PHY-685
QFT-1

Lecture - 30

When we apply the rules of angular momentum addition, we will get:

$$\left(\tfrac{1}{2},0\right) \otimes \left(\tfrac{1}{2},0\right) = (0,0)_A + (1,0)_S$$

$$\Rightarrow \hat{C}_{ab}(x) = \varepsilon_{ab}\hat{D}(x) + \hat{G}_{ab}(x)$$

$\varepsilon_{ab} = -\varepsilon_{ba}$ are anti-symmetric constants. The convention adapted is $\varepsilon_{12} = -\varepsilon_{21} = -1$. $\hat{D}(x)$ is a scalar field. $\equiv$ $\varepsilon_{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = i\sigma^2$

plugging this expression into the transformation rule of $\hat{C}_{ab}(x)$, we find

$$\varepsilon_{ab}\hat{D}(x) + \hat{G}_{ab}(x) = L_a{}^c(\Lambda) L_b{}^d(\Lambda)\, \hat{U}(\Lambda) \times \left[\varepsilon_{cd}\hat{D}(\Lambda^{-1}x) + \hat{G}_{cd}(\Lambda^{-1}x)\right] \hat{U}^{-1}(\Lambda)$$

$\hat{D}(x)$ is a scalar iff:

$$\boxed{\varepsilon_{ab} = L_a{}^c(\Lambda) L_b{}^d(\Lambda)\, \varepsilon_{cd}}.$$

This says that ε_{ab} is an invariant symbol of $\left(\tfrac{1}{2},0\right) \otimes \left(\tfrac{1}{2},0\right)$ representation and thus plays the role of Cartan metric of that representation, which can be used to raise and lower the indices.

The inverse of this metric $\varepsilon^{ab} \equiv (\varepsilon_{ab})^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

$$\Rightarrow \varepsilon^{12} = \varepsilon_{21} = +1, \quad \varepsilon^{21} = \varepsilon_{12} = -1, \quad \varepsilon^{\dot{1}\dot{2}} = \varepsilon_{\dot{2}\dot{1}} = +1, \quad \varepsilon^{\dot{2}\dot{1}} = \varepsilon_{\dot{1}\dot{2}} = -1$$

Hence we have $\varepsilon_{ab}\varepsilon^{bc} = \delta_a{}^c$ and $\varepsilon^{ab}\cdot\varepsilon_{bc} = \delta^a{}_c$

Now the raising and lowering is defined by:

$$\hat{\psi}^a(x) = \varepsilon^{ab}\hat{\psi}_b(x)$$

we also have $\hat{\psi}_a = \delta_a{}^c\hat{\psi}_c = \varepsilon_{ab}\varepsilon^{bc}\hat{\psi}_c = \varepsilon_{ab}\hat{\psi}^b$

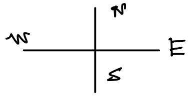
Now anti-symmetry of the metric (ε_{ab}) deserves some attention:

$$\hat{\psi}^a = \varepsilon^{ab}\hat{\psi}_b = -\varepsilon^{ba}\hat{\psi}_b = -\hat{\psi}_b\varepsilon^{ba} = \hat{\psi}_b\varepsilon^{ab}$$

$$\Rightarrow \boxed{\hat{\psi}^a = \varepsilon^{ab}\hat{\psi}_b = \hat{\psi}_b\varepsilon^{ab}}$$

$$\hat{\psi}^a\hat{\chi}_a = \varepsilon^{ab}\hat{\psi}_b\hat{\chi}_a = -\varepsilon^{ba}\hat{\psi}_b\hat{\chi}_a = -\hat{\psi}_b\hat{\chi}_a\varepsilon^{ba} = -\hat{\psi}_b\hat{\chi}^b$$

$$\Rightarrow \quad \boxed{\hat{\varphi}^a \hat{\chi}_a = - \hat{\varphi}_b \hat{\chi}^b}$$



How does this transform?

Similarly for $(0,\frac{1}{2}) \otimes (0,\frac{1}{2})$ representation, we have the invariant symbol $\varepsilon_{\dot{a}\dot{b}}$

We have $\hat{\varphi}^{\dagger}_{\dot{a}} = \varepsilon_{\dot{a}\dot{b}} \hat{\varphi}^{\dagger \dot{b}}$, $\hat{\varphi}^{\dagger \dot{b}} = \varepsilon^{\dot{b}\dot{c}} \hat{\varphi}^{\dagger}_{\dot{c}}$

Now let's consider a field of the form $\hat{A}_{a\dot{b}}(x) = \sigma^{\mu}_{a\dot{b}} \hat{A}_{\mu}(x)$. Here $\sigma^{\mu}_{a\dot{b}}$ is an invariant symbol and will be found to be $\sigma^{\mu}_{a\dot{b}} = (\mathbb{1}_{2\times 2}, \vec{\sigma})$. This is a singlet originating from

$$(\tfrac{1}{2},0) \otimes (0,\tfrac{1}{2}) \times (\tfrac{1}{2},\tfrac{1}{2}) = (0,0) \oplus \cdots$$

Similarly from the decomposition

$$(\tfrac{1}{2},\tfrac{1}{2}) \otimes (\tfrac{1}{2},\tfrac{1}{2}) = (0,0)_S \oplus (0,1)_A \oplus (1,0)_A \oplus (1,1)_S$$

symbol $\eta_{\mu\nu}$.

From $(\frac{1}{2},\frac{1}{2}) \otimes (\frac{1}{2},\frac{1}{2}) \otimes (\frac{1}{2},\frac{1}{2}) \otimes (\frac{1}{2},\frac{1}{2})$ we get the invariant quantity $\epsilon^{\mu\nu\rho\sigma}$, with $\epsilon^{0123} = +1$

Q. Show that $\epsilon^{\mu\nu\rho\sigma}$ is invariant under L.T

Let's look into the structure of the Weyl Fermions in a bit more detail:

$$\psi_L \equiv \phi_a = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \quad \psi_R \equiv \chi^{\dagger \dot{b}} = \begin{bmatrix} \chi^{\dagger \dot{1}} \\ \chi^{\dagger \dot{2}} \end{bmatrix}$$

Now we have the relation $\phi^a = \epsilon^{ab} \phi_b$. What does this relation looks like in term of matrix? Let's remember

$$\varepsilon^{ab} \equiv (\varepsilon_{ab})^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now $\phi\chi \equiv \phi^a \chi_a$ is an invariant object. If χ_a has a column structure, the ϕ^a must have the representation of a row. So that $\phi^a \chi_a$ produces an invariant through "dot product".

$$\phi^a \equiv \epsilon^{ab} \phi_b \Rightarrow \phi^1 = \epsilon^{12} \phi_2 = \phi_2 \quad \text{and} \quad \phi^2 = \epsilon^{21} \phi_1 = -\phi_1$$

similarly $\chi^\dagger_{\dot b} = \epsilon_{\dot b \dot c}\, \chi^{\dagger \dot c} \Rightarrow \chi^\dagger_{\dot 1} = \epsilon_{\dot 1 \dot 2}\, \chi^{\dagger \dot 2} = -\chi^{\dagger \dot 2}$

$$\chi^\dagger_{\dot 2} = \epsilon_{\dot 2 \dot 1}\, \chi^{\dagger \dot 1} = +\chi^{\dagger \dot 1}$$

$$\Rightarrow \chi^\dagger_{\dot b} = (\chi^\dagger_{\dot 1}, \chi^\dagger_{\dot 2}) = (-\chi^{\dagger \dot 2}, \chi^{\dagger \dot 1})$$

Now $(\phi_a)^\dagger = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}^\dagger = [\phi_1^* \quad \phi_2^*] \equiv \phi^\dagger_{\dot a}$

$$\Rightarrow \phi^\dagger_{\dot 1} = \phi_1^*, \quad \phi^\dagger_{\dot 2} = \phi_2^*$$

Now from the previous discussions, we know that ε_{ab}, $\varepsilon_{\dot c \dot d}$ and $\sigma^\mu_{a\dot b}$ etc. are invariant symbols. Hence contraction of two σ^μ's must be also an invariant.

This is found to $\boxed{\sigma^\mu_{a\dot b}\, \sigma_{\mu; c\dot d} = 2\, \varepsilon_{ac}\, \varepsilon_{\dot b \dot d}}$.

Using this identity, we can further check

$$\boxed{\varepsilon^{ab}\, \varepsilon^{\dot c \dot d}\, \sigma^\mu_{a\dot c}\, \sigma^\nu_{b\dot d} = 2\eta^{\mu\nu}}$$

Now let's define $\bar\sigma^{\mu; \dot a b} \equiv \varepsilon^{bc}\, \varepsilon^{\dot a \dot d}\, \sigma^\mu_{c\dot d}$

$$\Rightarrow \bar\sigma^\mu = (\sigma^0, -\vec\sigma)$$

similarly we have $\sigma^\mu_{a\dot b} = \varepsilon_{ac}\, \varepsilon_{\dot b \dot d}\, \bar\sigma^{\mu; \dot d c}$;

$$\varepsilon^{ab}\, \sigma^\mu_{b\dot c} = \varepsilon_{\dot c \dot d}\, \bar\sigma^{\mu \dot d a}, \quad \varepsilon^{\dot a \dot b}\, \sigma^\mu_{c\dot b} = \varepsilon_{cd}\, \bar\sigma^{\mu; \dot a d}$$

$$A_\mu B_\nu = \frac{1}{2}(A_\mu B_\nu - A_\nu B_\mu) + \frac{1}{2}(A_\mu B_\nu + A_\nu B_\mu)$$

$$= \frac{1}{D}(A\cdot B)\,\eta_{\mu\nu} + \frac{1}{2}(A_\mu B_\nu - A_\nu B_\mu) + \frac{1}{2}\left[A_\mu B_\nu + A_\nu B_\mu - \frac{2}{D}(A\cdot B)\,\eta_{\mu\nu}\right]$$

$$\downarrow$$

The decomposition of a Lorentz tensor.

Lagrangian for spinor fields:

1. It has to be at least quadratic in nature.
2. The corresponding Hamiltonian must be bounded from below.
3. It has to be real.

Here $\psi_a(x)$ is the eigenval
$\hat{\psi}_a(x)|\Psi\rangle = \psi_a(x)|\Psi\rangle$.

The default quadratic term is $\psi\psi + \psi^\dagger\psi^\dagger$.
For propagating D.O.F. we need derivative terms.
Let's try to replicate scalars

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\psi\,\partial^\mu\psi + h.c) \equiv \frac{1}{2}(\partial_\mu\psi\,\partial^\mu\psi + \partial_\mu\psi^\dagger\,\partial^\mu\psi^\dagger)$$

$$= \frac{1}{2}\left[\varepsilon^{cd}\,\partial_\mu\psi_d\,\partial^\mu\psi_c + \partial_\mu\psi^\dagger_{\dot{e}}\,\partial^\mu\psi^\dagger_{\dot{f}}\,\varepsilon^{\dot{f}\dot{e}}\right]$$

$$= \frac{1}{2}\left[\varepsilon^{cd}\,\partial_0\psi_d\,\partial_0\psi_c - \varepsilon^{cd}\,\partial_i\psi_d\,\partial_i\psi_c + \partial_0\psi^\dagger_{\dot{e}}\,\partial_0\psi^\dagger_{\dot{f}}\,\varepsilon^{\dot{f}\dot{e}} - \partial_i\psi^\dagger_{\dot{e}}\,\partial_i\psi^\dagger_{\dot{f}}\,\varepsilon^{\dot{f}\dot{e}}\right]$$

Lets compute the conjugate momentum of $\psi_a := \pi^a$

$$\pi^a = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi_a)} = \frac{1}{2}\left[\varepsilon^{cd}\,\delta^a_d\,\partial_0\psi_c - \varepsilon^{cd}\,\partial_0\psi_d\,\delta^a_c\right]$$

$$= \frac{1}{2}\left[-\varepsilon^{ac}\,\partial_0\psi_c - \partial_0\psi^a\right] = -\partial_0\psi^a$$

$$\pi^{\dagger\dot{a}} = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi^\dagger_{\dot{a}})} = \frac{1}{2}\left[\delta^{\dot{a}}_{\dot{e}}\,\partial_0\psi^\dagger_{\dot{f}}\,\varepsilon^{\dot{f}\dot{e}} - \partial_0\psi^\dagger_{\dot{e}}\,\delta^{\dot{a}}_{\dot{f}}\,\varepsilon^{\dot{f}\dot{e}}\right]$$

$$= \frac{1}{2}\left[\partial_0\psi^{\dagger\dot{a}} + \partial_0\psi^{\dagger\dot{a}}\right] = \partial_0\psi^{\dagger\dot{a}}$$

$$\Rightarrow \boxed{(\pi^a)^\dagger = (-\partial_0\psi^a)^\dagger = -\partial_0\psi^{\dagger\dot{a}} = -\pi^{\dagger\dot{a}}}$$

Now $\mathcal{H} = \pi^a\,\partial_0\psi_a + \pi^\dagger_{\dot{a}}\,\partial_0\psi^{\dagger\dot{a}} - \mathcal{L}$

$$= -\pi^a\pi_a + \pi^\dagger_{\dot{a}}\pi^{\dagger\dot{a}} - \frac{1}{2}\left[\varepsilon^{cd}\,\pi_d\pi_c - \varepsilon^{cd}\,\partial_i\psi_d\,\partial_i\psi_c + \pi^\dagger_{\dot{e}}\,\pi^{\dagger\dot{e}} - \partial_i\psi^\dagger_{\dot{e}}\,\partial_i\psi^{\dagger\dot{e}}\right]$$

$$\Rightarrow \boxed{\mathcal{H} = -\frac{3}{2}\pi^a \pi_a + \frac{1}{2}\pi^\dagger_{\dot{b}} \pi^{\dagger\dot{b}} + \frac{1}{2}\partial_i \psi^c \partial_i \psi_c + \frac{1}{2}\partial_i \psi^\dagger_{\dot{e}} \partial_i \psi^{\dagger\dot{e}}}$$

This Hamiltonian is un-bounded from below due to the 1st term. Hence $\frac{1}{2}\partial_\mu\psi\partial^\mu\psi$ + h.c. can't be a suitable candidate for a Lagrangian.

To get a bounded Hamiltonian, the kinetic term must contain both ψ and $\psi^\dagger$ and space-time derivative. A possible candidate is: $i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi$. This term is not Hermitian perse. However

$$(i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi)^\dagger = (i\psi^\dagger_{\dot{a}}\bar{\sigma}^{\mu;\dot{a}c}\partial_\mu\psi_c)^\dagger$$

$$= -i\,\partial^\mu\psi^\dagger_{\dot{c}}\,(\bar{\sigma}^{\mu;\dot{a}c})^*\,\psi_a$$

$$= -i\,\partial_\mu\psi^\dagger_{\dot{c}}\,\bar{\sigma}^{\mu;\dot{c}a}\,\psi_a$$

$$= -i\left[\partial_\mu\{\psi^\dagger_{\dot{c}}\bar{\sigma}^{\mu;\dot{c}a}\psi_a\} - \psi^\dagger_{\dot{c}}\bar{\sigma}^{\mu;\dot{c}a}\partial_\mu\psi_a\right]$$

$$= i\psi^\dagger_{\dot{c}}\bar{\sigma}^{\mu;\dot{c}a}\partial_\mu\psi_a - i\partial_\mu[\psi^\dagger\bar{\sigma}^\mu\psi_a]$$

$$= i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi - i\partial_\mu[\psi^\dagger\bar{\sigma}^\mu\psi_a]$$

So under the $\int d^Dx$, $i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi$ and h.c. are the same operator.

So a complete Lagrangian can be

$$\boxed{\mathcal{L} = i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi - \frac{1}{2}m\psi\psi - \frac{1}{2}m^*\psi^\dagger\psi^\dagger}$$

Now if m is complex, say $m = |m|e^{i\alpha}$, then we redefine $\psi = e^{-i\alpha/2}\tilde{\psi}$ and the phase is absorbed with K.E. unchanged! The mass parameter is determined upto a global phase.

So for the time being, lets take the mass m to be a real parameter.

The E.O.M. of ψ_a is given by $\delta S/\delta \psi^{\dagger \dot{a}} = 0$

$$\Rightarrow \boxed{i\, \bar{\sigma}^{\mu;\dot{a}b}\, \partial_\mu \psi_b - m\, \psi^{\dagger \dot{a}} = 0}$$

Taking Hermitian conjugate:

$$-i\,(\bar{\sigma}^{\mu;a\dot{c}})^* \partial_\mu \psi^\dagger_{\dot{c}} - m\, \psi^a = 0$$

$$\Rightarrow i\, \partial_\mu \psi^\dagger_{\dot{c}}\, \bar{\sigma}^{\mu;\dot{c}a} + m\, \psi^a = 0$$

$$\Rightarrow i\, \varepsilon_{\dot{c}\dot{d}}\, \partial_\mu \psi^{\dagger \dot{d}}\, \bar{\sigma}^{\mu;\dot{c}a} + m\,\psi^a = 0$$

$$\Rightarrow -i\, \varepsilon_{\dot{d}\dot{c}}\, \bar{\sigma}^{\mu;\dot{c}a}\, \partial_\mu \psi^{\dagger \dot{d}} + m\,\psi^a = 0$$

$$\Rightarrow -i\, \varepsilon^{ab}\, \sigma^\mu_{b\dot{d}}\, \partial_\mu \psi^{\dagger \dot{d}} + m\,\psi^a = 0$$

$$\Rightarrow \boxed{+i\, \sigma^\mu_{a\dot{d}}\, \partial_\mu \psi^{\dagger \dot{d}} - m\,\psi_a = 0}$$ [lowering the a index]

Lets try to combine the above two eqn

$$\boxed{\begin{pmatrix} -m\,\delta_a{}^c & i\,\sigma^\mu_{a\dot{c}}\,\partial_\mu \\ i\,\bar{\sigma}^{\mu;\dot{a}c} & -m\,\delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix} \begin{pmatrix} \psi_c \\ \psi^{\dagger \dot{c}} \end{pmatrix} = 0}$$

This is actually a 4×4 matrix acting on a 4×1 colm.

Here we introduce the notation $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{a\dot{b}} \\ \bar{\sigma}^{\mu;\dot{a}b} & 0 \end{pmatrix}$

Now the Pauli matrices satisfy the relation:

$$(\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu)_a{}^c = 2\eta^{\mu\nu}\delta_a{}^c, \quad (\bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu)^{\dot{a}}{}_{\dot{c}} = 2\eta^{\mu\nu} \times \delta^{\dot{a}}{}_{\dot{c}}$$

These two eqn's can be compactified into one algebra $\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}}$

We introduce a 4-component Majorana field:

$$\Psi_M = \begin{pmatrix} \psi_c \\ \psi^{\dagger c} \end{pmatrix}$$:= The upper and lower component are the same Weyl field.

The previous E.O.M. becomes $(i\gamma^\mu \partial_\mu - m \mathbb{1}_{4\times4}) \Psi_M = 0$

This is Dirac's eqn. for Majorana field.