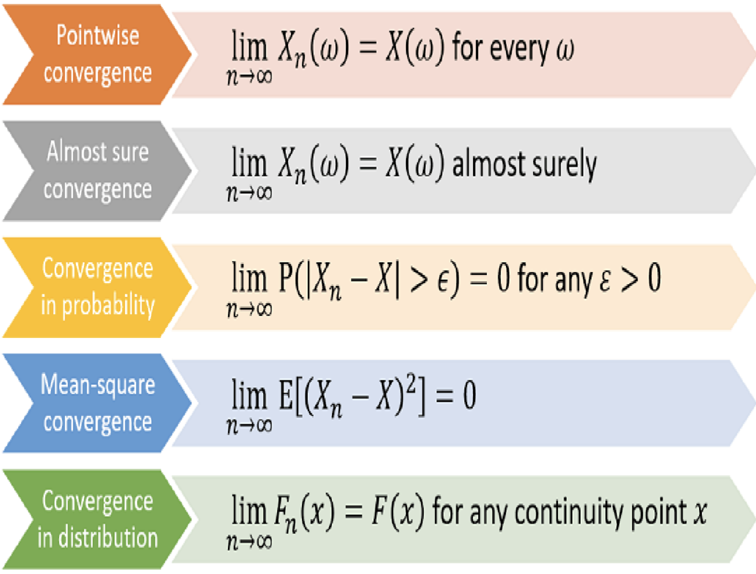
Probability and Statistics MA6.101

Tutorial 8

Topics Covered: Convergence, Central Limit Theorem

Summary



https://en.wikipedia.org/wiki/Convergence_of_random_variables

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Q1: Determine whether the following series' converges or diverges

(a) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(b) $\sum_{n=1}^{\infty} \left(\frac{3n+1}{4n}\right)^n$

Q2: Let U be a random variable having a uniform distribution on the interval $[0, 1]$. Now, define a sequence of random variables $\{X_n\}$ as follows:

$$
\begin{aligned}
&X_1 = \mathbf{1}_{\{U\in[0,1]\}},\\
&X_2 = \mathbf{1}_{\{U\in[0,1/2)\}}, \quad X_3 = \mathbf{1}_{\{U\in[1/2,1]\}},\\
&X_4 = \mathbf{1}_{\{U\in[0,1/4)\}}, \quad X_5 = \mathbf{1}_{\{U\in[1/4,2/4)\}}, \quad X_6 = \mathbf{1}_{\{U\in[2/4,3/4)\}}, \quad X_7 = \mathbf{1}_{\{U\in[3/4,1]\}},\\
&X_8 = \mathbf{1}_{\{U\in[0,1/8)\}}, \quad X_9 = \mathbf{1}_{\{U\in[1/8,2/8)\}}, \quad X_{10} = \mathbf{1}_{\{U\in[2/8,3/8)\}}, \quad \ldots\\
&X_{16} = \mathbf{1}_{\{U\in[0,1/16)\}}, \quad X_{17} = \mathbf{1}_{\{U\in[1/16,2/16)\}}, \quad X_{18} = \mathbf{1}_{\{U\in[2/16,3/16)\}}, \quad \ldots
\end{aligned}
$$

where $\mathbf{1}_{\{U\in[a,b)\}}$ is the indicator function of the event $\{U \in [a, b)\}$.

Find the probability limit (if it exists) of the sequence $\{X_n\}$.

Q3: If $X_n \xrightarrow{d} c$, where c is a constant, then show that $X_n \xrightarrow{p} c$.

Q4: Let $X_1, X_2, \cdots, X_n$ be i.i.d. with finite mean E(X) and variance Var(X). Then $S_n \to E(X)$ in m.s. Where $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. [Sohan]

Q5: It is possible for a sequence of discrete random variables to converge in distribution to a continuous one. For example, if Y_n is uniform on $\{1, \ldots, n\}$ and $X_n = Y_n/n$, then X_n converges in distribution to a random variable which is uniform on $[0, 1]$.

Q6: Assume that a test has a mean score of 75 and a standard deviation of 10. Assume the distribution of scores is approximately normal.

- What is the probability that a person chosen at random will make 100 or above on the test?
- In a group of 100 people, how many would you expect to score below 60?
- What is the probability that the mean of a group of 100 will score below 70?

Q7: Find

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i^2}{\sum_{i=1}^{n} X_i}.$$

where X_i are i.i.d. and $X_i \sim \mathrm{U}[0, 1]$.

Q8:

- Find the probability of getting more than 55 heads after tossing a fair coin 100 times
- Find the probability of getting more than 220 heads after tossing the same coin for 400 times.
- Are the probabilities of these events the same?. If not, why so?
- For a 100 flips of the same fair coin, find the probability of getting 40 to 60 heads.

Q9: A bank teller serves customers standing in the queue one by one. Suppose that the service time X_i for customer i has mean $\mathbb{E}[X_i] = 2$ minutes and $\mathrm{Var}(X_i) = 1$. Assume that service times for different customers are independent.

Let Y be the total time the bank teller spends serving 50 customers. Find $\mathbb{P}(90 < Y < 110)$.