Intend to formally capture the notion of proof that is commonly applied in other fields (e.g. mathematics)

A proof of a formula from a set of premises is a sequence of steps in which any step of the proof is:

- An axiom or premise
- A formula deduced from previous steps of the proof
- using some rule of inference

The last step of the proof should deduce the formula we wish to prove.

We say that S follows from (premises) α to denote that the set of formulae α "prove" the formula S.

Soundness, Completeness and Decidability

- Soundness: A logic is sound if it preserves truth (i.e. if a set of premises are all true, any conclusion drawn from those premises must also be true)

- Completeness:
  - (Semantic)A logic is complete if it is capable of proving any valid consequence
  - (Syntactic) Completeness requires that every sentence, or its negation, is provable (i.e., a theorem)

- Decidability: A logic is decidable if there is a mechanical procedure (computer program) to prove any given consequence
  - Decidability: there is a decision procedure (finite) for theorem

Inference rules

1. Modus Ponens: $P, P \rightarrow Q \Rightarrow Q$
2. Modus Tollens: $P \rightarrow Q, \neg Q \Rightarrow \neg P$
3. Hypothetical Syllogism: $P \rightarrow Q,\ Q \rightarrow R \Rightarrow P \rightarrow R$
4. And-Elimination: $P_1 \wedge P_2 \wedge \ldots \wedge P_n \Rightarrow P_i$
5. And-Introduction: $P_1, P_2, \ldots, P_n \Rightarrow P1 \wedge P2 \wedge \ldots \wedge Pn$
6. Or-Introduction: $P_i \Rightarrow P_1 \vee P_2 \vee \ldots \vee P_n$
7. Double-Negation Elimination: $\neg\neg P \Rightarrow P$
8. Unit Resolution: $P \vee Q, \neg Q \Rightarrow P$
9. Resolution: $P \vee Q, \neg Q \vee R \Rightarrow P \vee R$

Example of a Formal Proof

1. $A \vee (B \rightarrow D)$
2. $\neg C \rightarrow (D \rightarrow E)$
3. $A \rightarrow C$
4. $\neg C$ $\quad \therefore B \rightarrow E$
5. $\neg A$
6. $B \rightarrow D$
7. $D \rightarrow E$
8. $B \rightarrow E$

- *3,4 (Modus Tollens)*
- *1,5, (Unit resolution)*
- *2,4 (Modus Ponens)*
- *6,7 (Hypothetical Syllogism)*

Exercise

- Construct formal proof of validity for:
  If the investigation continues, then new evidence is brought to light. If new evidence is brought to light, then several leading citizens are implicated. If several leading citizens are implicated, then the newspapers stop publicizing the case. If continuation of the investigation implies that the newspapers stop publicizing the case, then the bringing to light of new evidence implies that the investigation continues. The investigation does not continue. Therefore, new evidence is not brought to light.
- *C*: The investigation continues. *N*: New evidence is brought to light. *I*: Several leading citizens are implicated. *S*: The newspapers stop publicizing the case.

$\alpha \Rightarrow \beta$.

α.

1. $C \to N$
2. $N \to I$
3. $I \to S$
4. $(C \to S) \to (N \to C)$
5. $\neg C$.

β $\neg N$ $\therefore \neg N$

Solution

1. $C \to N$	
2. $N \to I$	
3. $I \to S$	
4. $(C \to S) \to (N \to C)$	
5. $\neg C$. $\quad \therefore \neg N$	
6. $C \to I$	1,2 (Hypothetical Syllogism)
7. $C \to S$	6,3(Hypothetical Syllogism)
8. $N \to C$	
9. $\neg N$	8,5 (Modus Tollens)

Exercise

- If I study, I make good grades. If I do not study, I enjoy myself. Therefore, I make good grades or I enjoy myself.
- S, G, E


IIT HYDERABAD
1 S → G
α: 2 ¬S → E
β G V E

Solution

1. $S \rightarrow G$	
2. $\neg S \rightarrow E \therefore G \vee E$	
3. $\neg S \vee G$	1
4. $\neg\neg S \vee E$	2
5. $S \vee E$	4, (DoubleNegationEliminate)
6. $G \vee E$	3,5 (Resolution)

Conjunctive Normal Form (CNF)

- A literal is a propositional letter or the negation of a propositional letter

- A clause is a disjunction of literals

- Conjunctive Normal Form (CNF) – a conjunction of clauses
  - e.g. $(P \vee Q \vee \neg R) \wedge (\neg S \vee \neg R)$

- Disjunctive Normal Form (DNF) – a disjunction of conjunctions of literals
  - e.g. $(P \wedge Q \wedge \neg R) \vee (\neg S \wedge \neg R)$

- Every propositional logic formula can be converted to CNF and DNF

Converting to CNF

- Eliminate $\leftrightarrow$ rewriting $P \leftrightarrow Q$ as $(P \rightarrow Q) \wedge (Q \rightarrow P)$
- Eliminate $\rightarrow$ rewriting $P \rightarrow Q$ as $\neg P \vee Q$
- Use *De Morgan's laws* to push $\neg$ inwards:
- Rewrite $\neg(P \wedge Q)$ as $\neg P \vee \neg Q$
- Rewrite $\neg(P \vee Q)$ as $\neg P \wedge \neg Q$
- Eliminate double negations: rewrite $\neg\neg P$ as P
- Use the *distributive laws* to get CNF:
- Rewrite $(P \wedge Q) \vee R$ as $(P \vee R) \wedge (Q \vee R)$
- Rewrite $(P \vee Q) \wedge R$ as $(P \wedge R) \vee (Q \wedge R)$

Exercise: Convert $\neg(P \rightarrow (Q \wedge R))$ to CNF

Solution

$\neg(P \rightarrow (Q \wedge R))$
$\neg(\neg P \vee (Q \wedge R))$
$\neg\neg P \wedge \neg(Q \wedge R)$
$\neg\neg P \wedge (\neg Q \vee \neg R)$
$P \wedge (\neg Q \vee \neg R)$

Two clauses: P, $\neg Q \vee \neg R$