Part-2: Eigen Values and Eigen Vectors of M. ①

$$M = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix}$$

Eigen Values:

$$AV = \lambda V,$$

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

$$\det \begin{pmatrix} 1-\lambda & -4 & 2 \\ -4 & 1-\lambda & -2 \\ 2 & -2 & -2-\lambda \end{pmatrix}$$

$$\Rightarrow 1-\lambda \begin{pmatrix} 1-\lambda & -2 \\ -2 & -2-\lambda \end{pmatrix} - (-4) \begin{pmatrix} -4 & -2 \\ 2 & -2-\lambda \end{pmatrix} + 2 \begin{pmatrix} -4 & 1-\lambda \\ 2 & -2 \end{pmatrix}$$

$$\Rightarrow 1-\lambda (\lambda^2 + \lambda - 6) - (-4)(4\lambda + 12) + 2(2\lambda + 6)$$

$$\Rightarrow -\lambda^3 + 27\lambda + 54$$

$$\Rightarrow -\lambda^3 + 27\lambda + 54 = 0$$

$$-(\lambda+3)^2(\lambda-6)$$

$$\boxed{\lambda = -3, +6}$$

Eigen vectors:-

$$(A-\lambda I) = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix} - (-3)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -4 & 2 \\ 0 & 0 & 0 \\ 2 & -2 & 1 \end{pmatrix} R_2 = R_2 + R_1$$

$$= \begin{pmatrix} 4 & -4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R_3 = R_3 - \frac{1}{2} R_1$$

$$= \begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R_1 = \frac{1}{4} \cdot R_1$$

$\lambda = -3,$

$$(A+3I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x - y + \frac{1}{2}z = 0$$

$$v = \begin{pmatrix} y - \frac{1}{2}z \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}z \\ 0 \\ z \end{pmatrix} \quad y, z \neq 0$$

$y = 1,\ z = 1$

$$\boxed{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}}$$

$\lambda = 6$

$$\begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix} - 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & -4 & 2 \\ 0 & -9/5 & -18/5 \\ 2 & -2 & -8 \end{pmatrix} \quad R_2 = R_2 - \frac{4}{5} \cdot R_1$$

$$= \begin{pmatrix} -5 & -4 & 2 \\ 0 & -18/5 & -36/5 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} R_3 = R_3 + \frac{2}{5} \cdot R_1 \\ R_3 = R_3 - \frac{1}{2} \cdot R_2 \end{array}$$

$$= \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} R_2 = \frac{-5}{18} \cdot R_2 \\ R_1 = R_1 + 4 \cdot R_2 \\ R_1 = \frac{-1}{5} \cdot R_1 \end{array}$$

$$(A-6I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$x - 2z = 0$$

$$y + 2z = 0$$

$$x = 2z, \; y = -2z.$$

$$v = \begin{pmatrix} 2z \\ -2z \\ z \end{pmatrix} \quad z \neq 0$$

if $z = 1$

$$\boxed{= \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.}$$

$\therefore$ Eigen vectors are

$= \{ 1, 1, 0 \} \; (-1/2, 0, 1), \; (-2, 2, 1)$

Part-3.

Find the gradient of the $\nabla_A f(A)$ for the following:

$$A = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

$$f(A) = x_{11}^2 x_{22} x_{23} + x_{11} x_{12} x_{13} x_{31} - x_{33}^2 x_{32} x_{21}$$

$$\frac{d}{dx_{12}} \left(x_{11}^2 x_{22} x_{23} + x_{11} x_{12} x_{13} x_{31} - x_{21} x_{32} x_{33}^2 \right)$$

$$= \left(\frac{d}{dx_{12}} (x_{11}^2 x_{22} x_{23}) - \frac{d}{dx_{12}} (x_{21} x_{32} x_{33}^2) + \frac{d}{dx_{12}} (x_{11} x_{12} x_{13} x_{31}) \right)$$

$$= \frac{d}{dx_{12}}(x_{11} x_{12} x_{13} x_{31}) + \frac{d}{dx_{12}}(x_{11}^2 x_{22} x_{23})$$

$$- \frac{d}{dx_{12}}(x_{21} x_{32} x_{33}^2)$$

$$= x_{11} x_{13} x_{31} \frac{d}{dx_{12}}(x_{12}) + \frac{d}{dx_{12}}(x_{11}^2 x_{22} x_{23})$$

$$- \frac{d}{dx_{12}}(x_{21} x_{32} x_{33}^2)$$

$$= x_{11} x_{13} x_{31} \frac{d}{dx_{12}}(x_{12}) + \frac{d}{dx_{12}}(x_{11}^2 x_{22} x_{23})$$

$$- \frac{d}{dx_{12}}(x_{21} x_{32} x_{33}^2)$$

Apply Powe rule

$$= x_{11} x_{13} x_{31} \frac{d}{dx_{12}}(x_{12}) + \frac{d}{dx_{12}}(x_{11}^2 x_{22} x_{23})$$

$$- \frac{d}{dx_{12}}(x_{21} x_{32} x_{33}^2)$$

$$= x_{11} x_{13} x_{31} \cdot 1 + \frac{d}{dx_{12}} (x_{11}^2 x_{22} x_{23})$$

$$- \frac{d}{dx_{12}} (x_{21} x_{32} x_{33}^2).$$

Constant derivative is: 0

$$= x_{11} x_{13} x_{31} - \frac{d}{dx_{12}} (x_{21} x_{32} x_{33}^2) + \frac{d}{dx_{12}} (x_{11}^2 x_{22} x_{23})$$

$$= x_{11} x_{13} x_{31} - 0 + \frac{d}{dx_{12}} (x_{11}^2 x_{22} x_{23})$$

$$= x_{11} x_{13} x_{31} + 0.$$

$$\boxed{\therefore \frac{d}{dx_{12}} (x_{11}^2 x_{22} x_{23} + x_{11} x_{12} x_{13} x_{31} - x_{21} x_{32} x_{33}^2) = x_{11} x_{13} x_{31}}$$

Part - 4: Find the hessian matrix for:

$$g(x, y, z) = x^2 y + yz \sin(x) + xy^2 z^5$$

$$\frac{\partial g}{\partial x} = 3yx^2 + yz \cos(x) + y^2 z^5$$

$$\frac{\partial^2 g}{\partial x^2} = 6yx - yz \sin(x)$$

$$\frac{\partial g}{\partial y} = x^3 + z \sin(x) + 2xyz^5$$

$$\frac{\partial g^2}{\partial y^2} = 2xz^5$$

$$\frac{\partial g}{\partial z} = 0 + y \sin(x) + 5xy^2 z^4$$

$$\frac{\partial g^2}{\partial z^2} = 20xy^2 z^3$$

$$H_g = \begin{bmatrix} 6yx - yz\sin(x) & 3x^2 + z\cos(x) + 2z^5 y & 5y^2z^4 + y\cos(x) \\ 3x^2 + z\cos(x) + 2yz^5 & 2xz^5 & 10yxz^4 + \sin(x) \\ 5y^2z^4 + y\cos(x) & 10xz^4y + \sin(x) & 20xy^2z^3 \end{bmatrix}$$