Probabilistic Systems Analysis and Applied Probability

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摘要

This course introduces students to the modeling, quantification, and analysis of uncertainty. The tools of probability theory, and of the related field of statistical inference, are the keys for being able to analyze and make sense of data. These tools underlie important advances in many fields, from the basic sciences to engineering and management.

Instructor(s) : Prof. John Tsitsiklis

1 Lecture 1: Probability Models and Axioms

1.1 Reading

- Sections 1.1, 1.2

1.2 Lecture outline

- Probability as a mathematical framework for:
  - reasoning about uncertainty

    - developing approaches to inference problems

- Probabilistic models
    - sample space
    - probability law
- Axioms of probability
- Simple examples

1.3 Sample space Ω

- "List" (set) of possible outcomes
- List must be:
    - Mutually exclusive(互斥)
    - Collectively exhaustive(互补)
- Art: to be at the "right" granularity(粒度)

1.4 Sample space: Discrete example

- Two rolls of a tetrahedral(四面體) die
    - Sample space vs. sequential description
- See Figure 1

1.5 Sample space: Continuous example

- $\omega = \{(x, y) | 0 \leq x, y \leq 1\}$
- See Figure 2

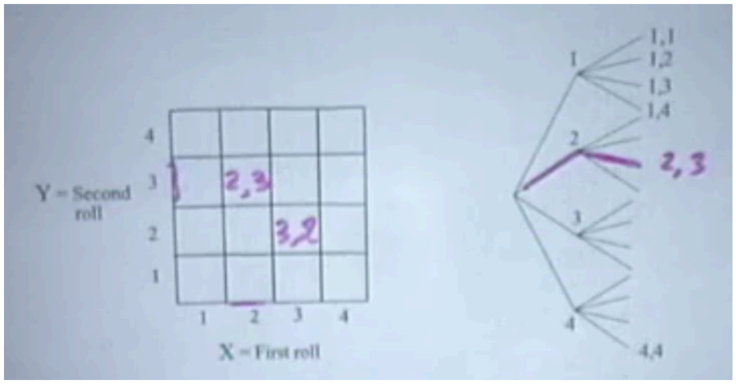

图 1: Sample space: Discrete example

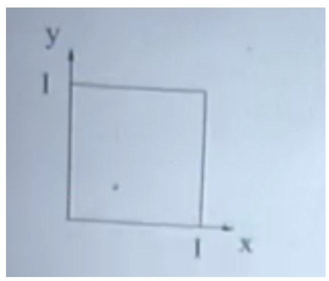

图 2: Sample space: Continuous example

1.6 Probability axioms(公理)

- **Event:** a subset of the sample space
- See Figure 3

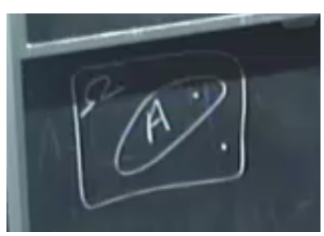

图 3: Event: a subset of the sample space

- Probability is assigned to events

- **Axioms:**
- 1. **Nonnegativity:** $P(A) \geq 0$
- 2. **Normalization:** $P(\Omega) = 1$
- 3. **Additivity:** If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$

- See Figure 4

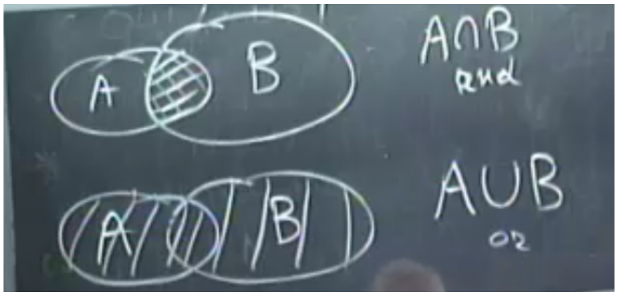

图 4: Axioms: Additivity

$$
\begin{aligned}
1 &\overset{(2)}{=} P(\Omega) = P(A \cup A^c) \\
&\overset{(3)}{=} P(A) + P(A^c) \\
P(A) &= 1 - P(A^c) \overset{(1)}{\leq} 1
\end{aligned}
$$

- See Figure 5

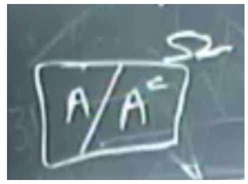

图 5: A and complement of A

$$
\begin{aligned}
&P(A \cup B \cup C) \\
&= P((A \cup B) \cup C) \\
&= P(A \cup B) + P(C) \\
&= P(A) + P(B) + P(C)
\end{aligned}
$$

If $A_1, A_2, ..., A_n$ disjoint. the $P(A_1 \cup ... \cup A_n) = P(A_1) + ... + P(A_n)$

- See Figure 6

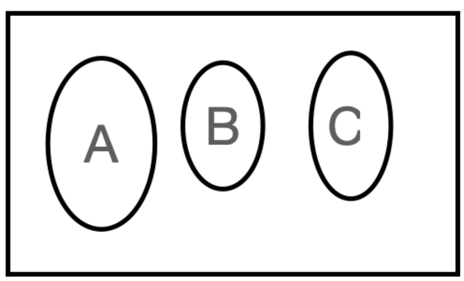

图 6: A,B,C in an *Omega*

$$
\begin{aligned}
P(\{s_1, s_2, \ldots, s_k\}) &= P(\{s_1\}) + \cdots + P(\{s_k\}) \\
&= P(s_1) + \cdots + P(s_k)
\end{aligned}
$$

- See Figure 7

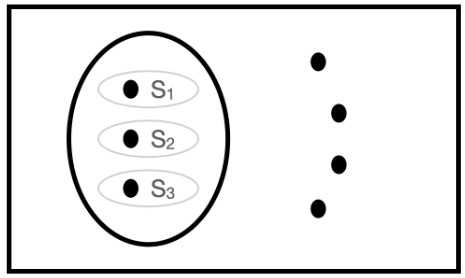

图 7: finite elements in an *Omega*

- Axiom 3 needs strengthening(加强)
- Do weird sets have probabilities?

1.7 Probability law: Example with finite sample space

- See Figure 8

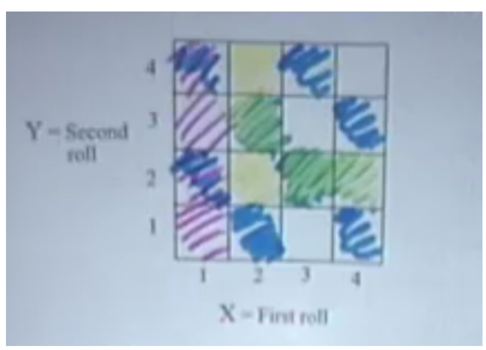

图 8: Probability law: Example with finite sample space

- Let every possible outcome have probability 1/16
  - $P((X, Y)\ is\ (1,1)\ or\ (1,2)) = 2/16$

- $P(X = 1) = 4/16$
- $P(X + Y \ is \ odd) = 8/16$
- $P(min(X, Y) = 2) = 5/16$

1.8 Discrete uniform law

- Let all outcomes be equally likely
- Then,
$$P(A) = \frac{number\ of\ elements\ of\ A}{total\ number\ of\ sample\ points}$$
- Computing probabilities $\equiv$ counting
- Defines fair coins, fair dice, well-shuffled card decks

1.9 Continuous uniform law

- Two "random" numbers is [0,1].
- See Figure 9

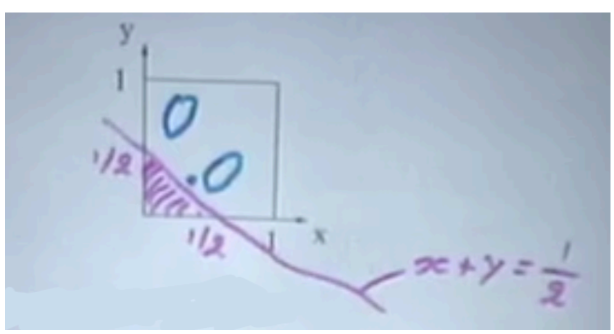

图 9: Continuous uniform law

- **Uniform** law: Probability $\equiv$ *Area*

- $P(X + Y \leq 1/2) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$
- $P((X, Y) = (0.5, 0.3)) = 0$

1.10 Probability law: Ex. w/countably infinite sample space

- Sample space: $\{1, 2, ...\}$
  - We are given $P(n) = 2^{-n}, n = 1, 2, ...$
  - Find $P(outcome\ is\ even)$
  - See Figure 10

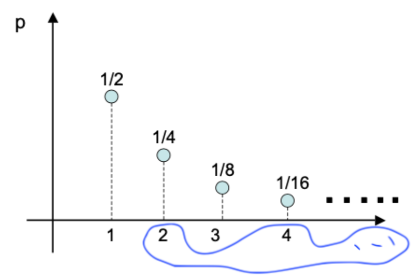

图 10: Probability law: Ex. w/countably infinite sample space

$$P(\{2, 4, 6, ...\}) = P(2) + P(4) + ... = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6}$$

- Countable Additivity axiom (needed for this calculation)(Santiago comment: It is **infinite** on this example instead of finite on before one): If $A_1, A_2, ...$ are disjoint events, then:

$$P(A_1 \cup A_2 \cup ...) = P(A_1) + P(A_2) + ...$$

2 Week 666666 - Transport Layer Overview

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