Solution to Linear Algebra Mid-Sem(2025)

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Problem 1

Prove that the union of two subspaces of a vector space V is a subspace of V if and only if one of the subspaces is contained in the other.
Solution:
Let U and W be subspaces of a vector space V.
We have to prove that $U \cup W$ is a subspace of V if and only if either $U \subseteq W$ or $W \subseteq U$.
Proof:
($\Rightarrow$) Suppose $U \cup W$ is a subspace of V.
BWOC, let us assume that $U \not\subseteq W$ and $W \not\subseteq U$

$$\implies \exists u_0 \in U \setminus W \text{ and } w_0 \in W \setminus U$$

.
Also, $u_0, w_0 \in U \cup W$
Since $U \cup W$ is a subspace, it is closed under addition

$$\implies u_0 + w_0 \in U \cup W$$

$$\implies u_0 + w_0 \in U \text{ or } u_0 + w_0 \in W$$

Now, if $u_0 + w_0 \in U$, and $\exists - u_0 \in U$ since U is a subspace

$$\implies u_0 + w_0 + -u_0 \in U$$
$$\implies w_0 \in U$$

contradicting $w_0 \notin U$.
Similarly, if $u_0 + w_0 \in W$, then $u_0 = (u_0 + w_0) - w_0 \in W$, contradicting $u_0 \notin W$.
Thus, either $U \subseteq W$ or $W \subseteq U$.
($\Leftarrow$) Suppose $U \subseteq W$ (WLOG). Then $U \cup W = W$, which is a subspace. Thus, the union is a subspace whenever one subspace is contained in the other.

Problem 2

We need to prove that the span of the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is the same as the span of the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$, where

$$\mathbf{u}_m = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_m, \quad \text{for } m = 1, 2, \ldots, k.$$

Step 1: Show that Span$(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) \subseteq$ Span$(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k)$

To prove this inclusion, we express each $\mathbf{v}_m$ in terms of the $\mathbf{u}_m$:

$$\mathbf{v}_1 = \mathbf{u}_1$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \mathbf{u}_1$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \mathbf{u}_2$$

$$\vdots$$

$$\mathbf{v}_k = \mathbf{u}_k - \mathbf{u}_{k-1}.$$

Since each $\mathbf{v}_m$ is a linear combination of the $\mathbf{u}_m$, it follows that every vector in $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$ can be written as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$. Thus,

$$\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) \subseteq \text{Span}(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k).$$

Step 2: Show that Span$(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k) \subseteq$ Span$(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$

Now, we express each $\mathbf{u}_m$ in terms of the $\mathbf{v}_m$:

$$\mathbf{u}_1 = \mathbf{v}_1$$

$$\mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_2$$

$$\mathbf{u}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$$

$$\vdots$$

$$\mathbf{u}_k = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k.$$

Since each $\mathbf{u}_m$ is a linear combination of the $\mathbf{v}_m$, every vector in $\text{Span}(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k)$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$. Thus,

$$\text{Span}(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k) \subseteq \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k).$$

Since we have shown both inclusions, we conclude that:

$$\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = \text{Span}(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k).$$

Thus, the two spans are equal, proving the required result. □

Problem 3

Let V be a subspace of $\mathbb{R}^3$ defined by:

$$V = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0,\ 2x + 3y + 4z = 0\}$$

Find a basis of V.

Solution:
We are given the following system of equations:

1. $x + y + z = 02.$ $\qquad\qquad$ $2x + 3y + 4z = 0$

Step 1: Express the system in matrix form
We write the system as an augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 3 & 4 & 0 \end{bmatrix}$$

Step 2: Perform row operations to reduce the matrix
Perform the row operation $R_2 \to R_2 - 2R_1$:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Step 3: Express variables in terms of free parameters
From the second row:

$$y + 2z = 0 \implies y = -2z$$

From the first row:

$$x + y + z = 0 \implies x - 2z + z = 0 \implies x = z$$

Let $z = t$ (free parameter), then:

$$(x, y, z) = (t, -2t, t) = t(1, -2, 1)$$

Step 4: Identify a basis
Since the solution space is one-dimensional, a basis for V is given by the vector:

$$\mathbf{v}_1 = (1, -2, 1)$$

Final Answer: A basis for V is:

$$\{(1, -2, 1)\}$$

Problem 4

Part - A

Proof that T is a Linear Transformation:
A function $T : V \to W$ is a linear transformation if it satisfies the property:

$$T(k\alpha + \beta) = kT(\alpha) + T(\beta)$$

for any scalar $k \in \mathbb{R}$ and matrices $\alpha, \beta \in M_{2\times 2}(\mathbb{R})$.

Step 1: Define α and β

Let

$$\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \beta = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Then, for a scalar $k \in \mathbb{R}$, we compute:

$$k\alpha + \beta = k \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Using matrix addition and scalar multiplication,

$$= \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ka+e & kb+f \\ kc+g & kd+h \end{bmatrix}$$

Step 2: Apply T

By definition,

$$T\left(\begin{bmatrix} ka+e & kb+f \\ kc+g & kd+h \end{bmatrix}\right) = ((ka+e)+(kd+h), (kb+f)+(kc+g))$$

$$= (ka+kd+e+h, kb+kc+f+g)$$

Step 3: Compute $kT(\alpha) + T(\beta)$

We separately compute:

$$T(\alpha) = (a+d, b+c), \quad T(\beta) = (e+h, f+g)$$

Multiplying $T(\alpha)$ by k:

$$cT(\alpha) = (k(a+d), k(b+c))$$

Adding $T(\beta)$:

$$kT(\alpha) + T(\beta) = (k(a+d)+(e+h), k(b+c)+(f+g))$$

$$= (ka+kd+e+h, kb+kc+f+g)$$

Step 4: Verify Equality

Since,

$$T(k\alpha+\beta) = (ka+kd+e+h, kb+kc+f+g)$$

$$kT(\alpha)+T(\beta) = (ka+kd+e+h, kb+kc+f+g)$$

we conclude:

$$T(k\alpha+\beta) = kT(\alpha)+T(\beta)$$

Thus, T satisfies linearity, proving it is a **linear transformation**. $\square$

Part - B

Statement of the Rank-Nullity Theorem

Let $T : V \to W$ be a **linear transformation** between two vector spaces, where the domain V is **finite-dimensional**. Then, we have:

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim V$$

where rank of T is the dimension of the range of T and the nullity of T is the dimension of the null space of T.

Mentioning Dimension

The domain of the transformation T is the space of all 2×2 real matrices:

$$M_{2\times 2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$$

Each matrix has 4 independent entries (a, b, c, d), so the space of all such matrices is isomorphic to $\mathbb{R}^4$. Thus, the dimension of the domain is:

$$\dim V = 4$$

Finding Rank

The transformation $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^2$ is given by:

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a + d, b + c)$$

- The output of T is a pair $(a+d, b+c)$, meaning that any matrix is mapped to some point in $\mathbb{R}^2$.
- The two expressions $(a + d)$ and $(b + c)$ are independent because they involve different combinations of a, b, c, d.
- Since there are exactly 2 independent outputs, the dimension of the image is 2.

$$\operatorname{rank}(T) = 2.$$

Finding Nullity using the formula

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim V$$

$$\operatorname{rank}(T) = 2$$

$$\dim(V) = 4$$

$$\text{Therefore, } \operatorname{nullity}(T) = 2$$

Problem 5

Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation represented by an $n \times n$ matrix A. Prove that T is invertible if and only if $\text{Rank}(A) = n$.

Theorem: Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation represented by an $n \times n$ matrix A. Then T is invertible if and only if $\text{Rank}(A) = n$.

Proof:

Forward Direction: Suppose T is invertible. Then there exists an inverse transformation U such that:

$$U(T(x)) = x \quad \text{for all } x \in \mathbb{R}^n.$$

We must show that U is a linear transformation. Let $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then:

$$U(x+y) = U(T(U(x)) + T(U(y))) = U(T(U(x) + U(y))) = U(x) + U(y),$$

where we use the linearity of T in the second step. Similarly,

$$U(cx) = U(T(U(cx))) = U(T(cU(x))) = cU(x).$$

Since U satisfies the properties of a linear transformation, it is a linear transformation.

Since $T(x) = Ax$ and $U(x) = Bx$, applying T and then U gives:

$$T(U(x)) = ABx = x, \quad U(T(x)) = BAx = x.$$

Thus, $AB = I$ and $BA = I$, so A has a two-sided inverse and is therefore invertible.

Since A is invertible, the equation $Ax = 0$ has only the trivial solution, which implies that the rows of A are linearly independent. Since the rank of A is equal to the row rank of matrix A, we conclude:

$$\text{Rank}(A) = n.$$

Since the rows are linearly independent, they form a basis for $\mathbb{R}^n$ of dimension n. Thus, the row rank of A is n.

Backward Direction: Suppose $\text{Rank}(A) = n$. Then the n rows of A are linearly independent, which means that A is row equivalent to the identity matrix I_n. Hence, A must be invertible, meaning there exists a matrix B such that:

$$AB = I, \quad BA = I.$$

Define $U : \mathbb{R}^n \to \mathbb{R}^n$ by $U(x) = Bx$. Then:

$$U(T(x)) = B(Ax) = (BA)x = x,$$

$$T(U(x)) = A(Bx) = (AB)x = x.$$

Thus, U is the inverse of T, confirming that T is invertible. $\square$

Problem 6

Let A be an $n \times p$ matrix and B be a $p \times m$ matrix. Suppose that the columns of A and the columns of B are linearly independent. We aim to determine whether the columns of the product matrix AB are also linearly independent.

Solution: Let $A = [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_p]$ be an $n \times p$ matrix where $\mathbf{a}_i$ are the columns of A, and let $B = [\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m]$ be a $p \times m$ matrix where $\mathbf{b}_j$ are the columns of B.

The product AB is an $n \times m$ matrix, and its jth column can be written as:

$$(AB)\mathbf{e}_j = A\mathbf{b}_j,$$

where $\mathbf{e}_j$ is the jth standard basis vector.

Thus, the jth column of AB is a linear combination of the columns of A with coefficients given by the entries of $\mathbf{b}_j$. Explicitly,

$$AB = [A\mathbf{b}_1 \; A\mathbf{b}_2 \; \dots \; A\mathbf{b}_m].$$

Suppose the columns of AB are linearly dependent. This means there exist scalars $c_1, c_2, \dots, c_m$ (not all zero) such that:

$$c_1(A\mathbf{b}_1) + c_2(A\mathbf{b}_2) + \dots + c_m(A\mathbf{b}_m) = \mathbf{0}.$$

By factoring out A, we obtain:

$$A(c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \dots + c_m\mathbf{b}_m) = \mathbf{0}.$$

Since the columns of A are linearly independent, the only solution to this equation is:

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \dots + c_m\mathbf{b}_m = \mathbf{0}.$$

We need to determine whether m linearly independent columns of B can remain independent after being transformed by A. When multiplying by A, the columns of B are projected into the subspace spanned by the columns of A. This subspace has dimension p.

- **If $m \leq p$:** The m columns of B can be independent because there is enough room in the p-dimensional span of A to accommodate m independent vectors, i.e. there is enough room to maintain the independence of the columns. Thus, the columns of AB will remain linearly independent.
- **If $m > p$:** This case is not possible since we are given that columns of B are linearly independent. For this case, the columns of B must fit within a p-dimensional space. Since there are more than p vectors, they must be linearly dependent as we can express the $m - p$ scalars as a non-zero combination of the remaining scalars. If m exceeds p, some columns of B must overlap, resulting in linear dependence, which contradicts our initial statement.

Problem 7

We are given two ordered bases, $\mathcal{B} = \{\vec{u}_1, \vec{u}_2\}$, $\mathcal{B}' = \{\vec{v}_1, \vec{v}_2\}$. We are also given that,

$$[\vec{v}_1]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, [\vec{v}_2]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}. \tag{1}$$

Let,

$$\vec{v} = x_1\vec{u}_1 + x_2\vec{u}_2 = x_1'\vec{v}_1 + x_2'\vec{v}_2. \tag{2}$$

Then (by notation), $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $[\vec{v}]_{\mathcal{B}'} = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$.

Also, $\vec{v}_1 = \frac{1}{\sqrt{2}}\vec{u}_1 + \frac{1}{\sqrt{2}}\vec{u}_2$ and $\vec{v}_2 = \frac{1}{\sqrt{3}}\vec{u}_1 - \frac{1}{\sqrt{3}}\vec{u}_2$. Using this in eq. 2 can write,

$$\begin{aligned} \vec{v} &= x_1'\vec{v}_1 + x_2'\vec{v}_2 && (3) \\ &= x_1'(\frac{1}{\sqrt{2}}\vec{u}_1 + \frac{1}{\sqrt{2}}\vec{u}_2) + x_2'(\frac{1}{\sqrt{3}}\vec{u}_1 - \frac{1}{\sqrt{3}}\vec{u}_2) && (4) \\ &= (\frac{x_1'}{\sqrt{2}} + \frac{x_2'}{\sqrt{3}})\vec{u}_1 + (\frac{x_1'}{\sqrt{2}} - \frac{x_2'}{\sqrt{3}})\vec{u}_2 && (5) \end{aligned}$$

Thus, $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} \frac{x_1'}{\sqrt{2}} + \frac{x_2'}{\sqrt{3}} \\ \frac{x_1'}{\sqrt{2}} - \frac{x_2'}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = P[\vec{v}]_{\mathcal{B}}'$, where, $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$.

And $P^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$.

Note:

1. P^{-1} **cannot be written directly** using *Cramer's rule i.e.* via determinants.