Real analysis
Assignment 2
Due: 9 November 2024 before 11:59 pm

1. (5 points) Find an example of a sequence of real numbers satisfying each set of properties:
   1. Cauchy but not monotone
   2. Monotone but not Cauchy
   3. Bounded but not Cauchy
2. (5 points) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{x^3}{1+x^2}$. Show that f is continuous on $\mathbb{R}$. Is f uniformly continuous on $\mathbb{R}$?
3. (5 points) Let (a_n) and (b_n) be bounded sequences of real numbers. Define a sequence (c_n) by $c_n = a_n b_n$. Show that if $\limsup a_n$ and $\limsup b_n$ are negative, then $\limsup c_n = \liminf(a_n) \cdot \liminf(b_n)$.
4. (5 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $k \in \mathbb{R}$. Prove that the set $f^{-1}(k)$ is closed.
5. (10 points) Let X be a metric space. Then show the following
   1. Any subset of a nowhere dense set is nowhere dense.
   2. The union of finitely many nowhere dense sets is nowhere dense.
   3. The closure of a nowhere dense set is nowhere dense.
   4. If X has no isolated points, then every finite set is nowhere dense.
6. (10 points) Let (a_n) be a sequence. Let (b_n) be a nondecreasing convergent sequence of positive numbers such that $|a_{n+1} - a_n| \leq b_{n+1} - b_n$. Show that (a_n) is a Cauchy sequence.
7. (10 points) If f is a continuous mapping of a metric space X into a metric space Y , prove that $f(\overline{E}) \subset \overline{f(E)}$ for every set $E \subset X$. (Here $\overline{A}$ denotes the closure of set A.)