Outline

Optimization Problems

Optimization problem (Standard Form):

$$
\begin{array}{rl}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, i = 1, \ldots, m \\
& h_i(x) = 0, i = 1, \ldots, p
\end{array}
$$

- Find an x that minimizes $f_0(x)$ among all x that satisfy
  - $f_i(x) \leq 0,\ i = 1, \ldots, m$
  - $h_i(x) = 0,\ i = 1, \ldots, p$
- x is called optimization variable
- $f_0 : \mathbb{R}^n \to \mathbb{R}$ is called objective function or cost function
- The inequalities $f_i(x) \leq 0$ are called inequality constraints
- $f_i : \mathbb{R}^n \to \mathbb{R}$ are called inequality constraint functions
- The equations $h_i(x)$ are called equality constraints
- The functions $h_i : \mathbb{R}^n \to \mathbb{R}$ are called equality constraint functions

Optimization Problems

Optimization problem (Standard Form):

$$
\begin{aligned}
\text{minimize} \quad & f_0(x) \\
\text{subject to} \quad & f_i(x) \leq 0, i = 1, \ldots, m \\
& h_i(x) = 0, i = 1, \ldots, p
\end{aligned}
$$

- If no constraints, i.e., $m = p = 0$, then called unconstrained problem
- Domain of opt. problem: where objective and constraint are defined

$$\mathcal{D} = \bigcap_{i=0}^{m} \textbf{dom } f_i \quad \cap \quad \bigcap_{i=1}^{p} \textbf{dom } h_i$$

- A point x is called feasible if it satisfies the constraints

$$
\begin{aligned}
f_i(x) \leq 0, & \quad i = 1, \ldots, m \\
h_i(x) = 0, & \quad i = 1, \ldots, p
\end{aligned}
$$

- The optimization problem is called feasible if there exists at-least one feasible point

Optimization Problems

Optimization problem (Standard Form):

$$\begin{array}{rl} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, i = 1, \ldots, m \\ & h_i(x) = 0, i = 1, \ldots, p \end{array}$$

- Feasible set or constraint set: set of all feasible points
- Optimal Value: The optimal value p^* defined as

$$p^* = \inf \{ f_0(x) \mid f_i(x) \leq 0, \ i = 1, \ldots, m, \ h_i(x) = 0, \ i = 1, \ldots, p \}$$

- p^* is allowed to take extended values $\pm\infty$
- Infeasible problem: problem is called infeasible when $p^* = \infty$
  - Note: we used the fact that $\inf \phi = \infty$
- Unbounded below: Problem is unbounded below if $f_0(x_k) \to -\infty$ as $k \to \infty$

Optimal and locally optimal

Optimization problem (Standard Form):

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \le 0, i = 1, \dots, m \\ & h_i(x) = 0, i = 1, \dots, p \end{aligned}$$

- Optimal Point: x^* is called optimal if it solves the given optimization problem, i.e.,
  - x^* is feasible point
  - $f(x^*) = p*$, that is, at x^* optimal value p^* is obtained
- Optimal Set: The set of all optimal points is called optimal set
- If there exists an optimal point, then we say that optimal value is achieved and the problem is solvable
- If the optimal set is empty, then we say that optimal value is not attained

Optimal and Locally Optimal Points

Optimization problem (Standard Form):

$$\begin{array}{rl} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, i = 1, \ldots, m \\ & h_i(x) = 0, i = 1, \ldots, p \end{array}$$

- $\epsilon-$suboptimal: $\bar{x}$ is $\epsilon-$suboptimal if $f(\bar{x}) \leq p^* + \epsilon$, $\epsilon > 0$
  - $\bar{x}$ is just ϵ more than optimal value p^*
- $\epsilon-$suboptimal set: set of all $\epsilon-$suboptimal points
- Active constraint: If x is feasible and $f_i(x) = 0$, then ith inequality is active
- Inactive constraint: If x is feasible and $f_i(x) < 0$, then this constraint is inactive
- Redundant constraint: A constraint is redundant if removing it does not change the feasible set

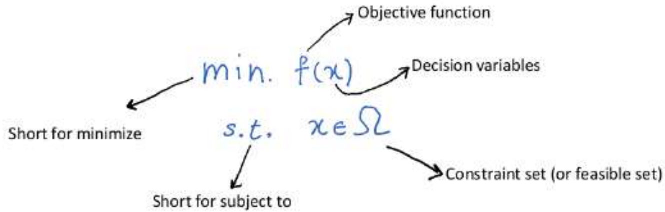
Optimization problem (Standard Form):

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, i = 1, \ldots, m \\ & h_i(x) = 0, i = 1, \ldots, p \end{aligned}$$

Define the feasible set

$$\Omega = \{x \mid f_i(x) \leq 0, \quad i = 1, \ldots, m, \quad h_i(x) = 0 \quad i = 1, \ldots, p\}$$

More compactly, we we can write:



Examples: $1/x$

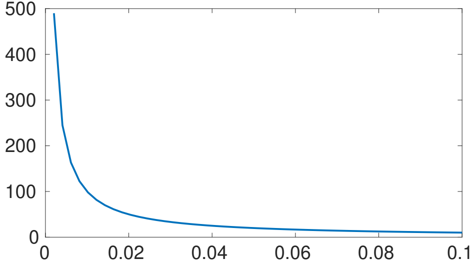
Consider the optimization problem:

$$\text{minimize } f_0(x) = 1/x,$$
$$\text{subject to } x \in \mathbb{R}$$

where $f_0 : \mathbb{R}_{++} \to \mathbb{R}$

Quiz: What is feasible set?
Quiz: What is p^*?
Quiz: Is the optimal value achieved?

Figure: Plot of $1/x$



Examples: $-\log x$

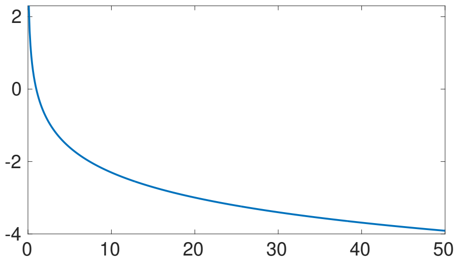
Consider the optimization problem:

$$\begin{aligned} \text{minimize } & f_0(x) = -\log x, \\ \text{subject to } & x \in \mathbb{R} \end{aligned}$$

where $f_0 : \mathbb{R}_{++} \to \mathbb{R}$

Quiz: What is feasible set?
Quiz: What is p^*?
Quiz: Is the optimal value achieved?
Quiz: Is this problem bounded below?

Figure: Plot of $-\log x$



Examples: $x \log x$

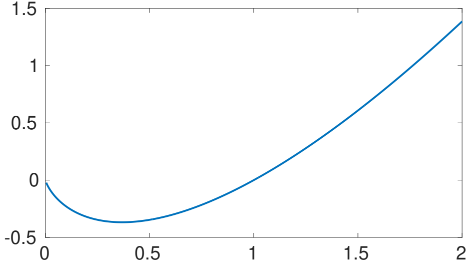
Consider the optimization problem:

$$\begin{aligned} \text{minimize } & f_0(x) = x \log x, \\ \text{subject to } & x \in \mathbb{R} \end{aligned}$$

where $f_0 : \mathbb{R}_{++} \to \mathbb{R}$

Quiz: What is feasible set?
Quiz: What is p^*?
Quiz: Is the optimal value acheived?
Quiz: Is this problem bounded below?
Quiz: What is optimal point?

Figure: Plot of $x \log x$



Expressing Problems in Standard Form

Optimization problem (Standard Form):

$$\begin{array}{rll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \le 0, & i = 1, \dots, m \\ & h_i(x) = 0, & i = 1, \dots, p \end{array}$$

Convention for standard form:

- right-hand side of the inequality and equality constraints are zero
  - For example, $g_i(x) = \tilde{g}_i(x)$ can be written as $h_i(x) = 0$
- $f_i(x) \ge 0$ as $-f_i(x) \le 0$

(Box Constraints). Consider the following

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & l_i \leq x_i \leq u_i, \quad i = 1, \ldots, n \end{array}$$

- constraints here are called variable bounds or box constraints

The problem above can be expressed in standard form:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & l_i - x_i \leq 0, \quad i = 1, \ldots, n \\ & x_i - u_i \leq 0, \quad i = 1, \ldots, n \end{array}$$

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, n \end{array}$$

where $f_i(x) = l_i - x_i$, $i = 1, \ldots, n$ and $f_i(x) = x_{i-n} - u_{i-n}$, $i = n+1, \ldots, 2n$

Maximization Problems Seen as Minimization Problems

Note: Maximization problem can be solved by minimization. Consider

$$\begin{array}{ll} \text{maximize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p, \end{array}$$

It can be solved by solving a corresponding minimization problem with $-f_0(x)$

Optimization problem (Standard Form):

$$\begin{array}{ll} \text{minimize} & -f_0(x) \\ \text{subject to} & f_i(x) \leq 0, i = 1, \ldots, m \\ & h_i(x) = 0, i = 1, \ldots, p \end{array}$$

- Note the constraints remain same!
- Obviously, the optimal value p^* is

$$p^* = \sup \{f_0(x) \mid \quad f_i(x) \leq 0, \; i = 1, \ldots, m, \quad h_i(x) = 0, \; i = 1, \ldots, p\}$$

Equivalent Optimization Problems

Two problems are equivalent if from a solution of the one, a solution of the other can be found

Example: Consider the following problem

$$\begin{aligned} \text{minimize} \quad & \alpha_0 f_0(x) \\ \text{subject to} \quad & \alpha_i f_i(x) \leq 0, \ i = 1, \ldots, m \\ & \beta_i h_i(x) = 0, \ i = 1, \ldots, p \end{aligned}$$

where $\alpha_i > 0, \ i = 0, \ldots, m, \quad \beta_i \neq 0, \ i = 1, \ldots, p.$

Note: α_i can be removed from inequality, and β_i can be removed from equality constraint. Minimizing $\alpha_0 f_0(x)$ is same as minimizing $f_0(x)$. Hence it is equivalent to

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, i = 1, \ldots, m \\ & h_i(x) = 0, i = 1, \ldots, p \end{aligned}$$

Equivalent Problems: Change of Variables

Suppose $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is one-one.

- Define $\tilde{f}_i$ as

$$\tilde{f}_i(z) = f_i(\phi(z)), \quad i = 0, \dots, m$$

- Define $\tilde{h}_i$ as

$$\tilde{h}_i(z) = h_i(\phi(z)),\ i = 1, \dots, p$$

Consider

$$\begin{array}{rl} \text{minimize} & \tilde{f}_0(z) \\ \text{subject to} & \tilde{f}_i(z) \leq 0,\ i = 1, \dots, m \\ & \tilde{h}_i(z) = 0,\ i = 1, \dots, p \end{array}$$

- If z solves above, then $x = \phi(z)$ solves the standard optimization problem.
- Similarly, if x solves original opt problem, then $z = \phi^{-1}(x)$ solves above

Equivalent Problems: Slack Variables

Given the optimization problem in standard form

Optimization problem (Standard Form):

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, i = 1, \ldots, m \\ & h_i(x) = 0, i = 1, \ldots, p \end{aligned}$$

Quiz: Is it possible to replace inequality constraints by equality constraints and non-negativity constraints?
Ans: Yes. Key observation is: $f_i(x) \leq 0$, if and only if there is an $s_i \geq 0$ such that $f_i(x) + s_i = 0$

$$\boxed{\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & s_i \geq 0, \quad i = 1, \ldots, m \\ & f_i(x) + s_i = 0, \ i = 1, \ldots, m \\ & h_i(x) = 0, \ i = 1, \ldots, p \end{aligned}}$$

Note: Here s_i are called slack variable. Is this equivalent?

Convex Optimization Problem in Standard Form

Convex Optimization Problem (Standard Form):

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \\
& a_i^T x = b_i, \quad i = 1, \dots, p
\end{array}
$$

where $f_0, \dots, f_m$ are convex functions.

Comparing this with the standard form

Optimization problem (Standard Form):

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \le 0, i = 1, \dots, m \\
& h_i(x) = 0, i = 1, \dots, p
\end{array}
$$

- objective function must be convex
- inequality constraint functions must be convex
- equality constraint functions $h_i(x) = a_i^T x - b_i$ must be affine

Convex Optimization Problem

Consider the following optimization problem with $x \in \mathbb{R}^2$

$$\begin{aligned} \text{maximize} \quad & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} \quad & f_1(x) = x_1/(1 + x_2^2) \leq 0 \\ & h_1(x) = (x_1 + x_2)^2 = 0, \end{aligned}$$

which is in standard form.
Quiz: Is this problem a convex optimization problem? Ans: No
Quiz: Can you rewrite this in convex optimization problem? Ans: Yes

$$\begin{aligned} \text{maximize} \quad & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} \quad & f_1(x) = x_1 \leq 0 \\ & h_i(x) = x_1 + x_2 = 0, \end{aligned}$$

Note: This is now a convex optimization problem

Outline

Optimality Criteria for Constraint Optimization Problem

Optimality criteria

Suppose f_0 in a convex optimization problem is differentiable so that for all $x, y \in \mathrm{dom}\ f_0$, we have

$$f_0(y) \geq f_0(x) + \nabla f_0(x)^T (y - x).$$

Let the feasible set X be

$$X = \{x \mid f_i(x) \leq 0, i = 1, \ldots, m, \quad h_i(x) = 0, i = 1, \ldots, p\}$$

Then x is optimal if and only if $x \in X$ and

$$\nabla f_0(x)^T (y - x) \geq 0 \quad \text{for all } y \in X.$$

Outline

1. Duality: Lagrange dual and KKT
   - Lagrangian dual function
   - Dual and conjugate functions
   - Weak Duality

Lagrangian Dual Function

Optimization problem (Standard Form):

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, i = 1, \ldots, m \\ & h_i(x) = 0, i = 1, \ldots, p \end{aligned}$$

Idea: Augment the objective $f_0(x)$ with a weighted sum of the constraint functions.

Lagrangian: Define Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ as

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x),$$

with dom $L = \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^p$. Here λ_i, ν_i are called Lagrange multipliers. Here λ and ν are called dual variables or Lagrange multiplier vectors.

Lagrange Dual Function

Lagrange Dual Function: Define the Lagrange dual function $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)$$

- If Lagrangian is unbounded below in x, the dual function takes on $-\infty$.
- The Lagrangian

$$L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)$$

  is affine as a function of λ and ν.
- Since $g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu)$, i.e., it is infimum of affine functions, $g(\lambda, \nu)$ is concave, **even when original problem is not convex!**

Lower Bounds on Optimal Value

Fact: The dual function yields lower bounds on the optimal value p^*. For any $\lambda \geq 0$ and any ν, we have

$$g(\lambda, \nu) \leq p^*.$$

Proof: Suppose $\tilde{x}$ is a feasible point, i.e., $f_i(\tilde{x}) \leq 0$ and $h_i(\tilde{x}) = 0$, and $\lambda \geq 0$. then

$$\sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{x}) \leq 0$$

$$\Longrightarrow \; L(\tilde{x}, \lambda, \nu) = f_0(\tilde{x}) + \sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{x}) \leq f_0(\tilde{x})$$

$$\Longrightarrow \; g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) \leq L(\tilde{x}, \lambda, \nu) \leq f_0(\tilde{x}), \quad \text{for any feasible } \tilde{x}$$

$$\Longrightarrow \; g(\lambda, \nu) \leq \inf_{x \in \mathcal{D}} f_0(x) = p^*$$

Examples: Least Squares Solution of Linear Equation

Problem-1: Consider the following optimization problem

$$\begin{aligned} &\text{minimize} \quad x^T x \\ &\text{subject to} \quad Ax = b, \quad A \in \mathbb{R}^{p \times n} \end{aligned}$$

- Find the Lagrangian function
- Find the dual function
- Check whether the dual function is concave
- Check whether the dual function is lower bound to p^*

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Answer: On chalkboard!

Examples: Standard Form LP

Problem-2: Consider an LP in standard form

$$
\begin{aligned}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & Ax = b \\
& x \geq 0
\end{aligned}
$$

- Find the Lagrangian
- Find the dual function
- Check whether the dual function gives lower bound to p^*

Two way Partitioning Problem

Problem-3:

$$\begin{array}{rl} \text{minimize} & x^T W x \\ \text{subject to} & x_i^2 = 1, \quad i = 1, \dots, n, \end{array}$$

where $W \in S^n$.

- Find the Lagrangian
- Find the dual function
- Check whether dual is a lower bound for p^*