Probability and Statistics: MA6.101

Homework 9

Topics Covered: Random Vectors

Q1: You are given the random vector $Y' = [Y_1, Y_2, Y_3, Y_4]$ with mean vector

$$\mu_Y = [5, -1, 4, -3]$$

and variance-covariance matrix

$$\Sigma_Y = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

(a) Find $E(BY)$, the mean of BY.
(b) Find $Cov(BY)$, the variances and covariances of BY.
(c) Which pairs of linear combinations have zero covariances?

Q2: Let X and Y are said to be bivariate normal if $aX + bY$ is normal for all a and b. If X and Y are bivariate normal with 0 mean, variance of 1, and ρ correlation, then their joint pdf is:

$$f(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(x^2 - 2\rho xy + y^2\right)\right)$$

Let $U = X + Y$ and $V = X - Y$. Find the joint pdf of U and V.

Q3: Let

$$Y = G(X) = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1 + X_2 \\ X_1 - X_2 \end{bmatrix}$$

where $X = (X_1, X_2)^T$ is a continuous random vector with joint pdf $f_X(x_1, x_2)$.

(a) Find the inverse transformation $H(Y)$ such that $X = H(Y)$.
(b) Compute the Jacobian determinant $|J|$ for the inverse transformation.
(c) Write the expression for the joint pdf $f_Y(y_1, y_2)$ in terms of $f_X(x_1, x_2)$.

Q4: Let

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

be a normal random vector with the following mean and covariance matrices

$$\mathbf{m} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}.$$

Let also

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

and

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \mathbf{AX} + \mathbf{b}.$$

(a) Find $P(X_2 > 0)$.

(b) Find expected value vector of $\mathbf{Y}$, $\mathbf{m}_Y = E[\mathbf{Y}]$.

(c) Find the covariance matrix of $\mathbf{Y}$, $\mathbf{C}_Y$.

(d) Find $P(Y_2 \leq 2)$.

Q5: The random vector $\mathbf{X} = [X_1, X_2, X_3]^T$ follows a multivariate normal distribution with mean $\mu = [1, 2, 3]^T$ and covariance matrix:

$$\Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

Find the conditional distribution of $[X_1, X_3]^T$ given that $X_2 = x_2$.

Q6: Let the random vector $\mathbf{Z} = [Z_1, Z_2]^T$ be normally distributed with mean $\mathbf{m} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and covariance $\mathbf{C} = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$. A new random vector is defined by the transformation $\mathbf{W} = A\mathbf{Z} + \mathbf{b}$, where $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$.

(a) Find the mean vector and covariance matrix of $\mathbf{W}$.

(b) Find the probability $P(W_2 > 1)$.

Q7: Let the random vector

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix},$$

have

$$E[\mathbf{X}] = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad Cov(\mathbf{X}) = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}.$$

Define two new random variables as

$$Y_1 = 2X_1 - X_2 + 3, \qquad Y_2 = X_1 + 4X_2.$$

(a) Compute $E[Y_1]$ and $E[Y_2]$.

(b) Compute $Var(Y_1)$, $Var(Y_2)$, and $Cov(Y_1, Y_2)$.

(c) Find the covariance matrix of the vector $\mathbf{Y} = [Y_1\ Y_2]^T$.

Q8: A random vector $\mathbf{X} = [X_1, X_2]^T$ has mean vector $\mu_X = [1, 2]^T$ and covariance matrix $\Sigma_X = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$. Let $Y_1 = X_1 + 2X_2$ and $Y_2 = 3X_1 - X_2$. Find the mean vector and covariance matrix of $\mathbf{Y} = [Y_1, Y_2]^T$.

Q9: Find the expectation of the quadratic form $(X_1 + X_2)^2$ where the random vector $\mathbf{X} = [X_1, X_2]^T$ has mean vector $\mu = [1, -1]^T$ and covariance matrix $\Sigma = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$.

Q10: Derive the Moment-Generating Function (MGF) for a k-dimensional multivariate normal random vector $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. That is, prove that:

$$M_{\mathbf{X}}(\mathbf{s}) = E[e^{\mathbf{s}^T\mathbf{X}}] = \exp\left(\mathbf{s}^T\boldsymbol{\mu} + \frac{1}{2}\mathbf{s}^T\boldsymbol{\Sigma}\mathbf{s}\right)$$

where $\mathbf{s}$ is a k-dimensional vector of real numbers.

Q11: Let $\mathbf{X}$ be a standard normal random vector in $\mathbb{R}^n$. Define a new random vector

$$\mathbf{Y} = A\mathbf{X} + b,$$

where A is a square, symmetric, and invertible matrix, and b is a constant shift vector.

(a) Derive an explicit expression for the probability density function of $\mathbf{Y}$ in terms of A, b, and y.

(b) Using your result, compute $f_{\mathbf{Y}}(y)$ numerically for

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad y = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$