Probability and Statistics

Tutorial 12

Q1: (Discrete Bayes) A factory produces "trick" coins. Your prior belief is that a coin is either a "Type A" (with $p(H) = 0.25$) or a "Type B" (with $p(H) = 0.75$). You believe $P(\text{Type A}) = 0.5$ and $P(\text{Type B}) = 0.5$. You pick one coin, flip it 3 times, and observe 2 Heads and 1 Tail (HHT).

a. What is the posterior probability that the coin was "Type A"?

b. What is the Maximum a Posteriori (MAP) estimate for $p(H)$?

Q2: (Continuous Bayes) Let the prior for a parameter p be the uniform distribution $p \sim U[0,1]$. We observe a single data point $y = 3$ from a Geometric distribution, which has the likelihood function $P(y \mid p) = p(1-p)^{y-1}$. Find the full posterior density function $f(p \mid y = 3)$.

Q3: (Gaussian-Gaussian Conjugate) A machine produces rods with length μ. The measurement error is $\sigma^2 = 1$, so a single measurement $x \sim N(\mu, \sigma^2 = 1)$. Your prior belief for the unknown mean μ is $N(\mu_0 = 10, \sigma_0^2 = 1)$. You take one measurement and get $x_1 = 12$. What is the posterior distribution for μ?

Q4: (Gamma-Poisson Conjugate) The number of cars arriving at a toll booth in an hour follows a Poisson(λ) distribution. Your prior belief for the unknown rate λ is $Gamma(\alpha = 2, \beta = 1)$. You observe the arrivals for $n = 3$ hours and count $x_1 = 2, x_2 = 3, x_3 = 4$ cars.

a. What is the posterior distribution for λ?

b. Find the Maximum a Posteriori (MAP) estimate $\hat{\lambda}_{MAP}$.

c. Find the Conditional Expectation (CE) estimate $\hat{\lambda}_{CE}$.

Q5: Suppose $D = \{x_1, \ldots, x_n\}$ is a data set consisting of independent samples of a Bernoulli random variable with unknown parameter θ, i.e., $f(x_i|\theta) = \theta^{x_i}(1-\theta)^{1-x_i}$ for $x_i \in \{0, 1\}$. We are also given that $\theta \sim U[0,1]$. Obtain an expression for the posterior distribution on θ. Using this, obtain $\hat{\theta}_{MAP}$ and the conditional expectation estimator $\hat{\theta}_{CE}$.

(Hint: $\int_0^1 \theta^m (1-\theta)^r d\theta = \frac{m!r!}{(m+r+1)!}$)

Q6: Let X be a continuous random variable with the following PDF:

$$f_X(x) = \begin{cases} 3x^2 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Also, suppose that $Y|X = x \sim Geometric(x)$. Find the MAP estimate of X given $Y = 5$.