Probability and Statistics

Homework 10

Q1: There are m classes offered by a particular department, and each year, the students rank each class from 1 to m, in order of difficulty, with rank m being the highest. Unfortunately, the ranking is completely arbitrary. In fact, any given class is equally likely to receive any given rank on a given year (two classes may not receive the same rank). A certain professor chooses to remember only the highest ranking his class has ever gotten.

(a) Find the transition probabilities of the Markov chain that models the ranking that the professor remembers.

(a) Find the recurrent and the transient states.

(a) Find the expected number of years for the professor to achieve the highest ranking given that in the first year he achieved the ith ranking.

Q2: **Ehrenfest model of diffusion.** We have a total of n balls, some of them black, some white. At each time step, we either do nothing, which happens with probability ϵ, where $0 < \epsilon < 1$, or we select a ball at random, so that each ball has probability $(1-\epsilon)/n > 0$ of being selected. In the latter case, we change the color of the selected ball (if white it becomes black, and vice versa), and the process is repeated indefinitely. What is the steady-state (stationary) distribution of the number of white balls?

Q3: Krrish and Krrish-3 are provided a 7-sided, fair die each. Krrish rolls the die until he observes two consecutive 6's. Krrish-3 rolls a die until he observes a 6 followed by a 7. Who is more likely to stop first?

(c) Provide an intuitive explanation to your answer.

(c) How could you use Markov chains to solve this?

(c) Provide mathematical proof.

Q4: Prove that a state i in a Markov chain is *recurrent* (i.e., $f_{ii} = 1$) if and only if

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty.$$

Explain the intuition behind this criterion using the concept of expected visits.

Q5: A spider has a web with 4 nodes: $1, 2, 3$, and a "Trap" state T. T is *absorbing* (once there, it never leaves).

- From 1: goes to 2 or T with equal probability.
- From 2: goes to 1, 3, or T with equal probability.
- From 3: goes to 2 or T with equal probability.

Set up the system of equations to find $\mu_i = \mathbb{E}[\text{time to reach } T \mid X_0 = i]$ for each non-absorbing state. Which state has the longest expected survival time, and why?

Q6: A random walk on the non-negative integers $\{0, 1, 2, \dots\}$ has transitions:

- From 0: always goes to 1.
- From $i > 0$: goes to 0 with probability $p_i = (1/2)^i$ (a "catastrophic fall"), or to $i + 1$ with probability $1 - p_i$.

(a) Will the chain eventually return to 0 with certainty? Is this chain recurrent or transient? **(b)** Write the recursive equation for $E[T_0 \mid X_0 = 1]$, where T_0 is the first return time to state 0. **(c)** If instead $p_i = 1/i$, how does the classification change?

Q7: Consider an irreducible Markov chain M_1 with transition matrix P on a finite state space $\mathcal{S}$. It has a unique stationary distribution π. We define the *lazy* version M_2 with transition matrix

$$Q = \tfrac{1}{2}P + \tfrac{1}{2}I.$$

At each step, the chain flips a fair coin: with heads, it follows P; with tails, it stays put. Does the lazy chain M_2 have a different stationary distribution than M_1?

Q8: A gambler starts with \$50 in a fair casino with the following rules:

If wealth is even, bet \$1. If wealth is odd, bet \$10.

The game ends when wealth reaches \$0 (broke) or \$100 (victory). What is the exact probability of reaching \$100 before \$0, despite the state-dependent betting rule?

Q9: Consider the following properties of accessibility and communication in Markov chains:

(a) Prove that if $i \to j$ and $j \to k$, then $i \to k$.

(b) Using the result of part (a), prove that the "communicate" relation is an equivalence relation. That is, show it is reflexive, symmetric, and transitive.

Q10: Let $\{\varepsilon_n, n \geq 0\}$ be independent random variables with

$$\mathbb{P}(\varepsilon_n = 1) = p, \qquad \mathbb{P}(\varepsilon_n = -1) = 1 - p,$$

and define

$$X_n = \varepsilon_{n+1}\varepsilon_n, \qquad n \geq 0.$$

For which values of p is the sequence $\{X_n\}$ a Markov chain?

Q11: **(Bonus)** Consider an $N \times N$ grid G where each cell (i, j) initially contains a distinct label

$$G_{i,j}(0) = iN + j.$$

At each step, perform the following random operation:

> Select uniformly at random one of the $2N$ lines (the N rows or N columns), and then shuffle the entries within the selected line uniformly at random.

Let $G_{r,c}(k)$ denote the label in cell (r, c) after k operations.

Devise an algorithm to compute the probability

$$\mathbb{P}\big(G_{r,c}(K) = rN + c\big)$$

that the label returns to its original position after K operations. Can it be made to run faster than $O(K)$ time?

Question Link: https://www.codechef.com/problems/SHUFGRID?tab=statement