Example: P be an $n\times n$ invertible matrix over F. $P_1, \dots, P_n$, the columns of P, form a basis for the space of column matrices, $F^{n\times 1}$. If X is a column matrix, $PX = x_1 P_1 + \dots + x_n P_n$.

$\because$ $PX = 0$ has only the trivial sol^n $X=0$, $\{P_1, \dots, P_n\}$ is a linearly independent set.

Spans $F^{n\times 1}$: Let Y be a column matrix. If $X = P^{-1}Y$, then $Y = PX$, i.e.,

$$Y = x_1 P_1 + \dots + x_n P_n.$$

So, $\{P_1, \dots, P_n\}$ is a basis for $F^{n\times 1}$.

Example: Let F be a subfield of $\mathbb{C}$. V be the space of polynomial f^{ns} over F;

$$f(x) = c_0 + c_1 x + \dots + c_n x^n.$$

Let $f_k(x) = x^k$, $k = 0, 1, 2, \dots$ The (infinite) set $\{f_0, f_1, f_2, \dots\}$ is a basis for V. Clearly, the set spans V, because the f^n f (above) is

$$f = c_0 f_0 + c_1 f_1 + \dots + c_n f_n.$$

<u>Thm</u>. Let V be a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2, \dots, \beta_m$. Then any independent set of vectors in V is finite and contains no more than m elements.

Proof: Show that any set of vectors with more than n elements is linearly dependent.

$\because \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m$ spans V, $\exists A_{ij} \in F$

s.t. $\bar{\alpha}_j = \sum_{i=1}^{m} A_{ij} \bar{\beta}_i \quad \forall \bar{\alpha}_j \in V$.

Consider $S = \{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$ be ~~set~~ of n distinct vectors. For any $x_1, x_2, \ldots, x_n \in F$,

$$x_1 \bar{\alpha}_1 + x_2 \bar{\alpha}_2 + \ldots + x_n \bar{\alpha}_n = \sum_{j=1}^{n} x_j \bar{\alpha}_j$$

$$= \sum_j x_j \sum_i A_{ij} \bar{\beta}_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij} \cdot x_j \right) \bar{\beta}_i$$

$\because n > m$, $\sum_{j=1}^{n} A_{ij} x_j = 0$ has a non-trivial sol^{n} for x_j's, i.e., not all x_j's are 0.

$\Rightarrow S$ is linearly dependent.

Corollary: If V is a finite-dimensional vector space, then any bases of V have the same (finite) number of elements.

Corollary: Let V be a finite-dim. vector space & let $\dim V = n$. Then, n is cardinality of basis &

ⓐ any subset of V which contains more than n ~~linear~~ vectors is linearly dependent.

ⓑ no subset of V which contains less than n vectors can span V.

Lemma: Let S be a linearly independent subset of a vector space V. Suppose $\beta \in V$ is not in a subspace spanned by S. Then the set obtained by adjoining β to S is linearly dependent.

<u>Thm</u>. If W is a subspace of a finite-dim. vector space V, every linearly independent subset of W is finite and is part of a (finite) basis for W.

Corollary: If W is a proper subspace of a finite-dim. V, then W is finite-dim. and $\dim W < \dim V$.

Corollary: Let $A_{n \times n}$ over F. Suppose the row vectors of A form a linearly independent set of vectors in F^n. Then A is invertible.

<u>Thm</u>. If W_1 & W_2 are finite-dim subspaces of a vector space V, then $W_1 + W_2$ is finite-dim &

$$\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2).$$

Coordinates

A basis $\mathcal{B}$ in an n-dim space V enables introduction of coordinates in V analogous to the 'natural coordinates' x_i of a $\bar{\alpha} = (x_1, \ldots, x_n) \in F^n$. Then, the coordinates of $\bar{\alpha} \in V$ relative to $\mathcal{B}$ will be scalars which serve to express $\bar{\alpha}$ as a linear combination of the vectors in the basis. The natural coordinates of $\bar{\alpha} \in F^n$ is defined by $\bar{\alpha}$ and the std basis for F^n. If $\bar{\alpha} = (x_1, \ldots, x_n) = \sum_i x_i \epsilon_i$ and $\mathcal{B}$ is the std. basis for F^n, how are the coordinates of $\bar{\alpha}$ determined by $\mathcal{B}$ & $\bar{\alpha}$?

<u>Def^n:</u> If V is a finite-dimensional vector space, an <u>ordered basis</u> for V is a finite sequence of vectors which is linearly independent and spans V.

If the sequence $\overline{\alpha}_1, \overline{\alpha}_2, \dots, \overline{\alpha}_n$ is an ordered basis for V, then the set $\{\overline{\alpha}_1, \dots, \overline{\alpha}_n\}$ is ~~the~~ a basis for V. The ordered basis is the set, together w/ the specified ordering. W/ slight abuse of notation: $\mathcal{B} = \{\overline{\alpha}_1, \dots, \overline{\alpha}_n\}$ is an ordered basis for V.

V is a finite-dim vector space over F and $\mathcal{B} = \{\overline{\alpha}_1, \dots, \overline{\alpha}_n\}$ is an ordered basis for V. For $\overline{\alpha} \in V$, $\overline{\alpha} = \sum_{i=1}^{n} x_i \overline{\alpha}_i$ for some ~~a~~ unique n-tupple $(x_1, x_2, \dots, x_n)$. It's unique because if $\overline{\alpha} = \sum_{i=1}^{n} y_i \overline{\alpha}_i$, then

$$\overline{\alpha} - \overline{\alpha} = \sum_{i=1}^{n} (x_i - y_i)\overline{\alpha}_i = \overline{0} \Rightarrow x_i = y_i \ \forall i.$$

x_i is called i^{th} coordinate of $\overline{\alpha}$ relative to an ordered basis $\mathcal{B}$.

If $\overline{\alpha} = \sum x_i \overline{\alpha}_i$ & $\overline{\beta} = \sum y_i \overline{\alpha}_i$, ~~w.r.t~~

then $\overline{\alpha} + \overline{\beta} = \sum_i \underbrace{(x_i + y_i)}_{i^{th} \text{ coordinate of } \overline{\alpha}+\overline{\beta} \text{ w.r.t } \mathcal{B}} \overline{\alpha}_i$.

Linear transformation preserves linear combinations; i.e., $T: V \longrightarrow W$

$\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n \in V, \quad c_1, c_2, \ldots, c_n \in F$

$$T(c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \ldots + c_n\bar{\alpha}_n) = c_1 T(\bar{\alpha}_1) + c_2 T\bar{\alpha}_2 + \ldots + c_n T\bar{\alpha}_n$$

__Thm__. Let V be a finite-dim. vector space over the field F. Let $\{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$ be an ordered basis for V. Let W be a vector space over the same field F. Let $\{\bar{\beta}_1, \ldots, \bar{\beta}_n\}$ be any vectors in W. Then there is precisely one linear transformation $T: V \longrightarrow W$ s.t. $T\bar{\alpha}_j = \bar{\beta}_j \quad j = 1, \ldots, n.$

Proof: First we prove $\exists\ T: V \longrightarrow W$ w/ $T\bar{\alpha}_j = \bar{\beta}_j$.

Given $\bar{\alpha} \in V$, $\exists$ unique n-tupple $(x_1, \ldots, x_n)$ s.t. $\bar{\alpha} = x_1\bar{\alpha}_1 + \ldots + x_n\bar{\alpha}_n$.

For $\bar{\alpha}$, we define

i^{th} coordinate of $c\bar{\alpha}$ is cx_i w.r.t. $\mathcal{B}$.

Note that every n-tupple $(x_1, \ldots, x_n) \in F^n$ is the n-tupple of coordinates of some vector in V, namely the vector

$$\sum_{i=1}^{n} x_i \bar{\alpha}_i .$$

I.e., each ordered basis for V determines a one-to-one correspondence

$$\bar{\alpha} \longrightarrow (x_1, \ldots, x_n)$$

b/w the set of all vectors in V & the set of all n-tupples in F^n. This correspondence has the property that the correspondent of $(\bar{\alpha} + \bar{\beta})$ is the sum in F^n of the correspondents of $\bar{\alpha}$ & $\bar{\beta}$, and that the correspondent of $c\bar{\alpha}$ is the product in F^n of the scalar c & the correspondent of $\bar{\alpha}$.

Coordinate matrix of $\bar{\alpha}$ w.r.t. the ordered basis $\mathcal{B}$, $[\alpha]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

$$\bar{\alpha} = \sum_{i=1}^{n} x_i \bar{\alpha}_i .$$

Suppose V is n-dim. and $\mathcal{B} = \{\bar{\alpha}_1, \bar{\alpha}_2, \ldots \bar{\alpha}_n\}$ and $\mathcal{B}' = \{\bar{\alpha}_1', \ldots, \bar{\alpha}_n'\}$ are two ordered bases for V. There are unique scalars P_{ij} s.t.

$$\bar{\alpha}_j' = \sum_{i=1}^{n} P_{ij} \alpha_i \quad , \quad 1 \le j \le n.$$

Let $x_1', \ldots, x_n'$ be the coordinates of a given $\bar{\alpha}$ in the ordered basis $\mathcal{B}'$. Then

$$\bar{\alpha} = x_1' \bar{\alpha}_1' + \ldots + x_n' \bar{\alpha}_n$$

$$= \sum_{j=1}^{n} x_j' \bar{\alpha}_j'$$

$$= \sum_{j=1}^{n} x_j' \sum_{i=1}^{n} P_{ij} \bar{\alpha}_i$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} (P_{ij} x_j') \right) \bar{\alpha}_i$$

$$\bar{\alpha} = \sum_{i} \left(\sum_{j} P_{ij} x_j' \right) \bar{\alpha}_i$$

Then $\bar{\alpha} = \sum_i x_i \bar{\alpha}_i$, $(x_1, \ldots, x_n)$ being coordinates w.r.t. $\mathcal{B}$,

$$x_i = \sum_j P_{ij} x_j', \quad 1 \le i \le n.$$

Let $P_{n\times n}$ whose i,j entry is the scalar P_{ij}, and let X and X' be the coordinate matrices of the vector $\bar{\alpha} \in V$ in the ordered bases $\mathcal{B}$ & $\mathcal{B}'$. Then,

$$x_i = \sum_{j=1}^{n} P_{ij} x_j' \quad \forall i \in \{1, \dots, n\}$$

can be expressed as $X = PX'$.

Since $\mathcal{B}$ & $\mathcal{B}'$ are linearly independent sets, $X = 0$ iff $X' = 0$. This implies, P is invertible. Hence, $X' = P^{-1}X$.

I.e.,
$$[\bar{\alpha}]_{\mathcal{B}} = P[\bar{\alpha}]_{\mathcal{B}'}$$
$$[\bar{\alpha}]_{\mathcal{B}'} = P^{-1}[\bar{\alpha}]_{\mathcal{B}}$$

<u>Thm</u>. Let V be an n-dim vector space over F. Let $\mathcal{B}$ & $\mathcal{B}'$ be two ordered bases ~~vector spaces~~ of V. Then there is a unique, necessarily invertible, $n\times n$ matrix P over F s.t.

(i) $[\bar{\alpha}]_{\mathcal{B}} = P[\bar{\alpha}]_{\mathcal{B}'}$

(ii) $[\bar{\alpha}]_{\mathcal{B}'} = P^{-1}[\bar{\alpha}]_{\mathcal{B}}$

$\forall \bar{\alpha} \in V$. Columns P_j of P are $P_j = [\bar{\alpha}_j']_{\mathcal{B}}$, for $j \in \{1, \dots, n\}$.

Thm. Suppose P is an $n \times n$ invertible matrix over F. Let V be an n-dim vector space over F, and let $\mathcal{B}$ be an ordered basis of V. Then there is a unique ordered basis $\mathcal{B}'$ of V s.t.

(i) $[\bar{\alpha}]_{\mathcal{B}} = P[\bar{\alpha}]_{\mathcal{B}'}$

(ii) $[\bar{\alpha}]_{\mathcal{B}'} = P^{-1}[\bar{\alpha}]_{\mathcal{B}}$

$\forall \bar{\alpha} \in V$.

$$\bar{\alpha} = \sum_{j=1}^{n} x_j' \alpha_j' = \sum_{i=1}^{n} x_i \bar{\alpha}_i$$

$$\bar{\alpha}_j' = \sum_{i=1}^{n} P_{ij} \bar{\alpha}_i$$

Example: R be real field & $\theta \in R$ is fixed.

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\mathcal{B}' = \{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\} \subset \mathbb{R}^2$$

$$[\bar{\alpha}]_{\mathcal{B}'} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$(X' = P^{-1}X)$$

Linear Transformations

Def^n: Let V & W be vector spaces over the field F. A linear transformation from V into W is a f^n T from V into W, i.e., $T: V \to W$, s.t.

$$T(c\bar{\alpha}+\bar{\beta}) = c(T\bar{\alpha}) + T\bar{\beta}$$

$\forall\ \bar{\alpha}, \bar{\beta} \in V$ and $\forall\ c \in F$.

Example: If V is any vector space, the identity transformation I, defined by $I\bar{\alpha} = \bar{\alpha}$, is a linear transformation from V into V. The zero transformation 0, defined by $0\bar{\alpha} = 0$, is a linear transformation from V into V.

Remark: If T is a linear transformation $T: V \to W$, then $T(0) = 0$.

$$T(0) = T(0+0) = T(0) + T(0) \Rightarrow T(0) = 0.$$

$$T\bar{\alpha} = x_1\bar{\beta}_1 + \dots + x_n\bar{\beta}_n.$$

Then T is a well-defined rule for associating w/ each vector $\bar{\alpha} \in V$ a vector $T\bar{\alpha} \in W$.

From def.^{n} $T\bar{\alpha}_j = \bar{\beta}_j$ for each j.

To check if T is linear, let

$$\bar{\beta} = y_1\bar{\alpha}_1 + \dots + y_n\bar{\alpha}_n \in V, \quad c \in F.$$

Now

$$c\bar{\alpha} + \bar{\beta} = (cx_1 + y_1)\bar{\alpha}_1 + \dots + (cx_n + y_n)\bar{\alpha}_n$$

& so by def.^{n}

$$T(c\bar{\alpha} + \bar{\beta}) = (cx_1 + y_1)\beta_1 + \dots + (cx_n + y_n)\bar{\beta}_n.$$

On the other hand,

$$c(T\bar{\alpha}) + T\bar{\beta} = c\sum_{i=1}^{n} x_i\bar{\beta}_i + \sum_{i=1}^{n} y_i\beta_i$$

$$= \sum_{i=1}^{n}(cx_i + y_i)\bar{\beta}_i$$

Thus, $T(c\bar{\alpha} + \bar{\beta}) = cT\bar{\alpha} + T\bar{\beta}$.

If U is a linear transformation $T: V \to W$ w/ $U\bar{\alpha}_i = \bar{\beta}_j$, $j = 1, \dots, n$, then for $\bar{\alpha} = \sum_{i=1}^{n} x_i\bar{\alpha}_i$

~~then for~~ we have,

$$U\bar{\alpha} = U\left(\sum_{i=1}^{n} x_i \alpha_i\right)$$

$$= \sum_{i=1}^{n} x_i (U\bar{\alpha_i})$$

$$= \sum_{i=1}^{n} x_i \bar{\beta_i},$$

so that U is exactly the rule T which we defined above. This shows that the linear transformation T w/ $T\bar{\alpha}_j = \bar{\beta}_j$ is unique.

$T : V \longrightarrow W$ Image of T is a subspace of W

Defn: Let V & W be vector spaces over the field F & let T be a linear trans. from V into W. The null space of T is the set of all vectors $\bar{\alpha} \in V$ s.t. $T\bar{\alpha} = 0$.

If V is fin.-dimensional, the rank of T is the dimension of the range of T & the nullity of T is the dim. of the null space of T.

<u>Thm.</u> Let V & W be vector spaces over the field F & $T: V \to W$ be a linear transformation. Suppose V is fin. dim. Then,

$$\text{rank}(T) + \text{nullity}(T) = \dim V.$$

<u>Thm.</u> If A is an $m \times n$ matrix w/ entries in the field F, then

$$\text{row rank}(A) = \text{column rank}(A).$$

<u>Thm.</u>

The Algebra of linear transformations

<u>Thm.</u> Let V & W be vector spaces over the field F. Let $T: V \to W$, $U: V \to W$ be linear transformations. The f^n $(T+U)$ defined by

$$(T+U)(\bar{\alpha}) = T(\bar{\alpha}) + U(\bar{\alpha})$$

is a linear transformation $(T+U): V \to W$.

If $c \in F$, the f^n (cT) defined by

$$(cT)(\bar{\alpha}) = c(T\bar{\alpha})$$

is a linear transformation $(cT): V \to W$.

The set of all linear transformations from V into W, together w/ the add^n & scalar multiplication defined above, is a vector space over the field F.

- The space of linear transformations $T: V \to W$ to be denoted as $L(V, W)$.

Thm. Let V be an n-dim. vector space over the field F, and let W be an m-dim vector space over F. Then the space $L(V,W)$ is finite dim. & has dim. mn ($=\dim V \times \dim W$).

Proof. Let $\mathcal{B} = \{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$ and $\mathcal{B}' = \{\bar{\beta}_1, \ldots, \bar{\beta}_m\}$ be ordered bases for V & W, resp. For each pair of integers (p,q) with $1 \le p \le m$ & $1 \le q \le n$, we define a linear transformation $E^{p,q}$ from V into W by

$$E^{p,q}(\bar{\alpha}_i) = \begin{cases} 0, & \text{if } i \neq q \\ \bar{\beta}_p, & \text{if } i = q \end{cases}$$

$$= \delta_{iq}\bar{\beta}_p.$$

A/c to a theorem earlier, $\exists$ a unique linear transformation from $V \to W$ satisfying these cond^{n}s. The claim is ~~that~~ the mn transformations $E^{p,q}$ form a basis for $L(V,W)$.

Thm. Let V, W, & Z be vector spaces over the field F. Let $T: V \to W$ & $U: W \to Z$ be linear transformations. Then the composed f^n UT defined by $(UT)(\bar{\alpha}) = U(T(\bar{\alpha}))$ is a linear trans.

$UT: V \longrightarrow Z$.

Proof: $(UT)(c\bar{\alpha}+\bar{\beta}) = U(T(c\bar{\alpha}+\bar{\beta}))$

$= U(cT\bar{\alpha}+T\bar{\beta})$

$= c\,UT\bar{\alpha} + UT\bar{\beta}$

$= c(UT)(\bar{\alpha}) + (UT)(\bar{\beta})$.

Def^n: If V is a vector space over the field F, a linear operator on V is a linear transformation from V to V.

$L(V,V)$ has a 'multiplication' defined on it by composition. Suppose $T: V \to V$ and $U: V \to V$ for $T, U \in L(V,V)$ are distinct, UT & TU are well-defined. However, in general $UT \neq TU$. $\underbrace{T \circ T \circ \cdots \circ T}_{n \text{ times}} = T^n$.

For $T \neq 0$, we define $T^0 = \mathbb{1}$.

Lemma: Let V be a vector space over the field F; let $U, T_1, T_2 \in L(V,V)$; let $c \in F$

ⓐ $\mathbb{1}U = U\mathbb{1} = U$;

ⓑ $U(T_1 + T_2) = UT_1 + UT_2$;

$(T_1 + T_2)U = T_1U + T_2U$;

ⓒ $c(UT_1) = (cU)T_1 = U(cT_1)$.

Proof: ⓐ obvious.

(b) $[U(T_1+T_2)](\bar{\alpha}) = U[(T_1+T_2)(\bar{\alpha})]$

$$= U(T_1\bar{\alpha} + T_2\bar{\alpha})$$
$$= U(T_1\bar{\alpha}) + U(T_2\bar{\alpha})$$
$$= (UT_1)(\bar{\alpha}) + (UT_2)(\bar{\alpha})$$

so that $U(T_1+T_2) = UT_1 + UT_2$.

$[(T_1+T_2)U](\bar{\alpha}) = (T_1+T_2)(U\bar{\alpha})$

$$= T_1(U\bar{\alpha}) + T_2(U\bar{\alpha})$$
$$= (T_1U)(\bar{\alpha}) + (T_2U)(\bar{\alpha})$$

so that $(T_1+T_2)U = T_1U + T_2U$.

Note that the proofs of these two distributive laws do not use the fact that T_1 & T_2 are linear, and the proof of the 2nd one does not use the fact that U is linear either.

(c) exercise.

For which linear operators $T: V \to V$ does there exist a linear operator T^{-1} s.t. $TT^{-1} = T^{-1}T = \mathbb{1}$?

F^n $T: V \to W$ is called invertible if $\exists$ a f^n $U: W \to V$ s.t. UT is the identity f^n on V & TU is the identity f^n on W. If T is invertible, the f^n U is unique & is denoted by T^{-1}.

Furthermore, T is invertible iff

1. T is 1:1, i.e., $T\bar{\alpha} = T\bar{\beta} \Rightarrow \bar{\alpha} = \bar{\beta}$.
2. T is onto, i.e., range$(T) = W$.

Thm. Let V & W be vector spaces over the field F. Let $T: V \to W$ be a linear trans. If T is invertible, then the inverse f^n T^{-1} is a linear transformation from W to V, i.e., $T^{-1}: W \to V$.

Proof: When T is one-one and onto, there is uniquely determined inverse f^n T^{-1} which maps W onto V s.t. $T^{-1}T$ is the identity f^n on V, and TT^{-1} is the identity f^n on W. We now prove here that if T is a linear f^n that is invertible, ~~the~~ the inverse T^{-1} is also linear.

Let $\bar{\beta}_1, \bar{\beta}_2 \in W$, $c \in F$. We need to show, $T^{-1}(c\bar{\beta}_1 + \bar{\beta}_2) = cT^{-1}\bar{\beta}_1 + T^{-1}\bar{\beta}_2$.

Let $\bar{\alpha}_i = T^{-1}\bar{\beta}_i \ \forall i \in \{1,2\}$, i.e., $\bar{\alpha}_i \in V$ is the unique vector s.t. $T\bar{\alpha}_i = \bar{\beta}_i$.

$\because$ T is linear,

$$T(c\bar{\alpha}_1 + \bar{\alpha}_2) = cT\bar{\alpha}_1 + T\bar{\alpha}_2$$
$$= c\bar{\beta}_1 + \bar{\beta}_2.$$

$\therefore$ $c\bar{\alpha}_1 + \bar{\alpha}_2 \in V$ is the unique vector which is sent by T into $c\bar{\beta}_1 + \bar{\beta}_2$,

& so $$T^{-1}(c\bar{\beta}_1 + \bar{\beta}_2) = c\bar{\alpha}_1 + \bar{\alpha}_2$$
$$= cT^{-1}\bar{\beta}_1 + T^{-1}\bar{\beta}_2$$

& T is linear.

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For invertible linear transformations:

$T: V \to W$, $U: W \to Z$,

we have

invertible linear transformation

$UT: V \to Z$ and

$(UT)^{-1} = T^{-1}U^{-1}$. Verification of

$(UT)^{-1} = T^{-1}U^{-1}$ requires that $T^{-1}U^{-1}$ is both a left and a right inverse of UT.

If T is linear, then $T(\bar{\alpha} - \bar{\beta}) = T\bar{\alpha} - T\bar{\beta}$. Hence, $T\bar{\alpha} = T\bar{\beta}$ iff $T(\bar{\alpha} - \bar{\beta}) = 0$.

→ Verifies that T is 1:1.

A linear transformation T is non-singular if $T\bar{\gamma} = 0$ implies $\bar{\gamma} = 0$, i.e., if the null space of T is $\{0\}$.

T is 1:1 iff T is non-singular.

<u>Thm</u>: Linear trans. $T: V \to W$. T is non-singular iff T carries each linearly independent subset of V onto a linearly indep. subset of W.

A linear transformation may be non-singular w/o being onto and maybe onto w/o being non-singular.

<u>Thm</u>. Let V & W be finite-dim. vector spaces over the field F s.t. $\dim V = \dim W$. If $T: V \to W$ is a linear transf., the following are equiv.

(i) T is invertible.

(ii) T is non-singular.

(iii) T is onto, i.e., $\text{range}(T) = W$.

(iv) If $\{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$ is a basis for V, then $\{T\bar{\alpha}_1, \ldots, T\bar{\alpha}_n\}$ is a basis for W.

(v) There is some basis $\{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$ for V s.t. $\{T\bar{\alpha}_1, \ldots, T\bar{\alpha}_n\}$ is a basis for W.

Isomorphism

V & W be vector spaces over the field F, any one-one linear transformation $T: V \to W$ is called an isomorphic of V onto W. If $\exists$ an isomorphism of V onto W, we say V is isomorphic to W.

Isomorphism is an equivalence rel^n on the class of vector spaces.

Thm. Every n-dim. vector space over the field F is isomorphic to the space F^n.

Proof: V be an n-dim. space over the field F. $\mathcal{B} = \{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$ be an ordered basis for V. Define a f^n $T: V \longrightarrow F^n$ as follows: If $\bar{\alpha} \in V$, let $T\bar{\alpha} = (x_1, \ldots, x_n)$, coordinates of $\bar{\alpha}$ relative to $\mathcal{B}$, i.e.,

$$\bar{\alpha} = x_1\bar{\alpha}_1 + \ldots + x_n\bar{\alpha}_n.$$

T is linear, one-one, and maps V onto F^n.

Isomorphism is dimension preserving.