

# Probability and Statistics

## Quiz 2 - Solutions

### Question 1

#### Problem Statement:

Suppose  $Z = X + U$ , where  $X$  is an exponential random variable with parameter  $\lambda$  and  $U$  is a Uniform(0,1) random variable. Derive the first two moments of  $Z$  using MGF. (You have to derive the MGF of  $X$  and  $U$  first.) Assume  $X$  and  $U$  are independent with respect to any other random variable.

**Note:** 0.5 marks will be deducted if the correct domain of  $t$  is not written.

#### Solution

##### Step 1: MGF of $X$ and $U$

For  $X \sim \text{Exponential}(\lambda)$ :

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \frac{\lambda}{\lambda - t}, \quad t < \lambda.$$

For  $U \sim \text{Uniform}(0, 1)$ :

$$M_U(t) = E[e^{tU}] = \int_0^1 e^{tu} du = \frac{e^t - 1}{t}, \quad t \neq 0, \quad M_U(0) = 1.$$

##### Step 2: MGF of $Z$

Since  $X$  and  $U$  are independent,

$$M_Z(t) = M_X(t) M_U(t) = \frac{\lambda}{\lambda - t} \cdot \frac{e^t - 1}{t}.$$

Define

$$A(t) = M_X(t) = \frac{\lambda}{\lambda - t}, \quad B(t) = M_U(t) = \frac{e^t - 1}{t},$$

so that  $M_Z(t) = A(t)B(t)$ .

##### Step 3: First Moment using MGF

Compute derivatives of  $A(t)$  and  $B(t)$ :

$$A'(t) = \frac{\lambda}{(\lambda - t)^2}, \quad A''(t) = \frac{2\lambda}{(\lambda - t)^3}.$$

Hence,

$$A(0) = 1, \quad A'(0) = \frac{1}{\lambda}, \quad A''(0) = \frac{2}{\lambda^2}.$$

Now expand  $B(t)$  using the Taylor series of  $e^t$ :

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \dots \Rightarrow \frac{e^t - 1}{t} = 1 + \frac{t}{2} + \frac{t^2}{6} + \dots$$

Thus,

$$B(0) = 1, \quad B'(0) = \frac{1}{2}, \quad B''(0) = \frac{1}{3}.$$

Now,

$$M'_Z(t) = A'(t)B(t) + A(t)B'(t).$$

At  $t = 0$ ,

$$M'_Z(0) = A'(0)B(0) + A(0)B'(0) = \frac{1}{\lambda} + \frac{1}{2}.$$

Hence,

$$E[Z] = M'_Z(0) = \frac{1}{\lambda} + \frac{1}{2}.$$

#### Step 4: Second Moment using MGF

Differentiate again:

$$M''_Z(t) = A''(t)B(t) + 2A'(t)B'(t) + A(t)B''(t).$$

At  $t = 0$ :

$$\begin{aligned} M''_Z(0) &= A''(0)B(0) + 2A'(0)B'(0) + A(0)B''(0) \\ &= \frac{2}{\lambda^2} + 2\left(\frac{1}{\lambda}\right)\left(\frac{1}{2}\right) + \frac{1}{3} \\ &= \frac{2}{\lambda^2} + \frac{1}{\lambda} + \frac{1}{3}. \end{aligned}$$

Thus,

$$E[Z^2] = M''_Z(0) = \frac{2}{\lambda^2} + \frac{1}{\lambda} + \frac{1}{3}.$$

#### Final Answers:

$$E[Z] = \frac{1}{\lambda} + \frac{1}{2}, \quad E[Z^2] = \frac{2}{\lambda^2} + \frac{1}{\lambda} + \frac{1}{3}.$$

*Note: Any other valid and correct derivation using the MGF of  $Z$  will be marked accordingly.*

## Question 2

### Problem Statement:

Let  $X_1, X_2, X_3, \dots$  be a sequence of Laplacian (double-exponential) random variables with density

$$f_{X_n}(x) = \frac{n}{2} e^{-n|x|}, \quad x \in \mathbb{R}.$$

Show that  $X_n \xrightarrow{P} 0$ .

### Proof 1: By Definition

We must show that for every  $\varepsilon > 0$ ,

$$\mathbb{P}(|X_n| > \varepsilon) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Use the density to compute the tail probability. For some arbitrary  $\varepsilon > 0$  we have,

$$\mathbb{P}(|X_n| > \varepsilon) = \int_{|x| > \varepsilon} \frac{n}{2} e^{-n|x|} dx = 2 \int_{\varepsilon}^{\infty} \frac{n}{2} e^{-nx} dx = n \int_{\varepsilon}^{\infty} e^{-nx} dx.$$

Now compute

$$n \int_{\varepsilon}^{\infty} e^{-nx} dx = n \left[ \frac{-1}{n} e^{-nx} \right]_{x=\varepsilon}^{\infty} = e^{-n\varepsilon}.$$

Hence for any fixed  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow \infty} e^{-n\varepsilon} = 0.$$

Therefore  $\mathbb{P}(|X_n| > \varepsilon) \rightarrow 0$ , so by definition  $X_n \xrightarrow{P} 0$ .

### Proof 2: Using Markov's Inequality

Markov's inequality states that for any nonnegative random variable  $Y$  and any  $\varepsilon > 0$ ,

$$\mathbb{P}(Y > \varepsilon) \leq \frac{\mathbb{E}[Y]}{\varepsilon}.$$

Since  $X_n$  is not nonnegative in general, apply the inequality to  $Y := |X_n|$ . Hence

$$\mathbb{P}(|X_n| > \varepsilon) \leq \frac{\mathbb{E}[|X_n|]}{\varepsilon}.$$

Therefore,

$$\mathbb{E}[|X_n|] = \int_{-\infty}^{\infty} |x| \frac{n}{2} e^{-n|x|} dx = 2 \cdot \frac{n}{2} \int_0^{\infty} x e^{-nx} dx = n \int_0^{\infty} x e^{-nx} dx.$$

Evaluate the integral (use  $\int_0^{\infty} x e^{-nx} dx = 1/n^2$ ):

$$\mathbb{E}[|X_n|] = n \cdot \frac{1}{n^2} = \frac{1}{n}.$$

Hence by Markov,

$$\mathbb{P}(|X_n| > \varepsilon) \leq \frac{\mathbb{E}[|X_n|]}{\varepsilon} = \frac{1}{n\varepsilon} \rightarrow 0.$$

Thus  $X_n \xrightarrow{P} 0$ .

### Proof 3: Using Chebyshev's Inequality

Chebyshev's inequality:  $\mathbb{P}(|X_n - \mathbb{E}[X_n]| > \varepsilon) \leq \text{Var}(X_n)/\varepsilon^2$ .

Since  $f_{X_n}$  is even, the integrand  $x f_{X_n}(x)$  is odd, so

$$\mathbb{E}[X_n] = \int_{-\infty}^{\infty} x f_{X_n}(x) dx = 0.$$

For the second moment:

$$\mathbb{E}[X_n^2] = \int_{-\infty}^{\infty} x^2 \frac{n}{2} e^{-n|x|} dx = 2 \cdot \frac{n}{2} \int_0^{\infty} x^2 e^{-nx} dx = n \int_0^{\infty} x^2 e^{-nx} dx.$$

Use  $\int_0^{\infty} x^2 e^{-nx} dx = \frac{2}{n^3}$  (gamma function property) to get

$$\mathbb{E}[X_n^2] = n \cdot \frac{2}{n^3} = \frac{2}{n^2}.$$

Thus  $\text{Var}(X_n) = 2/n^2$ . By Chebyshev,

$$\mathbb{P}(|X_n| > \varepsilon) \leq \frac{\text{Var}(X_n)}{\varepsilon^2} = \frac{2}{n^2 \varepsilon^2} \rightarrow 0,$$

so again  $X_n \xrightarrow{P} 0$ .

## Question 3

### Problem Statement:

Give a procedure to convert samples from an Exponential random variable to Uniform(0,1). Justify why the procedure is correct.

### Solution

We first recall the probability density function (pdf) and cumulative distribution function (cdf) of both distributions.

- **Uniform(0,1):**

$$f_U(u) = \begin{cases} 1, & 0 \leq u \leq 1, \\ 0, & \text{otherwise,} \end{cases} \quad F_U(u) = \begin{cases} 0, & u < 0, \\ u, & 0 \leq u \leq 1, \\ 1, & u > 1. \end{cases}$$

- **Exponential( $\lambda$ ):**

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0, \end{cases} \quad F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-\lambda x}, & x \geq 0. \end{cases}$$

### Part (a): Procedure

#### Property :

We can use the property **Universality of the Uniform**, which states:

*For any continuous random variable  $X$  with cdf  $F_X$ , the random variable  $F_X(X) \sim \text{Uniform}(0, 1)$ .*

#### Steps :

1. Draw a sample  $x$  from the exponential distribution  $X \sim \text{Exp}(\lambda)$ .
2. Compute  $u = F_X(x) = 1 - e^{-\lambda x}$ .
3. Return  $u$ . This gives a sample from Uniform(0, 1).

### Part (b): Justification

**Claim:** If  $X \sim \text{Exp}(\lambda)$  and  $U = F_X(X) = 1 - e^{-\lambda X}$ , then  $U \sim \text{Uniform}(0, 1)$ .

**Proof:**

$$\begin{aligned}P(U \leq u) &= P(F_X(X) \leq u) \\&= P(1 - e^{-\lambda X} \leq u) \\&= P(e^{-\lambda X} \geq 1 - u) \\&= P(-\lambda X \geq \ln(1 - u)) \\&= P\left(X \leq -\frac{1}{\lambda} \ln(1 - u)\right) \\&= F_X\left(-\frac{1}{\lambda} \ln(1 - u)\right) \\&= 1 - \exp\left(-\lambda \cdot \left(-\frac{1}{\lambda} \ln(1 - u)\right)\right) \\&= 1 - (1 - u) = u.\end{aligned}$$

Hence,  $P(U \leq u) = u$  for all  $u \in [0, 1]$ .

Therefore,  $U \sim \text{Uniform}(0, 1)$ .

**Note:**

Methods such as the **Accept–Reject method** or any other approach that relies on Uniform random variables **cannot be used here**, because we are only given access to Exponential random variables. We cannot use a Uniform random variable to generate itself. **Such methods will receive 0 marks.**

## Question 4

### Problem Statement:

Given the joint probability density function

$$f_{X_1, X_2}(x_1, x_2) = \begin{cases} 1, & 0 < x_1 < 1, 0 < x_2 < 1, \\ 0, & \text{otherwise,} \end{cases}$$

and the transformation

$$\begin{cases} Y_1 = X_1 - X_2, \\ Y_2 = X_1 + X_2. \end{cases}$$

Find the joint probability density function  $f_{Y_1, Y_2}(y_1, y_2)$ .

## Solution

### Finding the Inverse Transformation

From the given transformation, we solve for  $X_1$  and  $X_2$  in terms of  $Y_1$  and  $Y_2$ :

$$Y_1 + Y_2 = (X_1 - X_2) + (X_1 + X_2) = 2X_1 \quad \Rightarrow \quad X_1 = \frac{Y_1 + Y_2}{2},$$

$$Y_2 - Y_1 = (X_1 + X_2) - (X_1 - X_2) = 2X_2 \quad \Rightarrow \quad X_2 = \frac{Y_2 - Y_1}{2}.$$

Therefore, the inverse mapping is:

$$\boxed{x_1 = \frac{y_1 + y_2}{2}, \quad x_2 = \frac{y_2 - y_1}{2}}$$

### Calculating the Determinant of the Jacobian

The Jacobian matrix of the inverse transformation is:

$$J = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Computing the determinant:

$$\det(J) = \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) - \left(\frac{1}{2}\right) \left(-\frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Therefore,  $\boxed{|J| = \frac{1}{2}}.$

### Applying the Change of Variables Formula

By the transformation theorem for random variables:

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}\left(\frac{y_1 + y_2}{2}, \frac{y_2 - y_1}{2}\right) \cdot |J|.$$

Since  $f_{X_1, X_2}(x_1, x_2) = 1$  on the unit square, we have:

$$f_{Y_1, Y_2}(y_1, y_2) = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

whenever the point  $(y_1, y_2)$  maps to a point in the unit square.

### Determining the Support

The support of  $(Y_1, Y_2)$  consists of all points  $(y_1, y_2)$  such that:

$$0 < \frac{y_1 + y_2}{2} < 1 \quad \text{and} \quad 0 < \frac{y_2 - y_1}{2} < 1.$$

Multiplying through by 2:

$$\begin{cases} 0 < y_1 + y_2 < 2, \\ 0 < y_2 - y_1 < 2. \end{cases}$$

From these inequalities, we can derive:

$$\begin{aligned} y_2 - y_1 > 0 &\Rightarrow y_1 < y_2, \\ y_1 + y_2 > 0 &\Rightarrow y_1 > -y_2, \\ y_2 - y_1 < 2 &\Rightarrow y_1 > y_2 - 2, \\ y_1 + y_2 < 2 &\Rightarrow y_1 < 2 - y_2. \end{aligned}$$

Combining these constraints:

$$\max(-y_2, y_2 - 2) < y_1 < \min(y_2, 2 - y_2).$$

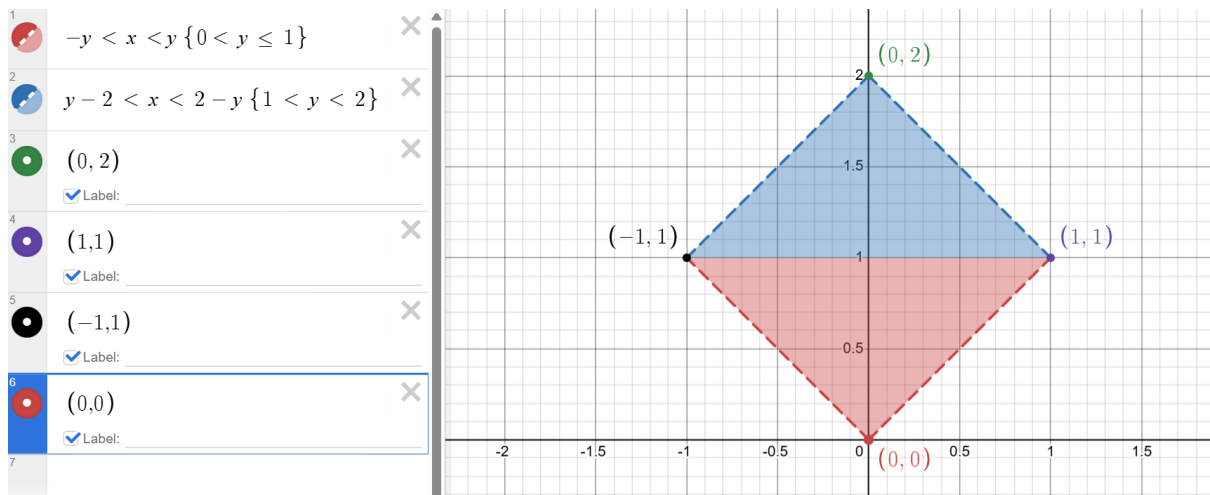
We can split this into two cases based on the value of  $y_2$ :

- **Case 1:** If  $0 < y_2 \leq 1$ , then  $-y_2 \geq y_2 - 2$  and  $y_2 \leq 2 - y_2$ , so:

$$-y_2 < y_1 < y_2.$$

- **Case 2:** If  $1 < y_2 < 2$ , then  $y_2 - 2 > -y_2$  and  $2 - y_2 < y_2$ , so:

$$y_2 - 2 < y_1 < 2 - y_2.$$



### Final Joint Density

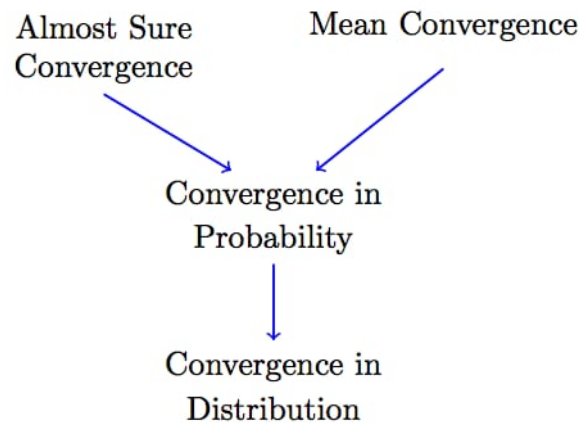
The joint probability density function of  $(Y_1, Y_2)$  is:

$$f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} \frac{1}{2}, & 0 < y_2 \leq 1, -y_2 < y_1 < y_2, \\ \frac{1}{2}, & 1 < y_2 < 2, y_2 - 2 < y_1 < 2 - y_2, \\ 0, & \text{otherwise.} \end{cases}$$

The shaded region demonstrates the probability density in the  $(y_1, y_2)$  space. The X-axis represents  $y_1$  and Y-axis represents  $y_2$ .

## Question 1b

2.5 marks for each part. No marks for correct answer without justification.



partial marks awarded for c,d if this relation has been used directly

### a. Convergence in Distribution

For  $y \geq 0$ :

$$\begin{aligned}
 F_{Y_n}(y) &= P(Y_n \leq y) \\
 &= P(nX_n \leq y) \\
 &= P(X_n \leq y/n)
 \end{aligned}$$

We now use the CDF of  $X_n$ ,  $F_{X_n}(x) = 1 - e^{-nx}$ , with  $x = y/n$ :

$$\begin{aligned}
 F_{Y_n}(y) &= F_{X_n}(y/n) \\
 &= 1 - e^{-n(y/n)} \\
 &= 1 - e^{-y}
 \end{aligned}$$

$$\lim_{n \rightarrow \infty} F_{Y_n}(y) = \lim_{n \rightarrow \infty} (1 - e^{-y}) = 1 - e^{-y} = F_Y(y)$$

Since the CDFs are identical for all  $n$ , convergence in distribution is shown.

### b. Convergence in Probability

For convergence in probability ( $Y_n \xrightarrow{P} Y$ ), we would need  $\lim_{n \rightarrow \infty} P(|Y_n - Y| > \epsilon) = 0$  for all  $\epsilon > 0$ .

From part (1), we know  $Y_n \sim \text{Exp}(1)$  for all  $n$ . We are given  $Y \sim \text{Exp}(1)$ . We assume  $Y_n$  and  $Y$  are independent random variables.

Let's compute the probability  $P(|Y_n - Y| > \epsilon)$ .

$$P(|Y_n - Y| > \epsilon) = P(Y_n - Y > \epsilon) + P(Y - Y_n > \epsilon)$$

We will compute each term by conditioning on one of the variables.

$$\begin{aligned}
 P(Y_n > Y + \epsilon) &= \int_0^\infty P(Y_n > y + \epsilon \mid Y = y) f_Y(y) dy \\
 &= \int_0^\infty P(Y_n > y + \epsilon) f_Y(y) dy \quad (\text{by independence}) \\
 &= \int_0^\infty (e^{-(y+\epsilon)}) \cdot (e^{-y}) dy \quad (\text{since } P(Y_n > x) = e^{-x} \text{ and } f_Y(y) = e^{-y}) \\
 &= \int_0^\infty e^{-y} e^{-\epsilon} e^{-y} dy \\
 &= e^{-\epsilon} \int_0^\infty e^{-2y} dy \\
 &= e^{-\epsilon} \left[ \frac{e^{-2y}}{-2} \right]_0^\infty \\
 &= e^{-\epsilon} \left( 0 - \left( \frac{e^0}{-2} \right) \right) = \frac{1}{2} e^{-\epsilon}
 \end{aligned}$$

The calculation is identical due to symmetry:

$$\begin{aligned}
 P(Y > Y_n + \epsilon) &= \int_0^\infty P(Y > x + \epsilon \mid Y_n = x) f_{Y_n}(x) dx \\
 &= \int_0^\infty P(Y > x + \epsilon) f_{Y_n}(x) dx \quad (\text{by independence}) \\
 &= \int_0^\infty (e^{-(x+\epsilon)}) \cdot (e^{-x}) dx \quad (\text{since } P(Y > x) = e^{-x} \text{ and } f_{Y_n}(x) = e^{-x}) \\
 &= e^{-\epsilon} \int_0^\infty e^{-2x} dx = \frac{1}{2} e^{-\epsilon}
 \end{aligned}$$

$$P(|Y_n - Y| > \epsilon) = \frac{1}{2} e^{-\epsilon} + \frac{1}{2} e^{-\epsilon} = e^{-\epsilon}$$

This probability is a positive constant that does not depend on  $n$ .

$$\lim_{n \rightarrow \infty} P(|Y_n - Y| > \epsilon) = \lim_{n \rightarrow \infty} e^{-\epsilon} = e^{-\epsilon}$$

Since  $e^{-\epsilon} \neq 0$  for any finite  $\epsilon > 0$ , the condition for convergence in probability is not met.

### c. Almost Sure Convergence

Theorem 7.5 (Borel Cantelli Lemma) can not be used to prove

$$Y_n \xrightarrow{a.s.} Y$$

since it is sufficient but not necessary

Using Theorem 7.6 from probababilitycourse.com (was mentioned as HW in slides).

## Borel Cantelli Lemma

Self-Study: Theorem 7.5 (probabilitycourse.com)

Consider a sequence of random variables  $X_1, X_2, \dots$ . If for all  $\epsilon$  we have

$$\sum_{n=1}^{\infty} P(|X_n - X| > \epsilon) < \infty$$

then  $X_n \rightarrow X$  a.s.

- ▶ This is only a sufficient condition for almost sure convergence!
- ▶ Thm 7.6 (HW) gives necessary and sufficient conditions.

### Theorem 7.6

Consider the sequence  $X_1, X_2, X_3, \dots$ . For any  $\epsilon > 0$ , define the set of events

$$A_m = \{|X_n - X| < \epsilon, \quad \text{for all } n \geq m\}.$$

Then  $X_n \xrightarrow{a.s.} X$  if and only if for any  $\epsilon > 0$ , we have

$$\lim_{m \rightarrow \infty} P(A_m) = 1.$$

$$\begin{aligned} P(A_m) &= P\left(\bigcap_{n=m}^{\infty} \{|Y_n - Y| < \epsilon\}\right) \\ &= \prod_{n=m}^{\infty} P(|Y_n - Y| < \epsilon) \quad (\text{independence}) \\ &= \prod_{n=m}^{\infty} (1 - P(|Y_n - Y| > \epsilon)) \end{aligned}$$

From part (2), we know  $P(|Y_n - Y| > \epsilon) = e^{-\epsilon}$ .

$$P(A_m) = \prod_{n=m}^{\infty} (1 - e^{-\epsilon})$$

$$\implies P(A_m) = 0$$

Now, we take the limit:

$$\lim_{m \rightarrow \infty} P(A_m) = \lim_{m \rightarrow \infty} 0 = 0$$

Since  $0 \neq 1$ , the condition for Theorem 7.6 is not met.

$$\implies Y_n \not\stackrel{q.s.}{\rightarrow} Y$$

#### d. Convergence in Mean ( $L^1$ )

Convergence in  $L^1$  requires  $\lim_{n \rightarrow \infty} E[|Y_n - Y|] = 0$ .

Let  $Z = Y_n - Y$ . We first derive the PDF of  $Z$ ,  $f_Z(z)$ , by finding its CDF,  $F_Z(z) = P(Z \leq z)$ .

- **Case 1:**  $z \geq 0$

$$F_Z(z) = P(Z \leq z) = P(Y_n - Y \leq z) = 1 - P(Y_n - Y > z).$$

From our calculation in part (2) (with  $z$  in place of  $\epsilon$ ), we know  $P(Y_n - Y > z) = \frac{1}{2}e^{-z}$ .

$$F_Z(z) = 1 - \frac{1}{2}e^{-z}.$$

- **Case 2:**  $z < 0$

$$F_Z(z) = P(Z \leq z) = P(Y_n - Y \leq z) = P(Y - Y_n \geq -z).$$

Let  $\epsilon = -z$ . Since  $z < 0$ ,  $\epsilon > 0$ .

$$F_Z(z) = P(Y - Y_n \geq \epsilon).$$

From our calculation in part (2), substituting  $\epsilon = -z$ , we get  $F_Z(z) = \frac{1}{2}e^{-(-z)} = \frac{1}{2}e^z$ .

- For  $z > 0$ :  $f_Z(z) = \frac{d}{dz}(1 - \frac{1}{2}e^{-z}) = \frac{1}{2}e^{-z}$ .
- For  $z < 0$ :  $f_Z(z) = \frac{d}{dz}(\frac{1}{2}e^z) = \frac{1}{2}e^z$ .

$$f_Z(z) = \frac{1}{2}e^{-|z|}$$

Now we can compute  $E[|Z|]$ :

$$\begin{aligned} E[|Y_n - Y|] &= E[|Z|] = \int_{-\infty}^{\infty} |z| f_Z(z) dz \\ &= \int_{-\infty}^{\infty} |z| \frac{1}{2} e^{-|z|} dz \\ &= 2 \int_0^{\infty} z \cdot \frac{1}{2} e^{-z} dz \quad (\text{by symmetry}) \\ &= \int_0^{\infty} z e^{-z} dz \end{aligned}$$

This integral is the definition of the expected value of an  $\text{Exp}(1)$  random variable, which is 1.

Since  $E[|Y_n - Y|] = 1$  for all  $n$ , the limit is:

$$\lim_{n \rightarrow \infty} E[|Y_n - Y|] = \lim_{n \rightarrow \infty} 1 = 1$$

Because the limit is  $1 \neq 0$ ,  $Y_n$  does not converge in  $L^1$  to  $Y$ .