< Stochastic Differential Equations. >

Def : SDE is a differential equation of the form

$$dX(t) = \mu(t, X(t))\,dt + \sigma(t, X(t))\,dB(t)$$

Goal is to find a stochastic process $X(t)$ satisfying the above.

i.e. $X(t) = \int \mu(s, X(s))\,ds + \int \sigma(s, X(s))\,dB(s)$

both dependent on current value $X(t)$

ex) $dX(t) = \underbrace{\mu X(t)\,dt}_{\text{drift}} + \underbrace{\sigma X(t)\,dB(t)}_{\text{error}}$ $\quad X(0) = x_0 \quad -\infty < \mu < \infty \quad \sigma > 0$

Guess $X(t) = f(t, B(t))$

Then $dX(t) = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \frac{\partial f}{\partial x}\,dB(t)$ $\quad$ x는 번 2개 변수.

① $\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = \mu X(t) = \mu f$

② $\frac{\partial f}{\partial x} = \sigma X(t) = \sigma f \rightarrow f = e^{\sigma x + a g(t)}$

$\frac{\partial f}{\partial t} = a g'(t) f \quad \frac{\partial^2 f}{\partial x^2} = \sigma^2 f \longrightarrow$ Into ① $\longrightarrow a g'(t) f + \frac{1}{2}\sigma^2 f = \mu f$

$a g'(t) = \mu - \frac{1}{2}\sigma^2 \quad g(t) = \left(\mu - \frac{1}{2}\sigma^2\right)\overset{t}{C_1} + C_2$

$\therefore f(x, t) = e^{\sigma x + (\mu - \frac{1}{2}\sigma^2)t + c} = x_0 e^{\sigma x + (\mu - \frac{1}{2}\sigma^2)t}$

$x(0) = e^c = x_0$

only the drift term is dependent on current state.

ex) $dX(t) = \underline{-\alpha X(t)\,dt} + \underline{\sigma\,dB(t)}$ $\quad X(0) = x_0 \quad \alpha > 0$ $\quad$: mean reverting stochastic process.

Ornstein – Uhlenbeck process

Guess $X(t) = a(t)\left(x_0 + \int_0^t b(s)\,dB(s)\right)$ $\quad a(0) = 1$

$dX(t) = a'(t)\frac{X(t)}{a(t)}\,dt + a(t)\,b(t)\,dB(t)$

① $-\alpha X(t) = a'(t)\frac{X(t)}{a(t)} \qquad -\alpha = a'(t)\frac{1}{a(t)} \qquad a(t) = e^{-\alpha t}$

② $a(t)\,b(t) = \sigma \qquad b(t) = \frac{\sigma}{a(t)} = \sigma e^{\alpha t}$

$$\Rightarrow X(t) = e^{-\alpha t}\left(x_0 + \int_0^t \sigma e^{\alpha s}\,dB(s)\right) = \underbrace{x_0 e^{-\alpha t}}_{\text{decay to zero}} + \underbrace{\int \sigma e^{\alpha(s-t)}\,dB(s)}_{\text{noise}}$$

<Side Notes>

What is $dB(t)$

- Infinitesimal increment of a Brownian motion (aka Weiner process) $B(t)$
- $B(t)$ is a continuous time stochastic process with

  a. $B(0) = 0$

  b. Independent normally distributed increments $B(t+\Delta t) - B(t) \sim N(0, \Delta t)$

  c. Continuous but nowhere differentiable.

How to interpret $dB(t)$?

→ random noise at time t with $dB(t) \sim N(0, dt)$

; mean of zero and variance of dt

Deterministic calculus : dx means a tiny deterministic change
Stochastic calculus : $dB(t)$ represents a tiny random change

Why can't we just write $\frac{dx}{dt}$

$\frac{dB(t)}{dt}$ $\nexists$ doesn't exist ; not differentiable → Ito Calculus.

* Deterministic Models (No noise)

Continuous time $\frac{dx}{dt} = f(x,t)$: describes change at every instant.

Discrete time $x_{k+1} = f(x_k, k)$: updates state at time steps.

* Stochastic Models (Noise)

Continuous-time

$$dx(t) = \underbrace{f(x(t),t)\,dt}_{\substack{\text{deterministic}\\ \text{dynamics}\\ \text{drift}}} + \underbrace{G(x(t),t)}_{\substack{\text{noise coefficient}\\ \text{diffusion}}} \underbrace{dB(t)}_{} \quad \text{Brownian motion increment}$$

Discrete-time

$$x_{k+1} = f(x_k, k) + w_k \qquad w_k \sim N(0, Q_k)$$

< Kalman Gain >

Actual.

$$\begin{cases} x_t = A_t x_{t-1} + w_t \\ y_t = H_t x_t + v_t \end{cases}$$

* Discrete time KF

I.C x_0 $P_{1|0}$

State propagation : $\hat{x}_{t|t-1} = A_t \hat{x}_{t-1}$ $\quad \hat{y}_t = H_t \hat{x}_{t-1|t} = H_t A_t \hat{x}_{t-1}$ $\boxed{\text{prediction}}$

$$\hat{x}_t = \hat{x}_{t|t-1} + K_t [y_t - \hat{y}_t] \qquad \boxed{\text{update}}$$

Covariance propagation $\quad P_{t+1|t} = A_t P_t A_t^T + G_t \underline{Q_t} G_t^T$ $\boxed{\text{prediction}}$

covariance of $x_{t+1|t}$ $\quad$ process var.

$$P_t = (I - K_t H_t) P_{t|t-1} \qquad \boxed{\text{update}}$$

$$K_t = P_{t|t-1} H_t^T (H_t P_{t|t-1} H_t^T + \underline{R_t})^{-1}$$

measurement var.

* Innovation covariance

The predicted output $\hat{y}$ is compared to the actual measurement y

Consider the covariance of this <u>output</u> error which is called <u>innovation covariance</u>.
measurement

$$\boxed{S_t} = E[(\hat{y}_t - y_t)(\hat{y}_t - y_t)^T]$$

This innovation covariance is involved in the Kalman Gain

$$K_t = P_{t|t-1} H_t^T \boxed{S_t}^{-1}$$

$$S_t = E\left[(H_t x_{t|t-1} - H_t x_t - v_t)(H_t x_{t|t-1} - H_t x_t - v_t)^T\right]$$

$$= E[(H_t e_{t|t-1} - v_t)(H_t e_{t|t-1} - v_t)^T] \qquad e_{t|t-1} = x_{t|t-1} - x_t$$

$$= H_t E(e_{t|t-1} e_{t|t-1}^T) H_t^T + E(v_t v_t^T) = H_t P_{t|t-1} H_t^T + R_t$$

* Interpretation of Kalman Gain.

$$K_t = P_{t|t-1} H_t^T (H_t P_{t|t-1} H_t^T + R_t)^{-1}$$

S_t

$$K_t S_t = P_{t|t-1} H_t^T \Leftrightarrow K_t H_t P_{t|t-1} H_t^T + K_t R_t = P_{t|t-1} H_t^T$$

$$\boxed{\Rightarrow K_t = P_t H_t^T R_t^{-1}}$$

using $P_t = (I - K_t H_t) P_{t|t-1}$ $\quad K_t H_t P_{t|t-1} = P_{t|t-1} - P_t$

$$(P_{t|t-1} - P_t) H_t^T + K_t R_t = P_{t|t-1} H_t^T \qquad P_t H_t^T = K_t R_t$$

measurement/output noise covar $\uparrow \rightarrow K_t \downarrow$

posteriori error covar $\uparrow \rightarrow K_t \uparrow$

< Kalman Filter : Continuous vs. Discrete Time. >

	Continuous	Discrete
State eq.	$\frac{dx}{dt} = F(t)x(t) + G(t)\underline{w(t)}_{?}$	$x_{t+1} = A_t x_t + (B_t u_t) + G_t w_t$
Measurement eq.	$y(t) = H(t)x(t) + v(t)$	$y_t = H_t x_t + v_t$
State update & propagation	$\frac{d\hat{x}(t)}{dt} = F(t)\hat{x}(t) + K(t)[y(t) - \hat{y}(t)]$	$\hat{x}_{t\mid t} = A_{t-1}\hat{x}_{t-1} + K_t[y_t - \hat{y}_t]$
Covariance update & propagation	Riccati Differential eq.	$P_t = (I - K_t H_t)P_{t\mid t-1}$ $P_{t+1\mid t} = A_t P_t A_t^T + G_t Q_t G_t^T$

$\dot{x} \cong \frac{x_{t+1} - x_t}{\Delta t}$

* $x \sim N(\mu, P)$

$E(Ax+B) = AE(x) + B = A\mu + B$

$E((Ax+B-A\mu-B)(Ax+B-A\mu-B)^T)$

$= E(A(x-\mu)(x-\mu)^T A^T) = \underline{APA^T}$

* Converting Discrete to Continuous.

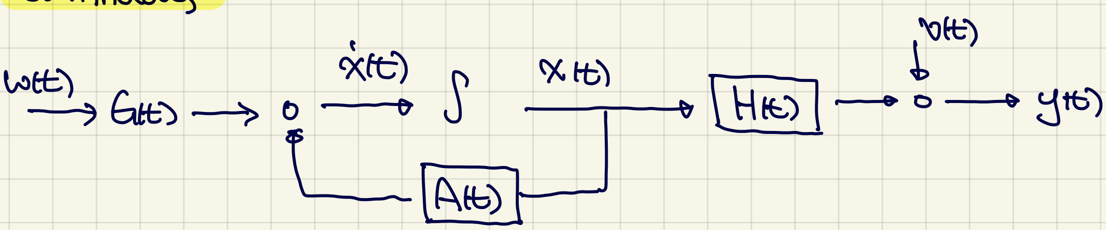
1. State eq.

$$\frac{x_{t+1} - x_t}{\Delta t} \approx \dot{x} = F(t)x(t) + G(t)w(t)$$

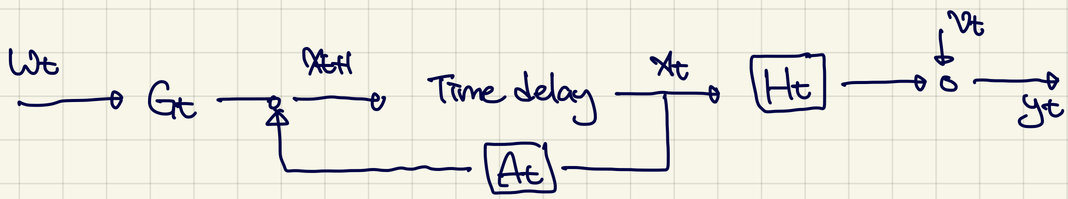
$$x_{t+1} = x_t + F(t)x(t)\Delta t + G(t)w(t)\Delta t = x_t(\underbrace{I + F(t)\Delta t}_{A_t}) + \underbrace{G(t)}_{G_t}\underbrace{\Delta t\, w(t)}_{w_t}$$

* modeling of noise.

Continuous



Discrete



process noise $\qquad w_t = \int_{t-\Delta t}^{t} w(\tau)\, d\tau = \overline{w(t)} \cdot \Delta t$

measurement noise $\qquad v_t = \frac{1}{\Delta t} \int_{t-\Delta t}^{t} v(\tau)\, d\tau = \bar{v}(t)$

* Noise Covariance

Measurement noise in discrete time v_t is the time average of continuous time noise $v(t)$ over sampling interval Δt

$$v_t = \frac{1}{\Delta t} \int_{t-\Delta t}^{t} v(\tau)\, d\tau = \bar{v}(t)$$

Based on this the covariance of measurement noise is related to the one in continuous time.

$$R_t = \mathbb{E}(v_t v_t^T) = \mathbb{E}\left(\frac{1}{\Delta t^2} \iint_{t-\Delta t}^{t} v(\tau) v(\tau')^T \, d\tau d\tau' \right) = \int \left[\int_{t-\Delta t}^{t} \underbrace{\mathbb{E}(v(\tau) v(\tau')^T)}_{R(\tau)\delta(\tau-\tau')} d\tau' \right] d\tau \frac{1}{\Delta t^2}$$

$$= \int_{t-\Delta t}^{t} R(\tau)\, d\tau \frac{1}{\Delta t^2} = \overline{R(t)}\, \Delta t \frac{1}{\Delta t^2} = \frac{\overline{R(t)}}{\Delta t}$$

when $\tau \neq \tau'$ $\mathbb{E}(v(\tau) v(\tau')^T) = 0$

Similarly relationship between process noise covariance in discrete time and continuous time.

$$Q_t = \mathbb{E}(w_t w_t^T) \cong \overline{Q(t)}\, \Delta t \quad \text{where} \quad \overline{Q(t)} = \frac{1}{\Delta t} \int_{t-\Delta t}^{t} Q(\tau)\, d\tau$$

$Q \in \mathbb{R}^{n\times n}$ process noise covariance matrix

SDE discretization rule

$$dx = f(x)\,dt + \underset{\substack{\uparrow \\ \text{standard deviation}}}{\Sigma}\, dB(t) \qquad \Sigma\Sigma^T = Q \qquad dB(t) \sim N(0, dt)$$

$$x_{k+1} = x_k + f(x_k)\,dt + \Sigma\sqrt{dt}\cdot N(0, I)$$

$\Sigma N(0, dt)$

SDE $\quad \dfrac{x_{k+1} - x_k}{dt} \approx \dot{x} = f(x_k) + \Sigma \dfrac{1}{\sqrt{dt}} N(0, I)$

$$\Sigma = \text{chol}(Q) \qquad \Sigma N(0, I) = N(0, Q)$$

Continuous $\quad \boxed{\text{SDE}} \qquad\qquad \boxed{\text{Simulation}}$

$$\boxed{dx(t) = f(x(t), t)\,dt + \Sigma\, dB(t)} \xrightarrow[\text{Simul}]{} x_{k+1} = x_k + f(x_k, t_k)\,\Delta t + \Sigma\sqrt{\Delta t}\, N(0, I)$$

Continuous

discrete

$$\boxed{x_{k+1} = f(x_k) + w_k \qquad w_k \sim N(0, Q)}$$

1. Role of Brownian Motion

   $dB(t) \sim N(0, dt)$

2. The structure of an SDE

   $dx = f(x, t)\,dt + \Sigma\, dB(t)$

3. Discretization with Euler Maruyama

   $t_k = k\,\Delta t \quad , \quad dB(t_k) \approx \Delta B_k = B(t_{k+1}) - B(t_k) \sim N(0, \Delta t)$

   $$x_{k+1} - x_k = f(x_k, t_k)\,\Delta t + \Sigma \boxed{\Delta B_k}$$

   $\Delta B_k = \sqrt{\Delta t}\, \xi_t \ , \ \xi_t \sim N(0, I)$

4. $\xi_t \sim N(0, I) \Rightarrow \sqrt{\Delta t}\, \xi_t \sim N(0, \Delta t I)$

   $\mathbb{E}((dB - 0)^2) = \mathbb{E}(dB^2) = dt$: covariance of $dB(t)$

5. Generalization with covariance matrix Q

   noise $\sim N(0, Q)$ noise at each moment not a process.

   $\Sigma\sqrt{\Delta t}\, \xi_k$ where $\Sigma\Sigma^T = Q$

   $Cov(\Sigma\sqrt{\Delta t}\, \xi_k) = Q\Delta t$

dynamics_with_noise_predefined.py

```python
def next_state(self, current_state, u, dt, dB=None, Q=None):
    """
    Simulate one step of the continuous-time stochastic dynamics using Euler-Maruyama.

    :param current_state: Current state vector (n,)
    :param u: Control input vector (u_dim,)
    :param dt: Time step
    :param noise: Optional pre-sampled noise (n,)
    :param Q: Optional process noise covariance matrix (n x n)
    :return: Next state vector (n,)
    """
    n = current_state.shape[0]

    if Q is None:
        Q = 0.01 * np.eye(n)  # default: isotropic noise

    # Cholesky decomposition of Q to get Sigma such that Q = Sigma @ Sigma.T
    Sigma = np.linalg.cholesky(Q)
```

$Q = \Sigma\Sigma^T$, Σ: standard deviation.

```python
    drift = self.f(current_state).T[0] + (self.g(current_state) @ np.array(u).reshape(self.u_dim, -1)).T[0]
    diffusion = np.sqrt(dt) * Sigma @ dB
```

$\sqrt{dt}\ \Sigma\ dB$

```python
    next_state = current_state + dt * drift + diffusion

    return next_state
```

$\boxed{\dot{x} = \text{drift} + \text{diffusion} : \text{Continuous time stochastic differential eq.}}$

$\text{drift} = \underset{3\times1}{f(x_k)} + \underset{3\times1}{g(x_k)}\ \underset{1\times1}{u}$

$\text{diffusion} = \sqrt{dt}\ \Sigma\ dB$ $\qquad dB \sim N(0, dt)$

$\dot{x} \approx \frac{x_{k+1} - x_k}{dt} = f(x_k) + g(x_k)\,u$ without diffusion

$x_{k+1} = x_k + [f(x_k) + g(x_k)\,u]\,dt$ ← Add accumulated noise during $\boxed{dt}$

$dB(t) \sim N(0, dt) \xrightarrow{Q} \underbrace{\sqrt{dt}\ \Sigma}_{\text{random variable}} \sim N(0, I)$

$\sqrt{dt} \sim N(0, I)$

$N = \frac{T}{dt}$

Simul Iteration #

sample time

```python
import numpy as np
def generate_process_noise(N, dt, dim, seed = None):
    if seed is not None:
        np.random.seed(seed)
    dB = np.random.normal(0.0, np.sqrt(dt), size = (N, dim))
    return dB
```

dim = 3 (= state dim)

mean: 0.0

std: np.sqrt(dt)

dB : normal distribution of mean 0 and

- Variance 0.02
- std $\sqrt{0.02}$

$\nabla h \cdot (\dot{x} + \alpha(h) \geq 0$

u "

$\bar{a}$

$C_{\dot{e}} + C_{\dot{e}}\,u$

"

$\dot{e}$ $\quad u$

$\dot{x} \geq u$

$\dot{x}(u)$

$\dot{e}(u)$

$\dot{e} \leftarrow \dot{x} \leftarrow \dot{x}$

$e^* = \quad \dot{\tilde{a}}\ \tilde{x}$

$C_1 \rightarrow$

$\nabla h \cdot (\dot{a} + \cancel{\alpha}\,\tilde{x}) \geq -1$

$\dot{e}(u)$

$$e^{x} = \dot{\lambda} + \dot{x}$$

$$\dot{\lambda} = \frac{\dot{\lambda}(\dot{x})\quad \dot{x}}{}$$

$$e^{x} = (\quad)\dot{x} + C$$

$$f(0)\, e^{r} p^{-r} e = \beta$$

$$0 \qquad (\quad x + C$$

$$f(\lambda, p, x) = e^{r} p^{r} e - \beta$$

$$\dot{f} = \frac{\partial f}{\partial \lambda}\cdot\dot{\lambda} + \frac{\partial f}{\partial p}\cdot \dot{p} + \frac{\partial f}{\partial x}\cdot\dot{x} = 0$$

$$\dot{\lambda} = \frac{\qquad \dot{x}}{\frac{\partial f}{\partial \lambda}}$$

$$\dot{\lambda}(\dot{x})$$

$\hat{x}_1$, $\underline{u_0}$

$\downarrow$

$\lambda^* =$ ~~sea~~ scalar eq root

$\downarrow$

$e^* = (H + 2\lambda P')q$, $q = 2H\underline{\hat{x}_1} + b$

$\downarrow$ $= f(\hat{x}(t)) + g(\hat{x}(t))u(t) + K(t)(y(t) - H(\hat{x}(t)))$

$\dot{\lambda} = \bigcirc \cdot \dot{\hat{x}} + \bigcirc$

$+ \bigcirc$

$\dot{e}^*(\lambda, P, \hat{x}) = (\quad)\dot{\lambda} + (\quad) + (\quad)\dot{\hat{x}}$

$\nabla h(\hat{x} + e^*) \cdot (\dot{\hat{x}} + \dot{e}^*) + \alpha(h(\hat{x} + e^*)) \geq 0$