

Approximation to PDE by continuous trial functions

- Defining the PDE problem
- Introduction to the PDE approximation
- Approximation by weighted residual
- Simultaneous approximation to the solution of the PDE in the domain and on its boundary.
- Natural boundary conditions
- Boundary solution methods
- Systems of PDE
- Nonlinear problems

Defining the PDE problem

$$A(\phi) = 0 \quad \text{in } \Omega$$

L : linear differential operator

$$p \neq p(\phi)$$

$$\mathsf{L}(\phi) = \frac{\partial}{\partial x} \left(\kappa \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\kappa \frac{\partial \phi}{\partial y} \right)$$

$$p = Q$$

$$\kappa \neq \kappa(\phi)$$

$$B(\phi) = \mathsf{M}(\phi) + \mathbf{r} = 0 \quad \text{in } \Gamma$$

$$\begin{cases} \mathsf{M}(\phi) = \phi & ; \quad r = -\bar{\phi} \quad \text{on } \Gamma_\phi \quad \text{DIRICHLET} \\ \mathsf{M}(\phi) = -\kappa \frac{\partial \phi}{\partial \eta} & ; \quad r = -\bar{q} \quad \text{on } \Gamma_q \quad \text{NEUMANN} \end{cases}$$

Introduction to the PDE approximation

How to choose the trial functions

$$\phi(x) \cong \hat{\phi}(x) = \psi(x) + \sum_m a_m N_m(x)$$

$$\text{on } \Gamma \begin{cases} \mathcal{M}(\psi) = -r \\ \mathcal{M}(N_m) = 0 \quad ; \quad m = 1, 2, \dots \end{cases}$$

$$\hat{\phi}(x) = \phi(x) \quad \text{for all } x \in \Gamma$$

Introduction to the PDE approximation

The trial functions and its derivatives

$$\phi(x) \cong \hat{\phi}(x) = \psi(x) + \sum_m a_m N_m(x)$$

$$\mathcal{L}\hat{\phi} = \mathcal{L}\psi + \sum_m a_m \mathcal{L}N_m$$

$$\frac{\partial \phi}{\partial x} \cong \frac{\partial \hat{\phi}}{\partial x} = \frac{\partial \psi}{\partial x} + \sum_m a_m \frac{\partial N_m}{\partial x}$$

$$\frac{\partial^2 \phi}{\partial x} \cong \frac{\partial^2 \hat{\phi}}{\partial x} = \frac{\partial^2 \psi}{\partial x} + \sum_m a_m \frac{\partial^2 N_m}{\partial x}$$

N_m has the regularity imposed by \mathcal{L}

Approximation by weighted residual

$$R_{\Omega} = A(\phi) = \mathcal{L}\hat{\phi} + p = \mathcal{L}\psi + \sum_m a_m \mathcal{L}N_m + p$$

$$\int_{\Omega} W_l R_{\Omega} d\Omega = \int_{\Omega} W_l \left(\mathcal{L}\psi + \sum_m a_m \mathcal{L}N_m + p \right) d\Omega = 0$$

$$\int_{\Omega} W_l \left(\sum_m a_m \mathcal{L}N_m \right) d\Omega = - \int_{\Omega} W_l (\mathcal{L}\psi + p) d\Omega$$

$$K_{lm} a_m = f_l$$

$$K_{lm} = \int_{\Omega} W_l \mathcal{L}N_m d\Omega$$

$$f_l = - \int_{\Omega} W_l (\mathcal{L}\psi + p) d\Omega \quad l, m = 1, 2, \dots, M$$

Point collocation

Weight function $W_l = \delta(x - x_l)$

$$\begin{aligned} \int_{\Omega} W_l R_{\Omega} d\Omega &= \int_{\Omega} \delta(x - x_l) (\mathcal{L}\hat{\phi} + p) d\Omega = \\ &= \int_{\Omega} \delta(x - x_l) \left(\left(\mathcal{L}\psi + \sum_m a_m \mathcal{LN}_m \right) + p \right) d\Omega \\ \therefore \quad \int_{\Omega} \delta(x - x_l) \left(\sum_m a_m \mathcal{LN}_m \right) d\Omega &= - \int_{\Omega} \delta(x - x_l) (\mathcal{L}\psi + p) d\Omega \\ K_{lm} \ a_m &= f_l \\ K_{lm} = \int_{\Omega} \delta(x - x_l) \mathcal{LN}_m \ d\Omega &= \mathcal{LN}_m(x = x_l) \\ f_l = - \int_{\Omega} \delta(x - x_l) (\mathcal{L}\psi + p) d\Omega &= -(\mathcal{L}\psi(x = x_l) + p(x = x_l)) \end{aligned}$$

Subdomain collocation

Weight function $W_l = \chi_{\Omega_l} = \begin{cases} 1 & x \in \Omega_l \\ 0 & x \notin \Omega_l \end{cases}$

$$\begin{aligned} \int_{\Omega} W_l R_{\Omega} d\Omega &= \int_{\Omega} \chi_{\Omega_l} (\mathcal{L}\hat{\phi} + p) d\Omega = \\ &= \int_{\Omega} \chi_{\Omega_l} \left(\left(\mathcal{L}\psi + \sum_m a_m \mathcal{L}N_m \right) + p \right) d\Omega \\ \therefore \quad \int_{\Omega} \chi_{\Omega_l} \left(\sum_m a_m \mathcal{L}N_m \right) d\Omega &= - \int_{\Omega} \chi_{\Omega_l} (\mathcal{L}\psi + p) d\Omega \end{aligned}$$

$$K_{lm} \ a_m = f_l$$

$$K_{lm} = \int_{\Omega} \chi_{\Omega_l} \mathcal{L}N_m d\Omega = \int_{\Omega_l} \mathcal{L}N_m d\Omega$$

$$f_l = - \int_{\Omega} \chi_{\Omega_l} (\mathcal{L}\psi + p) d\Omega = - \int_{\Omega_l} (\mathcal{L}\psi + p) d\Omega$$

Galerkin

Weight function $W_l = N_l$

$$\begin{aligned} \int_{\Omega} W_l R_{\Omega} d\Omega &= \int_{\Omega} N_l (\mathbf{L} \hat{\phi} + p) d\Omega = \\ &= \int_{\Omega} N_l \left(\left(\mathbf{L} \psi + \sum_m a_m \mathbf{L} N_m \right) + p \right) d\Omega \\ \therefore \quad \int_{\Omega} N_l \left(\sum_m a_m \mathbf{L} N_m \right) d\Omega &= - \int_{\Omega} N_l (\mathbf{L} \psi + p) d\Omega \end{aligned}$$

$$K_{lm} \ a_m = f_l$$

$$K_{lm} = \int_{\Omega} N_l \mathbf{L} N_m \ d\Omega$$

$$f_l = - \int_{\Omega} N_l (\mathbf{L} \psi + p) d\Omega$$

Example 1- Definition

Find $\hat{\phi}(x)$ solution of the following ODE

$$\frac{d^2\phi}{dx^2} - \phi = 0 \quad \text{in } \Omega : \{x; 0 \leq x \leq 1\}$$

$$\phi(x=0) = 0$$

$$\phi(x=1) = 1 \quad \text{in } \Gamma : \{x = 0, x = 1\}$$

$$\mathcal{L}(\phi) = \frac{d^2\phi}{dx^2} - \phi$$

$$p = 0$$

$$\mathcal{M}(\phi) + r = 0 \quad \text{in } \Gamma$$

$$\begin{cases} \mathcal{M}(\phi) = \phi & ; \quad r = 0 \quad \text{on } x = 0 \quad \text{DIRICHLET} \\ \mathcal{M}(\phi) = \phi & ; \quad r = -1 \quad \text{on } x = 1 \quad \text{DIRICHLET} \end{cases}$$

Example 1- Choosing the approximation

$$\phi(x) \cong \hat{\phi}(x) = \psi(x) + \sum_m a_m N_m(x)$$

$$\psi(x) = x$$

$$N_m(x) = \sin(m\pi x)$$

$$\psi(x=0) = \phi(x=0) = 0$$

$$N_m(x=0) = 0$$

$$\psi(x=1) = \phi(x=1) = 1$$

$$N_m(x=1) = 0$$

$$\mathcal{L}\hat{\phi} = \mathcal{L}\psi + \sum_m a_m \mathcal{L}N_m$$

$$\mathcal{L}\psi = \frac{d^2\psi}{dx^2} - \psi = 0 - x = -x$$

$$\mathcal{L}N_m = \frac{d^2N_m}{dx^2} - N_m = (m\pi)^2(-\sin(m\pi x)) - \sin(m\pi x)$$

Example 1- Weighted residual using collocation

$$K_{lm} a_m = f_l$$

$$K_{lm} = \mathcal{L}N_m(x = x_l)$$

$$\mathcal{L}N_m = \sin(m\pi x) \left(-1 - (m\pi)^2 \right)$$

$$K_{lm} = \sin(m\pi x_l) \left(-1 - (m\pi)^2 \right)$$

$$\mathcal{L}\psi = -x \quad ; \quad p = 0$$

$$f_l = -(-x_l + 0) = x_l$$

$$\therefore a_m = (K^{-1}f)_m$$

Example 1- Weighted residual using collocation (see routine Ej_2_1.m)

$$K_{lm} \ a_m = f_l$$

$$K_{lm} = -\sin(m\pi x_l)(1 + (m\pi)^2) \quad ; \quad f_l = x_l$$

$$\therefore K = - \begin{vmatrix} \sin(\pi x_1)(1 + \pi^2) & \sin(2\pi x_1)(1 + 4\pi^2) & \cdots & \sin(M\pi x_1)(1 + M^2\pi^2) \\ \sin(\pi x_2)(1 + \pi^2) & \sin(2\pi x_2)(1 + 4\pi^2) & \cdots & \sin(M\pi x_2)(1 + M^2\pi^2) \\ \vdots & \vdots & \ddots & \vdots \\ \sin(\pi x_M)(1 + \pi^2) & \sin(2\pi x_M)(1 + 4\pi^2) & \cdots & \sin(M\pi x_M)(1 + M^2\pi^2) \end{vmatrix}$$

$$f = \begin{vmatrix} x_1 & x_2 & \cdots & x_M \end{vmatrix}^T$$

Example 1- Weighted residual using Galerkin (see routine Ej_2_1.m)

$$K_{lm} a_m = f_l$$

$$K_{lm} = - \int_0^1 \sin(l\pi x) \sin(m\pi x) (1 + (m\pi)^2) dx ; \quad f_l = \int_0^1 x \sin(l\pi x) dx$$

$$\therefore K = \begin{vmatrix} \int_0^1 \sin(\pi x) \sin(\pi x) (1 + \pi^2) dx & \int_0^1 \sin(\pi x) \sin(2\pi x) (1 + 4\pi^2) dx & \cdots & \int_0^1 \sin(\pi x) \sin(M\pi x) (1 + M^2\pi^2) dx \\ \int_0^1 \sin(2\pi x) \sin(\pi x) (1 + \pi^2) dx & \int_0^1 \sin(2\pi x) \sin(2\pi x) (1 + 4\pi^2) dx & \cdots & \int_0^1 \sin(2\pi x) \sin(M\pi x) (1 + M^2\pi^2) dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^1 \sin(M\pi x) \sin(\pi x) (1 + \pi^2) dx & \int_0^1 \sin(M\pi x) \sin(2\pi x) (1 + 4\pi^2) dx & \cdots & \int_0^1 \sin(M\pi x) \sin(M\pi x) (1 + M^2\pi^2) dx \end{vmatrix}$$

$$f = \begin{bmatrix} \int_0^1 x \sin(\pi x) dx & \int_0^1 x \sin(2\pi x) dx & \cdots & \int_0^1 x \sin(M\pi x) dx \end{bmatrix}^T$$

$$\text{since } \int_0^1 \sin(l\pi x) \sin(m\pi x) dx = \frac{1}{2} \delta_{lm} \quad \text{by orthogonality}$$

$$\text{and } \int_0^1 x \sin(l\pi x) dx = (-1)^{l+1} \frac{1}{l\pi}$$

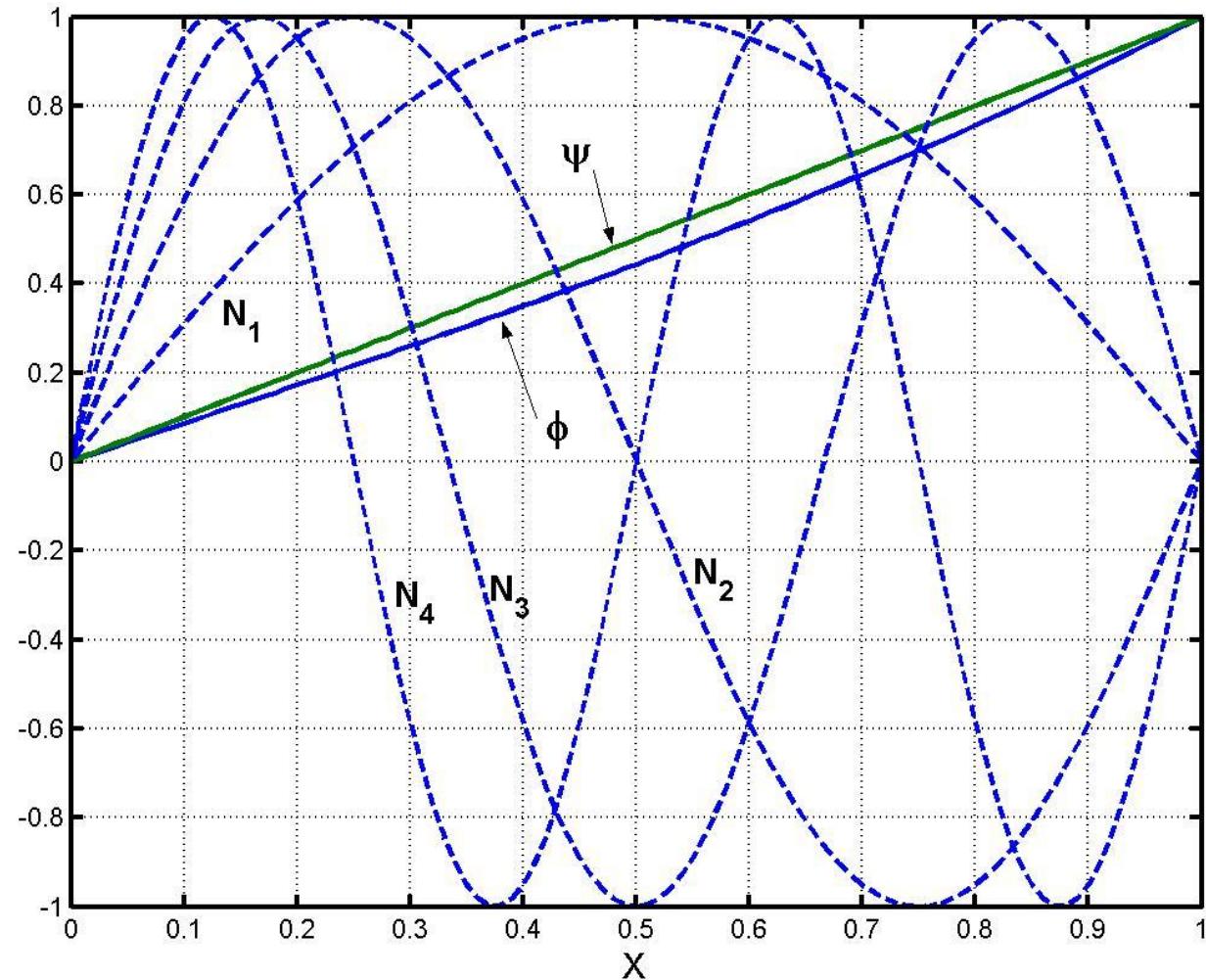
$$\therefore \frac{1}{2} \delta_{lm} (1 + m^2\pi^2) a_m = (-1)^m \frac{1}{m\pi} \Rightarrow a_m = (-1)^m \frac{2}{m\pi (1 + m^2\pi^2)}$$

Example 1- Exact solution and approximation basis

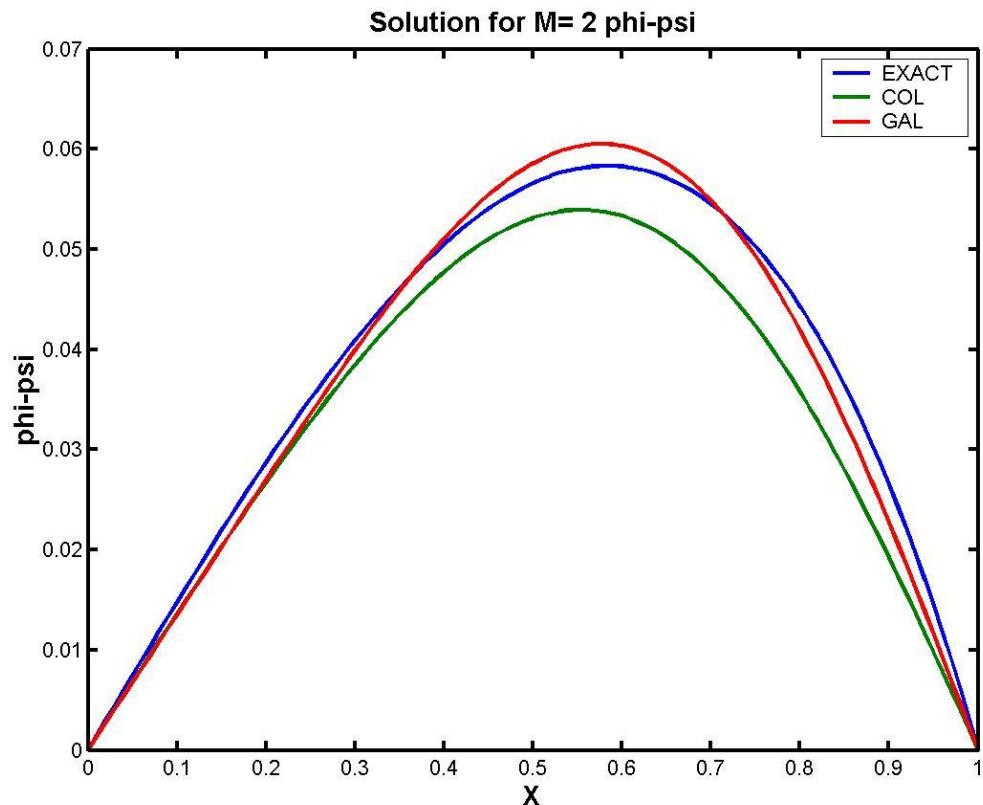
$$\phi^{\text{exact}} = \frac{e^x - e^{-x}}{e^1 - e^{-1}}$$

$$N_m = \sin(m\pi x)$$

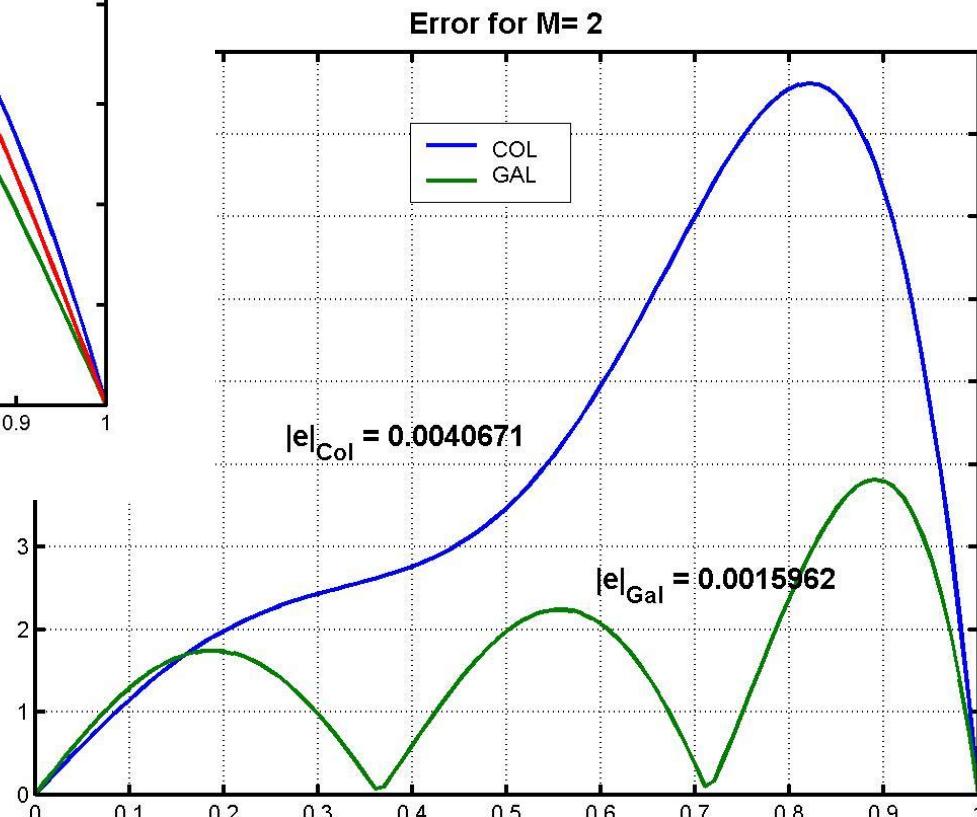
$$\psi = x$$



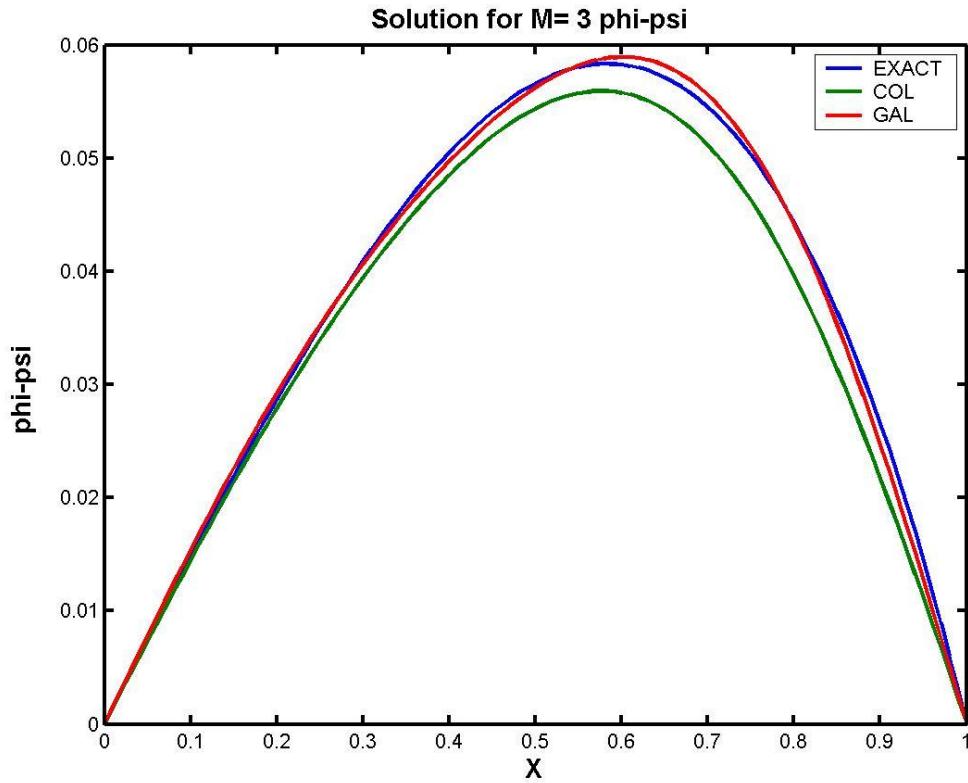
Example 1- Results using point collocation & Galerkin



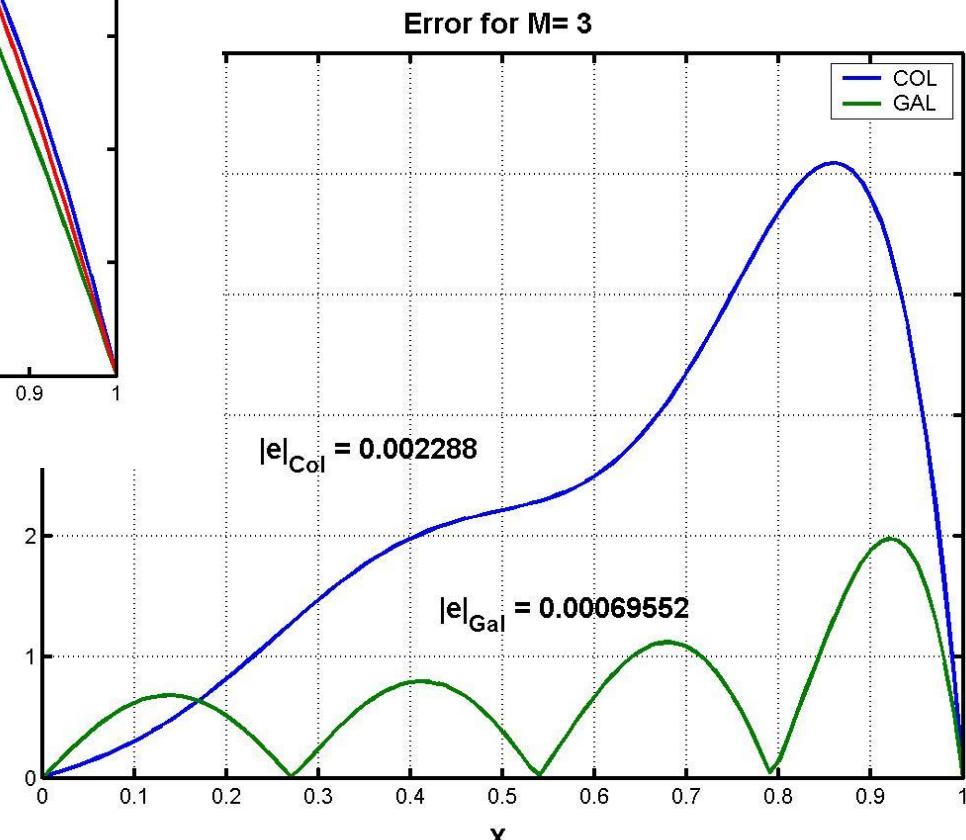
using $M = 2$



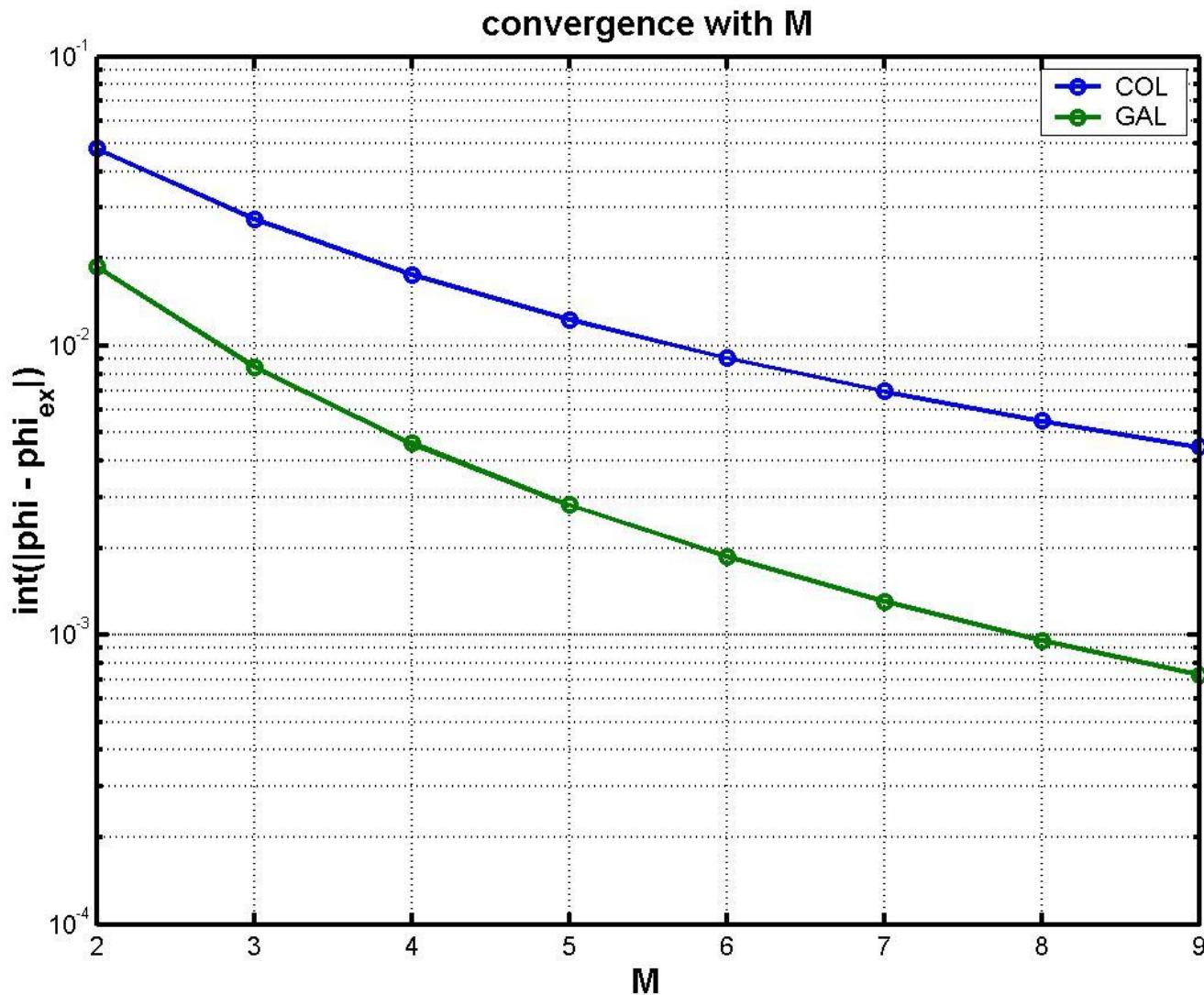
Example 1- Results using point collocation & Galerkin



using $M = 3$



Example 1- Results using point collocation & Galerkin



Example 2- A 2D scalar elastic problem - Definition

Find $\hat{\phi}(x)$ solution of the following PDE

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta \quad \text{in } \Omega : \{(x, y); -3 \leq x \leq 3; -2 \leq y \leq 2\}$$

$$\phi(x = -3) = \phi(x = 3) = 0$$

$$\phi(y = -2) = \phi(y = 2) = 0$$

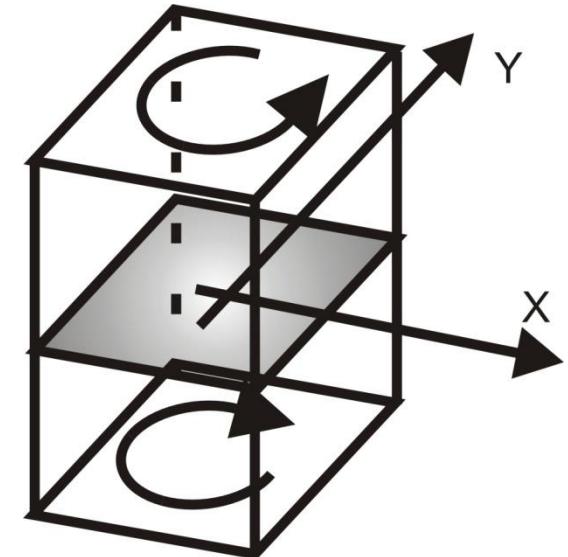
$$G\theta = 1 \quad G : \text{shear modulus} ; \theta : \text{torsion angle}$$

$$\mathcal{L}(\phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

$$p = 2$$

$$\mathcal{M}(\phi) = \phi \quad ; \quad r = 0$$

on $(x = -3) \cup (x = 3) \cup (y = -2) \cup (y = 2)$ DIRICHLET



Example 2- Choosing the approximation

$$\phi(x, y) \cong \hat{\phi}(x, y) = \psi(x, y) + \sum_m a_m N_m(x, y)$$

$\psi(x, y) = 0$ satisfying the boundary condition

$$N_m(x, y) = \cos(\alpha_x(m) \frac{\pi x}{6}) \cos(\alpha_y(m) \frac{\pi y}{4})$$

$$\alpha = (\alpha_x, \alpha_y) = \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 1 & 3 \end{pmatrix}$$

$$N_m(x, y) = 0 \quad \forall (x, y) \in \Gamma$$

$$R_\Omega = \mathcal{L}\hat{\phi} + p = \mathcal{L}\psi + \sum_m a_m \mathcal{L}N_m + p$$

$$= \sum_m a_m \mathcal{L}N_m + p$$

$$\mathcal{L}N_m = \frac{\partial^2 N_m}{\partial x^2} + \frac{\partial^2 N_m}{\partial y^2} = -\pi^2 \left(\left(\frac{\alpha_x}{6} \right)^2 + \left(\frac{\alpha_y}{4} \right)^2 \right) N_m$$

Example 2- Weighted residual using Galerkin (see routine Ej_2_2.m)

$$\int_{\Omega} W_l R_{\Omega} d\Omega = \int_{\Omega} W_l (\mathcal{L}\hat{\phi} + p) d\Omega = 0$$

$$\int_{\Omega} N_l \left(\sum_m a_m \mathcal{L}N_m + p \right) d\Omega = 0$$

$$\int_{\Omega} N_l \left(\sum_m a_m \left(-\pi^2 \left(\left(\frac{\alpha_x(m)}{6} \right)^2 + \left(\frac{\alpha_y(m)}{4} \right)^2 \right) N_m \right) + p \right) d\Omega = 0$$

$$K_{lm} a_m = f_l$$

$$\begin{aligned} K_{lm} &= -\pi^2 \left(\left(\frac{\alpha_x(m)}{6} \right)^2 + \left(\frac{\alpha_y(m)}{4} \right)^2 \right) \int_{\Omega} N_l N_m d\Omega \\ &= -\pi^2 \left(\left(\frac{\alpha_x(m)}{6} \right)^2 + \left(\frac{\alpha_y(m)}{4} \right)^2 \right) \int_{-2}^2 dy \int_{-3}^3 N_l N_m dx \end{aligned}$$

$$f_l = - \int_{\Omega} N_l p d\Omega = -2 \int_{\Omega} N_l d\Omega = -2 \int_{-2}^2 dy \int_{-3}^3 N_l dx$$

Example 2- Weighted residual using Galerkin (see routine Ej_2_2_symb.m with symbolic soft)

```
syms ax ay x y
```

```
ax=[1,3,1]
```

```
ay=[1,1,3]
```

```
N = [ cos(ax(1)*pi*x/6)*cos/ay(1)*pi*y/4);  
      cos(ax(2)*pi*x/6)*cos/ay(2)*pi*y/4);  
      cos(ax(3)*pi*x/6)*cos/ay(3)*pi*y/4)];
```

```
for l=1:3,
```

```
    for m=1:3,
```

```
        factor = pi^2*((ax(m)/6)^2+(ay(m)/4)^2);
```

```
        K(l,m) = factor*int(int(N(l)*N(m),x,-3,3),y,-2,2);
```

```
    end
```

```
    f(l) = -2*int(int(N(l),x,-3,3),y,-2,2);
```

```
end
```

Example 2- Weighted residual using Galerkin (see routine Ej_2_2_num.m)

```
global X Y II mm coef
```

```
% Number of terms to be used (M)
```

```
M = 3;
```

```
coef.p = 2;
```

```
% grid of samples
```

```
[X,Y]=meshgrid(-3:0.1:3,-2:0.1:2);
```

```
f = zeros(M,1);
```

```
K = zeros(M,M);
```

```
for II=1:M
```

```
    f(II) = gauss_integration('ffun_Ej_2_2_num_rhs',-3,3,-2,2);
```

```
    for mm=1:M
```

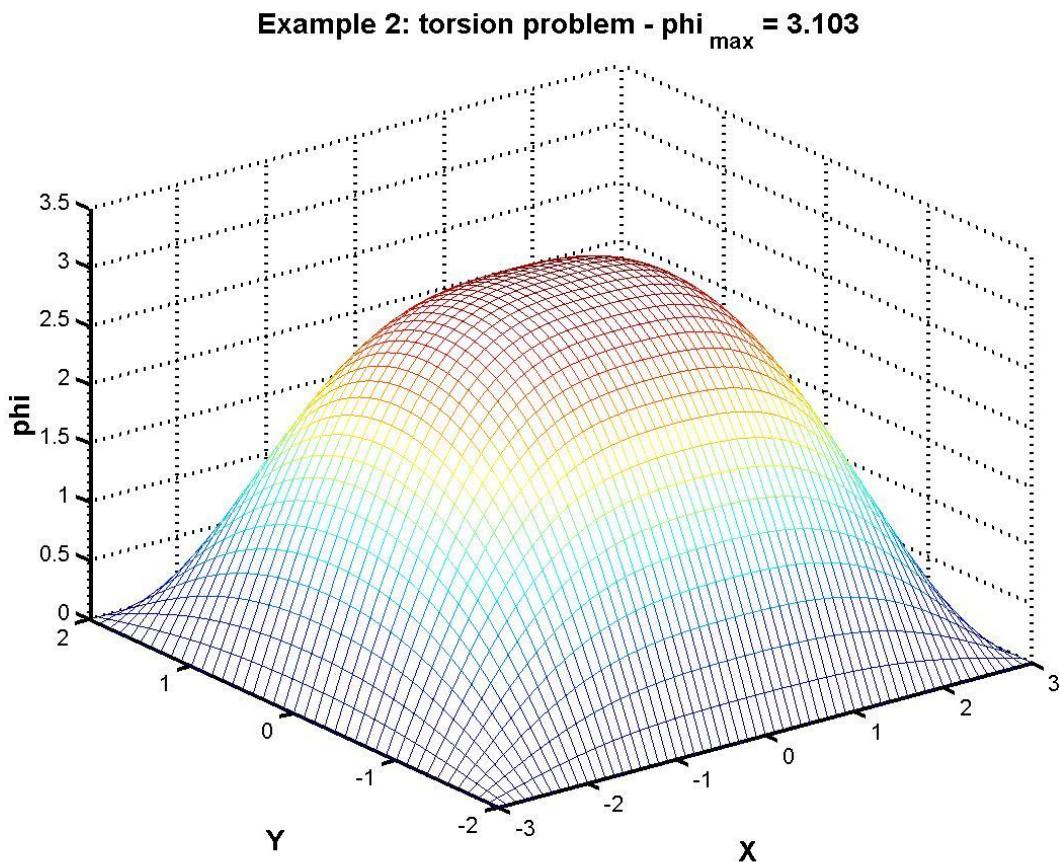
```
        K(II,mm) = gauss_integration('ffun_Ej_2_2_num_lhs',-3,3,-2,2);
```

```
    end
```

```
end
```

```
a = K\f;
```

Example 2- Weighted residual using Galerkin Results



$$a_1 = \frac{4608}{13 \pi^4} ; \quad a_2 = -\frac{4608}{135 \pi^4} ; \quad a_3 = -\frac{4608}{255 \pi^4}$$

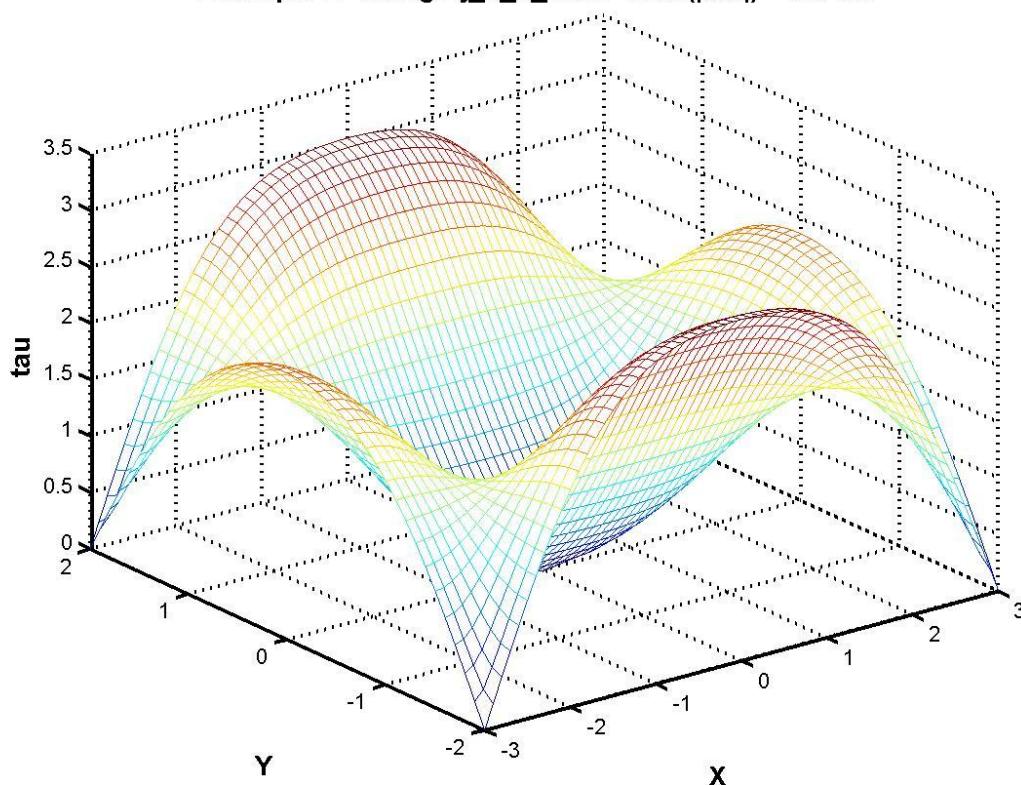
$$T = 2 \int_{-2}^2 dy \int_{-3}^3 \hat{\phi} dx = 74.265 \quad (\text{exact value } 76.4)$$

Example 2- Weighted residual using Galerkin Results

$$\tau_k = \frac{\partial \phi}{\partial x_k} \quad ; \quad |\tau| = \sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2}$$

$$|\hat{\tau}| = \sqrt{\left(\frac{\partial \hat{\phi}}{\partial x}\right)^2 + \left(\frac{\partial \hat{\phi}}{\partial y}\right)^2} = \sqrt{\left(\sum_m a_m \frac{\partial N_m}{\partial x}\right)^2 + \left(\sum_m a_m \frac{\partial N_m}{\partial y}\right)^2}$$

Example 2 - using Ej_2_2_num - max(|tau|) = 3.0199



3.02 (exact value 2.96)

Least square approximation to PDE

$$I(a_1, a_2, a_3, \dots, a_M) = \int_{\Omega} R_{\Omega}^2 d\Omega$$

$$\frac{\partial I}{\partial a_l} = 0 \quad l = 1, 2, \dots, M$$

$$\frac{\partial \int_{\Omega} R_{\Omega}^2 d\Omega}{\partial a_l} = \int_{\Omega} \frac{\partial}{\partial a_l} R_{\Omega}^2 d\Omega = \int_{\Omega} 2R_{\Omega} \frac{\partial R_{\Omega}}{\partial a_l} d\Omega = 0$$

$$\int_{\Omega} R_{\Omega} \frac{\partial R_{\Omega}}{\partial a_l} d\Omega = 0 \quad \Rightarrow W_l = \frac{\partial R_{\Omega}}{\partial a_l} = LN_l$$

$$\therefore \int_{\Omega} LN_l R_{\Omega} = 0$$

LEAST SQUARE APPROXIMATION \neq GALERKIN

Galerkin Least Square approximation to PDE (GLS)

$$\int_{\Omega} \mathsf{L}N_l R_{\Omega} = 0$$

$$\int_{\Omega} \mathsf{L}N_l \left\{ \left(\mathsf{L}\psi + \sum_m a_m \mathsf{L}N_m \right) + p \right\} = 0$$

$$\int_{\Omega} \mathsf{L}N_l \sum_m a_m \mathsf{L}N_m d\Omega = - \int_{\Omega} \mathsf{L}N_l (\mathsf{L}\psi + p) d\Omega$$

$$K_{lm} = \int_{\Omega} \mathsf{L}N_l \mathsf{L}N_m d\Omega$$

$$f_l = - \int_{\Omega} \mathsf{L}N_l (\mathsf{L}\psi + p) d\Omega$$

GALERKIN LEAST SQUARE APPROXIMATION

Simultaneous approximation to the solution of
the PDE in the domain and on its boundary.

$$R_\Omega = A(\hat{\phi}) = \mathbf{L}(\hat{\phi}) + \mathbf{p} = 0 \quad \text{in } \Omega$$

$$R_\Gamma = B(\hat{\phi}) = \mathbf{M}(\hat{\phi}) + \mathbf{r} = 0 \quad \text{in } \Gamma$$

$$\phi(x) \cong \hat{\phi}(x) = \sum_m a_m N_m(x) \quad \text{note that } \psi \text{ does not exist}$$

$$\int_{\Omega} W_l R_\Omega \, d\Omega + \int_{\Gamma} \overline{W}_l R_\Gamma \, d\Gamma = 0$$

Simultaneous approximation to the solution of the PDE in the domain and on its boundary.

$$\int_{\Omega} W_l R_{\Omega} \, d\Omega + \int_{\Gamma} \bar{W}_l R_{\Gamma} \, d\Gamma = 0$$

$$\int_{\Omega} W_l \left(\sum_m a_m \mathsf{L} N_m + p \right) d\Omega + \int_{\Gamma} \bar{W}_l \left(\sum_m a_m \mathsf{M} N_m + r \right) d\Gamma = 0$$

$$\int_{\Omega} W_l \left(\sum_m a_m \mathsf{L} N_m \right) d\Omega + \int_{\Gamma} \bar{W}_l \left(\sum_m a_m \mathsf{M} N_m \right) d\Gamma = - \int_{\Omega} W_l p \, d\Omega - \int_{\Gamma} \bar{W}_l r \, d\Gamma$$

$$K_{lm} a_m = f_l$$

$$K_{lm} = \int_{\Omega} W_l \mathsf{L} N_m \, d\Omega + \int_{\Gamma} \bar{W}_l \mathsf{M} N_m \, d\Gamma$$

$$f_l = - \int_{\Omega} W_l p \, d\Omega - \int_{\Gamma} \bar{W}_l r \, d\Gamma \quad l, m = 1, 2, \dots, M$$

Example 3 = Example 1- but solved without psi ψ

Find $\hat{\phi}(x)$ solution of the following ODE

$$\frac{d^2\phi}{dx^2} - \phi = 0 \quad \text{in } \Omega : \{x; 0 \leq x \leq 1\}$$

$$\phi(x=0) = 0$$

$$\phi(x=1) = 1 \quad \text{in } \Gamma : \{x = 0, x = 1\}$$

$$\mathcal{L}(\phi) = \frac{d^2\phi}{dx^2} - \phi$$

$$p = 0$$

$$\mathcal{M}(\phi) + r = 0 \quad \text{in } \Gamma$$

$$\begin{cases} \mathcal{M}(\phi) = \phi & ; \quad r = 0 \quad \text{on } x = 0 \quad \text{DIRICHLET} \\ \mathcal{M}(\phi) = \phi & ; \quad r = -1 \quad \text{on } x = 1 \quad \text{DIRICHLET} \end{cases}$$

Example 3 – weighted residual formulation

$$\int_{\Omega} W_l R_{\Omega} \, d\Omega + \int_{\Gamma} \bar{W}_l R_{\Gamma} \, d\Gamma = 0$$

$$\int_0^1 W_l R_{\Omega} \, dx + [\bar{W}_l R_{\Gamma}]_{x=0} + [\bar{W}_l R_{\Gamma}]_{x=1} = 0$$

$$W_l = N_l \quad \text{and} \quad \bar{W}_l = -N_l|_{\Gamma}$$

$$\int_0^1 N_l \left(\frac{d^2 \hat{\phi}}{dx^2} - \hat{\phi} \right) dx - [N_l \hat{\phi}]_{x=0} - [N_l (\hat{\phi} - 1)]_{x=1} = 0$$

using $N_m = x^{m-1}$, $m = 1, 2, 3, \dots$

$$K_{lm} a_m = f_l$$

$$K_{lm} = \int_0^1 N_l \left(\frac{d^2 N_m}{dx^2} - N_m \right) dx - [N_l N_m]_{x=0} - [N_l N_m]_{x=1}$$

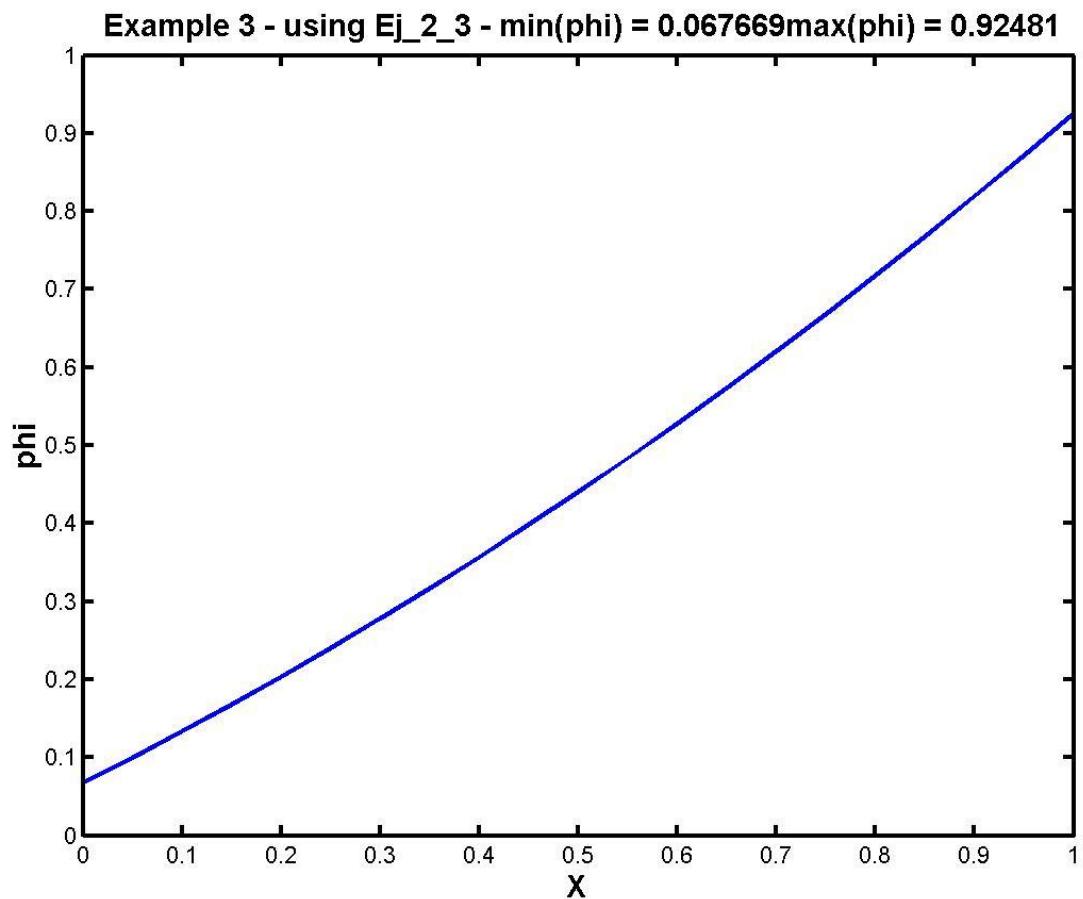
$$f_l = -[N_l]_{x=1} \quad l, m = 1, 2, \dots, M$$

Example 3 – algebraic system

$$K a = f$$

$$K = -\begin{vmatrix} 3 & \frac{3}{2} & -\frac{2}{3} \\ \frac{3}{2} & 4 & 1 \\ \frac{2}{3} & \frac{4}{3} & \frac{8}{15} \end{vmatrix}$$

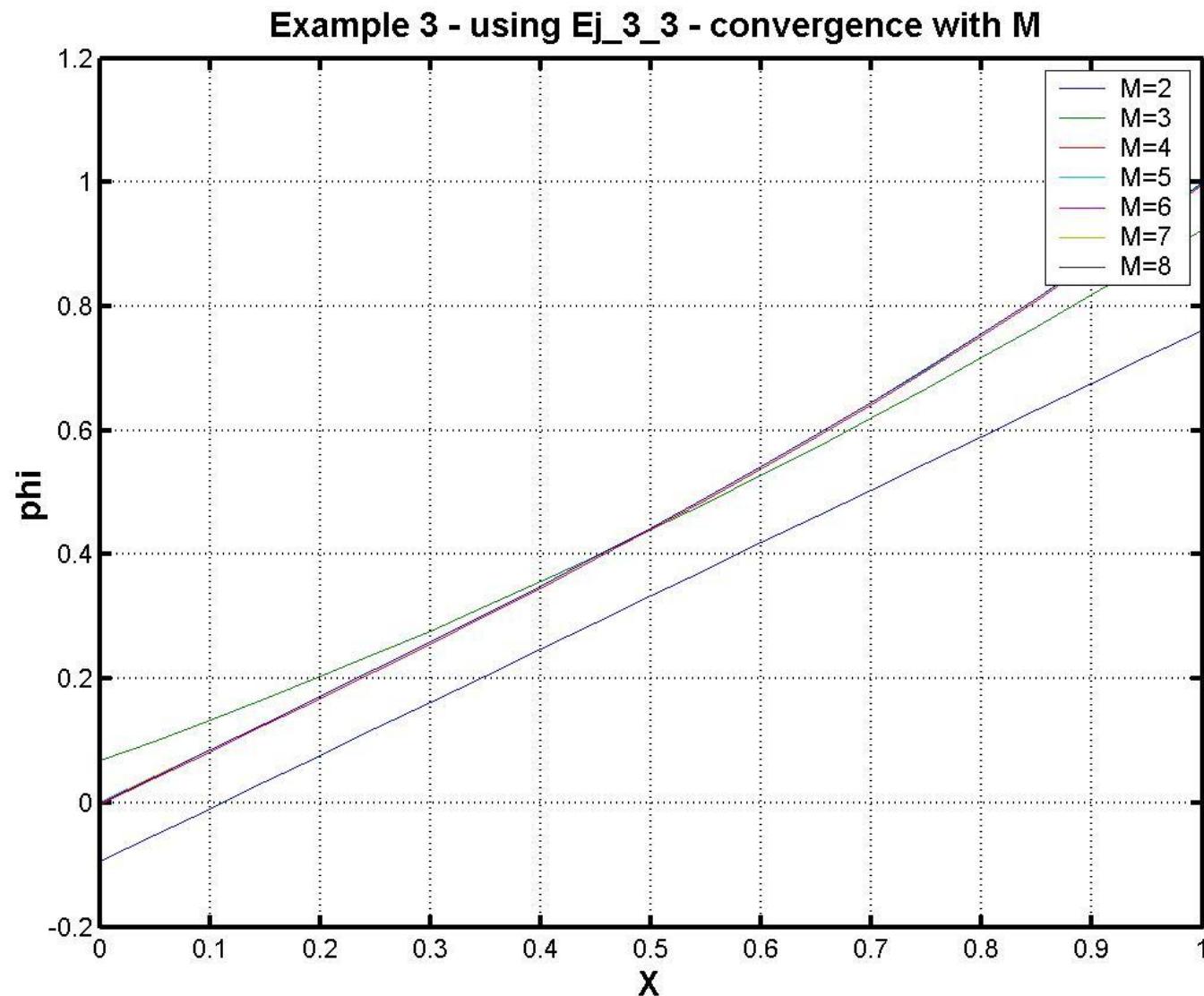
$$f = -|1 \quad 1 \quad 1|^T$$



Example 3 – Results

M	Phi (x = 0)	Phi (x = 1)
2.0000e+000 -	9.5238e-002	7.6190e-001
3.0000e+000	6.7669e-002	9.2481e-001
4.0000e+000	-3.1787e-003	9.9566e-001
5.0000e+000	5.6585e-004	9.9940e-001
6.0000e+000	-1.3409e-005	9.9998e-001
7.0000e+000	1.4329e-006	1.0000e+000

Example 3 – algebraic system



Example 4 = Example 2- but solved without ψ

$$N_j = \underbrace{\left(4 - y^2\right)}_{\text{satisfy } \mathcal{M}(\phi) + r = 0|_{y=\pm 2}} \times N_j^*$$

$$N_1^* = 1 \quad ; \quad N_2^* = x^2 \quad ; \quad N_3^* = y^2$$

$$N_4^* = x^2 y^2 \quad ; \quad N_5^* = x^4 \quad \text{and so on}$$

$$\hat{\phi} = \left(4 - y^2\right) \left(a_1 + a_2 x^2 + a_3 y^2 + a_4 x^2 y^2 + a_5 x^4\right)$$

$$\int_{-3}^3 \int_{-2}^2 W_l \left(\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} + 2 \right) dy \, dx + \int_{-2}^2 \overline{W}_l \left. \hat{\phi} \right|_{x=3} dy - \int_2^{-2} \overline{W}_l \left. \hat{\phi} \right|_{x=-3} dy = 0$$

Natural boundary conditions

$$\int_{\Omega} W_l R_{\Omega} d\Omega = \int_{\Omega} W_l (\mathcal{L}\hat{\phi} + p) d\Omega = 0$$

integration by parts allows to reduce the smoothness requirement over the functional space

$$\int_{\Omega} W_l (\mathcal{L}\hat{\phi} + p) d\Omega = \underbrace{\int_{\Omega} CW_l \mathcal{D}\hat{\phi} d\Omega}_{\text{strong form}} + \underbrace{\int_{\Gamma} W_l \mathcal{E}\hat{\phi} d\Gamma}_{\text{weak form}} + \int_{\Omega} W_l p d\Omega = 0$$

integration by parts

with $C, \mathcal{D}, \mathcal{E}$ linear operators of lower order than \mathcal{L}

Natural boundary condition using integration by parts or Gauss-Green theorem

Applying the Gauss - Green theorem

$$\int_{\Omega} \nabla \bullet \vec{v} \, d\Omega = \int_{\Gamma} \vec{v} \bullet \vec{\eta} \, d\Gamma ; \quad \vec{\eta} \text{ unit normal to surface } \Gamma$$

in general tensor form

$$\int_{\Omega} \nabla \bullet (\vec{u} \otimes \vec{v}) \, d\Omega = \int_{\Gamma} (\vec{u} \otimes \vec{v}) \bullet \vec{\eta} \, d\Gamma$$

in a reduced scalar situation

$$\int_{\Omega} d(u \ v) = \int_{\Omega} u dv + \int_{\Omega} v du = \int_{\Gamma} u \ v \, d\Gamma$$

and reduced to a 1D situation

$$\int_a^b d(u \ v) = \int_a^b u \ dv + \int_a^b v \ du = u \ v|_a^b$$

$$\text{if } u = W_l \text{ and } dv = \frac{\partial}{\partial x} \left(\kappa \frac{\partial \hat{\phi}}{\partial x} \right) + \frac{\partial}{\partial y} \left(\kappa \frac{\partial \hat{\phi}}{\partial y} \right)$$

therefore

$$\int_{\Omega} W_l \left(\frac{\partial}{\partial x} \left(\kappa \frac{\partial \hat{\phi}}{\partial x} \right) + \frac{\partial}{\partial y} \left(\kappa \frac{\partial \hat{\phi}}{\partial y} \right) \right) d\Omega + \int_{\Omega} \left(\frac{\partial W_l}{\partial x} \left(\kappa \frac{\partial \hat{\phi}}{\partial x} \right) + \frac{\partial W_l}{\partial y} \left(\kappa \frac{\partial \hat{\phi}}{\partial y} \right) \right) d\Omega = \int_{\Gamma} W_l \kappa \frac{\partial \hat{\phi}}{\partial \eta} d\Gamma$$

Natural boundary conditions

$$\underbrace{\int_{\Omega} W_l (\mathcal{L}\hat{\phi} + p) d\Omega}_{\text{strong form}} = \underbrace{\int_{\Omega} CW_l \mathcal{D}\hat{\phi} d\Omega + \int_{\Gamma} W_l \mathcal{E}\hat{\phi} d\Gamma}_{\text{weak form}} + \underbrace{\int_{\Omega} W_l p d\Omega}_{\text{integration by parts}} = 0$$

$$\begin{aligned} & \int_{\Omega} W_l (\mathcal{L}\hat{\phi} + p) d\Omega + \int_{\Gamma} \bar{W}_l (\mathcal{M}\hat{\phi} + r) d\Gamma = \\ &= \int_{\Omega} CW_l \mathcal{D}\hat{\phi} d\Omega + \int_{\Omega} W_l p d\Omega + \int_{\Gamma} W_l \mathcal{E}\hat{\phi} d\Gamma + \int_{\Gamma} \bar{W}_l (\mathcal{M}\hat{\phi} + r) d\Gamma = 0 \end{aligned}$$

combining the last two boundary integrals, is it possible to cancel out ?

This is only possible for certain selections of \bar{W}_l and boundary conditions

Example 5 – natural boundary conditions

Find $\hat{\phi}(x)$ solution of the following ODE

$$\frac{d^2\phi}{dx^2} - \phi = 0 \quad \text{in } \Omega : \{x; 0 \leq x \leq 1\}$$

$$\phi(x=0) = 0$$

$$\frac{d\phi}{dx}(x=1) = 20 \quad \text{in } \Gamma : \{x = 0, x = 1\}$$

using $N_m = x^m$

$$\int_0^1 W_l R_\Omega dx + [\bar{W}_l R_\Gamma]_{x=0} + [\bar{W}_l R_\Gamma]_{x=1} = 0$$

$$\int_0^1 W_l \left(\frac{d^2 \hat{\phi}}{dx^2} - \hat{\phi} \right) dx + [\bar{W}_l \hat{\phi}]_{x=0} + \left[\bar{W}_l \left(\frac{d\hat{\phi}}{dx} - 20 \right) \right]_{x=1} = 0$$

$$- \int_0^1 \frac{dW_l}{dx} \frac{d\hat{\phi}}{dx} dx - \int_0^1 W_l \hat{\phi} dx + \left[W_l \frac{d\hat{\phi}}{dx} \right]_{x=1} - \left[W_l \frac{d\hat{\phi}}{dx} \right]_{x=0} +$$

$$\underbrace{[\bar{W}_l \hat{\phi}]_{x=0}}_{=0 \text{ (DIRICHLET)}} + \left[\bar{W}_l \left(\frac{d\phi}{dx} - 20 \right) \right]_{x=1} = 0$$

Example 5 – natural boundary conditions

Choosing $\bar{W}_l \Big|_{x=1} = -W_l \Big|_{x=1}$ & $W_l \Big|_{x=0} = 0$

$$-\int_0^1 \frac{dW_l}{dx} \frac{d\hat{\phi}}{dx} dx - \int_0^1 W_l \hat{\phi} dx + \left[W_l \frac{d\hat{\phi}}{dx} \right]_{x=1} - \left[W_l \frac{d\hat{\phi}}{dx} \right]_{x=0} +$$

$$\underbrace{\left[\bar{W}_l \hat{\phi} \right]_{x=0}}_{=0 \text{ (DIRICHLET)}} + \left[\bar{W}_l \left(\frac{d\hat{\phi}}{dx} - 20 \right) \right]_{x=1} = 0$$

$$\Rightarrow \int_0^1 \frac{dW_l}{dx} \frac{d\hat{\phi}}{dx} dx + \int_0^1 W_l \hat{\phi} dx = [20 W_l]_{x=1}$$

there is no need to compute the $\frac{d\hat{\phi}}{dx} \Big|_{\Gamma}$

Gradient boundary conditions are natural
for such weak formulation

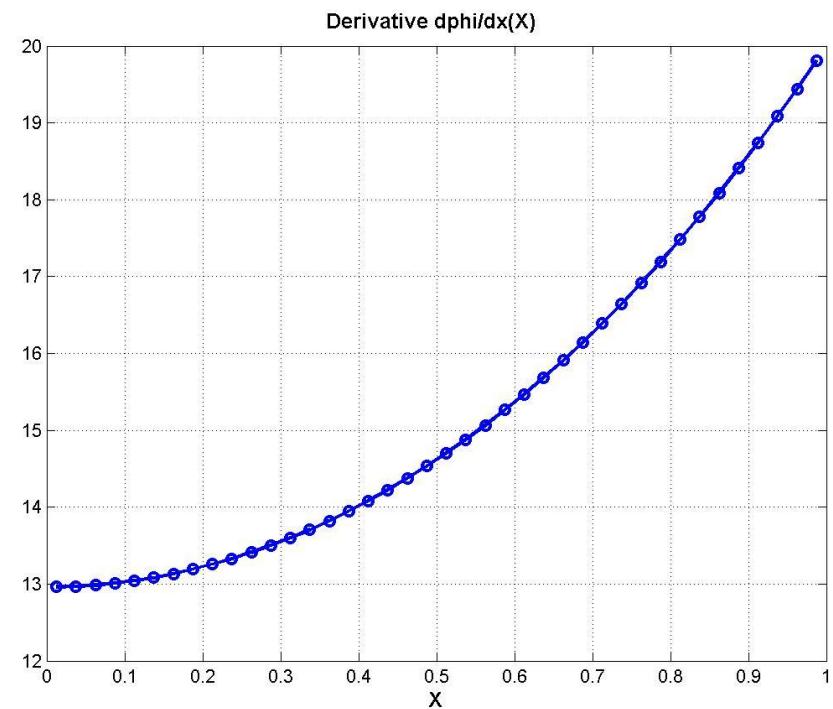
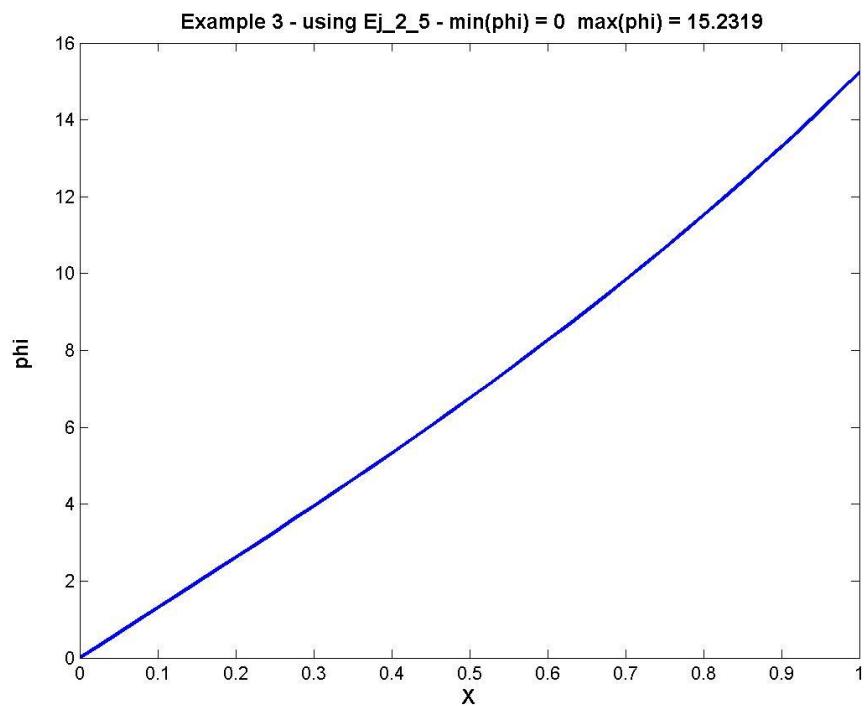
Moreover, if $\frac{d\hat{\phi}}{dx} \Big|_{\Gamma} = 0 \Rightarrow rhs = 0$

Example 5 – natural boundary conditions

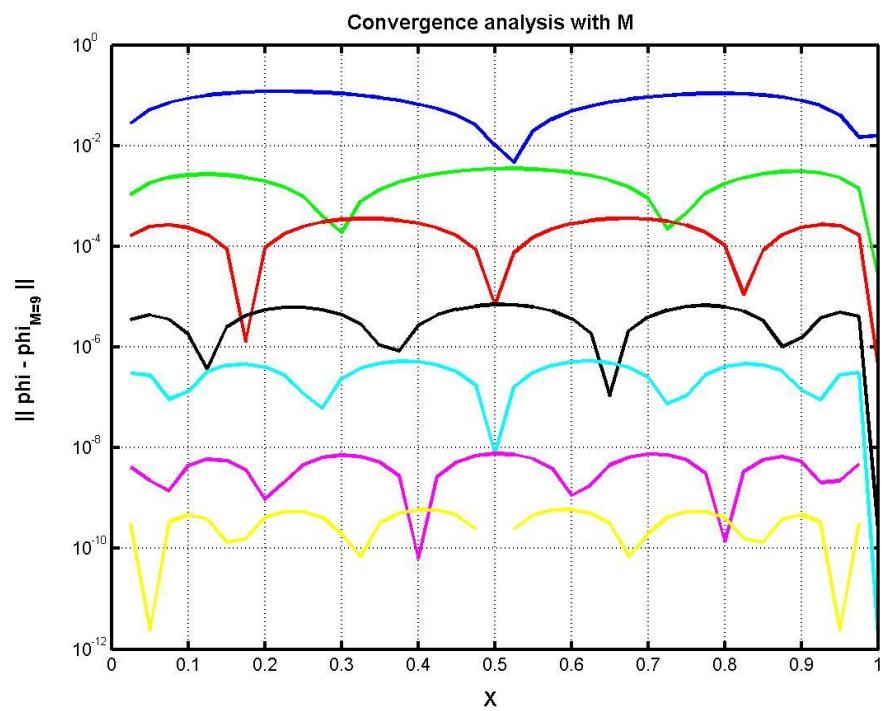
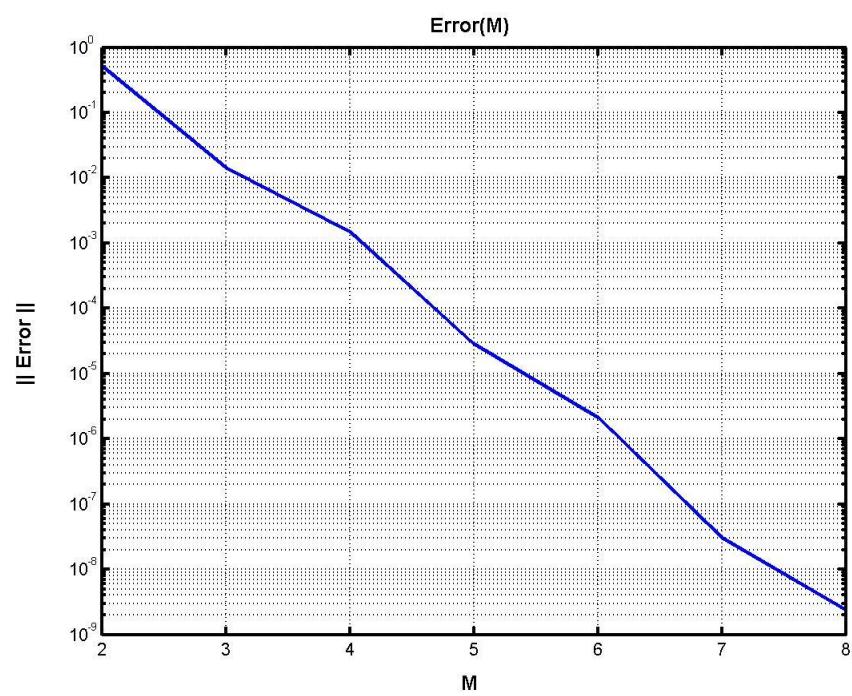
$$\int_0^1 \frac{dW_l}{dx} \frac{d\hat{\phi}}{dx} dx + \int_0^1 W_l \hat{\phi} dx = [20 W_l]_{x=1}$$
$$\Rightarrow K a = f$$

$$K_{lm} = \int_0^1 \frac{dW_l}{dx} \frac{dN_m}{dx} dx + \int_0^1 W_l N_m dx$$
$$f_l = 20 W_l \Big|_{x=1}$$

Example 5 – Results (routine Ej_2_5.m)



Example 5 – Results (routine Ej_2_5.m)



Natural boundary condition for heat equation

Find $\hat{\phi}(x)$ solution of the following PDE

$$\frac{\partial}{\partial x} \left(\kappa \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\kappa \frac{\partial \phi}{\partial y} \right) + Q = 0 \quad \text{in } \Omega : \{x; x \in \mathbb{R}^2\}$$

$$\phi = \bar{\phi} \quad \text{on } \Gamma_\phi$$

$$\kappa \frac{\partial \phi}{\partial \eta} = -\bar{q} \quad \text{on } \Gamma_q$$

$$\hat{\phi} = \psi + \sum_m a_m N_m$$

$$\psi = \bar{\phi} \quad \text{on } \Gamma_\phi$$

$$N_m = 0 \quad \text{on } \Gamma_\phi$$

$$\int_{\Omega} W_l \left(\frac{\partial}{\partial x} \left(\kappa \frac{\partial \hat{\phi}}{\partial x} \right) + \frac{\partial}{\partial y} \left(\kappa \frac{\partial \hat{\phi}}{\partial y} \right) + Q \right) d\Omega + \int_{\Gamma_q} \overline{W}_l \left(\kappa \frac{\partial \hat{\phi}}{\partial \eta} + \bar{q} \right) d\Gamma = 0$$

Natural boundary condition for heat equation

$$\int_{\Omega} W_l \left(\frac{\partial}{\partial x} \left(\kappa \frac{\partial \hat{\phi}}{\partial x} \right) + \frac{\partial}{\partial y} \left(\kappa \frac{\partial \hat{\phi}}{\partial y} \right) + Q \right) d\Omega + \int_{\Gamma_q} \bar{W}_l \left(\kappa \frac{\partial \hat{\phi}}{\partial \eta} + \bar{q} \right) d\Gamma = 0$$

Applying the Green's lemma

$$\begin{aligned} & \int_{\Omega} \left\{ - \left(\frac{\partial W_l}{\partial x} \left(\kappa \frac{\partial \hat{\phi}}{\partial x} \right) + \frac{\partial W_l}{\partial y} \left(\kappa \frac{\partial \hat{\phi}}{\partial y} \right) \right) + W_l Q \right\} d\Omega + \\ & + \int_{\Gamma_{\phi} \cup \Gamma_q} W_l \left(\kappa \frac{\partial \hat{\phi}}{\partial \eta} + \bar{q} \right) d\Gamma + \int_{\Gamma_q} \bar{W}_l \left(\kappa \frac{\partial \hat{\phi}}{\partial \eta} + \bar{q} \right) d\Gamma = 0 \end{aligned}$$

choosing the weighted functions so that

$$W_l = 0 \quad \text{on } \Gamma_{\phi}$$

$$\bar{W}_l = -W_l \quad \text{on } \Gamma_q$$

$$\int_{\Omega} \left(\frac{\partial W_l}{\partial x} \left(\kappa \frac{\partial \hat{\phi}}{\partial x} \right) + \frac{\partial W_l}{\partial y} \left(\kappa \frac{\partial \hat{\phi}}{\partial y} \right) - W_l Q \right) d\Omega + \int_{\Gamma_q} W_l \bar{q} d\Gamma = 0$$

Natural boundary condition for heat equation

$$K \ a = f$$

$$K_{lm} = \int_{\Omega} \left(\frac{\partial W_l}{\partial x} \left(\kappa \frac{\partial N_m}{\partial x} \right) + \frac{\partial W_l}{\partial y} \left(\kappa \frac{\partial N_m}{\partial y} \right) \right) d\Omega$$

$$f_l = \int_{\Omega} W_l Q \ d\Omega + \int_{\Gamma_q} W_l \bar{q} \ d\Gamma - \int_{\Omega} \left(\frac{\partial W_l}{\partial x} \left(\kappa \frac{\partial \psi}{\partial x} \right) + \frac{\partial W_l}{\partial y} \left(\kappa \frac{\partial \psi}{\partial y} \right) \right) d\Omega = 0$$

$$1 \leq l, m \leq M$$

using Galerkin approximation ($W_l = N_l$)

$$\therefore K_{lm} = K_{ml} \quad \text{symmetry}$$

$$\kappa \frac{\partial \phi}{\partial \eta} = -\bar{q} \quad \text{on } \Gamma_q \quad \Rightarrow \quad \text{there is no phi gradient at boundary terms}$$

$$\therefore \kappa \frac{\partial \phi}{\partial \eta} = -\bar{q} \quad \text{is the natural BC for heat equation}$$

Example 6 – natural BC for heat equation

Find $\hat{\phi}(x)$ solution of the following PDE in 2D

$$\frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial y} \right) = 0 \quad \text{in } \Omega : \{x, y; x, y \in (-1,1) \times (-1,1)\}$$

$$\phi = 0 \quad \text{on } \Gamma_\phi : \{x, y; y = \pm 1\}$$

$$\frac{\partial \phi}{\partial \eta} = \cos\left(\frac{\pi y}{2}\right) \quad \text{on } \Gamma_q : \{x, y; x = \pm 1\}$$

$$N_j = \underbrace{(1 - y^2)}_{\text{satisfy } \mathcal{M}(\phi) + r = 0|_{y=\pm 1}} \times N_j^*$$

$$N_1^* = 1 \quad ; \quad N_2^* = x^2 \quad ; \quad N_3^* = y^2$$

$$N_4^* = x^2 y^2 \quad ; \quad N_5^* = x^4 \quad \text{and so on}$$

$$\hat{\phi} = (1 - y^2) (a_1 + a_2 x^2 + a_3 y^2 + a_4 x^2 y^2 + a_5 x^4)$$

immediately satisfies the BC @ Γ_ϕ

$$\int_{-1}^1 \int_{-1}^1 W_l \left(\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} \right) dy dx + \int_{\Gamma_q} \overline{W}_l \left(\frac{\partial \hat{\phi}}{\partial \eta} - \cos\left(\frac{\pi y}{2}\right) \right) d\Gamma = 0$$

Example 6 – weak form

$$\int_{-1}^1 \int_{-1}^1 W_l \left(\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} \right) dy dx + \int_{\Gamma_q} \overline{W}_l \left(\frac{\partial \hat{\phi}}{\partial \eta} - \cos\left(\frac{\pi y}{2}\right) \right) d\Gamma = 0$$
$$\int_{-1}^1 \int_{-1}^1 \left(\frac{\partial W_l}{\partial x} \frac{\partial \hat{\phi}}{\partial x} + \frac{\partial W_l}{\partial y} \frac{\partial \hat{\phi}}{\partial y} \right) dy dx -$$
$$- \int_{\Gamma_q \cup \Gamma_\phi} W_l \frac{\partial \hat{\phi}}{\partial \eta} d\Gamma - \int_{\Gamma_q} \overline{W}_l \left(\frac{\partial \hat{\phi}}{\partial \eta} - \cos\left(\frac{\pi y}{2}\right) \right) d\Gamma = 0$$

Example 6 – weak form

$$W_l = N_l \quad (\text{Galerkin})$$

$$\Rightarrow \int_{\Gamma_\phi} W_l \frac{\partial \hat{\phi}}{\partial \eta} d\Gamma = 0 \quad \text{because } N_l|_{\Gamma_\phi} = 0$$

$$\begin{aligned} & \int_{-1}^1 \int_{-1}^1 \left(\frac{\partial N_l}{\partial x} \frac{\partial \hat{\phi}}{\partial x} + \frac{\partial N_l}{\partial y} \frac{\partial \hat{\phi}}{\partial y} \right) dy dx - \\ & - \int_{\Gamma_q} N_l \frac{\partial \hat{\phi}}{\partial \eta} d\Gamma - \int_{\Gamma_q} \bar{W}_l \left(\frac{\partial \hat{\phi}}{\partial \eta} - \cos\left(\frac{\pi y}{2}\right) \right) d\Gamma = 0 \end{aligned}$$

taking $\bar{W}_l|_{\Gamma_q} = -N_l|_{\Gamma_q}$ cancel out the derivative term @ Γ_q

$$\therefore \int_{-1}^1 \int_{-1}^1 \left(\frac{\partial N_l}{\partial x} \frac{\partial \hat{\phi}}{\partial x} + \frac{\partial N_l}{\partial y} \frac{\partial \hat{\phi}}{\partial y} \right) dy dx - \int_{\Gamma_q} N_l \cos\left(\frac{\pi y}{2}\right) \left(\frac{\partial \hat{\phi}}{\partial \eta} - \right) d\Gamma = 0$$

Example 6 – linear system

$$\therefore K_{lm} = \int_{-1}^1 \int_{-1}^1 \left(\frac{\partial N_l}{\partial x} \frac{\partial \hat{\phi}}{\partial x} + \frac{\partial N_l}{\partial y} \frac{\partial \hat{\phi}}{\partial y} \right) dy dx$$

$$f_l = \int_{-1}^1 N_l \Big|_{x=1} \cos\left(\frac{\pi y}{2}\right) dy - \int_1^{-1} N_l \Big|_{x=-1} \cos\left(\frac{\pi y}{2}\right) dy$$

equivalent to

$$K_{lm} = \int_0^1 \int_0^1 \left(\frac{\partial N_l}{\partial x} \frac{\partial \hat{\phi}}{\partial x} + \frac{\partial N_l}{\partial y} \frac{\partial \hat{\phi}}{\partial y} \right) dy dx$$

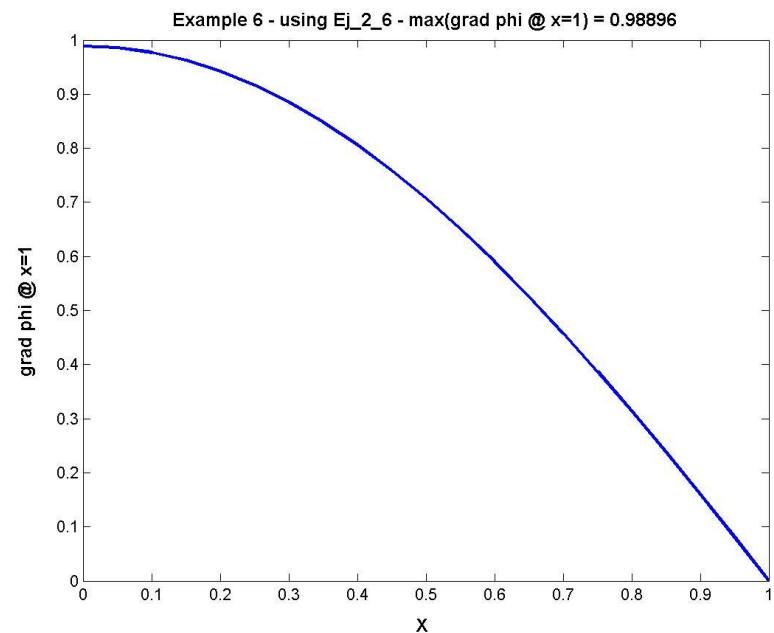
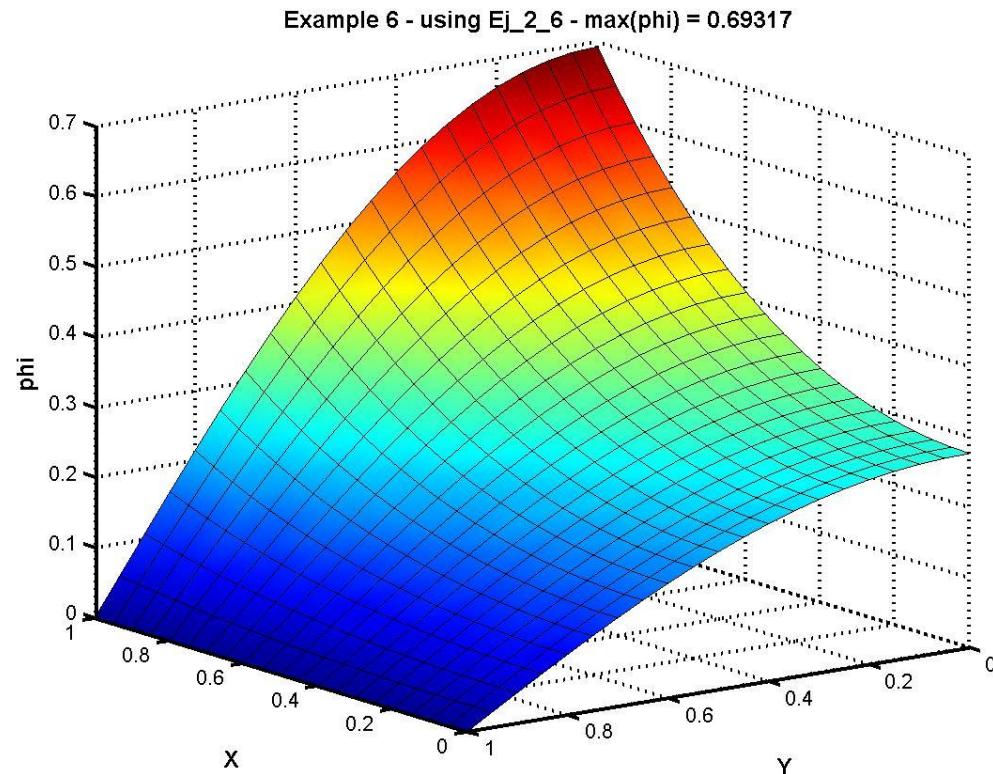
2D integral

$$f_l = \int_0^1 N_l \Big|_{x=1} \cos\left(\frac{\pi y}{2}\right) dy$$

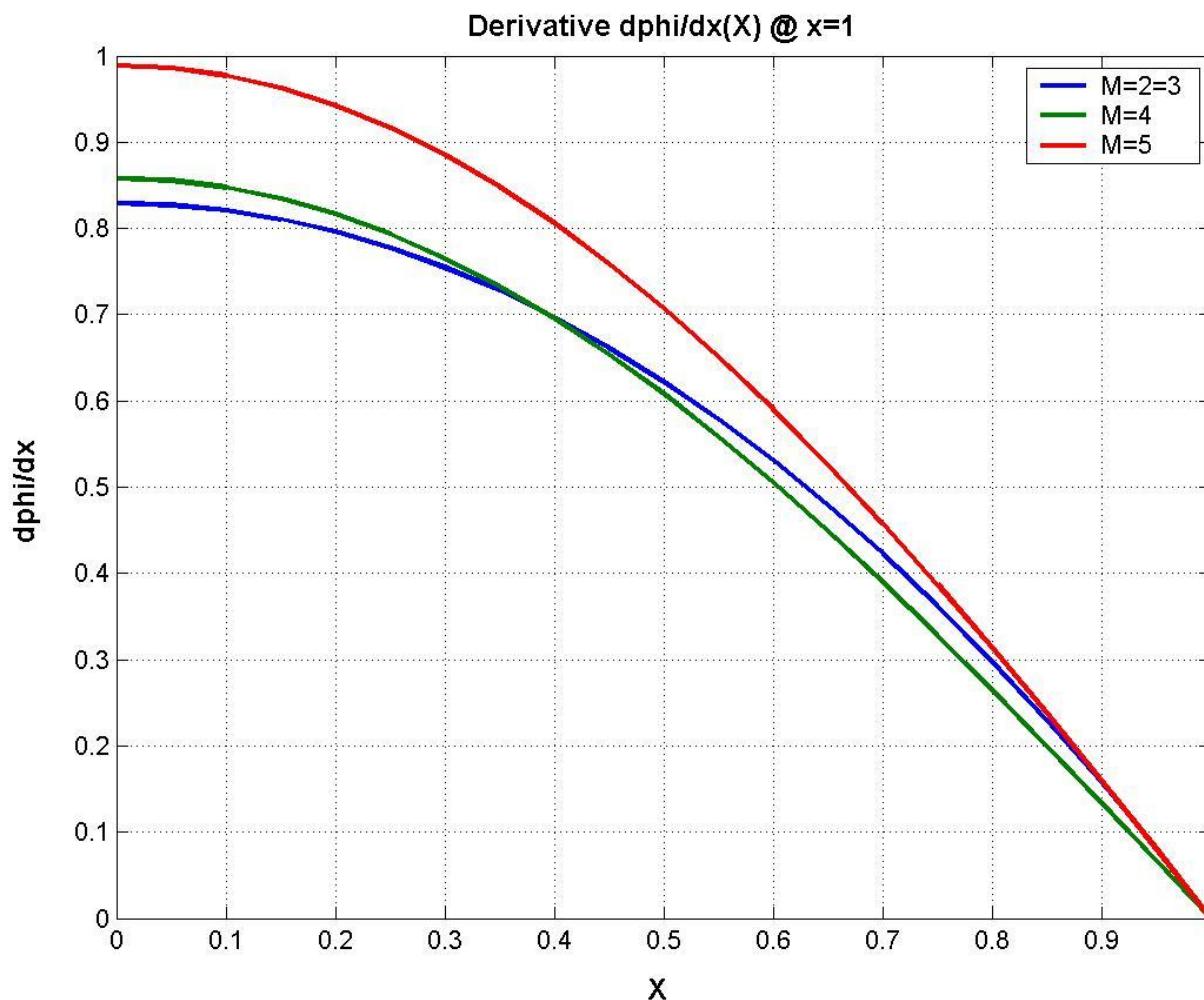
1D integral

```
f = zeros(M,1);
K = zeros(M,M);
for ll=1:M
    f(ll) =
        gauss_integration('ffun_Ej_2_6_rhs',1,c,d);
    for mm=1:M
        K(ll,mm) =
            gauss_integration('ffun_Ej_2_6_lhs',2,a,b,c,d);
    end
end
```

Example 6 – Results (routine Ej_2_6.m)



Example 6 – Results (routine Ej_2_6.m)



Example 6 – Results (routine Ej_2_6.m)

$K = [\dots$

```
1.3333e+000 4.4444e-001 2.6667e-001 8.8889e-002 2.6667e-001  
4.4444e-001 9.7778e-001 8.8889e-002 1.5492e-001 1.0438e+000  
2.6667e-001 8.8889e-002 4.1905e-001 1.3968e-001 5.3333e-002  
8.8889e-002 1.5492e-001 1.3968e-001 1.1767e-001 1.6000e-001  
2.6667e-001 1.0438e+000 5.3333e-002 1.6000e-001 1.3672e+000 ]
```

$f' = [5.1602e-001 5.1602e-001 7.0480e-002 7.0480e-002 5.1602e-001]$

$a' = [2.7631e-001 3.3925e-001 -5.8747e-002 -9.2206e-002 7.7616e-002]$

Boundary solution methods

[1] $\psi(x)$ to satisfy BC

$$\phi(x) \cong \hat{\phi}(x) = \psi(x) + \sum_m a_m N_m(x)$$

on Γ
$$\begin{cases} M(\psi) = -r \\ M(N_m) = 0 \quad ; \quad m = 1, 2, \dots \end{cases}$$

Weighted Residual exclusively over the domain

$$\int_{\Omega} W_l R_{\Omega} d\Omega = \int_{\Omega} W_l \left(L\psi + \sum_m a_m L N_m + p \right) d\Omega = 0$$

[2] give up $\psi(x)$ to satisfy BC and add a boundary residual term in the formulation

$$\phi(x) \cong \hat{\phi}(x) = \sum_m a_m N_m(x)$$

$$R_{\Omega} = A(\phi) = L(\phi) + p = 0 \quad \text{in } \Omega$$

$$R_{\Gamma} = B(\phi) = M(\phi) + r = 0 \quad \text{in } \Gamma$$

$$\int_{\Omega} W_l R_{\Omega} d\Omega + \int_{\Gamma} \bar{W}_l R_{\Gamma} d\Gamma = 0$$

Boundary solution methods

$$[1] \Rightarrow \int_{\Omega} W_l R_{\Omega} d\Omega = 0$$

$$[2] \Rightarrow \int_{\Omega} W_l R_{\Omega} d\Omega + \int_{\Gamma} \bar{W}_l R_{\Gamma} d\Gamma = 0$$

why don't propose to solve exclusively the boundary
including for the approximation the solution inside the domain ?
Probably it is difficult to know the analytical solution in the domain
Only for simple (linear) differential operators
But, the dimension of the problem is reduced in one (3D \rightarrow 2D)

$$[3] \Rightarrow \int_{\Gamma} \bar{W}_l R_{\Gamma} d\Gamma = 0$$

choosing N_m such that $A(N_m) = 0$ $\therefore R_{\Omega} = A(\phi) = \sum_m a_m A(N_m) = 0$

Boundary solution methods to Laplace equation

Any analytic function of the complex variable $z = x + i y$

$$f(z) = u + i v$$

with $u, v \in \Re$ satisfy

$$\frac{\partial^2 f}{\partial x^2} = f'' \quad , \quad \frac{\partial^2 f}{\partial y^2} = i^2 f'' = -f'' \quad , \quad f'' = \frac{d^2 f}{dz^2}$$

$$\therefore \nabla^2 f = \nabla^2 u + i \nabla^2 v = 0 \Rightarrow \nabla^2 u = \nabla^2 v = 0$$

$$\therefore \text{using } f(z) = z^n$$

$$n=1, \quad u=x, \quad v=y$$

$$n=2, \quad u=x^2 - y^2, \quad v=2xy$$

$$n=3, \quad u=x^3 - 3x y^2, \quad v=3x^2 y - y^3$$

$$n=4, \quad u=x^4 - 6x^2 y^2 + y^4, \quad v=4x^3 y - 4x y^3$$

Example 7- Torsion problem by BSM

Find $\hat{\phi}(x)$ solution of the following PDE

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2 \quad \text{in } \Omega : \{(x, y); -3 \leq x \leq 3; -2 \leq y \leq 2\}$$

$$\phi(x = -3) = \phi(x = 3) = 0$$

$$\phi(y = -2) = \phi(y = 2) = 0$$

equivalent to solve

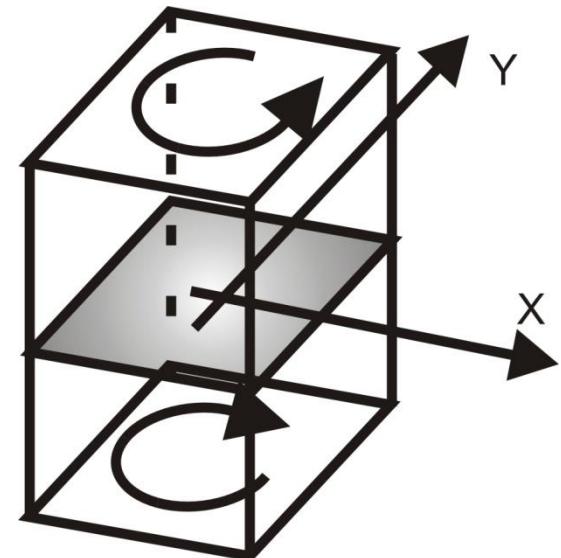
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2 \quad \text{in } \Omega : \{(x, y); 0 \leq x \leq 3; 0 \leq y \leq 2\}$$

$$\phi(x = 3) = 0$$

$$\phi(y = 2) = 0$$

$$\left. \frac{\partial \phi}{\partial x} \right|_{x=0} = 0$$

$$\left. \frac{\partial \phi}{\partial y} \right|_{y=0} = 0$$



Boundary solution methods - Example

Changing variables

$$\phi = \theta - \frac{1}{2}(x^2 + y^2)$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2 \Rightarrow \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

Approximating θ

by symmetric functions respect to x and y

$$\theta \approx \hat{\theta} = a_1 \underbrace{1}_{N_1} + a_2 \underbrace{(x^2 - y^2)}_{N_2} + a_3 \underbrace{(x^4 - 6x^2y^2 + y^4)}_{N_3}$$

$$\frac{\partial^2 \hat{\theta}}{\partial x^2} + \frac{\partial^2 \hat{\theta}}{\partial y^2} = 0 , \quad \forall (x, y) \in \Omega$$

$$\int_{\Gamma} \overline{W}_l R_{\Gamma} d\Gamma = 0$$

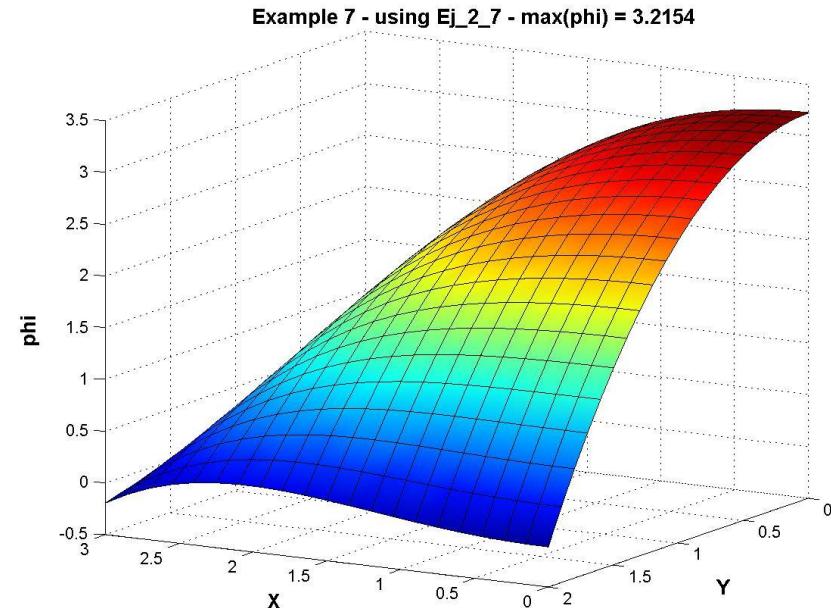
$$\Rightarrow \int_{\Gamma} N_l \Big|_{x=3} \left\{ \hat{\theta} \Big|_{x=3} - \frac{1}{2}(9 + y^2) \right\} dy + \int_{\Gamma} N_l \Big|_{y=2} \left\{ \hat{\theta} \Big|_{y=2} - \frac{1}{2}(x^2 + 4) \right\} dx = 0$$

Boundary solution methods - Example

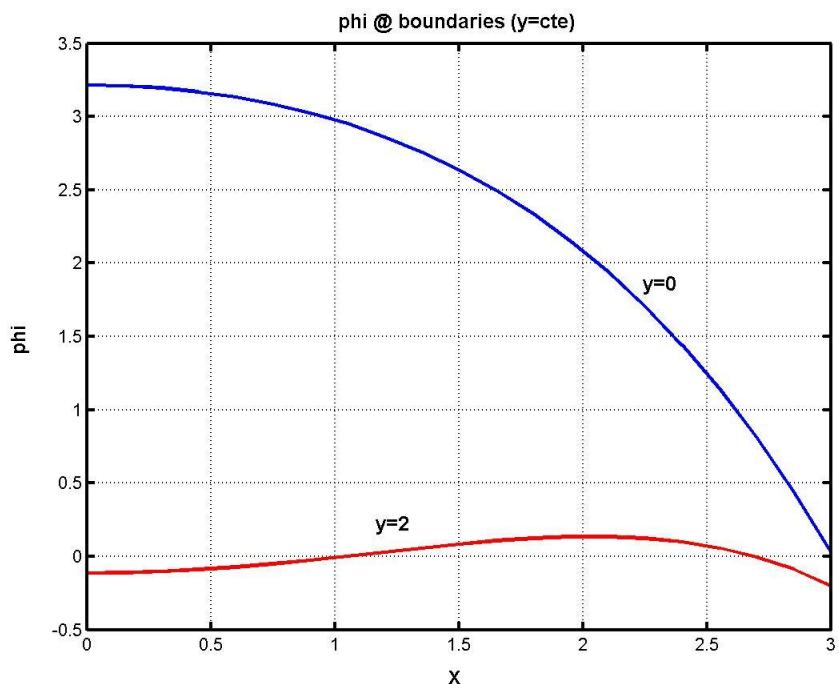
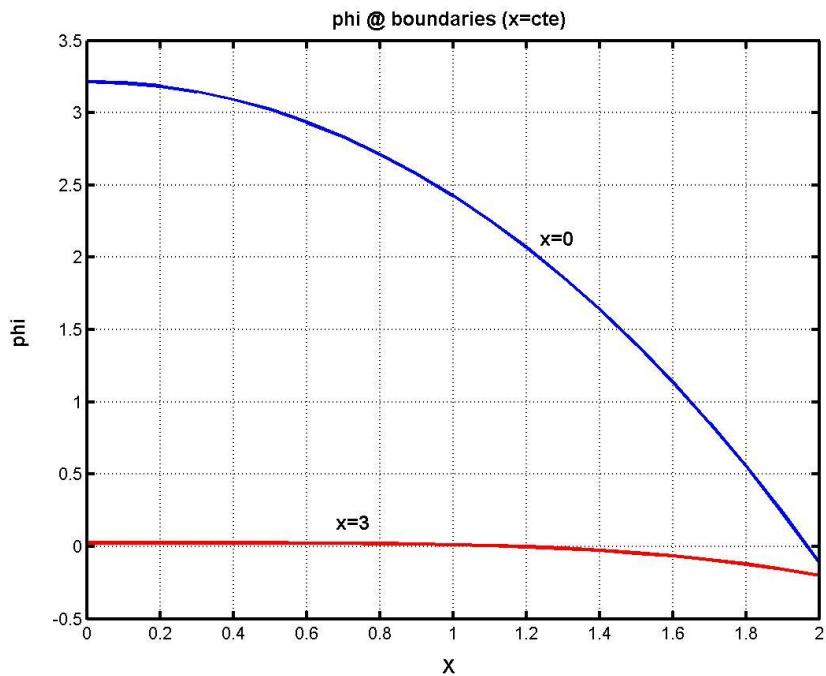
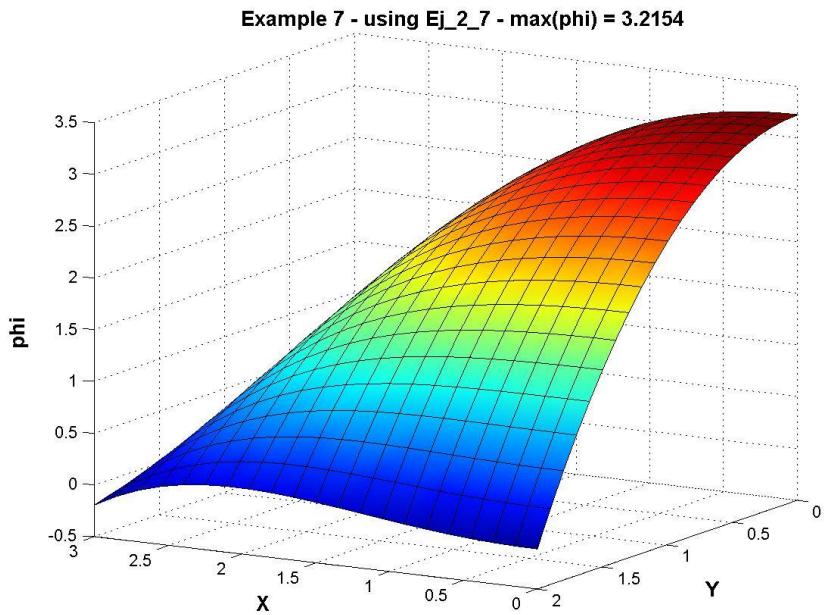
```
f = zeros(M,1);
K = zeros(M,M);
for ll=1:M
    f1 = gauss_integration('ffun_Ej_2_7_rhs_f1',1,c,d);
    f2 = gauss_integration('ffun_Ej_2_7_rhs_f2',1,a,b);
    f(ll) = -(f1+f2);
    for mm=1:M
        K1 = gauss_integration('ffun_Ej_2_7_lhs_f1',1,c,d);
        K2 = gauss_integration('ffun_Ej_2_7_lhs_f2',1,a,b);
        K(ll,mm) = K1+K2;
    end
end
```

$$\int_{\Gamma} N_l \left|_{x=3} \right. \left\{ \hat{\theta} \Big|_{x=3} - \frac{1}{2} (9 + y^2) \right\} dy +$$

$$\int_{\Gamma} N_l \left|_{y=2} \right. \left\{ \hat{\theta} \Big|_{y=2} - \frac{1}{2} (x^2 + 4) \right\} dx = 0$$



Boundary solution methods - Example



How the solution fits the right Dirichlet boundary values ?

Systems of differential equations

- Simultaneous resolution of PDEs coming from different applications
- 2D, axisymmetric or 3D linear elasticity
- 1D beam equations
- 1D Gas dynamics
- Fluid mechanics in general
- Maxwell equations for electromagnetism
- Scalar 2nd order PDEs transformed to a system of 1st order PDE

Systems of differential equations

$$\underline{\phi}^T = (\phi_1, \phi_2, \phi_3, \dots)$$

$$\begin{cases} A_1(\phi) = 0 \\ A_2(\phi) = 0 \\ \vdots \end{cases} \quad \text{in } \Omega \quad ; \quad A(\phi) = \begin{bmatrix} A_1(\phi) \\ A_2(\phi) \\ \vdots \end{bmatrix} = 0 \quad \text{in } \Omega$$

$$\begin{cases} B_1(\phi) = 0 \\ B_2(\phi) = 0 \\ \vdots \end{cases} \quad \text{in } \Gamma \quad ; \quad B(\phi) = \begin{bmatrix} B_1(\phi) \\ B_2(\phi) \\ \vdots \end{bmatrix} = 0 \quad \text{in } \Gamma$$

$$\phi_1(x) \cong \hat{\phi}_1(x) = \psi_1(x) + \sum_m a_{m,1} N_{m,1}(x)$$

$$\phi_2(x) \cong \hat{\phi}_2(x) = \psi_2(x) + \sum_m a_{m,2} N_{m,2}(x)$$

⋮

$$\underline{\phi}(x) \cong \hat{\underline{\phi}}(x) = \underline{\psi}(x) + \sum_m \underline{a}_m \underline{N}_m(x)$$

$$\underline{\psi}(x)^T = (\psi_1, \psi_2, \psi_3, \dots) \quad ; \quad \underline{a}_m^T = (a_{m,1}, a_{m,2}, a_{m,3}, \dots) \quad ; \quad \underline{\underline{N}}_m = \begin{bmatrix} N_{m,1} & & & 0 \\ & N_{m,2} & & \\ & & N_{m,3} & \\ & & & \ddots \end{bmatrix}$$

Systems of differential equations

$$\int_{\Omega} W_{l,1} A_1(\hat{\phi}) d\Omega + \int_{\Gamma} \bar{W}_{l,1} B_1(\hat{\phi}) d\Gamma = 0$$

$$\int_{\Omega} W_{l,2} A_2(\hat{\phi}) d\Omega + \int_{\Gamma} \bar{W}_{l,2} B_2(\hat{\phi}) d\Gamma = 0$$

⋮

⋮

$$W_l = \begin{bmatrix} W_{l,1} & & 0 \\ & W_{l,2} & \\ & & W_{l,3} \\ 0 & & \ddots \end{bmatrix} ; \quad \underline{W}_l = \begin{bmatrix} \bar{W}_{l,1} & & 0 \\ & \bar{W}_{l,2} & \\ & & \bar{W}_{l,3} \\ 0 & & \ddots \end{bmatrix}$$

$$\int_{\Omega} \underline{W}_l A(\hat{\phi}) d\Omega + \int_{\Gamma} \bar{W}_l B(\hat{\phi}) d\Gamma = 0$$

Example 8 : SoDE – 2nd order to 1st order

- Scalar heat equation (Poisson equation)
- In 1D
- 2nd order ODE (steady + 1D)
- Transformed to 1st order system of ODEs

$$\begin{cases} q + \kappa \frac{d\phi}{dx} = 0 \\ \frac{dq}{dx} - Q = 0 \end{cases} ; \quad A(\phi) = \begin{bmatrix} q + \kappa \frac{d\phi}{dx} \\ \frac{dq}{dx} - Q \end{bmatrix} = 0 \quad \text{in } \Omega$$

$$\underline{\phi^T} = (q, \phi)$$

$$\mathcal{L} \underline{\phi} + \underline{p} = \begin{bmatrix} 1 & \kappa \frac{d}{dx} \\ \frac{d}{dx} & 0 \end{bmatrix} \underline{\phi} + \begin{bmatrix} 0 \\ -Q \end{bmatrix} = 0$$

Example 8 : SoDE – 2nd order to 1st order

- Scalar heat equation (Poisson equation)
- In 1D
- 2nd order ODE (steady + 1D)
- Transformed to 1st order system of ODEs

$$\Omega: \{x; 0 \leq x \leq 1\} \quad ; \quad k = 1 \quad ; \quad Q = \begin{cases} 1 & x \leq \frac{1}{2} \\ 0 & x > \frac{1}{2} \end{cases}$$

$$\phi = 0 \quad @ x = 0 \quad ; \quad q = 0 \quad @ x = 1$$

$$\hat{q} = \psi_1 + \sum_m a_{m,1} N_{m,1}$$

$$\hat{\phi} = \psi_2 + \sum_m a_{m,2} N_{m,2}$$

$$\psi_1 = \psi_2 = 0 \quad ; \quad N_{m,1} = 0 \quad @ x = 1 \quad ; \quad N_{m,2} = 0 \quad @ x = 0$$

Example 8 : SoDE – 2nd order to 1st order

$$N_{m,1} = x^{m-1}(1-x) \quad ; \quad N_{m,2} = x^m$$

$$\int_0^1 \underline{W}_l \left(\underline{\mathcal{L}} \hat{\phi} + \underline{p} \right) dx = 0$$

$$K = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1M} \\ K_{21} & K_{22} & \cdots & K_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ K_{M1} & K_{M2} & \cdots & K_{MM} \end{bmatrix} ; \quad f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_M \end{bmatrix}$$

$$K_{lm} = \int_0^1 \underline{W}_l \underline{\mathcal{L}} \underline{N}_m dx \quad ; \quad f_l = - \int_0^1 \underline{W}_l \underline{p} dx$$

Galerkin typeweighting $\underline{W}_l = \underline{N}_l$

$$K_{lm} = \begin{bmatrix} \int_0^1 N_{l,1} N_{m,1} dx & \int_0^1 N_{l,1} \frac{dN_{m,2}}{dx} dx \\ \int_0^1 N_{l,2} \frac{dN_{m,1}}{dx} dx & 0 \end{bmatrix}$$

Example 8 : SoDE – 2nd order to 1st order

```
f = zeros(M,ndf); K = zeros(M,M,ndf,ndf);  
  
coef.ncol = ndf;  
for ll=1:M  
    coef.ncol = ndf;  
    f(ll,1:ndf) = gauss_integration('ffun_Ej_2_8_rhs',0,1);  
    for mm=1:M  
        coef.ncol = ndf*ndf;  
        vaux = gauss_integration('ffun_Ej_2_8_lhs',0,1);  
        K(ll,mm,1:ndf,1:ndf) = reshape(vaux,ndf,ndf);  
    end  
end
```

A
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```
K2 = zeros(ndf*M,ndf*M); f2 = zeros(ndf*M,1);  
for ll=1:M  
    ll2 = (ll-1)*ndf+(1:ndf);  
    f2(ll2) = reshape(f(ll,1:ndf),ndf,1);  
    for mm=1:M  
        mm2 = (mm-1)*ndf+(1:ndf);  
        K2(ll2,mm2) = reshape(K(ll,mm,1:ndf,1:ndf),ndf,ndf);  
    end  
end  
  
a = K2\f2;
```

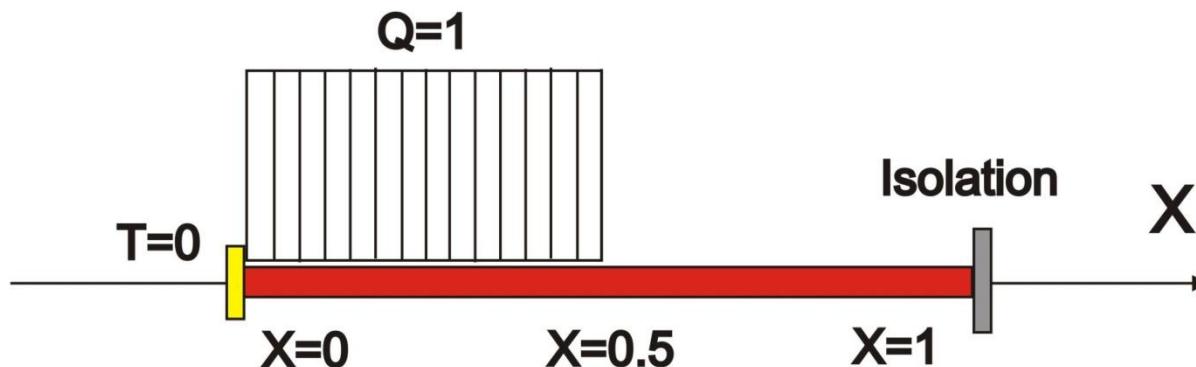
R
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Example 8 : SoDE – 2nd order to 1st order

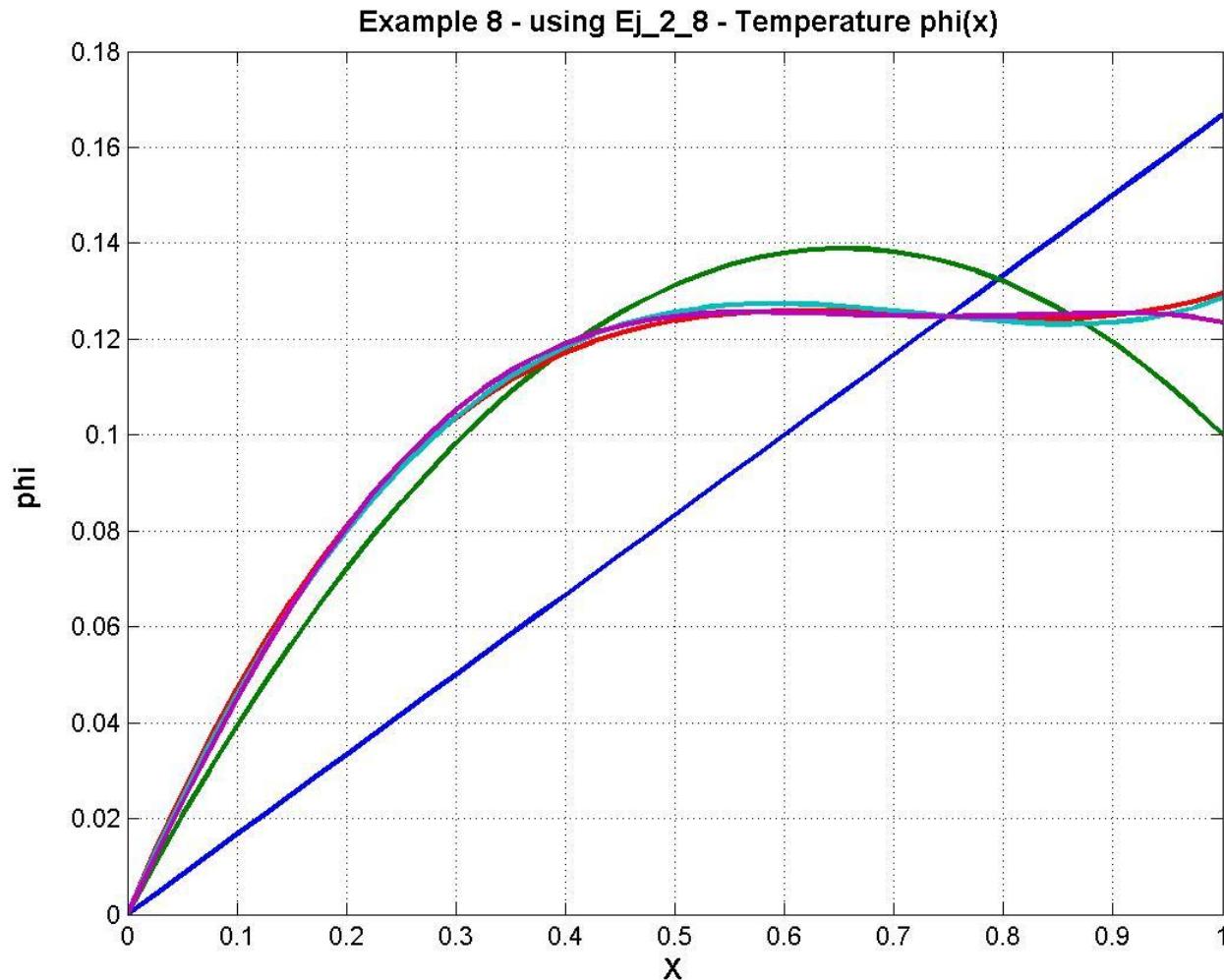
$$\frac{d}{dx} \left(\kappa \frac{d\phi}{dx} \right) - Q = 0$$

$$\Omega : \{x; 0 \leq x \leq 1\} \quad ; \quad \kappa = 1 \quad ; \quad Q = \begin{cases} 1 & x \leq \frac{1}{2} \\ 0 & x > \frac{1}{2} \end{cases}$$

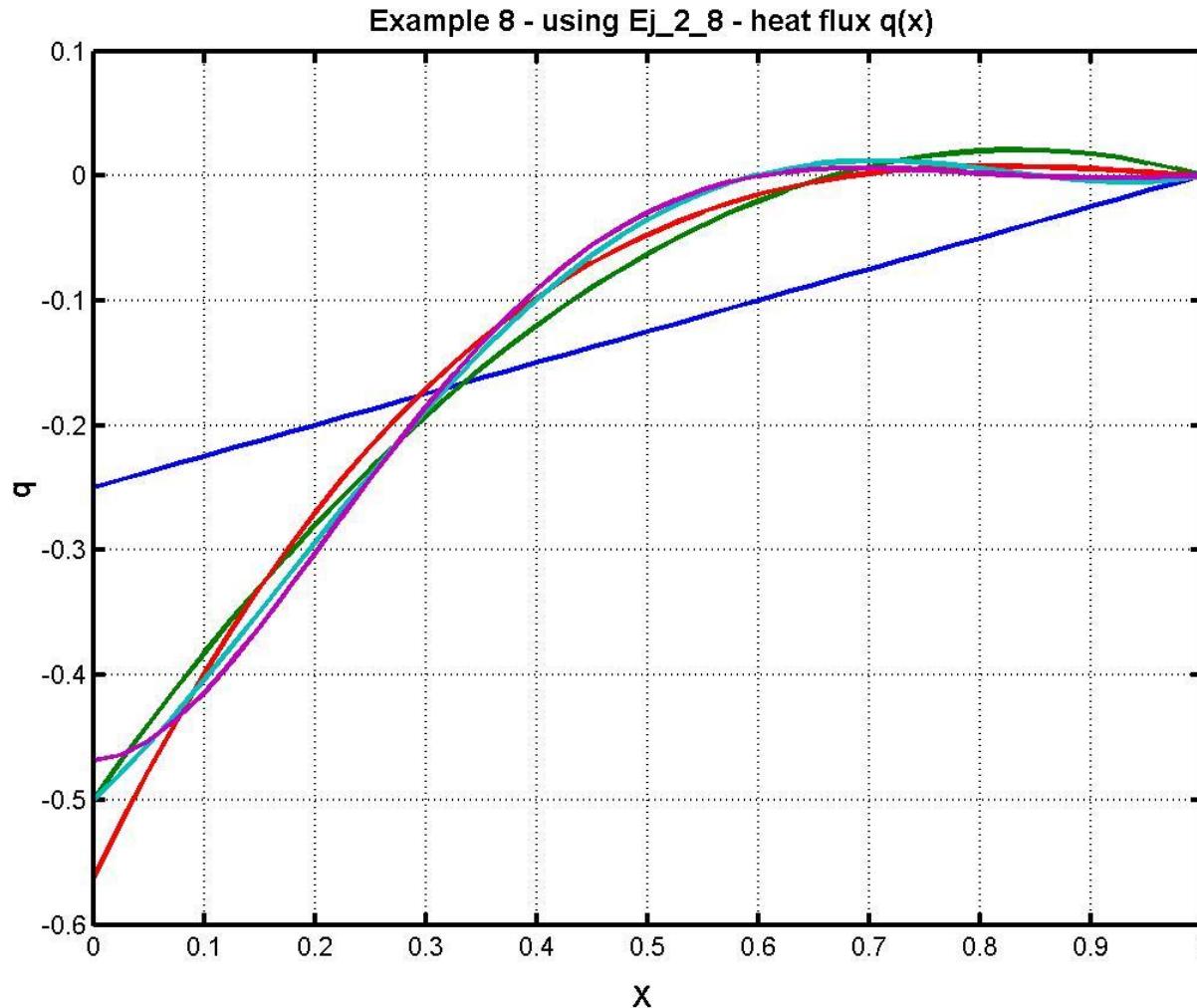
$$\phi = 0 \quad @ x = 0 \quad ; \quad q = -\kappa \frac{d\phi}{dx} = 0 \quad @ x = 1$$



Example 8 : SoDE – 2nd order to 1st order Results – Temperature convergence with M



Example 8 : SoDE – 2nd order to 1st order Results – heat flux convergence with M



Example 9 : SoDE – Linear elasticity in 2D

Plane stress in
2D elasticity

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \mathbf{L} \underline{\phi} \quad ; \quad \mathbf{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \boldsymbol{\varepsilon} = \mathbf{D} \boldsymbol{\varepsilon}$$

$$\underline{A}(\underline{\phi}) = \begin{bmatrix} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + X \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y \end{bmatrix} = \mathbf{L}^\top \mathbf{D} \mathbf{L} \underline{\phi} + \underline{X} = 0 \quad ; \quad \underline{X}^T = (X, Y)$$

$$\underline{B}(\underline{\phi}) = \begin{bmatrix} \sigma_x n_x + \sigma_{xy} n_y - \hat{t}_x \\ \sigma_{xy} n_x + \sigma_y n_y - \hat{t}_y \end{bmatrix} = 0 \quad \text{on } \Gamma_\sigma$$

$$\underline{B}(\underline{\phi}) = \begin{bmatrix} u - \bar{u} \\ v - \bar{v} \end{bmatrix} = 0 \quad \text{on } \Gamma_\phi$$

Example 9 : SoDE – Linear elasticity in 2D

Plane stress in
2D elasticity

$$\underline{\psi}_1 = \bar{u} \quad ; \quad \underline{\psi}_2 = \bar{v} \quad \text{on } \Gamma_\phi$$

$$\hat{\underline{\phi}} = \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = \underline{\psi} + \sum_m \underline{\underline{N}}_m \underline{a}_m \quad ; \quad \underline{\psi}^T = (\psi_1, \psi_2)$$

$$\underline{\underline{W}}_l = \begin{bmatrix} W_{l,1} & \\ & W_{l,2} \end{bmatrix} \quad ; \quad \overline{\underline{W}}_l = \begin{bmatrix} \overline{W}_{l,1} & \\ & \overline{W}_{l,2} \end{bmatrix}$$

$$\int_{\Omega} \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + X \right) W_{l,1} \, d\Omega + \int_{\Gamma_\sigma} (\sigma_x n_x + \sigma_{xy} n_y - \hat{t}_x) \overline{W}_{l,1} \, d\Gamma = 0$$

$$\int_{\Omega} \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y \right) W_{l,2} \, d\Omega + \int_{\Gamma_\sigma} (\sigma_{xy} n_x + \sigma_y n_y - \hat{t}_y) \overline{W}_{l,2} \, d\Gamma = 0$$

$$\underline{\sigma} = \mathbf{D} \mathbf{L} \underline{\phi}$$

Example 9 : SoDE – Linear elasticity in 2D

Gauss - Green or integration by parts

$$-\int_{\Omega} \left(\hat{\sigma}_x \frac{\partial W_{l,1}}{\partial x} + \hat{\sigma}_{xy} \frac{\partial W_{l,1}}{\partial y} - W_{l,1} X \right) d\Omega + \int_{\Gamma_{\sigma} \cup \Gamma_{\phi}} (\hat{\sigma}_x n_x + \hat{\sigma}_{xy} n_y) W_{l,1} d\Gamma$$

$$+ \int_{\Gamma_{\sigma}} (\hat{\sigma}_x n_x + \hat{\sigma}_{xy} n_y - \hat{t}_x) \overline{W_{l,1}} d\Gamma = 0$$

$$-\int_{\Omega} \left(\hat{\sigma}_{xy} \frac{\partial W_{l,2}}{\partial x} + \hat{\sigma}_y \frac{\partial W_{l,2}}{\partial y} - W_{l,2} Y \right) d\Omega + \int_{\Gamma_{\sigma} \cup \Gamma_{\phi}} (\hat{\sigma}_{xy} n_x + \hat{\sigma}_y n_y) W_{l,2} d\Gamma$$

$$+ \int_{\Gamma_{\sigma}} (\hat{\sigma}_{xy} n_x + \hat{\sigma}_y n_y - \hat{t}_y) \overline{W_{l,2}} d\Gamma = 0$$

Example 9 : SoDE – Linear elasticity in 2D

Choosing the weighted functions as

$$W_{l,1} = W_{l,2} = 0 \quad \text{on } \Gamma_\phi$$

$$\overline{W_{l,1}} = -W_{l,1}\Big|_{\Gamma_\sigma}$$

$$\overline{W_{l,2}} = -W_{l,2}\Big|_{\Gamma_\sigma}$$

$$-\int_{\Omega} \left(\hat{\sigma}_x \frac{\partial W_{l,1}}{\partial x} + \hat{\sigma}_{xy} \frac{\partial W_{l,1}}{\partial y} - W_{l,1} X \right) d\Omega + \int_{\Gamma_\sigma} \hat{t}_x W_{l,1} d\Gamma = 0$$

$$-\int_{\Omega} \left(\hat{\sigma}_{xy} \frac{\partial W_{l,2}}{\partial x} + \hat{\sigma}_y \frac{\partial W_{l,2}}{\partial y} - W_{l,2} Y \right) d\Omega + \int_{\Gamma_\sigma} \hat{t}_y W_{l,2} d\Gamma = 0$$

Example 9 : SoDE – Linear elasticity in 2D

Writing in a compact way

$$\underline{\mathcal{L}\hat{\phi}} = \underline{\mathcal{L}\psi} + \sum_m \underline{\mathcal{L}N}_m \underline{a}_m$$
$$\int_{\Omega} \left(\underline{\mathcal{L}W}_l \right)^T \underline{\hat{\sigma}} d\Omega - \int_{\Omega} \underline{W}_l \underline{X} d\Omega - \int_{\Gamma_{\sigma}} \underline{W}_l \underline{\hat{t}} d\Gamma = 0$$

$$\underline{\hat{t}} = (\hat{t}_x, \hat{t}_y)$$

expressing the stress in terms of displacements

$$\int_{\Omega} \left(\underline{\mathcal{L}W}_l \right)^T \mathcal{D} \underline{\hat{\phi}} d\Omega = \int_{\Omega} \underline{W}_l \underline{X} d\Omega + \underbrace{\int_{\Gamma_{\sigma}} \underline{W}_l \underline{\hat{t}} d\Gamma}_{\text{natural boundary condition}}$$

imposing $\underline{\hat{t}}$ is a natural boundary condition.

Example 9 : SoDE – Linear elasticity in 2D

using the approximation

$$\underline{\mathcal{L}}\hat{\underline{\phi}} = \underline{\mathcal{L}}\underline{\psi} + \sum_m \underline{\mathcal{L}}\underline{N}_m \underline{a}_m$$

$$\int_{\Omega} \left(\underline{\mathcal{L}}\underline{W}_l \right)^T \mathcal{D} \left(\underline{\mathcal{L}}\underline{\psi} + \sum_m \underline{\mathcal{L}}\underline{N}_m \underline{a}_m \right) d\Omega = \int_{\Omega} \underline{W}_l \underline{X} d\Omega + \int_{\Gamma_{\sigma}} \underline{W}_l \hat{\underline{t}} d\Gamma$$

$$\sum_m \left(\int_{\Omega} \left(\underline{\mathcal{L}}\underline{W}_l \right)^T \mathcal{D} \underline{\mathcal{L}}\underline{N}_m d\Omega \right) \underline{a}_m = \int_{\Omega} \underline{W}_l \underline{X} d\Omega + \int_{\Gamma_{\sigma}} \underline{W}_l \hat{\underline{t}} d\Gamma - \int_{\Omega} \left(\underline{\mathcal{L}}\underline{W}_l \right)^T \mathcal{D} \underline{\mathcal{L}}\underline{\psi} d\Omega$$

Example 9 : SoDE – Linear elasticity in 2D

linear system

$$\underline{\underline{K}} \underline{\underline{a}} = \underline{\underline{f}}$$

$$K_{lm} = \int_{\Omega} \left(\underline{\underline{\mathcal{L}W}}_l \right)^T \mathcal{D} \underline{\underline{\mathcal{LN}}}_m d\Omega$$

$$f_l = \int_{\Omega} \underline{\underline{W}}_l \underline{\underline{X}} d\Omega + \int_{\Gamma_\sigma} \underline{\underline{W}}_l \hat{\underline{\underline{t}}} d\Gamma - \int_{\Omega} \left(\underline{\underline{\mathcal{L}W}}_l \right)^T \mathcal{D} \underline{\underline{\mathcal{L}\psi}} d\Omega$$

Example 9 : SoDE – Linear elasticity in 2D

Remark :

$$\int_{\Omega} \left(\underline{\underline{L}} \underline{\underline{W}}_l \right)^T \hat{\sigma} d\Omega - \int_{\Omega} \underline{\underline{W}}_l \underline{\underline{X}} d\Omega - \int_{\Gamma_{\sigma}} \underline{\underline{W}}_l \hat{\underline{t}} d\Gamma = 0$$

is equivalent to the principle of virtual work

$$\int_{\Omega} \left(\underline{\underline{\varepsilon}}^* \right)^T \underline{\sigma} d\Omega = \int_{\Omega} \underline{\underline{\phi}}^{*T} \underline{\underline{X}} d\Omega + \int_{\Gamma_{\sigma}} \underline{\underline{\phi}}^{*T} \underline{t} d\Gamma$$

using as virtual displacement $\underline{\underline{\phi}}^*$ the following definition

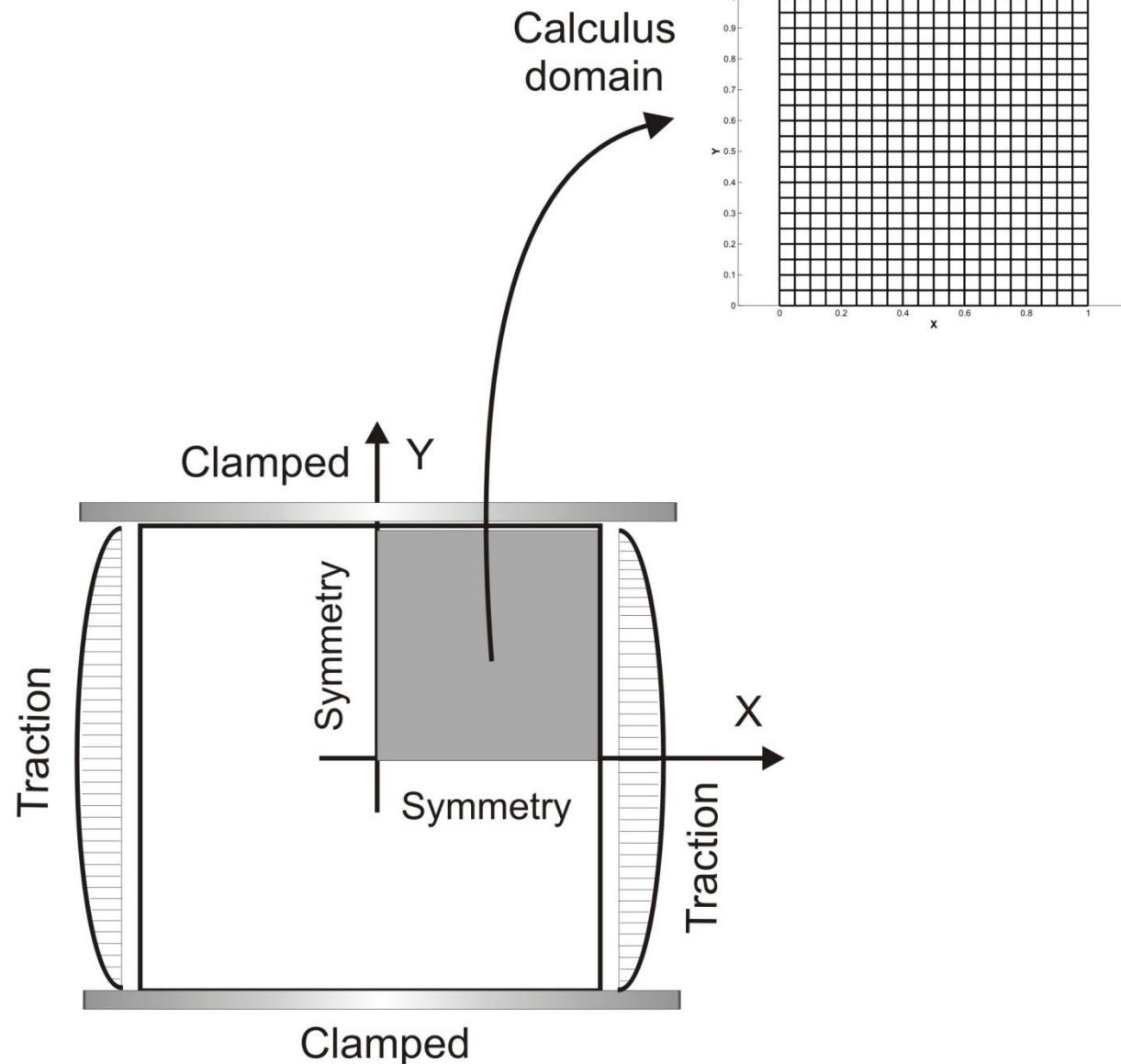
$$\underline{\underline{\phi}}^* = \underline{\underline{W}}_l \underline{\delta}_l^* \quad \text{with } \underline{\delta}_l^* \text{ arbitrary and} \quad \underline{\underline{\varepsilon}}^* = \underline{\underline{L}} \underline{\underline{\phi}}^*$$

Example 9 :

Linear elasticity

Problem definition

(see Ej_2_9.m
routine)



Example 9 : trial functions

```
Nxs =strvcat('(x)', '(x.^3)', '(x.*y.^2)');
dNxsdx =strvcat('1', '(3*x.^2)', '(y.^2)');
dNxsdy =strvcat('0', '0', '(2*y.*x)');
```

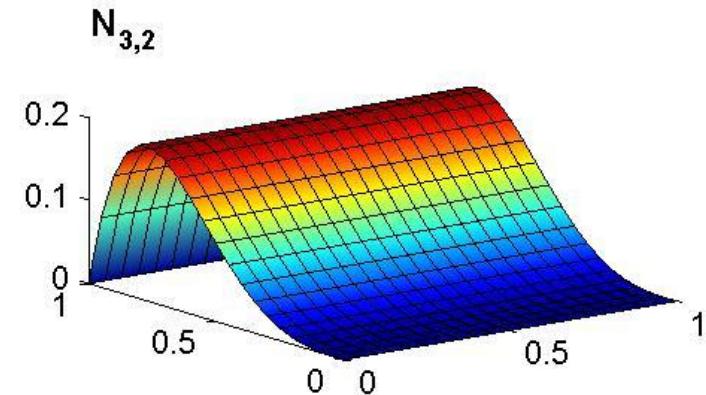
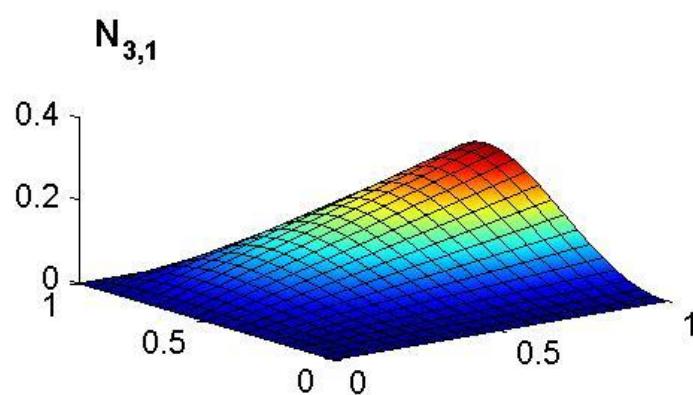
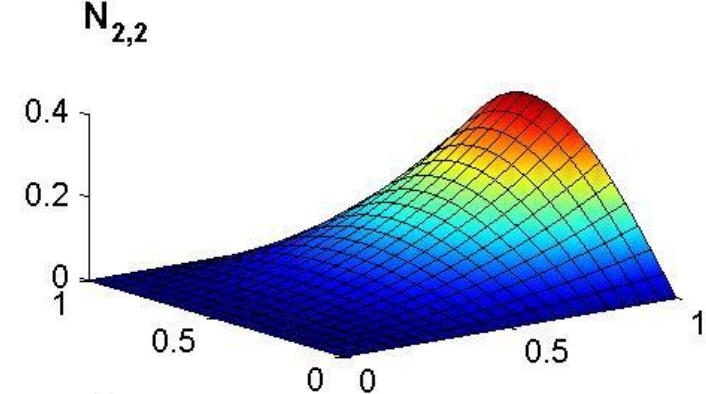
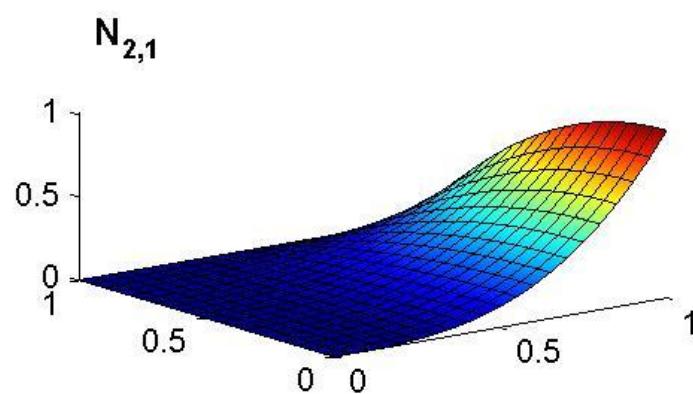
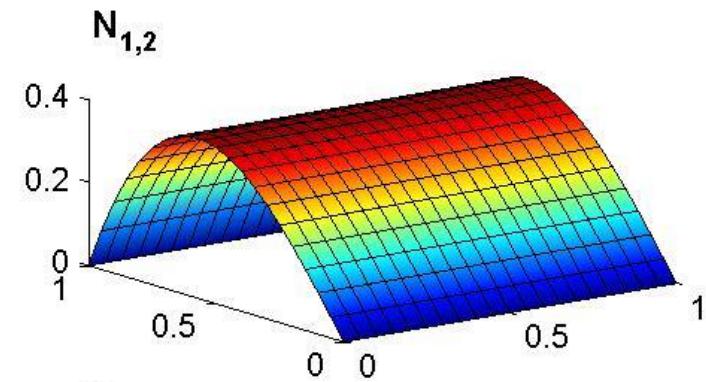
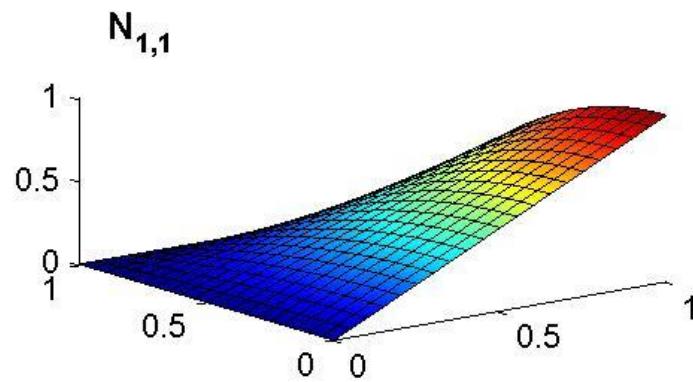
```
Nys =strvcat('(y)', '(y.*x.^2)', '(y.^3)');
dNysdx =strvcat('0', '(2*y.*x)', '0');
dNysdy =strvcat('1', '(x.^2)', '(3*y.^2)');
```

```
Nx = Nxs(mm,:);
dNx dx = dNxsdx(mm,:);
dNx dy = dNxsdy(mm,:);
```

```
Ny = Nys(mm,:);
dNy dx = dNysdx(mm,:);
dNy dy = dNysdy(mm,:);
```

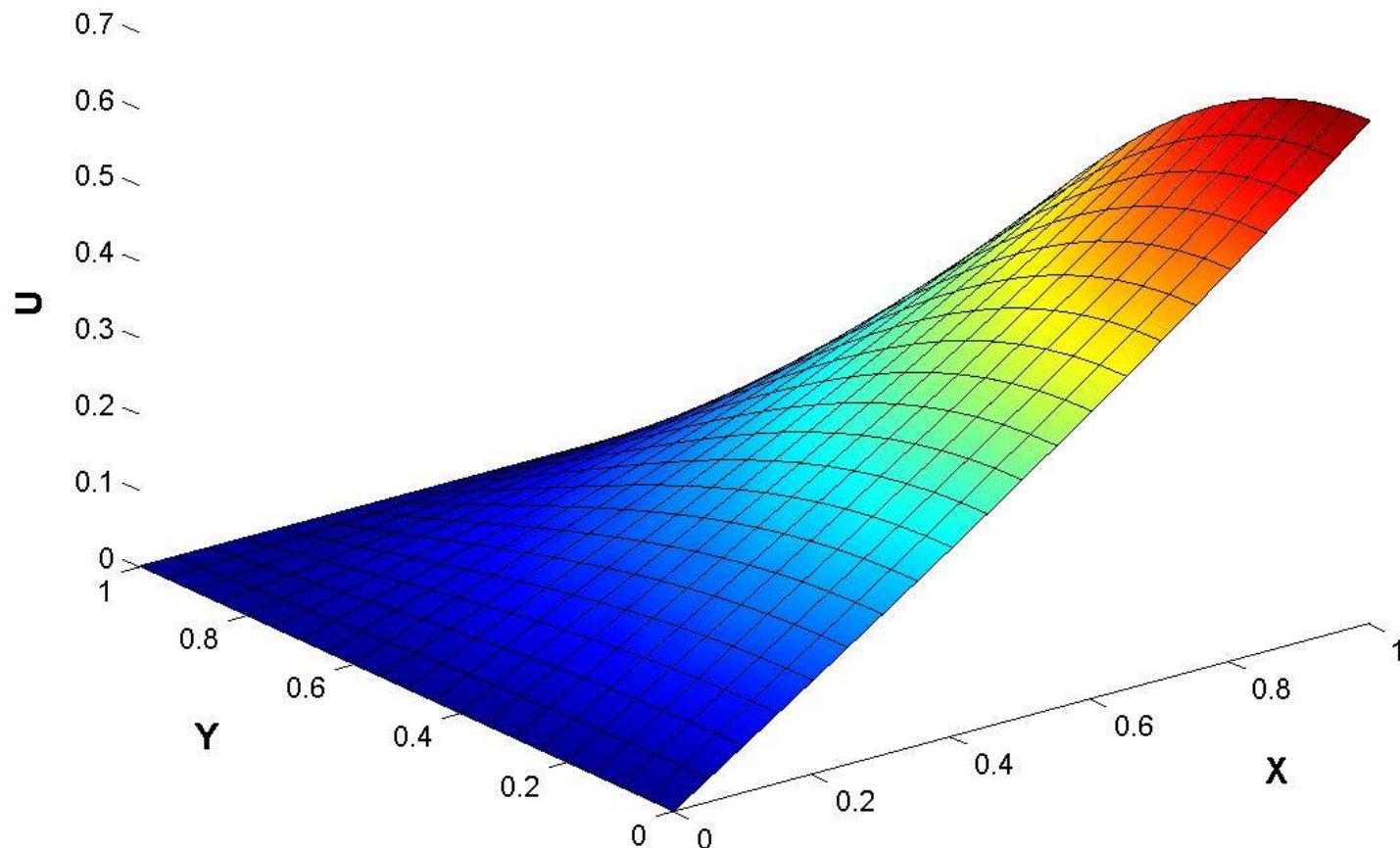
```
eval(['N1 = (1-y.^2).*' Nx ';'']);
eval(['N2 = (1-y.^2).*' Ny ';'']);
eval(['Lx_N1 = (1-y.^2).*' dNx dx ';'']);
eval(['Ly_N1 = (1-y.^2).*' dNx dy '-' Nx .* (2*y)' ';'']);
eval(['Lx_N2 = (1-y.^2).*' dNy dx ';'']);
eval(['Ly_N2 = (1-y.^2).*' dNy dy '-' Ny .* (2*y)' ';'']);
```

Example 9 : trial functions



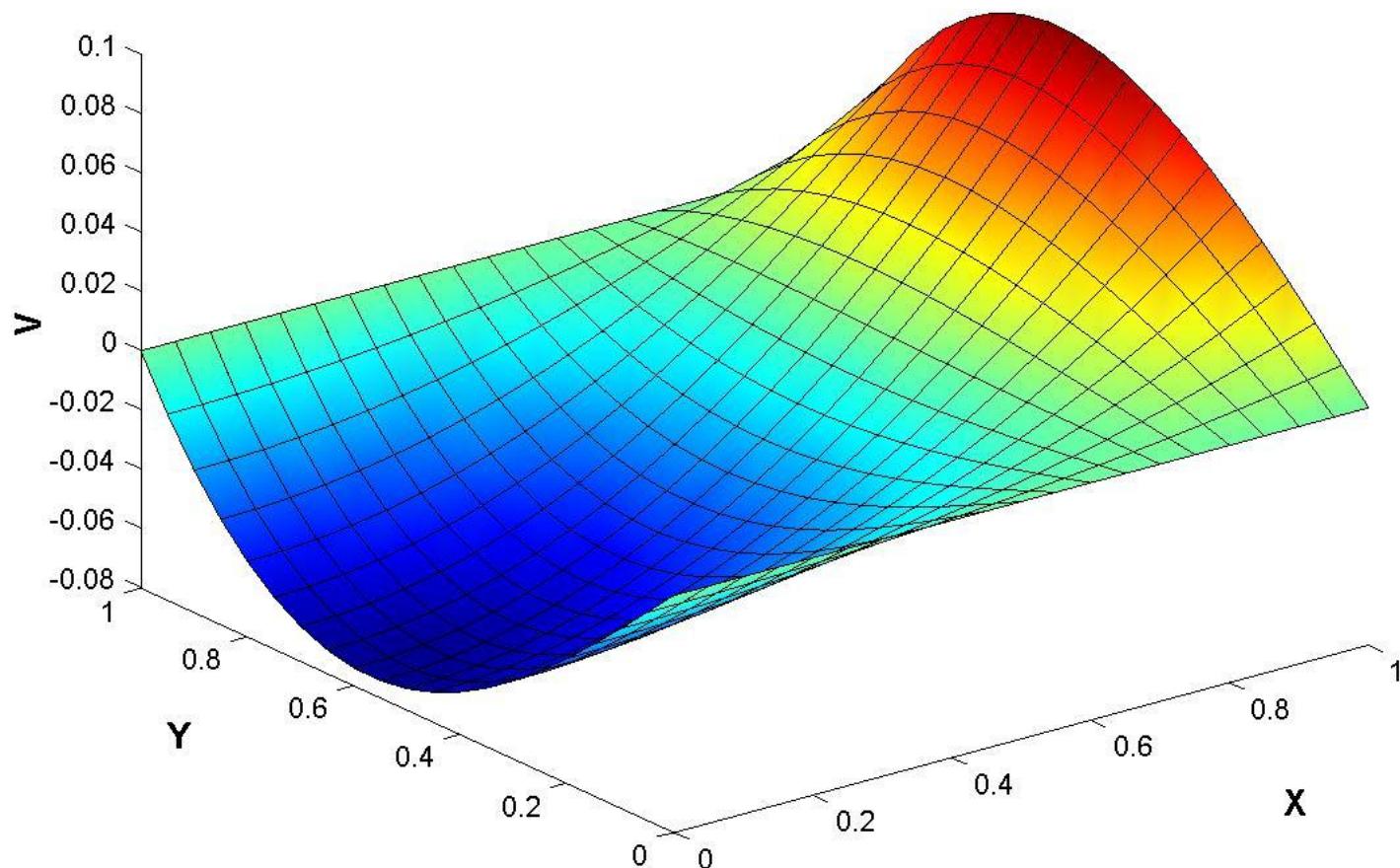
Example 9 : Linear elasticity – Results x-displacement

Example 9 - using Ej_2_9 - $\max(u) = 0.65749$



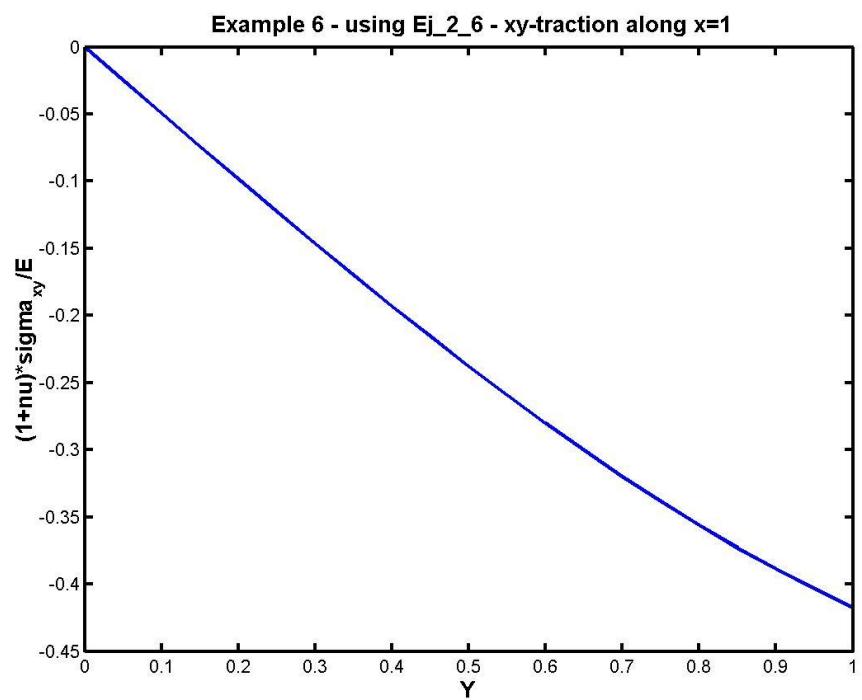
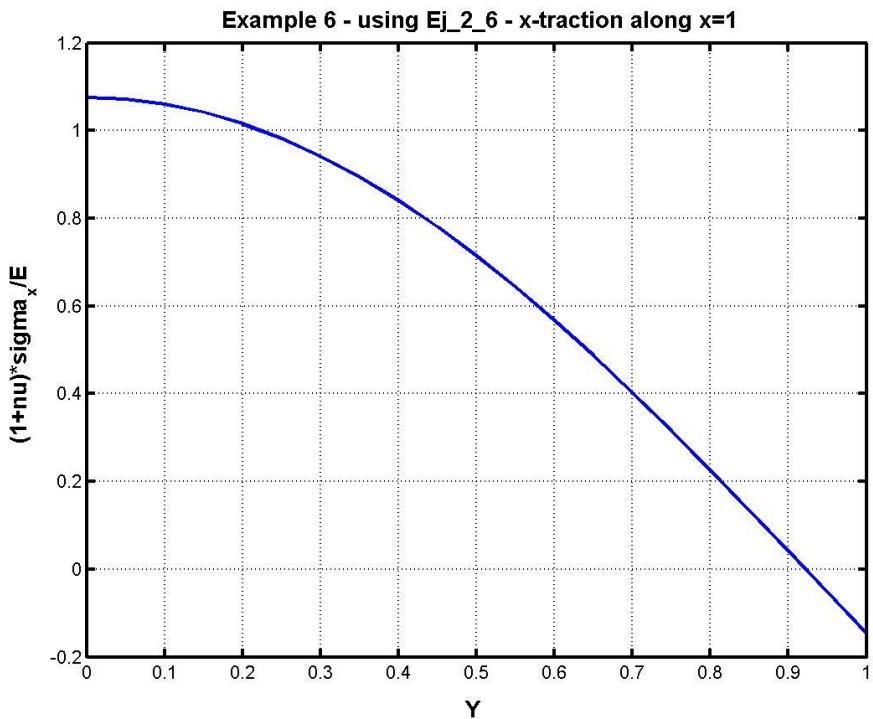
Example 9 : Linear elasticity – Results y-displacement

Example 9 - using Ej_2_9 - $\max(v) = 0.080562$



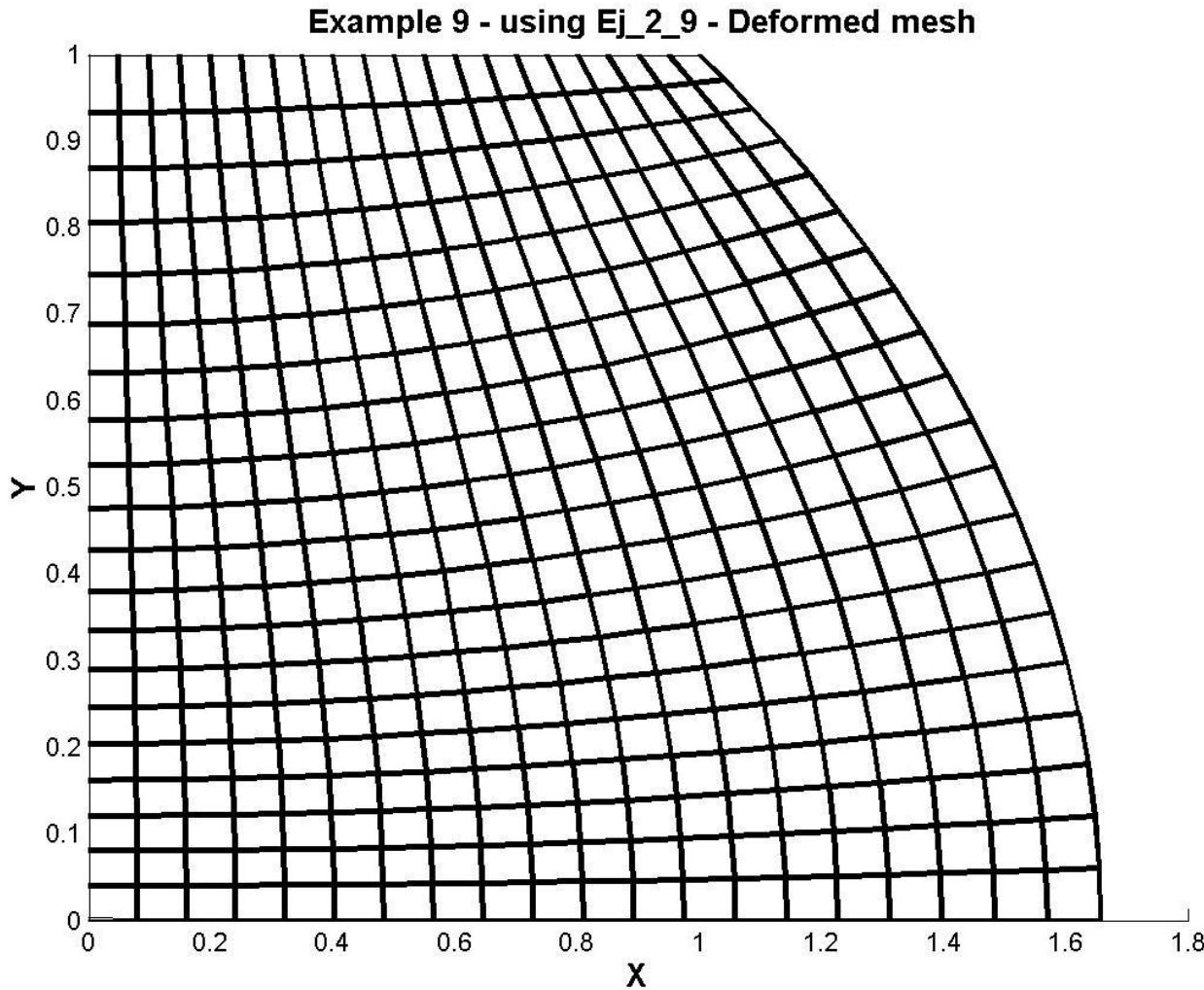
Example 9 : Linear elasticity – Results

Traction at boundary ($x=1$)



Example 9 : Linear elasticity – Results

Deformed mesh



Non linear problems

Non linear problems $\Rightarrow K(a) a = f$

Example : Find $\hat{\phi}(x)$ solution of the following PDE

$$\frac{\partial}{\partial x} \left(\kappa(\phi) \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\kappa(\phi) \frac{\partial \phi}{\partial y} \right) + Q = 0 \quad \text{in } \Omega : \{x; x \in \mathfrak{R}^2\}$$

$$\phi = \bar{\phi} \quad \text{on } \Gamma_\phi \quad ; \quad \kappa \frac{\partial \phi}{\partial \eta} = -\bar{q} \quad \text{on } \Gamma_q$$

$$\hat{\phi} = \psi + \sum_m a_m N_m$$

$$\psi = \bar{\phi} \quad \text{on } \Gamma_\phi \quad ; \quad N_m = 0 \quad \text{on } \Gamma_\phi$$

$$\int_{\Omega} W_l \left(\frac{\partial}{\partial x} \left(\kappa(\hat{\phi}) \frac{\partial \hat{\phi}}{\partial x} \right) + \frac{\partial}{\partial y} \left(\kappa(\hat{\phi}) \frac{\partial \hat{\phi}}{\partial y} \right) + Q \right) d\Omega + \int_{\Gamma_q} \overline{W}_l \left(\kappa(\hat{\phi}) \frac{\partial \hat{\phi}}{\partial \eta} + \bar{q} \right) d\Gamma = 0$$

Non linear problems

$$\int_{\Omega} W_l \left(\frac{\partial}{\partial x} \left(\kappa(\hat{\phi}) \frac{\partial \hat{\phi}}{\partial x} \right) + \frac{\partial}{\partial y} \left(\kappa(\hat{\phi}) \frac{\partial \hat{\phi}}{\partial y} \right) + Q \right) d\Omega + \int_{\Gamma_q} \overline{W}_l \left(\kappa(\hat{\phi}) \frac{\partial \hat{\phi}}{\partial \eta} + \bar{q} \right) d\Gamma = 0$$

integration by parts(Gauss - Green)

$$\begin{aligned} & \int_{\Omega} \left(\frac{\partial W_l}{\partial x} \left(\kappa(\hat{\phi}) \frac{\partial \hat{\phi}}{\partial x} \right) + \frac{\partial W_l}{\partial y} \left(\kappa(\hat{\phi}) \frac{\partial \hat{\phi}}{\partial y} \right) - W_l Q \right) d\Omega - \\ & - \int_{\Gamma_q \cup \Gamma_\phi} W_l \left(\kappa(\hat{\phi}) \frac{\partial \hat{\phi}}{\partial \eta} \right) d\Gamma - \int_{\Gamma_q} \overline{W}_l \left(\kappa(\hat{\phi}) \frac{\partial \hat{\phi}}{\partial \eta} + \bar{q} \right) d\Gamma = 0 \end{aligned}$$

$$\overline{W}_l = -W_l \Big|_{\Gamma_q} \quad ; \quad W_l \Big|_{\Gamma_\phi} = 0$$

$$\int_{\Omega} \left(\frac{\partial W_l}{\partial x} \left(\kappa(\hat{\phi}) \frac{\partial \hat{\phi}}{\partial x} \right) + \frac{\partial W_l}{\partial y} \left(\kappa(\hat{\phi}) \frac{\partial \hat{\phi}}{\partial y} \right) - W_l Q \right) d\Omega + \int_{\Gamma_q} W_l \bar{q} \, d\Gamma = 0$$

Non linear problems

$$K_{lm}(\underline{a}^0) = \int_{\Omega} \left(\frac{\partial W_l}{\partial \mathbf{x}} \left(\kappa(\hat{\phi}^0) \frac{\partial N_m}{\partial \mathbf{x}} \right) + \frac{\partial W_l}{\partial \mathbf{y}} \left(\kappa(\hat{\phi}^0) \frac{\partial N_m}{\partial \mathbf{y}} \right) \right) d\Omega$$

$$f_l(\underline{a}^0) = \int_{\Omega} W_l Q \, d\Omega - \int_{\Gamma_q} W_l \bar{q} \, d\Gamma -$$

$$- \int_{\Omega} \left(\frac{\partial W_l}{\partial \mathbf{x}} \left(\kappa(\hat{\phi}^0) \frac{\partial \psi}{\partial \mathbf{x}} \right) + \frac{\partial W_l}{\partial \mathbf{y}} \left(\kappa(\hat{\phi}^0) \frac{\partial \psi}{\partial \mathbf{y}} \right) \right) d\Omega +$$

In general, the iterations follow as :

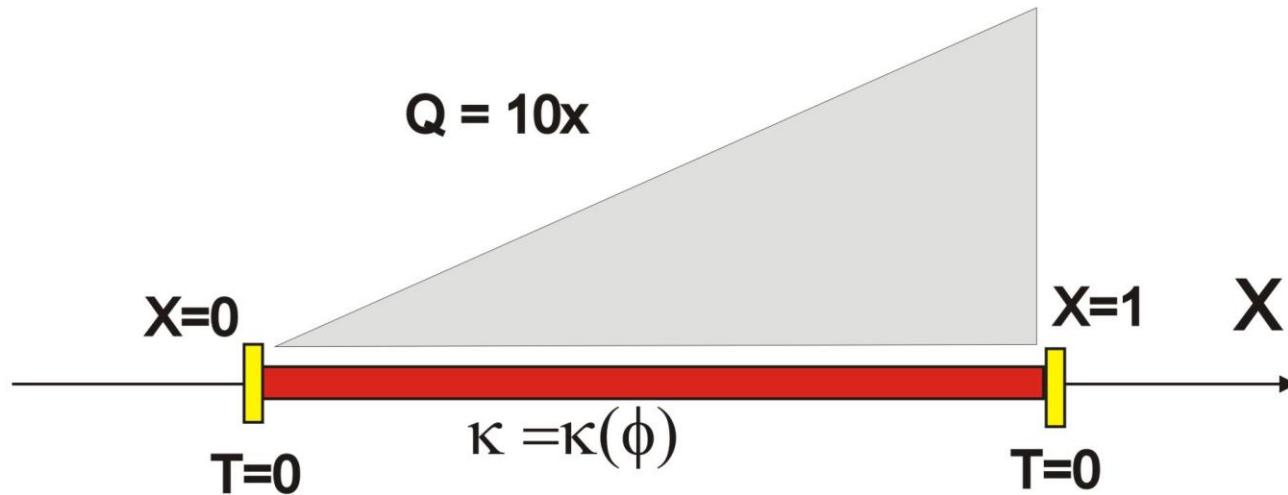
$$\mathbf{K}(\mathbf{a}^{n-1})\mathbf{a}^n = \mathbf{f}^{n-1}$$

Example 10 : Non linear heat equation

$$\frac{d}{dx} \left(\kappa \frac{d\phi}{dx} \right) + Q = 0$$

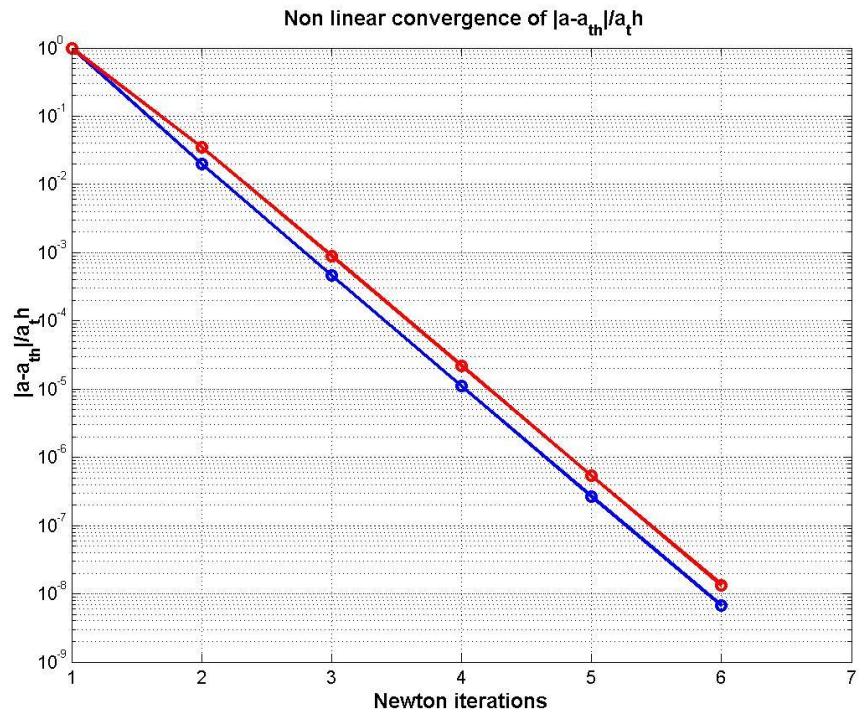
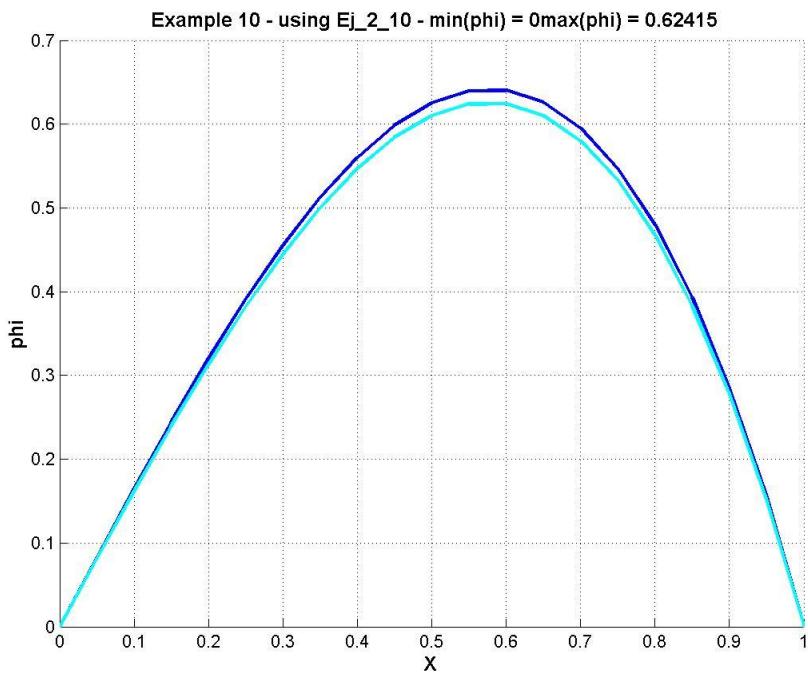
$$\Omega : \{x; 0 \leq x \leq 1\} \quad ; \quad \kappa = 1 + 0.1 \phi \quad ; \quad Q = 10x$$

$$\phi = 0 \quad @ x = 0 \quad ; \quad \phi = 0 \quad @ x = 1$$



Example 10 : Non linear convergence of solution

M=2 – convergence of coefficients **a** with Galerkin
using Ej_2_10.m routine



fin

