PHY-685
QFT-1

Lecture - 17

In last lecture we computed the observable

$$\langle \Omega | [\hat{\phi}(x), \hat{\phi}(y)] | \Omega \rangle :$$

for a free real scalar field $\hat{\phi}(x)$ and established that it vanishes if $(x-y)^2 < 0$. So the free particle theory and Hilbert space which we have developed doesn't go down in drain.

$$\langle \Omega | [\hat{\phi}(x), \hat{\phi}(y)] | \Omega \rangle$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right) \Big|_{p^0 = E_{\vec{p}}}$$

Now we concentrate on the 2nd-term:

$$\begin{aligned} p\cdot(x-y) &= E_{\vec{p}}(x^0 - y^0) - \vec{p}\cdot(\vec{x}-\vec{y}) \\ &= -\left[(-E_{\vec{p}})(x^0-y^0) - (-\vec{p})\cdot(\vec{x}-\vec{y})\right] \\ &= -\tilde{p}\cdot(x-y), \quad \text{where } \tilde{p} = -p \text{ and } \tilde{p}^0 = -E_{\vec{p}} = -E_{\tilde{\vec{p}}} \end{aligned}$$

Hence the 2nd term can be written as:

$$+ \int \frac{d^3\tilde{\vec{p}}}{(2\pi)^3} \frac{1}{2E_{\tilde{\vec{p}}}} \exp\left(-i\tilde{p}\cdot(x-y)\right) \Big|_{\tilde{p}^0 = -E_{\tilde{\vec{p}}}}$$

Hence we can write $\langle \Omega | [\hat{\phi}(x), \hat{\phi}(y)] | \Omega \rangle$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \left[\left\{ \frac{1}{2p^0} e^{-ip\cdot(x-y)} \right\}_{p^0 = E_{\vec{p}}} + \left\{ \frac{1}{2p^0} e^{-ip\cdot(x-y)} \right\}_{p^0 = -E_{\vec{p}}} \right]$$

$$= \theta(x^0 - y^0) \int \frac{d^3\vec{p}}{(2\pi)^3} \int \frac{dp^0}{2\pi i} \left(\frac{-1}{p^2 - m^2} \right) \exp\left(-ip\cdot(x-y)\right)$$

In obtaining the last term, we use Cauchy's integral theorem:

$$\frac{1}{2\pi i} \oint_c f(z)\, dz = \sum_{\substack{\text{enclosed by} \\ c}} \text{Residues}\left(f(z)\right)$$

We now investigate how this works out!

Let's consider the integral:

$$I = \int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \frac{e^{-ip^0(x^0-y^0)}}{(p^0)^2 - E_{\vec{p}}^2}$$

[Here w.r.t. the integration variable p^0, $E_{\vec{p}}$ is a constant]

There are poles on the line of integration at $p^0 = \pm E_{\vec{p}}$ and thus we have to elevate p^0 to a complex variable and carry out the integral on a closed contour which includes the real line $p^0 \in [-\infty, +\infty]$ and have to prove that the additional contribution from the analytically extended contour vanishes.

First we consider $x^0 > y^0$:

In the analytically continued complex p^0 plane, let's write $p^0 = \mathrm{Re}(p^0) + i\,\mathrm{Im}(p^0)$.

$$\Rightarrow -ip^0(x^0-y^0) = [-i\,\mathrm{Re}(p^0) + \mathrm{Im}(p^0)](x^0-y^0)$$

$$\Rightarrow \exp[-ip^0(x^0-y^0)] = \exp[-i\,\mathrm{Re}(p^0)(x^0-y^0)]\cdot\exp[\mathrm{Im}(p^0)(x^0-y^0)]$$

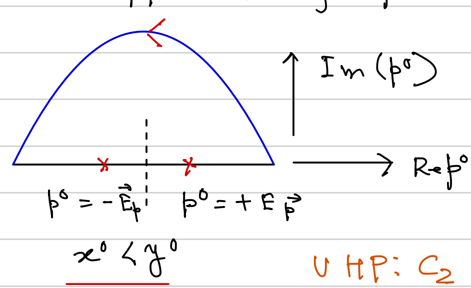
If the phase factor has to have a vanishing contribution from large $\mathrm{Im}(p^0)$ part, then the contour should be closed clock-wise in the Lower-Half-Plane (LHP), such that $\mathrm{Im}(p^0) < 0$.

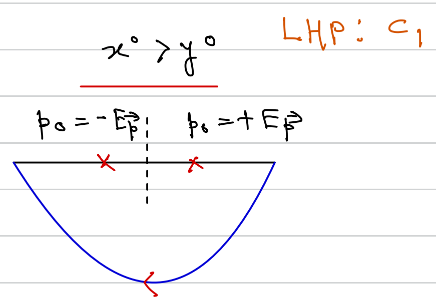
For the case $x^0 < y^0$:

It is quite evident from the expression of

$$\exp[-ip^0(x^0-y^0)] = \exp[-i\,\mathrm{Re}(p^0)(x^0-y^0)]\cdot\exp[\mathrm{Im}(p^0)(x^0-y^0)],$$

when $x^0 < y^0$, $\mathrm{Im}(p^0)$ must be +ve for the phase factor to be damped and thus the contour must be enclosed anti-clock-wise in the upper half plane

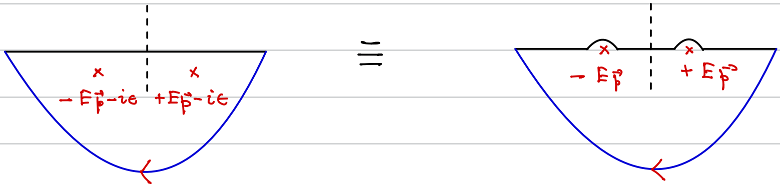




we combine both the cases at one go as:

$$I = \theta(x^0-y^0)\oint_{C_1}\frac{dp^0}{2\pi}\frac{e^{-ip^0(x^0-y^0)}}{(p^0)^2-E_{\vec{p}}^2} + \theta(y^0-x^0)\oint_{C_2}\frac{dp^0}{2\pi}\frac{e^{-ip^0(x^0-y^0)}}{(p^0)^2-E_{\vec{p}}^2}$$

For retarded case; i.e. $\theta(x^0-y^0)=1$, the poles have to be shifted to the LHP.



For the retarded case the poles are located at $p^0 = \pm E_{\vec{p}} - i\epsilon$ $(\epsilon>0)$

$\Rightarrow (p^0+i\epsilon)^2 - E_{\vec{p}}^2 = 0$. Hence

$$J_R = \theta(x^0-y^0)\oint_{C_1}\frac{dp^0}{2\pi}\frac{e^{-ip^0(x^0-y^0)}}{(p^0+i\epsilon)^2-E_{\vec{p}}^2} + \underbrace{\theta(y^0-x^0)\oint_{C_2}\frac{dp^0}{2\pi}\frac{e^{-ip^0(x^0-y^0)}}{(p^0+i\epsilon)^2-E_{\vec{p}}^2}}_{=0}$$

$$= \theta(x^0-y^0)\oint_{C_1}\frac{dp^0}{2\pi}\frac{e^{-ip^0(x^0-y^0)}}{(p^0+i\epsilon)^2-E_{\vec{p}}^2}$$

$$= \theta(x^0-y^0)\oint_{C_1}\frac{dp^0}{2\pi}e^{-ip^0(x^0-y^0)}\left\{\frac{1}{(p^0+i\epsilon)-E_{\vec{p}}}\;\frac{1}{(p^0+i\epsilon)+E_{\vec{p}}}\right\}$$

$$= \theta(x^0-y^0)\frac{1}{2\pi}(-2\pi i)\left[\frac{e^{-i(E_{\vec{p}}-i\epsilon)(x^0-y^0)}}{2E_{\vec{p}}} + \frac{e^{-i(-E_{\vec{p}}-i\epsilon)(x^0-y^0)}}{-2E_{\vec{p}}}\right]$$

$$= {}^1/_i\,\frac{\theta(x^0-y^0)}{2E_{\vec{p}}}\left[\exp\left(-iE_{\vec{p}}(x^0-y^0)\right) - \exp\left(+iE_{\vec{p}}(x^0-y^0)\right)\right]$$

Now lets look at the expression we conjectured:

$$\theta(x^0-y^0)\int\frac{d^3\vec{p}}{(2\pi)^3}\int\frac{dp^0}{2\pi i}\left(\frac{-1}{p^2-m^2}\right)\exp\left(-ip\cdot(x-y)\right)$$

$$=\int\frac{d^3\vec{p}}{(2\pi)^3}\exp\left(i\vec{p}\cdot(\vec{x}-\vec{y})\right)i\left[\theta(x^0-y^0)\int\frac{dp^0}{2\pi}\frac{1}{(p^0)^2-E_{\vec{p}}^2}e^{-ip^0(x^0-y^0)}\right]$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \exp\left(i\,\vec{p}\cdot(\vec{x}-\vec{y})\right) i \left[\frac{1}{i}\,\frac{\theta(x^0-y^0)}{2E_{\vec{p}}}\left\{e^{-iE_{\vec{p}}(x^0-y^0)} - e^{+iE_{\vec{p}}(x^0-y^0)}\right\}\right]$$

$$= \theta(x^0-y^0)\int \frac{d^3\vec{p}}{(2\pi)^3}\left[\left\{\frac{1}{2p^0}\,e^{-ip\cdot(x-y)}\right\}_{p^0=E_{\vec{p}}} + \left\{\frac{1}{2p^0}\,e^{-ip\cdot(x-y)}\right\}_{p^0=-E_{\vec{p}}}\right]$$

$$= \theta(x^0-y^0)\,\langle\Omega|[\hat{\phi}(x),\hat{\phi}(y)]|\Omega\rangle$$

So we have :

$$\theta(x^0-y^0)\,\langle\Omega|[\hat{\phi}(x),\hat{\phi}(y)]|\Omega\rangle$$

$$= \theta(x^0-y^0)\int\frac{d^4p}{(2\pi)^4}\,\frac{i}{(p^0+i\epsilon)^2-E_{\vec{p}}^2}\,\exp\left(-i\,p\cdot(x-y)\right)$$

$$= \theta(x^0-y^0)\int\frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\,\mathrm{sgn}(p^0)\cdot\epsilon}\,\exp\left(-i\,p\cdot(x-y)\right)$$

$$\Rightarrow \boxed{D_R(x-y) = \theta(x^0-y^0)\,D(x-y)}$$

Similarly one can shift the poles in UHP and integrate over C_2 to get

$$\boxed{D_A(x-y) = \theta(y^0-x^0)\,D(x-y)}$$

Now we verify something i.e. is well expected :
Retarded object $D_R(x-y)$ is a Green's function for the Klein-Gordon operator: (∂^2+m^2)

$$\text{i.e. } (\partial^2+m^2)\,D_R(x-y) \sim \delta^4(x-y)$$

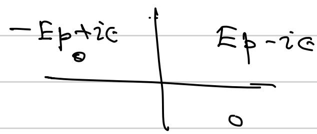
$$(\partial^2+m^2)\, D_R(x-y) = (\partial^2+m^2)\left[\theta(x^0-y^0)\,\langle\Omega|[\hat\phi(x),\hat\phi(y)]|\Omega\rangle\right]$$

$$= \left(\partial^2\theta(x^0-y^0)\right)\langle\Omega|[\hat\phi(x),\hat\phi(y)]|\Omega\rangle$$
$$+\, 2\left(\partial_\mu\theta(x^0-y^0)\right)\left(\partial^\mu\{\langle\Omega|[\hat\phi(x),\hat\phi(y)]|\Omega\rangle\}\right)$$
$$+\, \theta(x^0-y^0)\,(\partial^2+m^2)\,\langle\Omega|[\hat\phi(x),\hat\phi(y)]|\Omega\rangle$$

$$= \underbrace{-\delta(x^0-y^0)\,\langle\Omega|[\hat\pi(x),\hat\phi(y)]|\Omega\rangle}_{\text{follows from } \delta'[\phi] = -\delta[\phi'] = -\phi'[0]} + 2\,\delta(x^0-y^0)\,\langle\Omega|[\hat\pi(x),\hat\phi(y)]|\Omega\rangle + 0$$

$$= -i\,\delta^4(x-y)$$

Feynman Propagator: $\tilde{D}_F(p) = \dfrac{i}{p^2-m^2+i\epsilon}$



$$D_F(x-y) = D(x-y) \quad \text{for } x^0 > y^0$$
$$= D(y-x) \quad \text{for } x^0 < y^0$$

$$D_F(x-y) = \theta(x^0-y^0)\,\langle\Omega|\hat\phi(x)\,\hat\phi(y)|\Omega\rangle + \theta(y^0-x^0)\,\langle\Omega|\hat\phi(y)\,\hat\phi(x)|\Omega\rangle$$

$$= \langle\Omega|T\,\hat\phi(x)\,\hat\phi(y)|\Omega\rangle$$

Anti-time ordered $\tilde{D}_{\bar F}(p) = \dfrac{i}{p^2-m^2-i\epsilon}$

$$D_{\bar F}(x-y) = \theta(x^0-y^0)\,\langle\Omega|\hat\phi(y)\,\hat\phi(x)|\Omega\rangle$$
$$+\, \theta(y^0-x^0)\,\langle\Omega|\hat\phi(x)\,\hat\phi(y)|\Omega\rangle$$