PHY-685
QFT-1

Lecture - 19

In the last class we found that for a free scalar field theory:

$$Z_0[J] = Z_0[0]\exp\left[-\frac{1}{2}\int d^{D+1}x\, d^{D+1}x'\, J(x)\,\Delta_F(x-x')\,J(x')\right] \equiv Z_0[0]\exp\left[-\frac{1}{2}J\Delta J\right]$$

$$\text{where } \Delta_F(x-x') = \int \frac{d^{D+1}k}{(2\pi)^{D+1}}\frac{i}{k^2-m^2+i\epsilon}\, e^{-ik\cdot(x-x')}$$

using the explicit formula of time-ordered correlation function:

$${}_J\langle 0|\hat{T}\,\hat{\phi}(x_1)\,\hat{\phi}(x_2)|0\rangle_J = \frac{1}{Z_0[J]}\left(\frac{1}{i}\,\delta/\delta J(x_1)\right)\left(\frac{1}{i}\,\delta/\delta J(x_2)\right) Z_0[J]\Big|_{J=0}$$

$$= \frac{1}{Z_0[J]}\left(-\,\delta/\delta J(x_1)\right)\left\{\left[-\frac{1}{2}\int d^{D+1}x\, d^{D+1}x' \left\{\delta(x-x_2)\,\Delta_F(x-x')\,J(x') + J(x)\,\Delta_F(x-x')\,\delta^{D+1}(x'-x_2)\right\}\right] Z_0[J]\Big|_{J=0}\right\}$$

$$= \frac{1}{Z_0[J]}\left(+\,\delta/\delta J(x_1)\right)\left\{\left[\int d^{D+1}x'\,\Delta_F(x_2-x')\,J(x')\right] Z_0[J]\Big|_{J=0}\right\}$$

$$= \Delta_F(x_2-x_1) - \frac{1}{Z_0[J]}\left(\int d^{D+1}x'\,\Delta_F(x_2-x')\,J(x')\right)\left(\int d^{D+1}x''\,\Delta_F(x_1-x'')\,J(x'')\right) Z_0[J]\Big|_{J=0}$$

$$= \Delta_F(x_2-x_1)$$

Please note that 3-pt function vanishes.

The 4-pt fn. ${}_J\langle 0|\hat{T}\,\hat{\phi}(x_1)\,\hat{\phi}(x_2)\,\hat{\phi}(x_3)\,\hat{\phi}(x_4)|0\rangle_J$

$$= \frac{1}{Z_0[J]}\left\{\left(\frac{1}{i}\,\delta/\delta J(x_1)\right)\left(\frac{1}{i}\,\delta/\delta J(x_2)\right)\left(\frac{1}{i}\,\delta/\delta J(x_3)\right)\left(\frac{1}{i}\,\delta/\delta J(x_4)\right) Z_0[J]\Big|_{J=0}\right\}$$

$$= \frac{1}{Z_0[J]}\left(\left(1/i\;\delta/\delta J(x_1)\right)\left(1/i\;\delta/\delta J(x_2)\right)\right)\left[\Delta_F(x_3-x_4)\,Z_0[J]\Big|_{J=0} - \left\{\left(\int d^D x'\,\Delta(x_3-x')\,J(x')\right)\left(\int d^D x''\,\Delta(x_4-x'')\,J(x'')\right)\right\} Z_0[J]\right]\Big|_{J=0}$$

$$= \Delta_F(x_1-x_2)\,\Delta_F(x_3-x_4) + \frac{1}{Z_0[J]}\left(\delta/\delta J(x_1)\right)\Bigg\{\Bigg[\left(\int d^{D+1}x'\,\Delta_F(x_3-x')\,\delta^{D+1}(x'-x_2)\right)\left(\int d^{D+1}x''\,\Delta_F(x_4-x'')\,J(x'')\right) + \left(\int d^{D+1}x'\,\Delta_F(x_3-x')\,J(x')\right)\left(\int d^{D+1}x''\,\Delta_F(x_4-x'')\,\delta^D(x''-x_2)\right)\Bigg] \times Z_0[J]\Bigg\}\Bigg|_{J=0}$$

$$\left[\Delta_F(x_1-x_2)\,\Delta_F(x_3-x_4) + \Delta_F(x_3-x_2)\,\Delta_F(x_4-x_1) + \Delta_F(x_3-x_1)\,\Delta_F(x_4-x_2)\right]$$

[The term $\frac{1}{Z_0[J]}(\cdot\cdot)\,\frac{\delta}{\delta J(x_1)} Z_0[J]\Big|_{J=0}$ vanishes]

Similarly we have ${}_J\langle 0|\hat{T}\,\hat{\phi}(x_1)\cdots\hat{\phi}(x_{2n})|0\rangle_J$

$$= \sum_{\text{pairing}} \Delta_F(x_{i_1}-x_{i_2})\cdots\Delta_F(x_{i_{2n-1}}-x_{i_{2n}}) \Rightarrow \text{Wick's theorem}$$

The final missing piece in this formalism is that how do we compute $\Delta(x_1 - x_2)$. For that we need the help of Euclidean formalism.

$$\exp(iS) \sim \exp\left[i \int d^{D+1}x \left(\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - V(\phi) + J\phi \right) \right]$$

Now we define the Euclidean co-ordinates $x^0_E = i x^0$, $x^i_E = x^i$ for $i = 1, \dots, D$. Thus $i\, d^{D+1}x = d^{D+1}x_E$, $(\partial_t \phi)^2 = -(\partial_{t_E}\phi)^2$, $(\vec{\nabla}\phi)^2 = (\vec{\nabla}_E \phi)^2$

So $\exp(iS) = \exp(-S_E)$, where

$$S_E = \int d^{D+1}x_E \left(\mathcal{L}_E - J\phi \right), \text{ where } \mathcal{L}_E = \frac{1}{2}(\partial_0 \phi)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2$$

The Wick rotated p.f has the form

$$Z_E[J] = \int \mathcal{D}\phi \exp\left[- \int d^{D+1}x_E \left(\mathcal{L}_E - J\phi \right) \right].$$

The Euclidean Lagrangian $\mathcal{L}_E = \frac{1}{2}(\partial_E \phi)^2 + V(\phi)$ and is bounded from below, becomes large when ϕ is large or $\partial_E \phi$ is large.

In Euclidean field theory, we ask the question: what is the amplitude of starting from a time t_A and state $|\phi\rangle$ and at a later time t_B coming back to the same quantum state $|\phi\rangle$:

If we look at the space-time trajectory: the space-time interval $(x^0)^2 - \vec{x}^2 = -(x^0_E)^2 - \vec{x}_E^2 < 0$ and thus the correlation function we will compute are for space-like region. We have to analytically continue them back to Minkowskian correlator for time like separated regions. We compute the correlators from

$$Z'[J] = \int_{\substack{\text{periodic} \\ \text{B.C. on } \phi}} \mathcal{D}\phi \exp\left[-i \int d^{D+1}x\, H(\phi, \pi) \right]$$

$\longrightarrow$ the periodicity interval

$$= \int \mathcal{D}\phi \exp\left[- \int_0^\beta dx^0_E \int d^D\vec{x} \left(\mathcal{L}_E - J\phi \right) \right], \text{ with a B.C.}$$

in Euclidean time $\phi(x^0_E, \vec{x}) = \phi(x^0_E + \beta, \vec{x})$.

So $Z_E[J] = Z_E[0] \exp\left(\frac{1}{2} J \Delta J\right)$, with

$Z_E[0] = \int \mathcal{D}\xi \exp\left[-\frac{1}{2}\int d^{D+1}x \; \xi(x)\left[-\partial^2+m^2\right]\xi(x)\right]$. Lets compute this quantity:

we have the operator $(-\partial^2+m^2)$ on the spaces of function and let's say $\psi_n(x)$ are the eigenfunctions with eigenvalues A_n.

$$(-\partial^2+m^2)\psi_n(x) = A_n \psi_n(x)$$

If the field configurations $\phi(x)$ are expressed as

$\phi(x) = \sum_n c_n \psi_n(x)$, then the action becomes

$$S_0 = \int d^{D+1}x \, \mathcal{L}_E(\phi, \partial\phi) = \frac{1}{2}\sum_n A_n |c_n|^2 \Rightarrow Z_E[0] = \prod_n A_n^{-1/2}$$
$$= \left[\operatorname{Det}(-\partial^2+m^2)\right]^{-1/2}$$

Now $\ln\left[\operatorname{Det}(-\partial^2+m^2)\right] = \operatorname{Tr} \ln\left[-\partial^2+m^2\right]$

$$= V\int \frac{d^{D+1}p}{(2\pi)^{D+1}} \ln(p^2+m^2)$$

Euclidean S.T. dimension.

$$\Rightarrow \ln Z_E[0] = -\frac{V}{2}\int \frac{d^{D+1}p}{(2\pi)^{D+1}} \ln(p^2+m^2)$$ } scales linear with volume &.

$V \to \infty$, this quantity has a divergence. On top of that, there is an UV divergence from the momenta integral.

Now $Z_E[0] = \lim_{\beta\to\infty} \sum_n \exp(-\beta E_n) \sim \exp(-\beta E_0)$ [The ground s dominates]

$$\Rightarrow E_0 = -\lim_{\beta\to\infty} \frac{1}{\beta} \ln Z_E[0] = \frac{1}{2} V \int \frac{d^{D+1}p}{(2\pi)^{D+1}} \ln(p^2+m^2)$$

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under Wick rotation $(x^0)^2 - \vec{x}^2 \to -x_E^2 < 0$. To get correlation functi in real time, we need to do an analytic continuation back.

$$\underbrace{\langle \hat{\phi}_E(x_1)\cdots\hat{\phi}_E(x_n)\rangle}_{\text{Euclidean correlator}} = i \underbrace{\langle 0|\hat{T}\{\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)|0\rangle}_{\text{Minkowskian n-point fuction.}}$$

In the free theory (in D dim space-time)

$$\Delta_E^{(0)}(x-x') = \int \frac{d^D p}{(2\pi)^D} \Delta_E^{(0)}(p) \exp\left(i p_\mu (x-x')^\mu\right), \quad \text{where } \Delta_E^{(0)}(p) = 1/(p^2+m^2).$$

Now we use the identity: $1/A = \frac{1}{2}\int_0^\infty d\alpha \exp\left(-\frac{A}{2}\alpha\right)$, where $A>0$.

If we choose $A = p^2+m^2$, then

$$\Delta_E^{(0)}(x-x') = \frac{1}{2}\int_0^\infty d\alpha \int \frac{d^D p}{(2\pi)^D} \exp\left[-\frac{\alpha}{2}(p^2+m^2) + i p_\mu (x^\mu - x'^\mu)\right]$$

Now $\frac{\alpha}{2}(p^2+m^2) - i p_\mu (x-x')^\mu$

$$= \frac{1}{2}\left(\sqrt{\alpha}\, p_\mu - i \frac{x_\mu - x'_\mu}{\sqrt{\alpha}}\right)^2 - \frac{1}{2}\left(\frac{x_\mu - x'_\mu}{\sqrt{\alpha}}\right)^2$$

Now if we do the Gaussian integral:

$$\int \frac{d^D p}{(2\pi)^D} \exp\left[-\frac{1}{2}\left(\sqrt{\alpha}\, p_\mu - i \frac{x_\mu - x'_\mu}{\sqrt{\alpha}}\right)^2\right] = (2\pi\alpha)^{-\frac{D}{2}}.$$

$\Rightarrow$

$$\Delta_E^{(0)}(x-x') = \frac{1}{2}\frac{1}{(2\pi)^{D/2}} \int_0^\infty d\alpha\, \alpha^{-D/2} \exp\left(-\frac{|x-x'|^2}{2\alpha} - \frac{1}{2}m^2\alpha\right)$$

α was an arbitrary parameter, introduced in order to have an integral representation of $1/A$. Define a rescaling variable $\alpha = \lambda t$.

So $$\frac{|x-x'|^2}{2\alpha} + \frac{1}{2}m^2\alpha = \frac{|x-x'|^2}{2\lambda t} + \frac{1}{2}m^2\lambda t..$$

Notice that two parameters λ & t were introduced and hence only one of them can be independent. So we fix $\lambda = |x-x'|/m$.

Hence $$\frac{|x-x'|^2}{2\alpha} + \frac{1}{2}m^2\alpha = \frac{m|x-x'|}{2}\left(t + 1/t\right)$$

Now $\alpha = \lambda t \Rightarrow d\alpha\, \alpha^{-D/2}$

$$= \lambda\, dt\, \lambda^{-D/2} . t^{-D/2} = \lambda^{-(D/2-1)} t^{-D/2} dt$$

So $$\Delta_E^0(x-x') = \frac{1}{2}\frac{1}{(2\pi)^{D/2}} \int_0^\infty dt\, \lambda^{-(D/2-1)} t^{-D/2} \exp\left[-\frac{m}{2}|x-x'|\left(t+\frac{1}{t}\right)\right]$$

$$= \frac{1}{(2\pi)^{D/2}}\left(\frac{m}{|x-x'|}\right)^{D/2-1}\left[\frac{1}{2}\int_0^\infty dt\, t^{-(D/2-1)-1} \times \exp\left[-\frac{1}{2}(m|x-x'|)(t+1/t)\right]\right]$$

$$= \frac{1}{(2\pi)^{D/2}}\left(\frac{m}{|x-x'|}\right)^{D/2-1} K_{D/2-1}(m|x-x'|)$$

Here $K_\nu(z)$ is the modified Bessel function:

$$K_\nu(z) = \frac{1}{2}\int_0^\infty dt\; t^{\nu-1} \exp\left[-\frac{z}{2}(t + 1/t)\right]$$

Now we are interested in asymptotic limits of $\Delta_E^0(x-x')$.

A. Long distance behaviour: $z \gg 1$, i.e. $m|x-x'| \gg 1$. In this limit $K_\nu(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}\left[1 + O(1/z)\right] \Rightarrow$

$$\Delta_E^0(x-x') = \frac{1}{(2\pi)^{D/2}}\left(\frac{m}{|x-x'|}\right)^{D/2-1} K_{D/2-1}(m|x-x'|)$$

$$= \frac{1}{(2\pi)^{D/2}} \frac{m^{D-2}}{(m|x-x'|)^{D/2-1}} \sqrt{\frac{\pi}{2m|x-x'|}}\, \exp(-m|x-x'|)\left[1 + O\left(\frac{1}{m|x-x'|}\right)\right]$$

$$= \sqrt{\pi/2}\, \frac{m^{D-2}}{(m|x-x'|)^{\frac{D+1}{2}}} \exp(-m|x-x'|)\left[1 + O\left(\frac{1}{m|x-x'|}\right)\right]$$

So the 2 pt. correlation fn decays faster than an exponential.

B. Short distance approximation: $z \ll 1$.

For $z \ll 1$, $K_\nu(z) = \frac{\Gamma(\nu)}{2(z/2)^\nu} + O(1/z^{\nu-2})$.

So $$\Delta_E^{(0)}(x-x') = \frac{\Gamma(D/2-1)}{4\pi^{D/2}|x-x'|^{D-2}} + O(z^{3-D/2})$$

Leading order behaviour is mass independent.

For Minkowski S-T. $s^2 = (x^0 - x'^0)^2 - (\vec{x} - \vec{x}')^2$ and the Euclidean length $|x-x'| = \sqrt{(x-x')^2} = \sqrt{-s^2}$ [since $x^0 = -i x_4$]

So $\Delta_M^{(0)}(x-x') = i/4\pi^2\, (m/\sqrt{-s^2})\, K_1(m\sqrt{-s^2})$ (for $D=4$).

Now $K_1(z) = \sqrt{\frac{\pi}{2z}} \exp(-z)\left[1 + \frac{3}{8z} + \cdots\right]$ for $z \gg 1$

$= 1/z + \frac{z}{2}\left(\ln z + \gamma_E - 1/2\right) + \cdots$ for $z \ll 1$ ($\gamma_E \to$ Euler-Mascheroni cnst $= 0.577$)

For $s^2 < 0$:

$$\Delta_M^{(0)}(x-x') = i\,\frac{\sqrt{\pi/2}}{4\pi^2}\, \frac{m^2}{(m\sqrt{-s^2})^{3/2}} \exp(-m\sqrt{-s^2}) \quad \text{for } m\sqrt{-s^2} \gg 1$$

$$= \frac{i}{4\pi^2(-s^2)} \quad \text{for } m\sqrt{-s^2} \ll 1$$

for $m\sqrt{-s^2} \ll 1$

For $s^2 > 0$:

$$\Delta_M^{(0)}(x-x') = \sqrt{\pi/2}/4\pi^2\, \frac{m^2}{(m\sqrt{s^2})^{3/2}} \exp(im\sqrt{s^2}) \quad \text{for } m\sqrt{s^2} \gg 1$$

$$= 1/4\pi^2(s^2) \quad \text{for } m\sqrt{s^2} \ll 1$$