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Asuman G. Aksoy
Mohamed A. Khamsi

A Problem Book in Real Analysis

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Dedicated to Ercüment G. Aksoy and Anny Morrobel-Sosa

Contents

Preface

Education is an admirable thing, but it is well to remember from time to time that nothing worth knowing can be taught.

Oscar Wilde, "The Critic as Artist," 1890.

Analysis is a profound subject; it is neither easy to understand nor summarize. However, Real Analysis can be discovered by solving problems. This book aims to give independent students the opportunity to discover Real Analysis by themselves through problem solving.

The depth and complexity of the theory of Analysis can be appreciated by taking a glimpse at its developmental history. Although Analysis was conceived in the 17th century during the Scientific Revolution, it has taken nearly two hundred years to establish its theoretical basis. Kepler, Galileo, Descartes, Fermat, Newton and Leibniz were among those who contributed to its genesis. Deep conceptual changes in Analysis were brought about in the 19th century by Cauchy and Weierstrass. Furthermore, modern concepts such as open and closed sets were introduced in the 1900s.

Today nearly every undergraduate mathematics program requires at least one semester of Real Analysis. Often, students consider this course to be the most challenging or even intimidating of all their mathematics major requirements. The primary goal of this book is to alleviate those concerns by systematically solving the problems related to the core concepts of most analysis courses. In doing so, we hope that learning analysis becomes less taxing and thereby more satisfying.

The wide variety of exercises presented in this book range from the computational to the more conceptual and vary in difficulty. They cover the following subjects: Set Theory, Real Numbers, Sequences, Limits of Functions, Continuity, Differentiability, Integration, Series, Metric Spaces, Sequences and Series of Functions and Fundamentals of Topology. Prerequisites for accessing this book are a robust understanding of Calculus and Linear Algebra. While we define the concepts and cite theorems used in each chapter, it is best to use this book alongside standard analysis books such as: *Principles of Mathematical Analysis* by W. Rudin, *Understanding Analysis* by S. Abbott, *Elementary Classical Analysis* by J. E. Marsden and M. J. Hoffman, and *Elements of Real Analysis* by D. A. Sprecher. A list of analysis texts is provided at the end of the book.

Although *A Problem Book in Real Analysis* is intended mainly for undergraduate mathematics students, it can also be used by teachers to enhance their lectures or as an aid in preparing exams. The proper way to use this book is for students to first attempt to solve its problems without looking at solutions. Furthermore, students should try to produce solutions which are different from those presented in this book. It is through the search for a solution that one learns most mathematics.

Knowledge accumulated from many analysis books we have studied in the past has surely influenced the solutions we have given here. Giving proper credit to all the contributors is a difficult task that we have not undertaken; however, they are all appreciated. We also thank Claremont students Aaron J. Arvey, Vincent E. Selhorst-Jones and Martijn van Schaardenburg for their help with LaTeX. The source for the photographs and quotes given at the beginning of each chapter in this book are from the archive at http://www-history.mcs.st-andrews.ac.uk/

Perhaps Oscar Wilde is correct in saying "nothing worth knowing can be taught." Regardless, teachers can show that there are paths to knowledge. This book is intended to reveal such a path to understanding Real Analysis. *A Problem Book in Real Analysis* is not simply a collection of problems; it intends to stimulate its readers to independent thought in discovering Analysis.

Asuman Güven Aksoy
Mohamed Amine Khamsi
May 2009

Chapter 1

Elementary Logic and Set Theory

Reserve your right to think, for even to think wrongly is better than not to think at all.

Hypatia of Alexandria (370–415)

- If x belongs to a class A, we write $x \in A$ and read as "x is an element of A." Otherwise, we write $x \notin A$.
- If A and B are sets, then $A \subseteq B$ ("A is a subset of B" or "A is contained in B") means that each element of A is also an element of B. Sometimes we write $B \supseteq A$ ("B contains A") instead of $A \subseteq B$.
- We say two sets A and B are *equal*, written $A = B$, if $A \subseteq B$ and $B \subseteq A$.
- Any statement S has a *negation* $\sim S$ ("not S") defined by

$$\sim S \text{ is true if } S \text{ is false} \qquad \text{and} \qquad \sim S \text{ is false if } S \text{ is true.}$$

- Let $P(x)$ denote a *property* P of the object x. We write $\exists$ for the quantifier "*there exists.*" The expression

$$\exists\, x \in X \,:\, P(x)$$

means that "there exists (at least) one object x in the class X which has the property P." The symbol $\exists$ is called the *existential quantifier*.

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- We use the symbol $\forall$ for the quantifier "*for all.*" The expression

$$\forall x \in X \,:\, P(x)$$

has the meaning "for each object x in the class X, x has property P." The symbol $\forall$ is called the *universal quantifier* (or sometimes the *general quantifier*).

- We use the symbol $:=$ to mean "*is defined by.*" We take $x := y$ to mean that the object or symbol x is defined by the expression y .

- Note that for negation of a statement we have:

  (i) $\sim\sim A := \sim(\sim A) = A$
  (ii) $\sim(A \text{ and } B) \;=\; (\sim A) \text{ or } (\sim B)$
  (iii) $\sim(A \text{ or } B) \;=\; (\sim A) \text{ and } (\sim B)$
  (iv) $\sim(\forall x \in X \,:\, P(x)) = (\exists x \in X \,:\sim P(x))$
  (v) $\sim(\exists x \in X \,:\, P(x)) = (\forall x \in X \,:\sim P(x))$.

- Let A and B be statements. A *implies* B will be denoted by $A \Rightarrow B$. If A implies B, we take this to mean that if we wish to prove B, it suffices to prove A (A is a sufficient condition for B).

- The equivalence $A \Leftrightarrow B$ ("A and B are equivalent" or "A if and only if B," often written A iff B) of the statements A and B is defined by

$$(A \Leftrightarrow B) := (A \Rightarrow B) \text{ and } (B \Rightarrow A).$$

A is a necessary and sufficient condition for B, or vice versa.

- The statement $\sim B \Rightarrow \sim A$ is called the *contrapositive* of the statement $A \Rightarrow B$. In standard logic practices, any statement is considered equivalent to its contrapositive. It is often easier to prove a statement's contrapositive instead of directly proving the statement itself.

- To prove $A \Rightarrow B$ *by contradiction*, one supposes B is false (that $\sim B$ is true). Then, also assuming that A is true, one reaches a conclusion C which is already known to be false. This contradiction shows that if A is true $\sim B$ cannot be true, and hence B is true if A is true.

- Given two sets A and B, we define $A \cup B$ ("the union of A with B") as the set

$$A \cup B := \{x : x \in A \text{ or } x \in B \text{ or both}\}.$$

When speaking about unions, if we say $x \in A$ or $x \in B$ it also includes the possibility that x is in both A and B.

- We define $A \cap B$ ("the intersection of A with B") as the set

$$A \cap B := \{x : x \in A \text{ and } x \in B\}.$$

- Let A and B be subsets of X. Then
$$A \setminus B := \{x \in X : \ x \in A \text{ and } x \notin B\}$$
is the *relative complement* of B in A. When the set X is clear from the context we write also
$$A^c := X \setminus A$$
and call A^c the *complement* of A.

- If X is a set, then so is its *power set* $\mathcal{P}(X)$. The elements of $\mathcal{P}(X)$ are the subsets of X. Sometimes the power set is written 2^X for a reason which is made clear in Problem 2.8.

- Let $f : X \to Y$ be a function, then
$$im(f) := \{y \in Y ; \ \exists x \in X : y = f(x)\}$$
is called the *image of f*. We say f is *surjective* (or onto) if $im(f) = Y$, *injective* (or one-to-one) if $f(x) = f(y)$ implies $x = y$ for all $x, y \in X$, and f is *bijective* if f is both injective and surjective.

- If X and Y are sets, the *Cartesian product* $X \times Y$ of X and Y is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$.

- Let X be a set and $\mathcal{A} = \{A_i : \ i \in I\}$ be a family of sets and I is an index set. *Intersection and union of this family* are given by
$$\bigcap_{i \in I} A_i = \{x \in X ; \forall i \in I : \ x \in A_i\}$$
and
$$\bigcup_{i \in I} A_i = \{x \in X ; \exists i \in I : \ x \in A_i\}.$$

- Let $f : X \to Y$ be a function, and $A \subset X$ and $B \subset Y$ are subsets. *Image of A under f*, $f(A)$ defined as
$$f(A) = \{f(x) \in Y : \ x \in A\}.$$

- *Inverse image of B under f* (or pre-image of B), $f^{-1}(B)$ defined as
$$f^{-1}(B) = \{x \in X : \ f(x) \in B\}.$$
Note that we can form $f^{-1}(B)$ for a set $B \subset Y$ even though f might not be one-to-one or onto.

- We will use standard notation, $\mathbb{N}$ for the set natural numbers, $\mathbb{Z}$ for the set of integers , $\mathbb{Q}$ for the set rational numbers , and $\mathbb{R}$ for the set real numbers . We have the natural containments:
$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

- Two sets A and B have the same *cardinality* if there is a bijection from A to B. In this case we write $A \sim B$. We say A is *countable* if $\mathbb{N} \sim A$. An infinite set that is not countable is called an uncountable set.

- *Schröder–Bernstein Theorem*: Assume that there exists one-to-one function $f : A \to B$ and another one-to-one function $g : B \to A$. Then there exists a one-to-one, onto function $h : A \to B$ and hence $A \sim B$.

Problem 1.1 Consider the four statements

$$\begin{array}{lll} (a) & \exists x \in \mathbb{R}\ \forall y \in \mathbb{R} & x + y > 0; \\ (b) & \forall x \in \mathbb{R}\ \exists y \in \mathbb{R} & x + y > 0; \\ (c) & \forall x \in \mathbb{R}\ \forall y \in \mathbb{R} & x + y > 0; \\ (d) & \exists x \in \mathbb{R}\ \forall y \in \mathbb{R} & y^2 > x. \end{array}$$

1. Are the statements a, b, c, d true or false?
2. Find their negations.

Problem 1.2 Let $f : \mathbb{R} \to \mathbb{R}$. Find the negations of the following statements:

1. For any $x \in \mathbb{R}$ $f(x) \leq 1$.
2. The function f is increasing.
3. The function f is increasing and positive.
4. There exists $x \in \mathbb{R}^+$ such that $f(x) \leq 0$.
5. There exists $x \in \mathbb{R}$ such that for any $y \in \mathbb{R}$, if $x < y$ then $f(x) > f(y)$.

Problem 1.3 Replace … by the appropriate quantifier: $\Leftrightarrow$,$\Leftarrow$, or $\Rightarrow$.

1. $x \in \mathbb{R}$ $x^2 = 4$ …… $x = 2$;
2. $z \in \mathbb{C}$ $z = \overline{z}$ …… $z \in \mathbb{R}$;
3. $x \in \mathbb{R}$ $x = \pi$ …… $e^{2ix} = 1$.

Problem 1.4 Find the negation of: "Anyone living in Los Angeles who has blue eyes will win the Lottery and will take their retirement before the age of 50."

Problem 1.5 Find the negation of the following statements:

1. Any rectangular triangle has a right angle.
2. In all the stables, the horses are black.
3. For any integer $x \in \mathbb{Z}$, there exists an integer $y \in \mathbb{Z}$ such that, for any $z \in \mathbb{Z}$, the inequality $z < x$ implies $z < x + 1$.
4. $\forall \varepsilon > 0 \ \exists \alpha > 0 \ / \ |x - 7/5| < \alpha \Rightarrow |5x - 7| < \varepsilon$.

Problem 1.6 Show that $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ such that

$$(n \geq N \Rightarrow 2 - \varepsilon < \frac{2n+1}{n+2} < 2 + \varepsilon).$$

Problem 1.7 Let f, g be two functions defined from $\mathbb{R}$ into $\mathbb{R}$. Translate using quantifiers the following statements:

1. f is bounded above;
2. f is bounded;
3. f is even;
4. f is odd;
5. f is never equal to 0;
6. f is periodic;
7. f is increasing;
8. f is strictly increasing;
9. f is not the 0 function;
10. f does not have the same value at two different points;
11. f is less than g;
12. f is not less than g.

Problem 1.8 For two sets A and B show that the following statements are equivalent:

a) $A \subseteq B$

b) $A \cup B = B$

c) $A \cap B = A$

Problem 1.9 Establish the following set theoretic relations:

a) $A \cup B = B \cup A, \quad A \cap B = B \cap A$ (Commutativity)

b) $A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C$ (Associativity)

c) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributivity)

d) $A \subseteq B \quad \Leftrightarrow \quad B^c \subseteq A^c$

e) $A \backslash B = A \cap B^c$

f) $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$ (De Morgan's laws)

Problem 1.10 Suppose the collection $\mathcal{B}$ is given by $\mathcal{B} = \left\{ \left[1, 1 + \frac{1}{n}\right] : n \in \mathbb{N} \right\}$. Find $\bigcup_{B \in \mathcal{B}} B$ and $\bigcap_{B \in \mathcal{B}} B$.

Problem 1.11 Let A be a set and let $\mathcal{P}(A)$ denote the set of all subsets of A (i.e., the power set of A). Prove that A and $\mathcal{P}(A)$ do not have the same cardinality. (The term *cardinality* is used in mathematics to refer to the size of a set.)

Problem 1.12 If A and B are sets, then show that

a) $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$

b) $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$

Problem 1.13 Prove that for each nonempty set A, the function

$$f: \begin{array}{ccc} \mathcal{P}(A) & \longrightarrow & \{\chi_B\}_{B \in \mathcal{P}(A)}, \\ B & \longmapsto & \chi_B \end{array}$$

is bijective. Here the characteristic function χ_B of B is defined as

$$\chi_B: \begin{array}{rl} A \longrightarrow & \{0,1\}, \\ x \longmapsto & \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \in B^c. \end{cases} \end{array}$$

Problem 1.14 Give a necessary and sufficient condition for

$$A \times B = B \times A.$$

Problem 1.15 If A, B, C are sets, show that

a) $A \times B = \emptyset \quad \Leftrightarrow \quad A = \emptyset$ or $B = \emptyset$.

b) $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

c) $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

Problem 1.16 For an arbitrary function $f : X \longrightarrow Y$, prove that the following relations hold:

a) $f(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f(A_i)$.

b) $f(\bigcap_{i \in I} A_i) \subseteq \bigcap_{i \in I} f(A_i)$.

c) Give a counterexample to show that $f(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} f(A_i)$ is not always true.

Problem 1.17 Suppose $f : A \to B$ and $g : B \to C$ are functions, show that

a) If both f and g are one-to-one, then $g \circ f$ is one-to-one.

b) If both f and g are onto, then $g \circ f$ is onto.

c) If both f and g are bijection, then $g \circ f$ is bijection.

Problem 1.18 For a function $f : X \longrightarrow Y$, show that the following statements are equivalent:

a) f is one-to-one.

b) $f(A \cap B) = f(A) \cap f(B)$ holds for all $A, B \in \mathcal{P}(X)$.

Problem 1.19 For an arbitrary function $f : X \longrightarrow Y$, prove the following identities:

a) $f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i)$.

b) $f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i)$.

c) $f^{-1}(B^c) = \left[f^{-1}(B)\right]^c$.

Problem 1.20 Show that

1. $\mathbb{N} \sim \mathbb{E}$.
2. $\mathbb{N} \sim \mathbb{Z}$.
3. $(-1, 1) \sim \mathbb{R}$.

Problem 1.21 Show that any nonempty subset of a countable set is finite or countable.

Problem 1.22 Let A be an infinite set. Show that A is countable if and only if there exists $f : A \to \mathbb{N}$ which is 1-to-1. Use this to prove that $\mathbb{Z}$, $\mathbb{N} \times \mathbb{N}$, $\mathbb{N}^r$, for any $r \geq 1$, and $\mathbb{Q}$ are countable.

Problem 1.23 Show that the countable union of finite or countable sets is countable.

Problem 1.24 An algebraic number is a root of a polynomial, whose coefficients are rational. Show that the set of all algebraic numbers is countable.

Problem 1.25 Show that the set $\mathbb{R}$ is uncountable.

Problem 1.26 The power set of $\mathbb{N}$, i.e., $\mathcal{P}(\mathbb{N})$, is not countable as well as the sets $\mathbb{R}$, and $\{0,1\}^{\mathbb{N}}$ the set of all the sequences which takes values 0 or 1. Use this to show that the set of all irrationals is not countable.

Problem 1.27 Let A and B be two nonempty sets. Assume there exist $f : A \to B$ and $g : B \to A$ which are 1-to-1 (or injective). Then there exists a bijection $h : A \to B$.
This conclusion is known as the Schröder–Bernstein theorem.

Solutions

Solution 1.1

1. (a) is false. Since its negation $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\quad x + y \leq 0$ is true. Because if $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x + y \leq 0$. For example, we may take $y = -(x+1)$ which gives $x + y = x - x - 1 = -1 \leq 0$.

2. (b) is true. Indeed for $x \in \mathbb{R}$, one can take $y = -x + 1$ which gives $x + y = 1 > 0$. The negation of (b) is $\exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\quad x + y \leq 0$.

3. (c) : $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\quad x + y > 0$ is false. Indeed one may take $x = -1$, $y = 0$. The negation of (c) is $\exists x \in \mathbb{R}\ \exists y \in \mathbb{R}\ x + y \leq 0$.

4. (d) is true. Indeed one may take $x = -1$. The negation is: $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\quad y^2 \leq x$.

Solution 1.2

1. This statement may be rewritten as: (For every $x \in \mathbb{R}$) $(f(x) \leq 1)$. The negation of "(For every $x \in \mathbb{R}$)" is "There exists $x \in \mathbb{R}$" and the negation of "$(f(x) \leq 1)$" is "$f(x) > 1$." Hence the negation of the statement is: "There exists $x \in \mathbb{R}, f(x) > 1$."

2. First let us rewrite the statement "The function f is increasing": "for any real numbers (x_1, x_2), if $x_1 \leq x_2$ then $f(x_1) \leq f(x_2)$." This may be rewritten as: "(for any real numbers x_1 and x_2) ($x_1 \leq x_2$ implies $f(x_1) \leq f(x_2)$)." The negation of the first part is: "(there exists a pair of real numbers (x_1, x_2))" and the negation of the second part is: "($x_1 \leq x_2$ and $f(x_1) > f(x_2)$)". Hence the negation of the complete statement is: "There exist $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ such that $x_1 \leq x_2$ and $f(x_1) > f(x_2)$."

3. The negation is: the function f is not increasing or is not positive. We already did describe the statement "the function f is not increasing." Let us focus on "the function f is not positive." We get: "there exists $x \in \mathbb{R}, f(x) < 0$." Therefore the negation of the complete statement is: "there exist $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ such that $x_1 < x_2$ and $f(x_1) \geq f(x_2)$, or there exists $x \in \mathbb{R}, f(x) < 0$."

4. This statement may be rewritten as follows: "(there exists $x \in \mathbb{R}^+$) $(f(x) \leq 0)$." The negation of the first part is: "(for any $x \in \mathbb{R}^+$)," and for the second part:"$(f(x) > 0)$." Hence the negation of the complete statement is: "for any $x \in \mathbb{R}^+$, $f(x) > 0$."

5. This statement may be rewritten as follows: "$(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x < y \Rightarrow f(x) > f(y))$." The negation of the first part is: "$(\forall x \in \mathbb{R})$," for the second part: "$(\exists y \in \mathbb{R})$," and for the third part: "($x < y$ and $f(x) \leq f(y)$)." Hence the negation of the complete statement is: "$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x < y$ and $f(x) \leq f(y)$."

Solution 1.3

1. $\Leftarrow$
2. $\Leftrightarrow$
3. $\Rightarrow$

Solution 1.4

"There exists one person living in Los Angeles who has blue eyes who will not win the Lottery or will retire after the age of 50."

Solution 1.5

1. A triangle with no right angle, is not rectangular.
2. There exists a stable in which there exists at least one horse who is not black.
3. If we rewrite the statement in mathematical language:

$$\forall x \in \mathbb{Z} \;\; \exists y \in \mathbb{Z} \;\; \forall z \in \mathbb{Z} \quad (z < x \quad \Leftrightarrow \quad z < x+1),$$

   the negation is

$$\exists x \in \mathbb{Z} \;\; \forall y \in \mathbb{Z} \;\; \exists z \in \mathbb{Z} \quad (z < x \text{ and } z \geq x+1).$$

4. $\exists \varepsilon > 0 \;\; \forall \alpha > 0 \quad (|x - 7/5| < \alpha \text{ and } |5x - 7| \geq \varepsilon)$.

Solution 1.6

First note that for $n \in \mathbb{N}$, $\dfrac{2n+1}{n+2} \leq 2$ since $2n+1 \leq 2(n+2)$. Let $\varepsilon > 0$, we have

$$\forall n \in \mathbb{N} \quad \frac{2n+1}{n+2} < 2 + \varepsilon;$$

let us find a condition on n such that the inequality

$$2 - \varepsilon < \frac{2n+1}{n+2}$$

is true. We have

$$\begin{aligned} 2 - \varepsilon < \frac{2n+1}{n+2} &\Leftrightarrow (2-\varepsilon)(n+2) < 2n+1 \\ &\Leftrightarrow 3 < \varepsilon(n+2) \\ &\Leftrightarrow n > \frac{3}{\varepsilon} - 2. \end{aligned}$$

Here ε is given, let us pick $N \in \mathbb{N}$ such that $N > \frac{3}{\varepsilon} - 2$. Hence, for any $n \geq N$ we have $n \geq N > \frac{3}{\varepsilon} - 2$. Consequently $2 - \varepsilon < \frac{2n+1}{n+2}$. As a conclusion, for any $\varepsilon > 0$, we found $N \in \mathbb{N}$ such that for any $n \geq N$ we have $2 - \varepsilon < \frac{2n+1}{n+2}$ and $\frac{2n+1}{n+2} < 2 + \varepsilon$.

Solution 1.7

1. $\exists M \in \mathbb{R}\ \forall x \in \mathbb{R}\ f(x) \leq M$;
2. $\exists M \in \mathbb{R}\ \exists m \in \mathbb{R}\ \ \forall x \in \mathbb{R}\ m \leq f(x) \leq M$;
3. $\forall x \in \mathbb{R}\ f(x) = f(-x)$;
4. $\forall x \in \mathbb{R}\ f(x) = -f(-x)$;
5. $\forall x \in \mathbb{R}\ f(x) \neq 0$;
6. $\exists a \in \mathbb{R}^*\ \forall x \in \mathbb{R}\ f(x+a) = f(x)$;
7. $\forall (x, y) \in \mathbb{R}^2\ (x \leq y \Rightarrow f(x) \leq f(y))$;
8. $\forall (x, y) \in \mathbb{R}^2\ (x \leq y \Rightarrow f(x) > f(y))$;
9. $\exists x \in \mathbb{R}\ f(x) \neq 0$;
10. $\forall (x, y) \in \mathbb{R}^2\ (x \neq y \Rightarrow f(x) \neq f(y))$;
11. $\forall x \in \mathbb{R}\ f(x) \leq g(x)$;
12. $\exists x \in \mathbb{R}\ f(x) > g(x)$.

Solution 1.8

$(a \Rightarrow b)$

Suppose $A \subseteq B$. Let $x \in A \cup B$, then $x \in A$ or $x \in B$. If $x \in A$, then since $A \subseteq B$, we have $x \in B$. Thus, for any $x \in A \cup B$, $x \in B$, so $A \cup B \subseteq B$. Let $x \in B$, then $x \in A \cup B$ so $B \subseteq A \cup B$, and hence $A \cup B = B$.

$(b \Rightarrow c)$

Let $x \in A \cap B \Rightarrow x \in A$ so $A \cap B \subseteq A$. Let $x \in A \Rightarrow x \in A \cup B = B$ so $x \in B$ and therefore $x \in A \cap B$ so $A \subseteq A \cap B$. Thus $A \cap B = A$.

$(c \Rightarrow a)$

Let $x \in A$, then by hypothesis $x \in A \cap B$, which in turn implies that $x \in B$ as well. Thus $A \subseteq B$.

Solution 1.9

a) Follows directly from definitions.

b) Follows directly from definitions.

c) To establish this equality, note the following:

$$\begin{aligned} x \in A \cup (B \cap C) &\Leftrightarrow x \in A \text{ or } x \in B \cap C \\ &\Leftrightarrow x \in A \text{ or } (x \in B \text{ and } x \in C) \\ &\Leftrightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C) \\ &\Leftrightarrow x \in A \cup B \text{ and } x \in A \cup C \\ &\Leftrightarrow x \in (A \cup B) \cap (A \cup C). \end{aligned}$$

Similarly,

$$\begin{aligned} x \in A \cap (B \cup C) &\Leftrightarrow x \in A \text{ and } x \in B \cup C \\ &\Leftrightarrow x \in A \text{ and } (x \in B \text{ or } x \in C) \\ &\Leftrightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\ &\Leftrightarrow x \in A \cap B \text{ or } x \in A \cap C \\ &\Leftrightarrow x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

d) Let $A \subseteq B$, then $x \in B^c \Rightarrow x \notin B$ and so $x \notin A$ (i.e., $x \in A^c$), therefore, $B^c \subseteq A^c$. Conversely, if $B^c \subseteq A^c$ is true, then by the preceding case, $A = (A^c)^c \subseteq (B^c)^c = B$.

e) Note that

$$\begin{aligned} x \in A \backslash B &\Leftrightarrow x \in A \text{ and } x \notin B \\ &\Leftrightarrow x \in A \text{ and } x \in B^c \\ &\Leftrightarrow x \in A \cap B^c. \end{aligned}$$

f) Note that

$$\begin{aligned} x \in (A \cap B)^c &\Leftrightarrow x \notin A \cap B \\ &\Leftrightarrow x \notin A \text{ or } x \notin B \\ &\Leftrightarrow x \in A^c \text{ or } x \in B^c \\ &\Leftrightarrow x \in A^c \cup B^c. \end{aligned}$$

Similarly,

$$\begin{aligned} x \in (A \cup B)^c &\Leftrightarrow x \notin A \cup B \\ &\Leftrightarrow x \notin A \text{ and } x \notin B \\ &\Leftrightarrow x \in A^c \text{ and } x \in B^c \\ &\Leftrightarrow x \in A^c \cap B^c. \end{aligned}$$

Solution 1.10

Clearly, $\bigcup_{B\in\mathcal{B}} B = [1,2]$ and $\bigcap_{B\in\mathcal{B}} B = \{1\}$.

Solution 1.11

This is a proof by contradiction. Suppose they have the same cardinality, then there exists a bijection

$$T : A \to \mathcal{P}(A).$$

Let $K = \{x \in A : x \notin T(x)\}$. Since T is onto, there exists $y \in A$ such that $T(y) = K$. If $y \in K$, then by the definition of K we can conclude $y \notin T(y) = K$, so $y \notin K$. Similarly, if $y \notin K$, $y \notin T(y)$ so we conclude that $y \in K$. In both cases we have reached a contradiction.

Solution 1.12

a)

$$\begin{aligned} X \in \mathcal{P}(A) \cup \mathcal{P}(B) &\Rightarrow X \subseteq A \text{ or } X \subseteq B \\ &\Rightarrow X \subseteq A \cup B \\ &\Rightarrow X \in \mathcal{P}(A \cup B). \end{aligned}$$

b)

$$\begin{aligned} X \in \mathcal{P}(A) \cap \mathcal{P}(B) &\Leftrightarrow X \subseteq A \text{ and } X \subseteq B \\ &\Leftrightarrow X \subseteq A \cap B \\ &\Leftrightarrow X \in \mathcal{P}(A \cap B). \end{aligned}$$

Solution 1.13

Clearly, for any $B \in \mathcal{P}(A)$, $f(B) = \chi_B$. Note that f is one-to-one and onto.

Solution 1.14

Clearly, $A \times B = B \times A \quad \Leftrightarrow \quad A = B$.

Solution 1.15

a) Without loss of generality, let $A \neq \emptyset$ then if

$$\begin{aligned} B \neq \emptyset &\Leftrightarrow (\exists x \in A \text{ and } \exists y \in B) \\ &\Leftrightarrow (x, y) \in A \times B \\ &\Leftrightarrow A \times B \neq \emptyset. \end{aligned}$$

b) Suppose $(x, y) \in (A \cup B) \times C$ then $y \in C$ and $x \in A$ or $x \in B$. So $(x, y) \in A \times C$ or $(x, y) \in B \times C$, thus $(x, y) \in (A \times C) \cup (B \times C)$ and $(A \cup B) \times C \subseteq (A \times C) \cup (B \times C)$. Conversely, if $(x, y) \in (A \times C) \cup (B \times C)$, then $(x, y) \in A \times C$ or $(x, y) \in B \times C$, which means that $y \in C$ and $x \in A$ or $x \in B$; therefore, $(x, y) \in (A \cup B) \times C$ and $(A \times C) \cup (B \times C) \subseteq (A \cup B) \times C$.

c) Suppose $(x, y) \in (A \cap B) \times C$ then $y \in C$ and $x \in A$ and $x \in B$. So $(x, y) \in A \times C$ and $(x, y) \in B \times C$, thus $(x, y) \in (A \times C) \cap (B \times C)$ and $(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$. Conversely, if $(x, y) \in (A \times C) \cap (B \times C)$, then $(x, y) \in A \times C$ and $(x, y) \in B \times C$, which means that $y \in C$ and $x \in A$ and $x \in B$; therefore, $(x, y) \in (A \cap B) \times C$ and $(A \times C) \cap (B \times C) \subseteq (A \cap B) \times C$.

Solution 1.16

a) Note the following:

$$\begin{aligned} y \in f\left(\bigcup_{i \in I} A_i\right) &\Leftrightarrow \exists x \in \bigcup_{i \in I} A_i \text{ with } y = f(x) \\ &\Leftrightarrow \exists i \in I \text{ with } x \in A_i \text{ and } y = f(x) \\ &\Leftrightarrow \exists i \in I \text{ with } y \in f(A_i) \\ &\Leftrightarrow y \in \bigcup_{i \in I} f(A_i). \end{aligned}$$

b) Since $\bigcap_{i \in I} A_i \subseteq A_i$ we have $f(\bigcap_{i \in I} A_i) \subseteq f(A_i)$ for each i. We obtain that

$$f\left(\bigcap_{i \in I} A_i\right) \subseteq \bigcap_{i \in I} f(A_i).$$

c) Let $A_1 = \{0\}$ and $A_2 = \{1\}$ and $X = Y = \{0, 1\}$. Define $f : X \longrightarrow Y$ by $f(0) = f(1) = 0$, then $f(A_1) = \{0\} = f(A_2)$. Therefore, $f(A_1) \cap f(A_2) = \{0\}$, while $A_1 \cap A_2 = \emptyset$ and $f(A_1 \cap A_2) = \emptyset$.

Solution 1.17

a) Let $f : A \to B$ and $g : B \to C$ are given functions. Let $x_1, x_2 \in A$ with $x_1 \neq x_2$, since f is 1-1, we know that $f(x_1) \neq f(x_2)$. So now we have two distinct points $f(x_1)$ and $f(x_2)$ in B. Since g is 1-1, we also have $g(f(x_1)) \neq g(f(x_2))$. This is same as $(g \circ f)(x_1) \neq (g \circ f)(x_2)$.

b) Suppose $c \in C$, since g is onto, we know that there is a $b \in B$ with $g(b) = c$. Furthermore since f is also onto, there is some $a \in A$ with $f(a) = b$. But this means

$$(g \circ f)(a) = g(f(a)) = g(b) = c$$

so we have the required $a \in A$.

c) Since both f and g are bijective, they are both 1-1 and onto. So $g \circ f$ is 1-1 by part a) above and $g \circ f$ is onto by part b) above. Therefore $g \circ f$ is 1-1 and onto, in other words $g \circ f$ is a bijection.

Solution 1.18

$(a \Rightarrow b)$ If $y \in f(A) \cap f(B)$, then $\exists a \in A$ and $b \in B$ such that $y = f(a) = f(b)$. Since f is one-to-one we know $a = b \in A \cap B$ and therefore, $y \in f(A \cap B)$. Thus $f(A) \cap f(B) \subseteq f(A \cap B)$, but from the problem above $f(A \cap B) \subseteq f(A) \cap f(B)$.

$(b \Rightarrow a)$ If b holds, notice that if A and B were disjoint subsets of X, we then have $f(A) \cap f(B) = \emptyset$. Now let $f(a) = f(b)$. Then let $A = \{a\}$ and $B = \{b\}$. Thus $f(A \cap B) = f(A) \cap f(B) = f(a) \cap f(b) = f(a) \cap f(a) = f(a) \neq \emptyset$. So $A \cap B \neq \emptyset$, and therefore, $a = b$.

Solution 1.19

a)

$$\begin{aligned} x \in f^{-1}\left(\bigcup_{i \in I} B_i\right) \quad &\Leftrightarrow \quad f(x) \in \bigcup_{i \in I} B_i \\ &\Leftrightarrow \quad \exists i \in I \text{ such that } f(x) \in B_i \\ &\Leftrightarrow \quad \exists i \in I \text{ such that } x \in f^{-1}(B_i) \\ &\Leftrightarrow \quad x \in \bigcup_{i \in I} f^{-1}(B_i). \end{aligned}$$

b)

$$\begin{aligned} x \in f^{-1}\left(\bigcap_{i \in I} B_i\right) \quad &\Leftrightarrow \quad f(x) \in \bigcap_{i \in I} B_i \\ &\Leftrightarrow \quad f(x) \in B_i \quad \forall i \in I \\ &\Leftrightarrow \quad x \in f^{-1}(B_i) \quad \forall i \in I \\ &\Leftrightarrow \quad x \in \bigcap_{i \in I} f^{-1}(B_i). \end{aligned}$$

c)

$$\begin{aligned} x \in f^{-1}(B^c) \quad &\Leftrightarrow \quad f(x) \in B^c \\ &\Leftrightarrow \quad f(x) \notin B \\ &\Leftrightarrow \quad x \notin f^{-1}(B) \\ &\Leftrightarrow \quad x \in \left[f^{-1}(B)\right]^c. \end{aligned}$$

Solution 1.20

1. Let $f : \mathbb{N} \to \mathbb{E}$ given by $f(n) = 2n$. Clearly this map is one-to-one and onto.
2. This can be shown by defining a bijection $f : \mathbb{N} \to \mathbb{Z}$ as
$$f(n) = \begin{cases} \frac{1-n}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$
3. Using calculus one can show that the function
$$f : (-1, 1) \to \mathbb{R}$$
defined by
$$f(x) = \frac{x}{x^2 - 1}$$
is one-to-one and onto. In fact, $(a, b) \sim \mathbb{R}$ for any interval (a, b).

Solution 1.21

Let A be a countable set and $B \subset A$. Without loss of generality assume B not empty and not finite. Let us prove that B is countable. Since A is countable, there exists a bijection $f : A \to \mathbb{N}$. Consider the restriction of f to B, denoted by $f_B : B \to \mathbb{N}$. f_B is a bijection from B into $f(B) \subset \mathbb{N}$. Clearly $f(B)$ is not empty and is not finite. Let us prove that $f(B)$ is in bijection with $\mathbb{N}$. Indeed set $b_0 = \min f(B)$. Then define $b_1 = \min f(B) \setminus \{b_0\}$. Once b_n is built, we define $b_{n+1} = \min f(B) \setminus \{b_0, b_1, \ldots, b_n\}$. By induction, we build the set $\{b_n,\ n \in \mathbb{N}\} \subset f(B)$ where $b_n < b_{n+1}$, for $n \in \mathbb{N}$. Assume $f(B) \setminus \{b_n,\ n \in \mathbb{N}\} \neq \emptyset$. Let $b \in f(B) \setminus \{b_n,\ n \in \mathbb{N}\}$. Then we have $b_n < b$ for any $n \in \mathbb{N}$. This is a contradiction since any increasing sequence of elements in $\mathbb{N}$ is not bounded above. So $f(B) = \{b_n,\ n \in \mathbb{N}\}$. Define $g : f(B) \to \mathbb{N}$ by $g(b_n) = n$, for any $n \in \mathbb{N}$. g is a bijection. Clearly $g \circ f_B : B \to \mathbb{N}$ is a bijection.

Solution 1.22

It is clear that if A is countable, then there exists a bijection $f : A \to \mathbb{N}$ which is also 1-to-1. So assume there exists $f : A \to \mathbb{N}$ which is 1-to-1. Let us prove that A is countable. Since A is infinite and f is 1-to-1, then $f(A)$ is infinite. In the previous problem, we showed that $f(A)$ is countable. Since f restricted to A into $f(A)$ is a bijection, then one can construct a bijection from A into $\mathbb{N}$, i.e. A is countable. Let us complete the proof by showing that $\mathbb{Z}$, $\mathbb{N}^r$, for any $r \geq 1$, and $\mathbb{Q}$ are countable. Note that $\mathbb{N} \times \mathbb{N} = \mathbb{N}^r$ for $r = 2$. The map $f : \mathbb{Z} \to \mathbb{N}$ defined by
$$f(n) = \begin{cases} 2n + 1 & \text{if } \ n \in \mathbb{N}, \\ -2n & \text{if } \ n \notin \mathbb{N}, \end{cases}$$
is 1-to-1. The first part shows that $\mathbb{Z}$ is countable. In order to show that $\mathbb{N}^r$, for $r \geq 1$, is countable, consider the set of prime numbers $\mathcal{P}$. We know that $\mathcal{P}$ is infinite. So for any $r \geq 1$, consider any subset $\mathcal{P}_r$ of $\mathcal{P}$ with r elements. And write $\mathcal{P}_r = \{p_1, \ldots, p_r\}$. Define the map $f : \mathbb{N}^r \to \mathbb{N}$ by
$$f(n_1, \ldots, n_r) = p_1^{n_1} \times \cdots \times p_r^{n_r}.$$

The elementary number theory shows that f is 1-to-1. Hence the first part shows that $\mathbb{N}^r$ is countable. Finally define $f : \mathbb{Q} \to \mathbb{N}$ by

$$f\left(\frac{n}{m}\right) = 2^{\text{sign}(n)} \times 3^{|n|} \times 5^m.$$

It is easy to check that f is 1-to-1. Hence $\mathbb{Q}$ is countable.

Solution 1.23

Let $\{A_i\}_{i\in I}$ be a family of subsets of a set X such that A_i is finite or countable for any $i \in I$ and I is countable. Without loss of generality assume $I = \mathbb{N}$. Set $A = \underset{n\in\mathbb{N}}{\cup} A_n$. Let us prove that A is countable. Without loss of generality, assume A is not finite. For any $a \in A$, set $n_a = \min\{n \in \mathbb{N}; a \in A_n\}$. Since A_n is finite or countable, for any $n \in \mathbb{N}$, there exists $f_n : A_n \to \mathbb{N}$ which is 1-to-1. Define $f : A \to \mathbb{N}$ by $f(a) = f_{n_a}(a)$. Then it is easy to check that f is 1-to-1. This proves that A is countable.

Solution 1.24

Let $P_n[x]$ be the set of polynomial functions with rational coefficients. Obviously there is a bijection from $P_n[x]$ into $\mathbb{Q}^n$. Since $\mathbb{Q}$ is countable, there exists a bijection $f : \mathbb{Q} \to \mathbb{N}$. Hence the map $F : \mathbb{Q}^n \to \mathbb{N}^n$ defined by $F(r_1, \ldots, r_n) = (f(r_1), \ldots, f(r_n))$ is a bijection. Since $\mathbb{N}^n$ is countable, we conclude that $\mathbb{Q}^n$ is countable and consequently $P_n[x]$ is countable. The set

$$R_n = \bigcup_{P\in P_n[x]} \{x \in \mathbb{R}; P(x) = 0\}$$

is countable since it is a countable union of finite sets. Note that R_n is infinite since $\mathbb{Q} \subset R_n$, for any $n \geq 1$. Since the set of algebraic numbers $\mathcal{A}(\mathbb{R})$ is given by

$$\mathcal{A}(\mathbb{R}) = \bigcup_{n\geq 1} R_n,$$

then it is countable being a countable union of countable sets.

Solution 1.25

First notice that $(0, 1) \sim \mathbb{R}$, because the function defined by

$$f : (0, 1) \to \mathbb{R}$$

defined by

$$f(x) = \tan\left(\pi x - \frac{1}{2}\right)$$

is one-to-one and onto. Next we claim that $(0, 1)$ is uncountable. The proof of this claim uses a *diagonalization* argument due to Cantor. Suppose to the contrary that $(0, 1)$ is countable. Then all real numbers in $(0, 1)$ can be written as an *exhaustive* list $x_1, x_2, x_3, \ldots, x_k, \ldots$ where each x_k is given as a decimal expansion. There are certain real numbers in $(0, 1)$ that have two decimal expansions. For example, $\frac{1}{10}$ has the two representations

$$0.10000\cdots \quad\text{and}\quad 0.09999\cdots.$$

We can give preference to one of the representations, but it is not necessary to do so as can be seen in the following argument. Suppose all the real numbers in $(0, 1)$ are given by the following list:

$$x_1 = 0.a_{11}a_{12}a_{13}a_{14}\cdots$$
$$x_2 = 0.a_{21}a_{22}a_{23}a_{24}\cdots$$
$$x_3 = 0.a_{31}a_{32}a_{33}a_{34}\cdots$$
$$x_4 = 0.a_{41}a_{42}a_{43}a_{44}\cdots$$
$$\cdots$$
$$x_k = 0.a_{k1}a_{k2}a_{k3}a_{k4}\cdots$$
$$\cdots$$

Our goal is to write down another real number y in $(0, 1)$ which does not appear in the above list. Now let b_1 be a digit different from $0, a_{11}$, and 9; b_2 be a digit different from $0, a_{22}$, and 9; b_3 be a digit different from $0, a_{33}$, and 9; etc. Consider a number y with decimal representation

$$y = 0.b_1b_2b_3b_4\cdots$$

clearly $y \in (0, 1)$, furthermore y is not one of the numbers with two decimal representations, since $b_n \neq 0, 9$. Moreover $y \neq x_k$ for any k because the kth digit in the decimal representation for y and x_k are different. Therefore there is no list of *all* real numbers in $(0, 1)$, and thus $(0, 1)$ is not countable. Since $(0, 1) \sim \mathbb{R}$, $\mathbb{R}$ is uncountable too.

Solution 1.26

We have seen that for any set X, there does not exist an onto map from X into the power set $\mathcal{P}(X)$. Hence $\mathcal{P}(\mathbb{N})$ is an infinite set which is not in bijection with $\mathbb{N}$. Hence $\mathcal{P}(\mathbb{N})$ is not countable. First let us prove that $\{0, 1\}^{\mathbb{N}}$ is not countable. Assume not, then there exists a bijection $f : \mathbb{N} \to \{0, 1\}^{\mathbb{N}}$. Set $\varepsilon = (\varepsilon_n) \in \{0, 1\}^{\mathbb{N}}$ defined by $\varepsilon_n = 1 - f(n)_n$, where $f(n)_n$ is the nth term in the sequence $f(n) \in \{0, 1\}^{\mathbb{N}}$. Obviously $\varepsilon \neq f(n)$, for any $n \in \mathbb{N}$. Hence ε does not belong to the range of f contradicting the onto behavior of f. Hence $\{0, 1\}^{\mathbb{N}}$ is not countable. In order to prove that $\mathbb{R}$ is not countable, we find a subset of $\mathbb{R}$ which is not countable. Indeed, consider the set

$$C = \left\{ \sum_{n=0}^{\infty} \frac{\varepsilon_n}{3^n}; (\varepsilon_n) \in \{0, 1\}^{\mathbb{N}} \right\}.$$

C is the Cantor triadic set. It is clear that C is a subset of $[0, 1] \subset \mathbb{R}$, and it is in bijection with $\{0, 1\}^{\mathbb{N}}$. Hence C is not countable. Hence $\mathbb{R}$ is not countable. Since $\mathbb{R}$ is not countable and $\mathbb{Q}$ is countable, $\mathbb{R} \setminus \mathbb{Q}$ is not countable since the union of two countable sets is countable. Clearly the set $\mathbb{R} \setminus \mathbb{Q}$ is the set of all irrationals.

Solution 1.27

Let us first prove that if there exists a 1-to-1 map $f : A \to B$ with $B \subset A$, then there exists a bijection $h : A \to B$. Indeed set $B_0 = A \setminus B$, and $B_{n+1} = f(B_n)$. Note that the family $\{B_n\}$ is pairwise disjoint, i.e., $B_n \cap B_m = \emptyset$, for any $n \neq m$. Indeed we have $B_0 \cap B = \emptyset$ and $B_n \subset B$ for all $n \geq 1$. Hence $B_0 \cap B_n = \emptyset$ for any $n \geq 1$. Since f is 1-to-1, we get $f^m(B_0) \cap f^m(B_n) = \emptyset$, for any $n, m \in \mathbb{N}$. In other words we have $B_n \cap B_{n+m} = \emptyset$, for any $n, m \in \mathbb{N}$. This proves our claim. It is clear that we have $A \setminus \left(\bigcup_{n\geq 0} B_n \right) \subset B$. Indeed, if $a \in A \setminus \left(\bigcup_{n\geq 0} B_n \right)$, then $a \in A \setminus B_0 = B$. Define $h : A \to B$ by

$$h(a) = \begin{cases} f(a) & \text{if } \; a \in \bigcup_{n\geq 0} B_n, \\ a & \text{if } \; a \notin \bigcup_{n\geq 0} B_n. \end{cases}$$

We claim that h is a bijection. Indeed, it is straightforward that h is 1-to-1 since f is 1-to-1. Let us prove that h is onto (or surjective). Let $y \in B$. If $y \notin \bigcup_{n\geq 0} B_n$, then we have $h(y) = y$, i.e., y is in the range of h. Assume $y \in \bigcup_{n\geq 0} B_n$. Then there exists $n \geq 1$ such that $y \in B_n$. Note that $y \notin B_0$ because $y \in B$. Since $B_n = f(B_{n-1})$, then there exists $a \in B_{n-1}$ such that $f(a) = y$. But $f(a) = h(a)$. Hence y is in the range of h. This completes the proof that h is a bijection. In the general case, we do not assume $B \subset A$, but we do assume the existence of $f : A \to B$ and $g : B \to A$ which are 1-to-1. Clearly $g \circ f : A \to g(B)$ is 1-to-1, and $g(B) \subset A$. The first part of our proof shows the existence of a bijection $h_A : A \to g(B)$. Note that the restriction of g from B into $g(B)$ is a bijection. The map $g^{-1} \circ h_A : A \to B$ is a bijection.

Chapter 2

Real Numbers

Ah, but my Computations, People say,
Have Squared the Year to human Compass, eh?
If so, by striking from the Calendar
Unborn Tomorrow, and dead Yesterday

Omar Khayyam (1048–1123)

- *Mathematical induction* is a method of proof used to establish that a given statement is true for all natural numbers. Let $S(n)$ be a statement about the positive integer n. If

  1. $S(1)$ is true and
  2. for all $k \geq 1$, the truth of $S(k)$ implies the truth of $S(k+1)$,

  then $S(n)$ is true for all $n \geq 1$.
  Verifying $S(1)$ is true is called the *basis step*. The assumption that $S(k)$ is true for some $k \geq 1$ is called the *induction hypothesis*. Using the induction hypothesis to prove $S(k+1)$ is true is called the *induction step*. There are variants of mathematical induction used in practice, for example if one wants to prove a statement not for all natural numbers but only for all numbers greater than or equal to a certain number b, then

  1. Show $S(b)$ is true.
  2. Assume $S(m)$ is true for $m \geq b$ and show that truth of $S(m)$ implies the truth of $S(m+1)$.

  Another generalization, called *strong induction*, says that in the *inductive step* we may assume not only the statement holds for $S(k+1)$ but also that it is true for $S(m)$ for all $m \leq k+1$. In

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strong induction it is not necessary to list the *basis step*, it is clearly true that the statement holds for all previous cases. The *inductive step* of a strong induction in this case corresponds to the *basis step* in ordinary induction.

- Let A be a nonempty subset of $\mathbb{R}$. The number b is called an *upper bound* for A if for all $x \in A$, we have $x \leq b$. A number b is called a *least upper bound* of A if, first, b is an upper bound for A and, second, b is less than or equal to every upper bound for A. The *supremum* of A (also called least upper bound of A) is denoted by $\sup(A)$, $\sup A$ or $lub(A)$.
  If $A \subset \mathbb{R}$ is not bounded above , we say that $\sup A$ is infinite and write $\sup A = +\infty$.
- A *lower bound* for a set $A \subset \mathbb{R}$ is a number b such that $b \leq x$ for all $x \in A$. Also b is called a *greatest lower bound* if and only if it is a lower bound and for any lower bound c of A, $c \leq b$. The *infimum* of A (also called greatest lower bound of A) is denoted by $\inf(A)$, $\inf A$ or $glb(A)$. If $A \subset \mathbb{R}$ is not bounded below, we set $\inf A = -\infty$.
- *Well-Ordering Property*: If A is a nonempty subset of $\mathbb{N}$, then there is a smallest element in A, i.e., there is an $a \in A$ such that $a \leq x$ for every $x \in A$.
- *Archimedean Property*: If $x \in \mathbb{Q}$, then there is an integer n with $x < n$.
- For each $n \in \mathbb{N}$, let I_n be a nonempty closed interval in $\mathbb{R}$. The family $\{I_n : n \in \mathbb{N}\}$ is called a *nest of intervals* if the following conditions hold:
  - $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$.
  - For each $\varepsilon > 0$, there is some $n \in \mathbb{N}$ such that $|I_n| < \varepsilon$.

Problem 2.1 Prove that $\sqrt{2}$ is not rational.

Problem 2.2 Show that two real numbers x and y are equal if and only if $\forall \varepsilon > 0$ it follows that $|x - y| < \varepsilon$.

Problem 2.3 Use the induction argument to prove that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

for all natural numbers $n \geq 1$.

Problem 2.4 Use the induction argument to prove that

$$1^2 + 2^2 + \cdots + n^2 = \frac{2n^3 + 3n^2 + n}{6}$$

for all natural numbers $n \geq 1$.

Problem 2.5 Use the induction argument to prove that $n^3 + 5n$ is divisible by 6 for all natural numbers $n \geq 1$.

Problem 2.6 Use induction to prove that if $1 + x > 0$, then $(1 + x)^n \geq 1 + nx$ for all natural numbers $n \geq 0$. This is known as Bernoulli's inequality.

Problem 2.7 Consider the Fibonacci numbers $\{F_n\}$ defined by

$$F_1 = 1\,, F_2 = 1\,, \text{ and } F_{n+2} = F_{n+1} + F_n\,.$$

Show that

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}\,, n = 1, 2, \ldots\,.$$

Problem 2.8 Show by induction that if X is a finite set with n elements, then $\mathcal{P}(X)$, the power set of X (i.e., the set of subsets of X), has 2^n elements.

Problem 2.9 Let A be a nonempty subset of $\mathbb{R}$ bounded above. Set

$$B = \{-a;\ a \in A\}\,.$$

Show that B is bounded below and

$$\inf B = -\sup A\,.$$

Problem 2.10 Let S and T be nonempty bounded subsets of $\mathbb{R}$ with $S \subset T$. Prove that

$$\inf T \leq \inf S \leq \sup S \leq \sup T\,.$$

Problem 2.11 Let $x \in \mathbb{R}$ be positive, i.e., $x \geq 0$. Show that there exists $a \in \mathbb{R}$ such that $a^2 = x$.

Problem 2.12 Let x and y be two real numbers such that $x < y$. Show that there exists a rational number r such that $x < r < y$. (In this case we say $\mathbb{Q}$ is dense in $\mathbb{R}$.)Use this result to conclude that any open nonempty interval (a, b) contains infinitely many rationals.

Problem 2.13 Let x and y be two positive real numbers such that $x < y$. Show that there exists a rational number r such that $x < r^2 < y$, without using the square-root function.

Problem 2.14 Let $\omega \in \mathbb{R}$ be an irrational positive number. Set

$$A = \{m + n\omega : \ m + n\omega > 0 \text{ and } \ m, \ n \in \mathbb{Z}\} .$$

Show that $\inf A = 0$.

Problem 2.15 Show that the Cantor set

$$C = \Big\{\{e_n\}; e_n = 0 \text{ or } 1\Big\} = \{0,1\} \times \{0,1\} \times \cdots$$

is uncountable.

Problem 2.16 If $x \geq 0$ and $y \geq 0$, show that

$$\sqrt{xy} \leq \frac{x+y}{2} .$$

When do we have equality?

Problem 2.17 Let x, y, a, and b be positive real numbers not equal to 0. Assume that $\frac{x}{y} < \frac{a}{b}$. Show that

$$\frac{x}{y} < \frac{x+a}{y+b} < \frac{a}{b} .$$

Problem 2.18 Let x and y be two real numbers. Show that

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|} .$$

Problem 2.19 Let $r \in \mathbb{Q} \cap (0,1)$. Write $r = \dfrac{a}{b}$ where $a \geq 1$ and $b \geq 1$ are coprime natural numbers. Show that there exists a natural number $n \geq 1$ such that

$$\frac{1}{n+1} \leq \frac{a}{b} < \frac{1}{n}\,.$$

Use this to show that there exist natural numbers $n_1, \ldots, n_k$ such that

$$r = \frac{a}{b} = \frac{1}{n_1} + \cdots + \frac{1}{n_k}\,.$$

Problem 2.20 Let x and y be two different real numbers. Show that there exist a neighborhood X of x and a neighborhood Y of y such that $X \cap Y = \emptyset$.

Problem 2.21 Show that (a,b) is a neighborhood of any point $x \in (a,b)$.

Problem 2.22 (Young Inequality) Prove that for $p \in (1,\infty)$, we have $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$ for $x, y \in \mathbb{R}^+ := \{x \in \mathbb{R} : x \geq 0\}$, where $q := \frac{p}{p-1}$ is the Hölder conjugate of p determined by $\frac{1}{p} + \frac{1}{q} = 1$.

Problem 2.23 (Arithmetic and Geometric Means) Prove that for $n \in \mathbb{N}\backslash\{0\}$ and $x_j \in \mathbb{R}^+$ for $1 \leq j \leq n$, one has that

$$\sqrt[n]{\prod_{j=1}^{n} x_j} \leq \frac{1}{n}\sum_{j=1}^{n} x_j.$$

Problem 2.24 (Hölder Inequality) For $p \in (1,\infty)$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, define

$$|x|_p := \left(\sum_{j=1}^{n} |x_i|^p\right)^{\frac{1}{p}}.$$

Show that

$$\sum_{j=1}^{n} |x_j y_j| \leq |x|_p |y|_q \ \text{ for } x, y \in \mathbb{R}^n.$$

Note that in the case of $p = q = 2$, this reduces to the Cauchy–Schwartz Inequality.

Problem 2.25 (Minkowski Inequality) Show that for all $p \in (1,\infty)$, one has $|x+y|_p \leq |x|_p + |y|_p$ where $x, y \in \mathbb{R}^n$.

Problem 2.26 (The Nested Intersection Property) Let $\{I_n\}$ be a decreasing sequence of nonempty closed intervals in $\mathbb{R}$, i.e., $I_{n+1} \subset I_n$ for all $n \geq 1$. Show that $\bigcap_{n\geq 1} I_n$ is a nonempty closed interval. When is this intersection a single point?

Problem 2.27 (The Interval Intersection Property) Let $\{I_\alpha\}_{\alpha\in\Gamma}$ be a family of nonempty closed intervals in $\mathbb{R}$, such that $I_\alpha \cap I_\beta \neq \emptyset$ for any $\alpha, \beta \in \Gamma$. Show that $\bigcap_{\alpha\in\Gamma} I_\alpha$ is a nonempty closed interval.

Solutions

Solution 2.1

Assume not. Let $r \in \mathbb{Q}$ such that $r = \sqrt{2}$ or $r^2 = 2$. Without loss of generality, we may assume $r \geq 0$. And since $r^2 = 2$, we have $r > 0$. Since r is rational, there exist two natural numbers $n \geq 1$ and $m \geq 1$ such that

$$r = \frac{n}{m}\,.$$

Moreover one may assume that n and m are relatively prime, i.e., the only common divisor is 1. Since $r^2 = 2$, we get

$$\left(\frac{n}{m}\right)^2 = \frac{n^2}{m^2} = 2$$

which implies $2m^2 = n^2$. Therefore, n^2 is even, so it is a multiple of 2. Assume that n is not even, then $n = 2k + 1$ for some natural number k. Hence $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. In other words, if n is not even, then n^2 will not be even. Therefore, n is also even. Set $n = 2k$ for some natural number k. Then $n^2 = 4k^2$ and since $n^2 = 2m^2$, we deduce that $m^2 = 2k^2$. The same previous argument will imply that m is also even. So both n and m are even so both are multiples of 2. This is a contradiction with our assumption that both are relatively prime. Therefore, such a rational number r does not exist which completes the proof of our statement.

Solution 2.2

This is an if and only if statement, and we need to prove the implications in both directions.
($\Rightarrow$): If $x = y$, then $|x - y| = 0$ and thus $|x - y| < \varepsilon$ no matter what $\varepsilon > 0$ is chosen.
:($\Leftarrow$) We give a proof by contradiction. Assume $x \neq y$, then $\varepsilon_0 = |x - y| > 0$. However, the statements

$$|x - y| = \varepsilon_0 \text{ and } |x - y| < \varepsilon_0$$

cannot be both true, our assumption is wrong, thus $x = y$.

Solution 2.3

First note that

$$1 = \frac{1(1+1)}{2}$$

which implies that the desired identity holds for $n = 1$. Assume that it holds for n, and let us prove that it also holds for $n + 1$. We have

$$1 + 2 + \cdots + (n+1) = 1 + 2 + \cdots + n + (n+1)\,.$$

Using our assumption we get

$$1 + 2 + \cdots + (n+1) = \frac{n(n+1)}{2} + (n+1)\,.$$

Since

$$\frac{n(n+1)}{2} + (n+1) = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2},$$

we conclude that the identity is also valid for $n+1$. By induction we clearly showed that the above identity is valid for any natural number $n \geq 1$.

Solution 2.4

Since
$$1^2 = \frac{2+3+1}{6} ,$$
the above identity holds in the case $n = 1$. Assume that the identity holds for n and let us prove that it also holds for $n+1$. Since
$$1^2 + 2^2 + \cdots + (n+1)^2 = 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 ,$$
the induction assumption implies
$$1^2 + 2^2 + \cdots + (n+1)^2 = \frac{2n^3 + 3n^2 + n}{6} + (n+1)^2 .$$
Algebraic manipulations imply
$$\begin{aligned} \frac{2n^3 + 3n^2 + n}{6} + (n+1)^2 &= \frac{2n^3 + 3n^2 + n + 6(n+1)^2}{6} \\ &= \frac{2n^3 + 3n^2 + n + 3(n+1)^2 + 3(n+1)^2}{6} \\ &= \frac{2n^3 + 6n^2 + 7n + 3 + 3(n+1)^2}{6} . \end{aligned}$$
On the other hand, we have
$$\begin{aligned} \frac{2(n+1)^3 + 3(n+1)^2 + (n+1)}{6} &= \frac{2n^3 + 6n^2 + 6n + 2 + 3(n+1)^2 + (n+1)}{6} \\ &= \frac{2n^3 + 6n^2 + 7n + 3 + 3(n+1)^2}{6} \end{aligned}$$
which implies
$$1^2 + 2^2 + \cdots + (n+1)^2 = \frac{2(n+1)^3 + 3(n+1)^2 + (n+1)}{6} .$$
So our identity is also valid for $n+1$. By induction we clearly showed that the identity is valid for any natural number $n \geq 1$.

Solution 2.5

First take $n = 1$. Then $n^3 + 5n = 6$ which is a multiple of 6. Assume that $n^3 + 5n$ is divisible by 6 and let us prove that $(n+1)^3 + 5(n+1)$ is divisible by 6. But
$$(n+1)^3 + 5(n+1) = n^3 + 3n^2 + 3n + 1 + 5n + 5 = n^3 + 5n + 3(n^2 + n) + 6 .$$
Next note that $n^2 + n$ is always even or a multiple of 2. Indeed if n is even, then n^2 is also even and therefore $n^2 + n$ is even. Now assume n is odd, then n^2 is also odd. Since the sum of two odd

numbers is even we get that n^2+n is even. Hence $3(n^2+n)$ is a multiple of 6. Our induction assumption implies that n^3+5n is a multiple of 6. So $n^3+5n+3(n^2+n)+6$ is a multiple of 6 which implies $(n+1)^3+5(n+1)$ is a multiple of 6. This completes our proof by induction, i.e., n^3+5n is divisible by 6 (or multiple of 6) for all natural numbers $n \geq 1$.

Solution 2.6

It is clear that for $n=0$, both sides of the inequality are equal to 1. Now assume that we have $(1+x)^n \geq 1+nx$ and let us prove that $(1+x)^{n+1} \geq 1+(n+1)x$. We have

$$(1+x)^{n+1} = (1+x)^n(1+x) \geq (1+nx)(1+x)$$

and $(1+x)^n \geq 1+nx$. Since $(1+nx)(1+x) = 1+nx+x+nx^2 = 1+(n+1)x+nx^2$ and $nx^2 \geq 0$, we get

$$(1+x)^{n+1} \geq 1+(n+1)x+nx^2 \geq 1+(n+1)x.$$

Hence the inequality is also true for $n+1$. Therefore, by induction we have $(1+x)^n \geq 1+nx$ for all natural numbers $n \geq 0$.

Solution 2.7

The classical induction argument will not work here. The main reason is that in order to reach F_{n+2} one will need to make assumptions about F_{n+1} and F_n. Therefore, we will use a strong induction argument. Indeed, first it is obvious that

$$F_1 = F_2 = \frac{(1+\sqrt{5})^1 - (1-\sqrt{5})^1}{2^1\sqrt{5}}\,.$$

Next assume that

$$F_k = \frac{(1+\sqrt{5})^k - (1-\sqrt{5})^k}{2^k\sqrt{5}}\,, k = 1, 2, \ldots, n$$

and let us prove that

$$F_{n+1} = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}}\,.$$

By the definition of the Fibonacci numbers, we have

$$F_{n+1} = F_n + F_{n-1}\,.$$

Our induction assumption implies

$$F_{n+1} = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} + \frac{(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1}}{2^{n-1}\sqrt{5}}\,.$$

Algebraic manipulations will imply

$$F_{n+1} = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n + 2(1+\sqrt{5})^{n-1} - 2(1-\sqrt{5})^{n-1}}{2^n\sqrt{5}}$$

or

$$F_{n+1} = \frac{2(1+\sqrt{5})^n + 4(1+\sqrt{5})^{n-1} - 2(1-\sqrt{5})^n - 4(1-\sqrt{5})^{n-1}}{2^{n+1}\sqrt{5}}\,.$$

Note that
$$\begin{cases} (1+\sqrt{5})^2 &= 6+2\sqrt{5} &= 2(1+\sqrt{5})+4 \\ (1-\sqrt{5})^2 &= 6-2\sqrt{5} &= 2(1-\sqrt{5})+4\,. \end{cases}$$
This easily implies
$$\begin{cases} (1+\sqrt{5})^{n+1} &= 2(1+\sqrt{5})^n+4(1+\sqrt{5})^{n-1} \\ (1-\sqrt{5})^{n+1} &= 2(1-\sqrt{5})^n+4(1-\sqrt{5})^{n-1}\,. \end{cases}$$
From the above equations we get
$$F_{n+1} = \frac{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}}\,.$$
The induction argument then concludes that
$$F_n = \frac{(1+\sqrt{5})^{n}-(1-\sqrt{5})^{n}}{2^{n}\sqrt{5}}\,, n=1,2,\dots\,.$$
Note that the number
$$\Phi = \lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$$
is known as the golden ratio and is one of the roots of the quadratic equation $x^2 = x+1$.

Solution 2.8

Note that when $n = 0$, the set X is the empty set. In this case we have
$$\mathcal{P}(X) = \{\emptyset\}$$
with one element. Since $2^0 = 1$, the statement is true when $n = 0$. Assume that whenever a set X has n elements, then $\mathcal{P}(X)$ has 2^n elements. Now let us prove that whenever a set X has $n+1$ elements, then $\mathcal{P}(X)$ has 2^{n+1} elements. Indeed, let X be a set with $n+1$ elements. Fix $a \in X$ and set $Y = X \setminus \{a\}$. Then Y has n elements. Clearly, we have
$$\mathcal{P}(X) = \mathcal{P}(Y) \cup \mathcal{P}_a(Y)$$
where
$$\mathcal{P}_a(Y) = \big\{M \cup \{a\};\ M \in \mathcal{P}(Y)\big\}\,.$$
The map $T_a : \mathcal{P}(Y) \to \mathcal{P}_a(Y)$ defined by
$$T(M) = M \cup \{a\}$$
is a bijection. Hence $\mathcal{P}(Y)$ and $\mathcal{P}_a(Y)$ have the same number of elements. Since Y has n elements, our assumption implies that $\mathcal{P}(Y)$ has 2^n elements. Note that $\mathcal{P}(Y)$ and $\mathcal{P}_a(Y)$ have no common point, i.e.,
$$\mathcal{P}(Y) \cap \mathcal{P}_a(Y) = \emptyset,$$
so
$$\text{number of elements of } \mathcal{P}(X) = \text{number of elements of } \mathcal{P}(Y) + \text{number of elements of } \mathcal{P}_a(Y)$$
or
$$\text{number of elements of } \mathcal{P}(X) = 2^n + 2^n = 2\cdot 2^n = 2^{n+1}\,.$$

This proves our claim. So by induction we conclude that whenever a set X has n elements, then $\mathcal{P}(X)$ has 2^n elements, for any natural number $n \geq 0$.

Solution 2.9

Since A is bounded above, there exists $m \in \mathbb{R}$ such that

$$\forall\, a \in A \quad a \leq m\,.$$

Hence

$$\forall\, a \in A \quad -m \leq -a$$

which implies

$$\forall\, b \in B \quad -m \leq b\,.$$

So B is bounded below. Therefore $\inf B$ exists. Let us now complete the proof by showing that $\inf B = -\sup A$. By definition of $\sup A$, we know that

$$\forall\, a \in A \quad a \leq \sup A\,.$$

So

$$\forall\, a \in A \quad -\sup A \leq -a$$

or

$$\forall\, b \in B \quad -\sup A \leq b\,.$$

The definition of $\inf B$ implies $-\sup A \leq \inf B$. Next we have

$$\forall\, b \in B \quad \inf B \leq b$$

which implies

$$\forall\, b \in B \quad -b \leq -\inf B$$

or

$$\forall\, a \in A \quad a \leq -\inf B$$

since $A = \{-b;\ b \in B\}$. By the definition of $\sup A$ we get $\sup A \leq -\inf B$. Combining this conclusion with $-\sup A \leq \inf B$, we deduce that

$$\inf B = -\sup A\,.$$

Note that a similar proof will show that if A is bounded below, then B, defined as above, will be bounded above. Moreover we will have

$$\sup B = -\inf A\,.$$

In fact the above proof may be generalized to get the following result: if we set $k \cdot A = \{k \cdot a;\ a \in A\}$, for any $k \in \mathbb{R}$, then

$$\begin{cases} \sup(k \cdot A) &= \;\; k\ \sup A & \text{provided } k \geq 0, \\ \inf(k \cdot A) &= \;\; k\ \inf A & \text{provided } k \geq 0, \\ \sup(k \cdot A) &= \;\; -k\ \inf A & \text{provided } k \leq 0, \\ \inf(k \cdot A) &= \;\; -k\ \sup A & \text{provided } k \leq 0. \end{cases}$$

Solution 2.10

It is always the case that $\inf S \leq \sup S$ for any bounded nonempty subset of $\mathbb{R}$. So we need only to prove that $\inf T \leq \inf S$ and $\sup S \leq \sup T$. Let $x \in S$. Then $x \in T$ since $S \subset T$. So $\inf T \leq x$ by definition of $\inf T$. This implies that $\inf T$ is a lower bound for S because x was taken arbitrary in S. Since $\inf S$ is the greatest lower bound we get

$$\inf T \leq \inf S \,.$$

Similarly one can easily show that $\sup S \leq \sup T$.

Solution 2.11

Without loss of generality assume $x > 0$. Set

$$A = \{a \in \mathbb{R};\ a^2 \leq x\} \,.$$

Obviously we have $0 \in A$ which means that A is not empty. Next note that A is bounded above. Indeed, let $n \geq 1$ be a natural number such that $x \leq n$. We claim that n is an upper bound of A. Indeed let $a \in A$ and assume $n < a$. In particular, we have $0 < a$ which implies $n^2 < a^2$. Since $a \in A$ then we have $a^2 \leq x$ which implies $n^2 < x$. But $n \leq n^2$ which implies $n < x$, contradiction. So we must have $a \leq n$ for any $a \in A$. Since A is bounded above, then $\sup A$ exists. Set $y = \sup A$. Let us prove that $y^2 = x$ which will complete the proof of our problem. Since $0 \in A$, we get $y \geq 0$. Assume that $y^2 < x$. So the real number $\dfrac{2y+1}{x-y^2}$ is well defined. Let $n \geq 1$ be a natural number such that

$$\frac{2y+1}{x-y^2} \leq n \,,$$

which implies

$$\frac{2y+1}{n} \leq x - y^2 \,,$$

or

$$\frac{2y+1}{n} + y^2 \leq x \,.$$

Since $n \geq 1$ we know that $\dfrac{1}{n^2} \leq \dfrac{1}{n}$ which implies

$$y^2 + \frac{2y}{n} + \frac{1}{n^2} \leq y^2 + \frac{2y+1}{n} \leq x \,,$$

or

$$\left(y + \frac{1}{n}\right)^2 \leq x \,.$$

Hence $y + \dfrac{1}{n} \in A$, contradicting the fact that y is an upper bound of A. So we must have $x \leq y^2$. Assume that $y^2 \neq x$. So we have $x < y^2$. In particular we have $y > 0$. Let $n \geq 1$ be a natural number such that

$$\frac{2y}{y^2 - x} \leq n \,.$$

Similar calculations as above will yield

$$x \leq \left(y - \frac{1}{n}\right)^2 \,.$$

Since $y = \sup A$, there exists $a \in A$ such that $y - \frac{1}{n} < a$. Since

$$\frac{1}{y} < \frac{2y}{y^2 - x} ,$$

we get $y - \frac{1}{n} > 0$. So $\left(y - \frac{1}{n}\right)^2 < a^2$ which implies $x < a^2$ contradicting the fact that $a \in A$. So we must have $y^2 = x$.

Solution 2.12

We have $y - x > 0$. Since $\mathbb{R}$ is Archimedean, there exists a positive integer $N \geq 1$ such that

$$N > \frac{2}{y - x} .$$

So $N(y - x) > 2$ or $Nx + 2 < Ny$. Again because $\mathbb{R}$ is Archimedean, there exists a unique integer n such that

$$n \leq Nx < n + 1 .$$

We claim that $n + 1 \in (Nx, Ny)$. Indeed we have

$$Nx < n + 1 \leq Nx + 1 < Nx + 2 < Ny ,$$

and since $Nx < n+1$, we have our conclusion. $N \geq 1$ implies that $x < \frac{n+1}{N} < y$. Take $r = \frac{n+1}{N}$ which completes the proof of the first statement. Next assume that the nonempty interval (a, b) has finitely many rationals. Then

$$r^* = \min\{r \in \mathbb{Q},\ a < r < b\}$$

exists and is in (a, b), i.e., $a < r^* < b$. Using the previous statement, we know that there exists a rational number q such that $a < q < r^*$. Obviously we have $q \in (a, b)$ contradicting the definition of r^*. So the set $\{r \in \mathbb{Q},\ a < r < b\}$ is infinite.

Solution 2.13

Set $A = \{r \in \mathbb{Q}\,, 0 \leq r \text{ and } x < r^2\}$. Since A is not empty and bounded below, it has an infimum. Let $m = \inf A$. Clearly $0 \leq m$. We claim that $m^2 \leq x$. Assume not. Then $x < m^2$ which implies $m > 0$. Let

$$\varepsilon = \min\left(\frac{m^2 - x}{2m}, m\right) .$$

It is clear that $\varepsilon > 0$. Then we have $2m\varepsilon \leq m^2 - x$ which implies

$$x \leq m^2 - 2m\varepsilon < m^2 - 2m\varepsilon + \varepsilon^2 = (m - \varepsilon)^2 .$$

Since $m - \varepsilon < m$, there exists a rational number $r \in \mathbb{Q}$ such that $m - \varepsilon < r < m$. Note that r is positive because $\varepsilon \leq m$. This obviously implies $(m - \varepsilon)^2 < r^2$. In particular we have $x < (m - \varepsilon)^2 < r^2$. So $r \in A$ which contradicts $m = \inf A$. Hence our claim is valid, that is, $m^2 \leq x$. This implies $m^2 < y$. Let

$$\delta = \min\left(\frac{y - m^2}{2m + 1}, 1\right) .$$