Probability and Statistics

Tutorial 10

Q1: There are m classes offered by a particular department, and each year, the students rank each class from 1 to m, in order of difficulty, with rank m being the highest. Unfortunately, the ranking is completely arbitrary. In fact, any given class is equally likely to receive any given rank on a given year (two classes may not receive the same rank). A certain professor chooses to remember only the highest ranking his class has ever gotten.

(a) Find the transition probabilities of the Markov chain that models the ranking that the professor remembers.

(a) Find the recurrent and the transient states.

(a) Find the expected number of years for the professor to achieve the highest ranking given that in the first year he achieved the ith ranking.

Solution:

Let X_n be the state of the Markov chain in year n. The state is the highest ranking the professor's class has ever received up to that year. The state space is $S = \{1, 2, \ldots, m\}$.

In any given year, let R be the new rank. The problem states $P(R = k) = 1/m$ for any $k \in \{1, 2, \ldots, m\}$. The state transition is defined by $X_{n+1} = \max(X_n, R_{n+1})$.

(a) **Transition Probabilities**

We need to find $P_{ij} = P(X_{n+1} = j \mid X_n = i) = P(\max(i, R_{n+1}) = j)$.

- **Case 1:** $j < i$. $P_{ij} = P(\max(i, R_{n+1}) = j) = 0$. It is impossible for the new maximum rank to be less than the current maximum rank.
- **Case 2:** $j = i$. $P_{ii} = P(\max(i, R_{n+1}) = i)$. This happens if and only if the new rank R_{n+1} is less than or equal to i (i.e., $R_{n+1} \in \{1, 2, \ldots, i\}$). $P_{ii} = P(R_{n+1} \le i) = \sum_{k=1}^{i} P(R_{n+1} = k) = \sum_{k=1}^{i} \frac{1}{m} = \frac{i}{m}$.
- **Case 3:** $j > i$. $P_{ij} = P(\max(i, R_{n+1}) = j)$. For the new maximum to be j (which is greater than i), the new rank R_{n+1} must be exactly j. $P_{ij} = P(R_{n+1} = j) = \frac{1}{m}$.

Summary of transition probabilities:

$$P_{ij} = \begin{cases} \frac{i}{m} & \text{if } j = i \\ \frac{1}{m} & \text{if } j > i \\ 0 & \text{if } j < i \end{cases}$$

(a) **Recurrent and Transient States**

- **Transient States:** A state i is transient if there is a non-zero probability of leaving i and never returning. For any state $i \in \{1, 2, \ldots, m-1\}$, it is possible to move to a higher state $j > i$ (with probability $P_{ij} = 1/m$). Once the chain is in a state $j > i$, it can never return to i (since $P_{ji} = 0$). Therefore, states $\boldsymbol{\{1, 2, \ldots, m-1\}}$ **are transient**.

– **Recurrent States:** A state i is recurrent if, starting from i, the probability of returning is 1. Let's examine state m. The transition probability $P_{m,m} = \frac{m}{m} = 1$. This means state m is an **absorbing state**. Once the chain enters state m, it never leaves. By definition, an absorbing state is recurrent. Therefore, the only recurrent state is $\boldsymbol{\{m\}}$.

(a) **Expected Years to Achieve Highest Ranking**

We need to find E_i, the expected number of *additional* years to reach state m, given that the current state is i.

Base Case: If we are already in state m, the expected number of additional years is 0. $E_m = 0$.

Recursive Formula: For any state $i < m$, we use a first-step analysis:

$$E_i = 1 + \sum_{j=1}^{m} P_{ij} \cdot E_j$$

$$E_i = 1 + P_{ii} \cdot E_i + \sum_{j=i+1}^{m} P_{ij} \cdot E_j$$

$$E_i = 1 + \left(\frac{i}{m}\right) E_i + \sum_{j=i+1}^{m} \left(\frac{1}{m}\right) E_j$$

We solve for E_i by working backward from E_m.

For $i = m - 1$:

$$E_{m-1} = 1 + \left(\frac{m-1}{m}\right) E_{m-1} + P_{m-1,m} \cdot E_m$$

$$E_{m-1} = 1 + \left(\frac{m-1}{m}\right) E_{m-1} + \left(\frac{1}{m}\right)(0)$$

$$E_{m-1}\left(1 - \frac{m-1}{m}\right) = 1$$

$$E_{m-1}\left(\frac{1}{m}\right) = 1 \implies \boldsymbol{E_{m-1} = m}$$

For $i = m - 2$:

$$E_{m-2} = 1 + \left(\frac{m-2}{m}\right) E_{m-2} + P_{m-2,m-1} \cdot E_{m-1} + P_{m-2,m} \cdot E_m$$

$$E_{m-2} = 1 + \left(\frac{m-2}{m}\right) E_{m-2} + \left(\frac{1}{m}\right) E_{m-1} + \left(\frac{1}{m}\right) E_m$$

$$E_{m-2} = 1 + \left(\frac{m-2}{m}\right) E_{m-2} + \left(\frac{1}{m}\right)(m) + \left(\frac{1}{m}\right)(0)$$

$$E_{m-2} = 1 + \left(\frac{m-2}{m}\right) E_{m-2} + 1$$

$$E_{m-2}\left(1 - \frac{m-2}{m}\right) = 2$$

$$E_{m-2}\left(\frac{2}{m}\right) = 2 \implies \boldsymbol{E_{m-2} = m}$$

A clear pattern emerges: $E_i = m$ for all $i < m$.

Answer: The expected number of additional years to achieve the highest ranking, given the first year's ranking was i, is:

$$E_i = \begin{cases} \boldsymbol{m} & \text{if } i < m \\ \mathbf{0} & \text{if } i = m \end{cases}$$

Q2: **Ehrenfest model of diffusion.** We have a total of n balls, some of them black, some white. At each time step, we either do nothing, which happens with probability ϵ, where $0 < \epsilon < 1$, or we select a ball at random, so that each ball has probability $(1-\epsilon)/n > 0$ of being selected. In the latter case, we change the color of the selected ball (if white it becomes black, and vice versa), and the process is repeated indefinitely. What is the steady-state (stationary) distribution of the number of white balls?

Solution:

Let X_t be the number of white balls. The state space is $S = \{0, 1, \ldots, n\}$. Let's find the transition probabilities P_{ij}.

- **Move $i \to i-1$ (a white ball is picked and becomes black):** This requires selecting a ball (prob $1-\epsilon$) AND that ball being white (prob i/n). $P_{i,i-1} = (1-\epsilon)\frac{i}{n}$ (for $i \geq 1$)
- **Move $i \to i+1$ (a black ball is picked and becomes white):** This requires selecting a ball (prob $1-\epsilon$) AND that ball being black (prob $(n-i)/n$). $P_{i,i+1} = (1-\epsilon)\frac{n-i}{n}$ (for $i \leq n-1$)
- **Move $i \to i$ (no change):** This happens if we "do nothing" (prob ϵ). $P_{ii} = \epsilon$

The steady-state distribution π must satisfy $\pi = \pi P$, which means for any state j, the total probability of leaving j must equal the total probability of entering j.

$$\pi_j = \sum_i \pi_i P_{ij}$$

The only states that can transition *to* state j are $j-1$, j, and $j+1$. For any j (where $0 < j < n$):

$$\pi_j = \pi_{j-1}P_{j-1,j} + \pi_j P_{jj} + \pi_{j+1}P_{j+1,j}$$

Substitute the probabilities:

$$\pi_j = \pi_{j-1}\left((1-\epsilon)\frac{n-(j-1)}{n}\right) + \pi_j(\epsilon) + \pi_{j+1}\left((1-\epsilon)\frac{j+1}{n}\right)$$

Subtract $\pi_j(\epsilon)$ from both sides:

$$\pi_j(1-\epsilon) = \pi_{j-1}(1-\epsilon)\frac{n-j+1}{n} + \pi_{j+1}(1-\epsilon)\frac{j+1}{n}$$

Since $(1-\epsilon) \neq 0$, we divide by it:

$$\pi_j = \pi_{j-1}\frac{n-j+1}{n} + \pi_{j+1}\frac{j+1}{n}$$

Multiplying by n gives our central recurrence relation:

$$n\pi_j = \pi_{j-1}(n-j+1) + \pi_{j+1}(j+1)$$

The symmetry of the problem (black $\leftrightarrow$ white) suggests a symmetric solution. Let's guess the **Binomial distribution $B(n, 1/2)$**:

$$\pi_k = C \cdot \binom{n}{k} \quad \text{where } C \text{ is a normalization constant.}$$

Let's plug this guess into our central equation:

$$n\left(C\binom{n}{j}\right) \stackrel{?}{=} \left(C\binom{n}{j-1}\right)(n-j+1) + \left(C\binom{n}{j+1}\right)(j+1)$$

Divide by C:

$$n\binom{n}{j} \stackrel{?}{=} \binom{n}{j-1}(n-j+1) + \binom{n}{j+1}(j+1)$$

Expand the binomial coefficients on the right-hand side (RHS):

$$\binom{n}{j-1}(n-j+1) = \frac{n!}{(j-1)!(n-j+1)!} \cdot (n-j+1) = \frac{n!}{(j-1)!(n-j)!}$$
$$\binom{n}{j+1}(j+1) = \frac{n!}{(j+1)!(n-j-1)!} \cdot (j+1) = \frac{n!}{j!(n-j-1)!}$$

The RHS is:

$$\begin{aligned}
RHS &= \frac{n!}{(j-1)!(n-j)!} + \frac{n!}{j!(n-j-1)!} \\
&= \frac{n! \cdot j}{j \cdot (j-1)!(n-j)!} + \frac{n! \cdot (n-j)}{j! \cdot (n-j) \cdot (n-j-1)!} \\
&= \frac{n! \cdot j}{j!(n-j)!} + \frac{n! \cdot (n-j)}{j!(n-j)!} \\
&= \frac{n!(j+n-j)}{j!(n-j)!} = \frac{n \cdot n!}{j!(n-j)!} \\
&= n \cdot \left(\frac{n!}{j!(n-j)!}\right) = n\binom{n}{j}
\end{aligned}$$

The RHS equals the LHS, so the solution holds for $0 < j < n$.

Check Boundary Cases

- **For $j = 0$:** The balance equation is $\pi_0 = \pi_0 P_{00} + \pi_1 P_{10}$.

$$\pi_0 = \pi_0(\epsilon) + \pi_1\left((1-\epsilon)\frac{1}{n}\right) \implies \pi_0(1-\epsilon) = \pi_1(1-\epsilon)\frac{1}{n} \implies n\pi_0 = \pi_1$$

Test our solution: $n \cdot (C\binom{n}{0}) \stackrel{?}{=} C\binom{n}{1} \implies n \cdot C \cdot 1 = C \cdot n$. It holds.

- **For $j = n$:** The balance equation is $\pi_n = \pi_{n-1}P_{n-1,n} + \pi_n P_{nn}$.

$$\pi_n = \pi_{n-1}\left((1-\epsilon)\frac{1}{n}\right) + \pi_n(\epsilon) \implies \pi_n(1-\epsilon) = \pi_{n-1}(1-\epsilon)\frac{1}{n} \implies n\pi_n = \pi_{n-1}$$

Test our solution: $n \cdot (C\binom{n}{n}) \stackrel{?}{=} C\binom{n}{n-1} \implies n \cdot C \cdot 1 = C \cdot n$. It holds.

Our solution $\pi_k = C\binom{n}{k}$ works for all states. We find C by enforcing $\sum \pi_k = 1$:

$$\sum_{k=0}^{n} C\binom{n}{k} = 1 \implies C\sum_{k=0}^{n}\binom{n}{k} = 1$$

By the Binomial Theorem, $\sum_{k=0}^{n}\binom{n}{k} = 2^n$.

$$C \cdot 2^n = 1 \implies C = \frac{1}{2^n}$$

Therefore, the unique steady-state distribution is:

$$\pi_k = \binom{n}{k}\left(\frac{1}{2}\right)^n$$

This is the **Binomial $B(n, 1/2)$ distribution**.

Q3: Krrish and Krrish-3 are provided a 7 sided, fair die each. Krrish rolls the die until he observes 2 consecutive 6's. Krrish-3 rolls a die until he observes a 6 followed by a 7. Who is more likely to stop first?

(c) Provide an intuitive explanation to your answer.

(c) How could you use Markov chains to solve this?

(c) Provide mathematical proof.

Solution:

Krrish-3 is more likely to finish first.

(c) **Intuitive Explanation:**

The key difference is what happens when they have "partial progress" (i.e., they have just rolled one 6).

- For Krrish (goal "6-6"), any roll that is *not* a 6 (a 1, 2, 3, 4, 5, or 7) resets his progress completely back to the start. He is penalized for 6 out of 7 possible outcomes.
- For Krrish-3 (goal "6-7"), his progress only resets if he rolls a 1, 2, 3, 4, or 5. If he rolls a 7, he wins. Crucially, if he rolls *another 6*, his progress does not reset; he simply stays in the "just rolled a 6" state, ready to win on the next 7.

This "self-loop" on a 6 makes Krrish-3's game more efficient, as he is not penalized for one of the non-winning rolls. He only gets reset on 5/7 outcomes, while Krrish gets reset on 6/7 outcomes.

(c) **How to use Markov Chains:**

We can model each game as a separate Markov chain. The "winner" will be the player whose chain has the lower **expected number of steps (rolls)** to reach the "Win" state.

For each game, we would define three states:

(a) **State 0 (Start):** The initial state, where no progress has been made.

(b) **State 1 (Got one 6):** The state where the last roll was a 6.

(c) **State W (Win):** The absorbing "Win" state (e.g., "6-6" or "6-7" has been seen).

Let E_i be the expected number of *additional* rolls needed to win from any state i. We want to find E_0 for each player. We can set up a system of linear equations based on the transition probabilities between these states. For any state, the expected rolls from that state is 1 (for the current roll) plus the expected future rolls from the state you land in, weighted by the probability of landing there. Solving this system gives the expected number of total rolls, E_0.

(c) **Mathematical Proof:**

Analysis of Krrish (Goal: "6-6")

Let K be the number of rolls Krrish needs. We'll use states to find the expected value of K.

- **State 0:** The starting state (no progress). Let K_0 be the expected number of rolls needed from here. This is our goal.
- **State 1:** The last roll was a 6. Let K_1 be the expected additional rolls needed from here.
- **Win State:** The goal "6-6" is hit. The expected additional rolls are 0.

Let $p = 1/7$. We can set up a system of equations:

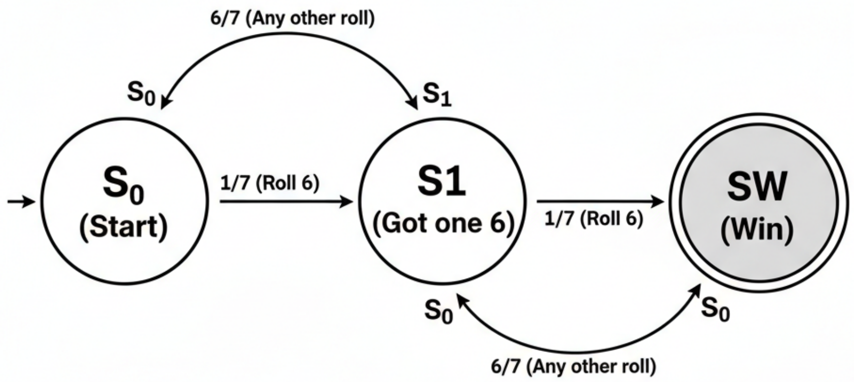

Figure 1: Markov chain for Krrish (Goal: 6-6)

From State 0 (K_0):

You roll once (cost = 1 roll). If you roll a 6 (prob p), you move to State 1. The remaining expected rolls are K_1. If you roll anything else (prob $1-p$), you stay in State 0. The remaining expected rolls are K_0.

Equation (1): $K_0 = 1 + p \cdot K_1 + (1-p) \cdot K_0$

From State 1 (K_1):

You roll once (cost = 1 roll). If you roll a 6 (prob p), you win. The remaining expected rolls are 0. If you roll anything else (prob $1-p$), your "6" streak is broken, and you go back to State 0. The remaining expected rolls are K_0.

Equation (2): $K_1 = 1 + p \cdot (0) + (1-p) \cdot K_0$

Solving the system:

Now, we solve this system with $p = 1/7$ and $1 - p = 6/7$. From (2): $K_1 = 1 + (6/7)K_0$

Substitute this into (1):

$$
\begin{aligned}
K_0 &= 1 + (1/7)K_1 + (6/7)K_0 \\
K_0 &= 1 + (1/7)\left(1 + (6/7)K_0\right) + (6/7)K_0 \\
K_0 &= 1 + 1/7 + (6/49)K_0 + (6/7)K_0 \\
K_0 &= 8/7 + (6/49)K_0 + (42/49)K_0 \\
K_0 &= 8/7 + (48/49)K_0 \\
K_0 - (48/49)K_0 &= 8/7 \\
(1/49)K_0 &= 8/7 \\
K_0 &= 49 \cdot (8/7) = \mathbf{56}
\end{aligned}
$$

The expected number of rolls for Krrish is **56**.

Analysis of Krrish-3 (Goal: "6-7")

Let J be the number of rolls Krrish-3 needs.

- **State 0:** The starting state. Let J_0 be the expected number of rolls.
- **State 1:** The last roll was a 6. Let J_1 be the expected additional rolls.
- **Win State:** Goal "6-7" is hit. Expected additional rolls are 0.

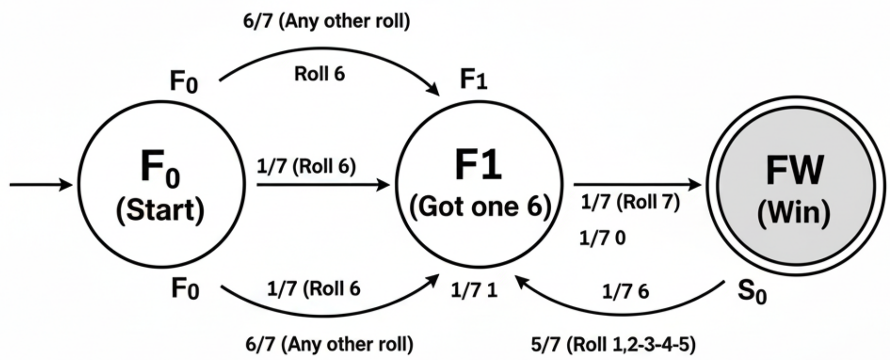

Figure 2: Markov chain for Krrish-3 (Goal: 6-7)

From State 0 (J_0):
This is identical to Krrish's start.
Equation (1): $J_0 = 1 + p \cdot J_1 + (1 - p) \cdot J_0$
From State 1 (J_1):
This is where the game differs. You roll once (cost = 1 roll). If you roll a 7 (prob p), you win. Remaining rolls = 0. If you roll a 6 (prob p), the sequence is "6-6". You still just rolled a 6, so you stay in State 1. Remaining rolls = J_1. If you roll anything else (prob $1 - 2p = 5/7$), the streak is broken. Go back to State 0. Remaining rolls = J_0.
Equation (2): $J_1 = 1 + p \cdot (0) + p \cdot J_1 + (1 - 2p) \cdot J_0$
Solving the system:
Now, we solve this system with $p = 1/7$, $1 - p = 6/7$, and $1 - 2p = 5/7$. From (1):

$$\begin{aligned} J_0 &= 1 + (1/7)J_1 + (6/7)J_0 \\ (1/7)J_0 &= 1 + (1/7)J_1 \\ J_0 &= 7 + J_1 \implies J_1 = J_0 - 7 \end{aligned}$$

From (2):

$$\begin{aligned} J_1 &= 1 + (1/7)J_1 + (5/7)J_0 \\ J_1 - (1/7)J_1 &= 1 + (5/7)J_0 \\ (6/7)J_1 &= 1 + (5/7)J_0 \\ 6J_1 &= 7 + 5J_0 \end{aligned}$$

Substitute $J_1 = J_0 - 7$ into this second equation:

$$\begin{aligned} 6(J_0 - 7) &= 7 + 5J_0 \\ 6J_0 - 42 &= 7 + 5J_0 \\ 6J_0 - 5J_0 &= 7 + 42 \\ J_0 &= \mathbf{49} \end{aligned}$$

The expected number of rolls for Krrish-3 is **49**.

Conclusion

- Krrish (6-6): Expected rolls = **56**
- Krrish-3 (6-7): Expected rolls = **49**

Since $49 < 56$, Krrish-3 is expected to finish his game in fewer rolls than Krrish. **Therefore, Krrish-3 is more likely to finish first.**

Q4: Prove that a state i in a Markov chain is *recurrent* (i.e., $f_{ii} = 1$) if and only if

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty.$$

Explain the intuition behind this criterion using the concept of expected visits.

Solution:

Let $N_i = \sum_{n=1}^{\infty} \mathbb{I}(X_n = i)$ be the total number of visits to state i after time 0. We compute its expectation:

$$E[N_i \mid X_0 = i] = E\left[\sum_{n=1}^{\infty} \mathbb{I}(X_n = i) \mid X_0 = i\right] = \sum_{n=1}^{\infty} E[\mathbb{I}(X_n = i) \mid X_0 = i] = \sum_{n=1}^{\infty} P_{ii}^{(n)}.$$

Alternative View: Geometric Trials. Every time the chain visits i, it "tries" to return again. If it returns (probability f_{ii}), it gets another visit; if not (probability $1 - f_{ii}$), it stops forever.

Hence:

$$P(N_i = k \mid X_0 = i) = (f_{ii})^k (1 - f_{ii}), \qquad k \geq 0.$$

Case 1: Transient ($f_{ii} < 1$). Then N_i is a proper geometric variable with finite mean:

$$E[N_i \mid X_0 = i] = \frac{f_{ii}}{1 - f_{ii}} < \infty.$$

Thus, $\sum_n P_{ii}^{(n)} < \infty$.

Case 2: Recurrent ($f_{ii} = 1$). Now $P(N_i = \infty \mid X_0 = i) = 1$. The chain visits i infinitely often, so

$$E[N_i \mid X_0 = i] = \infty.$$

Hence $\sum_n P_{ii}^{(n)} = \infty$.

Conclusion.

$$\boxed{\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty \quad \iff \quad f_{ii} = 1.}$$

Q5: A spider has a web with 4 nodes: $1, 2, 3$, and a "Trap" state T. T is *absorbing* (once there, it never leaves).

- From 1: goes to 2 or T with equal probability.
- From 2: goes to 1, 3, or T with equal probability.
- From 3: goes to 2 or T with equal probability.

Set up the system of equations to find $\mu_i = \mathbb{E}[\text{time to reach } T \mid X_0 = i]$ for each non-absorbing state. Which state has the longest expected survival time, and why?

Solution:

For each non-absorbing state i, where p_{ij} denotes the transition probability from state i to state j:

$$\mu_i = 1 + \sum_j p_{ij} \mu_j,$$

where the +1 accounts for the one step taken before moving.

Because T is absorbing, $\mu_T = 0$.

From state 1:

$$\mu_1 = 1 + \tfrac{1}{2}\mu_2 + \tfrac{1}{2}\mu_T = 1 + \tfrac{1}{2}\mu_2.$$

From state 2:

$$\mu_2 = 1 + \tfrac{1}{3}\mu_1 + \tfrac{1}{3}\mu_3 + \tfrac{1}{3}\mu_T = 1 + \tfrac{1}{3}(\mu_1 + \mu_3).$$

From state 3:

$$\mu_3 = 1 + \tfrac{1}{2}\mu_2 + \tfrac{1}{2}\mu_T = 1 + \tfrac{1}{2}\mu_2.$$

Final System:

$$\boxed{\begin{aligned} \mu_1 &= 1 + \tfrac{1}{2}\mu_2, \\ \mu_2 &= 1 + \tfrac{1}{3}(\mu_1 + \mu_3), \\ \mu_3 &= 1 + \tfrac{1}{2}\mu_2. \end{aligned}}$$

Intuition: Node 2 has a lower probability of going directly to T (1/3) compared to nodes 1 and 3 (each 1/2). This means node 2 delays absorption more often, having more opportunities to wander between states before being trapped. Hence,

$$\boxed{\mu_2 \text{ is the longest (node 2 has the longest expected survival time).}}$$

Q6: A random walk on the non-negative integers $\{0, 1, 2, \dots\}$ has transitions:

- From 0: always goes to 1.
- From $i > 0$: goes to 0 with probability $p_i = (1/2)^i$ (a "catastrophic fall"), or to $i + 1$ with probability $1 - p_i$.

Part (a): Will the chain eventually return to 0 with certainty? Is this chain recurrent or transient?

Part (b): Write the recursive equation for $E[T_0 \mid X_0 = 1]$, where T_0 is the first return time to state 0.

Part (c): If instead $p_i = 1/i$, how does the classification change?

Solution:

Part (a): The probability of climbing forever without falling is

$$\prod_{i=1}^{\infty} (1 - (1/2)^i),$$

which converges to a *non-zero constant*. Hence, there is a positive probability of climbing forever.

Conclusion: The chain is *not recurrent* — it is **transient**.

Part (b): Using first-step analysis:

$$E[T_0 \mid X_0 = 1] = 1 + (1 - p_1)E[T_0 \mid X_0 = 2] + p_1 \cdot 0.$$

Similarly, for $i \geq 2$:

$$E[T_0 \mid X_0 = i] = 1 + (1 - p_i)E[T_0 \mid X_0 = i + 1].$$

The solution to this system of equations is unbounded because the chain can climb arbitrarily high with nonzero probability before falling.

Part (c): Now, the fall probability decreases slowly ($1/i$). The chance of climbing forever becomes

$$\prod_{i=2}^{\infty}(1 - 1/i) = 0,$$

so a fall is guaranteed eventually.

Moreover, the expected time to fall becomes finite because the tail probability of large climbs decreases sufficiently fast.

Conclusion:

$$\boxed{p_i = (1/2)^i \Rightarrow \text{transient}, \quad p_i = 1/i \Rightarrow \text{positive recurrent.}}$$

Q7: Consider an irreducible Markov chain M_1 with transition matrix P on a finite state space $\mathcal{S}$. It has a unique stationary distribution π.

We define the *lazy* version M_2 with transition matrix

$$Q = \tfrac{1}{2}P + \tfrac{1}{2}I.$$

At each step, the chain flips a fair coin: with heads, it follows P; with tails, it stays put.

Does the lazy chain M_2 have a different stationary distribution than M_1?

Solution:

For M_1, we know $\pi P = \pi$. Compute:

$$\pi Q = \pi\big(\tfrac{1}{2}P + \tfrac{1}{2}I\big) = \tfrac{1}{2}(\pi P) + \tfrac{1}{2}(\pi I) = \tfrac{1}{2}\pi + \tfrac{1}{2}\pi = \pi.$$

Thus, π is stationary for Q.

Since Q is still irreducible (all transitions remain possible), the stationary distribution is unique. Therefore,

$$\boxed{\text{The lazy chain has the same stationary distribution } \pi.}$$

Remark: Adding laziness slows down convergence and removes periodicity but does not alter the stationary distribution.

Q8: A gambler starts with \$50 in a fair casino with the following rules:

If wealth is even, bet \$1. If wealth is odd, bet \$10.

The game ends when wealth reaches \$0 (broke) or \$100 (victory). These are absorbing states.

What is the exact probability of reaching \$100 before \$0, despite the state-dependent betting rule?

Solution:

Let X_n be the gambler's wealth after n rounds. Then

$$E[X_{n+1} \mid X_n] = X_n$$

(since every bet is fair). Hence, $\{X_n\}$ is a martingale.

Let T be the stopping time when the gambler first hits 0 or 100.

By the Optional Stopping Theorem (since wealth is bounded between 0 and 100),

$$E[X_T] = E[X_0] = 50.$$

Let $p = P(\text{reach 100 before 0})$. Then $E[X_T] = 100p + 0(1-p) = 100p$. Equating:

$$100p = 50 \quad \Rightarrow \quad \boxed{p = \tfrac{1}{2}.}$$

Conclusion: Even though the betting pattern changes with wealth, fairness ensures the expected wealth remains constant. Hence, the probability of reaching \$100 first is purely linear in starting wealth:

$$p = \frac{50}{100} = \frac{1}{2}.$$

Q9: Consider the following properties of accessibility and communication in Markov chains:

(a) a. Prove that if $i \to j$ and $j \to k$, then $i \to k$.

(b) b. Using the result of part (a), prove that the "communicate" relation is an equivalence relation. That is, show it is reflexive, symmetric, and transitive.

Solution:

(a) By definition, $i \to j$ means there exists an integer $n_1 \geq 0$ such that $P^{n_1}(i,j) > 0$, and $j \to k$ means there exists an integer $n_2 \geq 0$ such that $P^{n_2}(j,k) > 0$. Then, by the Chapman–Kolmogorov equation,

$$P^{n_1+n_2}(i,k) = \sum_r P^{n_1}(i,r)P^{n_2}(r,k) \geq P^{n_1}(i,j)P^{n_2}(j,k) > 0.$$

Hence, $i \to k$. □

(b) Define $i \leftrightarrow j$ ("i communicates with j") if $i \to j$ and $j \to i$.

- **Reflexive:** For any state i, $P^0(i,i) = 1 > 0$, so $i \to i$.
- **Symmetric:** If $i \leftrightarrow j$, then by definition both $i \to j$ and $j \to i$ hold, so symmetry follows immediately.
- **Transitive:** If $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \to j$ and $j \to k$ imply $i \to k$ by part (a), and similarly $k \to j \to i$ implies $k \to i$. Hence $i \leftrightarrow k$.

Therefore, the communication relation is reflexive, symmetric, and transitive — i.e., an **equivalence relation**.

Q10: Let $\{\varepsilon_n, n \geq 0\}$ be independent random variables with

$$\mathbb{P}(\varepsilon_n = 1) = p, \qquad \mathbb{P}(\varepsilon_n = -1) = 1 - p,$$

and define

$$X_n = \varepsilon_{n+1}\varepsilon_n, \qquad n \geq 0.$$

For which values of p is the sequence $\{X_n\}$ a Markov chain?

Proof.

The Markov property requires that

$$\mathbb{P}(X_{n+1} \mid X_n, X_{n-1}, \ldots, X_0) = \mathbb{P}(X_{n+1} \mid X_n),$$

for all n.

Note that $X_{n+1} = \varepsilon_{n+2}\varepsilon_{n+1}$. Since ε_{n+2} is *independent of the entire past*, the only relevant dependence comes from ε_{n+1}. Hence, to check the Markov property, it suffices to see whether the conditional law of ε_{n+1} given the past depends only on X_n.

$$\mathbb{P}(\varepsilon_{n+1} = 1 \mid X_n = 1) = \frac{\mathbb{P}(\varepsilon_n = 1, \varepsilon_{n+1} = 1)}{\mathbb{P}(X_n = 1)} = \frac{p^2}{p^2 + (1-p)^2}.$$

However, if we condition on an additional past value,

$$\mathbb{P}(\varepsilon_{n+1} = 1 \mid X_n = 1, X_{n-1} = 1) = \frac{\mathbb{P}(\varepsilon_{n-1} = \varepsilon_n = \varepsilon_{n+1} = 1)}{\mathbb{P}(X_{n-1} = 1, X_n = 1)} = \frac{p^3}{p^3 + (1-p)^3}.$$

For $\{X_n\}$ to be Markov, these two conditional probabilities must coincide:

$$\frac{p^2}{p^2 + (1-p)^2} = \frac{p^3}{p^3 + (1-p)^3}.$$

This equality holds only for $p \in \{0, 1, \frac{1}{2}\}$.

We have shown that the only possible values of p are 0,1,0.5. We now proceed to show that these indeed lead to markov property.

- If $p = 0$ or $p = 1$, all ε_n are constant, so $X_n \equiv 1$; the process is trivially Markov.
- If $p = \frac{1}{2}$, the ε_n are i.i.d. symmetric. Then $\mathbb{P}(\varepsilon_{n+1} = 1 \mid X_n) = \frac{1}{2}$, so

  $$\mathbb{P}(X_{n+1} = 1 \mid X_n) = \mathbb{P}(\varepsilon_{n+2} = \varepsilon_{n+1} \mid X_n) = \tfrac{1}{2},$$

  which depends only on X_n. Hence the process is Markov (in fact, i.i.d.).

$$\boxed{\text{Therefore, } \{X_n\} \text{ is Markov iff } p \in \{0, 1, \tfrac{1}{2}\}.}$$

Q11: Consider an $N \times N$ grid G where each cell (i, j) initially contains a distinct label

$$G_{i,j}(0) = iN + j.$$

At each step, perform the following random operation:

> Select uniformly at random one of the $2N$ lines (the N rows or N columns), and then shuffle the entries within the selected line uniformly at random.

Let $G_{r,c}(k)$ denote the label in cell (r, c) after k operations.

Devise an algorithm to compute the probability

$$\mathbb{P}\big(G_{r,c}(K) = rN + c\big)$$

that the label returns to its original position after K operations. Can it be made to run faster than $O(K)$ time.

Solution: Will be updated soon :)