Theory Assignment I

Discrete Structures Monsoon 2024, IIIT Hyderabad

August 16, 2024

Total Marks: 30 points

Due date: **24/08/24 11:59 pm**

<u>**General Instructions:**</u> All symbols have the usual meanings (example: $\mathbb{R}$ is the set of reals, $\mathbb{N}$ the set of natural numbers, and so on). Every problem is mapped to the following course outcomes: CO 1, CO 2, CO 3, and CO 4.

---

1. [5 points] Consider the following compound proposition:

$$((p \to q) \wedge (q \to r)) \to (p \to r)$$

   1. Construct a truth table for the proposition
   2. Prove that the expression is a tautology by using the following steps:
      (a) Express the proposition using only $\neg, \vee$ and $\wedge$
      (b) Apply De Morgan's laws and distributive properties to simplify the expression

2. [10 points] Consider the following propositional logic formula:

$$\neg(\neg(x_1 \to x_2) \vee \neg(x_1 \to x_3)) \wedge (\neg x_2 \to (\neg(x_4 \to x_3))) \wedge (\neg(\neg x_4 \wedge \neg x_5))$$

   a. Construct a truth table for all possible values of the clauses to find an assignment that satisfies this formula **(4.5 points)**

   b. We have a machine that can construct one row of the truth table in one step. It does this as follows: It generates binary strings by counting up in Binary from 00000 to 11111. A 0 in the i^{th} position represents a False value for the i^{th} propositional variable whereas a 1 represents True. How many steps would be required to reach an assignment of the propositional variables such that the whole formula evaluates to true? What about the worst case? What about a general formula with n propositions? **(1.5 points)**

   c. Now convert the propositional logic formula to Conjunctive Normal Form. Each part of the formula separated by a conjunction is called a clause. **(2 points)**

   d. Now enumerate the steps required for evaluating this new expression with a smarter machine. It generates bit strings in the same order, but any time one of the clauses becomes false, it skips all strings that have the assignment of variables that made the clause become false.
   So for example, if there was a clause of the form $(\neg x_3 \vee x_5)$ which would become false under the assignment "00100", all strings of the form "xx1x0" would be skipped.
   How many steps are required to solve the problem now? (Why does this work?) **(2 points)**

3. [3 points] Answer the following questions

a. Prove the exportation law, that is

$$((A \wedge B) \rightarrow C) \Leftrightarrow (A \rightarrow (B \rightarrow C))$$

b. You are given three assumptions as follows

- $A \vee (B \rightarrow C)$
- $[B \rightarrow (B \rightarrow C)] \rightarrow (D \vee E)$
- $(D \rightarrow A) \wedge (E \rightarrow F)$

Prove that the expression below is true

$$A \vee F$$

You are allowed to use any proof techniques and rules of inference, as long as they are valid. (**Hint :** You might need to use the exportation law).

4. [5 points] Assume your friend took out a corner square from your n $\times$ n chessboard. Is it still possible to use dominoes to cover the remaining squares? (A domino is a rectangular tile that covers two neighboring chessboard squares in a game.) What requirement does n have to meet? Describe the necessary and sufficient conditions (i.e., the precise values of n that work and those that don't). Prove your answer.

5. [3 points] Show that if n is an integer and $n^3 + 5$ is odd, then n is even using

   a a proof by contraposition.

   b a proof by contradiction.

6. [4 points] Let $a_1, a_2, ...a_n \in \mathbb{R}^+$. Prove using mathematical induction that

$$\frac{1}{n} \sum_{i=1}^{n} a_i \geq \left(\prod_{i=1}^{n} a_i \right)^{1/n} .$$