Probability and Statistics: MA6.101

Tutorial 9

Topics Covered: Random Vectors

Q1: Let $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ be a 2-dimensional random vector. The components X_1 and X_2 are independent random variables with the following properties:

- $E[X_1] = 2$, $Var(X_1) = 4$
- $E[X_2] = 5$, $Var(X_2) = 1$

Now, consider a new random vector Y defined by the linear transformation $Y = AX$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$$

Find the mean vector of Y and the covariance matrix of Y.

Q2: Let X_1 be a uniform random variable with support $R_{X_1} = [1, 2]$ and probability density function

$$f_{X_1}(x_1) = \begin{cases} 1 & \text{if } x_1 \in R_{X_1} \\ 0 & \text{if } x_1 \notin R_{X_1} \end{cases}$$

Let X_2 be a continuous random variable, independent of X_1, with support $R_{X_2} = [0, 2]$ and probability density function

$$f_{X_2}(x_2) = \begin{cases} \frac{3}{8}x_2^2 & \text{if } x_2 \in R_{X_2} \\ 0 & \text{if } x_2 \notin R_{X_2} \end{cases}$$

Let

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1^2 \\ X_1 + X_2 \end{bmatrix}$$

Find the joint probability density function of the random vector $\mathbf{Y}$.

Q3: Let X and Y be said to be bivariate normal if $aX + bY$ is normal for all a and b. If X and Y are bivariate normal with 0 mean, variance of 1, and ρ correlation, derive their joint PDF:

$$f(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(x^2 - 2\rho xy + y^2\right)\right)$$

Q4: Suppose $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Now consider $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$. Then we have

$$\mathbf{Y} \sim \mathcal{N}(\mathbf{A}E[\mathbf{X}] + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^T).$$

Prove this using Moment-Generating Functions (MGFs).

Q5: Let $\mathbf{X} = [X_1, X_2]^T$ be a bivariate normal random vector with mean $\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and covariance $\Sigma = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$. Find a constant a such that the random variables $Y_1 = X_1 + aX_2$ and $Y_2 = X_2$ are independent.

Q6: Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ belong to a bivariate normal distribution $\mathcal{N}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\right)$. Show that $x_1 | x_2 \sim \mathcal{N}(\mu_{1|2}, \Sigma_{1|2})$ where

$$\mu_{1|2} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)$$
$$\Sigma_{1|2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

Q7: Let $\mathbf{X} \in \mathbb{R}^n$ be a random vector with mean $\mu = E[\mathbf{X}]$ and covariance matrix $\Sigma = \mathrm{Cov}(\mathbf{X})$. Let $a, b \in \mathbb{R}^n$ be fixed (non-random) vectors.

(a) Show that

$$\mathrm{Var}(a^T\mathbf{X}) = a^T\Sigma a, \qquad \text{and} \qquad \mathrm{Cov}(a^T\mathbf{X},\, b^T\mathbf{X}) = a^T\Sigma b.$$

(b) Using

$$\Sigma = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}, \quad a = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

compute both $\mathrm{Var}(a^T\mathbf{X})$ and $\mathrm{Cov}(a^T\mathbf{X},\, b^T\mathbf{X})$ numerically.

Q8: Let $(\mathbf{X}, \mathbf{Y})$ be a jointly distributed random vector in $\mathbb{R}^{n+m}$, where

$$\mathbf{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \in \mathbb{R}^n, \qquad \mathbf{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_m \end{bmatrix} \in \mathbb{R}^m,$$

and assume all components have finite second moments. Define the conditional mean and conditional covariance of $\mathbf{Y}$ given $\mathbf{X}$ as

$$\boldsymbol{\mu}_{\mathbf{Y}|\mathbf{X}} := E[\mathbf{Y} \mid \mathbf{X}], \qquad \boldsymbol{\Sigma}_{\mathbf{Y}|\mathbf{X}} := Cov(\mathbf{Y} \mid \mathbf{X}) = E\big[(\mathbf{Y} - \boldsymbol{\mu}_{\mathbf{Y}|\mathbf{X}})(\mathbf{Y} - \boldsymbol{\mu}_{\mathbf{Y}|\mathbf{X}})^T \mid \mathbf{X}\big].$$

(a) Show that the unconditional mean of $\mathbf{Y}$ can be expressed as

$$E[\mathbf{Y}] = E[\boldsymbol{\mu}_{\mathbf{Y}|\mathbf{X}}] = E[E[\mathbf{Y} \mid \mathbf{X}]].$$

(b) Prove that the covariance matrix of $\mathbf{Y}$ satisfies

$$Cov(\mathbf{Y}) = E[\boldsymbol{\Sigma}_{\mathbf{Y}|\mathbf{X}}] + Cov(\boldsymbol{\mu}_{\mathbf{Y}|\mathbf{X}}).$$

(c) Suppose $\mathbf{Y}$ is given by the linear model

$$\mathbf{Y} = A\mathbf{X} + \mathbf{Z},$$

where A is an $(m \times n)$ constant matrix, and $\mathbf{X}$ and $\mathbf{Z}$ are independent random vectors satisfying

$$E[\mathbf{X}] = \mathbf{0}, \quad Cov(\mathbf{X}) = \boldsymbol{\Sigma}_{\mathbf{X}}, \qquad E[\mathbf{Z}] = \mathbf{0}, \quad Cov(\mathbf{Z}) = \boldsymbol{\Sigma}_{\mathbf{Z}}.$$

Using parts (a) and (b), derive expressions for $E[\mathbf{Y}]$ and $Cov(\mathbf{Y})$. Then compute $Cov(\mathbf{Y})$ numerically for

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad \boldsymbol{\Sigma}_{\mathbf{X}} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}, \qquad \boldsymbol{\Sigma}_{\mathbf{Z}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$