

# Probability and Statistics: MA6.101

## Homework 3

Topics Covered: Discrete Random Variables

Q1: Let  $X$  be a discrete random variable that can take the values  $\{-2, -1, 1, 2\}$ . The probability mass function (PMF) of  $X$  is given by  $p_X(X = x) = c(x + 4)$  for some constant  $c$ .

- (a) Find the value of the constant  $c$ .
- (b) Find the expected value of  $X$ ,  $E[X]$ .
- (c) Find the variance of  $X$ ,  $Var(X)$ .

**Answers:**

- (a)  $\frac{1}{16}$
- (b)  $\frac{5}{8}$
- (c)  $\frac{135}{64}$

Q2: A tech company manufactures microchips in batches of 20. In each batch, 5 chips come from a premium production line and are guaranteed to be flawless. The other 15 chips come from a standard production line, where each chip has a 10% chance of being defective, independent of the others. Let  $Z$  be the total number of flawless chips in a randomly selected batch.

- (a) Find the PMF of  $Z$ .
- (b) A batch is considered "high-quality" if it contains at least 18 flawless chips. What is the probability of this?

**Answers:**

- (a)  $p_Z(Z = k) = \binom{15}{k-5} (0.9)^{k-5} (0.1)^{20-k}$  for  $k \in \{5, 6, \dots, 20\}$ .
- (b) 0.816

Q3: You have come to attend the Probability and Statistics Lecture on Saturday. It is raining outside so you have kept your umbrellas outside H105. Suppose that the strength of the class is  $N$ . After the tutorial has ended, you have to find your umbrella, but you are running late for Kadamba Biryani, so you pick one umbrella randomly from the pile. Similarly everyone else picks an umbrella randomly independent from each other.

- (a) What is the probability that at least one of you receives his/her own umbrella?
- (b) Let  $X_N$  be the number of people who receive their own umbrella. What is

$$\lim_{n \rightarrow \infty} \mathbb{P}(X_N \geq 1)$$

- (c) Find the PMF of  $X_N$ .

**Answers:**

(a)  $1 - \frac{1}{2!} + \frac{1}{3!} - \dots + (-1)^{N-1} \frac{1}{N!}$ .

(b)  $1 - \frac{1}{e}$

(c)

$$p_{X_N}(k) = \begin{cases} \frac{1}{k!} \sum_{j=0}^{N-k} \frac{(-1)^j}{j!} & \text{for } k = 0, 1, \dots, N, \\ 0 & \text{otherwise.} \end{cases}$$

Q4: A bag contains 8 red balls and 7 blue balls (all identical within color). You draw 4 balls without replacement. Define the random variable

$$X = \begin{cases} \text{number of ways to arrange the drawn balls in a row such that} \\ \text{no two balls of the same color are adjacent.} \end{cases}$$

(a) Find the probability distribution of  $X$ , and  $E[X]$

(b) For general  $n$  red balls and  $m$  blue balls, where you draw  $k$  balls without replacement, provide a general formula for  $E[X]$

**Answers:**

(a)

$$X = \begin{cases} 2 & R = 2, \\ 0 & \text{otherwise} \end{cases}$$

where  $R$  is the number of red balls drawn from the bag.  $E[X] = \frac{56}{65}$

(b) Hint:  $E[X]$  depends on two cases: whether  $k$  is even or odd. For each case there is only one situation in which the valid arrangement is possible.

Q5: Let  $X \sim \text{Geometric}(\frac{1}{4})$ , and let  $Y = |X - 4|$ . Find the range and PMF of  $Y$ .

**Answers:**

(a) range:  $\{1, 2, \dots\}$

(b)

$$p_Y(y) = \begin{cases} \frac{27}{256} & y = 0, \\ \frac{1}{4} \left[ \left( \frac{3}{4} \right)^y + 3 \right] & y \in \{1, 2, 3\}, \\ \frac{1}{4} \left( \frac{3}{4} \right)^y & y \in \{4, 5, 6, \dots\}, \\ 0 & \text{otherwise.} \end{cases}$$

Q6: Refer to the median of a random variable definition in tutorial. Find the median of  $X$  if

(a) The PMF of  $X$  is given by

$$p_X(k) = \begin{cases} 0.4 & \text{for } k = 1, \\ 0.3 & \text{for } k = 2, \\ 0.3 & \text{for } k = 3, \\ 0 & \text{otherwise.} \end{cases}$$

(b)  $X \sim \text{Geometric}(p)$ , where  $p = \frac{1}{2}$ .

**Answers:**

- (a) 2
- (b)  $[1, 2]$

Q7: Let  $s > 0$  be a real parameter and let  $X$  be a non-negative integer valued random variable whose cumulative distribution function is given by

$$F(x) = \mathbb{P}(X \leq x) = \begin{cases} 0, & x < 0, \\ 1 - (\lfloor x \rfloor + 2)^{-s}, & x \geq 0. \end{cases}$$

Equivalently, for integer  $n \geq 0$ ,

$$F(n) = 1 - (n + 2)^{-s}.$$

- (a) Show that the PMF of  $X$  is

$$\mathbb{P}(X = n) = (n + 1)^{-s} - (n + 2)^{-s}, \quad n = 0, 1, 2, \dots$$

and verify that the probabilities sum to 1.

- (b) (i) Show that the tail probabilities satisfy

$$\mathbb{P}(X \geq k) = (k + 1)^{-s}, \quad k = 0, 1, 2, \dots$$

- (ii) Deduce the related strict-tail formula

$$\mathbb{P}(X > k) = (k + 2)^{-s}, \quad k = 0, 1, 2, \dots$$

- (c) Prove that for any nonnegative integer-valued random variable  $X$ ,

$$\mathbb{E}(X) = \sum_{k=0}^{\infty} \mathbb{P}(X > k),$$

i.e. the expectation equals the sum of its tail probabilities.

Q8: Let  $X$  be a random variable with mean  $\mathbb{E}[X] = \mu$ . Define the function  $f(\alpha)$  as

$$f(\alpha) = \mathbb{E}[(X - \alpha)^2].$$

Find the value of  $\alpha$  that minimizes  $f$ .

**Answer:**  $\mu$

Q9: You roll  $m$  independent fair  $n$ -sided dice.

- (a) What is the expected value of the minimum of the  $m$  rolls?
- (b) What will be the approximated value for large  $n$  for the expectation calculated in the question of tutorial?
- (c) What is the value for  $m = 3$  and  $n = 100$ .

**Answers:**

- (a)  $\frac{1}{n^m} \sum_{j=1}^n j^m$
- (b)  $\frac{n}{m+1}$
- (c) 25

Q10: A frog begins at position 100 on a number line. Every second, if the frog is at position  $x > 1$ , it jumps to a randomly chosen integer position between 1 and  $x - 1$  (inclusive). What is the expected number of seconds to reach to position 1?

**Answer:**

$$\mathbb{E}(100) = \sum_{j=1}^{99} \frac{1}{j} = H_{99} \approx 5.1776, \text{ where } H_{99} \text{ corresponds to the 99th harmonic number}$$

Q11: While roaming around in one of the gaming zones in your favorite mall, you come across an interesting game that might help you win some lunch money. It's a game of chance for a single player in which a fair coin is tossed at each stage. The pot starts at 1 rupee and is doubled every time a head appears. The first time a tail appears, the game ends and the player wins whatever is in the pot. Thus the player wins 1 rupee if a tail appears on the first toss, 2 rupees if a head appears on the first toss and a tail on the second, 4 rupees if a head appears on the first two tosses and a tail on the third, 8 rupees if a head appears on the first three tosses and a tail on the fourth, and so on. In short, the player wins  $2^{k-1}$  rupees if the coin is tossed  $k$  times until the first tail appears. What would be a fair price to pay the gaming zone for entering the game?

- (a) Let  $X$  be the amount of money (in rupees) that the player wins. Find  $\mathbb{E}[X]$ .
- (b) What is the probability that the player wins more than 65 rupees?
- (c) Now suppose that the casino only has a finite amount of money. Specifically, suppose that the maximum amount of the money that the casino will pay you is  $2^{30}$  rupees (around 1.07 billion rupees). That is, if you win more than  $2^{30}$  rupees, the casino is going to pay you only  $2^{30}$  rupees. Let  $Y$  be the money that the player wins in this case. Find  $\mathbb{E}[Y]$ .

**Answers:**

- (a)  $\infty$
- (b)  $\frac{1}{128}$
- (c) 16

Q12: Let  $X$  be a random variable with mean  $\mu$  and variance  $\sigma^2$ . For arbitrary constants  $a$  and  $b$ , show that  $\text{Var}(aX + b) = a^2\text{Var}(X)$ .

Q13: You are playing a game with an urn that contains 10 black balls and 1 white ball. The balls are drawn one by one from the urn without replacement. The game ends when the white ball is drawn.

- (a) For every black ball that is drawn before the game ends, you win \$5. What are your expected winnings?
- (b) Suppose the rules are changed. Now, if you draw  $k$  black balls before the game ends, your total payout is  $\$2^k$ . (Note: if you draw the white ball first,  $k = 0$  and your payout is  $2^0 = \$1$ ). What is the new expected payout?

**Answers:**

- (a) 25
- (b) \$186.09

Q14: In the coupon collector's problem from the tutorial, there are  $N$  different types of coupons. Each time you collect a coupon, it is equally likely to be any of the  $N$  types. Let  $X$  be the total number of coupons you need to collect to have at least one of each type. We can write  $X = \sum_{i=0}^{N-1} X_i$ , where  $X_i$  is the number of additional coupons needed to get a new type, given that you already have  $i$  distinct types. Each  $X_i$  is an independent geometric random variable,  $X_i \sim \text{Geometric}(p_i)$ , with success probability  $p_i = \frac{N-i}{N}$ . Find the variance of  $X$ .

**Answer:**  $N^2 \left( \sum_{k=1}^{N-1} \frac{1}{k^2} \right) - NH_{N-1}$

Q15: Rule a surface with parallel lines a distance  $d$  apart. What is the probability that a randomly dropped needle of length  $l \leq d$  crosses a line?

**Answer:**  $\frac{1}{\pi}$