Real analysis
Quiz 2 (Fall 2024)
Duration: 1 hour

Question 1: (8 marks) Define what it means for a function to be uniformly continuous on a set.

Solution:

A function $f : X \to Y$ is said to be uniformly continuous on X if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $x, y \in X$ that satisfies $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.
(note that alternate formulations using metric spaces are also acceptable).

(2 marks: "for every $\epsilon > 0$"; 2 marks: "for there exists a $\delta > 0$", 2 marks: "for every $x, y \in X$"; 1 mark: "$|x - y| < \delta$" and 1 mark: "$|f(x) - f(y)| < \epsilon$".

Question 2: (9 marks) Give examples, with justification, of each of the following.

1. A bounded sequence (x_n) for which $\lim_{n\to\infty} \sup x_n \neq \lim_{n\to\infty} \inf x_n$.
2. A function $f : [0, 1] \to \mathbb{R}$ which is discontinuous at each $x \in [0, 1]$.
3. A continuous function which is not uniformly continuous.

Solution:

1. $x_n = \frac{(-1)^n n}{n+1}$. (1 mark)

   This has $\lim_{n\to\infty} \sup x_n = 1$ and $\lim_{n\to\infty} \inf x_n = -1$. (2 marks)

2. Define f as follows:
$$f(x) = \left\{ \begin{array}{ll} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{array} \right\}$$

   Take $p \in [0, 1]$. Consider two sequences $\{x_n\} = p + \frac{1}{n}$ and $\{y_n\} = p + \frac{\sqrt{2}}{n}$.
   Then $x_n \to 1$ and $y_n \to 0$. This implies that the limit does not exist and hence f is not continuous for any p.

3. $f(x) = \frac{1}{x}$ for $x \in (0, 1)$. This is continuous (1 mark)

   However, it is not uniformly continuous.
   Choose $\epsilon = 1$. Set $y = \frac{x}{2}$. Then we will find an x such that this holds: $|x - y| = \frac{x}{2} < \delta$ and $|f(x) - f(y)| = \frac{1}{x} \geq 1$, which is equivalent to $x < \min[2\delta, 1]$. (2 marks)

Question 3: (8 marks)
Show the following statements:

1. A bounded monotone sequence is convergent.
2. Every sequence has a monotone subsequence.

Solution:

1. Suppose $\{a_n\}$ is monotone increasing. Define S to be the set of terms in $\{a_n\}$ and define $L = \sup(S)$ which exists since S is bounded. We claim $\{a_n\} \to L$. (1 mark)

   Let $\epsilon > 0$. Since $L - \epsilon$ is not an upper bound for S, there is some N such that $a_N > L - \epsilon$ and moreover since $\{a_n\}$ is increasing for all $n \geq N$ we have an $a_n > L - \epsilon$. (2 marks)

   Since L is an upper bound, we have $a_n \leq L < L + \epsilon$ as well and hence for all $n \geq N$ we have $|a_n - L| \leq \epsilon$. A similar argument works for a decreasing sequence. (1 mark)

2. Given $\{a_n\}$ we say say a_m is a "peak" if $a_n \leq a_m$ for all $n > m$. (1 mark)

   Case 1: $\{a_n\}$ has infinitely many peaks. List them: $a_{m_1}, a_{m_2}, \ldots$.. This is decreasing subsequence. (1 mark)

   Case 2: $\{a_n\}$ has finitely many peaks. Let a_{n_1} be the first element past the last peak. This point is not a peak so there is a $n_2 > n_1$ so that $x_{n_2} > x_{n_1}$. But x_{n_2} is not a peak either, so there is a $n_3 > n_2$ so that $x_{n_3} > x_{n_2}$. Continuing in this inductively gives an increasing sequence. (2 marks)

Question 4 (6 marks)
Let us say that a sequence $(c_n)_{n=1}^{\infty}$ of real numbers "*cervonges* to c" (where $c \in \mathbb{R}$) if and only if there is an $N \in \mathbb{N}$ such that, for all $n > N$ and all $\epsilon > 0$, we have $|c_n - c| < \epsilon$.

1. If a sequence (c_n) cervonges to c, does (c_n) converge to c? Explain, and if not, give an example.

2. If a sequence (c_n) converges to c, does (c_n) cervonge to c? Explain, and if not, give an example.

Solution:

1. The main difference between cervongent and convergent sequences is that, for a cervongent sequence, N does not depend on ϵ, while for convergent sequences it does. That is, for a cervongent sequence, the same N works for all $\epsilon > 0$. (2 marks)

   This can happen if and only if $c_n = c$ is constant for $n \geq N$. Thus a cervongent sequence is convergent. (1 mark)

2. In general, a convergent sequence is not cervongent. (1 mark)

   Any convergent sequence that is not eventually constant would work as an example. E.g.: $c_n = \frac{1}{n}$. (2 marks)

Question 5: (9 marks)
Let X be a metric space such that $X \subseteq Y$, where Y is a complete metric space. Let (x_n) be a Cauchy sequence in X such that (x_n) contains a convergent subsequence in X. Then (x_n) converges in X.

Solution:

Since $\{x_n\}$ is a Cauchy sequence in X, it is also a Cauchy sequence in Y. (2 marks)
Since Y is complete, $\{x_n\}$ converges to some $y \in Y$. (2 marks)
Hence every subsequence of $\{x_n\}$ also converges to y. (2 marks)

On the other hand, we are given that some subsequence $\{x_{n_k}\}$ converges to some $x \in X$. By uniqueness of the limit of a sequence, we must have $y = x$ and thus $y \in X$, so $\{x_n\}$ converges in X. (3 marks)

Question 6: (10 marks)
Let Z be a metric space and let Y be a dense subset of Z. Suppose that every Cauchy sequence in Y converges in Z. Prove that Z is complete.
Solution:

Let $\{z_n\}$ be an arbitrary sequence in Z. Since Y is dense in Z, we can find $y_n \in Y$ such that $d(y_n, z_n) < \frac{1}{n}$; in particular, $d(y_n, z_n) \to 0$ as $n \to \infty$. (2 marks)

Now assume that $\{z_n\}$ is Cauchy. We claim that in this case $\{y_n\}$ constructed above is also Cauchy. (2 marks)

Indeed, fix $\epsilon > 0$. Since $\{z_n\}$ is Cauchy, there exists $M_1 \in N$ such that $d(z_n, z_m) < \frac{\epsilon}{2}$ for all $n, m \geq N$. Choose $M_2 \in N$ such that $\frac{1}{M_2} < \frac{\epsilon}{4}$. Let $M = \max\{M_1, M_2\}$. We claim that $d(y_n, y_m) < \epsilon$ for M_2 for all $n, m \geq N$ (whence $\{y_n\}$ is Cauchy). Indeed, by quadrilateral inequality we have $d(y_n, y_m) \leq d(y_n, z_n) + d(z_n, z_m) + d(z_m, y_m) < \frac{1}{n} + \frac{\epsilon}{2} + \frac{1}{m} \leq \frac{\epsilon}{2} + \frac{2}{M_2} \leq \epsilon$. (3 marks)

Since $\{y_n\}$ is a Cauchy sequence in Y , by assumption it converges to some $z \in Z$, so $d(y_n, z) \to 0$ as $n \to \infty$. Since $0 \leq d(z_n, z) \leq d(z_n, y_n) + d(y_n, z)$ and $d(z_n, y_n) \to 0$ by construction, by the squeeze theorem we conclude that $d(z_n, z) \to 0$ as $n \to \infty$, so z_n converges to z. Thus, we proved that every Cauchy sequence in Z converges in Z, so by definition Z is complete. (3 marks)