PHY-685

QFT-1

Lecture-4

In the last few lectures we established the path integral formalism and showed that the evolution amplitude of a system $\langle \vec{q}_b | \hat{U}(t_b, t_a) | \vec{q}_a \rangle = \int \mathcal{D}\vec{q}\, \mathcal{D}\vec{p}\, \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} (\vec{p}\cdot\dot{\vec{q}} - H(\vec{q}, \vec{p}, t))\right]$

In the time independent Hamiltonian case, this is

$$\Theta(t_b - t_a) \sum_n \psi_n^*(\vec{q}_a)\, \psi_n(\vec{q}_b) \exp\left(-\frac{i}{\hbar} E_n (t_b - t_a)\right).$$

By reading of the coeffts on both the sides, we compute E_n, ψ_n.

Since we are also curious about relativity, we want to repeat this calculation for the most simple relativistic system: a free relativistic particle in $(3+1)$ dimension.
The Hamiltonian of the particle is $H = \sqrt{\vec{p}^2 + m^2}$.
Lets say at $t=0$, the particle is at $\vec{x}_0$ and hence have a state $|\vec{x}_0\rangle$, such that $\langle \vec{x} | \vec{x}_0 \rangle = \delta^3(\vec{x} - \vec{x}_0)$.
We want to compute $\langle \vec{x}, t ; \vec{x}_0, 0 \rangle \equiv \langle \vec{x} | e^{-i\hat{H}t} | \vec{x}_0 \rangle$
(Now on we are switching to $c=1$, $\hbar = 1$ convention).

$$\langle \vec{x} | \exp(-i\hat{H}t) | \vec{x}_0 \rangle = \int d^3\vec{p}\, \langle \vec{x} | \exp(-i\hat{H}t) | \vec{p} \rangle \langle \vec{p} | \vec{x}_0 \rangle$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \exp\left[i\left(\vec{p}\cdot(\vec{x} - \vec{x}_0) - E_{\vec{p}}\, t\right)\right], \text{ where } E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$$

$$= \int_0^\infty \frac{p^2\, dp}{(2\pi)^3} \exp(-iE_p t) \int_0^\pi \exp(i p r \cos\theta) \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty p\, dp\, \exp(-iE_p t) \left[\frac{e^{ipr} - e^{-ipr}}{ir}\right]$$

$$= \frac{1}{(2\pi)^2} \cdot \frac{1}{ir} \left[\int_0^\infty p\, dp\, \exp(-iE_p t)\, e^{ipr} + \int_\infty^0 p\, dp\, \exp(-iE_p t)\, e^{-ipr}\right]$$

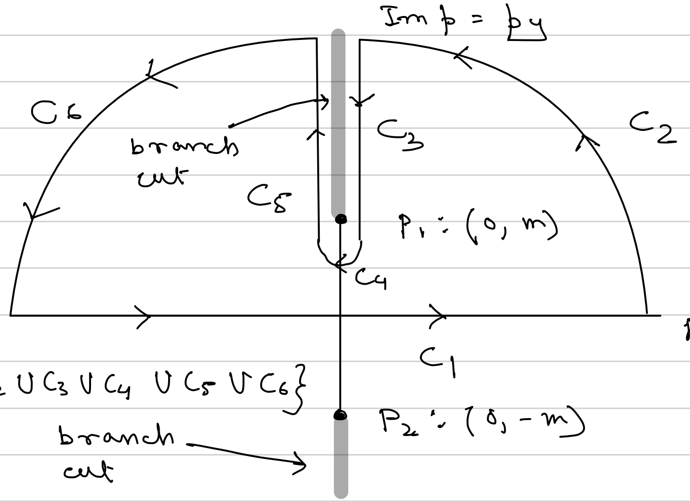
$$= -\frac{1}{(2\pi)^2} \frac{i}{r} \int_{-\infty}^\infty dp \cdot p\, \exp\left[i(pr - E_p t)\right]$$

Here $r = |\vec{x} - \vec{x}_0|$. This is a messy integral and we want to know if this amplitude vanishes or not for $r > t$.

To compute this integral, we analytically extend p to complex domain as $p \to \tilde{p} = p_x + i p_y$. Immediately we see that there are two points in the complex plane, viz. $P_1 : (p_x = 0,\ p_y = m)$ and $P_2 : (p_x = 0,\ p_y = -m)$ at which the integrand is multi-valued.
$E_p = \sqrt{p^2 + m^2}$ vanishes at these two points.

$$\tilde{p} \exp\left[i\left(\tilde{p} r - E_{\tilde{p}} t\right)\right]\Big|_{P_1} = i m \exp(-mr) = m e^{-mr} e^{i\pi/2}$$

$$\tilde{p} \exp\left[i\left(\tilde{p} r - E_{\tilde{p}} t\right)\right]\Big|_{P_2} = -i m \exp(+mr) = m e^{mr} e^{-i\pi/2}$$



Contour in the complex plane for evaluating the integral.

$C \equiv \{C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6\}$

We have chosen the contour of the integration as shown: anti-clockwise in the upper-half plane avoiding the branch-cut $(0,\ i(m+\alpha))$ with $\alpha \in \mathbb{R} > 0$. The integrand is an analytic function within this region enclosed by C.

The value of E_p should be discontinuous on the either side of the branch cut:

$$E_{\tilde{p}} = \begin{cases} i\sqrt{p_y^2 - m^2} & \text{for } p_x = 0^+ \\ -i\sqrt{p_y^2 - m^2} & \text{for } p_x = 0^- \end{cases}$$

We have $$\oint_C d\tilde{p} \cdot \tilde{p} \exp\left[i\left(\tilde{p} r - E_{\tilde{p}} t\right)\right] = 0$$

On the right arc we have $\tilde{p} = R\exp(i\theta)$ with $\theta \in [0, \frac{\pi}{2})$

So $\tilde{p} \exp\left[i\left(\tilde{p} r - \sqrt{\tilde{p}^2+m^2}\, t\right)\right]$

$$= R\exp(i\theta)\exp\left[R e^{i(\theta+\pi/2)} \cdot r - \sqrt{R^2 e^{i2\theta}+m^2}\; t\right]$$

$$= R\exp\left[i\theta + r\{-R\sin\theta + iR\cos\theta\} - t\sqrt{(R^2\cos 2\theta + m^2) + iR^2\sin 2\theta}\right]$$

$$\underset{R\to\infty}{\approx} R\exp\left[i\theta + r\{-R\sin\theta + iR\cos\theta\} - t(R\cos\theta + iR\sin\theta)\right]$$

$$\underset{R\to\infty}{\approx} R\exp\left[-R(r\sin\theta + t\cos\theta)\right]\exp\left[i\left\{\theta - R\left(r\sin\theta + t\cos\theta\right)\right\}\right]$$

≈ 0

similarly on the left arc $\theta \in (\pi/2, \pi]$ but R dependence remains same and hence the function vanishes.

On C_4, we can write $\tilde{p} = (0+im)\,R'\exp(i\delta)$ with $\delta \in [2\pi, \pi]$. But since $R' \to 0$, again the integrand vanishes.

Hence we have:
$$\left[\int_{C_1} + \int_{C_3} + \int_{C_5}\right] d\tilde{p}\cdot\tilde{p}\,\exp\left[i\left(\tilde{p}r - E_{\tilde{p}}t\right)\right] = 0$$

Integral on C_3:

$$\int_{C_3} d\tilde{p}\cdot\tilde{p}\,\exp\left[i\left(\tilde{p}r - E_{\tilde{p}}t\right)\right]$$

$$= \int_{\infty}^{m} i\,dp_y\,(ip_y)\exp\left[i\left\{(ip_y)r - i\sqrt{p_y^2-m^2}\,t\right\}\right]$$

$$= -\int_{\infty}^{m} p_y\,dp_y\,\exp\left[-p_y r + \sqrt{p_y^2-m^2}\;t\right]$$

$$= \int_{m}^{\infty} dp_y\,p_y\,\exp\left[-p_y r + \sqrt{p_y^2-m^2}\;t\right]$$

Integral on C_5:

$$\int_{C_5} d\tilde{p}\cdot\tilde{p}\,\exp\left[i\left(\tilde{p}r - E_{\tilde{p}}t\right)\right]$$

$$= \int_m^\infty i\, dp_y\, (ip_y) \exp\left[i \left\{ (ip_y) r + i\sqrt{p_y^2 - m^2}\, t \right\} \right]$$

$$= -\int_m^\infty dp_y\, p_y \exp\left[-p_y r - \sqrt{p_y^2 - m^2}\, t \right]$$

$$= -\int_m^\infty dp_y\, p_y \exp\left[-p_y r - \sqrt{p_y^2 - m^2}\, t \right]$$

So $\int_{C_3} + \int_{C_5}$

$$= \int_m^\infty dp_y\, p_y \exp\left[-p_y r + \sqrt{p_y^2 - m^2} \right] t$$

$$- \int_m^\infty dp_y\, p_y \exp\left[-p_y r - \sqrt{p_y^2 - m^2}\, t \right]$$

$$= \int_m^\infty dp_y\, p_y \exp(-p_y r) \left[\exp\left\{ \sqrt{p_y^2 - m^2}\, t \right\} - \exp\left\{ -\sqrt{p_y^2 - m^2}\, t \right\} \right]$$

$$= 2 \int_m^\infty dp_y\, p_y \exp(-p_y r) \sinh\left(\sqrt{p_y^2 - m^2}\, t \right)$$

So $\int_{-\infty}^{\infty} dp \cdot p \exp\left[i (p r - E_p t) \right]$

$$= \int_{C_1} d\tilde{p} \cdot \tilde{p} \exp\left[i (\tilde{p} r - E_{\tilde{p}} t) \right]$$

$$= -\left(\int_{C_3} + \int_{C_5} \right) d\tilde{p} \cdot \tilde{p} \exp\left[i (\tilde{p} r - E_{\tilde{p}} t) \right]$$

$$= -2 \int_m^\infty dp_y\, p_y \exp(-p_y r) \sinh\left(\sqrt{p_y^2 - m^2}\, t \right)$$

So $\langle \vec{x} | \exp(-i\hat{H} t) | \vec{x}_0 \rangle$

$$= -\frac{1}{(2\pi)^2} \frac{i}{r} \int_{-\infty}^{\infty} dp \cdot p \exp\left[i (p r - E_p t) \right]$$

$$= -\frac{i}{2\pi^2 r} \int_m^\infty dp_y\, p_y \exp(-p_y r) \sinh\left(\sqrt{p_y^2 - m^2}\, t \right)$$

This result tells us that we are dealing with a finite number even when $r \equiv |\vec{x} - \vec{x_0}| > t$. The propagator has a magnitude

$$|G(r,t)| = \frac{1}{2\pi^2 r} \int_m^\infty dp_y \, p_y \exp(-p_y r) \sinh\left(\sqrt{p_y^2 - m^2}\, t\right) \quad *$$

$$= \frac{1}{4\pi^2 r} \int_m^\infty dp_y \, p_y \exp(-p_y r) \left[\exp\left(\sqrt{p_y^2 - m^2}\, t\right) - \exp\left(-\sqrt{p_y^2 - m^2}\, t\right)\right]$$

$$< \frac{1}{4\pi^2 r} \int_m^\infty dp_y \, p_y \exp\left[-p_y (r-t)\right]$$

$$< \frac{1}{4\pi^2 r} \left[- p_y \frac{\exp\left[-p_y(r-t)\right]}{(r-t)} \Big|_m^\infty + \frac{1}{(r-t)} \int_m^\infty \exp\left[-p_y(r-t)\right] dp_y \right]$$

$$< \frac{1}{4\pi^2 r} \left[\frac{m \exp\left[-m(r-t)\right]}{(r-t)} + \frac{1}{(r-t)^2} \exp\left[-m(r-t)\right] \right]$$

$$\boxed{|G(r,t)| < \frac{1}{4\pi^2 r} \exp\left[-m(r-t)\right] \left[\frac{m}{r-t} + \frac{1}{(r-t)^2}\right]}$$

The propagator of a massive relativistic particle outside light cone is exponentially suppressed but NOT zero. Causality seems to be at a toss!!

The probability of finding a particle outside the forward light-cone is exponentially suppressed. Nonetheless its still is a violation of causality.

$$* \int_0^\infty x \exp\left[-\beta\sqrt{\gamma^2 + x^2}\right] \sin(bx)\, dx = \frac{b\beta\gamma^2}{\beta^2 + b^2} K_2\left(\gamma\sqrt{\beta^2 + b^2}\right)$$