Lecture 4 Vector spaces

Def^n. A vector space (or linear space) consists of the following:

Vector space over the field

1. a field F of scalars;
2. a set V of objects, called vectors;
3. a rule (or operation), called vector addition, which associates with each pair of vectors $\vec{\alpha}, \vec{\beta} \in V$ a vector $\vec{\alpha}+\vec{\beta} \in V$, called the sum of $\vec{\alpha}$ & $\vec{\beta}$, in such a way that
   ⓐ addition is commutative, $\vec{\alpha}+\vec{\beta} = \vec{\beta}+\vec{\alpha}$;
   ⓑ addition is associative, $\vec{\alpha}+(\vec{\beta}+\vec{\gamma}) = (\vec{\alpha}+\vec{\beta})+\vec{\gamma}$;
   ⓒ $\exists$ a unique vector $\vec{0} \in V$, called the zero vector, s.t. $\vec{\alpha}+\vec{0} = \vec{\alpha}$ $\forall$ $\vec{\alpha} \in V$;
   ⓓ for each $\vec{\alpha} \in V$ $\exists$ a unique vector $-\vec{\alpha} \in V$ s.t. $\vec{\alpha} + (-\vec{\alpha}) = 0$.
4. a rule, called scalar multiplication, which associates with each scalar $c \in F$ & $\vec{\alpha} \in V$ a vector $c\vec{\alpha} \in V$, called the product of c & $\vec{\alpha}$ s.t.
   ⓐ $1\vec{\alpha} = \vec{\alpha}$ $\forall$ $\vec{\alpha} \in V$;
   ⓑ $(c_1 c_2)\vec{\alpha} = c_1(c_2\vec{\alpha})$;
   ⓒ $c(\vec{\alpha}+\vec{\beta}) = c\vec{\alpha} + c\vec{\beta}$;
   ⓓ $(c_1 + c_2)\vec{\alpha} = c_1\vec{\alpha} + c_2\vec{\alpha}$.

A vector space is a composite object consisting of a field, a set of 'vectors', & two operations w/ certain properties.

Examples

① The n-tupple space, F^n. Let F be any field. Let V be the set of all n-tupples $\bar{\alpha} = (x_1, \dots, x_n)$ of scalars $x_i \in F$. If $\bar{\beta} = (y_1, y_2, \dots, y_n)$ w/ $y_i \in F$, the sum of $\bar{\alpha}$ & $\bar{\beta}$ is defined by

$$\bar{\alpha} + \bar{\beta} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \quad \text{——} (2.1)$$

The product of a scalar c and vector $\bar{\alpha}$ is defined by

$$c\bar{\alpha} = (cx_1, \dots, cx_n) \quad \text{——} (2.2)$$

② The space of m×n matrices, $F^{m\times n}$. Let F be any field and let m & n be +ve integers. Let $F^{m\times n}$ be the set of all m×n matrices over the field F.

$\bar{A}, \bar{B} \in F^{m\times n}$ then

$$(\bar{A} + \bar{B})_{ij} = A_{ij} + B_{ij}$$

$c \in F$, $\bar{A} \in F^{m\times n}$ then

$$(cA)_{ij} = cA_{ij}.$$

③ The space of functions from a set to a field. F be field, S be any non-empty set. V be the set of all f's from the set S into F.

For $\bar{f}, \bar{g} \in V$,

$$(f+g)(s) = f(s) + g(s).$$

For $c \in F$, $f \in V$,

$$(cf)(s) = cf(s).$$

(4) The space of polynomial f^ns over a field F.
Let F be a field and let V be the set of all f^ns f from F into F which have the rule of the form

$$f(x) = c_0 + c_1 x + \dots + c_n x^n,$$

where $c_0, c_1, \dots, c_n \in F$ are independent of x.

(5) The field $\mathbb{C}$ of complex nos. $\longrightarrow$ a vector space over the field $\mathbb{R}$ of real nos.

From the def^n of a vector space, we observe: ~~that~~ If for a scalar $c \in F$ and vector $\bar{\alpha} \in V$ we have $c\bar{\alpha} = 0$ then either $c = 0$ or $\bar{\alpha} = \bar{0}$.

For any $\bar{\alpha} \in V$, $-\bar{\alpha} \in V$ since $\bar{0} \in V$ and

$$\bar{0} = 0\bar{\alpha} = (1-1)\alpha = 1 \cdot \bar{\alpha} + (-1) \cdot \bar{\alpha} = \bar{\alpha} + (-1)\bar{\alpha},$$

$$(-1)\bar{\alpha} = -\bar{\alpha}.$$

For any $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \bar{\alpha}_4 \in V$, $(\bar{\alpha}_1 + \bar{\alpha}_2) + (\bar{\alpha}_3 + \bar{\alpha}_4)$
$= [\bar{\alpha}_2 + (\bar{\alpha}_1 + \bar{\alpha}_3)] + \bar{\alpha}_4.$

Def^n: A vector $\bar{\beta} \in V$ is said to be a <u>linear combination</u> of the vectors $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n \in V$ provided $\exists$ scalars $c_1, \ldots, c_n \in F$ s.t.

$$\bar{\beta} = c_1\bar{\alpha}_1 + \ldots + c_n\bar{\alpha}_n$$

$$= \sum_{i=1}^{n} c_i\bar{\alpha}_i .$$

Other extensions of the associative property of vector add^n & the distributive properties 4(c) and 4(d) of scalar multiplication apply to linear combinations:

$$\sum_{i=1}^{n} c_i\alpha_i + \sum_{i=1}^{n} d_i\alpha_i = \sum_{i=1}^{n} (c_i + d_i)\alpha_i$$

$$c\sum_{i=1}^{n} c_i\alpha_i = \sum_{i=1}^{n} (cc_i)\alpha_i .$$

Def^n: Let V be a vector space over the field F. A <u>subspace</u> of V is a subset W of V which is itself a vector space over F w/ the operations of vector addition and scalar multiplication on V.

Thm. A non-empty subset W of V is a subspace of V if and only if for each pair of vectors $\bar{\alpha}, \bar{\beta}$ in W and each scalar c in F the vector $c\bar{\alpha}+\bar{\beta}$ is again in W.

Proof. Let W be non-empty subset of V s.t. $c\bar{\alpha}+\bar{\beta} \in W$ $\forall$ $\bar{\alpha}, \bar{\beta} \in W$ and $\forall$ $c \in F$. $\because$ W is non-empty, $\exists$ $\bar{\gamma} \in W$, and hence $(-1)\bar{\gamma}+\bar{\gamma} = \bar{0} \in W$, ~~$(-1)\cdot\bar{\alpha} = -\bar{\alpha} \in W$~~ $\Rightarrow$ for any $\bar{\alpha} \in W, c \in F$, we have $c\bar{\alpha} = c\bar{\alpha}+\bar{0} \in W$. I.e., we also have, $(-1)\cdot\bar{\alpha} = -\bar{\alpha} \in W$. At last, $\bar{\alpha}, \bar{\beta} \in W$, then $\bar{\alpha}+\bar{\beta} = 1\bar{\alpha}+\bar{\beta} \in W$. Thus, W is subspace of V.

Conversely (easy part or trivial part) is obvious.

Examples

(a) If V is any vector space, V is a subspace of V; the subset ~~of~~ V consisting of the zero vector $\bar{0}$ alone is a subspace of V, called the zero subspace.

(Note: Field is non-empty set and has distinct additive identity 0 and multiplicative identity 1. Therefore, any Field at least always has 0 and 1 unlike def^n of vector space)

(b) An $n \times m$ matrix A over the field $\mathbb{C}$ of complex nos. is Hermitian (or self-adjoint) if

$$A_{jk} = \bar{A}_{kj}, \quad \text{(where } \bar{x} \text{ denotes complex conjugate of } x \in \mathbb{C}\text{)}$$

for each j,k. A 2×2 matrix is Hermitian if and only if it has the form

$$\begin{bmatrix} z & x+iy \\ x-iy & w \end{bmatrix}, \quad \text{where } x,y,z,w \in \mathbb{R}.$$

The set of all Hermitian matrices is not a subspace of the space of all $n\times n$ matrices over $\mathbb{C}$. (Why? How?)

What if the given vector space was to be defined over the field $\mathbb{R}$ of real nos.

Q. On $\mathbb{R}^n$, define two operations

$$\bar{\alpha} \oplus \bar{\beta} = \bar{\alpha} - \bar{\beta}$$
$$c \cdot \bar{\alpha} = -c\bar{\alpha}.$$

The operations on the right are the usual ones. Which of the axioms for a vector space are satisfied by $(\mathbb{R}^n, \oplus, \cdot)$?

Q. Let V be the set of pairs (x, y) of reals nos. & let F be the field of real nos. Define

$$(x, y) + (x_1, y_1) = (x + x_1, 0)$$
$$c\,(x, y) = (cx, 0).$$

Is V, with these operations, a vector space?

Q. Let V be the set of ~~pairs (x, y)~~ all complex valued f^ns f on the real line such that (for all $t \in \mathbb{R}$) $f(-t) = f^*(t)$.

* denotes complex conjugation. Show that V, w/ operations

$$(f+g)(t) = f(t) + g(t)$$
$$(cf)(t) = cf(t)$$

is a vector space over the field of real nos. Give an example of a f^n in V which is not real valued.

Lemma: If A is an $m \times n$ matrix over F and B, C are $n \times p$ matrices over F then

$$A(dB+C) = d(AB) + AC \text{ for each scalar } d \in F.$$

Proof:
$$\begin{aligned}[A(dB+C)]_{ij} &= \sum_k A_{ik}(dB+C)_{kj} \\ &= \sum_k (d A_{ik} B_{kj} + A_{ik} C_{kj}) \\ &= d \sum_k A_{ik} B_{kj} + \sum_k A_{ik} C_{kj} \\ &= d(AB)_{ij} + (AC)_{ij} \\ &= [d(AB) + AC]_{ij}.\end{aligned}$$

Similarly, one can show that $(dB+C)A = d(BA) + CA$, if matrix sums & products are defined.

__Thm__. Let V be a vector space over the field F. The intersection of any collection of subspaces of V is a subspace of V.

Proof: Let $\{W_a\}$ be a collect^n of subspaces of V, and let $W = \bigcap_a W_a$ be their intersection. Recall that W is defined as the set of all elements belonging to every W_a. Since each

W_a is a subspace, each contains the zero vector. Thus the zero vector is in the intersection W, and W is non-empty. Let α & β be vectors in W and let c be a scalar. By definition of W, both α & β ~~be vectors in W and~~ belong to each W_α, and because each W_a is a subspace, the vector $(c\alpha+\beta) \in W_\alpha \ \forall \alpha$. Thus, $(c\alpha+\beta)$ is again in W. ~~Thus, W is a~~ And, W is a subspace of V.

From aforementioned theorem it follows that if S is any collection of vectors in V, then there is a smallest subspace of V which contains S, i.e., a subspace which contains S and which is contained in every ~~subspace~~ other subspace containing S.

Defn. Let S be a set of vectors in a vector space V. The subspace spanned by S is defined to be the intersection W of all subspaces of V which contains S. When S is a finite set of vectors, $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, we shall simply call W the subspace spanned by the vectors $\alpha_1, \alpha_2, \dots, \alpha_n$.

Theorem. The subspace spanned by a non-empty subset S of a vector space V is the set of all linear combinations of vectors in S.

Proof. W be the subspace spanned by S. Each linear comb^n

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \dots + x_m\alpha_m$$

of vectors $\alpha_1, \alpha_2, \dots, \alpha_m \in S$ is clearly in W. Thus, W contains the set L of all linear comb^{n}s. of vectors in S. On the other hand, the set L contains S and is non-empty. If $\alpha, \beta \in L$ then α is linear comb^n

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \dots + x_m\alpha_m$$

of vectors $\alpha_i \in S$, and β is linear comb^n.

$$\beta = y_1\beta_1 + y_2\beta_2 + \dots + y_n\beta_n$$

of vectors $\beta_j \in S$. For each scalar c,

$$c\alpha + \beta = \sum_{i=1}^{m} (cx_i)\alpha_i + \sum_{j=1}^{n} y_j\beta_j.$$

$c\alpha + \beta \in L$. $\therefore$ L is subspace of V.

$\vdots$

Def^n: If $S_1, S_2, \ldots, S_k$ are subsets of a vector space V, the set of all sums

$$\bar{\alpha}_1 + \bar{\alpha}_2 + \ldots + \bar{\alpha}_k$$

of $\bar{\alpha}_i \in S_i$ is called the <u>sum</u> of the subsets $S_1, S_2, \ldots, S_k$ and is denoted by

$$S_1 + S_2 + \ldots + S_k$$

or by

$$\sum_{i=1}^{k} S_i .$$

If $W_1, W_2, \ldots, W_k$ are subspaces of V, then the sum

$$W = W_1 + W_2 + \ldots + W_k$$

is easily seen to be a subspace of V which contains each of the subspaces W_i. From this it follows that W is the subspace spanned by the union of $W_1, W_2, \ldots, W_k$.

Example. Let F be a subfield of $\mathbb{C}$.
Suppose, $\bar{\alpha}_1 = (1, 2, 0, 3, 0)$,
$\bar{\alpha}_2 = (0, 0, 1, 4, 0)$,
$\bar{\alpha}_3 = (0, 0, 0, 0, 1)$.

A vector $\bar{\alpha}$ is in the subspace W of F^5 spanned by $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3$ if and only if $\exists\ c_1, c_2, c_3 \in F$ s.t.

$$\bar{\alpha} = c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + c_3\bar{\alpha}_3 .$$

Thus, W consists of all vectors of the form $\bar{\alpha} = (c_1, 2c_1, c_2, 3c_1 + 4c_2, c_3)$, where c_1, c_2, c_3 are arbitray scalars in F.

Alternatively,

W can be described as the set of all 5-tupples $\bar{\alpha} = (x_1, x_2, x_3, x_4, x_5)$ w/ $x_i \in F$ s.t.

$$x_2 = 2x_1$$
$$x_4 = 3x_1 + 4x_3 .$$

Thus $(-3, -6, 1, -5, 2) \in W$, whereas $(2, 4, 6, 7, 8)$ is not.

Example. Let F be a subfield of $\mathbb{C}$, and let V be the vector space of all 2×2 matrices over F. Let W_1 be the subset of V consisting of all matrices of the form

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}, \quad x, y \in F \text{ are arbitrary.}$$

Then W_1 & W_2 are subspaces of V.

Also, $V = W_1 + W_2$

because $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$.

The subspace $W_1 \cap W_2$ consists of all matrices of the form $\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$.

Bases and Dimension

Def: Let V be a vector space over F. A subset S of V is said to be <u>linearly dependent</u> (or, <u>dependent</u>) if $\exists$ distinct vectors $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n \in S$ and scalars $c_1, c_2, \ldots, c_n \in F$, not all of which are $\bar{0}$, s.t.

$$c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \ldots + c_n\bar{\alpha}_n = \bar{0}.$$

A set which is not linearly dependent is called <u>linearly independent</u>. If the set S contains only finitely many $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n \in V$, we sometimes say that $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n$ are dependent (or independent) instead of saying S is dependent (or independent).

The following are easy consequences:

1. Any set which contains a linearly dependent set is linearly dependent.
2. Any subset of linearly independent set is linearly independent.
3. Any set which contains the $\bar{0}$ is linearly dependent.

4. A set S of vectors is linearly independent iff each finite subset of S is linearly independent, i.e., iff for any distinct vectors $\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_n \in S$, $c_1\bar{\alpha}_1 + \dots + c_n\bar{\alpha}_n = 0$ implies each $c_i = 0$.

Def^n: Let V be a vector space. A basis for V is a linearly indepent set of vectors in V which spans the space V. The space V is finite-dimensional if it has a finite basis.

Example: Let F be a subfield of $\mathbb{C}$.

In F^3 the vectors

$$\bar{\alpha}_1 = (3, 0, -3),$$
$$\bar{\alpha}_2 = (-1, 1, 2),$$
$$\bar{\alpha}_3 = (4, 2, -2),$$
$$\bar{\alpha}_4 = (2, 1, 1)$$

are linearly dependent, since

$$2\bar{\alpha}_1 + 2\bar{\alpha}_2 - \alpha_3 + 0 \cdot \bar{\alpha}_4 = 0.$$

The vectors

$$\bar{\varepsilon}_1 = (1, 0, 0),$$
$$\bar{\varepsilon}_2 = (0, 1, 0),$$
$$\bar{\varepsilon}_3 = (0, 0, 1)$$

are linearly dependent.

Example: Consider F^n over F. $S \subset F^n$ contains

$$\bar{\varepsilon}_1 = (1, 0, 0, \ldots, 0),$$
$$\bar{\varepsilon}_2 = (0, 1, 0, \ldots, 0),$$
$$\vdots$$
$$\bar{\varepsilon}_n = (0, 0, 0, \ldots, 1).$$

Let $x_1, x_2, \ldots, x_n \in F$ and put $\bar{\alpha} = x_1\bar{\varepsilon}_1 + x_2\bar{\varepsilon}_2 + \ldots + x_n\bar{\varepsilon}_n$.

Then, $\bar{\alpha} = (x_1, x_2, \ldots, x_n)$.

This shows that $\bar{\varepsilon}_1, \bar{\varepsilon}_2, \ldots, \bar{\varepsilon}_n$ span F^n.

$\therefore$ $\bar{\alpha} = \bar{0}$ iff $x_1 = x_2 = \ldots = x_n = 0$, $\bar{\varepsilon}_1, \bar{\varepsilon}_2, \ldots, \bar{\varepsilon}_n$ are linearly independent. Set $\underbrace{S = \{\bar{\varepsilon}_1, \bar{\varepsilon}_2, \ldots, \bar{\varepsilon}_n\}}_{\text{Standard basis of } F^n}$ is a basis of F^n.