MA6.101: Probability and Statistics

Mid-semester M25

Section A

1 Question 1 Solution

Question: Consider the following game with a fair die. You repeatedly throw a fair die until you get a 6. The game ends when 6 appears. The reward for each roll is the face value, except that a roll of 6 yields a reward of 0. Find the expected total reward from the game.

1.1 Law of Total Expectations

Let Y be the total reward from the game. We can express Y as a random sum of rewards:

$$Y = R_1 + R_2 + \cdots + R_N$$

where R_k is the reward on the k^{th} roll, and N is the (random) trial on which the first 6 appears.

By the law of total expectation, we can condition on N:

$$\mathbb{E}[Y] = \sum_{n=1}^{\infty} \mathbb{E}[Y \mid N = n] \cdot p_N(n)$$

The event $N = n$ means the first $n-1$ rolls are from $\{1, 2, 3, 4, 5\}$, and the n^{th} roll is a 6. Hence

$$p_N(n) = \left(\tfrac{5}{6}\right)^{n-1} \cdot \tfrac{1}{6}$$

If $N = n$, then the first $n-1$ rolls are i.i.d. from $\{1, 2, 3, 4, 5\}$ and the n^{th} roll contributes 0. Thus,

$$\mathbb{E}[Y \mid N = n] = (n-1) \cdot \mathbb{E}[\text{roll} \mid \text{not a } 6]$$

Since

$$\mathbb{E}[\text{roll} \mid \text{not a } 6] = \frac{1+2+3+4+5}{5} = 3$$

we have

$$\mathbb{E}[Y \mid N = n] = 3(n-1)$$

Therefore,

$$\mathbb{E}[Y] = \sum_{n=1}^{\infty} 3(n-1) \left(\tfrac{5}{6}\right)^{n-1} \tfrac{1}{6}$$

Factor constants:

$$\mathbb{E}[Y] = \frac{3}{6}\sum_{n=1}^{\infty}(n-1)\left(\tfrac{5}{6}\right)^{n-1}$$

Letting $m = n - 1$, this becomes

$$\mathbb{E}[Y] = \frac{1}{2}\sum_{m=0}^{\infty} m\left(\tfrac{5}{6}\right)^{m}$$

Using the formula

$$\sum_{m=0}^{\infty} mr^m = \frac{r}{(1-r)^2}, \quad |r| < 1$$

with $r = \frac{5}{6}$, we get

$$\sum_{m=0}^{\infty} m\left(\tfrac{5}{6}\right)^{m} = \frac{\frac{5}{6}}{(1-\frac{5}{6})^2} = \frac{\frac{5}{6}}{(\frac{1}{6})^2} = \frac{5}{6}\cdot 36 = 30$$

Thus,

$$\mathbb{E}[Y] = \frac{1}{2}\cdot 30 = 15$$

1.2 Law of Iterated Expectations

Let Y be the total reward and N be the random variable for the number of rolls until a 6 appears. The Law of Iterated Expectations states that $\mathbb{E}[Y] = \mathbb{E}_N[\mathbb{E}_Y[Y|N]]$.

Given that the game takes exactly n rolls ($N = n$), the first $n-1$ rolls were not a 6, and the n^{th} roll was a 6. The reward for the final roll is 0. The expected reward for any of the first $n-1$ rolls is the average of the non-6 outcomes:

$$\mathbb{E}[\text{reward of a non-6 roll}] = \frac{1+2+3+4+5}{5} = 3$$

By linearity of expectation, the total expected reward given $N = n$ is:

$$\mathbb{E}[Y|N=n] = (n-1)\times 3 = 3(n-1)$$

This implies that the random variable $\mathbb{E}_Y[Y|N]$ is equal to $3(N-1)$.

Now we take the expectation over all possible values of N:

$$\begin{aligned}\mathbb{E}[Y] &= \mathbb{E}_N[\mathbb{E}_Y[Y|N]]\\ &= \mathbb{E}_N[3(N-1)]\\ &= 3(\mathbb{E}[N]-1)\end{aligned}$$

The variable N follows a Geometric distribution with success probability $p = \frac{1}{6}$. The expectation of N is:

$$\mathbb{E}[N] = \frac{1}{p} = \frac{1}{1/6} = 6$$

Substituting this value back into our equation:

$$\mathbb{E}[Y] = 3(6-1) = 3(5) = 15$$

Thus, the expected total reward is 15.

2 Question 2 Solution

Question: Suppose X and Y are independent and identically distributed (i.i.d.) Uniform$[0, 1]$ random variables. Prove that $P(X < Y) = 0.5$.

2.1 Proof using Law of Total Probability and CDF

The Probability Density Function (PDF) and Cumulative Distribution Function (CDF) for a Uniform$[0, 1]$ random variable, Z.

$$f_Z(z) = \begin{cases} 1 & \text{if } 0 \leq z \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F_Z(z) = P(Z \leq z) = \begin{cases} 0 & \text{if } z < 0 \\ z & \text{if } 0 \leq z \leq 1 \\ 1 & \text{if } z > 1 \end{cases}$$

We can find $P(X < Y)$ by conditioning on Y and using the Law of Total Probability:

$$\begin{aligned}
P(X < Y) &= \int_{-\infty}^{\infty} P(X < Y|Y = y) f_Y(y)\, dy \\
&= \int_0^1 P(X < y|Y = y) \cdot 1\, dy \\
&= \int_0^1 P(X < y) \cdot 1\, dy \quad \text{(Independence)} \\
&= \int_0^1 F_X(y)\, dy \quad \text{(By definition of CDF)} \\
&= \int_0^1 y\, dy \quad \text{(Since } F_X(y) = y \text{ for } y \in [0, 1]\text{)} \\
&= \left[\frac{y^2}{2}\right]_0^1 \\
&= \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}
\end{aligned}$$

3 Question 3 Solution

Question: Let X, Y, Z be independent exponential random variables with parameters $\lambda_1, \lambda_2, \lambda_3$. Let $W = \min(X, Y, Z)$. Find the CDF and PDF of W.

3.1 Solution

Fix $w \in \mathbb{R}$. By definition,

$$F_W(w) = P(W \leq w).$$

Since W is the minimum of X, Y, and Z,

$$P(W > w) = P(X > w, Y > w, Z > w).$$

But

$$P(W > w) = 1 - P(W \leq w)$$

.

Therefore,

$$F_W(w) = P(W \leq w) = 1 - P(X > w, Y > w, Z > w)$$

Since X, Y, and Z are independent, we can rewrite this as

$$F_W(w) = 1 - P(X > w)P(Y > w)P(Z > w)$$

Now, for an exponential random variable with rate λ_i we have for any $w \in \mathbb{R}$

$$F_{X_i}(w) = P(X_i \leq w) = \begin{cases} 1 - e^{-\lambda_i w}, & w \geq 0, \\ 0, & w < 0, \end{cases}$$

so

$$P(X_i > w) = 1 - F_{X_i}(w) = \begin{cases} e^{-\lambda_i w}, & w \geq 0, \\ 1, & w < 0. \end{cases}$$

Substitute these into the product appearing above. For $w < 0$ we have $P(X > w) = P(Y > w) = P(Z > w) = 1$, hence

$$P(X > w)P(Y > w)P(Z > w) = 1 \quad \text{for } w < 0,$$

and therefore $F_W(w) = 0$ for $w < 0$.

For $w \geq 0$ we get

$$\begin{aligned} P(X > w)P(Y > w)P(Z > w) &= e^{-\lambda_1 w} \cdot e^{-\lambda_2 w} \cdot e^{-\lambda_3 w} \\ &= e^{-(\lambda_1+\lambda_2+\lambda_3)w}. \end{aligned}$$

Thus, for $w \geq 0$,

$$F_W(w) = 1 - P(X > w)P(Y > w)P(Z > w) = 1 - e^{-(\lambda_1+\lambda_2+\lambda_3)w}.$$

Combining the two regions of w, the full CDF of W is

$$F_W(w) = \begin{cases} 0, & w < 0, \\ 1 - e^{-(\lambda_1+\lambda_2+\lambda_3)w}, & w \geq 0. \end{cases}$$

Differentiate $F_W(w)$ for $w > 0$ to obtain the PDF:

$$f_W(w) = \frac{d}{dw} F_W(w) = (\lambda_1 + \lambda_2 + \lambda_3)\, e^{-(\lambda_1+\lambda_2+\lambda_3)w}, \qquad w \geq 0,$$

and $f_W(w) = 0$ for $w < 0$.

So,

$$f_W(w) = \begin{cases} 0, & w < 0, \\ (\lambda_1 + \lambda_2 + \lambda_3)\, e^{-(\lambda_1+\lambda_2+\lambda_3)w}, & w \geq 0. \end{cases}$$

Hence W is exponentially distributed with rate parameter $\lambda_1 + \lambda_2 + \lambda_3$:

$$W \sim \mathrm{Exp}(\lambda_1 + \lambda_2 + \lambda_3).$$

4 Question 4 Solution

Question: Show that for any two random variables X and Y,

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\ \mathrm{Cov}(X,Y).$$

4.1 Solution 1: Using Definition of Variance

By definition,

$$\mathrm{Var}(X+Y) = \mathbb{E}\left[(X+Y-\mathbb{E}[X+Y])^2\right].$$

Now, $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$. Substituting,

$$\mathrm{Var}(X+Y) = \mathbb{E}\left[((X-\mathbb{E}[X]) + (Y-\mathbb{E}[Y]))^2\right].$$

Expand the square:

$$= \mathbb{E}\left[(X-\mathbb{E}[X])^2\right] + \mathbb{E}\left[(Y-\mathbb{E}[Y])^2\right] + 2\,\mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])].$$

By definition,

$$\mathrm{Var}(X) = \mathbb{E}[(X-\mathbb{E}[X])^2], \quad \mathrm{Var}(Y) = \mathbb{E}[(Y-\mathbb{E}[Y])^2], \quad \mathrm{Cov}(X,Y) = \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])].$$

Hence,

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\ \mathrm{Cov}(X,Y).$$

4.2 Solution 2:

Recall the identity:

$$\mathrm{Var}(Z) = \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2.$$

Let $Z = X + Y$. Then,

$$\mathrm{Var}(X+Y) = \mathbb{E}[(X+Y)^2] - (\mathbb{E}[X+Y])^2.$$

Expand each term:

$$\mathbb{E}[(X+Y)^2] = \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2],$$

$$(\mathbb{E}[X+Y])^2 = (\mathbb{E}[X] + \mathbb{E}[Y])^2 = (\mathbb{E}[X])^2 + 2\mathbb{E}[X]\mathbb{E}[Y] + (\mathbb{E}[Y])^2.$$

So,

$$\text{Var}(X+Y) = (\mathbb{E}[X^2] - (\mathbb{E}[X])^2) + (\mathbb{E}[Y^2] - (\mathbb{E}[Y])^2) + 2(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]).$$

Which simplifies to:

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\ \text{Cov}(X, Y).$$

5 Question 5 Solution

Question: Consider a Gaussian random variable X with mean μ and variance σ^2. Let $Z = aX + b$ where $a, b \in \mathbb{R}$. Derive an expression for the probability density of Z and show that Z is also a Gaussian random variable. What is the mean and variance of Z?

5.1 Method 1

To find the PDF:

$$\begin{aligned}
&Z = g(X) = aX + b \\
&\implies f_Z(z) = \frac{1}{\left|g'(\frac{z-b}{a})\right|} f_X(\frac{z-b}{a}) \\
&\implies f_Z(z) = \frac{1}{|a|} f_X(\frac{z-b}{a}) \\
&\implies f_Z(z) = \frac{1}{|a|\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{\frac{z-b}{a} - \mu}{\sigma}\right)^2\right\} \\
&\implies f_Z(z) = \frac{1}{\sqrt{2\pi(a\sigma)^2}} \exp\left\{-\frac{1}{2}\left(\frac{z-(b+a\mu)}{a\sigma}\right)^2\right\}
\end{aligned}$$

From the PDF of Z, we can see that it is a Gaussian distribution with mean $b + a\mu$ and variance $a^2\sigma^2$.

5.2 Method 2

We find the PDF using the CDF:

- Case 1: $a > 0$

$$\begin{aligned}
F_Z(z) &= P(Z \leq z) \\
&= P(aX + b \leq z) \\
&= P\left(X \leq \frac{z-b}{a}\right) \\
&= F_X\left(\frac{z-b}{a}\right)
\end{aligned}$$

To find the PDF, we can take the derivative of F_Z:

$$\begin{aligned}
f_Z(z) &= \frac{d}{dz}F_Z(z) \\
&= \frac{d}{dz}F_X\left(\frac{z-b}{a}\right) \\
&= \frac{1}{a}F_X'\left(\frac{z-b}{a}\right) \\
&= \frac{1}{a}f_X\left(\frac{z-b}{a}\right) \\
&= \frac{1}{a\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2}\left(\frac{\frac{z-b}{a}-\mu}{\sigma}\right)^2\right\} \\
&= \frac{1}{\sqrt{2\pi(a\sigma)^2}}\exp\left\{-\frac{1}{2}\left(\frac{z-(b+a\mu)}{a\sigma}\right)^2\right\}
\end{aligned}$$

- Case 2: $a < 0$

$$\begin{aligned}
F_Z(z) &= P(Z \leq z) \\
&= P(aX + b \leq z) \\
&= P\left(X \geq \frac{z-b}{a}\right) \\
&= 1 - F_X\left(\frac{z-b}{a}\right)
\end{aligned}$$

To find the PDF, we can take the derivative of F_Z:

$$
\begin{aligned}
f_Z(z) &= \frac{d}{dz} F_Z(z) \\
&= -\frac{d}{dz} F_X\left(\frac{z-b}{a}\right) \\
&= -\frac{1}{a} F'_X\left(\frac{z-b}{a}\right) \\
&= \frac{1}{|a|} f_X\left(\frac{z-b}{a}\right) \\
&= \frac{1}{|a|\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{\frac{z-b}{a}-\mu}{\sigma}\right)^2\right\} \\
&= \frac{1}{\sqrt{2\pi(a\sigma)^2}} \exp\left\{-\frac{1}{2}\left(\frac{z-(b+a\mu)}{a\sigma}\right)^2\right\}
\end{aligned}
$$

From the PDF of Z in both cases, we can see that it is a Gaussian distribution with mean $b + a\mu$ and variance $a^2\sigma^2$.

Section B

6 Question 1 Solution

Given : The joint probability density function of two continuous random variables X and Y:

$$f_{X,Y}(x, y) = c(x + y), \quad 0 \le x \le 1, \quad 0 \le y \le 1,$$

and $f_{X,Y}(x, y) = 0$ otherwise.

(a) Find the constant c

For $f_{X,Y}(x, y)$ to be a valid joint probability density function (PDF), its integral over the entire support must be equal to 1.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx\, dy = 1$$

We set up the double integral over the region $0 \le x \le 1, \quad 0 \le y \le 1$:

$$\begin{aligned}
\int_0^1 \int_0^1 c(x+y)\,dx\,dy &= 1 \\
c \int_0^1 \left[\frac{x^2}{2} + yx \right]_{x=0}^{x=1} dy &= 1 \\
c \int_0^1 \left(\left(\frac{1^2}{2} + y \cdot 1 \right) - \left(\frac{0^2}{2} + y \cdot 0 \right) \right) dy &= 1 \\
c \int_0^1 \left(\frac{1}{2} + y \right) dy &= 1 \\
c \left[\frac{1}{2} y + \frac{y^2}{2} \right]_{y=0}^{y=1} &= 1 \\
c \left(\left(\frac{1}{2} \cdot 1 + \frac{1^2}{2} \right) - 0 \right) &= 1 \\
c \left(\frac{1}{2} + \frac{1}{2} \right) &= 1 \\
c \cdot 1 &= 1
\end{aligned}$$

Therefore, the constant $\mathbf{c = 1}$. The joint PDF is $f_{X,Y}(x,y) = x + y$ for $0 \le x \le 1$, $0 \le y \le 1$.

(b) Find the marginal density functions $f_X(x)$ and $f_Y(y)$

To find the marginal density function $f_X(x)$, we integrate the joint PDF with respect to y. For $0 \le x \le 1$:

$$\begin{aligned}
f_X(x) &= \int_0^1 f_{X,Y}(x,y)\,dy = \int_0^1 (x+y)\,dy \\
&= \left[xy + \frac{y^2}{2} \right]_{y=0}^{y=1} \\
&= \left(x \cdot 1 + \frac{1^2}{2} \right) - (0) = x + \frac{1}{2}
\end{aligned}$$

So, the marginal PDF for X is:

$$f_X(x) = \begin{cases} x + \frac{1}{2}, & 0 \le x \le 1, \\ 0, & otherwise. \end{cases}$$

Similarly, to find the marginal density function $f_Y(y)$, we integrate with respect to x. For

$0 \le y \le 1$:

$$\begin{aligned} f_Y(y) &= \int_0^1 f_{X,Y}(x,y)\,dx = \int_0^1 (x+y)\,dx \\ &= \left[\frac{x^2}{2} + yx\right]_{x=0}^{x=1} \\ &= \left(\frac{1^2}{2} + y \cdot 1)\right) - (0) = \frac{1}{2} + y \end{aligned}$$

So, the marginal PDF for Y is:

$$f_Y(y) = \begin{cases} y + \frac{1}{2}, & 0 \le y \le 1, \\ 0, & otherwise. \end{cases}$$

(c) Find the conditional density functions $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$

The conditional density function $f_{X|Y}(x|y)$ is given by:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\mathbf{x+y}}{\mathbf{y+1/2}}$$

for $0 \le x \le 1$ and $0 \le y \le 1$.

The final answer is:

$$f_{X|Y}(x|y) = \begin{cases} \frac{\mathbf{x+y}}{\mathbf{y+1/2}}, & 0 \le x \le 1, \quad 0 \le y \le 1, \\ 0, & otherwise. \end{cases}$$

The conditional density function $f_{Y|X}(y|x)$ is given by:

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\mathbf{x+y}}{\mathbf{x+1/2}}$$

for $0 \le x \le 1$ and $0 \le y \le 1$.

The final answer is:

$$f_{Y|X}(y|x) = \begin{cases} \frac{\mathbf{x+y}}{\mathbf{x+1/2}}, & 0 \le x \le 1, \quad 0 \le y \le 1, \\ 0, & otherwise. \end{cases}$$

(d) Compute the conditional expectation $E[X|Y = y]$

The conditional expectation $E[X|Y = y]$ is calculated as follows:

$$
\begin{aligned}
E[X|Y = y] &= \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\,dx \\
&= \int_0^1 x \left(\frac{x+y}{y+1/2}\right) dx \\
&= \frac{1}{y+1/2}\int_0^1 x(x+y)\,dx \\
&= \frac{1}{y+1/2}\int_0^1 (x^2+xy)\,dx \\
&= \frac{1}{y+1/2}\left[\frac{x^3}{3}+\frac{x^2y}{2}\right]_{x=0}^{x=1} \\
&= \frac{1}{y+1/2}\left(\frac{1}{3}+\frac{y}{2}\right) \\
&= \frac{\frac{2+3y}{6}}{\frac{2y+1}{2}} = \frac{2+3y}{6}\cdot\frac{2}{2y+1} = \frac{2+3y}{3(2y+1)}
\end{aligned}
$$

So, the conditional expectation is $\mathbf{E[X|Y = y] = \frac{2+3y}{3(2y+1)}}$.

(e) Are X and Y independent?

Two random variables X and Y are independent if and only if their joint PDF is the product of their marginal PDFs, i.e., $f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$.

We have:

- Joint PDF: $f_{X,Y}(x, y) = x + y$
- Product of marginals: $f_X(x) \cdot f_Y(y) = \left(x + \frac{1}{2}\right)\left(y + \frac{1}{2}\right) = xy + \frac{x}{2} + \frac{y}{2} + \frac{1}{4}$

Since $x + y \neq xy + \frac{x}{2} + \frac{y}{2} + \frac{1}{4}$, the condition for independence is not met.

Justification: No, **X and Y are not independent**. The joint density function $f_{X,Y}(x, y)$ is not equal to the product of the marginal density functions $f_X(x) \cdot f_Y(y)$.