Probability and Statistics: MA6.101

Homework 6

Topics Covered: Moment Generating Functions, Sums of Random Variables, Stochastic Simulation

Q1: Let $X \sim \mathcal{N}(\mu, \sigma^2)$ be a Gaussian random variable.

(a) Find the moment generating function (MGF) of X.

(b) Using the MGF, compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

A:
(a) Recall the PDF of X:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big), \qquad x \in \mathbb{R}.$$

The MGF is

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\Big(tx - \frac{(x-\mu)^2}{2\sigma^2}\Big)\,dx.$$

Combine exponents and complete the square. Expand the quadratic:

$$tx - \frac{(x-\mu)^2}{2\sigma^2} = -\frac{1}{2\sigma^2}\Big(x^2 - 2(\mu+\sigma^2 t)x + \mu^2\Big).$$

Write this as a completed square:

$$= -\frac{1}{2\sigma^2}\Big[(x-(\mu+\sigma^2 t))^2 - (\mu+\sigma^2 t)^2 + \mu^2\Big].$$

Thus the integrand becomes

$$\exp\Big(-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}\Big)\exp\Big(\mu t + \tfrac{1}{2}\sigma^2 t^2\Big).$$

Factor constants out of the integral:

$$M_X(t) = \exp\Big(\mu t + \tfrac{1}{2}\sigma^2 t^2\Big) \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\Big(-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}\Big)\,dx.$$

The integral equals 1 (it is the integral of a normal density with mean $\mu + \sigma^2 t$ and variance σ^2). Therefore

$$\boxed{M_X(t) = \exp\Big(\mu t + \tfrac{1}{2}\sigma^2 t^2\Big), \quad t \in \mathbb{R}.}$$

(b) Differentiate the MGF to obtain moments.
First derivative:

$$M_X'(t) = \frac{d}{dt}\Big(e^{\mu t + \frac{1}{2}\sigma^2 t^2}\Big) = (\mu + \sigma^2 t)\, e^{\mu t + \frac{1}{2}\sigma^2 t^2}.$$

Evaluate at $t = 0$:

$$\mathbb{E}[X] = M_X'(0) = \mu.$$

Second derivative:

$$M_X''(t) = \frac{d}{dt}\Big((\mu + \sigma^2 t)e^{\mu t + \frac{1}{2}\sigma^2 t^2}\Big) = \big(\sigma^2 + (\mu + \sigma^2 t)^2\big)\, e^{\mu t + \frac{1}{2}\sigma^2 t^2}.$$

Evaluate at $t = 0$:

$$M_X''(0) = \sigma^2 + \mu^2.$$

Therefore the variance is

$$\mathrm{Var}(X) = M_X''(0) - (M_X'(0))^2 = (\sigma^2 + \mu^2) - \mu^2 = \sigma^2.$$

$$\boxed{M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}, \quad \mathbb{E}[X] = \mu, \quad \mathrm{Var}(X) = \sigma^2.}$$

Q2: Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with common MGF $M_X(t)$.

(a) Show that the MGF of $Z = X_1 + X_2 + \cdots + X_n$ is $M_Z(t) = (M_X(t))^n$.

(b) Now suppose N is a positive integer-valued random variable, independent of the X_i's, and define

$$Z = X_1 + X_2 + \cdots + X_N.$$

Show that the MGF of Z is

$$M_Z(t) = \mathbb{E}\big[(M_X(t))^N\big].$$

(c) Express $M_Z(t)$ in terms of the MGF of N, $M_N(s)$, and prove that

$$M_Z(t) = M_N(\log M_X(t)).$$

A:

(a) By independence,

$$M_Z(t) = \mathbb{E}[e^{t(X_1+\cdots+X_n)}] = \mathbb{E}[e^{tX_1}\cdots e^{tX_n}] = \prod_{i=1}^{n} \mathbb{E}[e^{tX_i}] = (M_X(t))^n.$$

(b) Conditional on $N = n$, we have

$$M_Z(t \mid N = n) = (M_X(t))^n.$$

Therefore,

$$M_Z(t) = \mathbb{E}[M_X(t)^N] = \sum_{n=0}^{\infty} P(N = n)\,(M_X(t))^n.$$

(c) Note that the MGF of N is

$$M_N(s) = \mathbb{E}[e^{sN}] = \sum_{n=0}^{\infty} P(N = n)\, e^{sn}.$$

If we substitute $s = \log M_X(t)$, then

$$M_N(\log M_X(t)) = \sum_{n=0}^{\infty} P(N = n)\,(M_X(t))^n.$$

But this equals $M_Z(t)$. Hence

$$\boxed{M_Z(t) = M_N(\log M_X(t))}.$$

Q3: The moment-generating function of a discrete random variable X is given by:

$$M_X(t) = \frac{1}{6}e^{-2t} + \frac{2}{6}e^{t} + \frac{3}{6}e^{4t}$$

Determine the probability mass function (PMF) of X.

A:

The given MGF is:

$$M_X(t) = \frac{1}{6}e^{(-2)t} + \frac{2}{6}e^{(1)t} + \frac{3}{6}e^{(4)t}$$

By comparing the given MGF to the definition of MGF, we can identify the possible values of X from the exponents of e, and the corresponding probabilities from the coefficients.

- The term $\frac{1}{6}e^{-2t}$ corresponds to $p_X(-2) = \frac{1}{6}$.
- The term $\frac{2}{6}e^{t}$ corresponds to $p_X(1) = \frac{2}{6} = \frac{1}{3}$.
- The term $\frac{3}{6}e^{4t}$ corresponds to $p_X(4) = \frac{3}{6} = \frac{1}{2}$.

The PMF of X, $p_X(x)$, is:

$$p_X(x) = \begin{cases} \frac{1}{6} & \text{for } x = -2, \\ \frac{1}{3} & \text{for } x = 1, \\ \frac{1}{2} & \text{for } x = 4, \\ 0 & \text{otherwise.} \end{cases}$$

As a check, we can ensure the probabilities sum to 1: $\frac{1}{6} + \frac{2}{6} + \frac{3}{6} = \frac{6}{6} = 1$. This is a valid PMF.

Q4: The lifetime X of an electronic component follows a Weibull distribution, whose Cumulative Distribution Function (CDF) is given by:

$$F_X(x) = 1 - e^{-x^2}, \quad \text{for } x \geq 0$$

Derive a formula to generate a random sample from this distribution using the inverse transform method with a uniform random variable $U \sim \text{Uniform}[0,1]$.

A:

$$F_X(X) = U$$

Given the CDF is $F_X(x) = 1 - e^{-x^2}$, we set:

$$1 - e^{-X^2} = U$$

Now, we solve for X:

$$e^{-X^2} = 1 - U$$

Take the natural logarithm of both sides:

$$\ln(e^{-X^2}) = \ln(1 - U)$$

$$-X^2 = \ln(1 - U)$$

Multiply by -1:

$$X^2 = -\ln(1 - U)$$

Since $x \geq 0$ for this distribution, we take the positive square root:

$$X = \sqrt{-\ln(1 - U)}$$

Since U is a random variable uniformly distributed on [0, 1], the random variable $1 - U$ is also uniformly distributed on [0, 1]. Therefore, we can simplify the generation formula for computational purposes:

$$\mathbf{X} = \sqrt{-\ln(\mathbf{U})}$$

To generate a sample, one would generate a number u from $U[0, 1]$ and plug it into this formula.

Q5: Let X be an exponential random variable with parameter $\lambda = 2$, and let Y be a Bernoulli random variable with parameter $p = 0.5$. Assume X and Y are independent.

(a) Let $Z = X + Y$. Find the MGF of Z, denoted $M_Z(t)$.

(b) Use $M_Z(t)$ to find the expected value $\mathbb{E}[Z]$ and the variance $Var(Z)$.

A:

(a) First, we find the MGFs for X and Y.

- For $X \sim \text{Exp}(\lambda = 2)$, the MGF is $M_X(t) = \frac{\lambda}{\lambda - t} = \frac{2}{2-t}$ for $t < 2$.
- For $Y \sim \text{Ber}(p = 0.5)$, the MGF is $M_Y(t) = (1-p)e^{t(0)} + pe^{t(1)} = 1 - p + pe^t = 0.5 + 0.5e^t$.

Since X and Y are independent, the MGF of their sum $Z = X + Y$ is the product of their MGFs:

$$M_Z(t) = M_X(t) \cdot M_Y(t) = \left(\frac{2}{2-t}\right) \cdot (0.5 + 0.5e^t)$$

$$\mathbf{M_Z(t)} = \frac{\mathbf{1 + e^t}}{\mathbf{2 - t}}$$

for $t < 2$.

(b) We find the first and second derivatives of $M_Z(t)$ and evaluate them at $t = 0$. Using the quotient rule $\frac{d}{dt}\frac{f}{g} = \frac{f'g - fg'}{g^2}$:

$$M_Z'(t) = \frac{(e^t)(2-t) - (1+e^t)(-1)}{(2-t)^2} = \frac{2e^t - te^t + 1 + e^t}{(2-t)^2} = \frac{3e^t - te^t + 1}{(2-t)^2}$$

The expected value is $\mathbb{E}[Z] = M_Z'(0)$:

$$\mathbb{E}[Z] = \frac{3e^0 - 0 \cdot e^0 + 1}{(2-0)^2} = \frac{3 - 0 + 1}{4} = \mathbf{1}$$

Next, we find the second derivative, $M_Z''(t)$:

$$M_Z''(t) = \frac{(3e^t - e^t - te^t)(2-t)^2 - (3e^t - te^t + 1) \cdot 2(2-t)(-1)}{(2-t)^4}$$

$$M_Z''(t) = \frac{(2e^t - te^t)(2-t) + 2(3e^t - te^t + 1)}{(2-t)^3}$$

The second moment is $\mathbb{E}[Z^2] = M_Z''(0)$:

$$\mathbb{E}[Z^2] = \frac{(2e^0 - 0)(2-0) + 2(3e^0 - 0 + 1)}{(2-0)^3} = \frac{2(2) + 2(3+1)}{8} = \frac{4+8}{8} = \frac{12}{8} = \mathbf{1.5}$$

The variance is $Var(Z) = \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2$:

$$Var(Z) = 1.5 - (1)^2 = \mathbf{0.5}$$

Q6: A Geiger counter measures background radiation. The machine's true average rate of detection, λ, is known to be either low ($\lambda_1 = 0.1$ counts/sec) or high ($\lambda_2 = 0.5$ counts/sec). The prior probability of the rate being low is $P(\lambda = \lambda_1) = 0.8$. The number of counts K in a given second follows a Poisson distribution with parameter λ. You turn on the counter for one second and observe exactly $k = 2$ counts.

What is the updated (posterior) probability that the rate is low, given your observation? That is, find $P(\lambda = \lambda_1 | K = 2)$.

A:
We use Bayes' theorem to find the posterior probability. We want to calculate $P(\lambda = \lambda_1 | K = 2)$. The formula is:

$$P(\lambda = \lambda_1 | K = 2) = \frac{P(K = 2 | \lambda = \lambda_1) P(\lambda = \lambda_1)}{P(K = 2)}$$

The denominator is the total probability of observing $K = 2$, which can be expanded using the law of total probability:

$$P(K = 2) = P(K = 2 | \lambda = \lambda_1) P(\lambda = \lambda_1) + P(K = 2 | \lambda = \lambda_2) P(\lambda = \lambda_2)$$

We are given the prior probabilities:

- $P(\lambda = \lambda_1) = 0.8$
- $P(\lambda = \lambda_2) = 1 - 0.8 = 0.2$

The likelihoods are calculated from the Poisson PMF, $P(K = k) = \frac{e^{-\lambda}\lambda^k}{k!}$.

- Likelihood for $\lambda_1 = 0.1$:

$$P(K = 2|\lambda = 0.1) = \frac{e^{-0.1}(0.1)^2}{2!} = \frac{0.01e^{-0.1}}{2} = 0.005e^{-0.1}$$

- Likelihood for $\lambda_2 = 0.5$:

$$P(K = 2|\lambda = 0.5) = \frac{e^{-0.5}(0.5)^2}{2!} = \frac{0.25e^{-0.5}}{2} = 0.125e^{-0.5}$$

Now, substitute these into the Bayes' theorem formula:

$$P(\lambda = \lambda_1|K = 2) = \frac{(0.005e^{-0.1}) \cdot (0.8)}{(0.005e^{-0.1})(0.8) + (0.125e^{-0.5})(0.2)}$$

$$= \frac{0.004e^{-0.1}}{0.004e^{-0.1} + 0.025e^{-0.5}}$$

Now, we can approximate the values: $e^{-0.1} \approx 0.9048$ and $e^{-0.5} \approx 0.6065$.

$$\approx \frac{0.004 \cdot (0.9048)}{0.004 \cdot (0.9048) + 0.025 \cdot (0.6065)}$$

$$\approx \frac{0.003619}{0.003619 + 0.015163} = \frac{0.003619}{0.018782} \approx \mathbf{0.1927}$$

The observed number of counts (very high) moved our posterior belief towards the higher rate.

Q7: Let $X_1, \dots, X_n$ be independent $\text{Exp}(\lambda)$ random variables (rate $\lambda > 0$). Using MGFs, show that the sum

$$Y = X_1 + X_2 + \cdots + X_n$$

has the Gamma distribution $\text{Gamma}(n, \lambda)$.

A:
The MGF of an $\text{Exp}(\lambda)$ random variable X_i is

$$M_{X_i}(t) = \mathbb{E}[e^{tX_i}] = \frac{\lambda}{\lambda - t}, \qquad t < \lambda.$$

Because the X_i are independent, the MGF of the sum Y is the product of the individual MGFs:

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t) = \left(\frac{\lambda}{\lambda - t}\right)^n, \qquad t < \lambda.$$

Recall that the MGF $\big(\lambda/(\lambda - t)\big)^\alpha$ corresponds to a $\text{Gamma}(\alpha, \lambda)$ distribution. Comparing the forms we see that $M_Y(t)$ equals the Gamma MGF with shape parameter $\alpha = n$. Hence by uniqueness of the MGF, Y has the Gamma distribution with parameters (n, λ):

$$\boxed{Y \sim \text{Gamma}(n, \lambda).}$$

Optionally, one can read off the mean and variance:

$$\mathbb{E}[Y] = \frac{n}{\lambda}, \qquad \mathrm{Var}(Y) = \frac{n}{\lambda^2},$$

which agree with adding n independent $\mathrm{Exp}(\lambda)$ random variables.

Q8: Estimate the integral

$$I_2 = \int_0^\infty e^{-x} dx$$

using Monte Carlo integration with N random samples from $U[0, 1]$.

A:

To use samples from $U[0, 1]$, let

$$x = \frac{y}{1-y} \quad \text{with} \quad y \sim U[0, 1]$$

Then,

$$dx = \frac{dy}{(1-y)^2}$$

$$I_2 = \int_0^1 \frac{e^{-y/(1-y)}}{(1-y)^2} dy$$

$$I_2 \approx \frac{1}{N} \sum_{i=1}^{N} \frac{e^{-y_i/(1-y_i)}}{(1-y_i)^2}, \quad y_i \sim U[0, 1]$$

The steps to approximate the integral would be:

- Generate $y_1, y_2, \ldots, y_N \sim U[0, 1]$
- Compute $f(y_i) = \frac{e^{-y_i/(1-y_i)}}{(1-y_i)^2}$
- Average the results

Q9: Let X and Y be two random variables for which $Var(X) = 4$, $Var(Y) = 9$, and the covariance is $Cov(X, Y) = -2$.

(a) Find the variance of the random variable $Z = 3X - 2Y + 5$.

(b) If X and Y were independent, what would $Var(Z)$ be?

A:

(a) We use the properties of variance. First, adding a constant does not change the variance, so $Var(3X - 2Y + 5) = Var(3X - 2Y)$. The general formula for the variance of a linear combination of two random variables is:

$$Var(aX + bY) = a^2Var(X) + b^2Var(Y) + 2abCov(X, Y)$$

In our case, $a = 3$ and $b = -2$.

$$Var(3X - 2Y) = (3)^2Var(X) + (-2)^2Var(Y) + 2(3)(-2)Cov(X, Y)$$

$$Var(Z) = 9 \cdot Var(X) + 4 \cdot Var(Y) - 12 \cdot Cov(X, Y)$$

Substituting the given values:

$$Var(Z) = 9(4) + 4(9) - 12(-2) = 36 + 36 + 24 = \mathbf{96}$$

(b) If X and Y are independent, their covariance is zero: $Cov(X, Y) = 0$. The formula for variance simplifies to:

$$Var(Z) = a^2 Var(X) + b^2 Var(Y)$$

$$Var(Z) = 9(4) + 4(9) = 36 + 36 = \mathbf{72}$$

Q10: Let X be a discrete random variable with the following probability mass function (PMF):

$$p_X(-1) = 0.2, \quad p_X(0) = 0.5, \quad p_X(1) = 0.3.$$

(a) Find the Moment Generating Function (MGF) of X, denoted by $M_X(t)$.

(b) Using the MGF from part (a), compute the first and second moments of X, i.e., $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$.

(c) Use the results from part (b) to find the variance of X, $Var(X)$.

A:

(a) The moment generating function (MGF) is defined as $M_X(t) = \mathbb{E}[e^{tX}]$. For a discrete variable, this is $\sum_x e^{tx} p_X(x)$.

$$M_X(t) = e^{t(-1)} p_X(-1) + e^{t(0)} p_X(0) + e^{t(1)} p_X(1)$$

$$M_X(t) = 0.2e^{-t} + 0.5e^0 + 0.3e^t$$

$$\mathbf{M_X(t) = 0.2e^{-t} + 0.5 + 0.3e^t}$$

(b) To find the moments, we use the property $\mathbb{E}[X^r] = M_X^{(r)}(0)$, which is the r-th derivative of the MGF evaluated at $t = 0$.
For the first moment, $\mathbb{E}[X]$, we find the first derivative:

$$M_X'(t) = \frac{d}{dt}(0.2e^{-t} + 0.5 + 0.3e^t) = -0.2e^{-t} + 0.3e^t$$

Now, evaluate at $t = 0$:

$$\mathbb{E}[X] = M_X'(0) = -0.2e^0 + 0.3e^0 = -0.2 + 0.3 = \mathbf{0.1}$$

For the second moment, $\mathbb{E}[X^2]$, we find the second derivative:

$$M_X''(t) = \frac{d}{dt}(-0.2e^{-t} + 0.3e^t) = 0.2e^{-t} + 0.3e^t$$

Evaluate at $t = 0$:

$$\mathbb{E}[X^2] = M_X''(0) = 0.2e^0 + 0.3e^0 = 0.2 + 0.3 = \mathbf{0.5}$$

(c) The variance is given by the formula $Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.

$$Var(X) = 0.5 - (0.1)^2 = 0.5 - 0.01 = \mathbf{0.49}$$

Q11: Let X be a continuous random variable with the Gamma(α, λ) PDF

$$f_X(x) = \begin{cases} \dfrac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, & x > 0, \\ 0, & \text{otherwise}, \end{cases} \qquad \alpha > 0,\ \lambda > 0.$$

Find the moment generating function $M_X(t) = \mathbb{E}[e^{tX}]$ (specify the region of t for which it is finite).

By definition,

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_0^\infty e^{tx} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} \, dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\lambda-t)x} \, dx.$$

The last integral converges iff $\lambda - t > 0$, i.e. for $t < \lambda$. For such t we may evaluate the integral using the Gamma integral $\displaystyle\int_0^\infty x^{\alpha-1} e^{-cx} \, dx = \frac{\Gamma(\alpha)}{c^\alpha}$ for $c > 0$. Here $c = \lambda - t$, so

$$\int_0^\infty x^{\alpha-1} e^{-(\lambda-t)x} \, dx = \frac{\Gamma(\alpha)}{(\lambda - t)^\alpha}.$$

Substituting back,

$$M_X(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)}{(\lambda - t)^\alpha} = \left(\frac{\lambda}{\lambda - t} \right)^\alpha, \qquad t < \lambda.$$

Therefore

$$\boxed{M_X(t) = \left(\frac{\lambda}{\lambda - t} \right)^\alpha, \quad t < \lambda.}$$

Q12: Let $X \sim$ Poisson(λ) with $\lambda > 0$.

(a) Find the moment generating function $M_X(t)$.

(b) Use $M_X(t)$ to compute $\mathbb{E}[X]$ and Var(X).

A:
By definition the moment generating function for a discrete random variable is

$$M_X(t) = \mathbb{E}\left[e^{tX}\right] = \sum_{x \in \Omega'} e^{tx} p_X(x) = \sum_{x=0}^{\infty} e^{tx} p_X(x) = \sum_{x=0}^{\infty} e^{tx} e^{-\lambda} \frac{\lambda^x}{x!}.$$

Factor and recognize the exponential series:

$$M_X(t) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} \, e^{\lambda e^t} = \exp\big(\lambda(e^t - 1)\big).$$

Thus

$$\boxed{M_X(t) = \exp\big(\lambda(e^t - 1)\big)}, \qquad t \in \mathbb{R}.$$

To obtain moments differentiate $M_X(t)$. First derivative:

$$M_X'(t) = \lambda e^t \exp\big(\lambda(e^t - 1)\big) = \lambda e^t M_X(t).$$

Evaluate at $t = 0$ to get the mean:

$$\mathbb{E}[X] = M_X'(0) = \lambda \cdot 1 \cdot M_X(0) = \lambda,$$

since $M_X(0) = 1$.
Second derivative:

$$M_X''(t) = \frac{d}{dt}\big(\lambda e^t M_X(t)\big) = \lambda e^t M_X(t) + \lambda e^t \big(\lambda e^t M_X(t)\big) = \big(\lambda e^t + \lambda^2 e^{2t}\big) M_X(t).$$

Evaluate at $t = 0$:

$$\mathbb{E}[X^2] = M_X''(0) = \lambda + \lambda^2.$$

Now compute the variance:

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = (\lambda + \lambda^2) - \lambda^2 = \lambda.$$

$$\boxed{\mathbb{E}[X] = \lambda, \qquad \mathrm{Var}(X) = \lambda.}$$

Q13: Let X and Y be independent random variables with PMFs:

$$p_X(x) = \begin{cases} 1/3, & \text{if } x \in \{1, 2, 3\}, \\ 0, & \text{otherwise,} \end{cases} \qquad p_Y(y) = \begin{cases} 1/2, & \text{if } y = 0, \\ 1/3, & \text{if } y = 1, \\ 1/6, & \text{if } y = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Find the PMF of $Z = X + Y$, using the convolution formula.

A:
The possible values (the support) of Z are the sums of all possible values of X and Y.

- Minimum value of Z: $\min(X) + \min(Y) = 1 + 0 = 1$
- Maximum value of Z: $\max(X) + \max(Y) = 3 + 2 = 5$

So, the possible values for Z are the integers $\{1, 2, 3, 4, 5\}$. We will now compute the probability for each of these values.

- **For z = 1:**

$$\begin{aligned} p_Z(1) &= \mathbb{P}(X = 1)\mathbb{P}(Y = 0) \\ &= \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{6} \end{aligned}$$

- **For z = 2:**

$$\begin{aligned} p_Z(2) &= \mathbb{P}(X=1)\mathbb{P}(Y=1) + \mathbb{P}(X=2)\mathbb{P}(Y=0) \\ &= \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) \\ &= \frac{1}{9} + \frac{1}{6} = \frac{2+3}{18} = \frac{5}{18} \end{aligned}$$

- **For z = 3:**

$$\begin{aligned} p_Z(3) &= \mathbb{P}(X=1)\mathbb{P}(Y=2) + \mathbb{P}(X=2)\mathbb{P}(Y=1) + \mathbb{P}(X=3)\mathbb{P}(Y=0) \\ &= \left(\frac{1}{3}\right)\left(\frac{1}{6}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) \\ &= \frac{1}{18} + \frac{1}{9} + \frac{1}{6} = \frac{1+2+3}{18} = \frac{6}{18} = \frac{1}{3} \end{aligned}$$

- **For z = 4:**

$$\begin{aligned} p_Z(4) &= \mathbb{P}(X=2)\mathbb{P}(Y=2) + \mathbb{P}(X=3)\mathbb{P}(Y=1) \\ &= \left(\frac{1}{3}\right)\left(\frac{1}{6}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) \\ &= \frac{1}{18} + \frac{1}{9} = \frac{1+2}{18} = \frac{3}{18} = \frac{1}{6} \end{aligned}$$

- **For z = 5:**

$$\begin{aligned} p_Z(5) &= \mathbb{P}(X=3)\mathbb{P}(Y=2) \\ &= \left(\frac{1}{3}\right)\left(\frac{1}{6}\right) = \frac{1}{18} \end{aligned}$$

The PMF for $Z = X + Y$ is:

$$p_Z(z) = \begin{cases} 1/6, & \text{if } z = 1 \\ 5/18, & \text{if } z = 2 \\ 1/3, & \text{if } z = 3 \\ 1/6, & \text{if } z = 4 \\ 1/18, & \text{if } z = 5 \\ 0, & \text{otherwise.} \end{cases}$$

Q14: Use the convolution formula to establish that the sum of two independent Poisson random variables with parameters μ and λ respectively, is Poisson with parameter $\mu + \lambda$.

A: We want to show that if $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ are independent random variables, then their sum $Z = X + Y$ follows a Poisson distribution with parameter $\lambda + \mu$. For any non-negative integer k:

- For $X \sim \text{Poisson}(\lambda)$, the PMF is:

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

- For $Y \sim \text{Poisson}(\mu)$, the PMF is:

$$P(Y = k) = \frac{e^{-\mu}\mu^k}{k!}$$

The PMF of the sum $Z = X + Y$ can be found using the convolution formula for independent discrete random variables:

$$P(Z = n) = \sum_{k=-\infty}^{\infty} P(X = k)P(Y = n - k)$$

Since Poisson variables can only take non-negative integer values, we know that $k \geq 0$ and $(n - k) \geq 0$. This restricts the summation index k to the range $0 \leq k \leq n$.
Let's substitute the Poisson PMFs into the formula:

$$P(Z = n) = \sum_{k=0}^{n} \left(\frac{e^{-\lambda}\lambda^k}{k!}\right)\left(\frac{e^{-\mu}\mu^{n-k}}{(n-k)!}\right)$$

Now, we simplify the expression. We can factor out the exponential terms since they don't depend on the summation index k:

$$\begin{aligned} P(Z = n) &= e^{-\lambda}e^{-\mu}\sum_{k=0}^{n} \frac{\lambda^k\mu^{n-k}}{k!(n-k)!} \\ &= e^{-(\lambda+\mu)}\sum_{k=0}^{n} \frac{\lambda^k\mu^{n-k}}{k!(n-k)!} \end{aligned}$$

To make the summation look like a familiar formula, we can multiply and divide by $n!$:

$$P(Z = n) = \frac{e^{-(\lambda+\mu)}}{n!}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\lambda^k\mu^{n-k}$$

The term $\frac{n!}{k!(n-k)!}$ is the binomial coefficient $\binom{n}{k}$. The summation now perfectly matches the **binomial theorem**, which states that:

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Applying this theorem with $a = \lambda$ and $b = \mu$, we get:

$$\sum_{k=0}^{n} \binom{n}{k} \lambda^k\mu^{n-k} = (\lambda + \mu)^n$$

Substituting this back into our expression for $P(Z = n)$, we arrive at the final PMF:

$$P(Z = n) = \frac{e^{-(\lambda+\mu)}(\lambda + \mu)^n}{n!}$$

This is the PMF for a Poisson random variable with parameter $\lambda + \mu$. Therefore, we have established that the sum of two independent Poisson random variables is also a Poisson random variable whose parameter is the sum of the individual parameters.

Q15: We want to estimate $\theta = \mathbb{E}[\sqrt{X}]$ where X is a Normal random variable ($X \sim \mathcal{N}(0,1)$). Using importance sampling, formulate an estimator for θ by drawing N samples, $Y_1, \ldots, Y_N$, from a uniform distribution $Y \sim U[0,5]$.

A:
The goal is to estimate $\theta = \mathbb{E}_{X\sim f}[\sqrt{X}]$ using samples from $Y \sim g$, where $f(x)$ is the Normal PDF and $g(y)$ is the uniform PDF.
The PDFs are:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \text{for } x \in (-\infty, \infty) \quad \text{and} \quad g(y) = \begin{cases} \frac{1}{5} & \text{if } y \in [0,5] \\ 0 & \text{otherwise} \end{cases}$$

The importance sampling estimator is based on the principle $\mathbb{E}_f[h(X)] = \mathbb{E}_g\left[h(Y)\frac{f(Y)}{g(Y)}\right]$. Here, $h(Y) = \sqrt{Y}$.
The estimator $\hat{\theta}_N$ is the sample mean:

$$\hat{\theta}_N = \frac{1}{N}\sum_{i=1}^{N} \sqrt{Y_i}\frac{f(Y_i)}{g(Y_i)}$$

Substituting the PDFs for any $Y_i \in [0,5]$:

$$\frac{f(Y_i)}{g(Y_i)} = \frac{\frac{1}{\sqrt{2\pi}} e^{-Y_i^2/2}}{1/5} = \frac{5}{\sqrt{2\pi}} e^{-Y_i^2/2}$$

So, the final estimator is:

$$\hat{\theta}_N = \frac{1}{N}\sum_{i=1}^{N} \frac{5}{\sqrt{2\pi}} \sqrt{Y_i} e^{-Y_i^2/2}$$

where each Y_i is a sample drawn from $U[0,5]$.

Q16: Suppose, you want to generate samples from a Gaussian random variable with PDF $p(x)$ for $\mathcal{N}(\mu = 0.5, \sigma^2 = 0.04)$. Use the Accept-Reject method with a proposal distribution $q(x) = U(0,1)$.

(a) Find the smallest constant c such that $p(x) \leq c \cdot q(x)$ for all $x \in [0,1]$.
(b) Outline the algorithm to generate one sample.

A:

(a) The target PDF is $p(x) = \frac{1}{0.2\sqrt{2\pi}} \exp\left(-\frac{(x-0.5)^2}{2(0.04)}\right)$ and the proposal distribution is $q(x) = 1$ for $x \in [0,1]$. We need to find the smallest c such that $p(x) \leq c \cdot q(x)$ on the support of $q(x)$, which simplifies to finding $c \geq p(x)$ for $x \in [0,1]$.
The smallest value for c is the maximum value of $p(x)$ on the interval $[0,1]$. The maximum of a Gaussian PDF occurs at its mean, $\mu = 0.5$, which is inside the interval.

$$c = \max_{x\in[0,1]} p(x) = p(0.5) = \frac{1}{0.2\sqrt{2\pi}} \exp(0) = \frac{5}{\sqrt{2\pi}}$$

So, the smallest possible constant is $\mathbf{c} = \frac{\mathbf{5}}{\sqrt{\mathbf{2\pi}}} \approx \mathbf{1.995}$.

(b) The algorithm is as follows:

   i. Generate a candidate sample y from the proposal distribution $U(0,1)$.

   ii. Generate a random number u from a $U(0,1)$ distribution.

   iii. Check the acceptance condition: $u \leq \frac{p(y)}{c \cdot q(y)}$.

$$\frac{p(y)}{c \cdot q(y)} = \frac{\frac{1}{0.2\sqrt{2\pi}} \exp\left(-\frac{(y-0.5)^2}{0.08}\right)}{\left(\frac{1}{0.2\sqrt{2\pi}}\right) \cdot 1} = \exp\left(-\frac{(y-0.5)^2}{0.08}\right)$$

   The condition becomes:

$$u \leq \exp\left(-\frac{(y-0.5)^2}{0.08}\right)$$

   iv. If the condition is met, accept y as the sample. Otherwise, reject it and return to step 1.

Q17: Estimate the integral

$$I_1 = \int_0^a \sqrt{x}\, dx$$

using Monte Carlo integration with N random samples from the uniform distribution $U[0,1]$. Let $a > 0$ be a given constant. Also write the Monte Carlo estimate assuming X_i's are sampled according to some pdf $g(X_i)$ in the same domain.

A:

If $y \sim U[0,1]$, then let $x = ay$. Hence $x \sim U[0,a]$. By SLLN, we have:

$$I_1 \approx a \cdot \frac{1}{N} \sum_{i=1}^{N} \sqrt{ay_i}$$

If we are provided samples Y are sampled from some pdf $g(Y)$ then:

$$I_1 = \int_0^a \sqrt{x}\, dx = \int_0^a \frac{\sqrt{y}}{g(y)} \cdot g(y)\, dy = \mathbb{E}[\frac{f(y)}{g(y)}] \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(Y_i)}{g(Y_i)}$$

where $f(y) = \sqrt{y}$.

Q18: Suppose we want to generate a random variable X with a target pdf $f(x)$. We have access to a method for generating a random variable Y from a simpler proposal density function $g(x)$.

Let c be a constant such that the "envelope condition" $\frac{f(y)}{g(y)} \leq c$ holds for all y for which $g(y) > 0$. The rejection method algorithm is as follows:

(a) Generate a candidate sample Y from the proposal density $g(y)$.

(b) Generate a random number U from a Uniform$(0,1)$ distribution.

(c) **Acceptance Condition:** If $U \leq \frac{f(Y)}{cg(Y)}$, then set $X = Y$. Otherwise, reject the sample and return to step 1.

Prove the validity of this algorithm.

A: The core of the proof is to show that the cumulative distribution function (CDF) of an accepted sample X is equal to the target CDF, $F(x) = \int_{-\infty}^{x} f(v)dv$. An accepted sample is, by definition, a candidate sample Y that satisfies the acceptance condition.

Therefore, we must prove that the conditional probability $\mathbb{P}\left(Y \leq x \mid U \leq \frac{f(Y)}{cg(Y)}\right)$ is equal to $F(x)$. We will first calculate the unconditional probability of accepting a sample, which we call p.

Let A be the event that a sample is accepted in a single trial.

$$A = \left\{U \leq \frac{f(Y)}{cg(Y)}\right\}$$

The probability of this event, $p = \mathbb{P}(A)$, can be found by conditioning on the value of Y and integrating over all possible values.

$$\begin{aligned} p = \mathbb{P}(A) &= \int_{-\infty}^{\infty} \mathbb{P}\left(U \leq \frac{f(Y)}{cg(Y)} \mid Y = y\right) g(y)\,dy \\ &= \int_{-\infty}^{\infty} \mathbb{P}\left(U \leq \frac{f(y)}{cg(y)}\right) g(y)\,dy \end{aligned}$$

Since U is Uniform$(0,1)$, the probability $\mathbb{P}(U \leq k)$ is simply k (for $0 \leq k \leq 1$). The envelope condition ensures $\frac{f(y)}{cg(y)} \in [0,1]$.

$$\begin{aligned} p &= \int_{-\infty}^{\infty} \frac{f(y)}{cg(y)} g(y)\,dy \\ &= \frac{1}{c} \int_{-\infty}^{\infty} f(y)\,dy \\ &= \frac{1}{c} \quad \text{(since } f(y) \text{ is a pdf, its integral is 1)} \end{aligned}$$

The number of trials N needed to get one accepted sample follows a geometric distribution with success probability $p = 1/c$, so the expected number of trials is $\mathbb{E}[N] = 1/p = c$.

Now, let's find the CDF of an accepted sample. We want to compute $F_X(x) = \mathbb{P}(X \leq x)$. This is the conditional probability that $Y \leq x$ given that event A (acceptance) occurred.

$$F_X(x) = \mathbb{P}(Y \leq x \mid A) = \frac{\mathbb{P}(\{Y \leq x\} \cap A)}{\mathbb{P}(A)}$$

We already know the denominator is $\mathbb{P}(A) = 1/c$. Let's compute the numerator, which is the joint probability that the candidate sample is less than or equal to x and is accepted.

$$\begin{aligned}
\mathbb{P}(\{Y \leq x\} \cap A) &= \mathbb{P}\left(Y \leq x \text{ and } U \leq \frac{f(Y)}{cg(Y)}\right) \\
&= \int_{-\infty}^{x} \mathbb{P}\left(U \leq \frac{f(Y)}{cg(Y)} \mid Y = y\right) g(y)\,dy \\
&= \int_{-\infty}^{x} \frac{f(y)}{cg(y)} g(y)\,dy \\
&= \frac{1}{c} \int_{-\infty}^{x} f(y)\,dy \\
&= \frac{F(x)}{c}
\end{aligned}$$

Now we can compute the conditional probability:

$$\begin{aligned}
F_X(x) = \mathbb{P}(Y \leq x \mid A) &= \frac{\mathbb{P}(\{Y \leq x\} \cap A)}{\mathbb{P}(A)} \\
&= \frac{F(x)/c}{1/c} \\
&= F(x)
\end{aligned}$$

Since the CDF of the generated random variable X is $F(x)$, its probability density function must be $f(x)$. This completes the proof.