Data Structures and Algorithms (CS1.201)

Instructor: Kshitij Gajjar
Quiz 2 (3 Apr, 2025)
Total Marks: 25 [3+3+4, 1+1+1+2+3+3, 4]

[Please verify that this quiz sheet contains a total of 3 puzzles. Puzzles 1 and 3 are both MSQs (Multiple Select Questions), meaning they might have more than one correct answer amongst the set of options provided. For such puzzles, full marks will only be given if the explanations or proofs of your answer are correct. All the best!]

Puzzle 1. For this puzzle, assume the number of vertices n is an even number and $n \geq 4$. In this puzzle, we will partition the $n \times n$ adjacency matrix A of a graph G on n vertices into four sub-matrices of size $\left(\frac{n}{2} \times \frac{n}{2}\right)$:

- A_{TL} (the top-left sub-matrix of A);
- A_{TR} (the top-right sub-matrix of A);
- A_{BL} (the bottom-left sub-matrix of A);
- A_{BR} (the bottom-right sub-matrix of A).

For example, if $n = 8$ and the adjacency matrix A of a graph G is

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix},$$

then A can be partitioned into four sub-matrices of the same size:

$$A = \left[\begin{array}{cccc|cccc} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right].$$

And these four $\left(\frac{n}{2} \times \frac{n}{2}\right)$ matrices can be individually written as

$$A_{TL} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad A_{TR} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad A_{BL} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad A_{BR} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

[The above example is only intended to illustrate the definitions of $A_{TL}, A_{TR}, A_{BL}, A_{BR}$. In the puzzles that follow, you cannot make any assumptions about the graph G other than the fact that n is an even number and $n \geq 4$.]

Puzzle 1.1. Suppose the adjacency matrix A of a graph G has the property that $A_{TR} = 0$ and $A_{BL} = 0$. Then, what can you conclusively say about the graph G?

(a) G is definitely a tree.

(b) G is definitely a bipartite graph.

(c) G is definitely a disconnected graph.

(d) G is definitely not a complete graph.

Puzzle 1.2. Suppose the adjacency matrix A of a graph G has the property that $A_{TL} = 0$ and $A_{BR} = 0$. Then, what can you conclusively say about the graph G?

(a) G is definitely a tree.

(b) G is definitely a bipartite graph.

(c) G is definitely a disconnected graph.

(d) G is definitely not a complete graph.

Puzzle 1.3. Suppose each of the matrices $A_{TL}, A_{TR}, A_{BL}, A_{BR}$ has exactly one 1 in each column (the other $\left(\frac{n}{2}-1\right)$ entries in each column are 0), and exactly one 1 in each row (the other $\left(\frac{n}{2}-1\right)$ entries in each row are 0). Then, what can you conclusively say about the graph G?

(a) G is definitely a tree.

(b) G definitely contains a cycle.

(c) G is definitely either a bipartite graph or a disconnected graph.

(d) G is definitely not a complete graph.

Puzzle 2. For this puzzle, we will use the following definition of a tree. In this puzzle, you must use only this definition to conclude whether a graph is a tree or not.

Definition. *A tree is a graph in which each pair of vertices is connected by exactly one path.*

Now, suppose G is a graph on n vertices, where $n \geq 4$. Consider the 3 statements below. *[G need not satisfy all of these statements. The subset of these statements that are true for G is mentioned in the puzzles that follow.]*

(i) G is a connected graph.

(ii) G does not have any cycle.

(iii) G has $(n-1)$ edges.

Puzzle 2.1. Draw a graph G on $n \geq 4$ vertices that is NOT a tree and satisfies only statement (i).

Puzzle 2.2. Draw a graph G on $n \geq 4$ vertices that is NOT a tree and satisfies only statement (ii).

Puzzle 2.3. Draw a graph G on $n \geq 4$ vertices that is NOT a tree and satisfies only statement (iii).

Puzzle 2.4. Prove that if statements (*i*) and (*ii*) are true for a graph G, then you can decisively conclude that G is a tree.

Puzzle 2.5. Prove that if statements (*i*) and (*iii*) are true for a graph G, then you can decisively conclude that G is a tree.

Puzzle 2.6. Prove that if statements (*ii*) and (*iii*) are true for a graph G, then you can decisively conclude that G is a tree.

Puzzle 3. For this puzzle, if you want to prove that two graphs are isomorphic to each other, then you need to label both of them in exactly the same way. And if you want to prove that two graphs are not isomorphic to each other, then you need to prove that one of those graphs has a property that the other graph does not.

What can you say about the graphs G_1, G_2, G_3 shown below?

(a) G_1 and G_2 are isomorphic to each other.

(b) G_1 and G_3 are isomorphic to each other.

(c) G_2 and G_3 are isomorphic to each other.

(d) None of the three graphs are isomorphic to each other.