Midsem: Probability and Statistics (40 Marks)

Each question: 6 marks

1. Consider the following game with a fair die. You repeatedly roll a fair die until you get a six. The game ends, when 6 appears. The reward from each roll is the face value except that a roll of 6 yields reward 0. Find the expected total reward from the game.

2. Suppose X and Y are both Uniform[0,1] random variables. Then prove that $P(X < Y) = 0.5$.

3. Let X, Y and Z be independent exponential random variables with parameters λ_1, λ_2 and λ_3. Let $W = min(X, Y, Z)$. Find the cdf and pdf of W.

4. For two random variables X and Y, prove that $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$ where Var denotes variance and Cov denotes covariance.

5. Consider a Gaussian random variable X with mean μ and variance σ^2. Let $Z = aX + b$ where $a, b \in \mathbb{R}$. Derive an expression for the probability density of Z and show that Z is also a Gaussian random variable. What is the mean and variance of Z ?

Each question: 10 marks

1. Let the joint probability density function of two continuous random variables X and Y be

$$f_{X,Y}(x, y) = c\,(x + y), \qquad 0 \le x \le 1,\ 0 \le y \le 1,$$

and $f_{X,Y}(x, y) = 0$ otherwise.

(a) Find the constant c that makes $f_{X,Y}(x, y)$ a valid joint pdf.(2mks)

(b) Find the marginal density functions $f_X(x)$ and $f_Y(y)$.(2mks)

(c) Find the conditional density functions $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.(2mks)

(d) Compute the conditional expectation $\mathbb{E}[X \mid Y = y]$.(2mks)

(e) Are X and Y independent? Justify your answer.(2mks)