Theory Assignment I

Discrete Structures Monsoon 2024, IIIT Hyderabad

December 9, 2024

Total Marks: 30 points

1. [2+2+2+4 points] Professor Infiniti has a magical drawing board that extends forever in all directions. She challenges her students with a puzzle:

   "On this infinite board, I want you to draw as many shapes as possible without any of them touching or overlapping."

   She gives four different challenges:

   a) Draw straight line segments

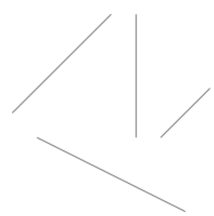


Figure 1: Filling up a plane with line segments

   b) Draw hollow squares (just the outlines)

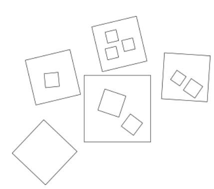


Figure 2: Filling up a plane with squares

c) Draw filled-in squares (hence now a square includes the outline as well as the area inside of it)

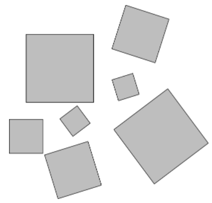


Figure 3: Filling up a plane with solid squares

d) Draw infinity symbols (∞)

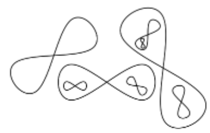


Figure 4: Filling up a plane with infinities

Note that you are free to choose any size for any of your shapes, you just need to ensure that the shapes don't touch or intersect with each other. For each shape, your task is to figure out:

a) Is it possible to draw an **uncountable** number of these shapes?

b) If yes, describe how you would do it (Just a simple construction would be enough, along with an explanation of why it works)

c) If no, explain why it's impossible. (Provide a rigorous proof)

Remember, Professor Infiniti's board is infinitely large, so you have all the space you need. But can you fit an "uncountable" number of each shape? (**Hint:** You might have to utilize the fact that rational numbers *are a dense subset* of the real numbers)

Solution:

a. **Uncountable**. Notice that we can draw parallel lines (or line segments) of the form $y = mx + c$. Note that for different c, that is, for $c_1 \neq c_2$. The lines do not intersect. Also, there exist a line for every such c. Also, c can take over all possible real numbers. Therefore we have a bijection from the set of real numbers to the possible lines that can be drawn. Hence we can draw an uncountable number of lines.

b. **Uncountable**. The same holds true for hollow squares as well, you can create a family of concentric squares whose corners are (a, a), $(a, -a)$, $(-a, a)$, $(-a, -a)$. Note that for different a, the hollow squares do not intersect/touch each other. Also there exists a hollow square for every such a. Since a can take up any real number, therefore we have a bijection from the set of real numbers to the hollow squares. Hence we can draw an uncountable number of hollow squares.

c. **Countable**. This is where the dense property comes into picture. A hollow square takes up an entire space in $\mathcal{R} \times \mathcal{R}$. By the density property we can argue that there exists a point

(x, y) inside of this square, such that both x and y are rational. We call this the rational point of the square. Therefore every square must have atleast one rational point. Also no two squares can share the same rational point, because that implies that the point is inside both squares, which then implies that the squares intersect or touch each other. Therefore we have a surjective mapping from the set of rational points (which in turn is a subset of $\mathcal{Q} \times \mathcal{Q}$ to the set of squares. We know that $\mathcal{Q} \times \mathcal{Q}$ is countable, therefore a subset of it must be countable as well. Also since this is a surjective mapping , therefore the size of the co-domain has to be smaller than the size of the domain. Which implies that the number of squares must be smaller than the size of $\mathcal{Q} \times \mathcal{Q}$. Thereby making the number of solid squares countable.

d. **Countable**. We can have a similar argument as (c.). The infinity symbol has two loops. By the density argument, both loops must have a rational point each. Let us call a **rational pair** as a pair of two rational points $\{(x, y), (x', y')\}$. Again, every infinity must have at least one rational pair, and two infinities cannot share the same rational pair, because then they intersect. Therefore we create a surjection from the set of rational pairs (which in turn is a subset of $\mathcal{Q}^4$. Therefore, the number of infinities that can be drawn is smaller than the size of $\mathcal{Q}^4$ which is countable. Hence the number of infinities is countable.

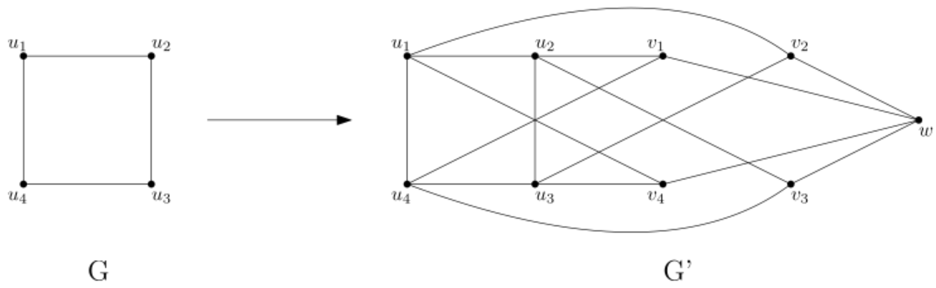
Rubric : 2 marks for each part, 1 mark for stating the correct number of figures (countable or uncountable) and 1 mark for writing the correct explanation. Cut 0.5 marks at max for silly mistakes.

2. [5 points] Given an undirected graph G, with n vertices $u_1, u_2 \dots u_n$, we construct a new graph G' as follows.

   1. Add $n + 1$ new vertices $v_1, v_2 \dots v_n$ and w
   2. Add all of the edges from G in G'
   3. In addition, for all v_i, connect it to all neighbours of u_i
   4. Finally, connect all of v_i to w

   Prove that if G does not have any cycle of length 3 (a triangle), then G' also does not have any cycle of length 3.

Given below is the construction of G' from G for an example graph



Solution: Solution for Question 2 From the construction, by process of elimination, a possible triangle would involve two vertices from set V and one from U (the newly added vertices). Call them v_1, v_2 and u_x. Now, $x \neq 1$ and $x \neq 2$ because in either case u_x would not be connected to v_1 or v_2 respectively. So, $x = 3$ say. For u_x to be connected to v_1 and v_2, both of them have to be neighbours of v_3. But, then, v_1, v_2, v_3 would form a triangle which is a contradiction to the problem statement.

Rubric:

- 2 Marks for ruling out the kind of triangles possible
- 3 Marks for rest of the proof by contradiction

3. [3 points] We define that a fraction is irreducible if its numerator and denominator are co-prime. For example $\frac{4}{7}$ is irreducible, however $\frac{4}{6}$ isn't irreducible, since 4 and 6 are not co-prime. Count the number of **proper irreducible** fractions, where the product of the numerator and denominator is 30!

Solution: We want to figure out the number of solutions of $p \times q = 30!$ where p and q are co-prime and $p < q$. Notice that since p and q are co-prime, therefore they cannot share any prime factors. Also observe that the prime factors of 30! are $2, 3, 5, 7, 11, 13, 17, 19, 23, 29$. What we can do is the following : For every prime factor either it goes to p or it goes to q (along with it's power). Every prime has 2 options, therefore we have 2^{10} solutions. However note that we also require the fraction to be proper as well. Notice that it is impossible for p and q to be equal, since they do not share any common factors. Therefore for any solution where $p > q$, we can just take the reciprocal instead. Therefore exactly half of the total number of solutions are proper fractions. Hence the total number of solutions is $2^{10}/2 = 2^9$

Rubric : 1.5 for writing down the number of irreducible fractions (2^{10}), and 1.5 for writing the final answer mentioning that the the total number of solutions is exactly half of this, so 2^9.

4. [2 points] If $A_1, A_2, \ldots, A_n$ are n sets in a universe Q of N elements, then prove that the number N_m of elements in exactly m sets and the number N_m^* of elements in at least m sets are given by

   (a) $N_m = S_m - \binom{m+1}{m} S_{m+1} + \binom{m+2}{m} S_{m+2} + \cdots + (-1)^{k-m} \binom{k}{m} S_k + \cdots + (-1)^{n-m} \binom{n}{m} S_n$

   (b) $N_m^* = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} + \cdots + (-1)^{k-m} \binom{k-1}{m-1} S_k + \cdots + (-1)^{n-m} \binom{n-1}{m-1} S_n$

Where S_m is the sum of the sizes of all m-tuple intersections of the A_is.

(**Hints:** (a) Use the fact that $\binom{k}{m}\binom{r}{k} = \binom{r}{m}\binom{r-m}{k-m}$. (b) Is there a relation between N_m and N_m^* that you can capitalize on?)

Solution:

(a) Let's start by counting N_m. We can first start by using the intersection of all m-tuple intersections. That is equal to S_m. But notice that we are also counting elements which are present in exactly $m+1$ sets as well (Consider this : Say you have four sets A, B, C, D. Then $A \cap B$ counts the elements which are exactly in A and B, and also elements which are in A, B and C and A, B, and D).

How many of these elements did you overcount. This can be given by taking S_{m+1}, also notice that all of the elements that were in $m+1$ sets must have been counted $\binom{m+1}{m}$ times. Therefore we need to remove $\binom{m+1}{m}S_{m+1}$.

Now, what about elements which were exactly in $m+2$ sets. The first term S_m overcounts this by a factor of $\binom{m+2}{m}$ times, whereas the second term $\binom{m+1}{m}S_{m+1}$ removes a total of $\binom{m+1}{m}\binom{m+2}{m+1}$.

Therefore the total net count we need to balance is $\binom{m+2}{m} - \binom{m+1}{m}\binom{m+2}{m+1} = -\binom{m+2}{m}$.

Hence we need to add $\binom{m+2}{m}$ to the tally. We keep doing this , in general We get that the the tally to be balanced for elements in exactly $m+k$ sets is

$$\begin{aligned}
&\binom{m+k}{k} - \binom{m+1}{m}\binom{m+k}{m+1} + \binom{m+2}{m}\binom{m+k}{m+2} + \cdots + \binom{m+k-1}{m}\binom{m+k}{m+k-1} \\
&= \binom{m+k}{m}\left\{\binom{k}{0} - \binom{k}{1} + \cdots + (-1)^{k-1}\binom{k}{k-1}\right\} \\
&= \binom{m+k}{m}\left\{\binom{k}{0} - \binom{k}{1} + \cdots + (-1)^{k-1}\binom{k}{k-1} + (-1)^k\binom{k}{k} - (-1)^k\binom{k}{k}\right\} \\
&= -(-1)^k\binom{m+k}{m}
\end{aligned}$$

Hence we need to add $(-1)^k\binom{m+k}{m}S_m$ to the RHS and keep doing for all k from 0 to $n-m$.

(b) Multiple Ways of doing this. You could plug in the value of N_m and solve it that way. Or you could also apply induction. Notice that the base case is true, that is when m is 1, N_1^* just counts the number of elements which are present in at least one set, which is just the size of the overall union, and the RHS just simplifies to the standard inclusion exclusion equation.

Now, Assume that this is true for some m. We need to prove that this is also true for $m+1$. Notice that the following is always true, by virtue of how we have defined N_m^*.

$$N_{m+1}^* = N_m^* - N_m$$

We can now plug in the formula for N_m^* (Our inductive hyptohesis), and N_m from the previous part. The Coefficient of S_k is given by