PHY-685
QFT-1

Lecture - 11

General transformation properties of a generic quantum field:

Until now we have seen that a single, spin-"j" free particle state $|p, j, \sigma, n\rangle$ is created out of vacuum by operation of a "creation" operator $|p, j, \sigma, n\rangle = \hat{a}^{\dagger}(p, j, \sigma, n)|\Omega\rangle$. When these states are "on shell" states, their momentum components obey a specific dispersion rule: $p^0 = \sqrt{\vec{p}^{\,2} + m^2} \equiv E_{\vec{p}}$, such that $p^{\mu} = (E_{\vec{p}}, \vec{p})$ and the corresponding single-particle state can be labelled as $|\vec{p}, j, \sigma, n\rangle$ which should be created from $|\Omega\rangle$ by action of $\hat{a}^{\dagger}(p, j, \sigma, n) \equiv \sqrt{2E_{\vec{p}}}\, \hat{a}^{\dagger}(\vec{p}, j, \sigma, n)$.

Now we apriori worked out the normalization of the states. For the on-shell states we can write

$$
\begin{aligned}
\langle p', j, \sigma', n | p, j, \sigma, n\rangle &= \langle \vec{p}', j, \sigma', n | \vec{p}, j, \sigma, n\rangle \\
&= \langle \Omega | \hat{a}(\vec{p}', j, \sigma', n)\left(\sqrt{2E_{\vec{p}'}}\right)\left(\sqrt{2E_{\vec{p}}}\right)\hat{a}^{\dagger}(\vec{p}, j, \sigma, n)|\Omega\rangle \\
&= \sqrt{2E_{\vec{p}}}\sqrt{2E_{\vec{p}'}}\,\langle\Omega|\left[\hat{a}(\vec{p}', j, \sigma', n), \hat{a}^{\dagger}(\vec{p}, j, \sigma, n)\right]|\Omega\rangle \\
&= 2E_{\vec{p}'}(2\pi)^3\,\delta^3(\vec{p}' - \vec{p})\,\delta_{\sigma'\sigma}.
\end{aligned}
$$

Here we used $\left[\hat{a}(\vec{p}', j, \sigma', n), \hat{a}^{\dagger}(\vec{p}, j, \sigma, n)\right] = (2\pi)^3\,\delta^3(\vec{p}' - \vec{p})$.
The extra $(2\pi)^3$ comes from "Coleman–Mantra": every inetgral over momentum components should be written as $dp_i/2\pi$.
Under a Poincare transformation, these operators transform as:

$$
\boxed{\hat{U}(\Lambda, \alpha)\,\hat{a}^{\dagger}(\vec{p}, j, \sigma, n)\,\hat{U}^{-1}(\Lambda, \alpha) = \exp\left[-i(\Lambda p)\cdot\alpha\right]\sqrt{(\Lambda p)^0/p^0}\,\sum_{\sigma'} D^{(j)}_{\sigma'\sigma}\left(W(\Lambda, p)\right)\hat{a}^{\dagger}(\vec{\Lambda p}, j, \sigma', n)}
$$

A generic quantum field is expandable in terms of the creation and annihilation operators as:

$$
\boxed{\hat{\phi}^{(j)}_{n n^c}(x) = \sum_{\sigma}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\Big[u_n(\vec{p}, \sigma)\,\hat{a}(\vec{p}, j, \sigma, n)\,e^{-ip\cdot x} + v_{n^c}(\vec{p}, \sigma)\,\hat{a}^{\dagger}(\vec{p}, j, \sigma, n^c)\,e^{ip\cdot x}\Big]}
$$

Here n^c denotes the charge-conjugate version of the particle n.

Given we have seen that for bunch of free single particle states, belonging to certain "j-value" do not mix with each other, we might be interested in looking into what kind of field is formed by ($\#a + \% a^\dagger$) for a particular j, i.e. classification of operators according to a particular spin?

A multi-component field $\hat{\phi}_A(x)$, under homogeneous Lorentz transformation (HLT) $x' = \Lambda x$, in general transforms as

$$\boxed{\hat{\phi}_A(x) = L_A{}^B(\Lambda)\, \hat{U}(\Lambda)\, \hat{\phi}_B(\Lambda^{-1}x)\, \hat{U}^{-1}(\Lambda)}$$

Here $\hat{\phi}_A(x)$ is not necessarily Hermitean.

The finite dimensional matrices $L_A{}^B(\Lambda)$ must obey the group composition law $\underbrace{L_A{}^B(\Lambda_1)}\, L_B{}^C(\Lambda_2) = L_A{}^C(\Lambda_1 \Lambda_2)$
$\longrightarrow$ representation of Lorentz group.

For an infinitesimal Lorentz transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu$, we have $L_A{}^B(1+\delta\omega) = \delta_A{}^B + \frac{i}{2}\delta\omega_{\mu\nu}\underbrace{(\bar{S}^{\mu\nu})_A{}^B}$

$U(1+\delta\omega) = \mathbb{1} + \frac{i}{2}\delta\omega_{\mu\nu}\hat{M}^{\mu\nu}$

Representation of the generator of Lorentz group in (A-B) repr.

For a single component field, where $L_A{}^B(\Lambda) = \delta_A{}^B$, we have a trivial transformation rule: $\boxed{\hat{\phi}(x) = \hat{U}(\Lambda)\,\hat{\phi}(\Lambda^{-1}x)\,\hat{U}^{-1}(\Lambda)}$

Let $\phi(x)$ and $|\phi\rangle$ are the eigen-value and eigen-vector of such an operator: $\hat{\phi}(x)|\phi\rangle = \phi(x)|\phi\rangle$. Plugging the above transformation rule, we get:

$$\hat{U}(\Lambda)\,\hat{\phi}(\Lambda^{-1}x)\,\hat{U}^{-1}(\Lambda)\,|\phi\rangle = \phi(x)\,|\phi\rangle$$

$$\Rightarrow \hat{\phi}(\Lambda^{-1}x)\underbrace{\hat{U}^{-1}(\Lambda)|\phi\rangle}_{|\phi'\rangle} = \phi(x)\underbrace{\hat{U}^{-1}(\Lambda)|\phi\rangle}_{|\phi'\rangle}$$

$$\Rightarrow \phi'(\Lambda^{-1}x)\,|\phi'\rangle = \phi(x)\,|\phi'\rangle$$

$$\Rightarrow \phi'(\Lambda^{-1}x) = \phi(x)$$

The eigenvalues are local functions of space-time, which follows the above invariance rule. These invariant local functions are called scalar fields and the corresponding operators are called scalar quantum fields.

Now lets take derivative on both sides of this operator eq.

$$\partial^\mu \hat{\Phi}(x) = \frac{\partial}{\partial x_\mu} \hat{\Phi}(x) = \hat{U}(\Lambda) \frac{\partial}{\partial x_\mu} \hat{\Phi}(\Lambda^{-1}x) \hat{U}^{-1}(\Lambda)$$

$$= \hat{U}(\Lambda) \frac{\partial}{\partial (\Lambda^{-1}x)_\nu} \left(\hat{\Phi}(\Lambda^{-1}x)\right) \cdot \frac{\partial (\Lambda^{-1}x)_\nu}{\partial x_\mu} \cdot \hat{U}^{-1}(\Lambda)$$

$$= \hat{U}\, \bar{\partial}^\nu \hat{\Phi}(\Lambda^{-1}x) \frac{\partial}{\partial x_\mu} \{(\Lambda^{-1})_\nu{}^\beta x_\beta\} \hat{U}^{-1}(\Lambda)$$

$$= \hat{U}\, \bar{\partial}^\nu \hat{\Phi}(\Lambda^{-1}x) (\Lambda^{-1})_\nu{}^\beta \delta^\mu_\beta \hat{U}^{-1}(\Lambda)$$

$$= \Lambda^\mu{}_\nu \hat{U}(\Lambda) \bar{\partial}^\nu \hat{\Phi}(\Lambda^{-1}x) \hat{U}^{-1}(\Lambda)$$

$$\Rightarrow \boxed{\partial^\mu \hat{\Phi}(x) = \Lambda^\mu{}_\nu \hat{U}(\Lambda) \bar{\partial}^\nu \hat{\Phi}(\Lambda^{-1}x) \hat{U}^{-1}(\Lambda)}$$

This result makes us postulate that a vector operator should evolve as $\boxed{\hat{A}^\mu(x) = \Lambda^\mu{}_\nu \hat{U}(\Lambda) \hat{A}^\nu(\Lambda^{-1}x) \hat{U}^{-1}(\Lambda)}$

A tensor field would transform as

$$\boxed{\hat{B}^{\mu\nu}(x) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \hat{U}(\Lambda) \hat{B}^{\rho\sigma}(\Lambda^{-1}x) \hat{U}^{-1}(\Lambda)}$$

How does $\mathrm{tr}(B)$ transforms like?

In general, if one is given a tensor field

$$\hat{B}^{\mu\nu}(x) = \hat{A}^{\mu\nu}(x) + \hat{S}^{\mu\nu}(x) + \overbrace{\alpha}^{\to 1/4 \text{ for } 4\text{-D}} \eta^{\mu\nu} \hat{T}(x),$$

where $\hat{A}^{\mu\nu} = -\hat{A}^{\nu\mu}$, $\hat{S}^{\mu\nu} = \hat{S}^{\nu\mu}$ and $\hat{T}(x) = \eta_{\mu\nu} \hat{B}^{\mu\nu}$, then $\hat{A}^{\mu\nu}$, $\hat{S}^{\mu\nu}$ and $\hat{T}(x)$ doesn't mix with each other under Lorentz transformation.

Can we further decompose these pieces, which do not mix with each other under Lorentz transformation?

If an object contains n-indices, how do we decompose that?

Are there any other kind of indices that can be assigned to fields and if yes, how do they transform under Lorentz transformation?

Plugging these expression in the transformation rule of $\hat{\Phi}_A(x)$,

$$\hat{\Phi}_A(x) = \left[\delta_A{}^B + \frac{i}{2}\delta\omega_{\mu\nu}(\bar{S}^{\mu\nu})_A{}^B\right]\left(\mathbb{1} + \frac{i}{2}\delta\omega_{\rho\sigma}\hat{M}^{\rho\sigma}\right)$$

$$\times \hat{\Phi}_B((1-\delta\omega)\cdot x)\left(\mathbb{1} - \frac{i}{2}\delta\omega_{\alpha\beta}\hat{M}^{\alpha\beta}\right)$$

$$= \left[\delta_A{}^B + \frac{i}{2}\delta\omega_{\mu\nu}(\bar{S}^{\mu\nu})_A{}^B\right]\left(\mathbb{1} + \frac{i}{2}\delta\omega_{\rho\sigma}\hat{M}^{\rho\sigma}\right)\hat{\Phi}_B(x - \delta\omega\cdot x)$$
$$\times\left(\mathbb{1} - \frac{i}{2}\delta\omega_{\alpha\beta}\hat{M}^{\alpha\beta}\right)$$

$$= \left[\delta_A{}^B + \frac{i}{2}\delta\omega_{\mu\nu}(\bar{S}^{\mu\nu})_A{}^B\right]\left(\mathbb{1} + \frac{i}{2}\delta\omega_{\rho\sigma}\hat{M}^{\rho\sigma}\right)\times$$
$$\left[\hat{\Phi}_B(x) - (\delta\omega\cdot x)_\gamma\,\partial^\gamma\hat{\Phi}_B(x)\right]\left(\mathbb{1} - \frac{i}{2}\delta\omega_{\alpha\beta}\hat{M}^{\alpha\beta}\right)$$

$$= \hat{\Phi}_A(x) + \frac{i}{2}\delta\omega_{\mu\nu}(\bar{S}^{\mu\nu})_A{}^B\hat{\Phi}_B(x) + \delta_A{}^B\cdot\frac{i}{2}\delta\omega_{\rho\sigma}\hat{M}^{\rho\sigma}\hat{\Phi}_B$$
$$- \delta_A{}^B\,\delta\omega_{\gamma\nu}x^\nu\partial^\gamma\hat{\Phi}_B(x) - \frac{i}{2}\delta_A{}^B\hat{\Phi}_B\,\delta\omega_{\alpha\beta}\hat{M}^{\alpha\beta} + \mathcal{O}((\delta\omega)^2)$$

$$\Rightarrow \frac{i}{2}\delta\omega_{\mu\nu}[\hat{\Phi}_A, \hat{M}^{\mu\nu}] = \frac{i}{2}\delta\omega_{\mu\nu}(\bar{S}^{\mu\nu})_A{}^B\hat{\Phi}_B(x)$$
$$-\frac{1}{2}\delta\omega_{\mu\nu}(x^\nu\partial^\mu - x^\mu\partial^\nu)\hat{\Phi}_A(x)$$

$$= \frac{i}{2}\delta\omega_{\mu\nu}(\bar{S}^{\mu\nu})_A{}^B\hat{\Phi}_B(x) - \frac{i^2}{2}\delta\omega_{\mu\nu}(x^\mu\partial^\nu - x^\nu\partial^\mu)\hat{\Phi}_A(x)$$

$$\Rightarrow [\hat{\Phi}_A(x), \hat{M}^{\mu\nu}] = -i(x^\mu\partial^\nu - x^\nu\partial^\mu)\hat{\Phi}_A(x) + (\bar{S}^{\mu\nu})_A{}^B\hat{\Phi}_B(x)$$

$$\Rightarrow \boxed{[\hat{\Phi}_A(x), \hat{M}^{\mu\nu}] = -\mathcal{L}^{\mu\nu}\hat{\Phi}_A(x) + (\bar{S}^{\mu\nu})_A{}^B\hat{\Phi}_B(x)},$$

where $\boxed{\mathcal{L}^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)}$

Both $\mathcal{L}^{\mu\nu}$ and $(S^{\mu\nu})_A{}^B$ should satisfy the same algebra of the Lorentz group. Remember that $\hat{M}^{\mu\nu}$ must be Hermitean but $(S^{\mu\nu})_A{}^B$ needn't be Hermitian.

Our task reduces to finding all finite dimensional representation $(S^{\mu\nu})_A{}^B$ to find non-trivial representations of Lorentz group.

Remember: with the definition $\hat{J}_i = \frac{1}{2} \epsilon_{ijk} \hat{M}^{jk}$ and $\hat{k}_i = \hat{M}_{0i}$, we had the algebra:

$$[\hat{J}_i, \hat{J}_j] = i\, \epsilon_{ijk}\, \hat{J}_k, \quad [\hat{J}_i, \hat{k}_j] = i\, \epsilon_{ijk}\, \hat{k}_k \quad \text{and}$$
$$[\hat{k}_i, \hat{k}_j] = -i\, \epsilon_{ijk}\, \hat{k}_k .$$

The first eqn of the algebra is an SU(2) algebra and represents angular momentum operator. So from Q.M. a spin-j representation corresponds to $(2j+1)\times(2j+1)$ size matrices $\hat{J}_1$, $\hat{J}_2$ and $\hat{J}_3$. The eigen-values of $\hat{J}_3$ are $(-j, \dots +j)$ and j can take values $0, \frac{1}{2}, 1, \frac{3}{2}, \dots$. These matrices constitute of all inequivalent, irreducible representations of SO(3). The half-integer representations are representations of SU(2).

The Lorentz algebra can be decomposed into two SU(2) algebra with the redefinition:

$$\hat{N}_i = \frac{1}{2}(\hat{J}_i - i\hat{k}_i), \quad \hat{N}_i^{\dagger} = \frac{1}{2}(\hat{J}_i + i\hat{k}_i), \text{ such that}$$

$$[\hat{N}_i, \hat{N}_j] = i\, \epsilon_{ijk}\, \hat{N}_k, \quad [\hat{N}_i^{\dagger}, \hat{N}_j^{\dagger}] = i\, \epsilon_{ijk}\, \hat{N}_k^{\dagger} \quad \text{and} \quad [\hat{N}_i, \hat{N}_j^{\dagger}] = 0$$

So we have two pieces of algebra which do not mix under L-T. The two representations are exchanged via Hermitean conjugation. They are also exchanged under parity.

That means the representation of Lorentz group can be labelled by a pair of SU(2) quantum numbers (j_1, j_2); j_1 corresponding to $\hat{N}_i$ and j_2 corresponding to $\hat{N}_i^{\dagger}$. The number of components of a representation is $(2j_1+1)(2j_2+1)$.

Now $\hat{J}_i = \hat{N}_i + \hat{N}_i^{\dagger}$ and hence for a (j_1, j_2) representation the allowed values of j is just the addition of angular momentum and can take the values (j_1+j_2) to $|j_1 - j_2|$.
Now $(j_1 = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots)$ and $(j_2 = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots)$.
Any pair of combination is a valid combination

The simplest representations are

$(j_1 = 0, j_2 = 0)$:= scalar, [SO(3) rep := 0]

$(j_1 = \frac{1}{2}, j_2 = 0)$:= Left-handed $\underbrace{\text{spin-}1/2}$ object (a.k.a. spinor)

$(j_1 = 0, j_2 = 1/2)$:= Right-handed spinor [SO(3) rep := 1/2]

$(j_1 = 1/2, j_2 = 1/2)$:= vector [SO(3) rep: $1 \oplus 0$]

$(j_1 = 1, j_2 = 0)$:= spin-1 [SO(3) rep: 1]

$(j_1 = 0, j_2 = 1)$:= spin-1 [SO(3) rep: 1]

$(j_1 = 1, j_2 = 1)$:= Tensor [SO(3) rep: $2 \oplus 1 \oplus 0$]