Outdated

Linear Algebra 4 Credits

Siddhartha Das
Indranil Chakrabarty

- Solving systems of linear equations.
  Row reduction, free variables, row reduced echeleon matrices.
- Vector spaces basics : Def^ns, subspaces, bases, dimension.
- Linear transformations : Def^ns, Effect of changes of basis on transformation, Rank of transformation.
  Range & Kernel of transformation. Rank-Nullity theorem.
- Determinants : Cofactor expansions. Multilinearity.
  Axiomatic approach. Physical meaning of determinants.
- Eigenvalues and Eigenvectors.
- Diagonalizability & Triangularizability
- Advanced spectral Theory.

Evaluation (Part I : 12-13 lectures)

50%.

1. Mid-sem (90 mins exam) : ~~15~~ ~~20~~% = 20%*
2. Quiz I (45 mins exam) : 10%
3. Assignments : 15%* ~~+ 5% = 20%~~*
4. In class light quizes : 5%*

* subject to change (possible).

Textbooks & References :

1. Linear Algebra by Hoffmann & Kunz
2. Algebra by Artin
3. Linear Algebra by Kumaresan
4. Introduction to Linear Algebra by Strang
5. https://textbooks.math.gatech.edu/ila
6. Linear Algebra by Jänich

Lecture 1: Linear Equations

Fields

Let us first list out properties of addition & multiplication.
Consider that F denotes the set of real nos. or the set of complex nos.

1. Addition is commutative, $x + y = y + x$, $\forall\ x, y \in F$.
2. Addition is associative, $x + (y + z) = (x + y) + z$, $\forall\ x, y, z \in F$.
3. $\exists$ unique element 0 (zero) in F s.t. $x + 0 = x$, $\forall\ x \in F$.
4. $\forall\ x \in F$, $\exists$ a unique element $(-x)$ in F s.t. $x + (-x) = 0$.
5. Multiplication is commutative, $x \cdot y = y \cdot x$, $\forall\ x, y \in F$.
6. Multiplication is associative, $x(yz) = (xy)z$, $\forall\ x, y, z \in F$.
7. There is a unique non-zero element 1 (one) in F s.t. $x \cdot 1 = x$ $\forall\ x \in F$.
8. To each non-zero x in F there corresponds a unique element x^{-1} (or $1/x$) in F s.t. $xx^{-1} = 1$.
9. Multiplication distributes over addition; i.e.,
$$x(y + z) = xy + xz, \forall\ x, y, z \in F.$$

A set F of objects $x, y, z, \ldots$ ~~and~~ along with two operations, addition and multiplication, satisfying conditions ①–⑨ above, is called a <u>Field</u>. $(F, +, \cdot)$ algebra

Elements of field $\rightarrow$ scalars or numbers

~~A subfield $(S, +, \cdot)$ is a subset of a field $F(+, \cdot)$ s.t. associated operations do not take elements of the subset~~

A subfield $(S, +, \cdot)$ is a subset of a field $(F, +, \cdot)$ in the sense that $S \subseteq F$ and $(S, +, \cdot)$ is also a field.

A subfield of the field $(\mathbb{C}, +, \cdot)$ is a set F of complex nos. which itself is a field under usual operations — add. & multip. That is, $0, 1 \in F$, and that if $x, y \in F$, so are $(x+y)$, $-x$, xy, and x^{-1} (if $x \neq 0$). For example, field $(\mathbb{R}, +, \cdot)$.

$\mathbb{Z}^+ \rightarrow$ not a ~~field~~ subfield of $(\mathbb{C}, +, \cdot)$
$\mathbb{Z} \rightarrow$ not a subfield of $(\mathbb{C}, +, \cdot)$ } $n \in \mathbb{Z}^+$ but $\frac{1}{n} \notin \mathbb{Z}$.

$\mathbb{Q}$ (set of rational nos.) $\longrightarrow$ subfield of $(\mathbb{C}, +, \cdot)$

Problems:

① Any subfield of $(\mathbb{C}, +, \cdot)$ must contain every rational no.

② Set of all complex no. of the form $x + y\sqrt{2}$, where $x, y \in \mathbb{Q}$, is a subfield of $\mathbb{C}$!

Characteristic of field:

The least n such that the sum of n 1's is 0 is called the characteristic of the field F. If it doesn't happen in F, then F is called a field of characteristic zero.

Systems of Linear Equations

F is a field. We consider the problem of finding n scalars (elements of F) $x_1, x_2, \ldots, x_n$ which satisfy the conditions:

$$\left.\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= y_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= y_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ A_{n1}x_1 + A_{n2}x_2 + \cdots + A_{n2}x_n &= y_n \end{aligned}\right\} \text{------} (1.1)$$

where $y_1, \ldots, y_n$ and A_{ij}, $1 \le i \le m$, $1 \le j \le n$, are given elements of F. (1.1) is called a system of m linear equations in n unknowns.

Any n-tuple $(x_1, \ldots, x_n)$ of elements of F which satisfies each of the eq^{n}s (1.1) is called a solution of the system.
If $y_i = 0$, $1 \le i \le m$, we say that the system is homogeneous or that each of the eq's is homogeneous.

- Finding the solutions of a system of linear eq^{n}s.

Technique of elimination.

Example: Consider homogenous system:

$$2x_1 - x_2 + x_3 = 0 \text{ —— } (a1)$$

$$x_1 + 3x_2 + 4x_3 = 0 \text{ —— } (a2)$$

$$-2 \times (a2) + (a1) \Rightarrow \quad -7x_2 - 7x_3 = 0 \text{ or, } x_2 = -x_3.$$

$$3 \times (a1) + (a2) \Rightarrow \quad x_1 = -x_3.$$

If (x_1, x_2, x_3) is a solution then $x_1 = x_2 = -x_3$.
Or, any such tripple i.e., $(x, x, -x)$ is a sol^n.
Thus, the set of sol^{n}s consists of all tripples $(x, x, -x)$.

For the general system (1·1), suppose we select m scalars $(c_1, \ldots, c_m)$, multiply the j^{th} eq^n by c_j and then add.. We have

$$(c_1 A_{11} + \ldots + c_m A_{m1}) x_1 + \ldots + (c_1 A_{1n} + \ldots + c_m A_{mn}) x_n$$
$$= c_1 y_1 + \ldots + c_m y_m. \qquad (1.1a)$$

we call it a <u>linear combination of the eq^{n}s in (1.1)</u>

Any sol^n of the entire system of eq^{n}s of (1.1) will also be a sol^n of (1.1a).

Fundamental idea of the elimination process.

Consider another system of linear eq^{n}s:

$$\left.\begin{array}{l} B_{11} x_1 + \ldots + B_{1n} x_n = z_1 \\ \vdots \qquad\qquad \vdots \\ B_{k1} x_1 + \ldots + B_{kn} x_n = z_k \end{array}\right\} \quad (1.2)$$

where each of the k eq^{n}s is a linear combination of the eq^{n}s in (1.1) then every sol^n of (1.1) is also a sol^n of (1.2). However, it may happen that some sol^{n}s of (1.2) are not sol^{n}s of (1.1).

Two systems of linear eq^{n}s are <u>equivalent</u> if each eq^n in each system is a <u>linear combination</u> of the eq^{n}s in ~~the~~ other system.

<u>Thm</u>: Equivalent systems of linear eq^{n}s have exactly the same solutions.