PHY-685
QFT-1

Lecture - 15

Global internal symmetries: An example with scalar fields:

Let's say we have two real scalar fields $\phi^a(x)$ $(a=1,2)$ and the Lagrangian density of the system is given by

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_a \partial^\mu \phi_a - m^2 \phi_a \phi_a\right)$$
$$= \frac{1}{2}\left(\partial_\mu \phi_1 \partial^\mu \phi_1 - m^2 \phi_1 \phi_1\right) + \frac{1}{2}\left(\partial_\mu \phi_2 \partial^\mu \phi_2 - m^2 \phi_2 \phi_2\right)$$

$\Rightarrow$ Sum of two Lagrangian densities with the same mass m.

This $\mathcal{L}$ has a symmetry: if we linearly superpose the fields

$$\phi_1'(x) = \phi_1(x)\cos\theta + \phi_2(x)\sin\theta$$

and
$$\phi_2'(x) = -\phi_1(x)\sin\theta + \phi_2(x)\cos\theta,$$

then we can verify that $\mathcal{L}(x, \phi_a(x), \partial\phi_a(x)) = \mathcal{L}(x, \phi_a'(x), \partial\phi_a'(x))$

Please note that here the explicit functional form of the Lagrangian density remains unchanged. One, in general, can add any higher order powers of $[\phi_a(x)\phi_a(x)]$ and the above fact will remain unchanged.

Now we want to compute Noether current and Noether charge due to above symmetry transformation.

Here we have only one transformation θ $(\equiv \omega^A$ & hence $A=1)$.

Under infinitesimal transformations:

$$\delta\phi_1(x) = \phi_2(x)\,\delta\theta \qquad \text{and} \qquad \delta\phi_2(x) = -\phi_1(x)\,\delta\theta$$

We also saw that after the transformation, there is no total derivative part in the Lagrangian density $\mathcal{L}$. Hence $K^\mu = 0$.

Now we plug back all of these in the formula:

$$J^\mu_A = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a(x))}\frac{\partial \phi^a(x)}{\partial \omega^A} - \frac{\partial K^\mu(x)}{\partial \omega^A}$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1(x))}\cdot\frac{\partial \phi_1}{\partial\theta} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2(x))}\frac{\partial \phi_2}{\partial\theta}$$

$$= \partial^\mu \phi_1(x)\,\phi_2(x) - \partial^\mu \phi_2(x)\,\phi_1(x)$$

$$= \phi_1(x)\overleftrightarrow{\partial^\mu}\phi_2(x) \qquad \left[\text{where } A\overleftrightarrow{\partial}B = (\partial A)B - A(\partial B)\right]$$

We can explicitly verify $\partial_\mu J^\mu = 0$ with the help of the E.O.M. for the two fields $(\partial^2 - m^2)\phi_1(x) = 0$ & $(\partial^2 - m^2)\phi_2(x) = 0$. These fields $\phi_a(x)$ are real fields and hence they must be eigen-values of Hermitean operators $\hat{\phi}_a(x)$, which has a mode expansion:

$$\hat{\phi}_a(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[\hat{a}_a(\vec{p})\, e^{-ip\cdot x} + \hat{a}_a^\dagger(\vec{p})\, e^{ip\cdot x} \right] \Big|_{p^0 = E_{\vec{p}}}$$

For a scalar operator as above and given that we have seen that the canonically conjugate momenta (for the free theory) is $\Pi_a(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)}$, we can write down

$$\hat{\Pi}_a(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} (-i) \sqrt{\frac{E_{\vec{p}}}{2}} \left(\hat{a}_a(\vec{p})\, e^{-ip\cdot x} - \hat{a}_a^\dagger(\vec{p})\, e^{ip\cdot x} \right)$$

$$= \frac{\delta L(t)}{\delta \dot{\hat{\phi}}_a(x)} = \frac{\partial}{\partial t} \hat{\phi}_a(x)$$

One can used the established bosonic algebra (we still need to prove it). $[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{p}\,')] = (2\pi)^3 \delta^3(\vec{p} - \vec{p}\,')$ to cross check:

$$[\hat{\phi}_a(\vec{x}, x^0), \hat{\Pi}_b(\vec{y}, x^0)] = i\,\delta_{ab}\, \delta^3(\vec{x} - \vec{y}).$$

In the theory with two free fields:

$$[\hat{a}_i(\vec{p}), \hat{a}_j(\vec{p}\,')] = 0 = [\hat{a}_i^\dagger(\vec{p}), \hat{a}_j^\dagger(\vec{p}\,')]$$

$$[\hat{a}_i(\vec{p}), \hat{a}_j^\dagger(\vec{p}\,')] = (2\pi)^3 \delta^3(\vec{p} - \vec{p}\,)\, \delta_{ij}$$

The Noether charge of the theory is:

$$\hat{Q} = \int d^3\vec{x}\, \hat{J}_0(x) = \int d^3\vec{x} \left[(\partial_0 \hat{\phi}_1(x))\, \hat{\phi}_2(x) - \hat{\phi}_1(x)(\partial_0 \hat{\phi}_2(x)) \right]$$

$$= i \int \frac{d^3\vec{p}}{(2\pi)^3} \left[\hat{a}_1^\dagger(\vec{p})\, \hat{a}_2(\vec{p}) - \hat{a}_2^\dagger(\vec{p})\, \hat{a}_1(\vec{p}) \right]$$

We know the Hamiltonian of the system:

$$\hat{H} = \int d^3\vec{x} \left(\hat{\Pi}_a \partial_0 \hat{\phi}_a - \hat{\mathcal{L}} \right) = \int d^3\vec{x} \left[\frac{1}{2} \hat{\Pi}_a \hat{\Pi}_a + \frac{1}{2} \vec{\nabla}\hat{\phi}_a \cdot \vec{\nabla}\hat{\phi}_a + \frac{1}{2} m^2 \phi_a \phi_a \right]$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} E_{\vec{p}} \left(\hat{a}_1^\dagger(\vec{p})\, \hat{a}_1(\vec{p}) + \hat{a}_2^\dagger(\vec{p})\, \hat{a}_2(\vec{p}) \right)$$

The momentum operator:

$$\hat{P}^i = \int d^3\vec{x}\; \hat{\pi}_a(x)\, \partial^i \hat{\phi}_a(x)$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3}\, p^i \left(\hat{a}_1^\dagger(\vec{p})\,\hat{a}_1(\vec{p}) + \hat{a}_2^\dagger(\vec{p})\,\hat{a}_2(\vec{p}) \right)$$

From these expression we see that $\hat{Q}|\Omega\rangle = 0$,

$[\hat{Q}, \hat{H}] = 0$, $[\hat{Q}, \hat{P}^i] = 0$

The operator $\hat{H}$, $\hat{Q}$ and $\hat{P}$ are all normal ordered. In terms of the basis operator $a_i(\vec{p})$ and $\hat{a}_i^\dagger(\vec{p})$, these operators have the form

$$\hat{P}^i = \int \frac{d^3\vec{p}}{(2\pi)^3}\, p^i \left[\hat{a}_1^\dagger(\vec{p}) \quad \hat{a}_2^\dagger(\vec{p}) \right] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} \hat{a}_1(\vec{p}) \\ \hat{a}_2(\vec{p}) \end{bmatrix}$$

$$\hat{H} = \int \frac{d^3\vec{p}}{(2\pi)^3}\, E_{\vec{p}} \left[\hat{a}_1^\dagger(\vec{p}) \quad \hat{a}_2^\dagger(\vec{p}) \right] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} \hat{a}_1(\vec{p}) \\ \hat{a}_2(\vec{p}) \end{bmatrix}$$

$$\hat{Q} = \int \frac{d^3\vec{p}}{(2\pi)^3}\, (i) \left[\hat{a}_1^\dagger(\vec{p}) \quad \hat{a}_2^\dagger(\vec{p}) \right] \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{bmatrix} \hat{a}_1(\vec{p}) \\ \hat{a}_2(\vec{p}) \end{bmatrix}$$

So $\hat{H}$ and $\hat{P}^i$ are diagonal but $\hat{Q}$ is <u>NOT</u>.

Infact using the algebra of $\hat{a}_i$ and $\hat{a}^\dagger_i$'s, we can check that

$$\boxed{\begin{aligned} [\hat{Q}, \hat{a}_i(\vec{p})] &= -i\,\epsilon_{ij}\,\hat{a}_j(\vec{p}) \\ \text{and} \quad [\hat{Q}, \hat{a}^\dagger_i(\vec{p})] &= -i\,\epsilon_{ij}\,\hat{a}^\dagger_j(\vec{p}) \end{aligned}}$$

Now we try to perform a linear mixing of the operators

$$\hat{b}(\vec{p}) = 1/\sqrt{2}\left[\hat{a}_1(\vec{p}) + i\,\hat{a}_2(\vec{p})\right] \qquad \hat{c}(\vec{p}) = 1/\sqrt{2}\left[\hat{a}_1(\vec{p}) - i\,\hat{a}_2(\vec{p})\right]$$

$$\hat{b}^\dagger(\vec{p}) = 1/\sqrt{2}\left[\hat{a}_1^\dagger(\vec{p}) - i\,\hat{a}_2^\dagger(\vec{p})\right] \qquad \hat{c}^\dagger(\vec{p}) = 1/\sqrt{2}\left[\hat{a}_1^\dagger(\vec{p}) + i\,\hat{a}_2^\dagger(\vec{p})\right]$$

The factor $1/\sqrt{2}$ ensures that

$[\hat{b}(\vec{p}), \hat{b}^\dagger(\vec{p}')] = (2\pi)^3 \delta^3(\vec{p} - \vec{p}')$ and $[\hat{c}(\vec{p}), \hat{c}^\dagger(\vec{p}')] = (2\pi)^3 \delta^3(\vec{p} - \vec{p}')$

creation and annihilation operators with different momentum labels commute trivially.

Now $[\hat{b}(\vec{p}), \hat{c}^\dagger(\vec{p}\,')]$

$$= \frac{1}{2}\left[\left(\hat{a}_1(\vec{p}) + i\,\hat{a}_2(\vec{p})\right), \left(\hat{a}_1^\dagger(\vec{p}) + i\,\hat{a}_2^\dagger(\vec{p}\,')\right)\right]$$

$$= \frac{1}{2}\left[\hat{a}_1(\vec{p}), \hat{a}_1^\dagger(\vec{p}\,')\right] - \frac{1}{2}\left[\hat{a}_2(\vec{p}), \hat{a}_2^\dagger(\vec{p}\,')\right]$$

$$= \frac{1}{2}\,\delta^3(\vec{p}-\vec{p}\,') - \frac{1}{2}\,\delta^3(\vec{p}-\vec{p}\,') = 0$$

Similarly we can show $[\hat{b}^\dagger(\vec{p}), \hat{c}(\vec{p}\,')] = 0$

We also have $\hat{b}(\vec{p})|\Omega\rangle = 0$ and $\hat{c}(\vec{p})|\Omega\rangle = 0$

We now have two kinds of particles, created by $\hat{b}^\dagger$ and $\hat{c}^\dagger$.

Now let's work out the number operator in the new basis

$$\hat{N}(\vec{p}) = \hat{a}_1^\dagger(\vec{p})\,\hat{a}_1(\vec{p}) + \hat{a}_2^\dagger(\vec{p})\,\hat{a}_2(\vec{p})$$

$$= \tfrac{1}{2}\left[\hat{b}^\dagger(\vec{p}) + \hat{c}^\dagger(\vec{p})\right]\left[\hat{b}(\vec{p}) + \hat{c}(\vec{p})\right] + \tfrac{1}{2}\left[\hat{b}^\dagger(\vec{p}) - \hat{c}^\dagger(\vec{p})\right]\left[\hat{b}(\vec{p}) - \hat{c}(\vec{p})\right]$$

$$= \hat{b}^\dagger(\vec{p})\,\hat{b}(\vec{p}) + \hat{c}^\dagger(\vec{p})\,\hat{c}(\vec{p})$$

So the Hamiltonian is $\hat{H} = \int \frac{d^3\vec{p}}{(2\pi)^3}\, E_{\vec{p}} \left(\hat{b}^\dagger(\vec{p})\hat{b}(\vec{p}) + \hat{c}^\dagger(\vec{p})\,\hat{c}(\vec{p})\right)$

and the momentum is $\hat{\vec{P}} = \int \frac{d^3\vec{p}}{(2\pi)^3}\, \vec{p} \left(\hat{b}^\dagger(\vec{p})\hat{b}(\vec{p}) + \hat{c}^\dagger(\vec{p})\,\hat{c}(\vec{p})\right)$

The charge operator is $\hat{Q} = \int \frac{d^3\vec{p}}{(2\pi)^3}\left[\hat{b}^\dagger(\vec{p})\,\hat{b}(\vec{p}) - \hat{c}^\dagger(\vec{p})\,\hat{c}(\vec{p})\right]$

$$= N_b - N_c$$

We also have the algebra of Q's:

$$[\hat{Q}, \hat{b}(\vec{p})] = -\hat{b}(\vec{p}), \quad [\hat{Q}, \hat{b}^\dagger(\vec{p})] = \hat{b}^\dagger(\vec{p})$$

$$[\hat{Q}, \hat{c}(\vec{p})] = \hat{c}(\vec{p}), \quad [\hat{Q}, \hat{c}^\dagger(\vec{p})] = -\hat{c}^\dagger(\vec{p})$$

So we have two kind of particles with charge +1 and −1.