

ESTIMATION AND APPROXIMATION OF TEMPERED STABLE
DISTRIBUTION

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Estimation and Approximation of Tempered Stable Distribution

Abstract

by

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Tempered stable random variables have a LePage like series representation, which was first introduced by Rosiński. In this dissertation, we study the accuracy of the Rosiński representation as determined by the convergence rates of the series. We also study estimators of parameters of certain tempered stable distributions and construct their confidence intervals. Finally, we present several simulation results for the Gamma-tempered random variable.

Keywords: Tempered stable distribution, convergence rate, parameter estimation.

Chapter 1

Introduction

1.1 Motivation

Stable laws, also called α -stable, or Lévy stable, had been first introduced by Paul Lévy in his study [8] of summation of independent random variables. They possess the property that the sum of two independent random variables having stable distribution with index α is again α -stable. Because of its "fat tail" property, stable distributions found applications in many fields where the normality assumption is not valid. Especially in the last 20 years, applications of α -stable distributions can be found in physics, finance, telecommunication, sociology, see, e.g., [2], [9]

An α -stable distribution has four parameters: index of stability $\alpha \in [0, 2]$, skewness parameter $\beta \in [-1, 1]$, scale parameter $\sigma > 0$, and location parameter $\mu \in \mathbb{R}$. The parameter α determines the decay rate of the tail; an α -stable distribution does not have a p -th moment for $p \geq \alpha$. With the exception of three cases, α -stable probability density functions (PDF) and cumulative distribution functions (CDF) do not have closed form expressions. The three special cases include Gaussian ($\alpha = 2$), Cauchy($\alpha = 1, \beta = 0$) and Lévy ($\alpha = 0.5, \beta = 1$) distributions. Hence the most convenient description of α -stable is by the characteristic function, but given the importance of the

α -stable distributions, several other representations for them have been developed. One of them is the so called LePage series representation, see, [5] and [7]. The convergence rate of LePage series was studied in [3] and [4].

Tempered stable distributions are distributions obtained by 'tempering' α -stable laws. Unlike α -stable distributions, tempered stable distributions have all the moments finite. The class was introduced by Jan Rosiński in [12], where, in particular, he derived a series representation for them, analogous to LePage representation. Some parametric families of tempered stable distributions were studied in [10] and [15].

In this dissertation, we study the convergence rate of the Rosiński's series representation and show that the error term decays as $n^{1-1/\alpha}$, for $\alpha < 1$. We also study other applications of the representation. Also, the issue of parametric estimation for some tempered stable distributions is addressed.

In chapter 1 we give a detailed introduction to the LePage series representation and the asymptotic properties of the LePage series. In Chapter 2, a method-of-moment estimation of some tempered stable parameters is introduced. Chapter 3 starts with the introduction of tempered stable distributions and their Lepage-Rosiński series representation. Several simulation results concerning the rate of convergence of the series can be found in Chapter 4.

1.2 Preliminaries: Asymptotics of Tails of Probability Distributions in Terms of Their Characteristic Functions

In this section, we collect well known [11] results on the relationships between asymptotic properties of the tails of probability distributions and behavior of their characteristic functions. Results of this type are called

Tauberian theorems. Because they are important for understanding our main body of work, we describe them systematically and provide detailed proofs.

Let $F(x)$ denote the cumulative distribution function of a random variable X . Write $H(x) = 1 - F(x) + F(-x)$, so that $H(x)$ is the sum of the tails of the distribution function. The characteristic function $\phi(t)$ of the distribution function $F(x)$ is defined as the inverse Fourier transform of $F(x)$, i.e., $\phi(t) = \int_{-\infty}^{+\infty} e^{itx} dF(x)$. Let $U(t)$ and $V(t)$ be the real and complex parts of $\phi(t)$, respectively, i.e., $\phi(t) = U(t) + iV(t)$. Then,

$$\frac{1 - U(t)}{t} = \int_0^{\infty} H(x) \sin tx \, dx,$$

and inversely,

$$H(x) = \frac{2}{\pi} \int_0^{\infty} \frac{1 - U(t)}{t} \sin xt \, dt.$$

One can verify the above identities by integration by parts of the expression

$$U(t) = \int_{-\infty}^0 \cos xt \, dF(x) + \int_0^{+\infty} \cos xt \, d[F(x) - 1].$$

Definition 1.2.1 *We say that a function $G(x)$ is of index k , as $x \rightarrow \infty$, if, for every $\lambda > 0$, $\frac{G(\lambda x)}{G(x)} \rightarrow \lambda^k$, as $x \rightarrow \infty$. Similarly, we say that a function $G(x)$ is of index k , as $x \downarrow 0$, if, for every $\lambda > 0$, $\frac{G(\lambda x)}{G(x)} \rightarrow \lambda^k$, as $x \downarrow 0$.*

What follows is a series of classical results (see [11]) that relate the asymptotic properties of the tail function H with those of the Fourier transform $U(t)$. For the sake of readability of this dissertation and its completeness we provide full proofs.

Lemma 1.2.1 *Let $G(w)$ be positive and bounded for $w > b$ and let h and c*

be greater than 0. If $G(w)$ is of index $-m$ as $w \rightarrow \infty$, where $m \geq 0$, then there exist A and B such that

$$\frac{G(\lambda w)}{G(w)} < \frac{A}{\lambda^{m+h}}, \quad w \geq B, 0 < \lambda \leq c,$$

and

$$\frac{G(\lambda w)}{G(w)} < \frac{A}{\lambda^{m-h}}, \quad w \geq B, \lambda \geq c.$$

Proof. It is easy to show that there exists some B such that $\frac{G(\lambda w)}{G(w)}$ is bounded for $w \geq B, \lambda \leq c$. Since $w^{m+h}G(w)$ is of index h , there exists an A such that

$$\frac{(\lambda w)^{m+h}G(\lambda w)}{w^{m+h}G(w)} < A.$$

Therefore

$$\frac{G(\lambda w)}{G(w)} < \frac{A}{\lambda^{m+h}}.$$

Proof for the second statement of the Lemma is similar. Q.E.D.

Lemma 1.2.2 *If $G(w)$ is monotonic for $w > a$ and $\int_a^w u^r G(u) du$ is of index k as $w \rightarrow \infty$, where $k > 0$, then $w^r G(w)$ is of index $(k - 1)$.*

Proof. Without loss of generality we assume that $G(w)$ is positive when $w > a$. We give the proof for $G(w)$ non-increasing, and $r \geq 0$.

For some $\mu > 1, \lambda > 0, \lambda w > a$, and $w > \mu a$, we have

$$\frac{\int_{\lambda w}^{\lambda \mu w} u^r G(u) du}{\int_{w/\lambda}^w u^r G(u) du} \leq \frac{\lambda \mu (\lambda \mu w)^r G(\lambda w)}{w / \mu^r G(w)} = \frac{\lambda^{r+1} \mu^{2r+1} G(\lambda w)}{G(w)}.$$

As $w \rightarrow \infty$,

$$\frac{\int_{\lambda w}^{\lambda \mu w} u^r G(u) du}{\int_{w/\lambda}^w u^r G(u) du} \rightarrow (\lambda \mu)^k.$$

Hence,

$$\liminf_{w \rightarrow \infty} \frac{G(\lambda w)}{G(w)} \geq \frac{(\lambda \mu)^k}{\lambda^{r+1} \mu^{2r+1}} = \lambda^{k-r-1} \mu^{k-2r-1}.$$

Letting $\mu \downarrow 1$, we obtain

$$\liminf_{w \rightarrow \infty} \frac{G(\lambda w)}{G(w)} \geq \lambda^{k-r-1}.$$

If we replace μ in above argument by $1/\mu$, we obtain

$$\limsup_{w \rightarrow \infty} \frac{G(\lambda w)}{G(w)} \leq \lambda^{k-r-1}.$$

Hence,

$$\lim_{w \rightarrow \infty} \frac{G(\lambda w)}{G(w)} = \lambda^{k-r-1}.$$

Thus G is of index $(k - r - 1)$, and $w^r G(w)$ is of index $(k - 1)$. Q.E.D.

The following result (see, e.g., [11]) will be used in the proof of Theorem 1.2.2.

Theorem 1.2.1 *If $H(x)$ is of index $-m$ when $x \rightarrow \infty$ and $0 < m < 2$, then*

$$1 - U(t) \xrightarrow{d} S(m)H(1/t)$$

when $t \downarrow 0$. Here, $S(m) = \int_0^\infty \frac{\sin x}{x^m} dx$.

Proof. Since

$$\frac{1 - U(t)}{t} = \int_0^\infty H(x) \sin tx \, dx,$$

we have

$$\frac{1 - U(t)}{H(1/t)} = \int_0^\infty \frac{H(x/t)}{H(1/t)} \sin x \, dx.$$

By Lemma 1.2.1, there exists x_0 such that for each x , $0 < x \leq x_0$ with $h > 0$ and t sufficiently small,

$$\frac{H(x/t)}{H(1/t)} |\sin x| \leq \frac{A |\sin x|}{x^{m+h}},$$

where A is a finite constant. Choose h so that $m + h < 2$. The last function is then integrable over the finite interval $(0, x_0)$. Since $H(x)$ is of index $-m$, for $x > 0$, we have, for $t \downarrow 0$, $\frac{H(x/t)}{H(1/t)} \rightarrow x^{-m}$. Therefore,

$$\int_0^p \frac{H(x/t)}{H(1/t)} \sin x \, dx \rightarrow \int_0^p \frac{\sin x}{x^m} \, dx.$$

By the Second Mean Value Theorem, there exist some q such that

$$\int_p^\infty \frac{H(x/t)}{H(1/t)} \sin x \, dx = \frac{H(p/t)}{H(1/t)} \int_p^q \sin x \, dx.$$

The right hand side has absolute value less than $\frac{2H(p/t)}{H(1/t)}$, and we know $\frac{2H(p/t)}{H(1/t)} \rightarrow \frac{2}{p^m}$ when $t \downarrow 0$. The limit can be arbitrarily small by choosing large p and small t . So it follows that when $t \downarrow 0$, the integral $\int_0^\infty \frac{H(x/t)}{H(1/t)} \sin x \, dx \rightarrow \int_0^\infty \frac{\sin x}{x^m} \, dx$. Q.E.D.

The final result provides the fundamental relationship between the limit behavior of the characteristic function at the origin and the tail function at infinity. Results of this type are known as Tauberian theorems.

Theorem 1.2.2 *If $(1 - U(t))$ is of index m as $t \downarrow 0$ and $0 < m < 2$, then*

$$H(x) \xrightarrow{d} \frac{1 - U(1/x)}{S(m)}, \quad \text{as } x \rightarrow \infty.$$

Proof. First, the idea is to show, first, that $H(x)$ is of index $-m$, and then

apply Theorem 1.2.1 to get the above statement.

For $x \geq 0$, define:

$$H_1(x) = \int_0^x uH(u) du,$$

and

$$H_2(x) = \int_0^x H_1(u) du.$$

Thus we have

$$H_1(x) = \frac{2}{\pi} \int_0^x \int_0^\infty \frac{1 - U(t)}{t} u \sin ut dt du.$$

Since each integrand is bounded, we can change the order of integration, which gives

$$\begin{aligned} H_1(x) &= \frac{2}{\pi} \int_0^\infty \int_0^x \frac{1 - U(t)}{t} u \sin ut du dt \\ &= \frac{2}{\pi} \int_0^\infty (1 - U(t)) \frac{\sin xt - xt \cos xt}{t^3} dt. \end{aligned}$$

We now calculate $H_2(x)$:

$$H_2(x) = \frac{2}{\pi} \int_0^x \int_0^\infty (1 - U(t)) \frac{\sin xt - xt \cos xt}{t^3} dt du.$$

The integral is absolutely convergent. By Fubini's theorem, we may change the integration order,

$$H_2(x) = \frac{2}{\pi} \int_0^\infty (1 - U(t)) \frac{2(1 - \cos xt) - xt \sin xt}{t^4} dt.$$

Hence,

$$\frac{H_2(x)}{x^3(1 - U(1/x))} = \frac{2}{\pi} \int_0^\infty \frac{1 - U(t/x)}{1 - U(1/x)} \frac{2(1 - \cos t) - t \sin t}{t^4} dt,$$

where $1 - U(1/x)$ is of index $-m$ as $x \rightarrow \infty$. By Lemma 1.2.1, when x is sufficiently large,

$$\frac{1 - U(t/x)}{1 - U(1/x)} < At^{m+h}, \quad t \geq 1;$$

$$\frac{1 - U(t/x)}{1 - U(1/x)} < At^{m-h}, \quad t < 1.$$

Given $0 < m < 2$, we can choose h so that $h < 1$ and $m + h < 2$. Then the integrand will be dominated by

$$A(t^{m+h} + t^{m-h}) \frac{1 - U(t/x)}{1 - U(1/x)} \frac{2(1 - \cos t) - t \sin t}{t^4},$$

which is integrable over $(0, \infty)$. Also, since $1 - U(t)$ is of index m as $t \downarrow 0$ when $x \rightarrow \infty$, we have

$$\frac{1 - U(t/x)}{1 - U(1/x)} \rightarrow t^m.$$

Therefore,

$$\frac{H_2(x)}{x^3(1 - U(1/x))} \rightarrow \frac{2}{\pi} \int_0^\infty \frac{2(1 - \cos t) - t \sin t}{t^{4-m}} dt.$$

Denote

$$g(m) = \frac{2}{\pi} \int_0^\infty \frac{2(1 - \cos t) - t \sin t}{t^{4-m}} dt.$$

It can be shown that

$$g(m) = \frac{1}{(3-m)(2-m)S(m)}.$$

Thus,

$$H_2(x) g(m) x^3 (1 - U(1/x))$$

for $x \rightarrow \infty$.

It is not hard to observe that $H_2(x)$ is of index $(3-m)$ as $x \rightarrow \infty$. Hence, by Lemma 1.2.2, $H_1(x)$ is of index $(2-m)$, and $xH(x)$ is of index $(1-m)$. Therefore, $H(x)$ is of index $-m$. Q.E.D.

1.3 Lévy α -Stable Distributions

There exist several representations of α -stable distributions. They can be found in [1], [13] and [16]. In this section, we will use the following definition in terms of the characteristic function representation.

Definition 1.3.1 *A random variable X is called α -stable (in brief $X \in S_\alpha(\sigma, \mu, \beta)$) if its characteristic function is given by the formula*

$$\phi(t) = \exp [it\mu - |\sigma t|^\alpha (1 - i\beta \operatorname{sgn}(t)\Phi)],$$

where $\operatorname{sgn}(t)$ is the sign of t , and

$$\Phi = \begin{cases} \tan(\pi\alpha/2), & \alpha \neq 1; \\ -(2/\pi) \log |t|, & \alpha = 1; \end{cases}$$

with $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $\sigma \geq 0$, and $\mu \in (-\infty, +\infty)$.

Lévy α -stable distributions do not have closed-form probability density functions (PDF) except in three special cases. These exceptions include Gaussian ($\alpha = 2$), Cauchy ($\alpha = 1, \beta = 0$), and Lévy ($\alpha = 0.5, \beta = 1$) laws.

The following well known result gives a precise description of the asymptotic behavior of the tail of an α -stable distribution.

Proposition 1.3.1 *Let $X \sim S_\alpha(\sigma, 0, 0)$, with $0 < \alpha < 2$. Then*

$$\lim_{\lambda \rightarrow \infty} \lambda^\alpha P\{|X| > \lambda\} = C_\alpha \sigma^\alpha,$$

where

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x \, dx \right)^{-1} = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha) \cos(\pi\alpha/2)}, & \alpha \neq 1; \\ 2/\pi, & \alpha = 1. \end{cases}$$

Proof. For $S_\alpha(\sigma, 0, 0)$, the characteristic function is $U(t) = e^{-\sigma t^\alpha}$. It is easy to show that $1 - U(t)$ is of index α , as $t \rightarrow 0$. By Theorem 1.2.2, we have:

$$P(|X| > x) \sim \frac{1 - U(1/x)}{S(\alpha)} = \frac{1 - e^{-\sigma^\alpha |1/x|^\alpha}}{S(\alpha)},$$

where $S(\alpha) = \int_0^\infty \frac{\sin x}{x^\alpha} dx$. Hence,

$$\begin{aligned} \lim_{x \rightarrow \infty} x^\alpha P(|X| > x) &= \lim_{x \rightarrow \infty} \frac{1 - e^{-\sigma^\alpha |x|^{-\alpha}}}{x^{-\alpha}} S^{-1}(\alpha) \\ &\stackrel{L}{=} \lim_{x \rightarrow \infty} \frac{-e^{-\sigma^\alpha |x|^{-\alpha}} \sigma^\alpha \alpha |x|^{-\alpha-1}}{-\alpha x^{-\alpha-1}} S^{-1}(\alpha) \\ &= \sigma^\alpha S^{-1}(\alpha) \\ &= C_\alpha \sigma^\alpha. \end{aligned}$$

Q.E.D.

Stable distributions are special cases of general infinitely divisible distributions (see, e.g. [14]). The latter are described by the following theorem.

Theorem 1.3.1 (Lévy-Khintchine formula) *The characteristic function of an infinitely divisible distribution in \mathbb{R}^d has the following representation,*

$$\phi(t) = \exp \left[i \langle t, \mu \rangle - \frac{1}{2} \langle \sigma t, t \rangle + \int_{\mathbb{R}^d \setminus \{0\}} (e^{i \langle t, x \rangle} - 1 - i \langle t, x \rangle I_{|x| < 1}) L(dx) \right],$$

where L is the Lévy measure satisfying the conditions $L(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (|x^2| \wedge 1) L(dx) < \infty$.

1.4 LePage Representation

In this section we describe the so-called LePage Series Representation for symmetric α -stable random variables. Several proofs of these results have

been proposed and, in view of the centrality of this result for our dissertation, we review the most important ones in detail.

Let

$$S = \sum_{k=1}^{\infty} \frac{1}{\Gamma_k^{1/\alpha}} Z_k, \quad (1.4.1)$$

where $\Gamma_k = E_1 + E_2 + \dots + E_k$, with E_i i.i.d, $\text{Exp}(1)$, and Z_k i.i.d, symmetric, with $E|Z_1| < \infty$, so that in particular $EZ_k = 0, k = 1, 2, \dots$. The sequences, $\{E_i\}$, and $\{Z_i\}$, are assumed to be independent.

Theorem 1.4.1: *The random series (1.4.1) converges a.s. to a symmetric α -stable random variable with the characteristic function*

$$\varphi_s(u) = e^{-c|u|^\alpha},$$

for some $c > 0$. More precisely; the series $\sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha} Z_k$ converges almost surely to a random variable $S \in S_\alpha((C_\alpha^{-1}E|Z_1|^\alpha)^{1/\alpha}, 0, 0)$, where C_α is the constant defined in Proposition 1.3.1

LePage's Proof: (see, [7]) We show that if the series S converges a.s., then the distribution of S is strictly stable with index α , i.e., if S_1, S_2, \dots, S_n are i.i.d. as S , then $S_1 + S_2 + \dots + S_n \stackrel{d}{=} n^{1/\alpha} S$, for all n .

Suppose that each S_i is of the form (1.4.1) with Γ_k, Z_k replaced by Γ_{ik}, Z_{ik} , where the sequence Γ_{ik}, Z_{ik} are mutually independent with the same distribution as Γ_k, Z_k . Then $\Gamma_{ik}, k \geq 1, i = 1, \dots, n$, may be regarded as the arrival times of events in n independent Poisson processes, each with unit arrival rate. Arrange Γ_{ik} in the increasing order and denote them by $\Gamma'_1, \Gamma'_2, \dots$. Then $\Gamma'_1, \Gamma'_2, \dots$ are the arrival times of a Poisson process with arrival rate n .

Notice that $\Gamma'_j = \sum_{i=1}^j E'_i, E'_i \sim \text{Exp}(n)$ and $nE'_i \sim \text{Exp}(1)$, and consequently,

$$n\Gamma'_j = \sum_{i=1}^j nE'_i \stackrel{d}{=} \sum_{i=1}^j E_i = \Gamma_j,$$

where E_i, Γ_j were defined in (1.4.1). So $n\Gamma'_j$ have the same distribution as Γ_j .

Similarly, let Z'_1, Z'_2, \dots denote Z_{ik} in the increasing order. Then Z'_1, Z'_2, \dots are i.i.d. as Z_1, Z_2, \dots , and independent of $\Gamma'_1, \Gamma'_2, \dots$.

Next, let

$$K_\epsilon = \sup\{k : \frac{1}{\Gamma'_k{}^{1/\alpha}} \geq \epsilon\}$$

and

$$K_{i,\epsilon} = \sup\{k : \frac{1}{\Gamma'_k{}^{1/\alpha}} \geq \epsilon\}$$

for $i = 1, \dots, n$ and $\epsilon > 0$. Then

$$\begin{aligned} \sum_{j=1}^{K_{1,\epsilon}} (\Gamma_{1j})^{-1/\alpha} Z_{1j} + \dots + \sum_{j=1}^{K_{n,\epsilon}} (\Gamma_{nj})^{-1/\alpha} Z_{nj} &= \sum_{j=1}^{K_\epsilon} (\Gamma'_j)^{-1/\alpha} Z'_j \\ &= n^{1/\alpha} \sum_{j=1}^{K_\epsilon} (n\Gamma'_j)^{-1/\alpha} Z'_j \\ &\stackrel{d}{\longrightarrow} n^{1/\alpha} \sum_{j=1}^K (\Gamma_j)^{-1/\alpha} Z_j \quad (1.4.2) \end{aligned}$$

for $\epsilon > 0$ and some K . As $\epsilon \rightarrow 0, K_{i,\epsilon} \rightarrow \infty$ a.s for all $i = 1, \dots, n$, and $K_\epsilon, K \rightarrow \infty$, so the left side of (1.4.2) converges to $(S_1 + \dots + S_n)$; the right side of (1.4.2) converges to a random variable having the same distribution as $n^{1/\alpha} S$. Q.E.D.

Samorodnitsky and Taqqu's Proof (see, [13]) The proof is carried out in three steps:

Step 1. Let U_1, U_2, \dots , be an i.i.d sequence of variables uniformly distributed on $(0,1)$, independent of the sequences $\{\epsilon_1, \epsilon_2, \dots\}$ and $\{Z_1, Z_2, \dots\}$. Set $Y_k = U_k^{-1/\alpha} Z_k, k = 1, 2, \dots$. Let $F_{|Z|}$ denote the distribution of $|Z|$. Then

for $\lambda > 0$,

$$\begin{aligned}
P(|Y_k| > \lambda) &= P(U_k^{-1/\alpha} |W_k| > \lambda) \\
&= P(U_k < \lambda^{-\alpha} |Z_k|^\alpha) \\
&= \int_0^\infty P(U_k < \lambda^{-\alpha} Z^\alpha) F_{|Z|}(dz) \\
&= \int_0^\lambda \lambda^{-\alpha} z^\alpha F_{|Z|}(dz) + \int_\lambda^\infty F_{|Z|}(dz) \\
&= \lambda^{-\alpha} \int_0^\lambda z^\alpha F_{|Z|}(dz) + P(|Z_k| > \lambda).
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{\lambda \rightarrow \infty} \lambda^\alpha P(|Y_k| > \lambda) &= E|Z_k|^\alpha + \lim_{\lambda \rightarrow \infty} \lambda^\alpha P(|Z_k| > \lambda) \\
&= E|Z_k|^\alpha.
\end{aligned}$$

Hence, Y_k is in the domain of (normal) attraction of a symmetric α -stable random variable, i.e., asymptotically,

$$\frac{1}{n^{1/\alpha}} \sum_{k=1}^n Y_k \stackrel{d}{=} X. \tag{1.4.3}$$

where $X \sim S_\alpha(\sigma, 0, 0)$, with $\sigma = (C_\alpha^{-1} E|Z_k|^\alpha)^{1/\alpha}$ by Proposition 1.3.1.

Step 2. Now, we want to show that $n^{-1/\alpha} \sum_{k=1}^n U_k^{-1/\alpha} Z_k$ has the same limiting distribution as $\sum_{k=1}^\infty \Gamma_k^{-1/\alpha} Z_k$.

Two observations: First, consider Γ_k as the arrival times in a Poisson process. Then, given Γ_{n+1} as the time of the $(n+1)^{th}$ arrival, the vector $(\Gamma_k/\Gamma_{n+1}, k = 1, 2, \dots, n)$ has the distribution of the order statistics of n i.i.d. $U(0, 1)$ random variables.

Second, any permutation of U_k 's in the sum $\sum_{k=1}^n U_k^{-1/\alpha} Z_k$ yields the same distribution of the sum since Z_k 's are i.i.d. This means we can use order statistics of U_k 's without affecting the distribution of the sum.

Consequently,

$$\begin{aligned} n^{-1/\alpha} \sum_{k=1}^n Y_k &= n^{-1/\alpha} \sum_{k=1}^n U_i^{-1/\alpha} Z_k \\ &\stackrel{d}{=} n^{-1/\alpha} \sum_{k=1}^n \left(\frac{\Gamma_k}{\Gamma_{n+1}} \right)^{-1/\alpha} Z_k, \end{aligned}$$

and by (1.4.3), we obtain that, asymptotically

$$\left(\frac{\Gamma_{n+1}}{n} \right)^{1/\alpha} \sum_{k=1}^n \Gamma_k^{-1/\alpha} Z_k \stackrel{d}{=} X_n \sim S_\alpha(\sigma, 0, 0). \quad (1.4.4)$$

Step 3. It remains to show that the L.H.S. of (1.4.4) converges a.s. to $\sum_{k=1}^\infty \Gamma_k^{-1/\alpha} Z_k$. Define the event $A = \{\lim_{n \rightarrow \infty} \frac{\Gamma_n}{n} = 1\} \cap \{\Gamma_1 > 0\}$. By the strong law of large numbers, $P(A) = 1$, since Γ_n is a sum of n i.i.d. exponential r.v's with mean 1. For each fixed sequence $\{\Gamma_k\}$ belonging to the event A , the summands $\frac{1}{\Gamma_k^{1/\alpha}} Z_k$ are independent and bounded by $C_1 k \leq \Gamma_k \leq C_2 k$, for some positive constants C_1 and C_2 , where $k = 1, 2, \dots$. Applying Kolmogorov's Three Series Theorem, we have, for each $\lambda > 0$,

(i)

$$\begin{aligned} \sum_{k=1}^\infty P\{|\Gamma_k^{-1/\alpha} Z_k| > \lambda\} &= \sum_{k=1}^\infty P\{|Z_k^\alpha| > \lambda^\alpha \Gamma_k\} \\ &\leq \sum_{k=1}^\infty P\{|Z_k^\alpha| > \lambda^\alpha C_1 k\} \\ &\leq \int_0^\infty P\{|Z_k^\alpha| > x\} dx \\ &= E|Z_k|^\alpha \\ &\leq \infty, \end{aligned}$$

since $E|Z_k|^\alpha < \infty$,

(ii)

$$\sum_{k=1}^{\infty} E(\Gamma_k^{-1/\alpha} Z_k 1\{|\Gamma_k^{-1/\alpha} Z_k| \leq \lambda\}) = 0,$$

since $EZ_k = 0$, and

(iii)

$$\begin{aligned} \sum_{k=1}^{\infty} E\left(\Gamma_k^{-1/\alpha} Z_k 1\{|\Gamma_k^{-1/\alpha} Z_k| \leq \lambda\}\right)^2 &\leq C_1^{-2/\alpha} \sum_{k=1}^{\infty} k^{-2/\alpha} \int_0^{\infty} z^2 1\{z \leq \lambda C_2^{1/\alpha} k^{1/\alpha}\} f(|z|) d|z| \\ &\leq C \int_0^{\infty} x^{-2/\alpha} dx \int_0^{\lambda C_2^{1/\alpha} x^{1/\alpha}} z^2 f(|z|) d|z| \\ &= C \int_0^{\infty} z^2 f(|z|) d|z| \int_{\lambda^{-\alpha} C_2^{-1} z^{\alpha}}^{\infty} x^{-2/\alpha} dx \\ &= C' \int_0^{\infty} z^{\alpha} f(|z|) d|z| \\ &< \infty. \end{aligned}$$

This completes the Samorodnitsky-Taquq's Proof. Q.E.D.

1.5 Accuracy of LePage Approximation

In this section we discuss the issue of the error of approximation of the LePage series by its finite sum. The issue was extensively studied by Bentkus et al, [4] in the variation distance. We will get back to this approach in chapters 3 and 4. But we also found an L^2 result (see, Theorem 1.5.2) stated in [6] without proof and we were unable to locate its proof elsewhere. So we produced our own proof which turned out to be more delicate than we thought initially. We start with an L^2 estimate for partial sums of the LePage

series.

Theorem 1.5.1 *Let $EZ_1^2 = C_z < \infty$ and*

$$S_N = \sum_{k=1}^N \frac{1}{\Gamma_k^{1/\alpha}} Z_k.$$

Then, for $N > M \geq \frac{2}{\alpha}$ and some constant $C < \infty$,

$$E[(S_N - S_M)^2] \leq C \frac{\alpha}{2 - \alpha} \left[\frac{1}{M^{\frac{2-\alpha}{\alpha}}} - \frac{1}{N^{\frac{2-\alpha}{\alpha}}} \right].$$

Proof: Observe that

$$\begin{aligned} E[(S_N - S_M)^2] &= E \left[\sum_{k=M+1}^N \frac{1}{\Gamma_k^{1/\alpha}} Z_k \right]^2 \\ &= E \sum_{k=M+1}^N \sum_{l=M+1}^N \frac{1}{\Gamma_k^{1/\alpha}} \frac{1}{\Gamma_l^{1/\alpha}} Z_k Z_l \\ &= \sum_{k=M+1}^N E \frac{1}{\Gamma_k^{2/\alpha}} E Z_k^2 \\ &= C_z \sum_{k=M+1}^N \frac{\Gamma(k - 2/\alpha)}{\Gamma(k)}. \end{aligned} \tag{1.5.1}$$

By Stirling's formula, $\Gamma(x) \sim \sqrt{2\pi(x-1)} \left(\frac{x-1}{e}\right)^{x-1}$, i.e.,

$$\lim_{x \rightarrow \infty} \frac{\Gamma(x)}{\sqrt{2\pi(x-1)}} \left(\frac{x-1}{e}\right)^{-(x-1)} = 1.$$

Hence for each $\epsilon > 0$, there exists an $N_\epsilon > \frac{2}{\alpha}$ such that for $k > N_\epsilon$,

$$1 - \epsilon < \frac{\Gamma(k)}{\sqrt{2\pi(k-1)}} \left(\frac{k-1}{e}\right)^{-(k-1)} < 1 + \epsilon.$$

So, for $N > M > N_\epsilon$, we can apply the bound in 1.5.1, and obtain the inequality

$$\begin{aligned}
E[(S_N - S_M)^2] &\leq C_z \sum_{k=M+1}^N \frac{(1+\epsilon) \sqrt{2\pi(k - \frac{2}{\alpha} - 1)} (\frac{k - \frac{2}{\alpha} - 1}{e})^{k - \frac{2}{\alpha} - 1}}{(1-\epsilon) \sqrt{2\pi(k-1)} (\frac{k-1}{e})^{k-1}} \\
&\leq C_z \frac{1+\epsilon}{1-\epsilon} e^{\frac{2}{\alpha}} \sum_{k=M+1}^N \frac{(k - \frac{2}{\alpha} - 1)^{k - \frac{2}{\alpha} - 1}}{(k-1)^{k-1}} \\
&= C_z \frac{1+\epsilon}{1-\epsilon} e^{\frac{2}{\alpha}} \sum_{k=M+1}^N (1 - \frac{2/\alpha}{k-1})^{k-1} (k - \frac{2+\alpha}{\alpha})^{-2/\alpha}.
\end{aligned}$$

Since

$$\lim_{k \rightarrow \infty} (1 - \frac{2/\alpha}{k-1})^{k-1} \longrightarrow e^{-2/\alpha},$$

we have

$$\begin{aligned}
E[(S_N - S_M)^2] &\leq C_z C_e \frac{1+\epsilon}{1-\epsilon} \sum_{k=M+1}^N (k - \frac{2+\alpha}{\alpha})^{-2/\alpha} \\
&\leq C_z C_e \frac{1+\epsilon}{1-\epsilon} \sum_{k=M+1}^N k^{-2/\alpha} \\
&\leq C_z C_e \frac{1+\epsilon}{1-\epsilon} \frac{\alpha}{\alpha-2} (N^{1-2/\alpha} - M^{1-2/\alpha}) \\
&= C_z C_e \frac{1+\epsilon}{1-\epsilon} \frac{\alpha}{2-\alpha} \left[\frac{1}{M^{\frac{2-\alpha}{\alpha}}} - \frac{1}{N^{\frac{2-\alpha}{\alpha}}} \right].
\end{aligned}$$

Selecting $C = 2 C_z C_e$ completes the proof. Q.E.D.

Remark 1.5.1 The result of Theorem 1.5.1 may seem strange since it deals with the "Cauchy-like" condition in the L^2 -norm although the series S itself, and the partial sums S_N and S_M have infinite second moments. The explanation is not difficult to see (for simplicity's sake, we take $P(Z_k = \pm 1) = \frac{1}{2}$).

Let $S \sim S_\alpha(\sigma, 0, 0)$, with $0 < \alpha < 2$. Then $P(S > x) \sim \frac{1}{x^\alpha}$ as $x \rightarrow \infty$.

In the k th LePage term, $\Gamma_k \sim \text{Gamma}(k, 1)$ with the density, $\frac{1}{\Gamma(k)}x^{k-1}e^{-x}$, where $x \geq 0$ and $k \geq 1$, so that

$$\begin{aligned} P(\Gamma_k^{-1/\alpha} > x) &= P(\Gamma_k < \frac{1}{x^\alpha}) \\ &= \int_0^{\frac{1}{x^\alpha}} \frac{1}{\Gamma(k)} z^{k-1} e^{-z} dz \\ &= 1 - e^{-\frac{1}{x^\alpha}} \sum_{i=0}^{k-1} \frac{\left(\frac{1}{x^\alpha}\right)^i}{i!}. \end{aligned}$$

Then,

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{P(\Gamma_k^{-1/\alpha} > x)}{\frac{1}{x^\alpha}} &= \lim_{x \rightarrow \infty} \frac{1 - e^{-\frac{1}{x^\alpha}} \sum_{i=0}^{k-1} \frac{\left(\frac{1}{x^\alpha}\right)^i}{i!}}{\frac{1}{x^\alpha}} \\ &\stackrel{y=\frac{1}{x^\alpha}}{=} \lim_{y \rightarrow 0} \frac{1 - e^{-y} \sum_{i=0}^{k-1} \frac{y^i}{i!}}{y} \\ &\stackrel{L}{=} \lim_{y \rightarrow 0} \left(e^{-y} \sum_{i=0}^{k-1} \frac{y^i}{i!} - e^{-y} \sum_{i=0}^{k-1} \frac{iy^{i-1}}{i!} \right) \\ &= \begin{cases} 1 & k = 1; \\ 0 & k > 1. \end{cases} \end{aligned}$$

As a result, the first term in the LePage representation has infinite second moment and is not in the L^2 .

By a similar calculation, using de l'Hospital formula, we obtain that

$$\lim_{x \rightarrow \infty} \frac{P(\Gamma_k^{-1/\alpha} > x)}{x^{\frac{1}{k\alpha}}} = \lim_{x \rightarrow \infty} \frac{1 - e^{\frac{1}{x^\alpha}} \sum_{i=0}^{k-1} \frac{1}{i!} \left(\frac{1}{x^\alpha}\right)^i}{x^{\frac{1}{k\alpha}}} = \frac{1}{k!} > 0,$$

so that only terms $\Gamma_k^{-1/\alpha}$ with $k > \frac{2}{\alpha}$ are in L^2 .

But under the restriction $N > M \geq 2/\alpha$ in Theorem 1.5.1, the terms with infinite second moments in the difference of S_N and S_M cancel out so that the quantity $S_N - S_M$ is in L^2 .

In view of the above remark we have to proceed cautiously passing to the limit, $N \rightarrow \infty$, in Theorem 1.5.1. Denote

$$N_\alpha = \lfloor 2/\alpha \rfloor + 1,$$

where $\lfloor a \rfloor$ is the integer part of a , and

$$S_{N_\alpha, N} = \sum_{k=N_\alpha+1}^N \frac{1}{\Gamma_k^{1/\alpha}} Z_k, \quad N > N_\alpha.$$

Then, $S_{N_\alpha, N}$ is in L^2 for each $N > N_\alpha$, and in view of Theorem 1.5.1, for $N > M > N_\alpha$,

$$\begin{aligned} E|S_{N_\alpha, N} - S_{N_\alpha, M}|^2 &= E|S_{N_\alpha, N} + S_{N_\alpha} - (S_{N_\alpha} + S_{N_\alpha, M})|^2 \\ &= E|S_N - S_M|^2 \\ &\leq C \frac{1}{2/\alpha - 1} \left[\frac{1}{M^{\frac{2}{\alpha}-1}} - \frac{1}{N^{\frac{2}{\alpha}-1}} \right] \end{aligned}$$

as $N, M \rightarrow \infty$. Thus, for $N > N_\alpha$, the sequence $S_{N_\alpha, N} \rightarrow 0$ satisfies the Cauchy condition in L^2 , and in view of completeness of L^2 , it has a limit, say $S_{N_\alpha, \infty}$, such that for each $N > N_\alpha$,

$$E|S_{N_\alpha, N} - S_{N_\alpha, \infty}|^2 \leq C \frac{1}{2/\alpha - 1} \frac{1}{N^{\frac{2}{\alpha}-1}}.$$

The convergence of $S_{N_\alpha, N}$ to $S_{N_\alpha, \infty}$ is also in probability, which implies that

$$S_N = S_{N_\alpha} + S_{N_\alpha, N} \rightarrow S_{N_\alpha} + S_{N_\alpha, \infty},$$

in probability as well as $N \rightarrow \infty$.

In view of the uniqueness of the limit in probability

$$S_{N_\alpha} + S_{N_\alpha, \infty} = S \quad a.s.$$

The above argument proves the following:

Theorem 1.5.2 For, $N > 2/\alpha$,

$$E(|S_N - S|^2) < C \frac{1}{N^{\frac{2}{\alpha}-1}},$$

where C is the constant in Theorem 1.5.1. Also, for $0 < p < 2$, we have

$$E|S_N - S|^p \leq (E|S_N - S|^2)^{p/2} \leq C \frac{1}{\left(N^{\frac{2}{\alpha}-1}\right)^{p/2}}.$$

Remark 1.5.2 It is easy to see that the above method can be applied to obtain the rate of convergence of $E|S_n - S|^p$ to 0, for $n > \frac{p}{\alpha}$, for arbitrary $p > 2$, which turns out to be of the order $N^{1-\frac{[p]+1}{\alpha}}$, $p > 2$.

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Chapter 2

Parameter Estimation for Tempered Stable Laws

2.1 Tempered Stable Laws

Tempered α -stable laws were first introduced by Rosiński in [5] and later studied by Terdik and Woyczynski in [7], Sztonyk in [6] and others. But there is a long history of studying similar probability distribution (see also, [5], for further details).

A tempered α -stable law is an infinitely divisible distribution with the Lévy measure of the form

$$L(B) = \int_{S^{d-1}} \int_0^\infty 1_B(ru) k(r|u) \frac{dr}{r^{\alpha+1}} \eta(du), \quad (2.1.1)$$

where $\alpha \in (0, 2)$ and $k(\cdot|u)$ is a completely monotone function with $k(\infty|u) = 0$ and $k(0+|u) = 1$, and $\eta(du)$ is a finite measure on the unit sphere S^{d-1} . The parameter α is called the *index*, and function k is called *tempering function*. Recall that a function $f(x)$ is said to be completely monotone (c.m.) if derivatives $f^{(n)}(x)$ exist for all $n = 1, 2, 3, \dots$ and if $(-1)^n f^{(n)}(x) \geq 0$ for all $x > 0$.

Any tempering function can be represented as the Laplace transform (see,

[4], [5])

$$k(r|u) = \int_0^\infty e^{-rs} Q(ds|u)$$

of a probability measure $Q(ds|u)$ on \mathbb{R}_+ . Then its Q -measure on R^d is defined as

$$Q(B) = \int_{S^{d-1}} \int_0^\infty 1_B(ru) Q(dr|u) \eta(du)$$

and its R -measure is defined by the formula

$$R(B) = \int_{R_0^d} 1_B\left(\frac{x}{\|x\|^2}\right) \|x\|^\alpha Q(dx)$$

with $R_0^d = R^d \setminus \{0\}$

In this context, Rosiński [5], proved that the characteristic function of a tempered stable random variable X with parameters α , b , and R (in brief, $X \in TS_\alpha(R, b)$) is uniquely defined by the formula

$$\phi(t) = e^{\int_{R_0^d} \psi_\alpha(\langle t, x \rangle) R(dx) + i \langle t, b \rangle}, \quad (2.1.2)$$

where

$$\psi_\alpha(r) = \begin{cases} \Gamma(-\alpha)[(1 - ir)^\alpha - 1], & 0 < \alpha < 1; \\ (1 - ir) \log(1 - ir) + ir, & \alpha = 1; \\ \Gamma(-\alpha)[(1 - ir)^\alpha - 1 + i\alpha r], & 1 < \alpha < 2. \end{cases} \quad (2.1.3)$$

Roughly speaking, tempered α -stable distributions have thin, exponential tails, and thus finite moments of all orders; but their behavior in the vicinity of 0 is similar to that of α -stable distributions.

2.2 Estimators for TS parameters and Their Asymptotic Distributions

Since tempered α -stable distributions have all moments finite, the parametric method of moment estimation is applicable for them. In this section, we study the moment estimators for tempered α -stable distributions, calculate the estimator's asymptotic variance, and construct their confidence intervals.

2.2.1 Cumulant Generating Functions and Cumulants

The cumulant generating function $\kappa_X(t)$ is defined as the logarithm of the moment-generating function, i.e.,

$$\kappa_X(t) = \log \left(E(e^{tX}) \right).$$

Cumulants can be directly calculated from moments, and vice versa. Let $\mu_k = E(X - EX)^k$ be the k th order central moment. It follows that

$$C_2(X) = \mu_2,$$

$$C_3(X) = \mu_3,$$

$$C_4(X) = \mu_4 - 3\mu_2^2,$$

$$C_5(X) = \mu_5 - 10\mu_2\mu_3,$$

$$C_6(X) = \mu_6 - 15\mu_2\mu_4 - 10\mu_3^2 + 30\mu_2^3,$$

etc., etc.

2.2.2 Estimators for 1-D Smoothly Truncated Lévy Flight (STLF)

Several infinitely divisible distributions which appeared in earlier studies are special classes of tempered stable distributions. One of such classes is the

Smoothly Truncated Lévy Flights (STLF) which was introduced by Koponen in [3] and, independently, by Hougaard in [2].

With 0-D unit 'sphere' $S^0 = \{\pm 1\}$, a *Smoothly Truncated Lévy Distribution* $STLD_\alpha(a, p, \lambda)$ is defined as a tempered α -stable distribution with the tempering function

$$k(r|\pm 1) = k(r) = \exp(-\lambda r),$$

with $\lambda > 0$, $r > 0$, and the measure,

$$\eta(\{-1\}) = ap, \quad \eta(\{1\}) = aq,$$

where $p + q = 1$, and $a > 0$.

The cumulant function of STLF (see, [7]) is

$$\kappa_X(t) = a\lambda^\alpha [p\psi_\alpha(-t/\lambda) + q\psi_\alpha(t/\lambda)] + iub,$$

where ψ_α is defined as in (2.1.3) with $0 < \alpha < 2$.

This corresponds to the selection of discrete measure concentrated at two symmetric points as the R -measure R . Without loss of generality, we can assume $b = 0$. So, the cumulants for $STLD_\alpha(a, p, \lambda)$ are of the form

$$C_m(X) = a\lambda^{\alpha-m} \Gamma(m - \alpha)(p(-1)^m + q).$$

More specifically, if $p \neq q$,

$$\begin{aligned} C_2(X) &= a\lambda^{\alpha-2} \Gamma(2 - \alpha), \\ C_3(X) &= a\lambda^{\alpha-3} \Gamma(2 - \alpha)(2 - \alpha)(q - p), \\ C_4(X) &= a\lambda^{\alpha-4} \Gamma(2 - \alpha)(2 - \alpha)(3 - \alpha), \\ C_5(X) &= a\lambda^{\alpha-5} \Gamma(2 - \alpha)(2 - \alpha)(3 - \alpha)(4 - \alpha)(q - p), \\ C_6(X) &= a\lambda^{\alpha-6} \Gamma(2 - \alpha)(2 - \alpha)(3 - \alpha)(4 - \alpha)(5 - \alpha), \end{aligned} \tag{2.2.1}$$

etc.,etc.

Note that

$$\frac{C_2(X) C_5(X)}{C_3(X) C_4(X)} = \frac{4 - \alpha}{2 - \alpha},$$

which gives the following estimator, for the parameter α :

$$\hat{\alpha} = \frac{2C_2(X) C_5(X) - 4C_3(X) C_4(X)}{C_2(X) C_5(X) - C_3(X) C_4(X)}.$$

This estimator has values in the interval $(0, 2)$. Indeed,

$$\frac{C_3(X) C_4(X)}{C_2(X) C_5(X)} \leq \frac{2C_3(X) C_4(X)}{C_2(X) C_5(X)} = \frac{4 - 2\alpha}{4 - \alpha} = 1 - \frac{\alpha}{4 - \alpha} < 1.$$

Therefore,

$$C_2(X)C_5(X) > 2C_3(X)C_4(X) > C_3(X)C_4(X).$$

Since $C_m(X) > 0$,

$$\hat{\alpha} = \frac{2C_2(X) C_5(X) - 4C_3(X) C_4(X)}{C_2(X) C_5(X) - C_3(X) C_4(X)} > 0,$$

and

$$\hat{\alpha} = 2 - \frac{2C_3(X) C_4(X)}{C_2(X) C_5(X) - C_3(X) C_4(X)} < 2.$$

Now, we are ready to calculate the MM-estimators for the remaining parameters. For λ , we exploit the identity

$$\frac{C_4(X)}{C_2(X)} = \frac{(3 - \alpha)(2 - \alpha)}{\lambda^2},$$

yielding the estimator

$$\hat{\lambda} = \left(\frac{C_2(X)}{C_4(X)} (3 - \hat{\alpha})(2 - \hat{\alpha}) \right)^{1/2}.$$

For parameter $(p - q)$, we have the equation

$$\frac{C_3(X)}{C_2(X)} = \frac{(2 - \alpha)(q - p)}{\lambda},$$

so that

$$\widehat{p - q} = \frac{C_3(X)}{C_2(X)} \left(\frac{\hat{\lambda}}{\hat{\alpha} - 2} \right).$$

Finally, for a , the second cumulant gives the identity

$$C_2(X) = a\lambda^{\alpha-2}\Gamma(2-\alpha),$$

resulting in the corresponding estimator

$$\hat{a} = \frac{C_2(X)\hat{\lambda}^{2-\hat{\alpha}}}{\Gamma(2-\hat{\alpha})}.$$

If we know, a priori, that $p = q = \frac{1}{2}$, odd cumulants are zero, so we can only use even cumulants to estimate $\hat{\alpha}$; other estimators do not change.

Hence,

$$\frac{C_4(X)}{C_2(X)} = \frac{(2-\alpha)(3-\alpha)}{\lambda^2}$$

which implies

$$\lambda^2 = (2-\alpha)(3-\alpha) \frac{C_2(X)}{C_4(X)}$$

and

$$\frac{C_6(X)}{C_4(X)} = \frac{\log(4-\alpha) + \log(5-\alpha)}{\lambda^2}.$$

This gives

$$\lambda^2 = (4-\alpha)(5-\alpha) \frac{\text{Cum}_4(X)}{\text{Cum}_6(X)}.$$

Therefore,

$$\alpha^2(C_2 C_6 - C_4^2) + \alpha(9C_4^2 - 5C_2 C_6) + (6C_2 C_6 - 20C_4^2) = 0,$$

so that

$$\hat{\alpha} = \frac{-(9C_4^2 - 5C_2 C_6)}{2(C_2 C_6 - C_4^2)} + \frac{\sqrt{(9C_4^2 - 5C_2 C_6)^2 - 4(C_2 C_6 - C_4^2)(6C_2 C_6 - 20C_4^2)}}{2(C_2 C_6 - C_4^2)}.$$

2.2.3 Asymptotic Variance of the Estimator for Parameter α and Its Confidence Intervals

Let $\mu_k = E(X - EX)^k$ be the k th order central moment. Recall that there exist a one-to-one relation between cumulants and moments:

$$\begin{aligned} C_2(X) &= \mu_2, \\ C_3(X) &= \mu_3, \\ C_4(X) &= \mu_4 - 3\mu_2^2, \\ C_5(X) &= \mu_5 - 10\mu_2\mu_3, \\ C_6(X) &= \mu_6 - 15\mu_2\mu_4 - 10\mu_3^2 + 30\mu_2^3, \end{aligned} \tag{2.2.2}$$

etc.,etc.

Let $m_i = \frac{1}{n} \sum_{j=1}^n (X_j - \bar{X}_n)^i$ be the empirical central moments which we can use to estimate cumulants, and hence to estimate parameters α , λ , $p - q$ and a .

To calculate the asymptotic variance of the estimators, we use Crámer's theorem (see [1]).

Theorem 2.2.1 (Crámer) *Let g be a mapping $g : \mathbb{R}^d \rightarrow \mathbb{R}^k$ such that $\dot{g}(x)$ is continuous in a neighborhood of $\mu \in \mathbb{R}^d$. If X_n is a sequence of d -dimensional random vectors such that $\sqrt{n}(X_n - \mu) \xrightarrow{d} X$, then $\sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{d} \dot{g}(\mu)X$. In particular, if $\sqrt{n}(X_n - \mu) \xrightarrow{d} N(0, \Sigma)$ where Σ is a $d \times d$ covariance matrix, then*

$$\sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{d} N(0, \dot{g}(\mu)\Sigma\dot{g}(\mu)^T).$$

Since cumulants can be expressed as a function of central moments, we need to calculate the distribution of the central moments first.

If we assume that our original data have known non-zero mean, say $EX =$

μ , then we define

$$k_i = \frac{1}{n} \sum_{j=1}^n (X_j - \mu)^i,$$

It follows that

$$k_1 = \frac{1}{n} \sum_{j=1}^n (X_j - \mu) = \bar{X} - \mu,$$

$$k_2 = \frac{1}{n} \sum_{j=1}^n (X_j - \mu)^2,$$

$$k_3 = \frac{1}{n} \sum_{j=1}^n (X_j - \mu)^3,$$

$$k_4 = \frac{1}{n} \sum_{j=1}^n (X_j - \mu)^4,$$

etc.,etc.

So we can write m_i in term of k_i . For example,

$$\begin{aligned} m_2 &= \frac{1}{n} \sum_{j=1}^n (X_j - \bar{X}_n)^2 \\ &= \frac{1}{n} \sum_{j=1}^n ((X_j - \mu) - (\bar{X}_n - \mu))^2 \\ &= k_2 - k_1^2, \end{aligned}$$

$$\begin{aligned}
m_3 &= \frac{1}{n} \sum_{j=1}^n (X_j - \bar{X}_n)^3 \\
&= \frac{1}{n} \sum_{j=1}^n ((X_j - \mu) - (\bar{X}_n - \mu))^3 \\
&= \frac{1}{n} \sum_{j=1}^n ((X_j - \mu)^3 - 3(X_j - \mu)^2(\bar{X}_n - \mu) + 3(X_j - \mu)(\bar{X}_n - \mu)^2 \\
&\quad - (\bar{X}_n - \mu)^3) \\
&= k_3 - 3k_1k_2 + 2k_1^3,
\end{aligned}$$

$$\begin{aligned}
m_4 &= \frac{1}{n} \sum_{j=1}^n (X_j - \bar{X}_n)^4 \\
&= \frac{1}{n} \sum_{j=1}^n ((X_j - \mu) - (\bar{X}_n - \mu))^4 \\
&= \frac{1}{n} \sum_{j=1}^n ((X_j - \mu)^4 - 4(X_j - \mu)^3(\bar{X}_n - \mu) + 6(X_j - \mu)^2(\bar{X}_n - \mu)^2 \\
&\quad - 4(X_j - \mu)(\bar{X}_n - \mu)^3 + (\bar{X}_n - \mu)^4) \\
&= k_4 - 4k_1k_3 + 6k_1^2k_2 - 3k_1^4,
\end{aligned}$$

$$\begin{aligned}
m_5 &= \frac{1}{n} \sum_{j=1}^n (X_j - \bar{X}_n)^5 \\
&= \frac{1}{n} \sum_{j=1}^n ((X_j - \mu) - (\bar{X}_n - \mu))^5 \\
&= \frac{1}{n} \sum_{j=1}^n ((X_j - \mu)^5 - 5(X_j - \mu)^4(\bar{X}_n - \mu) + 10(X_j - \mu)^3(\bar{X}_n - \mu)^2 \\
&\quad - 10(X_j - \mu)^2(\bar{X}_n - \mu)^3 + 5(X_j - \mu)(\bar{X}_n - \mu)^4 - (\bar{X}_n - \mu)^5) \\
&= k_5 - 5k_1k_4 + 10k_1^2k_3 - 10k_1^3k_2 + 4k_1^5,
\end{aligned}$$

etc.,etc.

By the Central Limit Theorem (CLT), we can find the asymptotic joint distribution of the random vector k_1, k_2, \dots, k_5 . Indeed,

$$\sqrt{n} \left[\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{pmatrix} - \begin{pmatrix} 0 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{pmatrix} \right] \xrightarrow{d} N \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \Sigma \right], \quad n \rightarrow \infty, \quad ,$$

where

$$\Sigma = [\text{Cov}((X - \mu)^i, (X - \mu)^j)]_{i,j=1}^5.$$

Based on the above distribution, we can calculate the limiting distribution of the empirical central moments m_i .

For m_2 , let $g(k_1, k_2) = k_2 - k_1^2$. Then we have

$$g(0, \mu_2) = \mu_2,$$

$$\dot{g}(k_1, k_2) = (-2k_1, 1),$$

$$\dot{g}(0, \mu_2) = (0, 1),$$

and, by Crámer's theorem:

$$\begin{aligned} \sqrt{n}(m_2 - \mu_2) &\xrightarrow{d} N(0, \dot{g}(0, \mu_2) \Sigma_{[2,2]} \dot{g}(0, \mu_2)) \\ &= N(0, (\mu_4 - \mu_2^2)), \end{aligned}$$

where $\Sigma_{[2,2]}$ is the 2 by 2 matrix located in the top left corner of Σ .

For m_3 , let $g(k_1, k_2, k_3) = k_3 - 3k_1k_2 + k_1^3$. Then we have

$$g(0, \mu_2, \mu_3) = \mu_3,$$

$$\dot{g}(k_1, k_2, k_3) = (-3k_2 + 6k_1^2, -3k_1, 1),$$

$$\dot{g}(0, \mu_2, \mu_3) = (-3\mu_2, 0, 1),$$

and, by Crámer's theorem:

$$\begin{aligned} \sqrt{n}(m_3 - \mu_3) &\xrightarrow{d} N(0, \dot{g}(0, \mu_2, \mu_3) \Sigma_{[3,3]} \dot{g}(0, \mu_2, \mu_3)) \\ &= N(0, (\mu_6 - \mu_3^2 - 6\mu_2\mu_4 + 9\mu_2^3)), \end{aligned}$$

where $\Sigma_{[3,3]}$ is the 3 by 3 matrix located in the top left corner of Σ .

For m_4 , let $g(k_1, k_2, k_3, k_4) = k_4 - 4k_1k_3 + 6k_1^2k_2 - 3k_1^4$. Then we have

$$\begin{aligned} g(0, \mu_2, \mu_3, \mu_4) &= \mu_4, \\ \dot{g}(k_1, k_2, k_3, k_4) &= (12k_1k_2 - 12k_1^3 - 4k_3, 6k_1^2, -4k_1, 1), \\ \dot{g}(0, \mu_2, \mu_3, \mu_4) &= (-4\mu_3, 0, 0, 1), \end{aligned}$$

and, by Crámer's theorem:

$$\begin{aligned} \sqrt{n}(m_4 - \mu_4) &\xrightarrow{d} N(0, \dot{g}(0, \mu_2, \mu_3, \mu_4) \Sigma_{[4,4]} \dot{g}(0, \mu_2, \mu_3, \mu_4)) \\ &= N(0, (16\mu_3^2\mu_2 - 8\mu_3\mu_5 + \mu_8 - \mu_4^2)), \end{aligned}$$

where $\Sigma_{[4,4]}$ is the 4 by 4 matrix located in the top left corner of Σ .

For m_5 , let $g(k_1, k_2, k_3, k_4, k_5) = k_5 - 5k_1k_4 + 10k_1^2k_3 - 10k_1^3k_2 + 4k_1^5$. Then we have

$$\begin{aligned} g(0, \mu_2, \mu_3, \mu_4, \mu_5) &= \mu_5, \\ \dot{g}(k_1, k_2, k_3, k_4, k_5) &= (20k_1k_3 - 30k_1^2k_2 + \\ &\quad 20k_1^4 - 5k_4, -10k_1^3, 10k_1^2, -5k_1, 1), \\ \dot{g}(0, \mu_2, \mu_3, \mu_4, \mu_5) &= (-5\mu_4, 0, 0, 0, 1), \end{aligned}$$

and, by Crámer's theorem:

$$\begin{aligned} \sqrt{n}(m_5 - \mu_5) &\xrightarrow{d} N(0, \dot{g}(0, \mu_2, \mu_3, \mu_4, \mu_5) \Sigma_{[5,5]} \dot{g}(0, \mu_2, \mu_3, \mu_4, \mu_5)) \\ &= N(0, (25\mu_4^2\mu_2 - 10\mu_4\mu_6 + \mu_10 - \mu_5^2)), \end{aligned}$$

where $\Sigma_{[5,5]}$ is the 5 by 5 matrix located in the top left corner of Σ .

Now we can write out the asymptotic joint distribution of (m_2, m_3, m_4, m_5) :

$$\sqrt{n} \left[\begin{pmatrix} m_2 \\ m_3 \\ m_4 \\ m_5 \end{pmatrix} - \begin{pmatrix} \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{pmatrix} \right] \xrightarrow{d} N \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \Sigma_m \right], \quad n \rightarrow \infty.$$

We still need to calculate the entries of the matrix Σ_m , i.e., the covariances between components of the random vector (m_2, m_3, m_4, m_5) . We claim that

$$\text{Cov}(m_i, m_j) \rightarrow \mu_{i+j} - \mu_i \mu_j, \quad n \rightarrow \infty, \quad ,$$

but just show that $\text{Cov}(m_2, m_3) \rightarrow \mu_5 - \mu_2 \mu_3$; other calculations are analogous.

Indeed, by Slutsky Theorem,

$$\text{Cov}(m_2, m_3) \rightarrow \text{Cov}(k_2, k_3) \rightarrow \mu_5 - \mu_2 \mu_3.$$

Therefore,

$$\Sigma_m = \begin{bmatrix} \mu_4 - \mu_2^2 & \mu_5 - \mu_2 \mu_3 & \mu_6 - \mu_2 \mu_4 & \mu_7 - \mu_2 \mu_5 \\ \mu_5 - \mu_2 \mu_3 & \mu_6 - \mu_3^2 - 6\mu_2 \mu_4 + 9\mu_2^2 & \mu_7 - \mu_3 \mu_4 & \mu_8 - \mu_3 \mu_5 \\ \mu_6 - \mu_2 \mu_4 & \mu_7 - \mu_3 \mu_4 & 16\mu_3^2 \mu_2 - 8\mu_3 \mu_5 + \mu_8 - \mu_4^2 & \mu_9 - \mu_4 \mu_5 \\ \mu_7 - \mu_2 \mu_5 & \mu_8 - \mu_3 \mu_5 & \mu_9 - \mu_4 \mu_5 & 25\mu_4^2 \mu_2 - 10\mu_4 \mu_6 + \mu_{10} - \mu_5^2 \end{bmatrix}.$$

Now we can write the estimator $\hat{\alpha}$ of the index of a Tempered Stable distribution as a function of m_i , i.e., of the empirical central moments. (calculations for the remaining parameters are similar).

Indeed,

$$\begin{aligned} \hat{\alpha} &= 2 + \frac{2C_3(X) C_4(X)}{C_3(X) C_4(X) - C_2(X) C_5(X)} \\ &= 2 + \frac{2m_3(m_4 - 3m_2^2)}{m_3(m_4 - 3m_2^2) - m_2(m_5 - 10m_2 m_3)} \\ &= 2 + \frac{2m_3 m_4 - 6m_3 m_2^2}{m_3 m_4 - m_2 m_5 + 7m_3 m_2^2}. \end{aligned} \tag{2.2.3}$$

Define the function

$$g(m_2, m_3, m_4, m_5) = 2 + \frac{2m_3 m_4 - 6m_3 m_2^2}{m_3 m_4 - m_2 m_5 + 7m_3 m_2^2}.$$

By Crámer's Theorem,

$$\lim_{n \rightarrow \infty} g(m_2, m_3, m_4, m_5) = g(\mu_2, \mu_3, \mu_4, \mu_5) \tag{2.2.4}$$

$$= 2 + \frac{2\mu_3 \mu_4 - 6\mu_3 \mu_2^2}{\mu_3 \mu_4 - \mu_2 \mu_5 + 7\mu_3 \mu_2^2}. \tag{2.2.5}$$

Also, we have

$$\begin{aligned} \dot{g}(m_2, m_3, m_4, m_5) = & \left(\frac{(m_3m_4 - m_2m_5 + 7m_3m_2^2)(-12m_3m_2) - (2m_3m_4 - 6m_3m_2^2)(-m_5 + 14m_3m_2)}{(m_3m_4 - m_2m_5 + 7m_3m_2^2)^2}, \right. \\ & \frac{(m_3m_4 - m_2m_5 + 7m_3m_2^2)(2m_4 - 6m_2^2) - (2m_3m_4 - 6m_3m_2^2)(m_4 + 7m_2^2)}{(m_3m_4 - m_2m_5 + 7m_3m_2^2)^2}, \\ & \frac{(m_3m_4 - m_2m_5 + 7m_3m_2^2)(2m_3) - (2m_3m_4 - 6m_3m_2^2)(m_3)}{(m_3m_4 - m_2m_5 + 7m_3m_2^2)^2}, \\ & \left. \frac{-(2m_3m_4 - 6m_3m_2^2)(-m_2)}{(m_3m_4 - m_2m_5 + 7m_3m_2^2)^2} \right). \end{aligned}$$

Thus the variance of the estimator $\hat{\alpha}$ as a function of empirical moments m_2, \dots, m_{10} is given by the formula

$$\begin{aligned} \text{Var}(\hat{\alpha}) &= \dot{g}(m_2, m_3, m_4, m_5) \Sigma_m \dot{g}(m_2, m_3, m_4, m_5) \quad (2.2.6) \\ &= 4(7m_2^2m_3 + m_3m_4 - m_2m_5)^{-2} \times \\ &\quad \left(1600m_2^5m_3^6 + 225m_2^7m_3^2m_4^2 - 100m_2^4m_3^4m_4^2 - 150m_2^5m_3^2m_4^3 \right. \\ &\quad + 400m_2^2m_3^4m_4^3 + 25m_2^3m_3^2m_4^4 - 1120m_2^4m_3^5m_5 + 60m_2^5m_3^3m_4m_5 \\ &\quad - 120m_2^3m_3^3m_4^2m_5 - 40m_2m_3^3m_4^3 + 81m_2^9m_5^2 - 9m_2^8m_3^2m_5^2 + 176m_2^3m_3^4m_5^2 \\ &\quad - 108m_2^7m_4m_5^2 - 111m_2^4m_3^2m_4m_5^2 + 45m_2^5m_4^2m_5^2 + 46m_2^2m_3^2m_4^2m_5^2 \\ &\quad - 6m_2^3m_4^3m_5^2 + m_3^2m_4^2m_5^2 + 18m_2^5m_3m_5^3 - 8m_2^2m_3^3m_5^3 - 2m_2m_3m_4^2m_5^3 \\ &\quad - 90m_2^6m_3^2m_4m_6 - 400m_2^3m_3^4m_4m_6 + 60m_2^4m_3^2m_4^2m_6 - 10m_2^2m_3^2m_4^3m_6 \\ &\quad + 60m_2^4m_3^3m_5m_6 + 60m_2^2m_3^3m_4m_5m_6 + 9m_2^6m_5^2m_6 - 6m_2^3m_3^2m_5^2m_6 \\ &\quad - 6m_2^4m_4m_5^2m_6 - 2m_2m_3^2m_4m_5^2m_6 + m_2^2m_4^2m_5^2m_6 + 120m_2^4m_3^3m_4m_7 \\ &\quad - 40m_2^2m_3^3m_4^2m_7 + 42m_2^5m_3^2m_5m_7 - 20m_2^3m_3^2m_4m_5m_7 + 2m_2m_3^2m_4^2m_5m_7 \\ &\quad - 6m_2^4m_3m_5^2m_7 + 2m_2^2m_3m_4m_5^2m_7 + 100m_2^4m_3^4m_8 - 18m_2^8m_3m_5m_8 \\ &\quad - 20m_2^3m_3^3m_5m_8 + 12m_2^4m_3m_4m_5m_8 - 2m_2^2m_3m_4^2m_5m_8 + m_2^2m_3^2m_5^2m_8 \\ &\quad - 60m_2^5m_3^2m_9 + 20m_2^3m_3^3m_4m_9 + 6m_2^4m_3^2m_5m_9 - 2m_2^2m_3^2m_4m_5m_9 \\ &\quad \left. + 9m_2^6m_3^2m_{10} - 6m_2^4m_3^2m_4m_{10} + m_2^2m_3^2m_4^2m_{10} \right). \end{aligned}$$

Proposition 2.2.1: *The estimator $\hat{\alpha}$ given by (2.2.3) is a consistent estimator for α .*

Proof: It is easy to verify that the right hand side of (2.2.4) is equal to α . By combining (2.2.1) and (2.2.2), we have

$$\begin{aligned}\mu_2 &= a\lambda^{\alpha-2}\Gamma(2-\alpha), \\ \mu_3 &= a\lambda^{\alpha-3}\Gamma(2-\alpha)(2-\alpha)(q-p), \\ \mu_4 &= a\lambda^{\alpha-4}\Gamma(2-\alpha)(2-\alpha)(3-\alpha) + 3a^2\lambda^{2\alpha-4}\Gamma(2-\alpha)^2, \\ \mu_5 &= a\lambda^{\alpha-5}\Gamma(2-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)(q-p) + \\ &\quad 10a^2\lambda^{2\alpha-5}\Gamma(2-\alpha)^2(2-\alpha)(q-p).\end{aligned}$$

Hence,

$$\begin{aligned}\mu_3\mu_2^2 &= a^3\lambda^{3\alpha-7}\Gamma(2-\alpha)^3(2-\alpha)(q-p), \\ \mu_2\mu_4 &= a^2\lambda^{2\alpha-7}\Gamma(2-\alpha)^2(2-\alpha)^2(3-\alpha)(q-p) + \\ &\quad 3a^3\lambda^{3\alpha-7}\Gamma(2-\alpha)(2-\alpha)(q-p), \\ \mu_2\mu_5 &= a^2\lambda^{2\alpha-7}\Gamma(2-\alpha)^2(2-\alpha)^2(2-\alpha)(3-\alpha)(q-p) + \\ &\quad 10a^3\lambda^{3\alpha-7}\Gamma(2-\alpha)^3(2-\alpha)(q-p).\end{aligned}\tag{2.2.7}$$

By putting (2.2.7) back into (2.2.5) we will get $g(\mu_2, \mu_3, \mu_4, \mu_5) = \alpha$. Q.E.D

Hence, the above discussion provides the proof of the following result concerning the limit behavior and the accuracy of the estimator $\hat{\alpha}$.

Theorem 2.2.1 *As $n \rightarrow \infty$, we have $\sqrt{n}(\hat{\alpha} - \alpha) \xrightarrow{d} N(0, \text{Var}(\hat{\alpha}))$, where $\text{Var}(\hat{\alpha})$ is given by the formula (2.2.6). In particular, the asymptotic confidence interval for parameter α , at confidence level c , is of the form*

$$\alpha = \hat{\alpha} \pm \Phi^{-1}\left(\frac{1+c}{2}\right) \sqrt{\frac{\text{Var}(\hat{\alpha})}{n}},$$

where Φ is the cumulative distribution function of the standard normal distribution.

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Chapter 3

Convergence Rate for the Series Representation of Tempered α -stable Random Variables

3.1 Error Estimates in the Total Variation

Bentkus et al (see [1] and [2]) studies the accuracy of the LePage series approximation of α -stable laws in terms of the total variation distance (for a discussion of other distances, such as Lévy and Kolmogorov-Smirnov distance, see Chapter 4). In this section we extend the Bentkus' method to address the problem of accuracy of Rosinski's series representation for tempered α -stable random variables. The partial sums in Rosiński's series representation for tempered α -stable random variables are of the form

$$S_n = \sum_{j=1}^n \left((\alpha \Gamma_j)^{-1/\alpha} \wedge e_j u_j^{1/\alpha} v_j^{-1} \right) \quad (3.1.1)$$

$$= (\alpha \Gamma_{n+1})^{-1/\alpha} \sum_{j=1}^n \left(\left(\frac{\Gamma_j}{\Gamma_{n+1}} \right)^{-1/\alpha} \wedge e_j u_j^{1/\alpha} v_j^{-1} (\alpha \Gamma_{n+1})^{1/\alpha} \right), \quad (3.1.2)$$

where for certain Q -measure, $\{v_j\}$ is an i.i.d. sequence with distribution $Q/Q(\mathbb{R})$, $\{u_j\}$ are i.i.d. uniform random variables on $(0, 1)$, and $\{e_j\}$ and

$\{e'_j\}$ are i.i.d exponential random variables with parameter 1. Put $\Gamma_j = e'_1 + \dots + e'_j$.

Denote the total variation distance between the distributions of S_n and S_N ,

$$\delta_{n,N}(\mathcal{M}) = \sup_{A \in \mathcal{M}} |P\{S_n \in A\} - P\{S_N \in A\}|,$$

and abbreviate

$$\delta_n(\mathcal{M}) = \delta_{n,n+1}(\mathcal{M}).$$

Our goal is to estimate

$$\delta_n(\mathcal{M}) = \sup_{A \in \mathcal{M}} |P\{S_n \in A\} - P\{S_{n+1} \in A\}|,$$

where \mathcal{M} denotes a class of subsets of \mathbb{R} that is invariant with respect to shift and multiplication by a positive constant, i.e.,

$$A \in \mathcal{M}, \lambda > 0, a \in \mathbb{R} \Rightarrow \lambda A \in \mathcal{M}, A + a \in \mathcal{M}.$$

In particular, the family $\mathcal{M} = \{(-\infty, x]; x \in \mathbb{R}\}$ satisfies the above condition.

In view of (3.1.1) we have

$$S_{n+1} = S_n + (\alpha \Gamma_{n+1})^{-1/\alpha} \wedge e_{n+1} u_{n+1}^{1/\alpha} v_{n+1}^{-1}.$$

Let ξ_i follow the Pareto distribution. Let $\xi_{n1} \geq \xi_{n2} \geq \dots \geq \xi_{nn}$ be the ordered statistics of ξ_1, \dots, ξ_n , $h = (\xi_{n1}, \dots, \xi_{nn})$, and

$$g = \left(\left(\frac{\Gamma_1}{\Gamma_{n+1}} \right)^{-1/\alpha}, \dots, \left(\frac{\Gamma_n}{\Gamma_{n+1}} \right)^{-1/\alpha} \right).$$

Then $\mathcal{L}(h) = \mathcal{L}(g)$ (see [4]).

Lemma 3.1.1 Denote $X_j = e_j u_j^{1/\alpha} v_j^{-1} (\alpha \Gamma_{n+1})^{1/\alpha}$. Let $Z_n = \sum_{j=1}^n \xi_j \wedge X_j$.

Then

$$\delta_n(\mathcal{M}) \leq \sup_{A \in \mathcal{M}} |P\{Z_n \in A\} - P\{Z_n + (1 \wedge X) \in A\}|$$

.

Proof. Observe that

$$\begin{aligned} \delta_n(\mathcal{M}) &= \sup_{A \in \mathcal{M}} |P\{S_n \in A\} - P\{S_{n+1} \in A\}| \\ &= \sup_{A \in \mathcal{M}} |P\{(\alpha\Gamma_{n+1})^{-1/\alpha} Z_n \in A\} - P\{(\alpha\Gamma_{n+1})^{-1/\alpha} (Z_n + 1 \wedge X_{n+1}) \in A\}|. \end{aligned}$$

By definition of \mathcal{M} and since X_j are i.i.d random variables with $\mathcal{L}(X_{n+1}) = \mathcal{L}(X)$,

$$\delta_n(\mathcal{M}) \leq \sup_{A \in \mathcal{M}} |P\{(Z_n \in A) - P\{Z_n + (1 \wedge X_{n+1}) \in A\}|.$$

This completes the proof of the Lemma. Q.E.D.

Let a_1, \dots, a_m be non-random constants such that $c_1 \leq a_j \leq c_2$, for all $1 \leq j \leq m$.

Lemma 3.1.2 Let $V_m = (\alpha m)^{-1/\alpha} \sum_{j=1}^m \xi_j \wedge a_j$, and

$$\tilde{\delta}_m(\mathcal{M}) = \sup_{A \in \mathcal{M}} \sup_a |P\{V_m \in A\} - P\{V_m + (\alpha m)^{-1/\alpha} (1 \wedge X) \in A\}|.$$

Also, define $p = P(c_1 \leq |X| \leq c_2)$, for $1 < c_1 \leq c_2 \leq \infty$, and $m = \lfloor pn/2 \rfloor$. If $p = 1$, then $\delta_n(\mathcal{M}) \leq \tilde{\delta}_n(\mathcal{M})$, and if $p < 1$, then $\delta_n(\mathcal{M}) \leq \tilde{\delta}_m(\mathcal{M}) + \exp\{-pn/8\}$.

Proof. If $p = 1$, just let $a_j = X_j$ and we get the result. If $p < 1$, introduce Bernoulli random variable $\eta_j = I\{c_1 \leq |X_j| \leq c_2\}$ for $j \geq 1$. Consider the event $B_n = \{\eta_1 + \dots + \eta_n > pn/2\}$. By Chernoff's inequality (see [3]),

$$P\{B_n^c\} \leq \exp\{-pn/8\}.$$

From Lemma 3.1.1 we have

$$\begin{aligned} \delta_n(\mathcal{M}) &\leq \sup_{A \in \mathcal{M}} |P\{Z_n \in A\} - P\{Z_n + (1 \wedge X) \in A\}| \\ &\leq \sup_{A \in \mathcal{M}} |(P\{Z_n \in A\} - P\{Z_n + (1 \wedge X) \in A\}) P\{B_n\}| + P\{B_n^c\} \\ &= \sup_{A \in \mathcal{M}} |E(I\{Z_n \in A\} - I\{Z_n + (1 \wedge X) \in A\}) I\{B_n\}| + P\{B_n^c\} \\ &\leq E_{\{X_1, \dots, X_n\}} \delta'_n(\mathcal{M}) + \exp\{-pn/8\}, \end{aligned}$$

where

$$\delta'_n(\mathcal{M}) = I\{B_n\} \sup_{A \in \mathcal{M}} |E_{\{\xi_1, \dots, \xi_n, X\}}(I\{Z_n \in A\} - I\{Z_n + (1 \wedge X) \in A\})|.$$

It suffices to show that $\delta'_n(\mathcal{M}) \leq \tilde{\delta}_m(\mathcal{M})$. If $I\{B_n\} = 0$, then it is trivial. When $I\{B_n\} = 1$, then $\eta_1 + \dots + \eta_n > pn/2$. Therefore, there exist l_1, \dots, l_k such that $k > pn/2 > m$ and $c_1 < |X_{l_j}| < c_2$ for $1 < j < k$.

Define random sets $M_1 = \{l_1 \dots l_m\}$, $M_2 = \{1 \dots n\} \setminus M_1$, and introduce the notation

$$U = \sum_{j \in M_1} \xi_j \wedge X_j, \quad V = \sum_{j \in M_2} \xi_j \wedge X_j,$$

and

$$E_M = E_{\{\xi_i, i \in M\}}.$$

Then

$$\begin{aligned} \delta'_n(\mathcal{M}) &= \sup_{A \in \mathcal{M}} |E_{M_1} E_{M_2} E_{\{X\}} (I\{U \in A - V\} - I\{U + (1 \wedge X) \in A - V\})| \\ &\leq E_{M_2} \sup_{A \in \mathcal{M}} |E_{M_1} E_{\{X\}} (I\{U \in A - V\} - I\{U + (1 \wedge X) \in A - V\})| \\ &= \sup_{A \in \mathcal{M}} |E_{M_1} E_{\{X\}} (I\{U \in A\} - I\{U + (1 \wedge X) \in A\})| \\ &\leq \sup_{A \in \mathcal{M}} \sup_a |E_{M_1} E_{\{X\}} (I\{U \in A\} - I\{U + (1 \wedge X) \in A\})| \\ &= \sup_{A \in \mathcal{M}} \sup_a |E_{M_1} E_{\{X\}} (I\{(\alpha m)^{-1/\alpha} U \in A\} - I\{(\alpha m)^{-1/\alpha} (U + (1 \wedge X)) \in A\})| \\ &= \tilde{\delta}_m(\mathcal{M}). \end{aligned}$$

This completes the proof of Lemma 3.1.2. Q.E.D.

To derive the error bound, we employ Lemma 3.1.2. Write

$$J = |P\{V_m \in A\} - P\{V_m + (\alpha m)^{-1/\alpha}(1 \wedge X) \in A\}|,$$

and introduce

$$Y = (1 \wedge X)I\{|1 \wedge X| \leq m^{1/\alpha}\}.$$

It follows that

$$\begin{aligned} J &= |P\{V_m \in A\} - P\{V_m + (\alpha m)^{-1/\alpha}(1 \wedge X) \in A\}P\{|1 \wedge X| \leq m^{1/\alpha}\} \\ &\quad - P\{V_m + (\alpha m)^{-1/\alpha}(1 \wedge X) \in A\}P\{|1 \wedge X| \geq m^{1/\alpha}\}| \\ &\leq |P\{V_m \in A\} - P\{V_m + (\alpha m)^{-1/\alpha}Y \in A\}| + P\{|1 \wedge X| \geq m^{1/\alpha}\}. \end{aligned}$$

Assume V_m defined in Lemma 3.1.2 has a differentiable density, say p_m and let

$$\begin{aligned} J_t &= |P\{V_m \in A\} - P\{V_m + (\alpha m)^{-1/\alpha}Y \in A\}| \\ &= |E_Y \int_A (p_m(x) - p_m(x - (\alpha m)^{-1/\alpha}Y)) dx|. \end{aligned}$$

Expand $p_m(x - (\alpha m)^{-1/\alpha}Y)$ in powers of $(\alpha m)^{-1/\alpha}Y$:

$$\begin{aligned} p_m(x - (\alpha m)^{-1/\alpha}Y) &= p_m(x) - (\alpha m)^{-1/\alpha}Y p'_m(x) + (\alpha m)^{-2/\alpha}Y^2 \frac{p''_m(x)}{2} \\ &\quad + O((\alpha m)^{-1/\alpha}Y). \end{aligned}$$

Then

$$\begin{aligned} J_t &\leq |E_Y \int_A \left((\alpha m)^{-1/\alpha}Y p'_m(x) - (\alpha m)^{-2/\alpha}Y^2 \frac{p''_m(x)}{2} \right) dx| \\ &\leq |E_Y (\alpha m)^{-1/\alpha}Y \int_A p'_m(x) dx| + |E_Y (\alpha m)^{-2/\alpha}Y^2 \int_A p''_m(x) dx| \\ &\leq (\alpha m)^{-1/\alpha} |E_Y| \sup_{A \in \mathcal{M}} \int_A |p'_m(x)| dx + (\alpha m)^{-2/\alpha} E_Y^2 \sup_{A \in \mathcal{M}} \int_A |p''_m(x)| dx. \end{aligned}$$

So it remains to estimate $\int_{\mathbb{R}} |p'_m(x)| dx$ and $\int_{\mathbb{R}} |p''_m(x)| dx$.

Lemma 3.1.3 *Consider a random variable $X = \xi \wedge a_j$, where ξ is a Pareto random variable and $a_j > 1$ is a constant so that X is a truncated Pareto random variable with density function*

$$f(x) = \frac{\alpha x^{-\alpha-1}}{1 - a_j^{-\alpha}} I\{1 < x < a_j\}.$$

Letting $f(t)$ be the characteristic function of X , we have the following inequalities:

$$\begin{aligned} |f(t)| &\leq \exp\{-C_1(\alpha, a_j)|t|^\alpha\}, & |t| \leq 1; \\ |f'(t)| &\leq C_2(\alpha, a_j)|t|^{\alpha-1}, & t \neq 0; \\ |f(t)| &\leq C_3(\alpha, a_j)|t|^{-1}, & t \neq 0; \\ |f(t)| &\leq C_4(\alpha, a_j, \delta), & |t| > \delta, \end{aligned}$$

where $C_1(\alpha, a_j), C_2(\alpha, a_j), C_3(\alpha, a_j)$ and $C_4(\alpha, a_j, \delta)$ are corresponding constants.

Proof. We assume $t > 0$. The case $t < 0$ can be proved similarly. We first prove the case where $|t| \leq 1$.

For a truncated Pareto, the characteristic function

$$\begin{aligned} f(t) &= \int_1^{a_j} \frac{\alpha x^{-\alpha-1}}{1 - a_j^{-\alpha}} e^{ixt} dx \\ &= \int_1^{a_j} \frac{\alpha x^{-\alpha-1}}{1 - a_j^{-\alpha}} \cos(xt) dx + i \int_1^{a_j} \frac{\alpha x^{-\alpha-1}}{1 - a_j^{-\alpha}} \sin(xt) dx. \end{aligned}$$

Let $f_1(t)$ and $f_2(t)$ be real and imaginary parts of $f(t)$. Then we have

$$\begin{aligned} f_1(t) &= 1 - \frac{\alpha}{1 - a_j^{-\alpha}} \int_1^{a_j} x^{-\alpha-1} (1 - \cos(xt)) dx \\ &= 1 - \frac{\alpha t^\alpha}{1 - a_j^{-\alpha}} \int_t^{a_j t} u^{-\alpha-1} (1 - \cos u) du. \end{aligned}$$

Since we can find some constant t_1 and t_2 such that $t < t_1 < t_2 < a_j t$ and since the integrand is strictly positive, we have

$$\begin{aligned} f_1(t) &\leq 1 - \frac{\alpha t^\alpha}{1 - a_j^{-\alpha}} \int_{t_1}^{t_2} u^{-\alpha-1} (1 - \cos u) du \\ &\leq 1 - C'_1(\alpha, a_j) t^\alpha. \end{aligned}$$

By integration by parts, we have,

$$f_2(t) \leq C''_1(\alpha, a_j)(t^\alpha + t).$$

Therefore, given $|t| \leq 1$, we have

$$\begin{aligned} |f(t)| &= (|f_1(t)|^2 + |f_2(t)|^2)^{1/2} \\ &\leq 1 - C_1(\alpha, a_j) t^\alpha \\ &\leq \exp(-C_1(\alpha, a_j) t^\alpha). \end{aligned}$$

For $|t| \geq \delta \geq 1$, notice that $f(t)$ is the Fourier transform of an integrable function, therefore, we can find some constant $C_4(\alpha, a_j, \delta)$ such that

$$|f(t)| \leq C_4(\alpha, a_j, \delta).$$

Thus, for all $t \neq 0$, we have

$$\begin{aligned} f_1(t) &= \frac{\alpha}{1 - a_j^{-\alpha}} \int_1^{a_j} x^{-\alpha-1} \cos(tx) dx \\ &= \frac{\alpha t^{-1}}{1 - a_j^{-\alpha}} \int_1^{a_j} x^{-\alpha-1} d \sin(tx) \\ &= \frac{\alpha t^{-1}}{1 - a_j^{-\alpha}} \left[x^{-\alpha-1} \sin(tx) \Big|_1^{a_j} - \int_1^{a_j} \sin(tx) (-\alpha - 1) x^{-\alpha-2} dx \right] \\ &= \frac{\alpha t^{-1}}{1 - a_j^{-\alpha}} \left[a_j^{-\alpha-1} \sin(a_j t) - \sin(t) + \int_1^{a_j} \sin(tx) (\alpha + 1) x^{-\alpha-2} dx \right] \\ &\leq C'_3(\alpha, a_j) t^{-1} \end{aligned}$$

and

$$\begin{aligned}
f_2(t) &= \frac{\alpha}{1 - a_j^{-\alpha}} \int_1^{a_j} x^{-\alpha-1} \sin(tx) dx \\
&= \frac{\alpha t^{-1}}{1 - a_j^{-\alpha}} \int_1^{a_j} -x^{-\alpha-1} d \cos(tx) \\
&= \frac{\alpha t^{-1}}{1 - a_j^{-\alpha}} \left[-x^{-\alpha-1} \cos(tx) \Big|_1^{a_j} - \int_1^{a_j} \cos(tx) (\alpha + 1) x^{-\alpha-2} dx \right] \\
&= \frac{\alpha t^{-1}}{1 - a_j^{-\alpha}} \left[-a_j^{-\alpha-1} \cos(a_j t) + \cos t - \int_1^{a_j} \cos(tx) (\alpha + 1) x^{-\alpha-2} dx \right] \\
&\leq C_3''(\alpha, a_j) t^{-1}.
\end{aligned}$$

Hence, for $|t| \neq 0$,

$$\begin{aligned}
|f(t)| &= (|f_1(t)|^2 + |f_2(t)|^2)^{1/2}, \\
&\leq C_3(\alpha, a_j) t^{-1}.
\end{aligned}$$

To estimate $|f'(t)|$, using the above information about $f_1(t)$ and $f_2(t)$, we have

$$\begin{aligned}
f_1'(t) &= - \frac{\alpha}{1 - a_j^{-\alpha}} \int_1^{a_j} x \sin(xt) x^{-\alpha-1} dx \\
&= - \frac{\alpha}{1 - a_j^{-\alpha}} \int_t^{a_j t} \frac{u}{t} \sin(u) \left(\frac{u}{t}\right)^{-\alpha-1} t^{-1} du \\
&= - \frac{\alpha t^{\alpha-1}}{1 - a_j^{-\alpha}} \int_t^{a_j t} \sin(u) u^{-\alpha} du
\end{aligned}$$

so that for all $t \neq 0$,

$$|f_1'(t)| \leq C_2'(\alpha, a_j) t^{\alpha-1}.$$

Similarly for $f_2'(t)$,

$$\begin{aligned} f_2'(t) &= -\frac{\alpha}{1-a_j^{-\alpha}} \int_1^{a_j} -x \cos(xt) x^{-\alpha-1} dx \\ &= \frac{\alpha}{1-a_j^{-\alpha}} \int_t^{a_j t} \frac{u}{t} \cos(u) \left(\frac{u}{t}\right)^{-\alpha-1} t^{-1} du \\ &= \frac{\alpha t^{\alpha-1}}{1-a_j^{-\alpha}} \int_t^{a_j t} \cos(u) u^{-\alpha} du, \end{aligned}$$

so that for all $t \neq 0$,

$$|f_2'(t)| \leq C_2''(\alpha, a_j) t^{\alpha-1}.$$

Therefore,

$$\begin{aligned} |f'(t)| &= (|f_1'(t)|^2 + |f_2'(t)|^2)^{1/2} \\ &\leq C_2(\alpha, a_j) t^{\alpha-1}, \end{aligned}$$

and the proof of Lemma 3.1.3 is complete. Q.E.D.

Actually, Lemma 3.1.3 can be also derived from the Tauberian theorems of section 1.2. However, we decided to provide direct proofs.

Lemma 3.1.4 *Assume that the distribution function of $V_m = (\alpha m)^{-1/\alpha} \sum_{j=1}^m \xi_j \wedge a_j$, where $c_1 \leq a_j \leq c_2$ are defined above, has a differentiable density, say p_m . Let*

$$K_1 = \sup_m \sup_a \int_{\mathbb{R}} |p_m'(x)| dx$$

and

$$K_2 = \sup_m \sup_a \int_{\mathbb{R}} |p_m''(x)| dx.$$

Then

$$K_2 \leq C(\alpha) K_1^2 \quad \text{and} \quad K_1^2 \leq C_K(\alpha, a).$$

Proof. First we show $K_2 \leq C(\alpha)K_1^2$. Let $k = \lfloor m/2 \rfloor$. Then we can split $V_m = U_1 + U_2$, where $U_1 = (\alpha m)^{-1/\alpha} \sum_{j=1}^k \xi_j \wedge a_j$, and $U_2 = V_m - U_1$. Define u_1 and u_2 as the densities of U_1 and U_2 , respectively. Then

$$p_m = u_1 * u_2,$$

and

$$|p_m''| = |u_1' * u_2'| \leq |u_1'| * |u_2'|.$$

By integrating both sides and taking suprema over m and a and using Fubini's Theorem, we obtain $K_2 \leq C(\alpha)K_1^2$.

In particular,

$$\int_{\mathbb{R}} |p_m''(x)| dx \leq \int_{\mathbb{R}} |u_1'| * |u_2'| (x) dx = \left(\int_{\mathbb{R}} |u_1'(x)| dx \right) \left(\int_{\mathbb{R}} |u_2'(x)| dx \right)$$

and

$$\sup_m \sup_a \int_{\mathbb{R}} |p_m''(x)| dx \leq \sup_m \sup_a \left(\int_{\mathbb{R}} |u_1'(x)| dx \right) \left(\int_{\mathbb{R}} |u_2'(x)| dx \right).$$

Hence,

$$K_2 \leq C(\alpha)K_1^2.$$

Now we show that $K_1^2 \leq C(\alpha, a)$. Let $\epsilon > 0$ denote an arbitrary fixed number. By Holder's inequality,

$$K_1^2 \leq C\epsilon^{-1} \sup_m \sup_a \int_{\mathbb{R}} (\epsilon^2 + x^2) (p_m'(x))^2 dx.$$

In view of the properties of the Fourier transform,

$$\int_{\mathbb{R}} (p_m'(x))^2 dx \leq C \int_{\mathbb{R}} t^2 |\hat{p}_m(t)|^2 dt,$$

where $\hat{p}_m(t)$ is the Fourier transform of $p_m(x)$ so that

$$\begin{aligned} \int_{\mathbb{R}} x^2 (p'_m(x))^2 dx &= \int_{\mathbb{R}} (xp'_m(x))^2 dx \\ &\leq C \int_{\mathbb{R}} |\hat{p}_m(t)|^2 dt + C \int_{\mathbb{R}} t^2 |\hat{p}'_m(t)|^2 dt. \end{aligned}$$

Therefore we can write

$$K_1^2 \leq C\epsilon^{-1} \sup_{a,m} I_0 + C\epsilon \sup_{a,m} I_2 + C\epsilon^{-1} \sup_{a,m} H,$$

where

$$I_k = \int_{\mathbb{R}} t^k |\hat{p}_m(t)|^2 dt \quad k = 0, 2,$$

and

$$H = \int_{\mathbb{R}} t^2 |\hat{p}'_m(t)|^2 dt.$$

We estimate I_k first. As defined in Lemma 3.1.3, $f_j(t)$ is the characteristic function of the truncated Pareto random variable $\xi_j \wedge a_j$. Let $g_j(t) = f_j((\alpha m)^{-1/\alpha} t)$. Then $\hat{p}_m = g_1 \dots g_m$. Applying the geometric-arithmetic mean inequality we have

$$|\hat{p}_m|^2 \leq m^{-1} \sum_{j=1}^m |g_j|^{2m}.$$

Therefore,

$$\begin{aligned} I_k &\leq m^{-1} \sum_{j=1}^m \sup_m \sup_a \int_{\mathbb{R}} t^k |g_j(t)|^{2m} dt \\ &= \sup_m \sup_{c_1 \leq a \leq c_2} \int_{\mathbb{R}} t^k |f_j((\alpha m)^{-1/\alpha} t)|^{2m} dt \\ &\leq \alpha^{-k-1} \sup_m \int_{\mathbb{R}} t^k |f_j(m^{-1/\alpha} t)|^{2m} dt. \end{aligned}$$

By Lemma 3.1.3, we have

$$\sup_{0 \leq k \leq 2} \sup_{m \geq 5} \int_{\mathbb{R}} t^k |f_j(m^{-1/\alpha} t)|^{2m-4} dt \leq C(\alpha, a).$$

Furthermore, if we split the real line \mathbb{R} into disjoint sets $\{|t| \leq m^{1/\alpha}\}$, $\{m^{1/\alpha} \leq |t| \leq m^{2/\alpha}\}$ and $\{|t| \geq m^{2/\alpha}\}$, and for each set, apply Lemma 3.1.3 (1st, 4th and 3rd inequalities, respectively), we get the above result. Therefore, we have

$$I_k \leq C_I(\alpha, a).$$

To estimate H , we use the product formula

$$\hat{p}'_m = \sum_{j=1}^m g'_j \prod_{1 \leq k \leq m, k \neq j} g_k.$$

Hence

$$(\hat{p}'_m)^2 = \sum_{j=1}^m (g'_j)^2 \prod_{1 \leq k \leq m, k \neq j} g_k^2 + 2 \sum_{1 \leq j < l \leq m} g'_j g'_l \left(\prod_{1 \leq k \leq m, k \neq j} g_k^2 \right) \left(\prod_{1 \leq k \leq m, k \neq l} g_k^2 \right).$$

By Holder's inequality, if we write $A = |g_3(t)|^2 \dots |g_m(t)|^2$, we have

$$\int t^2 |g'_1| |g'_2| A dt \leq \left(\sup_a \int t^2 |g'_1|^2 A dt \sup_a \int t^2 |g'_2|^2 A dt \right)^{1/2} \leq \sup_a \int t^2 |g'_1|^2 A dt.$$

Therefore we have a bound for H :

$$H \leq C m^2 \sup_{a, m \geq 3} \int_{\mathbb{R}} t^2 |g'_1(t)|^2 |g_2(t)|^{2m-4} dt.$$

Notice that g_m and f_m only differ in scale, so we can change variables and combine constants to get

$$H \leq C(\alpha) m^2 \sup_{a, m \geq 3} \int_{\mathbb{R}} t^2 |f'_1(m^{-1/\alpha} t)|^2 |f_2(m^{-1/\alpha} t)|^{2m-4} dt.$$

By the second inequality in Lemma 3.1.3, we have $|f'(t)| \leq C_2(\alpha, a_j)|t|^{\alpha-1}$ for all $|t| \neq 0$. Combining all above inequalities, we get

$$\begin{aligned} H &\leq C_H(\alpha, a_j) \sup_{m \geq 3} \sup_a \int_{\mathbb{R}} |t|^{2\alpha} |f_2(m^{-1/\alpha}t)|^{2m-4} dt \\ &\leq C'_H(\alpha, a_j). \end{aligned}$$

Since ϵ can be arbitrary, we have

$$K_1^2 \leq C_K(\alpha, a).$$

Now let us go back to definitions of J and J_t provided in Lemma 3.1.2. We already have

$$J \leq J_t + P\{|1 \wedge X| > m^{1/\alpha}\}$$

and

$$J_t \leq (\alpha m)^{-1/\alpha} EY K_1 + (\alpha m)^{-2/\alpha} EY^2 K_2,$$

where $Y = (1 \wedge X)I\{|1 \wedge X| \leq m^{1/\alpha}\}$. Hence,

$$J \leq C_J(\alpha, a) \left(P\{|1 \wedge X| > m^{1/\alpha}\} + (\alpha m)^{-1/\alpha} EY + (\alpha m)^{-2/\alpha} EY^2 \right).$$

Letting $t = m^{1/\alpha}$, observe that

$$1 - F(t) \equiv P\{|1 \wedge X| > t\} \leq C \exp(-t),$$

$$E|Y| \leq 1,$$

and

$$E|Y^2| \leq 1,$$

So we have

$$\begin{aligned} J &\leq C_J(\alpha, a) \left(C e^{-t} + t^{-1} EY + t^{-2} EY^2 \right) \\ &\leq C'_J(\alpha, a) t^{-1} \\ &= C'_J(\alpha, a) (m^{-1/\alpha}), \end{aligned}$$

where $m = n$ when $p = 1$ and $m = np/2$ when $p < 1$. The number p is defined in Lemma 3.1.2. The following theorems gives our final results:

Theorem 3.1.1: *Let $\alpha < 1$ and*

$$S_n = \sum_{j=1}^n \left((\alpha \Gamma_j)^{-1/\alpha} \wedge e_j u_j^{1/\alpha} v_j^{-1} \right),$$

where Γ_j , e_j , u_j and v_j are defined as in (3.1.1), then

$$\delta_n(\mathcal{M}) = \sup_{A \in \mathcal{M}} |P\{S_n \in A\} - P\{S_{n+1} \in A\}| \leq C(\alpha)(n^{-1/\alpha}), \quad (3.1.3)$$

Moreover

$$\sup_{N > n} \sup_{A \in \mathcal{M}} |P\{S_n \in A\} - P\{S_N \in A\}| \leq C'(\alpha)(n^{1-\frac{1}{\alpha}}). \quad (3.1.4)$$

Proof: The inequality (3.1.3) and (3.1.4) follows from the following estimate.

For $\delta_n(\mathcal{M})$, we have

$$\begin{aligned} \delta_n(\mathcal{M}) &= \sup_{A \in \mathcal{M}} |P\{S_n \in A\} - P\{S_{n+1} \in A\}| \\ &\leq \sup_{A \in \mathcal{M}} \sup_a J + \exp(-pn/8) \\ &\leq C(\alpha)(n^{-1/\alpha}). \end{aligned}$$

To estimate $\delta_{n,N}(\mathcal{M})$, integrate $\delta_x(\mathcal{M})$ in x over the interval (n, N) ,

$$\begin{aligned} \delta_{n,N} &\leq C(\alpha) \int_n^N (x^{-1/\alpha}) dx \\ &= C(\alpha) \frac{1}{1-1/\alpha} (N^{1-1/\alpha} - n^{1-1/\alpha}) \\ &\leq C'(\alpha)(n^{1-\frac{1}{\alpha}}). \end{aligned}$$

Q.E.D.

Theorem 3.1.2: *Let $\alpha < 1$ and*

$$G_\alpha = \mathcal{L} \left(\sum_{j=1}^{\infty} \left((\alpha \Gamma_j)^{-1/\alpha} \wedge e_j u_j^{1/\alpha} v_j^{-1} \right) \right),$$

then we have

$$\Delta_n = \sup_x |\mathbb{P}\{S_n \leq x\} - G_\alpha(x)| \leq \delta_{n,\infty} \leq C'(\alpha)(n^{1-1/\alpha})$$

for a certain constant $C'(\alpha)$ depending only on α .

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Chapter 4

Simulations

There exist numerous studies of computer simulations of α -stable random variables. Those studies include LePage-like series representation simulations [1], asymptotic expansions, [2], [4] and many others.

For the tempered stable distribution, we chose to experiment with the Rosiński series representation in [5]. The simulations discussed in this chapter preceded the theoretical work described in Chapter 3 and were meant to offer guidance in forming hypotheses about the rates of convergence of the Rosiński series representations in different metrics.

4.1 Lévy distance and Kolmogorov-Smirnov distance

In this section, we will give formal definitions for Lévy distance and Kolmogorov-Smirnov distance.

Lévy distance, was first introduced by P. Lévy in [3].

Definition 4.1.1 *Let $F, G : \mathbb{R} \rightarrow [0, +\infty)$ be two cumulative distribution functions. The Lévy distance between them is defined by the formula*

$$L(F, G) := \inf\{\epsilon > 0 \mid F(x - \epsilon) - \epsilon \leq G(x) \leq F(x + \epsilon) + \epsilon, \quad x \in \mathbb{R}\}.$$

Kolmogorov-Smirnov or uniform distance, which is used in the well known Kolmogorov-Smirnov test, has the definition below.

Definition 4.1.2 *Let F and G be two cumulative distribution functions, then the Kolmogorov-Smirnov distance is defined as*

$$D(F, G) := \sup_x |G(x) - F(x)|.$$

We will use Lévy distance and Kolmogorov-Smirnov distance between the empirical cumulative distribution function and the theoretical cumulative distribution function to measure the convergence rates in the Rosiński series representation for some tempered stable distributions.

4.2 Simulation of Tempered Stable Random Variable

In this section we use the Rosiński series representation in [5] to generate tempered stable random variables with given tempering function $k(r) = (1 + \lambda r)^{-\beta}$, which is called "Gamma tempering" in [7]. In this case, the corresponding density of Q -Measure is

$$q(x) = \frac{x^{\beta-1}}{\lambda^\beta \Gamma(\beta)} \exp(-x/\lambda).$$

The partial sums of the Rosiński series representation are of the form

$$S_n = \sum_{j=1}^n \left((\alpha \Gamma_j)^{-1/\alpha} \wedge e_j u_j^{1/\alpha} v_j^{-1} \right), \quad (4.2.1)$$

where $\{v_j\}$ are i.i.d. random variables with common distribution $Q/Q(\mathbb{R})$, $\{u_j\}$ are i.i.d. uniform random variables on $(0, 1)$, and $\{e_j\}$ and $\{e'_j\}$ are i.i.d. exponential random variables with parameter 1. Let $\Gamma_j = e'_1 + \dots + e'_j$ and recall that $x \wedge y := \min\{x, y\}$. Then the series converges a.s. to $S \sim TS_\alpha(R, 0)$

for $\alpha \in (0, 1)$.

The parameters in our simulation were chosen as follows: $\alpha = 0.8$, $\beta = 0.9$, $\lambda = 2$. Partial sums S_n were simulated for $n = 500, 1000, 1500, \dots, 50000$, giving us 100 sample points. For each n , we generated a random sample of size 200. Then, we produced the empirical cumulative distribution function of S_n from the above sample.

The theoretical cumulative distribution (CDF) of the Gamma tempering random variable was calculated using the following steps:

First, we determined the theoretical probability density function (PDF) of S via the numerical inverse Fourier transformation of its characteristic function. The inverse Fourier transformation was carried out via the FFT algorithm (see [6]). Then, we numerically integrate the PDF to get the CDF.

Some of our simulation results are shown in Fig. 4.2.1-7.

Plotted in the Log-Log scales, the simulations suggested the convergence rate of about $O(n^{-0.23})$ for $\alpha = 0.8$, and $O(n^{-0.03})$ for $\alpha = 0.9$. Interestingly, there was not much difference between the Lévy distance and the Kolmogorov-Smirnov distance. So, the theoretical rates $O(n^{1-1/\alpha})$ obtained for the uniform distance in Chapter 3 cannot be too far from the optimal rates in general. For $\alpha = 0.8$, they give the rate $O(n^{-0.25})$, and for $\alpha = 0.9$, they predict $O(n^{-0.11})$. However, our simulation results have to be taken with a grain of salt as they obviously are very noisy.

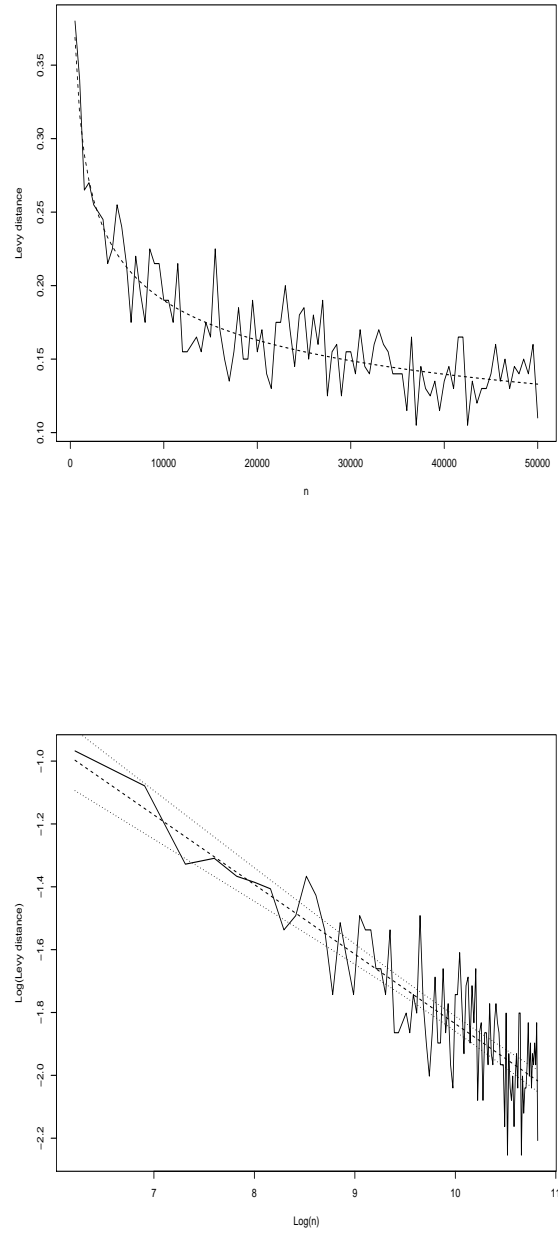


Figure 4.2.1: (Top) Plot of the Lévy distance $L(F_n, F)$, with $F_n \sim S_n$, $F \sim S$ as a function of n . (Bottom) Plot of $\log L(F_n, F)$ vs. $\log n$. In this simulation, $\alpha = 0.8$, and the slope of the linear best fit turned out to be -0.24 . Error bands correspond to the 95% confidence level.

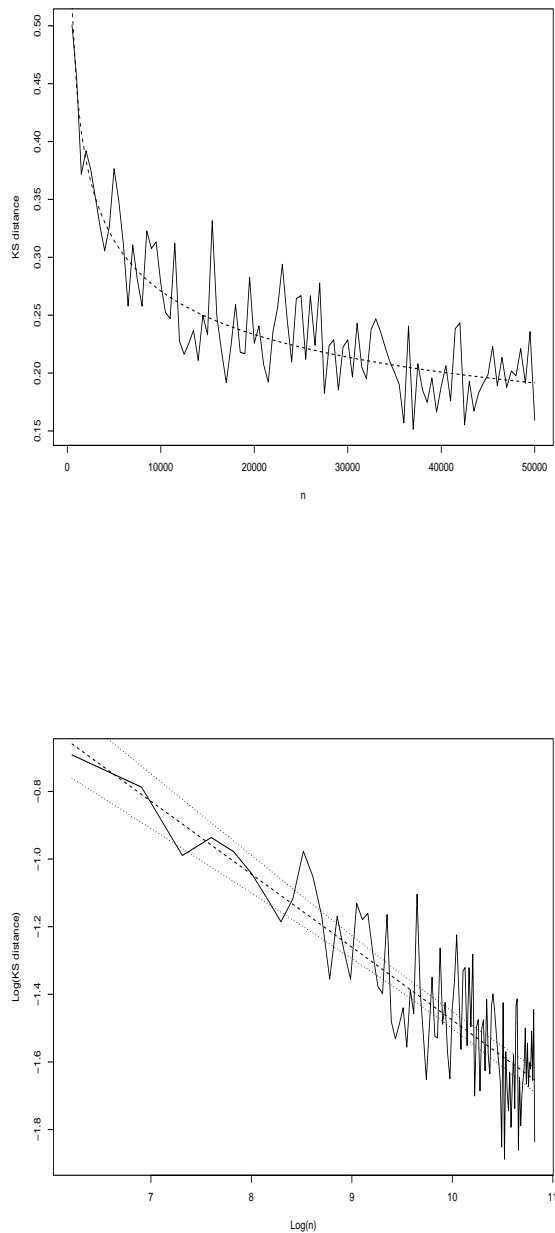


Figure 4.2.2: (Top) Plot of the Kolmogorov-Smirnov distance $D(F_n, F)$, with $F_n \sim S_n$, $F \sim S$, as a function of n . (Bottom) Plot of $\log D(F_n, F)$ vs. $\log n$. In this simulation, $\alpha = 0.8$, and the slope of the linear best fit turned out to be -0.23. Error bands correspond to the 95% confidence level.

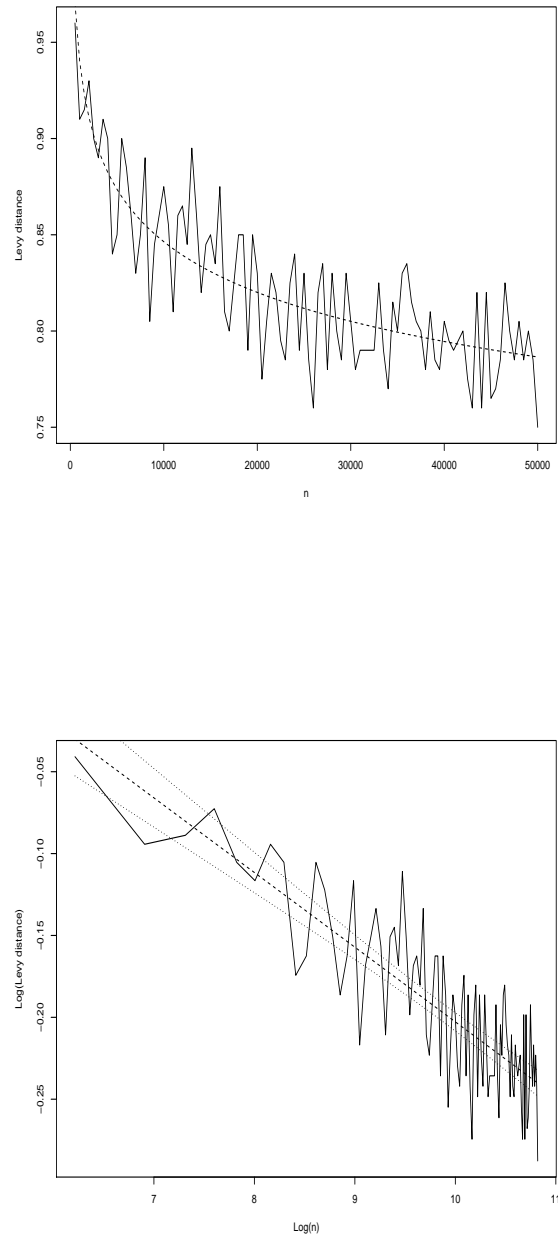


Figure 4.2.3: (Top) Plot of the Lévy distance $L(F_n, F)$, with $F_n \sim S_n$, $F \sim S$ as a function of n . (Bottom) Plot of $\log L(F_n, F)$ vs. $\log n$. In this simulation, $\alpha = 0.9$, and the slope of the linear best fit turned out to be -0.05. Error bands correspond to the 95% confidence level.

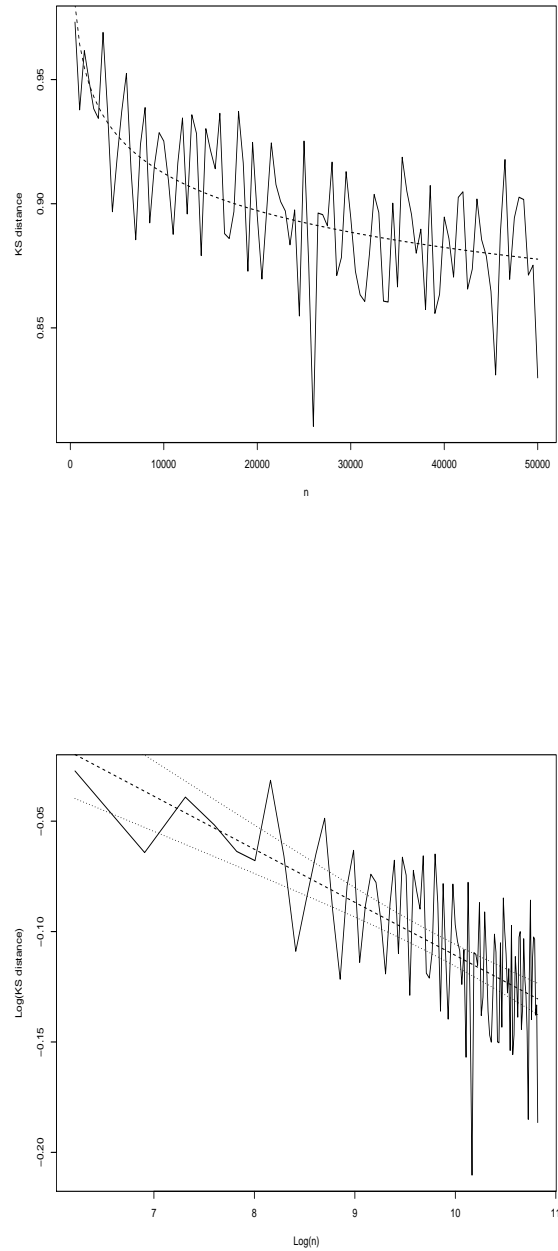


Figure 4.2.4: (Top) Plot of the Kolmogorov-Smirnov distance $D(F_n, F)$, with $F_n \sim S_n$, $F \sim S$, as a function of n . (Bottom) Plot of $\log D(F_n, F)$ vs. $\log n$. In this simulation, $\alpha = 0.9$, and the slope of the linear best fit turned out to be -0.03 . Error bands correspond to the 95% confidence level.

The last plot (Fig. 4.2.5) shows the empirical CDFs of S_n , for $n = 500, 2500, 7500, 15000, 20000, 30000, 40000$, and 50000 (dotted line) and the theoretical CDF of S (solid line) with $\alpha = 0.8$. The convergence is quite slow reflecting the low power convergence rate $n^{1-1/\alpha} = n^{-0.25}$

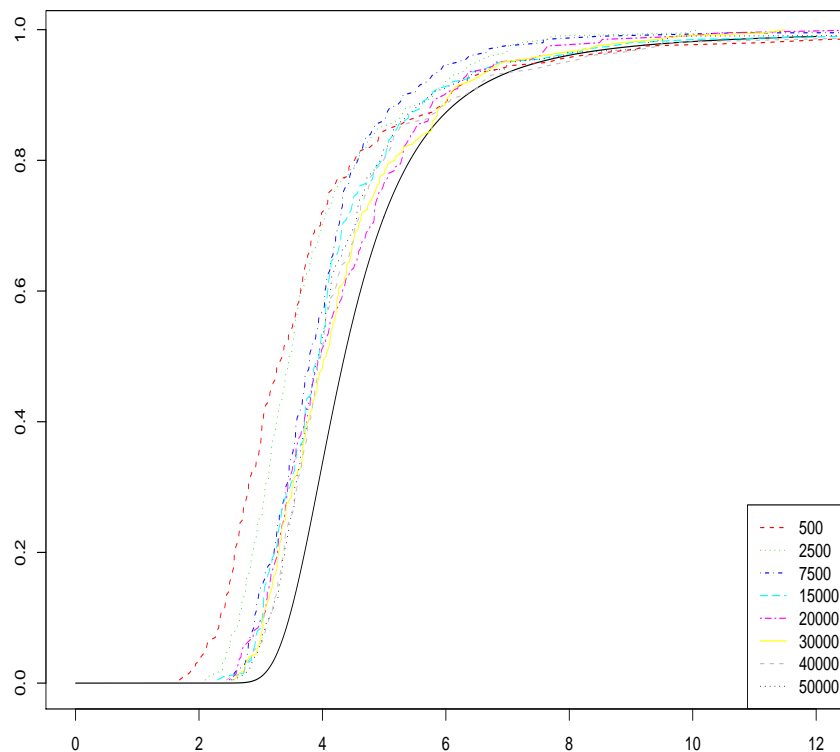


Figure 4.2.5: Empirical CDFs of S_n for $n = 500, 2500, 7500, 15000, 20000, 30000, 40000$, and 50000 (left to right, dotted lines) and the theoretical CDF of S (solid line) with $\alpha = 0.8$

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Chapter 5

Conclusions and Future Work

In this dissertation, a major effort is made to study the convergence rate of Rosiński's series representation for tempered α -stable random variables. We started by conducting simulations to estimate Log-Log regression slope between the Kolmogorov-Smirnov distance versus the number of summands. We ended by deriving the theoretical convergence rate for Rosiński's series as $O(n^{1-1/\alpha})$. The convergence rate calculated from theoretical results are close to the regression results from simulations which are described in Chapter 4.

However, our results are quite conservative. In Bentkus et al. (1996) (see [1]), the LePage series for α -stable random variables have convergence rate $O(n^{1-r/\alpha})$, $\alpha < r \leq \min\{1 + \alpha, 2\}$. Given specific values of α and r , e.g., $\alpha = 0.5$, $r = 2$, the Bentkus's convergence rate is $O(n^{-3})$. Our convergence rate is $O(n^{-1})$ for a general tempered stable distribution. But intuitively, the LePage series for tempered stable random variables should converge faster than the one for stable random variables because of all their moments being finite. Nevertheless, we were not able to prove a stronger result. The consolation is that the class of tempered stable distributions has much more variety than the relatively narrow class of α -stable distributions.

Other contributions of this dissertation include the asymptotic distribution of the MM estimator of parameter α for Smoothly Truncated Lévy Flights. The same derivation could be used to get the asymptotic distribution of the MM estimators of other parameters of tempered stable distributions.

We plan to do it in the future.

Our future work plan also includes finding better estimates of convergence rate for special classes of tempered stable random variables.

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