

Ordinary differential equations

A differential equation is an equation where the unknown (or unknowns) is a function f , and the equation **relates values of f at a point x** with values of derivatives of the function at **the same point x** . If the function has one variable only (as is the case in this chapter), one speaks of ordinary differential equations.

Theorem 2.1.6: $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a differentiable function of two variables. Let $x_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}^2$. Then the ODE $y' = F(x, y)$ has an **unique** solution f defined on a *largest* open interval I containing x_0 such that $f(x_0) = y_0$.

$\exists f : I : \mathbb{R} \text{ s.t. } \forall x \in I : f'(x) = F(x, y)$ and one cannot find a larger interval containing I with such a solution.

Separation of variables:

If the ODE can be rewritten as $\frac{dy}{dx} = f(x)g(y)$, then

$$\int \frac{dy}{g(y)} = \int f(x)dx.$$

Linear differential equations

Definition 2.2.1: Let $I \subset \mathbb{R}$ open interval and $k \geq 1$ an integer.

Homogeneous ODE of order k : $y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = 0$ where the coefficients a_0, \dots, a_{k-1} are complex-valued functions on I .

Inhomogeneous ODE of order k : $y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$ where $b : I \rightarrow \mathbb{C}$.

Theorem 2.2.3: Let $I \subset \mathbb{R}$ an open interval and $k \geq 1$ an integer,

$y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = 0$ a linear ODE over I with continuous coefficients.

1. Set S of k -times differentiable solutions $f : I \rightarrow \mathbb{C}$ is a **complex vector space** which is a subspace of complex-valued functions on I .

If the functions a_i are real-valued, the set of real-valued solutions is also a vector space.

2. The dimension of S is k , and for any choice of $x_0 \in I$ and any $(y_0, \dots, y_{k-1}) \in \mathbb{C}^k$, there exists a unique $f \in \mathbb{C}$ such that $f(x_0) = y_0, f'(x_0) = y_1, \dots, f^{(k-1)}(x_0) = y_{k-1}$.

3. Let b be a continuous function on I . There exists a solution f_0 to the inhomogeneous equation and the set S_b is the set of functions **$f + f_0$** where $f \in S$.
4. For any $x_0 \in I$ and any $(y_0, \dots, y_{k-1}) \in \mathbb{C}^k$, there exists a unique $f \in S_b$ such that $f(x_0) = y_0, f'(x_0) = y_1, \dots, f^{(k-1)}(x_0) = y_{k-1}$.

If $b \neq 0$, the set S_b is not a vector space!

Linear differential equations of order 1

Let $I \subset \mathbb{R}$ be an open interval. We consider here the linear differential equation **$y' + ay = b$** when a and b are general continuous functions defined on I .

Steps to solve:

1. Solve the **homogeneous equation**: $y' + ay = 0$ and obtain S .
2. Find solution f_0 to the **inhomogeneous equation**.
3. S_b will contain $f_0 + f$ where $f \in S$ and if some f_1 is a basis of S then the solutions are given by $f_0 + zf_1$ where $z \in \mathbb{C}$ are arbitrary.

If the initial value $f(x_0) = y_0$ is given, then one must solve $f_0(x_0) + zf_1(x_0) = y_0$ and determine the value of z .

Proposition 2.3.1: Any solution of **$y' + ay = 0$** is of the form **$f(x) = z \exp(-A(x))$** where A is a primitive of a . The unique solution with $f(x_0) = y_0$ is **$f(x) = y_0 \exp(A(x_0) - A(x))$** .

Linear differential equations with constant coefficients

Now let $k \geq 1$ an integer; a_0, \dots, a_{k-1} constant coefficients and b a continuous function. We consider the equation $y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$.

Solution of the homogeneous equation:

Let **$P(\lambda) = \lambda^k + a_{k-1}\lambda^{k-1} + \dots + a_1\lambda + a_0$** . Find the **roots** of this polynomial. If the roots are not real but the coefficients are, express the solution in terms of sin and cos using $e^{ix} = \cos x + i \sin x$.

Case 1: No multiple roots

Any solution $f \in S$ of solutions of the homo equation is of the form **$f(x) = z_1 e^{\alpha_1 x} + \dots + z_k e^{\alpha_k x}$** for arbitrary z_1, \dots, z_k . To find an unique solution to $f(x_0) = y_0, \dots, f^{(k-1)}(x_0) = y_{k-1}$ for given (y_0, \dots, y_{k-1}) just view z_1, \dots, z_k as unknowns.

To obtain the **real valued solutions** if the coefficients are real:

$$f(x) = x_1 e^{\alpha_1 x} + \dots + x_m e^{\alpha_m x} + x_{m+1} e^{\alpha_{m+1} x} \cos(b_{m+1} x) + y_{m+1} e^{\alpha_{m+1} x} \sin(b_{m+1} x) + \dots + x_k e^{\alpha_k x} \cos(b_k x) + y_k e^{\alpha_k x} \sin(b_k x)$$

with $\alpha_1, \dots, \alpha_m$ being the real solutions and $\alpha_{m+1}, \dots, \alpha_k$ the complex solutions $\alpha_j = a_j + ib_j$.

Case 2: Multiple roots

Assume α is a multiple root of order j of P then the solutions look as follows:

$$f_{\alpha,0}(x) = e^{\alpha x}, \quad f_{\alpha,1}(x) = x e^{\alpha x}, \quad \dots, \quad f_{\alpha,j-1}(x) = x^{j-1} e^{\alpha x}$$

That is, multiply the original solution from *Case 1* with x^{i-1} for $i = 1, \dots, j$.

Solution to the inhomogeneous equation:

1. **Ansatz method** (left side is $b(x)$ and the right side is Ansatz)

$$a e^{\alpha x} \rightarrow b e^{\alpha x}$$

$$a \sin(\beta x) \text{ or } a \cos(\beta x) \rightarrow c \sin(\beta x) + d \cos(\beta x)$$

$$a e^{\sin(\beta x)} \text{ or } a e^{\cos(\beta x)} \rightarrow e^{\alpha x} [c \sin(\beta x) + d \cos(\beta x)]$$

$$P_n(x) e^{\alpha x} \rightarrow R_n(x) e^{\alpha x}$$

$$P_n e^{\alpha x} \sin(\beta x) \text{ or } P_n e^{\alpha x} \cos(\beta x) \rightarrow e^{\alpha x} [R_n \sin(\beta x) + S_n \cos(\beta x)]$$

With $P_n(x), R_n(x), Q_n(x), S_n(x)$ being polynomials of degree n .

If $b(x)$ is a linear combination of the above functions, then one should try the corresponding linear combination of the *Ansatz* functions.

If $\lambda = \alpha + \beta i$ is a root of $P(\lambda)$ of multiplicity m , then the *Ansatz* function should be multiplied by x^m (otherwise the *Ansatz* would solve the homo solution again)

2. Variation of constants

Assume (f_1, \dots, f_k) is the basis of the space S of solutions of the homogeneous equation. Now we search for a solution of the **inhomogeneous equation** of the form $f(x) = z_1(x)f_1(x) + \dots + z_k(x)f_k(x)$ and impose the following **conditions**:

$$\begin{cases} z'_1(x)f_1(x) + \cdots + z'_k(x)f_k(x) = 0 \\ z'_1(x)f'_1(x) + \cdots + z'_k(x)f'_k(x) = 0 \\ \cdots \\ z'_1(x)f_1^{(k-2)}(x) + \cdots + z'_k(x)f_k^{(k-2)}(x) = 0 \end{cases}$$

Differential calculus in \mathbb{R}^n

Continuity in \mathbb{R}^n

The norm $\|x\|$ satisfies $\|x\| > 0$, $\|x\| = 0 \Leftrightarrow x = 0$, $\|tx\| = |t|\|x\|$ for all $t \in \mathbb{R}$, and $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

Definition 3.2.1: Let $(x_k)_{k \in \mathbb{N}}$ where $x_k \in \mathbb{R}^n$. $x_k = (x_{k,1}, \dots, x_{k,n})$. Let $y = (y_1, \dots, y_n)$. We say that the sequence (x_k) **converges to y** as $y \rightarrow +\infty$ if for all $\epsilon > 0$, there exists $N \geq 1$ such that for all $n \geq N$ we have $\|x_k - y\| < \epsilon$.

Lemma 3.2.2: The sequence (x_k) **converges to y** as $y \rightarrow +\infty \Leftrightarrow$ one of the following holds:

1. For each $1 \leq i \leq n$, the sequence $(x_{k,i})$ of real numbers **converges to y_i** .
2. The sequence of real numbers $\|x_k - y\|$ **converges to 0** as $y \rightarrow +\infty$.

Definition 3.2.3: Let $X \subset \mathbb{R}^n$ and $f : X \rightarrow \mathbb{R}^m$

1. Let $x_0 \in X$. f is continuous at x_0 if for all $\epsilon > 0$, there exists $\delta > 0$ such that, if $x \in X$ satisfies $\|x - x_0\| < \delta$, then $\|f(x) - f(x_0)\| < \epsilon$.
2. f is continuous on X if it is continuous at all $x_0 \in X$.

Proposition 3.2.4: Let $X \subset \mathbb{R}^n$ and $f : X \rightarrow \mathbb{R}^m$. Let $x_0 \in X$. f is **continuous at x_0** \Leftrightarrow for **every sequence** $(x_k)_{k \geq 1}$ in X such that $x_k \rightarrow x_0$ as $k \rightarrow +\infty$, the sequence $(f(x_k))_{k \geq 1}$ in \mathbb{R}^m **converges to $f(x_0)$** .

Definition 3.5.5: Let $X \subset \mathbb{R}^n$ and $f : X \rightarrow \mathbb{R}^m$. Let $x_0 \in X$ and $y \in \mathbb{R}^m$. f has a limit y as $x \rightarrow x_0$ with $x \neq x_0$ if for every $\epsilon > 0$, there exists $\delta > 0$, such that for all $x \in X$, $x \neq x_0$, such that $\|x - x_0\| < \delta : \|f(x) - y\| < \epsilon$. Then $\lim_{\substack{x \rightarrow x_0 \\ x \neq x_0}} f(x) = y$.

Proposition 3.2.7: Let $X \subset \mathbb{R}^n$ and $f : X \rightarrow \mathbb{R}^m$. Let $x_0 \in X$ and $y \in \mathbb{R}^m$.

$\lim_{\substack{x \rightarrow x_0 \\ x \neq x_0}} f(x) = y \Leftrightarrow$ for **every sequence** (x_k) in X such that $x_k \rightarrow x$ as $k \rightarrow +\infty$, and $x_k \neq x_0$, the sequence $(f(x_k))_{k \geq 1}$ in \mathbb{R}^m **converges to y** .

Proposition 2.3.9: Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ and $p \geq 1$ an integer. Let $f : X \rightarrow Y$ and $g : Y \rightarrow \mathbb{R}^l$ be **continuous functions**. Then the composition $f \circ g$ is **continuous**.

Cartesian product, linear maps, multiplication and addition of continuous functions are continuous.

If $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous then so is the function g defined by $g(x) = f(x, y_0)$ for a fixed y_0 . The converse is not true.

Definition 3.2.11: A subset $X \subset \mathbb{R}^n$ is

1. **Bounded** if the set of $\|x\|$ for $x \in X$ is bounded in \mathbb{R} .
2. **Closed** if for every sequence (x_k) in X that converges in \mathbb{R}^n to some vector $y \in \mathbb{R}^n$, we have $y \in X$.
3. **Compact** if it is bounded and closed.

$\{x \in \mathbb{R}^n : \|x - x_0\| = r, r \geq 0\}$ is closed (same for \mathbb{R}^3), $\{x \in \mathbb{R}^n : \|f(x)\| \leq r, r \geq 0\}$ is closed. **The union of open sets is open.**

Proposition 3.2.13: Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a continuous map. For any closed set $Y \subset \mathbb{R}^m$, the set $f^{-1}(Y) = \{x \in \mathbb{R}^n : f(x) \in Y\} \subset \mathbb{R}^n$ is closed.

Theorem 3.2.15: Let $X \subset \mathbb{R}^n$, **non empty compact set** and $f : X \rightarrow Y$ a **continuous function**. Then f is **bounded and achieves its maximum and minimum** ($\exists x_+$ and x_- in X such that $f(x_+) = \sup_x \in X$ and $f(x_-) = \inf_x \in X$).

Partial derivatives

Definition 3.3.1: A subset $X \subset \mathbb{R}^n$ is **open** if, for any $x = (x_1, \dots, x_n) \in X$, $\exists \delta > 0$ such that the set $\{y = (y_1, \dots, y_n) \in \mathbb{R}^n : |x_i - y_i| < \delta \text{ for all } i\}$ is **contained in X** .

Proposition 3.3.2: A set $X \subset \mathbb{R}^n$ is **open** \Leftrightarrow the **complement** $Y = \{x \in \mathbb{R}^n : x \notin X\}$ is **closed**.

Corollary 3.3.3: If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous and $Y \subset \mathbb{R}^m$ is open, then $f^{-1}(Y)$ is open in \mathbb{R}^n .

Definition 3.3.5: Let $X \subset \mathbb{R}^n$ be an **open set**. Let $f : X \rightarrow \mathbb{R}^m$. Let $1 \leq i \leq n$. f has a **partial derivative on X with respect to the i -th variable** if for all $x_0 = (x_{0,1}, \dots, x_{0,n}) \in X$, the function defined by $g(t) = f(x_{0,1}, \dots, x_{0,i-1}, t, x_{0,i+1}, \dots, x_{0,n})$ on the set $I = \{t \in \mathbb{R} : (x_{0,1}, \dots, x_{0,i-1}, t, x_{0,i+1}, \dots, x_{0,n}) \in X\}$ is **differentiable at $t = x_{0,i}$** . Its derivative $g'(x_{0,i})$ is denoted $\frac{\partial f}{\partial x_i}(x_0)$, $\partial_{x_i} f(x_0)$, $\partial_i f(x_0)$. $g'(t) = (g'_1(t), \dots, g'_n(t))$.

Proposition 3.3.7: $X \subset \mathbb{R}^n$ **open** and f, g functions from X to \mathbb{R}^n . $1 \leq i \leq n$. If f and g have **partial derivatives with respect to the i -th coordinate on X** , then:

1. **$f + g$ also does**, and $\partial_{x_i}(f + g) = \partial_{x_i}(f) + \partial_{x_i}(g)$
2. **if $m = 1$, fg also does** and $\partial_{x_i}(fg) = \partial_{x_i}(f)g + f\partial_{x_i}(g)$. Furthermore, if **$g(x) \neq 0$ for all $x \in X$** , then **$\frac{f}{g}$ has a partial derivative** with respect to the i -th coordinate on X , with $\partial_{x_i}(\frac{f}{g}) = \frac{\partial_{x_i}(f)g - f\partial_{x_i}g}{g^2}$.

Definition 3.3.9: Let $X \subset \mathbb{R}^n$ be an **open set** and $f : X \rightarrow \mathbb{R}^m$ with partial derivatives on X . $f(x) = (f_1(x), \dots, f_m(x))$. For any $x \in X$, the matrix $J_f(x) = (\partial_{x_j} f_i(x))_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ is the **Jacobi matrix** of f at x .

Definition 3.3.11: Let $X \subset \mathbb{R}^n$ be an **open set** and $f : X \rightarrow \mathbb{R}$. If **all partial derivatives** of f exists at x_0 , then the column vector $\nabla f(x_0) = \begin{pmatrix} \partial_{x_1} f(x_0) \\ \vdots \\ \partial_{x_n} f(x_0) \end{pmatrix}$ is the **Gradient of f at x_0** . The Jacobi matrix for $m = 1$.

The differential

Definition 3.4.3: Let $X \subset \mathbb{R}^n$ be an **open set** and $f : X \rightarrow \mathbb{R}^m$. Let u be a linear map $\mathbb{R}^n \rightarrow \mathbb{R}^m$ and $x_0 \in X$. f is **differentiable at x_0 with differential u** if

$$\lim_{\substack{x \rightarrow x_0 \\ x \neq x_0}} \frac{f(x) - f(x_0) - u(x - x_0)}{\|x - x_0\|} = 0$$

where the limit is in \mathbb{R}^m . Denote **$df(x_0) = u$** . If f is differentiable at every $x \in X$ then f is differentiable on X . **The existence of partial derivatives or directional derivatives does not guarantee continuity (only the existence of all partial derivatives and their continuity does).**

Proposition 3.4.4: Let $X \subset \mathbb{R}^n$ be an **open set** and $f : X \rightarrow \mathbb{R}^m$ **differentiable on X** . Then

1. f is **continuous on X** .
2. f admits **partial derivatives** with respect to each variable.
3. If $m = 1$, let $x_0 \in X$ and $u(x_1, \dots, x_n) = a_1 x_1, \dots, a_n x_n$ the differential of f at x_0 . Then **$\partial_{x_i} f(x_0) = a_i$** for $1 \leq i \leq n$.

Let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$, $h(x, y) = (f(x, y), g(x, y))$ and $m(u, v) = uv$, so that $m \circ h(x, y) = f(x, y)g(x, y)$. Therefore $\frac{\partial(fg)}{\partial x} = x\partial_x f + u\partial_x g$.

Let $f : I \rightarrow \mathbb{R}^m$ and $g : \mathbb{R}^m \rightarrow \mathbb{R}$, f differentiable on I and g differentiable on \mathbb{R}^m , then $g \circ f$ is differentiable on I and $(g \circ f)'(t) = dg(f(t))f'(t) = \nabla g(f(t)) \cdot f'(t)$.

Proposition 3.4.6: Let $X \subset \mathbb{R}^n$ be open and $f : X \rightarrow \mathbb{R}^m$, $g : X \rightarrow \mathbb{R}^m$ differentiable on X .

1. $f + g$ is differentiable with differential $d(f + g) = df + dg$ and if $m = 1$ then fg differentiable.
2. If $m = 1$ and $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is differentiable.

Proposition 3.4.7: Let $X \subset \mathbb{R}^n$ be open and $f : X \rightarrow \mathbb{R}^m$. If f has all partial derivatives on X , and if the partial derivatives are continuous on X , then f is differentiable on X . With $df(x_0) = J_f(x_0)$.

Proposition 3.4.9: Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be both open and $f : X \rightarrow Y$, $g : Y \rightarrow \mathbb{R}^p$ differentiable. Then $g \circ f : X \rightarrow \mathbb{R}^p$ is differentiable on X and for any $x_0 \in X$, its differential is given by the composition $d(g \circ f)(x_0) = dg(f(x_0)) \circ df(x_0)$, in particular $J_{g \circ f}(x_0) = J_g(f(x_0))J_f(x_0)$.

Definition 3.4.11: Let $X \subset \mathbb{R}^n$ be open and $f : X \rightarrow \mathbb{R}^m$ differentiable. Let $x_0 \in X$ and $u = df(x_0)$. The graph of the affine linear approximation $g(x) = f(x_0) + u(x - x_0)$ from \mathbb{R}^n to \mathbb{R}^m is called the tangent space at x_0 to the graph of f .

Definition 3.4.13: Let $X \subset \mathbb{R}^n$ be open and $f : X \rightarrow \mathbb{R}^m$. Let $v \in \mathbb{R}^n$ be non-zero vector and $x_0 \in X$. We say that f has a directional derivative $w \in \mathbb{R}^m$ in the direction v , if g defined on the set $I = \{t \in \mathbb{R} : x_0 + tv \in X\}$ by $g(t) = f(x_0 + tv)$ has a derivative at $t = 0$, and this is equal to w .

In other words: $\lim_{t \rightarrow 0, t \neq 0} \frac{f(x_0 + tv) - f(x_0)}{t} = w$.

Computing: Let $\varphi(t) = f(x + tv)$, then compute $\varphi'(0)$. If it doesn't exist, then the directional derivative does not exist.

Proposition 3.4.15: Let $X \subset \mathbb{R}^n$ be open and $f : X \rightarrow \mathbb{R}^m$ differentiable. Then for any $x \in X$ and non-zero $v \in \mathbb{R}^n$, f has a directional derivative at x_0 in the direction v equal to $df(x_0)(v)$. Suppose $m = 1$. Let $f : X \rightarrow \mathbb{R}$ be differentiable and $x_0 \in X$. The tangent space at x_0 to the graph of f is the set of $(x, y) \in \mathbb{R}^n \times \mathbb{R}$

such that $y = f(x_0) + \nabla f(x_0) \cdot (x - x_0)$. The vector $\nabla f(x_0)$ points to the direction of greatest increase.

The gradient is perpendicular to the level sets determined by an equation of the form $f(x) = c$.

$$0 = (f \circ \gamma)'(0) = \nabla f(x_0) \cdot \gamma'(0).$$

$$D_v f(\gamma(t)) = df(\gamma(t))v = \nabla f(\gamma(t)) \cdot v.$$

$\nabla f(x_0) \cdot v = \|\nabla f(x_0)\| \cos(\theta)$ where θ is the angle between the gradient and the direction v .

Higher derivatives

Definition 3.5.1: Let $X \subset \mathbb{R}^n$ be open and $f : X \rightarrow \mathbb{R}^m$. f is of class C^1 if f is differentiable on X and all its partial derivatives are continuous. The set of functions of class C^1 from X to \mathbb{R}^m is denoted $C^1(X; \mathbb{R}^m)$. Let $k \geq 2$, then f is of class C^k if it is differentiable and each partial derivative $\partial_{x_i} f : X \rightarrow \mathbb{R}^m$ is of class C^{k-1} . If $f \in C^k(C; \mathbb{R}^m)$ for all $k \geq 1$, then $f \in C^\infty$.

Proposition 3.5.4: Let $k \geq 2$, $X \subset \mathbb{R}^n$ be open and $f : X \rightarrow \mathbb{R}^m \in C^k$. Then the partial derivatives of order k are independent of the order in which the partial derivatives were taken: For any x and y , $\partial x, y f = \partial y, x f$. Same for more variables.

Definition 3.5.8: Let $X \subset \mathbb{R}^n$ be open and $f : X \rightarrow \mathbb{R} \in C^2$. For $x \in X$, the Hessian matrix of f at x is the symmetric square matrix $\text{Hess}_f(x) = (\partial_{x_i, x_j} f)_{\substack{1 \leq i \\ j \leq n}} = H_f(x)$.

$$\text{Hess}_f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

Change of variable

We have an open set $U \subset \mathbb{R}^n$ with (y_1, \dots, y_n) the new variables and a change of variable $g : U \rightarrow X$ which expresses the variables (x_1, \dots, x_n) in terms of the new variables. For a given function $f : X \rightarrow \mathbb{R}$, the composite $h = f \circ g : U \rightarrow \mathbb{R}$ is f expressed in terms of the new variables y .

$$\partial_{y_1} h = \frac{\partial f}{\partial x_1} \frac{\partial g_1}{\partial y_1} + \cdots + \frac{\partial f}{\partial x_n} \frac{\partial g_n}{\partial y_1}$$

Some common notation: $\partial_{y_1} h = \partial_{y_1} f$ since they are the same function with different coordinate systems. g_i is usually also replaced with x_i , s.t. $\partial_{y_1} f = \frac{\partial f}{\partial x_1} \frac{\partial g_1}{\partial y_1} + \cdots + \frac{\partial f}{\partial x_n} \frac{\partial g_n}{\partial y_1} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial y_1} + \cdots + \frac{\partial f}{\partial x_n} \frac{\partial x_n}{\partial y_1}$

Taylor polynomials

Definition 3.7.1: Let $k \geq 1$ be an integer and $f : X \rightarrow \mathbb{R}$ a function of class C^k on X and some fix $x_0 \in X$. The k -th Taylor polynomial of f at x_0 is the polynomial in n variables of degree $\leq k$ given by

$$T_k f(y; x_0) = f(x_0) + \sum_{i=1}^n \frac{\partial f}{\partial x_i} y_i + \cdots + \sum_{m_1 + \cdots + m_n = k} \frac{1}{m_1! \cdots m_n!} \frac{\partial^k f}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}}(x_0) y_1^{m_1} \cdots y_n^{m_n},$$

such that for $y = x - x_0 : f(x) = T_k f(x - x_0; x_0) + (\text{remainder})$.

$$\frac{1}{2} \sum_{i=1}^n \partial_{x_i^2}^2 f(x_0) y_i^2 + \sum_{1 \leq i < j \leq n} \partial_{x_i x_j}^2 f(x_0) y_i y_j = \frac{1}{2} y^t \text{Hess}_f(x_0) y \text{ hence}$$

$$T_2 f(y; x_0) = f(x_0) + \nabla f(x_0) \cdot y + \frac{1}{2} y^t \text{Hess}_f(x_0) \cdot y,$$

for $y \in \mathbb{R}$.

Proposition 3.7.3: Let $k \geq 1$ be an integer, $X \subset \mathbb{R}^n$ be open and $f : X \rightarrow \mathbb{R}$ a function of class C^k . For $x_0 \in X$, if we define $E_k f(x; x_0)$ by $f(x) = T_k f(x - x_0; x) + E_k f(x; x_0)$ then $\lim_{\substack{x \rightarrow x_0 \\ x \neq x_0}} \frac{E_k f(x; x_0)}{\|x - x_0\|^k} = 0$.

Critical points

Proposition 3.8.1: Let $X \subset \mathbb{R}^n$ be open and $f : X \rightarrow \mathbb{R}$ differentiable. If $x_0 \in X$ s.t.

$f(y) \leq f(x_0)$ for all y close enough to x_0 (local maximum at x_0)

or $f(y) \geq f(x_0)$ for all y close enough to x_0 (local minimum at x_0)

then $df(x_0) = 0$, $\nabla f(x_0) = 0$ or $\partial_{x_i} f(x_0) = 0$ equivalently for $1 \leq i \leq n$.

Definition: 3.8.2: Let $X \subset \mathbb{R}^n$ be open and $f : X \rightarrow \mathbb{R}$ differentiable. A point $x_0 \in X$ s.t. $\nabla f(x_0) = 0$ is called a critical point of f .

When f is defined on a compact set $\bar{X} = X \cup B$ with X an open set and B the boundary, then the maxima and minima should also be explicitly **computed at the boundary**.

Definition 3.8.6: Let $X \subset \mathbb{R}^n$ be open and $f : X \rightarrow \mathbb{R}$ of class C^2 . A critical point $x_0 \in X$ of f is **non-degenerate** if the **Hessian matrix has non-zero determinant**.

Corollary 3.8.7: Let $X \subset \mathbb{R}^n$ be open and $f : X \rightarrow \mathbb{R}$ of class C^2 , $x_0 \in X$ a non-degenerate critical point of f . Let p and q be the positive and negative eigenvalues of $\text{Hess}_f(x_0)$.

1. If $p = n$, f has a local **minimum** at x_0 .
2. If $q = n$, f has a local **maximum** at x_0 .
3. Otherwise, f has a **saddle point** at x_0 .

$p = n \Leftrightarrow H_f(x_0)$ is positive definite.

$A = (a_{i,j})_{\substack{1 \leq i \\ j \leq n}}^{\substack{1 \leq j \\ i \leq n}}$ is **positive definite** $\Leftrightarrow \det(A_k) > 0$ for $1 \leq k \leq n$. A is **negative definite** $\Leftrightarrow (-1)^k \det(A_k) > 0$ for $1 \leq k \leq n$.

Integration in \mathbb{R}^n

Line integrals

Definition 4.1.1:

1. Let $I = [a, b]$ a closed and bounded interval in \mathbb{R} . Let $f(t) = (f_1(t), \dots, f_n(t))$ continuous from I to \mathbb{R}^n . Then we define $\int_a^b f(t)dt = (\int_a^b f_1(t), \dots, \int_a^b f_n(t)) \in \mathbb{R}^n$.
2. A **parameterised curve** in \mathbb{R}^n is a continuous map $\gamma : [a, b] \rightarrow \mathbb{R}^n$ that is piecewise C^1 .
3. Let $\gamma : [a, b] \rightarrow \mathbb{R}^n$ be a parameterised curve. Let $X \subset \mathbb{R}^n$ be a subset containing the image of γ , and let $f : X \rightarrow \mathbb{R}^n$ continuous.

$$\int_a^b f(\gamma(t)) \cdot \gamma'(t)dt \in \mathbb{R}$$

is called the line integral of f along γ . It is denoted $\int_\gamma f(s)ds$ or $\int_\gamma f(s)d\vec{s}$.

For $X \subset \mathbb{R}^n$, $f : X \rightarrow \mathbb{R}^n$ is usually called a **vector field**.

Definition 4.1.4: Let $\gamma : [a, b] \rightarrow \mathbb{R}^n$ be a parameterised curve. An oriented reparametrisation of γ is a parameterised curve $\sigma : [c, d] \rightarrow \mathbb{R}^n$ such that $\sigma = \gamma \circ \varphi$ where $\varphi : [c, d] \rightarrow [a, b]$ is a continuous map, differentiable on

$]a, b[$, that is strictly increasing and satisfies $\varphi(a) = c$ and $\varphi(b) = d$.

Proposition 4.1.5: Let γ be a **parameterised curve** in \mathbb{R}^n and σ an **oriented reparametrisation** of γ . Let X be a set containing the image of γ , or equivalently the image of σ , and $f : X \rightarrow \mathbb{R}^n$ continuous. Then we have $\int_\gamma f(s) \cdot d\vec{s} = \int_\sigma f(s) \cdot d\vec{s}$.

Definition 4.1.8: Let $X \subset \mathbb{R}^n$ and $f : X \rightarrow \mathbb{R}^n$ a **continuous vector field**. If for any x_1, x_2 in X , the **line integral** $\int_\gamma f(c) \cdot d\vec{s}$ is **independent of the choice of a parameterised curve** γ in X from x_1 to x_2 , then we say that the vector field is **conservative**.

Equivalently, f is conservative $\Leftrightarrow \int_\gamma f(s) \cdot d\vec{s} = 0$ for any **closed** parameterised curve in X . Where a curve is closed if $\gamma(a) = \gamma(b)$.

Theorem 4.1.10: Let X be an open set and f a **conservative** vector field. Then there exists a C^1 function g on X such that **$f = \nabla g$** . If any two points of X are conjoined by a parameterised curve, then g is unique up to addition of a constant: if $\nabla g_1 = f$, then $g - g_1$ is constant on X .

Any two points x and y on X can be **joined by a parameterised curve** $\Leftrightarrow \exists \gamma : [a, b] \rightarrow X$ such that $\gamma(a) = x$ and $\gamma(b) = y$. Then X is **path connected**. This is also true whenever X is convex. If f is conservative, then a function g such that $\nabla g = f$ is called a **potential for f** .

$$\int_a^b f(\gamma(t)) \cdot \gamma'(t)dt \in \mathbb{R} = g(\gamma(b)) - g(\gamma(a))$$

A set X is **convex** if for any $x, y \in X$, the line segment joining x to y is contained in X .

Proposition 4.1.13: Let $X \subset \mathbb{R}^n$ be an open set and $f : X \rightarrow \mathbb{R}^n$ of class C^1 .

$f(x) = (f_1(x), \dots, f_n(x))$. If f is **conservative**, then we have $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ for any integers with $1 \leq i \neq j \leq n$.

Definition 4.1.15: A subset $X \subset \mathbb{R}^n$ is **star-shaped** if there exists $x_0 \in X$ such that, for all $x \in X$, the **line segment joining x_0 to x is contained in X** . We then say that X is star-shaped around x_0 .

Theorem 4.1.17: Let X be a **star-shaped** open subset of \mathbb{R}^n and f a C^1 vector field such that $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ on X

for all $i \neq j$ between 1 and n . Then the vector field f is **conservative**.

The requirement that X is star-shaped is not necessary. X should have no “hole” in the middle around which a circle can go without it being possible to contract it within X .

Definition 4.1.20: Let $X \subset \mathbb{R}^3$ be an open set and $f : X \rightarrow \mathbb{R}^3$ a C^1 **vector field**. Then the **curl** of f , denoted $\text{curl}(f)$, is the continuous vector field on X defined by

$$\text{curl}(f) = \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix}$$

where $f(x, y, z) = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$.

$$\text{curl}(f) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

with the rule that $\partial_x \cdot f_i = f_i \cdot \partial_x = \partial_x f_i$.

$\text{curl}(\nabla f) = 0$.

The Riemann integral in \mathbb{R}^n

The goal is to define the integral $\int_X f(x_1, \dots, x_n)dx$ for a closed and bounded set X and continuous function f , so that it has analogue properties to the Riemann integral for $n = 1$.

For a bounded and closed subset $X \subset \mathbb{R}^n$ and any continuous function $f : X \rightarrow \mathbb{R}$, one can define the integral of f over X denoted $\int_X f(x)dx$, which is a real number.

The integral satisfies the following properties:

1. **Compatibility:** If $n = 1$ and $X = [a, b]$ with $a \leq b$, then

$$\int_{[a,b]} f(x)dx = \int_a^b f(x)dx$$

2. **Linearity:** If f and g are continuous on X and $a, b \in \mathbb{R}$, then

$$\int_X (af_1(x) + bf_2(x))dx = a \int_X f_1(x)dx + b \int_X f_2(x)dx$$

3. **Positivity:** If $f \leq g$, then

$$\int_X f(x)dx \leq \int_X g(x)dx$$

and especially, if $f \geq 0$, then

$$\int_X f(x)dx \geq 0.$$

Moreover, if $Y \subset X$ is compact and $f \geq 0$, then

$$\int_Y f(x)dx \leq \int_X f(x)dx.$$

4. **Upper bound and triangle inequality:** In particular, since $-|f| \leq f \leq |f|$, we have

$$\left| \int_X f(x)dx \right| \leq \int_X |f(x)|dx,$$

and since $|f + g| \leq |f| + |g|$, we have

$$\left| \int_X (f(x) + g(x))dx \right| \leq \int_X |f(x)|dx + \int_X |g(x)|dx.$$

5. **Volume:** If $f = 1$, then the integral of f is the volume in \mathbb{R}^n of the set X , and if $f \geq 0$ in general, the integral of f is the volume of the set $\{(x, y) \in X \times \mathbb{R} : 0 \leq y \leq f(x)\} \subset \mathbb{R}^{n+1}$. In particular, if X is a bounded rectangle, say $X = [a_1, b_1] \times \dots \times [a_n, b_n] \subset \mathbb{R}^n$ and if $f = 1$, then

$$\int_X dx = (b_n - a_n) \dots (b_1 - a_1).$$

6. **Multiple integral or Fubini's Theorem:** If n_1 and n_2 are integers ≥ 1 such that $n = n_1 + n_2$, then for $x_1 \in \mathbb{R}^{n_1}$, let

$$Y_{x_1} = \{x_2 \in \mathbb{R}^{n_2} : (x_1, x_2) \in X\} \subset \mathbb{R}^{n_2}.$$

Let X_1 be the set of $x_1 \in \mathbb{R}^{n_1}$ such that Y_{x_1} is not empty. Then X_1 is compact in \mathbb{R}^{n_1} and Y_{x_1} is compact in \mathbb{R}^{n_2} for all $x_1 \in X_1$. If the function

$$g(x_1) = \int_{Y_{x_1}} f(x_1, x_2)dx_2$$

on X_1 is continuous, then

$$\int_X f(x_1, x_2)dx = \int_{X_1} g(x_1)dx_1$$

$$= \int_{X_1} \left(\int_{Y_{x_1}} f(x_1, x_2)dx_2 \right) dx_1.$$

Similarly, exchanging the role of x_1 and x_2 , we have

$$\int_X f(x_1, x_2)dx = \int_{X_2} \left(\int_{Z_{x_2}} f(x_1, x_2)dx_1 \right) dx_2$$

where $Z_{x_2} = \{x_1 : (x_1, x_2) \in X\}$, if the integral over x_1 is a continuous function.

If the variables are x_1, \dots, x_n we also write $\int_X f(x_1, \dots, x_n)dx_1, \dots, dx_n$

Definition 4.2.3:

1. Let $1 \leq m \leq n$ be an integer. A parameterised m -set in \mathbb{R}^n is a **continuous map** $f : [a_1, b_1] \times \dots \times [a_m, b_m] \rightarrow \mathbb{R}^n$ which is C^1 on $]a_1, b_1[\times \dots \times]a_m, b_m[$.
2. A subset $B \subset \mathbb{R}^n$ is **negligible** if there exists an integer $k \geq 0$ and parameterised m_i -sets $f_i : X_1 \rightarrow \mathbb{R}^n$, with $1 \leq i \leq k$ and $m_i < n$, such that $X \subset f_1(X_1) \cup \dots \cup f_k(X_k)$.

The **image** of a parameterised m -set f might be of **dimension smaller than m** (e.g. if f is constant, in which case the image is a single point, which is an object of dimension 0)

Proposition 4.2.5: Let $X \subset \mathbb{R}^n$ be compact. Assume that X is **negligible**. Then for any continuous function on X , we have **$\int_X f(x)dx = 0$** .

Improper integrals

Let $I \subset \mathbb{R}$ be a **bounded interval**, $J = [a, +\infty[$ for some $a \in \mathbb{R}$ and f a **continuous function** on $X = J \times I$. We say that it is **Riemann-integrable** on X if the limit

$$\begin{aligned} \lim_{x \rightarrow +\infty} \int_{[a, x] \times I} f(x, y)dx dy &= \lim_{x \rightarrow +\infty} \int_a^x \left(\int_I f(x, y)dy \right) dx \\ &= \lim_{x \rightarrow +\infty} \int_I \left(\int_a^x f(x, y)dx \right) dy \end{aligned}$$

exists. Denote this limit by

$$\int_{J \times I} f(x, y)dx dy.$$

Similarly let f be continuous on \mathbb{R}^2 . Assume that $f \geq 0$. f is Riemann-integrable on \mathbb{R}^2 , if the limit

$$\lim_{R \rightarrow +\infty} \int_{[-R, R]^2} f(x, y)dx dy$$

exists, which is called the integral of f over \mathbb{R}^2 and denoted

$$\int_{\mathbb{R}^2} f(x, y)dx dy.$$

This integral is also the limit of

$$\int_{D_R} f(x, y)dx dy$$

where D_R is the disc of radius R centred at 0.

The easiest way to prove that a certain **improper integral exists**: If $|f| \leq g$ (resp $0 \leq f \leq g$), and

$$\int_{J \times I} g(x, y)dx dy$$

or

$$\int_{\mathbb{R}^2} g(x, y)dx dy$$

exists, then so does

$$\int_{J \times I} f(x, y)dx dy \text{ or } \int_{\mathbb{R}^2} f(x, y)dx dy.$$

The change of variable formula

Let $\bar{X} \subset \mathbb{R}^n$ and $\bar{Y} \subset \mathbb{R}^n$ be **compact subsets**, $\varphi : \bar{X} \rightarrow \bar{Y}$ a **continuous map**. We assume that we can write $\bar{X} = X \cup B$, $\bar{Y} = Y \cup C$ where

1. X and Y are **open**
2. B and C are **negligible**
3. the restriction of φ to the open set X is a C^1 **bijection map from X to Y** .

In this situation, the Jacobian matrix $J_\varphi(x)$ is **invertible at all $x \in X$** .

Theorem 4.4.2: In the situation described above, for any continuous function f on \bar{Y} , we have **$\int_{\bar{X}} f(\varphi) |\det(J_\varphi(x))| dx = \int_{\bar{Y}} f(y) dy$** .

Cartesian to polar coordinates: **$x = r \cos(\theta)$, $y = r \sin(\theta)$** , then calculate all partial derivatives with respect to the new variables:

$\partial_r x = \cos \theta$, $\partial_r y = \sin \theta$, $\partial_\theta x = -r \sin \theta$, $\partial_\theta y = r \cos \theta$. Then **$J_{(r, \theta)} = r$, $dx dy \rightarrow r dr d\theta$** .

Geometric applications of integrals

1. Centre of mass:

Let X be a compact subset of \mathbb{R}^n , such that the volume of X is positive. The centre of mass of X is the point $\bar{x} \in \mathbb{R}^n$ such that $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$ with

$$\bar{x}_i = \frac{1}{\text{vol}(X)} \int_X x_i dx.$$

Intuitively, \bar{x}_i is the average over X of the i -th coordinate, and \bar{x} is the point where X is perfectly balanced.

Note that \bar{x} is not necessarily in X .

2. Surface area:

Consider the continuous function $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ which is C^1 on $]a, b[\times]c, d[$. Let

$$\Gamma = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in [a, b] \times [c, d], z = f(x, y)\}$$

$\subset \mathbb{R}^3$ be the graph of f . This is a surface and should have an area, which is given by

$$\int_a^b \int_c^d \sqrt{1 + (\partial_x f(x, y))^2 + (\partial_y f(x, y))^2} dx dy.$$

3. Length of a curve:

$$f : [a, b] \rightarrow \mathbb{R} : \int_a^b \sqrt{1 + f'(x)^2} dx.$$

The Green formula

The Green formula concerns the case of relating an integral over a subset X of \mathbb{R}^2 with a line integral over its boundary.

Definition 4.6.1: A simple closed parameterised curve $\gamma : [a, b] \rightarrow \mathbb{R}^2$ is a closed parameterised curve such that $\gamma(t) \neq \gamma(s)$ unless $t = s$ or $\{s, t\} = \{a, b\}$, and such that $\gamma'(t) \neq 0$ for $a < t < b$. If γ is only piecewise C^1 inside $]a, b[$, this condition only applies to where $\gamma'(t)$ exists.

Theorem 4.6.3: Let $X \subset \mathbb{R}^2$ compact with a boundary ∂X that is the union of finitely many simple closed parameterised curves $\gamma_1, \dots, \gamma_k$. Assume that $\gamma_i : [a_i, b_i] \rightarrow \mathbb{R}^2$ has the property that X lies always to the left of the tangent vector $\gamma_i'(t)$ based at $\gamma_i(t)$. Let $f = f(f_1, f_2)$ be a

vector field of class C^1 defined on some open set containing X . Then we have

$$\int_X \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx dy = \sum_{i=1}^k \int_{\gamma_i} f \cdot d\vec{s}.$$

Corollary 4.6.5: Let $X \subset \mathbb{R}^2$ compact with a boundary ∂X that is the union of finitely many simple closed parameterised curves $\gamma_1, \dots, \gamma_k$. Assume that $\gamma_i : [a_i, b_i] \rightarrow \mathbb{R}^2$ has the property that X lies always to the left of the tangent vector $\gamma_i'(t)$ based at $\gamma_i(t)$. Then we have

$$\text{Vol}(X) = \sum_{i=1}^k \int_{\gamma_i} x \cdot d\vec{s} = \sum_{i=1}^k \int_{a_i}^{b_i} \gamma_{i,1}(t) \gamma'_{i,2}(t) dt.$$

The Gauss-Ostrogradski formula

This formula is the analogue of the Green formula in \mathbb{R}^3 .

Definition 4.7.1: A parameterised surface $\Sigma : [a, b] \times [c, d] \rightarrow \mathbb{R}^3$ is a 2-set in \mathbb{R}^3 such that the rank of the Jacobian matrix is 2 at all $(s, t) \in]a, b[\times]c, d[$.

Definition 4.7.3: Let x and y be two linearly independent vectors in \mathbb{R}^3 . The vector product, or cross product $z = x \times y$ is the unique vector in \mathbb{R}^3 such that (x, y, z) is a basis of \mathbb{R}^3 with $\det(x, y, z) > 0$, and $\|z\| = \|x\| \|y\| \sin(\theta)$, where θ is the angle between x and y .

$$x \times y = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix} = \det \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$e_1 \times e_2 = e_3, \quad e_2 \times e_3 = e_1, \quad e_3 \times e_1 = e_2 \text{ and } y \times x = -x \times y$$

Theorem 4.7.6: Let $X \subset \mathbb{R}^3$ compact with a boundary ∂X that is a parameterised surface $\Sigma : [a, b] \times [c, d] \rightarrow \mathbb{R}^3$. Assume that Σ is injective in $]a, b[\times]c, d[$, and that Σ has the property that the normal vector \vec{n} points away from Σ at all points. Let $\vec{u} = \frac{\vec{n}}{\|\vec{n}\|}$ be the unit exterior normal vector. Let $f = (f_1, f_2, f_3)$ be a vector field of class C^1 defined on some open set containing X . Then we have

$$\int_X \text{div}(f) dx dy dz = \int_{\Sigma} (f \cdot \vec{u}) d\sigma.$$

For a vector field $f = (f_1, f_2, f_3)$ on $X \subset \mathbb{R}^3$, we denote the divergence of the field $\text{div}(f) = \partial_x f + \partial_y f + \partial_z f$.

For a parameterised surface $\Sigma : [a, b] \times [c, d] \rightarrow \mathbb{R}^3$ with exterior normal vector field $\vec{n} = (n_1, n_2, n_3) = \partial_s \Sigma \times \partial_t \Sigma$

and a function g defined on the image of Σ , we define the surface integral

$$\int_{\Sigma} g \, d\sigma = \int_a^b \int_c^d g(\Sigma(s, t)) \sigma(s, t) ds dt$$

where $\sigma(s, t) = \|\partial_s \Sigma \times \partial_t \Sigma\| = \|\vec{n}(s, t)\|$.

Like the integral for a parameterised curve, the surface integral is independent of the chosen parameterisation of the surface.

For a C^1 vector field $f = (f_1, f_2, f_3)$ on \mathbb{R}^3 , we define

$$\int_{\Sigma} (f \cdot \vec{n}) d\sigma = \int_{\Sigma} g \, d\sigma,$$

where

$$g(\Sigma(s, t)) = f(\Sigma(s, t)) \cdot \vec{u}(s, t) = \sum_{i=1}^3 u_i(s, t) f_i(\Sigma(s, t)).$$

This is called the flux of the vector field f through the surface Σ .

In the flux, $\vec{u}(s, t) \sigma(s, t) = \vec{n}(s, t)$.

Examples and other

Basic rules that you will forget :)

Complex Numbers

$$e^{i\varphi} = \cos(\varphi) + i \sin(\varphi), \quad e^{i\varphi} = \text{cis}(\varphi), \quad |e^{i\varphi}| = 1$$

z	$z = x + iy \begin{cases} \text{x: Real} \\ \text{y: Imaginary} \end{cases}$	$z = r \cdot e^{i\varphi} = r \cdot \text{cis}(\varphi)$
\bar{z}	$\bar{z} = x - iy$	$z = r \cdot e^{-i\varphi}$
$ z $	$ z = \sqrt{z \cdot \bar{z}} = \sqrt{x^2 + y^2}$	$ z = r = \sqrt{z \cdot \bar{z}}$
$z_1 + z_2$	$x_1 + x_2 + i(y_1 + y_2)$	
$z_1 - z_2$	$x_1 - x_2 + i(y_1 - y_2)$	
$z_1 \cdot z_2$	$(x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$	$r_1 \cdot r_2 \cdot e^{i(\varphi_1 + \varphi_2)}$
$\frac{z_1 \cdot \bar{z}_2}{ z_2 ^2}, z_2 \neq 0$	$\frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$	$\frac{r_1}{r_2} \cdot e^{i(\varphi_1 - \varphi_2)}$
$\frac{1}{z}, z \neq 0$	$\frac{\bar{z}}{ z ^2} = \frac{(x - iy)}{x^2 + y^2}$	$\frac{1}{r} \cdot e^{-i\varphi}$

z^n	$r^n \cdot (\cos(n \cdot \varphi) + i \sin(n \cdot \varphi)) = r^n \cdot e^{in\varphi}$
$\sqrt[n]{z}$	$\sqrt[n]{r} \cdot (\cos(\frac{\varphi + 2\pi k}{n}) + i \sin(\frac{\varphi + 2\pi k}{n})), \quad k = 0, 1, \dots, n - 1$

i^0	i^1	i^2	i^3
1	i	-1	$-i$

Abs:

$$|ab| = |a||b| \quad \left| \frac{a}{b} \right| = \frac{|a|}{|b|} \quad |a + b| \leq |a| + |b|$$

Power rules:

$$\begin{aligned}x &= \sqrt[n]{a} \Leftrightarrow (x^n = a \text{ and } x \geq 0) & \sqrt[n]{-a} &= -\sqrt[n]{a}, \quad a \geq 0 \\a^{-n} &= \frac{1}{a^n} = \left(\frac{1}{a}\right)^n & a^{\frac{1}{n}} &= \sqrt[n]{a} & \sqrt{ab} &= \sqrt{a}\sqrt{b} & a^{\frac{m}{n}} &= \sqrt[n]{a^m} \\ \sqrt{\frac{a}{b}} &= \frac{\sqrt{a}}{\sqrt{b}} & a^x &= e^{x \cdot \ln a} & \sqrt[n]{a^{-m}} &= \frac{1}{\sqrt[n]{a^m}} & a^m a^n &= a^{m+n} \\ a^{m+n} & \sqrt[n]{a^m} = \sqrt[n]{a^k} & \frac{a^m}{a^n} &= a^{m-n} & \sqrt[n]{\sqrt[k]{a}} &= \sqrt[nk]{a} \\ (a^m)^n &= a^{mn} & \sqrt[n]{a} \sqrt[n]{b} &= \sqrt[n]{ab} & a^n b^n &= (ab)^n & \frac{\sqrt[n]{a}}{\sqrt[n]{b}} &= \sqrt[n]{\frac{a}{b}} \\ \frac{a^n}{b^n} &= \left(\frac{a}{b}\right)^n\end{aligned}$$

Differentiation:

- **Sum** $(f(x) + g(x))' = f'(x) + g'(x)$
- **Factor** $(c \cdot f(x))' = c \cdot f'(x)$
- **Product** $(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x)$
- **Quotient** $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} \quad (g \neq 0)$
- **Chain Rule** $(f(g(x)))' = (f \circ g)' = f'(g(x))g'(x)$

Integration:

- $\int_a^b (f(x) + / - g(x))dx = \int_a^b f(x)dx + / - \int_a^b g(x)dx$
- $\int_a^b c \cdot f(x)dx = c \cdot \int_a^b f(x)dx$
- $\int_a^b f'(x) \cdot g(x)dx = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x)dx$
- $\int_{\phi(a)}^{\phi(b)} f(x)dx = \int_a^b f(\phi(t))\phi'(t)dt$
- $a + c, b + c \in I \quad \int_a^b f(t + c)dt = \int_{a+c}^{b+c} f(x)dx$
- $ca, cb \in I: \quad \int_a^b f(ct)dt = \frac{1}{c} \int f(x)dx$
- $\int \frac{f'(t)}{f(t)} dt = \log(|f(x)|), \quad \text{bzw.} \quad \int_a^b \frac{f'(t)}{f(t)} dt = \log(f(|b|)) - \log(f(|a|))$
- **Partial fraction:**
$$\frac{2x^6 - 4x^5 + 5x^4 - 3x^3 + x^2 + 3x}{(x-1)^3(x^2+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2}$$
- **Substitution:**
$$\int_a^b f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)du$$

$$\int_{\Omega} f(x, y) dx dy =$$

$$\int_{\Omega} f(g(u, v), h(u, v)) \left| \det \underbrace{\begin{pmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{pmatrix}}_{d\Phi = \nabla \Phi} \right| du dv$$

Coordinate transformations:

Polar Coordinates in \mathbb{R}^2

$$\begin{aligned}x &= r \cos \varphi & 0 \leq r < \infty & & dx dy &= r \cdot dr d\varphi \\ y &= r \sin \varphi & 0 \leq \varphi < 2\pi\end{aligned}$$

Elliptic Coordinates \mathbb{R}^2

$$\begin{aligned}x &= r \cdot a \cos \varphi & 0 \leq r < \infty & & dx dy &= a \cdot b \cdot r \cdot dr d\varphi \\ y &= r \cdot b \sin \varphi & 0 \leq \varphi < 2\pi\end{aligned}$$

Cylinder Coordinates \mathbb{R}^3

$$\begin{aligned}x &= r \cdot a \cos \varphi & 0 \leq r < \infty & & dx dy dz &= r \cdot dr d\varphi dz \\ y &= r \cdot b \sin \varphi & 0 \leq \varphi < 2\pi \\ z &= z & \infty \leq z < \infty\end{aligned}$$

Sphere Coordinates \mathbb{R}^3

$$\begin{aligned}x &= r \cdot \sin \theta \cos \varphi & 0 \leq r < \infty & & dx dy dz &= r^2 \sin \theta \cdot dr d\theta d\varphi \\ y &= r \cdot \sin \theta \sin \varphi & 0 \leq \theta < \pi \\ z &= r \cos \theta & 0 \leq \varphi < 2\pi\end{aligned}$$

Limits:

- $\lim_{x \rightarrow 0} \arctan(x) = 0, \quad \lim_{x \rightarrow \infty} \arctan(x) = \frac{\pi}{2}$
- $\lim_{x \rightarrow 0} \tan(x) = 0, \quad \lim_{x \rightarrow \infty} \tan(x) = \infty, \quad \lim_{x \rightarrow \frac{\pi}{2}} \tan(x) = \infty$
- $\lim_{x \rightarrow \infty} \cos(x) = [-1, 1], \quad \lim_{x \rightarrow \infty} \sin(x) = [-1, 1]$
- $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \quad \lim_{n \rightarrow \infty} \left(1 + n\right)^{\frac{1}{n}} = e$
- $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln(a) \quad \lim_{x \rightarrow 0} \frac{\log_a(1+x)}{x} = \frac{1}{\ln(a)} \quad \lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x} = 0$
- $\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x^2} = \frac{1}{2} \quad \lim_{x \rightarrow 0} \frac{\tan(x)}{x} = 1 \quad \lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1 = \frac{x}{\sin(x)}$
- $\lim_{x \rightarrow \infty} \frac{n!}{n^n} = 0 \quad \lim_{x \rightarrow 0} \frac{e^n - 1}{n} = 1 \quad \lim_{x \rightarrow \infty} \sqrt[n]{n!} = \infty \quad \lim_{x \rightarrow \infty} \sqrt[n]{n} = 1$

Obvious cases:

$$\frac{1}{0} = \infty \quad \frac{1}{\infty} = 0 \quad \infty + \infty = \infty \quad 0 + \infty = \infty \quad 0^\infty = 0 \quad \infty^\infty = \infty$$

Sums:

- $\sum_{k=0}^{\infty} aq^k = a + aq + aq^2 + \dots = \frac{a}{1-q}$ (Geometric Series)

- $\sum_{k=0}^{\infty} (k+1)q^k = 1 + 2q + 3q^2 + \dots = \frac{1}{(1-q)^2}, \quad |q| < 1$
- $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$
- $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$
- $\zeta(a) = \sum_{k=0}^{\infty} \frac{1}{k^a}$ ist konvergent $\Leftrightarrow a > 1$
- $\sum_{k=0}^{\infty} \frac{1}{k!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots = e$
- $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots = \frac{1}{e}$
- $\frac{1}{1 \pm x} = 1 \mp x + x^2 \mp x^3 + x^4 \mp \dots$
- $\frac{1}{(1 \pm x)^2} = 1 \mp 2x + 3x^2 \mp 4x^3 + 5x^4 \mp \dots$
- $\sqrt{1 \pm x} = 1 \pm \frac{x}{2} - \frac{1 \cdot 1}{2 \cdot 4} x^2 \pm \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} x^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} x^4 \pm \dots$
- $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
- $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$
- $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
- $\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$
- $\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$
- $\tan(x) = 1 + \frac{\phi^3}{3} + \frac{2\phi^5}{15} + \dots$
- $\tanh(z) = 1 - \frac{z^3}{3} + \frac{2z^5}{15} - \dots$
- $\ln(1+z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} z^k = z - \frac{z^2}{2} + \frac{z^3}{3} + \dots$
- $(1+z)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!} z^2 + \dots$

Random:

- Circle formula: $(x - x_0)^2 + (y - y_0)^2 = r^2$
- Ellipse formula: $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$
- Quadratic formula: $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
- Matrix is invertible iff. the Determinant $\neq 0$.
- Scalar Product: $x \cdot y = \sum_{i=1}^n x_i y_i$
- Crosss Product: $a \times b = (a_2 b_3 - a_3 b_2, \quad a_3 b_1 - a_1 b_3, \quad a_1 b_2 - a_2 b_1)^\top$

Let L be a **continuous linear function**, then if L is continuous at some point x_0 , it is continuous at every point.

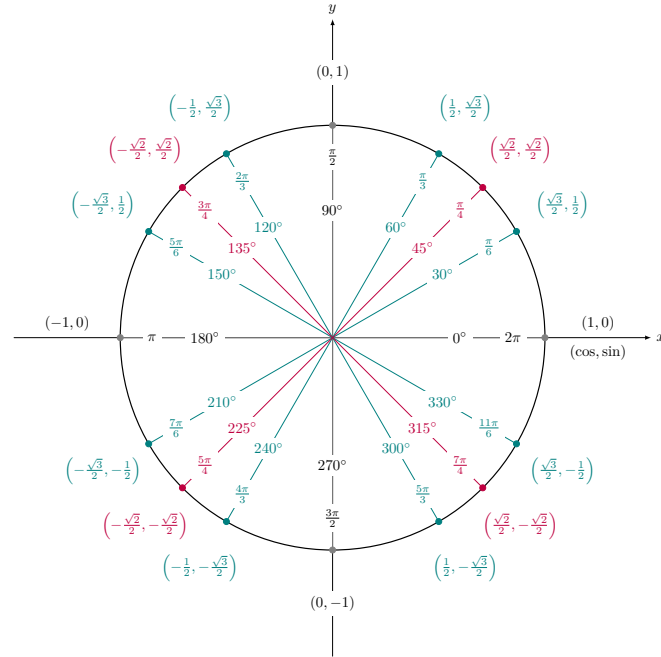
$$\lim_{x \rightarrow 0} ||L(a+x) - L(a)|| = \lim_{x \rightarrow 0} ||L(a) + L(x) - L(a)|| = \lim_{x \rightarrow 0} ||L(x)|| = \lim_{x \rightarrow 0} ||L(x_0 + x) - L(x_0)|| = 0.$$

Finding limits:

1. Different paths
2. If f is continuous then $\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x))$
3. Squeeze theorem
4. Taylor expansion
5. Polar coordinates:

$$x = r \cos(\phi), y = r \sin(\phi), \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{4x^2 + y^6} = \lim_{r \rightarrow 0} \frac{r^2}{4r^2 \cos^4(\phi) + r^6 \sin^6(\phi)}$$

Sin and Cos



Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°
φ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(\varphi)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\cos(\varphi)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\tan(\varphi)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\pm\infty$	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0

$$\sin(x) = x + o(x)$$

$$\sin(x+y) \sin(x-y) = \cos^2(y) - \cos^2(x) = \sin^2(x) - \sin^2(y)$$

$$\cos(x+y) \cos(x-y) = \cos^2(y) - \sin^2(x) = \cos^2(x) - \sin^2(y)$$

$$\sin x \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$$

$$\cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$$

$$\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$$

$$\cos(x)^2 + \sin(x)^2 = 1$$

$$\cos(\pi - x) = -\cos(x), \sin(\pi - x) = \sin(x)$$

$$\cos(x + \pi) = -\cos(x), \sin(x + \pi) = -\sin(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1$$

$$\sin(2x) = 2\sin(x) \cos(x) \quad \tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$$

$$\sin\left(\frac{x}{2}\right) = \sqrt{\frac{1 - \cos(x)}{2}} \quad \cos\left(\frac{x}{2}\right) = \sqrt{\frac{1 + \cos(x)}{2}}$$

$$\tan\left(\frac{x}{2}\right) = \frac{1 - \cos(x)}{\sin(x)} = \frac{\sin(x)}{1 + \cos(x)} \quad \cot\left(\frac{x}{2}\right) = \frac{1 + \cos(x)}{\sin(x)} = \frac{\sin(x)}{1 - \cos(x)}$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2} \quad \cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$\tan(\pi + x) = \tan(x)$$

$$-\sin(-x) = \sin(x), \cos(-x) = \cos(x), \tan(-x) = -\tan(x)$$

$$\text{For all } (a, b) \in \mathbb{R}^2, \text{ such that } a^2 + b^2 = 1, \text{ there is } x \in \mathbb{R}, \text{ such that } a = \cos(x), b = \sin(x).$$

$$\sin(x) = \frac{2\tan(x/2)}{1 + \tan^2(x/2)} \quad \cos(x) = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \quad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\int_0^{2\pi} \sin(t) \cdot \cos(t) dt = \int_0^{2\pi} \sin(t) dt = \int_0^{2\pi} \cos(t) dt = 0$$

$$\int \sin^2(x) dx = \frac{1}{2}(x - \sin(x) \cos(x))$$

$$\int \cos^2(x) dx = \frac{1}{2}(x + \sin(x) \cos(x))$$

$$\int x \sin(x) dx = \sin(x) - x \cos(x)$$

$$\int x \cos(x) dx = x \sin(x) + \cos(x)$$

$$\sin(\arccos(x)) = \sqrt{1 - x^2} \quad \sin(\arctan(x)) = \frac{x}{\sqrt{1 + x^2}}$$

$$\cos(\arctan(x)) = \frac{1}{\sqrt{1 + x^2}} \quad \cos(\arcsin(x)) = \sqrt{1 - x^2}$$

$$\tan(\arcsin(x)) = \frac{x}{\sqrt{1 - x^2}} \quad \tan(\arccos(x)) = \frac{\sqrt{1 - x^2}}{x}$$

$$\cosh(x) := \frac{e^x + e^{-x}}{2} \quad \sinh x := \frac{e^x - e^{-x}}{2} \quad \tanh x := \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\sinh(z) = -\sinh(-z) \quad \cosh(z) = \cosh(-z)$$

$$\sinh(z) = \sinh(z + 2\pi i) \quad \cosh(z) = \cosh(z + 2\pi i)$$

$$\sinh(z_1 \pm z_2) = \sinh(z_1) \cdot \cosh(z_2) \pm \sinh(z_2) \cdot \cosh(z_1)$$

$$\sinh(z_1 \pm z_2) = \cosh(z_1) \cdot \cosh(z_2) \pm \sinh(z_2) \cdot \sinh(z_1)$$

$$\tanh(z_1 \pm z_2) = \frac{\tanh(z_1) \pm \tanh(z_2)}{1 \pm \tanh(z_1) \cdot \tanh(z_2)}$$

Prove if f is differentiable at x_0

f continuous at x_0 ? No \Rightarrow Not differentiable.

\Downarrow Yes

Is f partially differentiable at x_0 , does $\partial_{x_i} f(x_0)$ exist for all i ? No \Rightarrow Not differentiable.

\Downarrow Yes

Are all partial derivatives continuous at x_0 ? Yes $\Rightarrow f$ is differentiable at x_0 with $df(x_0) = J_f(x_0)$.

\Downarrow No

Does a linear mapping u exist, such that $\lim_{\substack{x \rightarrow x_0 \\ x \neq 0}} \frac{f(x) - f(x_0) - u(x - x_0)}{\|x - x_0\|} = 0$?

Yes $\Rightarrow f$ is differentiable at x_0 .

No $\Rightarrow f$ is not differentiable at x_0 *cries*.

Primitives and Derivatives

$$\int (ax + b)^s dx = \frac{1}{a(s+1)}(ax + b)^{s+1} + C, s \neq -1$$

$$\int \frac{1}{ax + b} dx = \frac{1}{a} \log|ax + b| + C$$

$$\int (ax^p + b)^s x^{p-1} dx = \frac{(ax^p + b)^{s+1}}{ap(s+1)} + C, s \neq -1, a \neq 0$$

$$\int (ax^p + b)^{-1} x^{p-1} dx = \frac{1}{ap} \log|ax^p + b| + C, a \neq 0, p \neq 0$$

$$\int \frac{ax + b}{cx + d} dx = \frac{ax}{c} - \frac{ad - bc}{c^2} \log|cx + d| + C$$

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log\left|\frac{x - a}{x + a}\right|$$

$$\int \sqrt{a^2 + x^2} dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \log(x + \sqrt{a^2 + x^2}) + C$$

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin\left(\frac{x}{|a|}\right) + C$$

$$\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log|x + \sqrt{x^2 - a^2}| + C$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \log(x + \sqrt{a^2 + x^2}) + C$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \log|x + \sqrt{x^2 - a^2}| + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin\left(\frac{x}{|a|}\right) + C$$

$$\int e^{kx} dx = \frac{1}{k} e^{kx} + C$$

$$\int a^{kx} dx = \frac{1}{k * \log(a)} a^{kx} + C$$

$$\int e^{ax} p(x) dx = e^{ax} (a^{-1} p(x) - a^{-2} p'(x) + a^{-3} p''(x) - \dots + (-1)^n a^{-n-1} p^{(n)}(x)) + C, a \neq 0, p: \text{Polynome of degree } n$$

$$\int e^{kx} \sin(ax+b) dx = \frac{e^{kx}}{a^2 + k^2} \left(k \sin(ax+b) - a \cos(ax+b) \right) + C$$

$$\int e^{kx} \cos(ax+b) dx = \frac{e^{kx}}{a^2 + k^2} \left(k \cos(ax+b) + a \sin(ax+b) \right) + C$$

$$\int \log|x| dx = x(\log|x| - 1) + C$$

$$\int x^k \log(x) dx = \frac{x^{k+1}}{k+1} \left(\log(x) - \frac{1}{k+1} \right) + C, k \neq -1$$

$$\int x^{-1} \log(x) dx = \frac{1}{2} (\log(x))^2 + C$$

$$\int \sin(ax+b) dx = -\frac{1}{a} \cos(ax+b) + C$$

$$\int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b) + C$$

$$\int \sin^2(x) dx = \frac{1}{2} (x - \sin(x) \cos(x)) + C$$

$$\int \cos^2(x) dx = \frac{1}{2} (x + \sin(x) \cos(x)) + C$$

$$\int \tan^2(x) dx = \tan(x) - x + C$$

$$\int \frac{1}{\sin(x)} dx = \log \left| \tan \frac{x}{2} \right| + C$$

$$\int \frac{1}{\cos(x)} dx = \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{4} \right) \right| + C$$

$$\int \frac{1}{\tan(x)} dx = \log |\sin(x)| + C$$

$$\int_0^{2\pi} \sin(mx) \cos(nx) dx = 0, m, n \in \mathbb{Z}$$

$$\int_0^\infty \frac{\sin(ax)}{x} dx = \frac{\pi}{2}, a > 0$$

$$\int_0^\infty \sin(x^2) dx = \int_0^\infty \cos(x^2) dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}$$

$$\int_0^\infty e^{-ax} x^n dx = \frac{n!}{a^{n+1}}, a > 0$$

$$\int_0^\infty e^{-ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}}, a > 0$$

Differentiable \implies **Continuous** \implies **Integrable**

$f'(x)$	$f(x)$	$F(x)$
0	$c \ (c \in \mathbb{R})$	cx
c	cx	$\frac{c}{2} x^2$
$r \cdot x^{r-1}$	$x^r \ (r \in \mathbb{R} \setminus \{-1\})$	$\frac{x^{r+1}}{r+1}$
$\frac{-1}{x^2} = -x^{-2}$	$\frac{1}{x} = x^{-1}$	$\log x $
$\frac{1}{2\sqrt{x}} = -x^{-2}$	$\sqrt{x} = x^{\frac{1}{2}}$	$\frac{2}{3} x^{\frac{3}{2}}$
$\cos x$	$\sin x$	$-\cos x$
$-\sin x$	$\cos x$	$\sin x$
$1 + \tan^2 x = \frac{1}{\cos^2 x}$	$\tan x$	$-\log \cos x $
$-\frac{1}{\sin^2(x)}$	$\cot x$	$\log \sin x $
e^x	e^x	e^x
$c \cdot e^{cx}$	e^{cx}	$\frac{1}{c} e^{cx}$
$\log a \cdot a^x$	a^x	$\frac{a^x}{\log a}$
$\frac{1}{x}$	$\log x $	$x(\log x - 1)$
$\frac{1}{\log a \cdot x}$	$\log_a x $	$\frac{x}{\log a} (\log x - 1)$ $= x(\log_a x - \log_a e)$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$	$x \arcsin x + \sqrt{1-x^2}$
$-\frac{1}{\sqrt{1-x^2}}$	$\arccos x$	$x \arccos x - \sqrt{1-x^2}$
$\frac{1}{1+x^2}$	$\arctan x$	$x \arctan x - \frac{1}{2} \log(1+x^2)$
$\sinh(x)$	$\cosh(x)$	-
$\cosh(x)$	$\sinh(x)$	-
$\frac{1}{\cosh^2(x)}$	$\tanh(x)$	$\log(\cosh(x))$
$2 \sin(x) \cos(x)$	$\sin^2(x)$	$\frac{1}{2} (x - \sin(x) \cos(x))$
$-2 \sin(x) \cos(x)$	$\cos^2(x)$	$\frac{1}{2} (x + \sin(x) \cos(x))$
$\frac{2 \sin(x)}{\cos^3(x)}$	$\tan^2(x)$	$\tan(x) - x$
$\frac{1}{\sqrt{x^2+1}}$	$\operatorname{arsinh} x$	$x \operatorname{arsinh} x - \sqrt{x^2+1}$
$\frac{1}{\sqrt{x^2-1}} \ (x > 1)$	$\operatorname{arcosh} x$	$x \operatorname{arcosh} x - \sqrt{x^2-1}$
$\frac{1}{1-x^2} \ (x < 1)$	$\operatorname{artanh} x$	$x \operatorname{artanh} x + \frac{1}{2} \ln(1-x^2)$
$\frac{1}{1-x^2} \ (x > 1)$	$\operatorname{arcoth} x$	$x \operatorname{arcoth} x + \frac{1}{2} \ln(x^2-1)$