PHY-685

QFT-1

Lecture - 18

Path integral for field theories:

We have done the canonical quantization of a single free relativistic field and solved the full quantum theory using canonical quantization methods. Given the Hamiltonian, we can solve the full Hilbert state and can find out the states $|\phi\rangle$ which are eigen-states of the local quantum field operators $\hat{\phi}(x)|\phi\rangle = \phi(x)|\phi\rangle$. The states $|\phi\rangle$ forms a complete orthonormal basis and thus the identity operator on the Hilbert space can be constructed as:

$$\boxed{\hat{\mathbb{1}} = \int \mathcal{D}\phi \, |\phi\rangle\langle\phi|}$$

The functional integration measure is defined as:

$$\boxed{\mathcal{D}\phi \equiv \prod_x d\phi(x)}$$

Please note that for a continuous space-time it is an infinite product.

The orthonormality of the basis state is defined as:

$$\langle\phi|\phi'\rangle = \prod_x \delta(\phi(x) - \phi'(x)).$$

We have the canonically conjugate momentum operator:

$$\pi(x) = \partial\mathcal{L}/\partial(\partial_0\phi(x))$$

If we have an interacting Lagrangian density in the classical theory:

$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi)$, then the Hamiltonian operator is:

$$\boxed{\hat{H}(t) = \int d^D\vec{x}\left[\frac{1}{2}\hat{\pi}^2(t,\vec{x}) + \frac{1}{2}\left(\vec{\nabla}\hat{\phi}(t,\vec{x})\right)^2 + V(\hat{\phi}(t,\vec{x}))\right]}$$

In order to compute correlation functions, we need to couple the fields to some external source via the term $\mathcal{L}_{source} = J(x)\phi(x)$. We will impose the following boundary conditions on the source current: $\lim_{|\vec{x}|\to\infty} J(\vec{x}, x^0) = 0$; $\lim_{x^0\to\pm\infty} J(\vec{x}, x^0) = 0$.

The total Lagrangian density $\mathcal{L}(\phi, J) = \mathcal{L}(x) + J(x)\phi(x)$

For a real scalar field

$$\mathcal{L}(\phi, J) = \frac{1}{2}\left(\partial_t\phi(t,\vec{x})\right)^2 - \frac{1}{2}\left(\vec{\nabla}\phi(t,\vec{x})\right)^2 - V(\phi(t,\vec{x})) + J(x)\phi(x)$$

The action of the system in the phase space is given by

$$S[\phi,\pi] = \int dx^0 d^D\vec{x}\left[\pi(x)\partial_t\phi(x) - \frac{1}{2}\pi^2(x) - \frac{1}{2}(\vec{\nabla}\phi(x))^2 - V(\phi(x))\right]$$

In canonical quantization, the prescription was to impose an algebra

$$[\hat{\phi}(x), \hat{\pi}(x')] = i\,\delta^D(x-x')$$

In path integral formalism, we ask the following question:
A quantum system at time $t = t_A$ is at a state $|\{\phi\}_A\rangle$. At later time $t = t_B$ ($t_B > t_A$) the system evolves to a state $|\{\phi\}_B\rangle$.
Can we compute the matrix elements of the evolution operator: $\langle\{\phi\}_B|\hat{U}(t_B, t_A)|\{\phi\}_A\rangle$?
Here

$$\hat{U}(t_B, t_A) = \hat{T}\exp\left[-i\int_{t_A}^{t_B} dt\, \hat{H}(t)\right]$$

The field configurations of the system evolves with time with a boundary condition $\{\phi(t = t_A, \vec{x})\} = \{\phi_A(\vec{x})\}$ and $\{\phi(t = t_B, \vec{x})\} = \{\phi_B(\vec{x})\}$. Hence

$$\langle\phi_B|\hat{T}\exp\left[-i\int_{t_A}^{t_B}\hat{H}(t)\,dt\right]|\phi_A\rangle$$

$$= \lim_{N\to\infty}\langle\phi_B|\exp\left[-i\int_{t_N}^{t_B}\hat{H}(t)dt\right]\cdot\exp\left[-i\int_{t_{N-1}}^{t_N}\hat{H}(t)\,dt\right]\cdots$$

$$\exp\left[-i\int_{t_{n-1}}^{t_n}\hat{H}(t)\,dt\right]\cdots\exp\left[-i\int_{t_1}^{t_2}\hat{H}(t)\,dt\right]\cdot\exp\left[-i\int_{t_A}^{t_1}\hat{H}(t)\,dt\right]|\phi_A\rangle$$

$$= \lim_{N\to\infty}\langle\phi_B|\left[\hat{\mathbb{1}} - i\hat{H}(t_{N+1})\right]\cdots\left[\hat{\mathbb{1}} - i\hat{H}(t_1)\right]|\phi_A\rangle$$

Now we play the old trick which we adapted for harmonic oscillator:
For the piece $\hat{U}(t_{i+1}, t_i) = \exp\left[-i\int_{t_i}^{t_{i+1}}\hat{H}(t)\,dt\right]$, on the R.H.S we introduce $\hat{\mathbb{1}} = \int\mathcal{D}\phi_i\,|\phi_i\rangle\langle\phi_i|$. Again let's remind ourselves that ϕ_i is a dummy functional variable in this context. This is analogous to introducing $\int_{-\infty} d^D\vec{q}_n\,|\vec{q}_n\rangle\langle\vec{q}_n| = \hat{\mathbb{1}}$ in the case of multi-particle Q.M. We have to insert $\int\mathcal{D}\pi_i\,|\pi_i\rangle\langle\pi_i|$ to compute the expectation values.

So $\langle\phi_{i+1}|\left(\hat{\mathbb{1}} - i\,\delta t\,\hat{H}(t_i)\right)|\phi_i\rangle$

$$= \int\mathcal{D}\pi_i\,\langle\phi_{i+1}|\pi_i\rangle\langle\pi_i|\left(\hat{\mathbb{1}} - i\,\delta t\,\hat{H}(t_i)\right)|\phi_i\rangle$$

$$\approx \int \mathcal{D}\pi_i \exp\left[i \int d^D\vec{x}\, \pi_i(\vec{x}, t_i) \left[\phi_{i+1}(\vec{x}, t_{i+1}) - \phi_i(\vec{x}, t_i) \right] \right]$$
$$\times \exp\left[-i\, \delta t \left\{ \int d^D\vec{x} \left[\frac{1}{2}\pi_i^2(\vec{x}, t_i) + \frac{1}{2}\left(\vec{\nabla}\phi_i(\vec{x}, t_i)\right)^2 + V\left(\phi_i(\vec{x}, t_i)\right)\right] \right]$$

Hence,

$$\langle \phi_B | \hat{T} \exp\left[-i \int_{t_A}^{t_B} \hat{H}(t)\, dt \right] | \phi_A \rangle$$

$$= \lim_{N \to \infty} \left[\prod_{n=1}^{N} \mathcal{D}\phi_n \right] \left[\prod_{n=1}^{N+1} \mathcal{D}\pi_n \right] \exp\left[i \left\{ \sum_{n=1}^{N+1} \left(\pi_n \dot{\phi}_n - \mathcal{H}(\vec{x}, t_n) \right) \delta t \right\} \right]$$

$$= \int \mathcal{D}\phi\, \mathcal{D}\pi \exp\left[i\, S[\phi, \pi] \right]$$

If the system is probed with an external current $J(x)$, then

$$\langle \phi_B | \hat{U}(t_B, t_A) | \phi_A \rangle = \int_{b.c.} \mathcal{D}\phi\, \mathcal{D}\pi \exp\left[i \int d^{D+1}x \left(\dot{\phi}\pi - \mathcal{H}(\phi, \pi) + J\phi \right) \right]$$

When the $\mathcal{H}(\phi, \pi)$ is quadratic in π, one can carry out the π integral to write $\langle \phi_B | \hat{U}(t_B, t_A) | \phi_A \rangle = \mathcal{N} \int \mathcal{D}\phi \exp\left[i S(\phi, \partial_\mu \phi, J) \right]$

Now we are interested in the question what happens when we take $t_A \to -\infty$ and $t_B \to +\infty$? Let's say we turn on the source $J(x)$ at some intermediate time interval $t_A < t < t' < t_B$.

So $\langle \phi_B | \hat{U}(t_B, t_A) | \phi_A \rangle = \langle \phi_B | \hat{U}(t_B, t') \hat{U}(t', t) \hat{U}(t, t_A) | \phi_A \rangle$

$$= \int \mathcal{D}\phi\, \mathcal{D}\phi' \langle \phi_B | \hat{U}(t_B, t') | \phi' \rangle \langle \phi' | \hat{U}(t', t) | \phi \rangle_J \langle \phi | \hat{U}(t, t_A) | \phi_A \rangle$$

Now $\langle \phi | \hat{U}(t, t_A) | \phi_A \rangle = \sum_n \sum_m \langle \phi | \psi_n \rangle \langle \psi_n | \hat{U}(t, t_A) | \psi_m \rangle \langle \psi_m | \phi_A \rangle$

$$= \sum_n \sum_m \psi_n[\phi]\, \psi_m^*[\phi_A] \exp\left[-i E_m (t - t_A) \right] \delta_{nm}$$

$$= \sum_n \psi_n[\phi]\, \psi_n^*[\phi_A] \exp\left[-i E_n (t - t_A) \right]$$

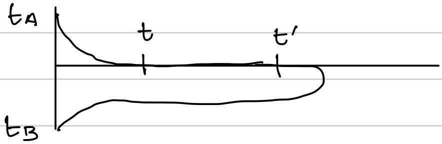
Similarly $\langle \phi_B | \hat{U}(t_B, t') | \phi' \rangle$

$$= \sum_m \psi_m^*[\phi']\, \psi_m[\phi_B] \exp\left[-i E_m (t_B - t') \right]$$

Here for simplicity we assume that the Hamiltonian has a discrete spectra: $(E_0, E_1, \cdots, E_n, \cdots)$, with $E_0 < E_1 < E_2 < \cdots < E_n < \cdots$

Now when we are interested in very early times ($t_A \to -\infty$) and very late time ($t_B \to +\infty$) the phase factors oscillates very rapidly.

To handle this issue we need to deform the time contour to complex plane as following:



$$\lim_{t_A \to +i\infty} \exp[-iE_0 t_A] \langle \phi | \hat{U}(t, t_A) | \phi_A \rangle$$

$$= \lim_{t_A \to +i\infty} \exp[-iE_0 t_A] \sum_n \psi_n[\phi] \psi_n^*[\phi_A] \exp[-iE_n(t - t_A)]$$

$$= \lim_{t_A \to +i\infty} \sum_n \psi_n[\phi] \psi_n^*[\phi_A] \exp(-iE_n t) \exp(+i(E_n - E_0) t_A)$$

$$= \psi_0[\phi] \psi_0^*[\phi_A] \exp(-iE_0 t)$$

Similarly

$$\lim_{t_B \to -i\infty} \exp[iE_0 t_B] \langle \phi_B | \hat{U}(t_B, t') | \phi' \rangle$$

$$= \lim_{t_B \to -i\infty} \exp[iE_0 t_B] \sum_m \psi_m^*[\phi'] \psi_m[\phi_B] \exp[-iE_m(t_B - t')]$$

$$= \lim_{t_B \to -i\infty} \sum_m \psi_m^*[\phi'] \psi_m[\phi_B] \exp[+iE_m t'] \exp[-i(E_m - E_0) t_B]$$

$$= \psi_0[\phi_B] \psi_0^*[\phi'] \exp[iE_0 t']$$

These results are known as Gellman-Low theorems.

Now
$$\lim_{t_B \to -i\infty} \lim_{t_A \to +i\infty} \frac{\langle \phi_B | \hat{U}(t_B, t_A) | \phi_A \rangle}{\exp[-iE_0(t_B - t_A)] \psi_0^*[\phi_A] \psi_0[\phi_B]}$$

$$\boxed{\hat{\tilde{U}}(t', t) = \hat{U}(t', t) \exp[iE_0(t' - t)]}$$

$$= \lim_{t_B \to -i\infty} \lim_{t_A \to +i\infty} \frac{\int \mathcal{D}\phi \, \mathcal{D}\phi' \langle \phi_B | \hat{U}(t_B, t') | \phi' \rangle \langle \phi' | \hat{\tilde{U}}(t', t) | \phi \rangle_J \langle \phi | \hat{U}(t, t_A) | \phi_A \rangle}{\exp[-iE_0(t_B - t_A)] \psi_0^*[\phi_A] \psi_0[\phi_B]}$$

$$= \int \mathcal{D}\phi \, \mathcal{D}\phi' \, \psi_0^*[\phi'] \psi_0[\phi] \langle \phi' | \hat{\tilde{U}}(t', t) | \phi \rangle_J \equiv {}_J\langle 0 | 0 \rangle_J$$

$$\boxed{\begin{aligned} Z[J] = {}_J\langle 0|0\rangle_J = \mathcal{N} \lim_{t_A\to+i\infty} \lim_{t_B\to-i\infty} \times \\ \int \Delta\phi\, \Delta\pi \, \exp\left[i \int_{t_A}^{t_B} dt \cdot \int d^D\vec{x} \, \left(\dot{\phi}\pi - \mathcal{H}(\phi,\pi) + J\phi\right)\right] \end{aligned}}$$

Extending the arguments of P.I. for Q.M. we can write down:

$${}_J\langle 0| \hat{T}\, \hat{\phi}(x_1)\cdots\hat{\phi}(x_N)|0\rangle_J = \frac{1}{(i)^N}\frac{1}{Z[J]} \left.\frac{\delta^N \, {}_J\langle 0|0\rangle_J}{\delta J(x_1)\cdots\delta J(x_N)}\right|_{J=0}$$

For free theory $Z_0[J] = \int \Delta\phi \, \exp\left[i\int d^{D+1}x \, (\mathcal{L}_0 + J\phi)\right]$, where $\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$. We define the Fourier modes as: $\tilde{\phi}(k) = \int d^{D+1}x \, e^{-ik\cdot x}\phi(x)$ & $\phi(x) = \int \frac{d^{D+1}k}{(2\pi)^{D+1}} e^{ikx}\tilde{\phi}(k)$

Now $kx = k^0 t - \vec{k}\cdot\vec{x} \Rightarrow \partial_\mu\phi$

$$= \partial_\mu\left[\int \frac{d^{D+1}k}{(2\pi)^{D+1}} e^{ik\cdot x}\tilde{\phi}(k)\right]$$

$$= \int \frac{d^{D+1}k}{(2\pi)^{D+1}} (ik_\mu)\, e^{ik\cdot x}\tilde{\phi}(k)$$

$$\Rightarrow S_0 = \frac{1}{2}\int \frac{d^{D+1}k}{(2\pi)^{D+1}}\left[\tilde{\phi}(k)(k^2-m^2)\tilde{\phi}(-k) + \tilde{J}(k)\tilde{\phi}(-k) + \tilde{J}(-k)\tilde{\phi}(k)\right]$$

We now change the P.I. variable

$\tilde{\chi}(k) = \tilde{\phi}(k) + \tilde{J}(k)/(k^2-m^2)$. The Jacobian of the measure is 1 and thus $\Delta\phi = \Delta\chi$.

Now let's expand the term within the $[\cdots]$ in S_0:

$$\tilde{\phi}(k)(k^2-m^2)\tilde{\phi}(-k) + \tilde{J}(k)\tilde{\phi}(-k) + \tilde{J}(-k)\tilde{\phi}(k)$$

$$= \left[\tilde{\chi}(k) - \tilde{J}(k)/(k^2-m^2)\right](k^2-m^2)\left[\tilde{\chi}(-k) - \tilde{J}(-k)/(k^2-m^2)\right] + \tilde{J}(k)\left[\tilde{\chi}(-k) - \tilde{J}(-k)/(k^2-m^2)\right] + \tilde{J}(-k)\left[\tilde{\chi}(k) - \tilde{J}(k)/(k^2-m^2)\right]$$

$$= \tilde{\chi}(k)(k^2-m^2)\tilde{\chi}(-k) - \tilde{J}(k)\tilde{\chi}(-k) - \tilde{\chi}(k)\tilde{J}(-k) + \tilde{J}(x)\tilde{J}(-k)/(k^2-m^2)$$
$$+ \tilde{J}(k)\tilde{\chi}(-k) - \tilde{J}(k)\tilde{J}(-k)/(k^2-m^2) + \tilde{J}(-k)\tilde{\chi}(k) - \tilde{J}(k)\tilde{J}(-k)/(k^2-m^2)$$

$$= \tilde{\chi}(k)(k^2-m^2)\tilde{\chi}(-k) - \tilde{J}(k)\tilde{J}(-k)/(k^2-m^2)$$

So $Z_0[J] = \int \mathcal{D}\phi \, \exp\left[i \int d^{D+1}x \, (\mathcal{L}_0 + J\phi)\right]$

$$= \int \mathcal{D}x \, \exp\left[\frac{i}{2}\int \frac{d^{D+1}k}{(2\pi)^{D+1}}\left[\tilde{x}(k)(k^2-m^2)\tilde{x}(-k) - \tilde{J}(k)\tilde{J}(-k)/(k^2-m^2)\right]\right]$$

$$= \exp\left[-\frac{i}{2}\int \frac{d^{D+1}k}{(2\pi)^{D+1}} \frac{\tilde{J}(k)\tilde{J}(-k)}{k^2-m^2+i\epsilon}\right]$$

$$\times \underbrace{\left[\int \mathcal{D}x \, \exp\left[\frac{i}{2}\int \frac{d^{D+1}k}{(2\pi)^{D+1}} \tilde{x}(k)(k^2-m^2)\tilde{x}(-k)\right]\right]}_{Z_0[J=0] \,=\, Z_0[0]}$$

So $Z_0[J] = Z_0[0] \exp\left[-\frac{i}{2}\int \frac{d^{D+1}k}{(2\pi)^{D+1}} \frac{\tilde{J}(k)\tilde{J}(-k)}{k^2-m^2+i\epsilon}\right]$

$$= Z_0[0] \exp\left[-\frac{1}{2}\int \frac{d^{D+1}k}{(2\pi)^{D+1}} \tilde{J}(k)\left(\frac{i}{k^2-m^2+i\epsilon}\right)\tilde{J}(-k)\right]$$

Now we write $\tilde{J}(k) = \int d^{D+1}x \, e^{-ikx} J(x)$

and $\tilde{J}(-k) = \int d^{D+1}x' \, e^{ikx'} J(x')$

So $\int \frac{d^{D+1}k}{(2\pi)^{D+1}} \tilde{J}(k)\left(\frac{i}{k^2-m^2+i\epsilon}\right)\tilde{J}(-k)$

$$= \int \frac{d^{D+1}k}{(2\pi)^{D+1}} \left[\int d^{D+1}x \, e^{-ik\cdot x} J(x)\right]\left(\frac{i}{k^2-m^2+i\epsilon}\right)$$
$$\times \left[\int d^{D+1}x' \, e^{ik\cdot x'} J(x')\right]$$

$$= \int d^{D+1}x \, J(x) \times$$
$$\int d^{D+1}x' \underbrace{\left[\int \frac{d^{D+1}k}{(2\pi)^{D+1}}\left(\frac{i}{k^2-m^2+i\epsilon}\right) e^{-ik\cdot(x-x')}\right]}_{\Delta_F(x-x')} J(x')$$

$$= \int d^{D+1}x \, d^{D+1}x' \, J(x)\,\Delta_F(x-x')\,J(x')$$

$$\Rightarrow \boxed{Z_0[J] = Z_0[0] \exp\left[-\frac{1}{2}\int d^{D+1}x \, d^{D+1}x' \, J(x)\,\Delta_F(x-x')\,J(x')\right]}$$