Problem - 2 :-

Find the Parameters? Return μ and σ.

μ - mean, σ - ~~Variance~~ Standard deviation

We will use moments generating function (M.G.F) to estimate μ and σ.

$M(t) = E[e^{tx}]$

$$= \int_{-\alpha}^{\alpha} e^{tx} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \int_{-\alpha}^{\alpha} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma^2}} (-2tx\sigma^2 + (x-\mu)^2)\, dx$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \text{ (or) } \exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}.$$

Taking 1^{st} derivative, and evaluate at $t=0$ gives mean μ.

Now, if $x \sim N(\mu_x, \sigma_x^2)$ and $y \sim N(\mu_y, \sigma_y^2)$ be 2 i.i.d normal Variables. Then, their Sum is also a normally distributed random Variables. $x+y = z \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$

The moment generating functions, of the Sum is the product of the moment generating functions of elements.

$M_z(t) = M_x(t)\, M_y(t)$

$= \exp\left\{ \mu_x t + \frac{1}{2} \sigma^2 t^2 \right\} \exp\left\{ \mu_y t + \frac{1}{2} \sigma^2_y t^2 \right\}$

$= \exp\left\{ (\mu_x + \mu_y) t + \frac{1}{2} (\sigma^2_x + \sigma^2_y) t^2 \right\}$

$\frac{d}{dt} (M_z(t))$

$= \frac{d}{dt} \left(\exp\left\{ (\mu_x + \mu_y) t + \frac{1}{2} (\sigma^2_x + \sigma^2_y) t^2 \right\} \right)$

$= e^{(\mu_x + \mu_y) t} (\mu_x + \mu_y) + e^{\frac{x^2 \sigma t^2 + y^2 \sigma t^2}{2}} \cdot t(x^2 \sigma + y^2 \sigma)$

at $t = 0$

$= e^0 (\mu_x + \mu_y) + e^0 \cdot 0$

$= \mu_x + \mu_y$

$\therefore$ Mean $= \mu_x + \mu_y$

Now, take 2^{nd} derivative

$$\frac{d^2}{dt^2}(M_z(t))$$

$$= e^{t(M_x + M_y)}(M_x + M_y)^2 +$$

$$(\sigma x^2 + \sigma y^2)\left(\sigma e^{\frac{x^2\sigma t^2 + \sigma y^2 t^2}{2}} \cdot t^2(x^2+y^2) + e^{\frac{\sigma x^2 t^2 + y^2\sigma t^2}{2}}\right)$$

At, $t=0$

$$= e^0 (M_x + M_y)^2 + (\sigma x^2 + \sigma y^2)(0 + e^0)$$

$$= (M_x + M_y)^2 + (\sigma x^2 + \sigma y^2)$$

Now, Variance $= M''(0) - (M'(0))^2$

$$= \cancel{(M_x + M_y)^2} + (\sigma x^2 + \sigma y^2) - \cancel{(M_x + M_y)^2}$$

$$= (\sigma x^2 + \sigma y^2)$$