State space model, dynamic model

Measurement noise : 센서에 포함된 잡음.
process noise : 상태 공간 방정식에 포함된 잡음.

measurement : 확률 변수, 확률분포는 확률 밀도 함수 (PDF)로 정의됨

Expected value : mean of measurements.

Bias, **systematic measurement error**, **accuracy of measurements**.

: offset between mean of measurements and true value.

measurement noise, **random measurement error**

: the dispersion of the distribution, measurement precision.

notation

x : true value

z_n : measured value of state at time n.

$\hat{x}_{n,n}$: estimate of x at time n (the estimate made after taking z_n)

$\hat{x}_{n+1,n}$: estimate of future state $(n+1)$ of x made at time n.
$\hat{x}_{n+1,n}$ is a predicted state or extrapolated state.

$\hat{x}_{n-1,n-1}$: estimate of x at time $n-1$ made after taking measurement z_{n-1}

$\hat{x}_{n,n-1}$: prior prediction, estimate of state at time n made at time $n-1$

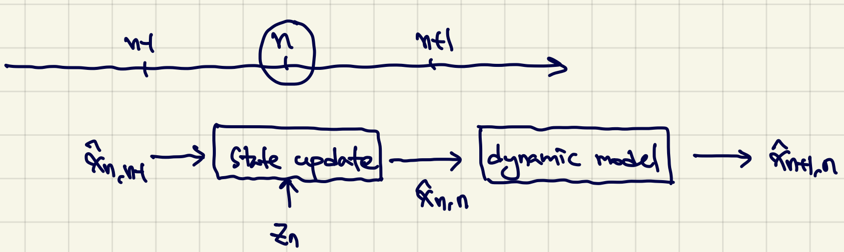
$\hat{x}_{n,n}$ 을 알기 위해 $z_n\ z_{n-1} \dots z_1$ 을 알아야 함. $\hat{x}_{n,n} = \frac{z_1+z_2+\dots+z_n}{n}$

매번 새로운 측정값 z_n 저장, 평균 계산시 계산량, 저장 공간 ↑

$\hat{x}_{n-1,n-1}$ 만 저장하고 새로운 관측값으로 업데이트

<Algorithm>

- estimate the current state based on measurement and prior prediction.
- predict the next state based on the current state estimate using the dynamic model



$$\hat{x}_{n,n} = \frac{1}{n}\sum_{i=1}^{n} z_i = \frac{1}{n}\left[\sum_{i=1}^{n-1} z_i + z_n\right] = \frac{n-1}{n-1}\cdot\frac{1}{n}\left[\sum_{i=1}^{n-1} z_i + z_n\right]$$

$$= \frac{n-1}{n}\cdot\frac{1}{n-1}\sum_{i=1}^{n-1} z_i + \frac{1}{n} z_n = \frac{n-1}{n}\hat{x}_{n-1,n-1} + \frac{1}{n} z_n = \hat{x}_{n-1,n-1} + \frac{1}{n}\left(z_n - \hat{x}_{n-1,n-1}\right)$$

?

$\hat{x}_{n-1,n-1}$: estimated state of x at $t=n-1$, based on measurement at $t=n-1$ z_{n-1}

$\hat{x}_{n-1,n-1} \longrightarrow \hat{x}_{n,n-1}$

Static system 이라면 dynamic model 로 $\hat{x}_{n,n-1} = \hat{x}_{n-1,n-1}$ $t=n-1$ 에서 예측한 $t=n$ 의 상태

$$\hat{x}_{n,n} = \hat{x}_{n-1,n-1} + \frac{1}{n}\left(z_n - \hat{x}_{n-1,n-1}\right) = \hat{x}_{n,n-1} + \frac{1}{n}\left(z_n - \hat{x}_{n,n-1}\right)$$

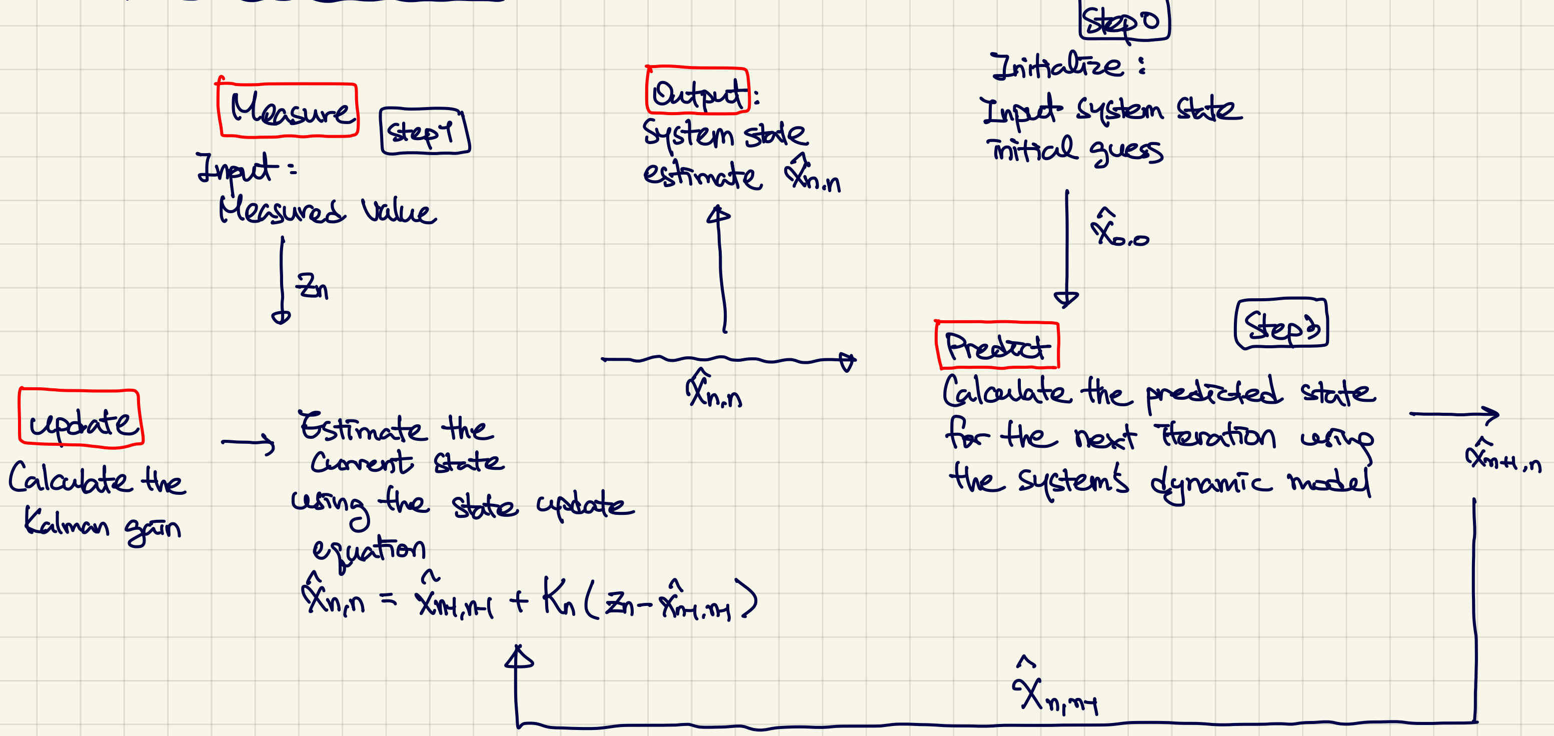
$$\boxed{\text{The estimate of the current state}} = \boxed{\text{predicted value of the current state}} + \underbrace{\boxed{\text{Factor}}}_{\text{Kalman gain } K_n}\underbrace{\left(\boxed{\text{Measurement}} - \boxed{\text{Predicted value of the current state}}\right)}_{\text{measurement residual, Innovation.}}$$

Factor $= \frac{1}{n}$ 이면 측정값에 대한 가중치 $\frac{1}{n}$ 은 점점 작아짐. 새 관측값 측정값에 대한 기여도는 점점 줄어듦.

Initial guess = first estimation.

Kalman Filter 는 Initial guess 가 필요함.

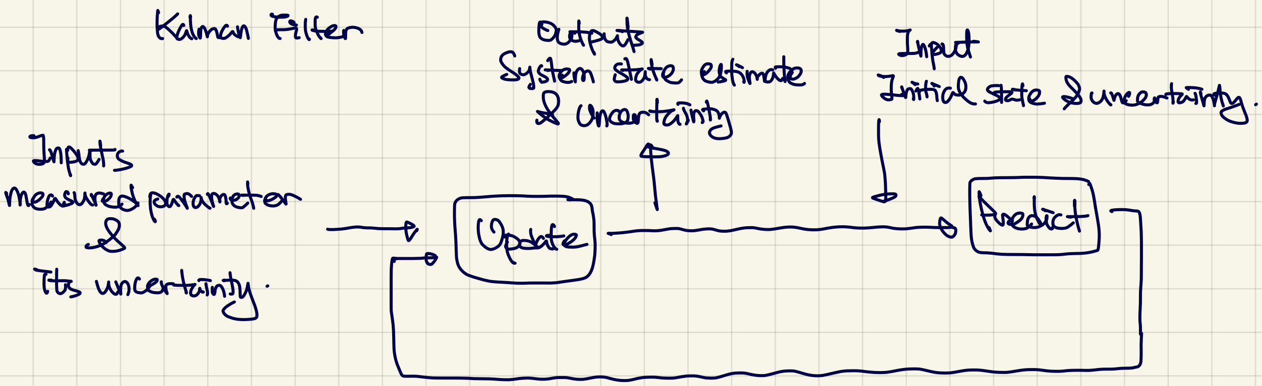
Estimation Algorithm



Two of Five Kalman Filter equations

- State update equation
- State extrapolation equation.

Kalman Filter treats ① measurements, ② current state estimation, ③ next state estimation (prediction) as normally distributed random variables. The random variable is described by mean and variance.



Estimate as a random variable

Measurement as a random variable : The measurement error is the difference between measurements and the true value
Measurement errors are random and can be described by variance (σ^2), standard deviation (σ) which is the measurement uncertainty.

One Dimensional Kalman Filter.

Current state estimate $\hat{x}_{n,n}$ $p_{n,n}$ ⟶ predict ⟶ Next state estimate $\hat{x}_{n+1,n}$ $p_{n+1,n}$

$*$ $X \sim N(\hat{x}, \sigma^2)$ $\quad AX \sim N(A\hat{x}, A^2\sigma^2)$?

$E(X) = \hat{x} \quad E([X - E(X)]^2) = \sigma$

$E(AX) = AE(X) = A\hat{x}$

$E([AX - E(AX)]^2) = E(A^2[X - E(X)]^2) = A^2 E([X - E(X)]^2) = A^2\sigma^2$

State update

To estimate the current state of the system, we combine two random variables

① The prior state estimate (the current state estimate predicted at the previous state)

② measurement

prior estimate

$\hat{x}_{n,n-1}$

$p_{n,n-1}$

State update $\longrightarrow$ Current state estimate $\hat{x}_{n,n}$ $p_{n,n}$

measurement

z_n, r_n

Kalman filter is an optimal filter. It combines prior state estimate with the measurement in a way that minimizes the uncertainty of the current state.

$$\begin{cases} \hat{x}_{n,n} = w_1 z_n + w_2 \hat{x}_{n,n-1} \qquad w_1 + w_2 = 1 \\ \phantom{\hat{x}_{n,n}} = w_1 z_n + (1-w_1)\hat{x}_{n,n-1} \\ p_{n,n} = w_1^2 r_n + (1-w_1)^2 p_{n,n-1} \end{cases}$$

$p_{n,n}$ is the variance of the optimum combined estimate $\hat{x}_{n,n}$

$p_{n,n-1}$ is the variance of the prior estimate $\hat{x}_{n,n-1}$

r_n is the variance of the measurement z_n

Since we want to minimize $p_{n,n}$

$$w_1^* = \arg\min_{w_1} p_{n,n}$$

$$\frac{dp_{n,n}}{dw_1} = 2w_1 r_n \overset{*}{-} 2(1-w_1)p_{n,n-1}$$

$$= 2w_1[r_n + p_{n,n-1}] - 2p_{n,n-1} = 0$$

$$w_1^* = \frac{p_{n,n-1}}{p_{n,n-1} + r_n}$$

$$* \quad \hat{x}_{n,n} = w_1 z_n + (1-w_1)\hat{x}_{n,n-1} = \left[\frac{p_{n,n-1}}{p_{n,n-1}+r_n}\right] z_n + \left[1 - \frac{p_{n,n-1}}{p_{n,n-1}+r_n}\right]\hat{x}_{n,n-1}$$

$$= \hat{x}_{n,n-1} + \underbrace{\frac{p_{n,n-1}}{p_{n,n-1}+r_n}}_{\text{Kalman gain.}} \underbrace{[z_n - \hat{x}_{n,n-1}]}_{\text{Innovation}}$$

$$* \; p_{n,n} = w_1^2 r_n + (1-w_1)^2 p_{n,n-1} = \left[\frac{p_{n,n-1}}{p_{n,n-1}+r_n}\right]^2 r_n + \left[\frac{-r_n}{p_{n,n-1}+r_n}\right]^2 p_{n,n-1}$$

$$= \frac{p_{n,n-1}^2 r_n}{(p_{n,n-1}+r_n)^2} + \frac{r_n^2 p_{n,n-1}}{(p_{n,n-1}+r_n)^2} = \frac{p_{n,n-1} r_n}{(p_{n,n-1}+r_n)^2}(p_{n,n-1}+r_n)$$

$$= \frac{p_{n,n-1} r_n}{p_{n,n-1}+r_n} = p_{n,n-1}(1-K_n)$$

$$K_n = \frac{p_{n,n-1}}{p_{n,n-1}+r_n} \qquad 1-K_n = \frac{r_n}{p_{n,n-1}+r_n}$$

$$\boxed{\begin{aligned} \hat{x}_{n,n} &= \hat{x}_{n,n-1} + \frac{p_{n,n-1}}{p_{n,n-1}+r_n}(z_n - \hat{x}_{n,n-1}) \\ p_{n,n} &= (1-K_n)\,p_{n,n-1} \end{aligned}}$$

$1-K_n \leq 1$ 이므로 매 반복 마다 분산이 감소함

$p_{n,n} = (1-K_n)\,p_{n,n-1}$: Covariance update equation

K_n : measurement weight (Kalman gain) $\quad 1-K_n$: prior prediction of current state estimation.

High Kalman gain : measurement uncertainty 가 작은 경우 Kalman gain 이 큰 것이다.
측정값에 대한 가중치를 높여 현재 상태에 대한 기여도 ↑

Low kalman gain : measurement uncertainty 가 큰 경우 Kalman gain 이 작은 것이다.
측정값에 대한 가중치를 낮춰 현재 상태에 대한 기여도 ↓

상태 변수가 둘 이상 → Multivariate Kalman Filter.

$$K_n = P_{n,n-1} H^T (H P_{n,n-1} H^T + R_n)^{-1}$$