Probability and Statistics: MA6.101

Homework 6

Topics Covered: Moment Generating Functions, Sums of Random Variables, Stochastic Simulation

Q1: Let $X \sim \mathcal{N}(\mu, \sigma^2)$ be a Gaussian random variable.

(a) Find the moment generating function (MGF) of X.

(b) Using the MGF, compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

Q2: Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with common MGF $M_X(t)$.

(a) Show that the MGF of $Z = X_1 + X_2 + \cdots + X_n$ is $M_Z(t) = (M_X(t))^n$.

(b) Now suppose N is a positive integer-valued random variable, independent of the X_i's, and define

$$Z = X_1 + X_2 + \cdots + X_N.$$

Show that the MGF of Z is

$$M_Z(t) = \mathbb{E}\big[(M_X(t))^N\big].$$

(c) Express $M_Z(t)$ in terms of the MGF of N, $M_N(s)$, and prove that

$$M_Z(t) = M_N(\log M_X(t)).$$

Q3: The moment-generating function of a discrete random variable X is given by:

$$M_X(t) = \frac{1}{6}e^{-2t} + \frac{2}{6}e^{t} + \frac{3}{6}e^{4t}$$

Determine the probability mass function (PMF) of X.

Q4: The lifetime X of an electronic component follows a Weibull distribution, whose Cumulative Distribution Function (CDF) is given by:

$$F_X(x) = 1 - e^{-x^2}, \quad \text{for } x \geq 0$$

Derive a formula to generate a random sample from this distribution using the inverse transform method with a uniform random variable $U \sim \mathrm{Uniform}[0, 1]$.

Q5: Let X be an exponential random variable with parameter $\lambda = 2$, and let Y be a Bernoulli random variable with parameter $p = 0.5$. Assume X and Y are independent.

(a) Let $Z = X + Y$. Find the MGF of Z, denoted $M_Z(t)$.

(b) Use $M_Z(t)$ to find the expected value $\mathbb{E}[Z]$ and the variance $Var(Z)$.

Q6: A Geiger counter measures background radiation. The machine's true average rate of detection, λ, is known to be either low ($\lambda_1 = 0.1$ counts/sec) or high ($\lambda_2 = 0.5$ counts/sec). The prior probability of the rate being low is $P(\lambda = \lambda_1) = 0.8$. The number of counts K in a given second follows a Poisson distribution with parameter λ. You turn on the counter for one second and observe exactly $k = 2$ counts.

What is the updated (posterior) probability that the rate is low, given your observation? That is, find $P(\lambda = \lambda_1 | K = 2)$.

Q7: Let $X_1, \ldots, X_n$ be independent $\text{Exp}(\lambda)$ random variables (rate $\lambda > 0$). Using MGFs, show that the sum

$$Y = X_1 + X_2 + \cdots + X_n$$

has the Gamma distribution $\text{Gamma}(n, \lambda)$.

Q8: Estimate the integral

$$I_2 = \int_0^\infty e^{-x} dx$$

using Monte Carlo integration with N random samples from $U[0, 1]$.

Q9: Let X and Y be two random variables for which $Var(X) = 4$, $Var(Y) = 9$, and the covariance is $Cov(X, Y) = -2$.

(a) Find the variance of the random variable $Z = 3X - 2Y + 5$.

(b) If X and Y were independent, what would $Var(Z)$ be?

Q10: Let X be a discrete random variable with the following probability mass function (PMF):

$$p_X(-1) = 0.2, \quad p_X(0) = 0.5, \quad p_X(1) = 0.3.$$

(a) Find the Moment Generating Function (MGF) of X, denoted by $M_X(t)$.

(b) Using the MGF from part (a), compute the first and second moments of X, i.e., $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$.

(c) Use the results from part (b) to find the variance of X, $Var(X)$.

Q11: Let X be a continuous random variable with the $\text{Gamma}(\alpha, \lambda)$ PDF

$$f_X(x) = \begin{cases} \dfrac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, & x > 0, \\ 0, & \text{otherwise}, \end{cases} \qquad \alpha > 0,\ \lambda > 0.$$

Find the moment generating function $M_X(t) = \mathbb{E}[e^{tX}]$ (specify the region of t for which it is finite).

Q12: Let $X \sim \text{Poisson}(\lambda)$ with $\lambda > 0$.

(a) Find the moment generating function $M_X(t)$.

(b) Use $M_X(t)$ to compute $\mathbb{E}[X]$ and $\text{Var}(X)$.

Q13: Let X and Y be independent random variables with PMFs:

$$p_X(x) = \begin{cases} 1/3, & \text{if } x \in \{1,2,3\}, \\ 0, & \text{otherwise,} \end{cases} \qquad p_Y(y) = \begin{cases} 1/2, & \text{if } y = 0, \\ 1/3, & \text{if } y = 1, \\ 1/6, & \text{if } y = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Find the PMF of $Z = X + Y$, using the convolution formula.

Q14: Use the convolution formula to establish that the sum of two independent Poisson random variables with parameters μ and λ respectively, is Poisson with parameter $\mu + \lambda$.

Q15: We want to estimate $\theta = \mathbb{E}[\sqrt{X}]$ where X is a Normal random variable ($X \sim \mathcal{N}(0,1)$). Using importance sampling, formulate an estimator for θ by drawing N samples, $Y_1, \ldots, Y_N$, from a uniform distribution $Y \sim U[0,5]$.

Q16: Suppose, you want to generate samples from a Gaussian random variable with PDF $p(x)$ for $\mathcal{N}(\mu = 0.5, \sigma^2 = 0.04)$. Use the Accept-Reject method with a proposal distribution $q(x) = U(0,1)$.

(a) Find the smallest constant c such that $p(x) \leq c \cdot q(x)$ for all $x \in [0,1]$.

(b) Outline the algorithm to generate one sample.

Q17: Estimate the integral

$$I_1 = \int_0^a \sqrt{x}\, dx$$

using Monte Carlo integration with N random samples from the uniform distribution $U[0,1]$. Let $a > 0$ be a given constant. Also write the Monte Carlo estimate assuming X_i's are sampled according to some pdf $g(X_i)$ in the same domain.

Q18: Suppose we want to generate a random variable X with a target pdf $f(x)$. We have access to a method for generating a random variable Y from a simpler proposal density function $g(x)$.

Let c be a constant such that the "envelope condition" $\frac{f(y)}{g(y)} \leq c$ holds for all y for which $g(y) > 0$. The rejection method algorithm is as follows:

(a) Generate a candidate sample Y from the proposal density $g(y)$.

(b) Generate a random number U from a Uniform$(0,1)$ distribution.

(c) **Acceptance Condition:** If $U \leq \frac{f(Y)}{cg(Y)}$, then set $X = Y$. Otherwise, reject the sample and return to step 1.

Prove the validity of this algorithm.