Quiz → Set I → Science - II

Q1) Ans :- Fixed points satisfy $\dot{N} = 0$

$$rN\left(1 - \frac{N^{\alpha}}{K}\right) = 0 \longrightarrow 1$$

Hence, $\boxed{N^* = 0}$, $\boxed{N^* = K^{1/\alpha}}$ ⇒ These are only fixed points. → 1

4Q) Ans :- ① Equilibrium States :- Only when $N(0)$ is exactly a fixed point $(0 \text{ or } K^{1/\alpha})$ is the rate of change exactly zero, meaning the population is 'at rest' and will not change over time.

0.5 Mark ← Sir → The population remains constant only when $\dot{N} = 0$, occurs only at Equilibrium points.

→ 0.5 Mark only at Equilibrium points.

② Non - Equilibrium States :- Only when $N(0)$ is exactly not a fixed point. (~~0 or K^{1/α}~~) then $f(N) \neq 0$, so population must change with time.

Because of $\frac{dN}{dt}$ ← $\frac{dN}{dt}$ = Velocity of the ~~change~~ Population change.

Population does not remain constant for an arbitrary nonzero initial condition

Quiz-1 Marking Scheme: Set I, Question 2

Question

For a scalar ordinary differential equation $\dot{x} = f(x)$, linearize the system about a fixed point x^* and derive the stability condition in terms of $f'(x^*)$.

Solution

1. Setup and Linearization

Let x^* be a fixed point such that $f(x^*) = 0$. Consider a small perturbation $\eta(t)$ such that:

$$x(t) = x^* + \eta(t) \quad \text{where } |\eta| \ll 1$$

Differentiating with respect to time (since $\dot{x^*} = 0$):

$$\dot{x} = \dot{\eta} = f(x^* + \eta)$$

Using the Taylor Series expansion about x^*:

$$\dot{\eta} = f(x^*) + f'(x^*)\eta + O(\eta^2)$$

Since $f(x^*) = 0$ and we neglect higher-order terms ($O(\eta^2) \approx 0$):

$$\dot{\eta} \approx f'(x^*)\eta$$

2. Derivation of Stability

The linearized equation is $\dot{\eta} = \lambda\eta$ where $\lambda = f'(x^*)$. The general solution is:

$$\eta(t) = \eta(0)e^{f'(x^*)t}$$

For the system to be stable, the perturbation $\eta(t)$ must decay to 0 as $t \to \infty$. This requires the exponential term to decay.

3. Condition

$$\lim_{t\to\infty} e^{f'(x^*)t} = 0 \iff f'(x^*) < 0$$

Thus, the fixed point is stable if $f'(x^*) < 0$.

Marking Scheme (Total: 2 Marks)

1. **Linearization (1 Mark)**
   - 0.5 marks: for correctly applying the Taylor Series expansion: $f(x^* + \eta) \approx f(x^*) + f'(x^*)\eta$.
   - 0.5 marks: for obtaining the correct linearized differential equation: $\dot{\eta} = f'(x^*)\eta$ (or equivalent $\dot{x} = f'(x^*)(x - x^*)$).
   - *Deduction:* If the perturbation term η (or $x - x^*$) is missing (i.e., writing $\dot{x} = f'(x^*)$), award 0 marks for this section.
2. **Derivation of Condition (0.5 Marks)**

- 0.5 marks: for showing the explicit time-dependent solution $\eta(t) = \eta_0 e^{\lambda t}$ OR explicitly stating that the solution is exponential in nature, which justifies why the sign of the derivative matters.
- *Note:* Simply stating "slope is negative" without reference to time evolution/decay is insufficient for the Derivation mark.

3. **Statement of Condition (0.5 Marks)**
   - 0.5 marks: for clearly stating the final stability criteria:
     - Stable if $f'(x^*) < 0$.
     - Unstable if $f'(x^*) > 0$.

Common Errors & Deductions

- *The Second Derivative Error:* Using $f''(x) < 0$ (concavity test) instead of $f'(x) < 0$. This is a fundamental conceptual error confusing optimization with dynamic stability. Score: 0 for the derivation and condition.
- *Missing Variable:* Writing $\dot{x} = f'(x^*)$ (constant velocity) instead of $\dot{x} = f'(x^*)(x - x^*)$. This fails to describe a dynamic system.

Note on Grading Appeals

The initial marking has been conducted with leniency. Please note the following regarding query submissions:

Strict Re-evaluation: If you decide to formally contest a grade or write a query, the question will be forwarded to the Professor's discretion for a final re-check. A formal re-check generally results in a more rigid application of the marking scheme than the initial grading.

Conceptual Errors: Fundamental errors, such as confusing the Second Derivative Test (calculus optimization) with Linear Stability Analysis (differential equations), will be marked strictly during a re-check.

Process: Submit queries formally through the designated form. Ensure that your query addresses a specific grading error rather than a general request for more marks.

5.

```
Initialize N = N0
Initialize t = 0
Choose time step dt

while t < T_final:
    N = N + dt * r * N * (1 - N^alpha / K)
    t = t + dt
```

0.5m Identify

State orthogonal transforms preserves Euclidean norms

0.5m Property

State and use $Q^T Q = I$

1m Proof

$$\|Qx\|^2 = (Qx)^T(Qx) = x^T Q^T Q x = x^T x$$
$$= \|x\|^2$$

non-negotiable in proof. needs to be there in some form