PHY-685

QFT-1

Lecture - 9 + 10

Mass +ve definite: The construction of the Wigner matrix

The little group W here is the 3-D rotation group R. It's unitary irreducible representation $D^{(j)}_{\sigma'\sigma}(R)$ with dimension $(2j+1)\times(2j+1)$ with $j=0,\frac{1}{2},1,\dots$ With infinitesimal rotations $R_{ik}=\delta_{ij}+\theta_{ik}$ (with $\theta_{ik}=-\theta_{ki}$).

$$\hat{D}^{(j)}_{\sigma'\sigma}(\mathbb{1}+\theta)=\delta_{\sigma'\sigma}\mathbb{1}+\frac{i}{2}\theta_{ik}\left(\hat{J}^{(j)}_{ik}\right)_{\sigma'\sigma}$$

Now

$$\left(\hat{J}^{(j)}_{23}\pm i\hat{J}^{(j)}_{31}\right)_{\sigma'\sigma}=\left(\hat{J}_1^{(j)}\pm i\hat{J}_2^{(j)}\right)_{\sigma'\sigma}$$
$$=\delta_{\sigma',\sigma\pm1}\sqrt{(j\mp\sigma)(j\pm\sigma+1)}$$

$$\left(\hat{J}^{(j)}_{12}\right)_{\sigma'\sigma}=\left(\hat{J}_3^{(j)}\right)_{\sigma'\sigma}=\sigma\,\delta_{\sigma'\sigma}$$

σ runs over the values $j, j-1,\dots,-j$.

$$\hat{U}(\Lambda)|\psi(p,\sigma)\rangle=\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\sigma'}D^{(j)}_{\sigma'\sigma}\left(W(\Lambda,p)\right)|\psi(\Lambda p,\sigma')\rangle$$

where $W(\Lambda,p)=L^{-1}(\Lambda p)\,\Lambda\,L(p)$. Since we have $k^\mu=(m,0,0,0)$, the matrix which converts this to p^μ is

$$L^i{}_k(p)=\delta^i{}_k+(\gamma-1)\hat{p}^i\hat{p}_k,\quad L^i{}_0(p)=L^0{}_i=\hat{p}_i\sqrt{\gamma^2-1}$$

and $L^0{}_0(p)=\gamma$, with $\hat{p}_i=p_i/|\vec{p}|$ and $\gamma=\sqrt{\vec{p}^2+m^2}/m$

Boost in a direction $\hat{\vec{p}}$ can be expressed as

$$L(p)=R(\hat{\vec{p}})\,B(|\vec{p}|)\,R^{-1}(\hat{\vec{p}}),$$

where $R(\hat{\vec{p}})$ is a rotation matrix which takes $(0,0,0,1)$ to $(0,\hat{p}_1,\hat{p}_2,\hat{p}_3)$. The boost matrix is

$$B(|\vec{p}|)=\begin{bmatrix}\gamma & 0 & 0 & \sqrt{\gamma^2-1}\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \sqrt{\gamma^2-1} & 0 & 0 & \gamma\end{bmatrix}=\begin{bmatrix}\cosh(\omega) & 0 & 0 & \sinh(\omega)\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \sinh(\omega) & 0 & 0 & \cosh(\omega)\end{bmatrix}$$

$\gamma\equiv\cosh\omega$

$v\gamma\equiv\sinh\omega$

$$W(\Lambda,p)=L^{-1}(\Lambda p)\,\Lambda\,L(p)$$

$$=\left[R(\hat{\vec{\Lambda p}})\,B^{-1}(|\vec{\Lambda p}|)\,R^{-1}(\hat{\vec{\Lambda p}})\right]\Lambda\left[R(\hat{\vec{p}})\,B(|\vec{p}|)\,R^{-1}(\hat{\vec{p}})\right]$$

For $j=0$ $D^{(j)}_{\sigma'\sigma}=\delta_{\sigma'0}\,\delta_{\sigma0}$.

In general (D+1) space-time dimension: The little group is $SO(D-1)$.

Zero Mass case:

Particles with $m=0$ are always moving with a velocity $=1$ and if such a particle is moving with initial energy E_0 at some direction, say $\hat{z}$, then the initial 4-momentum can be written as $k^\mu = E_0(1,0,0,1)$. Now let's say we want to change the energy of the particle to E and move it to some other direction $\hat{\vec{n}} = (n_1, n_2, n_3)$.

How do we achieve this?

1. First we perform a pure boost in the $(0,0,0,1)$ direction by a rapidity parameter $\omega = \ln(E/E_0)$.
2. Then we rotate the 3-momentum direction from $(0,0,1)$ to $\hat{\vec{n}}$.

So the Lorentz transformation matrix $L(p,k)$, which transforms the initial null vector $k = E_0(1,0,0,1)$ to final null vector $p = E(1, n_1, n_2, n_3)$ is given by

$$L(p,k) = R(\hat{\vec{n}})\, B\left(\omega = \ln\frac{E}{E_0}\right)$$

Now we want to figure out that what non-trivial transformations keep the reference vector k invariant? Clearly the trivial guess is 2-D rotations around the z-axis.

Let's define a 2-parameter $\vec{\xi} \equiv (\xi_1, \xi_2)$ transformation on space-time as:

$$T(\vec{\xi})^\mu{}_\nu = \begin{bmatrix} 1+\frac{1}{2}\vec{\xi}^2 & \xi_1 & \xi_2 & -\frac{1}{2}\vec{\xi}^2 \\ \xi_1 & 1 & 0 & -\xi_1 \\ \xi_2 & 0 & 1 & -\xi_2 \\ \frac{1}{2}\vec{\xi}^2 & \xi_1 & \xi_2 & 1-\frac{1}{2}\vec{\xi}^2 \end{bmatrix}$$

$\vec{\xi}^2 = \xi_1^2 + \xi_2^2$

It's easy to check that $k^\mu = T(\vec{\xi})^\mu{}_\nu k^\nu$.

The transformation forms an abelian subgroup:

$$T(\vec{\xi})\, T(\vec{x}) = T(\vec{\xi} + \vec{x}).$$

Hence this is like a translation in 2-D x-y plane where two translation commutes.

The 2-vector $\vec{\xi}$ transforms under rotation as:

$$R(\theta)\, T(\vec{\xi})\, R^{-1}(\theta) = T(R_\theta \vec{\xi}).$$

where $R_\theta \vec{\xi} = (\cos\theta\, \xi_1 + \sin\theta\, \xi_2,\ -\sin\theta\, \xi_1 + \cos\theta \cdot \xi_2)$

The restrictions imposed by the previous criteria shows that we are dealing with a subgroup of 2-D Euclidean group (2-translation + 1 rotation): $\boxed{W(\vec{\xi}, \theta) = T(\vec{\xi})\, R(\theta)}$

$W(\vec{\xi}, \theta)$ satisfies $W(\vec{\xi}, \theta)\, T(\vec{x})\, W^{-1}(\vec{\xi}, \theta) = T(R_\theta \vec{x})$ and $R(\theta_1) \cdot R(\theta_2) = R(\theta_1 + \theta_2)$.

The transformation $W(\vec{\xi}, \theta)$ doesn't have invariant abelian sub-group. This type of groups are called semi-simple.

$$W(\vec{\xi}, \theta)^{\mu}{}_{\nu} = \begin{bmatrix} 1 + \frac{1}{2}\vec{\xi}^2 & \xi_1 & \xi_2 & -\frac{1}{2}\vec{\xi}^2 \\ \xi_1 & 1 & 0 & -\xi_1 \\ \xi_2 & 0 & 1 & -\xi_2 \\ \frac{1}{2}\vec{\xi}^2 & \xi_1 & \xi_2 & 1 - \frac{1}{2}\vec{\xi}^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For infinitesimal transformation parameters ξ_1, ξ_2, θ

$$W(\xi, \theta) = \begin{bmatrix} 1 & \xi_1 & \xi_2 & 0 \\ \xi_1 & 1 & 0 & -\xi_1 \\ \xi_2 & 0 & 1 & -\xi_2 \\ 0 & \xi_1 & \xi_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta & 0 \\ 0 & -\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \xi_1 & \xi_2 & 0 \\ \xi_1 & 1 & \theta & -\xi_1 \\ \xi_2 & -\theta & 1 & -\xi_2 \\ 0 & \xi_1 & \xi_2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \xi_1 \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \xi_2 \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$+ \theta \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \xi_1 \left\{ +i \overbrace{\begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}^{k_1} + i \overbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}}^{J_2} \right\}$$

$$+ \xi_2 \left\{ +i \overbrace{\begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}^{k_2} - i \overbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}}^{J_1} \right\}$$

$$+ i\theta \overbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & +i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}^{J_3}$$

$$\equiv \mathbb{1}_{4\times4} + i\xi_1 (\underbrace{k_1 + J_2}_{A}) + i\xi_2 (\underbrace{k_2 - J_1}_{B}) + i\theta J_3$$

So this group has generators: $A = J_2 + k_1$, $B = -J_1 + k_2$ and J_3

The Lie-algebra of the group is

$$[\hat{J}_3, \hat{A}] = +i\hat{B}, \quad [\hat{J}_3, \hat{B}] = -i\hat{A}, \quad [\hat{A}, \hat{B}] = 0$$

Since $\hat{A}$ and $\hat{B}$ are commuting Hermitean operators, they can be simultaneously diagonalized by states $|\psi(k, c^2, \lambda)\rangle$ with $(\hat{A}^2 + \hat{B}^2)|\psi(k, c^2, \lambda)\rangle = c^2 |\psi(k, c^2, \lambda)\rangle$.

and $\hat{B}\,|\psi(k,a,b)\rangle = b\,|\psi(k,a,b)\rangle$.

Now if we consider a rotated state:

$$|\psi^{\theta}(k,a,b)\rangle = \hat{U}^{-1}(R(\theta))\,|\psi(k,a,b)\rangle,$$ we get the conserved 2-D momenta in the following way:

$$\left.\begin{aligned}\hat{U}(R(\theta))\,\hat{A}\,\hat{U}^{-1}(R(\theta)) &= \hat{A}\cos\theta - \hat{B}\sin\theta \\ \hat{U}(R(\theta))\,\hat{B}\,\hat{U}^{-1}(R(\theta)) &= \hat{A}\sin\theta + \hat{B}\cos\theta\end{aligned}\right\}$$

From these two operator relations, we have

$$\hat{A}\,|\psi^{\theta}(k,a,b)\rangle = (a\cos\theta - b\sin\theta)\,|\psi^{\theta}(k,a,b)\rangle$$
$$\hat{B}\,|\psi^{\theta}(k,a,b)\rangle = (a\cos\theta + b\sin\theta)\,|\psi^{\theta}(k,a,b)\rangle,$$

which means if we continuously rotate states, we will get a continuous spectra of (a,b) associated with translations in 2-D $\Rightarrow$ i.e. a conserved momenta.

Since we haven't seen any such conserved quantity phenomenologically, we set them to $0 \Rightarrow a=0=b$.

Now a general unitary operator for the Wigner transformation will be $\hat{U}(W(\vec{\xi},\theta))$ and it's action on the reference state $|\psi(k,\lambda)\rangle$ will be

$$\hat{U}(W(\vec{\xi},\theta))\,|\psi(k,\lambda)\rangle = \exp\left[i\{\xi_1\hat{A} + \xi_2\hat{B}\}\right]\exp\left[i\theta\,\hat{J}_3\right]|\psi(k,\lambda)\rangle$$

$$= \exp(i\theta\lambda)\,|\psi(k,\lambda)\rangle$$

So the "D-matrices" in this case have eigenvalues

$$D_{\lambda'\lambda}(W) = \exp(i\theta\lambda)\,\delta_{\lambda'\lambda}$$

So for arbitrary helicity state $|\psi(p,\lambda)\rangle$, the transformation rule is:

$$\boxed{\hat{U}(\Lambda)\,|\psi(p,\lambda)\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}}\,\exp\left[i\lambda\,\theta(\Lambda,p)\right]\,|\psi(\Lambda p,\lambda)\rangle}$$

$\Rightarrow$ For a massless particle state, the helicity λ doesn't change under Lorentz-transformation.

For 3-D $\exp[i(4\pi n)\lambda_n] = 1$

$$\Rightarrow \lambda_n = 0, \pm\tfrac{1}{2}, \pm 1, \pm\tfrac{3}{2}, \ldots$$