PHY-685
QFT-1

Lecture - 25+26

In the last lecture we found higher order correction to the 2-point correlation functions and found that there will be combination of connected and dis-connected pieces in the general expansion of:

$$\langle 0 | \hat{\phi}_I(x_1) \cdots \hat{\phi}_I(x_n) \exp\left[-i \int d^4x \, \mathcal{H}_{int}(\hat{\phi}_I(x))\right] | 0 \rangle .$$

The disconnected diagrams (or bubbles) are the ones which are only self contracted.

Now the above expansion, each term comes with an $\int d^4 z = \lim_{T \to \infty(1-\epsilon)} \int_{-T}^{T} dz^0 \int_{-\infty}^{\infty} d^3\vec{z}$. Does the imaginary limit over z^0 integral causes an issue? In the momentum space we have the following representation:

$$\lim_{T\to\infty(1-i\epsilon)} \int_{-T}^{T} dz^0 \int_{-\infty}^{\infty} d^3\vec{z} \; e^{-i(\sum p_i)\cdot z} (\cdots) = \int_{-T}^{T} dz^0 \int_{-\infty}^{\infty} d^3\vec{z}\, e^{-ip\cdot z} (\cdots)$$

$$= \lim_{T\to\infty(1-i\epsilon)} \int_{-T}^{T} dz^0 \int_{-\infty}^{\infty} d^3\vec{z} \; e^{-i\left(p^0 z^0 - \vec{p}\cdot\vec{z}\right)} \left(\cdots\right)$$

$$= \lim_{T\to\infty(1-i\epsilon)} \int_{-T}^{T} dz^0 \, e^{-ip^0 z^0} \int_{-\infty}^{\infty} d^3\vec{z}\; e^{i\vec{p}\cdot\vec{z}} \left(\cdots\right)$$

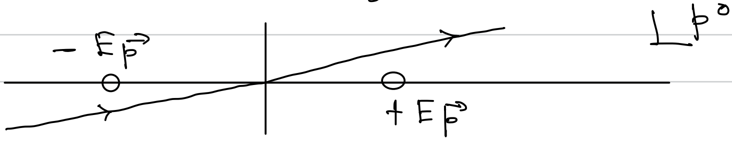
The momentum integrals are here

Now in $T \to \infty(1-i\epsilon)$, $e^{-ip^0 z^0} \sim \exp\left[-i(\mathrm{Re}\,p^0 + i\,\mathrm{Im}\,p^0)\,T(1-i\epsilon)\right]$

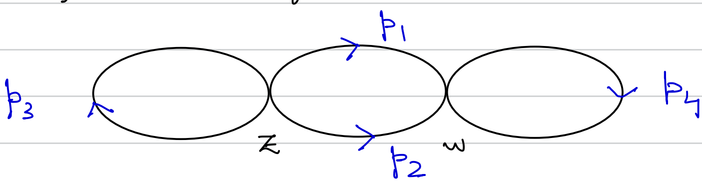
$$= \exp\left[(\mathrm{Re}\,p^0 + i\,\mathrm{Im}\,p^0)\,T\,(-\epsilon - i)\right] = \exp\left[-T\left\{(\epsilon\,\mathrm{Re}\,p^0 - \mathrm{Im}\,p^0) + i\,(\epsilon\,\mathrm{Im}\,p^0 + \mathrm{Re}\,p^0)\right\}\right]$$

$$= \exp\left[i\left(\mathrm{Re}\,p^0 + \epsilon\,\mathrm{Im}\,p^0\right)\right] \exp\left[+T\left\{(\mathrm{Im}\,p^0 - \epsilon\,\mathrm{Re}\,p^0)\right\}\right]$$

To ensure the amplitude is not divergent we have to assign $\mathrm{Im}\,p^0 = \epsilon\,\mathrm{Re}\,p^0$, or in other words we need to ensure that the contour of p^0 is $\propto (1+i\epsilon)$



The explicit T dependence in the above expression gets cancelled by choosing a p^0 contour which cancels the finite terms coming from $\epsilon \cdot T$ terms in $T \to \infty(1-i\epsilon)$ limit.
Hence connected diagrams are free from pathological issues.
Now what about the disconnected diagrams?
Lets consider the following diagram:



Following the Feynman's rule, the amplitude of such a diagram is:

$$A = \left(\frac{-i\lambda}{4!}\right)^2 \left(\int \prod_{i=1}^{4} \frac{d^4 p_i}{(2\pi)^4} \left[\frac{i}{p_3^2 - m^2 + i\epsilon}\right] \left\{(2\pi)^4 \delta^4(-p_3 + p_3 + p_1 + p_2)\right\} \right.$$

$$\times \left[\frac{i}{p_1^2 - m^2 + i\epsilon}\right] \left[\frac{i}{p_2^2 - m^2 + i\epsilon}\right] \left\{(2\pi)^4 \delta^4(-p_1 - p_2 + p_4 - p_4)\right\}$$

$$\left. \times \left[\frac{i}{p_4^2 - m^2 + i\epsilon}\right] \right) \times (4 \times 3 \times 3)$$

$$= -\frac{\lambda^2}{16} \left[\int \frac{d^4 p_3}{(2\pi)^4} \frac{i}{p_3^2 - m^2 + i\epsilon}\right]^2 \left[\int \frac{d^4 p}{(2\pi)^4} \left(\frac{i}{p^2 - m^2 + i\epsilon}\right)^2\right] \underbrace{(2\pi)^4 \delta^4(0)}$$

We want to understand the origin of the awkward singularity
If we write down the same amplitude using position space Feynman rule, we have:

$$A = -\lambda^2/16 \,[D_F(0)]^2 \int d^4z \, d^4w \,[D_F(z-w)]^2$$

$$= -\lambda^2/16 \,[D_F(0)]^2 \int d^4z \, d^4w \left[\int \frac{d^4p}{(2\pi)^4} D_F(p) e^{-ip\cdot(z-w)} \times \int d^4k \, D_F(k) \, e^{-ik\cdot(z-w)}\right]$$

$$= -\lambda^2/16 \,[D_F(0)]^2 \int \frac{d^4p}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} D_F(p) D_F(k) \int d^4z \, d^4w \, e^{-iz(p+k)} \times e^{+iw(p+k)}$$

$$= -\frac{\lambda^2}{16}\left[D_F(0)\right]^2 \int d^4z \int \frac{d^4p}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} D_F(p)\, D_F(k)\; e^{-iz(p+k)} \times (2\pi)^4 \delta^4(p+k)$$

$$= -\lambda^2/16 \left[D_F(0)\right]^2 \int d^4z \left[\int \frac{d^4p}{(2\pi)^4} D_F(p)\, D_F(-p)\right]$$

$$= \alpha \int d^4z \,, \text{ where } \alpha = -\lambda^2/16 \left[D_F(0)\right]^2 \times \left[\int \frac{d^4p}{(2\pi)^4} D_F(p)\, D_F(-p)\right]$$

$$= \lim_{T\to\infty(1-i\epsilon)} \alpha \int_{-T}^{T} dz^0 \int_{-\infty}^{\infty} d^3\vec{z}$$

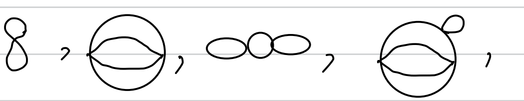
$$= \lim_{T\to\infty} (1-i\epsilon)\alpha \cdot (2T)\cdot(\text{volume of space}) = 2T\cdot V$$

This tells us that such a disconnected process can occur anywhere within the time $-T$ to $+T$. This $2T\cdot V$ is the hanging $(2\pi)^4\delta^4(0)$ factor when expressed in terms of momentum space Feynman rule. So there is an explicit dependence on the T parameter which we need to evaluate in the complex limit. What is the contribution of such pieces in the correlation function?

In a particular order of perturbation theory, when we have both connected amplitude and bubble diagrams, the total amplitude $A = A_C \times A_{DC}$

Amplitude of the connected piece   Amplitude of the bubble

For a given n-point correlation function, different diagrams can contribute to A_{DC}:

$S_B \in \{$  $, \ldots \}$

If an element in S_B has an amplitude V_i and in a general construction it appears n_i times, then it's contribution in A_{DC} is: $1/n_i!\,(V_i)^{n_i}$

So the total amplitude can be written as

$$A = A_C \times A_{DC} = A_C \times \prod_{i \in S_B} 1/n_i! \, (V_i)^{n_i}$$

Given an order in perturbation theory and a fixed n-point correlation function, we first construct the particular combination of the contraction with external legs; say α. So $A_C \equiv A_C(\alpha)$. The crux of the point is that once for a particular combination α, $A_C(\alpha)$ is computed, the A_{DC} (for the given order of perturbation theory) is independent of α.

$$\text{So } A = \left(\sum_\alpha A_C(\alpha)\right) \times \left[\sum_{\text{all } \{n_i\}} \left(\prod_i \frac{1}{n_i!} V_i^{n_i}\right)\right]$$

$$= \left(\sum_\alpha A_C(\alpha)\right) \times \left[\prod_i \left(\sum_{n_i=0}^{\infty} 1/n_i! \, V_i^{n_i}\right)\right]$$

This intercha-
-nge is possible
because integral over internal space-time points in different kinds of bubble diagrams are independent of each other.

$$= \left[\sum_\alpha A_C(\alpha)\right] \underbrace{\prod_i \exp(V_i)}_{\text{Contains all the spurious } (2\pi)^4 \delta^4(0) \text{ factors!}} = \left[\sum_\alpha A_C(\alpha)\right] \exp\left[\sum_i V_i\right]$$

Now this is not the end of the story. For the observable we have a denominator of the form:

$$\langle 0 | \exp\left[-i \int H_{int}(\hat{\phi}_I(x))\right] | 0 \rangle$$

This is exactly the sum of the bubble diagrams (with no external legs). So there is a cancellation and the physical observable is $= \left[\sum_\alpha A_C(\alpha)\right]$

$\Rightarrow$

$$\langle \Omega | \hat{T} \{ \hat{\phi}(x) \hat{\phi}(y) \} | \Omega \rangle$$

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