{i} > 0. \end{cases}$$

For $p \in \mathbb{P} \cup \{0\}$ we define

$$\mathbf{X}_p := \begin{cases} \{\chi_p\} & \text{if } p > 0, \text{ and} \\ \{\neg \chi_\ell \mid \ell \in \mathbb{P}\} & \text{if } p = 0. \end{cases}$$

For $p \in \mathbb{P}$ and $\mathfrak{I} \in \mathbb{N} \cup \{\infty\}$ we define

$$\mathbf{X}_{p,\mathfrak{I}} := \begin{cases} \mathbf{X}_p \cup \{\iota_{p,\mathfrak{I}}\} & \text{if } \mathfrak{I} < \infty, \text{ and} \\ \mathbf{X}_p \cup \{\neg \iota_{p,\leq \mathrm{i}} \mid \mathrm{i} \in \mathbb{N}\} & \text{if } \mathfrak{I} = \infty. \end{cases}$$

Let

$$\begin{array}{lll} \mathbf{F}_{p,\mathfrak{I}} & := & \mathbf{F} \cup \mathbf{X}_{p,\mathfrak{I}} \quad \text{for } (p, \mathfrak{I}) \in \mathbb{P} \times (\mathbb{N} \cup \{\infty\}), \text{ and} \\ \mathbf{F}_0 & := & \mathbf{F} \cup \mathbf{X}_0. \end{array}$$

Then $\mathbf{F}_{p,\mathfrak{I}}$ is the $\mathfrak{L}_{\mathrm{ring}}$-theory of fields of characteristic $p > 0$ and of elementary imperfection degree $\mathfrak{I}$, and $\mathbf{F}_0$ is the theory of fields of characteristic zero. These subscripts will be similarly applied to other theories of fields T in languages $\mathfrak{L} \supseteq \mathfrak{L}_{\mathrm{ring}}$: we write $T_0 = T \cup \mathbf{X}_0$, $T_p = T \cup \mathbf{X}_p$, and $T_{p,\mathfrak{I}} = T \cup \mathbf{X}_{p,\mathfrak{I}}$, for $(p, \mathfrak{I}) \in \mathbb{P} \times (\mathbb{N} \cup \{\infty\})$.

For $\mathfrak{L} = \mathfrak{L}_{\mathrm{val},\lambda}(\mathfrak{L}_{\mathbf{k}}, \mathfrak{L}_{\Gamma})$, let $\iota_{\mathbf{k}}$ denote the "standard" interpretation $\mathrm{Form}(\mathfrak{L}_{\mathbf{k}}) \to \mathrm{Form}(\mathfrak{L})$ of the residue field in a valued field, the latter viewed as an $\mathfrak{L}_{\mathrm{val}}$-structure, by relativising each $\mathrm{Form}(\mathfrak{L}_{\mathbf{k}})$ to the sort $\mathbf{k}$. Likewise let ι_{Γ} denote the standard interpretation $\mathrm{Form}(\mathfrak{L}_{\Gamma}) \to \mathrm{Form}(\mathfrak{L})$ for the value group on the sort Γ.

Definition 8.2.6. Let **IMP** (for "imperfection") be the $\mathfrak{L}_{\mathrm{ring}}$-fragment consisting of all Boolean combinations of the $\mathfrak{L}_{\mathrm{ring}}$-sentences χ_p and $\iota_{p,\leq \mathrm{i}}$. For $\mathfrak{L} = \mathfrak{L}_{\mathrm{val},\lambda}(\mathfrak{L}_{\mathbf{k}}, \mathfrak{L}_{\Gamma})$ a $(\mathbf{k}, \boldsymbol{\Gamma})$-expansion of $\mathfrak{L}_{\mathrm{val},\lambda}$, let $\mathsf{AKE\text{-}IMP}(\mathfrak{L})$ be the $\mathfrak{L}$-fragment generated by $\iota_{\mathbf{k}}\mathrm{Sent}(\mathfrak{L}_{\mathbf{k}})$ and $\iota_{\Gamma}\mathrm{Sent}(\mathfrak{L}_{\Gamma})$, and **IMP**. Then $\mathsf{AKE\text{-}IMP}$ is the fragment thus defined on the full subcategory of languages that are $(\mathbf{k}, \boldsymbol{\Gamma})$-expansions of $\mathfrak{L}_{\mathrm{val},\lambda}$.

Theorem 8.2.7. *Let $\mathfrak{L} = \mathfrak{L}_{\mathrm{val},\lambda}(\mathfrak{L}_{\mathbf{k}}, \mathfrak{L}_{\Gamma})$ be a $(\mathbf{k}, \boldsymbol{\Gamma})$-expansion of $\mathfrak{L}_{\mathrm{val},\lambda}$, and let $K, L \in \mathbf{Mod}_{\mathfrak{L}}(\mathbf{STVF}^{\mathrm{eq}})$. If $\mathrm{Th}_{\mathsf{AKE\text{-}IMP}}(K) = \mathrm{Th}_{\mathsf{AKE\text{-}IMP}}(L)$ then $\mathrm{Th}(K) = \mathrm{Th}(L)$.*

Proof. This is a reformulation of the $\mathrm{sAKE}^{\equiv}$ principle for $(\mathfrak{L}, \mathfrak{L}_{\mathbf{k}}, \mathfrak{L}_{\Gamma})$ from Theorem 8.2.3. □

The Hahn series fields $k((t^{\Gamma}))$, equipped with the t-adic valuation, are natural examples of tame valued fields of equal characteristic, with any given "suitable" pair of residue field k and value group Γ. By contrast, we lack such natural examples of separably tame valued fields with positive elementary imperfection degree $\mathfrak{I} > 0$. Nevertheless, the following lemma justifies the existence of some example, for each suitable pair k and Γ).

Lemma 8.2.8. *Let $(p, \mathfrak{I}) \in \mathbb{P} \times (\mathbb{N} \cup \{\infty\})$, let k be any perfect field of characteristic p, and let Γ be a p-divisible ordered abelian group. There exists $K \models \mathbf{STVF}^{\mathrm{eq}}_{p,\mathfrak{I}}$ with $Kv = k$ and $vK = \Gamma$.*

Proof. We consider the immediate extension $k((t^{\Gamma}))/k(t^{\Gamma})$, both equipped with the t-adic valuation. Let B be a transcendence basis of this field extension, let $B_0 \subseteq B$ be a subset of cardinality $\mathfrak{I}$ if $\mathfrak{I} < \infty$, or of cardinality $\aleph_0$ if $\mathfrak{I} = \infty$. We notice that B_0 is a p-basis of $L_0 := k(t^{\Gamma}, B_0)$. Let L be a separable tamification of L_0 taken inside $k((t^{\Gamma}))$, i.e. L is a fixed field inside the separable closure of L_0 of a complement of the ramification group inside the absolute Galois group of L_0. Then L is separably tame, with residue field k and value group G, and of elementary imperfection degree $\mathfrak{I}$. □

Theorem 8.2.9. *Let $\mathfrak{L} = \mathfrak{L}_{\mathrm{val},\lambda}(\mathfrak{L}_{\mathbf{k}}, \mathfrak{L}_{\Gamma})$ be a $(\mathbf{k}, \boldsymbol{\Gamma})$-expansion of $\mathfrak{L}_{\mathrm{val},\lambda}$. There is an "elimination" function $\epsilon : \mathrm{Sent}(\mathfrak{L}) \to \mathsf{AKE\text{-}IMP}(\mathfrak{L})$ such that $\mathbf{STVF}^{\mathrm{eq}} \models (\varphi \leftrightarrow \epsilon\varphi)$, for all $\varphi \in \mathrm{Sent}(\mathfrak{L})$. Moreover, if $\mathfrak{L}$ is computable then ϵ may also be chosen to be computable.*

Proof. We adopt the terminology of [AF26+a], and consider the bridge

$$B = ((\mathsf{AKE\text{-}IMP}(\mathfrak{L}), \mathbf{STVF}^{\mathrm{eq}}), (\mathrm{Form}(\mathfrak{L}), \mathbf{STVF}^{\mathrm{eq}}), \mathrm{id}).$$

First observe that the inclusion map $\iota : \mathsf{AKE\text{-}IMP}(\mathfrak{L}) \to \mathrm{Form}(\mathfrak{L})$ is an interpretation for B. Moreover B satisfies the monotonicity property "(mon)" by Theorem 8.2.7. By [AF26+a, Proposition 2.18], therefore, the required elimination ϵ exists.

Suppose now that $\mathfrak{L}$ is computable, then $\mathsf{AKE\text{-}IMP}(\mathfrak{L})$ and $\mathrm{Form}(\mathfrak{L})$ are computable $\mathfrak{L}$-fragments, and ι is computable. Moreover $\mathbf{STVF}^{\mathrm{eq}}$ is computably enumerable (even computable), in any case. Thus, again by [AF26+a, Proposition 2.18], the elimination ϵ may be chosen to be computable. □

8.3 Computability-theoretic reductions

Recall our Convention 8.0.2 that $\hat{p} = p$ if $p \in \mathbb{P}$, and $\hat{p} = 1$ if $p = 0$.

Define $\mathbf{STVF}^{\mathrm{eq}}(R, G, X) := (\mathbf{STVF}^{\mathrm{eq}} \cup \iota_{\mathbf{k}}R \cup \iota_{\Gamma}G \cup X)_{\mathsf{AKE\text{-}IMP}}$.

Theorem 8.3.1 (Fixed characteristic, uniform in imperfection degree)**.** *Let $\mathfrak{L} = \mathfrak{L}_{\mathrm{val},\lambda}(\mathfrak{L}_{\mathbf{k}}, \mathfrak{L}_{\Gamma})$ be a $(\mathbf{k}, \boldsymbol{\Gamma})$-expansion of $\mathfrak{L}_{\mathrm{val},\lambda}$, let $p \in \mathbb{P} \cup \{0\}$, let R be an $\mathfrak{L}_{\mathbf{k}}$-theory of fields of characteristic p, let G be an $\mathfrak{L}_{\Gamma}$-theory of $\hat{p}$-divisible ordered abelian groups, and let X be an **IMP**-theory extending $\mathbf{X}_p$. Suppose that $\mathfrak{L}$ is computable. Then*

- **(i)** $\mathbf{STVF}^{\mathrm{eq}}(R, G, X)^{\vdash} \simeq_T R^{\vdash} \oplus_T G^{\vdash} \oplus_T (\mathbf{F} \cup X)_{\mathsf{IMP}}$, *and*
- **(ii)** $\mathbf{STVF}^{\mathrm{eq}}(R, G, X)^{\vdash}$ *is decidable if and only if $R^{\vdash}$, $G^{\vdash}$, and $(\mathbf{F} \cup X)_{\mathsf{IMP}}$ are decidable.*

Proof. We begin just like in the proof of Theorem 8.2.9. Consider the bridge

$$B_p = ((\mathsf{AKE\text{-}IMP}(\mathfrak{L}), \mathbf{STVF}^{\mathrm{eq}}), (\mathrm{Form}(\mathfrak{L}), \mathbf{STVF}^{\mathrm{eq}}), \mathrm{id}).$$

Observe that $\mathfrak{L}$ is computable, so $\mathsf{AKE\text{-}IMP}(\mathfrak{L})$ and $\mathrm{Form}(\mathfrak{L})$ are computable, ι is computable, and $\mathbf{STVF}^{\mathrm{eq}}$ is computably enumerable (even computable). The bridge B_p satisfies "surjectivity" by Lemma 8.2.8, and B_p satisfies "monotonicity" by Theorem 8.2.7. We have verified the hypotheses of [AF26+a, Corollary 2.23] for the arch $A = (B_p, B_p, \iota)$. Applying that result, we obtain

(I) B_p admits a computable elimination (this already follows from Theorem 8.2.9).

(II) $\mathbf{STVF}^{\mathrm{eq}}(R, G, X)^{\vdash} \simeq_{\mathrm{m}} \mathbf{STVF}^{\mathrm{eq}}(R, G, X)_{\mathsf{AKE\text{-}IMP}}$

It's also rather clear that $\mathbf{STVF}^{\mathrm{eq}}(R, G, X)_{\mathsf{AKE\text{-}IMP}} \simeq_{\mathrm{m}} (R \sqcup G \sqcup X)_{\mathsf{AKE\text{-}IMP}}$, and weaking our sense of equivalence to that of Turing equivalence we have $(R \sqcup G \sqcup X)_{\mathsf{AKE\text{-}IMP}} \simeq_{\mathrm{T}} R^{\vdash} \oplus_T G^{\vdash} \oplus_T (\mathbf{F} \cup X)_{\mathsf{IMP}}$. Combining these with **(II)**, we obtain **(i)**. Finally, **(ii)** follows immediately from **(i)**. □

Corollary 8.3.2 (Fixed characteristic and fixed/arbitrary imperfection degree)**.** *Let $\mathfrak{L} = \mathfrak{L}_{\mathrm{val},\lambda}(\mathfrak{L}_{\mathbf{k}}, \mathfrak{L}_{\Gamma})$ be a $(\mathbf{k}, \Gamma)$-expansion of $\mathfrak{L}_{\mathrm{val},\lambda}$, let $(p, \mathfrak{I}) \in \{(0, 0)\} \cup (\mathbb{P} \times (\mathbb{N} \cup \{\infty\}))$, let R be an $\mathfrak{L}_{\mathbf{k}}$-theory of fields of characteristic p, and let G be an $\mathfrak{L}_{\Gamma}$-theory of $\hat{p}$-divisible ordered abelian groups. Suppose that $\mathfrak{L}$ is computable. Then*

(I) **(i)** $\mathbf{STVF}^{\mathrm{eq}}(R, G, \mathbf{X}_{p,\mathfrak{I}})^{\vdash} \simeq_{\mathrm{T}} R^{\vdash} \oplus_T G^{\vdash}$*, and*

(ii) $\mathbf{STVF}^{\mathrm{eq}}(R, G, \mathbf{X}_{p,\mathfrak{I}})^{\vdash}$ *is decidable if and only if* $R^{\vdash}$ *and* $G^{\vdash}$ *are decidable.*

(II) **(i)** $\mathbf{STVF}^{\mathrm{eq}}(R, G, \mathbf{X}_{p})^{\vdash} \simeq_{\mathrm{T}} R^{\vdash} \oplus_T G^{\vdash}$*, and*

(ii) $\mathbf{STVF}^{\mathrm{eq}}(R, G, \mathbf{X}_{p})^{\vdash}$ *is decidable if and only if* $R^{\vdash}$ *and* $G^{\vdash}$ *are decidable.*

Proof. Both $(\mathbf{F}_{p,\mathfrak{I}})_{\mathsf{IMP}}$ and $(\mathbf{F}_p)_{\mathsf{IMP}}$ are decidable. □

Taken together, Theorem 8.3.1 and Corollary 8.3.2 prove Main Theorem 6.

Question 8.3.3 (Problem 9.6)**.** How may we adapt Theorem 8.3.1 so that it is uniform in the characteristic p?

Chapter 9

Retour vers le futur

Oh, I've been smiling lately
Dreaming about the world as one
And I believe it could be
Some day it's going to come

Peace Train, Yusuf / Cat Stevens

We collect the questions and open problems raised in this mémoire:

Problem 9.1 (Question 3.2.14)**.** Use the S-maps to give a clear account of the structure of the automorphism groups of $C(k)$ and of $k(\!(t)\!)$.

Problem 9.2 (Question 3.2.15)**.** How many 1-types are there in $\mathbb{F}_p(\!(t)\!)$ of elements of value n?

This problem is a "work in progress' with Fehm. We have several bounds on the number of orbits in certain obvious classes

Problem 9.3 (Question 4.2.14)**.** Can the hypothesis $(\mathbb{Y})$ on C and k_K be replaced in Theorem 4.2.2 by an *a priori* weaker hypotheses?

Problem 9.4 (Question 7.1.7)**.** Are there at most 3 existentially $\emptyset$-definable henselian valuation rings on any field?

Problem 9.5 (Question 7.2.11)**.** What can we say about $\mathfrak{L}_{\mathrm{ring}}$-theories R that are fixed points of Φ_A?

Problem 9.6 (Question 8.3.3)**.** How may we adapt Theorem 8.3.1 so that it is uniform in the characteristic p?

A solution to Problem 9.6 must take into accout that the theories of valued group and residue field are not orthogonal: for example if the residue field is of characteristic 2 then the value group must be 2-divisible. However, this is really the *only* problem.

Problem 9.7. Give an account of the AKE theory of separably algebraically maximal Kaplansky ("SAMK") valued fields in all classical fragments, perhaps including quantifier elimination results, in line with the work of Soto Moreno in [SM25].

Problem 9.8. Give a unified account of the AKE theory of finitely ramified henselian, roughly tame, and separably tame valued fields, in all classical fragments, in line with the work in Chapter 8, building on [ADJ24].

Problem 9.9. Give an account of all of the above AKE theory in all "baroque" fragments, that is, for the class of fragments generated by $\mathsf{F}_0, \exists_n, \exists^n, \exists, \forall_n, \forall^n, \forall$.

Problem 9.10. Simplify and rewrite from first principles a complete proof of the Embedding Lemma (in a suitable form) for separably tame valued fields ("STVF").

This last problem is a work in progress, and I am optimistic of a clean proof. Problem 9.9 seems a lot harder but there is steady progress being made, for example see [AF25].

Sunrise doesn't last all morning
A cloudburst doesn't last all day

All things must pass, George Harrison

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