PHY-685
QFT-1

Lecture - 23 + 24

We have Wick's theorem to express time ordered product of the fields (in interaction picture) in terms of normal ordered product + contraction terms:

$$\hat{T}\{\hat{\phi}(x_1)\cdots\hat{\phi}(x_{2N})\} = N\{\hat{\phi}(x_1)\cdots\hat{\phi}(x_{2N})\}$$

$$+ N\{\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\cdots\hat{\phi}(x_{2N})\}$$

$$+ N\{\hat{\phi}(x_1)\cdots\hat{\phi}(x_i)\cdots\hat{\phi}(x_j)\cdots\hat{\phi}(x_{2N})\}$$

$$\vdots$$

$$+ \sum_{\{pm\}}\sum_{\{qn\}} N\{\hat{\phi}(x_1)\cdots\hat{\phi}(x_p)\cdots\hat{\phi}(x_q)\cdots\hat{\phi}(x_m)\cdots\hat{\phi}(x_n)\cdots\hat{\phi}(x_{2N})\}$$

$$\vdots$$

$$\sum_{\text{all possible pairings.}} \hat{\phi}(x_1)\cdots\hat{\phi}(x_{2N})$$

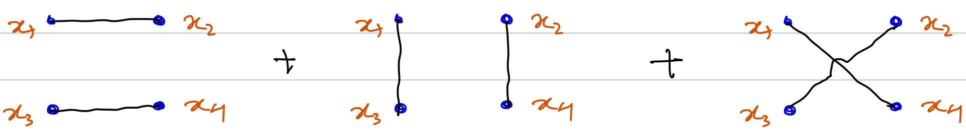
Let's remind ourselves that all the fields above are actually fields in interaction picture. $\hat{\phi}(x_i) \equiv \hat{\phi}_I(x_i)$. Now let's redo the stuff for four fields only and take a $\langle 0|\cdots|0\rangle$

$$\langle 0|\hat{T}\{\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)\}|0\rangle$$
$$= D_F(x_1-x_2)D_F(x_3-x_4) + D_F(x_1-x_3)D_F(x_2-x_4)$$
$$+ D_F(x_1-x_4)D_F(x_2-x_3)$$

Now this kind of expression can be pictorially represented using the following method: Represent $D_F(x-y)$ by a line starting from x and terminating on y:

$$D_F(x-y) = \underset{x}{\bullet}\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\underset{y}{\bullet}\,.$$

Hence: $\langle 0|\hat{T}\{\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)\}|0\rangle \equiv$



We are interested in expressions like:

$$\langle 0 | \hat{T} \left\{ \hat{\phi}(x_1) \hat{\phi}(x_2) \cdots \exp\left[-i \int_{-T}^{T} dt \, \hat{H}_I(t) \right] \right\} | 0 \rangle,$$

with $\int dt \, \hat{H}_I(t) = \int d^4x \, \mathcal{H}_{int}(\hat{\phi})$. This series can be evaluated when the size of $\mathcal{H}_{int}(\hat{\phi})$ is small, a.k.a. weakly coupled theory. In that case we can expand the exponential perturbatively and apply Wick's theorem. We do it for an example $\mathcal{H}_{int}$: THE LAMBDA PHI - FOUR: i.e. $\int dt \, \hat{H}_I(t) = \int dt \int d^3\vec{x} \left(\lambda/4! \, \phi^4(x) \right)$

$$= \int d^4x \left(\frac{\lambda}{4!} \phi^4(x) \right)$$

Here λ is the coupling term determining the strength of the interaction.

The local contact interaction term.

For weakly coupled theory, we can express the observables as power series expansion of λ.

Let's do the simplest expansion upto $O(\lambda)$, of the 2-pt. function. $\langle 0 | \hat{T} \{ \hat{\phi}(x) \hat{\phi}(y) \} | 0 \rangle$

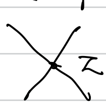
$$= \langle 0 | \hat{T} \left\{ \hat{\phi}(x) \hat{\phi}(y) (-i) \int dt \int d^3 z \, \frac{\lambda}{4!} \hat{\phi}^4 \right\} | 0 \rangle$$

$$= \langle 0 | \hat{T} \left\{ \hat{\phi}(x) \hat{\phi}(y) (-i) \int d^4 z \, \frac{\lambda}{4!} \hat{\phi}(z) \hat{\phi}(z) \hat{\phi}(z) \hat{\phi}(z) \right\} | 0 \rangle$$

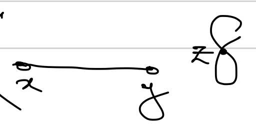
$$= 3 \left(-\frac{i\lambda}{4!} \right) D_F(x-y) \int d^4 z \, D_F(z-z) \, D_F(z-z)$$

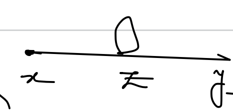
$$+ 12 \left(-\frac{i\lambda}{4!} \right) \int d^4 z \, D_F(x-z) \, D_F(y-z)$$

One should think the point z as an internal point which is integrated over. The $\phi^4(z)$ term of the potential is represented as



$:=$ 4-fields $\phi(z)$ at the same point.

3 (

) + 12 (

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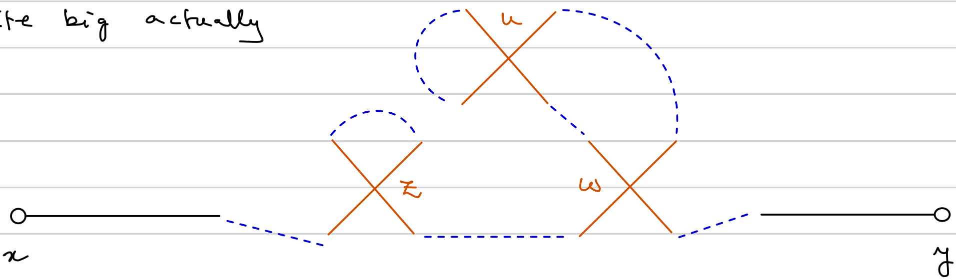
How this is done in practice? (the computation of the symmetry factor): Let's say we want to compute $O(\lambda^3)$ correction to the 2-pt. function:

$$\langle 0|\, \hat\phi(x)\,\hat\phi(y)\,\frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3 \int d^4z\, \hat\phi(z)\hat\phi(z)\hat\phi(z)\hat\phi(z) \int d^4w\, \hat\phi(w)\hat\phi(w)\hat\phi(w)\hat\phi(w) \int \hat\phi(u)\hat\phi(u)\hat\phi(u)\hat\phi(u)\,|0\rangle$$

$$= \frac{1}{3!}\left(-\frac{i\lambda}{4!}\right)^3 \int d^4z\, d^4w\, d^4u\; D_F(x-z) D_F(z-z)\, D_F(z-w)$$
$$\times\; D_F(y-w)\; D_F^2(w-u)\; D_F(u-u)$$

The number of different contraction that gives the same diagram is quite big actually



Now let's compute the pre-factor:

The whole term will come with a factor of $1/3!\left(-\frac{i\lambda}{4!}\right)^3$.

The z vertex:

$\hat\phi(x)$ can contract $\hat\phi(z)$ in 4 ways. Among rest of the available $\hat\phi(z)$ fields a $\hat\phi(z)\,\hat\phi(z)$ contraction can be done in $\binom{3}{2}$ ways. So total # of permutations $= 4 \times 3$

The w vertex:

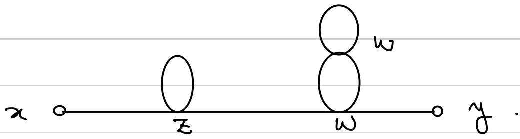
$\hat\phi(y)$ can contract with $\hat\phi(w)$ in 4 ways. The one left over $\hat\phi(z)$ can contract with $\hat\phi(w)$ in 3 ways. This keeps <u>two</u> $\hat\phi(w)$ left to be contracted with a total permutation of 4×3.

The u-vertex:

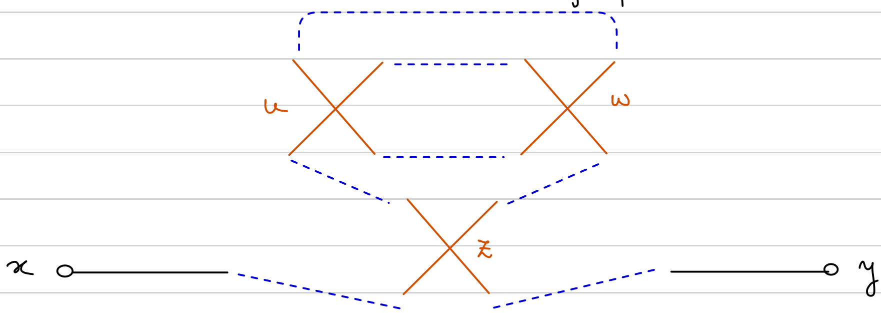
One $\hat\phi(w)$ can contract with $\hat\phi(u)$ in 4 ways. Then a $\hat\phi(u)\,\hat\phi(u)$ can happen in $\binom{3}{2} = 3$ ways. Then we are left with only one $\hat\phi(u)$ which contract with the one left over $\hat\phi(w)$. So again the number of permutations are 4×3

Since u, v, w are internal space-time points with random labelling, these can be permuted in $3!$ ways. This factor will be cancelled by the $(1/3!)$ coming from the $O(3)$ term of the exponential. So the final pre-factor which will sit infront of the diagram is: $\left(\frac{1}{4!}\right)^3 \times (4\times 3) \times (4\times 3) \times (4\times 3) = 1/8$.

So symmetry factor of the diagram is 8, with the following diagram



Now let's look into another topologically different diagram which can occur in same order of perturbation theory:



Again z, u, w are internal space-time points.

The Z vertex:

$\hat{\phi}(x)$ can contract with $\hat{\phi}(z)$ in $\underline{4 \text{ ways}}$. $\hat{\phi}(y)$ can contract with rest of the three $\hat{\phi}(z)$ in $\underline{3 \text{ ways}}$, keeping two legs open. The permutation factor is $\underline{4\times 3}$.

One of the $\hat{\phi}(z)$ can cotract with $\hat{\phi}(u)$ in $\underline{4 \text{ ways}}$ and another free $\hat{\phi}(z)$ can contract with $\hat{\phi}(w)$ in $\underline{4 \text{ ways}}$, completing the contraction of z-vertex.

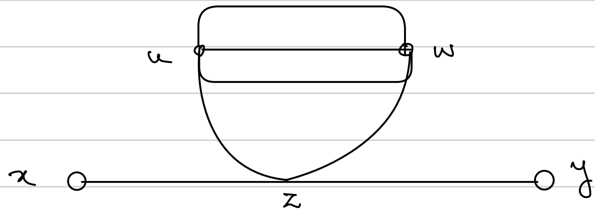
At this point both u and w vertex has 3 free lines each. One $\hat{\phi}(u)$ can be contracted with any of the $\hat{\phi}(w)$ in $\underline{3 \text{ ways}}$

Now we are left with 2 $\hat{\phi}(u)$ and 2 $\hat{\phi}(w)$ fields. One of the $\hat{\phi}(u)$ can contract available two $\hat{\phi}(w)$ in $\underline{2 \text{ ways}}$ and the enforces the left over $\hat{\phi}(u)$ and $\hat{\phi}(w)$ to contract with each other.

Hence the factor which sits infront of the term in perturbative expansion :

$$\left[(4\times 3) \times (4\times 4) \times (3\times 2) \right] / \left[4! \times 4! \times 4! \right] = 1/12$$

So the symmetry factor for the corresponding diagram is 12.
The diagram it gives is as following :

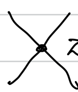


with the corresponding mathematical expression:

$$\frac{1}{12} (-i\lambda)^3 \int d^4z \, d^4u \, d^4w \; D_F(x-z)\, D_F(z-y)\, D_F(z-u)\, D_F(z-w) \times \left(D_F(u-w)\right)^3 .$$

This motivates us to write down the position space Feynman rules:

1. For each propagator, $= D_F(x-y)$;

2. For each vertex,  z $= (-i\lambda) \int d^4z$;

   The origin of the $(-i)$ is from $\exp(-i \int H_I(t) dt)$ of Dyson series.

3. For each external point :  $= 1$;

4. Divide by the symmetry factor.

One can go the Fourier domain and try to write down the momentum space Feynman rule for the same amplitude.

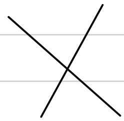
In momentum space:

$$\int d^4z\, \hat{\phi}^4(z) = \int d^4z\, e^{-ip_1 z} \cdot e^{-ip_2 z}\, e^{-ip_3 z}\, e^{-ip_4 z} \times \hat{\phi}(p_1)\, \hat{\phi}(p_2)\, \hat{\phi}(p_3)\, \hat{\phi}(p_4)$$

$$= (2\pi)^4\, \delta^4(p_1 + p_2 + p_3 + p_4)\, \hat{\phi}(p_1)\, \hat{\phi}(p_2)\, \hat{\phi}(p_3)\, \hat{\phi}(p_4).$$

In momentum space Feynman diagrams, the rules to impose are:

1. For each propagator $\longrightarrow \;= \dfrac{i}{p^2 - m^2 + i\epsilon}$

2. For each vertex  $= (-i\lambda)$

3. For each external point, $x \bullet\!\!\longleftarrow\!\!\!-\!\!\!-\!\!\!- \;= e^{-ip\cdot x}$

4. Impose momentum conservation at each vertex because of the $(2\pi)^4\, \delta^4(\Sigma p_i)$ factor we derived.

5. Integrate out all undetermined internal momenta $\int \frac{d^4p}{(2\pi)^4}$. The $1/(2\pi)^4$ factors will exactly cancel out the $(2\pi)^4$ factors appearing with momentum conserving delta functions at each vertex.

6. Divide by the symmetry factor.

Now let's remind ourselves one key point: when we defined the Dyson series it came in the form:

$$\langle 0| \cdots \exp\left[-i \int d^4z\, \mathcal{H}_{int}(\hat{\phi}(z))\right] |0\rangle$$

and every term in the series expansion is essentially terms in the perturbative expansion of this potential. Thus in the momentum space representation we have:

$$\lim_{T \to \infty(1-i\epsilon)} \int_{-T}^{T} dz^0 \int d^3\vec{z}\, e^{-i(\Sigma p_i)\cdot z} = \int_{-T}^{T} dz^0 \int d^3\vec{z}\, e^{-ip\cdot z}$$