Probability and Statistics

Homework 7 - Questions

Q1: Let $X_1, X_2, ...$ be independent random variables that are uniformly distributed over $[-1, 1]$. Show that the sequence $Y_1, Y_2, ...$ converges in probability to some limit, and identify the limit, for each of the following cases:

(a) $Y_n = X_n/n$

(b) $Y_n = (X_n)^n$

(c) $Y_n = X_1 \cdot X_2 \cdots X_n$

(d) $Y_n = \max\{X_1, ..., X_n\}$

Q2: Let $X_1, X_2, ...$ be independent, identically distributed random variables with (unknown but finite) mean μ and positive variance. For $i = 1, 2, ...$, let $Y_i = \frac{1}{3}X_i + \frac{2}{3}X_{i+1}$.

(a) Are the random variables Y_i independent?

(b) Are they identically distributed?

(c) Let $M_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$. Show that M_n converges to μ in probability.

Q3: Let $X_n \sim N(0, \frac{1}{n})$. Show that $X_n \longrightarrow 0$.

Q4: Let $Y_1, Y_2, \ldots$ be independent random variables, where Y_n is Bernoulli$(\frac{n}{n+1})$ for $n = 1, 2, 3, \ldots$. We define the sequence $\{X_n, n = 2, 3, 4, \ldots\}$ as $X_{n+1} = Y_1 Y_2 Y_3 \cdots Y_n$ for $n = 1, 2, 3, \ldots$. Show that $X_n \longrightarrow 0$ in probability.

Q5: Let $X_1, X_2, X_3, \ldots$ be continuous random variables with densities

$$f_{X_n}(x) = \frac{n}{2}e^{-n|x|}, \qquad x \in \mathbb{R},\ n = 1, 2, \ldots$$

(Thus X_n has the Laplace distribution with location 0 and rate n.) Show that $X_n \xrightarrow{p} 0$.

Q6: A sequence X_n of random variables is said to converge to a number c in the mean square if

$$\lim_{n\to\infty} \mathbb{E}\big[(X_n - c)^2\big] = 0.$$

(a) Show that convergence in the mean square implies convergence in probability.

(b) Give an example that shows that convergence in probability does not imply convergence in the mean square.

Q7: Let X be a random variable, and define $X_n = X + Y_n$, where

$$\mathbb{E}[Y_n] = \frac{1}{n}, \qquad \mathrm{Var}(Y_n) = \frac{\sigma^2}{n},$$

and $\sigma > 0$ is a constant. Show that $X_n \xrightarrow{p} X$.

Q8: What is the expected number of iterations to generate k random numbers from a distribution using the rejection method?