PHY-685
QFT-1

Lecture-1

A one liner for this course: We want to write down a quantum theory for every possible physical system in this universe. The underlying motivation being → every system is a quantum system and when and if we see a classical dynamics, it must be a limiting case of a quantum dynamics.

To elaborate, physical systems can be built out of multiple elementary degrees of freedom like a single point particle, a collection of point particles, waves (including longitudinal elastic waves + electromagnetic waves ---) and possibly some other unknown D.O.F. which we are yet to encounter.

Hence in the initial part of the course we will try to answer two major questions:

1) The Q.M. we have learned until now is mainly for single (or few) particle systems. Does the formalism hold good when we try to do Q.M. of infinitely many particles (i.e. continuous system)? Are there any tweak required to the formalism we have learned until now? (The example at hand are just too many!! EM waves, sound waves, fluid motion, ---)

2) The classical dynamics which we have quantized until now had a Galilean equivariance or invariance! But we know there exists the general space-time transformation of Poincaré group. How do we write Q.M. for such cases?

We will find out that we definitely need a change in the QM formalism which was done until QM-2 course. Simply because the 4-th postulate of Q.M. assumed a particular rule of time evolution, but for a Poincaré invariant theory we need to talk about space-time evolution!

Hence our journey begins by recapitulating the basic formalism of Classical Mechanics and Quantum Mechanics which we know!

The general formalism of Classical Mechanics with N D.O.F:

Let a system, with N time dependent generalized co-ordinates $q_1(t), \cdots, q_N(t)$; be described by a Lagrangian $L(\{q_i(t)\}, \{\dot{q}_i(t)\}, t)$.

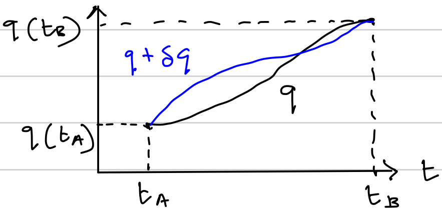
The action functional of the system for a particular configuration of $\{q_i\}$ is given by

$$A[\{q_i\}] = \int_{t_a}^{t_b} dt\, L(\{q_i(t)\}, \{\dot{q}_i(t)\}, t)$$

Now let's say we perturb each of the co-ordinates:

$q_i(t) \to q_i(t) + \delta q_i(t)$

We want to compute how much the action changes due to this deformation of the generalized co-ordinates?



$\delta A[\{q_i\}] = A[\{q_i + \delta q_i\}] - A[\{q_i\}]$

$$= \int_{t_A}^{t_B} dt\, L(\{q_i + \delta q_i\}, \{\dot{q}_i + \delta \dot{q}_i\}, t) - \int_{t_A}^{t_B} dt\, L(\{q_i\}, \{\dot{q}_i\}, t)$$

$$= \int_{t_A}^{t_B} \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right] dt$$

$$= \int_{t_A}^{t_B} \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{d}{dt} \left\{ \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right\} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right] dt$$

$$= \int_{t_A}^{t_B} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_i \, dt + \left[\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right]_{t_A}^{t_B}$$

This computation is done at $\mathcal{O}(\delta q)$. The extremum of the action corresponds to the classical path. If the path deformation vanishes at the two end points, then this leads to the Euler-Lagrange eqns

$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0}$$

The alternative formulation is the Hamiltonian formulation:

$$H = \sum p_i \dot{q}_i - L(\{q_i\}, \{\dot{q}_i\}, t) \equiv H(\{p_i\}, \{q_i\}, t),$$

where $p_i = \partial L / \partial \dot{q}_i$. $(i = 1, 2, \ldots, N)$

The E.O.M. in terms of H become $\dot{q}_i = \partial H / \partial p_i$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}$ with $\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$

This construction holds good iff the Hessian matrix $\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$ is non-singular

The general story of the classical dynamics framework:
Given an arbitrary phase space observable $O(\{p_i\}, \{q_i\}, t)$, it's time evolution is governed by the eqn

$$\frac{dO}{dt} = \frac{\partial O}{\partial t} + \frac{\partial O}{\partial p_i}\dot{p}_i + \frac{\partial O}{\partial q_i}\dot{q}_i$$
$$= \frac{\partial O}{\partial t} + \frac{\partial O}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial O}{\partial p_i}\frac{\partial H}{\partial q_i} = \frac{\partial O}{\partial t} + [H, O]_{PB}$$

where the Poisson bracket between two phase-space variables $A(\{p_i\}, \{q_i\}, t)$, $B(\{p_i\}, \{q_i\}, t)$ is given by

$$[A, B]_{PB} = \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} - \frac{\partial B}{\partial p_i}\frac{\partial A}{\partial q_i},$$ which have the following

property $[A, B]_{PB} = -[B, A]_{PB}$, $[A, [B, C]_{PB}]_{PB}$ + cyclic perm $= 0$ [Jacobi identity]

Take away: If you want to find E.O.M. for an observable, first try to compute the P.B. with the Hamiltonian (and try to solve if possible).
The phase space variables $q_i(t)$ and $p_j(t)$ satisfies the following P.B.

$$[p_i(t), q_j(t)]_{PB} = \delta_{ij}, \; [p_i(t), p_j(t)]_{PB} = 0 \text{ and } [q_i(t), q_j(t)]_{PB} = 0$$

Needless to say if an observable $O \equiv O(\{q_i\}, \{p_i\})$ and $[O, H]_{PB} = 0$, then $dO/dt = 0$ and its a constant of motion along the phase-space path.
The Lagrangian formalism is independent on the choice of a particular co-ordinate system. If one introduces a new set of coordinates $\{Q_j\}$ through a point transformation $q_i = f_i(\{Q_j\}, t)$. This relation has to be invertible i.e. $Q_i = f_i^{-1}(\{q_j\}, t)$ in the neighborhood of the classical path. This is true iff $\det(\partial f_i/\partial Q_j) \neq 0$.
In terms of these new co-ordinates, the Lagrangian becomes a new functional $L'(\{Q_i\}, \{\dot{Q}_i\}, t) = L(\{f_i(\{Q_j\}, t), \dot{f}_i(\{Q_j\}, t), t\})$
The E.L. eqn's retains their form under such point transformation.
Only a subset of transformation retains the Hamilton's E.O.M. Those are called canonical transformation.
For canonical transformation $P_j \equiv P_j(\{p_i\}, \{q_i\}, t)$, $Q_j \equiv Q_j(\{p_i\}, \{q_i\}, t)$ the action is invariant upto an arbitrary surface term:

$$\int_{t_a}^{t_b} dt\left[\sum_i p_i \dot{q}_i - H(\{p_i\}, \{q_i\}, t)\right]$$
$$= \int_{t_a}^{t_b} dt\left[\sum_j P_j \dot{Q}_j - H'(\{P_j\}, \{Q_j\}, t)\right] + F(\{P_j\}, \{Q_j\}, t)\Big|_{t_a}^{t_b}$$

What's the prescription to "QUANTIZE" a system whose "CLASSICAL" laws are just given? ⇒ The postulates of QM:

1. State of a particle is represented by a vector $|\psi(t)\rangle$ in a Hilbert space.

2. For every generalized co-ordinate q and generalized momentum p, there exists operators $\hat{Q}$ and $\hat{P}$, acting on the states in the Hilbert space, whose matrix elements are given by $\langle q|\hat{Q}|q'\rangle = q\,\delta(q-q')$ and $\langle q|\hat{P}|q'\rangle = -i\hbar \frac{\partial}{\partial q}\delta(q-q')$. [The def. of position basis]

For every dynamical phase space operator $\Omega(q,p)$ in classical mechanics, there exist an operator $\hat{\Omega}(\hat{Q},\hat{P}) \equiv \Omega(q\to\hat{Q}, p\to\hat{P})$ acting on the Hilbert space

3. If the operator $\hat{\Omega}$ has eigenvectors $|\omega_n\rangle$ and if a system is in some mixed state $|\psi\rangle$, then the probability that measurement of $\hat{\Omega}$ will yield one of the eigen-values ω_n will be $P(\omega_n) \propto |\langle\omega_n|\psi\rangle|^2$. If the outcome of a measurement is some value ω_n, then the system has collapsed to the state $|\omega_n\rangle$.

4. The time evolution of the state vector is governed by the Schrödinger's eqn:

$$\boxed{i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle}$$

You learned in your QM course what are the immediate barriers you face once you try to implement them! The core part is how do you compute time evolution of physical quantum systems?

i.e. if $|\psi(t=t_0)\rangle$ is given, how one computes $|\psi(t)\rangle$ for any $t > t_0$?

Ans: First figure out what are the eigenvectors $|E_n\rangle$ and corresponding eigen-values E_n for the Hamiltonian.

Find the coeffs. $c_n = \langle E_n|\psi(t_0)\rangle$.

Then $|\psi(t)\rangle = \sum_n c_n e^{-iE_n(t-t_0)}|E_n\rangle$

The time evolution operator:

If the Hamiltonian of a system is time independent, then S.E. can be integrated to find a general time evolution relation:

$$|\psi(t_b)\rangle = \exp\left[-\frac{i}{\hbar}(t_b - t_a)\hat{H}\right]|\psi(t_a)\rangle .$$

We identify the time evolution operator: $\hat{U}(t_b, t_a) \equiv \exp\left[-\frac{i}{\hbar}(t_b - t_a)\hat{H}\right]$ which acts on the states and initiates time translations. This operator satisfies the following differential eqn:

$$\boxed{i\hbar\,\partial_{t_b}\hat{U}(t_b, t_a) = \hat{H}\,\hat{U}(t_b, t_a)}$$

The following properties hold: $\hat{U}^{-1}(t_b, t_a) = \hat{U}(t_a, t_b)$

What happens if the Hamiltonian is an explicit function of time?

Here $\hat{H} \equiv \hat{H}(\hat{p}, \hat{x}, t)$. To solve Schrödinger eqn we slice the time interval $(t_b - t_a)$ into $(N+1)$ small pieces, of width $\epsilon = (t_b - t_a)/(N+1)$ so that n-th time interval $t_n = t_a + n\epsilon$. [Here $n = 0, --, N+1$].

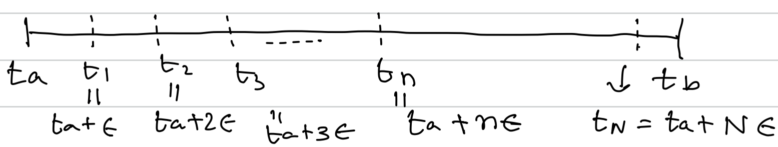
Now we iteratively try to solve S.E.:

$$|\psi(t_a+\epsilon)\rangle \approx \left(1 - i/\hbar \int_{t_a}^{t_a+\epsilon} dt\,\hat{H}(t)\right)|\psi(t_a)\rangle,$$

$$|\psi(t_a+2\epsilon)\rangle \approx \left(1 - i/\hbar \int_{t_a+\epsilon}^{t_a+2\epsilon} dt\,\hat{H}(t)\right)|\psi(t_a+\epsilon)\rangle,$$

$$\vdots$$

$$|\psi(t_a+(N+1)\epsilon)\rangle \approx \left(1 - i/\hbar \int_{t_a+N\epsilon}^{t_a+(N+1)\epsilon} dt\,\hat{H}(t)\right)|\psi(t_a+N\epsilon)\rangle$$



$t_b = t_a + (N+1)\epsilon$

Hence we have

$$|\psi(t_a+(N+1)\epsilon)\rangle \approx \left(1 - i/\hbar \int_{t_N}^{t_b} dt'_{N+1}\,\hat{H}(t'_{N+1})\right)$$
$$\times\left(1 - i/\hbar \int_{t_{N-1}}^{t_N} dt'_N\,\hat{H}(t'_N)\right)\cdots\left(1 - \frac{i}{\hbar}\int_{t_a}^{t_1} dt'_1\,\hat{H}(t'_1)\right)$$
$$\times\,|\psi(t_a)\rangle$$

Following the convention of the time evolution operator

$$\boxed{\hat{U}(t_b,t_a) = \left(\mathbb{1} - \frac{i}{\hbar}\int_{t_N}^{t_b} dt'_{N+1}\,\hat{H}(t'_{N+1})\right) \cdots \left(\mathbb{1} - \frac{i}{\hbar}\int_{t_a}^{t_1} dt'_1\,\hat{H}(t'_1)\right)}$$

$$= \mathbb{1} - \frac{i}{\hbar}\int_{t_a}^{t_b} dt'_1\,\hat{H}(t'_1)$$

$$+(-i/\hbar)^2\Big[\int_{t_N}^{t_b} dt'_{N+1}\,\hat{H}(t'_{N+1})\Big\{\int_{t_{N-1}}^{t_N} dt'_N\,\hat{H}(t'_N) + \int_{t_{N-2}}^{t_{N-1}} dt'_{N-1}\,\hat{H}(t'_{N-1})$$

$$+ \cdots \int_{t_a}^{t_1} dt'_1\,\hat{H}(t'_1)\Big\}$$

$$+ \int_{t_{N-1}}^{t_N} dt'_N\,\hat{H}(t'_N)\Big\{\int_{t_{N-2}}^{t_{N-1}} dt'_{N-1}\,\hat{H}(t'_{N-1}) + \cdots \int_{t_a}^{t_1} dt'_1\,\hat{H}(t'_1)\Big\}$$

$$\vdots$$

$$+ \int_{t_1}^{t_2} dt'_2\,\hat{H}(t'_2)\Big\{\int_{t_a}^{t_1} dt'_1\,\hat{H}(t'_1)\Big\}\Big]$$

$$+ (-i/\hbar)^3[\cdots] + \cdots$$

$$= \mathbb{1} - \frac{i}{\hbar}\int_{t_a}^{t_b} dt'_1\,\hat{H}(t'_1)$$

$$+\left(-\frac{i}{\hbar}\right)^2\Big[\left(\int_{t_N}^{t_b} dt'_{N+1}\,\hat{H}(t'_{N+1})\right)\left(\int_{t_a}^{t_N} dt'_N\,\hat{H}(t'_N)\right)$$

$$+\left(\int_{t_{N-1}}^{t_N} dt'_N\cdot\hat{H}(t'_N)\right)\left(\int_{t_a}^{t_{N-1}} dt'_{N-1}\,\hat{H}(t'_{N-1})\right)$$

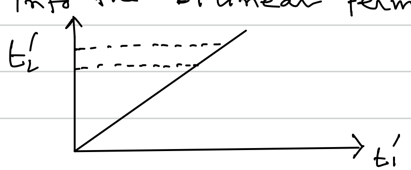
$$\vdots$$

$$+\left(\int_{t_1}^{t_2} dt'_2\,\hat{H}(t'_2)\right)\left(\int_{t_a}^{t_1} dt'_1\,\hat{H}(t'_1)\right)\Big] + \left(\frac{-i}{\hbar}\right)^3[\cdots] + \cdots$$

$$= \mathbb{1} - \frac{i}{\hbar}\int_{t_a}^{t_b} dt'_1\,\hat{H}(t'_1) + \left(\frac{-i}{\hbar}\right)^2\left[\sum_{i=1}^{N}\left(\int_{t_i}^{t_{i+1}} dt'_2\,\hat{H}(t'_2)\right)\left(\int_{t_a}^{t_i} dt'_1\,\hat{H}(t'_1)\right)\right]$$

$$+(-i/\hbar)^3[\cdots] + \cdots$$

we try to look into the bilinear term in a graphical way:



The sum in the bracket fills up the entire area above the diagonal in the $N \to \infty$ limit.