Probability and Statistics

Tutorial 7

Q1: Let $(\Omega, \mathcal{F}, P)$ be a probability space on which the random variable U is defined, where $U \sim \text{Uniform}[0,1]$. Define a sequence of random variables $(X_n)_{n\geq 1}$ by

$$X_n(\omega) = \begin{cases} n, & \text{if } U(\omega) \leq 1/n, \\ 0, & \text{if } U(\omega) > 1/n. \end{cases}$$

Investigate the convergence of the sequence X_n to 0 in the following modes.

(a) Does $X_n \xrightarrow{\text{d}} 0$?

(b) Does $X_n \xrightarrow{\text{P}} 0$

(c) Does $X_n \xrightarrow{\text{a.s.}} 0$. Can Borel-Cantelli Lemma be applied?

(d) Does $X_n \xrightarrow{\text{L1}} 0$ (Convergence in mean)

Q2: Let $\{X_n\}$ be a sequence of iid random variables all having a uniform distribution on the interval $[0,1]$. Define:

$$Y_n = n\left(1 - \max_{1\leq i\leq n} X_i\right).$$

Show that $Y_n \xrightarrow{d} Y$ where $Y \sim \text{Exp}(1)$.

Q3: Let X be a discrete random variable with support

$$R_X = \{0, 1\}$$

and probability mass function:

$$p_X(x) = \begin{cases} 1/3, & \text{if } x = 1, \\ 2/3, & \text{if } x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Consider a sequence of random variables $\{X_n\}$ whose generic term is:

$$X_n = \left(1 + \frac{1}{n}\right) X.$$

Show that $\{X_n\}$ converges in probability to X.

Q4: **Convergence of Discrete to continuous and vice versa**

(a) Let Y_n be uniformly distributed on the discrete set $\{1, 2, \ldots, n\}$, and define

$$X_n = \frac{Y_n}{n}.$$

Show that:

$$X_n \xrightarrow{d} X, \quad X \sim \text{U}(0,1).$$

(b) Let X_n be uniformly distributed on the interval $[-1/n, 1/n]$. Show that

$$X_n \xrightarrow{p} 0,$$

i.e., that the sequence of continuous random variables X_n converges in probability to the degenerate (discrete) random variable which is identically equal to zero. This also implies

$$X_n \xrightarrow{d} 0.$$

Q5: Given an integral $I = \int_0^{2\pi} f(x)dx$, solve the following:

(a) Write the Monte Carlo Estimate, assuming X_i's are sampled uniformly over the domain.

(b) Write the Monte Carlo Estimate, assuming X_i's are sampled according to some PDF $g(X_i)$.

(c) Prove that the Monte Carlo Estimates from the previous questions compute the right answer on average.

Q6: Let X be an exponential random variable with parameter λ and let Y be a random variable with the Gamma distribution $Y \sim \text{Gamma}(k, \theta)$.

(a) Show how to generate X using a uniform random variable U drawn from the interval $[0, 1]$.

(b) Show how to generate Y using k uniform random variables drawn from $[0, 1]$.

Q7: Use the rejection method to generate a random variable having the $Gamma(\frac{5}{2}, 1)$ density function.

Note: The pdf of $Gamma(k, \theta)$ is given by $f(x) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-x/\theta}$ and $\Gamma(\frac{5}{2}) = \frac{3}{4}\pi$.

Hint: You need to figure out an appropriate distribution you can already sample from to use in the rejection method.

Q8: **Important inequalities (use directly):**

- **Markov's inequality.** For any nonnegative random variable Z and any $t > 0$,

$$\mathbb{P}(Z \geq t) \leq \frac{\mathbb{E}[Z]}{t}.$$

- **Chebyshev's inequality.** For any random variable X with mean $\mu = \mathbb{E}[X]$ and variance $\sigma^2 = \text{Var}(X)$,

$$\mathbb{P}(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}, \qquad \forall \varepsilon > 0.$$

Problem. Let X be a real-valued random variable with finite mean $\mu = \mathbb{E}[X]$ and finite variance $\sigma^2 = \text{Var}(X)$. Use Markov's inequality to obtain Chebyshev's inequality. Then show the following consequence:

(a) Apply Markov's inequality to an appropriate nonnegative function of X to deduce Chebyshev's inequality.

(b) Let $X_1, X_2, \ldots$ be continuous random variables with densities

$$f_{X_n}(x) = \frac{n}{2} e^{-n|x|}, \qquad x \in \mathbb{R},\ n = 1, 2, \ldots.$$

Show that $X_n \xrightarrow{p} 0$.

Extra practice problems (recommended).

(1) Let $X_n \sim \text{Poisson}(n)$. Define $Y_n = X_n/n$. Show $Y_n \xrightarrow{p} 1$. (*Hint:* compute mean and variance of Y_n and use Chebyshev.)

(2) Let Y_n satisfy $\mathbb{E}[Y_n] = 1/n$ and $\text{Var}(Y_n) = 1/n^2$. Show $Y_n \xrightarrow{p} 0$. (*Hint:* Chebyshev.)

Resources for practice:

- Weak Law of Large Numbers (proof and discussion)
- Convergence in Probability (see Example 7.9)
- **From the textbook:** Try solving Example 5.6, Example 5.7, and Problem 6 and 7 in Chapter "Limit Theorems."

Q9: Suppose a sequence of random variables X_n satisfies $\lim_{n\to\infty} \mathbb{E}[|X_n - c|^\alpha] = 0$ for $\alpha > 0$.
Show that $X_n \xrightarrow{\text{p}} c$.

Q10: Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with $E[X] = 2$ and $\text{var}(X) = 9$, and let $Y_i = X_i/2^i$. We also define $T_n = \sum_{i=1}^n Y_i$ and $A_n = \frac{1}{n}\sum_{i=1}^n Y_i = \frac{T_n}{n}$.

(a) Evaluate the mean and variance of Y_n, T_n, and A_n.

(b) Does Y_n converge in probability? If so, to what value?

(c) Does T_n converge in probability? If so, to what value?

(d) Does A_n converge in probability? If so, to what value?