Vellore Institute of Technology
(Deemed to be University under section 3 of UGC Act, 1956)

E1+TE1

SCHOOL OF ADVANCED SCIENCES
CONTINUOUS ASSESSMENT TEST – I
Answer Key
FALL SEMESTER 2022-23

Programme Name & Branch: M.Sc. Data Science
Course Code: MAT 5012
Course Name: Probability theory and distributions
Faculty Name(s): Dr. Venkataramana B
Class Number(s): VL2022230105743 **Exam Duration: 90 minutes** **Maximum Marks: 50**

General instruction(s): **Scientific calculators are permitted.** **Answer all the questions** (**5 × 10 = 50**)

Q1	a) A lot of 140 semiconductor chips is inspected by choosing a sample of five chips. Assume 10 of the chips do not conform to customer requirements. (5M) (i) How many samples of five contain **exactly** one nonconforming chip? (ii) How many samples of five contain **at least** one nonconforming chip? b) The police plans to enforce speed limits by using radar traps at 4 different locations within the city limits. The radar traps at each of the locations L_1, L_2, L_3 and L_4 are operated 40%, 30% , 20% and 30% of the time. If a person who is speeding on his way to work has probabilities of 0.2, 0.1, 0.5 and 0.2 respectively, of passing through these locations, what is the probability that he will receive a speeding ticket ? (5M)

(a) A sample of 5-chips are selected in $\binom{140}{5}$ ways $= 416,965,528$

(i) 10 - bad chips
Exactly 1-bad chip & 4 - Good chips
$$\binom{130}{4}\binom{10}{1} = 113,588,800$$

(ii) At least 1 - or more bad chips of 5
(4g & 1b) + (3g & 2b) + (2g & 3b) + (1g & 4b)
(0g & 5b) $= \binom{130}{4}\binom{10}{1} + \binom{130}{3}\binom{10}{2} + \binom{130}{2}\binom{10}{3}$
$+ \binom{130}{1}\binom{10}{4} + \binom{130}{0}\binom{10}{5}$
$= 130,721,752$

or
$\binom{140}{5} - \binom{130}{5}$

(b) $P(L_1) = 0.2,\ P(L_2) = 0.1,\ P(L_3) = 0.5;\ P(L_4) = 0.2$

$P(T|L_1) = 0.4,\ P(T|L_2) = 0.3,\ P(T|L_3) = 0.2,$

$P(T|L_4) = 0.3$

$$P(T) = P(L_1 \cap T) + P(L_2 \cap T) + P(L_3 \cap T) + P(L_4 \cap T)$$

$$= P(L_1)\cdot P(T|L_1) + P(L_2)\cdot P(T|L_2) + P(L_3)\cdot P(T|L_3) + P(L_4)\cdot P(T|L_4)$$

$$= (0.2 * 0.4) + (0.1 * 0.3) + (0.5 * 0.2) + (0.2 * 0.3)$$

$$P(T) = 0.08 + 0.03 + 0.10 + 0.06 = 0.27$$

Q2

The distribution function of the random variable $\boldsymbol{X}$ is given by

$$F(x) = \begin{cases} 0 & ; \quad x < 0 \\ \frac{x}{2} & ; \quad 0 \le x < 1 \\ \frac{2}{3} & ; \quad 1 \le x < 2 \\ \frac{11}{12} & ; \quad 2 \le x < 3 \\ 1 & ; \quad 3 \le x \end{cases}$$

Compute $(i)\ P\{X < 3\}$, $(ii)\ P\{X = 1\}$, $(iii)\ P\left\{X > \frac{1}{2}\right\}$, $(iv)\ P\{2 < X \le 4\}$ and (v) Draw the graph of $\boldsymbol{F(x)}$.

(i) $P(X<3) = F(3) = \underset{n}{Lt}\ P(X \le 3 - 1/n) = \underset{n}{Lt}\ F(3 - 1/n)$

$= 11/12 = 0.9167$

(ii) $P(X=1) = P(X \le 1) - P(X < 1)$

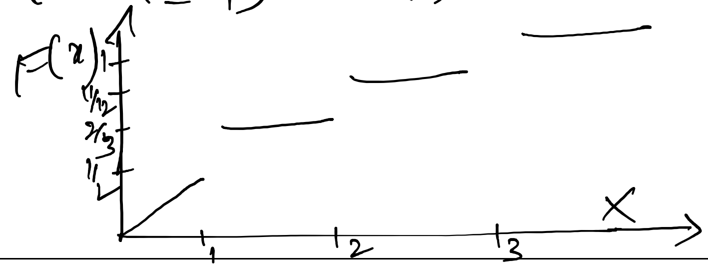
$= 2/3 - 1/2 = 1/6 = 0.167$

(iii) $P(X > 1/2) = 1 - P(X \le 1/2) = 1 - F(1/2) = 1 - 1/4 = 3/4$

$= 0.75$

(iv) $P(2 < X \le 4) = F(4) - F(2) = 1 - \frac{11}{12} = 1/12 = 0.083$

(v)



Q3 Let $\boldsymbol{X}$ be a random variable with p.d.f. :

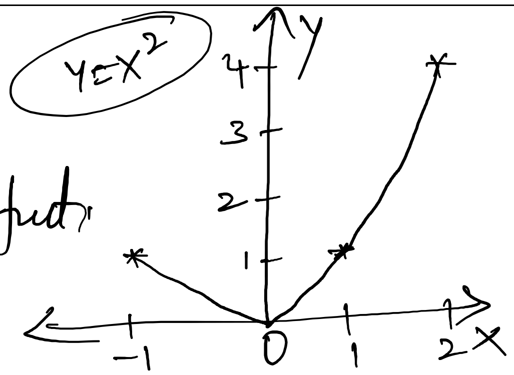
$$f(x)=\begin{cases}\frac{2}{9}\,(x+1)\ ;\ -1<x<2\\ \quad 0\ ,\ elsewhere\end{cases}$$

Find the pdf of $\boldsymbol{Z}=\boldsymbol{X}^{2}$ and hence compute mean of $\boldsymbol{Z}$.

Since $Z=x^2$ is not monotonic in $(-1,2)$

$\Rightarrow (-1,1)$ & $(1,2)$

Symmetrical Strictly increasing function



When $-1<x<1 \Rightarrow 0<y<1$

$\frac{dx}{dy}=\frac{d}{dy}(\sqrt{y})=\frac{1}{2\sqrt{y}}$

$[f(\sqrt{y})+f(-\sqrt{y})]$

$f(y)=f(x)\left|\frac{dx}{dy}\right|=\frac{\cancel{2}}{9}\left[(\cancel{\sqrt{y}}+1)+(1-\cancel{\sqrt{y}})\right]\frac{1}{\cancel{2}\sqrt{y}}$

$$\boxed{f(y)=\frac{2}{9\sqrt{y}}\ ;\ 0<y<1}$$

When $1<x<2 \Rightarrow 1<y<4$

$f(y)=f(x)\left|\frac{dx}{dy}\right|=\frac{\cancel{2}}{9}(\sqrt{y}+1)\cdot\frac{1}{\cancel{2}\sqrt{y}}$

$f(y)=\frac{1}{9}\left(1+\frac{1}{\sqrt{y}}\right)$

Q4 Consider the following probability distribution :

X \ Y	0	1	2	
0	0.1	0.2	0.1	$=0.4$
1	0.2	0.3	0.1	$=0.6$
	0.3	0.5	0.2	

(i) Calculate $E(X)$, $V(X)$ and $Cov(X,Y)$

(ii) Find $E(Y|X=0)$ and $E(Y|X=1)$.

(i) $E(X)=\Sigma x\,p(x)=0+0.6=0.6$; $E(Y)=0.5+(2*0.2)$
$=0.9$

$E(X^2)=\Sigma x^2 p(x)=0+0.6=0.6$

$V(X)=0.6-0.36=0.24$

$E(XY)=\Sigma xy\,p(x,y)=0.3+(2*0.1)=0.5$

$\Rightarrow Cov(X,Y)=0.5-\{(0.6)*(0.9)\}=-0.04$

	$E(Y\|X=0) = \sum y\, P(y\|X=0)$ $= 0 + (1)(0.2/0.4) + 2*(0.1/0.4) = 1$ $E(Y\|X=1) = \sum y\, P(Y\|X=1)$ $= 0 + (1)(0.3/0.6) + 2*(0.1/0.6) = 5/6 = 0.83$
Q5	If ***X*** is a random variable with $E(X) = 17$ and $E(X^2) = 298$, determine, **(a)** A lower bound for $P(10 < X < 24)$. **(b)** An upper bound for $P(\|X - 17\| \geq 16)$.
	$E(X)=17$, $E(X^2)=298$, $V(X)=9$ (a) $P(10<X<24) = P(\|X-17\|<7) \geq 1-\frac{\sigma^2}{k^2} = 1-\frac{9}{49} = \frac{40}{49}$ $P(10<X<24) \geq \frac{40}{49} = 0.8163$ (b) $P(\|X-17\| \geq 16) \leq \frac{\sigma^2}{k^2} = \frac{9}{256} = 0.0351$