Deutsch's Algorithm

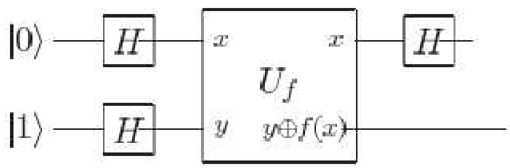
Note that if f is a function from {0,1} to {0,1} then f is one of the following four functions:

$0 \to 0 \text{ and } 1 \to 1,\ 0 \to 1 \text{ and } 1 \to 0,\ 0 \to 0 \text{ and } 1 \to 0,\ \text{ or } 0 \to 1 \text{ and } 1 \to 1$

Note that there are two possibilities - f is a constant function or f is not a constant function.

Deutsch's Algorithm gives you a way to use a quantum computer to determine whether x is constant or not in a "single" evaluation of f.

Here is the basic circuit where U_f is the linear operator as shown - that is, $U_f(|x>|y>) = |x>|y \oplus f(x)>$:



Note that H|0>=$\frac{1}{\sqrt{2}}(|0>+|1>)$ and H|1>=$\frac{1}{\sqrt{2}}(|0>-|1>)$ so the input to U_f is

$\frac{1}{\sqrt{2}}(|0>+|1>)\frac{1}{\sqrt{2}}(|0>-|1>)$. Let us look at $U_f\Big(\Big|x>\frac{1}{\sqrt{2}}(|0>-|1>)\Big)=\frac{1}{\sqrt{2}}U_f(|x>(|0>-|1>))=$

$\frac{1}{\sqrt{2}}(U_f(|x>|0>) - U_f(|x>|1>))=\frac{1}{\sqrt{2}}((|x>|0 \oplus f(x)>) - (|x>|1 \oplus f(x)>))$

Suppose f(x)=0. Then $0 \oplus 0 = 0 \text{ and } 1 \oplus 0 = 1$ so the right hand side becomes $\Big|x>\frac{1}{\sqrt{2}}(|0>-|1>)$ which can be written as $(-1)^{f(x)}\Big|x>\frac{1}{\sqrt{2}}(|0>-|1>)$. If f(x)=1 then the right hand side becomes $\Big|x>\frac{1}{\sqrt{2}}(|1>-|0>)$ which can be written as $(-1)^{f(x)}\Big|x>\frac{1}{\sqrt{2}}(|0>-|1>)$ where this time f(x)=1 implies $(-1)^{f(x)} = -1$.

Now suppose f(0)=f(1)=0. Then when we evaluate $U_f\Big(\frac{1}{\sqrt{2}}(|0>+|1>)\frac{1}{\sqrt{2}}(|0>-|1>)\Big)$ we get the result $(-1)^0\Big[\frac{|0>+|1>}{\sqrt{2}}\Big]\Big[\frac{|0>-|1>}{\sqrt{2}}\Big]$ and if f(0)=f(1)=1 we get the result $(-1)^1\Big[\frac{|0>+|1>}{\sqrt{2}}\Big]\Big[\frac{|0>-|1>}{\sqrt{2}}\Big]$. If we now apply the Hadamard gate to the first qubit of either of these we get $(-1)^{f(x)}H\Big[\frac{|0>+|1>}{\sqrt{2}}\Big]\Big[\frac{|0>-|1>}{\sqrt{2}}\Big]$ and since $H\Big[\frac{|0>+|1>}{\sqrt{2}}\Big] = \Big|0> \text{ we get } (-1)^{f(x)}|0>\Big[\frac{|0>-|1>}{\sqrt{2}}\Big]$ so if we measure the first qubit we get 0. If f(0)=0 and f(1)=1 we get $\Big[\frac{|0>-|1>}{\sqrt{2}}\Big]\Big[\frac{|0>-|1>}{\sqrt{2}}\Big]$ and if f(0)=1 and f(1)=0 we get $-\Big[\frac{|0>-|1>}{\sqrt{2}}\Big]\Big[\frac{|0>-|1>}{\sqrt{2}}\Big]$ and if we apply H to the first qubit we get $\pm|1>\Big[\frac{|0>-|1>}{\sqrt{2}}\Big]$ since $H\Big(\frac{|0>-|1>}{\sqrt{2}}\Big) = 1$ so if we measure the first qubit we get 1. Thus that measurement will tell us whether the function is constant (measured 0) or not constant (measured 1). Of course, this works because of quantum parallelism.