3

Mathematical Modeling of Mechanical Systems and Electrical Systems

3–1 INTRODUCTION

This chapter presents mathematical modeling of mechanical systems and electrical systems. In Chapter 2 we obtained mathematical models of a simple electrical circuit and a simple mechanical system. In this chapter we consider mathematical modeling of a variety of mechanical systems and electrical systems that may appear in control systems.

The fundamental law govering mechanical systems is Newton's second law. In Section 3–2 we apply this law to various mechanical systems and derive transfer-function models and state-space models.

The basic laws governing electrical circuits are Kirchhoff's laws. In Section 3–3 we obtain transfer-function models and state-space models of various electrical circuits and operational amplifier systems that may appear in many control systems.

3–2 MATHEMATICAL MODELING OF MECHANICAL SYSTEMS

This section first discusses simple spring systems and simple damper systems. Then we derive transfer-function models and state-space models of various mechanical systems.

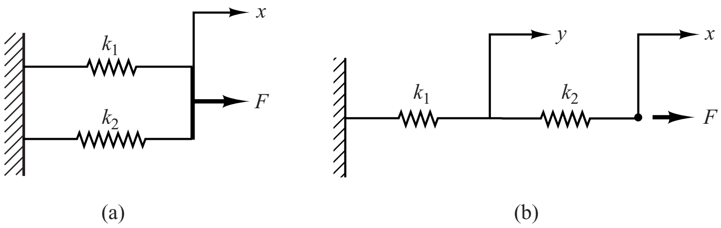


Figure 3–1
(a) System consisting of two springs in parallel;
(b) system consisting of two springs in series.

EXAMPLE 3–1 Let us obtain the equivalent spring constants for the systems shown in Figures 3–1(a) and (b), respectively.

For the springs in parallel [Figure 3–1(a)] the equivalent spring constant k_{eq} is obtained from

$$k_1 x + k_2 x = F = k_{eq} x$$

or

$$k_{eq} = k_1 + k_2$$

For the springs in series [Figure–3–1(b)], the force in each spring is the same. Thus

$$k_1 y = F, \qquad k_2(x - y) = F$$

Elimination of y from these two equations results in

$$k_2\left(x - \frac{F}{k_1}\right) = F$$

or

$$k_2 x = F + \frac{k_2}{k_1} F = \frac{k_1 + k_2}{k_1} F$$

The equivalent spring constant k_{eq} for this case is then found as

$$k_{eq} = \frac{F}{x} = \frac{k_1 k_2}{k_1 + k_2} = \frac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2}}$$

EXAMPLE 3–2 Let us obtain the equivalent viscous-friction coefficient b_{eq} for each of the damper systems shown in Figures 3–2(a) and (b). An oil-filled damper is often called a dashpot. A dashpot is a device that provides viscous friction, or damping. It consists of a piston and oil-filled cylinder. Any relative motion between the piston rod and the cylinder is resisted by the oil because the oil must flow around the piston (or through orifices provided in the piston) from one side of the piston to the other. The dashpot essentially absorbs energy. This absorbed energy is dissipated as heat, and the dashpot does not store any kinetic or potential energy.

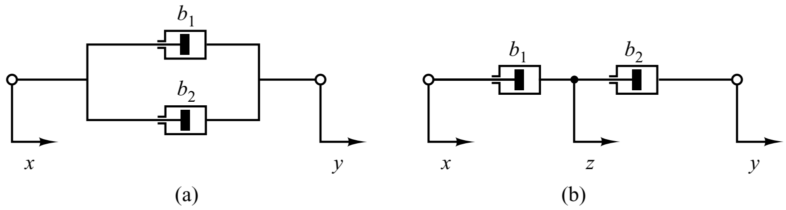


Figure 3–2
(a) Two dampers connected in parallel; (b) two dampers connected in series.

(a) The force f due to the dampers is

$$f = b_1(\dot{y} - \dot{x}) + b_2(\dot{y} - \dot{x}) = (b_1 + b_2)(\dot{y} - \dot{x})$$

In terms of the equivalent viscous-friction coefficient b_{eq}, force f is given by

$$f = b_{eq}(\dot{y} - \dot{x})$$

Hence

$$b_{eq} = b_1 + b_2$$

(b) The force f due to the dampers is

$$f = b_1(\dot{z} - \dot{x}) = b_2(\dot{y} - \dot{z}) \tag{3–1}$$

where z is the displacement of a point between damper b_1 and damper b_2. (Note that the same force is transmitted through the shaft.) From Equation (3–1), we have

$$(b_1 + b_2)\dot{z} = b_2\dot{y} + b_1\dot{x}$$

or

$$\dot{z} = \frac{1}{b_1 + b_2}(b_2\dot{y} + b_1\dot{x}) \tag{3–2}$$

In terms of the equivalent viscous-friction coefficient b_{eq}, force f is given by

$$f = b_{eq}(\dot{y} - \dot{x})$$

By substituting Equation (3–2) into Equation (3–1), we have

$$\begin{aligned} f = b_2(\dot{y} - \dot{z}) &= b_2\left[\dot{y} - \frac{1}{b_1 + b_2}(b_2\dot{y} + b_1\dot{x})\right] \\ &= \frac{b_1 b_2}{b_1 + b_2}(\dot{y} - \dot{x}) \end{aligned}$$

Thus,

$$f = b_{eq}(\dot{y} - \dot{x}) = \frac{b_1 b_2}{b_1 + b_2}(\dot{y} - \dot{x})$$

Hence,

$$b_{eq} = \frac{b_1 b_2}{b_1 + b_2} = \frac{1}{\dfrac{1}{b_1} + \dfrac{1}{b_2}}$$

EXAMPLE 3–3 Consider the spring-mass-dashpot system mounted on a massless cart as shown in Figure 3–3. Let us obtain mathematical models of this system by assuming that the cart is standing still for $t < 0$ and the spring-mass-dashpot system on the cart is also standing still for $t < 0$. In this system, $u(t)$ is the displacement of the cart and is the input to the system. At $t = 0$, the cart is moved at a constant speed, or $\dot{u} =$ constant. The displacement $y(t)$ of the mass is the output. (The displacement is relative to the ground.) In this system, m denotes the mass, b denotes the viscous-friction coefficient, and k denotes the spring constant. We assume that the friction force of the dashpot is proportional to $\dot{y} - \dot{u}$ and that the spring is a linear spring; that is, the spring force is proportional to $y - u$.

For translational systems, Newton's second law states that

$$ma = \sum F$$

where m is a mass, a is the acceleration of the mass, and $\sum F$ is the sum of the forces acting on the mass in the direction of the acceleration a. Applying Newton's second law to the present system and noting that the cart is massless, we obtain

$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y - u)$$

or

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku$$

This equation represents a mathematical model of the system considered. Taking the Laplace transform of this last equation, assuming zero initial condition, gives

$$(ms^2 + bs + k)Y(s) = (bs + k)U(s)$$

Taking the ratio of $Y(s)$ to $U(s)$, we find the transfer function of the system to be

$$\text{Transfer function} = G(s) = \frac{Y(s)}{U(s)} = \frac{bs + k}{ms^2 + bs + k}$$

Such a transfer-function representation of a mathematical model is used very frequently in control engineering.

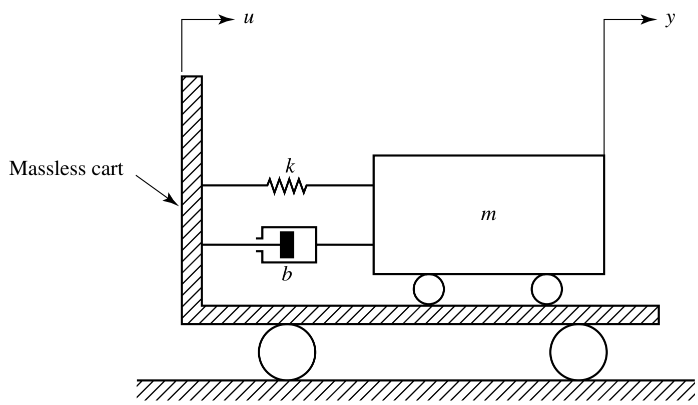


Figure 3–3 Spring-mass-dashpot system mounted on a cart.

Next we shall obtain a state-space model of this system. We shall first compare the differential equation for this system

$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = \frac{b}{m}\dot{u} + \frac{k}{m}u$$

with the standard form

$$\ddot{y} + a_1\dot{y} + a_2 y = b_0\ddot{u} + b_1\dot{u} + b_2 u$$

and identify a_1, a_2, b_0, b_1, and b_2 as follows:

$$a_1 = \frac{b}{m}, \qquad a_2 = \frac{k}{m}, \qquad b_0 = 0, \qquad b_1 = \frac{b}{m}, \qquad b_2 = \frac{k}{m}$$

Referring to Equation (3–35), we have

$$\beta_0 = b_0 = 0$$

$$\beta_1 = b_1 - a_1\beta_0 = \frac{b}{m}$$

$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = \frac{k}{m} - \left(\frac{b}{m}\right)^2$$

Then, referring to Equation (2–34), define

$$x_1 = y - \beta_0 u = y$$

$$x_2 = \dot{x}_1 - \beta_1 u = \dot{x}_1 - \frac{b}{m}u$$

From Equation (2–36) we have

$$\dot{x}_1 = x_2 + \beta_1 u = x_2 + \frac{b}{m}u$$

$$\dot{x}_2 = -a_2 x_1 - a_1 x_2 + \beta_2 u = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \left[\frac{k}{m} - \left(\frac{b}{m}\right)^2\right]u$$

and the output equation becomes

$$y = x_1$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{b}{m} \\ \frac{k}{m} - \left(\frac{b}{m}\right)^2 \end{bmatrix}u \tag{3–3}$$

and

$$y = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3–4}$$

Equations (3–3) and (3–4) give a state-space representation of the system. (Note that this is not the only state-space representation. There are infinitely many state-space representations for the system.)

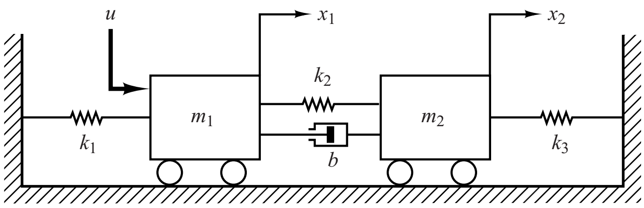


Figure 3–4
Mechanical system.

EXAMPLE 3–4 Obtain the transfer functions $X_1(s)/U(s)$ and $X_2(s)/U(s)$ of the mechanical system shown in Figure 3–4.

The equations of motion for the system shown in Figure 3–4 are

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2) + u$$

$$m_2\ddot{x}_2 = -k_3x_2 - k_2(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1)$$

Simplifying, we obtain

$$m_1\ddot{x}_1 + b\dot{x}_1 + (k_1 + k_2)x_1 = b\dot{x}_2 + k_2x_2 + u$$

$$m_2\ddot{x}_2 + b\dot{x}_2 + (k_2 + k_3)x_2 = b\dot{x}_1 + k_2x_1$$

Taking the Laplace transforms of these two equations, assuming zero initial conditions, we obtain

$$\left[m_1s^2 + bs + (k_1 + k_2)\right]X_1(s) = (bs + k_2)X_2(s) + U(s) \tag{3–5}$$

$$\left[m_2s^2 + bs + (k_2 + k_3)\right]X_2(s) = (bs + k_2)X_1(s) \tag{3–6}$$

Solving Equation (3–6) for $X_2(s)$ and substituting it into Equation (3–5) and simplifying, we get

$$\left[(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2\right]X_1(s)$$
$$= (m_2s^2 + bs + k_2 + k_3)U(s)$$

from which we obtain

$$\frac{X_1(s)}{U(s)} = \frac{m_2s^2 + bs + k_2 + k_3}{(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2} \tag{3–7}$$

From Equations (3–6) and (3–7) we have

$$\frac{X_2(s)}{U(s)} = \frac{bs + k_2}{(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2} \tag{3–8}$$

Equations (3–7) and (3–8) are the transfer functions $X_1(s)/U(s)$ and $X_2(s)/U(s)$, respectively.

EXAMPLE 3–5 An inverted pendulum mounted on a motor-driven cart is shown in Figure 3–5(a). This is a model of the attitude control of a space booster on takeoff. (The objective of the attitude control problem is to keep the space booster in a vertical position.) The inverted pendulum is unstable in that it may fall over any time in any direction unless a suitable control force is applied. Here we consider

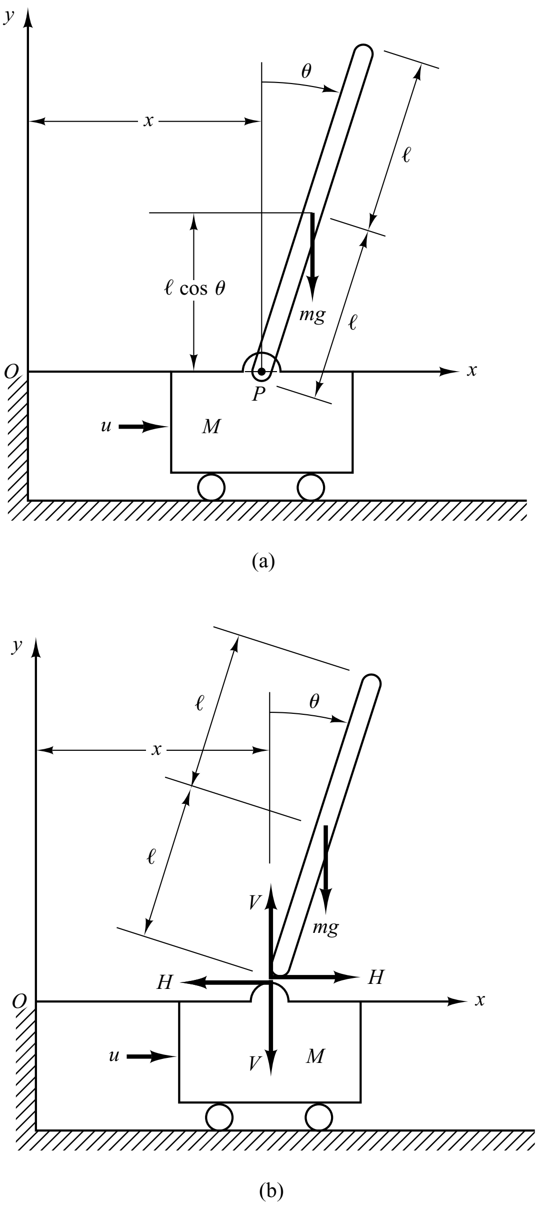


Figure 3–5
(a) Inverted pendulum system; (b) free-body diagram.

only a two-dimensional problem in which the pendulum moves only in the plane of the page. The control force u is applied to the cart. Assume that the center of gravity of the pendulum rod is at its geometric center. Obtain a mathematical model for the system.

Define the angle of the rod from the vertical line as θ. Define also the (x, y) coordinates of the center of gravity of the pendulum rod as (x_G, y_G). Then

$$x_G = x + l\sin\theta$$

$$y_G = l\cos\theta$$

To derive the equations of motion for the system, consider the free-body diagram shown in Figure 3–5(b). The rotational motion of the pendulum rod about its center of gravity can be described by

$$I\ddot{\theta} = Vl\sin\theta - Hl\cos\theta \tag{3–9}$$

where I is the moment of inertia of the rod about its center of gravity.

The horizontal motion of center of gravity of pendulum rod is given by

$$m\frac{d^2}{dt^2}(x + l\sin\theta) = H \tag{3–10}$$

The vertical motion of center of gravity of pendulum rod is

$$m\frac{d^2}{dt^2}(l\cos\theta) = V - mg \tag{3–11}$$

The horizontal motion of cart is described by

$$M\frac{d^2x}{dt^2} = u - H \tag{3–12}$$

Since we must keep the inverted pendulum vertical, we can assume that $\theta(t)$ and $\dot{\theta}(t)$ are small quantities such that $\sin\theta \doteqdot \theta$, $\cos\theta = 1$, and $\theta\dot{\theta}^2 = 0$. Then, Equations (3–9) through (3–11) can be linearized. The linearized equations are

$$I\ddot{\theta} = Vl\theta - Hl \tag{3–13}$$

$$m(\ddot{x} + l\ddot{\theta}) = H \tag{3–14}$$

$$0 = V - mg \tag{3–15}$$

From Equations (3–12) and (3–14), we obtain

$$(M + m)\ddot{x} + ml\ddot{\theta} = u \tag{3–16}$$

From Equations (3–13), (3–14), and (3–15), we have

$$\begin{aligned} I\ddot{\theta} &= mgl\theta - Hl \\ &= mgl\theta - l(m\ddot{x} + ml\ddot{\theta}) \end{aligned}$$

or

$$(I + ml^2)\ddot{\theta} + ml\ddot{x} = mgl\theta \tag{3–17}$$

Equations (3–16) and (3–17) describe the motion of the inverted-pendulum-on-the-cart system. They constitute a mathematical model of the system.

EXAMPLE 3–6 Consider the inverted-pendulum system shown in Figure 3–6. Since in this system the mass is concentrated at the top of the rod, the center of gravity is the center of the pendulum ball. For this case, the moment of inertia of the pendulum about its center of gravity is small, and we assume $I = 0$ in Equation (3–17). Then the mathematical model for this system becomes as follows:

$$(M + m)\ddot{x} + ml\ddot{\theta} = u \tag{3–18}$$

$$ml^2\ddot{\theta} + ml\ddot{x} = mgl\theta \tag{3–19}$$

Equations (3–18) and (3–19) can be modified to

$$Ml\ddot{\theta} = (M + m)g\theta - u \tag{3–20}$$

$$M\ddot{x} = u - mg\theta \tag{3–21}$$

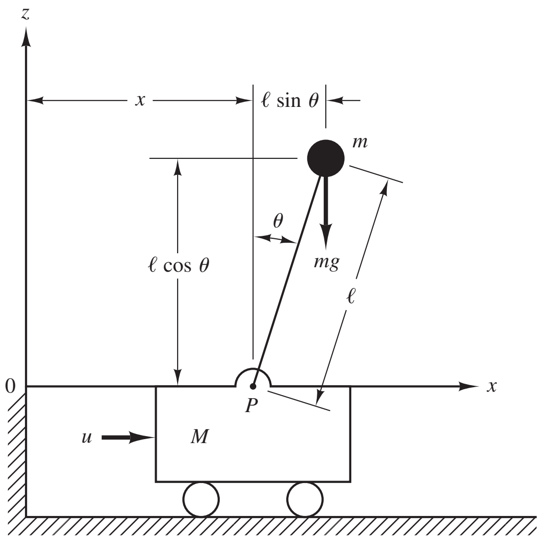


Figure 3–6 Inverted-pendulum system.

Equation (3–20) was obtained by eliminating $\ddot{x}$ from Equations (3–18) and (3–19). Equation (3–21) was obtained by eliminating $\ddot{\theta}$ from Equations (3–18) and (3–19). From Equation (3–20) we obtain the plant transfer function to be

$$\frac{\Theta(s)}{-U(s)} = \frac{1}{Mls^2 - (M+m)g} = \frac{1}{Ml\left(s + \sqrt{\frac{M+m}{Ml}g}\right)\left(s - \sqrt{\frac{M+m}{Ml}g}\right)}$$

The inverted-pendulum plant has one pole on the negative real axis $\left[s = -\left(\sqrt{M+m}/\sqrt{Ml}\right)\sqrt{g}\right]$ and another on the positive real axis $\left[s = \left(\sqrt{M+m}/\sqrt{Ml}\right)\sqrt{g}\right]$. Hence, the plant is open-loop unstable.

Define state variables x_1, x_2, x_3, and x_4 by

$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$
$$x_3 = x$$
$$x_4 = \dot{x}$$

Note that angle θ indicates the rotation of the pendulum rod about point P, and x is the location of the cart. If we consider θ and x as the outputs of the system, then

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \theta \\ x \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$$

(Notice that both θ and x are easily measurable quantities.) Then, from the definition of the state variables and Equations (3–20) and (3–21), we obtain

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{M+m}{Ml}gx_1 - \frac{1}{Ml}u$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -\frac{m}{M}gx_1 + \frac{1}{M}u$$

In terms of vector-matrix equations, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{M+m}{Ml}g & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{m}{M}g & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -\dfrac{1}{Ml} \\ 0 \\ \dfrac{1}{M} \end{bmatrix} u \tag{3–22}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{3–23}$$

Equations (3–22) and (3–23) give a state-space representation of the inverted-pendulum system. (Note that state-space representation of the system is not unique. There are infinitely many such representations for this system.)

3–3 MATHEMATICAL MODELING OF ELECTRICAL SYSTEMS

Basic laws governing electrical circuits are Kirchhoff's current law and voltage law. Kirchhoff's current law (node law) states that the algebraic sum of all currents entering and leaving a node is zero. (This law can also be stated as follows: The sum of currents entering a node is equal to the sum of currents leaving the same node.) Kirchhoff's voltage law (loop law) states that at any given instant the algebraic sum of the voltages around any loop in an electrical circuit is zero. (This law can also be stated as follows: The sum of the voltage drops is equal to the sum of the voltage rises around a loop.) A mathematical model of an electrical circuit can be obtained by applying one or both of Kirchhoff's laws to it.

This section first deals with simple electrical circuits and then treats mathematical modeling of operational amplifier systems.

***LRC* Circuit.** Consider the electrical circuit shown in Figure 3–7. The circuit consists of an inductance L (henry), a resistance R (ohm), and a capacitance C (farad). Applying Kirchhoff's voltage law to the system, we obtain the following equations:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e_i \tag{3–24}$$

$$\frac{1}{C}\int i\,dt = e_o \tag{3–25}$$

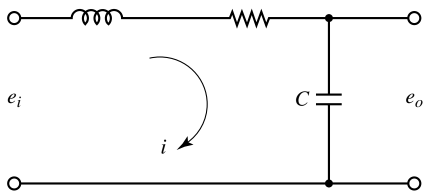


Figure 3–7
Electrical circuit.

Equations (3–24) and (3–25) give a mathematical model of the circuit.

A transfer-function model of the circuit can also be obtained as follows: Taking the Laplace transforms of Equations (3–24) and (3–25), assuming zero initial conditions, we obtain

$$LsI(s) + RI(s) + \frac{1}{C}\frac{1}{s}I(s) = E_i(s)$$

$$\frac{1}{C}\frac{1}{s}I(s) = E_o(s)$$

If e_i is assumed to be the input and e_o the output, then the transfer function of this system is found to be

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1} \tag{3–26}$$

A state-space model of the system shown in Figure 3–7 may be obtained as follows: First, note that the differential equation for the system can be obtained from Equation (3–26) as

$$\ddot{e}_o + \frac{R}{L}\dot{e}_o + \frac{1}{LC}e_o = \frac{1}{LC}e_i$$

Then by defining state variables by

$$x_1 = e_o$$
$$x_2 = \dot{e}_o$$

and the input and output variables by

$$u = e_i$$
$$y = e_o = x_1$$

we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix}u$$

and

$$y = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

These two equations give a mathematical model of the system in state space.

Transfer Functions of Cascaded Elements. Many feedback systems have components that load each other. Consider the system shown in Figure 3–8. Assume that e_i is the input and e_o is the output. The capacitances C_1 and C_2 are not charged initially.

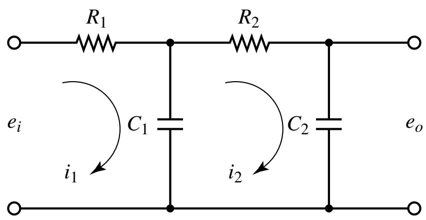


Figure 3–8 Electrical system.

It will be shown that the second stage of the circuit (R_2C_2 portion) produces a loading effect on the first stage (R_1C_1 portion). The equations for this system are

$$\frac{1}{C_1}\int (i_1 - i_2)\,dt + R_1 i_1 = e_i \tag{3–27}$$

and

$$\frac{1}{C_1}\int (i_2 - i_1)\,dt + R_2 i_2 + \frac{1}{C_2}\int i_2\,dt = 0 \tag{3–28}$$

$$\frac{1}{C_2}\int i_2\,dt = e_o \tag{3–29}$$

Taking the Laplace transforms of Equations (3–27) through (3–29), respectively, using zero initial conditions, we obtain

$$\frac{1}{C_1 s}\left[I_1(s) - I_2(s)\right] + R_1 I_1(s) = E_i(s) \tag{3–30}$$

$$\frac{1}{C_1 s}\left[I_2(s) - I_1(s)\right] + R_2 I_2(s) + \frac{1}{C_2 s} I_2(s) = 0 \tag{3–31}$$

$$\frac{1}{C_2 s} I_2(s) = E_o(s) \tag{3–32}$$

Eliminating $I_1(s)$ from Equations (3–30) and (3–31) and writing $E_i(s)$ in terms of $I_2(s)$, we find the transfer function between $E_o(s)$ and E_i(s) to be

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{(R_1C_1s + 1)(R_2C_2s + 1) + R_1C_2s}$$

$$= \frac{1}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2 + R_1C_2)s + 1} \tag{3–33}$$

The term R_1C_2s in the denominator of the transfer function represents the interaction of two simple RC circuits. Since $(R_1C_1 + R_2C_2 + R_1C_2)^2 > 4R_1C_1R_2C_2$, the two roots of the denominator of Equation (3–33) are real.

The present analysis shows that, if two RC circuits are connected in cascade so that the output from the first circuit is the input to the second, the overall transfer function is not the product of $1/(R_1C_1s + 1)$ and $1/(R_2C_2s + 1)$. The reason for this is that, when we derive the transfer function for an isolated circuit, we implicitly assume that the output is unloaded. In other words, the load impedance is assumed to be infinite, which means that no power is being withdrawn at the output. When the second circuit is connected to the output of the first, however, a certain amount of power is withdrawn, and thus the assumption of no loading is violated. Therefore, if the transfer function of this system is obtained under the assumption of no loading, then it is not valid. The degree of the loading effect determines the amount of modification of the transfer function.

Complex Impedances. In deriving transfer functions for electrical circuits, we frequently find it convenient to write the Laplace-transformed equations directly, without writing the differential equations. Consider the system shown in Figure 3–9(a). In this system, Z_1 and Z_2 represent complex impedances. The complex impedance $Z(s)$ of a two-terminal circuit is the ratio of $E(s)$, the Laplace transform of the voltage across the terminals, to $I(s)$, the Laplace transform of the current through the element, under the assumption that the initial conditions are zero, so that $Z(s) = E(s)/I(s)$. If the two-terminal element is a resistance R, capacitance C, or inductance L, then the complex impedance is given by R, $1/Cs$, or Ls, respectively. If complex impedances are connected in series, the total impedance is the sum of the individual complex impedances.

Remember that the impedance approach is valid only if the initial conditions involved are all zeros. Since the transfer function requires zero initial conditions, the impedance approach can be applied to obtain the transfer function of the electrical circuit. This approach greatly simplifies the derivation of transfer functions of electrical circuits.

Consider the circuit shown in Figure 3–9(b). Assume that the voltages e_i and e_o are the input and output of the circuit, respectively. Then the transfer function of this circuit is

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}$$

For the system shown in Figure 3–7,

$$Z_1 = Ls + R, \qquad Z_2 = \frac{1}{Cs}$$

Hence the transfer function $E_o(s)/E_i(s)$ can be found as follows:

$$\frac{E_o(s)}{E_i(s)} = \frac{\dfrac{1}{Cs}}{Ls + R + \dfrac{1}{Cs}} = \frac{1}{LCs^2 + RCs + 1}$$

which is, of course, identical to Equation (3–26).

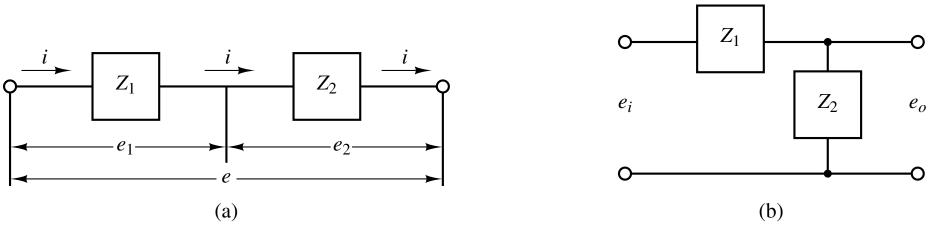


Figure 3–9
Electrical circuits.

EXAMPLE 3–7 Consider again the system shown in Figure 3–8. Obtain the transfer function $E_o(s)/E_i(s)$ by use of the complex impedance approach. (Capacitors C_1 and C_2 are not charged initially.)

The circuit shown in Figure 3–8 can be redrawn as that shown in Figure 3–10(a), which can be further modified to Figure 3–10(b).

In the system shown in Figure 3–10(b) the current I is divided into two currents I_1 and I_2. Noting that

$$Z_2 I_1 = (Z_3 + Z_4) I_2, \qquad I_1 + I_2 = I$$

we obtain

$$I_1 = \frac{Z_3 + Z_4}{Z_2 + Z_3 + Z_4} I, \qquad I_2 = \frac{Z_2}{Z_2 + Z_3 + Z_4} I$$

Noting that

$$E_i(s) = Z_1 I + Z_2 I_1 = \left[Z_1 + \frac{Z_2(Z_3 + Z_4)}{Z_2 + Z_3 + Z_4} \right] I$$

$$E_o(s) = Z_4 I_2 = \frac{Z_2 Z_4}{Z_2 + Z_3 + Z_4} I$$

we obtain

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2 Z_4}{Z_1(Z_2 + Z_3 + Z_4) + Z_2(Z_3 + Z_4)}$$

Substituting $Z_1 = R_1$, $Z_2 = 1/(C_1 s)$, $Z_3 = R_2$, and $Z_4 = 1/(C_2 s)$ into this last equation, we get

$$\frac{E_o(s)}{E_i(s)} = \frac{\frac{1}{C_1 s} \frac{1}{C_2 s}}{R_1\left(\frac{1}{C_1 s} + R_2 + \frac{1}{C_2 s}\right) + \frac{1}{C_1 s}\left(R_2 + \frac{1}{C_2 s}\right)}$$

$$= \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2) s + 1}$$

which is the same as that given by Equation (3–33).

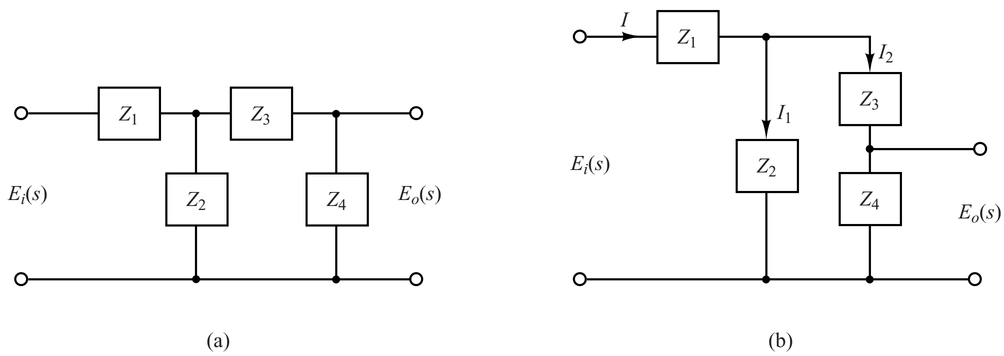


Figure 3–10 (a) The circuit of Figure 3–8 shown in terms of impedances; (b) equivalent circuit diagram.

Figure 3–11
(a) System consisting of two nonloading cascaded elements; (b) an equivalent system.

Transfer Functions of Nonloading Cascaded Elements. The transfer function of a system consisting of two nonloading cascaded elements can be obtained by eliminating the intermediate input and output. For example, consider the system shown in Figure 3–11(a). The transfer functions of the elements are

$$G_1(s) = \frac{X_2(s)}{X_1(s)} \qquad \text{and} \qquad G_2(s) = \frac{X_3(s)}{X_2(s)}$$

If the input impedance of the second element is infinite, the output of the first element is not affected by connecting it to the second element. Then the transfer function of the whole system becomes

$$G(s) = \frac{X_3(s)}{X_1(s)} = \frac{X_2(s)X_3(s)}{X_1(s)X_2(s)} = G_1(s)G_2(s)$$

The transfer function of the whole system is thus the product of the transfer functions of the individual elements. This is shown in Figure 3–11(b).

As an example, consider the system shown in Figure 3–12. The insertion of an isolating amplifier between the circuits to obtain nonloading characteristics is frequently used in combining circuits. Since amplifiers have very high input impedances, an isolation amplifier inserted between the two circuits justifies the nonloading assumption.

The two simple *RC* circuits, isolated by an amplifier as shown in Figure 3–12, have negligible loading effects, and the transfer function of the entire circuit equals the product of the individual transfer functions. Thus, in this case,

$$\frac{E_o(s)}{E_i(s)} = \left(\frac{1}{R_1C_1s + 1}\right)(K)\left(\frac{1}{R_2C_2s + 1}\right)$$
$$= \frac{K}{(R_1C_1s + 1)(R_2C_2s + 1)}$$

Electronic Controllers. In what follows we shall discuss electronic controllers using operational amplifiers. We begin by deriving the transfer functions of simple operational-amplifier circuits. Then we derive the transfer functions of some of the operational-amplifier controllers. Finally, we give operational-amplifier controllers and their transfer functions in the form of a table.

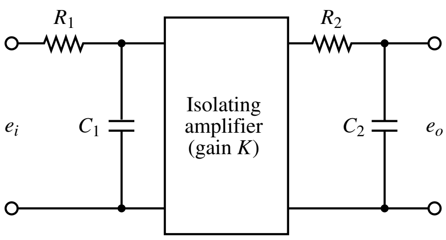


Figure 3–12
Electrical system.

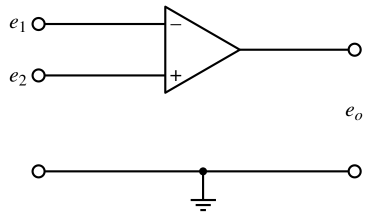


Figure 3–13 Operational amplifier.

Operational Amplifiers. Operational amplifiers, often called op amps, are frequently used to amplify signals in sensor circuits. Op amps are also frequently used in filters used for compensation purposes. Figure 3–13 shows an op amp. It is a common practice to choose the ground as 0 volt and measure the input voltages e_1 and e_2 relative to the ground. The input e_1 to the minus terminal of the amplifier is inverted, and the input e_2 to the plus terminal is not inverted. The total input to the amplifier thus becomes $e_2 - e_1$. Hence, for the circuit shown in Figure 3–13, we have

$$e_o = K(e_2 - e_1) = -K(e_1 - e_2)$$

where the inputs e_1 and e_2 may be dc or ac signals and K is the differential gain (voltage gain). The magnitude of K is approximately $10^5 \sim 10^6$ for dc signals and ac signals with frequencies less than approximately 10 Hz. (The differential gain K decreases with the signal frequency and becomes about unity for frequencies of 1 MHz $\sim$ 50 MHz.) Note that the op amp amplifies the difference in voltages e_1 and e_2. Such an amplifier is commonly called a differential amplifier. Since the gain of the op amp is very high, it is necessary to have a negative feedback from the output to the input to make the amplifier stable. (The feedback is made from the output to the inverted input so that the feedback is a negative feedback.)

In the ideal op amp, no current flows into the input terminals, and the output voltage is not affected by the load connected to the output terminal. In other words, the input impedance is infinity and the output impedance is zero. In an actual op amp, a very small (almost negligible) current flows into an input terminal and the output cannot be loaded too much. In our analysis here, we make the assumption that the op amps are ideal.

Inverting Amplifier. Consider the operational-amplifier circuit shown in Figure 3–14. Let us obtain the output voltage e_o.

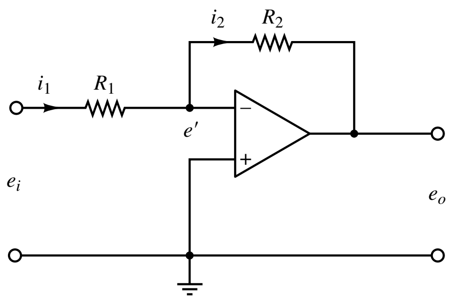


Figure 3–14 Inverting amplifier.

The equation for this circuit can be obtained as follows: Define

$$i_1 = \frac{e_i - e'}{R_1}, \qquad i_2 = \frac{e' - e_o}{R_2}$$

Since only a negligible current flows into the amplifier, the current i_1 must be equal to current i_2. Thus

$$\frac{e_i - e'}{R_1} = \frac{e' - e_o}{R_2}$$

Since $K(0 - e') = e_0$ and $K \gg 1$, e' must be almost zero, or $e' \doteqdot 0$. Hence we have

$$\frac{e_i}{R_1} = \frac{-e_o}{R_2}$$

or

$$e_o = -\frac{R_2}{R_1} e_i$$

Thus the circuit shown is an inverting amplifier. If $R_1 = R_2$, then the op-amp circuit shown acts as a sign inverter.

Noninverting Amplifier. Figure 3–15(a) shows a noninverting amplifier. A circuit equivalent to this one is shown in Figure 3–15(b). For the circuit of Figure 3–15(b), we have

$$e_o = K\left(e_i - \frac{R_1}{R_1 + R_2} e_o\right)$$

where K is the differential gain of the amplifier. From this last equation, we get

$$e_i = \left(\frac{R_1}{R_1 + R_2} + \frac{1}{K}\right) e_o$$

Since $K \gg 1$, if $R_1/(R_1 + R_2) \gg 1/K$, then

$$e_o = \left(1 + \frac{R_2}{R_1}\right) e_i$$

This equation gives the output voltage e_o. Since e_o and e_i have the same signs, the op-amp circuit shown in Figure 3–15(a) is noninverting.

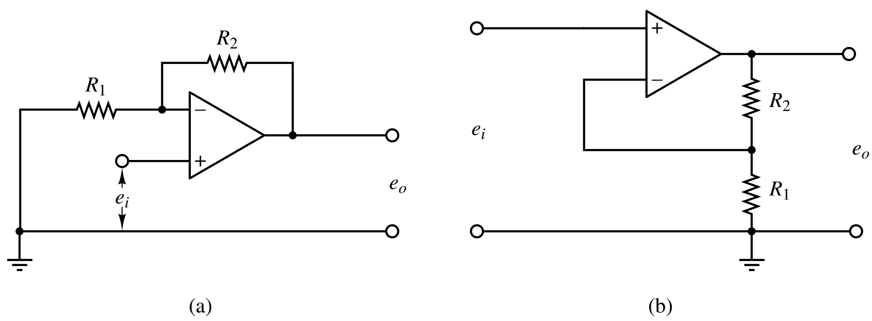


Figure 3–15
(a) Noninverting operational amplifier;
(b) equivalent circuit.

EXAMPLE 3–8 Figure 3–16 shows an electrical circuit involving an operational amplifier. Obtain the output e_o.

Let us define

$$i_1 = \frac{e_i - e'}{R_1}, \qquad i_2 = C\frac{d(e' - e_o)}{dt}, \qquad i_3 = \frac{e' - e_o}{R_2}$$

Noting that the current flowing into the amplifier is negligible, we have

$$i_1 = i_2 + i_3$$

Hence

$$\frac{e_i - e'}{R_1} = C\frac{d(e' - e_o)}{dt} + \frac{e' - e_o}{R_2}$$

Since $e' \doteqdot 0$, we have

$$\frac{e_i}{R_1} = -C\frac{de_o}{dt} - \frac{e_o}{R_2}$$

Taking the Laplace transform of this last equation, assuming the zero initial condition, we have

$$\frac{E_i(s)}{R_1} = -\frac{R_2Cs + 1}{R_2}E_o(s)$$

which can be written as

$$\frac{E_o(s)}{E_i(s)} = -\frac{R_2}{R_1}\frac{1}{R_2Cs + 1}$$

The op-amp circuit shown in Figure 3–16 is a first-order lag circuit. (Several other circuits involving op amps are shown in Table 3–1 together with their transfer functions. Table 3–1 is given on page 85.)

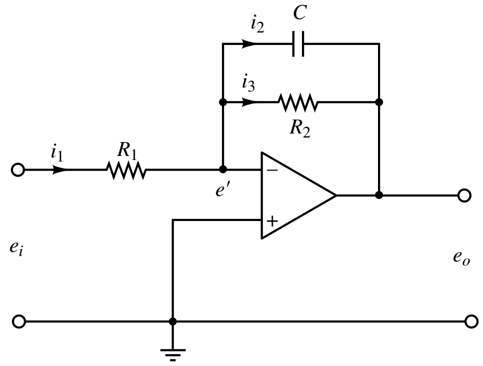


Figure 3–16
First-order lag circuit using operational amplifier.

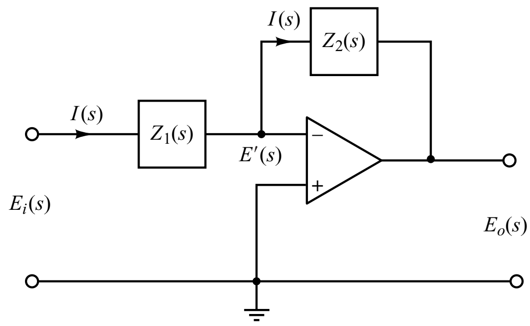


Figure 3–17 Operational-amplifier circuit.

Impedance Approach to Obtaining Transfer Functions. Consider the op-amp circuit shown in Figure 3–17. Similar to the case of electrical circuits we discussed earlier, the impedance approach can be applied to op-amp circuits to obtain their transfer functions. For the circuit shown in Figure 3–17, we have

$$\frac{E_i(s) - E'(s)}{Z_1} = \frac{E'(s) - E_o(s)}{Z_2}$$

Since $E'(s) \doteqdot 0$, we have

$$\frac{E_o(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)} \tag{3–34}$$

EXAMPLE 3–9 Referring to the op-amp circuit shown in Figure 3–16, obtain the transfer function $E_o(s)/E_i(s)$ by use of the impedance approach.

The complex impedances $Z_1(s)$ and $Z_2(s)$ for this circuit are

$$Z_1(s) = R_1 \qquad \text{and} \qquad Z_2(s) = \frac{1}{Cs + \dfrac{1}{R_2}} = \frac{R_2}{R_2Cs + 1}$$

The transfer function $E_o(s)/E_i(s)$ is, therefore, obtained as

$$\frac{E_o(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{R_2}{R_1}\frac{1}{R_2Cs + 1}$$

which is, of course, the same as that obtained in Example 3-8.

Lead or Lag Networks Using Operational Amplifiers. Figure 3–18(a) shows an electronic circuit using an operational amplifier. The transfer function for this circuit can be obtained as follows: Define the input impedance and feedback impedance as Z_1 and Z_2, respectively. Then

$$Z_1 = \frac{R_1}{R_1 C_1 s + 1}, \qquad Z_2 = \frac{R_2}{R_2 C_2 s + 1}$$

Hence, referring to Equation (3–34), we have

$$\frac{E(s)}{E_i(s)} = -\frac{Z_2}{Z_1} = -\frac{R_2}{R_1}\frac{R_1 C_1 s + 1}{R_2 C_2 s + 1} = -\frac{C_1}{C_2}\frac{s + \dfrac{1}{R_1 C_1}}{s + \dfrac{1}{R_2 C_2}} \tag{3–35}$$

Notice that the transfer function in Equation (3–35) contains a minus sign. Thus, this circuit is sign inverting. If such a sign inversion is not convenient in the actual application, a sign inverter may be connected to either the input or the output of the circuit of Figure 3–18(a). An example is shown in Figure 3–18(b). The sign inverter has the transfer function of

$$\frac{E_o(s)}{E(s)} = -\frac{R_4}{R_3}$$

The sign inverter has the gain of $-R_4/R_3$. Hence the network shown in Figure 3–18(b) has the following transfer function:

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2 R_4}{R_1 R_3}\frac{R_1 C_1 s + 1}{R_2 C_2 s + 1} = \frac{R_4 C_1}{R_3 C_2}\frac{s + \dfrac{1}{R_1 C_1}}{s + \dfrac{1}{R_2 C_2}}$$

$$= K_c \alpha \frac{Ts + 1}{\alpha T s + 1} = K_c \frac{s + \dfrac{1}{T}}{s + \dfrac{1}{\alpha T}} \tag{3–36}$$

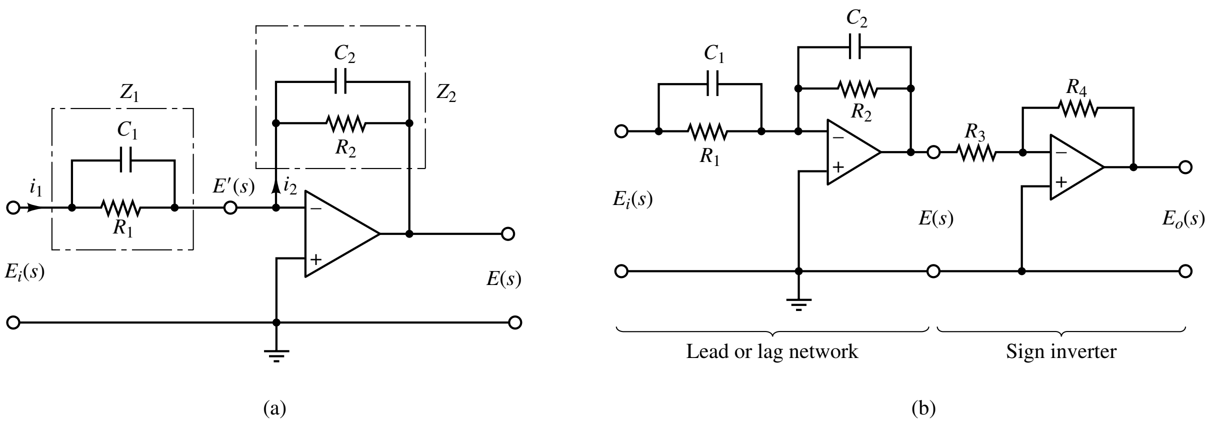


Figure 3–18
(a) Operational-amplifier circuit; (b) operational-amplifier circuit used as a lead or lag compensator.

where

$$T = R_1C_1, \qquad \alpha T = R_2C_2, \qquad K_c = \frac{R_4C_1}{R_3C_2}$$

Notice that

$$K_c\alpha = \frac{R_4C_1}{R_3C_2}\frac{R_2C_2}{R_1C_1} = \frac{R_2R_4}{R_1R_3}, \qquad \alpha = \frac{R_2C_2}{R_1C_1}$$

This network has a dc gain of $K_c\alpha = R_2R_4/(R_1R_3)$.

Note that this network, whose transfer function is given by Equation (3–36), is a lead network if $R_1C_1 > R_2C_2$, or $\alpha < 1$. It is a lag network if $R_1C_1 < R_2C_2$.

PID Controller Using Operational Amplifiers. Figure 3–19 shows an electronic proportional-plus-integral-plus-derivative controller (a PID controller) using operational amplifiers. The transfer function $E(s)/E_i(s)$ is given by

$$\frac{E(s)}{E_i(s)} = -\frac{Z_2}{Z_1}$$

where

$$Z_1 = \frac{R_1}{R_1C_1s + 1}, \qquad Z_2 = \frac{R_2C_2s + 1}{C_2s}$$

Thus

$$\frac{E(s)}{E_i(s)} = -\left(\frac{R_2C_2s + 1}{C_2s}\right)\left(\frac{R_1C_1s + 1}{R_1}\right)$$

Noting that

$$\frac{E_o(s)}{E(s)} = -\frac{R_4}{R_3}$$

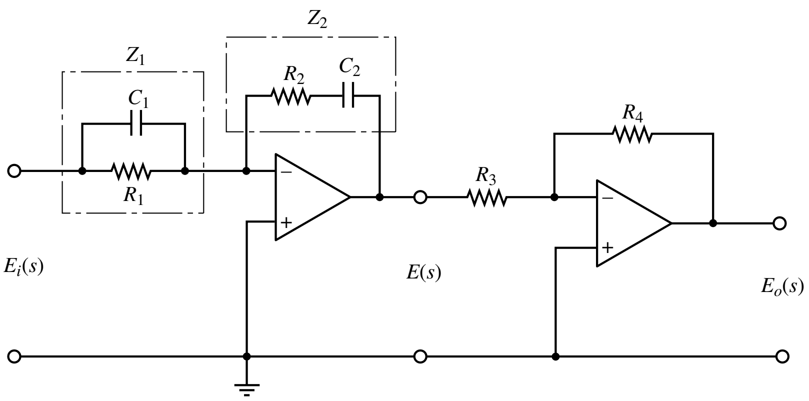


Figure 3–19 Electronic PID controller.

we have

$$\frac{E_o(s)}{E_i(s)} = \frac{E_o(s)}{E(s)}\frac{E(s)}{E_i(s)} = \frac{R_4 R_2}{R_3 R_1}\frac{(R_1C_1s + 1)(R_2C_2s + 1)}{R_2C_2s}$$

$$= \frac{R_4 R_2}{R_3 R_1}\left(\frac{R_1C_1 + R_2C_2}{R_2C_2} + \frac{1}{R_2C_2s} + R_1C_1s\right)$$

$$= \frac{R_4(R_1C_1 + R_2C_2)}{R_3R_1C_2}\left[1 + \frac{1}{(R_1C_1 + R_2C_2)s} + \frac{R_1C_1R_2C_2}{R_1C_1 + R_2C_2}s\right] \qquad (3\text{–}37)$$

Notice that the second operational-amplifier circuit acts as a sign inverter as well as a gain adjuster.

When a PID controller is expressed as

$$\frac{E_o(s)}{E_i(s)} = K_p\left(1 + \frac{T_i}{s} + T_d s\right)$$

K_p is called the proportional gain, T_i is called the integral time, and T_d is called the derivative time. From Equation (3–37) we obtain the proportional gain K_p, integral time T_i, and derivative time T_d to be

$$K_p = \frac{R_4(R_1C_1 + R_2C_2)}{R_3R_1C_2}$$

$$T_i = \frac{1}{R_1C_1 + R_2C_2}$$

$$T_d = \frac{R_1C_1R_2C_2}{R_1C_1 + R_2C_2}$$

When a PID controller is expressed as

$$\frac{E_o(s)}{E_i(s)} = K_p + \frac{K_i}{s} + K_d s$$

K_p is called the proportional gain, K_i is called the integral gain, and K_d is called the derivative gain. For this controller

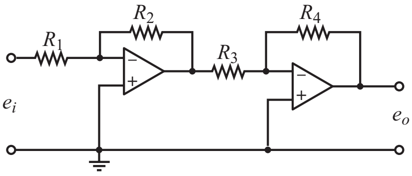
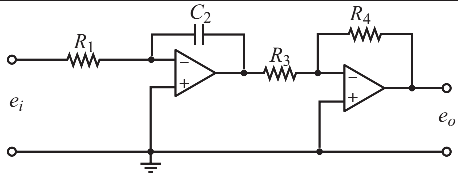
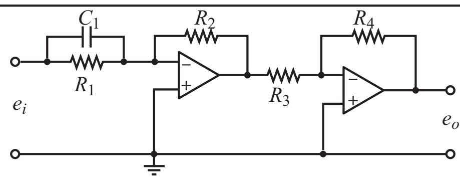
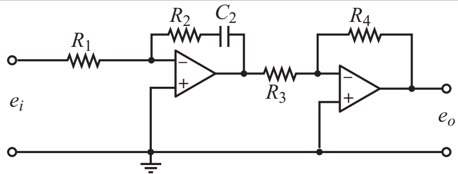
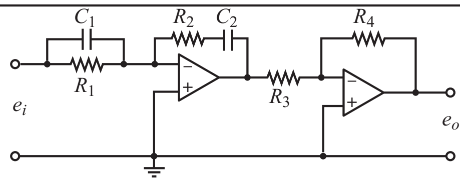
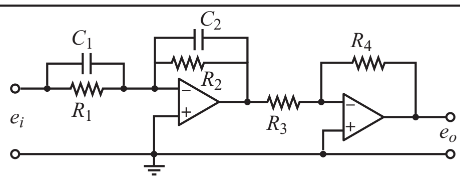
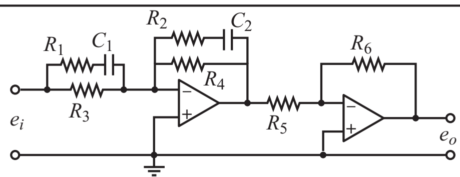
$$K_p = \frac{R_4(R_1C_1 + R_2C_2)}{R_3R_1C_2}$$

$$K_i = \frac{R_4}{R_3R_1C_2}$$

$$K_d = \frac{R_4R_2C_1}{R_3}$$

Table 3–1 shows a list of operational-amplifier circuits that may be used as controllers or compensators.

Table 3–1 Operational-Amplifier Circuits That May Be Used as Compensators

	Control Action	$G(s) = \dfrac{E_o(s)}{E_i(s)}$	Operational-Amplifier Circuits
1	P	$\dfrac{R_4}{R_3}\dfrac{R_2}{R_1}$	
2	I	$\dfrac{R_4}{R_3}\dfrac{1}{R_1C_2s}$	
3	PD	$\dfrac{R_4}{R_3}\dfrac{R_2}{R_1}(R_1C_1s + 1)$	
4	PI	$\dfrac{R_4}{R_3}\dfrac{R_2}{R_1}\dfrac{R_2C_2s + 1}{R_2C_2s}$	
5	PID	$\dfrac{R_4}{R_3}\dfrac{R_2}{R_1}\dfrac{(R_1C_1s + 1)\,(R_2C_2s + 1)}{R_2C_2s}$	
6	Lead or lag	$\dfrac{R_4}{R_3}\dfrac{R_2}{R_1}\dfrac{R_1C_1s + 1}{R_2C_2s + 1}$	
7	Lag–lead	$\dfrac{R_6}{R_5}\dfrac{R_4}{R_3}\dfrac{[(R_1 + R_3)\,C_1s + 1]\,(R_2C_2s + 1)}{(R_1C_1s + 1)\,[(R_2 + R_4)\,C_2s + 1]}$	

EXAMPLE PROBLEMS AND SOLUTIONS

A–3–1. Figure 3–20(a) shows a schematic diagram of an automobile suspension system. As the car moves along the road, the vertical displacements at the tires act as the motion excitation to the automobile suspension system. The motion of this system consists of a translational motion of the center of mass and a rotational motion about the center of mass. Mathematical modeling of the complete system is quite complicated.

A very simplified version of the suspension system is shown in Figure 3–20(b). Assuming that the motion x_i at point P is the input to the system and the vertical motion x_o of the body is the output, obtain the transfer function $X_o(s)/X_i(s)$. (Consider the motion of the body only in the vertical direction.) Displacement x_o is measured from the equilibrium position in the absence of input x_i.

Solution. The equation of motion for the system shown in Figure 3–20(b) is

$$m\ddot{x}_o + b(\dot{x}_o - \dot{x}_i) + k(x_o - x_i) = 0$$

or

$$m\ddot{x}_o + b\dot{x}_o + kx_o = b\dot{x}_i + kx_i$$

Taking the Laplace transform of this last equation, assuming zero initial conditions, we obtain

$$(ms^2 + bs + k)X_o(s) = (bs + k)X_i(s)$$

Hence the transfer function $X_o(s)/X_i(s)$ is given by

$$\frac{X_o(s)}{X_i(s)} = \frac{bs + k}{ms^2 + bs + k}$$

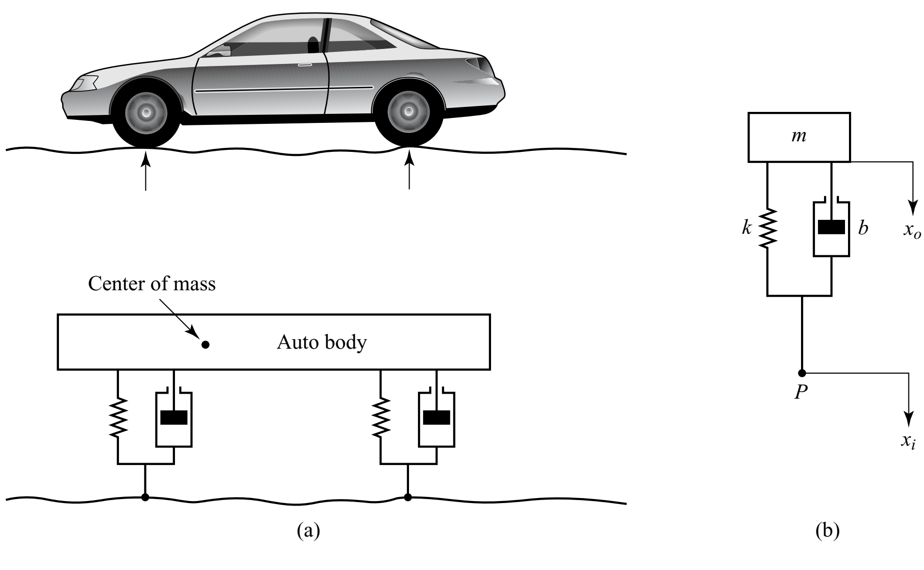


Figure 3–20 (a) Automobile suspension system; (b) simplified suspension system.

A–3–2. Obtain the transfer function $Y(s)/U(s)$ of the system shown in Figure 3–21. The input u is a displacement input. (Like the system of Problem **A–3–1**, this is also a simplified version of an automobile or motorcycle suspension system.)

Solution. Assume that displacements x and y are measured from respective steady-state positions in the absence of the input u. Applying the Newton's second law to this system, we obtain

$$m_1\ddot{x} = k_2(y - x) + b(\dot{y} - \dot{x}) + k_1(u - x)$$

$$m_2\ddot{y} = -k_2(y - x) - b(\dot{y} - \dot{x})$$

Hence, we have

$$m_1\ddot{x} + b\dot{x} + (k_1 + k_2)x = b\dot{y} + k_2 y + k_1 u$$

$$m_2\ddot{y} + b\dot{y} + k_2 y = b\dot{x} + k_2 x$$

Taking Laplace transforms of these two equations, assuming zero initial conditions, we obtain

$$\left[m_1 s^2 + bs + (k_1 + k_2)\right]X(s) = (bs + k_2)Y(s) + k_1 U(s)$$

$$\left[m_2 s^2 + bs + k_2\right]Y(s) = (bs + k_2)X(s)$$

Eliminating $X(s)$ from the last two equations, we have

$$(m_1 s^2 + bs + k_1 + k_2)\frac{m_2 s^2 + bs + k_2}{bs + k_2}Y(s) = (bs + k_2)Y(s) + k_1 U(s)$$

which yields

$$\frac{Y(s)}{U(s)} = \frac{k_1(bs + k_2)}{m_1 m_2 s^4 + (m_1 + m_2)bs^3 + \left[k_1 m_2 + (m_1 + m_2)k_2\right]s^2 + k_1 bs + k_1 k_2}$$

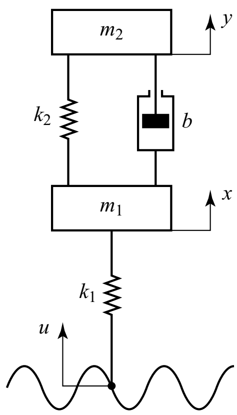


Figure 3–21
Suspension system.

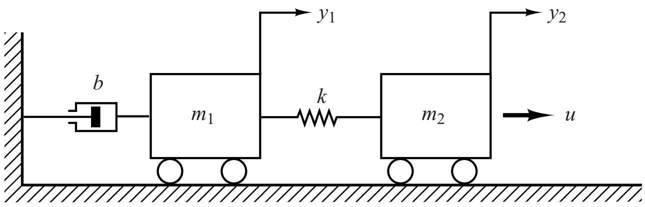


Figure 3–22
Mechanical system.

A–3–3. Obtain a state-space representation of the system shown in Figure 3–22.

Solution. The system equations are

$$m_1\ddot{y}_1 + b\dot{y}_1 + k(y_1 - y_2) = 0$$
$$m_2\ddot{y}_2 + k(y_2 - y_1) = u$$

The output variables for this system are y_1 and y_2. Define state variables as

$$x_1 = y_1$$
$$x_2 = \dot{y}_1$$
$$x_3 = y_2$$
$$x_4 = \dot{y}_2$$

Then we obtain the following equations:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{m_1}\left[-b\dot{y}_1 - k(y_1 - y_2)\right] = -\frac{k}{m_1}x_1 - \frac{b}{m_1}x_2 + \frac{k}{m_1}x_3$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = \frac{1}{m_2}\left[-k(y_2 - y_1) + u\right] = \frac{k}{m_2}x_1 - \frac{k}{m_2}x_3 + \frac{1}{m_2}u$$

Hence, the state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{k}{m_1} & -\dfrac{b}{m_1} & \dfrac{k}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k}{m_2} & 0 & -\dfrac{k}{m_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{m_2} \end{bmatrix} u$$

and the output equation is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

A–3–4. Obtain the transfer function $X_o(s)/X_i(s)$ of the mechanical system shown in Figure 3–23(a). Also obtain the transfer function $E_o(s)/E_i(s)$ of the electrical system shown in Figure 3–23(b). Show that these transfer functions of the two systems are of identical form and thus they are analogous systems.

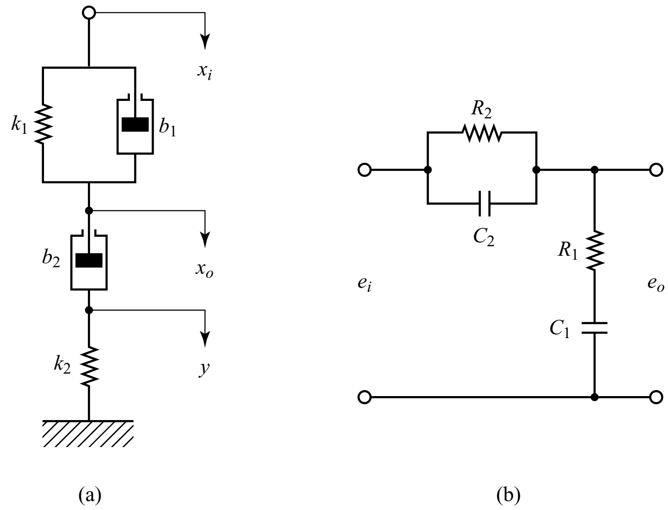


Figure 3–23
(a) Mechanical system;
(b) analogous electrical system.

Solution. In Figure 3–23(a) we assume that displacements x_i, x_o, and y are measured from their respective steady-state positions. Then the equations of motion for the mechanical system shown in Figure 3–23(a) are

$$b_1(\dot{x}_i - \dot{x}_o) + k_1(x_i - x_o) = b_2(\dot{x}_o - \dot{y})$$

$$b_2(\dot{x}_o - \dot{y}) = k_2 y$$

By taking the Laplace transforms of these two equations, assuming zero initial conditions, we have

$$b_1[sX_i(s) - sX_o(s)] + k_1[X_i(s) - X_o(s)] = b_2[sX_o(s) - sY(s)]$$

$$b_2[sX_o(s) - sY(s)] = k_2 Y(s)$$

If we eliminate $Y(s)$ from the last two equations, then we obtain

$$b_1[sX_i(s) - sX_o(s)] + k_1[X_i(s) - X_o(s)] = b_2 sX_o(s) - b_2 s\frac{b_2 sX_o(s)}{b_2 s + k_2}$$

or

$$(b_1 s + k_1)X_i(s) = \left(b_1 s + k_1 + b_2 s - b_2 s\frac{b_2 s}{b_2 s + k_2}\right)X_o(s)$$

Hence the transfer function $X_o(s)/X_i(s)$ can be obtained as

$$\frac{X_o(s)}{X_i(s)} = \frac{\left(\frac{b_1}{k_1}s + 1\right)\left(\frac{b_2}{k_2}s + 1\right)}{\left(\frac{b_1}{k_1}s + 1\right)\left(\frac{b_2}{k_2}s + 1\right) + \frac{b_2}{k_1}s}$$

For the electrical system shown in Figure 3–23(b), the transfer function $E_o(s)/E_i(s)$ is found to be

$$\frac{E_o(s)}{E_i(s)} = \frac{R_1 + \frac{1}{C_1 s}}{\frac{1}{(1/R_2) + C_2 s} + R_1 + \frac{1}{C_1 s}}$$

$$= \frac{(R_1 C_1 s + 1)(R_2 C_2 s + 1)}{(R_1 C_1 s + 1)(R_2 C_2 s + 1) + R_2 C_1 s}$$

A comparison of the transfer functions shows that the systems shown in Figures 3–23(a) and (b) are analogous.

A–3–5. Obtain the transfer functions $E_o(s)/E_i(s)$ of the bridged T networks shown in Figures 3–24(a) and (b).

Solution. The bridged T networks shown can both be represented by the network of Figure 3–25(a), where we used complex impedances. This network may be modified to that shown in Figure 3–25(b).

In Figure 3–25(b), note that

$$I_1 = I_2 + I_3, \qquad I_2 Z_1 = (Z_3 + Z_4) I_3$$

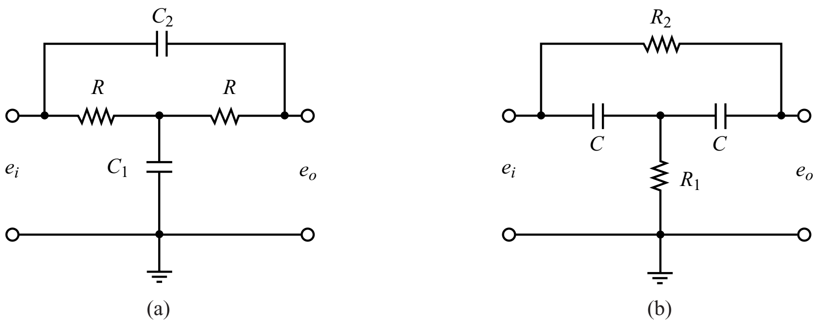


Figure 3–24
Bridged T networks.

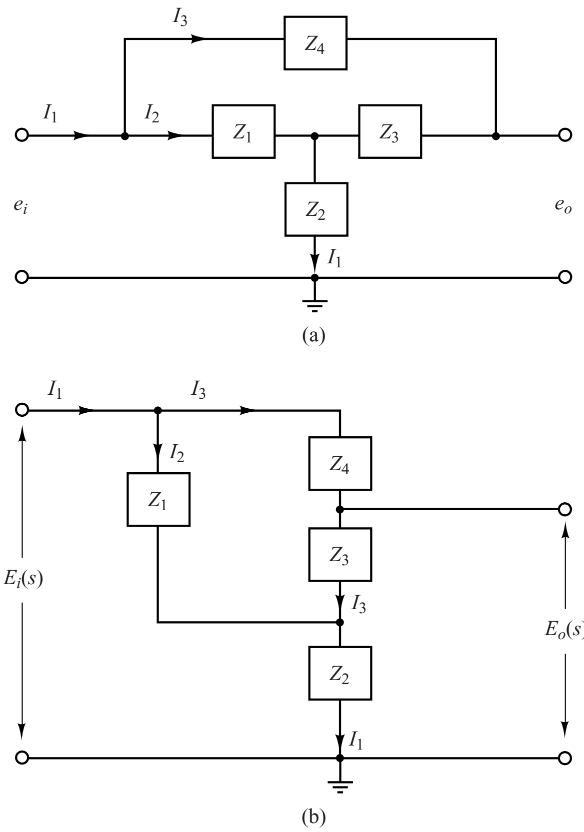


Figure 3–25
(a) Bridged T network in terms of complex impedances; (b) equivalent network.

Hence

$$I_2 = \frac{Z_3 + Z_4}{Z_1 + Z_3 + Z_4} I_1, \qquad I_3 = \frac{Z_1}{Z_1 + Z_3 + Z_4} I_1$$

Then the voltages $E_i(s)$ and $E_o(s)$ can be obtained as

$$\begin{aligned} E_i(s) &= Z_1 I_2 + Z_2 I_1 \\ &= \left[Z_2 + \frac{Z_1(Z_3 + Z_4)}{Z_1 + Z_3 + Z_4} \right] I_1 \\ &= \frac{Z_2(Z_1 + Z_3 + Z_4) + Z_1(Z_3 + Z_4)}{Z_1 + Z_3 + Z_4} I_1 \\ E_o(s) &= Z_3 I_3 + Z_2 I_1 \\ &= \frac{Z_3 Z_1}{Z_1 + Z_3 + Z_4} I_1 + Z_2 I_1 \\ &= \frac{Z_3 Z_1 + Z_2(Z_1 + Z_3 + Z_4)}{Z_1 + Z_3 + Z_4} I_1 \end{aligned}$$

Hence, the transfer function $E_o(s)/E_i(s)$ of the network shown in Figure 3–25(a) is obtained as

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_3 Z_1 + Z_2(Z_1 + Z_3 + Z_4)}{Z_2(Z_1 + Z_3 + Z_4) + Z_1 Z_3 + Z_1 Z_4} \tag{3–38}$$

For the bridged T network shown in Figure 3–24(a), substitute

$$Z_1 = R, \qquad Z_2 = \frac{1}{C_1 s}, \qquad Z_3 = R, \qquad Z_4 = \frac{1}{C_2 s}$$

into Equation (3–38). Then we obtain the transfer function $E_o(s)/E_i(s)$ to be

$$\begin{aligned} \frac{E_o(s)}{E_i(s)} &= \frac{R^2 + \dfrac{1}{C_1 s}\left(R + R + \dfrac{1}{C_2 s}\right)}{\dfrac{1}{C_1 s}\left(R + R + \dfrac{1}{C_2 s}\right) + R^2 + R\dfrac{1}{C_2 s}} \\ &= \frac{RC_1 RC_2 s^2 + 2RC_2 s + 1}{RC_1 RC_2 s^2 + (2RC_2 + RC_1)s + 1} \end{aligned}$$

Similarly, for the bridged T network shown in Figure 3–24(b), we substitute

$$Z_1 = \frac{1}{Cs}, \qquad Z_2 = R_1, \qquad Z_3 = \frac{1}{Cs}, \qquad Z_4 = R_2$$

into Equation (3–38). Then the transfer function $E_o(s)/E_i(s)$ can be obtained as follows:

$$\begin{aligned} \frac{E_o(s)}{E_i(s)} &= \frac{\dfrac{1}{Cs}\dfrac{1}{Cs} + R_1\left(\dfrac{1}{Cs} + \dfrac{1}{Cs} + R_2\right)}{R_1\left(\dfrac{1}{Cs} + \dfrac{1}{Cs} + R_2\right) + \dfrac{1}{Cs}\dfrac{1}{Cs} + R_2\dfrac{1}{Cs}} \\ &= \frac{R_1 C R_2 C s^2 + 2R_1 C s + 1}{R_1 C R_2 C s^2 + (2R_1 C + R_2 C)s + 1} \end{aligned}$$

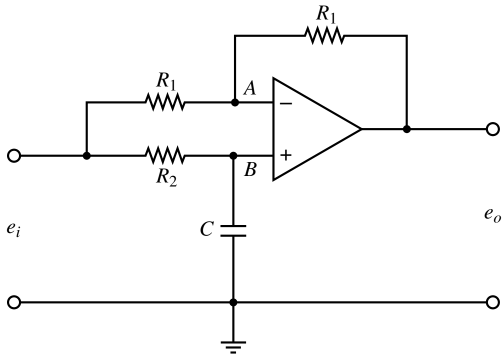


Figure 3–26 Operational-amplifier circuit.

A–3–6. Obtain the transfer function $E_o(s)/E_i(s)$ of the op-amp circuit shown in Figure 3–26.

Solution. The voltage at point A is

$$e_A = \frac{1}{2}(e_i - e_o) + e_o$$

The Laplace-transformed version of this last equation is

$$E_A(s) = \frac{1}{2}\left[E_i(s) + E_o(s)\right]$$

The voltage at point B is

$$E_B(s) = \frac{\frac{1}{Cs}}{R_2 + \frac{1}{Cs}} E_i(s) = \frac{1}{R_2Cs + 1} E_i(s)$$

Since $\left[E_B(s) - E_A(s)\right]K = E_o(s)$ and $K \gg 1$, we must have $E_A(s) = E_B(s)$. Thus

$$\frac{1}{2}\left[E_i(s) + E_o(s)\right] = \frac{1}{R_2Cs + 1} E_i(s)$$

Hence

$$\frac{E_o(s)}{E_i(s)} = -\frac{R_2Cs - 1}{R_2Cs + 1} = -\frac{s - \frac{1}{R_2C}}{s + \frac{1}{R_2C}}$$

A–3–7. Obtain the transfer function $E_o(s)/E_i(s)$ of the op-amp system shown in Figure 3–27 in terms of complex impedances Z_1, Z_2, Z_3, and Z_4. Using the equation derived, obtain the transfer function $E_o(s)/E_i(s)$ of the op-amp system shown in Figure 3–26.

Solution. From Figure 3–27, we find

$$\frac{E_i(s) - E_A(s)}{Z_3} = \frac{E_A(s) - E_o(s)}{Z_4}$$

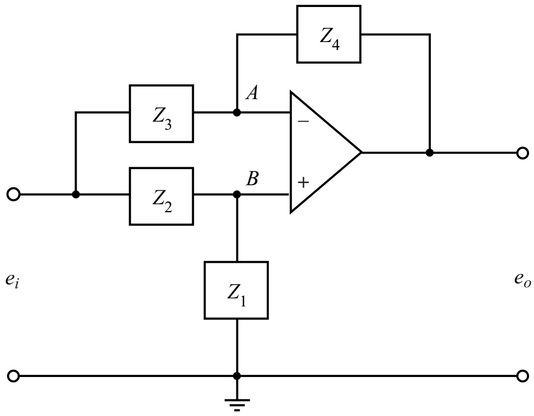


Figure 3–27 Operational-amplifier circuit.

or

$$E_i(s) - \left(1 + \frac{Z_3}{Z_4}\right)E_A(s) = -\frac{Z_3}{Z_4}E_o(s) \tag{3–39}$$

Since

$$E_A(s) = E_B(s) = \frac{Z_1}{Z_1 + Z_2}E_i(s) \tag{3–40}$$

by substituting Equation (3–40) into Equation (3–39), we obtain

$$\left[\frac{Z_4Z_1 + Z_4Z_2 - Z_4Z_1 - Z_3Z_1}{Z_4(Z_1 + Z_2)}\right]E_i(s) = -\frac{Z_3}{Z_4}E_o(s)$$

from which we get the transfer function $E_o(s)/E_i(s)$ to be

$$\frac{E_o(s)}{E_i(s)} = -\frac{Z_4Z_2 - Z_3Z_1}{Z_3(Z_1 + Z_2)} \tag{3–41}$$

To find the transfer function $E_o(s)/E_i(s)$ of the circuit shown in Figure 3–26, we substitute

$$Z_1 = \frac{1}{Cs}, \qquad Z_2 = R_2, \qquad Z_3 = R_1, \qquad Z_4 = R_1$$

into Equation (3–41). The result is

$$\frac{E_o(s)}{E_i(s)} = -\frac{R_1R_2 - R_1\dfrac{1}{Cs}}{R_1\left(\dfrac{1}{Cs} + R_2\right)} = -\frac{R_2Cs - 1}{R_2Cs + 1}$$

which is, as a matter of course, the same as that obtained in Problem **A–3–6**.

A–3–8. Obtain the transfer function $E_o(s)/E_i(s)$ of the operational-amplifier circuit shown in Figure 3–28.

Solution. We will first obtain currents $i_1, i_2, i_3, i_4,$ and i_5. Then we will use node equations at nodes A and B.

$$i_1 = \frac{e_i - e_A}{R_1}; \qquad i_2 = \frac{e_A - e_o}{R_3}, \qquad i_3 = C_1 \frac{de_A}{dt}$$

$$i_4 = \frac{e_A}{R_2}, \qquad i_5 = C_2 \frac{-de_o}{dt}$$

At node A, we have $i_1 = i_2 + i_3 + i_4$, or

$$\frac{e_i - e_A}{R_1} = \frac{e_A - e_o}{R_3} + C_1 \frac{de_A}{dt} + \frac{e_A}{R_2} \tag{3–42}$$

At node B, we get $i_4 = i_5$, or

$$\frac{e_A}{R_2} = C_2 \frac{-de_o}{dt} \tag{3–43}$$

By rewriting Equation (3–42), we have

$$C_1 \frac{de_A}{dt} + \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) e_A = \frac{e_i}{R_1} + \frac{e_o}{R_3} \tag{3–44}$$

From Equation (3–43), we get

$$e_A = -R_2 C_2 \frac{de_o}{dt} \tag{3–45}$$

By substituting Equation (3–45) into Equation (3–44), we obtain

$$C_1\left(-R_2C_2\frac{d^2e_o}{dt^2}\right) + \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)(-R_2C_2)\frac{de_o}{dt} = \frac{e_i}{R_1} + \frac{e_o}{R_3}$$

Taking the Laplace transform of this last equation, assuming zero initial conditions, we obtain

$$-C_1C_2R_2s^2E_o(s) + \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)(-R_2C_2)sE_o(s) - \frac{1}{R_3}E_o(s) = \frac{E_i(s)}{R_1}$$

from which we get the transfer function $E_o(s)/E_i(s)$ as follows:

$$\frac{E_o(s)}{E_i(s)} = -\frac{1}{R_1C_1R_2C_2s^2 + [R_2C_2 + R_1C_2 + (R_1/R_3)R_2C_2]s + (R_1/R_3)}$$

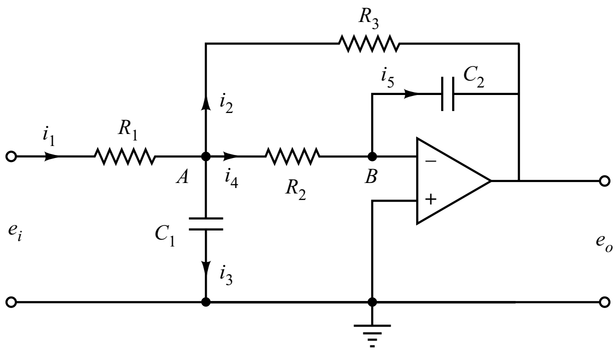


Figure 3–28
Operational-amplifier circuit.

A–3–9. Consider the servo system shown in Figure 3–29(a). The motor shown is a servomotor, a dc motor designed specifically to be used in a control system. The operation of this system is as follows: A pair of potentiometers acts as an error-measuring device. They convert the input and output positions into proportional electric signals. The command input signal determines the angular position r of the wiper arm of the input potentiometer. The angular position r is the reference input to the system, and the electric potential of the arm is proportional to the angular position of the arm. The output shaft position determines the angular position c of the wiper arm of the output potentiometer. The difference between the input angular position r and the output angular position c is the error signal e, or

$$e = r - c$$

The potential difference $e_r - e_c = e_v$ is the error voltage, where e_r is proportional to r and e_c is proportional to c; that is, $e_r = K_0 r$ and $e_c = K_0 c$, where K_0 is a proportionality constant. The error voltage that appears at the potentiometer terminals is amplified by the amplifier whose gain constant is K_1. The output voltage of this amplifier is applied to the armature circuit of the dc motor. A fixed voltage is applied to the field winding. If an error exists, the motor develops a torque to rotate the output load in such a way as to reduce the error to zero. For constant field current, the torque developed by the motor is

$$T = K_2 i_a$$

where K_2 is the motor torque constant and i_a is the armature current.

When the armature is rotating, a voltage proportional to the product of the flux and angular velocity is induced in the armature. For a constant flux, the induced voltage e_b is directly proportional to the angular velocity $d\theta/dt$, or

$$e_b = K_3 \frac{d\theta}{dt}$$

where e_b is the back emf, K_3 is the back emf constant of the motor, and θ is the angular displacement of the motor shaft.

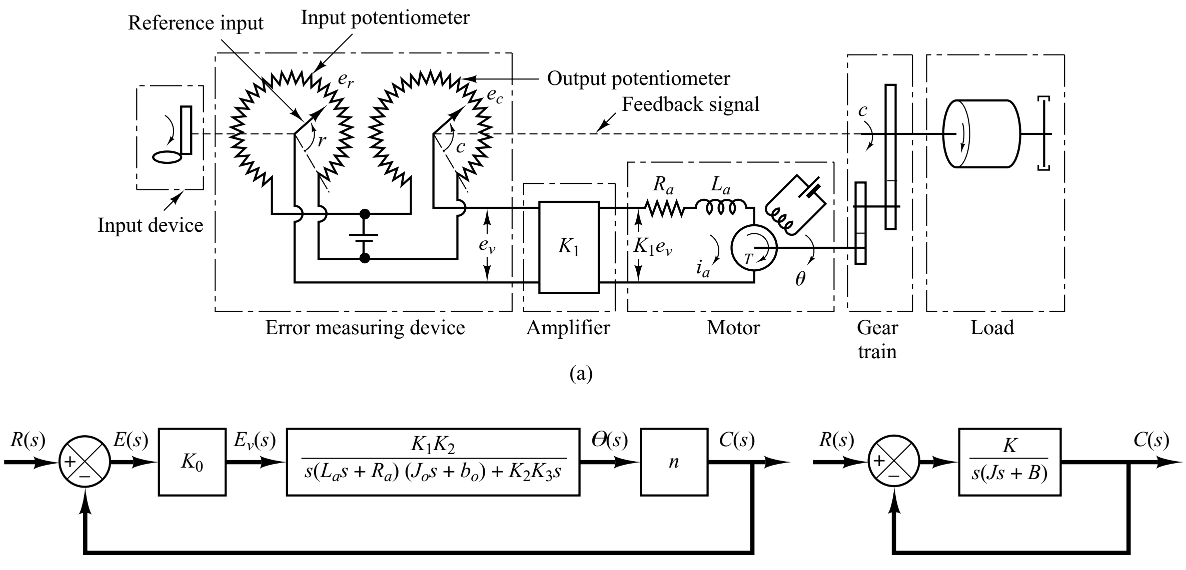


Figure 3–29
(a) Schematic diagram of servo system; (b) block diagram for the system; (c) simplified block diagram.

Obtain the transfer function between the motor shaft angular displacement θ and the error voltage e_v. Obtain also a block diagram for this system and a simplified block diagram when L_a is negligible.

Solution. The speed of an armature-controlled dc servomotor is controlled by the armature voltage e_a. (The armature voltage $e_a = K_1 e_v$ is the output of the amplifier.) The differential equation for the armature circuit is

$$L_a \frac{di_a}{dt} + R_a i_a + e_b = e_a$$

or

$$L_a \frac{di_a}{dt} + R_a i_a + K_3 \frac{d\theta}{dt} = K_1 e_v \tag{3–46}$$

The equation for torque equilibrium is

$$J_0 \frac{d^2\theta}{dt^2} + b_0 \frac{d\theta}{dt} = T = K_2 i_a \tag{3–47}$$

where J_0 is the inertia of the combination of the motor, load, and gear train referred to the motor shaft and b_0 is the viscous-friction coefficient of the combination of the motor, load, and gear train referred to the motor shaft.

By eliminating i_a from Equations (3–46) and (3–47), we obtain

$$\frac{\Theta(s)}{E_v(s)} = \frac{K_1 K_2}{s(L_a s + R_a)(J_0 s + b_0) + K_2 K_3 s} \tag{3–48}$$

We assume that the gear ratio of the gear train is such that the output shaft rotates n times for each revolution of the motor shaft. Thus,

$$C(s) = n\Theta(s) \tag{3–49}$$

The relationship among $E_v(s)$, $R(s)$, and $C(s)$ is

$$E_v(s) = K_0[R(s) - C(s)] = K_0 E(s) \tag{3–50}$$

The block diagram of this system can be constructed from Equations (3–48), (3–49), and (3–50), as shown in Figure 3–29(b). The transfer function in the feedforward path of this system is

$$G(s) = \frac{C(s)}{\Theta(s)} \frac{\Theta(s)}{E_v(s)} \frac{E_v(s)}{E(s)} = \frac{K_0 K_1 K_2 n}{s[(L_a s + R_a)(J_0 s + b_0) + K_2 K_3]}$$

When L_a is small, it can be neglected, and the transfer function $G(s)$ in the feedforward path becomes

$$G(s) = \frac{K_0 K_1 K_2 n}{s[R_a(J_0 s + b_0) + K_2 K_3]}$$

$$= \frac{K_0 K_1 K_2 n / R_a}{J_0 s^2 + \left(b_0 + \dfrac{K_2 K_3}{R_a}\right)s} \tag{3–51}$$

The term $[b_0 + (K_2 K_3/R_a)]s$ indicates that the back emf of the motor effectively increases the viscous friction of the system. The inertia J_0 and viscous friction coefficient $b_0 + (K_2 K_3/R_a)$ are referred to the motor shaft. When J_0 and $b_0 + (K_2K_3/R_a)$ are multiplied by $1/n^2$, the inertia and viscous-friction coefficient are expressed in terms of the output shaft. Introducing new parameters defined by

$$J = J_0/n^2 = \text{moment of inertia referred to the output shaft}$$

$$B = [b_0 + (K_2K_3/R_a)]/n^2 = \text{viscous-friction coefficient referred to the output shaft}$$

$$K = K_0K_1K_2/nR_a$$

the transfer function $G(s)$ given by Equation (3–51) can be simplified, yielding

$$G(s) = \frac{K}{Js^2 + Bs}$$

or

$$G(s) = \frac{K_m}{s(T_m s + 1)}$$

where

$$K_m = \frac{K}{B}, \qquad T_m = \frac{J}{B} = \frac{R_a J_0}{R_a b_0 + K_2 K_3}$$

The block diagram of the system shown in Figure 3–29(b) can thus be simplified as shown in Figure 3–29(c).

PROBLEMS

B–3–1. Obtain the equivalent viscous-friction coefficient b_{eq} of the system shown in Figure 3–30.

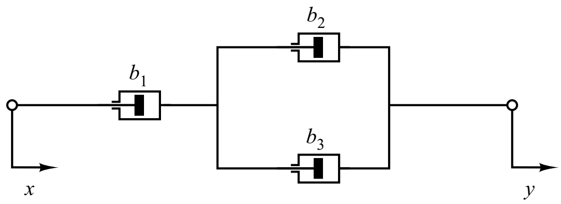


Figure 3–30
Damper system.

B–3–2. Obtain mathematical models of the mechanical systems shown in Figures 3–31(a) and (b).

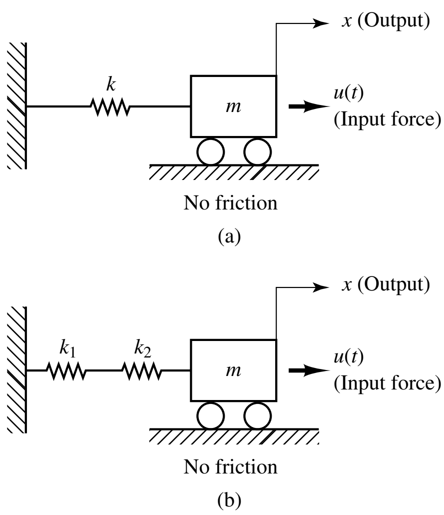


Figure 3–31
Mechanical systems.

B–3–3. Obtain a state-space representation of the mechanical system shown in Figure 3–32, where u_1 and u_2 are the inputs and y_1 and y_2 are the outputs.

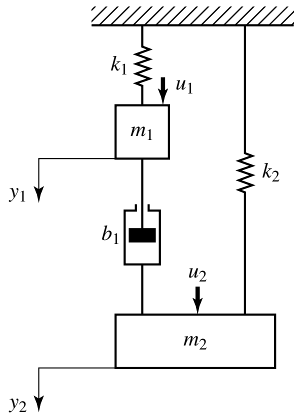


Figure 3–32 Mechanical system.

B–3–4. Consider the spring-loaded pendulum system shown in Figure 3–33. Assume that the spring force acting on the pendulum is zero when the pendulum is vertical, or $\theta = 0$. Assume also that the friction involved is negligible and the angle of oscillation θ is small. Obtain a mathematical model of the system.

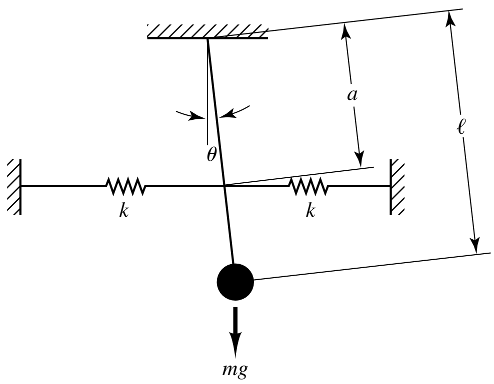


Figure 3–33 Spring-loaded pendulum system.

B–3–5. Referring to Examples 3–5 and 3–6, consider the inverted-pendulum system shown in Figure 3–34. Assume that the mass of the inverted pendulum is m and is evenly distributed along the length of the rod. (The center of gravity of the pendulum is located at the center of the rod.) Assuming that θ is small, derive mathematical models for the system in the forms of differential equations, transfer functions, and state-space equations.

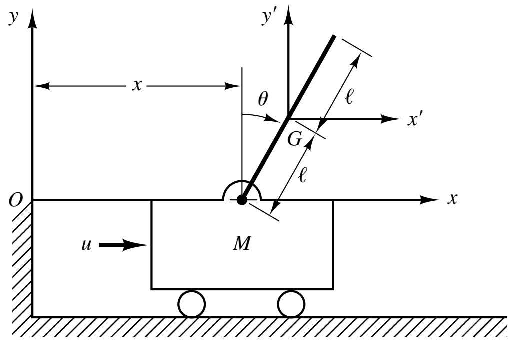


Figure 3–34 Inverted-pendulum system.

B–3–6. Obtain the transfer functions $X_1(s)/U(s)$ and $X_2(s)/U(s)$ of the mechanical system shown in Figure 3–35.

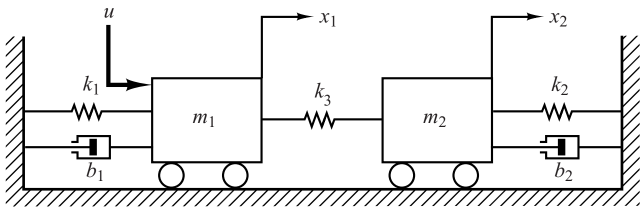


Figure 3–35 Mechanical system.

B–3–7. Obtain the transfer function $E_o(s)/E_i(s)$ of the electrical circuit shown in Figure 3–36.

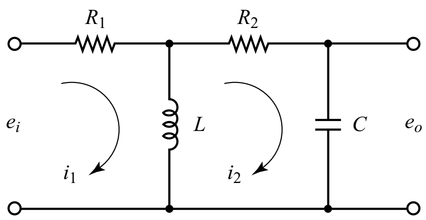


Figure 3–36 Electrical circuit.

B–3–8. Consider the electrical circuit shown in Figure 3–37. Obtain the transfer function $E_o(s)/E_i(s)$ by use of the block diagram approach.

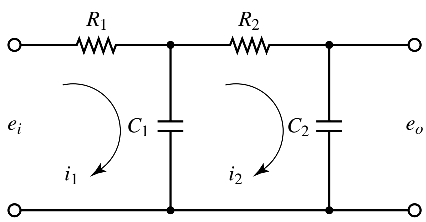


Figure 3–37 Electrical circuit.

B–3–9. Derive the transfer function of the electrical circuit shown in Figure 3–38. Draw a schematic diagram of an analogous mechanical system.

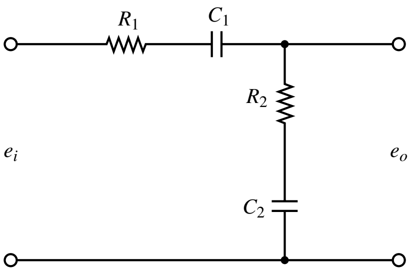


Figure 3–38 Electrical circuit.

B–3–10. Obtain the transfer function $E_o(s)/E_i(s)$ of the op-amp circuit shown in Figure 3–39.

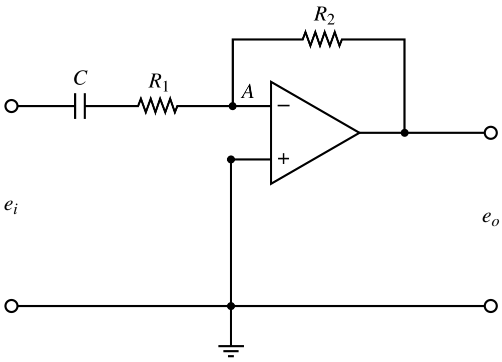


Figure 3–39 Operational-amplifier circuit.

B–3–11. Obtain the transfer function $E_o(s)/E_i(s)$ of the op-amp circuit shown in Figure 3–40.

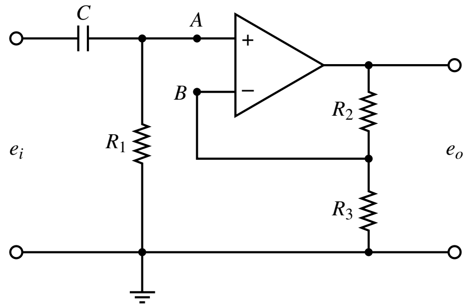


Figure 3–40 Operational-amplifier circuit.

B–3–12. Using the impedance approach, obtain the transfer function $E_o(s)/E_i(s)$ of the op-amp circuit shown in Figure 3–41.

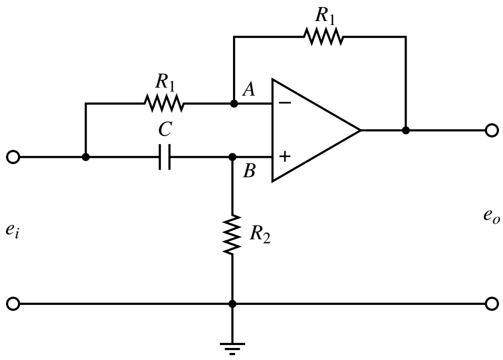


Figure 3–41 Operational-amplifier circuit.

B–3–13. Consider the system shown in Figure 3–42. An armature-controlled dc servomotor drives a load consisting of the moment of inertia J_L. The torque developed by the motor is T. The moment of inertia of the motor rotor is J_m. The angular displacements of the motor rotor and the load element are θ_m and θ, respectively. The gear ratio is $n = \theta/\theta_m$. Obtain the transfer function $\Theta(s)/E_i(s)$.

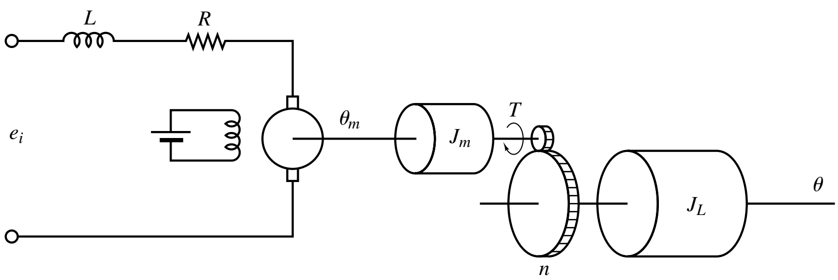


Figure 3–42 Armature-controlled dc servomotor system.