Quiz 2: Probability and Statistics (30 mks)

Each question: 5 marks

1. Suppose $Z = X + U$ where X is exponential random variable with parameter λ and U is Uniform $[0, 1]$ variable. **Derive the first two moments of Z using MGF. (You have to derive MGF of X and U first).**

2. Let $X_1, X_2, X_3, \ldots$ be a sequence of Laplacian (doubly exponential) continuous random variables such that
$$f_{X_n}(x) = 0.5ne^{-n|x|}, \quad x \in \mathbb{R}.$$
Show that X_n converges in probability to 0.

3. Give me a procedure to convert samples from an Exponential random variable to $Uniform[0, 1]$. Justify why the procedure is correct.

4. Let (X_1, X_2) be a continuous random vector with joint density
$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} 1, & 0 < x_1 < 1,\ 0 < x_2 < 1, \\ 0, & \text{otherwise.} \end{cases}$$
Define the transformation
$$Y_1 = X_1 - X_2, \qquad Y_2 = X_1 + X_2.$$
Obtain the joint density $f_{Y_1,Y_2}(y_1, y_2)$ in terms of $f_{X_1,X_2}(x_1, x_2)$.

$X = F_X^{-1}(U)$

$U = F_X($

U

10 marks

Let $X_n \sim \text{Exp}(n)$, i.e. its density is
$$f_{X_n}(x) = ne^{-nx}, \quad x \geq 0.$$
Define $Y_n = nX_n$.

1. Show that Y_n converge *in distribution* to $Y = Exp(1)$.
2. Does Y_n converge *in probability* to Y? Justify.
3. Does Y_n converge *almost surely* to Y? Justify.
4. Does Y_n converge *in mean* (i.e. in L^1)? Justify.

$A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$f_X\begin{pmatrix} y_1 + y_2 \\ y_1 - y_2 \end{pmatrix}$

$\to E[Y_n]$

$\lim_{n\to\infty} E[(X_n - X)^2] \to 0$