Probability and Statistics

Tutorial 7 Solutions

Q1: Let $(\Omega, \mathcal{F}, P)$ be a probability space on which the random variable U is defined, where $U \sim \text{Uniform}[0,1]$. Define a sequence of random variables $(X_n)_{n\geq 1}$ by

$$X_n(\omega) = \begin{cases} n, & \text{if } U(\omega) \leq 1/n, \\ 0, & \text{if } U(\omega) > 1/n. \end{cases}$$

Investigate the convergence of the sequence X_n to 0 in the following modes.

(a) Does $X_n \xrightarrow{\text{d}} 0$?

(b) Does $X_n \xrightarrow{\text{p}} 0$

(c) Does $X_n \xrightarrow{\text{a.s.}} 0$. Can Borel-Cantelli Lemma be applied?

(d) Does $X_n \xrightarrow{\text{L1}} 0$ (Convergence in mean)

Solution:

(a) **Convergence in Distribution: $X_n \xrightarrow{d} 0$.**
Let $F_n(x) = P(X_n \leq x)$ be the cdf of X_n. The candidate limit distribution is the degenerate (point mass) distribution at 0 whose cdf F is

$$F(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

We must check $F_n(x) \to F(x)$ at all continuity points of F, i.e. for all $x \neq 0$.
- If $x < 0$, then $F_n(x) = 0$ for every n
- If $x \geq 0$, then for all sufficiently large n we have $n > x$, so $F_n(x) = P(X_n \leq x) = P(X_n = 0) = 1 - 1/n \xrightarrow{n\to\infty} 1 = F(x)$.
Thus $F_n(x) \to F(x)$ for every continuity point $x \neq 0$, and therefore $X_n \xrightarrow{d} 0$.

(b) **Convergence in probability: $X_n \xrightarrow{p} 0$.**
Fix any $\varepsilon > 0$. For any $n > \varepsilon$ we have

$$P\{|X_n - 0| > \varepsilon\} = P(X_n > \varepsilon) = P(X_n = n) = 1/n.$$

Hence for each fixed $\varepsilon > 0$,

$$P(|X_n - 0| > \varepsilon) = \frac{1}{n} \xrightarrow{n\to\infty} 0,$$

so $X_n \to 0$ in probability.

(c) **Almost sure convergence:** $X_n \xrightarrow{\text{a.s.}} 0$**.**
By definition, we must find all $\omega \in \Omega$ such that

$$\lim_{n\to\infty} X_n(\omega) = 0.$$

For any outcome ω such that $U(\omega) > 0$, by the Archimedean property of the natural numbers, there exists an integer n_0 such that

$$\frac{1}{n_0} < U(\omega).$$

Then for all $n > n_0$, we have $1/n < 1/n_0 < U(\omega)$, hence $U(\omega) > 1/n$, and so

$$X_n(\omega) = 0.$$

Therefore, for all ω such that $U(\omega) > 0$, we have $X_n(\omega) \to 0$.
Since $P(U > 0) = 1$ (because U is uniform on $[0, 1]$), the set of ω satisfying this property has probability one. Hence,

$$X_n \xrightarrow{\text{a.s.}} 0.$$

About Borel–Cantelli: note that

$$\sum_{n=1}^{\infty} P(|X_n| > \epsilon) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

so it can not be invoked.

(d) **Convergence in Mean:** $X_n \xrightarrow{\mathbb{L}1} 0$**.**

$$\mathbb{E}[X_n] = \mathbb{E}[X_n] = n \cdot P(X_n = n) + 0 \cdot P(X_n = 0) = n \cdot \frac{1}{n} = 1$$

for every n. Since $\mathbb{E}|X_n| = 1 \not\to 0$, the sequence does *not* converge to 0 in L^1.

Q2: Let $\{X_n\}$ be a sequence of iid random variables all having a uniform distribution on the interval $[0, 1]$. Define:

$$Y_n = n\left(1 - \max_{1\le i\le n} X_i\right).$$

Show that $Y_n \xrightarrow{d} Y$ where $Y \sim \text{Exp}(1)$.

Solution:

The distribution function of Y_n is

$$\begin{aligned}
F_{Y_n}(y) &= P(Y_n \leq y) \\
&= P\left(n\left(1 - \max_{1\leq i\leq n} X_i\right) \leq y\right) \\
&= P\left(\max_{1\leq i\leq n} X_i \geq 1 - \frac{y}{n}\right) \\
&= 1 - P\left(\max_{1\leq i\leq n} X_i < 1 - \frac{y}{n}\right) \\
&= 1 - P\left(X_1 < 1 - \frac{y}{n}, X_2 < 1 - \frac{y}{n}, \ldots, X_n < 1 - \frac{y}{n}\right) \\
&= 1 - P\left(X_1 < 1 - \frac{y}{n}\right) \cdot P\left(X_2 < 1 - \frac{y}{n}\right) \cdots P\left(X_n < 1 - \frac{y}{n}\right) \\
&= 1 - P\left(X_1 \leq 1 - \frac{y}{n}\right) \cdot P\left(X_2 \leq 1 - \frac{y}{n}\right) \cdots P\left(X_n \leq 1 - \frac{y}{n}\right) \\
&= 1 - F_{X_1}\left(1 - \frac{y}{n}\right) \cdot F_{X_2}\left(1 - \frac{y}{n}\right) \cdots F_{X_n}\left(1 - \frac{y}{n}\right) \\
&= 1 - \left[F_{X_1}\left(1 - \frac{y}{n}\right)\right]^n,
\end{aligned}$$

where: in step (A) we have used the fact that the variables X_i are mutually independent; in step (B) we have used the fact that the variables X_i are absolutely continuous; in step (C) we have used the definition of distribution function; in step (D) we have used the fact that the variables X_i have identical distributions. Thus:

$$F_{Y_n}(y) = \begin{cases} 0, & \text{if } y < 0, \\ 1 - \left(1 - \frac{y}{n}\right)^n, & \text{if } 0 \leq y < n, \\ 1, & \text{if } y \geq n. \end{cases}$$

Since

$$\lim_{n\to\infty} \left(1 - \frac{y}{n}\right)^n = \exp(-y),$$

we have

$$\lim_{n\to\infty} F_{Y_n}(y) = F_Y(y) = \begin{cases} 0, & \text{if } y < 0, \\ 1 - \exp(-y), & \text{if } y \geq 0, \end{cases}$$

where $F_Y(y)$ is the distribution function of an exponential random variable. Therefore, the sequence $\{Y_n\}$ converges in distribution to an exponential distribution.

Q3: Let X be a discrete random variable with support

$$R_X = \{0, 1\}$$

and probability mass function:

$$p_X(x) = \begin{cases} 1/3, & \text{if } x = 1, \\ 2/3, & \text{if } x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Consider a sequence of random variables $\{X_n\}$ whose generic term is:

$$X_n = \left(1 + \frac{1}{n}\right) X.$$

Show that $\{X_n\}$ converges in probability to X.

Solution:

We want to prove that $X_n \xrightarrow{p} X$. Take any $\varepsilon > 0$. Note that:

$$\begin{aligned} |X_n - X| &= \left|\left(1 + \frac{1}{n}\right) X - X\right| \\ &= \left|\frac{1}{n} X\right| \\ &= \frac{1}{n} X, \end{aligned}$$

When $X = 0$, which happens with probability $\frac{2}{3}$, we have that

$$|X_n - X| = \frac{1}{n} X = 0,$$

and, of course, $|X_n - X| \leq \varepsilon$. When $X = 1$, which happens with probability $\frac{1}{3}$, we have that

$$|X_n - X| = \frac{1}{n} X = \frac{1}{n},$$

and $|X_n - X| \leq \varepsilon$ if and only if $\frac{1}{n} \leq \varepsilon$ (or $n \geq \frac{1}{\varepsilon}$). Therefore:

$$P(|X_n - X| \leq \varepsilon) = \begin{cases} 2/3, & \text{if } n < \frac{1}{\varepsilon}, \\ 1, & \text{if } n \geq \frac{1}{\varepsilon}, \end{cases}$$

and

$$P(|X_n - X| > \varepsilon) = 1 - P(|X_n - X| \leq \varepsilon) = \begin{cases} 1/3, & \text{if } n < \frac{1}{\varepsilon}, \\ 0, & \text{if } n \geq \frac{1}{\varepsilon}. \end{cases}$$

Thus, $P(|X_n - X| > \varepsilon)$ trivially converges to 0, because it is identically equal to 0 for all n such that $n > \frac{1}{\varepsilon}$. Since ε was arbitrary, we have obtained the desired result:

$$\lim_{n \to \infty} P(|X_n - X| > \varepsilon) = 0$$

for any $\varepsilon > 0$.

Q4: **Convergence of Discrete to continuous and vice versa**

(a) Let Y_n be uniformly distributed on the discrete set $\{1, 2, \ldots, n\}$, and define

$$X_n = \frac{Y_n}{n}.$$

Show that:

$$X_n \xrightarrow{d} X, \quad X \sim \mathrm{U}(0, 1).$$

(b) Let X_n be uniformly distributed on the interval $[-1/n, 1/n]$. Show that

$$X_n \xrightarrow{p} 0,$$

i.e., that the sequence of continuous random variables X_n converges in probability to the degenerate (discrete) random variable which is identically equal to zero. This also implies

$$X_n \xrightarrow{d} 0.$$

Solution:

(a) **Convergence in Distribution:** $X_n \xrightarrow{d} X$ **where** $X \sim$ **Uniform**$(0, 1)$**.**
Since Y_n is uniform on $\{1, 2, \ldots, n\}$, we have $P(Y_n = k) = 1/n$ for $k = 1, 2, \ldots, n$.
Let $F_n(x) = P(X_n \leq x)$ be the cdf of X_n. For $0 \leq x < 1$:

$$F_n(x) = P\left(\frac{Y_n}{n} \leq x\right) = P(Y_n \leq nx) = \frac{\lfloor nx \rfloor}{n}.$$

As $n \to \infty$, $\frac{\lfloor nx \rfloor}{n} \to x$.
The limit cdf is:

$$F(x) = \begin{cases} 0, & x < 0, \\ x, & 0 \leq x < 1, \\ 1, & x \geq 1, \end{cases}$$

which is the cdf of Uniform$(0, 1)$. Hence $X_n \xrightarrow{d} X$.

(b) **Convergence in Probability:** $X_n \xrightarrow{p} 0$**.**
As n tends to infinity, the probability density tends to become concentrated around the point $x = 0$. Therefore, it seems reasonable to conjecture that the sequence $\{X_n\}$ converges in probability to the constant random variable

$$X(\omega) = 0, \quad \forall \omega \in \Omega$$

To rigorously verify this claim we need to use the formal definition of convergence in probability. For any $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = \lim_{n\to\infty} P(|X_n - 0| > \varepsilon)$$

$$\begin{aligned}
&= \lim_{n\to\infty} [1 - P(-\varepsilon \leq X_n \leq \varepsilon)] \\
&= 1 - \lim_{n\to\infty} \int_{-\varepsilon}^{\varepsilon} f_{X_n}(x)\, dx \\
&= 1 - \lim_{n\to\infty} \int_{\max(-\varepsilon, -1/n)}^{\min(\varepsilon, 1/n)} \frac{n}{2}\, dx \\
&= 1 - \lim_{n\to\infty} \int_{-1/n}^{1/n} \frac{n}{2}\, dx \\
&= 1 - \lim_{n\to\infty} 1 \\
&= 0
\end{aligned}$$

where: in step $\boxed{A}$ we have used the fact that $1/n < \varepsilon$ when n becomes large.

Convergence in Distribution: Since convergence in probability implies convergence in distribution, we also have $X_n \xrightarrow{d} 0$.

Q5: Given an integral $I = \int_0^{2\pi} f(x)dx$, solve the following:

(a) Write the Monte Carlo Estimate, assuming X_i's are sampled uniformly over the domain.

(b) Write the Monte Carlo Estimate, assuming X_i's are sampled according to some PDF $g(X_i)$.

(c) Prove that the Monte Carlo Estimates from the previous questions compute the right answer on average.

Solution:

(a) We are given samples from $U \sim [0, 2\pi]$. Using SLLN,

$$I = 2\pi\mathbb{E}[f(U)] \approx \frac{2\pi}{N}\sum_{i=0}^{N} f(U_i)$$

(b) We are given samples of Y with PDF $g(\cdot)$.

$$I = \int_0^{2\pi} \frac{f(y)}{g(y)} g(y)dy = \mathbb{E}\left[\frac{f(Y)}{g(Y)}\right] \approx \frac{1}{N}\sum_{i=0}^{N} \frac{f(Y_i)}{g(Y_i)}$$

(c) We want to show that the expected value of the estimate of I is equal to I. Using linearity of expectation,

$$\mathbb{E}\left[\frac{2\pi}{N}\sum_{i=0}^{N} f(U_i)\right] = \frac{2\pi}{N}\sum_{i=0}^{N} \mathbb{E}[U_i] = 2\pi\mathbb{E}[U] = I$$

and

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=0}^{N} \frac{f(Y_i)}{g(Y_i)}\right] = \frac{1}{N}\sum_{i=0}^{N} \mathbb{E}\left[\frac{f(Y_i)}{g(Y_i)}\right] = \mathbb{E}\left[\frac{f(Y)}{g(Y)}\right] = I$$

Q6: Let X be an exponential random variable with parameter λ and let Y be a random variable with the Gamma distribution $Y \sim \text{Gamma}(k, \theta)$.

(a) Show how to generate X using a uniform random variable U drawn from the interval $[0, 1]$.

(b) Show how to generate Y using k uniform random variables drawn from $[0, 1]$.

Solution:

a) To generate X: The cumulative distribution function (CDF) of X is given by:

$$F_X(x) = 1 - e^{-\lambda x} \quad \text{for } x \geq 0$$

Let $U \sim \text{Uniform}(0,1)$. By the inverse transform method:

$$\begin{aligned} F_X(X) &= U \\ 1 - e^{-\lambda X} &= U \\ e^{-\lambda X} &= 1 - U \\ -\lambda X &= \ln(1-U) \\ X &= -\frac{1}{\lambda}\ln(1-U) \end{aligned}$$

Since $U \sim \text{Uniform}(0,1)$, the distribution of $1-U$ is also uniform, so $-\ln(1-U)$ has the same distribution as $-\ln(U)$. Therefore, we can simplify this to:

$$X = -\frac{1}{\lambda}\ln(U)$$

b) To generate $Y \sim \text{Gamma}(k,\theta)$: Generate k independent uniform random variables:

$$U_1, U_2, \ldots, U_k \sim \text{Uniform}(0,1)$$

Using the inverse transform method for each $X_i \sim \text{Exp}\left(\frac{1}{\theta}\right)$:

$$X_i = -\theta \ln(U_i), \quad i = 1, 2, \ldots, k$$

Sum the X_i's to obtain Y:

$$Y = \sum_{i=1}^{k} X_i = -\theta \sum_{i=1}^{k} \ln(U_i)$$

Q7: Use the rejection method to generate a random variable having the $Gamma(\frac{5}{2}, 1)$ density function.

Note: The pdf of $Gamma(k,\theta)$ is given by $f(x) = \frac{1}{\Gamma(k)\theta^k}x^{k-1}e^{-x/\theta}$ and $\Gamma(\frac{5}{2}) = \frac{3}{4}\pi$.

Hint: You need to figure out an appropriate distribution you can already sample from to use in the rejection method.

Solution:

We pick $exp(\lambda)$ as the distribution we'll be sampling from.

$$\begin{aligned} f(x) &= \frac{4}{3\sqrt{\pi}}x^{\frac{3}{2}}e^{-x}, x > 0 \\ g(x) &= \lambda e^{-\lambda x}, x > 0 \\ \implies \frac{f(x)}{g(x)} &= \frac{4}{3\lambda\sqrt{\pi}}x^{\frac{3}{2}}e^{(\lambda-1)x} \end{aligned}$$

We wish to find a c such that $\frac{f(x)}{g(x)} \leq c$ for all x

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = 0$$

Hence, $x = \frac{3}{2(1-\lambda)}$

We need to pick an appropriate λ such that $x > 0$. We pick $\lambda = \frac{2}{5}$.

$$c = \frac{10}{3\sqrt{\pi}} \left(\frac{5}{2}\right)^{\frac{3}{2}} e^{-\frac{3}{2}}$$

$$\frac{f(x)}{cg(x)} = \frac{x^{\frac{3}{2}} e^{-\frac{3}{5}x}}{\left(\frac{5}{2}\right)^{\frac{3}{2}} e^{-\frac{3}{2}}}$$

Now for to finally generate the required random number (i.e using the rejection method algorithm)

(a) Generate a random number U_1 and use that to generate a random number from $exp\left(\frac{2}{5}\right)$ $(Y = -\frac{5}{2}\log U_1)$

(b) Generate a random number U_2

(c) If $U_2 < \frac{Y^{\frac{3}{2}} e^{-\frac{3}{5}Y}}{\left(\frac{5}{2}\right)^{\frac{3}{2}} e^{-\frac{3}{2}}}$, set $X = Y$. Otherwise, execute the step (a).

Q8: Let X be a real-valued random variable with finite mean $\mu = \mathbb{E}[X]$ and finite variance $\sigma^2 = \mathrm{Var}(X)$. Prove the *Chebyshev inequality*

$$\mathbb{P}\big(|X - \mu| \geq \varepsilon\big) \leq \frac{\sigma^2}{\varepsilon^2} \qquad \text{for every } \varepsilon > 0.$$

(a) Markov's inequality: for any nonnegative random variable Z and $t > 0$, $\mathbb{P}(Z \geq t) \leq \mathbb{E}[Z]/t$. Apply Markov's inequality to an appropriate nonnegative function of X to obtain Chebyshev's inequality.

(b) Use Chebyshev's inequality to show: if $\{X_n\}$ is a sequence of random variables with $\mathbb{E}[X_n] \to a$ and $\mathrm{Var}(X_n) \to 0$, then $X_n \xrightarrow{p} a$.

Q9: Suppose a sequence of random variables X_n satisfies
$\lim_{n\to\infty}[|X_n - c|^\alpha] = 0$ for $\alpha > 0$.
Show that $X_n c$.

Proof. For any $\epsilon > 0$, we analyze the expectation:

$$\begin{aligned}
[|X_n - c|^\alpha] &= \int_\Omega |X_n - c|^\alpha dP \\
&= \int_{|X_n - c| \geq \epsilon} |X_n - c|^\alpha dP + \int_{|X_n - c| < \epsilon} |X_n - c|^\alpha dP \\
&\geq \int_{|X_n - c| \geq \epsilon} |X_n - c|^\alpha dP \quad \text{(since the second term is non-negative)} \\
&\geq \int_{|X_n - c| \geq \epsilon} \epsilon^\alpha dP \quad \text{(since } |X_n - c|^\alpha \geq \epsilon^\alpha \text{ on this domain)} \\
&= \epsilon^\alpha \int_{|X_n - c| \geq \epsilon} dP \\
&= \epsilon^\alpha P(|X_n - c| \geq \epsilon)
\end{aligned}$$

Rearranging the inequality gives:

$$0 \leq P(|X_n - c| \geq \epsilon) \leq \frac{[|X_n - c|^\alpha]}{\epsilon^\alpha}$$

Taking the limit as $n \to \infty$:

$$0 \leq \lim_{n\to\infty} P(|X_n - c| \geq \epsilon) \leq \lim_{n\to\infty} \frac{[|X_n - c|^\alpha]}{\epsilon^\alpha} = \frac{0}{\epsilon^\alpha} = 0$$

By the Sandwich Theorem, $\lim_{n\to\infty} P(|X_n - c| \geq \epsilon) = 0$. This is the definition of convergence in probability. □

Q10: Given i.i.d. random variables X_i with $E[X_i] = 2$ and $var(X_i) = 9$. Let $Y_i = X_i/2^i$, $T_n = \sum_{i=1}^n Y_i$, and $A_n = T_n/n$.

(a) **Mean and Variance**

- **For Y_n:**

$$E[Y_n] = E\left[\frac{X_n}{2^n}\right] = \frac{2}{2^n} = 2^{1-n}$$

$$var(Y_n) = var\left(\frac{X_n}{2^n}\right) = \frac{9}{(2^n)^2} = \frac{9}{4^n}$$

- **For T_n:**

$$E[T_n] = \sum_{i=1}^n E[Y_i] = \sum_{i=1}^n \frac{2}{2^i} = 2\left(\frac{1/2(1-(1/2)^n)}{1-1/2}\right) = 2(1-2^{-n})$$

$$var(T_n) = \sum_{i=1}^n var(Y_i) = \sum_{i=1}^n \frac{9}{4^i} = 9\left(\frac{1/4(1-(1/4)^n)}{1-1/4}\right) = 3(1-4^{-n})$$

- **For A_n:**

$$E[A_n] = \frac{1}{n}E[T_n] = \frac{2(1-2^{-n})}{n}$$

$$var(A_n) = \frac{1}{n^2}var(T_n) = \frac{3(1-4^{-n})}{n^2}$$

(b) **Convergence of Y_n**
Result: $Y_n 0$.

Proof. By Chebyshev's inequality, for any $\epsilon > 0$:

$$P(|Y_n - E[Y_n]| \geq \epsilon) \leq \frac{var(Y_n)}{\epsilon^2} = \frac{9}{4^n\epsilon^2}$$

As $n \to \infty$, the RHS goes to 0, so $Y_n - E[Y_n] 0$. Since $\lim_{n\to\infty} E[Y_n] = \lim_{n\to\infty} 2^{1-n} = 0$, it follows that $Y_n 0$. □

(c) **Convergence of T_n**
Result: T_n does not converge in probability to a constant.

Proof. Assume, for contradiction, that $T_n c$ for some constant c. This implies $\lim_{n\to\infty} E[T_n] = c$ and $\lim_{n\to\infty} E[T_n^2] = c^2$. From the mean:

$$c = \lim_{n\to\infty} E[T_n] = \lim_{n\to\infty} 2(1-2^{-n}) = 2$$

Now, we check the limit of the second moment:

$$\begin{aligned}
\lim_{n\to\infty} E[T_n^2] &= \lim_{n\to\infty} \left(\mathrm{var}(T_n) + (E[T_n])^2\right) \\
&= \lim_{n\to\infty} \left(3(1-4^{-n}) + (2(1-2^{-n}))^2\right) \\
&= 3(1) + (2(1))^2 = 3 + 4 = 7
\end{aligned}$$

The assumption requires the limit to be $c^2 = 2^2 = 4$. Since $7 \neq 4$, we have a contradiction. □

(d) **Convergence of A_n**
Result: $A_n 0$.

Proof. By Chebyshev's inequality, for any $\epsilon > 0$:

$$P(|A_n - E[A_n]| \geq \epsilon) \leq \frac{\mathrm{var}(A_n)}{\epsilon^2} = \frac{3(1-4^{-n})}{n^2\epsilon^2}$$

As $n \to \infty$, the RHS goes to 0, so $A_n - E[A_n] 0$. Since $\lim_{n\to\infty} E[A_n] = \lim_{n\to\infty} \frac{2(1-2^{-n})}{n} = 0$, it follows that $A_n 0$. □