PHY-685
QFT-1

Lecture - 28 + 29

Let's try to recall a discussion we had many moons back[*]

A generic quantum field of arbitrary spin has a transform-ation:

$$\hat{\phi}_A(x) = L_A{}^B(\Lambda)\, \hat{U}(\Lambda)\, \hat{\phi}_B(\Lambda^{-1}x)\, \hat{U}^{-1}(\Lambda),$$

under the action of homogeneous Lorentz group. $\hat{U}(\Lambda)$ are the unitary operators that acts on the Hilbert space, with generator $\hat{M}^{\mu\nu}$ (such that for small Lorentz transformation $\Lambda = (1+\delta\omega),\ \hat{U}(1+\delta\omega) = \mathbb{1} + i/2\,\delta\omega_{\mu\nu}\,\hat{M}^{\mu\nu}$). Similarly the "spin" representation matrices $L_A{}^B(\Lambda)$ have the decompo-sition $L_A{}^B(1+\delta\omega) = \delta_A{}^B + i/2\,\delta\omega_{\mu\nu}(\bar{S}^{\mu\nu})_A{}^B$.

We derived the following relation:

$$\boxed{\left[\,\hat{\phi}_A(x),\,\hat{M}^{\mu\nu}\,\right] = -\mathcal{L}^{\mu\nu}\,\hat{\phi}_A(x) + (\bar{S}^{\mu\nu})_A{}^B\,\hat{\phi}_B(x)},$$

where $\boxed{\mathcal{L}^{\mu\nu} = i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right)}$

we also found that both $\hat{M}^{\mu\nu}$ and $\bar{S}^{\mu\nu}$ satisfies the Lorentz group algebra.

The Lorentz algebra can be decomposed into two $SU(2)$ algebra with the redefinition: $\longrightarrow$ it's a "plus" and not a "dagger"

$$\hat{J}_i^- = \frac{1}{2}\left(\hat{J}_i - i\,\hat{K}_i\right),\quad \hat{J}_i^+ = \frac{1}{2}\left(\hat{J}_i + i\,\hat{K}_i\right),\ \text{such that}$$

$$[\hat{J}_i^-,\hat{J}_j^-] = i\,\epsilon_{ijk}\,\hat{J}_k^-,\quad [\hat{J}_i^+,\hat{J}_j^+] = i\,\epsilon_{ijk}\,\hat{J}_k^+ \ \text{and}\ [\hat{J}_i^-,\hat{J}_j^+] = 0$$

The full $SO(3,1)$ algebra breaks up into two pieces of $SU(2)$ and we call the corresponding quantum numbers as (j_1, j_2).

The simplest representations are

$(j_1 = 0,\ j_2 = 0)$: = scalar, [$SO(3)$ rep := 0]

$(j_1 = \frac{1}{2},\ j_2 = 0)$: = Left-handed spin-$\frac{1}{2}$ object (a.k.a. spinor)

$(j_1 = 0,\ j_2 = \frac{1}{2})$: = Right-handed spinor $\longrightarrow$ [$SO(3)$ rep := $\frac{1}{2}$]

$(j_1 = \frac{1}{2},\ j_2 = \frac{1}{2})$: = vector [$SO(3)$ rep: $1 \oplus 0$]

$(j_1 = 1,\ j_2 = 0)$: = spin-1 [$SO(3)$ rep: 1]

$(j_1 = 0,\ j_2 = 1)$: = spin-1 [$SO(3)$ rep: 1]

Left and Right handed Spinor fields:

The left-handed spinor field, a.k.a. left handed Weyl field, denoted by $\hat{\psi}_a(x)$, transforms as

$$\hat{\psi}_a(x) = L_a{}^b(\Lambda)\,\hat{U}(\Lambda)\,\hat{\psi}_b(\Lambda^{-1}x)\,\hat{U}^{-1}(\Lambda), \quad \text{where}$$

the representation follows: $L_a{}^c(\Lambda_1\Lambda_2) = L_a{}^b(\Lambda_1)\,L_b{}^c(\Lambda_2)$.

The dimension of the representation is $(2\cdot\tfrac{1}{2}+1)\cdot(2\cdot 0 +1) = 2$

so $a = 1, 2$

For an infinitesimal L.T. $L_a{}^b(1+\delta\omega) = \delta_a{}^b + \tfrac{i}{2}\,\delta\omega_{\mu\nu}\,(\bar{S}_L^{\mu\nu})_a{}^b$,

with $(\bar{S}_L^{\mu\nu})_a{}^b = -(\bar{S}_L^{\nu\mu})_a{}^b$ is a set of (2×2) matrices

They will satisfy the Lie algebra of the Lorentz group:

$$[\bar{S}_L^{\mu\nu}, \bar{S}_L^{\rho\sigma}] = i\left[\, g^{\mu\rho}\bar{S}_L^{\nu\sigma} - (\mu \leftrightarrow \nu)\right] - (\rho \leftrightarrow \sigma).$$

using $\hat{U}(1+\delta\omega) = 1 + \tfrac{i}{2}\,\delta\omega_{\mu\nu}\,\hat{M}^{\mu\nu}$, the transformation rule becomes

$$\delta\hat{\psi}_A(x) = [\hat{\psi}_a(x), \hat{M}^{\mu\nu}] = -\mathcal{L}^{\mu\nu}\hat{\psi}_a(x) + (\bar{S}_L^{\mu\nu})_a{}^b\,\hat{\psi}_b(x).$$

On R.H.S. the 1st term is there for a scalar field as well. The 2nd term appears for non-trivial representation.

Let's suppress the 1st-term by evaluating the whole eqn at $x=0$.

Taking $\mu = i$ and $\nu = j$ and using $\hat{M}^{ij} = \epsilon^{ijk}\hat{J}_k$, we get

$$\epsilon^{ijk}[\hat{\psi}_a(0), \hat{J}_k] = (\bar{S}_L^{ij})_a{}^b\,\hat{\psi}_b(0)$$

For $(\tfrac{1}{2}, 0)$ repr., it's a spin-$\tfrac{1}{2}$ particle and hence $\hat{J}_k = \tfrac{1}{2}\sigma_k$ and thus the above identity is satisfied if $\boxed{\bar{S}_L^{ij} = \tfrac{1}{2}\epsilon^{ijk}\sigma_k}$

The Pauli matrices are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For $(\tfrac{1}{2}, 0)$, $\hat{N}_i = \tfrac{1}{2}\sigma_i$ and $\hat{N}_i^\dagger = 0$

so $\hat{J}_k = \hat{N}_k + \hat{N}_k^\dagger = \tfrac{1}{2}\sigma_k$ and $\hat{k}_k = i(\hat{N}_k - \hat{N}_k^\dagger) = \tfrac{i}{2}\sigma_k$

This leads to $\bar{S}_L^{k0} = \tfrac{i}{2}\sigma_k$

$\Rightarrow$ for $(\tfrac{1}{2}, 0)$ we have $\boxed{\bar{S}_L^{ij} = \tfrac{1}{2}\epsilon^{ijk}\sigma_k \text{ and } \bar{S}_L^{k0} = \tfrac{i}{2}\sigma_k}$

The $(0, \tfrac{1}{2})$ representation is another independent spin-$\tfrac{1}{2}$ representation and we label the index as $\psi_{\dot{a}}^\dagger(x)$.

Thus we have $[\hat{\psi}_a(x)]^\dagger = \psi_{\dot{a}}^\dagger(x).$

The fields in the $(0, 1/2)$ representation transforms as

$$\hat{\psi}^{\dagger}_{\dot{a}}(x) = R_{\dot{a}}{}^{\dot{b}} \; \hat{U}(\Lambda) \; \hat{\psi}^{\dagger}_{\dot{b}}(x) \; \hat{U}^{-1}(\Lambda),$$

where $R_{\dot{a}}{}^{\dot{b}}(\Lambda)$ is a matrix that represents the $(0, 1/2)$ of the Lorentz group. Just like the case of $(1/2, 0)$ representation, we have the group composition rules

$$R_{\dot{a}}{}^{\dot{b}}(\Lambda_1) \; R_{\dot{b}}{}^{\dot{c}}(\Lambda_2) = R_{\dot{a}}{}^{\dot{c}}(\Lambda_1 \Lambda_2).$$

For an infinitesimal L.T. $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu}$,

$$R_{\dot{a}}{}^{\dot{b}}(1 + \delta\omega) = \delta_{\dot{a}}{}^{\dot{b}} + \frac{i}{2} \delta\omega_{\mu\nu} (\bar{S}_R{}^{\mu\nu})_{\dot{a}}{}^{\dot{b}}, \quad \text{where}$$

$\bar{S}_R{}^{\mu\nu}$ is anti-symmetric in (μ, ν) as expected.

Just like the previous exercise, we can derive

$$\left[\hat{\psi}^{\dagger}_{\dot{a}}(0), \hat{M}^{\mu\nu} \right] = (\bar{S}_R^{\mu\nu})_{\dot{a}}{}^{\dot{b}} \; \hat{\psi}^{\dagger}_{\dot{b}}(0).$$

under Hermitian conjugation:

$$\left[\hat{M}^{\mu\nu}, \hat{\psi}_{a}(0) \right] = \left[(\bar{S}^{\mu\nu}_R)_{\dot{a}}{}^{\dot{b}} \right]^{*} \hat{\psi}_{b}(0)$$

$$\Rightarrow \left[\hat{\psi}_{a}(0), \hat{M}^{\mu\nu} \right] = - \left[(\bar{S}^{\mu\nu}_R)_{\dot{a}}{}^{\dot{b}} \right]^{*} \hat{\psi}_{b}(0)$$

$$\equiv (S_L^{\mu\nu})_{a}{}^{b} \; \hat{\psi}_{b}(0)$$

$$\boxed{\Rightarrow (\bar{S}^{\mu\nu}_R)_{\dot{a}}{}^{\dot{b}} = - \left[(S^{\mu\nu}_L)_{a}{}^{b} \right]^{*}}$$

Now we address the question that how an object carrying two $(1/2, 0)$ indices should transform under Lorentz transformation?

$$\hat{C}_{ab}(x) = L_a{}^c(\Lambda) L_b{}^d(\Lambda) \; \hat{U}(\Lambda) \; C_{bd}(\Lambda^{-1}x) \; \hat{U}^{-1}(\Lambda)$$

Can the four component object $\hat{C}_{ab}$ decomposed into pieces which do not mix with each other under L.T.?

Remember: the previous exercise showed that for

$(1/2, 0) \Rightarrow \vec{J} = \frac{1}{2}\vec{\sigma}$ and $\vec{K} = \frac{i}{2}\vec{\sigma}$. Similarly for

$(0, 1/2) \Rightarrow \vec{J} = \frac{1}{2}\vec{\sigma}$ and $\vec{K} = -\frac{i}{2}\vec{\sigma}$.

Hermitean ↗ $\vec{J}$ anti Hermitean

we found that for

$(1/2, 0)$: we found that the generator are $\hat{J}_i^- = \frac{1}{2}\sigma_i$, $\hat{J}_i^+ = \vec{0}$

for $(0, 1/2)$　　　"　　　"　　　"　　　"　　　"　　　"　$\hat{J}_i^- = 0$, $\hat{J}_i^+ = \frac{1}{2}\sigma_i$

For $(1/2, 0)$: $\hat{\vec{J}}_L = \frac{1}{2}\vec{\sigma}$, $\hat{\vec{k}}_L = \frac{i}{2}\vec{\sigma}$

& $(0, 1/2)$: $\hat{\vec{J}}_R = 1/2\vec{\sigma}$, $\hat{\vec{k}}_R = -i/2\vec{\sigma}$.

$$\left|\begin{array}{l} J_i^- = \frac{1}{2}(J_i - i k_i) \\ J_i^+ = \frac{1}{2}(J_i + i k_i) \end{array}\right.$$

$$\Rightarrow J_i = \frac{1}{2}\left(J_i^+ + J_i^-\right)$$

$$k_i = \frac{1}{i}\left(J_i^+ - J_i^-\right)$$

$$= i\left(J_i^- - J_i^+\right)$$

Transformation of a Weyl Spinor.

$$\psi \to \psi' = \exp\left[i\left\{\theta_i \hat{J}_i + \beta_j \hat{k}_j\right\}\right]\psi$$

$$= \exp\left[+i\left\{\theta_i\left(J_i^+ + J_i^-\right) + \beta_i \ i\left(J_i^- - J_i^+\right)\right\}\right]$$

$$= \exp\left[-\beta_i\left(J_i^- - J_i^+\right) + i\theta_i\left(J_i^+ + J_i^-\right)\right]\psi$$

$$= \exp\left[i\left\{i\beta_i\left(J_i^- - J_i^+\right) + \theta_i\left(J_i^+ + J_i^-\right)\right\}\right]\psi$$

$$= \exp\left[i\left\{J_i^-\underbrace{(\theta_i + i\beta_i)}_{\theta_i^-} + J_i^+\underbrace{(\theta_i - i\beta_i)}_{\theta_i^+}\right\}\right]\psi$$

$$\psi_L \longrightarrow \exp\left[i\left(\frac{1}{2}\sigma_i\,\theta_i^-\right)\right]\psi_L$$

$$= \exp\left[+\frac{\sigma_i}{2}i\left(\theta_i + i\beta_i\right)\right]\psi_L$$

For small θ_i & β_i $\approx \left(\mathbb{1} + \frac{i}{2}\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_L$

$$\Rightarrow \psi_L^\dagger \to \psi_L^\dagger\left(\mathbb{1} - \frac{i}{2}\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)$$

$$\psi_R \to \exp\left[+i\frac{\sigma_i}{2}\left(\theta_i - i\beta_i\right)\right]\psi_R$$

$$\approx \left(\mathbb{1} + \frac{i}{2}\vec{\theta}\cdot\vec{\sigma} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_R \quad \left[\text{For small } \vec{\theta} \& \vec{\beta}\right]$$

$$\Rightarrow \psi_R^\dagger \to \psi_R^\dagger\left(\mathbb{1} - \frac{i}{2}\vec{\theta}\cdot\vec{\sigma} + \vec{\beta}\cdot\vec{\sigma}/2\right)$$

under finite transformation we can construct the represen-
-tations of $(j,0)$ (Left handed) and $(0,j)$ representation

$(j,0):$ $\hat{J}_L^i = J_i(j)$, $\hat{k}_L^i = i J_i(j)$

Boost eigenvalues are $e^{-m\eta}$ where $m = -j, \cdots, +j$

$(0,j):$ $\hat{J}_R^i = J_i(j)$, $\hat{k}_R^i = -i J_i(j)$

Boost eigen-values are $e^{+m\eta}$

Here $J_i(j)$ are the spin-j rotation matrices.

From these representations it is clear that for a fixed
spin representation, the angular momentum matrices are
Hermitian but boost matrices are anti-Hermitian.
Hence finite dimensional representation of Lorentz
group are NOT unitary.

Let say there is a boost eigen-state $|\vec{\beta}\rangle$ such
that $\hat{k}_i |\vec{\beta}\rangle = i\beta_i |\vec{\beta}\rangle$
$\longrightarrow$ eigenvalues are purely
imaginary for anti-Hermitian op.

under finite boost, clearly the norm of these
states are not preserved.