Probability and Statistics: MA6.101

Homework 9 Solutions

Topics Covered: Random Vectors

Q1: You are given the random vector $Y' = [Y_1, Y_2, Y_3, Y_4]$ with mean vector

$$\mu_Y = [5, -1, 4, -3]$$

and variance-covariance matrix

$$\Sigma_Y = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

(a) Find $E(BY)$, the mean of BY.
(b) Find $Cov(BY)$, the variances and covariances of BY.
(c) Which pairs of linear combinations have zero covariances?

Solution:

Consider the (4×1)-dimensional random vector

$$y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix},$$

whereby

$$\mu_Y = E(Y) = \begin{bmatrix} E(Y_1) \\ E(Y_2) \\ E(Y_3) \\ E(Y_4) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 4 \\ -3 \end{bmatrix}$$

and

$$\Sigma_Y = Cov(Y) = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Let

$$B = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -2 \end{bmatrix}.$$

So,

$$E(BY) = B\mu_Y \quad \text{and} \quad Cov(BY) = B\Sigma_Y B'.$$

Hence,

(a)

$$E(BY) = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 16 \\ 9 \\ 1 \end{bmatrix}.$$

(b)

$$Cov(BY) = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}.$$

Calculating this yields:

$$Cov(BY) = \begin{bmatrix} 21 & -6 & -10 \\ -6 & 13 & 8 \\ -10 & 8 & 21 \end{bmatrix}.$$

(c) Define $Z = BY = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}$, where

$$Z_1 = 2Y_1 - Y_2 + Y_4, \quad Z_2 = -Y_1 + 2Y_2 - Y_3, \quad Z_3 = Y_2 + Y_3 - 2Y_4.$$

Then, clearly,

$$Cov(BY) = \Sigma_Z,$$

and therefore, according to (b),

$$Cov(Z_i, Z_j) = 0 \quad \text{for } i \neq j, \text{ where } i, j = 1, 2, 3.$$

Q2: Let X and Y are said to be bivariate normal if $aX + bY$ is normal for all a and b. If X and Y are bivariate normal with 0 mean, variance of 1, and ρ correlation, then their joint pdf is:

$$f(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(x^2 - 2\rho xy + y^2\right)\right)$$

Let $U = X + Y$ and $V = X - Y$. Find the joint pdf of U and V.

Solution:

Since X and Y are bivariate normal with means 0, variances 1, and correlation ρ, the joint distribution of (X, Y) is:

$$f(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(x^2 - 2\rho xy + y^2\right)\right)$$

Step 1: Define the Transformation We define the transformation:

$$U = X + Y$$

$$V = X - Y$$

Step 2: Find the Inverse Transformation Solving for X and Y in terms of U and V:

$$X = \frac{U+V}{2} \quad \text{and} \quad Y = \frac{U-V}{2}$$

Step 3: Calculate the Jacobian The Jacobian of the transformation from (X, Y) to (U, V) is:

$$J = \begin{vmatrix} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

Thus, $|J| = \frac{1}{2}$. Step 4: Distribution of U and V Since X and Y are jointly normal with variances 1 and correlation ρ, we can derive the variances and covariance of U and V:

$$Var(U) = Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X, Y) = 1 + 1 + 2\rho = 2(1+\rho)$$

$$Var(V) = Var(X-Y) = Var(X) + Var(Y) - 2\,Cov(X, Y) = 1 + 1 - 2\rho = 2(1-\rho)$$

$$Cov(U, V) = Cov(X+Y, X-Y) = Var(X) - Var(Y) = 0$$

Therefore, U and V are independent with variances $2(1+\rho)$ and $2(1-\rho)$, respectively. Step 5: Joint pdf of U and V Since U and V are independent, their joint pdf is the product of their marginal pdfs:

$$f_{U,V}(u, v) = f_U(u) f_V(v)$$

where

$$f_U(u) = \frac{1}{\sqrt{2\pi \cdot 2(1+\rho)}} \exp\left(-\frac{u^2}{2 \cdot 2(1+\rho)}\right) = \frac{1}{\sqrt{4\pi(1+\rho)}} \exp\left(-\frac{u^2}{4(1+\rho)}\right)$$

and

$$f_V(v) = \frac{1}{\sqrt{2\pi \cdot 2(1-\rho)}} \exp\left(-\frac{v^2}{2 \cdot 2(1-\rho)}\right) = \frac{1}{\sqrt{4\pi(1-\rho)}} \exp\left(-\frac{v^2}{4(1-\rho)}\right)$$

Thus, the joint pdf of $U = X + Y$ and $V = X - Y$ is:

$$f_{U,V}(u, v) = \frac{1}{4\pi\sqrt{(1+\rho)(1-\rho)}} \exp\left(-\frac{u^2}{4(1+\rho)} - \frac{v^2}{4(1-\rho)}\right)$$

Q3: Let

$$Y = G(X) = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1 + X_2 \\ X_1 - X_2 \end{bmatrix}$$

where $X = (X_1, X_2)^T$ is a continuous random vector with joint pdf $f_X(x_1, x_2)$.

(a) Find the inverse transformation $H(Y)$ such that $X = H(Y)$.

(b) Compute the Jacobian determinant $|J|$ for the inverse transformation.

(c) Write the expression for the joint pdf $f_Y(y_1, y_2)$ in terms of $f_X(x_1, x_2)$.

A:

(a) From the given relations:

$$\begin{cases} Y_1 = X_1 + X_2, \\ Y_2 = X_1 - X_2. \end{cases}$$

Adding and subtracting these two equations, we obtain:

$$X_1 = \frac{Y_1 + Y_2}{2}, \quad X_2 = \frac{Y_1 - Y_2}{2}.$$

Hence, the inverse transformation $H(Y)$ is:

$$H(Y) = \begin{bmatrix} \frac{Y_1+Y_2}{2} \\ \frac{Y_1-Y_2}{2} \end{bmatrix}.$$

(b) We compute the Jacobian matrix of H as:

$$J = \begin{bmatrix} \dfrac{\partial X_1}{\partial Y_1} & \dfrac{\partial X_1}{\partial Y_2} \\ \dfrac{\partial X_2}{\partial Y_1} & \dfrac{\partial X_2}{\partial Y_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

Thus, the determinant is:

$$|J| = \left| \frac{1}{2} \cdot (-\frac{1}{2}) - \frac{1}{2} \cdot \frac{1}{2} \right| = \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2}.$$

(c) Using the change of variables formula for random vectors:

$$f_Y(y_1, y_2) = f_X\big(H(y_1, y_2)\big)\,|J|.$$

Substituting $H(Y)$ and $|J|$:

$$f_Y(y_1, y_2) = f_X\left(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}\right) \times \frac{1}{2}.$$

Q4: Let

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

be a normal random vector with the following mean and covariance matrices

$$\mathbf{m} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}.$$

Let also

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

and

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \mathbf{AX} + \mathbf{b}.$$

(a) Find $P(X_2 > 0)$.

(b) Find expected value vector of $\mathbf{Y}$, $\mathbf{m}_Y = E[\mathbf{Y}]$.

(c) Find the covariance matrix of $\mathbf{Y}$, $\mathbf{C}_Y$.

(d) Find $P(Y_2 \leq 2)$.

A:

We are given $\mathbf{X} \sim \mathcal{N}(\mathbf{m}, \mathbf{C})$ with $\mathbf{m} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$. The transformation is $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$ with $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 1 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

(a) Each component of a multivariate normal vector is normally distributed. For X_2, we extract its parameters from $\mathbf{m}$ and $\mathbf{C}$:
- Mean: $E[X_2] = \mu_{X_2} = \mathbf{m}_2 = 2$
- Variance: $\text{Var}(X_2) = \sigma^2_{X_2} = \mathbf{C}_{22} = 1$

So, $X_2 \sim \mathcal{N}(2, 1)$. We standardize to find the probability:

$$P(X_2 > 0) = P\left(\frac{X_2 - \mu_{X_2}}{\sigma_{X_2}} > \frac{0-2}{\sqrt{1}}\right) = P(Z > -2)$$

Where $Z \sim \mathcal{N}(0, 1)$. By symmetry, $P(Z > -2) = P(Z < 2) = \Phi(2)$.

$$P(X_2 > 0) = \Phi(2) \approx 0.9772$$

(b) Using the linearity of expectation, $E[\mathbf{A}\mathbf{X} + \mathbf{b}] = \mathbf{A}E[\mathbf{X}] + \mathbf{b} = \mathbf{A}\mathbf{m} + \mathbf{b}$.

$$\begin{aligned}
\mathbf{m}_Y &= \mathbf{A}\mathbf{m} + \mathbf{b} \\
&= \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} (2)(1) + (1)(2) \\ (-1)(1) + (1)(2) \\ (1)(1) + (3)(2) \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \\
\mathbf{m}_Y &= \begin{bmatrix} 3 \\ 1 \\ 8 \end{bmatrix}
\end{aligned}$$

(c) The covariance matrix of an affine transformation is $\text{Cov}(\mathbf{A}\mathbf{X} + \mathbf{b}) = \mathbf{A}\text{Cov}(\mathbf{X})\mathbf{A}^T = \mathbf{A}\mathbf{C}\mathbf{A}^T$. First, $\mathbf{A}^T = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$. Next, we compute $\mathbf{A}\mathbf{C}$:

$$\mathbf{A}\mathbf{C} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} (8+1) & (2+1) \\ (-4+1) & (-1+1) \\ (4+3) & (1+3) \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ -3 & 0 \\ 7 & 4 \end{bmatrix}$$

Now we multiply by $\mathbf{A}^T$:

$$\mathbf{C}_Y = (\mathbf{AC})\mathbf{A}^T = \begin{bmatrix} 9 & 3 \\ -3 & 0 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} (18+3) & (-9+3) & (9+9) \\ (-6+0) & (3+0) & (-3+0) \\ (14+4) & (-7+4) & (7+12) \end{bmatrix}$$
$$\mathbf{C}_Y = \begin{bmatrix} 21 & -6 & 18 \\ -6 & 3 & -3 \\ 18 & -3 & 19 \end{bmatrix}$$

(d) The transformed vector $\mathbf{Y}$ is also multivariate normal: $\mathbf{Y} \sim \mathcal{N}(\mathbf{m}_Y, \mathbf{C}_Y)$. The component Y_2 is normally distributed with parameters found in $\mathbf{m}_Y$ and $\mathbf{C}_Y$:

- Mean: $E[Y_2] = \mu_{Y_2} = (\mathbf{m}_Y)_2 = 1$
- Variance: $\text{Var}(Y_2) = \sigma^2_{Y_2} = (\mathbf{C}_Y)_{22} = 3$

So, $Y_2 \sim \mathcal{N}(1, 3)$. We standardize to find the probability:

$$P(Y_2 \leq 2) = P\left(\frac{Y_2 - \mu_{Y_2}}{\sigma_{Y_2}} \leq \frac{2-1}{\sqrt{3}}\right) = P\left(Z \leq \frac{1}{\sqrt{3}}\right)$$

Where $Z \sim \mathcal{N}(0, 1)$.

$$P(Y_2 \leq 2) = \Phi\left(\frac{1}{\sqrt{3}}\right) \approx \Phi(0.577) \approx 0.7181$$

Q5: The random vector $\mathbf{X} = [X_1, X_2, X_3]^T$ follows a multivariate normal distribution with mean $\mu = [1, 2, 3]^T$ and covariance matrix:

$$\Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

Find the conditional distribution of $[X_1, X_3]^T$ given that $X_2 = x_2$.

Solution:

This is a direct application of the formula for conditional multivariate normal distributions. We partition the vector $\mathbf{X}$, its mean μ, and covariance Σ. Let $\mathbf{X_a} = [X_1, X_3]^T$ and $\mathbf{X_b} = [X_2]$. The corresponding partitioned means are $\mu_a = [1, 3]^T$ and $\mu_b = [2]$. The partitioned covariance matrix is:

$$\Sigma = \begin{bmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{bmatrix}$$

Where:

$$\Sigma_{aa} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}, \quad \Sigma_{ab} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \Sigma_{ba} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad \Sigma_{bb} = [2]$$

The conditional distribution of $\mathbf{X_a}|\mathbf{X_b} = x_b$ is also normal, with a new mean $\mu_{a|b}$ and covariance $\Sigma_{a|b}$ given by the formulas:

$$\mu_{a|b} = \mu_a + \Sigma_{ab}\Sigma_{bb}^{-1}(x_b - \mu_b)$$
$$\Sigma_{a|b} = \Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba}$$

First, find $\Sigma_{bb}^{-1} = [2]^{-1} = [1/2]$. Now, calculate the conditional mean:

$$\mu_{a|b} = \begin{bmatrix}1\\3\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}[1/2](x_2 - 2) = \begin{bmatrix}1\\3\end{bmatrix} + \begin{bmatrix}0.5(x_2-2)\\0.5(x_2-2)\end{bmatrix} = \begin{bmatrix}1+0.5x_2-1\\3+0.5x_2-1\end{bmatrix} = \begin{bmatrix}0.5x_2\\2+0.5x_2\end{bmatrix}$$

Next, calculate the conditional covariance:

$$\Sigma_{a|b} = \begin{bmatrix}4&0\\0&3\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix}[1/2]\begin{bmatrix}1&1\end{bmatrix} = \begin{bmatrix}4&0\\0&3\end{bmatrix} - \begin{bmatrix}1/2\\1/2\end{bmatrix}\begin{bmatrix}1&1\end{bmatrix} = \begin{bmatrix}4&0\\0&3\end{bmatrix} - \begin{bmatrix}0.5&0.5\\0.5&0.5\end{bmatrix} = \begin{bmatrix}3.5&-0.5\\-0.5&2.5\end{bmatrix}$$

So, the conditional distribution is $\mathcal{N}(\mu_{a|b}, \Sigma_{a|b})$.

Q6: Let the random vector $\mathbf{Z} = [Z_1, Z_2]^T$ be normally distributed with mean $\mathbf{m} = \begin{bmatrix}1\\2\end{bmatrix}$ and covariance $\mathbf{C} = \begin{bmatrix}2&-1\\-1&3\end{bmatrix}$. A new random vector is defined by the transformation $\mathbf{W} = A\mathbf{Z} + \mathbf{b}$, where $A = \begin{bmatrix}1&1\\1&-1\\2&0\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}0\\0\\-2\end{bmatrix}$.

(a) Find the mean vector and covariance matrix of $\mathbf{W}$.

(b) Find the probability $P(W_2 > 1)$.

Solution:

(a) **Mean and Covariance of W:** The vector $\mathbf{W}$ is a linear transformation of a normal vector, so it is also normal. The mean vector is $\mathbf{m}_W = A\mathbf{m} + \mathbf{b}$.

$$\mathbf{m}_W = \begin{bmatrix}1&1\\1&-1\\2&0\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}0\\0\\-2\end{bmatrix} = \begin{bmatrix}1+2\\1-2\\2+0\end{bmatrix} + \begin{bmatrix}0\\0\\-2\end{bmatrix} = \begin{bmatrix}3\\-1\\2\end{bmatrix} + \begin{bmatrix}0\\0\\-2\end{bmatrix} = \begin{bmatrix}3\\-1\\0\end{bmatrix}$$

The covariance matrix is $\mathbf{C}_W = A\mathbf{C}A^T$.

$$\begin{aligned}
\mathbf{C}_W &= \begin{bmatrix}1&1\\1&-1\\2&0\end{bmatrix}\begin{bmatrix}2&-1\\-1&3\end{bmatrix}\begin{bmatrix}1&1&2\\1&-1&0\end{bmatrix}\\
&= \begin{bmatrix}2-1&-1+3\\2+1&-1-3\\4+0&-2+0\end{bmatrix}\begin{bmatrix}1&1&2\\1&-1&0\end{bmatrix}\\
&= \begin{bmatrix}1&2\\3&-4\\4&-2\end{bmatrix}\begin{bmatrix}1&1&2\\1&-1&0\end{bmatrix}\\
&= \begin{bmatrix}1+2&1-2&2+0\\3-4&3+4&6+0\\4-2&4+2&8+0\end{bmatrix} = \begin{bmatrix}3&-1&2\\-1&7&6\\2&6&8\end{bmatrix}
\end{aligned}$$

(b) **Probability** $P(W_2 > 1)$**:** From the mean vector $\mathbf{m}_W$ and covariance matrix $\mathbf{C}_W$ found in part (a), we know the marginal distribution of W_2. The component W_2 is normally distributed with mean $\mu_{W_2} = -1$ and variance $\sigma^2_{W_2} = 7$. We want to find $P(W_2 > 1)$. We standardize the variable:

$$P(W_2 > 1) = P\left(\frac{W_2 - \mu_{W_2}}{\sigma_{W_2}} > \frac{1-(-1)}{\sqrt{7}}\right) = P\left(Z > \frac{2}{\sqrt{7}}\right)$$

where $Z \sim \mathcal{N}(0, 1)$.

$$\frac{2}{\sqrt{7}} \approx \frac{2}{2.646} \approx 0.756$$

So we need $P(Z > 0.756) = 1 - P(Z \leq 0.756) = 1 - \Phi(0.756)$. Using a standard normal table or calculator, $\Phi(0.756) \approx 0.775$.

$$P(W_2 > 1) \approx 1 - 0.775 = 0.225$$

The probability is approximately 22.5%.

Q7: Let the random vector

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix},$$

have

$$E[\mathbf{X}] = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad Cov(\mathbf{X}) = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}.$$

Define two new random variables as

$$Y_1 = 2X_1 - X_2 + 3, \qquad Y_2 = X_1 + 4X_2.$$

(a) Compute $E[Y_1]$ and $E[Y_2]$.

(b) Compute $Var(Y_1)$, $Var(Y_2)$, and $Cov(Y_1, Y_2)$.

(c) Find the covariance matrix of the vector $\mathbf{Y} = [Y_1\ Y_2]^T$.

Solution:

(a) Since expectation is linear,

$$E[Y_1] = 2E[X_1] - E[X_2] + 3 = 2(1) - 2 + 3 = 3, \qquad E[Y_2] = E[X_1] + 4E[X_2] = 1 + 4(2) = 9.$$

$$\boxed{E[Y_1] = 3, \quad E[Y_2] = 9.}$$

(b) We can express $\mathbf{Y}$ as a linear transformation:

$$\mathbf{Y} = A\mathbf{X} + \mathbf{b}, \quad A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

Then,

$$Cov(\mathbf{Y}) = A\, Cov(\mathbf{X})\, A^T.$$

Compute step by step:

$$A\,Cov(\mathbf{X}) = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 8 & 9 \end{bmatrix}.$$

Then,

$$A\,Cov(\mathbf{X})\,A^T = \begin{bmatrix} 7 & 0 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 44 \end{bmatrix}.$$

Thus,

$$Cov(\mathbf{Y}) = \begin{bmatrix} Var(Y_1) & Cov(Y_1, Y_2) \\ Cov(Y_1, Y_2) & Var(Y_2) \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 44 \end{bmatrix}.$$

Hence,

$$\boxed{Var(Y_1) = 14, \quad Var(Y_2) = 44, \quad Cov(Y_1, Y_2) = 7.}$$

(c) Therefore, the covariance matrix of $\mathbf{Y}$ is

$$\boxed{Cov(\mathbf{Y}) = \begin{bmatrix} 14 & 7 \\ 7 & 44 \end{bmatrix}.}$$

Q8: A random vector $\mathbf{X} = [X_1, X_2]^T$ has mean vector $\mu_X = [1, 2]^T$ and covariance matrix $\Sigma_X = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$. Let $Y_1 = X_1 + 2X_2$ and $Y_2 = 3X_1 - X_2$. Find the mean vector and covariance matrix of $\mathbf{Y} = [Y_1, Y_2]^T$.

Solution:

We can express the transformation in matrix form as $\mathbf{Y} = A\mathbf{X}$, where:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$$

(a) **Mean Vector of Y** The mean of $\mathbf{Y}$ is given by the formula $E[\mathbf{Y}] = AE[\mathbf{X}]$.

$$\mu_Y = E[\mathbf{Y}] = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1(1) + 2(2) \\ 3(1) - 1(2) \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

(b) **Covariance Matrix of Y** The covariance matrix of $\mathbf{Y}$ is given by the formula $\Sigma_Y = A\Sigma_X A^T$. First, find the transpose of A:

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

Now, we compute the matrix product:

$$\begin{aligned}
\Sigma_Y &= \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \\
&= \left(\begin{bmatrix} 1(4) + 2(1) & 1(1) + 2(2) \\ 3(4) - 1(1) & 3(1) - 1(2) \end{bmatrix} \right) \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 6 & 5 \\ 11 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 6(1) + 5(2) & 6(3) + 5(-1) \\ 11(1) + 1(2) & 11(3) + 1(-1) \end{bmatrix} \\
&= \begin{bmatrix} 16 & 13 \\ 13 & 32 \end{bmatrix}
\end{aligned}$$

So, the mean vector of $\mathbf{Y}$ is $\mu_Y = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and its covariance matrix is $\Sigma_Y = \begin{bmatrix} 16 & 13 \\ 13 & 32 \end{bmatrix}$.

Q9: Find the expectation of the quadratic form $(X_1 + X_2)^2$ where the random vector $\mathbf{X} = [X_1, X_2]^T$ has mean vector $\mu = [1, -1]^T$ and covariance matrix $\Sigma = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$.

Solution:

We want to find $E[(X_1 + X_2)^2]$. We can expand the expression and use the linearity of expectation:

$$E[(X_1 + X_2)^2] = E[X_1^2 + 2X_1X_2 + X_2^2] = E[X_1^2] + 2E[X_1X_2] + E[X_2^2]$$

We need to find $E[X_1^2]$, $E[X_2^2]$, and $E[X_1X_2]$ from the given mean and covariance matrix. From the definition of variance and covariance:

- $Var(X_i) = E[X_i^2] - (E[X_i])^2 \implies E[X_i^2] = Var(X_i) + (E[X_i])^2$
- $Cov(X_1, X_2) = E[X_1X_2] - E[X_1]E[X_2] \implies E[X_1X_2] = Cov(X_1, X_2) + E[X_1]E[X_2]$

From the given μ and Σ:

- $E[X_1] = 1$, $Var(X_1) = 3$
- $E[X_2] = -1$, $Var(X_2) = 2$
- $Cov(X_1, X_2) = -1$

Now, we calculate the required expected values:

$$E[X_1^2] = Var(X_1) + (E[X_1])^2 = 3 + (1)^2 = 4$$

$$E[X_2^2] = Var(X_2) + (E[X_2])^2 = 2 + (-1)^2 = 3$$

$$E[X_1X_2] = Cov(X_1, X_2) + E[X_1]E[X_2] = -1 + (1)(-1) = -2$$

Finally, substitute these back into the expanded expression:

$$E[(X_1 + X_2)^2] = 4 + 2(-2) + 3 = 4 - 4 + 3 = 3$$

The expectation of the quadratic form is 3.

Q10: Derive the Moment-Generating Function (MGF) for a k-dimensional multivariate normal random vector $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. That is, prove that:

$$M_{\mathbf{X}}(\mathbf{s}) = E[e^{\mathbf{s}^T\mathbf{X}}] = \exp\left(\mathbf{s}^T\boldsymbol{\mu} + \frac{1}{2}\mathbf{s}^T\boldsymbol{\Sigma}\mathbf{s}\right)$$

where $\mathbf{s}$ is a k-dimensional vector of real numbers.

A:
The PDF of a k-dimensional multivariate normal distribution $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is given by:

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^k \det(\boldsymbol{\Sigma})}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$

The MGF is defined as $M_{\mathbf{X}}(\mathbf{s}) = E[e^{\mathbf{s}^T\mathbf{X}}]$. For a continuous random vector, this is:

$$M_{\mathbf{X}}(\mathbf{s}) = \int_{\mathbb{R}^k} e^{\mathbf{s}^T\mathbf{x}} f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}$$

Substituting the PDF:

$$M_{\mathbf{X}}(\mathbf{s}) = \frac{1}{\sqrt{(2\pi)^k \det(\boldsymbol{\Sigma})}} \int_{\mathbb{R}^k} \exp\left(\mathbf{s}^T\mathbf{x} - \frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right) d\mathbf{x}$$

Let's focus on the exponent, denoted E:

$$E = \mathbf{s}^T\mathbf{x} - \frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})$$

Expand the quadratic term:

$$\begin{aligned}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) &= \mathbf{x}^T\boldsymbol{\Sigma}^{-1}\mathbf{x} - \mathbf{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\mathbf{x} + \boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} \\ &= \mathbf{x}^T\boldsymbol{\Sigma}^{-1}\mathbf{x} - 2\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\mathbf{x} + \boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\end{aligned}$$

Substitute this back into E:

$$\begin{aligned}E &= \mathbf{s}^T\mathbf{x} - \frac{1}{2}(\mathbf{x}^T\boldsymbol{\Sigma}^{-1}\mathbf{x} - 2\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\mathbf{x} + \boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}) \\ &= \mathbf{s}^T\mathbf{x} - \frac{1}{2}\mathbf{x}^T\boldsymbol{\Sigma}^{-1}\mathbf{x} + \boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\mathbf{x} - \frac{1}{2}\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} \\ &= -\frac{1}{2}\mathbf{x}^T\boldsymbol{\Sigma}^{-1}\mathbf{x} + (\mathbf{s}^T + \boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1})\mathbf{x} - \frac{1}{2}\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\end{aligned}$$

To complete the square, we introduce a new mean vector $\boldsymbol{\nu}$. Let $\boldsymbol{\nu} = \boldsymbol{\mu} + \boldsymbol{\Sigma}\mathbf{s}$. The quadratic form for this new mean would be:

$$\begin{aligned}-\frac{1}{2}(\mathbf{x}-\boldsymbol{\nu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\nu}) &= -\frac{1}{2}(\mathbf{x}^T\boldsymbol{\Sigma}^{-1}\mathbf{x} - 2\boldsymbol{\nu}^T\boldsymbol{\Sigma}^{-1}\mathbf{x} + \boldsymbol{\nu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}) \\ &= -\frac{1}{2}\mathbf{x}^T\boldsymbol{\Sigma}^{-1}\mathbf{x} + \boldsymbol{\nu}^T\boldsymbol{\Sigma}^{-1}\mathbf{x} - \frac{1}{2}\boldsymbol{\nu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}\end{aligned}$$

Let's check the linear term $\boldsymbol{\nu}^T\boldsymbol{\Sigma}^{-1}\mathbf{x}$:

$$\begin{aligned}\boldsymbol{\nu}^T\boldsymbol{\Sigma}^{-1}\mathbf{x} &= (\boldsymbol{\mu}+\boldsymbol{\Sigma}\mathbf{s})^T\boldsymbol{\Sigma}^{-1}\mathbf{x} \\ &= (\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}+\mathbf{s}^T)\mathbf{x}\end{aligned}$$

This matches the linear term in our expanded exponent E. Therefore, we can write E as:

$$E = -\frac{1}{2}(\mathbf{x}-\boldsymbol{\nu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\nu}) + \frac{1}{2}\boldsymbol{\nu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu} - \frac{1}{2}\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$$

Now we evaluate the constant part $\frac{1}{2}\boldsymbol{\nu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu} - \frac{1}{2}\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$:

$$\begin{aligned}&\frac{1}{2}(\boldsymbol{\mu}+\boldsymbol{\Sigma}\mathbf{s})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}+\boldsymbol{\Sigma}\mathbf{s}) - \frac{1}{2}\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} \\ &= \mathbf{s}^T\boldsymbol{\mu} + \frac{1}{2}\mathbf{s}^T\boldsymbol{\Sigma}\mathbf{s}\end{aligned}$$

Thus, the exponent becomes:

$$E = -\frac{1}{2}(\mathbf{x}-(\boldsymbol{\mu}+\boldsymbol{\Sigma}\mathbf{s}))^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-(\boldsymbol{\mu}+\boldsymbol{\Sigma}\mathbf{s})) + \mathbf{s}^T\boldsymbol{\mu} + \frac{1}{2}\mathbf{s}^T\boldsymbol{\Sigma}\mathbf{s}$$

Substitute this back into the MGF integral:

$$\begin{aligned}M_{\mathbf{X}}(\mathbf{s}) &= \frac{1}{\sqrt{(2\pi)^k\det(\boldsymbol{\Sigma})}}\int_{\mathbb{R}^k}\exp\Big(-\tfrac{1}{2}(\mathbf{x}-(\boldsymbol{\mu}+\boldsymbol{\Sigma}\mathbf{s}))^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-(\boldsymbol{\mu}+\boldsymbol{\Sigma}\mathbf{s})) + \mathbf{s}^T\boldsymbol{\mu} + \tfrac{1}{2}\mathbf{s}^T\boldsymbol{\Sigma}\mathbf{s}\Big)\,d\mathbf{x} \\ &= \frac{\exp\big(\mathbf{s}^T\boldsymbol{\mu}+\frac{1}{2}\mathbf{s}^T\boldsymbol{\Sigma}\mathbf{s}\big)}{\sqrt{(2\pi)^k\det(\boldsymbol{\Sigma})}}\int_{\mathbb{R}^k}\exp\Big(-\tfrac{1}{2}(\mathbf{x}-(\boldsymbol{\mu}+\boldsymbol{\Sigma}\mathbf{s}))^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-(\boldsymbol{\mu}+\boldsymbol{\Sigma}\mathbf{s}))\Big)\,d\mathbf{x}\end{aligned}$$

The integral part, including the $\frac{1}{\sqrt{(2\pi)^k\det(\boldsymbol{\Sigma})}}$ factor, is the integral of a multivariate normal PDF (with mean $\boldsymbol{\mu}' = \boldsymbol{\mu}+\boldsymbol{\Sigma}\mathbf{s}$ and covariance $\boldsymbol{\Sigma}$). Since a PDF integrates to 1:

$$\frac{1}{\sqrt{(2\pi)^k\det(\boldsymbol{\Sigma})}}\int_{\mathbb{R}^k}\exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}')^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}')\right)\,d\mathbf{x} = 1$$

Therefore, the MGF simplifies to:

$$M_{\mathbf{X}}(\mathbf{s}) = \exp\left(\mathbf{s}^T\boldsymbol{\mu} + \frac{1}{2}\mathbf{s}^T\boldsymbol{\Sigma}\mathbf{s}\right)\cdot 1$$

$$M_{\mathbf{X}}(\mathbf{s}) = \exp\left(\mathbf{s}^T\boldsymbol{\mu} + \frac{1}{2}\mathbf{s}^T\boldsymbol{\Sigma}\mathbf{s}\right)$$

This completes the proof.

Q11: Let $\mathbf{X}$ be a standard normal random vector in $\mathbb{R}^n$. Define a new random vector

$$\mathbf{Y} = A\mathbf{X} + b,$$

where A is a square, symmetric, and invertible matrix, and b is a constant shift vector.

(a) Derive an explicit expression for the probability density function of $\mathbf{Y}$ in terms of A, b, and y.

(b) Using your result, compute $f_{\mathbf{Y}}(y)$ numerically for

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad y = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Solution:

The given transformation is

$$\mathbf{Y} = A\mathbf{X} + b.$$

Since A is invertible, the inverse map is

$$\mathbf{X} = A^{-1}(\mathbf{Y} - b).$$

The Jacobian of this transformation is constant:

$$J = \frac{\partial \mathbf{X}}{\partial \mathbf{Y}} = A^{-1},$$

so that

$$|\det(J)| = |\det(A^{-1})| = \frac{1}{|\det(A)|}.$$

By the change-of-variables rule for probability densities,

$$f_{\mathbf{Y}}(y) = f_{\mathbf{X}}(A^{-1}(y - b))\,|\det(J)|.$$

Given that $\mathbf{X}$ follows the standard normal distribution in $\mathbb{R}^n$,

$$f_{\mathbf{X}}(x) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}x^T x\right).$$

Substituting $x = A^{-1}(y - b)$, we obtain

$$f_{\mathbf{Y}}(y) = \frac{1}{(2\pi)^{n/2}|\det(A)|} \exp\left[-\frac{1}{2}(A^{-1}(y - b))^T(A^{-1}(y - b))\right].$$

Simplify the exponent:

$$(A^{-1}(y - b))^T(A^{-1}(y - b)) = (y - b)^T(A^{-T}A^{-1})(y - b) = (y - b)^T(AA^T)^{-1}(y - b).$$

Since A is symmetric, $A^T = A$ and hence $(AA^T)^{-1} = A^{-2}$. Therefore,

$$\boxed{f_{\mathbf{Y}}(y) = \frac{1}{(2\pi)^{n/2}|\det(A)|} \exp\left[-\frac{1}{2}(y - b)^T A^{-2}(y - b)\right].}$$

Numerical Example:

Let

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad y = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Then $\det(A) = 6$ and

$$A^{-2} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/9 \end{bmatrix}.$$

Compute

$$y - b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Therefore,

$$(y-b)^T A^{-2} (y-b) = \frac{2^2}{4} + \frac{3^2}{9} = 1 + 1 = 2.$$

Substituting these values,

$$f_{\mathbf{Y}}(y) = \frac{1}{(2\pi)^1 \cdot 6} \exp\left(-\frac{1}{2} \times 2\right) = \frac{1}{12\pi e}.$$

Final Answer:

$$\boxed{f_{\mathbf{Y}}(y) = \frac{1}{(2\pi)^{n/2} |\det(A)|} \exp\left[-\frac{1}{2}(y-b)^T A^{-2} (y-b)\right], \quad f_{\mathbf{Y}}(3,2) = \frac{1}{12\pi e}.}$$