PHY-685

QFT-1

Lecture - 21

We learned that the S-matrix operator can be constructed as : $\hat{S} = \hat{U}_I(+\infty, -\infty)$, where

$$\hat{U}_I(t_B, t_A) = \exp(i\hat{H}_0 t_B)\, e^{-i\hat{H}(t_B - t_A)} \exp(-i\hat{H}_0 t_A)$$

$$= \hat{T} \exp\left[-i \int_{t_A}^{t_B} dt\, \hat{H}_I(t)\right]$$

We saw that when interactions are local in space-time, the in and out states are equivalent to free states at $t = \mp\infty$. How do we now relate the interacting fields and vacuum of the theory to the free field and free vacuum?

The quantum fields $\hat{\phi}(x)$ evolves under time evolution as :

$$\hat{\phi}(t, \vec{x}) = \exp(i\hat{H}(t - t_0))\, \hat{\phi}(t_0, \vec{x}) \exp(-i\hat{H}(t - t_0)).$$

Lets say we set the initial operator at $t_0 = 0$ as

$$\hat{\phi}(t_0, \vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[\hat{a}(\vec{p})\, e^{i\vec{p}\cdot\vec{x}} + \hat{a}^\dagger(\vec{p})\, e^{-i\vec{p}\cdot\vec{x}}\right]$$

Now if we time evolve this operator with $\hat{H}_0$, we get a free theory evolved version of $\hat{\phi}(t, \vec{x})$, called interaction picture field $\hat{\phi}_I(t, \vec{x})$, given by

$$\boxed{\hat{\phi}_I(t, \vec{x}) = e^{i\hat{H}_0(t - t_0)}\, \hat{\phi}(t_0, \vec{x})\, e^{-i\hat{H}_0(t - t_0)}}$$

When we look into weakly coupled field theories, $\hat{\phi}_I(t, \vec{x})$ captures the most important part of the time evolution of $\hat{\phi}(t, \vec{x})$.

Plugging the expression of $\hat{\phi}(t_0, \vec{x})$ back:

$$\hat{\phi}_I(t, \vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(\hat{a}(\vec{p})\, e^{-ip\cdot x} + \hat{a}^\dagger(\vec{p})\, e^{ip\cdot x}\right)\Bigg|_{x^0 = t - t_0}$$

Now if we want to express the full Heisenberg operator $\hat{\phi}(t, \vec{x})$,

$$\hat{\phi}(t, \vec{x}) = \exp[i\hat{H}(t - t_0)] \exp[-i\hat{H}_0(t - t_0)]\, \hat{\phi}_I(t, \vec{x}) \exp[i\hat{H}_0(t - t_0)] . \exp[-i\hat{H}(t - t_0)]$$

$$\boxed{\hat{\phi}(t, \vec{x}) = \hat{U}_I^\dagger(t, t_0)\, \hat{\phi}_I(t, \vec{x})\, \hat{U}_I(t, t_0)}$$

we need to note that

$$\hat{H}_I(t) = e^{i\hat{H}_0(t-t_0)}\,\hat{H}_{int}\,e^{-i\hat{H}_0(t-t_0)}$$
$$= e^{i\hat{H}_0(t-t_0)}\,H_{int}(\hat{\phi})\,e^{-i\hat{H}_0(t-t_0)}$$
$$= H_{int}(\hat{\phi}_I).$$

Now we turn our attention to the question how the ground state $|\Omega\rangle$ of the $\hat{H}$ is related to the ground state of $\hat{H}_0$ i.e. $|0\rangle$. We will of course assume that $\langle\Omega|0\rangle \neq 0$ otherwise it means that the strength of the interaction is so large that it completely shifts the vacuum.

$$\exp[-i\hat{H}T]|0\rangle = \sum_n e^{-iE_nT}\,|n\rangle\langle n|0\rangle \quad \text{eigenstates of } \hat{H}$$
$$= e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n>0} e^{-iE_nT}|n\rangle\langle n|0\rangle$$

Here $E_0 = \langle\Omega|\hat{H}|\Omega\rangle$ and the zero-level of the energy is defined by the ground state expectation value of free Hamiltonian. i.e. $\langle 0|\hat{H}_0|0\rangle = 0$. Now we have ordered energy levels so that $E_0 < E_1 < \cdots < E_{n-1} < E_n < \cdots$. The contribution of the excited states dies out in the limit $T \to \infty(1-i\epsilon)$.

So $\lim_{T\to\infty(1-i\epsilon)} e^{-i\hat{H}T}|0\rangle = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle$

$$\Rightarrow |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-i\hat{H}T}|0\rangle$$

Now T is a very large number and thus we can shift it by a small constant t_0 to get

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1} e^{-i\hat{H}(T+t_0)}|0\rangle$$
$$= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} e^{-i\hat{H}(t_0-(-T))} \times e^{-i\hat{H}_0(-T-t_0)}|0\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} \hat{U}_I(t_0,-T)|0\rangle$$

similarly

$$\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \langle 0|\hat{U}_I(T,t_0)\left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1}$$

Now let's look at a quantity we have computed in the past for free theories: the 2-pt functions for $x^0 > y^0 > t_0$

$$\langle\Omega|\, \hat{\phi}(x)\, \hat{\phi}(y)\,|\Omega\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1} \langle 0|\,\hat{U}(T, t_0)$$

$$\times \left[\hat{U}_I^\dagger(x^0, t_0)\, \hat{\phi}_I(x)\, \hat{U}_I(x^0, t_0)\right]\left[\hat{U}_I^\dagger(y^0, t_0)\, \hat{\phi}_I(y)\, \hat{U}(y^0, t_0)\right]$$

$$\times \left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} \hat{U}_I(t_0, -T)\,|0\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \left(|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)}\right)^{-1} \times$$

$$\langle 0|\,\hat{U}_I(T, x^0)\, \hat{\phi}_I(x)\, \hat{U}_I(x^0, y^0)\, \hat{\phi}_I(y)\, \hat{U}_I(y^0, -T)\,|0\rangle$$

The overall factor in the front is just the normalization factor of $|\Omega\rangle$

$$\langle\Omega|\Omega\rangle = \left(|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)}\right)^{-1} \langle 0|\,\hat{U}_I(T, t_0)\, \hat{U}_I(t_0, -T)\,|0\rangle$$

$= 1$ [we impose that the interacting ground state is normalized

$$\Rightarrow \left(|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)}\right)^{-1} = \frac{1}{\langle 0|\,\hat{U}_I(T, t_0)\, \hat{U}_I(t_0, -T)\,|0\rangle}$$

$$= \frac{1}{\langle 0|\,\hat{U}_I(T, -T)\,|0\rangle}$$

Hence $\langle\Omega|\hat{\phi}(x)\hat{\phi}(y)|\Omega\rangle =$

$$\lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\,\hat{U}_I(T, x^0)\, \hat{\phi}_I(x)\, \hat{U}_I(x^0, y^0)\, \hat{\phi}_I(y)\, \hat{U}_I(y^0, -T)\,|0\rangle}{\langle 0|\,\hat{U}_I(T, -T)\,|0\rangle}$$

$$\boxed{\langle\Omega|\,\hat{T}\{\hat{\phi}(x)\,\hat{\phi}(y)\}\,|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\,\hat{T}\left\{\hat{\phi}_I(x)\,\hat{\phi}_I(y)\,\exp\left[-i\int_{-T}^{T} dt\, \hat{H}_I(t)\right]\right\}|0\rangle}{\langle 0|\,\hat{T}\left\{\exp\left[-i\int_{-T}^{T} dt\, \hat{H}_I(t)\right]\right\}|0\rangle}}$$