Real Analysis: Assignment 1

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Full marks: 100. You need to submit a pdf copy of your answer within 20th October. Your answer copy may also be a photocopy of your handwritten answer script. But it must be in pdf format. NO SUBMISSION AFTER 20TH OCTOBER WILL BE ALLOWED.

1. Given a sequence $\{a_n\}$, such that $a_n - a_{n-2} \to 0$ as $n \to \infty$, show that the sequence $b_n = \frac{a_n - a_{n-1}}{n}$ converges to 0. (10)

2. Prove that
$$\lim_{n \to \infty} \sum_{k=1}^{n} \left(\sqrt{1 + \frac{k}{n^2}} - 1 \right) = \frac{1}{4}$$
(10)

3. Consider a sequence $\{a_n\}$, such that $a_n \geq 1$ $\forall n$ and $a_n + \frac{1}{a_n}$ converges. Then prove that the sequence $\{a_n\}$ is convergent. (10)

4. Prove that
$$\lim_{x \to \infty} \left(1 + \frac{1}{x} \right)^x = e$$
[use the fact $(1+n^{-1})^n \to e$ as $n \to \infty$] (10)

5. Let a be a real number. Prove that
$$\lim_{x \to 0} \frac{(1 + x)^a - 1}{x} = a$$
(10)

6. Prove that : $29/18 < \sum_{n=1}^{\infty} \frac{1}{n^2} < 31/18$. (15)

7. Evaluate :
$$\sum_{n=1}^{\infty} \frac{n}{n^4 + n^2 + 1}$$

(15)

8. Study the convergence of the following series: a) $\sum_{n=1}^{\infty} \frac{a^n}{(n!)^{1/n}}$, b) $\sum_{n=1}^{\infty} a^n \left(1 + 1/n\right)^n$
with $a > 0$. $\qquad$ (10+10)