PHY-685
QFT-1

Lecture - 31

We have obtained the E.O.M. of a Weyl Fermion in 4-component notation.

We introduce a 4-component Majorana field:

$$\Psi_M = \begin{pmatrix} \psi_c \\ \psi^{\dagger c} \end{pmatrix}$$ := The upper and lower component are the same Weyl field.

The previous E.O.M. becomes $\left(i\gamma^\mu \partial_\mu - m\,\mathbb{1}_{4\times4}\right)\Psi_M = 0$

This is Dirac's eqn. for Majorana field.

Now let's consider two left handed Weyl field with an SO(2) symmetry:

$$\mathcal{L} = i\,\psi_i^\dagger \bar{\sigma}^\mu \partial_\mu \psi_i - \frac{1}{2} m\,\psi_i \psi_i - \frac{1}{2}\,\psi_i^\dagger \psi_i^\dagger$$

This lagrangian is invariant under an SO(2) transformation

$$\begin{pmatrix} \psi_1' \\ \psi_2' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

The SO(2) invariance appears as U(1) invariance when we write:

$$\chi = \frac{1}{\sqrt{2}}\left(\psi_1 + i\psi_2\right), \quad \xi = \frac{1}{\sqrt{2}}\left(\psi_1 - i\psi_2\right).$$

Plugging back to the original Lagrangian:

$$\mathcal{L} = i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi + i\,\xi^\dagger \bar{\sigma}^\mu \partial_\mu \xi - m\chi\xi - m\xi^\dagger\chi^\dagger.$$

This $\mathcal{L}$ is invariant under a global U(1):

$$\chi \to e^{-i\alpha}\chi, \quad \xi \to e^{i\alpha}\xi.$$

Now we have the E.O.M of $\chi$: $= \frac{\partial S}{\partial \chi^\dagger} = 0$

& " " $\xi = \partial S / \partial \xi^\dagger = 0$

Following the same algebraic steps, these two simultaneous eqns can be clubbed into:

$$\begin{pmatrix} -m\,\delta_a{}^c & i\,\sigma^\mu{}_{a\dot{c}}\,\partial_\mu \\ i\,\bar{\sigma}^{\mu;\dot{a}c} & -m\,\delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix} \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix} = 0$$

Now define a 4-component Dirac field $\Psi_D = \begin{pmatrix} \chi_a \\ \xi^{\dagger\dot{a}} \end{pmatrix}$

This also has the E.O.M as Dirac's eqn.

$\Psi^\dagger_D = ( \chi^\dagger_{\dot{a}} , \xi^a )$ : = Hermitian conjugation.

Then introduce: $\beta \equiv \begin{pmatrix} 0 & \delta^{\dot{a}}{}_{\dot{c}} \\ \delta_a{}^c & 0 \end{pmatrix}$. Using this

we define $\bar{\Psi}_D = \Psi^\dagger_D \beta = ( \chi^\dagger_{\dot{a}} , \xi^a ) \begin{pmatrix} 0 & \delta^{\dot{a}}{}_{\dot{c}} \\ \delta_a{}^c & 0 \end{pmatrix}$

$$= ( \xi^c , \chi^\dagger{}_{\dot{c}} )$$

$\beta$ and $\gamma^0$ are numerically same (modulo an $i$), but has different spinor structure.

So $\bar{\Psi}_D \Psi_D = ( \xi^c , \chi^\dagger{}_{\dot{c}} ) \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix} = \xi^c \chi_c + \chi^\dagger_{\dot{d}} \xi^{\dagger\dot{d}}$

Using these forms:

$$\bar{\Psi}_D \gamma^\mu \partial_\mu \Psi_D = \xi^a \sigma^\mu{}_{a\dot{c}} \partial_\mu \xi^{\dagger\dot{c}} + \chi^\dagger{}_{\dot{a}} \bar{\sigma}^{\mu;\dot{a}c} \partial_\mu \chi_c$$

Now $\xi^a \sigma^\mu{}_{a\dot{c}} \partial_\mu \xi^{\dagger\dot{c}} = \partial_\mu [ \xi^a \sigma^\mu{}_{a\dot{c}} \xi^{\dagger\dot{c}} ] - (\partial_\mu \xi^a) \sigma^\mu{}_{a\dot{c}} \xi^{\dagger\dot{c}}$

So the first term can be written as

$$-(\partial_\mu \xi^a)\,\sigma^\mu_{a\dot c}\,\xi^{\dagger\dot c} = -(\partial_\mu \varepsilon^{ab}\psi_b)\,\sigma^\mu_{a\dot c}\,\psi^\dagger_{\dot b}\,\varepsilon^{\dot b\dot c}$$

$$= +\psi^\dagger_{\dot b}\,\varepsilon^{\dot b\dot c}\,\varepsilon^{ab}\,\sigma^\mu_{a\dot c}\,(\partial_\mu\psi_b)$$

$$= +\psi^\dagger_{\dot b}\,\varepsilon^{\dot b\dot c}\,\varepsilon_{\dot c\dot d}\,\bar\sigma^{\mu;\dot d b}\,(\partial_\mu\psi_b)$$

$$= \psi^\dagger_{\dot b}\,\delta^{\dot b}{}_{\dot d}\,\bar\sigma^{\mu;\dot d b}\,(\partial_\mu\psi_b)$$

$$= \psi^\dagger_{\dot d}\,\bar\sigma^{\mu;\dot d b}\,\partial_\mu\psi_b$$

Hence

$$\bar\Psi_D\gamma^\mu\partial_\mu\Psi_D = \chi^\dagger\bar\sigma^\mu\partial_\mu\chi + \xi^\dagger\bar\sigma^\mu\partial_\mu\xi + \partial_\mu(\xi\sigma^\mu\xi^\dagger)$$

Hence upto a total divergence

$$\mathcal{L}_D = i\,\bar\Psi_D\gamma^\mu\partial_\mu\Psi_D - m\,\bar\Psi_D\Psi_D.$$

This is invariant under a global $U(1)$ $\Psi_D \to e^{-i\alpha}\Psi_D$ and $\bar\Psi_D \to e^{i\alpha}\bar\Psi_D$. (H.W. Compute the Nöther current).

There is an additional discrete symmetry called charge conjugation, which exchanges $\chi$ and $\xi$.

It operates as

$$C^{-1}\chi_a(x)\,C = \xi_a(x)$$
$$C^{-1}\xi_a(x)\,C = \chi_a(x)$$

One can show $\quad C^{-1}\mathcal{L}(x)\,C = \mathcal{L}(x)$

$$C^T = C^\dagger = C^{-1} = -C$$

$$C = \begin{pmatrix} -i\sigma^2 & \\ & i\sigma^2 \end{pmatrix}$$

$$C = \begin{pmatrix} \varepsilon_{ac} & 0 \\ 0 & \varepsilon^{\dot a\dot c} \end{pmatrix}$$

$$\bar\Psi_D^T = \begin{pmatrix} \xi^c \\ \chi^{\dagger\dot c} \end{pmatrix};\quad \Psi_D^c = C\,\bar\Psi_D^T = \begin{pmatrix} \varepsilon_{ac} & 0 \\ 0 & \varepsilon^{\dot a\dot c} \end{pmatrix}\begin{pmatrix} \xi^c \\ \chi^{\dagger\dot c} \end{pmatrix} = \begin{pmatrix} \xi_a \\ \chi^{\dagger\dot a} \end{pmatrix}$$

comparing side-by-side.

$$\psi_D = \begin{pmatrix} \chi_c \\ \xi^{\dagger \dot{c}} \end{pmatrix}, \quad \psi_D^c = \begin{pmatrix} \xi_c \\ \chi^{\dagger \dot{c}} \end{pmatrix}$$

Now $C^{-1} \psi_D(x) C$ $\quad | \quad C^{-1} = \begin{pmatrix} \varepsilon^{ba} & \\ & \varepsilon_{\dot{b}\dot{a}} \end{pmatrix}$

This leads to $C^{-1} \gamma^\mu C = -(\gamma^\mu)^T$.

For a Majorana field $\boxed{\psi_M^C = C^{-1} \psi_M C = \psi_M}$. Its analogous to the $\phi^\dagger = \phi$ for a real scalar field.

The original Lagrangian of the single left-handed Weyl fermion was

$$\mathcal{L} = i \psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi - \frac{1}{2} m \psi \psi + h.c. \; ..$$

$$= \frac{i}{2} \psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi + i/2 \, \psi \sigma^\mu \partial_\mu \psi^\dagger - \frac{1}{2} m \psi \psi - \frac{1}{2} m \psi^\dagger \psi^\dagger$$

Now $\bar{\psi}_M = (\psi^c, \psi^\dagger_{\dot{c}})$, so replacing $\chi \to \psi$ and $\xi \to \psi$ in the Dirac Lagrangian,

$$\mathcal{L}_M = i/2 \, \bar{\psi}_M \gamma^\mu \partial_\mu \psi_M - \frac{1}{2} m \bar{\psi}_M \psi_M$$

For Majorana fermions, $\bar{\psi} = \psi^T C$, so

$$\mathcal{L}_M = i/2 \, \psi_M^T C \gamma^\mu \partial_\mu \psi_M - \frac{1}{2} m \psi_M^T C \psi_M .$$

The projection operators are constructed using

$$\gamma_5 \equiv \begin{pmatrix} -\delta_a{}^c & 0 \\ 0 & +\delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix}$$

$$P_L = \frac{1}{2}(1 - \gamma_5) = \begin{pmatrix} \delta_a{}^c & 0 \\ 0 & 0 \end{pmatrix} \text{ and } P_R = \frac{1}{2}(1+\gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix}$$

This defines $\psi_L = P_L \psi = \begin{pmatrix} \chi_c \\ 0 \end{pmatrix}$ & $\psi_R = P_R \psi = \begin{pmatrix} 0 \\ \xi^{\dagger \dot{c}} \end{pmatrix}$

$$\gamma_5 = \frac{i}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$$

Now let's remember that if $\Psi_D = \begin{pmatrix} \chi_a \\ \xi^{\dagger \dot{a}} \end{pmatrix}$, then

$\bar{\Psi}_D = (\xi^{a}, \chi^{\dagger}_{\dot{a}})$. So when $\Psi_L = \begin{pmatrix} \chi_a \\ 0 \end{pmatrix}$, $\bar{\Psi}_L = (0, \chi^{\dagger}_{\dot{a}})$.

and $\Psi_R = \begin{pmatrix} 0 \\ \xi^{\dagger \dot{a}} \end{pmatrix} \Rightarrow \bar{\Psi}_R = (\xi^a, 0)$

Now $\mathcal{L}_D = i\chi^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\chi + i\xi^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\xi - m\chi\xi - m\xi^{\dagger}\chi^{\dagger}$.

So the Dirac fermion has mass term $m\bar{\Psi}_D\Psi_D$

$$= m\chi\xi + m\xi^{\dagger}\chi^{\dagger}$$
$$= m\xi^a\chi_a + m\chi^{\dagger}_{\dot{a}}\xi^{\dagger \dot{a}}$$
$$= m(\bar{\Psi}_R\Psi_L + \bar{\Psi}_L\Psi_R).$$

So $\Psi_L$ and $\Psi_R$ only mixes through mass terms and E.O.M.
Hence massless fermions are helicity eigen-states.

Now we have this grand equation of free fields
$$(i\gamma^{\mu}\partial_{\mu} - m)\Psi_D = 0$$

This is a "classical E.O.M" of free fields which transforms under $(1/2, 0)$ and $(0, 1/2)$ representation of $SL(2,\mathbb{C})$. First obvious question how do we solve it?
It's a matrix equation $\underbrace{(\quad)}_{4\times 4 \text{ diff operator}} \underbrace{(\psi)}_{(4\times 1)\text{ function}} = 0 \longrightarrow 4\times 1$ null matrix

First lets multiply from left by $(i\gamma^{\nu}\partial_{\nu} + m)$ to get:

$$[i\gamma^{\nu}\partial_{\nu} + m][i\gamma^{\mu}\partial_{\mu} - m]\psi = 0$$
$$\Rightarrow [-\partial_{\nu}\partial_{\mu}\gamma^{\nu}\gamma^{\mu} + im\gamma^{\mu}\partial_{\mu} - im\gamma^{\nu}\partial_{\nu} - m^2]\psi = 0$$

$$\Rightarrow [\partial_{\nu}\partial_{\mu}\gamma^{\nu}\gamma^{\mu} + m^2]\psi = 0$$
$$\Rightarrow [\partial_{\nu}\partial_{\mu}\left(\tfrac{1}{2}\{\gamma^{\nu}, \gamma^{\mu}\} + \tfrac{1}{2}[\gamma^{\nu}, \gamma^{\mu}]\right) + m^2]\psi = 0$$

$$\Rightarrow \left(\frac{1}{2}\,\partial_\nu \partial_\mu \cdot 2\eta^{\mu\nu} + m^2\right)\psi = 0$$

$$\Rightarrow \left(\partial^2 + m^2\right)\psi = 0 \;\Rightarrow\; \psi \text{ satisfy K-G. eqn. as well.}$$

We know that solution of these equations are plane waves $e^{-ip\cdot x}$ and $e^{ip\cdot x}$ where $p^0 = \sqrt{\vec{p}^2 + m^2}$.

Lets write the generic solution as linear superpositions of these two plane waves:

$$\psi_s(x) = \int \frac{d^4p}{(2\pi)^4}\, \delta^4(p^2 - m^2)\, \theta(p^0)\, u_s(p)\, e^{-ip\cdot x}$$

$$\text{and } \chi_s(x) = \int \frac{d^4p}{(2\pi)^4}\, \delta^4(p^2 - m^2)\, \theta(p^0)\, v_s(p)\, e^{ip\cdot x}$$

When we plug these solutions in Dirac's eqn we get

$$\begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} u_s(p) = 0 \quad \text{and} \quad \begin{pmatrix} -m & -p\cdot\sigma \\ -p\cdot\bar{\sigma} & -m \end{pmatrix} v_s(p) = 0$$

Lets look at this equation in the rest frame $p^\mu = (m, \vec{0}) \equiv \bar{p}^\mu$.

We have

$$m\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u_s(\bar{p}^\mu) = 0 \quad \text{and} \quad m\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} v_s(\bar{p}^\mu) = 0$$

So the solutions are constants $u_s = \sqrt{m}\begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}$, $v_s = \sqrt{m}\begin{pmatrix} \eta_s \\ -\eta_s \end{pmatrix}$

We conventionally normalize $\xi$ to have $\xi^\dagger \xi = 1$, same for $\eta$.

Dirac fermion is a 4-complex number object which can be written in the basis of

$$u_+ = \sqrt{m}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_- = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_+ = \sqrt{m}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_- = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

Now in a general boosted frame $p^\mu = (E, 0, 0, p_z)$

$$p\cdot\sigma = \begin{pmatrix} E - p_z & 0 \\ 0 & E + p_z \end{pmatrix} \quad \text{and} \quad p\cdot\bar{\sigma} = \begin{pmatrix} E + p_z & 0 \\ 0 & E - p_z \end{pmatrix},$$

Let $a = \sqrt{E - p_z}$ and $b = \sqrt{E + p_z}$, then $a^2 b^2 = E^2 - p_z^2 = m^2$

Now the eqn

$$\begin{pmatrix} -m\,\mathbb{1}_{2\times 2} & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m\,\mathbb{1}_{2\times 2} \end{pmatrix} u_s(p) = 0 \text{, becomes}$$

$$\begin{pmatrix} -m & 0 & E-p_z & 0 \\ 0 & -m & 0 & E+p_z \\ E+p_z & 0 & -m & 0 \\ 0 & E-p_z & 0 & -m \end{pmatrix} u_s(p) = 0$$

$$\Rightarrow \begin{pmatrix} -ab & 0 & a^2 & 0 \\ 0 & -ab & 0 & b^2 \\ b^2 & 0 & -ab & 0 \\ 0 & a^2 & 0 & -ab \end{pmatrix} u_s(p) = 0$$

The solutions are: $u_s = \begin{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \xi_s \\ \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} \xi_s \end{pmatrix}$

$$\Rightarrow u_s(p) = \begin{bmatrix} \begin{pmatrix} \sqrt{E-p_z} & 0 \\ 0 & \sqrt{E+p_z} \end{pmatrix} \xi_s \\ \begin{pmatrix} \sqrt{E+p_z} & 0 \\ 0 & \sqrt{E-p_z} \end{pmatrix} \xi_s \end{bmatrix} = \begin{bmatrix} \sqrt{p\cdot\sigma}\, \xi_s \\ \sqrt{p\cdot\bar{\sigma}}\, \xi_s \end{bmatrix}$$

$$v_s(p) = \begin{bmatrix} \begin{pmatrix} \sqrt{E-p_z} & 0 \\ 0 & \sqrt{E+p_z} \end{pmatrix} \eta_s \\ \begin{pmatrix} -\sqrt{E+p_z} & 0 \\ 0 & -\sqrt{E+p_z} \end{pmatrix} \eta_s \end{bmatrix} = \begin{bmatrix} \sqrt{p\cdot\sigma}\, \eta_s \\ -\sqrt{p\cdot\bar{\sigma}}\, \eta_s \end{bmatrix}$$

Check $\bar{u}_s(p)\, u_{s'}(p) = 2m\,\delta_{ss'}$, $\bar{v}_s(p)\, v_{s'}(p) = -2m\,\delta_{ss'}$

$\sum_s u_s \bar{u}_s = \not{p} + m$ and $\sum_s v_s \bar{v}_s = \not{p} - m$.