Probability and Statistics: MA6.101

Tutorial 3

Topics Covered: Discrete Random Variables

Q1: Let X be a Poisson random variable with parameter $\lambda > 0$. The probability mass function (PMF) is given by:

$$p_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad \text{for } k = 0, 1, 2, \ldots$$

(a) Show that $p_X(k)$ is consistent.

(b) Derive the mean (expected value) of X.

(c) Derive the variance of X.

Q2: The probability distribution of the discrete random variable X is given by

x	2	3	4
$p_X(x)$	$0.4 - a$	$2a$	$0.6 - a$

where a is a constant.

(a) State the range of the possible values of a.

(b) Two independent observations of X, denoted by X_1 and X_2, are considered. Determine the range of possible values of $\mathbb{P}(X_1 + X_2 = 6)$ as a varies over its allowable range.

Answers:

(a) $a \in [0, 0.4]$

(b) $\frac{47}{150} \leq \mathbb{P}(X_1 + X_2 = 6) \leq \frac{16}{25}$

Q3: Let X be the number of students waiting for a Teaching Assistant (TA) during office hours. Based on past observations, we have the following information:

- At any time, there are at most 3 students waiting for the TA.
- The probability of finding two students waiting is the same as the probability of finding one student.
- The probability that no one is waiting is the same as the probability of finding three students waiting.
- The probability of finding either one or two students is half the probability of finding the office hours either completely empty or completely full (with 3 students).

Find the PMF of X.
Answer:

$$p_X(k) = \begin{cases} \frac{1}{3} & \text{for } k = 0, \\ \frac{1}{6} & \text{for } k = 1, \\ \frac{1}{6} & \text{for } k = 2, \\ \frac{1}{3} & \text{for } k = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Q4: The median of a random variable X is defined as any number m that satisfies both of the following conditions:

$$\mathbb{P}(X \geq m) \geq \frac{1}{2} \quad \text{and} \quad \mathbb{P}(X \leq m) \geq \frac{1}{2}.$$

Note that the median of X is not necessarily unique. Find the median of X if X is the result of rolling a fair die.
Answer: $m \in [3, 4]$

Q5: The discrete random variable X has the probability mass function

$$p_X(X = x) = \begin{cases} kx & x = 2, 4, 6, \\ k(x-2) & x = 8, \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

(a) Show that $k = \frac{1}{18}$.

(c) Find the exact value of $\mathbb{E}(X)$.

(d) Find the exact value of $\mathbb{E}(X^2)$.

Answer:

b $\frac{52}{9}$

c $\frac{112}{3}$

Q6: Say there are 1000 households in Gachibowli. Specifically, there are 100 households with one member, 200 households with 2 members, 300 households with 3 members, 200 households with 4 members, 100 households with 5 members, and 100 households with 6 members. Thus, the total number of people living in Gachibowli is

$$n = 100 \cdot 1 + 200 \cdot 2 + 300 \cdot 3 + 200 \cdot 4 + 100 \cdot 5 + 100 \cdot 6 = 3300.$$

(a) We pick a household at random, and define the random variable X as the number of people in the chosen household. Find the PMF and the expected value of X.

(b) We pick a person in Gachibowli at random, and define the random variable Y as the number of people in the household where the chosen person lives. Find the PMF and the expected value of Y.

Answer:

(a)

$$p_X(k) = \begin{cases} 0.1 & \text{for } k = 1, \\ 0.2 & \text{for } k = 2, \\ 0.3 & \text{for } k = 3, \\ 0.2 & \text{for } k = 4, \\ 0.1 & \text{for } k = 5, \\ 0.1 & \text{for } k = 6, \\ 0 & \text{otherwise.} \end{cases}$$

$\mathbb{E}[X] = 3.3$

(b)

$$p_Y(k) = \begin{cases} \frac{1}{33} & \text{for } k = 1, \\ \frac{4}{33} & \text{for } k = 2, \\ \frac{9}{33} & \text{for } k = 3, \\ \frac{8}{33} & \text{for } k = 4, \\ \frac{5}{33} & \text{for } k = 5, \\ \frac{6}{33} & \text{for } k = 6, \\ 0 & \text{otherwise.} \end{cases}$$

$\mathbb{E}[Y] = 3.3$

Q7: Suppose that there are n different types of coupons. Each time you get a coupon, it is equally likely to be any of the n possible types. Let X be the number of coupons you will need to get before having observed each coupon at least once.

(a) Show that you can write $X = X_0 + X_1 + \cdots + X_{n-1}$ where $X_i \sim \text{Geometric}(\frac{n-i}{n})$

(b) Find $\mathbb{E}[X]$.

Answer:

b $\mathbb{E}[X] = nH_n$ where $H_n = \sum_{j=1}^{n} \frac{1}{j} \approx \ln(n) + \gamma$ where $\gamma \approx 0.5772$

Q8: You roll a single fair six-sided die, and you are paid an amount equal to the value shown. However, you have an option: you can refuse this payment and roll the die a second time. If you opt for the re-roll, you are paid an amount equal to the value of the second roll. What is the optimal strategy to maximize your winnings, and what is the expected payout if you follow this strategy?
Answer: the optimal strategy is:

- **Re-roll** if the first roll is a 1, 2, or 3 (since these are all less than 3.5).
- **Keep** the result if the first roll is a 4, 5, or 6 (since these are all greater than or equal to 3.5).

$\mathbb{E}[\text{Overall Payout}] = 4.25$

Q9: Suppose that n people attend a party, each wearing wigs. As part of a game, their wigs are mixed up, and each person randomly selects one. Let X be the number of people who select their own wig. Find the expected value and the variance of X. Answer: $\mathbb{E}[X] = \text{Var}[X] = 1$