Probability and Statistics

Homework 7 - Questions and Solutions

Q1: Let $X_1, X_2, ...$ be independent random variables that are uniformly distributed over $[-1, 1]$. Show that the sequence $Y_1, Y_2, ...$ converges in probability to some limit, and identify the limit, for each of the following cases:

(a) $Y_n = X_n/n$

(b) $Y_n = (X_n)^n$

(c) $Y_n = X_1 \cdot X_2 \cdots X_n$

(d) $Y_n = \max\{X_1, ..., X_n\}$

Solution:

(a) For any $\epsilon > 0$, we have $P(|Y_n| \geq \epsilon) = 0$, for all n with $1/n < \epsilon$, so $P(|Y_n| \geq \epsilon) \to 0$.

(b) For all $\epsilon \in (0, 1)$ we have

$$\begin{aligned} P(|Y_n| \geq \epsilon) &= P(|X_n|^n \geq \epsilon) \\ &= P(X_n \geq \epsilon^{1/n}) + P(X_n \leq -\epsilon^{1/n}) \\ &= 1 - \epsilon^{1/n} \end{aligned}$$

and the two terms in the right-hand side converge to 0, since $\epsilon^{1/n} \to 1$.

(c) Since $X_1, X_2, ...$ are independent random variables, we have $E[Y_n] = E[X_1] \cdots E[X_n] = 0$. Also

$$\begin{aligned} var(Y_n) &= E[Y_n^2] = E[X_1^2] \cdots E[X_n^2] \\ &= var(X_1)^n = (\frac{4}{12})^n \end{aligned}$$

so $var(Y_n) \to 0$. Since all Y_n have 0 as a common mean, from Chebyshev's inequality it follows that Y_n converges to 0 in probability.

(d) We have for all $\epsilon \in (0, 1)$, using the independence of $X_1, X_2, ...$,

$$\begin{aligned} P(|Y_n - 1| \geq \epsilon) &= P(\max\{X_1, ..., X_n\} \geq 1 + \epsilon) + P(\max\{X_1, ..., X_n\} \leq 1 - \epsilon) \\ &= P(X_1 \leq 1 - \epsilon, ..., X_n \leq 1 - \epsilon) \\ &= (P(X_1 \leq 1 - \epsilon))^n \\ &= (1 - \frac{\epsilon}{2})^n \end{aligned}$$

Hence $P(|Y_n - 1| \geq \epsilon) \to 0$.

Q2: Consider two sequences of random variables $X_1, X_2, ...$ and $Y_1, Y_2, ...$, which converge in probability to some constants. Let c be another constant. Show that $cX_n, X_n + Y_n, \max\{0, X_n\}, |X_n|$, and $X_n Y_n$ all converge in probability to corresponding limits.

Solution: Let x and y be the limits of X_n and Y_n, respectively. Fix some $\epsilon > 0$ and a constant c. If $c = 0$, then cX_n equals zero for all n, and convergence trivially holds. If $c \neq 0$ we observe that $P(|cX_n - cx| \geq \epsilon) = P(|X_n - x| \geq \epsilon/|c|)$ which converges to zero, thus establishing convergence in probability of cX_n.

We will now show that $P(|X_n + Y_n - x - y| \geq \epsilon)$ converges to zero, for any $\epsilon > 0$. To bound this probability, we note that for $|X_n + Y_n - x - y|$ to be as large as ϵ, we need either $|X_n - x|$ or $|Y_n - y|$ (or both) to be at least $\epsilon/2$. Therefore, in terms of events, we have

$$\{|X_n + Y_n - x - y| \geq \epsilon\} \subset \{|X_n - x| \geq \epsilon/2\} \cup \{|Y_n - y| \geq \epsilon/2\}$$

This implies that

$$P(|X_n + Y_n - x - y| \geq \epsilon) \leq P(|X_n - x| \geq \epsilon/2) + P(|Y_n - y| \geq \epsilon/2)$$

And

$$\lim_{n\to\infty} P(|X_n + Y_n - x - y| \geq \epsilon) \leq \lim_{n\to\infty} P(|X_n - x| \geq \epsilon/2) + \lim_{n\to\infty} P(|Y_n - y| \geq \epsilon/2) = 0$$

where the last equality follows since X_n and Y_n converge, in probability, to x and y, respectively.

By a similar argument, it is seen that the event $\{|\max\{0, X_n\} - \max\{0, x\}| \geq \epsilon\}$ is contained in the event $\{|X_n - x| \geq \epsilon\}$. Since $\lim_{n\to\infty} P(|X_n - x| \geq \epsilon) = 0$ this implies that $\lim_{n\to\infty} P(|\max\{0, X_n\} - \max\{0, x\}| \geq \epsilon) = 0$. Hence $\max\{0, X_n\}$ converges to $\max\{0, x\}$ in probability.

We have $|X_n| = \max\{0, X_n\} + \max\{0, -X_n\}$. Since $\max\{0, X_n\}$ and $\max\{0, -X_n\}$ converge, as shown earlier, it follows that their sum, $|X_n|$ converges to $\max\{0, x\} + \max\{0, -x\} = |x|$ in probability.

Finally, we have

$$\begin{aligned} P(|X_nY_n - xy| \geq \epsilon) &= P(|(X_n - x)(Y_n - y) + xY_n + yX_n - 2xy| \geq \epsilon) \\ &\leq P(|(X_n - x)(Y_n - y)| \geq \epsilon/2) + P(|xY_n + yX_n - 2xy| \geq \epsilon/2) \end{aligned}$$

Since xY_n and yX_n both converge to xy in probability, the last probability in the above expression converges to 0. The rest of the proof is similar to the earlier proof that $X_n + Y_n$ converges in probability.

Q3: Let $X_n \sim N(0, \frac{1}{n})$. Show that $X_n \longrightarrow 0$.

Solution: We will prove $\sum_{n=1}^{\infty} P(|X_n| > \epsilon) < \infty$, which implies $X_n \longrightarrow 0$. In particular, $P(|X_n| > \epsilon) = 2(1-\Phi(\epsilon\sqrt{n}))$. Using the inequality $1-\Phi(x) \leq \frac{1}{\sqrt{2\pi}}\frac{1}{x}e^{-\frac{x^2}{2}}$, we get

$$P(|X_n| > \epsilon) \leq \frac{1}{\sqrt{2\pi}}\frac{2}{\epsilon\sqrt{n}}e^{-\frac{\epsilon^2 n}{2}} \leq \frac{1}{\sqrt{2\pi}}\frac{2}{\epsilon}e^{-\frac{\epsilon^2 n}{2}}$$

Therefore,

$$\sum_{n=1}^{\infty} P(|X_n| > \epsilon) \leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{2}{\epsilon} e^{-\frac{\epsilon^2 n}{2}}$$
$$= \frac{1}{\sqrt{2\pi}} \frac{2}{\epsilon} \sum_{n=1}^{\infty} e^{-\frac{\epsilon^2 n}{2}}$$
$$= \frac{1}{\sqrt{2\pi}} \frac{2}{\epsilon} \frac{e^{-\frac{\epsilon^2}{2}}}{1 - e^{-\frac{\epsilon^2}{2}}} < \infty \quad \text{(geometric series)}$$

Q4: Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with (unknown but finite) mean μ and positive variance. For $i = 1, 2, \ldots$, let $Y_i = \frac{1}{3}X_i + \frac{2}{3}X_{i+1}$.

(a) Are the random variables Y_i independent?

(b) Are they identically distributed?

(c) Let $M_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$. Show that M_n converges to μ in probability.

Solution:

(a) No, the random variables Y_i are dependent.
The event $\{Y_1 = 1\}$ occurs if and only if $\{X_1 = 1, X_2 = 1\}$.
The event $\{Y_2 = 0\}$ occurs if and only if $\{X_2 = 0, X_3 = 0\}$.
$P(Y_2 = 0) = P(X_2 = 0, X_3 = 0) = (1-p)^2 > 0$.
$P(Y_2 = 0 | Y_1 = 1)$ is the probability of $\{X_2 = 0, X_3 = 0\}$ given that $\{X_1 = 1, X_2 = 1\}$.
Since the condition $X_2 = 1$ makes the event $X_2 = 0$ impossible, the conditional probability is 0.
Since $0 \neq (1-p)^2$, the variables are dependent.

(b) Yes, they are identically distributed. This can be seen by looking at the PDF of Y_i.

$$f_{Y_i}(y) = f_{X_i}(3y) * f_{X_{i+1}}(\frac{3}{2}y) = f_X(3y) * f_X(\frac{3}{2}y)$$

which is same for all values of i.

(c) We show convergence using Chebyshev's inequality by showing $E[M_n] = \mu$ and $Var(M_n) \to 0$. First, $E[Y_i] = E[\frac{1}{3}X_i + \frac{2}{3}X_{i+1}] = \frac{1}{3}E[X_i] + \frac{2}{3}E[X_{i+1}] = \mu$. So, $E[M_n] = \frac{1}{n}\sum E[Y_i] = \mu$. Next,

$$Var(M_n) = \frac{1}{n^2} Var(\sum Y_i) = \frac{1}{n^2}\left[\sum Var(Y_i) + 2\sum_{i<j} Cov(Y_i, Y_j)\right]$$

$Var(Y_i) = (\frac{1}{9} + \frac{4}{9})Var(X_i) = \frac{5}{9}\sigma^2$. The covariance is non-zero only for adjacent terms: $Cov(Y_i, Y_{i+1}) = \frac{2}{9}\sigma^2$. There are $n-1$ such terms.

$$Var(M_n) = \frac{1}{n^2}\left[n\frac{5}{9}\sigma^2 + 2(n-1)\frac{2}{9}\sigma^2\right]$$
$$= \frac{\sigma^2}{9n^2}[5n + 4n - 4] = \frac{\sigma^2(9n-4)}{9n^2}$$

As $n \to \infty$, $Var(M_n) \to 0$. By Chebyshev's inequality, M_n converges to μ in probability.

Q5: Let $Y_1, Y_2, \ldots$ be independent random variables, where Y_n is Bernoulli$(\frac{n}{n+1})$ for $n = 1, 2, 3, \ldots$. We define the sequence $\{X_n, n = 2, 3, 4, \ldots\}$ as $X_{n+1} = Y_1 Y_2 Y_3 \cdots Y_n$ for $n = 1, 2, 3, \ldots$. Show that $X_n \longrightarrow 0$ in probability.

Solution: We need to show that for any $\epsilon > 0$, $P(|X_n - 0| \geq \epsilon) \to 0$ as $n \to \infty$. The variable $X_n = Y_1 Y_2 \cdots Y_{n-1}$ can only take values 0 or 1. For any $\epsilon \in (0, 1]$, the event $|X_n| \geq \epsilon$ is equivalent to the event $X_n = 1$. The event $X_n = 1$ occurs if and only if $Y_1 = 1, Y_2 = 1, \ldots, Y_{n-1} = 1$. Since the Y_k are independent, we have:

$$P(X_n = 1) = \prod_{k=1}^{n-1} P(Y_k = 1)$$

Given $P(Y_k = 1) = \frac{k}{k+1}$, the product becomes:

$$P(X_n = 1) = \prod_{k=1}^{n-1} \frac{k}{k+1} = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n-1}{n}$$

This is a telescoping product which simplifies to $\frac{1}{n}$. Thus,

$$\lim_{n\to\infty} P(|X_n| \geq \epsilon) = \lim_{n\to\infty} P(X_n = 1) = \lim_{n\to\infty} \frac{1}{n} = 0$$

Therefore, X_n converges to 0 in probability.

Q6: Let $X_1, X_2, X_3, \ldots$ be continuous random variables with densities

$$f_{X_n}(x) = \frac{n}{2} e^{-n|x|}, \qquad x \in \mathbb{R},\ n = 1, 2, \ldots$$

(Thus X_n has the Laplace distribution with location 0 and rate n.) Show that $X_n \xrightarrow{p} 0$.

Solution: Fix $\varepsilon > 0$. Using the density we compute the tail probability

$$\begin{aligned}\mathbb{P}(|X_n| > \varepsilon) &= \int_{|x|>\varepsilon} f_{X_n}(x)\,dx = 2\int_{\varepsilon}^{\infty} \frac{n}{2} e^{-nx}\,dx = n\int_{\varepsilon}^{\infty} e^{-nx}\,dx \\ &= n\left[\frac{e^{-nx}}{-n}\right]_{x=\varepsilon}^{x=\infty} = e^{-n\varepsilon}.\end{aligned}$$

Since $e^{-n\varepsilon} \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \mathbb{P}(|X_n| > \varepsilon) = 0.$$

By the definition of convergence in probability, this shows $X_n \xrightarrow{p} 0$.

Alternate (variance) proof. For reference note that $E[X_n] = 0$ and

$$\mathrm{Var}(X_n) = \frac{2}{n^2} \xrightarrow{n\to\infty} 0.$$

By Chebyshev's inequality,

$$\mathbb{P}(|X_n| > \varepsilon) \leq \frac{\operatorname{Var}(X_n)}{\varepsilon^2} = \frac{2}{n^2\varepsilon^2} \to 0,$$

so again $X_n \xrightarrow{p} 0$. (This gives a cruder bound than the exact exponential tail above but is often useful.)

Q7: A sequence X_n of random variables is said to converge to a number c in the mean square if

$$\lim_{n\to\infty} \mathbb{E}\big[(X_n - c)^2\big] = 0.$$

(a) Show that convergence in the mean square implies convergence in probability.

(b) Give an example that shows that convergence in probability does not imply convergence in the mean square.

Solution: Let X_n converge to c in mean square, i.e. $\mathbb{E}[(X_n - c)^2] \to 0$.

Fix $\varepsilon > 0$. By Markov's (or Chebyshev's) inequality applied to the nonnegative r.v. $(X_n - c)^2$,

$$\mathbb{P}\big(|X_n - c| > \varepsilon\big) = \mathbb{P}\big((X_n - c)^2 > \varepsilon^2\big) \leq \frac{\mathbb{E}[(X_n - c)^2]}{\varepsilon^2}.$$

Since the right-hand side tends to 0 as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \mathbb{P}\big(|X_n - c| > \varepsilon\big) = 0.$$

This is exactly the definition of convergence in probability, so $X_n \xrightarrow{p} c$.

Define the sequence of random variables X_n by

$$X_n = \begin{cases} n, & \text{with probability } 1/n, \\ 0, & \text{with probability } 1 - 1/n. \end{cases}$$

(One can realize this on a standard probability space; the values and probabilities are as stated.)

First check convergence in probability to 0:

$$\mathbb{P}(|X_n - 0| > \varepsilon) = \mathbb{P}(X_n \neq 0) = \frac{1}{n} \xrightarrow{n\to\infty} 0,$$

for every fixed $\varepsilon \in (0, \infty)$. Hence $X_n \xrightarrow{p} 0$.

Now check mean square convergence to 0:

$$\mathbb{E}\big[(X_n - 0)^2\big] = \mathbb{E}[X_n^2] = n^2 \cdot \frac{1}{n} + 0 \cdot \Big(1 - \frac{1}{n}\Big) = n.$$

Since $n \to \infty$, we have $\mathbb{E}[(X_n - 0)^2] \not\to 0$ (in fact it diverges). Therefore X_n does *not* converge to 0 in mean square.

This example shows convergence in probability can occur without mean square convergence.

Q8: Let X be a random variable, and define $X_n = X + Y_n$, where

$$\mathbb{E}[Y_n] = \frac{1}{n}, \qquad \mathrm{Var}(Y_n) = \frac{\sigma^2}{n},$$

and $\sigma > 0$ is a constant. Show that $X_n \xrightarrow{p} X$.

Solution: We aim to show that X_n converges to X in probability, i.e.

$$\forall \varepsilon > 0, \quad \mathbb{P}(|X_n - X| \geq \varepsilon) \to 0 \quad \text{as } n \to \infty.$$

Since $X_n - X = Y_n$, we can write:

$$\mathbb{P}(|X_n - X| \geq \varepsilon) = \mathbb{P}(|Y_n| \geq \varepsilon).$$

By the triangle inequality, for all $a, b \in \mathbb{R}$, we have $|a + b| \leq |a| + |b|$. Setting $a = Y_n - \mathbb{E}[Y_n]$ and $b = \mathbb{E}[Y_n]$, we obtain:

$$|Y_n| \leq |Y_n - \mathbb{E}[Y_n]| + \frac{1}{n}.$$

Now, for any $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}(|Y_n| \geq \varepsilon) &\leq \mathbb{P}\Big(|Y_n - \mathbb{E}[Y_n]| + \frac{1}{n} \geq \varepsilon\Big) \\ &= \mathbb{P}\Big(|Y_n - \mathbb{E}[Y_n]| \geq \varepsilon - \frac{1}{n}\Big).\end{aligned}$$

Applying Chebyshev's inequality:

$$\mathbb{P}\Big(|Y_n - \mathbb{E}[Y_n]| \geq \varepsilon - \frac{1}{n}\Big) \leq \frac{\mathrm{Var}(Y_n)}{(\varepsilon - \frac{1}{n})^2} = \frac{\sigma^2/n}{(\varepsilon - \frac{1}{n})^2}.$$

As $n \to \infty$,

$$\frac{\sigma^2/n}{(\varepsilon - \frac{1}{n})^2} \longrightarrow 0.$$

Therefore,

$$\mathbb{P}(|X_n - X| \geq \varepsilon) \to 0,$$

which implies $X_n \xrightarrow{p} X$.

Q9: What is the expected number of iterations to generate k random numbers from a distribution using the rejection method?

A:

Let f be the pdf of the distribution we wish to sample from and g be the pdf of the distribution we sample from such that $support(f) \subseteq support(g)$.

Then for rejection sampling we have a c such that $\frac{f(y)}{g(y)} \leq c$ for all y.

We claim that $\mathbb{P}\left(U \leq \frac{f(Y)}{cg(Y)}\right) = \frac{1}{c}$

The last step comes from the fact that $support(f) \subseteq support(g)$.

Now notice that the number of iteration required to successfully generate one number is a geometric random variable with parameter $\frac{1}{c}$. So the expected number of iterations for generating one sample is c and for k samples is kc.