Probability and Statistics: MA6.101

Tutorial 6

Topics Covered: Moment Generating Functions, Sums of Random Variables, Stochastic Simulation

Q1: Let X and Y be the Cartesian coordinates of a randomly chosen point (according to a uniform PDF) in the triangle with vertices at $(0, 1)$, $(0, -1)$, and $(1, 0)$. Find the CDF and the PDF of $Z = |X - Y|$.

Q2: Suppose $X \sim \text{Binomial}(n_1, p)$ and $Y \sim \text{Binomial}(n_2, p)$ are two independent random variables.

(a) Find the MGF of X.

(b) Let $Z = X + Y$. Find the MGF of Z by using the MGFs of X and Y.

(c) By recognizing the form of the resulting MGF, identify the distribution of Z (including its parameters). What does this result imply about the sum of independent Binomial random variables with the same success probability?

Q3: Let X be a continuous random variable with the PDF

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|}, \qquad x \in \mathbb{R},\ \lambda > 0.$$

Find the MGF of X, $M_X(s)$.

Q4: Let $M_X(s)$ be finite for $s \in [-c, c]$ where $c > 0$. Prove:

(a)

$$\lim_{n\to\infty} \left[M_X(\frac{s}{n}) \right]^n = e^{sE[X]}$$

(b) Now assume $E[X] = 0$ and $Var[X] = 1$, then

$$\lim_{n\to\infty} \left[M_X\left(\frac{s}{\sqrt{n}}\right) \right]^n = e^{\frac{s^2}{2}}.$$

(c) We know that for $X \sim N(0, 1)$, we have $M_X(s) = e^{\frac{s^2}{2}}$. What can you say about the expression you derived above?

Q5: A discrete random variable X can take one of three values with the following probabilities:

$$p_X(1) = 0.4, \quad p_X(5) = 0.3, \quad p_X(10) = 0.3.$$

Describe, step-by-step, the inverse transform method to generate a random sample for X using a random number u drawn from a Uniform[0, 1] distribution. Provide the specific ranges of u that would correspond to each value of X.

Q6: Let the CDF of a continuous random variable X be

$$F(x) = \frac{1}{2}\left(1 + \frac{x}{\sqrt{1 + x^2}}\right), \qquad -\infty < x < \infty.$$

(a) Find the inverse CDF $F^{-1}(u)$ for $u \in (0, 1)$.

(b) Using the inverse transform method, compute the sample x when $U = 0.75$.

Q7: We want to estimate $\theta = \mathbb{E}[X^2]$ where X is an exponential random variable with rate $\lambda = 1$ ($X \sim \text{Exp}(1)$). Using importance sampling, formulate an estimator for θ by drawing N samples, $Y_1, \ldots, Y_N$, from a uniform distribution $Y \sim U[0, 5]$.

Q8: Let X and Y be two independent random variables with respective moment generating functions

$$M_X(t) = \frac{1}{1 - 3t}, \qquad M_Y(t) = \frac{1}{(1 - 3t)^2}, \qquad t < \tfrac{1}{3}.$$

Find $\mathbb{E}\big[(X + Y)^2\big]$.

Q9: You want to generate samples from a random variable with the PDF given by:

$$p(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Use the Accept-Reject method with a proposal distribution $q(x) = U(0, 1)$.

(a) Find the smallest constant c such that $p(x) \leq c \cdot q(x)$.

(b) Outline the algorithm to generate one sample.