Matrices and Elementary Row Operations

Eq^n (1.1) can be abbreviated as

$$AX = Y$$

where

$$A = \underbrace{\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}}_{\substack{\text{representation of} \\ \text{a matrix} \\ \text{(not a matrix itself)}}}, \quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

matrix of coefficients of the system

$$A_{[m\times n]} X_{[n\times 1]} = Y_{[m\times 1]}$$

An $m\times n$ matrix over the field F is a function A from the set of pairs of integers (i,j), $1\le i\le m$, $1\le j\le n$, into the field F. Entries of the matrix A are the scalars

$$A(i,j) = A_{ij}.$$

Plan to consider operations on the rows of the matrix A which correspond to forming linear combinations of the eq^{n}s in the system $AX=Y$.

We focus on 3 elementary row operations on an $m\times n$ matrix A over the field F:

1. multiplication of one row of A by a non-zero scalar c.
2. replacement of the r^{th} row of A by row r plus c times row s, c any scalar and $r\ne s$;
3. interchange of two rows of A.

An elementary row operation is thus a special type of function (rule) e which associated with each $m\times n$ matrix A an $m\times n$ matrix $e(A)$.

1. $e(A)_{ij} = A_{ij}$ if $i\ne r$, $e(A)_{rj} = cA_{rj}$, $c\ne 0$
2. $e(A)_{ij} = A_{ij}$ if $i\ne r$, $e(A)_{rj} = A_{rj} + cA_{sj}$, $r\ne s$
3. $e(A)_{ij} = A_{ij}$ if i is different from both r & s, $e(A)_{rj} = A_{sj}$, $e(A)_{sj} = A_{rj}$.

$\boxed{r,s \text{ constrained by } m.}$

Thm: To each elementary row operation e there corresponds an elementary row operation as e_1, such that $e_1(e(A)) = e(e_1(A)) = A$ for each A. I.e., the inverse operation (function) of an elementary row operation exists and is an elementary row operation of the same type.

Proof: ______

Def^{n}: If A & B are $m \times n$ matrices over the field F, we say that B is row-equivalent to A if B can be obtained from A by a finite sequence of elementary row operations.

Remark: Row-equivalence is an equivalence relation.

A binary relation $\sim$ on a set X is said to be an equivalence relation iff it is reflexive, symmetric and transitive. I.e., $\forall a, b, c \in X$:

1. $a \sim a$ (reflexivity)
2. $a \sim b$ iff $b \sim a$ (symmetry)
3. If $a \sim b$ and $b \sim c$ then $a \sim c$ (transitivity)

Equivalence class of a under $\sim$, denoted $[a]$ is defined as $[a] = \{x \in X : x \sim a\}$.

Thm: If A & B are row-equivalent $m \times n$ matrices, the homogenous systems of linear equations $AX=0$ & $BX=0$ have exactly the same solutions.

$\xrightarrow{e}$

Proof: $A = A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_k = B$. $\xleftarrow[e]{}$

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Example: Let F be the field of rational numbers, and

$$A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix}.$$ We perform elementary row operations.

$$\begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix} \xrightarrow{① - 2 × ②} \begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix} \xrightarrow{③ - 2 × ②} \begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 0 & -2 & -1 & +7 \end{bmatrix}$$

$$\xrightarrow{③/-2} \begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 0 & 1 & 1/2 & -7/2 \end{bmatrix} \xrightarrow{② - 4 × ③} \begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 0 & -2 & 13 \\ 0 & 1 & 1/2 & -7/2 \end{bmatrix} \xrightarrow{9 × ③ + ①} \begin{bmatrix} 0 & 0 & 15/2 & -55/2 \\ 1 & 0 & -2 & 13 \\ 0 & 1 & 1/2 & -7/2 \end{bmatrix}$$

$$\xrightarrow{① × 2/15} \begin{bmatrix} 0 & 0 & 1 & -11/3 \\ 1 & 0 & -2 & 13 \\ 0 & 1 & 1/2 & -7/2 \end{bmatrix} \xrightarrow{2 × ① + ②} \begin{bmatrix} 0 & 0 & 1 & -11/3 \\ 1 & 0 & 0 & 17/3 \\ 0 & 1 & 1/2 & -7/2 \end{bmatrix} \xrightarrow{-1/2 × ① + ③} \begin{bmatrix} 0 & 0 & 1 & -11/3 \\ 1 & 0 & 0 & 17/3 \\ 0 & 1 & 0 & -5/3 \end{bmatrix}$$

Row equivalence of A w/ the final matrix above tells us that the two systems are equivalent, i.e., have the same solutions.

$$\begin{aligned} 2x_1 - x_2 + 3x_3 + 2x_4 &= 0 \\ x_1 + 4x_2 \quad\quad - x_4 &= 0 \\ 2x_1 + 6x_2 - x_3 + 5x_4 &= 0 \end{aligned} \qquad \Bigg| \qquad \begin{aligned} x_3 - \tfrac{11}{3}x_4 &= 0 \\ x_1 \quad + \tfrac{17}{3}x_4 &= 0 \\ x_2 \quad - \tfrac{5}{3}x_4 &= 0. \end{aligned}$$

Def^n: An $m \times n$ matrix R is called <u>row-reduced</u> if

ⓐ the 1^{st} non-zero entry in each non-zero row of R is equal to 1;

ⓑ each column of R which contains the leading non-zero entry of some row has all its other entries 0.

<u>Thm</u>: Every $m \times n$ matrix over the field is row-equivalent to a row-reduced matrix.

Proof: __________ (as an exercise).

Row-Reduced Echelon Matrices

Def^n: An $m\times n$ matrix R is called a row-reduced echelon matrix if:

(a) R is row-reduced;

(b) every row of R which has all its entries 0 occurs below every row which has a non-zero entry;

(c) if rows $1,\dots,r$ are the non-zero rows of R, and if the leading non-zero entry of row i occurs in column k_i, $i=1,\dots,r$, ~~then~~ $k_1<k_2<\dots<k_r$.

I.e., Either every entry in R is 0, or $\exists\ r\in\mathbb{Z}^+$, $1\le r\le m$, and $k_1,k_2,\dots,k_r\in\mathbb{Z}^+$ with $1\le k_i\le n$ and

(a) $R_{ij}=0$ for $i>r$, $R_{ij}=0$ if $j<k_i$.

(b) $R_{ik_j}=\delta_{ij}$, $1\le i\le r$, $1\le j\le r$.

(c) $k_1<\dots<k_r$.

Examples: $I_{m\times n}$, $O_{n\times n}$, $\begin{bmatrix}0&1&-3&0&1/2\\0&0&0&1&2\\0&0&0&0&0\end{bmatrix}$.

Thm. Every $m \times n$ matrix A is row-equivalent to a row-reduced echelon matrix.

Now consider a homogenous system $RX=0$, wher R is a row-reduced echelon matrix. let $1, \ldots, r$ be non-zero rows of R, and let the leading non-zero entry of row i occurs in column k_i. The system $RX=0$ then consists of r non-trivial eq^{n}s. Also, the unknown x_{k_i} will occur (with non-zero coefficient) only in the i^{th} eq^{n}. Let $u_1, \ldots, u_{n-r}$ denote the $(n-r)$ unknowns which are different from $x_{k_1}, \ldots, x_{k_r}$, then the r non-trivial eq^{n}s in $RX=0$ are of the form

$$x_{k_1} + \sum_{j=1}^{n-r} c_{1j} u_j = 0$$

$$\vdots \qquad\qquad\qquad\qquad \textcircled{1.3}$$

$$x_{k_r} + \sum_{j=1}^{n-r} c_{rj} u_j = 0$$

Assign any values whatsoever to $u_1, \ldots, u_{n-r}$, to get corresponding values of $x_{k_1}, \ldots, x_{k_r} \longrightarrow$ sol^{n}s to the system.

For examples if $R = \begin{bmatrix} 0 & 1 & -3 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

in $RX=0$, then $r=2$, $k_1=2$, $k_2=4$ and two non-trivial eq^ns are

$$x_2 - 3x_3 + \tfrac{1}{2}x_5 = 0 \quad \text{or} \quad x_2 = 3x_3 - \tfrac{1}{2}x_5$$

$$x_4 + 2x_5 = 0 \quad \text{or} \quad x_4 = -2x_5.$$

Assign $x_1 = a$, $x_3 = b$, $x_5 = c$, then the sol^n is $(a, 3b - \frac{c}{2}, b, -2c, c)$.

Note: If the no. of non-zero rows, i.e., r, in R is less than n $(r<n)$ then $RX=0$ has a non-trivial sol^n. That is, a sol^n $(x_1, \ldots, x_n)$ in which not all x_j is 0. For, since, $r<n$, we may choose some x_j which is not among the r unknowns $x_{k_1}, \ldots, x_{k_r}$, & we can then construct a sol^n as above in which x_j is 1.

<u>Thm</u>: If A is an $m \times n$ matrix & $m<n$, then the homogenous system $AX=0$ has a non-trivial sol^n.

__Thm.__ If A is an $n \times n$ matrix, then A is row-equiv. to $I_{n\times n}$ iff the system $AX=0$ has only the trivial sol^{n}.

What about systems $AX = Y$? non-homogenous systems.

→ While $AX = 0$ always has a trivial sol^n,
systems $AX = Y$ for $Y \neq 0$ need not have a sol^n.

How to find solutions for $AX = Y$, $Y \neq 0$?

→ Form the augmented matrix A' of the system $AX = Y$. A' is the $m \times (n+1)$ matrix whose 1^{st} n columns are the columns of A and whose last column is Y.

$$A'_{ij} = A_{ij} \quad \forall j \leq n$$

$$A'_{i(n+1)} = y_i .$$

$$\begin{tikzcd}
A \arrow[d, "\substack{\text{sequence of elementary} \\ \text{row operations} \\ \text{(same for both } A \text{ \& } A')}"] & A' \arrow[d] & Y \arrow[d] \\
R & R' & Z
\end{tikzcd}$$

Then, $R' = \left[\begin{array}{c|c} R & Z_{m \times 1} \end{array} \right]$, $Z = \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix}$

$AX = Y$ and $RX = Z$ are equivalent and hence have same solutions.

Whether $RX=Z$ has any solutions? To determine all the sol^{n}s if any exist.

If R has r non-zero rows, with leading non-zero entry of row i occurring in column k_i, $i=1,\dots,r$, then the first r eq^{n}s of $RX=Z$ effectively express $x_{k_1},\dots,x_{k_r}$ in the terms of the $(n-r)$ remaining x_j and the scalars $z_1,\dots,z_r$. The last $(m-r)$ eq^{n}s are:

$$\begin{aligned} 0 &= z_{r+1} \\ &\vdots \\ 0 &= z_m \end{aligned}$$

and accordingly the cond^n for the system to have a sol^n is $z_i=0$ for $i>r$. If this cond^n is satisfied, all sol^{n}s to the system are found as in the homogenous case, by assigning arbitrary values to $(n-r)$ of the x_j and then computing x_{k_i} from the i^{th} eq^n.

Example: F be a field of $\mathbb{Q}$ and

$$A=\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 0 & 5 & -1 \end{bmatrix}.$$ Solve for $AX=Y=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

We perform a sequence of row operations on the augmented matrix A' which row-reduces A:

$$\begin{bmatrix} 1 & -2 & 1 & y_1 \\ 2 & 1 & 1 & y_2 \\ 0 & 5 & -1 & y_3 \end{bmatrix} \xrightarrow{2 - 2 \otimes 1} \begin{bmatrix} 1 & -2 & 1 & y_1 \\ 0 & 5 & -1 & (y_2 - 2y_1) \\ 0 & 5 & -1 & y_3 \end{bmatrix}$$

$$\Big\downarrow \text{③} - \text{①}$$

$$\begin{bmatrix} 1 & -2 & 1 & y_1 \\ 0 & 1 & -\frac{1}{5} & \frac{1}{5}(y_2 - y_1) \\ 0 & 0 & 0 & (y_3 - y_2 + 2y_1) \end{bmatrix} \xleftarrow{② \cdot \frac{1}{5}} \begin{bmatrix} 1 & -2 & 1 & y_1 \\ 0 & 5 & -1 & (y_2 - 2y_1) \\ 0 & 0 & 0 & (y_3 - y_2 + 2y_1) \end{bmatrix}$$

$$\Big\downarrow ① + 2 \otimes ②$$

$$\begin{bmatrix} 1 & 0 & \frac{3}{5} & \frac{1}{5}(y_1 + 2y_2) \\ 0 & 1 & -\frac{1}{5} & \frac{1}{5}(y_2 - 2y_1) \\ 0 & 0 & 0 & (y_3 - y_2 + 2y_1) \end{bmatrix}$$

Condⁿ that $AX = Y$ has a solⁿ is

$$2y_1 - y_2 + y_3 = 0$$

and if scalars y_i satisfy this condⁿ, all solⁿs are obtained by assigning a value c to x_3 & then computing

$$x_1 = -\frac{3}{5}c + \frac{1}{5}(y_1 + 2y_2)$$

$$x_2 = \frac{1}{5}c + \frac{1}{5}(y_2 - 2y_1)$$