Probability and Statistics

Tutorial 11

Q1: Given the sample mean and the maximum value of a random sample $X_1, X_2, \ldots, X_n$ from a uniform distribution on the interval $[0, \theta]$, derive two unbiased estimators for θ: one based on the sample mean and the other based on the maximum value. Compare their variances.

Q2: In the lecture, we saw that $\hat{\Theta}_1 = X_1$ (using only the first sample) is an unbiased estimator for the population mean μ. Is this estimator also a *consistent* estimator for μ? Justify your answer using the definition of consistency.

Q3: Given a random sample $X_1, X_2, \ldots, X_n$ from an exponential distribution with unknown parameter θ, the pdf is given by

$$f_X(x;\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the MLE for $\frac{1}{\theta}$.

Q4: Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2. We want to estimate the parameter μ^2. Consider the estimator $\hat{\Theta} = \overline{X}^2$. Show that this is a biased estimator for μ^2.

Q5: Let $X_1, X_2, \ldots, X_n$ ($n \geq 2$) be a random sample from a normal distribution $N(\mu, \sigma^2)$, where both μ and σ^2 are unknown. Let $(\hat{\mu}, \hat{\sigma}^2)$ denote the maximum likelihood estimators (M.L.E.) of (μ, σ^2).

- Show that the M.L.E. $\hat{\sigma}^2$ is not an unbiased estimator of σ^2.
- Find an unbiased estimator of σ^2 based on $\hat{\sigma}^2$.
- Among the estimators in the class

  $$\mathcal{D} = \left\{ \hat{\sigma}_c^2 : \hat{\sigma}_c^2 = c\,\hat{\sigma}^2 \right\},$$

  find the value of c that minimizes the mean squared error (MSE) of $\hat{\sigma}_c^2$.

Q6: Let $X_1, \ldots, X_n$ be a random sample from a distribution with mean θ and variance σ^2. Consider two estimators for the mean θ:

(a) $\hat{\Theta}_1 = X_1$

(b) $\hat{\Theta}_2 = \overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$

Compare these two estimators by finding the Mean Squared Error (MSE) for each. Which estimator is better for $n > 1$?

Q7: Given i.i.d. random variables $X_1, X_2, \ldots, X_n$ which follow the distribution with probability density function (pdf)

$$f_X(x) = \begin{cases} \frac{3\lambda}{2} e^{-\frac{3\lambda}{2}}, & 0 < x \\ \frac{\lambda}{2} e^{\frac{\lambda}{2}}, & \text{otherwise} \end{cases}$$

where $\lambda > 0$. Find the maximum likelihood estimator (MLE) for λ.

Q8: Let $X1, ..., Xn$ be a random sample from Bin(m, θ) distribution, where $\theta \in \Theta = (0, 1)$, (where Θ is called the parameter space) is unknown and m is a known positive integer.

(a) Find the MLE for θ.

(b) Does the MLE for $g(\theta) = \frac{1}{\theta}$ exist?