Lecture 3:
Matrix multiplication

Suppose B is an $n \times p$ matrix over a field F with rows $\beta_1, \dots, \beta_n$ and that from B we construct a matrix C with rows $\gamma_1, \dots, \gamma_m$ by forming certain linear combinations:

$$\gamma_i = A_{i1}\beta_1 + A_{i2}\beta_2 + \dots + A_{in}\beta_n \quad \text{--- (1.4)}$$

The rows of C are determined by the ~~$m \times n$~~ mn scalars A_{ij} which are themselves the entries of an $m \times n$ matrix A.

from (1.4),

$$(C_{i1} \; \dots \; C_{ip}) = \sum_{r=1}^{n} (A_{ir}B_{r1} \dots A_{ir}B_{rp}),$$

entries of C: $C_{ij} = \sum_{r=1}^{n} A_{ir}B_{rj}$.

Def^n^: Let A be an $m \times n$ matrix over the field F and let B be an $n \times p$ matrix over F. The product AB is the $m \times p$ matrix C whose i,j entry is

$$C_{ij} = \sum_{r=1}^{n} A_{ir}B_{rj}.$$

Example: (a) Consider $\begin{bmatrix} 5 & -1 & 2 \\ 0 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 5 & -1 & 2 \\ 15 & 4 & 8 \end{bmatrix}$

Here, $\gamma_1 = (5 \;\; -1 \;\; 2) = 1\cdot(5 \;\; -1 \;\; 2) + 0\cdot(15 \;\; 4 \;\; 8)$

$\gamma_2 = (0 \;\; 7 \;\; 2) = -3\cdot(5 \;\; -1 \;\; 2) + 1\cdot(15 \;\; 4 \;\; 8)$.

(b) $\begin{bmatrix} 0 & 6 & 1 \\ 9 & 12 & -8 \\ 12 & 62 & -3 \\ 3 & 8 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 3 \\ 5 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 & 1 \\ 3 & 8 & -2 \end{bmatrix}$

$\gamma_3 = 5\cdot(0 \;\; 6 \;\; 1) + 4\cdot(3 \;\; 8 \;\; -2)$

B is an $n \times p$ matrix, $B = [B_1, \ldots, B_p]$,

$$B_j = \begin{bmatrix} B_{1j} \\ \vdots \\ B_{nj} \end{bmatrix}, \quad 1 \le j \le p.$$

B_j is $1 \times n$ matrix. column matrix.

Check that $AB = [AB_1, \ldots, AB_p]$.

Thm: If A, B, C are matrices over the field F such that the products BC and $A(BC)$ are defined, then so are the products AB, $(AB)C$, and

$$(AB)C = A(BC).$$

Proof: ———

Remark: For a square matrix A, A^n is well-defined.

$$A^p A^q A^r = A^s A^t A^u \quad \text{for all } p+q+r = s+t+u = n.$$

$A(BC) = (AB)C \longrightarrow$ linear combinations of linear combinations of the rows of C are again linear combinations of the rows of C.

If $B \xrightarrow[\text{row operations}]{\text{elementary}} C$, then each row of C is a linear combination of the rows of B, and so $\exists$ a matrix A s.t. $AB = C$.

(There can be many such A's in general.)

Def^n: An $m \times m$ matrix is said to be an elementary matrix if it can be obtained from the $m \times m$ identity matrix $\mathbb{1}_{m \times n}$ by means of a single elementary row operation.

Example: 2×2 elementary matrices:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix},$$

$$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} \text{ for } c \neq 0, \quad \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \text{ for } c \neq 0.$$

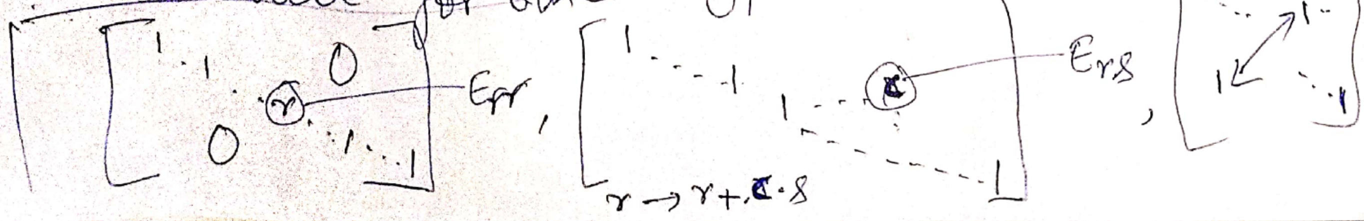
Thm: Let e be an elementary row operation and ~~let~~ $\mathcal{E}$ be the $m \times n$ elementary matrix $\mathcal{E} = e(\mathbb{1})$. Then, for every $m \times n$ matrix A,

$$e(A) = \mathcal{E} A.$$

Proof: Type (II) $\quad \mathcal{E}_{ik} = \begin{cases} \delta_{ik}, & i \neq r \\ \delta_{rk} + c\,\delta_{sk}, & i = r \end{cases}$ (To replace row r with row $r + c \cdot$ row s)

$$(\mathcal{E}A)_{ij} = \sum_{k=1}^{m} \mathcal{E}_{ik} A_{kj} = \begin{cases} A_{ik}, & i \neq r \\ A_{rj} + c A_{sj}, & i = r. \end{cases}$$

~~Check for other types:~~



Corollary: Let A and B be $m \times n$ matrices over the field F. Then B is row-equivalent to A if and only if $B = PA$, where P is a product of $m \times n$ elementary operation.

1.4 Invertible matrices.

Def^n: Let A be an $n \times n$ matrix over the field F. An $n \times n$ matrix B such that $BA = \mathbb{1}$ is called a left inverse of A; an $n \times n$ matrix B such that $AB = \mathbb{1}$ is called a right inverse of A. If $AB = BA = \mathbb{1}$ then B is called a two-sided inverse of A and A is said to be invertible.

Lemma: If A has a left inverse B and a right inverse C, then $B = C$.

Proof: $B = B\mathbb{1} = BAC = \mathbb{1}C = C$.

Thm: Let A and B be $n \times n$ matrices over the field F.

(I) If A is invertible, so is A^{-1} and $(A^{-1})^{-1} = A$.

(II) If both A and B are invertible, so is AB and $(AB)^{-1} = B^{-1}A^{-1}$.

Corollary: A product of invertible matrix is invertible.

Theorem: An elementary matrix is invertible.

Proof: ______

Thm. If A is an $n \times n$ matrix, the following are equivalent.

(I) A is invertible.

(II) A is row-equivalent to $\mathbb{1}_{n \times n}$.

(III) A is product of elementary operations.

Proof: ______

Thm: For an $n \times n$ matrix A, the following are equivalent.

(I) A is invertible.

(II) The homogenous system $AX=0$ has only the trivial solⁿ.

(III) The system of eqⁿs $AX=Y$ has a solⁿ X for each $n \times 1$ matrix Y.

Proof: ______

$AX = Y$

$X^T A^T = Y^T$

Column-equivalent

Column-reduced echelon matrix

~~Column~~ Elementary column operations: $A\underline{E}$

Corollary: A square matrix with either a left inverse or right inverse is invertible.

Proof: $A_{n\times n}$. Suppose, left inverse of A exists, $BA = \mathbb{1}$. Then, $AX = 0$ has only the trivial solⁿ, because $X = \mathbb{1}X = B(AX)$. $\therefore$ A is invertible.

If A has a right inverse, $AC = \mathbb{1}$. Then C has a left inverse & is therefore invertible. It then follows $A = C^{-1}$, so A is invertible w/ inverse C.

$$\begin{tikzcd} A \arrow[d, "\text{sequence of } e."'] & \mathbb{1} \arrow[d] \\ \mathbb{1} & A^{-1} \end{tikzcd}$$

$AX = Y$

$RX = Z$,

if $R = PA$

then $Z = PY$

$\downarrow$

invertible matrix.