Theory Assignment I

Discrete Structures Monsoon 2024, IIIT Hyderabad

December 5, 2024

Total Marks: 30 points

1. [5 points] Consider the following compound proposition:

$$((p \to q) \land (q \to r)) \to (p \to r)$$

   1. Construct a truth table for the proposition
   2. Prove that the expression is a tautology by using the following steps:
      (a) Express the proposition using only $\neg, \lor$ and $\land$
      (b) Apply De Morgan's laws and distributive properties to simplify the expression

Solution:

1. (2 Points if fully correct)

p	q	r	$p \to q$	$q \to r$	$((p \to q) \land (q \to r))$	$(p \to r)$	$((p \to q) \land (q \to r)) \to (p \to r)$
F	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	T	T	T	T	T	T	T
T	F	F	F	T	F	F	T
T	F	T	F	T	F	T	T
T	T	F	T	F	F	F	T
T	T	T	T	T	T	T	T

2. (a) (1 Point)

$$\begin{aligned}((p \to q) \land (q \to r)) \to (p \to r) &\equiv \neg ((p \to q) \land (q \to r)) \lor (p \to r) \\ &\equiv \neg ((\neg p \lor q) \land (\neg q \lor r)) \lor (\neg p \lor r)\end{aligned}$$

(b) (2 Points)

$$\begin{aligned}
((p \to q) \land (q \to r)) \to (p \to r) &\equiv \neg\left((\neg p \lor q) \land (\neg q \lor r)\right) \lor (\neg p \lor r) \\
&\equiv \neg(\neg p \lor q) \lor \neg(\neg q \lor r) \lor \neg p \lor r \\
&\equiv (p \land \neg q) \lor \neg p \lor (q \land \neg r) \lor r \\
&\equiv ((p \lor \neg p) \land (\neg q \lor \neg p)) \lor ((q \lor r) \land (\neg r \lor r)) \\
&\equiv (T \land (\neg q \lor \neg p)) \lor ((q \lor r) \land T) \\
&\equiv \neg q \lor \neg p \lor q \lor r \\
&\equiv T
\end{aligned}$$

2. [10 points] Consider the following propositional logic formula:

$$\neg(\neg(x_1 \to x_2) \lor \neg(x_1 \to x_3)) \land (\neg x_2 \to (\neg(x_4 \to x_3))) \land (\neg(\neg x_4 \land \neg x_5))$$

a. Construct a truth table for all possible values of the clauses to find an assignment that satisfies this formula **(4.5 points)**

b. We have a machine that can construct one row of the truth table in one step. It does this as follows: It generates binary strings by counting up in Binary from 00000 to 11111. A 0 in the i^{th} position represents a False value for the i^{th} propositional variable whereas a 1 represents True. How many steps would be required to reach an assignment of the propositional variables such that the whole formula evaluates to true? What about the worst case? What about a general formula with n propositions? **(1.5 points)**

c. Now convert the propositional logic formula to Conjunctive Normal Form. Each part of the formula separated by a conjunction is called a clause. **(2 points)**

d. Now enumerate the steps required for evaluating this new expression with a smarter machine. It generates bit strings in the same order, but any time one of the clauses becomes false, it skips all strings that have the assignment of variables that made the clause become false.
So for example, if there was a clause of the form $(\neg x_3 \lor x_5)$ which would become false under the assignment "00100", all strings of the form "xx1x0" would be skipped.
How many steps are required to solve the problem now? (Why does this work?) **(2 points)**

Solution:

1. 4.5 marks if fully correct. 0.25 marks lost for every row of the table that is wrong.

x_1	x_2	x$_3$	x_4	x_5	Formula
F	F	F	F	F	F
F	F	F	F	T	F
F	F	F	T	F	T
F	F	F	T	T	T
F	F	T	F	F	F
F	F	T	F	T	F
F	F	T	T	F	F
F	F	T	T	T	F
F	T	F	F	F	F
F	T	F	F	T	T
F	T	F	T	F	T
F	T	F	T	T	T
F	T	T	F	F	F
F	T	T	F	T	T
F	T	T	T	F	T
F	T	T	T	T	T
T	F	F	F	F	F
T	F	F	F	T	F
T	F	F	T	F	F
T	F	F	T	T	F
T	F	T	F	F	F
T	F	T	F	T	F
T	F	T	T	F	F
T	F	T	T	T	F
T	T	F	F	F	F
T	T	F	F	T	F
T	T	F	T	F	F
T	T	F	T	T	F
T	T	T	F	F	F
T	T	T	F	T	T
T	T	T	T	F	T
T	T	T	T	T	T

2. 0.5 marks for each answer.
   (a) 3 steps are required in the given case. If answer wrong due to truth table but logic explained correctly, can give upto 0.25.
   (b) 32 steps are required in the worst case.
   (c) 2^n steps are required in the worst case for a general proposition with n variables.
3. 1 mark for the final answer. 1 mark for the simplification process. Final answer for CNF:

$$(\neg x_1 \vee x_2) \wedge (\neg x_1 \vee x_3) \wedge (x_2 \vee x_4)(x_4 \vee x_5) \wedge (x_2 \vee \neg x_3)$$

3. 0.75 marks for number of steps (2 steps are required now). 1.25 marks for explanation (give 0.5 marks if explained the same as in question about skipping. If mentioned CNF give 0.25. If

mentioned that once a clause becomes false in CNF it can never be true again due to the nature of the machine give 0.5 more)

4. [3 points] Answer the following questions

   a. Prove the exportation law, that is

$$((A \land B) \to C) \Leftrightarrow (A \to (B \to C))$$

   b. You are given three assumptions as follows

   - $A \lor (B \to C)$
   - $[B \to (B \to C)] \to (D \lor E)$
   - $(D \to A) \land (E \to F)$

   Prove that the expression below is true

$$A \lor F$$

   You are allowed to use any proof techniques and rules of inference, as long as they are valid. (**Hint :** You might need to use the exportation law).

Solution:

a.

$$\begin{aligned} A \to (B \to C) &\Leftrightarrow \neg A \lor (B \to C) \\ &\Leftrightarrow \neg A \lor (\neg B \lor C) \\ &\Leftrightarrow (\neg A \lor \neg B) \lor C \\ &\Leftrightarrow \neg(A \land B) \lor C \\ &\Leftrightarrow (A \land B) \to C \end{aligned}$$

Truth Table solutions also work, but this would be the easiest way to show the equivalence.

b. We shall use an indirect proof technique. Assume that $A \lor F$ is not true. Therefore $\neg(A \lor F)$ is true. This implies that $\neg A \land \neg F$ is true. This implies that $\neg A$ has to be true (or A is false). Now from the first assumption we know that either A is true, or $(B \to C)$ is true. Since we already showed that A is false. Therefore $(B \to C)$ has to be true. We can rewrite $(B \to C)$ as $((B \land B) \to C)$. We can apply the exportation rule here, therefore we get that $(B \to (B \to C)$ has to be true. From the second assumption, we get that $(D \lor E)$ has to be true. Now from the third assumption, we see that $A \land F$ has to be true, since D implies A and E implies F, and we know that one of antecedents is true, therefore one of the consequents has to be true. This gives us a contradiction, since we assumed that $A \land F$ is false. Therefore, we have shown by contradiction that $A \land F$ has to be true.

5. [5 points] Assume your friend took out a corner square from your n $\times$ n chessboard. Is it still possible to use dominoes to cover the remaining squares? (A domino is a rectangular tile that covers

two neighboring chessboard squares in a game.) What requirement does n have to meet? Describe the necessary and sufficient conditions (i.e., the precise values of n that work and those that don't). Prove your answer.

Solution:

[0.5pts] for writing something relevant

A chessboard has n^2 squares. Domino covers 2 squares. Removing 1 square leaves n^2 -1 squares to cover.

[0.5pts] A key observation is that dominoes cover one white and one black square. Thus, for the board to be covered entirely with dominoes, the number of white and black squares must be equal.

Coloring the Chessboard Color the chessboard in a checkerboard pattern, with the top-left corner being black.

[0.5pts] An $n \times n$ board has:

- $\left\lceil \frac{n^2}{2} \right\rceil$ black squares if n is odd.
- $\frac{n^2}{2}$ black squares if n is even.

Effect of Removing a Corner Removing a corner square removes one black square if the corner is black and one white square if the corner is white.

Analysis [1.5 pts]

Even n**:** An $n \times n$ board has an equal number of black and white squares. Removing a corner will always disrupt this balance, resulting in $\frac{n^2}{2} - 1$ squares of one color and $\frac{n^2}{2}$ squares of the other color. Since dominoes require an equal number of each color, the board cannot be completely covered with dominoes.

Odd n**:** An $n \times n$ board has one more square of one color than the other, specifically $\left\lceil \frac{n^2}{2} \right\rceil = \frac{n^2+1}{2}$. Removing a corner, which will always remove one square of the more numerous color, results in an equal number of black and white squares, $\frac{n^2-1}{2}$ of each color. Thus, the board can be completely covered with dominoes.

Conclusion The board can be completely covered with dominoes if and only if n is odd. This is the necessary and sufficient condition for the board to be covered with dominoes after removing a corner square.

Proof [1.5 pts]

For even n**:**

$$\text{Number of squares} = n^2,$$
$$n^2 \equiv 0 \pmod{2}.$$

$$\text{Number of remaining squares} = n^2 - 1,$$
$$n^2 - 1 \equiv 1 \pmod{2}.$$

An odd number of squares cannot be fully covered with dominoes, each covering 2 squares.

For odd n:

$$\text{Number of squares} = n^2,$$
$$n^2 \equiv 1 \pmod{2}.$$

$$\text{Removing one square gives:}$$
$$n^2 - 1 \equiv 0 \pmod{2}.$$

This is even, and since we end up with an equal number of black and white squares, dominoes can cover the board completely.

[0.5pts]Therefore, only when n is odd can the board be covered with dominoes after removing a corner square.

6. [3 points] Show that if n is an integer and $n^3 + 5$ is odd, then n is even using

   a a proof by contraposition.

   b a proof by contradiction.

Solution:

a. (1.5 points - binary grading) We just need to show that if n is odd then, $n^3 + 5$ is even, we can do this as follows. Notice that if n is odd, then n^3 has to be odd. Now if n^3 is odd, then $n^3 + 5$ has to be even (Since odd + odd = even).

b. (1.5 points - binary grading) Assume that n is odd, therefore you can write it as $n = 2k+1$, now calculate $n^3 + 5 = (2k+1)^3 + 5 = 8k^3 + 12k^2 + 6k + 6 = 2(4k^3 + 6k^2 + 3k + 3)$ which is even, but we had to show that this was odd, therefore we have a contradiction, which implies that the assumption was wrong, therefore n has to be even.

7. [4 points] Let $a_1, a_2, ...a_n \in \mathbb{R}^+$. Prove using mathematical induction that

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geq \left(\prod_{i=1}^{n} a_i\right)^{1/n}.$$