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Modelling Tail Risk with Tempered Stable Distributions: An Overview

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Abstract

In this study, we investigate the performance of different parametric models with stable and tempered stable distributions for capturing the tail behaviour of log-returns (financial asset returns). First, we define and discuss the properties of stable and tempered stable random variables. We then show how to estimate their parameters and simulate them based on their characteristic functions. Finally, as an illustration, we conduct an empirical analysis to explore the performance of different models representing the distributions of log-returns for the S&P500 and DAX indexes.

Modelling Tail Risk with Tempered Stable Distributions: An Overview*

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Abstract

In this study, we investigate the performance of different parametric models with stable and tempered stable distributions for capturing the tail behaviour of log-returns (financial asset returns). First, we define and discuss the properties of stable and tempered stable random variables. We then show how to estimate their parameters and simulate them based on their characteristic functions. Finally, as an illustration, we conduct an empirical analysis to explore the performance of different models representing the distributions of log-returns for the S&P500 and DAX indexes.

Keywords: Lévy process, stable distribution, tail risk, tempered stable distribution.

JEL classification: C5, G12

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1 Introduction

Presently, it is widely accepted that the returns from financial assets are not normally distributed. A key feature of these returns is the existence of heavy tails on the distributions, i.e. extreme events occur with a much higher frequency compared with that predicted by a normal distribution. Hence, it is necessary to use other probability distributions that reflect this crucial feature to ensure appropriate risk management. Heavy tail random variables play important roles in the modelling of extreme events in finance and risk management but also in other fields, e.g., distributions of the size of web pages, file sizes in a computer system, and when modelling the distribution of loss in the context of insurance. Thus, when modelling a phenomenon with a heavy tail random variable, it is necessary to consider the occurrence of outliers (i.e., highly unexpected events) with extreme impacts accurately, but their prediction is inherently difficult.

Two main approaches are employed for modelling the tail risk: (i) using a parametric distribution for both the body and tail of the data, or (ii) identifying a threshold and then representing only the tail of the distribution.[1] Each approach has some advantages and drawbacks. The first approach uses a parametric distribution, so calculating a risk measure such as the value-at-risk or expected shortfall is straightforward.[2] However, imposing a structure on the data induces some model risk.[3] A key step when implementing the second approach is specifying the threshold and verifying the assumption that the tail follows a generalised Pareto distribution. In this study, we focus on the first approach where we assume that a parametric distribution can model the whole data set.

Several definitions of heavy tail or fat tail distributions have been provided in previous studies, but they are not all in agreement. In some cases, a heavy tail distribu-

[1]This approach has been investigated under the extreme value theory (EVT). Other approaches are used in the context of the EVT, such as block-maxima, peak-over-threshold, and limiting the distribution of the threshold, e.g., the generalised Pareto distribution. The application of the generalised Pareto distribution is canonical in the EVT. More information about the EVT was provided by McNeil et al. (2005)

[2]There are several methods for calculating the value-at-risk and expected shortfall: (1) variance-covariance approach; (2) Monte Carlo simulation, and; (3) historical simulation. Detailed information about these approaches was provided by Tompkins and D'Ecclesia (2006) and Cerqueti et al. (2018).

[3]Using a flexible distribution that captures all of the empirical regularities in the data can overcome this problem. Fallahgoul et al. (2018c) presented an approach for mitigating the model risk and they applied it to an asset allocation problem.

tion was defined as a distribution with a tail that is heavier than the exponential distribution, which is formally given as follows. Let X be a random variable with cumulative distribution function (cdf) F and $\overline{F}(x) = 1 - F(x)$ is its survival function, then $\forall\lambda > 0$, $\lim_{x\to+\infty} \bar{F}(x)e^{\lambda x} = +\infty$. Classic examples of heavy tail distributions are the Pareto, Student's t, Frechet, stable, and tempered stable distributions, as well as the log-normal distribution. Distributions with a probability density function (pdf) that behaves like a power law at infinity are sometimes called fat-tail distributions. A heavy tail distribution might have moments of any order, whereas a fat-tail distribution will have infinite moments at some point.

In this study, we employ a broader definition of the heavy-tail property by considering distributions that exhibit large *skewness* and *kurtosis* compared with the normal distribution.

Two main approaches can be used for identifying the heavy tail property: (i) statistical tests such as the Kolmogorov–Smirnov test, and (ii) graphical methods such as quantile-quantile (QQ)-plots. Both approaches are based on the fact that a heavy tail variable exhibits more extreme values than a normally distributed random variable with the same location and dispersion parameter.[4] When applied to a financial time series, this means that large (positive or negative) price variations are observed more often than would be expected if they were drawn from a normal distribution. Under the assumption of normality, the distribution is characterised entirely by its first two moments, whereas considering the heavy tail property requires knowledge of the higher order moments, particularly the third moment i.e., skewness as a measure of the distribution's asymmetry, and the fourth moment, i.e., kurtosis accounting for the fatness of the tail.

Tempered stable distributions can be selected to have both heavy tail properties, i.e., high kurtosis and an asymmetric property comprising high skewness. They have been used in numerous financial and risk management applications. The variance gamma (see, Carr et al. (2003)) and normal inverse Gaussian (see, Carr et al. (2003)) distributions are special cases of tempered stable distributions. The useful properties of these distribu-

[4]This is an informal definition of the heavy tail property but several others are available, e.g., see McNeil et al. (2005).

tions for applications in financial and risk management have been discussed in previous studies (e.g., Fallahgoul et al. (2016), Fallahgoul et al. (2018b), Fallahgoul et al. (2019), Kim et al. (2009)).

The first step when using a tempered stable distribution is estimating its parameters, which can be conducted in three ways: parametric, non-parametric, or based on simulation. In order to estimate the parameters of a distribution based on standard parametric methods we require either of the following: (1) a closed-form formula for the pdf or cdf, or (2) finite moments of some orders. However, there are some problems when estimating the parameters of stable and tempered stable distributions. First, closed-form formulae do not exist for the pdf and cdf of the stable and tempered stable distribution, and thus deriving the maximum likelihood estimation (MLE) function is not an easy task. Second, for stable distributions at least, the generalized method of moments (GMM), e.g. Hansen (1982), is not generally applicable due to the non-existence of moments at all orders. Some important studies have addressed these problems (e.g., Fama and Roll (1968, 1971), McCulloch (1986) and DuMouchel (1973, 1983)). Recently, an efficient method based on simulation was proposed by Fallahgoul et al. (2019) for estimating the different classes of tempered stable distributions.

In the following, we first review the definitions and some theoretical properties of stable and tempered stable distributions in Section 2. In Section 3, we explain how to simulate a heavy tail random variable and how to compute the pdf and cdf based on the characteristic function. Finally, we present two empirical examples of these computations based on financial time series (daily returns for the DAX and SPX index).

1.1 Literature Review

A tempered stable distribution can be constructed in two ways: (1) modifying the Lévy measure of a stable law with a function in order to temper/tilt both the left and right tails; or (2) changing the physical time of a Brownian motion with a subordinator.

By modifying the Lévy measure of a stable law, we can introduce the tempered stable family from both univariate and multivariate perspectives (Boyarchenko and Levendorskiĭ (2000) and Boyarchenko and Levendorskiĭ (2002)). Formal and elegant definitions of tempered stable distributions and processes were proposed in the seminal study by Rosiński (2007) who employed a completely monotone function to transform the Lévy measure of a stable distribution (various parametric classes were discussed by Terdik and Woyczynski (2006), Bianchi et al. (2010), Bianchi (2015), Rroji and Mercuri (2015), Grabchak (2016)). Two classes of distributions that are broader than the tempered stable class have been proposed, where Rosinski and Singlair (2010) introduced the generalised tempered stable class and Grabchak (2012) proposed the p-tempered stable class.

The subordination approach has been studied from a univariate perspective by Hurst et al. (1997) and Hurst et al. (1999), and from a multivariate perspective by Guillaume (2013), Tassinari and Bianchi (2014), Fallahgoul et al. (2016), Bianchi et al. (2016), Fallahgoul et al. (2018b), Hitaj et al. (2018), and Bianchi and Tassinari (2018). In the present study, we do not discuss the tempered stable distribution in a multivariate context.[5]

Simulating a tempered stable random variable is not a simple task and it has been investigated in many studies (e.g., Bianchi et al. (2017)). A series representation algorithm was proposed by Rosiński (2007) and empirically studied by Bianchi et al. (2010) and Imai and Kawai (2011). An efficient algorithm exists for drawing random samples from stable distributions (see Chambers et al. (1976)), so the problem of generating random numbers from a tempered stable law X can be solved by using a stable law Y with a probability density g that is similar to the probability density f of X. We can generate a value for Y and accept (reject) this value if a given condition is satisfied (not satisfied). More details of this method were provided byRachev et al. (2011). This acceptance–rejection simulation method has been studied widely (see Kawai and Masuda (2011a)), Kawai and Masuda (2011b), Kawai and Masuda (2012), Jelonek (2012), Grabchak (2018), and the references therein). By applying this algorithm, we can sample tempered stable random numbers in an exact/approximate manner if the tail index is less/greater than 1, where the computational cost depends strictly on the parameters. In the case when $\alpha < 1$, a double rejection sampling algorithm that does not depend upon the model parameters was proposed by Devroye (2009). Furthermore, we can use the method proposed by Rosiński

[5]However, it should be noted that all classes of the tempered stable distribution that are defined based on a subordinator can be extended to a multivariate model (e.g., see Bianchi et al. (2016), Fallahgoul et al. (2016), and Fallahgoul et al. (2018b)). In particular, if we replace the physical time of a multivariate Brownian motion with a subordinator, then we obtain a multivariate model.

(2001), where the ratio between the Lévy measures of a tempered stable and a stable law is used to construct an acceptance–rejection algorithm. In this method, the probability of the acceptance event also depends on the parameters needed to simulate the tempered stable distribution.

2 Stable and Tempered Stable Distributions

In the following, we discuss the definitions and some properties of stable and tempered stable random variables. None of these random variables have closed-form formulae for their pdf and cdf, and we are only interested in the representations of their characteristic functions. Furthermore, we discuss their cumulant generating functions (CGFs) in order to calculate their moments.

2.1 Lévy processes and the Lévy Khintchine formula

Stable and tempered stable random variables belong to the class of Lévy processes.[6] The characteristic function of a random variable X, which we denote as $\Phi_X(u;X)$, is the Fourier transform of its distribution. Therefore, the related pdf or cdf can be recovered with only one Fourier inversion.[7] The Lévy Khintchine formula states that characteristic function of any Lévy process, $X = (X_t)_{t\geq 0}$, defined by:

$$\Phi_X(u;X_t) = \mathbb{E}\left[e^{iuX_t}\right],$$

can be written in the following form:

$$\mathbb{E}\left[e^{iuX_t}\right] = \exp\left(t\left(aiu - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{iux} - 1 - iux\mathbf{I}_{|x|<1}\right)\nu(dx)\right)\right), \tag{1}$$

[6]We use a slight abuse of notation where a random variable is a snapshot at a given time for a random process.

[7]Detailed information about this approach was provided by Fallahgoul et al. (2016) and Fallahgoul et al. (2019).

where $a \in \mathbb{R}$, $\sigma \geq 0$, $\mathbf{I}$ is the indicator function, and ν is a Lévy measure of X that satisfies the property:

$$\int_{\mathbb{R}\setminus\{0\}} (1 \wedge x^2)\nu(dx) < \infty.$$

The measure ν describes the frequency of jumps of size x. Further details of Lévy processes were given by Cont and Tankov (2003) and Rachev et al. (2011). In particular, a well known case is when ν vanishes, and we find that X follows a Brownian motion with drift a and volatility σ.

2.2 Stable Distribution

Stable random variables were first introduced by Gnedenko and Kolmogorov (1954) in a study of the sum of random variables. However, the first formal definition of a stable random variable was given by Feller (1971). Stable distributions were also studied by Zolotarev (1986) and applied to finance by Rachev and Mittnik (2000).

There are several ways of defining a stable random variable based on its characteristic function.[8]

First, stable variables can be defined via their Lévy measure.[9] For any $\alpha < 2$, the Lévy measure of a stable process is given by:

$$\nu(dx) = \left(\frac{C_+}{x^{1+\alpha}} 1_{x>0} + \frac{C_-}{x^{1+\alpha}} 1_{x<0} \right) dx, \tag{2}$$

where $1_{x>0}$ is an indicator function. The calculation of the characteristic function of the stable distribution is based on the Lévy–Khintchine formula. After replacing the Lévy measure in (1) with (2), we can recover the characteristic function of S given by:

$$\begin{aligned} \Phi(u; S) &= \mathbb{E}[e^{iuS}] \\ &= \begin{cases} \exp\left(i\mu u - |\sigma u|^\alpha \left(1 - i\beta(\text{sign } u) \tan\left(\frac{\pi\alpha}{2}\right)\right)\right), & \alpha \neq 1 \\ \exp\left(i\mu u - \sigma|u| \left(1 + i\beta\frac{2}{\pi}(\text{sign } u) \ln|u|\right)\right), & \alpha = 1, \end{cases} \end{aligned}$$

[8]It should be noted that some random variables are a special case of a stable random variable such as Lévy and Gamma random variables, and they have closed-form formulae for their pdf and cdf.
[9]The arrival rate of jumps of size $x \in \mathbb{R}\setminus\{0\}$.

where

$$\text{sign}\, u = \begin{cases} 1, & u > 0 \\ 0, & u = 0 \\ -1, & u < 0, \end{cases}$$

$0 < \alpha \leq 2$, $\sigma \geq 0$, $-1 \geq \beta \geq 1$, and $\mu \in \mathbb{R}$. The parameter α is the index of stability and it controls the behaviour of the left and right tails. When α is close to 2, the tail becomes *thin*.[10] When moving down from 2 to 1.5, a stable random variable exhibits a heavier tail. For large values of x, the pdf of a stable variable behaves like a power law (sometimes called a Pareto tail):

$$f(x) \simeq \frac{C_{\pm}^{\alpha,\beta}}{|x|^{1+\alpha}},$$

where $C_{\pm}^{\alpha,\beta}$ are the parameters for large positive or negative values. In empirical applications, the estimated or calibrated value of α is usually between 1.3 and 1.9.[11] β, σ, and μ are the skewness, scale, and location parameters, respectively. When a random variable S follows a stable distribution, we denote it by $S_\alpha(\sigma, \beta, \mu)$.[12]

Special cases of a stable distribution where an analytic expression exists for the pdf are the Cauchy distribution ($\alpha = 1$):

$$f(x) = \frac{C}{x^2 + \pi^2 C^2},$$

and the Lévy distribution ($\alpha = 1/2$):

$$\sqrt{\frac{c}{2\pi}} \frac{e^{-\frac{c}{2x}}}{x^{3/2}}.$$

Detailed information about the characteristic function of a stable distribution was

[10] A Gaussian random variable is a special case of a stable variable when $\alpha = 2$ and $\beta = 0$.

[11] An empirical study by Kim et al. (2011) determined this range of α for stable and tempered stable distributions.

[12] Some studies refer to a stable random variable as $\alpha-$stable. In addition, different parameterisations are possible for a stable random variable. Detailed information regarding the parameterisation of a stable random variable was provided by Nolan (2003) and Rachev et al. (2011).

given by Samorodnitsky and Taqqu (1994).

Historically, the initial definition of a stable random variable is based on the sums of random variables. A random variable X is said to have a stable distribution if for any $n \geq 2$, a positive number C_n and real number D_n exist such that:

$$X_1 + X_2 + \cdots + X_n = C_n X + D_n,$$

where $X_1, X_2, \cdots, X_n$ are independent copies of X.[13] We can show that $C_n = n^{\frac{1}{\alpha}}$, where $\alpha \in (0, 2]$, i.e., a generalised central limit theorem (CLT) holds for the stable variables: a sum of independent and identically distributed variables with paretian tails that behave like $\frac{1}{|x|^{1+\alpha}}$ normalised by $n^{1/\alpha}$ will converge to a stable distribution (we recognise the usual CLT when $\alpha = 2$).

2.3 Variation and Existence of Moments

Lévy processes are said to have finite activity if:

$$\int_{\mathbb{R}\setminus\{0\}} \nu(dx) < \infty,$$

and infinite activity otherwise. Stable processes always have infinite activity, which means that an infinite number of jumps exist within any finite time interval (except in the normal case where there are no jumps at all).

The sample paths for a Lévy process have finite variation when:

$$\int_{\mathbb{R}\setminus\{0\}} (1 \wedge |x|)\nu(dx) < \infty,$$

and infinite variation otherwise.[14] Therefore, by (2), a stable process has finite variation if and only if $\alpha < 1$ (note that the Brownian motion, i.e. $\alpha = 2$, does not have finite variation).

[13] For more information about the connections among different definitions of a stable random variable, see Chapter 1 in the study by Samorodnitsky and Taqqu (1994).

[14] Loosely speaking, a process has finite variation when the *lengths* of its sample paths are locally finite almost surely.

Furthermore, the second moment of a stable process does not exist (except for $\alpha = 2$), and the moment of order p for $p \geq \alpha$ actually do not exist. To overcome this problem, we use a tempering function referred to as $t(x)$ to ensure that the second or higher moments do exist. This new process is then called a tempered stable process. The Lévy measure of the tempered stable process is given by $\nu_{ts}(dx) = t(x)\nu_s(dx)$, where $\nu_s(dx)$ is the Lévy measure of the stable process. The different choices of the tempering function come from the different classes of tempered stable process.[15]

2.4 Tempered Stable Distributions

In the following, we discuss the properties of different tempered stable distributions.[16] The characteristic functions and Lévy measures of all classes of tempered stable random variables are summarised in Appendix A.

2.4.1 Classical Tempered Stable (CTS)

A CTS random variable, which also referred to as a CGMY random variable based on the study by Carr et al. (2003), has a tempering function given by:

$$t(x) = \begin{cases} e^{-\lambda_+ x}, & x > 0 \\ e^{\lambda_- x}, & x < 0 \end{cases},$$

where λ_+ and λ_- are non-negative and the tempering parameters for the right and left tails, respectively.[17] Therefore, the Lévy measure of a CTS random variable is given by the following:

$$\nu(dx) = \left(C\frac{e^{-\lambda_+ x}}{x^{1+\alpha}} 1_{x>0} + C\frac{e^{\lambda_- x}}{x^{1+\alpha}} 1_{x<0} \right) dx.$$

The role of the parameter α is the same as that of a stable random variable, where it controls the behaviour of the left and right tails. The parameter C is the scale parameter

[15]Detailed information about tempered stable processes and their properties was given by Rachev et al. (2011), Fallahgoul et al. (2016), and Fallahgoul et al. (2019), and their references.
[16]Readers may to the study by Rachev et al. (2011) for detailed information about these classes of distributions and processes.
[17]In the study by Carr et al. (2003), they represented the tail parameter by Y instead of α, and the λ_+ and λ_- were given by G and M, respectively.

that controls the kurtosis of the distribution. The parameters λ_+ and λ_- control the rate of decay on the positive (right) and negative (left) tails, respectively. When $\lambda_+ = \lambda_-$, a CTS random variable is a symmetric random variable. We denote a CTS random variable by X_{CTS}.

Tempering the tail of a stable random variable allows the existence of higher moments. The moments of a random variable can be obtained from its CGF. Let $\Phi(u, X)$ be as defined in (2.1), and we let $\psi(u, X_{CTS}) = \log \Phi(u, X_{CTS})$, then a cumulant of order n is given by:

$$C_m(X_{CTS}) = \frac{1}{i^n} \frac{\partial^n \psi(u, X_{CTS})}{\partial u^n}|_{u=0} \quad n = 1, 2, \cdots,$$

where ψ is the CGF. The cumulants for a CTS random variable are equal to:

$$\begin{aligned} c_1(X_{CTS}) &= 0 \\ c_n(X_{CTS}) &= C\Gamma(n-\alpha)\left(\lambda_+^{\alpha-n} + (-1)^n \lambda_-^{\alpha-n}\right), \quad n = 2, 3, \cdots, \end{aligned}$$

where $\Gamma(n-\alpha) = (n-\alpha-1)!$.

There is a one-to-one relation between the CGF and a moment generating function (MGF). The MGF of a random variable X is given by:

$$M(u) = \mathbb{E}\left[e^{uX}\right].$$

The CGF is the logarithm of the MGF, i.e., $\psi(u) = \log(M(u))$. The first cumulant is the mean. The second and third cumulants are the second and third central moments, respectively. However, the higher cumulants are not moments or central moments.

2.4.2 Generalised Tempered Stable

A CTS random variable has the same scale parameter, i.e., C, for the left and right tails. An extension of the CTS random variable can be obtained by allowing more flexibility in the scale parameter. The new random variable is called a generalised CTS (GCTS) random variable. Several well-known tempered stable random variables such as the CGMY (Carr et al. (2003)), KoBoL (Boyarchenko and Levendorskiĭ (2002)), and Lévy flight are special cases of a GCTS random variable. We denote a GCTS random variable by X_{GCTS}.

The cumulants of X_{GCTS} are given by:

$$
\begin{aligned}
&c_1(X_{GCTS}) = 0 \\
&c_n(X_{GCTS}) = C_+\Gamma(n-\alpha_+)\lambda_+^{\alpha_+-n} + (-1)^n C_-\Gamma(n-\alpha_-)\lambda_-^{\alpha_--n}, \quad n = 1, 2, \cdots.
\end{aligned}
$$

The tempering functions of the next two tempered stable random variables are based on some hypergeometric functions.[18] This choice allows the finiteness of the higher order moments but also of some exponential moments, which might be useful for some applications in option pricing.

2.4.3 Modified Tempered Stable (MTS)

A MTS random variable is obtained by taking the Lévy measure of a symmetric stable random variable and multiplying it by the following tempering function:

$$
t(x) = (\lambda|x|)^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda|x|)
$$

on each half of the real axis, where $K_p(x)$ is the modified Bessel function of the second kind (see Appendix B).[19]

The Lévy measure of a MTS random variable is given by:

$$
\nu(dx) = C\left(\frac{\lambda_+^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_+ x)}{x^{\frac{\alpha+1}{2}}} 1_{x>0} + \frac{\lambda_-^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_- x)}{x^{\frac{\alpha+1}{2}}} 1_{x<0}\right) dx,
$$

where $C, \lambda_+, \lambda_- > 0$, and $\alpha \in (0,1) \cup (1,2)$. We denote a MTS random variable via X_{MTS}.[20]

[18] Detailed information about hypergeometric functions was given by Andrews (1992).
[19] Detailed information about the Bessel function was given by Andrews (1992). This random variable was introduced by Kim et al. (2008).
[20] Modified tempered stable and rapidly decreasing tempered stable distributions are parametric examples in the tempered infinitely divisible class introduced by Bianchi et al. (2010).

The cumulants of X_{MTS} are equal to:

$$c_n(X_{MTS}) = 2^{n-\frac{\alpha+3}{2}} C\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)\left(\lambda_+^{\alpha-n} + (-1)^n \lambda_-^{\alpha-n}\right)$$

for $n = 2, 3, 4, \cdots$.

Two important points should be noted concerning the MTS random variable. First, the use of the modified Bessel function of the second kind, i.e., $K_p(x)$, as a tempering function is related to the characteristic function of the MTS random variable. $K_p(x)$ allows a closed-form formula to exist for the characteristic function. Second, due to the complicated form of the tempering function, the asymptotic behaviour of the Lévy measure is not obvious for a MTS random variable. The asymptotic behaviour of the Lévy measure for the MTS random variable was explored by Kim et al. (2008). For $\alpha \in (0,2)\ \{1\}$, the Lévy measures for the stable and MTS, CTS, and GCTS random variables have the same asymptotic behaviour in the zero neighbourhood, but the tails of the MTS random variable are thinner (heavier) than those of the stable random variables (CTS and/or GCTS).

2.4.4 Rapidly Decreasing Tempered Stable (RDTS)

RDTS random variables were introduced by Kim et al. (2010b). The tempering function of the RDTS is given by:

$$t(x) = e^{-\frac{\lambda_+^2 x^2}{2}} 1_{x>0} + e^{-\frac{\lambda_-^2 x^2}{2}} 1_{|x|<0},$$

where $\lambda_+, \lambda_- > 0$, and $\alpha \in (0,1) \cup (1,2)$. Therefore, the Lévy measure of the RDTS is given by:

$$\nu(dx) = \left(C\frac{e^{-\frac{\lambda_+^2 x^2}{2}}}{x^{\alpha+1}} 1_{x>0} + C\frac{e^{-\frac{\lambda_-^2 x^2}{2}}}{x^{\alpha+1}} 1_{|x|<0} \right) dx,$$

where $C > 0$. We denote a RDTS random variable via X_{RDTS}.

2.5 Tempering with a Subordinator

The next two tempered stable random variables are obtained by replacing the physical time of the Brownian motion with a subordinator.[21] Constructing a tempered stable random variable in this manner has several advantages. First, we can control the rate of decay for the tails by choosing an appropriate subordinator. Secondly, the extension from a univariate tempered stable random variable to a multivariate one is straightforward (see Fallahgoul et al. (2016) and Fallahgoul et al. (2018b)).

2.5.1 Normal Tempered Stable (NTS)

A NTS random variable is obtained by replacing the physical time, i.e., t, by the CTS subordinator. The CTS subordinator is a Lévy process and its Lévy measure is the positive part of the Lévy measure of a CTS random variable. If T_t is the CTS subordinator, then the Lévy measure of T_t is given by the following:

$$\nu(dx) = \frac{Ce^{-\theta x}}{x^{\frac{\alpha}{2}+1}} 1_{x>0} dx,$$

where $C, \theta > 0$, and $0 < \alpha < 2$.[22] The unconditional expectation of T_t should be equal to t, i.e. $\mathbb{E}[T_t] = t$. To achieve this, we set $C = \frac{1}{\Gamma(1-\frac{\alpha}{2})\theta^{\frac{\alpha}{2}-1}}$, and we have:

$$\mathbb{E}[T_t] = tC\Gamma\left(1 - \frac{\alpha}{2}\right)\theta^{\frac{\alpha}{2}-1} = t.$$

The characteristic function of the subordinator T_t is given by the following:

$$\Phi(u; T_t) = exp\left(\Big(- \frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha}((\theta - iu)^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}})\Big)t\right).$$

Let $\mu, \beta \in \mathbb{R}$, $\sigma > 0$, B_t be Brownian motion, and T_t be the CTS subordinator. Then,

[21] Detailed information about a subordinator was given by Cont and Tankov (2003), Fallahgoul et al. (2016), and Fallahgoul et al. (2019).

[22] Previously, we have only discussed the unconditional tempered random variables; in particular, we have only assumed that the physical time is one.

the NTS process is defined by the following:

$$X_t = \mu t + \beta(T_t - t) + \sigma B_{T_t}.$$

Given that T_t and B_t are independent, the characteristic function of X_t is given by the following:

$$\Phi(u; X_t) = exp\left(iu(\mu - \beta)t - \frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha} \left((\theta - i\beta u + \frac{\sigma^2 u^2}{2})^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}} \right) t \right).$$

By using the CGF, the first two moments of X_t are equal to the following:

$$\begin{aligned} \mathbb{E}[X_t] &= \mu t \\ \mathbb{V}ar[X_t] &= \sigma^2 t + \beta^2 \left(\frac{2-\alpha}{2\theta} \right) t. \end{aligned}$$

2.5.2 Exponential Tilting Stable

An ETS process is obtained by replacing the physical time of a Brownian motion with an exponential tilting (ET) subordinator, which has a Lévy measure given by equation (4) (see Fallahgoul et al. (2018b)). The method that we employ for constructing the ETS process is similar to that proposed by Barndorff-Nielsen and Shephard (2001) who used the CTS subordinator to construct the NTS process (e.g., see Barndorff-Nielsen and Shephard (2001), Kim et al. (2008)). The ETS process, i.e., X_t, is given by the following:

$$X_t = \mu t + \beta \left(s_t - t \right) + \gamma B_{s_t}, \tag{3}$$

where $\mu, \beta \in \mathbb{R}$, $\gamma \in \mathbb{R}^+$, B_{s_t} is the Brownian motion, and s_t is the ET subordinator, which is independent of B_t. By rearranging (3), we can use the following properties of the arithmetic Brownian motion:

$$\begin{aligned} X_t &= (\mu - \beta)t + \beta s_t + B_{s_t} \\ &= (\mu - \beta)t + Y_t, \end{aligned}$$

where $(Y_t)_{t\geq 0}$ is a subordinated arithmetic Brownian motion with drift β and volatility γ. It should be noted that the characteristic exponent for the arithmetic Brownian motion is $\varphi_{B_t}(u) = i\beta u - \frac{1}{2}\sigma^2 u^2$.

The Lévy measure of an ET subordinator is given by the following:

$$\nu(dx) = \left(c \frac{e^{-\frac{\lambda x^2}{2}}}{x^{\alpha+1}} \mathbf{1}_{x>0} \right) dx, \tag{4}$$

where $\alpha \in (0,1)$ and $\lambda > 0$.[23] The characteristic function of the ET subordinator s_t is equal to the following:

$$\Phi(u; s_t) = \exp\left[\frac{\lambda}{2\Gamma\left(\frac{1-\alpha}{2}\right)} G\left(\alpha, \lambda; \frac{iu}{\lambda}\right) t \right].$$

The function G is based on the confluent hyper-geometric function.[24] By using the characteristic function of s_t, we can derive all of the cumulants for the ET subordinator, which are given by the following:

$$c_n(s_t) = \frac{2^{n-1}\Gamma\left(\frac{n-\alpha}{2}\right)}{\lambda^{n-1}\Gamma\left(\frac{1-\alpha}{2}\right)} t,$$

where $n = 1, 2, 3, \cdots$.

Let $(B_t)_{t\geq 0}$ be the Brownian motion on $\mathbb{R}$ with the characteristic exponent $i\beta u - \frac{1}{2}\sigma^2 u^2$, and let s_t be the ET subordinator with the parameters (α, λ). Then, the characteristic function for the ETS process X_t, i.e., equation (3), is as follows:

$$\Phi(u; X_t) = \exp\left[\frac{\lambda}{2\Gamma\left(\frac{1-\alpha}{2}\right)} G\left(\alpha, \lambda; \frac{i\beta u - \frac{1}{2}\sigma^2 u^2}{\lambda}\right) t + iu(\mu - \beta)t \right].$$

The mean and variance for the stochastic process X_t are equal to the following:

$$\mathbb{E}[X_t] = \mu t$$

[23]This Lévy measure is the positive side of the Lévy measure of a RDTS random variable.

[24]Detailed information about G can be found in the Online Appendix given by Fallahgoul et al. (2018b).

$$Var[X_t] = \gamma^2 t + \beta^2 \frac{2}{\lambda \Gamma\left(\frac{1-\alpha}{2}\right)} \Gamma\left(\frac{2-\alpha}{2}\right) t.$$

2.6 Non-linear dependency across assets

The dependency across assets is an important factor in financial analysis. Based on the dependency between returns, which can be linear or non-linear, we can decide the type of model that can be used for modelling. If the dependency is linear, the correlation coefficient precisely describes the movements of return series, but it is not always a good measure of dependency. Modelling the dependency among returns is more important during a financial crisis because of the excessive correlations observed during these periods.

Based on the linear dependency property, a multivariate normal distribution is a good candidate for modelling multivariate financial log-returns, where the major assumption is that the dependency is linear and that it can be explained by the variance–covariance matrix. However, many studies have rejected this claim (e.g., see McNeil et al. (2005), Rachev et al. (2011)). Thus, let us denote the returns of two assets by X and Y, and there is a dependency between them. If the dependency is linear, then $E[X|Y]$ can be expressed as a linear function of Y. However, if the conditional expectation is not a linear function, then a linear dependency measure such as *Pearson's correlation coefficient* ρ is not an adequate tool for capturing the dependency structure between these two random variables. To overcome this problem, we may use the copula technique.[25] It should be noted that we can also show the existence of the nonlinear dependency by using exceedance correlations methods.[26]

The tail dependency coefficient is a measure of non-linear dependency and it was designed to capture the dependency in tails. Tempered stable random variables have been used as a flexible class of models for calculating this coefficient.[27]

[25] See McNeil et al. (2005) and Fallahgoul et al. (2016).
[26] See Longin and Solnik (2001).
[27] See McNeil et al. (2005), Fallahgoul et al. (2016), and the references therein.

3 Evaluating the pdf and cdf Based on a Characteristic Function

3.1 Theoretical Setting

Let X be a random variable, then its characteristic function, i.e., $E[e^{iuX}]$, is the Fourier transform of its pdf. Consequently, we can recover the pdf by obtaining the inverse of the related Fourier transform. This approach was introduced by DuMouchel (1975) for calculating the pdf of a stable distribution.[28] Subsequently, this approach was applied to some classes of the tempered stable distribution by Kim et al. (2009), Fallahgoul et al. (2016), and Fallahgoul et al. (2018b).

3.1.1 PDF

Let X be a random variable for a class of the tempered stable distribution, then its characteristic function is given by:

$$\phi_X(u) = E[e^{iuX}] = \int_{-\infty}^{\infty} e^{iuX} f_X(X) dX,$$

where $f_X(X)$ is the pdf for the random variable X. Consequently, the pdf for the random variable X is given by:

$$f_X(X) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuX} \phi_X(u) dX, \tag{5}$$

where $\phi_X(u)$ is the characteristic function. Therefore, by calculating equation (5), we can obtain the pdf for the random variable X. The characteristic functions of the tempered stable distributions are complicated and an analytic formula does not exist for equation (5). Numerical approaches have been suggested based on numerical integration and the fast Fourier transform (FFT).[29]

The first step in the numerical approximation of equation (5) is the discrete Fourier transform (DFT), which involves transforming a vector $Y = (y_1, y_2, \cdots, y_n)$ into a vector

[28] This approach was also studied by Rachev and Mittnik (2000), Stoyanov and Racheva-Jotova (2004), and Scherer et al. (2012).

[29] For example, see Bailey and Swarztrauber (1994).

$X = (x_1, x_2, \cdots, x_n)$ using the DFT:

$$x_j = \sum_{k=1}^{n} y_i e^{\frac{2\pi(j-1)(k+1)}{n}}, \quad j = 1, 2, \cdots, n. \tag{6}$$

The computational cost of equation (6) is high and the FFT is a popular method for computing it more efficiently.

Let $a \in \mathbb{R}^+, q \in \mathbb{N}^+$, and for $j, k \in \{1, 2, \cdots, n = 2^q\}$, we define:

$$u_k = -a + \frac{2a}{n}(k-1), \quad u_k^* = \frac{u_{k+1} + u_k}{2}, \quad x_j = -\frac{n\pi}{2a} + \frac{\pi}{a}(j-1), \quad C_j = \frac{a}{n\pi}(-1)^{j-1} e^{i\frac{\pi(j-1)}{n}}.$$

An approximation of equation (5) at x_j is given by:

$$\begin{aligned} f_X(x_j) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux_j}\phi_X(u)du \\ &\sim \frac{1}{2\pi}\int_{-a}^{a} e^{-iux_j}\phi_X(u)du \\ &\sim C_j \sum_{i=1}^{n} (-1)^{k-1}\phi_X(u_k^*) e^{-i\frac{2\pi(j-1)(k-1)}{n}}, \end{aligned}$$

where the parameters a and $-a$ are the limits of integration for the Fourier transform.[30]

3.1.2 CDF

Evaluating the cdf for tempered stable distributions is similar to calculating the pdf, but there are some differences. The differences between computing the cdf and pdf are related to the form of the characteristic function. In this study, we employ the approach proposed Kim et al. (2010a) who showed that the cdf of a tempered stable distribution can be given by the following:

$$F_X(x) = \frac{e^{x\rho}}{\pi}\Re\left(\int_0^{\infty} e^{-iux}\frac{\phi_X(u+i\rho)}{c(\rho) - iu}du\right), \qquad x \in \mathbb{R}, \tag{7}$$

[30]More details regarding the accuracy of this approach for the stable distribution were given by Menn and Rachev (2006).

where $\Re$ and $\Im$ are the real and imaginary parts of a complex number, respectively, and for all Z, $|\phi_X(Z)| < \infty$ and $\Im(Z) = \rho > 0$.[31] Let $g_X(u) = \frac{\phi_X(u + i\rho)}{\rho - iu}$. In order to apply the DFT and FFT, let $a \in \mathbb{R}^+, q \in \mathbb{N}^+$ and for $j, k \in \{1, 2, \cdots, n = 2^q\}$, we define:

$$u_k = \frac{2a}{n}(k-1), \quad u_k^* = \frac{u_k + u_{k+1}}{2} = \frac{a}{n}(2k-1), \quad \text{and} \quad x_j = -\frac{n\pi}{2a} + \frac{\pi}{a}(j-1),$$

and thus:

$$\begin{aligned}\int_0^\infty e^{-ix_j u} g_X(u) du \quad &\sim \quad \int_0^a e^{-ix_j u} g_X(u) du, \\ &\sim \quad \sum_{k=1}^n e^{-ix_j u_k^*} g_X(u_k^*) \frac{2a}{n}.\end{aligned}$$

By a simple computation, we can show that for each x_j:

$$F_X(x_j) \quad \sim \quad \frac{e^{x_j \rho}}{\pi} \Re \left\{ \frac{2a}{n} e^{-i\frac{\pi}{n}(j-1)} \sum_{k=1}^n e^{i\pi\frac{(2k-1)}{2}} g_X(u_k^*) e^{-i\frac{2\pi}{n}(k-1)(j-1)} \right\}.$$

It should be noted that the accuracy of this approximation depends on the numerical integration method employed.[32]

3.1.3 Generating a Tempered Stable Random Variable

Generating a random variable is straightforward after calculating the cdf for a random variable. Let X be a random variable and F is its cdf, and let U be a uniformly distributed random variable on $[0, 1]$, then $F^{-1}(U)$ returns a random variable with the same rule as X. Consequently, in order to generate a tempered stable random variable by using the inversion of the cdf, we need to generate a vector of uniform independent random variables and calculate the inverse of the cdf for these values. The inverse of the cdf can be computed by applying any interpolation method. Therefore, generating a tempered stable random variable can be achieved according to the following steps:

- Given a set of y_i and x_i, compute the cdf at points x_i using the characteristic function

[31] For detailed information about the derivation of (7), see Proposition 1 given by Kim et al. (2010a).

[32] The accuracy and speed of this approximation can be increased by using different numerical integration methods (see Menn and Rachev (2006), Kim et al. (2010a), and Fallahgoul et al. (2018b)).

at points y_i;

- Generate a vector of uniform distributions;
- Compute the inverse of the cdf via interpolation at these points.

4 Empirical Analysis

We empirically investigated the performance of different types of distributions at capturing the empirical regularities of equity returns. We employed 18 years of daily prices for two major equity market indexes: (1) S&P500 and (2) DAX. Figure 1 shows the daily price processes for the S&P500 and DAX indexes.

4.1 GJR-GARCH Filtered Returns

Studies have shown that financial asset returns share common features such as a heavy tail, stochastic volatility, nonlinear dependency, leverage effect, and skewness. The leverage effect is the correlation between the returns and changes in the volatilities, which is likely to be negative for equities. However, the stochastic volatility can also be referred to as volatility clustering, where the standard deviation of the returns (or any other measure of their dispersion) follows a time evolving process.

Several methods can be employed to show that a historical time series of returns has a volatility clustering property: (i) visual inspection of the *autocorrelation function* plot; and (ii) statistical tests such as the *Ljung–Box test* and *Engle Ljung–Box test*.[33] The stochastic volatility and leverage effect lead to a time varying conditional distribution. We need to fit a suitable distribution to capture all of these "empirical regularities". Many studies have addressed this problem from discrete and continuous time viewpoints.[34] First, we filtered the data with a discrete-time model such as the GARCH process (see Bollerslev (1986) and Glosten et al. (1993)) to remove as much of the path-dependent effects as possible. In particular, we used the GJR-GARCH process introduced by Glosten et al. (1993).

Let r_t be a time series of an asset returns. Then, $r_t = \mu + \epsilon_t$, where $\mu \in \mathbb{R}$ is the mean of returns and ϵ_t is a zero-mean white noise. Empirically, the value of μ is very small (almost

[33]See McNeil et al. (2005) and Rachev et al. (2011).

[34]Detailed information can be found in the studies by Bates (2012), Fallahgoul et al. (2016), Fallahgoul et al. (2019), Fallahgoul et al. (2018a), and the references therein.

zero) for low/high frequency data, i.e., such as daily data. We say that $r_t - \mu$ is a GJR-GARCH(1,1) process if $\epsilon_t = r_t - \mu = \sigma_t z_t$, where z_t is a standard Gaussian distribution and σ_t follows:

$$\sigma_t^2 = \omega + \Big(\alpha + \gamma I_{t-1}\Big)\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2,$$

where:

$$I_{t-1} = \begin{cases} 0, & r_{t-1} \geq \mu \\ 1, & r_{t-1} < \mu. \end{cases}$$

To ensure that σ_t remains positive, we restrict it to non-negative values of $\omega, \alpha, \gamma, \beta$. We note that the usual GARCH process is a special case of GJR-GARCH when $\gamma = 0$ (see Bollerslev (1986)).

4.2 Removing Stochastic Volatility from Returns

Figure 2 shows the daily log-return processes for the S&P500 and DAX indexes. A quick visual inspection shows that stochastic volatility and volatility clustering are very likely to occur. It is known that the (unconditional) heavy tails generated by (conditional) volatility clustering can be mistakenly interpreted as evidence of fat tailed distributions. To prevent conditional volatility (and possible mean reversion), we filtered each return series with a GJR-GARCH model so the remaining tail dependencies were not conditional. To eliminate the volatility clustering feature, we fitted a GJR-GARCH model. The residuals of the GJR-GARCH process are independent and identically distributed, so we could use the tempered stable distributions as the unconditional distributions of these residuals.

Figure 3 shows the filtered daily log-returns for the S&P500 and DAX indexes. A visual comparison of Figure 2 and Figure 3 indicates that the filtered data did not exhibit volatility clustering. Thus, they could be treated as independent and identically distributed random variables.

4.3 Empirical Evidence

We used the MLE method to estimate the parameters of the stable and tempered stable distributions. First, we recovered the related pdfs by using the FFT, as explained in Section 3. Second, the log-likelihood function was constructed using the recovered pdfs based on their characteristic functions. Finally, we obtained the parameters of the model of interest by maximising the log-likelihood function over the unknown parameters.

We computed the skewness and kurtosis of both the raw and filtered returns. If a data set follows a normal distribution, it should have a zero skewness parameter due to the symmetric property of the normal distribution. The skewness values for the raw (filtered) daily returns from the S&P500 and DAX were $-0.2082(-0.4847)$ and $-0.0531(-0.3107)$, respectively, thereby indicating negative skewness and strongly suggesting a lack of normality.[35]he results obtained with the Kolmogorov–Smirnov test are shown in Table 1. In addition, the high kurtosis of a return series usually indicates that it has a heavy tail property.[36] Excessive kurtosis is defined as a kurtosis value above 3. A kurtosis value significantly higher than 3 is strong evidence against normality. The kurtosis values for the raw (filtered) daily returns from the S&P500 and DAX were $11.6076(4.7335)$ and $7.3591(3.9268)$, respectively, which shows that the kurtosis values for both the raw and filtered data indexes were higher than 3 (normality).

Statistical tests based on the skewness and kurtosis can be employed to test the assumption of normality, including the Jarque–Bera and Kolmogorov–Smirnov tests, where the latter is based on a minimum distance estimation by comparing a sample with a given probability distribution. The Kolmogorov–Smirnov test makes no assumption about the data distribution, so it is more flexible and easy to implement.

The estimated results (using MLE) are shown in Table 1 for the filtered SPY and DAX data. Panel A (B) in Table 1 shows the estimated results for the filtered SPY (DAX) data. Each cell in columns 2–5 is divided into two. The upper sub-cell is the estimated parameter and the lower sub-cell shows the estimated standard error. In column 6, the upper sub-cell is the Kolmogorov–Smirnov test statistic and the lower sub-cell is its associated p-value.

We can make three main conclusions based on Table 1. First, the small values for the

[35]T

[36]Most of the variance is predicted by the assumption of normality, where this is a strong but not perfect signal.

estimated standard errors show that the parameters were generally estimated precisely. Second, the null hypothesis of a correct distributional specification cannot be rejected at a significance level of 5% for the stable and tempered stable distributions. Moreover, the p-values obtained from the Kolmogorov–Smirnov test for the ETS were generally larger than those for the other tempered stable distributions, thereby suggesting that the ETS distribution provided a better fit. Third, we obtained strong evidence against the assumption of normality for the distribution. In particular, the large Kolmogorov–Smirnov statistics and small p-values for the normal distribution did not support the assumption of normality, which is the null hypothesis for the Kolmogorov–Smirnov test.

To implement the Kolmogorov–Smirnov test given the assumption of normality, we estimated the parameters μ and σ based on the observations, which led to some inaccuracy. The modified version of the Kolmogorov–Smirnov test called the Lilliefors test employs the same procedure as the Kolmogorov–Smirnov test but a correction is applied to obtain a more accurate approximation of the distribution of the test statistics.[37] The results obtained using the Lilliefors test were similar to those produced with the Kolmogorov–Smirnov test, as shown in Table 1, which is why the Lilliefors test results are not presented.

Finally, we conducted graphical tests. Several graphical methods can be used to detect the heavy tail property in data. These graphical methods do not allow precise statistical inference but they can show the nature of deviations in the data if any exist, and they can be used as a prototype model.

The QQ-plot is a widely used graphical tool for identifying the heavy tail property. A QQ-plot can be used to determine whether sample data are similar to a specific distribution or whether two data sets have similar distributions. The identification of the heavy tail property using QQ-plot involves comparing the sample data with some known heavy tail distribution, such as the Student's t, stable, or tempered stable distributions.

Figure 4 shows the QQ-plot for the S&P500 and DAX returns versus the fitted normal distribution. The x-axis shows the quantiles of the fitted normal distribution and the y-axis represents the empirical quantiles for the samples. If the quantiles of the sample

[37] See Lilliefors (1967).

data and fitted distribution follow the same probability rule, then the blue points should map onto the straight line. Clearly, the data above and below the observations deviate from normality. Our visualisation of data showed that the behaviours of the tails of both samples were totally different. We obtained better results by using a heavy tail model such as the Student's t distribution instead of the normal distribution, and the QQ-plot indicated the greater consistency of the Student's t distribution with the returns, as shown in the bottom panel in Figure 4.[38]

4.4 Selecting an Appropriate Tempered Stable Distribution

As discussed in Section 4.3, statistical tests such as the Kolmogorov–Smirnov and Lilliefors test can be employed to compare the fitting performance.[39] All of the statistical tests supported the applicability of tempered stable distributions. There was strong evidence in terms of the large test statistics and low p-values (almost zero) against the assumption of normality.

The problem of selecting one tempered stable model can be treated as a special case of the general problem. In fact, when several approaches can be used for modelling the sample data, the main problem is identifying the best model, i.e., is there a significant statistical difference among them? In econometrics, several tests have been introduced to address this question.

Figure 5 shows the detailed procedure employed for finding the best model/approach for fitting to the sample data when several candidates are available. The first step involves using goodness-of-fit tests such as the Kolmogorov–Smirnov, AD, Lilliefors, and χ^2 tests. However, we may not be able to reject any of the competing models with certainty. In particular, the differences in the p-values might not be significant and it is difficult to select one model compared with another.

After failing to select the best model in the first step, we can check the models/approaches that are nested or non-nested. If they are nested, the likelihood ratio test can be applied

[38] The same results were obtained using filtered data and they are available upon request. It should be noted that other graphical tests such as sequential moments, box plot, and histogram methods obtained the same results, where the former method is based on the extreme value theory and it can be used as a suitable alternative to QQ-plots (see McNeil et al. (2005)).

[39] Detailed information about these tests as well as the performance of stable and tempered stable distributions was given by Fallahgoul et al. (2016), Fallahgoul et al. (2019), and Fallahgoul et al. (2018b).

and a model/approach with the highest likelihood ratio will be selected. In this case, the best model can be a complex or simple model/approach. However, for the non-nested case, we can use Akaike?s information criterion or the Bayesian information criterion, where the model/approach with the smallest Akaike?s information criterion or Bayesian information criterion will be selected.

5 Conclusion

According to the statistical findings presented in Section 4.3, heavy tail models are crucial for capturing the empirical regularities of return distributions, particularly for effective assessments of the tail risk. A stable distribution plays an important role in modelling the tail risk as a heavy tail distribution, but some critical problems remain. First, closed-form formulae do not exist for the pdf and cdf, so the implementation of a MLE is difficult. Second, a stable distribution has an infinite $q-th$ moment for all $q \geq \alpha$, where $\alpha < 2$ controls the level of decay for the tails, which is why the tempered stable distribution was introduced. We provided details of the different classes of tempered stable distributions in Section 2. A tempered stable random variable can be constructed by multiplying the Lévy measure of a symmetric stable random variable by a tempering function or by a subordinator. Tempered stable distributions have a heavy tail property and their tails are thinner than those of the stable distributions but heavier than those of the normal distribution. However, the tempered stable distributions still have no closed-form formulae for their pdf and cdf, but their characteristic functions are available in closed-form. We can estimate the parameters of interest using the method described in Section 3. The results of our empirical study demonstrate the superior performance of the tempered stable distributions.

A Lévy Measure and Characteristic Function

In this section, we describe the Lévy measure and characteristic function for the stable and different classes of the tempered stable processes.

- The Lévy measure and characteristic function of a stable random variable X_S are given respectively by

$$\nu(dx) = \left(\frac{C_+}{x^{1+\alpha}} 1_{x>0} + \frac{C_-}{x^{1+\alpha}} 1_{x<0} \right) dx.$$

$$\begin{aligned}
\Phi(u; S) &= \mathbb{E}[e^{iuX_S}] \\
&= \begin{cases} \exp\left(i\mu u - |\sigma u|^\alpha \left(1 - i\beta(\operatorname{sign} u) \tan\left(\frac{\pi\alpha}{2}\right)\right)\right), & \alpha \neq 1 \\ \exp\left(i\mu u - \sigma|u| \left(1 + i\beta\frac{2}{\pi}(\operatorname{sign} u) \ln|u|\right)\right), & \alpha = 1 \end{cases}
\end{aligned}$$

where

$$\operatorname{sign} u = \begin{cases} 1, & u > 0 \\ 0, & u = 0 \\ -1, & u < 0, \end{cases}$$

$0 < \alpha \leq 2, \sigma \geq 0, -1 \geq \beta \geq 1$, and $\mu \in \mathbb{R}$.

- The Lévy measure and characteristic function of a Classical Tempered Stable (CTS) random variable X_{CTS} are given respectively by

$$\nu(dx) = \left(C\frac{e^{-\lambda_+ x}}{x^{1+\alpha}} 1_{x>0} + C\frac{e^{-\lambda_- x}}{x^{1+\alpha}} 1_{x<0} \right) dx.$$

$$\begin{aligned}
\Phi(u; X_{CTS}) &= \mathbb{E}[e^{iuX_{CTS}}] \\
&= \int_{\mathbb{R}} e^{iux} f_X(x) dx
\end{aligned}$$

$$= exp\left(-iuC\Gamma(1-\alpha)(\lambda_+^{\alpha-1}-\lambda_-^{\alpha-1}) \right.$$
$$\left. + C\Gamma(-\alpha)\left((\lambda_+ - iu)^\alpha - \lambda_+^\alpha + (\lambda_- + iu)^\alpha - \lambda_-^\alpha\right)\right).$$

- The Lévy measure and characteristic function of a Generalised Tempered Stable (GCTS) random variable are given respectively by

$$\nu(dx) = \left(C_+\frac{e^{-\lambda_+ x}}{x^{1+\alpha}}1_{x>0} + C_-\frac{e^{-\lambda_- x}}{x^{1+\alpha}}1_{x<0}\right)dx.$$

$$\begin{aligned}
\Phi(u;X_{GCTS}) &= \mathbb{E}[e^{iuX_{GCTS}}] \\
&= \int_{\mathbb{R}} e^{iux} f_X(x)dx \\
&= exp\left(-iu\Gamma(1-\alpha_+)(C_+\lambda_+^{\alpha_+-1} - C_-\lambda_-^{\alpha_--1})\right. \\
&\quad + C_+\Gamma(-\alpha_+)\left((\lambda_+ - iu)^{\alpha_+} - \lambda_+^{\alpha_+})\right) \\
&\quad \left. + C_-\Gamma(-\alpha_-)\left((\lambda_- + iu)^{\alpha_-} - \lambda_-^{\alpha_-})\right)\right.
\end{aligned}$$

where $\alpha_+, \alpha_- \in (0,1)\cup(1,2)$, C_+, C_-, λ_+, and, $\lambda_- > 0$.

- The Lévy measure and characteristic function of a Modified Tempered Stable (MTS)s random variable are given respectively by

$$\nu(dx) = C\left(\frac{\lambda_+^{\frac{\alpha+1}{2}}K_{\frac{\alpha+1}{2}}(\lambda_+ x)}{x^{\frac{\alpha+1}{2}}}1_{x>0} + \frac{\lambda_-^{\frac{\alpha+1}{2}}K_{\frac{\alpha+1}{2}}(\lambda_- x)}{x^{\frac{\alpha+1}{2}}}1_{x<0}\right)dx.$$

$$\begin{aligned}
\Phi(u;X_{MTS}) &= \mathbb{E}[e^{iuX_{MTS}}] \\
&= \int_{\mathbb{R}} e^{iux} f_X(x)dx \\
&= exp\Big(G_R(u;\alpha,C,\lambda_+,\lambda_-) + G_I(u;\alpha,C,\lambda_+,\lambda_-)\Big)
\end{aligned}$$

where for $u \in \mathbb{R}$

$$G_R(u;\alpha, C, \lambda_+, \lambda_-) = \sqrt{\pi}2^{-\frac{\alpha}{2}-\frac{3}{2}}C\Gamma(-\frac{\alpha}{2})\left((\lambda_+^2+u^2)^{\frac{\alpha}{2}} - \lambda_+^\alpha + (\lambda_-^2+u^2)^{\frac{\alpha}{2}} - \alpha_-^\alpha\right)$$

and

$$G_I(u;\alpha, C, \lambda_+, \lambda_-) = \frac{iuC\Gamma(\frac{1-\alpha}{2})}{2^{\frac{\alpha+1}{2}}}\left(\lambda_+^{\alpha-1}F\left(1, \frac{1-\alpha}{2}; \frac{3}{2}; -\frac{u^2}{\lambda_+^2}\right) - \lambda_-^{\alpha-1}F\left(1, \frac{1-\alpha}{2}; \frac{3}{2}; -\frac{u^2}{\lambda_-^2}\right)\right)$$

where F is the hypergeometric function.[40]

- The Lévy measure and characteristic function of a Rapidly Decreasing Tempered Stable (RDTS random) variable are given respectively by

$$\nu(dx) = \left(C\frac{e^{-\frac{\lambda_+^2x^2}{2}}}{x^{\alpha+1}}1_{x>0} + C\frac{e^{-\frac{\lambda_-^2x^2}{2}}}{x^{\alpha+1}}1_{|x|<0}\right)dx.$$

$$\begin{aligned}
\Phi(u; X_{RDTS}) &= \mathbb{E}[e^{iuX_{RDTS}}] \\
&= \int_{\mathbb{R}} e^{iux} f_X(x)dx \\
&= exp\Big(C(G(iu;\alpha,\lambda_+) + G(-iu;\alpha,\lambda_-))\Big)
\end{aligned}$$

where

$$\begin{aligned}
G(x;\alpha,\lambda) = 2^{-\frac{\alpha}{2}-1}\lambda^\alpha\Gamma\left(-\frac{\alpha}{2}\right)\left(M\left(-\frac{\alpha}{2}, \frac{1}{2}; \frac{x^2}{2\lambda^2}\right) - 1\right) \\
+ 2^{-\frac{\alpha}{2}-\frac{1}{2}}\lambda^{\alpha-1}\Gamma\left(\frac{1-\alpha}{2}\right)\left(M\left(\frac{1-\alpha}{2}, \frac{3}{2}; \frac{x^2}{2\lambda^2}\right) - 1\right)
\end{aligned}$$

and M is the confluent hypergeometric function.

- The Lévy measure and characteristic function of a Normal tempered stable (NTS

[40]Detailed information about this hypergeometric function can be found in Andrews (1992).

random) variable are given respectively by

$$\nu(dx) = \left(\frac{\sqrt{2}\theta^{1-\frac{\alpha}{2}}(\beta^2 + 2\gamma\lambda)^{\frac{\alpha+1}{4}}}{\sqrt{\pi}\gamma\Gamma(1-\frac{\alpha}{2})} \exp\left(\frac{x\gamma}{\gamma^2}\right) \frac{K_{\frac{\alpha+1}{2}}\left(\frac{|x|\sqrt{\beta^2+2\sigma^2\lambda}}{\gamma^2}\right)}{|x|^{\frac{\alpha+1}{2}}} \right) dx.$$

$$\Phi(u; X_t) = exp\left(iu(\mu-\beta)t - \frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha}\left((\theta - i\beta u + \frac{\sigma^2 u^2}{2})^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}} \right) t \right).$$

- The characteristic function of an exponential tilting stable (ETS random) variable is given by

$$\Phi(u; X_t) = exp\left(iu(\mu-\beta)t - \frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha}\left((\theta - i\beta u + \frac{\sigma^2 u^2}{2})^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}} \right) t \right).$$

The Lévy measure of the ETS random variable is not available in a closed-form (see, Fallahgoul et al. (2018b)).

B Modified Bessel function of the second kind

The modified Bessel function is the solution of a differential equation called modified Bessel's equation. It is given by

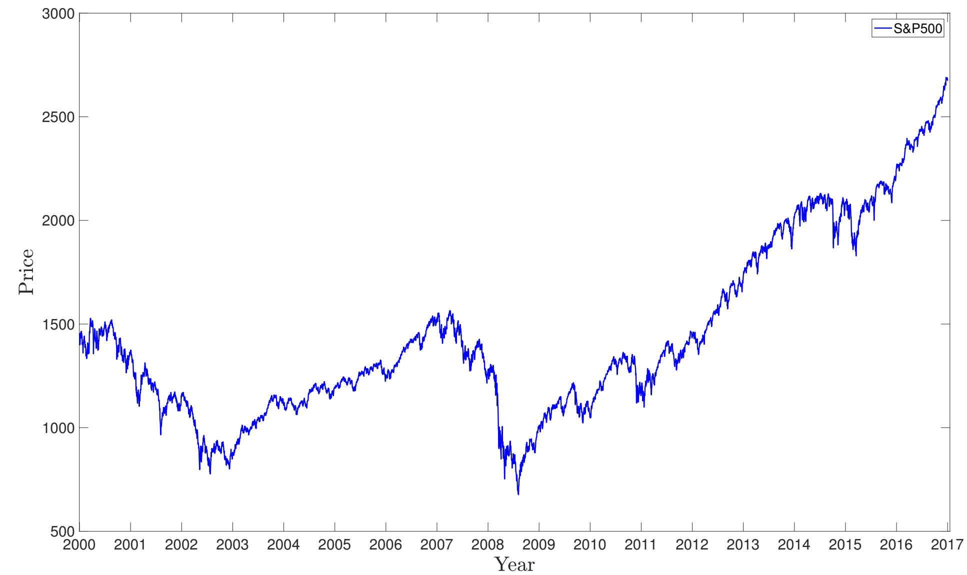
$$K_\nu(z) = \left(\frac{\pi}{2}\right) \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}$$

where $I_{-\nu}(z)$ and $I_\nu(z)$ are the fundamental solutions of the modifies Bessel's equation. $I_\nu(z)$ is given by

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{z^2}{4}\right)^k}{k!\Gamma(\nu + k + 1)}$$

where $\Gamma(a)$ is the Gamma function.

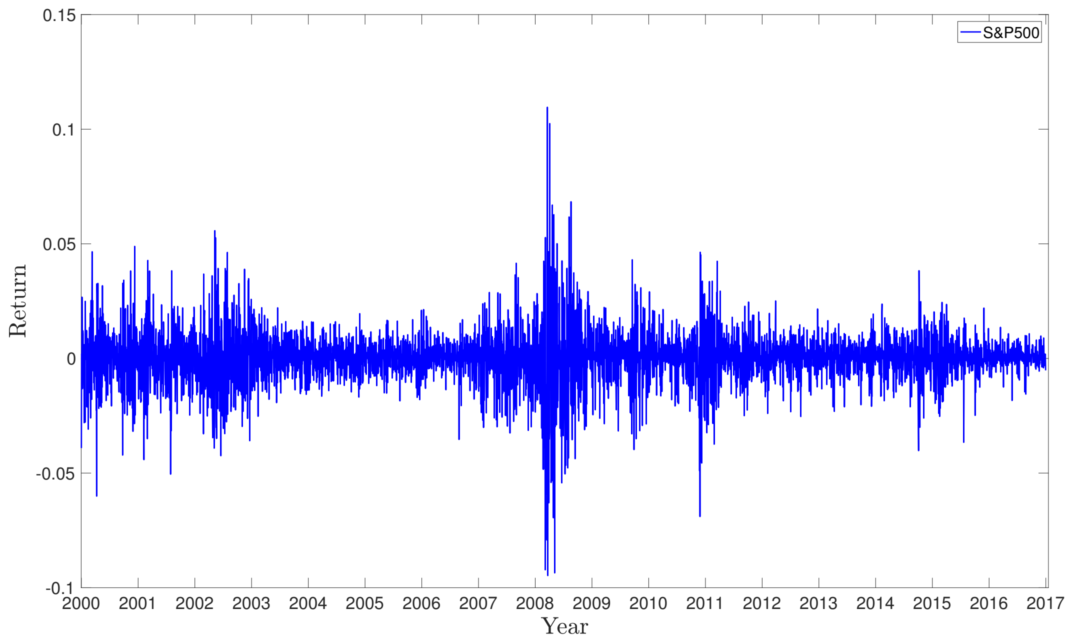
C Figures



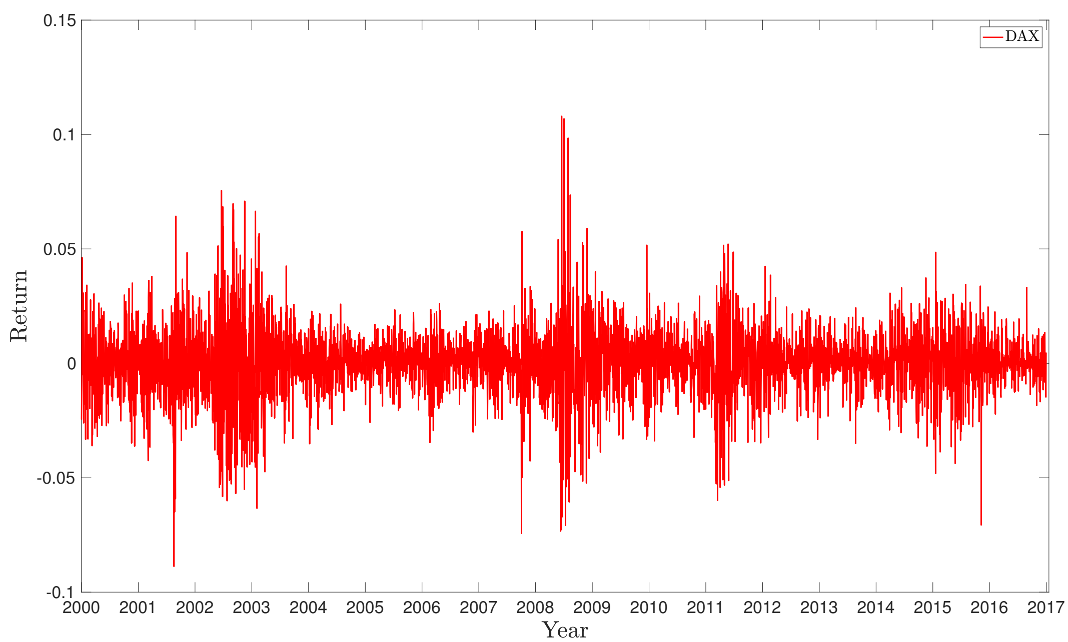
(a) Daily price process of S&P500 index.

(b) daily price process of DAX index.

Figure 1: Daily price process for S&P500 and DAX indexes from January 1, 2000 to December 29, 2017.

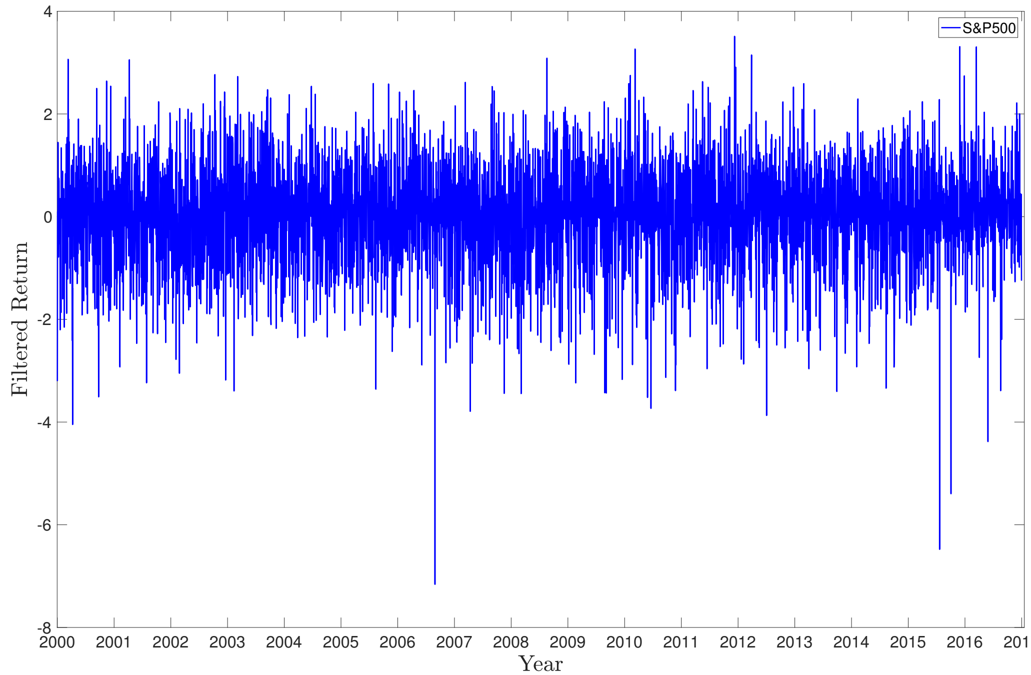


(a) Daily log-return process of S&P500 index.

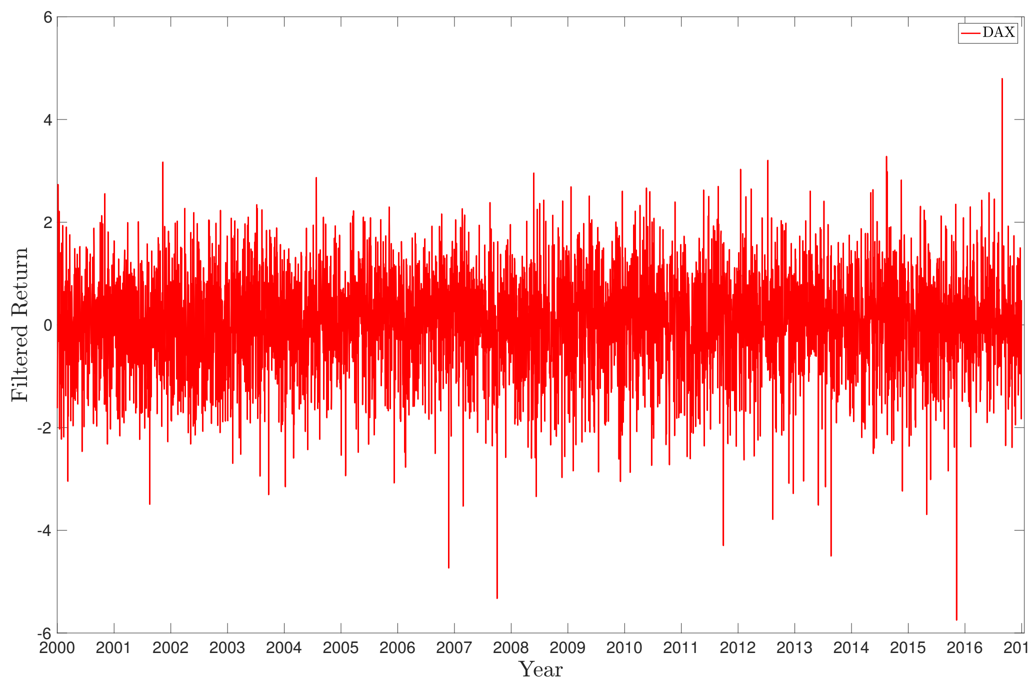


(b) Daily log-return process of DAX index.

Figure 2: Daily log-return process for S&P500 and DAX indexes from January 1, 2000 to December 29, 2017.

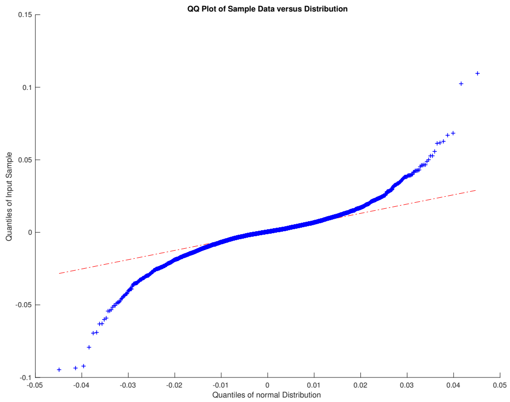


(a) Filtered daily log-return process of S&P500 index.

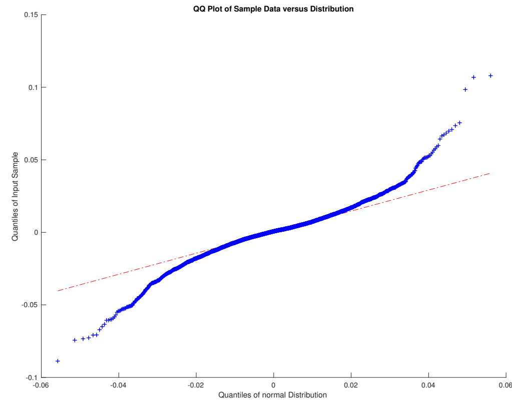


(b) Filtered daily log-return process of DAX index.

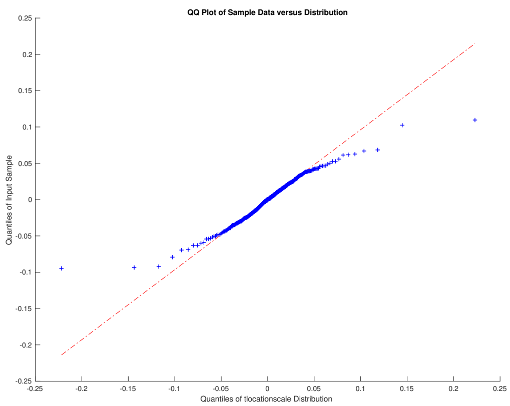
Figure 3: Filtered daily log-return process for S&P500 and DAX indexes from January 1, 2000 to December 29, 2017.



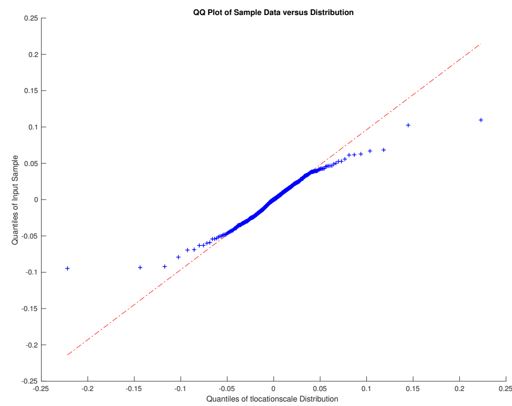
(a) S&P500



(b) DAX



(c) S&P500



(d) DAX

Figure 4: QQ-plots: quantile quantile plot. QQ-plots for daily returns of four indexes of both indexes: S&P500 and DAX. The x-axis shows the quantiles of fitted Normal and Student't. The y-axis shows the sample quantiles.

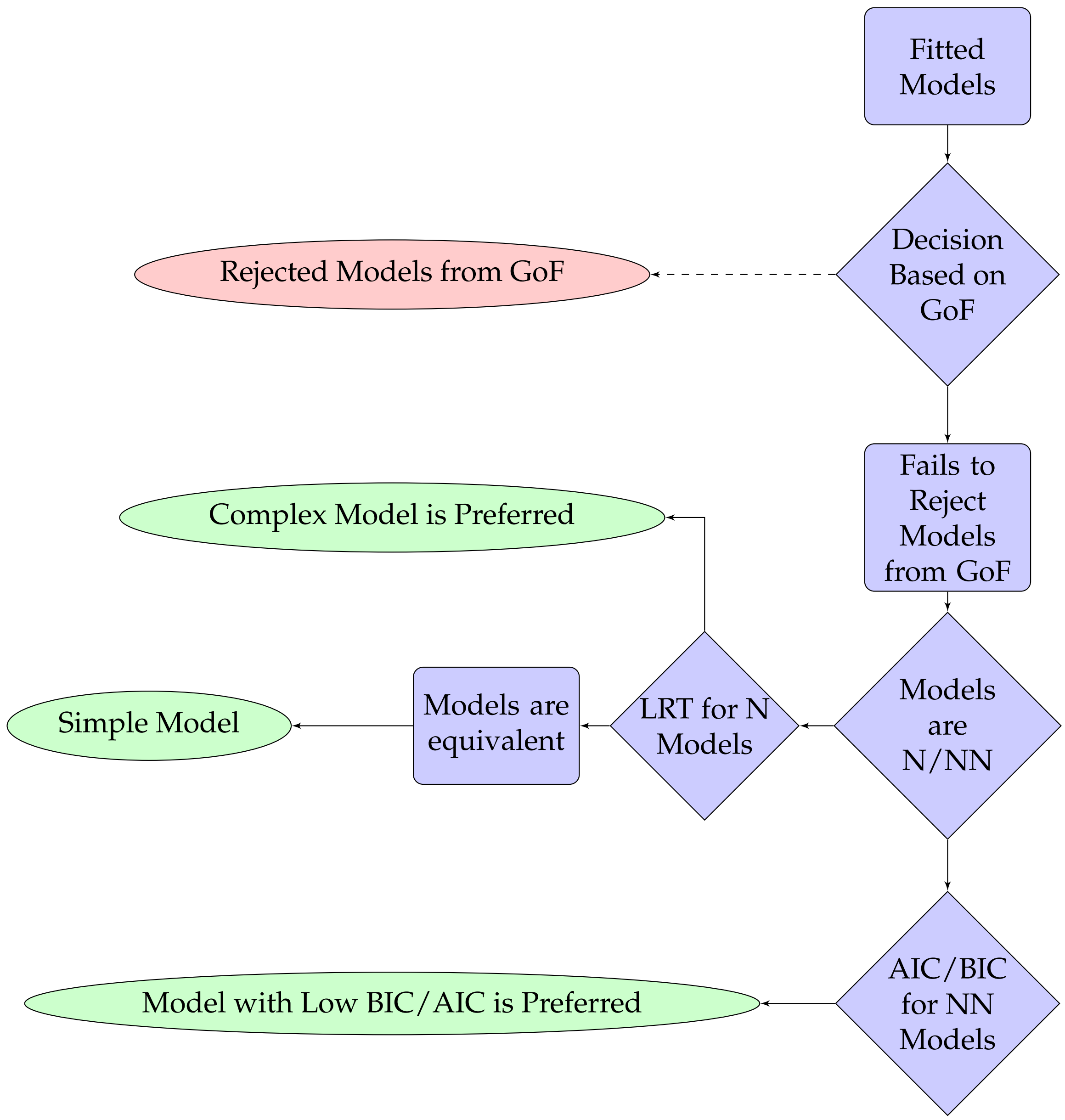


Figure 5: GoF: goodness-of-fit. LRT: Likelihood Ratio Test. AIC: Akaike Information Criteria. BIC: Bayesian Information Criteria. N: nested. NN: not nested. The algorithm for comparison two different fitted models,

D Tables

Table 1: Estimation results of all distributions for the filtered data.

This table shows the estimation results for all distributions. The estimated standard errors (p-values) are shown in parantesis under the estimated values (KS statistics). CTS: classical tempered stable. MTS: modified tempered stable. RDTS: rapidly decreasing tempered stable. NTS: normal tempered stable. ETS: exponentially tilting tempered stable. KS: Kolmogorov-Smirnov.

Panel A: SPY						
Distribution						KS
Normal	$\mu = 1.35 \times 10^{-4}$	$\sigma = 0.012$				0.023
	(0.000)	(0.004)				(0.007)
Stable	$\alpha = 1.904$	$\beta = 0.077$	$\sigma = 0.652$	$\mu = 0.001$		0.018
	(0.115)	(0.031)	(0.291)	(0.000)		(0.372)
CTS	$\alpha = 1.868$	$C = 0.099$	$\lambda_+ = 0.084$	$\lambda_- = 0.082$		0.015
	(0.109)	(0.006)	(0.015)	(0.009)		(0.509)
MTS	$\alpha = 1.782$	$C = 0.094$	$\lambda_+ = 0.171$	$\lambda_- = 0.159$		0.016
	(0.185)	(0.000)	(0.033)	(0.009)		(0.450)
RDTS	$\alpha = 1.902$	$C = 0.077$	$\lambda_+ = 0.471$	$\lambda_- = 0.396$		0.013
	(0.174)	(0.085)	(0.108)	(0.139)		(0.559)
NTS	$\alpha = 1.849$	$C = 0.440$	$\theta = 1.837$	$\beta = 0.033$	$\mu = -0.012$	0.016
	(0.130)	(0.093)	(0.292)	(0.000)	(0.000)	(0.472)
ETS	$\alpha = 0.893$	$C = 0.091$	$\lambda = 0.552$	$\beta = 0.061$	$\mu = 0.007$	0.012
	(0.011)	(0.000)	(0.004)	(0.000)	(0.000)	(0.633)
Panel B: DAX						
Distribution						KS
Normal	$\mu = 1.28 \times 10^{-4}$	$\sigma = 0.015$				0.030
	(0.000)	(0.000)				(0.008)
Stable	$\alpha = 1.893$	$\beta = 0.080$	$\sigma = 0.633$	$\mu = 0.001$		0.017
	(0.200)	(0.006)	(0.147)	(0.000)		(0.394)
CTS	$\alpha = 1.902$	$C = 0.074$	$\lambda_+ = 0.093$	$\lambda_- = 0.090$		0.016
	(0.194)	(0.003)	(0.071)	(0.039)		(0.489)
MTS	$\alpha = 1.714$	$C = 0.088$	$\lambda_+ = 0.281$	$\lambda_- = 0.277$		0.015
	(0.109)	(0.007)	(0.066)	(0.030)		(0.480)
RDTS	$\alpha = 1.890$	$C = 0.092$	$\lambda_+ = 0.539$	$\lambda_- = 0.447$		0.014
	(0.119)	(0.050)	(0.003)	(0.007)		(0.510)
NTS	$\alpha = 1.888$	$C = 0.571$	$\theta = 2.008$	$\beta = 0.062$	$\mu = -0.031$	0.015
	(0.192)	(0.082)	(0.506)	(0.006)	(0.000)	(0.517)
ETS	$\alpha = 0.910$	$C = 0.022$	$\lambda = 0.663$	$\beta = 0.039$	$\mu = 0.071$	0.012
	(0.045)	0.009	0.205	0.004	0.000	(0.704)

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