Probability and Statistics

Tutorial 10

Topics Covered: Markov Chains

Q1: $\{X_n\}_{n=1}^{\infty}$ is a Markov chain, prove that $\{Y_n\}_{n=2}^{\infty}$ is also a Markov chain, where $Y_n = [X_n, X_{n-1}]^T$ for some $k \leq n$.

Solution: We need to show that for all n,

$$P(Y_{n+1} = y_{n+1}|Y_n = y_n, Y_{n-1} = y_{n-1}, \ldots, Y_1 = y_1) = P(Y_{n+1} = y_{n+1}|Y_n = y_n).$$

By definition, we have

$$Y_n = \begin{bmatrix} X_n \\ X_{n-1} \end{bmatrix}.$$

Thus,

$$\begin{aligned}
&P(Y_{n+1} = y_{n+1}|Y_n = y_n, Y_{n-1} = y_{n-1}, \ldots, Y_1 = y_1) \\
&= P\left(\begin{bmatrix} X_{n+1} \\ X_n \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} \middle| \begin{bmatrix} X_n \\ X_{n-1} \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}, \ldots, \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} \right) \\
&= P(X_{n+1} = x_{n+1}, X_n = x_n|X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_1 = x_1) \\
&= P(X_{n+1} = x_{n+1}|X_n = x_n).
\end{aligned}$$

Now, we also have

$$\begin{aligned}
&P(Y_{n+1} = y_{n+1}|Y_n = y_n) \\
&= P\left(\begin{bmatrix} X_{n+1} \\ X_n \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} \middle| \begin{bmatrix} X_n \\ X_{n-1} \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} \right) \\
&= P(X_{n+1} = x_{n+1}, X_n = x_n|X_n = x_n, X_{n-1} = x_{n-1}) \\
&= P(X_{n+1} = x_{n+1}|X_n = x_n).
\end{aligned}$$

Thus, we have shown that

$$P(Y_{n+1} = y_{n+1}|Y_n = y_n) = P(Y_{n+1} = y_{n+1}|Y_n = y_n, Y_{n-1} = y_{n-1}, \ldots, Y_1 = y_1),$$

which shows that $\{Y_n\}$ is a Markov chain.

Q2: Given the Markov chain in Figure 1, find the set of stationary distributions. Note that since the Markov chain is reducible, there may be more than one stationary distribution.

Solution: Let the stationary distribution be $\pi = [\pi_1, \pi_2, \pi_3]^T$. The transition matrix P of the Markov chain is given by

$$P = \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}.$$

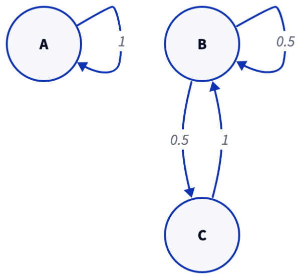

Figure 1: A reducible Markov chain

To find the stationary distribution, we need to solve the equation $\pi P = \pi$, along with the normalization condition $\pi_1 + \pi_2 + \pi_3 = 1$. This gives us the following system of equations:

$$\begin{aligned}
\pi_1 &= \pi_1, \\
\pi_2 &= 0.5\pi_2 + \pi_3, \\
\pi_3 &= 0.5\pi_2, \\
\pi_1 + \pi_2 + \pi_3 &= 1.
\end{aligned}$$

From the second and third equations, we can express π_3 in terms of π_2:

$$\pi_3 = 0.5\pi_2.$$

Substituting this into the second equation gives:

$$\pi_2 = 0.5\pi_2 + 0.5\pi_2 \implies \pi_2 = \pi_2,$$

which is always true. Thus, this equation does not provide any new information. Let $\pi_2 = x$. Then, we have:

$$\pi_3 = 0.5x.$$

Now, substituting these into the normalization condition:

$$\pi_1 + x + 0.5x = 1 \implies \pi_1 + 1.5x = 1 \implies \pi_1 = 1 - 1.5x.$$

Thus, the stationary distribution can be expressed as

$$\pi = \begin{bmatrix} 1 - 1.5x \\ x \\ 0.5x \end{bmatrix},$$

where $0 \leq x \leq \frac{2}{3}$ to ensure that all probabilities are non-negative.

Q3: Consider a Markov chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 0.2 & 0.8 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.3 & 0.7 \end{pmatrix}.$$

Suppose the chain starts in state $X_0 = 1$. Using the following sequence of independent Uniform$(0,1)$ random numbers:

$$0.76,\ 0.34,\ 0.15,\ 0.91,\ 0.40,$$

simulate the chain to obtain $(X_0, X_1, X_2, X_3, X_4, X_5)$.

Solution: Apply inverse transform sampling. For a current state i with transition probabilities p_{i1}, p_{i2}, p_{i3} we convert them into cumulative intervals on $[0,1)$:

$$[0, p_{i1}),\ [p_{i1}, p_{i1} + p_{i2}),\ [p_{i1} + p_{i2}, 1).$$

Given a uniform draw u, the next state is the index of the interval that contains u.

Write the rows of P and their cumulative intervals:

- From state 1: (0.2, 0.8, 0) so intervals are

$$1 : [0, 0.2), \qquad 2 : [0.2, 1.0), \qquad 3 : [1.0, 1.0).$$

  (State 3 has probability 0, hence an empty interval.)
- From state 2: (0.5, 0.5, 0) so intervals are

$$1 : [0, 0.5), \qquad 2 : [0.5, 1.0), \qquad 3 : [1.0, 1.0).$$

- From state 3: (0, 0.3, 0.7) so intervals are

$$1 : [0, 0), \qquad 2 : [0, 0.3), \qquad 3 : [0.3, 1.0).$$

Now simulate step-by-step starting at $X_0 = 1$.

Step 1: Current state $X_0 = 1$. Draw $u_1 = 0.76$. From state 1, intervals are $1 : [0, 0.2),\ 2 : [0.2, 1.0)$. Since $0.76 \in [0.2, 1.0)$, we set $X_1 = 2$.

Step 2: Current state $X_1 = 2$. Draw $u_2 = 0.34$. From state 2, intervals are $1 : [0, 0.5),\ 2 : [0.5, 1.0)$. Since $0.34 \in [0, 0.5)$, we set $X_2 = 1$.

Step 3: Current state $X_2 = 1$. Draw $u_3 = 0.15$. From state 1, $0.15 \in [0, 0.2)$, so $X_3 = 1$.

Step 4: Current state $X_3 = 1$. Draw $u_4 = 0.91$. From state 1, $0.91 \in [0.2, 1.0)$, so $X_4 = 2$.

Step 5: Current state $X_4 = 2$. Draw $u_5 = 0.40$. From state 2, $0.40 \in [0, 0.5)$, so $X_5 = 1$.

Putting it all together,

$$(X_0, X_1, X_2, X_3, X_4, X_5) = (1,\ 2,\ 1,\ 1,\ 2,\ 1).$$

Q4: Consider a Markov chain with state space $S = \{1, 2, 3\}$, transition matrix

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{pmatrix}.$$

and initial distribution $\pi_0 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Compute the probability of obtaining the trajectory (3, 2, 1, 1, 3).

Solution: The probability of a trajectory $(X_0, X_1, X_2, X_3, X_4)$ in a Markov chain is given by

$$P(X_0, X_1, X_2, X_3, X_4) = \pi_0(X_0) \prod_{t=1}^{4} P(X_t|X_{t-1}).$$

For the trajectory $(3, 2, 1, 1, 3)$, we have:

$$\begin{aligned} P(3,2,1,1,3) &= \pi_0(3) \cdot P(2|3) \cdot P(1|2) \cdot P(1|1) \cdot P(3|1) \\ &= \frac{1}{3} \cdot 0.8 \cdot 0.4 \cdot 0.6 \cdot 0.2 \\ &= \frac{1}{3} \cdot 0.0384 \\ &= 0.0128. \end{aligned}$$

Q5: Consider a discrete-time Markov chain with the transition probabilities $p_{ij} = 0$ for $i = j$. The initial distribution is given by $\mu = [\mu_1, \mu_2]$. Find the probability of head and tail in the n-th step, in terms of μ_1 and μ_2.

Solution: Lets begin by making a discrete-time Markov chain with states H and T each representing Heads and Tails.

Given in the question that $p_{HH} = 0$ and $p_{TT} = 0$ and we know that $p_{TH} + p_{HH} = 1$ and $p_{HT} + p_{TT} = 1$.

So Now we get the following markov chain.

We obtain the transition probabilities $p_{HT} = 1$ and $p_{TH} = 1$.

For $n = 1$, the probabilities are:

$$p_1(H) = p_H \mu_2 \quad \text{and} \quad p_1(T) = p_T \mu_1$$

$$p_1(H) = \mu_2 \quad \text{and} \quad p_1(T) = \mu_1$$

For $n = 2$, the probabilities are:

$$p_2(H) = p_1(T)p_H + p_1(H)p_{HH} \quad \text{and} \quad p_2(T) = p_T p_1(H) + p_1(T)p_{TT}$$

On substituting,

$$p_2(H) = \mu_1 \quad \text{and} \quad p_2(T) = \mu_2$$

So in general we can write, For even steps $(n = 2k)$, where $k \geq 1$, the probabilities are given by:

$$p_{2k}(H) = (p_T p_H)^k \mu_1 \quad \text{and} \quad p_{2k}(T) = (p_H p_T)^k \mu_2$$

$$p_{2k}(H) = \mu_1 \quad \text{and} \quad p_{2k}(T) = \mu_2$$

For odd steps ($n = 2k+1$), where $k \geq 0$, the probabilities are given by:

$$P_{2k+1}(H) = (p_{2k}(T)p_H) \quad \text{and} \quad P_{2k+1}(T) = (p_{2k}(H)p_T)$$

$$P_{2k+1}(H) = \mu_2 \quad \text{and} \quad P_{2k+1}(T) = \mu_1$$

Therefore we get the general formulas for even and odd steps are:

$$p_{2k}(H) = \mu_1 \quad \text{and} \quad p_{2k}(T) = \mu_2$$

$$p_{2k+1}(H) = \mu_2 \quad \text{and} \quad p_{2k+1}(T) = \mu_1$$

where $k \geq 0$

Q6: A gambler begins with an initial fortune of i dollars. Each time he plays, he has the possibility of winning 1 dollar with a probability p or losing 1 dollar with a probability $1-p$. The gambler will only stop playing if he either accumulates N dollars or loses all of his money.

A:

Let X_n be the amount of money after playing n times. Then $X_n = i + \Delta_1 + \cdots + \Delta_n$, where $\{\Delta_n\}$ is a random walk with step 1 and probability of going up p. Now let

$$\tau_i = \min_{n \geq 0}\{X_n \in \{0, N\} | X_0 = i\}$$

be the time at which the gambler stops playing. We want to calculate

$$P_i(N) = \mathbb{P}(X_{\tau_i} = N)$$

that is, the probability that the gambler accumulates N dollars starting with i dollars. Now if after the first play, if $\Delta_1 = 1$, then the gambler will finally win with a probability $P_{i+1}(N)$ by the markov property of the chain. Similar reasoning holds for $\Delta_1 = -1$. Thus we obtain the equation

$$P_i(N) = pP_{i+1}(N) + (1-p)P_{i-1}(N)$$

We also have the boundary probabilities $P_0(N) = 0, P_N(N) = 1$. Rearranging the terms of the equation, we have

$$P_{i+1}(N) - P_i(N) = \frac{q}{p}(P_i(N) - P_{i-1}(N))$$

where $q = 1-p$. Recursively backtracking using this equation, we get

$$\begin{aligned} P_{i+1}(N) - P_i(N) &= \left(\frac{q}{p}\right)^i (P_1(N) - P_0(N)) \\ &= \left(\frac{q}{p}\right)^i P_1(N) \end{aligned}$$

Using this equation, we can successively evaluate $P_i(N)$ as an expression of $P_1(N)$

$$P_2 = \left(1 + \frac{q}{p}\right) P_1$$

$$P_3 = \left(1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2\right) P_1$$

$$\vdots$$

$$P_i = \left(1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right) P_1$$

Setting $P_N(N) = 1$ we get

$$P_N(N) = 1 = \left(1 + \cdots + \left(\frac{q}{p}\right)^{N-1}\right) P_1(N)$$

$$P_N(N) = \begin{cases} \frac{1-(q/p)^N}{1-(q/p)} P_1(N) & q \neq p \\ N P_1(N) & q = p \end{cases}$$

Solving for $P_1(N)$ and substituting, we get

$$P_1(N) = \begin{cases} \frac{1-(q/p)}{1-(q/p)^N} & q \neq p \\ \frac{1}{N} & q = p \end{cases}$$

Q7: Consider a Markov chain with transition probability matrix:

$$P = \begin{bmatrix} 0.7 & 0.3 & 0.0 \\ 0.4 & 0.5 & 0.1 \\ 0.0 & 0.6 & 0.4 \end{bmatrix}.$$

(a) Write a Python function to simulate a state transition path of length N, starting from an initial state s_0, using a random variable $U \sim \text{Uniform}(0, 1)$ to decide transitions.

(b) Estimate the **empirical limiting distribution** by simulating the chain for a large number of steps.

(c) Compute the **stationary distribution** π in three different ways:

i. Using the **eigenvalue method**: find the left eigenvector of P corresponding to eigenvalue 1.

ii. By solving the linear system $\pi P = \pi$, $\sum_i \pi_i = 1$.

(d) Compute the **theoretical limiting distribution** by raising P to a large power (e.g. P^{1000}).

A:

```
import numpy as np

# Transition matrix
P = np.array([
    [0.7, 0.3, 0.0],
    [0.4, 0.5, 0.1],
    [0.0, 0.6, 0.4]
```

```
])

def simulate_markov_chain(P, s0, N):
    """
    Simulate a Markov chain of length N from initial state s0.
    Each transition is determined using a Uniform(0,1) random variable.
    """
    n_states = P.shape[0]
    states = [s0]
    for _ in range(N - 1):
        u = np.random.rand()
        cumulative = 0.0
        next_state = 0
        for j in range(n_states):
            cumulative += P[states[-1], j]
            if u <= cumulative:
                next_state = j
                break
        states.append(next_state)
    return np.array(states)

# (a) Generate a sample path
np.random.seed(42)
path = simulate_markov_chain(P, s0=0, N=100000)

# (b) Empirical limiting distribution
unique, counts = np.unique(path, return_counts=True)
empirical_limiting_dist = counts / len(path)
print("Empirical limiting distribution:", empirical_limiting_dist)

# (c1) Stationary distribution via eigenvalue method
eigvals, eigvecs = np.linalg.eig(P.T)
stat_vec = np.real(eigvecs[:, np.isclose(eigvals, 1)])
stationary_eig = stat_vec[:, 0] / np.sum(stat_vec)
print("Stationary (eigenvalue) distribution:", stationary_eig)

# (c2) Stationary distribution via solving linear equations
# Solve (P^T - I) * pi^T = 0, with sum(pi) = 1

n = P.shape[0]
A = P.T - np.eye(n)
A[-1, :] = np.ones(n)
b = np.zeros(n)
b[-1] = 1

stationary_solve = np.linalg.solve(A, b)
print("Stationary (direct solve) distribution:", stationary_solve)
```

```
# (d) Theoretical limiting distribution (matrix power)
P_power = np.linalg.matrix_power(P, 1000)
theoretical_limiting_dist = P_power[0, :]
print("Theoretical limiting distribution:", theoretical_limiting_dist)
```

Explanation

- The sample path is generated by drawing a uniform random number $u \in [0, 1]$ at each step and selecting the next state based on cumulative transition probabilities in that row of P.
- The empirical limiting distribution is estimated by simulating many transitions and counting the relative frequency of each state.
- In the eigenvalue method, the stationary distribution is the normalized eigenvector corresponding to eigenvalue $\lambda = 1$ of P^T:

$$\pi P = \pi \quad \Rightarrow \quad P^T \pi^T = \pi^T.$$

- `stat_vec = np.real(eigvecs[:, np.isclose(eigvals, 1)])` extracts the eigenvector(s) of P^T associated with eigenvalue $\lambda = 1$. Because eigenvalues can have small numerical imaginary parts (due to floating-point rounding), `np.isclose(eigvals, 1)` finds the index where the eigenvalue is effectively equal to 1. Then, `eigvecs[:, ...]` selects the corresponding eigenvector(s), and `np.real(...)` removes negligible imaginary components.
- `stationary_eig = stat_vec[:, 0] / np.sum(stat_vec)` normalizes the eigenvector so that its entries sum to 1, ensuring that it represents a valid probability distribution:

$$\pi_i = \frac{v_i}{\sum_j v_j}.$$

  If multiple eigenvectors correspond to eigenvalue 1 (which occurs in reducible chains), only the first one is selected here (`stat_vec[:, 0]`).
- In the direct linear solve method, we solve

$$(P^T - I)\pi^T = 0, \quad \sum_i \pi_i = 1,$$

  by replacing the last row of $P^T - I$ with $[1, 1, \ldots, 1]$ and the corresponding right-hand side with 1. This ensures a unique normalized solution. Using `np.linalg.solve` directly gives the stationary distribution without using least squares.
- The theoretical limiting distribution is obtained by computing P^k for a large k; for an ergodic chain, each row of P^k converges to the same stationary distribution.

Q8: Let $\alpha_0, \alpha_1, \cdots$ be a sequence of nonnegative numbers such that

$$\sum_{j=0}^{\infty} \alpha_j = 1.$$

Consider a Markov chain $X_0, X_1, X_2, \cdots$ with the state space $S = \{0, 1, 2, \cdots\}$ such that

$$p_{ij} = \alpha_j, \quad \text{for all } j \in S.$$

Show that $X_1, X_2, \cdots$ is a sequence of i.i.d random variables.

A: To show that $X_1, X_2, \cdots$ is a sequence of independent and identically distributed (i.i.d.) random variables, we must prove two conditions:

(a) **Identically Distributed:** $P(X_n = k) = P(X_m = k)$ for all $n, m \geq 1$ and $k \in S$.

(b) **Independent:** $P(X_1 = j_1, X_2 = j_2, \cdots, X_n = j_n) = \prod_{k=1}^{n} P(X_k = j_k)$ for any finite n and any sequence of states $j_1, \ldots, j_n$.

The transition probability is given as $p_{ij} = P(X_{n+1} = j \mid X_n = i) = \alpha_j$. The crucial observation here is that the probability of moving to state j (which is α_j) **does not depend on the current state** i.

Identically Distributed:
We need to show that $P(X_n = j)$ is the same for all $n \geq 1$. Let the initial distribution be $P(X_0 = i) = \pi_i^{(0)}$.
For X_1, using the law of total probability:

$$P(X_1 = j) = \sum_{i \in S} P(X_1 = j \mid X_0 = i) P(X_0 = i) = \sum_{i \in S} p_{ij} \pi_i^{(0)}$$

Using $p_{ij} = \alpha_j$:

$$P(X_1 = j) = \sum_{i \in S} \alpha_j \pi_i^{(0)} = \alpha_j \sum_{i \in S} \pi_i^{(0)} = \alpha_j \cdot 1 = \alpha_j$$

For X_2:

$$P(X_2 = j) = \sum_{i \in S} P(X_2 = j \mid X_1 = i) P(X_1 = i) = \sum_{i \in S} p_{ij} P(X_1 = i)$$

Using $p_{ij} = \alpha_j$ and $P(X_1 = i) = \alpha_i$:

$$P(X_2 = j) = \sum_{i \in S} \alpha_j \alpha_i = \alpha_j \sum_{i \in S} \alpha_i = \alpha_j \cdot 1 = \alpha_j$$

By induction, $P(X_n = j) = \alpha_j$ for all $n \geq 1$. Since $P(X_n = j) = \alpha_j$ for all $n \geq 1$, the random variables $X_1, X_2, \cdots$ are **identically distributed**.

Independent:
We need to show $P(X_1 = j_1, \ldots, X_n = j_n) = \prod_{k=1}^{n} P(X_k = j_k)$. From Part 1, the right-hand side is $\prod_{k=1}^{n} \alpha_{j_k}$.
Using the chain rule and the Markov property:

$$P(X_1 = j_1, \ldots, X_n = j_n) = P(X_1 = j_1) P(X_2 = j_2 \mid X_1 = j_1) \cdots P(X_n = j_n \mid X_{n-1} = j_{n-1})$$

$$P(X_1 = j_1, \ldots, X_n = j_n) = P(X_1 = j_1) \cdot p_{j_1, j_2} \cdot p_{j_2, j_3} \cdots p_{j_{n-1}, j_n}$$

Substitute $p_{ij} = \alpha_j$ and $P(X_1 = j_1) = \alpha_{j_1}$:

$$P(X_1 = j_1, \ldots, X_n = j_n) = \alpha_{j_1} \cdot \alpha_{j_2} \cdot \alpha_{j_3} \cdots \alpha_{j_n} = \prod_{k=1}^{n} \alpha_{j_k}$$

Since $P(X_k = j_k) = \alpha_{j_k}$ for each k, we have shown:

$$P(X_1 = j_1, \ldots, X_n = j_n) = \prod_{k=1}^{n} P(X_k = j_k)$$

This is the definition of **independence**.

Q9: Consider a Markovian Coin, $S = \{0, 1\}$. Where 0 denotes Head and 1 denotes Tails. Suppose that the transition matrix is given by

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix},$$

where a and b are two real numbers in the interval $[0, 1]$ such that $0 < a + b < 2$. Suppose that the system is in state 0 at time $n = 0$ with probability α, i.e.,

$$\pi^{(0)} = [P(X_0 = 0) \quad P(X_0 = 1)] = [\alpha \quad 1-\alpha],$$

where $\alpha \in [0, 1]$.

(a) How does transition matrix define the nature of the coin.

(b) Using induction (or any other method), show that

$$P^n = \frac{1}{a+b} \begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b} \begin{bmatrix} a & -a \\ -b & b \end{bmatrix}.$$

(c) Show that

$$\lim_{n\to\infty} P^n = \frac{1}{a+b} \begin{bmatrix} b & a \\ b & a \end{bmatrix}.$$

(d) Show that

$$\lim_{n\to\infty} \pi^{(n)} = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}.$$

A:

(a) Nature of the Markovian coin: The probability that there is a Heads to Tails transition is a, and the probability that there is a Tails to Heads transition is b. The probability that it retains its memory is $1 - \alpha$ where α is the probability that chain changes whatever state it is in.

(b) For $n = 1$, we have

$$P^1 = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} = \frac{1}{a+b} \begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{1-a-b}{a+b} \begin{bmatrix} a & -a \\ -b & b \end{bmatrix}.$$

Assuming that the statement of the problem is true for n, we can write P^{n+1} as

$$P^{n+1} = P^n P = \frac{1}{a+b}\left(\begin{bmatrix} b & a \\ b & a \end{bmatrix} + (1-a-b)^n \begin{bmatrix} a & -a \\ -b & b \end{bmatrix}\right) \cdot \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$
$$= \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^{n+1}}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix},$$

which completes the proof.

(c) By assumption $0 < a + b < 2$, which implies $-1 < 1 - a - b < 1$. Thus,

$$\lim_{n\to\infty} (1-a-b)^n = 0.$$

Therefore,

$$\lim_{n\to\infty} P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix}.$$

(d)

$$\begin{aligned}
\lim_{n\to\infty} \pi^{(n)} &= \lim_{n\to\infty} [\pi^{(0)} P^n] \\
&= \pi^{(0)} \lim_{n\to\infty} P^n \\
&= [\alpha \quad 1-\alpha] \cdot \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} \\
&= \left[\tfrac{b}{a+b} \quad \tfrac{a}{a+b}\right].
\end{aligned}$$

Q10: For the Markovian coin described above:

(a) Calculate the stationary distribution. What do you observe?

(b) Find the mean return times, r_0 and r_1, for this Markov chain. Do you observe anything?

(c) Can you intuitively explain the result above?

A:

(a) The stationary distribution is the same as the limiting distribution.

(b) To calculate r_0:

$$\begin{aligned}
r_0 &= E[R \mid X_1 = 0, X_0 = 0] \cdot P(X_1 = 0 \mid X_0 = 0) + E[R \mid X_1 = 1, X_0 = 0] \cdot P(X_1 = 1 \mid X
\\
&= E[R \mid X_1 = 0] \cdot (1-a) + E[R \mid X_1 = 1] \cdot a.
\end{aligned}$$

If $X_1 = 0$, then $R = 1$, so $E[R \mid X_1 = 0] = 1$.
If $X_1 = 1$, then $R \sim 1 + \text{Geometric}(b)$, so

$$E[R \mid X_1 = 1] = 1 + E[\text{Geometric}(b)] = 1 + \frac{1}{b}.$$

Therefore,

$$r_0 = 1 \cdot (1-a) + \left(1 + \frac{1}{b}\right) \cdot a = \frac{a+b}{b}.$$

Similarly, we can obtain the mean return time to state 1:

$$r_1 = \frac{a+b}{a}.$$

We can notice that:

$$r_0 = \frac{1}{\pi_o} \quad r_1 = \frac{1}{\pi_1}$$

(c) The larger the π_i, the smaller the r_i will be. For example, if $\pi_i = \frac{1}{4}$, we conclude that the chain is in state i one-fourth of the time. In this case, $r_i = 4$, which means that on average it takes the chain four time units to return to state i.