Probability and Statistics MA6.101

Homework 8

Topics Covered: Convergence, Central Limit Theorem

Q1: Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent random variables such that

$$X_n = \begin{cases} 0 & \text{with probability } 1 - \dfrac{1}{n} \\ 1 & \text{with probability } \dfrac{1}{n} \end{cases}$$

(a) Does this sequence converge to 0 in mean square? That is, does $X_n \xrightarrow{m.s.} 0$?

(b) Does this sequence converge to 0 almost surely? That is, does $X_n \xrightarrow{a.s.} 0$?

(c) Does this sequence converge in distribution to the degenerate random variable X that equals 0 with probability 1? That is, $X_n \xrightarrow{d} 0$?

Q2: A company measures how long each customer service call lasts. Historically, individual call durations have mean $\mu = 10$ minutes and standard deviation $\sigma = 10$ minutes, and call durations are roughly exponential.

(a) For a random sample of 50 calls, what is the probability that the sample average call duration exceeds 12 minutes?

(b) What is the minimum sample size n needed so that the probability the sample average exceeds 12 minutes is at most 0.05?

(c) For a sample of 80 calls, what is the probability the sample average lies between 9 minutes and 11 minutes?

Use:

$$\Phi^{-1}(0.95) = 1.6449, \qquad \Phi(\sqrt{2}) = 0.9214, \qquad \Phi(0.8944) = 0.8147.$$

Q3: A factory produces metal rods whose individual lengths historically have mean $\mu = 100$ mm and standard deviation $\sigma = 5$ mm, and lengths are approximately normal.

(a) For a single randomly chosen rod, what is the probability its length exceeds 110 mm?

(b) If a sample of 36 rods is taken, what is the probability that the sample mean length exceeds 103 mm?

Use:

$$\Phi(2) = 0.9772, \quad \Phi(3.6) = 0.9998$$

Q4: Let $X_n \sim \text{Uniform}\left(0, \frac{1}{n}\right)$ for $n = 1, 2, 3, \ldots$ Show that $X_n \xrightarrow{p} 0$ (i.e., X_n converges in probability to 0).

Q5: Let X be any random variable. For each positive integer n, let $Y_n \sim N(0, 1/n)$ independent of X and set $X_n = X + Y_n$. Show that X_n converges to X in mean-square:

$$X_n \xrightarrow{m.s.} X.$$

Q6: Let $X_n = \frac{1}{\sqrt{n}} Z$, where $\mathbb{E}(Z^2) < \infty$. Show that $X_n \to 0$ in distribution.

Q7: If $X_n \xrightarrow{L^r} X$ for some $r \geq 1$, then show that $X_n \xrightarrow{p} X$.

Q8: Before starting to play roulette in a casino you want to look for biases you can exploit. You therefore watch 100 independent rounds; each round yields a number in $\{1, \ldots, 36\}$, and you count the number of rounds for which the outcome is *odd*. If the count exceeds 55 you decide that the roulette is not fair.

Assuming the roulette is fair (so that odd and even are equally likely), find an approximation for the probability that you will make the wrong decision (i.e. the probability of deciding "not fair" when in fact the wheel is fair).

Remark / clarification about roulette: A standard (idealized) roulette without the green zero pocket would produce numbers 1 to 36 with equal probability; half of these are odd and half even. In that idealized model a *fair* wheel gives $\Pr(\text{odd}) = \frac{1}{2}$. The test described checks whether the observed count of odd outcomes in 100 spins is unusually large (strictly greater than 55).

Q9: We have a bag with n blue balls and n red balls, where $n \geq 10$. We randomly draw 10 balls without replacement. Let X_n be the number of blue balls drawn. Prove that:

$$X_n \xrightarrow{d} \text{Binomial}(10, 1/2)$$

Q10: Let $Y_1, Y_2, \ldots$ be independent random variables, where $Y_n \sim \text{Bernoulli}\left(\frac{n}{n+1}\right)$ for $n = 1, 2, 3, \ldots$. We define the sequence $\{X_n, n = 2, 3, 4, \ldots\}$ as

$$X_{n+1} = Y_1 Y_2 Y_3 \cdots Y_n, \quad \text{for } n = 1, 2, 3, \ldots$$

Show that $X_n \xrightarrow{\text{a.s.}} 0$.

Q11: Let $Y_1, Y_2, Y_3, \ldots$ be a sequence of i.i.d. random variables with mean $E[Y_i] = \mu$ and finite variance $Var(Y_i) = \sigma^2$. Define the sequence $\{X_n, n = 2, 3, \ldots\}$ as

$$X_n = \frac{Y_1 Y_2 + Y_2 Y_3 + \cdots + Y_{n-1} Y_n + Y_n Y_1}{n}, \quad \text{for } n = 2, 3, \ldots$$

Show that $X_n \xrightarrow{p} \mu^2$.

Q12: [**Bonus**] Show that in the limit of large N, the binomial distribution of n out of N objects becomes a Gaussian distribution without using MGFs. [Hint: Write $P(n) = \binom{N}{n} p^n q^{N-n}$. Take logarithms of both sides and use Stirling's approximation $\ln(N!) \approx N \ln(N) - N$. Then apply Taylor's series around the mean of the distribution].