Real analysis
Assignment 3
Due: 19 November 2024 before 11:59 pm

1. (5 points) Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of connected subsets of a space X. Suppose that $A_n \cap A_{n+1} \neq \emptyset$ for each n. Show that the union $\bigcup_n A_n$ is connected.
2. (5 points) Are closures and interiors of connected sets always connected? Justify with reason.
3. (5 points) Show that the union of a finite number of compact sets in a metric space (X, d) is compact.
4. (5 points) Let $f : A \to \mathbb{R}$ be continuous on A. If $K \subseteq A$ is compact, show that $f(K)$ is also compact.
5. (10 points) Give an example of an open cover of $(0, 1)$ which has no finite subcover.
6. (10 points) Find the pointwise limit of the sequence of functions $f_n(x) = x^n (n \in \mathbb{N})$ on the closed segment $[0, 1]$. Is this convergence uniform? Justify your answer.
7. (10 points) Show that there exist irrational numbers x such that x^n is irrational for all positive integers n.