

Approximation to PDE by narrow-base trial functions

- Defining the PDE problem
- Introduction to the PDE approximation
- Weak formulation and the Galerkin method
- Some 1D problems.
- Generalization to 2D and 3D problems
- 2D heat equation
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- Finite differences as a special case of finite elements

Defining the PDE problem

$$A(\phi) = 0 \quad \text{in } \Omega$$

L : linear differential operator

$$p \neq p(\phi)$$

$$L(\phi) = \frac{\partial}{\partial x} \left(\kappa \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\kappa \frac{\partial \phi}{\partial y} \right)$$

$$p = Q$$

$$\kappa \neq \kappa(\phi)$$

$$B(\phi) = M(\phi) + r = 0 \quad \text{in } \Gamma$$

$$\begin{cases} M(\phi) = \phi & ; \quad r = -\bar{\phi} & \text{on } \Gamma_{\phi} & \text{DIRICHLET} \\ M(\phi) = -\kappa \frac{\partial \phi}{\partial \eta} & ; \quad r = -\bar{q} & \text{on } \Gamma_q & \text{NEUMANN} \end{cases}$$

Introduction to the PDE approximation

How to choose the trial functions

$$\phi(x) \cong \hat{\phi}(x) = \sum_m \phi_m N_m(x)$$

$$\text{on } \Gamma \quad \begin{cases} M(\psi) = -r \\ M(N_m) = 0 \end{cases} \quad ; \quad m = 1, 2, \dots$$

$$\hat{\phi}(x) = \phi(x) \quad \text{for all } x \in \Gamma$$

Introduction to the PDE approximation

The trial functions and its derivatives

$$\phi(x) \cong \hat{\phi}(x) = \sum_m \phi_m N_m(x)$$

$$\mathbf{L}\hat{\phi} = \sum_m \phi_m \mathbf{L}N_m$$

$$\frac{\partial \phi}{\partial x} \cong \frac{\partial \hat{\phi}}{\partial x} = \sum_m \phi_m \frac{\partial N_m}{\partial x}$$

$$\frac{\partial^2 \phi}{\partial x^2} \cong \frac{\partial^2 \hat{\phi}}{\partial x^2} = \sum_m \phi_m \frac{\partial^2 N_m}{\partial x^2}$$

N_m needs the regularity imposed by \mathbf{L}

to relax it we redefine the problem in a weak form

Approximation by weighted residual Weak formulation and the Galerkin method

$$R_{\Omega} = A(\phi) = \mathbf{L} \hat{\phi} + p = \sum_m \phi_m \mathbf{L} N_m + p$$

$$R_{\Gamma} = B(\phi) = \mathbf{M}(\phi) + r = \sum_m \phi_m \mathbf{M} N_m + r$$

$$\int_{\Omega} W_l R_{\Omega} d\Omega + \int_{\Gamma} \bar{W}_l R_{\Gamma} d\Gamma = 0$$

$$\int_{\Omega} W_l \left(\sum_m \phi_m \mathbf{L} N_m + p \right) d\Omega + \int_{\Gamma} \bar{W}_l \left(\sum_m \phi_m \mathbf{M} N_m + r \right) d\Gamma = 0$$

$$\int_{\Omega} W_l \left(\sum_m \phi_m \mathbf{L} N_m \right) d\Omega + \int_{\Gamma} \bar{W}_l \left(\sum_m \phi_m \mathbf{M} N_m \right) d\Gamma = - \int_{\Omega} W_l p d\Omega - \int_{\Gamma} \bar{W}_l r d\Gamma$$

$$K_{lm} \phi_m = f_l$$

$$K_{lm} = \int_{\Omega} W_l \mathbf{L} N_m d\Omega + \int_{\Gamma} \bar{W}_l \mathbf{M} N_m d\Gamma$$

$$f_l = - \int_{\Omega} W_l p d\Omega - \int_{\Gamma} \bar{W}_l r d\Gamma \quad l, m = 1, 2, \dots, M$$

Weak formulation and functional regularity

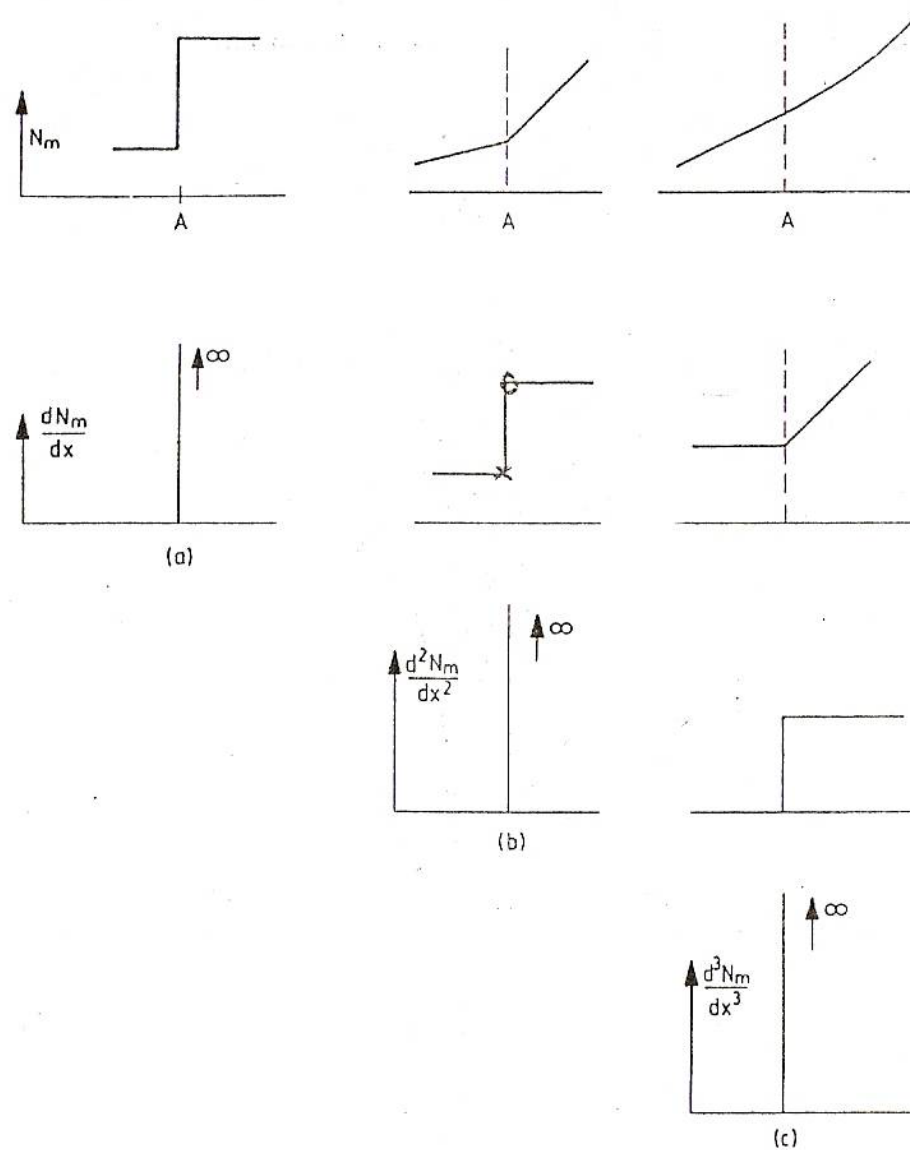


FIGURE 3.4. Behavior of three types of one-dimensional shape functions and their derivatives near the junction A of two elements.

Weak formulation and functional regularity

$$\underbrace{\int_{\Omega} W_l (\mathbf{L}\hat{\phi} + p) d\Omega}_{\text{strong form}} = \overbrace{\int_{\Omega} \mathbf{C} W_l \mathbf{D}\hat{\phi} d\Omega + \int_{\Gamma} W_l \mathbf{E}\hat{\phi} d\Gamma + \int_{\Omega} W_l p d\Omega}^{\text{integration by parts}} = 0$$

weak form

$$\begin{aligned} & \int_{\Omega} W_l (\mathbf{L}\hat{\phi} + p) d\Omega + \int_{\Gamma} \overline{W}_l (\mathbf{M}\hat{\phi} + \mathbf{r}) d\Gamma = \\ & = \int_{\Omega} \mathbf{C} W_l \mathbf{D}\hat{\phi} d\Omega + \int_{\Omega} W_l p d\Omega + \int_{\Gamma} W_l \mathbf{E}\hat{\phi} d\Gamma + \int_{\Gamma} \overline{W}_l (\mathbf{M}\hat{\phi} + \mathbf{r}) d\Gamma = 0 \end{aligned}$$

In this way it is feasible to reduce the regularity requirements for the functional spaces balancing the cost of the computations between weight and trial functions

Some 1D problems

Again solving example 5 now with FEM

Find $\hat{\phi}(x)$ solution of the following ODE

$$\frac{d^2\phi}{dx^2} - \phi = 0 \quad \text{in } \Omega : \{x; 0 \leq x \leq 1\}$$

$$\phi(x=0) = 0 \quad ; \quad \phi(x=1) = 1 \quad \text{in } \Gamma : \{x = 0, x = 1\}$$

$$\phi(x) \cong \hat{\phi}(x) = \sum_m \phi_m N_m(x) \quad \text{using for } N_m \text{ piecewise linear}$$

$$\int_0^1 W_l \left(\frac{d^2\phi}{dx^2} - \phi \right) dx + \left[\overline{W}_l R_\Gamma \right]_{x=0} + \left[\overline{W}_l R_\Gamma \right]_{x=1} = 0 \quad \text{for } l = 1, 2, \dots, M+1$$

$$\begin{aligned} & - \int_0^1 \frac{dW_l}{dx} \frac{d\hat{\phi}}{dx} dx - \int_0^1 W_l \hat{\phi} dx + \left[W \frac{d\hat{\phi}}{dx} \right]_{x=1} - \left[W \frac{d\hat{\phi}}{dx} \right]_{x=0} + \\ & + \left[\overline{W}_l (\hat{\phi} - 0) \right]_{x=0} + \left[\overline{W}_l (\hat{\phi} - 1) \right]_{x=1} = 0 \end{aligned}$$

Example 5 by piecewise linear Galerkin

Choosing $\overline{W_l}|_{\Gamma} = -W_l|_{\Gamma}$ & $W_l|_{\Gamma_\phi} = 0$

with $\Gamma = \Gamma_\phi \cup \Gamma_q$

$$\begin{aligned}
 & - \int_0^1 \frac{dW_l}{dx} \frac{d\hat{\phi}}{dx} dx - \int_0^1 W_l \hat{\phi} dx + \left[W_l \frac{d\hat{\phi}}{dx} \right]_{x=1} - \left[W_l \frac{d\hat{\phi}}{dx} \right]_{x=0} - \\
 & - \underbrace{\left[W_l \hat{\phi} \right]_{x=0}}_{=0 \text{ (DIRICHLET)}} - \underbrace{\left[W_l (\hat{\phi} - 1) \right]_{x=1}}_{=0 \text{ (DIRICHLET)}} = 0
 \end{aligned}$$

$$\Rightarrow \int_0^1 \frac{dW_l}{dx} \frac{d\hat{\phi}}{dx} dx + \int_0^1 W_l \hat{\phi} dx = \left[W_l \frac{d\hat{\phi}}{dx} \right]_{x=0}^{x=1}$$

the rhs term may be cancelled out because $W_l|_{\Gamma_\phi} = 0$

However it is possible to keep it including $\left. \frac{d\hat{\phi}}{dx} \right|_{x=0}^{x=1}$ as unknowns

Example 5 by piecewise linear Galerkin

$$\int_0^1 \frac{dW_l}{dx} \frac{d\hat{\phi}}{dx} dx + \int_0^1 W_l \hat{\phi} dx = \left[W_l \frac{d\hat{\phi}}{dx} \right]_{x=0}^{x=1}$$

$$\Rightarrow \underline{\underline{K}} \underline{\phi} = \underline{f}$$

$$K_{lm} = \int_0^1 \frac{dW_l}{dx} \frac{dN_m}{dx} dx + \int_0^1 W_l N_m dx$$

$$f_l = \left[W_l \frac{d\hat{\phi}}{dx} \right]_{x=0}^{x=1} ; \quad l, m = 1, 2, \dots, M, M+1$$

$$\underline{\phi} = \{\phi_1, \phi_2, \dots, \phi_M, \phi_{M+1}\}$$

Example 5 by piecewise linear Galerkin

$$N_i^e = \frac{h^e - \chi}{h^e} \quad ; \quad N_j^e = \frac{\chi}{h^e} \quad ; \quad \chi = x - x_i \quad ; \quad h^e = x_j - x_i$$

$$K_{lm} = \sum_{e=1}^E K_{lm}^e \quad ; \quad K_{lm}^e = \left\{ \begin{array}{l} 0 \quad \text{if } l, m \notin \text{element } e \\ K_{ij}^e = K_{ji}^e = \int_0^{h^e} \left(\frac{dN_i^e}{dx} \frac{dN_j^e}{dx} + N_i^e N_j^e \right) d\chi = \\ \quad = -\frac{1}{h^e} + \frac{h^e}{6} \\ K_{ii}^e = K_{jj}^e = \int_0^{h^e} \left(\left(\frac{dN_i^e}{dx} \right)^2 + (N_i^e)^2 \right) d\chi = \\ \quad = \frac{1}{h^e} + \frac{h^e}{3} \end{array} \right.$$

Example 5 by piecewise linear Galerkin

Assembling the system (gathering) $\Rightarrow \underline{\underline{K}} \underline{\phi} = \underline{f}$

$\underline{\underline{K}}$ is gathered as

$$\underbrace{\begin{bmatrix} \frac{1}{h^e} + \frac{h^e}{3} & -\frac{1}{h^e} + \frac{h^e}{6} & 0 & 0 \\ -\frac{1}{h^e} + \frac{h^e}{6} & \frac{1}{h^e} + \frac{h^e}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{ELEMENT 1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{h^e} + \frac{h^e}{3} & -\frac{1}{h^e} + \frac{h^e}{6} & 0 \\ 0 & -\frac{1}{h^e} + \frac{h^e}{6} & \frac{1}{h^e} + \frac{h^e}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{ELEMENT 2}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{h^e} + \frac{h^e}{3} & -\frac{1}{h^e} + \frac{h^e}{6} \\ 0 & 0 & -\frac{1}{h^e} + \frac{h^e}{6} & \frac{1}{h^e} + \frac{h^e}{3} \end{bmatrix}}_{\text{ELEMENT 3}}$$

\underline{f} is gathered as

$$\underbrace{\begin{bmatrix} \left. \frac{d\hat{\phi}}{dx} \right|_{x=0} & 0 & 0 & 0 \end{bmatrix}}_{\text{ELEMENT 1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{ELEMENT 2}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & \left. \frac{d\hat{\phi}}{dx} \right|_{x=1} \end{bmatrix}}_{\text{ELEMENT 3}}$$

the system may be solved for 2 unknowns $\{\phi_2, \phi_3\}$ or for 4 unknowns $\left\{ \frac{d\phi_1}{dx}, \phi_2, \phi_3, \frac{d\phi_4}{dx} \right\}$

Example 5 by piecewise linear Galerkin

with 4 unknowns

$$\begin{bmatrix} \frac{1}{h^e} + \frac{h^e}{3} & -\frac{1}{h^e} + \frac{h^e}{6} & 0 & 0 \\ -\frac{1}{h^e} + \frac{h^e}{6} & 2\left(\frac{1}{h^e} + \frac{h^e}{3}\right) & -\frac{1}{h^e} + \frac{h^e}{6} & 0 \\ 0 & -\frac{1}{h^e} + \frac{h^e}{6} & 2\left(\frac{1}{h^e} + \frac{h^e}{3}\right) & -\frac{1}{h^e} + \frac{h^e}{6} \\ 0 & 0 & -\frac{1}{h^e} + \frac{h^e}{6} & \frac{1}{h^e} + \frac{h^e}{3} \end{bmatrix} \begin{bmatrix} \phi_1 = 0 \\ \phi_2 \\ \phi_3 \\ \phi_4 = 1 \end{bmatrix} = \begin{bmatrix} -\frac{d\phi_1}{dx} \\ 0 \\ 0 \\ \frac{d\phi_4}{dx} \end{bmatrix}$$

with 2 unknowns

$$\begin{bmatrix} 2\left(\frac{1}{h^e} + \frac{h^e}{3}\right) & -\frac{1}{h^e} + \frac{h^e}{6} \\ -\frac{1}{h^e} + \frac{h^e}{6} & 2\left(\frac{1}{h^e} + \frac{h^e}{3}\right) \end{bmatrix} \begin{bmatrix} \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{h^e} + \frac{h^e}{6} \end{bmatrix}$$

and

$$\begin{bmatrix} -\frac{d\phi_1}{dx} \\ \frac{d\phi_4}{dx} \end{bmatrix} = \begin{bmatrix} \frac{1}{h^e} + \frac{h^e}{3} & -\frac{1}{h^e} + \frac{h^e}{6} & 0 & 0 \\ 0 & 0 & -\frac{1}{h^e} + \frac{h^e}{6} & \frac{1}{h^e} + \frac{h^e}{3} \end{bmatrix} \begin{bmatrix} \phi_1 = 0 \\ \phi_2 \\ \phi_3 \\ \phi_4 = 1 \end{bmatrix}$$

Example 5 by piecewise linear Galerkin (routine Ej_4_1.m)

```
Ke = zeros(numel,2,2);  
fe = zeros(numel,2);
```

Assembling

```
% loop over each element  
for k=1:numel  
    psi = (0:10)'/10;  
    node1 = icone(k,1);  
    node2 = icone(k,2);  
    xx = psi*(xnod(node2,1)-xnod(node1,1))+xnod(node1,1);  
    [N,L_N] = shape_function(xx);  
    for mi=1:2  
        for mj=1:2  
            Ke(k,mi,mj) = Ke(k,mi,mj) + trapz(xx,N(:,mi).*N(:,mj));  
            Ke(k,mi,mj) = Ke(k,mi,mj) + trapz(xx,L_N(:,mi).*L_N(:,mj));  
        end  
    end  
end  
end
```

Example 5 by piecewise linear Galerkin (routine Ej_4_1.m)

```
% gather Ke and fe in Kg and fg
```

```
Kg = zeros(Nx+1,Nx+1);
```

```
fg = zeros(Nx+1,1);
```

```
for k=1:numel
```

```
    for mi=1:nen
```

```
        for mj=1:nen
```

```
            Kg(icone(k,mi),icone(k,mj))=Kg(icone(k,mi),icone(k,mj)) + Ke(k,mi,mj);
```

```
        end
```

```
        fg(icone(k,mi),1) = fg(icone(k,mi),1) + fe(k,mi);
```

```
    end
```

```
end
```

Gathering

```
% reducing the global system for fixations
```

```
free = (2:Nx)';
```

```
fg(free,1) = fg(free,1) - Kg(free,1)*phi_lef - Kg(free,Nx+1)*phi_rig;
```

```
Kg = Kg(free,free);
```

```
fg = fg(free,1);
```

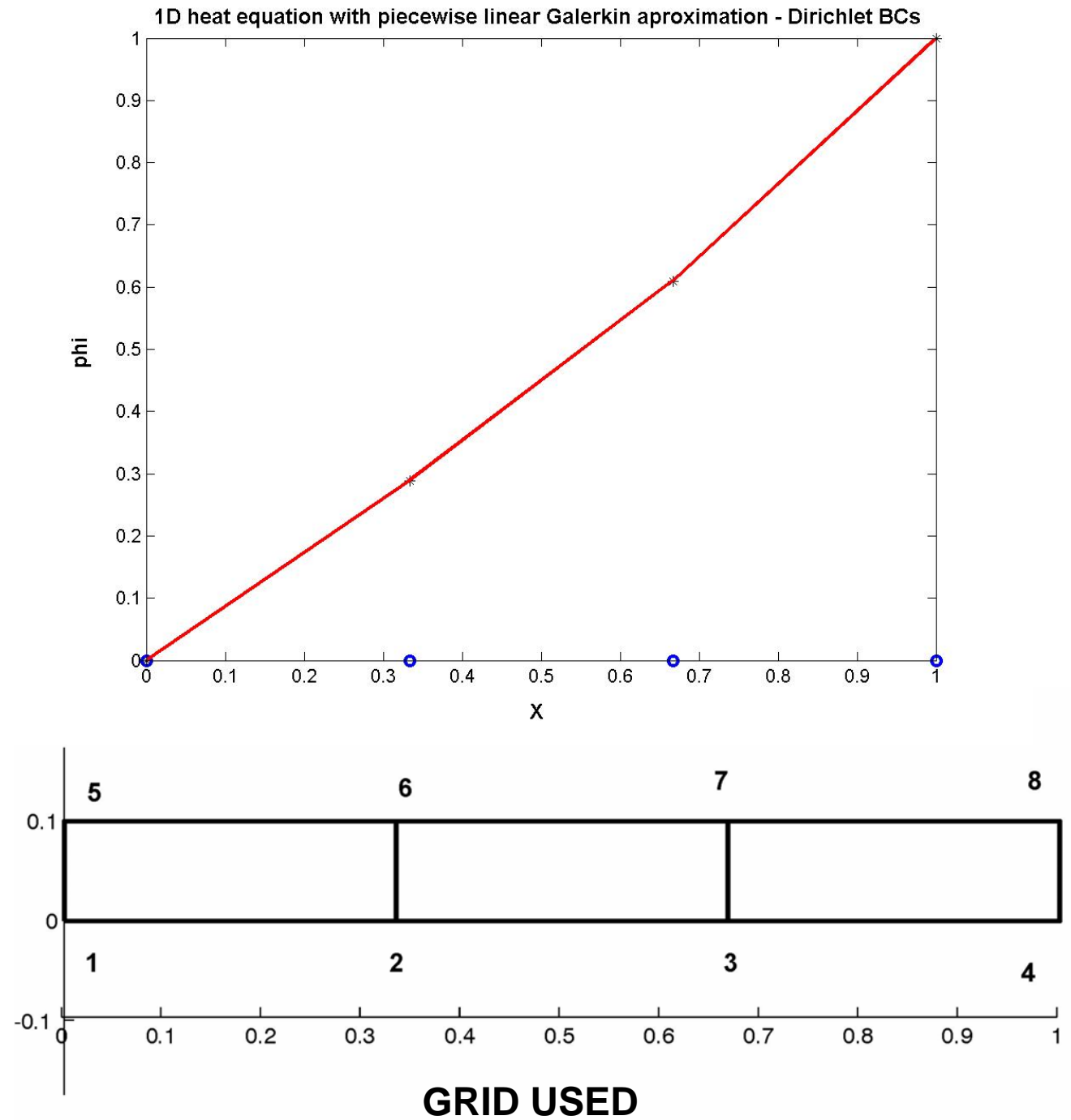
```
% solver
```

```
phi = Kg\fg;
```

```
phi = [phi_lef;phi;phi_rig];
```

**Applied BC
and solving**

Example 5: Heat equation in 1D with FEM Results



Exact solution to Example 5

In order to analyze the error we solve the ODE for the exact solution.

$$\frac{d^2\phi}{dx^2} - \phi = 0 \quad \text{admits solutions of the form } \phi = A_k e^{\lambda_k x}$$

replacing in the ODE we get

$$(\lambda_k^2 - 1)(A_k e^{\lambda_k x}) = 0 \Rightarrow \lambda = \pm 1$$

using $\phi(x=0) = 0$

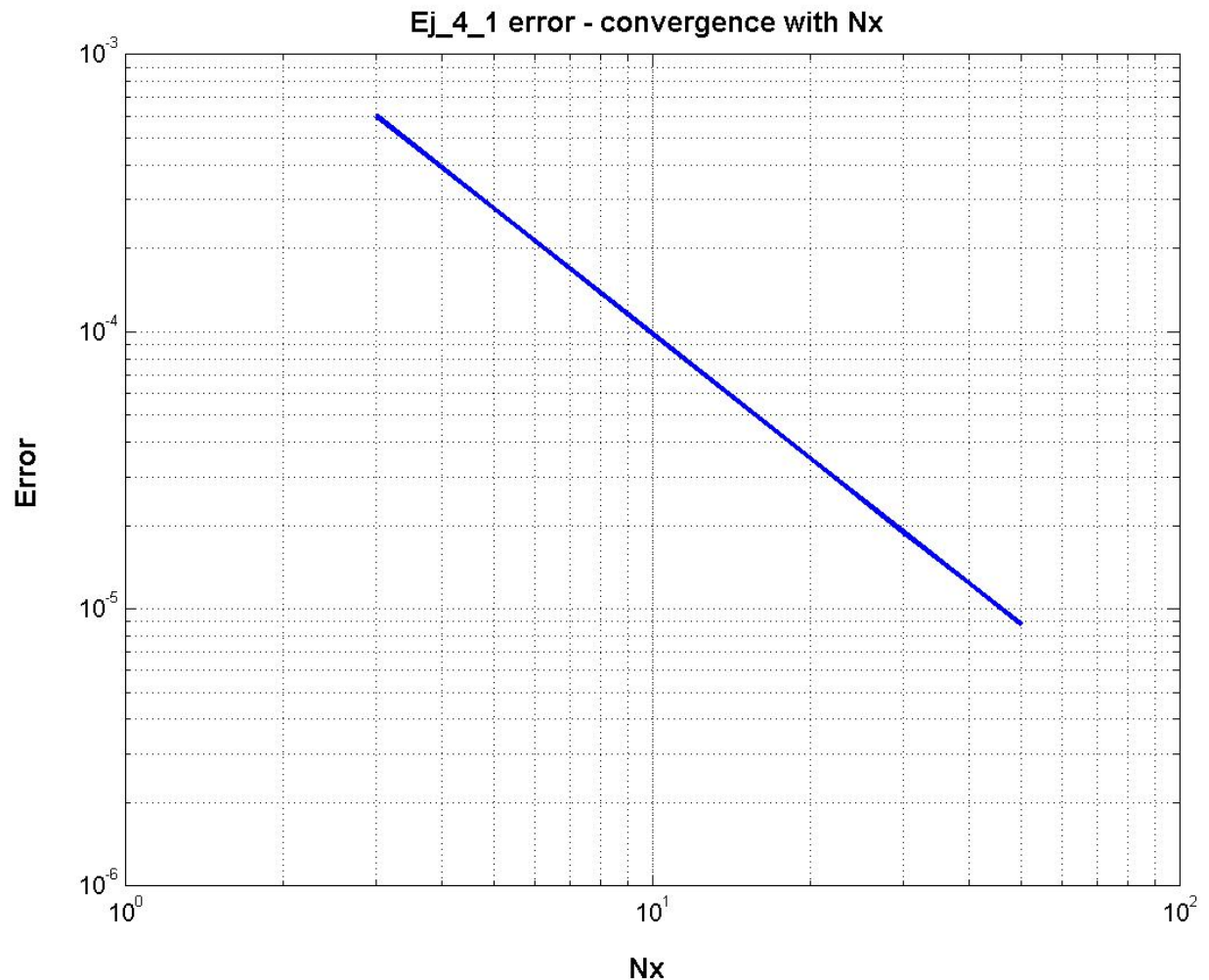
$$\phi(x=0) = A_1 e^x + A_2 e^{-x} \Big|_{x=0} = A_1 + A_2 = 0 \Rightarrow A_1 = -A_2$$

using $\phi(x=1) = 1$

$$\phi(x=1) = A_1 e^x + A_2 e^{-x} \Big|_{x=1} = A_1 e + A_2 e^{-1} = 1 \Rightarrow A_1 = \frac{1}{e - \frac{1}{e}}$$

$$\therefore \phi = \frac{1}{e - \frac{1}{e}} (e^x - e^{-x}) = \frac{e^x - e^{-x}}{e - \frac{1}{e}}$$

Example 5: Heat equation in 1D with FEM Results



```
[phi , phi_th] = [ ...  
                  0          0  
    2.885539967914507e-001  2.889212154417394e-001  
    6.097683214596928e-001  6.102431352945081e-001  
    1.000000000000000e+000  1.000000000000000e+000 ]
```

Some 1D problems

Example 5 now with Neumann BC at x=1

Find $\hat{\phi}(x)$ solution of the following ODE

$$\frac{d^2 \phi}{dx^2} - \phi = 0 \quad \text{in } \Omega : \{x; 0 \leq x \leq 1\}$$

$$\phi(x=0) = 0 \quad ; \quad \frac{d\phi}{dx}(x=1) = 1 \quad \text{in } \Gamma : \{x = 0, x = 1\}$$

$$\phi(x) \cong \hat{\phi}(x) = \sum_m \phi_m N_m(x) \quad \text{using for } N_m \text{ piecewise linear}$$

$$\int_0^1 W_l \left(\frac{d^2 \phi}{dx^2} - \phi \right) dx + [\overline{W}_l R_\Gamma]_{x=0} + [\overline{W}_l R_\Gamma]_{x=1} = 0 \quad \text{for } l = 1, 2, \dots, M+1$$

$$- \int_0^1 \frac{dW_l}{dx} \frac{d\hat{\phi}}{dx} dx - \int_0^1 W_l \hat{\phi} dx + \left[W \frac{d\hat{\phi}}{dx} \right]_{x=1} - \left[W \frac{d\hat{\phi}}{dx} \right]_{x=0} +$$

$$+ [\overline{W}_l (\hat{\phi} - 0)]_{x=0} + \left[\overline{W}_l \left(\frac{d\hat{\phi}}{dx} - 1 \right) \right]_{x=1} = 0$$

Example 5 by piecewise linear Galerkin

Choosing $\overline{W_l}|_{\Gamma} = -W_l|_{\Gamma}$ & $W_l|_{\Gamma_{\phi}} = 0$ with $\Gamma = \Gamma_{\phi} \cup \Gamma_q$

$$-\int_0^1 \frac{dW_l}{dx} \frac{d\hat{\phi}}{dx} dx - \int_0^1 W_l \hat{\phi} dx - W_l \frac{d\hat{\phi}}{dx} \Big|_{x=0} - \underbrace{[W_l \hat{\phi}]_{x=0}}_{=0 \text{ (DIRICHLET)}} + [W_l]_{x=1} = 0$$

$$\Rightarrow \int_0^1 \frac{dW_l}{dx} \frac{d\hat{\phi}}{dx} dx + \int_0^1 W_l \hat{\phi} dx + W_l \frac{d\hat{\phi}}{dx} \Big|_{x=0} - [W_l]_{x=1} = 0$$

Again we retain $\frac{d\hat{\phi}}{dx} \Big|_{x=0}$ as an unknown

$$K_{lm} = \int_0^1 \frac{dW_l}{dx} \frac{dN_m}{dx} dx + \int_0^1 W_l N_m dx$$

$$f_l = [W_l]_{x=1} - W_l \frac{d\hat{\phi}}{dx} \Big|_{x=0} ; \quad l, m = 1, 2, \dots, M, M+1$$

Exact solution to Example 5 with Neumann BC

$$\frac{d^2\phi}{dx^2} - \phi = 0 \quad \text{admits solutions of the form } \phi = A_k e^{\lambda_k x}$$

replacing in the ODE we get

$$(\lambda_k^2 - 1)(A_k e^{\lambda_k x}) = 0 \Rightarrow \lambda = \pm 1$$

using $\phi(x=0) = 0$

$$\phi(x=0) = A_1 e^x + A_2 e^{-x} \Big|_{x=0} = A_1 + A_2 = 0 \Rightarrow A_1 = -A_2$$

using $\frac{d\phi}{dx}(x=1) = 1$

$$\frac{d\phi}{dx}(x=1) = \lambda_k A_k e^{\lambda_k x} \Big|_{x=1} = A_1 e - A_2 e^{-1} = 1 \Rightarrow A_1 = \frac{1}{e + \frac{1}{e}}$$

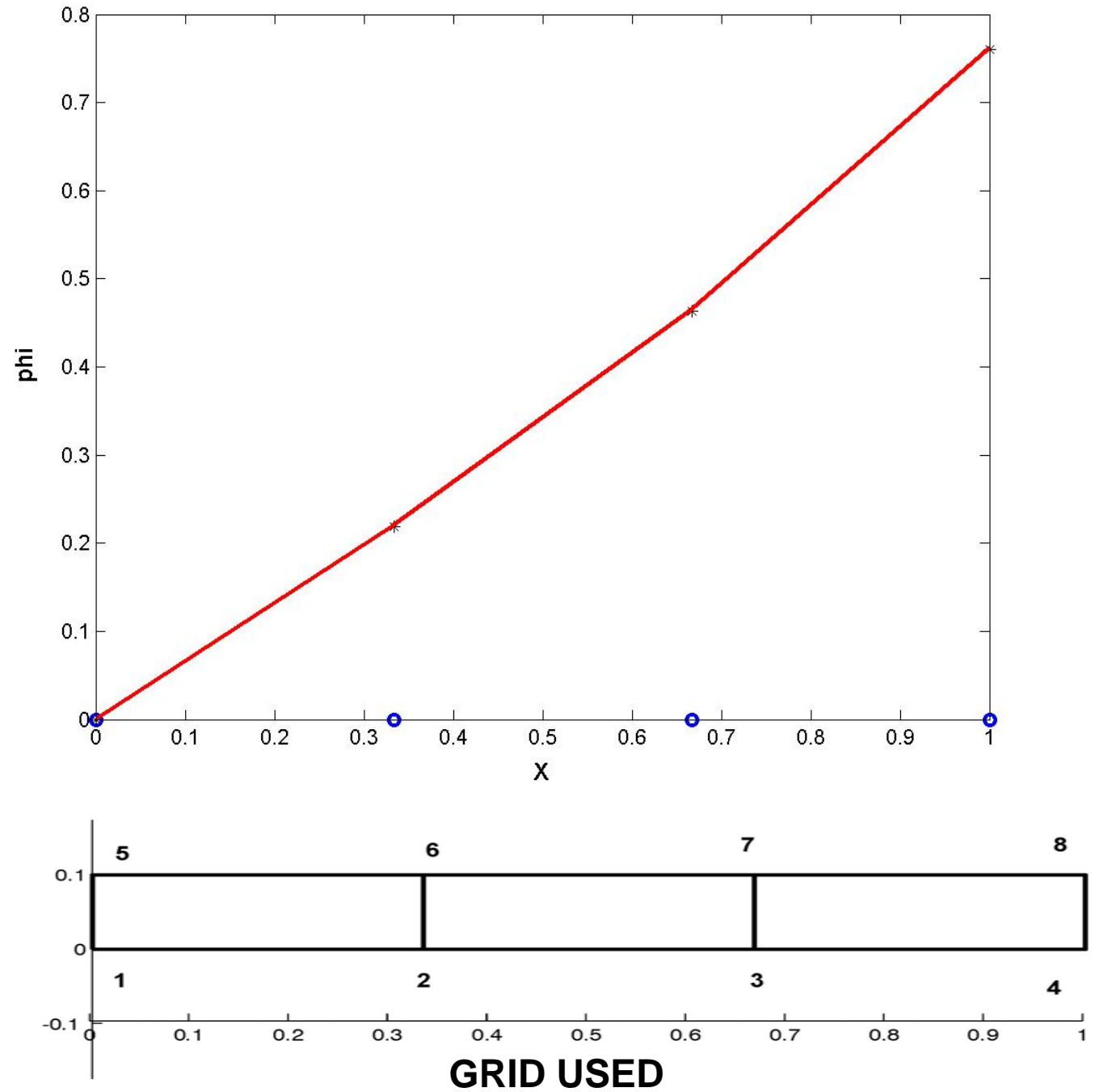
$$\therefore \phi = \frac{1}{e + \frac{1}{e}} (e^x - e^{-x}) = \frac{e^x - e^{-x}}{e + \frac{1}{e}}$$

Example 5

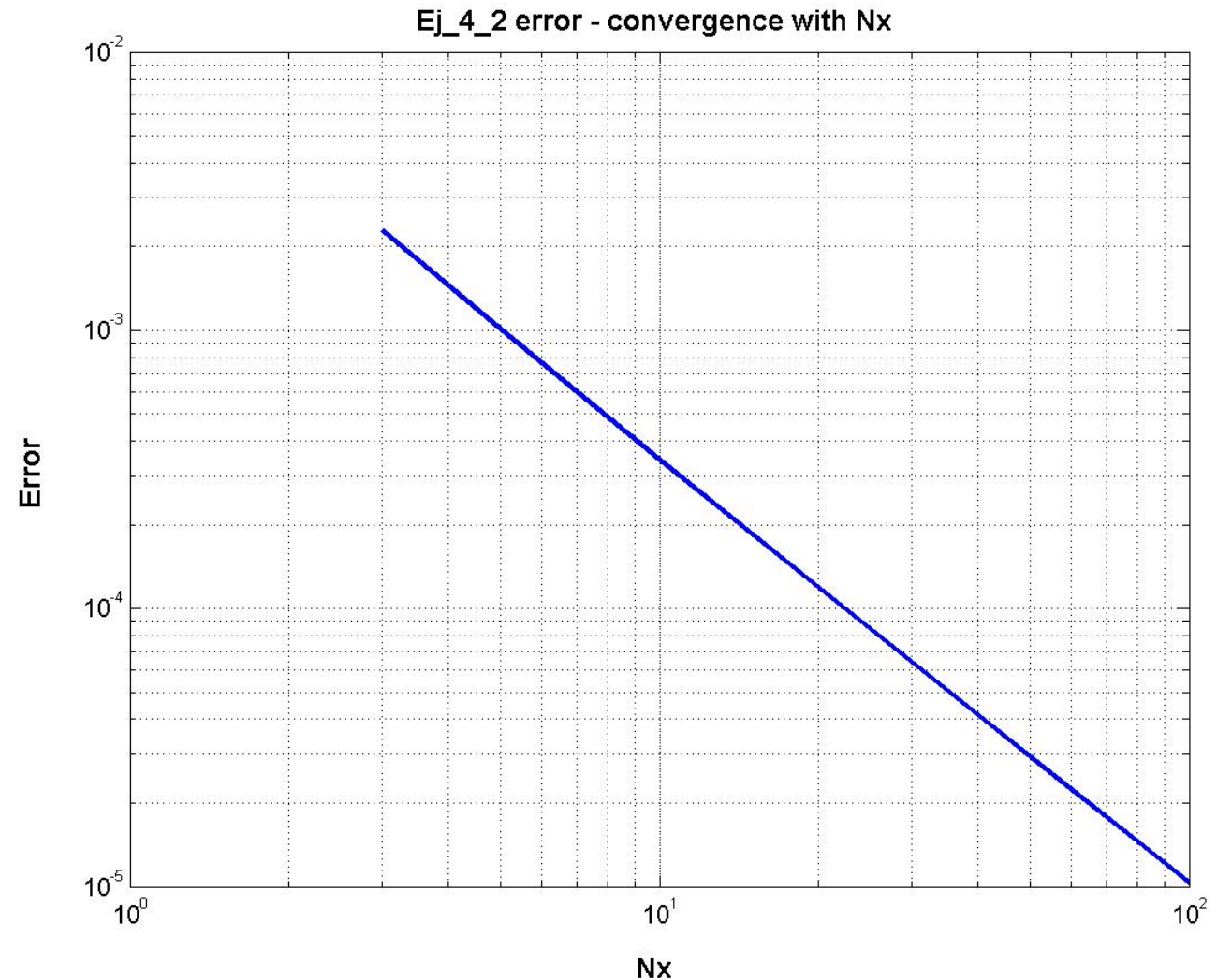
Results

with
Neumann
BC at $x=1$

1D heat equation with piecewise linear Galerkin approximation - Dirichlet/Neumann BCs



Example 5 Results with Neumann BC at $x=1$



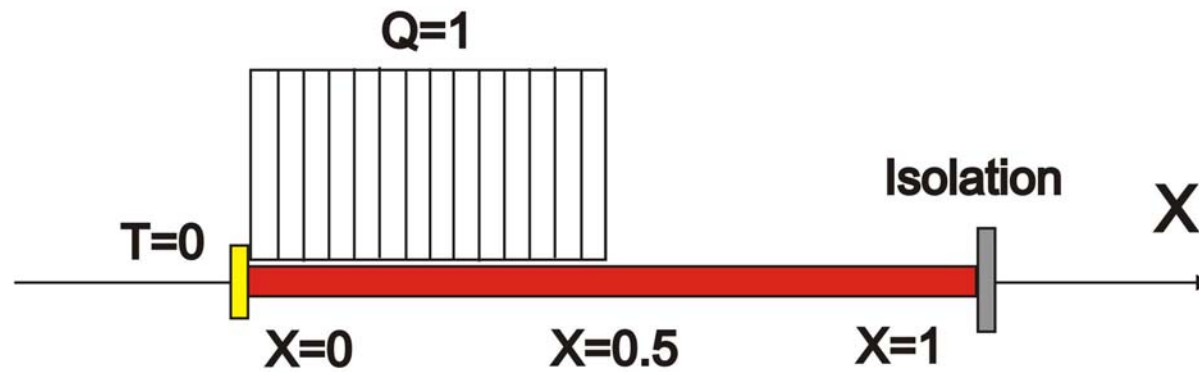
```
[phi , phi_th] = [ ...  
                  0          0  
    2.192827911335199e-001  2.200407092120653e-001  
    4.633853662097133e-001  4.647576055524206e-001  
    7.599367659842332e-001  7.615941559557650e-001 ]
```

Example 8 : SoDE – 2nd order to 1st order

$$\frac{d}{dx} \left(\kappa \frac{d\phi}{dx} \right) - Q = 0$$

$$\Omega: \{x; 0 \leq x \leq 1\} \quad ; \quad \kappa = 1 \quad ; \quad Q = \begin{cases} 1 & x \leq \frac{1}{2} \\ 0 & x > \frac{1}{2} \end{cases}$$

$$\phi = 0 \quad @ \quad x = 0 \quad ; \quad q = -\kappa \frac{d\phi}{dx} = 0 \quad @ \quad x = 1$$



Example 8 : two 1st order ODEs instead of one 2nd order ODE

Find $\hat{\phi}(x)$ solution of the following ODE $\frac{d}{dx}\left(\kappa \frac{d\phi}{dx}\right) - Q = 0$

as a 1st order system of ODEs with the vector of unknowns $\underline{\phi}^T = (q, \phi)$

the system to solve is
$$\begin{cases} q + \kappa \frac{d\phi}{dx} = 0 \\ \frac{dq}{dx} - Q = 0 \end{cases} \quad \text{in } \Omega$$

with the following approximation

$$\hat{q} = \sum_m q_m N_{m,q} \quad ; \quad \hat{\phi} = \sum_m \phi_m N_{m,\phi}$$

$$\therefore \int_0^1 N_{l,1} \kappa \frac{d\hat{\phi}}{dx} dx + \int_0^1 N_{l,1} \hat{q} dx = 0 \quad l = 1, 2, \dots, M_q$$

$$\int_0^1 N_{l,2} \frac{d\hat{q}}{dx} dx - \int_0^1 N_{l,2} Q dx = 0 \quad l = 1, 2, \dots, M_\phi$$

Example 8 : SoDE – 2nd order to 1st order

$$N_{l,1} = N_{l,2} = N_l \quad ; N_{m,q} = N_{m,\phi} = N_m$$

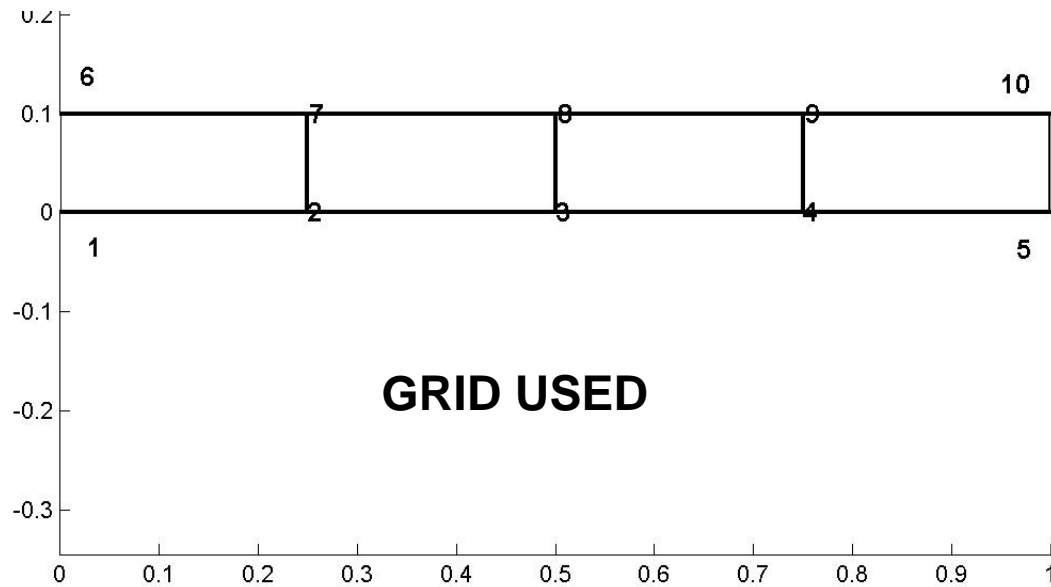
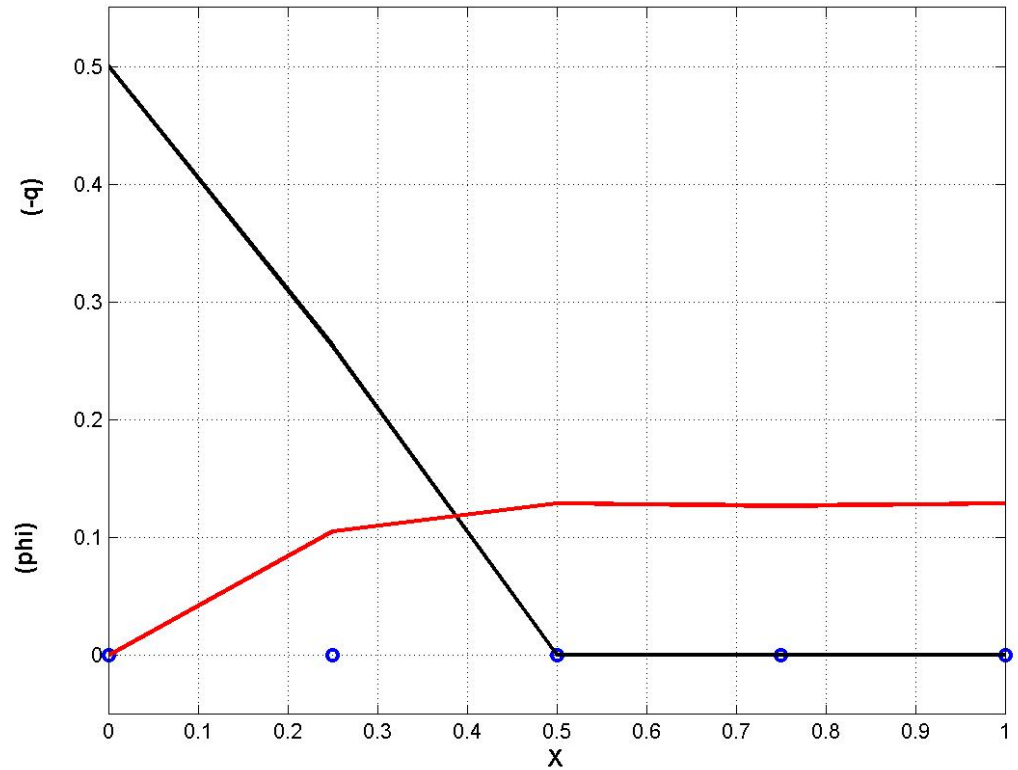
Contribution over each element

$$K^e = \int_0^1 \begin{bmatrix} N_{1,1}N_{1,q} & \kappa N_{1,1} \frac{dN_{1,\phi}}{dx} & N_{1,1}N_{2,q} & \kappa N_{1,1} \frac{dN_{2,\phi}}{dx} \\ N_{1,2} \frac{dN_{1,q}}{dx} & 0 & N_{1,2} \frac{dN_{2,q}}{dx} & 0 \\ N_{2,1}N_{1,q} & \kappa N_{2,1} \frac{dN_{1,\phi}}{dx} & N_{2,1}N_{2,q} & \kappa N_{2,1} \frac{dN_{2,\phi}}{dx} \\ N_{2,2} \frac{dN_{1,q}}{dx} & 0 & N_{2,2} \frac{dN_{2,q}}{dx} & 0 \end{bmatrix} dx$$

$$f^e = \int_0^1 \begin{bmatrix} 0 \\ N_{1,2} Q \\ 0 \\ N_{2,2} Q \end{bmatrix} dx$$

Example 8: Results

1D heat equation with piecewise linear Galerkin approximation as a system of 1st order ODEs



Example 8: Results from the book

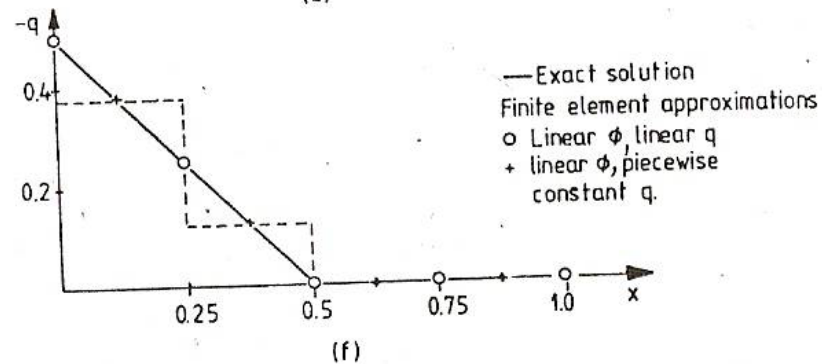
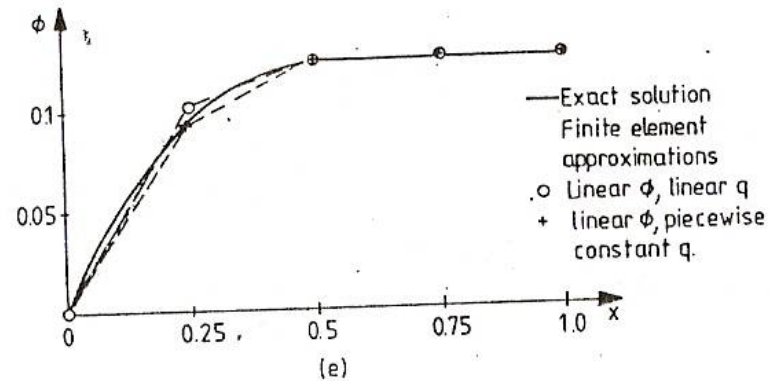
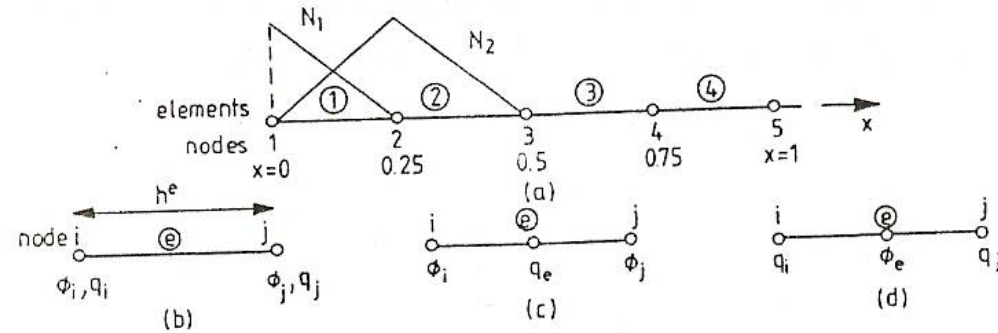


FIGURE 3.7. Solution of Example 3.3 showing (a) the node and element numbering system adopted with (b) piecewise linear ϕ and piecewise linear q . Also shown are the modifications which must be made for (c) piecewise linear ϕ , piecewise constant q , (d) piecewise constant ϕ , piecewise linear q . The graphs show a comparison of the approximations for (e) ϕ and (f) q .

Weak form and mixed formulation

**Some figure with weight and trial
Functions for different fields**

Example 8 : weak form and mixed formulation

Using a weak form for the second equation it is possible to change the approximation using piecewise constant for \hat{q} and piecewise linear for $\hat{\phi} \Rightarrow M_\phi > M_q$

$$\hat{q} = \sum_{m=1}^{M_q} q_m N_{m,q} \quad ; \quad \hat{\phi} = \sum_{m=1}^{M_\phi} \phi_m N_{m,\phi}$$

$$\therefore \int_0^1 N_{l,1} \kappa \frac{d\hat{\phi}}{dx} dx + \int_0^1 N_{l,1} \hat{q} dx = 0 \quad l = 1, 2, \dots, M_q$$

$$\int_0^1 N_{l,2} \frac{d\hat{q}}{dx} dx - \int_0^1 N_{l,2} Q dx =$$

$$- \int_0^1 \frac{dN_{l,2}}{dx} \hat{q} + \left[N_{l,2} \hat{q} \right]_{x=0}^{x=1} dx - \int_0^1 N_{l,2} Q dx = 0 \quad l = 1, 2, \dots, M_\phi$$

Contribution over each element

Weak form and mixed formulation

$$K^e = \int_0^{h^e} \begin{bmatrix} N_{1,1}N_{1,q} & \kappa N_{1,1} \frac{dN_{1,\phi}}{dx} & N_{1,1}N_{2,q} & \kappa N_{1,1} \frac{dN_{2,\phi}}{dx} \\ N_{1,2} \frac{dN_{1,q}}{dx} & 0 & N_{1,2} \frac{dN_{2,q}}{dx} & 0 \\ N_{2,1}N_{1,q} & \kappa N_{2,1} \frac{dN_{1,\phi}}{dx} & N_{2,1}N_{2,q} & \kappa N_{2,1} \frac{dN_{2,\phi}}{dx} \\ N_{2,2} \frac{dN_{1,q}}{dx} & 0 & N_{2,2} \frac{dN_{2,q}}{dx} & 0 \end{bmatrix} dx$$

at element level $M_q = 1$ (first equation) and $M_\phi = 2$ (second equation)

$\Rightarrow N_{2,1}$ does not exist anymore and $N_{2,q}$ either

$$\underline{\underline{K^e \phi^e}} = \int_0^{h^e} \begin{bmatrix} N_{1,1}N_{1,q} & \kappa N_{1,1} \frac{dN_{1,\phi}}{dx} & \kappa N_{1,1} \frac{dN_{2,\phi}}{dx} \\ N_{1,2} \frac{dN_{1,q}}{dx} & 0 & 0 \\ N_{2,2} \frac{dN_{1,q}}{dx} & 0 & 0 \end{bmatrix} dx \begin{bmatrix} q_0 \\ \phi_1 \\ \phi_2 \end{bmatrix}$$

$$\underline{\underline{K^e}} = \int_0^{h^e} \begin{bmatrix} 0 & N_q \frac{dN_{1,2}}{dx} & 0 \\ \kappa N_{2,1} \frac{dN_{1,\phi}}{dx} & N_{2,1}N_{2,q} & \kappa N_{2,1} \frac{dN_{2,\phi}}{dx} \\ 0 & N_q \frac{dN_{2,2}}{dx} & 0 \end{bmatrix} dx \begin{bmatrix} \phi_1 \\ q_0 \\ \phi_2 \end{bmatrix}$$

Weak form and mixed formulation

$$\underline{\underline{K}}^e = \int_0^{h^e} \begin{bmatrix} 0 & N_q \frac{dN_{1,2}}{dx} & 0 \\ \kappa N_{2,1} \frac{dN_{1,\phi}}{dx} & N_{2,1} N_{2,q} & \kappa N_{2,1} \frac{dN_{2,\phi}}{dx} \\ 0 & N_q \frac{dN_{2,2}}{dx} & 0 \end{bmatrix} dx \begin{bmatrix} \phi_1 \\ q_0 \\ \phi_2 \end{bmatrix}$$

$$\underline{f}^e = - \int_0^{h^e} \begin{bmatrix} N_{1,2} Q \\ 0 \\ N_{2,2} Q \end{bmatrix} dx$$

for the first and the last element the rhs should be modified as

$$\underline{f}^{e=1} = - \int_0^{h^e} \begin{bmatrix} N_{1,2} Q \\ 0 \\ N_{2,2} Q \end{bmatrix} dx + \begin{bmatrix} -\hat{q}|_{x=0} \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{f}^{e=M_q} = - \int_0^{h^e} \begin{bmatrix} N_{1,2} Q \\ 0 \\ N_{2,2} Q \end{bmatrix} dx + \begin{bmatrix} 0 \\ 0 \\ \hat{q}|_{x=1} \end{bmatrix}$$

Mixed formulation for heat equation

```
for k=1:numel
    psi = (0:20)'/20;
    node1 = icone(k,1);
    node2 = icone(k,3);
    xx = psi*(xnod(node2,1)-xnod(node1,1))+xnod(node1,1);
    eval(['Qx=' filename '(xx);' ]);
    [N,L_N] = shape_function(xx);

    Ke(k,1,2) = Ke(k,1,2) + trapz(xx,L_N(:,1));
    Ke(k,3,2) = Ke(k,3,2) + trapz(xx,L_N(:,2));
    Ke(k,2,1) = Ke(k,2,1) + kappa*trapz(xx,L_N(:,1));
    Ke(k,2,3) = Ke(k,2,3) + kappa*trapz(xx,L_N(:,2));
    Ke(k,2,2) = Ke(k,2,2) + 1*trapz(xx,ones(size(xx)));

    fe(k,1) = fe(k,1) - trapz(xx,Qx.*N(:,1));
    fe(k,3) = fe(k,3) - trapz(xx,Qx.*N(:,2));

end
```

Assembling

Mixed formulation for heat equation

```
for k=1:numel
    node_i = icone(k,1);
    node_j = icone(k,3);
    node_e = icone(k,2);
    Kg(node_i,node_e)=Kg(node_i,node_e) + Ke(k,1,2);
    Kg(node_j,node_e)=Kg(node_j,node_e) + Ke(k,3,2);
    Kg(node_e,node_i)=Kg(node_e,node_i) + Ke(k,2,1);
    Kg(node_e,node_j)=Kg(node_e,node_j) + Ke(k,2,3);
    Kg(node_e,node_e)=Kg(node_e,node_e) + Ke(k,2,2);

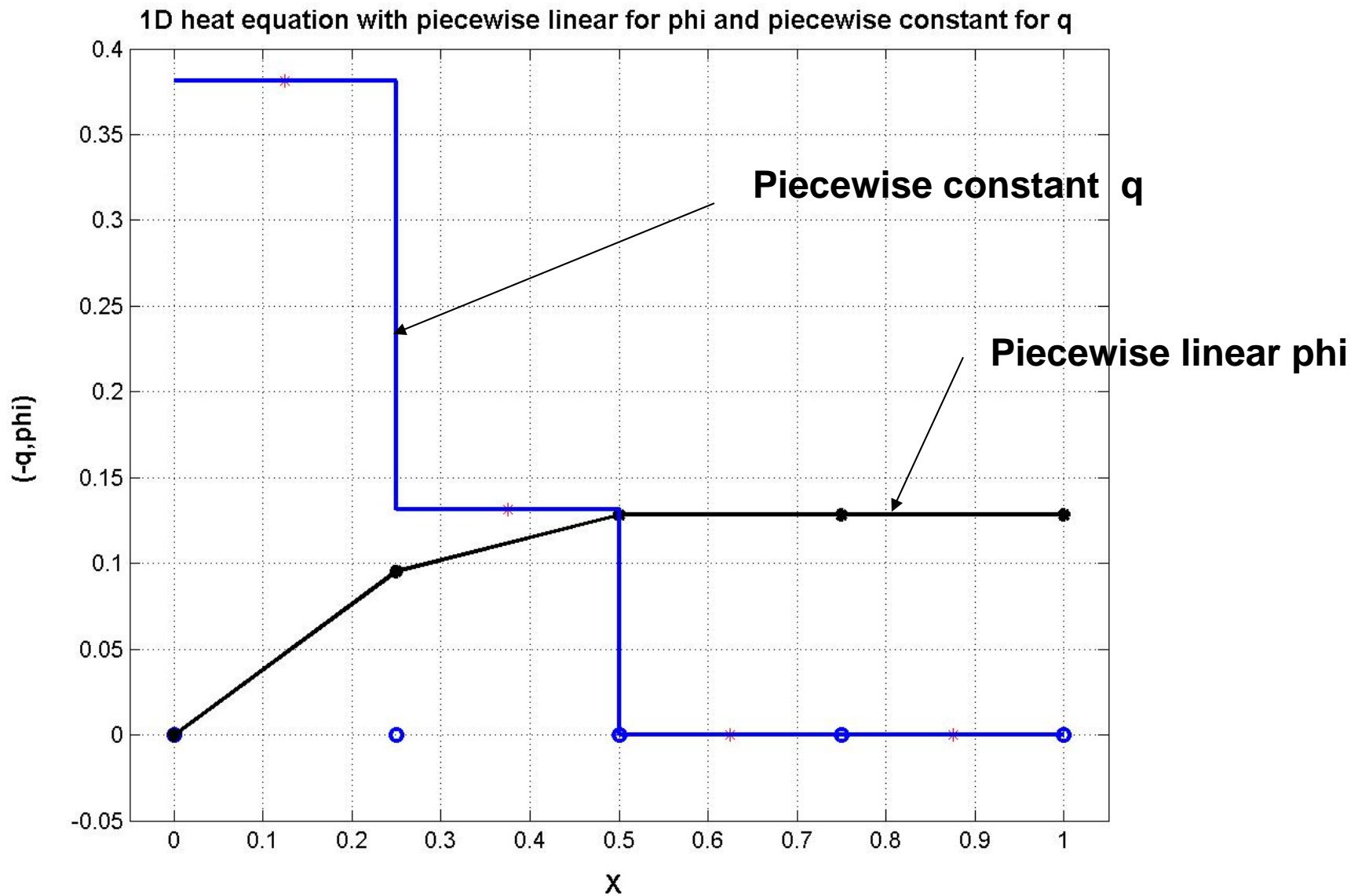
    fg(node_i,1) = fg(node_i,1) + fe(k,1);
    fg(node_e,1) = fg(node_e,1) + fe(k,2);
    fg(node_j,1) = fg(node_j,1) + fe(k,3);
end
```

Gathering

```
% reducing the global system for fixations
free = (1:2*Nx+1)'; free(1) = [];
fg(free,1) = fg(free,1) - Kg(free,1) * phi_lef ;
Kg = Kg(free,free);
fg = fg(free,1);
% solver
phi_new = Kg\fg;
phi = [phi_lef;phi_new];
```

Solving

Mixed formulation - Results

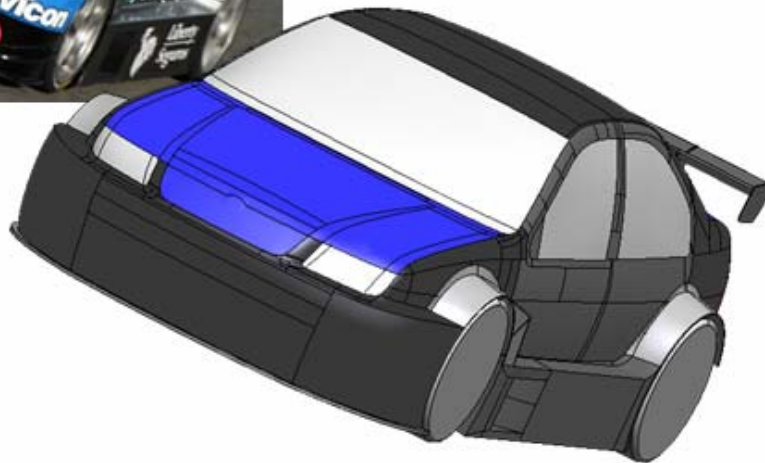


Generalization to 2D and 3D problems

- *1D problems are attractive to check numerical schemes (exact solution availability)*
- *1D problems have little practical (industrial) interests*
- *2D or 3D problems have analytical solution only for very simple geometries and boundary conditions*
- *In general only numerical methods may be used*
- *In 2D and 3D the choice of the shape functions depends on the equation to be solved and the type of element being used.*
- *Triangular and quadrangular types are commonly used (not necessarily restricted to these).*
- *In 3D tetrahedra are widely used because of mesh generators availability.*
- *Hexahedra are less common but they are attractive for their accuracy if the geometry allows for them.*
- *Prismatic elements close to bodies where boundary layer are expected may be included.*
- *2D & 3D the idea is to cover the whole domain with simple non-overlapping elements*

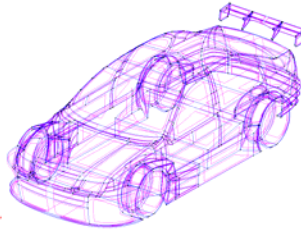
Example of 3D flow analysis

Aerodynamic analysis of a race car
The importance of computational geometry

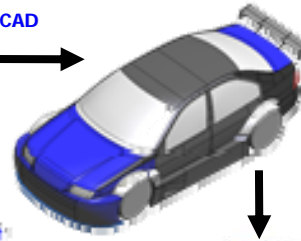




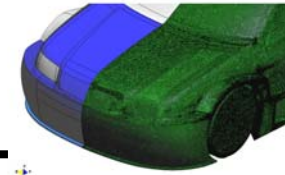
FOTOGRAMETRIA



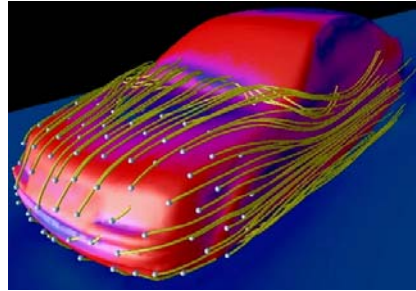
CAD



GENERACION DE MALLAS

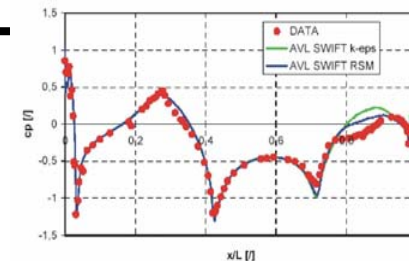


CFD & VISUALIZACION

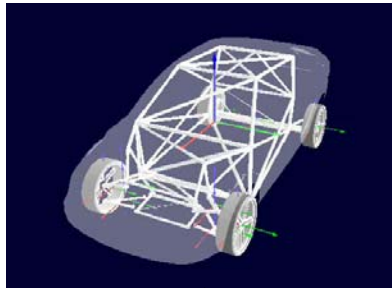


CARGAS AERODINAMICAS

CURVAS CFD



CAD
CHASIS, ESTRUCTURA &
SUSPENSION



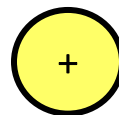
MODELO
DINAMICO



MANIOBRAS & CIRCUITOS

REALIMENTACION
AL MODELO

RESULTADOS
DINAMICOS

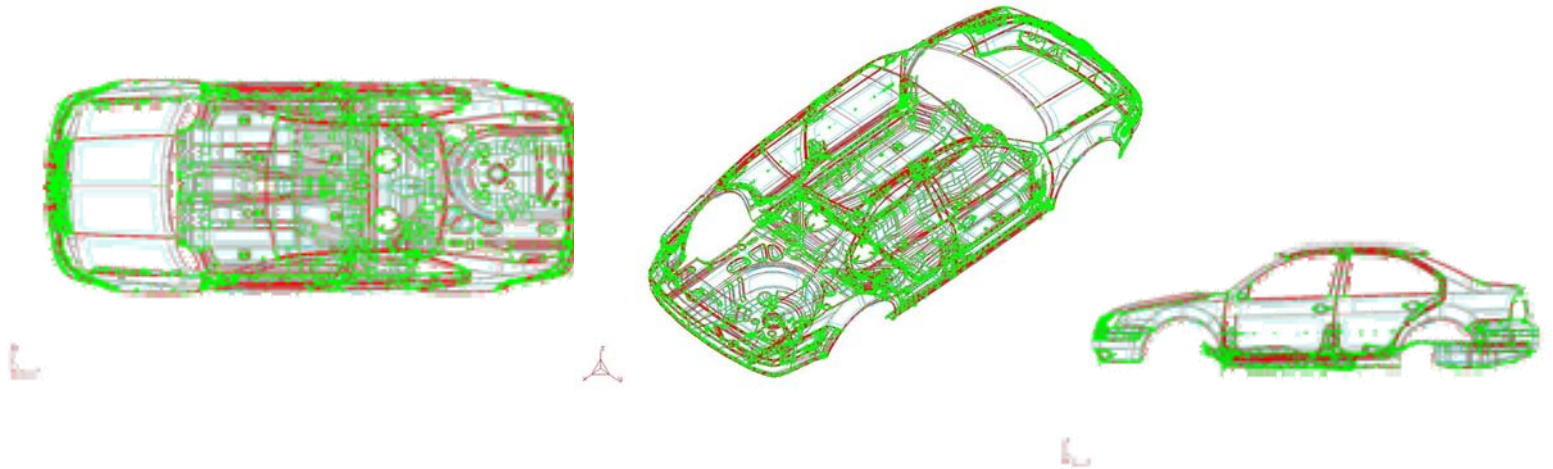


TELEMETRIA

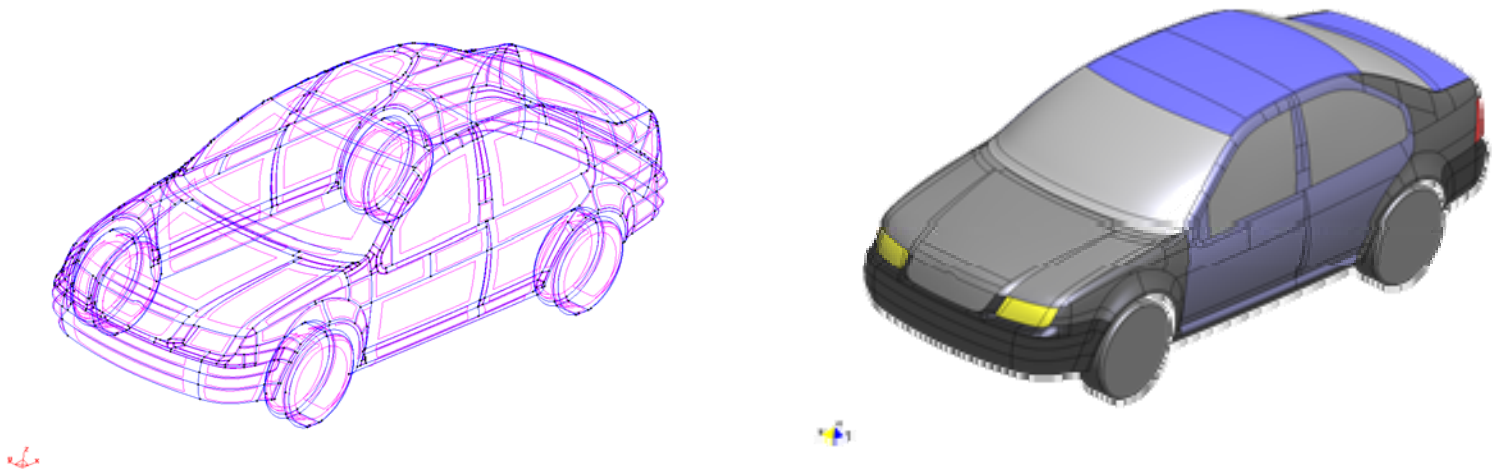


CAD – From production to analysis

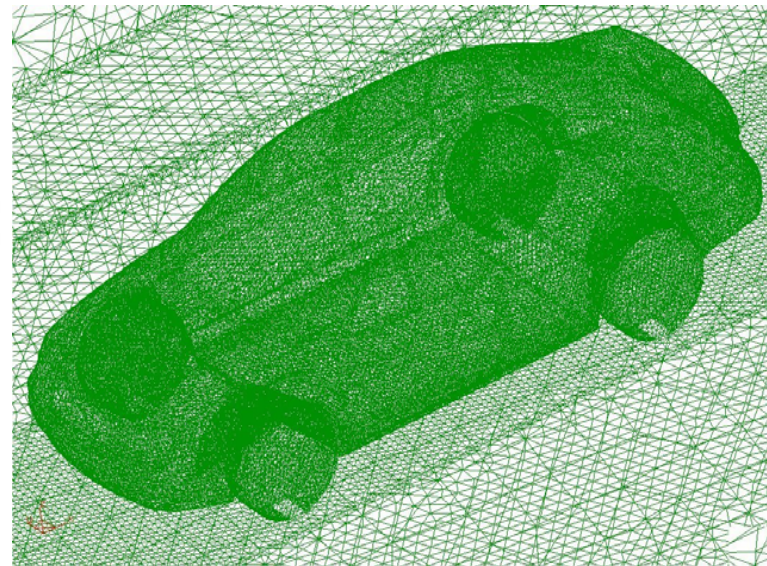
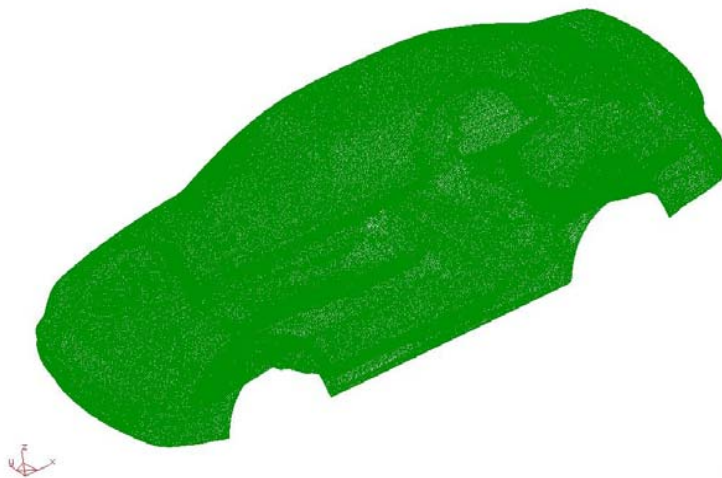
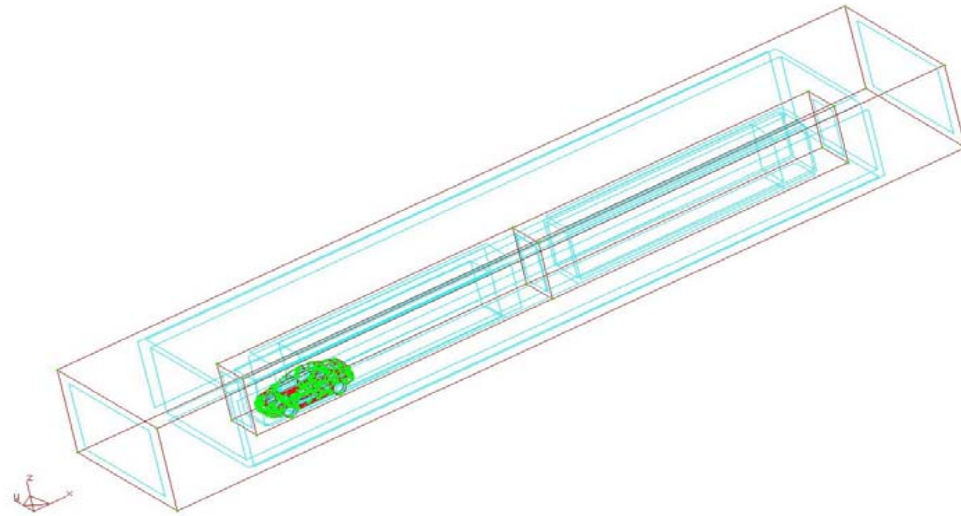
Geometría relevada



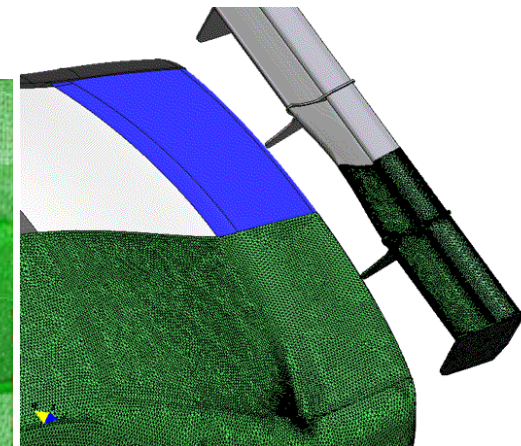
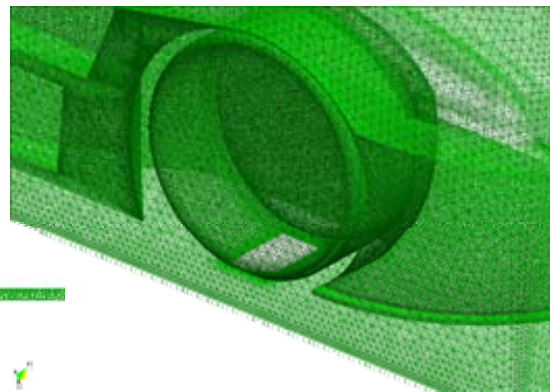
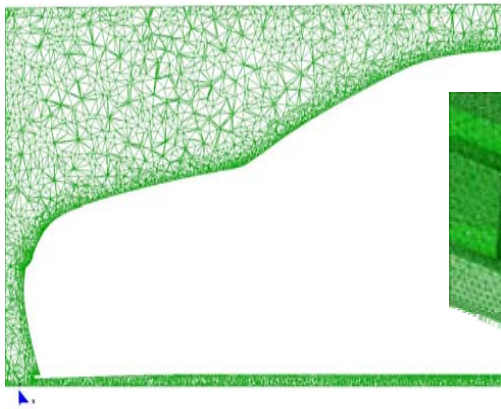
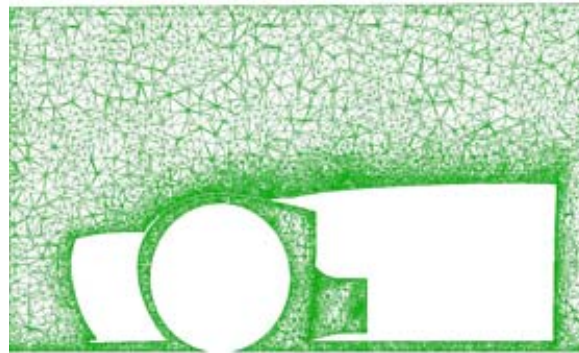
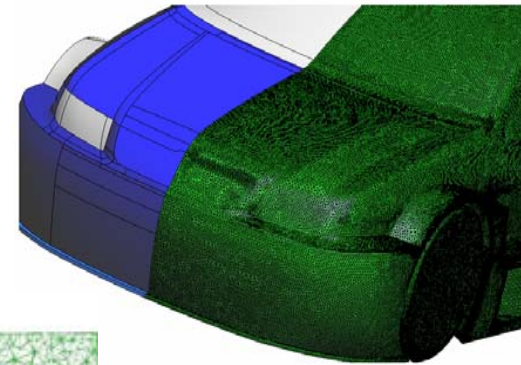
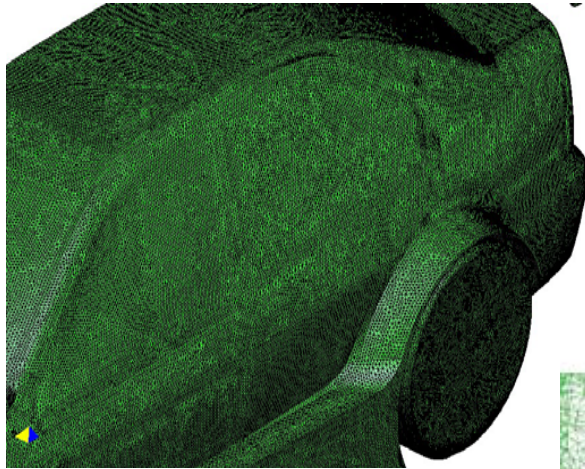
Geometría simplificada (limpia)



Geometrical model & mesh generation



mesh generation details



The linear triangle

Using piecewise linear over triangular elements

$$N_i^e(x, y) = \begin{cases} 1 & \text{for } (x, y) = (x, y)_i \\ 0 & \text{for } (x, y) = (x, y)_j = (x, y)_k \neq (x, y)_i \end{cases}$$

$$N_i^e(x, y) = \alpha_i^e + \beta_i^e x + \gamma_i^e y \quad \text{on element } e$$

How to get $\alpha_i^e; \beta_i^e; \gamma_i^e$?

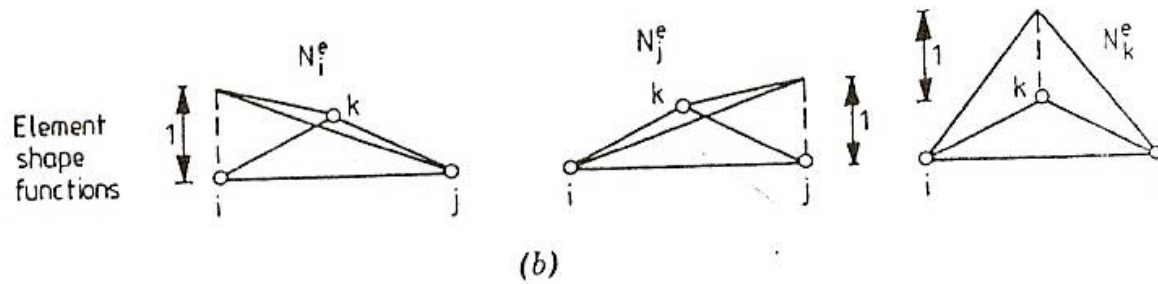
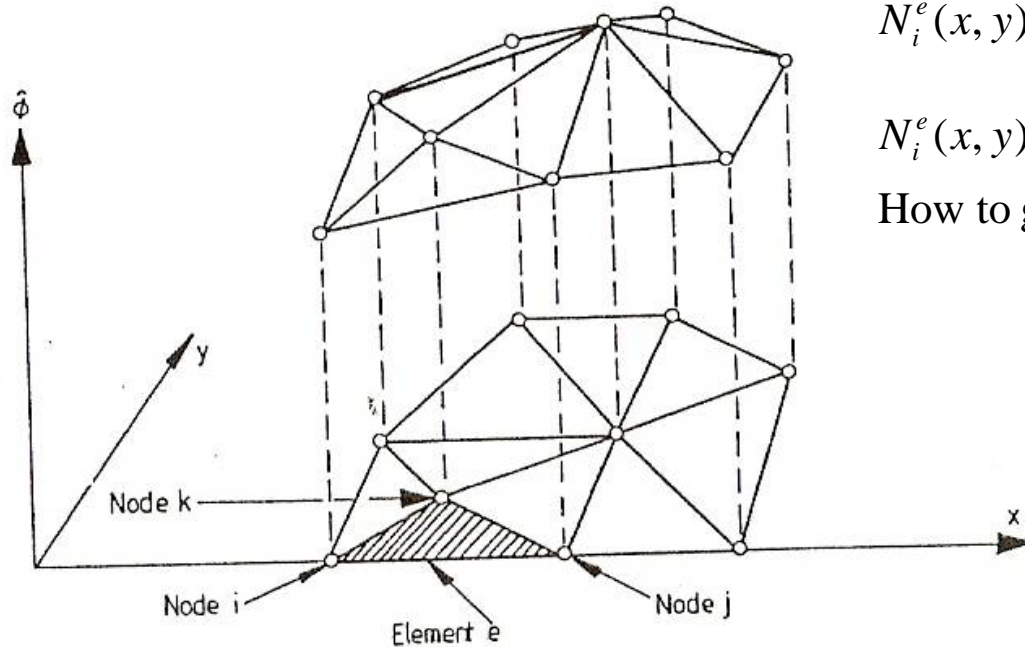


FIGURE 3.2. (continued).

The linear triangle

$$\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{bmatrix} \alpha_i^e \\ \beta_i^e \\ \gamma_i^e \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

with solution

$$\alpha_i^e = \frac{x_j y_k - x_k y_j}{2\Delta^e}$$

$$\beta_i^e = \frac{y_j - y_k}{2\Delta^e}$$

$$\gamma_i^e = \frac{x_k - x_j}{2\Delta^e}$$

$$2\Delta^e = \det \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} = 2A^e = 2(\text{area of element } e)$$

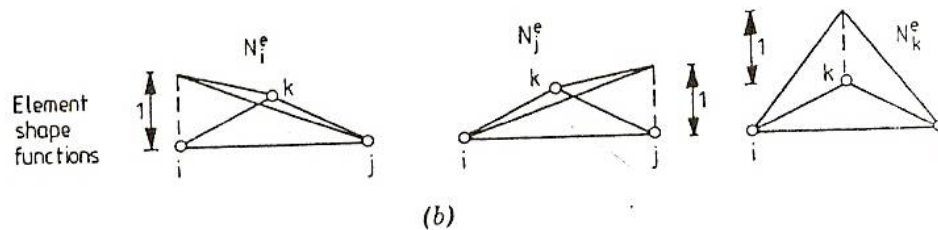
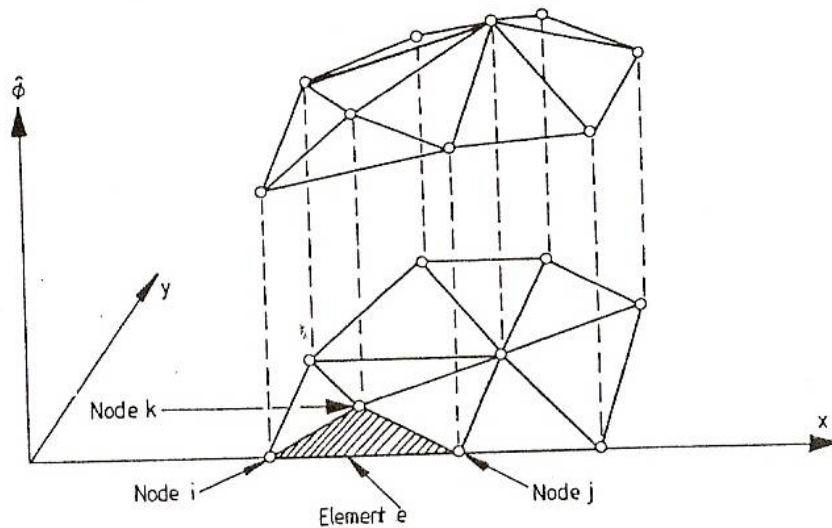
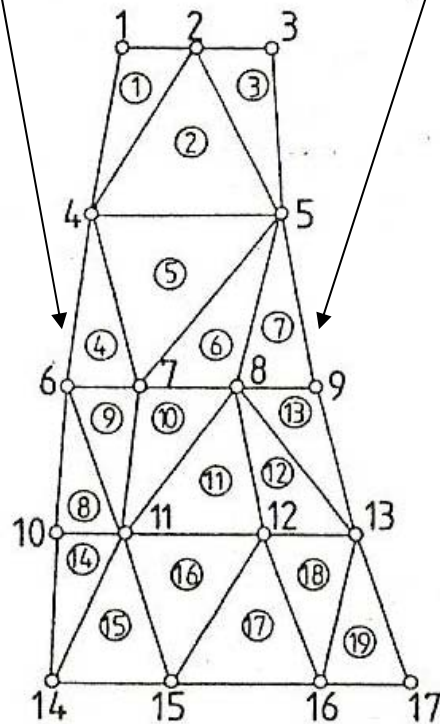


FIGURE 3.2. (continued).

The linear triangle – How the numbering impact on the algebraic structure of the stiffness matrix ?

Boundaries approximated by straight lines



* Non zeros terms

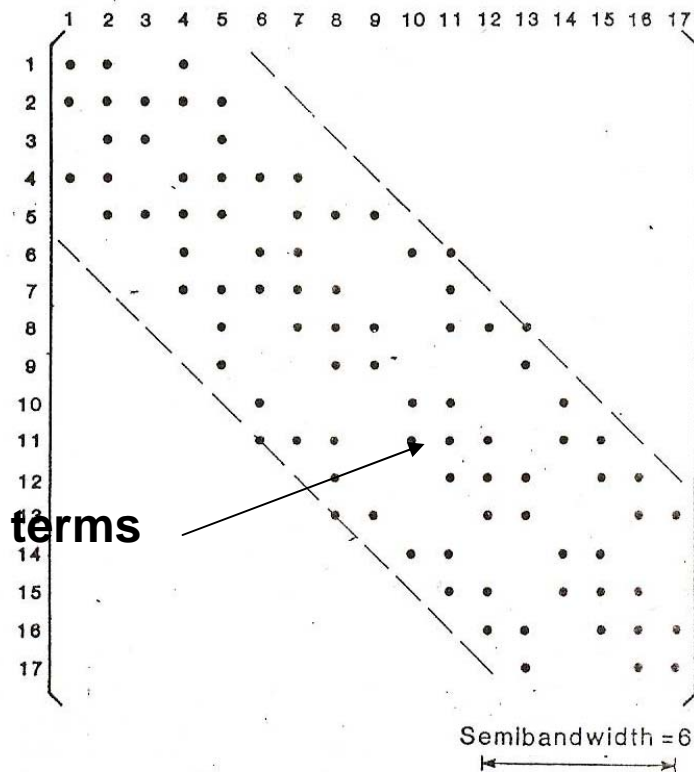


FIGURE 3.11. Position of the nonzero entries in the matrix K resulting from the assembly of the contributions from the elements shown in Fig. 3.10.

FIGURE 3.10. Triangular finite elements used to represent the profile of a dam.

The linear triangle – How the numbering impact on the algebraic structure of the stiffness matrix ?

- The numbering influences the bandwidth
- The bandwidth is closely related with the computational cost and memory requirements.
- Sparse matrices are found due to the narrow-based trial functions
- The computation may be divided in three main steps

- Element assembling
- Gathering
- Solver

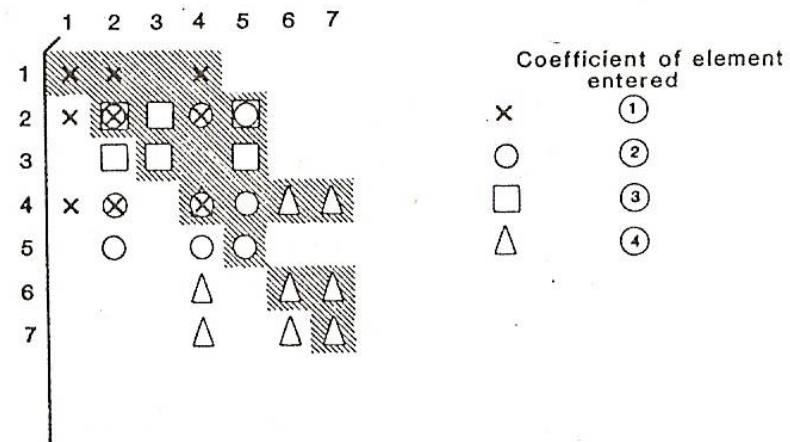


FIGURE 3.12. Process of assembly of elements 1–4 in the problem of Fig. 3.10. The shaded area represents the information that must be stored if the matrix is symmetric.

The linear triangle – How the numbering impact on the algebraic structure of the stiffness matrix ?

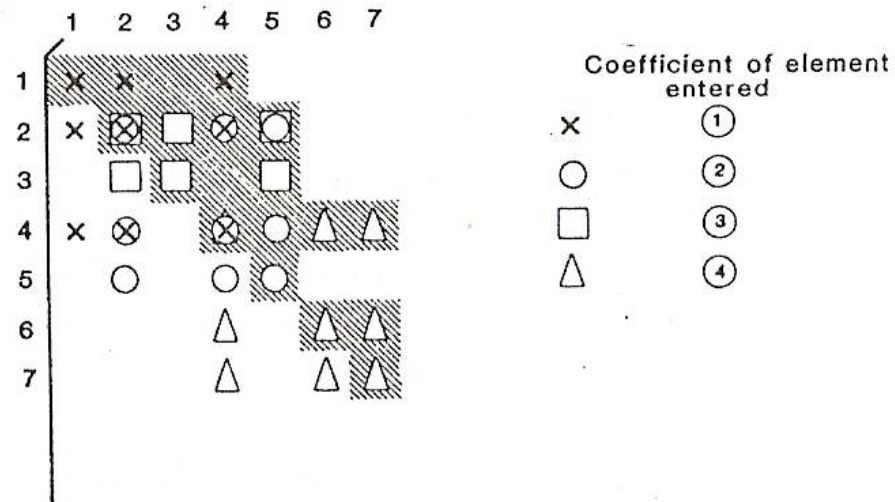
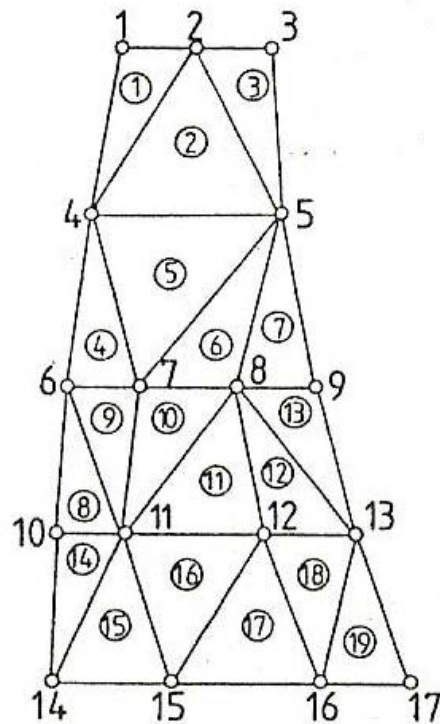


FIGURE 3.12. Process of assembly of elements 1-4 in the problem of Fig. 3.10. The shaded area represents the information that must be stored if the matrix is symmetric.

FIGURE 3.10. Triangular finite elements used to represent the profile of a dam.

The bilinear rectangle

A rectangle is a sort of cartesian product of linear segments

The cartesian product of linear trial functions in 1D

produces a bilinear 2D or a trilinear 3D shape functions

$$N_i^e(x, y) = \frac{(h_x - x)}{h_x} \frac{(h_y - y)}{h_y}$$

$$\therefore N_i^e(x, y) = \begin{cases} 1 & \text{for } (x, y) = (x, y)_i \\ 0 & \text{for } (x, y) = (x, y)_j = (x, y)_k = (x, y)_l \neq (x, y)_i \end{cases}$$

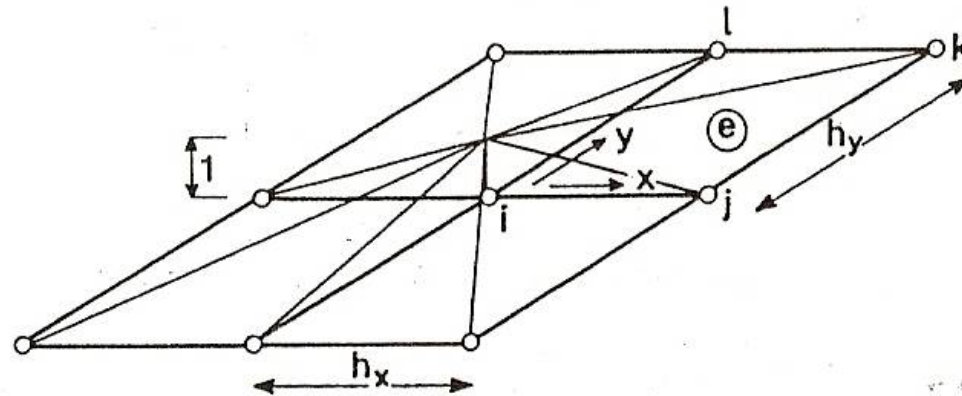


FIGURE 3.13. Bilinear shape function associated with node i of a typical four-noded rectangular element.

The bilinear rectangle

$$N_i^e(x, y) = \alpha_i^e + \beta_i^e x + \gamma_i^e y + \delta_i^e xy \quad \text{on element } e$$

How to get $\alpha_i^e; \beta_i^e; \gamma_i^e; \delta_i^e$?

$$\begin{bmatrix} 1 & x_i & y_i & x_i y_i \\ 1 & x_j & y_j & x_j y_j \\ 1 & x_k & y_k & x_k y_k \\ 1 & x_l & y_l & x_l y_l \end{bmatrix} \begin{bmatrix} \alpha_i^e \\ \beta_i^e \\ \gamma_i^e \\ \delta_i^e \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

**Low order (linear) and
low regularity (C0) of shape function**

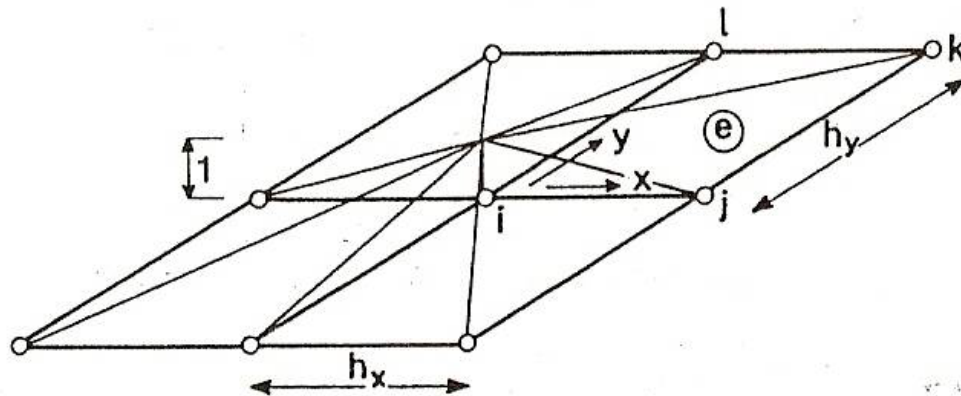


FIGURE 3.13. Bilinear shape function associated with node i of a typical four-noded rectangular element.

Approximation of curved boundaries

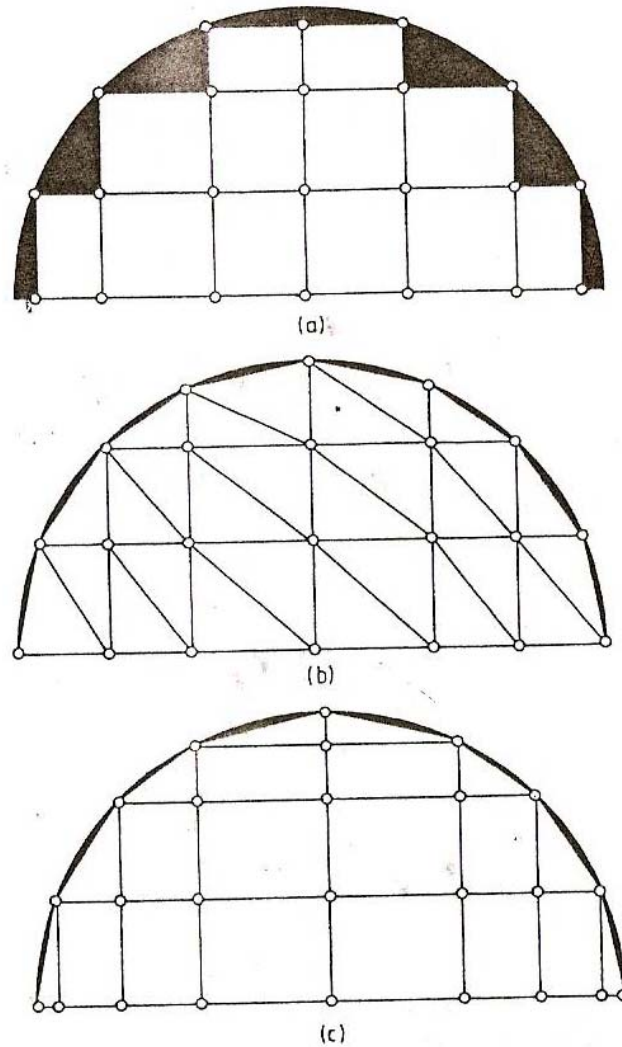


FIGURE 3.14. Finite element subdivision of a semicircular region using (a) rectangular elements only, (b) triangular elements with the same total number of nodes, (c) a combination of triangular and rectangular elements with additional boundary nodes. In each case the dark areas indicate the magnitude of the error made in the representation of the domain.

3D elements – tetrahedron and hexahedron

$$N_i^e(x, y, z) = \begin{cases} 1 & \text{for } (x, y, z) = (x, y, z)_i \\ 0 & \text{for } (x, y, z) = (x, y, z)_j = (x, y, z)_k = (x, y, z)_l \neq (x, y, z)_i \end{cases}$$

$$N_i^e(x, y, z) = \alpha_i^e + \beta_i^e x + \gamma_i^e y + \delta_i^e z \quad \text{on element } e$$

