PHY-685
QFT-1

Lecture-5

Since relativistic kinematics allows change in number of particles, lets try to see if we can address the quantization problem in relativity in a Fock space: a multiparticle Hilbert space. Lets try to construct a multiparticle relativistic Quantum theory of non-interacting particles.

Lets define the local creation / annihilation operators $\hat{a}^\dagger(\vec{x})$, $\hat{a}(\vec{x})$ which either creates or destroys "some" particles at the configuration $\vec{x}$. Here in the 2nd-quantization picture a state with zero particles are allowed.

The operators have the following algebra

$$[\hat{a}(\vec{x}), \hat{a}(\vec{x}')] = 0 \qquad [\hat{a}(\vec{x}), \hat{a}^\dagger(\vec{x}')] = \delta^D(\vec{x}-\vec{x}')$$

$$[\hat{a}^\dagger(\vec{x}), \hat{a}^\dagger(\vec{x}')] = 0$$

The vacuum is defined by

$$\hat{a}(\vec{x})|\Omega\rangle = 0 \ \forall \vec{x}.$$

Position-eigenstates are created from vacuum by

$\hat{a}^\dagger(\vec{x})|\Omega\rangle = |\vec{x}\rangle$ and the general wave fn can be written as

$$|\psi\rangle = \frac{1}{\sqrt{N!}} \int \psi(\vec{x}_1, \dots, \vec{x}_N)\, |\vec{x}_1, \dots, \vec{x}_N\rangle\, d^D\vec{x}_1 \dots d^D\vec{x}_N$$

The algebra is bosonic and hence we are talking about creation & annihilation of bosonic particles.

$$\hat{a}^\dagger(\vec{x}_1)\,\hat{a}^\dagger(x_2)|\Omega\rangle = \hat{a}^\dagger(\vec{x}_2)\,\hat{a}^\dagger(\vec{x}_1)|\Omega\rangle$$

The free Hamiltonian of such a system

$$\hat{H}_0 = \int d^D\vec{x}\, \hat{a}^\dagger(\vec{x}) \left[\sqrt{-\partial^2 + m^2}\right] \hat{a}^\dagger(\vec{x})$$

$$= \int \frac{d^D\vec{p}}{(2\pi)^D} \sqrt{\vec{p}^2 + m^2}\ \hat{a}^\dagger(\vec{p})\,\hat{a}(\vec{p}), \quad \text{where.}$$

$$\hat{a}(\vec{p}) = \int d^D\vec{x}\, \exp[-i\,\vec{p}\cdot\vec{x}]\,\hat{a}(\vec{x})$$

This shows that if we construct a state $\hat{a}^\dagger(\vec{p}_1) \dots \hat{a}^\dagger(\vec{p}_N)|\Omega\rangle$, this is an eigenstate of $\hat{H}_0$ with eigenvalue $\sum_{i=}^{N}\left[\sqrt{\vec{p}_i^{\,2} + m^2}\right]$.

So the propagator can be re-written as:

$$G(\vec{x}, t; \vec{x}_0, t_0) = \langle \vec{x} | \exp[-i\hat{H}_0(t-t_0)] | \vec{x}_0 \rangle$$

$$= \langle \Omega | \hat{a}(\vec{x}) \exp[-i\hat{H}_0(t-t_0)] \hat{a}^\dagger(x_0) | \Omega \rangle$$

$$= \langle \Omega | \hat{a}(x) \exp[-i\hat{H}_0 t] \exp[+i\hat{H}_0 t_0] \hat{a}^\dagger(\vec{x}_0) | \Omega \rangle$$

[Please verify that B·C·H doesn't give extra terms]

Let's define the Heisenberg operators

$$\hat{a}(\vec{x}, t) = \exp[+i\hat{H}_0 t] \, \hat{a}(\vec{x}) \exp[-i\hat{H}_0 t]$$

$$\Rightarrow \hat{a}(\vec{x}) = \exp[-i\hat{H}_0(t)] \, \hat{a}(\vec{x}, t) \exp[i\hat{H}_0 t]$$

So

$G(\vec{x}, t, \vec{x}_0, t_0)$

$$= \langle \Omega | \exp[-i\hat{H}_0(t)] \, \hat{a}(\vec{x}, t) \exp[i\hat{H}_0 t]$$

$$\times \exp[-i\hat{H}_0 t] \exp[i\hat{H}_0 t_0] \exp[-i\hat{H}_0 t_0] \hat{a}^\dagger(\vec{x}_0, t_0)$$

$$\times \exp[i\hat{H}_0 t_0] | \Omega \rangle$$

$$= \langle \Omega | \hat{a}(\vec{x}, t) \, \hat{a}^\dagger(\vec{x}_0, t_0) | \Omega \rangle$$

$$= \langle \Omega | [\hat{a}(\vec{x}, t), \hat{a}^\dagger(\vec{x}_0, t_0)] | \Omega \rangle$$

The algebra of $\hat{a}, a^\dagger$ ensures that at equal times $t = t_0$, the above propagator is a "delta function". Thus this second quantization picture restores causality by enforcing the algebra on the $\hat{a}$ and $\hat{a}^\dagger$.

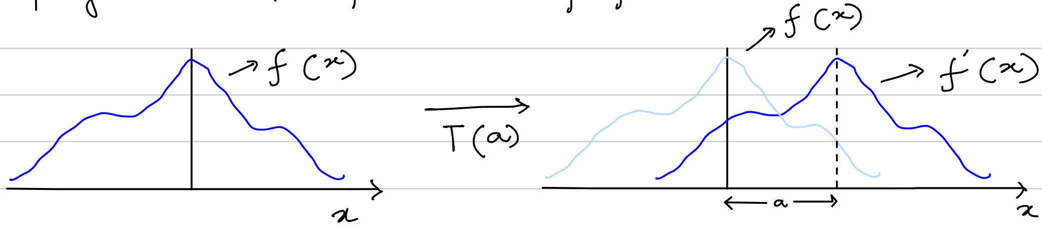
$\hat{N}(\vec{x}, t) = \hat{a}^\dagger(\vec{x}, t) \, \hat{a}(\vec{x}, t)$ doesn't commute with itself at spacelike points!

Now we ask the question that how can we introduce interactions to this many particle Hamiltonian? The way to do this is

$$\hat{V}(t) = \int d^D\vec{x} \, \hat{H}_{int}(t, \vec{x}).$$

An interaction term of this form should commute with itself at spatially seperated points. $\hat{H}_{int}(t, \vec{x})$ also has to be an invariant under space-time transformation

Recaping active transformations of fields:



without changing the shape of the function $f(x)$, it has been translated into a new position and thus its a new function; this is what is called active transformation.

If we define a new co-ordinate system $x' = x - a$, then the function will be called scalar (or invariant, since it's intrinsic shape didn't change) iff:

$$f'(x') = f(x)$$

Now $x' = T(a)\cdot x \equiv x - a \Rightarrow x = x' + a \equiv T^{-1}(a)\cdot x'$

Hence $f'(x') = f[T^{-1}(a)\cdot x']$

Now we relabel x' as x, since its a coordinate of an arbitrary point.

Hence for 1-D scalar functions: $\boxed{f'(x) = f(T^{-1}x)}$

Now lets extend this argument to arbitrary dimension $\mathbb{R}^n$.

Let $\phi(x) := \mathbb{R}^n \to \mathbb{R}$. Under active transformation, let the new function be $\phi'(x) := \mathbb{R}^n \to \mathbb{R}$. It will be called a scalar function iff $\phi'(x') = \phi(x)$, where $x' = \Lambda x + a$ is a transformed point on the manifold (and not co-ordinate relabelling)

Hence $\phi'(x') = \phi(\Lambda^{-1}(x' - a))$

$$\Rightarrow \boxed{\phi'(x) = \phi(\Lambda^{-1}(x - a))}$$ defn. of scalar under active transformation.

This relationship allows us to compute the new mapping ϕ' in terms of the old mapping ϕ

Now if there are unitary operators $\hat{U}(\Lambda, a)$ acting on the Hilbert space associated with the Poincaré group transformation:
$x' = \Lambda x + a$, such that $|x'\rangle = \hat{U}(\Lambda, a)|x\rangle$.
Hence the interaction term should undergo a similarity transformation: $\hat{U}(\Lambda, a)^{\dagger} \hat{H}_{int} \hat{U}(\Lambda, a) = \hat{H}_{int}(\Lambda^{-1}(x-a))$ in order to remain an invariant operator
Extending the analogy of classical mechanics if we ask: what are the possible elementary D.O.F. which we can be combined to form a polynomial, suitable to satisfy the properties of $\hat{H}_{int}$: 1> Poincaré scalar operator and 2> $\hat{H}_{int}(x)$ commutes with itself at space-like seperated points?
The solution just try to linearly combine $\hat{a}(x)$ and $\hat{a}^{\dagger}(x)$

$$\boxed{\hat{\phi}_i(x) = \sum_{\sigma, n} \int \frac{d^D \vec{p}}{(2\pi)^D} \Big[u_i(x; \vec{p}, \sigma; n)\, \hat{a}(\vec{p}, \sigma; n) + v_i(x; \vec{p}, \sigma; n) \times \hat{a}^{\dagger}(\vec{p}, \sigma, n) \Big]}$$

$\hat{\phi}_i(x)$ are <u>local</u> operators, acting on Hilbert space which has a vanishing algebra for space-like separated points.

$$[\hat{\phi}_i(x), \hat{\phi}_j(y)]_{\pm} = 0, \quad [\hat{\phi}_i(x), \hat{\phi}_j^{\dagger}(y)]_{\pm} = 0 \text{ for}$$
$(x-y)^2 < 0$.
These objects are called "Quantum Fields" and for a Poincaré invariant theory, they have an active transformation rule:

$$\boxed{\hat{U}(\Lambda, a)^{\dagger} \hat{\phi}_i(x) \hat{U}(\Lambda, a) = \sum_j D_{ij}(\Lambda) \hat{\phi}_j(\Lambda^{-1}(x-a))}$$

$D_{ij}(\Lambda)$ are representation of the Lorentz group in the sense that

$$\sum_j D_{ij}(\Lambda_1) D_{jk}(\Lambda_2) = D_{ik}(\Lambda_1 \Lambda_2)$$