“Quantitative Macroeconomics & Social Insurance - TA 3”

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1 Consumption, Saving, and Portfolio Choice

Consider the household problem

$$
\begin{gathered}
v_t(w_t) = \max_{c_t,\alpha_t,w_{t+1}} \{u(c_t) + \beta \mathrm{E}_t v_{t+1}(w_{t+1})\}, \\
\text{s.t.} \quad w_{t+1} = (w_t - c_t) R^p_{t+1}, \\
R^p_{t+1} = R^f + \alpha_t (R_{t+1} - R^f), \;\; \ln R_{t+1} \sim N(\mu, \sigma^2) \\
w_0 \;\text{ given}, w_{T+1} = 0.
\end{gathered}
$$

- Finite-horizon $t = 0, 1, 2, ..., T$.
- Non-tradable asset h_t equal to present value of deterministic income $\{y_t\}$
- Total wealth w_t is the sum of cash-on-hand $x_t = a_t R^p_{t+1} + y_t$ and h_t
- Natural debt limit $a_t \geq -y_t - h_t$ if binding then $c_t = 0, \forall t$

1. Life cycle numerical examples in partial equilibrium.

- Individuals start to work at age 20, retire at 65, and exit the model at 80.
- Solve the problem with VFI or by guess and verify (see lecture notes)
- The portfolio choice reduces to $\max_{\alpha_t} g(R^p_{t+1})$
  - Exploit log-normality to rewrite g, approximate R^p, solve FOC.
- Solution:

$$
\begin{aligned}
v_t &= (1-\theta)^{-1} m_t^{-\theta} w_t^{1-\theta}, \\
c_t &= m_t w_t, \\
m_t &= (1+b_t)^{-1}, \ \ \forall t \in [0, T-1], m_T = 1, \\
b_t &= [\beta \mathrm{E}_t (R^p_{t+1}(\alpha)^{1-\theta}) m_{t+1}^{-\theta}]^{1/\theta}, \\
\alpha &= \frac{\ln(1+\mu) - \ln R^f + \sigma^2/2}{\theta \sigma^2}.
\end{aligned}
$$

Set up:

- Time $s = 1, ..., 46, ..., 61$ and initial condition at $s = 1$
- Fix parameters values and the income stream $(y_0, ..., y_T)$

Implementation:

1. Compute policy α, and also m_t solving backward b_t given $m_T = 1$
2. Solve backward $h_{t+1} = Rh_t - y_{t+1}$ given $h_T = 0$
3. Simulate $(\ln R_0, ..., \ln R_T)$ from $\ln(1 + \mu) + \sigma\epsilon,\ \epsilon \sim N(0, 1)$
4. Solve forward budget constraint given R^p_{t+1} and $w_0 = c_0 + h_0, c_0 = 1$
   - $c_t = m_t w_t, x_t = w_t - h_t$
   - $s_t = w_t - c_t, s^f_t = x_t - c_t, \hat{\alpha}_t = \alpha(s_t/s^f_t), w_{t+1} = s_t R^p_{t+1}$
5. Simulate N times and compute aggregates $c_t = N^{-1} \sum_{i=1}^N c^i_t$

2. Consumption over the life-cycle.

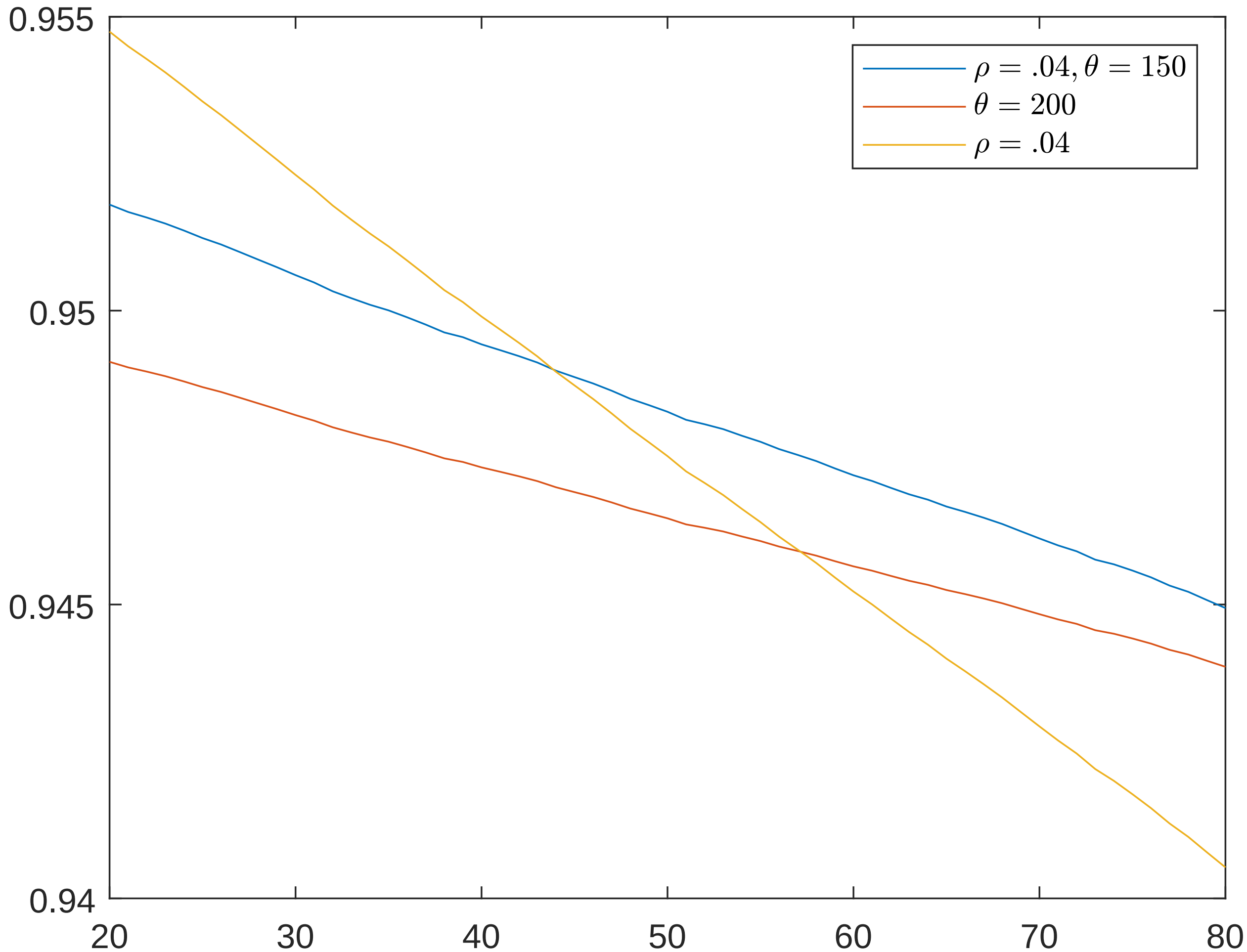

Figure 1: Average consumption

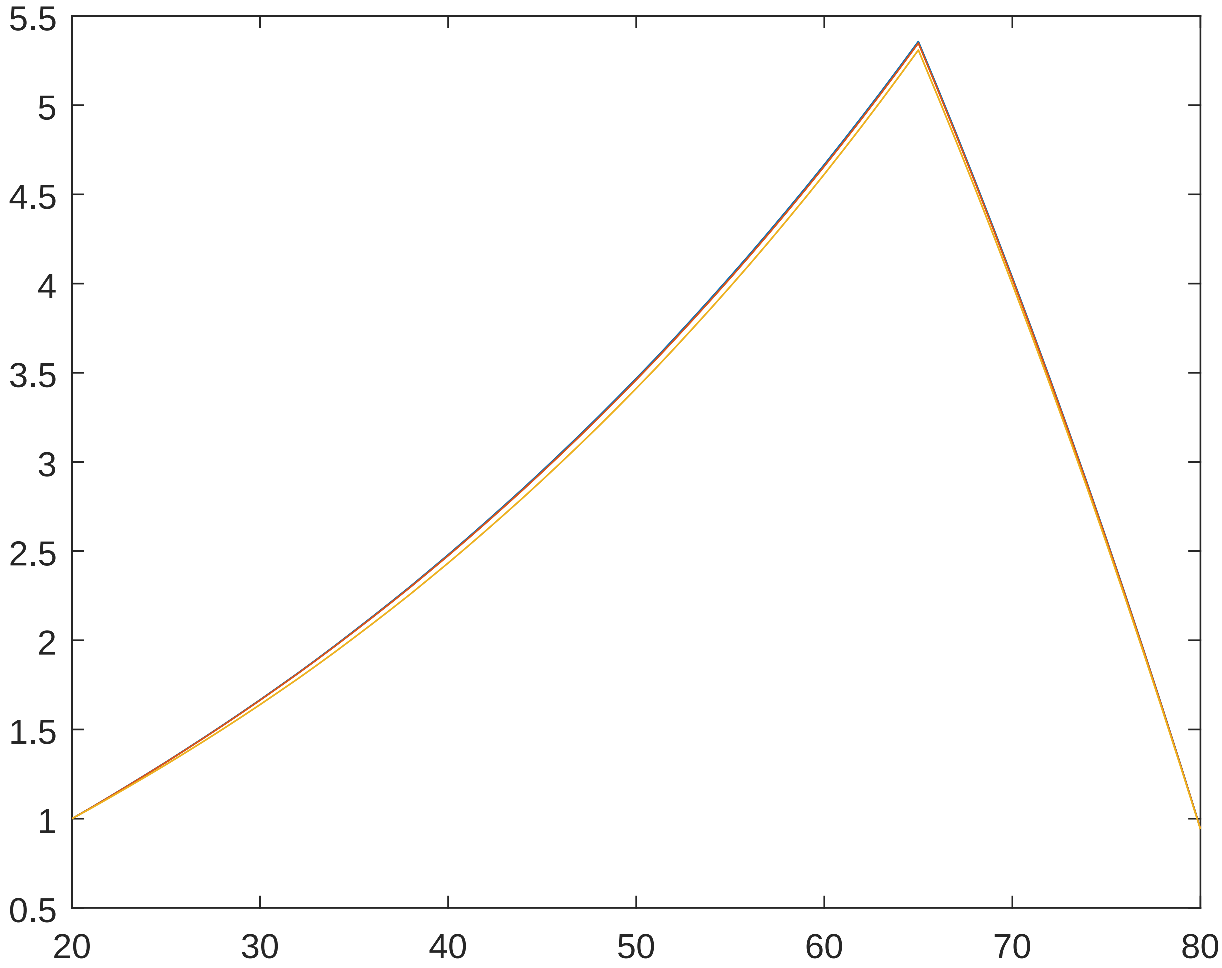

Figure 2: Average cash on hand

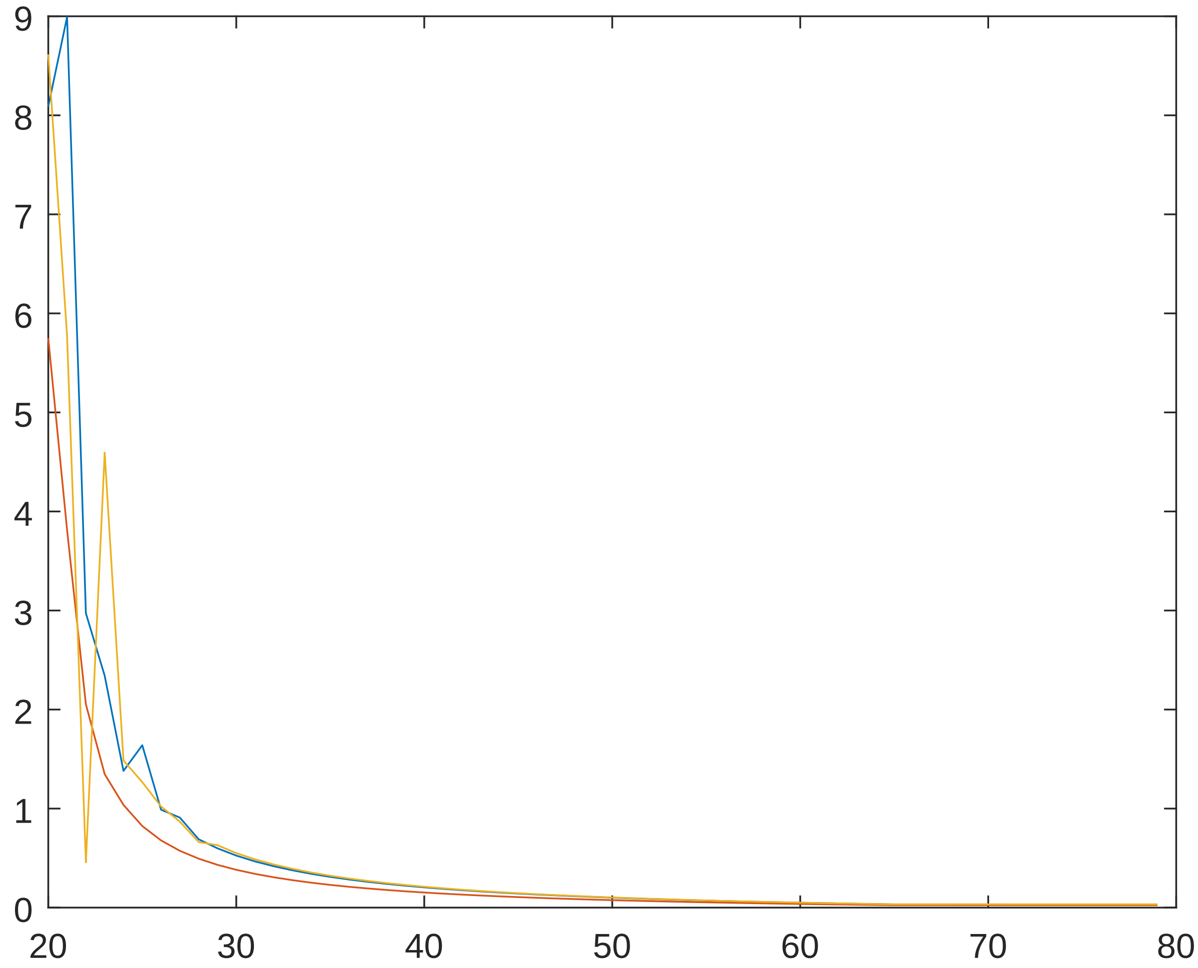

Figure 3: Average risky share

Remarks:

- $\hat{\alpha}_t = \alpha(1 + h_t/s_t^f)$ where h is decreasing and s^f increasing over time
- Since $\downarrow h_t, \uparrow$ risk-free asset, α_t is decreasing over time
- Borrow from future income to invest in risky asset $\alpha_t > 1$
- Consumption decreases over time due to the precautionary saving motive.
- In the data consumption is hump-shaped with a peak around forty/fifty.
- Literature often set θ around 2. Using 150 lead to much less volatility
- $\uparrow \theta$ (high RRA, lower IES), $\downarrow \alpha_t$, lower and less steep consumption profile.
- $\uparrow \rho$ (more impatience), steeper consumption profile, almost no effect on α_t.

3. Rule of thumb portfolio.

$$\bar{a}_j = 1 - \frac{j}{100},$$
$$\bar{a}_{20} = .8,$$
$$\bar{a}_{65} = .2,$$
$$\bar{a}_{80} = .35.$$

Compared to Figure 3 this allocation is clearly suboptimal:

- Very flat over life-cycle, below 1 when young and above 0 when old.

4. Epstein-Zin-Weil (EZW) preferences

Remarks:

- Typical RRA estimates put θ around 2 and IES η below 1
- However, some studies find very low IES, but then $\theta = \eta^{-1} = 10$

Consider the household problem

$$v_t(w_t) = \max_{c_t, \alpha_t, w_{t+1}} \left\{ (1-\beta) c_t^{\frac{1-\theta}{\gamma}} + \beta \mathrm{E}_t (v_{t+1}(w_{t+1})^{1-\theta})^{\frac{1}{\gamma}} \right\}^{\frac{\gamma}{1-\theta}},$$

$$\text{s.t.} \quad w_{t+1} = (w_t - c_t) R^p_{t+1},$$

where

$$\gamma = \frac{1-\theta}{1-\frac{1}{\psi}},$$

θ is the relative risk aversion coefficient and ψ is the intertemporal elasticity of substitution. If $\gamma = 1 \iff \theta = 1/\psi$. We solve this by value function iteration.

Notice that $w_{T+1} = 0 \Rightarrow c_T = w_T$. Therefore,

$$v_T(w_T) = \left[(1-\beta)c_T^{\frac{1-\theta}{\gamma}}\right]^{\frac{\gamma}{1-\theta}} = \left[(1-\beta)(w_{T-1} - c_{T-1})^{\frac{1-\theta}{\gamma}}(R_T^p)^{\frac{1-\theta}{\gamma}}\right]^{\frac{\gamma}{1-\theta}}$$

Hence,

$$\begin{aligned} v_{T-1}(w_{T-1}) &= \max_{c_{T-1},\alpha_{T-1},w_T}\left\{(1-\beta)c_{T-1}^{\frac{1-\theta}{\gamma}} + \beta\Big(\mathbf{E}_{T-1}(v_T(w_T)^{1-\theta})^{\frac{1}{\gamma}}\Big)\right\}^{\frac{\gamma}{1-\theta}} \\ &= \max_{c_{T-1}}\left\{(1-\beta)c_{T-1}^{\frac{1-\theta}{\gamma}} + \beta(1-\beta)(w_{T-1} - c_{T-1})^{\frac{1-\theta}{\gamma}}\max_{\alpha_{T-1}}\Big(\mathbf{E}_{T-1}[(R_T^p)^{1-\theta}]^{\frac{1}{\gamma}}\Big)\right\}^{\frac{\gamma}{1-\theta}} \end{aligned}$$

Notice that if $V = \max_{x,y}(A(x) + B(x,y))^\rho$. Then,

$$\frac{\partial V}{\partial y} = \rho(A+B)^{\rho-1}\frac{\partial B}{\partial y} = 0 \iff \frac{\partial B}{\partial y} = 0.$$

Assuming without loss of generality $\rho(A+B)^{\rho-1} \neq 0$.

Given the optimal share α we solve for c_{T-1} using the first order condition

$$\left(\frac{1-\theta}{\gamma}\right)(1-\beta)c_{T-1}^{\frac{1-\theta}{\gamma}-1}-\left(\frac{1-\theta}{\gamma}\right)\beta(1-\beta)(w_{T-1}-c_{T-1})^{\frac{1-\theta}{\gamma}-1}A_{T-1}=0.$$

$$c_{T-1}^{\frac{1-\theta}{\gamma}-1}=\beta A_{T-1}(w_{T-1}-c_{T-1})^{\frac{1-\theta}{\gamma}-1},$$

$$c_{T-1}=(\beta A_{T-1})^{\frac{1}{\frac{1-\theta}{\gamma}-1}}(w_{T-1}-c_{T-1}).$$

$$c_{T-1}=\frac{1}{1+B_{T-1}}w_{T-1}=m_{T-1}w_{T-1},$$

where $B_{T-1}^{-1}:=(\beta A_{T-1})^{\frac{1}{\frac{1-\theta}{\gamma}-1}}, A_{T-1}:=\mathrm{E}_{T-1}[(R_T^p(\alpha))^{1-\theta}]^{\frac{1}{\gamma}}$.

Going back to the value function

$$
\begin{aligned}
v_{T-1}(w_{T-1}) &= \left\{(1-\beta)\left[\frac{w_{T-1}}{1+B_{T-1}}\right]^{\frac{1-\theta}{\gamma}} + \beta(1-\beta)\left(w_{T-1} - \frac{w_{T-1}}{1+B_{T-1}}\right)^{\frac{1-\theta}{\gamma}} A_{T-1}\right\}^{\frac{\gamma}{1-\theta}} \\
&= \left\{(1-\beta)\left(\frac{1}{1+B_{T-1}}\right)^{\frac{1-\theta}{\gamma}} + \beta(1-\beta)\left(\frac{B_{T-1}}{1+B_{T-1}}\right)^{\frac{1-\theta}{\gamma}} A_{T-1}\right\}^{\frac{\gamma}{1-\theta}} w_{T-1} \\
&= \left\{(1-\beta)(1+B_{T-1})\left(\frac{1}{1+B_{T-1}}\right)^{\frac{1-\theta}{\gamma}}\right\}^{\frac{\gamma}{1-\theta}} w_{T-1} \\
&= (1-\beta)^{\frac{\gamma}{1-\theta}} m_{T-1}^{\left(\frac{1-\theta}{\gamma}-1\right)\left(\frac{\gamma}{1-\theta}\right)} w_{T-1} \\
&= (1-\beta)^{\frac{\gamma}{1-\theta}} m_{T-1}^{\omega} w_{T-1},
\end{aligned}
$$

where $\beta A_{T-1} = B_{T-1}^{-(\frac{1-\theta}{\gamma}-1)}$, $m = 1/(1+B)$, $\eta = 1/\psi$, $\omega = -\eta/(1-\eta)$.

Proceeding backwards yields

$$v_t = (1-\beta)^{\frac{\gamma}{1-\theta}} m_t^{\omega} w_t,$$

$$c_t = m_t w_t,$$

$$m_t = (1+B_t)^{-1},\ \ \forall t \in [0, T-1], m_T = 1,$$

$$B_t = [\beta(1-\beta)(\mathrm{E}_t(R_{t+1}^p(\alpha)^{1-\theta} m_{t+1}^{\omega(1-\theta)})^{\frac{1}{\gamma}}]^{-\gamma/(1-\theta-\gamma)},$$

$$\alpha = \frac{\ln(1+\mu) - \ln R^f + \sigma^2/2}{\theta\sigma^2}.$$

Assuming iid returns $\mathrm{E}_t(m_{t+1}R_{t+1}^p) = \mathrm{E}_t(m_{t+1})\mathrm{E}_t(R_{t+1}^p)$

$$\max_{\alpha_t}\left\{\left(\mathrm{E}_t\big((m_{t+1}^{\omega} R_{t+1}^p)^{1-\theta}\big)\right)^{\frac{1}{\gamma}}\right\} \Rightarrow \frac{\partial \mathrm{E}_t(R_{t+1}^p(\alpha_t)^{1-\theta})}{\partial \alpha_t} = 0.$$

Notice that we can solve the portfolio choice using the standard approximation.