Advanced Optimization Techniques

Linear Programming Problem

- **Linear programming** (LP, also called **linear optimization**) is a method to achieve the best outcome (such as maximum profit or lowest cost) in a mathematical model whose **requirements are represented by linear**

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$$

$$\text{which minimizes } f(\mathbf{X}) = \sum_{i=1}^{n} c_i x_i$$

subject to the constraints

$$\sum_{i=1}^{n} a_{ij} x_i = b_j, \quad j = 1, 2, \ldots, m$$

$$x_i \geq 0, \quad i = 1, 2, \ldots, n$$

where c_i, a_{ij}, and b_j are constants.

Linear Programming Problem

- Suppose that a farmer has **a piece of farm land**, say **L km square**, to be planted with **either wheat or barley** or some combination of the two. The farmer has a limited amount of **fertilizer, F kilograms**, and **pesticide, P kilograms**. Every square kilometer of **wheat requires F1 kilograms of fertilizer** and **P1 kilograms of pesticide**, while every square kilometer of **barley requires F2 kilograms of fertilizer** and **P2 kilograms of pesticide**. Let **S1** be the selling price of **wheat per square kilometer**, and **S2** be the selling price of **barley**. If we denote the area of land planted with wheat and barley by **x1 and x2** respectively, then **profit can be maximized** by choosing optimal values for **x1 and x2**.

Linear Programming Problem

Maximize: $S_1 \cdot x_1 + S_2 \cdot x_2$ (maximize the revenue—revenue is the "objective function")

Subject to: $x_1 + x_2 \leq L$ (limit on total area)

$F_1 \cdot x_1 + F_2 \cdot x_2 \leq F$ (limit on fertilizer)

$P_1 \cdot x_1 + P_2 \cdot x_2 \leq P$ (limit on pesticide)

$x_1 \geq 0, x_2 \geq 0$ (cannot plant a negative area).

Nonlinear Programming Problem

- If **any of the functions** among the objective and constraint functions is **nonlinear**, the problem is called a nonlinear programming (NLP) problem.
- This is the **most general programming problem** and all other problems can be considered as **special cases** of the NLP problem.

Nonlinear Programming Problem

- **Nonlinear System:** **C**hange of the output is **not proportional** to the change of the input.
- The behavior of a nonlinear system is described in mathematics by a **nonlinear system of equations.**
- **Nonlinear system of equations** : unknowns appear as variables of a **polynomial of degree higher than one**.
- In a nonlinear system of equations, the equation(s) to be solved cannot be written as a **linear combination** of the unknown variables or functions that appear in them.

Nonlinear Programming Problem

- A simple problem can be defined by the constraints

$$x_1 \geq 0$$

$$x_2 \geq 0$$

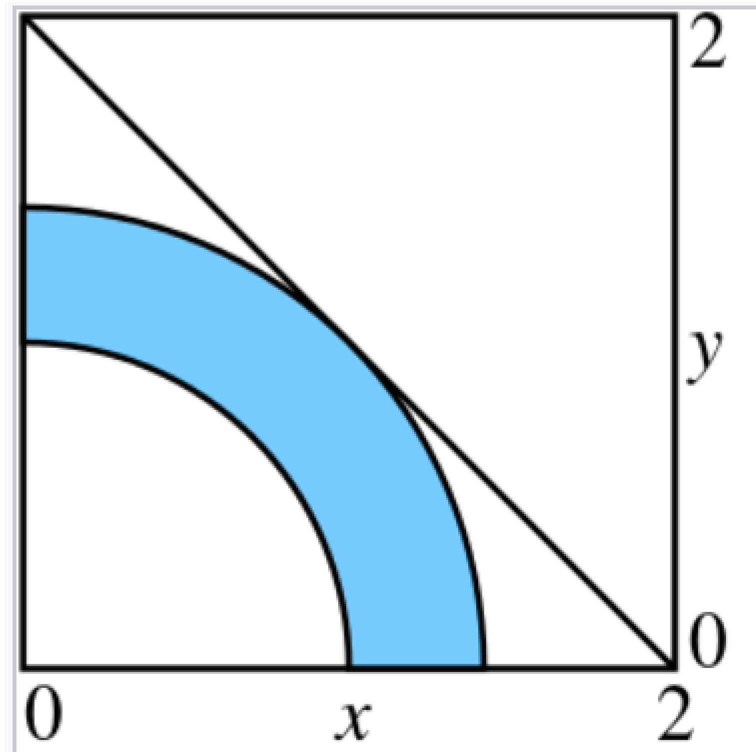
$$x_1^2 + x_2^2 \geq 1$$

$$x_1^2 + x_2^2 \leq 2$$

with an objective function to be maximized

$$f(\mathbf{x}) = x_1 + x_2$$

where $\mathbf{x} = (x_1, x_2)$.



Nonlinear Programming Problem

- Another simple problem can be defined by the constraints

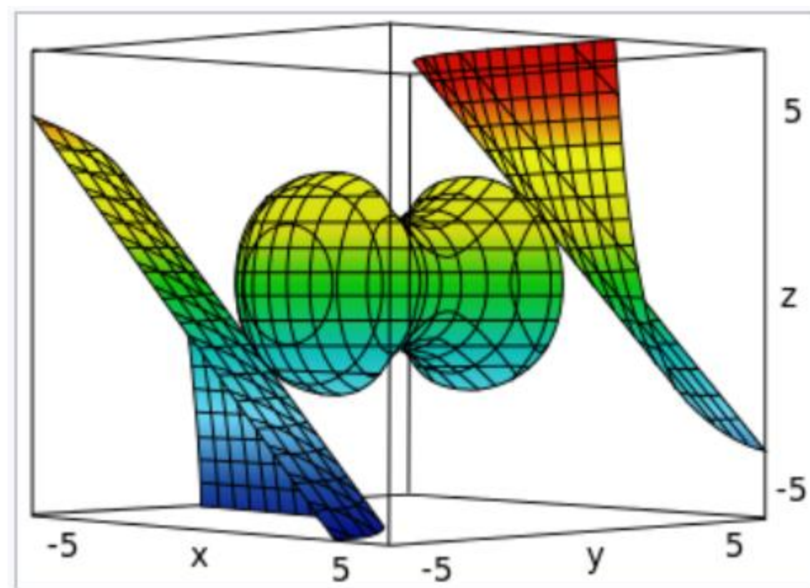
constraints

$$x_1^2 - x_2^2 + x_3^2 \leq 2$$
$$x_1^2 + x_2^2 + x_3^2 \leq 10$$

with an objective function to be maximized

$$f(\mathbf{x}) = x_1x_2 + x_2x_3$$

where $\mathbf{x} = (x_1, x_2, x_3)$.



Classical Optimization Techniques

Nonlinear Programming Problem

- The classical methods of optimization are useful in finding the optimum solution of **continuous and differentiable functions**.
- Since some of the practical problems involve **objective functions that are not continuous and/or differentiable**, the classical optimization techniques have **limited scope in practical applications**.

SINGLE-VARIABLE OPTIMIZATION

A function of one variable $f(x)$ is said to have a *relative* or *local minimum* at $x = x^*$ if $f(x^*) \leq f(x^* + h)$ for all sufficiently small positive and negative values of h. Similarly, a point x^* is called a *relative* or *local maximum* if $f(x^*) \geq f(x^* + h)$ for all values of h sufficiently close to zero. A function $f(x)$ is said to have a *global* or *absolute minimum* at x^* if $f(x^*) \leq f(x)$ for all x, and not just for all x close to x^*, in the domain over which $f(x)$ is defined. Similarly, a point x^* will be a global maximum of $f(x)$ if $f(x^*) \geq f(x)$ for all x in the domain. Figure 2.1 shows the difference between the local and global optimum points.

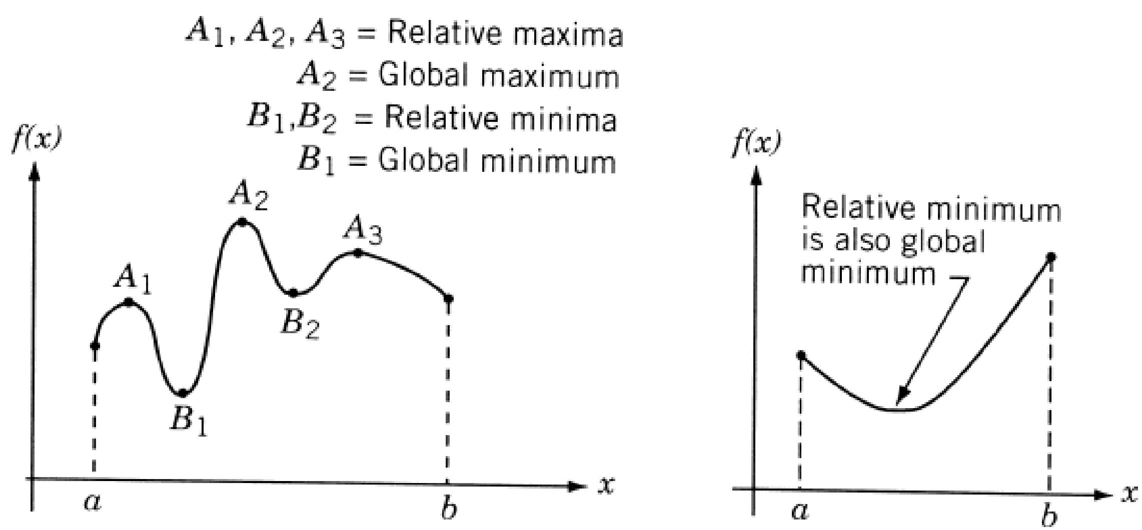


Figure 2.1 Relative and global minima.

SINGLE-VARIABLE OPTIMIZATION

A *single-variable optimization problem* is one in which the value of $x = x^*$ is to be found in the interval $[a, b]$ such that x^* minimizes $f(x)$.

Theorem 2.1 Necessary Condition If a function $f(x)$ is defined in the interval $a \leq x \leq b$ and has a relative minimum at $x = x^*$, where $a < x^* < b$, and if the derivative $df(x)/dx = f'(x)$ exists as a finite number at $x = x^*$, then $f'(x^*) = 0$.

Proof: It is given that

$$f'(x^*) = \lim_{h \to 0} \frac{f(x^* + h) - f(x^*)}{h}$$

exists as a definite number, which we want to prove to be zero. Since x^* is a relative minimum, we have

$$f(x^*) \leq f(x^* + h)$$

SINGLE-VARIABLE OPTIMIZATION

Theorem 2.1 Necessary Condition If a function $f(x)$ is defined in the interval $a \leq x \leq b$ and has a relative minimum at $x = x^*$, where $a < x^* < b$, and if the derivative $df(x)/dx = f'(x)$ exists as a finite number at $x = x^*$, then $f'(x^*) = 0$.

for all values of h sufficiently close to zero. Hence

$$\frac{f(x^* + h) - f(x^*)}{h} \geq 0 \quad \text{if } h > 0$$

$$\frac{f(x^* + h) - f(x^*)}{h} \leq 0 \quad \text{if } h < 0$$

Thus Eq. (2.1) gives the limit as h tends to zero through positive values as

$$f'(x^*) \geq 0$$

while it gives the limit as h tends to zero through negative values as

$$f'(x^*) \leq 0$$

The only way to satisfy both Eqs. (2.2) and (2.3) is to have

$$f'(x^*) = 0$$

This proves the theorem.

SINGLE-VARIABLE OPTIMIZATION

Theorem 2.1 Necessary Condition If a function $f(x)$ is defined in the interval $a \leq x \leq b$ and has a relative minimum at $x = x^*$, where $a < x^* < b$, and if the derivative $df(x)/dx = f'(x)$ exists as a finite number at $x = x^*$, then $f'(x^*) = 0$.

This theorem can be proved even if x^* is a relative maximum.

The theorem does not say what happens if a minimum or maximum occurs at a point x^* where the derivative fails to exist. For example, in Fig. 2.2,

$$\lim_{h \to 0} \frac{f(x^* + h) - f(x^*)}{h} = m^+ \text{(positive) or } m^- \text{(negative)}$$

depending on whether h approaches zero through positive or negative values, respectively. Unless the numbers m^+ and m^- are equal, the derivative $f'(x^*)$ does not exist. If $f'(x^*)$ does not exist, the theorem is not applicable.

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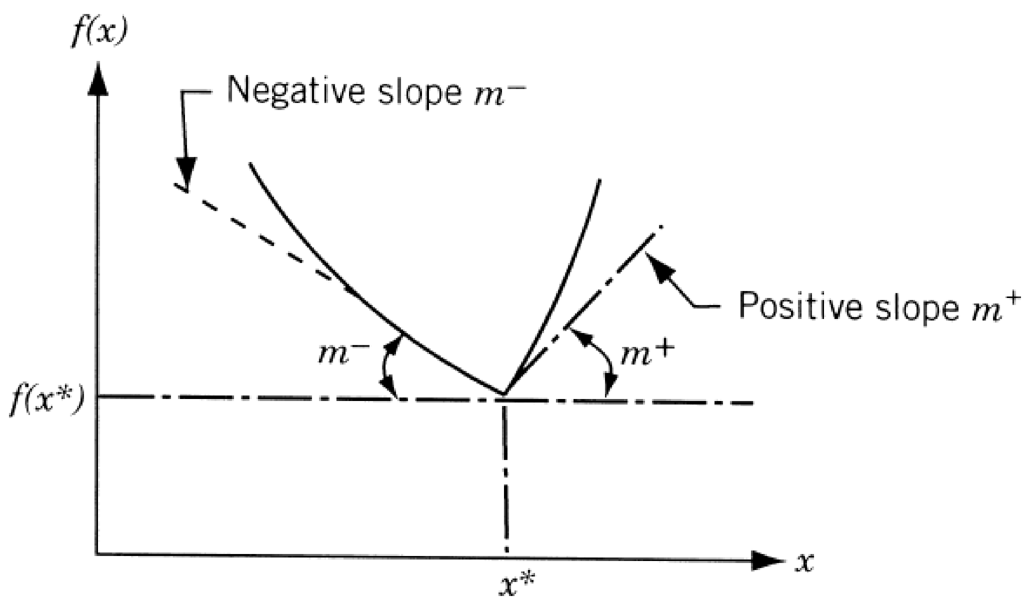


Figure 2.2 Derivative undefined at x^*.

SINGLE-VARIABLE OPTIMIZATION

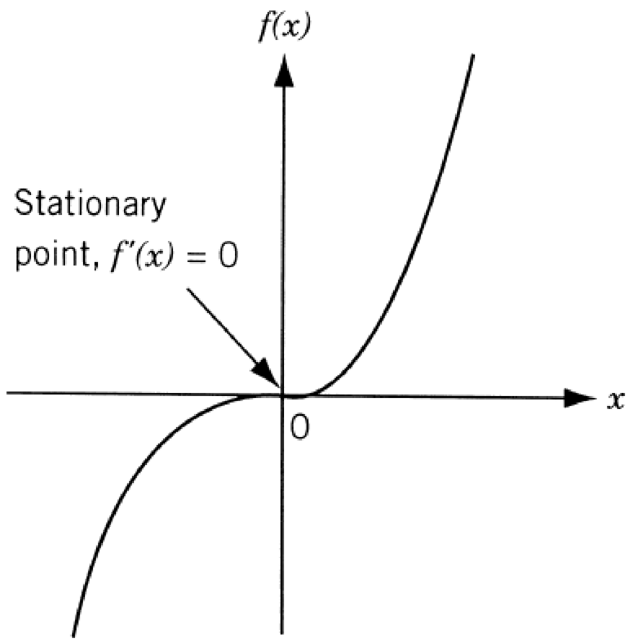
The theorem does not say what happens if a minimum or maximum occurs at an endpoint of the interval of definition of the function. In this case

$$\lim_{h \to 0} \frac{f(x^* + h) - f(x^*)}{h}$$

exists for positive values of h only or for negative values of h only, and hence the derivative is not defined at the endpoints.

SINGLE-VARIABLE OPTIMIZATION

The theorem does not say that the function necessarily will have a minimum or maximum at every point where the derivative is zero. For example, the derivative $f'(x) = 0$ at $x = 0$ for the function shown in Fig. 2.3. However, this point is neither a minimum nor a maximum. In general, a point x^* at which $f'(x^*) = 0$ is called a *stationary point*.



Stationary (inflection) point.

SINGLE-VARIABLE OPTIMIZATION

Theorem 2.2 Sufficient Condition Let $f'(x^*) = f''(x^*) = \cdots = f^{(n-1)}(x^*) = 0$, but $f^{(n)}(x^*) \neq 0$. Then $f(x^*)$ is (i) a minimum value of $f(x)$ if $f^{(n)}(x^*) > 0$ and n is even; (ii) a maximum value of $f(x)$ if $f^{(n)}(x^*) < 0$ and n is even; (iii) neither a maximum nor a minimum if n is odd.

Taylor's Theorem. If f is a function continuous and n times differentiable in an interval $[x, x+h]$, then there exists some point in this interval, denoted by $x + \lambda h$ for some $\lambda \in [0, 1]$, such that

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \cdots$$
$$\cdots + \frac{h^{(n-1)}}{(n-1)!} f^{(n-1)}(x) + \frac{h^n}{n!} f^n(x + \lambda h).$$

SINGLE-VARIABLE OPTIMIZATION

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Proof: Applying Taylor's theorem with remainder after n terms, we have

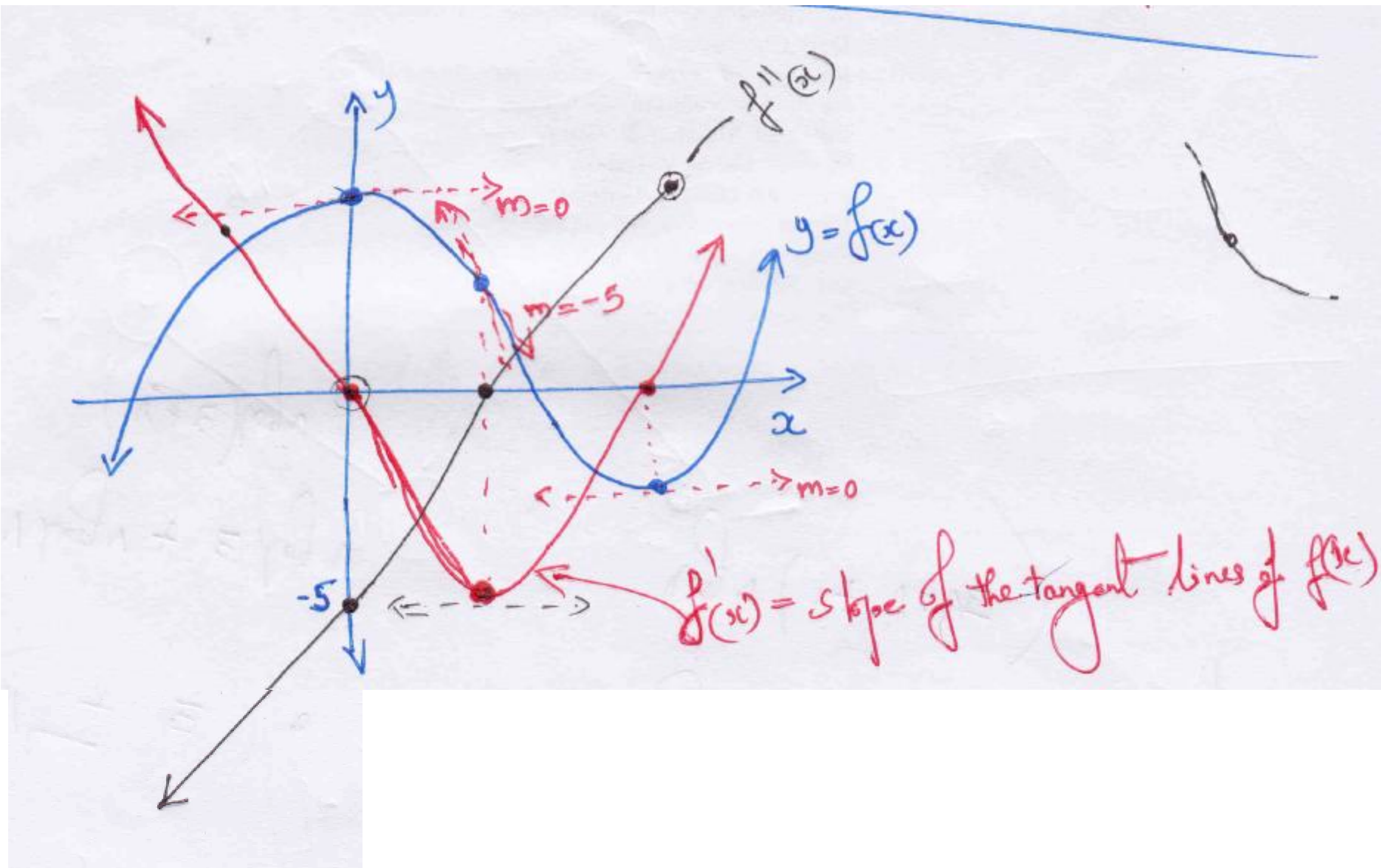
$$f(x^* + h) = f(x^*) + hf'(x^*) + \frac{h^2}{2!} f''(x^*) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x^*)$$
$$+ \frac{h^n}{n!} f^{(n)}(x^* + \theta h) \qquad \text{for} \quad 0 < \theta < 1 \tag{2.5}$$

Since $f'(x^*) = f''(x^*) = \cdots = f^{(n-1)}(x^*) = 0$, Eq. (2.5) becomes

$$f(x^* + h) - f(x^*) = \frac{h^n}{n!} f^{(n)}(x^* + \theta h)$$

As $f^{(n)}(x^*) \neq 0$, there exists an interval around x^* for every point x of which the nth derivative $f^{(n)}(x)$ has the same sign, namely, that of $f^{(n)}(x^*)$. Thus for every point $x^* + h$ of this interval, $f^{(n)}(x^* + \theta h)$ has the sign of $f^{(n)}(x^*)$. When n is even, $h^n/n!$ is positive irrespective of whether h is positive or negative, and hence $f(x^* + h) - f(x^*)$ will have the same sign as that of $f^{(n)}(x^*)$. Thus x^* will be a relative minimum if $f^{(n)}(x^*)$ is positive and a relative maximum if $f^{(n)}(x^*)$ is negative. When n is odd, $h^n/n!$ changes sign with the change in the sign of h and hence the point x^* is neither a maximum nor a minimum. In this case the point x^* is called a *point of inflection*.

SINGLE-VARIABLE OPTIMIZATION



SINGLE-VARIABLE OPTIMIZATION

Example 2.1 Determine the maximum and minimum values of the function

$$f(x) = 12x^5 - 45x^4 + 40x^3 + 5$$

SINGLE-VARIABLE OPTIMIZATION

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$$f(x) = 12x^5 - 45x^4 + 40x^3 + 5$$

SOLUTION Since $f'(x) = 60(x^4 - 3x^3 + 2x^2) = 60x^2(x - 1)(x - 2)$, $f'(x) = 0$ at $x = 0$, $x = 1$, and $x = 2$. The second derivative is

$$f''(x) = 60(4x^3 - 9x^2 + 4x)$$

At $x = 1$, $f''(x) = -60$ and hence $x = 1$ is a relative maximum. Therefore,

$$f_{\max} = f(x = 1) = 12$$

At $x = 2$, $f''(x) = 240$ and hence $x = 2$ is a relative minimum. Therefore,

$$f_{\min} = f(x = 2) = -11$$

At $x = 0$, $f''(x) = 0$ and hence we must investigate the next derivative:

$$f'''(x) = 60(12x^2 - 18x + 4) = 240 \quad \text{at} \quad x = 0$$

Since $f'''(x) \neq 0$ at $x = 0$, $x = 0$ is neither a maximum nor a minimum, and it is an inflection point.

SINGLE-VARIABLE OPTIMIZATION

Assume the following relationship for revenue and cost functions.
Find out at what level of output x, where x is measured in tons per week, profit is maximum.

$$R(x) = 1000x - 2x^2$$

And

$$C(x) = x^3 - 59x^2 + 1315x + 5000$$

SINGLE-VARIABLE OPTIMIZATION

Assume the following relationship for revenue and cost functions. Find out at what level of output x, where x is measured in tons per week, profit is maximum.

Solution: The profit function is

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 1000x - 2x^2 - x^3 + 59x^2 - 1315x - 5000 \\ &= -x^3 + 57x^2 - 315x - 5000 \end{aligned}$$

Differentiating both sides of (1) with respect to x, we get

$$\frac{dP}{dx} = -3x^2 + 114x - 315$$

For maxima and minima, we have

$$\frac{dP}{dx} = 0$$

SINGLE-VARIABLE OPTIMIZATION

Assume the following relationship for revenue and cost functions. Find out at what level of output x, where x is measured in tons per week, profit is maximum.

$$\Rightarrow \qquad -3x^2 + 114x - 315 = 0$$

$$\Rightarrow \qquad x = 3, 35.$$

Differentiating both sides of (2) again with respect to x, we get

$$\frac{d^2P}{dx^2} = -6x + 114$$

At $x = 3$, $\frac{d^2P}{dx^2} = 96 > 0$, i.e., P is minimum at $x = 3$.

At $x = 35$, $\frac{d^2P}{dx^2} = 96 < 0$, i.e., P is maximum at $x = 35$.

Hence, the profit is maximum at $x = 35$ tons per week.