Quiz: Numerical Algorithms

Instructions

Answer all questions. Show work clearly.

Q1. (Long Answer) **[13 pts]**

Higher-order numerical differentiation (4th-order central difference).

Let

$$f(x) = \ln(1+x), \qquad x > -1.$$

Consider the 5-point, 4th-order central-difference estimator of $f'(x)$:

$$D_4(h) = \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h}.$$

(a) Using Taylor expansions about x up to the necessary order, show that the truncation error of $D_4(h)$ is $\mathcal{O}(h^4)$. (Show at least **three** explicit expansion/cancellation steps.)

(b) Compute $D_4(h)$ at $x = 0.5$ using $h = 10^{-2}$ and $h = 10^{-4}$ (report at least 3 significant digits).

(c) Compute the exact derivative $f'(0.5)$ and the absolute errors for both step sizes.

(d) Briefly explain (2–3 sentences) why taking h *too small* can still degrade the estimate in floating-point arithmetic (mention cancellation/round-off).

Q2. (Long Answer) **[12 pts]**

Numerical integration via higher-order Gauss–Legendre quadrature.

Approximate

$$I = \int_0^1 \frac{1}{1+x^2}\, dx$$

using the **3-point** Gauss–Legendre rule on $[0,1]$. Recall the 3-point Gauss–Legendre nodes and weights on $[-1,1]$:

$$t_1 = -\sqrt{\frac{3}{5}}, \quad t_2 = 0, \quad t_3 = \sqrt{\frac{3}{5}}, \qquad w_1 = \frac{5}{9}, \quad w_2 = \frac{8}{9}, \quad w_3 = \frac{5}{9}.$$

(a) Write the affine map $x(t) = \frac{1}{2}t + \frac{1}{2}$ from $[-1,1]$ to $[0,1]$, and the transformed formula

$$\int_0^1 f(x)\, dx = \frac{1}{2}\int_{-1}^{1} f\left(\frac{t+1}{2}\right) dt.$$

(b) Compute the mapped nodes $x_i = \frac{t_i+1}{2}$ for $i = 1, 2, 3$.

(c) Form the Gauss approximation

$$Q_3 = \frac{1}{2}\sum_{i=1}^{3} w_i \frac{1}{1+x_i^2},$$

and evaluate it numerically (at least 3 significant digits).

(d) Compute the absolute error using the exact value $I = \arctan(1) - \arctan(0) = \frac{\pi}{4}$.

Q3. **(Long Answer)** **[13 pts]**

Gradient descent: 3 explicit steps (1 digit precision after decimal) on a quadratic.

Consider

$$f(x) = \frac{1}{2}x^\top Q x - b^\top x, \qquad Q = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

and gradient descent with constant stepsize $\eta > 0$:

$$x^{(k+1)} = x^{(k)} - \eta \nabla f(x^{(k)}).$$

(a) Derive $\nabla f(x)$.

(b) Starting from $x^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ with stepsize $\eta = 0.1$, compute $x^{(1)}, x^{(2)}, x^{(3)}$. For each iteration, show at least **three** explicit steps: (i) compute $\nabla f(x^{(k)})$, (ii) compute $\eta \nabla f(x^{(k)})$, (iii) update $x^{(k+1)}$.

(c) Compute $f(x^{(1)})$ and $f(x^{(3)})$ and state whether the objective decreased over these steps.

$(A - A^*)^*$

$\nabla f(x) = Qx - b$

$(A^* - A)$

$P \to Q$

$\cancel{P \to \neg Q}$ $\quad Q \to \neg R$

$P \to \neg R \qquad S_1, S_2 \to S_3$

$S_3 \quad P \to \neg R$

$S_4 \quad (R \to \neg Q) \to (P \to \neg R)$