PHY-685

QFT-1

Lecture-8

We continue our discussion on unitary representation of little group of the Poincaré group.

Let's recall : $|\psi(p,\sigma)\rangle \equiv N(p)\,\hat{U}(L(p))\,|\psi(k,\sigma)\rangle$

and $\hat{U}(\Lambda)|\psi(p,\sigma)\rangle = \dfrac{N(p)}{N(\Lambda p)} \sum_{\sigma'} D_{\sigma'\sigma}\big(W(\Lambda,p)\big)|\psi(\Lambda p,\sigma')\rangle$

Now we want to study $\langle\psi(p',\sigma')|\psi(p,\sigma)\rangle$ in order to fix the normalization :

$$= \langle\psi(p',\sigma')|\Big(N(p)\,\hat{U}(L(p))\,|\psi(k,\sigma)\rangle\Big)$$

$$= N(p)\Big(\langle\psi(p',\sigma')|\{\hat{U}^{-1}(L(p))\}^{\dagger}\Big)|\psi(k,\sigma)\rangle$$

Now using the relation

$$\hat{U}(\Lambda)|\psi(p,\sigma)\rangle = N(p)\sum_{\sigma'} D_{\sigma'\sigma}\big(W(\Lambda,p)\big)\,\hat{U}(L(\Lambda p))|\psi(k,\sigma')\rangle$$

Replacing p by p', σ by σ' and Λ with $L^{-1}(p)$, we get

$$\hat{U}^{-1}(L(p))|\psi(p',\sigma')\rangle = N(p')\sum_{\sigma''} D_{\sigma''\sigma'}\big(W(L^{-1}(p),p')\big)$$
$$\times\,\hat{U}\big(L\cdot(\underbrace{L^{-1}(p)\cdot p'}_{=k'})\big)|\psi(k',\sigma'')\rangle$$

$$= N(p')\sum_{\sigma''} D_{\sigma''\sigma'}\big(W(L^{-1}(p),p')\big)\sum_{\sigma_2''} D_{\sigma_2''\sigma''}\big(W(L^{-1}(p),p'\big)$$
$$|\psi(k',\sigma_2'')\rangle$$

$$= N(p')\sum_{\sigma_2''} D_{\sigma_2''\sigma'}\big(W(L^{-1}(p),p'\big)|\psi(k',\sigma_2'')\rangle$$

$$\Rightarrow \langle\psi(p',\sigma')|\{\hat{U}^{-1}(L(p))\}^{\dagger}$$

$$= N^{*}(p')\sum_{\sigma_2''} D^{*}_{\sigma_2''\sigma'}\big(W(L^{-1}(p),p')\big)\langle\psi(k',\sigma_2'')|$$

$$\langle\psi(p',\sigma')|\psi(p,\sigma)\rangle = N(p)N^{*}(p')\sum_{\sigma_2''} D^{*}_{\sigma_2''\sigma'}\big(W(L^{-1}(p),p')\big)$$
$$\times\langle\psi(k',\sigma_2'')|\psi(k,\sigma)\rangle$$

$\Rightarrow \langle \psi(p', \sigma') | \psi(p, \sigma) \rangle$

$$= N(p)\, N^*(p') \sum_{\sigma_2''} D^*_{\sigma_2'' \sigma'}\left(W(L^{-1}(p), p')\right) \times \delta^3(\vec{k}' - \vec{k})\, \delta_{\sigma_2'' \sigma}$$

$$= N(p)\, N^*(p')\, D^*_{\sigma\sigma'}\left(W(L^{-1}(p), p')\right) \delta^3(\vec{k}' - \vec{k})$$

Now $k' - k = L^{-1}(p)(p' - p) \Rightarrow \delta^3(\vec{k}' - \vec{k}) \propto \delta^3(\vec{p}' - \vec{p})$

For $p' = p$

$$\langle \psi(p', \sigma') | \psi(p, \sigma) \rangle$$
$$= |N(p)|^2 D^*_{\sigma\sigma'}\left(\underbrace{W(L^{-1}(p), p)}_{\mathbb{1}_{4\times 4}}\right) \delta^3(\vec{k}' - \vec{k})$$
$$= |N(p)|^2 \delta_{\sigma\sigma'}\, \delta^3(\vec{k}' - \vec{k})$$

To find out the proportionality constant we notice the following fact: For any function $f(p)$,

$$\int_{-\infty}^{\infty} d^4p\, \delta(p^2 - m^2)\, \theta(p^0)\, f(p)$$
$$= \int_{-\infty}^{\infty} d^3\vec{p}\, dp^0\, \delta\Big(\underbrace{(p^0)^2 - \vec{p}^2 - m^2}_{g(p^0)}\Big)\, \theta(p^0)\, f(p^0, \vec{p})$$

Now $\delta(g(x)) = \sum_{\bar{x}} \frac{\delta(x - \bar{x})}{|g'(x)|_{x = \bar{x}}}$, where $\bar{x}$ are the solutions of $g(x) = 0$. Here $g(p^0)$ has two roots at:

$$p^0_{\pm} = \pm\sqrt{\vec{p}^2 + m^2}$$

Also $g'(p^0) = 2p^0$. Hence $\delta(g(p^0))$

$$= \left[\frac{\delta(p^0 - \sqrt{\vec{p}^2 + m^2})}{2\sqrt{\vec{p}^2 + m^2}} + \frac{\delta(p^0 + \sqrt{\vec{p}^2 + m^2})}{2\sqrt{\vec{p}^2 + m^2}} \right]$$

Hence $\int_{-\infty}^{\infty} d^4p \; \delta(p^2-m^2)\,\theta(p^0)\, f(p^0,\vec{p})$

$$= \int_{-\infty}^{\infty} d^3\vec{p} \left[\int_{-\infty}^{\infty} dp^0 \left\{ \frac{\delta(p^0 - \sqrt{\vec{p}^2+m^2})}{2\sqrt{\vec{p}^2+m^2}} + \frac{\delta(p^0 + \sqrt{\vec{p}^2+m^2})}{2\sqrt{\vec{p}^2+m^2}} \right\} \times \theta(p^0)\, f(p^0,\vec{p}) \right]$$

$$= \int_{-\infty}^{\infty} d^3\vec{p} \; \frac{1}{2\sqrt{\vec{p}^2+m^2}} \; f\left(\sqrt{\vec{p}^2+m^2}, \vec{p}\right)$$

$$= \int_{-\infty}^{\infty} \frac{d^3\vec{p}}{2E_{\vec{p}}} \; f(E_{\vec{p}}, \vec{p}) \qquad \text{where } E_{\vec{p}} = \sqrt{\vec{p}^2+m^2}.$$

When we try to do a phase space integration "on-shell", then the Lorentz Invariant Phase Space (LIPS) measure is $d^3\vec{p}/2E_{\vec{p}}$

The "delta-function" is given by

$$F(\vec{p}) = \int F(\vec{p}\,')\, \delta^3(\vec{p}-\vec{p}\,')\, d^3\vec{p}\,'$$

$$= \int F(\vec{p}\,') \left(2E_{\vec{p}\,'}\, \delta^3(\vec{p}-\vec{p}\,')\right) \frac{d^3\vec{p}\,'}{2E_{\vec{p}\,'}}$$

So the invariant delta-function is $\sqrt{\vec{p}^2+m^2}\; \delta^3(\vec{p}-\vec{p}\,')$

Hence $p^0\, \delta^3(\vec{p}\,'-\vec{p}) = k^0\, \delta^3(\vec{k}\,'-\vec{k})$ and therefore,

$$\langle \Psi(p',\sigma') | \Psi(p,\sigma) \rangle = |N(p)|^2\, \delta_{\sigma'\sigma} \left(\frac{p^0}{k^0}\right) \delta^3(\vec{p}\,'-\vec{p}).$$

If we set the normalization $N(p) = \sqrt{k^0/p^0}$. Then we have

$$\langle \Psi(p',\sigma') | \Psi(p,\sigma) \rangle = \delta_{\sigma'\sigma}\, \delta^3(\vec{p}\,'-\vec{p}).$$

Now we discuss the transformation of massive and massless states under Wigner rotations.