PHY-685

QFT-1

Lecture - 14

In the previous lecture we have obtained the expression for a generalized Noether current J^μ_A, under the global transformation $\delta\omega^A$ as:

$$\boxed{J^\mu_A = \frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \phi^a(x))}\left\{\frac{\partial_T \phi^a(x)}{\partial \omega^A} - (\partial_\nu \phi^a(x))\frac{\partial x^\nu}{\partial \omega^A}\right\} + \mathcal{L}\frac{\partial x^\mu}{\partial \omega^A}}$$

under the symmetry transformation, we have $\partial_\mu J^\mu = 0$. Integrating this over the volume:

$$0 = \int_V d^3\vec{x}\, \partial_\mu J^\mu_A = \int_V d^3\vec{x}\, [\partial_0 J^0_A(x) + \partial_i J^i_A(x)]$$

$$= \frac{d}{dx^0}\left[\int_V d^3\vec{x}\, J^0_A(x) + \underbrace{\int_{\partial V} d^2\sigma\, n_i J^i_A(x)}\right]$$ falls of sufficiently fast and hence vanishes.

$$\Rightarrow \frac{d}{dx^0}\Big[\underbrace{\int_V d^3\vec{x}\, J^0_A(x)}_{Q_A(x^0)}\Big] = 0$$

$Q_A(x^0) \longrightarrow$ Noether charge

$$\boxed{Q_A(x^0) = \int_V d^3\vec{x}\left[\frac{\delta \mathcal{L}}{\delta(\partial_0\phi^a)}\left\{\frac{\delta_T \phi^a}{\delta\omega^A} - \partial_\nu \phi^a \frac{\delta x^\nu}{\delta \omega^A}\right\} + \mathcal{L}\frac{\delta x^0}{\delta \omega^A}\right]}$$

Space-time translations:

Under a pure space-time translation of the system (and no active transformation on the system: $\delta x^\mu = c^\mu$

In this case

$$\delta S = \int d^4x\, \partial_\mu\left[\eta^{\mu\nu}\mathcal{L} - \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)}\left(\partial^\nu\phi^a - \frac{\delta_T\phi^a}{\delta x_\nu}\right)\right] c_\nu .$$

When there is no variation of the physical system and only the origin of the co-ordinate system is shifted, then the change in the action of the system must be zero, i.e. $\delta S = 0$.

$$\Rightarrow \boxed{\begin{aligned} &\partial_\mu T^{\mu\nu} = 0, \text{ where} \\ &T^{\mu\nu} \equiv -\eta^{\mu\nu}\mathcal{L} + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)}\left(\partial^\nu\phi^a - \frac{\delta_T\phi}{\delta x_\nu}\right)\end{aligned}}$$

ENERGY - MOMENTUM - TENSOR.

$T^{\mu\nu}(x)$ is a locally conserved quantity.

Lets define $P^\nu(x^0) = \int_V d^3\vec{x}\, T^{0\nu}(\vec{x}, x^0)$, a constant of motion.

In particular $P^0 = \int_V d^3\vec{x}\, T^{00}(\vec{x}, x^0) = \int_V d^3\vec{x} \Big[\underbrace{-\mathcal{L} + \frac{\partial \mathcal{L}}{\delta(\partial_0 \phi^a)}\left(\partial^0 \phi^a - \frac{\delta^T \phi}{\delta x_0}\right)}_{\text{Hamiltonian density } \mathcal{H}} \Big]$

The space-time components are:

$$P^i = \int_V d^3\vec{x}\, T^{0i} = \int_V d^3\vec{x} \left[-\eta^{0i}\mathcal{L} + \frac{\partial \mathcal{L}}{\delta(\partial_0 \phi^a)}\left(\partial^i \phi^a - \frac{\delta^T \phi^a}{\delta x_i}\right)\right]$$

$$= \int_V d^3\vec{x} \left[\pi_a \left(\partial^i \phi^a - \frac{\delta^T \phi}{\delta x_i}\right)\right]$$

Just worth stressing is that canonical momenta and total momenta are completely different quantities.

Lorentz transformations;

For Lorentz transformations, we have $\delta x^\mu = \delta\omega^\mu{}_\nu x^\nu$.

Also for scalar fields we have: $\delta^T \phi = 0$. Hence

$$\delta S = \int d^4x\, \partial_\mu \left[\mathcal{L}\,\delta x^\mu - \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^a)}\left(\partial_\nu \phi^a - \frac{\delta^T \phi^a}{\delta x^\nu}\right)\delta x^\nu \right]$$

$$= \int d^4x\, \partial_\mu \left[\left(\mathcal{L}\,\delta^\mu{}_\nu - \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^a)}\left(\partial_\nu \phi^a - \frac{\delta^T \phi^a}{\delta x^\nu}\right)\right)\delta x^\nu \right]$$

$$= \int d^4x\, \partial_\mu \left[\left\{\mathcal{L}\,\delta^\mu{}_\nu - \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^a)}\left(\partial_\nu \phi^a - \frac{\delta^T \phi^a}{\delta x^\nu}\right)\right\}\delta\omega^\nu{}_\rho\, x^\rho \right]$$

$$= \int d^4x\, \partial_\mu \left[\left\{\mathcal{L}\,\eta^{\mu\nu} - \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^a)}\left(\partial^\nu \phi^a - \frac{\delta^T \phi^a}{\delta x_\nu}\right)\right\}\omega_{\nu\rho}\, x^\rho \right]$$

$$= -\int d^4x\, \partial_\mu \left[T^{\mu\nu} x^\rho\, \omega_{\nu\rho} \right]$$

$$= \int d^4x\, \partial_\mu \left[\frac{1}{2}\omega_{\nu\rho} \underbrace{\left\{ T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu \right\}}_{M^{\mu\nu\rho}} \right]$$

The conservation equation is $\partial_\mu M^{\mu\nu\rho} = 0$

For pure rotations $\delta x^0 = 0$, $\delta x^i = \delta\omega^i{}_j x^j$. where $\delta\omega^i{}_j = \epsilon^i{}_{jk}\theta^k$ and θ^k $(k=1,2,3)$ are the Euler angles.

The local current conservation leads to global conserved tensor: $L^{\nu\rho}(x^0) = \int_V d^3\vec{x}\; M^{0\nu\rho}(\vec{x}, x^0)$, with $L^{\nu\rho} = -L^{\rho\nu}$

So $$L_{jk}(x^0) = \int_V d^3\vec{x}\; M_{0jk}(\vec{x}, x^0)$$

$$= \int_V d^3\vec{x}\left[T_{0j}(x)\, x_k - T_{0k}(x)\, x_j\right]$$

$$= \int_V d^3\vec{x}\left[P_j(x)\, x_k - P_k(x)\, x_j\right]$$

$$L_i = \frac{1}{2}\epsilon_{ijk} L_{jk}(x^0) = \int_V d^3\vec{x}\; \epsilon_{ijk}\; x_j\, P_k(x)$$

Now $0 = \partial_\mu M^{\mu\nu\rho} = \partial_\mu\left[T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu\right]$

$= (\partial_\mu T^{\mu\nu})x^\rho + T^{\mu\nu}\delta_\mu^\rho - (\partial_\mu T^{\mu\rho})x^\nu - T^{\mu\rho}\delta_\mu^\nu$

$\Rightarrow \quad T^{\rho\nu} - T^{\nu\rho} \Rightarrow \boxed{T^{\mu\nu} = T^{\nu\mu}}$ Not by construction but with the imposition of angular momentum conservation

Now if we perform a translation $x^\mu \to x^\mu + a^\mu$, then $L^{\mu\nu} \to L^{\mu\nu} + a^\mu P^\nu - a^\nu P^\mu$. So to make a translation invariant quantity one constructs:

$$\boxed{W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\frac{L^{\nu\rho}P^\sigma}{\sqrt{P_\alpha P^\alpha}}.}$$

This in the rest frame reduces to angular momentum.

We have seen that $T^{\mu\nu}$ by construction is not symmetric. Now if we add a quantity $\partial_\lambda K^{\mu\nu\lambda}$ such that

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\mu\nu\lambda}.$$

If we demand $\partial_\mu \tilde{T}^{\mu\nu} = 0$ when $\partial_\mu T^{\mu\nu} = 0$ and $\tilde{T}^{\mu\nu} = \tilde{T}^{\nu\mu}$, then $K^{\mu\nu\lambda}$ is antisymmetric in (μ, λ) and (ν, λ).

$\tilde{T}^{\mu\nu}$ is called Belinfante Energy-Momentum tensor.
Gravity couples to $\tilde{T}^{\mu\nu}$ and not $T^{\mu\nu}$

Lets take example of a scalar field Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi)$$

$$T^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial^\nu \phi = -\eta^{\mu\nu}\mathcal{L} + \partial^\mu\phi\,\partial^\nu\phi$$

The conserved E-M 4-vector:

$$p^\mu = \int d^3\vec{x}\,\left(-\eta^{0\mu}\mathcal{L} + \partial^0\phi\,\partial^\mu\phi\right)$$

So $$p^0 = \int d^3\vec{x}\,(\pi\,\partial_0\phi - \mathcal{L}) = \int d^3\vec{x}\left[\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi)\right]$$

$$p^i = \int d^3\vec{x}\;\pi(x)\,\partial_i\phi(x)$$

Example of internal symmetry

Lets take a complex scalar field:

$$\mathcal{L} = \partial_\mu\phi^*\,\partial^\mu\phi - m^2\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2$$

say $\phi = (\phi_1 + i\phi_2)/\sqrt{2} \Rightarrow \phi^*\phi = \frac{1}{2}(\phi_1^2 + \phi_2^2)$

$$\partial_\mu\phi^* = \frac{1}{\sqrt{2}}(\partial_\mu\phi_1 - i\,\partial_\mu\phi_2)$$

$$\Rightarrow \partial_\mu\phi^*\,\partial^\mu\phi = \frac{1}{2}\left[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2\right]$$

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi_1\,\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\,\partial^\mu\phi_2 - \frac{m^2}{2}(\phi_1^2 + \phi_2^2) - \frac{\lambda}{16}(\phi_1^2 + \phi_2^2)^2$$