Probability and Statistics

Tutorial 10

Topics Covered: Markov Chains

Q1: $\{X_n\}_{n=1}^{\infty}$ is a Markov chain, prove that $\{Y_n\}_{n=2}^{\infty}$ is also a Markov chain, where $Y_n = [X_n, X_{n-1}]^T$ for some $k \leq n$.

Q2: Given the Markov chain in Figure 1, find the set of stationary distributions. Note that since the Markov chain is reducible, there may be more than one stationary distribution.

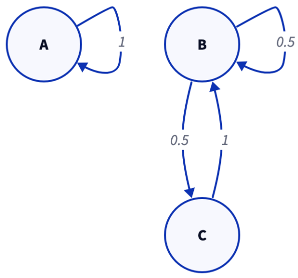

Figure 1: A reducible Markov chain

Q3: Consider a Markov chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 0.2 & 0.8 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.3 & 0.7 \end{pmatrix}.$$

Suppose the chain starts in state $X_0 = 1$. Using the following sequence of independent Uniform$(0, 1)$ random numbers:

$$0.76,\ 0.34,\ 0.15,\ 0.91,\ 0.40,$$

simulate the chain to obtain $(X_0, X_1, X_2, X_3, X_4, X_5)$.

Q4: Consider a Markov chain with state space $S = \{1, 2, 3\}$, transition matrix

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{pmatrix}.$$

and initial distribution $\pi_0 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Compute the probability of obtaining the trajectory (3, 2, 1, 1, 3).

Q5: Consider a discrete-time Markov chain with the transition probabilities $p_{ij} = 0$ for $i = j$. The initial distribution is given by $\mu = [\mu_1, \mu_2]$. Find the probability of head and tail in the n-th step, in terms of μ_1 and μ_2.

Q6: A gambler begins with an initial fortune of i dollars. Each time he plays, he has the possibility of winning 1 dollar with a probability p or losing 1 dollar with a probability $1 - p$. The gambler will only stop playing if he either accumulates N dollars or loses all of his money.

Q7: Consider a Markov chain with transition probability matrix:

$$P = \begin{bmatrix} 0.7 & 0.3 & 0.0 \\ 0.4 & 0.5 & 0.1 \\ 0.0 & 0.6 & 0.4 \end{bmatrix}.$$

(a) Write a Python function to simulate a state transition path of length N, starting from an initial state s_0, using a random variable $U \sim \text{Uniform}(0, 1)$ to decide transitions.

(b) Estimate the **empirical limiting distribution** by simulating the chain for a large number of steps.

(c) Compute the **stationary distribution** π in three different ways:

i. Using the **eigenvalue method**: find the left eigenvector of P corresponding to eigenvalue 1.

ii. By solving the linear system $\pi P = \pi$, $\sum_i \pi_i = 1$.

(d) Compute the **theoretical limiting distribution** by raising P to a large power (e.g. P^{1000}).

Q8: Let $\alpha_0, \alpha_1, \cdots$ be a sequence of nonnegative numbers such that

$$\sum_{j=0}^{\infty} \alpha_j = 1.$$

Consider a Markov chain $X_0, X_1, X_2, \cdots$ with the state space $S = \{0, 1, 2, \cdots\}$ such that

$$p_{ij} = \alpha_j, \quad \text{for all } j \in S.$$

Show that $X_1, X_2, \cdots$ is a sequence of i.i.d random variables.

Q9: Consider a Markovian Coin, $S = \{0, 1\}$. Where 0 denotes Head and 1 denotes Tails. Suppose that the transition matrix is given by

$$P = \begin{bmatrix} 1 - a & a \\ b & 1 - b \end{bmatrix},$$

where a and b are two real numbers in the interval $[0, 1]$ such that $0 < a + b < 2$. Suppose that the system is in state 0 at time $n = 0$ with probability α, i.e.,

$$\pi^{(0)} = [P(X_0 = 0) \quad P(X_0 = 1)] = [\alpha \quad 1 - \alpha],$$

where $\alpha \in [0, 1]$.

(a) How does transition matrix define the nature of the coin.

(b) Using induction (or any other method), show that

$$P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}.$$

(c) Show that

$$\lim_{n\to\infty} P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix}.$$

(d) Show that

$$\lim_{n\to\infty} \pi^{(n)} = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}.$$

Q10: For the Markovian coin described above:

(a) Calculate the stationary distribution. What do you observe?

(b) Find the mean return times, r_0 and r_1, for this Markov chain. Do you observe anything?

(c) Can you intuitively explain the result above?