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$$i = 1;\ A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + \cdots A_{1N} = \beta_1$$

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In matrix form it should be

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Question: How can we solve the N dimensions equations? What will be the caveats?

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Given a linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, a sequence $s_1, s_2, \ldots, s_n$ of n numbers is called a **solution** to the equation if

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$$

that is, if the equation is satisfied when the substitutions $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ are made. A sequence of numbers is called **a solution to a system** of equations if it is a solution to every equation in the system.

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Rudiment of Matrix

Symmetric and Transpose

$\boldsymbol{A}_{ij}^{T} = \boldsymbol{A}_{ji}$

If $A^T = A$, then A can be called as symmetric

Triangular Matrix

A triangular matrix is a type of square matrix that has all values in the upper-right or lower-left of the matrix with the remaining elements filled with zero values.

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Diagonal Matrix

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Identity Matrix

An identity matrix is a square matrix that does not change a vector when multiplied.

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Skew Symmetric*

$\boldsymbol{A}^{\boldsymbol{T}} = -\boldsymbol{A}$

Hermitian

A matrix is said to be hermitian if and only it is equal to the transpose of its conjugate matrix.

Rudiment of Matrix

Idempotent

$$A^k = A$$
$$k \geq 2$$

Nilpotent

$$A^k = 0$$

A square matrix A of order n is nilpotent if and only if $A^k = O$ for some $k \leq n$.

Orthogonal

$$AA^T = I$$

Involutory Matrix

$$A^2 = I$$

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Formally, consider an $n \times n$ matrix $A=(a_{i,j})$. If all matrix elements are zero outside a diagonally bordered band whose range is determined by constants k_1 and k_2:

$$a_{i,j} = 0 \quad \text{if} \quad j < i - k_1 \quad \text{or} \quad j > i + k_2; \quad k_1, k_2 \geq 0.$$

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- A band matrix with $k_1 = k_2 = 0$ is a diagonal matrix
- A band matrix with $k_1 = k_2 = 1$ is a tridiagonal matrix
- For $k_1 = k_2 = 2$ one has a pentadiagonal matrix and so on.
- Triangular matrices
  - For $k_1 = 0$, $k_2 = n-1$, one obtains the definition of an upper triangular matrix
  - similarly, for $k_1 = n-1$, $k_2 = 0$ one obtains a lower triangular matrix.

Matlab:
Issymmetric,
inv,
A',
A*A; A*.A;

Addition

Let $\boldsymbol{X}$ represent a matrix, $\boldsymbol{X_{ij}}$ denote the entry that is in the ith row and jth column of $\boldsymbol{X}$.

$(\boldsymbol{A} + \boldsymbol{B})_{ij} = A_{ij} + B_{ij}$

Rudiments of Matrix algebra

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Multiplicative

Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{B} \in \mathbb{R}^{n\times k}$

$(\boldsymbol{AB})_{ij} = \sum_{l=1}^{n} A_{il}B_{lj}$

In general, matrix multiplication is not commutative.

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Properties

- *Associativity:*
  $\forall \boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{B} \in \mathbb{R}^{n\times p}, \boldsymbol{C} \in \mathbb{R}^{p\times q}$:
  $(AB)C = A(BC)$
- *Distributivity:*
  $\forall \boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m\times n}, \boldsymbol{C}, \boldsymbol{D} \in \mathbb{R}^{n\times p}$,
  $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC}+\boldsymbol{BC}$,
  $\boldsymbol{A}(\boldsymbol{C}+\boldsymbol{D}) = \boldsymbol{AC}+\boldsymbol{AD}$

Rudiment of Matrix algebra

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  $\boldsymbol{A}(\boldsymbol{C}+\boldsymbol{D}) = \boldsymbol{AC} + \boldsymbol{AD}$

Multiplication By Identity Matrix:
$\forall \boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{I_m A} = \boldsymbol{A I_n} = \boldsymbol{A}$, where $\boldsymbol{I_m}$ is an $m \times m$ matrix such that it has 1s on the diagonal and 0s everywhere else. It is known as the identity matrix.

Properties of Matrix Transpose

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

Rudiments of Matrix algebra

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Matrix Inverse

Consider a square matrix $A \in \mathbb{R}^{n \times n}$. Let matrix $B \in \mathbb{R}^{n \times n}$ have the property that $AB = I_n = BA$. B is called the inverse of A and is denoted by A^{-1}.

Properties of Matrix Inverse

- $(A + B)^{-1} \neq A^{-1} + B^{-1}$
- $(AB)^{-1} = B^{-1} A^{-1}$

Viewing a Matrix – 4 Ways

V Vector times Vector – 2 Ways

Mv Matrix times Vector – 2 Ways

vM Vector times Matrix – 2 Ways

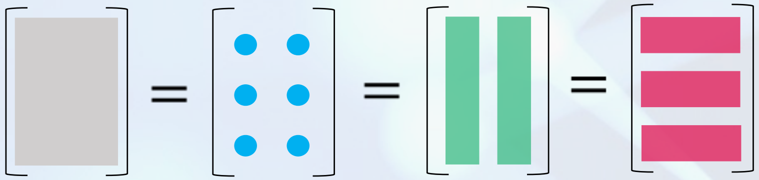
MM Matrix times Matrix – 4 Ways

P Practical Patterns

The Five Matrix Factorizations

- $CR, LU, QR, Q\Lambda Q^{\mathrm{T}}, U\Sigma V^{\mathrm{T}}$

Viewing a Matrix – 4 Ways



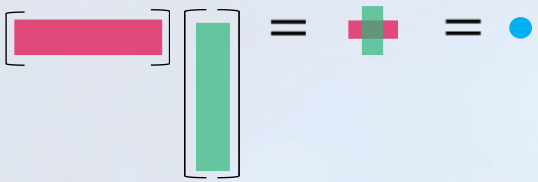
$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

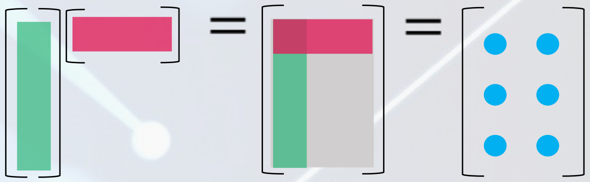
1 matrix — 6 numbers — 2 column vectors with 3 numbers — 3 row vectors with 2 numbers

$$A = \begin{bmatrix} a_{11} a_{12} \\ a_{21} a_{22} \\ a_{31} a_{32} \end{bmatrix} = \begin{bmatrix} | & | \\ \boldsymbol{a_1} & \boldsymbol{a_2} \\ | & | \end{bmatrix} = \begin{bmatrix} -\boldsymbol{a}_1^* - \\ -\boldsymbol{a}_2^* - \\ -\boldsymbol{a}_3^* - \end{bmatrix}$$

Here, column vectors are in bold as $\boldsymbol{a_1}$, row vectors are with * as $\boldsymbol{a}_1^*$.
And transposed vectors/matrices are with T on the shoulders as $\boldsymbol{a}^{\mathrm{T}}$, A^{T}

v Vector times Vector – 2 Ways

v1  Dot product (number)

v2  Rank 1 Matrix

Dot product ($\boldsymbol{a} \cdot \boldsymbol{b}$) is expressed as $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}$ in matrix language and yields a number.

$\boldsymbol{ab}^{\mathrm{T}}$ is a matrix ($\boldsymbol{ab}^{\mathrm{T}} = A$). If neither a, b are 0, the result A is a rank 1 matrix.

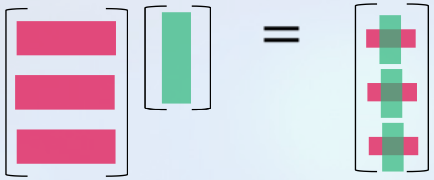
$$\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_1 + 2x_2 + 3x_3$$

$$\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}x & y\end{bmatrix} = \begin{bmatrix}x & y\\2x & 2y\\3x & 3y\end{bmatrix}$$

Mv

Matrix times Vector – 2 Ways

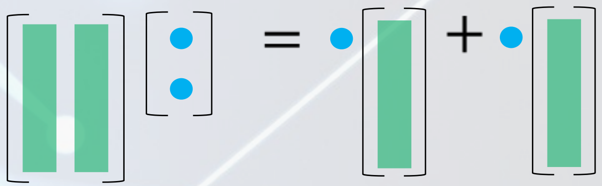
Mv 1



The row vectors of A are multiplied by a vector $\boldsymbol{x}$ and become the three dot-product elements of $A\boldsymbol{x}$.

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (x_1+2x_2) \\ (3x_1+4x_2) \\ (5x_1+6x_2) \end{bmatrix}$$

Mv 2



The product $A\boldsymbol{x}$ is a linear combination of the column vectors of A.

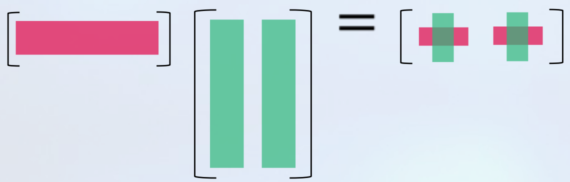
$$A\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

At first, you learn (Mv1). But when you get used to viewing it as (Mv2), you can understand $A\boldsymbol{x}$ as a linear combination of the columns of A. Those products fill the column space of A denoted as $\mathbf{C}(A)$.
The solution space of $A\boldsymbol{x} = \mathbf{0}$ is the nullspace of A denoted as $\mathbf{N}(A)$.

vM

Vector times Matrix – 2 Ways

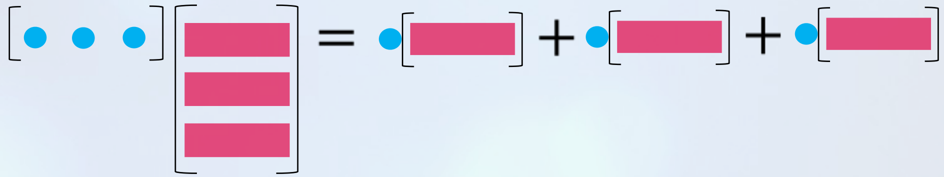
vM 1



$$\boldsymbol{y}A = [y_1 \quad y_2 \quad y_3]\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix} = [(y_1 + 3y_2 + 5y_3) \quad (2y_1 + 4y_2 + 6y_3)]$$

A row vector $\boldsymbol{y}$ is multiplied by the two column vectors of A and become the two dot-product elements of $\boldsymbol{y}A$.

vM 2



$$\boldsymbol{y}A = [y_1 \quad y_2 \quad y_3]\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix} = y_1[1 \quad 2] + y_2[3 \quad 4] + y_3[5 \quad 6]$$

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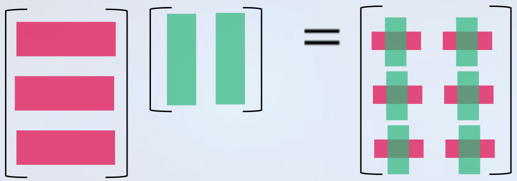
MM

Matrix times Matrix – 4 Ways

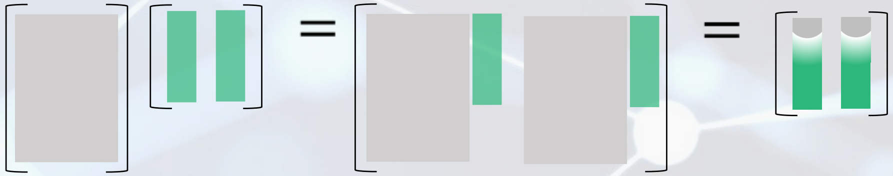
MM 1



Every element becomes a dot product of row vector and column vector.

$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}x_1 & y_1\\x_2 & y_2\end{bmatrix}=\begin{bmatrix}(x_1+2x_2) & (y_1+2y_2)\\(3x_1+4x_2) & (3y_1+4y_2)\\(5x_1+6x_2) & (5y_1+6y_2)\end{bmatrix}$$

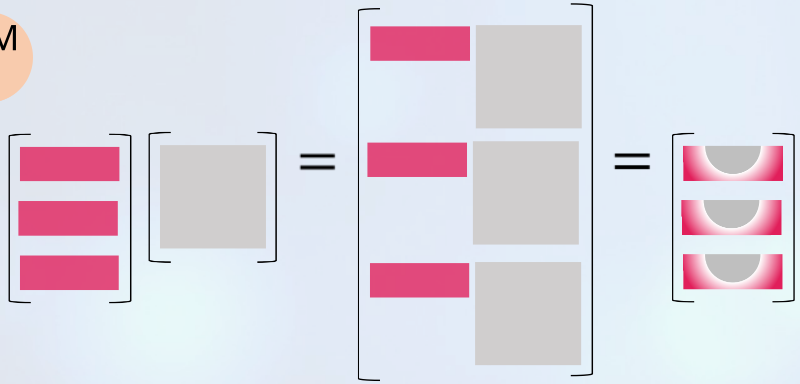
MM 2



$A\boldsymbol{x}$ and $A\boldsymbol{y}$ are linear combinations of columns of A.

$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}x_1 & y_1\\x_2 & y_2\end{bmatrix}=A[\boldsymbol{x} \quad \boldsymbol{y}]=[A\boldsymbol{x} \quad A\boldsymbol{y}]$$

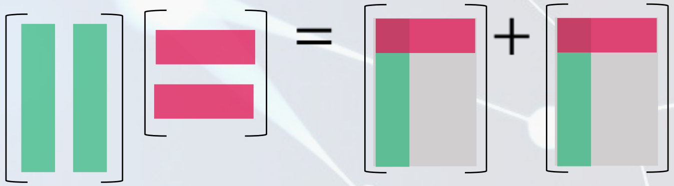
MM 3



The produced rows are linear combinations of rows.

$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}x_1 & y_1\\x_2 & y_2\end{bmatrix}=\begin{bmatrix}\boldsymbol{a}_1^*\\\boldsymbol{a}_2^*\\\boldsymbol{a}_3^*\end{bmatrix}X=\begin{bmatrix}\boldsymbol{a}_1^*X\\\boldsymbol{a}_2^*X\\\boldsymbol{a}_3^*X\end{bmatrix}$$

MM 4

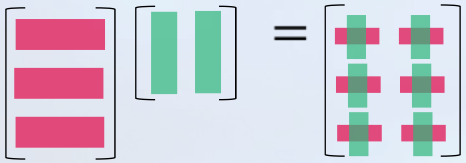


Multiplication AB is broken down to a sum of rank 1 matrices.

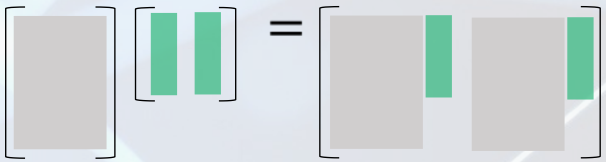
$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}b_{11} & b_{12}\\b_{21} & b_{22}\end{bmatrix}=[\boldsymbol{a}_1 \quad \boldsymbol{a}_2]\begin{bmatrix}\boldsymbol{b}_1^*\\\boldsymbol{b}_2^*\end{bmatrix}=\boldsymbol{a}_1\boldsymbol{b}_1^*+\boldsymbol{a}_2\boldsymbol{b}_2^*$$

$$=\begin{bmatrix}1\\3\\5\end{bmatrix}[b_{11} \quad b_{12}]+\begin{bmatrix}2\\4\\6\end{bmatrix}[b_{21} \quad b_{22}]=\begin{bmatrix}b_{11} & b_{12}\\3b_{11} & 3b_{12}\\5b_{11} & 5b_{12}\end{bmatrix}+\begin{bmatrix}2b_{21} & 2b_{22}\\4b_{21} & 4b_{22}\\6b_{21} & 6b_{22}\end{bmatrix}$$

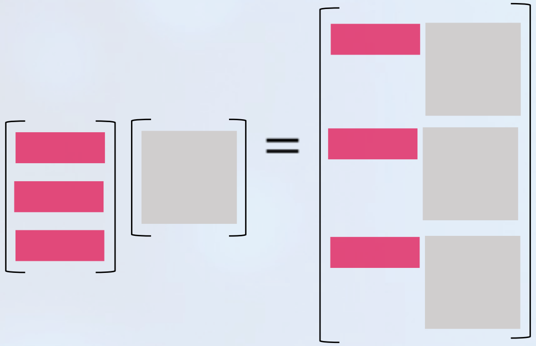
Four Ways to Multiply AB



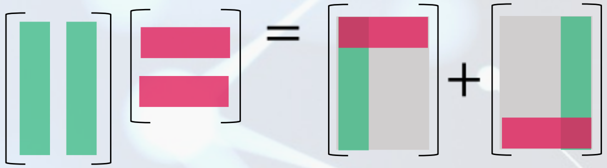
(Rows of $\boldsymbol{A}$) • (Columns of $\boldsymbol{B}$)



$\boldsymbol{A}$ times (Columns of $\boldsymbol{B}$)



(Rows of $\boldsymbol{A}$) times $\boldsymbol{B}$



Sum of (Columns of $\boldsymbol{A}$) (Rows of $\boldsymbol{B}$)

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Given $m \times n$ matrix $\boldsymbol{A}$ and m-vector $\boldsymbol{b}$, find unknown n-vector $\boldsymbol{x}$ satisfying

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System of equations asks "Can $\boldsymbol{b}$ be expressed as linear combination of columns of $\boldsymbol{A}$?"

Solution may or may not exist, and may or may not be unique

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We can also talk about non-square systems where A is $m \times n$, b is $m \times 1$, and x is $n \times 1$

- *Overdetermined* if $m>n$:
  "more equations than unknowns"
- *Underdetermined* if $n>m$:
  "more unknowns than equations"

Singularity and nonsingularity

$n \times n$ matrix $\boldsymbol{A}$ is *nonsingular* if it has any of following equivalent properties:

1. Inverse of $\boldsymbol{A}$, denoted by $\boldsymbol{A}^{-1}$, exists

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1. Inverse of $\boldsymbol{A}$, denoted by $\boldsymbol{A}^{-1}$, exists
2. $\det(\boldsymbol{A}) \neq 0$
3. $\operatorname{rank}(\boldsymbol{A}) = n$
4. For any vector $\boldsymbol{z} \neq \boldsymbol{o}$, $\boldsymbol{A}\boldsymbol{z} \neq \boldsymbol{o}$

Singularity and nonsingularity

$n \times n$ matrix A is *nonsingular* if it has any of following equivalent properties:

1. Inverse of A, denoted by A^{-1}, exists
2. $\det(A) \neq 0$
3. $\operatorname{rank}(A) = n$
4. For any vector $z \neq o$, $Az \neq o$

A is singular if some row is linear combination of other rows.

Singular systems can be underdetermined:

$$2x_1 + 3x_2 = 5$$
$$4x_1 + 6x_2 = 10$$

or inconsistent:

$$2x_1 + 3x_2 = 5$$
$$4x_1 + 6x_2 = 11$$

Example

Singularity and nonsingularity

Solvability of $\boldsymbol{Ax} = \boldsymbol{b}$ depends on whether $\boldsymbol{A}$ is singular or nonsingular

If $\boldsymbol{A}$ is nonsingular, then $\boldsymbol{Ax} = \boldsymbol{b}$ has unique solution for any $\boldsymbol{b}$

If $\boldsymbol{A}$ is singular, then number of solutions is determined by $\boldsymbol{b}$

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If $\boldsymbol{A}$ is singular and $\boldsymbol{Ax} = \boldsymbol{b}$, then $\boldsymbol{A}(\boldsymbol{x}+\gamma\boldsymbol{z}) = \boldsymbol{b}$ for any scalar γ, where $\boldsymbol{Az} = \boldsymbol{o}$ and $\boldsymbol{z} \neq \boldsymbol{o}$, so solution not unique

One solution:	nonsingular
No solution:	singular
∞ many solutions:	singular

Solving Linear Equations

How to solve?

Solving Linear Equations
Gauss-Jordan Elimination

Step 1: Write the Augmented Matrix

For a system of equations:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

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Write its augmented matrix:

$$\begin{pmatrix} a_1 & b_1 & c_1 & | \; d_1 \\ a_2 & b_2 & c_2 & | \; d_2 \\ a_3 & b_3 & c_3 & | \; d_3 \end{pmatrix}$$

Solving Linear Equations
Gauss-Jordan Elimination

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Write its augmented matrix:

$$\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right)$$

Step 2: Convert to Row Echelon Form (Upper Triangular)

Use row operations to make the leading coefficient (pivot) of each row equal to **1** and make all elements below it **0**.

Row operations allowed:

•Swap two rows.

•Multiply a row by a nonzero scalar.

•Add or subtract a multiple of one row from another.

Solving Linear Equations
Gauss-Jordan Elimination

Step 3: Convert to Reduced Row Echelon Form (RREF)
•Make all elements **above** each pivot **0**, so the left part of the augmented matrix becomes an **identity matrix**.

Step 4: Extract the Solution
The last column of the matrix represents the values of the variables.

A "leading entry" is the first nonzero element in a row.
Definition: A matrix is in **echelon form** (or row echelon form) if it has the following three properties:
1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

Solving Linear Equations
Gauss-Jordan Elimination

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The last column of the matrix represents the values of the variables.

Key Takeaways

•The **Gauss-Jordan method** reduces the system to **RREF**.
•If all rows are nonzero and form an identity matrix, a **unique solution** exists.
•If a row of zeros appears with a nonzero number in the augmented column, the system is **inconsistent** (no solution).
•If there are free variables, the system has **infinitely many solutions**.

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Definition: A matrix is in **echelon form** (or row echelon form) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
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3. All entries in a column below a leading entry are zeros.

Solving Linear Equations
Gauss-Jordan Elimination

Exploring the Augmented Matrix
1. Understanding Row Operations
The augmented matrix allows us to apply **elementary row operations** directly:

1. **Swapping rows** (if needed for numerical stability).
2. **Multiplying a row by a scalar** (to make leading coefficients 1).
3. **Adding/subtracting multiples of one row to another** (to eliminate variables).

- If any row reduces to **all zeros except the last column** the system is **inconsistent** (no solution).
- If a row becomes all **zeros**, and there are fewer independent equations than unknowns, the system has **infinitely many solutions**.
- If the coefficient matrix reduces to an **identity matrix**, the system has a **unique solution**.

Solving Linear Equations
Gauss-Jordan Elimination
Pseudo code

```
for i = 1 to n do
   // Step 1: Make the pivot non-zero
   if A[i][i] == 0 then
      for k = i+1 to n do
         if A[k][i] ≠ 0 then
            swap row i with row k
            break
         end if
      end for
   end if

   // Step 2: Normalize the pivot row
   pivot = A[i][i]
   for j = 1 to n+1 do
      A[i][j] = A[i][j] / pivot
   end for

   // Step 3: Eliminate all other entries in column i
   for k = 1 to n do
      if k ≠ i then
         factor = A[k][i]
         for j = 1 to n+1 do
            A[k][j] = A[k][j] - factor * A[i][j]
         end for
      end if
   end for
end for
```

Solving Linear Equations
Gauss-Jordan Elimination
Pseudo code

2. Check for **inconsistency (No solution)**:
   - If a row has all zeros in A but a nonzero value in b → **No solution (inconsistent system).**
     Example: `0x + 0y + 0z = 5` (Contradiction)

3. Check for **infinitely many solutions**:
   - If at least one variable is **free** (i.e., a column without a pivot), there are **infinitely many solutions**.

4. If neither inconsistency nor infinite solutions, extract solution x from last column.

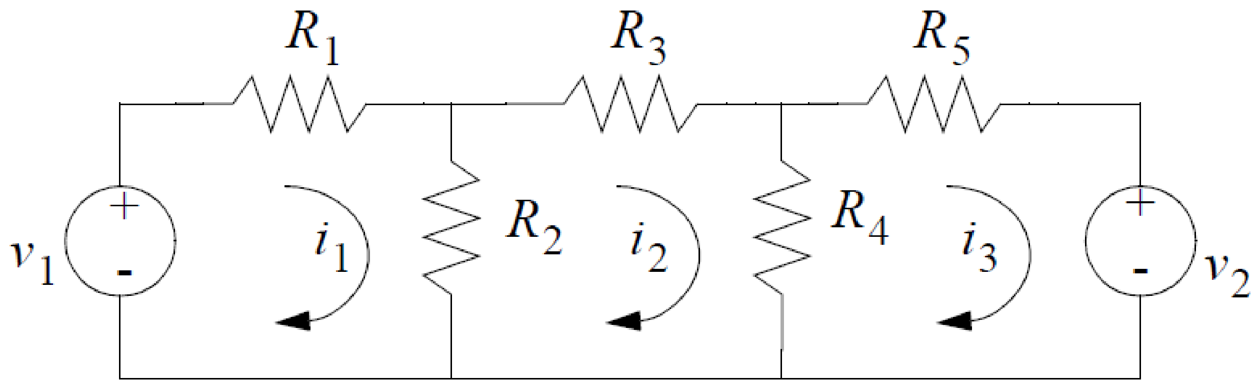
End

Solving Linear Equations
Gauss-Jordan Elimination

1. **Row Reduction (Forward Elimination)**:
    1. Converts the system into **upper triangular form**.
    2. Uses **row swaps** to prevent division by zero.
    3. Normalizes pivot rows and makes elements below the pivot zero.
2. **Backward Elimination (RREF conversion)**:
    1. Ensures the identity matrix is formed on the left.
    2. Eliminates nonzero elements above each pivot.
3. **Extracting the Solution**:
    1. Checks for inconsistencies.
    2. Identifies free variables for **infinite solutions**.
    3. Extracts unique solutions when possible.

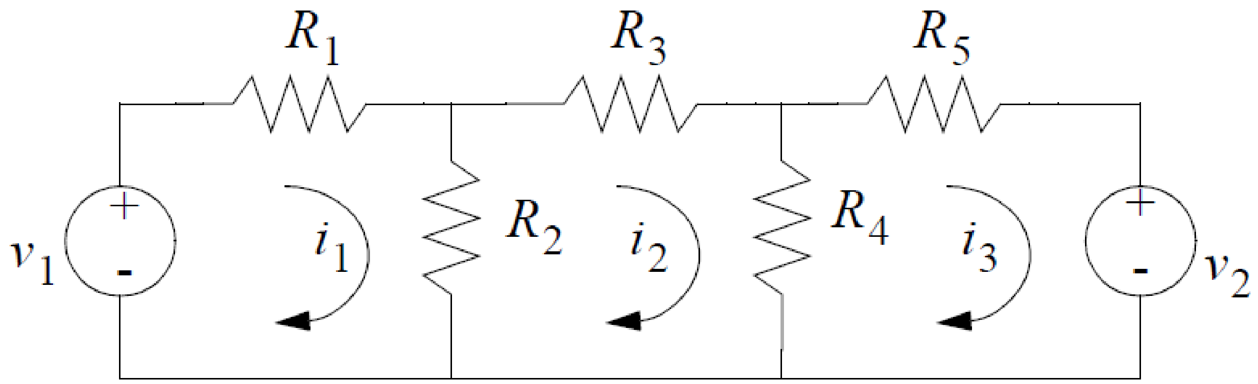
```
% Check for inconsistencies (e.g., [0 0 0 | k] where k ≠ 0)
```

Electrical Networks



Write the Equations

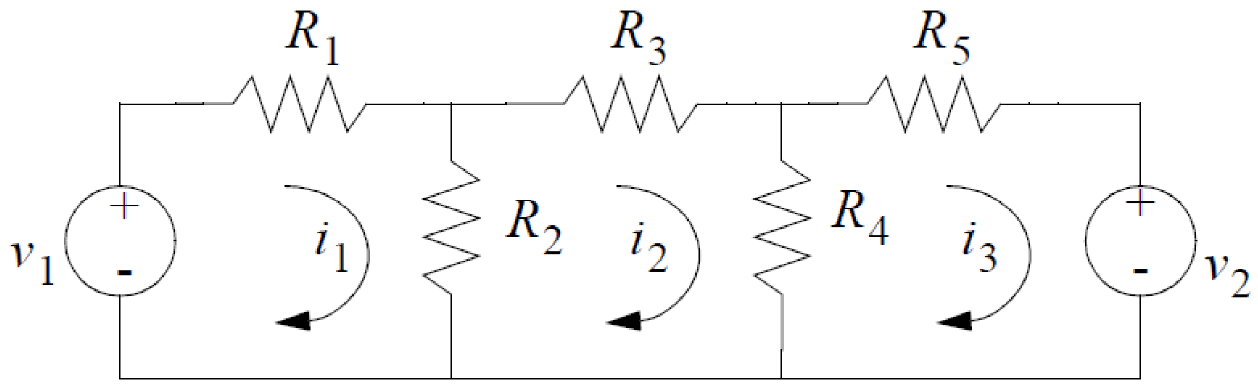
Electrical Networks



- The first loop equation has a voltage source and two resistors; the resistor R_2 has current i_1 flowing from top to bottom and current i_2 flowing from bottom to top

$$-v_1 + R_1 i_1 + (i_1 - i_2)R_2 = 0$$

Electrical Networks



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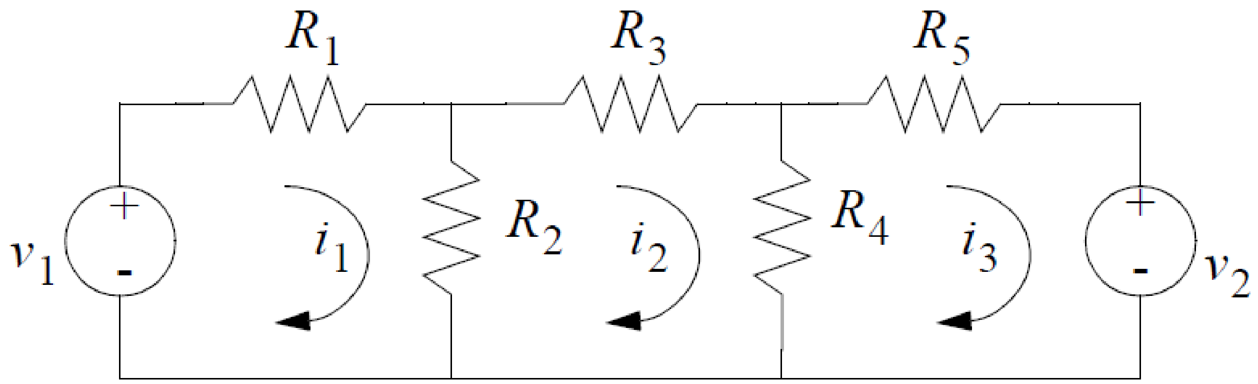
$$(R_1 + R_2)i_1 - R_2 i_2 = v_1$$

Electrical Networks

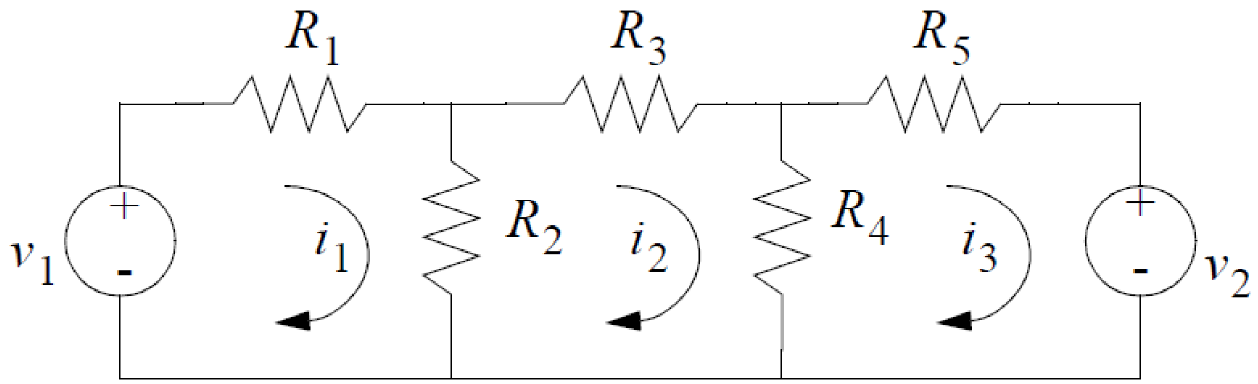


$$(R_1 + R_2)i_1 - R_2 i_2 = v_1$$

The second loop equation involves three resistors, but all three loop currents appear in the equation

$$(i_2 - i_1)R_2 + i_2 R_3 + (i_2 - i_3)R_4 = 0$$

Electrical Networks



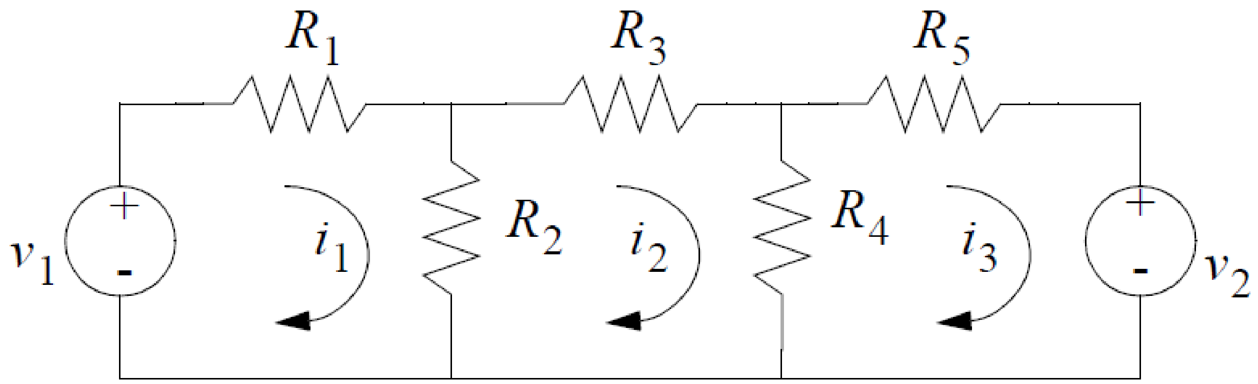
$$(R_1 + R_2)i_1 - R_2 i_2 = v_1$$

$$(i_2 - i_1)R_2 + i_2 R_3 + (i_2 - i_3)R_4 = 0$$

$$(i_3 - i_2)R_4 + i_3 R_5 + v_2 = 0$$

$$-R_4 i_2 + (R_4 + R_5)i_3 = -v_2$$

Electrical Networks



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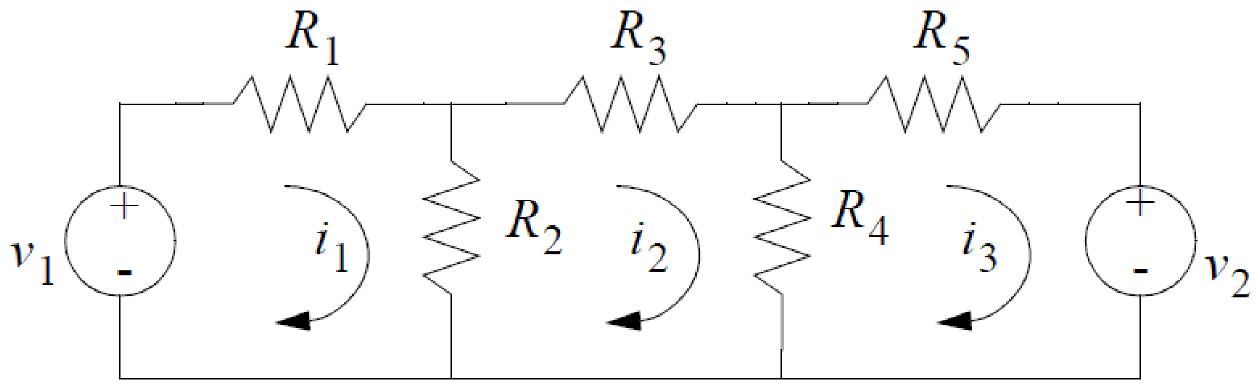
$$-R_4 i_2 + (R_4 + R_5)i_3 = -v_2$$

$$\begin{bmatrix} (R_1 + R_2) & -R_2 & 0 \\ -R_2 & (R_2 + R_3 + R_4) & -R_4 \\ 0 & -R_4 & (R_4 + R_5) \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \\ -v_2 \end{bmatrix}$$

Electrical Networks



$$
\begin{aligned}
(R_1 + R_2)i_1 - R_2 i_2 &= v_1 \\
(i_2 - i_1)R_2 + i_2 R_3 + (i_2 - i_3)R_4 &= 0 \\
-R_4 i_2 + (R_4 + R_5)i_3 &= -v_2
\end{aligned}
$$

$$
\begin{bmatrix} (R_1+R_2) & -R_2 & 0 \\ -R_2 & (R_2+R_3+R_4) & -R_4 \\ 0 & -R_4 & (R_4+R_5) \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \\ -v_2 \end{bmatrix}
$$

$$R_1 = R_2 = R_3 = R_4 = R_5 = 1\,\text{ohm}$$

$$v_1 = 5 \text{ volts and } v_2 = -6 \text{ volts}$$

$$
\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}
$$

```
x =

    3.8750 % Units of amps
    2.7500 % Units of amps
    4.3750 % Units of amps
```

Balancing Chemical Reactions

$$aO_2 + bC_6H_{12}O_6 \rightarrow cH_2O + dCO_2$$

$$O: 2a + 6b = 1c + 2d \longrightarrow 2a + 6b - 1c = 2d,$$

$$C: 0a + 6b = 0c + 1d \longrightarrow 0a + 6b + 0c = 1d,$$

$$H: 0a + 12b = 2c + 0d \longrightarrow 0a + 12b - 2c = 0d.$$

Balancing Chemical Reactions

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$$H: 0a + 12b = 2c + 0d \longrightarrow 0a + 12b - 2c = 0d.$$

$$\boxed{6O_2 + C_6H_{12}O_6 \rightarrow 6H_2O + 6CO_2}$$

Tridiagonal Systems

- Common special case:

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & 0 & 0 & \cdots & b_1 \\ a_{21} & a_{22} & a_{23} & 0 & \cdots & b_2 \\ 0 & a_{32} & a_{33} & a_{34} & \cdots & b_3 \\ 0 & 0 & a_{43} & a_{44} & \cdots & b_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{array}\right]$$

- Only main diagonal + 1 above and 1 below

Solving Tridiagonal Systems

- When solving using Gauss-Jordan:
  - Constant # of multiplies/adds in each row
  - Each row only affects 2 others

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & 0 & 0 & \cdots & b_1 \\ a_{21} & a_{22} & a_{23} & 0 & \cdots & b_2 \\ 0 & a_{32} & a_{33} & a_{34} & \cdots & b_3 \\ 0 & 0 & a_{43} & a_{44} & \cdots & b_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{array}\right]$$

Triangular Systems

- Another special case: A is lower-triangular

$$\left[\begin{array}{ccccc|c} a_{11} & 0 & 0 & 0 & \cdots & b_1 \\ a_{21} & a_{22} & 0 & 0 & \cdots & b_2 \\ a_{31} & a_{32} & a_{33} & 0 & \cdots & b_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots & b_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{array}\right]$$

Triangular Systems

- Solve by forward substitution

$$\left[\begin{array}{ccccc|c} a_{11} & 0 & 0 & 0 & \cdots & b_1 \\ a_{21} & a_{22} & 0 & 0 & \cdots & b_2 \\ a_{31} & a_{32} & a_{33} & 0 & \cdots & b_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots & b_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{array}\right]$$

$$x_1 = \frac{b_1}{a_{11}}$$

Triangular Systems

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$$\left[\begin{array}{ccccc|c} a_{11} & 0 & 0 & 0 & \cdots & b_1 \\ a_{21} & a_{22} & 0 & 0 & \cdots & b_2 \\ a_{31} & a_{32} & a_{33} & 0 & \cdots & b_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots & b_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{array}\right]$$

$$x_2 = \frac{b_2 - a_{21}x_1}{a_{22}}$$

Triangular Systems

- Solve by forward substitution

$$\left[\begin{array}{ccccc|c} a_{11} & 0 & 0 & 0 & \cdots & b_1 \\ a_{21} & a_{22} & 0 & 0 & \cdots & b_2 \\ a_{31} & a_{32} & a_{33} & 0 & \cdots & b_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots & b_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{array}\right]$$

$$x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

What type of transformation makes the solution unchanged?

What type of transformation of the linear system leaves solution unchanged?

Claim

We can premultiply (from left) both sides of linear system $Ax = b$ by any nonsingular matrix M without affecting solution

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$$MAx = Mb$$

What type of transformation of the linear system leaves solution unchanged?

Claim

We can premultiply (from left) both sides of linear system $Ax = b$ by any nonsingular matrix M without affecting solution. Prove it.

$$MAx = Mb$$

$$x = (MA)^{-1}Mb$$

$$\boldsymbol{x = A^{-1}M^{-1}Mb = A^{-1}b}$$

Example

Permutation matrix P

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- One 1 in each row.
- $P^{-1} = P$

Example

Permutation matrix P

- One 1 in each row.
- $P^{-1} = P$

- Premultiplying both sides of system by permutation matrix, $PAx = Pb$ reorders row, but solutions remain unchanged.

What type of linear system is easy to solve?

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- If this pattern continues, with only one new solution component per equation, then all components of solution can be computed in succession.

System with this property called triangular

Forward and Backward Substitution

$$\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

Last equation, $4x_3 = 8$, can be solved directly to obtain $x_3 = 2$

x_3 then substituted into second equation to obtain $x_2 = 2$

Finally, both x_3 and x_2 substituted into first equation to obtain $x_1 = -1$

Code: Backward Substitution (Upper Triangular matrix)

Input: Upper triangular matrix U (n × n), column vector b (n × 1) Output: Solution vector x (n × 1).

1. Initialize x as an empty vector of size n.
2. For i = n down to 1:

Compute x[i] = (b[i] - sum(U[i, j] * x[j] for j = i+1 to n)) / U[i, i]

3. Return x

Task

Forward and Backward Substitution

$$\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

LU decomposition and solving linear equations

1. *Decomposition*:

$$A = LU$$

2. *Forward substitution*: solve

$$L\mathbf{y} = \mathbf{b}.$$

3. *Backward substitution*: solve

$$U\mathbf{x} = \mathbf{y}.$$

LU decomposition and solving linear equations

in order to solve for x. The advantage is that L captures the transformation (using Gauss elimination) from the original matrix A to the upper diagonal matrix U. That is, if L and U are stored, the steps in the Gauss elimination are also stored. Then, if we have to solve the equation for different values of b, we could use the stored values of L and U, instead of doing the elimination once again.

LU decomposition

```
%%function
[L,A]=LU_factor_using_Gaussian_1a(A1,n)
clear all
%%% A matrix is decomposed here in L and U form
A1=[2 4 -2 ; 4 9 -3; -2 -3 7]; n=3;


%A1=[1 2 2 ; 4 4 2; 4 6 4]; n=length(A1);
A=A1;
L=eye(n);
for k=1:n
if (A(k,k) == 0)
Error('Pivoting is needed!'); end
L(k+1:n,k)=A(k+1:n,k)/A(k,k);
for j=k+1:n
A(j,:)=A(j,:)-L(j,k)*A(k,:);
end
end
[l u]=lu(A1);
```

LU decomposition and solving linear equations

Task

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

When Gauss was around 17 years old, he developed a method for working with inconsistent linear systems, called the method of *least squares*. A few years later (at the advanced age of 24) he turned his attention to a particular problem in astronomy.
In 1801 the Sicilian astronomer Piazzi discovered a (dwarf) planet, which he named Ceres, in honor of the patron goddess of Sicily. Piazzi took measurements of Ceres' position for 40 nights, but then lost track of it when it passed behind the sun. Piazzi had only tracked Ceres through about 3 degrees of sky.

Gauss however *then succeeded in calculating the orbit of Ceres*, even though the task seemed hopeless on the basis of so few observations. His computations were so

In the course of his computations **Gauss had to solve systems of 17 linear equations.** Since Gauss at first refused to reveal the methods that led to this amazing accomplishment, some even accused him of sorcery. Eight years later, in 1809, Gauss revealed his methods of orbit computation in his book *Theoria Motus Corporum Coelestium.*

Transforming general linear system into triangular form

- You need to replace selected nonzero entries of matrix by zeros.
- This can be done by using the linear combination of rows.

Transforming general linear system into triangular form

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- Let the vector be $v = [v_1 \ \ v_2]^T$
- Find M such $v \rightarrow [v_1 \ 0]$

Transforming general linear system into triangular form

- You need to replace selected nonzero entries of matrix by zeros.

- This can be done by using the linear combination of rows.

- Let the vector be $v = [v_1 \ \ v_2]^T$
- Find M such $Mv \rightarrow [v_1 \ 0]$

- $M = \begin{bmatrix} 1 & 0 \\ -\frac{v_2}{v_1} & 1 \end{bmatrix}$

Transforming general linear system into triangular form

More generally, can annihilate *all* entries below kth position in n-vector $\boldsymbol{a}$ by transformation

$\boldsymbol{M}_k\boldsymbol{a} =$

$$\begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & -m_{k+1} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -m_n & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ a_{k+1} \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $m_i = a_i/a_k$, $i = k+1, \ldots, n$

Transforming general linear system into

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where $m_i = a_i/a_k$, $i = k+1, \ldots, n$

Divisor a_k, called *pivot*, must be nonzero

Matrix $\boldsymbol{M}_k$, called *elementary elimination matrix*, adds multiple of row k to each subsequent row, with multipliers m_i chosen so that result is zero

Transforming general linear system into triangular form

If $a = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$

Find M_1, such that

$$M_1 a = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Find M_2, such that

$$M_2 a = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

Transforming general linear system into triangular form

If $\boldsymbol{a} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$

Find M_1, such that

$$M_1 a = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

More generally, can annihilate *all* entries below kth position in n-vector $\boldsymbol{a}$ by transformation $M_k\boldsymbol{a} =$

$$\begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & -m_{k+1} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -m_n & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ a_{k+1} \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $m_i = a_i/a_k$, $i = k+1, \ldots, n$

Find M_2, such that

$$M_2 a = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}$$

Transforming general linear system into triangular form

Note that

$$L_1 = M_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$L_2 = M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix}$$

Also, find out $L_1 L_2$ and $M_1 M_2$

Transforming general linear system into triangular form

$$MAx = M_{n-1} \cdots M_1 Ax = M_{n-1} \cdots M_1 b = Mb$$

can be solved by back-substitution to obtain solution to original linear system $Ax = b$

Gaussian Elimination using Upper/Lower triangular matrix

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

Gaussian Elimination using Upper/Lower triangular matrix

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

To annihilate subdiagonal entries of first column of $\boldsymbol{A}$, $\boldsymbol{M}_1\boldsymbol{A} =$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

Gaussian Elimination using Upper/Lower triangular

To annihilate subdiagonal entries of first column of $\boldsymbol{A}$, $\boldsymbol{M}_1\boldsymbol{A} =$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

$$\boldsymbol{M}_1\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 12 \end{bmatrix}$$

Gaussian Elimination using Upper/Lower triangular matrix

To annihilate subdiagonal entry of second column of $M_1 A$, $M_2 M_1 A =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$M_2 M_1 b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

Pseudo code (finding L)

Input: Square matrix A of size n × n
Output: Lower triangular matrix L

1. Initialize L as an n × n identity matrix (ones on diagonal, zeros elsewhere).
2. For i from 1 to n: (Loop over rows)

a. For j from i+1 to n: (Compute lower triangular entries)

```
    L[j][i] = A[j][i]
        For k from 1 to i-1:
        L[j][i] = L[j][i] - (L[j][k] * A[k][i])
         End For
L[j][i] = L[j][i] / A[i][i] (Normalize)
```

3. Return L

Gaussian Elimination using Upper/Lower triangular matrix

Pseudo code (finding L and U together)

```
Input: Matrix A (n × n)
Output: Matrices L (n × n) and U (n × n) such that A = LU

1. Initialize L as an identity matrix of size n × n.
2. Initialize U as a zero matrix of size n × n.

3. For i = 1 to n:  # Loop over rows
     a. For j = i to n:  # Compute elements of U
          U[i, j] = A[i, j] - sum(L[i, k] * U[k, j] for k = 1 to i-1)

     b. For j = i+1 to n:  # Compute elements of L
          L[j, i] = (A[j, i] - sum(L[j, k] * U[k, i] for k = 1 to i-1)) /
U[i, i]

4. Return L, U
```

- **Only works for square matrices** ($n \times n$).
- **Does not require row swapping** (pivoting is needed if A is singular or nearly singular).
- **Time Complexity**: $O(n^3)$ due to nested loops.