PHY-685
QFT-1

Lecture-3

In the last class we have defined the path-integral which computes the transition amplitude: $\langle \vec{q}_b, t_b | \vec{q}_a, t_a \rangle$. Today we want to compute it for simplest of systems and want to see where we may run into murky water!

Lets first see when we have a non-relativistic Hamiltonian $H = \vec{p}^2/2m + V(\vec{q}, t)$. We want to write the P.I. for this Hamiltonian. So

$$U(\vec{q}_b, t_b; \vec{q}_a, t_a) = \langle \vec{q}_b | \hat{U}(t_b, t_a) | \vec{q}_a \rangle \qquad \left[\vec{\bar{q}}_n = (\vec{q}_n + \vec{q}_{n-1})/2 \right]$$

$$= \left[\prod_{n=1}^{N} \int_{-\infty}^{\infty} d^D \vec{q}_n \right] \left[\prod_{n=1}^{N+1} \langle \vec{q}_n | \hat{U}(t_n, t_{n-1}) | \vec{q}_{n-1} \rangle \right]$$

$$= \lim_{N\to\infty} \left[\prod_{n=1}^{N} \int_{-\infty}^{\infty} d^D \vec{q}_n \right] \left[\prod_{n=1}^{N+1} \int_{-\infty}^{\infty} \frac{d^D \vec{p}_n}{(2\pi\hbar)^D} \exp\left[\frac{i}{\hbar} \sum_{n=1}^{N+1} \left\{ \vec{p}_n \cdot (\vec{q}_n - \vec{q}_{n-1}) - \epsilon H(\vec{\bar{q}}_n, \vec{p}_n, t_n) \right\} \right] \right]$$

Putting $H(\vec{\bar{q}}_n, \vec{p}_n, t_n) = \vec{p}_n^{\,2}/2m + V(\vec{\bar{q}}_n, t_n)$, we get the factor in the exponent as:

$$\sum_{n=1}^{N+1} \left[\vec{p}_n \cdot (\vec{q}_n - \vec{q}_{n-1}) - \epsilon \left\{ \frac{\vec{p}_n^{\,2}}{2m} + V(\vec{\bar{q}}_n, t_n) \right\} \right]$$

$$= \sum_{n=1}^{N+1} \left[-\frac{\epsilon}{2m} \left\{ \vec{p}_n^{\,2} - \frac{2m}{\epsilon} \vec{p}_n \cdot (\vec{q}_n - \vec{q}_{n-1}) \right\} - \epsilon V(\vec{\bar{q}}_n, t_n) \right]$$

$$= \sum_{n=1}^{N+1} \left[-\frac{\epsilon}{2m} \left\{ \vec{p}_n - \frac{m}{\epsilon} (\vec{q}_n - \vec{q}_{n-1}) \right\}^2 + \frac{m}{2\epsilon} (\vec{q}_n - \vec{q}_{n-1})^2 - \epsilon V(\vec{\bar{q}}_n, t_n) \right]$$

Putting this back

$$U(\vec{q}_b, t_b; \vec{q}_a, t_a) = \lim_{N\to\infty} \left[\prod_{n=1}^{N} \int_{-\infty}^{\infty} d^D \vec{q}_n \right] \left[\prod_{n=1}^{N+1} \left(\exp\left[\frac{i}{\hbar} \left\{ \frac{m}{2\epsilon} (\vec{q}_n - \vec{q}_{n-1})^2 - \epsilon V(\vec{\bar{q}}_n, t_n) \right\} \right] \right) \right]$$

$$\times \left[\prod_{n=1}^{N+1} \left(\underbrace{\int_{-\infty}^{\infty} \frac{d^D \vec{p}_n}{(2\pi\hbar)^D} \exp\left[-\frac{i}{\hbar} \cdot \frac{\epsilon}{2m} \left\{ \vec{p}_n - \frac{m}{\epsilon} (\vec{q}_n - \vec{q}_{n-1}) \right\}^2 \right]}_{I_n} \right) \right]$$

$$I_n = \int_{-\infty}^{\infty} \frac{d^D \vec{p}_n}{(2\pi\hbar)^D} \exp\left[-\frac{i}{\hbar}\frac{\epsilon}{2m} \left\{ \vec{p}_n - \frac{m}{\epsilon}\left(\vec{q}_n - \vec{q}_{n-1}\right) \right\}^2 \right]$$

putting $a = -\frac{\epsilon}{\hbar m}$ and $\vec{p}_n - \frac{m}{\epsilon}\left(\vec{q}_n - \vec{q}_{n-1}\right) = \vec{k}_n$, we have

$$I_n = \int_{-\infty}^{\infty} \frac{d^D \vec{k}_n}{(2\pi\hbar)^D} \exp\left[i \frac{a}{2} \vec{k}_n^2 \right]$$

$$= \left[\int_{-\infty}^{\infty} \frac{dk}{2\pi\hbar} \exp\left[i\frac{a}{2} k^2 \right] \right]^D$$

$$= \left[\sqrt{\frac{2\pi}{|a|}} \frac{1}{\sqrt{i}} \cdot \frac{1}{2\pi\hbar} \right]^D = \left(\frac{1}{2\pi\hbar} \sqrt{\frac{2\pi\hbar m}{\epsilon i}} \right)^D = \left(\frac{m}{2\pi i \hbar \epsilon} \right)^{D/2}$$

Here we have used the Fresnel integral formula

$$\int_{-\infty}^{\infty} dx \exp\left(i \frac{a}{2} x^2 \right) = \sqrt{\frac{2\pi}{|a|}} \begin{cases} \sqrt{i} & \text{if } a > 0 \\ 1/\sqrt{i} & \text{if } a < 0 \end{cases}$$

So we get $U(\vec{q}_b, t_b; \vec{q}_a, t_a)$

$$= \lim_{N\to\infty} \left[\prod_{n=1}^{N} \int_{-\infty}^{\infty} d^D \vec{q}_n \right] \left[\exp\left[\frac{i}{\hbar} \epsilon \sum_{n=1}^{N+1} \left\{ \frac{m}{2} \left(\frac{\vec{q}_n - \vec{q}_{n-1}}{\epsilon} \right)^2 - V(\vec{q}_n, t_n) \right\} \right] \right]$$

$$\times \left(\frac{m}{2\pi i \hbar \epsilon} \right)^{(N+1)\frac{D}{2}}$$

$$= C \int_{\vec{q}(t=t_a)=\vec{q}_a}^{\vec{q}(t=t_b)=\vec{q}_b} \Delta \vec{q} \exp\left[\frac{i}{\hbar} \int_{t_a}^{t_b} L(\vec{q}, \dot{\vec{q}}, t)\, dt \right], \quad C = \left(\frac{m}{2\pi i \hbar \epsilon} \right)^{\frac{(N+1)D}{2}}$$

Now it may seem that in the limit $N \to \infty$ ($\epsilon \to 0$), the constant part c diverges. So is this computation unphysical? It turns out that this divergence is combined with other contributions coming from the rest of the terms. To see how it works let's try to evaluate the integral for a free theory, i.e. $V(\vec{q}, t) = 0$.

In that case

$$U(\vec{q}_b, t_b; \vec{q}_a, t_a) = \lim_{N\to\infty} \left[\prod_{n=1}^{N} \int_{-\infty}^{\infty} d^D \vec{q}_n \right] \left[\exp\left[\frac{i\epsilon}{\hbar} \sum_{n=1}^{N+1} \frac{m}{2} \left(\frac{\vec{q}_n - \vec{q}_{n-1}}{\epsilon} \right)^2 \right] \right]$$

$$\times C$$

$$= \lim_{N\to\infty} C \int d^D\vec{q}_1 \cdots d^D\vec{q}_N \exp\left[\frac{im}{2\hbar\epsilon}\sum_{n=1}^{N+1}(\vec{q}_n - \vec{q}_{n-1})^2\right]$$

Let's introduce some scaling variable $\vec{y}_n = \left(\frac{m}{2\hbar\epsilon}\right)^{D/2}\vec{q}_n$

So the integral becomes

$$\lim_{N\to\infty} C\left(\frac{m}{2\hbar\epsilon}\right)^{-\frac{ND}{2}}\int d^D\vec{y}_1 \cdots d^D y_N \exp\left[i\sum_{n=1}^{N+1}(\vec{y}_n - \vec{y}_{n-1})^2\right]$$

These are set of coupled gaussian integrals. For example if we look into a single integration, coming from two consecutive time intervals, we have

$$\int d^D\vec{y}_1 \exp\left[i\left\{(\vec{y}_1 - \vec{y}_0)^2 + (\vec{y}_2 - \vec{y}_1)^2\right\}\right]$$

$$= \int d^D\vec{y}_1 \exp\left[i\left\{2\left(\vec{y}_1 - \frac{\vec{y}_0 + \vec{y}_2}{2}\right)^2 + \frac{1}{2}(\vec{y}_2 - \vec{y}_0)^2\right\}\right]$$

$$= \left(\frac{i\pi}{2}\right)^{D/2}\exp\left[\frac{i}{2}(\vec{y}_2 - \vec{y}_1)^2\right]$$

If we look into first three time intervals and carry out the integration, then we have

$$\int d^D\vec{y}_1\, d^D\vec{y}_2 \exp\left[i\left\{(\vec{y}_1 - \vec{y}_0)^2 + (\vec{y}_2 - \vec{y}_1)^2 + (\vec{y}_3 - \vec{y}_2)^2\right\}\right]$$

$$= (i\pi/2)^{D/2}\int d^D\vec{y}_2 \exp\left[i\left\{\tfrac{1}{2}(\vec{y}_2 - \vec{y}_0)^2 + (y_3 - y_2)^2\right\}\right]$$

$$= (i\pi/2)^{\frac{D}{2}}\int d^D\vec{y}_2 \exp\left[i\left\{\frac{3}{2}\left(\vec{y}_2 - \frac{\vec{y}_0 + 2\vec{y}_3}{2}\right)^2 + \frac{1}{3}(\vec{y}_3 - \vec{y}_0)^2\right\}\right]$$

$$= (i\pi/2)^{\frac{D}{2}}\left(\frac{2i\pi}{3}\right)^{D/2}\exp\left[\frac{i}{3}(\vec{y}_3 - \vec{y}_0)^2\right]$$

So following the multiplicative pattern that emerges, when we do the full integral

$$U(\vec{q}_b, t_b; \vec{q}_a, t_a) = C\left(\frac{m}{2\hbar\epsilon}\right)^{-\frac{ND}{2}}\left(\frac{(i\pi)^N}{N+1}\right)^{D/2} \times \exp\left[\frac{i}{N+1}(\vec{y}_{N+1} - \vec{y}_0)^2\right]$$

$$= \lim_{N\to\infty}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{(N+1)\frac{D}{2}}\left(\frac{m}{2\hbar\epsilon}\right)^{-\frac{ND}{2}}(i\pi)^{ND/2}\frac{1}{(N+1)^{D/2}} \times \exp\left[\frac{im}{2\hbar(N+1)\epsilon}(\vec{q}_b - \vec{q}_a)^2\right]$$

$$= \lim_{N\to\infty} \left[\left(\frac{m}{2\pi i\hbar\epsilon}\right)\times\left(\frac{2\hbar\epsilon}{m}\right)\times(i\pi)\right]^{ND/2} \cdot \left[\left(\frac{m}{2\pi i\hbar\epsilon}\right)\left(\frac{1}{N+1}\right)\right]^{D/2}$$

$$\times \exp\left[\frac{im}{2\hbar(N+1)\epsilon}(\vec{q}_b-\vec{q}_a)^2\right]$$

$U(\vec{q}_b, t_b; \vec{q}_a, t_a)$

$$= \left[\frac{m}{2\pi i\hbar(t_b-t_a)}\right]^{D/2} \cdot \exp\left[\frac{i}{\hbar}\,\frac{m(\vec{q}_b-\vec{q}_a)^2}{2(t_b-t_a)}\right]$$

In the limit $t_b \to t_a \longrightarrow \delta^D(\vec{q}_b - \vec{q}_a)$

$$\int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} \exp\left[i \frac{a}{2} p^2 \right] = \frac{1}{\sqrt{|a|}} \begin{cases} \sqrt{i} & a > 0 \\ 1/\sqrt{i} & a < 0 \end{cases}$$