PHY-685
QFT-1

Lecture - 22

The problem of computing n-point correlation function has been reduced to the problem of computing the free-vacuum expectation values of the form: $\langle 0 | \hat{T}\, \hat{\phi}_I(x_1) \cdots \hat{\phi}_I(x_n) | 0 \rangle$,

where $\hat{\phi}_I(t, \vec{x})$ has an exactly computable time evolution

$$\hat{\phi}_I(t, \vec{x}) = e^{i\hat{H}_0 (t - t_0)}\, \hat{\phi}(t_0, \vec{x})\, e^{-iH_0(t-t_0)},$$

from an initially prepared interacting field $\hat{\phi}(\overset{=0}{t_0}, \vec{x})$, which is prepared from the creation, annihilation operators of the free theory:

$$\hat{\phi}(t_0, \vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \Big[f(\vec{p})\, \hat{a}(\vec{p})\, e^{+i\vec{p}\cdot x} + f^*(\vec{p})\, \hat{a}^\dagger(\vec{p})\, e^{-i\vec{p}\cdot\vec{x}} \Big]$$

Let's consider the 2-point fn $\langle 0 | \hat{T}\, \hat{\phi}_I(x)\, \hat{\phi}_I(y) | 0 \rangle$. We write
$\hat{\phi}_I(x) = \hat{\phi}_I^+(x) + \hat{\phi}_I^-(x)$, where

$$\hat{\phi}_I^+(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} f(\vec{p})\, \hat{a}(\vec{p})\, e^{-ip\cdot x} \quad \text{and}$$

$$\hat{\phi}_I^-(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} f^*(\vec{p})\, \hat{a}^\dagger(\vec{p})\, e^{ip\cdot x}$$

Remember that $\hat{a}$ and $\hat{a}^\dagger$ are coming from free theory

So we have $\hat{\phi}_I^+(x)|0\rangle = 0$ and $\langle 0 | \hat{\phi}_I^-(x) = 0$. Now let's consider the case $x^0 > y^0$. Then:

$$\hat{T}\{\hat{\phi}_I(x)\, \hat{\phi}_I(y)\} = \hat{T}\Big\{ \big(\hat{\phi}_I^+(x) + \hat{\phi}_I^-(x)\big)\big(\hat{\phi}_I^+(y) + \hat{\phi}_I^-(y)\big) \Big\}$$

$$= \hat{\phi}_I^+(x)\hat{\phi}_I^+(y) + \hat{\phi}_I^+(x)\hat{\phi}_I^-(y) + \hat{\phi}_I^-(x)\hat{\phi}_I^+(y) + \hat{\phi}_I^-(x)\hat{\phi}_I^-(y)$$

$$= \hat{\phi}_I^+(x)\hat{\phi}_I^+(y) + \hat{\phi}_I^-(x)\hat{\phi}_I^+(y) + \hat{\phi}_I^-(x)\hat{\phi}_I^+(y) + \hat{\phi}_I^-(x)\hat{\phi}_I^-(y)$$
$$+ \big[\hat{\phi}_I^+(x), \hat{\phi}_I^-(y)\big]$$

In every term, except the commutator, the $\hat{a}^\dagger$ are on left and the $\hat{a}$'s are on the right. This ordering is called normal ordering.

$$N\big(\hat{a}(\vec{p}_1)\, \hat{a}^\dagger(\vec{p}_2)\, \hat{a}(\vec{p}_3)\big) \equiv \hat{a}^\dagger(\vec{p}_2)\, \hat{a}(\vec{p}_1)\, \hat{a}(\vec{p}_3)$$

If we had taken $y^0 > x^0$, then
$$\hat{T}\{\hat{\phi}_I(x)\,\hat{\phi}_I(y)\} = (\cdots) + [\hat{\phi}_I^+(y), \hat{\phi}_I^-(x)]$$

We define the contraction of two fields to capture this extra piece in time ordering
$$\overbrace{\hat{\phi}(x)\quad\hat{\phi}(y)} = \begin{cases}[\hat{\phi}^+(x), \hat{\phi}^-(y)] & \text{for } x^0 > y^0\\ [\hat{\phi}^+(y), \hat{\phi}^-(x)] & \text{for } y^0 < x^0\end{cases}$$

For free theory this is exactly the Feynman propagator:
$$\overbrace{\hat{\phi}(x)\,\hat{\phi}(y)} = D_F(x-y)\,.\,\hat{\mathbb{1}}$$

For interaction picture, where $\hat{\phi}(x) \equiv \hat{\phi}_I(x)$,
$$\overbrace{\hat{\phi}_I(x)\quad\hat{\phi}_I(y)} = \int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\epsilon}\,|f(\vec{p})|^2\,e^{-ip\cdot(x-y)}\,\hat{\mathbb{1}}$$

Now dropping the "I" subscript on the interaction fields because we are going to work only with them. Whenever a distinc--tion between full-time evolved quantum fields, vs. fields in interaction picture is required we will make it explicit.

We found that $\hat{T}\{\hat{\phi}(x)\,\hat{\phi}(y)\} = N\{\hat{\phi}(x)\,\hat{\phi}(y) + \overbrace{\hat{\phi}(x)\,\hat{\phi}(y)}\}$

$$\equiv N\{\hat{\phi}(x)\hat{\phi}(y)\} + D_F(x-y)\,\hat{\mathbb{1}}$$

The generalization to arbiliary many fields are:
$$\hat{T}\{\hat{\phi}(x_1)\cdots\hat{\phi}(x_m)\} = N\{\phi(x_1)\cdots\phi(x_m) + \text{all possible contractions}\}$$

This is known as Wick's theorem [a path-integral version we have already seen]

We prove it by the method if induction. on N

Lets say we have $N+1$ fields $\hat{\phi}(x_1), \cdots, \hat{\phi}(x_{N+1})$ and we

assume Wick's theorem holds for the operators $\hat{\phi}(x_2) \cdots \hat{\phi}(x_{N+1})$.
We will assume $(x^1)^0 \geq (x^2)^0 \geq \cdots \geq (x^{N+1})^0$ without any loss of generality. Denote $\hat{\phi}_i \equiv \hat{\phi}(x_i)$

So $\hat{T}\{\hat{\phi}_1 \cdots \hat{\phi}_{N+1}\} = \hat{\phi}_1 \cdots \hat{\phi}_{N+1}$

$$= \hat{\phi}_1 \hat{T}\{\hat{\phi}_2 \cdots \hat{\phi}_{N+1}\} = \hat{\phi}_1 N\{\hat{\phi}_2 \cdots \hat{\phi}_{N+1} + (\text{all possible contractions not involving } \hat{\phi}_1)\}$$

$$= (\hat{\phi}_1^+ + \hat{\phi}_1^-) N\{\hat{\phi}_2 \cdots \hat{\phi}_{N+1} + (\text{all possible contractions not involving } \hat{\phi}_1)\}$$

For $\hat{\phi}_1^-$ term (with $\hat{a}^\dagger$) we can push it inside the normal ordering

Now $\hat{\phi}_1^+ N\{\hat{\phi}_2 \cdots \hat{\phi}_{N+1}\}$

$$= N\{\hat{\phi}_2 \cdots \hat{\phi}_{N+1}\}\hat{\phi}_1^+ + [\hat{\phi}_1, N\{\hat{\phi}_2 \cdots \hat{\phi}_{N+1}\}]$$

$$= N(\hat{\phi}_1^+ \hat{\phi}_2 \cdots \hat{\phi}_{N+1}) + N\left([\hat{\phi}_1^+, \hat{\phi}_2^-]\hat{\phi}_3 \cdots \hat{\phi}_{N+1} + \hat{\phi}_2 [\hat{\phi}_1^+, \hat{\phi}_3^-] \phi_4 \cdots \hat{\phi}_{N+1}\right)$$

$$= N(\hat{\phi}_1^+ \hat{\phi}_2 \cdots \hat{\phi}_{N+1}) + N(\text{all contractions of } \hat{\phi}_1 \cdots \hat{\phi}_{N+1})$$