“Quantitative Macroeconomics & Social Insurance - TA 5”

Valerio Pieroni

1 Finite Horizone Income Fluctuations

Households solve

$$v_t(x, z) = \max_{c,x'} \{u(c) + \beta \sum_{z'} P(z'|z) v_{t+1}(x', z')\},$$
$$\text{s.t.} \quad x' = (x - c)(1 + r) + yz',$$
$$a' \geq -\phi.$$

Note that $a' = (a + y - c)(1 + r) = s(1 + r) \Rightarrow s = a'(1 + r)^{-1} \geq -\phi(1 + r)^{-1}$.

The first order condition fo this problem is

$$u'(c) \geq \beta R \sum_{z'} P(z'|z) D_{x'} v_{t+1}(x', z'), \quad \text{equality if } s > 0. \tag{1}$$

1.1 Policy Function Iteration on the Euler Equation

Define:

- States $\{z_1, z_2, ..., z_J\}$, initial distribution T_0, and a $J \times J$ transition matrix T.
- A grid for cash-on-hand $G_X = \{x_1, x_2, ..., x_I\}$.
- Let $n = T-1, ..., 1$, $i = 1, ..., I$, $j = 1, ..., J$, $q = 1, ..., J$.

Algorithm

1. Solve backward (1) with equality for c_{ij}^n given $v'_{x'} = v_{x,ijq}^{n+1}$
2. If borrowing limit is violated $c_{ij}^n > x_i + \phi R^{-1}$ set $c_{ij} = x_i + \phi R^{-1}$
3. Given c_{ij}^n compute $s_{ij}^n = x_i - c_{ij}^n$, $ap_{ij}^n = s_{ij} R$ and v_{ij}^n.
4. Update using the envelope condition $v_{x,ij}^n = u'(c_{ij}^n)$ and iterate from step 1.

Remarks:

- We need to interpolate $(x_i, v_{x,ij}^{n+1})$ over $x' = xp_{ijq}$ to find $v_{x,ijq}^{n+1}$.
- We need an inner loop to solve numerically a nonlinear equation $\forall(i, j)$.

Endogenous grid method:

- Speed up inner solution of the Euler equation.
- Fix a grid for cash-on-hand $G_X^T = \{x_1, x_2, ..., x_I\}$ at time T.
- Construct a grid over saving $G_S = \{s_1, ..., s_I\}$ where $s_1 = -\phi(1+r)^{-1}$.

Algorithm

1. Compute $M_{ij} = \beta R \sum_{q=1}^{J} T(z'_q, z_j) u'(c_{iq}^{n+1}), \forall (i, j)$.
2. Solve analytically the Euler equation $c_{ij}^n = (u')^{-1}(M_{ij})$.
3. Find the endogenous grid $x_{ij}^n = s_i + c_{ij}^n, ap_i^n = s_i R$.
4. At s_1 if $x_{2j} > s_1 + \epsilon$ set $x_{1j}^n = s_1 + \epsilon$ else $x_{1j}^n = .9x_{2n}$ and $c_{1j}^n = x_{1j}^n - s_1$
5. Update and iterate from step 1.

Alternative methods (personally favoured):

- Discrete or continuous value function iteration.
- Projection methods, e.g. collocation or least squares.
- Local perturbation methods.
- If you really want PFI working on the Euler directly can be more intuitive.

1.2 Distribution Dynamics in Discrete Time

Remarks and notation:

- See also Chapter 10 in lecture notes (pag. 182).
- We will analyze the Chapman-Kolmogorov equation.
- In continuous time this is the Kolmogorov forward (Fokker-Planck) eqaution.
- Let $(X, \mathcal{X})$ be a measurable space where $X = A \times S = [0, \infty) \times \{l_l, l_h\}$.
- Let $P(S)$ power set of S, $\mathcal{B}(A)$ Borel σ-algebra of A, $\mathcal{A} \in \mathcal{B}(A), \mathcal{S} \in P(S)$.
- Let $\mathcal{X} = \sigma(\mathcal{B}(A) \times P(S)) = \mathcal{B}(A) \otimes P(S)$ be a product σ-algebra.
- Let $a_{t+1} = g^a(a_t, l_t)$ be the policy function of next period assets.

Definition. A transition function $Q : (X, \mathcal{X}) \to [0,1]$ for the process $\{a_t, l_t\}$ is

$$
\begin{aligned}
Q((a,l),(\mathcal{A},\mathcal{S})) &= 1_{[g^a(a,l)\in\mathcal{A}]} Q_l(l,\mathcal{S}) \\
&= 1_{[g^a(a,l)\in\mathcal{A}]} \sum_{l'\in\mathcal{S}} P_{l'|l}(l'|l) = \begin{cases} \sum_{l'\in\mathcal{S}} P_{l'|l}(l'|l) & \text{if } g^a(a,l)\in\mathcal{A} \\ 0 & \text{otherwise} \end{cases} ,
\end{aligned}
$$

for all $(\mathcal{A},\mathcal{S}) \in \mathcal{X}$.

- Q gives the probability of $a' \in \mathcal{A}, l' \in \mathcal{S}$ for someone with current states a, l.
- The transition function induces a sequence of probability distributions:

$$P_{a',l'}((\mathcal{A},\mathcal{S})) = \int_X Q((a,l),(\mathcal{A},\mathcal{S}))dP_{a,l}(a,l).$$

Let M be the set of probability measures on $(X, \mathcal{X})$. It is convenient to use an implicit form for the previous law of motion $H : M \to M$,

$$P_{a',l'} = H(P_{a,l}).$$

Numerical implementation:

- Let $f_t := P_{a_t,l_t}$ be a probability measure.
- Compute initial distribution $f_0(x_t, z_t)$ using the initial distribution of l.
- Solve forward H for $f_t(x_t, z_t)$ iterating over

$$P_{a',l'}(a', l') = \sum_{\{(a,l):a'=g^a(a,l)\}} P_{a,l}(a, l)P_{l'|l}(l'|l)$$

for each (a', l') on the grid.

- In general, the policy function a' will be on the grid. Once we selected the relevant points (a_i, l_j) for $i = 1, ..., I, j = 1, ..., J$ the integral reduces to $P_{t+1} = T_t' P_t$ where P_{t+1} is $I \times J$ and T_t is $(I \times J) \times (I \times J)$. This is not an approximation but is due to the discrete nature of the problem.
- Back to our example using cash on hand: If x' is not on the grid we might need interpoaltion. We also adjust for the survival risk multiplying by the fraction of households living in each period.

Modifications and simulations:

- Merging the steps of first constructing TT and then looping forward on Phi speed up the algorithm a bit (see code on github).
- Deterministic model without survival risk.
- Deterministic model with survival risk.
- Income and survival risk.
- If $\beta R < 1$ agents have an incentive to frontload consumption.

1.3 Data and Calibration

Calibrate the deterministic component y of income yz' (see github code):

- We set $y = \hat{y}_t$ for $t = 1, ..., 45$ or $20, ..., 64$.
- Load the vector of estimates $\hat{y}_t$ in matlab.
- Linearly extrapolate at the bounds $t < 25, t > 60$.
- Normalize $\hat{y}_t / \hat{y}_1$.

Calibrate income risk (see github code):

- Analytic formulae to approximate AR(1) with a 2 state Markov process.
- Using Corr$(\tilde{y}_t, \tilde{y}_{t-1}) = \rho(1 + \sigma_\epsilon^2/\sigma_z^2)^{-1}$.

1.4 Analysis

Remarks:

- EV: How much income will give me the same utility that would occur under a different allocation?
- CRRA preferences
$$\left(\frac{v_t^{post}(x,z)}{v_t(x,z)}\right)^{\frac{1}{1-\theta}} - 1.$$
- Welfare gains decline with cash on hand and age.
- The former is due to the distortion induced by the borrowing limit.
- The latter is due to the positive correlation between age and assets.