International Institute of Information Technology, Hyderabad

MA2.101: Linear Algebra

Midsem Exam (Spring 2025)

March 1, 2025

Time duration: 90 mins. **Total points: 50**

Answer all the following questions. Notations and assumptions are as discussed in the lectures. Show all the steps, make appropriate assumptions wherever (and only if) necessary. Clearly state the theorems or facts if you are using them. The points for each question is indicated at the right. 50 points is equivalent to 20 marks out of total 100 for this course.

Problem 1. Prove that the union of two subspaces of a vector space V is a subspace of V if and only if one of the subspaces is contained in the other. [6 points]

Problem 2. Suppose $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is a set of vectors. For $m \in \{1, 2, \dots, k\}$, let $\vec{u}_m = \vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_m$. Prove that $\text{Span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k) = \text{Span}(\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k)$. [6 points]

Problem 3. Let V be a subspace of $\mathbb{R}^3$ defined as

$$V = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0, \quad 2x + 3y + 4z = 0\}.$$

Find a basis of V. [6 points]

Problem 4. Let $M_{2\times2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$, i.e., $M_{2\times2}$ is a set of all 2×2 matrices over $\mathbb{R}$. Consider a linear transformation $T : M_{2\times2}(\mathbb{R}) \to \mathbb{R}^2$ such that

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a + d, b + c).$$

(a) Show that T is a linear transformation. [4 points]

(b) State the rank and nullity theorem. Using that find the rank and the nullity of T. [8 points]

Problem 5. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation represented by an $n \times n$ matrix A. Prove that T is invertible if and only if $\text{Rank}(A) = n$. [8 points]

Problem 6. Consider an $n \times p$ matrix A and a $p \times m$ matrix B. Assume that the columns of A and the columns of B are linearly independent. Are the columns of the product AB linearly independent as well? Provide proper reasoning. [4 points]

Problem 7. Suppose $\mathcal{B}' = \{\vec{v}_1, \vec{v}_2\}$ and $\mathcal{B} = \{\vec{u}_1, \vec{u}_2\}$ be ordered bases of a vector space V defined over $\mathbb{R}$. Let $[\vec{v}_1]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $[\vec{v}_2]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}$. Find matrices P and P^{-1} such that $[\vec{v}]_{\mathcal{B}} = P[\vec{v}]_{\mathcal{B}'}$ for all $\vec{v} \in V$. [8 points]