Probability and Statistics: MA6.101

Tutorial 6

Topics Covered: Moment Generating Functions, Sums of Random Variables, Stochastic Simulation

Q1: Suppose $X \sim \text{Binomial}(n_1, p)$ and $Y \sim \text{Binomial}(n_2, p)$ are two independent random variables.

(a) Find the MGF of X.

(b) Let $Z = X + Y$. Find the MGF of Z by using the MGFs of X and Y.

(c) By recognizing the form of the resulting MGF, identify the distribution of Z (including its parameters). What does this result imply about the sum of independent Binomial random variables with the same success probability?

A:

(a) The Moment Generating Function (MGF) is defined as $M_X(t) = \mathbb{E}[e^{tX}]$. For a discrete random variable like the Binomial, we compute this using its probability mass function (PMF), $p_X(k) = \binom{n_1}{k} p^k (1-p)^{n_1-k}$.

$$M_X(t) = \sum_{k=0}^{n_1} e^{tk} p_X(k) = \sum_{k=0}^{n_1} e^{tk} \binom{n_1}{k} p^k (1-p)^{n_1-k}$$

We can group the terms with exponent k:

$$M_X(t) = \sum_{k=0}^{n_1} \binom{n_1}{k} (pe^t)^k (1-p)^{n_1-k}$$

This summation is in the form of the binomial theorem, which states that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$. By setting $a = pe^t$ and $b = (1-p)$, we can simplify the expression:

$$\mathbf{M_X(t) = (pe^t + 1 - p)^{n_1}}$$

(b) The MGFs for X and Y are:

$$M_X(t) = (pe^t + 1 - p)^{n_1}$$

$$M_Y(t) = (pe^t + 1 - p)^{n_2}$$

For a sum of independent random variables $Z = X + Y$, the MGF is the product of their individual MGFs: $M_Z(t) = M_X(t) M_Y(t)$.

$$M_Z(t) = (pe^t + 1 - p)^{n_1} \cdot (pe^t + 1 - p)^{n_2}$$

$$\mathbf{M_Z(t) = (pe^t + 1 - p)^{n_1+n_2}}$$

(b) The resulting MGF, $M_Z(t) = (pe^t + 1 - p)^{n_1+n_2}$, is the MGF of a Binomial distribution with parameters $n = n_1 + n_2$ and p.
Therefore, the distribution of Z is Binomial$(n_1 + n_2, p)$.
This implies that the sum of two (or more) independent Binomial random variables that share the same success probability p is also a Binomial random variable. Its number of trials is the sum of the individual numbers of trials.

Q2: A discrete random variable X can take one of three values with the following probabilities:

$$p_X(1) = 0.4, \quad p_X(5) = 0.3, \quad p_X(10) = 0.3.$$

Describe, step-by-step, the inverse transform method to generate a random sample for X using a random number u drawn from a Uniform[0, 1] distribution. Provide the specific ranges of u that would correspond to each value of X.

A:
The inverse transform method works by mapping the output of a Uniform[0, 1] random number generator to the values of the desired random variable by partitioning the interval [0, 1] according to the cumulative probabilities.

Step 1: Calculate the Cumulative Distribution Function (CDF) of X.

- The first cumulative probability is $F_X(1) = p_X(1) = 0.4$.
- The next is $F_X(5) = p_X(1) + p_(5) = 0.4 + 0.3 = 0.7$.
- The final is $F_X(10) = p_X(1) + p_X(5) + p_X(10) = 0.7 + 0.3 = 1.0$.

Step 2: Define the sampling algorithm. The algorithm is as follows:

(a) Generate a random number u from a Uniform[0, 1] distribution.
(b) Compare u to the cumulative probabilities to determine the value of X.

Step 3: Specify the ranges for u. Based on the CDF, we assign the value of X according to which interval u falls into:

- If $\mathbf{0 \leq u < 0.4}$, then the generated sample is $\mathbf{X = 1}$.
- If $\mathbf{0.4 \leq u < 0.7}$, then the generated sample is $\mathbf{X = 5}$.
- If $\mathbf{0.7 \leq u < 1.0}$, then the generated sample is $\mathbf{X = 10}$.

Q3: Let X be a continuous random variable with the PDF

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|}, \qquad x \in \mathbb{R},\ \lambda > 0.$$

Find the MGF of X, $M_X(s)$.

A:
The moment generating function is defined as

$$M_X(s) = \mathbb{E}[e^{sX}] = \int_{-\infty}^{\infty} e^{sx} f_X(x)\, dx = \int_{-\infty}^{\infty} e^{sx} \frac{\lambda}{2} e^{-\lambda|x|}\, dx.$$

We split the integral into two parts:

$$M_X(s) = \frac{\lambda}{2}\left(\int_{-\infty}^{0} e^{sx}e^{\lambda x}\,dx + \int_{0}^{\infty} e^{sx}e^{-\lambda x}\,dx\right).$$

$$= \frac{\lambda}{2}\left(\int_{-\infty}^{0} e^{(s+\lambda)x}\,dx + \int_{0}^{\infty} e^{(s-\lambda)x}\,dx\right).$$

Case 1: For the first integral ($x < 0$), we require $s + \lambda > 0$ for convergence:

$$\int_{-\infty}^{0} e^{(s+\lambda)x}\,dx = \frac{1}{s+\lambda}.$$

Case 2: For the second integral ($x > 0$), we require $s < \lambda$ for convergence:

$$\int_{0}^{\infty} e^{(s-\lambda)x}\,dx = \frac{1}{\lambda - s}.$$

Thus, the MGF is

$$M_X(s) = \frac{\lambda}{2}\left(\frac{1}{s+\lambda} + \frac{1}{\lambda - s}\right),$$

for $-\lambda < s < \lambda$

Simplify:

$$M_X(s) = \frac{\lambda}{2} \cdot \frac{(\lambda - s) + (\lambda + s)}{(\lambda + s)(\lambda - s)} = \frac{\lambda^2}{\lambda^2 - s^2}.$$

$$\boxed{M_X(s) = \frac{\lambda^2}{\lambda^2 - s^2}}, \qquad |s| < \lambda.$$

Q4: Let $M_X(s)$ be finite for $s \in [c, c]$ where $c > 0$. Prove:

(a)

$$\lim_{n\to\infty}\left[M_X(\frac{s}{n})\right]^n = e^{sE[X]}$$

(b) Now assume $E[X] = 0$ and $Var[X] = 1$, then

$$\lim_{n\to\infty}\left[M_X\left(\frac{s}{\sqrt{n}}\right)\right]^n = e^{\frac{s^2}{2}}.$$

(c) We know that for $X \sim N(0, 1)$, we have $M_X(s) = e^{\frac{s^2}{2}}$. What can you say about the expression you derived above?

A:

(a) Let L be the limit.

$$L = \lim_{n\to\infty}\left[M_X\left(\frac{s}{n}\right)\right]^n$$

As $n \to \infty$, $\frac{s}{n} \to 0$. Since $M_X(t)$ is continuous at $t = 0$ and $M_X(0) = E[e^0] = 1$, the limit is of the indeterminate form 1^∞. To resolve this, we first take the natural logarithm of L.

$$\ln(L) = \lim_{n\to\infty} n \ln\left[M_X\left(\frac{s}{n}\right)\right]$$

This is of the form $\infty \cdot \ln(1) = \infty \cdot 0$. We rearrange the expression to get the indeterminate form $\frac{0}{0}$ so we can apply L'Hôpital's Rule.

$$\ln(L) = \lim_{n\to\infty} \frac{\ln\left[M_X\left(\frac{s}{n}\right)\right]}{\frac{1}{n}}$$

Now, we differentiate the numerator and the denominator with respect to n.

- Numerator derivative: $\frac{d}{dn}\left(\ln\left[M_X\left(\frac{s}{n}\right)\right]\right) = \frac{1}{M_X\left(\frac{s}{n}\right)} \cdot M_X'\left(\frac{s}{n}\right) \cdot \left(-\frac{s}{n^2}\right)$
- Denominator derivative: $\frac{d}{dn}\left(\frac{1}{n}\right) = -\frac{1}{n^2}$

Applying L'Hôpital's Rule:

$$\begin{aligned}\ln(L) &= \lim_{n\to\infty} \frac{\frac{1}{M_X(s/n)} \cdot M_X'(s/n) \cdot (-s/n^2)}{-1/n^2} \\ &= \lim_{n\to\infty} \frac{s \cdot M_X'(s/n)}{M_X(s/n)}\end{aligned}$$

Now we can evaluate the limit as $n \to \infty$:

$$\ln(L) = \frac{s \cdot M_X'(0)}{M_X(0)} = \frac{s \cdot E[X]}{1} = sE[X]$$

Since $\ln(L) = sE[X]$, we have $L = e^{sE[X]}$.

$$\therefore \lim_{n\to\infty} \left[M_X\left(\frac{s}{n}\right)\right]^n = e^{sE[X]}$$

(b) Let L be the limit.

$$L = \lim_{n\to\infty} \left[M_X\left(\frac{s}{\sqrt{n}}\right)\right]^n$$

This is again the indeterminate form 1^∞. Taking the logarithm gives the form $\infty \cdot 0$.

$$\ln(L) = \lim_{n\to\infty} n \ln\left[M_X\left(\frac{s}{\sqrt{n}}\right)\right] = \lim_{n\to\infty} \frac{\ln\left[M_X\left(\frac{s}{\sqrt{n}}\right)\right]}{\frac{1}{n}}$$

This is the indeterminate form $\frac{0}{0}$. Let's change variables to simplify differentiation. Let $t = \frac{1}{\sqrt{n}}$, so $n = \frac{1}{t^2}$. As $n \to \infty$, $t \to 0^+$.

$$\ln(L) = \lim_{t\to 0^+} \frac{\ln[M_X(st)]}{t^2}$$

This is still in the form $\frac{0}{0}$. We apply L'Hôpital's Rule with respect to t.

$$\ln(L) = \lim_{t \to 0^+} \frac{\frac{d}{dt}\left(\ln[M_X(st)]\right)}{\frac{d}{dt}(t^2)} = \lim_{t \to 0^+} \frac{\frac{1}{M_X(st)} \cdot M_X'(st) \cdot s}{2t} = \lim_{t \to 0^+} \frac{sM_X'(st)}{2tM_X(st)}$$

As $t \to 0^+$, the limit approaches $\frac{sM_X'(0)}{0 \cdot M_X(0)} = \frac{s \cdot 0}{0} = \frac{0}{0}$, since we are given $E[X] = M_X'(0) = 0$. This means we must apply L'Hôpital's Rule a second time.

$$\begin{aligned}
\ln(L) &= \lim_{t \to 0^+} \frac{\frac{d}{dt}(sM_X'(st))}{\frac{d}{dt}(2tM_X(st))} \\
&= \lim_{t \to 0^+} \frac{s \cdot (M_X''(st) \cdot s)}{2M_X(st) + 2t(M_X'(st) \cdot s)} \quad \text{(using the product rule in the denominator)} \\
&= \lim_{t \to 0^+} \frac{s^2 M_X''(st)}{2M_X(st) + 2stM_X'(st)}
\end{aligned}$$

Now, we can evaluate the limit by substituting $t = 0$:

$$\ln(L) = \frac{s^2 M_X''(0)}{2M_X(0) + 0}$$

We are given $E[X] = 0$ and $\text{Var}[X] = 1$. The required moments are:

- $M_X(0) = 1$
- $M_X''(0) = E[X^2] = \text{Var}[X] + (E[X])^2 = 1 + 0^2 = 1$

Substituting these values:

$$\ln(L) = \frac{s^2(1)}{2(1)} = \frac{s^2}{2}$$

Since $\ln(L) = \frac{s^2}{2}$, we have $L = e^{\frac{s^2}{2}}$.

$$\therefore \lim_{n \to \infty} \left[M_X\left(\frac{s}{\sqrt{n}}\right)\right]^n = e^{\frac{s^2}{2}}$$

(c) The result from part (b) is a proof of the **Central Limit Theorem (CLT)** using moment generating functions.
Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d.) random variables with the same distribution as X, where $E[X] = 0$ and $\text{Var}[X] = 1$. The standardized sum is defined as $Z_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}$.
Let's find the MGF of this standardized sum, Z_n:

$$\begin{aligned}
M_{Z_n}(s) &= E\left[e^{s \cdot Z_n}\right] = E\left[e^{\frac{s}{\sqrt{n}}(X_1 + \cdots + X_n)}\right] \\
&= E\left[e^{\frac{s}{\sqrt{n}}X_1} \cdot e^{\frac{s}{\sqrt{n}}X_2} \cdots e^{\frac{s}{\sqrt{n}}X_n}\right] \\
&= \left(E\left[e^{\frac{s}{\sqrt{n}}X}\right]\right)^n \quad \text{(due to the i.i.d. assumption)} \\
&= \left[M_X\left(\frac{s}{\sqrt{n}}\right)\right]^n
\end{aligned}$$

The calculation in part (b) shows that the limit of this MGF is:

$$\lim_{n\to\infty} M_{Z_n}(s) = e^{\frac{s^2}{2}}$$

We recognize $e^{\frac{s^2}{2}}$ as the MGF of a standard normal distribution, $N(0,1)$.

Q5: Let the CDF of a continuous random variable X be

$$F(x) = \frac{1}{2}\left(1 + \frac{x}{\sqrt{1+x^2}}\right), \qquad -\infty < x < \infty.$$

(a) Find the inverse CDF $F^{-1}(u)$ for $u \in (0,1)$.

(b) Using the inverse transform method, compute the sample x when $U = 0.75$.

A:
(a) Start from the definition $u = F(x)$. For $u \in (0,1)$,

$$u = \frac{1}{2}\left(1 + \frac{x}{\sqrt{1+x^2}}\right).$$

Rearrange:

$$2u - 1 = \frac{x}{\sqrt{1+x^2}}.$$

Set $y = 2u - 1$. Note $y \in (-1,1)$. Then

$$y = \frac{x}{\sqrt{1+x^2}} \quad \Longrightarrow \quad y^2 = \frac{x^2}{1+x^2}.$$

Solve for x^2:

$$y^2(1+x^2) = x^2 \quad \Longrightarrow \quad y^2 = x^2(1-y^2) \quad \Longrightarrow \quad x^2 = \frac{y^2}{1-y^2}.$$

Taking the (signed) square root and using the sign of y (which matches the sign of x from the original relation), we get

$$x = \frac{y}{\sqrt{1-y^2}}.$$

Note that x and y have the same sign. Substitute back $y = 2u - 1$:

$$\boxed{F^{-1}(u) = \frac{2u-1}{\sqrt{1-(2u-1)^2}}, \qquad 0 < u < 1.}$$

an alternate (equivalent) form is

$$\boxed{F^{-1}(u) = \frac{2u-1}{2\sqrt{u(1-u)}}, \qquad 0 < u < 1.}$$

(b) Now plug in $U = 0.75$. Compute $2u - 1$ and $u(1-u)$:

$$2u - 1 = 2(0.75) - 1 = 0.5, \qquad u(1-u) = 0.75 \cdot 0.25 = 0.1875 = \frac{3}{16}.$$

Thus

$$F^{-1}(0.75) = \frac{0.5}{2\sqrt{0.1875}} = \frac{0.5}{2 \cdot \frac{\sqrt{3}}{4}} = \frac{0.5}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} \approx 0.577350269.$$

$$\boxed{x \approx 0.577350269 \quad \text{when } U = 0.75.}$$

Q6: Let X and Y be the Cartesian coordinates of a randomly chosen point (according to a uniform PDF) in the triangle with vertices at $(0, 1)$, $(0, -1)$, and $(1, 0)$. Find the CDF and the PDF of $Z = |X - Y|$.

A:

Let's first find the distribution of the difference, $W = X - Y$. We will then find the distribution of $Z = |W|$. The range of values for W in the triangle is from -1 to 1.

The CDF of W is defined as $F_W(w) = \mathbb{P}(W \leq w) = \mathbb{P}(X - Y \leq w)$. To find this probability, we must find the area of the region within the triangle where the inequality $x - y \leq w$ holds. Since the total area of the triangle is 1, this area is equal to the probability.

It's easier to calculate the area of the complementary region, where $x - y > w$, and subtract it from the total area of 1.

$$F_W(w) = 1 - \mathbb{P}(X - Y > w)$$

The region where $x - y > w$ within the triangle is a quadrilateral with vertices at $(1, 0)$, $\left(\frac{1+w}{2}, \frac{1-w}{2}\right)$, $(0, -w)$, and $(0, -1)$. The area of this quadrilateral is:

$$\mathbb{P}(X - Y > w) = \text{Area} = \frac{3 - 2w - w^2}{4}$$

Now, we can find the CDF of W:

$$\begin{aligned} F_W(w) &= 1 - \frac{3 - 2w - w^2}{4} \\ &= \frac{4 - (3 - 2w - w^2)}{4} \\ &= \frac{1 + 2w + w^2}{4} = \frac{(1+w)^2}{4} \end{aligned}$$

This CDF is valid for $w \in [-1, 1]$.

To find the PDF of W, we differentiate its CDF with respect to w:

$$f_W(w) = \frac{d}{dw} F_W(w) = \frac{d}{dw}\left(\frac{(1+w)^2}{4}\right) = \frac{2(1+w)}{4} = \frac{1+w}{2}$$

So, $f_W(w) = \frac{1+w}{2}$ for $w \in [-1, 1]$.
Now we find the distribution of $Z = |W|$. The range for Z is $[0, 1]$. The CDF of Z is:

$$F_Z(z) = \mathbb{P}(Z \le z) = \mathbb{P}(|W| \le z) = \mathbb{P}(-z \le W \le z)$$

We can express this in terms of the CDF of W:

$$F_Z(z) = F_W(z) - F_W(-z)$$

Substituting our expression for $F_W(w)$:

$$\begin{aligned} F_Z(z) &= \frac{(1+z)^2}{4} - \frac{(1-z)^2}{4} \\ &= \frac{(1+2z+z^2) - (1-2z+z^2)}{4} \\ &= \frac{4z}{4} = z \end{aligned}$$

So, the CDF of Z for $z \in [0, 1]$ is $F_Z(z) = z$.
Finally, we differentiate the CDF to get the PDF of Z:

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz}(z) = 1$$

The final CDF and PDF for $Z = |X - Y|$ are:

- **CDF:** $F_Z(z) = \begin{cases} 0 & z < 0 \\ z & 0 \le z \le 1 \\ 1 & z > 1 \end{cases}$
- **PDF:** $f_Z(z) = \begin{cases} 1 & 0 \le z \le 1 \\ 0 & \text{otherwise} \end{cases}$

Q7: We want to estimate $\theta = \mathbb{E}[X^2]$ where X is an exponential random variable with rate $\lambda = 1$ ($X \sim \text{Exp}(1)$). Using importance sampling, formulate an estimator for θ by drawing N samples, $Y_1, \ldots, Y_N$, from a uniform distribution $Y \sim U[0, 5]$.

A:
The goal is to estimate $\theta = \mathbb{E}_{X\sim f}[X^2]$ using samples from $Y \sim g$, where $f(x)$ is the exponential PDF and $g(y)$ is the uniform PDF.
The PDFs are:

$$f(x) = e^{-x} \quad \text{for } x \ge 0 \quad \text{and} \quad g(y) = \begin{cases} \frac{1}{5} & \text{if } y \in [0, 5] \\ 0 & \text{otherwise} \end{cases}$$

The importance sampling estimator is based on the principle $\mathbb{E}_f[h(X)] = \mathbb{E}_g\left[h(Y)\frac{f(Y)}{g(Y)}\right]$. Here, $h(Y) = Y^2$.
The estimator $\hat{\theta}_N$ is the sample mean:

$$\hat{\theta}_N = \frac{1}{N} \sum_{i=1}^{N} Y_i^2 \frac{f(Y_i)}{g(Y_i)}$$

Substituting the PDFs for any $Y_i \in [0,5]$:

$$\frac{f(Y_i)}{g(Y_i)} = \frac{e^{-Y_i}}{1/5} = 5e^{-Y_i}$$

So, the final estimator is:

$$\hat{\theta}_N = \frac{1}{N}\sum_{i=1}^{N} 5Y_i^2 e^{-Y_i}$$

where each Y_i is a sample drawn from $U[0,5]$.

Q8: Let X and Y be two independent random variables with respective moment generating functions

$$M_X(t) = \frac{1}{1-3t}, \qquad M_Y(t) = \frac{1}{(1-3t)^2}, \qquad t < \tfrac{1}{3}.$$

Find $\mathbb{E}\big[(X+Y)^2\big]$.

A:
Let $W = X + Y$. By independence, the MGF of W is the product of the individual MGFs:

$$M_W(t) = M_X(t)\,M_Y(t) = \frac{1}{(1-3t)^3}, \qquad t < \tfrac{1}{3}.$$

We use the fact that $\mathbb{E}[W^2] = M_W''(0)$. Differentiate M_W:

$$M_W'(t) = \frac{d}{dt}\big((1-3t)^{-3}\big) = -3(1-3t)^{-4}\cdot(-3) = 9(1-3t)^{-4},$$

$$M_W''(t) = \frac{d}{dt}\big(9(1-3t)^{-4}\big) = 9\cdot(-4)(1-3t)^{-5}\cdot(-3) = 108(1-3t)^{-5}.$$

Evaluating at $t = 0$ yields

$$\mathbb{E}\big[(X+Y)^2\big] = M_W''(0) = 108.$$

$$\boxed{\mathbb{E}\big[(X+Y)^2\big] = 108}$$

Q9: You want to generate samples from a random variable with the PDF given by:

$$p(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Use the Accept-Reject method with a proposal distribution $q(x) = U(0,1)$.

(a) Find the smallest constant c such that $p(x) \le c \cdot q(x)$.

(b) Outline the algorithm to generate one sample.

A:

(a) The proposal distribution is $q(x) = 1$ for $x \in [0, 1]$. We need to find the smallest c such that $p(x) \leq c \cdot q(x)$, which simplifies to $2x \leq c$ for $x \in [0, 1]$. The maximum value of $p(x) = 2x$ on the interval $[0, 1]$ occurs at $x = 1$.

$$c = \max_{x \in [0,1]} p(x) = 2(1) = 2$$

So, the smallest possible constant is $\mathbf{c = 2}$.

(b) The algorithm is as follows:

i. Generate a candidate sample y from the proposal distribution $U(0, 1)$.

ii. Generate a random number u from a $U(0, 1)$ distribution.

iii. Check the acceptance condition: $u \leq \frac{p(y)}{c \cdot q(y)}$.

$$u \leq \frac{2y}{2 \cdot 1} \implies u \leq y$$

iv. If $u \leq y$, accept y as the sample. Otherwise, reject it and return to step 1.