“Quantitative Macroeconomics & Social Insurance - TA 1”

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1 CRRA Preferences

1. Consider the utility function

$$u(c) = \begin{cases} \frac{c^{1-\sigma}}{1-\sigma}, & \sigma \neq 1, \\ \ln c, & \sigma = 1. \end{cases}.$$

Then,

$$u'(c) = \begin{cases} c^{-\sigma}, \\ 1/c. \end{cases}, \qquad u''(c) = \begin{cases} -\sigma c^{-(1+\sigma)}, \\ -1/c^2. \end{cases}.$$

The Arrow-Pratt coefficient is

$$\sigma(c) = -\eta_{u',c} = -\frac{u''(c)c}{u'(c)} = \sigma$$

The coefficient σ measures the curvature of the utility function.

2. Consider the monotonic transformation

$$\frac{c^{1-\sigma}}{1-\sigma} + x, \quad \forall x \in \mathbb{R}.$$

We preserve the ordering, i.e.

$$u(c_1) > u(c_2) \Rightarrow u(c_1) + x > u(c_2) + x.$$

3. Note that

$$\lim_{\sigma \to 1} \frac{c^{1-\sigma} - 1}{1-\sigma} = \frac{0}{0},$$

l'Hopital's rule yields

$$\lim_{\sigma \to 1} \frac{c^{1-\sigma} \ln c(-1)}{-1} = \ln c.$$

4. Let $\sigma \neq 1$. If

$$u((1+g)c_t) = (1+g)^{1-\sigma} u(c_t),$$

then u is called homogenoeus of degree $1-\sigma$.

Example:

$$u(c) = \frac{c_0^{1-\sigma}}{1-\sigma} + \beta \frac{c_1^{1-\sigma}}{1-\sigma}.$$

Then,

$$\begin{aligned} u =& \frac{((1+g)c_0)^{1-\sigma}}{1-\sigma} + \beta \frac{((1+g)c_1)^{1-\sigma}}{1-\sigma} = \frac{(1+g)^{1-\sigma} c_0^{1-\sigma}}{1-\sigma} + \beta \frac{(1+g)^{1-\sigma} c_1^{1-\sigma}}{1-\sigma} \\ &= (1+g)^{1-\sigma} \left(\frac{c_0^{1-\sigma}}{1-\sigma} + \beta \frac{c_1^{1-\sigma}}{1-\sigma} \right) \end{aligned}$$

Example:
$$u(c) = \frac{c_0^{1-\sigma} - 1}{1-\sigma} + \beta\frac{c_1^{1-\sigma} - 1}{1-\sigma}.$$
Then,
$$\begin{aligned} u =& \frac{((1+g)c_0)^{1-\sigma} - 1}{1-\sigma} + \beta\frac{((1+g)c_1)^{1-\sigma} - 1}{1-\sigma} = \frac{(1+g)^{1-\sigma}c_0^{1-\sigma}}{1-\sigma} + \beta\frac{(1+g)^{1-\sigma}c_1^{1-\sigma}}{1-\sigma} \\ &= (1+g)^{1-\sigma}\left(\frac{c_0^{1-\sigma}}{1-\sigma} + \beta\frac{c_1^{1-\sigma}}{1-\sigma}\right) - \frac{1+\beta}{1-\sigma}. \end{aligned}$$

If for any monotonic transformation $f(u)$ is homogeneous of degree 1 then u is homothetic. Note that CRRA is homothetic,
$$f(u(c)) = \frac{1}{1-\sigma}\left((c_0^{1-\sigma} + \beta c_1^{1-\sigma})^{1/(1-\sigma)}\right)^{1-\sigma} - \frac{1+\beta}{1-\sigma}.$$
The function $f(x) = (1/(1-\sigma))x^{1-\sigma} - (1+\beta)/(1-\sigma)$ is increasing $\sigma > 0$ and $x = (c_0^{1-\sigma} + \beta c_1^{1-\sigma})^{1/(1-\sigma)}$ is homogeneous of degreee 1.

2 The Negishi Method

1. Given income streams

$$y_t^1 = \begin{cases} 2, & t = \text{Even} \\ 0 & t = \text{Odd.} \end{cases}, \qquad y_t^2 = \begin{cases} 0, & t = \text{Even} \\ 2 & t = \text{Odd.} \end{cases}.$$

Consider the social planner's problem

$$\max_{\{c_t^1, c_t^2\}} \sum_{t=0}^{\infty} \beta^t (\alpha \ln c_t^1 + (1-\alpha) \ln c_t^2),$$

$$\text{s.t.} \quad c_t^1 + c_t^2 = y_t^1 + y_t^2, \quad \forall t \tag{SP}$$

FOCs $\forall t$:

$$\beta^t \alpha / c_t^1 = \mu_t / 2, \tag{1}$$

$$\beta^t (1-\alpha) / c_t^1 = \mu_t / 2, \tag{2}$$

$$c_t^1 + c_t^2 = 2. \tag{3}$$

Remarks:

- We can normalize the lagrangian multiplier to $\mu_t/2$ since utility is ordinal.
- Check if FOCs are sufficient (this is the case under transversality conditions).
- No inequality constraints because of Inada conditions and monotonicity.

From (1) and (2) we have $c_t^1/c_t^2 = \alpha/(1-\alpha)$, from (3)

$$c_t^1 = 2\alpha,$$
$$c_t^2 = 2(1-\alpha).$$

Substituting this back in (1) or (2) yields

$$\mu_t = \beta^t.$$

The transfers are given by

$$
\begin{aligned}
t^1 &= \sum_{t=0}^{\infty} \mu_t (c_t^1 - y_t^1) = \sum_{t=0}^{\infty} \beta^t c_t^1 - \sum_{t=0}^{\infty} \beta^t y_t^1 \\
&= \frac{2\alpha}{1-\beta} - \frac{2}{1-\beta^2}.
\end{aligned}
$$

$$
\begin{aligned}
t^2 &= \sum_{t=0}^{\infty} \mu_t (c_t^2 - y_t^2) = \sum_{t=0}^{\infty} \beta^t c_t^2 - \sum_{t=0}^{\infty} \beta^t y_t^2 \\
&= \frac{2(1-\alpha)}{1-\beta} - \frac{2}{1-\beta^2}.
\end{aligned}
$$

Remarks:

- $\sum_{t=0}^{\infty} x^t = 1/(1-x)$ **if** $|x| < 1$.
- $\sum_{t=0}^{\infty} \beta^t y_t^1 = 2 \sum_{t=0}^{\infty} \beta^{2t} = 2/(1-\beta^2)$.
- $\sum_{t=0}^{\infty} \beta^t y_t^1 = 2 \sum_{t=0}^{\infty} \beta^{2t+1} = 2\beta/(1-\beta^2)$.
- $\mu_t = p_t > 0$.

Finally,

$$t^1 = 0 \Rightarrow \frac{2\alpha}{1-\beta} = \frac{2}{1-\beta^2} \Rightarrow \alpha = \frac{1-\beta}{1-\beta^2} = \frac{1-\beta}{(1+\beta)(1-\beta)} = \frac{1}{1+\beta}.$$

Remarks:

- $\sum_{t=0}^{k-1} x^t = (1-x^k)/(1-x)$.
- $0 < \beta < 1 \Rightarrow c_t^1 > c_t^2$.
- If $c_t = c_{t+1} = c$ then $u(c) > \beta u(c)$. Individuals are impatient.
- The planner weights more 1 because 1 has positive income before 2.

2. A stochastic economy is given by

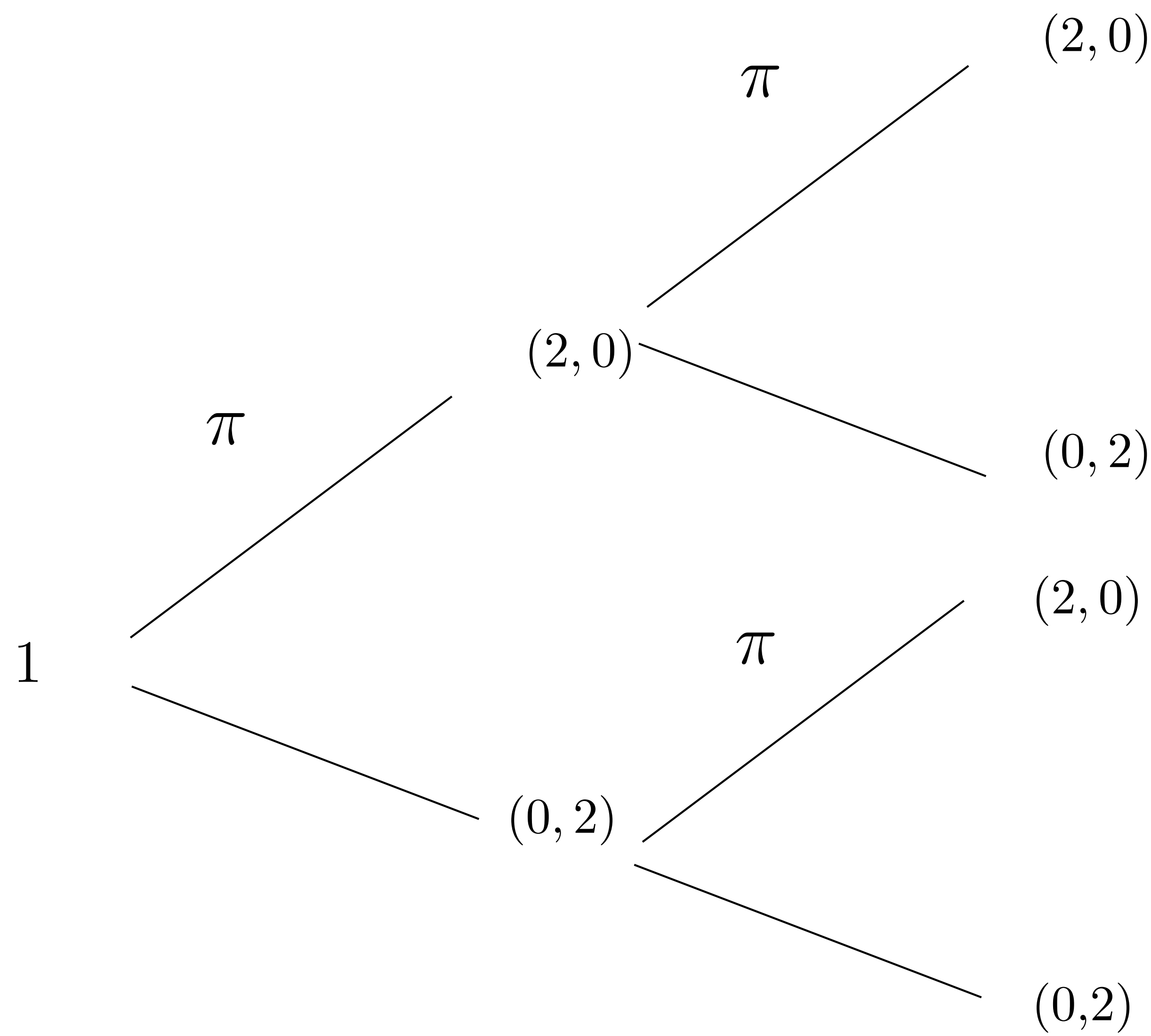
- Deterministic initial condition s_0 and a terminal period T.
- Probability space over complete histories $(S^T, \mathcal{S}^T, \Pr)$
- Measurable space of endogenous variables $(X, \mathcal{X})$, e.g. $x = c, p$.
- Filtration $\mathrm{F} = \{\mathcal{S}^t\}$ and a stochastic process $\{x_t(s^t)\}$ adapted to F.
- Probability over events $B_t \in \mathcal{S}^t$

$$P(B_t) = \sum_{s^t \in B_t} \Pr(s^t).$$

Remarks:

- $s^t = (s_1, ..., s_t) \in S^t, \forall t \geq 0$ where s_t are states. $\Pr : \mathcal{S}^T \rightarrow [0,1]$.
- A filtration is a collection of σ-algebras: $\mathcal{S}^1 \subset \mathcal{S}^2 \subset ... \subset \mathcal{S}^T$.
- We often use x_t as short hand notation for $x_t(s^t)$.
- We often write E_t for the expected value conditional on $\mathcal{S}^t$.
- With rational expectations we assume that everybody knows Pr.

Example: Let $s_0 = 1, s_t \in \mathbb{R}^2, \forall t > 0$. Then, $S^0 = 1, S^1 = \{(2,0),(0,2)\}, S^2 = \{(2,0),(0,2),((2,0),(2,0)),((2,0),(0,2)),((0,2),(0,2)),((0,2),(2,0))\}$.



Consider the social planner's problem

$$\max_{\{c_t^1, c_t^2\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t P(s^t)(\alpha \ln c_t^1(s^t) + (1-\alpha) \ln c_t^2(s^t)),$$
$$\text{s.t.} \quad c_t^1(s^t) + c_t^2(s^t) = y_t^1(s^t) + y_t^2(s^t), \quad \forall t, \forall s^t \tag{SP}$$

FOCs $\forall t, \forall s^t$:

$$\beta^t P\alpha / c_t^1 = \mu_t / 2, \tag{4}$$
$$\beta^t P(1-\alpha) / c_t^2 = \mu_t / 2, \tag{5}$$
$$c_t^1 + c_t^2 = 2. \tag{6}$$

Hence,

$$c_t^1 = 2\alpha,$$
$$c_t^2 = 2(1-\alpha),$$
$$\mu_t = \beta^t P.$$

Consider the lagrangian in an AD economy

$$L = \sum_{t=0}^{\infty} \sum_{s^t} [\beta^t u(c_t^i(s^t))P(s^t) - \lambda^i(p(s^t)c_i(s^t) - p(s^t)y_t^i(s^t))].$$

FOCs:

$$(\beta^t P)/(p_t c_t^i) = \lambda^i, \forall t, \forall s^t$$

If $\lambda^i = 1/2\alpha^i$ then

$$p_t = \beta^t P = \mu_t,$$

$$\mu_0(1) = 1,$$
$$\mu_1((2,0)) = \beta\pi, \mu_1((0,2)) = \beta(1-\pi),$$
$$\mu_2((2,0),(2,0)) = \beta^2\pi^2, \mu_2((0,2),(0,2)) = \beta^2(1-\pi)^2,$$
$$\mu_2((2,0),(0,2)) = \mu_2((0,2),(2,0)) = \beta^2\pi(1-\pi).$$

The transfer is given by

$$
\begin{aligned}
t^1 &= \sum_t \sum_{s^t} \beta^t P(c_t^1 - y_t^1) \\
&= 1(2\alpha - 1) + \beta\pi(2\alpha - 2) + \beta(1-\pi)(2\alpha - 0) \\
&+ \beta^2\pi^2(2\alpha - 2) + \beta^2(1-\pi)^2(2\alpha - 0) \\
&+ \beta^2\pi(1-\pi)(2\alpha - 0) + \beta^2(1-\pi)\pi(2\alpha - 2) = 0.
\end{aligned}
$$

Collecting terms yields

$$2\alpha(1 + \beta + \beta^2\pi^2 + \beta^2(1-\pi)^2 + 2\beta^2\pi(1-\pi)) - 1 - 2\pi(\beta + \beta^2) = 0.$$

Using $2\pi(1-\pi) = 1 - (1-\pi)^2$ yields

$$2\alpha = \frac{1 + 2\pi(\beta + \beta^2)}{1 + \beta + \beta^2}.$$

- If $\pi = 1/2$ then $\alpha = 1/2$.
- With equal odds of being the first with positive income β does not matter.
- $\uparrow \pi$, $\uparrow$ expected income and utility for 1. Hence, the SP weights more 1.

3. Consider the lagrangian with SM

$$L = \sum_{t=0}^{\infty} \sum_{s^t} [\beta^t u(c_t^i(s^t)) P(s^t)$$
$$- \lambda_t(s^t)((c_t^i(s^t) + \sum_{s_{t+1} \in S} q_t(s^t, s_{t+1}) a_{t+1}^i(s^t, s_{t+1}) - y_t^i(s^t) - a_t^i(s^t))].$$

FOCs $\forall t, \forall s^t$:

$$(\beta^t P)/c_t^i = \lambda_t,$$
$$-q_t \lambda_t + \lambda_{t+1} = 0$$

Hence,

$$q_t = \beta \frac{c_t^i}{c_{t+1}^i} \frac{P(s_{t+1}, s^t)}{P(s^t)} = \beta \frac{c_t^i}{c_{t+1}^i} P(s_{t+1} | s^t).$$

Consider the lagrangian in an AD economy

$$L = \sum_{t=0}^{\infty} \sum_{s^t} [\beta^t u(c_t^i(s^t))P(s^t) - \lambda^i(p(s^t)c_i(s^t) - p(s^t)y_t^i(s^t))].$$

FOCs $\forall t, \forall s^t$:

$$(\beta^t P)/(p_t c_t^i) = \lambda^i.$$

Using $\lambda^i = (\beta^{t+1}P)/(p_{t+1}c_{t+1}^i)$ yields

$$\frac{p_{t+1}}{p_t} = \beta \frac{c_t^i}{c_{t+1}^i} P(s_{t+1}|s^t).$$

Writing this explicitly

$$q_0((2,0)) = \beta\pi, q_0((0,2)) = \beta(1-\pi),$$

$$q_1((2,0),(2,0)) = \frac{\beta^2\pi^2}{\beta\pi} = \beta\pi, q_1((0,2),(0,2)) = \frac{\beta^2(1-\pi)^2}{\beta(1-\pi)} = \beta(1-\pi),$$

$$q_1((0,2),(2,0)) = \frac{\beta^2\pi(1-\pi)}{\beta(1-\pi)} = \beta\pi, q_1((2,0),(0,2)) = \frac{\beta^2(1-\pi)\pi}{\beta\pi} = \beta(1-\pi).$$