UNIVERSITY NAME

DOCTORAL THESIS

Non-linear structure formation in models beyond LCDM

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in the

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Department or School Name

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Declaration of Authorship

I, Santiago CASAS CASTRO, declare that this thesis titled, "Non-linear structure formation in models beyond LCDM" and the work presented in it are my own. I confirm that:

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“Thanks to my solid academic training, today I can write hundreds of words on virtually any topic without possessing a shred of information, which is how I got a good job in journalism.”

Dave Barry

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Abstract

Faculty Name
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Doctor in Physics

Non-linear structure formation in models beyond LCDM

by Santiago CASAS CASTRO

The Thesis Abstract is written here (and usually kept to just this page). The page is kept centered vertically so can expand into the blank space above the title too…

Acknowledgements

The acknowledgments and the people to thank go here, don't forget to include your project advisor…

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LAH	**L**ist **A**bbreviations **H**ere
WSF	**W**hat (it) **S**tands **F**or

Physical Constants

Speed of Light $c_0 = 2.997\,924\,58 \times 10^8\,\mathrm{m\,s^{-1}}$ (exact)

List of Symbols

a	distance	m
P	power	W $(\mathrm{J\,s^{-1}})$
ω	angular frequency	rad

For/Dedicated to/To my…

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In a homogeneous and isotropic universe, the distance element assumes the form:

$$ds^2 = a(\tau)[-(1+2\Psi)d\tau^2 + (1-2\Phi)\delta_{ij}dx^i dx^j] \, , \tag{2.1}$$

where the conformal time τ is related to the cosmic time t by $d\tau = dt/a(\tau)$ and Ψ and Φ are the two scalar gauge invariant gravitational potentials, related (modulo sign) to the Bardeen potentials.

Horndeski models include (almost) all theories with second order equations of motion in the fields and can be generically described with five functions of time. They include a variety of particular models studied in literature, characterised by a universal coupling to matter fields.

The Lagrangian can be written as ...

where ... is the matter Lagrangian and ... are arbitrary functions of the scalar field ϕ, whose canonical kinetic term is $X = -\frac{1}{2}\phi^{;\mu}\phi_{;\mu}$; the latter is also coupled to the Ricci scalar and the Einstein tensor.

In the following, we will adopt the conformal Newtonian gauge in the notation of (**MaBertschinger**), so that the gauge invariant potentials Ψ and Φ are also equal to the gravitational potentials in the Newtonian limit. In this formulation, the cold dark matter (CDM, c) perturbation equations for the density contrast δ and the velocity divergence θ can be written in Fourier space in the usual way:

$$\dot{\delta}(\tau, \mathbf{k}) = -(\theta(\tau, \mathbf{k}) - 3\Phi(\tau, \mathbf{k})) \; , \tag{2.2}$$

$$\dot{\theta}(\tau, \mathbf{k}) = -H\theta(\tau, \mathbf{k}) + k^2\Psi(\tau, \mathbf{k}) \; . \tag{2.3}$$

Derivating the first of the above equations with respect to τ and eliminating $\dot{\theta}$ and θ using both equations, one obtains the usual second order equation in time for the density contrast:

$$\ddot{\delta}(\tau, \mathbf{k}) = -(H\dot{\delta}(\tau, \mathbf{k}) - 3H\dot{\Phi}(\tau, \mathbf{k}) + k^2\Psi(\tau, \mathbf{k}) - 3\ddot{\Phi}(\tau, \mathbf{k})) \; . \tag{2.4}$$

In the following, we will stay within the quasi-static limit, i.e restrict to scales much smaller than the cosmological horizon ($k/aH \gg 1$) and well inside the Jeans length of the scalar field $c_s k \gg 1$, so that terms containing k dominate over terms containing time derivatives. In this case, Eq. 2.4 reduces to:

$$\ddot{\delta}(\tau, \mathbf{k}) + H\dot{\delta}(\tau, \mathbf{k}) + k^2\Psi(\tau, \mathbf{k}) = 0\,. \tag{2.5}$$

The gravitational potential is then obtained from the Poisson equation, that relates the gravitational potential to matter and relativistic species in the universe, by perturbing the Einstein field equations to first order.

In Horndeski theories, in the quasi-static limit, the deviation of the gravitational potentials from General Relativity has a known scale dependence in the Poisson equation, given by (**...**):

$$\mu(k, a) \equiv -\frac{2k^2\Psi}{3\Omega_m\delta_m} = h_1\left(\frac{1 + (k/k_*)^2 h_5}{1 + (k/k_*)^2 h_3}\right) \tag{2.6}$$

where h_1, h_3, h_5 are functions of time only. A similar expression, with different time dependent coefficients h_2, h_4, holds for the gravitational slip η:

$$\eta(k, a) \equiv -\frac{\Phi}{\Psi} = h_2\left(\frac{1 + (k/k_*)^2 h_4}{1 + (k/k_*)^2 h_5}\right)\,. \tag{2.7}$$

In this work we will consider all h_i functions as constants, for simplicity, although in general they are all functions of time. k_* is an arbitrary pivot scale, chosen at large scales close to the horizon, in order to fulfil the quasi-static limit.

As it can be seen from eqns. 2.6 and 2.7, the functions h_5 and h_3 are degenerate as well as h_4 and h_5. ΛCDM case is recovered when $\mu = \eta = 1$, which implies $h_5 = h_4 = h_3$ and an amplitude $h_1 = h_2 = 1$.

In the following, we will focus on CDM perturbations and deal only with the function $\mu(k, a) = \mu(k)$ (although perturbations require two functions of time and scale to fully specify the model). In particular, we will test the following cases:

1. Scale independent μ with modified amplitude: $h_5 = h_3$ and $h_1 > 1$ or $h_1 < 1$.
2. Scale dependent μ with unity amplitude: $h_1 = 1$ and $h_5 > h_3$ or $h_5 < h_3$.
3. Scale dependent μ with modified amplitude: $h_1 \neq 1$ and $h_5 > h_3$ or $h_5 < h_3$.

In logarithmic space $\mu(k)$ is very close to a step function, as shown in figure **??**, so that we will only use values for the h_i in which the difference of the minimum and the maximum is of the order of 15%, since otherwise we would impose a rather unrealistic structure formation.

2.1.4 Effective Field Theory of Dark Energy

2.2 Non-universal coupling

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Chapter 4

The non-linear evolution of matter perturbations

4.0.0.1 The non-linear standard equations in Fourier space

For studying large scale structure formation we will treat the Dark Matter distribution as a perfect fluid coupled to gravity, described by the continuity, Euler and Poisson's equations. This amounts to neglecting higher moments of the Vlasov equation, such as the stress tensor and velocity dispersions **bernardeau_large-scale_2001** This is equivalent to the single stream approximation and is one of the main limitations of Eulerian perturbation theory, since the theory breaks down as soon as shell crossing and multi-streaming start being important.

In the Einstein frame we have the following fluid equations for a general modification of gravity or a coupling to other scalar degrees of freedom **pietroni_flowing_2008**

$$\frac{\partial \delta_c}{\partial \tau} + \nabla \cdot [(1+\delta_c)\,\mathbf{v}] = 0 \tag{4.1}$$

$$\frac{\partial \mathbf{v}}{\partial \tau} + \mathcal{H}(\mathbf{v} + [\mathcal{A}\mathbf{v}]) + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\nabla\Psi \tag{4.2}$$

$$\nabla^2\Psi = \frac{3}{2}\mathcal{H}^2\Omega_c(\tau)\,(\delta_c + [\mathcal{B}\delta_c]) \tag{4.3}$$

Here τ is the conformal time,$\delta_c(\mathbf{x},\tau)$ and $\mathbf{v}(\mathbf{x},\tau)$ are respectively the matter density contrast and the peculiar velocity, $\mathcal{H} = \mathrm{d}\log a/\mathrm{d}\tau$ is the conformal Hubble parameter, $\Psi(\mathbf{x},\tau)$ the gravitational potential and the functions$\mathcal{A}(\mathbf{x},\tau)$ and $\mathcal{B}(\mathbf{x},\tau)$ are general functions of space and time that parametrize different cosmologies, when particles' geodesics are modified, i.e. the Poisson and Euler equations. This can happen due to couplings with a scalar field or more general modifications of gravity (MG), such as any Horndeski theory or massive bigravity theories **amendola_observables_2012** However, in the Jordan frame (where most MG theories are formulated) the Euler equation is not modified and there is only a space-time dependent modification to the Poisson equation, which can be expressed as in eqn. 2.6 once we go to Fourier space. In this work $\Omega_m(\tau) \equiv \Omega_c(\tau)$ are the time functions representing the cold dark matter (CDM) or the full matter density of the Universe, since we are not considering baryonic matter nor non-universal couplings.

The last term in the left hand side of eqns. 4.1 and 4.2 is what causes the nonliniearities in the evolution of the fluid. This can be seen in Fourier space as a mode-mode coupling or expressed as a nonlocality of the equations. This is the property that describes how a specific Fourier mode is able to depend on another mode at a different wave vector. We now go to Fourier space and will express equations 4.1-4.3 in a compact form introduced by **scoccimarro_new_2000**

The density is a scalar degree of freedom and the velocity can be decomposed such that:

$$\mathbf{v}(\mathbf{k}) = \mathbf{v}_\theta(\mathbf{k}) + \mathbf{v}_\omega(\mathbf{k}) \tag{4.4}$$

where:

$$\begin{aligned} \mathbf{k} \cdot \mathbf{v}_\omega(\mathbf{k}) &= 0 \\ \mathbf{k} \times \mathbf{v}_\theta(\mathbf{k}) &= 0 \end{aligned}$$

According to linear theory the vorticity component $\mathbf{v}_\omega(\mathbf{k})$ decays with the expansion of the Universe as a^{-1}, so if we assume non-vortical initial conditions, we can neglect this term and look only at the velocity divergence $\theta = i\mathbf{k} \cdot \mathbf{v}$, which is now a scalar function. Inverting this last relation to get: $\mathbf{v} = -\frac{i\mathbf{k}}{k^2}\theta$, allows us to perform the Fourier transforms explicitly:

$$\begin{aligned} FT\left\{\nabla \cdot (\delta_m \mathbf{v})\right\} &= +i\mathbf{k} \int \mathrm{d}^3q \mathrm{d}^3p \delta_D(\mathbf{p}+\mathbf{q}-\mathbf{k}) \frac{-i\mathbf{p}}{p^2} \delta_m(\mathbf{q},\tau)\theta(\mathbf{p},\tau) \\ &= \int \mathrm{d}^3q \mathrm{d}^3p \delta_D(\mathbf{k}-\mathbf{p}-\mathbf{q}) \underbrace{\frac{(\mathbf{p}+\mathbf{q})\cdot\mathbf{p}}{p^2}}_{\alpha(\mathbf{p},\mathbf{q})} \delta_m(\mathbf{q},\tau)\theta(\mathbf{p},\tau) \end{aligned}$$

$$\begin{aligned} FT\left\{\nabla \cdot [(\mathbf{v}\cdot\nabla)\cdot\mathbf{v}]\right\} &= i\mathbf{k} \cdot \int \mathrm{d}^3q \mathrm{d}^3p \delta_D(\mathbf{q}+\mathbf{p}-\mathbf{k}) \left(\frac{-i\mathbf{q}}{q^2}\cdot i\mathbf{p}\right) \frac{-i\mathbf{p}}{p^2}\theta(\mathbf{q},\tau)\theta(\mathbf{p},\tau) \\ &= \mathbf{k} \cdot \int \mathrm{d}^3q \mathrm{d}^3p \delta_D(\mathbf{k}-\mathbf{q}-\mathbf{p}) \left(\frac{\mathbf{p}\cdot\mathbf{q}}{p^2q^2}\right) \mathbf{p}\, \theta(\mathbf{q},\tau)\theta(\mathbf{p},\tau) \\ &= \int \mathrm{d}^3q \mathrm{d}^3p \delta_D(\mathbf{k}-\mathbf{q}-\mathbf{p}) \underbrace{\frac{(\mathbf{p}\cdot\mathbf{q})^2\mathbf{p}\cdot\mathbf{q}}{2p^2q^2}}_{\beta(\mathbf{q},\mathbf{p})} \theta(\mathbf{q},\tau)\theta(\mathbf{p},\tau) \end{aligned}$$

where in the last step we used the symmetry between $\mathbf{p}$ and $\mathbf{q}$. The terms marked with an underbrace, are the ones responsible for the mode-mode coupling:

$$\alpha(\mathbf{q},\mathbf{p}) = \frac{(\mathbf{p}+\mathbf{q})\cdot\mathbf{q}}{q^2} = \alpha(-\mathbf{q},-\mathbf{p});\; \beta(\mathbf{q},\mathbf{p}) = \frac{(\mathbf{p}+\mathbf{q})^2\mathbf{p}\cdot\mathbf{q}}{2p^2q^2} = \beta(-\mathbf{q},-\mathbf{p}) \tag{4.5}$$

4.0.0.2 The field notation

In the perturbation theory literature **bernardeau_large-scale_2001; crocce_renormalized_2005; pietroni_flowing_2008** eqns. 4.1-4.3 are written in a more compact way (also called field notation):

$$\partial_\eta \varphi_a(\mathbf{k},\eta) = -\Omega_{ab}(\mathbf{k},\eta)\varphi_b(\mathbf{k},\eta) + e^\eta \gamma_{abc}(\mathbf{k},-\mathbf{p},-\mathbf{q})\varphi_b(\mathbf{p},\eta)\varphi_c(\mathbf{q},\eta) \quad (4.6)$$

Where the field doublet is defined as:

$$\varphi_a(\mathbf{k},\eta) = e^{-\eta}\begin{pmatrix}\delta_m(\mathbf{k},\eta)\\ -\theta(\mathbf{k},\eta)/\mathcal{H}\end{pmatrix} \quad (4.7)$$

and we define a new time variable which will prove to be very convenient in this notation

$$\eta \equiv \log \frac{a}{a_{in}}$$

On the l.h.s. of eqn. 4.6 the second term represents all the nonlinearities and non-localities, if we neglect this term we go back to linear perturbation theory. The γ_{abc} functions in this formalism can be understood as interaction vertices and its only non-vanishing components are precisely given by the mode-mode coupling functions 4.5 :

$$\gamma_{121}(\mathbf{k},\mathbf{p},\mathbf{q}) = \frac{1}{2}\delta_D(\mathbf{k}+\mathbf{p}+\mathbf{q})\alpha(\mathbf{p},\mathbf{q}) \qquad \gamma_{121}(\mathbf{k},\mathbf{q},\mathbf{p}) = \gamma_{112}(\mathbf{k},\mathbf{p},\mathbf{q}) \quad (4.8)$$
$$\gamma_{222}(\mathbf{k},\mathbf{p},\mathbf{q}) = \delta_D(\mathbf{k}+\mathbf{p}+\mathbf{q})\beta(\mathbf{p},\mathbf{q})$$

in this notation and throughout this work, an integration over $\mathbf{p},\mathbf{q}$ is understood and $\mathbf{k}$ is always the "external" momentum. The integral signs will be added only when there could be a confusion.

Due to mode-mode coupling, there is a loss of information about the initial conditions in the limit of high-k, since density perturbations get mixed **crocce_memory_2006** This has as a consequence that highly-nonlinear corrections to the power spectrum become cosmology-independent in a rough way.

The information about the cosmological background is contained in:

$$\Omega_{ab}(\mathbf{k},\eta) = \begin{pmatrix} 1 & -1 \\ -\frac{3}{2}\Omega_m(\eta)(1+\mathcal{B}(\mathbf{k},\eta)) & 2+\frac{\mathcal{H}'}{\mathcal{H}}+\mathcal{A}(\mathbf{k},\eta)\end{pmatrix} \quad (4.9)$$

Where we have the usual definitions: $\mathcal{H} = aH,\ a\mathrm{d}\tau = \mathrm{d}t,\ a' = \frac{\mathrm{d}a}{\mathrm{d}\ln a} = a,\ \frac{\mathcal{H}'}{\mathcal{H}} = \frac{H'}{H}+1$, where $' := \frac{\mathrm{d}}{\mathrm{d}\ln a}$ and $N \equiv \ln a$ is what is typically called the e-foldings time.

For details on how to go back to the usual fluid equations 4.1-4.3 in Fourier space starting from the field equation 4.6, see Appendix **??**.

Equation 4.6 is the starting point for all renormalization and resummation methods **crocce_renormalized_2005**; **bernardeau_evolution_2013**; **bernardeau_constructing_2012**; **valageas_matter_2013**; **anselmi_nonlinear_2012**; **anselmi_next–leading_2010** The linear part can be easily solved and the function relating the initial primordial density perturbations to the final one, is called the linear propagator and will be studied in section 4.0.1. The nonlinear part, cannot be solved exactly analytically, nor numerically. But a perturbative approach using all tools from Quantum Field Theory can be used to regularize its divergences which are caused by the fact that the density perturbations (which is at the same time the perturbation variable) grows with time and increasing wave vector. For this part we will use the resummation technique of **anselmi_nonlinear_2012** and this will be explained in detail in section 4.1.

Renormalized perturbation theory (RPT) can help in finding the evolution of the nonlinear power spectrum at small scales and late times, but even if we could calculate exactly its result at all loop orders, there are still intrinsic limitations given by the starting equations 4.1-4.3. Apart from neglecting vorticity in the later stages of evolution, the initial equations are derived in the single-stream-approximation. This means that at a single point in space, there can be only one velocity direction. This clearly breaks down in the virialization regime and even before. The RPT approach can be extended and improved by including these other sources of density power into the equations in an effective way, see the discussion in **manzotti_coarse_2014**; **pietroni_coarse-grained_2011** and recent results in the effective field theory of large scale structures **baumann_cosmological_2012**; **pajer_renormalization_2013**; **senatore_ir-resummed_2014**; **carrasco_effective_2012**

4.0.1 Solving for the general scale dependent linear propagator

As was explained above, in eqn. 4.6 the non-linearity of the Vlasov-Poisson system of equations is fully encoded in the vertex γ_{abc} which represents the mode-mode coupling. Without this term, we recover the linear equation:

$$\partial_\eta \varphi_a(\mathbf{k},\eta) = -\Omega_{ab}(\mathbf{k},\eta)\varphi_b(\mathbf{k},\eta) \tag{4.10}$$

which is valid for a fully scale and time dependent Ω_{ab}.

The linear propagator is just the function that connects the initial density perturbations with the final ones, or in other words solves the above equation **crocce_renormalized_2005** If this equation has solutions of the form **pietroni_flowing_2008**

$$\varphi_{sol}(\mathbf{k},\eta) = \begin{pmatrix} 1 \\ f(\mathbf{k},\eta) \end{pmatrix} \varphi(\mathbf{k},\eta)$$

then we can find an equation that describes the evolution of the growth of perturbations.

For the index $a = 1$:

$$\partial_\eta \varphi = -\Omega_{11}\varphi - \Omega_{12} f\varphi = -(\Omega_{11} + \Omega_{12} f)\varphi \tag{4.11}$$

For the index $a = 2$:

$$\begin{aligned}
&\partial_\eta (f\varphi) = f\partial_\eta \varphi + \varphi \partial_\eta f = -\Omega_{21}\varphi - \Omega_{22} f\varphi \\
&\Rightarrow \partial_\eta f = \Omega_{12} f^2 + (\Omega_{11} - \Omega_{22})f - \Omega_{21} \qquad (4.12)\\
&\Rightarrow \partial_\eta f = \Omega_{12}(f - \bar{f}_+)(f - \bar{f}_-) \qquad (4.13)
\end{aligned}$$

Equation 4.12 is what we usually know as the growth rate equation for $f = \mathrm{d}\ln D/\mathrm{d}\ln a$, being D the growth factor of density perturbations and in this general case it can have a time and scale dependent solution.

The zeros of eqn. 4.13 are given by:

$$\bar{f}_{\pm}(\mathbf{k},\eta) = \frac{(\Omega_{22}-\Omega_{11}) \mp \sqrt{(\Omega_{22}-\Omega_{11})^2 + 4\Omega_{21}\Omega_{12}}}{2\Omega_{12}} \tag{4.14}$$

The solution of eqns. 4.11 and 4.13 is then given by:

$$\begin{aligned}
\varphi(\eta) &= e^{-\int_{\eta'}^{\eta}(\Omega_{11}+\Omega_{12}f)\mathrm{d}x}\varphi(\eta') \\
f(\eta)\varphi(\eta) &= e^{-\int_{\eta'}^{\eta}(\Omega_{21}+\Omega_{22}f)\mathrm{d}x} f(\eta')\varphi(\eta') \\
&= e^{-\int_{\eta'}^{\eta}(\Omega_{11}+\Omega_{12}f)\mathrm{d}x}\varphi(\eta')f(\eta')\frac{f(\eta)}{f(\eta')}
\end{aligned}$$

One can identify the basis solutions by setting their initial conditions as:

$$f_{\pm}^{in} = \bar{f}_{\pm}(\mathbf{k},\eta_i)$$

where η_i is an initial time that can be set at high redshift where the Universe is approximately Einstein-deSitter (E-dS) or strongly matter dominated.

For E-dS, $\Omega_m = 1$ we have very simple background quantities: $\mathcal{H}'\backslash\mathcal{H} = -1/2 - 3w_{eff}/2 = -1/2$, so that the Ω_{ab} from equation 4.9 is simply:

$$\Omega_{ab} = \begin{pmatrix} 1 & -1 \\ -\frac{3}{2} & \frac{3}{2} \end{pmatrix}$$

which gives the following initial conditions for the growing u and decaying v modes:

$$\begin{aligned}
u_a &= \begin{pmatrix} 1 \\ f_+^{in} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
v_a &= \begin{pmatrix} 1 \\ f_-^{in} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix}
\end{aligned}$$

The growing mode will be the mode of interest that we will use in section 4.1, when we want to calculate the evolution of the power spectrum.

The instantaneous projectors on the two basis solutions are defined as:

$$\begin{aligned}
\mathbf{M}^+(\mathbf{k},\eta)\begin{pmatrix} 1 \\ f_+(\mathbf{k},\eta) \end{pmatrix} &= \begin{pmatrix} 1 \\ f_+(\mathbf{k},\eta) \end{pmatrix} \\
\mathbf{M}^+(\mathbf{k},\eta)\begin{pmatrix} 1 \\ f_-(\mathbf{k},\eta) \end{pmatrix} &= 0 \\
\mathbf{M}^-(\mathbf{k},\eta)\begin{pmatrix} 1 \\ f_-(\mathbf{k},\eta) \end{pmatrix} &= \begin{pmatrix} 1 \\ f_-(\mathbf{k},\eta) \end{pmatrix} \\
\mathbf{M}^-(\mathbf{k},\eta)\begin{pmatrix} 1 \\ f_+(\mathbf{k},\eta) \end{pmatrix} &= 0
\end{aligned}$$

The growing projector can be written explicitly by subtracting the decaying projector from the unity matrix:

$$\mathbf{M}^+(\mathbf{k},\eta) = \mathbb{1} - \mathbf{M}^-(\mathbf{k},\eta) = \frac{1}{f_- - f_+}\begin{pmatrix} f_- & -1 \\ f_- f_+ & -f_+ \end{pmatrix}$$

For the Einstein-deSitter case, the projectors do not evolve in time, since u_a and v_a are constant, and they are given by:

$$\mathbf{M}^+ = \frac{1}{5}\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}$$
$$\mathbf{M}^- = \frac{1}{5}\begin{pmatrix} 2 & -2 \\ -3 & 3 \end{pmatrix}$$

The linear propagator is defined as the operator giving the linear evolution of the field φ_a, or in other words, solves eqn.4.10 :

$$\varphi_a(\mathbf{k},\eta) = g_{ab}(\mathbf{k},\eta,\eta')\varphi_b(\mathbf{k},\eta)$$

and it has to fulfill following properties:

$$\partial_\eta g_{ab}(\mathbf{k},\eta,\eta') = -\Omega_{ac}(\mathbf{k},\eta)\cdot g_{cb}(\mathbf{k},\eta,\eta')$$
$$\lim_{\eta'\to\eta} g_{ab}(\mathbf{k},\eta,\eta') = \mathbb{K}_{ab}$$
$$g_{ab}(\mathbf{k},\eta,\eta')\cdot g_{bc}(\mathbf{k},\eta',\eta'') = g_{ac}(\mathbf{k},\eta,\eta'')$$

Using these properties and the projectors, the propagator can be written in general as:

$$g(\mathbf{k},\eta,\eta') = \Theta(\eta-\eta')\left[e^{-\int_{\eta'}^{\eta}(\Omega_{11}+\Omega_{12}f_+)\mathrm{d}x}\begin{pmatrix} 1 & 0 \\ 0 & \frac{f_+(\mathbf{k},\eta)}{f_+(\mathbf{k},\eta')} \end{pmatrix}\mathbf{M}^+(\mathbf{k},\eta') + e^{-\int_{\eta'}^{\eta}(\Omega_{11}+\Omega_{12}f_-)\mathrm{d}x}\begin{pmatrix} 1 & 0 \\ 0 & \frac{f_-(\mathbf{k},\eta)}{f_-(\mathbf{k},\eta')} \end{pmatrix}\mathbf{M}^-(\mathbf{k},\eta')\right] \tag{4.15}$$

In the E-dS case, this would reduce to simply (since there is no k-dependence in any quantity and the growing mode is constant):

$$g(\eta,\eta') = \Theta(\eta-\eta')\left[\frac{1}{5}\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 2 & -2 \\ -3 & 3 \end{pmatrix}e^{-5/2(\eta-\eta')}\right]$$

Using the same procedure, we will calculate the linear propagator for the Horndeski case, in which the growth factor D is now scale and time dependent. Then the linear propagator can be used to calculate its fully nonlinear renormalized version, which then is a crucial ingredient of the evolution equation in section 4.1.

4.0.2 The linear propagator in the Horndeski case

Using eqn. 2.6 as the modification of the Poisson equation we can write the general Ω_{ab} matrix as:

$$\Omega_{ab}(\mathbf{k},\eta) = \begin{pmatrix} 1 & -1 \\ -\frac{3\mu(k)\Omega_m(\eta)}{2} & \frac{\mathcal{H}'}{\mathcal{H}}+2 \end{pmatrix} \tag{4.16}$$

in this case, the initial conditions from eqn.4.14 are:

$$f_-^{in} = -\frac{1}{2}(\Sigma + \omega)$$
$$f_+^{in} = \frac{1}{2}(\Sigma - \omega) \tag{4.17}$$

where $\omega = 1 + \frac{\mathcal{H}'}{\mathcal{H}} = \frac{1}{2} - \frac{3}{2}w_{eff}$ and $\Sigma = \sqrt{6Y\Omega_m + \omega^2}$ are quite general functions of scale and time. Inserting this into eqn.4.15, one can find the most general form of the propagator for the Horndeski theory. However, this is not so practical, since we would have to solve for a scale and time dependent growth rate f and evaluate the $\mathcal{H}$ function at each step.

For models which are close to ΛCDM, it is more convenient to change the time variable from $\eta = \ln \frac{a}{a_{in}}$ to $\mathcal{X} = \ln \frac{D(\tau)}{D(\tau=\tau_{in})}$, where $D(\tau)$ is the growth function usually written as the growing solution of the linear density perturbation equation : $\delta_c(\tau) = D_+(\tau)\delta_c^{in}$. In our case, however, the time variable $\mathcal{X}$ would itself be scale dependent, since the growth rate equation in the Horndeski case is scale dependent, so that $\mathcal{X}(k) = \ln \frac{D(\tau,k)}{D(\tau=\tau_{in},k)}$. Using $\frac{\partial}{\partial\eta} = \frac{\mathrm{d}\ln D}{\mathrm{d}\ln a}\frac{\partial}{\partial \ln D} = f\frac{\partial}{\partial\mathcal{X}}$, we can rewrite eqns.4.11-4.12 as (where $' := \frac{\partial}{\partial\mathcal{X}}$):

$$f(\mathbf{k},\mathcal{X})\delta_c'(\mathbf{k},\mathcal{X}) + \frac{\theta(\mathbf{k},\mathcal{X})}{\mathcal{H}} + \frac{\mathbf{k}\cdot\mathbf{q}}{q^2}\delta_c(\mathbf{q},\mathcal{X})\frac{\theta(\mathbf{p},\mathcal{X})}{\mathcal{H}} = 0 \tag{4.18}$$

$$f(\mathbf{k},\mathcal{X})\frac{\theta(\mathbf{k},\mathcal{X})'}{\mathcal{H}} + \frac{\theta(\mathbf{k},\mathcal{X})}{\mathcal{H}} + \frac{1}{2}\frac{\mathbf{k}^2\mathbf{q}\cdot\mathbf{p}}{q^2p^2}\frac{\theta(\mathbf{q},\mathcal{X})}{\mathcal{H}}\frac{\theta(\mathbf{p},\mathcal{X})}{\mathcal{H}} = -\frac{3}{2}\Omega_m(\mathcal{X})\mu(\mathbf{k})\mathcal{H}^2\delta_c(\mathbf{k},\mathcal{X}) \tag{4.19}$$

The doublet 4.7 can now be redefined as:

$$\tilde{\varphi}_a = \begin{pmatrix}\tilde{\varphi}_1\\ \tilde{\varphi}_2\end{pmatrix} = \begin{pmatrix} e^{-\mathcal{X}}\delta_c \\ -e^{-\mathcal{X}}\frac{\theta}{\mathcal{H}f}\end{pmatrix} \tag{4.20}$$

Substituting in the previous equations the values for δ_c, θ and $\delta_c' = e^{\mathcal{X}}(\varphi_1' + \varphi_1)$, $\theta' = -e^{\mathcal{X}}f(\mathcal{X})\mathcal{H}\left(\tilde{\varphi}_2\left(1 + \frac{f'(\mathcal{X})}{f(\mathcal{X})} + \frac{\mathcal{H}'}{\mathcal{H}}\right) + \tilde{\varphi}_2'\right)$ we get:

$$\tilde{\varphi}_1' + \tilde{\varphi}_1 - \tilde{\varphi}_2 - \alpha e^{\mathcal{X}}\tilde{\varphi}_1\tilde{\varphi}_2 = 0 \tag{4.21}$$

$$-\tilde{\varphi}_2' - \tilde{\varphi}_2(1 + \frac{f'}{f} + \frac{1}{f} + \frac{\mathcal{H}'}{\mathcal{H}}) + \beta e^{\mathcal{X}}\tilde{\varphi}_2\tilde{\varphi}_2 = -\frac{3}{2}\Omega_m\mu\frac{\tilde{\varphi}_1}{f^2}$$
$$\Rightarrow -\tilde{\varphi}_2' - \frac{3}{2}\frac{\Omega_m\mu}{f^2}\tilde{\varphi}_2 + \frac{3}{2}\frac{\Omega_m\mu}{f^2}\tilde{\varphi}_1 + \beta e^{\mathcal{X}}\tilde{\varphi}_2\tilde{\varphi}_2 = 0 \tag{4.22}$$

where we have omitted for notational simplicity the momentum and time dependence. In the last step we used the growth rate equation $\ddot{\delta}_m + \dot{\delta}_m\left(1 + \frac{\dot{\mathcal{H}}}{\mathcal{H}}\right) = \frac{3}{2}\Omega_m\delta_m\mu$, where in this case an overdot represents a derivative with respect to $\eta = \ln\frac{a}{a_{in}}$, since the $'$-symbol is now reserved for the $\mathcal{X}$ time variable.

Comparing eqns. 4.21-4.22 with 4.6, we get the following Ω matrix:

$$\tilde{\Omega}(\mathbf{k},\mathcal{X}) = \begin{pmatrix} 1 & -1 \\ -\frac{3}{2}\frac{\Omega_m(\mathcal{X})}{f_+^2(\mathcal{X})}\mu(k) & \frac{3}{2}\frac{\Omega_m(\mathcal{X})}{f_+^2(\mathcal{X})}\mu(k)\end{pmatrix} \tag{4.23}$$

It is of great importance to note that in the Horndeski case, the quantity $\frac{\Omega_m(\mathcal{X})}{f_+^2(\mathcal{X})}$ cannot be approximated to 1 as it is done in LCDM for all times, since in this case it can have a much greater or lower value during structure formation, but more importantly it is scale-dependent through the scale dependence of $\mathcal{X}(k)$. This can be seen clearly in plot **??**.

4.0.3 Assuming a constant $\mu \neq 1$ for the 1-loop quantities

Assuming a constant μ different from 1 and the approximation that $\frac{\Omega_m(\mathcal{X})}{f_+^2(\mathcal{X})} = 1$ throughout the history of the Universe (which is valid only for E-dS but turns out to be a good approximation (much more accurately than 1%) for the nonlinear ΛCDM power spectrum), we can follow the steps given above in section 4.0.1 and obtain the initial growing and decaying modes:

$$u_a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$v_a = \frac{-2}{3Y} \begin{pmatrix} 1 \\ -\frac{3Y}{2} \end{pmatrix}$$

and with this the projectors:

$$\mathbf{M}^+ = \frac{1}{2+3Y} \begin{pmatrix} 3Y & 2 \\ 3Y & 2 \end{pmatrix}$$
$$\mathbf{M}^- = \frac{1}{2+3Y} \begin{pmatrix} 2 & -2 \\ -3Y & 3Y \end{pmatrix}$$

The propagator would then look like:

$$g(\mathcal{X}, \mathcal{X}') = \Theta(\mathcal{X}-\mathcal{X}') \left[\frac{1}{2+3Y} \begin{pmatrix} 3Y & 2 \\ 3Y & 2 \end{pmatrix} + \frac{1}{2+3Y} \begin{pmatrix} 2 & -2 \\ -3Y & 3Y \end{pmatrix} e^{-\frac{(2+3Y)}{2}(\mathcal{X}-\mathcal{X}')} \right] \tag{4.24}$$

For the rest of this work, eqn. 4.24 will be the linear propagator we will use, even if we are treating more general cases where $\mu(k)$ is an arbitrary function and $\Omega_m/f^2 \neq 1$. This approximation can be justified better in the next section, when we will see that g only enters in the evolution equation of the power spectrum inside the integral of the self-energy and mode-coupling 1-loop quantities, which should contribute only sub-dominantly in the final power spectrum.

4.1 The Evolution Equation for the Power Spectrum

We are interested in solving the general scale and time dependent evolution equation from the resummation method presented in **anselmi_nonlinear_2012** (eqn. 53):

$$\begin{aligned}\partial_{\mathcal{X}} \tilde{P}_{ab}(k;\mathcal{X}) &= -\tilde{\Omega}_{ac}(\mathbf{k};\mathcal{X})\tilde{P}_{cb}(\mathbf{k};\mathcal{X}) - \tilde{\Omega}_{bc}(\mathbf{k};\mathcal{X})\tilde{P}_{ac}(\mathbf{k};\mathcal{X}) \\ &+ H_{\mathbf{a}}(k;\mathcal{X},\mathcal{X}_{in})\tilde{P}_{\mathbf{a}b}(\mathbf{k};\mathcal{X}) + H_{\mathbf{b}}(k;\mathcal{X},\mathcal{X}_{in})\tilde{P}_{a\mathbf{b}}(\mathbf{k};\mathcal{X}) \\ &+ \int \mathrm{d}s \left[\tilde{\Phi}_{ad}(k;\mathcal{X},s) G^{eik}_{bd}(k;\mathcal{X},s) + G^{eik}_{ad}(k;\mathcal{X},s)\tilde{\Phi}_{db}(k;\mathcal{X},s)\right]\end{aligned} \tag{4.25}$$

where $\mathcal{X} = \ln(D(a)/D(a_{in}))$ (notice that in ref. **anselmi_nonlinear_2012** η is the time variable and represents the growth factor). The first line of this equation corresponds to the linear evolution equation of the power spectrum already discussed before. The second and third lines contain the 1PI (one-particle-irreducible) functions, the self-energy Σ_{ab} and the mode-coupling term $\tilde{\Phi}G^{AB}_{ab}$ accounting for the contributions at the large- and small-k limit at 1-loop order.

Massimo: Can you check from eqns. 33-36?

If we transform this equation to $\eta = \ln \frac{a}{a_{in}}$, using the variable transformation $\partial\mathcal{X}/\partial\eta = \frac{\mathrm{d}\ln D(a)}{\mathrm{d}\ln a} = f(\eta)$, where $f(\eta) \equiv f(N(\eta))$ and the relation between N and η is given by $N = \eta + \ln(a_{in})$, we have to transform also the power spectrum since the field has been redefined (see 4.20):

$$\tilde{P}_{ab} = e^{-2\mathcal{X}(\eta)} e^{2\eta}(\delta_{a1} + \frac{1}{f(\eta)}\delta_{a2})(\delta_{b1} + \frac{1}{f(\eta)}\delta_{b2}) P_{ab}(\eta) \tag{4.26}$$

where we can call this transformation Ξ_{ab}:

$$\tilde{P}_{ab} = \Xi_{ab} P_{ab}(\eta)$$

and its inverse would be:

$$\Xi^{-1}_{ab} = e^{2\mathcal{X}(\eta)} e^{-2\eta}(\delta_{a1} + f(\eta)\delta_{a2})(\delta_{b1} + f(\eta)\delta_{b2}) \tag{4.27}$$

However, eqn. 4.25 is only invariant under the transformation:

$$\Upsilon_{ab}(\eta) = f(\eta)\Xi^{-1}_{ab} = f(\eta) e^{2\mathcal{X}(\eta)} e^{-2\eta}(\delta_{a1} + f(\eta)\delta_{a2})(\delta_{b1} + f(\eta)\delta_{b2}) \tag{4.28}$$

since we also have to transform the derivatives and the Ω_{ab} functions.

Inserting these transformations into eqn. 4.25, and generalizing to the case where the growth rate is k-dependent, $f(\eta, k)$,we obtain:

$$\begin{aligned}\partial_\eta P_{ab}(k;\eta) &= -\Omega_{ac}(\mathbf{k};\eta)P_{cb}(\mathbf{k};\eta) - \Omega_{bc}(\mathbf{k};\eta)P_{ac}(\mathbf{k};\eta) \\ &+ [H_{\mathbf{a}}(k;\mathcal{X}(\eta,k),\mathcal{X}(\eta_{in},k)) f(\eta,k) P_{\mathbf{a}b}(k;\eta) \\ &+ H_{\mathbf{b}}(k;\mathcal{X}(\eta,k),\mathcal{X}(\eta_{in},k)) f(\eta,k) P_{a\mathbf{b}}(k;\eta)] \\ &+ \Upsilon_{ab}(\eta,k) \times \int \mathrm{d}s \left[\tilde{\Phi}_{ad}(k;\mathcal{X}(\eta,k),s) G^{eik}_{bd}(k;\mathcal{X}(\eta,k),s) + G^{eik}_{ad}(k;\mathcal{X}(\eta,k),s)\tilde{\Phi}_{db}(k;\mathcal{X}(\eta,\end{aligned} \tag{4.29}$$

The quantities such as $H_{\mathbf{a}}(k;\eta,\eta_{in})$ and $\tilde{\Phi}_{ad}(k;\eta,s)$, shown in eqns 4.33-4.36, which are actually calculated in the $\mathcal{X}$ variable, are now indirectly a function of η through the relation $\mathcal{X}(\eta) = \ln(D(N(\eta))/D(N(\eta_{in})))$.

The evolution equation as it is written in eqn. 4.25 relies on three different assumptions: the power spectrum is well behaved for $k \to 0$, which is in our cosmology a good assumption, since it behaves just a power law k^n; there is a clear separation of scales between "hard" and "soft" modes, or in other words, the eikonal limit is fulfilled. This means that the modes k we are interesting in are much bigger than the internal coupling modes p, q. The third assumption is of course the single-stream approximation, which is used in all forms of resummed and renormalized perturbation theories in cosmology as was explained already in section 4.0.0.2.

In this work we want to solve eqn. **??** for the simple Horndeski models stated above. We will test two apporaches, first we will include inside the $\Omega_{ac}(\mathbf{k};\eta)$ functions, the full scale and time dependence of the parametrized Horndeski models. These terms will have a dominant effect on the evolution of $P(\mathbf{k})$ and we will calculate the 1-loop integrals using the usual ΛCDM model. Since the $H_{\mathbf{a}}(k;\eta,\eta_{in})$ and $\tilde{\Phi}_{ab}(k;\eta,s)$ functions are calculated for a scale-dependent growth factor, we will have to test if this scale dependence has a big effect on the result.

As a second approach we will calculate the 1PI functions using the linear Horndeski propagator in the case that μ is a constant different from one. Once these integrals are calculated, we can then solve the evolution equation completely. In this case we have to test how much a change in $\mu(k)$ affects the result, i.e. if assuming a constant μ is a good approximation or not. In the next section we will show the explicit expressions for these integrals.

4.1.1 The 1PI functions in the Horndeski constant $\mu \neq 1$ case

The Horndeski variable is Y for compatibility with the code for now, but it can be changed to μ later on.

Now we give the general expression for for the 1PI functions $\Sigma^{(1)}_{ab}(k;\mathcal{X},\mathcal{X}')$ and $\Phi^{(1)}_{ab}(k;\mathcal{X},\mathcal{X}')$ computed at 1-loop in eRPT. The general expression for $\Sigma^{(1)}_{ab}(k;\mathcal{X},\mathcal{X}')$ is given by:

$$\Sigma^{(1)}_{ab}(k;\mathcal{X},\mathcal{X}') = \\ 4e^{\mathcal{X}+\mathcal{X}'}\int \mathrm{d}^3q\gamma_{acd}(\mathbf{k},-\mathbf{q},\mathbf{q}-\mathbf{k})u_cP^0(q)u_e\gamma_{feb}(\mathbf{k}-\mathbf{q},\mathbf{q},-\mathbf{k})g_{df}(\mathcal{X},\mathcal{X}') \tag{4.30}$$

If we insert for g_{ab} the linear propagator from 4.24 and the coupling vertices γ_{abc} from 4.8 we have to perform an the angular integration of d^3q, see Appendix **??** for the explicit terms.

The $H_1(k;\mathcal{X},-\infty)$, $H_2(k;\mathcal{X},-\infty)$ functions are a time integration of the $\Sigma^{(1)}_{ab}(k;\mathcal{X},\mathcal{X}')$ quantities in the internal time from minus infinity to the external time η:

$$H_1(k;\mathcal{X},-\infty) = \int_{-\infty}^{\eta}\mathrm{d}s\Sigma^{(1)}_{1b}(k;\mathcal{X},\mathcal{X}')u_b \tag{4.31}$$

$$H_2(k;\mathcal{X},-\infty) = \int_{-\infty}^{\eta}\mathrm{d}s\Sigma^{(1)}_{2b}(k;\mathcal{X},\mathcal{X}')u_b \tag{4.32}$$

They have the following form after performing the time integration:

$$H_{1Y}(k;\mathcal{X},-\infty) = -\frac{\pi k^3 e^{2\mathcal{X}}}{3(3Y+4)}\int \mathrm{d}r \left[16 + 3Y\left(3r^4 - 8r^2 + 1\right) - \frac{9Y}{2r}\left(r^2-1\right)^3 \log\left|\frac{1+r}{1-r}\right|\right] P^0(kr) \tag{4.33}$$

$$H_{2Y}(k;\mathcal{X},-\infty) = -\frac{\pi k^3 e^{2\mathcal{X}}}{3(3Y+4)}\int \mathrm{d}r \left[-\frac{9Y}{r^2} + 9r^2\mu + 4(9Y+4) - \frac{9Y}{2r^3}\left(r^2-1\right)^3 \log\left|\frac{1+r}{1-r}\right|\right] P^0(kr) \tag{4.34}$$

The third line of the evoution equation 4.25 contains the mode-coupling function, which is obtained by integrating the counter-term 1-loop quantity:

$$\tilde{\Phi}^{(1)}_{ad}(k;\mathcal{X},\mathcal{X}') = 2e^{\mathcal{X}+\mathcal{X}'}\int \mathrm{d}^3q\,\gamma_{acd}(\mathbf{k},-\mathbf{q},\mathbf{p})u_c P^0(q) u_d \tag{4.35}$$

$$\times\, u_e P^0(p) u_f \gamma_{bef}(\mathbf{k},-\mathbf{q},\mathbf{p}) \tag{4.36}$$

where $\mathbf{p} = \mathbf{k} - \mathbf{q}$, together with the renormalized propagator from Crocce-Scoccimarro $\bar{G}^L_{bd}(k;\eta,s)$:

$$\int \mathrm{d}s \left[\tilde{\Phi}_{ad}(k;\mathcal{X},s)\bar{G}^L_{bd}(k;\mathcal{X},s) + \bar{G}^L_{ad}(k;\mathcal{X},s)\tilde{\Phi}_{db}(k;\mathcal{X},s)\right] = \tilde{\Phi}G^A_{ab}(k;\mathcal{X}) + \tilde{\Phi}G^B_{ab}(k;\mathcal{X}) \tag{4.37}$$

where for all a, b, the B terms are:

$$\tilde{\Phi}G^B_{ab}(k;\mathcal{X}) = u_a u_b y^2\left(\frac{\sqrt{\pi}}{2}\left(2y^2+1\right)\mathrm{Erf}(y) + \left(e^{-y^2}-2\right)y\right)P^0(k) \tag{4.38}$$

The $\tilde{\Phi}G^B_{ab}(k;\eta)$ expression has to be switched off in the small k-limit since it contains 2-loop expressions valid only at large k, therefore it has to be "filtered" by a momentum-cutoff function:

$$\tilde{\Phi}G^A_{ab}(k;\eta) + \frac{(k/\bar{k})^4}{1+(k/\bar{k})^4}\tilde{\Phi}G^B_{ab}(k;\eta)$$

The $\bar{k}$ quantity can be set to a reasonable scale at which the large scale expression starts to be applicable, usually we can set here $\bar{k} = 0.2h/$Mpc. The A terms read (a new variable $W = \frac{3}{2}Y$ has been defined for simplicity):

$$\tilde{\Phi}G^A_{11} = y(\Phi^{(1)}_{11} - \Phi^{(1)}_{11})\mathcal{B}(y^2;W) \tag{4.39}$$

$$+ \frac{\sqrt{\pi}\mathrm{Erf}(y)}{W+1}(\Phi^{(1)}_{11}W + \Phi^{(1)}_{12})$$

$$\tilde{\Phi}G^A_{12} = \frac{y\mathcal{B}_{12}(y^2;W)}{(1+W)^2(2+W)}(\Phi^{(1)}_{12} - \Phi^{(1)}_{22} - W(\Phi^{(1)}_{11} - \Phi^{(1)}_{12})) \tag{4.40}$$

$$+ \frac{\sqrt{\pi}\mathrm{Erf}(y)}{2(W+1)}(W(\Phi^{(1)}_{11} + \Phi^{(1)}_{12}) + \Phi^{(1)}_{12} + \Phi^{(1)}_{22}) \tag{4.41}$$

$$\tilde{\Phi}G^{A}_{22} = \frac{yW}{(W+1)^2}(\Phi^{(1)}_{22} - \Phi^{(1)}_{12})\mathcal{B}(y^2;W) + \frac{\sqrt{\pi}}{(W+1)}\text{Erf}(y)(\Phi^{(1)}_{12}W + \Phi^{(1)}_{22}) \tag{4.42}$$

and where we have used the combination of generalized Hypergeometric functions:

$$\begin{aligned}\mathcal{B}(y^2;W) = {}& {}_2F_2\left(\frac{1}{2},1;\frac{W}{2}+1,\frac{W}{2}+\frac{3}{2};-y^2\right)\\ &+\frac{W+1}{W+2}\,{}_2F_2\left(\frac{1}{2},1;\frac{W}{2}+\frac{3}{2},\frac{W}{2}+2;-y^2\right)\\ &-\frac{1}{W+2}\,{}_2F_2\left(1,\frac{3}{2};\frac{W}{2}+\frac{3}{2},\frac{W}{2}+2;-y^2\right)\end{aligned}$$

$$\begin{aligned}\mathcal{B}_{12}(y^2;W) = {}&(2+W)\,{}_2F_2\left(\frac{1}{2},1;\frac{W}{2}+1,\frac{W}{2}+\frac{3}{2};-y^2\right)\\ &-{}_2F_2\left(1,\frac{3}{2};\frac{W}{2}+\frac{3}{2},\frac{W}{2}+2;-y^2\right)\end{aligned}$$

Assuming a general scale-dependent function $\mu(k)$ would turn out impossible an analaytic calculation of the intermediate quantities, which would render a practical implementation of this method to be numerically prohibitive. Therefore, using the fact that the μ function behaves as a step function in $\log k$, having a well defined minimum and maximum value we can use simply a constant and check its effect on the 1-loop and 2-loop corrections. Besides, in any viable model compatible with observations μ can differ from 1 by about 15-20%. So that we can be sure that the overall effect will be just a perturbation around the LCDM value and should reduce to it in the limit $\mu \to 1$.

Issue: In the LCDM case from Massimo's paper, all three $\tilde{\Phi}G^{A}_{ab}$ functions can be written using the same combination of Hypergeometric Functions. In this case, this is only possible for $\tilde{\Phi}G^{A}_{11}$ and $\tilde{\Phi}G^{A}_{22}$.

4.1.2 Method of implementation

Equation **??** is the final and main equation we want to solve and it represents a differential equation in time, for each external momentum k that we are interested in. This means that if we want to compute the power spectrum on a grid with 100 points in k-space, we need to solve 100 times the same differential equation. That is why parallelism plays an important role here to speed up the computational calulation.

First we compute the linear growth function for a specific Horndeski model, using the linear part of 4.19. With this we obtain the growth rate and the growth factor, which are then used to calculate the loop integrals.

The initial power spectrum is obtained from CAMB and its evaluated at 100 points in k-space. This is used to compute analytically eqns. 4.33, 4.34. These integrals can be stored as long as the cosmological parameters are not modified.

Finally we compute the 4.25 for each k mode in parallel, which using 8 cores on a normal machine, takes about 15 seconds until redshift $z = 0$.

Chapter 5

Conclusions

Appendix A

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