Real analysis
Final solutions (Fall 2024)
Duration: 2 hours
Maximum marks: 100

Question 1: (6 marks) Give two examples of sets that are connected but not path connected (No justification required; just examples).

Solution:

1. Topologist's sine curve:

$$T = \{(x, \sin(\frac{1}{x}) : x \in (0,1]\} \cup \{(0,0)\}$$

   (3 marks)

2. Infinite broom: All closed line segments joining the origin to the point $(1, \frac{1}{n})$ as n varies over all positive integers, together with the interval $(\frac{1}{2}, 1]$ on the x-axis. (3 marks)

Question 2: (10 marks) Let $A \subseteq \mathbb{R}$ and $(f_n(x))_{n\in\mathbb{N}}$ be a sequence of functions from $A \to \mathbb{R}$.

1. What does it mean to say that the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges pointwise on A to a function $f : A \to \mathbb{R}$?

2. What does it mean to say that the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges uniformly on A to a function $f : A \to \mathbb{R}$?

Solution:

1. For every $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for every $n \geq N_0$, $|f_n(x) - f(x)| < \epsilon$ for every $x \in A$. (5 marks)

2. For every $\epsilon > 0$, for every $x \in A$, there exists $N_0 \in \mathbb{N}$ such that for every $n \geq N_0$, $|f_n(x) - f(x)| < \epsilon$. (5 marks)

Question 3: (12 marks)
Let $A, B \subset \mathbb{R}^n$. State True or False with justification:

1. A is compact and B is closed implies $A \cap B$ is compact.

2. A is open and B is closed implies $A \cap B$ is closed.

3. Let X and Y be metric spaces, and $f : X \to Y$ a continuous function. If $B \subseteq X$ is bounded, then $f(B)$ is bounded.

4. Let $(f_n(x))_{n\in\mathbb{N}}$ be a sequence of continuous functions that converge pointwise to a function $f : X \to \mathbb{R}$. If f is continuous, then the functions must converge uniformly.

Solution:

1. True (1 mark)

   Indeed if A compact and B closed, then A is closed, so $A \cap B$ is also closed. Then $A \cap B$ is a closed subset of A which is compact, hence $A \cap B$ is compact (2 marks)

2. False (1 mark).

   Indeed, let $A = (0, 2)$ and $B = [1, 2]$. Then $A \cap B = [1, 2)$ is neither open nor closed. (2 marks)

3. False (1 mark)

   Indeed consider $f(x) = \frac{1}{x}$ which is continuous on $(0, 1)$. Let $B = (0, 1)$. Then $f(B)$ is not bounded.

4. False (1 mark)

   For example, consider the function

$$\begin{aligned} f_n &: (0,1) \to \mathbb{R} \\ f_n(x) &= x^n. \end{aligned} \tag{1}$$

   Then the sequence converges to the constant function $f(x) = 0$, but convergence is not uniform.

Question 4 (10 marks)
Find the pointwise limit of the sequence $f_n(x) = \frac{e^x}{n} (n \in \mathbb{N})$ on $\mathbb{R}$. Is this convergence uniform?

Solution:

For fixed $x \in \mathbb{R}$, $\lim_{n \to \infty} \dfrac{e^x}{n} = 0$. So the pointwise limit of this sequence if $f(x) = 0$. (3 marks).
The convergence is not uniform (2 mark)
Let $\epsilon = 1$, then for all $n \in \mathbb{N}$, there exists $x_N = \log(N) \in \mathbb{R}$ and $n (= N) \in \mathbb{N}$ such that $|f_n(x_N) - f_0(x_N)| = |\frac{e^{x_N}}{N} - 0| = |1 - 0| = 1 \geq \epsilon$. (5 marks)

Question 5: (10 marks)
Let $f : A \to \mathbb{R}$ be continuous on A. If $K \subseteq A$ is compact, show that $f(K)$ is also compact.

Solution:

Let $\{V_\alpha\}_{\alpha \in I}$ be an open cover of $f(K)$. Thus, $f(K) \subseteq \bigcup_{\alpha \in I} V_\alpha$. (2 marks).

This implies that $K \subseteq f^{-1}(\bigcup_{\alpha \in I} V_\alpha) = \bigcup_{\alpha \in I} f^{-1}(V_\alpha)$. (3 marks)

Since f is continuous, each $f^{-1}(V_\alpha)$ is an open subset of X (1 mark).

Since K is compact and $K \subseteq f^{-1}(\bigcup_{\alpha \in I} V_\alpha)$, there exists $n \in \mathbb{N}$, with $K \subseteq f^{-1}(\bigcup_{j=1}^{n} V_{\alpha_j})$ for some $\alpha_1, \alpha_2, \cdots, \alpha_n \in I$. (3 marks)
Hence $f(K) \subseteq \bigcup_{j=1}^{n} V_{\alpha_j}$ and $f(K)$ is compact. (1 mark)

Question 6: (12 marks)
Consider the set $S = [0, 1) \cup [2, 3)$. Classify each of the points $\{0\}, \{1\}, \{2\}, \{3\}, \{1.5\}$ as

1. Boundary

2. Interior

3. Accumulation
4. Adherent

Solution:

1. $\{0\}$: Boundary, Accumulation, Adherent.
2. $\{1\}$: Boundary, Accumulation, Adherent.
3. $\{2\}$: Boundary, Accumulation, Adherent.
4. $\{3\}$: Boundary, Accumulation, Adherent.
5. $\{1.5\}$: Neither of the options.

Question 7: (10 marks)
Given a metric space X and $D \subset X$. Then prove that D is dense if and only if every point of X is an adherent point of D.

Solution:

Let D be dense in X. We will show that every point of X is an adherent point of D. Let $x \in X$. Let U be any non-empty open set containing x. Then since D is dense in X, we get $U \cap D \neq \emptyset$. This implies that x is an adherent point of D. (5 marks).

Let us assume that every point of X is an adherent point of D. We will show that D is dense in X. Let $x \in X$. Then x is an adherent point. This implies that every open set containing x intersects a point of D. This implies that D is dense in X. (5 marks)

Question 8: (15 marks)
Let x^* be an accumulation point of a set S. Prove that every neighbourhood of x^* contains infinitely many points of S.

Solution:

For contradiction, assume that there exists a ϵ-neighbourhood of x^* that contains only finitely many points of S. Let these points be $s_1, s_2, \cdots, s_k \neq x$. (2 marks)

Let $\epsilon' = \min(|x - s_1|, |x - s_2|, \cdots, |x - s_k|)$. Take any z such that $|z - x| < \epsilon' < \epsilon$. (4 marks)

Note that $|x - s_i| \geq \epsilon'$. Therefore $z \neq \{s_1, s_2, \cdots, s_k\}$. This implies that $z \notin S$. (6 marks).

Therefore there are not points in S that are within ϵ' distance of x^* and therefore x^* cannot be an accumulation point of set S. (3 marks)

Question 9: (15 marks)
Show that union of two connected sets is connected if their intersection is nonempty.

Solution:

Let $A = A_1 \cup A_2$ be the union of two disjoint connected sets A_1 and A_2. For contradiction, assume that A is disconnected. (2 marks)
This implies that there exists open sets G_1, G_2 such that

1. $G_1 \cap G_2 \neq \emptyset$.

2. $A \subset G_1 \cup G_2$.

3. $G_1 \cap A \neq \emptyset$.

4. $G_2 \cap A \neq \emptyset$.

(3 marks)
Note that $A_1 \subseteq A \subseteq G_1 \cup G_2$. (1 mark)

If $A_1 \cap G_1 \neq \emptyset$ and $A_1 \cap G_2 \neq \emptyset$, then A_1 is not connected. (2 marks).

Both $A_1 \cap G_1$ and $A_1 \cap G_2$ cannot be empty. (1 mark)

Then exactly $A_1 \cap G_1 = \emptyset$ or $A_2 \cap G_1 = \emptyset$. We consider the case when $A_1 \cap G_1 = \emptyset$. This implies that $A_1 \subseteq G_2$. If $A_2 \subseteq G_1$, then this is not possible since A_1 and A_2 are disjoint. This implies that $A_2 \subseteq G_2$. Therefore $A = A_1 \cup A_2 \subseteq G_2$. But this implies that $A \cap G_1 = \emptyset$, a contradiction. (6 marks).