Ordinary differential equations

A differential equation is an equation where the unknown (or unknowns) is a function f, and the equation **relates values of f at a point x** with values of derivatives of the function at **the same point x**. If the function has one variable only (as is the case in this chapter), one speaks of ordinary differential equations.

Theorem 2.1.6: $F : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function of two variables. Let $x_0 \in \mathbb{R}$ and $y_o \in \mathbb{R}^2$. Then the ODE $y' = F(x, y)$ has an **unique** solution f defined on a *largest* open interval I containing x_0 such that $f(x_0) = y_0$.

$\exists\ f : I : \mathbb{R}$ s.t. $\forall x \in I : f'(x) = F(x, y)$ and one cannot find a larger interval containing I with such a solution.

Separation of variables:
If the ODE can be rewritten as $\frac{dy}{dx} = f(x)g(y)$, then

$$\int \frac{dy}{g(y)} = \int f(x)dx.$$

Linear differential equations

Definition 2.2.1: Let $I \subset \mathbb{R}$ open interval and $k \geq 1$ an integer.

Homogeneous ODE of order k: $y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_1y' + a_0y = 0$ where the coefficients $a_0, \ldots, a_{k-1}$ are complex-valued functions on I.

Inhomogeneous ODE of order k: $y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_1y' + a_0y = b$ where $b : I \to \mathbb{C}$.

Theorem 2.2.3: Let $I \subset \mathbb{R}$ an open interval and $k \geq 1$ an integer,

$y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_1y' + a_0y = 0$ a linear ODE over I with continuous coefficients.

1. Set S of k-times differentiable solutions $f : I \to \mathbb{C}$ is a **complex vector space** which is a subspace of complex-valued functions on I.

   If the functions a_i are real-valued, the set of real-valued solutions is also a vector space.
2. The dimension of S is k, and for any choice of $x_0 \in I$ and any $(y_0, \ldots, y_{k-1}) \in \mathbb{C}^k$, there exists a unique $f \in \mathbb{C}$ such that $f(x_0) = y_0, f'(x_0) = y_1, \ldots, f^{(k-1)} = y_{k-1}$.
3. Let b be a continuous function on I. There exists a solution f_0 to the inhomogeneous equation and the set S_b is the set of functions $\boldsymbol{f + f_0}$ where $f \in S$.
4. For any $x_o \in I$ and any $(y_0, \ldots, y_{k-1}) \in \mathbb{C}^k$, there exists a unique $f \in S_b$ such that $f(x_0) = y_0, f'(x_0) = y_1, \ldots, f^{(k-1)} = y_{k-1}$.

If $b \neq 0$, the set S_b is not a vector space!

Linear differential equations of order 1

Let $I \subset \mathbf{R}$ be an open interval. We consider here the linear differential equation $\boldsymbol{y' + ay = b}$ when a and b are general continuous functions defined on I.

Steps to solve:
1. Solve the **homogeneous equation**: $y' + ay = 0$ and obtain S.
2. Find solution f_0 to the **inhomogeneous equation**.
3. S_b will contain $f_0 + f$ where $f \in S$ and if some f_1 is a basis of S then the solutions are given by $f_0 + zf_1$ where $z \in \mathbb{C}$ are arbitrary.

If the initial value $f(x_0) = y_0$ is given, then one must solve $f_0(x_0) + zf_1(x_0) = y_0$ and determine the value of z.

Proposition 2.3.1: Any solution of $\boldsymbol{y' + ay = 0}$ is of the form $\boldsymbol{f(x) = z\exp(-A(x))}$ where A is a primitive of a. The unique solution with $f(x_0) = y_0$ is $\boldsymbol{f(x) = y_0 \exp(A(x_0) - A(x))}$.

Linear differential equations with constant coefficients

Now let $k \geq 1$ an integer; $a_0, \ldots, a_{k-1}$ constant coefficients and b a continuous function. We consider the equation $y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_1y' + a_0y = b$.

Solution of the homogeneous equation:
Let $\boldsymbol{P(\lambda) = \lambda^k + a_{k-1}\lambda^k + \cdots + a_1\lambda + a_0}$. Find the **roots** of this polynomial. If the roots are not real but the coefficients are, express the solution in terms of sin and cos using $e^{ix} = \cos x + i \sin x$.

Case 1: No multiple roots
Any solution $f \in S$ of solutions of the homo equation is of the form $\boldsymbol{f(x) = z_1e^{\alpha_1x} + \cdots + z_ke^{\alpha_kx}}$ for arbitrary $z_1, \ldots, z_k$. To find an unique solution to $f(x_0) = y_0, \ldots, f^{(k-1)}(x_0) = y_{k-1}$ for given $(y_0, \ldots, y_{k-1})$ just view $z_1, \ldots, z_k$ as unknowns.

To obtain the **real valued solutions** if the coefficients are real:

$f(x) = x_1e^{\alpha_1x} + \cdots + x_me^{\alpha_mx} + x_{m+1}e^{a_{m+1}x}\cos(b_{m+1}x) + y_{m+1}e^{a_{m+1}x}\sin(b_{m+1}x) + \cdots + x_ke^{a_kx}\cos(b_kx) + y_ke^{a_kx}\sin(b_kx)$

with $\alpha_1, \ldots, \alpha_m$ being the real solutions and $\alpha_{m+1}, \ldots, \alpha_k$ the complex solutions $\alpha_j = a_j + ib_j$.

Case 2: Multiple roots
Assume α is a multiple root of order j of P then the solutions look as follows:
$f_{\alpha,0}(x) = e^{\alpha x}$, $f_{\alpha,1}(x) = xe^{\alpha x}$, $\ldots$, $f_{\alpha,j-1}(x) = x^{j-1}x^{\alpha x}$

That is, multiply the original solution from *Case 1* with x^{i-1} for $i = 1, \ldots, j$.

Solution to the inhomogeneous equation:

1. ***Ansatz* method** (left side is $b(x)$ and the right side is Ansatz)

   $ae^{\alpha x} \to be^{\alpha x}$

   $a\sin(\beta x)$ or $a\cos(\beta x) \to c\sin(\beta x) + d\cos(\beta)$

   $ae^{\sin(\beta x)}$ or $ae^{\cos(\beta x)} \to e^{\alpha x}[c\sin(\beta x) + d\cos(\beta x)]$

   $P_n(x)e^{\alpha x} \to R_n(x)e^{\alpha x}$

   $P_ne^{\alpha x}\sin(\beta x)$ or $P_ne^{\alpha x}\cos(\beta x) \to e^{\alpha x}[R_n\sin(\beta x) + S_n\cos(\beta x)]$

   With $P_n(x), R_n(x), Q_n(x), S_n(x)$ being polynomials of degree n.

   If $b(x)$ is a linear combination of the above functions, then one should try the corresponding linear combination of the *Ansatz* functions.

   If $\lambda = \alpha + \beta i$ is a root of $P(\lambda)$ of multiplicity m, then the *Ansatz* function should be multiplied by x^m (otherwise the *Ansatz* would solve the homo solution again)
2. **Variation of constants**

   Assume $(f_1, \ldots, f_k)$ is the basis of the space S of solutions of the homogeneous equation. Now we search for a solution of the **inhomogeneous equation** of the form $f(x) = z_1(x)f_1(x) + \cdots + z_k(x)f_k(x)$ and impose the following **conditions**:

$$\begin{cases} z_1'(x)f_1(x) + \cdots + z_k'(x)f_k(x) = 0 \\ z_1'(x)f_1'(x) + \cdots + z_k'(x)f_k'(x) = 0 \\ \cdots \\ z_1'(x)f_1^{(k-2)}(x) + \cdots + z_k'(x)f_k^{(k-2)}(x) = 0 \end{cases}$$

Differential calculus in $\mathbb{R}^n$

Continuity in $\mathbb{R}^n$

The norm $||x||$ satisfies $||x|| > 0$, $||x|| = 0 \Leftrightarrow x = 0$, $||tx|| = |t|||x||$ for all $t \in \mathbb{R}$, and $||x+y|| \leq ||x|| + ||y||$ (triangle inequality).

Definition 3.2.1: Let $(x_k)_{k\in\mathbb{N}}$ where $x_k \in \mathbb{R}^n$. $x_k = (x_{k,1}, \ldots, x_{k,n})$. Let $y = (y_1, \ldots, y_n)$. We say that the sequence (x_k) **converges to y** as $y \to +\infty$ if for all $\epsilon > 0$, there exists $N \geq 1$ such that for all $n \geq N$ we have $||x_k - y|| < \epsilon$.

Lemma 3.2.2: The sequence (x_k) **converges to y** as $y \to +\infty \Leftrightarrow$ one of the following holds:

1. For each $1 \leq i \leq n$, the sequence $(x_{k,i})$ of real numbers **converges to y_i**.
2. The sequence of real numbers **$||x_k - y||$ converges to 0** as $y \to +\infty$.

Definition 3.2.3: Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$

1. Let $x_0 \in X$. f is continuous at x_0 if for all $\epsilon > 0$, there exists $\delta > 0$ such that, if $x \in X$ satisfies $||x - x_0|| < \delta$, then $||f(x) - f(x_0)|| < \epsilon$.
2. f is continuous on X if it is continuous at all $x_0 \in X$.

Proposition 3.2.4: Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$. Let $x_0 \in X$. f is **continuous at x_0** $\Leftrightarrow$ for **every sequence** $(x_k)_{k\geq 1}$ in X such that $x_k \to x_0$ as $k \to +\infty$, the sequence $(f(x_k))_{k\geq 1}$ in $\mathbb{R}^n$ **converges to $f(x_0)$**.

Definition 3.5.5: Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$. Let $x_0 \in X$ and $y \in \mathbb{R}^m$. f has a limit y as $x \to x_0$ with $x \neq x_0$ if for every $\epsilon > 0$, there exists $\delta > 0$, such that for all $x \in X$, $x \neq x_0$, such that $||x - x_0|| < \delta : ||f(x) - y|| < \epsilon$. Then $\lim_{\substack{x\to x_0 \\ x\neq x_0}} f(x) = y$.

Proposition 3.2.7: Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$. Let $x_0 \in X$ and $y \in \mathbb{R}^m$.

$\lim_{\substack{x\to x_0 \\ x\neq x_0}} f(x) = y \Leftrightarrow$ for **every sequence (x_k)** in X such that $x_k \to x$ as $k \to +\infty$, and $x_k \neq x_0$, the sequence $(f(x_k))_{k\geq 1}$ in $\mathbb{R}^n$ **converges to y**.

Proposition 3.2.9: Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ and $p \geq 1$ an integer. Let $f : X \to Y and g : Y \to \mathbb{R}^{\prime}$ be **continuous functions**. Then the composition $f \circ g$ is **continuous**.

Cartesian product, linear maps, multiplication and addition of continuous functions are continuous.

If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous then so is the function g defined by $g(x) = f(x, y_0)$ for a fixed y_0. The converse is not true.

Definition 3.2.11: A subset $X \subset \mathbb{R}^n$ is

1. **Bounded** if the set of $||x||$ for $x \in X$ is bounded in $\mathbb{R}$.
2. **Closed** if for every sequence (x_k) in X that converges in $\mathbb{R}^n$ to some vector $y \in \mathbb{R}^n$, we have $y \in X$.
3. **Compact** if it is bounded and closed.

$\{x \in \mathbb{R}^n : ||x - x_0|| = r, r \geq 0\}$ is closed (same for $\mathbb{R}^3$), $\{x \in \mathbb{R}^n : |f(x)| \leq r, r \geq 0\}$ is closed. **The union of open sets is open**.

Proposition 3.2.13: Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous map. For any closed set $Y \subset \mathbb{R}^m$, the set $f^{-1}(Y) = \{x \in \mathbb{R}^n : f(x) \in Y\} \subset \mathbb{R}^n$ is closed.

Theorem 3.2.15: Let $X \subset \mathbb{R}^n$, **non empty compact set** and $f : X \to Y$ a **continuous** function. Then f is **bounded and achieves its maximum and minimum** ($\exists\, x_+$ and x_- in X such that $f(x_+) = \sup_x \in X$and $f(x_-) = \inf_x \in X$).

Partial derivatives

Definition 3.3.1: A subset $X \subset \mathbb{R}^n$ is **open** if, for any $x = (x_1, \ldots, x_n) \in X$, $\exists \delta > 0$ such that the set $\{y = (y_1, \ldots, y_n) \in \mathbb{R}^n : |x_i - xy_i| < \delta$ for all $i\}$ is **contained in X**.

Proposition 3.3.2: A set $X \subset \mathbb{R}^n$ is **open** $\Leftrightarrow$ the **complement** $Y = \{x \in \mathbb{R}^n : x \notin X\}$ is **closed**.

Corollary 3.3.3: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $Y \subset \mathbb{R}^n$ is open, then $f^{-1}(Y)$ is open in $\mathbb{R}^n$.

Definition 3.3.5: Let $X \subset \mathbb{R}^n$ be an **open set**. Let $f : X \to \mathbb{R}^m$. Let $1 \leq i \leq n$. f has a **partial derivative on X with respect to the i-th variable** if for all $x_0 = (x_{0,1}, \ldots, x_{0,n}) \in X$, the function defined by $g(t) = f(x_{0,1}, \ldots, x_{0,i-1}, t, x_{0,i+1}, \ldots, x_{0,n})$ on the set $I = \{t \in R : (x_{0,1}, \ldots, x_{0,i-1}, t, x_{0,i+1}, \ldots, x_{0,n}) \in X\}$ **is differentiable at $t = x_{0,i}$**. Its derivative $g'(x_{0,i})$ is denoted $\frac{\partial f}{\partial x_i}(x_0)$, $\partial_{x_i} f(x_0)$, $\partial_i f(x_0)$. $g'(t) = (g'_1(t), \ldots, g'_n(t))$.

Proposition 3.3.7: $X \subset \mathbb{R}^n$ **open** and f, g functions from X to $\mathbb{R}^n$. $1 \leq i \leq n$. If f and g have **partial derivatives with respect to the i-th coordinate on X**, then:

1. **$f + g$ also does**, and $\partial x_i(f + g) = \partial x_i(f) + \partial x_i(g)$
2. **if $m = 1$, fg also does** and $\partial_{x_i}(fg) = \partial_{x_i}(f)g + f\partial_{x_i}(g)$. Furthermore, if **$g(x) \neq 0$ for all $x \in X$**, then **$\frac{f}{g}$ has a partial derivative** with respect to the i-th coordinate on X, with $\partial_{x_i}(\frac{f}{g}) = \frac{\partial_{x_i}(f)g - f\partial_{x_i}g}{g^2}$.

Definition 3.3.9: Let $X \subset \mathbb{R}^n$ be an **open set** and $f : X \to \mathbb{R}^m$ with partial derivatives on X. $f(x) = (f_1(x), \ldots, f_m(x))$. For any $x \in X$, the matrix $J_f(x) = (\partial_{x_j} f_i(x))_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}$ is the **Jacobi matrix** of f at x.

Definition 3.3.11: Let $X \subset \mathbb{R}^n$ be an **open set** and $f : X \to \mathbb{R}$. If **all partial derivatives** of f exists at , then the column vector $\nabla f(x_0) = \begin{pmatrix} \partial_{x_1} f(x_0) \\ \ldots \\ \partial_{x_n} f(x_0) \end{pmatrix}$ is the **Gradient of f at x_0**. The Jacobi matrix for $m = 1$.

The differential

Definition 3.4.3: Let $X \subset \mathbb{R}^n$ be an **open set** and $f : X \to \mathbb{R}^m$. Let u be a linear map $\mathbb{R}^n \to \mathbb{R}^m$ and $x_0 \in X$. f is **differentiable at x_0 with differential u** if

$$\lim_{\substack{x\to x_0 \\ x\neq 0}} \frac{f(x) - f(x_0) - u(x - x_0)}{||x - x_0||} = 0$$

where the limit is in $\mathbb{R}^m$. Denote **$df(x_0) = u$**. If f is differentiable at every $x \in X$ then f is differentiable on X.**The existsence of partial derivatives or directional derivatives does not guarantee continuity (only the existsence of all partial derivatives and their continuity does).**

Proposition 3.4.4: Let $X \subset \mathbb{R}^n$ be an **open set** and $f : X \to \mathbb{R}^m$ **differentiable on X**. Then

1. f is **continuous on X**.
2. f admits **partial derivatives** with respect to each variable.
3. If $m = 1$, let $x_0 \in X$ and $u(x_1, \ldots, x_n) = a_1x_1, \ldots, a_nx_n$ the differential of f at x_0. Then **$\partial x_i f(x_0) = a_i$** for $1 \leq i \leq n$.

Let $f, g : \mathbb{R}^n \to \mathbb{R}$, $h(x, y) = (f(x, y), g(x, y))$ and $m(u, v) = uv$, so that $m \circ h(x, y) = f(x, y)g(x, y)$. Therefore $\frac{\partial(fg)}{\partial x} = x\partial_x f + u\partial_x g$.

Let $f : I \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}$, f differentiable on I and g differentiable on $\mathbb{R}^m$, then $g \circ f$ is differentiable on I and **$(g \circ f)'(t) = dg(f(t))f'(t) = \nabla g(f(t)) \cdot f'(t)$**.

Proposition 3.4.6: Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}^m$, $g : X \to \mathbb{R}^m$ differentiable on X.

1. **$f+g$ is differentiable** with differential $d(f+g)=df+dg$ and if $m=1$ then **fg differentiable**.
2. If $m=1$ and $g(x)\neq 0$ for all $x\in X$, then **$\frac{f}{g}$ is differentiable**.

Proposition 3.4.7: Let $X\subset\mathbb{R}^n$ be open and $f:X\to\mathbb{R}^m$. If f has **all partial derivatives** on X, and if the **partial derivatives are continuous on X**, then f is **differentiable on X**. With **$df(x_0)=J_f(x_0)$**.

Proposition 3.4.9: Let $X\subset\mathbb{R}^n$ and $Y\subset\mathbb{R}^m$ be **both open** and $f:X\to Y$, $g:Y\to\mathbb{R}^p$ **differentiable**. Then $g\circ f:X\to\mathbb{R}^p$ is differentiable on X and for any $x_0\in X$, its differential is given by the composition $d(g\circ f)(x_0)=dg(f(x_0))\circ df(x_0)$, in particular **$J_{g\circ f}(x_0)=J_g(f(x_0))J_f(x_0)$**.

Definition 3.4.11: Let $X\subset\mathbb{R}^n$ be open and $f:X\to\mathbb{R}^m$ differentiable. Let $x_0\in X$ and $u=df(x_0)$. The graph of the affine linear approximation **$g(x)=f(x_0)+u(x-x_0)$** from $\mathbb{R}^n$ to $\mathbb{R}^m$ is called the **tangent space at x_0 to the graph of f**.

Definition 3.4.13: Let $X\subset\mathbb{R}^n$ be open and $f:X\to\mathbb{R}^m$. Let $v\in\mathbb{R}^n$ be non-zero vector and $x_0\in X$. We say that f has a **directional derivative w** $\in\mathbb{R}^m$ in the **direction v**, if g defined on the set $I=\{t\in\mathbb{R}:x_0+tv\in X\}$ by $g(t)=f(x_0+tv)$ has a derivative at $t=0$, and this is equal to w.

In other words: $\lim_{\substack{t\to 0\\ t\neq 0}}\frac{f(x_0+tv)-f(x_0)}{t}=\boldsymbol{w}$.

Computing: Let $\varphi(t)=f(x+tv)$, then compute $\varphi'(0)$. If it doesn't exists, then the directional derivative does not exists.

Proposition 3.4.15: Let $X\subset\mathbb{R}^n$ be open and $f:X\to\mathbb{R}^m$ **differentiable**. Then for any $x\in X$ and non-zero $v\in\mathbb{R}^n$, f has a **directional derivative at x_0 in the direction v equal to $df(x_0)(v)$**. Suppose $m=1$. Let $f:X\to\mathbb{R}$ be differentiable and $x_0\in X$. The **tangent space at x_0** to the graph of f is the set of $(x,y)\in\mathbb{R}^n\times\mathbb{R}$ such that **$y=f(x_0)+\nabla f(x_0)\cdot(x-x_0)$**. The vector **$\nabla f(x_0)$ points to the direction of greatest increase**.

The **gradient is perpendicular to the level sets** determined by an equation of the form $f(x)=c$.

$0=(f\circ\gamma)'(0)=\nabla f(x_0)\cdot\gamma'(0)$.

$D_vf(\gamma(t))=df(\gamma(t))v=\nabla f(\gamma(t))\cdot v$.

$\nabla f(x_0)\cdot v=||\nabla f(x_0)||\cos(\theta)$ where θ is the angle between the gradient and the direction v.

Higher derivatives

Definition 3.5.1: Let $X\subset\mathbb{R}^n$ be open and $f:X\to\mathbb{R}^m$. f is of class **C^1** if f is **differentiable on X** and **all its partial derivatives are continuous**. The set of functions of class C^1 from X to $\mathbb{R}^m$ is denoted $C^1(X;\mathbb{R}^m)$. Let $k\geq 2$, then f is of class C^k if it is differentiable and each partial derivative $\partial x_i f:X\to\mathbb{R}^m$ is of class C^{k-1}. If $f\in C^k(C;\mathbb{R}^m)$ for all $k\geq 1$, then $f\in C^\infty$.

Proposition 3.5.4: Let $k\geq 2$, $X\subset\mathbb{R}^n$ be open and $f:X\to\mathbb{R}^m\in C^k$. Then the partial derivatives of order k are **independent of the order in which the partial derivatives were taken**: For any x and y, $\partial x,yf=\partial y,xf$. Same for more variables.

Definition 3.5.8: Let $X\subset\mathbb{R}^n$ be open and $f:X\to\mathbb{R}\in C^2$. For $x\in X$, the **Hessian matrix** of f at x is the **symmetric square matrix** $\mathrm{Hess}_f(x)=(\partial_{x_i,x_j}f)_{\substack{1\leq i\\ j\leq n}}=H_f(x)$.

$$\mathrm{Hess}_f(x)=\begin{pmatrix}\frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 x_n}\\ \vdots & \ddots & \vdots\\ \frac{\partial^2 f}{\partial x_n x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2}\end{pmatrix}$$

Change of variable

We have an open set $U\subset\mathbb{R}^n$ with $(y_1,\dots,y_n)$ the new variables and a change of variable $g:U\to X$ which expresses the variables $(x_1,\dots,x_n)$ **in terms of the new variables**. For a given function $f:X\to\mathbb{R}$, the composite $h=f\circ g:U\to\mathbb{R}$ is f expressed in terms of the new variables y.

$$\partial_{y_1}h=\frac{\partial f}{\partial x_1}\frac{\partial g_1}{\partial y_1}+\cdots+\frac{\partial f}{\partial x_n}\frac{\partial g_n}{\partial y_1}$$

Some common notation: $\partial_{y_1}h=\partial_{y_1}f$ since they are the same function with different coordinate systems. g_i is usually also replaced with x_i, s.t. $\partial_{y_1}f=\frac{\partial f}{\partial x_1}\frac{\partial g_1}{\partial y_1}+\cdots+\frac{\partial f}{\partial x_n}\frac{\partial g_n}{\partial y_1}=\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_1}+\cdots+\frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial y_1}$

Taylor polynomials

Definition 3.7.1: Let $k\geq 1$ be an integer and $f:X\to\mathbb{R}$ a function of class C^k on X and some fix $x_0\in X$. The k-th Taylor polynomial of f at x_0 is the polynomial in n variables of degree $\leq k$ given by

$T_kf(y;x_0)=f(x_0)+\sum_{i=1}^n\frac{\partial f}{\partial x_i}y_i+\cdots+\sum_{m_1+\cdots+m_n=k}\frac{1}{m_1!\cdots m_n!}\frac{\partial^k f}{\partial x_1^{m_1}\cdots\partial x_n^{m_n}}(x_0)y_1^{m_1}\cdots y_n^{m_n}$, such that for $y=x-x_0: f(x)=T_kf(x-x_0;x_0)+(\text{remainder})$.

$\frac{1}{2}\sum_{i=1}^n\partial^2_{x_i^2}f(x_0)\,y_i^2+\sum_{1\leqslant i<j\leqslant n}\partial^2_{x_ix_j}f(x_0)\,y_iy_j=\frac{1}{2}y^t\,\mathrm{Hess}_f(x_0)\,y$ hence

$$\boldsymbol{T_2f(y;x_0)=f(x_0)+\nabla f(x_0)\cdot y+\frac{1}{2}y^t\,\mathrm{Hess}_f(x_0)\cdot y,}$$

for $y\in X$.

Proposition 3.7.3: Let $k\geq 1$ be an integer, $X\subset\mathbb{R}^n$ be open and $f:X\to\mathbb{R}$ a function of class C^k. For $x_0\in X$, if we define $E_kf(x;x_0)$ by $f(x)=T_kf(x-x_0;x)+E_kf(x;x_0)$ then $\lim_{\substack{x\to x_0\\ x\neq x_0}}\frac{E_kf(x;x_0)}{||x-x_0||^k}=0$.

Critical points

Proposition 3.8.1: Let $X\subset\mathbb{R}^n$ be open and $f:X\to\mathbb{R}$ differentiable. If $x_0\in X$ s.t.

$f(y)\leq f(x_0)$ for all y close enough to x_0 (local maximum at x_0)

or $f(y)\geq f(x_0)$ for all y close enough to x_0 (local minimum at x_0)

then **$df(x_0)=0$** , $\nabla f(x_0)=0$ or $\partial_{x_i}f(x_0)=0$ equivalently for $1\leq i\leq n$.

Definition: 3.8.2: Let $X\subset\mathbb{R}^n$ be open and $f:X\to\mathbb{R}$ differentiable. A point $x_0\in X$ s.t. **$\nabla f(x_0)=0$** is called a **critical point** of f.

When f is defined on a compact set $\bar{X}=X\cup B$ with X an open set and B the boundary, then the maxima and minima should also be explicitly **computed at the boundary**.

Definition 3.8.6: Let $X\subset\mathbb{R}^n$ be open and $f:X\to\mathbb{R}$ of class C^2. A critical point $x_0\in X$ of f is **non-degenerate** if the **Hessian matrix has non-zero determinant**.

Corollary 3.8.7: Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$ of class C^2, $x_0 \in X$ a non-degenerate critical point of f. Let p and q be the positive and negative eigenvalues of $\text{Hess}_f(x_0)$.

1. If $p = n$, f has a local **minimum** at x_0.
2. If $q = n$, f has a local **maximum** at x_0.
3. Otherwise, f has a **saddle point** at x_0.

bg

$p = n \Leftrightarrow H_f(x_0)$ is positive definite.

$A = (a_{i,j})_{\substack{1\le i\\ j\le n}}$ is **positive definite** $\Leftrightarrow$ $\det(A_k) > 0$ for $1 \le k \le n$. A is **negative definite** $\Leftrightarrow$ $(-1)^k \det(A_k) > 0$ for $1 \le k \le n$.

Integration in $\mathbb{R}^n$

Line integrals

Definition 4.1.1:

1. Let $I = [a, b]$ a closed and bounded interval in $\mathbb{R}$. Let $f(t) = (f_1(t), \ldots, f_n(t))$ continuous from I to $\mathbb{R}^n$. Then we define $\int_a^b f(t)dt = (\int_a^b (f_1(t), \ldots, f_n(t)) \in \mathbb{R}^n$.
2. A **parameterised curve** in $\mathbb{R}^n$ is a continuous map $\gamma : [a, b] \to \mathbb{R}^n$ that is piecewise C^1.
3. Let $\gamma : [a, b] \to \mathbb{R}^n$ be a parameterised curve. Let $X \subset \mathbb{R}^n$ be a subset containing the image of γ, and let $f : X \to \mathbb{R}^n$ continuous.

$$\int_a^b f(\gamma(t)) \cdot \gamma'(t)dt \in \mathbb{R}$$

is called the line integral of f along γ. It is denoted $\int_\gamma f(s)ds$ or $\int_\gamma f(s)d\vec{s}$.

For $X \subset \mathbb{R}^n$, $f : X \to \mathbb{R}^n$ is usually called a **vector field**.

Definition 4.1.4: Let $\gamma : [a, b] \to \mathbb{R}^n$ be a parameterised curve. An oriented reparametrisation of γ is a parameterised curve $\sigma : [c, d] \to \mathbb{R}^n$ such that $\sigma = \gamma \circ \varphi$ where $\varphi : [c, d] \to [a, b]$ is a continuous map, differentiable on $]a, b[$, that is strictly increasing and satisfies $\varphi(a) = c$ and $\varphi(b) = d$.

Proposition 4.1.5: Let γ be a **parameterised curve** in $\mathbb{R}^n$ and σ an **oriented reparametrisation** of γ. Let X be a set containing the image of γ, or equivalently the image of σ, and $f : X \to \mathbb{R}^n$ continuous. Then we have $\int_\gamma f(s) \cdot d\vec{s} = \int_\sigma f(s) \cdot d\vec{s}$.

Definition 4.1.8: Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^n$ a **continuous vector field**. If for any x_1, x_2 in X, the **line integral** $\int_\gamma f(c) \cdot d\vec{s}$ is **independent of the choice of a parameterised curve** γ in X from x_1 to x_2, then we say that the vector field is **conservative**.

Equivalently, f is conservative $\Leftrightarrow$ $\int_\gamma f(s) \cdot d\vec{s} = 0$ for any **closed** parameterised curve in X. Where a curve is closed if $\gamma(a) = \gamma(b)$.

Theorem 4.1.10: Let X be an open set and f a **conservative** vector field. Then there exists a C^1 function g on X such that $f = \nabla g$. If any two points of X are conjoined by a parameterised curve, then g is unique up to addition of a constant: if $\nabla g_1 = f$, then $g - g_1$ is constant on X.

Any two points x and y on X can be **joined by a parameterised curve** $\Leftrightarrow \exists\gamma : [a, b] \to X$ such that $\gamma(a) = x$ and $\gamma(b) = y$. Then X is **path connected**. This is also true whenever X is convex. If f is conservative, then a function g such that $\nabla g = f$ is called a **potential for f**.

$$\int_a^b f(\gamma(t)) \cdot \gamma'(t)dt \in \mathbb{R} = g(\gamma(b)) - g(\gamma(a))$$

A set X is **convex** if for any $x, y \in X$, the line segment joining x to y is contained in X.

Proposition 4.1.13: Let $X \subset \mathbb{R}^n$ be an open set and $f : X \to \mathbb{R}^n$ of class C^1.

$f(x) = (f_1(x), \ldots, f_n(x))$. If f is **conservative**, then we have $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ for any integers with $1 \le i \ne j \le n$.

Definition 4.1.15: A subset $X \subset \mathbb{R}^n$ is **star-shaped** if there exists $x_0 \in X$ such that, for all $x \in X$, the **line segment joining x_0 to x is contained in X**. We then say that X is star-shaped around x_0.

Theorem 4.1.17: Let X be a **star-shaped** open subset of $\mathbb{R}^n$ and f a C^1 vector field such that $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ on X for all $i \ne j$ between 1 and n. Then the vector field f is **conservative**.

The requirement that X is star-shaped is not necessary. X should have no "hole" in the middle around which a circle can go without it being possible to contract it within X.

Definition 4.1.20: Let $X \subset \mathbb{R}^3$ be an open set and $f : X \to \mathbb{R}^3$ a C^1 **vector field**. Then the **curl** of f, denoted $\text{curl}(f)$, is the continuous vector field on X defined by

$$\text{curl}(f) = \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix}$$

where $f(x, y, z) = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$.

$$\text{curl}(f) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

with the rule that $\partial_x \cdot f_i = f_i \cdot \partial_x = \partial_x f_i$.

$\text{curl}(\nabla f) = 0$.

The Riemann integral in $\mathbb{R}^n$

The goal is to define the integral $\int_X f(x_1, \ldots, x_n)dx$ for a closed and bounded set X and continuous function f, so that it has analogue properties to the Riemann integral for $n = 1$.

For a bounded and closed subset $X \subset \mathbb{R}^n$ and any continuous function $f : X \to \mathbb{R}$, one can define the integral of f over X denoted $\int_X f(x)dx$, which is a real number.

The integral satisfies the following properties:

1. **Compatibility:** If $n = 1$ and $X = [a, b]$ with $a \le b$, then
$$\int_{[a,b]} f(x)dx = \int_a^b f(x)dx$$
.
2. **Linearity:** If f and g are continuous on X and $a, b \in \mathbb{R}$, then
$$\int_X (af_1(x) + bf_2(x))dx = a\int_X f_1(x)dx + b\int_X f_2(x)dx$$
.
3. **Positivity:** If $f \le g$, then
$$\int_X f(x)dx \le \int_X g(x)dx$$
and especially, if $f \ge 0$, then
$$\int_X f(x)dx \ge 0.$$
Moreover, if $Y \subset X$ is compact and $f \ge 0$, then
$$\int_Y f(x)dx \le \int_X f(x)dx.$$

4. **Upper bound and triangle inequality:** In particular, since $-|f| \leq f \leq |f|$, we have
$$\left|\int_X f(x)dx\right| \leq \int_X |f(x)|dx,$$
and since $|f+g| \leq |f|+|g|$, we have
$$\left|\int_X (f(x)+g(x))dx\right| \leq \int_X |f(x)|dx + \int_X |g(x)|dx.$$
5. **Volume:** If $f=1$, then the integral of f is the volume in $\mathbb{R}^n$ of the set X, and if $f \geq 0$ in general, the integral of f is the volume of the set $\{(x,y) \in X \times \mathbb{R} : 0 \leq y \leq f(x)\} \subset \mathbb{R}^{n+1}$. In particular, if X is a bounded rectangle, say $X = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ and if $f=1$, then
$$\int_X dx = (b_n - a_n)\cdots(b_1 - a_1).$$
6. **Multiple integral or Fubini's Theorem:** If n_1 and n_2 are integers ≥ 1 such that $n = n_1 + n_2$, then for $x_1 \in \mathbb{R}^n$, let
$$Y_{x_1} = \{x_2 \in \mathbb{R}^{n_2} : (x_1, x_2) \in X\} \subset \mathbb{R}^{n_2}.$$
Let X_1 be the set of $x_1 \in \mathbb{R}^n$ such that Y_{x_1} is not empty. Then X_1 is compact in $\mathbb{R}^{n_1}$ and Y_{x_1} is compact in $\mathbb{R}^{n_2}$ for all $x_1 \in X_1$. If the function
$$g(x_1) = \int_{Y_{x_1}} f(x_1, x_2)dx_2$$
on X_1 is continuous, then
$$\int_X f(x_1, x_2)dx = \int_{X_1} g(x_1)dx_1$$
$$= \int_{X_1}\left(\int_{Y_{x_1}} f(x_1, x_2)dx_2\right)dx_1.$$
Similarly, exchanging the role of x_1 and x_2, we have
$$\int_X f(x_1, x_2)\,dx = \int_{X_2}\left(\int_{Z_{x_2}} f(x_1, x_2)\,dx_1\right)dx_2$$
where $Z_{x_2} = \{x_1 : (x_1, x_2) \in X\}$, if the integral over x_1 is a continuous function.

If the variables are $x_1, \ldots, x_n$ we also write $\int_X f(x_1, \ldots, x_n)dx_1, \ldots, dx_n$

Definition 4.2.3:

1. Let $1 \leq m \leq n$ be an integer. A parameterised m-set in $\mathbb{R}^n$ is a **continuous map** $f : [a_1, b_1] \times \cdots \times [a_m, b_m] \to \mathbb{R}^n$ which is C^1 on $]a_1, b_1[\times \cdots \times]a_m, b_m[$.
2. A subset $B \subset \mathbb{R}^n$ is **negligible** if there exists an integer $k \geq 0$ and parameterised m_i-sets $f_i : X_1 \to \mathbb{R}^n$, with $1 \leq i \leq k$ and $m_i < n$, such that $X \subset f_1(X_1) \cup \ldots \cup f_k(X_k)$.

The **image** of a parameterised m-set f might be of **dimension smaller than m** (e.g. if f is constant, in which case the image is a single point, which is an object of dimension 0)

Proposition 4.2.5: Let $X \subset \mathbb{R}^n$ be compact. Assume that X is **negligible**. Then for any continuous function on X, we have $\boldsymbol{\int_X f(x)dx = 0}$.

Improper integrals

Let $I \subset \mathbb{R}$ be a **bounded interval**, $J = [a, +\infty[$ for some $a \in \mathbb{R}$ and f a **continuous function** on $X = J \times I$. We say that it is **Riemann-intrgrable** on X if the limit
$$\lim_{x\to+\infty}\int_{[a,x]\times I} f(x,y)dxdy = \lim_{x\to+\infty}\int_a^x \left(\int_I f(x,y)dy\right)dx$$
$$= \lim_{x\to+\infty}\int_I \left(\int_a^x f(x,y)dx\right)dy$$
exists. Denote this limit by
$$\int_{J\times I} f(x,y)dxdy.$$
Similarly let f be continuous on $\mathbb{R}^2$. Assume that $f \geq 0$. f is Riemann-integrable on $\mathbb{R}^2$, if the limit
$$\lim_{R\to+\infty}\int_{[-R,R]^2} f(x,y)dxdy$$
exists, which is called the integral of f over $\mathbb{R}^2$ and denoted
$$\int_{R^2} f(x,y)dxdy.$$
This integral is also the limit of
$$\int_{D_R} f(x,y)dxdy$$
where D_R is the disc of radius R centred at 0.

The easiest way to prove that a certain **improper integral exists**: If $|f| \leq g$ (resp $0 \leq f \leq g$), and
$$\int_{J\times I} g(x,y)dxdy$$
or
$$\int_{\mathbf{R}^2} g(x,y)dxdy$$
exists, then so does
$$\int_{J\times I} f(x,y)dxdy \text{ or } \int_{R^2} f(x,y)dxdy.$$

The change of variable formula

Let $\bar{X} \subset \mathbb{R}^n$ and $\bar{Y} \subset \mathbb{R}^n$ be **compact subsets**, $\varphi : \bar{X} \to \bar{Y}$ a **continuous map**. We assume that we can write $\bar{X} = X \cup B, \bar{Y} = Y \cup C$ where

1. X and Y are **open**
2. B and C are **negligible**
3. the restriction of φ to the open set X is a C^1 **bijective map from X to Y**.

In this situation, the Jacobian matrix $J_\varphi(x)$ is **invertible at all $x \in X$**.

Theorem 4.4.2: In the situation described above, for any continuous function f on $\bar{Y}$, we have $\boldsymbol{\int_{\bar{X}} f(\varphi)|\det(J_\varphi(x))|dx = \int_{\bar{Y}} f(y)dy}$.

Cartesian to polar coordinates: $\boldsymbol{x = r\cos(\theta), y = r\sin(\theta)}$, then calculate all partial derivatives with respect to the new variables:
$\partial_r x = \cos\theta, \partial_r y = \sin\theta, \partial_\theta x = -r\sin\theta, \partial_\theta y = r\cos\theta$. Then $\boldsymbol{J_{(r,\theta)} = r, dxdy \to rdrd\theta}$.

Polar coordinates: (r, Θ) are useful for integrating over a disc in $\mathbb{R}^2$ centered at 0, or more generally over a disc sector $\Delta = \Delta(a, b, R)$ defined by

$$0 \leq r \leq R \qquad -\pi < a \leq \Theta \leq b \leq \pi$$

for some parameters (a, b, R). One gets the general formula:

$$\int_\Delta f(x,y)dxdy = \int_0^R \int_a^0 f(r,cos\Theta, rsin\Theta)r\, d\Theta dr$$

Taking r to vary between $0 < r_0 \leq r \leq R$ we obtain an annulus.

Spherical coordinates: (r, Θ, Φ) in $\mathbb{R}^3$ are useful for integrating over balls centered at 0, or parts of them. We computed the Jacobian and its determinant $-r^2 sin(\Phi)$. So, to integrate a function f over a ball B of radius R in $\mathbb{R}^3$, we have:
$\int_B f(x,y,z)\, dxdydz =$
$\int_0^R \int_0^{2\pi} \int_0^\pi f(rcos\Theta sin\Phi, rsin\Theta cos\Phi, rcos\Phi) r^2 sin\Phi \, d\Phi d\Theta dr$

Geometric applications of integrals

1. **Centre of mass:**

   Let X be a compact subset of $\mathbb{R}^n$, such that the volume of X is positive. The centre of mass of X is the point $\bar{x} \in \mathbb{R}^n$ such that $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ with

   $$\bar{x}_i = \frac{1}{\text{vol}(X)} \int_X x_i dx.$$

   Intuitively, $\bar{x}_i$ is the average over X of the i-th coordinate, and $\bar{x}$ is the point where X is perfectly balanced.

   Note that $\bar{x}$ is not necessarily in X.

2. **Surface area:**

   Consider the continuous function $f : [a, b] \times [c, d] \to \mathbb{R}$ which is C^1 on $]a, b[\times]c, d[$. Let

   $$\Gamma = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in [a, b] \times [c, d], z = f(x, y)\}$$

   $\subset \mathbb{R}^3$ be the graph of f. This is a surface and should have an area, which is given by

   $$\int_a^b \int_c^d \sqrt{1 + (\partial_x f(x,y))^2 + (\partial_y f(x,y))^2} dxdy.$$

3. **Length of a curve:**

   $$f : [a, b] \to \mathbb{R} : \int_a^b \sqrt{1 + f'(x)^2} dx.$$

The Green formula

The Green formula concerns the case of **relating an integral over a subset** X of $\mathbb{R}^2$ with a **line integral over its boundary**.

Definition 4.6.1: A **simple closed parameterised curve** $\gamma : [a, b] \to \mathbb{R}^2$ is a closed parameterised curve such that $\boldsymbol{\gamma(t) \neq \gamma(s)}$ unless $t = s$ or $\{s, t\} = \{a, b\}$, and such that $\gamma'(t) \neq 0$ for $a < t < b$. If γ is only piecewise C^1 inside $]a, b[$, this condition only applies to where $\gamma'(t)$ exists.

Theorem 4.6.3: Let $X \subset \mathbb{R}^2$ **compact with a boundary** $\boldsymbol{\partial X}$ that is the union of finitely many simple closed parameterised curves $\gamma_1, \ldots, \gamma_k$. Assume that $\gamma_i : [a_i, b_i] \to \mathbb{R}^2$ has the property that X lies always **to the left** of the tangent vector $\gamma_i'(t)$ based at $\gamma_i(t)$. Let $f = f(f_1, f_2)$ be a vector field of class C^1 defined on some open set containing X. Then we have

$$\int_X \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dxdy = \sum_{i=1}^k \int_{\gamma_i} f \cdot d\vec{s}.$$

Corollary 4.6.5: Let $X \subset \mathbb{R}^2$ compact with a boundary ∂X that is the union of finitely many simple closed parameterised curves $\gamma_1, \ldots, \gamma_k$. Assume that $\gamma_i : [a_i, b_i] \to \mathbb{R}^2$ has the property that X lies always **to the left** of the tangent vector $\gamma_i'(t)$ based at $\gamma_i(t)$. Then we have

$$\text{Vol}(X) = \sum_{i=1}^k \int_{\gamma_i} x \cdot d\vec{s} = \sum_{i=1}^k \int_{a_i}^{b_i} \gamma_{i,1}(t) \gamma_{i,2}'(t) dt.$$

The Gauss-Ostrogradski formula

This formula is the analogue of the Green formula in $\mathbb{R}^3$.

Definition 4.7.1: A **parameterised surface** $\Sigma : [a, b] \times [c, d] \to \mathbb{R}^3$ is a 2-set in $\mathbb{R}^3$ such that the **rank of the Jacobian matrix is 2** at all $(s, t) \in]a, b[\times]c, d[$.

Definition 4.7.3: Let x and y be two **linearly independent** vectors in $\mathbb{R}^3$. The vector product, or **cross product** $z = x \times y$ is the unique vector in $\mathbb{R}^3$ such that (x, y, z) is a basis of $\mathbb{R}^3$ with $\det(x, y, z) > 0$, and $||z|| = ||x||\,||y|| \sin(\theta)$, where θ is the angle between x and y.

$$x \times y = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix} = \det \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$e_1 \times e_2 = e_3,\ e_2 \times e_3 = e_1,\ e_3 \times e_1 = e_2$ and $y \times x = -x \times y$

Theorem 4.7.6: Let $X \subset \mathbb{R}^3$ **compact with a boundary** $\boldsymbol{\partial X}$ that is a parameterised surface $\Sigma : [a, b] \times [c, d] \to \mathbb{R}^3$. Assume that Σ is injective in $]a, b[\times]c, d[$, and that Σ has the property that the normal vector $\vec{n}$ points away from Σ at all points. Let $\vec{u} = \frac{\vec{n}}{||\vec{n}||}$ be the unit exterior normal vector. Let $f = (f_1, f_2, f_3)$ be a vector field of class C^1 defined on dome open set containing X. Then we have

$$\int_X \text{div}(f) dxdydz = \int_\Sigma (f \cdot \vec{u}) d\sigma.$$

For a vector field $f = (f_1, f_2, f_3)$ on $X \subset \mathbb{R}^3$, we denote the **divergence of the field** $\text{div}(f) = \partial_x f + \partial_y f + \partial_z f$.

For a parameterised surface $\Sigma : [a, b] \times [c, d] \to \mathbb{R}^3$ with exterior normal vector field $\vec{n} = (n_1, n_2, n_3) = \partial_s \Sigma \times \partial_t \Sigma$ and a function g defined on the image of Σ, we define the **surface integral**

$$\int_\Sigma g\, d\sigma = \int_a^b \int_c^d g(\Sigma(s,t)) \sigma(s,t) dsdt$$

where $\sigma(s, t) = ||\partial_s \Sigma \times \partial_t \Sigma|| = ||\vec{n}(s, t)||$.

Like the integral for a parameterised curve, the surface integral is **independent of the chosen parmeterisation of the surface**.

For a C^1 vector field $f = (f_1, f_2, f_3)$ on $\mathbb{R}^3$, we define

$$\int_\Sigma (f \cdot \vec{n}) d\sigma = \int_\Sigma g\, d\sigma,$$

where

$$g(\Sigma(s,t)) = f(\Sigma(s,t)) \cdot \vec{u}(s,t) = \sum_{i=1}^3 u_i(s,t) f_i(\Sigma(s,t)).$$

This is called the **flux of the vector field** f through the surface Σ.

In the flux, $\vec{u}(s, t)\sigma(s, t) = \vec{n}(s, t)$.

Examples and other

Basic rules that you will forget :)

Complex Numbers

$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$, $e^{i\varphi} = \text{cis}(\varphi)$, $|e^{i\varphi}| = 1$

z	$z = x + iy \begin{cases} \text{x: Real} \\ \text{y: Imaginary} \end{cases}$	$z = r \cdot e^{i\varphi} = r \cdot \text{cis}(\varphi)$
$\overline{z}$	$\overline{z} = x - iy$	$z = r \cdot e^{-i\varphi}$
$\|z\|$	$\|z\| = \sqrt{z \cdot \overline{z}} = \sqrt{x^2 + y^2}$	$\|z\| = r = \sqrt{z \cdot \overline{z}}$
$z_1 + z_2$	$x_1 + x_2 + i(y_1 + y_2)$	
$z_1 - z_2$	$x_1 - x_2 + i(y_1 - y_2)$	
$z_1 \cdot z_2$	$(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$	$r_1 \cdot r_2 \cdot e^{i(\varphi_1+\varphi_2)}$
$\frac{z_1}{z_2}, z_2 \neq 0$	$\frac{z_1 \cdot \overline{z_2}}{\|z_2\|^2} = \frac{(x_1x_2+y_1y_2)+i(x_2y_1-x_1y_2)}{x_2^2+y_2^2}$	$\frac{r_1}{r_2} \cdot e^{i(\varphi_1-\varphi_2)}$
$\frac{1}{z}, z \neq 0$	$\frac{\overline{z}}{\|z\|^2} = \frac{(x)+iy}{x^2+y^2}$	$\frac{1}{r} \cdot e^{-i\varphi}$

z^n	$r^n \cdot (\cos(n \cdot \varphi) + i\sin(n \cdot \varphi)) = r^n \cdot e^{in\varphi}$
$\sqrt[n]{z}$	$\sqrt[n]{r} \cdot (\cos(\frac{\varphi+2\pi k}{n})) + i\sin(\frac{\varphi+2\pi k}{n}),\ k = 0, 1, ..., n-1$

i^0	i^1	i^2	i^3
1	i	-1	$-i$

Abs:

$|ab| = |a||b|$ $|\frac{a}{b}| = \frac{|a|}{|b|}$ $|a + b| \leq |a| + |b|$

Power rules:

$x = \sqrt[n]{a} \Leftrightarrow (x^n = a \text{ and } x \geq 0)$ $\sqrt[n]{-a} = -\sqrt[n]{a},\ a \geq 0$
$a^{-n} = \frac{1}{a^n} = (\frac{1}{a})^n$ $a^{\frac{1}{n}} = \sqrt[n]{a}$ $\sqrt{ab} = \sqrt{a}\sqrt{b}$ $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ $a^x = e^{x \cdot \ln a}$ $\sqrt[n]{a^{-m}} = \frac{1}{\sqrt[n]{a^m}}$ $a^m a^n = a^{m+n}$
$\sqrt[n]{a^m} = \sqrt[kn]{a^{km}}$ $\frac{a^m}{a^n} = a^{m-n}$ $\sqrt[n]{\sqrt[k]{a}} = \sqrt[nk]{a}$ $(a^m)^n = a^{mn}$
$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$ $a^n b^n = (ab)^n$ $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$ $\frac{a^n}{b^n} = (\frac{a}{b})^n$

Differentiation:

- **Sum** $(f(x) + g(x))' = f'(x) + g'(x)$
- **Factor** $(c \cdot f(x))' = c \cdot f'(x)$
- **Product** $(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x)$
- **Quotient** $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$ $(g \neq 0)$
- **Chain Rule** $(f(g(x)))' = (f \circ g)' = f'(g(x))g'(x)$

Integration:

- $\int_a^b (f(x) + / - g(x))xd = \int_a^b f(x) + / - \int_a^b g(x)$
- $\int_a^b c \cdot f(x)dx = c \cdot \int_a^b f(x)dx$
- $\int_a^b f'(x) \cdot g(x)dx = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x)$
- $\int_{\phi(a)}^{\phi(b)} f(x)dx = \int_a^b f(\phi(t))\phi'(t)dt$
- $a + c, b + c \in I$ $\int_a^b f(t + c)dt = \int_{a+c}^{b+c} f(x)dx$
- $ca, cb \in I$: $\int_a^b f(ct)dt = \frac{1}{c} \int f(x)dx$
- $\int \frac{f'(t)}{f(t)}dt = \log(|f(x)|)$, bzw. $\int_a^b \frac{f'(t)}{f(t)}dt = \log(f(|b|)) - \log(f(|a|))$
- **Partial fraction:** $\frac{2x^6-4x^5+5x^4-3x^3+x^2+3x}{(x-1)^3(x^2+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2}$
- **Substitution:** $\int_a^b f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)du$

$$\int_\Omega f(x, y)dxdy =$$

$$\int_{\widetilde{\Omega}} f(g(u, v),\ h(u, v)) \left| \det \underbrace{\begin{pmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{pmatrix}}_{d\Phi=\nabla\Phi} \right| dudv$$

Coordinate transformations:

Polar Coordinates in $\mathbb{R}^2$

$$\begin{aligned} x &= r\cos\varphi & 0 \leq r < \infty & \quad dxdy = r \cdot drd\varphi \\ y &= r\sin\varphi & 0 \leq \varphi < 2\pi & \end{aligned}$$

Elliptic Coordinates $\mathbb{R}^2$

$$\begin{aligned} x &= r \cdot a\cos\varphi & 0 \leq r < \infty & \quad dxdy = a \cdot b \cdot r \cdot drd\varphi \\ y &= r \cdot b\sin\varphi & 0 \leq \varphi < 2\pi & \end{aligned}$$

Cylinder Coordinates $\mathbb{R}^3$

$$\begin{aligned} x &= r \cdot a\cos\varphi & 0 \leq r < \infty & \quad dxdydz = r \cdot drd\varphi dz \\ y &= r \cdot b\sin\varphi & 0 \leq \varphi < 2\pi & \\ z &= z & \infty \leq z < \infty & \end{aligned}$$

Sphere Coordinates $\mathbb{R}^3$

$$\begin{aligned} x &= r \cdot \sin\theta\cos\varphi & 0 \leq r < \infty & \quad dxdydz = r^2\sin\theta \cdot drd\theta d\varphi \\ y &= r \cdot \sin\theta\sin\varphi & 0 \leq \theta < \pi & \\ z &= r\cos\theta & 0 \leq \varphi < 2\pi & \end{aligned}$$

Limits:

- $\lim_{x\to 0} \arctan(x) = 0$, $\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$
- $\lim_{x\to 0} \tan(x) = 0$, $\lim_{x\to\infty} \tan(x) = \infty$, $\lim_{x\to\frac{\pi}{2}} \tan(x) = \infty$
- $\lim_{x\to\infty} \cos(x) = [-1, 1]$, $\lim_{x\to\infty} \sin(x) = [-1, 1]$
- $\lim_{n\to\infty} (1 + \frac{x}{n})^n = e^x$ $\lim_{n\to\infty} (1 + \frac{1}{n})^n = \lim_{n\to\infty} (1 + n)^{\frac{1}{n}} = e$
- $\lim_{x\to 0} \frac{a^x-1}{x} = \ln(a)$ $\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \frac{1}{\ln(a)}$ $\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$
- $\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$ $\lim_{x\to 0} \frac{\tan(x)}{x} = 1$ $\lim_{x\to 0} \frac{\sin(x)}{x} = 1 = \frac{x}{\sin(x)}$
- $\lim_{x\to\infty} \frac{n!}{n^n} = 0$ $\lim_{x\to 0} \frac{e^n-1}{n} = 1$ $\lim_{x\to\infty} \sqrt[n]{n!} = \infty$ $\lim_{x\to\infty} \sqrt[n]{n} = 1$

Obvious cases:

$\frac{1}{0} = \infty$ $\frac{1}{\infty} = 0$ $\infty + \infty = \infty$ $0 + \infty = \infty$ $0^\infty = 0$ $\infty^\infty = \infty$

Sums:

- $\sum_{k=0}^\infty aq^k = a + aq + aq^2 + ... = \frac{a}{1-q}$ (Geometric Series)
- $\sum_{k=0}^\infty (k + 1)q^k = 1 + 2q + 3q^2 + ... = \frac{1}{(1-q)^2}$, $|q| < 1$
- $\sum_{k=0}^\infty \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}$
- $\sum_{k=1}^\infty \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ... = \frac{\pi^2}{6}$
- $\zeta(a) = \sum_{k=0}^\infty \frac{1}{k^a}$ ist konvergent $\iff a > 1$
- $\sum_{k=0}^\infty \frac{1}{k!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... = e$
- $\sum_{k=0}^\infty \frac{(-1)^k}{k!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ... = \frac{1}{e}$
- $\frac{1}{1\pm x} = 1 \mp x + x^2 \mp x^3 + x^4 \mp ...$
- $\frac{1}{(1\pm x)^2} = 1 \mp 2x + 3x^2 \mp 4x^3 + 5x^4 \mp ...$
- $\sqrt{1 \pm x} = 1 \pm \frac{x}{2} - \frac{1 \cdot 1}{2 \cdot 4}x^2 \pm \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}x^4 \pm ...$
- $\exp(x) = \sum_{n=0}^\infty \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$
- $\sin(x) = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$
- $\cos(x) = \sum_{n=0}^\infty (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$
- $\sinh(x) = \sum_{n=0}^\infty \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$
- $\cosh(x) = \sum_{n=0}^\infty \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$

- $\tan(x) = 1 + \frac{\phi^3}{3} + \frac{2\phi^5}{15} + ...$
- $\tanh(z) = 1 - \frac{z^3}{3} + \frac{2z^5}{15} - ...$
- $\ln(1+z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} z^k = z - \frac{z^2}{2} + \frac{z^3}{3} + ...$
- $(1+z)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!} z^2 + ...$

Random:

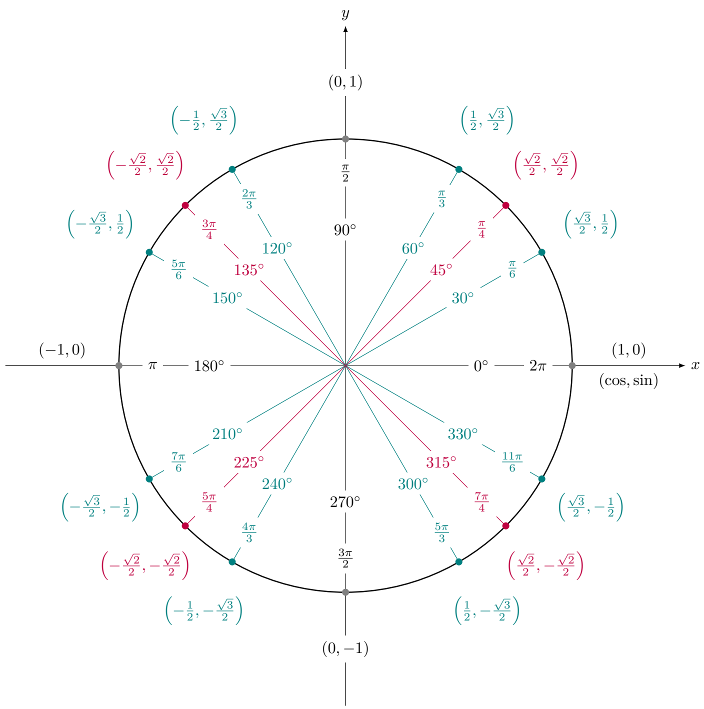
- Circle formula: $(x-x_0)^2 + (y-y_0)^2 = r^2$
- Ellipse formula: $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$
- Quadratic formula: $x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
- Determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
- Matrix is invertible iff. the Determinant $\neq 0$.
- Scalar Product: $x \cdot y = \sum_{i=1}^{n} x_i y_i$
- Crosss Product: $a \times b = (a_2 b_3 - a_3 b_2, \quad a_3 b_1 - a_1 b_3, \quad a_1 b_2 - a_2 b_1)^\top$

Let L be a **continuous linear function**, then if L is continuous at some point x_0, it is continuous at every point. $\lim_{x\to 0} ||L(a+x) - L(a)|| = \lim_{x\to 0} ||L(a) + L(x) - L(a)|| = \lim_{x\to 0} ||L(x)|| = \lim_{x\to 0} ||L(x_o + x) - L(x_0)|| = 0$.

Finding limits:

1. Different paths
2. If f is continuous then $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$
3. Squeeze theorem
4. Taylor expansion
5. Polar coordinates:
   $x = r\cos(\phi), y = r\sin(\phi), \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{4x^2+y^6} = \lim_{r\to 0} \frac{r^2}{4r^2\cos^4(\phi) + r^6 \sin^6(\phi)}$

Sin and Cos



Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°
φ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(\varphi)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\cos(\varphi)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\tan(\varphi)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\pm\infty$	-$\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0

- $\sin(x) = x + o(x)$
- $\sin(x+y)\sin(x-y) = \cos^2(y) - \cos^2(x) = \sin^2(x) - \sin^2(y)$
- $\cos(x+y)\cos(x-y) = \cos^2(y) - \sin^2(x) = \cos^2(x) - \sin^2(y)$
- $\sin x \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$
- $\cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$
- $\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$
- $\cos(x)^2 + \sin(x)^2 = 1$
- $\cos(\pi - x) = -\cos(x)$, $\sin(\pi - x) = \sin(x)$
- $\cos(x+\pi) = -\cos(x)$, $\sin(x+\pi) = -\sin(x)$
- $\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1$
- $\sin(2x) = 2\sin(x)\cos(x) \quad \tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$
- $\sin(\frac{x}{2}) = \sqrt{\frac{1-\cos(x)}{2}} \quad \cos(\frac{x}{2}) = \sqrt{\frac{1+\cos(x)}{2}}$
- $\tan(\frac{x}{2}) = \frac{1-\cos(x)}{\sin(x)} = \frac{\sin(x)}{1+\cos(x)} \quad \cot(\frac{x}{2}) = \frac{1+\cos(x)}{\sin(x)} = \frac{\sin(x)}{1-\cos(x)}$
- $\sin^2(x) = \frac{1-\cos(2x)}{2} \quad \cos^2(x) = \frac{1+\cos(2x)}{2}$
- $\tan(\pi + x) = \tan(x)$
- $-\sin(-x) = \sin(x), \cos(-x) = \cos(x), \tan(-x) = -\tan(x)$
- For all $(a,b) \in \mathbb{R}^2$, such that $a^2 + b^2 = 1$, there is $x \in \mathbb{R}$, such that $a = \cos(x)$, $b = \sin(x)$.
- $\sin(x) = \frac{2\tan(x/2)}{1+\tan^2(x/2)} \quad \cos(x) = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$
- $\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \quad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}$
- $\int_0^{2\pi} \sin(t)\cdot\cos(t)dt = \int_0^{2\pi} \sin(t)dt = \int_0^{2\pi} \cos(t)dt = 0$
- $\int \sin^2(x)dx = \frac{1}{2}(x - \sin(x)\ \cos(x))$
- $\int \cos^2(x)dx = \frac{1}{2}(x + \sin(x)\ \cos(x))$
- $\int x\sin(x)dx = \sin(x) - x\cos(x)$
- $\int x\cos(x)dx = x\sin(x) + \cos(x)$
- $\sin(\arccos(x)) = \sqrt{1-x^2} \quad \sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}$
- $\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}} \quad \cos(\arcsin(x)) = \sqrt{1-x^2}$
- $\tan(\arcsin(x)) = \frac{x}{\sqrt{1-x^2}} \quad \tan(\arccos(x)) = \frac{\sqrt{1-x^2}}{x}$
- $\cosh(x) := \frac{e^x + e^{-x}}{2} \quad \sinh x := \frac{e^x - e^{-x}}{2} \quad \tanh x := \frac{e^x - e^{-x}}{e^x + e^{-x}}$
- $\sinh(z) = -\sinh(-z) \quad \cosh(z) = \cosh(-z)$
- $\sinh(z) = \sinh(z + 2\pi i) \quad \cosh(z) = \cosh(z + 2\pi i)$
- $\sinh(z_1 \pm z_2) = \sinh(z_1)\cdot\cosh(z_2) \pm \sinh(z_2)\cdot\cosh(z_1)$
- $\sinh(z_1 \pm z_2) = \cosh(z_1)\cdot\cosh(z_2) \pm \sinh(z_2)\cdot\sinh(z_1)$
- $\tanh(z_1 \pm z_2) = \frac{\tanh(z_1) \pm \tanh(z_2)}{1 \pm \tanh(z_1)\cdot\tanh(z_2)}$
- $sin^2(x)cos^2(x) = \frac{1-cos(4x)}{8}$

Prove if f is differentiable at x_0

f continuous at x_0? No $\Rightarrow$ Not differentiable.
$\Downarrow$ Yes

Is f partially differentiable at x_0, does $\partial_{x_i} f(x_0)$ exist for all i? No $\Rightarrow$ Not differentiable.
$\Downarrow$ Yes
Are all partial derivatives continuous at x_0? Yes $\Rightarrow$ f is differentiable at x_0 with $df(x_0) = J_f(x_0)$.
$\Downarrow$ No
Does a linear mapping u exist, such that $\lim\limits_{\substack{x \to x_0 \\ x \neq 0}} \dfrac{f(x) - f(x_0) - u(x - x_0)}{||x - x_0||} = 0$?
Yes $\Rightarrow$ f is differentiable at x_0.
No $\Rightarrow$ f is not differentiable at x_0 *cries*.

Primitives and Derivatives

$$\int (ax+b)^s \, dx = \frac{1}{a(s+1)}(ax+b)^{s+1} + C, \; s \neq -1$$
$$\int \frac{1}{ax+b} dx = \frac{1}{a} \, log|ax+b| + C$$
$$\int (ax^p+b)^s x^{p-1} \, dx = \frac{(ax^p+b)^{s+1}}{ap(s+1)} + C, \; s \neq -1, a \neq 0$$
$$\int (ax^p+b)^{-1} x^{p-1} \, dx = \frac{1}{ap} log|ax^p+b| + C, \; a \neq 0, p \neq 0$$
$$\int \frac{ax+b}{cx+d} dx = \frac{ax}{c} - \frac{ad-bc}{c^2} log|cx+d| + C$$
$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} arctan(\frac{x}{a}) + C$$
$$\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} log\left|\frac{x-a}{x+a}\right|$$
$$\int \sqrt{a^2+x^2} \, dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2} log(x+\sqrt{a^2+x^2}) + C$$
$$\int \sqrt{a^2-x^2} \, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} arcsin\left(\frac{x}{|a|}\right) + C$$
$$\int \sqrt{x^2-a^2} \, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2} log|x+\sqrt{x^2-a^2}| + C$$
$$\int \frac{1}{\sqrt{x^2-a^2}} \, dx = log(x+\sqrt{a^2+x^2}) + C$$
$$\int \frac{1}{\sqrt{x^2-a^2}} \, dx = log|x+\sqrt{x^2-a^2}| + C$$
$$\int \frac{1}{\sqrt{a^2-x^2}} \, dx = arcsin\left(\frac{x}{|a|}\right) + C$$
$$\int e^{kx} \, dx = \frac{1}{k} e^{kx} + C$$
$$\int a^{kx} \, dx = \frac{1}{k * log(a)} a^{kx} + C$$
$$\int e^{ax} p(x) \, dx = e^{ax}(a^{-1}p(x) - a^{-2}p'(x) + a^{-3}p''(x) - \ldots$$
$+(-1)^n a^{-n-1} p^{(n)}(x)) + C$, $a \neq 0$, p: Polynome of degree n

$$\int e^{kx} sin(ax+b) \, dx = \frac{e^{kx}}{a^2+k^2}\Big(k \; sin(ax+b) - a \; cos(ax+b)\Big) + C$$
$$\int e^{kx} cos(ax+b) \, dx = \frac{e^{kx}}{a^2+k^2}\Big(k \; cos(ax+b) ... + a \; sin(ax+b)\Big) + C$$
$$\int log|x| \, dx = x(log|x| - 1) + C$$
$$\int x^k log(x) \, dx = \frac{x^{k+1}}{k+1}\left(log(x) - \frac{1}{k+1}\right) + C, \; k \neq -1$$
$$\int x^{-1} log(x) \, dx = \frac{1}{2}(log(x))^2 + C$$
$$\int sin(ax+b) \, dx = -\frac{1}{a} cos(ax+b) + C$$
$$\int cos(ax+b) \, dx = \frac{1}{a} sin(ax+b) + C$$
$$\int sin^2(x) \, dx = \frac{1}{2}(x - sin(x)cos(x)) + C$$
$$\int cos^2(x) \, dx = \frac{1}{2}(x + sin(x)cos(x)) + C$$
$$\int tan^2(x) \, dx = tan(x) - x + C$$
$$\int \frac{1}{sin(x)} \, dx = log\left|tan\frac{x}{2}\right| + C$$
$$\int \frac{1}{cos(x)} \, dx = log\left|tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$$
$$\int \frac{1}{tan(x)} \, dx = log|sin(x)| + C$$

$$\int_0^{2\pi} sin(mx)cos(nx) \, dx = 0, \; m, n \in \mathbb{Z}$$
$$\int_0^{\infty} \frac{sin(ax)}{x} \, dx = \frac{\pi}{2}, \; a > 0$$
$$\int_0^{\infty} sin(x^2) \, dx = \int_0^{\infty} cos(x^2) \, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$
$$\int_0^{\infty} e^{-ax} x^n \, dx = \frac{n!}{a^{n+1}}, \; a > 0$$
$$\int_0^{\infty} e^{-ax^2} \, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}, \; a > 0$$

Differentiable $\Longrightarrow$ **Continuous** $\Longrightarrow$ **Integrable**

$\mathbf{f'(x)}$	$\mathbf{f(x)}$	$\mathbf{F(x)}$
0	c $(c \in \mathbb{R})$	cx
c	cx	$\frac{c}{2}x^2$
$r \cdot x^{r-1}$	$x^r (r \in \mathbb{R} \backslash \{-1\}$	$\frac{x^{r+1}}{r+1}$
$\frac{-1}{x^2} = -x^{-2}$	$\frac{1}{x} = x^{-1}$	$\log\|x\|$
$\frac{1}{2\sqrt{x}} = -x^{-2}$	$\sqrt{x} = x^{\frac{1}{2}}$	$\frac{2}{3}x^{\frac{3}{2}}$
$\cos x$	$\sin x$	$-\cos x$
$-\sin x$	$\cos x$	$\sin x$
$1 + \tan^2 x = \frac{1}{\cos^2 x}$	$\tan x$	$-\log\|\cos x\|$
$-\frac{1}{\sin^2(x)}$	$\cot x$	$\log\|\sin x\|$
e^x	e^x	e^x
$c \cdot e^{cx}$	e^{cx}	$\frac{1}{c}e^{cx}$
$\log a \cdot a^x$	a^x	$\frac{a^x}{\log a}$
$\frac{1}{x}$	$\log\|x\|$	$x(\log\|x\| - 1)$
$\frac{1}{\log a \cdot x}$	$\log_a\|x\|$	$\frac{x}{\log a}(\log\|x\| - 1)$
		$= x(\log_a\|x\| - \log_a e)$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$	$x \arcsin x + \sqrt{1-x^2}$
$-\frac{1}{\sqrt{1-x^2}}$	$\arccos x$	$x \arccos x - \sqrt{1-x^2}$
$\frac{1}{1+x^2}$	$\arctan x$	$x \arctan x - \frac{1}{2}\log(1+x^2)$
$\sinh(x)$	$\cosh(x)$	-
$\cosh(x)$	$\sinh(x)$	-
$\frac{1}{\cosh^2(x)}$	$\tanh(x)$	$\log(\cosh(x))$
$2\sin(x)\cos(x)$	$\sin^2(x)$	$\frac{1}{2}(x - \sin(x)\cos(x))$
$-2\sin(x)\cos(x)$	$\cos^2(x)$	$\frac{1}{2}(x + \sin(x)\cos(x))$
$\frac{2\sin(x)}{\cos^3(x)}$	$\tan^2(x)$	$\tan(x) - x$
$\frac{1}{\sqrt{x^2+1}}$	$\operatorname{arsinh} x$	$x \operatorname{arsinh} x - \sqrt{x^2+1}$
$\frac{1}{\sqrt{x^2-1}} \quad (x > 1)$	$\operatorname{arcosh} x$	$x \operatorname{arcosh} x - \sqrt{x^2-1}$
$\frac{1}{1-x^2} \quad (\|x\| < 1)$	$\operatorname{artanh} x$	$x \operatorname{artanh} x + \frac{1}{2}\ln\left(1-x^2\right)$
$\frac{1}{1-x^2} \quad (\|x\| > 1)$	$\operatorname{arcoth} x$	$x \operatorname{arcoth} x + \frac{1}{2}\ln\left(x^2-1\right)$