PHY-685

QFT-1

Lecture - 20

We have solved the free field theory until now and exactly solved the corresponding quantum theory. However, in reality we see that the number of particles and types are changing under time evolution. This phenomena is possible iff there is some interactions are present. Hence we need to look back to the formalism that we have developed and ask which parts of the quantum theory changes when we turn on interactions and how do we use the methods, developed until now, to compute observables in an interacting theory?

Let's remind ourselves that when we talk about time evolutions in Q.M, we have three formalisms:

1> Schrödinger picture: States are evolving with time and operators don't carry the time label:

$$|\psi(t)\rangle_s = \hat{U}(t,t')|\psi(t')\rangle_s$$ and $|\psi(t)\rangle_s$ satisfies the Schrödinger eqn. $i\, d/dt\, |\psi(t)\rangle_s = \hat{H}(\hat{p}_s, \hat{q}_s, t)|\psi(t)\rangle_s$

$$\Rightarrow \quad i\,\partial/\partial t\, \hat{U}(t,t') = \hat{H}(\hat{p}_s, \hat{q}_s, t)\,\hat{U}(t,t')$$

We have solved the above differential eqn. before and know that the soln is given by $\hat{U}(t_B,t_A) = \hat{T}\exp\left[-i\int_{t_A}^{t_B} dt\, \hat{H}\right]$

$$\hat{U}(t_B,t_A) = \hat{U}^{-1}(t_A,t_B)$$

2> Heisenberg Picture: The operators evolve with time : $\hat{O}_H(t)$

$$d/dt(\hat{O}_H) = i[\hat{H}, \hat{O}_H], \quad \text{with}$$

$$\hat{O}_H(t) = \hat{U}^{\dagger}(t,0)\,\hat{O}_H(t=0)\,\hat{U}(t,0)$$

3> Dirac's interaction picture: Here states evolves via the action of

$$|\psi(t)\rangle_I = \exp(i\hat{H}_0 t)|\psi(t)\rangle_s = \exp(i\hat{H}_0(t))\exp(-i\hat{H}t)|\psi(t=0)\rangle$$

So the time evolution operator:

$$\hat{U}_I(t,0) = \exp(i\hat{H}_0 t)\exp(-i\hat{H}t)$$
$$= \exp(i\hat{H}_0 t)\,\hat{U}(t,0)$$

The evolution operator $\hat{U}_I(t,t')$ satisfies

$$i\,\partial/\partial t\left[\hat{U}_I(t,t')\right] = \hat{H}_I(t)\,\hat{U}_I(t,t')$$

and $\hat{U}_I(t_B,t_A) = \hat{T}\exp\left[-i\int_{t_A}^{t_B}\hat{H}_I(t)\,dt\right]$

Scattering theory :

We want to compute how the quantum state of a system evolves from far past to far future, when some finite interactions are turned on in the intermediate times?

Let's say we have created a wave packet $|\psi(t)\rangle$ in the far past whose time evolution is governed by the free Hamiltonian $\hat{H}_0$. Now the operator $\hat{H}_0$ has its own Hilbert space, say $\tilde{\mathcal{H}}_0$. $|\psi(t)\rangle \in \tilde{\mathcal{H}}_0$. Now the full Hamiltonian $\hat{H}$ will have its own Hilbert space $\tilde{\mathcal{H}}$. Now we put the following ansatz:

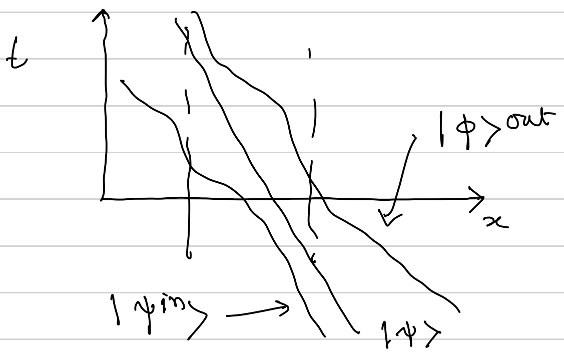
There is a state $|\psi(t)\rangle^{in} \in \tilde{\mathcal{H}}$ that looks like $|\psi(t)\rangle \in \tilde{\mathcal{H}}_0$ in the far past.

i.e.
$$\boxed{\lim_{t\to-\infty} \left\| e^{-i\hat{H}_0 t}|\psi(t=0)\rangle - e^{-i\hat{H}t}|\psi(t=0)\rangle^{in}\right\| = 0}$$

The operation of associating a state $|\psi(t)\rangle^{in} \in \tilde{\mathcal{H}}$ to a state $|\psi(t)\rangle \in \tilde{\mathcal{H}}_0$ in the limit $t\to-\infty$ is called "in-ing". Similarly there exists a concept called "out-ing".

If $|\phi(t)\rangle^{out} \in \tilde{\mathcal{H}}$ and $|\phi(t)\rangle \in \tilde{\mathcal{H}}_0$, then "out-ing" is defined as:

$$\boxed{\lim_{t\to+\infty} \left\| e^{-i\hat{H}_0 t}|\phi(t=0)\rangle - e^{-i\hat{H}t}|\phi(t)\rangle^{out}\right\| = 0}$$



We are interested in the complex number

$${}^{out}\langle\phi(t)|\psi(t)\rangle^{in} \Big\}\text{ time independent}$$
$$= \langle\phi(t)|\hat{S}|\psi(t)\rangle$$
$\langle\phi(t)| \in \tilde{\mathcal{H}}_0$, $|\psi(t)\rangle \in \tilde{\mathcal{H}}_0$

So the scattering operator: $\hat{S}$ is an operator on $\tilde{\mathcal{H}}_0$ (the Hilbert space of the free theory).

* A quantum theory with full interacting Hamiltonian $\hat{H}$ has a scattering description if it has a complete set of "in-state" eigenstate

$|\alpha, -\rangle$ and a complete set of out state $|\beta, +\rangle$ both with same eigenvalues $\hat{H}|\alpha, \mp\rangle = E_\alpha |\alpha, \mp\rangle$,
where E_α are the eigenvalues of $\hat{H}_0$ with eigen-state $|\alpha\rangle$. Hence:

$$\boxed{\lim_{t\to\mp\infty} \int d\alpha \, g(\alpha) \, e^{-iE_\alpha t} |\alpha, \mp\rangle = \lim_{t\to\mp\infty} \int d\alpha \, g(\alpha) \, e^{-iE_\alpha t} |\alpha\rangle}$$

$\hookrightarrow$ This implies that one can always create asymptotic states in the interacting theory by using the basis of the free theory.
Now it follows that if we compute the inner product of the asymptotic states in the interacting theory, then we can compute the results using free theory:

$$\langle \beta, \mp | \alpha, \mp \rangle = \delta(\beta - \alpha)$$

Following earlier definition of "S-matrix": $S_{\beta\alpha} = \langle \beta, + | \alpha, - \rangle$
Now let's look into an important property of S-matrix:

$$\int d\beta \, S^*_{\beta\alpha} S_{\beta\gamma} = \int d\beta \, \langle \alpha, - | \beta, + \rangle \langle \beta, + | \gamma, - \rangle$$
$$= \langle \alpha, - | \gamma, - \rangle = \delta(\alpha - \gamma) .$$

So S-matrix operators are unitary! $\hat{S}^\dagger \hat{S} = \mathbb{1}$

Now lets consider the following situation: We have an "M-particle" in-state and an "N-particle" out state. Then the S-matrix element is given by:

$$\langle p_1', \sigma_1', n_1'; p_2', \sigma_2', n_2'; \dots; p_N', \sigma_N', n_N', + | p_1, \sigma_1, n_1; p_2, \sigma_2, n_2; \dots; p_M, \sigma_M, n_M, - \rangle$$

under Lorentz transformation this quantity evolves as:

$$\langle p_1', \sigma_1', n_1'; p_2', \sigma_2', n_2'; \dots; p_N', \sigma_N', n_N', + | p_1, \sigma_1, n_1; p_2, \sigma_2, n_2; \dots; p_M, \sigma_M, n_M, - \rangle$$
$$= \prod_{i=1}^{N} \left(\sqrt{\frac{(\Lambda p_i')^0}{p_i'^0}} \sum_{\bar{\sigma}_i} \left(D^{n_i'}_{\bar{\sigma}_i', \sigma_i'}\right)^* (W_p) \right) \prod_{j=1}^{M} \left(\sqrt{\frac{(\Lambda p_j)^0}{p_j^0}} \sum_{\bar{\sigma}_j} D^{n_j}_{\bar{\sigma}_j \sigma_j} (W_p) \right)$$
$$\times \langle \Lambda p_1', \bar{\sigma}_1', n_1'; \dots; \Lambda p_N', \bar{\sigma}_N', n_N', + | \Lambda p_1, \bar{\sigma}_1, n_1; \dots, \Lambda p_M, \bar{\sigma}_M, n_M, - \rangle$$

where W_p is the good old Wigner-rotation matrix:

$$W(p) = L^{-1}(\Lambda p, k) \, \Lambda \, L(p, k)$$

If we make a square on both sides:

$$\sum_{\{\sigma',\sigma\}} \left| \langle \{p'_i, \sigma'_i, n'_i\}_N + | \{p_j, \sigma_j, n_j\}_M - \rangle \right|^2$$

$$= \prod_{i=1}^{N} \left(\frac{(\Lambda p'_i)^0}{p'^0_i} \right) \prod_{j=1}^{M} \left(\frac{(\Lambda p_j)^0}{p_j^0} \right)$$

$$\times \sum_{\{\bar{\sigma}',\bar{\sigma}\}} \left| \langle \{\Lambda p'_i, \bar{\sigma}'_i, n'_i\}, + | \{\Lambda p_j, \bar{\sigma}_j, n_j\}, - \rangle \right|^2$$

So the normalized S-matrix element

$$\tilde{S}_{\beta\alpha} = \left(\prod_{i=1}^{N_\beta} \sqrt{2E_{\beta,i}} \right) \left(\prod_{j=1}^{N_\alpha} \sqrt{2E_{\alpha,j}} \right) S_{\beta\alpha},$$

then $\sum_{\text{spin/helicity}} |\tilde{S}_{\beta\alpha}|^2$ is Lorentz invariant.

Now if we represent the in-states and out-states as time evolved version

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Lets revisit the interaction picture in a bit more details. Let us denote a prepared state at $t_0 = 0$ as $|\Psi_H\rangle$. This state is defined for a particular time $t_0 = 0$ (NOW!) and has no notion of time evolution. As mentioned before, the Hamiltonian eigenstates in the Schrödinger picture is defined as

$$|\Psi_S(t)\rangle = \underbrace{\hat{U}(t, t_0)}_{\text{full time evolution operator.}} |\Psi_H\rangle \qquad \hat{U}(t,t_0) = \hat{T}\exp\left[-i\int_{t_0}^{t} \hat{H}(t')dt'\right]$$

The time dependent states in the interaction picture are given by

$$|\Psi_I(t)\rangle = \exp[i\hat{H}_0 t]\, |\Psi_S(t)\rangle$$

$$= \exp[i\hat{H}_0 t]\, \hat{U}(t, t_0)\, |\Psi_H\rangle .$$

The operators in the interaction pictures undergo a time evolution as following:

$$\hat{O}_I(t) = \exp[i\hat{H}_0 t]\, \hat{O}_S \exp[-i\hat{H}_0 t]$$

$$= \exp[i\hat{H}_0 t] \exp[-i\hat{H} t]\, \hat{O}_H \exp[i\hat{H} t] \times \exp[-i\hat{H}_0 t]$$

$$\Rightarrow \boxed{\hat{O}_H(t) = e^{i\hat{H}t} e^{-i\hat{H}_0 t}\, \hat{O}_I(t)\, e^{i\hat{H}_0 t} e^{-i\hat{H}t}}$$

Now again look back to the eqn $|\psi_I(t)\rangle = e^{i\hat{H}_0 t}|\psi_S(t)\rangle$

$$\Rightarrow i\frac{\partial}{\partial t}|\psi_I(t)\rangle = -\hat{H}_0|\psi_I(t)\rangle + e^{i\hat{H}_0 t}\, i\frac{\partial}{\partial t}|\psi_S(t)\rangle$$

$$= -\hat{H}_0|\psi_I(t)\rangle + e^{i\hat{H}_0 t}\hat{H}_S|\psi_S(t)\rangle$$

$$= -\hat{H}_0|\psi_I(t)\rangle + e^{i\hat{H}_0 t}\hat{H}_S e^{-i\hat{H}_0 t}|\psi_I(t)\rangle$$

$$= -\hat{H}_0|\psi_I(t)\rangle + \hat{H}_I(t)|\psi_I(t)\rangle$$

$$= \hat{V}_I(t)|\psi_I(t)\rangle$$ [where we assume $\hat{H}_0$ has no time dependence]

Now $\hat{V}_I(t) = e^{i\hat{H}_0 t}\hat{V}_S e^{-i\hat{H}_0 t}$

The solution to the above differential eqn is written as: $|\psi_I(t)\rangle = \hat{U}_I(t,t_0)|\psi_I(t_0)\rangle$, where

$$\hat{U}_I(t,t_0) = \hat{T}\exp\left[-i\int_{t_0}^{t}\hat{V}_I(t')\,dt'\right]$$

Now $|\psi_S(t)\rangle = \hat{U}(t,t_0)|\psi_S(t_0)\rangle$

$$= \exp\left[-i\hat{H}(t-t_0)\right]|\psi_S(t_0)\rangle$$

Now writing $|\psi_S(t)\rangle = e^{-i\hat{H}_0 t}|\psi_I(t)\rangle$, we get

$$e^{-i\hat{H}_0 t}|\psi_I(t)\rangle = \exp\left[-i\hat{H}(t-t_0)\right]e^{-i\hat{H}_0 t_0}|\psi_I(t_0)\rangle$$

$$\Rightarrow |\psi_I(t)\rangle = e^{i\hat{H}_0 t}\exp\left[-i\hat{H}(t-t_0)\right]e^{-i\hat{H}_0 t_0}|\psi_I(t_0)\rangle$$

$$\equiv \hat{U}_I(t,t_0)|\psi_I(t_0)\rangle$$

$$\Rightarrow \boxed{\hat{U}_I(t,t_0) = \exp\left[i\hat{H}_0 t\right]\exp\left[-i\hat{H}(t-t_0)\right]\exp\left[-i\hat{H}_0 t_0\right]}$$

$$\Rightarrow \hat{U}_I(t,t_0) = \exp\left[i\hat{H}_0 t\right]\hat{U}(t,t_0)\,e^{-i\hat{H}_0 t_0}$$

Now $\hat{U}(t, t_0) = \exp[-i\hat{H}(t-t_0)]$

$$\Rightarrow \hat{U}(t, 0) = \exp[-i\hat{H}t]$$

and $\hat{U}_I(t, 0) = \exp[i\hat{H}_0 t]\,\hat{U}(t, 0)$

$$= \exp[i\hat{H}_0 t]\,\exp[-i\hat{H}t]$$

$$\Rightarrow \hat{U}_I^{-1}(t, 0) = \hat{U}_I^{\dagger}(t, 0) = \exp[i\hat{H}t]\,\exp[-i\hat{H}_0 t].$$

Recalling the relation:

$$\hat{O}_H(t) = e^{i\hat{H}t}e^{-i\hat{H}_0 t}\,\hat{O}_I(t)\,e^{i\hat{H}_0 t}e^{-i\hat{H}t}$$

$$= \hat{U}_I^{-1}(t, 0)\,\hat{O}_I(t)\,\hat{U}_I(t, 0)$$

Now from the discussion we know that "in" and "out" states can be used to construct the S-matrix as following

$$\lim_{\substack{t_A \to -\infty \\ t_B \to +\infty}} \langle \phi_I(t_B)|\,\hat{U}_I^{\dagger}(t, t_B)\cdot\hat{U}_I(t, t_A)\,|\psi_I(t_A)\rangle$$

$$= \lim_{\substack{t_A \to -\infty \\ t_B \to +\infty}} \langle \phi_I(t_B)|\,\hat{U}_I(t_B, t)\,\hat{U}_I(t, t_A)\,|\psi_I(t_A)\rangle$$

Denoting $\lim_{t_A \to -\infty} |\psi_I(t_A)\rangle \equiv |\alpha\rangle$

and $\lim_{t_B \to +\infty} |\phi_I(t_B)\rangle \equiv |\beta\rangle$, we get

$$S_{\beta\alpha} = \langle \beta|\,\hat{U}_I(+\infty, -\infty)\,|\alpha\rangle$$

So $$\boxed{\hat{S} = \lim_{\substack{t_A \to -\infty \\ t_B \to +\infty}} \hat{U}_I(t_B, t_A)}$$