PHY-685
QFT-1

Lecture - 16

In the last lecture, we worked with a toy theory of two real scalar fields ϕ_1 and ϕ_2, with equal mass m. We found that there is a one parameter global internal symmetry, under which the Lagrangian density is invariant.

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_a \partial^\mu \phi_a - m^2 \phi_a \phi_a\right).$$

This Lagrangian density has full $O(2)$ invariance, which includes reflections. This is true even if we add some interaction terms like $g(\phi_a \phi_a)^n$.

Since reflections about arbitrary planes can always be decomposed into reflection about some fixed plane and a rotation. So in the spaces of the fields lets consider the following reflection:

$\phi_1 \to \phi_1$ and $\phi_2 \to -\phi_2$. Does there exist an unitary operator $\hat{U}$ acting on the Hilbert space serve this purpose, such that:

$$\hat{U}^\dagger \hat{\phi}_1 \hat{U} = \phi_1 \text{ and } \hat{U}^\dagger \hat{\phi}_2 \hat{U} = -\phi_2 \text{ ?}$$

Using the mode expansion of the fields $\hat{\phi}_1$ and $\hat{\phi}_2$, we get these criteria should be equivalent to:

$$\hat{a}_1(\vec{p}) \longrightarrow \hat{U}^\dagger \hat{a}_1(\vec{p}) \hat{U} = \hat{a}_1(\vec{p}) \text{ and } \hat{a}_2(\vec{p}) \to \hat{U}^\dagger \hat{a}_2(\vec{p}) \hat{U} = -\hat{a}_2(\vec{p}).$$

$$\hat{a}_1^\dagger(\vec{p}) \longrightarrow \hat{U}^\dagger \hat{a}_1^\dagger(\vec{p}) \hat{U} = \hat{a}_1^\dagger(\vec{p}) \text{ and } \hat{a}_2^\dagger(\vec{p}) \to \hat{U}^\dagger \hat{a}_2^\dagger(\vec{p}) \hat{U} = -\hat{a}_2^\dagger(\vec{p}).$$

In this picture, the operators with definite charges are:

$$\hat{\psi} = 1/\sqrt{2}\left(\hat{\phi}_1 + i\hat{\phi}_2\right) \text{ and } \hat{\psi}^\dagger = 1/\sqrt{2}\left(\hat{\phi}_1 - i\hat{\phi}_2\right).$$

Under the action of the charge-conjugation operation:

$\hat{\psi} \longrightarrow \hat{\psi}^\dagger$ and $\hat{\psi}^\dagger \longrightarrow \hat{\psi}$. This can be equivalently put in the language: $\hat{b}(\vec{p}) = 1/\sqrt{2}\left(\hat{a}_1(\vec{p}) + i\,\hat{a}_2(\vec{p})\right) \longrightarrow 1/\sqrt{2}\left(\hat{a}_1(\vec{p}) - i\,\hat{a}_2(\vec{p})\right) = \hat{c}(\vec{p})$

$$\hat{U}^\dagger \hat{Q} \hat{U} = -\hat{Q}.$$

Applying charge conjugation operator, one can convert a "b-type" particle into a "c-type" particle.

$$\boxed{\hat{\psi}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[\hat{b}(\vec{p})\, e^{-ip\cdot x} + \hat{c}^\dagger(\vec{p})\, e^{ip\cdot x}\right]}$$

When $\hat{\phi}(x)$ is real, the interpretation of $\hat{\phi}(\vec{x})|\Omega\rangle$ is that it is a position eigen-state $|\vec{x}\rangle$, as we can compute

$$\langle \Omega | \hat{\phi}(\vec{x}) | \vec{p}\rangle = e^{i\vec{p}\cdot\vec{x}}.$$

Causality Again:

We recall that we went to a multi-particle picture in the case of a relativistic free particle's time evolution, in order to retain the notion of causality.

We have seen that for a real free scalar field, acting on the vacuum, creates a position eigen-state: $|x\rangle = \hat{\phi}(x)|\Omega\rangle$.

The amplitude of the state propagating from y to x is then

$$D(x-y) = \langle x|y\rangle = \langle\Omega|\hat{\phi}(x)\,\hat{\phi}(y)|\Omega\rangle$$

$$= \langle\Omega|\left[\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(\hat{a}(\vec{p})\,e^{-ip\cdot x} + \hat{a}^\dagger(\vec{p})\,e^{ip\cdot x}\right)\right]$$

$$\left[\int\frac{d^3\vec{p}'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}'}}}\left(\hat{a}(\vec{p}')\,e^{-ip'\cdot y} + \hat{a}^\dagger(\vec{p}')\,e^{ip'\cdot y}\right)\right]|\Omega\rangle$$

$$= \langle\Omega|\left[\iint\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\cdot\frac{d^3\vec{p}'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}'}}}\,\hat{a}(\vec{p})\,\hat{a}^\dagger(\vec{p}')\,e^{-ip\cdot x + ip'\cdot y}\right]|\Omega\rangle$$

$$= \langle\Omega|\left[\iint\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\cdot\frac{d^3\vec{p}'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}'}}}\left[\hat{a}(\vec{p}),\hat{a}^\dagger(\vec{p}')\right]e^{-ip\cdot x + ip'\cdot y}\right]|\Omega\rangle$$

$$= \langle\Omega|\left[\iint\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\cdot\frac{d^3\vec{p}'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}'}}}\left((2\pi)^3\delta^3(\vec{p}-\vec{p}')\right)e^{-ip\cdot x + ip'\cdot y}\right]|\Omega\rangle$$

$$= \int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\,e^{-ip\cdot(x-y)} = \int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\exp\left[-iE_p\left(x^0-y^0\right)\right]\times\exp\left[+i\vec{p}\cdot(\vec{x}-\vec{y})\right]$$

where $E_{\vec{p}} = \sqrt{\vec{p}^{\,2}+m^2}$. We put $x^0-y^0 = t$ and $\vec{x}-\vec{y} = \vec{r}$.

$D(x-y)$ by construction is Lorentz invariant amplitude.

Amazingly, the structure of the integral is similar to the first attempt of computing unitary time evolution amplitude. We just have the correct Lorentz-Invariant Phase-Space (LIPS) integral.

If we proceed in similar way of analytic continuation, we get:

$$D(x-y) = -\frac{i}{4\pi^2 r}\int_m^\infty d\rho\,\frac{\rho\exp(-\rho r)}{\sqrt{\rho^2-m^2}}\left[\exp\left(\sqrt{\rho^2-m^2}\,t\right) - \exp\left(-\sqrt{\rho^2-m^2}\cdot t\right)\right]$$

This will give that outside light-cone the propagation amplitude is exponentially small but non-vanishing.

The real question of causality is can measurement at two distinct point x and y can affect each other even when they are space-like separated?

To establish causality, the correct observable to consider should be $\langle \Omega | [\hat{\phi}(x), \hat{\phi}(y)] | \Omega \rangle$.

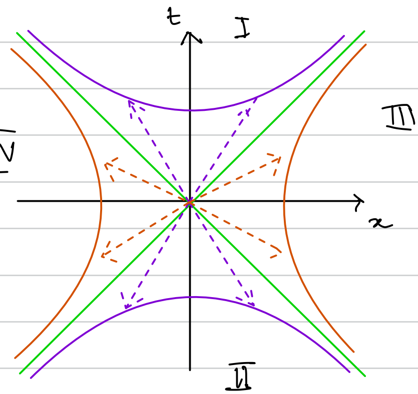
$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right]$$

$$= \underbrace{D(x-y)}_{} - \underbrace{D(y-x)}_{}$$

Each of these two pieces are separately Lorentz invariant.

Causality demands that the above quantity must vanish if $(x-y)$ is a space-like separation.

To prove this, first we look back at basic properties of Lorentz transformations:



The 1+1-D case:

The points in region I and II, which are time-like connected to origin, a point (t, x) can be taken to $(t, -x)$ via a L.T.

In III and IV a point (t, x) can be taken to $(-t, x)$.

But in none of the regions at point can be taken from (t, x) to $(-t, -x)$

For 2+1-D or higher:

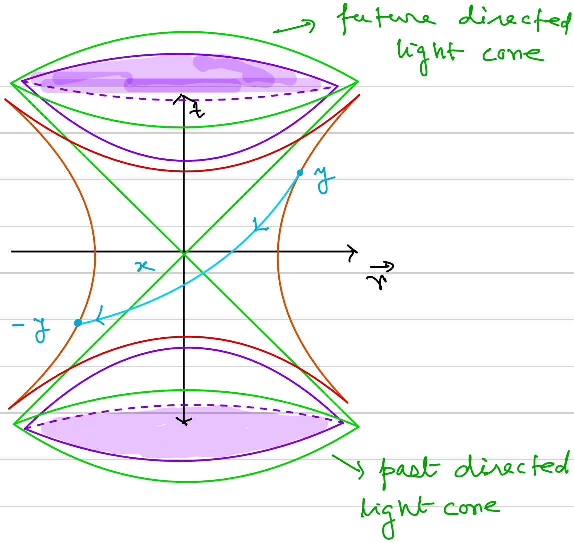
The space-like points are on a hyperboloid. So a point with $(t, \vec{x})$ can be taken to $(-t, -\vec{x})$ if it is space-like separated. So $\langle \Omega | [\hat{\phi}(x), \hat{\phi}(y)] | \Omega \rangle = 0$.

For complex ϕ, $\hat{\phi}(x)|\Omega\rangle$ will create +vely charged particle at x and destroy -vely charged particle at x. $\hat{\phi}(x)$ will do the opposite $\langle \Omega | \hat{\phi}(x) \hat{\phi}^\dagger(y) | \Omega \rangle - \langle \Omega | \hat{\phi}^\dagger(y) \hat{\phi}(x) | \Omega \rangle$

propagation of -vely ch particle from y to x

propagation of +vely ch particle from x to y

For (D+1) dimensions, where D > 1, the points y, which are within the future directed lightcone of the point x, always moves on a hyperboloid $(x-y)^2 > 0$. However, via a proper orthocronus transformation, the point y can never be taken somewhere in the past oriented light cone of x.



When $(y-x)^2 < 0$, i.e. they are space-like separated, the point y can be taken to its anti-podal position, w.r.t. x via a continuous Lorentz transformation. (Shown by the blue line in the figure).

Hence we can identify $D(y-x)$ with $D(x-y)$.

For the case of complex scalar field we have the charge conjugated particles, viz. anti-particles.

So $\langle \Omega | [\hat{\phi}(x), \hat{\phi}^\dagger(y)] | \Omega \rangle = D(x-y) - D^*(y-x)$

$$= \sum_p \left(\underbrace{(x \text{-----} \leftarrow p \text{-----} y)}_{\text{particle}} - \underbrace{(x \text{-----} p \rightarrow \text{-----} y)}_{\text{anti-particle}} \right)$$

Causality demands the existance of anti-particles. Both the particle and anti-particle have to have exactly the same mass.