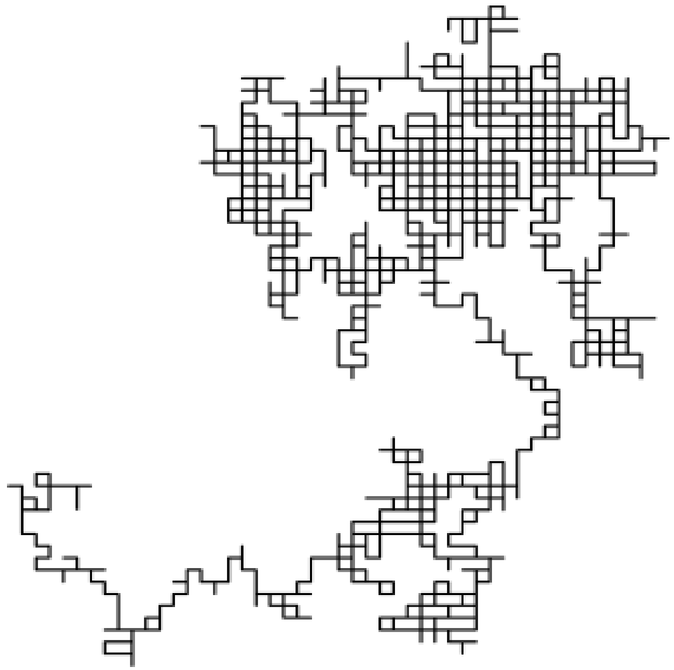
Markov Chains

Introduction to Stochastic processes

- Stochastic process $\{X(t), t \in T\}$ is a collection of random variables defined such that for every $t \in T$ we have $X(t) : \Omega \to \mathcal{S}$.
- These random variables could be dependent and need not have identical distribution.
- T is the parameter space (often resembles time) and $\mathcal{S}$ is the state space.
- When T is countable, we have a discrete time process.
- If T is a subset of real line, we have a continuous time process.
- State space could be integers or real numbers

Examples of Stochastic Processes

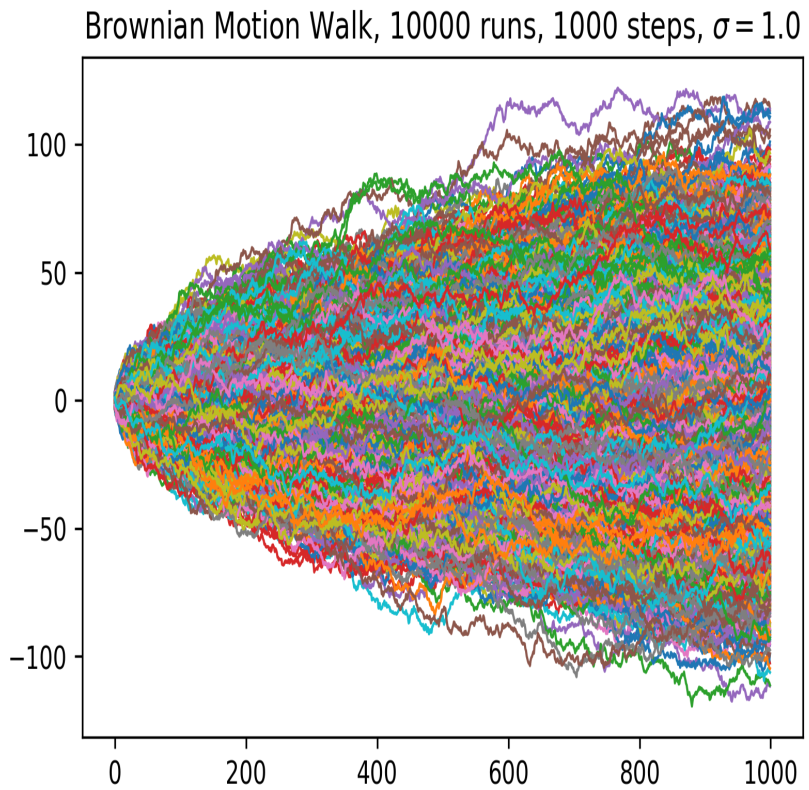
- Sequence $\{X_i\}$ of i.i.d random variables.
- General random walk: If $X_1, X_2, \ldots$ is a sequence i.i.d of random variables, then $S_n = \sum_{i=1}^{n} X_i$ is a random walk.
- 1D Random walks can have positive, negative or no drift depending on the sign of $E[X]$.
- A trajectory of 2D random walk



https://upload.wikimedia.org/wikipedia/commons/f/f3/Random_walk_2500_animated.svg

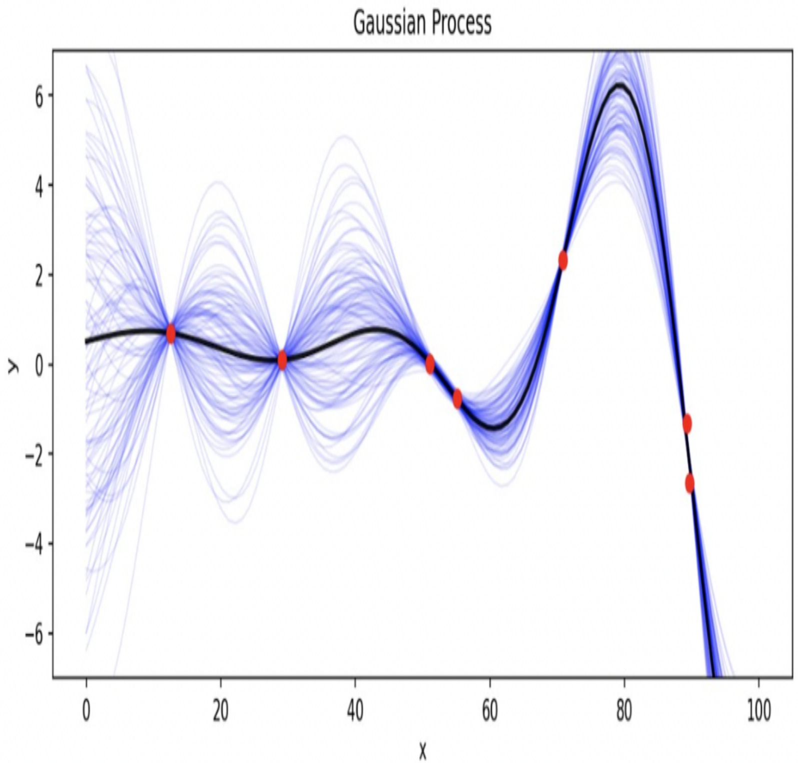
Examples of Stochastic Processes

- Weiner process: $\{X(t), t \geq 0\}$ is a Weiner process if
  1. for every $t > 0$, $X(t) \sim \mathcal{N}(0, t)$.
  2. Often called as Brownian Motion as it was used by Robert Brown to describe motion of particle suspended in liquid.
  3. It is a scaling limit of a random walk (zoomed out BM).
  4. Trajectories are continuous but not differntiable (Financial modeling)
  5. Limit of Functional CLT (CLT for Stochastic processes)



Examples of Stochastic Processes

- Gaussian Process: A continuous time stochastic process $\{X_t, t \in T\}$ is a gausssian process if and only if for any finite set of indices $t_1, \ldots t_k$, $[X_{t_1}, \ldots, X_{t_k}]$ is a multivariate Gaussian vector.



- $\{X_n, n \geq 0\}$ is a martingale if $E[X_{n+1}|X_1, \ldots, X_n)] = X_n$. (Applications in Finance, Optimal Stopping, pricing)

Discrete time Markov Chains (DTMC)

- A stochastic process $\{X_n, n \in \mathbb{Z}_+\}$ is a discrete time Markov chain if for any n we have

$$\boxed{P(X_n = j | X_1 = x_1, .., X_{n-1} = x_{n-1}) = P(X_n = j | X_{n-1} = x_{n-1})}$$

- This is called as the Markov property.
- $P(next\ state | past\ states,\ present\ state) = P(next\ state |\ present\ state)$
- Why Chain? You can view the successive random variables as a chain of states being visited in a sequence and where the next state visited depends only on the current state.
- We will throughout assume that the state space $\mathcal{S}$ is countable.

Running example: Coin with memory!

- In a Markovian coin with memory, the outcome of the next toss depends on the current toss.
- $X_n = 1$ for heads and $X_n = -1$ otherwise. $\mathcal{S} = \{+1, -1\}$.
- Sticky coin : $P(X_{n+1} = 1 | X_n = 1) = 0.9$ and $P(X_{n+1} = -1 | X_n = -1) = 0.8$ for all n.
- Flippy Coin: $P(X_{n+1} = 1 | X_n = 1) = 0.1$ while $P(X_{n+1} = -1 | X_n = -1) = 0.3$ for all n.
- This can be represented by a transition diagram (see board)
- The one step transition probability matrix P for the two cases is $P_s = \begin{bmatrix} 0.9 & .1 \\ 0.2 & 0.8 \end{bmatrix}$ and $P_f = \begin{bmatrix} 0.1 & 0.9 \\ 0.7 & 0.3 \end{bmatrix}$
- The row corresponds to present state and the column corresponds to next state.

Running example: Dice with memory!

- In a markovian dice with memory, the outcome of the next roll depends on the current roll.
- $X_n = i$ for $i \in \mathcal{S}$ where $\mathcal{S} = \{1, \ldots, 6\}$.
- Example one-step transition probability matrix

$$P = \begin{bmatrix} 0.9 & .1 & 0 & 0 & 0 & 0 \\ 0 & .9 & .1 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0.1 \\ 0.1 & 0 & 0 & 0 & 0 & 0.9 \end{bmatrix}$$

- State transition diagram on board
- Consider $S_n = \sum_{i=1}^{n} X_i$ and $\hat{\mu}_n = \frac{S_n}{n}$. What is $\lim_{n\to\infty} \hat{\mu}_n$?
- Cannot invoke SLLN as $\{X_i\}$ are not i.i.d.
- We will see later SLLN for Markov chains!

Finite dimensional distributions

- Consider a Markov dice with transition probability P.
- What is $P(X_0 = 4, X_1 = 5, X_2 = 6)$?
- $= P(X_2 = 6 | X_1 = 5, X_0 = 4) P(X_1 = 5 | X_0 = 4) P(X_0 = 4)$
- $= p_{65} p_{54} P(X_0 = 4)$.
- What is $P(X_0 = 4)$?
- This probability of starting in a particular state is called initial distribution of the markov chain.

Finite dimensional distributions

- Consider a DTMC $\{X_n, n \geq 0\}$ with transition matrix P.
- We assume M states and X_0 denotes the initial state.
- You can start in any starting state or may pick your starting state randomly.
- Let $\bar{\mu} = (\mu_1, \ldots, \mu_M)$ denote the initial distribution, i.e., $P(X_0 = x_0) = \mu_{x_0}$.
- How does one obtain the finite dimensional distribution $P(X_0 = x_0, X_1 = x_1, , X_2 = x_2)$?
- $P(X_0 = x_0, X_1 = x_1, , X_2 = x_2) = p_{x_1,x_2} p_{x_0,x_1} \mu_{x_0}$.
- In general, $P(X_0 = x_0, X_1 = x_1, \ldots X_k = x_k) = p_{x_{k-1},x_k} \times \ldots \times p_{x_0,x_1} \mu_{x_0}$