PHY-685
QFT-1

Lecture-6

We have a candidate operator acting on the Hilbert space as following: $\hat{\phi}_i(x) = \sum_{\sigma, n} \int \frac{d^D\vec{p}}{(2\pi)^D} \Big[u_i(x; \vec{p}, \sigma, n)\, \hat{a}(\vec{p}, \sigma, n) + v_i(x; \vec{p}, \sigma, n) \times \hat{a}^\dagger(\vec{p}, \sigma, n) \Big]$,

which can preserve causality: $[\hat{\phi}_i(x), \hat{\phi}_j(y)] = 0$, $[\hat{\phi}_i(x), \hat{\phi}_j^\dagger(y)] = 0$ for $(x-y)^2 < 0$ (and of course we are still talking about free theories only).

We also learned that the basis states are multi-particle momentum eigenstate $|\vec{p}_1, \dots, \vec{p}_N\rangle \equiv \hat{a}^\dagger(\vec{p}_1) \cdots \hat{a}^\dagger(\vec{p}_N)|\Omega\rangle$

In this causal theory thus if we want to introduce an operator representing time-dependent potential energy, a suitable candidate at the operator level is a product of these operators $\hat{\phi}(x)$.

Lets start talking about how relativity plays a role in this whole story. Until now relativity contributed only at two places, viz. the free particle Hamiltonian is $\mathcal{H} = \sqrt{\vec{p}^2 + m^2}$ and imposing a causality criteria. Now we would like to ask the following question: I have some quantum excitation at a point x, which can be measured by an observable $\hat{\phi}_i(x)$. I perform an active trans-formation on the system so that the physical point of interest shifts to a new point $x' = \Lambda x + a$. How will I construct the operator at x', i.e. $\hat{\phi}(x')$ in terms of $\hat{\phi}(x)$?

The answer is it's possible. If $\hat{U}(\Lambda, a)$ are the unitary operators corresponding to the Poincare' transformation on the space-time co-ordinates, such that $|x'\rangle = \hat{U}(\Lambda, a)|x\rangle$. Then in the Heisenberg picture the operators in Hilbert space must undergo some similarity transformation: $\hat{\phi}(x') = \hat{U}(\Lambda, a)\, \hat{\phi}(x)\, \hat{U}^{-1}(\Lambda, a)$

Now $x = \Lambda^{-1}(x' - a) \Rightarrow$

This introduces operator transformation rule:

$$\boxed{\hat{U}(\Lambda, a)^{-1}\, \hat{\phi}_i(x)\, \hat{U}(\Lambda, a) = \sum_j L_i{}^j(\Lambda, a)\, \hat{\phi}_j\big(\Lambda^{-1}(x - a)\big)}$$

In order to exactly know this transformation rules, we need to know how to construct these matrices $L_{i\bar{\jmath}}$? For that we need to dive a bit deeper to know about the general structure of $\hat{U}(\Lambda, a)$.

<u>Lorentz transformations:</u> In the (3+1)-D Minkowski space-time L.T.'s are bunch of linear set of co-ordinate transformations that preserves infinitesimal distances. If S and S' are two inertial frames, with co-ordinates x^μ and x'^μ ($\mu = 0, 1, 2, 3$) respectively, then we have the invariant differential distance as: $\eta_{\mu\nu}\, dx^\mu dx^\nu = \eta_{\rho\sigma}\, dx'^\rho dx'^\sigma$.

$$\Rightarrow \eta_{\mu\nu}\left(\partial x'^\mu/\partial x^\rho\right)\left(\partial x'^\nu/\partial x^\sigma\right) = \eta_{\rho\sigma}. \quad [\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)].$$

The solution to this differential eqn is $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$.
In matrix notation the length-squared of a 4-vector x is given by $x^T \eta x$. The invariance of this norm gives

$$x'^T \eta x' = x^T \eta x \Rightarrow x^T \Lambda^T \eta \Lambda x = x^T \eta x \to \Lambda^T \eta \Lambda = \eta.$$

The same we see in the infinitesimal version:

$$\begin{aligned}
\eta_{\mu\nu}\, dx^\mu dx^\nu &= \eta_{\rho\sigma}\, dx'^\rho dx'^\sigma \\
&= \eta_{\rho\sigma}\left(\Lambda^\rho{}_\mu dx^\mu\right)\left(\Lambda^\sigma{}_\nu dx^\nu\right) \\
&= \eta_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \, dx^\mu dx^\nu \\
\Rightarrow \eta_{\mu\nu} &= \Lambda^\rho{}_\mu\, \eta_{\rho\sigma}\Lambda^\sigma{}_\nu = (\Lambda^T)_\mu{}^\rho\, \eta_{\rho\sigma}\Lambda^\sigma{}_\nu \\
&= (\Lambda^T \eta \Lambda)_{\mu\nu} \Rightarrow \boxed{\eta = \Lambda^T \eta \Lambda}
\end{aligned}$$

If $\eta^{\mu\nu}$ are the matrix elements of the inverse matrix η^{-1}, then we have an analogous identity: $\Lambda^\nu{}_\sigma \Lambda^k{}_\tau \eta^{\sigma\tau} = \eta^{\nu k}$

Two successive transformation with parameters (Λ_1, a_1) and (Λ_2, a_2) are equivalent to one transformation

$$T(\Lambda_2, a_2)\, T(\Lambda_1, a_1) = T(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)$$

we also have $\det(\Lambda) = \pm 1$. $[T(\Lambda, a)]^{-1} = T(\Lambda^{-1}, -\Lambda^{-1}a)$

Inverse of Λ: $(\Lambda^{-1}\Lambda)^\mu{}_\nu = \delta^\mu{}_\nu \Rightarrow (\Lambda^{-1})^\mu{}_\rho \Lambda^\rho{}_\nu = \delta^\mu_\nu \Rightarrow$

Now $(\Lambda^{-1})^\mu{}_\rho = \Lambda_\rho{}^\mu = \eta_{\rho\alpha}\eta^{\mu\beta}\Lambda^\alpha{}_\beta$

$$\Lambda^T \eta \Lambda = \eta \Rightarrow \Lambda^T \eta = \eta \Lambda^{-1} \Rightarrow \Lambda^{-1} = \eta^{-1}\Lambda^T \eta$$

$$\begin{aligned}
\Rightarrow (\Lambda^{-1})^\mu{}_\rho &= (\eta^{-1}\Lambda^T\eta)^\mu{}_\rho = (\eta^{-1})^{\mu\beta}(\Lambda^T)_\beta{}^\alpha \eta_{\alpha\rho} \\
&= \eta^{\mu\beta}\eta_{\rho\alpha}\Lambda^\alpha{}_\beta .
\end{aligned}$$

The transformation $T(\Lambda, a)$ on the Minkowski space-time $(\mathbb{R}^4, \eta)$

induces an unitary linear transformation on the Hilbert space as:

$$|\psi'\rangle = \hat{U}(\Lambda, a)|\psi\rangle.$$

The operator $\hat{U}$ satisfies the group composition rules:

$$\hat{U}(\Lambda_2, a_2)\,\hat{U}(\Lambda_1, a_1) = \exp[i\Phi(T_1, T_2)]\;\hat{U}(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)$$

For projective representations this $\Phi(T_1, T_2)$ is important but we will drop this overall phase factor for the time being!

The Poincaré group, also known as inhomogeneous Lorentz group, has a subgroup with the translation parameter $a = 0$. This is called homogeneous Lorentz group: $T(\Lambda_2, 0)\,T(\Lambda_1, 0) = T(\Lambda_2\Lambda_1, 0)$.

Now we have the matrix eqn.

$$\Lambda^T \eta \Lambda = \eta$$

$$\begin{aligned}
\eta_{00} &= (\Lambda^T)_0{}^\alpha\, \eta_{\alpha\beta}\, \Lambda^\beta{}_0 \\
&= (\Lambda^T)_0{}^0\, \eta_{00}\, \Lambda^0{}_0 + (\Lambda^T)_0{}^i\, \eta_{ii}\, \Lambda^i{}_0 \\
&= (\Lambda^0{}_0)^2 - (\Lambda^i{}_0)^2
\end{aligned}$$

$$\Rightarrow (\Lambda^0{}_0)^2 = 1 + (\Lambda^i{}_0)(\Lambda^i{}_0) = 1 + (\Lambda^0{}_i)(\Lambda^0{}_i).$$

So $(\Lambda^0{}_0)^2 \geq 1 \Rightarrow \Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq -1$.

Since $\det(\Lambda)$ doesn't change sign under similarity transformation, $\det(\Lambda) = +1$ forms a sub-group. (called "proper" transformation).

Let's say Λ_1, Λ_2 are two consecutive transformations.

$$(\Lambda_2\Lambda_1)^0{}_0 = (\Lambda_2)^0{}_\mu(\Lambda_1)^\mu{}_0 = (\Lambda_2)^0{}_0(\Lambda_1)^0{}_0 + (\Lambda_2)^0{}_1(\Lambda_1)^1{}_0 + (\Lambda_2)^0{}_2(\Lambda_1)^2{}_0 + (\Lambda_2)^0{}_3(\Lambda_1)^3{}_0.$$

Now let's consider two 3-vectors:

$$\vec{A} = \left((\Lambda_1)^1{}_0, (\Lambda_1)^2{}_0, (\Lambda_1)^3{}_0\right) \text{ with a length } \sqrt{[(\Lambda_1)^0{}_0]^2 - 1}$$

$$\vec{B} = \left((\Lambda_2)^0{}_1, (\Lambda_2)^0{}_2, (\Lambda_2)^0{}_3\right) \text{ with a length } \sqrt{[(\Lambda_2)^0{}_0]^2 - 1}$$

Now $\vec{A}\cdot\vec{B} \leq |\vec{A}||\vec{B}|$

$$\Rightarrow (\Lambda_2)^0{}_1(\Lambda_1)^1{}_0 + (\Lambda_2)^0{}_2(\Lambda_1)^2{}_0 + (\Lambda_2)^0{}_3(\Lambda_1)^3{}_0$$

$$\leq \sqrt{[(\Lambda_1)^0{}_0]^2 - 1}\;\sqrt{[(\Lambda_2)^0{}_0]^2 - 1}$$

$$\Rightarrow (\Lambda_2)^0{}_0(\Lambda_1)^0{}_0 + (\Lambda_2)^0{}_1(\Lambda_1)^1{}_0 + (\Lambda_2)^0{}_2(\Lambda_1)^2{}_0 + (\Lambda_2)^0{}_3(\Lambda_1)^3{}_0$$

$$\geq (\Lambda_2)^0{}_0(\Lambda_1)^0{}_0 - \sqrt{[(\Lambda_1)^0{}_0]^2 - 1}\;\sqrt{[(\Lambda_2)^0{}_0]^2 - 1} \geq 1$$

Thus we proved that if we have two L.T. Λ_1 and Λ_2, with $(\Lambda_1)^0{}_0 \geq 1$ and $(\Lambda_2)^0{}_0 \geq 1$, then the combined Lorentz transformation $(\tilde{\Lambda})^0{}_0 = (\Lambda_2 \Lambda_1)^0{}_0 \geq 1$. These subgroup of transformation

The subset of transformation where $\det(\Lambda) = +1$ and $\Lambda^0{}_0 \geq 1$ forms a group called "proper orthocronous"

Any L.G. can be written as P, T and a proper orthocronus group.

The Poincare' algebra:

To understand the property of Lie group, one needs to concentrate near identity. So we choose the parameters of the transformation to be very small.

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu \quad \text{and} \quad a^\mu = \epsilon^\mu, \quad \text{where}$$

$|\delta\omega^\mu{}_\nu|$ and $|\epsilon^\mu| << 1$.

Now we have the invariance property of the flat metric:

$$\eta_{\rho\sigma} = \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma$$
$$= \eta_{\mu\nu} (\delta^\mu{}_\rho + \delta\omega^\mu{}_\rho)(\delta^\nu{}_\sigma \; \delta\omega^\nu{}_\sigma)$$
$$= \eta_{\mu\nu} \delta^\mu{}_\rho \delta^\nu{}_\sigma + \eta_{\mu\nu} \delta^\mu{}_\rho \delta\omega^\nu{}_\sigma + \eta_{\mu\nu} \delta\omega^\mu{}_\rho \delta^\nu{}_\sigma + \eta_{\mu\nu} \delta\omega^\mu{}_\rho \delta\omega^\nu{}_\sigma .$$

$$= \eta_{\rho\sigma} + \eta_{\rho\nu} \delta\omega^\nu{}_\sigma + \eta_{\mu\sigma} \delta\omega^\mu{}_\rho + O((\delta\omega)^2)$$

$$\Rightarrow 0 = \eta_{\rho\nu} \delta\omega_{\alpha\beta} \eta^{\nu\alpha} \delta^\beta{}_\sigma + \eta_{\mu\sigma} \delta\omega_{\beta\alpha} \eta^{\beta\mu} \delta^\alpha{}_\rho$$
$$\Rightarrow \delta^\alpha{}_\rho \delta^\beta{}_\sigma \delta\omega_{\alpha\beta} + \delta^\alpha{}_\rho \delta^\beta{}_\sigma \delta\omega_{\beta\alpha} = 0$$
$$\Rightarrow \delta^\alpha{}_\rho \delta^\beta{}_\sigma (\delta\omega_{\alpha\beta} + \delta\omega_{\beta\alpha}) = 0$$

$$\Rightarrow \boxed{\delta\omega_{\alpha\beta} = - \delta\omega_{\beta\alpha}}$$

Hence in $(D+1)$ dimension, the dimension of Poincare group is $D(D-1)/2 + D = D(D+1)/2$.

In the spaces of unitary operators $\hat{U}$, we identify $\hat{U}(\mathbb{1}, 0)$ with the identity matrix. When $\delta\omega^\mu{}_\nu$ and ϵ^μ are small, we expect $\hat{U}$ to be $\hat{U}(\mathbb{1}, 0)$ + linear terms in $\delta\omega^\mu{}_\nu$ & ϵ^μ.