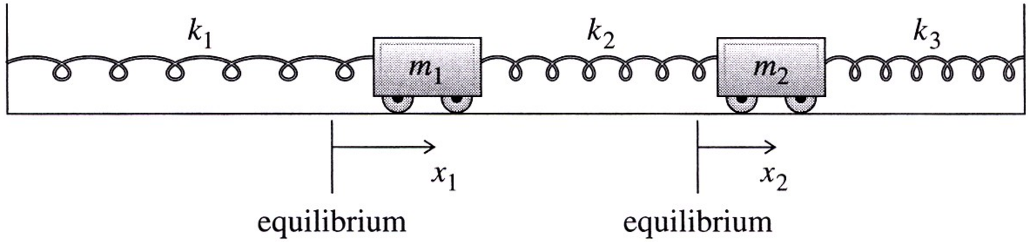
- Eigenvalue and eigenvector
- Leading Eigenvector of a graph
- Normal Mode analysis
- Hessian Matrix

Two Masses and Three Springs

- Here consider a system with two oscillators coupled together. The coupling can be strong or weak.
- We will limit the discussion to oscillators obeying Hooke's Law, and without friction. It is a special case, but one with a wide application.
- Consider the situation shown in the figure at right. There are two cars of masses m_1 and m_2, and three springs of spring constants k_1, k_2 and k_3, and we want to obtain the equations of motion for the two cars.



- We could use the Lagrangian formalism, but let's use the Newtonian approach first. The equilibrium positions of the two cars are shown by the lines, and we will use coordinates x_1 and x_2 relative to those.
- The forces on m_1 are k_1x_1 to the left, and $k_2(x_2 - x_1)$ to the right, so its equation of motion is Likewise:

$$m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1)$$
$$= -(k_1 + k_2)x_1 + k_2x_2,$$

$$m_2\ddot{x}_2 = k_2x_1 - (k_2 + k_3)x_2.$$

https://mathlets.org/mathlets/coupled-oscillators/

http://spiff.rit.edu/classes/phys283/lectures/n_coupled/n_coupled.html

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Two Masses and Three Springs

- The two **coupled** equations of motion:

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$$m_2\ddot{x}_2 = k_2 x_1 - (k_2 + k_3)x_2,$$

can be written more compactly using matrix notation, as $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x},$ where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \qquad \text{and} \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}.$$

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- Notice that this is a generalization of the single oscillator, which you can see by setting k_2 and $k_3 = 0$. You then get a single equation. Note also that if the coupling spring, $k_2 = 0$, then the two equations become uncoupled and describe two separate oscillators.

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Two Masses and Three Springs

- The two **coupled** equations of motion:

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- We will find complex solutions $z(t) = \boldsymbol{a}e^{iwt}$, but you can imagine that we might have more than one frequency of oscillation, since we have two *m*s and 3 *k*s.
- It turns out that we only need to assume one frequency $\omega = \sqrt{k/m},$ initially,

but we will arrive at an equation for *w* that is satisfied by more than one frequency.

- Let's try the solutions: $\mathbf{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega t} = \mathbf{a}e^{i\omega t},$ where $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 e^{-i\delta_1} \\ \alpha_2 e^{-i\delta_2} \end{bmatrix}.$

https://mathlets.org/mathlets/coupled-oscillators/
http://spiff.rit.edu/classes/phys283/lectures/n_coupled/n_coupl

Two Masses and Three Springs

- Here we will keep only the real part of the solution, i.e. $\mathbf{x}(t) = \mathrm{Re}\,\mathbf{z}(t)$.

https://mathlets.org/mathlets/coupled-oscillators/

http://spiff.rit.edu/classes/phys283/lectures/n_coupled/n_coupled.html

Two Masses and Three Springs

- Here we will keep only the real part of the solution, i.e. $\mathbf{x}(t)$ = Re $\mathbf{z}(t)$.

- When we substitute $\mathbf{z}(t)$ into $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x},$ we find the relation $-\omega^2\mathbf{M}\mathbf{a}e^{i\omega t} = -\mathbf{K}\mathbf{a}e^{i\omega t},$

  so the following equation must be satisfi $\boxed{(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0.}$

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  so the following equation must be satisfi $\boxed{(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0.}$

- Clearly, if we ignore the trivial solution $\mathbf{a} = 0$ (no motion at all), we must $\boxed{\det(\mathbf{K} - \omega^2\mathbf{M}) = 0.}$

  This is a generalization of the eigenvalue problem.

- This implies that there are two frequencies at which our system will oscillate, and these are called **normal modes** of the system. Lets deal with simpler systems

Equal Masses and Identical Springs

- For this case, the matrices **M** and **K** become:

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \qquad \text{and} \quad \mathbf{K} = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}.$$

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- Thus

$$\mathbf{K} - \omega^2\mathbf{M} = \begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{bmatrix},$$

Find the determinant

Equal Masses and Identical Springs

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- Thus

$$\mathbf{K} - \omega^2\mathbf{M} = \begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{bmatrix},$$

- To obtain the eigen value one may

$$\det(\mathbf{K} - \omega^2\mathbf{M}) = (2k - m\omega^2)^2 - k^2 = (k - m\omega^2)(3k - m\omega^2),$$

$$\boxed{\omega = \sqrt{\frac{k}{m}} = \omega_1, \qquad \text{and} \quad \omega = \sqrt{\frac{3k}{m}} = \omega_2.}$$

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0.$$

First Normal Modes

- First insert $\omega_1 = \sqrt{k/m}$, so that the relation

$$\mathbf{K} - \omega_1^2 \mathbf{M} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}.$$

- As a check, you can see that the determinant of this matrix is zero. The solutions are then

$$(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{a} = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \quad \text{or} \quad \begin{aligned} a_1 - a_2 &= 0 \\ -a_1 + a_2 &= 0 \end{aligned}.$$

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- These are both the same equation, and simply says that $a_1 = a_2 = Ae^{-i\delta}$. Since

$$\mathbf{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega_1 t} = \begin{bmatrix} A \\ A \end{bmatrix} e^{i(\omega_1 t - \delta)},$$

we finally have

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A \\ A \end{bmatrix} \cos(\omega_1 t - \delta).$$

- That is, both carts move in un

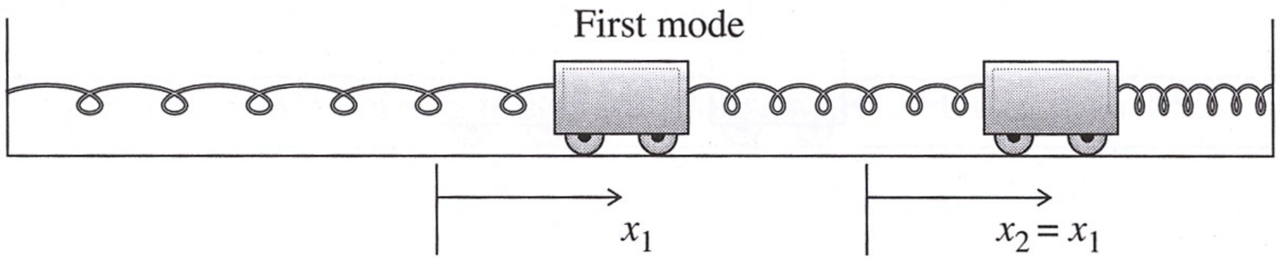
$$\boldsymbol{x_1(t) = A\cos(\omega_1 t - \delta)}$$
$$\boldsymbol{x_2(t) = A\cos(\omega_1 t - \delta)}$$

[first normal mode].

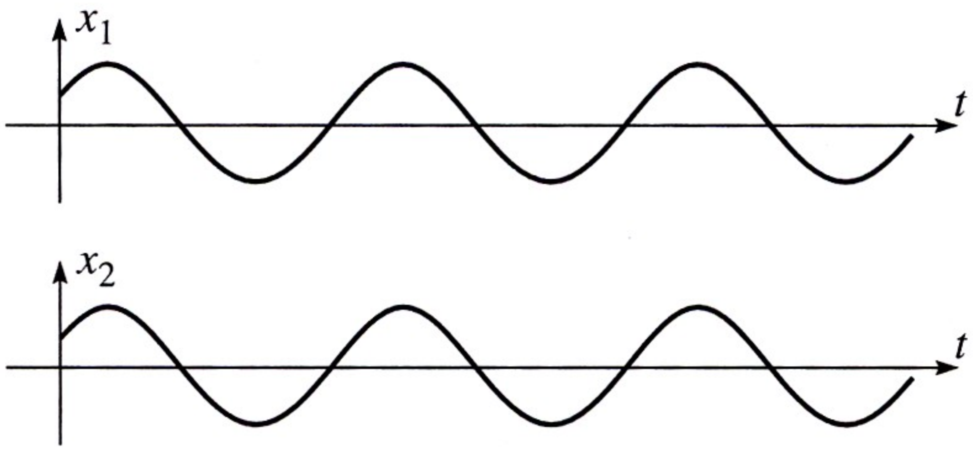
Plot, for t=[0 10]

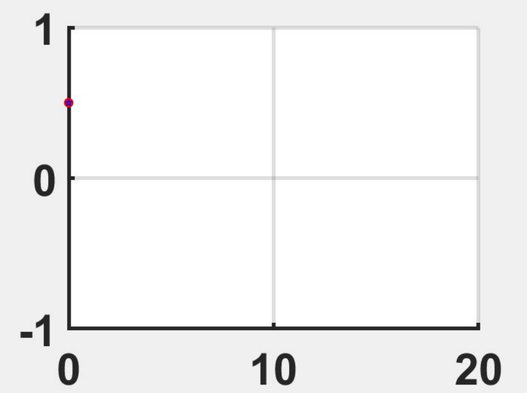
First Normal Modes

- The motion is shown in the figure below. Notice that the spring between the two carts does not stretch or contract at all.



- When we plot the motion, it looks like this (identical motions, in phase).





First Normal Modes

```
tspan=20;
k1=1; k2=1; m1=1 ; m2=1.0; k3=1;
%%%%% Ode solve %%%
x1=0.5;
x1_prime=0;
x2=0.5;
x2_prime=0;
y0=[x1 x1_prime x2 x2_prime];
[t,y]=ode45(@(t,y) spring_mass_model(t,y,k1,k2,k3,m1,m2), [0,tspan], y0);
%%%%%%%%%%%% theoretical calculation %%%%
%k1=1; k2=1; m1=1 ; m2=1.0; k3=1;
A1=0.5; A2=0.5; delta_1=0.0; delta_2=0;
omega_1=1;% t=0:01:100;
omega_2=sqrt(3)*omega_1;
%%First Normal Mode
sp1=A1*cos(omega_1*t-delta_1);
sp2=A2*cos(omega_1*t-delta_2);
```

```
function [yprime] =
spring_mass_model(t,y,k1,k2,k3,m1,m2)
y_prime=zeros(4,1);
yprime(1) = y(2);
yprime(3)= y(4) ;
yprime(2)= (1/m1)*(-k1*y(1)+k2*(y(3)-y(1)));
yprime(4)= (1/m2)*(-k3*y(3)+k2*(y(1)-y(3)));
yprime=yprime';
end
```

First Normal Modes

```
tspan=20;
k1=1; k2=1; m1=1 ; m2=1.0; k3=1;
%%%%% Ode solve %%%
x1=0.5;
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x2_prime=0;
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sp1=A1*cos(omega_1*t-delta_1);
sp2=A2*cos(omega_1*t-delta_2);
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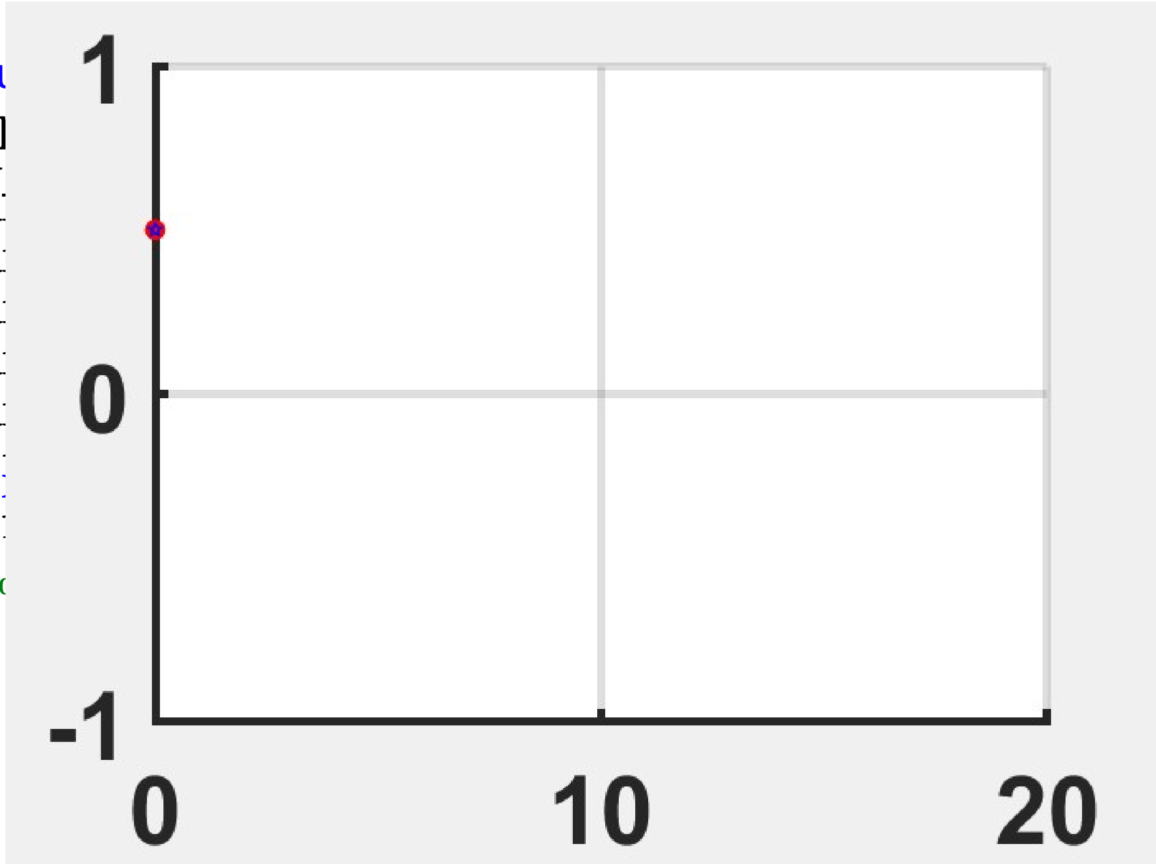

Second Normal Modes

- insert $\omega_2 = \sqrt{3k/m}$, so that the relation $\mathbf{K} - \omega_1^2 \mathbf{M} = \begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix}.$

- Thus

$$(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{a} = -k \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \quad \text{or} \quad \begin{aligned} a_1 + a_2 &= 0 \\ a_1 + a_2 &= 0 \end{aligned}.$$

$$a_1 = -a_2 = Ae^{-\delta}.$$

Second Normal Modes

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- Thus $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = -k\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$ or $\begin{aligned} a_1 + a_2 &= 0 \\ a_1 + a_2 &= 0 \end{aligned}.$

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- Since

$$\mathbf{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega_2 t} = \begin{bmatrix} A \\ -A \end{bmatrix} e^{i(\omega_2 t - \delta)},$$

we finally have $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A \\ -A \end{bmatrix} \cos(\omega_2 t - \delta).$

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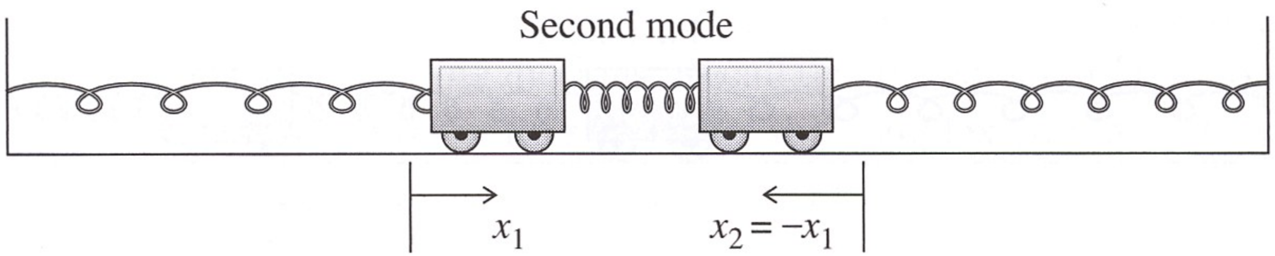
Plot, for t=[0 10]

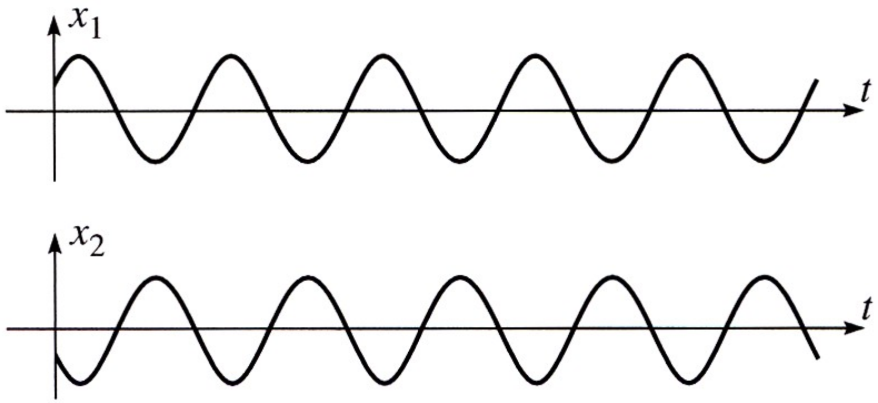
That is, both carts move oppositely: $\begin{matrix} x_1(t) = A\cos(\omega_2 t - \delta) \\ x_2(t) = -A\cos(\omega_2 t - \delta) \end{matrix}$ [2nd normal mode].

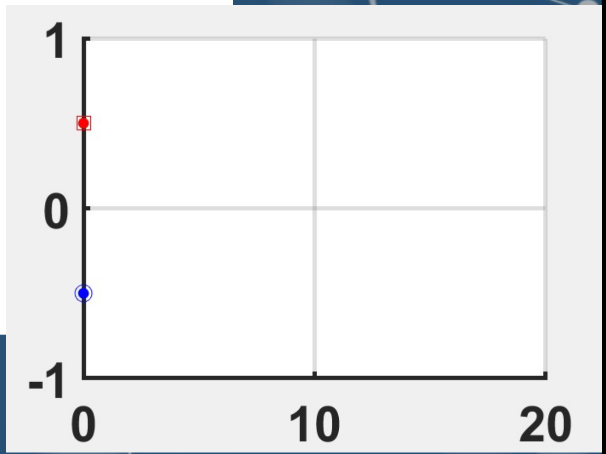
Second Normal Modes

- The motion is shown in the figure below. Notice that the spring between the two carts stretches and contracts, contributing to the higher force, and hence, higher frequency



- When we plot the motion, it looks like this (identical motions, in phase, but at a higher frequency ω_2)





Second Normal Modes

```
tspan=20;
k1=1; k2=1; m1=1 ; m2=1.0; k3=1;
%%%%% Ode solve %%%
x1=0.5;
x1_prime=0;
x2=-0.5;
x2_prime=0;
y0=[x1 x1_prime x2 x2_prime];
```

```
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```

```
[t,y]=ode45(@(t,y) spring_mass_model(t,y,k1,k2,k3,m1,m2), [0,tspan], y0);
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%k1=1; k2=1; m1=1 ; m2=1.0; k3=1;
A1=0.5; A2=0.5; delta_1=0.0; delta_2=0;
omega_1=1;% t=0:01:100;
omega_2=sqrt(3)*omega_1;
%%First Normal Mode
sp1=A1*cos(omega_2*t-delta_1);
sp2=-A2*cos(omega_2*t-delta_2);
```

General Motion

- It is important to realize that, although these are the only two normal modes for the oscillation, the general oscillation is a combination of these two modes, with possibly different amplitudes and phases depending on initial conditions.

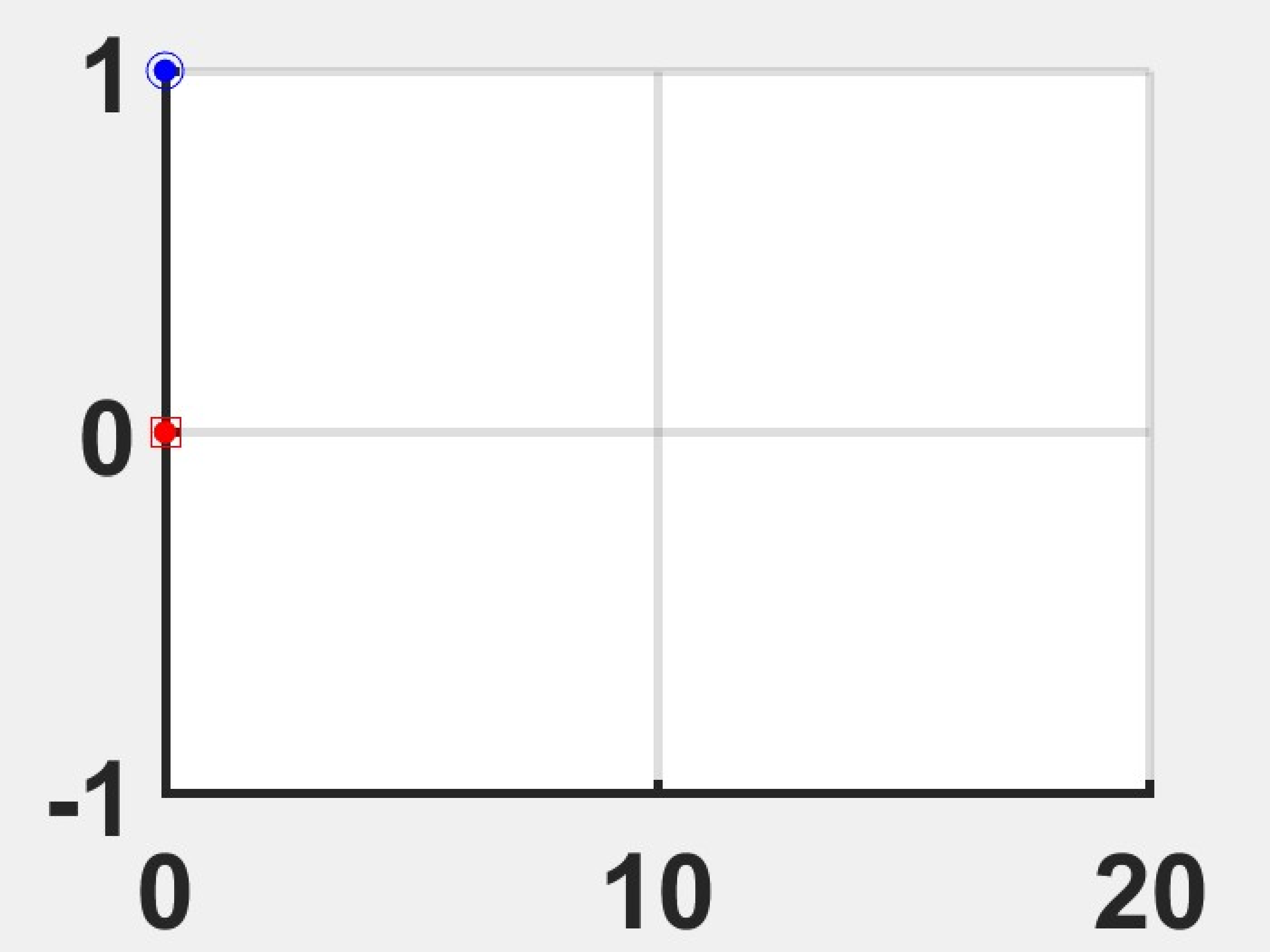
$$\mathbf{x}(t) = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t - \delta_1) + A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_2 t - \delta_2).$$

General Motion

- It is importa… des for the oscillation, t… ith possibly different am…

- The resultin… use $\omega_2 = \sqrt{3}\omega_1$

  is an iratio… at either A_1 or A_2 = 0.



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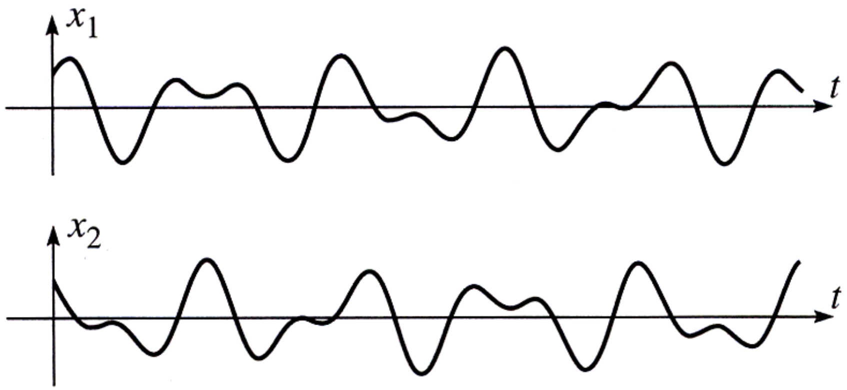
- The resulting motion is surprisingly complicated, but deterministic. Because $\omega_2 = \sqrt{3}\omega_1$ is an irrational ratio, the motion never repeats itself, except in the case that either A_1 or A_2 = 0.

Motion for
$A_1 = 1,\ \delta_1 = 0$
$A_2 = 0.7,\ \delta_2 = \pi/2$



- m_i: Mass of the i-th particle.
- k_j: Spring constant of the j-th spring.
- x_i: Displacement of the i-th mass from equilibrium.

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- k_j: Spring constant of the j-th spring.
- x_i: Displacement of the i-th mass from equilibrium.

Equations of Motion (Using Newton's Second Law)

For each mass i, we write the equation of motion:

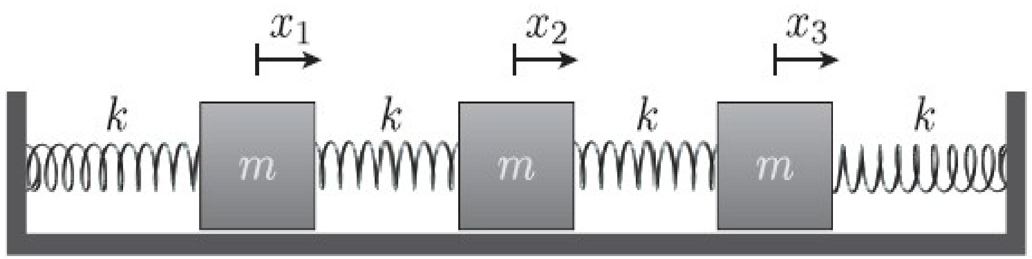
$$m_i \ddot{x}_i = -k_{i-1}(x_i - x_{i-1}) + k_i(x_{i+1} - x_i)$$

where:

- The term $-k_{i-1}(x_i - x_{i-1})$ is the restoring force from the left spring.
- The term $+k_i(x_{i+1} - x_i)$ is the restoring force from the right spring.

For the first and last masses, we adjust the equation accordingly to account for boundary conditions.

$$\begin{pmatrix} (-m\omega^2+2k) & -k & 0 \\ -k & (-m\omega^2+2k) & -k \\ 0 & -k & (-m\omega^2+2k) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$$



$$\begin{vmatrix} -m\omega^2+2k & -k & 0 \\ -k & -m\omega^2+2k & -k \\ 0 & -k & -m\omega^2+2k \end{vmatrix} = 0.$$

Expanding about the top row,

$$(-m\omega^2+2k)[(-m\omega^2+2k)^2-k^2]+k(-k(-m\omega^2+2k))=0.$$

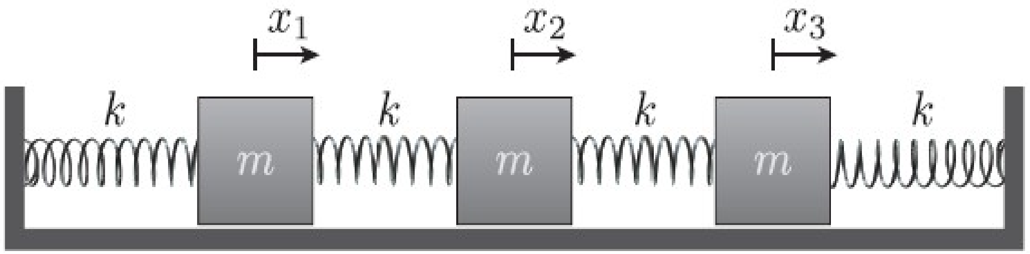
Factoring,

$$\omega = \omega_1 = \sqrt{\frac{2k}{m}}.$$

$$(-m\omega^2+2k)[(-m\omega^2+2k)^2-2k^2]=0,$$

$$\omega_2 = \sqrt{\frac{(2-\sqrt{2})k}{m}} \quad \text{and} \quad \omega_3 = \sqrt{\frac{(2+\sqrt{2})k}{m}}.$$

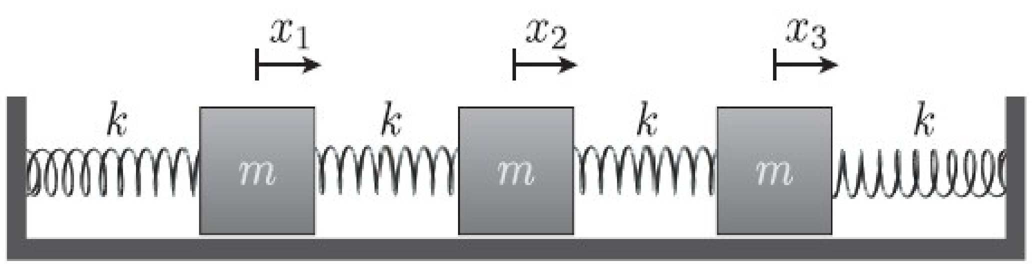
$$\begin{pmatrix} (-m\omega^2+2k) & -k & 0 \\ -k & (-m\omega^2+2k) & -k \\ 0 & -k & (-m\omega^2+2k) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$$



As with the two-block problem, we can find the three normal mode motions either intuitively or mathematically. Intuitively, it is clear that ω_1 corresponds to the frequency when the center block remains at rest and the outer blocks oscillate oppositely to one another, both moving outwards and then both moving inwards, etc. Each is connected to two springs whose opposite ends stay at rest, so the frequency should be $\omega = \sqrt{2k/m}$. This motion can be verified by substituting ω_1 into the algebraic equations, confirming that $A_3 = -A_1$ and $A_2 = 0$. The eigenfrequency ω_2 corresponds to the two outer blocks moving together in phase, with the middle block moving in the same direction with a different amplitude; and the eigenfrequency ω_3 corresponds to the two outer blocks moving together in phase, with the middle block moving always in the opposite direction, and with a different amplitude.

$$\omega = \omega_1 = \sqrt{\frac{2k}{m}}.$$

$$\begin{pmatrix} (-m\omega^2+2k) & -k & 0 \\ -k & (-m\omega^2+2k) & -k \\ 0 & -k & (-m\omega^2+2k) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$$



Let us find the exact motion for the second eigenfrequency ω_2. The algebraic equations then become

$$\begin{aligned} (-(2-\sqrt{2})k+2k)A_1 - kA_2 + 0\cdot A_3 &= 0 \\ -kA_1 + (-(2-\sqrt{2})+2k)A_2 - kA_3 &= 0 \\ 0\cdot A_1 - kA_2 + (-(2-\sqrt{2}+2k)A_3 &= 0, \end{aligned}$$

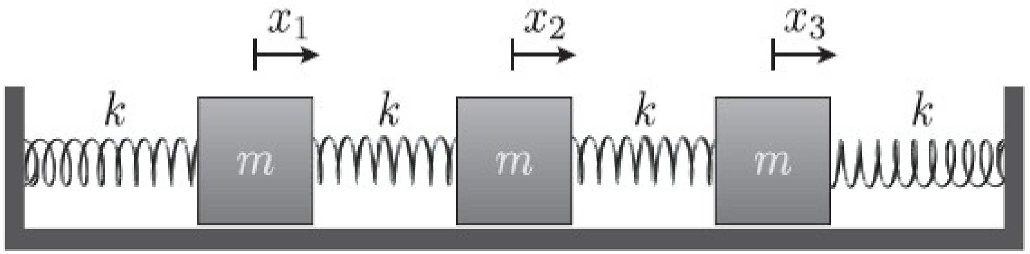
$$\omega_2 = \sqrt{\frac{(2-\sqrt{2})k}{m}}$$

so

$$A_2 = \sqrt{2}A_1, \qquad -kA_1 + \sqrt{2}A_2 - kA_3 = 0, \qquad A_2 = \sqrt{2}A_3$$

which reduce to $A_2 = \sqrt{2}A_1 = \sqrt{2}A_3$. So in this mode the two outer blocks oscillate in phase with equal amplitudes $A_3 = A_1$, while the middle block oscillates with the *same* phase, but with an amplitude that is larger by the factor $\sqrt{2}$.

$$\begin{pmatrix} (-m\omega^2+2k) & -k & 0 \\ -k & (-m\omega^2+2k) & -k \\ 0 & -k & (-m\omega^2+2k) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$$



Let us find the exact motion for the second eigenfrequency ω_2. The algebraic equations then become

$$\omega_2 = \sqrt{\frac{(2-\sqrt{2})k}{m}} \quad \text{and} \quad \omega_3 = \sqrt{\frac{(2+\sqrt{2})k}{m}}.$$

In a similar way we find for the third eigenfrequency $\omega_3 = \sqrt{(2+\sqrt{2})k/m}$ that the two outer blocks oscillate in phase with $A_3 = A_1$, while the middle block oscillates with the *opposite* phase, with an amplitude $A_2 = -\sqrt{2}A_1$.